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# On the stability of port-Hamiltonian descriptor systems <sup>\*</sup>

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## Abstract

We study stable differential-algebraic equations. Besides characterizing the stability in terms of a generalized Lyapunov inequality, we show that these systems can always be rewritten as port-Hamiltonian systems on the subspace where the solutions evolve.

*Keywords:* Descriptor systems, port Hamiltonian systems, stability, differential-algebraic equations, linear matrix inequalities.

## 1 Introduction

In this note, we consider generalized port-Hamiltonian systems of the form

$$\begin{aligned} \frac{d}{dt}Ex(t) &= (J - R)Qx(t) + (B - P)u(t), \\ y(t) &= (B + P)^T Qx(t) + (S - N)u(t), \end{aligned} \quad (1)$$

where  $E, J, R, Q \in \mathbb{R}^{n \times n}$ ,  $B, P \in \mathbb{R}^{n \times k}$  and  $S, N \in \mathbb{R}^{k \times k}$  and

$$\begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0, \quad \begin{bmatrix} J & B \\ -B^T & N \end{bmatrix} = - \begin{bmatrix} J & B \\ -B^T & N \end{bmatrix}^T, \quad (2)$$

see e.g. [MM19]. In comparison to classical port-Hamiltonian systems where the coefficient  $E$  is the identity, here it might be singular. Besides the explicit formulation (1) of generalized port-Hamiltonian DAEs there is also an implicit definition of generalized pH-systems given in [vdSM18] via so called Lagrangian and Dirac subspaces. This geometric formulation of pH-systems was studied extensively for classical systems, i.e.,  $E = I_n$ , in [vdSJ14]. Recently, the authors compared this geometric and the explicit formulation (1) in [GHR21] but for DAEs

$$\frac{d}{dt}Ex(t) = (J - R)Qx(t). \quad (3)$$

In this case the conditions (2) simplify to

$$D + D^T \leq 0, \quad \text{with} \quad D := J - R, \quad Q^T E = E^T Q,$$

and such matrices  $D$  are called *dissipative*.

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The main objective of this paper is to show that stable DAEs can be rewritten as pH-DAEs (3). We show that the matrix  $Q$  in (3) can be obtained from the solution of a generalized Lyapunov equation. For this rewriting we have to restrict to the smallest subspace where the solutions evolve, also called the *system space*, see [RRV15].

The second aim of the paper is to show how systems of the form (1) fit into the geometric port-Hamiltonian framework from [vdSM18].

Outline of the paper is as follows. After recalling some terminologies on matrix pencils in Section 2, we study the behavior and determine the system space of differential-algebraic systems in Section 3. The stability of DAEs in behavioral sense is recalled in Section 4. We provide sufficient conditions for stability, rewrite stable DAEs as pH-DAEs and give an additional assumption under which pH-DAEs are stable. Finally, in Section 5 we show how the pH-systems (1) can be embedded into the geometric framework from [vdSM18].

## 2 Preliminaries

The matrix pencil  $sE - A \in \mathbb{R}[s]^{n \times n}$  is called *regular* if  $\det(sE - A)$  is non-zero. The *spectrum*  $\sigma(E, A)$  of a matrix pencil  $sE - A$  is set of all complex numbers  $\lambda$  for which  $\lambda E - A$  is not invertible. An eigenvalue is called simple if  $\ker(\lambda E - A)$  has dimension one and semi-simple if the dimension of this subspace coincides with the multiplicity of  $\lambda$  as a root of  $\det(sE - A)$ .

Recall that every pencil  $sE - A \in \mathbb{R}[s]^{n \times m}$  can be transformed to Kronecker canonical form, see e.g. [Gan59, Chapter XII], i.e., there exists invertible  $S \in \mathbb{C}^{n \times n}$  and  $T \in \mathbb{C}^{m \times m}$  such that  $S(sE - A)T$  is a block diagonal with the following four types of blocks

$$\begin{aligned} sI_{n_i} - J_{n_i}(\lambda) &= \begin{bmatrix} s-\lambda & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda \end{bmatrix} \in \mathbb{R}[s]^{n_i \times n_i}, \\ sN_{\alpha_i} - I_{\alpha_i} &= \begin{bmatrix} -1 & s & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix} \in \mathbb{R}[s]^{\alpha_i \times \alpha_i} \\ sK_{\beta_i} - L_{\beta_i} &= \begin{bmatrix} s & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & s \\ & & & -1 \end{bmatrix} \in \mathbb{R}[s]^{(\beta_i-1) \times \beta_i}, \\ sK_{\gamma_i}^T - L_{\gamma_i}^T &\in \mathbb{R}[s]^{\gamma_i \times (\gamma_i-1)}. \end{aligned} \tag{4}$$

It is also possible to choose  $S, T \in \mathbb{R}^{n \times n}$  but then the blocks  $sI_{n_i} - J_{n_i}(\lambda)$  have to be replaced by blocks in real Jordan form, see e.g. [HJ13]. The indices in the above blocks are collected in  $\alpha = (\alpha_1, \dots, \alpha_{\ell_\alpha}) \in \mathbb{N}^{\ell_\alpha}$ ,  $\beta \in \mathbb{N}^{\ell_\beta}$  and  $\gamma \in \mathbb{N}^{\ell_\gamma}$  and  $|\alpha| := \sum_{k=1}^{\ell_\alpha} \alpha_k$ . Then the Kronecker form can be briefly written as

$$\begin{aligned} &S(sE - A)T \\ &= \text{diag}(sI_{n_0} - J, sN_\alpha - I_{|\alpha|}, sK_\beta - L_\beta, sK_\gamma^T - L_\gamma^T) \end{aligned} \tag{5}$$

where  $N_\alpha := \text{diag}(N_{\alpha_1}, \dots, N_{\alpha_{\ell_\alpha}})$ , and  $J, K_\beta, L_\beta, K_\gamma, L_\gamma$  are defined accordingly. The indices  $\beta_i$  and  $\gamma_i$  are called the *row* and *column minimal indices*. Note that we do not exclude  $\beta_i = 1$  and  $\gamma_i = 1$ . In this case the diagonal operator will add a zero column and a zero row to the matrix, respectively. The largest  $\alpha_i$  is called the *index* of the DAE. Furthermore, we write  $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \text{Re } z > 0\}$ ,  $\mathbb{C}_- := \{z \in \mathbb{C} \mid \text{Re } z < 0\}$  and  $\overline{\mathbb{C}_-} := \mathbb{C} \setminus \mathbb{C}_+$ .

### 3 Solutions, behavior and system space

In this section, we study the solutions of the system

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \quad Ex(0) = Ex_0, \quad (6)$$

with  $sE - A \in \mathbb{R}[s]^{n \times n}$  regular,  $B \in \mathbb{R}^{n \times k}$  and  $k, n \in \mathbb{N}_0$ . The set of these systems is denoted by  $\Sigma_{n,k}$  and we write  $[E, A, B] \in \Sigma_{n,k}$ . If the system (6) has no input variables  $u$ , i.e.,  $k = 0$ , then we write  $[E, A] \in \Sigma_n$ .

Here we consider solutions in the behavioral sense, see [PW97]. The *behavior* of  $[E, A, B] \in \Sigma_{n,k}$  is given by

$$\mathfrak{B}_{[E,A,B]}^\infty := \{(x, u) \in C^\infty(\mathbb{R}, \mathbb{R}^n) \times C^\infty(\mathbb{R}, \mathbb{R}^k) \mid \frac{d}{dt}Ex = Ax + Bu\}.$$

Here we consider only smooth trajectories. A more general definition using weak derivatives can be found in [BR13].

In the following, we will characterize the behavior and the set of initial values using the transformation to a certain canonical form. Recall that the DAEs  $[E, A, B], [\hat{E}, \hat{A}, \hat{B}] \in \Sigma_{n,k}$  are called *feedback equivalent*, see [RRV15, Definition 2.7], if there exist invertible  $S, T \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{k \times n}$  such that

$$[s\hat{E} - \hat{A}, \hat{B}] = S[sE - A, B] \begin{bmatrix} T & 0 \\ -FT & I_k \end{bmatrix}. \quad (7)$$

A system can then be transformed to feedback equivalence form, see [RRV15, Proposition 2.9], i.e., there exist invertible  $S, T \in \mathbb{R}^{n \times n}$  and  $F \in \mathbb{R}^{k \times n}$  such that

$$S[sE - A, B] \begin{bmatrix} T & 0 \\ -FT & I_k \end{bmatrix} = \begin{bmatrix} sI_{n_1} - A_1 & 0 & 0 & B_1 \\ 0 & I_{n_2} & sE_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}, \quad (8)$$

where  $n_1 + n_2 + n_3 = n$  and  $E_{33}$  is nilpotent.

If  $k = 0$  then  $[E, A]$  and  $[\hat{E}, \hat{A}]$  are called *equivalent* if  $[\hat{E}, \hat{A}] = [SET, SAT]$  for some invertible  $S, T \in \mathbb{R}^{n \times n}$ . The behavior transforms as follows

$$\begin{aligned} \mathfrak{B}_{[SET, SAT]}^\infty &= T^{-1} \mathfrak{B}_{[E, A]}^\infty, \\ \mathfrak{B}_{[SET, SAT, SB]}^\infty &= \begin{bmatrix} T^{-1} & 0 \\ F & I_k \end{bmatrix} \mathfrak{B}_{[E, A, B]}^\infty. \end{aligned} \quad (9)$$

Based on this, we define the *system space* as follows

$$\begin{aligned} \mathcal{V}_{\text{sys}}^{[E, A, B]} &:= \{(x(0), u(0)) \in \mathbb{R}^{n+k} \mid (x, u) \in \mathfrak{B}_{[E, A, B]}^\infty\}, \\ \mathcal{V}_{\text{sys}}^{[E, A]} &:= \{x(0) \in \mathbb{R}^n \mid x \in \mathfrak{B}_{[E, A, B]}^\infty\}. \end{aligned}$$

Another equivalent definition of system space was given in [RRV15, Prop. 3.3] based on Wong-sequences of subspaces.

Furthermore, the space of *consistent initial differential variables* is given by

$$\mathcal{V}_{\text{diff}}^{[E, A, B]} := \{x_0 \in \mathbb{R}^n \mid \exists (x, u) \in \mathfrak{B}_{[E, A, B]}^\infty, Ex(0) = Ex_0\}.$$

Below, we describe this subspace using the feedback equivalence form, see also [RRV15, p. 162].

**Lemma 3.1.** *Let  $[E, A] \in \Sigma_n$  and let  $S, T \in \mathbb{R}^{n \times n}$  such that  $S(sE - A)T$  is in Kronecker form (4). Then*

$$\mathcal{V}_{\text{sys}}^{[E, A]} = T(\mathbb{R}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \{0\}^{|\gamma| - \ell_\gamma}). \quad (10)$$

Let  $[E, A, B] \in \Sigma_{n,k}$  and let  $S, T \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{k \times n}$  transform  $[E, A, B]$  to feedback equivalence form (8) then

$$\mathcal{V}_{\text{sys}}^{[E,A,B]} = \begin{bmatrix} T & 0 \\ -FT & I_k \end{bmatrix} (\mathbb{R}^{n_1} \times \text{ran } B_2 \times \{0\}^{n_3} \times \mathbb{R}^k). \quad (11)$$

*Proof.* To prove (10) we can apply (9) and hence assume that  $sE - A$  is already given in Kronecker form (4). The formula (10) follows then by computing the solutions for each of the four types of blocks. Clearly,  $\mathcal{V}_{\text{sys}}^{[I_{n_0}, J]} = \mathbb{R}^{n_0}$  and a short calculation reveals  $\mathfrak{B}_{[N_\alpha, I_{|\alpha|}]}^\infty = \{0\}^{|\alpha|}$ . Next, we consider  $\frac{d}{dt}K_{\beta_i}x(t) = L_{\beta_i}x(t)$  with  $\beta_i \geq 2$ . Using integration it is immediate that for every  $x(0) \in \mathbb{R}^{|\beta_i|}$  there exists  $x \in \mathfrak{B}_{[E,A]}^\infty$ . If  $\beta_i = 1$  then, by definition, we add a zero column in (5) and hence the system space is also the whole space. Finally, the DAE  $\frac{d}{dt}K_{\gamma_i}^T x(t) = L_{\gamma_i}^T x(t)$  for  $\gamma_i \geq 2$  has only the trivial solution. For  $\gamma_i = 1$ , by definition, a zero row is added in (5) which leads to a zero component in the system space. This altogether implies (10). The equation (11) was obtained in the proof of [RRV15, Prop. 3.3].  $\square$

## 4 Relation between stable and pH DAEs

In this section, we focus on the case where the input matrix  $B$  is not present in the system (6), i.e.,

$$\frac{d}{dt}Ex(t) = Ax(t), \quad Ex(0) = Ex_0. \quad (12)$$

The aim is to characterize when there exists  $Q \in \mathbb{R}^{n \times n}$  such that this system can be rewritten as a pH-DAE of the form

$$\frac{d}{dt}Ex(t) = DQx(t), \quad D + D^T \leq 0, \quad Q^T E \geq 0. \quad (13)$$

These DAEs are also called *dissipative Hamiltonian* in [MMW20].

We say that  $[E, A]$  is *stable* if

$$\forall x \in \mathfrak{B}_{[E,A]}^\infty \exists M > 0 : \sup_{t \geq 0} \|x(t)\| \leq M.$$

Below, the stability is characterized either in terms of the eigenvalues or a Lyapunov inequality on the system space. To this end, for a subspace  $\mathcal{L} \subseteq \mathbb{R}^n$  and symmetric matrices  $M, N \in \mathbb{R}^{n \times n}$  we say that

$$\begin{aligned} M \geq_{\mathcal{L}} N & \quad :\Leftrightarrow \quad x^T M x \geq x^T N x \quad \forall x \in \mathcal{L}, \\ M >_{\mathcal{L}} N & \quad :\Leftrightarrow \quad x^T M x > x^T N x \quad \forall x \in \mathcal{L} \setminus \{0\}. \end{aligned}$$

**Proposition 4.1.** *Let  $[E, A] \in \Sigma_n$  then the following statements are equivalent.*

- (a)  $[E, A]$  is stable.
- (b) The pencil  $sE - A$  is regular and there exists symmetric  $X \in \mathbb{R}^{n \times n}$  with  $X >_{E\mathcal{V}_{\text{sys}}^{[E,A]}} 0$  and

$$A^T X E + E^T X A \leq_{\mathcal{V}_{\text{sys}}^{[E,A]}} 0. \quad (14)$$

- (c) The pencil  $sE - A$  is regular,  $\sigma(E, A) \subseteq \overline{\mathbb{C}_-}$  and the eigenvalues on the imaginary axis are semi-simple.

*Proof.* Since the above conditions are invariant under multiplication from left and right with invertible matrices, we can assume that  $sE - A$  is already given in Kronecker form (4). If the DAE is stable then by definition  $\beta$  cannot be present in the Kronecker form. Since  $sE - A$  is square also  $\gamma$  is not present and therefore  $sE - A$  is regular. Furthermore, computing the solutions for the ODEs  $[I_{n_i}, J_{n_i}(\lambda)]$  shows  $\sigma(E, A) \subseteq \overline{\mathbb{C}_-}$  and that all eigenvalues on the imaginary axis are semi-simple. This proves (c). Clearly, (c) implies (a). To prove the equivalence of (b) and (c), we assume that  $sE - A$  is regular. Then Lemma 3.1 implies that

$$\mathcal{V}_{\text{sys}}^{[E, A]} = \mathbb{R}^{n_0} \times \{0\}^{|\alpha|}, \quad E\mathcal{V}_{\text{sys}}^{[E, A]} = \mathcal{V}_{\text{sys}}^{[E, A]}.$$

Furthermore, (14) holds if and only if

$$\begin{bmatrix} J^T & 0 \\ 0 & I_{|\alpha|} \end{bmatrix} X \begin{bmatrix} I_{n_0} & 0 \\ 0 & N \end{bmatrix} + \begin{bmatrix} I_{n_0} & 0 \\ 0 & N^T \end{bmatrix} X \begin{bmatrix} J & 0 \\ 0 & I_{|\alpha|} \end{bmatrix} \leq_{\mathbb{R}^{n_0} \times \{0\}^{|\alpha|}} 0$$

which is equivalent to the existence of some  $X_1 \in \mathbb{R}^{n_0 \times n_0}$  with  $X_1 > 0$  such that

$$J^T X_1 + X_1 J \leq 0$$

and hence to  $\sigma(J) \subseteq \overline{\mathbb{C}_-}$  and semi-simple eigenvalues on the imaginary axis.  $\square$

**Remark 4.2.** Oftentimes asymptotically stable DAEs are of interest which are defined by

$$\forall x \in \mathfrak{B}_{[E, A]}^\infty : \lim_{t \rightarrow \infty} \|x(t)\| = 0.$$

Analogously to Proposition 4.1, asymptotically stable DAEs can be characterized by a strict inequality in (14) or by having eigenvalues only in  $\mathbb{C}_-$ . Related characterizations with Lyapunov equations were previously given in [Lew86, Sty02], see also the recent study of stability of pH-DAEs and its connection to hypercoercivity by [AAM21] and in particular Theorem 4 therein for a related Lyapunov equation.

If  $[E, A]$  is stable, then the symmetric solution  $X$  of the Lyapunov-like inequality (14) can be used to rewrite a stable DAE in pH-form (13). To this end, we consider positive semi-definite matrix  $\hat{X}x := Xx$  for all  $x \in E\mathcal{V}_{\text{sys}}^{[E, A]}$  and  $\hat{X}x = 0$  for  $x \in (E\mathcal{V}_{\text{sys}}^{[E, A]})^\perp$ . Further, let  $Q := \hat{X}EP_{\mathcal{V}_{\text{sys}}^{[E, A]}}$ , where  $P_{\mathcal{V}_{\text{sys}}^{[E, A]}}$  is the orthogonal projector onto the system space. In the following, we restrict the coefficients  $E, A$  to the system space  $\mathcal{V}_{\text{sys}}^{[E, A]}$ . Note that  $E : \mathcal{V}_{\text{sys}}^{[E, A]} \rightarrow E\mathcal{V}_{\text{sys}}^{[E, A]}$  is invertible and therefore  $Q : \mathcal{V}_{\text{sys}}^{[E, A]} \rightarrow E\mathcal{V}_{\text{sys}}^{[E, A]}$  is also invertible. Hence we can define  $D := AQ^{-1}$  which is dissipative according to (14) and leads to

$$Q^T E = E^T \hat{X} E >_{\mathcal{V}_{\text{sys}}^{[E, A]}} 0, \quad A =_{\mathcal{V}_{\text{sys}}^{[E, A]}} AQ^{-1}Q = DQ.$$

This provides a port-Hamiltonian formulation (13) on the system space.

The second possibility is to use the pseudo-inverse  $Q^\dagger$  of  $Q$ . We have by definition that

$$QQ^\dagger = P_{\text{ran } Q} = P_{E\mathcal{V}_{\text{sys}}^{[E, A]}}, \quad Q^\dagger Q = P_{\text{ran } Q^T} = P_{\mathcal{V}_{\text{sys}}^{[E, A]}}$$

and if we define  $D := AQ^\dagger$  we obtain  $DQ = AQ^\dagger Q = AP_{\mathcal{V}_{\text{sys}}^{[E, A]}} =_{\mathcal{V}_{\text{sys}}^{[E, A]}} A$ .

The following example from [MMW18] shows that not every port-Hamiltonian DAE given by (13) is stable. Consider

$$sE - DQ = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}, \quad D := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

which is port-Hamiltonian and has a Jordan block of size 2 at zero. The above example is an ordinary differential equation and the quadratic form  $x \mapsto x^T Q^T E x$  is not a Lyapunov function for the system. Thus, no stability can be concluded.

It will be shown below, that an additional assumption which guarantees the stability of pH-DAEs is  $\ker Q^T E \subseteq \ker E$ . Together with trivial inclusion  $\ker E \subseteq \ker Q^T E$  the above inclusion is equivalent to

$$\ker Q^T E = \ker E. \quad (15)$$

Below we describe the row minimal indices of pencils  $sE - Q$  under the additional assumption (15). For more detailed results on the Kronecker structure without this assumption we refer to [MMW18, Sec. 3].

**Lemma 4.3.** *Let  $sE - Q \in \mathbb{R}[s]^{n \times n}$  be a matrix pencil with  $Q^T E = E^T Q$  and (15). Then the row minimal indices are at most zero (if there are any).*

*Proof.* Assume that  $sE - Q$  has a row minimal index greater than one. Then there exist  $x, y, z_1, z_2 \in \mathbb{R}^n$ ,  $z_1 \neq 0$ , such that

$$(E^T x, Q^T x) = (z_2, z_1), \quad (E^T y, Q^T y) = (z_1, 0). \quad (16)$$

This implies

$$\begin{aligned} (QE^T x, QQ^T x) &= (Qz_2, Qz_1), \\ (QE^T y, QQ^T y) &= (Qz_1, 0). \end{aligned}$$

Hence  $y \in \ker Q^T$ . Thus,  $QE^T = EQ^T$  implies that  $Qz_1 = QE^T y = 0$ . Hence, using (16) we find  $z_1 = Q^T x = 0$  which is a contradiction.  $\square$

**Proposition 4.4.** *Given a pH-DAE (13) with  $\ker Q^T E = \ker E$ . Then the following holds:*

- (a) *If  $sE - Q$  is regular then  $Q$  is invertible.*
- (b) *Let  $Q$  be invertible. Then the pH-DAE (13) is stable if and only if  $\ker D \cap (Q \ker E) = \{0\}$ . this may not always occur.*

*Proof.* If  $sE - Q$  is regular then  $\ker E \cap \ker Q = \{0\}$ . Further,

$$\ker E = \ker Q^T E = \ker E^T Q \supset \ker Q.$$

This together implies  $\ker Q = \{0\}$  and hence  $Q$  is invertible. This proves (a).

Let  $Q$  be invertible. To characterize the stability of (13), we use Proposition 4.1 (c). If (13) is stable then  $sE - DQ$  is regular. Hence,

$$(Q \ker E) \cap \ker D = \ker EQ^{-1} \cap \ker D = \{0\}.$$

Conversely, if  $\ker D \cap (Q \ker E) = \{0\}$  then Corollary 5.1 in [GHR21] yields the regularity of  $sE - DQ$ . As a consequence of Lemma 4.3,  $sE - Q$  has row minimal indices at most zero. Therefore, [MMW18, Theorem 4.3] implies that  $\sigma(E, A) \subseteq \overline{\mathbb{C}_-}$  and the eigenvalues on the imaginary axis except for 0 are semi-simple. To deduce the stability of (13) from Proposition 4.1 (c), it remains to show that zero is semi-simple. Assume the converse. Then after examining the Kronecker form (4), there exist  $x, y \in \mathbb{R}^n$  and  $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$  such that

$$(Ex, Ax) = (z_2, z_1), \quad (Ey, Ay) = (z_1, 0).$$

This implies

$$\begin{aligned} (Q^T Ex, Q^T Ax) &= (Q^T z_2, Q^T z_1), \\ (Q^T Ey, Q^T Ay) &= (Q^T z_1, 0). \end{aligned} \quad (17)$$

Since  $sQ^T E - Q^T A = sQ^T E - Q^T DQ$  is positive real the eigenvalue 0 is semi-simple for this pencil [AV73, Sec. 2.7]. This together with (17) implies either  $Q^T z_1 = 0$  or  $Q^T z_2 = 0$ . In the first case,  $Q^T E y = 0$  which implies  $E y = 0$  and hence  $z_1 = 0$ . In the second case, i.e.,  $Q^T z_2 = 0$ , we must have  $Q^T z_1 = 0$ , because otherwise there exists a column minimal index exceeding one. Thus,  $z_1 = 0$  which is a contradiction. Hence, using Proposition 4.1 (c), the DAE (13) is stable.  $\square$

Below we provide a sufficient condition for a regular DAE to be stable with index at most two directly in terms of the coefficients.

**Proposition 4.5.** *Let  $[E, A] \in \Sigma_n$  such that  $sE - A \in \mathbb{R}[s]^{n \times n}$  is regular. Then  $[E, A]$  is stable with index at most one if*

$$A^T E + E^T A \leq 0. \quad (18)$$

Further, if

only sufficient

$$A^T E + E^T A \leq_{\text{ran } E} 0, \quad (19)$$

then  $[E, A]$  is stable with index at most two.

*Proof.* The condition (18) implies that  $\sigma(E, A) \subset \overline{\mathbb{C}_-}$ . Furthermore, we have for all  $\lambda \in \mathbb{C}_+$  from (18) the estimate

$$\begin{aligned} 0 &\geq \text{Re}(Ax)^T E x \\ &= \text{Re}(Ax)^T E x - \text{Re } \lambda (Ex)^T E x + \text{Re } \lambda (Ex)^T E x \\ &= \text{Re}((A - \lambda E)x)^T E x + \text{Re } \lambda \|Ex\|^2. \end{aligned}$$

Replacing  $y = (A - \lambda E)^{-1}x$  and estimating with Cauchy-Bunjakowski and  $\|x\| = 1$  yields for all  $\lambda \in \mathbb{C}_+$

$$\|E(A - \lambda E)^{-1}\| \leq \frac{1}{\text{Re } \lambda}. \quad (20)$$

Hence, all poles on the imaginary axis have order at most one. Thus, by Proposition 4.1 (c),  $[E, A]$  is stable. Furthermore, considering (20) as  $\lambda \rightarrow \infty$  shows that no blocks with  $\alpha_i \geq 2$  in (4) are present. Hence, the index is at most one. If (19) then one can show analogously that

$$\|E(A - \lambda E)^{-1}x\| \leq \frac{\|x\|}{\text{Re } \lambda} \quad \text{for all } x \in \text{ran } E. \quad (21)$$

This implies that the pole order at eigenvalues on the imaginary axis is at most one. Furthermore,  $\text{ran } E = \text{ran } E(A - \mu E)^{-1}$  for some  $\mu \in \mathbb{C} \setminus \sigma(E, A)$  with  $\lambda \neq \mu$  then (21) can be reformulated using the resolvent identity for regular matrix pencils

$$(\lambda - \mu)(A - \mu E)^{-1}E(A - \lambda E)^{-1} = (A - \lambda E)^{-1} - (A - \mu E)^{-1}$$

and  $x = E(A - \mu E)^{-1}y$  as follows

$$\begin{aligned} &\frac{\|E(A - \mu E)^{-1}y\|}{\text{Re } \lambda} \\ &\geq \|E(A - \lambda E)^{-1}E(A - \mu E)^{-1}y\| \\ &= \|E(\lambda - \mu)^{-1}((A - \lambda E)^{-1} - (A - \mu E)^{-1})y\| \end{aligned}$$

This implies

$$\begin{aligned} & \|E(A - \lambda E)^{-1}y\| \\ & \leq |\lambda - \mu| \frac{\|E(A - \mu E)^{-1}y\|}{\operatorname{Re} \lambda} + \|E(A - \mu E)^{-1}y\| \end{aligned}$$

for all  $\lambda \in \mathbb{C}_+$  and all  $y \in \mathbb{R}^n$ . Hence, no blocks in the Kronecker form (4) with  $\alpha_i \geq 3$  are present, implying that the index of  $[E, A]$  is at most two.  $\square$

The inequalities (18) and (19) are closely related to the Lyapunov inequality (14) given in Proposition 4.1 (b). If (18) holds then (14) holds with  $X = I_n$ . However the assumption (18) cannot be used to obtain bounds on the index of the DAE.

Finally, we would like to remark that similar considerations can be done analogously for systems which are stable in reversed time, i.e.,

$$\forall x \in \mathfrak{B}_{[E, A]}^\infty \exists M > 0 : \sup_{t \leq 0} \|x(t)\| \leq M. \quad (22)$$

Since  $[E, A]$  fulfills (22) if and only if  $[-E, A]$  is stable, a characterization similar to Proposition 4.1 is straightforward. As a consequence every DAE which as semi-simple eigenvalues on the imaginary axis can be rewritten on the system space  $\mathcal{V}_{\text{sys}}^{[E, A]}$  in the form (13) but with  $Q^T E = E^T Q$  instead of  $Q^T E \geq 0$ .

## 5 Geometric formulation of port-Hamiltonian systems

In this section we show how systems of the form (1) fit into the geometric formulation of generalized pH-systems introduced in [vdSM18].

In comparison to (1), the equations are given here implicitly. To this end, let  $\mathcal{L}$  and  $\mathcal{D}$  be subspaces of  $\mathbb{R}^n \times \mathbb{R}^n$  with *range representations*  $\mathcal{L} = \operatorname{ran} \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$  and  $\mathcal{D} = \operatorname{ran} \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$  for some  $L_1, L_2, D_1, D_2 \in \mathbb{R}^{n \times n}$ . Then  $\mathcal{L}$  is *Lagrangian* and  $\mathcal{D}$  is *maximally dissipative* if

$$L_1^T L_2 - L_2^T L_1 = 0, \quad \dim \mathcal{L} = n, \quad \text{and} \quad (23)$$

$$D_2^T D_1 + D_1^T D_2 \leq 0, \quad \dim \mathcal{D} = n. \quad (24)$$

These  $\mathcal{D}$  and  $\mathcal{L}$  can be used to implicitly define a DAE by demanding that the system trajectories described by  $z, e : \mathbb{R} \rightarrow \mathbb{R}^n$  satisfy

$$(e(t), -\frac{d}{dt}z(t)) \in \mathcal{D}, \quad (z(t), e(t)) \in \mathcal{L}. \quad (25)$$

These systems were called generalized pH in [vdSM18]. Therein,  $\mathcal{D}$  was assumed to be a so-called *Dirac structure*, i.e., (24) holds with equality. Here, for simplicity, we include the port and the resistive variables already in the state.

The DAE is then explicitly given by the range representation of the following linear relation

$$\begin{aligned} \mathcal{DL} &:= \{(x, z) \mid \exists e \in \mathbb{R}^n : (x, e) \in \mathcal{L}, (e, z) \in \mathcal{D}\} \\ &= \operatorname{ran} \begin{bmatrix} E \\ A \end{bmatrix} \end{aligned}$$

for some  $E, A \in \mathbb{R}^{n \times n}$ . Moreover, the functions which fulfill  $\frac{d}{dt}Ex(t) = Ax(t)$  and (25) with  $z(t) = Ex(t)$  coincide, see [GHR21, Section 4].

To show that the system equations (1) can be rewritten as (25), we define

$$\mathcal{D} := \operatorname{gr} D = \operatorname{gr} \begin{bmatrix} J-R & B-P \\ (B+P)^T & S-N \end{bmatrix}, \quad \mathcal{L} := \operatorname{ran} \begin{bmatrix} E & 0 \\ 0 & I_k \\ Q & 0 \\ 0 & I_k \end{bmatrix}.$$

It follows from (2) that  $D$  is dissipative and, hence,  $\mathcal{D}$  fulfills (24). Moreover,  $\mathcal{L}$  is Lagrangian if and only if  $Q^T E = E^T Q$  and  $sE - Q$  is regular, see [GHR21, Corollary 5.1].

The system (1) is then implicitly given by

$$(z(t), u(t), -\frac{d}{dt}z(t), -y(t)) \in \mathcal{DL} = \text{ran} \begin{bmatrix} \hat{E} \\ \hat{A} \end{bmatrix},$$

with  $\hat{E} := \begin{bmatrix} E & 0 \\ 0 & I_k \end{bmatrix}$ ,  $\hat{A} := D \begin{bmatrix} Q & 0 \\ 0 & I_k \end{bmatrix}$ .

Under the additional assumptions  $\ker Q^T E = \ker E$  and  $Q^T E \geq 0$ , Proposition 4.4 implies that  $Q$  is invertible that the underlying DAE  $\frac{d}{dt}Ex(t) = (J - R)Qx(t)$  for  $u = 0$  is stable.

A more general characterization of the eigenvalues of the index of port-Hamiltonian DAEs which are given by maximally dissipative subspaces that are not necessarily graphs of dissipative matrices can be found in [GHR21, Sec. 6].

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