Received: 20 June 2020 | Accepted: 28 July 2020

DOI: 10.1002/cta.2870

WILEY

RESEARCH ARTICLE

Dynamic iteration schemes and port-Hamiltonian formulation in coupled differential-algebraic equation circuit simulation

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Summary

Electric circuits are usually described by charge/flux-oriented modified nodal analysis. Here, we derive models as port-Hamiltonian systems on several levels: overall systems, multiply coupled systems, and systems within dynamic iteration procedures. To this end, we introduce new classes of port-Hamiltonian differential-algebraic equations. Thereby, we additionally allow for nonlinear dissipation on a subspace of the state space. Both, each subsystem and the overall system possess a port-Hamiltonian structure. A structural analysis is performed for the new setups. Dynamic iteration schemes are investigated, and we show that the Jacobi approach as well as an adapted Gauss-Seidel approach lead to port-Hamiltonian differential-algebraic equations.

KEYWORDS

dynamic iteration, differential-algebraic equations, electrical circuits, port-Hamiltonian systems

JEL CLASSIFICATION

34A09; 37J05; 65L80; 94C05; 94C15

1 | INTRODUCTION

Models for electric circuits are based on a collection of basic electric components. These form branches of a directed graph. The directed graph represents the interconnection structure, which is represented by the incidence matrix A that enables to formulate Kirchhoff's voltage law (KVL) and Kirchhoff's current law (KCL). Electric components describe a certain electric effect. In our case, these are resistances, capacitances, inductances, independent current, and independent voltage sources.

Often, modified nodal analysis (MNA) is used as modeling approach for electric circuits. For charge and flux conservation, this is extended to the charge/flux-oriented form; see Günther and Feldmann.[1] Now, KVL allows the assignment of node potentials, where an identified ground node has a given value. The node potentials, apart from ground, from the unknowns e. Further unknowns of the charge/flux-oriented network model are all currents through inductances J_L and currents through voltages sources J_V, the charges q_C at the capacitances, and the magnetic fluxes ϕ_L at the inductances. Thus, the vector of unknowns reads $x^\top(t) = \left(e^\top(t), J_L^\top(t), J_V^\top(t), q_C^\top(t), \phi_L^\top(t)\right) \in R^d$, where the time

t evolves in a specified operation interval $\mathcal{I} := [0, t_e] \subseteq R$. The circuit equations in *charge/flux-oriented modified nodal analysis (MNA)* read

$$A_C \frac{\mathrm{d}}{\mathrm{d}t} q_C + A_R g\left(A_R^\top e\right) + A_{LJL} + A_{VJV} + A_I \imath(t) = 0, \tag{1a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_L - A_L^\top e = 0, \tag{1b}$$

$$A_V^\top e - v(t) = 0, \tag{1c}$$

$$q_C - q(A_C^\top e) = 0, \tag{1d}$$

$$\phi_L - \phi(J_L) = 0, \tag{1e}$$

where we have component-specific incidence matrices $A_\star$. Moreover, we use for the component relations: $q(v)$ for capacitances, $g(v)$ for resistances, and $\phi(J_L)$ for inductances. And we have given time-dependent functions $v(t)$ for independent voltage sources and $\imath(t)$ for independent current sources. The involved matrices and functions are further specified in the forthcoming Section 2.

One aim of this paper is to model the MNA as port-Hamiltonian DAE. Port-Hamiltonian systems form a joint structure of systems in various physical domains. This approach has its roots in analytical mechanics and starts from the principle of least action and proceeds towards the Hamiltonian equations of motion. Dynamic systems, which result from variational principles, can usually be modeled by a port-Hamiltonian system. A system theoretical and geometric treatment of port-Hamiltonian ordinary differential systems goes back to van der Schaft, and there is by now a well-established theory (see van der Schaft[2] and Jeltsema & van der Schaft[3] for an overview), which has been applied to electrical circuits, Gernandt et al.[4] Only recently, the concept has been generalized to port-Hamiltonian differential-algebraic systems, that is, ordinary differential equations with algebraic constrains (see van der Schaft[5] and Maschke and van der Schaft[6,7]). In Beattie et al.,[8] linear time-varying port-Hamiltonian DAEs have been studied, and the notion has been generalized to quasilinear systems in Mehrmann and Morandin.[9]

Now, we extend the class even further in order to allow for nonlinear dissipation on a subspace of the state space. We introduce two circuit models throughout this article, which are slightly different from the charge/flux oriented MNA (1). Both models are formulated as a port-Hamiltonian DAE. Furthermore, we investigate multiply coupled circuits and extend our definitions in this respect to multiply coupled port-Hamiltonian DAEs. In fact, we show that port-Hamiltonian DAEs can be coupled in such a way that the overall system is a port-Hamiltonian DAE as well. This is applied to our circuits models.

A further novelty of this paper is the study of dynamic iteration schemes in the context of port-Hamiltonian systems. For an overview on dynamic iteration schemes for ODEs, see Burrage.[10] These schemes have also been studied for DAEs, where convergence cannot be generally guaranteed; see, for example, Lelarsemee et al.,[11] Jackiewicz and Kwapisz,[12] and Arnold and Günther.[13] Here, we investigate dynamic iteration schemes for coupled systems composed by k subsystems with a dedicated coupling equation. For these type of systems, we show that both Jacobi- and Gauss-Seidel-type schemes can be interpreted as port-Hamiltonian systems. In order to achieve this goal, we have to modify slightly the modeling of the interconnections. Again, as an example, we study electric circuits.

The outline of the paper is as follows: Section 2 addresses the mathematical modeling background for the charge/flux oriented circuit equations. In the following Section 3, the various port-Hamiltonian formulations are introduced. Then, a DAE index analysis is performed for our models (Section 4). Section 5 introduces structural properties for coupled circuits, and Section 6 merges the port-Hamiltonian formulation with the dynamic iteration schemes. Finally, there are conclusions.

2 | CIRCUIT EQUATIONS—A STRUCTURAL ANALYSIS

We will consider special variants of the charge/flux-oriented MNA equations (1), suitable for the port-Hamiltonian setting. To this end, we first present some fundamentals on circuit equations. An electrical circuit is described by the properties of its components together with the interconnection structure. The latter is modeled by a directed and finite graph which has no self-loops. Moreover, many properties of the circuit equations such as soundness, passivity, and DAE-index depend both on topological conditions of the underlying graph, as for instance about the absence of certain component-specific loops and and cutsets (see, e.g., Bartel et al.[14] and Bartel and Günther[15]). Circuits consist of a *branch set* E, a *node set* V, and two *incidence mappings* init, ter : $E \to V$ assigning to each branch $\bar{e}$ an *initial node* init($\bar{e}$) and a

terminal node $\mathrm{ter}(\bar{e})$. (A branch $\bar{e}$ are also referred to as edge.) The sets E and V are assumed to be finite, and the property that the graph does not have any *self-loops* means that $\mathrm{init}(\bar{e}) \neq \mathrm{ter}(\bar{e})$ for all $\bar{e} \in E$. For the notions of *connected*, *loop*, and *cutset*, we refer to previous study.[16-18]

Finite directed graphs without self-loops allow the association of a special matrix; see Andrásfai.[16], Sec. 3.2 Namely, for $E = \{\bar{e}_1, ..., \bar{e}_m\}$ and $V = \{v_1, ..., v_n\}$, the *all-node incidence matrix* is $A_0 \in R^{n \times m}$ with

$$a_{jk} = \begin{cases} 1 & \mathrm{init}(\bar{e}_k) = v_j, \\ -1 & \mathrm{ter}(\bar{e}_k) = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If the graph is connected, then the co-rank of A_0 equals one and each column sum up to zero. Hence, the deletion of an arbitrary row leads to a matrix with full row rank.[16], p140 In the context of electrical circuits, this corresponds to the grounding of this node.

Let be given an incidence matrix A of a finite and loop-free directed graph (with branch set E). For a subset of branches $E' \subset E$, it is possible to obtain an incidence matrix corresponding to E' by deleting all columns corresponding to branches of $E \setminus E'$. By rearranging the columns, it follows that the matrix A is of the form

$$A = [A_{E \setminus E'} A_{E'}]. \tag{2}$$

Next, we collect some auxiliary results on incidence matrices corresponding to subgraphs from Estévez Schwarz and Tischendorf.[18] The first statement of the following proposition can be inferred from the fact that incidence matrices of connected (sub)graphs have full row rank. The further assertions are shown in Estévez Schwarz and Tischendorf.[18]

Proposition 1 *(Estévez Schwarz and Tischendorf[18], Thm. 2.2). Let a finite and loop-free connected graph be given with edge set* E*, incidence matrix* A*, and let* $\mathrm{E}'', \mathrm{E}' \subset \mathrm{E}$ *with* $\mathrm{E}'' \subset \mathrm{E}'$*. Assume that the incidence matrix is partitioned as in (2) and, likewise, the edges in* E' *are arranged such that* $A_{E'} = [A_{E' \setminus E''} A_{E''}]$*. Then the following holds:*

(**i**) *There do not exist any cutsets only consisting of branches in* E' *if, and only if,* $\ker A_{E \setminus E'}^{\top} = \{0\}$.
(**ii**) *There do not exist any loops only consisting of branches in* E' *if, and only if,* $\ker A_{E'} = \{0\}$.
(**iii**) *There do not exist any loops only consisting of branches in* E' *except for loops only consisting of branches in* E'' *if, and only if, for some (and hence any) matrix* $\mathrm{Z}_{E''}$ *with* $\mathrm{im}\, Z_{E''} = \ker A_{E''}^{\top}$*, it holds* $\ker Z_{E''}^{\top} A_{E' \setminus E''} = \{0\}$.

When considering an electrical circuit as a graph, we can split the incidence matrix into submatrices, which represent capacitances, resistances, inductances, voltage sources, and current sources, that is, $(A_C A_R A_L A_I A_V)$. We declare the following way of speaking: For $\mathcal{M} \subset \{C, R, L, I, V\}$, we say that the circuit has a $\mathcal{M}$-loop (resp. $\mathcal{M}$-cutset), if the graph corresponding to the circuit has a loop (cutset) only consisting of of branches to components in $\mathcal{M}$. For instance, a *CV*-loop is a loop only consisting of branches of capacitances and/or voltage sources, whereas an *I*-cutset is a cutset only consisting of branches of current sources. Now we are able to formulate our assumptions on the circuit.

Assumption 1 (**a**) **Soundness.** The circuit graph has at least one branch and is connected. The circuit graph further neither contains *V*-loops nor *I*-cutsets. Equivalently, by Proposition 1, A_V and $(A_C A_R A_L A_V)^{\top}$ have full column rank.
(**b**) **Passivity.** The functions q, ϕ, and g fulfill

(**i**) $q : R^{n_C} \to R^{n_C}$ and $\phi : R^{n_L} \to R^{n_L}$ are bijective, continuously differentiable, and their Jacobians

$$C(u_C) := \frac{dq}{du_C}(u_C),\ L(J_L) := \frac{d\phi}{d_{J_L}}(J_L)$$

are symmetric and positive definite for all $u_C \in R^{n_C}$, $J_L \in R^{n_L}$.

(**ii**) $g : R^{n_R} \to R^{n_R}$ is continuously differentiable, and its Jacobian has the property that $\frac{dg}{du_R}(u_R) + \frac{dg}{du_R}(u_R)^{\top}$ is positive definite for all $u_R \in R^{n_R}$.

The condition on the charge and flux functions implies that there exist certain scalar-valued functions that will later on be shown to be expressing the energy of an electrical circuit.

Proposition 3 If $q : R^{n_C} \to R^{n_C}$ and $\phi : R^{n_L} \to R^{n_L}$ fulfill Assumption 1 (bii), then there exist twice continuously differentiable functions $V_C : R^{n_C} \to R$, $V_L : R^{n_L} \to R$ with the following properties:

(**a**) $V_C : R^{n_C} \to R$, $V_L : R^{n_L} \to R$ are strictly convex, that is,

$$\forall \lambda \in [0,1], \forall q_{C,1}, q_{C,2} \in R^{n_C} : V_C(\lambda q_{C,1} + (1-\lambda) q_{C,2}) < \lambda V_C(q_{C,1}) + (1-\lambda) V_C(q_{C,2}),$$
$$\forall \lambda \in [0,1], \forall \phi_{L,1}, \phi_{L,2} \in R^{n_L} : V_L(\lambda \phi_{L,1} + (1-\lambda) \phi_{L,2}) < \lambda V_L(\phi_{L,1}) + (1-\lambda) V_L(\phi_{C,2}),$$

(**b**) The gradients of V_C and V_L are, respectively, the inverse functions of q and ϕ. That is,

$$\forall q_C \in R^{n_C} : \nabla V_C(q_C) = q^{-1}(q_C), \forall \phi_L \in R^{n_L} : \nabla V_L(\phi_L) = \phi^{-1}(\phi_L).$$

(**c**) V_C and V_L take, except for one $q_C^* \in R^{n_C}$ (resp. $\phi_L^* \in R^{n_L}$), positive values. That is, there exist $q_C^* \in R^{n_C}$, $\phi_L^* \in R^{n_L}$ such that $V_C(q_C) > 0$ and $V_L(\phi_L) > 0$ for all $q_C \in R^{n_C} \setminus \{q_C^*\}$ and $\phi_L \in R^{n_L} \setminus \{\phi_L^*\}$.

Proof By changing the roles of fluxes and charges, it suffices to prove the statement only for the charge function. Since q is bijective and its derivative is, by positive definiteness of $C(u_C)$, invertible, the inverse function of q is continuously differentiable as well, and the Jacobian reads $\frac{dq^{-1}}{dq_C}(q_C) = C(q^{-1}(q_C))^{-1}$. In particular, the Jacobian of q^{-1} is pointwise symmetric and positive definite as well. This together with the trivial fact that R^{n_C} is simply connected implies that there exists some twice differentiable function $V_C : R^{n_C} \to R$ with $\nabla V_C(q_C) = q^{-1}(q_C)$ for all $q_C \in R^{n_C}$. The pointwise positive definiteness of $\frac{dq^{-1}}{dq_C}(q_C)$ implies that V_C is strictly convex. Hence, V_C has a unique minimum $q_C^* \in R^{n_C}$. Now replacing V_C with the difference of V_C and $V_C(q_C^*)$, this function has the desired properties, and the proof is complete.

Remark 4 (**a**) If $n_C = n_L = n_R = 1$, then the conditions on q, ϕ and g imply that these functions are strictly monotonically increasing with

$$\lim_{u_C \to \pm\infty} q(u_C) = \pm\infty, \lim_{J_L \to \pm\infty} \phi(J_L) = \pm\infty, \lim_{u_R \to \pm\infty} g(u_R) = \pm\infty.$$

(**b**) Bijectivity of q, ϕ might by difficult to check. A sufficient condition can be inferred from the Hadamard-Levy Theorem,[19] which gives bijectivity of q and ϕ, if the conditions

$$\int_0^\infty \min_{\|u_C\| = r} \|C(u_C)^{-1}\|^{-1} dr = \infty, \int_0^\infty \min_{\|J_L\| = r} \|L(J_L)^{-1}\|^{-1} dr = \infty,$$

are fulfilled. By using the positive definiteness of $C(u_C)$ and $L(J_L)$, the latter is equivalent to

$$\int_0^\infty \min_{\|u_C\| = r} \lambda_{\min}(C(u_C)) dr = \infty, \int_0^\infty \min_{\|J_L\| = r} \lambda_{\min}(L(J_L)) dr = \infty,$$

where $\lambda_{\min}$ denotes the smallest eigenvalue of a matrix.

We will discuss two circuit models throughout this article, which are slightly different from the charge/flux oriented MNA (1). Both models are formulated such that they fit into the PH-DAE framework introduced in Section 3.

For the first model, we apply ϕ^{-1} to the equation $\phi_L - \phi(J_L) = 0$ to obtain $J_L = \phi^{-1}(\phi_L)$ which is further eliminated. Likewise, q^{-1} is applied to the equation $q_C - q(A_C^\top e)$ for the charges, which results into $A_C^\top e - q^{-1}(q_C) = 0$ (which is not eliminated from the system of equations). Summing up, we get

$$\frac{d}{dt} \begin{pmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_C \\ \phi_L \\ e \\ J_V \end{pmatrix} = \begin{pmatrix} 0 & -A_L & 0 & -A_V \\ A_L^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \end{pmatrix} - \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - q^{-1}(q_C) \\ 0 \end{pmatrix} + \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \imath(t) \\ v(t) \end{pmatrix}, \tag{3a}$$

and output equation

$$y = \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}^\top \begin{pmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \end{pmatrix} = \begin{pmatrix} -A_I^\top e \\ -J_V \end{pmatrix}. \tag{3b}$$

In the second model, we further add the variable J_C and the equation $\frac{d}{dt}q_C = J_C$ to the model (3). Moreover, the expression $\frac{d}{dt}q_C$ in the first equation of (3) is replaced by J_C, which results into

$$\frac{d}{dt}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ J_C \\ q_C \\ \phi_L \\ J_V \end{pmatrix} = \begin{pmatrix} 0 & -A_C & 0 & -A_L & -A_V \\ A_C^\top & 0 & -I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ A_L^\top & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ J_C \\ q^{-1}(q_C) \\ \phi^{-1}(\phi) \\ J_V \end{pmatrix} - \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \imath(t) \\ v(t) \end{pmatrix}, \tag{4a}$$

which is again completed by the output

$$y = \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}^\top \begin{pmatrix} e \\ J_C \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \end{pmatrix} = \begin{pmatrix} -A_I^\top e \\ -J_V \end{pmatrix}. \tag{4b}$$

Both models will be shown to fit into the port-Hamiltonian framework which will be presented in the forthcoming section. The first model contains less equations and unknowns and shares the index analysis results with those for the charge/flux-oriented MNA equations from Estévez Schwarz[18] as shown in Section 4, whereas the second model is slightly higher structured than the first one.

3 | PORT-HAMILTONIAN FORMULATION OF ELECTRIC CIRCUITS

In this section, we introduce the class of nonlinear port-Hamiltonian DAEs, for short *PH-DAE*, used in this paper. The following system class is a modification of a class of port-Hamiltonian differential-algebraic equations introduced by Mehrmann and Morandin.[9] We will show that our circuit models (3) and (4) fit into this framework. Furthermore, in the second part of this section, we look into multiply coupled PH-DAEs.

3.1 | Port-Hamiltonian for an overall system

Definition 5 (Port-Hamiltonian differential-algebraic equation [PH-DAE]**).** A differential-algebraic equation of the form

$$\begin{aligned} \frac{d}{dt}Ex(t) &= Jz(x(t)) - r(z(x(t))) + Bu(t), \\ y(t) &= B^\top z(x(t)) \end{aligned} \tag{5}$$

is called a *port-Hamiltonian differential-algebraic equation (PH-DAE)*, if the following holds:

1. $E \in R^{k\times n}$, $J \in R^{n\times n}$ and $B \in R^{n\times m}$, $z, r : R^n \to R^k$.
2. There exists a subspace $\mathcal{V} \subset R^n$ with the following properties:

(**i**) for all intervals $\mathcal{I} \subset R$ and functions $u : \mathcal{I} \to R^m$ such that (5) has a solution $x : \mathcal{I} \to R^n$, it holds $z(x(t)) \in \mathcal{V}$ for all $t \in \mathcal{I}$.

(**ii**) J is skew-symmetric on $\mathcal{V}$. That is, $v^\top J w = -w^\top J v$ for all $v, w \in \mathcal{V}$.

(**iii**) r is accretive on $\mathcal{V}$. That is, $v^\top r(v) \geq 0$ for all $v \in \mathcal{V}$.

3. There exists some function $H \in C^1(R^n, R)$ such that $\nabla H(x) = E^\top z(x)$ for all $x \in z^{-1}(\mathcal{V})$.

Port-Hamiltonian systems fulfill an energy balance. In doing so, notice that the total energy of a PH-DAE at time t is given by $H(x(t))$, whereas the power inflow is realized by the inner product of input and output.

Lemma 6 (Energy balance). The PH-DAE (5) system provides the usual energy balance

$$\frac{d}{dt} H(x(t)) \leq y(t)^\top u(t) \tag{6}$$

of port-Hamiltonian systems.

Proof By using that for any solution $(x, u, y) : \mathcal{I} \to R^n \times R^m \times R^m$ of (5), the following holds: First notice that, for a projector P onto $\mathrm{im} E^\top$, we have that $Px : \mathcal{I} \to R^n$ is differentiable. Furthermore, by $\nabla H(x) = E^\top z(x)$ for all $x \in z^{-1}(\mathcal{V})$, we have

$$\begin{aligned} \frac{d}{dt} H(x(t)) &= (\nabla H(x(t)))^\top \frac{d}{dt} Px(t) = z(x(t))^\top E \frac{d}{dt} Px(t) = z(x(t))^\top \frac{d}{dt} EPx(t) = z(x(t))^\top \frac{d}{dt} Ex(t) \\ &= \underbrace{z(x(t))^\top J z(x(t))}_{=0} \underbrace{- z(x(t))^\top r(z(x(t)))}_{\leq 0} + \underbrace{z(x(t))^\top B u(t)}_{= (B^\top z(x(t)))^\top u(t) = y(t)^\top u(t)} \leq y(t)^\top u(t). \end{aligned}$$

This completes the proof.

Remark 7 (**a**) The function r is responsible for energy dissipation. If $r = 0$, then the energy balance (6) becomes an equation. In particular, the energy of the system is conserved, if $r = 0$ and $u = 0$.

(**b**) Our definition of a port-Hamiltonian differential-algebraic equation differs from the one by Mehrmann and Morandin,[9] which is more general in the sense that time-varying port-Hamiltonian differential-algebraic systems are considered, and the matrices E and J may depend on the state x. However, the definition of a differential-algebraic port-Hamiltonian system in Mehrmann and Morandin[9] does not involve a (possibly proper) subspace $\mathcal{V} \subset R^n$ on which $z(x(\cdot))$ evolves and the function r is assumed to be linear in z. We note that Definition 5 can also be extended to the time-varying situation and to the case of z dependent matrices E and J.

(**c**) The space $\mathcal{V} \subset R^n$ may be proper because of linear (hidden) algebraic constraints. For instance, if for some matrix $K \in R^{k \times n}$ holds $KE = 0$, $KB = 0$ and $Kr(z) = 0$ for all $z \in R^n$, then a multiplication of (5) from the left with K leads to $KJz(x(t)) = 0$. This means that the solutions of (5) fulfill $z(x(t)) \in \ker KJ$ for all $t \in \mathcal{I}$.

3.2 | Electric circuits—a PH-DAE description

We show that the above models (3) and (4) of the electric circuit equations, which are based on the charge/flux-oriented MNA circuit equations, match with the PH-DAE definition.

Proposition 9 Let Assumption 1 hold. Moreover, let V_C and V_L be defined as in Proposition 3. Then the following holds:

(**a**) The model (3) is a PH-DAE with

$$u(t) = \begin{pmatrix} \imath(t) \\ v(t) \end{pmatrix}, y(t) = \begin{pmatrix} -A_I^\top e(t) \\ -J_V(t) \end{pmatrix}, x(t) = \begin{pmatrix} q_C(t) \\ \phi_L(t) \\ e(t) \\ J_V(t) \end{pmatrix}, z(x) = \begin{pmatrix} e \\ J_L \\ u_C \\ J_V \end{pmatrix} = \begin{pmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \end{pmatrix},$$

$$r \begin{pmatrix} e \\ J_L \\ u_C \\ J_V \end{pmatrix} = \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - u_C \\ 0 \end{pmatrix}, E = \begin{pmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & -A_L & 0 & -A_V \\ A_L^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 \end{pmatrix} B = \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix},$$

subspace $\mathcal{V} = \left\{ (e, J_L, u_C, J_V)^\top \in R^n \middle| A_C^\top e = u_C \right\}$ and Hamiltonian $H(x) = V_C(q_C) + V_L(\phi_L)$.

(**b**) The model (4) is a PH-DAE with $u(t)$, $y(t)$ as in (a), and

$$x(t) = \begin{pmatrix} e(t) \\ J_C(t) \\ q_C(t) \\ \phi_L(t) \\ J_V(t) \end{pmatrix}, z(x) = \begin{pmatrix} e \\ J_C \\ u_C \\ J_L \\ J_V \end{pmatrix} = \begin{pmatrix} e \\ J_C \\ q^{-1}(q_C) \\ \phi^{-1}(\phi_L) \\ J_V \end{pmatrix}, r \begin{pmatrix} e \\ J_C \\ u_C \\ J_L \\ J_V \end{pmatrix} = \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, J = \begin{pmatrix} 0 & -A_C & 0 & -A_L & -A_V \\ A_C^\top & 0 & -I & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ A_L^\top & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix},$$

and, for n_v being the number on non-grounded nodes, subspace $\mathcal{V} = R^{n_v} \times R^{n_C} \times R^{n_C} \times R^{n_L} \times R^{n_V}$ and Hamiltonian $H(x) = V_C(q_C) + V_L(\phi_L)$.

Proof (**a**) Since (3) contains $A_C^\top e(t) - u_C(t) = 0$, we see that any solution fulfills $z(x(t)) \in \mathcal{V}$ pointwise. The skew-symmetry of J is obvious. By the assumption that the Jacobian of g has positive definite real part, we obtain that g is accretive. This directly implies that r is accretive on $\mathcal{V}$. Moreover, by using Proposition 3, we compute

$$\nabla H(x) = \begin{pmatrix} \nabla V(q_C) \\ \nabla V(\phi_L) \\ 0 \\ 0 \end{pmatrix} \overset{\text{Prop.3}}{=} \begin{pmatrix} q^{-1}(q_C) \\ \phi^{-1}(\phi_L) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_C \\ J_L \\ 0 \\ 0 \end{pmatrix} \overset{z \in \mathcal{V}}{=} \begin{pmatrix} A_C^\top e \\ J_L^\top \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^\top \begin{pmatrix} e \\ J_L \\ u_C \\ J_V \end{pmatrix} = E^\top z(x).$$

(**b**) The space $\mathcal{V} = R^{n_v} \times R^{n_C} \times R^{n_C} \times R^{n_L} \times R^{n_V}$ trivially has the property that all solutions evolve in $\mathcal{V}$. Moreover, J is skew-symmetric, and the accretivity of r follows from the accretivity of g, where the latter can be concluded by the argumentation as in (a). Finally, for the gradient of the Hamiltonian, we compute

$$\nabla H(x) = \begin{pmatrix} 0 \\ 0 \\ \nabla V(q_C) \\ \nabla V(\phi_L) \\ 0 \end{pmatrix} \overset{\text{Prop.3}}{=} \begin{pmatrix} 0 \\ 0 \\ q^{-1}(q_C) \\ \phi^{-1}(\phi_L) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u_C \\ J_L \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}^\top \begin{pmatrix} e \\ J_C \\ u_C \\ J_V \\ J_V \end{pmatrix} = E^\top z(x)$$

3.3 | Port-Hamiltonian system formulation for multiple subsystems

Next, we generalize the above monolithic setting of Definition 5 to the case of $k \geq 2$ subsystems. To couple several PH-DAEs, we first setup some notation, to address different types of input and output: internal and coupling quantities.

Definition 6 (Multiply coupled PH-DAE**).** We consider k copies of PH-DAEs (5)

$$\begin{aligned} \frac{d}{dt} E_i x_i(t) &= J_i z_i(x_i(t)) - r_i(z_i(x_i(t))) + B_i u_i(t) \\ y_i(t) &= B_i^\top z_i(x_i(t)) \end{aligned} \tag{7}$$

with associated Hamiltonians H_i $(i = 1, ..., k)$. We call these k copies of PH-DAEs a multiply coupled PH-DAE if the following prerequisites are satisfied: The input u_i and the output y_i are split into

$$u_i(t) = \begin{pmatrix} \hat{u}_i(t) \\ \bar{u}_i(t) \end{pmatrix}, y_i(t) = \begin{pmatrix} \hat{y}_i(t) \\ \bar{y}_i(t) \end{pmatrix}, \tag{8}$$

where bar-accents refer to external inputs and outputs, that is, quantities, which are not communicated to other subsystems, and the hat-accents refer to input/output data used for coupling of the k subsystems. Moreover, the port matrix is split accordingly: $B_i = \left(\hat{B}_i \, \bar{B}_i\right)$. The subsystems are coupled via topological coupling matrices $\hat{C}_{i,j} \in \{-1, 0, 1\}^{m_i \times m_i}$

$$\hat{u}_i + \sum_{j=1, j \neq i}^{k} \hat{C}_{i,j} \hat{y}_j = 0 \, (\text{for} \, i = 1, ..., k), \, \hat{C} = \begin{pmatrix} 0 & \hat{C}_{1,2} & ... & \hat{C}_{1,k} \\ \hat{C}_{2,1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{C}_{k-1,k} \\ \hat{C}_{k,1} & ... & \hat{C}_{k,k-1} & 0 \end{pmatrix}$$

with $\hat{C}$ skew symmetric.

Now, we can deduce for the overall system described in Definition 6:

Corollary 10 (Multiply skew-symmetric coupling structure preserving interconnection**).** We consider a multiply coupled PH-DAE with k subsystems. The overall system is obtained by aggregation of vector quantities and matrices:

$$\begin{aligned} v^\top = (v_1^\top, ..., v_k^\top) \, \text{for} \, v \in \{x, u, \hat{u}, \bar{u}, y, \hat{y}, \bar{y}\}, \, F = \text{diag}(F_1, ..., F_k) \, \text{for} \, F \in \{E, J, \hat{B}, \bar{B}\}, \\ r^\top(z(x)) = \left(r_1(z_1(x_1))^\top, ..., r_k(z_k(x_k))^\top\right), \, z^\top(x) = \left((z_1(x_1)^\top, ..., z_k(x_k)^\top\right), \end{aligned}$$

and it reads (with coupling equation $\hat{u} + \hat{C}\hat{y} = 0$ in the third block equation)

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}E & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\ \hat{u}\\ \hat{y}\end{pmatrix} &= \begin{pmatrix}J & \hat{B} & 0\\ -\hat{B}^\top & 0 & I\\ 0 & -I & -\hat{C}\end{pmatrix}\begin{pmatrix}z\\ \hat{u}\\ \hat{y}\end{pmatrix} - \begin{pmatrix}r\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\bar{B}\\ 0\\ 0\end{pmatrix}\bar{u},\\ \bar{y} &= \begin{pmatrix}\bar{B}^\top 0 0\end{pmatrix}\begin{pmatrix}z\\ \hat{u}\\ \hat{y}\end{pmatrix}.\end{aligned} \tag{9}$$

Then this system is a PH-DAE with Hamiltonian $H = H_1 + ... + H_k$.

Proof In order to simply superpose the subsystems, we rewrite the ith subsystem (7) in a matrix format. To this end, we use split input and output: both comprise coupling terms and external terms. Thereby, the coupling terms will belong to the internal description of the overall systems. Only external input/output will form the input/output of the overall systems. Subsystem (7) can be equivalently written as

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}E_i & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_i\\ \hat{u}_i\end{pmatrix} &= \begin{pmatrix}J_i & \hat{B}_i\\ -\hat{B}_i^\top & 0\end{pmatrix}\begin{pmatrix}z_i(x_i)\\ \hat{u}_i\end{pmatrix} - \begin{pmatrix}r_i(z_i(x_i))\\ 0\end{pmatrix} + \begin{pmatrix}\bar{B}_i & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\bar{u}_i\\ \hat{y}_i\end{pmatrix}\\ \begin{pmatrix}\bar{y}_i\\ \hat{d}_i\end{pmatrix} &= \begin{pmatrix}\bar{B}_i^\top & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}z_i(x_i)\\ \hat{u}_i\end{pmatrix},\end{aligned} \tag{10}$$

where we use the additional dummy output $\hat{d}_i = \hat{u}_i$. Then, the extended system (10) is again a PH-DAE with

$$\tilde{E}_i := \begin{pmatrix}E_i & 0\\ 0 & 0\end{pmatrix}, \tilde{J}_i := \begin{pmatrix}J_i & \hat{B}_i\\ -\hat{B}_i^\top & 0\end{pmatrix}, \tilde{B}_i := \begin{pmatrix}\bar{B}_i & 0\\ 0 & I\end{pmatrix}$$

(extended matrices). Now, we discuss every block of equations in the joint system (9). First, the aggregation $F = \mathrm{diag}(F_1, ..., F_k)$ for $F \in \{E, J, \hat{B}, \bar{B}\}$ of (10.1) yields directly (9.1) padded with zeros for the variable $\hat{y}$. For the second block, we aggregate and move $\hat{y}$ from the output position to internal variables. Thereby, the vector $(x^\top, \hat{u}^\top)$ and $(z^\top, \hat{u}^\top)$ are extended. Then, the aggregated structure preserving interconnection $\hat{u} = -\hat{C}\hat{y}$ gives the third block. Finally, the output equation of (10) yields the output equation by aggregation, dropping the dummy part and adding a padding of zeros. The properties of the terms are inherited from the respective definition of the subsystems.

Remark 11 This transfers the result from Mehrmann and Morandin[9] to circuits with nonlinearities. Furthermore, no additional variables are introduced. Moreover, the structure matrix of the overall system (9) is identified as

$$J^{\mathrm{tot}} := \begin{pmatrix}J & \hat{B} & 0\\ -\hat{B}^\top & 0 & I\\ 0 & -I & -\hat{C}\end{pmatrix}.$$

Remark 12 **(i)** System (9) can be condensed to a PH-DAE (by removing internal input $\hat{u}$ and output $\hat{y}$)

$$\frac{d}{dt}Ex = \hat{J}z - r + \bar{B}\bar{u}, \tag{11a}$$

$$\bar{y} = \bar{B}^\top z \tag{11b}$$

with the skew-symmetric matrix $\hat{J}$ given by $\hat{J} = J - \hat{B}\hat{C}\hat{B}^\top$. This follows directly from $Jz + \hat{B}\hat{u} = Jz - \hat{B}\hat{C}\hat{y} = (J - \hat{B}\hat{C}\hat{B}^\top)z$. Thereby the PH-DAE structure is kept.

(ii) Note that the change in the Hamiltonian H of (9) from time t to $t + h$ is given by

$$\int_t^{t+h} -z(x(\tau))^\top r(z(x(\tau))) + \bar{u}(\tau)^\top \bar{y}(\tau)\mathrm{d}\tau = \int_t^{t+h} -z(x(\tau))^\top r(z(x(\tau))) + \bar{u}(\tau)^\top \bar{B}^\top z(x(\tau))\mathrm{d}\tau. \tag{12}$$

This holds also for its condensed version (11).

3.4 | Electric circuits with multiple subsystems—a PH-DAE description

Large integrated circuits are usually designed in blocks which may comprise even different functional units. Then these subcircuits are put together in an overall system by connecting respective terminals. In this way, a substructure may be already given by the circuit design; see, for example, Figure 1 (left) with respective inputs $\bar{u}$ and outputs $\bar{y}$. To form separate models of the subcircuits, one can artificially double the nodes of the subsystems' terminals by inserting a voltage source which provides a voltage drop of zero (artificial voltage source). This amounts to further inputs and outputs for the subsystems, which state the coupling $\hat{u}$ and $\hat{y}$; see Figure 1 (right).

Let the overall circuit (with given Assumption 1) consists of subcircuits $i = 1, ..., k$. We use the index i to identify the quantities of the ith subcircuit; for example, we use $e_i(t) \in R^{n_{u_i}}$ for the node potentials and so on. Moreover, we assume that we have n_λ coupling branches linking the k subcircuits in the overall setting. Then, we have associated branch currents $\lambda(t) \in R^{n_\lambda}$ and n_λ artificial voltage source. Now, let the ith subsystem have the respective incidence matrix $A_{\lambda_i} \in \{-1, 0, 1\}^{n_{u_i} \times n_\lambda}$ for the artificial voltage sources. Thus, the coupling amounts to (i) an additional term in the KCL (ith circuit) for the coupling branch/current: $A_{\lambda_i}\lambda$. In fact, one can model this by adding this contribution to the current source term (A_I):

$$A_{I_i}(A_{I_i}A_{\lambda_i}), 1_i \begin{pmatrix} 1_i \\ \lambda \end{pmatrix}.$$

Due to the virtuality of the coupling voltage sources, one has (ii) to guarantee that the node potentials at the boundaries coincide, as done in (13d); see below.

In the end, the circuit equations for the k coupled circuit DAEs are comprised by the subsystems $i = 1, ..., k$:

$$0 = A_{C_i}\frac{d}{dt}q_{C_i}\left(A_{C_i}^\top e_i\right) + A_{R_i}g_i\left(A_{R_i}^\top e_i\right) + A_{L_i}J_{L_i} + A_{V_i}J_{V_i} + A_{I_i}1_i(t) + A_{\lambda_i}\lambda \tag{13a}$$

$$0 = \frac{d}{dt}\phi_{L_i}(J_{L_i}) - A_{L_i}^\top e_i \tag{13b}$$

$$0 = A_{V_i}^\top e_i - v_i(t) \tag{13c}$$

together with a set of linear coupling equations

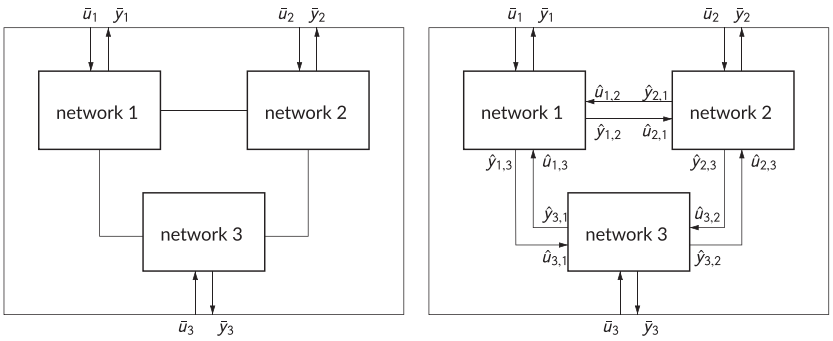

FIGURE 1 Input/output for distributed circuits: monolithic view (left), coupled circuits view (right)

$$0 = \sum_{i=1}^{k} A_{\lambda_i}^{\top} e_i. \tag{13d}$$

These coupled DAE circuit equations can be written as k multiply coupled PH-DAE system according to Definition 6. The only ambiguity is the handling of the coupling condition (13d). The simplest approach is to add the coupling condition to one of the subsystem, without loss of generality to the last one. It holds

Lemma 13 (PH-DAE formulation of mutually coupled DAEs**).** The coupled circuit DAEs (13) define k multiply coupled PH-DAE systems according to Definition 6.

Proof For $i = 1, \ldots, k-1$, we set

$$x_i = \begin{pmatrix} q_{C_i} \\ \phi_{L_i} \\ e_i \\ J_{V_i} \end{pmatrix}, z_i = \begin{pmatrix} e_i \\ J_{L_i} \\ u_{C_i} \\ J_{V_i} \end{pmatrix}, \bar{u}_i = \begin{pmatrix} \mathrm{i}_i(t) \\ v_i(t) \end{pmatrix}, \hat{u}_i + \hat{y}_k = 0, E_i = \begin{pmatrix} A_{C_i} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14a}$$

$$J_i = \begin{pmatrix} 0 & -A_{L_i} & 0 & -A_{V_i} \\ A_{L_i}^{\top} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{V_i}^{\top} & 0 & 0 & 0 \end{pmatrix}, r_i = \begin{pmatrix} A_{R_i} g_i\left(A_{R_i}^{\top} e_i\right) \\ 0 \\ A_{C_i}^{\top} e_i - u_{C_i} \\ 0 \end{pmatrix}, \hat{B}_i = \begin{pmatrix} A_{\lambda_i} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \bar{B}_i = \begin{pmatrix} -A_{I_i} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}, \tag{14b}$$

and for $i = k$, the definition

$$x_k = \begin{pmatrix} q_{C_k} \\ \phi_{L_k} \\ e_k \\ J_{V_k} \\ \lambda \end{pmatrix}, z_k = \begin{pmatrix} e_k \\ J_{L_k} \\ u_{C_k} \\ J_{V_k} \\ \lambda \end{pmatrix}, \bar{u}_k = \begin{pmatrix} \mathrm{i}_k(t) \\ v_k(t) \end{pmatrix}, \hat{u}_k - \sum_{i=1}^{k-1} \hat{y}_i = 0, E_k = \begin{pmatrix} A_{C_k} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{15a}$$

$$J_k = \begin{pmatrix} 0 & -A_{L_k} & 0 & -A_{V_k} & -A_{\lambda_k} \\ A_{L_k}^{\top} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{V_k}^{\top} & 0 & 0 & 0 & 0 \\ A_{\lambda_k}^{\top} & 0 & 0 & 0 & 0 \end{pmatrix}, r_k = \begin{pmatrix} A_{R_k} g_k\left(A_{R_k}^{\top} e_k\right) \\ 0 \\ A_{C_k}^{\top} e_k - u_{C_k} \\ 0 \\ 0 \end{pmatrix}, \hat{B}_k = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix}, \bar{B}_k = \begin{pmatrix} -A_{I_k} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{pmatrix}, \tag{15b}$$

completes the proof.

The overall circuit, seen as a single system, has a PH-DAE formulation, too:

Lemma 14 (PH-DAE formulation of coupled circuit DAEs**).** The coupled circuit equations (13), written as a single system, can be represented as PH-DAE in the condensed form (11).

Proof Here, we set

$$x := \begin{pmatrix} q_C \\ \phi_L \\ e \\ J_V \\ \lambda \end{pmatrix}, z(x) := \begin{pmatrix} e \\ J_C \\ u_C \\ J_L \\ J_V \end{pmatrix} = \begin{pmatrix} e \\ J_C \\ q^{-1}(q_C) \\ \phi^{-1}(\phi_L) \\ J_V \end{pmatrix}, E := \begin{pmatrix} A_C & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16a}$$

$$r := \begin{pmatrix} A_R g\left(A_R^\top e, t\right) \\ 0 \\ A_C^\top e - u_C \\ 0 \\ 0 \end{pmatrix}, \hat{J} := \begin{pmatrix} 0 & -A_L & 0 & -A_V & -A_\lambda \\ A_L^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \\ A_\lambda^\top & 0 & 0 & 0 & 0 \end{pmatrix}, \bar{B} := \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{pmatrix}, \bar{u} = \begin{pmatrix} 1 \\ v \end{pmatrix}, \tag{16b}$$

where we have used aggregated matrices

$$\begin{aligned} A_R g &= \left(A_{R_1} g_1\left(A_{R_1}^\top x_1, t\right), ..., A_{R_k} g_k\left(A_{R_k} x_k, t\right)\right)^\top, A_\lambda^\top = \left(A_{\lambda_1}^\top, ..., A_{\lambda_k}^\top\right), \\ A_P &:= \operatorname{diag}\left(A_{P_1}, ..., A_{P_k}\right) \text{ for} P \in \{C, R, L, V\} \end{aligned}$$

and aggregated quantities

$$w^\top = \left(w_1^\top, ..., w_k^\top\right) \text{ for } w \in \{q_C, \phi_L, e, u_C, J_V, J_C, J_L\}.$$

The Hamiltonian is given as in Proposition 9 as the sum of the Hamiltonians of the k subsystems.

4 | INDEX ANALYSIS OF CIRCUIT EQUATIONS

In the field of DAEs, there exist several index concepts, which quantify the distance to the case of ODEs. This can be done with respect to derivatives needed to transform a DAE into an ODE, that is, the differentiation index.[20] On the other hand, the perturbation index[20] quantifies the distance of the solutions to a perturbed system, with respect to the number of derivatives of the perturbation (which may enter the solution). A third concept is the tractability index,[21,22] which is based on a matrix change and reveals the respective components with the minimal regularity required. In this work, we focus on the differentiation index, which we refer to as *index* throughout this article.

Definition 15 (Derivative array, differentiation index**[22], Def. 3.72).** Let $U, V \subset R^n$ be open and $\mathcal{I} \subset R$ be an interval. Let $\nu \in N$, $\mathcal{F} : U \times V \times I \to R^k$, and a DAE

$$\mathcal{F}(\dot{x}(t), x(t), t) = 0 \tag{17}$$

be given. Then the νth derivative array of (17) is given by the first ν formal derivatives of (17) with respect to time, that is

$$\mathcal{F}_\nu(x^{(\nu+1)}(t), x^{(\nu)}(t), ..., \dot{x}(t), x(t), t) = \begin{pmatrix} \mathcal{F}(\dot{x}(t), x(t), t) \\ \frac{d}{dt}\mathcal{F}(\dot{x}(t), x(t), t) \\ \vdots \\ \frac{\mathrm{d}^\nu}{\mathrm{d}t^\nu}\mathcal{F}(\dot{x}(t), x(t), t) \end{pmatrix} = 0. \tag{18}$$

The DAE (17) is said to have *(differentiation) index* $\nu \in N$ if for all $(x,t) \in V \times I$, there exists a unique $\dot{x} \in U$ such that $x^{(2)}, \dots, x^{(\nu+1)} \in U$ and $\mathcal{F}_\nu(x^{(\nu+1)}, x^{(\nu)}, \dots, \dot{x}, x(t), t) = 0$. In this case, there exists some function $f: V \times I \to V$ with $(x,t) \mapsto \dot{x}$ for t, x and $\dot{x}$ with the above properties, which describes the so-called *inherent ODE of* (17):

$$\dot{x}(t) = f(x(t), t) \tag{19}$$

Next, we characterize the index of the circuit Equations (3a) and (4a) by means of the properties of the subgraphs corresponding to specific electric components.

Theorem 16 Assumption 1 shall hold. For the index ν of the circuit DAEs (3a) and (4a), we have:

(**i**) DAE (3a) fulfills $\nu = 1$ if, and only if, it neither contains *LI*-cutsets nor *CV*-loops except for *C*-loops. Otherwise, $\nu = 2$.
(**ii**) DAE (4a) fulfills: $\nu = 1$ if, and only if, it neither contains *LI*-cutsets nor *CV*-loops. Otherwise, $\nu = 2$.

Remark 17 (**a**) There is a small but nice difference between the indices of DAEs (3a) and (4a): Whereas *C*-loops lead to an index $\nu = 2$ of (4a), this is not necessarily the case for the DAE (3a).
(**b**) Let Z_C be a matrix with full column rank and $\mathrm{im} Z_C = \ker A_C^\top$. Then, by Proposition 1, the absence of *CV*-loops except for *C*-loops is equivalent to

$$\ker Z_C^\top A_V = \{0\}, \tag{20}$$

whereas the circuit does not contain any *LI*-cutsets if, and only if,

$$\ker (A_C A_R A_V)^\top = \{0\}. \tag{21}$$

Theorem 16 shows that the index is a structural invariant of the circuit equation. That is, it depends on the interconnection properties of the circuit rather than on parameter values. Notice that our index results are a slight modification of those in Estévez Schwarz and Tischendorf,[18] where an index analysis for the modified nodal analysis and charge-oriented modified nodal analysis has been performed. A combination of the results from Estévez Schwarz and Tischendorf[18] with Theorem 16 yields that the circuit DAE (3) has index two if, and only if, the MNA equations being subject of Estévez Schwarz and Tischendorf[18] have index two.

Proof We start with the index result for the DAE (4a). To this end, notice that the diffeomorphism

$$\begin{pmatrix} e \\ J_C \\ q_C \\ \phi_L \\ J_V \end{pmatrix} \mapsto \begin{pmatrix} e \\ u_C \\ J_L \\ J_C \\ J_V \end{pmatrix} = \begin{pmatrix} e \\ q^{-1}(q_C) \\ \phi^{-1}(\phi_L) \\ J_C \\ J_V \end{pmatrix}$$

applied to the unknown of the DAE (4a) does not change the index, and, by a suitable permutation of the equations, results in the DAE

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & C(u_C) & 0 & 0 & 0 \\ 0 & 0 & L(J_L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{e} \\ \dot{u}_C \\ \dot{J}_L \\ \dot{J}_C \\ \dot{J}_V \end{pmatrix} = \begin{pmatrix} 0 & 0 & -A_L & -A_C & -A_V \\ 0 & 0 & 0 & I & 0 \\ A_L^\top & 0 & 0 & 0 & 0 \\ A_C^\top & -I & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ u_C \\ J_L \\ J_C \\ J_V \end{pmatrix} - \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \mathrm{i}(t) \\ v(t) \end{pmatrix}, \tag{22}$$

Then, Assumption 1 yields that we are in the situation of Reis,[23, Thm. 6.6] which yields that the index ν of (22) fulfills

- $\nu = 0$ if, and only if, the matrix in front of the derivative of the state is invertible. That is, the vectors of potentials, capacitive currents, and currents of voltage sources are void.
- $\nu = 1$ if, and only if, $\nu \neq 0$ and

$$\ker \begin{pmatrix} 0 & A_R & -A_C & -A_V \\ C(u_C) & 0 & I & 0 \end{pmatrix}^\top = \{0\} \wedge \tag{23}$$

$$\ker \begin{pmatrix} 0 & 0 \\ 0 & C(u_C) \end{pmatrix} \times \{0\} \times \{0\} = \ker \begin{pmatrix} 0 & 0 & -A_C & -A_V \\ 0 & C(u_C) & I & 0 \end{pmatrix} \tag{24}$$

- $\nu = 2$ otherwise.

The soundness assumption that the circuit has at least one branch implies that the vector of potentials is non-void. Hence, the index of the circuit equations (22) is not equal to zero.
Furthermore, since (23) is equivalent to $(A_C A_R A_V)$ having full row rank and (24) is equivalent to the full column rank property of $(A_C A_V)$, we obtain from Proposition 1that $\nu = 1$ is equivalent to the absence of *CV*-loops as well as *LI*-cutsets. This completes the proof for the circuit equations (4a).

To prove the index result (3a), we first notice that the characterization for $\nu = 0$ follows by the same argumentation as for (4a). Moreover, we notice that a multiplication of (4a) from the left with a suitable invertible matrix T and a re-ordering of the state components leads to the DAE

$$\frac{d}{dt}\begin{pmatrix} A_C & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_C \\ \phi_L \\ e \\ J_V \\ u_C \end{pmatrix} = \begin{pmatrix} 0 & -A_L & 0 & -A_V & 0 \\ A_L^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \\ u_C \end{pmatrix} - \begin{pmatrix} A_R g(A_R^\top e) \\ 0 \\ A_C^\top e - q^{-1}(q_C) \\ 0 \\ A_C^\top e - u_C \end{pmatrix} + \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i}(t) \\ v(t) \end{pmatrix}. \tag{25}$$

The upper four equations represent exactly the DAE (3a) whereas the variable u_C appears explicitly in the last equation. It can now be inferred from Definition 15 that the index of (3a) does not exceed that of (4a). By the already proven results for (4a), this implies that $\nu \leq 2$. Hence, it suffices to prove that the absence of *LI*-cutsets and *CV*-loops except for *C*-loops is necessary and sufficient for $\nu \leq 1$: To this end, consider matrices Z_C, Z'_C with full column rank and $\mathrm{im} Z_C = \ker A_C^\top$, $\mathrm{im} Z'_C = \mathrm{im} A_C$. Then, $[Z_C Z'_C]$ is an invertible matrix. Now, we multiply the first equation in (3a) from the left with $Z_C^\top$ and $(Z'_C)^\top$ to obtain an equivalent DAE

$$\begin{aligned} Z_C^\top A_L \phi^{-1}(\phi_L) + Z_C^\top A_V J_V + Z_C^\top A_R g(A_R^\top e) + Z_C^\top A_I \mathrm{i}(t) &= 0, \\ \frac{d}{dt}(Z'_C)^\top A_C q_C + (Z'_C)^\top A_L \phi^{-1}(\phi_L) + (Z'_C)^\top A_V J_V + (Z'_C)^\top A_R g(A_R^\top e) + (Z'_C)^\top A_I \mathrm{i}(t) &= 0, \\ \frac{d}{dt}\phi_L - A_L^\top e &= 0, \\ -A_C^\top e + q^{-1}(q_C) &= 0, \\ -A_V^\top e + v(t) &= 0. \end{aligned}$$

The first, forth, and fifth equations are now purely algebraic and will be differentiated in the next step. Using the differentiation rule for inverse functions, we obtain that for *C* and *L*

$$\frac{d}{dt} q^{-1}(q_C) = C(q^{-1}(q_C))^{-1} \frac{d}{dt} q_C, \; \frac{d}{dt} \phi^{-1}(\phi_L) = L(\phi^{-1}(\phi_L))^{-1} \frac{d}{dt} \phi_L.$$

(as in Assumption 1). We further abbreviate $C = C(q^{-1}(q_C))$, $L = L(\phi^{-1}(\phi_L))$ and $G = \frac{\mathrm{d}g}{\mathrm{d}u_R}(A_R^\top e)$. A differentiation of the algebraic equations now gives

$$\begin{pmatrix} 0 & -Z_C^\top A_L^\top L^{-1} & -Z_C^\top A_R G A_R^\top & -Z_C^\top A_V \\ (Z_C')^\top A_C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ C^{-1} & 0 & A_C^\top & 0 \\ 0 & 0 & A_V^\top & 0 \end{pmatrix} \begin{pmatrix} \dot{q}_C \\ \dot{\phi}_L \\ \dot{e} \\ \dot{J}_V \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(Z_C')^\top A_L & 0 & -(Z_C')^\top A_V \\ A_L^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \phi^{-1}(\phi_L) \\ q^{-1}(q_C) \\ J_V \end{pmatrix}$$

$$- \begin{pmatrix} 0 \\ (Z_C')^\top A_R g(A_R^\top e) \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & (Z_C')^\top A_I & 0 \\ -(Z_C')^\top A_I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} 1(t) \\ \frac{d}{dt}1(t) \\ \frac{d}{dt}v(t) \end{pmatrix}.$$

The definition of the index implies that $\nu \le 1$ if and only if the matrix in front of the derivative is invertible. By applying elementary row operations to that matrix, we see that

$$\nu \le 1 \Leftrightarrow \ker \underbrace{\begin{pmatrix} 0 & 0 & -Z_C^\top A_R G A_R^\top & -Z_C^\top A_V \\ 0 & 0 & -(Z_C')^\top A_C C A_C^\top & 0 \\ 0 & I & 0 & 0 \\ C^{-1} & 0 & A_C^\top & 0 \\ 0 & 0 & A_V^\top & 0 \end{pmatrix}}_{=:\tilde{E}} = \{0\}. \tag{26}$$

If the circuit contains *CV*-loops except for *C* loops or *LI*-cutsets, then, by Proposition 1 and Remark 17, $\ker Z_C^\top A_V \neq \{0\}$. Both lead to $\ker \tilde{E} \neq \{0\}$ and thus, by (26), to $\nu > 1$.

To prove the converse, we assume that the circuit neither contains *LI*-cutsets nor *CV*-loops except for *C*-loops. Taking an accordingly partitioned vector $x = (x_1^\top x_2^\top x_3^\top x_4^\top)^\top \in \ker \tilde{E}$, we see immediately that $x_2 = 0$ holds. We obtain from the positive definiteness and the fact that $\ker(Z_C')^\top$ equals to the orthogonal complement of $\mathrm{im} A_C$ that

$$\ker(Z_C')^\top A_C C A_C^\top = \ker A_C^\top.$$

Hence, $x_3 \in \ker A_C^\top$, which leads to $x_3 = Z_C w_3$ for some real vector w_3 of suitable size. In particular, $\tilde{E}x = 0$ leads to $A_V^\top Z_C w_3 = 0$, whence $w_3 = Z_{V-C} z_3$ for a real vector z_3 and a matrix Z_{V-C} with full column rank and $\mathrm{im} Z_{V-C} = \ker A_V^\top Z_C$. A multiplication of the first row of $\tilde{E}x = 0$ with $Z_{V-C}^\top$ gives, by using $Z_{V-C}^\top Z_C^\top A_V = 0$,

$$0 = Z_{V-C}^\top Z_C^\top A_R G A_R^\top x_3 = Z_{V-C}^\top Z_C^\top A_R G A_R^\top Z_C Z_{V-C} z_3$$

and the positive definiteness of $G + G^\top$ (which holds by Assumption 1) leads to $A_R^\top Z_C Z_{V-C} z_3 = 0$.

By Proposition 1 and Remark 17, the absence of the aforementioned loops and cutsets leads to $\ker(A_C A_R A_V)^\top = \{0\}$ or $\ker Z_C^\top A_V = \{0\}$. The first condition yields $z_3 = 0$ and thus $x_3 = 0$, and the second one $x_4 = 0$. With $x_3 = 0$, the positive-definiteness of *C* then finally leads to $x_1 = 0$. Summing up, we obtain $x = 0$, and the index of (4a) equals to one.

5 | MODELING OF COUPLED CIRCUIT DAES AND DYNAMIC ITERATION SCHEMES

Regarding the coupled circuit DAEs (13) discussed in Section 3.4, we can take three different perspectives with respect to the input. We will formulate the corresponding circuit equations as PH-DAE systems of type (3). Note that a modification of the considerations in this section with respect to the alternative circuit model (4) is straightforward.

Different views on coupled electrical circuits are possible:

(**C1**) Here all k subsystems, together with the coupling equation, are considered as one system, the PH-DAE system (16) with state $x^\top := \left(q_C^\top, \phi_L^\top, e^\top, J_V^\top, \lambda^\top\right)$, and given input $(\imath^\top, v^\top)^\top$.

(**C2**) We consider the ith subsystem separately, with term $\hat{u}_i = -\lambda$ arising from the virtual voltage source regarded as an additional input to the system, that is, the PH-DAE system (14) with state $x_i{}^\top := \left(q_{C_i}^\top, \phi_{L_i}^\top, e_i^\top, J_{V_i}^\top\right)$, and given input $(\lambda^\top, \imath^\top, v^\top)^\top$.

(**C3**) We consider the ith subsystem separately together with the coupling condition, that is, the PH-DAE system (15) with state $x_i{}^\top := \left(q_{C_i}^\top, \phi_{L_i}^\top, e_i^\top, J_{V_i}^\top, \lambda^\top\right)$. Now the node potentials $e_1, \dots e_{k-1}$ add to the input $\hat{u}_k = \sum_{i=1}^{k-1} A_{\lambda_i}^\top e_i$.

5.1 | Structural properties

In the following, we investigate the index properties of the k coupled electric circuits, where each subcircuit is assumed to fulfill Assumption 1. In particular, each subcircuit is connected, and the component matrices have the property that C_i, L_i and $G_i + G_i^\top$ of each subsystem ($i = 1, \dots, k$) are pointwise positive definite.

We can have different points of view: either regarding the overall system as one joint system or regard just a subsystem with given input, potentially linked to the coupling system or to a part of it. This amounts to certain index assumptions on the overall system (C1) as well as for the subsystems (C2) and (C3). More precisely, we will assume that the systems (C1), (C2), and (C3) have index one. Note that even in the case that both system (C1) and all subsystems (C2) have index one, system (C3) may not have index one. However, having (C3) satisfied for all subsystems implies that system (C1) is index one. Of course, it is not necessary for system (C1) being of index one.

5.1.1 | Monolithic perspective (C1)

For the overall system (13), the virtual voltage sources extend the set of voltage sources. Thus, Theorem 16 yields that the coupled system (13) has index one if, and only if, the circuit neither *LI*-cutsets nor *CV*-loops except for *C*-loops. By Proposition 1 and Remark 17, this is equivalent to both matrices

$$\left(\begin{pmatrix} A_{C_1} & & \\ & \ddots & \\ & & A_{C_k} \end{pmatrix}, \begin{pmatrix} A_{R_1} & & \\ & \ddots & \\ & & A_{R_k} \end{pmatrix}, \begin{pmatrix} A_{V_1} & & A_{\lambda_1} \\ & \ddots & \vdots \\ & A_{V_k} & A_{\lambda_k} \end{pmatrix}\right)^\top, \begin{pmatrix} Z_{C_1}^\top & & \\ & \ddots & \\ & & Z_{C_k}^\top \end{pmatrix} \cdot \begin{pmatrix} A_{V_1} & & A_{\lambda_1} \\ & \ddots & \vdots \\ & A_{V_k} & A_{\lambda_k} \end{pmatrix}$$

having full column rank. The latter is equivalent to the full column rank of

$$\begin{pmatrix} Z_{V_1-C_1}^\top Z_{C_1}^\top A_{\lambda_1} \\ \vdots \\ Z_{V_k-C_k}^\top Z_{C_k}^\top A_{\lambda_k}, \end{pmatrix}$$

with Z_{C_i} and $Z_{V_i-C_i}$ being matrices with full column rank and $\operatorname{im} Z_{C_i} = \ker A_{C_i}^\top$, $\operatorname{im} Z_{V_i-C_i} = \ker A_{V_i}^\top Z_{C_i}$.

5.1.2 | Single subsystem perspective (C2)

We can apply Theorem 16 to the ith subsystem (13a)–(13c) to obtain that its index is one if, and only if, the subcircuit neither contains *LI*-cutsets nor *CV*-loops except for *C*-loops. By Proposition 1 and Remark 17, this is equivalent to the full column rank property of the matrices

$$Z_{C_i}^{\top} A_{V_i}, (A_{C_i}, A_{R_i}, A_{V_i})^{\top}. \tag{27}$$

5.1.3 | Subsystem plus coupling equation (C3)

This DAE has index one if, and only if, the subcircuit neither contains *LI*-cutsets nor *CV*λ-loops except for *C*-loops, where λ stands for virtual voltage sources. By Proposition 1 and Remark 17, this is equivalent to the property that the subsequent two matrices have full column rank:

$$(A_{C_i}, A_{R_i}, A_{V_i}, A_{\lambda_i})^{\top}, Z_{C_i}^{\top}(A_{V_i}, A_{\lambda_i}). \tag{28}$$

5.2 | Dynamic iteration perspective on modeling

Dynamic iteration schemes exploit the coupling structure of system (13) by solving subsystems independently and by defining a suitable information update. Let us assume that a numerical approximation $(\tilde{q}, \tilde{\phi}, \tilde{e}, \tilde{J}_V, \tilde{\lambda})$ is given for a time window $[t_{n-1}, t_n]$, then a new approximate for the next time window $[t_n, t_{n+1}]$ can be iteratively derived:

(i) *Extrapolation step:* the approximate solution $(\tilde{q}, \tilde{\phi}, \tilde{e}, \tilde{J}_V, \tilde{\lambda})$ is extrapolated into the current time window $[t_n, t_{n+1}]$. This defines initial waveforms (approximate solutions) $(q^{(0)}, \phi^{(0)}, e^{(0)}, J_V{}^{(0)}, \lambda^{(0)})$ on $[t_n, t_{n+1}]$ for the following iteration process.

(ii) *Iteration step for* $l = 0, \ldots, l_{\max}$*:*

- The first $k-1$ DAE-IVP subsystems (where the constituents are given in (14)) are solved separately with respect to the variables $(q_i, \phi_i, e_i, J_{V_i}) := (q_i{}^{(l+1)}, \phi_i{}^{(l+1)}, e_i{}^{(l+1)}, J_{V_i}{}^{(l+1)})$. Thereby, the input of the ith subsystem is the coupling current $\lambda^{(l)}$; this quantity is given from the previous iteration; that is, we have $\hat{u}_i := -\lambda^{(l)}$. In principle, this part can be done in parallel, since these subsystems are decoupled.

- The last system (number k) can be computed in two different ways:
  - **(a)** *Jacobi-type approach:* here, one solves the DAE-IVP (15) with respect to the variables $(q_k, \phi[k], e[k], J_{V_k}, \lambda) := (q_k{}^{(l+1)}, \phi[k]^{(l+1)}, e[k]^{(l+1)}, J_{V_k}{}^{(l+1)}, \lambda^{(l+1)})$. Thereby the input is given by the coupling potentials $e_1^{(l)}, \ldots, e[k-1]^{(l)}$ from the previous iteration, that is, $\hat{u}_k := \sum_{i=1}^{k-1} A_{\lambda_i}^{\top} e_i^{(l)}$. In this case, the calculation of the last system could be performed in parallel with the computation of the first $k-1$ systems.
  - **(b)** *Gauss-Seidel-type approach:* The only difference to the Jacobi-type approach above is the assignment of the input. Here we employ as input the coupling node potentials $e_1^{(l+1)}, \ldots, e[k-1]^{(l+1)}$ from the current iteration instead of the previous one; that is, we set $\hat{u}_k := \sum_{i=1}^{k-1} A_{\lambda_i}^{\top} e_i^{(l+1)}$.

Remark 18 Notice that this iteration process is based on the perspective (C2) for the first $k-1$ subsystems and perspective (C3) for the last subsystem.

In the following, we will see that the k different subsystems in the dynamic iteration scheme can be interpreted as port-Hamiltonian systems, too.

6 | PORT-HAMILTONIAN FORMULATION OF COUPLED DAE CIRCUIT EQUATIONS—THE DYNAMIC ITERATION PERSPECTIVE

We study the Jacobi approach and the Gauss-Seidel method for a number of k coupled DAEs. To cope with port-Hamiltonian systems arising in this context, we have to modify slightly the interconnections. This is treated in the first part. Second, we map the formulation to the electric circuit case.

6.1 | The dynamic iteration PH-DAE setup

We give a modified version of Definition 6 for the dynamic iteration context. Thereby, we have to introduce the iteration count (l) and the interconnection needs to map outputs of the last iterate to inputs of the current iterate:

Definition 19 (Multiply coupled PH-DAE—the dynamic iteration perspective**).** We consider the complete Definition 6 (multiply coupled PH-DAE with k subsystems) apart from the assumption that $\hat{C}$ is skew symmetric. We add the iteration count: the state variables x_i, inputs u_i and outputs y_i in (7) are labelled with an iteration number $l+1$: $x_i^{(l+1)}, u_i^{(l+1)}, y_i^{(l+1)}$. In the case of a Jacobi-type iteration, the ith subsystem reads (for $i=1,\dots,k$)

$$\frac{d}{dt}E_i x_i^{(l+1)} = J_i z_i^{(l+1)} - r_i\left(z_i^{(l+1)}\right) + \begin{pmatrix}\hat{B}_i & \bar{B}_i\end{pmatrix}\begin{pmatrix}\hat{u}_i^{(l+1)}\\ \bar{u}_i^{(l+1)}\end{pmatrix} \tag{29a}$$

$$\begin{pmatrix}\hat{y}_i^{(l+1)}\\ \bar{y}_i^{(l+1)}\end{pmatrix} = \begin{pmatrix}\hat{B}_i & \bar{B}_i\end{pmatrix}^\top z_i^{(l+1)} \tag{29b}$$

together with the shorthand $z_i^{(l+1)} = z\left(x_i^{(l+1)}\right)$ and the input (of ith subsystem) in the current iteration $(l+1)$ is linked to the output of the previous iteration (l) by

$$0 = \hat{u}_i^{(l+1)} + \sum_{j=1, j\neq i}^{k} \hat{C}_{i,j}\hat{y}_j^{(l)}. \tag{29c}$$

And we require the Schur complement $\hat{B}\hat{C}\hat{B}^\top$ (of the interconnect matrix $\hat{C}$) to be skew symmetric. For the case of a Gauss-Seidel type iteration, only (29c) is replaced by

$$0 = \hat{u}_i^{(l+1)} + \sum_{j=1}^{i-1}\hat{C}_{i,j}\hat{y}_j^{(l+1)} + \sum_{j=i+1}^{k}\hat{C}_{i,j}\hat{y}_j^{(l)}. \tag{30}$$

Remark 21 In contrast to Definition 6, we do not require the interconnection matrix $\hat{C}$ in (29)c to be skew-symmetric in the overall. We only require $\hat{B}(\hat{C}+\hat{C}^\top)\hat{B}^\top = 0$.

Now, we have the analogous result to Corollary 10:

Corollary 21 (Multiply skew-symmetric structure-preserving interconnection, Jacobi approach**).** In the case of dynamic iteration, the assumption of Jacobi-type coupling (29c) gives

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}E & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}x^{(l+1)}\\ \hat{u}^{(l+1)}\\ \hat{y}^{(l+1)}\end{pmatrix} &= \begin{pmatrix}J & \hat{B} & 0\\ -\hat{B}^\top & 0 & I\\ 0 & -I & 0\end{pmatrix}\begin{pmatrix}z(x^{(l+1)})\\ \hat{u}^{(l+1)}\\ \hat{y}^{(l+1)}\end{pmatrix} - \begin{pmatrix}r(z(x^{(l+1)}))\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\bar{B} & 0\\ 0 & 0\\ 0 & -\hat{C}\end{pmatrix}\begin{pmatrix}\bar{u}^{(l+1)}\\ \hat{y}^{(l)}\end{pmatrix},\\ \bar{y}^{(l+1)} &= \begin{pmatrix}\bar{B}^\top & 0 & 0\\ 0 & 0 & -\hat{C}^\top\end{pmatrix}\begin{pmatrix}z(x^{(l+1)})\\ \hat{u}^{(l+1)}\\ \hat{y}^{(l+1)}\end{pmatrix},\end{aligned} \tag{31}$$

which is a PH-DAE

$$\begin{aligned} \frac{d}{dt}E^{\text{tot}}x^{\text{tot}} &= J^{\text{tot}}z^{\text{tot}} - r^{\text{tot}}(z^{\text{tot}}) + B^{\text{tot}}u^{\text{tot}}, \\ y^{\text{tot}} &= B^{\text{tot}\top}z^{\text{tot}} \end{aligned} \tag{32}$$

with

$$x^{\text{tot}} = \begin{pmatrix} x^{(l+1)} \\ \hat{u}^{(l+1)} \\ \hat{y}^{(l+1)} \end{pmatrix}, z^{\text{tot}} = \begin{pmatrix} z(x^{(l+1)}) \\ \hat{u}^{(l+1)} \\ \hat{y}^{(l+1)} \end{pmatrix}, y^{\text{tot}} = \bar{y}^{(l+1)}, r^{\text{tot}}(z^{\text{tot}}) = \begin{pmatrix} r(z(x^{(l+1)})) \\ 0 \\ 0 \end{pmatrix},$$
$$u^{\text{tot}} = \begin{pmatrix} \bar{u}^{(l+1)} \\ \hat{y}^{(l)} \end{pmatrix}, E^{\text{tot}} = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, J^{\text{tot}} = \begin{pmatrix} J & \hat{B} & 0 \\ -\hat{B}^{\top} & 0 & I \\ 0 & -I & 0 \end{pmatrix}, B^{\text{tot}} = \begin{pmatrix} \bar{B} & 0 \\ 0 & 0 \\ 0 & -\hat{C} \end{pmatrix}.$$

For the Gauss-Seidel coupling (30), a PH-DAE (32) can be formulated with

$$J^{\text{tot}} = \begin{pmatrix} J & \hat{B} & 0 \\ -\hat{B}^{\top} & 0 & I \\ 0 & -I & -\hat{C} \end{pmatrix}, B^{\text{tot}} = \begin{pmatrix} \bar{B} & 0 \\ 0 & 0 \\ 0 & \hat{C}_1 \end{pmatrix}, u^{\text{tot}} = \begin{pmatrix} \bar{u}^{(l+1)} \\ \Delta\hat{y}^{(l+1)} \end{pmatrix}$$

instead of $J^{\text{tot}}, B^{\text{tot}}$, and u^{tot} above, provided that $\hat{C}$ is skew-symmetric. Here, we have used $\hat{u}^{(l+1)} + C_1\hat{y}^{(l)} + C_2\hat{y}^{(l+1)} = 0$.

Remark 22 **(i)** The only difference to the setting of Corollary 10 is the following: $\hat{y}^{(l)}$ defines a new input variable, and correspondingly, the coupling matrix $\hat{C}$ (Jacobi-type approach) and $\hat{C}_1$ (Gauss-Seidel type approach), resp., is shifted from the structure matrix J^{tot} to the port matrix B^{tot}.

(ii) In the dynamic iteration case (31), Jacobi-type approach, the system can be condensed to

$$\frac{d}{dt}Ex^{(l+1)} = Jz(x^{(l+1)}) - r(z(x^{(l+1)})) - \hat{B}\hat{C}\hat{B}^{\top}z^{(l)} + \bar{B}\bar{u}^{(l+1)}, \tag{33a}$$

$$\bar{\bar{y}}^{(l+1)} = -\left(\hat{B}\hat{C}\hat{B}^{\top}\right)^{\top}z(x^{(l+1)}) = \left(\hat{B}\hat{C}\hat{B}^{\top}\right)z(x^{(l+1)}), \tag{33b}$$

$$\bar{y}^{(l+1)} = \bar{B}^{\top}z(x^{(l+1)}) \tag{33c}$$

with an extra output $\bar{\bar{y}}^{(l+1)} = \hat{B}\hat{C}\hat{B}^{\top}z^{(l+1)}$. Note that in the Gauss-Seidel-type approach, the same PH-DAE (33) holds, with $\hat{B}\hat{C}\hat{B}^{\top}$ replaced by $\hat{B}\hat{C}_1\hat{B}^{\top}$.

(iii) Here, the change in the Hamiltonian is given by

$$-\int_t^{t+h} z(x^{(l+1)}(\tau))^{\top}r\left(z(x^{(l+1)}(\tau)\right)d\tau + \int_t^{t+h}\left(\bar{u}^{(l+1)}(\tau)^{\top}\bar{B}^{\top}z(x^{(l+1)}(\tau)) - (\Delta z(x^{(l+1)}(\tau))^{\top}\hat{B}\hat{C}\hat{B}^{\top}z(x^{(l)}(\tau))\right)d\tau. \tag{34}$$

We note that $\Delta z^{(l+1)} = z^{(l+1)} - z^{(l)}$ is the dynamic iteration update. Moreover, we point out that the third term in (34), which is additional to the first two terms already known from (12), decays with the converging dynamic iteration.

(iv) For a Gauss-Seidel iteration in Corollary 21, the input $\hat{y}^{(l)}$ needs to be split into old and new iterates.

6.2 | The dynamic iteration perspective for multiply coupled electric circuits

We study k coupled circuits, all described in charge oriented form (13). In the perspective of k copies of the PH-DAE model from Proposition 9, the respective constituents are already given in the proofs of Lemma 13 (in (14) for the systems $1,...,k-1$, and of (15) for system k). Now, we include the dynamic iteration process. First, in the $l+1$st iteration, say, we solve (successively or in parallel) the subsystems $i=1,...,k-1$. These subsystems read in the PH-DAE notation (cf. Corollary 21) for both the Jacobi and the Gauss-Seidel approach as follows:

$$\frac{d}{dt}\begin{pmatrix} A_{C_i} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_{[i]}^{(l+1)} \\ \phi_{[i]}^{(l+1)} \\ e_i^{(l+1)} \\ J_{V_i}^{(l+1)} \end{pmatrix} = \begin{pmatrix} 0 & -A_{L_i} & 0 & -A_{V_i} \\ A_{L_i}^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ A_{V_i}^\top & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_i^{(l+1)} \\ J_{L_i}^{(l+1)} \\ u_{C_i}^{(l+1)} \\ J_{V_i}^{(l+1)} \end{pmatrix}$$

$$-\begin{pmatrix} A_{R_i} g_i(A_{R_i}^\top e_i^{(l+1)}) \\ 0 \\ A_{C_i}^\top e_i^{(l+1)} - u_{C_i}^{(l+1)} \\ 0 \end{pmatrix} + \begin{pmatrix} A_{\lambda_i} \\ 0 \\ 0 \\ 0 \end{pmatrix}\hat{u}_i + \begin{pmatrix} -A_{I_i} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \mathfrak{1}_i(t) \\ v_i(t) \end{pmatrix}, \tag{35a}$$

$$\hat{y}_i^{(l+1)} = \begin{pmatrix} A_{\lambda_i} \\ 0 \\ 0 \\ 0 \end{pmatrix}^\top \begin{pmatrix} e_i^{(l+1)} \\ J_{L_i}^{(l+1)} \\ u_{C_k}^{(l+1)} \\ J_{V_i}^{(l+1)} \end{pmatrix},\ \bar{y}_i = \begin{pmatrix} -A_{I_i} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I_i \end{pmatrix}^\top \begin{pmatrix} e_i^{(l+1)} \\ J_{L_i}^{(l+1)} \\ u_{C_k}^{(l+1)} \\ J_{V_i}^{(l+1)} \end{pmatrix}, \tag{35b}$$

with $J_{L_i}^{(l+1)} = \phi_i^{-1}(\phi_{L,i}^{(l+1)})$, $u_{C,i}^{(l+1)} = q_i^{-1}(q_{C,i}^{(l+1)}))$ (within z_i). Finally, the kth subsystem (last) reads for both approaches

$$\frac{d}{dt}\begin{pmatrix} A_{C_k} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} q_k^{(l+1)} \\ \phi[k]^{(l+1)} \\ e_k^{(l+1)} \\ J_{V_k}^{(l+1)} \\ \lambda_k^{(l+1)} \end{pmatrix} = \begin{pmatrix} 0 & -A_{L_k} & 0 & -A_{V_k} & -A_{\lambda_k} \\ A_{L_k}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_{V_k}^\top & 0 & 0 & 0 & 0 \\ A_{\lambda_k}^\top & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_k^{(l+1)} \\ J_{L_k}^{(l+1)} \\ u_{C_k}^{(l+1)} \\ J_{V_k}^{(l+1)} \\ \lambda_k^{(l+1)} \end{pmatrix}$$

$$-\begin{pmatrix} A_{R_k} g_k(A_{R_k}^\top e_k^{(l+1)}) \\ 0 \\ A_{C_k}^\top e_k^{(l+1)} - u_{C_k}^{(l+1)} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix}\hat{u}_k + \begin{pmatrix} -A_{I_k} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathfrak{1}_k(t) \\ v_k(t) \end{pmatrix}, \tag{36a}$$

$$\hat{y}_k^{(l+1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \end{pmatrix}^\top \begin{pmatrix} e_k^{(l+1)} \\ J_{L_k}^{(l+1)} \\ u_{C_k}^{(l+1)} \\ J_{V_k}^{(l+1)} \\ \lambda_k^{(l+1)} \end{pmatrix},\ \bar{y}_k = \begin{pmatrix} -A_{I_k} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I_k \\ 0 & 0 \end{pmatrix}^\top \begin{pmatrix} e_k^{(l+1)} \\ J_{L_k}^{(l+1)} \\ u_{C_k}^{(l+1)} \\ J_{V_k}^{(l+1)} \\ \lambda_k^{(l+1)} \end{pmatrix}. \tag{36b}$$

The relation of outputs and inputs are the same for the first $k-1$ iterates, where these inputs are connected to the output of the last system (number k) from the previous iteration step (l):

$$\hat{u}_i = \hat{u}_i^{(l)} = -\hat{y}_k^{(l)},\ i = 1,...,k-1 \tag{37}$$

Only for the last iterate, the relation of outputs and inputs differs: for the Jacobi case, we have a relation to the previous iterates:

$$\hat{u}_k = \hat{u}_k^{(l)} = \sum_{i=1}^{k-1} \hat{y}_i^{(l)}; \tag{39}$$

and in the Gauss-Seidel case, the current iterates need to be used:

$$\hat{u}_k = \hat{u}_k^{(l+1)} = \sum_{i=1}^{k-1} \hat{y}_i^{(l+1)}. \tag{40}$$

In both cases, after aggregation, the k subsystems can be written as a joint PH-circuit-DAE system.

Lemma 23 For the Jacobi approach, system (35)+(36) with both input-output relation (37) + (39) is in the overall a PH-DAE of type (31).

Proof Via aggregation, we identify the terms in (31):

$$x^{(l+1)} = \begin{pmatrix} q^{(l+1)} \\ \phi^{(l+1)} \\ e^{(l+1)} \\ J_V^{(l+1)} \\ \lambda^{(l+1)} \end{pmatrix}, \bar{u}^{(l+1)} = \begin{pmatrix} 1(t) \\ v(t) \end{pmatrix}, \hat{u} = \hat{u}^{(l)} = \begin{pmatrix} \hat{u}_1^{(l)} \\ \vdots \\ \hat{u}_k^{(l)} \end{pmatrix}, E = \begin{pmatrix} A_C & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, r(z^{(l+1)}) = \begin{pmatrix} A_R g[R](A_R^\top e^{(l+1)}) \\ 0 \\ A_C^\top e - u_C \\ 0 \\ 0 \end{pmatrix},$$

$$\hat{y}^{(l+1)} = \begin{pmatrix} \hat{y}_1^{(l+1)} \\ \vdots \\ \hat{y}_k^{(l+1)} \end{pmatrix}, J = \begin{pmatrix} 0 & -A_L & 0 & -A_V & -\hat{A}_\lambda \\ A_L^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ A_V^\top & 0 & 0 & 0 & 0 \\ \tilde{A}_\lambda{}^\top & 0 & 0 & 0 & 0 \end{pmatrix}, \hat{A}_\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A_{\lambda_k} \end{pmatrix}, \hat{B} = \begin{pmatrix} \hat{A}_\lambda \\ 0 \\ 0 \\ 0 \\ F \end{pmatrix}, \hat{A}_\lambda = \mathrm{diag}(A_{\lambda_1},...,A_{\lambda_{k-1}},0),$$

$$F = (0,...,0,I), \bar{B} = \begin{pmatrix} -A_I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -I \\ 0 & 0 \end{pmatrix}, \hat{C} = \begin{pmatrix} 0 & ... & 0 & I_{n_\lambda} \\ \vdots & & \vdots & \vdots \\ 0 & ... & 0 & I_{n_\lambda} \\ -I_{n_\lambda} & ... & -I_{n_\lambda} & 0 \end{pmatrix},$$

where F is split analogously to $\hat{A}_\lambda$.

Remark 24 (i) Note that the change in the Hamiltonian according to (34) is given by

$$\begin{aligned} &\int_t^{t+h} \Big(-e^{(l+1)}(\tau)^\top A_R g(A_R^\top e^{(l+1)}(\tau)) - 1(\tau)^\top A_I^\top e^{(l+1)}(\tau) - v(\tau)^\top J_V^{(l+1)}(\tau) \\ &+ \sum_{i=1}^{k-1} \Big[(\Delta e_i{}^{(l+1)})^\top A_{\lambda_i} \lambda^{(l+1)} - (\Delta \lambda^{(l+1)})^\top (A_{\lambda_i})^\top e_i{}^{(l+1)} \Big] \Big) d\tau. \end{aligned} \tag{41}$$

(ii) The Schur complement part for the condensed version, compare (33), reads:

$$\hat{B}\hat{C}\hat{B}^\top = \begin{pmatrix} 0 & 0 & 0 & 0 & \bar{A}_\lambda \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\bar{A}_\lambda^\top & 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \bar{A}_\lambda^\top = (A_{\lambda_1}^\top, ..., A_{\lambda_{k-1}}^\top, 0).$$

(iii) We state explicitly the matrix $\hat{B}$ with dimensions:

$$\hat{B}^\top = \begin{array}{c} \\ n_\lambda \\ \vdots \\ n_\lambda \\ n_\lambda \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} n_{e,1} & \cdots & n_{e,k-1} & n_{e,k} & n_{L,1} & \cdots & n_{L,k} & n_{u_C,1} & \cdots & n_{u_C,k} & n_{V,1} & \cdots & n_{V,k} & n_\lambda \end{array} \\ \begin{pmatrix} A_{\lambda,1}^\top & & & & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ & \ddots & & & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ & & A_{\lambda,k-1}^\top & & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & 0 \\ & & & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & I \end{pmatrix} \end{array}.$$

Remark 25 By Remark 22(ii), the condensed PH-DAE description for the Gauss-Seidel iteration reads

$$\begin{aligned} \frac{d}{dt}Ex^{(l+1)} &= Jz^{(l+1)} - r(z^{(l+1)}) - \hat{B}\hat{C}_1\hat{B}^\top z^{(l)} + \bar{B}\bar{u}^{(l+1)} \\ \bar{y}^{(l+1)} &= -(\hat{B}\hat{C}_1\hat{B}^\top)^\top z^{(l+1)} \left(= -A_{\lambda,k}^\top e_k^{(l+1)} \right) \\ \bar{y}^{(l+1)} &= \bar{B}^\top z^{(l+1)}. \end{aligned}$$

The error in the Hamiltonian is given by

$$-\int_t^{t+h} \left(e^{(l+1}(\tau)^\top A_R g(A_R^\top e^{(l+1)}(\tau)) + \imath^\top(\tau) A_I^\top x^{(l+1)}(\tau) + v^\top(\tau) J_V^{(l+1)}(\tau) + + (\Delta\lambda^{(l+1)}(\tau))^\top A_{\lambda_k}^\top e_k^{(l+1)}(\tau) \right) d\tau.$$

7 | CONCLUSIONS

We have introduced several PH-DAE formulations, where all cases correspond to dedicated perspectives: overall systems, multiply coupled DAE systems, and systems within a dynamic iteration process. We proved that versions of the charge-oriented electric circuit models (based on MNA) fall into these classes. Furthermore, we showed that dynamic iteration processes of such PH-DAE systems yield in a certain setup again PH-DAEs. The splitting error enters the respective Hamiltonian as an additional term.

In particular, we included nonlinear dissipative terms in the PH-DAE setup, and we added DAE-specific subspace restrictions. Moreover, dissipativity of electric circuits is here treated very generally by assuming the existence of according gradient fields. A discussion on structural properties (in our case with respect to the differential index) reveals that known index results translate to our new PH-DAE settings.

We believe that our concepts of PH-DAEs can be applied also to other DAEs, in particular to DAEs stemming from multibody systems and flow networks. The next steps include the development of discretizations, which respect the PH-DAE structure and preserve in this way the energy in order to enable fully discrete systems with the same properties.

ACKNOWLEDGMENT
Open access funding enabled and organized by Projekt DEAL.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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How to cite this article: Günther M, Bartel A, Jacob B, Reis T. Dynamic iteration schemes and port-Hamiltonian formulation in coupled differential-algebraic equation circuit simulation. *Int J Circ Theor Appl*. 2020;1–23. https://doi.org/10.1002/cta.2870