On the stability analysis of arbitrarily high-index singular systems with multiple delays

Ha Phi[a], Vu Hoang Linh[a,*]

[a] *Faculty of Mathematics, Mechanics, and Informatics, VNU University of Science, Vietnam National University, Hanoi, Vietnam.*

Abstract

This paper is devoted to the stability analysis for linear, singular systems with multiple delays of arbitrarily high-index. By transforming a given system to an equivalent regular, impulse-free system, the global exponential stability problem is studied by using both approaches: spectral method and Lyanpunov-Krasovskii functional method. Characterizations for the stability are developed in both the spectral condition and the linear matrix inequality (LMI) setting. Numerical examples are presented to illustrate the advantages of the proposed results.

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1. Introduction

In this paper we study the stability analysis for linear (continuous time) singular system with multiple delays of the following form

$$E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t - \tau_i), \quad \text{for all } t \in [t_0, \infty), \tag{1}$$

$$x(t) = \phi(t), \text{ for all } t_0 - \tau_m \leqslant t \leqslant t_0, \tag{2}$$

where $E \in \mathbb{R}^{n,n}$ is allowed to be singular. Here the state is $x : [t_0 - \tau_m, \infty) \to \mathbb{R}^n$, and the (constant) time-delays satisfy $0 < \tau_1 < \tau_2 < ... < \tau_m$. The capital letters are real-valued matrices of appropriate dimensions. Time-delayed singular systems of the form (1) are a combination of delay-differential equations along with (implicitly hidden) difference/algebraic constraints. They play important roles in the mathematical modeling of real-life problems arising in a wide variety of scientific and engineering applications, e.g. human balance control and multibody control systems [1], circuits analysis including power systems [2, 3] and lossless transmission lines [4], chemical engineering [5, 6], fluid dynamics [7, 8], etc. Due to the singular structure of the system, the existence of these time-delays can affect profoundly the behavior of the systems, and many interesting properties, which occur neither for non-delayed singular systems nor for

* Corresponding author
Email addresses: haphi.hus@vnu.edu.vn (Ha Phi), linhvh@vnu.edu.vn (Vu Hoang Linh)

regular delay differential equations, have been observed, e.g. [9, 10, 11, 12]. For these reasons time-delayed singular systems has attracted significant attention of many researchers in the last two decades, in particular about the stability analysis. This topic has been studied by many methods, and we name here two popular methods namely (1) the *spectral method*, e.g. [13, 14, 10, 11, 15, 16], and (2) the *Lyapunov-Krasovskii functional method*, e.g. [17, 18, 19, 20, 21]. Due to our best knowledge, most of the existing stability results are established for impulse-free (which is also known as strangeness-free, index-1) systems, see Lemma 2.4. Nevertheless, singular systems of high-index are frequently occur in many applications, for example in multibody dynamics, electric circuits, chemical engineering or fluid dynamic, e.g. [22, 23, 24]. A stability analysis for these systems (including time-delayed) is still missing. We notice, that the index reduction procedure, which is used to reduce high-index to low-index systems, can be applied for certain classes of time-delayed singular system, as has been done in the solvability analysis, e.g. [25, 12]. However, from the theoretical viewpoint, till now there is not any investigation about the affection of the index reduction procedure to the system spectrum. Therefore, additional difficulties may arise while applying spectral method for system (1), e.g. [10].

This paper aim to addresses the exponential stability of arbitrarily high-index systems. The main idea is to transform a given system to a regular, impulse-free system, while still keeping both the solution and the spectrum invariant. Based on this technique, the stability problem is studied by using both the spectral method and the Lyanpunov-Krasovskii functional method. The rest of the paper is organized as follows. In Section 2, some definitions concerning about the solution and the system classification are recalled. Auxiliary Lemmas about the solution's presentation and the non-advanced test are also presented. In Section 3, the preservation of the system spectrum under the index-reduction technique is discussed. Based on this, characterization for the exponential stability of arbitrarily high-index systems is given in term of a spectral condition (Theorem 3.4). Then, the advantages of the index-reduction technique for the Lyapunov-Krasovskii functional method has been proven in the linear matrix inequality (LMI) setting (Theorem 3.8). Numerical examples are presented to confirm the theoretical results. Finally, in Section 4, conclusion and outlook are given.

2. Preliminaries

To keep the brevity of this research, we refer the interested readers to [26, 9, 27, 28, 12] for the solvability analysis of the IVP (1).

Definition 2.1. *The null solution $x = 0$ of the free system* (1) *is called* exponentially stable *if there exist positive constants δ and γ such that for any consistent initial function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, the solution $x = x(t, \varphi)$ of the corresponding IVP to* (1) *satisfies*

$$\|x(t)\| \leqslant \delta e^{-\gamma t} \|\varphi\|_\infty, \quad \textit{for every} \ \ t \geqslant 0.$$

Definition 2.2. *i) Consider the DDAE* (1)*. The matrix pair* (E, A_0) *is called* regular *if the* polynomial $\det(\lambda E - A_0)$ *is not identically zero.*
ii) The sets $\sigma(E, A_0, ..., A_m) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A_0 - e^{-\lambda\tau_i}A_i) = 0\}$ *is called the* spectrum *of system* (1)*.*
iii) The real number $\rho(E, A_0, ..., A_m) := \max\{Re(\lambda),\ \lambda \in \sigma(E, A_0, ..., A_m)\}$ *is called the* spectral abcissa *of system* (1)*.*

Definition 2.3. *Two matrix tuples* $(E, A_0, ..., A_m)$ *and* $(\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m)$ *are called* (strongly) equivalent *if there exist non-singular matrices* $W, T \in \mathbb{R}^{n,n}$ *such that*

$$\tilde{E} = WET,\ \ \tilde{A}_i = WA_iT, \quad \textit{for all } i = 0, ..., m.$$

If this is the case, we write $(E, A_0, ..., A_m) \sim (\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m)$*.*

It is well-known that the relation introduced in Definition 2.3 is an equivalence relation, i.e., it is is reflexive, symmetric, and transitive. Provided that the pair (E, A_0) is regular, we can transform them to the Kronecker-Weierstraß canonical form as follows.

Lemma 2.4. *([29, 23]) Provided that the matrix pair* (E, A_0) *is regular, then there exist nonsingular matrices* $W, T \in \mathbb{R}^{n,n}$ *such that*

$$(WET, WA_0T) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{3}$$

where N *is a nilpotent, upper triangular matrix of nilpotency index* ν*. We also say that the pair* (E, A_0) *has an* index ν*, i.e.,* $\mathrm{ind}(E, A_0) = \nu$*. Furthermore, the system* (1) *is called* impulse-free (index 1, or strangeness-free) *if* $N = 0$*.*

Remark 1. Due to our best knowledge, if the system has high index, i.e. bigger than one, then in many cases it **may have** impulsive behavior. This is the reason why index 1 is also called impulse-free. Consequently, till now most of the research on the stability analysis are devoted to impulse-free systems, see e.g. [16, 19, 30, 31, 17, 32, 18, 33, 34, 35]. Nevertheless, there are plenty singular systems of high-index arising from various applications, for example in multibody dynamics, electric circuits, chemical engineering or fluid dynamic, see e.g. [22, 23, 24]. A stability analysis for these systems (including time-delayed) is still missing. In the following example, we illustrate that there exist systems with arbitrarily high-index (and hence, not impulse-free) which are stable.

Example 2.5. *Let us consider the (single) time-delayed incompressible Navier-Stokes equation on an open, bounded domain* $\Omega \subset \mathbb{R}^k$*,* $k = 2$ *or* 3*, of the form*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu\Delta u - \nabla p - (u \cdot \nabla)u + f(t, u(t-\tau), p(t-\tau)), \\ \nabla \cdot u &= 0, \\ u \mid_{\partial\Omega} &= 0, \\ u \mid_{t=0} &= u_0, \end{aligned}$$

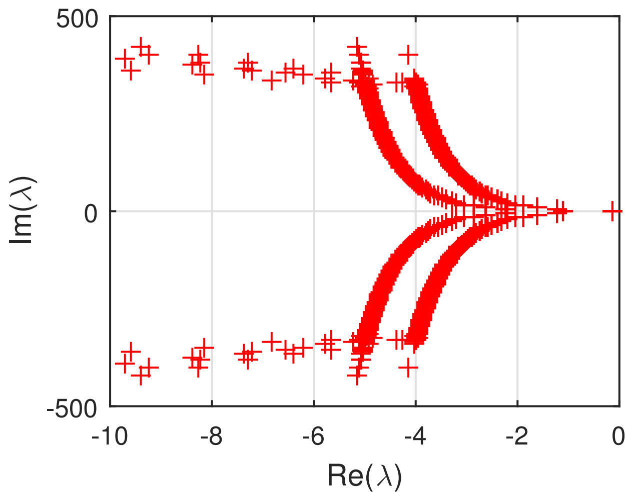

Figure 1: Spectrum of the system (5) using the MATLAB Toolbox TDS_STABIL ([37]).

where $\nu > 0$ is the viscosity, $u = u(t, \xi)$ is the velocity field which is a function of the time t and the position ξ, p is the pressure, f is the external force or disturbance. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [36], one obtains the delay singular system

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} K & -C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} + \begin{bmatrix} F(t, U(t-\tau), P(t-\tau)) \\ 0 \end{bmatrix}, \tag{4}$$

where $U(t)$, $P(t)$ approximate the velocity $u(t, \xi)$ and the pressure $p(t, \xi)$, respectively. Furthermore, in many applications, the matrix $C^T M^{-1} C$ is nonsingular. It is well-known that system (4) has index 2. Let us choose the matrix coefficients and the external force such that the system (4) reads

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{U}_1 \\ \dot{U}_2 \\ \dot{P} \end{bmatrix} = \left[\begin{array}{cc|c} 0.11 & 0.6 & 0 \\ 0.6 & 1 & -1 \\ \hline 0 & 1 & 0 \end{array}\right] \begin{bmatrix} U_1 \\ U_2 \\ P \end{bmatrix} + \left[\begin{array}{cc|c} -0.2 & 0 & -1 \\ -0.01 & 0 & -0.8 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} U_1(t-\tau) \\ U_2(t-\tau) \\ P(t-\tau) \end{bmatrix}. \tag{5}$$

Here $\tau = 1.5$. From the last equation, we see that $U_2 = 0$. Thus, by crossing out the coefficients associated with U_2, we obtain an impulse-free system for the variable $\left[\begin{smallmatrix} U_1 \\ P \end{smallmatrix}\right]$. The spectrum of system (4), given in Figure 1, implies that this equation is stable.

Lemma 2.6. *([38]) For a nilpotent, upper triangular matrix N of nilpotency index ν, the matrix $I - \lambda N$ is invertible for all $\lambda \in \mathbb{C}$, and $\det(I - \lambda N) = 1$. Furthermore, the following identity holds true.*

$$(I - \lambda N)^{-1} = I + \sum_{i=1}^{\nu} (\lambda N)^i.$$

2.1. System classification

It is well-known (see e.g. [39, 40]) that in general, time-delayed systems has been classified into three different types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0\dot{x}(t) + a_1\dot{x}(t-\tau) + b_0x(t) + b_1x(t-\tau) = f(t)$$

is retarded if $a_0 \neq 0$ and $a_1 = 0$; is neutral if $a_0 \neq 0$, $a_1 \neq 0$; is advanced if $a_0 = 0$, $a_1 \neq 0$, $b_0 \neq 0$. This classification is based on the smoothness comparison between $x(t)$ and $x(t-\tau)$. In literature, not only the theoretical but also the numerical solution has been studied mainly for retarded and neutral systems, due to their appearance in various applications. For this reason, in [28, 12, 41] the authors proposed a concept of *non-advancedness* for the free system (see Definition 2.7 below). We also notice, that even though not clearly proposed, due to the author's knowledge, so far results for delay-descriptor are only obtained for certain classes of non-advanced systems, e.g. [26, 27, 42, 43, 16, 21, 44, 45, 46].

Definition 2.7. *A regular delay-descriptor system* (1) *is called* non-advanced *if for any consistent and continuous initial function φ, there exists a continuous, piecewise differentiable solution $x(t)$.*

From Definition (2.7), we can directly see three important consequences as follows. Firstly, in the solution's formula of $x(t)$, only $x(t-\tau_i)$ but not its derivative $x^{(j)}(t-\tau_i)$, $j \geqslant 1$ can appear. Otherwise $x(t)$ will no-longer be continuous. Secondly, non-advancedness is a necessary condition for the classical Lyapunov stability in Definition 2.1. Finally, the non-advancedness of a system is invariant under equivalence transformation in Definition 2.3.

Making use of Lemma 2.4, we change the variable $x = Ty$ and scale the whole system (1) with W to obtain the transformed system

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{y}(t) = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} y(t) + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t-\tau_i), \tag{6}$$

where $WA_iT = \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}$ for all $i = 1, ..., m$. The following lemma gives us the necessary and sufficient condition for the non-advancedness of system (1).

Lemma 2.8. *i) System* (1) *is non-advanced if and only if the matrix coefficients of the transformed system* (6) *satisfy*

$$N \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \textit{for all} \;\; i = 1, \ldots, m. \tag{7}$$

ii) Consequently, system (6) *has exactly the same solution as the so-called* index-reduced system

$$\tilde{E}\dot{y}(t) = \tilde{A}_0 y(t) + \sum_{i=1}^{m} \tilde{A}_i y(t-\tau_i), \tag{8}$$

where $\tilde{E} := \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix}$, $\tilde{A}_0 := \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}$, $\tilde{A}_i := \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}$, $i = 1, ..., m$.

PROOF. Partitioning $y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ conformably, we can rewrite system (6) as follows

$$\dot{y}_1 = Jy_1 + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \end{bmatrix} y(t-\tau_i),$$
$$N\dot{y}_2 = y_2 + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t-\tau_i), \tag{9}$$

The second equation has a unique solution

$$y_2(t) = -\begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t-\tau_i) - \sum_{j=1}^{\nu} \sum_{i=1}^{m} N^i \begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y^{(j)}(t-\tau_i). \tag{10}$$

Since the system (1) is non-advanced, then so is system (6). Consequently, $y(t)$ must not depend on $y^{(j)}(t-\tau_i)$ for all $j \geqslant 1$ and $i = 1, ..., m$. This implies the identity (7). Then, equation (10) becomes

$$y_2(t) = -\begin{bmatrix} \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t-\tau_i),$$

and hence, the second claim is trivially followed.

Remark 2. From Lemma 2.8 ii), we see that if system (1) is non-advanced, then there is a linear, bijective mapping $x \mapsto y = T^{-1}x$ (where T is the matrix given in the Kronecker-Weierstraß form (3)) between the solution set of the high-index system (1) and the impulse-free system (8). This will play the key role in the stability analysis in Section 3.

3. Main results

3.1. Stability via the spectral method

In this method, the stability analysis of the null solution of (1) is based on a spectrum determined growth property of the solutions, which allows us to infer stability information from the location of the characteristic roots. For instance, exponential stability will be related to a strictly negative spectral abscissa.

Proposition 3.1. *([16, 10]) Consider the linear, homogeneous DDAE* (1)*. Furthermore, assume that it is regular, impulse-free. Then it is stable if and only if the corresponding spectrum of this system lies entirely on the left half plane and is bounded away from the imaginary axis.*

As have seen in Example 2.5, system of the form (1) can be stable without the impulse-free assumption. However, the non-advancedness is necessary. Without it, the system can not be stable, even if the spectrum lies entirely on the left half plane and is bounded away from the imaginary axis. The following example is taken from [11].

Example 3.2. *Consider the following system on the time interval* $[0, \infty)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \\ x_3(t-\tau) \end{bmatrix}. \quad (11)$$

This system has an index $\nu(E, A_0) = 2$ *and hence, it is not impulse-free. Direct computation yields that the spectrum is* $\sigma(E, A_0, A_1) = \{-1\} \subseteq \mathbb{C}_-$. *Nevertheless, by checking condition* (7), *we see that system* (11) *is advanced, so it is unstable. We can also see the unstability of* (11) *by taking derivative of* $x_3(t)$ *from the third equation, and then substituting it into the second one to obtain*

$$0 = x_2(t) - \dot{x}_1(t-\tau)\ . \quad (12)$$

This implies that system (11) *is not stable, since on the interval* $[0, \tau]$ *we have* $x_2(t) = \dot{\varphi}_1(t-\tau)$ *can be arbitrarily large.*

The following lemma plays the key role in the proof of the main Theorem 3.4 below.

Lemma 3.3. *Consider the linear, homogeneous DDAE* (1). *Furthermore, assume that it is non-advanced. Then system* (1) *has the same spectrum (without counting multiplicity) as the index-reduced system* (8).

Proof. We will show that both systems (1) and (8) have the same spectrum (without counting multiplicity) as the system (6). Due to the variable transformation $x = Ty$ and the identity

$$W\left(\lambda E - A_0 - e^{-\lambda\tau_i} A_i\right) T = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - e^{-\lambda\tau_i} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix},$$

it is straightforward that

$$\sigma(E, A_0, ..., A_m) = \sigma\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ \tilde{A}_{1,3} & \tilde{A}_{1,4} \end{bmatrix}, ..., \begin{bmatrix} \tilde{A}_{m,1} & \tilde{A}_{m,2} \\ \tilde{A}_{m,3} & \tilde{A}_{m,4} \end{bmatrix}\right). \quad (13)$$

Now let us consider the right hand side of (13), due to Lemma 2.6 we see that for an arbitrary $\lambda \in \mathbb{C}$

$$\begin{aligned} &\det\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \sum_{i=1}^{m} e^{-\lambda\tau_i} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} I & 0 \\ 0 & (I-\lambda N)^{-1} \end{bmatrix} \cdot \begin{bmatrix} I - J - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,1} & -\sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,2} \\ -\sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,3} & \lambda N - I - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,4} \end{bmatrix}\right). \end{aligned}$$

Due to Lemma 2.6 and the identity (7), we have

$$\begin{aligned} &(I + \sum_{i=1}^{\nu} (\lambda N)^i) \cdot \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,3} = \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,3}, \\ &(I + \sum_{i=1}^{\nu} (\lambda N)^i) \cdot \left(\lambda N - I - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,4}\right) = -I - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,4}. \end{aligned}$$

Hence, it follows that for any $\lambda \in \mathbb{C}$

$$
\begin{aligned}
&\det\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \sum_{i=1}^{m} e^{-\lambda\tau_i} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} I - J - \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,1} & -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,2} \\ -\sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,3} & -I - \sum_{i=1}^{m} e^{-\lambda\tau_i}\tilde{A}_{i,4} \end{bmatrix}\right),
\end{aligned}
$$

which yields that

$$
\sigma\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ \tilde{A}_{1,3} & \tilde{A}_{1,4} \end{bmatrix}, ..., \begin{bmatrix} \tilde{A}_{m,1} & \tilde{A}_{m,2} \\ \tilde{A}_{m,3} & \tilde{A}_{m,4} \end{bmatrix}\right) = \sigma(\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m). \tag{14}
$$

From (13) and (14) we have $\sigma(E, A_0, ..., A_m) = \sigma(\tilde{E}, \tilde{A}_0, ..., \tilde{A}_m)$. $\square$

Theorem 3.4. *Consider the free system* (1)*. Furthermore, we assume that the matrix pair* (E, A_0) *is regular. Then,* (1) *is exponentially stable if and only if the following assertions hold.*

i) System (1) *is non-advanced.*

ii) The spectrum $\sigma(E, A_0, \ldots, A_m)$ *lies entirely on the left half plane and it is bounded away from the imaginary axis.*

Proof. " $\Rightarrow$ " Assume that system (1) is exponentially stable. Clearly, it is non-advanced, so we only need to prove ii). Furthermore, due to Lemma 2.8ii), system (1) is stable if and only if the index-reduced system (8) is also stable. Thus, the spectrum $\sigma(\tilde{E}, \tilde{A}_0, \ldots, \tilde{A}_m)$ lies entirely on the left half plane and it is bounded away from the imaginary axis, and hence, due to Lemma 3.3 we obtain the desired claim.

" $\Leftarrow$ " Since the index-reduced system (8) is impulse-free, Proposition 3.1 applied to it implies that the index-reduced system (8) is exponentially stable, and so is system (1). This completes the proof. $\square$

From the numerical viewpoint, the numerical computation of the Kronecker-Weierstraß form (3) is complicated and unstable (see [47]), so Lemma 2.8 has more theoretical than numerical meaning for checking the non-advancedness and/or the stability of (1). Instead, we will make use of the (reordered) QZ-decomposition ([48]) of the matrix pair (E, A_0) as follows

$$
QEZ^T = \begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & N_E \end{bmatrix}, \; QA_0Z^T = \begin{bmatrix} J_A & \hat{A}_2 \\ 0 & \Sigma_A \end{bmatrix}, \; QA_iZ^T = \begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix}, \tag{15}
$$

for all $i = 1, ..., m$. Here Q and Z are orthogonal matrices, Σ_E and Σ_A are nonsingular, upper triangular matrices, N_E is a nilpotent, upper triangular matrix.

Using the same argument as in Lemma 2.8, we have the following lemma.

Lemma 3.5. *Consider the free system* (1) *and the QZ-decomposition* (15)*. Then, the following assertions hold true.*
i) System (1) *is non-advanced if and only if the following identities hold.*

$$N_E \Sigma_A^{-1} \begin{bmatrix} \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix} = 0 \text{ for all } i = 1, ..., m. \tag{16}$$

ii) If this is the case, then there is a linear, bijective mapping $x \mapsto y = Zx$ (where Z is the matrix given in (15)*) between the solution set of the high-index system* (1) *and the following index-reduced system*

$$\underbrace{\begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & \mathbf{0} \end{bmatrix}}_{\hat{E}} \dot{y}(t) = \underbrace{\begin{bmatrix} J_A & \hat{A}_{02} \\ 0 & \Sigma_A \end{bmatrix}}_{\hat{A}_0} y(t) + \sum_{i=1}^{m} \underbrace{\begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix}}_{\hat{A}_i} y(t - \tau_i) \ . \tag{17}$$

iii) The spectrum (without counting multiplicity) of two systems (1) *and* (17) *coincide.*

Proof. The proof is essentially the same as the proof of Lemmas 2.8 and 3.3, so it will be omitted to keep the brevity of this research.

To determine the stability (or the decay rate of the solution) of system (1), we will make use of the spectral discretization approach in [16]. Nevertheless, since this method has only been developed for impulse-free (or index-1) system, we will apply it for the index-reduced system (17).

Example 3.6. *To illustrate the advantage of the proposed method, we consider the following system, motivated from [19].*

$$\begin{bmatrix} -1 & 2 & 0.2648 \\ -2 & 4 & 0.8476 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 4.7 & 0.4 & 0.1192 \\ -4.9 & 0.8 & 1.1783 \\ 0 & 0 & 0.6473 \end{bmatrix} x(t) + \begin{bmatrix} 0.7 & -0.95 & 0.6456 \\ 1.1 & -1.75 & 1.7706 \\ 0 & 0 & 0 \end{bmatrix} x(t - 0.2)$$
$$+ \begin{bmatrix} 1 & -0.8 & 0.6393 \\ 1.4 & -1.3 & 1.8234 \\ 0 & 0 & 0 \end{bmatrix} x(t - 2). \tag{18}$$

We notice that the matrix pair (E, A_0) *in system* (18) *has index* $\nu = 2$*, and hence the system is not impulse-free. Using the MATLAB Toolbox TDS_STABIL ([37, 16]) we obtain the dominant eigenvalues of the original system* (18) *and that of the index-reduced system* (17)*, as shown in Figure 2. Clearly, we see that without the index reduction step, the spectrum is not properly computed and hence, is not reliable to determine the stability of system* (18)*.*

3.2. Lyapunov-Krasovskii functional method

Adopting the Lyapunov-Krasovskii approach, (sufficient) stability conditions for many classes of singular systems with different types of delays (single, multiple, time-varying, etc.) have been proposed, see for example, [19, 30, 31, 17, 32, 18, 33, 34, 35]. We, again, notice that all the conditions on the references mentioned above are only valid for impulse-free system. We recall one important result in the following proposition.

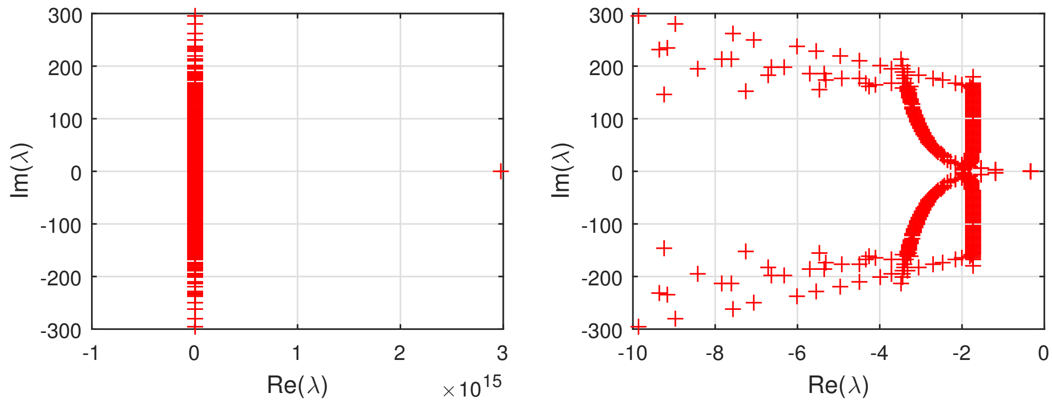

Figure 2: Spectrum of the system (18) (left) and the index-reduced system (17) (right), using the MATLAB Toolbox TDS_STABIL ([37]).

Proposition 3.7. *([18, 17]) Consider the linear, homogeneous DDAE* (1)*. Furthermore, assume that it is regular, impulse-free. Then it is stable if there exist matrix P and matrices $Q_i \succ 0$, $i = 1, ..., m$ such that following linear matrix inequialities (LMIs) are satisfied*

$$EP^T = P^T E \succ 0,$$

$$M := \left[\begin{array}{c|ccc} A_0P^T + PA_0^T + \sum_{i=1}^m Q_i & A_1P^T & \dots & A_mP^T \\ \hline PA_1^T & -Q_1 & & \\ \vdots & & \ddots & \\ PA_m^T & & & -Q_m \end{array}\right] \prec 0 \ .$$

Similarly, in order to generalize this result for arbitrarily-high index systems, first we need to transform system (1) to the index-reduced form (17). The following theorem is a direct consequence of Lemma 3.5 and Proposition 3.7.

Theorem 3.8. *Consider the time-delayed singular system* (1)*, the index-reduced system* (17)*. Furthermore, assume that system* (1) *is non-advanced and the matrix pair* (E, A_0) *is regular. Then system* (1) *is stable if there exist matrix P and matrices $Q_i \succ 0$, $i = 1, ..., m$ such that following LMIs are satisfied*

$$\hat{E}P^T = P^T \hat{E} \succ 0,$$

$$M := \left[\begin{array}{c|ccc} \hat{A}_0P^T + P\hat{A}_0^T + \sum_{i=1}^m Q_i & \hat{A}_1P^T & \dots & \hat{A}_mP^T \\ \hline P\hat{A}_1^T & -Q_1 & & \\ \vdots & & \ddots & \\ P\hat{A}_m^T & & & -Q_m \end{array}\right] \prec 0 \ . \qquad (19)$$

We illustrate the advantage of this strategy in the following example.

Example 3.9. *Motivated from [45], let us consider the following system whose matrix coefficients are*

$$E = \begin{bmatrix} -11 & 1 & 0 \\ 0 & 0 & 0.127 \\ 0 & 0 & 0 \end{bmatrix}, \ A_0 = \begin{bmatrix} 0.2 & 0.61 & 0.1891 \\ -1 & 0.6 & 0.5607 \\ 0 & 0 & 0.2998 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -0.2 & -1.597 \\ -0.8 & -0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{20}$$

The system is not impulse-free and having an index $\nu(E, A) = 2$. If we directly apply the MATLAB LMI-Toolbox or the package CVX [49, 50] to the system (18) *then the obtained matrix M (defined by* (19)*) is not negative definite. Nevertheless, by transforming the system to the index-reduced form* (17) *which reads*

$$\begin{bmatrix} -4.802 & -9.9469 & -0.7885 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0.626 & -0.1423 & -0.1891 \\ 0 & 1.1662 & 0.5607 \\ 0 & 0 & 0.2998 \end{bmatrix} x(t) + \\ + \begin{bmatrix} -0.686 & -0.7546 & 1.597 \\ 0.4202 & 0.6808 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-\tau), \tag{21}$$

then both the MATLAB LMI-Toolbox or the package CVX work properly. The matrices P, Q_1 are

$$P = \begin{bmatrix} -15.3413 & 2.2457 & -2.6746 \\ 8.5630 & -4.1706 & 0.4629 \\ 0.1705 & -0.0140 & -0.8608 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 5.9597 & -1.1767 & 0.2239 \\ -1.1767 & 2.9958 & 0.3278 \\ 0.2239 & 0.3278 & 0.3995 \end{bmatrix}.$$

Thus, we can conclude that system (21) *is stable.*

We summarize the results above in the following algorithm for cheking the exponential stability of system (1).

Algorithm 1 Checking the exponential stability of system (1)

Input: The time-delayed singular system (1).
Output: The stability of (1) and the decay rate γ.

1: Set *is_stable* $= 0$, *is_advanced* $= 0$.
2: Transform the system coefficiens via QZ-decomposition as in (15).
3: Check the non-advancedness of the system.
4: **if** condition (16) does not hold true **then** set *is_advanced* $= 1$ and STOP.
5: **else** transform the system to the index-reduced form (17).
6: **end if**
7: Choose 1 of 2 following approaches: (1) Spectral method (GO TO STEP 8) ; (2) Lyapunov-Krasovskii functional method (GO TO STEP 12).
8: Compute the spectrum $\sigma(E, A_0, ..., A_m)$ (or dominant eigenvalues) of system (17) using TDS_STABIL.
9: **if** $\rho(E, A_0, ..., A_m) < 0$ **then** set *is_stable* $= 1$, $\gamma = \rho(E, A_0, ..., A_m)$.
10: **else** *is_stable* $= 0$, i.e. the system is not exponentially stable.
11: **end if**
12: Solve the LMI system (19).
13: **if** the solution P, Q exist **then** set *is_stable* $= 1$.
14: **else** give a warning "LMIs do not have a solution. The system may not be exponentially stable."
15: **end if**

---

4. Conclusion and Outlook

The exponential stability of arbitrarily high index, time-delayed singular systems has been investigated. The preservation of the system spectrum under the index-reduction technique is proven, and numerically verifiable criteria for the exponential stability are given using both the spectral method and the Lyapunov-Krosovskii functional methods. Further study on the robust stability and stabilization (of high-index systems) under this index-reduction technique is still an open problem for future research.

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Appendix A. A second test for checking non-advancedness

Below we will construct a second test for checking non-advancedness without using the system transformations as in Lemmas 2.8, 3.5. Assume that the pair (E, A_0) is regular with index $\operatorname{ind}(E, A_0) = \nu$. We want to give a simple check whether the system (1) is non-advanced or not. In analoguous to the case of

DAEs, see e.g. [22, 23], we aim to extract the so-called *underlying delay equation* of the form

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \sum_{i=1}^{m} \mathbf{A}_i x(t-\tau_i) + \sum_{i=1}^{m} \mathbf{F}_i \dot{x}(t-\tau_i), \tag{A.1}$$

from an augmented system consisting of system (1) and its derivatives, which read in details

$$\frac{d^j}{dt^j}\left(E\dot{x}(t) - A_0 x(t) - \sum_{i=1}^{m} A_i x(t-\tau_i)\right) = 0, \text{ for all } j = 0, 1, \ldots, \nu.$$

We rewrite these equations into the so-called *inflated system*

$$\underbrace{\begin{bmatrix} E & & & \\ -A_0 & E & & \\ & \ddots & \ddots & \\ & & -A_0 & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{\mathcal{A}_0} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix}$$

$$+ \sum_{i=1}^{m} \underbrace{\begin{bmatrix} A_i & & & \\ & A_i & & \\ & & \ddots & \\ & & & A_i \end{bmatrix}}_{\mathcal{A}_i} \begin{bmatrix} x(t-\tau_i) \\ \dot{x}(t-\tau_i) \\ \vdots \\ x^{(\nu)}(t-\tau_i) \end{bmatrix}. \tag{A.2}$$

Here the matrix coefficients are $\mathcal{E}, \mathcal{A}_0, \mathcal{A}_i \in \mathbb{R}^{(\nu+1)n,(\nu+1)n}$ for all $i = 1, ..., m$. For the reader's convenience, below we will use MATLAB notations. An underlying delay system (A.1) can be extracted from (A.2) if and only if there exists a matrix $P = \begin{bmatrix} P_0 & P_1 & \ldots & P_\nu \end{bmatrix}^T$ in $\mathbb{R}^{(\nu+1)n,n}$ such that

$$\begin{aligned} P^T \mathcal{E} &= \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}, \\ P^T \mathcal{A}_i &= \begin{bmatrix} * & * & 0_{n,(\nu-1)n} \end{bmatrix}, \text{ for all } i = 1, ..., m, \end{aligned}$$

where $*$ stands for an arbitrary matrix. Consequently, P is the solution to the following linear systems

$$\begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:, 2n+1 : end)^T \\ \vdots \\ \mathcal{A}_m(:, 2n+1 : end)^T \end{bmatrix} P = \begin{bmatrix} \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}^T \\ 0_{(\nu-1)n,n} \\ \vdots \\ 0_{(\nu-1)n,n} \end{bmatrix}.$$

Therefore, making use of Crammer's rule we directly obtain the simple check for the non-advancedness of system (1) in the following theorem.

Theorem Appendix A.1. *Consider the zero-input descriptor system* (1) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = \nu$. *Then, this system is non-advanced if and only if the following rank condition is satisfied*

$$\operatorname{rank}\begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:,2n{+}1:end)^T \\ \vdots \\ \mathcal{A}_m(:,2n{+}1:end)^T \end{bmatrix} = \operatorname{rank}\left[\begin{array}{c|c} \mathcal{E}^T & \begin{bmatrix} I_n & 0_{n,\nu n}\end{bmatrix}^T \\ \mathcal{A}_1(:,2n{+}1:end)^T & 0_{(\nu-1)n,n} \\ \vdots & \vdots \\ \mathcal{A}_m(:,2n{+}1:end)^T & 0_{(\nu-1)n,n} \end{array}\right]. \tag{A.3}$$

Theorem Appendix A.1 applied to the index two case straightly gives us the following corollary.

Corollary Appendix A.2. *Consider the zero-input descriptor system* (1) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = 2$. *Then, system* (1) *is non-advanced if and only if the following identity hold true.*

$$\operatorname{rank}\left[\begin{array}{ccc} E^T & -A_0^T & 0 \\ 0 & E^T & -A_0^T \\ 0 & 0 & E^T \\ \hline 0 & 0 & A_1^T \\ \vdots & \vdots & \vdots \\ 0 & 0 & A_m^T \end{array}\right] = n + \operatorname{rank}\left[\begin{array}{cc} E^T & -A_0^T \\ 0 & E^T \\ \hline 0 & A_1^T \\ \vdots & \vdots \\ 0 & A_m^T \end{array}\right]. \tag{A.4}$$