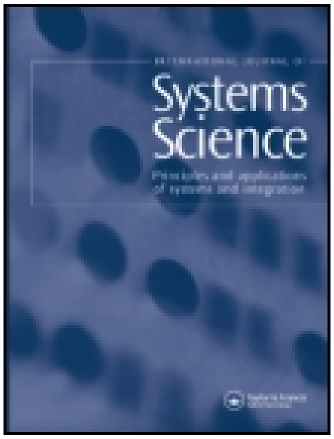

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Positive solutions of a discrete-time descriptor system

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Positive solutions of a discrete-time descriptor system

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Discrete-time linear descriptor systems with restrictions over their trajectory are considered. In this work, some conditions required to achieve nonnegative output-response are studied. State–feedback and dynamic compensators are constructed to achieve the desired property. Finally, some applications to the Leontief economic model are given.

Keywords: Descriptor control system; Non-negative matrix; Positive solution; Holdability; Feedback; Compensator

1. Introduction

Many biological, economic, engineering and chemical processing have been treated as difference equations in discrete-time, whose variables have to be non-negative. That is, each one of these problems can be modelled by a standard linear control system such as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(x) &= Cx(k) + Du(k), \end{aligned}$$

where the trajectory of the system is non-negative, and it is called positive system.

In the characterization of positive systems the coefficient matrices play an important role. An initial analysis about structural properties of positive standard systems is given in Coxson and Shapiro (1987) and Rumchev and James (1989), and some canonical forms are constructed in Bru *et al*. (2000). Recently, in Farina and Rinaldi (2000) a good study on positive linear systems is given, and a survey about structural properties can be found in Rumchev and Caccetta (2000) and Bru *et al*. (2002a).

Sometimes the dynamic process cannot be represented by a standard system and for modelling this process it is necessary to use a descriptor system. This kind of system is given by

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(x) &= Cx(k), \end{aligned} \tag{1}$$

where the matrix E is not necessarily invertible and where $E,\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n},\ k \in \mathbb{Z}_+$. This system is denoted by (E, A, B, C). When the outputs are equal to the states, the system is denoted by (E, A, B). If $E = I$, the system (1) is called standard.

It is well-known (Dai 1989, Kaczorek 1992) that if the pair (E, A) is regular, that is, there exists at least $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$, then the system has a solution.

The Leontief economic model is one example of descriptor system (see for instance Gandolfo 1985, De Lima 1990). The Leontief model aims to determine and analyse the structural and the technological changes of an economy by including an intertemporal mechanism of capital accumulation. This system can be described by

$$Cx(k+1) = (I - P + C)x(k) - Du(k) \tag{2}$$

where C is the capital coefficient matrix, P is the technological coefficient matrix, D is the demand coefficient matrix (excluding investment), $x(k)$ is the

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production level vector and $u(k)$ is the demand level vector.

Since the Leontief model is an economic model, $C \geq 0$, $P \geq 0$ and $D \geq 0$. This kind of system is a dynamic descriptor system because the capital coefficient matrix C can be singular. The entries of the coefficient capital matrix are the required capital per unit of production per sector. The singularity of this matrix arises because no output from one sector is used in the production of some products. By the nature of the technological matrix P, we can suppose p_{ij} such that $0 \leq p_{ij} < 1$, and

$$\sum_{i=1}^{n} p_{ij} < 1, \quad j = 1, \ldots, n.$$

In this case, the Leontief matrix $(I - P)$ is invertible. The pair $(C, I - P + C)$ is regular since there exists at least $\lambda = 1$, such that $\det(\lambda C - (I - P + C)) = \det(-I + P) \neq 0$ and thus, the system (2) has a solution. But unfortunately, some components of this solution can be negative. This means that the solution is not good for the economic problem. Sometimes, a special kind of solution is desired, in which each sector increases by a constant percentage per unit of time. These solutions are called balanced growth solutions. Conditions for a balanced growth solution for a Leontief model without demand are given in Szyld (1985) and Zeng (2001).

An initial characterisation for positive descriptor systems is given in Bru *et al.* (2002b).

When the positiveness of the solution of the system cannot be assured, it is worthwhile studying if from an initial non-negative state and under an appropriate control sequence it is possible to obtain a non-negative trajectory of the solution. In Berman and Stern (1987), this problem was discussed for standard systems and it was treated as the holdability problem of $\mathbb{R}^n_+$.

In this article, we focus our attention on the holdability problem of $\mathbb{R}^n_+$ for descriptor systems. We discuss some issues concerning the solution of descriptor systems and we use feedbacks to obtain the desired property. Furthermore, a forward–backward descriptor system is examined and a dynamic compensator such that the trajectory of the closed-loop system that is non-negative is constructed. Finally, some economic applications for the Leontief model are studied.

Consider the system (E, A, B). Given an initial state $x(0) \in \mathcal{X}_0$, where $\mathcal{X}_0$ is the set of admissible initial conditions, and a sequence control $u(j)$, $j = 0, 1, \ldots, k + q - 1$ and denoting by M^D the Drazin inverse of a matrix M, the state solution of the system (1) at time k is given by (Kaczorek 1992)

$$x(k) = \left(\widehat{E}^D \widehat{A}\right)^k \widehat{E}^D \widehat{E} x(0) + \sum_{i=0}^{k-1} \widehat{E}^D \left(\widehat{E}^D \widehat{A}\right)^{k-i-1} \widehat{B} u(i) - \left(I - \widehat{E}^D \widehat{E}\right) \sum_{i=0}^{q-1} \left(\widehat{E} \widehat{A}^D\right)^i \widehat{A}^D \widehat{B} u(k+i) \tag{3}$$

where $\widehat{E} = (\lambda E - A)^{-1} E$, $\widehat{A} = (\lambda E - A)^{-1} A$ and $\widehat{B} = (\lambda E - A)^{-1} B$, and $q = \operatorname{ind}(\widehat{E})$ is the smallest non-negative integer such that rank $(\widehat{E}^q) = \operatorname{rank}(\widehat{E}^{q+1})$.

Note that the matrices $\widehat{E}$ and $\widehat{A}$ satisfy $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$, which is a basic property to obtain the above explicit solution in terms of the respective Drazin inverses.

Given the discrete-time descriptor system (1), applying the coordinates transformation $x(k) = P\left[\begin{smallmatrix} x_1(k) \\ x_2(k) \end{smallmatrix}\right]$, where P is the non-singular matrix transformation used to obtain the Jordan canonical form of matrix $\widehat{E}$ (Dai 1989), we can obtain the equivalent forward–backward system $(\tilde{E}, \tilde{A}, \tilde{B})$ given as the following two subsystems

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 u(k) \\ N x_2(k+1) &= x_2(k) + B_2 u(k), \end{aligned}$$

with $\widetilde{E} = QEP = \operatorname{diag}(I_{n1}, N)$, $\widetilde{A} = QAP = \operatorname{diag}(A_1, I_{n_2})$ and $\widetilde{B} = QB = \left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$, where $n_1 + n_2 = n$, $A_1 \in \mathbb{R}^{n_1 \times n_2}$ and $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent. The solution of initial system (E, A, B) is obtained starting from the equivalent system $(\widetilde{E}, \widetilde{A}, \widetilde{B})$. The solution in this case is given by

$$\begin{aligned} x_1(k) &= A_1^k x_1(0) + \sum_{i=0}^{k-1} A_1^{k-i-1} B_1 u(i) \\ x_2(k) &= -\sum_{i=0}^{q-1} N^i B_2 u(k+i), \ k \in \mathbb{Z}_+. \end{aligned}$$

2. Attracting to $\mathbb{R}^n_+$

In this section, the influence of feedbacks to obtain a non-negative trajectory is analysed when a non-negative initial state and non-negative controls are considered.

First, the following well-known definitions and results are given.

Definition 1: A nonempty set $\Gamma \subset \mathbb{R}^n$ is called holdable with respect to system (1) if for each $x(0) \in \mathcal{X}_0 \cap \Gamma$, there exists a control sequence $u(\cdot)$ such that the trajectory of the system belongs to Γ, that is, $x(k) = x(k, x_0, u(\cdot)) \in \Gamma$, $\forall k \geq 0$ (Berman and Stern 1987).

Definition 2: The system (1) is positive if, for every $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$, and for every non-negative control sequence $u(\cdot) \geq 0$, the state trajectory belongs to $\mathbb{R}^n_+$, that is, $x(k) = x(k, x_0, u(\cdot)) \in \mathbb{R}^n_+, \forall k \geq 0$, (Bru *et al.* 2002b).

A characterisation of positive descriptor system is given in the following proposition.

Proposition 1: *Consider a discrete-time descriptor system* (E, A, B). *Suppose that* $EE^D \geq 0$, $EA = AE$ *and* $\operatorname{Ker} E \cap \operatorname{Ker} A = \{0\}$. *The system is positive if, and only if,* $E^D A \geq 0$, $E^D B \geq 0$ *and* $(I - E^D E)\,(EA^D)^i A^D B \leq 0$, $i = 0, 1 \ldots, q-1$, *where* $q = \operatorname{ind}(E)$ (Bru *et al.* 2002b).

Note that, if the system (E, A) satisfies $E^D E \geq 0$ and $EA = AE$, then the holdability of $\mathbb{R}^n_+$ with respect to (E, A) is equivalent to the positiveness of the system (E, A).

In the next result, the holdability of $\mathbb{R}^n_+$ using proportional feedbacks is discussed.

Theorem 1: *Consider a descriptor system* (E, A, B), *with* $E^D E \geq 0$, $EA = AE$ *and* $\operatorname{Ker} E \cap \mathit{Ker}\, A = \{0\}$.

(i) *If there exists a state-feedback* $u(k) = Fx(k)$ *such that* $E^D\,(A + BF) \geq 0$ *and*
  (a) $EBF = BFE$, *or*
  (b) $(I - EE^D)A^D BF = 0$,
  then the set $\mathbb{R}^n_+$ *is holdable with respect to the system* (E, A, B).
(ii) *If the set* $\mathbb{R}^n_+$ *is holdable with respect to the system* (E, A, B) *then there exists a state–feedback* $u(k) = Fx(k)$ *such that* $E^D\,(A + BF) \geq 0$.

Proof: (i)

(a) Applying the feedback $u(k) = Fx(k)$, the closed-loop autonomous system is

$$Ex(k+1) = (A + BF)x(k).$$

As $EBF = BFE$, the matrices E and $A + BF$ commute and the trajectory of the system is given by

$$x(k) = (E^D(A + BF))^k E^D Ex(0).$$

By hypothesis $E^D\,(A + BF) \geq 0$, then $x(k) \geq 0$, for all $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$ and $k \geq 0$. Thus, the set $\mathbb{R}^n_+$ is holdable.

(b) Consider

$$x(k) = EE^D x(k) + (I - EE^D)x(k).$$

From equation (3)

$$EE^D x(k) = (E^D A)^k E^D Ex(0) + \sum_{i=0}^{k-1} E^D (E^D A)^{k-i-1} Bu(i)$$

and

$$(I - EE^D)x(k) = -(I - E^D E)\sum_{i=0}^{q-1} (EA^D)^i A^D Bu(k+i),$$

Applying a state-feedback $u(k) = Fx\,(k)$ such that $(I - EE^D)A^D BF = 0$, then $(I - EE^D)x(k) = 0$ and it is only necessary to prove that $EE^D x(k) \geq 0$ using the condition $E^D\,(A + BF) \geq 0$, for all $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$.
If $k = 1$,

$$\begin{aligned} x(1) &= EE^D x(1) = (E^D A)E^D Ex(0) + E^D BFx(0) \\ &= E^D(A + BF)x(0), \end{aligned}$$

then $x(1) \geq 0$.
For $k = 2$,

$$\begin{aligned} x(2) &= EE^D x(2) = (E^D A)^2 E^D Ex(0) \\ &\quad + \sum_{i=0}^{1} E^D (E^D A)^{1-i} BFx(i) \\ &= E^D AE^D(A + BF)x(0) + E^D BFx(1) \\ &= E^D(A + BF)x(1), \end{aligned}$$

then $x(2) \geq 0$. By induction on k and using the Drazin inverse properties, it is easy to prove that

$$x(k) = EE^D x(k) = E^D(A + BF)x(k-1)$$

and since $E^D(A + BF) \geq 0$ then $x(k) \geq 0$.
(ii) Consider $x(0) \in \mathcal{X}_0 \cap \mathbb{R}^n_+$ and the generator set of $\mathbb{R}^n_+$ given by the canonical vectors $\{e_1, \ldots, e_n\}$. As $EE^D \geq 0$ then $x^i(0) = EE^D e_i \geq 0$, for each $i = 1, \ldots, n$, is an admissible initial state. Using that $\mathbb{R}^n_+$ is holdable there exists a control sequence $u_i\,(j), j = 0, 1, \ldots, k + q - 1$, such that the trajectory of the system is non-negative.
Constructing the state-feedback matrix,

$$F = [f_1, f_2, \ldots, f_n]$$

where $f_i = u_i\,(0)$, that is, $Fe_i = u_i\,(0)$. In particular for $k = 1$, as $\mathbb{R}^n_+$ is holdable and $EE^D \geq 0$, then $EE^D x\,(1) \geq 0$. Thus,

$$\begin{aligned} EE^D x(1) &= E^D(Ax(0) + Bu_i(0)) \\ &= E^D AEE^D e_i + E^D Bu_i(0) \\ &= E^D(A + BF)e_i \geq 0. \end{aligned}$$

Hence, $E^D(A + BF)e_i \geq 0$. In a similar way, the same result is obtained for the different columns of matrix $E^D(A + BF)$ using $x^i(0) = EE^D e_i \geq 0$, for all $I = 1, \ldots, n$. □

Note that in the case (b) the solution of the system is only given by the forward part.

The following example shows that the system can be holdable in spite of the fact that conditions (a) and (b) do not hold.

Example 1: Consider the system (E, A, B) with

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

It is easy to check that $E^D = E$ and $A^D = A$. Moreover, this system satisfies $E^D E \geq 0$, $EA = AE$ and $\operatorname{Ker} E \cap \operatorname{Ker} A = \{0\}$. For $F = (1\ 1\ 0)$, $E^D(A + BF) \geq 0$,

$$E^D(A + BF) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

but neither $EBF = BFE$ nor $(I - E^D E)\ A^D BF = 0$. The trajectory of the system is given by

$$x(k) = (E^D A)^k E^D E x(0) + \sum_{i=0}^{k-1} E^D (E^D A)^{k-i-1} BFx(i) - (I - E^D E) A^D BFx(k+i)$$

Then, for all $x(0) \geq 0$

$$x(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x(0) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \sum_{i=0}^{k-2} Fx(i) + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} Fx(k-1) - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} Fx(k).$$

Thus, the system has a positive trajectory given by

$$\begin{aligned} x_1(k) &= x_1(0) + \sum_{i=0}^{k-1} (x_1(i) + x_2(i)), \\ x_2(k) &= x_1(k-1) + x_2(k-1), \\ x_3(k) &= x_1(k) + x_2(k). \end{aligned}$$

In the following example, it is shown that the condition $E^D(A + BF) \geq 0$ for some F is not sufficient for the positiveness of the closed-loop system obtained using the state-feedback $u(x) = Fx(k)$.

Example 2: Consider the system defined by

$$E = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

This system satisfies $E^D E \geq 0$ and $EA = AE$, $E^D = E, A^D = A, q = 1$ and $\operatorname{Ker} E \cap \operatorname{Ker} A = \{0\}$. Then, applying the proportional feedback $u(k) = Fx(k)$ the solution is given by

$$\begin{aligned} &(I + (I - EE^D) A^D BF) x(k) \\ &\quad = (E^D A)^k E^D E x(0) + \sum_{i=0}^{k-1} E^D (E^D A)^{k-i-1} BFx(i). \end{aligned}$$

Choosing $F = (-1, -1, 0)$, the matrix $I + (I - EE^D)$ $A^D BF$ is the invertible matrix given by

$$\begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$E^D(A + BF) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0.$$

Then for $k = 1$,

$$\begin{aligned} x(1) &= \begin{pmatrix} -1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} E^D(A + BF) x(0) \\ &= \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} x(0) = \begin{pmatrix} -x_1(0) - x_2(0) \\ 0 \\ x_1(0) + x_2(0) \end{pmatrix}. \end{aligned}$$

It is clear that with this feedback some components of the trajectory vector are negative.

2.1 *On the forward-backward systems*

Now, the holdability of $\mathbb{R}^n_+$ is analysed when the descriptor system is given in the forward–backward form.

Note that in the autonomous case, the state of the system (E, A) at time k is given by

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} A_1^k x_1(0) \\ 0 \end{bmatrix} \geq 0.$$

Then, the backward subsystem has no influence in the trajectory of system. Taking as initial condition

$x_1^i(0) = e_i$. $i = 1, \ldots, n_1$, the state of the system at time $k = 1$ is

$$x_1(1) = A_1 x_1(0) = A_1 e_i \geq 0,$$

and thus A_1 is a non-negative matrix. Using this assertion we give the following result.

Proposition 2: *The forward–backward system* (E, A) *is positive if and only if* $\mathbb{R}_+^n$ *is holdable with respect to the system* (E, A).

Now, we give in the following proposition a new sufficient condition to the holdability of $\mathbb{R}_+^n$ with respect to a forward–backward system.

Proposition 3: *Consider a forward–backward system* (E, A, B). *If there exists a feedback* $u(k) = [F_1 \ 0] \ x(k)$ *such that* $A_1 + B_1 F_1 \geq 0$ *and* $B_2 F_1 = 0$, *then* $\mathbb{R}_+^n$ *is holdable with respect to the system* (E, A, B).

Proof: Suppose that there exists a feedback $u(k) = [F_1 \ 0] \ x(k)$ such that $A_1 + B_1 F_1 \geq 0$ and $B_2 F_1 = 0$. The forward closed-loop subsystem is given by

$$\begin{aligned} x_1(k+1) &= A_1 x_1(k) + B_1 F_1 x_1(k) \\ &= (A_1 + B_1 F_1) x_1(k), \end{aligned}$$

and since $A_1 + B_1 F_1 \geq 0$, we have that $x_1(k) \geq 0$. The backward closed-loop subsystem is given by

$$N x_2(k+1) = x_2(k),$$

and

$$\begin{aligned} x_2(k) &= -\sum_{i=0}^{q-1} N^i B_2 u(k+i) \\ &= -\sum_{i=0}^{q-1} N^i B_2 F_1 x_1(k+i), \quad k \in \mathbb{Z}_+, \end{aligned}$$

then $x_2(k) = 0$, since $B_2 F_1 = 0$. Thus, $x(k) \geq 0$ and then $\mathbb{R}_+^n$ is holdable with respect to the system (E, A, B). □

Under certain conditions, the state is not available to construct a state-feedback. In this case, it is reasonable to design a dynamic compensator to obtain satisfactory results. For that, we need to impose that the reachability and observability properties of the forward–backward system are at hold. Remember that the forward–backward system (E, A, B, C) is *reachable* when, for all state $x \in \mathbb{R}^n$ there exists a finite control sequence $u(\cdot)$ transferring the state of the system from the origin at time 0 to x. Therefore, if the system is reachable then it is *stabilisable*, that is, there exists a state-feedback such that the closed-loop system has all poles into the open unit disk. The concept of observability and detectability are defined as the dual concepts of reachability and stabilisability (for more details see Dai 1989).

Theorem 2: *Consider a forward–backward system* (E, A, B, C) *that is reachable and observable. If there exist two matrices* X *and* Y, *such that* $BX = \operatorname{diag}(I_{n_1}, 0)$ *and* $YC = \operatorname{diag}(I_{n_1}, 0)$, *then there exists a compensator such that the closed-loop system is non-negative.*

Proof: Consider the forward–backward system (E, A, B, C) that is reachable and observable. First, we construct the observer

$$E x_c(k+1) = A x_c(k) + B u(k) + G(y(k) - C x_c(k)).$$

Since the system is reachable and observable, it is stabilisable and detectable, respectively. Then, denoting $\varepsilon(k) = x_c(k) - x(k)$,

$$\lim_{k \to \infty} \varepsilon(k) = 0.$$

The compensator of the system is given by

$$\begin{aligned} E x_c(k+1) &= A x_c(k) + B u(k) + G(y(k) - C x_c(k)) \\ u(k) &= F x_c(k), \end{aligned}$$

and the closed-loop system is as follows

$$\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} x(k+1) \\ \varepsilon(k+1) \end{bmatrix} = \begin{bmatrix} A + BF & BF \\ 0 & A - GC \end{bmatrix} \begin{bmatrix} x(k) \\ \varepsilon(k) \end{bmatrix}.$$

We choose two matrices F and G such that the closed-loop system is non-negative. We select $F = [X\widetilde{F}_1 \ 0]$ such that $A_1 + \widetilde{F}_1 \geq 0$, and $G = \left[\begin{smallmatrix} \widetilde{G}_1 Y \\ 0 \end{smallmatrix}\right]$ such that $A_1 + \widetilde{G}_1 \geq 0$.

If we permute suitably the rows and columns of this system, we give a similar system with a forward–backward structure. The new system is given by

$$\widehat{E}\widehat{x}(k+1) = \widehat{A}\widehat{x}(k)$$

where

$$\widehat{x}(k) = \begin{bmatrix} x_1(k+1) \\ \varepsilon_1(k+1) \\ x_2(k+1) \\ \varepsilon_2(k+1) \end{bmatrix}, \quad \widehat{E} = \begin{bmatrix} I_{n_1} & O & O & O \\ O & I_{n_1} & O & O \\ O & O & N_2 & O \\ O & O & O & N_2 \end{bmatrix},$$

$$\widehat{A} = \begin{bmatrix} A_1 + \widehat{F}_1 & \widehat{F}_1 & O & O \\ O & A_1 - \widehat{G}_1 & O & O \\ O & O & I_{n2} & O \\ O & O & O & I_{n2} \end{bmatrix},$$

with

$$\begin{bmatrix} A_1 + \widetilde{F}_1 & \widetilde{F}_1 \\ O & A_1 - \widetilde{G}_1 \end{bmatrix} \geq 0.$$

Then, the forward–backward closed-loop system is positive.

3. Some applications to the Leontief economic model

Consider the Leontief model (DLM) given by

$$Cx(k+1) = (I - P + C)x(k) - Du(k). \tag{4}$$

3.1 *Balanced growth solution*

In economic problems, it is important to assure the existence of a solution whose output of each sector increases by a constant percentage per unit of time. This kind of solution is called balanced growth solution, and it satisfies

$$x(k) = (1+\delta)^k x_0 >> 0$$

with the balanced growth rate $\delta > 0$. When the matrix B is equal to the zero matrix, the problem to obtain a balanced growth solution can be completely resolved in Zeng (2001). The interest to obtain balanced growth solutions for some dynamic model is necessary to do the following question: is it possible to obtain a feedback such that the closed-loop system has a balanced growth solution? The answer to this question is given in the following result.

Proposition 4: *Consider the Leontief model* (4). *The closed-loop system has a balanced growth solution if and only if there exists a matrix F such that $(I-P)^{-1}$ $(C-DF)$ has a positive eigenvalue with a positive eigenvector.*

Proof: Consider $x(k) = (1+\delta)^k x_0 >> 0$, with $\delta > 0$ a balanced growth solution of the Leontief model (DLM)

$$(I-P)x(k) = C(x(k+1) - x(k)) + Du(k).$$

Take a state-feedback defined by

$$u(k) = -\delta F x(k).$$

Replacing the solution $x(k) = (1+\delta)^k$ x_0 in the closed-loop system

$$\begin{aligned}(I-P)&(1+\delta)^k x_0\\ &= C\big((1+\delta)^{k+1}x_0 - (1+\delta)^k x_0\big) - \delta(1+\delta)^k DFx_0,\end{aligned}$$

we have

$$(I-P)x_0 = (C-DF)\delta x_0$$

and

$$\big(\lambda I - (I-P)^{-1}(C-DF)\big)x_0 = 0,$$

where $\lambda = 1/\delta$. Hence, x_0 is a positive eigenvector of $(I-P)^{-1}$ $(C-DF)$ corresponding to the eigenvalue λ.

Reciprocally, consider F such that there exists a positive eigenvalue with a positive eigenvector of the matrix $(I-P)^{-1}$ $(C-DF)$. Then,

$$\big(\lambda I - (I-P)^{-1}(C-DF)\big)x_0 = 0,$$

and taking $\delta = 1/\lambda$ we have

$$(I-P)x_0 = (C-DF)\delta x_0.$$

Considering the state-feedback $u(k) = -\delta Fx(k)$, then $x(k) = (1+\delta)^k x_0$ is a balanced growth solution of the Leontief model (DLM). □

The economic meaning of the existence of a positive eigenvector can be given, for instance, when there is one economy where each sector of this economy depends on all others directly or indirectly for either its intermediate product or its capital. We refer to Zeng (2001) for more details of it and in the case when the economy can be divided into several subeconomies.

3.2 *Positive solution for two compartmental sectors*

Now, consider the DLM system (4). Multiplying this system by $(I-P)^{-1}$ we obtain a descriptor system given by

$$Ex(k+1) = Ax(k) - Bu(k), \tag{5}$$

where $E = (I-P)^{-1}$ $C, A = (I+E)$ and $B = (I-P)^{-1}$ D. Note that $E \geq 0$ because $(I-P)$ is an M matrix, matrices E and A commute and $\operatorname{Ker} E \cap \operatorname{Ker} A = \{0\}$.

When the economy is divided into several groups of sectors such that none of the sectors of one group delivers its output to any sector of some other group of industries, then the matrix E is reducible (Szyld 1985). That is, there exists a permutation matrix S such that

$$S^{-1}ES = \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}$$

where E_{11} and E_{22} are square matrices.

In the particular case that the model has two compartmental sectors: the first includes only the star products and the second the uninteresting products, the matrix E has a nilpotent submatrix, because the uninteresting situation has to finish in a certain period of time. Thus, the matrix E can be written using a permutation matrix, as $QEQ^{-1} = \operatorname{diag}(E_1, N)$, with E_1 an invertible matrix and N a nilpotent matrix. In this case, we are assuming that E is completely reducible.

Note that this model only represents two kind of products, the star and the uninteresting one, but the model can be extended to a more complex structure, using new goods and considering changes in the production. For example, the situation of a star good can be changed to uninteresting good, for a period of time, that is considered and new goods can be introduced. In these situations the capital and the technological matrices can be variables in the time. Some changes in the technological matrix can be seen as technological changes or introduction of new technologies in some sectors (Cantó *et al.* 2003).

Consider a Leontief model (5) with $E = \mathrm{diag}(E_1, N)$, that is

$$\begin{bmatrix} E_1 & O \\ O & N \end{bmatrix} x(k+1) = \begin{bmatrix} I_{n_1} + E_1 & O \\ O & I_{n_2} + N \end{bmatrix} x(k) - \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} u(k). \tag{6}$$

The state solution of this system gives a lot of information, because it informs that future input production is needed to supply the future market demand.

Given the system (6) and multiplying this equation by $\mathrm{diag}(E_1^{-1}, I_{n_2})$, we have the following equivalent system

$$\begin{bmatrix} I_{n_1} & O \\ O & N \end{bmatrix} x(k+1) = \begin{bmatrix} E_1^{-1} + I_{n_1} & O \\ O & I_{n_2} + N \end{bmatrix} x(k) - \begin{bmatrix} E_1^{-1} D_1 \\ D_2 \end{bmatrix} u(k).$$

This system is a forward–backward system. This dynamic model includes the stocks and flows of capital goods explicitly. If we consider an initial state $x(0) \in \mathcal{X}_0$ and a sequence control $u(j)$, $j = 0, 1, \ldots, k+q-1$, the state solution of the system at time k is given by

$$\begin{aligned} x(k) &= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &= \begin{bmatrix} I_{n1} \\ O \end{bmatrix} \left((E_1^{-1} + I_{n1})^k x_1(0) - \sum_{i=0}^{k-1} (E_1^{-1} + I_{n1})^{k-i-1} E_1^{-1} D_1 u(i) \right) \\ &\quad + \begin{bmatrix} O \\ I_{n_2} \end{bmatrix} \sum_{i=0}^{q-1} (N(I_{n_2} + N)^{-1})^i (I_{n_2} + N)^{-1} D_2 u(k+i). \end{aligned}$$

And using the characterisation given in Proposition 1, the system is positive if $(E_1^{-1} + I_{n1}) \geq 0$, $-E_1^{-1} D_1 \geq 0$ and $\left(N(I_{n_2} + N)^{-1}\right)^i (I_{n_2} + N)^{-1} D_2 \geq 0$, $i = 0, \ldots, q-1$. Note that normally in economic problems the initial stock must be positive, then it is usual that

$$0 < (E_1^{-1} + I_{n_1})^k x_1(0) \geq \sum_{i=0}^{k-1} (E_1^{-1} + I_{n_1})^{k-i-1} E_1^{-1} D_1 u(i), \tag{7}$$

then it is sufficient that $\left(N(I_{n_2} + N)^{-1}\right)^i (I_{n_2} + N)^{-1} D_2 \geq 0$, $i = 0, \ldots, q-1$ and with an initial state satisfying (7).

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