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Preliminary Results on Positizable Systems

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Abstract: Great achievements have been made in the field of positive systems in the past two decades and many elegant properties of positive systems have been revealed. Among all the nice properties, stability of delayed positive systems is of great interest: It is not affected by the magnitude of delays. Though positive systems have very nice properties, these properties cannot be applied widely since positive systems are just a little portion among all dynamic systems. This paper tries to consider a more broad class of systems-positizable systems whose stability is also not affected by the size of delays. Positizable systems means a class of systems which are non-positive themselves but can be transformed into positive systems by means of an invertible linear transformation. Generally speaking, both judging if a system is positizable and realizing positization are very difficult. This paper first discusses these two questions and focuses on some preliminary cases. Some necessary and sufficient conditions for judging if a system is diagonally positizable are established, several cases are also proposed where systems are not positizable.

Key Words: Positive Systems, Positizable Systems, Sign-Preserving Systems, Stability.

1 Introduction

During the past two decades, it has been witnessed that great achievements were made in the field of positive systems [1–3]. By positive systems, we means such class of systems whose states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative. The states of positive systems are confined within a “cone” located in the positive orthant rather than in the whole space $\mathbb{R}^n$. This feature makes the analysis and synthesis of positive systems a challenging and interesting job [4, 5]. Positive systems have found applications in mathematics, ecosystems, mechanical system, and chemical systems, see references for details [6–10].

Positive systems possess many important and elegant properties. For example, they have the property of preserving signs, which means that if the initial condition is nonnegative, then the trajectory will always keep nonnegative and never cross the axes [2]; for a stable positive systems with constant delays, the trajectory will monotonically decrease [11] if the initial condition is properly chosen. Among all the properties, stability of positive systems with time-varying delays is most attractive: The stability is totally determined by system matrices and has nothing to do with the size of delays [12–15]. More surprisingly, if a positive system is asymptotically stable for certain constant delay, then it is exponentially stable for all bounded time-varying delays [16]. Meanwhile, the methods of stability analysis are quite different from those used to analyze general dynamic systems. For example, copositive Lyapunov function or functional methods are frequently employed in this field and have been proven to be powerful tools [17–20].

It should be pointed out, in spite of nice properties of positive systems, these properties have not been applied widely, since positive systems are just a special class of dynamic systems and it seems unreasonable to apply them to systems which are non-positive. However, it is the seemingly unreasonable idea that motivates the current work.

There might exist a class of systems which are non-positive systems themselves but can be changed into positive systems by means of an invertible linear transformation. Such systems are called positizable systems. It is widely known that invertible linear transformations do not change the stability property of a dynamic system. So, if we can identify positizable systems, then can apply the results on positive systems to handle their stability. In this paper, we will try to determine a class of positizable systems.

Moreover, other sign-preserving systems rather than positive systems are also of great interest. Since if a system has been identified to be a sign-preserving system, then it is always possible to use copositive Lyapunov functions or functionals to deal with its stability issue. Clearly, this problem is attractive.

Technically, the key work in this framework is not how to apply theory of positive systems to positizable systems, but how to identify if a system is positizable, which is a hard work in general case. So, this paper will first deal with the problem and start from some simple cases.

The main contribution of this paper lies in following aspects. First, we explore the background for studying positizable systems and completely formulate the framework. Second, a necessary and sufficient condition is provided, which determines whether or not a system is positizable by means of diagonal matrix. Note that if a system can be positized by a diagonal matrix, then it is a sign-preserving system. A necessary and sufficient exponential stability criterion is proposed for such class systems with time-varying delays. Finally, some cases are listed where the matrices are not positizable.

The remainder of this paper is organized as follows. In Section 2, notation, necessary preliminaries, and problems to be addressed are presented. Subsection 3 treats the class of systems which is diagonally positizable. Section 4 deals with stability of positizable systems, Section 5 discusses the case where matrices are not positizable, Section 6 points out some important work in the future, and Section 7 concludes this paper.

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2 Preliminaries and Problem Statements

Nomenclatures

$A \succeq 0 (\preceq 0)$:	All elements of matrix A are nonnegative (nonpositive)
$A \succ 0 (\prec 0)$:	All elements of matrix A are positive (negative)
$A^{\mathrm{T}}(A^{-1})$:	The transpose (inverse) of matrix A
I:	Unit matrix of appropriate dimension
$\det(A)$:	The determinant of matrix A
$\mathbb{R}^n(\mathbb{R}^n_+)$:	The sef of n-dimensional real (positive) vector
$\mathbb{R}^{n\times n}$:	The set of all real matrices of $n \times n$ dimension
$\mathrm{diag}\,(a_1, \cdots, a_n)$:	A diagonal matrix with diagonal elements $a_1, \cdots, a_n$
$\mathbb{N}$:	$\{1, 2, 3, \cdots\}$
$\mathbb{N}_0$:	$\{0\} \bigcup \mathbb{N}$
$\underline{m}$:	$\{1, 2, \cdots, m\}$ with m being an arbitrary positive integer
$\|\boldsymbol{x}\|$:	Any norm of vector $\boldsymbol{x}$

Throughout this paper, the dimensions of matrices and vectors will not be explicitly mentioned if clear from context.

The following definitions will be used repeatedly.

Definition 1. *A real matrix is said to be nonnegative if all its elements are nonnegative, to be a Metzler matrix if all its off-diagonal elements are nonnegative.*

Definition 2 (Positization of matrix)**.** *Fix a matrix A. If there exists an invertible matrix P such that $PAP^{-1} \succeq 0$, then we say that matrix A is a positizable and the matrix P is a positizing matrix of A. If P is a diagonal or triangular matrix, then we say that A is diagonally or triangularly positizable, respectively.*

Matrix A is Metzlerizable if there exists a matrix P such that PAP^{-1} is a Metzler matrix. Diagonal and triangular Metzlerization can be defined analogously.

A set of matrices $A_1, A_2, \ldots, A_m$ is said to be simultaneously positizable (Metzlerizable) if there exists a common matrix P such that $PA_iP^{-1} \succeq 0$ (PA_iP^{-1} is Metzler matrix) holds for any $i \in \underline{m}$.

Remark 1. Clearly, any nonnegative matrix is positizable and Metzlerizable, and a set of nonnegative matrices is simultaneously positizable and Metzlerizable. A matrix A is always Metzlerizable since it is similar to its Jordan standard form.

For a given $A \in \mathbb{R}^{n\times n}$, denote

$$\mathcal{P}\,(A) = \left\{P : PAP^{-1} \succeq 0, P \in \mathbb{R}^{n\times n}\right\}$$

the set of all positizing matrices of A and

$$\mathcal{M}\,(A) = \left\{P : PAP^{-1} \text{ is Metzler matrix}, P \in \mathbb{R}^{n\times n}\right\}$$

the set of all Metzlerizing matrices of A.

Consider the following two systems:

$$\begin{aligned} \boldsymbol{x}(k+1) =& A\boldsymbol{x}(k) + B\boldsymbol{x}(k-\tau(k)), \quad k \in \mathbb{N}_0 \\ \boldsymbol{x}(k) =& \boldsymbol{\varphi}(k), k \in \{-\tau, \ldots, 0\} \end{aligned} \tag{1}$$

and

$$\begin{aligned} \dot{\boldsymbol{x}}(t) =& A\boldsymbol{x}(t) + B\boldsymbol{x}(t-\tau(t)), \quad t \geq 0 \\ \boldsymbol{x}(t) =& \boldsymbol{\phi}(t), t \in [-\tau, 0] \end{aligned} \tag{2}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is state variable, A and B are system matrices, $\tau(k) \geq 0$ and $\tau(t) \geq 0$ are delays with the maximum τ, $\boldsymbol{\varphi}(k)$ and $\boldsymbol{\phi}(t)$ are initial conditions.

Definition 3. *System* (1) *is positizable, if there exists a matrix P such that $P \in \mathcal{P}\,(A), P \in \mathcal{P}\,(B)$. System* (2) *is positizable, if there exists a matrix P such that $P \in \mathcal{M}\,(A)$ and $P \in \mathcal{P}\,(B)$. Systems $\boldsymbol{x}(k+1) = A\boldsymbol{x}(k)$ is positizable if $\mathcal{P}\,(A) \neq \emptyset$, and by Remark 1, $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t)$ is always positizable.*

Problems: In this paper, we will first explore the conditions under which systems (1) and (2) are positizable, and then analyze stability of positizable systems (1) and (2).

3 Diagonal Positization and Positizable Systems

This section focuses on positizability of matrix by means of diagonal matrix. First extend the concept of sign pattern of matrix in [21] in the following definition.

Definition 4 (Sign pattern of matrix)**.** *The sign pattern, or sign matrix of matrix A is $\mathbf{S}(A) = [s_{ij}]$, where $s_{ij} = \mathrm{sgn}\,(a_{ij})$ and*

$$\mathrm{sgn}\,(a_{ij}) = \begin{cases} +, & a_{ij} > 0 \\ -, & a_{ij} < 0 \\ 0, & a_{ij} = 0 \end{cases}$$

Hence, for a given matrix, its corresponding sign matrix is unique. The sign pattern of a vector can be defined similarly.

Definition 5. *Signs "$+$" and "$-$" are incompatible with each other, "$+$" is compatible with "$+$" and "0", and "$-$" is compatible with "$-$" and "0". "$+$", "$-$", and "0" are strictly compatible with "$+$", "$-$", and "0", respectively.*

A is (strictly) compatible with B in sign (for simplicity, A is (strictly) compatible with B) if the sign matrices of A and B are entrywise (strictly) compatible with each other.

A matrix $A \in \mathbb{R}^{n\times n}$ is of (strictly) symmetric sign pattern if s_{ij} is (strictly) compatible with s_{ji} for any $1 \leq i < j \leq n$, where $s_{ij} = \mathrm{sgn}\,(a_{ij})$.

Remark 2. In Definitions 4 and 5, $+, -, 0$ can be seen as $1, -1, 0$ anywhere in this paper, respectively.

Note that if $P \in \mathcal{P}\,(A)$, then for $\alpha \neq 0$, it holds that $(\alpha P)\,A\,(\alpha P)^{-1} = PAP^{-1} \succeq 0$. Hence, we have the following proposition:

Proposition 1. *If $P \in \mathcal{P}\,(A)$, then $\alpha P \in \mathcal{P}\,(A)$ holds for any nonzero scalar α.*

The following fact clearly holds.

Proposition 2. *A matrix A containing negative diagonal elements can never be positized by any diagonal matrix.*

Definition 6. *Denote*

$$\mathcal{E} = \{\boldsymbol{e} : \boldsymbol{e} \in \mathbb{R}^n, \textit{ all the elements of } \boldsymbol{e} \textit{ are } 1 \textit{ or } -1\}$$

Clearly, for any $\boldsymbol{e} \in \mathcal{E}$, the sign pattern of the matrix $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ contains only $+$ and $-$. For fixed $\boldsymbol{e} \in \mathcal{E}$, define $E(\boldsymbol{e}) = \mathrm{diag}\,(e_1, e_2, \ldots, e_n)$ with e_i being the ith element of $\boldsymbol{e}$.

Theorem 1. *Matrix $A \in \mathbb{R}^{n\times n}$ is diagonally positizable if and only if there exists a matrix $E(\boldsymbol{e}) \in \mathcal{P}(A)$ with $\boldsymbol{e} \in \mathcal{E}$.*

Proof. The sufficiency is straightforward. It is enough to show the necessity.

Suppose that there exists a positizing matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$ of A, where $d_i \neq 0, \forall i \in \underline{n}$. Then

$$D^{-1} = \text{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right)$$

and

$$DAD^{-1} = \begin{bmatrix} d_1 a_{11}\frac{1}{d_1} & \ldots & d_1 a_{1n}\frac{1}{d_n} \\ \vdots & \ddots & \vdots \\ d_n a_{n1}\frac{1}{d_1} & \ldots & d_n a_{nn}\frac{1}{d_n} \end{bmatrix} \succeq 0$$

that is,

$$d_i a_{ij}\frac{1}{d_j} \geq 0, \quad \forall i, j \in \underline{n}$$

which is equivalent to

$$\text{sgn}(d_i)\,|d_i|\,a_{ij}\,\text{sgn}(d_j)\left|\frac{1}{d_j}\right| \geq 0, \quad \forall i, j \in \underline{n}$$

or

$$\text{sgn}(d_i)a_{ij}\,\text{sgn}(d_j) \geq 0, \quad \forall i, j \in \underline{n} \tag{3}$$

Let $\boldsymbol{e} = [\text{sgn}(d_1), \text{sgn}(d_2), \ldots, \text{sgn}(d_n)]^{\mathrm{T}} \in \mathcal{E}$. Note that $E(\boldsymbol{e}) = (E(\boldsymbol{e}))^{-1}$, and (3) means that

$$E(\boldsymbol{e})A\,(E(\boldsymbol{e}))^{-1} \succeq 0$$

The proof is completed. □

Theorem 2. *Matrix $A \in \mathbb{R}^{n\times n}$ is diagonally positizable if and only if there exists a matrix $E(\boldsymbol{e})$ with $\boldsymbol{e} \in \mathcal{E}$ such that A is compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$.*

Proof. By Theorem 1, A is diagonally positizable if and only if there exists a positizable matrix $E(\boldsymbol{e})$ with $\boldsymbol{e} \in \mathcal{E}$. Hence, we only need to show that $E(\boldsymbol{e})AE(\boldsymbol{e}) \succeq 0$ if and if A is compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$.

Let $E(\boldsymbol{e}) = \text{diag}(e_1, e_2, \ldots, e_n)$. Then,

$$E(\boldsymbol{e})AE(\boldsymbol{e}) = \begin{bmatrix} e_1 a_{11} e_1 & \ldots & e_1 a_{1n} e_n \\ \vdots & \ddots & \vdots \\ e_n a_{n1} e_1 & \ldots & e_n a_{nn} e_n \end{bmatrix} \succeq 0$$

that is,

$$e_i a_{ij} e_j = (e_i e_j)\,a_{ij} \geq 0, \quad \forall i, j \in \underline{n} \tag{4}$$

Note that $e_i e_j$ is the element of $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ located at the ith row and jth column.

If A is compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$, then for any $i, j \in \underline{n}$, there are the following two possible cases:

$$\begin{cases} e_i e_j > 0, & a_{ij} \geq 0 \\ e_i e_j < 0, & a_{ij} \leq 0 \end{cases}$$

for any case, (4) always holds. On the contrary, if A is not compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$, then there exists at least a pair (i, j) such that one of the following two cases holds:

$$\begin{cases} e_i e_j > 0, & a_{ij} < 0 \\ e_i e_j < 0, & a_{ij} > 0 \end{cases}$$

for each case, (4) does not hold for (i, j), hence A is not diagonally positizable. □

For $\boldsymbol{e} \in \mathcal{E}$, define

$$\boldsymbol{e}\boldsymbol{e}_*^{\mathrm{T}} = \begin{bmatrix} \Delta & e_1 e_2 & \ldots & e_1 e_n \\ e_2 e_1 & \Delta & \ldots & e_2 e_n \\ \vdots & \vdots & \ddots & \vdots \\ e_n e_1 & \ldots & e_n e_{n-1} & \Delta \end{bmatrix}$$

where the symbol Δ on the diagonal of $\boldsymbol{e}\boldsymbol{e}_*^{\mathrm{T}}$ means the corresponding element has no restriction on its sign.

Parallel to Theorems 1 and 2, we have following results:

Corollary 1. *Matrix $A \in \mathbb{R}^{n\times n}$ is diagonally Metzlerizable if and only if there exists a matrix $E(\boldsymbol{e}) \in \mathcal{M}(A)$ with $\boldsymbol{e} \in \mathcal{E}$.*

Corollary 2. *Matrix $A \in \mathbb{R}^{n\times n}$ is diagonally Metzlerizable if and only if there exists a matrix $E(\boldsymbol{e})$ with $\boldsymbol{e} \in \mathcal{E}$ such that A is compatible with $\boldsymbol{e}\boldsymbol{e}_*^{\mathrm{T}}$.*

Suppose that systems (1) and (2) are not positive systems, that is, in (1), either A or B is not nonnegative matrix, and in (2), either A is not a Metzler matrix or B is not nonnegative matrix.

Definition 7 (Sign-preserving systems)**.** *System* (1) *is a sign-preserving system if there exists a vector $\boldsymbol{e} \in \mathcal{E}$ such that initial condition $\boldsymbol{\varphi}(k)$ ($k \in \{-\tau, \ldots, 0\}$) is compatible with $\boldsymbol{e}$ implies that $\boldsymbol{x}(k)$ is also compatible with $\boldsymbol{e}$ for each $\forall k > 0$.*

Similarly, system (2) *is a sign-preserving system if there exists an $\boldsymbol{e} \in \mathcal{E}$ such that any initial condition $\boldsymbol{\varphi}(t)$ ($t \in [t-\tau, 0]$) is compatible with $\boldsymbol{e}$ implies that $\boldsymbol{x}(t)$ is also compatible with $\boldsymbol{e}$ for all $\forall t > 0$.*

Remark 3. Clearly, positive systems are a class of sign-preserving systems.

Theorem 3. *System* (1) *is sign-preserving system if there exists a vector $\boldsymbol{e} \in \mathcal{E}$ such that both A and B are compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ in sign.*

Proof. Suppose that there exists a vector $\boldsymbol{e} \in \mathcal{E}$ such that both A and B are compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$. By Theorem 2, system (1) is diagonally positizable. Take the transformation $\boldsymbol{y}(k) = E\boldsymbol{x}(k)$, and system (1) changes into

$$\boldsymbol{y}(k+1) = E(\boldsymbol{e})AE(\boldsymbol{e})\boldsymbol{y}(k) + E(\boldsymbol{e})AE(\boldsymbol{e})\boldsymbol{y}(k-\tau(k)) \tag{5}$$

which is a positive system. In other words, if the initial condition is nonnegative, then $\boldsymbol{y}(k) \succeq 0, k \in \mathbb{N}$. By $\boldsymbol{y}(k) = E(\boldsymbol{e})\boldsymbol{x}(k)$, we have that $\boldsymbol{x}(k) = E(\boldsymbol{e})\boldsymbol{y}(k), \forall k \in \{-\tau, \ldots, 0\} \cup \mathbb{N}$, which implies that system (1) is a sign-preserving system. □

Corollary 3. *System* (2) *is a sign-preserving system if there exists a vector* $\boldsymbol{e} \in \mathcal{E}$ *such that* A *is compatible with* $\boldsymbol{e}\boldsymbol{e}_*^{\mathrm{T}}$ *and* B *is compatible with* $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$.

4 Stability Analysis of Positizable Systems

The following fact implicitly states the reason why we can analyze stability of a given system via its transformed system.

Fact 1. *Consider system* (1). *Take a transformation* $\boldsymbol{y}(k) = P\boldsymbol{x}(k)$ *with* P *being a nonsingular matrix, and system* (1) *changes into* (5). *Then system* (1) *is stable, asymptotically stable, or exponentially stable if and only if so is system* (5).

For system (2), *take a transformation* $\boldsymbol{y}(t) = P\boldsymbol{x}(t)$ *with* P *being a nonsingular matrix, then system* (2) *is recast into*

$$\dot{\boldsymbol{y}}(t) = E(\boldsymbol{e})AE(\boldsymbol{e})\boldsymbol{y}(t) + E(\boldsymbol{e})AE(\boldsymbol{e})\boldsymbol{y}(t-\tau(t)) \quad (6)$$

And system (2) *is stable, asymptotically stable, or exponentially stable if and only if so is system* (6).

As a matter of fact, by transformation $\boldsymbol{y}(t) = P\boldsymbol{x}(t), \forall t \geq 0$, if $\boldsymbol{y}(t)$ is stable in the sense of Lyapunov, then for any $\epsilon > 0$, there exists a corresponding δ such that $\sup_{-\tau\leq s\leq 0}\|\boldsymbol{\phi}(s)\| \leq \delta$ implies that $\|\boldsymbol{y}(t)\| \leq \epsilon$. Hence, if $\sup_{-\tau\leq s\leq 0}\|\boldsymbol{\phi}(s)\| \leq \frac{\delta}{\|P^{-1}\|}$, then $\|\boldsymbol{x}(t)\| \leq \|\boldsymbol{y}(t)\|\left\|P^{-1}\right\| \leq \epsilon$. That is, system (1) is stable in the sense of Lyapunov. We can similarly show that the cases of asymptotically stable and exponentially stable are also true. Clearly, the same conclusion holds for system (1).

Remark 4. Fact 1 clearly holds for systems without delays.

By Fact 1, if a non-positive system is positizable, then we can apply theory of positive systems to handle the stability property of the non-positive system.

Theorem 4. *Suppose that both* A *and* B *are compatible with* $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ *for a common* $\boldsymbol{e} \in \mathcal{E}$. *Then system* (1) *is exponentially stable for any bounded delay if and only if there exists a vector* $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ *such that*

$$E(\boldsymbol{e})\,(A + B - I)\,E(\boldsymbol{e})\boldsymbol{\lambda} \prec 0 \quad (7)$$

Proof. Note that both A and B are compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ for a common $\boldsymbol{e} \in \mathcal{E}$. By Theorem 2, A and B are simultaneously positizable by $E(\boldsymbol{e})$. Take the transformation $\boldsymbol{y}(k) = E(\boldsymbol{e})\boldsymbol{x}(k)$, and systems (1) is equivalent to system (5) regarding stability (by Fact 1). Since system (5) is positive, by [16, Theorem 3.4], it is exponentially stable for all bounded delay if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that

$$\left(E(\boldsymbol{e})AE(\boldsymbol{e}) + E(\boldsymbol{e})BE(\boldsymbol{e}) - I\right)\boldsymbol{\lambda} \prec 0$$

which is equivalent to (7) since

$$\begin{aligned} &E(\boldsymbol{e})AE(\boldsymbol{e}) + E(\boldsymbol{e})BE(\boldsymbol{e}) - I \\ =&E(\boldsymbol{e})AE(\boldsymbol{e}) + E(\boldsymbol{e})BE(\boldsymbol{e}) - E(\boldsymbol{e})E(\boldsymbol{e}) \\ =&E(\boldsymbol{e})\,(A + B - I)\,E(\boldsymbol{e}) \end{aligned}$$

This proof is completed. □

Corollary 4. *Consider system* (2). *Suppose that there exists a vector* $\boldsymbol{e} \in \mathcal{E}$ *such that* A *is compatible with* $\boldsymbol{e}\boldsymbol{e}_*^{\mathrm{T}}$ *in sign and* B *is compatible with* $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$. *Then system* (2) *is exponentially stable for any bounded delay if and only if there exists a vector* $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ *such that*

$$E(\boldsymbol{e})\,(A + B)\,E(\boldsymbol{e})\boldsymbol{\lambda} \prec 0.$$

Proof. Following a similar line in the proof of Theorem 4 and using Corollary 3 in this paper and [16, Theorem 3.12], the conclusion immediately follows. □

5 Positization by Arbitrary Nonsingular Matrix

In Section 3, the diagonal positization issue has been discussed. In some actual situations, however, it may happen that a matrix cannot be positized by any diagonal matrix, but can be positized by some other matrices, see the following example.

Example 1. Let $A = \begin{bmatrix} -100.25 & 256.375 \\ -41.5 & 106.25 \end{bmatrix}$. Clearly, A is not compatible with $\boldsymbol{e}\boldsymbol{e}^{\mathrm{T}}$ for any $\boldsymbol{e} \in \mathcal{E}$, so it is not diagonally positizable (Theorem 2). However, there exists a matrix $P = \begin{bmatrix} 0.5 & -0.75 \\ -1 & 2.5 \end{bmatrix}$ satisfying

$$PAP^{-1} = \begin{bmatrix} 2 & 20 \\ 1 & 4 \end{bmatrix} \succeq 0.$$

Clearly, it is very difficult to determine if a matrix, especially a matrix of higher dimension, can be positized by an arbitrary square invertible matrix. And even if it is positizable, how to figure out such a positizing matrix is also a hard task. Therefore, we only consider some special cases here.

Lemma 1. *Matrix* $A \in \mathbb{R}^{n\times n}$ *is positizable if and only if there exists a matrix* P *such that* $PAP^{-1} \succeq 0$ *and* $\det(P) = \pm 1$.

Proof. Sufficiency is clear. It suffices to show the necessity.

Suppose that there exists a positizing matrix Q of A. Clearly, $\det(Q) \neq 0$. Let $P = \mathrm{sgn}(\det(Q))\,|\det(Q)|^{\frac{1}{n}}\,Q$, and $\det(P) = \pm 1$. Then,

$$\begin{aligned} PAP^{-1} =& \left(\mathrm{sgn}(\det(Q))\,|\det(Q)|^{\frac{1}{n}}\right) QA\times \\ &\left(\mathrm{sgn}(\det(Q^{-1}))\,\left|\det(Q^{-1})\right|^{\frac{1}{n}}\,Q^{-1}\right) \\ =&\,|\det(Q)|^{\frac{1}{n}}\,QA\,\left|\det(Q^{-1})\right|^{\frac{1}{n}}\,Q^{-1} \\ =&QAQ^{-1} \\ \succeq&0 \end{aligned}$$

The proof is completed. □

Remark 5. The significance of Lemma 1 lies in: When we try to figure out a positizing matrix of A, the number of free variables (elements in positizing matrix) will decrease by 1.

Next, we will work with the cases where the considered matrix is not positizable.

The following fact is clear.

Fact 2. *Any negative scalar (matrix of dimension* 1×1*) is not positizable.*

Definition 8. *A diagonal matrix in which all of the diagonal elements are equal is called scalar matrix.*

Fact 3. *Any scalar matrix A whose diagonal elements are negative is not positizable.*

It is straightforward to see Fact 3 holds since for any non-singular matrix P, PAP^{-1} is just equal to A.

However, regarding the diagonal matrix with all diagonal elements negative, the situation seems much more difficult to handle than the case in Fact 3. Next, we consider positization of 2×2-dimensional matrix.

Theorem 5. *Any diagonal $\mathbb{R}^{2\times 2}$ matrix with negative diagonal elements is not positizable.*

Proof. Let $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a < 0, b < 0, a \neq b$ (since A is not positizable if $a = b < 0$). Suppose there exists some positizing matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ such that $PAP^{-1} \succeq 0$. Clearly, P cannot be a diagonal matrix.

Suppose that P takes an upper triangular form. By Lemma 1, $P = \begin{bmatrix} s & 0 \\ t & s^{-1} \end{bmatrix}$ or $P = \begin{bmatrix} s & 0 \\ t & -s^{-1} \end{bmatrix}$ for some $s, t \neq 0$. If $P = \begin{bmatrix} s & 0 \\ t & s^{-1} \end{bmatrix}$, then $P^{-1} = \begin{bmatrix} s^{-1} & 0 \\ -t & s \end{bmatrix}$, and $PAP^{-1} = \begin{bmatrix} a & 0 \\ \diamond & b \end{bmatrix}$, where the symbol $\diamond$ denotes an element that does not matter. Hence P is not a positizing matrix. Similarly, we can show that $P = \begin{bmatrix} s & 0 \\ t & -s^{-1} \end{bmatrix}$, $P = \begin{bmatrix} 0 & s \\ s^{-1} & t \end{bmatrix}$, $P = \begin{bmatrix} 0 & s \\ -s^{-1} & t \end{bmatrix}$, $P = \begin{bmatrix} s & t \\ 0 & s^{-1} \end{bmatrix}$, $P = \begin{bmatrix} s & t \\ 0 & -s^{-1} \end{bmatrix}$, $P = \begin{bmatrix} t & s \\ s^{-1} & 0 \end{bmatrix}$, and $P = \begin{bmatrix} t & s \\ -s^{-1} & 0 \end{bmatrix}$ cannot serve as a positizing matrix. That is, A is not positizable using any triangular matrix.

Suppose that there exists some positizing matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ such that $PAP^{-1} \succeq 0$ and $p_{ij} \neq 0, i, j = 1, 2$. By Lemma 1, we assume $\det(P) = \pm 1$. Consider only the case $\det(P) = -1$. The case $\det(P) = 1$ is similar to $\det(P) = -1$.

Since $\det(P) = -1$, one has that $P^{-1} = \begin{bmatrix} -p_{22} & p_{12} \\ p_{21} & -p_{11} \end{bmatrix}$ and

$$PAP^{-1} = \begin{bmatrix} bp_{12}p_{21} - ap_{11}p_{22} & (a-b)p_{11}p_{12} \\ (b-a)p_{22}p_{21} & ap_{12}p_{21} - bp_{11}p_{22} \end{bmatrix} = \begin{bmatrix} (1,1) & (1,2) \\ (2,1) & (2,2) \end{bmatrix}$$

where

$$(1,1) = bp_{12}p_{21} - ap_{11}p_{22}, (1,2) = (a-b)p_{11}p_{12}$$
$$(2,1) = (b-a)p_{22}p_{21}, (2,2) = ap_{12}p_{21} - bp_{11}p_{22}$$

Since P has no zero element, it may have 16 sign patterns.

Consider the pattern $\begin{bmatrix} + & + \\ + & + \end{bmatrix}$ first. In this case, $p_{11}p_{12} > 0, p_{22}p_{21} > 0$, and either $a - b > 0$ or $b - a > 0$, hence one of $(1,2)$ and $(2,1)$ is negative, so P cannot take this sign pattern. Similarly, we claim that P cannot have the pattern $\begin{bmatrix} - & - \\ - & - \end{bmatrix}$.

Suppose P has the pattern $\begin{bmatrix} + & + \\ - & - \end{bmatrix}$. Then $p_{11}p_{12} > 0, p_{22}p_{21} > 0$, so one of $(1,2)$ and $(2,1)$ is negative. As a result, P cannot take the pattern $\begin{bmatrix} + & + \\ - & - \end{bmatrix}$. Similarly, P cannot take these patterns: $\begin{bmatrix} - & - \\ + & + \end{bmatrix}$, $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$, $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$, $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$, and $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$.

Clearly, P cannot take these patterns: $\begin{bmatrix} + & - \\ + & + \end{bmatrix}$, $\begin{bmatrix} + & + \\ - & + \end{bmatrix}$, $\begin{bmatrix} - & + \\ - & - \end{bmatrix}$, and $\begin{bmatrix} - & - \\ + & - \end{bmatrix}$, since it is assumed that $\det(p) = -1$.

Now consider the pattern $\begin{bmatrix} - & + \\ + & + \end{bmatrix}$. Note that $a \neq b$. If $a > b$ then $(2,1) < 0$, hence we necessarily have $a < b < 0$. Since $p_{11}p_{22} - p_{12}p_{21} = -1$, it holds that $p_{11}p_{22} = p_{12}p_{21} - 1$. Therefore, $(1,1) > 0$ means that $(b-a)p_{12}p_{21} > -a$ and $(2,2) > 0$ implies that $(a-b)p_{12}p_{21} > -a$. However, this is impossible because $(a-b)p_{12}p_{21} < 0$ or $(b-a)p_{12}p_{21} < 0$. Using the same approach, it is not difficult to show that P cannot take the following patterns: $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$, $\begin{bmatrix} + & - \\ - & - \end{bmatrix}$, and $\begin{bmatrix} - & - \\ - & + \end{bmatrix}$.

In a word, if $\det(P) < 0$, then P cannot take any possible sign patterns, therefore P is not a positizing matrix and A is not positizable.

The same conclusion also holds for the case of $\det(P) > 0$. Thus, the proof is completed. □

Using a similar way in Theorem 5, we have

Corollary 5. *Any diagonal $\mathbb{R}^{2\times 2}$ matrix of form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}$ is not positizable, where $a < 0$.*

Note that a diagonal matrix with some negative elements may be positizable, see the example below.

Example 2.

$$\begin{bmatrix} 1 & -1 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1.5 & 1.5 \end{bmatrix}.$$

6 Important Topics in the Future

The reason for discussing diagonally positization lies in the following facts: It is easy to determine if a matrix or a set of matrices is (simultaneously) diagonally positizable, and if it is diagonally positizable, the positization can be easily realized. As an extension of diagonal positization, triangular positization will be considered in the future.

Essentially, (simultaneous) positization by an arbitrary square matrix amounts to solving a non-convex optimization problem, which is an NP hard problem. However, there might exist some cases where the issue of positization by an arbitrary square matrix may be relatively simple. Therefore, exploring the cases where positization can be handled easily is a direction in the future.

We have defined the sigh-preserving systems and proposed a sufficient condition to check if a system is sigh-preserving system. From the viewpoint of stability, this class of systems is of interest: We can use the copositive Lyapunov function or functional to handle their stability. Hence, how to completely characterize sigh-preserving systems is an important topic.

7 Conclusions

We have proposed a series of concepts: Positization and Metzlerization of matrices and systems. Some necessary and sufficient condition are provided, which completely solves the issue of diagonal Positization of systems. A necessary and sufficient condition is also provided to check the exponential stability of a class of sign-preserving systems. Some situations have been revealed where the considered matrices are not positizable. Some important topics have been pointed out which give some new directions in the future.

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