International Journal of Control

ISSN: 0020-7179 (Print) 1366-5820 (Online) Journal homepage: http://www.tandfonline.com/loi/tcon20

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INTERNATIONAL JOURNAL OF CONTROL, 2018
https://doi.org/10.1080/00207179.2018.1436773

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ABSTRACT

The pole assignment problem for descriptor systems is a classical inverse algebraic eigenvalue problem, which has attracted attention for decades. In this paper, we propose a direct method to solve the problem with the application of the proportional plus derivative state feedback, intending to obtain a robust closed-loop system. Theorems on the feasibility of our method will be presented. Numerical examples show that our method yields poles of high relative accuracy.

ARTICLE HISTORY

Received 15 August 2017

Accepted 23 January 2018

KEYWORDS

Descriptor system; robust pole assignment; proportional plus derivative state feedback; generalised real Schur form

1. Introduction

Consider the linear time-invariant dynamical system

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and the input, respectively. The system is referred to as a descriptor system and has found a variety of applications, such as chemical processes and electrical network control (Dai, 1989; Kumar & Daoutidis, 1999). With a generally singular E, Equation (1) is an algebraic-differential equation which attracts much recent interest. Introduced in Luenberger (1977), studies of Equation (1) include several meaningful mathematical problems, motivated intrinsically from the associated engineering design, such as its controllability, regularisation, pole assignment (PA), and so on. Please refer to Bunse-Gerstner, Mehrmann, and Nichols (1992), Byers, Geerts, and Mehrmann (1997), Chaabane, Bachelier, Souissi, and Mehdi (2006), Duan (1998), Duan and Patton (1997), Duan and Patton (1998), Duan and Patton (1999), Hou (2004), Kautsky, Nichols, and Chu (1989), Kumar and Daoutidis (1999), Lewis (1986), Li and Chu (2008), Luenberger (1977), Miminis (1993), Ren and Zhang (2013), Syrmos and Lewis (1992), Varga (2003), Yip and Sincovec (1981), Zhang (2013), Zhang (2011) and the references therein for more information.

If the infinity index $\mathrm{ind}_\infty(A, E)$, the maximal size of the Jordan blocks in the Weierstrass canonical form of the matrix pair (A, E), is 0 or 1 and (A, E) is regular, the algebraic part (the associated redundant variables) in Equation (1) can be eliminated, resulting in a standard linear system (with a nonsingular E) of reduced order. In contrary, systems with $\mathrm{ind}_\infty(A, E) > 1$ might lose causality for some insufficiently smooth inputs. So one hopes to obtain a regular closed-loop system with an infinity index 0 or 1 after applying feedback. Fortunately, Bunse-Gerstner et al. (1992) tells that, if (E, A, B) is strongly controllable (S-controllable), a proportional plus derivative state feedback (PD-SF) exists for such a design goal.

Regarding the PA problem, which is of some importance for system design, the dynamical behaviour of Equation (1) fundamentally depends on the eigen-structure of (A, E), especially the eigenvalues (Bunse-Gerstner et al., 1992). If only the state is available, the proportional state feedback will be adopted (Kautsky et al., 1989); if the derivative of the state can be measured, we may apply the derivative state feedback (Varga, 2003). When both are procurable, a PD-SF is employed (Duan & Patton, 1999). All these state feedback designs are applicable for output feedback.

It is worthwhile to point out that a state feedback involving derivatives has some advantages over the one without. More specifically, by modifying E to $E + BG$ for some $G \in \mathbb{R}^{m\times n}$, we could regularise the closed-loop descriptor system, assigning $\mathrm{rank}([E\ B])$ finite poles meanwhile shifting some infinite ones. Consequently, Equation (1) may be converted into a standard one of reduced order, under certain conditions, eliminating the algebraic part.

For the multi-input system (i.e. $m > 1$), many different PD-SFs lead to regular closed-loop systems, which has an infinity index no higher than 1 and the finite eigenvalues are r specified complex numbers (closed under complex conjugation), with $\mathrm{rank}(E) \le r \le \mathrm{rank}([E\ B])$. In applications, one may prefer a PD-SF which produces a robust closed-loop system. Applying the regularisation results in Bunse-Gerstner et al. (1992), we will focus on the *robust PA problem via the proportional plus derivative state feedback (RPA-PDSF)*, stated as follows:

RPA-PDSF: For given $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ with (E, A, B) S-controllable, and an arbitrarily set $\mathfrak{L} = \{(\alpha_1, 0), \ldots, (\alpha_{n-r}, 0); (\alpha_{n-r+1}, \beta_{n-r+1}), \ldots, (\alpha_n, \beta_n)\}$, closed under complex conjugation, where $\beta_j \neq 0$ for $j = n - r + 1, \ldots, n$, with $q - m \le r \le q$, $q \triangleq \mathrm{rank}([E\ B])$, find a pair of matrices $G, F \in \mathbb{R}^{m\times n}$, such that $(A + BF, E + BG)$ is regular, $\mathrm{ind}_\infty(A +$

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$BF, E + BG) \leq 1$, the spectrum $\lambda(A + BF, E + BG) = \mathfrak{L}$ and the obtained closed-loop system is robust, that is, the eigenvalues of $(A + BF, E + BG)$ are as insensitive to perturbations on $(A + BF, E + BG)$ as possible.

Here we represent an eigenvalue $\lambda_j = \alpha_j/\beta_j$ by the ordered-pair (α_j, β_j), eliminating the distinction between finite and infinite eigenvalues. Note that (α_j, β_j) is a representative of an equivalence class defined by the relation $\sim$, where $(\alpha_i, \beta_i) \sim (\alpha_j, \beta_j) \Leftrightarrow \alpha_i\beta_j = \alpha_j\beta_i$. *Without loss of generality, we set* $\alpha_1 = \cdots = \alpha_{n-r} = 1$.

Adopting different measures of robustness, different methods were proposed to solve the *RPA-PDSF*. Two frequently used measures are the condition number of the eigenvectors matrix (Varga, 2003) and the departure from normality (Li & Chu, 2008). When Equation (1) is completely controllable (C-controllable), adopting the condition numbers of the left and right eigenvector matrices of $(A + BF, E + BG)$ as the measure, Varga (2003) solved the *RPA-PDSF* through a series of generalised Sylvester equations and the Weierstrass canonical form of $(A + BF, E + BG)$. Arbitrary pole placement were permitted, under the harsh assumption that the sizes of all the Jordan blocks (for both finite and infinite eigenvalues) are known *a priori*. Computing the Weierstrass canonical form would also cause some numerical instability in general. Furthermore, the accuracy in solving the generalised Sylvester equations relies on a wide separation between $\lambda(A, E)$ and $\mathfrak{L} = \lambda(A + BF, E + BG)$, thus placing an unreasonable demand. (After all, why should some well-behaved poles not be allowed to remain?) Recently, a Schur–Newton method was proposed in Li and Chu (2008), minimising the departure from normality of $(A + BF, E + BG)$. With the generalised Schur form $(A + BF, E + BG) = (XSP, XTP)$, where $X, P \in \mathbb{R}^{n\times n}$ are non-singular, $S, T \in \mathbb{R}^{n\times n}$ are upper quasi-triangular and all finite poles are real, the method in Li and Chu (2008) generates an orthogonal P. For complex conjugate poles, the acquired P is usually not orthogonal, implying that it virtually does not optimise the departure from normality of $(A + BF, E + BG)$. Both methods are iterative and convergence are not proved.

In Duan and Patton (1999), the proportional plus partial derivative state feedback was employed, where the closed-loop system in the form $(A + BF, E + BGC)$ for $C \in \mathbb{R}^{l\times n}$ is the output matrix. Adopting a sum of the condition numbers of individual eigenvalues as the robust measure, the method assigns n distinct finite poles, requiring the existence of $G \in \mathbb{R}^{m\times l}$ with $\operatorname{rank}(E + BGC) = n$. However, no sufficient and necessary condition is offered to guarantee such an existence. Besides, the method essentially computes the right coprime polynomial matrices $N(s)$ and $D(s)$ such that $(A - sE)N(s) + BD(s) = 0$, which is theoretically elegant yet numerically difficult to implement.

Inspired by the algorithms schur (Chu, 2007) and Schur-rob (Guo, Cai, Qian, & Xu, 2015), we propose a direct method to solve the *RPA-PDSF*, utilising the generalised real Schur form of $(A + BF, E + BG)$ in this paper. We shall adopt a robustness measure which is closely related to the departure from normality. All poles will be placed in turn, and in each step (which assigns an infinite pole, a real pole or a pair of complex conjugate poles), we minimise the robust measure in an optimisation subproblem. When assigning an infinite pole, we merely need to solve some linear equations; while assigning a real pole, only a singular value decomposition (SVD) is required. When assigning a pair of complex conjugate poles, similarly to Schur-rob, an efficient solution of the corresponding optimisation subproblem is proposed. In addition, theorems will be proved to guarantee the feasibility of our method. Abundant amount of numerical results will show the efficiency of our method, producing robust closed-loop systems with highly accurate finite poles.

The paper is organised as follows. In Section 2, we present some preliminaries. Our method is developed in Section 3. Numerical results are reported in Section 4. Some concluding remarks are then made in Section 5.

2. Notations and preliminaries

In this paper, for any matrix M, we denote its null space by $\mathcal{N}(M)$, its range space by $\mathcal{R}(M)$, its strictly upper triangular part by $\operatorname{offdiag}(M)$, and its sub-matrix comprised by rows k to l and columns s to t by $M(k{:}\ l, s{:}\ t)$. For any $\lambda \in \mathbb{C}$, define $D_\delta(\lambda) \equiv \begin{bmatrix} \Re(\lambda) & \delta\Im(\lambda) \\ -\delta^{-1}\Im(\lambda) & \Re(\lambda)\end{bmatrix}$ with some $0 \neq \delta \in \mathbb{R}$.

Lemma 2.1 (Guo, 2016)**:** *For any regular matrix pair* (A, E), $A, E \in \mathbb{R}^{n\times n}$, *there exist a non-singular matrix* $X \in \mathbb{R}^{n\times n}$ *and an orthogonal matrix* $P \in \mathbb{R}^{n\times n}$ *such that* $X^{-1}AP = S$ *and* $X^{-1}EP = T$ *are both upper quasi-triangular with* 1×1 *or* 2×2 *diagonal blocks. Moreover, writing S and T in partitioned form, i.e.*

$$S = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1k} \\ & \ddots & \vdots \\ & & \Phi_{kk}\end{bmatrix}, \quad T = \begin{bmatrix} \Psi_{11} & \cdots & \Psi_{1k} \\ & \ddots & \vdots \\ & & \Psi_{kk}\end{bmatrix},$$

then the 1×1 *diagonal blocks are* $\Phi_{jj} = \alpha/\sqrt{\alpha^2 + \beta^2}$ *and* $\Psi_{jj} = \beta/\sqrt{\alpha^2 + \beta^2}$, *corresponding to real eigenvalues* (α, β) ($\beta = 0$ *indicates an infinite eigenvalue). The* 2×2 *diagonal blocks corresponding to complex conjugate eigenvalues* $\{(\alpha, \beta), (\bar\alpha, \bar\beta)\}$ *are* $\Phi_{jj} = I_2$, $\Psi_{jj} = D_\delta(\sigma + i\tau)$ *if* $|\alpha| \geq |\beta|$, *or* $\Phi_{jj} = D_\delta(\tilde\sigma + i\tilde\tau)$, $\Psi_{jj} = I_2$ *if* $|\alpha| < |\beta|$, *where* $\sigma = \Re(\bar\alpha\beta)/|\alpha|^2$, $\tau = \Im(\bar\alpha\beta)/|\alpha|^2$, $\tilde\sigma = \Re(\bar\beta\alpha)/|\beta|^2$, $\tilde\tau = \Im(\bar\beta\alpha)/|\beta|^2$, $0 \neq \delta \in \mathbb{R}$.

Remark 2.1: Suppose that $\operatorname{ind}_\infty(A, E) \leq 1$, or the infinite eigenvalues of (A, E) are semi-simple. Then when all the diagonal block parts corresponding to the infinite eigenvalues are collected together: $\Psi_j = \cdots = \Psi_{j+l} = 0$, we have $\Psi_{pq} = 0$ for $p = j, \ldots, j + l$, $q = p + 1, \ldots, j + l$.

When $\Psi_{jj} = D_{\delta_j}(\sigma_j + i\tau_j)$ and $\Phi_{ll} = D_{\delta_l}(\tilde\sigma_l + i\tilde\tau_l)$, let $\Psi_{jj} = W_{jj}^*\begin{bmatrix}\sigma_j + i\tau_j & \zeta_j \\ & \sigma_j - i\tau_j\end{bmatrix}W_{jj}$ and $\Phi_{ll} = \widetilde{W}_{ll}^*\begin{bmatrix}\tilde\sigma_l + i\tilde\tau_l & \tilde\zeta_l \\ & \tilde\sigma_l - i\tilde\tau_l\end{bmatrix}\widetilde{W}_{ll}$ be the Schur decompositions of Φ_{jj} and Ψ_{ll}, respectively. Direct calculations show that $\zeta_j = (\delta_j - \delta_j^{-1})\tau_j$ and $\tilde\zeta_l = (\delta_l - \delta_l^{-1})\tilde\tau_l$. Now define $Z \equiv \operatorname{diag}(Z_{11}, \ldots, Z_{kk}) \in \mathbb{C}^{n\times n}$, $D \equiv \operatorname{diag}(D_{11}, \ldots, D_{kk}) \in \mathbb{C}^{n\times n}$, where (i) $Z_{jj} = 1$, $D_{jj} = 1$ if the size of Φ_{jj} equals to 1; (ii) $Z_{jj} = W_{jj}$, $D_{jj} = (1 + \sigma_j^2 + \tau_j^2)^{-1/2}I_2$ if $\Phi_{jj} = I_2$; or (iii) $Z_{jj} = \widetilde{W}_{jj}$, $D_{jj} = (1 + \tilde\sigma_j^2 + \tilde\tau_j^2)^{-1/2}I_2$ if $\Psi_{jj} = I_2$. It then holds that $DZSZ^*$ and $DZTZ^*$ are both upper triangular with the diagonal elements satisfying $|(DZSZ^*)_{jj}|^2 + |(DZTZ^*)_{jj}|^2 = 1$ for $j = 1, \ldots,$

n. Moreover, it follows from the definition of Z and D that

$$\begin{aligned}\varpi &:= \|\operatorname{offdiag}(DZSZ^*)\|_F^2 + \|\operatorname{offdiag}(DZTZ^*)\|_F^2\\ &= \sum_{1\times 1\text{blocks}} \sum_{l>j} (|\Phi_{jl}|^2 + |\Psi_{jl}|^2)\\ &\quad + \sum_{2\times 2\text{blocks}}^{|\alpha_j|\ge|\beta_j|} \frac{\sum_{l>j}(\|\Phi_{jl}\|_F^2 + \|\Psi_{jl}\|_F^2) + (\delta_j^2-1)^2\delta_j^{-2}\tau_j^2}{1+\sigma_j^2+\tau_j^2}\\ &\quad + \sum_{2\times 2\text{blocks}}^{|\alpha_j|<|\beta_j|} \frac{\sum_{l>j}(\|\Phi_{jl}\|_F^2 + \|\Psi_{jl}\|_F^2) + (\delta_j^2-1)^2\delta_j^{-2}\tilde{\tau}_j^2}{1+\tilde{\sigma}_j^2+\tilde{\tau}_j^2}.\end{aligned}$$

For those $D_\delta(\sigma_j + i\tau_j)$, it holds that $\sigma_j^2 + \tau_j^2 = |\beta_j|^2/|\alpha_j|^2 \le 1$, leading to $1 \le 1 + \sigma_j^2 + \tau_j^2 \le 2$. Analogously, we also have $1 \le 1 + \tilde{\sigma}_j^2 + \tilde{\tau}_j^2 \le 2$. Now write

$$\begin{aligned}\Delta_F^2(A,E) &\equiv \|S - \operatorname{diag}(\Phi_{jj})\|_F^2 + \|T - \operatorname{diag}(\Psi_{jj})\|_F^2\\ &\quad + \sum_{2\times 2\text{blocks}}^{|\alpha_j|\ge|\beta_j|} \tau_j^2(\delta_j - \delta_j^{-1})^2\\ &\quad + \sum_{2\times 2\text{blocks}}^{|\alpha_j|<|\beta_j|} \tilde{\tau}_j^2(\delta_j - \delta_j^{-1})^2. \qquad (2)\end{aligned}$$

Then $\frac{1}{2}\Delta_F^2(A,E) \le \varpi \le \Delta_F^2(A,E)$. From the Henrici-type theorem (Stewart & Sun, 1990) for the sensitivity of generalised eigenvalues, Lemma 2.1 suggests that $\Delta_F^2(A+BF, E+BG)$ will be an appropriate robust measure for the corresponding *RPA-PDSF*.

Next we quote the solvability result for the *RPA-PDSF*.

Lemma 2.2 (Bunse-Gerstner et al., 1992)**:** *For the S-controllable descriptor system* (E, A, B), *there exists* $G, F \in \mathbb{R}^{m\times n}$ *which solve the RPA-PDSF.*

For such $G, F \in \mathbb{R}^{m\times n}$ in Lemma 2.2, it follows from Lemma 2.1 that there exist a non-singular matrix $X_{G,F} \in \mathbb{R}^{n\times n}$ and an orthogonal matrix $P_{G,F} \in \mathbb{R}^{n\times n}$ such that

$$X_{G,F}^{-1}(A+BF)P_{G,F} = S = \begin{array}{cc} \begin{array}{cc} n-r & r \end{array} & \\ \begin{bmatrix} S_{11} & S_{12}\\ \mathbf{0} & S_{22}\end{bmatrix}, & \begin{array}{c} n-r\\ r\end{array}\end{array} \qquad (3)$$

$$X_{G,F}^{-1}(E+BG)P_{G,F} = T = \begin{array}{cc} \begin{array}{cc} n-r & r \end{array} & \\ \begin{bmatrix} \mathbf{0} & T_{12}\\ \mathbf{0} & T_{22}\end{bmatrix}, & \begin{array}{c} n-r\\ r\end{array}\end{array} \qquad (4)$$

here all diagonal elements in S_{11} are 1 and $\lambda(S_{22}, T_{22}) = \{(\alpha_{n-r+1}, \beta_{n-r+1}), \ldots, (\alpha_n, \beta_n)\}$. The choice of $T_{11} = 0$ in Equation (4) is justified in Note 2.1. We shall utilise the decomposition in Equations (3) and (4) to solve the *RPA-PDSF*.

Let $P_{G,F} = [p_1 \cdots p_n]$, $X_{G,F} = [x_1 \cdots x_n]$, and define $N_S \equiv [\breve{v}_{1,S} \cdots \breve{v}_{n,S}]$ and $N_T \equiv [\breve{v}_{1,T} \cdots \breve{v}_{n,T}]$ as the strictly upper quasi-triangular parts of S and T, respectively. In other words, we have $\breve{v}_{j,S} = [v_{j,S}^\top\ 0]^\top$ with $v_{j,S} \in \mathbb{R}^{j-1}$ or $\mathbb{R}^{j-2}$ $(j = 2, \ldots, n)$, and $\breve{v}_{j,T} = [v_{j,T}^\top\ 0]^\top$ with $v_{j,T} \in \mathbb{R}^{j-1}$ or $\mathbb{R}^{j-2}$ $(j = n-r+1, \ldots, n)$. Write $P_j = P_{G,F}(1{:}\,n, 1{:}\,j)$, $X_j = X_{G,F}(1{:}\,n, 1{:}\,j)$, $S_j = S(1{:}\,j, 1{:}\,j)$, and $T_j = T(1{:}\,j, 1{:}\,j)$.

3. Solving the RPA-PDSF via the generalised real Schur form

Without loss of generality, assume that B *is of full column rank*. Let $B = Q[R^\top\ 0]^\top = Q_1R$ be the QR factorisation of B, where $Q = [Q_1\ Q_2] \in \mathbb{R}^{n\times n}$ is orthogonal, $Q_1 \in \mathbb{R}^{n\times m}$, and $R \in \mathbb{R}^{m\times m}$ is non-singular and upper triangular. Substituting the QR factorisation of B into Equations (3) and (4), we deduce that $F = R^{-1}Q_1^\top(X_{G,F}SP_{G,F}^\top - A)$, $G = R^{-1}Q_1^\top(X_{G,F}TP_{G,F}^\top - E)$, where

$$Q_2^\top(AP_{G,F} - X_{G,F}S) = 0, \qquad (5a)$$

$$Q_2^\top(EP_{G,F} - X_{G,F}T) = 0. \qquad (5b)$$

Consequently, once we obtain an orthogonal $P_{G,F}$, a non-singular $X_{G,F}$ and a pair of upper quasi-triangular S and T satisfying Equation (5), a solution (G, F) to the PA problem can be computed directly.

3.1 Assigning the infinite pole $(\alpha_j, 0)$

Provided that there exist some infinite poles in $\mathfrak{L}$, we shall show how to assign all infinite poles $(\alpha_j, 0)$ for $j = 1, \ldots, n-r$. Suppose $j-1$ infinite poles $(j \ge 1)$ have already been placed, suggesting that P_{j-1}, $\Xi_{j-1} \equiv Q_2^\top X_{j-1}$ and S_{j-1} have been acquired. We are going to compute the jth column of $P_{G,F}$, $Q_2^\top X_{G,F}$ and S when assigning $(\alpha_j, 0)$. We emphasise that Ξ_{j-1} (not X_{j-1}) is known, in the computation of $\Xi_j \equiv Q_2^\top X_j$ (not X_j).

It is simple to show that $\operatorname{rank}([E\ B]) = m + \operatorname{rank}(Q_2^\top E)$, indicating that $l \triangleq \dim(\mathcal{N}(Q_2^\top E)) \ge n-r$. Let the columns of $Z \in \mathbb{R}^{n\times l}$ be an orthonormal basis of $\mathcal{N}(Q_2^\top E)$. By Equation (5b), $Q_2^\top EP_{n-r} = 0$, so $P_{j-1} = ZW_{j-1}$ for some $W_{j-1} \in \mathbb{R}^{l\times(j-1)}$ with $W_{j-1}^\top W_{j-1} = I_{j-1}$ and $p_j = Zw_j$ with a normalised $w_j \in \mathbb{R}^l$ to be specified. From the jth column of Equation (5a), noting that the jth diagonal element of S is 1 and P is orthogonal, we have

$$Q_2^\top AZw_j = \Xi_{j-1}v_{j,S} + Q_2^\top x_j, \quad W_{j-1}^\top w_j = 0.$$

The definition of $\Delta_F^2(A,E)$ in Equation (2) asks to solve this:

$$\min \quad \|v_{j,S}\|_2^2 \qquad (6a)$$

$$\text{s.t.} \quad M_{j-1}[w_j^\top\ x_j^\top\ v_{j,S}^\top]^\top = 0, \qquad (6b)$$

where $M_{j-1} = \begin{bmatrix} Q_2^\top AZ & -Q_2^\top & -\Xi_{j-1}\\ W_{j-1}^\top & \mathbf{0} & \mathbf{0}\end{bmatrix}$.

Theorem 3.1 in Section 3.5 demonstrates that M_{j-1} is of full row rank and there exists some vector $[w^\top\ x^\top\ v_S^\top]^\top \in \mathcal{N}(M_{j-1})$ with $w \in \mathbb{R}^l$, $x \in \mathbb{R}^n$, $v_S \in \mathbb{R}^{j-1}$, such that $w \ne 0$. Denote $W_\perp \in \mathbb{R}^{l\times(l-j+1)}$ satisfying $W_{j-1}^\top W_\perp = 0$ and $W_\perp^\top W_\perp = I_{l-j+1}$, and then the columns of

$$\begin{bmatrix} W_\perp & \mathbf{0} & \mathbf{0}\\ Q_2Q_2^\top AZW_\perp & -Q_2\Xi_{j-1} & Q_1\\ \mathbf{0} & I_{j-1} & \mathbf{0}\end{bmatrix}$$

form a basis of $\mathcal{N}(M_{j-1})$, suggesting $w_j = W_\perp u_j$ for some normalised $u_j \in \mathbb{R}^{l-j+1}$ and $x_j = Q_1y_j + Q_2Q_2^\top AZw_j -$

$Q_2\Xi_{j-1}v_{j,S}$ for some $y_j \in \mathbb{R}^m$. Accordingly, $p_j = ZW_\perp u_j$. Apparently, Equation (6) obtains its minimum when $v_{j,S} = 0$, leading to $x_j = Q_1 y_j + Q_2 Q_2^\top A p_j$, with y_j to be specified in Section 3.4. Consequently, we obtain $Q_2^\top x_j = Q_2^\top A p_j$ which is sufficient for the assigning process to continue. Note in the definition of M_{j-1} and the assigning procedure for finite poles later that only Ξ_{j-1} (not X_{j-1}) is required.

Once u_j is determined, P_j, Ξ_j and S_j will be updated as $P_j = [\,P_{j-1}\ p_j\,]$, $\Xi_j = [\,\Xi_{j-1}\ Q_2^\top A p_j\,]$ and $S_j = \begin{bmatrix} S_{j-1} & 0 \\ 0 & 1\end{bmatrix}$.

Provided that all infinite poles and some finite poles have already been assigned, where the complex conjugate poles are placed together. Note P_j, $\Xi_j = Q_2^\top X_j$, S_j and T_j that are already acquired satisfy $Q_2^\top A P_j = \Xi_j S_j$, $Q_2^\top E P_j = \Xi_j T_j$ $(j \geq n-r)$. The details of the PA for the finite real pole $(\alpha_{j+1}, \beta_{j+1})$ and the finite complex conjugate poles $\{(\alpha_{j+1}, \beta_{j+1}), (\bar\alpha_{j+1}, \bar\beta_{j+1})\}$ will be presented. The $(j+1)$th column, or the $(j+1)$th and $(j+2)$th columns, of $P_{G,F}$, $Q_2^\top X_{G,F}$, S and T are computed in the assignment process.

3.2 Assigning the finite real pole $(\alpha_{j+1}, \beta_{j+1})$

Obviously the $(j+1)$th diagonal elements of S and T are $\alpha_{j+1}/\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}$ and $\beta_{j+1}/\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}$, respectively. The $(j+1)$th columns in Equation (5) are

$$\begin{cases} Q_2^\top A p_{j+1} - \Xi_j v_{j+1,S} - \dfrac{\alpha_{j+1} Q_2^\top x_{j+1}}{\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}} = 0, \\ Q_2^\top E p_{j+1} - \Xi_j v_{j+1,T} - \dfrac{\beta_{j+1} Q_2^\top x_{j+1}}{\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}} = 0, \end{cases}$$

which are the conditions p_{j+1}, x_{j+1}, $v_{j+1,S}$ and $v_{j+1,T}$ have to meet. From the definition of $\Delta_F^2(A,E)$ in Equation (2) and the orthogonality of $P_{G,F}$, it is natural to consider this:

$$\min_{\|p_{j+1}\|_2=1} \|v_{j+1,S}\|_2^2 + \|v_{j+1,T}\|_2^2 \tag{7a}$$

$$\text{s.t.}\quad M_j[\,p_{j+1}^\top\ x_{j+1}^\top\ v_{j+1,S}^\top\ v_{j+1,T}^\top\,]^\top = 0, \tag{7b}$$

where $M_j = \begin{bmatrix} Q_2^\top A & -\alpha_{j+1}/\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}Q_2^\top & -\Xi_j & 0 \\ Q_2^\top E & -\beta_{j+1}/\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2}Q_2^\top & 0 & -\Xi_j \\ P_j^\top & 0 & 0 & 0 \end{bmatrix}$. Theorem 3.2 in Section 3.5 shows that $\dim(\mathcal{N}(M_j)) = 2m+j$ and there exists $[\,p_{j+1}^\top\ x_{j+1}^\top\ v_{j+1,S}^\top\ v_{j+1,T}^\top\,]^\top \in \mathcal{N}(M_j)$ such that $p_{j+1} \neq 0$, which guarantees the solvability of Equation (7). Next, we shall consider the solution of Equation (7).

Case i ($|\alpha_{j+1}| \geq |\beta_{j+1}|$) Define

$$\widetilde{M}_j = \left[\begin{array}{c|c} (-\alpha_{j+1}/\sqrt{|\alpha_{j+1}|^2+|\beta_{j+1}|^2})Q_2^\top & Q_2^\top A\ \ -\Xi_j\ \ 0 \\ \hline 0 & \widetilde{M}_{2,j} \end{array}\right]$$

with $\widetilde{M}_{2,j} = \begin{bmatrix} Q_2^\top(E-(\beta_{j+1}/\alpha_{j+1})A) & (\beta_{j+1}/\alpha_{j+1})\Xi_j & -\Xi_j \\ P_j^\top & 0 & 0\end{bmatrix}$. Then Equation (7b) $\Leftrightarrow \widetilde{M}_j[\,x_{j+1}^\top\ p_{j+1}^\top\ v_{j+1,S}^\top\ v_{j+1,T}^\top\,]^\top = 0$. Equivalently, we have $\widetilde{M}_{2,j}[\,p_{j+1}^\top\ v_{j+1,S}^\top\ v_{j+1,T}^\top\,]^\top = 0$, $\alpha_{j+1}(|\alpha_{j+1}|^2 + |\beta_{j+1}|^2)^{-1/2}Q_2^\top x_{j+1} = Q_2^\top A p_{j+1} - \Xi_j v_{j+1,S}$. Evidently, $\operatorname{rank}(\widetilde{M}_{2,j}) = n-m+j$, implying that $\dim(\mathcal{N}(\widetilde{M}_{2,j})) = m+j$. Let the columns of $[\,Z_1^\top\ Z_3^\top\ Z_4^\top\,]^\top$ be an orthonormal basis of $\mathcal{N}(\widetilde{M}_{2,j})$, where $Z_1 \in \mathbb{R}^{n\times(m+j)}$, $Z_3, Z_4 \in \mathbb{R}^{j\times(m+j)}$. Then the columns of

$$\begin{bmatrix} 0 & Q_1^\top & 0 & 0 \\ Z_1^\top & \alpha_{j+1}^{-1}\sqrt{\alpha_{j+1}^2+\beta_{j+1}^2}(AZ_1 - X_j Z_3)^\top Q_2 Q_2^\top & Z_3^\top & Z_4^\top \end{bmatrix}^\top$$

form a basis of $\mathcal{N}(M_j)$.

Consequently, Equation (7) can be reduced to

$$\min_{\|Z_1 u\|_2 = 1} u^\top(Z_3^\top Z_3 + Z_4^\top Z_4)u, \tag{8}$$

where $u \in \mathbb{R}^{m+j}$. Furthermore, since $Z_1^\top Z_1 + Z_3^\top Z_3 + Z_4^\top Z_4 = I_{m+j}$, Equation (8) is further reduced to

$$\min_{\|Z_1 u\|_2 = 1} u^\top u, \tag{9}$$

whose solution is u, an eigenvector of $Z_1^\top Z_1$ corresponding to its greatest eigenvalue with $\|Z_1 u\| = 1$. Once such u is obtained, p_{j+1}, $v_{j+1,S}$ and $v_{j+1,T}$ can be retrieved by $p_{j+1} = Z_1 u$, $v_{j+1,S} = Z_3 u$ and $v_{j+1,T} = Z_4 u$, respectively. Also $x_{j+1} = Q_1 y_{j+1} + \alpha_{j+1}^{-1}(\alpha_{j+1}^2+\beta_{j+1}^2)^{1/2}Q_2(Q_2^\top A p_{j+1} - \Xi_j v_{j+1,S})$ for some $y_{j+1} \in \mathbb{R}^m$ to be determined. Clearly, $Q_2^\top x_{j+1} = \alpha_{j+1}^{-1}\sqrt{\alpha_{j+1}^2+\beta_{j+1}^2}(Q_2^\top A p_{j+1} - \Xi_j v_{j+1,S})$, which can be computed and added to $\Xi_{j+1} \equiv Q_2^\top X_{j+1} = [\,\Xi_j\ Q_2^\top x_{j+1}\,]$. By the definition of M_j, it is Ξ_{j+1}, rather than y_{j+1} or X_{j+1}, that is required when assigning the finite real pole $(\alpha_{j+2}, \beta_{j+2})$. Similar comments hold for the case of $(\alpha_{j+2}, \beta_{j+2}) \in \mathbb{C}\times\mathbb{C}$, which will be discussed later. The choice of y_{j+1} will be discussed in Section 3.4.

Case ii ($|\alpha_{j+1}| < |\beta_{j+1}|$) Analogously to **Case i**, let the columns of $[\,Z_1^\top\ Z_3^\top\ Z_4^\top\,]^\top$, where $Z_1 \in \mathbb{R}^{n\times(m+j)}$, $Z_3, Z_4 \in \mathbb{R}^{j\times(m+j)}$, be an orthonormal basis of

$$\widetilde{M}_{2,j} = \begin{bmatrix} Q_2^\top(A - (\alpha_{j+1}/\beta_{j+1})E) & -\Xi_j & (\alpha_{j+1}/\beta_{j+1})\Xi_j \\ P_j^\top & 0 & 0 \end{bmatrix},$$

where $\dim(\mathcal{R}([\,Z_1^\top\ Z_3^\top\ Z_4^\top\,]^\top)) = m+j$ is guaranteed by $\operatorname{rank}(\widetilde{M}_{2,j}) = (n-m+j)$. Besides, the columns of

$$\begin{bmatrix} 0 & Q_1^\top & 0 & 0 \\ Z_1^\top & \beta_{j+1}^{-1}\sqrt{\alpha_{j+1}^2+\beta_{j+1}^2}(EZ_1 - X_j Z_4)^\top Q_2 Q_2^\top & Z_3^\top & Z_4^\top \end{bmatrix}^\top$$

form a basis of $\mathcal{N}(M_j)$, leading to $p_{j+1} = Z_1 u$, $v_{j+1,S} = Z_3 u$, $v_{j+1,T} = Z_4 u$, where u is the solution to Equation (9), and $x_{j+1} = Q_1 y_{j+1} + \beta_{j+1}^{-1}(\alpha_{j+1}^2+\beta_{j+1}^2)^{1/2}Q_2(Q_2^\top E p_{j+1} - \Xi_j v_{j+1,T})$ for some $y_{j+1} \in \mathbb{R}^m$. Similarly to **Case i**, y_{j+1} and x_{j+1} will be specified in Section 3.4.

Concisely, once Equation (9) is solved, P_j, Ξ_j, S_j and T_j would be undated as $P_{j+1} = [\,P_j\ p_{j+1}\,]$, $\Xi_{j+1} = [\,\Xi_j\ Q_2^\top x_{j+1}\,]$, $S_{j+1} = \begin{bmatrix} S_j & v_{j+1,S} \\ 0 & \alpha_{j+1}(\alpha_{j+1}^2+\beta_{j+1}^2)^{-1/2}\end{bmatrix}$ and $T_{j+1} = \begin{bmatrix} T_j & v_{j+1,T} \\ 0 & \beta_{j+1}(\alpha_{j+1}^2+\beta_{j+1}^2)^{-1/2}\end{bmatrix}$.

Remark 3.1: If there is no infinite pole, i.e. $r = n$, some minor modifications are required for the first to be placed real finite pole (α_1, β_1). Specifically, since there is no contribution from the first columns of S and T to $\Delta_F^2(A+BF, E+BG)$, Equation (7) is degenerate. We just need to select p_1 and x_1 from Equation (7b). Lemma 3.2 implies that M_0 is of full row rank, thus the feasibility of Equation (7b). We can select a normalised p_1 and have $x_1 = Q_1 y_1 + Q_2(Q_2^\top x_1)$, where $y_1 \in \mathbb{R}^m$, $Q_2^\top x_1 = \alpha_1^{-1}\sqrt{\alpha_1^2+\beta_1^2}Q_2^\top A p_1$ (for $|\alpha_1| \geq |\beta_1|$) or $Q_2^\top x_1 = \beta_1^{-1}\sqrt{\alpha_1^2+\beta_1^2}Q_2^\top E p_1$ (for $|\alpha_1| < |\beta_1|$). Way to choose y_1 is given in Section 3.4.

3.3 Assigning the finite complex pole $(\alpha_{j+1}, \beta_{j+1})$

With the 2×2 diagonal blocks in S and T specified in Lemma 2.1, assigning the complex conjugate pair $\{(\alpha_{j+1}, \beta_{j+1}), (\bar{\alpha}_{j+1}, \bar{\beta}_{j+1})\}$ involves two different cases, when $|\alpha_{j+1}| \geq |\beta_{j+1}|$ or otherwise.

3.3.1 Situation I: $|\alpha_{j+1}| \geq |\beta_{j+1}|$

By noting $S(j+1{:}\,j+2, j+1{:}\,j+2) = I_2$ and $T(j+1{:}j+2, j+1{:}j+2) = D_{\delta_{j+1}}(\gamma_{j+1})$ with $\gamma_{j+1} \equiv \sigma_{j+1} + i\tau_{j+1}$ here, hence the $(j+1)$th and $(j+2)$th columns of Equation (5) can be expanded to

$$\begin{aligned} &Q_2^\top A\big[p_{j+1}\ p_{j+2}\big] - \Xi_j\big[v_{j+1,S}\ v_{j+2,S}\big] - Q_2^\top\big[x_{j+1}\ x_{j+2}\big] = 0, \\ &Q_2^\top E\big[p_{j+1}\ p_{j+2}\big] - \Xi_j\big[v_{j+1,T}\ v_{j+2,T}\big] \\ &\quad - Q_2^\top\big[x_{j+1}\ x_{j+2}\big]D_{\delta_{j+1}}(\gamma_{j+1}) = 0. \end{aligned} \tag{10}$$

By defining $\delta_{j+1} = \varsigma_1/\varsigma_2$ with $\varsigma_1, \varsigma_2 \in \mathbb{R}$ and $\varsigma_2 \neq 0$, $\tilde{p}_{j+l} = \varsigma_l p_{j+l}$, $\tilde{x}_{j+l} = \varsigma_l x_{j+l}$, $\tilde{v}_{j+l,S} = \varsigma_l v_{j+l,S}$ and $\tilde{v}_{j+l,T} = \varsigma_l v_{j+l,T}$ for $l = 1, 2$. Then since

$$D_{\delta_{j+1}}(\gamma_{j+1}) = \frac{1}{2}\begin{bmatrix}\varsigma_1 & \\ & \varsigma_2\end{bmatrix}\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}\begin{bmatrix}\gamma_{j+1} & \\ & \bar{\gamma}_{j+1}\end{bmatrix}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\begin{bmatrix}\varsigma_1 & \\ & \varsigma_2\end{bmatrix}^{-1},$$

it is simple to show that Equation (10) is equivalent to

$$\begin{cases} Q_2^\top A(\tilde{p}_{j+1} + i\tilde{p}_{j+2}) - \Xi_j(\tilde{v}_{j+1,S} + i\tilde{v}_{j+2,S}) \\ \quad -Q_2^\top(\tilde{x}_{j+1} + i\tilde{x}_{j+2}) = 0, \\ Q_2^\top E(\tilde{p}_{j+1} + i\tilde{p}_{j+2}) - \Xi_j(\tilde{v}_{j+1,T} + i\tilde{v}_{j+2,T}) \\ \quad -\gamma_{j+1}Q_2^\top(\tilde{x}_{j+1} + i\tilde{x}_{j+2}) = 0, \end{cases}$$

which are the conditions for $\tilde{p}_{j+l}$, $\tilde{x}_{j+l}$, $\tilde{v}_{j+l,S}$ and $\tilde{v}_{j+l,T}$ ($l = 1, 2$), in addition to the constraint $\big[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\big]^\top\big[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\big] = \mathrm{diag}(\varsigma_1^2, \varsigma_2^2)$ (so that $\big[p_{j+1}\ p_{j+2}\big]^\top\big[p_{j+1}\ p_{j+2}\big] = I_2$).

Recalling the definition $\Delta_F^2(A, E)$ in Equation (2), we then select the $(j+1)$th and $(j+2)$th columns of $P_{G,F}$, $X_{G,F}$, S and T while minimising their contributions to $\Delta_F^2(A+BF, E+BG)$. In other words, we solve

$$\min_{\substack{\varsigma_l,\ \tilde{v}_{j+l,S},\ \tilde{v}_{j+l,T},\\ l=1,2}} \sum_{l=1}^{2}\frac{\|\tilde{v}_{j+l,S}\|_2^2 + \|\tilde{v}_{j+l,T}\|_2^2}{\varsigma_l^2} + \tau_{j+1}^2\left(\frac{\varsigma_1}{\varsigma_2} - \frac{\varsigma_2}{\varsigma_1}\right)^2 \tag{11a}$$

$$\begin{aligned} \text{s.t.}\quad &\big[\tilde{p}_{j+1}^\top + i\tilde{p}_{j+2}^\top,\ \tilde{x}_{j+1}^\top + i\tilde{x}_{j+2}^\top, \\ &\tilde{v}_{j+1,S}^\top + i\tilde{v}_{j+2,S}^\top,\ \tilde{v}_{j+1,T}^\top + i\tilde{v}_{j+2,T}^\top\big]^\top \in \mathcal{N}(M_j), \end{aligned} \tag{11b}$$

$$\big[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\big]^\top\big[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\big] = \mathrm{diag}(\varsigma_1^2, \varsigma_2^2), \tag{11c}$$

where $M_j = \begin{bmatrix} Q_2^\top A & -Q_2^\top & -\Xi_j & \mathbf{0} \\ Q_2^\top E & -(\sigma_{j+1} + i\tau_{j+1})Q_2^\top & \mathbf{0} & -\Xi_j \\ P_j^\top & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}$.

Once a solution to Equation (11) is acquired, the $(j+l)$th columns of $P_{G,F}$, $X_{G,F}$, S and T, $l = 1, 2$, can be retrieved via normalisation:

$$p_{j+l} = \frac{\tilde{p}_{j+l}}{\|\tilde{p}_{j+l}\|_2}, \quad x_{j+l} = \frac{\tilde{x}_{j+l}}{\|\tilde{p}_{j+l}\|_2}, \quad v_{j+l,S} = \frac{\tilde{v}_{j+l,S}}{\|\tilde{p}_{j+l}\|_2},$$
$$v_{j+l,T} = \frac{\tilde{v}_{j+l,T}}{\|\tilde{p}_{j+l}\|_2}.$$

To solve Equation (11), we first consider Equation (11b). Define $\widetilde{M}_{2,j} = \begin{bmatrix} Q_2^\top(E - \gamma_{j+1}A) & \gamma_{j+1}\Xi_j & -\Xi_j \\ P_j^\top & \mathbf{0} & \mathbf{0}\end{bmatrix}$, $\widetilde{M}_j = \left[\begin{array}{c|ccc} -Q_2^\top & Q_2^\top A & -\Xi_j & \mathbf{0} \\ \hline \mathbf{0} & & \widetilde{M}_{2,j} & \end{array}\right]$,

and then we have $M_j\big[z_1^\top\ z_2^\top\ z_3^\top\ z_4^\top\big]^\top = 0$ if and only if $\widetilde{M}_j\big[z_2^\top\ z_1^\top\ z_3^\top\ z_4^\top\big]^\top = 0$ with $z_1, z_2 \in \mathbb{C}^n$, $z_3, z_4 \in \mathbb{C}^j$. Furthermore, it follows from Theorem 3.2 in Section 3.5 that M_j, thus $\widetilde{M}_{2,j}$, are of full row rank. Now let the columns of $[\,Z_1^\top\ Z_3^\top\ Z_4^\top\,]^\top$ be an orthonormal basis of $\mathcal{N}(\widetilde{M}_{2,j})$, where $Z_1 \in \mathbb{C}^{n\times(m+j)}$, $Z_3, Z_4 \in \mathbb{C}^{j\times(m+j)}$, and then the columns of $\begin{bmatrix}\mathbf{0} & Q_1^\top & \mathbf{0} & \mathbf{0} \\ Z_1^\top & (AZ_1 - X_jZ_3)^\top Q_2Q_2^\top & Z_3^\top & Z_4^\top\end{bmatrix}^\top$ constitute a basis of $\mathcal{N}(M_j)$. We can then select

$$\begin{aligned} \tilde{p}_{j+1} + i\tilde{p}_{j+2} &= Z_1 b, \\ \tilde{v}_{j+1,S} + i\tilde{v}_{j+2,S} &= Z_3 b, \quad \tilde{v}_{j+1,T} + i\tilde{v}_{j+2,T} = Z_4 b, \\ \tilde{x}_{j+1} + i\tilde{x}_{j+2} &= Q_1 y + Q_2Q_2^\top(AZ_1 - X_jZ_3)b, \end{aligned}$$

for some $0 \neq b \in \mathbb{C}^{m+j}$ and $y \in \mathbb{C}^m$. Accordingly, Equation (11) is reduced to choosing some suitable b such that $\tilde{p}_{j+1}^\top\tilde{p}_{j+2} = 0$, while solving Equation (11a). It is worthwhile to point out that Theorem 3.2 in Section 3.5 guarantees that $Z_1 \neq 0$ and there exist some nontrivial b such that $\tilde{p}_{j+1}$ and $\tilde{p}_{j+2}$ are linearly independent, which is necessary for $\{\tilde{p}_{j+1}, \tilde{p}_{j+2}\}$ to be orthogonal. In what follows, we consider how b is selected, in two distinct cases.

Denote $Z_1 = U_{Z_1}\Sigma_{Z_1}V_{Z_1}^*$ as the SVD of Z_1, where $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{r_{Z_1}} > 0$ are its non-zero singular values. Note that $Z_1^*Z_1 + Z_3^*Z_3 + Z_4^*Z_4 = I_{m+j}$, implying that $Z_3^*Z_3 + Z_4^*Z_4 = V_{Z_1}(I_{m+j} - \Sigma_{Z_1}^\top\Sigma_{Z_1})V_{Z_1}^*$.

Case i ($\mathrm{rank}(Z_1) = 1$) Here there exists a unique nonzero singular value ν_1 for Z_1, with the corresponding left-singular vector f_1. Define

$$\begin{aligned} \mathcal{N}_1(\widetilde{M}_{2,j}) \equiv \Big\{\big[\,f_1^\top\ z_3^\top\ z_4^\top\,\big]^\top &: z_3 = Z_3 b, z_4 = Z_4 b, \\ &b = V_{Z_1}\big[\,\tfrac{1}{\nu_1}\ \eta_2 \cdots \eta_{m+j-1}\,\big]^\top, \eta_2,\ \ldots,\ \eta_{m+j-1} \in \mathbb{C}\Big\}. \end{aligned}$$

Then with proper scaling, $\mathcal{N}_1(\widetilde{M}_{2,j})$ is the unique subset of $\mathcal{N}(\widetilde{M}_{2,j})$ that contains $\big[z_1^\top\ z_3^\top\ z_4^\top\big]^\top$ with $0 \neq z_1 \in \mathbb{C}^n$, $z_3, z_4 \in \mathbb{C}^j$. Moreover, it follows from Theorem 3.2 in Section 3.5 that $\Re(f_1)$ and (f_1) are linearly independent.

We then select $\tilde{p}_{j+1}$ and $\tilde{p}_{j+2}$ as the vectors generated by the Jacobi orthogonal transformation on $\Re(f_1)$ and (f_1):

$\left[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\right] = \left[\Re(f_1)\ \Im(f_1)\right]\begin{bmatrix} c & s \\ -s & c\end{bmatrix}$, with c and s selected to enforce $\tilde{p}_{j+1}^\top \tilde{p}_{j+2} = 0$. Note that $D_{\delta_{j+1}}(\sigma_{j+1} + i\tau_{j+1})$ is already determined with $\delta_{j+1} = \|\tilde{p}_{j+1}\|_2/\|\tilde{p}_{j+2}\|_2$, for $\varsigma_1 = \|\tilde{p}_{j+1}\|_2$, $\varsigma_2 = \|\tilde{p}_{j+2}\|_2$. Accordingly, $\tilde{v}_{j+l,S}$, $\tilde{v}_{j+l,T}$, $l = 1, 2$ will be selected from

$$\left[\tilde{v}_{j+1,S}\ \tilde{v}_{j+2,S}\right] = \left[\Re(Z_3 b)\ \Im(Z_3 b)\right]\begin{bmatrix} c & s \\ -s & c\end{bmatrix},$$
$$\left[\tilde{v}_{j+1,T}\ \tilde{v}_{j+2,T}\right] = \left[\Re(Z_4 b)\ \Im(Z_4 b)\right]\begin{bmatrix} c & s \\ -s & c\end{bmatrix}, \qquad (12)$$

where $b = V_{Z_1}\left[\frac{1}{\nu_1}\ \eta_2 \cdots \eta_{m+j-1}\right]^\top$, with $\eta_2, \ldots, \eta_{m+j-1} \in \mathbb{C}$ to be determined. Then our goal is to choose appropriate η's to minimise the first term in Equation (11a).

Define $\left[w\ W\right] \equiv \left[Z_3^\top\ Z_4^\top\right]^\top V_{Z_1}$ with $w \in \mathbb{C}^{2j}$, $W \in \mathbb{C}^{2j\times(m+j-1)}$, $K_1 \equiv \left[\Re(W)\ -\Im(W)\right]$, $K_2 \equiv \left[\Im(W)\ \Re(W)\right]$ and $g = \Re(g) + i\Im(g) \equiv \left[\eta_2 \cdots \eta_{m+j-1}\right]^\top$. Then the first term in Equation (11a) equals

$$\sum_{l=1}^{2}\left[\varsigma_l^{-2}\|\tilde{v}_{j+l,S}\|_2^2 + \varsigma_l^{-2}\|\tilde{v}_{j+l,T}\|_2^2\right]$$
$$= \left\{\left[\Re(g)^\top\ \Im(g)^\top\right]H + h^\top\right\}\left[\Re(g)^\top\ \Im(g)^\top\right]^\top + \zeta, \qquad (13)$$

where

$$\begin{aligned}
H &= \frac{1}{\|\tilde{p}_{j+1}\|_2^2}(cK_1 - sK_2)^\top(cK_1 - sK_2) \\
&\quad + \frac{1}{\|\tilde{p}_{j+2}\|_2^2}(sK_1 + cK_2)^\top(sK_1 + cK_2), \\
h &= \frac{2}{\nu_1}\left(\frac{c^2}{\|\tilde{p}_{j+1}\|_2^2} + \frac{s^2}{\|\tilde{p}_{j+2}\|_2^2}\right)K_1^\top\Re(w) \\
&\quad + \frac{2}{\nu_1}\left(\frac{s^2}{\|\tilde{p}_{j+1}\|_2^2} + \frac{c^2}{\|\tilde{p}_{j+2}\|_2^2}\right)K_2^\top\Im(w) \\
&\quad + \frac{2cs}{\nu_1}\left(\frac{1}{\|\tilde{p}_{j+2}\|_2^2} - \frac{1}{\|\tilde{p}_{j+1}\|_2^2}\right)(K_2^\top\Re(w) + K_1^\top\Im(w)), \\
\zeta &= \left(\frac{c^2}{\|\tilde{p}_{j+1}\|_2^2} + \frac{s^2}{\|\tilde{p}_{j+2}\|_2^2}\right)\frac{\|\Re(w)\|_2^2}{\nu_1^2} \\
&\quad + \left(\frac{s^2}{\|\tilde{p}_{j+1}\|_2^2} + \frac{c^2}{\|\tilde{p}_{j+2}\|_2^2}\right)\frac{\|\Im(w)\|_2^2}{\nu_1^2} \\
&\quad + \frac{2cs}{\nu_1^2}\left(\frac{1}{\|\tilde{p}_{j+2}\|_2^2} - \frac{1}{\|\tilde{p}_{j+1}\|_2^2}\right)\Re(w)^\top\Im(w).
\end{aligned}$$

Obviously, $H \succeq 0$. In fact, $H \succ 0$, for if $He = 0$ with $e \in \mathbb{R}^{2m+2j-2}$, and then $K_1 f = K_2 e = 0$ by the definition of H. On the other hand, it follows from the definitions of K_1, K_2 and W that $K_1^\top K_1 + K_2^\top K_2 = I_{2(m+j-1)}$. Hence $e = 0$, and then H is nonsingular. Thus, the minimiser of Equation (13) is given by $\left[\Re(g)^\top\ \Im(g)^\top\right]^\top = -\frac{1}{2}H^{-1}h$.

Once we obtain $g \in \mathbb{C}^{m+j-1}$, $\tilde{v}_{j+1,S}$, $\tilde{v}_{j+2,S}$, $\tilde{v}_{j+1,T}$ and $\tilde{v}_{j+2,T}$ can be computed via Equation (12). Also, we observe that $b = (c + is)V_{Z_1}\left[\frac{1}{\nu_1}\ g^\top\right]^\top$ here, with c and s from the Jacobi orthogonal transformation on $\left[\Re(f_1)\ \Im(f_1)\right]$.

We still need to determine $\tilde{x}_{j+1}$ and $\tilde{x}_{j+2}$, where $\tilde{x}_{j+1} + i\tilde{x}_{j+2} = Q_1 y + Q_2 Q_2^\top A(\tilde{p}_{j+1} + i\tilde{p}_{j+2}) - Q_2\Xi_j(\tilde{v}_{j+1,S} + i\tilde{v}_{j+2,S})$ for some $y \in \mathbb{C}^m$, which gives $x_{j+l} = \|\tilde{p}_{j+l}\|_2^{-1}(Q_1\Re(y) + Q_2 Q_2^\top A\tilde{p}_{j+l} - Q_2\Xi_j\tilde{v}_{j+l,S})$ for $l = 1, 2$. This implies $Q_2^\top x_{j+l} = Q_2^\top A p_{j+l} - \Xi_j v_{j+l,S}$. Again, as stated previously, only $\Xi_{j+2} = Q_2^\top X_{j+2} = Q_2^\top\left[X_j\ x_{j+1}\ x_{j+2}\right]$ is required for the assigning procedure to continue. To compute x_{j+1} and x_{j+2}, we rewrite $y_{j+1} = \frac{\Re(y)}{\|\tilde{p}_{j+1}\|_2}$, $y_{j+2} = \frac{\Im(y)}{\|\tilde{p}_{j+2}\|_2}$, which will be selected in Section 3.4.

Case ii (rank$(Z_1) \geq 2$) Here we shall employ the strategy for placing complex conjugate pairs in Guo et al. (2015). It produces reasonably good suboptimal solution for Equation (11). See Guo et al. (2015) for the details of the sketchy process in the following.

We set $b = V_{Z_1}\left[\frac{e_1}{\nu_1}\ \frac{e_2}{\nu_2}\right]\begin{bmatrix}\gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2\end{bmatrix}$, where $\gamma_1, \gamma_2, \zeta_1, \zeta_2 \in \mathbb{R}$ are to be determined with $\gamma_1^2 + \gamma_2^2 + \zeta_1^2 + \zeta_2^2 = 1$. By f_1 and f_2 denote the left singular vectors of Z_1 corresponding to its two largest singular values ν_1 and ν_2, respectively. It then follows that

$$\tilde{p}_{j+1} + i\tilde{p}_{j+2} = \left[f_1\ f_2\right]\begin{bmatrix}\gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2\end{bmatrix},$$
$$\begin{bmatrix}\tilde{v}_{j+1,S} + i\tilde{v}_{j+2,S} \\ \tilde{v}_{j+1,T} + i\tilde{v}_{j+2,T}\end{bmatrix} = \left[w_1\ w_2\right]\begin{bmatrix}\gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2\end{bmatrix},$$

where $w_l = \frac{1}{\nu_l}\left[V_{Z_1}^\top Z_3^\top\ V_{Z_1}^\top Z_4^\top\right]^\top e_l$ for $l = 1, 2$. In the case of $\Re(f_1)^\top(f_1) = 0$ and $\|\Re(f_1)\|_2 = \|\Im(f_1)\|_2 = 1/\sqrt{2}$, we simply take $\gamma_1 = 1$, $\zeta_1 = \gamma_2 = \zeta_2 = 0$, yielding $\tilde{p}_{j+1} = \Re(f_1)$ and $\tilde{p}_{j+2} = \Im(f_1)$. This actually gives the objective function in Equation (11a) its minimum $2(1 - \nu_1^2)/\nu_1^2$. In general, there are two simple possibilities. One is to apply the Jacobi orthogonal transformation on $\left[\Re(f_1)\ \Im(f_1)\right]$ to produce $\left[\tilde{p}_{j+1}\ \tilde{p}_{j+2}\right]$. This postulates that $\Re(f_1)$ and (f_1) are linearly independent, and the value of the objective function in Equation (11a) equals

$$\begin{aligned}
\varrho_1 &= \frac{\|c\Re(w_1) - s\Im(w_1)\|_2^2}{\|\tilde{p}_{j+1}\|_2^2} + \frac{\|s\Re(w_1) + c\Im(w_1)\|_2^2}{\|\tilde{p}_{j+2}\|_2^2} \\
&\quad + \tau_{j+1}^2\left(\frac{\|\tilde{p}_{j+1}\|_2}{\|\tilde{p}_{j+2}\|_2} - \frac{\|\tilde{p}_{j+2}\|_2}{\|\tilde{p}_{j+1}\|_2}\right)^2 \leq \frac{(\nu_1^{-2} - 1) + \tau_{j+1}^2}{\min\{\|\tilde{p}_{j+1}\|_2^2,\ \|\tilde{p}_{j+2}\|_2^2\}}.
\end{aligned}$$

The other possibility makes use of the following spectral decomposition of the Hamiltonian matrix (Guo et al., 2015):

$$\begin{bmatrix} K_R^\top K_R - K_I^\top K_I & -(K_R^\top K_I + K_I^\top K_R) \\ -(K_R^\top K_I + K_I^\top K_R) & K_I^\top K_I - K_R^\top K_R\end{bmatrix}$$
$$= \Omega\,\mathrm{diag}(\phi_1, \phi_2, -\phi_1, -\phi_2)\Omega^\top,$$

with $K_R = \left[\Re(f_1)\ \Re(f_2)\right]$, $K_I = \left[\Im(f_1)\ \Im(f_2)\right]$, $\phi_1 \geq \phi_2 > 0$. Some γ_1, γ_2, ζ_1, ζ_2 are chosen (essentially determined by ϕ_1, ϕ_2) such that $\tilde{p}_{j+1}^\top\tilde{p}_{j+2} = 0$ and $\|\tilde{p}_{j+1}\|_2 = \|\tilde{p}_{j+2}\|_2 = \frac{1}{\sqrt{2}}$. This eventually gives the objective function in Equation (11a) the value

$$\varrho_2 = 2\sum_{l=1}^{2}\left(\nu_l^{-2} - 1\right)\left(\gamma_l^2 + \zeta_l^2\right) \leq 2\left(\nu_2^{-2} - 1\right).$$

Then we take the possibility corresponding to the minimum of ϱ_1 and ϱ_2, choosing the $(j+1)$th and $(j+2)$th columns of $P_{G,F}$, S and T accordingly. As in **Case i**, we also need to determine, for $l = 1, 2$:

$$x_{j+l} = Q_1 y_{j+l} + Q_2 Q_2^\top A p_{j+l} - Q_2 \Xi_j v_{j+l,S},$$
$$Q_2^\top x_{j+l} = Q_2^\top A p_{j+l} - \Xi_j v_{j+l,S},$$

for some $y_{j+l} \in \mathbb{R}^m$ to be determined in Section 3.4.

Ultimately, we shall update P_j, Ξ_j, S_j and T_j as $P_{j+2} = \left[P_j \; p_{j+1} \; p_{j+2}\right]$, $\Xi_{j+2} = \left[\Xi_j \; Q_2^\top x_{j+1} \; Q_2^\top x_{j+2}\right]$, $S_{j+2} = \left[\begin{array}{c|c} S_j & v_{j+1,S} \; v_{j+2,S} \\ \hline \mathbf{0} & I_2 \end{array}\right]$ and $T_{j+2} = \left[\begin{array}{c|c} T_j & v_{j+1,T} \; v_{j+2,T} \\ \hline \mathbf{0} & D_{\delta_{j+1}(\sigma_{j+1}+i\tau_{j+1})} \end{array}\right]$.

3.3.2 Situation II: $|\alpha_{j+1}| < |\beta_{j+1}|$

Contrasting *Situation I*, here we have $S(j+1:j+2,\, j+1:j+2) = D_{\delta_{j+1}}(\tilde{\sigma}_{j+1} + i\tilde{\tau}_{j+1})$ and $T(j+1:j+2,\, j+1:j+2) = I_2$. Similarly to Equation (10), the $(j+1)$th and $(j+2)$th columns of $P_{G,F}$, $X_{G,F}$, S and T satisfy

$$Q_2^\top A\left[p_{j+1} \; p_{j+2}\right] - \Xi_j\left[v_{j+1,S} \; v_{j+2,S}\right] - Q_2^\top\left[x_{j+1} \; x_{j+2}\right] D_{\delta_{j+1}}(\tilde{\sigma}_{j+1} + i\tilde{\tau}_{j+1}) = 0,$$
$$Q_2^\top E\left[p_{j+1} \; p_{j+2}\right] - \Xi_j\left[v_{j+1,T} \; v_{j+2,T}\right] - Q_2^\top\left[x_{j+1} \; x_{j+2}\right] = 0.$$

As in *Situation I*, after defining $\delta_{j+1} = \varsigma_1/\varsigma_2$ with $\varsigma_1, \varsigma_2 \in \mathbb{R}$ and $\varsigma_2 \neq 0$, we need to solve a constrained optimisation subproblem similar to Equation (11), just with τ_{j+1} replaced by $\tilde{\tau}_{j+1}$ and M_j changed into $M_j = \begin{bmatrix} Q_2^\top A & -(\tilde{\sigma}_{j+1} + i\tilde{\tau}_{j+1})Q_2^\top & -\Xi_j & \mathbf{0} \\ Q_2^\top E & -Q_2^\top & \mathbf{0} & -\Xi_j \\ P_j^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$. Definitely, such proposed optimisation problem can be treated similarly as Equation (11). We skip the details here.

Remark 3.2: Analogously to Section 3.2 when assigning the finite real poles, we need to pay some attention when $j = 0$. Suppose that no infinite poles exist and the first finite poles to be assigned are (α_1, β_1) and $(\bar{\alpha}_1, \bar{\beta}_1)$. Following from the structure of S and T, we just need to compute the first two columns of $P_{G,F}$, $Q_2^\top X_{G,F}$ and S_2, T_2.

If $|\alpha_1| \geq |\beta_1|$ (and neglecting the complementary case), p_1, p_2, x_1, x_2 should be chosen to satisfy

$$\begin{cases} Q_2^\top A\left[p_1 \; p_2\right] - Q_2^\top\left[x_1 \; x_2\right] = 0, \\ Q_2^\top E\left[p_1 \; p_2\right] - Q_2^\top\left[x_1 \; x_2\right] D_{\delta_1}(\sigma_1 + i\tau_1) = 0, \\ \left[p_1 \; p_2\right]^\top\left[p_1 \; p_2\right] = I_2, \end{cases} \tag{14}$$

so that $(\delta_1 - \delta_1^{-1})^2\tau_1^2$ is minimised. This is obviously achieved when $\delta_1 = 1$, where Equation (14) is reduced to Equations (11b) and (11c) with $j = 0$, $\tilde{v}_{j+l,S}, \tilde{v}_{j+l,T}$ empty, $\tilde{p}_{j+l}, \tilde{x}_{j+l}$ replaced by p_l, x_l respectively, and $\varsigma_l = 1$ ($l = 1, 2$). Let the columns of $Z \in \mathbb{C}^{n\times m}$ be an orthonormal basis of $\mathcal{N}(M)$, where $M \equiv Q_2^\top(E - (\sigma_1 + i\tau_1)A)$ is of full row rank by Lemma 3.2 in Section 3.5. Then the columns of $\begin{bmatrix} \mathbf{0} & Z \\ Q_1 & Q_2Q_2^\top AZ \end{bmatrix}$ construct a basis of $\mathcal{N}(M_0)$. Furthermore, there exist $p, x \in \mathbb{C}^n$ with $\{\Re(p), (p)\}$ linearly independent such that $\left[p^\top \; x^\top\right]^\top \in \mathcal{N}(M_0)$. Obviously, $p_1 = \left[\Re(Z) \; -\Im(Z)\right]\left[u_1^\top \; u_2^\top\right]^\top$, $p_2 = \left[\Im(Z) \; \Re(Z)\right]\left[u_1^\top \; u_2^\top\right]^\top$ for some $u_1, u_2 \in \mathbb{R}^m$. Adopting the method in Guo et al. (2015), two Hamiltonian matrices would be constructed and their spectral decompositions lead to $p_1^\top p_2 = 0$ and $\|p_1\|_2 = \|p_2\|_2 = 1$. Also $x_l = Q_1 y_l + Q_2 Q_2^\top A p_l$ for some $y_l \in \mathbb{R}^m$ (from Section 3.4), leading to $Q_2^\top x_l = Q_2^\top A p_l$ for $l = 1, 2$. This is sufficient for the process to continue.

3.4 Determining $X_{G,F}$

We have $X_{G,F} = Q_1 Y + Q_2(Q_2^\top X_{G,F})$, where $Q_2^\top X_{G,F}$ has been computed and $Y = \left[y_1 \cdots y_n\right] \in \mathbb{R}^{m\times n}$ is to be determined. This last gap is to be filled in this section.

Lemma 3.1: *$Q_2^\top X_{G,F}$ is of full row rank.*

Proof: Since the descriptor system (E, A, B) is S-controllable, $\left[E \; AN_\infty \; B\right]$ is of full row rank, where $\mathcal{R}(N_\infty) = \mathcal{N}(E)$. This leads to $\mathrm{rank}(\left[Q_2^\top E \; Q_2^\top AN_\infty\right]) = n - m$. Then $\left[Q_2^\top E \; Q_2^\top A\right]$ is of full row rank, which is equivalent to $\left[Q_2^\top EP_{G,F} \; Q_2^\top AP_{G,F}\right]$ of having full row rank. Also, it follows from Equation (5) that $\left[Q_2^\top X_{G,F} T \; Q_2^\top X_{G,F} S\right]$ is of full row rank, yielding the same for $Q_2^\top X_{G,F}$. □

Rewrite $X_{G,F} = \left[Q_1 \; Q_2\right]\left[Y^\top \; (Q_2^\top X_{G,F})^\top\right]^\top$. It is nonsingular with $Y^\top$ not deficient in the complementary subspace of $\mathcal{R}(X_{G,F}^\top Q_2)$. Moreover, $X_{G,F}$ is hoped to be as well conditioned as possible. Thus, we should choose $Y^\top$ whose orthonormal columns span the complementary subspace of $\mathcal{R}(X_{G,F}^\top Q_2)$. From the QR factorisation $X_{G,F}^\top Q_2 = Q_X\left[R_X^\top \; 0\right]^\top = Q_{1,X} R_X$ with $Q_X = \left[Q_{1,X} \; Q_{2,X}\right] \in \mathbb{R}^{n\times n}$ orthogonal, $Q_{1,X} \in \mathbb{R}^{n\times(n-m)}$ and $R_X \in \mathbb{R}^{(n-m)\times(n-m)}$ non-singular upper triangular, we then select $Y = Q_{2,X}^\top$, leading to $X_{G,F} = Q_1 Q_{2,X}^\top + Q_2(Q_2^\top X_{G,F})$.

3.5 Supporting theorems

Lemma 3.2: $\mathrm{rank}(Q_2^\top(\lambda E - A)) = n - m$ *for any* $\lambda \in \mathbb{C}$.

Proof: Note that $\mathrm{rank}([\lambda E - A \; B]) = n$ for all $\lambda \in \mathbb{C}$ and $\left[Q_1 \; Q_2\right]^\top\left[\lambda E - A \; B\right] = \begin{bmatrix} Q_1^\top(\lambda E - A) & R \\ Q_2^\top(\lambda E - A) & \mathbf{0} \end{bmatrix}$. □

Theorem 3.1: *For an S-controllable descriptor system (E, A, B), assume j infinite poles $(0 \leq j \leq (n - r - 1))$ have already been assigned with $P_j = ZW_j$ and $Q_2^\top X_j$ computed, where the orthonormal columns of $Z \in \mathbb{R}^{n\times l}$ span $Q_2^\top E$, $l = n -$ $\mathrm{rank}(Q_2^\top E)$ and $W_j \in \mathbb{R}^{l\times j}$ satisfies $W_j^\top W_j = I_j$. Define $M = \begin{bmatrix} Q_2^\top AZ & -Q_2^\top & -Q_2^\top X_j \\ W_j^\top & \mathbf{0} & \mathbf{0} \end{bmatrix}$. Then*

(*a*) $\dim(\mathcal{N}(M)) = m + l$; *and*
(*b*) *there exist nonzero $w \in \mathbb{R}^l$, $x \in \mathbb{R}^n$, $v_S \in \mathbb{R}^j$ such that* $\left[w^\top \; x^\top \; v_S^\top\right]^\top \in \mathcal{N}(M)$.

Proof: We first consider *(a)*, which is equivalent to M possessing full row rank. Let $f \in \mathbb{R}^{n-m}$, $h \in \mathbb{R}^j$ be vectors satisfying $\left[f^\top \; h^\top\right]M = 0$, we have $f^\top Q_2^\top AZ + h^\top W_j^\top = 0$ and $f^\top Q_2^\top = 0$. Hence,f and h vanish for Q_2 and W_j are of full column rank, implying the result.

For *(b)*, assume the contrary and we have $\mathrm{rank}(Q_2^\top) = \mathrm{rank}(\left[Q_2^\top \; Q_2^\top X_j\right]) = (n - m) + (j - l)$, implying $j = l$. Since $\mathrm{rank}(\left[E \; B\right]) = m + \mathrm{rank}(Q_2^\top E)$, $l = n - \mathrm{rank}(Q_2^\top E) = n - (\mathrm{rank}(\left[E \; B\right]) - m) \geq n - r$ for $\mathrm{rank}(\left[E \; B\right]) - m \leq r$. On the other hand, $j \le n - r - 1$ since there exists at least one infinite pole that is not placed. Thus, we get a contradiction and *(b)* holds. □

Theorem 3.2: *For an S-controllable descriptor system (E, A, B), assume all infinite poles $\{(\alpha_1, 0), \ldots, (\alpha_{n-r}, 0)\}$ and j finite poles $\{(\alpha_{n-r+1}, \beta_{n-r+1}), \ldots, (\alpha_{n-r+j}, \beta_{n-r+j})\}$ have already been assigned with the computed P_{n-r+j}, $Q_2^\top X_{n-r+j}$, S_{n-r+j} and T_{n-r+j} satisfying*

$$\begin{cases} Q_2^\top AP_{n-r+j} = Q_2^\top X_{n-r+j}S_{n-r+j}, \\ Q_2^\top EP_{n-r+j} = Q_2^\top X_{n-r+j}T_{n-r+j}, \end{cases} \tag{15}$$

where $j < r$ if there is at least one unassigned finite real pole, or $(j + 1) < r$ if there is at least a pair of unassigned complex conjugate poles. Assume that $(\alpha, \beta) \in \mathfrak{L}$ is the finite real pole or $\{(\alpha, \beta), (\bar{\alpha}, \bar{\beta})\} \subseteq \mathfrak{L}$ are the complex conjugate poles to be assigned. Denote

$$M = \begin{bmatrix} Q_2^\top A & -\epsilon_1 Q_2^\top & -Q_2^\top X_{n-r+j} & \mathbf{0} \\ Q_2^\top E & -\epsilon_2 Q_2^\top & \mathbf{0} & -Q_2^\top X_{n-r+j} \\ P_{n-r+j}^\top & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where (i) $\epsilon_1 = \alpha/\sqrt{|\alpha|^2 + |\beta|^2}$, $\epsilon_2 = \beta/\sqrt{|\alpha|^2 + |\beta|^2}$ for $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$; (ii) $\epsilon_1 = 1$ and $\epsilon_2 = \sigma + i\tau$ for $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ and $|\alpha| \ge |\beta|$, with $\sigma = \Re(\bar{\alpha}\beta)/|\alpha|^2$, $\tau = \Im(\bar{\alpha}\beta)/|\alpha|^2$; or (iii) $\epsilon_1 = \tilde{\sigma} + i\tilde{\tau}$, $\epsilon_2 = 1$ for $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$ and $|\alpha| < |\beta|$, with $\sigma = \Re(\bar{\beta}\alpha)/|\beta|^2$ and $\tau = \Im(\bar{\beta}\alpha)/|\beta|^2$. Let the columns of Z be an orthonormal basis of $\mathcal{N}(M)$. Then we have

- *(a)* $\dim(\mathcal{R}(Z)) = 2m + (n - r + j)$;
- *(b)* $Z_1 \neq 0$, *where* $Z = \begin{bmatrix} Z_1^\top & Z_2^\top \end{bmatrix}^\top$ *with* $Z_1 \in \mathbb{R}^{n\times(2m+n-r+j)}$; *and*
- *(c) for $(\alpha, \beta) \in \mathbb{C} \times \mathbb{C}$, there exist $0 \neq p = \Re(p) + i\Im(p) \in \mathbb{C}^n$ with $\{\Re(p), (p)\}$ linearly independent, $x \in \mathbb{C}^n$, $v_S \in \mathbb{C}^{n-r+j}$ and $v_T \in \mathbb{C}^{n-r+j}$ such that $\begin{bmatrix} p^\top & x^\top & v_S^\top & v_T^\top \end{bmatrix}^\top \in \mathcal{R}(Z)$.*

Proof: Suppose $z, y \in \mathbb{C}^{n-m}$ and $w \in \mathbb{C}^{n-r+j}$ satisfy $\begin{bmatrix} z^\top & y^\top & w^\top \end{bmatrix} M = 0$, which is equivalent to

$$z^\top Q_2^\top A + y^\top Q_2^\top E + w^\top P_{n-r+j}^\top = 0, \tag{16a}$$

$$\epsilon_1 z^\top Q_2^\top + \epsilon_2 y^\top Q_2^\top = 0, \tag{16b}$$

$$z^\top Q_2^\top X_{n-r+j} = y^\top Q_2^\top X_{n-r+j} = 0. \tag{16c}$$

Post-multiplying P_{n-r+j} on both sides of Equation (16a) gives $z^\top Q_2^\top AP_{n-r+j} + y^\top Q_2^\top EP_{n-r+j} + w^\top = 0$. Together with Equations (15) and (16c), we get $z^\top Q_2^\top AP_{n-r+j} = 0$ and $y^\top Q_2^\top EP_{n-r+j} = 0$, leading to $w = 0$. Thus, $z^\top Q_2^\top A + y^\top Q_2^\top E = z^\top (Q_2^\top A - \frac{\epsilon_1}{\epsilon_2} Q_2^\top E) = 0$ follows from Equation (16a) and $\epsilon_2 \neq 0$ for all three cases. Furthermore $\operatorname{rank}(Q_2^\top (A - \frac{\epsilon_1}{\epsilon_2} E)) = n - m$ (Lemma 3.2) implies $y = z = 0$. So M is of full row rank, hence *(a)* holds.

Denote $M_2 = \begin{bmatrix} -\epsilon_1 Q_2^\top & -\Xi_{n-r+j} & \mathbf{0} \\ -\epsilon_2 Q_2^\top & \mathbf{0} & -\Xi_{n-r+j} \end{bmatrix}$ with $\Xi_{n-r+j} \equiv Q_2^\top X_{n-r+j}$. To prove *(b)*, we assume the contrary that $Z_1 = 0$. Since Q_2 is of full column rank and

$$\begin{bmatrix} \mathbf{0} & -\Xi_{n-r+j} \, \frac{\epsilon_1}{\epsilon_2} \Xi_{n-r+j} \\ -\epsilon_2 Q_2^\top & \mathbf{0} \quad\quad \mathbf{0} \end{bmatrix} = \begin{bmatrix} I_{n-m} & -\frac{\epsilon_1}{\epsilon_2} I_{n-m} \\ \mathbf{0} & I_{n-m} \end{bmatrix} M_2 \begin{bmatrix} I_n & \mathbf{0} & -\frac{1}{\epsilon_2} X_{n-r+j} \\ \mathbf{0} & I_{n-r+j} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{n-r+j} \end{bmatrix}, \tag{17}$$

we deduce $\operatorname{rank}(\begin{bmatrix} -\Xi_{n-r+j} & \frac{\epsilon_1}{\epsilon_2} \Xi_{n-r+j} \end{bmatrix}) = \operatorname{rank}(\Xi_{n-r+j}) = (n - r + j) - m$, which forces $(n - r + j) \ge m$. If $(n - r + j) < m$, obviously $Z_1 \neq 0$; now consider the complementary case that $(n - r + j) \ge m$. Let $H \in \mathbb{R}^{(n-r+j)\times m}$ satisfy $\Xi_{n-r+j} H = 0$, $H^\top H = I_m$, and write $Y_{n-r+j} = Q\begin{bmatrix} H & \Xi_{n-r+j}^\top \end{bmatrix}^\top$. Then $\operatorname{rank}(\begin{bmatrix} H & \Xi_{n-r+j}^\top \end{bmatrix}^\top) = n - r + j$ and $Q_2^\top Y_{n-r+j} = \Xi_{n-r+j}$. From Equation (15), there exist $W_A, W_E \in \mathbb{R}^{m\times(n-r+j)}$ such that $AP_{n-r+j} = Y_{n-r+j}S_{n-r+j} + BW_A$, $EP_{n-r+j} = Y_{n-r+j}T_{n-r+j} + BW_E$. Moreover, it follows from $Y_{n-r+j}H = Q\begin{bmatrix} H & \Xi_{n-r+j}^\top \end{bmatrix}^\top H = Q_1 = BR^{-1}$ that $B = Y_{n-r+j}HR$. Thus, by denoting $A_{11} = S_{n-r+j} + HRW_A$ and $E_{11} = T_{n-r+j} + HRW_E$, it holds that

$$AP_{n-r+j} = Y_{n-r+j}A_{11}, \; EP_{n-r+j} = Y_{n-r+j}E_{11}. \tag{18}$$

Define $K \in \mathbb{R}^{n\times(r-j)}$ satisfying $K^\top Y_{n-r+j} = 0$ and $K^\top K = I_{r-j}$. Then pre-multiplying

$$L = \begin{bmatrix} (Y_{n-r+j}^\top Y_{n-r+j})^{-1} & \\ & I_{r-j} \end{bmatrix} \begin{bmatrix} Y_{n-r+j}^\top \\ K^\top \end{bmatrix}$$

on both sides of Equation (18) yields $LAP_{n-r+j} = \begin{bmatrix} A_{11}^\top & \mathbf{0} \end{bmatrix}^\top$ and $LEP_{n-r+j} = \begin{bmatrix} E_{11}^\top & \mathbf{0} \end{bmatrix}^\top$. Define $P := \begin{bmatrix} P_{n-r+j} & P_\perp \end{bmatrix}$ where $P^\top P = I_n$. Then

$$LAP = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix}, \; LEP = \begin{bmatrix} E_{11} & E_{12} \\ \mathbf{0} & E_{22} \end{bmatrix}, \; LB = \begin{bmatrix} HR \\ \mathbf{0} \end{bmatrix},$$

where $\begin{bmatrix} A_{12}^\top & A_{22}^\top \end{bmatrix}^\top = LAP_\perp$ and $\begin{bmatrix} E_{12}^\top & E_{22}^\top \end{bmatrix}^\top = LEP_\perp$. Thus, for the system (LEP, LAP, LB) (equivalent to the system (E, A, B)), there are at most j finite poles assignable for the *RPA-PDSF*, obviously contradicting Lemma 2.2 since $j < r$, hence *(b)* holds.

Regarding *(c)*, we just give the proof when $\epsilon_1 = 1$ and $\epsilon_2 = \sigma + i\tau$. (When $\epsilon_1 = \tilde{\sigma} + i\tilde{\tau}$ and $\epsilon_2 = 1$, the proof is similar and ignored.)

If $\operatorname{rank}(M_2) \ge (n - m) + (n - r + j - m) + 2$, then there exist $\begin{bmatrix} \tilde{p}_l^\top & \tilde{x}_l^\top & \tilde{v}_{l,S}^\top & \tilde{v}_{l,T}^\top \end{bmatrix}^\top$, $l = 1, 2$, where $\tilde{p}_l, \tilde{x}_l \in \mathbb{C}^n$ and $\tilde{v}_{l,S}, \tilde{v}_{l,T} \in \mathbb{C}^{n-r+j}$, such that $\tilde{p}_1$ and $\tilde{p}_2$ are linearly independent. Let $\begin{bmatrix} p^\top & x^\top & v_S^\top & v_T^\top \end{bmatrix}^\top = (\xi_1 + i\eta_1)\begin{bmatrix} \tilde{p}_1^\top & \tilde{x}_1^\top & \tilde{v}_{1,S}^\top & \tilde{v}_{1,T}^\top \end{bmatrix}^\top + (\xi_2 + i\eta_2)\begin{bmatrix} \tilde{p}_2^\top & \tilde{x}_2^\top & \tilde{v}_{2,S}^\top & \tilde{v}_{2,T}^\top \end{bmatrix}^\top$, and then we can always find suitable $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R}$ such that the real and imaginary parts of the resulting $p \in \mathbb{C}^n$ are linearly independent.

If $\operatorname{rank}(M_2) = (n - m) + (n - r + j - m) + 1$, it follows from Equation (17) that $\operatorname{rank}(\Xi_{n-r+j}) = \operatorname{rank}(\begin{bmatrix} -\Xi_{n-r+j} & \frac{\epsilon_1}{\epsilon_2} \Xi_{n-r+j} \end{bmatrix}) = (n - r + j) - (m - 1)$. Thus, it is necessary that $(n - r + j) \ge (m - 1)$, which we assume from now on. If *(c)* does not hold, then there exist vectors $0 \neq p \in \mathbb{R}^n$, $\Re(x) + i\Im(x) = x \in \mathbb{C}^n$, $\Re(v_S) + i\Im(v_S) = v_S \in \mathbb{C}^{n-r+j}$,

$\Re(v_T) + i\Im(v_T) = v_T \in \mathbb{C}^{n-r+j}$ such that $\left[p^\top\ x^\top\ v_S^\top\ v_T^\top\right]^\top \in \mathcal{N}(M)$, which is equivalent to

$$Q_2^\top Ap = Q_2^\top \Re(x) + \Xi_{n-r+j}\Re(v_S), \tag{19a}$$

$$Q_2^\top Ep = Q_2^\top(\sigma\Re(x) - \tau\Im(x)) + \Xi_{n-r+j}\Re(v_T), \tag{19b}$$

$$Q_2^\top \Im(x) = -\Xi_{n-r+j}\Im(v_S), \tag{19c}$$

$$Q_2^\top(\sigma\Im(x) + \tau\Re(x)) + \Xi_{n-r+j}\Im(v_T) = 0. \tag{19d}$$

By Equations (19b) and (19c), we have $Q_2^\top Ep = \sigma Q_2^\top \Re(x) + \Xi_{n-r+j}(\tau\Im(v_S) + \Re(v_T))$; by Equations (19c) and (19d), we get $Q_2^\top\Re(x) = \Xi_{n-r+j}(\frac{\sigma}{\tau}\Im(v_S) - \frac{1}{\tau}\Im(v_T))$. Consequently, writing $\widehat{Q}^\top \equiv \left[\Xi_{n-r+j}\ \ Q_2^\top\Re(x)\right]$, we obtain

$$Q_2^\top A\left[P_{n-r+j}p\right] = \widehat{Q}^\top\widehat{S},\ Q_2^\top E\left[P_{n-r+j}p\right] = \widehat{Q}^\top\widehat{T}, \tag{20}$$

where

$$\widehat{S} = \left[\begin{array}{c|c} S_{n-r+j} & \Re(v_S) \\ \hline & 1 \end{array}\right],\ \widehat{T} = \left[\begin{array}{c|c} T_{n-r+j} & \tau\Im(v_S) + \Re(v_T) \\ \hline & \sigma \end{array}\right],$$

and $\mathrm{rank}(\widehat{Q}^\top) = \mathrm{rank}(\Xi_{n-r+j}) = (n-r+j) - m + 1$. Now let $H \in \mathbb{R}^{(n-r+j+1)\times m}$ satisfy $\widehat{Q}^\top H = 0$ and $H^\top H = I_m$. Define $Y_{n-r+j+1} = Q\left[H\ \widehat{Q}\right]^\top$, which is of full column rank, and let $K \in \mathbb{R}^{n\times(r-j-1)}$ be the matrix satisfying $K^\top Y_{n-r+j+1} = 0$ and $K^\top K = I_{r-j-1}$. Then it follows from Equation (20) and $Q_2^\top Y_{n-r+j+1} = \widehat{Q}^\top$ that there exist $L_A, L_E \in \mathbb{R}^{m\times(n-r+j+1)}$ such that $A\left[P_{n-r+j}p\right] = Y_{n-r+j+1}\widehat{S} + BL_A$, $E\left[P_{n-r+j}p\right] = Y_{n-r+j+1}\widehat{T} + BL_E$. Furthermore, it is easy to verify that $B = Y_{n-r+j+1}HR$. Now let $P_\perp \in \mathbb{R}^{n\times(r-j-1)}$ satisfy $P_\perp^\top\left[P_{n-r+j}\ p\right] = 0$ and $P_\perp^\top P_\perp = I_{r-j-1}$. Denoting $A_1 = \widehat{S} + HRL_A$, $E_1 = \widehat{T} + HRL_E$, and writing $Y_{n-r+j+1}^\dagger = (Y_{n-r+j+1}^\top Y_{n-r+j+1})^{-1}Y_{n-r+j+1}^\top$, then simple manipulations show that

$$\begin{bmatrix} Y_{n-r+j+1}^\dagger \\ K^\top \end{bmatrix} A\left[P_{n-r+j}\ p\,\middle|\,P_\perp\right] = \begin{bmatrix} A_1 & A_{12} \\ \mathbf{0} & A_2 \end{bmatrix} \equiv \hat{A},$$
$$\begin{bmatrix} Y_{n-r+j+1}^\dagger \\ K^\top \end{bmatrix} E\left[P_{n-r+j}\ p\,\middle|\,P_\perp\right] = \begin{bmatrix} E_1 & E_{12} \\ \mathbf{0} & E_2 \end{bmatrix} \equiv \hat{E},$$

$$\begin{bmatrix} cY_{n-r+j+1}^\dagger \\ K^\top \end{bmatrix} B = \begin{bmatrix} HR \\ \mathbf{0} \end{bmatrix} \equiv \hat{B},$$

where $A_{12} = Y_{n-r+j+1}^\dagger AP_\perp$, $A_2 = K^\top AP_\perp$, $E_{12} = Y_{n-r+j+1}^\dagger EP_\perp$, $E_2 = K^\top EP_\perp$. Apparently, for the descriptor system $(\hat{E}, \hat{A}, \hat{B})$, which is equivalent to (E, A, B), there are at most $(j+1)$ finite poles assignable. This contradicts the fact that at least $(j+2)$ finite poles are assignable, hence *(c)* holds. □

4. Numerical examples

In this section, we illustrate the performance of our method, denoted by DRSchurS, by applying it to several examples, some from various references and others generated randomly. The DRSchurS algorithm first assigns all infinite poles, and then the finite ones.

Similar to the definition of the precision of the assigned poles in Guo et al. (2015), we define precs $= \max_{n-r+1\le j\le n}(\log|1 - \hat{\lambda}_j/\lambda_j|)$ to characterise the precision of the assigned finite poles, where $\lambda_j = \alpha_j/\beta_j \in \mathbb{C}$ and $\hat{\lambda}_j \in \mathbb{C}$ are the computed eigenvalues of $(A + BF,\ E + BG)$. Implicitly, we expect the number of computed finite eigenvalues to be identical to that of those to be placed. Apparently, smaller 'precs' indicates more

Table 1. Numerical results for Example 4.1 (compare with the method in Varga (2003)).

	Method in Varga (2003)						DRSchurS					
Ex. No.	*precs*	Δ_F^2	$\lVert F\rVert_F$	$\lVert G\rVert_F$	$\kappa_F(Y)$	$\kappa_F(X)$	*precs*	Δ_F^2	$\lVert F\rVert_F$	$\lVert G\rVert_F$	$\kappa_F(X_{G,F})$	$\kappa_F(X)$
1[a]	−4.1	7.80(0)	1.43(0)	1.09(0)	1.57(0)	3.75(0)	−15.65	2.18(0)	1.71(0)	7.28(− 1)	6.11(0)	1.21(1)
2[b]	−12.19	7.55(5)	5.26(2)	1.34(2)	2.19(1)	6.18(1)	−13.49	5.42(0)	3.00(0)	9.95(− 1)	6.90(0)	5.80(2)
3[c]	−14.26	6.13(1)	6.24(0)	6.27(0)	1.49(0)	1.15(1)	−14.61	4.23(0)	5.11(0)	2.59(0)	3.12(0)	6.23(0)
4[d]	−9.09	9.52(0)	1.96(− 2)	9.20(− 3)	1.17(6)	4.88(6)	−11.08	1.75(1)	3.23(0)	2.34(0)	7.95(0)	2.71(4)
5[e]	−6.55	1.78(6)	2.48(1)	6.41(1)	9.72(2)	5.74(0)	−7.34	1.62(1)	9.89(0)	1.59(1)	1.56(2)	4.40(1)
6[e]	−10.63	1.13(8)	5.49(3)	9.65(3)	1.14(1)	3.79(1)	−10.53	3.40(7)	4.67(3)	1.41(3)	4.31(1)	1.47(5)
7[e]	−12.53	8.38(4)	5.54(2)	1.14(3)	5.72(− 1)	5.16(− 1)	−14.23	5.65(1)	1.18(2)	2.97(1)	1.29(1)	1.11(2)
8[e]	−12.45	1.16(4)	6.77(0)	2.76(1)	2.48(2)	3.34(2)	−14.19	2.31(0)	6.18(0)	5.08(1)	9.22(1)	2.60(1)

[a]Kautsky et al. (1989) [b]Duan and Patton (1999) [c]Ren and Zhang (2013) [d]Miminis (1993) [e]Li and Chu (2008)

Table 2. Numerical results for Example 4.1.

Ex. No.	9[f]	9a[f]	10[g]	11[h]	12[h]	12a[h]	13[i]	14[j]	15[k]	16[l]	17[l]	18[m]
precs	−15.86	−17	−15.48	−15.78	−8.52	−17.00	−5.09	−15.09	−15.48	−15.35	−14.81	−5.63
Δ_F^2	7.80(− 1)	2.47(− 32)	2.97(− 1)	2.80(0)	5.03(− 1)	5.00(− 1)	1.18(1)	4.67(− 1)	1.84(0)	2.45(0)	2.72(0)	9.02(0)
$\lVert F\rVert_F$	2.16(0)	1.58(0)	1.23(0)	1.12(0)	1.54(0)	1.27(0)	2.28(0)	4.90(0)	1.92(0)	1.89(0)	2.88(0)	4.41(0)
$\lVert G\rVert_F$	1.45(0)	1.58(0)	1.37(0)	1.89(0)	1.12(0)	1.35(0)	4.41(0)	2.39(0)	3.32(− 2)	1.15(0)	1.46(0)	1.39(0)
$\kappa_F(X_{G,F})$	3.00(0)	3.16(0)	4.09(0)	3.00(0)	3.05(0)	3.06(0)	6.76(0)	3.49(0)	4.00(0)	4.30(0)	6.39(0)	6.41(0)
$\kappa_F(X)$	8.38(15)	3.00(0)	6.40(0)	4.57(0)	1.86(8)	4.00(0)	1.29(11)	3.29(0)	1.53(1)	2.79(1)	3.06(1)	2.12(11)

[f] Duan (1998) [g]Fletcher (1988) [h]Duan and Patton (1998) [i]Duan and Patton (1997) [j]Chaabane et al. (2006) [k]Syrmos and Lewis (1992) [l]Zhang (2013) [m]Zhang (2011)

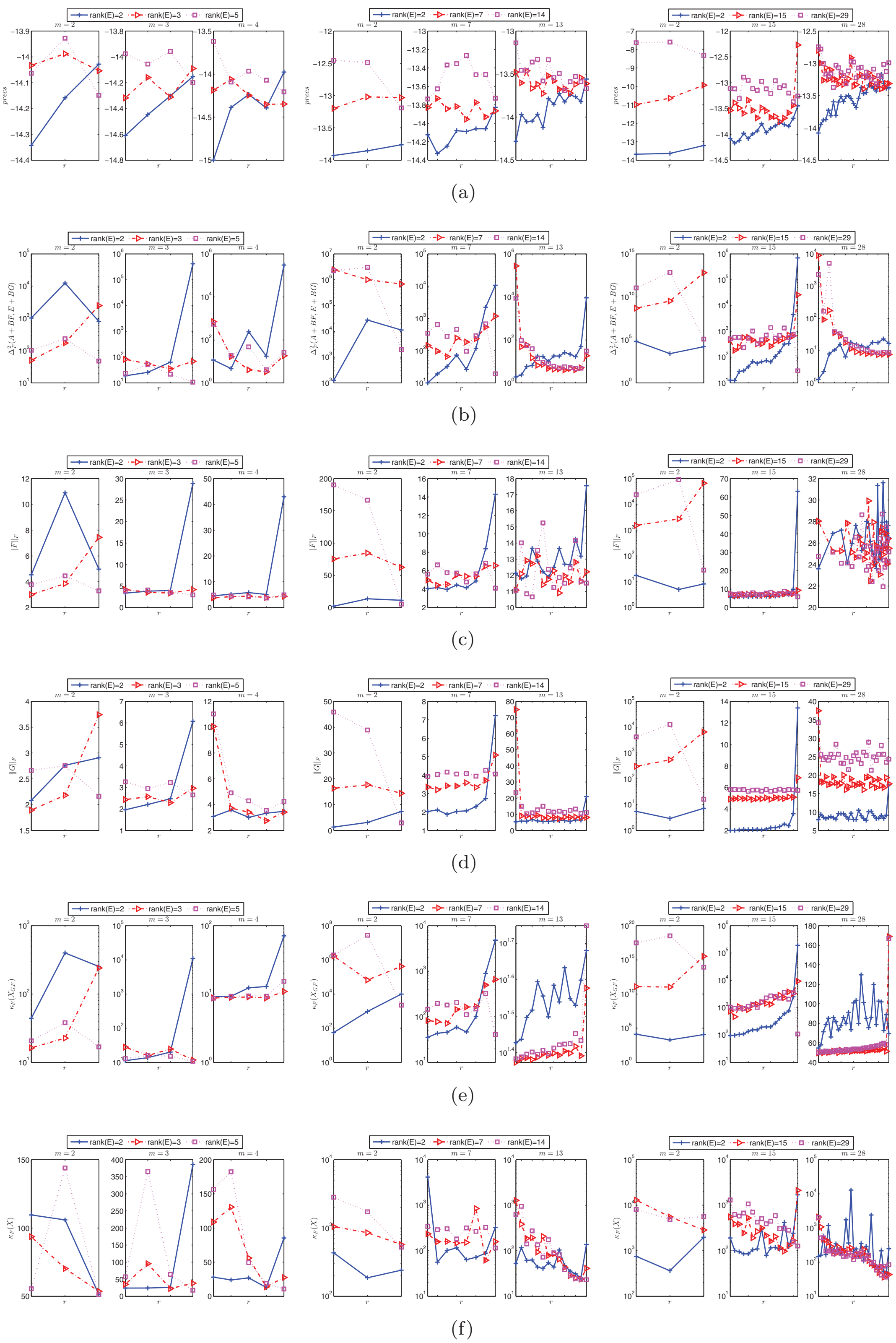

Figure 1. Results for Example 4.2: each row shows three cases for $n = 6, 15, 30$ respectively; (a) Precision of the assigned finite poles; (b) Numerical results for Δ_F^2; (c) Norm of the proportional part F; (d) Norm of the derivative part G; (e) Condition number of $X_{G,F}$; (f) Condition number of X.

accurately computed finite eigenvalues. To reveal the robustness of the closed-loop system, in addition to $\Delta_F^2(A + BF, E + BG)$ in Equation (2) (abbreviated as Δ_F^2 in the following tables and figures), the condition number of the closed-loop generalised eigenvectors matrix will also be displayed. Specifically, assume that $A + BF = Y \operatorname{diag}(\alpha_1, \ldots, \alpha_n)X$, $E + BG = Y \operatorname{diag}(\beta_1, \ldots, \beta_n)X$, where Y and X are non-singular, the Bauer-Fike type theorem then shows that $\kappa_F(X) = \|X\|_F\|X^{-1}\|_F$

measures the sensitivity of the eigenvalues relative to perturbations on the matrix pair $(A + BF, E + BG)$. When determining the non-singular $X_{G,F}$, it is hoped that $X_{G,F}$ would be well-conditioned. Accordingly, $\kappa_F(X_{G,F})$ is given explicitly for all examples. In addition, $\|F\|_F$ and $\|G\|_F$, representing the energy involved in the feedback control, are also displayed.

Example 4.1: This illustrative set includes the examples in Kautsky et al. (1989), Duan and Patton (1999), Duan (1998), Fletcher (1988), Ren and Zhang (2013), Duan and Patton (1998), Duan and Patton (1997), Miminis (1993), Li and Chu (2008), Chaabane et al. (2006), Syrmos and Lewis (1992), Zhang (2013), Zhang (2011), some of which are employed to compare the efficiency of the method proposed in Varga (2003) and DRSchurS here. Tables 1 and 2 present the numerical results, where $\alpha(k) := \alpha \times 10^k$ to save space.

Table 1 shows that DRSchurS produces comparable or occasionally better results than the method proposed in Varga (2003). For Ex. No. 5, the relative accuracy 'precs' produced by DRSchurS is not that high, probably because some poles are close to the imaginary axis. This is possibly a weakness of our algorithm. Note also that the numerical results corresponding to Ex. No. 6 is not that satisfactory, probably due to the difference in magnitudes of the entries in A. DRSchurS produces nice numerical results for Ex. No. 4 except for $\kappa_F(X)$, indicating that it would not be wise to access an algorithm on only one criterion.

For all examples in Table 2, the method put forward in Varga (2003) is not tested since it would fail. The reason may be one of the following — $\lambda(A, E) \cap \mathfrak{L} \neq \emptyset$ or the sizes of the Jordan blocks corresponding to some repeated poles cannot be determined. Note that there are two more tests, in Ex. No. 9a and Ex. No. 12a, which correspond to the inputs in Ex. No. 1 and Ex. No. 5, respectively, but with all finite poles are placed before the infinite ones. Though we cannot prove the feasibility of DRSchurS when all infinite poles are assigned lastly, numerical results demonstrate better performance for certain examples. DRSchurS produces fairly low relative accuracy 'precs' and very large $\kappa_F(X)$ for Ex. No. 13 and Ex. No. 18, both possessing repeated finite poles with algebraic multiplicities greater than m. The treatment of repeated finite poles deserves further investigation.

Example 4.2: This test set contains 255 random examples, where $n = 6, 15, 30$, $\mathrm{rank}(E) = 2, \lfloor \frac{n}{2} \rfloor, n - 1$, and $m = 2, \lfloor \frac{n}{2} \rfloor, n - 2$. For each triple $(n, \mathrm{rank}(E), m)$, the number of the finite poles, denoted by r, increases from $\mathrm{rank}([\,E\ B\,]) - m$ to $\mathrm{rank}([\,E\ B\,])$ in increment of 1. Note that for randomly generated examples, we usually have $\mathrm{rank}([\,E\ B\,]) = \min\{n, \mathrm{rank}(E) + m\}$, bringing $r = \mathrm{rank}(E), \mathrm{rank}(E) + 1, \ldots, \mathrm{rank}(E) + m$ or $r = (n - m), (n - m) + 1, \ldots, n$. All examples are generated randomly as follows. For a fixed quadruple $(n, \mathrm{rank}(E), m, r)$, we first randomly generate five matrices $A, E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $W, Y \in \mathbb{R}^{r\times r}$ by the MATLAB function randn, and set the finite poles as $\mathfrak{L}_1 = \mathrm{eig}(W, Y)$ and $\mathfrak{L} = \{\underbrace{\infty, \ldots, \infty}_{n-r}, \mathfrak{L}_1\}$. Then compute the QR factorisation $E = Q_E R_E$, reset the $(n - \mathrm{rank}(E)) \times (n - \mathrm{rank}(E))$ principal sub-matrix of R_E to 0, and reassign $E = Q_E R_E Q_E^{-1}$. Taking the resulting A, E, B and $\mathfrak{L}$ as the inputs, we apply DRSchurS.

All numerical results are plotted in the following figures. Specifically, with the triple $(n, \mathrm{rank}(E), m)$ fixed, precs, Δ_F^2, $\|F\|_F$, $\|G\|_F$, $\kappa_F(X_{G,F})$ and $\kappa_F(X)$, with respect to r, are displayed in Figure 1(a–f), respectively. For each fixed n, the three sub-figures correspond to $m = 2, \lfloor \frac{n}{2} \rfloor$ and $n - 2$, respectively, where the three different lines match the three distinct $\mathrm{rank}(E) = 2, \lfloor \frac{n}{2} \rfloor, n - 1$. The x-axis represents r, which varies from $\mathrm{rank}(E)$ to $(\mathrm{rank}(E) + m)$ or $(n - m)$ to n, and the values on the y-axis are mean values over 50 trials for a certain quadruple $(n, \mathrm{rank}(E), m, r)$.

Figure 1(a) reveals that DRSchurS can produce high relative accuracy of the assigned finite poles for all the examples except the special case when $n = 30$, $\mathrm{rank}(E) = 29$, $m = 2$. In that case, $r = 28, 29, 30$, and the decline of the relative accuracy can be attributed to the differences between the number of the finite poles and m. In addition, when $\mathrm{rank}(E) = 2$, the value of precs exhibits an ascending trend with respect to r, probably due to the low rank of E. As for Δ_F^2, $\|F\|_F$ and $\|G\|_F$, they all display an ascending trend for $\mathrm{rank}(E) = 2$, but an oscillatory or a downward trend for $\mathrm{rank}(E) = \lfloor \frac{n}{2} \rfloor$, $n - 1$. It shows that $\kappa_F(X_{G,F})$ decreases with respect to m since the freedom in $X_{G,F}$ increases with respect to m.

5. Conclusions

Based on the remarkable results in Bunse-Gerstner et al. (1992), a new direct method DRSchurS for the *RPA-PDSF* is proposed in this paper. Using the generalised real Schur form of the closed-loop system matrix pair, DRSchurS is capable of minimising a robust measure, which is closely related to the departure from normality of the closed-loop system matrix pair, via solving some standard eigen-problems. Several numerical examples demonstrate that DRSchurS solves *RPA-PDSF*, producing robust closed-loop systems with highly accurate finite poles.

For future work, we may further investigate the assignment of repeated finite poles, as well as how the freedom in the first eigenvector for the finite poles and the order of poles in $\mathfrak{L}$ can be best exploited.

Disclosure statement

No potential conflict of interest was reported by the author.

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