SIAM J. CONTROL OPTIM.
Vol. 56, No. 2, pp. 1463–1490

GENERICITY OF THE STRONG OBSERVABILITY FOR SAMPLED SYSTEMS*

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Abstract. In this paper we prove that, generically, a sampled data system is observable provided that the number of outputs is greater than the number of inputs plus one.

Key words. observability, sampled systems, genericity, transversality

AMS subject classifications. 93B07, 93C57

DOI. 10.1137/16M1084961

1. Introduction.

In this paper we deal with the genericity of the observability of sampled data systems. Consider a controlled continuous time system written as

$$\begin{cases} \dot{x} = f(x,u) \\ y = h(x). \end{cases} \tag{1}$$

Given a time T, to system (1), we relate the continuous discrete time system

$$\begin{cases} \dot{x}(t) = f(x(t), u_k), t \in [kT, (k+1)T) \\ \quad y_k = h(x(kT)), \end{cases} \tag{2}$$

where the control u is maintained constant on the intervals $[kT, (k+1)T)$ and the measurements of the state are made only at each of the times $0, T, 2T, \dots$ System (2) is called the sampled data system related to (1).

Many physical processes or industrial devices can be modeled by a system of continuous time differential equations as (1). From a mathematical viewpoint, the time and the state of this system can vary continuously, but in practice, a controlled process is regulated by a digital computer which is not able to record a continuum of data. This is why control decisions are restricted to be taken at fixed times $0, T, 2T, \dots$; here T is called the sampling time and is a (generally small) parameter which depends on the instrumentation of the process, on the computing power, and on other parameters. For a continuous time system, the resulting situation can be modeled through the restriction that the applied inputs are constant on the intervals $[0, T)$, $[T, 2T)$, $\dots$ and the state is (partially) measured only at those fixed times $0, T, 2T, \dots$; that is to say, we access to the values of the observation function only at times $0, T, \dots$

For the sake of clarity, the precise assumptions that we make on these systems are stated in section 1.1, but we recall here the notion of observability. Regarding system (2), an input u^0 is a sequence $(u_k)_{k\geq 0}$ with $u_k \in U$ (U, the input space). An input u^0 being given, we denote by $x(t)$ and $\bar{x}(t)$ the solutions of (2) starting from

*Received by the editors July 15, 2016; accepted for publication (in revised form) November 16, 2017; published electronically April 17, 2018.
http://www.siam.org/journals/sicon/56-2/M108496.html

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x_0 and $\bar{x}_0$, respectively; we say that system (2) is *observable for* u^0 if for every pair of initial conditions $(x_0, \bar{x}_0)$ there exists an integer k such that $h(x(kT)) \neq h(\bar{x}(kT))$. The aim of this paper is to prove that, under some conditions on the respective dimensions of the inputs and the output, generically, the sampled system obtained from a continuous time system is observable.

Two questions can be investigated about observability and sampled systems. The first one is the problem of the preservation of the observability: If the continuous time system (1) is observable for any inputs, is it also the case for the sampled system (2)? The second question is the subject of this paper: Given a sampling time T, how many are the continuous time systems (1) such that the sampled system (2) is observable?

Concerning the first question, the answer for linear systems is well known (see, e.g., [16] and also [13] when the sampling time is not constant). If we deal with nonlinear systems, one would think that the observability of the continuous time system involves the observability of the sampled one, at least if the sampling time is chosen small enough. Surprisingly, this is not the case: a counterexample can be found in [6]; in order to get the observability of the sampled system, to this natural condition (the observability of the continuous time system), we have to add the condition of infinitesimal observability (see [11]) together with a technical condition bearing on the sequence of controls $u_0, u_1, \ldots$

The aim of this paper lies on a more "philosophical plane." Due to the importance of the notion of observability, it is of interest to know "how many" continuous time systems give rise to observable sampled data systems. To be more precise, in [6] (systems given on a compact manifold) and in [3, 5] (systems given on $\mathbf{R}^n$), we gave some natural sufficient conditions bearing on the continuous time system and under which the sampled system is observable. So, these works have a practical interest: For a class of continuous time systems, our result allows us to decide on the observability of the sampled system. The present paper intends to prove that the set of continuous time systems which admit an observable sampled system is everywhere dense. Knowing that the set of rational numbers is dense in the set of real numbers does not permit us to decide if a particular given number is rational; in the same way, the result proven in this paper does not permit us to decide if a particular sampled system is observable. Moreover, this result cannot be deduced from the above-mentioned papers because, while the observability of the continuous time system is generic (see [11]), the additional conditions in [6, 3, 5] are not; also, in these papers, the observability is ensured only for sufficiently small sampling time T.

The genericity of the observability has been the subject of some research in the last decades. As regards continuous time systems, the first paper on the subject was about the genericity of the observability for uncontrolled systems [9]; this work was generalized to controlled systems by J.-P. Gauthier and I. Kupka. In [10], these authors proved the genericity of differential observability for systems with more outputs than inputs. A reference book on this subject is [11]. A related issue is the problem of the identifiability. In [8], the authors deal with general nonlinear systems which contain an unknown function, and they prove that these (uncontrolled) systems are generically identifiable if the number of observations is at least three. Regarding the discrete-time systems, the first paper on the subject was from Aeyels [2]. We can cite also [18] for the uncontrolled case and [7, 4] for the controlled case. In all of these papers, it is proved that the observability is a generic property provided that the number of outputs is greater than the number of inputs. Surprisingly, this result is no more valid for the systems considered in this paper: In the next sections, we shall prove that if the number of inputs is one and the number of outputs is two, the set of pairs (f, h) such that system (2) is observable is not dense. Concerning the subject of this paper, we have also to cite [12], where the authors prove also a result of genericity of the observability for sampled data systems; the systems considered in this paper are uncontrolled, and the sampling time is not constant but depends on the sate of the system.

The tools used to prove our main result are essentially the same (but applied to different situations) as the ones used in the above-mentioned papers, that is to say, the major theorem of the transversality theory.

The paper is organized as follows: In the next section, we state the precise formulation of the problem, and we deal with and recall some useful facts from the transversality theory. In section 2, we state the main result of the paper and introduce the lists L_{2n} and $\bar{L}_{2n}$, which are built from two initial conditions x_0 and $\bar{x}_0$ and their images under the iterates of f; we then introduce five possible configurations for these lists (cf. section 2.1). We then prove our main theorem for each of these configurations: The corresponding results are stated in the Propositions 5, 9, and 10; the conjunction of these propositions gives the main result. Finally, we give a counterexample in order to prove that the observability of the sampled system is not generic in the case $d_u = d_y + 1$ (cf. section 5).

1.1. Problem formulation. We consider two compact manifolds X and U; we let $n = \dim X$ and $d_u = \dim U$. As usual we denote by $\mathrm{T}_x X$ the tangent space to X at x and by $\mathrm{T}X$ the tangent bundle. A parametrized vector field will be a C^∞ mapping defined from $X \times U$ into $\mathrm{T}X$ such that for every $u \in U$, $f(\cdot, u)$ is a vector field defined on X. The set of parametrized vector fields defined on X will be denoted by $\Gamma_U(X)$. If f belongs to $\Gamma_U(X)$, we denote by φ_t^u the flow generated by the vector field $f(\cdot, u)$ (the parameter u being fixed); so for every $x \in X$, every $u \in U$, and every $t \geq 0$, we have

$$\varphi_0^u(x) = x \qquad \text{and} \qquad \frac{\mathrm{d}\varphi_t^u(x)}{\mathrm{d}t} = f(\varphi_t^u(x), u)\,.$$

Let $u_0, u_1, \ldots$ be a sequence of controls (i.e., a sequence of elements of U), for $k \geq 1$, we denote by $\underline{u_k}$ the finite sequence $\underline{u_k} = (u_0, \ldots, u_{k-1})$.

Let $\psi : M \to N$ be a differentiable mapping between two manifolds M and N; the notation $\mathrm{d}\psi(x)$ will stand for the differential of ψ at x. Let $\xi \in \mathrm{T}_x M$ be a tangent vector; $\mathrm{d}\psi(x) \cdot \xi$ will denote the image of ξ under $\mathrm{d}\psi(x)$.

Hereafter, together with a parametrized vector field, we consider a C^∞ mapping h from X to $\mathbf{R}^{d_y}$ and, given a sampling time $T > 0$, consider the mapping $\Theta_T^{f,h}$ defined as

$$\begin{array}{llll} \Theta_T^{f,h}: & X \times U^{2n} & \longrightarrow & \mathbf{R}^{(2n+1)d_y} \times U^{2n} \\ & (x, \underline{u_{2n}}) & \longmapsto & \big(h(x_0), h(x_1), \ldots, h(x_{2n}), \underline{u_{2n}}\big), \end{array} \tag{3}$$

where the sequence $(x_0, x_1, \ldots, x_{2n})$ is defined recursively by $x_0 = x$ and $x_{k+1} = \varphi_T^{u_k}(x_k)$. Also, we denote by y_i and $\bar{y}_i$, the values at x_i and $\bar{x}_i$ under h: $y_i = h(x_i)$ and $\bar{y}_i = h(\bar{x}_i)$.

DEFINITION 1. *We shall say that the sampled data system* (2) *is* strongly observable *if the mapping* $\Theta_T^{f,h}$ *defined above is one-to-one.*

We shall show that, generically, system (2) is strongly observable; to be more precise, we endow $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ with the Whitney topology, and we shall prove that the set of pairs (f, h) such that the mapping $\Theta_T^{f,h}$ is injective is a residual subset of $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ provided that $d_y \geq d_u + 2$ (case $d_u > 0$) or $d_y \geq 1$ (case

$d_u = 0$). The tools used in this paper come from the transversality theory; hereafter, we recall the notion of transversality as well as Abraham's theorem of density [1], which will be used intensively in the proof of our main result.

Definition 2 (transversality). *Let f be a smooth mapping between two smooth manifolds X and Y, W a submanifold of Y, and x a point in X. We shall say that f is* transversal *to W at x if either*

- $f(x) \notin W$ *or*
- $f(x) \in W$ *and* $\mathrm{T}_{f(x)}Y = \mathrm{T}_{f(x)}W + \mathrm{d}f_x(\mathrm{T}_x X)$.

We shall say that f is transversal to W if it is transversal to W at every $x \in X$. We shall use the symbol $\pitchfork$ to denote the transversality.

Concerning this definition, some elementary conditions show that the second equality cannot be satisfied if $\operatorname{codim} W > \dim X$. Therefore, if $\operatorname{codim} W > \dim X$, transversality means nonmembership: In this case saying that f is transverse to W amounts to saying that $f(x) \notin W$ for every $x \in X$. This trick will be used later in the proofs of Propositions 9 and 10.

We recall also the notion of representation: Let $\mathscr{A}, X$, and Y be C^r manifolds and ρ a map from $\mathscr{A}$ to $C^r(X, Y)$. For $a \in \mathscr{A}$, $\rho_a : X \to Y$ is the map defined as $\rho_a(x) = \rho(a)(x)$. We say that ρ is a C^r representation if the evaluation map

$$\begin{array}{rcl} \mathrm{ev}_\rho : \mathscr{A} \times X & \longrightarrow & Y \\ (a, x) & \longmapsto & \rho_a(x) = \rho(a)(x) \end{array}$$

is a C^r map from $\mathscr{A} \times X$ to Y.

Theorem 3 (transversal density theorem). *Let $\mathscr{A}, X, Y$ be C^r manifolds, $\rho : \mathscr{A} \to C^r(X, Y)$ a C^r representation, $W \subset Y$ a submanifold (not necessarily closed), and $\mathrm{ev}_\rho : \mathscr{A} \times X \to Y$ the evaluation map. Define $\mathscr{A}_W \subset \mathscr{A}$ by*

$$\mathscr{A}_W = \{a \in \mathscr{A} \mid \rho_a \pitchfork W\}.$$

Assume that

1. *X has a finite dimension n, and W has a finite codimension q in Y;*
2. *$\mathscr{A}$ and X are second countable;*
3. *$r > \max(0, n - q)$;*
4. *$\mathrm{ev}_\rho \pitchfork W$.*

Then $\mathscr{A}_W$ is residual in $\mathscr{A}$.

Notice that manifold $\mathscr{A}$ is not necessarily finite dimensional; it may be a Banach space or an open subset of a Banach space.

2. Main result.

Theorem 4. *Assume that $d_y \geq d_u + 2$ or that $d_u = 0$ and $d_y \geq 1$, and let $T > 0$ a given sampling time. Then the set of pairs (f, h) such that system* (2) *is strongly observable is a residual subset of $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$.*

In order to prove this theorem, we need some preliminary results, namely, Propositions 5, 9, and 10 stated in sections 3 and 4.3–4.4. In these propositions, different possible configurations, denoted hereafter by $\mathbf{C_0}$ through $\mathbf{C_4}$, of the lists $(x_0, x_1, \ldots, x_{2n})$ and $(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{2n})$ are considered (the $\bar{x}_i$'s are defined as the x_i's; cf. (3)). For each of this configuration $\mathbf{C_k}$ ($k = 0, \ldots, 4$), we prove that, generically, $\Theta_T^{f,h}(x, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n}})$ if $(x, \underline{u_{2n}}), (\bar{x}, \underline{u_{2n}})$ is under $\mathbf{C_k}$ configuration.

To be more precise, given two different initial conditions x and $\bar{x}$ and an integer $s \leq 2n$, we shall consider the two lists

$$L_s = (x_0, \ldots, x_s) \qquad \text{and} \qquad \bar{L}_s = (\bar{x}_0, \ldots, \bar{x}_s)\,, \tag{4}$$

where the $\bar{x}_i$'s are defined as the x_i's. Concerning the two lists L_{2n} and $\bar{L}_{2n}$, we shall examine all the possible situations: The elements of these lists can be pairwise distinct, some equalities can occur among the elements of the first list while the ones of the second are pairwise distinct, etc. In the following section, we define five possible configurations and prove that, necessarily, the above-mentioned lists belong to one of five configurations.

2.1. The different configurations. Hereafter we shall give an exhaustive list of all the possibilities concerning the equalities between the elements of the lists L_{2n} and $\bar{L}_{2n}$; in the following, we shall say that the equalities $x_i = \bar{x}_j$ and $x_{i'} = \bar{x}_{j'}$ between elements of L_{2n} and elements of $\bar{L}_{2n}$ are in the same direction if the differences $i - j$ and $i' - j'$ have the same sign. Take $x \neq \bar{x} \in X$, even if we have to invert the roles of L_{2n} and $\bar{L}_{2n}$, the possible configurations for these lists are the following:

$\mathbf{C_0}$ The elements of $\bar{L}_{2n}$ are pairwise different; moreover, the only possible equalities between elements of L_{2n} and elements of $\bar{L}_{2n}$ are all in the "same direction." That is to say, let

$$I = \{\, 0 \le i \le 2n \mid \exists j \in \{0, \ldots, 2n\}, x_i = \bar{x}_j \,\}$$

for $i \in I$, and let $E_i = \{\, 0 \le j \le 2n \mid x_i = \bar{x}_j \,\}$; then either for every $i \in I$, for every $j \in E_i$, $j < i$, or for every $i \in I$, for every $j \in E_i$, $j > i$.

$\mathbf{C_1}$ There exist some subscripts $0 \le i < p \le 2n$ and $0 \le j, m \le 2n$ such that
- $x_p = x_i$ and $\bar{x}_m = x_j$;
- there is no equality between the elements of L_{p-1} ;
- letting $q = \max(j, p, m)$, there is no equality between the elements of $\bar{L}_{q-1}$;
- the equalities between elements of L_{q-1} and $\bar{L}_{q-1}$ have the same direction.

$\mathbf{C_2}$ There exist some subscripts $0 \le i < p \le 2n$ and $0 \le j < m \le 2n$ with $m \ge p$ and such that
- $x_p = \bar{x}_m$ and $x_i = \bar{x}_j$ with $p - i \neq m - j$;
- there are no equalities between the elements of L_p or between the elements of $\bar{L}_m$;
- the only possible equalities between elements of L_{m-1} and $\bar{L}_{m-1}$ are all in the same direction; moreover, if these equalities write $x_{i_1} = \bar{x}_{j_1}, \ldots, x_{i_r} = \bar{x}_{j_r}$, the differences $i_1 - j_1, \ldots, i_r - j_r$ are equal;
- if $x_{i'} = \bar{x}_{j'}$ with $i', j' \le m$, then $i' \ge i$ and $j' \ge j$.

$\mathbf{C_3}$ There exist some subscripts $0 \le i < p \le 2n$, and $0 \le m < j \le 2n$ with $p \le j$ and such that
- $x_p = \bar{x}_m$ and $x_i = \bar{x}_j$;
- there are no equalities between the elements of L_p or between the elements of $\bar{L}_j$;
- the only possible equalities between elements of L_{j-1} and $\bar{L}_{j-1}$ are all in the same direction; moreover, if these equalities write $x_{i_1} = \bar{x}_{j_1}, \ldots, x_{i_r} = \bar{x}_{j_r}$, the differences $i_1 - j_1, \ldots, i_r - j_r$ are equal;
- if $x_{i'} = \bar{x}_{j'}$ with $i', j' \le j$, then $i' \ge p$ and $j' \ge m$.

$\mathbf{C_4}$ There exist some subscripts $0 \le i < p \le 2n$ and $0 \le j < m \le 2n$ with $m \ge p$ and such that
- $x_p = x_i$ and $\bar{x}_m = \bar{x}_j$;
- there is no equality between one element of L_{m-1} and one element of $\bar{L}_{m-1}$.

Denote by e_s the number of equalities between the elements of $L_s \cup \bar{L}_s$ if $e_{2n} = 0$ or 1; then we are under the $\mathbf{C_0}$ configuration.

Now, assume that $e_{2n} \geq 2$; if we go from $L_s \cup \bar{L}_s$ to $L_{s-1} \cup \bar{L}_{s-1}$, we lose zero, one, or two equalities. Thus, if $e_{2n} \geq 2$, there exists a subscript $m \leq 2n$ such that there exist exactly two or exactly three equalities between the elements of $L_m \cup \bar{L}_m$; we denote by s the minimal subscript with this property. Notice that if e_s is exactly three, x_s *and* $\bar{x}_s$ must be equal to some elements of $L_{s-1} \cup \bar{L}_{s-1}$ (because we always have $x_s \neq \bar{x}_s$).

Assume first that $e_s = 3$, and denote by $x_s = z_1$, $\bar{x}_s = z_2$, and $z_3 = z_4$ the three equalities; then

- if $z_1 \in L_s$, and $z_3, z_4 \in L_s$, then L_{s-1} and $\bar{L}_s$ are under $\mathbf{C_1}$ (if $z_2 \in L_s$) or $\mathbf{C_4}$ configuration (if $z_2 \in \bar{L}_s$);
- if $z_1 \in L_s$, $z_3 \in L_s$, and $z_4 \in \bar{L}_s$, then L_s and $\bar{L}_{s-1}$ are under $\mathbf{C_1}$ configuration;
- if $z_1 \in L_s$, $z_3 \in \bar{L}_s$, and $z_4 \in \bar{L}_s$, then L_s and $\bar{L}_{s-1}$ are under $\mathbf{C_4}$ configuration;
- if $z_1 \notin L_s$, then L_s and $\bar{L}_{s-1}$ are under $\mathbf{C_1}$, $\mathbf{C_2}$ or $\mathbf{C_3}$ configuration (we could have to invert the roles of L_s and $\bar{L}_s$).

If $e_s = 2$ and if the lists L_s and $\bar{L}_s$ are not under $\mathbf{C_1}$–$\mathbf{C_4}$ configurations, we have two cases to consider.

In the first case, there exists some subscripts $0 \leq t < s$ and $0 \leq t' < s'$ such that $x_t = x_s$ and $x_{t'} = x_{s'}$. If $s' < s$, we set $i = t'$ and $p = s'$; if $s' = s$, we set $i = \min(t, t')$ and $p = \max(t, t')$. As noticed above, when we go from lists L_s and $\bar{L}_s$ to the lists L_{s+1} and $\bar{L}_{s+1}$, we gain zero, one, or two equalities. Denote by σ the least subscript greater than s such that there exist three or four equalities among the elements of L_σ and $\bar{L}_\sigma$; if such a subscript fails to exist, the lists L_{2n} and $\bar{L}_{2n}$ are under $\mathbf{C_0}$ configuration. Otherwise, assume that there exists exactly three equalities between the elements of $L_\sigma \cup \bar{L}_\sigma$; this additional equality can be one of the following:

- $\bar{x}_\sigma = x_j$ (with $j < \sigma$)—the lists L_{2n} and $\bar{L}_{2n}$ are then under $\mathbf{C_1}$ configuration;
- $\bar{x}_\sigma = \bar{x}_j$ (with $j < \sigma$)—the lists L_p and $\bar{L}_\sigma$ are under $\mathbf{C_4}$ configuration;
- $x_\sigma = \bar{x}_m$ (with $m < \sigma$)—the lists L_{2n} and $\bar{L}_{2n}$ are then under $\mathbf{C_1}$ configuration;
- $x_\sigma = x_j$—in this case we seek for the least subscript $\sigma' > \sigma$ (if any) such that one get one or two additional equalities by going from $L_\sigma \cup \bar{L}_\sigma$ to $L_{\sigma'} \cup \bar{L}_{\sigma'}$.

If we are in the case where there exist exactly four equalities between the elements of $L_\sigma \cup \bar{L}_\sigma$, these equalities can be

- $x_\sigma = x_{j_1}$ and $\bar{x}_\sigma = x_{j_2}$ (with $j_1, j_2 < \sigma$)—in this case the lists L_{2n} and $\bar{L}_{2n}$ are then under $\mathbf{C_1}$ configuration;
- $x_\sigma = x_{j_1}$ and $\bar{x}_\sigma = \bar{x}_{j_2}$—the lists L_p and $\bar{L}_\sigma$ are then under $\mathbf{C_4}$ configuration;
- $x_\sigma = \bar{x}_{j_1}$ and $\bar{x}_\sigma = x_{j_2}$—the lists L_p and $\bar{L}_{j_1}$ are then under $\mathbf{C_1}$ configuration;
- $x_\sigma = \bar{x}_{j_1}$ and $\bar{x}_\sigma = \bar{x}_{j_2}$—the lists L_p and $\bar{L}_\sigma$ are then under $\mathbf{C_4}$ configuration.

In the second case there exist some subscripts $0 \leq t < s$ and $0 \leq t' < s' < s$ such that $s - t = s' - t'$, $x_s = \bar{x}_t$, and $x_{s'} = \bar{x}_{t'}$ (without loss of generality, we can exchange the roles of x and $\bar{x}$). Proceeding as in the first case, we prove either that the lists L_{2n} and $\bar{L}_{2n}$ are under $\mathbf{C_0}$ configuration or there exists a subscript $s < \sigma \leq 2n$ such that L_σ and $\bar{L}_\sigma$ are under one of the configurations $\mathbf{C_1}$, $\mathbf{C_2}$, or $\mathbf{C_3}$.

Let the pair $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$ be fixed; hereafter, we shall say that the configuration of the triplet $(x, \bar{x}, \underline{u_{2n}})$ $(x \neq \bar{x})$ is $\mathbf{C}_{\boldsymbol{i}}$ if the configuration of the lists L_{2n} and $\bar{L}_{2n}$ related to x and $\bar{x}$ is $\mathbf{C}_{\boldsymbol{i}}$. In the following, we shall assume that all the function spaces (such that $C^\infty(X, \mathbf{R}^p)$,...) as well as the spaces $\Gamma(X)$, $\Gamma_U(X)$, and $\mathscr{G}_2^U(a)$ are endowed with the C^r topology, where $r \in \mathbf{N}^*$.

2.2. Outline of the proof of Theorem 4. Without going into the technique details, we shall explain our strategy for the proof. Take $(f,h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$, and let $x_0 \neq \bar{x}_0$ be two different initial conditions and $\underline{u_{2n+1}}$ a sequence of controls. The equality $\Theta_T^{f,h}(x_0, \underline{u_{2n+1}}) = \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n+1}})$ can be formulated in geometric terms. To show where are the difficulties, we make a first attempt by considering the simplest way to make this formulation. Consider the mapping, denoted by $r_{f,h}$, related to the pair (f,h) and defined as

$$r_{f,h}: \begin{array}{ccc} X^{(2)} \times U^{2n} & \longrightarrow & \mathbf{R}^{(2n+1)d_y} \\ (x, \bar{x}, \underline{u_{2n+1}}) & \longmapsto & (y_0 - \bar{y}_0, \dots, y_{2n} - \bar{y}_{2n}) \end{array},$$

where $y_i = h(x_i)$, $\bar{y}_i = f(\bar{x}_i)$, and the x_i's and the $\bar{x}_i$'s defined as in (3). The equality $\Theta_T^{f,h}(x_0, \underline{u_{2n+1}}) = \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n+1}})$ means that $r_{f,h}(x_0, \bar{x}_0, \underline{u_{2n}})$ belongs to the submanifold $W = \{0\} \subset \mathbf{R}^{(2n+1)d_y)}$. Notice that the codimension of W is equal to $(2n+1)d_y$ and is greater than $2n(d_u+1)$, the dimension of the domain of $r_{f,h}$. If $r_{f,h}$ is transverse to W, this inequality on $\operatorname{codim} W$ implies that $r_{f,h}(x_0, \bar{x}_0, \underline{u_{2n}})$ does not belong to W and therefore that $\Theta_T^{f,h}(x_0, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n+1}})$. Assume now that we are able to prove that, generically, $r_{f,h}$ is transversal to W, that is to say, assume that there exists a residual set $\mathscr{R}$ such that $r_{f,h}$ is transversal to W whenever (f,h) belongs to $\mathscr{R}$; we then have proved that, generically, $\Theta_T^{f,h}$ is one-to-one. In order to prove that, generically, $r_{f,h}$ is transversal to W, we could try to apply Theorem 3 by proving that the evaluation map related to the representation r from $\Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ to $C^r(X^{(2)} \times U^{2n}, \mathbf{R}^{(2n+1)d_y)}$ is transversal to W. This evaluation map ev_r is defined as $\mathrm{ev}_r(f, h, x, \bar{x}, \underline{u_{2n}}) = r_{f,h}(x, \bar{x}, \underline{u_{2n}})$; as it is linear with respect to h, its differential with respect to h is given by $\eta \mapsto (\eta(x_0) - \eta(\bar{x}_0), \dots, \eta(x_{2n}) - \eta(\bar{x}_{2n}))$, and if we were able to show that there exists η in $C^\infty(X, \mathbf{R}^{d_y})$ such that $\eta(x_i) - \eta(\bar{x}_i) = \mathfrak{Y}_i$, for arbitrary vectors $\mathfrak{Y}_0, \dots, \mathfrak{Y}_{2n}$ of $\mathbf{R}^{d_y}$, then we would be done. The existence of such an η is generally not ensured but is certainly true if the elements $x_0, \dots, x_{2n}, \bar{x}_0, \dots, \bar{x}_{2n}$ are all different or, more generally, if the two lists L_{2n} and $\bar{L}_{2n}$ are under $\mathbf{C}_0$ configuration. In this case, modifying $r_{f,h}$ and the definition of W as explained in the next section, we can prove that, generically, $r_{f,h}$ is transversal to W. Now the two lists are not always under such a configuration; the points x_0 and $\bar{x}_0$ could be located on a periodic trajectory of f or could be singular points. The case of singular points shows that we cannot disregard the cases where the two lists L_{2n} and $\bar{L}_{2n}$ are not under configuration $\mathbf{C_0}$. Assume that x_0 and $\bar{x}_0$ are singular points for the vector field $f(\cdot, u_0)$, and take a sequence of identical controls: $u_0 = u_1 = \dots, = u_{2n}$; then we cannot argue as in the case of $\mathbf{C_0}$ configuration: A mapping η as above fails to exist because $x_0 = \cdots = x_{2n}$ and $\bar{x}_0 = \dots \bar{x}_{2n}$. Notice that this situation is unavoidable: On some manifolds, *every* vector field has at least one singular point; this means that the particular configurations $\mathbf{C_1}$–$\mathbf{C_4}$ cannot be eliminated by using an argument of density.

The outline of the rest of this section will be the following: For each configuration $\mathbf{C_0}$–$\mathbf{C_4}$, we shall prove that there exists a residual subset of $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$ (endowed with the C^r topology), denoted by $\mathscr{C}_k^r$ $(k = 0, \dots, 4)$, such that if $(f,h) \in \mathscr{C}_k^r$ and if $(x_0, \bar{x}_0, \underline{u_{2n}})$ is in configuration $\mathbf{C_k}$, then $\Theta_T^{f,h}(x_0, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n}})$. Consider the intersection $\mathscr{C}^r \triangleq \mathscr{C}_0^r \cap \cdots \cap \mathscr{C}_4^r$, which also is a residual subset, and take a pair (f,h) in $\mathscr{C}^r$; let $x_0 \neq \bar{x}_0$ be two different initial conditions and $\underline{u_{2n}}$ a finite sequence of controls; as $(x_0, \bar{x}_0, \underline{u_{2n}})$ must be in one of the $\mathbf{C_0}$–$\mathbf{C_4}$ configurations, we have $\Theta_T^{f,h}(x_0, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}_0, \underline{u_{2n}})$. Now taking the intersections of the sets $\mathscr{C}^r$ for

$r = 1, 2 \dots$, we obtain a residual $\mathscr{R}$ of $\Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y})$ endowed with the C^∞ topology; this residual is such that if (f, h) belongs to $\mathscr{R}$, then $\Theta_T^{f,h}$ is one-to-one.

In the following section, the existence of the residual sets $\mathscr{C}_0^r$–$\mathscr{C}_4^r$ is stated in Propositions 5, 9, and 10. For the proofs of these propositions, our strategy will be the following: We shall consider some submanifold W together with some representation ρ, the choice of W and ρ being related to the considered configuration of the lists L_{2n} and $\bar{L}_{2n}$. Concerning W and ρ, we shall prove the following results:

- By applying the transversal density theorem [1], we shall see that the set of pairs $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ which are transversal to W is dense.
- We shall prove also that the codimension of W is greater than the dimension of the domain of $\rho_{f,h}$, which implies that the range of $\rho_{f,h}$ does not intersect W.
- Due to our choice of W, saying that $\rho_{f,h}(x, \bar{x}, \underline{u_{2n}}) \notin W$ will imply that $\Theta_T^{f,h}(x, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n}})$.

We shall provide a detailed proof for the $\mathbf{C_0}$ and $\mathbf{C_1}$ configurations; the proofs for the $\mathbf{C_2}$ and $\mathbf{C_3}$ configurations will be omitted because they are very similar to the previous ones. Concerning the configurations $\mathbf{C_0}$–$\mathbf{C_3}$, we only need the assumption $d_y \geq d_u + 1$ to prove the existence of the residual subsets $\mathscr{C}_0$–$\mathscr{C}_3$. We have to consider apart the case of $\mathbf{C}_4$ configuration because to prove the existence of the residual subset $\mathscr{C}_4$, we need the assumption $d_y \geq d_u + 2$.

3. The triplet $(x, \bar{x}, \underline{u_{2n}})$ is under configuration $\mathbf{C_0}$.

In this section, we deal first with the simplest case: the $\mathbf{C_0}$ configuration.

PROPOSITION 5. *Assume that $d_y > d_u$ (the number of observations is greater than the number of controls). Denote by $\mathscr{C}_0^r$ the set of pairs $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ such that $\Theta_T^{f,h}(x, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n+1}})$ whenever the triplet $(x, \bar{x}, \underline{u_{2n}})$ (with $x \neq \bar{x}$) is in configuration $\mathbf{C_0}$. Then $\mathscr{C}_0^r$ contains a residual for the C^r topology.*

Proof. We consider the representation

$$\begin{array}{llll} \rho: & \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y}) & \longrightarrow & C^r\big(X^{(2)} \times U^{2n}, \, (X^{2n+1})^2 \times \mathbf{R}^{(2n+1)d_y}\big) \\ & (f, h) & \longmapsto & \rho_{f,h}, \end{array}$$

where $\rho_{f,h}$ is the mapping

$$\begin{array}{llll} \rho_{f,h}: & X^{(2)} \times U^{2n} & \longrightarrow & X^{2n+1} \times X^{2n+1} \times \mathbf{R}^{(2n+1)d_y} \\ & (x, \bar{x}, \underline{u_{2n}}) & \longmapsto & (x_0, \dots, x_{2n}, \bar{x}_0, \dots, \bar{x}_{2n}, y_0 - \bar{y}_0, \dots, y_{2n} - \bar{y}_{2n}) \end{array}$$

and where $y_i = h(x_i)$ and $\bar{y}_i = h(\bar{x}_i)$, the x_i's and the $\bar{x}_i$'s being defined above. We consider the submanifold W_0 included in $(X^{2n+1})^2 \times \mathbf{R}^{(2n+1)d_y}$ defined as follows. Submanifold W_0 is the set of those elements $(a_0, \dots, a_{2n}, \bar{a}_0, \dots, \bar{a}_{2n}, 0, \dots, 0)$ such that

- the elements $\bar{a}_0, \dots, \bar{a}_{2n}$ are pairwise distinct;
- we have $a_k \neq \bar{a}_l$ if $k > l$.

Notice that the codimension of W_0 is equal to $(2n+1)d_y$; as $d_y > d_u$, we have $(2n+1)d_y \geq (2n+1)d_u + 2n + 1$, so the codimension of W_0 is greater than the dimension of $X^{(2)} \times U^{2n+1}$, the domain of $\rho_{f,h}$.

Recall that the evaluation map ev_ρ is defined as

$$\mathrm{ev}_\rho(f, h, x, \bar{x}, \underline{u_{2n}}) = \rho_{f,h}(x, \bar{x}, \underline{u_{2n}}) \,.$$

We shall see that ev_ρ is transversal to W_0 at every given point

$$\mathscr{X} \triangleq (f, h, x, \bar{x}, \underline{u_{2n}}) \in \Gamma_U(X) \times C^r(X, \mathbf{R}^{d_y}) \times X^{(2)} \times U^{2n} \,.$$

Consider a point $\mathscr{X}$ such that $\mathrm{ev}_\rho(\mathscr{X}) \in W_0$ and a vector $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$ that is tangent to the codomain of ev_ρ with $\mathfrak{X}_i \in \mathrm{T}_{x_i}X$, $\bar{\mathfrak{X}}_i \in \mathrm{T}_{\bar{x}_i}X$, and $\mathfrak{Y}_i \in \mathbf{R}^{d_y}$ $(i = 0, \dots, 2n)$; we have to prove that there exist $\phi \in \Gamma_U(X)$, $\eta \in C^\infty(X, \mathbf{R}^{d_y})$, $\xi \in \mathrm{T}_x X$, $\bar{\xi} \in \mathrm{T}_{\bar{x}}X$, $\nu_i \in \mathrm{T}_{u_i}U$ (for $i = 0, \dots, 2n$) and a vector ζ in the tangent space to W_0 at $\mathrm{ev}_\rho(\mathscr{X})$ such that

$$(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = \mathrm{d}(\mathrm{ev}_\rho)(\mathscr{X}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu) + \zeta\,. \tag{5}$$

We shall prove this relation with $\phi = 0$, $\xi = 0$, $\bar{\xi} = 0$, and $\nu = 0$. We denote by

$$\alpha_0, \dots, \alpha_{2n}, \bar{\alpha}_0, \dots, \bar{\alpha}_{2n}, 0, \dots, 0$$

the components of ζ. In the right-hand member of (5), the $2(2n+1)$ first components are equal to $\alpha_0, \dots, \alpha_{2n}, \bar{\alpha}_0, \dots, \bar{\alpha}_{2n}$, and can be chosen such that $\alpha_i = \mathfrak{X}_i$ and $\bar{\alpha}_i = \bar{\mathfrak{X}}_i$ $(i = 0, \dots, 2n)$. The $2n + 1$ last equations in (5) are

$$\eta_0 - \bar{\eta}_0 = \mathfrak{Y}_0, \qquad \dots, \qquad \eta_{2n} - \bar{\eta}_{2n} = \mathfrak{Y}_{2n}, \tag{6}$$

where we let $\eta_i \triangleq \eta(x_i)$ and $\bar{\eta}_i \triangleq \eta(\bar{x}_i)$. We consider this system as a linear system whose unknown are $\eta_0, \dots, \eta_{2n}, \bar{\eta}_0, \dots, \bar{\eta}_{2n}$. If $\mathrm{ev}_\rho(\mathscr{X})$ belongs to the submanifold W_0, the points $\bar{x}_0, \dots, \bar{x}_{2n}$ are pairwise distinct, so the unknown $\bar{\eta}_0, \dots, \bar{\eta}_{2n}$ can be arbitrarily and independently chosen. There could be some equalities between the elements of the list L_{2n} and between an element of L_{2n} and an element of $\bar{L}_{2n}$. If an equality such that $x_k = \bar{x}_l$ exists, then, as $\mathrm{ev}_\rho(\mathscr{X}) \in W_0$, we necessarily have $k < l$ and $\eta_k = \bar{\eta}_l$. The matrix of system (6) is then

$$M = \begin{pmatrix} A & | & -I_{(2n+1)d_y} + B \end{pmatrix},$$

where $I_{(2n+1)d_y}$ is the $(2n+1)d_y$ dimensional identity matrix and matrix B is a block matrix, whose blocks are 0 or d_y dimensional identity matrices. Moreover, matrix B is upper triangular: If we have an equality like $x_k = \bar{x}_l$, then we find in B the block I_{d_y} (identity matrix) at position (k, l) with $l > k$. From the form of matrix B, we can conclude that the rank of M is equal to $(2n + 1)d_y$, which implies that we can find $\bar{\eta}_0, \dots, \bar{\eta}_{2n}, \eta_0, \dots, \eta_{2n}$ such that system (6) has a solution. Denote by $x_{k_1}, \dots, x_{k_p}$ the list of the elements of L_{2n} which are not equal to an element of $\bar{L}_{2n}$. It is possible to find a solution of (6) such that $\eta_{k_1} = \cdots = \eta_{k_p} = 0$; for such a solution, it is then possible to find a mapping η such that $\eta(x_i) = \eta_i$ and $\eta(\bar{x}_i) = \bar{\eta}_i$ $(i = 0, \dots, 2n)$.

We have shown that ev_ρ is transversal to W_0. The conclusion of the proposition now follows from the application of the transversal density theorem [1]: The set $\mathscr{O}_0^r$ of pairs $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$ such that $\rho_{f,h}$ is transversal to W_0 is open and dense in the C^r topology (for every $r > 0$). Take now a pair (f, h) in this set $\mathscr{O}_0^r$, and take two initial conditions $x \neq \bar{x}$ and a finite sequence of controls $\underline{u_{2n}}$ such that the triplet $(x, \bar{x}, \underline{u_{2n}})$ is in configuration $\mathbf{C_0}$ and the mapping $\rho_{f,h}$ is transversal to W_0 at $(x, \bar{x}, \underline{u_{2n}})$, but as $\operatorname{codim} W_0 > \dim(X^{(2)} \times U^{2n})$, transversality means that $\rho_{f,h}(x, \bar{x}, \underline{u_{2n}}) \notin W_0$, which implies that at least one of the equalities $y_j = \bar{y}_j$ is not satisfied, and so $\Theta_T^{f,h}(x, \underline{u_{2n}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n}})$. ∎

4. The case of the $\mathbf{C_1}$–$\mathbf{C_4}$ configurations. In the proofs of the next Propositions 9 and 10, we shall have to consider the derivative of $\varphi_T(x)$ with respect to the vector field f (we are no more able to take $\phi = 0$ as in the proof of Proposition 5). This is why we need to state a technical lemma, which bears on the computation of these derivatives. To prove this lemma, we have to take into account the periodic trajectories of a vector field; these trajectories have some generic properties—which we intend to use in the proof of Lemma 8—which are stated in the Kupka–Smale theorem. Now the Kupka–Smale theorem has not been stated for parametrized vector fields, so we will show that it can be generalized to those vector fields.

4.1. Periodic trajectories and the Kupka–Smale theorem. Take a parametrized vector field $f \in \Gamma_U(X)$, $u \in U$, $x \in X$, and assume that x belongs to a periodic trajectory of the vector field $f(\cdot, u)$. Then there exists $\pi_0 > 0$ such that $\varphi^u_{\pi_0}(x) = x$; this implies that $\mathrm{d}\varphi^u_{\pi_0}(x) \cdot f(x,u) = f(x,u)$, so 1 is an eigenvalue of $A \triangleq \mathrm{d}\varphi^u_{\pi_0}(x)$. In the following, we shall have to consider expressions like $\mathrm{Id} + A + \cdots + A^k$, and we shall need that this sum of linear mappings be invertible; this is certainly true if, apart from 1, the other eigenvalues have modulus different from 1. The theorem of Kupka–Smale [14, 15] asserts that this is generically the case for a vector field. Let $a > 0$; hereafter, we denote by $\mathscr{G}_2(a)$ the subset of $\Gamma(X)$ of those vector fields f such that

- if x is a singular point of f (i.e., $f(x) = 0$), then for every $t \neq 0$, $\mathrm{d}\varphi_t(x) : \mathrm{T}_x X \to \mathrm{T}_x X$ has no complex eigenvalue of modulus 1;
- if x belongs to a periodic trajectory of f with period $0 < \pi_0 \leq a$, then, denoting by $1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of $\mathrm{d}\varphi_{\pi_0}(x)$, we have $|\lambda_i| \neq 1$ for $i = 2, \ldots, n$.

Hereafter, recall that the manifolds X and U are assumed to be compact. We have the following.

THEOREM 6 (Kupka–Smale). *Let $a > 0$; the set $\mathscr{G}_2(a)$ is residual; moreover, for the C^r topology ($r < +\infty$), $\mathscr{G}_2(a)$ is open and dense.*

This theorem can be generalized to parametrized vector fields; namely, we have the following.

THEOREM 7. *Let $a > 0$; the set $\mathscr{G}^U_2(a)$ of parametrized vector fields such that $f(\cdot, u) \in \mathscr{G}_2(a)$ for every $u \in U$ is a residual; moreover, $\mathscr{G}^U_2(a)$ is open and dense for the C^r topology.*

This theorem can be proved by adapting the steps of the proof of Kupka–Smale's theorem, which can be found in [1]. Owing to lack of space, we do not write here the proof of Theorem 7, but we refer the reader to [17], where this result is proved in the case where the dimension of U is 1; moreover, a sketch of this proof can also be found at https://hal.inria.fr/hal-01630461 .

4.2. Technical lemma. The proofs of Propositions 9 and 10 below will follow the same scheme as the proof of Proposition 5. Nevertheless, in the following propositions, in order to prove that the mapping ev_ρ is transversal to some submanifold W, we shall have to consider the derivative of the flow with respect to a vector field. Thus, before going further, we recall a result which will be used in the proof of Lemma 8 and Propositions 9 and 10. Take two vector fields f and ϕ defined on X, and denote by φ_t and φ^λ_t ($\lambda \in \mathbf{R}$) the flows related to f and $f + \lambda\phi$, respectively. In [1, perturbation theorem], the following formula is proved: For every $x \in X$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \varphi^\lambda_t(x) \bigg|_{\lambda=0} = \int_0^t \mathrm{d}\varphi_\sigma \circ \phi \circ \varphi_{t-\sigma}(x) \mathrm{d}\sigma \,. \tag{7}$$

Obviously, this formula can be extended to the case of parametrized vector fields. Consider f and ϕ in $\Gamma_U(X)$, and denote by $\varphi^{u,\lambda}_t$ the flow generated by the vector field $f(\cdot, u) + \lambda\, \phi(\cdot, u)$ (with u fixed). Starting from an initial condition x_0, consider now the sequence $x_0^\lambda, x_1^\lambda, \dots$ defined recursively as $x_0^\lambda = x_0$ and $x_{i+1}^\lambda = \varphi_T^{u_i,\lambda}(x_i^\lambda)$; then, applying formula (7), we deduce easily that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} x_{i+1}^\lambda \bigg|_{\lambda=0} = J_i + \delta_i(J_{i-1}) + \cdots + \delta_1(J_0),$$

where

$$J_k = \int_0^T \mathrm{d}\varphi_\sigma^{u_k}(\varphi_{T-\sigma}^{u_k}(x_k)) \cdot \phi(\varphi_{T-\sigma}^{u_k}(x_k), u_k)\mathrm{d}\sigma\,;$$

the integral J_k belongs to the tangent space of X at $\varphi_T^{u_k}(x_k) = x_{k+1}$. Moreover, the δ_k's are the mappings defined as

$$\delta_k = \mathrm{d}(\varphi_T^{u_i} \circ \cdots \circ \varphi_T^{u_k})(x_k)\,.$$

We state now a preliminary result which will be used in the proofs of Propositions 9 and 10. We shall say that these lists are under $\mathbf{C_3'}$ (resp., $\mathbf{C_4'}$) configuration if

- they are under $\mathbf{C_3}$ (resp. $\mathbf{C_4}$) configuration and if
- there exists a subscript $k \in \{i, \dots, p-1\} \cup \{m, \dots, j-1\}$ (resp., $k \in \{i, \dots, p-1\} \cup \{j, \dots, m-1\}$) such that $u_k \neq u_{p-1}$.

The proof of the following lemma is postponed until the appendix.

Lemma 8. *Let $\mathfrak{X}_p \in \mathrm{T}_{x_p}X$ be an arbitrary tangent vector to X at x_p. Assume that the lists L_{2n} and $\bar{L}_{2n}$ are under $\mathbf{C_1}$ configuration; then one can find a vector field $\phi \in \Gamma_U(X)$ such that we have*

$$\frac{\mathrm{d}x_p^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \mathfrak{X}_p, \qquad\qquad \frac{\mathrm{d}x_i^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0\,.$$

If these lists are under $\mathbf{C_2}$, $\mathbf{C_3'}$, or $\mathbf{C_4'}$ configuration, then one can find a vector field $\phi \in \Gamma_U(X)$ such that we have

$$\frac{\mathrm{d}x_p^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = \mathfrak{X}_p, \qquad \frac{\mathrm{d}x_i^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0, \qquad \frac{\mathrm{d}\bar{x}_m^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0, \qquad \frac{\mathrm{d}\bar{x}_j^\lambda}{\mathrm{d}\lambda}\bigg|_{\lambda=0} = 0\,.$$

4.3. The triplet $(x, \bar{x}, \underline{u_{2n}})$ is under one of the configurations $\mathbf{C_1}$, $\mathbf{C_2}$, or $\mathbf{C_3}$.

Proposition 9. *Assume that $d_y > d_u$ (the number of observations is greater than the number of controls). For $k = 1, 2, 3$, denote by $\mathscr{C}_k^r$ the subset of pairs $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$ such that $\Theta_T^{f,h}(x, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n+1}})$ whenever the triplet $(x, \bar{x}, \underline{u_{2n}})$ (with $x \neq \bar{x}$) is in configuration $\mathbf{C_k}$. Then each subset $\mathscr{C}_k^r$ contains a residual for the C^r topology.*

Hereafter, we write the proof of this proposition only in the case of $\mathbf{C_1}$ configuration, the case proofs for the other configurations being very similar.

4.3.1. Proof of the proposition in the case of $\mathbf{C_1}$ configuration. If the triplet $(x, \bar{x}, \underline{u_{2n}})$ is in the $\mathbf{C_1}$ configuration, there exist subscripts i, j, p, m such that $x_p = x_i$ and $\bar{x}_m = x_j$.

We choose four subscripts $0 \leq i, j, p, m \leq 2n$ such that $i < p$; letting $q = \max(i, j, p, m)$, we consider the representation ρ

$$\begin{array}{rccl}\rho: & \mathscr{G}_2^U(a)\times C^r(X,\mathbf{R}^{d_y}) & \longrightarrow & C^r\big(X^{(2)}\times U^q\,,\,X^{q+1}\times X^{q+1}\times\mathbf{R}^{q\,d_y}\big)\\ & (f,h) & \longmapsto & \rho_{f,h}\end{array}$$

defined through the mapping $\rho_{f,h}$

$$\begin{array}{rccl}\rho_{f,h}: & X^{(2)}\times U^q & \longrightarrow & X^{q+1}\times X^{q+1}\times\mathbf{R}^{q\,d_y}\\ & (x,\bar{x},\underline{u_q}) & \longmapsto & (x_0,\dots,x_q,\bar{x}_0,\dots,\bar{x}_q,y_0-\bar{y}_0,\dots,y_{q-1}-\bar{y}_{q-1})\,.\end{array}$$

We fix two lists of subscripts (possibly empty) $i_1<\cdots<i_r<q$ and $j_1<\cdots<j_r<q$ such that the signs of $i_1-j_1,\dots,i_r-j_r$ are the same; if $j,m<q$, one has $i_k=j$ and $j_k=m$ for some subscript k. We consider also the submanifold

$$V_m^{j,i,p}\subset X^{q+1}\times X^{q+1}\times\mathbf{R}^{(2n+1)d_y}$$

defined as follows: $V_m^{j,i,p}$ is the set of those elements

$$(a_0,\dots,a_q,\bar{a}_0,\dots,\bar{a}_q,0,\dots,0)$$

such that

- we have the equalities $\bar{a}_m=a_j$ and $a_i=a_p$;
- the elements of $\{\bar{a}_0,\dots,\bar{a}_{q-1}\}$ are pairwise different;
- $a_{i'}\neq\bar{a}_{j'}$ if $(i',j')\neq(i_k,j_k)$ $(i',j'<q,\ k=1,\dots,r)$.

Notice that the number of submanifold having the above properties is finite; moreover, the codimension of $V_m^{j,i,p}$ is equal to $2n+q\,d_y$ and is greater than or equal to $2n+q(d_u+1)>2n+q\,d_u$, so the codimension of $V_m^{j,i,p}$ is greater than the dimension of the domain of $\rho_{f,h}$.

We shall show that ev_ρ is transversal to $V_m^{j,i,p}$. Let $\mathscr{X}\triangleq(f,h,x,\bar{x},\underline{u_q})$ be a point such that $\mathrm{ev}_\rho(\mathscr{X})\in V_m^{j,i,p}$, and take a vector $(\mathfrak{X},\bar{\mathfrak{X}},\mathfrak{Y})$ which is tangent to the codomain of ev_ρ at $\mathrm{ev}_\rho(\mathscr{X})$ with $\mathfrak{X}_k\in T_{x_k}X$, $\bar{\mathfrak{X}}_k\in T_{\bar{x}_k}X$ $(k=0,\dots,q)$, and $\mathfrak{Y}_k\in\mathbf{R}^{d_y}$ $(k=0,\dots,q-1)$. We have to prove that there exist $\phi\in\Gamma_U(X)$, $\eta\in C^r(X,\mathbf{R}^{d_y})$, $\xi\in\mathrm{T}_xX$, $\bar{\xi}\in\mathrm{T}_{\bar{x}}X$, $\nu_l\in\mathrm{T}_{u_l}U$ (for $l=0,\dots,m-1$) and a vector ζ in the tangent space to $V_m^{j,i,p}$ at $\mathrm{ev}_\rho(\mathscr{X})$ such that

$$(\mathfrak{X},\bar{\mathfrak{X}},\mathfrak{Y})=\mathrm{d}(\mathrm{ev}_\rho)(\mathscr{X})\cdot(\phi,\eta,\xi,\bar{\xi},\nu)+\zeta\,. \tag{8}$$

We shall prove this relation with $\xi=0$ and $\nu=0$. We denote by

$$\alpha_0,\dots,\alpha_q,\bar{\alpha}_0,\dots,\bar{\alpha}_q,\beta_0,\dots,0,\dots,0$$

the components of ζ.

Among the $2q+2$ first equations in (8), the ones corresponding to subscripts different from j, i, and p (first $q+1$ equations) or m (last $q+1$ equations) are trivial because the corresponding tangent vectors α_k $(k\neq j,i,p)$ and $\bar{\alpha}_k$ $(k\neq m)$ can be arbitrarily chosen. Thus, we focus on the following four equations:

$$\alpha_j+A_j=\mathfrak{X}_j,\qquad \alpha_i+A_i=\mathfrak{X}_i,\qquad \alpha_p+A_p=\mathfrak{X}_p,\qquad \bar{\alpha}_m+\bar{A}_m+\bar{\xi}_m=\mathfrak{X}_m\,. \tag{9}$$

Here A_l (resp., $\bar{A}_l$), $l=0,\dots,q$, denotes the derivative of x_l^λ (resp., $\bar{x}_l$) with respect to λ evaluated at $\lambda=0$, while $\bar{\xi}_j$ is defined recursively as

$$\bar{\xi}_0=\bar{\xi},\qquad\qquad \bar{\xi}_{k+1}=\mathrm{d}\varphi_T^{u_k}(\bar{x}_k)\cdot\bar{\xi}_k,\qquad\qquad k=0,\dots,2n-1\,. \tag{10}$$

Notice also that, as ζ is a tangent vector to $W_r^{j,i,p}$, necessarily, $\alpha_i = \alpha_p$ and $\alpha_j = \bar{\alpha}_m$. As we are under $\mathbf{C_1}$ configuration, we can apply the preliminary Lemma 8; thus, we can find a vector field ϕ such that $A_i = 0$, while A_p is equal to an arbitrary tangent vector in $\mathrm{T}_{x_p}X$ (notice that we cannot guarantee that $A_j = 0$ because we do not know the position of j with respect to i and p). Taking into account that $\alpha_p = \alpha_i$ and $\bar{\alpha}_m = \alpha_j$, the equations (9) then rewrite as

$$\alpha_j + A_j = \mathfrak{X}_j, \qquad \alpha_i = \mathfrak{X}_i, \qquad \alpha_i + A_p = \mathfrak{X}_p, \qquad \alpha_j + \bar{A}_m + \bar{\xi}_m = \mathfrak{X}_m; \tag{11}$$

a solution to system (11) is then

$$\alpha_i = \mathfrak{X}_{\mathrm{i}}, \qquad A_p = \mathfrak{X}_p - \mathfrak{X}_i, \qquad \alpha_j = \mathfrak{X}_{\mathrm{j}} - A_j, \qquad \bar{\xi}_m = \mathfrak{X}_m - \bar{A}_m - \mathfrak{X}_j + A_j.$$

Notice that once A_p has been chosen, A_j and $\bar{A}_m$ are fixed (they depend on $\mathfrak{X}_p - \mathfrak{X}_i$); moreover, $\bar{\xi}$ can be chosen in such a way that $\bar{\xi}_m$ is equal to an arbitrary tangent vector.

As for the last $2q$ equations, they can be written as

$$\eta_0 - \bar{\eta}_0 - \bar{\chi}_0 = \mathfrak{Y}_0, \qquad \ldots, \qquad \eta_{q-1} - \bar{\eta}_{q-1} - \bar{\chi}_{q-1} = \mathfrak{Y}_{q-1}, \tag{12}$$

where $\bar{\chi}_k = \mathrm{d}h(x_k) \cdot \bar{\xi}_k$, $\eta_k = \eta(x_k)$, and $\bar{\eta}_k = \eta(\bar{x}_k)$ $(k = 0, \ldots, q-1)$. We regard this system as a linear system. As $\bar{x}_m = x_j$ is the only equality between the x_k's and the $\bar{x}_k$'s, we can consider that the unknowns for this system are $\eta_0, \ldots, \eta_{q-1}$ and $\bar{\eta}_0, \ldots, \bar{\eta}_{q-1}$; the matrix of this system then writes as

$$\begin{pmatrix} A & | & -\mathrm{I}_{q\,d_y} + B \end{pmatrix};$$

here $\mathrm{I}_{q\,d_y}$ denotes the $q\,d_y$ dimensional identity matrix, while B is a $d_y \times d_y$ block matrix which is a strictly upper or strictly lower triangular matrix. Thus, $\mathrm{I}_{q\,d_y} - B$ is an upper- or lower-triangular block matrix, the elements of the diagonal being equal to the $d_y \times d_y$ identity matrix; hence, $-\mathrm{I}_{q\,d_y} + B$ is nonsingular, which proves that system (12) admits a solution.

This proves that ev_ρ is transversal to $V_m^{j,i,p}$; we achieve the proof of Proposition 9 as the one of Proposition 5.

4.4. The triplet $(x, \bar{x}, \underline{u_{2n}})$ is under one configuration $\mathbf{C_4}$. We examine now the case of $\mathbf{C_4}$ configuration; in this case, the assumption $d_y \geq d_u + 1$ is no more sufficient.

PROPOSITION 10. *Assume that $d_u = 0$ and $d_y \geq 1$ or $d_u > 0$ and $d_y \geq d_u + 2$ (the number of observations is greater than the number of controls plus one). Denote by $\mathscr{C}_4^r$ the set of pairs $(f, h) \in \Gamma_U(X) \times C^\infty(X, \mathbf{R}^p)$ such that $\Theta_T^{f,h}(x, \underline{u_{2n+1}}) \neq \Theta_T^{f,h}(\bar{x}, \underline{u_{2n+1}})$ whenever the triplet $(x, \bar{x}, \underline{u_{2n}})$ (with $x \neq \bar{x}$) is in configuration $\mathbf{C_4}$. Then $\mathscr{C}_4^r$ contains a residual for the C^r topology.*

Proof of Proposition 10. In the following, we shall say that the lists L_{2n} and $\bar{L}_{2n}$ are under $\mathbf{C_4''}$ configuration if they are under $\mathbf{C_4}$ configuration but not under $\mathbf{C_4'}$ configuration. We shall consider these two subcases separately.

Configuration $\mathbf{C_4'}$. In this case there exist subscripts $0 \leq i < p \leq 2n$ and $0 \leq j < m \leq 2n$ such that $x_i = x_p$ and $\bar{x}_j = \bar{x}_m$; moreover, there exists a subscript $k \in \{i, \ldots, p-1\} \cup \{j, \ldots, m-1\}$ such that $u_k \neq u_{p-1}$. Without loss of generality, we can assume that $m \geq p$.

We choose four subscripts $0 \leq i < p \leq 2n$ and $0 \leq j < m \leq 2n$ $(m \geq p)$ as well as a subscript $k_0 \in \{i, \ldots, p-1\} \cup \{j, \ldots, m-1\}$. We consider the representation ρ defined on $\mathscr{G}_2^U(a) \times C^r(X, \mathbf{R}^{d_y})$ through the mapping $\rho_{f,h}$ as

$$\begin{array}{rccl}\rho_{f,h}: & X^{(2)} \times U^m_{(k_0,p-1)} & \longrightarrow & X^{m+1} \times X^{m+1} \times \mathbf{R}^{m\,d_y} \\ & (x, \bar{x}, \underline{u_m}) & \longmapsto & (x_0, \dots, x_m, \bar{x}_0, \dots, \bar{x}_m, y_0 - \bar{y}_0, \dots, y_{m-1} - \bar{y}_{m-1})\,.\end{array}$$

Together with ρ, we consider the submanifold

$$Z^{i,p}_{j,m} \subset X^{m+1} \times X^{m+1} \times \mathbf{R}^{m\,d_y}$$

defined as the set of those elements

$$(a_0, \dots, a_m, \bar{a}_0, \dots, \bar{a}_m, 0, \dots, 0)$$

such that

- we have the equalities $a_i = a_p$ and $\bar{a}_j = \bar{a}_m$;
- the above equalities are the only ones between the elements of $L_p \cup \bar{L}_m$.

Notice first that the codimension of $Z^{i,p}_{j,m}$ is equal to $2n + m\,d_y$ and is greater than $2n + m\,d_u$, which is greater than the dimension of the domain of $\rho_{f,h}$.

We shall show that ev_ρ is transversal to $Z^{i,p}_{j,m}$. Let $\mathscr{X} \triangleq (f, h, x, \bar{x}, \underline{u_m})$ be a point such that $\mathrm{ev}_\rho(\mathscr{X}) \in Z^{i,p}_{j,m}$, and take a tangent vector $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$ with $\mathfrak{X}_k \in T_{x_k}X$, $\bar{\mathfrak{X}}_k \in T_{\bar{x}_k}X$ $(k = 0, \dots, m)$, and $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$ $(k = 0, \dots, m-1)$. We have to prove that there exist $\phi \in \Gamma_U(X)$, $\eta \in C^r(X, \mathbf{R}^{d_y})$, $\xi \in \mathrm{T}_x X$, $\bar{\xi} \in \mathrm{T}_{\bar{x}}X$, $\nu_k \in \mathrm{T}_{u_k}U$ (for $j = 0, \dots, m-1$), and a vector ζ in the tangent space to $Z^{i,p}_{j,m}$ at $\mathrm{ev}_\rho(\mathscr{X})$ such that

$$(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = \mathrm{d}(\mathrm{ev}_\rho)(f, h, x, \bar{x}, \underline{u_m}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu) + \zeta\,. \tag{13}$$

We shall prove this relation with $\xi = 0$, $\bar{\xi} = 0$, and $\nu = 0$. We denote by

$$\alpha_0, \dots, \alpha_m, \bar{\alpha}_0, \dots, \bar{\alpha}_m, 0, \dots, 0$$

the components of ζ; notice that, as ζ is a tangent vector to $Z^{i,p}_{j,m}$, we have $\alpha_i = \alpha_p$ and $\bar{\alpha}_j = \bar{\alpha}_m$.

To prove that (13) admits a solution, the reasoning is analogous to the one made in the proofs of the previous propositions, but here we have to apply twice Lemma 8. Hereafter, given a vector field ϕ, we denote by $x_i^{\lambda,\phi}$, the sequence defined recursively as

$$x_0^{\lambda,\phi} = x_0, \qquad\qquad x_{i+1}^{\lambda,\phi} = \varphi_T^{u_i,\lambda}(x_i^{\lambda,\phi}),$$

where $\varphi_t^{u_i,\lambda}$ denotes the flow related to the vector field $f(\cdot, u_i) + \lambda\phi$. According to Lemma 8, there exists a vector field ϕ_0 such that the derivatives of x_i^{λ,ϕ_0}, $\bar{x}_j^{\lambda,\phi_0}$, and $\bar{x}_m^{\lambda,\phi_0}$ with respect to λ are all zero, while the derivative of x_p^{λ,ϕ_0} can be arbitrarily chosen. As there exists k_0 such that $u_{k_0} \neq u_{p-1}$, we can also apply this lemma by replacing u_{p-1} by u_{k_0} in the lemma. For example, if $j \le k_0 < m$ and assuming without loss of generality that k_0 is the greatest subscript less than m such that $u_{k_0} \neq u_{p-1}$, we deduce from Lemma 8 that there exists a vector field ϕ_1 such that the derivatives of x_i^{λ,ϕ_1}, x_p^{λ,ϕ_1}, and $\bar{x}_j^{\lambda,\phi_1}$ with respect to λ are all zero, while the derivative of $\bar{x}_{k_0+1}^{\lambda,\phi_1}$ can be arbitrarily chosen. Noticing that

$$\bar{x}_m^{\lambda,\phi_1} = \begin{cases} \bar{x}_{k_0+1}^{\lambda,\phi_1} & \text{if } k_0 = m-1\,, \\ \varphi^{u_{p-1}}_{(m-k_0)T}(\bar{x}_{k_0}^{\lambda,\phi_1}) & \text{if } k_0 < m-1\,, \end{cases}$$

we see that the derivative of $\bar{x}_m^{\lambda,\phi_1}$ with respect to λ can be arbitrarily chosen.

We chose now the vector field $\phi \in \Gamma_U(X)$ as follows:

- $\phi(\cdot, u_{p-1}) = \phi_0$ and $\phi(\cdot, u_{k_0}) = \phi_1$;
- $\phi(\cdot, u_k) \equiv 0$ if $u_k \notin \{u_{p-1}, u_{k_0}\}$.

Clearly, the derivatives of $x_i^{\lambda,\phi}$ and $\bar{x}_j^{\lambda,\phi}$ with respect to λ are zero, while the derivatives of $x_p^{\lambda,\phi}$ and $\bar{x}_m^{\lambda,\phi}$ can be arbitrarily chosen.

As in the previous configurations, as regards the first $2m+2$ equations in (13), we have to consider only the four following ones:

$$\alpha_i + A_i = \mathfrak{X}_i, \qquad \alpha_p + A_p = \mathfrak{X}_p, \qquad \bar{\alpha}_j + \bar{A}_j = \bar{\mathfrak{X}}_j, \qquad \bar{\alpha}_m + \bar{A}_m = \bar{\mathfrak{X}}_m. \tag{14}$$

Here the A_k's (resp., the $\bar{A}_k$'s) denote the derivatives of the $x_k^{\lambda,\phi}$ (resp., of the $\bar{x}_k^{\lambda,\phi}$) with respect to λ, so we have $A_i = 0$ and $\bar{A}_j = 0$. Notice also that, from the definition of $Z_{j,m}^{i,p}$, it follows that $\alpha_i = \alpha_p$ and $\bar{\alpha}_j = \bar{\alpha}_m$. Taking into account these equalities, the solution to the equations (14) is then

$$\alpha_i = \mathfrak{X}_i, \qquad A_p = \mathfrak{X}_p - \mathfrak{X}_j, \qquad \bar{\alpha}_j = \bar{\mathfrak{X}}_j, \qquad \bar{A}_m = \bar{\mathfrak{X}}_m - \bar{\mathfrak{X}}_j\,.$$

As regards the last m equalities in (13), the proof is the same as the one of Proposition 9: The two lists L_{m-1} and $\bar{L}_{m-1}$ are disjoint, and the elements of $\bar{L}_{m-1}$ are pairwise distinct, so one can find a function $\eta \in C^\infty(X, \mathbf{R}^{d_y})$ such that $\eta(x_k) = 0$ for $k = 0, \dots, m-1$, while the values of η at $\bar{x}_k$ ($k = 0, \dots, m-1$) can be chosen arbitrarily.

Configuration $\mathbf{C_4''}$. In this case, we have $u_i = \dots u_{p-1} = u_j = \dots u_{m-1}$; the equalities $x_i = x_p$ and $\bar{x}_j = \bar{x}_m$ then imply that the trajectories of the vector field $f(\cdot, u_{p-1})$ are periodic.

We choose some subscripts $0 \le i < p \le 2n$ and $0 \le j < m \le 2n$; without loss of generality, we assume that $m \ge p$. We consider first t representation ρ defined on $\mathscr{G}_2^U(a) \times C^r(X, \mathbf{R})$ through the mapping $\rho_{f,h}$ defined as

$$\begin{array}{llll} \rho_{f,h}: & X^{(2)} \times U^m_{(i,p,j,m)} \times \mathbf{R}_+^* & \longrightarrow & X^{m+1} \times X^{m+1} \times \mathbf{R}^{m\,d_y} \\ & (x, \bar{x}, \underline{u_m}, t) & \longmapsto & (x_0, \dots, x_{p-1}, \varphi_t^{u_{p_1}}(x_i), x_{p+1}, \dots, x_m, \\ & & & \quad \bar{x}_0, \dots, \bar{x}_m, y_0 - \bar{y}_0, \dots, y_{m-1} - \bar{y}_{m-1})\,, \end{array}$$

where $U^m_{(i,p,j,m)}$ is the submanifold of U^m defined as the set of those $\underline{u_m}$ such that $u_i = \cdots = u_{p-1} = u_j = \cdots = u_{m-1}$.

Together with ρ, we consider the submanifold $Z_{j,m}^{i,p}$ defined as in the previous case. We shall prove that ev_ρ is transversal to $Z_{j,m}^{i,p}$. Let $\mathscr{X} \triangleq (f, h, x, \bar{x}, \underline{u_m}, t)$ be a point such that $\mathrm{ev}_\rho(\mathscr{X}) \in Z_{j,m}^{i,p}$, and take a tangent vector $(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y})$ with $\mathfrak{X}_k \in T_{x_k}X$, $k \in \{0, \dots, m\} \setminus \{p\}$, $\mathfrak{X}_p \in \mathrm{T}_{x_i}X$, $\bar{\mathfrak{X}}_k, \in \mathrm{T}_{\bar{x}_k}X$, $k = 0, \dots, m$, and $\mathfrak{Y}_k \in \mathbf{R}^{d_y}$, $k = 0, \dots, m-1$. Notice that $\mathrm{ev}_\rho(\mathscr{X}) \in Z_{j,m}^{i,p}$ implies that x_i and $\bar{x}_j$ belong to periodic trajectories of the vector field $f(\cdot, u_{p-1})$.

We have to prove that there exist $\phi \in \Gamma_U(X)$, $\eta \in C^r(X, \mathbf{R}^{d_y})$, $\xi \in \mathrm{T}_x X$, $\bar{\xi} \in \mathrm{T}_{\bar{x}} X$, $\nu_k \in \mathrm{T}_{u_k} U$, $\tau \in \mathbf{R}$, and a vector ζ in the tangent space to $W_{j,m}^{i,p}$ at $\mathrm{ev}_\rho(\mathscr{X})$ such that

$$(\mathfrak{X}, \bar{\mathfrak{X}}, \mathfrak{Y}) = \mathrm{d}(\mathrm{ev}_\rho)(\mathscr{X}) \cdot (\phi, \eta, \xi, \bar{\xi}, \nu, \tau) + \zeta\,. \tag{15}$$

We shall prove this relation with $\nu = 0$ and $\bar{\xi} = 0$. We denote by

$$\alpha_0, \dots, \alpha_m, \bar{\alpha}_0, \dots, \bar{\alpha}_m, 0, \dots, 0$$

the components of ζ; notice that, as ζ is a tangent vector to $Z_{j,m}^{i,p}$, we have $\alpha_i = \alpha_p$ and $\bar{\alpha}_j = \bar{\alpha}_m$.

As in the previous cases, in order to prove that the first $2m+2$ equations in (15) can be satisfied, it is sufficient to focus our attention to the four following equations:

$$\alpha_i + \xi_i + A_i = \mathfrak{X}_i, \qquad \alpha_p + \xi_p + A_p + \tau f(x_i, u_{p-1}) = \mathfrak{X}_p,$$
$$\bar\alpha_j + \bar A_j = \bar{\mathfrak{X}}_j, \qquad \bar\alpha_m + \bar A_m = \bar{\mathfrak{X}}_m,$$

The notations are the same as in the previous cases except for A_p and ξ_p:

$$A_p = \left.\frac{\mathrm{d}\varphi_t^{u_{p-1},\lambda}(x_i)}{\mathrm{d}\lambda}\right|_{\lambda=0}, \qquad \xi_p = \mathrm{d}\varphi_t^{u_{p-1}}(x_i)\cdot\xi_i\,.$$

We can apply Lemma 8: There exists ϕ such that $\bar A_j = 0$, while $\bar A_m$ can be arbitrarily chosen. Here we cannot ensure that we also have $A_i = A_p = 0$ because x_i could belong to the periodic trajectory of $f(\cdot, u_{p-1})$ passing through $\bar x_j$, so we first choose $\bar\alpha_j = \bar{\mathfrak{X}}_j$ and $\bar A_m = \bar{\mathfrak{X}}_m - \bar{\mathfrak{X}}_j$. Now as $\varphi_t^{u_{p-1}}(x_i) = x_i$, we have $t = q\,\pi_0$, where π_0 denotes the prime period of the periodic trajectory of $f(\cdot, u_{p-1})$ passing through x_i; thus, $\mathrm{d}\varphi_t^{u_{p-1}}(x_i) = (\mathrm{d}\varphi_{\pi_0}^{u_{p-1}}(x_i))^q$. As before, due to the fact that f belongs to $\mathscr{G}_2^U(a)$, the linear mapping

$$(\xi_i, \tau) \mapsto \big((\mathrm{d}\varphi_{\pi_0}^{v}(x_i))^q - \mathrm{Id}\big)\cdot\xi_i + \tau f(x_i, v)$$

is onto. Thus, we can find ξ_i and τ such that

$$\big((\mathrm{d}\varphi_{\pi_0}^{v}(x_i))^q - \mathrm{Id}\big)\cdot\xi_i + \tau f(x_i, v) = \mathfrak{X}_p - \mathfrak{X}_i + A_p - A_i\,.$$

We take also $\alpha_i = \mathfrak{X}_i - A_i$; with these choices of α_i, ξ_i, and τ, we see that the two first equations are also satisfied.

As regards the last m equations in (15), we argue as in the previous case.

At this point, the application of the transversal density theorem shows that $\mathscr{R}_4$, the set of pairs (f, h) in $\mathscr{G}_2^U(a)\times C^r(X, \mathbf{R}^{d_y})$ such that $\rho_{f,h}$ is transversal to the finite set of submanifolds $Z_{j,m}^{i,p}$, is a residual. Now, we have to compute the codimension of $Z_{j,m}^{i,p}$; it is equal to $2n + m\,d_y$ and is greater than or equal to $2n + m\,d_u + m$. This codimension is greater than the dimension of the domain of $\rho_{f,h}$ if $m \geq 2$ or $d_y \geq d_u + 2$. In this case, to be transversal to $Z_{j,m}^{i,p}$ means nonmembership, and we can conclude the proof of Proposition 10 as for the previous propositions. If $m = 1$, $d_u = 0$, and $d_y \geq 1$, ev_ρ is still transversal to submanifold $Z_{j,m}^{i,p}$, but $\mathrm{codim}(Z_{0,1}^{0,1})$ is then equal to the dimension of the domain of $\rho_{f,h}$, so we need an additional argument to conclude in this case. Hereafter, we shall see that, in this particular case, although the codimension of $Z_{0,1}^{0,1}$ is equal to the dimension of the domain of $\rho_{f,h}$, transversality implies nonmembership.

Case where $m = 1$, $d_u = 0$, and $d_y \geq 1$. Take $(f, h) \in \mathscr{R}_4^r$; then $\rho_{f,h}$ is transversal to $W_{0,1}^{0,1}$. Assume that $x_0 \neq \bar x_0$ are two points of X such that

- there exists some $t > 0$ such that $\varphi_t(x_0) = x_0$, $\varphi_T^u(\bar x_0) = \bar x_0$;
- $h(x_0) = h(\bar x_0)$.

Then there exist $\xi_0 \in \mathrm{T}_{x_0}X$, $\bar\xi_0 \in \mathrm{T}_{\bar x_0}X$, $\tau \in \mathbf{R}$, α_0 and $\bar\alpha_0$ in the tangent spaces to X at x_0 and $\bar x_0$ such that the following equations are satisfied:

$$\text{(16)}\qquad \begin{cases} \alpha_0 + \xi_0 = \mathfrak{X}_0, \quad \alpha_0 + \mathrm{d}\varphi_t(x_0)\cdot\xi_0 + \tau f(x_0, u) = \mathfrak{X}_1,\\ \bar\alpha_0 + \bar\xi_0 = \bar{\mathfrak{X}}_0, \qquad \bar\alpha_0 + \mathrm{d}\varphi_T(\bar x_0)\cdot\bar\xi_0 = \bar{\mathfrak{X}}_1\\ \mathrm{d}h(x_0)\cdot\xi_0 - \mathrm{d}h(\bar x_0)\cdot\bar\xi_0 = \mathfrak{Y}\end{cases}$$

for all $\mathfrak{X}_0$, $\mathfrak{X}_1$, $\bar{\mathfrak{X}}_0$, $\bar{\mathfrak{X}}_1$, and $\mathfrak{Y}$ vectors tangent to the appropriate spaces. Clearly, the four first equations in this system are equivalent to the two following ones:

$$(\mathrm{d}\varphi_t(x_0) - \mathrm{Id}) \cdot \xi_0 + \tau f(x_0, u) = \mathfrak{X}_1 - \mathfrak{X}_0,$$
$$(\mathrm{d}\varphi_T(\bar{x}_0) - \mathrm{Id}) \cdot \bar{\xi}_0 = \bar{\mathfrak{X}}_1 - \bar{\mathfrak{X}}_1 .$$

As 1 is an eigenvalue of the linear mapping $\mathrm{d}\varphi_T(\bar{x}_0)$, clearly the second equation of this system cannot be satisfied. This implies that if $\varphi_T(x_0) = x_0$ and $\varphi_T(\bar{x}_0) = \bar{x}_0$, then the point $\rho_{f,h}(x_0, \bar{x}_0)$ cannot belong to $W_{0,1}^{0,1}$, which means that we must have $h(x_0) \neq h(\bar{x}_0)$. This achieve the proof of Proposition 10. ∎

5. The case $d_u \geq 1$ and $d_y = d_u + 1$. Counterexample. We shall exhibit here a simple counterexample which shows that if $d_y = 2$ and $d_u = 1$, then the conclusion of our main result (Theorem 4) is no more true. That it to say, we exhibit a pair (f_0, h_0) such that for every (f, h) in some neighborhood of (f_0, h_0), the related mapping $\Theta_T^{f,h}$ is not injective. We recall hereafter, the isotopy theorem [1], which will be used to prove some optimality of our main result.

THEOREM 11 (transversal isotopy theorem). *Let $\mathscr{A}$, Z, and Y be C^{r+1} manifolds ($r \geq 1$), $\rho : \mathscr{A} \to C^{r+1}(Z, Y)$ a C^{r+1} representation, $W \subset Y$ a submanifold, and $a_0 \in \mathscr{A}$ a point. For $a \in \mathscr{A}$, let $W_a = \rho_a^{-1}(W)$. Assume that*
1. *W is closed in Y;*
2. *Z is compact and C^{r+3};*
3. *ρ_{a_0} is transversal to W.*

Then there is an open neighborhood N of a_0 in $\mathscr{A}$ such that, for $a \in N$, there is a C^r diffeomorphism $F_a : Z \to Z$ such that $F_a(W_{a_0}) = W_a$ and F_a is C^r isotopic to the identity.

We shall apply the transversal isotopy Theorem 11 to the following situation: We take
- $\mathscr{A} = \Gamma_U(X) \times C^\infty(X, \mathbf{R}^{d_y})$, where X and U are compact manifolds, $\dim U = d_u > 0$ with $d_y = d_u + 1$;
- $Z = X^2 \times U \times S^1$;
- $Y = X^3 \times \mathbf{R}^{d_y}$.

We consider also a representation ρ which is slightly different from the one which has been used in the proof of Proposition 10; we define this representation through $\rho_{f,h}$ as

$$\begin{array}{rccl} \rho_{f,h} : & X^2 \times U \times S^1 & \longrightarrow & X^3 \times \mathbf{R}^{d_y} \\ & (x, \bar{x}, u, s) & \longmapsto & (x, \varphi_T^u(x), \varphi_{\gamma(s)}^u(\bar{x}), h(x) - h(\bar{x})); \end{array}$$

here $\gamma \in C^\infty(S^1, \mathbf{R})$.

The submanifold $W \subset X^3 \times \mathbf{R}^{d_y}$ is then defined as

$$W = \{ (z_1, z_2, z_3, 0) \in X^3 \times \mathbf{R}^{d_y} \mid z_1 = z_2 = z_3 \} .$$

Consider a pair (f_0, h_0) such that ρ_{f_0,h_0} is transversal to W and such that W_{f_0,h_0} is nonempty. Applying Theorem 11, we deduce that there exists a neighborhood N of (f_0, h_0) such that if $(f, h) \in N$, $W_{f,h} = F(W_{f_0,h_0})$ with F a diffeomorphism from Z to Z. Thus, $W_{f,h}$ is nonempty. Notice that we can assume that $\rho_{f,h}$ is transversal to W for every pair $(f, h) \in N$; this is a direct consequence of the openness of transversal intersection theorem [1] applied to this situation with $K = Z$. Let $(x, \bar{x}, u, s)$ be an element of $W_{f,h}$; we shall show that $x \neq \bar{x}$: Arguing by contradiction, we shall see that if we have the equality $x = \bar{x}$, then $\rho_{f,h}$ cannot be transversal to W at $(x, \bar{x}, u, s)$. We introduce some notations:

$$A = \mathrm{d}\varphi_T^u(x), \qquad \bar{A} = \mathrm{d}\varphi_{\gamma(s)}^u(\bar{x}),$$
$$B = \left.\frac{\partial \varphi_T^v(x)}{\partial v}\right|_{v=u}, \qquad \bar{B} = \left.\frac{\partial \varphi_{\gamma(s)}^v(\bar{x})}{\partial v}\right|_{v=u},$$
$$C = \mathrm{d}h(x), \qquad \bar{C} = \mathrm{d}h(\bar{x}).$$

Consider now the following "matrix" M:

$$M = \begin{pmatrix} A - \mathrm{Id} & 0 & B & 0 \\ -\mathrm{Id} & \bar{A} & \bar{B} & f(x,u) \\ C & -\bar{C} & 0 & 0 \end{pmatrix}.$$

Arguing as in the proof of Proposition 10, we see that the transversality of $\rho_{f,h}$ to W at $(x, \bar{x}, u, s)$ is equivalent to the invertibility of the square matrix M. Since we assume that $x = \bar{x}$, we have $C = \bar{C}$, which implies that the determinant of M is equal to the one of the following matrix M':

$$M' = \begin{pmatrix} A - \mathrm{Id} & 0 & B & 0 \\ \bar{A} - \mathrm{Id} & \bar{A} & \bar{B} & f(x,u) \\ 0 & -\bar{C} & 0 & 0 \end{pmatrix}.$$

Now as $f(x,u)$ belongs to the kernels of $A - \mathrm{Id}$ and $\bar{A} - \mathrm{Id}$ (because $\varphi_{\gamma(t)}^u(\bar{x}) = x$), the vector $(f(x,u), 0, 0, 0)^{\mathrm{T}}$ belongs to the kernel of M', which implies that the determinant of M' is zero: We have reached a contradiction. As a consequence, if $(f,h) \in N$, then there exists $x \neq \bar{x}$, u and s such that $h(x) = h(\bar{x})$; moreover, the trajectory of $f(\cdot, u)$ passing through x is periodic, and $\bar{x}$ belongs to this trajectory, so we have $\varphi_{kT}^u(x) = x$ and $\varphi_{kT}^u(\bar{x}) = \bar{x}$.

We give now an explicit example of a pair (f_0, h_0) such that ρ_{f_0,h_0} is transversal to W. In what follows, for the sake of simplicity and without loss of generality, we assume that $T = 2\pi$. The manifold X will be equal to the circle S^1, and the set of controls U will also be equal to the circle S^1. We denote by u_1 and u_2 (resp., s_1 and s_2) the components of u (resp., of s) and consider the vector field

$$f_0(x,u) = R_1 \cdot x + u_1 \, R_2 \cdot x$$

with R_1 and R_2 the skew-symmetric matrices

$$R_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix},$$

with $\beta \in (0,1)$.

Mapping h_0 is defined by

$$\begin{array}{rccl} h_0 : & X & \longrightarrow & \mathbf{R}^2 \\ & x & \longmapsto & (x_1, x_1 x_2), \end{array}$$

while function γ is defined by

$$\gamma(s) = \left(-\frac{1}{2} + \frac{1}{4}\, s_1\right) 2\pi.$$

The set $\boldsymbol{W_{f_0,h_0}}$. Let $x = (x_1, x_2)$ and $\bar{x} = (\bar{x}_1, \bar{x}_2)$ be two points in S^1 and $u = (u_1, u_2) \in S^1$ a control and $s \in S^1$; assume that $\rho_{f_0,h_0}(x, \bar{x}, u, s) \in W$. Letting $z = x_1 + ix_2 \in \mathbf{C}$, the equality $\varphi_T^u(x) = x$ is equivalent to $e^{i(1+u_1\beta)2\pi} z = z$, which is equivalent to $1 + u_1\beta = k$ with $k \in \mathbf{Z}$. Now, as $\beta \in (0,1)$, we have $|u_1\beta| < 1$, and so the equality $u_1\beta = k - 1 \in \mathbf{Z}$ is possible if and only if $u_1 = 0$. From the definition of h, we can easily see that the equality $h(x) = h(\bar{x})$ amounts to $x = \bar{x}$ or $x_1 = \bar{x}_1 = 0$.

If $x = \bar{x}$, as $u_1 = 0$, the equality $\varphi^u_{\gamma(s)}(\bar{x}) = x$ is possible only if $\gamma(s) \in \mathbf{Z}$, but from the definition of γ, we have $\gamma(s) \in [-3/4, -1/4]$. Thus, $x \neq \bar{x}$ and $x_1 = \bar{x}_1 = 0$, so, taking into account that $u_1 = 0$, the equality $\varphi^u_{\gamma(s)}(\bar{x}) = x$ is equivalent to the following ones:

$$-\sin\bigl(\gamma(s)\bigr)\bar{x}_2 = 0, \qquad\qquad \cos\bigl(\gamma(s)\bigr)\bar{x}_2 = x_2 = -\bar{x}_2\,.$$

These two equalities are true if and only if $\gamma(s) = (2k+1)\pi$ with $k \in \mathbf{Z}$, which is equivalent to $s_1 = 4k + 4$; as $|s_1| \leq 1$, this is possible only if $s_1 = 0$.

In conclusion, W_{f_0,h_0} is the set consisting of the eight elements

$$W_{f_0,h_0} = \bigl\{\bigl((0,\varepsilon_0),(0,-\varepsilon_0),(0,\varepsilon_1),(0,\varepsilon_2)\bigr)\bigr\},$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{-1, 1\}$.

The transversality of $\boldsymbol{\rho_{f_0,h_0}}$. Take $(x, \bar{x}, u, s) \in X \times X \times U \times S^1$ an element such that $\rho_{f_0,h_0}(x, \bar{x}, u, s)$ belongs to W. Thus, we have

$$x = (0,\varepsilon_0), \qquad \bar{x} = (0,-\varepsilon_0), \qquad u = (0,\varepsilon_1), \qquad s = (0,\varepsilon_2) \tag{17}$$

with $\varepsilon_i \in \{-1, 1\}$ $(i = 0, 1, 2)$. Let $(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{Y})$ be a tangent vector to $X^3 \times \mathbf{R}^2$ at $\rho_{f_0,h_0}(x, \bar{x}, u, s)$. Thus, $\mathfrak{X}_1$, $\mathfrak{X}_2$, and $\mathfrak{X}_3$ are tangent to S^1 at $(0, \varepsilon_0)$, while $\mathfrak{Y}$ is a vector in $\mathbf{R}^2$. We write a tangent vector to submanifold W as $\zeta = (\alpha, \alpha, \alpha, 0)$, where α is a tangent vector to S^1 at $(0, \varepsilon_0)$. We have to prove that there exists ξ (resp., $\bar{\xi}$) in the tangent space to S^1 at $(0, \varepsilon_0)$ (resp., $(0, -\varepsilon_0)$) as well as ν a vector in the tangent space to S^1 at $(0, \varepsilon_1)$ and σ a vector tangent to S^1 at s such that the following equalities are satisfied:

$$\mathfrak{X}_1 = \xi + \alpha \tag{18}$$

$$\mathfrak{X}_2 = \mathrm{d}\varphi^u_T(x)\cdot\xi + \frac{\partial\varphi^u_T(x)}{\partial u}\cdot\nu + \alpha \tag{19}$$

$$\mathfrak{X}_3 = \mathrm{d}\varphi^u_{\gamma(s)}(\bar{x})\cdot\bar{\xi} + \frac{\partial\varphi^u_{\gamma(s)}(\bar{x})}{\partial u}\cdot\nu + \frac{\partial\varphi_{\gamma(s)}}{\partial s}\cdot\sigma + \alpha \tag{20}$$

$$\mathfrak{Y} = \mathrm{d}h(x)\cdot\xi - \mathrm{d}h(\bar{x})\cdot\bar{\xi}\,. \tag{21}$$

We examine first the fourth equation (21); taking into account that $\xi = (\xi_1, 0)^{\mathrm{T}}$ and $\bar{\xi} = (\bar{\xi}_1, 0)^{\mathrm{T}}$, it writes as

$$\begin{pmatrix} 1 & 0 \\ \varepsilon_0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ -\varepsilon_0 & 0 \end{pmatrix}\begin{pmatrix} \bar{\xi}_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathfrak{Y}_1 \\ \mathfrak{Y}_2 \end{pmatrix}.$$

Clearly, the solution of this equation is

$$\xi_1 = \frac{\varepsilon_0\,\mathfrak{Y}_2 + \mathfrak{Y}_1}{2}\,, \qquad\qquad \bar{\xi}_1 = \frac{\varepsilon_0\,\mathfrak{Y}_2 - \mathfrak{Y}_1}{2}\,. \tag{22}$$

Taking into account that $\mathfrak{X}_i = (\mathfrak{X}_i^1, 0)^{\mathrm{T}}$ and $\alpha = (\alpha_1, 0)^{\mathrm{T}}$, the solution of the first equation (18) is then given by (22) and by the equality

$$\alpha_1 = \mathfrak{X}_1^1 - \frac{\varepsilon_0\,\mathfrak{Y}_2 + \mathfrak{Y}_1}{2}\,. \tag{23}$$

Concerning the second equation (19), we notice that $\mathrm{d}\varphi^u_T(x) = \mathrm{Id}$ and that

$$\frac{\partial \varphi_T^u(x)}{\partial u} = \begin{pmatrix} -2\pi\,\beta\,\varepsilon_0 & 0 \\ 0 & 0 \end{pmatrix},$$

so the solution of equation (19) is given by (22) and (23) and by

$$\nu_1 = -\frac{\mathfrak{X}_2^1 - \mathfrak{X}_1^1}{2\pi\,\beta\,\varepsilon_0}\,. \tag{24}$$

So far, the values of ξ, $\bar{\xi}$, and ν are fixed, therefore showing that equation (20) comes down showing that the third term in the right-hand member of (20) can take arbitrary value. This third term writes as

$$\frac{\partial \varphi_{\gamma(s)}}{\partial s} \cdot \sigma = \frac{\pi}{2}\,\sigma_1 f(\varphi^u_{\gamma(s)}(x)) = \frac{\pi}{2}\,\sigma_1 \begin{pmatrix} \varepsilon_0 \\ 0 \end{pmatrix}\,,$$

obviously; thanks to a suitable choice of σ_1, this expression can be made equal to an arbitrary tangent vector $\tilde{\mathfrak{X}}_3$ of S^1 at x.

This achieves the proof of the transversality of ρ_{f_0,h_0} to W.

Appendix. Proof of Lemma 8.

The derivative of x_p^λ with respect to λ can be written as

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda} x_p^\lambda\right|_{\lambda=0} = A_p + B_p\,, \tag{25}$$

where A_p is zero or a sum of terms of the form $\delta_{j_k}^p(J_{j_k})$, where J_{j_k} is an integral that can be written as

$$J_{j_k} = \int_0^T \mathrm{d}\varphi_\sigma^{u_{p-1}}(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{p-1}}(x_{j_k}), u_{p-1})\mathrm{d}\sigma\,, \tag{26}$$

where the subscripts j_k in A_p are less than or equal to $p-1$ and are such that $u_{j_k} = u_{p-1}$; moreover, $\delta_{j_k}^p$ is the mapping defined as

$$\delta_{j_k}^a = \begin{cases} \mathrm{Id} & \text{if } j_k = p-1\,, \\ \mathrm{d}\big(\varphi_T^{u_{p-1}} \circ \cdots \circ \varphi_T^{u_{j_k+1}}\big)(x_{j_k+1}) & \text{if } j_k < p-1\,. \end{cases}$$

As for the term B_p, it is zero or the sum of terms $\delta_{j_k}^p(J'_{j_k})$, where the J'_{j_k} are integrals that we write as

$$J'_{j_k} = \int_0^T \mathrm{d}\varphi_\sigma^{u_{j_k}}(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k})) \cdot \phi(\varphi_{T-\sigma}^{u_{j_k}}(x_{j_k}), u_{j_k})\mathrm{d}\sigma$$

with $u_{j_k} \neq u_{p-1}$.

We write the derivative of x_i^λ and, if we are under $\mathbf{C_2}$, $\mathbf{C_3'}$, or $\mathbf{C_4'}$ configurations, the ones of $\bar{x}_j^\lambda$ and $\bar{x}_m^\lambda$, in the same way; we denote by $\bar{J}_{j_k}$ and $\bar{J}'_{j_k}$ the integrals appearing in the derivatives of $\bar{x}_j^\lambda$ and $\bar{x}_m^\lambda$. We shall see that ϕ can be chosen such that

- the terms B_p, B_i, $\bar{B}_j$, and $\bar{B}_m$ are zero;
- all the integrals $\bar{J}_{j_k}$ that occur in the terms $\bar{A}_j$ and $\bar{A}_m$ are zero;
- all the integrals J_{j_k} are zero except the one corresponding to the subscript $j_k = p-1$, which can be arbitrarily chosen.

To this end, the perturbation ϕ that we shall consider will be zero outside some neighborhood of u_{p-1}; to be more precise, given ϕ_0 a vector field defined on X, it is possible to find $\phi \in \Gamma_U(X)$ such that

- $\phi(\cdot, u_{p-1}) = \phi_0$;
- for $j = 0, \ldots, 2n$, $\phi(\cdot, u_j) \equiv 0$ as soon as $u_j \neq u_{p-1}$.

With this choice of ϕ, we have $B_p = 0$, and $B_i = 0$ as well as $\bar{B}_j = 0$, and $\bar{B}_m = 0$ if we are under $\mathbf{C_2}$, $\mathbf{C'_3}$, or $\mathbf{C'_4}$ configuration.

We split the points x_{j_k} (resp., $\bar{x}_{j_k}$) that appear under the integrals J_{j_k} (resp., $\bar{J}_{j_k}$) into two classes: The first class, denoted by $\mathscr{P}_1$, contains the points x_{j_k} and $\bar{x}_{j_k}$ that belong to the trajectory of the vector field $f(\cdot, u_{p-1})$ passing through x_{p-1} (and so $\mathscr{P}_1$ contains the point x_{p-1} itself); The second class, denoted by $\mathscr{P}_2$, contains the points x_{j_k} and $\bar{x}_{j_k}$ that do not belong to this trajectory. Denote by $\mathscr{T}_1$ and $\mathscr{T}_2$ the union of these trajectories restricted to the interval $[0, T]$, namely,

$$\mathscr{T}_i = \{ \varphi_t^{u_{p-1}}(z) \mid t \in [0,T], z \in \mathscr{P}_i \} \quad i = 1, 2 \,.$$

The sets $\mathscr{T}_1$ and $\mathscr{T}_2$ being disjoint and compact, let $\mathscr{U}_1$ and $\mathscr{U}_2$ be two open sets of X such that $\mathscr{T}_i \subset \mathscr{U}_i$ $(i = 1, 2)$ and $\mathscr{U}_1 \cap \mathscr{U}_2 = \varnothing$. In the following, we shall assume that the vector field ϕ_0 is zero when restricted to $\mathscr{U}_2$; this implies that the integrals J_{j_k} (resp., $\bar{J}_{j_k}$) such that x_{j_k} (resp., $\bar{x}_{j_k}$) belongs to $\mathscr{P}_2$ are zero. Denote by $j_1, \ldots, j_a$ $(0 \leq j_1 < \cdots < j_a = p-1)$ the subscripts such that x_{j_k} belongs to $\mathscr{P}_1$, and let $t_1, \ldots, t_a$ be such that $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$ $(k = 1, \ldots, a)$. Denote also by $l_1 < \cdots < l_b < \max(m, j)$ the subscripts such that $\bar{x}_{l_k}$ belongs to $\mathscr{P}_1$, and let $t'_1, \ldots, t'_b$ be such that $\bar{x}_{l_k} = \varphi_{t'_k}^{u_{p-1}}(x_{p-1})$ $(k = 1, \ldots, b)$.

Notice that, except when $j_k = j_a$, we cannot have $t_k = 0$ since this would imply that $x_{p-1} = x_{j_k}$ with $j_k < p-1$, which is not possible under $\mathbf{C_1}$–$\mathbf{C_4}$ configuration. Also, all the t'_k $(k = 1, \ldots, b)$ are nonzero because if there existed a subscript k such that $t'_k = 0$, then, as $u_{l_k} = u_{p-1}$, we would have $\bar{x}_{l_k+1} = x_p$, which implies that we are under $\mathbf{C_2}$ or $\mathbf{C'_3}$ configuration and that $l_k + 1 = m$. Thus, we have $\bar{x}_{m-1} = x_{p-1}$; if we are under $\mathbf{C'_2}$ configuration, then this implies $(m-1) - (p-1) = i - j$, and so $m - p = i - j$, which contradicts the definition of the $\mathbf{C_2}$ configuration. If we are under $\mathbf{C'_3}$ configuration, we found a pair $(i', j') \triangleq (p-1, m-1)$ such that $x_{i'} = \bar{x}_{j'}$ with $i' - j' = p - m$ and $i' < p$, which contradicts the definition of $\mathbf{C_3}$ configuration.

The trajectory related to the vector field $f(\cdot, u_{p-1})$ passing through x_{p-1} may be periodic or aperiodic; we have to distinguish between these two cases.

The trajectory passing through $\boldsymbol{x_{p-1}}$ *is not periodic.* Assume that the trajectory of the vector field $f(\cdot, u_{p-1})$ passing through x_{p-1} is not periodic. Taking into account that, with our notations, $\phi(\cdot, u_{p-1}) = \phi_0$, the terms J_{j_k} appearing in A_p and (possibly) in A_i can also be written as

$$J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{T-t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma \,.$$

In the same way, the integrals $\bar{J}_{l_k}$ appearing (possibly) in $\bar{A}_m$ and $\bar{A}_j$ write as

$$\bar{J}_{l_k} = \mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p) \cdot \int_{-t'_k}^{T-t'_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1}) \mathrm{d}\sigma \,.$$

We set

$$\begin{aligned} T_{\min} &= \min\bigl(\{ t_k \mid k = 1, \ldots, a \} \cup \{ t'_k \mid k = 1, \ldots, b \}\bigr) \\ T_{\max} &= \max\bigl(\{ T + t_k \mid k = 1, \ldots, a \} \cup \{ T + t'_k \mid k = 1, \ldots, b \}\bigr), \end{aligned}$$

and we introduce the set $\mathcal{T} = \{ \varphi_t^{u_{p-1}}(x_{p-1}) \mid T_{\min} \leq t \leq T_{\max} \}$. For $z = \varphi_t^{u_{p-1}}(x_{p-1})$ in $\mathcal{T}$, we define $\phi_0(z)$ as

$$\phi_0(z) = \mu(t)\,\mathrm{d}\varphi_{t-T}^{u_{p-1}}(x_p)\cdot \mathfrak{X}_p,$$

where μ is a smooth function defined on $[T_{\min}, T_{\max}]$ and $\mathfrak{X}_p$ is an arbitrary vector tangent to X at x_p. As the trajectory passing through x_{p-1} is not periodic, ϕ_0 is unambiguously defined on $\mathcal{T}$; moreover, ϕ_0 extends to a smooth vector field defined on the whole manifold X (and which is zero on $\mathscr{U}_2$). With this choice of ϕ_0, the integrals occurring in A_i and A_p write as

$$J_{j_k} = \left(\int_{-t_k}^{T-t_k} \mu(T-\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p)\cdot\mathfrak{X}_p = \left(\int_{t_k}^{T+t_k} \mu(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p)\cdot\mathfrak{X}_p,$$

while the integrals occurring in the terms $\bar{A}_j$ and $\bar{A}_m$ write as

$$\bar{J}_{l_k} = \left(\int_{-t'_k}^{T-t'_k} \mu(T-\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p)\cdot\mathfrak{X}_p = \left(\int_{t'_k}^{T+t'_k} \mu(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}\varphi_{t'_k}^{u_{p-1}}(x_p)\cdot\mathfrak{X}_p\,;$$

here all the t_k's are nonzero but t_a and all the t'_k's are nonzero. Choose now a smooth function M defined on $[T_{\min}, T_{\max}]$ and such that

- $M(T) = M(2T) = \cdots = M(cT) = 1$; here c denotes the integer part of $T_{\max}/T$;
- $M(T+t_k) = M(t_k) = M(T+t'_{k'}) = M(t'_{k'}) = 0$, $k = 1,\ldots,a-1$, $k' = 1,\ldots,b$, where $t_k, t'_k \neq \alpha T$, $\alpha = 1,\ldots,c$;
- $M(0) = 0$.

We take now $\mu(t) = \dfrac{\mathrm{d}M(t)}{\mathrm{d}t}$; with this choice of μ, all the integrals J_{j_k} and $\bar{J}_{l_k}$ are zero except J_{p-1}, which is equal to $\mathfrak{X}_p$.

Point x_{p-1} is singular. In this case, we have $x_{p-1} = x_p$, so necessarily we are under $\mathbf{C_1}$ or $\mathbf{C'_4}$ configuration. The vector field ϕ_0 is then chosen such that ϕ_0 is zero outside an open neighborhood $\mathcal{N}$ of x_{p-1}, this neighborhood being chosen so small that the trajectories of $f(\cdot, u_{p-1})$ passing through the points x_{j_k} ($j_k \neq p-1$) and $\bar{x}_{l_k}$ do not cross $\mathcal{N}$. With this choice of ϕ_0, all the integrals J_{j_k} and $\bar{J}_{l_k}$ are zero except the one which corresponds to $j_k = p-1$, which is equal to

$$J_{p-1} = \int_0^T \mathrm{d}\varphi_s^{u_{p-1}}(x_{p-1})\cdot\phi_0(x_{p-1})\,\mathrm{d}s\,.$$

Now $\mathrm{d}\varphi_s^{u_{p-1}}(x_{p-1}) = e^{sA}$, where A is the differential of f at x_{p-1}; notice that, as $f \in \mathscr{G}_2^U(a)$, A does not have any purely imaginary eigenvalue (and so is invertible). Hence, we can compute explicitly integral J_{p-1}:

$$J_{p-1} = A^{-1}(e^{TA} - \mathrm{Id})\cdot\phi_0(x_{p-1})\,;$$

as e^{TA} does not admit 1 as an eigenvalue, $e^{TA} - \mathrm{Id}$ is invertible, which proves that J_{p-1} can be made equal to any tangent vector of $T_{x_{p-1}}$ thanks to an appropriate choice of $\phi_0(x_{p-1})$.

The trajectory passing through $\boldsymbol{x_{p-1}}$ is periodic. We shall show now the same result in the case when the trajectory passing through x_{p-1} is periodic; in other words, we assume that the mapping $t \mapsto \varphi_t^{u_{p-1}}(x_{p-1})$ is periodic, and we denote by π_0 its prime period. In this case, the function μ which appears in the definition of ϕ_0 must be periodic. Writing $T = q\pi_0 + \tau$ with $q \in \mathbf{N}$ and $0 \leq \tau < \pi_0$, we have

$$
\begin{aligned}
J_{j_k} &= \int_0^T \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma \\
&= \sum_{l=0}^{q-1} \int_{l\pi_0}^{(l+1)\pi_0} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma + \int_{q\pi_0}^{T} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma \\
&= \sum_{l=0}^{q-1} \mathrm{d}\varphi_{l\,\pi_0}^{u_{p-1}}(\varphi_T^{u_{p-1}}(x_{j_k})) \cdot \int_0^{\pi_0} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma \\
&\qquad + \mathrm{d}\varphi_{q\pi_0}^{u_{p-1}}(\varphi_T^{u_{p-1}}(x_{j_k})) \int_0^{\tau} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma\,.
\end{aligned}
\tag{27}
$$

Now the x_{j_k}'s in the above integrals are such that $x_{j_k} = \varphi_{t_k}^{u_{p-1}}(x_{p-1})$. Notice that, due to the periodicity of the trajectory passing through x_{p-1}, we can assume that $0 \le t_k < \pi_0$. Writing the x_{j_k}'s in terms of x_{p-1}, the above integrals between 0 and π_0 can be rewritten as

$$
\int_0^{\pi_0} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\pi_0 - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1})\mathrm{d}\sigma\,.
$$

In the same way, the integral between 0 and τ in (27) can be written as

$$
\int_0^{\tau} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{j_k})\mathrm{d}\sigma = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \int_{-t_k}^{\tau - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{T-\sigma}^{u_{p-1}}(x_{p-1})\mathrm{d}\sigma\,.
$$

It follows from these considerations and from the equality $\varphi_T^{u_{p-1}}(x_{p-1}) = x_p$ that we can write J_{j_k} under the form

$$
\begin{aligned}
J_{j_k} &= \sum_{l=0}^{q-1} \mathrm{d}\varphi_{l\,\pi_0 + t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\pi_0 - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p)\mathrm{d}\sigma \\
&\qquad + \mathrm{d}\varphi_{q\pi_0 + t_k}^{u_{p-1}}(x_p) \cdot \int_{-t_k}^{\tau - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p)\mathrm{d}\sigma \\
&= \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \Bigg(Q \cdot \int_{-t_k}^{\pi_0 - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p)\mathrm{d}\sigma \\
&\qquad + \delta^q \cdot \int_{-t_k}^{\tau - t_k} \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p)\mathrm{d}\sigma \Bigg),
\end{aligned}
\tag{28}
$$

where we let

$$
\delta = \mathrm{d}\varphi_{\pi_0}^{u_{p-1}}(x_p) \qquad \text{and} \qquad Q = \sum_{l=0}^{q-1} \delta^l\,. \tag{29}
$$

We introduce also the following notation: Let $t \in [0, \pi_0]$, and denote by I_t the integral

$$
I_t = \int_0^t \mathrm{d}\varphi_\sigma^{u_{p-1}} \circ \phi_0 \circ \varphi_{-\sigma}^{u_{p-1}}(x_p)\mathrm{d}\sigma\,.
$$

We now rewrite J_{j_k} in terms of integrals I_t; from (28), we get

- if $t_k < \tau$,

$$
J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \big((Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_k} + \delta^q \cdot I_{\tau - t_k} \big)\,; \tag{30}
$$

- if $\tau \le t_k$,

$$J_{j_k} = \mathrm{d}\varphi_{t_k}^{u_{p-1}}(x_p) \cdot \left(Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_k} + \delta^{q-1} \cdot I_{\pi_0 + \tau - t_k}\right). \tag{31}$$

Notice that, in each case, $\pi_0 - t_k$ and $\pi_0 + \tau - t_k$ belong to $[0, \pi_0]$.

As regards the integrals $\bar{J}_{l_k}$ ($k = 1, \dots, b$), analogous computations lead to the same above formulas (30) and (31).

We choose now ϕ_0 as follows: For $z = \varphi_{-\sigma}^{u_{p-1}}(x_p)$ with $\sigma \in [0, \pi_0]$, we define $\phi_0(z)$ as

$$\phi_0(z) = \mathrm{d}\varphi_{-\sigma}^{u_{p-1}}(x_p) \cdot \vartheta(\sigma), \tag{32}$$

where $\vartheta : \mathbf{R} \to \mathrm{T}_{x_p} X$ is a π_0-periodic mapping to be determined; notice first that, as we want ϕ_0 to be C^r, we must have $\vartheta(0) = \vartheta(\pi_0) = 0$ as well as $\vartheta^{(l)}(0) = \vartheta^{(l)}(\pi_0) = 0$ for $l = 1, \dots, r$.

We deal first with the special case $\tau = 0$; in this case, $T = q\,\pi_0$, and we have $x_{p-1} = x_p$, so the lists L_p and $\bar{L}_m$ must be in the $\mathbf{C_1}$ or $\mathbf{C_4'}$ configuration. If we are under $\mathbf{C_4'}$ configuration and if there exists a subscript $j_1 < p-1$ such that $u_{j_1} = u_{p-1}$ and $x_{j_1} = \varphi_{t_1}^{u_{p-1}}(x_{p-1})$, then $x_{j_1} = x_{j_1+1}$, which is impossible from the definition of the $\mathbf{C_4'}$ configuration. If there exists a subscript $l_1 < m$ such that $u_{l_1} = u_{p-1}$ and $\bar{x}_{l_1} = \varphi_{t_1'}^{u_{p-1}}(x_{p-1})$, then we would have $\bar{x}_{l_1} = \bar{x}_{l_1+1}$; as the lists L_p and $\bar{L}_m$ are in the $\mathbf{C_4}$ configuration, this implies that $l_1 = m-1$ and $u_{m-1} = u_{p-1}$, which is incompatible with the definition of the $\mathbf{C_4'}$ configuration. We conclude that the terms A_{p-1}, $\bar{A}_j$, and $\bar{A}_m$ are zero in this case and that A_p is equal to $Q \cdot I_{\pi_0}$; so, with our choice of ϕ_0, we have

$$A_p = Q \cdot \int_0^{\pi_0} \vartheta(\sigma)\,\mathrm{d}\sigma,$$

and it is obviously possible to find a periodic function ϑ satisfying the above constraints and whose integral over the interval $[0, \pi_0]$ is equal to $Q^{-1} \cdot \mathfrak{X}_p$; clearly, this choice of ϑ is also possible if we are under $\mathbf{C_1}$ configuration.

We assume now that $\tau \neq 0$; we deal first with the case of $\mathbf{C_1}$ configuration. We introduce the following sets:

$$\mathbf{T}_\alpha = \{\, t_k,\ k = 1, \dots, a-1 \mid t_k \equiv -\alpha\,\tau \pmod{\pi_0} \,\}.$$

If $\mathbf{T}_1 = \varnothing$, we set $\alpha_0 = 0$; if not, we denote by α_0 the largest integer α such that $\mathbf{T}_1 \neq \varnothing, \mathbf{T}_2 \neq \varnothing, \dots, \mathbf{T}_\alpha \neq \varnothing$. If $\alpha_0 \neq 0$, we introduce the integers $\gamma_\alpha = [\alpha\,\tau/\pi_0] + 1$ (where $[x]$ denotes the integer part of x); so, if $t_k \in \mathbf{T}_\alpha$, we have $t_k = \gamma_\alpha \pi_0 - \alpha\,\tau$.

It could happen that π_0 is divisible by τ; hereafter, we distinguish two cases.

First, we assume that there does not exist any $\alpha \le \alpha_0$ such that $(\alpha+1)\tau \equiv 0 \pmod{\pi_0}$; this assumption implies that for every pair $0 \le \alpha < \alpha' \le \alpha_0$, we have $\alpha'\tau - \alpha\tau \not\equiv 0 \pmod{\pi_0}$. Thus, there exists a mapping $V : \mathbf{R} \to \mathrm{T}_{x_p} X$ such that

- V is π_0-periodic and $V(0) = 0$, $V^{(l)}(0) = 0$ for $l = 1, \dots, r+1$;
- $V(\tau) = \delta^{-q} \cdot \mathfrak{X}_p$ and

$$V(2\tau) = \begin{cases} \delta^{-q-1} \cdot V(\tau) & \text{if } \pi_0 < 2\tau \\ \delta^{-q} \cdot V(\tau) & \text{if } \pi_0 > 2\tau\,, \end{cases}$$

$$\vdots$$

$$V((\alpha_0+1)\tau) = \begin{cases} \delta^{-q-1} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 < (\alpha_0+1)\tau \\ \delta^{-q} \cdot V(\alpha_0\tau) & \text{if } \gamma_{\alpha_0}\pi_0 > (\alpha_0+1)\tau\,; \end{cases}$$

- $V(\pi_0 - t_k) = 0$, $V(\tau - t_k) = 0$ (case $t_k < \tau$) or $V(\pi_0 + \tau - t_k) = 0$ (case $\tau \le t_k$) if $t_k \notin \mathbb{T}_1 \cup \cdots \cup \mathbb{T}_{\alpha_0}$ (notice that, for such a t_k, $\pi_0 - t_k$ and $\pi_0 + \tau - t_k$ do no more belong to $\mathbb{T}_1 \cup \cdots \cup \mathbb{T}_{\alpha_0}$).

We take then the mapping ϑ equal to the derivative of V; with this choice, all the integrals J_{j_k} are zero except J_{p-1}, which is equal to $\mathfrak{X}_p$; this implies that A_p is equal to $\mathfrak{X}_p$ while we have $A_i = 0$.

Case where τ divides π_0. Assume now that there exists $\alpha \le \alpha_0$ such that $(\alpha + 1)\tau \equiv 0 \pmod{\pi_0}$; then necessarily $\alpha = \alpha_0$ (if $\alpha < \alpha_0$, the time $t_a = 0$ would belong to $\mathbf{T}_{\alpha+1}$). There exists a subscript j_k such that $x_{j_k} = \varphi^{u_{p-1}}_{-\alpha_0\tau}(x_{p-1}) = \varphi^{u_{p-1}}_{\tau}(x_{p-1}) = x_p$; as $j_k < p$, from this equality and from the definition of configuration $\mathbf{C_1}$, we deduce that $j_k = i$ and that $u_i = u_{p-1}$. Arguing by induction, assume that, for some $0 \le r < p-1-i$, we have $u_i = \cdots = u_{i+r} = u_{p-1}$ and $x_i = \varphi^{u_{p-1}}_{-\alpha_0\tau}(x_{p-1}), \ldots, x_{i+r} = \varphi^{u_{p-1}}_{-(\alpha_0-r)\tau}(x_{p-1})$. There exists a subscript j_k such that $x_{j_k} = \varphi^{u_{p-1}}_{-(\alpha_0-r-1)\tau}(x_{p-1}) = \varphi^{u_{p-1}}_{\tau}(x_{i+r}) = x_{i+r+1}$; as above, this equality implies that $u_{i+r+1} = u_{p-1}$. We have proved that $u_i = u_{i+1} = \cdots = u_{p-1}$ and that $\alpha_0 = p - i - 1$. As $x_p = \varphi^{u_{p-1}}_{(p-i)T}(x_i)$, we rewrite A_p as follows:

$$A_p = \int_0^{(p-i)T} \mathrm{d}\varphi^{u_{p-1}}_{\sigma} \circ \phi_0 \circ \varphi^{u_{p-1}}_{-\sigma}(x_p)\,\mathrm{d}\sigma + \mathrm{d}\varphi^{u_{p-1}}_{(p-i)T}(x_p) \cdot A_i\,.$$

As $(\alpha_0 + 1)\tau \equiv 0 \pmod{\pi_0}$, there exists $r \in \mathbf{N}$ such that $(p-i)T = q'\pi_0$; thus, the first term in this new expression of A_p can be rewritten as

$$\int_0^{(p-i)T} \mathrm{d}\varphi^{u_{p-1}}_{\sigma} \circ \phi_0 \circ \varphi^{u_{p-1}}_{-\sigma}(x_p)\,\mathrm{d}\sigma = Q' \cdot I_{\pi_0},$$

where $Q' = \mathrm{Id} + \delta + \ldots \delta^{q'-1}$. We shall see that ϕ_0 can be chosen such that $A_i = 0$ and A_p equals any tangent vector field. Hereafter, we call chain of length c ($c \ge 2$) a sequence of c pairs $((x_{j_{k_1}}, t_{k_1}), \ldots, (x_{j_{k_c}}, t_{k_c}))$ such that

- the points $x_{j_{k_1}}, \ldots, x_{j_{k_c}}$ belong to $\mathscr{P}_1$ and the subscripts $j_{k_1}, \ldots j_{k_c}$ are pairwise distinct;
- the times $t_{k_1}, \ldots, t_{k_c}$ belong to $\mathbf{T}_1 \cup \cdots \cup \mathbf{T}_{\alpha_0} \cup \{0\}$ and are such that

$$t_{k_2} \equiv t_{k_1} + \tau \pmod{\pi_0},\ \ldots\ , t_{k_c} \equiv t_{k_1} + (c-1)\tau \pmod{\pi_0}\,.$$

Notice that two chains are either disjoint or equal. The chain

$$\mathfrak{c}_0 \triangleq ((x_i, \tau), (x_{i+1}, 2\tau), \ldots, (x_{p-1}, (\alpha_0 + 1)\tau))$$

has a length equal to $\alpha_0 + 1$; the lengths of all the other chains are less than $\alpha_0 + 1$ because, otherwise, we could find at least two equalities between the elements of L_p. We choose vector field ϕ_0 as in the case where $(\alpha_0 + 1)\tau \not\equiv 0 \pmod{\pi_0}$, but the mappings ϑ and V are chosen as follows:

- $\vartheta(t) = V'(t)$ and $V(t) = V_0(t) + tv_0$, where V_0 is π_0-periodic with $V_0(\pi_0) = V_0(0) = 0$; notice that this choice of ϑ implies $I_t = V_0(t) + tv_0$;
- $V_0'(0) = V_0'(\pi_0) = -v_0$ and $V_0^{(k)} = V_0^{(k)}(\pi_0) = 0$ with $k = 2, \ldots, r+1$;
- $v_0 = \frac{1}{\pi_0} \cdot (Q')^{-1}\mathfrak{X}_p$;
- if $((x_{j_{k_1}}, t_{k_1}), \ldots, (x_{j_{k_c}}, t_{k_c}))$ is a chain of maximal length distinct from the chain $\mathfrak{c}_0$, then we must have

$$(33)\qquad \begin{cases} (Q \circ \delta^{-1} + \delta^{q-1}) \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^q \cdot I_{\tau - t_{k_s}} = 0, & \text{if } t_{k_s} < \tau, \\ Q \circ \delta^{-1} \cdot I_{\pi_0} - \delta^{-1} \cdot I_{\pi_0 - t_{k_s}} + \delta^{q-1} \cdot I_{\pi_0 + \tau - t_{k_s}} = 0, & \text{if } \tau \le t_{k_s}. \end{cases}$$

These equalities can also be written as

$$\begin{cases} (Q\circ\delta^{-1}+\delta^{q-1})\cdot I_{\pi_0}-\delta^{-1}\cdot(V_0(-t_{k_s})+(\pi_0-t_{k_s})v_0) \\ \qquad\qquad +\delta^{q}\cdot(V_0(\tau-t_{k_s})+(\tau-t_{k_s})v_0)=0, \quad \text{if } t_{k_s}<\tau, \\ Q\circ\delta^{-1}\cdot I_{\pi_0}-\delta^{-1}\cdot(V_0(-t_{k_s})+(\pi_0-t_{k_s})v_0) \\ \qquad\qquad +\delta^{q-1}\cdot(V_0(\tau-t_{k_s})+(\pi_0+\tau-t_{k_s})v_0=0, \qquad \text{if } \tau\le t_{k_s}. \end{cases}$$

As $c<\alpha_0+1$, the numbers $\tau-t_{k_1}$, $-t_{k_1},\dots,-t_{k_1}-(c-1)\tau$ are pairwise distinct modulo π_0, and the values of V_0 at these points can be chosen independently from each other; this proves that the above equalities can be achieved;

- if t_k does not belong to a chain, nor is the case for $\tau-t_k$, so equality (33) can be satisfied thanks to an appropriate choice of $V(-t_k)$ and $V(\tau-t_k)$.

With this choice of ϑ, taking into account the formulas (30) and (31), we see that we have $A_p=\mathfrak{X}_p$ and $A_i=0$.

If we are under the configuration $\mathbf{C_2}$, $\mathbf{C_3'}$, or $\mathbf{C_4'}$, the sets $\mathbf{T}_\alpha$ are defined as follows:

$$\mathbf{T}_\alpha=\{\,t_k,\ k=1,\dots,a-1 \mid t_k\equiv-\alpha\,\tau \pmod{\pi_0}\,\} \\ \cup\{\,t_k',\ k=1,\dots,b \mid t_k'\equiv-\alpha\,\tau \pmod{\pi_0}\,\},$$

the integer α_0 being defined as above. If there does not exist any $\alpha\le\alpha_0$ such that $(\alpha+1)\tau\equiv0 \pmod{\pi_0}$, the reasoning is exactly the same as that in the the case of $\mathbf{C_1}$ configuration: Choosing ϕ_0 as explained above, we get $A_i=0$, $\bar A_m=0$, and $\bar A_j=0$ while A_p can be arbitrarily chosen.

Assume now that there exists α such that $(\alpha+1)\tau\equiv0 \pmod{\pi_0}$; then, as above, necessarily $\alpha=\alpha_0$. As we are under $\mathbf{C_2}$, $\mathbf{C_3'}$, or $\mathbf{C_4'}$ configuration, the definition of a chain takes into account the elements of $\bar L_{2n}$: We call chain of length c a sequence $((z_1,\mathfrak{t}_{k_1}),\dots,(z_c,\mathfrak{t}_{k_c}))$ such that

- the points $z_1,\dots,z_c$ belong to $\mathscr{P}_1$;
- the times $\mathfrak{t}_{k_1},\dots,\mathfrak{t}_{k_c}$ belong to $\mathbf{T}_1\cup\cdots\cup\mathbf{T}_{\alpha_0}\cup\{0\}$ and are such that

$$\mathfrak{t}_{k_2}\equiv\mathfrak{t}_{k_1}+\tau \pmod{\pi_0},\ \dots\ ,\mathfrak{t}_{k_c}\equiv\mathfrak{t}_{k_1}+(c-1)\tau \pmod{\pi_0};$$

- if $\mathfrak{t}_k=t_k$ (resp., $\mathfrak{t}_k=t_k'$), then $z_k=x_{j_k}$ (resp., $z_k=\bar x_{l_k}$); moreover, the subscripts j_k (resp., l_k) related to those x_{j_k} (resp., $\bar x_{l_k}$) are pairwise distinct.

If we are under configuration $\mathbf{C_4'}$, the only possible chain of length α_0+1 is either

$$\mathfrak{c}_0=((x_i,\tau),\dots,(x_{p-1},(\alpha_0+1)\tau))$$

or

$$\bar{\mathfrak{c}}_0=((\bar x_j,\tau),\dots,(\bar x_{m-1},(\alpha_0+1)\tau))\,.$$

Notice that, from the definition of $\mathbf{C_4'}$ configuration, these two chains, $\mathfrak{c}_0$ and $\bar{\mathfrak{c}}_0$, cannot coexist. All the other chains have a length less than α_0+1. To show this fact, let $((z_1,\mathfrak{t}_{k_1}),\ \dots,(z_c,\mathfrak{t}_{k_c}))$ be a chain such that $c=\alpha_0+1$. All the elements $(z_1,\dots,z_c)$ belong either to L_{p-1} or to $\bar L_{m-1}$ (if not, we could find an equality between an element of L_{p-1} and an element of $\bar L_{m-1}$). If all the z_k's belong to L_{p-1}, then we have $z_{c+1}=z_1$; from the definition of $\mathbf{C_4'}$ configuration, it follows that $z_1=x_i$ and $z_{c+1}=x_p$. If all the z_k's are in $\bar L_{m-1}$, we obtain that $z_1=\bar x_j$, $z_{c+1}=\bar x_m$ and $u_j=\cdots=u_{m-1}=u_{p-1}$. As all the chains except $\mathfrak{c}_0$ have a length less than α_0+1, we can conclude, as for $\mathbf{C_1}$ configuration, that there exists a function $\mathscr{V}$ defining a vector field ϕ_0 which ensures that $A_i = 0$, $\bar{A}_j = 0$, and $\bar{A}_m = 0$, while A_p can be arbitrarily chosen.

We shall see that a chain of length $\alpha_0 + 1$ is not possible under $\mathbf{C_2}$ or $\mathbf{C_3'}$ configuration. Assume that we are not under $\mathbf{C_4}$ configuration, and denote by

$$\mathfrak{c}_0 = ((z_1, \mathfrak{t}_1), \dots, (z_{\alpha_0+1}, \mathfrak{t}_{\alpha_0+1}))$$

a chain with length $\alpha_0 + 1$; notice that, from the definition of a chain, we have

$$z_{k+1} = \varphi_\tau^{u_{p-1}}(z_k) = \varphi_T^{u_{p-1}}(z_k)\,. \tag{34}$$

There exists $0 \le \alpha_1 \le \alpha_0$ such that $\mathfrak{t}_{k_1} \equiv -\alpha_1\tau \pmod{\pi_0}$, so, from the definition of a chain, we have

$$\mathfrak{t}_{k_1} \equiv -\alpha_1\tau \pmod{\pi_0}, \qquad \mathfrak{t}_{k_2} \equiv -(\alpha_1 - 1)\tau \pmod{\pi_0}, \dots,$$
$$\mathfrak{t}_{k_{\alpha_0+1}} \equiv -(\alpha_1 - \alpha_0)\tau \pmod{\pi_0};$$

therefore, $\mathfrak{t}_{k_{\alpha_1+1}} \equiv 0 \pmod{\pi_0}$ and $z_{\alpha_1+1} = x_{p-1}$. If $\alpha_1 < \alpha_0$, this equality implies $z_{\alpha_1+2} = x_p$; if $\alpha_1 = \alpha_0$, we have $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_\tau^{u_{p-1}}(x_{p-1}) = x_p$. Thus, in chain $\mathfrak{c}_0$, there exists an element z_i equal to x_p; this implies that we cannot be under $\mathbf{C_2}$ configuration because, in this case, we can have neither $z_i = x_{j_k}$ because $j_k < p$ nor $z_i = x_p = \bar{x}_m = \bar{x}_{l_k}$ because $l_k < m$. Thus, we are under $\mathbf{C_3'}$ configuration; reordering the elements of the chain, we can assume that

$$\mathfrak{t}_{k_1} \equiv -\alpha_0\tau \pmod{\pi_0}, \mathfrak{t}_{k_2} \equiv -(\alpha_0 - 1)\tau \pmod{\pi_0}, \dots, \mathfrak{t}_{k_{\alpha_0+1}} \equiv 0 \pmod{\pi_0}\,.$$

We have $z_1 = \varphi_{-\alpha_0\tau}^{u_{p-1}}(x_{p-1}) = \varphi_\tau^{u_{p-1}}(x_{p-1}) = x_p$, so, as all the subscripts j_k are less than p, we have $z_1 = \bar{x}_m \in \bar{L}_{2n}$. Let r be the greatest subscript such that $z_1, \dots, z_r \in \bar{L}_{2n}$; from (34), we have $z_r = \bar{x}_{m+r-1}$ and $z_{r+1} = x_{j_k}$ for some subscript $j_k < p$, so $x_{j_k} = \bar{x}_{m+r}$; from the definition of configuration $\mathbf{C_3'}$, this implies that we cannot have $m + r < j$, so $m + r = j$ and $z_{r+1} = x_i$. Let s be the greatest subscript greater than or equal to $r + 1$ such that $z_{r+1}, \dots, z_s \in L_{2n}$. We have $z_s = x_{s+i-r-1}$, and we claim that z_s is the last element of $\mathfrak{c}_0$ because if there exist $z_{s+1} \in \bar{L}_{2n}$, then we have $z_{s+1} = \varphi_\tau^{u_{p-1}}(z_s) = x_{s+i-r}$; from the definition of configuration $\mathbf{C_3'}$, this implies that $s + i - r = p$ and $z_{s+1} = \bar{x}_m$, which contradicts the definition of a chain. The element z_s being the last element of the chain, we have $z_s = \varphi_{\alpha_0\tau}^{u_{p-1}}(z_1) = \varphi_{-\tau}^{u_{p-1}}(\bar{x}_m) = x_{p-1}$; therefore, chain $\mathfrak{c}_0$ can be written as

$$\mathfrak{c}_0 = ((\bar{x}_m, -\alpha_0\tau), \dots, (\bar{x}_{j-1}, -(\alpha_0 - j + m + 1)\tau), (x_i, -(\alpha_0 - j + m)\tau), \dots, (x_{p-1}, 0)),$$

from which we deduce that $u_i = \cdots = u_{p-1} = u_m = \cdots = u_{j-1}$, which contradicts the definition of $\mathbf{C_3'}$.

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