A characterization of the distance between controllable and uncontrollable LTI systems

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***Abstract*—The controllability distance for a linear time-invariant (LTI) system is defined as the norm of the smallest perturbation rendering the system uncontrollable. This is a widely used concept in control theory and provides a measure of the robustness of a system. Previous investigations have shown that the controllability distance can be characterized by a optimization problem involving singular values of extended matrices. This characterization has been established for general first-order systems and a certain class of higher-order systems. In this paper, we develop an analogous characterization of the controllability distance for a more general family of LTI systems, where controllability is formulated in a behavioral framework.**

I. Introduction

LTI systems of the form

$$x'(t) = Ax(t) + Bu(t) \qquad (t \geq 0), \tag{1}$$

whith $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, are said to be *state-controllable* (in the sense of Kalman's notion of controllability), if for any initial state $x(0) = x_{\text{ini}} \in \mathbb{R}^n$ and any desired final state $x_{\text{fin}} \in \mathbb{R}^n$, there exist $T > 0$ and a measurable control function u such that $x(T) = x_1$. Hautus [1] proved that the controllability of (1) is equivalent to the condition that the augmented matrices $[A - \lambda I, B]$ satisfy

$$\forall \lambda \in \mathbb{C}\colon\ \operatorname{rank}\left([A - \lambda I, B]\right) = n \tag{2}$$

Kalman [2] characterized controllability of (1) with his well-known rank condition for the controllability matrix $[B, AB, \dots, A^{n-1}B]$ by

$$\operatorname{rank}\left([B, AB, \dots, A^{n-1}B]\right) = n. \tag{3}$$

The characterization of controllability (2) can be extended to higher-order systems of the form

$$A_k x^{(k)}(t) + \dots + A_1 x'(t) + A_0 x(t) = Bu(t) \qquad (t \geq 0), \tag{4}$$

with $k \in \mathbb{N}$, $B \in \mathbb{R}^{n\times m}$, $A_0, \dots, A_k \in \mathbb{R}^{n\times n}$. Mengi [3] showed that (4) is controllable if and only if

$$\forall \lambda \in \mathbb{C}\colon\ \operatorname{rank}\left([P(\lambda), B]\right) = n \tag{5}$$

with $P(\lambda) := \sum_{i=0}^k A_i \lambda^i$ for $\lambda \in \mathbb{C}$. The controllability criteria (2), (3) and (5) are rank determination problems. Since any matrix is arbitrarily close to a matrix of full rank, an uncontrollable system is arbitrarily close to a controllable one [3]–[6]. On the other hand, controllability is a robust property and the controllability distance [4] of a controllable system measures the norm of the smallest perturbation yielding an uncontrollable system. Eising [7], [8] proved that the controllability distance of system (1) is equal to

$$\min_{\lambda \in \mathbb{C}} \sigma_{\min}\left([\lambda I - A, B]\right), \tag{6}$$

where $\sigma_{\min}(M)$ denotes the smallest singular value of a matrix M. This approach was extended by Mengi [3] to higher-order systems (4). Computational techniques to approximate the controllability distance based on the optimization problem (6) for singular values can be found in [9]–[13].

In this paper, we consider a further generalization of (1) and (4), given by

$$\begin{aligned} &A_k x^{(k)}(t) + \dots + A_1 x'(t) + A_0 x(t) \\ &= B_\ell u^{(\ell)}(t) + \dots + B_1 u'(t) + B_0 u(t) \qquad (t \geq 0), \end{aligned} \tag{7}$$

whith $\ell \in \mathbb{N}$ and $B_0, \dots, B_\ell \in \mathbb{R}^{n\times m}$. Controllability of system (7) is fully understood and the main results can be found e.g. in [14], [15]. However, the notion of controllability distance has not been fully explored for this type of systems yet. In this paper we present in Theorem III.1 a formula for the controllability distance of (7) based on an optimization problem for singular values.

II. Basic Setup and Notation

Let $\mathbb{R}[\xi]$ denote the ring of polynomials in the indeterminate ξ with coefficients in $\mathbb{R}$ and let $\mathbb{R}^{p\times q}[\xi]$ denote the set of p by q matrices with entries in $\mathbb{R}[\xi]$. An element $R(\xi) \in \mathbb{R}^{p\times q}[\xi]$ will be referred to as a polynomial matrix. In this setup, we consider throughout this paper systems of the form

$$P\left(\frac{d}{dt}\right) x = Q\left(\frac{d}{dt}\right) u, \tag{8}$$

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whith $P(\xi) \in \mathbb{R}^{n\times n}[\xi]$ and $Q(\xi) \in \mathbb{R}^{n\times m}[\xi]$. Note that systems (1), (4) and (7) belong to the class of systems defined by (8). In our discussion, we assume that the polynomial matrices in (8) satisfy the following conditions (see [14, Section 3.3]):

- $\det(P(\xi)) \neq 0$, i.e. $\det(P(\xi))$ is not the zero polynomial in $\mathbb{R}[\xi]$.
- The transfer matrix $T(\xi) := P(\xi)^{-1}Q(\xi)$ is a matrix of proper rational functions, i.e., in each entry of $T(\xi)$ the degree of the numerator does not exceed the degree of the denominator.

A pair of locally integrable functions $(u,x) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$ satisfying (8) (weakly) is said to belong to the *behavior $\mathcal{B}$ of* (8), i.e.

$$\mathcal{B} := \big\{(u,x) \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) \times \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\colon \\ (u,x) \text{ is a weak solution of (8)}\big\},$$

see [16]. This is a key concept in the behavioral approach of mathematical control theory [14]. System (8) is called *controllable*, if an arbitrary past trajectory can be steered so as to be concatenated with an arbitrary future trajectory, i.e. if

$$\forall w_1, w_2 \in \mathcal{B}\ \exists w \in \mathcal{B}, t_0 \geq 0\colon w(t) = \begin{cases} w_1(t), & t \leq 0, \\ w_2(t-t_0), & t \geq t_0. \end{cases} \tag{9}$$

One could interpret w_1 as a "natural" response of the system and w_2 as the "desired" trajectory to which the system should be steered after a delay time $t_0 \geq 0$.

If system (8) satisfies $\deg(P(\xi)) \geq \deg(Q(\xi))$, its controllability is characterized by the condition

$$\forall \lambda \in \mathbb{C}\colon \operatorname{rank}\big([Q(\lambda), -P(\lambda)]\big) \text{ is constant}, \tag{10}$$

see [14, Theorem 5.2.10]. As mentioned above, rank conditions to verify controllability are not convenient from a computational point of view, as wrong conclusions may be drawn due to rounding errors. Therefore, in this paper we introduce a suitable notion of controllability distance for system (8). Borrowing the notation of Eising [7], [8], we define the set of uncontrollable systems

$$\mathrm{UNCO} := \big\{(P(\xi),Q(\xi)) \in \mathbb{R}^{n\times n}[\xi] \times \mathbb{R}^{n\times m}[\xi]\colon \\ \deg(P(\xi)) \geq \deg(Q(\xi)) \text{ and} \\ \mathbb{C} \ni \lambda \mapsto \operatorname{rank}\big([Q(\lambda), -P(\lambda)]\big) \text{ is not constant}\big\}.$$

Another important notion in our discussion is that of a distance between two systems of the form (8), i.e. a metric on the Cartesian product $\mathbb{R}^{n\times n}[\xi] \times \mathbb{R}^{n\times m}[\xi]$. To introduce this, let $P(\xi), \widetilde{P}(\xi) \in \mathbb{R}^{n\times n}[\xi]$, $Q(\xi), \widetilde{Q}(\xi) \in \mathbb{R}^{n\times m}[\xi]$. Then for any $k, \ell \in \mathbb{N}$ with

$$k \geq \max\big\{\deg(P(\xi)), \deg(\widetilde{P}(\xi))\big\}, \\ \ell \geq \max\big\{\deg(Q(\xi)), \deg(\widetilde{Q}(\xi))\big\}, \tag{11}$$

there exist unique $A_i, \widetilde{A}_i \in \mathbb{R}^{n\times n}$, $i \in \{0,\dots k\}$ and $B_i, \widetilde{B}_i \in \mathbb{R}^{n\times m}$, $i \in \{0,\dots,\ell\}$, such that

$$P(\xi) = \sum_{i=0}^{k} A_i\xi^i, \qquad \widetilde{P}(\xi) = \sum_{i=0}^{k} \widetilde{A}_i\xi^i,$$
$$Q(\xi) = \sum_{i=0}^{\ell} B_i\xi^i, \qquad \widetilde{Q}(\xi) = \sum_{i=0}^{\ell} \widetilde{B}_i\xi^i.$$

The distance between the pairs $\big(P(\xi),Q(\xi)\big)$ and $\big(\widetilde{P}(\xi),\widetilde{Q}(\xi)\big)$, or the corresponding systems (8), is defined as

$$d\Big(\big(P(\xi),Q(\xi)\big),\big(\widetilde{P}(\xi),\widetilde{Q}(\xi)\big)\Big) \\ := \Big\|\Big[A_0-\widetilde{A}_0,\dots,A_k-\widetilde{A}_k,B_0-\widetilde{B}_0,\dots,B_\ell-\widetilde{B}_\ell\Big]\Big\|_2, \tag{12}$$

where $\|M\|_2 = \sqrt{\sum_{i,j}|m_{i,j}|^2} = \sqrt{\sigma_1^2+\cdots+\sigma_r^2}$ denotes the Frobenius norm of a matrix $M = (m_{i,j})$ with singular values $\sigma_1,\dots,\sigma_r$, $r = \operatorname{rank} M$. Note that (12) is well-defined, since varying k,ℓ in (11) yields alternative representations of $P(\xi), Q(\xi), \widetilde{P}(\xi), \widetilde{Q}(\xi)$ which differ only by zero matrixes and the value of the norm in (12) remains unchanged.

We are now in a position to define the controllability distance for system (8).

Definition II.1 (Controllability distance)**.** The *controllability distance* $\tau\colon\ \mathbb{R}^{n\times n}[\xi]\times\mathbb{R}^{n\times m}[\xi] \to \mathbb{R}\cup\{\infty\}$ of (8) is defined as

$$\tau(P(\xi),Q(\xi)) \\ := \inf_{(\widetilde{P}(\xi),\widetilde{Q}(\xi))\in\mathrm{UNCO}} d\Big(\big(P(\xi),Q(\xi)\big),\big(\widetilde{P}(\xi),\widetilde{Q}(\xi)\big)\Big). \tag{13}$$

III. Characterization of the Controllability Distance

Theorem III.1 (Controllability distance)**.** *Let $P(\xi) \in \mathbb{R}^{n\times n}[\xi]$ and $Q(\xi) \in \mathbb{R}^{n\times m}[\xi]$ with $\det P(\xi) \neq 0$ and $\deg P(\xi) \geq \deg Q(\xi)$. Suppose that system* (8) *is controllable. Define $R(\lambda) := [Q(\lambda), -P(\lambda)]$ for $\lambda \in \mathbb{C}$ and $k := \deg P(\xi)$, $\ell := \deg Q(\xi)$. Then*

$$\tau(P(\xi),Q(\xi)) \\ = \begin{cases} 0, & \textit{if } \operatorname{rank} R(\lambda) < n \textit{ for all } \lambda \in \mathbb{C}, \\ \inf_{\lambda\in\mathbb{C},\tilde{k}\geq k,\tilde{\ell}\geq\ell} \sigma_{\min}\left(\left[\frac{Q(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{\ell}}|\lambda|^{2i}}},\ -\frac{P(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{k}}|\lambda|^{2i}}}\right]\right), \\ \qquad \textit{if } \operatorname{rank} R(\lambda) = n \textit{ for all } \lambda \in \mathbb{C}. \end{cases}$$

Proof. There exist $A_0,\dots,A_k \in \mathbb{R}^{n\times n}$, $B_0,\dots,B_\ell \in \mathbb{R}^{n\times m}$ with

$$P(\xi) = \sum_{i=0}^{k} A_i\xi^i, \qquad Q(\xi) = \sum_{i=0}^{\ell} B_i\xi^i. \tag{14}$$

Since (8) is controllable, $r := \operatorname{rank} R(\lambda)$ is constant for $\lambda \in \mathbb{C}$ by (10). We distinguish the following two cases:

$$\text{(i) } r < n \qquad \text{and} \qquad \text{(ii) } r = n.$$

(i) $r < n$. Setting $\lambda = 0$, we obtain that $r = \operatorname{rank} R(0) = \operatorname{rank}[B_0, -A_0]$. By assumption $\det P(\xi) \neq 0$, and hence $k = \deg P(\xi) > 0$. Let $\varepsilon > 0$. Using the fact that the set of nonsingular matrices in $(\mathbb{R}^{n\times n}, \|\cdot\|_2)$ is dense (see e.g. [17, Problem 5.6.P8]), there exists a matrix $\Delta \in \mathbb{R}^{n\times n}$ with $\|\Delta\| \le \varepsilon$ such that $A_k + \Delta$ is nonsingular, i.e. $\det(A_k + \Delta) \neq 0$. Define

$$\widetilde{P}(\xi) := \sum_{i=0}^{k-1} A_i \xi^i + (A_k + \Delta)\xi^k \quad \text{and} \quad \widetilde{Q}(\xi) := Q(\xi).$$

Then $\widetilde{P}(\xi) \in \mathbb{R}^{n\times n}[\xi]$, $\widetilde{Q}(\xi) \in \mathbb{R}^{n\times m}[\xi]$ and $\deg(\widetilde{P}(\xi)) \ge \deg(\widetilde{Q}(\xi))$. In order to see that $(\widetilde{P}(\xi), \widetilde{Q}(\xi)) \in \mathrm{UNCO}$, we need to show that

$$\mathbb{C} \ni \lambda \mapsto \operatorname{rank}\big([\widetilde{Q}(\lambda), -\widetilde{P}(\lambda)]\big) \text{ is not constant.}$$

To this end, note that

$$\operatorname{rank}\big([\widetilde{Q}(0), -\widetilde{P}(0)]\big) = \operatorname{rank}[B_0, -A_0] = r,$$

and use the fact that

$$\begin{aligned}
&\lim_{\lambda\to\infty} \det\left(\frac{1}{\lambda^k}\widetilde{P}(\lambda)\right) \\
&= \lim_{\lambda\to\infty} \det\left(\frac{1}{\lambda^k}\sum_{i=0}^{k-1} A_i\lambda^i + A_k + \Delta\right) \\
&= \det(A_k + \Delta) \neq 0
\end{aligned}$$

to conclude that there exists a $\lambda_0 \in \mathbb{R}$ such that $\widetilde{P}(\lambda_0)$ is nonsingular. Hence

$$\operatorname{rank}\big([\widetilde{Q}(\lambda_0), -\widetilde{P}(\lambda_0)]\big) = n > r,$$

proving that $(\widetilde{P}(\xi), \widetilde{Q}(\xi)) \in \mathrm{UNCO}$. Consequently,

$$\begin{aligned}
\tau(P(\xi), Q(\xi)) &\le d\Big(\big(P(\xi), Q(\xi)\big), \big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big)\Big) \\
&\le \|\Delta\|_2 \le \varepsilon,
\end{aligned}$$

and since $\varepsilon > 0$ was arbitrary, $\tau(P(\xi), Q(\xi)) = 0$ follows.

(ii) $r = n$. We first prove that

$$\begin{aligned}
&\tau(P(\xi), Q(\xi)) \\
&\ge \inf_{\lambda\in\mathbb{C}, \tilde{k}\ge k, \tilde{\ell}\ge\ell} \sigma_{\min}\left(\left[\frac{Q(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda|^{2i}}}, \quad -\frac{P(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda|^{2i}}}\right]\right).
\end{aligned} \tag{15}$$

For this purpose, let $\big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big) \in \mathrm{UNCO}$. Then, there exists $\lambda_0 \in \mathbb{C}$ such that

$$\operatorname{rank}\big([\widetilde{Q}(\lambda_0), -\widetilde{P}(\lambda_0)]\big) < n. \tag{16}$$

Using (14), for $\tilde{k}, \tilde{\ell} \in \mathbb{N}$ with

$$\begin{aligned}
\tilde{k} &\ge \max\{k, \deg(\widetilde{P}(\xi))\}, \\
\tilde{\ell} &\ge \max\{\ell, \deg(\widetilde{Q}(\xi))\},
\end{aligned}$$

there exist $\Delta_0, \ldots, \Delta_{\tilde{k}} \in \mathbb{C}^{n\times n}$, $\Lambda_0, \ldots, \Lambda_{\tilde{\ell}} \in \mathbb{C}^{n\times m}$, such that

$$\widetilde{P}(\xi) = \sum_{i=0}^{k}(A_i + \Delta_i)\xi^i + \sum_{i=k+1}^{\tilde{k}} \Delta_i\xi^i,$$

$$\widetilde{Q}(\xi) = \sum_{i=0}^{\ell}(B_i + \Lambda_i)\xi^i + \sum_{i=\ell+1}^{\tilde{\ell}} \Lambda_i\xi^i.$$

By (16), there exists a unit vector $v \in \mathbb{C}^n$ such that

$$\begin{bmatrix} \widetilde{Q}(\lambda_0)^\top \\ -\widetilde{P}(\lambda_0)^\top \end{bmatrix} v = 0.$$

Hence,

$$\begin{bmatrix} \frac{Q(\lambda_0)^\top}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}} \\ -\frac{P(\lambda_0)^\top}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}} \end{bmatrix} v = \begin{bmatrix} -\frac{\sum_{i=0}^{\tilde{\ell}} \Lambda_i^\top \lambda_0^i}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}} \\ \frac{\sum_{i=0}^{\tilde{k}} \Delta_i^\top \lambda_0^i}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}} \end{bmatrix} v,$$

which in turn implies that

$$\begin{aligned}
&\sigma_{\min}\left(\left[\frac{Q(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}}, \quad -\frac{P(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}}\right]\right) \\
&\le \left\|\left[-\frac{\sum_{i=0}^{\tilde{\ell}} \Lambda_i \lambda_0^i}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}}, \quad \frac{\sum_{i=0}^{\tilde{k}} \Delta_i \lambda_0^i}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}}\right]\right\|_2 \\
&\le \Big\|\Big[\Lambda_{\tilde{\ell}}, \ldots, \Lambda_0, \Delta_{\tilde{k}}, \ldots, \Delta_0\Big]\Big\|_2.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&\sigma_{\min}\left(\left[\frac{Q(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}}, \quad -\frac{P(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}}\right]\right) \\
&\le d\Big(\big(P(\xi), Q(\xi)\big), \big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big)\Big),
\end{aligned}$$

proving (15). To show the claim, it remains to prove that

$$\begin{aligned}
&\tau(P(\xi), Q(\xi)) \\
&\le \inf_{\lambda\in\mathbb{C}, \tilde{k}\ge k, \tilde{\ell}\ge\ell} \sigma_{\min}\left(\left[\frac{Q(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda|^{2i}}}, \quad -\frac{P(\lambda)}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda|^{2i}}}\right]\right).
\end{aligned} \tag{17}$$

For this purpose, let $\lambda_0 \in \mathbb{C}$, $\tilde{k} \ge k$, $\tilde{\ell} \ge \ell$, and set

$$s_0 := \sigma_{\min}\left(\left[\frac{Q(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}}, \quad -\frac{P(\lambda_0)}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}}\right]\right).$$

Then there exists a unit vector $v \in \mathbb{C}^n$ and

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathbb{C}^n \times \mathbb{C}^m$$

such that

$$\begin{bmatrix} \frac{Q(\lambda_0)^T}{\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}} \\ -\frac{P(\lambda_0)^T}{\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}} \end{bmatrix} v = s_0 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

which implies that the matrix

$$\left[Q(\lambda_0) - s_0\sqrt{\sum_{i=0}^{\tilde{\ell}} |\lambda_0|^{2i}}\, v u_1^\top, \quad -P(\lambda_0) - s_0\sqrt{\sum_{i=0}^{\tilde{k}} |\lambda_0|^{2i}}\, v u_2^\top\right]$$

is singular. Define

$$\Delta_i := -\frac{s_0 \overline{\lambda}_0^i v u_2^\top}{\sqrt{\sum_{i=0}^{k} |\lambda_0|^{2i}}} \qquad (i \in \{0, \ldots, \tilde{k}\}),$$

$$\Lambda_i := \frac{s_0 \overline{\lambda}_0^i v u_1^\top}{\sqrt{\sum_{i=0}^{\ell} |\lambda_0|^{2i}}} \qquad (i \in \{0, \dots, \tilde{\ell}\}),$$

and

$$\widehat{P}(\xi) := \sum_{i=0}^{k} (A_i + \Delta^i)\xi^i + \sum_{i=k+1}^{\tilde{k}} \Delta^i \xi^i,$$
$$\widehat{Q}(\xi) := \sum_{i=0}^{\ell} (B_i + \Lambda_i)\xi^i + \sum_{i=\ell+1}^{\tilde{\ell}} \Lambda^i \xi^i.$$

Then

$$\left[\widehat{Q}(\lambda_0), \quad -\widehat{P}(\lambda_0)\right]$$

is singular and

$$d\Big(\big(P(\xi), Q(\xi)\big), \big(\widehat{P}(\xi), \widehat{Q}(\xi)\big)\Big) = s_0.$$

Consequently, (17) is proved in case

$$\big(\widehat{P}(\xi), \widehat{Q}(\xi)\big) \notin \mathrm{UNCO}\,.$$

In the remaining case, i.e. if

$$\big(\widehat{P}(\xi), \widehat{Q}(\xi)\big) \in \mathrm{UNCO},$$

using part (i), we obtain that for any $\varepsilon > 0$ there exists

$$\big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big) \in \mathrm{UNCO}$$

such that

$$d\Big(\big(\widehat{P}(\xi), \widehat{Q}(\xi)\big), \big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big)\Big) \le \varepsilon.$$

Consequently,

$$\begin{aligned}
\tau(P(\xi), Q(\xi)) &\le d\Big(\big(P(\xi), Q(\xi)\big), \big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big)\Big) \\
&\le d\Big(\big(P(\xi), Q(\xi)\big), \big(\widehat{P}(\xi), \widehat{Q}(\xi)\big)\Big) \\
&\quad + d\Big(\big(\widehat{P}(\xi), \widehat{Q}(\xi)\big), \big(\widetilde{P}(\xi), \widetilde{Q}(\xi)\big)\Big) \\
&\le s_0 + \varepsilon,
\end{aligned}$$

proving (17), and the proof is complete. □

IV. Concluding Remarks

Numerical approximations of rank conditions characterizing controllability are sensitive to rounding errors. The controllability distance of a control system is the norm of the smallest perturbation which makes the system uncontrollable. It serves as a measure of controllability. We provide a numerically accessible formula for the controllability distance based on an optimization problem for singular values of an associated system matrix.

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