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DISTRIBUTIONAL CONVERGENCE OF BDF APPROXIMATIONS TO SOLUTIONS OF DESCRIPTOR SYSTEMS*

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Abstract. It has been observed that the backward differentiation approximation to the solutions of $Ex' + Fx = f$ can fail to converge even pointwise in an initial boundary layer. This paper shows that the approximations converge in a distributional sense even if the exact solution is also distributional.

1. Introduction

In a fundamental series of papers [3]-[5], Cobb investigated the distributional solutions of the linear time-invariant descriptor system

$$Ex' + Fx = f \tag{1}$$

and showed that the distributional solutions were the limits, in a distributional sense, of the solutions of the regularized system $(E + \varepsilon F)x' + Fx = f$ as $\varepsilon \to 0$. These impulsive solutions were also studied via transfer functions by Verghese [9]. One of the first class of numerical methods, other than the reduction of the pencil $\lambda E + F$, to be applied to (1) were backward differentiation formulas (BDF) [7], [8]. In [8] it was shown that the BDF approximations converge to the true solution outside of an initial boundary layer of nonconvergence.

The simplest of the BDF methods is the implicit Euler's method with fixed stepsize h which is

$$E\frac{x_{i+1} - x_i}{h} + Fx_{i+1} = f_{i+1}, \tag{2}$$

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where $t_i = t_0 + ih$, $f_i = f(t_i)$, and x_i is the approximation of $x(t_i)$. In [2] it is observed that the boundary-layer errors look a lot like approximations of distributions. This paper shows that the BDF approximations actually converge in a distributional sense on the entire interval of definition to the solution of (1) even if that solution is distributional.

2. Results

We assume that E, F are $n \times n$ matrices and that $\lambda E + F$ is a regular pencil so that $\det(\lambda E + F) \not\equiv 0$. We also assume that E is singular, and, in order to avoid trivial special cases, that the index of (1) is $k \geq 2$. That is, the order of the pole at infinity is at least two. Equivalently, the solutions of (1) involve $k-1$ derivatives of the forcing function f. We assume that f is at least k times continuously differentiable. In order to establish convergence for general fixed-step BDF methods, it suffices to consider (2). Notation is simplified by taking the initial time t_0 to be zero.

Given a sequence $a = \{x_i\}_{i=0}^{\infty}$ we identify it with the function

$$x_a(t) = \sum_{i=0}^{\infty} x_i [H(t-ih) - H(t-(i+1)h)],$$

where H is the unit step (Heaviside) function. From [8] it is known that $x_a(t)$ gives an $O(h)$ approximation to the true solution $x(t)$ on the interval $[kh\ \ T]$ with fixed T. We are interested in what happens on $[0\ \ kh]$.

By using the Kronecker structure of the pencil, and the well-known theory of singular systems [1], [2], it suffices to consider the special case

$$Nx' + x = f, \tag{3}$$

where N is a $k \times k$ nilpotent Jordan block

$$N = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Then (2) becomes

$$(N + hI)x_{i+1} = Nx_i + hf_{i+1}. \tag{4}$$

We first consider the asociated homogeneous equation for (3):

$$Nx' + x = 0, \qquad x(0) = x_0. \tag{5}$$

The solution of (5) is

$$x = \sum_{i=0}^{k-2} (-1)^i \delta^{(i)} N^{i+1} x_0, \tag{6}$$

where $\delta^{(i)}$ is the ith derivative of the delta function $\delta(t)$. The backward Euler approximation for (5), given by (4), is

$$x_{i+1}=(N+hI)^{-1}Nx_i=[(N+hI)^{-1}N]^{i+1}x_0 \tag{7}$$

which is zero for $i+1\geq k$. Let $p=1/h$. Then

$$(N+hI)^{-1}N=\begin{bmatrix} 0 & p & -p^2 & \cdots & (-1)^k p^{k-1} \\ 0 & 0 & p & \ddots & \vdots \\ \vdots & \vdots & \vdots & & -p^2 \\ \vdots & \vdots & \vdots & \ddots & p \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

and, for $1\leq i\leq k-1$,

$$[(N+hI)^{-1}N]^i=\begin{bmatrix} \underset{i\text{ zeros}}{0\cdots 0} & p^i & \cdots & (-1)^{k-i+1}\binom{k-2}{i-1}p^{k-1} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & p^i \\ 0 & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{bmatrix}, \tag{8}$$

where the $(i+j+1)$th superdiagonal of (8), for $0\leq j\leq k-1-i$, has entries $(-1)^{i+j+1}\binom{i+j-1}{i-1}p^{i+j}$.

Using (8) and comparing (7) with (6) and equating the coefficients of like coordinates of x_0, we find that we need to show that

$$z_h^{[i]}\to(-1)^i\delta^{(i)} \tag{9}$$

as $h\to 0^+$ where

$$z_h^{[i]}=\frac{1}{h^{i+1}}\sum_{j=0}^{i}(-1)^j\binom{i}{j}[H(t-(j+1)h)-H(t-(j+2)h)]$$

and (9) is taken in the weak distributional sense. That is,

$$\lim_{h\to 0}\int_0^\infty z_h^{[i]}(t)g(t)\,dt=\int_0^\infty \delta^{(i)}(t)g(t)\,dt=(-1)^i g^{(i)}(0) \tag{10}$$

for every infinitely differentiable test function g on $[0\ \infty)$. Let $w'(t)=g(t)$ so that

$$\begin{aligned}\int_0^\infty z_h^{[i]}(t)g(t)\,dt&=\frac{1}{h^{i+1}}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_{(j+1)h}^{(j+2)h}g(t)\,dt\\ &=\frac{1}{h^{i+1}}\sum_{j=0}^{i}(-1)^j\binom{i}{j}[w((j+2)h)-w((j+1)h)]. \end{aligned} \tag{11}$$

Define the operators S, Δ by $Su_i = u_{i+1}$ and $\Delta u_i = u_{i+1} - u_i$ so that $\Delta = S - I$. Then (11) becomes

$$= \frac{1}{h^{i+1}} \sum_{j=0}^{i} (-1)^j \binom{i}{j} (S^{j+2} - S^{j+1}) w$$

$$= \frac{1}{h^{i+1}} \sum_{j=0}^{i} (-1)^j \binom{i}{j} \Delta S^{j+1} w$$

$$= \frac{1}{h^{i+1}} \Delta \left[\sum_{j=0}^{i} (-1)^j \binom{i}{j} S^j \right] Sw$$

$$= \frac{1}{h^{i+1}} \Delta (I - S)^i Sw$$

$$= \frac{(-1)^i}{h^{i+1}} \Delta^{i+1} Sw.$$

But $h^{-i-1} \Delta^{i+1} w$ converges to $w^{(i+1)} = g^{(i)}$ [6] and thus (9) holds.

Now consider the nonhomogeneous problem (3). By linearity the solution of (3) is made up of a smooth solution on $[0 \ \ T]$ and (possibly) an initial impulse satisfying (3) with $f = 0$. The preceding argument shows that the approximation for the distributional part converges. It suffices then to consider the approximation of the smooth solution. Let x_{0T} be the exact initial value of the smooth solution of (3). Simple examples show that for the first k steps the error in using (4) can be unbounded as h goes to 0 even if $x_0 = x_{0T}$. We wish to show that, in fact, the error goes to zero weakly in the distributional sense. By iterating (4) backward, it can be shown that there is an initial condition x_{0h} such that taking $x_0 = x_{0h}$ leads to a solution of (4) which gives a uniformly $O(h)$ approximation to the smooth solution of (3) on $[0 \ \ T]$ and also

$$x_{0h} = x_{0T} + h\phi(h), \tag{12}$$

where ϕ has a series expansion $\phi(h) = \sum_{i=0}^{k^2} \phi_i h^i + O(h^{k^2})$. Now let $x_i^{[0]}$ be the solution of (4) with $x_0 = x_{0h}$, while $x_i^{[T]}$ is the solution of the associated homogeneous equation (7) with $x_0 = x_{0T}$ and $x_i^{[pi]}$ is the solution of the associated homogeneous equation with $x_0 = \phi_i$. Then

$$x_i^{[0]} = x_i^{[T]} + h \sum_{i=0}^{k^2} h^i x_i^{[pi]} + O(h).$$

But $x_i^{[0]}$ is a uniformly $O(h)$ approximation of the unique smooth solution of (3). From the argument used to prove (9) we have that the $x_i^{[pi]}$ converge to distributions, so that $h^{i+1} x_i^{[pi]}$ converges to zero as a distribution as h goes to zero. In summary, we have shown:

Theorem 1. *Assume that $\lambda E + F$ is a regular pencil and x_0 is an arbitrary initial condition. Let $\{x_i\}$ be the backward Euler approximation using (2).*

Then this approximation converges weakly in the distributional sense to the distributional solution of $Ex' + Fx = f$, $x(t_0) = x_0$.

3. Comments

We consider the main result of this paper to be primarily of theoretical interest. However, it is interesting to note that if we have the system (1) and are interested in the possible impulsive behavior, then impulsive behavior can be modeled by using (2). This simulation is much quicker, and easier to program, than a code to compute the pencil decomposition. Also note that if the quantities of interest involve weighted integrals of the solution, then these quantities can also be estimated using backward differentiation formulas.

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