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Stability analysis of descriptor systems with multiple commensurate time-delays

Fatemeh Zahedi, Mohammad Haeri*

Department of Electrical Engineering, Sharif University of Technology, Tehran, Iran

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Abstract

A new method is proposed to analyze stability of descriptor systems with multiple commensurate time-delays. In this method, purely imaginary roots of the system are calculated, asymptotic behavior of root path at these roots is studied, and unstable regions of the system in the space of delays are determined. Moreover, the number of roots in each unstable region is calculated. Also necessary and sufficient conditions for τ-stabilizability of these systems are presented. Examples are provided to illustrate the effectiveness and validity of the proposed method.

1. Introduction

Descriptor systems with time-delay (also called singular, generalized state-space, semi-state systems with time-delay or delay differential-algebraic equations depending on the area of application) have been widely investigated over the past few years. This representation is more comprehensive than the normal state-space and neutral models and has been employed in many practical applications including economics, power systems, vibrational and structural analysis, spacecraft control, electrical networks, and robotics control [1–6]. Existence of time-delay and singularity not only produces complex behaviors such as oscillation, instability, and degraded performance (which delay may cause), but also demands examining of regularity and impulse freeness.

* Corresponding author.
E-mail addresses: haeri@sharif.ir, haeri@ee.sharif.edu (M. Haeri).

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Stability analysis of the descriptor systems with time-delay is studied either in time or frequency domain. Time-domain approaches generally use Lyapunov–Krasovskii or Lyapunov–Razumikhin function and express the stability criteria in the form of linear matrix inequalities (LMIs). For instance, practical stability of descriptor systems with time-delays has been presented in [7]. Delay-dependent robust stability criteria have been investigated in [8] for two classes of descriptor time-delay systems with norm-bounded uncertainty in terms of linear matrix inequalities. Exponential stability of singular systems with multiple time-varying delays has also been studied [9]. Some approaches (e.g. [10]) deal with the stability of these systems by representing them in neutral forms. However, these studies are based on delay-dependent stability analyses, and none of them are able to determine all stable/unstable regions in delay space. They only presented algorithms to maximize this region and make it closer to the real one. Thus, presenting an approach to determine all stable/unstable regions in delay space for the descriptor systems with time-delay could be of great importance.

Frequency-domain approaches study the stability in different ways (e.g., [11,12]). An approach presented in [13] determines delay values that maintain the stability of time-delayed LTI systems. A similar method was developed in [14] that determines all possible stable intervals of time-delay for neutral type time-delayed LTI systems. Invariance properties of the double imaginary roots of time-delay systems have been investigated in [15]. The main purpose of those researches in [16,17] is to separate stable and unstable regions in delay space. A new method for stability analysis of fractional delay systems with multiple commensurate delays has been proposed in [18]. However, due to the special features and differences in the descriptor systems with time-delay, the mentioned approaches are either inaccurate or inapplicable.

The main purpose of this paper is to present a practical approach to tackle the exact stability analysis of descriptor systems with multiple commensurate time-delays. To claim a comprehensive stability analysis, different aspects should be considered. In this paper, important aspects such as determining stable/unstable regions in the delay space and considering small changes of delay have been investigated, while none of the existing works has studied these systems in such a comprehensive way, to say nothing that none of them could determine exact stable/unstable regions in the delay space. The proposed approach in this paper not only determines stable/unstable regions in the delay space for the studied systems, but also calculates the number of roots in each unstable region. This approach is applicable to normal state-space systems with multiple commensurate time-delays as a special case as well.

The organization of the rest of this paper is as follows: Section 2 states problem formulation, some definitions, and preliminaries. Section 3 includes the main results of the paper. Section 4 provides some illustrative examples, and finally Section 5 concludes the paper.

Notations: $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{n\times m}$, $\mathbb{C}$, $\mathbb{C}_-$, and $\mathbb{C}_+$ represent the real numbers, $n\times 1$ real column vectors, $n\times m$ real matrices, complex numbers, complex numbers with negative real parts, and complex number with positive real parts, respectively. Also, T_n is a vector space with dimension $n\times n$, A is a matrix, $\rho(A)$ is spectral radius of A defined as $\max\{|\lambda_k|,\ \lambda_k$ is eigenvalue of $A\}$, and . is a matrix norm on T_n. τ and T are the delay value and the parameter of the Rekasius substitution, respectively.

2. Preliminaries and problem formulation

In this section, first some well-known fundamental and useful concepts regarding stability of delayed systems are presented, and then the problem is formulated.

2.1. Rekasius substitution

The characteristic equation of delay systems is generally expressed as follows

$$f(s,\tau)=a_0(\mathrm{s})+a_1(s)e^{-\tau s}+\cdots+a_k(s)e^{-k\tau s}=0. \tag{1}$$

The existence of exponential terms results in infinite number of finite roots. It is due to this feature that the analysis of such systems is challenging. One way to overcome this problem is to use the Rekasius substitution [19]. This substitution is defined as

$$e^{-\tau s}=\frac{1-Ts}{1+Ts},\ \ T\in\mathbb{R}, \tag{2}$$

which is an exact substitution at $s=j\omega,\ \ \omega\in\mathbb{R}$ or imaginary axis in $\mathbb{C}$. The relationship between τ and T is given by

$$\tau_l=\frac{2}{\omega}\left[\tan^{-1}(\omega T)\mp l\pi\right],\ \ l=0,\ 1,\ 2,\ \ldots,\ \infty. \tag{3}$$

This substitution removes the exponential terms in Eq. (1) and results the following relation.

$$a_0(s)+a_1(s)\left(\frac{1-Ts}{1+Ts}\right)+\cdots+a_k(s)\left(\frac{1-Ts}{1+Ts}\right)^k=0 \tag{4}$$

Calculating the roots of this algebraic equation is easier and more practical. It is obvious that purely imaginary roots of Eq. (4) coincide exactly with those of Eq. (1) [20]. Based on Eq. (3), it can be seen that for a T and $s=j\omega$ satisfying Eq. (4), infinite number of τ can be found from Eq. (3) that satisfy Eq. (1) at the same s [13,20].

Since purely imaginary roots of a system's characteristic equation determine the boundary of stability/instability regions, calculating such roots could be of great importance.

2.2. Puiseux series

Studying a system's behavior at purely imaginary roots can help one to analyze its stability. Suppose pair $(\omega_c,\ \tau_c)$ is a critical or a purely imaginary root of Eq. (1). The system behavior in such a root can be achieved through perturbation analysis for a small variation in τ_c. For a simple imaginary root $s=j\omega_c,\ \ \tau=\tau_c$, the asymptotic behavior can be expressed by a Taylor series in the following form [21].

$$\Delta s=\frac{ds}{d\tau}\Delta\tau+\frac{d^2s}{d\tau^2}(\Delta\tau)^2+\frac{d^3s}{d\tau^3}(\Delta\tau)^3+\cdots, \tag{5}$$

where $\Delta s=s-j\omega_c$ and $\Delta\tau=\tau-\tau_c$. For the case of multiple critical roots, the Puiseux series can be used instead to study the asymptotic behavior.

A critical root $s=j\omega_c,\ \ \tau=\tau_c$ is multiple with multiplicity $m_1\in\mathbb{N}_+$ for $s=j\omega_c$ and corresponding index $m_2\in\mathbb{N}_+$ for $\tau=\tau_c$, when

$$f_{s^0}=\cdots=f_{s^{m_1-1}}=0,\ \ f_{s^{m_1}}\neq 0, \tag{6}$$

$$f_{\tau^0}=\cdots=f_{\tau^{m_2-1}}=0,\ \ f_{\tau^{m_2}}\neq 0. \tag{7}$$

Lemma 1 [22]. *Suppose that pair $(s_c,\ \tau_c)$ represents a purely imaginary root with multiplicity m_1 for Eq. (1), where $s_c\neq 0$. If τ is perturbed at τ_c, s varies around s_c in accordance to the* *following Puiseux series*

$$\Delta s = \sum_{i=1}^{\infty} \mu_i (\Delta\tau)^{\frac{i}{N}} \tag{8}$$

where μ_is are complex coefficients and N is an integer.

Lemma 2 [22]. *If s_c is a purely imaginary root with multiplicity m_1 for Eq. (1) and $m_2 = 1$, the Puiseux series is expressed as follows*

$$\Delta s_i = \mu_i (\Delta\tau)^{\frac{1}{m_1}} + o\Big((\Delta\tau)^{\frac{1}{m_1}}\Big), \ \ i = 1, \ \ldots, \ m_1 \tag{9}$$

where $\mu_i = (-m_1! \frac{f_\tau}{f_{s^{m_1}}})^{\frac{1}{m_1}}$.

To obtain the series for $m_2 \neq 1$ is rather complicated. The following algorithm is presented for the general case.

Algorithm 1 [22].

Step 1: Set $a_0 = 0, \ b_0 = m_2$.
Step 2: Define $q^* = \max\{q = (b_0 - b)/(a - a_0) | f_{s^a \tau^b} \neq 0, \ q > 0, \ a > a_0, \ b < b_0\}$.
Step 3: If there is a q^*, go to Step 4. Otherwise, go to Step 5.
Step 4: Obtain all values $L_{ab} = ((f_{s^a \tau^b}/(a+b)!)\binom{a+b}{a}) \neq 0$ in which a and b satisfy $q^* = (b_0 - b)/(a - a_0)$, and form the following set.
$\left\{\left(\frac{f_{s^{a_1}\tau^{b_1}}}{(a_1+b_1)!}\binom{a_1+b_1}{a_1}\right)\Delta s^{a_1} \Delta\tau^{b_1}, \ \left(\frac{f_{s^{a_2}\tau^{b_2}}}{(a_2+b_2)!}\binom{a_2+b_2}{a_2}\right)\Delta s^{a_2} \Delta\tau^{b_2}, \ \ldots\ldots\right\}$,
with the order $a_1 > a_2 > \cdots$. Then, define a set of Puiseux series as $\Delta s_i = \mu_i (\Delta\tau)^{q^*} + o((\Delta\tau)^{q^*})$, $i = 1, \ldots, \ a_1 - a_0$, where $\mu_i, \ i = 1, \ldots, \ a_1 - a_0$ are the solutions of $L_{a_1 b_1} \mu^{a_1 - a_0}$ $+ L_{a_2 b_2} \mu^{a_2 - a_0} + \cdots + L_{a_0 b_0} = 0$.
Step 5: Set $a_0 = a_1, \ b_0 = b_1$, and return to Step 2.
Step 6: End.

Lemma 3 [22]. *If s_c is a purely imaginary root with multiplicity m_1 for Eq. (1), while $s_c \neq 0$, all its Puiseux series can be obtained by Algorithm 1.*

2.3. Problem formulation

Consider the following general form of v-order descriptor systems with multiple commensurate time-delays

$$M_0 z^{(v)}(t) + \sum_{i=0}^{p} \sum_{j=1}^{v} M_{i\tau j} z^{(v-j)}(t - i\tau) = 0, \tag{10}$$

where z is $n_z \times \ 1$ vector and M_0 and $M_{i\tau j}$ are $n_z \times n_z$ matrices with proper dimension and $\tau \geq 0$ is time-delay. This form of v-order descriptor systems can be restated in the first order systems by defining a new vector as follows

$$x(t) = \mathrm{col}\{z(t), \ \dot{z}(t), \ldots, \ z^{(v-1)}(t)\}. \tag{11}$$

In this way, Eq. (10) is expressed as follows

$$E\dot{x}(t) = \sum_{i=0}^{p} A_{i\tau} x(t - i\tau). \tag{12}$$

$x(t) \in \mathbb{R}^n$ $(n = n_z v)$ is the state vector and E, $A_{i\tau} \in \mathbb{R}^{n\times n}$ are known constant matrices, where $\text{rank}(E) = r \leq n$. Eq. (12) represents descriptor systems with multiple commensurate time-delays. The characteristic equation of system (12) is given by

$$P(s,\ \tau) = \det\left(sE - \sum_{i=0}^{p} A_{i\tau} e^{-i\tau s}\right) = 0, \tag{13}$$

which can be written in the following form as well.

$$P(s,\ \tau) = \sum_{k=0}^{p(n-r)} a_{1k}(s)e^{-k\tau s} + \sum_{k=p(n-r)+1}^{pn} a_{2k}(s)e^{-k\tau s} = 0 \tag{14}$$

$a_{1k}(s)$ and $a_{2k}(s)$ are polynomials in terms of s. All $a_{1k}(s)$ have the largest power of s as $\text{rank}(E) = r$. In normal state-space systems, only $a_{10}(s)$ has the largest power of s.

Definition 1 [1]**.** The pair (E, A) is regular, if $\det(sE - A)$ is not identically zero and is impulse free if $\deg\{\det(sE - A)\} = \text{rank } E$.

Definition 2. System (12) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.

Assumption 1. System (12) is assumed to be regular and impulse free in the remainder of the paper.

3. Main result

Unlike the delay free case, descriptor systems with time-delay behave differently regarding the stability. One of the substantial differences (which is also seen in neutral time-delayed systems [14]) is known as small delay effect [23]. Indeed, these systems are divided into two classes based on the continuity of their characteristic roots. According to [24], descriptor systems with time-delay which satisfy Assumption 1 have continuous characteristic roots. However, in this paper, it is shown that even these systems though are impulse free do not exempt this feature. In other words the discontinuity feature of the characteristic roots at $\tau = 0$ can be seen.

In the first class the number of unstable roots does not change by varying τ from 0 to 0^+ which indicates the continuity feature of the characteristic roots at $\tau = 0$. This feature makes it possible for such systems to have a chance to come back to stability for some values of $\tau > 0$. We call this class "τ-stabilizable" the same name which is used in [14]. In the second class the number of unstable roots at $\tau = 0$ and $\tau \to 0^+$ is different. In other words, the continuity does not hold at $\tau = 0$. It is possible that by varying τ from 0 to 0^+, infinite number of unstable roots appear, even though the system is stable for $\tau = 0$. Such systems cannot come back to stability for any value of $\tau > 0$. This is mainly because it is impossible to return infinite number of roots from $\mathbb{C}_+$ to $\mathbb{C}_-$ with a finite value of τ. This class of systems is called "τ-non-stabilizable".

Since these two classes of systems behave differently from the stability point of view, it would be desirable to study them separately.

First consider case $p = 1$, where Eq. (14) could be written as

$$E\dot{x}(t) = Ax(t) + A_\tau x(t - \tau) \tag{15}$$

The following two lemmas are used in the next discussions.

Lemma 4 [25]**.** *Inequality $\rho(A) \leq A$ holds for any $A \in T_n$.*

Lemma 5 [25]**.** *For $A \in T_n$ and an $\varepsilon > 0$, there exists a matrix norm . such that $A \leq \rho(A) + \varepsilon$.*

Theorem 1. *The descriptor system in Eq. (15) is τ-stabilizable, if for an $\tilde{\varepsilon} > 0$, the following inequality holds.*

$$\rho(A_\tau) - \rho(A) < \tilde{\varepsilon} \tag{16}$$

Proof. Considering Eq. (13), the characteristic equation of the system becomes as follows

$$P(s, \sigma) = \det(sE - A - A_\tau\sigma) = \det(A + A_\tau\sigma)\prod_{k=1}^{n}\left(-1 + s\lambda_k\left[(A + A_\tau\sigma)^{-1}E\right]\right) = 0, \tag{17}$$

where $\sigma = e^{-\tau s}$. The discontinuity will happen if $s = \infty$ is a root of Eq. (17). Considering Assumption 1 $s = \infty$ cannot be a root of Eq. (17) for $\tau > 0$, except for $\tau \to 0^+$. Let us consider this exception. Assume

$$A + A_\tau\sigma = 0, \tag{18}$$

holds for $|\sigma| < 1$, then Eq. (17) is expressed as follows

$$P(s, \sigma) = \det(sE) = 0. \tag{19}$$

Due to singularity of E, $s = \infty$ would be a root of Eq. (17) for $\tau \to 0^+$. However, when Eq. (18) holds for $|\sigma| > 1$, even for $\tau \to 0^+$, $s = \infty$ cannot be a root of Eq. (17). Therefore, system (15) will be τ-stabilizable, if Eq. (18) holds only for $|\sigma| > 1$. Thus, the following equation holds for τ-stabilizable systems.

$$\frac{A}{A_\tau} = |\sigma| > 1. \tag{20}$$

According to Lemmas 4 and 5, Eq. (20) can be restated as $\rho(A_\tau) - \rho(A) < \tilde{\varepsilon}$. Consequently, system (15) is τ-stabilizable if Eq. (16) holds. ■

Consider the characteristic equation of descriptor systems with multiple commensurate time-delays (12) which is in the form of Eq. (14). After replacing Eq. (2) into Eq. (14), the characteristic equation of system converts to

$$P_a(s,\ T) = \sum_{k=0}^{p(n-r)} a_{1k}(s)(1 + Ts)^{pn-k}(1 - Ts)^k + \sum_{k=p(n-r)+1}^{pn} a_{2k}(s)(1 + Ts)^{pn-k}(1 - Ts)^k = 0, \tag{21}$$

which is a polynomial without transcendentality.

Theorem 2. *Descriptor system with multiple commensurate time-delays (12) has finite number of purely imaginary characteristic roots $\pm j\omega_c$ for all possible $\tau \in \mathbb{R}^+$.*

Proof. The coefficients of $P_a(s,\ T)$ are polynomials in terms of T. Thus, after forming Routh–Hurwitz array for $P_a(s,\ T)$, the elements of its first column become rational polynomials in terms of T, which have finite number of roots. Then, the number of sign changes in the first column, that illustrates the number of purely imaginary roots of Eq. (21) is finite. Since

$P_a(s,\ T)$ and $P(s,\ \tau)$ possess the same purely imaginary roots, system (12) has finite number of purely imaginary characteristic roots. ■

Through Theorem 1, τ-stabilizability of descriptor systems with time-delay in the form of Eq. (15) can be checked using the Rekasius substitution and forming Routh–Hurwitz array. In the following theorem, necessary and sufficient conditions for general form of descriptor systems with multiple commensurate time-delays (12) are provided.

Theorem 3. *A descriptor system with multiple commensurate time-delays in the form of Eq. (12) is τ-stabilizable, if and only if the following conditions*

$$N_c\left(T \to 0^-\right) - N_c\left(T \to 0^+\right) = pn \text{ and } N_c\left(T \to 0^+\right) = N_{u0} \tag{22}$$

hold, where N_c is the number of sign changes in the Routh–Hurwitz array formed for $P_a(s,\ T)$ in Eq. (21) and N_{u0} is the number of unstable roots of system (12) for $\tau = 0$.

Proof. Continuity feature at $\tau = 0$ would hold, if the numbers of unstable roots at $\tau = 0$ and $\tau \to 0^+$ were the same. On the other hand, the number of sign changes in the Routh–Hurwitz array for $T \to 0^+$ determines the number of unstable roots at $\tau = 0^+$. This is why $N_c(T \to 0^+) = N_{u0}$ should hold for the continuity. Additionally, based on the Riemann sphere, by varying T from $T \to 0^-$ to $T \to 0^+$, pn infinity roots of Eq. (21) move from $+\infty$ to $-\infty$. As a consequence, $N_c(T \to 0^-) - N_c(T \to 0^+) = pn$ should hold for continuity of the characteristic roots. ■

Consider the following definitions

$$\text{Root Tendency} = \mathrm{RT}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}} = \mathrm{sgn}\left(\mathrm{Re}\left\{\frac{ds}{d\tau}\right\}\right)\Big|_{\substack{\tau=\tau_{cl}\\ s=j\omega_c}},\ \ l = 0,\ 1,\ \ldots \tag{23}$$

$$\mathrm{RT}_2\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}} = \mathrm{sgn}\left(\mathrm{Re}\left\{\frac{d^2s}{d\tau^2}\right\}\right)\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}},\ \ l = 0,\ 1,\ \ldots. \tag{24}$$

Theorem 4. *Let $\pm j\omega_c$ be a simple root of $P(s,\ \tau)$ for τ_{cl}. By shifting the delay value from $\tau_{cl} - \varepsilon$ to $\tau_{cl} + \varepsilon$, if $RT > (<)0$ then the root path enters $\mathbb{C}_+(\mathbb{C}_-)$, and if $RT = 0$ and $RT_2 \neq 0$ then the root path is tangent to the imaginary axis and does not cross it.*

Proof. Consider the following Taylor series expansion.

$$s = j\omega_c + \frac{ds}{d\tau}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}}(\tau - \tau_{cl}) + \frac{1}{2}\frac{d^2s}{d\tau^2}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}}(\tau - \tau_{cl})^2 + \cdots \tag{25}$$

For $\tau > \tau_{cl}$,

$$s = j\omega_c + \mathrm{Re}\left\{\frac{ds}{d\tau}\right\}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}}(\tau - \tau_{cl}) + \mathrm{Im}\left\{\frac{ds}{d\tau}\right\}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}}(\tau - \tau_{cl}) + \frac{1}{2}\frac{d^2s}{d\tau^2}\Big|_{\substack{\tau=\tau_{cl},\\ s=j\omega_c}}(\tau - \tau_{cl})^2 + \cdots,$$

then if $\mathrm{sgn}(\mathrm{Re}\{ds/d\tau\})|_{\tau\,=\,\tau_{cl},\ s\,=\,j\omega_c} > 0$, the root path enters $\mathbb{C}_+$, and if $\mathrm{sgn}(\mathrm{Re}\{ds/d\tau\})|_{\tau\,=\,\tau_{cl},\ s\,=\,j\omega_c} < 0$, the root path enters $\mathbb{C}_-$, and if $\mathrm{sgn}(\mathrm{Re}\{ds/d\tau\})|_{\tau\,=\,\tau_{cl},\ s\,=\,j\omega_c} = 0$, then the root path is tangent to imaginary axis and $\mathrm{RT}_2 = \mathrm{sgn}(\mathrm{Re}\{d^2s/d\tau^2\})|_{\tau\,=\,\tau_{cl},\ s\,=\,j\omega_c}$ determines the side of imaginary axis touched by the root path. However, if $\mathrm{RT}_2 = 0$, the root path is not necessarily tangent to the imaginary axis, and it might cross it. This is where the higher order of Taylor series should be checked. ■

Theorem 5. *The value of RT for any simple root of $P(s,\ \tau)$ is independent from l. Therefore, it would be the same for all τ_{cl}.*

Proof. Consider the following equation

$$\frac{ds}{d\tau} = -\left(\frac{\partial P}{\partial s}\right)^{-1}\left(\frac{\partial P}{\partial \tau}\right) = \frac{\sum_{k=0}^{p(n-r)} ksa_{1k}(s)e^{-k\tau s}+}{\sum_{k=0}^{p(n-r)}\left(\frac{da_{1k}(s)}{ds} - k\tau a_{1k}(s)\right)e^{-k\tau s}} + \frac{\sum_{k=p(n-r)+1}^{pn} a_{2k}(s)e^{-k\tau s}}{\sum_{k=p(n-r)+1}^{pn}\left(\frac{da_{2k}(s)}{ds} - k\tau a_{2k}(s)\right)e^{-k\tau s}}, \tag{26}$$

where a_{1k} and a_{2k} are polynomials in ω_c and $e^{-\tau s}$ is replaced by $\frac{1-Ts}{1+Ts}$. Hence, $\frac{ds}{d\tau}$ does not depend on τ, and thus, does not change by l. Then RT is independent from τ_{cl}. ■

Theorem 6. *Let $j\omega_c$ be a multiple root of $P(s,\ \tau)$ with multiplicity m_1 for $\tau = \tau_{c0}$ and $P_{\tau^{m_2}} \neq 0$. By calculating $RT_{n_i}^{+}$ and $RT_{n_i}^{-}$, $i = 1,\ \cdots,\ m_1$ which are defined as follows*

$$RT_{n_i}^{+} = sgn\left(Re\left\{C_i^{+}\right\}\right)\ for\ \Delta\tau = 0^{+} and\ RT_{n_i}^{-} = sgn\left(Re\left\{C_i^{-}\right\}\right)\ for\ \Delta\tau = 0^{-} \tag{27}$$

the direction of movement for each branch of root path can be determined. C_i, $i = 1,\ \ldots\cdots,\ m_1$ are coefficients of the Puiseux series obtained by Algorithm 1. This direction is obtained in a way that if $RT_{n_i}^{-} < 0$ and $RT_{n_i}^{+} > 0$, then the ith branch of root path shifts from $\mathbb{C}_-$ to $\mathbb{C}_+$ and vice versa. *Also, if $RT_{n_i}^{-} > 0$ and $RT_{n_i}^{+} > 0$, then the ith branch of root path stays in $\mathbb{C}_+$ and is tangent to imaginary axis and* vice versa.

Proof. The variation of a multiple root of $P(s,\ \tau)$ can be expressed by the Puiseux series in the following form.

$$s_i = j\omega_c + C_i(\Delta\tau)^q + o((\Delta\tau)^q),\ \ i = 1,\ \ \ldots,\ \ m_1, \tag{28}$$

where $\Delta\tau = \tau - \tau_{c0}$. C_i and q are calculated based on Algorithm 1. The value of $C_i^{+} = C_i$ is obtained for $\Delta\tau = 0^{+}$ and $C_i^{-} = C_i$ is obtained for $\Delta\tau = 0^{-}$. In this way, the signs of $\mathrm{RT}_{\mathrm{n}_i}^{-}$ and $\mathrm{RT}_{\mathrm{n}_i}^{+}$ determine the direction of movement for all branches of root path at any purely imaginary root. Also, the number of unstable roots changes around that purely imaginary root is obtained. ■

Remark 1. For $m_2 = 1$, coefficients of Eq. (28) can be calculated based on Lemma 2. Then, $\mathrm{RT}_{\mathrm{n}_i}^{+}$ or $\mathrm{RT}_{\mathrm{n}_i}^{-}$ can be calculated as

$$\mathrm{RT}_{\mathrm{n}_i}^{+}\text{or } \mathrm{RT}_{\mathrm{n}_i}^{-} = \mathrm{sgn}\left(Re\left\{\left(-m_1!\frac{P_\tau}{P_{s^{m_1}}}\right)^{\frac{1}{m_1}}\right\}\right). \tag{29}$$

Remark 2. System in Eq. (12) also covers normal systems with multiple commensurate time-delays. The largest power of polynomials in the characteristic equation of these systems is for

$k=0$. On the other hand, when $\deg(a_0(s)) > \deg(a_i(s))$, $i=1, \ldots, k$ holds, it is proven that $m_2=1$. Thus, the Puiseux series can be obtained based on Lemma 2 [22,26]. However, this is not true for descriptor systems and $m_2=1$ does not necessarily hold for these systems. This in turn demonstrates the importance and completeness of Theorem 6.

Corollary 1. *When $j\omega_c$ is a multiple root of $P(s, \tau)$ for $\tau=\tau_{c0}$ with multiplicity 2 and $m_2=1$, the number of unstable roots of $P(s, \tau)$ in a small vicinity of this root does not change by changing the time-delay.*

Proof. Due to $m_2=1$, the Puiseux series in the vicinity of this root are as follow

$$\left[\Delta s_i = \left(-2!\frac{P_\tau}{P_{s^{m_1}}}\right)^{0.5}(\tau-\tau_{c0})^{0.5} + o\left((\tau-\tau_{c0})^{0.5}\right)\right]\Bigg|_{s=j\omega_c, \tau=\tau_{c0}}, \quad i=1, 2. \tag{30}$$

Considering the nth root of a complex number z as

$$z^{1/n} = |z|^{1/n}e^{j(2(k-1)\pi+\angle z)/n}, \quad k=1, \ldots, n, \tag{31}$$

$\mathrm{RT}^+_{\mathrm{n}_i}$ and $\mathrm{RT}^-_{\mathrm{n}_i}$ are calculated as follow

$$\mathrm{RT}^+_{\mathrm{n}_i} = \mathrm{sgn}\{\cos(\frac{2(i-1)\pi + \measuredangle(-2!\frac{P_\tau}{P_{s^{m_1}}})^{0.5}}{2})\}, \quad i=1,2, \tag{32}$$

$$\mathrm{RT}^-_{\mathrm{n}_i} = \mathrm{sgn}\{\sin(\frac{2(i-1)\pi + \measuredangle(-2!\frac{P_\tau}{P_{s^{m_1}}})^{0.5}}{2})\}, \quad i=1,2. \tag{33}$$

According to Eqs. (32) and (33), $\mathrm{RT}^+_{\mathrm{n}_i}$ and $\mathrm{RT}^-_{\mathrm{n}_i}$ are opposite for each branch. Thus, each branch enters different side of the imaginary axis and this confirms that the number of unstable roots will not be changed. ■

Theorem 7. *Let $j\omega_c$ be a multiple root of $P(s, \tau)$ for $\tau=\tau_{c0}$ with a multiplicity of 2 and $m_2=1$, then $j\omega_c$ is a simple root of $P(s, \tau)$ for $\tau=\tau_{cl}$ where $l\neq 0$ and also*

$$\frac{ds}{d\tau}\Big|_{s=j\omega_c, \ \tau=\tau_{cl}} = -\frac{j\omega_c^2}{2\pi l} \tag{34}$$

Proof. Since $P(s, \tau)|_{s=j\omega_c, \ \tau=\tau_{cl}}=0$ for $l\neq 0$, the following equations hold.

$$\frac{ds}{d\tau} = -\left(\frac{\partial P}{\partial s}\right)^{-1}\left(\frac{\partial P}{\partial \tau}\right), \ \frac{d^2 s}{d\tau^2} = -\left(\frac{\partial P}{\partial s}\right)^{-1}\left(\frac{\partial^2 P}{\partial \tau^2} + \left(\frac{ds}{d\tau}\right)^2\frac{\partial^2 P}{\partial s^2} + 2\frac{ds}{d\tau}\frac{\partial^2 P}{\partial s\partial \tau}\right) \tag{35}$$

The derivatives of $P(s, \tau)$ at τ_{cl} can be calculated by its derivatives at τ_{c0}. The derivative vector of $P(s, \tau)$ is defined as follows

$$D_P = \begin{bmatrix} P & P_s & P_\tau & P_{s^2} & P_{s\tau} & P_{\tau^2}\end{bmatrix}^T. \tag{36}$$

Let us define a new function to highlight the periodicity of $P(s,\ \tau)$ regarding τ.

$$P_b\left(s,\ e^{-\tau s}\right) = P(s,\ \tau) \tag{37}$$

The following equation is obvious.

$$P_b\left(j\omega_c,\ e^{-j\tau_{cl}\omega_c}\right) = P_b\left(j\omega_c,\ e^{-j\tau_{c0}\omega_c}\right) \tag{38}$$

By $D_P = H(s,\ \tau)D_{P_b}$, the relation between $P_b(s,\ e^{-\tau s})$ and $P(s,\ \tau)$ is obtained, where $D_{P_b} = \left[P \quad P_s \quad P_{e^{-\tau s}} \quad P_{s^2} \quad P_{se^{-\tau s}} \quad P_{e^{-2\tau s}}\right]^T$.

Assuming $\sigma = e^{-\tau s}$, $H(s,\ \tau)$ is defined as follows

$$H(s,\ \tau) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\tau\sigma & 0 & 0 & 0 \\ 0 & 0 & -s\sigma & 0 & 0 & 0 \\ 0 & 0 & \tau^2\sigma & 1 & -2\tau\sigma & 0 \\ 0 & 0 & (\tau s - 1)\sigma & 0 & -s\sigma & \tau s\sigma^2 \\ 0 & 0 & s^2\sigma & 0 & 0 & s^2\sigma^2 \end{bmatrix}. \tag{39}$$

Since $e^{-\tau_{c0}s} = e^{-\tau_{cl}s}$, D_P for any τ_{cl} is computed as follows

$$D_P(j\omega_c,\ \tau_{cl}) = H(j\omega_c,\ \tau_{cl})H(j\omega_c,\ \tau_{c0})^{-1}D_P(j\omega_c,\ \tau_{c0}), \tag{40}$$

which is equivalent to

$$D_P(j\omega_c,\ \tau_{cl}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{2\pi l}{j\omega_c^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{4\pi l}{\omega_c^3} + \frac{4l^2\pi^2 j}{\omega_c^3} + \frac{4l\pi\tau_{c0}j}{\omega_c^2} & 1 & \frac{4\pi l}{j\omega_c^2} & \frac{4\pi l\tau_{c0}}{\omega_c^3} \\ 0 & 0 & 0 & 0 & 1 & \frac{2\pi l}{j\omega_c^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} D_P(j\omega_c,\ \tau_{c0}). \tag{41}$$

Based on Eq. (41), it follows that

$$P_{s|_{s=j\omega_c,\ \tau=\tau_{cl}}} = P_{s|_{s=j\omega_c,\ \tau=\tau_{c0}}} + P_{\tau|_{s=j\omega_c,\ \tau=\tau_{c0}}}\frac{j(\tau_{c0} - \tau_{cl})}{\omega_c}. \tag{42}$$

Since $P_{s|_{s=j\omega_c,\ \tau=\tau_{c0}}} = 0$ and $P_{\tau|_{s=j\omega_c,\ \tau=\tau_{c0}}} \neq 0$, $s = j\omega_c$ is a simple root for any other delay in the sequence $\tau_{cl} = \tau_{c0} + 2\pi l/\omega_c$. By substituting Eq. (41) in Eq. (35), $ds/d\tau$ can be computed as follows

$$\frac{ds}{d\tau}|_{s=j\omega_c,\ \tau=\tau_{cl}} = -\left(\frac{\partial P}{\partial s}\right)^{-1}\left(\frac{\partial P}{\partial \tau}\right) = -\frac{j\omega_c^2}{2\pi l}.$$

Based on the lemmas and theorems in this section, a hands-on algorithm (Algorithm 2) is proposed to analyze the stability of descriptor systems with multiple commensurate time-delays such that the stable and unstable regions in the space of delay as well as the number of roots in each unstable region are determined. Furthermore, necessary and sufficient conditions for τ-stabilizability of these systems are explored.

Algorithm 2 .

Step 1: Obtain characteristic equation of descriptor systems with multiple commensurate time-delays (12), which is in the form of (14).
Step 2: Use the Rekasius substitution in Eq. (14) and obtain the algebraic polynomial in the form of Eq. (21).
Step 3: Form the Routh-Hurwitz array for Eq. (21) and then calculate the critical sets of Ts and ωs, and then by using Eq. (3), obtain critical set of τ_ls for each member of T and ω sets.
Step 4: For $p = 1$, check inequality (16). If it holds, then the system is τ-stabilizable and the continuity feature of the characteristic roots in the neighborhood of $\tau = 0$ exists (go to Step 6). Otherwise the system is τ-non-stabilizable and cannot be studied thus stop the algorithm.
Step 5: For $p \neq 1$ check condition (22). If it holds, the system is τ-stabilizable and the continuity feature of the characteristic roots in the neighborhood of $\tau = 0$ exists (go to Step 6. Otherwise the system is τ-non-stabilizable and cannot be studied thus stop the algorithm.
Step 6: Consider the multiplicity of calculated purely imaginary (critical) roots. If the root is simple, go to Step 6–1, otherwise go to Step 6–2.
Step 6–1: Compute RT and RT_2. Define the root path according to Theorem 4. Based on Theorem 5 investigating τ_0 is sufficient (τ_l for $l \neq 0$ behaves similar to τ_0).
Step 6–2: Compute the root path by using Theorem 6, Corollary 1, and Remark 1. If $m_2 = 1$, then based on Theorem 7, roots are simple for $l \neq 0$. To define the root paths return to Step 6–1.
Step 7: Form the stability analysis table from the computed critical roots of the system and their root paths. This table determines stable/unstable regions in space of delay and provides the number of roots in each unstable region.
Step 8: End.

4. Illustrative examples

In this section, two numerical and one practical examples are presented to illustrate the superiority and effectiveness of the proposed method.

Example 1. Consider the following delayed descriptor system.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 2 & -12 \\ 1 & -1 & 1 \\ -2 & 3 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & -4 & 24 \\ -12 & 2 & -2 \\ 8 & -6 & 4 \end{bmatrix} x(t-\tau) \tag{43}$$

This system is stable for $\tau = 0$ and has two roots at -4.43 and -33.07. The characteristic equation of the system is obtained as follows

$$P(s,\tau) = \left(2s^2 - 23s + 11\right) + \left(4s^2 - 144s + 348\right)e^{-\tau s} + (-196s + 1260)e^{-2\tau s} + 1216e^{-3\tau s} = 0. \tag{44}$$

The Rekasius substitution is used to compute purely imaginary roots of the system. However, before computing these roots, the condition in Theorem 1 is checked.

$\rho(A_\tau) - \rho(A) = 20.6595 - 7.4296 = 13.23$

As it can be seen the sufficient condition does not hold, so condition (22) should be checked.

$N_c(T \to 0^-) - N_c(T \to 0^+) = 2 - 1 = 1 \neq 3$ and $N_c(T \to 0^+) = N_u(\tau = 0) \to 1 \neq 0$

This condition does not hold either. As a result, without considering other steps it can be claimed that this system is τ-non-stabilizable. If this system is analyzed based on the method in [13], $\tau = [0,\ 0.0335)$ will be obtained as a stable region in the space of delay.

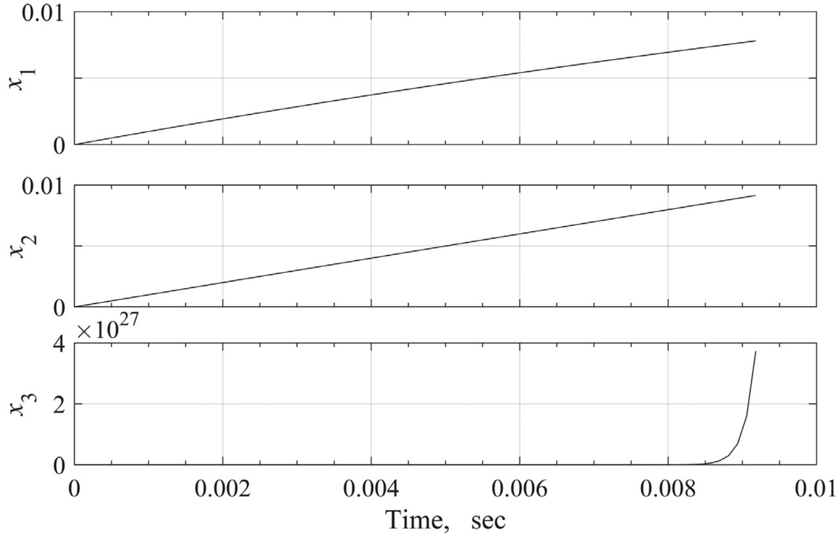

Fig. 1. Free response of system (43) for $\tau = 0.0001$.

Fig. 1 shows the free response of the system for $\tau = 0.0001$. As it is seen the system is unstable, which confirms the validity of the proposed method. This also demonstrates the superiority and comprehensiveness of this method over [13].

Example 2. Consider the following descriptor system with multiple commensurate time-delays.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1.41562 & 4 \\ 4 & -10 \end{bmatrix} x(t) + \begin{bmatrix} -0.481256 & 1 \\ -1.17989 & -2 \end{bmatrix} x(t-\tau) + \begin{bmatrix} -0.162767 & 1 \\ -1.769367 & 4.7268 \end{bmatrix} x(t-2\tau) \tag{45}$$

This system is stable for $\tau = 0$ and has a root at -1.19284. The characteristic equation of the system is obtained as

$$P(s,\tau) = (10s - 1.8438) + (2s + 8.3634)e^{-\tau s} + (-4.7268s + 0.1562)e^{-2\tau s} + e^{-3\tau s} + e^{-4\tau s} = 0. \tag{46}$$

The Rekasius substitution is used to compute the purely imaginary roots of the system. However, before computing these roots, the condition in Theorem 3 is checked.

$N_c(T \to 0^-) - N_c(T \to 0^+) = 4 - 0 = 4$ and $N_c(T \to 0^+) = N_u(\tau = 0) \to 0 = 0$

This condition holds. Thus, this system is τ-stabilizable and has the continuity feature at $\tau = 0$.

This system has a simple root at $s = \pm 0.5588j$ $(\tau = 1.5464)$ and a multiple root with multiplicity $m_1 = 2$ at $\pm 0.5j$ $(\tau = \pi)$, while $m_2 = 1$. The calculated stability map of the system is given in Table 1. This table shows the stable/unstable regions of the system in delay space and the number of roots in each unstable region, as it has been claimed. It can be seen that the system is stable in $\tau \in [0,\ 1.5464)$. Considering the variety in critical roots of this system, the importance of the presented theorems can be seen.

Table 1
Stability map of system (43).

τ	ω	Direction of branches in critical roots	Number of unstable roots
0			0
0^+			
1.5464	0.5588	$\mathrm{RT} = +1$	2
3.1416	0.5	$\mathrm{RT}^-_{n_1} = -1,\ \mathrm{RT}^+_{n_1} = +1,\ \mathrm{RT}^-_{n_2} = +1,\ \mathrm{RT}^+_{n_2} = -1$	2
12.7913	0.5588	$\mathrm{RT} = +1$	4
15.708	0.5	$\mathrm{RT} = 0,\ \mathrm{RT}_2 = -1$	4
24.0362	0.5588	$\mathrm{RT} = +1$	8
28.274	0.5	$\mathrm{RT} = 0,\ \mathrm{RT}_2 = -1$	8
⋮	⋮	⋮	⋮

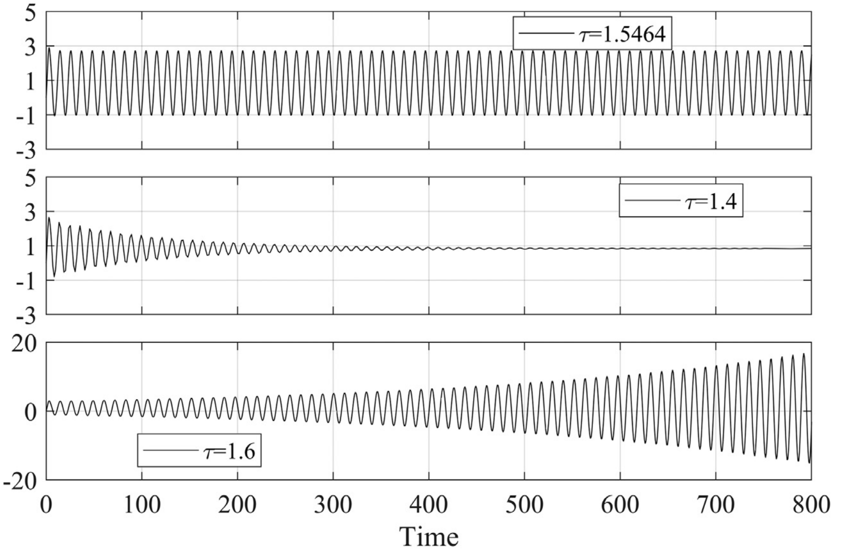

Fig. 2. Free responses of system (43) at $\tau = 1.5464$, $\tau = 1.4$ and $\tau = 1.6$.

The free responses of system (43) at $\tau = 1.5464$ and around this delay are shown in Fig. 2. As it can be seen, the system oscillates exactly at $\tau = 1.5464$, and it is stable for delay less than this value and is unstable for larger delay. This also illustrates the validity of the proposed method.

Example 3. Consider a small partial element equivalent circuit (PEEC) model for metal strip [27–29]. This model has an important role in a large number of practical applications such as power electronic, antenna design, and signal integrity analysis. The PEEC model can be written as follows

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} L & I_3 \\ 0 & -I_3 \end{bmatrix}x(t) + \begin{bmatrix} 0 & 0 \\ M+NL & N \end{bmatrix}x(t-\tau), \tag{47}$$

where

Table 2
Stability map of system (47) for various β.

$\beta = -2.105$		$\beta = -2.103$		$\beta = -2.1$	
τ	Number of unstable roots	τ	Number of unstable roots	τ	Number of unstable roots
0	0	0	0	0	0
0^+	0	0^+	0	0^+	0
1.785	0	0.6039	0	0.3936	2
5.371	0	1.827	0	1.1963	4
8.956	0	3.05	0	1.999	8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 3
Comparison of maximum τ using different methods.

Method	$\beta = -2.105$	$\beta = -2.103$	$\beta = -2.1$
[29]	1.0874	0.3707	0.2433
[28]	1.1413	0.3892	0.2553
[30]	1.6567	0.5599	0.3647
[31]	1.7399	0.5888	0.3839
The proposed method	1.785	0.6039	0.3936

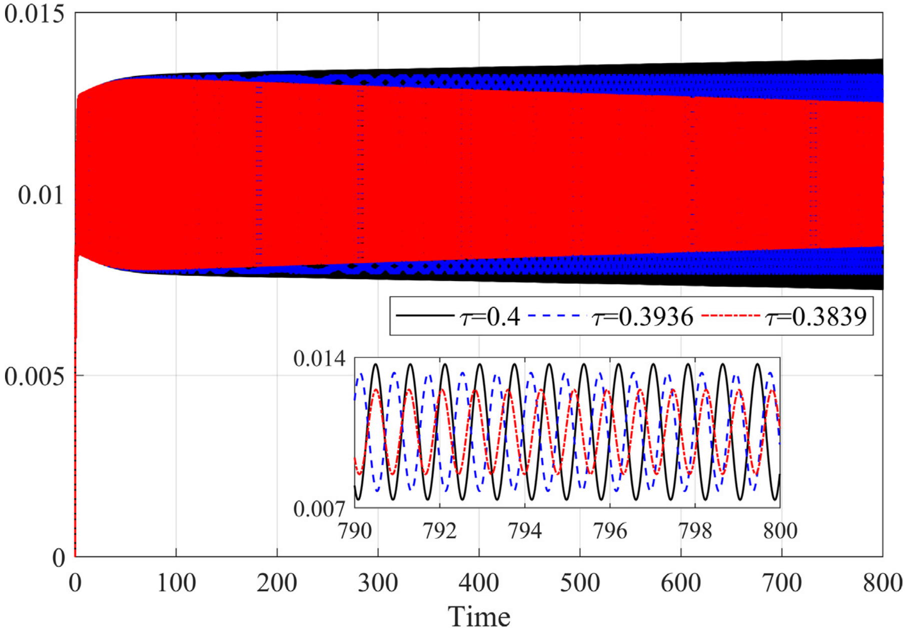

Fig. 3. Free responses of system (47) for $\beta = -2.1$ at $\tau = 0.3839$, $\tau = 0.3936$, and $\tau = 0.4$.

$$L = 100\begin{bmatrix} \beta & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \quad M = 100\begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}, \quad N = \frac{1}{72}\begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}.$$

The stability of the system is analyzed for three values of β. This system is stable at $\tau = 0$ for all three βs. In addition, condition of Theorem 1 holds for all these βs. Thus, this system is τ-stabilizable with continuity at $\tau = 0$ for the considered βs.

The stability map of the system is given in Table 2. By utilizing the proposed method, all stable and unstable regions in the delay space are determined for the given βs. Also the number of unstable roots in each region is obtained.

Table 3 compares the proposed method with other methods in the obtainment of the maximum τ in which the system remains stable. According to this table, it is obvious that the proposed method obtains more accurate results than other methods in the literature.

The free responses of the system for $\beta = -2.1$ at $\tau = 0.3936$ and also a vicinity of this delay are shown in Fig. 3. This system oscillates exactly at $\tau = 0.3936$ and is stable for lesser delays and is unstable for larger delays. This in turn illustrates the validity and precision of the proposed method.

5. Conclusion

In this paper, a practical method for stability analysis of descriptor systems with multiple commensurate time-delays is presented. This method can determine the stable and unstable regions of the system in space of delays, and also provide the number of unstable roots in each unstable region. Furthermore, by utilizing this method, τ-stabilizability and τ-non-stabilizability of the system can be determined. All proposed lemmas, corollaries and theorems are also valid for normal time-delayed systems.

In the process of stability analysis of these systems only the first term of Puiseux series is considered for multiple roots, while there may be some degenerate cases which need to consider the higher-order terms of the Puiseux series. Therefore, one of the future research directions for this paper can be considering these higher-order terms in the analysis. Additionally, in this method, invariance property is considered only for simple roots and multiple roots with multiplicity two. The other future research direction can be investigation of invariance property for multiple roots with higher multiplicities.

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