International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

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Published online: 29 May 2012.

To cite this article: He Kong , Graham Goodwin & Maria Seron (2012) A revisit to inverse optimality of linear systems, International Journal of Control, 85:10, 1506-1514, DOI: 10.1080/00207179.2012.691180

To link to this article: http://dx.doi.org/10.1080/00207179.2012.691180

International Journal of Control
Vol. 85, No. 10, October 2012, 1506–1514

A revisit to inverse optimality of linear systems

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(*Received 19 December 2011; final version received 2 May 2012*)

In this article, we revisit the problem of inverse optimality for linear systems. By applying certain explicit formulae for coprime matrix fraction descriptions (CMFD) of linear systems, we propose a necessary and sufficient condition for a given stable state feedback law to be optimal for some quadratic performance index. Compared to existing results in the literature, the proposed condition is simpler to check and interpret. Moreover, it reduces the redundancy in the solutions of the associated algebraic Riccati equation (ARE). As a direct application of the proposed results, we consider the problem of inverse optimality of observer-based state feedback. To be specific, for the case where the state is not fully known, we consider the inverse optimality problem of an observer-based state feedback for the closed-loop system augmented by an observer. More precisely, it is shown that the observer-based state feedback is inverse optimal for the closed-loop system with some general forms of cost functions, only if the original state feedback is inverse optimal for the original system with certain cost functions, irrespective of the choice of the observer. This coincides with existing results in the literature. Some other applications of the proposed results are also discussed. We also illustrate the proposed results through an example.

Keywords: inverse optimality; coprime matrix fraction description; algebraic Riccati equation; observer

1. Introduction

This article considers the inverse optimal control problem (IOCP) for the following continuous linear time-invariant (LTI) system

$$\dot{x} = Ax + Bu \tag{1}$$

with a quadratic cost of the form

$$J = \int_0^\infty (x^{\mathrm{T}} Q x + u^{\mathrm{T}} u)\mathrm{d}t, \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ denotes the state and control vectors, respectively; A, B and Q are real matrices of appropriate dimensions with (A, B) controllable and Q symmetric non-negative definite ($Q \geq 0$). The IOCP is as follows: given a stable state feedback control law

$$u = Kx, \tag{3}$$

find a condition on A, B, K such that the control law (3) minimises the cost (2) for some (unknown) $Q \geq 0$ and determine all such $Q \geq 0$.

In the seminal work Kalman (1964), the above question was firstly raised and it was shown that, for single-input systems, the so-called circle criterion is a necessary and sufficient condition for this problem. In Anderson (1966) and Anderson and Moore (1971), the results in Kalman (1964) have been generalised to the multi-input case. In the latter work, separate necessary and sufficient conditions (but not a necessary and sufficient) were proposed. Results have also been presented in Kreindler and Jameson (1972), Molinari (1973) and Yokoyama and Kinnen (1972) covering special cases of the original IOCP, by making some simplifying assumptions (such as a cross-product term is allowed between the state and control, admissible control laws are restricted to stabilising ones or the non-negative definiteness requirement on the weighting matrix Q is removed). It should be noted that such assumptions make the revised inverse problem easier to solve. In Yokoyama and Kinnen (1972), the inverse problem of the optimal regulator was considered for a general class of multi-input systems with specified feedback control laws and a certain form of integral-type performance indices. Necessary and sufficient conditions were given based on a canonical form of the considered system. In Buelens and Hellinckx (1974), for LTI systems with single-input and quadratic performance index, a procedure was given for computing the weighting matrices in the performance index for which an existing state feedback control law is optimal. However, these results have not been extended to the multi-input case. In Casti (1980), necessary and sufficient conditions were developed for the general

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ISSN 0020–7179 print/ISSN 1366–5820 online

http://dx.doi.org/10.1080/00207179.2012.691180
http://www.tandfonline.com

inverse problem of optimal control from a dynamic programming point of view. It was also shown that the results for the linear quadratic case in Jameson and Kreindler (1973) are a special case of the general results in Casti (1980). In Fujii and Narazaki (1982, 1984), after a detailed discussion on the results in Kalman (1964), Anderson (1966), Anderson and Moore (1971), Kreindler and Jameson (1972), Molinari (1973) and Jameson and Kreindler (1973), a complete optimality condition, in terms of a geometric condition and the well-known return difference condition, for the standard IOCP was proposed, without making simplifying assumptions as in Kreindler and Jameson (1972), Molinari (1973) and Jameson and Kreindler (1973). While the result in Fujii and Narazaki (1982, 1984) provides a better insight into the sensitivity reduction property of optimal control, it is computationally expensive to apply in testing the optimality of a given state feedback law. In Sugimoto and Yamamoto (1987), a necessary and sufficient condition for optimality of a stable state feedback law was proposed based on the generalised Kalman equation in polynomial matrix form. Connections with the results in Fujii and Narazaki (1982, 1984) were also discussed in Sugimoto and Yamamoto (1987). In Fujii (1987), the linear quadratic regulator (LQR) of interest was designed without specifying a quadratic cost. It was proven that it minimises some quadratic cost; hence, it possesses some desirable properties such as reduced sensitivity and robust stability (see, e.g. Anderson and Moore 1971 and the references therein). There are also some results on the IOCP of discrete-time linear systems (see Wu and Schroeder 1973; Park and Lee 1975; Willems and van de Voorde 1973, 1978; Sugimoto and Yamamoto 1988; Mehdi, Darouach, and Zasadzinski 1994 and the references therein). Moreover, some of the results on inverse optimal control have been extended to optimal pole placement (see Haddad and Bernstein 1992; Hu, Lin, and Lam 2004; Wu and Lee 2005; Zhou, Li, Duan, and Wang 2009 and the references therein) and the inverse optimality of static output feedback (see, e.g. Zheng 1989; Gu 1990; Kučera and De Souza 1995; Gao, Liu, Sreeram, and Teo 2000 and the references therein). Besides, the IOCP for nonlinear systems has also attracted much attention in the literature (see Yokoyama and Kinnen 1972; Moylan and Anderson 1973; Freeman and Kokotović 1996; Krstić and Li 1998; Hamzi 2001 and the references therein).

The approach proposed in this article to the IOCP is based on the use of comprime matrix fraction descriptions (CMFD) of linear systems. For fundamentals and applications of CMFD in system analysis and design, one can refer to Kailath (1980), Goodwin, Graebe, and Salgado (2001), Duan (2010), Kong, Duan, and Zhou (2009) and Kong, Zhou, and Zhang (2010). Based on some explicit formulae for CMFD (Nett, Jacobson, and Balas 1984; Meyer and Franklin 1987), we propose a necessary and sufficient condition for a given stable state feedback law to be optimal for some quadratic performance index. What is more, all such quadratic performance indices (Q) are characterised in the proposed condition. Compared to the results in Fujii and Narazaki (1982, 1984) and Sugimoto and Yamamoto (1987), the proposed condition is simpler to check and interpret. Moreover, it reduces the redundancy in the solutions of the algebraic Riccati equation (ARE) while maintaining the advantages of the results in Sugimoto and Yamamoto (1987). Also note that we do not make the simplifying assumptions of Kreindler and Jameson (1972), Molinari (1973) and Jameson and Kreindler (1973).

The second problem of interest in this article is the IOCP incorporating an observer. It is known that the optimal control law for a linear system (1) with a cost function as in (2) is given by state feedback as in (3). However, in practice, the state information is usually not fully available for feedback. Thus it becomes necessary to estimate the state variables by constructing an observer (see, e.g. Luenberger 1971; O' Reilly 1983; Goodwin and Middleton 1989 and the references therein). The implementation using an observer clearly raises the question of how is the original performance index affected by such a treatment of the optimal law. Apart from that, since there is a certain degree of freedom in designing an observer, another question of interest and importance is which observer should be chosen according to the design freedom. There are some existing results on these two questions (see Bongiorno and Youla 1968, 1970; Anderson and Moore 1971; Arimoto and Hino 1973, 1974; Suda and Fujii 1981 and the references therein). As was discussed in Anderson and Moore (1971), the observer-based optimal state feedback control law minimises some quadratic performance for the closed-loop system under two assumptions namely that $Q > 0$ and that the observer is a state observer. It was stated in Bongiorno and Youla (1968, 1970) that the use of observers in regulator control systems results in an increase of the cost index and the cost increment depends on the choice of the observer. In Arimoto and Hino (1973, 1974), for a few limited cases, conditions for the performance deterioration to be made arbitrarily small were derived. In Suda and Fujii (1981), it was proven that the observer-based optimal state feedback law is an optimal control law for the closed-loop system only if the original state feedback law is a solution to the IOCP for the original system. Then conditions were given for the observer-based optimal control law to be inverse optimal for the closed-loop system, independently of the choice of the observer. The results for the second problem considered in this article is pertinent to the results in Suda and Fujii (1981). However, we will consider the problem based on CMFD. We will show that the inverse optimality of the original state feedback control law for the original system with a given cost function is a necessary condition for the inverse optimality of the observer-based state feedback for the closed-loop system with some general forms of cost function, independent of the choice of the observer. This conclusion reinforces the results in Suda and Fujii (1981) but is proven in a more direct way.

The remainder of this article is organised as follows. Section 2 gives the problem formulation and some preliminaries. Section 3 contains the main results of this article. Here the IOCP is solved by using CMFD. The IOCP incorporating an observer is then addressed. Section 5 concludes this article.

Notation: The notation used in this article is standard. We use A^{T} to denote the transpose of matrix A and $\mathbf{R}^n$ to denote the n-dimensional Euclidean space. $A(s)$ denotes a polynomial matrix with appropriate dimensions. The notation $M>0$ (≥ 0) means that M is real symmetric and positive definite (semi-definite). I stands for identity matrices of appropriate dimensions. For $M>0$ (≥ 0), $M^{1/2}$ denotes the matrix square root. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for the associated algebraic operations. λ (A) denotes all eigenvalues of a given matrix $A \in \mathbf{R}^{n\times n}$.

2. Problem formulation and preliminaries

Two polynomial matrix pairs $(N(s),\ D(s)) \in \mathbb{R}^{n\times m}[s] \times \mathbb{R}^{m\times m}[s]$ and $(H(s),\ L(s)) \in \mathbb{R}^{n\times n}[s] \times \mathbb{R}^{n\times m}[s]$ are respectively said to be a right coprime polynomial matrix pair and a left coprime polynomial pair if (see, e.g. Kailath 1980; Goodwin et al. 2001; Duan 2010)

$$\operatorname{rank}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = m, \quad \operatorname{rank}\begin{bmatrix} H(s) & L(s) \end{bmatrix} = n, \quad \forall s \in \mathbb{C}.$$

The first problem to be considered in this article is stated as follows.

Problem 2.1: Given a stable state feedback control law

$$u = Kx \tag{4}$$

for the linear system

$$\dot{x} = Ax + Bu \tag{5}$$

with a quadratic cost of the form

$$J = \int_0^\infty (x^{\mathrm{T}} H^{\mathrm{T}} H x + u^{\mathrm{T}} u)\mathrm{d}t, \tag{6}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ denote the state vector and control input, respectively; A, B and H are real matrices of appropriate dimensions with (A, B) controllable. Find a condition on A, B, K such that the control law (4) minimises the cost (6) with some (unknown) $H^{\mathrm{T}}H \geq 0$. Determine all such H.

It is well-known that the solution to the optimal control problem for the system (5) with cost (6) is given in the form of a state feedback law as in (4). In practice, this control law is often implemented in an observer form, because usually the only information about the system is the output measurement

$$y = Cx, \tag{7}$$

where $C \in \mathbb{R}^{m\times n}$ is a constant matrix. Assume that an observer of the following form is constructed for estimating the state feedback (4):

$$\begin{cases} \dot{z} = Fz + Gy + Ju \\ w = Ez + Dy \end{cases}, \tag{8}$$

where $z \in \mathbb{R}^p$ with w satisfying

$$\lim_{t\to\infty}(w - Kx) = 0, \tag{9}$$

independently of the initial conditions of the system (5), the observer (8) and the control input (4). We assume that (A, B, C) is both controllable and observable with $\operatorname{rank}(B) = r$ and $\operatorname{rank}(C) = m$. Then from Luenberger (1971), it is known that for the control input (4), there exists an observer of the form (8) satisfying (9) if and only if

$$\operatorname{Re}\lambda(F) < 0 \tag{10}$$

and there exists a matrix $T \in \mathbb{R}^{p\times n}$ such that

$$\begin{cases} TB = J \\ TA - FT = GC\,. \\ ET + DC = K \end{cases} \tag{11}$$

Note that there is a certain degree of freedom in designing an observer since the solutions to (10) and (11) are not unique. The closed-loop system consisting of the sub-systems (5), (7) and (8) can be written as

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ GC & F \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ J \end{bmatrix} u \\ \begin{bmatrix} y \\ w \end{bmatrix} = \begin{bmatrix} C & 0 \\ DC & E \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} \end{cases}. \tag{12}$$

Denote $e = Tx - z$, $y_{cl} = \begin{bmatrix} y \\ w \end{bmatrix}$ and $x_{cl} = \begin{bmatrix} x \\ e \end{bmatrix}$, then the closed-loop system (12) can be rewritten as

$$\begin{cases} \dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}u \\ y_{cl} = \begin{bmatrix} C_{cl} \\ K_{cl} \end{bmatrix} x_{cl} \end{cases}, \tag{13}$$

where

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \tag{14}$$

$$C_{cl} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad K_{cl} = \begin{bmatrix} K & -E \end{bmatrix}. \tag{15}$$

Then the observer-based input w can be represented as a state feedback for the system (13)

$$u = w = K_{cl}x_{cl.} \tag{16}$$

For the IOCP incorporating an observer, we consider system (13) with a quadratic cost function in the following form:

$$J_{cl} = \int_0^\infty (\|Hx + Le\|^2 + e^{\mathrm{T}}Se + u^{\mathrm{T}}u)\mathrm{d}t \tag{17}$$

where $S \geq 0$. We also assume that (A, H) is observable. We would like to find conditions that guarantee the optimality of the state feedback (16) for the system (13) with cost function (17). The second problem to be addressed then is stated as follows.

Problem 2.2: For optimality of state feedback (16) for system (13) with cost function (17), is it necessary for state feedback (4) to be optimal for system (5) with cost function (6)?

Remark 1: In Problems 2.1 and 2.2, it is without loss of generality to assume that the weighting matrix R for the control input is I. Other cases with $R \neq I$ can be treated similarly by replacing B with $BR^{-1/2}$, if R is known.

3. Main results

3.1 *Solutions to the IOCP*

We first present the following lemma which is part of the results of Nett et al. (1984).

Lemma 3.1: *Given a controllable matrix pair* (A, B). *Denote* $G(s) = (sI - A)^{-1}B$. *If* K *is chosen such that* $A + BK$ *has all its eigenvalues with negative real parts, then it holds that* $G(s)$ *can be expressed as*

$$G(s) = N(s)D^{-1}(s)$$

where

$$N(s) = (sI - A - BK)^{-1}B, \tag{18}$$

$$D(s) = K(sI - A - BK)^{-1}B + I, \tag{19}$$

i.e. $(N(s), D(s))$ *is a stable right CMFD for* (A, B).

Remark 1: Similar results can be obtained if (A, B) is only stabilisable. The results in the remainder of this article can also be obtained if we relax controllability and observability of certain matrix pairs to stabilisability and detectability, respectively (Anderson and Moore 1971).

Our solution to Problem 2.1 is given in the follow theorem.

Theorem 3.2: *Given a stable state feedback control law* (4) *for the linear system* (5) *with a quadratic cost* (6) *with* (A, B) *controllable and* (A, H) *observable, the control law* (4) *minimises the cost* (6) *if and only if*

$$N^{\mathrm{T}}(-s)H^{\mathrm{T}}HN(s) + D^{\mathrm{T}}(-s)D(s) = I, \tag{20}$$

where $N(s)$ *and* $D(s)$ *are given by* (18) *and* (19), *respectively.*

Proof: For the necessity part, one can refer to Theorem 1 in Meyer and Franklin (1987). We next establish the sufficiency. Since $A + BK$ is stable and $H^{\mathrm{T}}H + K^{\mathrm{T}}K \geq 0$, there exists a solution $P \geq 0$ to the following Lyapunov function (Kailath 1980):

$$(A + BK)^{\mathrm{T}}P + P(A + BK) = -H^{\mathrm{T}}H - K^{\mathrm{T}}K, \tag{21}$$

which can be rewritten as

$$P(sI - A - BK) + [-sI - (A + BK)^{\mathrm{T}}]P = H^{\mathrm{T}}H + K^{\mathrm{T}}K.$$

Pre- and postmultiplying the above equation by $N^{\mathrm{T}}(-s)$ and $N(s)$, respectively, we obtain

$$N^{\mathrm{T}}(-s)PB + B^{\mathrm{T}}PN(s) = N^{\mathrm{T}}(-s)(H^{\mathrm{T}}H + K^{\mathrm{T}}K)N(s).$$

The above equation, together with (20) yields

$$N^{\mathrm{T}}(-s)PB + B^{\mathrm{T}}PN(s) = N^{\mathrm{T}}(-s)K^{\mathrm{T}}KN(s) + I - D^{\mathrm{T}}(-s)D(s). \tag{22}$$

From (18)–(19), we have

$$D(s) = KN(s) + I.$$

Substituting the above equation into (22), we have

$$N^{\mathrm{T}}(-s)(PB + K^{\mathrm{T}}) + (B^{\mathrm{T}}P + K)N(s) = 0.$$

Denote

$$X(s) = (B^{\mathrm{T}}P + K)N(s),$$

then we obtain

$$X(s) + X^{\mathrm{T}}(-s) = 0.$$

Since $N(s) \neq 0$, then it is necessary for the above equation to hold, that

$$B^{\mathrm{T}}P + K = 0,$$

i.e.

$$K = -B^{\mathrm{T}}P. \tag{23}$$

If we substitute Equation (23) into Equation (21), we obtain

$$A^{\mathrm{T}}P + PA + H^{\mathrm{T}}H - PBB^{\mathrm{T}}P = 0, \tag{24}$$

which is in fact an ARE (Anderson and Moore 1971). Hence we conclude that there exists a solution $P \geq 0$ to the ARE (24) such that $K = -B^{\mathrm{T}}P$. Also note (A, H) is observable, then it can be concluded that the positive semi-definite solution to the ARE (24) is unique and the state feedback $K = -B^{\mathrm{T}}P$ is optimal (Kučera 1972). The proof is completed. □

Remark 2: A few comments on the result in Theorem 3.2 are presented below.

(i) Equation (20) is similar to the so-called generalised Kalman equation in polynomial matrix form that discussed in Sugimoto and Yamamoto (1987). In Sugimoto and Yamamoto (1987), a general right CMFD ($N(s)$, $D(s)$) for the system (5) with $D(s)$ row-reduced is applied to deduce the results. In Theorem 3.2, on the other hand, we utilise the explicit formulae for ($N(s)$, $D(s)$) in (18)–(19). By doing so, we simplify the results in Sugimoto and Yamamoto (1987), since the denominator polynomial matrix of the closed-loop system under state feedback (4) becomes I. Thus we do not have to perform symmetric extraction as in Sugimoto and Yamamoto (1987). And also connections with results in Fujii and Narazaki (1984) can be obtained (see Remark 3.2 in Sugimoto and Yamamoto 1987).

(ii) It is known that for a given stable state feedback (4), if it is inverse optimal for cost functions in the form of (6), there may exist several $H^{\mathrm{T}}H \geq 0$ for which the ARE (24) has a solution P satisfying $K = -B^{\mathrm{T}}P$, i.e. there is redundancy in the solutions of the ARE (see Kreindler and Hedrick 1970; Molinari 1971; Martin 1973; Majumdar 1984 and the references therein). However, in Theorem 3.2, all such H are characterised by Equation (20). Thus Theorem 3.2 maintains the advantages of the results in Sugimoto and Yamamoto (1987) while simplifying them. This proves to be useful for the development of the solutions to Problem 2.2 (see next subsection).

(iii) The result in Theorem 3.2 holds for the multi-input case, and can be readily extended to consider the problem of inverse optimality of stable static output feedback for linear systems.

(iv) The result in Theorem 3.2 (and also the results in the next section) can be extended to discrete-time linear systems (along the lines of Sugimoto and Yamamoto 1988; Meyer 1990; O' Reilly 1983).

(v) For discrete-time linear systems, there has been recent interest in the problem of tuning a model predictive control (MPC) controller such that it behaves as an existing linear controller when the constraints are not active (see Hartley and Maciejowski 2009; Di Cairano and Bemporad 2010; Foo and Weyer 2011; Kong, Goodwin, and Seron 2011; Kong, Goodwin, and Seron 2012a,b). Once this is achieved, the resulting MPC controller inherits the small-signal properties of the linear control design, when the constraints are not active, and still optimally deals with constraints during transients (see, e.g. Rossiter 2003; Goodwin, Seron, and Doná 2005; Wang 2009 and the references therein). The result in Theorem 3.2 can be used to determine whether an existing stabilising state feedback law is inverse optimal for linear systems with some infinite horizon cost function. If the answer to the former question is 'yes', i.e. the state feedback law is an LQR gain, then some standard techniques (see, e.g. Goodwin et al. 2005; Di Cairano and Bemporad 2010 and the references therein) can be applied to solve the MPC matching problem. Moreover, in Di Cairano and Bemporad (2010), the MPC tuning techniques presented for static state-feedback controllers have also been extended to dynamic output-feedback controllers, based on non-minimal state-space (NMSS) realisation of the corresponding process dynamics. When the state information is not fully available and if the existing state feedback controller is implemented in an observer form, it would be interesting to consider the MPC matching problem for the closed-loop system augmented by the observer-based state feedback. This is the subject of our current and future work.

(vi) In recent MPC literature, there has also been interest in proposing MPC schemes based on NMSS realisations of the process dynamics (see Wang and Young 2006; González, Perez, and Odloak 2009; Wang, Young, Gawthrop, and Taylor 2009 and the references therein). It has been shown in Wang and Young (2006), Wang et al. (2009), González et al. (2009) that, without the need of observers, the proposed MPC controllers may have better performance than those MPC controllers that make use of observers to estimate the current states. It should be noted that the proposed result in Theorem 1 could also be applied to NMSS representations, if the NMSS model is reachable (or stabilisable), which, in fact, is the baseline of the results in Wang and Young (2006), Wang et al. (2009) and González et al. (2009).

3.2 *Solutions to the IOCP incorporating observers*

We now apply the result in Theorem 3.2 to solve Problem 2.2. Clearly, (17) can be rewritten as

$$J_{cl} = \int_0^\infty (x_{cl}^{\mathrm{T}} Q_{cl} x_{cl} + u^{\mathrm{T}} u)\mathrm{d}t, \tag{25}$$

where x_{cl} is as in (13) and

$$Q_{cl} = \begin{bmatrix} H^{\mathrm{T}}H & H^{\mathrm{T}}L \\ L^{\mathrm{T}}H & L^{\mathrm{T}}L + S \end{bmatrix} \geq 0 \tag{26}$$

in which $S \geq 0$. The ARE associated with system (13) and cost (25) is as below

$$A_{cl}^{\mathrm{T}} P_{cl} + P_{cl} A_{cl} + Q_{cl} - P_{cl} B_{cl} B_{cl}^{\mathrm{T}} P_{cl} = 0, \tag{27}$$

where P_{cl} is the corresponding solution. Then we have the following solution to Problem (2.2).

Theorem 3.3: *Assume that* (A, B) *and* (A, H) *are controllable and observable, respectively. The state feedback control law* (16) *is optimal for the system* (13) *with quadratic cost* (25) *only if the state feedback law* (4) *is optimal for the linear system* (5) *with quadratic cost* (6), *independently of the choice of the observer.*

Proof: It can be easily verified that $\mathrm{Re}\,\lambda(A_{cl} + B_{cl}K_{cl}) < 0$ such that we have (A_{cl}, B_{cl}) is stabilisable. Denote $(N_{cl}(s), D_{cl}(s))$ a right CMFD of (A_{cl}, B_{cl}), $(sI - A_{cl})^{-1}B_{cl} = N_{cl}(s)D_{cl}^{-1}(s)$. Then from Lemma 3.1, we have

$$N_{cl}(s) = (sI - A_{cl} - B_{cl}K_{cl})^{-1}B_{cl}, \tag{28}$$

$$D_{cl}(s) = K_{cl}(sI - A_{cl} - B_{cl}K_{cl})^{-1}B_{cl} + I. \tag{29}$$

Substituting expressions of A_{cl}, B_{cl}, K_{cl} of (14)–(15) into (28)–(29) gives

$$N_{cl}(s) = \begin{bmatrix} N(s) \\ 0 \end{bmatrix}, \quad D_{cl}(s) = D(s), \tag{30}$$

where $N(s)$ and $D(s)$ are given as in (18)–(19). Moreover, (A_{cl}, Q_{cl}) is detectable as proved in Sugimoto and Yamamoto (1987), i.e. $(A_{cl}, Q_{cl}^{1/2})$ is also detectable (Anderson and Moore 1971). From Theorem 3.2, (16) is optimal for the system (13) with a quadratic cost (25) only if

$$N_{cl}^{\mathrm{T}}(-s)Q_{cl}N_{cl}(s) + D_{cl}^{\mathrm{T}}(-s)D_{cl}(s) = I.$$

If we substitute Equations (26) and (30) into the above equation, we obtain

$$N^{\mathrm{T}}(-s)H^{\mathrm{T}}HN(s) + D^{\mathrm{T}}(-s)D(s) = I.$$

From Theorem 3.2, (4) is optimal for the system (5) with quadratic cost (6). Since the above steps hold irrespective of the choice of the observer, this completes the proof. □

Remark 3: The result in Theorem 3.3 coincides with the corresponding result in Sugimoto and Yamamoto (1987), where the deduction is based on connections between the solutions to the AREs (24) and (27). We believe the proof given above is more concise and direct.

4. Illustrative example

We use a simple example to illustrate the proposed results. We consider the following harmonic oscillator

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

A stabilising controller (4) with

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

is designed by the pole assignment technique with closed-loop poles $\{-0.5000 \pm 0.8660\mathrm{i}\}$. We would like to know whether this stabilising feedback control law is optimal for some quadratic costs of the form (6) with some $H^{\mathrm{T}}H \geq 0$. And if it is, we desire to determine all such H. From Theorem 1, the above control law minimises a quadratic cost of the form (6) with some $H^{\mathrm{T}}H \geq 0$ if and only if

$$N^{\mathrm{T}}(-s)H^{\mathrm{T}}HN(s) + D^{\mathrm{T}}(-s)D(s) = 1, \tag{31}$$

where

$$N(s) = \frac{1}{s^2 + s + 1}\begin{bmatrix} 1 \\ s \end{bmatrix}, \quad D(s) = \frac{s^2 + 1}{s^2 + s + 1}.$$

Denote

$$H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}.$$

Substitute $N(s)$, $D(s)$ and H into Equation (31) and rearrange it in terms of s, we obtain

$$h_1^2 - 2h_1h_2s + (1 - h_2^2)s^2 = 0.$$

Clearly, the above equation is satisfied for all s if and only if $h_1 = 0$ and $h_2 = \pm 1$. It can be concluded that the considered stabilising feedback gain is optimal for some quadratic costs of the form (6) with $H = \begin{bmatrix} 0 & \pm 1 \end{bmatrix}^{\mathrm{T}}$. If we solve the ARE (24), it can be verified that $K = -B^{\mathrm{T}}P$. This further confirms the proposed results.

We next consider another stabilising controller (4) with

$$K = \begin{bmatrix} -0.5 & -0.25 \end{bmatrix}$$

which is also designed by the pole assignment technique with closed-loop poles $\{-0.1250 \pm 1.2183\mathrm{i}\}$. Following the above procedure, we obtain the following equation in terms of s, h_1 and h_2:

$$15(s^2 + 1)^2 + (15 + s^2)h_2^2 + 20 - h_1^2 = 0.$$

Obviously, there is no H satisfying the above equation. Therefore, this stabilising controller is not optimal for any quadratic cost of the form (6), i.e. it is not an LQR gain.

5. Conclusion

By applying certain explicit formulae for CMFD of linear systems, a new necessary and sufficient condition for the problem of inverse optimality for linear systems is given. The proposed condition is simpler to apply while maintaining the advantages of several existing results. Applications of the proposed results are also discussed.

Acknowledgements

The work of the authors was partially supported by University of Newcastle and also by the Australian Research Council through Discovery Projects. The authors would like to thank the anonymous reviewers for their helpful comments and suggestions which have helped us to improve the quality of this article.

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