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CONDENSED FORMS FOR LINEAR PORT-HAMILTONIAN DESCRIPTOR SYSTEMS*

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Abstract. Motivated by the structure which arises in the port-Hamiltonian formulation of constraint dynamical systems, structure preserving condensed forms for skew-adjoint differential-algebraic equations (DAEs) are derived. Moreover, structure preserving condensed forms under constant rank assumptions for linear port-Hamiltonian differential-algebraic equations are developed. These condensed forms allow for the further analysis of the properties of port-Hamiltonian DAEs and to study, e.g., existence and uniqueness of solutions or to determine the index. It can be shown that under certain conditions for regular port-Hamiltonian DAEs the strangeness index is bounded by $\mu \leq 1$.

Key words. Port-Hamiltonian system, Descriptor system, Differential-algebraic equation, Strangeness index, System transformation, Skew-adjoint pair of matrix functions, Condensed form.

AMS subject classifications. 34H05, 93A30, 93B11, 93B17, 93C05, 93C15.

1. Introduction.

In this paper, we study linear variable coefficient descriptor systems of the form

$$E\dot{x} = [(J - R)Q - EK]x + (B - P)u, \tag{1.1a}$$

$$y = (B + P)^T Qx + (S + N)u, \tag{1.1b}$$

where $J, R, K \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $Q, E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $B, P \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, and $S, N \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$ on a real time interval $\mathbb{I} = [t_0, t_f]$ with $S(t) = S(t)^T$, $N(t) = -N(t)^T$ for all $t \in \mathbb{I}$. Here, $C^\ell(\mathbb{I}, \mathbb{R}^{n,m})$ denotes the ℓ-times continuously differentiable functions from $\mathbb{I}$ to the real $n \times m$ matrices. Moreover, $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ (or from an appropriate subspace) denotes the *state* of the system, $u \in C^0(\mathbb{I}, \mathbb{R}^m)$ denotes the m-dimensional *input* of the system and $y \in C^0(\mathbb{I}, \mathbb{R}^m)$ denotes the m-dimensional *output* of the system. Note that for simplicity we omit the argument t in all matrix and vector valued functions. Systems of the form (1.1) have been investigated in [2] as a new modeling framework of port-Hamiltonian systems with constrained dynamics.

Definition 1.1. [2] A linear descriptor system of the form (1.1) is called *linear port-Hamiltonian descriptor system* or *linear port-Hamiltonian differential-algebraic equations (pHDAE)* if

(i) the differential-algebraic operator $\mathcal{L} : \mathbb{D} \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)$ defined by

$$\mathcal{L}(x) = Q^T E \frac{\mathrm{d}}{\mathrm{d}t} x - (Q^T JQ - Q^T EK)x$$

is skew-adjoint, i.e., for all $t \in \mathbb{I}$ it holds that $Q^T(t)E(t) = E^T(t)Q(t)$, and

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q^T E) = Q^T [EK - JQ] + [EK - JQ]^T Q; \tag{1.2}$$

(ii) the matrix function $Q^T E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ is bounded from below by a constant symmetric matrix $H_0 \in \mathbb{R}^{n,n}$, i.e., $Q^T(t)E(t) - H_0 \geq 0$ for all $t \in \mathbb{I}$;

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(iii) the matrix function

$$W := \begin{bmatrix} Q^T RQ & Q^T P \\ P^T Q & S \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{n+m,n+m})$$

is symmetric positive semi-definite, i.e., $W(t) = W^T(t) \geq 0$ for all $t \in \mathbb{I}$.

The associated *Hamiltonian* is defined as

$$\mathcal{H}(x) := \frac{1}{2} x^T Q^T E x. \tag{1.3}$$

Descriptor systems of the form (1.1) arise in *energy based modeling* where underlying physical properties (such as conservation laws) are directly encoded into the structure of the system model. With this respect, statement (iii) in Definition 1.1 is related to the stability and passivity of the system, while the Hamiltonian (1.3) describes the total energy of the system, cf. [2]. The assumption that $Q^T E$ is bounded by a constant matrix from below implies that the Hamiltonian $\mathcal{H}$ is bounded from below by a constant in order to guarantee that the Hamiltonian can be interpreted as energy. In most of the cases assumption (ii) can be replaced by the stronger condition that $Q^T E$ is positive semi-definite on $\mathbb{I}$.

Theorem 1.2. [2] *A linear time-varying pHDAE (1.1) has the following properties:*

1. *If* $W \equiv 0$*, then* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = u^T y$*. In particular, if* $u \equiv 0$ *and* $W \equiv 0$*, then* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = 0$ *(*conservation of energy*).*
2. *The system satisfies the* dissipation inequality

$$\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^T u(t)\, dt. \tag{1.4}$$

Linear port-Hamiltonian DAEs of the described form can be seen as generalization of linear port-Hamiltonian and gyroscopic systems, see, e.g., [1, 7, 8, 16, 17], where $E = Q = I_n$ and $K = 0$ such that we get

$$\begin{aligned} \dot{x} &= (J - R)x + (B - P)u, \\ y &= (B + P)^T x + (S + N)u, \end{aligned}$$

and (1.2) reduces to the condition that J has to be (pointwise) skew-symmetric. In this case, J is referred to as structure matrix describing energy flux among energy storage elements within the system, $R = R^T$ is the dissipation matrix describing energy dissipation/loss in the system, $B \pm P$ are port matrices, describing the manner in which energy enters and exits the system, and $S + N$ describes the direct feed-through from input to output. In general, port-Hamiltonian systems generalize Hamiltonian systems in the sense that the conservation of energy for Hamiltonian systems is replaced by the dissipation inequality (1.4) that shows that the dynamical system is passive, see also [4].

The presented definition of a linear pHDAE is based on the concept of skew-adjoint differential-algebraic operators. In this paper, we will derive condensed forms for skew-adjoint pairs of matrix functions as well as for linear port-Hamiltonian DAEs that will serve as theoretical basis and main tool for the further analysis of port-Hamiltonian DAEs. We will see that the derived condensed forms require certain constant rank assumptions that are often required in the theory of general DAEs, see [11]. Condensed forms for structured DAE systems have also been considered in [20]. In [12], we have considered condensed forms for linear self-adjoint DAE systems that arise, e.g., in the necessary optimality conditions for linear optimal control

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problems. Based on the condensed forms a further analysis of the system properties as, e.g., existence and uniqueness of solution or the index of the DAE is possible. It can be shown that for regular pHDAEs (under certain assumptions) the index of the system is bounded such that the strangeness index is always less or equal 1 (the differentiation index is always less or equal 2). Non-regular pHDAEs, however, can have arbitrary high index (see Example 4.14.)

The remainder of this paper is organized as follows. After introducing some preliminary results in Section 2, we present condensed forms for skew-adjoint pairs of matrix functions under orthogonal and general congruence transformations using certain constant rank assumptions in Section 3. Next, we derive condensed forms for linear port-Hamiltonian DAEs in Section 4. The derived condensed forms allow us to analyze the corresponding DAE systems and, in particular, to draw conclusions regarding existence and uniqueness of solutions and on the index of the DAE. We close with some concluding remarks in Section 5.

2. Preliminaries.

We consider linear differential-algebraic equations (DAEs)

$$\mathcal{E}\dot{x} = \mathcal{A}x + f, \tag{2.5}$$

where $\mathcal{E}, \mathcal{A} : \mathbb{I} \to \mathbb{R}^{n,n}$ are continuous matrix-valued functions, $\mathbb{I} = [t_0, t_f] \subset \mathbb{R}$, $x : \mathbb{I} \to \mathbb{R}^n$ is a continuously differentiable unknown function, and $f : \mathbb{I} \to \mathbb{R}^n$ is a given continuous function. For a differentiable time depending function x, the derivative of x with respect to t is denoted by $\dot{x}(t) = \mathrm{d}x(t)/\mathrm{d}t$. The same notation is used for the derivative of matrix-valued functions. For a matrix $A \in \mathbb{R}^{n,n}$, A^T denotes the transposed of A, $\operatorname{rank} A$ denotes the rank of A, and a real symmetric matrix A that is positive definite or positive semi-definite is denoted by $A > 0$ or $A \geq 0$, respectively.

At first, we gather some facts about linear skew-adjoint differential-algebraic operators and recall known condensed forms for general linear DAEs of the form (2.5) under time-varying equivalence transformations. For a more detailed discussion we refer to [2] and [11].

Definition 2.1. [2] A differential-algebraic operator

$$\mathcal{L} = \mathcal{E}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A} : \mathbb{D} \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n) \tag{2.6}$$

with coefficient functions $\mathcal{E} \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $\mathcal{A} \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ is called *skew-adjoint* if

$$\mathcal{E}(t)^T = \mathcal{E}(t), \quad \dot{\mathcal{E}}(t) = -(\mathcal{A}(t) + \mathcal{A}(t)^T) \quad \text{for all } t \in \mathbb{I}. \tag{2.7}$$

Lemma 2.2. [2] *Consider a skew-adjoint differential-algebraic operator (2.6). Let $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, then the operator $\mathcal{L}_V$ defined by*

$$\mathcal{L}_V(x) := V^T\mathcal{E}V\frac{\mathrm{d}}{\mathrm{d}t}x - (V^T\mathcal{A}V - V^T\mathcal{E}\dot{V})x$$

is again skew-adjoint.

It should be noted that we have $\mathcal{L}_V(x(t)) = V^T(t)\mathcal{L}(V(t)x(t))$ for all $x \in \mathbb{D}$ and all $t \in \mathbb{I}$, which corresponds to an equivalence transformation of the underlying homogeneous DAE system $\mathcal{E}\dot{x} = \mathcal{A}x$, if V is pointwise invertible on $\mathbb{I}$. For general linear DAEs of the form (2.5) condensed forms under time-varying equivalence transformation have been presented in [10], see also [11]. To introduce these condensed forms, at first, we let $\hat{t} \in \mathbb{I}$ be fixed and consider

a matrix T which columns form a basis of $\operatorname{kernel} \mathcal{E}(\hat{t})$,

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a matrix Z whose columns form a basis of kernel $\mathcal{E}^T(\hat{t})$,

a matrix T' whose column form a basis of range $\mathcal{E}^T(\hat{t})$, and

a matrix V whose columns form a basis of corange $(Z^T\mathcal{A}(\hat{t})T)$,

as well as the characteristic quantities

$$r = \operatorname{rank}\mathcal{E}(\hat{t}), \quad a = \operatorname{rank}(Z^T\mathcal{A}(\hat{t})T), \quad s = \operatorname{rank}(V^TZ^T\mathcal{A}(\hat{t})T')$$

with the convention $\operatorname{rank}\emptyset = 0$. A necessary assumption for deriving condensed forms for (2.5) according to [10, 11] is that the quantities r, a and s are constant for all $t \in \mathbb{I}$, i.e.,

$$r(t) \equiv r, \quad a(t) \equiv a, \quad s(t) \equiv s \quad \text{for all } t \in \mathbb{I}. \tag{2.8}$$

Theorem 2.3. [10, Theorem 17] *Let $\mathcal{E}$, $\mathcal{A}$ in (2.5) be sufficiently smooth, and let (2.8) hold. Then, there exists pointwise nonsingular matrix functions $P \in C(\mathbb{I}, \mathbb{R}^{n,n})$ and $Q \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the pair $(\mathcal{E}, \mathcal{A})$ is (globally) equivalent to a pair of matrix functions $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = (P\mathcal{E}Q, P\mathcal{A}Q - P\mathcal{E}\dot{Q})$ of the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{A}_{12}(t) & 0 & \mathcal{A}_{14}(t) \\ 0 & 0 & 0 & \mathcal{A}_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \begin{array}{c} s \\ d \\ a \\ s \\ v \end{array} \tag{2.9}$$

with $d = r - s$, $v = n - d - a - s$ and the last block column in both matrices has width $u = n - r - a - s$.

Remark 2.4. A transformation of a pair of matrix functions $(\mathcal{E}, \mathcal{A})$ of the form $(P\mathcal{E}Q, P\mathcal{A}Q - P\mathcal{E}\dot{Q})$ with pointwise nonsingular matrix functions P and Q is also known as *global equivalence transformation.*

Passing from (2.9) to a pair of matrix functions

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathcal{A}_{12}(t) & 0 & \mathcal{A}_{14}(t) \\ 0 & 0 & 0 & \mathcal{A}_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) \tag{2.10}$$

corresponds to eliminating the 'strangeness' by inserting the derivative of the equation corresponding to the fourth block row of (2.9) into the equation corresponding to the first block row of (2.9). Inductively, we can define a sequence of pairs $(\mathcal{E}_i(t), \mathcal{A}_i(t))$, $i \in \mathbb{N}_0$, where $(\mathcal{E}_0(t), \mathcal{A}_0(t)) = (\mathcal{E}(t), \mathcal{A}(t))$ and $(\mathcal{E}_{i+1}(t), \mathcal{A}_{i+1}(t))$ is derived from $(\mathcal{E}_i(t), \mathcal{A}_i(t))$ by bringing it into the form (2.9) and passing then to (2.10). Assuming (2.8) for every occurring pair of matrices, we get sequences r_i, a_i, s_i, $i \in \mathbb{N}$, of nonnegative integers, which are characteristic for the given pair $(\mathcal{E}(t), \mathcal{A}(t))$. The sequence stops after finitely many steps with $s_i = 0$. The quantity $\mu = \min\{i \in \mathbb{N} \,|\, s_i = 0\}$ is called the *strangeness index* of the pair $(\mathcal{E}(t), \mathcal{A}(t))$ (or of the DAE (2.5)). The information obtained in this inductive procedure, in particular the characteristic quantities and the strangeness index, can be encoded directly in a global condensed form of the original pair of matrix functions as the following theorem shows.

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THEOREM 2.5. [11, Theorem 3.21] *Let the strangeness index μ be well defined for the pair $(\mathcal{E}(t), \mathcal{A}(t))$ of smooth matrix functions and let r_i, a_i, s_i, $i \in \{0, \dots, \mu\}$ be the related characteristic values as above. Define*

$$\begin{aligned} b_0 &= a_0, \quad b_i = \operatorname{rank}[\mathcal{A}_{14}^{(i-1)}(t)] \\ c_0 &= a_0 + s_0, \quad c_i = \operatorname{rank}[\mathcal{A}_{12}^{(i-1)}(t) \quad \mathcal{A}_{14}^{(i-1)}(t)], \\ w_0 &= v_0, \quad w_i = v_i - v_{i-1}, \quad i = l, \dots, \mu. \end{aligned}$$

Then, we have

$$\begin{aligned} c_i &= b_i + s_i, \quad i = 0, ..., \mu, \\ w_i &= s_{i-1} - c_i, \quad i = 1, ..., \mu, \end{aligned}$$

and there exists pointwise nonsingular matrix functions $P \in C(\mathbb{I}, \mathbb{R}^{n,n})$ and $Q \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the pair $(\mathcal{E}, \mathcal{A})$ is (globally) equivalent to a pair of matrix functions $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = (P\mathcal{E}Q, P\mathcal{A}Q - P\mathcal{E}\dot{Q})$ of the form

$$(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = \left(\begin{bmatrix} I_{d_\mu} & 0 & W \\ 0 & 0 & F \\ 0 & 0 & G \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{a_\mu} \end{bmatrix} \right), \tag{2.11}$$

with

$$F = \begin{bmatrix} 0 & F_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & F_1 \\ & & & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 & G_\mu & & * \\ & \ddots & \ddots & \\ & & \ddots & G_1 \\ & & & 0 \end{bmatrix},$$

where F_i and G_i have sizes $w_i \times c_{i-1}$ and $c_i \times c_{i-1}$, respectively, and $W = \begin{bmatrix} 0 & * & \cdots & * \end{bmatrix}$ is partitioned accordingly, such that F_i and G_i together have full row rank, i.e.,

$$\operatorname{rank} \begin{bmatrix} F_i \\ G_i \end{bmatrix} = c_i + w_i = s_{i-1} \le c_{i-1}.$$

In Theorem 2.5, for convenience, we denote unspecified blocks in a matrix by $*$. The condensed form (2.11) allows to read off the strangeness index of the system, as well as conditions for existence and uniqueness of solutions. In particular, the strangeness index determines the number of differentiations that are required to solve the system, e.g., the equations corresponding to the last block row of (2.11) can be solved by differentiating and backward substitution due to the nilpotent structure of the block matrix G. A DAE system is called *strangeness-free* if $\mu = 0$. The second block row in (2.11) is of size $v_\mu = w_0 + w_1 + \cdots + w_\mu$ and corresponds to the overdetermined part of the system (e.g., redundancies or vanishing equations). If $v_\mu > 0$, then there always exist inhomogeneities f for which the corresponding DAE system is not solvable. On the other hand, if $v_\mu = 0$, then every consistent initial condition fixes a unique solution. In the latter case, we call the DAE system *regular*.

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3. Condensed forms for skew-adjoint pairs of matrix functions. Motivated by the observation that a linear pHDAE is related to a skew-adjoint DAE operator (see Definition 1.1), we consider general linear DAEs

$$\mathcal{E}\dot{x} = \mathcal{A}x + f \tag{3.12}$$

with skew-adjoint pair $(\mathcal{E}, \mathcal{A})$ of matrix functions that satisfy condition (2.7). For the analysis of such systems we will derive condensed forms for skew-adjoint pairs of matrix functions in this section.

In view of Lemma 2.2, we know that we have to restrict to *congruence transformations* in order to preserve the skew-adjointness of the pair of matrix functions. For matrix pairs $(\mathcal{E}, \mathcal{A})$, with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$, $\mathcal{E} = \mathcal{E}^T$ and $\mathcal{A} = -\mathcal{A}^T$, the canonical form under congruence, i.e., $(V^T\mathcal{E}V, V^T\mathcal{A}V)$ with nonsingular V is well known, see e. g. [14, 15]. If the transformation matrices are restricted to be real orthogonal matrices, then the resulting staircase form has been developed in [3], modifying the staircase form of [19].

For self-adjoint pairs of matrix functions $(\mathcal{E}, \mathcal{A})$ where $\mathcal{E} = -\mathcal{E}^T$ and $\dot{\mathcal{E}} = \mathcal{A}^T - \mathcal{A}$ that arise, e.g., in the necessary optimality conditions for linear quadratic optimal control problems, a condensed form under congruence transformations as well as global condensed forms have been derived in [12]. In this paper, we will present the corresponding results for skew-adjoint pairs of matrix functions. Under some additional assumptions, namely regularity of the DAE and positive semi-definiteness of the leading matrix, we can draw some important conclusions in Theorem 3.5 that are specific for the skew-adjoint setting.

In [12], we have derived a staircase form for self-adjoint pairs of matrix functions and also presented a recursive procedure for its construction assuming that certain matrix functions have constant rank in the given interval $\mathbb{I}$. These constant rank assumptions are equivalent to the well-definedness of the strangeness index of $(\mathcal{E}, \mathcal{A})$ (according to (2.8) for the sequence of characteristic quantities r_i, a_i, s_i as in Theorem 2.5). For skew-adjoint pairs $(\mathcal{E}, \mathcal{A})$ the construction of a staircase form follows exactly the same lines and a similar procedure as in [12] can be constructed leading to the following result.

Theorem 3.1. *Consider a skew-adjoint pair $(\mathcal{E}, \mathcal{A})$ of matrix functions and assume that the strangeness index is well-defined. Then, there exists a congruence transformation with a pointwise orthogonal $U \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, leading to a congruent matrix pair*

$$U^T\mathcal{E}U =$$
$$\left[\begin{array}{cccc|c|cccc}
\mathcal{E}_{11} & \cdots & \cdots & \mathcal{E}_{1,\omega} & \mathcal{E}_{1,\omega+1} & \mathcal{E}_{1,\omega+2} & \cdots & \mathcal{E}_{1,2\omega} & 0 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \mathcal{E}_{\omega-1,\omega+2} & \iddots & & \\
\mathcal{E}_{1,\omega}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega} & \mathcal{E}_{\omega,\omega+1} & 0 & & & \\
\hline
\mathcal{E}_{1,\omega+1}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega+1}^T & \mathcal{E}_{\omega+1,\omega+1} & & & & \\
\hline
\mathcal{E}_{1,\omega+2}^T & \cdots & \mathcal{E}_{\omega-1,\omega+2}^T & 0 & & & & & \\
\vdots & \iddots & \iddots & & & & & & \\
\mathcal{E}_{1,2\omega}^T & \iddots & & & & & & & \\
0 & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ q_2 \\ q_1
\end{array}$$

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$$
U^T\mathcal{A}U - U^T\mathcal{E}\dot{U} =
\left[\begin{array}{cccc|c|cccc}
\mathcal{A}_{11} & \cdots & \cdots & \mathcal{A}_{1,\omega} & \mathcal{A}_{1,\omega+1} & \mathcal{A}_{1,\omega+2} & \cdots & \cdots & \mathcal{A}_{1,2\omega+1} \\
\vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\
\mathcal{A}_{\omega,1} & \cdots & \cdots & \mathcal{A}_{\omega,\omega} & \mathcal{A}_{\omega,\omega+1} & \mathcal{A}_{\omega,\omega+2} & & & \\
\hline
\mathcal{A}_{\omega+1,1} & \cdots & \cdots & \mathcal{A}_{\omega+1,\omega} & \mathcal{A}_{\omega+1,\omega+1} & & & & \\
\hline
\mathcal{A}_{\omega+2,1} & \cdots & \cdots & \mathcal{A}_{\omega+2,\omega} & & & & & \\
\vdots & & \iddots & & & & & & \\
\vdots & \iddots & & & & & & & \\
\mathcal{A}_{2\omega+1,1} & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ \vdots \\ q_1
\end{array},
$$

where $q_1 \geq s_1 \geq q_2 \geq s_2 \geq \cdots \geq q_\omega \geq s_\omega$, $b := \ell_1 + \ell_2$,

$$
\begin{aligned}
\mathcal{E}_{j,2\omega+1-j} &\in C^1(\mathbb{I}, \mathbb{R}^{s_j, q_{j+1}}), \qquad 1 \leq j \leq \omega - 1, \\
\mathcal{E}_{\omega+1,\omega+1} &= \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = \Delta^T \in C^1(\mathbb{I}, \mathbb{R}^{\ell_1,\ell_1}), \\
\mathcal{E}_{j,j} &= \mathcal{E}_{j,j}^T, \qquad j = 1, \ldots, \omega, \\
\mathcal{A}_{j,2\omega+2-j} &= -\mathcal{A}_{2\omega+2-j,j}^T = \begin{bmatrix} \Gamma_j & 0 \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_j}),\ \Gamma_j \in C^0(\mathbb{I}, \mathbb{R}^{s_j,s_j}),\ 1 \leq j \leq \omega, \\
\mathcal{A}_{\omega+1,\omega+1} &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ -\Sigma_{12}^T & \Sigma_{22} \end{bmatrix},\ \Sigma_{11} = -\Sigma_{11}^T - \dot{\Delta} \in C^0(\mathbb{I}, \mathbb{R}^{\ell_1,\ell_1}), \\
\Sigma_{22} &= -\Sigma_{22}^T \in C^0(\mathbb{I}, \mathbb{R}^{\ell_2,\ell_2}),
\end{aligned}
$$

and the blocks Σ_{22}, Δ *and* Γ_j, $j = 1, \ldots, \omega$ *are pointwise nonsingular, implying that* Σ_{22} *has even dimension* $\ell_2 = 2\ell$. *Furthermore, each of the first* ω *block columns (block rows) of the matrix* $U^T\mathcal{E}U$ *has full column rank (full row rank).*

Proof. The proof is analogous to the proof of Theorem 4.4 in [12] for self-adjoint matrix pairs. We also refer to [13] where the corresponding recursive procedure for the construction has be adapted to the skew-adjoint case. ∎

With nonsingular congruence transformations it is possible to reduce the system even further.

Corollary 3.2. *Consider a skew-adjoint pair* $(\mathcal{E}, \mathcal{A})$ *of matrix functions for which the strangeness index is well-defined. Then, there exists a congruence transformation with a pointwise nonsingular* $T \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, *leading to the congruent matrix pair*

$$
T^T\mathcal{E}T =
\left[\begin{array}{cccc|c|cccc}
\mathcal{E}_{11} & \cdots & \cdots & \mathcal{E}_{1,\omega} & \mathcal{E}_{1,\omega+1} & \mathcal{E}_{1,\omega+2} & \cdots & \mathcal{E}_{1,2\omega} & 0 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \mathcal{E}_{\omega-1,\omega+2} & \iddots & & \\
\mathcal{E}_{1,\omega}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega} & \mathcal{E}_{\omega,\omega+1} & 0 & & & \\
\hline
\mathcal{E}_{1,\omega+1}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega+1}^T & \mathcal{E}_{\omega+1,\omega+1} & & & & \\
\hline
\mathcal{E}_{1,\omega+2}^T & \cdots & \mathcal{E}_{\omega-1,\omega+2}^T & 0 & & & & & \\
\vdots & \iddots & \iddots & & & & & & \\
\mathcal{E}_{1,2\omega}^T & \iddots & & & & & & & \\
0 & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ q_2 \\ q_1
\end{array}
$$

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$$
T^T\mathcal{A}T - T^T\mathcal{E}\dot{T} =
$$
$$
\left[\begin{array}{cccc|c|cccc}
\mathcal{A}_{1,1} & \cdots & \cdots & \mathcal{A}_{1,\omega} & \mathcal{A}_{1,\omega+1} & \mathcal{A}_{1,\omega+2} & \cdots & \cdots & \mathcal{A}_{1,2\omega+1} \\
\vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\
\mathcal{A}_{\omega,1} & \cdots & \cdots & \mathcal{A}_{\omega,\omega} & \mathcal{A}_{\omega,\omega+1} & \mathcal{A}_{\omega,\omega+2} & & & \\
\hline
\mathcal{A}_{\omega+1,1} & \cdots & \cdots & \mathcal{A}_{\omega+1,\omega} & \mathcal{A}_{\omega+1,\omega+1} & & & & \\
\hline
0 & \cdots & 0 & \mathcal{A}_{\omega+2,\omega} & & & & & \\
\vdots & & \iddots & & & & & & \\
0 & \iddots & & & & & & & \\
\mathcal{A}_{2\omega+1,1} & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ \vdots \\ q_1
\end{array},
$$

where $q_1 \geq s_1 \geq q_2 \geq s_2 \geq \cdots \geq q_\omega \geq s_\omega$, $b := \ell_1 + \ell_2$,

$$
\begin{aligned}
\mathcal{E}_{j,2\omega+1-j} &\in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_{j+1}}), \qquad 1 \leq j \leq \omega - 1, \\
\mathcal{E}_{\omega+1,\omega+1} &= \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = \Delta^T \in C^1(\mathbb{I}, \mathbb{R}^{\ell_1,\ell_1}), \\
\mathcal{A}_{j,2\omega+2-j} &= -\mathcal{A}^T_{2\omega+2-j,j} = \begin{bmatrix} I_{s_j} & 0 \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_j}), \quad 1 \leq j \leq \omega, \\
\mathcal{A}_{i,j} &= -\dot{\mathcal{E}}_{i,j}, \quad i = 1, \ldots, \omega - 1,\ j = \omega + 2, \ldots, 2\omega + 1 - i, \\
\mathcal{A}_{\omega+1,\omega+1} &= \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}, \ \Sigma_{11} = -\Sigma_{11}^T \in C^0(\mathbb{I}, \mathbb{R}^{\ell_1,\ell_1}), \\
\Sigma_{22} &= -\Sigma_{22}^T \in C^0(\mathbb{I}, \mathbb{R}^{2\ell,2\ell}),
\end{aligned}
$$

and the block Σ_{22} *is pointwise nonsingular. Furthermore, each of the first* ω *block columns (block rows) of the matrix* $T^T\mathcal{E}T$ *has full column rank (full row rank).*

Proof. The proof is similar to the proof of Corollary 4.6 in [12]. Starting from the staircase form in Theorem 3.1 we can first perform a congruence transformation

$$
(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = (T_1^T U^T \mathcal{E} U T_1, T_1^T U^T \mathcal{A} U T_1 - T_1^T U^T \mathcal{E} \tfrac{\mathrm{d}}{\mathrm{d}t}(UT_1))
$$

with $T_1^T = \operatorname{diag}(\Gamma_1^{-1}, \ldots, \Gamma_\omega^{-1}, L, I_{q_\omega}, \ldots, I_{q_1})$ where

$$
L = \begin{bmatrix} I_{\ell_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{\ell_1} \end{bmatrix}.
$$

Then, with block-Gaussian congruence transformations, we can eliminate all elements above the block anti-diagonal of $\tilde{\mathcal{A}}$ in block-columns $1, \ldots, \omega$. ∎

Remark 3.3. If all eigenvalues of the matrix function $\Delta \in C^1(\mathbb{I}, \mathbb{R}^{\ell_1,\ell_1})$ in the central block $\mathcal{E}_{\omega+1,\omega+1}$ are distinct, then the matrix function Δ can be diagonalized with a smooth congruence transformation. If there are multiple eigenvalues, then such a transformation might only be possible with some loss of smoothness (depending on the order of coalescing of the eigenvalues), see [6].

Note that neither the orthogonal staircase form in Theorem 3.1 nor the condensed form in Corollary 3.2 is a normal form in the algebraic sense, since there is still further refinement possible using congruence transformations. For the purpose of analyzing systems of differential-algebraic equations, however, these condensed forms are sufficient.

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COROLLARY 3.4. *Consider a skew-adjoint pair $(\mathcal{E},\mathcal{A})$ of matrix functions and suppose that the assumptions of Theorem 3.1 hold so that there exists a congruence transformation with a pointwise orthogonal $U \in C^1(\mathbb{I},\mathbb{R}^{n,n})$ to the staircase form in Theorem 3.1.*

i) The differential-algebraic equation (3.12) is regular if and only if $s_j = q_j$ for all $j = 1,\dots,\omega$ in the staircase form in Theorem 3.1.
ii) If $\omega = 0$, then the DAE (3.12) is regular and strangeness-free.
iii) If $\omega > 0$, then $\mu \le 2\omega-1$ differentiations will be necessary to solve the system if $\ell_2 = 0$, and $\mu \le 2\omega$ differentiations will be necessary otherwise.

Proof. i) If $s_j = q_j$ for $j = 1,\dots,\omega$, then we can successively solve the equation by backward substitution in a unique way, thus the system is regular. Conversely, if $q_1 > s_1$ the DAE is non-regular, because then it has a zero row, and hence, the problem is not solvable for every smooth right hand side. If $s_j = q_j$ for $j = 1,\dots,k-1$ but $q_k > s_k$, then we can successively solve the equation from the bottom up in a unique way, until we reach the remaining system with a non-square block $\mathcal{A}_{2\omega+2-k,k} = -\mathcal{A}_{k,2\omega+2-k}^T = \begin{bmatrix}-\Gamma_k & 0\end{bmatrix}^T$. Then again, the last $q_k - s_k$ equations associated with this block are not solvable for every smooth right hand side and, hence, the problem is not regular.

ii) If $\omega = 0$, then the associated staircase form has the form

$$\left(\begin{bmatrix}\Delta & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{12}\\ \mathcal{A}_{21} & \mathcal{A}_{22}\end{bmatrix}\right)$$

with $\mathcal{A}_{22}$ pointwise nonsingular and it is well known already from the unstructured case, see [9, 11], that the associated DAE is regular and strangeness-free.

iii) Using the condensed form in Corollary 3.2, we can apply backward substitution starting with the last block row. Then we have to differentiate the right hand side at most ω times until we reach the middle block. If after backward substitution the middle block contains an algebraic part, then we continue with at most ω further differentiations. If the middle block has no algebraic part, then at most $\omega-1$ further differentiations are necessary. ◻

If the pair $(\mathcal{E},\mathcal{A})$ is in the condensed form of Corollary 3.2, and the associated DAE (3.12) is regular, then we can permute and re-arrange the condensed form to

$$\left(\begin{bmatrix}\tilde{\mathcal{E}}_{11} & \tilde{\mathcal{E}}_{12} & \tilde{\mathcal{E}}_{13} & \tilde{\mathcal{E}}_{14}\\ \tilde{\mathcal{E}}_{12}^T & \tilde{\mathcal{E}}_{22} & 0 & 0\\ \tilde{\mathcal{E}}_{13}^T & 0 & 0 & 0\\ \tilde{\mathcal{E}}_{14}^T & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}\tilde{\mathcal{A}}_{11} & \tilde{\mathcal{A}}_{12}-\dot{\tilde{\mathcal{E}}}_{12} & \tilde{\mathcal{A}}_{13}-\dot{\tilde{\mathcal{E}}}_{13} & I_s-\dot{\tilde{\mathcal{E}}}_{14}\\ -\tilde{\mathcal{A}}_{12}^T & \tilde{\mathcal{A}}_{22} & 0 & 0\\ -\tilde{\mathcal{A}}_{13}^T & 0 & \tilde{\mathcal{A}}_{33} & 0\\ -I_s & 0 & 0 & 0\end{bmatrix}\right),$$

where $s = \sum_{i=1}^{\omega} s_i$, and $\tilde{\mathcal{E}}_{22} = \Delta = \Delta^T$ as well as $\tilde{\mathcal{A}}_{33} = \Sigma_{22} = -\Sigma_{22}^T$ are invertible. Moreover, $\tilde{\mathcal{E}}_{14}$ is block upper-triangular with square diagonal blocks, which are zero matrices. Performing some further block-Gaussian elimination congruence transformations, we can eliminate the block to the left of $\tilde{\mathcal{A}}_{33}$. Then, due to the skew-adjoint structure, the part $\tilde{\mathcal{A}}_{13}$ above the block $\tilde{\mathcal{A}}_{33}$ is eliminated as well, while the part $\dot{\tilde{\mathcal{E}}}_{13}$ remains. In the same way, the block above and to the left of $\tilde{\mathcal{E}}_{22}$ can be eliminated. One further block permutation (exchanging the first two block rows and columns), partitioning the blocks further, and

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renaming the blocks, finally leads to the form

$$\left(\begin{bmatrix}\Delta & 0 & 0 & 0\\ 0 & \mathcal{E}_{33} & \mathcal{E}_{34} & \mathcal{E}_{35}\\ 0 & \mathcal{E}_{34}^T & 0 & 0\\ 0 & \mathcal{E}_{35}^T & 0 & 0\end{bmatrix}, \begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{13} & 0 & 0\\ -\mathcal{A}_{13}^T & \mathcal{A}_{33} & -\dot{\mathcal{E}}_{34} & I_s-\dot{\mathcal{E}}_{35}\\ 0 & 0 & \mathcal{A}_{44} & 0\\ 0 & -I_s & 0 & 0\end{bmatrix}\right), \tag{3.13}$$

with $\mathcal{A}_{44} = -\mathcal{A}_{44}^T$ invertible (and of even dimension), and $\mathcal{E}_{35}$ block upper-triangular with square diagonal blocks, which are zero matrices. These results are similar to the corresponding results for the self-adjoint case presented in [12]. However, the following is specific to the skew-adjoint setting leading to an important result in Theorem 3.5.

In our original motivation the leading matrix $\mathcal{E}$ of the skew-adjoint differential-algebraic system is given by $\mathcal{E} = Q^TE$, where Q and E are coefficient functions of a linear pHDAE (1.1) (see Definition 1.1). We will now assume that the matrix function $\mathcal{E} = Q^TE$ is positive semi-definite on $\mathbb{I}$, a somewhat stronger assumption than (ii) in Definition 1.1, but often satisfied in physical applications (for examples see [13]), and furthermore that the DAE (3.12) is regular. Under these assumptions we can transform the pair $(\mathcal{E},\mathcal{A})$ to the condensed form (3.13), where due to the positive semi-definiteness of $\mathcal{E}$ we have that Δ is positive definite as well as $\mathcal{E}_{34}\equiv 0$ and $\mathcal{E}_{35}\equiv 0$. Thus, the condensed form (3.13) reduces to

$$\left(\begin{bmatrix}\Delta & 0 & 0 & 0\\ 0 & \mathcal{E}_{33} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}\mathcal{A}_{11} & \mathcal{A}_{13} & 0 & 0\\ -\mathcal{A}_{13}^T & \mathcal{A}_{33} & 0 & I_s\\ 0 & 0 & \mathcal{A}_{44} & 0\\ 0 & -I_s & 0 & 0\end{bmatrix}\right) \begin{matrix}\ell_1\\ s\\ 2\ell\\ s\end{matrix} \tag{3.14}$$

with $\dot{\Delta} = -(\mathcal{A}_{11}+\mathcal{A}_{11}^T)$ on $\mathbb{I}$, $\mathcal{A}_{44}(t) = -\mathcal{A}_{44}^T(t)$ for all $t\in\mathbb{I}$ and $\mathcal{A}_{44}$ of even dimension 2ℓ is pointwise nonsingular, as well as $\dot{\mathcal{E}}_{33} = -(\mathcal{A}_{33}+\mathcal{A}_{33}^T)$ on $\mathbb{I}$. Moreover, we know that $\mathcal{E}_{33}$ of size $s\times s$ is pointwise nonsingular due to the full column rank condition in Corollary 3.2, and also positive definite. The corresponding DAE takes the form

$$\begin{aligned}\Delta\dot{x}_1 &= \mathcal{A}_{11}x_1 + \mathcal{A}_{13}x_2 + f_1,\\ \mathcal{E}_{33}\dot{x}_2 &= -\mathcal{A}_{13}^Tx_1 + \mathcal{A}_{33}x_2 + x_4 + f_2,\\ 0 &= \mathcal{A}_{44}x_3 + f_3,\\ 0 &= -x_2 + f_4,\end{aligned} \tag{3.15}$$

for a sufficiently smooth inhomogeneity $f = [f_1, f_2, f_3, f_4]^T$. The last two equations in (3.15) can be solved for x_2 and x_3 giving $x_2 = f_4$ and $x_3 = -\mathcal{A}_{44}^{-1}f_3$. Differentiating the relation for x_2 and inserting it into the second equation of (3.15) gives

$$\begin{aligned}\Delta\dot{x}_1 &= \mathcal{A}_{11}x_1 + \mathcal{A}_{13}f_4 + f_1,\\ x_4 &= \mathcal{E}_{33}\dot{f}_4 + \mathcal{A}_{13}^Tx_1 - \mathcal{A}_{33}f_4 - f_2,\end{aligned}$$

i.e., an ordinary differential equation for x_1 and subsequently the solution for x_4. Thus, we require at most one differentiation of equations to obtain the unique solution of the system. Also we see that x_1 is the only differential component in the system which is related to the dynamics, while x_2, x_3 and x_4 are algebraic components related to algebraic constraints on the dynamics. The algebraic component x_2 and its coupling to the second equation in (3.15) results in an index greater than 0. We formulate this result in the following theorem.

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THEOREM 3.5. *Consider a regular DAE (3.12) with coefficient functions $(\mathcal{E}, \mathcal{A})$ that form a skew-adjoint pair. Suppose that the assumptions of Theorem 3.1 hold and that $\mathcal{E}$ is positive semi-definite for all $t \in \mathbb{I}$. Then the DAE (3.12) has strangeness index $\mu \leq 1$. In particular, the DAE (3.12) has strangeness index $\mu = 1$ if and only if $s > 0$ in the reduced condensed form (3.14). If $s = \ell = 0$ in the reduced condensed form (3.14), then the system (3.12) is an ODE.*

Thus, linear regular skew-adjoint DAEs with well-defined strangeness index and positive semi-definite leading matrix always have a strangeness index less or equal than 1 (which is equivalently to a differentiation index less or equal than 2, cf. [11]). Here, regularity of the DAE (3.12) is an assumption that should always be satisfied in reasonable models. Non-regular DAE systems usually result from modeling or discretization errors, and in this case, the system should be regularized in a preprocessing step (e.g., by using feedback regularization [5]). In the context of port-Hamiltonian DAEs we will see that undetermined components can also be reinterpreted as port variables, see Lemma 4.9.

4. Condensed forms for linear port-Hamiltonian DAEs. In this section, we derive condensed forms for linear pHDAEs (1.1) under equivalence transformations. In order to shorten the notation, we introduce the matrix function $D = S + N$ in the sequel. The development of the condensed forms is based on the following result.

THEOREM 4.1. [2] *Consider a linear pHDAE system (1.1) with Hamiltonian (1.3). Let $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ be pointwise invertible on $\mathbb{I}$. Then the transformed system*

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \\ y &= (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + Du, \end{aligned}$$

with

$$\begin{aligned} &\tilde{E} = U^T E V, \quad \tilde{Q} = U^{-1} Q V, \quad \tilde{J} = U^T J U, \quad \tilde{R} = U^T R U, \\ &\tilde{B} = U^T B, \quad \tilde{P} = U^T P, \quad \tilde{K} = V^{-1} K V + V^{-1}\dot{V}, \quad \tilde{x} = V^{-1} x \end{aligned}$$

is again a pHDAE with the same Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{Q}^T \tilde{E}\tilde{x} = \frac{1}{2} x^T Q^T E x = \mathcal{H}(x)$.

Thus, we can define equivalence of linear pHDAEs in the following way.

DEFINITION 4.2. Two pHDAE systems of the form (1.1) defined by the tuples of matrix functions $(E_i, J_i, R_i, Q_i, K_i, B_i, P_i, D_i)$, $i = 1, 2$ are called *equivalent* if there exist pointwise invertible matrix functions $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that

$$\begin{aligned} &E_2 = U^T E_1 V, \quad Q_2 = U^{-1} Q_1 V, \quad J_2 = U^T J_1 U, \quad R_2 = U^T R_1 U, \\ &B_2 = U^T B_1, \quad P_2 = U^T P_1, \quad K_2 = V^{-1} K_1 V + V^{-1}\dot{V}, \quad D_2 = D_1. \end{aligned} \tag{4.16}$$

It should be noted that the matrix K is required to describe equivalence transformations in the time-varying setting, see also Example 4.14. However, we can always find an equivalence transformation that eliminates K in the pHDAE (1.1) due to the following result.

LEMMA 4.3. [2] *Consider a linear pHDAE*

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \\ y &= (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + Du, \end{aligned}$$

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with Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T\tilde{Q}^T\tilde{E}\tilde{x}$, *where* $\tilde{K} \in C(\mathbb{I}, \mathbb{R}^{n,n})$. *If* $V_{\tilde{K}}$ *is a pointwise invertible solution of the matrix differential equation* $\dot{V} = -\tilde{K}V$ *with initial condition* $V(t_0) = I_n$, *then defining* $E = \tilde{E}V_{\tilde{K}}^{-1}$, $Q = \tilde{Q}V_{\tilde{K}}^{-1}$, $x = V_{\tilde{K}}\tilde{x}$, $J = \tilde{J}$, $R = \tilde{R}$, $B = \tilde{B}$, $P = \tilde{P}$, *the system*

$$\begin{aligned} E\dot{x} &= (J-R)Qx + (B-P)u, \\ y &= (B+P)^TQx + Du, \end{aligned}$$

is again a pHDAE with $\mathcal{H}(x) = \frac{1}{2}x^TQ^TEx = \tilde{\mathcal{H}}(\tilde{x})$.

Remark 4.4. The matrix differential equation for $V_{\tilde{K}}$ in Lemma 4.3 can be solved numerically, and if $\tilde{K} = -\tilde{K}^T$, then the resulting solution is pointwise orthogonal. On the other hand, if we restrict to equivalence transformations using only pointwise orthogonal matrices U and V in Definition 4.2, then we will always get a skew-symmetric matrix function $\tilde{K}$.

We start our investigations by restricting to equivalence transformations with orthogonal matrix functions U and V such that $U^{-1} = U^T$ and $V^{-1} = V^T$ in (4.16). In this case, we can assume that $K = -K^T$ (see Remark 4.4). As before, to derive the condensed form, we have to assume constant rank of certain matrix functions. Then, we can use the following result.

Theorem 4.5. [11, Theorem 3.9] *Let* $E \in C^\ell(\mathbb{I}, \mathbb{R}^{m,n})$, $\ell \in \mathbb{N}_0 \cup \{\infty\}$, *with* rank $E(t) = r$ *for all* $t \in \mathbb{I}$. *Then there exist pointwise real orthogonal matrix functions* $U \in C^\ell(\mathbb{I}, \mathbb{R}^{m,m})$ *and* $V \in C^\ell(\mathbb{I}, \mathbb{R}^{n,n})$, *such that*

$$U^TEV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

with pointwise nonsingular $\Sigma \in C^\ell(\mathbb{I}, \mathbb{R}^{r,r})$.

Assuming that $\operatorname{rank} E(t) = r$ for all $t \in \mathbb{I}$ we can at first compute a factorization of E as in Theorem 4.5, i.e., there exist pointwise orthogonal matrix functions U_1 and V_1 of size $n \times n$ such that

$$\tilde{E}_1 := U_1^TEV_1 = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix},$$

with Σ_r of size $r \times r$ pointwise nonsingular. The other matrix functions are transformed according to (4.16) into

$$\tilde{Q}_1 := U_1^TQV_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

and $\tilde{J}_1 := U_1^TJU_1$, $\tilde{R}_1 := U_1^TRU_1$, $\tilde{K}_1 := V_1^TKV_1 + V_1^T\dot{V}_1$, $\tilde{B}_1 := U_1^TB$, $\tilde{P}_1 := U_1^TP$, partitioned accordingly. Since the transformed system is again a pHDAE, we have that $\tilde{Q}_1^T\tilde{E}_1 = \tilde{E}_1^T\tilde{Q}_1$, such that we get $Q_{12} \equiv 0$, as well as $Q_{11}^T\Sigma_r = \Sigma_r^TQ_{11}$ on $\mathbb{I}$. Thus, we have

$$\tilde{Q}_1 = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{4.17}$$

Assuming that

$$\operatorname{rank} Q_{22}(t) = q \quad \text{for all } t \in \mathbb{I}, \tag{R1}$$

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again using Theorem 4.5, there exist pointwise orthogonal matrix functions U_{22} and V_{22}, both of size $(n-r)\times(n-r)$, such that

$$U_{22}^T Q_{22} V_{22} = \begin{bmatrix} \Sigma_q & 0 \\ 0 & 0 \end{bmatrix},$$

where Σ_q of size $q\times q$ is pointwise nonsingular. With $U_2 = \begin{bmatrix} I_r & 0 \\ 0 & U_{22} \end{bmatrix}$, $V_2 = \begin{bmatrix} I_r & 0 \\ 0 & V_{22} \end{bmatrix}$ we get the transformed matrix functions

$$\tilde{E}_2 := U_2^T \tilde{E}_1 V_2 = \begin{bmatrix} \Sigma_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_2 := U_2^T \tilde{Q}_1 V_2 = \begin{bmatrix} Q_{11} & 0 & 0 \\ Q_{21} & \Sigma_q & 0 \\ Q_{31} & 0 & 0 \end{bmatrix}$$
$$\tilde{J}_2 := U_2^T \tilde{J}_1 U_2 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$

as well as $\tilde{R}_2 := U_2^T \tilde{R}_1 U_2$, $\tilde{K}_2 := V_2^T \tilde{K}_1 V_2 + V_2^T \dot{V}_2$, $\tilde{B}_2 := U_2^T \tilde{B}_1$, $\tilde{P}_2 := U_2^T \tilde{P}_1$, partitioned accordingly. The skew-adjointness condition (1.2) now takes the form

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}(Q_{11}^T \Sigma_r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * & Q_{11}^T \Sigma_r K_{13} \\ * & -\Sigma_q^T (J_{22} + J_{22}^T) \Sigma_q & 0 \\ K_{13}^T \Sigma_r^T Q_{11} & 0 & 0 \end{bmatrix},$$

and, thus, gives $J_{22}(t) = -J_{22}^T(t)$ for all $t \in \mathbb{I}$, as well as $Q_{11}^T \Sigma_r K_{13} \equiv 0$ on $\mathbb{I}$. Assuming that

$$\text{(R2)} \qquad \operatorname{rank} Q_{31}(t) = w \quad \text{for all } t \in \mathbb{I},$$

there exist pointwise orthogonal matrix functions U_{31} of size $(n-r-q)\times(n-r-q)$ and V_{31} of size $r\times r$ such that

$$U_{31}^T Q_{31} V_{31} = \begin{bmatrix} \Sigma_w & 0 \\ 0 & 0 \end{bmatrix},$$

where Σ_w of size $w\times w$ is pointwise nonsingular for all $t \in \mathbb{I}$. Finally, we obtain the following condensed form for linear pHDAEs.

THEOREM 4.6. *Consider a linear pHDAE (1.1) that is defined by the tuple of matrix functions (E, J, R, Q, K, B, P, D) with $r = \operatorname{rank} E(t)$ for all $t \in \mathbb{I}$. Under the constant rank assumptions (R1) and (R2) there exist pointwise orthogonal matrix functions $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the system is equivalent to a pHDAE described by*

$$(4.18) \qquad U^T E V = \begin{array}{c} \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} w \\ d \\ q \\ w \\ v \end{matrix} \\ \begin{matrix} w & \; d & \; q & \; w & \; v \end{matrix} \end{array}, \quad U^T Q V = \begin{array}{c} \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & \Sigma_q & 0 & 0 \\ \Sigma_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} w \\ d \\ q \\ w \\ v \end{matrix} \\ \begin{matrix} w & \; d & \; q & \; w & \; v \end{matrix} \end{array},$$

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where $d = r - w$, $v = n - r - q - w$, *with pointwise nonsingular blocks* Σ_q, Σ_w *and* $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$, *and*

$$U^T JU = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} \end{bmatrix}, \quad U^T RU = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix},$$

$$U^T B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}, ; U^T P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}, \; \tilde{K} := V^T KV + V^T \dot{V} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix},$$

partitioned accordingly, with $J_{33}(t) = -J_{33}^T(t)$, $\tilde{K}^T(t) = -\tilde{K}(t)$ *for all* $t \in \mathbb{I}$, *and*

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} K_{14} & K_{15} \\ K_{24} & K_{25} \end{bmatrix} \equiv 0$$

on $\mathbb{I}$.

Proof. The proof follows directly from the previous discussion using the skew-adjointness condition (1.2). ☐

If we also allow non-orthogonal transformations, then we can further simplify the matrices in (4.18).

THEOREM 4.7. *Consider a linear pHDAE (1.1) that is defined by the tuple of matrix functions* (E, J, R, Q, K, B, P, D) *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$. *Under the assumptions of Theorem 4.6 there exist pointwise nonsingular matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by the tuple of matrix functions* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, D)$ *given by*

$$\tilde{E} = U^T EV = \begin{array}{c} \begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} w \\ d \\ q \\ w \\ v \end{matrix} \\ \begin{matrix} w & d & q & w & v \end{matrix} \end{array}, \quad \tilde{Q} = U^{-1} QV = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} w \\ d \\ q \\ w \\ v \end{matrix} \\ \begin{matrix} w & d & q & w & v \end{matrix} \end{array}, \tag{4.19a}$$

where $d = r - w$, $v = n - r - q - w$, *and* $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{d,d}$ *for all* $t \in \mathbb{I}$, *and*

$$\tilde{J} = U^T JU = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} \end{bmatrix}, \; \tilde{R} = U^T RU = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix}$$

$$\tilde{K} = V^{-1} KV + V^{-1} \dot{V} = \begin{bmatrix} 0 & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & 0 & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & 0 & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & 0 & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & 0 \end{bmatrix}, \tag{4.19b}$$

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$$\tilde{B} = U^T B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}, \quad \tilde{P} = U^T P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},$$

partitioned accordingly, with

$$\begin{aligned}
&J_{44} = -J_{44}^T,\ J_{33} = -J_{33}^T,\ J_{43} = -J_{34}^T,\\
&R_{33} = R_{33}^T,\ R_{44} = R_{44}^T,\ R_{32} = R_{23}^T,\ R_{42} = R_{24}^T,\ R_{43} = R_{34}^T,\\
&Q_{22}K_{24} \equiv 0,\ Q_{22}K_{25} \equiv 0,\ Q_{22}(J_{23} - K_{23}) = -Q_{22}J_{32}^T,\ Q_{22}(J_{24} - K_{21}) = -Q_{22}J_{42}^T,\\
&\dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T)Q_{22},
\end{aligned}$$

as well as

$$\begin{bmatrix} R_{44} & R_{24}^T Q_{22} & R_{34}^T \\ Q_{22}R_{24} & Q_{22}R_{22}Q_{22} & Q_{22}R_{23} \\ R_{34} & R_{23}^T Q_{22} & R_{33} \end{bmatrix} \geq 0$$

on $\mathbb{I}$.

Proof. Using the results of Theorem 4.6, we consider a linear pHDAE with matrix functions (E, J, R, Q, K, B, P, D) given in condensed form (4.18). For the pair of matrix functions (E, Q) we can perform a sequence of equivalence transformations according to (4.16) yielding

$$\begin{aligned}
&\left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & \Sigma_q & 0 & 0 \\ \Sigma_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}$$

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$$\sim \left(\begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & Q_{12} & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right),$$

where $d = r - w$ and J, R, K, B, P are transformed and partitioned accordingly. From the symmetry of $Q^T E$ we get that $Q_{12} \equiv 0$ as well as $Q_{22}^T(t) = Q_{22}(t)$ for all $t \in \mathbb{I}$. Finally, we can determine pointwise nonsingular matrix functions V_{ii}, $i = 1, \ldots, 5$, such that

$$\dot{V}_{ii} = -K_{ii} V_{ii}, \quad V_{ii}(t_0) = I, \quad i = 1, \ldots, 5$$

in order to eliminate the diagonal blocks in $K = [K_{ij}]_{i,j=1,\ldots,5}$, resulting in a tuple of matrix functions with a block structure as in (4.19). The skew-adjointness condition (1.2) and condition (iii) in Definition 1.1 yield the remaining conditions for the block matrices in (4.19). ∎

Remark 4.8. The total energy of the pHDAE (1.1) is given by

$$\mathcal{H}(x) = \tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{E}^T \tilde{Q} \tilde{x} = \frac{1}{2}\tilde{x}_2^T Q_{22} \tilde{x}_2,$$

for the transformed state vector $\tilde{x} = V^{-1}x$ partitioned according to the condensed form (4.19). Thus, the only contribution to the total energy comes from the component $\tilde{x}_2$ and the matrix function Q_{22}. The remaining components of the state vector belong to algebraic constraints that have no energy contribution to the system or to undetermined components that should have been considered as port variables in the modeling (see also Lemma 4.9).

If there are undetermined components of the state vector in a pHDAE (1.1) this usually means that a modeling error has occurred. Such undetermined components should be reinterpreted as port variables. For a pHDAE given in condensed form (4.19) such a reinterpretation can be easily performed.

Lemma 4.9. *Consider a linear pHDAE (1.1) given in condensed form (4.19) and let the state vector $\tilde{x} = [\tilde{x}_1^T, \tilde{x}_2^T, \tilde{x}_3^T, \tilde{x}_4^T, \tilde{x}_5^T]^T$ be partitioned according to the block structure of (4.19). If $v > 0$, then there are undetermined components of the state vector that can be reinterpreted as port variables of a DAE system*

$$\begin{aligned} \hat{E}\dot{\hat{x}} &= [(\hat{J} - \hat{R})\hat{Q} - \hat{E}\hat{K}]\hat{x} + (\hat{B} - \hat{P})\hat{u}, \\ \hat{y} &= (\hat{B} + \hat{P})^T \hat{Q}\hat{x} + \hat{D}\hat{u}, \end{aligned} \tag{4.20}$$

where

$$\hat{E} = \begin{bmatrix} I_w & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{Q} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ I_w & 0 & 0 & 0 \end{bmatrix}, \hat{K} = \begin{bmatrix} 0 & K_{12} & K_{13} & K_{14} \\ K_{21} & 0 & K_{23} & K_{24} \\ K_{31} & K_{32} & 0 & K_{34} \\ K_{41} & K_{42} & K_{43} & 0 \end{bmatrix},$$

$$\hat{J} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}, \hat{R} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix}, \quad \hat{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} B_1 & \frac{1}{2}(J_{51}^T - R_{51}^T - K_{15}) \\ B_2 & \frac{1}{2}(J_{52}^T - R_{52}^T - K_{25}) \\ B_3 & \frac{1}{2}(J_{53}^T - R_{53}^T) \\ B_4 & \frac{1}{2}(J_{54}^T - R_{54}^T) \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} P_1 & \frac{1}{2}(J_{51}^T - R_{51}^T + K_{15}) \\ P_2 & \frac{1}{2}(J_{52}^T - R_{52}^T + K_{25}) \\ P_3 & \frac{1}{2}(J_{53}^T - R_{53}^T) \\ P_4 & \frac{1}{2}(J_{54}^T - R_{54}^T) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u \\ \tilde{x}_5 \end{bmatrix},$$

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$$\hat{D} = \begin{bmatrix} D & 0 \\ B_5 - P_5 & I \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y \\ \tilde{x}_5 \end{bmatrix}.$$

If $\| \begin{bmatrix} J_{51} - R_{51} & \cdots & J_{54} - R_{54} \end{bmatrix} \|$ *and* $\|B_5 - P_5\|$ *is sufficiently small, then the system* (4.20) *is again a pHDAE.*

Proof. In a pHDAE in condensed form (4.19), the components $\tilde{x}_5$ of dimension v are undetermined components of the state vector $\tilde{x}$. These components can be reinterpreted as port variables by rewriting the system as

$$\begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_{14} & L_{12}Q_{22} - K_{12} & L_{13} - K_{13} & -K_{14} \\ L_{24} - K_{21} & L_{22}Q_{22} & L_{23} - K_{23} & -K_{24} \\ L_{34} & L_{32}Q_{22} & L_{33} & 0 \\ L_{44} & L_{42}Q_{22} & L_{43} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} B_1 - P_1 & -K_{15} \\ B_2 - P_2 & -K_{25} \\ B_3 - P_3 & 0 \\ B_4 - P_4 & 0 \end{bmatrix} \begin{bmatrix} u \\ \tilde{x}_5 \end{bmatrix},$$

$$\begin{bmatrix} y \\ \tilde{x}_5 \end{bmatrix} = \begin{bmatrix} (B_1 + P_1)^T & (B_2 + P_2)^T & (B_3 + P_3)^T & (B_4 + P_4)^T \\ L_{51} & L_{52} & L_{53} & L_{54} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ I_w & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} + \begin{bmatrix} D & 0 \\ B_5 - P_5 & I \end{bmatrix} \begin{bmatrix} u \\ \tilde{x}_5 \end{bmatrix},$$

where we define $L_{ij} := J_{ij} - R_{ij}$ for $i, j = 1, \ldots, 5$. By using the above definitions it can be easily checked that the properties (i) and (ii) in Definition 1.1 of a pHDAE are satisfied for (4.20). For condition (iii) in Definition 1.1 we need that $\hat{W} = \hat{W}^T \geq 0$, where $\hat{W}$ is given by

$$\hat{W} = \begin{bmatrix} \hat{Q}^T \hat{R} \hat{Q} & \hat{Q}^T \hat{P} \\ \hat{P}^T \hat{Q} & \hat{S} \end{bmatrix}$$

and $\hat{S} = \begin{bmatrix} S & \frac{1}{2}(B_5^T - P_5^T) \\ \frac{1}{2}(B_5 - P_5) & I \end{bmatrix}$ with S given from $D = S + N$. Using the properties of the Schur complement it follows that $\hat{W} \geq 0$ if

$$\begin{bmatrix} \hat{Q}^T \hat{R} \hat{Q} - \frac{1}{4} \hat{Q}^T \begin{bmatrix} J_{51}^T - R_{51}^T \\ \vdots \\ J_{54}^T - R_{54}^T \end{bmatrix} \begin{bmatrix} J_{51} - R_{51} & \cdots & J_{54} - R_{54} \end{bmatrix} \hat{Q} & \hat{Q}^T \begin{bmatrix} P_1 \\ \vdots \\ P_4 \end{bmatrix} - \frac{1}{4} \hat{Q}^T \begin{bmatrix} J_{51}^T - R_{51}^T \\ \vdots \\ J_{54}^T - R_{54}^T \end{bmatrix} (B_5 - P_5) \\ \begin{bmatrix} P_1^T & \cdots & P_4^T \end{bmatrix} \hat{Q} - \frac{1}{4}(B_5^T - P_5^T) \begin{bmatrix} J_{51} - R_{51} & \cdots & J_{54} - R_{54} \end{bmatrix} \hat{Q} & S - \frac{1}{4}(B_5^T - P_5^T)(B_5 - P_5) \end{bmatrix} \geq 0$$

and if $J_{51}, \ldots, J_{54} = 0$, $R_{51}, \ldots, R_{54} = 0$ and $B_5 = 0$, $P_5 = 0$, this condition is trivially satisfied, since the original system was assumed to be a pHDAE (fulfilling (iii) of Definition 1.1). Otherwise, if $\| \begin{bmatrix} J_{51} - R_{51} & \cdots & J_{54} - R_{54} \end{bmatrix} \|$ and $\|B_5 - P_5\|$ is sufficiently small (e.g., if the undetermined part results from (small) modeling or approximation errors), this condition is satisfied as well. ∎

In many practical examples and applications, not only $Q^T E$ but also the product EQ^T is symmetric, i.e.,

$$E(t)Q^T(t) = Q(t)E^T(t) \quad \text{for all } t \in \mathbb{I}$$

(for examples see [13]). In this case, the condensed form (4.19) simplifies as follows.

Corollary 4.10. *Consider a linear pHDAE* (1.1) *that is defined by the tuple of matrix functions* (E, J, R, Q, K, B, P, D) *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$, *which satisfies* $E(t)Q^T(t) = Q(t)E^T(t)$ *for all* $t \in \mathbb{I}$. *Under the assumption of Theorem 4.6 there exist pointwise nonsingular matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$

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and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, D)$ *given by*

$$\tilde{E} := U^T E V = \begin{array}{cc} \begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} r \\ q \\ v \end{array} \\ \begin{array}{ccc} r & q & v \end{array} & \end{array}, \quad \tilde{Q} := U^{-1} Q V = \begin{array}{cc} \begin{bmatrix} Q_{22} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} r \\ q \\ v \end{array} \\ \begin{array}{ccc} r & q & v \end{array} & \end{array}, \tag{4.21a}$$

where $v = n - r - q$, *with* $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{r,r}$ *for all* $t \in \mathbb{I}$, *and*

$$\tilde{J} := U^T J U = \begin{bmatrix} J_{22} & J_{23} & J_{25} \\ J_{32} & J_{33} & J_{35} \\ J_{52} & J_{53} & J_{55} \end{bmatrix}, \quad \tilde{R} := U^T R U = \begin{bmatrix} R_{22} & R_{23} & R_{25} \\ R_{32} & R_{33} & R_{35} \\ R_{52} & R_{53} & R_{55} \end{bmatrix}, \tag{4.21b}$$

$$\tilde{K} := V^{-1} K V + V^{-1} \dot{V} = \begin{bmatrix} 0 & K_{23} & K_{25} \\ K_{32} & 0 & K_{35} \\ K_{52} & K_{53} & 0 \end{bmatrix}, \tag{4.21c}$$

$$\tilde{B} := U^T B = \begin{bmatrix} B_2 \\ B_3 \\ B_5 \end{bmatrix}, \quad \tilde{P} := U^T P = \begin{bmatrix} P_2 \\ P_3 \\ P_5 \end{bmatrix},$$

partitioned accordingly, with

$$\begin{aligned} &J_{33} = -J_{33}^T, \quad R_{33} = R_{33}^T, \quad R_{32} = R_{23}^T, \\ &Q_{22} K_{25} \equiv 0, \quad Q_{22}(J_{23} - K_{23}) = -Q_{22} J_{32}^T, , \\ &\dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T) Q_{22}, \end{aligned}$$

as well as

$$\begin{bmatrix} Q_{22} R_{22} Q_{22} & Q_{22} R_{23} \\ R_{23}^T Q_{22} & R_{33} \end{bmatrix} \geq 0$$

on $\mathbb{I}$.

Proof. From the symmetry condition $E(t) Q^T(t) = Q(t) E^T(t)$ for all $t \in \mathbb{I}$ we get that $Q_{21} \equiv 0$ in (4.17), and consequently $w = 0$. The rest follows from Theorem 4.7. ∎

Another additional property that occurs frequently in practical applications is the case that the matrix function Q is pointwise invertible (see again [13]). This assumption can possibly be met after reinterpretation of state variables as port variables (as in Lemma 4.9).

COROLLARY 4.11. *Consider a linear pHDAE (1.1) that is defined by the tuple of matrix functions* (E, J, R, Q, K, B, P, D) *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$ *and pointwise invertible matrix function* Q. *Under the assumption of Theorem 4.6 there exist pointwise nonsingular matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by the tuple* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, D)$ *given by*

$$\tilde{E} := U^T E V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q} := U^{-1} Q V = \begin{bmatrix} Q_{22} & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{4.22a}$$

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with $Q_{22}(t) = Q_{22}^T(t) > 0$ *for all* $t \in \mathbb{I}$ *and*

$$\begin{aligned}&\tilde{J} := U^T J U = \begin{bmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{bmatrix},\ \tilde{R} := U^T R U = \begin{bmatrix} R_{22} & R_{23} \\ R_{23}^T & R_{33} \end{bmatrix}, \\ &\tilde{K} := V^{-1} K V + V^{-1}\dot{V} = \begin{bmatrix} 0 & K_{23} \\ K_{32} & 0 \end{bmatrix},\ \tilde{B} := U^T B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix},\ \tilde{P} := U^T P = \begin{bmatrix} P_2 \\ P_3 \end{bmatrix},\end{aligned} \tag{4.22b}$$

partitioned accordingly, with

$$\begin{aligned}&J_{33} = -J_{33}^T, \quad K_{23} = J_{32}^T + J_{23}, \quad R_{22} = R_{22}^T, \quad R_{33} = R_{33}^T, \\ &\dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T)Q_{22},\end{aligned}$$

as well as $\tilde{R} \geq 0$ *on* $\mathbb{I}$.

Proof. From the condition that Q is pointwise nonsingular we get that Q_{22} in (4.17) is pointwise nonsingular and, thus, $q = n - r$. It follows that $d = r$, and $w = 0$ as well as $v = 0$ in the condensed form (4.19). ◻

If the matrix function Q in (1.1) is pointwise nonsingular, the results from Theorem 3.5 can be applied.

Corollary 4.12. *Consider a linear pHDAE* (1.1) *with pointwise invertible matrix function* Q. *Suppose that the assumptions of* Theorem 3.1 *hold for the skew-adjoint pair of matrix functions* $(Q^T E, Q^T J Q - Q^T E K)$. *Then, the undamped and uncontrolled DAE system* (1.1a) *with* $R \equiv 0$ *and* $u \equiv 0$ *has strangeness index* $\mu \leq 1$. *In particular, the undamped and uncontrolled system is strangeness-free if and only if the matrix function* J_{33} *of size* $(n-r) \times (n-r)$ *in the condensed form* (4.22) *is pointwise invertible for all* $t \in \mathbb{I}$ *(and, thus, is of even dimension).*

Proof. If Q is pointwise invertible, then the index of the DAE (1.1a) with $R \equiv 0$ and $u \equiv 0$ is the same as the index of the skew-adjoint DAE

$$Q^T E \dot{x} = Q^T J Q x - Q^T E K x.$$

Moreover, using the condensed form (4.22) we see that the matrix function $Q^T E$ is symmetric positive semi-definite for all $t \in \mathbb{I}$ such that the result follows from Theorem 3.5. ◻

The condensed forms (4.19),(4.21) and (4.22) are very useful if we want to determine the strangeness index of a pHDAE (1.1a). To simplify the notation we write a pHDAE (1.1a) in condensed form (4.19) as

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \end{bmatrix} x + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{4.23}$$

with

$$\begin{aligned}&A_{11} := \begin{bmatrix} L_{14} & L_{12}Q_{22} - K_{12} \\ L_{24} - K_{21} & L_{22}Q_{22} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,r}), \quad A_{12} := \begin{bmatrix} L_{13} - K_{13} \\ L_{23} - K_{23} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,q}), \\ &A_{13} := \begin{bmatrix} -K_{14} & -K_{15} \\ -K_{24} & -K_{25} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,w+v}) \\ &A_{21} := \begin{bmatrix} L_{34} & L_{32}Q_{22} \\ L_{44} & L_{42}Q_{22} \\ L_{54} & L_{52}Q_{22} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{q+w+v,r}), \quad A_{22} := \begin{bmatrix} L_{33} \\ L_{43} \\ L_{53} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{q+w+v,q}),\end{aligned}$$

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where we use the definition $L_{ij} := J_{ij} - R_{ij}$ for $i, j = 1, \dots, 5$ and

$$f_1 := \begin{bmatrix} B_1 - P_1 \\ B_2 - P_2 \end{bmatrix} u, \quad f_2 := \begin{bmatrix} B_3 - P_3 \\ B_4 - P_4 \\ B_5 - P_5 \end{bmatrix} u$$

are seen as given input functions (or inhomogeneities). Note that $r = w + d$. Then the pair of matrix functions corresponding to the DAE system (4.23) can be transformed into an equivalent pair of matrix functions in a similar manner as before. We get

$$\begin{aligned}
\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \end{bmatrix} \right) &\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \Sigma_a & 0 \\ \tilde{A}_{31} & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} E_{11} & E_{21} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ \hat{A}_{31} & \hat{A}_{32} & \Sigma_a & 0 \\ \Sigma_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_{r-s} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ \hat{A}_{31} & \hat{A}_{32} & \Sigma_a & 0 \\ \Sigma_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}$$

with pointwise nonsingular matrix functions Σ_a of size $a \times a$ and Σ_s of size $s \times s$, assuming that $\operatorname{rank} A_{22}(t) = a$ for all $t \in \mathbb{I}$, as well as $\operatorname{rank} \tilde{A}_{31}(t) = s$ for all $t \in \mathbb{I}$.

Finally, we obtain the following result.

Theorem 4.13. *Consider a linear pHDAE (1.1) given in condensed form (4.19) and assume that*

$$\operatorname{rank} \begin{bmatrix} J_{33}(t) - R_{33}(t) \\ J_{43}(t) - R_{43}(t) \\ J_{53}(t) - R_{53}(t) \end{bmatrix} = a \leq q \quad \text{for all } t \in \mathbb{I}.$$

Furthermore, let Z be a matrix function of size $(q + w + v) \times (q - a)$ and pointwise maximal rank such that

$$Z(t)^T \begin{bmatrix} J_{33}(t) - R_{33}(t) \\ J_{43}(t) - R_{43}(t) \\ J_{53}(t) - R_{53}(t) \end{bmatrix} = 0$$

for all $t \in \mathbb{I}$ and assume that

$$\operatorname{rank} Z^T(t) \begin{bmatrix} J_{34}(t) - R_{34}(t) & (J_{32}(t) - R_{32}(t))Q_{22}(t) \\ J_{44}(t) - R_{44}(t) & (J_{42}(t) - R_{42}(t))Q_{22}(t) \\ J_{54}(t) - R_{54}(t) & (J_{52}(t) - R_{52}(t))Q_{22}(t) \end{bmatrix} = s \quad \text{for all } t \in \mathbb{I}.$$

Then, the following hold:

1. *the pHDAE (1.1) is an ODE if and only if $q = w = v = 0$;*

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2. *the pHDAE (1.1) is strangeness-free if and only if $s = 0$;*
3. *the pHDAE (1.1) is regular and strangeness-free if and only if $s = w = v = 0$ and $q = a$. In this case, the pHDAE in condensed form (4.19) reduces to*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} (J_{22}-R_{22})Q_{22} & J_{23}-R_{23}-K_{23} \\ (J_{32}-R_{23}^T)Q_{22} & J_{33}-R_{33} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_2-P_2 \\ B_3-P_3 \end{bmatrix} u,$$
$$y = \begin{bmatrix} B_2+P_2 \\ B_3+P_3 \end{bmatrix}^T \begin{bmatrix} Q_{22} & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + Du,$$

with $J_{33}(t) = -J_{33}^T(t)$, $R_{33}(t) = R_{33}^T(t)$, $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{r,r}$ for all $t \in \mathbb{I}$, where the matrix function $J_{33} - R_{33}$ is pointwise invertible on $\mathbb{I}$ and

$$0 = Q_{22}\left[J_{32}^T + J_{23} - K_{23}\right], \quad \dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T)Q_{22},$$

as well as

$$\begin{bmatrix} Q_{22}R_{22}Q_{22} & Q_{22}R_{23} \\ R_{23}^TQ_{22} & R_{33} \end{bmatrix} \geq 0$$

on $\mathbb{I}$;
4. *if the pHDAE (1.1) is regular with $Q = I$ in the condensed form (4.19), then the strangeness index μ is less or equal 1 (the differentiation index is less or equal 2).*

Proof. The first three statements of the theorem follow directly from the previous discussion. In order to prove 4, we consider the corresponding system

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} J_{22}-R_{22} & -J_{32}^T-R_{23} \\ J_{32}-R_{23}^T & J_{33}-R_{33} \end{bmatrix} x - (B-P)u,$$

where $J_{22} = -J_{22}^T$, $J_{33} = -J_{33}^T$, and $R = R^T = \begin{bmatrix} R_{22} & R_{23} \\ R_{23}^T & R_{33} \end{bmatrix} \geq 0$. To analyze the index only the triple of matrix functions

$$(E, J, R) = \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_{22} & -J_{32}^T \\ J_{32} & J_{33} \end{bmatrix}, \begin{bmatrix} R_{22} & R_{23} \\ R_{23}^T & R_{33} \end{bmatrix} \right)$$

has to be considered, and we can use structure preserving congruence transformations of the form

$$(\tilde{E}, \tilde{J}, \tilde{R}) = (U^TEU, U^TJU, U^TRU)$$

with pointwise nonsingular matrix function $U \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ (as in Theorem 4.1 with $V = U$) to transfer this triple into a more suitable representation. Note that $\tilde{E}\tilde{K} = U^TE\dot{U}$ is not necessarily equal to zero.

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At first, we can obtain

$$
\begin{aligned}
&(E, J, R)\\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{J}_{11} & -\tilde{J}_{21}^T & -\tilde{J}_{31}^T \\ \tilde{J}_{21} & \tilde{J}_{22} & -\tilde{J}_{32}^T \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & \tilde{R}_{13} \\ \tilde{R}_{12}^T & \Sigma_R & 0 \\ \tilde{R}_{13}^T & 0 & 0 \end{bmatrix} \right)\\
&= \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{J}_{11} & -\tilde{J}_{21}^T & -\tilde{J}_{31}^T \\ \tilde{J}_{21} & \tilde{J}_{22} & -\tilde{J}_{32}^T \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & 0 \\ \tilde{R}_{12}^T & \Sigma_R & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{J}_{11} & -\tilde{J}_{21}^T & -\tilde{J}_{31}^T & -\tilde{J}_{41}^T \\ \tilde{J}_{21} & \tilde{J}_{22} & -\tilde{J}_{32}^T & -\tilde{J}_{42}^T \\ \tilde{J}_{31} & \tilde{J}_{32} & \Delta_J & 0 \\ \tilde{J}_{41} & \tilde{J}_{42} & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & 0 & 0 \\ \tilde{R}_{12}^T & \Sigma_R & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)\\
&\sim \left(\begin{bmatrix} I_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{J}_{11} & -\tilde{J}_{21}^T & -\tilde{J}_{31}^T & -\tilde{J}_{41}^T & -\tilde{J}_{51}^T & -\tilde{J}_{61}^T \\ \tilde{J}_{21} & \tilde{J}_{22} & -\tilde{J}_{32}^T & -\tilde{J}_{42}^T & -\Sigma_{52}^T & 0 \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} & -\tilde{J}_{43}^T & 0 & 0 \\ \tilde{J}_{41} & \tilde{J}_{42} & \tilde{J}_{43} & \Delta_J & 0 & 0 \\ \tilde{J}_{51} & \Sigma_{52} & 0 & 0 & 0 & 0 \\ \tilde{J}_{61} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} & \tilde{R}_{13} & 0 & 0 & 0 \\ \tilde{R}_{12}^T & \tilde{R}_{22} & \tilde{R}_{23} & 0 & 0 & 0 \\ \tilde{R}_{13}^T & \tilde{R}_{23}^T & \tilde{R}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}
$$

with $\Sigma_R = \Sigma_R^T$, $\Delta_J = -\Delta_J^T$ and Σ_{52} pointwise nonsingular, using a number of transformation as in Theorem 4.5. Here, in the first step, the block $\tilde{R}_{13}$ has to be identical to zero due to the positive semi-definiteness of R. In the above sequence of transformations, we always have $\tilde{E}\tilde{K} = 0$. In a last step, we can transform the block $\tilde{J}_{61}$ into the form $\begin{bmatrix} \Sigma_{61} & 0 \end{bmatrix}$, with pointwise nonsingular matrix function Σ_{61}. Note that, due to the regularity of the system, the block $\tilde{J}_{61}$ needs to have full row rank. Using the corresponding transformation on the whole triple of matrix functions we get

$$
\begin{aligned}
(E, J, R) \sim \left(\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & 0 & 0 & 0 & 0 & 0 \\ \tilde{E}_{21} & \tilde{E}_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & -\tilde{J}_{31}^T & -\tilde{J}_{41}^T & -\tilde{J}_{51}^T & -\tilde{J}_{61}^T & -\Sigma_{61}^T \\ * & * & -\tilde{J}_{32}^T & -\tilde{J}_{42}^T & -\tilde{J}_{52}^T & -\tilde{J}_{62}^T & 0 \\ \tilde{J}_{31} & \tilde{J}_{32} & \tilde{J}_{33} & -\tilde{J}_{43}^T & -\tilde{J}_{53}^T & -\Sigma_{52}^T & 0 \\ \tilde{J}_{41} & \tilde{J}_{42} & \tilde{J}_{43} & \tilde{J}_{44} & -\tilde{J}_{54}^T & 0 & 0 \\ \tilde{J}_{51} & \tilde{J}_{52} & \tilde{J}_{53} & \tilde{J}_{54} & \Delta_J & 0 & 0 \\ \tilde{J}_{61} & \tilde{J}_{62} & \Sigma_{52} & 0 & 0 & 0 & 0 \\ \Sigma_{61} & 0 & 0 & 0 & 0 & 0 & \end{bmatrix}, \right.\\
\left. \begin{bmatrix} * & * & \tilde{R}_{13} & \tilde{R}_{14} & 0 & 0 & 0 \\ * & * & \tilde{R}_{23} & \tilde{R}_{24} & 0 & 0 & 0 \\ \tilde{R}_{13}^T & \tilde{R}_{23}^T & \tilde{R}_{33} & \tilde{R}_{34} & 0 & 0 & 0 \\ \tilde{R}_{14}^T & \tilde{R}_{24}^T & \tilde{R}_{34}^T & \tilde{R}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right).
\end{aligned}
$$

In this last transformation $\tilde{E}\tilde{K} \neq 0$, so that the structure in the upper left 2×2-blocks of $\tilde{J}$ and $\tilde{R}$ is not preserved (the corresponding blocks are denoted with $*$). However, from the above representation we can now draw conclusions about the index of the corresponding DAE system: the equation belonging to the last

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block row, an algebraic relation for $\tilde{x}_1$ (using a splitting of the transformed state vector $\tilde{x}$ according to the block structure above), is responsible for strangeness index $\mu > 0$. Since the first block equations must be used for the determination of $\tilde{x}_7$, as the system is assumed to be regular, there cannot be further (higher index) coupling between the first and second block equations, such that the strangeness index has to be bounded by $\mu \leq 1$. Note also, that $J_{44} - R_{44}$ has to be regular, in order to be able to uniquely determine $\tilde{x}_4$. ◻

From Theorem 4.13 we see that a linear pHDAE (1.1) with pointwise nonsingular matrix function Q given in condensed form (4.22) is regular and strangeness-free if and only if $J_{33} - R_{33}$ in (4.22) is pointwise nonsingular on $\mathbb{I}$. Moreover, a regular port-Hamiltonian DAE with pointwise nonsingular matrix function Q will always be of strangeness index less than or equal to 1. For linear non-regular pHDAEs (1.1) the strangeness index can also be larger than 1 (differentiation index larger than 2) as the following example shows.

Example 4.14. Consider the system

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\dot{\tilde{x}} = \begin{bmatrix}0&1&0&0\\0&0&1&0\\-1&0&0&0\\-1&0&0&0\end{bmatrix}\tilde{x} + \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}u, \quad \tilde{x}(0) = \tilde{x}_0 \tag{4.24}$$
$$y = \begin{bmatrix}1&1&0&0\end{bmatrix}\tilde{x}$$

of strangeness index $\mu = 2$. This system can be written as pHDAE in condensed form (4.19) with $w = 2$, $d = q = v = 0$ and

$$\tilde{E} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \quad \tilde{Q} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix}, \quad \tilde{J} = \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&\frac{1}{2}\\-1&0&-\frac{1}{2}&0\end{bmatrix}, \quad \tilde{R} = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&1&\frac{1}{2}\\0&0&\frac{1}{2}&0\end{bmatrix},$$
$$\tilde{K} = \begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \quad \tilde{P} = \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}.$$

It can easily be checked that the conditions in Definition 1.1 are satisfied. Following Lemma 4.3, system (4.24) is equivalent to the system

$$\begin{bmatrix}1&0&0&0\\0&1&-t&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\dot{x} = \begin{bmatrix}0&1&-t&0\\0&0&0&0\\-1&0&0&0\\-1&0&0&0\end{bmatrix}x + \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}u, \quad x(0) = x_0$$
$$y = \begin{bmatrix}1&1&-t&0\end{bmatrix}x$$

with vanishing matrix $K = 0$, using a transformation $x = V_{\tilde{K}}\tilde{x}$, $x_0 = V_{\tilde{K}}(0)\tilde{x}_0$ where $V_{\tilde{K}} = \begin{bmatrix}1&0&0&0\\0&1&t&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$,

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and

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -t & 0 \end{bmatrix}, \quad J = \tilde{J}, \quad R = \tilde{R}, \quad B = \tilde{B}, \quad P = \tilde{P}.$$

In the above example, we can also see that $v = 0$ in the condensed form (4.19) does not necessarily imply that the system is regular.

For linear pHDAEs that are not strangeness-free an index reduction is necessary. Usually, the differentiation and elimination step used in the index reduction procedure proposed in [11] will destroy the port-Hamiltonian structure of the system. However, a modification of the regularization procedure has been presented in [2] that preserve the pHDAE structure and, under some (local) constant rank assumption, allows us to reformulated any linear pHDAE as an implicitly defined standard port-Hamiltonian system plus an algebraic constraint.

5. Conclusions. We have considered linear port-Hamiltonian DAEs that arise in energy-based modeling of constrained dynamical systems and the corresponding skew-adjoint differential-algebraic operator that is related to this structure. For skew-adjoint differential-algebraic equations we have developed structure preserving condensed forms under orthogonal and non-orthogonal congruence transformations in Theorem 3.1 and Corollary 3.2. These condensed forms require some constant rank assumptions that are usually satisfied in the common applications and are trivially satisfied for systems with constant coefficients. The constant rank restriction can also be removed by considering the system in a piecewise manner, see [11]. Based on the derived condensed forms an analysis of existence and uniqueness of solutions and of the index of skew-adjoint DAEs is possible (Corollary 3.4). In particular, for linear regular skew-adjoint DAEs with well-defined strangeness index and positive semi-definite leading matrix we have shown that the strangeness index is always less than or equal to 1 (Theorem 3.5).

In the second part of the paper, we have derived condensed forms for linear port-Hamiltonian DAEs under orthogonal and non-orthogonal equivalence transformations (Theorem 4.6 and Theorem 4.7). Under additional structural properties these condensed forms can be further simplified (Corollary 4.10 and Corollary 4.11). Again, the derived condensed forms allow us to analyze existence and uniqueness of solutions as well as the index of linear port-Hamiltonian DAEs. The additional structural properties (the symmetry of the product EQ^T and the pointwise regularity of the matrix function Q) are satisfied e. g. in the index-2 formulation of the equations of motions of linear constrained multibody systems or in the port-Hamiltonian formulation of linear electrical RLC-circuits, see [13]. It was stated in [18] that port-Hamiltonian DAEs are of differentiation index at most one (i.e., strangeness-free). That this is not the case can be seen from the preceding results and in the examples presented in [13].

As a main result we have obtained that regular linear port-Hamiltonian DAEs with pointwise nonsingular matrix function Q always have a strangeness index less than or equal to 1 (Corollary 4.12 and Theorem 4.13). For linear non-regular port-Hamiltonian DAEs (1.1) the strangeness index can also be larger than 1 (differentiation index larger than 2). In case of higher-index pHDAEs a regularization procedure is necessary. We refer to [2], where a structure preserving regularization procedure for port-Hamiltonian DAEs has been presented. Moreover, the derived condensed forms allow an easy reinterpretation of undetermined components of the state vector as port variables (Lemma 4.9).

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