Acta Appl Math (2012) 122:3–17
DOI 10.1007/s10440-012-9722-2

On the Tractability Index of a Class of Partial Differential-Algebraic Equations

Giuseppe Alì · Nella Rotundo

Received: 16 January 2012 / Accepted: 2 April 2012 / Published online: 23 May 2012

Abstract We consider a class of nonlinear partial-differential-algebraic equations, where the nonlinearity is present only in the PDEs and in the coupling conditions, and some additional structural conditions hold. For this special class of PDAEs, we introduce and characterize simple algebraic conditions which lead to a notion of extended tractability index, and exemplify its application to coupled systems arising from microelectronics.

Keywords Tractability index · PDAE's equations · Drift-diffusion equations

1 Introduction

Coupling effects arise naturally in many physical problems and industrial applications. Strictly speaking, "coupling" arises from complexity. A typical example of this, is the occurrence of coupling effects in microelectronics, in the modeling of integrated circuits [1, 7, 15, 17]. In this case, several levels of complexity can be retained. At a basic level, it is sufficient to consider an electric network description, by using Kirchhoff laws, in terms of applied voltages at contacts and currents through branches. At a higher level, it is necessary to add detailed models for the passive components, in particular for semiconductor devices. Sometimes it is also required a detailed description of electromagnetic fields, of interaction with substrate, of cross-talking effects, and so on. Since the full model, which includes all the previous descriptions, is not computationally feasible, it is replaced by a combination of the previous models, at different levels of complexity. These models are assembled together by means of appropriate coupling conditions.

G. Alì (✉) · N. Rotundo
Dipartimento di Matematica, Università della Calabria, Via Pietro Bucci, 87036 Arcavacata di Rende (CS), Italy
e-mail: g.ali@mat.unical.it

N. Rotundo
e-mail: nella.rotundo@unical.it

G. Alì
INFN, Gruppo collegato di Cosenza, Arcavacata di Rende, Italy

This paper deals with a class of nonlinear coupled systems, for which the dynamics is ruled by a linear differential-algebraic equation (DAE), nonlinearly coupled with a system of nonlinear partial differential equations (PDEs), which act as differential constraints. Several authors deal with the definition of appropriate index concepts for general classes of PDAEs, e.g. [5, 8, 11]. For applications to PDAE models of integrated circuits, we mention also [6, 7, 15]. We limit ourselves to proposing algebraic conditions so that the PDE variables play the role of purely algebraic variables, in the sense used for DAEs, according to the usual definition of tractability index [10, 16]. Thus, as far as the index is concerned, we regard the PDEs as perturbations of the DAEs to which they are coupled, and we attempt to find conditions so that the decoupling procedure for DAEs is not "perturbed" by the coupling with the PDEs. The approach followed here, extends the one used for elliptic systems of PDAEs arising in microelectronics [2–4].

The paper is organized as follows. In Sect. 2, we present the class of nonlinear systems under consideration. In Sect. 3, we discuss the notion of tractability index for DAEs, introducing appropriate sequences of matrices and projectors, and proving some relevant identities for chains of projectors. In Sect. 4, we introduce a notion of tractability index suitable for the nonlinear systems under examination. Finally, in Sect. 5, we show an application of the theory to coupled models in microelectronics.

2 Nonlinear Elliptic PDAEs

We consider the following coupled problem:

$$\begin{cases} E\dot{x} = Ax + \sigma + b(t), & t \in (t_0, t_1) \subset \mathbb{R}, \\ x(t_0) = x_0, \end{cases} \tag{1a}$$

$$\begin{cases} \mathcal{F}(x, u, \nabla u, \nabla^2 u) = 0, & x \in \Omega \subset \mathbb{R}^d, \\ \mathcal{B}(x, u, \partial u/\partial \nu, \eta) = 0, & x \in \partial\Omega, \end{cases} \tag{1b}$$

$$\sigma = U s(u), \tag{1c}$$

$$\eta = V^\top x. \tag{1d}$$

The first subproblem (1a) is a linear system of DAEs for the unknown $x : [t_0, t_1] \to \mathbb{R}^n$, in a functional space $\mathcal{W}_x$, with initial data x_0. The dot denotes the time derivative, $\mathrm{d}/\mathrm{d}t$, and E, $A \in \mathbb{R}^{n\times n}$. The vector function $b(t) \in \mathbb{R}^n$ is the external input data. The vector $\sigma \in \mathbb{R}^n$ is the internal DAE-input data, which will connect the differential-algebraic part with the partial-differential part of the full problem.

The subproblem (1b) is a boundary value problem for a system of PDEs in $\Omega \subset \mathbb{R}^d$, for the unknown $u : \Omega \to \mathbb{R}^m$, in a functional space $\mathcal{W}_u$. So, $\mathcal{F}$ is a vector-valued function, which takes values in $\mathbb{R}^m$. The dimension d may coincide with the physical dimension of a distributed component of the system, or may be the sum of several physical dimensions of different distributed components of the system. The boundary data in (1b) depend on a vector $\eta \in \mathbb{R}^k$, which will represent the internal PDE-input data, connecting the partial-differential part with the differential-algebraic part. We notice that the time t may enter problem (1b) only parametrically, through η.

The condition (1c) is the PDE-to-DAE coupling condition, which represents how the PDE unknown u affects the DAE problem. We assume that U is a matrix in $\mathbb{R}^{n\times \ell}$ and s is a vector-valued function, $s : \mathcal{W}_u \to \mathbb{R}^\ell$, which in general might be nonlinear.

The condition (1d) is the DAE-to-PDE coupling condition, which represents how the DAE unknown x affects the PDE problem. We assume that V is a matrix in $\mathbb{R}^{n\times k}$. Thus, this is a linear coupling condition.

If the system (1b) admits a solution, uniquely determined by the boundary data η, then, using the coupling condition (1d), it is possible to express the solution u as a function of the unknown vector x of the DAE system (1a), and we can write: $u = \tilde{u}(V^\top x)$. Thus, the coupling condition (1c), becomes

$$\sigma = \varrho(x) := Ur\big(V^\top x\big), \quad \text{with } r\big(V^\top x\big) := s\big(\tilde{u}\big(V^\top x\big)\big). \tag{2}$$

A special case of our system occurs when the PDE subsystem (1b) reduces to an algebraic system,

$$\mathcal{G}(u,\eta) = 0, \qquad \mathcal{G} : D \to \mathbb{R}^m, \quad D \subset \mathbb{R}^m \times \mathbb{R}^k, \tag{3}$$

so that $\varrho(x)$ reduces to an algebraic relation. In this case, assuming that $\mathcal{G}$ is differentiable in D, if we have

$$\det \frac{\partial \mathcal{G}(u,\eta)}{\partial u} \neq 0, \quad \forall (u,\eta) \in D,$$

then it follows that, globally in D, (3) defines u as a function of η, and we can write $u = \tilde{u}(\eta)$. Then, we can define $\rho(x)$ as in (2). If the previous condition is not satisfied, and $\partial\mathcal{G}/\partial u$ is not invertible for some point in D, then we may find several branches of solutions. Assuming that we have only a finite number of branches, we end up with a multivalued version of (2). Nevertheless, the initial data and the coupling with the evolutionary subsystem (1a) might be sufficient to determine a unique branch of solution. A similar situation should occur also for the general case under study, when (1b) truly is a system of PDEs.

Definition 1 We say that the coupled problem (1a)–(1d) admits a solution if there exist $(x,u) \in \mathcal{W}_x \times C^0([t_0,t_1];\mathcal{W}_u)$ which satisfy (1a)–(1d).

This definition holds also when the solution u of (1b), regarded as a decoupled system of PDEs, is not uniquely determined by the boundary data η.

3 Tractability Index

The solutions of a differential-algebraic system behave differently, at variance with the tractability index of the matrix pencil associated to the system. The notion of tractability index is equivalent to the notion of Kronecker index, but it leads to a weaker normal form, which does not require the construction of a block-diagonal matrix with blocks in Jordan form. Moreover, it is sometimes more convenient for theoretical purposes. This section is based on the work of März [10, 16]. We consider DAEs with constant coefficients:

$$E\dot{x} = Ax + b, \tag{4}$$

with associated matrix pencil $A - E\lambda$, which we assume to be regular, that is,

$$\det(A - E\lambda) \not\equiv 0, \text{ as a polynomial in } \mathbb{R}[\lambda].$$

We define, by iteration, a sequence of matrices. We set the initial matrices

$$E_0 := E, \qquad A_0 := A. \tag{5}$$

We assume to know the matrices E_i, A_i, $i = 0, \dots, k-1$, together with the projector Q_i onto the $\ker E_i$, that is, $\operatorname{im} Q_i = \ker E_i$, $Q_i^2 = Q_i$, and its complementary projector $P_i = I - Q_i$, $i = 0, \dots, k-1$, with

$$Q_i Q_j = O, \quad i > j \geq 0 \quad (\text{if } k \geq 1). \tag{6}$$

Then we define the matrices

$$E_k = E_{k-1} - A_{k-1} Q_{k-1}, \qquad A_k = A_{k-1} P_{k-1}. \tag{7}$$

The condition (6) ensures that $\ker Q_i$ contains $\operatorname{im} Q_j$, for $j \leq i$, thus the dimension of $\ker Q_i$ is greater than or equal to the dimension of $\ker Q_{i-1}$, that is, the dimension of $\ker E_i = \operatorname{im} Q_i$ is less than or equal to the dimension of $\ker E_{i-1} \equiv \operatorname{im} Q_{i-1}$. After a finite number of iterations, we get a nonsingular matrix E_μ, with E_k singular for $k < \mu$. Then $E_{\mu+i} = E_\mu$ for all $i \geq 0$.

Definition 2 (Tractability Index [10]) The tractability index of the regular matrix pencil $A - E\lambda$ is the minimum index μ such that $E_{\mu+1} = E_\mu$. The tractability index of system (4) is the tractability index of the matrix pencil $A - E\lambda$.

Next we consider the implications of the iterative construction of E_μ, A_μ to system (4). Using the property (6), and the fact that

$$E_i P_{i-1} = E_{i-1}, \qquad -E_i Q_{i-1} = A_{i-1} Q_{i-1},$$

it is possible to prove [10] that (4) is equivalent to

$$E_k (P_{k-1} \cdots P_0 \dot{x} + Q_0 x + \cdots + Q_{k-1} x) = A_k x + b,$$

for any k. In particular, for $k = \mu$, since E_μ is nonsingular, we can write

$$P_{\mu-1} \cdots P_0 \dot{x} + Q_0 x + \cdots + Q_{\mu-1} x = E_\mu^{-1} (A_\mu x + b). \tag{8}$$

This equation can be projected to obtain a purely differential equation for a part of the solution, and purely algebraic equations for other appropriate parts of the solution. To see this, we need some notation. We set

$$|P_i\rangle_k := \begin{cases} P_i P_{i+1} \cdots P_k, & \text{if } i < k, \\ P_k, & \text{if } i = k, \\ P_i P_{i-1} \cdots P_k, & \text{if } i > k, \end{cases}$$

and use the short notation $\langle P_k| := |P_0\rangle_k$, $|P_i\rangle := |P_i\rangle_0$. We set also

$$\langle Q_i| := \begin{cases} Q_0, & \text{if } i = 0, \\ P_0 \cdots P_{i-1} Q_i, & \text{if } i > 0, \end{cases} \qquad |Q_i\rangle_k := \begin{cases} Q_i P_{i+1} \cdots P_k, & \text{if } 0 \leq i < k, \\ Q_k, & \text{if } i = k, \end{cases}$$

$\langle Q_0\rangle_k := |Q_0\rangle_k$, $\langle Q_i\rangle_k := P_0\cdots P_{i-1}Q_iP_{i+1}\cdots P_k$, $0<i<k$, $\langle Q_k\rangle_k := \langle Q_k|$, and the short notation $|Q_i\rangle := |Q_i\rangle_{\mu-1}$, $\langle Q_i\rangle := \langle Q_i\rangle_{\mu-1}$. Finally, we set:

$$\langle Q_iQ_j| := \begin{cases} P_0\cdots P_{i-1}Q_iQ_{i+1}, & \text{if } j=i+1,\\ P_0\cdots P_{i-1}Q_iP_{i+1}\cdots P_{j-1}Q_j, & \text{if } j>i+1.\end{cases}$$

Condition (6) ensures that some relevant combinations of the projectors Q_i, P_i are still projectors. Also it allows to establish certain useful decompositions of the identity. The main tool is the following absorption property, easily derived from (6):

$$Q_iP_j = Q_i, \qquad P_iQ_j = Q_j, \quad \text{for } i>j. \tag{9}$$

Proposition 1 *Let us consider a sequence of projectors $\{Q_i\}_{i\geq 0}$, satisfying condition* (6), *and let $\{P_i\}_{i\geq 0}$ be their complementary projectors. Then*:

(i) $\langle P_i|$, $\langle Q_i|$, $\langle Q_i\rangle_k$, *for* $0\leq i\leq k$, *are projectors.*
(ii) *The following decompositions of the identity hold*:

$$\langle P_k| + \sum_{i=0}^{k}\langle Q_i| = I, \tag{10}$$

$$|P_k\rangle + \sum_{i=0}^{k} Q_i = I. \tag{11}$$

Proof Using the absorption property (9), for $j\leq i$ we have

$$Q_j\langle P_i| = Q_jP_0\cdots P_j\cdots P_i = Q_jP_j\cdots P_i = O.$$

Then we find $\langle P_i|^2 = \langle P_{i-1}|(I-Q_i)\langle P_i| = \langle P_{i-1}|\langle P_i|$, and iterating the procedure we get $\langle P_i|^2 = \langle P_i|$. Using similar arguments it is possible to prove that $\langle Q_i|$ and $\langle Q_i\rangle_k$ are projectors.

Next, we observe that $\langle P_i| + \langle Q_i| = \langle P_{i-1}|(P_i+Q_i) = \langle P_{i-1}|$. Thus,

$$\langle P_k| + \sum_{i=0}^{k}\langle Q_i| = \langle P_k| + \sum_{i=0}^{k-1}\langle Q_i| + \langle Q_k| = \langle P_{k-1}| + \sum_{i=0}^{k-1}\langle Q_i|,$$

and iterating the procedure we get (10). Also, using property (9), we find that $|P_i\rangle + Q_i = (I-Q_i)|P_{i-1}\rangle + Q_i = |P_{i-1}\rangle$. Then we get

$$|P_k\rangle_0 + \sum_{i=0}^{k} Q_i = P_k|P_{k-1}\rangle_0 + \sum_{i=0}^{k-1} Q_i + Q_k = |P_{k-1}\rangle_0 + \sum_{i=0}^{k-1} Q_i,$$

and iterating the procedure we find (11). □

Proposition 2 *Let us consider two sequences of projectors $\{Q_i\}_{i\leq 0}$, $\{P_i\}_{i\leq 0}$, as in Proposition* 1. *Then*:

(i) *For the projector $\langle P_k|$, it holds*:

$$\langle P_k||P_k\rangle = |P_k\rangle\langle P_k| = \langle P_k|, \tag{12}$$

$$\langle P_k|Q_j = Q_j\langle P_k| = O, \quad j \le k. \tag{13}$$

(ii) *For the projectors* $\langle Q_i\rangle_k$, *with* $i \le k$, *it holds*:

$$\langle Q_i\rangle_k|P_k\rangle = \langle Q_i\rangle_k - \langle Q_i|, \quad i \le k, \tag{14}$$

$$\langle Q_i\rangle_k Q_j = \langle Q_i|\delta_{ij}, \quad i, j \le k. \tag{15}$$

(iii) *For* $i < k$, *it holds*:

$$\langle Q_i\rangle_k = \langle Q_i| - \sum_{j=i+1}^{k} \langle Q_i Q_j|. \tag{16}$$

(iv) *For any* $i, j \ge 0$ *it holds*:

$$Q_i\langle Q_j| = Q_i\delta_{ij}. \tag{17}$$

Proof (i) For $j = k$, we verify immediately that $\langle P_k|Q_k = O$. For $j < k$, using the absorption property (9), we have $\langle P_k|Q_j = \langle P_{k-1}|P_k Q_j = \langle P_{k-1}|Q_j$, and iterating the procedure we find $\langle P_k|Q_j = \langle P_j|Q_j = O$. Similarly, for $j = 0$ we find $Q_0\langle P_k| = Q_0|P_0\rangle_k = O$. For $0 < j \le k$, using again (9), we get $Q_j|P_0\rangle_k = Q_j P_0|P_1\rangle_k = Q_j|P_1\rangle_k$. Iterating the procedure, we find $Q_j|P_0\rangle_k = Q_j|P_j\rangle_k = O$, which proves (13). Using (11) we get (12). (ii) The proof of (15) is perfectly analogous to the proof of (13), and (14) follows by means of (11). (iii) We have

$$\langle Q_i\rangle_k = \langle Q_i\rangle_{k-1}(I - Q_k) = \langle Q_i\rangle_{k-1} - \langle Q_i Q_k|,$$

and iterating $k - i$ times we get (16). (iv) If $i < j$, using property (9), we find $Q_i\langle Q_j| = Q_i P_i \cdots P_{j-1} Q_j = O$. If $i \ge j$, using again (9), we get $Q_i\langle Q_j| = Q_i Q_j$, which is zero if $i > j$. □

In particular, using (14), (16) and (17), Proposition (2) yields

$$\langle Q_i\rangle|P_{\mu-1}\rangle = \langle Q_i\rangle - \langle Q_i| = -\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j| = -\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j|\langle Q_j|. \tag{18}$$

Proposition 1 implies that $\langle P_{\mu-1}|$, $\langle Q_i\rangle$, $i = 0, \dots, \mu - 1$, are projectors, although they do not form a partition of the identity. Left-multiplying equation (8) by these projectors, and using Proposition 2 it is possible to obtain a differential equation for the variable $y := \langle P_{\mu-1}|x$, and μ algebraic equations for the variables $z_i := \langle Q_i|x$, $i = 0, 1, \dots, \mu - 1$:

$$\begin{cases} \dot{y} = \langle P_{\mu-1}|E_\mu^{-1}(A_\mu y + b), \\ z_i = \langle Q_i\rangle E_\mu^{-1}(A_\mu y + b) + \displaystyle\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j|\dot{z}_j, \quad i = 0, \dots, \mu - 2, \\ z_{\mu-1} = \langle Q_{\mu-1}\rangle E_\mu^{-1}(A_\mu y + b). \end{cases} \tag{19}$$

We can get the differential part y by solving the first equation in (19), which is a purely differential equation, and then use the solution in the remaining algebraic equations. The algebraic parts z_i can be computed recursively, starting from the last one, $z_{\mu-1}$, to the first one, z_0. The full solution can be recovered from the decomposition of the identity (10), which implies

$$x = y + \sum_{i=0}^{\mu-1} z_i.$$

Thus, the solution contains the first $\mu - 1$ derivatives of the input vector b, since z_i depends linearly on the time-derivative of z_j, with $j > i$, and $z_{\mu-1}$ depends linearly on b.

4 Tractability Index of a Class of Nonlinear PDAEs

In this section we turn our attention to the nonlinear system of PDAEs (1a)–(1d), and we introduce additional conditions which ensure that the decomposition procedure for getting the differential and algebraic components, presented in the previous section, is preserved in presence of the nonlinear term.

We start from the nonlinear system

$$\begin{cases} E\dot{x} = Ax + \varrho(x) + b(t), \\ \varrho(x) = Ur(V^\top x), \end{cases} \tag{20}$$

where the nonlinear term $\varrho(x)$ is, at the moment, unspecified, and could come from the coupling with a system of PDEs. We consider the decomposition procedure associated to the tractability index for the linear part of system (20), with matrix pencil $A - E\lambda$, treating the nonlinear term $\varrho(x)$ as a perturbation of the input term b. In this way, we obtain

$$\begin{cases} \dot{y} = \langle P_{\mu-1}| E_\mu^{-1}(A_\mu y + \varrho(x) + b), \\ z_i = \langle Q_i \rangle E_\mu^{-1}(A_\mu y + \varrho(x) + b) + \displaystyle\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | \dot{z}_j, \quad i = 0, \ldots, \mu-2, \\ z_{\mu-1} = \langle Q_{\mu-1} \rangle E_\mu^{-1}(A_\mu y + \varrho(x) + b), \end{cases} \tag{21}$$

for the variables $y := \langle P_{\mu-1}|x$, $z_i := \langle Q_i|x$, $i = 0, 1, \ldots, \mu-1$, where the matrices E_μ, A_μ, and the projectors P_i, Q_i, $i = 0, \ldots, \mu-1$, are defined in the previous section, and we assume $\mu \geq 1$.

At the right-hand side of the projected equations (21), in the argument of the nonlinear term ϱ, there appears the full unknown x, which can be reconstructed as follows:

$$\begin{aligned} x &= y + \sum_{i=0}^{\mu-1} z_i = y + \sum_{i=0}^{\mu-1} \langle Q_i \rangle E_\mu^{-1}\big(A_\mu y + \varrho(x) + b\big) + \sum_{i=0}^{\mu-2} \sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | \dot{z}_j \\ &= M_\mu^* y + M_\mu b + \sum_{i=0}^{\mu-1} \langle Q_i \rangle E_\mu^{-1} \varrho(x) + \sum_{i=0}^{\mu-2} \sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | \dot{z}_j \end{aligned}$$

with $M_\mu^* = I + M_\mu A_\mu$, $M_\mu = \sum_{i=0}^{\mu-1} \langle Q_i \rangle_{\mu-1} E_\mu^{-1}$. To avoid recursion in the nonlinear term $\varrho(x) = Ur(V^\top x)$, it is sufficient to assume the conditions

$$V^\top \sum_{i=0}^{\mu-1} \langle Q_i \rangle E_\mu^{-1} U = O, \quad \text{if } \mu \geq 1, \tag{22}$$

$$V^\top \sum_{i=0}^{\mu-2} \sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | = O, \quad \text{if } \mu \geq 2. \tag{23}$$

If (22)–(23) holds, we can write the nonlinear term in the following way:

$$\varrho(x) = Ur\big(V^\top M_\mu^* y + V^\top M_\mu b\big) = \varrho\big(M_\mu^* y + M_\mu b\big).$$

Then the projected system becomes

$$\begin{cases} \dot{y} = \langle P_{\mu-1} | E_\mu^{-1}\big(A_\mu y + \varrho\big(M_\mu^* y + M_\mu b\big) + b\big), \\ z_i = \langle Q_i \rangle E_\mu^{-1}\big(A_\mu y + \varrho\big(M_\mu^* y + M_\mu b\big) + b\big) + \displaystyle\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | \dot{z}_j, \\ \quad i = 0, \dots, \mu - 2, \\ z_{\mu-1} = \langle Q_{\mu-1} \rangle E_\mu^{-1}\big(A_\mu y + \varrho\big(M_\mu^* y + M_\mu b\big) + b\big), \end{cases} \tag{24}$$

which retains the same structure of (19). Again, we can get the differential part y by solving the first equation in (24), which is a differential equation, and we can compute the algebraic parts z_i recursively, from the remaining algebraic equations.

Definition 3 The extended tractability index of the nonlinear system (20) is μ if the following conditions hold:

(i) the matrix pencil $A - E\lambda$ has tractability index μ;
(ii) the matrices U, V satisfy the additional conditions (22)–(23);
(iii) equation $(24)_1$ is solvable for y according to Definition 1, and the solution admits $\mu - 1$ time derivatives.

The same definition applies also to the extended tractability index of the system of PDAEs (1a)–(1d). We remark that the condition (iii) poses some severe restrictions on the PDE (1b). Its validity depends on the well-posedness of the coupled problem made of the ODE $(24)_1$ and the PDE (1b), coupled by (1d). In the next section we will discuss briefly this condition for a specific elliptic system of PDEs.

Next, we will concentrate on the additional conditions (22)–(23), determining and characterizing simpler conditions which imply them. We will replace (22) with one of the following conditions:

$$V^\top \sum_{i=0}^{\mu-1} \langle Q_i \rangle = O, \tag{25}$$

$$\sum_{i=0}^{\mu-1} \langle Q_i \rangle E_\mu^{-1} U = O. \tag{26}$$

Proposition 3 *Let $A - E\lambda \in \mathbb{R}^{n\times n}[\lambda]$ be a regular matrix pencil with tractability index $\mu \geq 1$, and let $V \in \mathbb{R}^{n\times \ell}$. Then, the following two conditions are equivalent to condition* (25):

$$V^\top \langle Q_j | = O, \quad j = 0, 1, \dots, \mu - 1; \tag{27}$$

$$V^\top Q_j = O, \quad j = 0, 1, \dots, \mu - 1. \tag{28}$$

Proof Condition (27) can be derived from (25), after right-multiplying the latter by Q_j, $j = 1, \ldots, \mu - 1$, and using (15). On the other hand, (27) implies (25), since by definition $\langle Q_i \rangle_{\mu-1} = \langle Q_i || Q_i \rangle_{\mu-1}$, thus (25) and (27) are equivalent. To prove the equivalence of (27) and (28) we proceed by induction. For $j = 0$, the two conditions both reduce to $V^\top Q_0 = O$, so they are equivalent. Let us assume that they are equivalent for $j = 0, \ldots, k$. Then, the equivalence for $j = 0, \ldots, k+1$ follows from the identity $\langle Q_{k+1}| = Q_{k+1} - \sum_{j=0}^{k} \langle Q_j | Q_{k+1}$, which can be obtained by using the definition of $\langle Q_{k+1}|$ and identity (10). □

Corollary 1 *Let $A - E\lambda \in \mathbb{R}^{n\times n}[\lambda]$ be a regular matrix pencil with tractability index $\mu \geq 1$, and let $V \in \mathbb{R}^{n\times\ell}$ satisfy condition* (25). *Then V satisfies also* (22)–(23).

Proof It is sufficient to observe that (25) implies (23), because of (27). □

When the nonlinear term $\rho(x)$ in (20) is algebraic, the above definition of extended tractbility index should coincide with the usual definition of tractability index of a nonlinear system of DAEs, as given in [12]. This is not the case, but we can prove that condition (25) for V implies the coincidence of the two notions of index.

Theorem 1 *Let $A - E\lambda \in \mathbb{R}^{n\times n}[\lambda]$ be a regular matrix pencil with tractability index $\mu \geq 1$, let $\rho(x) = Ur(V^\top x)$, with r differentiable function from $\mathbb{R}^\ell$ to $\mathbb{R}^k$, and let $V \in \mathbb{R}^{n\times\ell}$ satisfy condition* (25). *Then the nonlinear system* (20) *has tractability index μ.*

Proof Adapting to systems of the form (20) the definition of tractability index given in [12], we can introduce a matrix sequence which extends the one defined by (5)–(7). We start with $G_0 := E \equiv E_0$, $B_0(x) := A + Ur_\xi(V^\top x)V^\top \equiv A_0 + Ur_\xi(V^\top x)V^\top$, where r_ξ is the Jacobian matrix of r with respect to its argument. After introducing a projector Q_0 onto $\ker G_0$, and $P_0 = I - Q_0$, we set $G_1 := G_0 + B_0(x)Q_0$. Using $V^\top Q_0 = O$, we have $G_1 = E_0 + A_0 Q_0 \equiv E_1$. Since the projector Q_1 onto $\ker G_1$ does not depend on x and t, as well as P_1, we can define $B_1(x) := B_0(x)P_0 \equiv A_1 + Ur_\xi(V^\top x)V^\top P_0$. Then, we set $G_2 := G_1 + B_1(x)Q_1 \equiv E_1 + A_1 Q_1 + Ur_\xi(V^\top x)V^\top P_0 Q_1 = E_2$. since $V^\top \langle Q_1| = O$. Recursively, for $i > 1$, we assume to have the matrices $B_{i-1}(x) \equiv A_{i-1} + Ur_\xi(V^\top x)V^\top \langle P_{i-2}|$, $G_i \equiv E_i$, and we define $B_i(x) := B_{i-1}(x)P_i \equiv A_i + Ur_\xi(V^\top x)V^\top \langle P_{i-1}|$, $G_{i+1} := G_i + B_i(x)Q_i$. We find $G_{i+1} = E_i + A_i Q_i + Ur_\xi(V^\top x)V^\top \langle Q_i| = E_{i+1}$, since $V^\top \langle Q_i| = O$. In conclusion, the sequence $\{G_i\}_{i\geq 0}$ coincide with the sequence $\{E_i\}_{i\geq 0}$, and the nonlinear tractability index is μ. □

Next, we consider condition (26). Using (16) and (10) we find

$$\sum_{i=0}^{\mu-1} \langle Q_i \rangle = \big(I - \langle P_{\mu-1}|\big) - \sum_{i=0}^{\mu-2} \sum_{j=i+1}^{\mu-1} \langle Q_i Q_j|.$$

It follows that, if (23) holds, then (26) is equivalent to

$$\big(I - \langle P_{\mu-1}|\big) E_\mu^{-1} U = O, \quad \text{if } \mu \geq 1. \tag{29}$$

The following two propositions characterize conditions (29) and (23).

Proposition 4 *Let $A - E\lambda \in \mathbb{R}^{n\times n}[\lambda]$ be a regular matrix pencil with tractability index $\mu \geq 1$, and let $U \in \mathbb{R}^{n\times\ell}$. Then the following two conditions are equivalent to condi-*

tion (29):

$$Q_i E_\mu^{-1} U = O, \quad i = 0, 1, \dots, \mu - 1; \tag{30}$$

$$U = E_i E_\mu^{-1} U, \quad i = 0, 1, \dots, \mu - 1. \tag{31}$$

Proof First we prove that (29) is equivalent to (30). Using the property (13), we have $Q_i(I - \langle P_{\mu-1}|)E_\mu^{-1}U = Q_i E_\mu^{-1} U$, thus (30). On the other hand, (30) implies condition (29) because of identity (10). Next, we observe that (30) is equivalent to $U = E_\mu P_i E_\mu^{-1} U$, $i = 0, 1, \dots, \mu - 1$. For $i < \mu - 1$, using the absorption property (9), we have

$$\begin{aligned} E_\mu P_i E_\mu^{-1} U &= (E_{\mu-1} - A_{\mu-1} Q_{\mu-1}) P_i E_\mu^{-1} U \\ &= E_{\mu-1} P_i E_\mu^{-1} U - A_{\mu-1} Q_{\mu-1} E_\mu^{-1} U = E_{\mu-1} P_i E_\mu^{-1} U. \end{aligned}$$

Iterating $\mu - 1 - i$ times, we arrive to $E_\mu P_i E_\mu^{-1} U = E_{i+1} P_i E_\mu^{-1} U$. Then (31) follows from the identity

$$E_{i+1} P_i = (E_i - A_i Q_i) P_i = E_i, \quad i = 0, 1, \dots, \mu - 1.$$

For $i = \mu - 1$, (31) follows directly from the previous identity. Thus, we have proved that (30) implies (31). On the other hand, if (31) holds we get

$$\begin{aligned} U &= E_i E_\mu^{-1} U = E_{i+1} P_i E_\mu^{-1} U \\ &= (E_{i+1} - A_{i+1} Q_{i+1}) P_i E_\mu^{-1} U - A_{i+1} Q_{i+1} E_\mu^{-1} U \\ &= E_{i+2} P_i E_\mu^{-1} U - A_{i+1} Q_{i+1} E_\mu^{-1} U, \quad \text{for } i < \mu - 1. \end{aligned}$$

Iterating the above procedure, we find

$$U = E_\mu P_i E_\mu^{-1} U - \sum_{j=i+1}^{\mu-1} A_j Q_j E_\mu^{-1} U, \quad \text{for } i < \mu - 1. \tag{32}$$

Similarly, for $i = \mu - 1$ we find

$$U = E_\mu P_{\mu-1} E_\mu^{-1} U,$$

which is equivalent to $Q_{\mu-1} E_\mu^{-1} U = O$. Using this condition in (32), proceeding recursively from $i = \mu - 2$ to $i = 0$, we find (30). □

Proposition 5 *Let $A - E\lambda \in \mathbb{R}^{n\times n}[\lambda]$ be a regular matrix pencil with tractability index $\mu \geq 2$, and let $V \in \mathbb{R}^{n\times \ell}$. Then the following condition is equivalent to* (23):

$$V^\top \sum_{i=0}^{j-1} \langle Q_i Q_j | = O, \quad j = 1, \dots, \mu - 1. \tag{33}$$

Proof Condition (33) can be derived from (23), after right-multiplying the latter by Q_j, $j = 1, \dots, \mu - 1$, and using (17). Condition (23) follows from (33) because of the identity

$$V^\top \sum_{i=0}^{\mu-2} \sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | = \sum_{j=1}^{\mu-1} V^\top \sum_{i=0}^{j-1} \langle Q_i Q_j |.$$

□

Propositions 4 and 5 lead to simpler checks of the index conditions (29) and (23). In particular, condition (31) implies that (29) holds if and only if

$$\hat{Q}_i^\top U = O, \quad i = 0, 1, \dots, \mu - 1, \tag{34}$$

for any projector $\hat{Q}_i$ onto $\ker E_i^\top$. Finally, by using the above propositions, if U and V satisfy (29) and (33), the projected system (24) reduces to:

$$\begin{cases} \dot{y} = \langle P_{\mu-1} | E_\mu^{-1}\big(A_\mu y + \varrho\big(M_\mu^* y + M_\mu b\big) + b\big), \\ z_i = \langle Q_i \rangle E_\mu^{-1}(A_\mu y + b) + \displaystyle\sum_{j=i+1}^{\mu-1} \langle Q_i Q_j | \dot{z}_j, \quad i = 0, \dots, \mu - 2, \\ z_{\mu-1} = \langle Q_{\mu-1} \rangle E_\mu^{-1}(A_\mu y + b). \end{cases} \tag{35}$$

We notice that the nonlinear term ϱ does not enter explicitly the algebraic equations in (35).

We conclude this section by comparing the additional condition (25), on one side, and (26), (23), on the other side, for index-1 and index-2 systems.

If the matrix pencil $A - E\lambda$ has tractability index 1, then E_1 is nonsingular, and the additional condition (25) is equivalent to

$$V^\top Q_0 = O, \tag{36}$$

while the additional conditions (26) and (23) are equivalent to (34), that is,

$$\hat{Q}_0^\top U = O, \tag{37}$$

for any projector $\hat{Q}_0$ onto $\ker E_0^\top$. In the case $U \equiv V$, the previous two conditions coincide if and only if $\ker E_0^\top = \ker E_0$.

If the matrix pencil $A - E\lambda$ has tractability index 2, then E_1 is singular and E_2 is nonsingular, and the additional condition (25) is equivalent to

$$V^\top Q_0 = O, \qquad V^\top Q_1 = O, \tag{38}$$

while the additional conditions (26) and (23) are equivalent to

$$\hat{Q}_0^\top U = O, \qquad \hat{Q}_1^\top U = O, \qquad V^\top Q_0 Q_1 = O, \tag{39}$$

for any projector $\hat{Q}_0$ onto $\ker E_0^\top$, and $\hat{Q}_1$ onto $\ker E_1^\top$. In the case $U \equiv V$, the condition (39) simplifies if $\ker E_0^\top = \ker E_0$, since the third condition in it is implied by the first one. The equivalence of (38) and (39) requires also $\ker E_1^\top = \ker E_1$.

5 Applications to PDAE Models of Integrated Circuits

In this section we apply the concept of tractability index to PDAE models of integrated circuits. An integrated circuit can be modeled by an electric network, described by a system of DAEs, with distributed components, like semiconductor devices, described by systems of PDEs [1, 7, 17]. The final model takes the form (1a)–(1d). We describe separately the network equations (1a), the semiconductor device equations (1b), and the coupling conditions (1c), (1d).

5.1 Network Models for Electric Circuits

We model an electric circuit by an RLC network, that is a set of nodes connected by branches containing only resistances R, inductances L, conductances C, and in addition branches with independent current $i(t) \in \mathbb{R}^{n_I}$ and voltage sources $v(t) \in \mathbb{R}^{n_V}$. An electric voltage is associated to each node but one, the mass node, where the voltage is zero, and an electric current is associated to each branch. Using the formalism of the Modified Nodal Analysis [14], the unknowns of the problem are the node potentials, $e(t) \in \mathbb{R}^{n_e}$, the currents through inductors, $i_L(t) \in \mathbb{R}^{n_L}$, and the currents through voltage sources, $i_V(t) \in \mathbb{R}^{n_V}$. The vector $x = [e^\top\ i_L^\top\ i_V^\top]^\top \in \mathbb{R}^n$, $n = n_e + n_L + n_V$, satisfies the network equation (1a), with:

$$E = \begin{bmatrix} A_C C A_C^\top & O & O \\ O & L & O \\ O & O & O \end{bmatrix}, \qquad A = -\begin{bmatrix} A_R G A_R^\top & A_L & A_V \\ -A_L^\top & O & O \\ -A_V^\top & O & O \end{bmatrix}, \qquad b = -\begin{bmatrix} A_I i(t) \\ O \\ v(t) \end{bmatrix}. \tag{40}$$

The matrices $A_C \in \mathbb{R}^{n_e \times n_C}$, $A_R \in \mathbb{R}^{n_e \times n_R}$, $A_L \in \mathbb{R}^{n_e \times n_L}$, $A_V \in \mathbb{R}^{n_e \times n_V}$, $A_I \in \mathbb{R}^{n_e \times n_I}$, are incidence matrices, which describe the topology of the network. The capacitance, inductance and conductance matrices, $C \in \mathbb{R}^{n_C \times n_C}$, $L \in \mathbb{R}^{n_L \times n_L}$ and $G \in \mathbb{R}^{n_G \times n_G}$, are assumed to be and positive definite. The term $\sigma \in \mathbb{R}^n$ describes the coupling with the devices, and comprises the currents exchanged by the electric circuit and the devices.

5.2 Distributed Models for Semiconductor Devices

For simplicity, we consider one semiconductor device with $k = \ell + 1$ terminals, modeled by steady-state drift-diffusion equations [13]. The unknowns of the problem are the electric potential $\phi(\mathrm{x}, t)$, and the electron and hole quasi-Fermi potentials, $\phi_n(\mathrm{x}, t)$, $\phi_p(\mathrm{x}, t)$, with x in the domain $\Omega \subset \mathbb{R}^d$, which describes the physical device. The time dependence is induced by time-dependent boundary conditions. We assume that the boundary $\partial\Omega$ can be split in two parts, $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, where $\Gamma_D = \bigcup_{i=0}^{\ell} \Gamma_i$ is union of all the device terminals (Ohmic contacts), where we assume Dirichlet boundary conditions, and Γ_N is the union of all the insulated portions, where we assume homogeneous Neumann boundary conditions. The unknown $u = (\phi, \phi_n, \phi_p)$ satisfies the steady-state drift-diffusion model, that is, an equation of the form (1b):

$$\begin{cases} -\nabla \cdot (\epsilon \nabla \phi) = q(N + p - n), & \\ \nabla \cdot j_n = qR(n, p), & j_n = -q\mu_n n \nabla \phi_n, \\ \nabla \cdot j_p = -qR(n, p), & j_p = -q\mu_p p \nabla \phi_p, \end{cases} \tag{41a}$$

with boundary conditions

$$\begin{cases} \phi - \phi_{\mathrm{bi}} = \phi_n = \phi_p = e_{D,i}(t), & \text{on } \Gamma_i, i = 0, \dots, \ell, \\ \partial\phi/\partial\nu = \partial\phi_n/\partial\nu = \partial\phi_p/\partial\nu = 0, & \text{on } \Gamma_N, \\ [e_{D,0} \cdots e_{D,\ell}]^\top =: \eta \in \mathbb{R}^k. & \end{cases} \tag{41b}$$

In system (41a), (41b), the electron and hole number densities are related to the potentials by the Maxwell-Boltzmann relations, $n = n_{\mathrm{i}} \exp((\phi - \phi_n)/U_T)$, $p = n_{\mathrm{i}} \exp((\phi_p - \phi)/U_T)$, $j_n(\mathrm{x}, t)$, $j_p(\mathrm{x}, t)$ are the current densities for electrons and holes and q is the positive elementary electric charge. Moreover, $N(\mathrm{x})$ is the doping profile, $\epsilon(\mathrm{x})$ is the dielectric constant, $R(n, p)$ is the recombination-generation term, $\mu_n(n, p, \mathrm{x})$, $\mu_p(n, p, \mathrm{x})$ are electron and hole mobilities, n_{i} is the intrinsic concentration and $\phi_{\mathrm{bi}}(\mathrm{x})$ is the built-in potential. The term $\eta \in \mathbb{R}^k$ is responsible for the coupling with the electric network.

5.3 Coupling conditions

The coupling of the electric network with the device is described by a selection matrix $S_D = (s_{D,ij}) \in \mathbb{R}^{n_e \times k}$, where $s_{D,ij}$ is equal to 1 if the node i is connected to the terminal j, and equal to 0 otherwise. Also, we introduce the incidence matrix A_D for the network branches containing the device, defined by $A_D = S_D \widehat{A} \in \mathbb{R}^{n_e \times \ell}$, with

$$\widehat{A} = \begin{bmatrix} -1 & \cdots & -1 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{k \times \ell}.$$

The coupling conditions (1c) and (1d) hold with the matrices

$$V = \begin{bmatrix} S_D \\ O \\ O \end{bmatrix} \in \mathbb{R}^{n \times k}, \qquad U = \begin{bmatrix} A_D \\ O \\ O \end{bmatrix} = V\widehat{A} \in \mathbb{R}^{n \times \ell}, \tag{42}$$

and the vector-valued function

$$s(u) = \begin{bmatrix} j_{D,1} \\ \vdots \\ j_{D,k} \end{bmatrix}, \quad j_{D,i}(t) = -\int_{\Gamma_i} \nu \cdot \big(j_n(\mathrm{x},t) + j_p(\mathrm{x},t)\big)\, \mathrm{d}\Sigma, \tag{43}$$

which comprises the currents through the device's terminals. The current $j_{D,0}$ through Γ_0 does not appear in $s(u)$, because it can be expressed in terms of the other currents, $\sum_{i=0}^{k} j_{D,i} = 0$. This relation explains the form of the matrix $\widehat{A}$, which connects U to V.

5.4 Tractability Index for Elliptic PDAEs for Integrated Circuits

The analysis of the tractability index of the system described in this section is based on the tractability index of the matrix pencil $A - E\lambda$, which has been extensively studied in literature [9, 18]. In particular, Tischendorf has shown that this index has a topological character, that is, it depends only on the incidence matrices. Moreover, for the matrices described in subsection 5.1 the tractability index (or topological index) cannot exceed 2.

Proposition 6 *The matrix pencil $A - E\lambda$, with E, A given in Sect. 5.1, has tractability index* 1 *if and only if*

$$\ker(A_C, A_V, A_R)^\top = \{0\}, \qquad \ker\big(Q_C^\top A_V\big) = \{0\}, \tag{44}$$

where Q_C is a projector onto the $\ker A_C^\top$. *The matrix pencil has tractability index* 2 *if and only if at least one of the conditions in* (44) *is violated.*

To study the tractability index of the full coupled model, we need to check the additional conditions for the matrices U, V discusses in the previous section. The tractability condition

(25) can be easily checked by using the equivalent condition (28). The alternative conditions (26), (22) can be checked by using the equivalent condition (34).

If $A - E\lambda$ has tractability index 1, the check of the additional conditions requires only the knowledge of the projector Q_0, which can be expressed in terms of Q_C [9]. In this case it holds $\ker E_0^\top = \ker E_0$, so we can choose $\hat{Q}_0 = Q_0$. It is possible to prove that:

$$V^\top Q_0 = O \quad \Longleftrightarrow \quad S_D^\top Q_C = O, \tag{45}$$

$$Q_0^\top U = O \quad \Longleftrightarrow \quad A_D^\top Q_C = O. \tag{46}$$

The following existence result for index-1 elliptic PDAEs for integrated circuits has been proved in [4].

Theorem 2 *The problem* (1a)–(1b), *with* (40)–(43), *and topological conditions* (44), (46), *admits a solution*,

$$(x, u) \in C^0\big([t_0, t_1]\big) \times C^0\big([t_0, t_1]; H^1(\Omega) \cap L_\infty(\Omega)\big),$$

with $P_0 x \in C^1([t_0, t_1])$. *Moreover, any solution satisfies the estimates*:

$$\begin{aligned}
&\big|P_0 x(t)\big|^2 \le c_y e^{k(t-t_0)}\big(|y_0|^2 + \|b\|^2_{L^2([t_0,t_1])}\big),\\
&\big|Q_0 x(t)\big|^2 \le c_z\big(\big|P_0 x(t)\big|^2 + \big|b(t)\big|^2\big),\\
&\inf_{\Gamma_D} \phi_{\mathrm{bi}} + \min_i e_{D,i} \le u \le \sup_{\Gamma_D} \phi_{\mathrm{bi}} + \max_i e_{D,i},\\
&\min_i e_{D,i} \le \phi_n \le \max_i e_{D,i}, \quad \min_i e_{D,i} \le \phi_p \le \max_i e_{D,i},
\end{aligned}$$

where y_0 *is the initial data for the differential part of* x, *for some positive constants* c_y, c_z *and* k *depending only on* E, A.

In conclusion, the conditions (44), (46) imply that the system (1a)–(1b), with (40)–(43), has extended tractability index 1, according to Definition 3.

For index-2 matrix pencils $A - E\lambda$, that is, such that at least one of the conditions in (44) is violated, the check of the additional conditions for U, V requires the knowledge of Q_0, as well as of Q_1 and $\hat{Q}_1$. It is possible to express Q_1 in terms of Q_{CVR}, projector onto the $\ker(A_C, A_V, A_R)^\top$, and Q^*_{CV}, projector onto the $\ker(Q_C^\top A_V)$, but the expression involves the inversion of a matrix [18]. We can give sufficient conditions for the validity of (25):

$$Q_C^\top S_D = O, \quad Q_{CVR}^\top S_D = O \quad \Longrightarrow \quad V^\top Q_0 = O, \quad V^\top Q_1 = O. \tag{47}$$

A simple computation shows that $\hat{Q}_1$ has a simple expression, in terms of Q_{CVR} and Q^*_{CV}, and we find that:

$$Q_0^\top U = O, \quad \hat{Q}_1^\top U = O \quad \Longleftrightarrow \quad Q_C^\top A_D = O, \quad Q_{\mathrm{CVR}}^\top A_D = O. \tag{48}$$

The condition (23), that is, $V^\top Q_0 Q_1 = O$, is more difficult to check. We just notice that it is implied by the first condition in (47), that is, $Q_C^\top S_D = O$.

To conclude that the system (1a)–(1b), with (40)–(43), has extended tractability index 2, according to Definition 3, we need a result similar to Theorem 2. The proof of Theorem 2 relies essentially on a priori estimates for the DAE variables independently of the PDE variables, on one hand, and on some technical results for the PDEs (41a), (41b). Since the a priori estimates depend on the passivity property $x^\top \sigma \leq 0$, we expect a similar result to hold also for index-2 elliptic PDAEs, and will be given in a forthcoming paper, together with a full characterization of the conditions for tractability index 2.

References

1. Alì, G.: PDAE models of integrated circuits. Math. Comput. Model. **51**, 915–926 (2010)
2. Alì, G., Bartel, A., Günther, M., Tischendorf, C.: Elliptic partial differential-algebraic multiphysics models in electrical network design. Math. Models Methods Appl. Sci. **13**(9), 1261–1278 (2003)
3. Alì, G., Bartel, A., Günther, M.: Existence and uniqueness for an elliptic PDAE model of integrated circuits. SIAM J. Appl. Math. **70**(5), 1587–1610 (2010)
4. Alì, G., Rotundo, N.: An existence result for elliptic partial differential-algebraic equations arising in semiconductor modeling. Nonlinear Anal. **72**, 4666–4681 (2010)
5. Rang, J., Angermann, L.: Perturbation index of linear partial differential-algebraic equations. Appl. Numer. Math. **53**, 437–456 (2005)
6. Bodestedt, M.: Perturbation analysis of refined models in circuit simulation. PhD thesis, Technical University of Berlin (2007)
7. Bodestedt, M., Tischendorf, C.: PDAE models of integrated circuits and index analysis. Math. Comput. Model. Dyn. Syst. **13**, 1–17 (2007)
8. Campbell, S.L., Marszalek, W.: The index of an infinite dimensional implicit system. Math. Comput. Model. Dyn. Syst. **5**, 18–42 (1999)
9. Estévez Schwarz, D., Tischendorf, C.: Structural analysis of electric circuits and consequences for MNA. Int. J. Theor. Appl. **28**, 131–162 (2000)
10. Griepentrog, E., März, R.: Basic properties of some differential-algebraic equations. Z. Anal. Anwend. **8**(1), 25–40 (1989)
11. Lucht, W., Strehmel, K.: Discretization based indices for semilinear partial differential algebraic equations. Appl. Numer. Math. **28**, 371–386 (1998)
12. März, R.: Characterizing differential algebraic equations without the use of derivative arrays. Comput. Math. Appl. **50**, 1141–1156 (2005)
13. Markowich, P.A.: The Stationary Semiconductor Device Equations. Springer, Wien (1986)
14. McCalla, W.J.: Fundamentals of Computer Aided Circuit Simulation. Kluwer, Dordrecht (1988)
15. Reis, T.: Systems theoretic aspects of PDAEs and applications to electrical circuits. PhD thesis, Fachbereich Mathematik, Technische Universität Kaiserslautern, Kaiserslautern (2006)
16. Riaza, R.: Differential-Algebraic Systems. Analytical Aspects and Circuit Applications. World Scientific, Singapore (2008)
17. Tischendorf, C.: Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis. Habilitationsschrift, Inst. für Math., Humboldt-Univ. zu Berlin (2004)
18. Tischendorf, C.: Topological index calculation of differential-algebraic equations in circuit simulation. Surv. Math. Ind. **8**, 187–199 (1999)