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Characterizations of Dichotomies of Evolution Equations on the Half-Line

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In this paper we investigate the characterization of dichotomies of an evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ of bounded linear operators on Banach space $\mathbf{X}$. We introduce operators I_0 and $I_{\mathbf{X}}$ on subspaces of $L_p(\mathbf{R}_+, \mathbf{X})$ using the integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi$. The exponential and ordinary dichotomies of $\mathcal{U}$ are characterized by properties of $I_0, I_{\mathbf{X}}$.
Key Words: evolution equations; evolution family; exponential dichotomy; ordinary dichotomy; integral equation.

1. INTRODUCTION AND PRELIMINARIES

In his famous paper [14], Perron gave a characterization of exponential dichotomy of the solutions to the linear differential equation

$$\frac{dx}{dt} = A(t)x, \qquad t \in [0, +\infty), x \in \mathbf{R}^n,$$

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where $A(t)$ is a matrix continuous function, in terms of the surjectiveness of the differential operator $dx/dt - A(t)$ as an operator in the space $BC(\mathbf{R}_+, \mathbf{R}^n)$ of $\mathbf{R}^n$-valued bounded continuous functions on the half-line $\mathbf{R}_+$. This result serves as a starting point for numerous works on the qualitative theory of solutions of differential equations. We refer the reader to the book by Massera and Schäffer [6] and Daleckii and Krein [2] for more information on the bounded case and [4] for the extension to the infinite dimensional case for equations defined on the whole line. Note that a similar characterization of exponential stability can be made by using the differential operator $dx/dt - A(t)$ in suitable function spaces (see [2, 3, 6]). In the infinite dimensional case to characterize the exponential dichotomy of solutions to differential equations on the half-line, apart from the surjectiveness of the differential operator $dx/dt - A(t)$ one needs additional conditions, namely the complementedness of the stable subspaces (see [2, 6, 8]).

Recently there has been an increasing interest in the asymptotic behavior of solutions of differential equations in Banach spaces, in particular, in the unbounded case (see, e.g., [10, 12]). In this direction, we would like to mention a recent paper [8] in which a new characterization of exponential dichotomy was given in Hilbert spaces using only conditions on $dx/dt - A(t)$ (more precisely, its closure). These conditions are closely related to the so-called evolution semigroups associated with an evolutionary process $\mathcal{U} = U(t, s)_{t \geq s \geq 0}$ on the half-line, defined as

$$[T(t)f](\xi) = \begin{cases} U(\xi, \xi - t)f(\xi - t); & \text{for } \xi \geq t \geq 0, \\ U(\xi, 0)f(0) & \text{for } \ 0 \leq \xi \leq t, \end{cases}$$

where f is an element of suitable function space. Note that the characterization of exponential dichotomy in [8] was studied in the space of bounded continuous functions. Technically speaking, this function space is more convenient than the function spaces $L_p(\mathbf{R}_+, \mathbf{X})$ to define appropriate operators used in the proving process as well as to apply available results of stability of semigroups of linear operators. We refer the reader to the recent papers [9, 11] for related results concerned with applications of the operator $d/dt - A(t)$ to the admissibility theory of function spaces.

In this paper we first aim at characterizing the exponential dichotomy of evolution families using the function spaces of $L_p(\mathbf{R}_+, \mathbf{X})$. We will overcome the above mentioned difficulty by not using available results in stability theory of semigroups. Furthermore, we extend our method to study the ordinary dichotomy of evolution families. Our main results are contained in Theorems 3.1, 4.2, and Corollary 3.1. We note that in [15] a similar problem has been discussed, and it seems that there is a gap in the proof of the characterization of exponential dichotomy.

Below we recall some notions.

DEFINITION 1.1. A family of operators $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on a Banach space $\mathbf{X}$ is a *(strongly continuous, exponential bounded) evolution family* on the half-line if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$,

(ii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in \mathbf{X}$,

(iii) There are constants $K, \alpha \geq 0$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then $\omega(\mathcal{U}) := \inf\{\alpha \in \mathbf{R} :$ there is $K \geq 0$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)},\ t \geq s \geq 0\}$ is called the *growth bound* of $\mathcal{U}$.

This notion of evolution families arises naturally from the theory of evolution equations which are well-posed (see, e.g., [13]). In fact, in terminology of [13], as an evolution family we can take the evolution operator generated by the following well-posed evolution equation

$$\frac{du(t)}{dt} = A(t)u(t), \qquad t \geq 0,$$

where $A(t)$ is in general the unbounded linear operator for every fixed t. Note that due to the above general setting, in general the function $U(t,s)x$, as a function of t, is not differentiable. Moreover, we are concerned here with the notion of evolution families rather than that of evolution equations involving concrete "differential equations." This restriction will constitute a considerable difficulty in dealing with the problems stated above. However, as shown below, we can overcome it by using operators I_0, $I_{\mathbf{X}}$ defined below.

Throughout this paper we will use the following function spaces (endowed with the norm $\|.\|_p = (\int_0^\infty \|.\|^p)^{1/p},\ 1 \leq p < \infty$):

$$L_p(\mathbf{R}_+, \mathbf{X}) := \left\{ v : \mathbf{R}_+ \to \mathbf{X} : \int_0^{+\infty} \|v(t)\|^p dt < \infty \right\} =: L_p$$

$$L_p([t_0,\infty), \mathbf{X}) := \left\{ v|_{[t_0,\infty)} : v \in L_p(\mathbf{R}_+, \mathbf{X}) \right\}; \qquad t_0 > 0.$$

Let $D^+ = \{(t,s) \in \mathbf{R}^2 \colon t \geq s \geq 0\}$. We shall consider the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi, \qquad t \geq s \geq 0. \tag{1}$$

We also consider the space (endowed with the sup-norm)

$$C(\mathbf{R}_+; \mathbf{X}) := \{v : \mathbf{R}_+ \to \mathbf{X} \colon v \text{ is continuous and bounded}\} =: C,$$

and the space $E_X := L_p \cap C$ with norm $\|v\|_{E_X} := \max\{\|v\|_p; \|v\|_C\}$. Then E_X is a Banach space. Next, we define an operator $I_{\mathbf{X}} : E_X \to L_p$ as follows: If $u \in E_X$, $f \in L_p$ satisfy Eq. (1) for a.e. $(t,s) \in D^+$ we set $I_{\mathbf{X}}u := f$ with

$$D(I_{\mathbf{X}}) := \big\{u \in E_X \colon \text{ there exists } f \in L_p \text{ such that } u, f \text{ satisfy Eq. (1) for a.e. } (t,s) \in D^+\big\}.$$

LEMMA 1.1. *The operator $I_{\mathbf{X}}$ is a well-defined, closed, and linear operator.*

Proof. Let

$$u(t) := U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)d\xi \qquad \text{for a.e. } (t,s) \in D^+;$$

$$v(t) := U(t,s)v(s) + \int_s^t U(t,\xi)g(\xi)d\xi \qquad \text{for a.e. } (t,s) \in D^+;$$

and $u(t) = v(t)$ for t a.e. in $\mathbf{R}_+$. Then,

$$\int_s^t U(t,\xi)g(\xi)d\xi = \int_s^t U(t,\xi)f(\xi)d\xi.$$

Hence,

$$\int_s^t U(t,\xi)[f(\xi) - g(\xi)]d\xi = 0.$$

Thus,

$$\frac{1}{(s-t)} \int_s^t U(t,\xi)[f(\xi) - g(\xi)]d\xi = 0.$$

Let $t - s \to 0$. We obtain that

$$f(t) = g(t) \qquad \text{for } t \text{ a.e. in } \mathbf{R}_+;$$

therefore, $I_{\mathbf{X}}$ is well-defined.

Let $\{v_n\}$ be a sequence in $D(I_{\mathbf{X}})$, such that $\lim_{n\to\infty} \|v_n - v\|_{E_X} = 0$ for some $v \in E_X$ and

$$\exists f \in L_p \text{ such that } \lim_{n\to\infty} \|I_{\mathbf{X}} v_n - f\|_p = 0. \tag{2}$$

Hence,

$$\lim_{n\to\infty} \|v_n(t) - v(t)\|_{\mathbf{X}} = 0 \qquad \text{for fixed } t \in \mathbf{R}_+. \tag{3}$$

Now we prove that $v \in D(I_{\mathbf{X}})$ and $I_{\mathbf{X}} v = f$. In fact, we have

$$v_n(t) = U(t,s)v_n(s) + \int_s^t U(t,\xi)I_{\mathbf{X}} v_n(\xi)d\xi \qquad \text{for a.e. } (t,s) \in D^+. \tag{4}$$

For fixed (t,s) a.e. in D^+, from the Hölder inequality we have

$$\left\| \int_s^t U(t,\xi)I_{\mathbf{X}} v_n(\xi)d\xi - \int_s^t U(t,\xi)f(\xi)d\xi \right\|$$
$$\leq \int_s^t \|U(t,\xi)\| \|I_{\mathbf{X}} v_n(\xi) - f(\xi)\| d\xi$$
$$\leq N_1 \int_s^t \|I_{\mathbf{X}} v_n(\xi) - f(\xi)\| d\xi \leq N_1 (t-s)^{1-\frac{1}{p}} \|I_{\mathbf{X}} v_n - f\|_p.$$

From this and (2) we obtain

$$\lim_{n\to\infty}\left\|\int_s^t U(t,\xi)I_{\mathbf{X}}v_n(\xi)d\xi-\int_s^t U(t,\xi)f(\xi)d\xi\right\|_{\mathbf{X}}=0. \tag{5}$$

The equalities (3), (4), and (5) yield

$$v(t)=U(t,s)v(s)+\int_s^t U(t,\xi)f(\xi)d\xi \qquad \text{for a.e. } (t,s)\in D^+.$$

Therefore, $v\in D(I_{\mathbf{X}})$ and $I_{\mathbf{X}}v=f$. ∎

Similarly, we define an operator I_0 related to the equation

$$u(t)=\int_0^t U(t,\xi)f(\xi)d\xi \tag{6}$$

as follows: for I_0: $E_X\to L_p$ if $u\in E_X, f\in L_p$ satisfy Eq. (6) then we set $I_0u=f$ with

$$D(I_0):=\big\{u\in E_X:\ \text{there exists } f\in L_p \\ \text{such that } u,f \text{ satisfy Eq. (6) for a.e. } t\in\mathbf{R}_+\big\}.$$

By the same method as in the proof of Lemma 1.1 we can prove that I_0 is a well-defined, closed, and linear operator. Next we define a subspace $X_0(t_0)$ of $\mathbf{X}$ as

$$X_0(t_0):=\left\{x\in\mathbf{X}:\ \int_{t_0}^{\infty}\|U(t,t_0)x\|^p dt<\infty\right\} \qquad \text{for } t_0\geq 0,$$

and $X_0(t_0)$ is called *the stable subspace* of $\mathbf{X}$ corresponding to t_0 and $\mathcal{U}$.

Remark 1.1. (i) $\ker I_{\mathbf{X}}:=\{u\in D(I_{\mathbf{X}}): u(t)=U(t,t_0)u(0)\}$,

(ii) *It is easy to verify that* $D(I_0):=\{v\in D(I_{\mathbf{X}}): v(0)=0\}$ *and* $I_{\mathbf{X}}v=I_0v$ *whenever* $v\in D(I_0)$. *Therefore* $I_{\mathbf{X}}$ *is an extension of* I_0.

LEMMA 1.2. *Let* χ: $[t_0,t_1)\to(0,\infty)$ *be a continuous function and let* $c>0$ *and* $K,\alpha\geq 0$ *be constants such that* $\chi(t)\leq Ke^{\alpha(t-t_0)}$ *and* $\int_{t_0}^t\chi(t)\chi(\xi)^{-1}d\xi<c$ *with* $t\in[t_0,t_1)$. *Then*

$$\chi(t)\leq\max(cK,K)e^{\alpha+\frac{1}{c}}e^{-\frac{1}{c}(t-t_0)}.$$

Proof. If $t\in[t_0,t_1)$ and $t\leq t_0+1\Rightarrow t-t_0\leq 1$, then $\chi(t)\leq Ke^{\alpha}\leq Ke^{\alpha+1/c}e^{-(1/c)(t-t_0)}$. Now let $t\in[t_0,t_1)$ and $t>t_0+1$. Set $\phi(s):=\int_{t_0}^s\chi^{-1}(\xi)d\xi$ for $s\in[t_0,t_1)$. By our assumption $\phi(s)\leq c\phi'(s)$, and hence $\phi(s_1)\geq\phi(s_0)e^{(1/c)(s_1-s_0)}$ for $t_1>s_1\geq s_0>t_0$. Thus

$$\chi(t)=\frac{1}{\phi'(t)}\leq\frac{c}{\phi(t_0+1)}e^{-\frac{1}{c}(t-t_0-1)}\leq cKe^{\alpha+\frac{1}{c}}e^{-\frac{1}{c}(t-t_0)}.$$ ∎

LEMMA 1.3. *Let* $\chi\colon [0,\infty) \to (0,\infty)$ *be a continuous function and let* $c, K, \alpha > 0$ *be constants such that* $\chi(\tau) \leq Ke^{\alpha(\tau-t)}\chi(t)$, $\tau \geq t \geq 0$, *and* $\int_t^\infty \chi(t)\chi(\tau)^{-1}d\tau \leq c$, $t \geq 0$. *Then there exists* $N > 0$ *which depends only on* K, α, c *such that* $\chi(t) \geq Ne^{\frac{1}{c}(t-s)}\chi(s)$ *for* $t \geq s \geq 0$.

Proof. Let $\phi(t) := \int_t^\infty \chi(\tau)^{-1}d\tau$, $t \geq 0$. By our assumption $\phi(t) \leq -c\phi'(t)$. Thus $\phi(t) \leq \phi(s)e^{-(1/c)(t-s)}$ for $t \geq s \geq 0$. On the other hand the exponential estimate of χ yields

$$\chi(t)\phi(t) = \chi(t)\int_t^\infty \frac{1}{\chi(\tau)}d\tau \geq K^{-1}\int_t^\infty e^{-\alpha(\tau-t)}d\tau$$

$$\geq K^{-1}\sum_{k=1}^\infty \int_{t+k-1}^{t+k} e^{-\alpha(t+k-t)}d\tau = K^{-1}\sum_{k=1}^\infty e^{-\alpha k} := N_1.$$

Moreover by our assumption, $\chi(s)\phi(s) \leq c$, $t \geq s \geq 0$. Thus

$$\chi(t) \geq \frac{N_1}{\phi(t)} \geq \frac{N_1}{\phi(s)}e^{\frac{1}{c}(t-s)} \geq \frac{N_1}{c}e^{\frac{1}{c}(t-s)}\chi(s) \qquad \text{for } t \geq s \geq 0.$$ ∎

2. EXPONENTIAL STABILITY OF L_P-BOUNDED ORBITS

In this section we will give a sufficient condition for stability of L_p-bounded orbits of an evolution family. The obtained results will be used in the next section to characterize the exponential dichotomy of evolutionary processes.

DEFINITION 2.1. Let A and B be Banach spaces endowed with the norm $\|.\|_A$ and $\|.\|_B$, respectively. Then an operator $T\colon A \to B$ is said to be *correct* if there exists a constant $\nu > 0$ such that

$$\|Tv\|_B \geq \nu\|v\|_A \qquad \text{for } v \in D(T).$$

The following theorem connects the exponential dichotomy of L_p-bounded orbits to the correctness of the operator I_0.

THEOREM 2.1. *Let* $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ *be an evolution family on the Banach space* $\mathbf{X}$ *such that* I_0 *is correct. Then every* L_p*-bounded orbit of* $\mathcal{U}$ *is exponentially stable. Precisely, if*

$$\int_{t_0}^{+\infty} \|U(t,t_0)x\|^p dt < \infty,$$

with $x \in \mathbf{X}$ *and* $t_0 > 0$, *then there exist positive constants* N, ν *independent of* x *and* t_0 *such that*

$$\|U(t,t_0)x\| \leq Ne^{-\nu(t-s)}\|U(s,t_0)x\|, \qquad t \geq s \geq t_0.$$

Proof. Let us start by proving that

$$\|U(t, t_0)x\| \leq Ne^{-\nu(t-t_0)}\|x\|, \qquad t \geq t_0.$$

Without loss of generality we may assume that $\|x\| = 1$. Since I_0 is correct there exists a constant $\nu > 0$ such that

$$\|I_0 v\|_p \geq \nu\|v\|_{E_X} \geq \nu\|v\|_p \qquad \text{for } v \in D(I_0).$$

Let $u(t) := U(t, t_0)x$, $t \geq t_0$, $t_1 := \sup\{t \geq t_0\colon U(t, t_0)x \neq 0\}$. The exponential boundedness of $\mathcal{U}$ yields

$$\|u(t)\| \leq Ke^{\alpha(t-t_0)}; \qquad t \geq t_0,$$

where K, α are positive constants. Take

$$v(t) = u(t)\int_0^t \chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(s)\|u(s)\|^{-1}ds$$
$$f(t) = \chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(t)\frac{u(t)}{\|u(t)\|},$$

where Δ_1 and Δ_2 are positive constants such that $t_0 + \Delta_1 \leq t_0 + \Delta_2 \leq t_1$ and

$$\chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(s) = \begin{cases} 0 & \text{for } s \notin [t_0+\Delta_1, t_0+\Delta_2], \\ 1 & \text{for } s \in [t_0+\Delta_1, t_0+\Delta_2]. \end{cases}$$

Then $v \in E_X$, and $f \in L_p$. They satisfy Eq. (6). It follows that

$$I_0 v = f \qquad \text{hence } \|f\|_p \geq \nu\|v\|_p.$$

Putting $\Delta := \Delta_2 - \Delta_1$ we have

$$\left(\int_{t_0+\Delta_1}^{t_0+\Delta_2} \|u(t)\|^p\left(\int_{t_0}^t \chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(s)\|u(s)\|^{-1}ds\right)^p dt\right)^{1/p}$$
$$\leq \|v\|_p \leq \nu^{-1}\|f\|_p = \nu^{-1}\left(\int_0^{+\infty} \chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(s)ds\right)^{1/p} = \nu^{-1}\Delta^{1/p}.$$

Thus

$$\frac{1}{\Delta}\int_{t_0+\Delta_1}^{t_0+\Delta_2} \|u(t)\|^p\left(\int_{t_0}^t \chi_{[t_0+\Delta_1,\, t_0+\Delta_2]}(s)\|u(s)\|^{-1}ds\right)^p dt$$
$$= \frac{1}{\Delta}\int_{t_0+\Delta_1}^{t_0+\Delta_2} \|u(t)\|^p\left(\int_{t_0}^t \|u(s)\|^{-1}ds\right)^p dt \leq \nu^{-p}.$$

From the arbitrary nature of Δ_1, Δ_2, and the continuity of $\|u(t)\|$ with respect to t it follows that

$$\|u(t)\|\int_{t_0}^t \|u(s)\|^{-1}ds \leq \nu^{-1}.$$

By Lemma 1.2 there is a positive constant N dependent only on ν, K, and α such that

$$\|U(t,t_0)x\| = \|u(t)\| \le Ne^{-\nu(t-t_0)}; \quad t \ge t_0.$$

Now we fix $s \ge t_0$ and set $y := U(s,t_0)x$. Then $\int_s^\infty \|U(t,s)y\|^p dt < \infty$, and

$$\|U(t,t_0)x\| = \|U(t,s)y\| \le Ne^{-\nu(t-s)}\|y\| = Ne^{-\nu(t-s)}\|U(s,t_0)x\|, \, t \ge s. \quad \blacksquare$$

As an immediate consequence of this theorem we obtain the following corollary:

COROLLARY 2.1. *Under the conditions of Theorem* 2.1 *the space*

$$\begin{aligned} X_0(t_0) &:= \{x \in \mathbf{X}\colon \int_{t_0}^\infty \|U(t,t_0)x\|^p dt < \infty\} \\ &= \{x \in \mathbf{X}\colon \|U(t,t_0)x\| \le Ne^{-\nu(t-t_0)}\|x\|\}, \qquad t_0 \ge 0, \end{aligned}$$

for certain positive constants N, ν, is a closed linear subspace of $\mathbf{X}$.

3. EXPONENTIAL DICHOTOMY

We will characterize the exponential dichotomy of evolution families by using the operators $I_0, I_\mathbf{X}$. In particular, applying Corollary 2.1 we will get necessary and sufficient conditions for exponential dichotomy in Hilbert spaces. Before doing so we now make precise the notion of exponential dichotomy in the following definition.

DEFINITION 3.1. An evolution family $\mathcal{U} = (U(t,s))_{t\ge s\ge 0}$ on the Banach space $\mathbf{X}$ is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t), t \ge 0$, on $\mathbf{X}$ and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s), t \ge s \ge 0$,

(b) the restriction $U(t,s)_{|}\colon \ker P(s) \to \ker P(t), t \ge s \ge 0$ is an isomorphism (and we denote its inverse by $U(s,t)_{|}\colon \ker P(t) \to \ker P(s)$),

(c) $\|U(t,s)x\| \le Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)\mathbf{X}, t \ge s \ge 0$,

(d) $\|U(s,t)_{|}x\| \le Ne^{-\nu(t-s)}\|x\|$ for $x \in \ker P(t), t \ge s \ge 0$.

The following lemma can be proved by using the same arguments as used in [8]. By $\mathcal{L}(\mathbf{X},\mathbf{Y})$ we denote the space of bounded linear operators between the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$.

LEMMA 3.1. *Let $\mathcal{U}$ be an evolution family having exponential dichotomy on $[0,+\infty)$ with the corresponding family of projections $P(t), t \ge 0$ and constants $N > 0, \nu > 0$. Then*

(a) $M := \sup_{t\ge 0}\|P(t)\| < \infty$,

(b) $[0, t] \ni s \mapsto U(s,t)_| \in \mathcal{L}(\ker P(t), \mathbf{X})$ *is strongly continuous for* $t \geq 0$,

(c) $t \mapsto P(t)$ *is strongly continuous.*

Now we come to our first main result. It characterizes the exponential dichotomy of an evolution family by properties of the operator $I_{\mathbf{X}}$.

THEOREM 3.1. *Let* $\mathcal{U} = U(s,t)_{t \geq s \geq 0}$ *be an evolution family on the Banach space* $\mathbf{X}$. *Then the following assertions are equivalent*:

(i) $\mathcal{U}$ *has an exponential dichotomy on* $[0, +\infty)$,

(ii) $I_{\mathbf{X}}: D(I_{\mathbf{X}}) \to L_p$ *is surjective and* $X_0(0)$ *is complemented in* $\mathbf{X}$.

Proof. (i)⇒(ii). Let $(P(t))_{t \geq 0}$ be the family of projections determined by the exponential dichotomy. Then $X_0(0) = P(0)X$, and hence $X_0(0)$ is complemented. If $f \in L_p(\mathbf{R}_+, X)$ define $v: [0, \infty) \longrightarrow X$ by

$$v(t) = \int_0^t U(t,\xi)P(\xi)f(\xi)d\xi - \int_t^\infty U(t,\xi)_|(Id - P(\xi))f(\xi)d\xi. \qquad (7)$$

It is easy to see (see [6]) that $v \in E_X$ and is a solution of Eq. (1). By the definition of I_X we have $I_X v = f$. Therefore $I_X: D(I_X) \to L_p$ is surjective.

(ii)⇒(i). We prove this in several steps.

(A) Let $Z \subseteq X$ be a complement of $X_0(0)$ in X, i.e., $X = X_0(0) \oplus Z$. Set $X_1(t) = U(t,0)Z$. Then

$$U(t,s)X_0(s) \subseteq X_0(t), \qquad U(t,s)X_1(s) = X_1(t), t \geq s \geq 0. \qquad (8)$$

(B) There are constants $N, \nu > 0$ such that

$$\|U(t,0)x\| \geq Ne^{\nu(t-s)}\|U(s,0)x\| \qquad \text{for } x \in X_1(0), t \geq s \geq 0. \qquad (9)$$

In fact, let $Y := \{v \in D(I_X): v(0) \in X_1(0)\}$ endowed with graph norm $\|v\|_{I_X} = \|v\|_{E_X} + \|I_X v\|_p$. Then Y is a closed subspace of the Banach space $(D(I_X), \|.\|_{I_X})$, and hence Y is complete. By Remark 1.1 we have $\ker I_X := \{v \in D(I_X): v(t) = u(t,0)x \text{ for some } x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and I_X is surjective we obtain that

$$I_X: Y \to L_p(\mathbf{R}_+, X)$$

is bijective and hence an isomorphism. Thus there is a constant $\nu > 0$ such that

$$\|I_X v\|_p \geq \nu\|v\|_{I_X} \geq \nu\|v\|_{E_X} \geq \nu\|v\|_p, \qquad \text{for } v \in Y. \qquad (10)$$

Let $0 \neq x \in X_1(0)$ and set $u(t) := U(t,0)x$, $t \geq 0$. By Remark 1.1 we have $u(t) \neq 0$ for all $t \geq 0$. For $\tau > 0$, Δ_1, Δ_2 are real positive constants such that $\tau > \Delta_1 > \Delta_2$. Take

$$v_\tau(t) = -u(t) \int_t^\infty \chi_{[\tau-\Delta_1,\, \tau-\Delta_2]}(s) \|u(s)\|^{-1} ds$$

$$f_\tau(t) = \chi_{[\tau-\Delta_1,,\tau-\Delta_2]}(t) \frac{u(t)}{\|u(t)\|}$$

$$\chi_{[\tau-\Delta_1,\, \tau-\Delta_2]}(s) = \begin{cases} 0 & \text{for } s \notin [\tau-\Delta_1, \tau-\Delta_2], \\ 1 & \text{for } s \in [\tau-\Delta_1, \tau-\Delta_2]. \end{cases}$$

Then $v_\tau \in Y$ and $f_\tau \in L_p$. They satisfy Eq. (1). It follows that

$$I_X v_\tau = f_\tau \Rightarrow \|f_\tau\|_p \geq \nu \|v_\tau\|_{I_X} \geq \nu \|v_\tau\|_p.$$

Putting $\Delta := \Delta_1 - \Delta_2$ we come to

$$\begin{aligned} &\left(\int_{\tau-\Delta_1}^{\tau-\Delta_2} \|u(t)\|^p \left(\int_t^\infty \chi_{[\tau-\Delta_1,\, \tau-\Delta_2]}(s) \|u(s)\|^{-1} ds \right)^p dt \right)^{1/p} \\ &\quad \leq \|v_\tau\|_p \leq \nu^{-1} \|f_\tau\|_p = \nu^{-1} \left(\int_0^{+\infty} \chi_{[\tau-\Delta,\, \tau]}(s) ds \right)^{1/p} \\ &\quad = \nu^{-1} \Delta^{1/p}. \end{aligned}$$

Thus for every $\Delta_1 > \Delta_2 > 0$

$$\begin{aligned} &\frac{1}{\Delta} \int_{\tau-\Delta_1}^{\tau-\Delta_2} \|u(t)\|^p \left(\int_t^\infty \chi_{[\tau-\Delta,\, \tau]}(s) \|u(s)\|^{-1} ds \right)^p dt \\ &\qquad = \frac{1}{\Delta} \int_{\tau-\Delta_1}^{\tau-\Delta_2} \|u(t)\|^p \left(\int_t^\tau \|u(s)\|^{-1} ds \right)^p dt \leq \nu^{-p}. \end{aligned}$$

From the arbitrary nature of Δ_1, Δ_2 and the continuity of $\|u(t)\|$ with respect to t it follows that

$$\int_t^\tau \|u(s)\|^{-1} ds \leq \nu^{-1} \|u(t)\|^{-1}.$$

Thus letting $\tau \to \infty$ we obtain

$$\int_t^\infty \|u(s)\|^{-1} ds \leq \nu^{-1} \|u(t)\|^{-1}.$$

Therefore the exponential boundedness of $\mathcal{U}$ and Lemma 1.3 imply that there is a constant $N > 0$ independent of x such that

$$\|u(t)\| \geq N e^{\nu(t-t_0)} \|u(s)\|; \qquad t \geq s \geq 0.$$

(C) $X = X_0(t) \oplus X_1(t)$, $t \geq 0$. Let $Y \subset D(I_X)$ be as in **(B)**. Then by Remark 1.1, $D(I_0) \subset Y$. From this and (10) we have $\|I_0 v\|_p \geq \nu \|v\|_{E_X}$, for $v \in D(I_0)$. Thus, I_0 is correct. By Corollary 2.1, $X_0(t)$ is closed. From (8), (9), and the closedness of $X_1(0)$ we can easily derive that $X_1(t)$ is closed and $X_1(t) \cap X_0(t) = \{0\}$ for $t \geq 0$.

Finally, fix $t_0 > 0$, and $x \in X$. Set

$$v(t) = \int_t^\infty \chi_{[t_0,\, t_0+1]}(s)ds.U(t, t_0)x,$$

$$f(t) = -\chi_{[t_0,\, t_0+1]}(t).U(t, t_0)x, \qquad t \geq t_0.$$

Then v, f solve Eq. (1) with $t \geq s \geq t_0 \geq 0$, $v \in L_p([t_0, \infty), \mathbf{X})$. Extend f to $[0,\infty)$ by setting $f\,|_{[0,\, t_0)} = 0$. Then $f \in L_p(\mathbf{R}_+, \mathbf{X})$ by assumption and there exists $w \in D(I_X)$ such that $I_X w = f$. By the definition of I_X, w is a solution of Eq. (1). In particular, $w\,|_{[t_0,\, \infty)}$ also satisfies (1). Thus,

$$\begin{aligned} v(t) - w(t) &= U(t, t_0)(v(t_0) - w(t_0)) \\ &= U(t, t_0)(x - w(t_0)), \qquad t \geq t_0. \end{aligned}$$

Since $v - w\,|_{[t_0,\, \infty)} \in L_p([t_0,\, \infty), \mathbf{X})$ this implies $x - w(t_0) \in X_0(t_0)$. On the other hand $w(0) = w_0 + w_1$ with $w_k \in X_k(0)$. Then $w(t_0) = U(t_0, 0)w_0 + U(t_0, 0)w_1$ and by (8) we have $U(t, t_0)w_k \in X_k(t_0)$, $k = 0, 1$. Hence $x = x - w(t_0) + w(t_0) \in X_0(t_0) + X_1(t_0)$. This proves **(C)**.

(D) Let $P(t)$ be the projections from X onto $X_0(t)$ with kernel $X_1(t)$, $t \geq 0$. Then (8) implies that $P(t)U(t, s) = U(t, s)P(s)$, $t \geq s \geq 0$. From (8), (9) we obtain that $U(t, s)_{|}\ker P(s) \to \ker P(t)$, $t \geq s \geq 0$ is an isomorphism. Finally, by (9), Theorem 2.1, and the assumption that I_0 is correct, there exist constants $N, \nu > 0$ such that

$$\|U(t, s)x\| \leq Ne^{-\nu(t-s)}\|x\| \qquad \text{for } x \in P(s)X,\ t \geq s \geq 0$$

$$\|U(s, t)x\| \leq Ne^{-\nu(t-s)}\|x\| \qquad \text{for } x \in \ker P(t),\ t \geq s \geq 0.$$

Thus $\mathcal{U}$ has an exponential dichotomy. ∎

If X is a Hilbert space we need only the closedness of $X_0(0)$. Therefore, we have

COROLLARY 3.1. *If X is a Hilbert space then the conditions that I_0 is correct and I_X is surjective are necessary and sufficient for exponential dichotomy of an evolution family.*

Proof. The corollary is obvious in view of Corollary 2.1 and Theorem 3.1. ∎

4. ORDINARY DICHOTOMY

In this section we consider the characterization of an evolution family having ordinary dichotomy by virtue of the operator I_1 defined almost the same as the operator $I_{\mathbf{X}}$. However, the domain of I_1 is contained in $C(\mathbf{R}_+, \mathbf{X})$ and its range is in $L_1(\mathbf{R}_+, \mathbf{X})$.

DEFINITION 4.1. An evolution equation family $\mathcal{U} = (U(t,s))_{t \geq s \geq 0}$ on the Banach space $\mathbf{X}$ is said to have an *ordinary dichotomy* if there exist bounded linear projections $P(t)$, $t \geq 0$, on $\mathbf{X}$ and positive constants M such that

(i) $P(t)U(t,s) = U(t,s)P(s); t \geq s \geq 0$,

(ii) $U(t,s)_{|}\colon \ker P(s) \to \ker P(t); t \geq s \geq 0$ is an isomorphism (and we denote its inverse by $U(s,t)_{|}\colon \ker P(t) \to \ker P(s)$),

(iii) $\|U(t,s)x\| \leq M\|x\|$ for $x \in P(s)\mathbf{X}; t \geq s \geq 0$,

(iv) $\|U(s,t)_{|}x\| \leq M\|x\|$ for $x \in \ker P(t)$, $t \geq s \geq 0$,

(v) $\sup_{t \geq 0} \|P(t)\| < \infty$.

In the following lemma we collect some properties of the family $P(t)$, $t \geq 0$.

LEMMA 4.1. *Let $\mathcal{U}$ be an evolution family having ordinary dichotomy with corresponding family of projections $P(t)$, $t \geq 0$, and constants $M > 0$. Then*

(a) $[0,t] \ni s \mapsto U(s,t)_{|} \in \mathcal{L}(\ker P(t), \mathbf{X})$ *is strongly continuous for* $t \geq 0$,

(b) $t \mapsto P(t)$ *is strongly continuous*,

(c) *Condition* (v) *in the definition of ordinary dichotomy is equivalent to the condition that there is a positive constant γ_0 such that for $0 \neq x_0 \in P(s)X$ and $0 \neq x_1 \in \ker P(s)$ we have*

$$\gamma[x_0, x_1] = \left\| \frac{x_0}{\|x_0\|} + \frac{x_1}{\|x_1\|} \right\| \geq \gamma_0 > 0.$$

Proof. This lemma has been essentially proved in [8]. ∎

Next we give some necessary definitions for later use. First, we define the operator I_1: For $v \in C$ and $f \in L_1$ such that they satisfy Eq. (1) we set $I_1 v := f$ with

$$D(I_1) := \big\{ v \in C\colon \text{ there exists } f \in L_1 \text{ such that } v, f \text{ satisfy (1) for a.e. } (t,s) \in D^+ \big\}.$$

It is easy to verify that I_1 is a linear, well-defined, and closed operator. Then for each t_0 we denote by $X_0(t_0)$ a linear manifold in $\mathbf{X}$ defined as

$$X_0(t_0) := \left\{ x \in \mathbf{X}\colon \sup_{t \geq t_0} \|U(t,t_0)x\| < \infty \right\}.$$

Now we come to our main results of this section.

THEOREM 4.1. *If the evolution family $\mathcal{U}$ has an ordinary dichotomy then for each $f \in L_1$ Eq.* (1) *has at least one solution $u \in C$.*

Proof. Let $P(t)$, $t \geq 0$, be the family of projections given by the ordinary dichotomy. Then from Lemma 4.1 we have that there exists $K > 0$ such that $\|P(t)\| \leq K$, for all $t \in \mathbf{R}_+$.

For each $f \in L_1$ we define $v : [0, \infty) \to \mathbf{X}$ by

$$v(t) = \int_0^t U(t,\xi)P(\xi)f(\xi)d\xi - \int_t^\infty U(t,\xi)_|(Id - P(\xi))f(\xi)d\xi. \tag{11}$$

Then

$$\|v(t)\| \leq M.K \int_0^t \|f(\xi)\|d\xi + M(1+K)\int_t^\infty \|f(\xi)\|d\xi \leq M(1+K)\|f\|_{L_1}.$$

Hence, $v \in C$, and it is easy to verify that v, f satisfy Eq. (1). ∎

THEOREM 4.2. *Let $\mathcal{U} = U(t,s)$ be an evolution family on the Banach space $\mathbf{X}$. If for each $f \in L_1$ Eq.* (1) *has at least one solution $v(.) \in C$, and $X_0(0)$ is a complemented subspace of $\mathbf{X}$, then $\mathcal{U}$ has an ordinary dichotomy.*

Proof. The assumption means that I_1 is surjective. We prove the theorem in several stages.

(A) Let $Z \subseteq \mathbf{X}$ be a complement of $X_0(0)$ in $\mathbf{X}$, i.e., $\mathbf{X} = X_0(0) \oplus Z$. Set $X_1(t) := U(t,s)Z$. Then

$$U(t,s)X_0(s) \subseteq X_0(t), \qquad U(t,s)X_1(s) = X_1(t), t \geq s. \tag{12}$$

(B) There is a constant $M > 0$ such that

$$\|U(t,0)x\| \geq M^{-1}\|U(s,0)x\|, \qquad \text{for } x \in X_1(0). \tag{13}$$

In fact, let $Y := \{v \in D(I_1) : v(0) \in X_1(0)\}$ endowed with graph norm $\|v\|_{I_1} = \|v\|_C + \|I_1 v\|_{L_1}$. Then Y is a closed subspace of the Banach space $(D(I_1), \|.\|_{I_1})$ and hence is complete. It can be seen that $\ker I_1 := \{v \in D(I_1), v(t) = U(t,0)x$ for some $x \in X_0(0)\}$.

Let $\widehat{I_1}$ be the restriction of I_1 in Y, i.e., $D(\widehat{I_1}) = Y$ and $\widehat{I_1}(v) = I(v)$ for $v \in Y$. Since $\mathbf{X} = X_0(0) \oplus X_1(0)$ and I_1 is surjective, we can obtain that

$$\widehat{I_1} : Y \to L_1$$

is bijective and hence an isomorphism. Put $M := \|\widehat{I_1}^{-1}\|$. We have

$$M\|I_1 v\|_{L_1} \geq \|v\|_{I_1} \geq \|v\|_C \qquad \text{for } v \in Y.$$

Let $0 \neq x \in X_1(0)$. Set $u(t) = U(t,0)x$, $t \geq 0$; then we have $u(t) \neq 0$ for all $t \geq 0$. Indeed, if $u(t_0) = 0$ for some $t_0 > 0$, then $u(t) = 0$ for all $t \geq t_0$.

Hence, $u \in C$, so $x \in X_0(0)$, that means $x \in X_1(0) \cap X_0(0) = \{0\}$. It contradicts the fact that $x \neq 0$. For each $\tau \geq 0, \Delta > 0$, take

$$v_\tau(t) = -u(t) \int_t^\infty \chi_{[\tau, \tau+\Delta]}(\xi) \|u(\xi)\|^{-1} d\xi$$

$$f_\tau(t) = \chi_{[\tau, \tau+\Delta]}(t) \frac{u(t)}{\|u(t)\|}.$$

Then $v_\tau \in Y$; $f_\tau \in L_1$ and they satisfy Eq. (1). It follows that $I_1 v_\tau = f_\tau$; hence, $M\|f_\tau\| \geq \|v_\tau\|$. Therefore

$$\|v_\tau\|_C \leq M \int_0^\infty \|\chi_{[\tau, \tau+\Delta]}(\xi) u(\xi) \|u(\xi)\|^{-1}\| d\xi = M\Delta.$$

For every fixed $0 \leq t \leq \tau$ we have

$$\begin{aligned} \|v_\tau(t)\| &= \|u(t)\| \int_\tau^{\tau+\Delta} \|u(\xi)\|^{-1} d\xi \\ &\leq \sup_{s \in \mathbf{R}_+} \|v_\tau(s)\| \\ &\leq M\Delta. \end{aligned}$$

Hence,

$$\frac{1}{\Delta} \int_\tau^{\tau+\Delta} \|u(\xi)\|^{-1} d\xi \leq M \|u(t)\|^{-1}.$$

Letting $\Delta \to 0$, we arrive at $\|u(\tau)\|^{-1} \leq M\|u(t)\|^{-1}$, $\forall t \leq \tau$. Consequently, for all $t \geq s$ and $x \in X_1(0)$

$$\|U(t, 0)x\| \geq M^{-1} \|U(s, 0)x\|.$$

(C) $\|U(t, t_0)x\| \leq M\|x\|$ for $x \in X_0(t_0)$, with $M := \|\widehat{I_1}^{-1}\|$ as mentioned above. Without loss of generality we can assume that $\|x\| = 1$. Let $u(t) = U(t, t_0)x$, $t \geq t_0$, and $t_1 := \sup\{t > t_0\colon u(t) \neq 0\}$, $t_1 > t_0$. Take

$$v(t) := u(t) \int_0^t \chi_{[t_0, t_0+\Delta]}(\xi) d\xi$$

$$f(t) := \chi_{[t_0, t_0+\Delta]} \frac{u(t)}{\|u(t)\|}.$$

Here $\Delta > 0$ and $t_0 + \Delta \leq t_1$. Therefore, we have $v \in Y$, $f \in L_1$ and they solve Eq. (1). Hence, from **(B)** we can derive

$$\|v\|_C \leq M\|f\|_{L_1} = M\Delta.$$

By the same way as done in part **(B)** we come to

$$\|u(t)\| \leq \frac{M}{\Delta} \int_{t_0}^{t_0+\Delta} \|u(\xi)\| d\xi,$$

for $t > t_0 \geq 0$ and $\Delta > 0$ such that $t > t_0 + \Delta$. Letting $\Delta \to 0$ we obtain $\|u(t)\| \leq M\|u(t_0)\|$.

For fixed $s \geq t_0$, setting $y := U(s, t_0)x$, we get

$$\sup_{t \geq s}\{\|U(t, s)y\|\} = \sup_{t \geq t_0}\{\|U(t, t_0)x\|\} < \infty.$$

Therefore,

$$\|U(t, t_0)x\| = \|U(t, s)y\| \leq M\|y\| = M\|U(s, t_0x)\|.$$

(D) $\mathbf{X} = X_0(t) \oplus X_1(t)$, $t \geq 0$. From **(B)** we have $X_0(t)$ is closed. From (12), (13) and the closeness of $X_1(0)$ it is easy to derive that $X_1(t)$ is close and $X_1(t) \cap X_0(t) = \{0\}$ for $t \geq 0$.

Finally, fix t_0 >0, and $x \in \mathbf{X}$. Set

$$v(t) = \int_t^\infty \chi_{[t_0,\, t_0+1]}(s)ds.U(t, t_0)x$$
$$f(t) = -\chi_{[t_0,\, t_0+1]}(t).U(t, t_0)x,\ t \geq t_0.$$

Then v, f solve Eq. (1) with $t \geq s \geq t_0 \geq 0$ and $v \in L_p([t_0, \infty), \mathbf{X})$. Extend f to $[0, \infty)$ by setting $f\mid_{[0,\, t_0)} = 0$. Hence $f \in L_p(\mathbf{R}_+, \mathbf{X})$ by assumption and there exists $w \in D(I_1)$ such that $I_1 w = f$. By the definition of I_1, w is a solution of Eq. (1). In particular, $w\mid_{[t_0,\, \infty)}$ also satisfies (1). Thus

$$v(t) - w(t) = U(t, t_0)(v(t_0) - w(t_0)) = U(t, t_0)(x - w(t_0)), \qquad \forall t \geq t_0.$$

Since

$$v - w\mid_{[t_0,\, \infty)} \in L_p([t_0, \infty), \mathbf{X})$$

this implies $x - w(t_0) \in X_0(t_0)$. On the other hand $w(0) = w_0 + w_1$ with $w_k \in X_k(0)$. Hence, $w(t_0) = U(t_0, 0)w_0 + U(t_0, 0)w_1$ and by (12) we have $U(t, t_0)w_k \in X_k(t_0)$, $k = 0, 1$. Therefore, $x = x - w(t_0) + w(t_0) \in X_0(t_0) + X_1(t_0)$. This proves **C**).

(E) Let $P(t)$ be the projections from $\mathbf{X}$ onto $X_0(t)$ with kernel $X_1(t)$, $t \geq 0$. Then (12) implies that $P(t)U(t, s) = U(t, s)P(s)$, $t \geq s \geq 0$. From (12), (13) we obtain that $U(t, s) \mid \ker P(s) \to \ker P(t)$, $t \geq s \geq 0$, is an isomorphism. Finally, from **(B)**, **(C)**, and **(D)** there exists $M > 0$ such that

$$\|U(t, s)x\| \leq M\|x\| \qquad \text{for } x \in P(s)\mathbf{X},\ t \geq s \geq 0$$
$$\|U(s, t)_{|}x\| \leq M\|x\| \qquad \text{for } x \in \ker P(t),\ t \geq s \geq 0.$$

(F) By Lemma 4.1 to prove that $\sup_{t\in\mathbf{R}_+}\|P(t)\| < \infty$ we will show that for $0 \neq x_0 \in P(s)\mathbf{X}$; $0 \neq x_1 \in \ker P(s)$ the following holds

$$\left\| \frac{x_0}{\|x_0\|} + \frac{x_1}{\|x_1\|} \right\| \geq M. \tag{14}$$

Indeed, for $\tau > s$ put $u(\tau) = U(\tau, s)x_0$; $v(\tau) = U(\tau, s)x_1$ and

$$y(t) = \frac{U(t, s)x_0}{\|u(\tau)\|}, \ z(t) = \frac{U(t, s)x_1}{\|v(\tau)\|}; \qquad w(t) = y(t) + z(t).$$

From (12) and (**E**) we have $w(t) \neq 0$ for all $t \in \mathbf{R}_+$.

Put

$$x(t) = y(t)\int_0^t \chi_{[\tau,\,\tau+\Delta]}(\xi)\|w(\xi)\|^{-1}d\xi - z(t)\int_t^\infty \chi_{[\tau,\,\tau+\Delta]}(\xi)\|w(\xi)\|^{-1}d\xi.$$

Hence, $x(\cdot)$ is a solution of (1) for $f(t) = \frac{w(t)}{\|w(t)\|}\chi_{[\tau,\,\tau+\Delta]}(t)$ and $x(\cdot) \in Y$. Therefore, for $M := \|\widehat{I_1}^{-1}\|$ we have $\|x(\cdot)\|_C \leq M\|f\|_{L_1} = M\Delta$. Thus, $\|x(\cdot)\|_C \leq M\Delta$. Hence, we have

$$\begin{aligned}\int_\tau^{\tau+\Delta} \|w(\xi)\|^{-1}d\xi = \|x(\tau)\| \\ &= \|z(\tau)\| \int_\tau^{\tau+\Delta} \|w(\xi)\|^{-1}d\xi \\ &= \|x(\tau)\| \\ &\leq \sup_{\xi\in\mathbf{R}_+} \|x(\xi)\| = \|x(\cdot)\|_C \\ &\leq M\Delta.\end{aligned}$$

Thus, by letting Δ to zero

$$\|w(\tau)\|^{-1} \leq M.$$

This proves (14), finishing the proof of the theorem. ∎

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