NORTH-HOLLAND

Digraph Characterization of Structural Controllability for Linear Descriptor Systems

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ABSTRACT

For linear descriptor systems of the form $E\dot{x} = Ax + Bu$, the different kinds of controllability are analyzed by graph-theoretic means. Starting from known algebraic criteria, digraph conditions for structural r-controllability, structural impulse controllability, and structural complete controllability are derived. A nontrivial electrical example system illustrates the application of the results. © 1997 Elsevier Science Inc.

1. INTRODUCTION

In this paper we consider linear descriptor systems

$$E\dot{x} = Ax + Bu \tag{1.1}$$

with $x(t) \in \mathbb{R}^n$ the descriptor vector, $u(t) \in \mathbb{R}^m$ the input vector, and real matrices $E \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. The $n \times n$ matrix E is possibly singular, whereas the matrix pencil $(sE - A)$ is assumed to be nondegenerate, i.e.,

$$\det(sE - A) \neq \text{constant}. \tag{1.2}$$

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Descriptor systems (1.1) behave more in a complicated way than linear systems in the standard state-space description that formally result from (1.1) on replacing the matrix E with the unit matrix I (as in [7] and many other works). In particular, we have to distinguish between different kinds of controllability ([2, 3, 14, 17] etc.). This paper deals with structural conditions for the different kinds of controllability. For this purpose only the "structure" of the matrices E, A, and B is taken into account. The real matrices E, A, and B are mapped into binary matrices $[E]$, $[A]$, and $[B]$, which are associated with digraphs. Then digraph conditions for the structural controllability of the class of structurally equivalent systems defined by $[E, A, B]$ may be derived. The procedure is a generalization of that used in [8] to prove structural controllability criteria for state-space systems.

The main results are formulated in Theorems 3.1–3.3. The essential contents of these theorems were presented (without proof) at an ILAS meeting [9].

Depending on the questions to be answered, sometimes other types of graph representation can be more suited for descriptor systems than the digraph representation used in this paper (see [6, 10, 13]). Bipartite-graph criteria equivalent to Theorems 3.1 and 3.2 were proved by K. Murota in his monograph [6]. Readers interested in other books on the graph-theoretic approach to linear systems are referred to [1, 5, 11, 15].

The paper is organized as follows: In Section 2 we remind the reader of the controllability conditions for numerically given systems (1.1) and introduce some graph-theoretic notions. In Section 3 graph-theoretic conditions for structural r-controllability, impulse controllability, and complete controllability are proved. In Section 4 we discuss the derived results, comparing them with the bipartite-graph approach. Finally, in Section 5 a nontrivial electrical example system illustrates the application of the results.

2. PRELIMINARIES

Typical features of descriptor systems (1.1), which are unknown in the realm of state-space systems, are possible impulsive responses to nonimpulsive excitations as well as provision for the consistency of initial conditions. The subset of $\mathbb{R}^n$ comprising all consistent initial values $x(0)$ is called the *reachable set* [17]. The different kinds of controllability have been defined as follows (cf. [17, 14] etc.):

DEFINITION 2.1. A descriptor system (1.1) is said to be

(1) *r-controllable* if it is controllable within the reachable set,

(2) *impulse-controllable* if all impulsive modes can be excited by suitably chosen nonimpulsive inputs,

(3) (*completely*) *controllable* if it is controllable within $\mathbb{R}^n$.

Necessary and sufficient conditions for controllability have been proved (see, e.g., [3]):

LEMMA 2.1. *A descriptor system* (1.1) *is*

(1) *r-controllable iff*

$$\operatorname{rank}(sE - A, B) = n \qquad \textit{for all} \quad s \in \mathbb{C}, \tag{2.1}$$

(2) *impulse-controllable iff*

$$\operatorname{rank}\begin{pmatrix} E & 0 & 0 \\ A & E & B \end{pmatrix} = n + \operatorname{rank} E, \tag{2.2}$$

(3) (*completely*) *controllable iff both*

$$\operatorname{rank}(E, B) = n \tag{2.3}$$

and

$$\operatorname{rank}(sE - A, B) = n \qquad \textit{for all} \quad s \in \mathbb{C}.$$

Now, let us suppose the entries of the matrices A, B, and E are not precisely known. More exactly, we distinguish between two types of entries: entries that are fixed at zero and entries that are assumed to be mutually independent. In this way the real matrices A, B, and E are replaced by binary structure matrices $[A]$, $[B]$, and $[E]$ of the same size.

DEFINITION 2.2. The entries of a *structure matrix* $[M]$ either are fixed at zero or have indeterminate values. By fixing all the indeterminate entries of $[M]$ at some particular real values we obtain an *admissible realization* M of the binary structure matrix $[M]$; for short, we write $M \in [M]$. Two matrices $M' \in [M]$ and $M'' \in [M]$ are called *structurally equivalent*.

Each admissible realization $M \in [M]$ where $[M]$ possesses $h > 0$ indeterminate entries can be interpreted as an element of a vector space $\mathbb{R}^h$. We say that a matrix property holds structurally for $[M]$ if this property holds for almost all $M \in \mathbb{R}^h$. Here "almost all" means "for all except for those in some proper algebraic variety in $\mathbb{R}^h$" (cf. [16]). For example, the *structural rank* of $[M]$ is a very important structural property of the set of structurally equivalent matrices. It is defined by [1, 6]

$$s - \operatorname{rank}[M] = \max_{M \in [M]} \operatorname{rank} M. \tag{2.4}$$

The following fact, which is easy to see, should be mentioned: If we know at least one realization, say $\tilde{M}$, with

$$\operatorname{rank} \tilde{M} = \max_{M \in [M]} \operatorname{rank} M = s - \operatorname{rank}[M], \tag{2.5}$$

then

$$\operatorname{rank} M = \operatorname{rank} \tilde{M}$$

holds for almost all $M \in [M]$.

Any $n \times n$ structure matrix $[M]$ can be represented by a digraph $G[M]$ formed by n vertices named $1, 2, \ldots, n$ as well as edges leading from the initial vertex j to the final vertex i if $m_{ij} \neq 0$ $(i, j = 1, 2, \ldots, n)$.

In the following Section 3, a few graph-theoretic concepts are needed:

A *path* is a sequence of edges such that the initial vertex of the succeeding edge is the final vertex of the preceding edge.

A path is called a *cycle* if the initial vertex of the first edge and the final vertex of the last edge are the same and no other vertex is reached more than once in going along the path.

Cycles consisting of one edge only are called *self-cycles*.

A set of vertex-disjoint cycles is said to be a *cycle family*. The number of edges contained in a cycle family defines the *length* of this cycle family. A cycle family the length of which equals the number of vertices contained in the digraph is called a *spanning-cycle family*.

Two vertices i and j are called *strongly connected* if a path exists from vertex i to vertex j as well as a path from vertex j to vertex i. It is easy to realize that the subset of vertices which are strongly connected to a given vertex i forms an equivalence class $C(i)$ within the vertex set of $G[M]$. Such an *equivalence class of strongly connected vertices*, together with all edges incident only with these vertices, constitutes a subgraph associated with a square submatrix of $[M]$.

The structure of the descriptor system (1.1) can also be represented by a digraph $G[sE - A, B]$. Its vertex set consists of n *descriptor vertices* $1, 2, \ldots, n$ and m *input vertices* $I_1, I_2, \ldots, I_m$. Its edge set consists of *A-edges*, *E-edges*, and *B-edges*: If $a_{ij} \neq 0$ (or $e_{ij} \neq 0$), then an A-edge (or an E-edge) leading from j to i exists, and if $b_{ik} \neq 0$, then a B-edge leading from I_k to i exists. A descriptor vertex i is called *input-connected* if there is a path in $G[sE - A, B]$ starting in an input vertex and terminating in i.

Sometimes it is helpful to take *descriptor feedback* into consideration. Thus we obtain an augmented system description. Essentially all the information contained in this augmented system is reflected by the square system matrix $\begin{pmatrix} sE - A & B \\ F & 0 \end{pmatrix}$ and the associated digraph $G(\begin{bmatrix} sE - A \\ [F] \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix})$. The number of descriptor vertices contained in a cycle family of $G(\begin{bmatrix} sE - A \\ [F] \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix})$ is called the *width* of this cycle family. The following statement has been shown in [8], page 36:

LEMMA 2.2. *If $[F]$ is an $m \times n$ structure matrix without structural zeros then the structural rank s −* rank$[A, B]$ *equals n if and only if there exists a cycle family of width n in $G(\begin{bmatrix} A \\ [F] \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix})$.*

There are many other possibilities to graph-theoretically represent the pencil $[sE - A, B]$. In particular, the bipartite-graph representation should be mentioned. Any $s \times t$ structure matrix $[M]$ can be represented by a bipartite graph $G_b[M]$ as follows: the rows of $[M]$ are associated to s vertices $r_1, r_2, \ldots, r_s$ of a vertex set R, and the columns of $[M]$ to t vertices $c_1, c_2, \ldots, c_t$ of a vertex set C, respectively. Every entry $m_{ij} \neq 0$ corresponds to an edge leading from c_j to r_i.

To obtain a bipartite-graph representation of (1.1) we set $[M] := [sE - A, B]$. There is a vertex set R consisting of n elements as well as a vertex set C consisting of $n + m$ vertices, and every nonvanishing element e_{ij}, a_{ij}, and $b_{i, j-n}$, is associated to an E-edge, A-edge, and B-edge from vertex c_j to vertex r_i.

A subset of edges is said to be a *matching* if any two edges of it do not have a common vertex. The number of edges is called the *cardinality* of the matching. A matching of maximal cardinality is a *maximum matching*. In this context, Lemma 2.2 has the following counterpart:

LEMMA 2.3. (cf. e.g. [6]). *The structural rank s-*rank$[A, B]$ *is equal to the number of edges involved in a maximum matching of $G_b[A, B]$.*

Instead of the decomposition into equivalence classes of strongly connected vertices in digraphs, there exists a unique decomposition of a bipartite graph into partially ordered subgraphs often referred to as a *Dulmage-Mendelsohn decomposition*, or, for short, *DM-decomposition* ([4], cf. [6] and references cited there). A *consistent DM-component* of the bipartite graph can be assigned to a strongly connected subgraph of the corresponding digraph.

3. STRUCTURAL CONTROLLABILITY CONDITIONS

The different kinds of structural controllability may be defined as follows:

DEFINITION 3.1. A class of descriptor systems (1.1) given by its structure matrices $[E, A, B]$ is said to be *structurally (completely) controllable (r-controllable, impulse-controllable)* if at least one realization $(E, A, B) \in [E, A, B]$ is (completely) controllable (r-controllable, impulse-controllable) in the usual numerical sense exists.

As we are dealing with nondegenerate pencils $(sE - A)$, the rows of $[E, A, B]$ may always be reordered in such a way that no main-diagonal element of $[sE - A]$ vanishes (see [10]). Therefore, without loss of generality, we will assume nonzero main-diagonal elements of $[sE - A]$.

First of all, the structural r-controllability is investigated. We start with two lemmas from which necessary and sufficient conditions for structural r-controllability may easily be derived.

LEMMA 3.1. *Let $[M]$ and $[N]$ be two $n \times n$ structure matrices such that none of the main-diagonal elements of $[sM - N]$ vanishes. The digraph $G[sM - N]$ is assumed to be strongly connected and to have two spanning cycle families, each of which contains a different number of M-edges. Let $[r]$ be an $n \times 1$ nonzero structure matrix. Then*

$$\operatorname{rank}(sM - N, r) = n \qquad \textit{for all} \quad s \in \mathbb{C} \setminus \{0\} \tag{3.1}$$

holds for almost all admissible realizations $(M, N, r) \in [M, N, r]$.

Proof. Each spanning-cycle family yields a nonvanishing summand to $\det(sM - N)$. The number of M-edges contained in such a cycle family defines the degree in s of the associated determinantal summand. By assumption there are at least two spanning-cycle families with a different number of M-edges. Consequently, the determinant $\det(sM - N)$ is a nondegenerate polynomial with simple nonvanishing roots for almost all realizations $(M, N) \in [M, N]$. If s_0 is such a root of $\det(sM - N)$ then $\operatorname{rank}(s_0 M - N) = n - 1$.

A certain column of $s_0 M - N$ may be replaced by $r \in [r]$. Then, the modified square matrix has full rank for almost all admissible r, which implies (3.1).

This can be seen as follows: Assume M, N and $s \neq 0$ to be fixed so that all main-diagonal elements are different and $\operatorname{rank}(sM - N) = n - 1$. That means there are two or more spanning-cycle families in $G(sM - N)$ which numerically cancel each other out.

It may happen that one main-diagonal element vanishes. In this case, the column belonging to the vanishing main-diagonal element is referred to as the kth column. Otherwise, the kth column denotes a column of $sM - N$ that is linearly dependent upon the remaining $n - 1$ columns.

Removing the kth column of $sM - N$ means eliminating of all edges in $G[sM - N]$ with the initial vertex k. The subsequent replacement by r is associated with newly introduced edges starting in k and leading to vertices according to the structure pattern of $[r]$. Since $[r]$ is a nonzero structure column, there is at least one newly introduced edge from vertex k to, say, vertex i. Since all n vertices of the original digraph $G[sM - N]$ are strongly connected, a cycle formed by the new edge from k to i and a "backward" path from i to k exists in the modified digraph. That is, the modified digraph contains a spanning-cycle family consisting of the new cycle just explained and self-cycles associated with the vertices not involved in the new cycle. The numerical weight of that cycle family depends on the numerical realization of $r \in [r]$. For almost all $r \in [r]$ the spanning-cycle families within the modified digraph do not numerically cancel each other out. In other words, for the chosen realizations M, N, $s \neq 0$, and for almost all $r \in [r]$ we have $\operatorname{rank}(sM - N, r) = n$. This completes the proof. ■

Now, let us turn back to nondegenerate descriptor systems (1.1). Every subset of strongly connected vertices (together with the subset of edges incident only with vertices of the subset under consideration) defines a strongly connected subgraph within $G[sE - A]$.

LEMMA 3.2. *Consider a class of descriptor systems* (1.1) *characterized by the* $n \times (2n + m)$ *structure matrix* $[E, A, B]$ *whose rows have been ordered so that none of the main-diagonal elements of* $[sE - A]$ *vanishes.*

Then the following conditions are equivalent:

(a) *One has*

$$\operatorname{rank}(sE - A, B) = n \qquad \textit{for all} \quad s \in \mathbb{C} \setminus \{0\} \tag{3.2}$$

for almost all $(E, A, B) \in [E, A, B]$.

(b) *Every strongly connected subgraph of* G$[sE - A]$ *that has two spanning-cycle families comprising different numbers of involved E-edges is input-connected.*

Proof. As a preparatory step, we rename the vertices of G$[sE - A]$. Let k be the number of strongly connected subgraphs within G$[sE - A]$. These subgraphs $G_1, G_2, \dots, G_k$ may be enumerated so that there is no path from G_i to G_j if $i < j$. The vertices of G_1 are named $1, 2, \dots, n_1$, the vertices of G_2 are named $n_1 + 1, n_1 + 2, \dots, n_1 + n_2$, the vertices of G_3 are named $n_1 + n_2 + 1, n_1 + n_2 + 2, \dots, n_1 + n_2 + n_3$, and so on. This kind of enumeration is reflected by an upper block-triangular matrix representation of $[sE - A]$. Every strongly connected subgraph G_i comprising n_i vertices is associated with an $n_i \times n_i$ main-diagonal block $[(sE - A)_{ii}]$. By assumption, all the individual diagonal entries are nonzero elements.

The block-triangular representation implies a factorization of the characteristic polynomial

$$\det(sE - A) = \prod_{i=1}^{k} \det(sE - A)_{ii}.$$

For almost all $(E, A) \in [E, A]$, every nonvanishing root s_0 of $\det(sE - A) = 0$ is simple, and s_0 appears as a simple root of exactly one subdeterminant, say, of $\det(sE - A)_{\nu\nu}$.

Now, we can proceed with the proof of Lemma 3.2.

Assume a subgraph G_ν that possesses two spanning-cycle families with different numbers of E-edges to be input-connected. The associated main-diagonal block $(sE - A)_{\nu\nu}$ has a nondegenerate determinant $\det(sE - A)_{\nu\nu}$ with a simple root $s_\nu \neq 0$, i.e., $\operatorname{rank}(s_\nu E - A)_{\nu\nu} = n_\nu - 1$. As for the other main-diagonal blocks,

$$\operatorname{rank}(s_\nu E - A)_{ii} = n_i \qquad \text{for} \quad i = 1, 2, \dots, k, \quad i \neq \nu, \tag{3.3}$$

holds.

Since G_ν is input-connected, there is a shortest path from the subset I of input vertices to G_ν. If this path has length 1, then the B-part within the hyperrow of $(sE - A, B)$, which lies to the right of the main-diagonal block $(sE - A)_{\nu\nu}$, contains a nonvanishing column b_ν. The main-diagonal block $(sE - A)_{\nu\nu}$ and the column b_ν may be interpreted as a matrix $sM - N$ and a column r in the sense of Lemma 3.1. Applying this lemma, we get

$$\operatorname{rank}\big((s_\nu E - A)_{\nu\nu}, b_\nu\big) = n_\nu .$$

In other words, the hyperrow headed by $(sE - A)_{\nu\nu}$ has full row rank for $s = s_\nu$. In conjunction with (3.3) we conclude

$$\operatorname{rank}(s_\nu E - A, B) = n.$$

Provided the subgraph G_ν cannot be reached from inputs by a path of length 1, then G_ν may be reached via a chain of other subgraphs, say $G_\mu, G_\lambda, \dots, G_\kappa$. That is, there is a path $I \to G_\kappa \to \cdots \to G_\lambda \to G_\mu \to G_\nu$.

In this case the hyperrow headed by $(sE - A)_{\nu\nu}$ has a nonvanishing column r_μ that lies in the part of the hyperrow common with the hypercolumn headed by $(sE - A)_{\mu\mu}$. Applying Lemma 3.1 to the matrix $(sE - A)_{\nu\nu}$ and the column r_μ, we obtain

$$\operatorname{rank}\big((s_\nu E - A)_{\nu\nu}, r_\mu\big) = n_\nu .$$

The column r_μ belongs to a vertex v_μ of the subgraph G_μ. Its corresponding diagonal block $(s_\nu E - A)_{\mu\mu}$ is regular. Since the column of $sE - A$ from which r_μ was taken must not be used for a second time, we replace this column of $(sE - A)_{\mu\mu}$ by a nonvanishing column r_λ in the common part of the hyperrow headed by $(sE - A)_{\mu\mu}$ and the hypercolumn headed by $(sE - A)_{\lambda\lambda}$. Applying Lemma 3.1 to the matrix $(sE - A)_{\mu\mu}$ and the column r_λ, we get

$$\operatorname{rank}\big((s_\nu E - A)_{\mu\mu}, r_\lambda\big) = n_\mu .$$

Proceeding in the same manner, we eventually arrive at a subgraph G_κ which is adjacent to an input vertex associated with a nonvanishing column b_κ situated in the hyperrow headed by $(sE - A)_{\kappa\kappa}$. Applying Lemma 3.1 again, we conclude

$$\operatorname{rank}\big((s_\nu E - A)_{\kappa\kappa}, b_\kappa\big) = n_\kappa .$$

Hence, each of the hyperrows headed by $(sE - A)_{\nu\nu}$, $(sE - A)_{\mu\mu}$, $(sE - A)_{\lambda\lambda}, \dots, (sE - A)_{\kappa\kappa}$ has full row rank. Altogether, the corresponding $n_\nu + n_\mu + n_\lambda + \cdots + n_\kappa$ rows of $(sE - A, B)$ have rank $n_\nu + n_\mu + n_\lambda + \cdots + n_\kappa$. Considering the diagonal blocks not touched during the process just described and taking the equations (3.3) into account, we conclude: The wanted regular $n \times n$ submatrix of $(s_\nu E - A, B)$ is yielded by the $n + 1$ columns covering the main-diagonal blocks $(s_\nu E - A)_{ii}$ for $i = 1, 2, \dots, k$ and the B-column containing b_κ. This completes the first part of the proof.

Now, let us assume there exists a strongly connected subgraph G_ν which is not input-connected. We shall show that this contradicts the condition (3.2).

If G_ν is connected neither to the input set I nor to any other subgraph G_i $(i > \nu)$, then the hyperrow of $(sE - A, B)$ headed by $(sE - A)_{\nu\nu}$ has nonzero elements only in the main-diagonal block $(sE - A)_{\nu\nu}$. For $s_\nu \neq 0$ solving $\det(s_\nu E - A)_{\nu\nu} = 0$, the rows of the hyperrow under consideration are linearly dependent. Consequently, Equation (3.2) cannot be satisfied.

Suppose G_ν can be reached from the subgraphs $G_\mu, G_\lambda, \dots, G_\kappa$, where $\nu < \mu < \lambda < \cdots < \kappa$, that are also not input-connected. Consider the hyperrows of $(sE - A, B)$ headed by $(sE - A)_{\nu\nu}$, $(sE - A)_{\mu\mu}$, $(sE - A)_{\lambda\lambda}, \dots, (sE - A)_{\kappa\kappa}$:

$$(sE - A, B)_{\mathrm{aux}}$$

$$:= \left(\begin{array}{ccccc|c} (sE - A)_{\nu\nu} & \times & \times & \cdots & \times & B_\nu \\ 0 & (sE - A)_{\mu\mu} & \times & \cdots & \times & B_\mu \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \times & \vdots \\ 0 & 0 & \cdots & 0 & (sE - A)_{\kappa\kappa} & B_\kappa \end{array}\right).$$

All the submatrices B_ν, B_μ, $B_\lambda, \dots, B_\kappa$ are zero blocks because the subgraphs $G_\nu, G_\mu, G_\lambda, \dots, G_\kappa$ are not input-connected. Let $s_\nu \neq 0$ be a root of $\det(s_\nu E - A)_{\nu\nu} = 0$. Then the rows of $(sE - A)_{\mathrm{aux}}$ are linearly dependent for $s = s_\nu$. Therefore $\operatorname{rank}(sE - A, B) < n$, contrary to the condition (3.2). ∎

Lemma 3.2 and Lemma 2.2 imply the next theorem.

THEOREM 3.1. *Consider a class of descriptor systems* (1.1) *characterized by the* $n \times (2n + m)$ *structure matrix* $[E, A, B]$ *whose rows have been ordered so that all the main diagonal elements of* $[sE - A]$ *are nonzero. Let* $[F]$ *be an* $m \times n$ *structure matrix. Then a class* $[E, A, B]$ *of descriptor systems* (1.1) *is structurally r-controllable iff*

(a) *there exists a cycle family of width n in* $\mathrm{G}(\left[\begin{smallmatrix} A \\ F \end{smallmatrix}\right] \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right])$, *and*

(b) *every strongly connected subgraph of* $\mathrm{G}[sE - A]$ *that involves at least one E-edge is input-connected.*

Proof. The criterion (2.1) for r-controllability of descriptor systems (1.1) can be split into two parts:

$$\operatorname{rank}(A, B) = n \tag{3.4}$$

and

$$\operatorname{rank}(sE - A, B) = n \qquad \text{for all} \quad s \in \mathbb{C} \setminus \{0\}. \tag{3.5}$$

To ensure the structural r-controllability, the two conditions (3.4) and (3.5) must be valid for almost all realizations $(E, A, B) \in [E, A, B]$. Graph-theoretic conditions ensuring (3.4) and (3.5) for almost all realizations $(E, A, B) \in [E, A, B]$ have already been shown in Lemma 2.2 and Lemma 3.2, respectively.

Instead of condition (b) of Lemma 3.2, condition (b) of Theorem 3.1 can be used. This can be seen in the following manner: Consider a strongly connected subgraph G_i that is input-connected and contains at least one E-edge. Since each vertex of G_i is equipped with a self-cycle, it is obvious that all edges of G_i belong to at least one spanning cycle family. Therefore, either all spanning-cycle families have the same number ($\neq 0$) of involved E-edges or not. In the former case, the corresponding subdeterminant $(sE - A)_{ii}$ has roots only at $s = 0$. For $s = 0$, however, the rank condition (3.4) is crucial. Equation (3.4) holds for almost all $(A, B) \in [A, B]$ iff condition (a) of Theorem 3.1 is fulfilled. The latter case implies that the corresponding subdeterminant $(sE - A)_{ii}$ has roots at $s \neq 0$. Hence, two spanning-cycle families with different numbers of involved E-edges exist in G_i. That is, the subgraph G_i meets condition (b) of Lemma 3.2. This was to be proved. ∎

To check the structural (complete) controllability of descriptor systems (1.1), the structural rank of $[E, B]$ has additionally to be examined (cf. Lemma 2.2).

THEOREM 3.2. *Consider a class of descriptor systems* (1.1) *characterized by the* $n \times (2n+m)$ *structure matrix* $[E, A, B]$ *whose rows have been ordered so that all the main-diagonal elements of* $[sE - A]$ *are nonzero. Let* $[F]$ *be an* $m \times n$ *structure matrix. Then a class* $[E, A, B]$ *of descriptor systems* (1.1) *is structurally* (*completely*) *controllable iff*

(a) *there exists a cycle family of width n in* $\mathrm{G}\left(\begin{smallmatrix}[E] & [B] \\ [F] & 0\end{smallmatrix}\right)$, *and*

(b) *there exists a cycle family of width n in* $\mathrm{G}\left(\begin{smallmatrix}[A] & [B] \\ [F] & 0\end{smallmatrix}\right)$, *and*

(c) *every strongly connected subgraph of* $\mathrm{G}[sE - A]$ *that involves at least one E-edge is input-connected.*

Finally, we deal with a graph-theoretic criterion for structural impulse controllability.

THEOREM 3.3. *Let* $[E, A, B]$ *be a class of descriptor systems* (1.1), $[F]$ *an* $m \times n$ *structure matrix, and* $t :=$ s-rank$[E]$. *Then* $[E, A, B]$ *is structurally impulse controllable iff a cycle family having width n and involving t E-edges exists in*

$$\mathrm{G}\begin{pmatrix}[sE - A] & [B] \\ [F] & 0\end{pmatrix}. \tag{3.6}$$

Proof. Suppose (3.6) holds. Consider such a cycle family. It consists of t E-edges, n_A A-edges, and $n - t - n_A$ B-edges as well as $n - t - n_A$ F-edges. Each of the involved E-, A-, or B-edges terminates in a different descriptor vertex of the whole set $1, 2, \ldots, n$, and each of the involved A- or E-edges starts in another descriptor vertex. We mark the entries of the structure matrix

$$\begin{pmatrix}[E] & 0 & 0 \\ [A] & [E] & [B]\end{pmatrix}$$

that correspond to E-, A-, or B-edges involved in the cycle family under consideration. These $n + t$ entries lie both in different rows and in different columns; in other words,

$$\text{s-rank}\begin{pmatrix} E & 0 & 0 \\ A & E & B \end{pmatrix} = t + n.$$

Because of the condition (2.2) of Lemma 2.1, the descriptor system (1.1) is structurally impulse-controllable. Now, assume (2.2) to be valid for an admissible realization $(E, A, B) \in [E, A, B]$. Then a set of t E-entries, n_A A-entries, and $n - t - n_A$ B-entries exists, which constitute a set of $t + n$ entries lying both in different rows and in different columns of $(\begin{smallmatrix}[E] & 0 & 0\\ [A] & [E] & [B]\end{smallmatrix})$. The corresponding n edges form a subgraph

$$G\begin{pmatrix} [sE - A] & [B] \\ [F] & 0 \end{pmatrix}$$

consisting of $n - t - n_A$ vertex-disjoint paths and, possibly, cycles. These $n - t - n_A$ paths start in $n - t - n_A$ different input vertices. The final vertices of each path may be connected to their initial vertices by F-paths to complete $n - t - n_A$ feedback cycles. Thus, we have generated a cycle family in $G(\begin{smallmatrix}[sE - A] & [B]\\ [F] & 0\end{smallmatrix})$ having width n and involving t E-edges. This completes the proof. ■

4. DISCUSSION

As mentioned in the Introduction, the results of Theorem 3.1 and Theorem 3.2 are closely related to that of [6], where bipartite graphs are used. For convenience, the conditions derived there are reformulated in the following lemmas:

LEMMA 4.1. *Consider a class of descriptor systems* (1.1) *characterized by the* $n \times (2n + m)$ *structure matrix* $[E, A, B]$. *Then a class* $[E, A, B]$ *of* *descriptor systems* (1.1) *is structurally r-controllable iff*

(a) *there exists a matching of cardinality* n *in* $G_b[A, B]$, *and*
(b) *no consistent DM component of* $G_b[sE - A, B]$ *contains E-edges.*

LEMMA 4.2. *Consider a class of descriptor systems* (1.1) *characterized by the* $n \times (2n + m)$ *structure matrix* $[E, A, B]$. *Then a class* $[E, A, B]$ *of descriptor systems* (1.1) *is structurally (completely) controllable iff*

(a) *there exists a matching of cardinality* n *in* $G_b[E, B]$, *and*
(b) *there exists a matching of cardinality* n *in* $G_b[A, B]$, *and*
(c) *no consistent part of* $G_b[sE - A, B]$ *contains E-edges.*

In fact, the conditions obtained in Section 3 could be derived from the lemmas above by purely graph-theoretic arguments. Both the criteria of Section 3 and of Section 4 exploit the partitioning of the graph under consideration into irreducible subgraphs. These partitionings are related to each other in the following manner: If all main-diagonal elements of $[sE - A]$ are occupied, then each strongly connected subgraph of $G[sE - A, B]$ that is not input-connected corresponds uniquely to a consistent DM component of $G_b[sE - A, B]$ and vice versa.

The bipartite-graph representation of a matrix M is invariant to any permutation of rows and columns, whereas the digraph representation of M is invariant only to permutations achieved by a permutation transformation $P^{-1}MP = P^{\mathsf{T}}MP$ where P is a permutation matrix.

The block-triangular representation used in the proof of Lemma 3.2 depends on the choice of the main-diagonal elements. Another choice of the main-diagonal elements results in another block-triangular representation where the individual main-diagonal blocks of the first representation may be obtained from the corresponding main-diagonal blocks of the second representation by permutation transformation. The determinants of the main-diagonal block yield the same absolute value for both representations. Omitting the sign, the determinants associated with the subgraphs that are not input-connected are equal to the determinants of the corresponding consistent DM components.

Using the correspondence between bipartite graphs and digraphs, and keeping additionally Lemma 2.2 and Lemma 2.3 in mind, the equivalence of Theorem 3.1 and Lemma 4.1, as well as of Theorem 3.2 and Lemma 4.2, is readily checked.

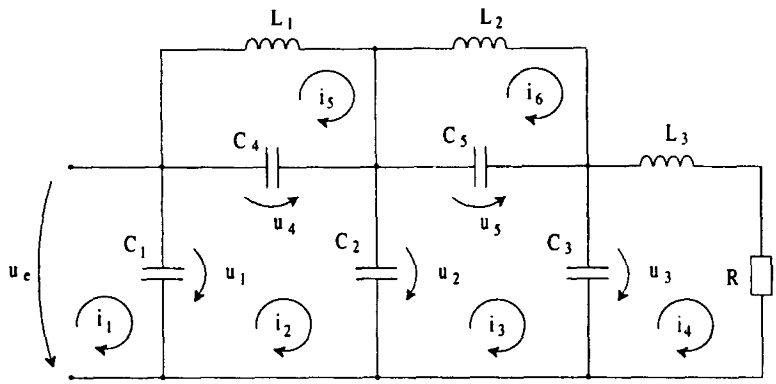

FIG. 1. An electrical example system.

5. EXAMPLE

Consider the electrical network depicted in Figure 1. Let us use a descriptor vector $x = (i_1\ i_5\ i_6\ i_2\ i_3\ i_4\ u_1\ u_2\ u_3\ u_4\ u_5)^T$, where $i_1, i_2, \dots, i_6$ symbolize mesh currents and $u_1, u_2, \dots, u_5$ capacitor voltages as indicated in Figure 1. One possible system description reads

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & C_1 & 0 & 0 & 0 & 0 \\
0 & L_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & L_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_5 \\
0 & 0 & 0 & 0 & 0 & L_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \dot{x}$$

$$= \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -R & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} x$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u_e .$$

Here, the equations are ordered so that no diagonal element of $(sE - A)$ vanishes. The digraph $G[sE - A, B]$ is shown in Figure 2. Full lines mean A-edges, dashed lines mean E-edges. The dotted lines encircle the subgraphs whose vertices are strongly connected.

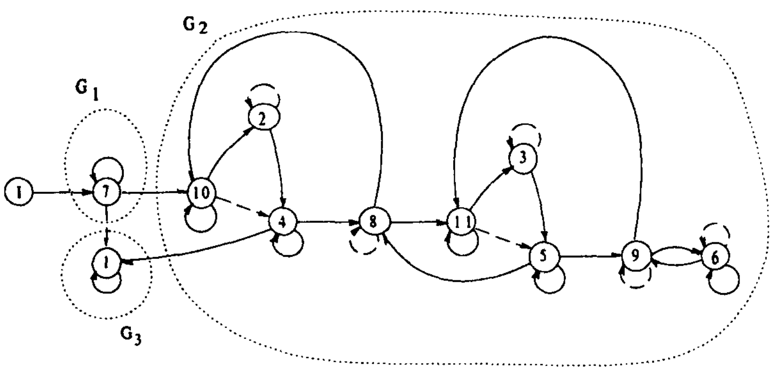

FIG. 2. The digraph $G[sE - A]$.

Let us investigate the different kinds of controllability for the example system. As for structural r-controllability, we apply Theorem 3.1:

Condition (a): There are cycle families of width 11 in

$$G\begin{pmatrix} [A] & [B] \\ [F] & 0 \end{pmatrix},$$

e.g., $2 \to 4 \to 8 \to 10 \to 2$, $3 \to 5 \to 9 \to 11 \to 3$, $6 \to 6$, $7 \to 7$, $1 \to 1$, or $I \to 7 \to 10 \to 2 \to 4 \to 8 \to 11 \to 3 \to 5 \to 9 \to 6 \to I$, $1 \to 1$ (cf. Figure 3).

Condition (b): The strongly connected subgraph G_2, which contains E-edges, is input-connected.

Both conditions are fulfilled, i.e., the example system is structurally r-controllable.

To ensure structural (complete) controllability, condition (a) of Theorem 3.2 must be met additionally. We have to look for a cycle family of width $n = 11$ in $G(\left[\begin{smallmatrix} E \\ F \end{smallmatrix}\right] \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right])$. It is easy to see from Figure 4 that such a cycle family does not exist. Consequently, the example system is not structurally (completely) controllable.

At last, we turn to the structural impulse controllability. For the example system, $t = s - \operatorname{rank}[E] = 8$ holds. According to Theorem 3.3, the question is whether or not a cycle family of width 11 with eight E-edges exists in

$$G\begin{pmatrix} [sE - A] & [B] \\ [F] & 0 \end{pmatrix}.$$

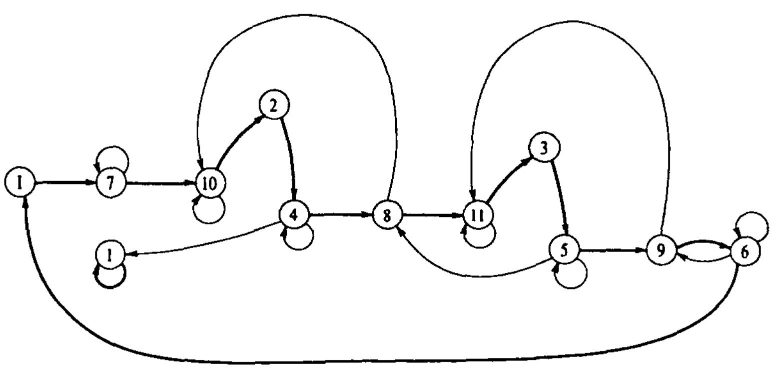

FIG. 3. A cycle family of width 11 in $G(\left[\begin{smallmatrix} A \\ F \end{smallmatrix}\right] \left[\begin{smallmatrix} B \\ 0 \end{smallmatrix}\right])$.

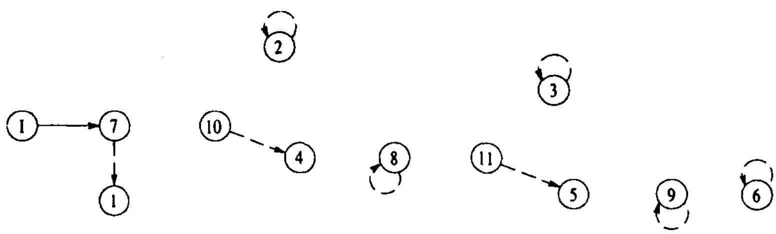

FIG. 4. The digraph $G([E],[B])$.

Obviously, such a cycle family should contain the eight dashed edges in Figure 4. It is impossible to supplement the graph of Figure 4 by A-edges and F-edges such that the augmented graph has width 11. This means that the example system is not structurally impulse-controllable.

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