Applied Mathematics Letters 85 (2018) 118–124

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Representation of solution of a Riemann–Liouville fractional differential equation with pure delay[☆]

Mengmeng Li[a], JinRong Wang[a,b,*]

[a] *Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China*
[b] *School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China*

A R T I C L E   I N F O

Article history:
Received 11 April 2018
Accepted 5 June 2018
Available online 15 June 2018

Keywords:
Linear delay differential equation
Riemann–Liouville fractional derivative
Delayed Mittag-Leffler type matrix function
Solution

A B S T R A C T

This paper derives a representation of a solution to the initial value problem for a linear fractional delay differential equation with Riemann–Liouville derivative. We apply the method of variation of constants to obtain the representation of a solution via a delayed Mittag-Leffler type matrix function.

1. Introduction

Recently, Khusainov and Shuklin [1] and Diblík and Khusainov [2,3] introduce a new concept, delayed exponential matrix function, which is used to seek a representation of a solution to a linear time-invariant continuous/discrete delay equation. For more contributions of the representation of solutions, the stability, and the control theory for linear time-invariant continuous/discrete delay systems, we refer to [4–9] and the references.

Inspired by [7,8], we seek a representation of a solution to a linear fractional delay differential equation with Riemann–Liouville derivative whose the initial condition involving a singular kernel that is different from the standard initial condition for a Caputo fractional delay differential equation.

In this paper, we study a fractional delay differential equation of the form:

$$\begin{cases} (\mathbb{D}^{\alpha}_{-\tau^+}y)(x) = By(x-\tau) + f(x),\ B \in \mathbb{R}^{n\times n},\ x \in (0,T],\ \tau > 0, \\ y(x) = \omega(x),\ \omega(x) \in \mathbb{R}^n\ \ -\tau \le x \le 0, \\ (\mathbb{I}^{1-\alpha}_{-\tau^+}y)(-\tau^+) = \omega(-\tau),\ \omega(-\tau) \in \mathbb{R}^n, \end{cases} \tag{1}$$

[☆] This work was supported by the National Natural Science Foundation of China (11661016), Training Object of High Level and Innovative Talents of Guizhou Province ((2016)4006) and Science and Technology Program of Guizhou Province ([2017]5788).

* Corresponding author at: Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, China.
E-mail addresses: Lmm0424@126.com (M. Li), jrwang@gzu.edu.cn (J. Wang).

https://doi.org/10.1016/j.aml.2018.06.003
0893-9659/© 2018 Published by Elsevier Ltd.

where $\mathbb{D}^{\alpha}_{-\tau^+}y$ denotes the Riemann–Liouville derivative of order $\alpha \in (0,1)$ (see Definition 2.1), $\mathbb{I}^{1-\alpha}_{-\tau^+}y$ denotes the Riemann–Liouville fractional integral of order $1-\alpha$ (see Definition 2.2), $T = k^*\tau$ for a fixed $k^* \in \mathbb{N}^+ := \{1, 2, \ldots\}$, τ is a fixed delay time, $f \in C([-\tau, T], \mathbb{R}^n)$, and ω is an arbitrary Riemann–Liouville differentiable vector function, i.e., $\mathbb{D}^{\alpha}_{-\tau^+}\omega$ exists.

The main contributions are stated as follows.

We find a fundamental matrix for homogeneous problem of (3) and then derive its general solution. Next, we derive a special solution for (1) with zero initial condition. Finally, we give a representation of a solution of (1) via the superposition principle.

2. Preliminaries

Let $a, b \in \mathbb{R}, a < b$ and $C((a, b], \mathbb{R}^n)$ be the Banach space of vector-valued continuous function from $(a, b]$ into $\mathbb{R}^n$. Let Θ and I be the zero and identity matrices, respectively.

We recall some definitions and lemmas as follows.

Definition 2.1 (*See [10]*)**.** The Riemann–Liouville derivative of order $0 < \alpha < 1$ for a function $f : [a, \infty) \to \mathbb{R}$ can be written as $(\mathbb{D}^{\alpha}_{a^+}y)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} y(t)dt,\ x > a.$

Definition 2.2 (*See [10]*)**.** The Riemann–Liouville fractional integral of order $0 < \alpha < 1$ for a function $f : [a, \infty) \to \mathbb{R}$ can be written as $(\mathbb{I}^{\alpha}_{a^+}y)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)dt,\ x > a.$

Definition 2.3 (*See [7, Definition 2.5]*)**.** Set $0 < \alpha < 1$ and $\beta > 0$. Delayed two parameters Mittag-Leffler type matrix $\mathbb{Z}^{B\cdot^{\alpha}}_{\tau,\beta} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$\mathbb{Z}^{Bx^{\alpha}}_{\tau,\beta} = \begin{cases} \Theta, & -\infty < x \le -\tau, \\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)} & -\tau < x \le 0, \\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)} + B\dfrac{x^{2\alpha-1}}{\Gamma(\alpha+\beta)} + B^2\dfrac{(x-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\beta)} + \cdots + B^k\dfrac{(x-(k-1)\tau)^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}, & \\ & (k-1)\tau < x \le k\tau,\ k \in \mathbb{N}^+. \end{cases} \tag{2}$$

Lemma 2.4 (*See [7, Lemma 2.6]*)**.** *For any $(k-1)\tau < x \le k\tau$, $0 \le s \le t$ and $k \in \mathbb{N}^+$ is a fixed number, we have $\int_{(k-1)\tau+s}^x (x-t)^{-\alpha}(t-(k-1)\tau-s)^{k\alpha-1}dt = (x-(k-1)\tau-s)^{(k-1)\alpha}\mathbb{B}[1-\alpha, k\alpha]$, where $\mathbb{B}[\xi, \eta] = \int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ is a Beta function.*

Lemma 2.5 (*See [8, Lemma 2.5]*)**.** *For any $(k-1)\tau < x \le k\tau$ and $k \in \mathbb{N}^+$, we have $\int_{(k-1)\tau}^x (x-t)^{-\alpha}(t-(k-1)\tau)^{(k+1)\alpha-1}dt = (x-(k-1)\tau)^{k\alpha}\mathbb{B}[1-\alpha, (k+1)\alpha]$.*

Lemma 2.6. *Let $(k-1)\tau < x \le k\tau$, $-\tau \le s \le t$ and $k \in \mathbb{N}^+$ is a fixed number, we have $\int_s^x (x-t)^{-\alpha}\mathbb{Z}^{B(t-\tau-s)^{\alpha}}_{\tau,\alpha}dt = \sum_{i=0}^k \int_{i\tau+s}^x (x-t)^{-\alpha}B^i\frac{(t-i\tau-s)^{(i+1)\alpha-1}}{\Gamma(i\alpha+\alpha)}dt.$*

Proof. The proof is similar to [7, Lemma 2.7], so we omit it here. $\square$

3. Main results

Firstly, we use (2) to construct the explicit formula of solutions of

$$\begin{cases}(\mathbb{D}^{\alpha}_{-\tau^+}y)(x)=By(x-\tau),\ y(x)\in\mathbb{R}^n,\ x\in(0,T],\ \tau>0,\\ y(x)=\omega(x),\ \omega(x)\in\mathbb{R}^n\ \ -\tau\le x\le 0,\\ (\mathbb{I}^{1-\alpha}_{-\tau^+}y)(-\tau^+)=\omega(-\tau),\ \omega(-\tau)\in\mathbb{R}^n.\end{cases}\tag{3}$$

Theorem 3.1. *For delayed Mittag-Leffler type matrix* $\mathbb{Z}^{B\cdot^{\alpha}}_{\tau,\alpha}:\mathbb{R}\to\mathbb{R}^{n\times n}$*, one has* $(\mathbb{D}^{\alpha}_{-\tau^+}\mathbb{Z}^{Bt^{\alpha}}_{\tau,\alpha})(x)=B\mathbb{Z}^{B(x-\tau)^{\alpha}}_{\tau,\alpha}$*, i.e.,* $\mathbb{Z}^{Bx^{\alpha}}_{\tau,\alpha}$ *is a solution of* $(\mathbb{D}^{\alpha}_{-\tau^+}y)(x)=By(x-\tau)$*, that satisfies initial conditions* $\mathbb{Z}^{Bx^{\alpha}}_{\tau,\alpha}=I\frac{(\tau+x)^{\alpha-1}}{\Gamma(\alpha)}$*,* $-\tau\le x\le 0$.

Proof. For $x\in((k-1)\tau,k\tau]$ and $k\in\mathbb{N}^+$. We adopt the mathematical induction to prove our result.

(i) For $k=1,0<x\le\tau$, we have

$$y(x)=\mathbb{Z}^{Bx^{\alpha}}_{\tau,\alpha}=I\frac{(\tau+x)^{\alpha-1}}{\Gamma(\alpha)}+B\frac{x^{2\alpha-1}}{\Gamma(\alpha+\alpha)}.\tag{4}$$

By using Definition 2.1 for $\mathbb{Z}^{B\cdot^{\alpha}}_{\tau,\alpha}$ via (4) and Lemma 2.5, we obtain

$$\begin{aligned}(\mathbb{D}^{\alpha}_{-\tau^+}\mathbb{Z}^{Bt^{\alpha}}_{\tau,\alpha})(x)&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_{-\tau}^{0}(x-t)^{-\alpha}y(t)dt+\int_{0}^{x}(x-t)^{-\alpha}y(t)dt\right)\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_{-\tau}^{x}(x-t)^{-\alpha}I\frac{(\tau+t)^{\alpha-1}}{\Gamma(\alpha)}dt+\int_{0}^{x}(x-t)^{-\alpha}B\frac{t^{2\alpha-1}}{\Gamma(\alpha+\alpha)}dt\right)\\ &=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(I\frac{\mathbb{B}[1-\alpha,\alpha]}{\Gamma(\alpha)}+\frac{B}{\Gamma(2\alpha)}x^{\alpha}\mathbb{B}[1-\alpha,2\alpha]\right)=B\frac{x^{\alpha-1}}{\Gamma(\alpha)}.\end{aligned}$$

(ii) For $k=2,\tau<x\le 2\tau$, we have

$$y(x)=\mathbb{Z}^{Bx^{\alpha}}_{\tau,\alpha}=I\frac{(\tau+x)^{\alpha-1}}{\Gamma(\alpha)}+B\frac{x^{2\alpha-1}}{\Gamma(\alpha+\alpha)}+B^2\frac{(x-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\alpha)}.\tag{5}$$

By using Definition 2.1 for $\mathbb{Z}^{B\cdot^{\alpha}}_{\tau,\alpha}$ via (5) and Lemma 2.5, we obtain

$$\begin{aligned}(\mathbb{D}^{\alpha}_{-\tau^+}\mathbb{Z}^{Bt^{\alpha}}_{\tau,\alpha})(x)&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_{-\tau}^{\tau}(x-t)^{-\alpha}y(t)dt+\int_{\tau}^{x}(x-t)^{-\alpha}y(t)dt\right)\\ &=B\frac{x^{\alpha-1}}{\Gamma(\alpha)}+\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{\tau}^{x}(x-t)^{-\alpha}B^2\frac{(t-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\alpha)}dt\\ &=B\frac{x^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}.\end{aligned}$$

(iii) Suppose $k=n,(n-1)\tau<x\le n\tau$ and $n\in\mathbb{N}^+$, the following relation holds:

$$(\mathbb{D}^{\alpha}_{-\tau^+}\mathbb{Z}^{Bt^{\alpha}}_{\tau,\alpha})(x)=B\frac{x^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^n\frac{(x-(n-1)\tau)^{n\alpha-1}}{\Gamma(n\alpha)}.$$

Next, for $k=n+1,n\tau<x\le(n+1)\tau$, we have

$$y(x)=\mathbb{Z}^{Bx^{\alpha}}_{\tau,\alpha}=I\frac{(\tau+x)^{\alpha-1}}{\Gamma(\alpha)}+B\frac{x^{2\alpha-1}}{\Gamma(\alpha+\alpha)}+\cdots+B^{n+1}\frac{(x-n\tau)^{(n+2)\alpha-1}}{\Gamma((n+1)\alpha+\alpha)}.\tag{6}$$

By using Definition 2.1 for $\mathbb{Z}_{\tau,\alpha}^{B\cdot^{\alpha}}$ via (6) and Lemma 2.5, we obtain

$$
\begin{aligned}
&(\mathbb{D}_{-\tau^+}^{\alpha}\mathbb{Z}_{\tau,\alpha}^{Bt^{\alpha}})(x)\\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\int_{-\tau}^{0}(x-t)^{-\alpha}y(t)dt+\int_{0}^{\tau}(x-t)^{-\alpha}y(t)dt+\cdots+\int_{n\tau}^{x}(x-t)^{-\alpha}y(t)dt\right)\\
&= B\frac{x^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^n\frac{(x-(n-1)\tau)^{n\alpha-1}}{\Gamma(n\alpha)}\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\left(\frac{B^{n+1}}{\Gamma((n+1)\alpha+\alpha)}\int_{n\tau}^{x}(x-t)^{-\alpha}(t-n\tau)^{(n+2)\alpha-1}dt\right)\\
&= B\frac{x^{\alpha-1}}{\Gamma(\alpha)}+B^2\frac{(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^{n+1}\frac{(x-n\tau)^{(n+1)\alpha-1}}{\Gamma((n+1)\alpha)}.
\end{aligned}
$$

Therefore, for any $(k-1)\tau < x \le k\tau$ and $k \in \mathbb{N}^+$, according to the mathematical induction, we have

$$
(\mathbb{D}_{-\tau^+}^{\alpha}\mathbb{Z}_{\tau,\alpha}^{Bt^{\alpha}})(x) = B\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)}+B\frac{(x-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}+\cdots+B^{k-1}\frac{(x-(k-1)\tau)^{k\alpha-1}}{\Gamma(k\alpha)}\right) = B\mathbb{Z}_{\tau,\alpha}^{B(x-\tau)^{\alpha}}.
$$

The proof is completed. □

Theorem 3.2. *Let $(n-1)\tau < x \le n\tau$ for all $n \in \{0,1,2,\ldots,k^*\}$. A solution $y \in C(X,\mathbb{R}^n)$ of (3) can be expressed by the following formula*

$$
y(x) = \mathbb{Z}_{\tau,\alpha}^{Bx^{\alpha}}\omega(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}}(\mathbb{D}_{-\tau^+}^{\alpha}\omega)(s)ds, \tag{7}
$$

where either $X = ((n-1)\tau, n\tau]$ for $0 < \alpha < 1/(n+1)$ or $X = [(n-1)\tau, n\tau]$ for $\alpha \ge 1/(n+1)$.

Proof. Let matrix $Y_0(x) = \mathbb{Z}_{\tau,\alpha}^{Bx^{\alpha}}$ satisfy Theorem 3.1 and any solution of (3) satisfy initial conditions $y(x) = \omega(x),\ -\tau \le x \le 0$ should search in the form

$$
y(x) = Y_0(x)C+\int_{-\tau}^{0}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}}z(s)ds. \tag{8}
$$

where C is a vector of unknown constant, $z(\cdot)$ is an unknown Riemann–Liouville differentiable vector function. According to the matrix $Y_0(x)$ is a solution of (3). Therefore, we choose C satisfying $(\mathbb{I}_{-\tau^+}^{1-\alpha}y)(-\tau^+) = \omega(-\tau)$.

Let us assume $x = -\tau$, by using Definition 2.3, we obtain $\mathbb{Z}_{\tau,\alpha}^{B(-2\tau-s)^{\alpha}} = \Theta$ with $-\tau \le s \le 0$. For $-\tau < x \le 0$, we have

$$
\begin{aligned}
\omega(-\tau) &= (\mathbb{I}_{-\tau^+}^{1-\alpha}y)(-\tau^+) = \lim_{x\to-\tau^+}(\mathbb{I}_{-\tau^+}^{1-\alpha}y)(x)\\
&= \lim_{x\to-\tau^+}\left(\frac{1}{\Gamma(1-\alpha)}\int_{-\tau}^{x}(x-t)^{-\alpha}Y_0(t)Cdt\right)\\
&= \lim_{x\to-\tau^+}\frac{C}{\Gamma(1-\alpha)}\left(\int_{-\tau}^{x}(x-t)^{-\alpha}(\tau+t)^{\alpha-1}dt\right) = \lim_{x\to-\tau^+}C = C,
\end{aligned}
$$

which implied the formula (8) takes a form

$$
y(x) = \mathbb{Z}_{\tau,\alpha}^{Bx^{\alpha}}\omega(-\tau)+\int_{-\tau}^{0}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}}z(s)ds.
$$

Since $-\tau \le x \le 0$, one should divide interval into two subintervals, we have

(i) For $-\tau \leq s \leq x$, and $-\tau \leq x-\tau-s \leq x$, the delayed Mittag-Leffler type matrix $\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}} = I\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}$.

(ii) For $x \leq s \leq 0$, and $x-\tau \leq x-\tau-s \leq -\tau$, the delayed Mittag-Leffler type matrix $\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}} = \Theta$. Thus on the interval $-\tau \leq x \leq 0$, we have

$$\omega(x) = I\frac{(\tau+x)^{\alpha-1}}{\Gamma(\alpha)}\omega(-\tau) + \int_{-\tau}^{x} I\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds. \tag{9}$$

Having Riemann–Liouville fractional differentiation on both sides of (9), we have

$$\begin{aligned}(\mathbb{D}_{-\tau^+}^{\alpha}\omega)(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}(x-t)^{-\alpha}\left(I\frac{(\tau+t)^{\alpha-1}}{\Gamma(\alpha)}\omega(-\tau) + \int_{-\tau}^{x} I\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds\right)dt \\ &= \frac{I}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}\frac{z(s)}{\Gamma(\alpha)}\left(\int_{s}^{x}(x-t)^{-\alpha}(t-s)^{\alpha-1}dt\right)ds = \frac{d}{dx}\int_{-\tau}^{x}z(s)ds = z(x).\end{aligned}$$

The proof is completed. □

Now we are ready to derive a special solution of (1) with the zero initial conditions.

Theorem 3.3. *A solution $\widetilde{y} \in C([-\tau,T],\mathbb{R}^n)$ of (1) satisfying initial conditions $y(x) = 0$, $x \in [-\tau,0]$ has a form*

$$\widetilde{y}(x) = \int_{-\tau}^{x}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}f(t)dt,\ \ x \in [0,T].$$

Proof. Having used method of variation of constants, solution of nonhomogeneous system $\widetilde{y}(x)$ will be search in the form

$$\widetilde{y}(x) = \int_{-\tau}^{x}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}c(t)dt, \tag{10}$$

where $c(\cdot)$, $-\tau \leq t \leq x$ is an unknown vector function and $\widetilde{y}(0) = 0$.

Taking the Riemann–Liouville fractional differentiate for formula (10), we obtain

(i) For $k = 1$ and $0 < x \leq \tau$, according to (1), we have

$$\begin{aligned}(\mathbb{D}_{-\tau^+}^{\alpha}\widetilde{y})(x) &= B\widetilde{y}(x-\tau) + f(x) = B\int_{-\tau}^{x-\tau}\mathbb{Z}_{\tau,\alpha}^{B(x-2\tau-t)^{\alpha}}c(t)dt + f(x) \\ &= B\int_{-\tau}^{x-\tau} I\frac{(x-\tau-t)^{\alpha-1}}{\Gamma(\alpha)}c(t)dt + f(x).\end{aligned}$$

However, according to Definition 2.1 and Lemma 2.6, we have

$$\begin{aligned}&(\mathbb{D}_{-\tau^+}^{\alpha}\widetilde{y})(x) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}(x-t)^{-\alpha}\left(\int_{-\tau}^{t}\mathbb{Z}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}c(s)ds\right)dt \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}c(s)\left(\int_{s}^{x}(x-t)^{-\alpha}\mathbb{Z}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dt\right)ds \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}c(s)\left(\int_{s}^{x}(x-t)^{-\alpha}I\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dt + \int_{\tau+s}^{x}(x-t)^{-\alpha}B\frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}dt\right)ds \\ &= \frac{d}{dx}\int_{-\tau}^{x}c(s)ds + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x-\tau}c(s)\left(\int_{\tau+s}^{x}(x-t)^{-\alpha}B\frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}dt\right)ds \\ &= c(x) + \frac{B}{\Gamma(\alpha+1)}\frac{d}{dx}\int_{-\tau}^{x-\tau}c(s)(x-\tau-s)^{\alpha}ds = c(x) + B\int_{-\tau}^{x-\tau}\frac{(x-\tau-s)^{\alpha-1}}{\Gamma(\alpha)}c(s)ds.\end{aligned}$$

Hence, we obtain $c(x) = f(x)$.

(ii) For $(k-1)\tau < x \le k\tau$ and $k \in \mathbb{N}^+$, according to (1), we have

$$\begin{aligned}(\mathbb{D}^{\alpha}_{-\tau^+}\widetilde{y})(x) &= B\widetilde{y}(x-\tau) + f(x) = B\int_{-\tau}^{x-\tau} \mathbb{Z}_{\tau,\alpha}^{B(x-2\tau-t)^{\alpha}} c(t)dt + f(x)\\ &= \sum_{i=1}^{k}\int_{-\tau}^{x-i\tau} B^i \frac{(x-i\tau-t)^{i\alpha-1}}{\Gamma(i\alpha)} c(t)dt + f(x).\end{aligned}$$

However, according to Definition 2.1, we have

$$\begin{aligned}(\mathbb{D}^{\alpha}_{-\tau^+}\widetilde{y})(x) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}(x-t)^{-\alpha}\left(\int_{-\tau}^{t}\mathbb{Z}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}c(s)ds\right)dt\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x}c(s)\left(\int_{s}^{x}(x-t)^{-\alpha}\mathbb{Z}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dt\right)ds.\end{aligned}$$

According to Lemmas 2.4 and 2.6, we obtain

$$\begin{aligned}&(\mathbb{D}^{\alpha}_{-\tau^+}\widetilde{y})(x)\\ &= \frac{d}{dx}\int_{-\tau}^{x}c(s)ds + \sum_{i=1}^{k}\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\tau}^{x-i\tau}c(s)\left(\int_{i\tau+s}^{x}(x-t)^{-\alpha}B^i\frac{(t-i\tau-s)^{(i+1)\alpha-1}}{\Gamma(i\alpha+\alpha)}dt\right)ds\\ &= c(x) + \sum_{i=1}^{k}\frac{B^i}{\Gamma(i\alpha+1)}\frac{d}{dx}\int_{-\tau}^{x-i\tau}c(s)(x-i\tau-s)^{i\alpha}ds\\ &= c(x) + \sum_{i=1}^{k}\int_{-\tau}^{x-i\tau}B^i\frac{(x-i\tau-t)^{i\alpha-1}}{\Gamma(i\alpha)}c(t)dt.\end{aligned}$$

Hence, we obtain $c(x) = f(x)$. The proof is completed. □

Linking Theorems 3.2 and 3.3 via the superposition principle, we have the following result.

Theorem 3.4. *A solution $y \in C(X,\mathbb{R}^n) \cap C([-\tau,T],\mathbb{R}^n)$ of (1) satisfying initial conditions $y(x) = \omega(x),\ -\tau \le x \le 0$ has a form*

$$y(x) = \mathbb{Z}_{\tau,\alpha}^{Bx^{\alpha}}\omega(-\tau) + \int_{-\tau}^{0}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-s)^{\alpha}}(\mathbb{D}^{\alpha}_{-\tau^+}\omega)(s)ds + \int_{-\tau}^{x}\mathbb{Z}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}f(t)dt.$$

Finally, an example is presented to illustrate the above theoretical results.

Let $\alpha = 0.3$, $\tau = 0.4$, $k^* = 4$ and $T = 1.6$. Consider

$$\begin{cases}\mathbb{D}^{0.3}_{0^+}y(x) = By(x-0.4) + f(x),\ x \in (0,1.6],\\ \omega(x) = (0.1, 0.2)^{\top},\ -0.4 \le x \le 0,\\ (\mathbb{I}^{0.7}_{-0.4^+}y)(-0.4^+) = \omega(-0.4),\end{cases} \tag{11}$$

where

$$y(x) = \begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix},\quad B = \begin{pmatrix}0.2 & 0.1\\ 0.3 & 0.5\end{pmatrix},\quad f(x) = \begin{pmatrix}x\\ x^2\end{pmatrix}.$$

By Theorem 3.4, for every $x \in (0.4(k-1), 0.4k],\ k \in \{1,2,3,4\}$, the explicit solution of (11) can be expressed by

$$y(x) = \mathbb{Z}_{0.4,0.3}^{Bx^{0.3}}\omega(-0.4) + \int_{-0.4}^{0}\mathbb{Z}_{0.4,0.3}^{B(x-0.4-s)^{0.3}}(\mathbb{D}^{0.3}_{-0.4^+}\omega)(s)ds + \int_{-0.4}^{x}\mathbb{Z}_{0.4,0.3}^{B(x-0.4-t)^{0.3}}f(t)dt,$$

where

$$\begin{aligned}
&\int_{-0.4}^{0} \mathbb{Z}_{0.4,0.3}^{B(x-0.4-s)^{0.3}} (\mathbb{D}_{-0.4^+}^{0.3}\omega)(s)ds \\
&= \frac{1}{\Gamma(0.7)} \int_{-0.4}^{0} \mathbb{Z}_{0.4,0.3}^{B(x-0.4-s)^{0.3}} \left(\frac{d}{ds} \int_{-0.4}^{s} \begin{pmatrix} 0.1(s-t)^{-0.3} \\ 0.2(s-t)^{-0.3} \end{pmatrix} dt \right) ds \\
&= \frac{1}{\Gamma(0.7)} \int_{-0.4}^{0} \mathbb{Z}_{0.4,0.3}^{B(x-0.4-s)^{0.3}} \begin{pmatrix} 0.1(s+\tau)^{-0.3} \\ 0.2(s+\tau)^{-0.3} \end{pmatrix} ds,
\end{aligned}$$

and

$$\mathbb{Z}_{0.4,0.3}^{Bx^{0.3}} \varphi(-0.4) = \sum_{i=0}^{k} B^i \frac{(x-(i-1)0.4)^{0.3(i+1)-1}}{\Gamma((i+1)0.3)}, \ k \in \{1,2,3,4\}.$$

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