5. Sampled-data Systems

In this chapter we consider sampled-data systems with zero-order hold. We first convert them to jump systems and then solve the H_∞ control problem with initial uncertainty and the H_2 control problem.

5.1 Jump System Approach

We shall show how to transform the sampled-data systems to jump systems. Then we apply the results in Chapter 4 on stability, H_2 and H_∞ norms, disturbance attenuation problems and quadratic control.

5.1.1 Transformation to Jump Systems

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T),\ 0 \le N\tau \le T < (N+1)\tau \end{aligned} \tag{5.1}$$

with initial condition

$$x(0) = Hh,\ h \in \mathbf{R}^{n_1}$$

where $x \in \mathbf{R}^n$ is the state, $(w, w_d) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_{1d}}$ is the disturbance, $\tilde{u} \in \mathbf{R}^{m_2}$ is the control input realized through a zero-order hold,

$$\tilde{u}(t) = u(k),\ k\tau < t \le (k+1)\tau,$$

$(z_1, z) \in \mathbf{R}^q \times \mathbf{R}^{p_1+m_2}$ is the controlled output, $y \in \mathbf{R}^{p_2}$ is the sampled observation, τ is a sampling period, $C_1 \in \mathbf{R}^{p_1 \times n}$, $D_{12} \in \mathbf{R}^{m_2 \times m_2}$ and other matrices are of compatible dimensions. For the system $\mathbf{G}_s$ we introduce discrete-time controllers of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k) \end{aligned} \tag{5.2}$$

where $\hat{x} \in \mathbf{R}^{\hat{n}}$ and all matrices are of compatible dimensions. Since the system $\mathbf{G}_s$ is essentially a continuous-time system and the controller is a discrete-time system, we need two devices, a zero-order hold and a sampler, to connect these two systems.

We first express this system as a jump system of the form (4.42). Since the control $\tilde{u}(t)$ is constant between two sampling instants, i.e., $k\tau < t \leq (k+1)\tau$, we can introduce the following state space representation:

$$\dot{\bar{x}} = 0, \;\; \bar{x}(k\tau^+) = u(k), \;\; k\tau < t \leq (k+1)\tau.$$

Then clearly $\tilde{u}(t) = \bar{x}(t)$. Let

$$x_e(t) = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}(t)$$

be the new state variable. Then the sampled-data system $\mathbf{G}_s$ is equivalent to the following system with jumps (denoted by $\mathbf{G}_e$):

$$\begin{aligned} \dot{x}_e(t) &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} x_e(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(t), \;\; k\tau < t < (k+1)\tau, \\ x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \;\; k = 0, 1, 2, ..., \\ z &= \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = \begin{bmatrix} [\,C_1 \;\; 0\,]\, x_e(t) \\ \sqrt{\tau} D_{12} u(k) \end{bmatrix}, \\ y(k) &= [\,C_2 \;\; 0\,]\, x_e(k\tau) + D_{21} w_d(k), \\ z_1 &= [\,F \;\; 0\,]\, x_e(T) \end{aligned} \tag{5.3}$$

and

$$x_e(0) = [\,H \;\; 0\,] \begin{bmatrix} h \\ 0 \end{bmatrix}.$$

Here $z_d = \sqrt{\tau} D_{12} u(k)$ comes from

$$\int_0^\infty \mid D_{12}\tilde{u}(t) \mid^2 = \sum_{k=0}^\infty \int_0^\tau \mid D_{12} u(k) \mid^2 = \sum_{k=0}^\infty \mid \sqrt{\tau} D_{12} u(k) \mid^2, u(\cdot) \in l^2.$$

Since the system $\mathbf{G}_e$ is a jump system, we can solve the H_2 and H_∞ control problems for the sampled-data systems using results in Chapter 4.

For the systems $\mathbf{G}_s$ and $\mathbf{G}_e$, we can easily obtain the following result.

Lemma 5.1 *If (A, B_1, C_1) is stabilizable and detectable for the system $\mathbf{G}_s$, then the jump system*

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right], \left[\begin{bmatrix} B_1 \\ 0 \end{bmatrix}, 0 \right], [[\,C_1 \;\; 0\,], 0] \right)$$

is stabilizable and detectable.

Proof. Since (A, B_1) is stabilizable, there exists a matrix K such that the system

$$\dot{\xi} = (A + B_1 K)\xi$$

is exponentially stable. Then the system

$$\begin{aligned}
\dot{\xi}_e(t) &= \left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} K & 0 \end{bmatrix}\right) \xi_e(t) = \begin{bmatrix} A + B_1 K & B_2 \\ 0 & 0 \end{bmatrix} \xi_e(t), \\
& \qquad k\tau < t \le (k+1)\tau, \\
\xi_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \xi_e(k\tau)
\end{aligned}$$

is obviously exponentially stable and hence we have the assertion. We can show the detectability in a similar manner. ∎

However, stabilizability and detectability of (A, B_2, C_2) does not imply that of

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ I \end{bmatrix}\right], [0, [C_2 \quad 0]]\right)$$

under the special sampling period. We shall show this in the next section.

5.1.2 Comments on the Sampling Period

Consider the system

$$\dot{x} = Ax + B_2\tilde{u}, \quad y(k) = C_2 x(k\tau) \tag{5.4}$$

where

$$\tilde{u}(t) = u(k), \ k\tau < t \le (k+1)\tau$$

and assume that (A, B_2, C_2) is stabilizable and detectable in the usual sense. As we see in the previous section, the system (5.4) is equivalent to the following jump system

$$\begin{aligned}
\dot{x}_e(t) &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} x_e(t), \ k\tau < t < (k+1)\tau, \\
x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \ k = 0, 1, 2, ..., \\
y(k) &= [C_2 \quad 0]\, x_e(k\tau)
\end{aligned} \tag{5.5}$$

and moreover

$$\begin{aligned}
x((k+1)\tau) &= e^{A\tau} x(k\tau) + \Pi(\tau) B_2 u(k), \\
y(k) &= C_2 x(k\tau)
\end{aligned} \tag{5.6}$$

where $\Pi(t) = \int_0^t e^{Ar} dr$. Note that (5.5) is stabilizable and detectable if and only if (5.6) is stabilizable and detectable. We now introduce an important notion about a sampling period τ.

Definition 5.1 *The sampling period τ is called pathological (with respect to A) if A has two eigenvalues, say λ and $\tilde{\lambda}$, such that*

$$\lambda = \sigma + j\omega, \quad \tilde{\lambda} = \sigma + j\tilde{\omega}$$

with $\mid \omega - \tilde{\omega} \mid = k\frac{2\pi}{\tau}$ for some positive integer k. Otherwise τ is called non-pathological.

The following example [8] shows that if the sampling period is pathological, the stabilizability and detectability of (A, B_2, C_2) does not necessarily imply that of $(e^{A\tau}, \Pi(\tau)B_2, C_2)$.

Example 5.1 ([8]) Consider the sampled-data system with sampling period τ:

$$\begin{aligned} \dot{x} &= Ax + b_2\tilde{u},\ A = \begin{bmatrix} 0 & 1 \\ -(\frac{2\pi}{\tau})^2 & 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ y(k) &= c_2 x(k\tau),\ c_2 = [0 \quad 1] \end{aligned} \tag{5.7}$$

(A, b_2, c_2) is obviously controllable and observable (and hence stabilizable and detectable). Note that eigenvalues of A are

$$\lambda = 0 + j\omega,\ 0 + j\tilde{\omega},\ \omega = \frac{2\pi}{\tau},\ \tilde{\omega} = -\frac{2\pi}{\tau}.$$

Since $\omega - \tilde{\omega} = 2\frac{2\pi}{\tau}$, the sampling period τ is pathological. Now

$$\begin{aligned} e^{At} &= \begin{bmatrix} \cos(\frac{2\pi}{\tau}t) & \frac{\tau}{2\pi}\sin(\frac{2\pi}{\tau}t) \\ -\frac{2\pi}{\tau}\sin(\frac{2\pi}{\tau}t) & \cos(\frac{2\pi}{\tau}t) \end{bmatrix}, \\ e^{A\tau} &= \begin{bmatrix} \cos(2\pi) & \frac{\tau}{2\pi}\sin(2\pi) \\ -\frac{2\pi}{\tau}\sin(2\pi) & \cos(2\pi) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \Pi(\tau)b_2 &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Hence the discrete-time system obtained from the sampled-data system (5.7) in neither stabilizable nor detectable.

However, if the sampling period is nonpathological, stabilizability and detectability are preserved [8].

Lemma 5.2 *Assume that the sampling period τ is nonpathological. Then (A, B_2, C_2) is stabilizable and detectable if and only if $(e^{A\tau}, \Pi(\tau)B_2, C_2)$ is stabilizable and detectable.*

Proof. Note that $e^{\lambda_i\tau}$ is an eigenvalue of $e^{A\tau}$ if λ_i is an eigenvalue of A. Using the Taylor expansion of $e^{s\tau} - e^{\lambda_i\tau}$ we can write

$$e^{s\tau} - e^{\lambda_i\tau} = g(s)(s - \lambda_i).$$

Hence

$$\begin{bmatrix} e^{A\tau} - e^{\lambda_i \tau} I \\ C_2 \end{bmatrix} = \begin{bmatrix} g(A) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda_i I \\ C_2 \end{bmatrix}.$$

Now we shall show that $g(A)$ is nonsingular. It is enough to show that 0 is not an eigenvalue of $g(A)$. Since the eigenvalues of $g(A)$ are $g(\lambda_k)$, we shall show $g(\lambda_k) \neq 0$ for any eigenvalue λ_k of A. If $\lambda_k \neq \lambda_i$, then $g(\lambda_k) \neq 0$ since otherwise

$$e^{\lambda_k \tau} - e^{\lambda_i \tau} = g(s)(\lambda_k - \lambda_i) = 0$$

which contradicts the nonpathological assumption of τ. Moreover, by direct calculation

$$g(\lambda_i) = \tau e^{\lambda_i \tau} \neq 0.$$

Hence

$$\operatorname{rank} \begin{bmatrix} e^{A\tau} - e^{\lambda_i \tau} I \\ C_2 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A - \lambda_i I \\ C_2 \end{bmatrix}$$

for any eigenvalue λ_i of A. Since $| e^{\lambda_i \tau} | \geq 1$ if and only if $Re\lambda_i \geq 0$, detectability is preserved. Considering the adjoint of the original system we can show that stabilizability is also preserved. ∎

5.1.3 Stability

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \; x(0) = Hh, \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12} \tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T), \; 0 \leq N\tau \leq T < (N+1)\tau \end{aligned}$$

and the discrete-time controllers (5.2)

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k) \end{aligned}$$

where we assume that (A, B_2, C_2) is stabilizable and detectable and the sampling period is nonpathological. Since the system $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the controller (5.2) is equivalent to the following jump system

$$\begin{aligned} \dot{\hat{x}} &= 0, \; k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= \hat{A}\hat{x}(k\tau) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k\tau) + \hat{D}y(k), \end{aligned} \tag{5.8}$$

the closed-loop system $\mathbf{G}_s$ and (5.2) (and hence $\mathbf{G}_e$ and (5.8)) is given by

$$\begin{aligned}
\dot{x}_{cl} &= A_{cl}x_{cl} + B_{cl}w, \; k\tau < t < (k+1)\tau, \\
x_{cl}(k\tau^{+}) &= A_{dcl}x_{cl}(k\tau) + B_{dcl}w_d(k), \qquad (5.9) \\
z_c(t) &= C_{cl}x_{cl}, \\
z_d(k) &= C_{dcl}x_{cl}(k\tau) + D_{dcl}w_d(k), \\
z_1 &= F_{cl}x_{cl}(T) \qquad (5.10)
\end{aligned}$$

and

$$x_{cl}(0) = H_{cl}\begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \qquad (5.11)$$

which is the jump system of the form (4.4) where $x_{cl} = [x_e' \quad \hat{x}']'$ and

$$\begin{aligned}
A_{cl} &= \begin{bmatrix} A & B_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & A_{dcl} &= \begin{bmatrix} I & 0 & 0 \\ \hat{D}C_2 & 0 & \hat{C} \\ \hat{B}C_2 & 0 & \hat{A} \end{bmatrix}, \\
B_{cl} &= \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, & B_{dcl} &= \begin{bmatrix} 0 \\ \hat{D}D_{21} \\ \hat{B}D_{21} \end{bmatrix}, \\
C_{cl} &= [\, C_1 \quad 0 \quad 0 \,], & C_{dcl} &= [\, \sqrt{\tau}D_{12}\hat{D}C_2 \quad 0 \quad \sqrt{\tau}D_{12}\hat{C} \,], \\
D_{dcl} &= \sqrt{\tau}D_{12}D_{21}, & & \\
F_{cl} &= [\, F \quad 0 \quad 0 \,], & H_{cl} &= [\, H \quad 0 \quad 0 \,].
\end{aligned}$$

Hence we can consider stability, H_2, H_∞ norms and the disturbance attenuation problems of the sampled-data feedback systems using the system (5.9) and the results in Chapter 4.

Lyapunov Equations

By applying Propositions 4.2, 4.3 and Corollary 4.1 to the homogeneous system of (5.9), we have the following result.

Proposition 5.1 *The following statements are equivalent.*
(a) The feedback system (5.9) is exponentially stable.
(b) There exists a τ-periodic symmetric matrix $X(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$ such that

$$\begin{aligned}
&(i) \quad c_1 I \le X(t) \le c_2 I, \; {}^{\forall}t \ge 0 \text{ for some } c_i > 0, \; i = 1,2, \\
&(ii) \quad -\dot{X} = A_{cl}'X + XA_{cl} + I, \; k\tau < t < (k+1)\tau, \\
&\qquad\quad X(k\tau^{-}) = A_{dcl}'X(k\tau)A_{dcl} + I.
\end{aligned}$$

(c) There exists a symmetric matirx $Y(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$ and a

$0 < \delta < \tau$ *such that*

$$\begin{array}{ll} (i) & 0 < Y(t),\ {}^{\forall}t \geq 0 \text{ and } c_1 I \leq Y(t),\ {}^{\forall}t \geq \delta \text{ for some } c_1 > 0, \\ (ii) & Y(t) \leq c_2 I,\ 0 \leq\ {}^{\forall}t < \infty \text{ for some } c_2 > 0, \\ (iii) & \dot{Y} = A_{cl}Y + YA'_{cl} + I,\ k\tau < t < (k+1)\tau, \\ & Y(k\tau^+) = A_{dcl}Y(k\tau)A'_{dcl} + I, \\ & Y(0) = 0. \end{array}$$

(d) There exists a τ-periodic symmetric solution $Y_\tau(t)$ of (iii) in (c) without $Y(0) = 0$ such that $c_1 I \leq Y_\tau(t) \leq c_2 I$ for some c_1, $c_2 > 0$.

H_2 and H_∞ Norms

Now we assume that the sampled-data feedback system (5.9) is exponentially stable and $h = 0$. Then we can define its H_2-norm as in Definition 4.7 and calculate it using Theorem 4.1.

Proposition 5.2 *Let $\| G \|_2$ be the H_2-norm of the system (5.9). Then*

$$\| G \|_2^2 = \frac{1}{\tau}\int_0^\tau tr.B'_{cl}X(s)B_{cl}\ ds + tr.[B'_{dcl}X(0)B_{dcl} + D'_{dcl}D_{dcl}]$$

where X is a τ-periodic nonnegative solution of

$$\begin{aligned} -\dot{X} &= A'_{cl}X + XA_{cl} + C'_{cl}C_{cl},\ k\tau < t < (k+1)\tau, \\ X(k\tau^-) &= A'_{dcl}X(k\tau)A_{dcl} + C'_{dcl}C_{dcl}. \end{aligned}$$

We can also define the H_∞-norm of the sampled-data feedback system (5.9) as in Definition 4.8.

Disturbance Attenuation Problems

Let G_T be the input-output operator of the sampled-data feedback system (5.9)-(5.11) on $[0, T]$. Then by Theorem 4.6 we have the following result.

Proposition 5.3 *The following statements are equivalent.*
(a) $\| G_T \| < \gamma$.
(b) There exists a nonnegative solution $X(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$, $t \in [0, T]$ to (4.29)-(4.33) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.
(b) There exists a nonnegative solution $Y(t) \in \mathbf{R}^{(n+m_2+\hat{n})\times(n+m_2+\hat{n})}$, $t \in [0, T]$ to (4.34)-(4.38) with A, B and etc replaced by A_{cl}, B_{cl} and etc, respectively.

Next we consider the system (5.9) and (5.11) on $[0, \infty)$. We assume that (A_{cl}, A_{dcl}) is exponentially stable. Let G be the input-output operator of the system (5.9) and (5.11). Then by Theorem 4.7 we have the following result.

Proposition 5.4 *The following statements are equivalent.*
(a) $\| G \| < \gamma$.
(b) There exists a τ*-periodic nonnegative stabilizing solution* $X(t)$, $t \in [0, \infty)$ *to (4.29)-(4.31) satisfying (4.33) with* A, B *and etc replaced by* A_{cl}, B_{cl} *and etc, respectively.*
(b) There exists a τ*-periodic nonnegative stabilizing solution* $Y(t)$, $t \in [0, \infty)$ *to (4.34)-(4.37) with* A, B *and etc replaced by* A_{cl}, B_{cl} *and etc, respectively.*

5.1.4 Quadratic Control

Consider the system

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \ \tilde{u}(t) = u(k), \ k\tau < t \le (k+1)\tau, \\ x(0) &= x_0 \end{aligned} \tag{5.12}$$

and the functional to be minimized

$$\begin{aligned} J_T(\tilde{u}; x_0) &= \int_0^T [| Cx(t) |^2 + | \tilde{u}(t) |^2] dt + | Fx(T) |^2, \\ & \qquad 0 \le N\tau \le T < (N+1)\tau \end{aligned} \tag{5.13}$$

where $x \in \mathbf{R}^n$, $\tilde{u} \in \mathbf{R}^{m_2}$, $C \in \mathbf{R}^{p_2 \times n}$ and other matrices are of compatible dimensions. Since the system (5.12) and the functional (5.13) are equivalent to the jump system

$$\begin{aligned} \dot{x}_e &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x_e, \ k\tau < t < (k+1)\tau, \\ x_e(k\tau^+) &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_e(k\tau) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(k), \ x_e(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \\ z &= \begin{bmatrix} z_c(t) \\ z_d(k) \end{bmatrix} = \begin{bmatrix} [C \quad 0] x_e \\ \sqrt{\tau} u(k) \end{bmatrix} \end{aligned} \tag{5.14}$$

and the functional

$$\tilde{J}_T(u; x_0) = \int_0^T | z_c(t) |^2 \, dt + \sum_{k=0}^{N} | z_d(k) |^2 + | [F \quad 0] x_e(T) |^2$$

we can apply Theorems 4.2 and 4.3. Let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \ X_1 \in \mathbf{R}^{n \times n}, \ X_{12} \in \mathbf{R}^{n \times m}, \ X_2 \in \mathbf{R}^{m \times m}$$

be the solution of the Riccati equation (4.12)-(4.14) with

$$T_2(k) = I + B_d' X(k\tau) B_d$$

replaced by

$$T_2(k) = \tau I + B_d' X(k\tau) B_d.$$

Then we obtain for $k\tau < t < (k+1)\tau$

$$\begin{aligned} -\dot{X}_1 &= A'X_1 + X_1 A + C'C, \\ -\dot{X}_{12} &= X_{12}A' + B'X_1, \\ -\dot{X}_2 &= B'X_{12} + X_{12}'B \end{aligned} \tag{5.15}$$

and at $t = k\tau$

$$\begin{aligned} X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\ X_{12}(k\tau^-) &= 0, \\ X_2(k\tau^-) &= 0 \end{aligned} \tag{5.16}$$

with

$$\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix} (T) = \begin{bmatrix} F'F & 0 \\ 0 & 0 \end{bmatrix} \tag{5.17}$$

and by Theorem 4.2 we have the following result.

Theorem 5.1 *There exists a unique nonnegative solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$, $X_1 \in \mathbf{R}^{n\times n}$, $X_{12} \in \mathbf{R}^{n\times m}$, $X_2 \in \mathbf{R}^{m\times m}$ *to the Riccati equation (5.15)-(5.17). Moreover, the state feedback law*

$$\bar{\tilde{u}}(t) = \bar{u}(k), \;\; \bar{u}(k) = -[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau) x(k\tau), \;\; k\tau < t \le (k+1)\tau$$

is optimal and

$$J_T(\bar{\tilde{u}}; x_0) = \tilde{J}_T(\bar{u}; x_0) = x_0' X_1(0^-) x_0.$$

Next we consider the infinite horizon problem

$$\begin{aligned} \dot{x} &= Ax + B\tilde{u}, \;\; x(0) = x_0, \\ J(\tilde{u}; x_0) &= \int_0^\infty [| Cx(t) |^2 + | \tilde{u}(t) |^2] dt \end{aligned}$$

where $\tilde{u} \in L^2(0,\infty; \mathbf{R}^{m_2})$ is admissible if its response $x \in L^2(0,\infty;\mathbf{R}^n)$ and $\lim_{t\to\infty} x(t) = 0$. This problem is again equivalent to

$$\tilde{J}(u; x_0) = \int_0^\infty | z_c(t) |^2 \, dt + \sum_{k=0}^{\infty} | z_d(k) |^2$$

subject to the jump system (5.14) where $u \in l^2(0,\infty;\mathbf{R}^{m_2})$ is admissible if its response $x_e \in L^2(0,\infty;\mathbf{R}^{n+m_2})$ and $\lim_{t\to\infty} x_e(t) = 0$.

Now we assume that (A,B) is stabilizable and the sampling period is nonpathological for the system (5.12). Then by Lemma 5.2

$$\left(\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right], \left[0, \begin{bmatrix} 0 \\ I \end{bmatrix}\right]\right)$$

is stabilizable and the condition **RJ** in Section 4.1.3 for the system (5.14) is satisfied. If we further assume that (C,A) is detectable, then by Lemma 5.1

$$\left([[C \quad 0], 0]\left[\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right]\right)$$

is detectable. Hence we can apply Theorem 4.3 to the system (5.14) (and hence (5.12)). We say that $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ is a stabilizing solution of (5.15) and (5.16) if

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ -[\tau I + X_2(k\tau)]^{-1}X_{12}'(k\tau) & 0 \end{bmatrix}\right)$$

is exponentially stable, which is equivalent to the stability of the system

$$\begin{aligned} \dot{x} &= Ax + B\hat{u}(t), \\ \hat{u}(t) &= -[\tau I + X_2(k\tau)]^{-1}X_{12}'(k\tau)x(k\tau), \ k\tau < t < (k+1)\tau \end{aligned}$$

and equivalently that of the discrete-time system

$$x((k+1)\tau) = \left\{e^{A\tau} - \Pi(\tau)B[\tau I + X_2(k\tau)]^{-1}X_{12}'(k\tau)\right\}x(k\tau).$$

Summing up we have the following result.

Theorem 5.2 *Suppose (C,A,B) is stabilizable and detectable and the sampling period τ is nonpathological. Then there exists a τ-periodic nonnegative stabilizing solution $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ to (5.15) and (5.16). Moreover, the state feedback law*

$$\begin{aligned} \bar{\tilde{u}}(t) &= \bar{u}(k), \\ \bar{u}(k) &= -[\tau I + X_2(0)]^{-1}X_{12}'(0)x(k\tau), \ k\tau < t \le (k+1)\tau \end{aligned}$$

is optimal and

$$J_T(\bar{\tilde{u}}; x_0) = \tilde{J}_T(\bar{u}; x_0) = x_0'X_1(0^-)x_0.$$

Example 5.2 Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\tilde{u}(t), \ \tilde{u}(t) = u(k), \ k < t \le k+1$$

and the functional

$$J(\tilde{u}; x_0) = \int_0^\infty [\| x_1(t) |^2 + | \tilde{u}(t) |^2] dt$$

For this system and the functional the assumptions of Theorem 5.2 are satisfied. Then there exists a periodic nonnegative stabilizing solution $X(t) = [X_{ij}(t)]$, $i, j = 1, 2, 3$ with period 1 of the Riccati equation (5.15) and (5.16). Figure 5.1 shows the periodic solution $X(t)$. Figure 5.2 shows the response of the closed-loop system with $x_1(0) = 1$ and $x_2(0) = 0$ to the optimal state feedback.

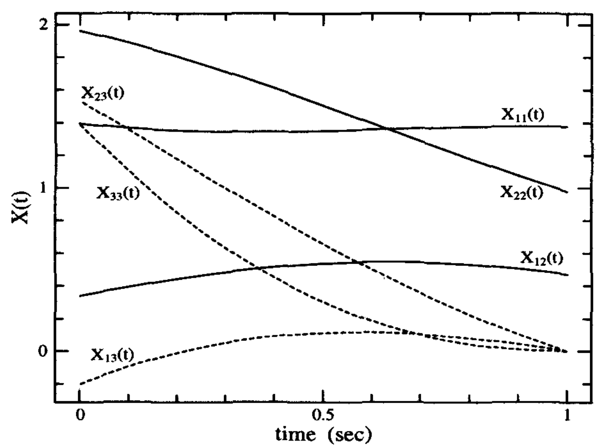

Figure 5.1: The periodic nonnegative solution $X(t)$

5.2 H_∞ Control

Here we consider the H_∞ control problem initial uncertainty. We apply the results in Section 4.2 to the jump systems obtained from the sampled-data systems.

5.2.1 Finite Horizon Problems

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12} \tilde{u}(t) \end{bmatrix}, \qquad (5.18) \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k), \\ z_1 &= Fx(T),\ 0 \le N\tau \le T < (N+1)\tau \qquad (5.19) \end{aligned}$$

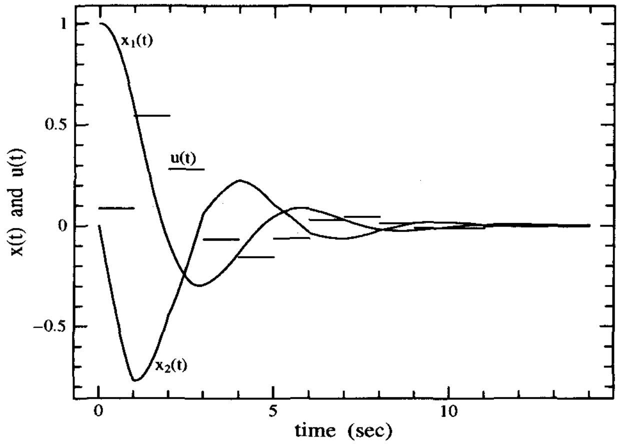

Figure 5.2: Simulation result

with initial condition

$$x(0) = Hh \tag{5.20}$$

and a discrete-time controller of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k). \end{aligned} \tag{5.21}$$

For the system $\mathbf{G}_s$ we assume

$$\mathbf{S1} : D_{12}'D_{12} = I, \ D_{21}D_{21}' = I.$$

Consider the sampled-data system $\mathbf{G}_s$ and a discrete-time controller $u = Ky$ of the form (5.21) on $[0, T]$. Define the input-output operator of the closed-loop system by

$$\begin{pmatrix} z_1 \\ z \end{pmatrix} = G \begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.$$

Then

$$\begin{aligned} G \ \in \ & \mathcal{L}(\mathbf{R}^{n_1} \times L^2(0,T;\mathbf{R}^{m_1}) \times l^2(0,N;\mathbf{R}^{m_{1d}}); \\ & \mathbf{R}^q \times L^2(0,T;\mathbf{R}^{p_1}) \times l^2(0,N;\mathbf{R}^{p_{1d}})). \end{aligned}$$

The H_∞-problem for the sampled-data system $\mathbf{G}_s$ is to find necessary and sufficient conditions for the existence of a discrete-time controller such that $\| G \| < \gamma$, i.e.,

$$| z_1 |^2 + \| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \|_{L^2 \times l^2}^2 \leq d^2(| h |^2 + \| \begin{pmatrix} w \\ w_d \end{pmatrix} \|_{L^2 \times l^2}^2) \text{ for some } 0 < d < \gamma.$$

Such a controller is called γ-suboptimal. Since the sampled-data system $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the assumption **S1** implies the conditions **J1** for the system $\mathbf{G}_e$, we can apply Theorems 4.8 and 4.9 to the system $\mathbf{G}_e$ and hence for the system $\mathbf{G}_s$.

Remark 5.1 The standard way to solve the H_∞ and H_2 problems for $\mathbf{G}_s$ with $D_{21} = 0$ is to use the lifting technique which converts periodic systems to discrete-time systems with infinite dimensional input and output spaces and to reduce the original problems to those for ordinary discrete-time systems [2, 3, 8, 25, 76, 88]. We shall show that their lifted system is directly obtained from $\mathbf{G}_e$. In fact for $k\tau < t \le (k+1)\tau$ we have

$$x_e(t) = e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(t-k\tau)} x_e(k\tau^+) + \int_{k\tau}^{t} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(t-r)} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} w(r)dr.$$

Since

$$e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}t} = \begin{bmatrix} e^{At} & \Pi(t)B_2 \\ 0 & I \end{bmatrix},$$

we have for $k\tau < t \le (k+1)\tau$

$$\begin{aligned} x_e(t) &= \begin{bmatrix} x \\ \bar{x} \end{bmatrix}(t) \\ &= \begin{bmatrix} e^{A(t-k\tau)}x(k\tau^+) + \Pi(t-k\tau)B_2u(k) + \int_{k\tau}^{t} e^{A(t-r)}B_1w(r)dr \\ u(k) \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} x((k+1)\tau) &= e^{A\tau}x(k\tau^+) + \Pi(\tau)B_2u(k) + \int_{k\tau}^{(k+1)\tau} e^{A[(k+1)\tau - r]}B_1w(r)dr \\ &= e^{A\tau}x(k\tau) + \Pi(\tau)B_2u(k) + \int_0^{\tau} e^{A(\tau-s)}B_1w(s+k\tau)ds \\ &= e^{A\tau}x(k\tau) + \Pi(\tau)B_2u(k) + \int_0^{\tau} e^{A(\tau-s)}B_1\hat{w}_k(s)ds \end{aligned}$$

where $\hat{w}_k(s) = w(s+k\tau)$. We also have

$$\begin{aligned} z_c(t) &= C_1e^{A(t-k\tau)}x(k\tau) + C_1\int_0^{t-k\tau-s} e^{A(t-k\tau-s)}B_1\hat{w}_k(s)ds \\ &\quad + C_1\Pi(t-k\tau)B_2u(k), \\ z_d(k) &= \sqrt{\tau}D_{12}u(k). \end{aligned}$$

Hence the system $\mathbf{G}_s$ is equivalent to the following lifted system (denoted by $\tilde{\mathbf{G}}$)

$$\hat{x}(k+1) = e^{A\tau}\hat{x}(k) + \int_0^{\tau} e^{A(\tau-s)}B_1\hat{w}_k(s)ds + \Pi(\tau)B_2u(k),$$

$$\begin{aligned}
z_c(t) &= C_1 e^{A(t-k\tau)}\hat{x}(k) + C_1 \int_0^{t-k\tau-s} e^{A(t-k\tau-s)} B_1 \hat{w}_k(s) ds \\
&\qquad + C_1 \Pi(t-k\tau) B_2 u(k), \\
z_d(k) &= \sqrt{\tau} D_{12} u(k), \\
y(k) &= C_2 \hat{x}(k) + D_{21} w_d(k).
\end{aligned}$$

Contrary to the discrete-time representation $\tilde{\mathbf{G}}$ of the sampled-data system $\mathbf{G}_s$, the jump system $\mathbf{G}_e$ is a natural state space representation of $\mathbf{G}_s$ in the following sense.

(a) The genuine control input to $\mathbf{G}_s$ is $u(k)$ rather than $\tilde{u}(t)$.
(b) Original signals and parameters of $\mathbf{G}_s$ are maintained in the system $\mathbf{G}_e$ [37, 65].
(c) The H_∞ and H_2 problems can be treated in a unified manner as in Chapters 1-3. Hence it is easy to introduce the theory to those who are not familiar with sampled-data systems.
(d) The jump system approach to sampled-data control can be easily extended to more general cases of delayed observation [45], a first-order hold [32, 34] and infinite dimensions [33] (see Chapter 6).

Let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \; \bar{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}' & Y_2 \end{bmatrix}$$

be the solutions of the Riccati equations (4.46)-(4.49) and (4.50)-(4.52), respectively with $T_2(k) = I + B_2' X(k\tau) B_2$ replaced by $T_2(k) = \tau I + B_2' X(k\tau) B_2$, where X_1, $Y_1 \in \mathbf{R}^{n\times n}$, X_{12}, $Y_{12} \in \mathbf{R}^{n\times m_2}$ and X_2, $Y_2 \in \mathbf{R}^{m_2\times m_2}$ and n and m_2 are the dimensions of x and u respectively. Then from the Riccati equation (4.46)-(4.49) we obtain

$$\begin{aligned}
-\dot{X}_1 &= A'X_1 + X_1 A + C_1' C_1 + \frac{1}{\gamma^2} X_1 B_1 B_1' X_1, \\
-\dot{X}_{12} &= A' X_{12} + X_1 B_2 + \frac{1}{\gamma^2} X_1 B_1 B_1' X_{12}, \\
-\dot{X}_2 &= B_2' X_{12} + X_{12}' B_2 + \frac{1}{\gamma^2} X_{12}' B_1 B_1' X_{12}
\end{aligned} \tag{5.22}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$, $k = 0, 1, , 2, ..$

$$\begin{aligned}
X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\
X_{12}(k\tau^-) &= 0, \\
X_2(k\tau^-) &= 0
\end{aligned} \tag{5.23}$$

with

$$\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(T) = \begin{bmatrix} F'F & 0 \\ 0 & 0 \end{bmatrix} \tag{5.24}$$

and

$$H'X_1(0^-)H \le d^2 I \text{ for some } 0 < d < \gamma. \tag{5.25}$$

The second equation is written

$$\begin{aligned}
\dot{Y}_1 &= AY_1 + Y_1A' + B_1B_1' + \frac{1}{\gamma^2}Y_1C_1'C_1Y_1 + B_2Y_{12}' + Y_{12}B_2', \\
\dot{Y}_{12} &= AY_{12} + B_2Y_2 + \frac{1}{\gamma^2}Y_1C_1'C_1Y_{12}, \\
\dot{Y}_2 &= \frac{1}{\gamma^2}Y_{12}C_1'C_1Y_{12}
\end{aligned}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$, $k = 1, 2, ..$

$$\begin{aligned}
Y_1(k\tau^+) &= Y_1(k\tau) - Y_1(k\tau)C_2'(I + C_2Y_1(k\tau)C_2')^{-1}C_2Y_1(k\tau), \\
Y_{12}(k\tau^+) &= 0, \\
Y_2(k\tau^+) &= 0, \qquad\qquad k = 0, 1, ..
\end{aligned}$$

with

$$\begin{bmatrix} Y_1(0) & Y_{12}(0) \\ Y_{12}'(0) & Y_2(0) \end{bmatrix} = \begin{bmatrix} HH' & 0 \\ 0 & 0 \end{bmatrix}.$$

Since Y_{12} and Y_2 form a homogeneous system, we conclude $Y_{12} = 0$ and $Y_2 = 0$. Hence $\bar{Y}$ is of the form $\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$, where $Y \in \mathbf{R}^{n\times n}$ is the solution of

$$\begin{aligned}
\dot{Y} &= AY + YA + B_1B_1' + \frac{1}{\gamma^2}YC_1'C_1Y, && (5.26) \\
&\qquad k\tau < t < (k+1)\tau, \\
Y(k\tau^+) &= Y(k\tau) - Y(k\tau)C_2'(I + C_2Y(k\tau)C_2')^{-1}C_2Y(k\tau), && (5.27) \\
Y(0) &= HH'. && (5.28)
\end{aligned}$$

Replacing Z by $(I - \frac{1}{\gamma^2}YX)^{-1}Y$ in (4.57), we obtain a γ-suboptimal controller

$$\begin{aligned}
\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ 0 & 0 \end{bmatrix}(t)\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \ k\tau < t < (k+1)\tau, && (5.29) \\
\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau^+) &= \begin{bmatrix} \hat{A}_d & 0 \\ \hat{C}_1 & 0 \end{bmatrix}(k)\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \begin{bmatrix} \hat{B}_1 \\ \hat{D}_{11} \end{bmatrix}(k)y(k) \\
&\qquad + \begin{bmatrix} \hat{B}_2 \\ \hat{D}_{12} \end{bmatrix}(k)v(k), \\
u(k) &= [\hat{C}_1(k) \quad 0]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\
r(k) &= [\hat{C}_2(k) \quad 0]\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) + \hat{D}_{21}(k)y(k), \\
v &= Qr, \ Q \in Q_\gamma
\end{aligned}$$

where

$$
\begin{aligned}
\hat{A}_1(t) &= A + \frac{1}{\gamma^2} B_1 B_1' X_1(t), \\
\hat{A}_2(t) &= B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t), \\
\hat{A}_d(k) &= (I + W(k) Y(k\tau) C_2' C_2)^{-1}, \\
\hat{B}_1(k) &= \hat{A}_d(k) W(k) Y(k\tau) C_2', \\
\hat{B}_2(k) &= \frac{1}{\gamma} \hat{A}_d(k) W(k) Y(k\tau) X_{12}(k\tau) E^{-\frac{1}{2}}(k) \Xi^{-\frac{1}{2}}(k), \\
\hat{C}_1(k) &= -E^{-1}(k) X_{12}'(k\tau) \hat{A}_d(k), \\
\hat{C}_2(k) &= -(I + C_2 W(k) Y(k\tau) C_2')^{-\frac{1}{2}} C_2, \\
\hat{D}_{11}(k) &= -E^{-1}(k) X_{12}'(k\tau) \hat{A}_d(k) W(k) Y(k\tau) C_2', \\
\hat{D}_{12}(k) &= \frac{1}{\gamma} E^{-\frac{1}{2}}(k) \Xi^{\frac{1}{2}}(k), \\
\hat{D}_{21}(k) &= (I + C_2 W(k) Y(k\tau) C_2')^{-\frac{1}{2}}, \\
W(k) &= [I - \frac{1}{\gamma^2} Y(k\tau) X_1(k\tau^-)]^{-1}, \\
E(k) &= \tau I + X_2(k\tau), \\
\Xi(k) &= \gamma^2 I - E^{-\frac{1}{2}}(k) X_{12}'(k\tau) \hat{A}_d(k) W(k) Y(k\tau) X_{12}(k\tau) E^{-\frac{1}{2}}(k)
\end{aligned}
\tag{5.30}
$$

and

$$
\begin{aligned}
Q_\gamma = \{ & Q \in \mathcal{L}(l^2(0, N; \mathbf{R}^{p_2}); l^2(0, N; \mathbf{R}^{m_2})) : \\
& \text{Q is of the form (4.45) and } \| Q \| < \gamma \}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\dot{\hat{x}}_2 &= 0, \ k\tau < t < (k+1)\tau, \\
\hat{x}_2(k\tau^+) &= \hat{C}_1(k) \hat{x}_1(k\tau) + \hat{D}_{11}(k) y(k) + \hat{D}_{12}(k) v(k),
\end{aligned}
$$

we can rewrite (5.29) as

$$
\begin{aligned}
\dot{\hat{x}} &= \hat{A}_1(t) \hat{x} + \hat{A}_2(t) \tilde{s}(t), \ k\tau < t < (k+1)\tau, \\
\hat{x}(k\tau^+) &= \hat{A}_d(k) \hat{x}(k\tau) + \hat{B}_1(k) y(k) + \hat{B}_2(k) v(k), \\
u(k) &= \hat{C}_1(k) \hat{x}(k\tau) + \hat{D}_{11}(k) y(k) + \hat{D}_{12}(k) v(k), \\
r(k) &= \hat{C}_2(k) \hat{x}(k\tau) + \hat{D}_{21}(k) y(k), \\
v &= Qr, \ Q \in Q_\gamma
\end{aligned}
\tag{5.31}
$$

where $\tilde{s}$ is given by

$$
\tilde{s}(t) = u(k), \ k\tau < t \le (k+1)\tau.
$$

Summing up we have the following result.

Theorem 5.3 *Assume* **S1** *and consider the system* $\mathbf{G}_s$.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the following hold:*
(i) There exists a nonnegative solution $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t)$, $t \in [0,T]$, $X_1 \in \mathbf{R}^{n\times n}$, $X_{12} \in \mathbf{R}^{n\times m_2}$, $X_2 \in \mathbf{R}^{m_2\times m_2}$ *to the Riccati equation (5.22)-(5.25).*
(ii) There exists a nonnegative solution Y *to the Riccati equation (5.26)-(5.28).*
(iii) $\rho(\begin{bmatrix} X_1 Y \\ X_{12} Y \end{bmatrix}(t)) \le d^2$, $t \in [0,T]$, *for some* $0 < d < \gamma$.
(b) In this case the set of all γ*-suboptimal controllers of the form (4.45) is given by (5.31).*

We now convert the controller (5.31) to the usual discrete one. Let $S(\cdot,\cdot)$ be the state transition matrix of $\hat{A}_1$. Then $\hat{x}((k+1)\tau)$ in (5.31) is given by

$$\hat{x}((k+1)\tau) = S((k+1)\tau, k\tau)\hat{x}(k\tau^+) + \int_{k\tau}^{(k+1)\tau} S((k+1)\tau, r)A_2(r)\tilde{s}(r)dr.$$

Since $\tilde{s}(t)$, $k\tau < t \le (k+1)\tau$ is given by

$$\tilde{s}(t) = \hat{C}_1(k)\hat{x}(k\tau) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k),$$

we have

$$\hat{x}((k+1)\tau) = A_D(k)\hat{x}(k\tau) + B_{1D}(k)y(k) + B_{2D}(k)s(k)$$

where

$$\begin{aligned} A_D(k) &= S((k+1)\tau, k\tau)\hat{A}_d(k) + (\Theta\hat{C}_1)(k), \\ B_{1D}(k) &= S((k+1)\tau, k\tau)\hat{B}_1(k) + (\Theta\hat{D}_{11})(k), \\ B_{2D}(k) &= S((k+1)\tau, k\tau)\hat{B}_2(k) + (\Theta\hat{D}_{12})(k) \end{aligned}$$

and

$$\Theta(k) = \int_{k\tau}^{(k+1)\tau} S((k+1)\tau, r)\hat{A}_2(r)dr. \tag{5.32}$$

Hence the controller (5.31) is equivalent to the following discrete-time controller:

$$\begin{aligned} \hat{x}(k+1) &= A_D(k)\hat{x}(k) + B_{1D}(k)y(k) + B_{2D}(k)s(k), \\ u(k) &= \hat{C}_1(k)\hat{x}(k) + \hat{D}_{11}(k)y(k) + \hat{D}_{12}(k)v(k), \\ r(k) &= \hat{C}_2(k)\hat{x}(k) + \hat{D}_{21}(k)y(k), \\ v &= Qr,\ Q \in Q_\gamma^D \end{aligned} \tag{5.33}$$

where

$$\begin{aligned} Q_\gamma^D &= \{Q \in \mathcal{L}(l^2(0,N;\mathbf{R}^{p_2}); l^2(0,N;\mathbf{R}^{m_2})) : \\ &\qquad\qquad Q \text{ is of the form (5.21) and } \| Q \| < \gamma\}. \end{aligned}$$

Hence we have the following result.

Theorem 5.4 *Assume* **S1** *and consider the system* $\mathbf{G}_s$.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the conditions (i)-(iii) in Theorem 5.3 hold.*
(b) In this case the set of all γ*-suboptimal controllers of the form (5.21) is given by (5.33).*

5.2.2 The Infinite Horizon Problem

Next we consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \ \ x(0) = Hh, \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned}$$

on $[0,\infty)$ and a controller $u = Ky$ of the form (5.21) where we assume **S1** and

S2 : (A, B_1, C_1) is stabilizable and detectable,
S3 : (A, B_2, C_2) is stabilizable and detectable,
S4 : The sampling period τ is nonpathological.

Assumptions **S1-S4** imply **J1-J4** for G_e. If the controller is IO-stabilizing (or internally stabilizing), then the closed-loop system is defined by

$$z = G\begin{pmatrix} h \\ w \\ w_d \end{pmatrix}.$$

Then

$$\begin{aligned} G \ \in \ & \mathcal{L}(\mathbf{R}^{n_1} \times L^2(0,\infty;\mathbf{R}^{m_1}) \times l^2(0,\infty;\mathbf{R}^{m_{1d}}); \\ & \qquad L^2(0,\infty;\mathbf{R}^{p_1}) \times l^2(0,\infty;\mathbf{R}^{p_{1d}})). \end{aligned}$$

The H_∞-problem on $[0,\infty)$ is to find necessary and sufficient conditions for the existence of a γ-suboptimal controller, i.e., an internally stabilizing discrete-time controller such that $\| G \| < \gamma$, i.e.,

$$\left\| \begin{pmatrix} z_c \\ z_d \end{pmatrix} \right\|^2_{L^2 \times l^2} \leq d^2\left(| h |^2 + \left\| \begin{pmatrix} w \\ w_d \end{pmatrix} \right\|^2_{L^2 \times l^2}\right) \text{ for some } 0 < d < \gamma.$$

Such a controller is called γ-suboptimal.

To give the solution of this problem, we first consider the stabilizing solutions of the Riccati equations (5.22), (5.23), (5.26) and (5.27). If $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ is a stabilizing solution of the Riccati equation (5.22) and (5.23), then

$$\left(\begin{bmatrix} A + \frac{1}{\gamma^2} B_1 B_1' X_1(t) & B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t) \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ -E^{-1}(k) X_{12}'(k\tau) & 0 \end{bmatrix} \right)$$

is exponentially stable. So the system

$$\begin{aligned} \dot{\xi} &= [A + \frac{1}{\gamma^2} B_1 B_1' X_1(t)]\xi + [B_2 + \frac{1}{\gamma^2} B_1 B_1' X_{12}(t)]\tilde{v}(t), \quad (5.34) \\ \tilde{v}(k) &= -E^{-1}(k) X_{12}'(k\tau)\xi(k\tau), \; k\tau < t \le (k+1)\tau \end{aligned}$$

is exponentially stable. Similarly if Y is a stabilizing solution, then

$$\left(\begin{bmatrix} A + \frac{1}{\gamma^2} Y(t) C_1' C_1 & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I - Y(k\tau) C_2'(I + C_2 Y(k\tau) C_2')^{-1} C_2 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

is exponentially stable, which is equivalent to the exponential stability of the system

$$\begin{aligned} \dot{\xi} &= [A + \frac{1}{\gamma^2} Y(t) C_1' C_1]\xi, \; k\tau < t < (k+1)\tau, \quad (5.35) \\ \xi(k\tau^+) &= [I - Y(k\tau) C_2'(I + C_2 Y(k\tau) C_2')^{-1} C_2]\xi(k\tau). \end{aligned}$$

Remark 5.2 (a) If $X(t) = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t)$ is a τ-periodic nonnegative stabilizing solution (5.22) and (5.23), then the exponential stability of the system (5.34) is equivalent to that of the following discrete-time system

$$\begin{aligned} \xi(k+1) &= [S((k+1)\tau, k\tau) - \Theta(k) E^{-1}(k) X_{12}(k\tau)]\xi(k) \\ &= [S(\tau, 0) - \Theta(0) E^{-1}(0) X_{12}(0)]\xi(k) \end{aligned}$$

where $S(\cdot,\cdot)$ is the state transition matrix of $A + \frac{1}{\gamma^2} B_1 B_1' X_1$ and $\Theta(k)$ is defined by (5.32).
(b) The exponential stability of the system (5.35) is equivalent to that of the following time-varying discrete-time system

$$\xi(k+1) = S_Y((k+1)\tau, k\tau)[I - Y(k\tau) C_2'(I + C_2 Y(k\tau) C_2')^{-1} C_2]\xi(k) \quad (5.36)$$

where $S_Y(\cdot,\cdot)$ is the state transition matrix of $A + \frac{1}{\gamma^2} Y C_1' C_1$. If $Y(t)$ is τ-periodic, then the system (5.36) becomes

$$\xi(k+1) = S_Y(\tau, 0)[I - Y(0) C_2'(I + C_2 Y(0) C_2')^{-1} C_2]\xi(k)$$

which is time-invariant.

Again we define Q_γ and Q_γ^D as

$$\begin{aligned} Q_\gamma &= \{Q \in \mathcal{L}(l^2(0,\infty;\mathbf{R}^{p_2}); l^2(0,\infty;\mathbf{R}^{m_2})) : \\ &\qquad \text{Q is of the form (4.45) and internally stable with } \| Q \| < \gamma\}, \\ Q_\gamma^D &= \{Q \in \mathcal{L}(l^2(0,\infty;\mathbf{R}^{p_2}); l^2(0,\infty;\mathbf{R}^{m_2})) : \\ &\qquad Q \text{ is of the form (5.21) and internally stable with } \| Q \| < \gamma\}. \end{aligned}$$

Then we have the following results.

Theorem 5.5 *Consider the system* $\mathbf{G}_s$ *on* $[0,\infty)$ *with the assumptions* **S1-S4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *of the form (5.21) if and only if the following hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ *to the Riccati equation (5.22), (5.23) and (5.25).*
(ii) There exists a bounded nonnegative stabilizing solution Y *to the Riccati equation (5.26)-(5.28).*
(iii) $\rho(\begin{bmatrix} X_1 Y \\ X_{12} Y \end{bmatrix}(t)) \le d^2$, $t \in [0,\infty)$, *for some* $0 < d < \gamma$.
(b) The set of all γ*-suboptimal controllers of the form (4.45) is also given by (5.31) with* Q *internally stable.*
(c) In this case the set of all γ*-suboptimal controllers of the form (5.21) is given by (5.33).*
Moreover the $\lim_{n\to\infty} Y(t+n\tau)$ *exists (denoted by* Y_τ*) and* Y_τ *is a* τ*-periodic nonnegative stabilizing solution to (5.26) and (5.27).*

Since the solution Y in (ii) is not τ-periodic, γ-suboptimal controllers (5.31) and (5.33) are in general time-varying. However applying Corollary 4.8 we also obtain τ-periodic controllers and time-invariant discrete-time controllers.

Theorem 5.6 *Consider the system* $\mathbf{G}_s$ *with* $h = 0$ *on* $[0,\infty)$ *and assume* **S1-S4**.
(a) There exists a γ*-suboptimal controller* $u = Ky$ *on* $[0,\infty)$ *of the form (5.21) if and only if the following hold:*
(i) There exists a τ*-periodic nonnegative stabilizing solution* $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ *to the Riccati equation (5.22) and (5.23).*
(ii) There exists a τ*-periodic nonnegative stabilizing solution* Y_τ *to the Riccati equation (5.26) and (5.27).*
(iii) $\rho(\begin{bmatrix} X_1 Y_\tau \\ X_{12} Y_\tau \end{bmatrix}(t)) \le d^2$, $t \in [0,\tau)$, *for some* $0 < d < \gamma$.
(b) In this case the following controllers are γ*-suboptimal:*

$$\dot{\hat{x}} = \hat{A}_1(t) + \hat{A}_2(t)\tilde{s}(t), \;\; k\tau < t < (k+1)\tau,$$

$$\begin{aligned}\hat{x}(k\tau^+) &= \hat{A}_d(0)\hat{x}(k\tau) + \hat{B}_1(0)y(k) + \hat{B}_2(0)v(k), \\ u(k) &= \hat{C}_1(0)\hat{x}(k\tau) + \hat{D}_{11}(0)y(k) + \hat{D}_{12}(0)v(k), \\ r(k) &= \hat{C}_2(0)\hat{x}(k\tau) + \hat{D}_{21}(0)y(k), \\ v &= Qr, \ Q \in Q_\gamma\end{aligned} \tag{5.37}$$

where $\hat{A}_d(0)$, $\hat{B}_1(0)$ are defined by (5.30) with Y replaced by Y_τ. Moreover, controllers given by (5.37) with τ-periodic Q are τ-periodic.
(c) Discrete-time controllers given by (5.33) with Y replaced by Y_τ are γ-suboptimal. Moreover, if we restrict $Q \in Q_\gamma^D$ to be time-invariant, the controllers (5.33) are time-invariant.

Remark 5.3 Some comments on the comparison of the lifting technique and the jump system approach to sampled-data H_∞ control are found in [64, 77].

Example 5.3 Consider the system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{u}(t), \\ z &= \begin{bmatrix} [1 \quad 0]x \\ \tilde{u}(t) \end{bmatrix}, \\ y(k) &= [1 \quad 0]x(k) + w_d(k)\end{aligned}$$

where $\tilde{u}(t) = u(k)$, $k < t \le k+1$. For this system all the assumptions **S1-S4** are satisfied. For all $\gamma \ge 2.1$, the conditions (i)-(iii) of Theorems 5.5 and 5.6 are satisfied. Figure 5.3 shows the periodic solution $X(t) = [X_{ij}(t)]$, $i, j = 1, 2, 3$ of the Riccati equation (5.22) and (5.23) with $\gamma = 2.1$ and period 1. Figure 5.4 shows the bounded nonnegative stabilizing solution $Y(t) = [Y_{ij}(t)]$, $i, j = 1, 2$ of the Riccati equation (5.26)-(5.28) which converges to a periodic solution. Figure 5.5 shows that the condition (iii) of both Theorems 5.5 and 5.6 are satisfied. In this case a central discrete-time controller is given by

$$\begin{aligned}\hat{x}(k+1) &= \begin{bmatrix} -0.3683 & 0.5812 \\ -0.0282 & 0.0313 \end{bmatrix} \hat{x}(k) + \begin{bmatrix} 0.9707 \\ -0.7417 \end{bmatrix} y(k), \\ \hat{u}(k) &= [0.4982 \quad -0.7709]\hat{x}(k) - 0.3729y(k).\end{aligned}$$

Figure 5.6 shows the simulation result of the closed-loop system with the central discrete-time controller where $\gamma = 2.1$ and the disturbances $w(t) = 10e^{-10t}\sin 10t$ and $w_d(k) = 0$.

5.3 H_2 Control

As in the previous section we apply the H_2 theory for jump systems to the sampled-data systems.

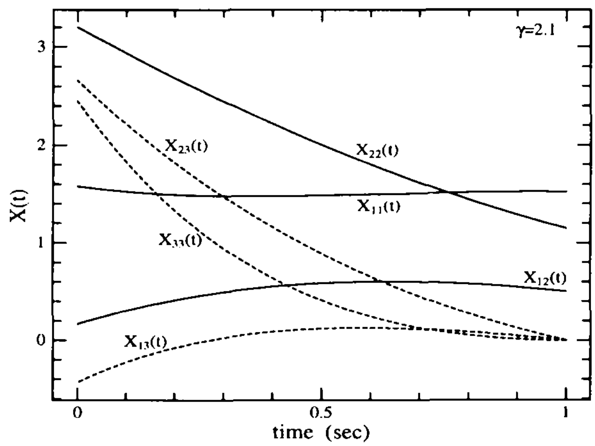

Figure 5.3: The periodic nonnegative solution $X(t)$

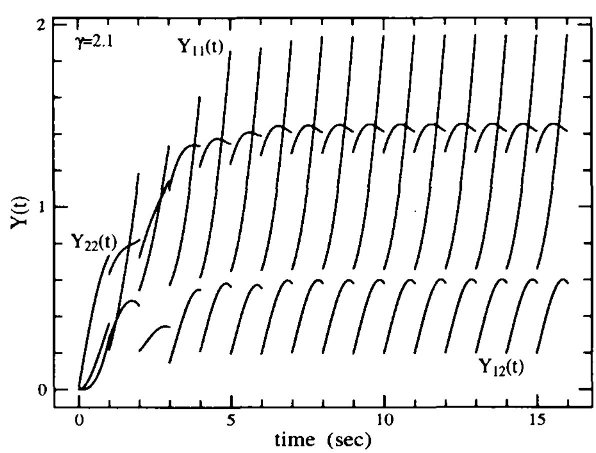

Figure 5.4: The bounded nonnegative solution $Y(t)$

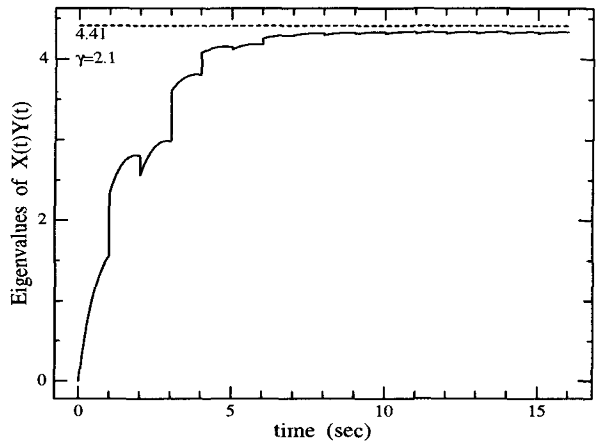

Figure 5.5: Eigenvalues of $X(t)Y(t)$

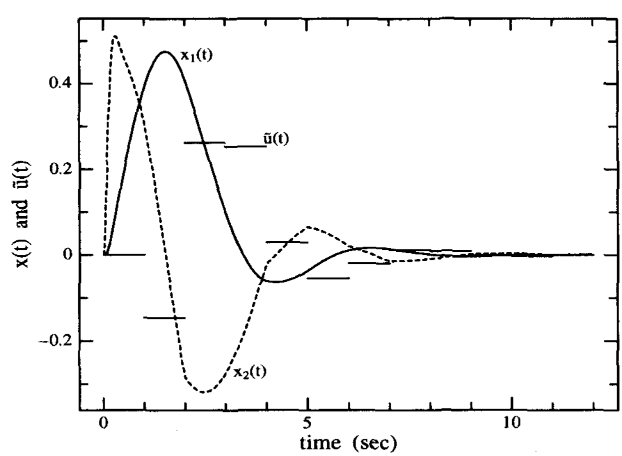

Figure 5.6: Simulation result

Consider the sampled-data system $\mathbf{G}_s$:

$$\begin{aligned} \dot{x} &= Ax(t) + B_1 w(t) + B_2 \tilde{u}(t), \\ z(t) &= \begin{bmatrix} C_1 x(t) \\ D_{12}\tilde{u}(t) \end{bmatrix}, \\ y(k) &= C_2 x(k\tau) + D_{21} w_d(k) \end{aligned} \tag{5.38}$$

and a discrete-time controller of the form

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k) + \hat{D}y(k). \end{aligned} \tag{5.39}$$

We assume **S1-S4**. To formulate the H_2-problem for $\mathbf{G}_s$ we introduce the following set of controllers

$$\mathbf{K} = \{K : K \text{ is of the form (5.39) and internally stabilizes the system } \mathbf{G}_s\}.$$

The H_2 control problem for the system $\mathbf{G}_s$ is to find an internally stabilizing controller which minimizes $\| G \|_2$, where G is the input-output operator of the closed-loop system defined by

$$z = G\begin{pmatrix} w \\ w_d \end{pmatrix}.$$

Since $\mathbf{G}_s$ is equivalent to the jump system $\mathbf{G}_e$ and the assumptions **S1** – **S4** imply the assumptions **J1** – **J4** for $\mathbf{G}_e$, we can apply Theorem 4.24 to the system $\mathbf{G}_e$.

As in the H_∞ control problem, let

$$X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}, \ \bar{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}' & Y_2 \end{bmatrix}$$

be the solutions of the Riccati equations (4.163)-(4.166) respectively, with $T_2(k) = I + B_2' X(k\tau) B_2$ replaced by $T_2(k) = \tau I + B_2' X(k\tau) B_2$, where X_1, $Y_1 \in \mathbf{R}^{n\times n}$, X_{12}, $Y_{12} \in \mathbf{R}^{n\times m_2}$ and X_2, $Y_2 \in \mathbf{R}^{m_2\times m_2}$ and n and m_2 are the dimensions of x and u respectively. Then from the first Riccati equation (4.163) and (4.164), we obtain for $k\tau < t < (k+1)\tau$

$$-\begin{bmatrix} \dot{X}_1 & \dot{X}_{12} \\ \dot{X}_{12}' & \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}' X + X \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C_1' C_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.40}$$

and at $t = k\tau$, $k = 0, 1, 2, ..$

$$\begin{aligned} X_1(k\tau^-) &= X_1(k\tau) - X_{12}(k\tau)[\tau I + X_2(k\tau)]^{-1} X_{12}'(k\tau), \\ X_{12}(k\tau^-) &= 0, \\ X_2(k\tau^-) &= 0. \end{aligned} \tag{5.41}$$

The second Riccati equation (4.165) and (4.166) is written

$$\begin{aligned}\dot{Y}_1 &= AY_1 + Y_1A' + \frac{1}{\tau}B_1B_1' + B_2Y_{12}' + Y_{12}'B_2,\\ \dot{Y}_{12} &= AY_{12} + B_2Y_2',\\ \dot{Y}_2 &= 0\end{aligned} \tag{5.42}$$

for $k\tau < t < (k+1)\tau$ and at $t = k\tau$

$$\begin{aligned}Y_1(k\tau^+) &= Y_1(k\tau) - Y_1(k\tau)C_2'(I + C_2Y_1(k\tau)C_2')^{-1}C_2Y_1(k\tau),\\ Y_{12}(k\tau^+) &= 0,\\ Y_2(k\tau^+) &= 0.\end{aligned} \tag{5.43}$$

Note that

$$\bar{Y}(t) = \lim_{n\to\infty} \hat{Y}(t + n\tau),$$

as we see in Remark 4.6 where $\hat{Y} = \begin{bmatrix} \hat{Y}_1 & \hat{Y}_{12} \\ \hat{Y}_{12}' & \hat{Y}_2 \end{bmatrix}$ is the solution of (5.42) and (5.43) with $\hat{Y}(0) = 0$. Since $\hat{Y}_{12}(t)$, $\hat{Y}_2(t)$ form a homogeneous system with $\hat{Y}_{12}(0) = 0$ and $\hat{Y}_2(0) = 0$, we conclude $\hat{Y}_{12}(t) = 0$ and $\hat{Y}_2(t) = 0$ for all $t \geq 0$ and $\bar{Y}$ has the form $\begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}$ where $Y \in \mathbf{R}^{n\times n}$ is the solution of

$$\dot{Y} = AY + YA' + \frac{1}{\tau}B_1B_1', \; k\tau < t < (k+1)\tau, \tag{5.44}$$

$$Y(k\tau^+) = Y(k\tau) - Y(k\tau)C_2'(I + C_2Y(k\tau)C_2')^{-1}C_2Y(k\tau). \tag{5.45}$$

If $X = \begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}$ is a τ-periodic nonnegative stabilizing solution,

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I & 0 \\ F_s & 0 \end{bmatrix}\right)$$

is exponentially stable where

$$F_s = -[\tau I + X_2(0)]^{-1}X_{12}'(0).$$

So the system

$$\dot{\xi} = A\xi + B_2\tilde{v}(t), \;\; \tilde{v}(t) = F_s\hat{\xi}(k\tau), \; k\tau < t \leq (k+1)\tau \tag{5.46}$$

is exponentially stable. Similarly if Y is a τ-periodic stabilizing solution,

$$\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I + H_sC_2 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

is exponentially stable where

$$H_s = -Y(0)C_2'[I + C_2Y(0)C_2']^{-1},$$

which is equivalent to the exponentially stability of the system

$$\begin{aligned} \dot{\xi} &= A\xi, \ k\tau < t < (k+1)\tau, \\ \xi(k\tau^+) &= (I + H_s C_2)\xi(k\tau). \end{aligned} \tag{5.47}$$

Remark 5.4 The exponential stability of the systems (5.46) and (5.47) are equivalent respectively to that of the following time-invariant discrete-time systems:

$$\begin{aligned} \xi(k+1) &= (\tilde{A} + \tilde{B}_2 F_s)\xi(k), \ \tilde{A} = e^{A\tau}, \ \tilde{B}_2 = \int_0^\tau e^{At} dt B_2, \\ \xi(k+1) &= (\tilde{A} + \tilde{H}_s C_2)\xi(k), \ \tilde{H}_s = e^{A\tau} H_s. \end{aligned}$$

From (4.167) the controller given by

$$\begin{aligned} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} &= \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}, \ k\tau < t < (k+1)\tau, \\ \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau^+) &= \begin{bmatrix} I + H_s C_2 & 0 \\ F_s(I + H_s C_2) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) - \begin{bmatrix} H_s \\ F_s H_s \end{bmatrix} y(k), \\ u(k) &= \begin{bmatrix} F_s(I + H_s C_2) & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}(k\tau) - F_s H_s y(k) \end{aligned} \tag{5.48}$$

is optimal. Since

$$\begin{aligned} \dot{\hat{x}}_2 &= 0, \ k\tau < t < (k+1)\tau, \\ \hat{x}_2(k\tau^+) &= F_s(I + H_s C_2)\hat{x}_1(k\tau) - F_s H_s y(k), \end{aligned}$$

we can rewrite (5.48) as

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 \tilde{v}(t), \ k\tau < t < (k+1)\tau, \\ \hat{x}(k\tau^+) &= (I + H_s C_2)\hat{x}(k\tau) - H_s y(k), \\ u(k) &= F_s(I + H_s C_2)\hat{x}(k\tau) - F_s H_s y(k) \end{aligned} \tag{5.49}$$

where $\tilde{v}$ is given by $\tilde{v}(t) = F_s(I + H_s C_2)\hat{x}(k\tau) - F_s H_s y(k)$, $k\tau < t \leq (k+1)\tau$. The controller (5.49) is equivalent to the following discrete-time controller

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}\hat{x}(k) + \hat{B}y(k), \\ u(k) &= \hat{C}\hat{x}(k\tau) + \hat{D}y(k) \end{aligned} \tag{5.50}$$

where

$$\begin{aligned} \hat{A} &= \tilde{A} + \tilde{B}_2 F_s + \tilde{H}_s C_2 + \tilde{B}_2 F_s H_s C_2, \\ \hat{B} &= -(\tilde{H}_s + \tilde{B}_2 F_s H_s), \\ \hat{C} &= F_s + F_s H_s C_2, \\ \hat{D} &= -F_s H_s. \end{aligned}$$

Finally the optimal value is given by

$$
\begin{aligned}
&\min_{K\in\mathbf{K}} \| G \|_2^2 \\
= \; & \frac{1}{\tau}\int_0^\tau tr.\begin{bmatrix} B_1' & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(s)\begin{bmatrix} B_1 \\ 0 \end{bmatrix} ds \\
& + tr.[\tau I + X_2(0)]^{\frac{1}{2}}\begin{bmatrix} F_s & 0 \end{bmatrix}\begin{bmatrix} [I+Y(0)C_2'C_2]^{-1}Y(0) & 0 \\ 0 & 0 \end{bmatrix} \\
& \qquad \times \begin{bmatrix} F_s' \\ 0 \end{bmatrix}[\tau I + X_2(0)]^{\frac{1}{2}} \\
= \; & \frac{1}{\tau}\int_0^\tau tr.B_1'X_1(s)B_1 ds + tr.[\tau I + X_2(0)]F_s[I+Y(0)C_2'C_2]^{-1}Y(0)F_s'.
\end{aligned}
$$

Since $Y(0^+) = [I+Y(0)C_2'C_2]^{-1}Y(0)$ we can rewrite $\min_{K\in\mathbf{K}} \| G \|_2^2$ as

$$
\begin{aligned}
\min_{K\in\mathbf{K}} \| G \|_2^2 \;=\; & \frac{1}{\tau}\int_0^\tau tr.B_1'X_1(s)B_1 ds \\
& + tr.[\tau I + X_2(0)]F_sY(0^+)F_s'. \qquad (5.51)
\end{aligned}
$$

Summing up we have the following.

Theorem 5.7 *Assume* **S1-S4** *and consider the H_2-problem for* $\mathbf{G}_s$*. Then the controller (5.49) (and hence (5.50)) is optimal and the minimum H_2 norm is given by (5.51).*

We now compare our results with the known results in [2, 8, 50]. By (5.40) for $k\tau < t \le (k+1)\tau$ we have

$$
\begin{aligned}
&\begin{bmatrix} X_1 & X_{12} \\ X_{12}' & X_2 \end{bmatrix}(t) \\
= \; & \int_t^{(k+1)\tau} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}'(r-t)}\begin{bmatrix} C_1'C_1 & 0 \\ 0 & 0 \end{bmatrix} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}(r-t)} dr \\
& + e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}'((k+1)\tau-t)}\begin{bmatrix} X_1((k+1)\tau^-) & 0 \\ 0 & 0 \end{bmatrix} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}((k+1)\tau-t)}.
\end{aligned}
$$

Since

$$
e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} t} = \begin{bmatrix} e^{At} & \int_0^t e^{At}dtB_2 \\ 0 & I \end{bmatrix},
$$

we have

$$
\begin{aligned}
X_1(t) \;=\; & e^{A'((k+1)\tau-t)}X_1((k+1)\tau^-)e^{A((k+1)\tau-t)} \\
& + \int_t^{(k+1)\tau} e^{A'(r-t)}C_1'C_1e^{A(r-t)}dr,
\end{aligned}
$$

$$
\begin{aligned}
X_{12}(t) &= e^{A'((k+1)\tau-t)}X_1((k+1)\tau^-)\int_0^{(k+1)\tau-t} e^{As}dsB_2 \\
&\quad + \int_t^{(k+1)\tau} e^{A'(r-t)}C_1'C_1\int_0^{r-t} e^{As}dsB_2dr, \\
X_2(t) &= B_2'\int_0^{(k+1)\tau-t} e^{A's}dsX_1((k+1)\tau^-)\int_0^{(k+1)\tau-t} e^{As}dsB_2 \\
&\quad + \int_t^{(k+1)\tau} B_2'\int_0^{r-t} e^{A's}dsC_1'C_1\int_0^{r-t} e^{As}dsB_2dr
\end{aligned}
$$

and

$$
\begin{aligned}
X_1(k\tau) &= \tilde{A}'X_1((k+1)\tau^-)\tilde{A} + \tilde{C}_1'\tilde{C}_1, \\
X_{12}(k\tau) &= \tilde{A}'X_1((k+1)\tau^-)\tilde{B}_2 + \tilde{C}_1\tilde{D}_{12}, \\
X_2(k\tau) &= \tilde{B}_2'X_1((k+1)\tau^-)\tilde{B}_2 + \int_0^\tau B_2'\int_0^t e^{A's}dsC_1'C_1\int_0^t e^{As}dsB_2dt
\end{aligned}
$$

where

$$
\begin{bmatrix} \tilde{C}_1' \\ \tilde{D}_{12}' \end{bmatrix}\begin{bmatrix} \tilde{C}_1 & \tilde{D}_{12} \end{bmatrix} = \int_0^\tau e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix}' t}\begin{bmatrix} C_1' \\ D_{12}' \end{bmatrix}\begin{bmatrix} C_1 & D_{12} \end{bmatrix} e^{\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} t} dt.
$$

Hence the Riccati equation (5.40) and (5.41) is equivalent to

$$
\begin{aligned}
X(t) &= e^{A'((k+1)\tau-t)}\tilde{X}e^{A((k+1)\tau-t)} + \int_t^{(k+1)\tau} e^{A'(r-t)}C_1'C_1e^{A(r-t)}dr, \\
&\qquad k\tau \le t < (k+1)\tau, \\
\tilde{X} &= \tilde{A}'\tilde{X}\tilde{A} + \tilde{C}_1'\tilde{C}_1 \\
&\quad -(\tilde{A}'\tilde{X}\tilde{B}_2 + \tilde{C}_1'\tilde{D}_{12})(\tilde{D}_{12}'\tilde{D}_{12} + \tilde{B}_2'\tilde{X}\tilde{B}_2)^{-1}(\tilde{B}_2'\tilde{X}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1)
\end{aligned}
$$

where $\tilde{X} = X_1(k\tau^-) = X_1(0^-)$. Similarly the Riccati equation (5.44) and (5.45) is equivalent to

$$
\begin{aligned}
Y(t) &= e^{A(t-k\tau)}Y(k\tau)e^{A'(t-k\tau)} + \int_{k\tau}^t e^{A(t-r)}(\frac{1}{\tau}B_1B_1')e^{A'(t-r)}dr, \\
&\qquad k\tau < t \le (k+1)\tau, \\
Y((k+1)\tau) &= \tilde{A}Y(k\tau)\tilde{A}' + \tilde{B}_1\tilde{B}_1' \\
&\quad -\tilde{A}'Y(k\tau)C_2'(I + C_2Y(k\tau)C_2')^{-1}C_2Y(k\tau)\tilde{A}
\end{aligned}
$$

where $\tilde{B}_1\tilde{B}_1' = \frac{1}{\tau}\int_0^\tau e^{At}B_1B_1'e^{A't}dt$. We also have

$$
\begin{aligned}
F_s &= -(\tilde{D}_{12}'\tilde{D}_{12} + \tilde{B}_2'\tilde{X}\tilde{B}_2)^{-1}(\tilde{B}_2'\tilde{X}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1), \\
\tilde{H}_s &= \tilde{A}H_s = -\tilde{A}Y^0C_2'(I + C_2Y^0C_2')^{-1}\ Y^0 = Y(0).
\end{aligned}
$$

Remark 5.5 The optimal controller (5.50) is obtained via the two algebraic Riccati equations:

$$\begin{aligned}
\tilde{X} &= \tilde{A}'\tilde{X}\tilde{A} + \tilde{C}_1'\tilde{C}_1 \\
&\quad -(\tilde{A}'\tilde{X}\tilde{B}_2 + \tilde{C}_1'\tilde{D}_{12})(\tilde{D}_{12}'\tilde{D}_{12} + \tilde{B}_2'\tilde{X}\tilde{B}_2)^{-1}(\tilde{B}_2'\tilde{X}\tilde{A} + \tilde{D}_{12}'\tilde{C}_1), \\
Y^0 &= \tilde{A}Y^0\tilde{A}' + \tilde{B}_1\tilde{B}_1' - \tilde{A}'Y^0C_2'(I + C_2Y^0C_2')^{-1}C_2Y^0\tilde{A}.
\end{aligned}$$

5.4 Notes and References

This chapter is based on [36, 37].

The transformation of sampled-data system into jump systems was introduced in [68]. The notion of pathological sampling periods and its related results can be found in [8]. The H_∞ problem for sampled-data systems is originally solved using the so-called lifting technique [3, 8, 25, 76, 88]. An advantage of the jump system approach is that original systems are maintained in the formulation and the results are regarded as an extension of those for continuous- and discrete-time systems. The jump system approach is adopted in [64, 68, 77]. Necessary and sufficient conditions for the existence of a γ-suboptimal controller are given in [68]. The equivalence of the jump system approach and the lifting technique is discussed in [64].

The H_2 problem in Section 5.3 can be solved using lifting or FR-operator approach [2, 8, 24, 50]. The solution using the jump system representation is found in [37]. The H_2 and H_∞ problems can be considered within the same framework if the jump system approach is adopted.