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Stability and Stabilization of Aperiodic Sampled-Data Control Systems: An Approach Using Robust Linear Matrix Inequalities

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***Abstract*— Stability analysis of an aperiodic sampled-data control system is considered for application to network and embedded control. The stability condition is described in a linear matrix inequality to be satisfied for all possible sampling intervals. Although this condition is numerically intractable, a tractable sufficient condition can be constructed with the mean value theorem. Special attention is paid to tightness of the sufficient condition for less conservative stability analysis. A region-dividing technique for reduction of conservatism and generalization to stabilization are also discussed. Examples show the efficacy of the approach.**

***Keywords*—sampled-data control, robust linear matrix inequality, conservatism, asymptotic exactness, semidefinite programming, adaptive division**

I. Introduction

Sampled-data control is mature research area and established methodology is available both for analysis and design [2]. However, most of the existing results assume constant sampling interval and cannot be applied to network and embedded control systems, whose sampling interval is uncertain and varying with time.

For analysis and design of such an aperiodic sampled-data control system, several approaches have been proposed. Some of them are based on the continuous-time or hybrid framework [6], [14], [15]. The stability conditions presented there are rather conservative though applicable to general systems. Recent approaches such as [7], [8], [9], [23] provide less conservative stability conditions in the discrete-time framework. Hetel–Daafouz–Iung [9] gave a stability condition by approximately evaluating the effect of aperiodic sampling with a polynomial. Increase of the degree of the polynomial reduces conservatism of the result. On the other hand, Fujioka [7], [8] and Suh [23] gave stability conditions based on division of the region where the uncertain sampling interval takes a value. Here, increase of the resolution of the division leads us to a less conservative result. Skaf–Boyd [21] used such division to evaluate degradation of the optimal quadratic performance of an aperiodic sampled-data control system.

The approach in this paper inherits some ideas from Fujioka [7], [8] and Suh [23] but is improved with the following three techniques. First, the stability condition is based on the delta-operator representation [13] and converges to the continuous-time stability condition as the sampling interval goes to zero. Due to this property, stability analysis can be made without numerical difficulty even when the sampling interval is small. Second, the effect of aperiodic sampling is modeled as parametric uncertainty rather than matrix uncertainty for less conservative stability analysis. Third, a technique of adaptive division is introduced for suppression of the computational cost. In this paper, the stability condition is in the form of a linear matrix inequality (LMI, in short) to be satisfied for all possible sampling intervals. In general, a parameter-dependent LMI to be satisfied for all possible parameter values is called a *robust LMI*. Recently, intensive investigation has been made on a robust LMI whose parameter dependence is polynomial or rational ([1], [3], [5], [11], [19], [20] for example). Since our stability condition exponentially depends on the sampling interval, we take a different approach using the mean value theorem. This is an adaptation of the technique of Chesi–Hung [4].

This paper is organized as follows. In Section II, our problem is provided. In Section III, the stability condition is presented in a robust LMI and its tractable sufficient condition is given. Section IV introduces a region-dividing technique for less conservative stability analysis. The brief Section V is for generalization to stabilization. Illustrating examples are presented in Section VI and extension of the approach is discussed in Section VII. Details omitted in this paper can be found in the technical report [18].

Notation is standard. The symbols O and I denote the zero matrix and the identity matrix of appropriate size. For symmetric matrices A and B, the inequalities $A \succ B$ and $A \succeq B$ mean that $A - B$ is positive definite and positive semidefinite, respectively.

II. Problem

We consider a continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with the n-dimensional state $x(t)$ and its stabilization by state-feedback control with a constant gain F. The state is measured only at discrete time instants $0 = t_0 < t_1 < t_2 < \cdots$ such that $\lim_{k\to\infty} t_k = \infty$ and the input is a piecewise signal determined as

$$u(t) = Fx(t_k) \quad (t_k \le t < t_{k+1})$$

for each $k = 0, 1, 2, \ldots$. We refer to this control system by S henceforth.

The control system S is different from a conventional sampled-data control system in that the sampling interval $t_{k+1} - t_k$ is not necessarily constant but may vary with k. We assume availability of its bounds $\underline{h}$ and $\overline{h}$ such that

$$\underline{h} \le t_{k+1} - t_k \le \overline{h} \quad (k = 0, 1, 2, \ldots).$$

Our problem is to verify stability of the control system S. We present an approach using a robust LMI in the following sections.

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III. Proposed Stability Condition

A. Formulation into a robust LMI

We use the criterion of quadratic stability for stability analysis of the control system S. Note first in S that the states at two adjacent sampling instants are related by

$$x(t_{k+1}) = \Phi(t_{k+1} - t_k)x(t_k) \quad (k = 0, 1, 2, \dots)$$

with

$$\Phi(h) = \mathrm{e}^{Ah} + \int_0^h \mathrm{e}^{At}\mathrm{d}t\, BF = I + \int_0^h \mathrm{e}^{At}\mathrm{d}t(A + BF).$$

Hence, the exponential stability of S follows if there exists a symmetric matrix Q such that

$$Q \succ O, \quad Q - \Phi(h)Q\Phi(h)^{\mathrm{T}} \succ O \quad (\underline{h} \le h \le \overline{h}).$$

This stability condition has two drawbacks to be utilized for stability analysis. First, the condition is in the form of a robust LMI and we need to find Q that satisfies the inequality for infinitely many values of h. Second, the matrix $\Phi(h)$ is close to identity when h is small, which makes the condition difficult to handle numerically. In the following, we consider the second drawback first and then the first.

In order to avoid the numerical drawback of $\Phi(h)$, we use the following stability condition equivalent to the previous one.

Proposition 1: The control system S is exponentially stable if there exists a symmetric matrix Q such that

$$\begin{aligned} &Q \succ O,\\ &-\Psi(h)Q - Q\Psi(h)^{\mathrm{T}} - h\Psi(h)Q\Psi(h)^{\mathrm{T}} \succ O\\ &\qquad\qquad (\underline{h} \le h \le \overline{h}), \end{aligned} \tag{1}$$

where

$$\Psi(h) = \frac{1}{h}(\Phi(h) - I) = \frac{1}{h}\int_0^h \mathrm{e}^{At}\,\mathrm{d}t(A + BF). \tag{2}$$

Proof. Substitution of $\Psi(h) = (\Phi(h)-I)/h$ to (1) gives the quadratic stability condition above. The proposition hence follows. □

In the limit of $h \to 0$, the matrix $\Psi(h)$ converges to $A + BF$, which is the system matrix of the continuous-time control system where a continuous-time state-feedback control $u(t) = Fx(t)$ is applied to $\dot{x}(t) = Ax(t)+Bu(t)$. In the same limit, the stability condition of Proposition 1 assures stability of this continuous-time control system. Hence, this condition has no numerical drawback discussed above even for small h.

The condition of Proposition 1 is again a robust LMI to be satisfied for infinitely many values of h. Since it is difficult to find Q satisfying the robust LMI, we consider its sufficient condition expressed by finitely many LMIs. We can solve those LMIs using the standard interior-point method. Once a solution Q is found, the same Q serves as a solution of the original robust LMI. In this approach, it is critical to use a tight sufficient condition. Motivated by this, we assume availability of the real Jordan canonical form of A and consider a sufficient condition based on it.

We consider the case of $\underline{h} > 0$ in Section III-B and the case of $\underline{h} = 0$ in Section III-C. The equality $\underline{h} = 0$ means that a positive lower bound is not available for the sampling interval. This latter case can be considered because our stability condition is usable with small h.

B. The case of $\underline{h} > 0$

The function $\Psi(h)$ in Proposition 1 has exponential dependence on the uncertain parameter h. Since exponential parameter dependence is difficult to handle, we replace $\Psi(h)$ with a new function dependent not only on h but also on an additional uncertain parameter θ. This new function is multi-affine in h and θ, that is, it is affine in h for any fixed θ and *vice versa*. This multi-affinity leads us to the desired sufficient condition expressed by finitely many LMIs.

Definition of the new function uses the real Jordan canonical form of A [10, Section 3.4]. For the real $n \times n$ matrix A, there exists a real nonsingular matrix T such that

$$A = T\begin{pmatrix} A^{(1)} & & & \\ & A^{(2)} & & \\ & & \ddots & \\ & & & A^{(r)} \end{pmatrix}T^{-1}$$

(the omitted elements are equal to zero). Here, each $A^{(i)}$ is a matrix having one of the following two forms:

$$\begin{pmatrix} \lambda^{(i)} & 1 & & \\ & \lambda^{(i)} & \ddots & \\ & & \ddots & 1 \\ & & & \lambda^{(i)} \end{pmatrix}, \quad \begin{pmatrix} P^{(i)} & I_2 & & \\ & P^{(i)} & \ddots & \\ & & \ddots & I_2 \\ & & & P^{(i)} \end{pmatrix}, \tag{3}$$

where $\lambda^{(i)}$ is a real number, $P^{(i)}$ is a 2×2 real matrix of the form $P^{(i)} = \begin{pmatrix} p^{(i)} & q^{(i)} \\ -q^{(i)} & p^{(i)} \end{pmatrix}$, and I_2 is the 2×2 identity matrix.

The function for replacement of $\Psi(h)$ is now defined as

$$\Psi_{\widehat{h}}(h, \theta) = \frac{1}{h}\Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\,\mathrm{d}t + (h - \widehat{h})TE(\theta)T^{-1}\Big](A + BF). \tag{4}$$

Here, $\widehat{h}$ is any fixed number such that $\underline{h} \le \widehat{h} \le \overline{h}$. The function $\Psi_{\widehat{h}}(h, \theta)$ results from expansion of $\Psi(h)$ around this $\widehat{h}$ as seen in the proof of Lemma 2 below. Also in (4), $E(\theta)$ is an affine function of a newly introduced n-dimensional uncertain parameter θ and has the block-diagonal form

$$E(\theta) = \begin{pmatrix} E^{(1)}(\theta) & & & \\ & E^{(2)}(\theta) & & \\ & & \ddots & \\ & & & E^{(r)}(\theta) \end{pmatrix}$$

consistent with the real Jordan canonical form of A. More precisely, when $A^{(i)}$ is an $n^{(i)} \times n^{(i)}$ matrix of the left form of (3), the corresponding $E^{(i)}(\theta)$ is

$$\begin{pmatrix} \theta_1^{(i)} & \theta_2^{(i)} & \cdots & \theta_{n^{(i)}}^{(i)} \\ & \theta_1^{(i)} & \ddots & \vdots \\ & & \ddots & \theta_2^{(i)} \\ & & & \theta_1^{(i)} \end{pmatrix}$$

with $n^{(i)}$ uncertain parameters $\theta_j^{(i)}$ $(j = 1, 2, \dots, n^{(i)})$, which are elements of θ. When $A^{(i)}$ is a $2m^{(i)} \times 2m^{(i)}$ matrix of the right form of (3), the corresponding $E^{(i)}(\theta)$ is

$$\begin{pmatrix} \Xi_1^{(i)} & \Xi_2^{(i)} & \cdots & \Xi_{m^{(i)}}^{(i)} \\ & \Xi_1^{(i)} & \ddots & \vdots \\ & & \ddots & \Xi_2^{(i)} \\ & & & \Xi_1^{(i)} \end{pmatrix},$$

where $\Xi_j^{(i)} = \begin{pmatrix} \xi_j^{(i)} & \eta_j^{(i)} \\ -\eta_j^{(i)} & \xi_j^{(i)} \end{pmatrix}$ and $\xi_j^{(i)}$, $\eta_j^{(i)}$ $(j = 1, 2, \dots, m^{(i)})$ are $2m^{(i)}$ uncertain parameters contained again in θ. Hence, θ is a vector consisting of the parameters $\theta_j^{(i)}$, $\xi_j^{(i)}$, and $\eta_j^{(i)}$, whose dimension sums up to n. The domain of the parameters $\underline{\theta}_j^{(i)} \le \theta_j^{(i)} \le \overline{\theta}_j^{(i)}$, $\underline{\xi}_j^{(i)} \le \xi_j^{(i)} \le \overline{\xi}_j^{(i)}$, and $\underline{\eta}_j^{(i)} \le \eta_j^{(i)} \le \overline{\eta}_j^{(i)}$ are defined as

$$\begin{aligned}
\underline{\theta}_j^{(i)} &= \min_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{\lambda^{(i)} h}, \\
\overline{\theta}_j^{(i)} &= \max_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{\lambda^{(i)} h}, \\
\underline{\xi}_j^{(i)} &= \min_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, \\
\overline{\xi}_j^{(i)} &= \max_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \cos q^{(i)} h, \\
\underline{\eta}_j^{(i)} &= \min_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h, \\
\overline{\eta}_j^{(i)} &= \max_{\underline{h} \le h \le \overline{h}} \frac{h^{j-1}}{(j-1)!} \mathrm{e}^{p^{(i)} h} \sin q^{(i)} h
\end{aligned}$$

with the convention $0! = 1$. The domain of θ is hence a box-shaped set, which will be denoted by Θ.

Properties of $\Psi_{\widehat{h}}(h, \theta)$ and Θ are summarized in the next lemma.

Lemma 2: Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. The function $\Psi_{\widehat{h}}(h, \theta)$ and the domain Θ defined above have the following properties: (i) for any $\underline{h} \le h \le \overline{h}$, there exists $\theta \in \Theta$ such that $\Psi(h) = \Psi_{\widehat{h}}(h, \theta)$; (ii) $h\Psi_{\widehat{h}}(h, \theta)$ is multi-affine in h and θ; (iii) $\Psi_{\widehat{h}}(h, \theta)$ is independent of θ at $h = \widehat{h}$.

Proof. We prove the property (i) in the special case that A is a 2×2 matrix of the form

$$A = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{-1}.$$

The proof in the general case is similar.

From the definition (2) of $\Psi(h)$ it follows that

$$\begin{aligned}
\Psi(h) &= \frac{1}{h} \int_0^h \mathrm{e}^{At}\, \mathrm{d}t (A + BF) \\
&= \frac{1}{h} \Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\, \mathrm{d}t + T \begin{pmatrix} \int_{\widehat{h}}^h \mathrm{e}^{\lambda_1 t}\, \mathrm{d}t & 0 \\ 0 & \int_{\widehat{h}}^h \mathrm{e}^{\lambda_2 t}\, \mathrm{d}t \end{pmatrix} T^{-1} \Big] \\
&\qquad \times (A + BF) \\
&= \frac{1}{h} \Big[\int_0^{\widehat{h}} \mathrm{e}^{At}\, \mathrm{d}t + T \begin{pmatrix} (h - \widehat{h})\mathrm{e}^{\lambda_1 h_1} & 0 \\ 0 & (h - \widehat{h})\mathrm{e}^{\lambda_2 h_2} \end{pmatrix} T^{-1} \Big] \\
&\qquad \times (A + BF).
\end{aligned}$$

The last equality is implied by the mean value theorem with h_1 and h_2 being some numbers between $\widehat{h}$ and h. Since $h_1, h_2 \in [\underline{h}, \overline{h}]$, we have

$$\min_{\underline{h} \le h \le \overline{h}} \mathrm{e}^{\lambda_i h} \le \mathrm{e}^{\lambda_i h_i} \le \max_{\underline{h} \le h \le \overline{h}} \mathrm{e}^{\lambda_i h} \quad (i = 1, 2),$$

which means $(\mathrm{e}^{\lambda_1 h_1} \quad \mathrm{e}^{\lambda_2 h_2})^{\mathrm{T}} \in \Theta$. The property (i) hence follows.

The properties (ii) and (iii) are obvious from the definition of $\Psi_{\widehat{h}}(h, \theta)$. □

We replace $\Psi(h)$ by $\Psi_{\widehat{h}}(h, \theta)$ in the condition of Proposition 1. We will see below that the resulting LMI has the vertex property due to the multi-affinity of $h\Psi_{\widehat{h}}(h, \theta)$ (Lemma 2 (ii)). Namely, the LMI holds for all $\underline{h} \le h \le \overline{h}$ and $\theta \in \Theta$ if and only if the same LMI holds only at the vertices. We hence obtain a sufficient condition consisting of a finite number of LMIs, which makes the stability analysis tractable. Let $\mathrm{ver}\, \Theta$ denote the set of the vertices of the box-shaped set Θ.

Theorem 3: Suppose $\underline{h} > 0$. Let $\widehat{h}$ be any number such that $\underline{h} \le \widehat{h} \le \overline{h}$. Define $\Psi_{\widehat{h}}(h, \theta)$ and Θ as above. Then, the control system S is exponentially stable if there exists a symmetric matrix Q such that

$$\begin{aligned}
&Q \succ O, \\
&-\Psi_{\widehat{h}}(h, \theta) Q - Q \Psi_{\widehat{h}}(h, \theta)^{\mathrm{T}} - h \Psi_{\widehat{h}}(h, \theta) Q \Psi_{\widehat{h}}(h, \theta)^{\mathrm{T}} \succ O \\
&\qquad\qquad (h = \underline{h}, \overline{h};\ \theta \in \mathrm{ver}\, \Theta). \qquad (5)
\end{aligned}$$

Proof. Suppose that the condition of the theorem is satisfied. It is equivalent to

$$\begin{pmatrix} -h\Psi_{\widehat{h}}(h, \theta) Q - hQ\Psi_{\widehat{h}}(h, \theta)^{\mathrm{T}} & h\Psi_{\widehat{h}}(h, \theta) Q \\ hQ\Psi_{\widehat{h}}(h, \theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O$$
$$(h = \underline{h}, \overline{h};\ \theta \in \mathrm{ver}\, \Theta).$$

Here, $h\Psi_{\widehat{h}}(h, \theta)$ is multi-affine in h and θ by Lemma 2 (ii). Fix h at either $\underline{h}$ or $\overline{h}$. Affinity of $h\Psi_{\widehat{h}}(h, \theta)$ in θ implies convexity of the above inequality. Hence, this inequality holds for all $\theta \in \Theta$. Next, fix θ at any vector in Θ and use affinity of $h\Psi_{\widehat{h}}(h, \theta)$ in h. Consequently, the same inequality holds for all $\underline{h} \le h \le \overline{h}$ and $\theta \in \Theta$. By Lemma 2 (i), the stability condition of Proposition 1 is now satisfied. Hence, the desired exponential stability follows. □

The choice of $\widehat{h}$ is up to the user. In particular, the choice $\widehat{h} = \underline{h}$ or $\widehat{h} = \overline{h}$ is computationally attractive. Indeed, by Lemma 2 (iii), the inequality (5) is independent of θ either at $h = \underline{h}$ or $h = \overline{h}$ with this choice, which decreases the number of LMIs. When $\underline{h}$ is close to zero, the choice $\widehat{h} = \underline{h}$ is preferable because the choice $\widehat{h} = \overline{h}$ makes $(h - \widehat{h})/h$ large at $h = \underline{h}$, which results in large effect of θ in $\Psi_{\widehat{h}}(\underline{h}, \theta)$.

C. The case of $\underline{h} = 0$

We next consider the case of $\underline{h} = 0$. The basic idea is the same as in the previous case, that is, we introduce a new function $\Psi_{\widehat{h}}(h,\theta)$ based on expansion of $\Psi(h)$ around $\widehat{h}$ and use it in place of $\Psi(h)$ in the stability condition of Proposition 1. We choose $\widehat{h} = 0$ in the present case for boundedness of $\Psi_{\widehat{h}}(h,\theta)$. With Θ being the same as before but $\underline{h} = 0$, the function $\Psi_0(h,\theta)$ has the property of Lemma 2 (i), that is, for any $0 \le h \le \overline{h}$, there exists $\theta \in \Theta$ such that $\Psi(h) = \Psi_0(h,\theta)$. One difference from the previous case is that

$$\Psi_0(h,\theta) = TE(\theta)T^{-1}(A+BF)$$

is actually independent of h. This independence leads us to the stability condition in the present case, which is also obtained by putting formally $\underline{h} = 0$ and $\widehat{h} = 0$ in Theorem 3.

Theorem 4: Suppose $\underline{h} = 0$. Define $\Psi_0(h,\theta)$ and Θ as in the previous section. Then, the control system S is exponentially stable if there exists a symmetric matrix Q such that

$$\begin{aligned} & Q \succ O, \\ & -\Psi_0(h,\theta)Q - Q\Psi_0(h,\theta)^{\mathrm{T}} - h\Psi_0(h,\theta)Q\Psi_0(h,\theta)^{\mathrm{T}} \succ O \\ & \qquad\qquad (h = 0, \overline{h};\ \theta \in \operatorname{ver}\Theta). \end{aligned} \tag{6}$$

Proof. The condition of the theorem is equivalent to

$$\begin{pmatrix} -\Psi_0(h,\theta)Q - Q\Psi_0(h,\theta)^{\mathrm{T}} & \sqrt{h}\Psi_0(h,\theta)Q \\ \sqrt{h}Q\Psi_0(h,\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad (h = 0, \overline{h};\ \theta \in \operatorname{ver}\Theta).$$

Fix h at either 0 or $\overline{h}$. Since the above inequality is convex in θ, it holds for all $\theta \in \Theta$ and so does the inequality (6). Fix θ at any vector in Θ next. Since $\Psi_0(h,\theta)$ is independent of h, the inequality (6) is convex in h and hence it holds for all $0 \le h \le \overline{h}$. In summary, this inequality holds for all $0 \le h \le \overline{h}$ and $\theta \in \Theta$. By the property of Lemma 2 (i), the stability condition of Proposition 1 is now satisfied and assures the desired exponential stability. □

IV. A Region-Dividing Technique

A. Division of the region of the sampling interval

The proposed condition can be considerably conservative when the region of the sampling interval, $[\underline{h}, \overline{h}]$, is large because the condition is based on the mean value theorem there. In such a case, we can reduce conservatism of the condition by division of the region of the sampling interval.

In the sequel, a *division* means a set of subregions $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1,2,\ldots,J\}$ such that

$$\underline{h} = \underline{h}^{[1]} < \overline{h}^{[1]} = \underline{h}^{[2]} < \overline{h}^{[2]} = \underline{h}^{[3]} < \cdots < \overline{h}^{[J]} = \overline{h}.$$

Given such a division Δ, we let $\widehat{h}^{[j]}$ be any number in each subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ for $j = 1,2,\ldots,J$. For j with $\underline{h}^{[j]} = 0$, if any, we let $\widehat{h}^{[j]} = 0$. We define $\Theta^{[j]}$ as Θ but with replacing $\underline{h}$, $\overline{h}$, $\widehat{h}$ by $\underline{h}^{[j]}$, $\overline{h}^{[j]}$, $\widehat{h}^{[j]}$, respectively. We now have the following stability condition.

Theorem 5: Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1,2,\ldots,J\}$ be a division of $[\underline{h}, \overline{h}]$. For each $j = 1,2,\ldots,J$, define $\widehat{h}^{[j]}$ and $\Theta^{[j]}$ as above. Define $\Psi_{\widehat{h}}(h,\theta)$ as (4). Then, the control system S is exponentially stable if there exists a symmetric matrix Q such that

$$\begin{aligned} & Q \succ O, \\ & -\Psi_{\widehat{h}^{[j]}}(h,\theta)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} \\ & \qquad - h\Psi_{\widehat{h}^{[j]}}(h,\theta)Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} \succ O \\ & \qquad (h = \underline{h}^{[j]}, \overline{h}^{[j]};\ \theta \in \operatorname{ver}\Theta^{[j]};\ j = 1,2,\ldots,J). \end{aligned}$$

Proof. For each subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$, the discussion in the proof of Theorem 3 or 4 is applicable depending on whether $\underline{h}^{[j]} > 0$ or $\underline{h}^{[j]} = 0$. Consequently, the condition of Proposition 1 is satisfied for any h in $\cup_{j=1,2,\ldots,J}[\underline{h}^{[j]}, \overline{h}^{[j]}] = [\underline{h}, \overline{h}]$. The exponential stability of S hence follows. □

B. Asymptotic exactness

The stability condition of Theorem 5 is asymptotically exact when the choice $\widehat{h}^{[j]} = \underline{h}^{[j]}$ is adopted for all the subregions. Namely, if there exists Q satisfying the original stability condition of Proposition 1, the same Q satisfies the condition of Theorem 5 for a sufficiently fine division Δ. Hence, conservatism of our tractable stability condition can be reduced to any degree at the cost of increased computational complexity. Such asymptotic exactness has been discussed on existing approaches to aperiodic sampled-data control [8], [21].

For precise statement, we call $|\overline{h}^{[j]} - \underline{h}^{[j]}|$ the *width* of the subregion $[\underline{h}^{[j]}, \overline{h}^{[j]}]$ and $\max_{j=1,2,\ldots,J} |\overline{h}^{[j]} - \underline{h}^{[j]}|$ the *maximum width* of the division $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1,2,\ldots,J\}$, which is denoted by $\overline{\operatorname{wid}}\,\Delta$.

Theorem 6: Suppose that there exists Q satisfying the condition of Proposition 1. Then, the same Q satisfies the condition of Theorem 5 for a division Δ having sufficiently small $\overline{\operatorname{wid}}\,\Delta$ when the choice $\widehat{h}^{[j]} = \underline{h}^{[j]}$ is adopted for any j.

Idea of the proof. The condition of Proposition 1 is equivalent to

$$\begin{pmatrix} -\Psi(h)Q - Q\Psi(h)^{\mathrm{T}} & \sqrt{h}\Psi(h)Q \\ \sqrt{h}Q\Psi(h)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad (\underline{h} \le h \le \overline{h});$$

The condition of Theorem 5 is equivalent to

$$\begin{pmatrix} -\Psi_{\widehat{h}^{[j]}}(h,\theta)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} & \sqrt{h}\Psi_{\widehat{h}^{[j]}}(h,\theta)Q \\ \sqrt{h}Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} & Q \end{pmatrix} \succ O \quad (h = \underline{h}^{[j]}, \overline{h}^{[j]};\ \theta \in \operatorname{ver}\Theta^{[j]};\ j = 1,2,\ldots,J).$$

The left-hand sides of these inequalities differ by

$$\begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h,\theta) - \Psi(h) \\ O \end{pmatrix} \begin{pmatrix} -Q & \sqrt{h}Q \end{pmatrix} + \begin{pmatrix} -Q \\ \sqrt{h}Q \end{pmatrix} \begin{pmatrix} \Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} - \Psi(h)^{\mathrm{T}} & O \end{pmatrix}.$$

Here, the maximum singular value of $\Psi_{\widehat{h}^{[j]}}(h,\theta) - \Psi(h)$ maximized over $h \in [\underline{h}^{[j]}, \overline{h}^{[j]}]$ and $\theta \in \Theta^{[j]}$ is bounded by some number proportional to the width $|\overline{h}^{[j]} - \underline{h}^{[j]}|$ when $\widehat{h}^{[j]} = \underline{h}^{[j]}$. Hence, if Q satisfies the condition of Proposition 1 and the maximum width $\overline{\mathrm{wid}}\,\Delta$ is sufficiently small, the same Q satisfies the condition of Theorem 5 as well. □

C. Formulation into an SDP problem

While fine division gives a less conservative stability condition, it increases the computational cost of the stability analysis. It is hence desirable that fine division is made only in an important region of h. To this aim, we introduce in Section IV-D the technique of *adaptive division*. As its preparation, we show here that our stability condition can be restated in a semidefinite programming (SDP, in short) problem. An SDP problem is an optimization problem having LMIs as its constraints. This fact is important also because it enables us to test our stability condition with the softwares for an SDP problem.

Theorem 7: Let $\Delta = \{[\underline{h}^{[j]}, \overline{h}^{[j]}] \,|\, j = 1, 2, \ldots, J\}$ be a division of $[\underline{h}, \overline{h}]$. Define $\widehat{h}^{[j]}$ and $\Theta^{[j]}$ as in Section IV-A and define $\Psi_{\widehat{h}}(h, \theta)$ as (4). Then, the control system S is exponentially stable if the following SDP problem has a maximum value unbounded from above:

$$\begin{aligned} \text{maximize} \quad & x \\ \text{subject to} \quad & Q \succeq I, \\ & -\Psi_{\widehat{h}^{[j]}}(h,\theta)Q - Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} \\ & \qquad - h\Psi_{\widehat{h}^{[j]}}(h,\theta)Q\Psi_{\widehat{h}^{[j]}}(h,\theta)^{\mathrm{T}} \succeq xI \\ & (h = \underline{h}^{[j]}, \overline{h}^{[j]};\ \theta \in \operatorname{ver}\Theta^{[j]};\ j = 1, 2, \ldots, J). \end{aligned}$$

Proof. If the SDP problem has a maximum value unbounded from above, it is feasible with a positive x. Then, all the left-hand side matrices of the inequalities are positive definite. This implies the existence of Q that satisfies the condition of Theorem 5. Hence, the exponential stability of S follows. □

Note that the SDP problem of Theorem 7 is linear in the design variables x and Q. Hence, unboundedness of its maximum value is actually equivalent to feasibility with a positive x.

D. Adaptive division

In this section, we consider adaptive division of the region of sampling intervals for suppression of the computational cost. The corresponding technique is used in [16], [17].

Suppose that we construct the SDP problem of Theorem 7 for some division and obtain the (bounded) nonpositive maximum value. Since the stability of the control system S is not assured in this case, we are to refine the division. We here notice an *active constraint*, which is an LMI constraint such that the discrepancy between its two sides has a zero eigenvalue with the obtained maximum solution. Since an active constraint prevents the maximum value from being improved, we may be able to make improvement by subdividing a subregion having an active constraint. Based on this idea, we have the following algorithm for adaptive division, which is expected to produce an efficient division, that is, a division that gives a less conservative result with small amount of computation. We here mean by an *active subregion* a subregion having an active constraint. When the maximum value is not attained, an active constraint or an active subregion is not defined.

Algorithm 8:

0. Prepare a coarse division.
1. Solve the SDP problem of Theorem 7 corresponding to the current division.
2. Stop if the problem has a maximum value unbounded from above.
3. If the maximum value is attained, find and subdivide an active subregion. Otherwise, find and subdivide a subregion of the maximum width.
4. Go back to Step 1 unless the number of subregions exceeds the prescribed number. □

Algorithm 8 appears contradictory with Theorem 6 because the produced non-uniform division is not efficient for reduction of the maximum width of a division. We can resolve this contradiction by introducing a notion of the *maximum active width*, which means the maximum width over all active subregions in Δ when the maximum value is attained in the SDP problem for Δ. Algorithm 8 aims at reduction of this maximum active width because it repeats subdivision of an active subregion. Now, it is possible to show that reduction of the maximum active width is no inferior to reduction of the maximum width for improvement of the maximum value of the considered SDP problem. This result justifies the use of Algorithm 8. More details are found in [18].

V. Design of a State-Feedback Gain

Our approach can be generalized to design of a state-feedback gain. The conditions of Theorems 3 and 4 contain the product $\Psi_{\widehat{h}}(h, \theta)Q$, which includes the factor $(A + BF)Q$. If we replace it by $AQ + BG$ and solve the inequalities for Q and G, we can obtain a stabilizing feedback gain by $F = GQ^{-1}$. The region-dividing technique is effective also in this case.

VI. Examples

The proposed approach is applied to the sampled-data control system with

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -0.1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}, \quad F = (-3.75 \quad -11.5).$$

Stability of this control system has been analyzed with various approaches. For example, the approach of [15] verified stability with the sampling interval h varying in the region $(0, 1.1137]$; the approach of [14] with $h \in (0, 1.3659]$; the approach of [8] with $h \in [0.01, 1.72]$; the approach of [23] with $h \in [0.5, 1.729]$. This system is known to be unstable for the constant sampling interval $h = 1.7295$.

We chose the region of the sampling interval with $\underline{h} = 0$ and $\overline{h} = 1.7294$ (*i.e.*, $h \in (0, 1.7294]$) and successfully verified the stability with the approach of Section IV. This shows efficacy of the present approach because the existing approaches do not allow such large region of sampling intervals. The division was adaptively constructed with Algorithm 8 and consisted of 9 subregions when it assured the

TABLE I
STABILITY ANALYSIS WITH ADAPTIVE DIVISION

dividing points	comp. time (s)	max. value
0, 1.7294	0.337	-0.665
0, 0.8647, 1.7294	0.449	-0.147
0, 0.8647, 1.2971, 1.7294	0.502	-0.0353
0, 0.8647, 1.2971, 1.5133, 1.7294	0.524	-0.00870
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.7294	0.593	-0.00214
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7294	0.605	-5.17×10^{-4}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7294	0.668	-1.11×10^{-4}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7159, 1.7294	0.702	-9.81×10^{-6}
0, 0.8647, 1.2971, 1.5133, 1.6214, 1.6754, 1.7024, 1.7159, 1.7227, 1.7294	0.493	$+\infty$

stability. The process of the construction is summarized in Table I, where the stability is assured at the bottom line. The region near to 1.7294 was divided finely, which was considered important for less conservative analysis. Here, we chose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for all the subregions. The SDP problems were solved with the SDP solver SeDuMi [22] and the modeling language YALMIP [12]. The used computer was equipped with Intel Core 2 Duo U7500 ($1.06\,\mathrm{GHz}$) and memory of $2\,\mathrm{GBytes}$.

The result became even better when we chose $\widehat{h}^{[j]} = \overline{h}^{[j]}$ for all the subregions except for the one including the origin. Namely, the stability was assured for a division consisting of only two subregions $[0, 0.8647]$ and $[0.8647, 1.7294]$. The computational time was $0.368\,\mathrm{s}$.

We next designed the state-feedback gain F for the A and B above. We took the approach in Section V with $\underline{h} = 0$ and $\overline{h} = 50$. We chose $\widehat{h}^{[j]} = \underline{h}^{[j]}$ for all the subregions. As a result, a stabilizing gain $F = (-0.0115 \;\; 0.0905)$ was obtained with the division consisting of $[0, 25]$ and $[25, 50]$. The computational time was $0.434\,\mathrm{s}$. The choice $\widehat{h}^{[j]} = \overline{h}^{[j]}$ did not make significant difference this time.

VII. EXTENSION

In this paper, we assumed availability of the real Jordan canonical form of the system matrix A in order to reduce the problem to a robust LMI with parametric uncertainty. This assumption may not be practical when the system dimension n is large. In such a case, an approach with matrix uncertainty may be useful because it does not need the real Jordan canonical form. The result will be reported elsewhere.

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