EXPONENTIAL DICHOTOMY OF DIFFERENCE EQUATIONS AND APPLICATION TO EVOLUTION EQUATIONS ON THE HALF-LINE

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Abstract. For a sequence of bounded linear operator $\{A_n\}_{n=0}^{\infty}$ on a Banach space X we investigate the characterization of exponential dichotomy of the difference equations $v_{n+1} = A_n v_n$. We characterize the exponential dichotomy of difference equations in terms of the existence of solutions to the equations $v_{n+1} = A_n v_n + f_n$ in l_∞ space. Then we apply the results to study the exponential dichotomy of evolution families generated by evolution equations.

1. INTRODUCTION AND PRELIMINARIES

In this paper we are concerned with differrence equations of the form

$$x_{n+1} = A_n x_n, \ n \in \mathbf{N} \tag{1.1}$$

and

$$x_{n+1} = A_n x_n + f_n, \ n \in \mathbf{N}, \tag{1.2}$$

where $A_n, n = 0, 1, 2, ...$, is a sequence of bounded linear operators on a given Banach space X, $x_n, f_n \in X$.

One of the central interests in the asymptotic behavior of solutions to Eq. (1.1) is to find conditions for solutions of Eq. (1.1) to be stable, unstable, and especially to have an exponential dichotomy (see e.g. [10], [3], [8], [19], [2] and the references therein for more details on the history of this problem). In the infinite dimensional case, a sufficient condition for Eq. (1.1) to have an exponential dichotomy is a *a priori* condition that the stable space is complemented (see e.g. [3]). In our recent paper (see [12]) in the case evolution equations we have replaced this condition by a rather Perron-styled one. As a result, we have obtained a necessary and sufficient condition for an evolution equation to have an exponential dichotomy. As is known, there is an analogy between difference equations and differential equations. The central purpose of this paper is to provide for linear difference equations the analogues of the most central results for linear evolution equations. Moreover, we will

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show that using the obtained results one can find sufficient conditions for linear evolution equations to have an exponential dichotomy.

To describe more detailedly our construction we will use the following notations: In this paper X denotes a given complex Banach space. As usual, we denote by $\mathbf{N}$, $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{C}$ the set of natural, real, nonnegative real and complex numbers, respectively. Throughout of this paper we shall consider the following sequence spaces:

$$\begin{aligned} l_\infty(\mathbf{N},X) &:= \{v=\{v_n\}_{n\in\mathbf{N}} : v_n\in X : \sup_{n\in\mathbf{N}} v_n<\infty\} := l_\infty \\ l^0_\infty(\mathbf{N},X) &:= \{v=\{v_n\} : v\in l_\infty;\ v_0=0\} := l^0_\infty \\ l_\infty([n_0,\infty),X) &:= \{v=\{v_n\} : v\in l_\infty;\ 0<n_0\le n\in\mathbf{N}\}. \end{aligned}$$

Let $\{A_n\}_{n\in\mathbf{N}}$ be a sequence of bounded linear operators from X to X which is uniformly bounded. That means that there exists $M>0$ such that $\|A_nx\|\le M\|x\|$ for all $n\in\mathbf{N}$ and $x\in X$. Next we define a discrete evolution family $\mathcal{U}=(U_{n,m})_{n\ge m\ge 0}$ associated with the sequence $\{A_n\}_{n\in\mathbf{N}}$ as follows:

$$\begin{aligned} U_{m,m} &= Id \ \ (\text{ the identity operator in } X) \\ U_{n,m} &= A_{n-1}A_{n-2}...A_m \text{ for } n>m. \end{aligned}$$

The uniform boundedness of $\{A_n\}$ yields the exponential boundedness of the evolution family $(U_{n,m})_{n\ge m\ge 0}$. That is, there exist positive constants K,α such that $\|U_{n,m}x\|\le Ke^{\alpha(n-m)}\|x\|;\ x\in X; n\ge m\ge 0$.

Definition 1.1. The equation (1.1) is said to have an *exponential dichotomy* if there exist a sequence of projection $(P_n)_{n\in\mathbf{N}}$ on X and positive constants N,ν such that:

(1) $A_nP_n=P_{n+1}A_n$.
(2) $A_n:\ker P_n\to\ker P_{n+1}$ is an isomorphism and we denote its inverse by $A^{-1}_{|n}$.
(3) $\|U_{n,m}x\|\le Ne^{-\nu(n-m)}\|x\|; n\ge m\ge 0; x\in P_mX$
(4) Denote by $U_{|m,n}=A^{-1}_{|m}.A^{-1}_{|m+1}..A^{-1}_{|n-1};\ n>m$ and $U_{|m,m}=Id$ then

$$\|U_{|m,n}x\|\le Ne^{-\nu(n-m)}\|x\|;\ n\ge m\ge 0;\ x\in\ker P_n.$$

We define an operator $T: l_\infty\to l_\infty$ as follows: If $u=\{u_n\}$, $f=\{f_n\}\in l_\infty$ satisfy the equation (1.2) set:

$$Tu := f.$$

For $u=\{u_n\}\in l_\infty$, take $f=\{f_n\}$ where $f_n=u_{n+1}-A_nu_n$ we have $\|f_n\|\le(1+M)\|u\|_{l_\infty}$, hence $f\in l_\infty$ and $Tu=f$. That means $D(T)=l_\infty$. It is easy to derive that operator T is a well-defined, bounded linear operator. We denote the restriction of T on l^0_∞ by T_0. From the definition of T the following are obvious:

Remark 1.2. i) $\ker T=\{u=\{u_n\}\in l_\infty : u_n=U_{n,0}u_0\},\ n\in\mathbf{N}$

ii) It is easy to verify that T_0 is injective. Indeed, let $u = \{u_n\},\ v = \{v_n\} \in l^0_\infty$ and $T_0u = T_0v$. Then we have $u_0 = v_0 = 0, u_1 = (T_0v)_0 = v_1, u_2 = A_1u_1 + (T_0u)_1 = A_1v_1 + (T_0v)_1 = v_2, .., u_{n+1} = A_nu_n + (T_0u)_n = A_nv_n + (T_0v)_n = v_{n+1}$, for all $n \in \mathbf{N}$. Hence, $u = v$.

iii) $D(T_0) = l^0_\infty$. Indeed, For $u = \{u_n\} \in l^0_\infty$, take $f = \{f_n\}$ where $f_n = u_{n+1} - A_nu_n$ we have $\|f_n\| \le (1 + M)\|u\|_{l_\infty}$, hence $f \in l_\infty$ and $T_0u = f$. That means $D(T_0) = l^0_\infty$.

Recall that for an operator B on a Banach space Y the approximate point spectrum $A\sigma(B)$ of B is the set of all complex numbers λ such that for every $\epsilon > 0$ there exists $y \in D(B)$ with $\|y\| = 1$ and $\|(\lambda - B)y\| \le \epsilon$. The following lemmas will be needed in the sequel:

Lemma 1.3. *Let $\{\chi_n\}_{n_1>n\ge n_0}$ be positive real numbers and let $c > 1$ and $K, \alpha > 0$ be constants such that $\chi_n \le Ke^{\alpha(n-n_0)}$ and $\sum_{k=n_0}^{n} \chi_n\chi_k^{-1} \le c$ with $n_0 \le n < n_1$. Then there exist N, ν dependent only on K, c, α such that $\chi_n \le Ne^{-\nu(n-n_0)}$ for $n_0 \le n < n_1$.*

Proof. Put $S_n = \sum_{k=n_0}^{n} \frac{1}{\chi_k}$. From $\chi_n.S_n \le c$ we have

$$\frac{-1}{\chi_nS_n} \le -c^{-1}.$$

Hence,

$$S_{n-1} = S_n - \chi_n^{-1} = S_n(1 - \frac{1}{\chi_nS_n}) \le S_n(1 - c^{-1}).$$

Therefore $1/S_n \le (1 - c^{-1})/S_{n-1}$. Thus,

$$\chi_n \le \frac{c}{S_n} \le c\frac{(1 - c^{-1})}{S_{n-1}} \le ... \le c\frac{(1 - c^{-1})^{n-n_0}}{S_{n_0}} = c(1 - c^{-1})^{n-n_0}\chi_{n_0}$$

$$\le Kc\left(\frac{c-1}{c}\right)^{n-n_0}$$

By choosing $N = Kc$; $\nu = \ln\frac{c}{c-1}$ we complete the proof. □

Lemma 1.4. *Let $\{\chi_n\}_{n\in\mathbf{N}}$ be a sequence of positive real numbers. Assume that there are constants $c > 1$ and $K, \alpha \ge 0$ such that $\chi_n \le Ke^{\alpha(n-m)}\chi_m$ and*

$$\sum_{k=m}^{n} \chi_m\chi_k^{-1} \le c,\ \forall\ \ n \ge m \ge 0.$$

Then there exist N, ν dependent only on K, c, α such that $\chi_n \ge Ne^{\nu(n-m)}\chi_m$ for $n \ge m \ge 0$.

Proof. Put $S_m = \sum_{k=m}^{n} \frac{1}{\chi_k}$. From $\chi_m.S_m \le c$ we have

$$\frac{-1}{\chi_nS_n} \le -c^{-1}.$$

Hence,

$$S_n = S_{n-1} - \chi_n^{-1} = S_{n-1}(1 - \frac{1}{\chi_{n-1}S_{n-1}}) \leq S_{n-1}(1 - c^{-1}).$$

Therefore $1/S_{n-1} \leq 1 - c^{-1}/S_n$. Thus,

$$\chi_m \leq \frac{c}{S_m} \leq c\frac{(1-c^{-1})}{S_{m+1}} \leq ... \leq c\frac{(1-c^{-1})^{n-m}}{S_n} = c(1-c^{-1})^{n-m}\chi_n.$$

To finish the proof we can choose $N = \frac{1}{c}$; $\nu = \ln \frac{c}{c-1}$. □

2. EXPONENTIAL STABILITY OF DISCRETE BOUNDED ORBITS

In this section we will give a sufficient condition for stability of bounded orbits of a discrete evolution family $\mathcal{U}$. The obtained results will be used in the next section to characterize the exponential dichotomy of the equation (1.1).

Theorem 2.1. *Let the operator T_0 defined as above satisfy the condition $0 \notin A\sigma(T_0)$. Then every discrete bounded orbit of $\mathcal{U}$ is exponentially stable. Precisely, if*

$$\sup_{n_0 \leq n \in \mathbf{N}} \|U_{n,n_0}x\| < \infty,$$

with $x \in X$ and $n_0 > 0$, then there exist positive constants N, ν independent of x and n_0 such that:

$$\|U_{n,n_0}x\| \leq Ne^{-\nu(n-s)}\|U_{s,n_0}x\|, n \geq s \geq n_0.$$

Proof. Let us start by proving that:

$$\|U_{n,n_0}x\| \leq Ne^{-\nu(n-n_0)}\|x\|, \ \forall \ n \geq n_0.$$

Without loss of generality we may assume that $\|x\| = 1$. Since $0 \notin A\sigma(T_0)$ there exists a constant $\delta > 0$ such that $\|T_0 v\| \geq \delta\|v\|$, for $v \in l_\infty^0$. Replacing δ by a smaller one if necessary we can assume that $\delta < 1$. Let $u_n = U_{n,n_0}x$ for $n \geq n_0$; $n_1 := \sup\{n \geq n_0 : U_{n,n_0}x \neq 0\}$. The exponential boundedness of $\mathcal{U}$ yields

$$\|u_n\| \leq Ke^{\alpha(n-n_0)}; n \geq n_0,$$

where K, α are positive constants. For any natural number $n_2 < \infty$ such that $n_0 \leq n_2 \leq n_1$ take

$$v = \{v_n\} \text{ with } v_n = \begin{cases} 0 & \text{for } \ 0 \leq n < n_0 \\ u_n \sum_{k=n_0}^{n} \frac{1}{\|u_k\|} & \text{for } \ n_0 \leq n \leq n_2, \\ u_n \sum_{k=n_0}^{n_2} \frac{1}{\|u_k\|} & \text{for } \ n > n_2 \end{cases}$$

$$f = \{f_n\} \text{ with } f_n = \begin{cases} 0, & \text{for } \ 0 \leq n < n_0 - 1 \\ \frac{u_{n+1}}{\|u_{n+1}\|} & \text{for } \ n_0 - 1 \leq n < n_2 \\ 0 & \text{for } \ n \geq n_2. \end{cases}$$

Then $v_{n+1} = A_n v_n + f_n$ and $v \in l^0_\infty, f \in l_\infty$. It follows that $T_0 v = f$ and $\|f\| \geq \delta \|v\|$. That means

$$\delta \sup_n \|u_n\| \sum_{k=n_0}^{n} \frac{1}{\|u_k\|} \leq \|f\|_{l_\infty} = 1, \text{ or } \|u_n\| \sum_{k=n_0}^{n} \frac{1}{\|u_k\|} \leq \frac{1}{\delta}.$$

Lemma 1.3 yields the exitences of $N, \nu > 0$ such that $\|u_n\| \leq N e^{-\nu(n-n_0)}$

Now we fix $s \geq n_0$, set $y := U_{s,n_0} x$. Then $\sup_{n \geq s} \|U_{n,s} y\| < \infty$, and

$$\|U_{n,n_0} x\| = \|U_{n,s} y\| \leq N e^{-\nu(n-s)} \|y\| = N e^{-\nu(n-s)} \|U_{s,n_0} x\|, n \geq s.$$

□

From this theorem we obtain the following corollary:

Corollary 2.2. Under the conditions of Theorem 2.1 we have

$$\begin{aligned} X_0(n_0) &:= \{x \in X : \sup_{n \geq n_0} \|U_{n,n_0} x\| < \infty\} \\ &= \{x \in X : \|U_{n,n_0} x\| \leq N e^{-\nu(n-n_0)} \|x\|;\ n \geq n_0 \geq 0\}. \end{aligned}$$

for certain positive constants N, ν, is a closed linear subspace of X.

3. EXPONENTIAL DICHOTOMY

We will characterize the exponential dichotomy of the equation (1.1) by using the operators T_0, T. In particular, applying Corollary 2.2 we will get necessary and sufficient conditions for exponential dichotomy in Hilbert spaces and finite dimensional spaces.

Lemma 3.1. *Assume that the equation (1.1) has an exponential dichotomy with corresponding family of projections $P_n, n \geq 0$ and constants $N > 0, \nu > 0$, then $M := \sup_{n \geq 0} \|P_n\| < \infty$.*

Proof. Fix $n_0 > 0$, and set $P^0 := P_{n_0}; P^1 := Id - P_{n_0}$, $X_k = P^k X, k = 0, 1$. Set $\gamma_0 := \inf\{\|x^0 + x^1\| : x^k \in X_k,\ \|x^0\| = \|x^1\| = 1\}$. If $x \in X$ and $P^k x \neq 0$, then

$$\begin{aligned} \gamma_{n_0} &\leq \left\|\frac{P^0 x}{\|P^0 x\|} + \frac{P^1 x}{\|P^1 x\|}\right\| \leq \frac{1}{\|P^0 x\|} \left\|P^0 x + \frac{\|P^0 x\|}{\|P^1 x\|} P^1 x\right\| \\ &\leq \frac{1}{\|P^0 x\|} \left\|x + \frac{\|P^0 x\| - \|P^1 x\|}{\|P^1 x\|} P^1 x\right\| \leq \frac{2\|x\|}{\|P^0(x)\|}. \end{aligned}$$

Hence, $\|P^0\| < 2/\gamma_{n_0}$. It remains to show that there is constant $c > 0$ (independent of n_0) such that $\gamma_{n_0} \geq c$. For this fix $x^k \in X_k,\ k = 0.1.$ with $\|x^k\| = 1$. By the exponential boundedness of $\mathcal{U}$ we have $\|U_{n,n_0}(x^0 + x^1)\| \leq K e^{\alpha(n-n_0)} \|x^0 + x^1\|$ for $n \geq n_0$ and constants $K,\ \alpha \geq 0$. Thus,

$$\begin{aligned} \|x^0 + x^1\| &\geq K^{-1} e^{-\alpha(n-n_0)} \|U_{n,n_0} x^0 + U_{n,n_0} x^1\| \\ &\geq K^{-1} e^{-\alpha(n-n_0)} (N^{-1} e^{\nu(n-n_0)} - N e^{-\nu(n-n_0)}) =: c_{n-n_0}, \end{aligned}$$

and hence $\gamma_{n_0} \geq c_{n-n_0}$. Obviously $c_m > 0$ for m sufficiently large. Thus $0 < c_m \leq \gamma_{n_0}$. □

Now we come to our first main result. It characterizes the exponential dichotomy of the equation (1.1) by properties of the operator T.

Theorem 3.2. *Let $\{A_n\}_{n\in\mathbf{N}}$ be a family of bounded linear and uniformly bounded operators on the Banach space X. Then the following assertions are equivalent:*

i) *The equation (1.1) has an exponential dichotomy*
ii) *T is surjective and $X_0(0)$ is complemented in X.*

Proof. (i)⇒(ii): Let $(P_n)_{n\geq 0}$ be the family of projections determined by the exponential dichotomy. Then $X_0(0) = P_0X$, and hence $X_0(0)$ is complemented. If $f \in l_\infty$ define $v = \{v_n\}_{n\in\mathbf{N}}$ by

$$v_n = \begin{cases} \sum_{k=1}^{n} U_{n,k}P_kf_{k-1} - \sum_{k=n+1}^{\infty} U_{|n,k}(Id - P_k)f_{k-1} & \text{for } n \geq 1 \\ -\sum_{k=1}^{\infty} U_{|0,k}(Id - P_k)f_{k-1} & \text{for } n = 0 \end{cases} \tag{3.1}$$

then $v_{n+1} = A_nv_n + f_n$ and $v \in l_\infty$. By the definition of T we have $Tv = f$. Therefore $T : l_\infty \to l_\infty$ is surjective.

(ii)⇒(i):

A) Let $Z \subseteq X$ be a complement of $X_0(0)$ in X i.e.:$X = X_0(0) \oplus Z$. Set $X_1(n) = U_{n,0}Z$. Then

$$U_{n,s}X_0(s) \subseteq X_0(n),\ \ U_{n,s}X_1(s) = X_1(n),\ n \geq s \geq 0. \tag{3.2}$$

B) There are constants $N,\ \nu > 0$ such that

$$\|U_{n,0}x\| \geq Ne^{\nu(n-s)}\|U_{s,0}x\| \text{ for } x \in X_1(0)\ , n \geq s \geq 0. \tag{3.3}$$

In fact, let $Y := \{(v_n)_{n\in\mathbf{N}} \in l_\infty : v_0 \in X_1(0)\}$ endowed with l_∞-norm. Then Y is a closed subspace of the Banach space l_∞ and hence Y is complete. By Remark 1.2 we have $\ker T := \{v \in l_\infty : v_n = U_{n,0}x$ for some $x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and T is surjective we obtain

$$T : Y \to l_\infty$$

is bijective and hence an isomorphism. Thus there is a constant $\delta > 0$ such that

$$\|Tv\|_{l_\infty} \geq \delta\|v\|_{l_\infty},\ \text{for } v \in Y. \tag{3.4}$$

Let $0 \neq x \in X_1(0)$, set $u_n := U_{n,0}x, n \geq 0$. By Remark 1.2 we have $u_n \neq 0$ for all $n \geq 0$. For a natural large number $\tau > 0$ take $v = \{v_n\}, f = \{f_n\}$, where

$$\begin{aligned} v_n &= \begin{cases} u_n \sum_{k=n+1}^{\tau} \frac{1}{\|u_k\|} & \text{for } 0 \leq n < \tau \\ 0 & \text{for } n \geq \tau \end{cases} \\ f_n &= \begin{cases} -\frac{u_{n+1}}{\|u_{n+1}\|} & \text{for } 0 \leq n < \tau \\ 0, & \text{for } n \geq \tau \end{cases} \end{aligned}$$

Then $v \in Y$, and $f \in l_\infty$ which satisfy the equation $v_{n+1} = A_n v_n + f_n$. It follows that

$$Tv = f \Rightarrow \|f\|_{l_\infty} \geq \delta \|v\|_{l_\infty}.$$

Hence,

$$1 \geq \delta \|u_n\| \sum_{k=n+1}^{\tau} \frac{1}{\|u_k\|} \Rightarrow \|u_n\| \sum_{k=n}^{\tau} \frac{1}{\|u_k\|} \leq \frac{1}{\delta} + 1.$$

Therefore the exponential boundedness of $\mathcal{U}$ and Lemma 1.4 imply that there are constants $N, \nu > 0$ independent of x such that

$$\|u_n\| \geq N e^{\nu(n-s)} \|u_s\|;\ n \geq s \geq 0.$$

C) $X = X_0(n) \oplus X_1(n)$, $n \in \mathbf{N}$.

Let $Y \subset l_\infty$ be as in **B)**. Then by Remark 1.2 $l_\infty^0 \subset Y$. From this and (3.4) we have $\|T_0 v\|_{l_\infty} \geq \nu \|v\|_{l_\infty}$, for $v \in l_\infty^0$. Thus, $0 \notin A\sigma(T_0)$ and Corollary 2.2 imply that $X_0(n)$ is closed. From (3.2) , (3.3) and the closedness of $X_1(0)$ we can easily derive that $X_1(n)$ is closed and $X_1(n) \cap X_0(n) = \{0\}$ for $n \geq 0$.

Finally, fix $n_0 > 0$, and $x \in X$. For large natural number n_1 set

$$v = \{v_n\} \text{ with } v_n = \begin{cases} (n - n_0 + 1) U_{n,n_0} x, & \text{for } n_0 \leq n \leq n_1 \\ 0, & \text{for } n > n_1. \end{cases}$$

$$f = \{f_n\} \text{ with } f_n = \begin{cases} U_{(n+1),n_0} x, & \text{for } n_0 \leq n < n_1 \\ -(n_1 - n_0 + 1) U_{(n+1),n_0} x & \text{for } n = n_1 \\ 0, & \text{for } n > n_1. \end{cases}$$

Then v_n, f_n solve the equation (1.2) with $n \geq n_0 > 0$ and $v \in l_\infty([n_0, \infty), X)$. Set $f_n = 0$ for $0 \leq n < n_0$. Then $f \in l_\infty(\mathbf{N}, X)$ by assumption there exists $w \in l_\infty$ such that $Tw = f$. By the definition of T, w_n is a solution of the equation (1.2). In particular, $\{w_n\}_{n_0 \leq n < \infty}$ satisfies (1.2) as well. Thus,

$$v_n - w_n = U_{n,n_0}(v_{n_0} - w_{n_0}) = U_{n,n_0}(x - w_{n_0}),\ n \geq n_0.$$

Since for $n_0 \leq n < \infty$ we have $\{v_n - w_n\}_{n \geq n_0} \in l_\infty([n_0, \infty), X)$. This implies $x - w_{n_0} \in X_0(n_0)$. On the other hand, since $w_0 = w^0 + w^1$ with $w^k \in X_k(0)$, $w_{n_0} = U_{n_0,0} w^0 + U_{n_0,0} w^1$ and by (3.2) we have $U_{n,n_0} w^k \in X_k(n_0)$, $k = 0, 1$. Hence $x = x - w_{n_0} + w_{n_0} \in X_0(n_0) + X_1(n_0)$. This proves **C)**.

D) Let P_n be the projections from X onto $X_0(n)$ with kernel $X_1(n)$, $n \geq 0$. Then (3.2) implies that $P_{n+1} U_{(n+1)n} = U_{(n+1)n} P_n$, or $A_n P_n = P_{n+1} A_n$ for $n \geq 0$. From (3.2), (3.3) and $A_n = U_{(n+1)n}$ we obtain that $A_n : \ker P_n \to \ker P_{n+1},\ n \geq 0$ is an isomorphism. Finally, by (3.3), Theorem 2.1 and the assumption $0 \notin A\sigma(T_0)$ there exist constants $N, \nu > 0$ such that

$$\|U_{n,m} x\| \leq N e^{-\nu(n-m)} \|x\| \ \text{ for } \ x \in P_m X \ , n \geq m \geq 0$$

$$\|U_{|m,n} x\| \leq N e^{-\nu(n-m)} \|x\| \text{ for } x \in \ker P_n \ , n \geq m \geq 0.$$

Thus the equation (1.1) has an exponential dichotomy. □

If X is a Hilbert space we need only the closedness of $X_0(0)$. Therefore, we have

Corollary 3.3. If X is a Hilbert space then the conditions that $0 \notin A\sigma(T_0)$ and T is surjective are necessary and sufficient for the equation (1.1) to have an exponential dichotomy.

This can be restated as follows:

If X is a Hinbert space then the condition that for all $f \in l_\infty$ there exists a solution $x \in l_\infty$ of the equation (1.2) and there exists constant $c > 0$ such that all of bounded solution $x = \{x_n\}$ (with $x_0 = 0$ and $f \in l_\infty$) of the equation (1.2) satisfies $\sup_{n\in\mathbf{N}} \|x_n\| \leq c \sup_{n\in\mathbf{N}} \|f_n\|$ are necessary and sufficient for the equation (1.1) to have an exponential dichotomy.

Proof. The corollary is obvious in view of Corollary 2.2 and Theorem 3.2. □

If X is a finite dimensional space then every subspace of X is closed and complemented. Hence, by Theorem 3.2 we have

Corollary 3.4. If X is a finite dimensional space, then the condition that T is surjective is necesary and sufficient for existence of exponential dichotomy of the equation (1.1).

4. APPLICATION TO EVOLUTION FAMILIES

In this section we shall consider evolution families $\mathcal{U} = U(t,s)_{t\geq s\geq 0}$ defined as below. We shall characterize the exponential dichotomy of $\mathcal{U}$ by discretizing the evolution family and using the results obtained in previous sections.

Definition 4.1. A family of operators $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on a Banach space X is said to be a *(strongly continuous, exponential bounded) evolution family* on the half line if

i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s \geq 0$,
ii) The map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
iii) There are constants $K \geq 0$ and $\alpha \in \mathbf{R}$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)}$ for $t \geq s \geq 0$.

Then $\omega(\mathcal{U}) := \inf\{\alpha \in \mathbf{R} :$ there is $K \geq 0$ such that $\|U(t,s)\| \leq Ke^{\alpha(t-s)}, \quad t \geq s \geq 0\}$ is called the *growth bound* of $\mathcal{U}$. The notion of evolution families arises naturally when we are concerned with "well-posed" evolution equations of the form

$$\frac{du(t)}{dt} = A(t)u(t), \ t \geq 0,$$

where $A(t)$, for fixed t, is in general unbounded linear operator. For more details on this notion, conditions for the existence of such families and applications to partial differential equations we refer the reader to [8], [16].

For an evolution family $\mathcal{U}$ and each $t_0 \in \mathbf{R}_+$ we consider the sequence of uniformly bounded operators $\{A_n(t_0)\}_{n\in\mathbf{N}}$ with $A_n(t_0) = U(t_0+n+1, t_0+n)$ and the following difference equations:

$$x_{n+1} = A_n(t_0)x_n,\ n \in \mathbf{N} \tag{4.1}$$

and

$$x_{n+1} = A_n(t_0)x_n + f_n,\ n \in \mathbf{N} \tag{4.2}$$

We shall define the two concepts of exponential dichotomy *exponential dichotomy* and *discrete exponential dichotomy.*

Definition 4.2. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on the Banach space X is said to have an *exponential dichotomy* if there exist bounded linear projections $P(t),\ t \geq 0$ on X and positive constants $N,\ \nu$ such that

a) $U(t,s)P(s) = P(t)U(t,s),\ t \geq s \geq 0,$
b) the restriction $U(t,s)_{|} : \ker P(s) \to \ker P(t),\ t \geq s \geq 0$ is an isomorphism (and we denote its inverse by $U_{|}(s,t) : \ker P(t) \to \ker P(s)$),
c) $\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X,\ t \geq s \geq 0,$
d) $\|U_{|}(s,t)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \ker P(t),\ t \geq s \geq 0.$

Definition 4.3. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ on the Banach space X is said to have a *discrete exponential dichotomy* if for each $t_0 \in \mathbf{R}_+$ the equation (4.1) has exponential dichotomy with family of projection $(P_n(t_0))_{n\in\mathbf{N}}$ and positive constants $N(t_0),\ \nu(t_0)$.

Definition 4.4. An evolution family $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ is said to have an exponential dichotomy (a discrete exponential dichotomy, respectively) in the sense of Sacker and Sell if and only if it has an exponential dichotomy (a discrete exponential dichotomy, respectively) and $dim \ker P(t) = k < \infty$ for all $t \geq 0$ ($dim \ker P_n(t_0) = k < \infty$ for all $t_0 \geq 0$ and $n \in \mathbf{N}$, respectively).

From Theorem 3.2 we obtain

Theorem 4.5. *Let $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ be an evolution family on the Banach space X. Then the following assertions are equivalent:*

i) *$\mathcal{U}$ has a discrete exponential dichotomy for each $t_0 \in \mathbf{R}_+$*
ii) *For each $t_0 \in \mathbf{R}_+,\ f \in l_\infty$ the equation (4.2) has at least a solution $u \in l_\infty$ and the spaces*

$$X_0(t_0)(0) := \{x \in X : \sup_{n\in\mathbf{N}} \|U(t_0+n, t_0)x\| < \infty\}$$

is complemented in X.

In what follows we will need the fact that the constants $N,\ \nu$ in Definition 4.3 are independent of t_0. The following lemma supplies a criterion for this.

Lemma 4.6. *Let the evolution family $\mathcal{U}$ $=(U(t,s))_{t\geq s\geq 0}$ on the Banach space X have a discrete exponential dichotomy. We define a bounded linear operator $S(t_0): l_\infty \to l_\infty$ as follows: for $x=\{x_n\}\in l_\infty$ put $(S(t_0)x)_n = A_n(t_0)x_n$. We denote a complement of $X_0(t_0)(0)$ by $X_1(t_0)(0)$ and $Y(t_0) := \{(v_n)_{n\in\mathbf{N}} \in l_\infty : v_0 \in X_1(t_0)(0)\}$. If there exists a constant $\gamma > 1$ such that $\|S(t_0)x\| \geq \gamma\|x\|$ for all $t_0 \in \mathbf{R}_+,\ x \in Y(t_0)$, then the constants $N,\ \nu$ determined by the discrete exponential dichotomy are independent of t_0.*

Proof. For each $t_0 \in \mathbf{R}$ we define the operator $L : Y(t_0) \to l_\infty$ as follows: For $x = \{x_n\} \in Y(t_0)$ take

$$(Lx)_n = x_{n+1} \text{ (the shift operator) }.$$

Then $\sup\{\|x_0\|, \|x_1\|, ..., \|x_n\|, ..\} \geq \sup\{\|x_1\|, ..., \|x_n\|, ..\}$, so $\|x\| \geq \|Lx\|$. Therefore, for $x \in Y(t_0)$:

$$\|Tx\| = \|(L - S(t_0))x\| \geq \|S(t_0))x\| - \|Lx\| \geq (\gamma - 1)\|x\|.$$

Thus, the constant δ in the equality (3.4) can be replaced by $\gamma - 1$ which is independent of t_0. That means the constants $N,\ \nu$ are independent of t_0. □

Theorem 4.7. *Let $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ be an evolution family on the Banach space X. If $\mathcal{U}$ has an exponential dichotomy then $\mathcal{U}$ has a discrete exponential dichotomy for each $t_0 \in \mathbf{R}_+$ with projections $P_n(t_0) = P(t_0 + n)$ and positive constants $N,\ \nu$ independent of t_0.*

Proof. **A)** $A_n(t_0)P_n(t_0) = P_{n+1}(t_0)A_n(t_0)$.

In fact,

$$A_n(t_0)P_n(t_0) = U(t_0 + n + 1, t_0 + n)P(t_0 + n) =$$
$$P(t_0 + n + 1)U(t_0 + n + 1, t_0 + n) = P_{n+1}(t_0)A_n(t_0).$$

B) $A_n(t_0) : \ker P_n(t_0) \to \ker P_{n+1}(t_0)$ is an isomorphism. We denote its inverse by $A^{-1}_{|n}(t_0)$.

This can be derived from the fact that $U(t_0+n+1, t_0+n) : \ker P(t_0+n) \to \ker P(t_0+n+1)$ is an isomorphism.

C) If we put $U_{n,m} = A_{n-1}(t_0)A_{n-2}(t_0)..A_m(t_0)$ for $n > m$ and $U_{m,m} = Id$, then $U_{n,m} = U(t_0 + n, t_0 + m)$ for $n \geq m \geq 0$. Hence,

$$\|U_{n,m}x\| = \|U(t_0 + n, t_0 + m)x\| \leq Ne^{-\nu(n-m)}\|x\|$$

for $x \in P_m(t_0)X$.

D) Denote by $U_{|m,n} = A^{-1}_{|m}(t_0)A^{-1}_{|m+1}(t_0)..A^{-1}_{|n-1}(t_0)$ for $n > m$ and $U_{|m,m} = Id$ we have $U_{|m,n} = U_|(t_0 + m, t_0 + n)$. Therefore,

$$\|U_{|m,n}x\| = \|U_|(t_0 + m, t_0 + n)x\| \leq Ne^{-\nu(n-m)}\|x\|$$

for $x \in \ker P_n(t_0)$. □

Theorem 4.8. *Let $\mathcal{U} = (U(t,s))_{t\geq s\geq 0}$ be an evolution family on the Banach space X. If for each $t_0 \in \mathbf{R}_+$, $\mathcal{U}$ has a discrete exponential dichotomy in the sense of Sacker and Sell with positive constants $N,\ \nu$ independent of t_0 then $\mathcal{U}$ has an exponential dichotomy in the sense of Sacker and Sell.*

Proof. We define the family of projections on X as follows: $P(t_0) = P_0(t_0)$ for all $t_0 \in \mathbf{R}_+$.

A) There exist $N_1,\ \nu_1 > 0$ such that $\|U(t,s)x\| \leq N_1 e^{-\nu_1(t-s)}\|x\|$ for $x \in P(s)X$.
In fact, let $n \in \mathbf{N}$ be such that $n \leq t-s < n+1$. Then, for $x \in P(s)X$

$$\|U(t,s)x\| = \|U(t,s+n)U(s+n,s)x\| \leq Ke^{\alpha}Ne^{-\nu n}\|x\| \leq KNe^{\alpha+\nu}e^{-\nu(t-s)}\|x\|.$$

Hence, put $N_1 := KNe^{\alpha+\nu}$; $\nu_1 := \nu$ the claim is proved.

B) Let $X_0(t_0) := \{x \in X : \sup_{t\geq t_0}\|U(t,t_0)x\| < \infty\}$,then $X_0(t_0) = P(t_0)X$.
In fact, from **A)** we have $P(t_0)X \subseteq X_0(t_0)$. Set $x \notin P(t_0)X$. Then $x = P(t_0)x + (Id - P(t_0))x$ with $(Id - P(t_0))x \neq 0$. Therefore,

$$\begin{aligned}\|U(t_0+n,t_0)x\| &= \|U(t_0+n,t_0)P(t_0)x + U(t_0+n,t_0)(Id-P(t_0))x\| \\ &\geq \|U(t_0+n,t_0)(Id-P(t_0))x\| - \|U(t_0+n,t_0)P(t_0)x\|.\end{aligned}$$

Since $\mathcal{U}$ has a discrete exponential dichotomy we have $\|U(t_0+n,t_0)P(t_0)x\| \to 0$ and $\|U(t_0+n,t_0)(Id-P(t_0))x\| \to \infty$ when $n \to \infty$. Hence, $\|U(t_0+n,t_0)x\| \to \infty$ when $n \to \infty$. So $x \notin X_0(t_0)$. Thus, $X_0(t_0) \subseteq P(t_0)X$. Therefore, $P(t_0)X = X_0(t_0)$

C) $U(t,t_0)P(t_0)X \subseteq P(t)X$.
In fact, let $x \in P(t_0)X \Leftrightarrow P(t_0)x = x$ then $\|U(t,t_0)x\|$ bounded for $t \geq t_0$. We have,

$$\sup_{s\geq t}\|U(s,t)U(t,t_0)x\| \leq \sup_{s\geq t_0}\|U(s,t_0)x\| < \infty.$$

Hence, from **B)** we get $U(t,t_0)x \in P(t)X$.

D) $U(t,t_0)_{|}\ker P(t_0)$ is one to one.
In fact, for the purpose of contradiction let $0 \neq x \in \ker P(t_0) : U(t,t_0)x = 0$. Taking $n \in \mathbf{N}$ such that $t_0 + n > t$ we have $U(t_0+n,t)U(t,t_0)x = 0$ or $U(t_0+n,t_0)x = 0$. This contradicts to the fact that $U(t_0+n,t_0) : \ker P(t_0) \to \ker P_n(t_0)$ is isomorphism.

E) Because a complement of a complemented subpace of Banach space X is not unique, the family $(P_n(t_0))_{n\in\mathbf{N}}$ (precisely, the family of spaces $(\ker P_n(t_0))_{n\in\mathbf{N}}$) for each $t_0 \geq 0$ is not unique. However, we shall point out that for each $t_0 \geq 0$ there exists a family $(P_n(t_0))_{n\in\mathbf{N}}$ such that:

$$U(t_1,t_0)\ker P_0(t_0) = \ker P_0(t_1) \text{ for } t_1 \geq t_0 \geq 0.$$

Firstly we prove that $U(t_1,t_0)\ker P_0(t_0)$ is a closed subspace of X. Indeed, take $n \in N$ such that $n+1 \geq t_1 - t_0 \geq n$, for all $y \in \ker P_0(t_0)$ we have

$$\begin{aligned}Ke^{\alpha}\|U(t_1,t_0)y\| &\geq \|U(t_0+n+1,t_1)U(t_1,t_0)y\| \\ &= \|U(t_0+n+1,t_0)y\| \geq Ne^{(n+1)\nu}\|y\|.\end{aligned}$$

Hence,

$$\|U(t_1,t_0)y\| \geq \frac{N}{K}e^{-\alpha}e^{(n+1)\nu}\|y\| \geq \frac{N}{K}e^{-\alpha}e^{\nu(t_1-t_0)}\|y\|. \tag{4.3}$$

From this inequality and the closedness of $\ker P_0(t_0)$ we easily derive that the space $U(t_1,t_0)\ker P_0(t_0)$ is a closed subspace of X.

Now we prove that $U(t_1,t_0)\ker P_0(t_0)\cap X_0(t_1)=0$.

Indeed, suppose that $x\in U(t_1,t_0)\ker P_0(t_0)\cap X_0(t_1)$. Then from definition of $X_0(t_1)$ we have that $\sup_{t\geq t_1}\|U(t,t_1)x\|=M<\infty$. Since $x\in U(t_1,t_0)\ker P_0(t_0)$, there exists $y\in\ker P_0(t_0)$ such that $x=U(t_1,t_0)y$. By the inequality (4.3) we have

$$M\geq\|U(t,t_1)x\|=\|U(t,t_1)U(t_1,t_0)y\|=\|U(t,t_0)y\|\geq\frac{N}{K}e^{-\alpha}e^{\nu(t-t_0)}\|y\|$$

for all $t\geq t_1\geq t_0$. Therefore, $y=0$, thus $x=0$.

Since $U(t_1,t_0)_{|}\ker P_0(t_0)$ is one to one we have

$$dim U(t_1,t_0)\ker P_0(t_0)=dim\ker P_0(t_0)=k=dim\ker P_0(t_1).$$

That means we have

$$X=X_0(t_1)\oplus U(t_1,t_0)\ker P_0(t_0).$$

Hence, we can take $P_0(t_1)$ as the projection on to $X_0(t_1)$ with

$$kerP_0(t_1)=U(t_1,t_0)\ker P_0(t_0).$$

Therefore,

$$U(t_1,t_0)\ker P_0(t_0)=\ker P_0(t_1)\text{ for }t_1\geq t_0\geq 0.$$

From **D)** and **E)** we have $U(t,t_0):\ker P(t_0)\to\ker P(t)$ is an isomorphism and we denote its inverse by $U_{|}(t_0,t):\ker P(t)\to\ker P(t_0)$, for $t\geq t_0\geq 0$.

F) $\|U_{|}(t_0,t)x\|\leq N_1e^{-\nu_1(t-t_0)}\|x\|$ for $x\in\ker P(t)$ and $t\geq t_0\geq 0$.

In fact, firstly we prove that for $t\geq s\geq 0$ with $0\leq t-s\leq 1$ there exists $0<M<\infty$ such that $\|U_{|}(s,t)x\|\leq M\|x\|$ for $x\in\ker P(t)$. Indeed, since $U_{|}(s,t)x\in\ker P(s)$ for $x\in\ker P(t)$ we have

$$Ke^{\alpha}\|x\|\geq\|U(s+1,t)x\|=\|U(s+1,s)U_{|}(s,t)x\|\geq Ne^{\nu}\|U_{|}(s,t)x\|.$$

Hence, $\|U_{|}(s,t)x\|\leq\frac{K}{N}e^{\alpha-\nu}\|x\|$ for $x\in\ker P(t)$, so we may take $M:=\frac{K}{N}e^{\alpha-\nu}$. Now, let $n\in\mathbf{N}$ such that $n\leq t-t_0\leq n+1$ then,

$$\|U_{|}(t_0,t)x\|=\|U_{|}(t_0,t_0+n)U_{|}(t_0+n,t)x\|\leq Ne^{-\nu n}M\|x\|\leq NMe^{\nu}e^{-\nu(t-t_0)}\|x\|.$$

Take $N_1:=NMe^{\nu}$; $\nu_1=\nu$ then the claim is proved .

G) $U(t,s)P(s)=P(t)U(t,s)$.

In fact, for $x\in\ker P(s):U(t,s)P(s)x=P(t)U(t,s)x=0$ and

$$x\in P(s)X:U(t,s)(Id-P(s))x=(Id-P(t))U(t,s)x=0.$$

Thus,

$$U(t,s)P(s)x=P(t)U(t,s)x.$$

Therefore, for $x \in X$ we have $x = x_1 + x_2$ with $x_1 \in \ker P(s)$ and $x_2 \in P(s)X$. So,

$$U(t,s)P(s)x = U(t,s)P(s)(x_1 + x_2) = P(t)U(t,s)(x_1 + x_2) = P(t)U(t,s)x.$$

□

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