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CONTROLLABILITY AND OBSERVABILITY OF SECOND ORDER DESCRIPTOR SYSTEMS*

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Abstract. We analyze controllability and observability conditions for second order descriptor systems and show how the classical conditions for first order systems can be generalized to this case. We show that performing a classical transformation to first order form may destroy some controllability and observability properties. As an example, we demonstrate that the loss of impulse controllability in constrained multibody systems is due to the representation as a first order system. To avoid this problem, we will derive a canonical form and new first order formulations that do not destroy the controllability and observability properties.

Key words. descriptor system, impulse controllability, impulse observability, second order system, order reduction, index reduction, complete controllability, strong controllability, complete observability, strong observability

AMS subject classifications. 93B05, 93B07, 93B10

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1. Introduction.

We study linear second order constant coefficient descriptor control problems of the form

$$M\ddot{x} + G\dot{x} + Kx = Bu, \tag{1.1}$$

$$y = Cx, \tag{1.2}$$

$$x(0) = x_0,\ \dot{x}(0) = \dot{x}_0 \tag{1.3}$$

with coefficients M, G, $K \in \mathbb{R}^{n,n}$, $C \in \mathbb{R}^{p,n}$, and $B \in \mathbb{R}^{n,m}$. Here $\mathbb{R}^{n,\ell}$ denotes the vector space of $n \times \ell$ real matrices, x is the state, u is the input or control, and y is the output of the system. In particular, we study descriptor systems, where the matrix M is singular, and, despite the fact that formally $\ddot{x}$ and $\dot{x}$ occur in (1.1), we require that $\ddot{x}$ only has to exist outside the kernel of M and that $\dot{x}$ has to exist outside the kernel of G.

All of the results in this paper also carry over to the complex case, and they can also be easily extended to systems of higher than second order, but, for ease of notation and because this is the most important case in practice, we restrict ourselves to the real second order case.

In the following we denote by I or I_n the identity matrix of size $n \times n$ and by A^T the transpose of a matrix A. We denote a matrix with orthonormal columns spanning the right null space of the matrix M by $S_\infty(M)$ and a matrix with orthonormal columns spanning the left null space of M by $T_\infty(M)$. These matrices are not uniquely determined, although the corresponding spaces are. Nevertheless, for simplicity, we speak of these matrices as the corresponding spaces.

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Second order descriptor systems arise in the control of constrained mechanical systems (see, e.g., [14, 19, 23, 36, 38, 39, 40]) in the control of electrical and electromechanical systems [2, 3], and in particular in heterogeneous systems, where different models are coupled together [37].

Usually, in the classical theory of ordinary differential equations and *classical state space systems* (i.e., descriptor systems where the leading coefficient is the identity), second order systems are turned into first order systems by introducing new variables for the first derivative. This gives rise to linear first order *descriptor* (or *generalized state space*) systems of the form

$$E\dot{\xi} = A\xi + B_1 u, \tag{1.4}$$

$$y = C_1\xi, \tag{1.5}$$

$$\xi(0) = \xi_0. \tag{1.6}$$

Let us briefly recall some results for first order descriptor systems; see, e.g., [4, 9, 12, 44]. In contrast to classical state space systems, where $E = I$, the response of a descriptor system can consist of step functions or can be discussed only in a distributional setting [10, 18, 43], if the input function u is not sufficiently smooth. But here we are interested only in classical solutions in the sense that $M\ddot{x}$ and $G\dot{x}$ exist, and we explicitly want to avoid impulsive terms in the solution; thus, we do not use this formulation.

The response of system (1.4) can be described in terms of the eigenstructure of the matrix pencil $\alpha E - \beta A$. The pencil and the corresponding system (1.4)–(1.5) are said to be *regular* if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Regular systems are *solvable* in the sense that (1.4) admits a classical solution $\xi : \mathbb{R} \to \mathbb{R}^n$, with ξ differentiable in the image of E for all sufficiently smooth controls u and consistent initial conditions ξ_0 [9, 12, 44].

For regular pencils, *generalized eigenvalues* are the pairs $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ for which $\det(\alpha E - \beta A) = 0$. If $\beta \neq 0$, then the pair represents the finite eigenvalue $\lambda = \alpha/\beta$. If $\beta = 0$, then (α, β) represents an "infinite" eigenvalue. In the following, for simplicity, we use the notation with λ.

The solution and many properties of the *free descriptor system* (with $u = 0$) can be characterized in terms of the Weierstraß canonical form (WCF) for regular matrix pencils.

THEOREM 1.1 (see [16]). *If $\lambda E - A$ is a regular pencil, then there exist nonsingular matrices $X = \begin{bmatrix} X_r & X_\infty \end{bmatrix} \in \mathbb{R}^{n,n}$ and $Y = \begin{bmatrix} Y_r & Y_\infty \end{bmatrix} \in \mathbb{R}^{n,n}$ for which*

$$Y^T E X = \begin{bmatrix} Y_r^T \\ Y_\infty^T \end{bmatrix} E \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \tag{1.7}$$

and

$$Y^T A X = \begin{bmatrix} Y_r^T \\ Y_\infty^T \end{bmatrix} A \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \tag{1.8}$$

where J is a matrix in real Jordan canonical form whose eigenvalues are the finite eigenvalues of the pencil and N is a nilpotent matrix, also in Jordan form. J and N are unique up to permutation of Jordan blocks.

Usually, the index of nilpotency ν of the nilpotent matrix N in (1.7) is called the *differentiation index* or *index* of the system, and if E is nonsingular, then the pencil is said to be of index zero. In recent years the theory of descriptor systems has been extended to rectangular, time-varying, and even nonlinear systems, and different index concepts, in particular the *strangeness index*, have been introduced; see [30] for a recent textbook. The strangeness index generalizes the index of a linear descriptor system to over- and underdetermined linear and nonlinear systems, and it uses a slightly different counting; i.e., systems of the form (1.4) with an index of at most one have a strangeness index of zero and are called *strangeness-free*. For all other systems where the differentiation index is defined, it is the strangeness index plus 1. Since we restrict ourselves to square systems, we will use only the differentiation index ν and call it the *index of the system.*

For first order systems (1.4) the index describes the degree of differentiability of the input function that is needed to achieve a continuous solution with the property that Ex is continuously differentiable. In analogy we define the *index of the second order system* (1.1) to be the degree of differentiability of the input function that is needed to achieve a continuous solution with the property that Mx is twice and Gx is once continuously differentiable.

In the notation of (1.7)–(1.8), classical solutions of (1.4) take the form

$$\xi(t) = X_r z_1(t) + X_\infty z_2(t),$$

where

$$\begin{aligned} \dot{z}_1 &= J z_1 + Y_r^T B_1 u, \\ N \dot{z}_2 &= z_2 + Y_\infty^T B_1 u. \end{aligned} \tag{1.9}$$

This system admits the explicit solution

$$\begin{aligned} z_1(t) &= e^{tJ} z_1(t_0) + \int_{t_0}^{t} e^{(t-s)J} Y_r^T B_1 u(s)\, ds, \\ z_2(t) &= -\sum_{i=0}^{\nu-1} \frac{d^i}{dt^i} \left(N^i Y_\infty^T B_1 u(t) \right), \end{aligned} \tag{1.10}$$

where ν is the index of the system. Equation (1.10) shows that, for regular systems that are not of index at most one, in order to have classical, continuous solutions, the input u has to be sufficiently smooth, and, to ensure a smooth response for every continuous input u, the system must be regular and of index at most one. Under certain further requirements that we discuss below, this property may, however, be achieved by feedback. If this is the case, then the system is said to be *regularizable.*

Equation (1.10) also shows that the initial condition ξ_0 is restricted. For a given input function u, the set of *consistent* initial conditions is given by

$$\mathcal{S} = \left\{ X_r z_1 + X_\infty z_2 \;\middle|\; z_1 \in \mathbb{R}^r,\ z_2 = -\sum_{i=0}^{\nu-1} \left(\frac{d^i}{dt^i} (N^i Y_\infty^T B_1 u)(0) \right) \right\}. \tag{1.11}$$

The set $\mathcal{R}$ of *reachable* states or *reachable set* of (1.4) from the set $\mathcal{S}$ of consistent initial conditions is $\mathcal{S}$ itself [44].

Coming back to second order descriptor systems and their first order representations, one should note first that there is no unique way of performing this transformation to first order; see [33] for large vector spaces of first order formulations in the context of eigenvalue problems. As a consequence, the solution space and the set of admissible controls may be different for different first order formulations.

This has recently been shown in the context of the numerical solution of higher order differential-algebraic systems [35, 41]. There, it also has been demonstrated that the classical first order formulations may even lead to false results if certain smoothness conditions are not met or if the initial conditions are not chosen properly.

Let us illustrate these difficulties with the well-known example of mechanical multibody systems.

Example 1.2. Consider a simplified, linearized model of a two-dimensional, three-link mobile manipulator [22]. The Lagrangian equations of motion in its linearized form are given by a linear second order system

$$\begin{aligned} M_0\ddot{z} + G_0\dot{z} + K_0 z &= B_0 u - H_0^T\phi, \\ H_0 z &= 0, \end{aligned}$$

where M_0 represents the nonsingular mass matrix, G_0 the coefficient matrix associated with damping, centrifugal, gravity, and Coriolis forces, K_0 the stiffness matrix, and H_0 the constraint, whereas ϕ is a vector of Lagrange multipliers.

By setting $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} z \\ \phi \end{smallmatrix}\right]$, and adding an output equation

$$y = Cz = \begin{bmatrix} C_0 & 0 \end{bmatrix} x,$$

we obtain a descriptor system of the form (1.1)–(1.2) given by

$$\begin{aligned} \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix}\ddot{x} + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x &= \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} C_0 & 0 \end{bmatrix} x. \end{aligned}$$

If one would follow the usual approach for ordinary differential equations, then one would introduce a new state vector, often called a *descriptor vector*,

$$\xi = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} = \begin{bmatrix} \dot{z} \\ \dot{\phi} \\ z \\ \phi \end{bmatrix}.$$

Under the usual assumptions that M_0 is invertible and that H_0 has full row rank, it is easy to check that the resulting descriptor system has blocks of size 4 in the Weierstraß form associated with the eigenvalue ∞ and thus an index $\nu = 4$. It follows that the input functions have to be at least three times continuously differentiable to obtain a continuous solution.

As a simple example, consider the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ddot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{1.12}$$

which has the structure of a constrained and damped mechanical system. The classical first order version yields

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

A transformation $(\hat{E}, \hat{A}) = P(E, A)Q$, with

$$P = \begin{bmatrix} 0 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

yields the Weierstraß canonical form

$$(\hat{E}, \hat{A}) = \left(\left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right] \right),$$

which consists only of one block associated with the eigenvalue ∞ of size 4.

This classical approach, however, is usually not taken in practice, since on one hand it would introduce the unnecessary derivative of the Lagrange multiplier ϕ, which may not be differentiable, and also this approach would require extra initial values associated with $\dot{\phi}(t_0)$ which usually are not available. In practice, one therefore uses the knowledge about the structure of the system and introduces the reduced descriptor vector

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \dot{z} \\ z \\ \phi \end{bmatrix}.$$

In this way one obtains a first order descriptor system of the form

$$\begin{aligned} \begin{bmatrix} M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \dot{\xi} &= \begin{bmatrix} -G_0 & -K_0 & -H_0^T \\ 0 & -H_0 & 0 \\ I & 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} B_0 \\ 0 \\ 0 \end{bmatrix} u, \\ y &= \begin{bmatrix} 0 & C_0 & 0 \end{bmatrix} \xi, \end{aligned} \tag{1.13}$$

which has index $\nu = 3$.

For the simple system (1.12) the first order formulation (1.13) has

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

If we use the transformation $(\hat{E}, \hat{A}) = P(E, A)Q$, with

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix},$$

then we obtain the Weierstraß canonical form

$$(\hat{E}, \hat{A}) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$

which has only one block of size 3.

From this example we see that different first order formulations lead to different indexes and therefore to different differentiability requirements for the input functions u which we assume to be at least piecewise continuous functions.

But there is a second difficulty which both first order formulations in Example 1.2 share that is connected to the controllability and observability of the descriptor system and its first order formulations.

To describe this second difficulty we return again to our review of results for first order descriptor systems (1.4)–(1.5). Typically one or more of the following conditions is essential for most classical control design aims:

$$\begin{aligned} &\mathbf{C0}: \quad \operatorname{rank}[\alpha E - \beta A,\, B_1] = n \text{ for all } (\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\};\\ &\mathbf{C1}: \quad \operatorname{rank}[\lambda E - A,\, B_1] = n \text{ for all } \lambda \in \mathbb{C};\\ &\mathbf{C2}: \quad \operatorname{rank}[E,\, AS_\infty(E),\, B_1] = n. \end{aligned} \tag{1.14}$$

A regular first order descriptor system is called *completely controllable or C-controllable* if **C0** holds [44] and *controllable in the reachable set or $\mathcal{R}$-controllable* if condition **C1** holds. The system is called *strongly controllable or S-controllable* if **C1** and **C2** hold [5]. C-controllability ensures that for any given initial and final states ξ_0, ξ_f there exists a piecewise continuous control u that transfers the system from ξ_0 to ξ_f in finite time [44], while S-controllability ensures the same for any given initial and final states in the reachable set, i.e., $\xi_0,\, \xi_f \in \mathcal{R}$. Systems that satisfy condition **C2** are called *controllable at infinity, impulse-controllable*, or *I-controllable* [11, 27, 43]. For these systems, impulsive modes that arise from a high index of (E, A) can be avoided by a suitable linear feedback; see [4, 5]. It has been shown in [12] that a first order descriptor system is C-controllable if and only if it is $\mathcal{R}$-controllable and rank $[E \quad B_1] = n$. To have S-controllability, however, the condition that rank $[E \quad B_1] = n$ is not needed; see [4, 5].

Observability for descriptor systems is the dual of controllability. Consider the following conditions:

$$\begin{aligned} &\mathbf{O0}: \quad \operatorname{rank}\begin{bmatrix} \alpha E - \beta A \\ C_1 \end{bmatrix} = n \text{ for all } (\alpha,\beta) \in \mathbb{C}^2 \setminus \{(0,0)\};\\ &\mathbf{O1}: \quad \operatorname{rank}\begin{bmatrix} \lambda E - A \\ C_1 \end{bmatrix} = n \text{ for all } \lambda \in \mathbb{C};\\ &\mathbf{O2}: \quad \operatorname{rank}\begin{bmatrix} E \\ T_\infty^T(E)A \\ C_1 \end{bmatrix} = n. \end{aligned} \tag{1.15}$$

A regular descriptor system is called *completely observable or C-observable* if condition **O0** holds, *observable in the reachable set or $\mathcal{R}$-observable* if condition **O2** holds, and *strongly observable or S-observable* if conditions **O1** and **O2** hold. A system that satisfies condition **O2** is called *observable at infinity, impulse-observable, or I-observable*. Analogous to the controllable case, a system is C-observable if and only if it is $\mathcal{R}$-observable and rank $\left[\begin{smallmatrix} E \\ C_1 \end{smallmatrix}\right] = n$; see [12].

Note that the conditions (1.14) are preserved under equivalence transformations of the system and under state and output feedback. Analogous properties hold for (1.15).

Classical design approaches in control require the system to be at least S-controllable and S-observable; see [12, 30, 34]. But it is well known that in many practical examples, e.g., in the context of constrained mechanical systems, the resulting system in neither of the first order formulations as described in Example 1.2 is I-controllable and I-observable.

Example 1.3. Consider the first order formulation (1.13) in Example 1.2 with

$$E = \begin{bmatrix} M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -G_0 & -K_0 & -H_0^T \\ 0 & -H_0 & 0 \\ I & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_0 \\ 0 \\ 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0 & C_0 & 0 \end{bmatrix}.$$

Here we immediately see that

$$\begin{bmatrix} E & AS_\infty(E) & B_1 \end{bmatrix} = \begin{bmatrix} M_0 & 0 & 0 & -H_0^T & B_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \end{bmatrix}$$

does not have full row rank if constraints are present, and hence the system is not I-controllable. Similarly,

$$\begin{bmatrix} E \\ T_\infty^T(E)A \\ C_1 \end{bmatrix} = \begin{bmatrix} M_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & -H_0 & 0 \\ 0 & C_0 & 0 \end{bmatrix}$$

does not have full column rank either; i.e., the system is not I-observable. Furthermore, neither $[E \quad B_1]$ has full row rank, nor $\left[\begin{smallmatrix} E \\ C_1 \end{smallmatrix}\right]$ has full column rank, and hence the system is neither C-controllable nor C-observable.

It should be noted that a first order system that is regular and of index at most one is always I-controllable, since already $\operatorname{rank}[E, AS_\infty(E)]$ is full, which follows directly from the Weierstraß canonical form.

Since the conditions of I-controllability and I-observability are so important, it has been discussed, for the first order case in [6] for linear systems with constant coefficients and in [7, 26, 29, 31] for linear variable coefficient and nonlinear systems (see also [30]), how systems that are not I-controllable can be modified by a combination of index reduction and feedback to have this property. It has also been argued in [6] that if the system is not I-observable, then the modeling of the system should be reconsidered, since this means that the solution can be formulated only with the help of distributions, but these impulsive parts are not observed.

In view of all of these difficulties it is a natural question to ask whether the choice of the first order formulation may be the reason for the described difficulties with the I-controllability and I-observability. To analyze this question is the topic of the present paper which is organized as follows.

In section 2 we derive normal forms that allow us to check the controllability and observability conditions and the construction of adequate first order formulations. In sections 3 and 4 we then derive the controllability and observability conditions for second order systems analogous to **C0**, **C1**, **C2** and **O1**, **O2**, **O3**. We demonstrate that we can always find first order formulations which are guaranteed to be I-controllable and I-observable, so that the described difficulties can be avoided. We finish with some conclusions.

2. Normal forms. In this section we will discuss partial normal forms for matrix triples. The general results for matrix tuples can be found in [35].

DEFINITION 2.1. *Two second order descriptor systems of the form* (1.1) *with system matrices* (M, G, K, B)*, and* $(\hat{M}, \hat{G}, \hat{K}, \hat{B})$ *are called* strongly equivalent *if there exist nonsingular matrices* $P \in \mathbb{R}^{n,n}$*,* $Q \in \mathbb{R}^{n,n}$*, and* $V \in \mathbb{R}^{m,m}$ *such that*

$$\hat{M} = PMQ, \qquad \hat{G} = PGQ, \qquad \hat{K} = PKQ, \qquad \hat{B} = PBV. \tag{2.1}$$

We write $(M, G, K, B) \sim (\hat{M}, \hat{G}, \hat{K}, \hat{B})$.

Canonical forms under strong equivalence are known only for the case of matrix pairs, giving the Weierstraß and Kronecker canonical forms [15, 16]. For matrix triples or larger tuples, such canonical forms are not known. Condensed forms which present partial information about the invariants associated with the eigenvalue ∞ and the singular chains have recently been given in [35]. We will recall and extend these results below.

Another class of equivalence transformations that is studied in matrix polynomials is unimodular transformations such as adding the λa multiple of one row to another (or the same for columns) without increasing the degree of the polynomial. The analogue of these transformations in the context of descriptor systems is well studied [30] and has been discussed in the context of higher order systems in [35]. We reformulate these transformations by using the concept of differential polynomials; see, e.g., [25]. Let $\mathbb{R}[D_i]$ be the set of *ith order differential polynomials with coefficients in* $\mathbb{R}$, i.e.,

$$\mathbb{R}[D_i] := \left\{ a_0 + a_1 \frac{d}{dt} + a_2 \frac{d^2}{dt^2} + \ldots + a_i \frac{d^i}{dt^i} \;\middle|\; a_k \in \mathbb{R},\ k = 0, 1, \ldots, i \right\}.$$

Since we do not want to increase the order of the polynomial, we consider only the following restricted transformations.

DEFINITION 2.2. *Systems* $M\ddot{x} + G\dot{x} + Kx = Bu$ *and* $\hat{M}\ddot{x} + \hat{G}\dot{x} + \hat{K}x = \hat{B}u$*, with* $M, G, K, \hat{M}, \hat{G}, \hat{K} \in \mathbb{R}^{n,n}, B, \hat{B} \in \mathbb{R}^{n,m}$*, are called* order-preserving unimodularly equivalent, or opu-equivalent, *if there exists* $P \in \mathbb{R}[D_2]^{n,n}$ *with a constant nonzero determinant such that*

$$P(M\ddot{x} + G\dot{x} + Kx - Bu) = \hat{M}\ddot{x} + \hat{G}\dot{x} + \hat{K}x - \hat{B}u.$$

The concept of opu-equivalence requires that the order of differentiation in x, u does not increase. In section 4 we will make use of analogous transformations which do not increase the order of differentiation in x but allow derivatives of the input function u to be introduced. To distinguish these two types of transformations we call the latter ones *state order-preserving unimodularly equivalences, or sopu-equivalences*.

Note that the set of consistent initial conditions is not altered by opu- and sopu-equivalence transformations, since the solution set is not altered.

In the following we will show that, as in the first order case, regularization and index reduction can be obtained via a combination of unimodular equivalence transformations and appropriate feedback transformations.

DEFINITION 2.3. *Systems* $M\ddot{x} + G\dot{x} + Kx = Bu$ *and* $M\ddot{x} + \hat{G}\dot{x} + \hat{K}x = B\hat{u}$ *are called* equivalent under proportional feedback *if there exists a matrix* F_0 *of appropriate dimension such that* $\hat{K} = K + BF_0$.

They are called equivalent under first order derivative feedback *if there exists a matrix* F_1 *of appropriate dimension such that* $\hat{G} = G + BF_1$.

After introducing the definitions, we now describe a condensed form under strong equivalence.

THEOREM 2.4. *Consider the system* (1.1). *Then there exist nonsingular matrices* $P, Q \in \mathbb{R}^{n,n}$ *such that the coefficients in the transformed system*

$$\hat{M}\ddot{\hat{x}} + \hat{G}\dot{\hat{x}} + \hat{K}\hat{x} - \hat{B}u = PMQ\ddot{\hat{x}} + PGQ\dot{\hat{x}} + PKQ\hat{x} - PBu \tag{2.2}$$

that are obtained by setting $x = Q\hat{x}$ *have the following form:*

$$\left(\begin{bmatrix} I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & I_{s^{(0,1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 \\ I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left.\begin{bmatrix} 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 \\ 0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 \\ I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4 \\ \hat{B}_5 \\ \hat{B}_6 \\ \hat{B}_7 \\ \hat{B}_8 \\ \hat{B}_9 \\ \hat{B}_{10} \\ \hat{B}_{11} \\ \hat{B}_{12} \\ \hat{B}_{13} \end{bmatrix}\right), \tag{2.3}$$

where $s^{(0,1,2)}, s^{(1,2)}, s^{(0,2)}, s^{(0,1)}, d^{(2)}\, d^{(1)}, a,$ *and* v *are nonnegative integers and the blocks denoted by* $*$ *are not specified.*

Proof. This result follows directly from Theorem 12 in [35] with $f = Bu$. ∎

Based on Theorem 2.4 we can then show the following result.

Theorem 2.5. *Consider system* (1.1)–(1.2). *Then there exists a sequence of strong and opu-equivalence transformations such that the transformed system*

$$\hat{M}\ddot{\hat{x}} + \hat{G}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{B}\hat{u}$$

has the form

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ddot{\hat{x}} + \begin{bmatrix} \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & I & 0 & 0 \\ \hat{G}_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\hat{x}} + \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ \hat{K}_{31} & \hat{K}_{32} & \hat{K}_{33} & 0 \\ \hat{K}_{41} & \hat{K}_{42} & \hat{K}_{43} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \hat{x}$$
$$= \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4 \\ \hat{B}_5 \\ 0 \end{bmatrix} \hat{u},$$
$$y = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 & \hat{C}_3 & \hat{C}_4 \end{bmatrix} \hat{x},$$

where

$$\hat{x} = \begin{bmatrix} \hat{x}_1^T & \hat{x}_2^T & \hat{x}_3^T & \hat{x}_4^T \end{bmatrix}^T,$$

and, furthermore, $\hat{B}_3, \hat{B}_4$, *and* $\hat{B}_5$ *have full row rank.*

Proof. A detailed constructive proof is given in Appendix A of [32]. ∎

Theorem 2.6. *Consider system* (1.1)–(1.2). *Then there exists a sequence of strong and opu-equivalence transformations, as well as proportional feedbacks and first order derivative feedbacks such that the transformed system*

$$\hat{M}\ddot{\hat{x}} + \hat{G}\dot{\hat{x}} + \hat{K}\hat{x} = \hat{B}\hat{u}$$

has the form

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ddot{\hat{x}} + \begin{bmatrix} \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{\hat{x}} + \begin{bmatrix} \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \hat{x}$$
$$= \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ 0 \end{bmatrix} \hat{u},$$
$$y = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 & \hat{C}_3 & \hat{C}_4 \end{bmatrix} \hat{x}, \tag{2.4}$$

where

$$\hat{x} = \begin{bmatrix} \hat{x}_1^T & \hat{x}_2^T & \hat{x}_3^T & \hat{x}_4^T \end{bmatrix}^T,$$

and $\hat{B}_3$ *has full row rank.*

Proof. A detailed constructive proof is given in Appendix B of [32]. □

Remark 2.7. It is important to note that, in general, a combination of all of the described types of transformations in Theorem 2.6 is needed to achieve the condensed form (2.4). Furthermore, even though the proofs to Theorems 2.5 and 2.6 as well as that of the condensed form (2.3) are constructive (see [32, 35]), in general they cannot be implemented in a numerically reliable way. As an alternative way, for matrix pencils, staircase algorithms have been constructed that determine the structural information in the condensed forms via orthogonal transformations; see, e.g., [13, 42].

We can use the normal form (2.4) to derive a first order formulation which, as we will show later, avoids the difficulties of other first order formulations.

COROLLARY 2.8. *Consider system* (1.1)–(1.2). *Then there exists a bijective map between the solutions of* (1.1) *and the components* $\xi_2, \dots, \xi_5$ *of the first order system* $\hat{E}\dot{\xi} = \hat{A}\xi + \tilde{B}_1\hat{u}$ *given by*

$$\left[\begin{array}{c|cccc} I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 \end{array}\right] \dot{\xi} = \left[\begin{array}{c|cccc} 0 & -\hat{K}_{11} & -\hat{K}_{12} & -\hat{K}_{13} & 0 \\ 0 & -\hat{K}_{21} & -\hat{K}_{22} & -\hat{K}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \\ \hline I & 0 & 0 & 0 & 0 \end{array}\right] \xi + \left[\begin{array}{c} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ 0 \\ \hline 0 \end{array}\right] \hat{u},$$

$$y = \left[\begin{array}{ccccc} 0 & \hat{C}_1 & \hat{C}_2 & \hat{C}_3 & \hat{C}_4 \end{array}\right] \xi, \tag{2.5}$$

where $\hat{B}_3$ *has full row rank,*

$$\xi = \left[\begin{array}{ccccc} \xi_1^T & \xi_2^T & \xi_3^T & \xi_4^T & \xi_5^T \end{array}\right]^T = \left[\begin{array}{ccccc} \dot{\hat{x}}_1^T & \hat{x}_1^T & \hat{x}_2^T & \hat{x}_3^T & \hat{x}_4^T \end{array}\right]^T,$$

and $\hat{x} = [\hat{x}_1^T \quad \hat{x}_2^T \quad \hat{x}_3^T \quad \hat{x}_4^T]^T$ *is a solution of* (2.4).

Furthermore, the first order system (2.5) *is I-controllable.*

Proof. By solving for ξ_1 in the last block row of (2.5) we obtain (2.4), which is equivalent to (1.1)–(1.2).

The I-controllability of (2.5) follows immediately from the definition, since in this case

$$[E, AS_\infty(E), B] = \left[\begin{array}{ccccc|cc|c} I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 & -\hat{K}_{13} & 0 & \hat{B}_1 \\ 0 & 0 & I & 0 & 0 & -\hat{K}_{23} & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -I & 0 \\ \hline 0 & I & 0 & 0 & 0 & -\hat{G}_{13} & 0 & 0 \end{array}\right]$$

has full rank. □

Remark 2.9. The bijectivity of the map in Corollary 2.8 follows from the fact that strong equivalence transformations form a bijection and that the linear combination of derivatives of equations that do contain the output (opu-equivalence) does not change the solution sets of classical solutions. Note that the relationship between u and $\hat{u}$ is just a change of basis.

If the system is considered in the distributional setting, then one has to be careful with opu-equivalences, since then the impulse order may change but the smooth parts of the solution are still mapped in a bijective way. See [17, 30] for a detailed discussion of this issue.

Let us illustrate these results with some examples.

Example 2.10. Consider the artificial second order descriptor system (1.1) with

$$M = 0, \qquad G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Since this is in fact a first order system, we can check its I-controllability using condition **C2** with $E = G$, $A = -K$, $B_1 = B$ and see that

$$\operatorname{rank} \begin{bmatrix} E & AS_\infty(E) & B \end{bmatrix} = \operatorname{rank} \left[\begin{array}{cc|c|c} 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] = 2.$$

But if we perform the classical transformation to first order, then we obtain

$$\tilde{E} = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad \tilde{A} = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{array}\right], \qquad \tilde{B} = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ 0 \end{array}\right].$$

In this case

$$\begin{bmatrix} \tilde{E} & \tilde{A}S_\infty(\tilde{E}) & \tilde{B} \end{bmatrix} = \left[\begin{array}{cccc|cc|c} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

does not have full row rank, and hence the system is not I-controllable.

In this system we can easily reduce the classical first order formulation to one which is *I*-controllable by carrying out an index reduction procedure on the first order formulation.

The previous example seems artificial, but a similar phenomenon arises for constrained mechanical systems.

Example 2.11. Consider again the system (1.12). The first order version (1.13) yields

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

which is obviously not I-controllable.

If, however, we use the construction to the normal form (2.4), then we obtain a system with

$$\hat{M} = 0, \qquad \hat{G} = 0, \qquad \hat{K} = K, \qquad \hat{B} = B,$$

and this system is I-controllable.

We see from these examples that the choice of the first order formulation is important. While the classical first order formulation of the system may not be I-controllable, the normal form (2.4) allows one to obtain a first order formulation that is I-controllable.

Remark 2.12. A natural question that arises is whether we could not just formally first introduce a first order formulation and then perform index reduction via transformation to normal form, including opu-equivalence transformations. This is indeed possible if the original triple is regular, i.e., $\det(\lambda^2 M + \lambda G + K) \neq 0$, since it is then known that the length of chains associated with the eigenvalue ∞ is kept invariant under the classical companion formulation [20]. This is not true any longer if $\det(\lambda^2 M + \lambda G + K)$ vanishes identically as the following example of [8] shows. For the singular matrix polynomial

$$P(\lambda) = \begin{bmatrix} \lambda^2 + \lambda & 0 \\ 1 & 0 \end{bmatrix},$$

the right null space is $x(\lambda) = e_2$, which creates a chain of length 1, and the left null space is $y(\lambda) = [-1 \quad \lambda^2 + \lambda]$, which gives $y_0 = -e_1,\ y_1 = e_2$, and $y_2 = e_2$, and thus the chain has length 3.

By considering the first companion linearization, we get

$$L(\lambda) = \lambda \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] + \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right].$$

The right and left null space vector polynomials are

$$x(\lambda) = \left[\begin{array}{c} 0 \\ \lambda \\ \hline 0 \\ 1 \end{array}\right], \qquad y(\lambda) = \left[\begin{array}{c} 1 \\ -\lambda^2 - \lambda \\ \hline \lambda + 1 \\ 0 \end{array}\right],$$

and clearly the right chain does not have the same length as in the original matrix polynomial.

Furthermore, performing a first order formulation first may substantially change the sensitivity of the problem, in particular when computing the index reduction; see [35] for an illustrative example.

Thus it is generally preferable to perform index reductions and reformulations on the original data which is the second order pencil.

In the next section we will show how we can use the condensed forms of this section to check the different controllability and observability conditions directly for second order systems and how to derive first order formulations that preserve these conditions.

3. Controllability for second order systems.

For a descriptor system (1.1)–(1.3), the following definitions extend the concepts of C-controllability and C-observability to second order descriptor systems.

Definition 3.1. *Consider a system as in* (1.1)–(1.3). *A set $\mathcal{R} \subseteq \mathbb{R}^n$ is called* reachable from x_0, $\dot{x}_0$ *if for every $x_f \in \mathcal{R}$ there exists a piecewise continuous input function u that transfers the system in finite time from $x(t_0) = x_0$ to x_f.*

A set $\mathcal{R} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is called $\mathcal{R}2$-reachable from x_0, $\dot{x}_0$ *if for every $x_f, \dot{x}_f \in \mathcal{R}$ there exists a piecewise continuous input function u that transfers the system in finite time from $x(t_0) = x_0$, $\dot{x}(t_0) = \dot{x}_0$ to $x_f, \dot{x}_f$. The system is called*

(i) C-controllable *if for any x_0 and $\dot{x}_0$ and any $x_f \in \mathbb{R}^n$ there exist a time t_f and a piecewise continuous input function $u : [t_0, t_f] \to \mathbb{R}^m$ such that $x(t_f) = x_f$;*

(ii) strongly C2-controllable *if for any $x_0, \dot{x}_0$ and any $x_f, \dot{x}_f \in \mathbb{R}^n$ there exist a time t_f and a piecewise continuous input function $u : [t_0, t_f] \to \mathbb{R}^m$ such that $x(t_f) = x_f,\ \dot{x}(t_f) = \dot{x}_f$;*

(iii) $\mathcal{R}$-controllable *if any state* x_f *in the reachable set* $\mathcal{R}$ *can be reached from any admissible* $x_0, \dot{x}_0$ *in finite time;*

(iv) $\mathcal{R}2$-controllable *if any state and derivative* $(x_f, \dot{x}_f)$ *in the* $\mathcal{R}2$*-reachable set can be reached from any admissible* $x_0, \dot{x}_0$ *in finite time.*

By using the normal form (2.4) we get a variation of C2-controllability, which is better adapted to the problem.

DEFINITION 3.2. *A system in normal form* (2.4) *is called* C2-controllable *if for any* $\hat{x}(0), \dot{\hat{x}}_1(0)$ *and any* $\hat{x}_f \in \mathbb{R}^n, \hat{x}_{1,f} \in \mathbb{R}^{\dim(\hat{x}_1)}$ *there exists a time* t_f *and a piecewise continuous input function* $u : [t_0, t_f] \to \mathbb{R}^m$ *such that* $\hat{x}(t_f) = \hat{x}_f,\ \dot{\hat{x}}(t_f) = \dot{\hat{x}}_{1,f}$.

We immediately see that a strongly C2-controllable second order descriptor system is also C2-controllable, that a C2-controllable second order descriptor system is also C-controllable, and that an $\mathcal{R}2$-controllable second order descriptor system is also $\mathcal{R}$-controllable.

For the analysis of controllability conditions let us first discuss the case that M is invertible; i.e., we have an implicitly defined second order ordinary differential equation. Then it is known that C-controllability is equivalent to C2-, strong C2-, $\mathcal{R}2$-, and $\mathcal{R}$-controllability, and all five are characterized by the Hautus criterion [1, 24],

$$\operatorname{rank} \begin{bmatrix} \lambda^2 M + \lambda G + K & B \end{bmatrix} = n \text{ for all } \lambda \in \sigma(M, G, K), \tag{3.1}$$

where $\sigma(M, G, K)$ denotes *the spectrum* of the matrix polynomial $P(\lambda) = \lambda^2 M + \lambda G + K$, i.e., the roots of $\det P(\lambda)$.

Strong C2-controllability and $\mathcal{R}2$-controllability is trivially characterized via the classical (companion) first order form.

COROLLARY 3.3. *Consider a second order descriptor system* (1.1) *and its classical (companion) first order form*

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \dot{z} = \begin{bmatrix} -G & -K \\ I & 0 \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u. \tag{3.2}$$

(i) *System* (1.1) *is strongly C2-controllable if and only if* (3.2) *is C-controllable.*

(ii) *System* (1.1) *is* $\mathcal{R}2$*-controllable if and only if* (3.2) *is* $\mathcal{R}$*-controllable.*

To characterize the other controllability conditions for second order descriptor systems, we make use of the condensed forms of section 2. From (2.4) we see that, for a consistent initial condition in the variables that occurs only in first order, we can prescribe only initial values and not initial derivatives. This immediately implies the following corollary.

COROLLARY 3.4.

(i) *A second order system in normal form* (2.4) *is C2-controllable if and only if the associated first order system* (2.5) *is C-controllable.*

(ii) *A second order system in normal form* (2.4) *is* $\mathcal{R}2$*-controllable if and only if the associated first order system* (2.5) *is* $\mathcal{R}$*-controllable.*

We can illustrate the difference between strong C2-controllability and C2-controllability by the following example.

Example 3.5. Consider the following C2-controllable second order system in normal form (2.4):

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

The first order formulation given in Corollary 2.8 is given by

$$\left[\begin{array}{c|cc} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right] \dot{\xi} = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] \xi + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \end{array}\right] u.$$

This system is C-controllable, since

$$\operatorname{rank} \left[\begin{array}{cc} \alpha E - \beta A & B_1 \end{array}\right] = \left[\begin{array}{ccc|c} \alpha & 0 & 0 & 1 \\ 0 & -\beta & \alpha & 0 \\ -\beta & \alpha & 0 & 0 \end{array}\right] = 3 \text{ for all } (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}.$$

But the classical first order formulation

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \dot{z} = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] z + \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] u$$

is not C-controllable, since

$$\operatorname{rank} \left[\begin{array}{cc} \alpha E - \beta A & B_1 \end{array}\right] = \left[\begin{array}{cccc|c} \alpha & 0 & 0 & 0 & 1 \\ 0 & -\beta & \beta & 0 & 0 \\ \beta & 0 & \alpha & 0 & 0 \\ 0 & \beta & 0 & \alpha & 0 \end{array}\right] \neq 4 \text{ for } \beta = 0,$$

and hence the original system is not strongly C2-controllable. It is easily checked that both first order formulations are $\mathcal{R}$-controllable.

In order to study I-controllability we will make use of different types of feedback.

DEFINITION 3.6. *Consider a second order descriptor system* (1.1)–(1.2).

(i) *The system is called* proportionally I-controllable *if there exists a state feedback* $u = \hat{u} - F_0 x$ *such that the resulting system with coefficients* $(M, G, K + BF_0)$ *is regular and of index at most one.*

(ii) *The system is called* differentially I-controllable *if there exists a first order derivative feedback* $u = \hat{u} - F_1 \dot{x}$ *such that the resulting system with coefficients* $(M, G + BF_1, K)$ *is regular and of index at most one.*

(iii) *The system is called* proportionally and differentially I-controllable *or just* I-controllable *if there exist a proportional and a first order derivative feedback* $u = \hat{u} - F_0 x - F_1 \dot{x}$ *such that the resulting system given by* $(M, G + BF_1, K + BF_0)$ *is regular and of index at most one.*

It is straightforward to show that strong equivalence transformations preserve all types of controllability for second order descriptor systems.

The same is true for proportional and first order derivative feedback. On the other hand, opu-equivalence transformations preserve C-, C2-, and strong C2-controllability as well as $\mathcal{R}$- and $\mathcal{R}$2-controllability but may turn a system that is not I-controllable into one that is I-controllable.

Example 3.7. Consider the first order descriptor system (1.4) given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\xi} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xi = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u.$$

The system is not I-controllable, since

$$\operatorname{rank}\left[\begin{array}{ccc} E & AS_\infty(E) & B_1 \end{array}\right] = \operatorname{rank}\left[\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = 2,$$

but if we apply the opu-equivalence transformation of multiplying by

$$P = \begin{bmatrix} 1 & 0 & -\frac{d}{dt} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

from the left, then we obtain the I-controllable system

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\xi} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xi = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u.$$

As we will see below, a combination of opu-equivalence transformations and proportional and first order derivative feedbacks together always allows one to make a second order system regular and of index at most one, which then implies I-controllability. See also [30, 31] for similar results in the first order case.

In the following we derive algebraic characterizations for the different controllability conditions. We begin with systems in normal form (2.4).

THEOREM 3.8. *Consider a second order descriptor system* (1.1) *in normal form* (2.4), *and let* (2.5) *be the first order system derived from this normal form.*

(i) *The first order system* (2.5) *is $\mathcal{R}$-controllable if the system matrices of the normal form* (2.4) *satisfy* (3.1).

(ii) *System* (2.5) *is I-controllable.*

(iii) *System* (2.5) *is C-controllable if and only if it is $\mathcal{R}$-controllable and the* 4*th row in* (2.4) *is void.*

Proof. Let n_1 be the size of the component ξ_1. To see that (i) holds, we observe that (2.5) is $\mathcal{R}$-controllable if and only if

$$\operatorname{rank}\begin{bmatrix} \lambda I & \lambda\hat{G}_{11} - \hat{K}_{11} & -\hat{K}_{12} & \lambda\hat{G}_{13} - \hat{K}_{13} & 0 & \hat{B}_1 \\ 0 & -\hat{K}_{21} & \lambda I - \hat{K}_{22} & -\hat{K}_{23} & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & 0 & -I & 0 \\ -I & -\lambda I & 0 & 0 & 0 & 0 \end{bmatrix} = n + n_1,$$

and this is the case if and only if

$$\operatorname{rank}\begin{bmatrix} 0 & -\lambda^2 I + \lambda\hat{G}_{11} - \hat{K}_{11} & -\hat{K}_{12} & \lambda\hat{G}_{13} - \hat{K}_{13} & 0 & \hat{B}_1 \\ 0 & -\hat{K}_{21} & \lambda I - \hat{K}_{22} & -\hat{K}_{23} & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & 0 & -I & 0 \\ I & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = n + n_1,$$

which holds if and only if

$$\operatorname{rank}\begin{bmatrix} -\lambda^2 I + \lambda\hat{G}_{11} - \hat{K}_{11} & -\hat{K}_{12} & \lambda\hat{G}_{13} - \hat{K}_{13} & 0 & \hat{B}_1 \\ -\hat{K}_{21} & \lambda I - \hat{K}_{22} & -\hat{K}_{23} & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & -I & 0 \end{bmatrix} = n.$$

By comparison with (2.4) we see that this holds if and only if $\operatorname{rank}[-\lambda^2 M + \lambda G - K \quad B] = n$ for all $\lambda \in \mathbb{C}$, which proves the assertion.

(ii) We first carry out a strong equivalence transformation by a change of basis that eliminates $\hat{G}_{11}$, $\hat{G}_{13}$ and turn (2.5) to the form

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 \end{bmatrix} \dot{\hat{\xi}} + \begin{bmatrix} 0 & \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ 0 & \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \end{bmatrix} \hat{\xi} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ 0 \\ 0 \end{bmatrix} \hat{u}.$$

This system is I-controllable if the matrix

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & \hat{K}_{13} & 0 & \hat{B}_1 \\ 0 & 0 & I & 0 & 0 & \hat{K}_{23} & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 & 0 & \hat{G}_{13} & 0 & 0 \end{bmatrix}$$

has full row rank. But this follows since $\hat{B}_3$ has full row rank.

(iii) Since (see, e.g., [4]) a first order descriptor system is C-controllable if it is $\mathcal{R}$-controllable and $[E \quad B_1]$ has full row rank, we can just check this rank condition. In the given case this matrix has the form

$$\begin{bmatrix} I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 & \hat{B}_1 \\ 0 & 0 & I & 0 & 0 & \hat{B}_2 \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix},$$

and since $\hat{B}_3$ has full row rank, this matrix has full row rank if and only if the 4th block row is void. By considering system (2.5), we see that this holds if and only if the part ξ_5 is void. But $\xi_5 = \hat{x}_4$, and thus we have finished the proof. ∎

THEOREM 3.9. *Consider a second order descriptor system* (1.1) *in normal form* (2.4)*. Then the* 4*th component* $\hat{x}_4$ *in* (2.4) *is void if and only if* $\operatorname{rank}[M \quad G \quad B] = n$.

Proof. From the proof of Theorem 2.6 we see that the component $\hat{x}_4$ is void if and only if there is no rank deficit in $[\hat{B}_9^T \quad \hat{B}_{10}^T \quad \hat{B}_{11}^T \quad \hat{B}_{12}^T \quad \hat{B}_{13}^T]^T$, with $\hat{B}_9, \ldots, \hat{B}_{13}$ as in (2.3).

It remains to show that this is equivalent to $\operatorname{rank}[M \quad G \quad B] = n$. Since this rank is invariant under strong equivalence transformations, it remains to show that $\operatorname{rank}[\hat{M} \quad \hat{G} \quad \hat{B}] = n$ in (2.3) if and only if $[\hat{B}_9^T \quad \hat{B}_{10}^T \quad \hat{B}_{11}^T \quad \hat{B}_{12}^T \quad \hat{B}_{13}^T]^T$ has full row rank. But

$$
\begin{aligned}
&\operatorname{rank}\left[\begin{array}{ccc} \hat{M} & \hat{G} & \hat{B} \end{array}\right] \\
&= \operatorname{rank}\left[\begin{array}{ccccccccccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & \hat{B}_1 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & \hat{B}_2 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & \hat{B}_3 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & \hat{B}_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & I & 0 & 0 & 0 & \hat{B}_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & \hat{B}_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{13}
\end{array}\right] \\
&= \operatorname{rank}\left[\begin{array}{ccccccccc}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{13}
\end{array}\right],
\end{aligned}
$$

and hence the assertion follows. ∎

THEOREM 3.10. *Consider a second order descriptor system* (1.1) *and the corresponding classical (companion) first order form* (3.2). *Then* (3.2) *is C-controllable if and only if* (1.1) *is* $\mathcal{R}2$*-controllable and* $\operatorname{rank}[M, B] = n$.

Proof. System (3.2) is C-controllable if and only if

$$\operatorname{rank}\left[\begin{array}{ccc} \alpha M - \beta G & -\beta K & B \\ \beta I & \alpha I & 0 \end{array}\right] = 2n.$$

Setting $\beta = 0$ gives $\operatorname{rank}[M, B] = n$, and $\beta \neq 0$ gives the $\mathcal{R}2$-controllability condition. ∎

Obviously, for a second order descriptor system (1.1), $\operatorname{rank}[\lambda^2 M + \lambda G + K \quad B]$ is invariant under strong equivalence transformations, proportional and first order derivative feedback, and opu-equivalence transformations. Thus, we can combine these results with Corollaries 3.3 and 3.4.

COROLLARY 3.11. *A second order descriptor system of the form* (1.1) *is*

(i) $\mathcal{R}2$*-controllable if and only if*

$$\operatorname{rank}\left[\begin{array}{cc} \lambda^2 M + \lambda G + K & B \end{array}\right] = n \quad \text{for all } \lambda \in \mathbb{C};$$

(ii) $\mathcal{C}2$*-controllable if and only if it is* $\mathcal{R}2$*-controllable and*

$$\operatorname{rank}\left[\begin{array}{ccc} M & G & B \end{array}\right] = n;$$

(iii) *strongly* $\mathcal{C}2$*-controllable if and only if it is* $\mathcal{R}2$*-controllable and*

$$\operatorname{rank}\begin{bmatrix} M & B \end{bmatrix} = n.$$

Let us illustrate this result with an example.

Example 3.12. By continuing with the data of Example 2.11, we obtain

$$\operatorname{rank}\begin{bmatrix} \lambda^2 M + \lambda G + K & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \lambda^2+\lambda+1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = 2 \text{ for all } \lambda \in \mathbb{C}.$$

Hence, the system is $\mathcal{R}2$-controllable, while

$$\operatorname{rank}\begin{bmatrix} M & G & B \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = 1,$$

and thus the system is not C2-controllable.

To characterize I-controllability we use the condensed form (2.3) which can be obtained by using only strong equivalence transformations.

THEOREM 3.13. *Consider a second order descriptor system* (1.1) *in condensed form* (2.3). *The system is*

(i) *proportionally I-controllable if and only if in* (2.3) *the* 7*th and* 8*th block rows are void and the matrix* $[\hat{B}_{10}^T \quad \dots \quad \hat{B}_{13}^T]^T$ *has full row rank;*

(ii) *first order derivative I-controllable if and only if in* (2.3) *the* 10*th to* 12*th block rows are void and the matrix* $[\hat{B}_7^T \quad \hat{B}_8^T \quad \hat{B}_{13}^T]^T$ *has full row rank;*

(iii) *proportional and first order derivative I-controllable if and only if in* (2.3) *the matrix* $[\hat{B}_7^T \quad \hat{B}_8^T \quad \hat{B}_{10}^T \quad \dots \quad \hat{B}_{13}^T]^T$ *has full row rank.*

Proof. From the proof of Theorem 2.6 we observe the following:

(a) If in (2.3) the 7th and 8th block rows are void, then we do not need a first order derivative feedback to make the system regular and of index at most one. If these are not void, then proportional feedback is not enough to achieve this.

(b) Similarly, if in (2.3) the 10th to 12th block rows are void, then we do not need a proportional feedback to make the system regular and of index at most one. If these are not void, then first order derivative feedback is not enough to achieve this.

(c) If in (2.3) the matrix

$$\begin{bmatrix} \hat{B}_7^T & \hat{B}_8^T & \hat{B}_{10}^T & \dots & \hat{B}_{13}^T \end{bmatrix}^T$$

has full row rank, then we do not need opu-equivalence transformations to make the system regular and of index at most one. If there is a rank deficit, then proportional and first order derivative feedback is not sufficient to make the system regular and of index at most one.

Then with (c) we obtain (iii), with (a) and (c) we obtain (i), and with (b) and (c) we get (ii). $\square$

Theorem 3.13 shows that the condensed form (2.3) and the canonical from (2.4) allow one to check the different controllability properties for second order descriptor systems. For mathematical elegance and a simpler description it would also be nice to have a coordinate-free algebraic characterization. This is given in the following theorem.

THEOREM 3.14. *Consider a second order descriptor system* (1.1) *and its condensed form* (2.3), *and let* $s^{(0,1,2)}$, $s^{(0,2)}$, $s^{(0,1)}$, *and* $s^{(1,2)}$ *be the integer quantities defined in Theorem* 2.4. *Then the system is*

(i) *proportionally and first order derivative I-controllable if and only if*

$$\operatorname{rank} \begin{bmatrix} M & GS_\infty^1 & KS_\infty^2 & B \end{bmatrix} = n, \tag{3.3}$$

where the columns of the matrix S_∞^1 *form a basis of* $\operatorname{kernel} M$*, the columns of* S_∞^2 *form a basis of*

$$\operatorname{kernel} \begin{bmatrix} M \\ Z_1^T G \end{bmatrix} \setminus \operatorname{kernel} \begin{bmatrix} M \\ Z_1^T G \\ Z_3^T K \end{bmatrix},$$

the columns of Z_1 *form a basis of* $\operatorname{kernel} M^T$*, and those of* Z_3 *form a basis of* $\operatorname{kernel} [\begin{smallmatrix} M^T \\ G^T \end{smallmatrix}]$*;*

(ii) *proportionally I-controllable if and only if it satisfies* (i) *and* $s^{(0,1,2)} = s^{(1,2)} = 0$*;*

(iii) *first order derivative I-controllable if and only if it satisfies* (i) *and* $s^{(0,1,2)} = s^{(0,2)} = s^{(0,1)} = 0$.

Proof. (i) In the condensed form (2.3) we have

$$\begin{bmatrix} M & GS_\infty^1 & KS_\infty^2 & B \end{bmatrix}$$
$$= \left[\begin{array}{cccccccc|cccc|c|c}
I_{s^{(0,1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & \hat{B}_1 \\
0 & I_{s^{(1,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & \hat{B}_2 \\
0 & 0 & I_{s^{(0,2)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & \hat{B}_3 \\
0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & \hat{B}_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{s^{(0,1)}} & 0 & 0 & 0 & 0 & \hat{B}_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0 & 0 & \hat{B}_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_a & \hat{B}_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \hat{B}_{13}
\end{array}\right].$$

Thus, $\operatorname{rank}[M \quad GS_\infty^1 \quad KS_\infty^2 \quad B] = n$ if and only if the matrix $sfif[\hat{B}_7^T \quad \hat{B}_8^T \quad \hat{B}_{10}^T \quad \dots \quad \hat{B}_{13}^T]^T$ has full row rank. Then by Theorem 3.13(iii) the system is proportionally and first order derivative I-controllable.

It remains to show that $\operatorname{rank} \begin{bmatrix} M & GS_\infty^1 & KS_\infty^2 & B \end{bmatrix}$ is invariant under strong equivalence. For this let

$$\tilde{M} = PMQ, \qquad \tilde{G} = PGQ, \qquad \tilde{K} = PKQ, \qquad \tilde{B} = PBV,$$

and let $\tilde{Z}_1$, $\tilde{Z}_3$, $\tilde{S}_\infty^1$, $\tilde{S}_\infty^2$ be the corresponding subspaces. Since $\tilde{M}v = 0$ if and only if $PMQv = 0$ if and only if $MQv = 0$, we obtain $Q\tilde{S}_\infty^1 = S_\infty^1$ and analogously in Theorem $P^T\tilde{Z}_1 = Z_1$ and $P^T\tilde{Z}_3 = Z_3$. Since, furthermore,

$$\begin{bmatrix} \tilde{M} \\ \tilde{Z}_1^T\tilde{G} \end{bmatrix} v = 0 \Leftrightarrow \begin{bmatrix} PMQ \\ Z_1^T P^{-1} PGQ \end{bmatrix} v = 0 \Leftrightarrow \begin{bmatrix} M \\ Z_1^T G \end{bmatrix} Qv = 0$$

and

$$\begin{bmatrix} \tilde{M} \\ \tilde{Z}_1^T\tilde{G} \\ \tilde{Z}_3^T\tilde{K} \end{bmatrix} v = 0 \Leftrightarrow \begin{bmatrix} PMQ \\ Z_1^T P^{-1} PGQ \\ Z_3^T P^{-1} PKQ \end{bmatrix} v = 0 \Leftrightarrow \begin{bmatrix} M \\ Z_1^T G \\ Z_3^T K \end{bmatrix} Qv = 0,$$

we have $Q\tilde{S}^2_\infty = S^2_\infty$. Thus, altogether we have

$$\begin{aligned}
&\operatorname{rank}[\tilde{M}, \tilde{G}\tilde{S}^1_\infty, \tilde{K}\tilde{S}^2_\infty, \tilde{B}] \\
&\quad = \operatorname{rank}[PMQ, PGQQ^{-1}S^1_\infty, PKQQ^{-1}S^2_\infty, PBV] \\
&\quad = \operatorname{rank}[M, GS^1_\infty, KS^2_\infty, B].
\end{aligned}$$

This finishes the proof of (i). Parts (ii) and (iii) then follow from Theorem 3.13. □

Remark 3.15. If in Theorem 3.14 we have $M = 0$, then $S^1_\infty = I$, $Z_1 = I$, Z_3 is a basis of kernel G^T, and S^2_∞ is a basis of

$$\operatorname{kernel} G \setminus \operatorname{kernel} \begin{bmatrix} G \\ Z_3 K \end{bmatrix}.$$

Thus, $\operatorname{rank}[M \quad GS^1_\infty \quad KS^2_\infty \quad B] = \operatorname{rank}[G \quad KS^2_\infty \quad B]$. In this case, the condensed form is

$$\left(0, \begin{bmatrix} I_{s^{(0,1)}} & 0 & 0 & 0 \\ 0 & I_{d^{(1)}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & I_a & 0 \\ I_{s^{(0,1)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{B}_5 \\ \hat{B}_6 \\ \hat{B}_9 \\ \hat{B}_{10} \\ \hat{B}_{13} \end{bmatrix} \right),$$

and thus

$$\begin{aligned}
\operatorname{rank} \begin{bmatrix} G & KS^2_\infty & B \end{bmatrix} &= \operatorname{rank} \left[\begin{array}{cccc|c|c} I_{s^{(0,1)}} & 0 & 0 & 0 & 0 & \hat{B}_5 \\ 0 & I_{d^{(1)}} & 0 & 0 & 0 & \hat{B}_6 \\ 0 & 0 & 0 & 0 & I_a & \hat{B}_9 \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_{10} \\ 0 & 0 & 0 & 0 & 0 & \hat{B}_{13} \end{array} \right] \\
&= \operatorname{rank} \begin{bmatrix} G & KS_\infty(G) & B \end{bmatrix}.
\end{aligned}$$

This shows that Theorem 3.14 is a direct generalization of the I-controllability results for first order systems.

Example 3.16. By continuing with Example 2.10, we obtain that the system is proportionally and first order derivative I-controllable if and only if rank $[G \quad KS_\infty(G) \quad B] = n$, which we have seen already. Since $M = 0$ we have $s^{(0,1,2)} = s^{(1,2)} = s^{(0,2)} = 0$, and, thus, the system is proportionally I-controllable as well as first order derivative I-controllable.

This example also demonstrates that condition (3.3) in Theorem 3.14(i) is not equivalent to condition **C2** in (1.14), since the classical first order companion form does not satisfy **C2** but (3.3). Other examples with $M \neq 0$ are easily constructed.

Example 3.17. In Example 1.2 we have

$$GS^1_\infty = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad Z_1^T G = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad Z_3^T K = \begin{bmatrix} H_1 & 0 \end{bmatrix},$$

and

$$\operatorname{kernel} \begin{bmatrix} M \\ Z_1^T G \end{bmatrix} \setminus \operatorname{kernel} \begin{bmatrix} M \\ Z_1^T G \\ Z_3^T K \end{bmatrix} = \emptyset.$$

Then $\operatorname{rank}[M \quad GS^1_\infty \quad KS^2_\infty \quad B] = 3 < n = 5$; i.e., the system is not I-controllable.

We also have similar $\mathcal{R}$-controllability.

THEOREM 3.18. *Consider a second order descriptor system* (1.1) *and its first order formulation* (2.5). *Let $\mathcal{R}$ be the reachable set of* (2.5), *and let*

$$E_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

be partitioned as $\hat{E}$ in (2.5). *Then the following are equivalent:*

(i) *The system is C-controllable.*

(ii) *In the first order formulation* (2.5) *for $\xi_2(t_0), \dots, \xi_5(t_0)$ and $\xi_{2f}, \dots, \xi_{5f}$, there exist t_f and an input function $u : [t_0, t_f] \to \mathbb{R}^m$ such that $\xi_2(t_f) = \xi_{2f}, \dots, \xi_5(t_f) = \xi_{5f}$.*

(iii) *The system is $\mathcal{R}$-controllable and* $\operatorname{Im}(E_0) \subset \mathcal{R}$.

(iv) *The system is $\mathcal{R}$-controllable and* $\operatorname{rank}\begin{bmatrix} M & G & B \end{bmatrix} = n$.

Proof. The equivalence of (i) and (ii) is obvious. To prove the other equivalences, consider the first order system (2.5). By carrying out a strong equivalence transformation with

$$P = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -I \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} I & -\hat{G}_{11} & 0 & -\hat{G}_{13} & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

from left and right, respectively, we obtain the system

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\hat{\xi}} + \begin{bmatrix} 0 & \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ -I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \hat{\xi} = \begin{bmatrix} \hat{B}_1 \\ 0 \\ \hat{B}_2 \\ \hat{B}_3 \\ 0 \end{bmatrix} \hat{u},$$

where $\xi = Q\hat{\xi}$. Since $\hat{B}_3$ has full row rank, we can compress its columns and eliminate with the full-rank part upwards. This gives the system

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\hat{\xi}} + \begin{bmatrix} 0 & \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ -I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \hat{\xi} = \begin{bmatrix} 0 & \tilde{B}_1 \\ 0 & 0 \\ 0 & \tilde{B}_2 \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

By choosing a proportional feedback $u_1 = v_1 - \hat{\xi}_4$, $u_2 = v_2$, which does not change the $\mathcal{R}$-controllability or the reachable set $\mathcal{R}$, we obtain a closed loop system $E\dot{\hat{\xi}} + A\hat{\xi} = Bv$ of the form

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{\hat{\xi}} + \begin{bmatrix} 0 & \hat{K}_{11} & \hat{K}_{12} & \hat{K}_{13} & 0 \\ -I & \hat{G}_{11} & 0 & \hat{G}_{13} & 0 \\ 0 & \hat{K}_{21} & \hat{K}_{22} & \hat{K}_{23} & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \hat{\xi} = \begin{bmatrix} 0 & \tilde{B}_1 \\ 0 & 0 \\ 0 & \tilde{B}_2 \\ I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

By eliminating further in the second coefficient matrix, we get

$$\left[\begin{array}{ccc|cc} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \dot{\hat{\xi}} + \left[\begin{array}{ccc|cc} 0 & \hat{K}_{11} & \hat{K}_{12} & 0 & 0 \\ -I & \hat{G}_{11} & 0 & 0 & 0 \\ 0 & \hat{K}_{21} & \hat{K}_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{array}\right] \hat{\xi} = \left[\begin{array}{cc} -\hat{K}_{13} & \tilde{B}_1 \\ -\hat{G}_{13} & 0 \\ -\hat{K}_{23} & \tilde{B}_2 \\ \hline I & 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This system has the form

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{\hat{\xi}} + \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \hat{\xi} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} v. \tag{3.4}$$

By following [12], we can determine the reachable set as

$$\mathcal{R} = \mathbb{R}^{n_1} \oplus \mathcal{K}(N, \bar{B}_2),$$

where $n_1 = \operatorname{rank}(E)$, $n_2 = n - n_1$ and

$$\mathcal{K}(N, \bar{B}_2) = \operatorname{Im}[\bar{B}_2, N\bar{B}_2, N^2\bar{B}_2, \ldots, N^{n_2-1}\bar{B}_2].$$

Since $N = 0$ we obtain

$$\mathcal{R} = \operatorname{Im} \left[\begin{array}{c|c} I & 0 \\ \hline 0 & \bar{B}_2 \end{array}\right] = \operatorname{Im} \left[\begin{array}{ccc|cc} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

By incorporating the change of variables in the beginning, it remains to show that

$$\operatorname{Im} \begin{bmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \subset \operatorname{Im} \begin{bmatrix} I & -\hat{G}_{11} & 0 & -\hat{G}_{13} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.5}$$

if and only if $\operatorname{rank}[M \quad G \quad B] = n$. But (3.5) holds if and only if the last row is void, which is by Theorem 3.9 the case if and only if $\operatorname{rank} \begin{bmatrix} M & G & B \end{bmatrix} = n$. ∎

THEOREM 3.19. *The second order descriptor system* (1.1) *is* $\mathcal{R}$*-controllable if and only if for the corresponding first order system* (3.4) *the matrix*

$$\begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} [\bar{B}_1, A_1\bar{B}_1, A_1^2\bar{B}_1, \ldots, A_1^{n_1-1}\bar{B}_1]$$

has full row rank.

Proof. From [12] it is known that for a first order system in the form (3.4) the reachable set is $\mathcal{R} = \mathbb{R}^{n_1} \oplus \operatorname{Im}(\mathcal{K}(N, \bar{B}_2))$ and the reachable set from $\xi_0 = 0$ is $\mathcal{R}(0) = \operatorname{Im}(\mathcal{K}(A_1, \bar{B}_1)) \oplus \operatorname{Im}(\mathcal{K}(N, \bar{B}_2))$. Thus, the first order system is $\mathcal{R}$-controllable if and only if $\operatorname{Im}(\mathcal{K}(A_1, \bar{B}_1)) = \mathbb{R}^{n_1}$. The second order descriptor system has in its state only the variables $\xi_2, \ldots, \xi_4$; the other variables come from the transformation to first order and are not relevant. Hence the proof follows. ∎

We conclude this section with a summary of the obtained results. We have shown that natural extensions of the rank conditions **C0**, **C1**, **C2** allow one to characterize C-, C2-, strong C2- $\mathcal{R}$-, $\mathcal{R}$2-, and I-controllability for second order systems but that the common transformations to first order form may destroy the I-controllability. This implies two possible routes for second order descriptor systems. Either one works directly with the second order form and avoids the transformation to first order, or one performs a transformation to first order that preserves the I-controllability. The latter approach would require the computation of the normal form (2.4). If a first order formulation is desirable, then, however, it is essential to first regularize the system and to reduce the index to at most one.

4. Observability of second order descriptor systems. In this section we derive the corresponding observability conditions for second order descriptor systems and analyze, in particular, the duality between controllability and observability. For this we will need the subspace spanned by the right eigenvectors and principal vectors corresponding to the finite eigenvalues of $\lambda^2 M + \lambda G + K$; see [21]. We call this space the *right finite eigenspace of* $\lambda^2 M + \lambda G + K$ and denote by $P_{r,2}$ the projection onto this space.

DEFINITION 4.1. *Consider a system as in* (1.1)–(1.2). *The system is called*

(i) C-observable *if from an output $y = 0$ for the input $u = 0$ it already follows that the system has only the trivial solution $x = 0$;*

(ii) $\mathcal{R}$-observable *if from an output $y = 0$ for the input $u = 0$ it already follows that the solution x satisfies $P_{r,2}x = 0$;*

(iii) I-observable *if the impulsive behavior of the solution is uniquely determined by the impulsive behavior of the output y and the jump behavior of the input u.*

Remark 4.2. Since for the trivial solution also its derivative vanishes, it makes no sense to define a concept like C2-observability.

Because the transformation from (2.4) to (2.5) leaves input and output unchanged and the impulsive behavior of the newly introduced variables is uniquely determined by the impulsive behavior of the old variables, I-observability of second order systems is a direct generalization of I-observability for first order systems. Thus, it follows immediately that a system (2.4) is I-observable if and only if the corresponding first order system (2.5) is I-observable.

THEOREM 4.3. *Consider a second order descriptor system* (1.1)–(1.2)*, in normal form* (2.4)*, and let* (2.5) *be the first order system derived from this normal form. Then the system* (2.4) *is $\mathcal{R}$-observable if and only if the first order system* (2.5) *is $\mathcal{R}$-observable.*

Proof. Let $\hat{P}_{r,2}$ be the projection onto the right finite eigenspace of $\lambda^2\hat{M}+\lambda\hat{G}+\hat{K}$, with $\hat{M}$, $\hat{G}$, $\hat{K}$ as in (2.4), and let $\hat{P}_{r,1}$ be the projection onto the right finite eigenspace of $\lambda\hat{E} + \hat{A}$, with $\hat{E}$, $\hat{A}$ as in (2.5). If we choose the partitioning as in (2.4), then

$$\hat{P}_{r,2} = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So, if (2.4) is $\mathcal{R}$-observable and if we set $u = 0$ and $y = 0$, then it follows that $\hat{x}_1 = 0$ and thus also $\dot{\hat{x}}_1 = 0$. From the fifth block row of (2.5) it then follows that

$\xi_1 = \dot{\hat{x}}_1 = 0$. Accordingly ξ has the form $\xi = [0, \hat{x}^T]^T$. Because

$$\hat{P}_{r,1} = \begin{bmatrix} I & 0 \\ 0 & \hat{P}_{r,2} \end{bmatrix}, \tag{4.1}$$

it follows that $\hat{P}_{r,1}\xi = 0$, and so (2.5) is $\mathcal{R}$-observable. For the converse, observe that the solution ξ of (2.5) has the form $\xi = [\begin{smallmatrix}\xi_1 \\ \hat{x}\end{smallmatrix}]$, where $\hat{x}$ is the solution of (2.4). From $\hat{P}_{r,1}\xi = 0$ and (4.1) it then follows immediately that $\hat{P}_{r,2}\hat{x} = 0$. $\square$

It is again straightforward to show that strong equivalence preserves all types of observability for second order descriptor systems. The same is true for opu-equivalence transformations. Proportional or first order derivative feedback, on the other hand, may change the observability properties.

Example 4.4. The second order descriptor system $M\ddot{x} + G\dot{x} + Kx = Bu,\ y = Cx$, with

$$M = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \qquad K = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

is clearly C-observable, because from $u = 0$ one obtains $x_3 = 0$ and from $y = 0$ one gets $x_1 = x_2 = 0$. For the proportional feedback $u = v + x_3$ and the closed loop system with input v, we obtain

$$\hat{M} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \qquad \hat{G} = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \qquad \hat{K} = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Here we can no longer make any statement about x_3. Similar examples can be constructed by using first order derivative feedback. Analogously one can also show that $\mathcal{R}$-observability is not invariant.

To see the noninvariance of I-observability, consider a modification of system (1.12)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ddot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\dot{x} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix}x. \tag{4.2}$$

By using as input the Heaviside function $H(t)$, which is 0 for $t < 0$ and 1 for $t \geq 0$, the solution is $x_1 = H(t)$ and $x_2 = -H(t) - \dot{H} - \ddot{H}$, but this impulsive solution is not observed in the output $y = x_1$. By choosing the proportional feedback $u = -x_2 + x_1 + v$, we obtain $x_2 = v$ and x_1 solves the second order differential equation $\ddot{x}_1 + \dot{x}_1 + x_1 + x_2 = 0$. A jump in the input v will be integrated, and hence the output cannot contain impulsive parts if the input is piecewise continuous; i.e., all potential impulsive parts of the solution (of which there are none) are observed in the output.

The noninvariance under proportional or first order derivative feedback poses a problem insofar as we cannot use Theorem 2.6 to construct a system that can be correctly transformed to first order. For this reason we proceed in a different way and make use of Theorem 14 in [35], which implies the following result.

THEOREM 4.5. *Consider a second order descriptor system* (1.1)–(1.2) *with differentiation index* ν*, and suppose that* Bu *is* $\nu-1$ *times continuously differentiable. Then there exists a sequence of strong equivalence transformations and sopu-equivalence transformations such that the transformed system has the coefficients*

$$
\begin{aligned}
&(\hat{M},\ \hat{G},\ \hat{K},\ \hat{B})\\
&= \left(\begin{bmatrix} I_{d^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{G}_{1,1} & 0 & 0 & \tilde{G}_{1,4} \\ 0 & I_{d^1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.\\
&\left. \begin{bmatrix} \tilde{K}_{1,1} & \tilde{K}_{1,2} & 0 & \tilde{K}_{1,4} \\ \tilde{K}_{2,1} & \tilde{K}_{2,2} & 0 & \tilde{K}_{2,4} \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \hat{B}_3 \\ \hat{B}_4 \end{bmatrix} \right),
\end{aligned}
$$

where $\hat{M},\hat{G},\hat{K} \in \mathbb{R}^{n\times n}$ *and* $\hat{B} \in \mathbb{R}[D_\mu]^{n\times m}$*.*

Note here that we allow sopu-equivalences, which do not increase the differentiation order of x but may introduce derivatives of u.

Remark 4.6. In contrast to Theorem 2.6, the transformed system with coefficients as in Theorem 4.5 requires derivatives of u. But since we consider $u = 0$ only to check $\mathcal{R}$- and C-observability, this is not a problem.

Now that we have a transformation to normal form that preserves $\mathcal{R}$-, C-, and I-observability, we immediately observe that the first order duality of controllability and observability [12, 28] also holds in the second order case if the particular output $y = Cx$ is used, since transposing and changing the roles of B and C^T can be carried out also in the specific reduction order given by (2.5). Thus we have the following immediate consequences for the dual system to (1.1) given by

$$M^T\ddot{x} + G^T\dot{x} + K^T x = C^T u. \tag{4.3}$$

THEOREM 4.7. *Consider a second order descriptor system* (1.1)–(1.2)*. The system is* C-observable *if and only if the dual system* (4.3) *is* C2-controllable.

Proof. Let (1.1)–(1.2) be in normal form (2.4). The system is C-observable if and only if the corresponding first order system (2.5) is C-observable. This, however, is the case if and only if the dual first order system is C-controllable; see, e.g., [12]. But the dual first order system is C-controllable if and only if the dual second order system is C2-controllable. ∎

The result for $\mathcal{R}$-observability is analogous.

THEOREM 4.8. *Consider a second order descriptor system* (1.1)–(1.2)*. The system is* $\mathcal{R}$*-observable if and only if the dual system* (4.3) *is* $\mathcal{R}$*2-controllable.*

Proof. By using Theorem 4.3 the proof is analogous to that of Theorem 4.7. ∎

THEOREM 4.9. *Consider a second order descriptor system* (1.1)–(1.2)*. The system is I-observable if and only if the dual system* (4.3) *is proportionally and first order derivative I-controllable.*

Proof. The proof is analogous to that of Theorem 4.7. ∎

For completeness we will also present coordinate-free algebraic conditions that can be immediately derived from the duality between controllability and observability.

COROLLARY 4.10. *A second order descriptor system* (1.1)–(1.2) *is*

(i) $\mathcal{R}$-observable *if and only if*

$$\operatorname{rank}\begin{bmatrix} \lambda^2 M + \lambda G + K \\ C \end{bmatrix} = n;$$

(ii) C-observable *if and only if it is $\mathcal{R}$-observable and*

$$\operatorname{rank}\begin{bmatrix} M \\ G \\ C \end{bmatrix} = n;$$

(iii) I-observable *if and only if*

$$\operatorname{rank}\begin{bmatrix} M \\ T_\infty^1 G \\ T_\infty^2 K \\ C \end{bmatrix} = n,$$

where the rows of the matrix T_∞^1 form a basis of cokernel M and the rows of T_∞^2 form a basis of

$$\operatorname{cokernel}\begin{bmatrix} M & GZ_2 \end{bmatrix} \setminus \operatorname{cokernel}\begin{bmatrix} M & GZ_2 & KZ_5 \end{bmatrix},$$

the columns of Z_2 form a basis of kernel M, *and those of Z_5 form a basis of* kernel $\left[\begin{smallmatrix} M \\ G \end{smallmatrix}\right]$.

Remark 4.11. In the output equation (1.2) we could have also considered a term $C_1\dot{x}$. If such a term is present, then we can still transform to the form (2.4) and investigate the observability. In this case, however, the duality may be lost if derivatives of $\hat{x}_2, \ldots, \hat{x}_5$ occur.

5. Conclusion. We have shown how to extend the analysis of controllability and observability conditions to second order descriptor systems. We have demonstrated that the straightforward idea of using a classical first order formulation and then applying the first order results does not work, because in particular I-controllability and I-observability are not invariant under this transformation to first order. We have derived normal forms which can be used to check the controllability and observability conditions and from which we can obtain new first order formulations which preserve I-controllability and I-observability.

All of the presented results can be extended to nonreal, rectangular, and also higher order descriptor systems.

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