Applied Mathematics and Computation 378 (2020) 125167

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Admissibility criteria for nonuniform dichotomic behavior of nonautonomous systems on the whole line

Adina Luminiţa Sasu [a,b], Bogdan Sasu [a,b,*]

[a] *Department of Mathematics, West University of Timişoara, V. Pârvan Blvd. 4, Timişoara, Romania*
[b] *Academy of Romanian Scientists, Splaiul Independenţei 54, Bucharest, Romania*

ARTICLE INFO

Article history:
Received 11 November 2019
Revised 3 February 2020
Accepted 17 February 2020

MSC:
34D09
93C25
93D25

Keywords:
Exponential dichotomy
Admissibility
Evolution family
Integral equation

ABSTRACT

We give new criteria for nonuniform dichotomy of nonautonomous systems on the whole line in terms of admissibility relative to an integral equation. In our approach the input space $I(\mathbb{R}, X)$ is an intersection of spaces that can be successively minimized and the output space $C(\mathbb{R}, X)$ can be one of some well-known spaces of continuous functions. Using computational arguments, we show that the admissibility of $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ leads to a nonuniform exponential dichotomy. We expose a complete analysis of the connections between admissibility and nonuniform dichotomy on the whole line and we also discuss several interesting consequences. Moreover, we obtain the explicit expression of the growth rates for dichotomy in terms of the initial exponential growth and the norm of the input-output operator. Finally, we present a direct application of the main result in the case of evolution families which admit uniform exponential growth.

1. Introduction

The exponential dichotomy is one of the most important asymptotic behaviors, with a notable progress over the past decades, having a major significance in the qualitative theory of dynamical systems (see [1–26,28–38,40–56,58–62]). The theories developed around exponential dichotomy have a strong foundation in the remarkable books of Massera and Schäffer [31], Daleckiĭ and Kreĭn [19] and Coppel [18]. In the last decades, a substantial number of studies were devoted to dichotomy from various perspectives among we mention the methods for detecting the existence of a dichotomy (see Alhalawa and Dragičević [1], Aulbach and Minh [2], Barreira et al. [3,4], Barreira and Valls [5], Battelli and Palmer [6], Bătăran et al. [7], Ben-Artzi and Gohberg [8], Ben-Artzi et al., [9], Berezansky and Braverman [10,11], Biriş and Megan [12], Braverman and Zhukovskiy [13], Chow and Leiva [15], Dragičević [20,21], Hai [26], Huy and Phi [29], Megan et al. [33,34], Minh et al. [35], Minh [36], Minh and Huy [37], Palmer [40–42], Preda and Megan [45], Sasu and Sasu [46–48,53,55], Sasu [49], Sasu [50–52], Sasu et al. [54], Zhou and Zhang [61], Zhou et al. [62]) and respectively the analysis of qualitative properties of diverse classes of dynamical systems with dichotomies such as robustness, Sacker-Sell spectrum or existence of invariant manifolds (see Barreira et al. [3,4], Chang et al. [14], Chow and Leiva [15–17], Dragičević [20,21], Elaydi and Hájek [24], Elaydi and Janglajew [25], Huy [28], Naulin and Pinto [38], Palmer [41], Pötzsche [43,44], Zhang et al.[58], Zhang et al. [59], Zhou et al. [60], Zhou and Zhang [61]). Recently, in [22], Dragičević and Zhang developed an interesting robustness method,

* Corresponding author at: Department of Mathematics, West University of Timişoara, V. Pârvan Blvd. 4, Timişoara, Romania.
E-mail addresses: adina.sasu@e-uvt.ro (A.L. Sasu), bogdan.sasu@e-uvt.ro (B. Sasu).

https://doi.org/10.1016/j.amc.2020.125167
0096-3003/

proving new criteria for roughness of nonuniform exponential dichotomy of nonautonomous difference equations and, as consequences, they obtained conditions for stability of Lyapunov exponents. Moreover, in the case of evolution families, the authors deduced new robustness criteria for nonuniform semi-strong exponential dichotomy.

A very interesting class of criteria used to investigate the asymptotic properties of dynamical systems is landmarked by the so-called admissibility methods. These methods have a fascinating history that dates back to the innovative article of Perron [39] and to the outstanding works of Massera and Schäffer [30,31], Daleckiĭ and Kreĭn [19], Coppel [18] and Palmer [40]. More precisely, considering the system

$$(\mathcal{A}) \qquad \dot{x}(s) = A(s)x(s), \quad s \in \mathbb{J}$$

in a space X, with $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, the central idea was to find conditions for its stability and dichotomy using various properties of the associated operator

$$\mathcal{P}x(s) = \dot{x}(s) - A(s)x(s), \quad s \in \mathbb{J}$$

on certain spaces of X-valued functions (see Massera and Schäffer [30,31], Daleckiĭ and Kreĭn [19], Coppel [18], Palmer [40], Ben-Artzi and Gohberg [8], Ben-Artzi et al. [9]). For instance, the existence of an exponential dichotomy was expressed using the Fredholm properties of $\mathcal{P}$ or in terms of its invertibility properties (see Palmer [40], Ben-Artzi and Gohberg [8], Ben-Artzi et al. [9] and the references included). Thus, it turned out that the asymptotic properties of $(\mathcal{A})$ could be studied in terms of the existence of solutions (uniquely determined or not) of the nonhomogeneous system

$$\dot{x}(s) = A(s)x(s) + w(s), \quad s \in \mathbb{J} \tag{1.1}$$

in certain function spaces - called *output spaces*, for every input w in a well-chosen function space - called *input space*. Such a solvability property is called *admissibility* and the pair of function spaces, that play the roles of input and output spaces, form the so-called *admissible pair*.

Later, the asymptotic properties of nonautonomous systems were expressed in terms of their propagators - evolution families $\mathcal{U} = \{U(t,r)\}_{t \geq r, t, r \in \mathbb{J}}$ - and one considered an associated integral equation

$$f(t) = U(t,r)f(r) + \int_r^t U(t,\zeta)w(\zeta)d\zeta, \quad t, r \in \mathbb{J}, t \geq r \tag{1.2}$$

having the input function w in certain function spaces and imposed the condition that the solution f (unique or not) belong to some well chosen output spaces (see e.g. Hai [27], Megan et al. [33,34], Minh et al. [35], Minh [36], Minh and Huy [37], Sasu and Sasu [46–48,53,55], Sasu [49], Sasu [50–52], Sasu et al. [54]). A notable work in this direction, that substantially marked the development of the admissibility theory in the case $\mathbb{J} = \mathbb{R}_+$, is due to Minh, Räbiger and Schnaubelt [35], where the authors proved input-output criteria for stability, instability/expansiveness and also for dichotomy on the half-line, considering spaces of continuous functions in the admissible pairs. Later, Minh and Huy succeeded to extend the admissibility techniques in [37] using p-integrable functions. In the following years, the uniform exponential dichotomy on the half-line was characterized by using the admissibility of various couples of Banach function or sequence spaces (see Sasu and Sasu [48,53,55] and the references therein). Working with inputs of the form $w(\zeta)/\zeta$, Hai considered in [27] other admissibility properties relative to an integral equation and obtained input-output criteria for uniform polynomial stability and polynomial expansiveness. The methods developed to study the exponential dichotomies on the whole line were distinct and emphasized major differences compared with the half-line and led to general results with spectacular applications (see Aulbach and Minh [2], Barreira et al. [3,4], Minh [36], Palmer [41,42], Sasu and Sasu [46,47], Sasu [50–52], Zhou and Zhang [61], Zhou et al. [62]). For example, a major difference between the half-line versus the whole line relies on the fact that every study on the half-line begins by choosing a closed (invariant) complement for the initial stable subspace - thus one already has the projections at the initial moment and the following steps depend on the initial choice of unstable subspace (see Megan et al. [33], Minh et al. [35], Minh and Huy [37], Sasu and Sasu [48,53,55], Sasu [49], Sasu et al. [54]). In the studies concerning the dichotomic behavior on the whole line the existence of the projections should be deduced at every moment and their structure plays an important role (see e.g. Sasu and Sasu [46,47], Sasu [51]). Moreover, the techniques used on the whole line can be extended to trichotomy or to the variational case (see Chow and Leiva [15], Elaydi and Hájek [23], Elaydi and Janglajew [25], Sasu and Sasu [56,57]).

An important aspect that should be mentioned is that a substantial part of the studies devoted to the asymptotic behavior of nonautonomous systems was focused on the case of evolution families $\mathcal{U} = \{U(t,r)\}_{t \geq r, t, r \in \mathbb{J}}$ with uniform growth, i.e. those for which there exist two constants $M \geq 1$, $\omega > 0$ such that $||U(t,r)|| \leq Me^{\omega(t-r)}$, for all $t, r \in \mathbb{J}, t \geq r$ (see Minh et al. [35], Minh and Huy [37], Sasu and Sasu [46–48,55,56], Sasu [50–52]). Nevertheless, in many significant cases the propagator of a nonautonomous system may not satisfy a uniform exponential growth property. Consequently, the asymptotic properties cannot be described in the uniform setting, but the system may exibit a nonuniform asymptotic behavior. A special nonuniform case is represented by the pointwise nonuniform stability and instability of differential equations and was studied by Massera and Schäffer in [30,31]. Two concepts of nonuniform dichotomy for evolutionary processes on the half-line were studied by Preda and Megan in [45]. In the case of variational dynamical systems described by skew-product flows, a specific nonuniform phenomenon was described by Chow and Leiva in [15] through pointwise exponential dichotomy. Admissibility conditions for pointwise exponential dichotomy were obtained by Chow and Leiva in [15], Huy and Phi in [29] and Megan et al., in [34]. A property of (non)uniform exponential dichotomy for skew-product semiflows, expressed in terms of Lyapunov norms, was considered by Bătăran et al., in [7]. An important study devoted to the nonuniform asymptotic behavior of random dynamical systems was developed by Zhou et al., in [60], the authors investigating the property of tempered exponential dichotomy. Coming back to the case of nonautonomous dynamical systems, it should be mentioned that the nonuniform asymptotic behavior is strongly motivated by the theory of nonuniform hyperbolicity. Thus, in the last years, contributions to the theory of nonuniform dichotomies were developed by Alhalawa and Dragičević [1], Barreira et al. [3,4], Dragičević [20,21], Dragičević and Zhang [22], Megan et al. [33], Sasu et al. [54], Zhou and Zhang [61], Zhou et al. [62]. A special case of nonuniform dichotomic behavior, called (h, k)-dichotomy, was introduced by Naulin and Pinto [38]. Their dichotomy concept was recently extended to the property of (μ, ν)-dichotomy by Chang et al., in [14] and respectively to (h, k, μ, ν)-dichotomy by Zhang et al., in [58] and by Zhang et al., in [59]. Recently, Barreira and Valls studied the ρ-exponential dichotomy in [5], considering both discrete and continuous time behavior. Despite the progress made in the past years, there are still many open problems regarding the nonuniform (dichotomic) behaviors. Some open questions are related to the detection of the nonuniform behaviors on the whole line by using input-output arguments and explore in particular the issue of finding a direct solvability of suitable input-output systems between some function or sequence spaces. Thus, one of the first aims would be to identify what type of spaces can be considered as input or as output spaces and to optimize the requirements regarding their structures.

The aim of this paper is to present a new study concerning the nonuniform asymptotic behavior of nonautonomous systems on the whole line. We propose a direct and computational method in the nonuniform setting and we obtain new criteria for nonuniform dichotomy in a very general context. The paper is organized as follows: to an evolution family on a Banach space X, with a nonuniform exponential growth, we associate an input-output (integral) equation. We define a new admissibility property such that the input space $I(\mathbb{R}, X)$ can be successively minimized, the output space $C(\mathbb{R}, X)$ can be selected from various spaces of continuous functions and for every input function in $I(\mathbb{R}, X)$ there exists a unique solution of the integral equation in $C(\mathbb{R}, X)$ (see Definition 2.3 below). In several steps, we prove that this admissibility implies all the properties that completely describe the nonuniform dichotomic behavior. First, we show that the admissibility of $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ gives the splitting of the space X into the direct sum between the stable subspace and the unstable subspace at every moment. Next, under the admissibility assumption, we use computational arguments to deduce the nonuniform stability on stable subspaces and respectively the nonuniform expansiveness on unstable subspaces. The central result will show that the admissibility of $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ provides the nonuniform exponential dichotomy of evolution families defined on the whole line. Moreover, we deduce the expression of the growth function for nonuniform dichotomy on the whole line in terms of the initial growth functions and of the norm of the input-output operator.

The next natural step is to establish whether the converse implication remains valid and if so, which are the underlying hypotheses. For a better understanding, we expose an illustrative example and we provide a complete answer in the case of nonuniform behavior. In addition, we present an application of the main results in the uniform case, generalizing the previous approaches in this framework. Finally we emphasize that the admissibility property considered herein leads to the existence of an exponential dichotomy whose (non)uniformity depends on the initial exponential growth of the evolution family.

2. Conditions for nonuniform dichotomy on the whole line

Let X be a Banach space. We consider $\mathcal{B}(X)$ - the Banach algebra of all bounded linear operators on X. In this paper the norm on X and respectively on $\mathcal{B}(X)$ will be denoted by $||\cdot||$. We denote by I_d the identity operator on the space X. We begin with some known definitions and notations.

Definition 2.1. Let $\mathcal{U} = \{U(t, r)\}_{t \geq r} \subset \mathcal{B}(X)$. We say that $\mathcal{U}$ is an *evolution family* on X if it satisfies:

(i) $U(r, r) = I_d$, for all $r \in \mathbb{R}$ and $U(t, r)U(r, r_0) = U(t, r_0)$, for all $t \geq r \geq r_0$;
(ii) for each $r_0 \in \mathbb{R}$ and each $x \in X$, the mappings $(-\infty, r_0] \ni s \mapsto U(r_0, s)x$ and $[r_0, \infty) \ni s \mapsto U(s, r_0)x$ are continuous;
(iii) there are $M : \mathbb{R} \to [1, \infty)$ and $\omega : \mathbb{R} \to (0, \infty)$, two continuous mappings, with the property

$$||U(t, r)|| \leq M(r) e^{\omega(r)(t-r)}, \quad \forall t \geq r.$$

Remark 2.1. If the mappings $M : \mathbb{R} \to [1, \infty)$, $\omega : \mathbb{R} \to (0, \infty)$ are bounded, then setting

$$M := \sup_{t \in \mathbb{R}} M(t) \quad \text{and} \quad \omega := \sup_{t \in \mathbb{R}} \omega(t)$$

we obtain

$$||U(t, r)|| \leq M\, e^{\omega(t-r)}, \quad \forall t \geq r.$$

In this case, $\mathcal{U}$ admits a *uniform exponential growth*. If at least one of the functions M or ω is unbounded, then $\mathcal{U}$ has a *nonuniform exponential growth*.

Remark 2.2. An operator $P \in \mathcal{B}(X)$ is a projection if $P^2 = P$. A projection has the property that $X = Range\, P \oplus Ker\, P$.

Let $\mathcal{U} = \{U(t, r)\}_{t \geq r}$ be an evolution family on X.

Definition 2.2. We say that $\mathcal{U}$ has a *nonuniform exponential dichotomy* if there are a family of projections $\{P(t)\}_{t \in \mathbb{R}}$, a function $N : \mathbb{R} \to [1, \infty)$ and a constant $\nu > 0$ such that the following properties hold:

(i) $U(t,r)P(r) = P(t)U(t,r)$, for all $t \geq r$;
(ii) the operator $U(t, r)_|$: $Ker\,P(r) \to Ker\,P(t)$ is invertible, for all $t \geq r$;
(iii) $||U(t,r)x|| \leq N(r)e^{-\nu(t-r)}||x||$, for all $x \in Range\,P(r)$ and $t \geq r$;
(iv) $N(t)||U(t,r)y|| \geq e^{\nu(t-r)}||y||$, for all $y \in Ker\,P(r)$ and $t \geq r$.

Remark 2.3. If the function $N : \mathbb{R} \to [1, \infty)$ from Definition 2.2 is bounded, then we denote by

$$N := \sup_{t\in\mathbb{R}} N(t)$$

and the inequalities (*iii*) and (*iv*) become

(iii)' $\| U(t,r)x \| \leq N\,e^{-\nu(t-r)} \| x \|$, for all $x \in Range\,P(r)$ and $t \geq r$;
(iv)' $||U(t,r)y|| \geq \frac{1}{N}\,e^{\nu(t-r)}||y||$, for all $y \in Ker\,P(r)$ and $t \geq r$.

In this case, $\mathcal{U}$ has a *uniform exponential dichotomy* (see e.g. [47,50–52]).

Remark 2.4. (i) An interesting concept of nonuniform exponential dichotomy was studied by Barreira et al., and it corresponds to

$$N(t) = Ce^{\epsilon|t|}, \quad \forall t \in \mathbb{R} \tag{2.1}$$

for some $\epsilon > 0$ and $C > 0$ (see Definition 5.5 in [4]). This notion is equivalent with the property of exponential dichotomy with respect to a well-chosen family of norms $\{||\cdot||_t\}_{t\in\mathbb{R}}$ (see Definition 5.6 and Proposition 5.6 in [4]). For differential equations and (invertible) evolution families, the property of *strong exponential dichotomy* relative to a family of norms was introduced and studied by Barreira et al., in [3].

(ii) An important nonuniform behavior is described by the *nonuniform semi-strong exponential dichotomy* and has been recently studied by Dragičević and Zhang in [22] for general evolution families on the whole line (see Section 5 in [22]).

An interesting open problem regarding the nonuniform exponential dichotomy on $\mathbb{R}$ is whether an admissibility condition expressed in terms of certain properties of solutions of an integral equation of the form (1.2) can provide the existence of the nonuniform exponential dichotomy. In addition, considering that in any input-output criteria a central aim is to minimize the input space and to consider an output space as general as possible, the question would be how could one define such an admissibility property in the nonuniform case and what would be the connections with the nonuniform exponential dichotomy. In what follows we will answer these questions.

We will work with some usual function spaces whose notations will be presented below. Indeed, we consider $C_b(\mathbb{R}, X) := \{f : \mathbb{R} \to X : f$ is continuous and bounded$\}$ with the norm

$$||f||_\infty := \sup_{t\in\mathbb{R}} ||f(t)||.$$

The space $C_b(\mathbb{R}, X)$ is complete. We consider three closed subspaces denoted as follows:

$$C_{b0}(\mathbb{R}, X) := \Big\{f \in C_b(\mathbb{R}, X) : \lim_{t\to\infty} f(t) = 0\Big\},$$

$$C_{0b}(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X) : \lim_{t\to-\infty} f(t) = 0\}$$

and

$$C_{00}(\mathbb{R}, X) := \Big\{f \in C_b(\mathbb{R}, X) : \lim_{t\to\infty} f(t) = \lim_{t\to-\infty} f(t) = 0\Big\}.$$

With respect to the norm $||\cdot||_\infty$, $C_{b0}(\mathbb{R}, X)$, $C_{0b}(\mathbb{R}, X)$ and $C_{00}(\mathbb{R}, X)$ are Banach spaces.

For each $p \in (1, \infty)$, we consider $L^p(\mathbb{R}, X) = \{ f : \mathbb{R} \to X : f$ Bochner measurable with $\int_{\mathbb{R}} \| f(\zeta)\|^p \, d\zeta < \infty \}$ with the norm

$$||f||_p := \left(\int_{\mathbb{R}} ||f(\zeta)||^p \, d\zeta\right)^{1/p}.$$

For every $f : \mathbb{R} \to X$ we denote by $supp\ f := \{t \in \mathbb{R} : f(t) \neq 0\}$.

In what follows, let $C(\mathbb{R}, X)$ denote one of the spaces $C_b(\mathbb{R}, X)$, $C_{b0}(\mathbb{R}, X)$, $C_{0b}(\mathbb{R}, X)$ or $C_{00}(\mathbb{R}, X)$. Hence, we have that

$$C_{00}(\mathbb{R}, X) \subseteq C(\mathbb{R}, X) \subseteq C_b(\mathbb{R}, X).$$

Let $l \in \mathbb{N}$, $l \geq 2$ and let $p_1, \ldots, p_l \in (1, \infty)$ with $p_1 < p_2 < \cdots < p_l$. We consider

$$I_0^{p_1,\ldots,p_l}(\mathbb{R}, X) := C_{00}(\mathbb{R}, X) \cap L^{p_1}(\mathbb{R}, X) \cap \ldots \cap L^{p_l}(\mathbb{R}, X)$$

which is a Banach space with the norm

$$||w||_{I_0^{p_1,\ldots,p_l}(\mathbb{R},X)} := \max\{||w||_{p_1}, \ldots, ||w||_{p_l}, ||w||_\infty\}.$$

$I_0^{p_1,\dots,p_l}(\mathbb{R},X)$ will play the role of *input space* throughout our study and will be denoted simply by $I(\mathbb{R},X)$. The way how $I(\mathbb{R},X)$ is defined shows that one can work with input spaces becoming smaller and smaller.

Definition 2.3. We say that the pair $(C(\mathbb{R},X), I(\mathbb{R},X))$ is *admissible* for $\mathcal{U}$ if for every function $w \in I(\mathbb{R},X)$ the equation

$$(\mathrm{E}_{\mathcal{U}}) \qquad f(t) = U(t,r)f(r) + \int_r^t U(t,\zeta)w(\zeta)\,d\zeta, \quad \forall t \geq r$$

admits a unique solution $f \in C(\mathbb{R},X)$.

Remark 2.5. In the study of nonuniform dichotomic behaviors of evolution families on the whole line, an interesting admissibility concept with respect to an integral equation - called *weak admissibility* - was introduced by Barreira, Dragičević and Valls (see [3,4]). In their admissibility method, the authors work with a family of norms $||\cdot||_t$ with specific properties and the spaces in the admissible pair as well as the norms on the admissible spaces are constructed using the family of norms mentioned above. Thus, the authors obtained criteria for nonuniform exponential dichotomies in terms of admissibilities (see Section 4 in [3] and Section 5.3.2 in [4]).

Remark 2.6. If $(C(\mathbb{R},X), I(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then we consider the linear operator

$$\mathcal{P} : I(\mathbb{R},X) \to C(\mathbb{R},X), \quad \mathcal{P}(w) = f$$

where f is the unique function in $C(\mathbb{R},X)$ such that (f, w) satisfies the integral equation $(E_{\mathcal{U}})$.

Lemma 2.1. *If $(C(\mathbb{R},X), I(\mathbb{R},X))$ is admissible for $\mathcal{U}$, then the operator $\mathcal{P}$ is bounded.*

Proof. Using a similar argumentation like in the proof of Proposition 4.3 in [47] it follows that the operator $\mathcal{P}$ is closed, so $\mathcal{P}$ is bounded. □

For every $(t_0, x) \in \mathbb{R} \times X$, let

$$q_{t_0,x} : \mathbb{R} \to X, \quad q_{t_0,x}(t) = \begin{cases} U(t,t_0)x, & \text{if} \quad t \geq t_0 \\ e^{t-t_0}x, & \text{if} \quad t < t_0 \end{cases}.$$

Obviously $q_{t_0,x}$ is continuous.

For every $t_0 \in \mathbb{R}$ we consider

$$\mathcal{F}_{t_0}(\mathbb{R},X) = \{f \in C(\mathbb{R},X) : \ f(r) = U(r,\tau)f(\tau), \text{ for all } \tau \leq r \leq t_0\}.$$

Remark 2.7. For each $s \in \mathbb{R}$, $\mathcal{F}_s(\mathbb{R},X)$ is a linear subspace and $\mathcal{F}_t(\mathbb{R},X) \subset \mathcal{F}_s(\mathbb{R},X)$, for all $t \geq s$.

We also consider

$$\mathcal{S}(t_0) = \{x \in X : \ q_{t_0,x} \in C(\mathbb{R},X)\}$$

and

$$\mathcal{U}(t_0) := \{x \in X : \ \text{there is } h \in \mathcal{F}_{t_0}(\mathbb{R},X) \text{ with } h(t_0) = x\}.$$

Both are linear subspaces. In addition, $\mathcal{S}(t_0)$ is called *the stable subspace* and $\mathcal{U}(t_0)$ is called *the unstable subspace* at the moment t_0.

Lemma 2.2. *For each $t \geq t_0$, the following invariance properties are satisfied:*

(i) $U(t,t_0)\mathcal{S}(t_0) \subset \mathcal{S}(t)$;
(ii) $U(t,t_0)\mathcal{U}(t_0) = \mathcal{U}(t)$.

Proof. (i) Let $t > t_0$ and let $x \in \mathcal{S}(t_0)$. We set $y = U(t,t_0)x$ and we notice that

$$q_{t,y}(s) = \begin{cases} U(s,t)y, & s \geq t \\ e^{s-t}y, & s < t \end{cases} = \begin{cases} U(s,t_0)x, & s \geq t \\ e^{s-t}U(t,t_0)x, & s < t \end{cases}. \tag{2.2}$$

From (2.2) it follows that

$$q_{t,y}(s) = q_{t_0,x}(s), \quad \forall s \geq t \tag{2.3}$$

and

$$||q_{t,y}(s)|| \leq e^{t_0-t}||U(t,t_0)||\,||q_{t_0,x}(s)||, \quad \forall s \leq t_0. \tag{2.4}$$

Since $q_{t_0,x} \in C(\mathbb{R},X)$ from (2.3) and (2.4) we obtain that $q_{t,y} \in C(\mathbb{R},X)$, so $y \in \mathcal{S}(t)$.

(ii) Let $t > t_0$ and let $x \in \mathcal{U}(t_0)$. Then there is $h \in \mathcal{F}_{t_0}(\mathbb{R},X)$ with $h(t_0) = x$. Let $y = U(t,t_0)x$ and

$$\lambda : \mathbb{R} \to X, \quad \lambda(s) = \begin{cases} h(s), & s < t_0 \\ U(s,t_0)x, & s \in [t_0,t] \\ e^{-(s-t)}y, & s > t \end{cases}.$$

We have that λ is continuous.

Since $h \in C(\mathbb{R}, X)$ we obtain that $\lambda \in C(\mathbb{R}, X)$. Moreover, by using the property that

$$h(r) = U(r, s)h(s), \quad \forall s \le r \le t_0$$

it is easy to deduce that

$$\lambda(r) = U(r, s)\lambda(s), \quad \forall s \le r \le t.$$

It follows that $\lambda \in \mathcal{F}_t(\mathbb{R}, X)$. This implies that $y = \lambda(t) \in \mathcal{U}(t)$.

Let $z \in \mathcal{U}(t)$. Thus, there is $f \in \mathcal{F}_t(\mathbb{R}, X)$ with $f(t) = z$. Then

$$f(t) = U(t, t_0)f(t_0). \tag{2.5}$$

Using Remark 2.7 we have $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$. This implies $f(t_0) \in \mathcal{U}(t_0)$. Then, from relation (2.5) it follows that $z = f(t) \in U(t, t_0)\mathcal{U}(t_0)$. □

Theorem 2.1. *If $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ is admissible for $\mathcal{U}$, then for each $t_0 \in \mathbb{R}$, the following properties are satisfied:*

(i) *$\mathcal{S}(t_0)$ is closed;*
(ii) *$\mathcal{U}(t_0)$ is closed;*
(iii) *$\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$;*
(iv) *$\mathcal{S}(t_0) \oplus \mathcal{U}(t_0) = X$.*

Proof. Let $M : \mathbb{R} \to [1, \infty), \omega : \mathbb{R} \to (0, \infty)$ be such that

$$||U(t, r)|| \le M(r)e^{\omega(r)(t-r)}, \quad \forall t \ge r. \tag{2.6}$$

We consider

$$\delta : \mathbb{R} \to \left[0, \frac{3}{2}\right], \quad \delta(s) = \begin{cases} 6s(1-s), & s \in [0, 1] \\ 0, & s \in \mathbb{R} \setminus [0, 1] \end{cases}.$$

Then δ is continuous, $\int_0^1 \delta(\zeta)\, d\zeta = 1$ and

$$||\delta||_p \le \frac{3}{2}, \quad \forall p \in [1, \infty]. \tag{2.7}$$

Let $t_0 \in \mathbb{R}$.

(i) Let $(x_n) \subset \mathcal{S}(t_0)$, $x_n \underset{n\to\infty}{\longrightarrow} x$. For each $n \in \mathbb{N}$ let

$$w_n : \mathbb{R} \to X, \quad w_n(s) = \delta(s - t_0)U(s, t_0)x_n$$

$$f_n : \mathbb{R} \to X, \quad f_n(s) = \begin{cases} (\int_{t_0}^s \delta(\zeta - t_0)\, d\zeta)U(s, t_0)x_n, & s \ge t_0 \\ 0, & s < t_0 \end{cases}.$$

Then f_n and w_n are continuous functions. Moreover, we have that

$$w_n(s) = 0, \quad \forall s \in \mathbb{R} \setminus [t_0, t_0 + 1], \forall n \in \mathbb{N}. \tag{2.8}$$

From relation (2.8) we obtain in particular that $w_n \in I(\mathbb{R}, X)$.

We note that

$$\int_{t_0}^s \delta(\zeta - t_0)\, d\zeta = \int_{t_0}^{t_0+1} \delta(\zeta - t_0)\, d\zeta = 1, \quad \forall s \ge t_0 + 1.$$

This implies that

$$f_n(s) = \begin{cases} 0, & s < t_0 \\ (\int_{t_0}^s \delta(\zeta - t_0)\, d\zeta)U(s, t_0)x_n, & s \in [t_0, t_0 + 1) \\ U(s, t_0)x_n, & s \ge t_0 + 1 \end{cases}, \quad \forall n \in \mathbb{N}.$$

In particular, we have

$$||f_n(s)|| \le ||q_{t_0, x_n}(s)||, \quad \forall s \in \mathbb{R}. \tag{2.9}$$

Since $x_n \in \mathcal{S}(t_0)$ from relation (2.9) we obtain that $f_n \in C(\mathbb{R}, X)$. Moreover (f_n, w_n) satisfies the equation $(E_{\mathcal{U}})$, so $f_n = \mathcal{P}(w_n)$.

We define

$$w : \mathbb{R} \to X, \quad w(s) = \delta(s - t_0)U(s, t_0)x.$$

The function w is continuous with

$$w(s) = 0, \quad \forall s \in \mathbb{R} \setminus [t_0, t_0 + 1]. \tag{2.10}$$

In particular $w \in I(\mathbb{R}, X)$. Taking $f = \mathcal{P}(w)$, we have

$$||f_n - f||_\infty \le ||\mathcal{P}||\,||w_n - w||_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

From relations (2.6), (2.8) and (2.10) we obtain

$$||w_n(s) - w(s)|| \le \delta(s - t_0) M(t_0) e^{\omega(t_0)} ||x_n - x||, \quad \forall s \in \mathbb{R}, \forall n \in \mathbb{N}. \tag{2.12}$$

From (2.12) and (2.7) it follows

$$\| w_n - w \|_{I(\mathbb{R},X)} \le \frac{3}{2} M(t_0) e^{\omega(t_0)} \ \| x_n - x \|, \quad \forall n \in \mathbb{N}. \tag{2.13}$$

From relations (2.11) and (2.13) we have $f_n \underset{n\to\infty}{\longrightarrow} f$ in $C(\mathbb{R}, X)$, so

$$f(s) = \lim_{n\to\infty} f_n(s) = \lim_{n\to\infty} U(s, t_0) x_n = q_{t_0,x}(s), \quad \forall s \ge t_0 + 1. \tag{2.14}$$

Since $f \in C(\mathbb{R}, X)$, using (2.14) we deduce that $q_{t_0,x} \in C(\mathbb{R}, X)$. This shows that $x \in \mathcal{S}(t_0)$, so $\mathcal{S}(t_0)$ is closed.

(ii) Let $(y_n) \subset \mathcal{U}(t_0)$, $y_n \underset{n\to\infty}{\longrightarrow} y$. For each $n \in \mathbb{N}$, since $y_n \in \mathcal{U}(t_0)$ there is $h_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $h_n(t_0) = y_n$.

For each $n \in \mathbb{N}$ let

$$w_n : \mathbb{R} \to X, \quad w_n(s) = -\delta(s - t_0) U(s, t_0) y_n$$

and

$$f_n : \mathbb{R} \to X, \quad f_n(s) = \begin{cases} h_n(s), & s \le t_0 \\ (\int_s^{t_0+1} \delta(\zeta - t_0)\, d\zeta) U(s, t_0) y_n, & s \in (t_0, t_0 + 1) \\ 0, & s \ge t_0 + 1 \end{cases}.$$

Then $w_n \in I(\mathbb{R}, X)$. In addition, since $h_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we also have $f_n \in C(\mathbb{R}, X)$. Moreover, (f_n, w_n) satisfies the equation $(E_{\mathcal{U}})$, so $f_n = \mathcal{P}(w_n)$.

Let

$$w : \mathbb{R} \to X, \quad w(s) = -\delta(s - t_0) U(s, t_0) y.$$

Hence $w \in I(\mathbb{R}, X)$ and setting $f = \mathcal{P}(w)$ we obtain that

$$||f_n - f||_\infty \le ||\mathcal{P}||\,||w_n - w||_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{2.15}$$

Using a similar estimation as in (2.14), we immediately deduce that $\| w_n - w \|_{I(\mathbb{R},X)} \underset{n\to\infty}{\to} 0$. Then, from (2.15) it follows that, in $C(\mathbb{R}, X)$, $f_n \underset{n\to\infty}{\longrightarrow} f$, so

$$f_n(s) \underset{n\to\infty}{\longrightarrow} f(s), \quad \forall s \in \mathbb{R}. \tag{2.16}$$

Since $h_n \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ we have that

$$h_n(t) = U(t, r) h_n(r), \quad \forall r \le t \le t_0, \forall n \in \mathbb{N}. \tag{2.17}$$

For $n \to \infty$ in (2.17) and using (2.16) we deduce that $f \in \mathcal{F}_{t_0}(\mathbb{R}, X)$. From

$$f(t_0) = \lim_{n\to\infty} f_n(t_0) = \lim_{n\to\infty} y_n = y$$

we have that $y \in \mathcal{U}(t_0)$, so $\mathcal{U}(t_0)$ is closed.

(iii) Let $z \in \mathcal{S}(t_0) \cap \mathcal{U}(t_0)$. Since $z \in \mathcal{U}(t_0)$ there is $h \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ with $h(t_0) = z$. We consider

$$\alpha : \mathbb{R} \to X, \quad \alpha(t) = \begin{cases} U(t, t_0) z, & t > t_0 \\ h(t), & t \le t_0 \end{cases}.$$

We have that α is continuous. From $h \in \mathcal{F}_{t_0}(\mathbb{R}, X)$ and $z \in \mathcal{S}(t_0)$ we obtain that $\alpha \in C(\mathbb{R}, X)$. Moreover, the pair $(\alpha, 0)$ satisfies $(E_{\mathcal{U}})$. This implies that $\alpha = \mathcal{P}(0) = 0$. In particular $z = \alpha(t_0) = 0$, so $\mathcal{S}(t_0) \cap \mathcal{U}(t_0) = \{0\}$.

(iv) We prove that $\mathcal{S}(t_0) + \mathcal{U}(t_0) = X$. Let $x \in X$ and

$$w : \mathbb{R} \to X, \quad w(s) = \delta(s - t_0) U(s, t_0) x.$$

Then $w \in I(\mathbb{R}, X)$. Let $f = \mathcal{P}(w)$. Hence $f \in C(\mathbb{R}, X)$ and

$$f(t) = U(t, r) f(r) + \int_r^t U(t, \zeta) w(\zeta)\, d\zeta, \quad \forall t \ge r. \tag{2.18}$$

In particular, from (2.18) we obtain $f(s) = U(s, t_0)(f(t_0) + x)$, for every $s \ge t_0 + 1$. From $f \in C(\mathbb{R}, X)$, using the previous estimation we deduce that $f(t_0) + x \in \mathcal{S}(t_0)$.

On the other side from (2.18) we have that

$$f(s) = U(s, r) f(r), \quad \forall r \le s \le t_0. \tag{2.19}$$

From $f \in C(\mathbb{R}, X)$ and (2.19) we deduce that $f(t_0) \in \mathcal{U}(t_0)$. This implies

$$x = (f(t_0) + x) - f(t_0) \in \mathcal{S}(t_0) + \mathcal{U}(t_0).$$

Since $x \in X$ was arbitrary, from the previous argumentation we obtain that $\mathcal{S}(t_0) + \mathcal{U}(t_0) = X$. Finally, from $(i) - (iii)$ and the above decomposition we have that $\mathcal{S}(t_0) \oplus \mathcal{U}(t_0) = X$ and the proof is complete. □

Remark 2.8. According to the previous theorem we deduce that the admissibility gives the decomposition of the space X at every $t_0 \in \mathbb{R}$, so in what follows we should establish the asymptotic behavior on the corresponding subspaces.

First, we need a technical lemma:

Lemma 2.3. *The following inequalities hold:*

(i) $1 \leq x^{1/x} < 2$, *for all* $x \in [1, \infty)$;

(ii) *if* $l \in \mathbb{N}$, $l \geq 2$ *and* $1 < p_1 < \cdots < p_l$, *then for every* $\lambda \in (0, 2^{-\frac{p_1 p_2}{p_2 - p_1}}]$ *we have that*

$$\frac{1}{(\lambda p_k)^{1/p_k}} \leq \frac{1}{(\lambda p_1)^{1/p_1}}, \quad \forall k \in \{1, \ldots, l\}.$$

Proof. (i) The function $\varphi : [1, \infty) \to \mathbb{R}_+$, $\varphi(x) = x^{1/x}$ is increasing on $[1, e)$ and decreasing on $[e, \infty)$ with $\lim\limits_{x \to \infty} \varphi(x) = 1$. This implies that

$$1 \leq \varphi(x) \leq \varphi(e) < 2, \quad \forall x \geq 1.$$

(ii) Let $\lambda \in (0, 2^{-\frac{p_1 p_2}{p_2 - p_1}}]$ and let $k \in \{2, \ldots, l\}$. Then using (i) we successively have that

$$\lambda^{\frac{1}{p_1} - \frac{1}{p_k}} \leq \lambda^{\frac{1}{p_1} - \frac{1}{p_2}} = \lambda^{\frac{p_2 - p_1}{p_1 p_2}} \leq \frac{1}{2} \leq \frac{(p_k)^{1/p_k}}{2} < \frac{(p_k)^{1/p_k}}{(p_1)^{1/p_1}}$$

which implies that

$$\frac{1}{(\lambda p_k)^{1/p_k}} < \frac{1}{(\lambda p_1)^{1/p_1}}, \quad \forall k \in \{2, \ldots, l\}. \qquad \square$$

Theorem 2.2. *If* $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ *is admissible for* $\mathcal{U}$, *then there exist* $\nu > 0$ *and a continuous function* $N : \mathbb{R} \to [1, \infty)$ *with*

$$||U(t, t_0)x|| \leq N(t_0) e^{-\nu(t - t_0)} ||x||, \quad \forall x \in \mathcal{S}(t_0), \forall t \geq t_0. \tag{2.20}$$

Proof. We consider $\mathcal{P}$ the operator introduced in Remark 2.6. Let

$$m := \min\left\{\frac{1}{p_1},\ 2^{-\frac{p_1 p_2}{p_2 - p_1}},\ ||\mathcal{P}||^{-\frac{p_1}{p_1 - 1}}\right\}$$

and let $\lambda \in (0, m)$. Since $\lambda < 1/p_1$ we obtain that

$$1 < \frac{1}{(\lambda p_1)^{1/p_1}}. \tag{2.21}$$

In addition, from Lemma 2.3 we have that

$$\frac{1}{(\lambda p_k)^{1/p_k}} \leq \frac{1}{(\lambda p_1)^{1/p_1}}, \quad \forall k \in \{1, \ldots, l\}. \tag{2.22}$$

Setting

$$\gamma := \frac{(\lambda p_1)^{1/p_1}}{||\mathcal{P}||} \tag{2.23}$$

and using Lemma 2.3 (i) we deduce that

$$\lambda = \lambda^{\frac{1}{p_1}} \lambda^{\frac{p_1 - 1}{p_1}} < \lambda^{\frac{1}{p_1}} m^{\frac{p_1 - 1}{p_1}} \leq \frac{\lambda^{\frac{1}{p_1}}}{||\mathcal{P}||} \leq \frac{\lambda^{\frac{1}{p_1}} p_1^{\frac{1}{p_1}}}{||\mathcal{P}||} = \gamma.$$

This implies that

$$\nu := \gamma - \lambda > 0. \tag{2.24}$$

Let $M : \mathbb{R} \to [1, \infty), \omega : \mathbb{R} \to (0, \infty)$ be given by the property (iii) from Definition 2.1, i.e.

$$||U(t, r)|| \leq M(r) e^{\omega(r)(t - r)}, \quad \forall t \geq r. \tag{2.25}$$

Let $t_0 \in \mathbb{R}$ and $x \in \mathcal{S}(t_0) \setminus \{0\}$. There are two possible situations:

Case 1. $U(t_0+1, t_0)x \neq 0$. In this case, let

$$A_{t_0,x} = \{r \geq t_0 : U(r, t_0)x \neq 0\}$$

and then $[t_0, t_0+1] \subset A_{t_0,x}$.

Let $t \in A_{t_0,x}$, $t > t_0$, and let $n_t \in \mathbb{N}$ be with the property

$$\frac{2}{t-t_0} < n_t. \tag{2.26}$$

For each $n \in \mathbb{N}$, $n \geq n_t$, let $\delta_n : \mathbb{R} \to [0, 1]$ be a continuous map with $\delta_n(s) = 1$, for all $s \in [t_0 + \frac{1}{n}, t - \frac{1}{n}]$ and $supp\, \delta_n \subset (t_0, t)$. From (2.26) we have that δ_n is correctly defined, for all $n \geq n_t$.

For each $n \in \mathbb{N}$, $n \geq n_t$, we consider the functions

$$w_n : \mathbb{R} \to X, \quad w_n(s) = \delta_n(s)\, e^{\lambda(s-t_0)} \frac{U(s, t_0)x}{||U(s, t_0)x||}$$

$$f_n : \mathbb{R} \to X, \quad f_n(s) = \begin{cases} \left(\int_{t_0}^{s} \frac{\delta_n(\zeta)}{||U(\zeta,t_0)x||}\, e^{\lambda(\zeta-t_0)}\, d\zeta\right) U(s, t_0)x, & s \geq t_0 \\ 0, & s < t_0 \end{cases}.$$

Then f_n, w_n are continuous, for all $n \geq n_t$. Since $supp\, w_n \subset (t_0, t)$ we have that $w_n \in I(\mathbb{R}, X)$. We set

$$\rho_n := \int_{t_0}^{t} \frac{\delta_n(\zeta) e^{\lambda(\zeta-t_0)}}{||U(\zeta, t_0)x||}\, d\zeta$$

and then we have that

$$f_n(s) = \rho_n U(s, t_0)x, \quad \forall s \geq t. \tag{2.27}$$

Since $x \in \mathcal{S}(t_0)$, from relation (2.27) we deduce that $f_n \in C(\mathbb{R}, X)$. In addition, (f_n, w_n) satisfies the equation $(E_{\mathcal{U}})$, so

$$f_n = \mathcal{P}(w_n), \quad \forall n \geq n_t.$$

This implies that

$$||f_n||_\infty \leq ||\mathcal{P}||\, ||w_n||_{I(\mathbb{R},X)}, \quad \forall n \geq n_t.$$

In particular

$$||f_n(t)|| \leq ||\mathcal{P}||\, ||w_n||_{I(\mathbb{R},X)}, \quad \forall n \geq n_t. \tag{2.28}$$

Let $n \geq n_t$. We observe that

$$||w_n(s)|| = \delta_n(s) e^{\lambda(s-t_0)}, \quad \forall s \in \mathbb{R}. \tag{2.29}$$

From (2.29) and (2.21) it follows that

$$\| w_n\|_\infty = \sup_{s\in[t_0,t]} \| w_n(s) \| < e^{\lambda(t-t_0)} < \frac{e^{\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}. \tag{2.30}$$

In addition, for every $k \in \{1, \dots, l\}$, using (2.29) and (2.22) we successively deduce that

$$\| w_n\|_{p_k} = \left(\int_{\mathbb{R}} \delta_n^{p_k}(s) e^{\lambda p_k(s-t_0)}\, ds\right)^{1/p_k} < \left(\int_{t_0}^{t} e^{\lambda p_k(s-t_0)}\, ds\right)^{1/p_k} < \frac{e^{\lambda(t-t_0)}}{(\lambda p_k)^{1/p_k}} \leq \frac{e^{\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}. \tag{2.31}$$

From relations (2.30) and (2.31) we have

$$||w_n||_{I(\mathbb{R},X)} = \max\{||w_n||_{p_1}, \dots, ||w_n||_{p_l}, ||w_n||_\infty\} \leq \frac{e^{\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}, \quad \forall n \geq n_t. \tag{2.32}$$

Let $a_{t_0,x} := \sup A_{t_0,x} \in (t_0+1, \infty]$. Then $A_{t_0,x} = [t_0, a_{t_0,x})$. Let

$$\psi : [t_0, a_{t_0,x}) \to \mathbb{R}_+, \quad \psi(s) = \frac{e^{\lambda(s-t_0)}}{||U(s, t_0)x||}.$$

Then, from (2.28), (2.32) and (2.23) we obtain that

$$\left(\int_{t_0}^{t} \delta_n(\zeta)\psi(\zeta)\, d\zeta\right) ||U(t, t_0)x|| \leq \frac{1}{\gamma} e^{\lambda(t-t_0)}, \quad \forall n \geq n_t. \tag{2.33}$$

From (2.33) it follows

$$\int_{t_0}^{t} \psi(\zeta)\, \delta_n(\zeta)\, d\zeta \leq \frac{1}{\gamma}\, \psi(t), \quad \forall n \geq n_t. \tag{2.34}$$

For $n \to \infty$ in (2.34) we have that

$$\int_{t_0}^{t} \psi(\zeta)\, d\zeta \le \frac{1}{\gamma}\, \psi(t).$$

Since $t \in A_{t_0,x}$ was arbitrary, we obtain

$$\int_{t_0}^{t} \psi(\zeta)\, d\zeta \le \frac{1}{\gamma}\, \psi(t), \quad \forall t \in A_{t_0,x}. \tag{2.35}$$

We define

$$\alpha : [t_0, a_{t_0,x}) \to \mathbb{R}_+, \quad \alpha(t) = e^{-\gamma t} \int_{t_0}^{t} \psi(\zeta)\, d\zeta.$$

Using the inequality (2.35) we deduce that $\dot{\alpha}(s) \ge 0$, for all $s \in (t_0, a_{t_0,x})$. So α is nondecreasing on $[t_0, a_{t_0,x})$. Then in particular

$$\alpha(t_0+1) \le \alpha(t), \quad \forall t \in [t_0+1, a_{t_0,x}). \tag{2.36}$$

We set $K_{t_0} := M(t_0)e^{\omega(t_0)}$. Using (2.25) we have

$$||U(r, t_0)x|| \le M(t_0)e^{\omega(t_0)(r-t_0)}||x|| \le K_{t_0}||x||, \quad \forall r \in [t_0, t_0+1]. \tag{2.37}$$

Using relation (2.37) we deduce that

$$\alpha(t_0+1) > e^{-\gamma(t_0+1)} \int_{t_0}^{t_0+1} \frac{1}{\| U(r,t_0)x \|}\, dr \ge \frac{e^{-\gamma(t_0+1)}}{K_{t_0} \| x \|}. \tag{2.38}$$

Successively, from (2.38) and respectively (2.36) and (2.35), we have that

$$\frac{e^{-\gamma(t_0+1)}}{K_{t_0} \| x \|} < \alpha(t_0+1) \le \alpha(t) \le \frac{e^{-\gamma t}}{\gamma}\, \psi(t), \quad \forall t \in [t_0+1, a_{t_0,x})$$

which implies

$$||U(t, t_0)x|| \le \frac{e^{\gamma}}{\gamma} K_{t_0}\, e^{-(\gamma-\lambda)(t-t_0)}||x||, \quad \forall t \in [t_0+1, a_{t_0,x}). \tag{2.39}$$

From relations (2.24), (2.39) and using the way how $a_{t_0,x}$ is defined, we obtain

$$||U(t, t_0)x|| \le \frac{e^{\gamma}}{\gamma} K_{t_0}\, e^{-\nu(t-t_0)}||x||, \quad \forall t \ge t_0+1. \tag{2.40}$$

In addition, using relation (2.25) we have

$$||U(t, t_0)x|| \le M(t_0)e^{\omega(t_0)}\, ||x|| \le e^{\nu}K_{t_0}\, e^{-\nu(t-t_0)}||x||, \quad \forall t \in [t_0, t_0+1]. \tag{2.41}$$

Then, setting $\mu := \max\{(e^{\gamma}/\gamma), e^{\nu}\}$ and $N_{t_0} := \mu K_{t_0}$ from relations (2.40) and (2.41) it follows that

$$||U(t, t_0)x|| \le N_{t_0}e^{-\nu(t-t_0)}\, ||x||, \quad \forall t \ge t_0. \tag{2.42}$$

Case 2. $U(t_0+1, t_0)x = 0$. In this case $U(r, t_0)x = 0$, for every $r \ge t_0+1$. Using the same estimations as in (2.41), we also have that

$$||U(t, t_0)x|| \le N_{t_0}e^{-\nu(t-t_0)}||x||, \quad \forall t \ge t_0. \tag{2.43}$$

Then, from relations (2.42) and (2.43) we obtain that in both cases

$$||U(t, t_0)x|| \le N_{t_0}e^{-\nu(t-t_0)}||x||, \quad \forall t \ge t_0.$$

Because $t_0 \in \mathbb{R}$ and $x \in \mathcal{S}(t_0) \setminus \{0\}$ were arbitrary it follows

$$||U(t, t_0)x|| \le N_{t_0}e^{-\nu(t-t_0)}||x|| \quad \forall t \ge t_0, \forall x \in \mathcal{S}(t_0), \forall t_0 \in \mathbb{R}.$$

Considering

$$N : \mathbb{R} \to [1, \infty), \quad N(t) = \mu M(t)e^{\omega(t)}$$

we have that N is continuous and then relation (2.20) is satisfied. □

Theorem 2.3. *If* $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ *is admissible for* $\mathcal{U}$, *then there exist* $\nu > 0$ *and a continuous function* $N : \mathbb{R} \to (0, \infty)$ *such that*

$$N(t)\, ||U(t, t_0)x|| \ge e^{\nu(t-t_0)}||x||, \quad \forall x \in \mathcal{U}(t_0), \forall t \ge t_0. \tag{2.44}$$

Proof. We consider $\mathcal{P}$ the operator introduced in Remark 2.6 and let

$$m := \min\left\{\frac{1}{p_1},\ 2^{-\frac{p_1p_2}{p_2-p_1}},\ ||\mathcal{P}||^{-\frac{p_1}{p_1-1}}\right\}.$$

Let $\lambda \in (0, m)$, let

$$\gamma := \frac{(\lambda p_1)^{1/p_1}}{||\mathcal{P}||} \tag{2.45}$$

and let $\nu := \gamma - \lambda$. Using an argumentation like in the proof of Theorem 2.2 it follows

$$1 \ < \ \frac{1}{(\lambda p_1)^{1/p_1}} \tag{2.46}$$

$$\frac{1}{(\lambda p_k)^{1/p_k}} \le \frac{1}{(\lambda p_1)^{1/p_1}}, \quad \forall k \in \{1,\dots,l\} \tag{2.47}$$

and also $\nu > 0$.

Let $M : \mathbb{R} \to [1,\infty), \omega : \mathbb{R} \to (0,\infty)$ be given by the property (iii) in Definition 2.1, i.e.

$$||U(t,r)|| \le M(r)e^{\omega(r)(t-r)}, \quad \forall t \ge r. \tag{2.48}$$

Let $t_0 \in \mathbb{R}$ and let $x \in \mathcal{U}(t_0) \setminus \{0\}$. According to Theorem 2.1 (iii), $U(t,t_0)x \neq 0$, for all $t \ge t_0$.

Let $t \ge t_0$. For $n \in \mathbb{N}, n \ge 2$, let $\delta_n : \mathbb{R} \to [0,1]$ be continuous, with $\delta_n(s) = 1$, for $s \in [t+\frac{1}{n}, t+n]$ and $supp\,\delta_n \subset (t, t+n+1)$.

For each $n \in \mathbb{N}, n \ge 2$, let

$$w_n : \mathbb{R} \to X, \quad w_n(s) = -\delta_n(s)e^{-\lambda(s-t_0)}\frac{U(s,t_0)x}{||U(s,t_0)x||}.$$

Then w_n is continuous and $supp\,w_n \subset (t, t+n+1)$, so, in particular, $w_n \in I(\mathbb{R},X)$, for all $n \in \mathbb{N}, n \ge 2$.

For each $n \in \mathbb{N}, n \ge 2$, let

$$a_n := \int_t^{t+n+1} \frac{\delta_n(\zeta)e^{-\lambda(\zeta-t_0)}}{||U(\zeta,t_0)x||}\,d\zeta. \tag{2.49}$$

Since $x \in \mathcal{U}(t_0)$ there is a function $h \in \mathcal{F}_{t_0}(\mathbb{R},X)$ with $h(t_0) = x$. We also define

$$f_n : \mathbb{R} \to X, \quad f_n(s) = \begin{cases} \int_s^\infty \frac{\delta_n(\zeta)e^{-\lambda(\zeta-t_0)}}{||U(\zeta,t_0)x||}\,d\zeta\, U(s,t_0)x, & s \ge t_0 \\ a_n h(s), & s < t_0 \end{cases}.$$

We have that f_n is continuous and moreover $f_n(s) = 0$, for all $s \ge t+n+1$.

Since $h \in \mathcal{F}_{t_0}(\mathbb{R},X)$, in particular $h \in C(\mathbb{R},X)$. This implies that $f_n \in C(\mathbb{R},X)$, for all $n \in \mathbb{N}$, $n \ge 2$. Moreover, the pair (f_n, w_n) satisfies the equation ($E_{\mathcal{U}}$), so $f_n = \mathcal{P}(w_n)$, for all $n \in \mathbb{N}$, $n \ge 2$. Then we obtain that

$$||f_n(t)|| \le ||f_n||_\infty \le ||\mathcal{P}||\,||w_n||_{I(\mathbb{R},X)}, \quad \forall n \ge 2. \tag{2.50}$$

Let $n \in \mathbb{N}, n \ge 2$. Observing that

$$||w_n(s)|| = \delta_n(s)e^{-\lambda(s-t_0)}, \quad \forall s \in \mathbb{R} \tag{2.51}$$

we deduce that

$$\|w_n\|_\infty = \sup_{s\in[t,t+n+1]} \delta_n(s)e^{-\lambda(s-t_0)} \ < \ e^{-\lambda(t-t_0)}.$$

Then, from relation (2.46) it follows that

$$\|w_n\|_\infty \ < \ \frac{e^{-\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}. \tag{2.52}$$

In addition, using relations (2.47) and (2.51) we obtain that

$$\|w_n\|_{p_k} = \left(\int_{\mathbb{R}} \delta_n^{p_k}(s)e^{-\lambda p_k(s-t_0)}\,ds\right)^{1/p_k} < \left(\int_t^\infty e^{-\lambda p_k(s-t_0)}\,ds\right)^{1/p_k} = \frac{e^{-\lambda(t-t_0)}}{(\lambda p_k)^{1/p_k}} \le \frac{e^{-\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}, \quad \forall k \in \{1,\dots,l\}. \tag{2.53}$$

From relations (2.52) and (2.53) we deduce that

$$\|w_n\|_{I(\mathbb{R},X)} = \max\left\{\|w_n\|_{p_1},\dots,\|w_n\|_{p_l},\|w_n\|_\infty\right\} \ \le \ \frac{e^{-\lambda(t-t_0)}}{(\lambda p_1)^{1/p_1}}, \quad \forall n \in \mathbb{N},\ n \ge 2. \tag{2.54}$$

We consider

$$\varphi\ :\ [t_0,\infty)\to\mathbb{R}_+,\quad \varphi(s)=\frac{e^{-\lambda(s-t_0)}}{\parallel U(s,t_0)x\parallel}.$$

From (2.50) and (2.54) it follows

$$\left(\int_t^\infty \delta_n(\zeta)\varphi(\zeta)\,d\zeta\right)||U(t,t_0)x||\le\frac{||\mathcal{P}||}{(\lambda p_1)^{1/p_1}}e^{-\lambda(t-t_0)},\quad \forall n\in\mathbb{N}, n\ge 2. \tag{2.55}$$

Using (2.45), from relation (2.55) we observe in particular that

$$\int_{t+\frac{1}{n}}^{t+n}\varphi(\zeta)\,d\zeta\le\frac{1}{\gamma}\varphi(t),\quad \forall n\in\mathbb{N},\ n\ge 2. \tag{2.56}$$

For $n\to\infty$ in (2.56) we deduce that

$$\int_t^\infty\varphi(\zeta)\,d\zeta\le\frac{1}{\gamma}\varphi(t). \tag{2.57}$$

Since $t\ge t_0$ was arbitrary, from (2.57) we have

$$\int_t^\infty\varphi(\zeta)\,d\zeta\le\frac{1}{\gamma}\varphi(t),\quad \forall t\ge t_0. \tag{2.58}$$

Let

$$\beta\ :\ [t_0,\infty)\to\mathbb{R}_+,\quad \beta(s)=e^{\gamma(s-t_0)}\int_s^\infty\varphi(\zeta)\,d\zeta.$$

From (2.57) we have that $\dot\beta(s)\le 0$, for $s>t_0$. This implies that β is decreasing on $[t_0,\infty)$, so $\beta(t)\le\beta(t_0)$, for all $t\ge t_0$. Then, using (2.57) we deduce that

$$e^{\gamma(t-t_0)}\int_t^\infty\varphi(\zeta)\,d\zeta\le\int_{t_0}^\infty\varphi(\zeta)\,d\zeta\le\frac{1}{\gamma\,||x||},\quad \forall t\ge t_0. \tag{2.59}$$

From relation (2.48) we have

$$||U(r,t_0)x||\le M(t)e^{\omega(t)(r-t)}||U(t,t_0)x||,\quad \forall r\ge t\ge t_0. \tag{2.60}$$

Let $L_t:=M(t)(\lambda+\omega(t))$. For $t\ge t_0$, using relation (2.60) it follows

$$\int_t^\infty\frac{e^{-\lambda(r-t)}}{||U(r,t_0)x||}\,dr\ge\frac{1}{L_t||U(t,t_0)x||}. \tag{2.61}$$

From (2.59) and (2.61) we have

$$\frac{e^{\nu(t-t_0)}}{L_t\ \parallel U(t,t_0)x\parallel}\ \le\ e^{(\gamma-\lambda)(t-t_0)}\int_t^\infty\frac{e^{-\lambda(r-t)}}{\parallel U(r,t_0)x\parallel}\,dr\ =\ e^{\gamma(t-t_0)}\int_t^\infty\varphi(r)\,dr\ \le\ \frac{1}{\gamma\ \parallel x\parallel},\quad \forall t\ge t_0. \tag{2.62}$$

Relation (2.62) implies that

$$\frac{L_t}{\gamma}||U(t,t_0)x||\ge e^{\nu(t-t_0)}||x||,\quad \forall t\ge t_0. \tag{2.63}$$

Because $t_0\in\mathbb{R}$ and $x\in\mathcal{U}(t_0)$ were arbitrary, we deduce that (2.63) holds true for every $t_0\in\mathbb{R}$ and every $x\in\mathcal{U}(t_0)$.

We define

$$N:\mathbb{R}\to(0,\infty),\quad N(t)=\frac{M(t)(\lambda+\omega(t))}{\gamma}. \tag{2.64}$$

Obviously N is continuous. From (2.63) and (2.64) it follows that the inequality (2.44) holds true. □

The central result of this paper is:

Theorem 2.4. *If the pair $(C(\mathbb{R},X),I(\mathbb{R},X))$ is admissible for an evolution family $\mathcal{U}=\{U(t,r)\}_{t\ge r}$, then $\mathcal{U}$ has a nonuniform exponential dichotomy.*

Proof. Let $M\ :\ \mathbb{R}\to[1,\infty)$, $\omega\ :\ \mathbb{R}\to(0,\infty)$ be given by the property (iii) in Definition 2.1, i.e.

$$||U(t,r)||\le M(r)e^{\omega(r)(t-r)},\quad \forall t\ge r. \tag{2.65}$$

Let $\mathcal{P}$ be the operator introduced in Remark 2.6 and let

$$m:=\min\left\{\frac{1}{p_1},\ 2^{-\frac{p_1p_2}{p_2-p_1}},\ ||\mathcal{P}||^{-\frac{p_1}{p_1-1}}\right\}$$

and let $\lambda \in (0, m)$. Setting

$$\gamma := \frac{(\lambda p_1)^{1/p_1}}{||\mathcal{P}||} \quad \text{and} \quad \nu := \gamma - \lambda$$

from Theorem 2.2 we have that $\nu > 0$. Let $\mu := \max\{\frac{e\gamma}{\gamma}, e^{\nu}\}$. We define the functions

$$N : \mathbb{R} \to [1, \infty), \quad N(t) = \mu M(t) e^{\omega(t)}$$

and

$$\tilde{N} : \mathbb{R} \to (0, \infty), \quad \tilde{N}(t) = \frac{M(t)(\lambda + \omega(t))}{\gamma}.$$

Then N and $\tilde{N}$ are continuous functions. Moreover from Theorem 2.2 we have

$$||U(t, r)x|| \leq N(r) e^{-\nu(t-r)} ||x||, \quad \forall x \in \mathcal{S}(r), \forall t \geq r \tag{2.66}$$

and from Theorem 2.3 we obtain

$$\tilde{N}(t) \, ||U(t, r)y|| \geq e^{\nu(t-r)} ||y||, \quad \forall y \in \mathcal{U}(r), \forall t \geq r. \tag{2.67}$$

We observe that

$$\lambda < m \leq \frac{1}{p_1} < 1$$

and $\mu > (1/\gamma)$. Then

$$\mu \, e^{\omega(t)} > \frac{e^{\omega(t)}}{\gamma} > \frac{1 + \omega(t)}{\gamma} > \frac{\lambda + \omega(t)}{\gamma}, \quad \forall t \in \mathbb{R}.$$

This implies that

$$N(t) > \tilde{N}(t), \quad \forall t \in \mathbb{R}. \tag{2.68}$$

Then, from (2.67) and (2.68) we deduce that

$$N(t) ||U(t, r)y|| \geq e^{\nu(t-r)} ||y||, \quad \forall y \in \mathcal{U}(r), \forall t \geq r. \tag{2.69}$$

From Theorem 2.1 we have that $\mathcal{S}(r) \oplus \mathcal{U}(r) = X$, for each $r \in \mathbb{R}$. Let $P(r)$ be the projection with

$$\mathcal{S}(r) = Range\, P(r) \quad \text{and} \quad \mathcal{U}(r) = Ker\, P(r). \tag{2.70}$$

From Lemma 2.2 we obtain

$$P(t) U(t, r) = U(t, r) P(r), \quad \forall t \geq r. \tag{2.71}$$

Let $t > r$. From Lemma 2.2 and relation (2.70) we have that the restricted operator $U(t, r)_|: Ker\, P(r) \to Ker\, P(t)$ is well defined and surjective. Using (2.69) and (2.70) we also obtain that $U(t, r)_|$ is injective. So $U(t, r)_|: Ker\, P(r) \to Ker\, P(t)$ is invertible.

Finally, from relations (2.66), (2.69) and (2.71) we conclude that $\mathcal{U}$ has a nonuniform exponential dichotomy with the projections $\{P(t)\}_{t \in \mathbb{R}}$. □

Remark 2.9. Compared with the previous studies devoted to admissibility criteria for nonuniform dichotomies of evolution families on the whole line, in the present method all the estimates are done in the original norms of the spaces we work with, giving a direct approach from computational point of view.

Remark 2.10. The method presented above allows one to determine the expressions of the mapping N and of the exponent ν, which describe the nonuniform dichotomic behavior. We note that function N depends on the mappings $M(\cdot)$, $\omega(\cdot)$ and also on the norm of the input-output operator $\mathcal{P}$.

3. Connections between admissibility and dichotomy: nonuniform case versus uniform case

Let X be a Banach space (real or complex). Let $\mathcal{U} = \{U(t, r)\}_{t \geq r}$ be an evolution family on X. Let $l \in \mathbb{N}$, $l \geq 2$ and let $p_1, \ldots, p_l \in (1, \infty)$ with $p_1 < p_2 < \cdots < p_l$. We consider

$$I(\mathbb{R}, X) := C_{00}(\mathbb{R}, X) \cap L^{p_1}(\mathbb{R}, X) \cap \ldots \cap L^{p_l}(\mathbb{R}, X)$$

with the following norm

$$||w||_{I(\mathbb{R}, X)} := \max\{||w||_{p_1}, \ldots, ||w||_{p_l}, ||w||_{\infty}\}.$$

The main result in the previous section provides a sufficient condition for the existence of a nonuniform exponential dichotomy of nonautonomous systems on the whole axis. An interesting question in this context would be whether the converse implication remains true and, if so, in what hypotheses.

We start with a first answer in the nonuniform case. By an example we will show that generally if the evolution family $\mathcal{U}$ has a nonuniform exponential growth then the property of nonuniform exponential dichotomy does not necessarily imply the admissibility of the pair $(C(\mathbb{R}, X), I(\mathbb{R}, X))$.

Example 3.1. Let $X = \mathbb{R}^2$ with the norm $||(x, y)|| = |x| + |y|$. Let

$$\alpha : \mathbb{R} \to \mathbb{R}, \quad \alpha(t) = 2t - t\sin t.$$

For each $t, r \in \mathbb{R}, t \geq r$, we define

$$U(t, r)(x, y) = (e^{\alpha(r)-\alpha(t)}x, e^{\alpha(t)-\alpha(r)}y), \quad \forall (x, y) \in X.$$

It is easy to see that the family $\mathcal{U} = \{U(t, r)\}_{t \geq r}$ satisfies the properties (i) and (ii) from Definition 2.1. In addition, we observe that

$$\alpha(r) - \alpha(t) \leq 2|r| + 3(t - r), \quad \forall t \geq r \tag{3.1}$$

respectively

$$\alpha(t) - \alpha(r) \leq 2|r| + 3(t - r), \quad \forall t \geq r. \tag{3.2}$$

Using (3.1) and (3.2) we deduce that

$$||U(t, r)|| \leq e^{2|r|} e^{3(t-r)}, \quad \forall t \geq r$$

which shows that $\mathcal{U}$ is an evolution family which has a nonuniform exponential growth.

We consider the projection

$$P : \mathbb{R}^2 \to \mathbb{R}^2, \quad P(x, y) = (x, 0)$$

and let $P(t) = P$, for all $t \in \mathbb{R}$. We show that $\mathcal{U}$ admits a nonuniform exponential dichotomy relative to the projections $\{P(t)\}_{t\in\mathbb{R}}$.

Indeed, we consider the function

$$N : \mathbb{R} \to [1, \infty), \quad N(t) = e^{2|t|}$$

and by direct computation we successively deduce that

$$||U(t, r)x|| \leq N(r)\, e^{-(t-r)}||x||, \quad \forall x \in Range\, P(r), \forall t \geq r \tag{3.3}$$

and respectively

$$N(t)\, ||U(t, r)y|| \geq e^{t-r}||y||, \quad \forall y \in Ker\, P(r), \forall t \geq r. \tag{3.4}$$

Taking into account that the properties (i) and (ii) from Definition 2.2 obviously hold, from (3.3) and (3.4) we obtain that $\mathcal{U}$ has a nonuniform exponential dichotomy.

We prove now that $(C(\mathbb{R}, X), I(\mathbb{R}, X))$ is not admissible for $\mathcal{U}$. We take

$$w : \mathbb{R} \to X, \quad w(t) = (0, e^{-|t|}).$$

Then $w \in I(\mathbb{R}, X)$.

Suppose to the contrary that there is $f = (f_1, f_2) \in C(\mathbb{R}, X)$ such that

$$f(t) = U(t, r)f(r) + \int_r^t U(t, \zeta)w(\zeta)\, d\zeta, \quad \forall t \geq r. \tag{3.5}$$

From (3.5) it follows that

$$f_2(t) = e^{\alpha(t)-\alpha(r)} f_2(r) + \int_r^t e^{\alpha(t)-\alpha(\zeta)} e^{-|\zeta|}\, d\zeta, \quad \forall t \geq r. \tag{3.6}$$

In particular, for $t = 0$ in (3.6), we deduce that

$$f_2(r) = e^{2r - rsinr}\left[f_2(0) - \int_r^0 e^{-\tau+\tau sin\tau}\, d\tau\right], \quad \forall r \leq 0. \tag{3.7}$$

Let $n \in \mathbb{N}^*$. We successively have that

$$\int_{-2n\pi-\frac{\pi}{2}}^{-2n\pi-\frac{\pi}{6}} e^{-\tau+\tau\sin\tau}\, d\tau \;>\; e^{2n\pi}\int_{-2n\pi-\frac{\pi}{2}}^{-2n\pi-\frac{\pi}{6}} e^{\tau\sin\tau}\, d\tau \;>\; e^{2n\pi}\int_{-2n\pi-\frac{\pi}{2}}^{-2n\pi-\frac{\pi}{6}} e^{-\frac{\tau}{2}}\, d\tau \;>\; \frac{\pi}{3}\, e^{3n\pi}.$$

Then

$$\int_r^0 e^{-\tau+\tau\sin\tau}\, d\tau \;>\; \frac{\pi}{3}\, e^{3n\pi}, \quad \forall r \;\leq -2n\pi - \frac{\pi}{2}, \;\forall n \in \mathbb{N}^*. \tag{3.8}$$

From (3.7) and (3.8) we deduce

$$f_2\Big(-2n\pi - \frac{3\pi}{2}\Big) = e^{-2n\pi - \frac{3\pi}{2}} \left[f_2(0) - \int_{-2n\pi - \frac{3\pi}{2}}^{0} e^{-\tau + \tau \sin \tau}\, d\tau \right] < e^{-2n\pi - \frac{3\pi}{2}} \left[f_2(0) - \frac{\pi}{3}\, e^{3n\pi} \right]. \tag{3.9}$$

From (3.9) it follows that

$$f_2\Big(-2n\pi - \frac{3\pi}{2}\Big) < e^{-2n\pi - \frac{3\pi}{2}} f_2(0) - \frac{\pi}{3}\, e^{n\pi - \frac{3\pi}{2}}, \quad \forall n \in \mathbb{N}^*. \tag{3.10}$$

For $n \to \infty$ in relation (3.10) we obtain that

$$f_2\Big(-2n\pi - \frac{3\pi}{2}\Big) \longrightarrow -\infty, \quad \text{as } n \to \infty,$$

which contradicts the hypothesis that $f \in C(\mathbb{R}, X)$.

In conclusion, $\mathcal{U}$ has a nonuniform exponential dichotomy, but, for all that $(C(\mathbb{R}, X),\ I(\mathbb{R}, X))$ is not admissible for $\mathcal{U}$. $\square$

So, in the case of evolution families for which at least one of the functions that describes the exponential growth is unbounded, the converse implication of Theorem 2.4 does not hold true. In conclusion, in the nonuniform nonautonomous case, the admissibility concept introduced in this paper is not implied by a nonuniform exponential dichotomy.

To provide a complete answer to the problem formulated in the beginning of this section we will prove that the admissibility criteria obtained in Theorem 2.4 is a necessary and sufficient condition for *uniform* exponential dichotomy, in the particular case of evolution families which admit a uniform growth.

Remark 3.1. If an evolution family $\mathcal{U} = \{U(t, r)\}_{t \ge r}$ admits a uniform exponential growth and has a uniform exponential dichotomy relative to the projections $\{P(t)\}_{t \in \mathbb{R}}$, then the family of projections is uniformly bounded, i.e. $\sup_{t\in\mathbb{R}} ||P(t)|| < \infty$ (see Proposition 4.1 (i) in [46]). This property will be essential in the necessity part of the next result.

Theorem 3.1. *Let $\mathcal{U} = \{U(t, r)\}_{t \ge r}$ be an evolution family with uniform exponential growth. Then $\mathcal{U}$ has a uniform exponential dichotomy if and only if the pair $(C(\mathbb{R}, X),\ I(\mathbb{R}, X))$ is admissible for it.*

Proof. *Necessity.* According to our hypothesis there exist $N, \nu > 0$ and the projections $\{P(t)\}_{t\in\mathbb{R}}$ such that the properties described in Definition 2.2 and Remark 2.3 are fulfilled. Hence

$$||U(t, r)x|| \le N e^{-\nu(t-r)} ||x||, \quad \forall x \in Range\, P(r), \forall t \ge r \tag{3.11}$$

and respectively

$$||U(t, r)y|| \ge \frac{1}{N}\, e^{\nu(t-r)} ||y||, \quad \forall y \in Ker\, P(r), \forall t \ge r. \tag{3.12}$$

Since for each $t \ge r$ the restriction $U(t, r)_|$: $Ker\, P(r) \to Ker\, P(t)$ is invertible, we denote its inverse by $U(t, r)_|^{-1} : Ker\, P(t) \to Ker\, P(r)$. We also denote by

$$\rho := \sup_{t\in\mathbb{R}} ||P(t)||.$$

From Remark 3.1 we have that $\rho < \infty$.

Let $w \in I(\mathbb{R}, X)$. Then $f : \mathbb{R} \to X$ defined as

$$f(t) = \int_{-\infty}^{t} U(t, r)P(r)w(r)\, dr - \int_{t}^{\infty} U(r, t)_|^{-1} (I_d - P(r))w(r)\, dr \tag{3.13}$$

is a solution of $(E_{\mathcal{U}})$.

From (3.11)–(3.13) we deduce that

$$||f(t)|| \le N\rho \int_{-\infty}^{t} e^{-\nu(t-r)} ||w(r)||\, dr + N(1+\rho) \int_{t}^{\infty} e^{-\nu(r-t)} ||w(r)||\, dr, \quad \forall t \in \mathbb{R}. \tag{3.14}$$

Since in particular $w \in C_{00}(\mathbb{R}, X)$, also the functions

$$\varphi_1 : \mathbb{R} \to \mathbb{R}_+, \quad \varphi_1(t) = \int_{-\infty}^{t} e^{-\nu(t-r)} ||w(r)||\, dr$$

and

$$\varphi_2 : \mathbb{R} \to \mathbb{R}_+, \quad \varphi_2(t) = \int_{t}^{\infty} e^{-\nu(r-t)} ||w(r)||\, dr$$

belong to $C_{00}(\mathbb{R}, \mathbb{R})$. Then, from (3.14) we obtain $f \in C_{00}(\mathbb{R}, X)$, so it belongs to $C(\mathbb{R}, X)$. It remains only to prove that f is unique.

Let $g \in C(\mathbb{R}, X)$ be such that (g, w) satisfies $(E_{\mathcal{U}})$. Then $h := f - g \in C(\mathbb{R}, X)$ and

$$h(t) = U(t, r)h(r), \quad \forall t \ge r. \tag{3.15}$$

We consider $h_1, h_2 : \mathbb{R} \to X$ given by

$$h_1(t) = P(t)h(t) \quad \text{and} \quad h_2(t) = (I_d - P(t))h(t).$$

Then $h_1, h_2 \in C(\mathbb{R}, X)$, $h = h_1 + h_2$ and from (3.15) it follows that

$$h_k(t) = U(t,r)h_k(r), \quad \forall t \geq r, \forall k \in \{1, 2\}. \tag{3.16}$$

Let $t_0 \in \mathbb{R}$. From (3.12) and (3.16) we immediately deduce that

$$\frac{1}{N} e^{\nu(t-t_0)} ||h_2(t_0)|| \leq ||h_2(t)|| \leq (1+\rho) \, ||h(t)|| \leq (1+\rho) \, ||h||_\infty, \quad \forall t \geq t_0.$$

This implies that

$$\| \, h_2(t_0) \, \| \quad \leq \quad N(1+\rho) \, \| \, h\|_\infty \, e^{-\nu(t-t_0)}, \quad \forall t \geq t_0. \tag{3.17}$$

For $t \to \infty$ in relation (3.17) we obtain that $h_2(t_0) = 0$. In addition, from (3.11) and (3.16) we have that

$$\| \, h_1(t_0) \, \| \quad \leq N e^{-\nu(t_0-s)} \, \| \, h_1(s) \, \| \leq N\rho \, \ \| \, h\|_\infty \, e^{-\nu(t_0-s)}, \quad \forall s \leq t_0. \tag{3.18}$$

For $s \to -\infty$ in relation (3.18) it follows that $h_1(t_0) = 0$. In conclusion

$$h(t_0) = h_1(t_0) + h_2(t_0) = 0, \quad \forall t_0 \in \mathbb{R}$$

so f is uniquely determined.

Sufficiency. Suppose that $||U(t,r)|| \leq M e^{\omega(t-r)}$, for all $t \geq r$ (see Remark 2.1).

Let $\mathcal{P}$ be the operator introduced in Remark 2.6 and let

$$m := \min\left\{ \frac{1}{p_1}, \, 2^{-\frac{p_1 p_2}{p_2 - p_1}}, \, ||\mathcal{P}||^{-\frac{p_1}{p_1 - 1}} \right\}. \tag{3.19}$$

Let $\lambda \in (0, m)$. We consider

$$\gamma := \frac{(\lambda p_1)^{1/p_1}}{||\mathcal{P}||}, \quad \nu := \gamma - \lambda \quad \text{and} \quad \mu := \max\{\frac{e^\gamma}{\gamma}, e^\nu\}. \tag{3.20}$$

According to the conclusions in the proof of Theorem 2.4 we have that $\mathcal{U}$ has an exponential dichotomy with the projections $\{P(t)\}_{t\in\mathbb{R}}$, such that

$$Range\, P(t) = \mathcal{S}(t) \quad \text{and} \quad Ker\, P(t) = \mathcal{U}(t), \quad \forall t \in \mathbb{R}$$

and respectively with the constant exponent $\nu > 0$ and the mapping

$$N : \mathbb{R} \to \mathbb{R}_+, \quad N(t) = \mu M e^\omega. \tag{3.21}$$

Then it is sufficient to consider $N := \mu M e^\omega$ and from the proof of Theorem 2.4 we have

$$||U(t,r)x|| \leq N e^{-\nu(t-r)} ||x||, \quad \forall x \in Range\, P(r), \forall t \geq r$$

and respectively

$$||U(t,r)y|| \geq \frac{1}{N} e^{\nu(t-r)} ||y||, \quad \forall y \in Ker\, P(r), \forall t \geq r.$$

This shows that $\mathcal{U}$ has a uniform exponential dichotomy. □

Remark 3.2. The method presented above allows one to determine the expressions of the growth rates as relations (3.19) - (3.21) show.

Remark 3.3. The admissibility property introduced in this paper implies the existence of an exponential dichotomy whose (non)uniformity depends directly on the initial exponential growth of the evolution family, i.e. on its (non)uniformity. The reverse implication is true provided that the evolution family has a *uniform* exponential growth.

Acknowledgment

The authors would like to thank the referees for carefully reading the paper and for their suggestions and comments, which led to the improvement of the paper.

References

[1] M.A. Alhalawa, D. Dragičević, New conditions for (non)uniform behaviour of linear cocycles over flows, J. Math. Anal. Appl. 473 (2019) 367–381.
[2] B. Aulbach, N.V. Minh, The concept of spectral dichotomy for linear difference equations II, J. Differ. Equ. Appl. 2 (1996) 251–262.
[3] L. Barreira, D. Dragičević, C. Valls, Strong and weak (L^p, L^q)-admissibility, Bull. Sci. Math. 138 (2014) 721–741.
[4] L. Barreira, D. Dragičević, C. Valls, Admissibility and Hyperbolicity, in: Springer Briefs in Mathematics, Springer International Publishing, 2018.
[5] L. Barreira, C. Valls, General exponential dichotomies: from finite to infinite time, Adv. Oper. Theory 4 (2019) 215–225.
[6] F. Battelli, K.J. Palmer, Criteria for exponential dichotomy for triangular systems, J. Math. Anal. Appl. 428 (2015) 525–543.
[7] F. Bătăran, C. Preda, P. Preda, Extending some results of L. Barreira and C. Valls to the case of linear skew-product semiflows, Results Math. 72 (2017) 965–978.
[8] A. Ben-Artzi, I. Gohberg, Dichotomies of systems and invertibility of linear ordinary differential operators, Oper. Theory Adv. Appl. 56 (1992) 90–119.
[9] A. Ben-Artzi, I. Gohberg, M.A. Kaashoek, Invertibiiity and dichotomy of differential operators on the half-line, J. Dynam. Differ. Equ. 5 (1993) 1–36.
[10] L. Berezansky, E. Braverman, On exponential dichotomy, Bohl-Perron type theorems and stability of difference equations, J. Math. Anal. Appl. 304 (2005) 511–530.
[11] L. Berezansky, E. Braverman, On exponential dichotomy for linear difference equations with bounded and unbounded delay, Differ. Differen. Equ. Appl. (2006) 169–178.
[12] L. Biriş, M. Megan, On a concept of exponential dichotomy for cocycles of linear operators in Banach spaces, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 59 (2016) 217–223.
[13] E. Braverman, S. Zhukovskiy, The problem of a lazy tester, or exponential dichotomy for impulsive differential equations revisited, Nonlinear Anal. Hybrid Syst. 2 (2008) 971–979.
[14] X. Chang, J. Zhang, J. Qin, Robustness of nonuniform (μ, ν)-dichotomies in Banach spaces, J. Math. Anal. Appl. 387 (2012) 582–594.
[15] S.N. Chow, H. Leiva, Existence and roughness of the exponential dichotomy for linear skew-product semiflows in Banach spaces, J. Differen. Equ. 120 (1995) 429–477.
[16] S.N. Chow, H. Leiva, Two definitions of exponential dichotomy for skew-product semiflow in Banach spaces, Proc. Amer. Math. Soc. (124) (1996) 1071–1081.
[17] S.N. Chow, H. Leiva, Unbounded perturbation of the exponential dichotomy for evolution equations, J. Differen. Equ. 129 (1996) 509–531.
[18] W.A. Coppel, Dichotomies in stability theory, in: Lecture Notes in Math., 629, Springer-Verlag, Berlin, 1978.
[19] J.L. Daleckiĭ, M.G. Kreĭn, Stability of differential equations in Banach space, American Mathematical Society, Providence, RI, 1974.
[20] D. Dragičević, Strong nonuniform behaviour: a Datko type characterization, J. Math. Anal. Appl. 459 (2018) 266–290.
[21] D. Dragičević, Admissibility and nonuniform polynomial dichotomies, Math. Nachr. 293 (2020) 226–243.
[22] D. Dragičević, W. Zhang, Asymptotic stability of nonuniform behavior, Proc. Amer. Math. Soc. 147 (2019) 2437–2451.
[23] S. Elaydi, O. Hájek, Exponential trichotomy of differential systems, J. Math. Anal. Appl. 129 (1988) 362–374.
[24] S. Elaydi, O. Hájek, Exponential dichotomy and trichotomy of nonlinear differential equations, Differ. Integral Equ. 3 (1990) 1201–1224.
[25] S. Elaydi, K. Janglajew, Dichotomy and trichotomy of difference equations, J. Differ. Equ. Appl. 3 (1998) 417–448.
[26] P.V. Hai, The relation between the uniform exponential dichotomy and the uniform admissibility of the pair (l^p, l^q) on $\ominus$, Asian-Eur. J. Math. 3 (2010) 593–605.
[27] P.V. Hai, On the polynomial stability of evolution families, Appl. Anal. 95 (2016) 1239–1255.
[28] N.T. Huy, Invariant manifolds of admissible classes for semi-linear evolution equations, J. Differen. Equ. 246 (2009) 1820–1844.
[29] N.T. Huy, H. Phi, Discretized characterizations of exponential dichotomy of linear skew-product semiflows over semiflows, J. Math. Anal. Appl. 362 (2010) 46–57.
[30] J.L. Massera, J.J. Schäffer, Linear differential equations and functional analysis IV, Math. Ann. 139 (1960) 287–342.
[31] J.L. Massera, J.J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New-York, 1966.
[32] M. Megan, L. Buliga, Functionals on normed function spaces and exponential instability of linear skew-product semiflows, Bull. Belg. Math. Soc. Simon Stevin 14 (2007) 355–362.
[33] M. Megan, B. Sasu, A.L. Sasu, On nonuniform exponential dichotomy of evolution operators in Banach spaces, Integral Equ. Operat. Theory 44 (2002) 71–78.
[34] M. Megan, A.L. Sasu, B. Sasu, Perron conditions for pointwise and global exponential dichotomy of linear skew-product semiflows, Integral Equ. Operat. Theory 50 (2004) 489–504.
[35] N.V. Minh, F. Räbiger, R. Schnaubelt, Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half line, Integral Equ. Operat. Theory 32 (1998) 332–353.
[36] N.V. Minh, On the proof of characterisations of the exponential dichotomy, Proc. Amer. Math. Soc. 127 (1999) 779–782.
[37] N.V. Minh, N.T. Huy, Characterizations of dichotomies of evolution equations on the half-line, J. Math. Anal. Appl. 261 (2001) 28–44.
[38] R. Naulin, M. Pinto, Roughness of (h, k)-dichotomies, J. Differen. Equ. 118 (1995) 20–35.
[39] O. Perron, Die stabilitätsfrage bei differentialgleischungen, Math. Z. 32 (1930) 703–728.
[40] K.J. Palmer, Exponential dichotomies and Fredholm operators, Proc. Amer. Math. Soc. 104 (1988) 149–156.
[41] K.J. Palmer, Shadowing in Dynamical Systems, in: Mathematics and Its Applications, 501, Kluwer Academic Publishers, 2000.
[42] K.J. Palmer, Exponential dichotomy and expansivity, Ann. Mat. Pura Appl. 185 (2006) 171–185.
[43] C. Pötzsche, Smooth roughness of exponential dichotomies, revisited, Discrete Contin. Dyn. Syst. Ser. B 20 (2015) 853–859.
[44] C. Pötzsche, E. Russ, Continuity and invariance of the Sacker-Sell spectrum, J. Dynam. Differen. Equ. 28 (2016) 533–566.
[45] P. Preda, M. Megan, Nonuniform dichotomy of evolutionary processes in Banach spaces, Bull. Aust. Math. Soc. 27 (1983) 31–52.
[46] B. Sasu, A.L. Sasu, Exponential trichotomy and p-admissibility for evolution families on the real line, Math. Z. 253 (2006) 515–536.
[47] A.L. Sasu, B. Sasu, Exponential dichotomy on the real line and admissibility of function spaces, Integral Equ. Operat. Theory 54 (2006) 113–130.
[48] B. Sasu, A.L. Sasu, Exponential dichotomy and (ℓ^p, ℓ^q)-admissibility on the half-line, J. Math. Anal. Appl. 316 (2006) 397–408.
[49] B. Sasu, Uniform dichotomy and exponential dichotomy of evolution families on the half-line, J. Math. Anal. Appl. 323 (2006) 1465–1478.
[50] A.L. Sasu, Exponential dichotomy for evolution families on the real line, Abstr. Appl. Anal. (2006) 1–16. Article ID 31641.
[51] A.L. Sasu, Integral equations on function spaces and dichotomy on the real line, Integral Equ. Operat. Theory 58 (2007) 133–152.
[52] A.L. Sasu, Pairs of function spaces and exponential dichotomy on the real line, Adv. Differen. Equ. (2010) 1–15. Article ID 347670.
[53] A.L. Sasu, B. Sasu, Integral equations, dichotomy of evolution families on the half-line and applications, Integral Equ. Operat. Theory 66 (2010) 113–140.
[54] A.L. Sasu, M.G. Babuţia, B. Sasu, Admissibility and nonuniform exponential dichotomy on the half-line, Bull. Sci. Math. 137 (2013) 466–484.
[55] B. Sasu, A.L. Sasu, On the dichotomic behavior of discrete dynamical systems on the half-line, Discrete Contin. Dyn. Syst. 33 (2013) 3057–3084.
[56] A.L. Sasu, B. Sasu, Discrete admissibility and exponential trichotomy of dynamical systems, Discrete Contin. Dyn. Syst. 34 (2014) 2929–2962.
[57] A.L. Sasu, B. Sasu, Admissibility and exponential trichotomy of dynamical systems described by skew-product flows, J. Differen. Equ. 260 (2016) 1656–1689.

[58] J. Zhang, X. Chang, J. Wang, Existence and robustness of nonuniform (h, k, μ, ν)-dichotomies for nonautonomous impulsive differential equations, J. Math. Anal. Appl. 400 (2013) 710–723.
[59] J. Zhang, M. Fan, H. Zhu, Nonuniform (h, k, μ, ν)-dichotomy with applications to nonautonomous dynamical systems, J. Math. Anal. Appl. 452 (2017) 505–551.
[60] L. Zhou, K. Lu, W. Zhang, Roughness of tempered exponential dichotomies for infinite-dimensional random difference equations, J. Differen. Equ. 254 (2013) 4024–4046.
[61] L. Zhou, W. Zhang, Admissibility and roughness of nonuniform exponential dichotomies for difference equations, J. Funct. Anal. 271 (2016) 1087–1129.
[62] L. Zhou, K. Lu, W. Zhang, Equivalences between nonuniform exponential dichotomy and admissibility, J. Differen. Equ. 262 (2017) 682–747.