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Brief Paper

Robust pole placement for second-order linear systems using velocity-plus-acceleration feedback

Taha Helmy S. Abdelaziz[1,2]

[1] Department of Mechanical Engineering, Faculty of Engineering, Helwan University, 1 Sherif Street, 11792 Helwan, Cairo, Egypt
[2] Department of Industrial Engineering, Faculty of Engineering, Northern Border University, Arar, KSA
E-mail: tahahelmy@helwan.edu.eg; tahahelmy@yahoo.com

Abstract: This paper considers the robust pole assignment problem using combined velocity and acceleration feedbacks for matrix second-order linear systems. The necessary and sufficient conditions which ensure solvability are derived. Based on recently developed parametric solution to the eigenstructure assignment problem by velocity-plus-acceleration feedback, a new technique is described to perform robust pole placement for second-order systems. The available degrees-of-freedom offered by the velocity-plus-acceleration feedback in selecting the associated eigenvectors are utilised to improve robustness of the closed-loop system. The main advantage of the described approach is that the problem is tackled in the second-order form without transformation into the first-order form. Finally, several examples are introduced to demonstrate the effectiveness of the proposed approach.

1 Introduction

Many dynamical systems can be described by matrix second-order, time-invariant, differential equations of the form

$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Cu(t)$$
$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0 \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$ is the generalised coordinate vector and $u(t) \in \mathbb{R}^r$ is the vector of applied forcing. The matrices M, D, $K \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times r}$ are, respectively, the mass, damping, stiffness and control influence matrices. In practical applications, the matrix M is symmetric positive definite, $M > 0$. This system arises in many engineering applications, including the control of mechanical multi-body systems, robotics control, earthquake engineering, electrical circuit simulation and vibration control in structural dynamics. Consequently, control design for second-order systems has attracted much attention in the last three decades; see for instance [1–23] and the references therein.

Concerning the control of matrix second-order linear systems, it is customary to use combined displacements and velocities measurements in order to achieve a desired closed-loop system behaviour. Consequently, several techniques have been proposed for the design of control laws for second-order systems using the proportional-plus-derivative feedback, $u(t) = -F_d x(t) - F_v \dot{x}(t)$; see [4–19]. There have been various approaches concerning pole placement [5–7], robust pole placement [8–11], eigenstructure assignment [12–14], robust eigenstructure assignment [4], robust partial pole placement [15–17] and optimal control [18, 19] for matrix second-order linear systems. However, the displacements, especially for flexible structures, are not as easily and accurately obtainable as accelerations, which have been commonly obtained through the use of accelerometers. From the viewpoint of measurement, using accelerometer as sensor is favourable and more reliable than using position sensor to measure the dynamic response of structures. It is well-known that, direct measurement of absolute accelerations is an inexpensive and reliable method of measuring. Acceleration is often easier to measure than displacement or velocity, particularly when the structure is stiff [22]. Consequently, the acceleration feedback has been recognised by previous investigators to improve several system performances; see for instance [24–29]. There have been many published papers, for example, [23, 30–32], describing acceleration feedback for controlled vibration of structures. From measured accelerations, it is possible to reconstruct velocities from integration with reasonable accuracy. As a result, the available signals for feedback are accelerations and velocities. So, the velocity and acceleration variables can be utilised and new techniques for controlling such practical systems should be developed.

Recently, the derivative feedback methodology that utilises combined acceleration and velocity variables of first-order systems has attracted growing worldwide interest; see [23, 33–39]. The researchers have been recognised the importance of using derivative feedback on diverse practical engineering fields, such as vibration suppression in mechanical systems and flexible structures where the accelerometer is the only sensor. Hence, the velocity-plus-acceleration feedback control is interesting and useful for linear systems from the viewpoint of both mathematics and applications.

Consequently, in this paper, we discuss the robust pole placement problem for second-order systems using velocity-plus-acceleration feedback

$$u(t) = -F_v\dot{x}(t) - F_a\ddot{x}(t) \tag{1.2}$$

where $F_v \in \mathbb{R}^{r\times n}$ and $F_a \in \mathbb{R}^{r\times n}$ are, respectively, velocity and acceleration feedback gain matrices. To the best of our knowledge, there has been little research deal with this kind of feedback in the literature [1, 20, 21]. Consequently, the closed-loop system can be obtained

$$(M + CF_a)\ddot{x}(t) + (D + CF_v)\dot{x}(t) + Kx(t) = 0 \tag{1.3}$$

By robust pole placement, we mean to seek matrices F_v and F_a so that the closed-loop eigenvalues are as insensitive as possible to parameter perturbation in the closed-loop system matrices $(M + CF_a)$, $(D + CF_v)$ and K. The problem of maintaining the stability of second-order system by proportional-plus-derivative feedback subjected to parameter perturbations has been an active area of research; see [8–11]. Furthermore, the robust pole assignment problem for first-order systems have been well-studied in literature; see [37, 40–43]. The parametric pole assignment for continuous-time case and discrete-time case is presented in [44–46].

This work is based on a recent article by Abdelaziz [1] where the author investigates the eigenstructure assignment problem for matrix second-order linear systems using velocity-plus-acceleration feedback. It is well known that the eigenstructure assignment provides the whole set of admissible closed-loop eigenvectors. Consequently, the robust pole placement is closely related to the eigenstructure assignment. More specifically, the closed-loop eigenvalue sensitivity measures, which are essential for robust pole placement design, are determined by the closed-loop eigenvectors. So, the previous eigenstructure assignment work is utilised to solve the robust pole placement problem. We want to emphasise that the robust solution to this problem is important since there often exist parameter variations or perturbations in the practical second-order models. Therefore it is desirable that the feedback controller not only assigns specified eigenvalues to the second-order closed-loop system but also that the system is robust or insensitive to perturbations.

The main contribution in this paper is to introduce a design procedure that solves the robust pole placement problem for matrix second-order linear systems using combined velocity and acceleration variables. The explicit necessary and sufficient conditions that ensure solvability for the proposed problem are introduced. The parametric expressions for the feedback gain controllers and the eigenvector matrices are presented which describe the available degrees-of-freedom offered by the velocity-plus-acceleration feedback in selecting the associated eigenvectors from an admissible class. These freedoms are utilised to improve robustness of the closed-loop system. Based on these parametric expressions, a composite performance index is proposed to utilise the available degrees-of-freedom to obtain the robust solution. Consequently, the robust pole placement problem is considered and an effective method that finds the robust gain controller is obtained. Finally, illustrative examples are given to show that our results are effective.

2 Problem statement

In this section, the robust pole placement problem for matrix second-order systems using velocity and acceleration variables is formulated.

Since matrix M is assumed invertible, then pre-multiplying (1.1) by M^{-1}, the second-order system can be rewritten as

$$\ddot{x}(t) - D_1\dot{x}(t) - K_1x(t) = C_1u(t) \tag{2.1}$$

where $D_1 \triangleq -M^{-1}D$, $K_1 \triangleq -M^{-1}K$ and $C_1 \triangleq M^{-1}C$. So, a monic quadratic polynomial pencil corresponding to system (2.1) is $P_m(\lambda) \triangleq \lambda^2 I_n - \lambda D_1 - K_1$, where I_n is the $n\times n$ identity matrix. The zeros of $\det(P_m(\lambda))$ are known as the characteristic frequencies of the system and play an important role in system stability.

For analysis and design purposes, the second-order system (1.1) can be transformed to the familiar first-order model as

$$\dot{z}(t) = Az(t) + Bu(t) \tag{2.2}$$

$$u(t) = -F\dot{z}(t), \quad F \triangleq [F_v \quad F_a] \in \mathbb{R}^{r\times 2n} \tag{2.3}$$

where

$$z(t) \triangleq \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} \in \mathbb{R}^{2n}, \quad A \triangleq \begin{pmatrix} 0_{n,n} & I_n \\ K_1 & D_1 \end{pmatrix} \in \mathbb{R}^{2n\times 2n},$$

$$B \triangleq \begin{pmatrix} 0_{n,r} \\ C_1 \end{pmatrix} \in \mathbb{R}^{2n\times r}$$

Remark 2.1: In the majority of methods that have been proposed to design controllers for second-order models, system (1.1) is usually rearranged into first-order form (2.2). Hence, several techniques for treating the feedback design problem using state-derivative feedback (2.3) can be applied; see [23, 33–39]. However, for large systems the resulting model suffers from increased dimension. As a result, computational efficiency is lost [3]. This concern favours tackling the problem in second-order form directly.

Substituting the feedback law (1.2) into system (2.1), the closed-loop system can be obtained

$$(I_n + C_1F_a)\ddot{x}(t) - (D_1 - C_1F_v)\dot{x}(t) - K_1x(t) = 0 \tag{2.4}$$

Consequently, the closed-loop quadratic polynomial pencil is

$$P_c(\lambda) \triangleq \lambda^2(I_n + C_1F_a) - \lambda(D_1 - C_1F_v) - K_1 \tag{2.5}$$

The pencil $P_c(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$ has $2n$ eigenvalues and the closed-loop system (2.4) is called regular if and only if $\det(P_c(\lambda)) \neq 0$, $\forall\lambda \in \mathbb{C}$.

Let $\Gamma = \{\lambda_i \in \mathbb{C},\ i = 1, 2, \ldots, 2n\}$ be a desired, self-conjugate, set. Then, the corresponding right eigenvectors $v_i \in \mathbb{C}^n$ are satisfying

$$(\lambda_i^2(I_n + C_1F_a) - \lambda_i(D_1 - C_1F_v) - K_1)v_i = 0, v_i \neq 0, \quad i = 1, 2, \ldots, 2n. \tag{2.6}$$

Denote

$$V \triangleq [v_1, v_2, \ldots, v_{2n}] \in \mathbb{C}^{n\times 2n},$$
$$\Lambda \triangleq \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n}\} \in \mathbb{C}^{2n\times 2n} \tag{2.7}$$

where the columns of V comprise the right eigenvectors of the quadratic polynomial $P_c(\lambda)$ and Λ is in Jordan canonical

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form with the eigenvalues of $P_c(\lambda)$ on the diagonal. Then, there exist full rank matrix V that satisfy

$$(I_n + C_1F_a)V\Lambda^2 - (D_1 - C_1F_v)V\Lambda - K_1V = 0 \quad (2.8)$$

It is well known that the behaviour of closed-loop system (2.4) is governed by the eigenstructure of the corresponding pencil $P_c(\lambda)$. Therefore, if the system response needs to be altered by feedback, both eigenvalue assignment as well as eigenvector assignment should be considered. A desirable property is that the eigenvalues should be insensitive to perturbations in the system coefficient matrices. This criterion is used to restrict the degrees-of-freedom in the assignment problem and to produce a well-conditioned or robust solution to the problem.

We now formulate the robust pole placement problem as

Problem: Given second-order linear system (1.1) and the self-conjugate complex set Γ, find real velocity and acceleration gain controllers in the form of (1.2) such that the closed-loop system (1.3) is regular and the closed-loop eigenvalues λ_i, $i = 1, 2, \ldots, 2n$, are as insensitive as possible to parameter perturbations in the closed-loop system matrices $(M + CF_a)$, $(D + CF_v)$ and K.

The aim now is to develop a simple algorithm, which solves this problem, for second-order linear systems.

3 Necessary and sufficient conditions for solvability

In this section, we will provide the explicit necessary and sufficient conditions which ensure solvability for the proposed problem such that the closed-loop system (2.4) is regular in terms of the original system parameters and the desired eigenvalues.

Definition 3.1: The second-order linear system (2.1) is called regularisable by velocity-plus-acceleration feedback if there exists a controller (1.2) such that the resulted closed-loop system (2.4) is uniquely solvable for every consistent initial vectors x_0 and $\dot{x}_0$, and any given control input $u(t)$.

A second-order linear system (2.1) is regularisable by velocity-plus-acceleration feedback, if there exists a feedback $u(t)$ such that the associated quadratic matrix polynomial $P_c(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$ is regular, that is, the characteristic polynomial $\det(P_c(\lambda))$ is not identically zero, $\forall \lambda \in \mathbb{C}$. Without loss of generality, if the pencil $P_c(\lambda)$ is regular, then the roots of the characteristic polynomial $\det(P_c(\lambda))$ are $2n$ finite eigenvalues.

It is well known that the second-order linear system (2.1) is controllable if and only if the following condition is satisfied

$$\operatorname{rank}([\lambda^2 I_n - \lambda D_1 - K_1 \quad C_1]) = n \quad (3.1)$$

for every eigenvalue λ of the quadratic pencil $P_m(\lambda)$ [2].

Lemma 3.1: If the open-loop system (2.1) is controllable, then there exist the feedback gain matrices $F_a \in \mathbb{R}^{r\times n}$ and $F_v \in \mathbb{R}^{r\times n}$ so that the resulted closed-loop system (2.4) is also controllable.

Proof: For the resulted closed-loop system (2.4), we can obtain

$$\begin{aligned}&[\lambda^2(I_n + C_1F_a) - \lambda(D_1 - C_1F_v) - K_1 \quad C_1]\\ &\quad = [\lambda^2 I_n - \lambda D_1 - K_1 \quad C_1]\begin{pmatrix} I_n & 0\\ \lambda^2 F_a + \lambda F_v & I_r\end{pmatrix}, \quad \forall\lambda \in \mathbb{C}\end{aligned}$$

Then, one can deduce that

$$\begin{aligned}&\operatorname{rank}([\lambda^2(I_n + C_1F_a) - \lambda(D_1 - C_1F_v) - K_1 \quad C_1])\\ &\quad = \operatorname{rank}([\lambda^2 I_n - \lambda D_1 - K_1 \quad C_1]), \quad \forall\lambda \in \mathbb{C} \quad (3.2)\end{aligned}$$

Therefore the conclusion holds. □

Complete controllability is a necessary condition for the existence of gain matrices F_v and F_a such that the closed-loop pencil has a spectrum that can be assigned arbitrarily. To establish the solvability conditions for the above described control problem, we make the following assumptions on system (2.1).

Assumption 1: The second-order system (2.1) is controllable.

Assumption 2: The real pair (D_1, C_1) is controllable, that is, $\operatorname{rank}([D_1 - \lambda I_n \quad C_1]) = n, \quad \forall\lambda \in \mathbb{C}$ [1].

Assumption 3: $\operatorname{rank}(C) = r$.

Definition 3.2: The second-order linear system (2.1) is called normalisable by velocity-plus-acceleration feedback if a proposed controller (1.2) can be found such that the resulted closed-loop system (2.4) is normal, that is, it does not involve infinite eigenvalues.

In this work, we study the necessary and sufficient conditions in order that the system (2.1) can be normalised by means of a velocity-plus-acceleration feedback. The following lemma establishes the necessary condition to guarantee that the closed-loop system is normalisable.

Lemma 3.2: Consider second-order linear system (2.1) satisfying Assumptions 1–3, then the system is normalisable by means of a velocity-plus-acceleration feedback controller (1.2) if and only if the term $(I_n + C_1F_a)$ is non-singular.

Proof: The characteristic polynomial of degree $2n$ for the closed-loop system (2.4) can be expanded as

$$\det(P_c(\lambda)) = a_{2n}\lambda^{2n} + a_{2n-1}\lambda^{2n-1} + \cdots + a_1\lambda + a_0 \quad (3.3)$$

where a_i, $i = 0, 1, \ldots, 2n$, are the coefficients of characteristic polynomial and $a_{2n} = \det(I_n + C_1F_a)$. It is worth to observe that the pencil $P_c(\lambda)$ is regular and has $2n$ finite eigenvalues over the complex field if and only if the leading matrix coefficient $(I_n + C_1F_a)$ is non-singular. Otherwise, if the term $(I_n + C_1F_a)$ is singular, then $\det(P_c(\lambda)) = 0$ and the degree of $\det(P_c(\lambda))$ is $d < 2n$ ($P_c(\lambda)$ has $2n - d$ infinite eigenvalues). Therefore the acceleration gain F_a is restricted to ensure that $\det(I_n + C_1F_a) \neq 0$. This guarantees the existence and uniqueness of a solution for a design system. □

Remark 3.1: It is worth to note that if the term $(I_n + C_1F_a)$ is singular, then the pencil $P_c(\lambda)$ is said to have infinite eigenvalues which may be identified as the zero eigenvalues of the reverse or dual polynomial $\lambda^2 K_1 + \lambda(D_1 - C_1 F_v) - (I_n + C_1 F_a)$. Let n_f and n_∞ denote the finite eigenvalues counting algebraic multiplicities and the eigenvalue at infinity of algebraic multiplicity, respectively, then $n_f + n_\infty = 2n$.

As a matter of fact, the normalisability of closed-loop system (2.4) involves only the coefficient matrices C_1 and F_a. The constraint $\det(I_n + C_1 F_a) \neq 0$ is not strict. In the following, we will investigate the conditions guaranteeing that the closed-loop system (2.4) is normalisable.

Theorem 3.1: Consider second-order linear system (2.1) satisfying Assumptions 1–3, then the system is normalisable by velocity-plus-acceleration feedback (1.2) if there exists an acceleration gain F_a such that

$$\operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} = n + r \tag{3.4}$$

Proof: Obviously, the following expression

$$\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} = \begin{pmatrix} I_n & C_1 \\ 0_{r,n} & I_r \end{pmatrix} \begin{pmatrix} I_n + C_1 F_a & 0_{n,r} \\ 0_{r,n} & I_r \end{pmatrix} \begin{pmatrix} I_n & 0_{n,r} \\ -F_a & I_r \end{pmatrix}$$

shows that the rank of the matrix is

$$\operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} = r + \operatorname{rank}(I_n + C_1 F_a)$$

Hence,

$$\operatorname{rank}(I_n + C_1 F_a) = \operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} - r.$$

Consequently, the condition (3.4) is necessary to guarantee that the system is normalisable. This completes the proof. □

The following is an immediate consequence of Theorem 3.1.

Corollary 3.1: For single-input systems ($r = 1$), the closed-loop system (2.4) is normalisable if and only if the following condition is satisfied

$$F_a C_1 \neq -1 \tag{3.5}$$

Proof: First, it is easy to prove that

$$\operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} = \operatorname{rank}\left(\begin{pmatrix} I_n & 0_{n,r} \\ -F_a & I_r \end{pmatrix}\begin{pmatrix} I_n & 0_{n,r} \\ 0_{r,n} & I_r + F_a C_1 \end{pmatrix} \times \begin{pmatrix} I_n & C_1 \\ 0_{r,n} & I_r \end{pmatrix}\right) = n + \operatorname{rank}(I_r + F_a C_1)$$

Hence, one can deduce that

$$\operatorname{rank}(I_r + F_a C_1) = \operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} - n$$

Utilising (3.4), then the system is normalisable if the following condition

$$\operatorname{rank}(I_r + F_a C_1) = r \tag{3.6}$$

is satisfied. In particular, for single-input systems ($r = 1$), without loss of generality, $F_a C_1 \neq -1$ guarantee that the closed-loop system is normalisable. Thus, we complete the proof. □

The usefulness of this result is that it is easy to check the normalisabilty for single-input linear systems using a simple equation.

Lemma 3.3: For normalisable second-order linear system (2.4), if matrix K_1 is non-singular, then all the desired closed-loop eigenvalues λ_i, $i = 1, 2, \ldots, 2n$, should be non-zero.

Proof: Suppose that λ_i, $i = 1, 2, \ldots, 2n$, are the eigenvalues of the characteristic polynomial $\det(P_c(\lambda))$. When the pencil $P_c(\lambda)$ is normalisable, then the following relation holds

$$\prod_{i=1}^{2n} \lambda_i = \frac{a_0}{a_{2n}} \tag{3.7}$$

where the coefficients $a_{2n} = \det(I_n + C_1 F_a) \neq 0$ and $a_0 = \det(K_1)$. One can observe that if matrix K_1 is non-singular, then all the desired eigenvalues λ_i, $i = 1, 2, \ldots, 2n$, should be non-zero to ensure that $a_0 \neq 0$. Otherwise, when $\det(K_1) = 0$, then at least one desired eigenvalue should be equal zero ($a_0 = 0$). This means that all coefficients of $\det(P_c(\lambda))$, a_i, $i = 0, 1, \ldots, 2n$, can be arbitrarily selected by non-zero values with the exception when K_1 is singular ($a_0 = 0$). This completes the proof. □

It is well known that the characteristic polynomial $\det(P_c(\lambda))$ is Hurwitz stable if all its roots lie in the open left-half of the complex plane and non-zero. As a consequence of Lemma 3.3, we consider the case of non-singular stiffness matrices.

Assumption 4: $\operatorname{rank}(K) = n$.

Example 3.1: Consider a simple dynamic system with only two degrees-of-freedom ($n = 2$, $r = 1$)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ddot{x}(t) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\dot{x}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}x(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t)$$

This system satisfies Assumptions 1–4 and the eigenvalues of open-loop system are $\{1,\ 1,\ -1,\ -1\}$. Furthermore, the feedback controller can be defined as

$$u(t) = -F_v\dot{x}(t) - F_a\ddot{x}(t),\ F_v \triangleq [f_{v1} \quad f_{v2}],\ F_a \triangleq [f_{a1} \quad f_{a2}]$$

Consequently, the closed-loop quadratic polynomial pencil is computed as

$$P_c(\lambda) = \begin{pmatrix} \lambda^2 + 1 & \lambda \\ f_{a1}\lambda^2 + f_{v1}\lambda & (f_{a2} + 1)\lambda^2 + f_{v2}\lambda + 1 \end{pmatrix}$$

Moreover, the characteristic polynomial for the closed-loop system is

$$\begin{aligned} \det(P_c(\lambda)) &= (\lambda^2 + 1)((f_{a2} + 1)\lambda^2 + f_{v2}\lambda + 1) \\ &\quad - \lambda(f_{a1}\lambda^2 + f_{v1}\lambda) \\ &= (f_{a2} + 1)\lambda^4 + (f_{v2} - f_{a1})\lambda^3 \\ &\quad + (f_{a2} - f_{v1} + 2)\lambda^2 + f_{v2}\lambda + 1 \end{aligned}$$

Therefore it is easily verified that the closed-loop system is normalisable for all gains provided that the acceleration gain

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element $f_{a2} \neq -1$ (the degree of $\det(P_c(\lambda))$ is $2n$). Based on this choice, one can check that

$$F_aC_1 = f_{a2} \neq -1, \ \operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} = 3 \text{ and } \det(I_2 + C_1F_a) = \det\begin{pmatrix} 1 & 0 \\ f_{a1} & 1+f_{a2} \end{pmatrix} = 2$$

Obviously, the conditions in Theorem 3.1 and Corollary 3.1 are satisfied, therefore the closed-loop system is normalisable (the system has $2n$ finite eigenvalues).

Otherwise, if the acceleration gain element $f_{a2} = -1$, then

$$F_aC_1 = f_{a2} = -1, \ \operatorname{rank}\begin{pmatrix} I_n & C_1 \\ -F_a & I_r \end{pmatrix} < 3, \quad \det(I_2 + C_1F_a) = \det\begin{pmatrix} 1 & 0 \\ f_{a1} & 0 \end{pmatrix} = 0$$

and

$$\det(P_c(\lambda)) = (f_{v2} - f_{a1})\lambda^3 + (1 - f_{v1})\lambda^2 + f_{v2}\lambda + 1.$$

Based on this choice, the degree of $\det(P_c(\lambda))$ is $3 < 2n$ whenever the velocity gain element $f_{v2} \neq f_{a1}$ (the system has one infinite eigenvalue). In addition, suppose that the gain elements are given as: $f_{a2} = -1$, $f_{v2} = f_{a1}$ and $f_{v1} \neq 1$ then $\det(P_c(\lambda)) = (1 - f_{v1})\lambda^2 + f_{v2}\lambda + 1$ (degree of $\det(P_c(\lambda))$ is $2 < 2n$). Consequently, the system has two infinite eigenvalues.

4 Parametric expressions for the feedback gain controllers

In this section, the parametric expressions for gain controllers and closed-loop eigenvector matrices are presented in terms of the desired eigenvalues and the parameter vectors.

Denote,

$$T = -[F_v \quad F_a]\tilde{V}, \quad T \triangleq [t_1, t_2, \ldots, t_{2n}] \in \mathbb{C}^{r\times 2n} \tag{4.1}$$

and

$$\tilde{V} \triangleq \begin{pmatrix} V \\ V\Lambda \end{pmatrix}, \quad \tilde{V} \triangleq [\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{2n}] \in \mathbb{C}^{2n\times 2n} \tag{4.2}$$

It is straightforward to put relation (2.8) in the following form

$$V\Lambda^2 - D_1V\Lambda - K_1V = C_1T\Lambda \tag{4.3}$$

The above equation can be equivalently written as

$$\lambda_i^2 v_i - \lambda_i D_1 v_i - K_1 v_i = \lambda_i C_1 t_i, \quad i = 1, 2, \ldots, 2n \tag{4.4}$$

Since the real pair (D_1, C_1) is assumed controllable, then there exist two unimodular polynomial matrices $P(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$ and $Q(\lambda) \in \mathbb{R}^{(n+r)\times(n+r)}[\lambda]$ satisfying the following relation

$$P(\lambda)[D_1 - \lambda I_n \quad C_1]Q(\lambda) = [I_n \quad 0_{n,r}],$$
$$Q(\lambda) \triangleq \left(\begin{array}{c:c} Q_{11}(\lambda) & Q_{12}(\lambda) \\ \hdashline \multicolumn{2}{c}{Q_2(\lambda)} \end{array}\right), \quad \forall\lambda \in \mathbb{C} \tag{4.5}$$

where

$$Q_{11}(\lambda) \in \mathbb{R}^{n\times n}[\lambda], \quad Q_{12}(\lambda) \in \mathbb{R}^{n\times r}[\lambda] \text{ and } Q_2(\lambda) \in \mathbb{R}^{r\times(n+r)}[\lambda]$$

Furthermore, there exist two unimodular polynomial matrices $H(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$ and $L(\lambda) \in \mathbb{R}^{(n+r)\times(n+r)}[\lambda]$ satisfying the following relation

$$H(\lambda)[Q_{11}(\lambda)P(\lambda)K_1 + \lambda I_n \quad -Q_{12}(\lambda)]L(\lambda) = [I_n \quad 0_{n,r}], \quad \forall\lambda \in \mathbb{C} \quad [1] \tag{4.6}$$

Based on these results, the parametric solutions for (4.4) are presented in the following theorem.

Theorem 4.1: Consider the second-order linear system (1.1) satisfying Assumptions 1–4, then the parametric expressions for feedback gain controllers, right eigenvectors v_i and associated vectors t_i are expressed by

$$[F_v \quad F_a] = -T\tilde{V}^{-1} \tag{4.7}$$

$$v_i = [I_n \quad 0_{n,r}]L(\lambda_i)\begin{pmatrix} 0_n \\ g_i \end{pmatrix}, \quad i = 1, 2, \ldots, 2n \tag{4.8}$$

$$t_i = \frac{1}{\lambda_i}Q_2(\lambda_i)\begin{pmatrix} -P(\lambda_i)K_1 & 0_{n,r} \\ 0_{r,n} & I_r \end{pmatrix}L(\lambda_i)\begin{pmatrix} 0_n \\ g_i \end{pmatrix}, \quad i = 1, 2, \ldots, 2n \tag{4.9}$$

where $g_i \in \mathbb{C}^r$, $i = 1, 2, \ldots, 2n$, are parameter vectors satisfying the following two constraints

Constraint 1: $g_j = g_i^*$ whenever $\lambda_j = \lambda_i^*$, $\forall i, j$;

Constraint 2: $\det(\tilde{V}) \neq 0$.

Proof: See [1]. □

Remark 4.1: It is worth to note that the resulted closed-loop system is regular if $\det(P_c(\lambda))$ is not identically zero. Furthermore, the system (1.1) can be normalised by velocity-plus-acceleration feedback if $\det(I_n + C_1F_a) \neq 0$ or $\det(I_r + F_aC_1) \neq 0$ is satisfied. Thus, all the eigenvalues of the closed-loop system are finite.

In the following, an approach to obtain the feedback gain is presented if the desired eigenvalues are prescribed. One can find the constant matrices $\tilde{P}_i \in \mathbb{C}^{n\times n}$, $\tilde{Q}_i \in \mathbb{C}^{(n+r)\times(n+r)}$, $\tilde{H}_i \in \mathbb{C}^{n\times n}$ and $\tilde{L}_i \in \mathbb{C}^{(n+r)\times(n+r)}$ satisfying the following equations

$$\tilde{P}_i[D_1 - \lambda_i I_n \quad C_1]\tilde{Q}_i = [I_n \quad 0_{n,r}],$$
$$\tilde{Q}_i \triangleq \begin{pmatrix} Q_{i,11} & Q_{i,12} \\ \hline \multicolumn{2}{c}{Q_{i,2}} \end{pmatrix}, \quad Q_{i,11} \in \mathbb{C}^{n\times n}, \quad Q_{i,12} \in \mathbb{C}^{n\times r},$$
$$i = 1, 2, \ldots, 2n \tag{4.10}$$

and

$$\tilde{H}_i[Q_{i,11}\tilde{P}_i K_1 + \lambda_i I_n \quad -Q_{i,12}]\tilde{L}_i = [I_n \ \ 0_{n,r}], \quad i = 1, 2, \ldots, 2n \tag{4.11}$$

using the singular value decomposition (SVD) as

$$[D_1 - \lambda_i I_n \quad C_1] = X_i[\Sigma_i \quad 0_{n,r}]\tilde{Q}_i^*, \quad i = 1, 2, \ldots, 2n$$

and

$$[Q_{i,11}\Sigma_i^{-1}X_i^T K_1 + \lambda_i I_n \quad -Q_{i,12}] = Y_i[\Pi_i \quad 0_{n,r}]\tilde{L}_i^*,$$
$$i = 1, 2, \ldots, 2n$$

where X_i, $\tilde{Q}_i$, Y_i, $\tilde{L}_i$ are the orthogonal and Σ_i, $\Pi_i \in \mathbb{C}^{n\times n}$ are the non-singular diagonal matrices. Consequently, the feedback gain controller can be obtained using (4.7) and the complete parametric expressions for right eigenvectors v_i and associated vectors t_i are obtained analogous to (4.8) and (4.9) as

$$v_i = [I_n \quad 0_{n,r}]\tilde{L}_i \begin{pmatrix} 0_n \\ g_i \end{pmatrix}, \quad i = 1, 2, \ldots, 2n \tag{4.12}$$

$$t_i = \begin{pmatrix} 0_{r,n} & \frac{1}{\lambda_i} I_r \end{pmatrix} \tilde{Q}_i \begin{pmatrix} -\tilde{P}_i^{-1}K_1 & 0_{n,r} \\ 0_{r,n} & I_r \end{pmatrix} \tilde{L}_i \begin{pmatrix} 0_n \\ g_i \end{pmatrix}, \quad i = 1, 2, \ldots, 2n \tag{4.13}$$

Remark 4.2: For single-input system ($r = 1$), the vectors g_i, $i = 1, 2, \ldots, 2n$, are reduced to scalars and accordingly the feedback gain is unique regardless of the choice of g_i. In multi-input system ($r > 1$), there are extra degrees-of-freedom in the design that can be specified so as to optimise a measure of robustness of the system. These freedoms in the gain matrices are reflected precisely by the degrees-of-freedom available for assigning the eigenvectors. These freedoms can be used for robustness enhancement because of parameter perturbations.

5 Solution to the robust pole placement problem

In many practical problems, there often exist parameter variations or perturbations in the design system. Additionally, the presence of uncertainty in the model severely affects the performance and stability of the closed-loop system. Consequently, the robust problem is considered and an effective method that finds a robust controller can be obtained. The robust pole placement methods try to utilise the freedom over the closed-loop eigenvectors to arrange them such that the closed-loop system becomes insensitive to parameter variations. It is shown that if the eigenvectors of a system are assigned to match exactly a set of mutually orthogonal vectors, then the corresponding eigenvalues will have the minimum sensitivity to the perturbations and parameter variations. To utilise the available freedom for multi-input systems to obtain an optimal solution is next discussed. The performance index can be defined in many different ways depending on the chosen robustness measure. In the following, three robustness measures are discussed and an algorithm is suggested to design the robust controller.

For an insensitive solution, the eigenvectors are chosen to be as mutually orthogonal as possible. The spectral condition number of the eigenvector matrix $\tilde{V}$ is the most widely accepted measure of eigenvalue sensitivity. A measure of the sensitivity of a solution is the conditioning of eigenvector matrix given by $\kappa_2(\tilde{V}) \triangleq \|\tilde{V}\|_2\|\tilde{V}^{-1}\|_2$, where $\|.\|_2$ denotes the Euclidean norm. Another possible measure is the Frobenius condition number $\kappa_F(\tilde{V}) \triangleq \|\tilde{V}\|_F\|\tilde{V}^{-1}\|_F$, where $\|.\|_F$ denotes the Frobenius norm.

Consequently, the performance function can be expressed by

$$J_1 = \left\| \begin{pmatrix} V \\ V\Lambda \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} V \\ V\Lambda \end{pmatrix}^{-1} \right\|_2 \tag{5.1}$$

Another minimisation problem can be considered that is based on the minimisation of the norm of gain controller

$$J_2 = \|F_v\|_2 + \|F_a\|_2 \tag{5.2}$$

This representing the amount of energy required for the corresponding control action. A reduction of gain controllers $\|F_v\|_2$ and $\|F_a\|_2$ results in the average magnitude of control being reduced; therefore it is an attractive design objective. This is important because large gain matrix entries will result in proportionally greater inputs being applied to the system by the controller.

The notion of robustness to the quadratic eigenstructure assignment problem has been introduced in [4]. The next lemma presents an explicit condition number of the simple, finite, non-zero eigenvalue λ_i.

Lemma 5.1: Let λ_i be a simple eigenvalue of the closed-loop quadratic polynomial $P_c(\lambda)$ with corresponding right eigenvector v_i and left eigenvector w_i. Then, the explicit condition number $c(\lambda_i)$ of the eigenvalues λ_i is given by

$$c(\lambda_i) = \frac{\sqrt{|\lambda_i|^4 + |\lambda_i|^2 + 1}\,\|w_i^*(I_n + C_1F_a)\|_2\|v_i\|_2}{|w_i^*(2\lambda_i(I_n + C_1F_a) - (D_1 - C_1F_v))v_i|},$$
$$i = 1, 2, \ldots, 2n \tag{5.3}$$

For a non-zero eigenvalue, a measure of the relative sensitivity is given by the condition number $\kappa(\lambda_i)$ defined as

$$\kappa(\lambda_i) = \frac{c(\lambda_i)}{|\lambda_i|} = \frac{\sqrt{|\lambda_i|^4 + |\lambda_i|^2 + 1}\,\|w_i^*(I_n + C_1F_a)\|_2\|v_i\|_2}{|\lambda_i||w_i^*(2\lambda_i(I_n + C_1F_a) - (D_1 - C_1F_v))v_i|},$$
$$i = 1, 2, \ldots, 2n. \tag{5.4}$$

Proof: See [4]. □

The condition number $c(\lambda_i)$ given by (5.3) measures the sensitivity of the eigenvalue λ_i to perturbations in the closed-loop quadratic polynomial pencil $P_c(\lambda)$ in an absolute sense.

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The closed-loop left eigenvectors $w_i \in \mathbb{C}^n$ are satisfying

$$w_i^*(\lambda_i^2(I_n + C_1F_a) - \lambda_i(D_1 - C_1F_v) - K_1) = 0, \quad w_i \neq 0, \quad i = 1, 2, \ldots, 2n$$

Therefore the left eigenvector matrix W of the quadratic polynomial $P_c(\lambda)$ can be constructed as

$$W \triangleq [w_1, w_2, \ldots, w_{2n}] \in \mathbb{C}^{n\times 2n}$$

It is worth to note that the matrix W can be obtained by Nichols and Kautsky [4]

$$W^* = \begin{pmatrix} V \\ V\Lambda \end{pmatrix}^{-1} \begin{pmatrix} 0_{n,n} \\ I_n \end{pmatrix} (I_n + C_1F_a)^{-1} \tag{5.5}$$

As mentioned above, the matrices $\tilde{V}$ and $(I_n + C_1F_a)$ are assumed non-singular.

Remark 5.1: It is well-known that the problem is ill-conditioned when its condition number is large. The condition number of a problem is necessarily related to the distance from ill-posed instances because the inverse of the condition number is zero on ill-posed instances.

Based on Lemma 5.1, the following performance index can be considered

$$J_3 = \sum_{i=1}^{2n} \omega_i^2 c(\lambda_i)^2 \tag{5.6}$$

where the eigenvalues λ_i are assumed to be non-defective and the positive weights ω_i, $i = 1, 2, \ldots, 2n$, are satisfying $\sum_{i=1}^{2n} \omega_i^2 = 1$ with $\omega_i = \omega_j$ if $\lambda_i = \lambda_j$, $\forall i, j$ [4].

The three performance indices are quite different, so the trade-off between J_1, J_2 and J_3 can be made by forming the composite performance index as

$$J = \varphi \left\| \begin{pmatrix} V \\ V\Lambda \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} V \\ V\Lambda \end{pmatrix}^{-1} \right\|_2 + \zeta \|F_v\|_2 + \gamma \|F_a\|_2 + \sum_{i=1}^{2n} \omega_i^2 c(\lambda_i)^2 \tag{5.7}$$

where φ, ζ and γ are positive scalars representing the weighting factors on $\kappa_2(\tilde{V})$, $\|F_v\|_2$ and $\|F_a\|_2$, respectively. One can observe that the gain controllers F_v and F_a are calculated using the matrices $\tilde{V}$ and T. Moreover, matrix $\tilde{V}$ is determined by the parameter vectors g_i, $i = 1, 2, \ldots, 2n$. Consequently, the robust pole placement problem can be explicitly expressed by the following optimisation problem

$$\min_{g_i} J$$

Subject to: matrix $\tilde{V}$ is non-singular

the term $(I_n + C_1F_a)$ is non-singular

the parameter vector $g_j = g_i^*$ whenever $\lambda_j = \lambda_i^*, \quad \forall i, j$

Our aim is to minimise the composite performance index J. Consequently, a straightforward procedure for solving the robust pole placement problem can be summarised in the following algorithm.

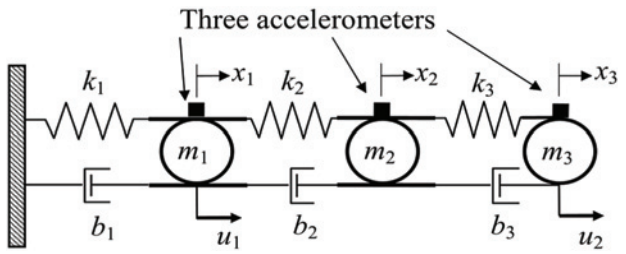

Fig. 1 *Three mass-spring-dashpots system*

5.1 Algorithm

Input: Given the second-order linear system (1.1) satisfying Assumptions 1–4 and a non-zero, self-conjugate, set Γ.

Step 1: Choose the initial values of the parameter vectors g_{i0}, $\forall i$, satisfying Constraints 1–2 and the positive weights ω_i, $\forall i$, satisfying $\sum_{i=1}^{2n} \omega_i^2 = 1$ with $\omega_i = \omega_j$ if $\lambda_i = \lambda_j$.
Step 2: Construct the polynomial matrices $P(\lambda)$, $Q(\lambda)$, $H(\lambda)$ and $L(\lambda)$ satisfying (4.5) and (4.6) or compute the constant matrices $\tilde{P}_i$, $\tilde{Q}_i$, $\tilde{H}_i$ and $\tilde{L}_i$, $\forall i$, satisfying (4.10) and (4.11) using the SVD.
Step 3: Compute the optimal parameter vectors g_i, $\forall i$, which minimise the composite performance index J and satisfy Constraints 1–2.
Step 4: Calculate the right eigenvectors v_i and the associated vectors t_i, $\forall i$, using (4.8) and (4.9) or using (4.12) and (4.13) based on the optimal parameter vectors g_i determined in step 3. Moreover, construct the matrices V, T and $\tilde{V}$.
Step 5: Compute the real gain controller using (4.7).

Remark 5.2: It should be emphasised that there are more than one local minimum of the composite performance index J. The choice of the initial values of parameter vectors, g_{i0}, plays an important role for obtaining the robust solution. Thus, several initial parameters should be considered.

6 Illustrative examples

In this section, we present two numerical examples to illustrate the feasibility and effectiveness of the proposed technique.

Example 6.1: Consider the three degrees-of-freedom mechanical system shown in Fig. 1 [1]. The vibration of this system is governed by a second-order equation

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \ddot{x}(t) + \begin{pmatrix} b_1 + b_2 & -b_2 & 0 \\ -b_2 & b_2 + b_3 & -b_3 \\ 0 & -b_3 & b_3 \end{pmatrix} \dot{x}(t) + \begin{pmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix} x(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u(t)$$

where $x(t) \triangleq [x_1, x_2, x_3]^{\mathrm{T}}$ and $u(t) \triangleq [u_1, u_2]^{\mathrm{T}}$. For this system, the accelerations of three masses can be measured directly using three accelerometers $\ddot{x}(t) \triangleq [\ddot{x}_1, \ddot{x}_2, \ddot{x}_3]^{\mathrm{T}}$ and the velocity components can then be obtained from integration $\dot{x}(t) \triangleq [\dot{x}_1, \dot{x}_2, \dot{x}_3]^{\mathrm{T}}$. When the model parameters are taken as: $k_1 = k_2 = 5\,\mathrm{N/m}$, $k_3 = 20\,\mathrm{N/m}$, $b_1 = b_3 = 2\,\mathrm{N\,s/m}$, $b_2 = 0.5\,\mathrm{N\,s/m}$, $m_1 = m_2 = m_3 = 1\,\mathrm{kg}$. Then, the spectrum for open-loop pencil is located at $\{-2.1629 \pm 6.1939\mathrm{i},\ -1.1859 \pm 3.0278\mathrm{i},\ -0.1512 \pm 1.0372\mathrm{i}\}$. In this simulation, the desired eigenvalues are selected as $\Gamma = \{-1, -2, -3, -4, -5, -6\}$. Thus, the Assumptions 1–4 are satisfied.

Below, numerical simulations are carried out to verify the proposed control methods presented in this paper for controlling the model.

6.1 Non-robust solutions

In this simulation, the parameter vectors can be chosen as

$$g_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$g_4 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 5 \\ 4 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$$

Solution 1: This solution is based on the elementary transformations of polynomial matrices. First, the polynomial matrices $P(\lambda)$, $Q(\lambda)$, $H(\lambda)$ and $L(\lambda)$ satisfying (4.5) and (4.6) can be obtained as

$$P(\lambda) = I_3$$

$$Q(\lambda) = \left(\begin{array}{ccc|cc} 0 & 2 & 0 & \lambda + 2.5 & -2 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \\ \hline 1 & 2\lambda + 5 & 0 & \lambda^2 + 5\lambda + 6 & -2\lambda - 5 \\ 0 & 0 & 1 & -1 & 0.5\lambda + 1 \end{array}\right)$$

$$H(\lambda) = \begin{pmatrix} -0.005 & 0.01(\lambda + 2.5) & -0.02 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L(\lambda) = \begin{pmatrix} 2\lambda - 15 & 0 & 0 & 2\lambda^2 + 5\lambda + 50 & -4 \\ 1 & 0 & 0 & \lambda + 10 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 2\lambda & 2 & 0 & 2\lambda(\lambda + 10) & 0 \\ 0 & 0 & 2 & 0 & 2\lambda \end{pmatrix}$$

If the parameter vectors are $g_i = \binom{\alpha_i}{\beta_i}$, $i = 1, 2, \ldots, 6$, then all the parametric solution for vectors v_i and t_i are obtained as (see equations at the bottom of the page)

Finally, the feedback gain controllers are

$$F_v = \begin{pmatrix} 20.7357 & -50.6779 & 83.9705 \\ -9.5739 & 21.5117 & -34.7679 \end{pmatrix}$$
$$F_a = \begin{pmatrix} 1.5046 & 14.3064 & 21.0463 \\ -1.0787 & -7.0744 & -9.7870 \end{pmatrix}$$

Note that, as discussed before, the obtained solution F_a is such that $\det(I_3 + C_1 F_a) \neq 0$ (it is equal to 0.6944). So, the system is normalisable by velocity-plus-acceleration feedback. The resulting errors of closed-loop pencil are obtained as: $10^{-10} * \{0.3967, -0.7198, 0.4230, -0.0942, 0.0040, 0.0002\}$. In addition, the system's response is presented in Fig. 2 using the following initial state conditions $x_0 = [-0.01, -0.02, 0.01]^T$ m, $\dot{x}_0 = [0.03, 0.02, 0.03]^T$ m/s.

Solution 2: The second solution is based on the SVD. After applying the SVD to matrices $[D_1 - \lambda_i I_n \;\; C_1]$ and $[Q_{i,11}\tilde{P}_i K_1 + \lambda_i I_n \;\; -Q_{i,12}]$, one can compute the vectors v_i and t_i using (4.12) and (4.13), respectively, for the same eigenvalues and parameter vectors g_i. Consequently, the feedback gain controllers are

$$F_v = \begin{pmatrix} 1.7924 & 10.6492 & -10.0903 \\ -0.6479 & -8.5106 & 10.9190 \end{pmatrix}$$
$$F_a = \begin{pmatrix} -0.2509 & 0.9311 & -1.9486 \\ -0.3299 & -0.5449 & 0.7852 \end{pmatrix}$$

Accordingly, the errors of closed-loop eigenvalues are $10^{-11} * \{-0.0180, 0.2047, -0.0605, -0.0096, 0.0014, 0.0009\}$. Since, $\mathrm{rank}(I_3 + C_1 F_a) = 3$, the system is normalisable.

6.2 Robust solutions

Here, the non-uniqueness of the eigenvector matrix is exploited to optimise the robustness of the closed-loop system. In this simulation, the 'fmincon.m' constrained minimisation function available in the optimisation toolbox of MATLAB has been employed to minimise the performance index J. In this case, the initial parameter vectors g_{i0} and the positive weights ω_i are taken, respectively, as

$$g_{10} = \begin{pmatrix} 0.5 \\ 4 \end{pmatrix}, \quad g_{20} = \begin{pmatrix} 0.5 \\ 8 \end{pmatrix}, \quad g_{30} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$$
$$g_{40} = \begin{pmatrix} -2 \\ 7 \end{pmatrix}, \quad g_{50} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \quad g_{60} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$
$$\omega_1 = 0.6856, \quad \omega_2 = 0.3000, \quad \omega_3 = 0.4690$$
$$\omega_4 = 0.3138, \quad \omega_5 = 0.3464, \quad \omega_6 = 0.0387$$

Solution 3: For the same model parameters, the optimal parameter vectors based on the minimisation of the condition number $\kappa_2(\tilde{V})$ ($\varphi = 1$, $\zeta = 0$ and $\gamma = 0$) are obtained as

$$g_1 = \begin{pmatrix} -1.1732 \\ 3.7819 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0.0187 \\ 8.2346 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 4.4081 \\ -1.7755 \end{pmatrix}$$
$$g_4 = \begin{pmatrix} -1.2606 \\ 6.8552 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 6.0564 \\ -2.3044 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 2.5535 \\ 14.5359 \end{pmatrix}$$

$$v_i = \begin{pmatrix} (2\lambda_i^2 + 5\lambda_i + 50)\alpha_i - 4\beta_i \\ (\lambda_i + 10)\alpha_i \\ \beta_i \end{pmatrix}, \quad \forall i$$

$$t_i = \begin{pmatrix} \left(2\lambda_i^3 + 10\lambda_i^2 + 82\lambda_i + 165 + \dfrac{450}{\lambda_i}\right)\alpha_i - \left(4\lambda_i + 10 + \dfrac{40}{\lambda_i}\right)\beta_i \\ -\left(2\lambda_i + 40 + \dfrac{200}{\lambda_i}\right)\alpha_i + \left(\lambda_i + 2 + \dfrac{20}{\lambda_i}\right)\beta_i \end{pmatrix}, \quad \forall i$$

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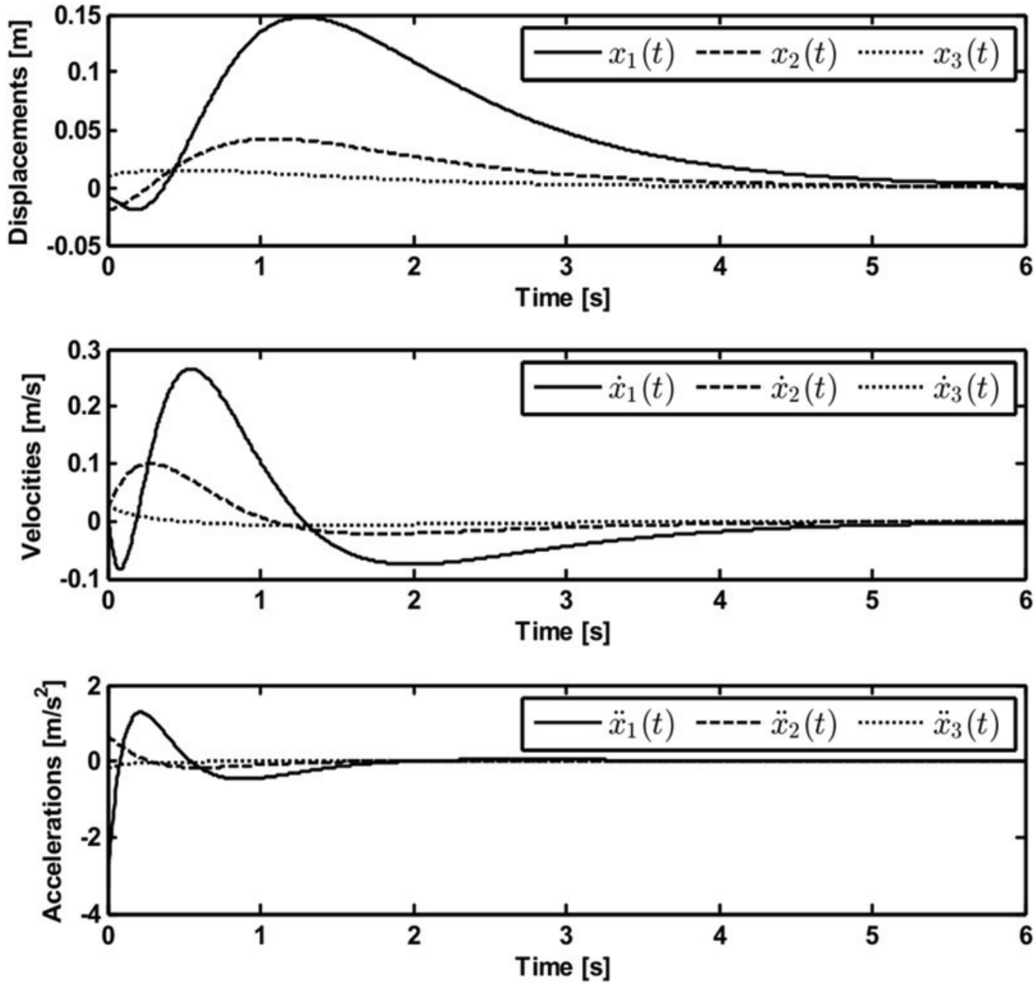

Fig. 2 *System's response for the simulation example using solution 1*

So, the robust feedback gain matrices are

$$F_v = \begin{pmatrix} 0.8042 & 5.3366 & -8.1166 \\ 1.0021 & -9.1475 & 12.6750 \end{pmatrix}$$

$$F_a = \begin{pmatrix} -0.5870 & -1.0734 & -1.0148 \\ 0.0539 & 0.6971 & 0.5491 \end{pmatrix}$$

The resulting errors of closed-loop pencil are $10^{-12} * \{0.0036, 0.0089, 0.0826, -0.0115, -0.1168, 0.0315\}$. Here, $\text{rank}(I_3 + C_1F_a) = 3$.

Solution 4: The optimal parameter vectors based on the minimisation of the composite performance index J ($\varphi = 0.5$, $\zeta = 0.1$ and $\gamma = 0.2$) are obtained as

$$g_1 = \begin{pmatrix} -0.4183 \\ 2.8285 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0.2798 \\ 6.5153 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 2.4920 \\ 2.3071 \end{pmatrix}$$

$$g_4 = \begin{pmatrix} -1.6360 \\ 5.3562 \end{pmatrix}, \quad g_5 = \begin{pmatrix} 6.4199 \\ -3.7415 \end{pmatrix}, \quad g_6 = \begin{pmatrix} 1.1172 \\ 11.7647 \end{pmatrix}$$

Therefore, the robust gain matrices are

$$F_v = \begin{pmatrix} 1.6421 & 2.0050 & -4.1468 \\ 0.7939 & -7.7604 & 10.8785 \end{pmatrix}$$

$$F_a = \begin{pmatrix} -0.4932 & -0.6301 & -0.5210 \\ 0.0431 & 0.4704 & 0.3260 \end{pmatrix}$$

So, $\text{rank}(I_3 + C_1F_a) = 3$ and the resulting errors in closed-loop eigenvalues are obtained as: $10^{-12}\{0.0187, 0.0631, 0.0862, -0.0115, -0.1021, 0.0204\}$. Furthermore, the simulation result of closed-loop system is displayed in Fig. 3 for the same initial state conditions x_0 and $\dot{x}_0$.

Finally, it is interesting to compare these results with previous results using the state-derivative feedback technique which was proposed in [37]. In this simulation, the system (1.1) is transformed to the first-order model as (2.2). For the same system parameters and the desired eigenvalues, the robust gain using the rank-one update method is obtained as (see equations at the bottom of the page)

$$[F_v \quad F_a] = \left(\begin{array}{ccc|ccc} -3.3135 & 28.0763 & -30.5475 & -0.9926 & -2.1949 & -3.9420 \\ 2.4554 & -18.2782 & 21.8381 & 0.1710 & 1.1511 & 1.7567 \end{array}\right)$$

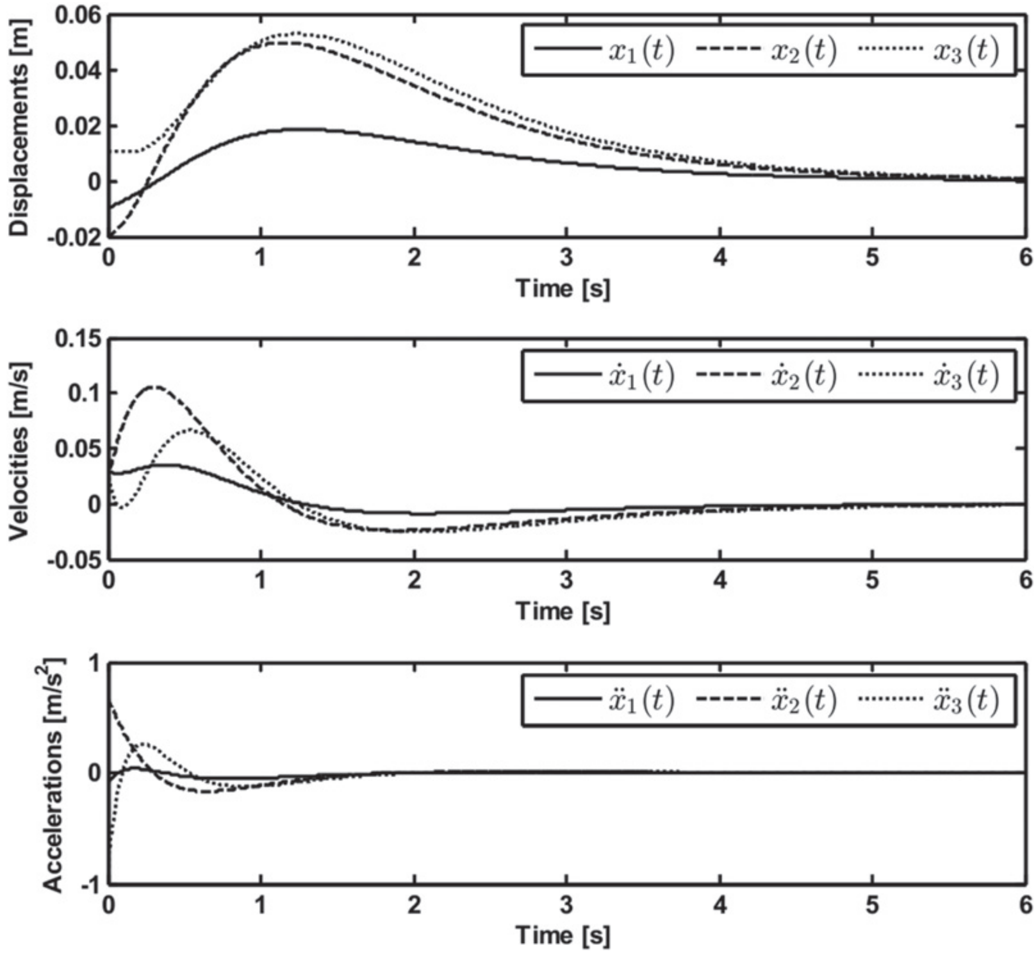

Fig. 3 *System's response for the simulation example using solution 4*

Table 1 Numerical results

Solutions		$\|\tilde{V}\|_2$	$\kappa_2(\tilde{V})$	$\kappa_F(\tilde{V})$	$\|F_v\|_2$	$\|F_a\|_2$
non-robust solutions	solution 1	3887.5514	8086.2541	9193.9596	108.6812	28.2254
	solution 2	4.1685	1922.0716	1995.9865	20.1904	2.3595
robust solutions	solution 3	4.8597	300.7670	342.0800	18.4050	1.8035
	solution 4	3.9579	313.7998	358.5913	14.1171	1.0901
	rank-one update method	1.9142	535.1414	685.2221	50.4800	5.0692
	rank-two update method	1.9145	533.8753	683.5019	50.7176	5.0963

Furthermore, the robust state-derivative feedback gain matrix using the rank-two update method is obtained as (see equations at the bottom of the page)

In conclusion, we can summarise some of the values related to the obtained solutions. Table 1 gives the following numerical results $\|\tilde{V}\|_2$, $\kappa_2(\tilde{V})$, $\kappa_F(\tilde{V})$, $\|F_v\|_2$ and $\|F_a\|_2$ for all solutions. Table 2 lists the closed-loop eigenvalue sensitivity measures $c(\lambda_i)$, $\forall i$, and $\sum_{i=1}^{2n}\omega_i^2 c(\lambda_i)^2$ for all solutions. It is clear that the proposed robust solutions here obtain better performance compared with the previous results proposed in [37].

6.3 Perturbed system

It is very important to study the sensitivity of the solution with respect to perturbations in the system data in order to analyse the validity of the computed results and the robustness of the proposed solutions. Perturbations can lead to severe degradation in performance and even instability. The open-loop system perturbations can be defined as ΔM, ΔD and ΔK. Consequently, the open-loop system matrices M, D and K can be perturbed to $(M + \Delta M)$, $(D + \Delta D)$ and $(K + \Delta K)$, respectively. Accordingly, the

$$[F_v \quad F_a] = \left(\begin{array}{ccc|ccc} -3.3549 & 28.2147 & -30.6951 & -0.9980 & -2.2102 & -3.9598 \\ 2.4786 & -18.3616 & 21.9244 & 0.1740 & 1.1590 & 1.7673 \end{array}\right)$$

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Table 2 Eigenvalue sensitivities of the closed-loop system

Solutions		$c(\lambda_1), c(\lambda_2), \ldots, c(\lambda_6)$	$\sum_{i=1}^{2n} \omega_i^2 c(\lambda_i)^2$
non-robust solutions	solution 1	$10^3 * \{0.0159, 0.0464, 0.5650, 2.235, 2.7600, 1.1047\}$	$1.4779 * 10^6$
	solution 2	3.0633, 19.5751, 40.0148, 129.8120, 250.6640, 169.8581	9634.1653
robust solutions	solution 3	4.1982, 17.5148, 4.7697, 47.9894, 12.7574, 49.1495	290.8947
	solution 4	4.3307, 19.0824, 9.6186, 43.7439, 14.3896, 48.5297	278.8037
	rank-one update method	2.9355, 9.0220, 38.9666, 72.4067, 40.6859, 63.4365	1066.5092
	rank-two update method	2.9437, 9.0594, 38.8619, 72.1952, 40.6669, 63.3502	1061.5858

Table 3 Eigenvalues of perturbed closed-loop system and norm of errors

Solutions		$\lambda_{1p}, \lambda_{2p}, \ldots, \lambda_{6p}$	Error norm
non-robust solutions	solution 1	$-1.0034, -2.1213, -2.3891, -4.2025 \pm 1.4585i, -7.2134$	9.2199
	solution 2	$-1.0021, -1.9908, -3.0380, -3.8022, -5.5604 \pm 0.4275i$	8.3332
robust solutions	solution 3	$-1.0023, -1.9865, -3.0081, -4.0354, -4.9768, -5.9630$	0.0584
	solution 4	$-1.0024, -1.9843, -3.0174, -4.0220, -4.9806, -5.9672$	0.0499
	rank-one update method	$-1.0018, -1.9989, -2.9573, -4.0948, -5.0168, -5.8894$	0.1527
	rank-two update method	$-1.0018, -1.9988, -2.9575, -4.0945, -5.0166, -5.8895$	0.1524

perturbed closed-loop linear system can be described by the following equation

$$(M + \Delta M + CF_a)\ddot{x}_p(t) + (D + \Delta D + CF_v)\dot{x}_p(t) + (K + \Delta K)x_p(t) = 0 \tag{6.1}$$

Consequently, the perturbed closed-loop quadratic polynomial pencil can be obtained as

$$P_{cp}(\lambda) \triangleq \lambda^2(M + \Delta M + CF_a) + \lambda(D + \Delta D + CF_v) + K + \Delta K \tag{6.2}$$

Therefore the perturbed closed-loop system is normalisable provided that the leading matrix coefficient $(M + \Delta M + CF_a)$ is non-singular. Consequently, the displacement, velocity and acceleration error trajectories are denoted, respectively, by

$$e(t) \triangleq x(t) - x_p(t), \quad \dot{e}(t) \triangleq \dot{x}(t) - \dot{x}_p(t), \quad \ddot{e}(t) \triangleq \ddot{x}(t) - \ddot{x}_p(t) \tag{6.3}$$

Suppose that the three masses are perturbed to 1.001 kg and all the other model parameters are not changed. Consequently, the mass matrix M is perturbed to $M_p \triangleq (M + \Delta M)$, where the perturbation ΔM is defined as

$$\Delta M \triangleq \mathrm{diag}(\Delta m_1, \Delta m_2, \Delta m_3) = 0.001 I_3$$

Here, the system matrices D, K and C are not perturbed. For all solutions, the corresponding eigenvalues of perturbed closed-loop quadratic polynomial pencil $P_{cp}(\lambda)$, $\{\lambda_{1p}, \lambda_{2p}, \ldots, \lambda_{6p}\}$, and the norm of errors in poles because of perturbation are illustrated in Table 3. One can observe that the eigenvalues of perturbed closed-loop system for robust solutions are considerably smaller compared with the non-robust one. This clearly indicated that minimising the condition number $c(\lambda)$ generally gives smaller deviation from the nominal pole positions. Figs. 4 and 5 describe the displacement, velocity and acceleration error trajectories for the non-robust solution #1 and the robust solution #4 with the same initial state conditions x_0 and $\dot{x}_0$. The results indicate that the errors for robust solutions are considerably smaller compared with the non-robust one.

Example 6.2: Consider the analysis of the oscillations of a wing in an air stream [10]. The system matrices M, D, K and C are given by

$$M = \begin{pmatrix} 17.600 & 1.280 & 2.890 \\ 1.280 & 0.824 & 0.413 \\ 2.890 & 0.413 & 0.725 \end{pmatrix}$$

$$D = \begin{pmatrix} 7.660 & 2.450 & 2.100 \\ 0.230 & 1.040 & 0.223 \\ 0.600 & 0.656 & 0.658 \end{pmatrix}$$

$$K = \begin{pmatrix} 121.000 & 18.900 & 15.900 \\ 0.000 & 2.700 & 0.145 \\ 11.900 & 3.640 & 15.500 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The system is unstable and the open-loop eigenvalues are located at

$$\{0.0947 \pm 2.5229i, -0.8848 \pm 8.4415i, -0.9180 \pm 1.7606i\}$$

In this simulation, the desired eigenvalues are selected as $\{-1 \pm i, \ -2 \pm 2i, \ -4 \pm 3i\}$. Thus, the Assumptions 1–4 are satisfied. Furthermore, the positive weights ω_i, $i = 1, 2, \ldots, 6$, are taken as

$$\omega_1 = 0.5099, \quad \omega_2 = 0.5477, \quad \omega_3 = 0.3742$$
$$\omega_4 = 0.4123, \quad \omega_5 = 0.2646, \quad \omega_6 = 0.2449$$

In the following, the non-robust and robust solutions are discussed and compared.

6.4 Non-robust solution

In this simulation, the parameter vectors can be chosen as

$$g_1 = g_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad g_3 = g_4 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad g_5 = g_6 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

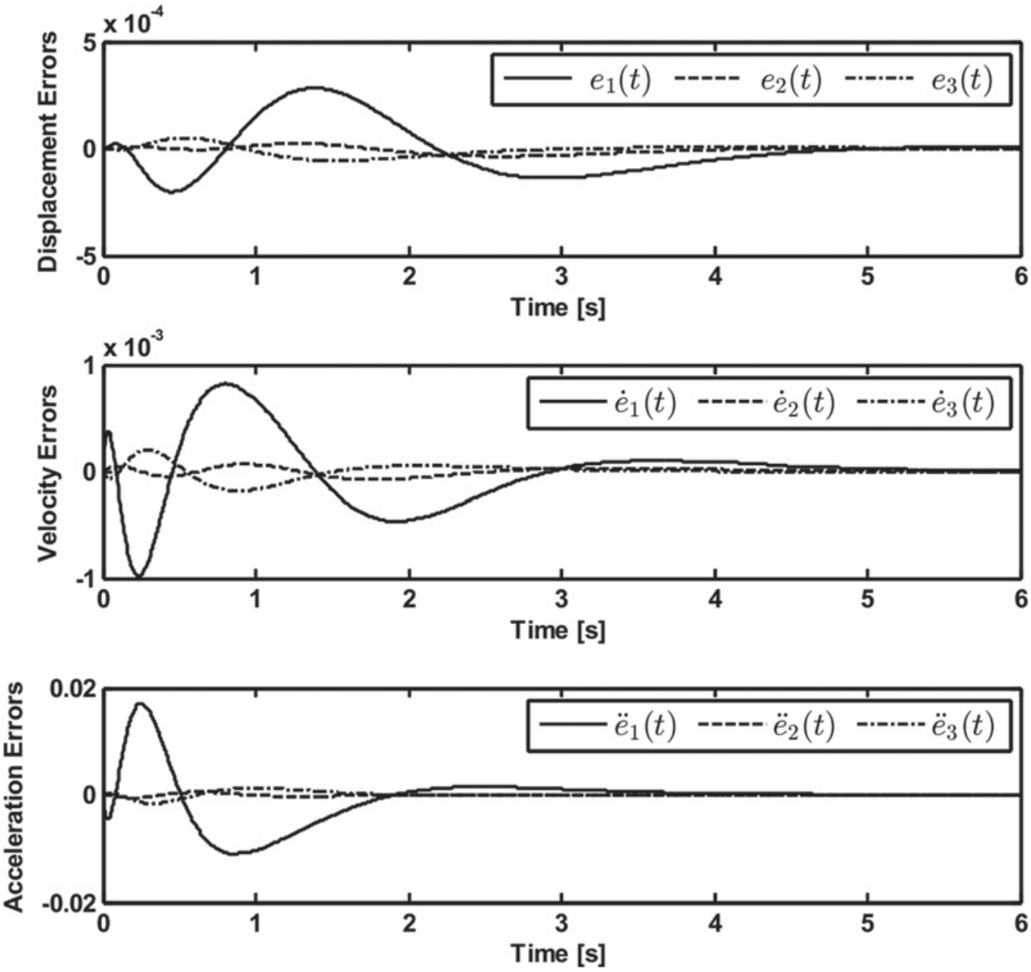

Fig. 4 *Displacement, velocity and acceleration error trajectories for perturbed closed-loop system using solution 1*

Clearly, Constraint 1 is satisfied. Based on the SVD, the computed feedback gain controllers are

$$F_v = \begin{pmatrix} 47.0317 & 0.0927 & 6.2289 \\ 96.8689 & 30.2359 & 27.0192 \end{pmatrix}$$

$$F_a = \begin{pmatrix} 9.4317 & 4.2167 & 2.3996 \\ 31.5233 & 15.4920 & 9.3189 \end{pmatrix}$$

For the feedback gain matrices obtained by the procedure, the computed parameters are $\|\tilde{V}\|_2 = 3.2000$, $\kappa_2(\tilde{V}) = 180.3047$, $\kappa_F(\tilde{V}) = 231.0399$, $\|F_v\|_2 = 114.4090$ and $\|F_a\|_2 = 37.8527$. Here, $\mathrm{rank}(I_3 + C_1F_a) = 3$, so the system is normalisable. Moreover, the errors of closed-loop eigenvalues are $10^{-11} * \{0.0046 \pm 0.0017i,\ 0.0128 \pm 0.0037i, -0.2331 \pm 0.0371\mathrm{i}\}$. In addition, the computed condition numbers $c(\lambda_i)$ of the assigned eigenvalues λ_i, $i = 1, 2, \ldots, 6$, are given by

$$c(\lambda_1) = c(\lambda_2) = 14.6676, \quad c(\lambda_3) = c(\lambda_4) = 25.8456$$
$$c(\lambda_5) = c(\lambda_6) = 171.7289$$

and

$$\sum_{i=1}^{6} \omega_i^2 c(\lambda_i)^2 = 4.1614 \times 10^3$$

6.5 Robust solution

In this simulation, the MATLAB function 'fmincon.m' has been employed to minimise the composite performance index J. Moreover, the initial parameter vectors g_{i0} are taken as the parameters for non-robust case. Consequently, the optimal parameter vectors based on the minimisation of the composite performance index J ($\varphi = 4$, $\zeta = 0.5$ and $\gamma = 0.2$) are obtained as

$$g_1 = g_2 = \begin{pmatrix} 15.1343 \\ 2.4528 \end{pmatrix}, \quad g_3 = g_4 = \begin{pmatrix} 14.0689 \\ 1.7856 \end{pmatrix}$$
$$g_5 = g_6 = \begin{pmatrix} -15.6225 \\ -3.5551 \end{pmatrix}$$

Therefore the robust gain matrices are

$$F_v = \begin{pmatrix} 100.6625 & 27.2543 & 23.4250 \\ 25.6788 & 11.1021 & 11.9984 \end{pmatrix},$$
$$F_a = \begin{pmatrix} 28.2127 & 16.8108 & 6.9149 \\ 11.2116 & 6.4616 & 3.6378 \end{pmatrix}$$

With these feedback gain matrices, the resulting errors in the closed-loop eigenvalues are obtained as: $10^{-13} * \{-0.0488 \pm 0.0133i,\ 0.1998 \pm 0.2043i,\ 0.2487 \pm 0.8971i\}$. In this case, $\det(I_3 + C_1F_a) = 4.5257 \neq 0$. Moreover, one can obtain that $\|\tilde{V}\|_2 = 1.4147$, $\kappa_2(\tilde{V}) = 16.8897$, $\kappa_F(\tilde{V}) = 32.7851$, $\|F_v\|_2 = 110.9355$ and $\|F_a\|_2 = 36.1433$. In addition, the

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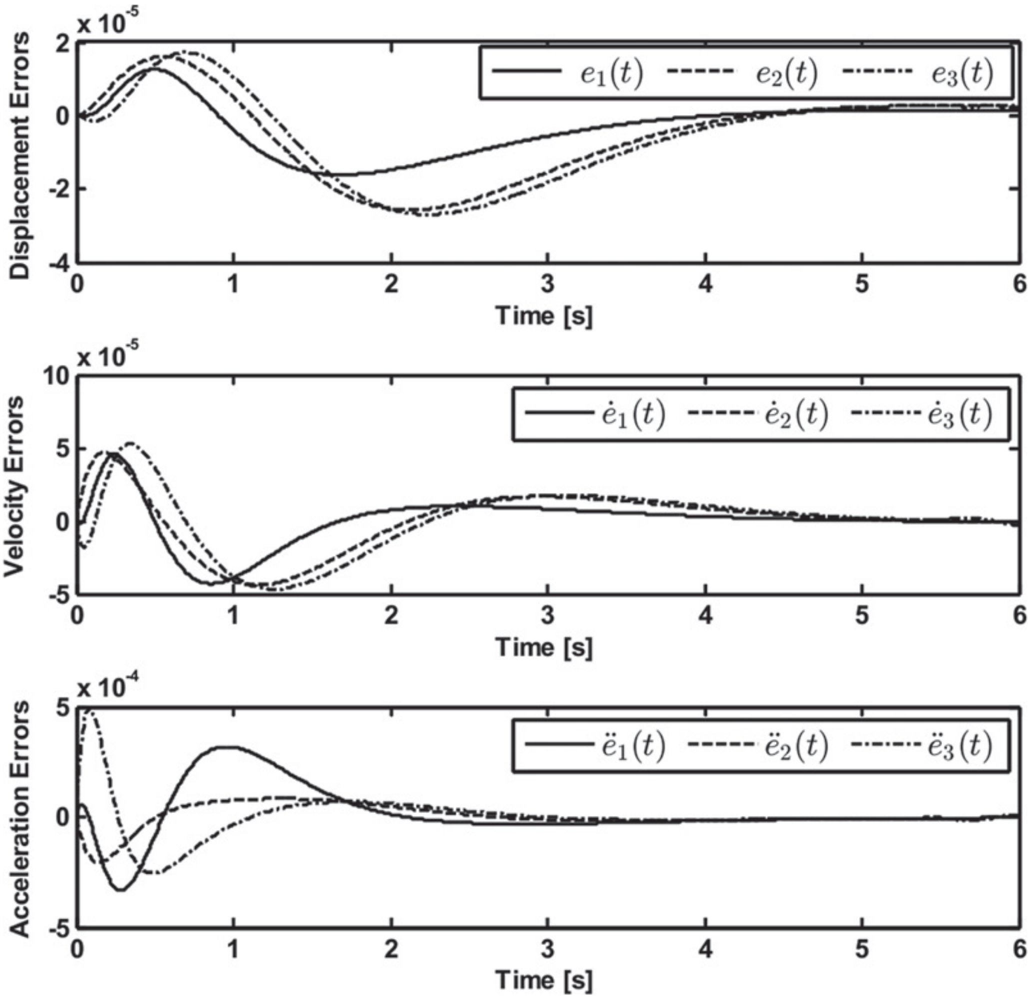

Fig. 5 *Displacement, velocity and acceleration error trajectories for perturbed closed-loop system using solution 4*

closed-loop eigenvalue sensitivity measures $c(\lambda_i)$, $i = 1, 2, \ldots, 6$, are computed as

$$c(\lambda_1) = c(\lambda_2) = 4.3717, \quad c(\lambda_3) = c(\lambda_4) = 9.4411$$
$$c(\lambda_5) = c(\lambda_6) = 14.9023$$

and

$$\sum_{i=1}^{6} \omega_i^2 c(\lambda_i)^2 = 67.2048$$

Finally, one can observe that the desired eigenvalues are more accurately assigned, the eigenvector matrix is more well-conditioned, the condition numbers of the assigned eigenvalues are considerably reduced and the components of the feedback gain matrices are significantly smaller in magnitude.

6.6 Perturbed system

In order to test the robustness of the two solutions, suppose that the system matrices M, D and K are perturbed and the perturbations ΔM, ΔD and ΔK are defined as

$$\Delta M = 0.01M, \quad \Delta D = 0.01D, \quad \Delta K = 0.01K$$

while matrix C is not changed. For the non-robust and robust solutions, the norms of errors in poles because of perturbation are computed, respectively, as 0.0732 and 0.0468. Thus, the robust solution possesses good numerical stability.

7 Conclusions

In this paper, a technique for solving the robust pole placement for matrix second-order linear systems using velocity-plus-acceleration feedback is elaborated such that the sensitivity of the spectrum of closed-loop system to perturbations in the system is minimised. The explicit necessary and sufficient conditions which ensure solvability for the proposed problem are derived. Based on the parametric expressions for the feedback gain controllers, the minimisation problem is formulated. The parametric solution describes the available degrees-of-freedom offered by the velocity-plus-acceleration feedback in selecting the associated eigenvectors from an admissible class. These freedoms are exploited to improve the closed-loop robustness against perturbations. The main result of this work is an efficient algorithm for obtaining the robust gain controllers for second-order systems.

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