On the stability analysis of arbitrarily high-index singular systems with multiple delays

Ha Phi[a,1,*]

[a]*Faculty of Mathematics, Mechanics, and Informatics, VNU University of Science, Vietnam National University, Hanoi, Vietnam.*

Abstract

This paper deals with the class of continuous-time singular linear systems with multiple time-varying delays in a range. The global exponential stability problem of this class of systems is addressed. Delay range-dependent sufficient conditions such that the system is regular, impulse-free and α-stable are developed in the linear matrix inequality (LMI) setting. Moreover, an estimate of the convergence rate of such stable systems is presented. A numerical example is employed to show the usefulness of the proposed results.

Keywords: Singular systems, Delay, LMIs, Spectral, Stabilization, Feedback.
2000 MSC: 34D20, 93D05, 93D20

1. Introduction

Consider the linear singular time-delay system of the form

$$E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \tag{1}$$

{delay-descriptor}

$$x(t) = \phi(t), \text{ for all } t_0 - \tau_m \leqslant t \leqslant t_0, \tag{2}$$

where $E \in \mathbb{R}^{n,n}$ is allowed to be singular. Here the state is $x : [t_0 - \tau_m, \infty) \to \mathbb{R}^n$, and the (constant) time-delays satisfy $0 < \tau_1 < \tau_2 < \ldots < \tau_m$. The capital letters are real-valued matrices of appropriate dimensions. The system is called *free (or DDAE)* if we let $u \equiv 0$, i.e., the system reads

$$E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i)\ . \tag{3}$$

{free system}

*Corresponding author
Email address: haphi.hus@vnu.edu.vn (Ha Phi)

[1]Footnote of first author.

Preprint submitted to Applied Mathematics Letters *July 25, 2021*

The motivation for the system description 1 in the context of designing controllers lies in its generality in modelling interconnected systems.

The rest of the paper is organized as follows. In Section 2, some definitions concerning about the solution and the system classification are stated. Auxiliary Lemmas about the solution's presentation and the non-advanced test are also recalled. In Section 3, our first main results about the stability of arbitrarily high-index system are given, making use of both approaches above. In Sections 4, we discuss the stabilization problem via the Lyapunov-Krasovskii functional method. Finally, in Section 5, a numerical example and the conclusion are given.

2. Preliminaries

To keep the brevity of this research, we refer the interested readers to Ascher and Petzold [1], Campbell [2], Shampine and Gahinet [3], Ha [4], Ha and Mehrmann [5] for the solvability analysis of the IVP (1).

Definition 1. *The null solution $x = 0$ of the free system* (3) *is called* exponentially stable *if there exist positive constants δ and γ such that for any consistent initial function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, the solution $x = x(t, \varphi)$ of the corresponding IVP to* (3) *satisfies*

$$\|x(t)\| \leqslant \delta e^{-\gamma t} \|\varphi\|_\infty, \text{ for every } t \geqslant 0.$$

Definition 2. *i) Consider the DDAE* (1). *The matrix pair (E, A_0) is called* regular *if the* polynomial $\det(\lambda E - A_0)$ *is not identically zero.*
ii) The sets $\sigma(E, A_0, ..., A_m) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A_0 - e^{-\lambda \tau_i} A_i) = 0\}$ is called the spectrum *of* (1).

Provided that the pair (E, A_0) is regular, we can transform them to the Kronecker-Weierstraß canonical form as follows.

Lemma 3. *(Dai [6], Kunkel and Mehrmann [7]) Provided that the matrix pair (E, A_0) is regular, then there exist regular matrices $W, T \in \mathbb{R}^{n,n}$ such that*

$$(WET, WA_0T) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{4}$$

{KW form}

where N is a nilpotent, upper triangular matrix of nilpotency index ν. We also say that the pair (E, A_0) has an index *ν, i.e.,* $\operatorname{ind}(E, A_0) = \nu$. *Furthermore, the system* (1) *is called* impulse-free (index 1, or strangeness-free) *if $N = 0$.*

Remark 1. In general, the two concepts index and stability are independent. In fact, Examples 5 in Ha and Nam [8] has illustrated that there exist systems with arbitrarily high-index (and hence, not impulse-free) which are stable.

Lemma 4. *For a nilpotent, upper triangular matrix N of nilpotency index ν, the matrix $I - \lambda N$ is invertible for all $\lambda \in \mathbb{C}$, and* $\det(I - \lambda N) = 1$. *Furthermore, the following identity holds true.*

$$(I - \lambda N)^{-1} = I + \sum_{i=1}^{\nu} (\lambda N)^i.$$

PROOF. The proof is simple and can be found in classical matrix theory textbooks, for example [9].

2.1. System classification

It is well-known (see e.g. Bellman and Cooke [10], Hale and Lunel [11]) that in general, time-delayed systems has been classified into three different types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0 \dot{x}(t) + a_1 \dot{x}(t - \tau) + b_0 x(t) + b_1 x(t - \tau) = f(t)$$

is retarded if $a_0 \neq 0$ and $a_1 = 0$; is neutral if $a_0 \neq 0$, $a_1 \neq 0$; is advanced if $a_0 = 0$, $a_1 \neq 0$, $b_0 \neq 0$. This classification is based on the smoothness comparison between $x(t)$ and $x(t - \tau)$. In literature, not only the theoretical but also the numerical solution has been studied mainly for retarded and neutral systems, due to their appearance in various applications. For this reason, in Ha [4], Ha and Mehrmann [5], Unger [12] the authors proposed a concept of *non-advancedness* for the free system (see Definition 5 below). We also notice, that even though not clearly proposed, due to the author's knowledge, so far results for delay-descriptor are only obtained for certain classes of non-advanced systems, e.g. Ascher and Petzold [1], Shampine and Gahinet [3], Zhu and Petzold [13, 14], Michiels [15], Phat and Sau [16], Sau et al. [17], Cui et al. [18], Ngoc [19].

Definition 5. *A regular delay-descriptor system* (1) *is called* non-advanced *if for any consistent and continuous initial function φ, there exists a continuous, piecewise differentiable solution $x(t)$.*

Making use of Lemma 3, we change the variable $x = Ty$ and scale the whole system (3) with W to obtain the transformed system

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{y}(t) = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} y(t) + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t - \tau_i), \tag{5}$$

{eq9}

where $WA_iT = \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}$ for all $i = 1, ..., m$. The following lemma gives us the necessary and sufficient condition for the non-advancedness of system (3).

Lemma 6. *i) System* (3) *is non-advanced if and only if the matrix coefficients of the transformed system* (5) *satisfy*

$$N \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \text{ for all } i = 1, \ldots, m. \tag{6}$$ {non-advanced cond.}

ii) Consequently, system (5) *has exactly the same solution as the so-called* index-reduced system

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{=:\tilde{E}} \dot{y}(t) = \underbrace{\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}}_{=:\tilde{A}_0} y(t) + \sum_{i=1}^{m} \underbrace{\begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}}_{=:\tilde{A}_i} y(t - \tau_i). \tag{7}$$ {index reduced system

PROOF. Partitioning $y := \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ conformably, we can rewrite system (5) as follows

$$\begin{aligned} \dot{y}_1 &= Jy_1 + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \end{bmatrix} y(t - \tau_i), \\ N\dot{y}_2 &= y_2 + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t - \tau_i), \end{aligned} \tag{8}$$ {eq14.2}

The second equation has a unique solution

$$y_2(t) = -\begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t - \tau_i) - \sum_{j=1}^{\nu} \sum_{i=1}^{m} N^i \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y^{(j)}(t - \tau_i).$$

Since the system (3) is non-advanced, then so is system (5). Consequently, $y(t)$ must not depend on $y^{(j)}(t - \tau_i)$ for all $j = 1, ..., \nu$ and $i = 1, ..., m$, which implies the identity (12). Then, the second claim is trivially followed.

Remark 2. From Lemma 6 ii), we see that if system (3) is non-advanced, then there is a linear, bijective mapping $x \mapsto y = T^{-1}x$ (where T is the matrix given in the Kronecker-Weierstraß form (4)) between the solution set of the high-index system (3) and the impulse-free system (7). This will play the key role in the stability analysis in Section 3.

Remark 3. Since the numerical computation of the Kronecker-Weierstraß form (4) is quite complicated and unstable (see [20]), Lemma 6 has more theoretical than numerical meaning for checking the non-advancedness of (3). Below we will construct another test, which is more practical.

Assume that the pair (E, A_0) is regular with index $\operatorname{ind}(E, A_0) = \nu$. We want to give a simple check whether the system (3) is non-advanced or not. In analogous to the case of DAEs, see e.g. Brenan et al. [21], Kunkel and Mehrmann [7], we aim to extract the so-called *underlying delay equation* of the form

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \sum_{i=1}^{m} \mathbf{A}_i x(t-\tau_i) + \sum_{i=1}^{m} \mathbf{F}_i \dot{x}(t-\tau_i), \tag{9}$$

{underlying DDEs}

from an augmented system consisting of system (3) and its derivatives, which read in details

$$\frac{d^j}{dt^j}\left(E\dot{x}(t) - A_0 x(t) - \sum_{i=1}^{m} A_i x(t-\tau_i)\right) = 0, \text{ for all } j = 0, 1, \ldots, \nu.$$

We rewrite these equations into the so-called *inflated system*

$$\underbrace{\begin{bmatrix} E & & & \\ -A_0 & E & & \\ & \ddots & \ddots & \\ & & -A_0 & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{\mathcal{A}_0} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix}$$
$$+ \sum_{i=1}^{m} \underbrace{\begin{bmatrix} A_i & & & \\ & A_i & & \\ & & \ddots & \\ & & & A_i \end{bmatrix}}_{\mathcal{A}_i} \begin{bmatrix} x(t-\tau_i) \\ \dot{x}(t-\tau_i) \\ \vdots \\ x^{(\nu)}(t-\tau_i) \end{bmatrix}. \tag{10}$$

{inflated}

Here the matrix coefficients are $\mathcal{E}, \mathcal{A}_0, \mathcal{A}_i \in \mathbb{R}^{(\nu+1)n,(\nu+1)n}$ for all $i = 1, ..., m$. For the reader's convenience, below we will use MATLAB notations. An underlying delay system (9) can be extracted from (10) if and only if there exists a matrix $P = \begin{bmatrix} P_0 & P_1 & \ldots & P_\nu \end{bmatrix}^T$ in $\mathbb{R}^{(\nu+1)n,n}$ such that

$$\begin{aligned} P^T \mathcal{E} &= \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}, \\ P^T \mathcal{A}_i &= \begin{bmatrix} * & * & 0_{n,(\nu-1)n} \end{bmatrix}, \text{ for all } i = 1, ..., m, \end{aligned}$$

where $*$ stands for an arbitrary matrix. Consequently, P is the solution to the following linear systems

$$\begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:,2n+1:end)^T \\ \vdots \\ \mathcal{A}_m(:,2n+1:end)^T \end{bmatrix} P = \begin{bmatrix} \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}^T \\ 0_{(\nu-1)n,n} \\ \vdots \\ 0_{(\nu-1)n,n} \end{bmatrix}.$$

Therefore, making use of Crammer's rule we directly obtain the simple check for the non-advancedness of system (3) in the following theorem.

Theorem 7. *Consider the zero-input descriptor system* (3) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = \nu$*. Then, this system is non-advanced if and only if the following rank condition is satisfied*

$$\operatorname{rank} \begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:,2n+1:end)^T \\ \vdots \\ \mathcal{A}_m(:,2n+1:end)^T \end{bmatrix} = \operatorname{rank} \left[\begin{array}{c|c} \mathcal{E}^T & \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}^T \\ \mathcal{A}_1(:,2n+1:end)^T & 0_{(\nu-1)n,n} \\ \vdots & \vdots \\ \mathcal{A}_m(:,2n+1:end)^T & 0_{(\nu-1)n,n} \end{array}\right]. \tag{11}$$

{adv. check eq.}

Theorem 7 applied to the index two case straightly gives us the following corollary.

Corollary 8. *Consider the zero-input descriptor system* (3) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = 2$*. Then, system* (3) *is non-advanced if and only if the following identity hold true.*

$$\operatorname{rank} \left[\begin{array}{ccc} E^T & -A_0^T & 0 \\ 0 & E^T & -A_0^T \\ 0 & 0 & E^T \\ \hline 0 & 0 & A_1^T \\ \vdots & \vdots & \vdots \\ 0 & 0 & A_m^T \end{array}\right] = n + \operatorname{rank} \left[\begin{array}{cc} E^T & -A_0^T \\ 0 & E^T \\ \hline 0 & A_1^T \\ \vdots & \vdots \\ 0 & A_m^T \end{array}\right]. \tag{12}$$

{check advanced}

3. Stability

3.1. Spectral approach

The stability analysis of the null solution of (1) in this work is based on a spectrum determined growth property of the solutions, which allows us to infer stability information from the location of the characteristic roots. For instance, exponential stability will be related to a strictly negative spectral abscissa (the supremum of the real parts of the characteristic roots). As we shall see, the spectral abscissa of (1) may not be a continuous function of the delays. Moreover, this may lead to a situation where infinitesimal delay perturbations destabilise an exponentially stable system. These properties are very similar to the spectral properties of neutral equations (see, e.g. [2, Section 2]), which are known to be closely related to DDAEs [3].

Proposition 9. *([15, 22]) Consider the linear, homogeneous DDAE* (3). *Furthermore, assume that it is regular, impulse-free. Then it is stable if and only if the corresponding spectrum of this system lies entirely on the left half plane and it is bounded away from the imaginary axis.*

The following lemma plays the key role in the proof of the main Theorem 11 below.

Lemma 10. *Consider the linear, homogeneous DDAE* (3). *Furthermore, assume that it is non-advanced. Then system* (3) *has the same spectrum (without counting multiplicity), and also the same stability property as the index-reduced system* (7).

PROOF. Firstly, we will show that both systems (3) and (7) have the same spectrum (without counting multiplicity) as the system (5). Due to the variable transformation $x = Ty$ and the identity

$$W\left(\lambda E - A_0 - e^{-\lambda\tau_i}A_i\right)T = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - e^{-\lambda\tau_i}\begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix},$$

it is straightforward that

$$\sigma(E, A_0, ..., A_m) = \sigma\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ \tilde{A}_{1,3} & \tilde{A}_{1,4} \end{bmatrix}, ..., \begin{bmatrix} \tilde{A}_{m,1} & \tilde{A}_{m,2} \\ \tilde{A}_{m,3} & \tilde{A}_{m,4} \end{bmatrix}\right). \tag{13}$$

{eq11}

Now let us consider the right hand side of (13), due to Lemma 4 we see that for an arbitrary $\lambda \in \mathbb{C}$

$$\begin{aligned} &\det\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \sum_{i=1}^{m} e^{-\lambda\tau_i}\begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} I & 0 \\ 0 & (I-\lambda N)^{-1} \end{bmatrix} \cdot \begin{bmatrix} I - J - \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,1} & -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,2} \\ -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,3} & \lambda N - I - \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,4} \end{bmatrix}\right). \end{aligned}$$

Due to Lemma 4 and (6), we have

$$\begin{aligned}
(I+\sum_{i=1}^{\nu}(\lambda N)^i)\cdot\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,3} &= \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,3},\\
(I+\sum_{i=1}^{\nu}(\lambda N)^i)\cdot\left(\lambda N - I - \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,4}\right) &= -I-\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,4}.
\end{aligned}$$

Hence, it follows that for any $\lambda \in \mathbb{C}$

$$\begin{aligned}
&\det\left(\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix} - \begin{bmatrix} J & 0\\ 0 & I\end{bmatrix} - \sum_{i=1}^{m} e^{-\lambda\tau_i}\begin{bmatrix}\tilde{A}_{i,1} & \tilde{A}_{i,2}\\ \tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix}\right)\\
&= \det\left(\begin{bmatrix} I-J-\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,1} & -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,2}\\ -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,3} & -I-\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,4}\end{bmatrix}\right),
\end{aligned}$$

which yields that

$$\sigma\left(\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}, \begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}, \begin{bmatrix}\tilde{A}_{1,1} & \tilde{A}_{1,2}\\ \tilde{A}_{1,3} & \tilde{A}_{1,4}\end{bmatrix}, ..., \begin{bmatrix}\tilde{A}_{m,1} & \tilde{A}_{m,2}\\ \tilde{A}_{m,3} & \tilde{A}_{m,4}\end{bmatrix}\right) = \sigma(\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m). \tag{14}$$

{eq12}

From (13) and (14) we have $\sigma(E, A_0, ..., A_m) = \sigma(\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m)$.
Now let us consider the stability of (3). Clearly, system (3) is stable if and only if system (5) is also stable, and vice versa. Hence, due to Lemma 6ii) we obtain the desired claim. □

Lemma 10 shows that a non-advanced system (3) has the same spectrum and the same stability property as the index-reduced system (7), which is impulse-free. Therefore, Proposition 9 applied to system (7) leads us to the following theorem.

Theorem 11. *Consider the free system* (3)*. Furthermore, we assume that the matrix pair* (E, A_0) *is regular. Then,* (3) *is exponentially stable if and only if the following assertions hold.*

i) System (3) *is non-advanced.*

ii) The spectrum $\sigma(E, A_0, \ldots, A_m)$ *lies entirely on the left half plane and it is bounded away from the imaginary axis.*

Remark 4. Again, we notice that due to the complication in computing the Kronecker-Weierstraß form (4), we will not compute the spectrum $\sigma(E, A_0, \ldots, A_m)$ based on (4). Instead, we refer the reader to the spectral discretisation approach in [15]. Nevertheless, since this method has only been developed for impulse-free (or index-1) system, we need the pre-processing step as in Lemma 12 below.

Let us consider the (reordered) QZ-decomposition ([23]) of the matrix pair (E, A_0) as follows

$$QEZ^T = \begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & N_E \end{bmatrix},\ QA_0Z^T = \begin{bmatrix} J_A & \hat{A}_2 \\ 0 & \Sigma_A \end{bmatrix},\ QA_iZ^T = \begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix}, \quad (15)$$

{eq15}

where Σ_E, Σ_A are nonsingular, upper triangular matrices, N_E is a nilpotent, upper triangular matrix.

Using the same argument as in Lemma 6, we have the following lemma.

Lemma 12. *Consider the free system* (3) *and the QZ-decomposition* (15)*. Then, the following assertions hold true.*
i) System (3) *is non-advanced if and only if* $N_E\Sigma_A^{-1}\begin{bmatrix} \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix} = 0$ *for all* $i = 1, ..., m$.
ii) If this is the case, then there is a linear, bijective mapping $x \mapsto y = Zx$ *(where* Z *is the matrix given in* (15)*) between the solution set of the high-index system* (3) *and the following impulse-free system*

$$\begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & \mathbf{0} \end{bmatrix} \dot{y}(t) = \begin{bmatrix} J_A & \hat{A}_2 \\ 0 & \Sigma_A \end{bmatrix} y(t) + \sum_{i=1}^{m} \begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix} y(t-\tau_i)\,. \quad (16)$$

{impulse free system}

PROOF. The proof is essentially the same as the proof of Lemma 6 and will be omitted to keep the brevity of this research.

3.2. Lyapunov-Krasovskii aproach

4. Robust stabilization

5. Conclusion and Outlook

Acknowledgment The author would like to thank the anonymous referee for his suggestions to improve this paper.

References

[1] U. M. Ascher and L. R. Petzold. The numerical solution of delay-differential algebraic equations of retarded and neutral type. *SIAM J. Numer. Anal.*, 32: 1635–1657, 1995.

[2] S. L. Campbell. Nonregular 2D descriptor delay systems. *IMA J. Math. Control Appl.*, 12:57–67, 1995.

[3] L. F. Shampine and P. Gahinet. Delay-differential-algebraic equations in control theory. *Appl. Numer. Math.*, 56(3-4):574–588, March 2006. ISSN 0168-9274. doi: 10.1016/j.apnum.2005.04.025. URL `http://dx.doi.org/10.1016/j.apnum.2005.04.025`.

[4] Phi Ha. *Analysis and numerical solutions of delay differential-algebraic equations*. Dissertation, Institut für Mathematik, TU Berlin, Berlin, Germany, 2015.

[5] Phi Ha and Volker Mehrmann. Analysis and numerical solution of linear delay differential-algebraic equations. *BIT*, 56:633 – 657, 2016.

[6] L. Dai. *Singular Control Systems*. Springer-Verlag, Berlin, Germany, 1989.

[7] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations – Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

[8] Phi Ha and Phan Thanh Nam. Stability analysis of arbitrarily high-index, positive delay-descriptor systems. In preparation, July 2021.

[9] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990. ISBN 9780521386326.

[10] Richard Bellman and Kenneth L. Cooke. *Differential-difference equations*. Mathematics in Science and Engineering. Elsevier Science, 1963.

[11] J.K. Hale and S.M.V. Lunel. *Introduction to Functional Differential Equations*. Springer, 1993.

[12] Benjamin Unger. Discontinuity propagation in delay differential-algebraic equations. *The Electronic Journal of Linear Algebra*, 34:582–601, Feb 2018. ISSN 1081-3810.

[13] Wenjie Zhu and Linda R. Petzold. Asymptotic stability of linear delay differential-algebraic equations and numerical methods. *Appl. Numer. Math.*, 24:247 – 264, 1997. doi: http://dx.doi.org/10.1016/S0168-9274(97)00024-X.

[14] Wenjie Zhu and Linda R. Petzold. Asymptotic stability of Hessenberg delay differential-algebraic equations of retarded or neutral type. *Appl. Numer. Math.*, 27(3):309 – 325, 1998. ISSN 0168-9274. doi: http://dx.doi.org/10.

1016/S0168-9274(98)00008-7. URL http://www.sciencedirect.com/science/article/pii/S0168927498000087.

[15] W. Michiels. Spectrum-based stability analysis and stabilisation of systems described by delay differential algebraic equations. *IET Control Theory Appl.*, 5(16):1829–1842, 2011. ISSN 1751-8644. doi: 10.1049/iet-cta.2010.0752.

[16] V.N. Phat and N.H. Sau. On exponential stability of linear singular positive delayed systems. *Applied Mathematics Letters*, 38:67–72, 2014. ISSN 0893-9659. doi: https://doi.org/10.1016/j.aml.2014.07.003. URL https://www.sciencedirect.com/science/article/pii/S0893965914002250.

[17] Nguyen H. Sau, P. Niamsup, and Vu N. Phat. Positivity and stability analysis for linear implicit difference delay equations. *Linear Algebra and its Applications*, 510:25–41, 2016. ISSN 0024-3795. doi: https://doi.org/10.1016/j.laa.2016.08.012. URL https://www.sciencedirect.com/science/article/pii/S0024379516303391.

[18] Yukang Cui, Jun Shen, Zhiguang Feng, and Yong Chen. Stability analysis for positive singular systems with time-varying delays. *IEEE Transactions on Automatic Control*, 63(5):1487–1494, 2018. doi: 10.1109/TAC.2017.2749524.

[19] Pham Huu Anh Ngoc. Exponential stability of coupled linear delay time-varying differential–difference equations. *IEEE Transactions on Automatic Control*, 63(3):843–848, 2018. doi: 10.1109/TAC.2017.2732064.

[20] P. Van Dooren. The computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. Appl.*, 27:103–141, 1979.

[21] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*. SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[22] Nguyen Huu Du, Vu Hoang Linh, Volker Mehrmann, and Do Duc Thuan. Stability and robust stability of linear time-invariant delay differential-algebraic equations. *SIAM J. Matr. Anal. Appl.*, 34(4):1631–1654, 2013.

[23] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, 3rd edition, 1996.