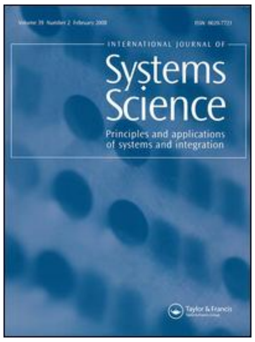

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Fractional variable order discrete-time systems, their solutions and properties

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ABSTRACT
In this paper, fractional variable order discrete state-space systems based on different definitions of fractional variable order difference are investigated.The general solution of these systems is derived. Moreover, necessary and sufficient conditions for reachability and observability are given and proven. The sufficient conditions for controllability are proposed too.

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Controllability and reachability; linear systems; discrete time systems; fractional calculus; variable order

1. Introduction

Analysis of dynamics for the case when fractional order varies in time is more complicated than for constant orders. There is a number of definitions of variable order derivatives that are characterised by different behaviour. In Lorenzo and Hartley (2002) and Valerio and da Costa (2011), three general types of variable order derivative definitions can be found. Moreover, in Ramirez and Coimbra (2010), more definitions were presented. In Sierociuk, Malesza, and Macias (2015a), Sierociuk, Malesza, and Macias (2015b) and Sierociuk, Macias, and Malesza (2013a), clear categorisation according to switching strategies was presented. These switching strategies allow to better understand the mechanism of order changing, and choosing proper type of definition in more predictable, intuitive way. Moreover, these switching schemes allow to build analogue models, whose experimental validation have been presented in Sierociuk et al. (2015a), Sierociuk et al. (2015b), Sierociuk et al. (2013a) and Sierociuk, Macias, and Malesza (2013b). A very important property – duality between variable order definitions – was introduced in Sierociuk and Twardy (2014).

Although, the description of the variable order systems is much more complex, using variable order calculus enables more accurate modelling many of systems, for example in situation, when the structure of object is changed. In Sakrajda and Sierociuk (2016), modelling of heat transfer process in a medium that structure is changed in time is presented. In article Sheng, Sun, Coopmans, Chen, and Bohannan (2010), experimental studies of an electrochemical example of physical fractional variable order system are presented. In Ramirez and Coimbra (2011), the variable order equations are used to describe a history of drag expression. In paper Sierociuk, Podlubny, and Petras (2013), the variable order interpretation of the analogue realisation of fractional orders integrators, realised as domino ladders, was presented. Although, this approach gives more accurate solutions, the description of the system when the order is changing in time is much more complicated than for constant order case.

Comprehensive introduction to discrete fractional order systems is presented in Ostalczyk (2016) and Goodrich and Peterson (2015). Results of reachability, controllability and observability problems for constant fractional order discrete state-space system were presented in Dzielinski and Sierociuk (2007), Mozyrska and Pawłuszewicz (2015), Mozyrska, Pawłuszewicz, and Wyrwas (2017), Kaczorek (2011), Kaczorek and Rogowski (2015) and Balachandran, Matar, and Trujillo (2016). Similar preliminary results for one particular type of fractional variable order discrete fractional order system were given in paper Sierociuk (2012).

In this article, fractional variable order state-space systems based on different types of variable order differences will be taken into consideration. For such systems, the necessary and sufficient conditions for reachability and observability will be given and proven. The sufficient conditions for controllability will be proposed too. The results obtained will demonstrate an influence of time-varying order on the system properties.

The paper is organised as follows. In Section 2, definitions of fractional constant order and variable order differences are presented. In Section 3, the general discrete fractional variable order state-space systems (GDFVOSS) for different types of variable order difference definitions are recalled and the general form of the solution for these systems is introduced. Section 4 presents the reachability definition and necessary and sufficient condition for GDFVOSS. Section 5 presents the definition and sufficient conditions for the controllability of GDFVOSS. In Section 6, the definition and conditions for observability of GDFVOSS are introduced. Finally, in Section 7 numerical examples are given.

2. Fractional variable order differences

The following definition constitutes a starting point for generalisation of constant fractional order difference operators onto a variable order case. A constant fractional order difference

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operator is defined in the following way

$$ {}_0\Delta_k^{\alpha} x_k \equiv \sum_{j=0}^{k} w(\alpha, j) x_{k-j}\,, \qquad (1)$$

where

$$ w(\alpha, j) = \frac{1}{h^{\alpha}} (-1)^j \binom{\alpha}{j}, \qquad (2)$$

and

$$\binom{\alpha}{j} \equiv \begin{cases} 1 & \text{for} \quad j = 0\,, \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j > 0\,. \end{cases}$$

Definitions for variable order case presented below exhibit different behaviour, however, for constant order all of them are equivalent to Definition 1. The first one is obtained by replacing a constant order α by variable order α_k.

Definition 2.1: The $\mathcal{A}$-type of fractional variable order difference is given by

$$ {}_0^{\mathcal{A}}\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^{k} {}^{\mathcal{A}}w(\alpha(\cdot), k, j) x_{k-j}\,, \qquad (3)$$

where

$$ {}^{\mathcal{A}}w(\alpha(\cdot), k, j) = \frac{(-1)^j}{h^{\alpha_k}} \binom{\alpha_k}{j}.$$

The second definition assumes that coefficients for past samples are obtained for order that was present for these samples. The particular switching scheme corresponding to this definition was presented in Sierociuk et al. (2015b).

Definition 2.2: The $\mathcal{B}$-type of fractional variable order difference is given by

$$ {}_0^{\mathcal{B}}\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^{k} {}^{\mathcal{B}}w(\alpha(\cdot), k, j) x_{k-j}\,, \qquad (4)$$

where

$$ {}^{\mathcal{B}}w(\alpha(\cdot), k, j) = \frac{(-1)^j}{h^{\alpha_{k-j}}} \binom{\alpha_{k-j}}{j}.$$

The third definition is less intuitive and assumes that coefficients for the newest samples are obtained, respectively, for the oldest orders.

Definition 2.3: The $\mathcal{C}$-type of fractional variable order difference is given by

$$ {}_0^{\mathcal{C}}\Delta_k^{\alpha_k} x_k \equiv \sum_{j=0}^{k} {}^{\mathcal{C}}w(\alpha(\cdot), k, j) x_{k-j}\,, \qquad (5)$$

where

$$ {}^{\mathcal{C}}w(\alpha(\cdot), k, j) = \frac{(-1)^j}{h^{\alpha_j}} \binom{\alpha_j}{j}. \qquad (6)$$

Besides of presented above iterative definitions, we use also the following recursive type of variable order difference definitions.

Definition 2.4: The $\mathcal{D}$-type of fractional variable order difference is given by

$$ {}_0^{\mathcal{D}}\Delta_k^{\alpha_k} x_k \equiv \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k} (-1)^j \binom{-\alpha_k}{j} {}^{\mathcal{D}}\Delta^{\alpha_{k-j}} x_{k-j}. \qquad (7)$$

Using matrix representation of (7), which is presented and in details described in Sierociuk et al. (2015a), the $\mathcal{D}$-type difference in recursive form given in (7) can be rewritten in the following iterative form:

$$ {}_0^{\mathcal{D}}\Delta_k^{\alpha_k} x_k = \sum_{j=0}^{k} {}^{\mathcal{D}}w(\alpha(\cdot), k, j) x_{k-j}, \qquad (8)$$

where

$$ {}^{\mathcal{D}}w(\alpha(\cdot), k, j) = \begin{cases} {}^{\mathcal{D}}q_k\big({}^{\mathcal{D}}w(\alpha(\cdot), 0, j-k), \\ \quad \ldots, {}^{\mathcal{D}}w(\alpha(\cdot), k-1, j-1)\big)^T & \text{for } j > 0, \\ h^{-\alpha_k} & \text{for } j = 0, \\ 0 & \text{for } j < 0, \end{cases}$$

and for $r = 1, \ldots, k$

$$ {}^{\mathcal{D}}q_r = \big(-{}^{\mathcal{D}}v_{-\alpha_r, r}, \ldots, -{}^{\mathcal{D}}v_{-\alpha_r, 1}\big) \in \mathbb{R}^{1\times r},$$

where for $s = 1, \ldots, r$

$$ \mathcal{D}v_{-\alpha_r, s} = (-1)^s \binom{-\alpha_r}{s}. \qquad (9)$$

Definition 2.5: The $\mathcal{E}$-type of fractional variable order difference is given by

$$ {}^{\mathcal{E}}\Delta^{\alpha_l} x_k = \frac{x_k}{h^{\alpha_k}} - \sum_{j=1}^{k} (-1)^j \binom{-\alpha_{k-j}}{j} \frac{h^{\alpha_{k-j}}}{h^{\alpha_k}} {}^{\mathcal{E}}\Delta^{\alpha_{k-j}} x_{k-j}. \quad (10)$$

Using matrix representation of (10), which is presented and in details described in Macias and Sierociuk (2014), the $\mathcal{E}$-type difference in recursive form given in (10) can be rewritten in the following iterative form:

$$ {}_0^{\mathcal{E}}\Delta_k^{\alpha_k} x_k = \sum_{j=0}^{k} {}^{\mathcal{E}}w(\alpha(\cdot), k, j) x_{k-j}, \qquad (11)$$

where

$$ {}^{\mathcal{E}}w(\alpha(\cdot), k, j) = \begin{cases} {}^{\mathcal{E}}q_k\big({}^{\mathcal{E}}w(\alpha(\cdot), 0, j-k), \\ \quad \ldots, {}^{\mathcal{E}}w(\alpha(\cdot), k-1, j-1)\big)^T & \text{for } j > 0, \\ h^{-\alpha_k} & \text{for } j = 0, \\ 0 & \text{for } j < 0, \end{cases}$$

and for $r = 1, \ldots, k$

$$^{\mathcal{E}}q_r = \left(-^{\mathcal{E}}\upsilon_{-\alpha_0,r}, \ldots, -^{\mathcal{E}}\upsilon_{-\alpha_{r-1},1}\right) \in \mathbb{R}^{1\times r},$$

where for $s = 1, \ldots, r$

$$^{\mathcal{E}}\upsilon_{-\alpha_{r-s},s} = (-1)^s \binom{-\alpha_{r-s}}{s} \frac{h^{\alpha_{r-s}}}{h^{-\alpha_r}}. \tag{12}$$

For simplicity and shortness of notations these five definitions of fractional variable order differences can be written in the unified form with parameter $\mathbb{T}$ being the type of used definition, i.e. for $\mathbb{T} \in (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E})$, we have

$$^{\mathbb{T}}_{0}\Delta^{\alpha_k}_{k} x_k = \sum_{j=0}^{k} {}^{\mathbb{T}}w(\alpha(\cdot), k, j)x_{k-j}. \tag{13}$$

Remark 2.1: For any $\mathbb{T}$, any i and $j = 0$ the following holds

$$\left({}^{\mathbb{T}}w(\alpha(\cdot), i, 0)\right)^{-1} = {}^{\mathbb{T}}w(-\alpha(\cdot), i, 0).$$

3. Discrete fractional variable order state-space systems

In this section, we introduce the following system in general form, for unified fractional variable order difference given by (13).

Definition 3.1: The general linear Discrete Fractional Variable Order System in State-space representation (GDFVOSS), for different types of variable fractional order difference definitions, is given as follows:

$$^{\mathbb{T}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} = Ax_k + Bu_k \tag{14a}$$

$$x_{k+1} = w(-\alpha(\cdot), k+1, 0)^{\mathbb{T}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} - \sum_{j=1}^{k+1} {}^{\mathbb{T}}w(-\alpha(\cdot), k+1, 0)^{\mathbb{T}}w(\alpha(\cdot), k+1, j)x_{k-j+1} \tag{14b}$$

$$y_k = Cx_k, \tag{14c}$$

where $\mathbb{T}$ is a type of used definition ($\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ or $\mathcal{E}$).

Above, the state space vector $x_k \in \mathbb{R}^n$, input vector $u_k \in \mathbb{R}^m$, output vector $y_k \in \mathbb{R}^p$, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$.

System (14) can be rewritten in the following form:

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k - \sum_{j=1}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j)x_{k-j+1} \tag{15a}$$

$$y_k = Cx_k, \tag{15b}$$

where

$$\tilde{A} = {}^{\mathbb{T}}w(-\alpha(\cdot), k+1, 0)A \tag{16}$$

and

$$\tilde{B} = {}^{\mathbb{T}}w(-\alpha(\cdot), k+1, 0)B \tag{17}$$

and

$$^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j) = {}^{\mathbb{T}}w(-\alpha(\cdot), k+1, 0)^{\mathbb{T}}w(\alpha(\cdot), k+1, j). \tag{18}$$

Remark 3.1: The matrices $\tilde{A}$ and $\tilde{B}$ of system (15), given respectively by (16) and (17), may depend on time. They are constant only for $\mathbb{T} = \mathcal{C}$, because, by (6), the coefficient $^{\mathcal{C}}w(-\alpha(\cdot), k+1, 0) = \text{const}$ for any $k \in \mathbb{N}$.

Remark 3.2: For recursive $\mathbb{T}$-type differences (for $\mathbb{T}$ being $\mathcal{D}$, $\mathcal{E}$), system (14) can be rewritten in the following equivalent form

$$\begin{aligned}
^{\mathbb{T}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} &= Ax_k + Bu_k \\
x_{k+1} &= h^{\alpha_{k+1}}\,{}^{\mathbb{T}}_{0}\Delta^{\alpha_{k+1}}_{k+1} x_{k+1} \\
&\quad + \sum_{j=1}^{k+1} {}^{\mathbb{T}}\bar{w}h^{\alpha_{k+1}}\,{}^{\mathbb{T}}_{0}\Delta^{\alpha_{k+1-j}}_{k+1-j} x_{k-j+1} \\
y_k &= Cx_k,
\end{aligned}$$

where

$$^{\mathbb{T}}\bar{w} = \begin{cases} (-1)^j \binom{-\alpha_{k+1}}{j} & \text{for } \mathbb{T} = \mathcal{D}, \\ (-1)^j \binom{-\alpha_{k+1-j}}{j} \frac{h^{\alpha_{k+1-j}}}{h^{\alpha_{k+1}}} & \text{for } \mathbb{T} = \mathcal{E}. \end{cases}$$

Theorem 3.1: *Solution of GDFVOSS given by* (14) *is*

$$x_k = {}^{\mathbb{T}}\Phi(k)x_0 + \sum_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k, k-j-1)\tilde{B}u_j, \tag{19}$$

where the 1*-parameter transition matrix is*

$$^{\mathbb{T}}\Phi(k+1) = (\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1))^{\mathbb{T}}\Phi(k) - \sum_{j=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j)^{\mathbb{T}}\Phi(k+1-j), \tag{20}$$

such that $^{\mathbb{T}}\Phi(0) = I$, *and* 2*-parameter transition matrix*

$$^{\mathbb{T}}\Phi(k, l) = (\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k, 1))^{\mathbb{T}}\Phi(k-1, l-1) - \sum_{j=2}^{l} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k, j)^{\mathbb{T}}\Phi(k-j, l-j), \quad \text{for } k > l, \tag{21}$$

such that $^{\mathbb{T}}\Phi(k, 0) = I$, *and* $^{\mathbb{T}}\Phi(k, l) = 0$ for $l < 0$.

Remark 3.3: Since $^{\mathbb{T}}\Phi(k, l) = 0$ for $l < 0$, the 2-parameter transition matrix (21) can be rewritten as

$$^{\mathbb{T}}\Phi(k, l) = (\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k, 1))^{\mathbb{T}}\Phi(k-1, l-1) - \sum_{j=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k, j)^{\mathbb{T}}\Phi(k-j, l-j). \tag{22}$$

Proof of Theorem 3.1: We will show that (19) satisfies (14), which can be rewritten as (15), i.e.

$$x_{k+1} = \tilde{A}x_k - \sum_{j=1}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j)x_{k-j+1} + \tilde{B}u_k. \qquad (23)$$

Evaluate (19) at $k+1$ obtaining

$$\begin{aligned} x_{k+1} &= {}^{\mathbb{T}}\Phi(k+1)x_0 + \sum_{j=0}^{k} {}^{\mathbb{T}}\Phi(k+1, k-j)\tilde{B}u_j \\ &= {}^{\mathbb{T}}\Phi(k+1)x_0 + \sum_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k+1, k-j)\tilde{B}u_j + \tilde{B}u_k. \end{aligned} \qquad (24)$$

Substituting (20) and (22) into (24) we get

$$\begin{aligned} x_{k+1} = &\left(\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1)\right){}^{\mathbb{T}}\Phi(k)x_0 + \tilde{B}u_k \\ &- \sum_{j=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j){}^{\mathbb{T}}\Phi(k+1-j)x_0 \\ &+ \sum_{j=0}^{k-1} (\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1)){}^{\mathbb{T}}\Phi(k, k-j-1)\tilde{B}u_j \\ &- \sum_{j=0}^{k-1}\sum_{i=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, i){}^{\mathbb{T}}\Phi(k+1-i, k-j-i)\tilde{B}u_j. \end{aligned}$$

After rearranging terms

$$\begin{aligned} x_{k+1} = &\left(\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1)\right) \\ &\times \underbrace{\left({}^{\mathbb{T}}\Phi(k)x_0 + \sum\nolimits_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k, k-j-1)\tilde{B}u_j\right)}_{x_k} \\ &+ \tilde{B}u_k - \sum_{j=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, j){}^{\mathbb{T}}\Phi(k+1-j)x_0 \\ &- \sum_{i=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, i)\sum_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k+1-i, k-j-i)\tilde{B}u_j \end{aligned}$$

and

$$\begin{aligned} x_{k+1} = &\left(\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1)\right)x_k + \tilde{B}u_k \\ &- \sum_{m=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, m){}^{\mathbb{T}}\Phi(k+1-m)x_0 \\ &- \sum_{m=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, m) \\ &\times \sum_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k+1-m, k-j-m)\tilde{B}u_j \end{aligned}$$

we get

$$\begin{aligned} x_{k+1} = &\left(\tilde{A} - I^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, 1)\right)x_k + \tilde{B}u_k \\ &- \sum_{m=2}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, m) \\ &\times \underbrace{\left({}^{\mathbb{T}}\Phi(k+1-m)x_0 + \sum\nolimits_{j=0}^{k-1} {}^{\mathbb{T}}\Phi(k+1-m, k-j-m)\tilde{B}u_j\right)}_{x_{k+1-m}}. \end{aligned}$$

Finally,

$$x_{k+1} = \tilde{A}x_k - \sum_{m=1}^{k+1} {}^{\mathbb{T}}\tilde{w}(\alpha(\cdot), k+1, m)x_{k+1-m} + \tilde{B}u_k,$$

which fits (23) and ends the proof. ■

Remark 3.4: For constant order $\alpha = \text{const}$, any $\mathbb{T} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$, the coefficients ${}^{\mathbb{T}}\tilde{w}(\alpha, i, j)$ do not depend on i, i.e.

$${}^{\mathbb{T}}\tilde{w}(\alpha, i, j) \equiv \tilde{w}(\alpha, j), \quad i = 0, 1, \ldots,$$

where $\tilde{w}(\alpha, j) = h^{\alpha}w(\alpha, j)$, and $w(\alpha, j)$ is given by (2).

Remark 3.5: For constant order α the GDFVOSS takes the form of traditional fractional constant order discrete-time system

$$\begin{aligned} {}_0\Delta^{\alpha}_{k+1}x_{k+1} &= Ax_k + Bu_k \\ x_{k+1} &= h^{\alpha}{}_0\Delta^{\alpha}_{k+1}x_{k+1} - h^{\alpha}\sum_{j=1}^{k+1} w(\alpha, j)x_{k-j+1} \\ y_k &= Cx_k, \end{aligned}$$

or, in equivalent form,

$$\begin{aligned} x_{k+1} &= \tilde{A}x_k + \tilde{B}u_k - h^{\alpha}\sum_{j=1}^{k+1} w(\alpha, j)x_{k-j+1} \\ y_k &= Cx_k, \end{aligned}$$

where

$$\tilde{A} = h^{\alpha}A, \quad \tilde{B} = h^{\alpha}B,$$

because ${}^{\mathbb{T}}w(-\alpha, 0) = h^{\alpha}$.

Remark 3.6: For constant order $\alpha = \text{const}$, the 2-parameter transition matrix (21) simplifies to the 1-parameter transition matrix, i.e.

$$\Phi(l) = (\tilde{A} - I\tilde{w}(\alpha, 1)){}^{\mathbb{T}}\Phi(l-1) - \sum_{j=2}^{l} \tilde{w}(\alpha, j)\Phi(l-j),$$

or

$$\Phi(l+1) = (A - I\tilde{w}(\alpha, 1))\Phi(l) - \sum_{j=2}^{l+1} \tilde{w}(\alpha, j)\Phi(l-j+1).$$

Therefore, the solution of system (14) for $\alpha = \text{const}$ is

$$x_k = \Phi(k)x_0 + \sum_{j=0}^{k-1} \Phi(k-j-1)\tilde{B}u_j, \tag{25}$$

where the 1-parameter transition matrix is

$$\Phi(k+1) = (\tilde{A} - I\tilde{\omega}(\alpha,1))\Phi(k) - \sum_{j=2}^{k+1} \tilde{\omega}(\alpha,j)\Phi(k+1-j), \tag{26}$$

such that $\Phi(0) = I$.

4. Reachability

The definition of reachability for GDFVOSS is identical to a classical reachability definition.

Definition 4.1: We say that GDFVOSS is reachable, if for an arbitrary final state $x_f \in \mathbb{R}^n$ there exists a number $q \in \mathbb{N}$ and an input sequence $\{u_0, u_1, \ldots, u_{q-1}\}$, which carries the system from the initial state $x_0 = 0$ to the desired final state $x_q = x_f$.

Theorem 4.1: *The following conditions are equivalent for GDFVOSS given by* (14)*:*

(*i*) *GDFVOSS is reachable;*
(*ii*) *the reachability matrix $\mathcal{S}$ is of full rank n, that is*

$$\operatorname{rank}\mathcal{S} = \operatorname{rank}\left(B, {}^{\mathbb{T}}\Phi(n,1)B, \ldots, {}^{\mathbb{T}}\Phi(n,n-1)B\right) = n,$$

where ${}^{\mathbb{T}}\Phi(n,l)$, $l = 0, \ldots, n-1$, is the transition matrix given by (21)*;*
(*iii*) *the reachability matrix $\mathcal{S}_I$ is of full rank n, that is*

$$\operatorname{rank}\mathcal{S}_I = \operatorname{rank}\left(B, AB, A^2B, \ldots, A^{n-1}B\right) = n.$$

Proof: (*i*)⇒(*ii*) Using solution (19) of system (14) for zero initial conditions $x_0 = 0$ and final state x_n we obtain the following relation between control sequence and final state:

$$x_n = \sum_{j=0}^{n-1} {}^{\mathbb{T}}\Phi(n,n-j-1)\tilde{B}u_j = \left(\tilde{B}, {}^{\mathbb{T}}\Phi(n,1)\tilde{B}, \ldots, {}^{\mathbb{T}}\Phi(n,n-1)\tilde{B}\right)\begin{pmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{pmatrix}. \tag{27}$$

Since system (14) is reachable and the relation between control sequence and final state is given by (27), any element x_n of whole $\mathbb{R}^n$ can be obtained using a sequence of control-values $\{u_0, u_1, \ldots, u_{n-1}\}$ only if $\operatorname{Im}\mathcal{S} = \mathbb{R}^n$, which implies $\operatorname{rank}\mathcal{S} = n$ (with relation between $\tilde{B}$ and B given by (17)).

(*ii*)⇒(*iii*) The transition matrix $\Phi(n,l)$, $l = 0, \ldots, n-1$, can be rewritten as the following polynomial of the matrix A

$${}^{\mathbb{T}}\Phi(n,l) = f_{l,l}A^l + f_{l,l-1}A^{l-1} + \cdots + f_{l,1}A + f_{l,0}I,$$

where $f_{l,j}$, $j = 0, \ldots, l-1$, are proper coefficients that follow from (21). Therefore,

$$\mathcal{S} = \left(B, \left(f_{1,1}A + f_{1,0}I\right)B, \left(f_{2,2}A^2 + f_{2,1}A + f_{2,0}I\right)B, \ldots, \left(f_{n-1,n-1}A^{n-1} + f_{n-1,n-2}A^{n-2} + \cdots + f_{n-1,1}A + f_{n-1,0}I\right)B\right). \tag{28}$$

Elements of matrix (28) can be obtained from the following matrix

$$\left(B, AB, A^2B, \ldots, A^{n-1}B\right)$$

by elementary matrix operations on its columns. Since elementary matrix operations do not change the rank of matrix, we get $\operatorname{rank}\mathcal{S}_I = \operatorname{rank}\mathcal{S}$.

(*iii*)⇒(*i*) From (27), given arbitrary $x_n \in \mathbb{R}^n$, we can calculate a sequence $\{u_0, u_1, \ldots, u_{n-1}\}$ only if $\operatorname{rank}\mathcal{S} = n$. Since $\operatorname{Im}\mathcal{S}_I = \operatorname{Im}\mathcal{S} = \mathbb{R}^n$, it implies that $\operatorname{rank}\mathcal{S}_I = n$. ■

Remark 4.1: The Kalman condition (iii) in Theorem 4.1 is equivalent to the well-known Hautus condition.

5. Controllability

The definition of controllability for GDFVOSS is identical to a classical controllability definition.

Definition 5.1: We say that GDFVOSS is controllable, if there exists a number $q \in \mathbb{N}$ and an input sequence $\{u_0, u_1, \ldots, u_{q-1}\}$, which carries the system from an arbitrary initial state $x_0 \in \mathbb{R}^n$ to the desired final state $x_q = 0$.

Theorem 5.1: *The GDFVOSS given by* (14) *is controllable* (*in n steps*) *if the following condition is satisfied:*

$$\operatorname{rank}\mathcal{S} = \operatorname{rank}\left(B, {}^{\mathbb{T}}\Phi(n,1)B, \ldots, {}^{\mathbb{T}}\Phi(n,n-1)B\right) = n,$$

where ${}^{\mathbb{T}}\Phi(n,l)$, $l = 0, \ldots, n-1$, is the transition matrix given by (21)*, and $\mathcal{S}$ is the controllability matrix.*

Proof: The system solution given by (19) for zero final conditions $x_n = 0$ has the following form:

$$0 = {}^{\mathbb{T}}\Phi(n)x_0 + \sum_{j=0}^{n-1} {}^{\mathbb{T}}\Phi(n,n-j-1)\tilde{B}u_j = {}^{\mathbb{T}}\Phi(n)x_0 + \left(\tilde{B}, {}^{\mathbb{T}}\Phi(n,1)\tilde{B}, \ldots, {}^{\mathbb{T}}\Phi(n,n-1)\tilde{B}\right)\begin{pmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{pmatrix},$$

which can be rewritten as follows

$${}^{\mathbb{T}}\Phi(n)x_0 = -\left(\tilde{B}, {}^{\mathbb{T}}\Phi(n,1)\tilde{B}, \ldots, {}^{\mathbb{T}}\Phi(n,n-1)\tilde{B}\right)\begin{pmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{pmatrix}.$$

This makes the set of n equations with n unknowns u_j, $j = 0, \ldots, n-1$. The solution of the set of equations exists if the condition given by Theorem 5.1 is satisfied (taking into account the relation between B and $\tilde{B}$ given by (17), which does not affect the rank condition). ■

In general case, the condition given by Theorem 5.1 is only sufficient. However, if the matrix ${}^{\mathbb{T}}\Phi(n) = 0$ then using zero input sequence $u_0 = u_1 = \cdots = u_{n-1} = 0$ the system can be transferred to the zero final state.

Analogically to Theorem 4.1, the condition given by Theorem 5.1 can be simplified as follows:

Theorem 5.2: *The GDFVOSS is controllable (in n steps) if the following condition is satisfied:*

$$\operatorname{rank} \mathcal{S}_I = \operatorname{rank}\left(B, AB, A^2B, \ldots, A^{n-1}B\right) = n,$$

where $\mathcal{S}_I$ is the controllability matrix in the equivalent form.

Proof: The proof is analogous to the proof of Theorem 4.1. ■

6. Observability

The following definition of observability will be used in sequel (Dzielinski & Sierociuk, 2007).

Definition 6.1: The GDFVOSS given by (14) is observable on $[0, k_f]$ if $x_0 \in \mathbb{R}^n$ can be determined from the knowledge of y_k and u_k for $k \in [0, k_f]$.

The necessary and sufficient condition for observability of GDFVOSS is given in the following form.

Theorem 6.1: *The GDFVOSS given by* (14) *is observable (in n steps) if and only if the following condition is satisfied:*

$$\operatorname{rank} \mathcal{O} = \operatorname{rank}\begin{pmatrix} C \\ C^{\mathbb{T}}\Phi(1) \\ C^{\mathbb{T}}\Phi(2) \\ \vdots \\ C^{\mathbb{T}}\Phi(n-1) \end{pmatrix} = n,$$

where n is the number of state equations, ${}^{\mathbb{T}}\Phi(j), j = 0, \ldots, n-1$, is the transition matrix given by (20), *and $\mathcal{O}$ is the observability matrix.*

Proof: Let us substitute the solution of GDFVOSS given by (19) to the output equation (14c) for first $n-1$ steps. This makes a set of equations with unknown x_0:

$$\begin{aligned} y_0 &= Cx_0 \\ y_1 &= C^{\mathbb{T}}\Phi(1)x_0 + C^{\mathbb{T}}\Phi(1,0)\tilde{B}u_0 \\ y_2 &= C^{\mathbb{T}}\Phi(2)x_0 + C^{\mathbb{T}}\Phi(2,1)\tilde{B}u_0 + C^{\mathbb{T}}\Phi(2,0)\tilde{B}u_1 \\ &\vdots \\ y_{n-1} &= C^{\mathbb{T}}\Phi(n-1)x_0 + C\sum_{j=0}^{n-2}{}^{\mathbb{T}}\Phi(n-1, n-j-2)\tilde{B}u_j. \end{aligned}$$

This set of equations can be rewritten as follows:

$$\begin{pmatrix} C \\ C^{\mathbb{T}}\Phi(1) \\ C^{\mathbb{T}}\Phi(2) \\ \vdots \\ C^{\mathbb{T}}\Phi(n-1) \end{pmatrix} x_0 = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} - O_{\text{obs}} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix},$$

where

$$O_{\text{obs}} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ C^{\mathbb{T}}\Phi(1,0)\tilde{B} & 0 & \ldots & 0 \\ C^{\mathbb{T}}\Phi(2,1)\tilde{B} & C^{\mathbb{T}}\Phi(2,0)\tilde{B} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ C^{\mathbb{T}}\Phi(n-1,n-2)\tilde{B} & C^{\mathbb{T}}\Phi(n-1,n-3)\tilde{B} & \ldots & 0 \end{pmatrix}.$$

This set of equations has a solution if and only if the condition given by Theorem 6.1 is satisfied. ■

This condition can be rewritten in a simpler form in the same way as controllability (reachability) conditions.

Theorem 6.2: *The GDFVOSS given by* (14) *is observable (in n steps) if and only if following condition is satisfied:*

$$\operatorname{rank} \mathcal{O}_I = \operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n, \tag{29}$$

where n is the number of state equations, and $\mathcal{O}_I$ is the observability matrix in the equivalent form.

Proof: Using the same argumentation as in Theorem 6.1, the following condition is obtained:

$$\operatorname{rank}\begin{pmatrix} C \\ C(f_{1,1}A + If_{1,0}) \\ C(f_{2,2}A^2 + f_{2,1}A + f_{2,0}I) \\ \vdots \\ C\left(f_{n-1,n-1}A^{n-1} + f_{n-1,n-2}A^{n-2} + \cdots + f_{n-1,1}A + f_{n-1,0}I\right) \end{pmatrix} = n.$$

Elements of this matrix can be obtained by the elementary matrix operations on the rows of the following matrix:

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

Elementary matrix operations do not change the rank of matrices, so the condition given by Theorem 6.2 is equivalent to the condition given by Theorem 6.1. ■

7. Numerical examples

Let us now illustrate the results presented by the following examples of system based on the $\mathcal{A}$-type type of variable fractional order difference definition with step-time $h = 1$. For simplicity of exposition, we limit the variability of order to switching only between tree different values.

Example 7.1: In this example, we will check whether the system given by (14) is controllable (reachable):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \alpha_2 = 0.6, \alpha_3 = 0.7.$$

Since $n = 3$, using condition of Theorem 5.1, the following transition matrices ${}^{\mathcal{A}}\Phi(3, 1)$, ${}^{\mathcal{A}}\Phi(3, 2)$ have to be calculated. Thus,

$$\begin{aligned} {}^{\mathcal{A}}\Phi(3, 1) &= A - I^{\mathcal{A}}w(\alpha(\cdot), 3, 1) = A + I\binom{\alpha_3}{1} \\ &= \begin{pmatrix} 0.7 & 0 & 1 \\ 1 & 0.7 & 1 \\ 0 & 1 & 1.7 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} {}^{\mathcal{A}}\Phi(3, 2) &= (A - I^{\mathcal{A}}w(\alpha(\cdot), 3, 1))^{\mathcal{A}}\Phi(2, 1) - I^{\mathcal{A}}w(\alpha(\cdot), 3, 2) \\ &= (A + I\binom{\alpha_3}{1})(A + I\binom{\alpha_2}{1}) - I\binom{\alpha_3}{2} \\ &= \begin{pmatrix} 0.525 & 1 & 2.3 \\ 1.3 & 1.525 & 3.3 \\ 1 & 2.3 & 3.825 \end{pmatrix}. \end{aligned}$$

The controllability (reachability) matrix has the following form:

$$\mathcal{S} = \left(B, {}^{\mathcal{A}}\Phi(3, 1)B, {}^{\mathcal{A}}\Phi(3, 2)B\right) = \begin{pmatrix} 1 & 0.7 & 0.525 \\ 0 & 1 & 1.3 \\ 0 & 0 & 1 \end{pmatrix},$$

and then, since $\operatorname{rank} \mathcal{S} = 3$, the system is controllable and reachable.

Let us check now its properties by using condition given by Theorem 5.2:

$$\operatorname{rank} \mathcal{S}_I = \operatorname{rank}\left(B, AB, A^2B\right) = \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3.$$

Using condition from Theorem 5.2, the same result as given by Theorem 5.1 is obtained (however, the equivalent controllability (reachability) matrix form is not identical to original controllability (reachability) matrix).

Example 7.2: In this example we will check whether system (14) is observable:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \; C = (1, 0, 0), \; \alpha_1 = 0.5, \; \alpha_2 = 0.6.$$

Using the condition of Theorem 6.1, we conclude that we have to calculate the transition matrices ${}^{\mathcal{A}}\Phi(1)$ and ${}^{\mathcal{A}}\Phi(2)$, which are

$$\begin{aligned} {}^{\mathcal{A}}\Phi(1) &= A - I^{\mathcal{A}}w(\alpha(\cdot), 1, 1) = A + I\binom{\alpha_1}{1} \\ &= \begin{pmatrix} 0.5 & 0 & 1 \\ 1 & 0.5 & 1 \\ 0 & 1 & 1.5 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} {}^{\mathcal{A}}\Phi(2) &= (A - I^{\mathcal{A}}w(\alpha(\cdot), 2, 1))^{\mathcal{A}}\Phi(1) - I^{\mathcal{A}}w(\alpha(\cdot), 2, 2) \\ &= (A + I\binom{\alpha_2}{1})(A + I\binom{\alpha_1}{1}) - I\binom{\alpha_2}{2} \\ &= \begin{pmatrix} 0.42 & 1 & 2.1 \\ 1.1 & 1.42 & 3.1 \\ 1 & 2.1 & 3.52 \end{pmatrix}. \end{aligned}$$

The observability matrix has the following form

$$\mathcal{O} = \begin{pmatrix} C \\ C^{\mathcal{A}}\Phi(1) \\ C^{\mathcal{A}}\Phi(2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0 & 1 \\ 0.42 & 1 & 2.1 \end{pmatrix},$$

and then, since $\operatorname{rank} \mathcal{O} = 3$, the system is observable.

The observability checked by using condition given by Theorem 6.2:

$$\operatorname{rank}\begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3$$

gives the same result as given by Theorem 6.1 (however, the equivalent observability matrix form is not identical to original observability matrix).

8. Conclusions

In the paper, for discrete-time fractional variable order state-space system, with different types of variable order difference definitions, the general form of the solution was proposed and proven. The necessary and sufficient conditions for reachability and observability of GDFVOSS have been formulated and proven. Only sufficient conditions for controllability of such a system have also been given. The obtained results are the same as for integer order systems, and the studied properties do not depend on the type of variable order difference. Theoretical results have been illustrated with numerical examples.

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