Journal of Computational and Applied Mathematics 236 (2012) 4027–4041

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Analytical and numerical methods for the stability analysis of linear fractional delay differential equations

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ARTICLE INFO

Article history:
Received 14 December 2011
Received in revised form 7 March 2012

Dedicated to the memory of our wonderful colleague, collaborator and friend Professor Donato Trigiante and his wife, Valeria

Keywords:
Fractional differential equation
Method of steps
Laplace transform
Asymptotic stability
BIBO stability
Argument Principle

ABSTRACT

In this paper, several analytical and numerical approaches are presented for the stability analysis of linear fractional-order delay differential equations. The main focus of interest is asymptotic stability, but bounded-input bounded-output (BIBO) stability is also discussed. The applicability of the Laplace transform method for stability analysis is first investigated, jointly with the corresponding characteristic equation, which is broadly used in BIBO stability analysis. Moreover, it is shown that a different characteristic equation, involving the one-parameter Mittag-Leffler function, may be obtained using the well-known method of steps, which provides a necessary condition for asymptotic stability. Stability criteria based on the Argument Principle are also obtained. The stability regions obtained using the two methods are evaluated numerically and comparison results are presented. Several key problems are highlighted.

1. Introduction

Fractional calculus and its applications to the sciences and engineering is a recent focus of interest to many researchers. The generalization of dynamical equations using fractional derivatives proved to be useful and more accurate in mathematical modeling related to many interdisciplinary areas. Applications of fractional calculus and fractional-order differential equations include, but are not limited to modeling real world phenomena such as: dielectric relaxation phenomena in polymeric materials [1], viscoelastic behavior [2], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [3], transport dynamics in systems governed by anomalous diffusion [4,5], long-time memory in financial time series [6], self-similar processes such as protein dynamics [7]. Recently, even fractional-order models of happiness [8] and love [9] have been developed and are claimed to give a better representation than the integer-order dynamical systems approach.

Highly remarkable scientific books which provide the main theoretical tools for the qualitative analysis of fractional-order dynamical systems, and at the same time, show the interconnection as well as the contrast between classical differential equations and fractional differential equations, are [10–12].

It is important to emphasize that in addition to natural similarities that can be drawn between fractional- and integer-order derivatives and fractional- and integer-order dynamical systems, very important differences arise as well. For instance, the fractional-order derivative of a non-constant periodic function cannot be a periodic function of the same

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doi:10.1016/j.cam.2012.03.010

period [13], while the integer-order derivative of a periodic function is indeed a periodic function of the same period. As a consequence, periodic solutions do not exist in a wide class of fractional-order dynamical systems. Therefore, since in many cases, qualitative properties of integer-order dynamical systems cannot be extended by generalization to fractional-order dynamical systems, the analysis of fractional-order dynamical systems is a very important field of research.

Time delays are present inherently in many interconnected real systems due to transportation of energy and materials. Feedback control systems containing time delays and fractional-order processes and/or controllers lead to fractional delay systems. While integer-order delay differential equations have been thoroughly investigated during the past decades [14,15], there is no general stability theory for fractional-order differential equations with delay. A very good survey of the results concerning the stability of fractional differential equations can be found in [16], which focuses especially on the non-delayed case. Finite-time stability for fractional differential equations with time-delays has been studied in [17,18]. Several results related to the bounded-input bounded-output stability of fractional-order control systems with delays have been reported in [19–21]. The stability of some particular linear fractional-order differential systems has been investigated in [22,23].

The aim of this paper is to present several analytical and numerical approaches for the stability analysis of linear fractional delay differential equations of fractional order $q \in (0, 1)$. The main focus of interest is asymptotic stability, but bounded-input bounded-output (BIBO) stability will also be discussed. In Section 2, we review the main asymptotic stability results in the case of integer-order linear delay differential equations and fractional-order differential equations without delay. In Section 3, the applicability of the Laplace transform method for stability analysis is investigated, jointly with the corresponding characteristic equation, which is broadly used in BIBO stability analysis. Problems related to initialization response and to the multivalued nature of the complex power function appearing in the characteristic equation are pointed out. In Section 4, a necessary condition is obtained for the asymptotic stability of the considered fractional delay differential equation using the method of steps, which leads to a different characteristic equation that involves the one-parameter Mittag-Leffler function. It needs to be emphasized that the characteristic equations obtained by the two methods coincide in the integer-order case $q = 1$. The stability regions obtained by the two methods are evaluated numerically and comparison results are presented. Stability criteria based on the Argument Principle are also obtained. Some conclusions are formulated in Section 5.

2. Asymptotic stability in linear fractional-order differential equations with delay

2.1. Preliminaries

In general, three different definitions of fractional derivatives are widely used: the Grünwald–Letnikov derivative, the Riemann–Liouville derivative and the Caputo derivative. These three definitions are in general non-equivalent. However, the main advantage of the Caputo derivative is that it only requires initial conditions given in terms of integer-order derivatives, representing well-understood features of physical situations and thus making it more applicable to real world problems.

Definition 1. For a continuous function f, with $f' \in L^1_{\text{loc}}(\mathbb{R}^+)$, the Caputo fractional-order derivative of order $q \in (0, 1)$ of f is defined by

$$^cD^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} f'(s)ds.$$

Remark 1. When $q \to 1$, the fractional order derivative $^cD^q f(t)$ converges to the integer-order derivative $f'(t)$.

We consider the linear fractional-order differential equation with delay:

$$^C D^q y(t) = Ay(t) + By(t-\tau), \quad t > 0 \tag{1}$$

where $A, B \in \mathbb{R},\ \tau > 0$ represents the time-delay and $q \in (0, 1)$ is the fractional order. The initial condition associated to this equation is

$$y(t) = \phi(t), \quad \forall t \in [-\tau, 0] \tag{2}$$

where $\phi \in L^\infty([-\tau, 0], \mathbb{R})$.

In the following, we will denote by $y(t; \phi)$ the solution of Eq. (1) with the initial condition (2), i.e. $y(t; \phi) = \phi(t)$, for any $t \in [-\tau, 0]$.

Definition 2. The null solution of Eq. (1) is asymptotically stable if and only if there exists $\delta > 0$ such that

$$\lim_{t\to\infty} y(t; \phi) = 0, \quad \text{for any } \phi \in L^\infty([-\tau, 0], \mathbb{R}) \text{ such that } \|\phi\|_\infty = \sup_{t\in[-\tau,0]} |\phi(t)| < \delta.$$

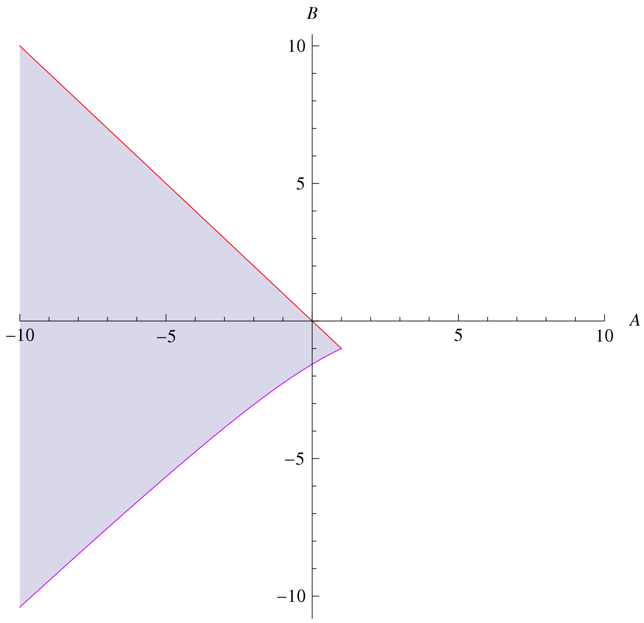

Fig. 1. Asymptotic stability region $S_{1,1}$ (in this case, $\tau = 1$).

Definition 3. The set $S_{q,\tau}$ of parameters $(A, B) \in \mathbb{R}^2$, for which the null solution of Eq. (1) is asymptotically stable, is called the *asymptotic stability region* of Eq. (1).

In the following two subsections, we will review a few well-known results related to the asymptotic stability region $S_{q,\tau}$, in two particular cases: the integer order case ($q = 1$) and in the non-delayed case ($\tau = 0$), respectively.

2.2. The integer-order case $q = 1$

From the theory of delay differential equations (see for example [15]), it is well-known that in the case of integer-order equations of the form

$$y'(t) = Ay(t) + By(t - \tau), \quad \forall\, t > 0, \tag{3}$$

the asymptotic stability region $S_{1,\tau}$ is characterized by the roots of the characteristic equation

$$s - A - Be^{-s\tau} = 0. \tag{4}$$

This characteristic equation may be obtained by seeking exponentially growing solutions of (3) of the form $y(t) = c\, e^{st}$, where $c, s \in \mathbb{C}$, $c \neq 0$.

More precisely, we can express the region $S_{1,\tau}$ as

$$S_{1,\tau} = \{(A, B) \in \mathbb{R}^2 : s - A - Be^{-s\tau} \neq 0, \forall\, s \in \mathbb{C},\ \Re(s) \geq 0\}.$$

The asymptotic stability region $S_{1,\tau}$ (see Fig. 1) is bounded above by the part of the straight line $A + B = 0$, with $A \leq 1$, and bounded below by the curve:

$$C_1 : \begin{cases} A = w \cot(w\tau) \\ B = -w \csc(w\tau), \end{cases} \quad w \in \left[0, \frac{\pi}{\tau}\right),$$

which meet at the point $(A, B) = (1, -1)$.

2.3. The non-delayed case

We first consider the case of non-delayed fractional-order differential equations ($B = 0$),

$${}^{C}D^{q}y(t) = Ay(t), \quad t > 0. \tag{5}$$

The solution of this linear fractional-order equation is (see for example [24]):

$$y(t) = y(0) \cdot E_q(At^q), \quad t > 0, \tag{6}$$

where the one-parameter Mittag-Leffler function E_q is defined by

$$E_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(qk+1)}.$$

When $q = 1$, we have $E_q(z) = e^z$.

Based on the properties of the one-parameter Mittag-Leffler function, it is known that the null solution of (5) is asymptotically stable if and only if $A < 0$. However, if $A < 0$, it has to be emphasized [25] that when $q \in (0, 1)$, the solution $y(t)$ of Eq. (5) exhibits an algebraic decay to 0, while in the integer-order case $q = 1$, the solution $y(t) = y(0) \cdot e^{At}$ decays exponentially to 0. Due to this observation, a special type of non-exponential asymptotic stability has been defined for fractional-order differential equations [26], called Mittag-Leffler stability, which in our context, can be expressed as follows:

Definition 4. The null solution of Eq. (1) is said to be Mittag-Leffler stable if and only if there exist $a > 0$, $b > 0$ and $\delta > 0$ and a locally Lipschitz function $m : (-\delta, \delta) \to [0, \infty)$, $m(0) = 0$, such that for any initial condition $\phi \in L^{\infty}([-\tau, 0], \mathbb{R})$ such that $\|\phi\|_{\infty} < \delta$, we have

$$|y(t; \phi)| \le \left\{ m\left(\|\phi\|_{\infty}\right) \cdot E_q(-at^q) \right\}^b, \quad \forall\, t > 0.$$

According to the results presented above, it can be easily seen that

$$R^- = \{(A, 0) \in \mathbb{R}^2 : A < 0\} \subset S_{q,\tau}, \quad \forall\, q \in (0, 1),\ \tau > 0$$

and

$$R^+ = \{(A, 0) \in \mathbb{R}^2 : A \ge 0\} \subset \mathbb{R}^2 \setminus S_{q,\tau}, \quad \forall\, q \in (0, 1),\ \tau > 0.$$

In the following sections, our main aim is to characterize the asymptotic stability domain $S_{q,\tau}$, using characteristic equations. Two different approaches will be explored: the Laplace transform method (in Section 3) and the method of steps (in Section 4).

3. Considerations on the applicability of the Laplace transform method

Having reviewed the main results related to the asymptotic stability of integer-order linear delay differential equations and non-delayed linear fractional-order differential equations, it is now time to direct our attention towards obtaining a characteristic equation for the general case of Eq. (1).

It is well-known that the characteristic equation of an integer-order delayed differential equation may be deduced using the Laplace transform method. In this section, we attempt to apply this method with the aim of determining a characteristic equation for (1). It will be pointed out that the characteristic equation that is obtained in this way plays an important role in bounded-input bounded-output (BIBO) stability analysis. However, it is yet unclear what is the link between this characteristic equation and the asymptotic stability of (1). Moreover, several questions will be raised regarding the applicability of the final value theorem of the Laplace transform, as well as the problem of initialization response in fractional-order systems.

3.1. Applications of the Laplace transform method and the final value theorem

Denoting by $Y(s) = \mathcal{L}[y](s)$ the Laplace transform of the function $y(t) = y(t; \phi)$, we can evaluate

$$\mathcal{L}[y(t-\tau)](s) = \int_0^{\infty} e^{-st} y(t-\tau)dt = \int_{-\tau}^{\infty} e^{-s(t+\tau)} y(t)dt = e^{-s\tau} Y(s) + e^{-s\tau} \int_{-\tau}^{0} e^{-st} \phi(t)dt.$$

Denoting $\Phi(s) = \int_{-\tau}^{0} e^{-st}\phi(t)dt$ and taking into consideration that the Laplace transform of the Caputo derivative, when $q \in (0, 1)$, is given by [10]:

$$\mathcal{L}\left[{}^{C}D^q y(t)\right](s) = s^q \mathcal{L}[y](s) - s^{q-1} y(0)$$

we obtain by applying the Laplace transform to (1):

$$s^q Y(s) - s^{q-1}\phi(0) = AY(s) + B\left(e^{-s\tau} Y(s) + e^{-s\tau}\Phi(s)\right)$$

which is equivalent to

$$Y(s) = Y_{\phi}(s) = \frac{s^{q-1}\phi(0) + Be^{-s\tau}\Phi(s)}{s^q - A - Be^{-s\tau}}, \quad \text{where } \Phi(s) = \int_{-\tau}^{0} e^{-st}\phi(t)dt. \tag{7}$$

Obviously, the behavior of the function $Y_\phi(s)$ depends on the initial condition $\phi(t),\ t \in [-\tau, 0]$. There are two classes of poles for the function $Y_\phi(s)$: the poles of the function $\Phi(s)$ and the roots of the equation

$$s^q - A - Be^{-s\tau} = 0. \tag{8}$$

Therefore, if we attempt to use the final value theorem of the Laplace transform [27], with the aim of proving the convergence of $y(t;\phi)$ to 0 as $t \to \infty$, we need to take into consideration all the poles of $Y_\phi(s)$: not only the roots of Eq. (8), but also the poles of the function $\Phi(s)$, as it can be seen in the following example.

Example 1. Considering a constant initial condition $\phi(t) = \alpha \in \mathbb{R}$, for any $t \in [-\tau, 0]$, it follows that $\Phi(s) = \alpha s^{-1}(e^{s\tau} - 1)$. Therefore, in this case:

$$Y_\phi(s) = \alpha \cdot \frac{s^q + B(1 - e^{-s\tau})}{s(s^q - A - Be^{-s\tau})},$$

and hence, in general, besides all the roots of Eq. (8), $s = 0$ is a single pole of $Y_\phi(s)$ as well.

For the initial condition $\phi(t) = \beta \sin(t)$, for any $t \in [-\tau, 0]$, where $\beta \in \mathbb{R}$, it is easy to compute $\Phi(s) = \beta(1 + s^2)^{-1}[e^{s\tau}(\cos(\tau) - s\sin(\tau)) - 1]$ and hence

$$Y_\phi(s) = \beta \cdot \frac{B\,[\cos(\tau) - s\sin(\tau) - e^{-s\tau}]}{(s^2 + 1)(s^q - A - Be^{-s\tau})}.$$

In this case, $\pm i$ are poles of $Y_\phi(s)$ as well, besides all the roots of Eq. (8).

Based on the previous observations, the following result can be formulated.

Proposition 1. *If all the poles of the function $Y_\phi(s)$ given by (7) are either in the open left half-plane ($\Re(s) < 0$) or at the origin, and $Y_\phi(s)$ has at most a single pole at the origin, then the solution $y(t;\phi)$ of Eq. (1) converges to 0 as $t \to \infty$.*

Proof. A simple application of the final value theorem of the Laplace transform [27] yields

$$\lim_{t\to\infty} y(t;\phi) = \lim_{s\to 0} sY_\phi(s) = \lim_{s\to 0} \frac{s^q\phi(0) + Bse^{-s\tau}\Phi(s)}{s^q - A - Be^{-s\tau}} = 0. \quad \square$$

Remark 2. It has to be emphasized that a correct application of the final value theorem of the Laplace transform, as in Proposition 1, clearly does not provide a general sufficient condition for asymptotic stability of the following form: "*If all the roots of the* Eq. (8) *are in the open left half-plane, then the null solution of* (1) *is asymptotically stable*".

Recently, in [23], sufficient conditions have been presented for the asymptotic stability of the null solution of a linear system of fractional differential equations with delays, using the final value theorem of the Laplace transform. However, in their proof, the authors failed to take into consideration the importance of the poles of the functions $\Phi_i(s)$ that result from the initial conditions $\phi_i(t),\ t \in [-\tau, 0]$ associated to the system. In light of all our previous observations, the proof of the main Theorem 1 of [23] is incorrect.

Remark 3. In [22], the stability of the linear fractional-order differential equation of the form

$${}^C D^q y(t) = By(t - \tau) \tag{9}$$

has been investigated (see Eq. (10) in [22]). However, a very important hypothesis considered by the authors was that all initial values are zero. This means that in our setting, the initial condition is $\phi_0(t) = 0$, for all $t \in [-\tau, 0)$, and possibly, $\phi_0(0) \neq 0$. Therefore, the Laplace transform of the solution $y(t;\phi_0)$ of (9) is

$$Y_{\phi_0}(s) = \frac{s^{q-1}\phi(0)}{s^q - Be^{-s\tau}}.$$

Indeed, in this case, the final value theorem provides that if all the roots of the equation

$$s^q - Be^{-s\tau} = 0$$

have negative real part, then the solution $y(t;\phi_0)$ of (9) converges to zero as $t \to \infty$. However, as a special type of initial conditions has been considered, it is clear that these results cannot be extended to the problem of asymptotic stability. Instead, the results presented in [22] are related to the concept of BIBO stability.

3.2. Some remarks on BIBO stability analysis

In control theory, the concept of bounded-input, bounded-output (BIBO) stability (external stability) plays a very important role.

Definition 5. The linear control system

$$\begin{cases} {}^{C}D^{q}y(t) = Ay(t) + By(t-\tau) + u(t) \\ z(t) = Cy(t), \end{cases} \quad t > 0 \tag{10}$$

where $A, B, C \in \mathbb{R}$, is called *BIBO stable* if for zero initial conditions ($\phi(t) = 0$, for any $t \in [-\tau, 0]$), a bounded input $u(t)$ always evokes a bounded output $z(t)$.

The transfer function of the control system (10) is

$$G(s) = \frac{C}{s^q - A - Be^{-s\tau}}.$$

It follows from [19] that the system (10) is BIBO stable if and only if all the roots of the characteristic equation (8) are in the open left half-plane.

In the following, we denote the BIBO-stability region of the control system (10) by

$$R_{q,\tau} = \{(A, B) \in \mathbb{R}^2 : s^q - A - Be^{-s\tau} \neq 0,\ \forall s \in \mathbb{C},\ \Re(s) \geq 0\}. \tag{11}$$

In the integer-order case $q = 1$, the region $R_{1,\tau}$ coincides with the stability region $S_{1,\tau}$, i.e. $R_{1,\tau} = S_{1,\tau}$, for any $\tau > 0$. However, to the best of our knowledge, for the fractional-order case $q \in (0, 1)$, the relationship between the BIBO-stability region $R_{q,\tau}$ and the asymptotic stability region $S_{q,\tau}$ is not known.

Moreover, when $q \in (0, 1)$, the complex power function s^q is a multivalued function, whose domain can be seen as a Riemann surface, which has a finite number of sheets only when $q \in \mathbb{Q} \cap (0, 1)$. In the case when $q \in \mathbb{Q} \cap (0, 1)$, only the main sheet defined by $-\pi < \arg(s) < \pi$ is considered to have physical meaning in BIBO-stability analysis. However, in the case when q is irrational, the Riemann surface has an infinity of sheets.

With the aim of determining the boundary of $R_{q,\tau}$ in the general case, we assume that Eq. (8) has a solution $s = i\omega$, $\omega \in \mathbb{R}$, on the imaginary axis. Hence, we obtain

$$(i\omega)^q = A + Be^{-i\omega\tau}. \tag{12}$$

For $q \in (0, 1)$, we write

$$(i\omega)^q = e^{q \ln(i\omega)} = e^{q(\mathrm{Ln}(i\omega) + 2ik\pi)}, \quad k \in \mathbb{Z},$$

where Ln represents the principal branch of the complex logarithm function ln. Therefore:

$$\mathrm{Ln}(i\omega) = \ln|i\omega| + i\mathrm{Arg}(i\omega) = \ln|\omega| + i\,\mathrm{sign}(\omega)\frac{\pi}{2},$$

and hence

$$(i\omega)^q = e^{q(\ln|\omega| + i\,\mathrm{sign}(\omega)\frac{\pi}{2} + 2ik\pi)} = |\omega|^q e^{iq\pi(\mathrm{sign}(\omega)/2 + 2k)}, \quad k \in \mathbb{Z}.$$

Equating the real and imaginary parts in Eq. (12), we obtain

$$A + B\cos(\omega\tau) = |\omega|^q \cos(q\pi(\mathrm{sign}(\omega)/2 + 2k))$$
$$-B\sin(\omega\tau) = |\omega|^q \sin(q\pi(\mathrm{sign}(\omega)/2 + 2k)).$$

If we assume $\omega\tau = m\pi$, with $m \in \mathbb{Z}$, from the second equation we obtain that $\omega = 0$, and hence, from the first equation we get $A + B = 0$. Therefore, we observe that $\omega \notin \left\{\frac{m\pi}{\tau},\ m \in \mathbb{Z}^\star\right\}$ and we obtain the following parametric equations

$$\Gamma^q_{k,m} : \begin{cases} A = |\omega|^q \csc(\omega\tau)\sin(\omega\tau + q\pi(\mathrm{sign}(\omega)/2 + 2k)) \\ B = -|\omega|^q \csc(\omega\tau)\sin(q\pi(\mathrm{sign}(\omega)/2 + 2k)), \end{cases} \quad \omega \in \left(\frac{m\pi}{\tau}, \frac{(m+1)\pi}{\tau}\right),\ m \in \mathbb{Z},\ k \in \mathbb{Z}$$

defining the curves $\Gamma^q_{k,m}$ in the (A, B)-plane, where $k, m \in \mathbb{Z}$.

It can be easily seen that for rational values of q, i.e. $q = \frac{n_1}{n_2} \in \mathbb{Q} \cap (0, 1)$, where $n_1, n_2 \in \mathbb{Z}_+$ such that $\gcd(n_1, n_2) = 1$, it is enough to take into consideration the curves $\Gamma^q_{k,m}$ with $k \in \{0, 1, \ldots, n_2 - 1\}$. The curves $\Gamma^q_{0,m}$, $m \in \mathbb{Z}$ correspond to the main Riemann sheet (see Fig. 2). The curve $\Gamma^q_{0,0}$ constitutes the lower bound of the BIBO-stability domain $R_{q,\tau}$, while the line $A + B = 0$ represents the upper bound. However, for irrational values of q, i.e. $q \in (0, 1) \setminus \mathbb{Q}$, an infinity of values for $k \in \mathbb{Z}$ have to be considered.

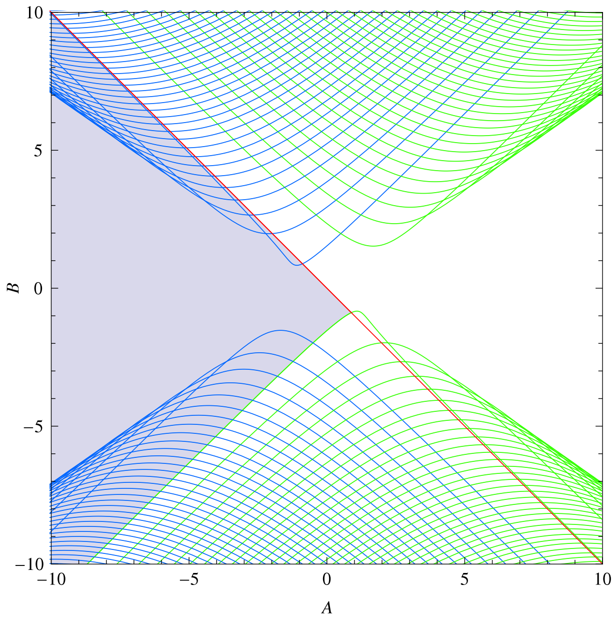

Fig. 2. For $q = \frac{1}{2}$ and $\tau = 1$, the curves $\Gamma^q_{0,m}$ (green) and $\Gamma^q_{1,m}$ (blue) (where $m \in \{0, 1, \dots, 70\}$), the line $A + B = 0$ (red) and the BIBO-stability region $R_{q,\tau}$ (shaded). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

3.3. Problems related to initialization response

Hartley and Lorenzo [28–30] have repeatedly pointed out in several papers that due to the importance of historical effects, a time-varying initialization response needs to be taken into account in fractional-order systems. In [30], they showed that the initialization response depends on the history function of the fractional-order derivative. Moreover, they have showed that the Laplace transform of the Caputo derivative provides the initialization response for history functions that turn on at time $t = -\infty$. Therefore, it provides only an approximation to the exact solutions which usually have finite turn-on times for their history functions.

The Laplace transform of Caputo derivative of the function $y(t)$ given in [10] is

$$\mathcal{L}\left[{}^{C}D^q y(t)\right](s) = s^q \mathcal{L}[y](s) - \sum_{k=0}^{\lfloor q\rfloor} s^{q-k-1} y^{(k)}(0),$$

which gives, for $q \in (0, 1)$,

$$\mathcal{L}\left[{}^{C}D^q y(t)\right](s) = s^q \mathcal{L}[y](s) - s^{q-1} y(0) = s^q \mathcal{L}[y](s) - s^{q-1}\phi(0).$$

In our case, the initial function (2) associated to Eq. (1) is given on the finite interval $[-\tau, 0]$. But the Laplace transform of the Caputo derivative transforms it to only at $t = 0$, and the term $s^{q-1}\phi(0)$ plays an important role. However, the initial function $\phi(t),\ t \in [-\tau, 0)$ does not influence the Laplace transform of the Caputo derivative of $y(t)$, unless it is constantly equal to $\phi(0)$. The fractional delay differential equation (1) has a delay term on the right hand side and because of this, when applying the Laplace transform, we get an additional contribution of the initial function.

On the other hand, it is interesting to see the that Laplace transform of the Riemann–Liouville derivative [10] gives

$$\mathcal{L}\left[{}^{RL}D^q y(t)\right](s) = s^q \mathcal{L}[y](s) - \sum_{k=0}^{\lfloor q\rfloor} s^k \left[{}^{RL}D^{q-k-1} y(t)\right]_{t=0},$$

which, in our case $q \in (0, 1)$, becomes

$$\mathcal{L}\left[{}^{RL}D^q y(t)\right](s) = s^q \mathcal{L}[y](s) - \left[{}^{RL}D^{q-1} y(t)\right]_{t=0} = s^q \mathcal{L}[y](s) - \frac{1}{\Gamma(1-q)}\left[\int_0^t (t-s)^{-q} y(s)ds\right]_{t=0}.$$

Moreover, if we assume that the function $y(t)$ is bounded in a neighborhood of the origin, $|y(t)| < M$, for any $t \in [0, \varepsilon]$, it follows that

$$\left|\int_0^t (t-s)^{-q} y(s)ds\right| \le M \int_0^t (t-s)^{-q} ds = -\frac{M}{1-q}\ (t-s)^{1-q}\Big|_0^t = \frac{M}{1-q} t^{1-q} \xrightarrow{t\to 0} 0.$$

Hence, the Laplace transform of the Riemann–Liouville derivative is, in our case:

$$\mathcal{L}\left[{}^{RL}D^q y(t)\right](s) = s^q \mathcal{L}[y](s),$$

which does not take into account the history function $\phi(t)$, $t \in [-\tau, 0]$ at all, or it may be considered that it corresponds to null initial conditions.

Based on the above observations, the Laplace transform method should be used with caution. Further investigations are required to address the applicability of the Laplace transforms for the initialized fractional derivative provided in [29], in the framework of the analysis of Eq. (1), but this will be the subject of a future paper.

4. A characteristic equation obtained by the method of steps

Recently, Kalmár-Nagy [31] has shown that the characteristic equation of an integer-order linear delay differential equation can also be obtained using the method of steps. In this section, we generalize the results from [31] to obtain a necessary condition for the asymptotic stability of the null solution of (1), using the method of steps.

Another recent application of the method of steps has been presented in [18], as a tool for proving existence and uniqueness theorems for the initial value problem of a linear non-autonomous fractional-order time-delay system, and for investigating finite-time stability.

4.1. A necessary condition for asymptotic stability obtained by the method of steps

The procedure of evaluating the solutions of (1) using this method of steps is presented below.

Let $y(t) = y(t;\phi)$ be the solution of (1) with the initial condition (2). We denote, for every $n \in \mathbb{Z}_+$:

$$y_n(t) = \begin{cases} y(t + (n-1)\tau), & \text{if } t \in [0, \tau] \\ 0, & \text{otherwise .} \end{cases}$$

Eq. (1) is equivalent to

$${}^C D^q y(t + n\tau) = Ay(t + n\tau) + By(t + (n-1)\tau), \quad \forall\, t \in [0, \tau),\ n \in \mathbb{Z}_+^\star$$

or

$${}^C D^q y_n(t) = Ay_n(t) + By_{n-1}(t), \quad \forall\, t \geq 0,\ n \in \mathbb{Z}_+^\star.$$

Applying the Laplace transform, we obtain

$$s^q Y_n(s) - s^{q-1} y_n(0) = AY_n(s) + BY_{n-1}(s)$$

and hence

$$Y_n(s) = \frac{BY_{n-1}(s) + s^{q-1} y_n(0)}{s^q - A}, \quad \forall\, n \in \mathbb{Z}_+^\star.$$

We can evaluate

$$Y_0(s) = \int_0^\infty e^{-st} y_0(t)dt = \int_0^\tau e^{-st} y(t-\tau)dt = e^{-s\tau} \int_{-\tau}^0 e^{-st}\phi(t)dt.$$

Denoting $g(s) = \frac{B}{s^q - A}$ and $f(s) = \frac{s^{q-1}}{s^q - A}$ we get

$$Y_n(s) = g(s)Y_{n-1}(s) + f(s)y_n(0)$$

and by mathematical induction, it follows that

$$Y_n(s) = g(s)^n Y_0(s) + f(s) \sum_{k=1}^n g(s)^{n-k} y_k(0)$$

or

$$Y_n(s) = \left(\frac{B}{s^q - A}\right)^n Y_0(s) + \frac{s^{q-1}}{s^q - A} \sum_{k=1}^n \left(\frac{B}{s^q - A}\right)^{n-k} y_k(0).$$

Applying the inverse Laplace transform we obtain

$$y_n(t) = \mathcal{L}^{-1}[g(s)^n Y_0(s)](t) + \sum_{k=1}^n \mathcal{L}^{-1}[f(s)g(s)^{n-k}](t) y_k(0).$$

On one hand, we have (see Eq. (1.80) in [10]):

$$\mathcal{L}^{-1}[g(s)^n](t) = B^n\mathcal{L}^{-1}\left[\frac{1}{(s^q-A)^n}\right](t) = \frac{B^n}{(n-1)!}t^{qn-1}E_{q,q}^{(n-1)}(At^q),$$

where $E_{\alpha,\beta}$ denotes the two-parameter Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}[g(s)^nY_0(s)](t) &= \left(\mathcal{L}^{-1}[g(s)^n]*y_0\right)(t)\\ &= \frac{B^n}{(n-1)!}\int_0^t u^{qn-1}E_{q,q}^{(n-1)}(Au^q)y_0(t-u)du\\ &= \frac{B^n}{(n-1)!}\int_{t-\tau}^t u^{qn-1}E_{q,q}^{(n-1)}(Au^q)\phi(t-\tau-u)du\\ &= \frac{B^n}{(n-1)!}\int_0^\tau (t-u)^{qn-1}E_{q,q}^{(n-1)}(A(t-u)^q)\phi(u-\tau)du.\end{aligned}$$

On the other hand, we can also see that

$$\mathcal{L}^{-1}[f(s)g(s)^j](t) = \mathcal{L}^{-1}\left[\frac{B^js^{q-1}}{(s^q-A)^{j+1}}\right](t) = \frac{B^j}{j!}t^{qj}E_q^{(j)}(At^q).$$

Hence, for any $t\in[0,\tau]$ and $n\in\mathbb{Z}_+^\star$ we have

$$y_n(t) = \frac{B^n}{(n-1)!}\int_0^\tau (t-u)^{qn-1}E_{q,q}^{(n-1)}(A(t-u)^q)\phi(u-\tau)du + \sum_{k=1}^n\frac{B^{n-k}}{(n-k)!}t^{q(n-k)}E_q^{(n-k)}(At^q)y_k(0). \tag{13}$$

We remark that Eq. (13) can also be obtained without the use of Laplace transforms, via the Volterra integral representation of Eq. (1) and mathematical induction.

Based on all the facts described above, we can formulate the following necessary condition for the global asymptotic stability of (1):

Theorem 1. *If the null solution of the linear fractional-order delay differential equation (1) is globally asymptotically stable, then all the roots of the characteristic equation*

$$\lambda - E_q\left[\left(A+\frac{B}{\lambda}\right)\tau^q\right] = 0 \tag{14}$$

are inside the unit circle.

Proof. If we assume that the null solution of Eq. (1) is globally asymptotically stable, i.e. $\lim_{t\to\infty}y(t)=0$, it follows that $\lim_{n\to\infty}y(n\tau)=\lim_{n\to\infty}y_{n+1}(0)=0$. Considering $t=\tau$ in (13) we obtain

$$y_{n+1}(0) = y_n(\tau) = \frac{B^n}{(n-1)!}\int_0^\tau(\tau-u)^{qn-1}E_{q,q}^{(n-1)}(A(\tau-u)^q)\phi(u-\tau)du + \sum_{k=1}^n\frac{B^{n-k}}{(n-k)!}\tau^{q(n-k)}E_q^{(n-k)}(A\tau^q)y_k(0).$$

In this equation, we may neglect the first term, because stability should not depend on the form of the initial function, i.e. ϕ can be chosen so as to make this term negligible. Denoting $z_k=y_{k+1}(0)$, we obtain the difference equation

$$z_n \simeq \sum_{k=1}^n\frac{B^{n-k}}{(n-k)!}\tau^{q(n-k)}E_q^{(n-k)}(A\tau^q)z_{k-1}$$

or equivalently

$$z_{n+1} \simeq \sum_{k=0}^n\frac{B^k}{k!}\tau^{qk}E_q^{(k)}(A\tau^q)z_{n-k}. \tag{15}$$

Denoting

$$C_k = \frac{B^k}{k!}\tau^{qk}E_q^{(k)}(A\tau^q),\quad \forall\, k\in\mathbb{Z}_+,$$

Eq. (15) can be written as

$$z_{n+1} = (C \star z)(n)$$

where $\star$ denotes the discrete convolution. Based on the previous considerations, it is clear that the solution of the difference equation (15) must converge to 0 as $n \to \infty$, for any $z_0 = y(0)$. According to the theory of scalar linear difference equations of convolution type (Volterra difference equations) [32], it follows that the null solution of (15) is asymptotically stable if and only if all the roots of the characteristic equation

$$\lambda - \hat{C}(\lambda) = 0$$

are inside the unit circle, where $\hat{C}$ denotes the Z-transform of C_k. Taking into account the Taylor series representation of the Mittag-Leffler function $E_q(t)$ (in this case $t \in \mathbb{R}$), we can easily compute

$$\hat{C}(\lambda) = \sum_{k=0}^{\infty} C_k \lambda^{-k} = \sum_{k=0}^{\infty} \frac{B^k}{k!} \tau^{qk} E_q^{(k)}(A\tau^q)\lambda^{-k} = \sum_{k=0}^{\infty} \frac{E_q^{(k)}(A\tau^q)}{k!} \left(\frac{B\tau^q}{\lambda}\right)^k = E_q\left[\left(A + \frac{B}{\lambda}\right)\tau^q\right].$$

We finally obtain the characteristic equation

$$\lambda - E_q\left[\left(A + \frac{B}{\lambda}\right)\tau^q\right] = 0.$$

Hence, if the null solution of (1) is asymptotically stable, then all the roots of the characteristic equation (14) are inside the unit circle. □

Remark 4. For the particular case $q = 1$, Eq. (14) becomes

$$\lambda - e^{\left(A + \frac{B}{\lambda}\right)\tau} = 0$$

which is consistent with Eq. (4.32) from [31]. Denoting $s = A + \frac{B}{\lambda}$, we deduce $\lambda = \frac{B}{s-A}$, and obtain

$$\frac{B}{s-A} = e^{s\tau},$$

or equivalently

$$s - A - Be^{-s\tau} = 0,$$

which is the well-known characteristic equation (4) for integer-order delay differential equations.

Denoting

$$U_{q,\tau} = \left\{(A, B) \in \mathbb{R}^2 : \lambda - E_q\left[\left(A + \frac{B}{\lambda}\right)\tau^q\right] \neq 0,\ \forall\, \lambda \in \mathbb{C},\ |\lambda| \geq 1\right\}$$

from Theorem 1, it follows that

$$S_{q,\tau} \subset U_{q,\tau}, \quad \forall q \in (0, 1),\ \tau > 0.$$

4.2. *Estimating the boundary of $U_{q,\tau}$*

In what follows, our aim is to estimate the boundary of the region $U_{q,\tau}$. First, based on the properties of the Mittag-Leffler function, we observe that $\lambda = 1$ is a solution of the characteristic equation (14) if and only if $A + B = 0$. Therefore, similarly as in the integer-order case $q = 1$, we expect that the upper bound of the region $U_{q,\tau}$ will be given by the straight line $A + B = 0$. As for the lower bound, it can be determined by imposing that Eq. (14) has a solution on the unit circle, more precisely, a solution of the form $\lambda = e^{iw}$, where $w \in (0, \pi)$. Hence, we obtain

$$e^{iw} = E_q\left((A + Be^{-iw})\tau^q\right)$$

which leads to the system of nonlinear equations

$$C_q : \begin{cases} \cos(w) = \Re\left[E_q\left((A + Be^{-iw})\tau^q\right)\right] \\ \sin(w) = \Im\left[E_q\left((A + Be^{-iw})\tau^q\right)\right], \end{cases} \quad w \in (0, \pi) \tag{16}$$

that defines implicitly a curve C_q in the (A, B)-plane. With the aim of plotting this curve, we choose a finite set of values of w from the interval $(0, \pi)$. For each w from this finite set of values, we solve the nonlinear system (16) numerically, with respect to the unknowns (A, B), using a numerical method such as Newton's or Broyden's method (see Algorithm 1). Finally, we obtain the equation of the curve C_q by interpolation (polynomial or spline interpolation may be used).

Algorithm 1 Numerical estimation of the curve C_q

Input: Fractional order $q \in (0, 1)$; precisions p_1 and p_2.
Output: Function f which gives the explicit equation $B = f(A)$ of the curve C_q.
Let L be an empty list.
Set $(A_0, B_0) = (1, -1)$.
for all $w \in [p_1, \pi)$ increasing with step p_1 **do**
    Use a numerical method (with initial guess (A_0, B_0)) to derive the solution (A, B) of the system (16) with precision p_2.
    Append (A, B) to L.
    Set $(A_0, B_0) = (A, B)$.
end for
$f \leftarrow Interpolation(L)$.

In all the numerical results presented in this paper (using Wolfram Mathematica 8), for the implementation of the Mittag-Leffler function E_q, the numerical algorithm proposed in [33] has been applied, which uses Taylor series, exponentially improved asymptotic series, and integral representations to obtain optimal stability and accuracy.

For $q \in \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$ and $\tau = 1$, the regions $U_{q,\tau}$ have been plotted in Figs. 3–5. Comparing with the BIBO-stability region $R_{q,\tau}$, the following conclusions can be drawn:

- When $q = \frac{1}{2}$ (Fig. 3), the lower bounds of the regions $U_{q,\tau}$ and $R_{q,\tau}$, i.e. the curves C_q and $\Gamma^q_{0,0}$, respectively, intersect at the point $(0.58257, -1.07094)$.
- When $q = \frac{1}{3}$ (Fig. 4), we have $R_{q,\tau} \subset U_{q,\tau}$, because the curve $\Gamma^q_{0,0}$ is above the curve C_q.
- When $q = \frac{2}{3}$ (Fig. 5), we have $U_{q,\tau} \subset R_{q,\tau}$, because the curve $\Gamma^q_{0,0}$ is below the curve C_q. In this case, from Theorem 1 we have: $S_{q,\tau} \subset U_{q,\tau} \subset R_{q,\tau}$, and hence, asymptotic stability implies BIBO-stability.

All numerical simulations suggest that in certain cases, the curves $\Gamma^q_{0,0}$ and C_q may be considered as each other's approximations.

4.3. Stability criteria based on the Argument Principle

It can be easily seen (denoting $\lambda^{-1} = \mu$) that

$$U_{q,\tau} = \left\{(A, B) \in \mathbb{R}^2 : \mu E_q\left[(A + B\mu)\,\tau^q\right] - 1 \neq 0,\ \forall\, \mu \in \mathbb{C},\ |\mu| \leq 1\right\}.$$

Considering the function

$$F_{A,B}(z) = zE_q\left[(A + Bz)\,\tau^q\right] - 1, \quad \forall\, z \in \mathbb{C} \tag{17}$$

we first remark that since the Mittag-Leffler function E_q is an entire function, it follows that the function $F_{A,B}$ is holomorphic on the whole complex plane. Hence, the number of zeros of the function $F_{A,B}$ inside the unit circle (denoted in the following by γ) can be determined using the Argument Principle:

$$N_\gamma = \frac{1}{2\pi i}\oint_\gamma \frac{F'_{A,B}(z)}{F_{A,B}(z)}dz = \frac{1}{2\pi}\Delta_\gamma \arg F_{A,B}(z) = W(F_{A,B}(\gamma), 0),$$

where $\Delta_\gamma \arg F_{A,B}(z)$ represents the change in the argument of the function $F_{A,B}$ over γ and $W(F_{A,B}(\gamma), 0)$ denotes the winding number of the curve $F_{A,B}(\gamma)$ around the origin.

Therefore, the parameters (A, B) belong to the region $U_{q,\tau}$ if and only if $W(F_{A,B}(\gamma), 0) = 0$. Taking into account Theorem 1, the following result can be easily formulated:

Proposition 2.

a. *If the null solution of* (1) *is asymptotically stable, then* $W(F_{A,B}(\gamma), 0) = 0$.
b. *If* $W(F_{A,B}(\gamma), 0) \neq 0$ *then the null solution of* (1) *is not asymptotically stable.*

We remark that these types of results are rooted in the Mikhailov stability criterion [34], which is essentially a geometrical interpretation of the Argument Principle, and its modifications which are well-known in the stability theory of integer-order systems. Similar frequency domain methods for the BIBO-stability analysis of linear fractional-order systems with delays of the retarded type have been given in [21].

Using the following formula for the derivative of the Mittag-Leffler function E_q:

$$E'_q(z) = \frac{E_{q,0}(z)}{qz},$$

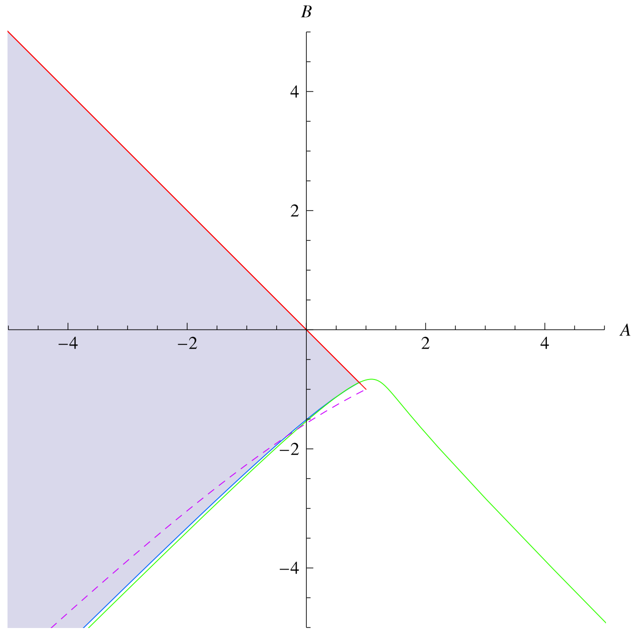

Fig. 3. The region $U_{q,\tau}$ for $\mathbf{q} = \frac{1}{2}$ and $\tau = 1$. The upper bound of this region is $A + B = 0$ (red) and the lower bound is the curve C_q (blue). The green curve $\Gamma^q_{0,0}$ constitutes the lower bound of the BIBO-stability region $R_{q,\tau}$ (see Fig. 2). The intersection point of C_q and $\Gamma^q_{0,0}$ is $(0.58257, -1.07094)$. The dashed line (purple) represents the curve C_1 (see Fig. 1). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

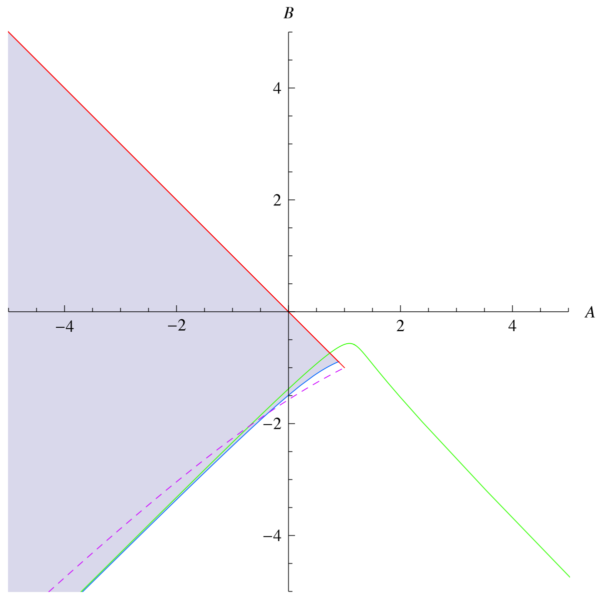

Fig. 4. The region $U_{q,\tau}$ for $\mathbf{q} = \frac{1}{3}$ and $\tau = 1$. In this case, the region $U_{q,\tau}$ includes the BIBO-stability region $R_{q,\tau}$, as the curve $\Gamma^q_{0,0}$ (green) is above the curve C_q (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

based on (17), the winding number $W(F_{A,B}(\gamma), 0)$ can be evaluated as

$$W(F_{A,B}(\gamma), 0) = \frac{1}{2\pi i} \oint_\gamma \frac{F'_{A,B}(z)}{F_{A,B}(z)} dz$$
$$= \frac{1}{2\pi i} \oint_\gamma \frac{E_q[(A+Bz)\tau^q] + B\tau^q z E'_q[(A+Bz)\tau^q]}{zE_q[(A+Bz)\tau^q] - 1} dz$$

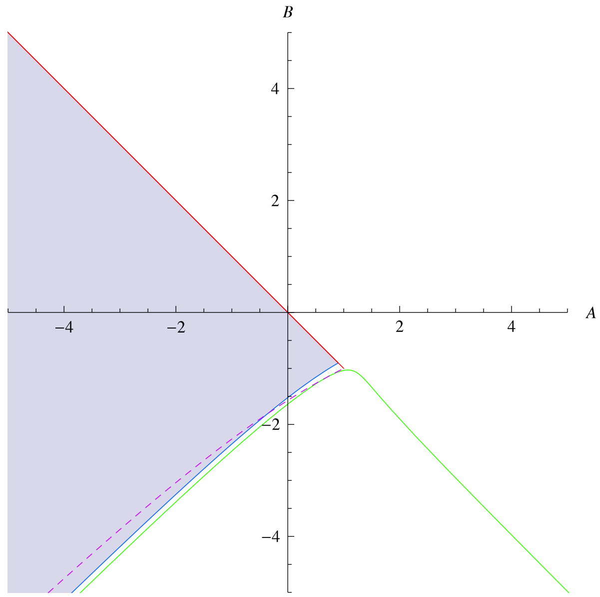

Fig. 5. The region $U_{q,\tau}$ for $\mathbf{q} = \frac{2}{3}$ and $\tau = 1$. In this case, the region $U_{q,\tau}$ is included in the BIBO-stability region $R_{q,\tau}$, as the curve $\Gamma^q_{0,0}$ (green) is below the curve C_q (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$= \frac{1}{2\pi i}\oint_\gamma \frac{E_q[(A+Bz)\tau^q] + Bq^{-1}z(A+Bz)^{-1}E_{q,0}[(A+Bz)\tau^q]}{zE_q[(A+Bz)\tau^q]-1}dz$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{E_q[(A+Be^{i\omega})\tau^q] + Bq^{-1}e^{i\omega}(A+Be^{i\omega})^{-1}E_{q,0}[(A+Be^{i\omega})\tau^q]}{e^{i\omega}E_q[(A+Be^{i\omega})\tau^q]-1}e^{i\omega}d\omega$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{E_q[(A+Be^{i\omega})\tau^q] + Bq^{-1}(Ae^{-i\omega}+B)^{-1}E_{q,0}[(A+Be^{i\omega})\tau^q]}{E_q[(A+Be^{i\omega})\tau^q]-e^{-i\omega}}d\omega.$$

This last integral can be computed using a numerical integration scheme, where the Mittag-Leffler functions E_q and $E_{q,0}$ may be implemented using the algorithm presented in [33].

The following graphical test can also be used: if for fixed values of the parameters (A, B), the graph of the closed curve $F_{A,B}(\gamma) = F_{A,B}(e^{i\omega})$, $\omega \in [0, 2\pi]$, encircles the origin, then the null solution of Eq. (1) cannot be asymptotically stable. On the other hand, if the graph of $F_{A,B}(\gamma)$ does not encircle the origin, the null solution of (1) may be asymptotically stable.

Example 2. In the case $q = \frac{1}{2}$, $\tau = 1$ and $A = -1$ fixed, the curve $F_{A,B}(\gamma)$ has been plotted in Fig. 6 for different values of the parameter B in the range $[-3, 1.5]$. Based on the numerical evaluation of the explicit equation of the curve C_q presented in the previous subsection, we obtain that $(A, B) = (-1, B) \in C_q$ if and only if $B = -2.38544$. Since the upper bound of the region $U_{q,\tau}$ is the line $A + B = 0$, we obtain that $(A, B) \in U_{q,\tau}$ if and only if $B \in (-2.38544, 1)$. In Fig. 6, it can be seen that the winding number $W(F_{A,B}(\gamma), 0)$ is equal to zero for the values of $B \in (-2.38544, 1)$. For $B = 1.5$, the winding number is 1, and hence there is one root of the function $F_{A,B}$ inside the unit circle. When $B \in [-3, -2.38544)$, the winding number is 2, and the function $F_{A,B}$ has two roots inside the unit circle.

5. Conclusions

Analytical and numerical approaches have been investigated for the stability analysis of linear fractional-order delay differential equations, focusing mainly on asymptotic stability, but mentioning BIBO-stability as well. We have shown that the Laplace transform method and the well-known method of steps lead to different characteristic equations that have to be associated to the linear fractional-order delay differential equation of order $q \in (0, 1)$. The characteristic equation obtained by the first approach contains the multivalued complex power function and the exponential function, while the one obtained by the second method contains the one-parameter Mittag-Leffler function. However, the two characteristic equations coincide in the integer order case $q = 1$. Using the characteristic equation involving the Mittag-Leffler function, a necessary condition for asymptotic stability has been obtained, and other stability criteria based on the Argument Principle have also been explored. At this moment, it is still an open question whether the characteristic equation obtained by the method of steps can also provide a sufficient condition for the asymptotic stability of a linear fractional-order delay differential equation.

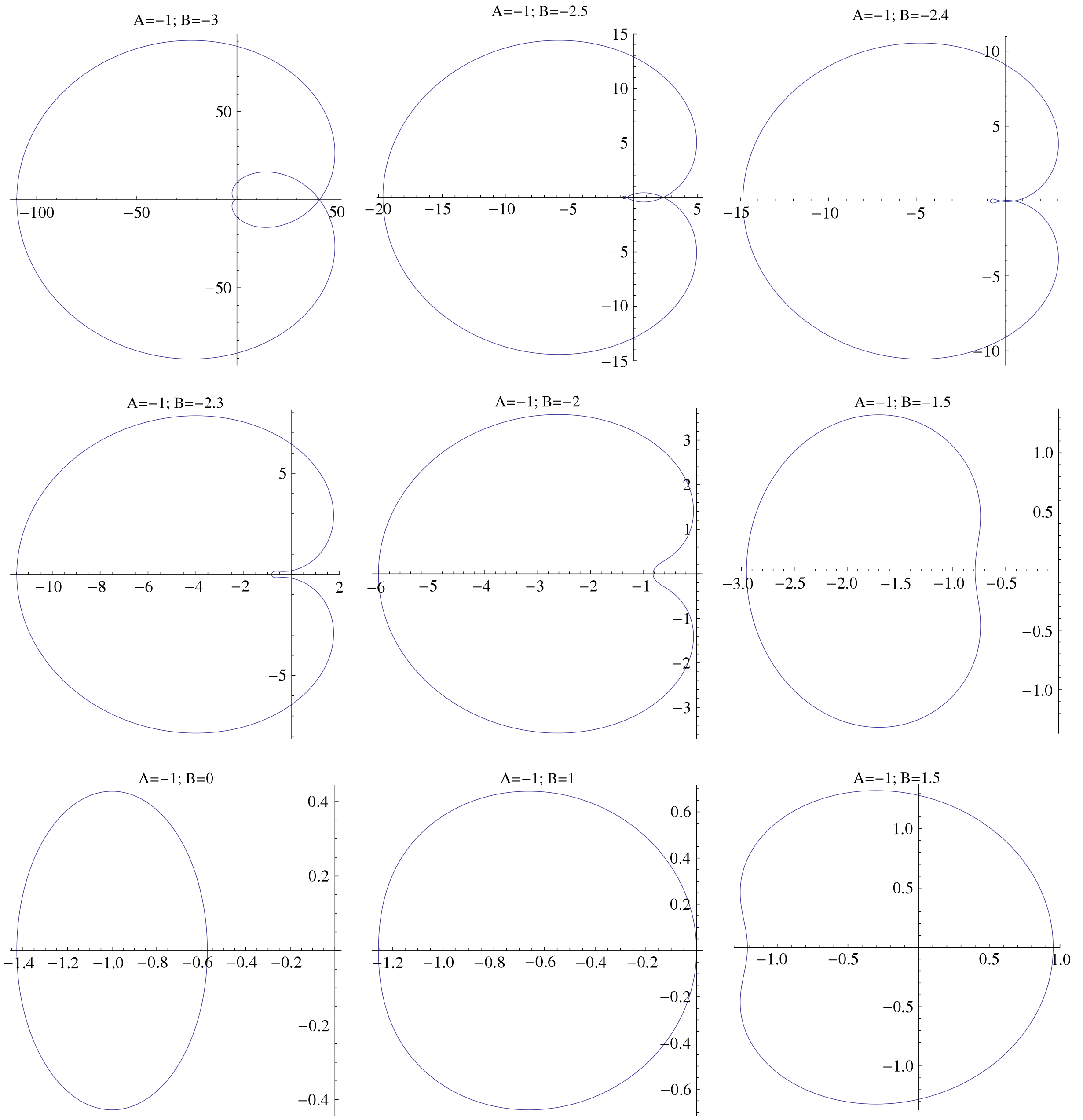

Fig. 6. Curve $F_{A,B}(e^{i\omega})$, $\omega \in [0, 2\pi]$ in the complex plane, for $q = \frac{1}{2}$ and $\tau = 1$, considering $A = -1$ fixed and increasing values of B in the interval $[-3, 1.5]$. The winding number $W(F_{A,B}(\gamma), 0)$ is zero for $B \in (-2.38544, 1)$.

At the same time, several questions have been raised concerning the applicability of the Laplace transform method and the final value theorem, and a few problems related to initialization response have also been pointed out. The use of the Laplace transform in combination with the Caputo and Riemann–Liouville fractional derivatives needs further investigation, and these will be studied in detail with examples in a forthcoming paper.

Acknowledgment

This work was supported by CNCS-UEFISCDI, project number PN-II-RU-PD-145/2010 (Advanced impulsive and fractional-order neural network models).

References

[1] E. Reyes-Melo, J. Martinez-Vega, C. Guerrero-Salazar, U. Ortiz-Mendez, Application of fractional calculus to the modeling of dielectric relaxation phenomena in polymeric materials, Journal of Applied Polymer Science 98 (2005) 923–935.
[2] N. Heymans, J.-C. Bauwens, Fractal rheological models and fractional differential equations for viscoelastic behavior, Rheologica Acta 33 (1994) 210–219.
[3] R. Schumer, D. Benson, Eulerian derivative of the fractional advection–dispersion equation, Journal of Contaminant 48 (2001) 69–88.
[4] B. Henry, S. Wearne, Existence of Turing instabilities in a two-species fractional reaction–diffusion system, SIAM Journal on Applied Mathematics 62 (2002) 870–887.
[5] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Physics Reports 339 (2000) 1–77.
[6] S. Picozzi, B.J. West, Fractional langevin model of memory in financial markets, Physical Review E 66 (2002) 46–118.
[7] W. Glockle, T. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics, Biophysical Journal 68 (1995) 46–53.
[8] L. Song, S. Xu, J. Yang, Dynamical models of happiness with fractional order, Communications in Nonlinear Science and Numerical Simulation 15 (2010) 616–628.
[9] R. Gu, Y. Xu, Chaos in a fractional-order dynamical model of love and its control, in: S. Li, X. Wang, Y. Okazaki, J. Kawabe, T. Murofushi, L. Guan (Eds.), Nonlinear Mathematics for Uncertainty and its Applications, in: Advances in Intelligent and Soft Computing, vol. 100, Springer, Berlin, Heidelberg, 2011, pp. 349–356.
[10] I. Podlubny, Fractional Differential Equations, Academic Press, 1999.
[11] A. Kilbas, H. Srivastava, J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.
[12] V. Lakshmikantham, S. Leela, J.V. Devi, Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, 2009.
[13] E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions, Nonlinear Analysis. Real World Applications 13 (2012) 1489–1497.
[14] J.K. Hale, S.M.V. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, 1993.
[15] H. Smith, An Introduction to Delay Differential Equations with Applications to the Life Sciences, Springer, 2011.
[16] C. Li, F. Zhang, A survey on the stability of fractional differential equations, The European Physical Journal Special Topics 193 (2011) 27–47.
[17] M.P. Lazarević, D.L. Debeljković, Robust finite time stability of nonlinear fractional order time delay systems, International Journal of Information and Systems Sciences 4 (2008) 301–315.
[18] X. Zhang, Some results of linear fractional order time-delay system, Applied Mathematics and Computation 197 (2008) 407–411.
[19] C. Bonnet, J.R. Partington, Stabilization of fractional exponential systems including delays, Kybernetika 37 (2001) 345–353.
[20] C. Hwang, Y.-C. Cheng, A numerical algorithm for stability testing of fractional delay systems, Automatica 42 (2006) 825–831.
[21] M. Busłowicz, Stability of linear continuous-time fractional order systems with delays of the retarded type, Bulletin of the Polish Academy of Sciences 56 (2008) 319–324.
[22] Y. Chen, K.L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, Nonlinear Dynamics 29 (2002) 191–200.
[23] W. Deng, C. Li, J. Lu, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dynamics 48 (2007) 409–416.
[24] K. Diethelm, The Analysis of Fractional Differential Equations, Springer, 2004.
[25] R. Gorenflo, F. Mainardi, Fractional calculus, integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, in: CISM Courses and Lecture Notes, vol. 378, Springer Verlag, Wien, 1997, pp. 223–276.
[26] Y. Li, Y. Chen, I. Podlubny, Mittag-Leffler stability of fractional order nonlinear dynamic systems, Automatica 45 (2009) 1965–1969.
[27] J. Chen, K.H. Lundberg, D.E. Davison, D.S. Bernstein, The final value theorem revisited: infinite limits and irrational functions, IEEE Control Systems Magazine 27 (2007) 97–99.
[28] T. Hartley, C. Lorenzo, Dynamics and control of initialized fractional-order systems, Nonlinear Dynamics 29 (2002) 201–233.
[29] C. Lorenzo, T. Hartley, Initialization of fractional-order operators and fractional differential equations, Journal of Computational and Nonlinear Dynamics 3 (2008) 021101:1–021101:9.
[30] T. Hartley, C. Lorenzo, The error incurred in using the caputo-derivative Laplace-transform, in: Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference 2009, DETC2009, vol. 4, pp. 271–278.
[31] T. Kalmár-Nagy, Stability analysis of delay-differential equations by the method of steps and inverse Laplace transform, Differential Equations and Dynamical Systems 17 (2009) 185–200.
[32] S.N. Elaydi, An Introduction to Difference Equations, Springer-Verlag, New York, 1996.
[33] H. Seybold, R. Hilfer, Numerical algorithm for calculating the generalized Mittag-Leffler function, SIAM Journal on Numerical Analysis 47 (2008) 69–88.
[34] A. Mikhailov, Method of harmonic analysis in control theory, Automatika i Telemehanika 3 (1938) 27–81.