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Stability analysis by Krasnoselskii's fixed point theorem for nonlinear fractional differential equations

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ABSTRACT

This paper is concerned with the stability analysis of nonlinear fractional differential equations of order $\alpha(1<\alpha<2)$. Our main results are obtained by using Krasnoselskii's fixed point theorem in a weighted Banach space. An example and its corresponding simulation are presented to illustrate the main results.

1. Introduction

Our aim in this paper is to study the asymptotical stability of the following nonlinear fractional differential equations (FDEs)

$$ {}^{C}D_{0+}^{\alpha}x(t)=f(t,x(t)),\quad t\geqslant 0, \tag{1.1}$$

$$ x(0)=x_0, x'(0)=x_1, \tag{1.2}$$

where $1<\alpha<2, x_0, x_1\in\mathbb{R}, \mathbb{R}^+=[0,\infty), f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is a continuous function and $f(t,0)\equiv 0, {}^{C}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative and we denote the solution of the system (1.1) and (1.2) by $x(t,x_0,x_1)$.

Spurred by the extensive applicability of FDEs, the subject of fractional calculus, emerged as new branch of applied mathematics, has been applied widely in a variety of mathematical models in science and engineering in the last three decades. The theory of FDEs have been extensively studied by many authors [1–4]. However, to the best of our knowledge, the investigation on stability theory of nonlinear FDEs is still in the initial stage and there is a great deal of work that needs to be done. More recently, several methods are introduced to study the stability of nonlinear FDEs. For example, the generalized Gronwall-Bellman inequality is employed in [5] to study the stability of nonlinear FDEs and the fractional-order comparison principle is used in [6]. And in [7,8], the Mittag–Leffler stabilities of nonlinear FDEs of order $\alpha(0<\alpha<1)$ are introduced. Moreover, by utilizing the generalized singular Gronwall inequalities, the Ulam stability for impulsive FDEs of order $\alpha(0<\alpha<1)$ is discussed in [9]. However, those methods introduced above are mostly implemented to the stability analysis of FDEs of order $\alpha(0<\alpha<1)$ and it is not easy to use them to discuss the stability of nonlinear FDEs of order $\alpha(1<\alpha<2)$. For this reason, as a meaningful attempt, the purpose of this paper is to find another effective method to study the stability of nonlinear FDEs of order $\alpha(1<\alpha<2)$.

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As we all know, the fixed-point theorems have been used to study the stability of integer-order differential systems by many authors, notably Burton[10–15] and Becker[15,16]. Moreover, we note that there are a few contributions to study stability of FDEs by fixed-point theorems. In [10], the author derived a stability criterion for a Volterra equation which is based on the contraction mapping principle and Becker's form resolvent theory. The stability of Caputo type FDEs of order $\alpha(0<\alpha<1)$ is considered by Burton and Zhang in [14], where the stability of the FDEs in a weighted Banach space is studied via resolvent theory and fixed point theorems.

In this paper, motivated by those valuable contributions mentioned above, we mainly discuss the stability of the nonlinear FDEs of order $\alpha(1<\alpha<2)$. To this end, we first transform the fractional differential equation into a first-order ordinary differential equation with a fractional integral perturbation, then the equivalent integral equations of (1.1) and (1.2) are obtained by means of the variation of constants formula and some analytical skills. Furthermore, we investigate the stability of the problems (1.1) and (1.2) by Krasnoselskii's fixed point theorem without considering the resolvent theory and results are given in a simplified way.

This paper is organized as follows: in the next section we present some preliminaries and lemmas that will be used to prove our main results. In Section 3 we give and prove our main results. Finally an application of the main results is presented.

2. Preliminaries

For convenience, we introduce some necessary definitions and lemmas which will be used in this paper. For more details, see [1–3,12,17].

Definition 2.1 [1,2]. The fractional integral of order $\alpha>0$ of a function $x:\mathbb{R}^+\to\mathbb{R}$ is given by

$$I_{0+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}x(s)ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 [1,2]. The Caputo fractional derivative of order $\alpha>0$ of a function $x:\mathbb{R}^+\to\mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha}x(t)=I_{0+}^{n-\alpha}x^{(n)}(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}x^{(n)}(s)ds,$$

where $n=[\alpha]+1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.1 [1,2]. *Let $\Re(\alpha)>0$. Suppose $x(t)\in C^{n-1}[0,+\infty)$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k.$$

In particular, when $0<\Re(\alpha)<1$, $(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x)(t)=x(t)-x(0)$.

Remark 2.1. From Definition 2.1, 2.2 and Lemma 2.1, it is easy to see that

(1) Let $\Re(\alpha)>0$. If $x(t)$ is continuous on $\mathbb{R}^+$, then $D_{0+}^{\alpha}I_{0+}^{\alpha}x(t)=x(t)$ holds for all $t\in\mathbb{R}^+$.
(2) The Caputo derivative of a constant is equal to zero.

The following Banach space plays a fundamental role in our discussion.

Let $g:\mathbb{R}^+\to[1,+\infty)$ be a strictly increasing continuous function with $g(0)=1, g(t)\to\infty$ as $t\to\infty, g(s)g(t-s)\leqslant g(t)$ for all $0\leqslant s\leqslant t\leqslant\infty$. Let

$$E\triangleq\{x(t)\in C[0,+\infty):\sup_{t\geqslant0}|x(t)|/g(t)<\infty\}.$$

Then E is a Banach space equipped with the norm $\|x\|=\sup_{t\geqslant0}\frac{|x(t)|}{g(t)}$. For more properties of this Banach space, see [3,12]. Moreover, let

$$\|\varphi\|_t=\max\{|\varphi(s)|:0\leqslant s\leqslant t\},$$

for any $t\geqslant0$, any given $\varphi\in C(\mathbb{R}^+)$ and let $\Im(\varepsilon)=\{x:x\in E,\|x\|\leqslant\varepsilon\}$ for any $\varepsilon>0$.

Lemma 2.2. *Let $r(t)\in C[0,+\infty)$. Then $x(t)\in C[0,+\infty)$ is a solution of the Cauchy type problem*

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$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = r(t), t \in \mathbb{R}^{+}, 1 < \alpha < 2, \\ x(0) = x_0, x'(0) = x_1, \end{cases} \tag{2.1}$$

if and only if $x(t)$ *is a solution of the Cauchy type problem*

$$\begin{cases} x'(t) = I_{0+}^{\alpha-1}r(t) + x_1, \\ x(0) = x_0. \end{cases} \tag{2.2}$$

Proof. To begin with, we claim that for any $0 < \gamma < 1$, if $\psi \in C[0, +\infty)$, then $(I_{0+}^{\gamma}\psi)(0) = 0$. In fact, since

$$I_{0+}^{\gamma}\psi(t) = \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\psi(s)ds,$$

we can conclude that

$$|I_{0+}^{\gamma}\psi(t)| = \frac{1}{\Gamma(\gamma)}\left|\int_0^t (t-s)^{\gamma-1}\psi(s)ds\right| \leqslant \frac{\|\psi\|_t}{\Gamma(\gamma+1)}t^{\gamma} \to 0 \ as \ t \to 0.$$

(1) Let $x \in C[0, +\infty)$ be a solution of the problem (2.1).

For any $t \in \mathbb{R}^{+}$, Definition 2.2 shows that

$${}^{C}D_{0+}^{\alpha}x(t) = ({}^{C}D_{0+}^{\alpha-1}D^{1}x)(t) = r(t).$$

According to Lemma 2.1, we have

$$x'(t) = x'(0) + I_{0+}^{\alpha-1}r(t) = I_{0+}^{\alpha-1}r(t) + x_1,$$

which means that $x(t)$ is a solution of the problem (2.2).

(2) Let $x(t)$ be a solution of the problem (2.2).

For any $t \in \mathbb{R}^{+}$, by Remark 2.1, it is easy to see that

$${}^{C}D_{0+}^{\alpha}x(t) = {}^{C}D_{0+}^{\alpha-1}x'(t) = ({}^{C}D_{0+}^{\alpha-1}I_{0+}^{\alpha-1}r)(t) + {}^{C}D_{0+}^{\alpha-1}x_1 = r(t).$$

Besides, note that $r(t) \in C[0, +\infty)$, we have $x'(0) = I_{0+}^{\alpha-1}r(0+) + x_1 = x_1$. □

Lemma 2.2 shows that the system (1.1) and (1.2) is equivalent to the system

$$x'(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}f(s, x(s))ds + x_1, x(0) = x_0. \tag{2.3}$$

For convenience, setting $\omega(t-s) := \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + k$ for $0 \leqslant s \leqslant t < +\infty$ and $F(t, x(t)) := I_{0+}^{\alpha-1}[f(t, x(t)) - x(t)] + x_1$, where $k \in \mathbb{R}$ satisfies that there exists a constant $\beta_1 \in (0, 1)$ such that

$$e^{-kt}/g(t) \in BC[0+\infty) \cap L^{1}[0+\infty), |k|\int_0^{\infty} e^{-kt}/g(t)ds \leqslant \beta_1 < 1. \tag{2.4}$$

Then (2.3) can be equivalently written as

$$\begin{cases} x'(t) = -kx(t) + \frac{d}{dt}\int_0^t \omega(t-s)x(s)ds + F(t, x(t)), \\ x(0) = x_0. \end{cases} \tag{2.5}$$

By the variation of constants formula, we have

$$\begin{aligned} x(t) &= e^{-kt}x_0 + \int_0^t e^{-k(t-s)}\left[\frac{d}{ds}\int_0^s \omega(s-u)x(u)du + F(s, x(s))\right]ds \\ &= e^{-kt}x_0 + e^{-k(t-s)}\int_0^s \omega(s-u)x(u)du\Big|_{s=0}^{t} - k\int_0^t e^{-k(t-s)} \times \int_0^s \omega(s-u)x(u)duds + \int_0^t e^{-k(t-s)}F(s, x(s))ds \\ &= e^{-kt}x_0 + \int_0^t \omega(t-s)x(s)ds - k\int_0^t e^{-k(t-s)}\int_0^s \omega(s-u)x(u)duds + \int_0^t e^{-k(t-s)}\left[I_{0+}^{\alpha-1}f(s, x(s)) - I_{0+}^{\alpha-1}x(s) + x_1\right]ds \end{aligned}$$

$$
\begin{aligned}
&= e^{-kt}x_0 + \int_0^t \omega(t-s)x(s)ds - k\int_0^t\int_u^t e^{-k(t-s)}\omega(s-u)dsx(u)du + \int_0^t e^{-k(t-s)}\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}f(u,x(u))duds \\
&\quad + \int_0^t e^{-k(t-s)}x_1 ds - \int_0^t e^{-k(t-s)}\int_0^s \frac{(s-u)^{\alpha-2}}{\Gamma(\alpha-1)}x(u)duds \\
&= e^{-kt}x_0 + \int_0^t \omega(t-s)x(s)ds - k\int_0^t\int_u^t e^{-k(t-s)}\omega(s-u)dsx(u)du + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}dsf(u,x(u))du \\
&\quad + \frac{(1-e^{-kt})}{k}x_1 - \int_0^t\int_u^t e^{-k(t-s)}\frac{\partial\omega(s-u)}{\partial s}dsx(u)du \\
&= e^{-kt}x_0 + \int_0^t \omega(t-u)x(u)du - k\int_0^t\int_u^t e^{-k(t-s)}\omega(s-u)dsx(u)du + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}dsf(u,x(u))du \\
&\quad + \frac{(1-e^{-kt})}{k}x_1 - \int_0^t\left[e^{-k(t-s)}\omega(s-u)|_{s=u}^t - k\int_u^t e^{-k(t-s)}\omega(s-u)ds\right]x(u)du \\
&= e^{-kt}x_0 + \frac{(1-e^{-kt})}{k}x_1 + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}dsf(u,x(u))du + k\int_0^t e^{-k(t-u)}x(u)du.
\end{aligned}
$$

On the other hand, if

$$
x(t) = e^{-kt}x_0 + \frac{(1-e^{-kt})}{k}x_1 + k\int_0^t e^{-k(t-u)}x(u)du + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}dsf(u,x(u))du, \tag{2.6}
$$

holds, we have $x(0) = x_0$ and

$$
\begin{aligned}
ke^{kt}x(t) + e^{kt}x'(t) = (e^{kt}x(t))' &= \left[x_0 + \frac{(e^{kt}-1)x_1}{k} + k\int_0^t e^{ku}x(u)du + \int_0^t e^{ku}I_{0+}^{\alpha-1}f(u,x(u))du\right]' \\
&= e^{kt}\left(x_1 + I_{0+}^{\alpha-1}f(t,x(t))\right) + ke^{kt}x(t).
\end{aligned}
$$

Based on the argument above, we get that the system (1.1) and (1.2) can be equivalently written as (2.6). Then our following study will focus on the system (2.6).

Definition 2.3. The trivial solution $x \equiv 0$ of (1.1) and (1.2) is said to be

(i) stable in Banach space E, if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x_0| + |x_1| \leqslant \delta$ implies that the solution $x(t) = x(t, x_0, x_1)$ exists for all $t \geqslant 0$ and satisfies $\|x\| \leqslant \varepsilon$.
(ii) asymptotically stable, if it is stable in Banach space E and there exists a number $\sigma > 0$ such that $|x_0| + |x_1| \leqslant \sigma$ implies $\lim_{t\to\infty}\|x(t)\| = 0$.

Lemma 2.3 Krasnoselskii [17]. *Let Ω be a non-empty closed convex subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map Ω into S such that*

(i) *$Ax + By \in \Omega$ for all $x, y \in \Omega$,*
(ii) *A is continuous and $A\Omega$ is contained in a compact set of S,*
(iii) *B is a contraction with constant $l < 1$.*

Then there is a $x \in \Omega$ with $Ax + Bx = x$.

In order to prove our main results, the following modified compactness criterion is needed.

Lemma 2.4 [3]. *Let $\mathcal{M}$ be a subset of the Banach space E. Then $\mathcal{M}$ is relatively compact in E if the following conditions are satisfied:*

(i) *$\{x(t)/g(t) : x(t) \in \mathcal{M}\}$ is uniformly bounded;*
(ii) *$\{x(t)/g(t) : x(t) \in \mathcal{M}\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$;*
(iii) *$\{x(t)/g(t) : x(t) \in \mathcal{M}\}$ is equiconvergent at infinity. i.e. for any given $\varepsilon > 0$, there exists a $T_0 > 0$ such that for all$x \in \mathcal{M}$ and $t_1, t_2 > T_0$, it holds $|x(t_2)/g(t_2) - x(t_1)/g(t_1)| < \varepsilon$.*

3. Main results

In this section, we shall present and prove our main results.

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Theorem 3.1. *Suppose that (2.4) holds and there exist constants* $\eta > 0, \beta_2 \in (0, 1-\beta_1)$*and a continuous function* $\tilde{f} : \mathbb{R}^+ \times (0, \eta] \to \mathbb{R}^+$ *such that*

$$\frac{|f(t, vg(t))|}{g(t)} \leqslant \tilde{f}(t, |v|), \tag{3.1}$$

holds for all $t \geqslant 0, 0 < |v| \leqslant \eta$ *and*

$$\sup_{t\geqslant 0} \int_0^t \frac{K(t-u)}{g(t-u)} \frac{\tilde{f}(u,r)}{r} du \leqslant \beta_2 < 1 - \beta_1, \tag{3.2}$$

holds for every $0 < r \leqslant \eta$*,where* β_1 *is defined in (2.4),* $\tilde{f}(t,r)$ *is nondecreasing in r for fixed t and* $\tilde{f}(t,r) \in L^1[0, +\infty)$ *in t for fixed r, and*

$$K(t-u) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds, & t-u \geqslant 0, \\ 0, & t-u < 0. \end{cases} \tag{3.3}$$

Then the trivial solution $x \equiv 0$ of (1.1) and (1.2) is stable in Banach space E.

Proof. For any given $\varepsilon > 0$, we first prove the existence of $\delta > 0$ such that $|x_0| + |x_1| < \delta$ implies $\|x\| \leqslant \varepsilon$. In fact, according to (2.4), there exists a constant $M_1 > 0$ such that

$$\frac{e^{-kt}}{g(t)} \leqslant M_1. \tag{3.4}$$

Let $0 < \delta \leqslant \frac{(1-\beta_1-\beta_2)|k|}{M_1|k|+1+M_1} \varepsilon$. Consider the non-empty closed convex subset $\Im(\varepsilon) \subseteq E$, for $t \geqslant 0$, we define two mappings A, B on $\Im(\varepsilon)$ as follows:

$$Ax(t) = \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds f(u, x(u)) du = \int_0^t K(t-u) f(u, x(u)) du, \tag{3.5}$$

$$Bx(t) = e^{-kt} x_0 + \frac{x_1(1-e^{-kt})}{k} + k \int_0^t e^{-k(t-u)} x(u) du. \tag{3.6}$$

Obviously, for $x \in \Im(\varepsilon)$, both Ax and Bx are continuous functions on $\mathbb{R}^+$. Furthermore, for $x \in \Im(\varepsilon)$, by (2.4), (3.1), (3.2) for any $t \geqslant 0$, we have

$$\frac{|Ax(t)|}{g(t)} \leqslant \int_0^t \frac{K(t-u)}{g(t-u)} \frac{|f(u, x(u))|}{g(u)} du \leqslant \int_0^t \frac{K(t-u)}{g(t-u)} \tilde{f}(u, \frac{|x(u)|}{g(u)}) du \leqslant \beta_2 \|x\| \leqslant \beta_2 \varepsilon < \infty \tag{3.7}$$

and

$$\begin{aligned} \frac{|Bx(t)|}{g(t)} &= \left| \frac{e^{-kt}}{g(t)} x_0 + \frac{(1-e^{-kt})}{kg(t)} x_1 + k \int_0^t \frac{e^{-k(t-u)}}{g(t)} x(u) du \right| \\ &\leqslant M_1|x_0| + \frac{1+M_1}{|k|}|x_1| + |k| \int_0^{\in} fty \frac{e^{-ku}}{g(u)} du \|x\| \\ &\leqslant M_1|x_0| + \frac{1+M_1}{|k|}|x_1| + \frac{|k|}{k}\varepsilon < \infty. \end{aligned} \tag{3.8}$$

Then $A\Im(\varepsilon) \subseteq E$ and $B\Im(\varepsilon) \subseteq E$. Next, we shall use Lemma 2.3 to prove there exists at least one fixed point of the operator $A + B$ in $\Im(\varepsilon)$. Here, we divide the proof into three steps.

Step 1. We prove that $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

For any $x, y \in \Im(\varepsilon)$, from (3.7) and (3.8), we obtain that

$$\begin{aligned} \sup_{t\geqslant 0} \frac{|Ax + By|}{g(t)} &= \sup_{t\geqslant 0} \left\{ \left| \frac{e^{-kt}}{g(t)} x_0 + \frac{(1-e^{-kt})}{kg(t)} x_1 + k \int_0^t \frac{e^{-k(t-u)}}{g(t)} y(u) du \int_0^t \frac{K(t-u) f(u, x(u))}{g(t)} du \right| \right\} \\ &\leqslant M_1|x_0| + \frac{1+M_1}{|k|}|x_1| + |k| \int_0^\infty \frac{e^{-ku}}{g(u)} du \|y\| + \beta_2 \|x\| \\ &\leqslant \frac{M_1|k| + 1 + M_1}{|k|} \delta + \beta_1 \varepsilon + \beta_2 \varepsilon \leqslant \varepsilon, \end{aligned}$$

which implies $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

Step 2. It is easy to see that A is continuous. Now we only prove that $A\Im(\varepsilon)$ is a relatively compact in E.

In fact, from (3.7), we get that $\{x(t)/g(t) : x(t) \in \Im(\varepsilon)\}$ is uniformly bounded in E. Moreover, a classical theorem states the fact that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Then we conclude that for $t-u \geqslant 0$, we have

$$0 \leqslant \lim_{t\to\infty} \frac{K(t-u)}{g(t-u)} \leqslant \lim_{t\to\infty} \frac{1}{\Gamma(\alpha-1)} \int_u^t \frac{e^{-k(t-s)}}{g(t-s)} \frac{(s-u)^{\alpha-2}}{g(s-u)} ds = \lim_{t\to\infty} \frac{1}{\Gamma(\alpha-1)} \int_0^t \frac{e^{-k(t-u-s)}}{g(t-u-s)} \frac{s^{\alpha-2}}{g(s)} ds = 0, \tag{3.9}$$

due to the fact $\lim_{t\to\infty} \frac{t^{\alpha-2}}{g(t)} = 0$. Together with the continuity of $K(t)$ and $g(t)$, we get that there exists a constant $M_2 > 0$ such that

$$\left|\frac{K(t-u)}{g(t-u)}\right| \leqslant M_2, \tag{3.10}$$

and for any $T_0 \in \mathbb{R}^+$, the function $K(t-u)g(u)/g(t)$ is uniformly continuous on $\{(t,u) : 0 \leqslant u \leqslant t \leqslant T_0,\}$. For any $x \in \Im(\varepsilon)$ and any $t_1, t_2 \in [0, T_0], t_1 < t_2$, we have

$$\begin{aligned}
\left|\frac{Ax(t_2)}{g(t_2)} - \frac{Ax(t_1)}{g(t_1)}\right| &= \left|\int_0^{t_2} \frac{K(t_2-u)}{g(t_2)} f(u,x(u))du - \int_0^{t_1} \frac{K(t_1-u)}{g(t_1)} f(u,x(u))du\right| \\
&\leqslant \int_0^{t_1} \left|\frac{K(t_2-u)}{g(t_2)} - \frac{K(t_1-u)}{g(t_1)}\right| |f(u,x(u))|du + \int_{t_1}^{t_2} \frac{K(t_2-u)}{g(t_2-u)} \tilde{f}(u,\varepsilon)du \\
&\leqslant \int_0^{t_1} \left|\frac{K(t_2-u)g(u)}{g(t_2)} - \frac{K(t_1-u)g(u)}{g(t_1)}\right| \tilde{f}(u,\varepsilon)du + M_2 \int_{t_1}^{t_2} \tilde{f}(u,\varepsilon)du \to 0,
\end{aligned}$$

as $t_2 \to t_1$, which means that $\{x(t)/g(t) : x(t) \in \Im(\varepsilon)\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$. By Lemma 2.4, in order to show that $A\Im(\varepsilon)$ is a relatively compact set of E, we only need to prove that $\{x(t)/g(t) : x(t) \in \Im(\varepsilon)\}$ is equiconvergent at infinity. In fact, for any $\varepsilon_1 > 0$, there exists a $L > 0$ such that

$$M_2 \int_L^\infty \tilde{f}(u,\varepsilon)du \leqslant \frac{\varepsilon_1}{3}.$$

According to (3.9), we get that

$$\lim_{t\to\infty} \sup_{u\in[0,L]} \frac{K(t-u)}{g(t-u)} \leqslant \max\left\{\lim_{t\to\infty} \frac{K(t-L)}{g(t-L)}, \lim_{t\to\infty} \frac{K(t)}{g(t)}\right\} = 0.$$

Thus, there exists $T > L$ such that for $t_1, t_2 \geqslant T$, we have

$$\sup_{s\in[0,L]} \left|\frac{K(t_2-u)g(u)}{g(t_2)} - \frac{K(t_1-u)g(u)}{g(t_1)}\right| \leqslant \sup_{s\in[0,L]} \left|\frac{K(t_2-u)}{g(t_2-u)}\right| + \sup_{s\in[0,L]} \left|\frac{K(t_1-u)}{g(t_1-u)}\right| \leqslant \frac{\varepsilon_1}{3} \left(\int_0^\infty \tilde{f}(u,\varepsilon)du\right)^{-1}.$$

Therefore, for $t_1, t_2 \geqslant T$,

$$\begin{aligned}
\left|\frac{Ax(t_2)}{g(t_2)} - \frac{Ax(t_1)}{g(t_1)}\right| &= \left|\int_0^{t_2} \frac{K(t_2-u)}{g(t_2)} f(u,x(u))du - \int_0^{t_1} \frac{K(t_1-u)}{g(t_1)} f(u,x(u))du\right| \\
&\leqslant \int_0^L \left|\frac{K(t_2-u)g(u)}{g(t_2)} - \frac{K(t_1-u)g(u)}{g(t_1)}\right| \tilde{f}(u,\varepsilon)du \\
&\quad + \int_L^{t_2} \frac{K(t_2-u)}{g(t_2-u)} \tilde{f}(u,\varepsilon)du + \int_L^{t_1} \frac{K(t_1-u)}{g(t_1-u)} \tilde{f}(u,\varepsilon)du \\
&\leqslant \frac{\varepsilon_1}{3} + 2M_2 \int_L^\infty \tilde{f}(u,\varepsilon)du \leqslant \varepsilon_1.
\end{aligned}$$

Hence the required conclusion is true.

Step 3. We claim that $B : \Im(\varepsilon) \to E$ is a contraction mapping.

In fact, for any $x, y \in \Im(\varepsilon)$, from (2.4), we obtain that

$$\begin{aligned}
\sup_{t\geqslant 0} \left|\frac{Bx(t)}{g(t)} - \frac{By(t)}{g(t)}\right| &= \sup_{t\geqslant 0} \left|\frac{k\int_0^t e^{-k(t-u)}x(u)du}{g(t)} - \frac{k\int_0^t e^{-k(t-u)}y(u)du}{g(t)}\right| \leqslant \sup_{t\geqslant 0} |k| \int_0^t \frac{e^{-k(t-u)}}{g(t-u)} \frac{|x(u)-y(t)|}{g(u)} du \\
&\leqslant |k| \int_0^t \frac{e^{-k(t-u)}}{g(t-u)} du \|x-y\| \leqslant \beta_1 \|x-y\| < \|x-y\|.
\end{aligned}$$

By Lemma 2.3, we know that there exists at least one fixed point of the operator $A+B$ in $\Im(\varepsilon)$.

Finally, for any $\varepsilon_2 > 0$, if $0 < \delta_1 \leqslant \frac{(1-\beta_1-\beta_2)|k|}{M_1|k|+1+M_1}\varepsilon_2$, then $|x_0| + |x_1| \leqslant \delta_1$ implies that

$$\begin{aligned}\|x\| &= \sup_{t\geqslant 0}\left\{\left|\frac{e^{-kt}}{g(t)}x_0 + \frac{(1-e^{-kt})}{kg(t)}x_1 + k\int_0^t \frac{e^{-k(t-u)}}{g(t)}x(u)du + \int_0^t \frac{K(t-u)f(u,x(u))}{g(t)}du\right|\right\} \\ &\leqslant \sup_{t\geqslant 0}\left\{\frac{e^{-kt}}{g(t)}|x_0| + \frac{|1-e^{-kt}|}{|k|g(t)}|x_1| + |k|\int_0^t \frac{e^{-k(t-u)}}{g(t-u)}\frac{|y(u)|}{g(u)}du + \int_0^t \frac{K(t-u)}{g(t-u)}\frac{|f(u,x(u))|}{g(u)}du\right\} \\ &\leqslant M_1\delta_1 + \frac{1+M_1}{|k|}\delta_1 + \beta_1\|x\| + \beta_2\|x\| \leqslant \frac{M_1|k|+1+M_1}{(1-\beta_1-\beta_2)|k|}\delta_1 \leqslant \varepsilon_2.\end{aligned}$$

Thus, we know that the trivial solution of the system (1.1) and (1.2) is stable in Banach space E. $\square$

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 are satisfied.*

$$\lim_{t\to\infty} e^{-kt}/g(t) = 0, \tag{3.11}$$

and for any $r > 0$, there exists a function $\varphi_r(t) \in L^1[0, +\infty)$, $\varphi_r(t) > 0$ such that $|u| \leqslant r$ implies

$$|f(t,u)|/g(t) \leqslant \varphi_r(t), \text{ a.e. } t \in [0, +\infty). \tag{3.12}$$

Then the trivial solution of (1.1) and (1.2) is asymptotically stable.

Proof. First, it follows from Theorem 3.1 that the trivial solution of (1.1) and (1.2) is stable in the Banach space E. Next, we shall show that the trivial solution $x \equiv 0$ of (1.1) and (1.2) is attractive. For any $r > 0$, defining

$$\mathfrak{I}_*(r) = \{x : x \in \mathfrak{I}(r), \lim_{t\to\infty} x(t)/g(t) = 0\}.$$

We only need to prove that $Ax + By \in \mathfrak{I}_*(r)$ for any $x, y \in \mathfrak{I}_*(r)$, i.e. $\frac{Ax(t)+By(t)}{g(t)} \to 0$ as $t \to \infty$, where

$$Ax + By = e^{-kt}x_0 + \frac{x_1(1-e^{-kt})}{k} + k\int_0^t e^{-k(t-u)}y(u)du + \int_0^t K(t-u)f(u,x(u))du.$$

In fact, for $x, y \in \mathfrak{I}_*(r)$, based on the fact that used in the proof of Theorem 3.1 (Step 2), it follows from (2.4) and (3.11) that

$$\int_0^t \frac{e^{-k(t-u)}}{g(t-u)}\frac{y(u)}{g(u)}du \to 0 \text{ and } \frac{K(t-u)}{g(t-u)} = \frac{\int_u^t \frac{e^{-k(t-s)}}{g(t-u)}(s-u)^{\alpha-2}ds}{\Gamma(\alpha-1)} \to 0,$$

as $t \to \infty$. Together with the hypothesis $\varphi_r(t) \in L^1[0, +\infty)$, we obtain that

$$\int_0^t \frac{K(t-u)}{g(t-u)}\frac{|f(u,x(u))|}{g(u)}du \leqslant \int_0^t \frac{K(t-u)}{g(t-u)}\varphi_r(u)du \to 0$$

as $t \to \infty$. Thus we get the conclusion. $\square$

By the similar arguments in Theorem 3.2, we give the following corollary and omit the details.

Corollary 3.1. *Suppose that all conditions of Theorem 3.1 and the hypothesis (3.11) are satisfied. There exist two functions $a, b : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $a(t) \in L^1(0, +\infty)$, $\lim_{r\to 0} b(r)/r = 0$ such that*

$$|f(t,u)|/g(t) \leqslant a(t)b(|u|). \tag{3.13}$$

Then the trivial solution of (1.1) and (1.2) is asymptotically stable.

4. Example

Let us consider the following nonlinear fractional initial value problems

$$\begin{cases} {}^C D_{0+}^{\frac{3}{2}} x(t) = \mu\left(\frac{t^2x^2}{e^{(\lambda+1)t}} + \frac{x^{3/2}}{(1+t^2)e^{\lambda t/2}}\right), \\ x(0) = x_0, x'(0) = x_1, \end{cases} \tag{4.1}$$

where $\lambda > 1, \mu > 0$. Suppose $0 < |k| \leqslant \frac{\lambda-1}{2}$, let $g(t) = e^{\lambda t}, \beta_1 = \frac{|k|}{\lambda+k}$, then (2.4) holds and the Banach space is

$$E^* \triangleq \{x(t) \in C[0, +\infty) : \sup_{t\geqslant 0}|x(t)|/e^{\lambda t} < \infty\},$$

equipped with the norm $\|x\| = \sup_{t\geqslant 0}\frac{|x(t)|}{e^{\lambda t}}$. Let $\tilde{f}(t,r) = \mu\left(r^2t^2e^{-t} + \frac{r^{3/2}}{1+t^2}\right)$. Then (3.1) holds and $\tilde{f}(t,r) \in L^1[0,\infty)$ in t for fixed r. Note that

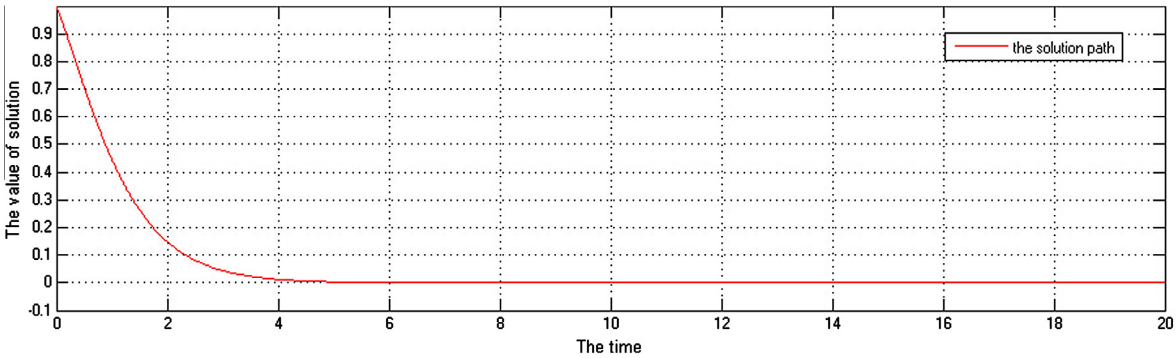

Fig. 1. A solution path in Example 4.1 with $[\lambda = 1.5 > 1, \mu = 0.001$ and $x_0 = x_1 = 1]$.

$$\frac{K(t-u)}{e^{\lambda(t-u)}} = \frac{1}{\Gamma(1/2)}\int_u^t \frac{1}{e^{(\lambda+k)(t-s)}}\frac{(s-u)^{-1/2}}{e^{\lambda(s-u)}}ds \leqslant \frac{\int_u^t \frac{(s-u)^{-1/2}}{e^{\lambda(s-u)}}ds}{\Gamma(1/2)} = \frac{\int_0^{t-u}\frac{\tau^{-1/2}}{e^{\lambda\tau}}ds}{\Gamma(1/2)} \leqslant \lambda^{1/2},$$

for all $t \geqslant 0$, if there exists $\eta \geqslant 0$ such that

$$\mu \leqslant \frac{1}{2(2\eta + \frac{\pi}{2}\eta^{1/2})(\lambda+k)\lambda^{1/2} + 1}, \tag{4.2}$$

then

$$\int_0^t \frac{K(t-u)}{g(t-u)}\frac{\tilde{f}(u,r)}{r}du = \mu\int_0^t \frac{K(t-u)}{g(t-u)}\left(rt^2e^{-t} + \frac{r^{1/2}}{1+t^2}\right)du \leqslant \frac{1/2}{\lambda+k} < 1 - \beta_1,$$

for all $t \geqslant 0, 0 \leqslant r \leqslant \eta$. Thus the trivial solution of (4.1) is stable in E^* follows from Theorem 3.1.

Moreover, let $\varphi_r(t) = \mu\left(\frac{t^2r^2}{e^{(\lambda+1)t}} + \frac{r^{3/2}}{(1+t^2)e^{\lambda t/2}}\right) \in L^1[0,\infty)$. For any bounded $r > 0$, we get that $|f(t,u)| \leqslant \varphi_r(t)$ and

$$\lim_{t\to 0} e^{-kt}/g(t) \leqslant \lim_{t\to 0} e^{-\frac{\lambda t}{2}} = 0.$$

Then, by Theorem 3.2, we get that the trivial solution of (4.1) is asymptotically stable.

On the other hand, in order to explain the validity of our main results, the corresponding simulation is also added (see Fig. 1).

5. Concluding remark

In this paper, we try to study the stability analysis of nonlinear FDEs of order $\alpha(1 < \alpha < 2)$ by using Krasnoselskii's fixed point theorem in a weighted Banach space. And it is a meaningful attempt to investigate a FDE of Caputo type by transforming it into a first-order ordinary differential equation with a fractional integral perturbation and then establishing its equivalent integral equations by means of the variation of constants formula, together with some analytical skills. Finally, two main theorems are established on the stability of the problems (1.1) and (1.2) by Krasnoselskii's fixed point theorem without considering the resolvent theory and results are given in a simplified way. However, the conditions of our main results (Theorem 3.1) are a little strong and complex, and we shall optimize them in our forthcoming papers. Furthermore, we hope to employ this method to study the Mittag–Leffler stability, Ulam stability and other stabilty of FDEs of order $\alpha(1 < \alpha < 2)$.

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