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On the linearization theorem for nonautonomous differential equations

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This paper presents an improvement of Palmer's linearization theorem in [10]. Palmer's linearization theorem extended the Hartman–Grobman theorem to the nonautonomous case. It requires two essential conditions: (i) the nonlinear term is bounded and Lipschitzian; (ii) the linear system as a whole possesses an exponential dichotomy. The main purpose of this paper is to weaken assumptions (i) and (ii). Also in this paper we prove that the topologically equivalent function $H(t,x)$ in the linearization theorem is always Hölder continuous (and has a Hölder continuous inverse), so as a result we generalize and improve Palmer's linearization theorem.

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1. Introduction and motivation

Linear equations are mathematically well-understood but nonlinear systems are relatively difficult to investigate. For this reason, linearization of differential equations is very important. A basic contribution to the linearization problem for autonomous differential equations is the Hartman–Grobman theorem (see [6] and [7]). Some improvements of the Hartman–Grobman theorem can be found in Lu [9], Pugh [11] and Reinfelds [12]. Palmer successfully generalized the Hartman–Grobman theorem to non-autonomous differential equations (see [10]). To weaken the conditions of Palmer's linearization theorem, some improvements were given in Barreira et al. [1], Fenner and Pinto [5], Jiang [8], Reinfelds and Sermone [13], Sermone [14], Shi and Zhang [15], Xia et al. [16,17,19] and Yuan et al. [18].

Palmer's linearization theorem requires two essential conditions: (i) the nonlinear term f is bounded and Lipschitzian; (ii) the linear system as a whole possesses an exponential dichotomy. The above mentioned works also need the boundedness of f. In this paper we extend Palmer's linearization theorem in two ways: (1) we consider nonlinear terms f which may be unbounded or not Lipschitzian (see Example 2.1); (2) we prove that it is enough to assume that the linear system partially possesses an exponential dichotomy. More precisely, if we divide the linear system into two subsystems

$$\begin{cases} \dot{x_1}(t) = A_1(t)x_1(t), \\ \dot{x_2}(t) = A_2(t)x_2(t), \end{cases}$$

it suffices to assume that one of these subsystems requires an exponential dichotomy.

The Hölder regularity of the topologically equivalent function H was not discussed in Palmer's linearization theorem. In this paper we give a rigorous proof of the Hölder regularity of the topologically equivalent function H under our assumptions. In particular, we show that the topologically equivalent function H is always Hölder continuous (and has a Hölder continuous inverse).

2. Main results

Consider the following two nonautonomous systems

$$\dot{x} = f(t, x), \tag{2.1}$$

and

$$\dot{y} = g(t, y), \tag{2.2}$$

where $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$.

Definition 2.1. Suppose that there exists a function $H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

(i) for each fixed t, $H(t, \cdot)$ is a homeomorphism of $\mathbb{R}^n$ into $\mathbb{R}^n$;

(ii) $|H(t,x) - x|$ is uniformly bounded with respect to t;
(iii) assume that $G(t,\cdot) = H^{-1}(t,\cdot)$ also has property (ii);
(iv) if $x(t)$ is a solution of the system (2.1), then $H(t,x(t))$ is a solution of the system (2.2); and if $y(t)$ is a solution of the system (2.2), then $G(t,y(t))$ is a solution of the system (2.1).

If such a map H_t ($H_t := H(t,\cdot)$) exists, then the system (2.1) is topologically conjugated to the system (2.2) and the transformation $H(t,x)$ is called an equivalent function.

Definition 2.2. The linear system $\dot{x} = A_1(t)x$ is said to possess an exponential dichotomy[3] (Coppel [4]), if there exists a projection P and constants $K > 0$, $\alpha > 0$ such that

$$\begin{cases} \left|U(t)PU^{-1}(s)\right| \leq K\exp\{-\alpha(t-s)\} & (t \geq s), \\ \left|U(t)(I-P)U^{-1}(s)\right| \leq K\exp\{\alpha(t-s)\} & (t \leq s), \end{cases} \tag{2.3}$$

hold; here $U(t)$ is a fundamental matrix of the linear system $\dot{x} = A_1(t)x$.

Consider the system

$$\begin{cases} x_1' = A_1(t)x_1 + f(t,x_1,x_2), \\ x_2' = A_2(t)x_2, \end{cases} \tag{2.4}$$

where $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, $A_1(t)$, $A_2(t)$ are $n\times n$, $m\times m$ continuous and bounded matrices defined on $\mathbb{R}$ respectively. Let $V_i = C_i.\{1-\exp(-\alpha)\}^{-1}$ $(i = 1,2)$, where C_i are positive constants.

Theorem 2.1. *Suppose that the subsystem $x_1' = A_1(t)x_1$ has an exponential dichotomy of the form (2.3) on $\mathbb{R}$, and there exist nonnegative local integrable functions $\mu(t)$, $r(t)$ such that $f(t,x_1,x_2)$ satisfies*

$$\begin{cases} \left|f(t,x_1,x_2)\right| \leq \mu(t), \\ \left|f(t,x_1,x_2) - f(t,\bar{x}_1,\bar{x}_2)\right| \leq r(t)\left[|x_1-\bar{x}_1| + |x_2-\bar{x}_2|\right], \\ 2KV_2 < 1, \end{cases} \tag{2.5}$$

where $\mu(t)$, $r(t)$ satisfy $\int_t^{t+1}\mu(s)ds \leq C_1$ and $\int_t^{t+1} r(s)ds \leq C_2$.

Then system (2.4) is topologically conjugated to its linear system

$$\begin{cases} x_1' = A_1(t)x_1, \\ x_2' = A_2(t)x_2, \end{cases} \tag{2.6}$$

and the equivalent function $H(t,x)$, $x = (x_1,x_2)^T$ satisfies $|H(t,x) - x| \leq 2KV_1$.

[3] This is classical definition of uniform dichotomy in sense of Coppel [4,20]. For the concept of nonuniform hyperbolicity, one can refer to [2,3].

Theorem 2.2. *Suppose that the conditions in Theorem 2.1 are satisfied. Then when $|x-\bar{x}|<1$, there exist positive constants p, q, $\bar{p}$, $\bar{q}$ such that for all t*

$$\left|H(t,x)-H(t,\bar{x})\right| \le p|x-\bar{x}|^q \quad \text{and} \quad \left|G(t,y)-G(t,\bar{y})\right| \le \bar{p}|x-\bar{x}|^{\bar{q}}. \tag{2.7}$$

Namely, the equivalence function $H(t,x)$ is always Hölder continuous (and has a Hölder continuous inverse), where $G_t = H_t^{-1}$.

Remark 2.1. Note when $A_2(t) \equiv 0$, $\mu(t) \equiv \mu$ and $r(t) \equiv r$, Theorem 2.1 reduces to the classical Palmer linearization theorem. We note that Palmer did not give a conclusion on the Hölder continuity of H_t. It should be noted that $f(t,x_1,x_2)$ in our theorem could be unbounded or not Lipschitzian. Note $\mu(t)$, $r(t)$ are locally integrable (satisfying $\int_t^{t+1}\mu(s)ds \le C_1$ and $\int_t^{t+1} r(s)ds \le C_2$), so they could be unbounded.

Example 2.1. We construct a continuous function $f(t,x_1,x_2)$ which is unbounded, not Lipschitzian, but locally integrable. Considering $[0,+\infty)$, for any positive integer m, let

$$\bar{g}(t) = \begin{cases} 0, & t \in [0,1), \\ cm^2t - cm^3, & t \in [m, m+\frac{1}{2m}), \\ -cm^2t + cm^3 + cm, & t \in [m+\frac{1}{2m}, m+\frac{1}{m}), \\ 0, & t \in [m+\frac{1}{m}, m+1), \end{cases}$$

where c is a positive constant. Note $\bar{g}(t)$ is continuous on $[0,+\infty)$. Let

$$g(t) = \begin{cases} \bar{g}(t), & t \ge 0, \\ \bar{g}(-t), & t < 0, \end{cases}$$

so $g(t)$ is continuous on $\mathbb{R}$. Thus,

$$f(t,x_1,x_2) = g(t)\sin kx_1 \cos x_2, \quad k \ne 0 \text{ is a constant},$$

is continuous on $\mathbb{R}\times\mathbb{R}^2$. It is easy to see that for any $(t,x_1,x_2), (t,\tilde{x}_1,\tilde{x}_2) \in \mathbb{R}\times\mathbb{R}^2$,

$$\begin{aligned} \left|f(t,x_1,x_2) - f(t,\tilde{x}_1,\tilde{x}_2)\right| &\le |k|g(t)|x_1-\tilde{x}_1| \\ &\le |k|g(t)\big[|x_1-\tilde{x}_1| + |x_2-\tilde{x}_2|\big], \end{aligned}$$

and

$$\left|f(t,x_1,x_2)\right| \le g(t), \qquad \int_t^{t+1} g(s)ds \le c.$$

However, we see that

$$g\left(m+\frac{1}{2m}\right)\to+\infty,\quad \text{as } m\to\infty,$$

which implies that $g(t)$ is unbounded. Consequently, $f(t,x_1,x_2)$ is not only unbounded, but also $f(t,x_1,x_2)$ is not Lipschitzian. This example shows that in some cases $f(t,x_1,x_2)$ could be unbounded or not Lipschitzian, and still fulfill the conditions of our theorem. In this sense, we generalize and improve Palmer's linearization theorem. In this paper we try to reduce the Lipschitzian condition in Palmer's linearization theorem to locally integrable functions.

3. Proof of main results

In what follows, we always suppose that the conditions of Theorem 2.1 are satisfied. Let $\begin{bmatrix} X_1(t,t_0,x_{10},x_{20}) \\ X_2(t,t_0,x_{10},x_{20}) \end{bmatrix}$ be a solution of the system (2.4) satisfying the initial condition $\begin{bmatrix} X_1(t_0) \\ X_2(t_0) \end{bmatrix}=\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$; $\begin{bmatrix} Y_1(t,t_0,y_{10},y_{20}) \\ Y_2(t,t_0,y_{10},y_{20}) \end{bmatrix}$ is a solution of the system (2.6) satisfying the initial condition $\begin{bmatrix} Y_1(t_0) \\ Y_2(t_0) \end{bmatrix}=\begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}$. To prove the main results, we divide our proof into several lemmas.

Lemma 3.1. *If $\mu(t)$ is a nonnegative local integrable function on $\mathbb{R}$, and there exists a constant $C_1>0$, satisfying $\int_t^{t+1}\mu(s)ds\le C_1$, then we have*

$$\int_{-\infty}^{t}\mu(s)\exp\{-\alpha(t-s)\}ds\le V_1,\qquad \int_{t}^{+\infty}\mu(s)\exp\{\alpha(t-s)\}ds\le V_1. \tag{3.1}$$

Proof. We prove the first inequality (the other one is similar). For each natural number m, we obtain

$$\begin{aligned}\int_{t-(m+1)}^{t-m}\mu(s)\exp\{-\alpha(t-s)\}ds&\le\int_{t-(m+1)}^{t-m}\mu(s)\exp\{-\alpha m\}ds\\&\le C_1\exp\{-\alpha m\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{-\infty}^{t}\mu(s)\exp\{-\alpha(t-s)\}ds&=\sum_{m\in[0,+\infty)}\int_{t-(m+1)}^{t-m}\mu(s)\exp\{-\alpha(t-s)\}ds\\&\le\sum_{m\in[0,+\infty)}C_1.\exp\{-\alpha m\}\\&=C_1.\{1-\exp(-\alpha)\}^{-1}\\&=V_1.\qquad\square\end{aligned}$$

Lemma 3.2. *For each (τ, ξ, η), the system*

$$Z' = A_1(t)Z - f\big(t, X_1(t, \tau, \xi, \eta), X_2(t, \tau, \xi, \eta)\big) \tag{3.2}$$

has a unique bounded solution $h(t, (\tau, \xi, \eta))$ with $|h(t, (\tau, \xi, \eta))| \le 2KV_1$.

Proof. For any fixed (τ, ξ, η), let

$$\begin{aligned} h\big(t, (\tau, \xi)\big) = &-\int_{-\infty}^{t} U(t)PU^{-1}(s)f\big(s, X_1(s, \tau, \xi, \eta), X_2(s, \tau, \xi, \eta)\big)ds \\ &+ \int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)f\big(s, X_1(s, \tau, \xi, \eta), X_2(s, \tau, \xi, \eta)\big)ds. \end{aligned}$$

Differentiating it, it is easy to see that $h(t, (\tau, \xi))$ is a solution of the system (3.2). It follows from (2.3) and (2.5) that

$$\begin{aligned} \big|h\big(t, (\tau, \xi)\big)\big| &\le \int_{-\infty}^{t} K\mu(s)\exp\big\{-\alpha(t-s)\big\}ds + \int_{t}^{+\infty} K\mu(s)\exp\big\{\alpha(t-s)\big\}ds \\ &\le 2KV_1, \end{aligned}$$

which implies that $h(t, (\tau, \xi))$ is a bounded solution of the system (3.2). We claim that the bounded solution is unique. In fact, for any fixed (τ, ξ, η), the system (3.2) is linearly inhomogeneous, and its linear system $Z' = A_1(t)Z$ has an exponential dichotomy. This implies that the bounded solution of (3.2) is unique. □

Lemma 3.3. *For each (τ, ξ, η), the system*

$$Z' = A_1(t)Z + f\big(t, Y_1(t, \tau, \xi, \eta) + Z, Y_2(t, \tau, \xi, \eta)\big) \tag{3.3}$$

has a unique bounded solution $g(t, (\tau, \xi, \eta))$, and $|g(t, (\tau, \xi, \eta))| \le 2KV_1$.

Proof. Let $\mathbf{B}$ be the set of all the continuous bounded functions $Z(t)$ with $|Z(t)| \le 2KV_1$. For each (τ, ξ, η) and any $Z(t) \in \mathbf{B}$, define a mapping $\mathcal{T}$ as follows,

$$\begin{aligned} \mathcal{T}Z(t) = &\int_{-\infty}^{t} U(t)PU^{-1}(s)f\big(s, Y_1(s, \tau, \xi, \eta) + Z(s), Y_2(s, \tau, \xi, \eta)\big)ds \\ &- \int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)f\big(s, Y_1(s, \tau, \xi, \eta) + Z(s), Y_2(s, \tau, \xi, \eta)\big)ds. \end{aligned}$$

A simple computation leads to

$$\|\mathcal{T}Z(t)\| \leq 2KV_1,$$

which implies that $\mathcal{T}\mathbf{B} \subset \mathbf{B}$. For any $Z_1(t), Z_2(t) \in \mathbf{B}$,

$$|\mathcal{T}Z_1(t) - \mathcal{T}Z_2(t)| \leq 2KV_2\|Z_1 - Z_2\|.$$

Now since $2KV_2 < 1$, then $\mathcal{T}$ has a unique fixed point, namely $Z_0(t)$, and

$$\begin{aligned} Z_0(t) = {} & \int_{-\infty}^{t} U(t)PU^{-1}(s)f\big(s, Y_1(s,\tau,\xi,\eta) + Z_0(s), Y_2(s,\tau,\xi,\eta)\big)ds \\ & - \int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)f\big(s, Y_1(s,\tau,\xi,\eta) + Z_0(s), Y_2(s,\tau,\xi,\eta)\big)ds. \end{aligned}$$

It is easy to see that $Z_0(t)$ is a bounded solution of the system (3.3). From an argument similar to that in [15], we see that the bounded solution is unique. We may call the unique solution $g(t,(\tau,\xi,\eta))$. From the above proof, it is easy to see that $|g(t,(\tau,\xi,\eta))| \leq 2KV_1$. □

By differentiation and similar arguments, we have the following lemmas.

Lemma 3.4. *Let $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ be any solution of the system (2.4). Then $Z(t) = 0$ is the unique bounded solution of the system*

$$Z' = A_1(t)Z + f\big(t, x_1(t) + Z, x_2(t)\big) - f\big(t, x_1(t), x_2(t)\big). \tag{3.4}$$

Now we define two functions as follows

$$\begin{bmatrix} H_1(t,x_1,x_2) \\ H_2(t,x_1,x_2) \end{bmatrix} = \begin{bmatrix} x_1 + h(t,(t,x_1,x_2)) \\ x_2 \end{bmatrix}, \tag{3.5}$$

$$\begin{bmatrix} G_1(t,y_1,y_2) \\ G_2(t,y_1,y_2) \end{bmatrix} = \begin{bmatrix} y_1 + g(t,(t,y_1,y_2)) \\ y_2 \end{bmatrix}. \tag{3.6}$$

Lemma 3.5. *For any fixed (t_0, x_{10}, x_{20}), $\begin{bmatrix} H_1(t,X_1(t,t_0,x_{10},x_{20}),X_2(t,t_0,x_{10},x_{20})) \\ H_2(t,X_1(t,t_0,x_{10},x_{20}),X_2(t,t_0,x_{10},x_{20})) \end{bmatrix}$ is a solution of the system (2.6).*

Lemma 3.6. *For any fixed (t_0, y_{10}, y_{20}), $\begin{bmatrix} G_1(t,(t,Y_1(t,t_0,y_{10},y_{20}),Y_2(t,t_0,y_{10},y_{20}))) \\ G_2(t,(t,Y_1(t,t_0,y_{10},y_{20}),Y_2(t,t_0,y_{10},y_{20}))) \end{bmatrix}$ is a solution of the system (3.7).*

Lemma 3.7. *For any $t \in \mathbb{R}$, $y_1 \in \mathbb{R}^n$, $y_2 \in \mathbb{R}^m$, $\begin{bmatrix} H_1(t,G_1(t,y_1,y_2),G_2(t,y_1,y_2)) \\ H_2(t,G_1(t,y_1,y_2),G_2(t,y_1,y_2)) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.*

Proof. Let $y(t)=\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}$ be any solution of the linear system (2.6). From Lemma 3.6, $G(t,y(t))=\begin{bmatrix} G_1(t,y_1(t),y_2(t))\\ G_2(t,y_1(t),y_2(t))\end{bmatrix}$ is a solution of the system (2.4). Then by Lemma 3.5, we see that $H(t,G(t,y(t)))$ is a solution of the system (2.6), written as $\begin{bmatrix} \bar y_1(t)\\ \bar y_2(t)\end{bmatrix}$. Let

$$J(t)=\begin{bmatrix} J_1(t)\\ J_2(t)\end{bmatrix}=\bar y(t)-y(t)=\begin{bmatrix} \bar y_1(t)-y_1(t)\\ \bar y_2(t)-y_2(t)\end{bmatrix}.$$

To prove this conclusion, we need to show that $J(t)\equiv 0$. To this end, it suffices to prove that $J_1(t)\equiv 0$ and $J_2(t)\equiv 0$. Firstly we show that $J_1(t)=\bar y_1(t)-y_1(t)=0$. In fact, differentiating J_1, we have

$$\begin{aligned} J_1'(t)&=\bar y_1'(t)-y_1'(t)\\ &=A_1(t)\bar y_1(t)-A_1(t)y_1(t)\\ &=A_1(t)J_1(t), \end{aligned}$$

which implies that $J_1(t)$ is a solution of the system $Z'=A_1(t)Z$. From Lemma 3.2 and Lemma 3.3, it follows that

$$\begin{aligned} \left|J_1(t)\right|&=\left|\bar y_1(t)-y_1(t)\right|\\ &=\left|H_1\big(t,G_1\big(t,\big(t,y_1(t),y_2(t)\big)\big),G_2\big(t,\big(t,y_1(t),y_2(t)\big)\big)\big)-y_1(t)\right|\\ &\le \left|H_1\big(t,G_1\big(t,\big(t,y_1(t),y_2(t)\big)\big),G_2\big(t,\big(t,y_1(t),y_2(t)\big)\big)\big)-G_1\big(t,\big(t,y_1(t),y_2(t)\big)\big)\right|\\ &\quad+\left|G_1\big(t,\big(t,y_1(t),y_2(t)\big)\big)-y_1(t)\right|\\ &\le 2KV_1+2KV_1\\ &=4KV_1. \end{aligned}$$

This implies that $J_1(t)$ is a bounded solution of the system $Z'=A_1(t)Z$. However, the linear system $Z'=A_1(t)Z$ has no nontrivial bounded solution. Hence $J_1(t)\equiv 0$, that is $\bar y_1(t)=y_1(t)$.

Now we show that $J_2(t)\equiv 0$. In fact, by the definition of H and the second equality of (3.5), we see that

$$\bar y_2(t)=H_2\big(t,G_1\big(t,y_1(t),y_2(t)\big),G_2\big(t,y_1(t),y_2(t)\big)\big)=G_2\big(t,y_1(t),y_2(t)\big)=y_2(t).$$

Thus, $J(t)\equiv 0$, that is,

$$\bar y(t)=\begin{bmatrix} \bar y_1(t)\\ \bar y_2(t)\end{bmatrix}=\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix}=y(t),\quad \text{or}\quad H\big(t,G\big(t,y(t)\big)\big)\equiv y(t).$$

Since $y(t)$ is an arbitrary solution of the linear system (2.6), the proof of Lemma 3.7 is complete. □

Lemma 3.8. *For any* $t \in \mathbb{R}$, $x_1 \in \mathbb{R}^n$, $x_2 \in \mathbb{R}^m$, *we have*

$$\begin{bmatrix} G_1(t, H_1(t, x_1, x_2), H_2(t, x_1, x_2)) \\ G_2(t, H_1(t, x_1, x_2), H_2(t, x_1, x_2)) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Proof. The proof is similar to that in Lemma 3.7. □

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1. Now we show that $H(t, \cdot)$ satisfies the four conditions of Definition 2.1.

For any fixed t, it follows from Lemmas 3.7 and 3.8 that $H(t, \cdot)$ is homeomorphism and $G(t, \cdot) = H^{-1}(t, \cdot)$. Thus, *Condition (i) is satisfied.*

From (2.5) and Lemma 3.2, we derive $|H(t, x) - x| = |h(t, (t, x))| \leq 2KV_1$. Note $|H(t, x)| \to \infty$ as $|x| \to \infty$, uniformly with respect to t. Thus, *Condition (ii) is satisfied.*

From (2.6) and Lemma 3.3, we derive $|G(t, y) - y| = |g(t, (t, y))| \leq 2KV_1$. Note $|G(t, y)| \to \infty$ as $|x| \to \infty$, uniformly with respect to t. Thus, *Condition (iii) is satisfied.*

From Lemmas 3.5 and 3.6, we know that *Condition (iv) is true.*

Hence, the system (2.4) and its linear system (2.6) are topologically conjugated. This completes the proof of Theorem 2.1. □

Lemma 3.9. *Suppose that* $\sup_{t\in\mathbb{R}}|A_1(t)| + \sup_{t\in\mathbb{R}}|A_2(t)| = M$. *Then we have*

$$\begin{aligned} &\big|X_1(t, t_0, x_{10}, x_{20}) - X_1(t, t_0, \bar{x}_{10}, \bar{x}_{20})\big| + \big|X_2(t, t_0, x_{10}, x_{20}) - X_2(t, t_0, \bar{x}_{10}, \bar{x}_{20})\big| \\ &\quad \leq \exp\{C_2\}\big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \exp\big\{(M + C_2)|t - t_0|\big\}, \end{aligned}$$

$$\big|Y_1(t, t_0, y_{10}, y_{20}) - Y_1(t, t_0, \bar{y}_{10}, \bar{y}_{20})\big| \leq |y_{10} - \bar{y}_{10}| \exp\big\{M|t - t_0|\big\},$$
$$\big|Y_2(t, t_0, y_{10}, y_{20}) - Y_2(t, t_0, \bar{y}_{10}, \bar{y}_{20})\big| \leq |y_{20} - \bar{y}_{20}| \exp\big\{M|t - t_0|\big\}.$$

Proof. In view of $\int_t^{t+1} r(s)ds \leq C_2$,

$$\begin{aligned} \int_{t_0}^{t} r(s)ds &= \int_{t_0}^{t_0+1} r(s)ds + \int_{t_0+1}^{t_0+2} r(s)ds + \int_{t_0+2}^{t_0+3} r(s)ds + \ldots + \int_{t-1}^{t} r(s)ds \\ &\leq \big([t - t_0] + 1\big) \cdot C_2, \end{aligned}$$

where $[t - t_0]$ means taking the largest integer not greater than $t - t_0$. By the variation formula, we get

$$\begin{aligned} \begin{bmatrix} X_1(t, t_0, \xi, \eta) \\ X_2(t, t_0, \xi, \eta) \end{bmatrix} = \begin{bmatrix} \xi \\ \eta \end{bmatrix} \\ + \int_{t_0}^{t} \begin{bmatrix} A_1(s)X_1(s, t_0, \xi, \eta) + f(s, X_1(s, t_0, \xi, \eta), X_2(s, t_0, \xi, \eta)) \\ A_2(s)X_2(s, t_0, \xi, \eta) \end{bmatrix} ds. \end{aligned}$$

For any initial conditions $(x_{10}, x_{20})^T, (\bar{x}_{10}, \bar{x}_{20})^T \in \mathbb{R}^{n+m}$, the above equality leads to

$$\begin{aligned}
&\left|X_1(t,t_0,x_{10},x_{20}) - X_1(t,t_0,\bar{x}_{10},\bar{x}_{20})\right| + \left|X_2(t,t_0,x_{10},x_{20}) - X_2(t,t_0,\bar{x}_{10},\bar{x}_{20})\right| \\
&\quad \le |x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}| \\
&\qquad + \left| \int_{t_0}^{t} \big(M + r(s)\big)\big(\left|X_1(t,t_0,x_{10},x_{20}) - X_1(t,t_0,\bar{x}_{10},\bar{x}_{20})\right| \right. \\
&\qquad \left. + \left|X_2(t,t_0,x_{10},x_{20}) - X_2(t,t_0,\bar{x}_{10},\bar{x}_{20})\right|\big) ds \right|.
\end{aligned}$$

Using the Bellman inequality and Lemma 3.1, we obtain

$$\begin{aligned}
&\left|X_1(t,t_0,x_{10},x_{20}) - X_1(t,t_0,\bar{x}_{10},\bar{x}_{20})\right| + \left|X_2(t,t_0,x_{10},x_{20}) - X_2(t,t_0,\bar{x}_{10},\bar{x}_{20})\right| \\
&\quad \le \big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \exp\left\{ \int_{t_0}^{t} \big(M + r(s)\big) ds \right\} \\
&\quad = \big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \left[\exp\left\{ \int_{t_0}^{t} M ds \right\} . \exp\left\{ \int_{t_0}^{t} r(s) ds \right\} \right] \\
&\quad \le \big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \exp\left\{ \int_{t_0}^{t} \left[M|t - t_0| + \big([t - t_0] + 1\big) \cdot C_2\right] \right\} \\
&\quad \le \big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \exp\left\{ \int_{t_0}^{t} \big(M|t - t_0| + \big(|t - t_0| + 1\big) \cdot C_2\big) \right\} \\
&\quad \le \exp\{C_2\}\big[|x_{10} - \bar{x}_{10}| + |x_{20} - \bar{x}_{20}|\big] \exp\big\{(M + C_2)|t - t_0|\big\}.
\end{aligned}$$

The proof of the other inequality is similar, so we omit it. □

Now we are in a position to prove Theorem 2.2.

Proof of Theorem 2.2.

Step 1. *We show that there exist constants $p, q > 0$ such that if $|x - \bar{x}| < 1$, we have $|H(t,x) - H(t,\bar{x})| \le p|x - \bar{x}|^q$.*

From Lemma 3.2, it follows that

$$\begin{aligned}
h\big(t, (t, \xi, \eta)\big) = &-\int_{-\infty}^{t} U(t)PU^{-1}(s) f\big(s, X_1(s,\tau,\xi,\eta), X_2(s,\tau,\xi,\eta)\big) ds \\
&+ \int_{t}^{+\infty} U(t)(I - P)U^{-1}(s) f\big(s, X_1(s,\tau,\xi,\eta), X_2(s,\tau,\xi,\eta)\big) ds.
\end{aligned}$$

Thus we get

$$h\big(t,(t,\xi,\eta)\big)-h\big(t,(t,\bar{\xi},\bar{\eta})\big)=-\int_{-\infty}^{t}U(t)PU^{-1}(s)\big[f\big(s,X_1(s,t,\xi,\eta),X_2(s,t,\xi,\eta)\big)$$
$$-f\big(s,X_1(s,t,\bar{\xi},\bar{\eta}),X_2(s,t,\bar{\xi},\bar{\eta})\big)\big]ds$$
$$+\int_{t}^{+\infty}U(t)(I-P)U^{-1}(s)\big[f\big(s,X_1(s,t,\xi,\eta),X_2(s,t,\xi,\eta)\big)$$
$$-f\big(s,X_1(s,t,\bar{\xi},\bar{\eta}),X_2(s,t,\bar{\xi},\bar{\eta})\big)\big]ds.$$

We suppose that $0<|\xi-\bar{\xi}|+|\eta-\bar{\eta}|<1$, and $\tau=\frac{1}{M+C_1}\ln\frac{1}{|\xi-\bar{\xi}|+|\eta-\bar{\eta}|}$.

Now divide I_1, I_2 into two parts:

$$I_1=\int_{-\infty}^{t-\tau}+\int_{t-\tau}^{t}\triangleq I_{11}+I_{12},$$
$$I_2=\int_{t}^{t+\tau}+\int_{t+\tau}^{+\infty}\triangleq I_{21}+I_{22}.$$

Then, using (2.3), (2.5) and Lemma 3.1, we have

$$|I_{11}|\le\int_{-\infty}^{t-\tau}K\exp\big\{-\alpha(t-s)\big\}2\mu(s)ds$$
$$=\sum_{m\in[0,+\infty)}2K\int_{t-\tau-(m+1)}^{t-\tau-m}\mu(s)\exp\big\{-\alpha(t-s)\big\}ds$$
$$\le\sum_{m\in[0,+\infty)}2K\int_{t-\tau-(m+1)}^{t-\tau-m}\mu(s)\exp\big\{-\alpha(t-t+\tau+m)\big\}ds$$
$$\le\sum_{m\in[0,+\infty)}2KC_1\exp\big\{-\alpha(\tau+m)\big\}$$
$$=2KC_1\exp\{-\alpha\tau\}\big(1-\exp\{-\alpha\}\big)^{-1}$$
$$=2KC_1\big(1-\exp\{-\alpha\}\big)^{-1}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}},$$

and

$$|I_{22}|\le\int_{t+\tau}^{+\infty}K\exp\big\{\alpha(t-s)\big\}2\mu(s)ds$$

$$
\begin{aligned}
&= \sum_{m\in[0,+\infty)} 2K \int_{t+\tau+m}^{t+\tau+m+1} \mu(s)\exp\{\alpha(t-s)\}ds \\
&\le \sum_{m\in[0,+\infty)} 2K \int_{t+\tau+m}^{t+\tau+m+1} \mu(s)\exp\{-\alpha(\tau+m)\}ds \\
&= 2KC_1\exp\{-\alpha\tau\}\big(1-\exp\{-\alpha\}\big)^{-1} \\
&= 2KC_1\big(1-\exp\{-\alpha\}\big)^{-1}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}}.
\end{aligned}
$$

Furthermore, it follows from (2.3), (2.5), and Lemma 3.9 that

$$
\begin{aligned}
|I_{12}| &\le \int_{t-\tau}^{t} K\exp\{-\alpha(t-s)\}r(s)\big[\big|X_1(s,t,\xi,\eta)-X_1(s,t,\bar{\xi},\bar{\eta})\big| \\
&\quad + \big|X_2(s,t,\xi,\eta)-X_2(s,t,\bar{\xi},\bar{\eta})\big|\big]ds \\
&\le \int_{t-\tau}^{t} K\exp\{-\alpha(t-s)\}r(s)\exp\{C_2\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\exp\big\{(M+C_2)|t-s|\big\}ds \\
&\le \int_{t-\tau}^{t} K\exp\big\{(-\alpha+M+C_2)(t-s)\big\}r(s)\exp\{C_2\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le \sum_{m\in[0,[\tau]]} K\exp\{C_2\} \int_{t-\tau+m}^{t-\tau+m+1} \exp\big\{(M+C_2-\alpha)(t-s)\big\}r(s)\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le \sum_{m\in[0,[\tau]]} C_2K\exp\{C_2\}\exp\big\{(M+C_2-\alpha)(\tau-m)\big\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le \sum_{m\in[0,\tau]} C_2K\exp\{C_2\}\exp\big\{(M+C_2-\alpha)(\tau-m)\big\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le C_2K\exp\{C_2\}\exp\big\{(M+C_2-\alpha)\tau\big\} \\
&\quad \times \frac{1-\exp\{(M+C_2-\alpha)(-\tau)\}}{1-\exp\{-M+C_2-\alpha\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big] \\
&\le C_2K\exp\{C_2\}\exp\big\{(M+C_2-\alpha)\tau\big\}\frac{1}{1-\exp\{-(M+C_2-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big] \\
&\le \frac{C_2K\exp\{C_2\}}{1-\exp\{-(M+C_2-\alpha)\}} \\
&\quad \times \exp\bigg\{(M+C_2-\alpha).\frac{1}{M+C_1}\ln\frac{1}{|\xi-\bar{\xi}|+|\eta-\bar{\eta}|}\bigg\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big] \\
&= \frac{C_2K\exp\{C_2\}}{1-\exp\{-(M+C_2-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{21}| &\le \int_t^{t+\tau} K\exp\{\alpha(t-s)\}r(s)\big[\big|X_1(s,t,\xi,\eta)-X_1(s,t,\bar{\xi},\bar{\eta})\big| \\
&\quad + \big|X_2(s,t,\xi,\eta)-X_2(s,t,\bar{\xi},\bar{\eta})\big|\big]ds \\
&\le \int_t^{t+\tau} K\exp\{\alpha(t-s)\}r(s)\exp\{C_2\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\exp\{(M+C_2)|t-s|\}ds \\
&\le \int_t^{t+\tau} K\exp\{\alpha(t-s)\}r(s)\exp\{C_2\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\exp\{(M+C_2)(s-t)\}ds \\
&\le \sum_{m\in[0,[\tau]]} K\exp\{C_2\}\int_{t+\tau-m-1}^{t+\tau-m} \exp\{(\alpha-M-C_2)(t-s)\}.r(s)\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le \sum_{m\in[0,[\tau]]} C_2K\exp\{C_2\}\exp\{(\alpha-M-C_2)(m-\tau)\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds \\
&\le C_2K\exp\{C_2\}\exp\{(M+C_2-\alpha)\tau\}\frac{1}{1-\exp\{-(M+C_2-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big] \\
&= \frac{C_2K\exp\{C_2\}}{1-\exp\{-(M+C_2-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}}.
\end{aligned}
$$

By the definition of $H(t,x)$, if $|x-\bar{x}|<1$,

$$
\begin{aligned}
\big|H(t,x)-H(t,\bar{x})\big| &\le |x-\bar{x}| + \left[\frac{4KC_1}{1-\exp\{-\alpha\}} + \frac{2C_2K\exp\{C_2\}}{1-\exp\{-(M+C_2-\alpha)\}}\right]|x-\bar{x}|^{\frac{\alpha}{M+C_1}} \\
&\le \left(1+\frac{4KC_1}{1-\exp\{-\alpha\}} + \frac{2C_2K\exp\{C_2\}}{1-\exp\{-(M+C_2-\alpha)\}}\right)|x-\bar{x}|^{\frac{\alpha}{M+C_1}} \\
&\equiv p|x-\bar{x}|^q.
\end{aligned}
$$

This completes the proof of Step 1.

Step 2. *There exist constants $\bar{p},\bar{q}>0$ such that if $|y-\bar{y}|<1$, we have $|G(t,y)-G(t,\bar{y})|\le \bar{p}|y-\bar{y}|^{\bar{q}}$.*

Proof. From Lemma 3.3, we know that $g(t,(\tau,\xi))$ is a fixed point of the map $\mathcal{T}$:

$$
\begin{aligned}
\mathcal{T}Z(t) &= \int_{-\infty}^{t} U(t)PU^{-1}(s)f\big(s,Y_1(s,\tau,\xi,\eta)+Z(s),Y_2(s,\tau,\xi,\eta)\big)ds \\
&\quad - \int_t^{+\infty} U(t)(I-P)U^{-1}(s)f\big(s,Y_1(s,\tau,\xi,\eta)+Z(s),Y_2(s,\tau,\xi,\eta)\big)ds.
\end{aligned}
$$

Let $g_0(t,(\tau,\xi,\eta)) \equiv 0$ and recursively define

$$\begin{aligned} g_{m+1}\big(t,(\tau,\xi,\eta)\big) = & \int_{-\infty}^{t} U(t)PU^{-1}(s)f\big(s, Y_1(s,t,\xi,\eta) + g_m\big(t,(\tau,\xi,\eta)\big), Y_2(s,t,\xi,\eta)\big)ds \\ & - \int_{t}^{+\infty} U(t)(I-P)U^{-1}f\big(s, Y_1(s,t,\xi,\eta) \\ & + g_m\big(t,(\tau,\xi,\eta)\big), Y_2(s,t,\bar{\xi},\bar{\eta})\big)ds. \end{aligned} \tag{3.7}$$

It is not hard to conclude that

$$g_m\big(t,(\tau,\xi,\eta)\big) \to g\big(t,(\tau,\xi,\eta)\big), \quad \text{as } m \to +\infty, \tag{3.8}$$

uniformly with respect to τ, ξ, η.

Note that

$$\begin{aligned} Y_1(t,\tau,\xi,\eta) &= Y_1\big(t,t,Y_1(t,\tau,\xi,\eta),Y_2(t,\tau,\xi,\eta)\big), \\ Y_2(t,\tau,\xi,\eta) &= Y_2\big(t,t,Y_1(t,\tau,\xi,\eta),Y_2(t,\tau,\xi,\eta)\big), \\ g_0\big(t,(\tau,\xi,\eta)\big) &= g_0\big(t,\big(t,Y_1(t,\tau,\xi,\eta)\big),Y_2(t,\tau,\xi,\eta)\big) \equiv 0. \end{aligned}$$

Therefore, by induction we easily see that for all m, we have

$$g_m\big(t,(\tau,\xi,\eta)\big) = g_m\big(t,\big(t,Y_1(t,\tau,\xi,\eta)\big),Y_2(t,\tau,\xi,\eta)\big). \tag{3.9}$$

Choose $\lambda > 0$ sufficiently large and $\bar{q} > 0$ sufficiently small such that

$$\begin{aligned} & \lambda > \frac{3}{1-\exp\{-\alpha\}} + \frac{3}{2(1-\exp\{\alpha - M\})} \\ & \bar{q} < \frac{\alpha}{M + C_1} \\ & 0 < \frac{2KC_2}{1-\exp\{-(\alpha - M\bar{q})\}} < \frac{1}{3}. \end{aligned} \tag{3.10}$$

Now we show that if $0 < |\xi - \bar{\xi}| + |\eta - \bar{\eta}| < 1$ for all n, we have

$$\big|g_n\big(t,(t,\xi,\eta)\big) - g_n\big(t,(t,\bar{\xi},\bar{\eta})\big)\big| < \lambda\big[|\xi - \bar{\xi}| + |\eta - \bar{\eta}|\big]^{\bar{q}}. \tag{3.11}$$

Note inequality (3.11) holds when $n = 0$. Now assume inequality (3.11) holds when $n = m$. We now show that inequality (3.11) holds when $n = m + 1$. Note

$$
\begin{aligned}
&g_{m+1}\big(t,(t,\xi,\eta)\big)-g_{m+1}\big(t,(t,\bar{\xi},\bar{\eta})\big)\\
&\quad=\int_{-\infty}^{t} U(t)PU^{-1}(s)\big[f\big(s,Y_1(s,t,\xi,\eta)+g_m\big(t,(\tau,\xi,\eta)\big),Y_2(s,t,\xi,\eta)\big)\\
&\qquad-f\big(s,Y_1(s,t,\bar{\xi},\bar{\eta})+g_m\big(s,(t,\xi,\eta)\big),Y_2(s,t,\bar{\xi},\bar{\eta})\big)\big]ds\\
&\qquad-\int_{t}^{+\infty} U(t)(I-P)U^{-1}(s)\big[f\big(s,Y_1(s,t,\xi,\eta)+g_m\big(t,(\tau,\xi,\eta)\big),Y_2(s,t,\xi,\eta)\big)\\
&\qquad-f\big(s,Y_1(s,t,\bar{\xi},\bar{\eta})+g_m\big(s,(t,\xi,\eta)\big),Y_2(s,t,\bar{\xi},\bar{\eta})\big)\big]ds\\
&\quad\triangleq J_1+J_2.
\end{aligned}
$$

Suppose that $0<|\xi-\bar{\xi}|+|\eta-\bar{\eta}|<1$ and $\tau=\frac{1}{M+C_1}\ln\frac{1}{|\xi-\bar{\xi}|+|\eta-\bar{\eta}|}$.

Now divide J_1, J_2 into two parts:

$$
J_1=\int_{-\infty}^{t-\tau}+\int_{t-\tau}^{t}\triangleq J_{11}+J_{12},
$$

$$
J_2=\int_{t}^{t+\tau}+\int_{t+\tau}^{+\infty}\triangleq J_{21}+J_{22}.
$$

Then using (2.3) and (2.5), we deduce that

$$
\begin{aligned}
|J_{11}|&\le\int_{-\infty}^{t-\tau} K\exp\{-\alpha(t-s)\}2\mu(s)ds\\
&\le\frac{2KC_1}{1-\exp\{-\alpha\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}},\\
|J_{22}|&\le\int_{t+\tau}^{+\infty} K\exp\{-\alpha(t-s)\}2\mu(s)ds\\
&\le\frac{2KC_1}{1-\exp\{-\alpha\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{\alpha}{M+C_1}},
\end{aligned}
$$

when $0<|\xi-\bar{\xi}|+|\eta-\bar{\eta}|<1$, $s\in[t-\tau,t]$, and from Lemma 3.9, we get

$$
\begin{aligned}
&\big|Y_1(s,t,\xi,\eta)-Y_1(s,t,\bar{\xi},\bar{\eta})\big|+\big|Y_2(s,t,\xi,\eta)-Y_2(s,t,\bar{\xi},\bar{\eta})\big|\\
&\quad\le\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\exp\{M|t-s|\}\\
&\quad\le\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\exp\{M\tau\}
\end{aligned}
$$

$$\begin{aligned}&\le \big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\frac{M}{M+C_1}}\\&<1.\end{aligned}$$

Furthermore, it follows from (2.3), (2.5), and Lemma 3.9, that

$$\begin{aligned}&\big|g_m\big(s,(t,\xi,\eta)\big)-g_m\big(s,(t,\bar{\xi},\bar{\eta})\big)\big|\\&\quad=\big|g_m\big(s,\big(s,Y_1(s,t,\xi,\eta),Y_2(s,t,\xi,\eta)\big)\big)-g_m\big(s,\big(s,Y_1(s,t,\bar{\xi},\bar{\eta}),Y_2(s,t,\bar{\xi},\bar{\eta})\big)\big)\big|\\&\quad\le\lambda\big[\big|Y_1(s,t,\xi,\eta)-Y_1(s,t,\bar{\xi},\bar{\eta})\big|+\big|Y_2(s,t,\xi,\eta)-Y_2(s,t,\bar{\xi},\bar{\eta})\big|\big]^{\bar{q}}\\&\quad\le\lambda\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}\exp\big\{M\bar{q}|t-s|\big\},\end{aligned}$$

and

$$\begin{aligned}|J_{12}|&\le\int_{t-\tau}^{t}K\exp\big\{-\alpha(t-s)\big\}r(s)\big[\big(|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big)\\&\qquad\cdot\exp\big\{M(t-s)\big\}+\lambda\big(|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big)^{\bar{q}}.\exp\big\{M\bar{q}(t-s)\big\}\big]ds\\&=\int_{t-\tau}^{t}K\exp\big\{(M-\alpha)(t-s)\big\}r(s)\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds\\&\qquad+\int_{t-\tau}^{t}\lambda K\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.r(s).\exp\big\{(M\bar{q}-\alpha)(t-s)\big\}ds\\&=\sum_{m\in[0,[\tau]]}K\int_{t-\tau+m}^{t-\tau+m+1}\exp\big\{(M-\alpha)(t-s)\big\}r(s)\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]ds\\&\qquad+\sum_{m\in[0,[\tau]]}K\lambda\int_{t-\tau+m}^{t-\tau+m+1}\exp\big\{(M\bar{q}-\alpha)(t-s)\big\}r(s)\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}ds\\&\le\sum_{m\in[0,[\tau]]}KC_2\exp\big\{(M-\alpha)(\tau-m)\big\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\\&\qquad+\sum_{m\in[0,[\tau]]}C_2K\lambda\exp\big\{(M\bar{q}-\alpha)\tau\big\}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}\exp\big\{(M\bar{q}-\alpha)(-m-1)\big\}\\&\le C_2K\exp\big\{(M-\alpha)\tau\big\}\frac{1}{1-\exp\{-(M-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]\\&\qquad+KC_2\lambda\exp\big\{(M\bar{q}-\alpha)\tau\big\}\frac{\exp\{\alpha-M\bar{q}\}}{1-\exp\{\alpha-M\bar{q}\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}\\&\le C_2K\exp\bigg\{(M-\alpha)\frac{-1}{M+C_1}\ln\big(|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big)\bigg\}\end{aligned}$$

$$\times \frac{1}{1-\exp\{-(M-\alpha)\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]$$
$$+ KC_2\lambda \exp\big\{(M\bar{q}-\alpha)\tau\big\}\frac{\exp\{\alpha-M\bar{q}\}}{1-\exp\{\alpha-M\bar{q}\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}$$
$$= C_2K\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}]^{\frac{\alpha+C_1}{M+C_1}}\cdot\frac{1}{1-\exp\{-(M-\alpha)\}}$$
$$+ KC_2\lambda\frac{1}{1-\exp\{-(\alpha-M\bar{q})\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.$$

Note that $M-\alpha>0$, $-\alpha+M\bar{q}<0$ imply that $\exp\{(M\bar{q}-\alpha)\tau\}<1$ and $\bar{q}<\frac{\alpha+C_1}{M+C_1}$. Then

$$|J_{12}| \le KC_2\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}\cdot\frac{1}{1-\exp\{-(M-\alpha)\}}$$
$$+ KC_2\lambda\frac{1}{1-\exp\{-(\alpha-M\bar{q})\}}\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}$$
$$= KC_2\left[\frac{1}{1-\exp\{\alpha-M\}}+\frac{\lambda}{1-\exp\{-(\alpha-M\bar{q})\}}\right]\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.$$

Similar arguments lead to

$$|J_{21}| \le KC_2\left[\frac{1}{1-\exp\{\alpha-M\}}+\frac{\lambda}{1-\exp\{-(\alpha-M\bar{q})\}}\right]\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.$$

Therefore, it follows from (3.10) that

$$\begin{aligned}
&\big|g_{m+1}\big(t,(t,\xi,\eta)\big)-g_{m+1}\big(t,(\tau,\bar{\xi},\bar{\eta})\big)\big| \\
&\quad \le |J_{11}|+|J_{12}|+|J_{21}|+|J_{22}| \\
&\quad \le \left[\frac{4KC_1}{1-\exp\{-\alpha\}}+\frac{2KC_2}{1-\exp\{\alpha-M\}}+\frac{2KC_2\lambda}{1-\exp\{-(\alpha-M\bar{q})\}}\right] \\
&\qquad \times \big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}} \\
&\quad \le \lambda\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.
\end{aligned}$$

Thus inequality (3.11) holds for all n. Letting $n\to+\infty$, we have

$$\big|g\big(t,(t,\xi,\eta)\big)-g\big(t,(t,\bar{\xi},\bar{\eta})\big)\big| \le \lambda\big[|\xi-\bar{\xi}|+|\eta-\bar{\eta}|\big]^{\bar{q}}.$$

Thus, according to the definition of $G(t,x)$, if $0<|y-\bar{y}|<1$, then we have

$$\big|G(t,y)-G(t,\bar{y})\big| \le |y-\bar{y}|+\lambda|y-\bar{y}|^{\bar{q}} \le (1+\lambda)|y-\bar{y}|^{\bar{q}}. \tag{3.12}$$

This completes the proof of Theorem 2.2. □

Conflict of interest statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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