DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **8**, Number **4**, October **2002**

Website: http://AIMsciences.org
pp. **815–833**

# COMPUTING INERTIAL MANIFOLDS

James C. Robinson

Mathematics Institute,
University of Warwick,
Coventry CV4 7AL.
U.K.

(Communicated by Roger Temam)

**Abstract.** This paper discusses two numerical schemes that can be used to approximate inertial manifolds whose existence is given by one of the standard methods of proof. The methods considered are fully numerical, in that they take into account the need to interpolate the approximations of the manifold between a set of discrete gridpoints. As all the discretisations are refined the approximations are shown to converge to the true manifold.

1. **Introduction & summary.** Since their introduction by Foias *et al.* [12], inertial manifolds have been an active area of research. These finite-dimensional, exponentially attracting, positively invariant manifolds offer a way of reducing an *a priori* infinite-dimensional partial differential equation to a finite set of ordinary differential equations, via an asymptotic "slaving rule" (cf. Haken [12]).

However, all the current existence results (see later for references) are non-constructive, and hence it is of interest to develop methods that can provide more concrete representations of these manifolds. One approach has given rise to the theory of "approximate inertial manifolds" (Foias *et al.* [7], [10], [11]; Sell [29]), which seeks an explicit manifold that lies close to the global attractor (and is useful even when an exact inertial manifold is not known to exist).

In this paper we develop two computational methods for calculating to within an arbitrary degree of accuracy any inertial manifold whose existence can be proved when a standard spectral gap condition holds (see later). The advantage of such an approach is that one can theoretically relate the approximate manifold obtained to the dynamics of a particular discrete scheme, and (given sufficient computing time) the approximation can be refined with no extra analytical effort. The disadvantage is, as with any numerical scheme, the lack of a closed functional form for the approximation.

Note, however, the convergence of these approximations requires that a slightly strengthened form of the spectral gap condition holds. Since the most interesting questions in fact concern how inertial manifolds vary towards the limits of the standard gap condition (and beyond) these methods are clearly not ideal. However, it is hoped that they will serve as a first step towards more generally applicable methods. We discuss one possibility in the final section.

---

1991 *Mathematics Subject Classification.* 35B40, 35K22, 65M12.
*Key words and phrases.* Inertial manifolds, approximate inertial manifolds.
JCR is a Royal Society University Research Fellow.

1.1. **Preliminaries.** Many interesting partial differential equations have a dissipativity property, meaning that all solutions will eventually enter a bounded set in the phase space. By truncating all the nonlinear terms to be zero outside this bounded absorbing set, the equation is rendered easier to analyse while preserving all its interesting asymptotic behaviour (see Foias *et al.* [9], Mallet-Paret & Sell [16], Robinson [27], or Temam [30]).

In such a way we arrive at the following general dissipative evolution equation on some Hilbert space $H$ (with norm $|\cdot|$),

$$\mathrm{d}u/\mathrm{d}t + Au + f(u) = 0. \tag{1.1}$$

Here $A$ is a positive linear self-adjoint operator with a compact inverse: we denote by $D(A^\alpha)$ the domain of $A^\alpha$, and by $|\cdot|_\alpha$ the norm in this space, i.e.

$$|u|_\alpha = |A^\alpha u|.$$

We assume that the nonlinear term $f$

- has bounded support in $D(A^\alpha)$,

$$\operatorname{supp}(f) \subset \{u : |u|_\alpha \le \rho\}, \tag{1.2}$$

- is globally bounded

$$|f(u)| \le C_0 \qquad u \in D(A^\alpha), \tag{1.3}$$

- and satisfies the global Lipschitz estimate

$$|f(u) - f(v)| \le C_1 |u - v|_\alpha \qquad u, v \in D(A^\alpha). \tag{1.4}$$

For technical reasons we take $0 \le \alpha \le 1/2$, although this restriction can be relaxed (see Chow *et al.* [1] or Rodriguez Bernal [28]: in fact one can take $f : D(A^\alpha) \to D(A^\beta)$ with $0 \le \alpha - \beta < 1$).

In particular these conditions imply that corresponding to each initial condition $u_0 \in D(A^\alpha)$ there exists a unique solution of (1.1) – see Henry [13] and Miklavčič [18].

Since $A$ is self-adjoint and $A^{-1}$ is compact, $A$ has a set of eigenvalues $\lambda_j$ and corresponding eigenfunctions $w_j$,

$$Aw_j = \lambda_j w_j \qquad \lambda_{j+1} \ge \lambda_j$$

(Renardy & Rogers [20]). The generalised Fourier basis $\{w_j\}_{j=1}^\infty$ gives rise to a sequence of finite-dimensional projection operators $P_N$ defined by

$$P_N u = \sum_{j=1}^N (u, w_j) w_j.$$

For an $N$ with $\lambda_{N+1} \ne \lambda_N$ we define the orthogonal complement of $P_N$, $Q_N$,

$$Q_N u = \sum_{j=N+1}^\infty (u, w_j) w_j.$$

All current existence results give the inertial manifold as the graph of some Lipschitz function $\phi : P_n H \to Q_n H \cap D(A^\alpha)$ with

$$|\phi(p_1) - \phi(p_2)|_\alpha \le |p_1 - p_2|_\alpha. \tag{1.5}$$

An inertial manifold of this form is known to exist provided that there exist large enough gaps in the spectrum of the linear term $A$, i.e. if the spectral gap condition

$$\lambda_{n+1} - \lambda_n > 2C_1(\lambda_n^\alpha + \lambda_{n+1}^\alpha) \tag{1.6}$$

holds.

This paper presents two numerical methods that can be used to approximate the function $\phi$ that gives the inertial manifold. There are three stages to the "fully numerical" method presented here. First the spatial dependence is discretised using a Galerkin truncation (section 2); then the time is made discrete (two simple time-stepping methods are introduced in section 3); and finally some interpolation of $\phi$ (and its approximations along the way) is needed: one possible interpolation scheme is also introduced in section 3.

In section 4 a numerical scheme based on the Lyapunov-Perron fixed point method is shown to converge, while in section 5 we analyse a procedure based on the Hadamard graph transform.

## 2. **The Galerkin truncation.**

The first step is to make the spatial dependence discrete by using a Galerkin truncation, which also has the happy effect of turning the original, infinite-dimensional, problem into a finite-dimensional one.

Setting all modes higher than $N$ to zero gives

$$\mathrm{d}u_N/\mathrm{d}t + Au_N + P_N f(u_N) = 0 \qquad u_N \in P_N H. \tag{2.1}$$

Under the gap condition (1.6) these truncations possess inertial manifolds given as the graphs of functions $\phi_N : \mathbb{R}^n \to \mathbb{R}^{N-n}$,

$$\mathcal{M}_N = \mathcal{G}[\phi_N] \equiv \{p + \phi(p_N) : p \in \mathbb{R}^n\}.$$

One can show by various methods (Foias *et al.* [9]; Robinson [25]; Temam [30]) that $\phi_N \to \phi$ in the $C^0$ topology as $N \to \infty$. (In Jones & Stuart [14] and Jones *et al.* [15], the much stronger result of $C^1$ convergence of the manifolds is obtained.)

Rather than write (2.1) repeatedly, in the remainder of this paper we consider the finite-dimensional system of ordinary differential equations on $\mathbb{R}^N$,

$$\mathrm{d}u/\mathrm{d}t + Au + f(u) = 0, \qquad u \in \mathbb{R}^N \tag{2.2}$$

where $A$ is a positive symmetric $N \times N$ matrix, and $f$ satisfies

$$\begin{aligned} |f(u)| &\le C_0 & u &\in \mathbb{R}^N, \\ |f(u) - f(v)| &\le C_1|u - v| & u, v &\in \mathbb{R}^N, \quad \text{and} \\ \mathrm{supp}(f) &\subset \Omega_\rho \equiv \{u : \ |u| \le \rho\}. \end{aligned}$$

In this case the spectral gap condition (1.6) becomes (with $\alpha = 0$)

$$\lambda_{n+1} - \lambda_n > 4C_1. \tag{2.3}$$

Our task is to approximate $\phi_N$ computationally.

## 3. **Elements of the numerical approach.**

In this section we analyse two aspects of making the problem discrete. First we introduce two discrete time-stepping schemes, and prove that they converge as the timestep is refined. Secondly we consider notions of interpolation.

We will use the notation $\mathcal{F}_l^n$ (often abbreviated $\mathcal{F}_l$ when the value of $n$ is clear) to denote globally bounded $l$-Lipschitz functions from $\mathbb{R}^n$ into $\mathbb{R}^{N-n}$,

$$\begin{gathered} |\phi(p_1) - \phi(p_2)| \le l|p_1 - p_2|, \\ \|\phi\| \equiv \sup_{p \in \mathbb{R}^n} |\phi(p)| < \infty, \text{ and} \\ \mathrm{supp}(\phi) \subset P_n\Omega_\rho. \end{gathered} \tag{3.1}$$

3.1. **Discrete time-stepping.** In this section the time evolution of the equation

$$\mathrm{d}u/\mathrm{d}t + Au + f(u) = 0 \qquad u \in \mathbb{R}^N \tag{3.2}$$

is approximated using a simple one-step method.

We give a brief derivation of the methods based on the variation of constants formula,

$$u(t+h) = \mathrm{e}^{-Ah}u(t) - \int_t^{t+h} \mathrm{e}^{-A(t+h-s)} f(u(s))\,\mathrm{d}s.$$

The idea is to retain the first term unchanged and to approximate the integrand by $f(u(t))$ or $f(u(t+h))$ to give

$$u_{k+1} = \mathrm{e}^{-Ah}u_k - hf(u_k) \tag{3.3}$$

or

$$u_{k+1} = \mathrm{e}^{-Ah}u_k - hf(u_{k+1}) \tag{3.4}$$

(a simple application of the contraction mapping theorem to the map $\mathcal{J}x = \mathrm{e}^{-Ah}y + hf(x)$ shows that this method has is well-defined). The first of these two methods is akin to that of Demengel & Ghidaglia [4], who consider

$$u_{k+1} = (I + Ah)^{-1}[u_k - hf(u_k)] :$$

it is clear that these methods agree to first order in $h$. However, the two methods above are more suitable for what follows.

The following proposition shows that both these methods are accurate to $O(h)$ on bounded time intervals.

**Proposition 3.1.** *For both (3.3) and (3.4) there exists a constant $K = K(T, X)$ such that*

$$|u(kh) - u_k| \le Kh,$$

*for all $0 \le kh \le T$, and all $u_0 \in X$, a bounded set in $\mathbb{R}^N$. I.e. the schemes converge uniformly on bounded time intervals and bounded sets in $\mathbb{R}^N$.*

*Proof.* The exact solution is given by

$$u((k+1)h) = \mathrm{e}^{-Ah}u(kh) - \int_0^h \mathrm{e}^{-A(h-s)} f(u(kh+s))\,\mathrm{d}s$$

and the one-step method has

$$u_{k+1} = \mathrm{e}^{-Ah}u_k - \int_0^h f(u_\bullet)\,\mathrm{d}s,$$

where $u_\bullet$ represents $u_k$ or $u_{k+1}$ depending on whether the method is explicit or implicit.

Thus, writing $\epsilon_k = u(kh) - u_k$,

$$\begin{aligned}\epsilon_{k+1} &= \mathrm{e}^{-Ah}\epsilon_k + \int_0^h f(u_\bullet) - \mathrm{e}^{-A(h-s)} f(u(kh+s))\,\mathrm{d}s \\ &= \mathrm{e}^{-Ah}\epsilon_k + \int_0^h f(u_\bullet) - f(u(kh+s))\,\mathrm{d}s + \int_0^h (I - \mathrm{e}^{-A(h-s)}) f(u(kh+s))\,\mathrm{d}s.\end{aligned}$$

The norm of $\mathrm{e}^{As} - 1$ is bounded by $\mathrm{e}^{\lambda_N s} - 1 \le 2\lambda_N h$ for sufficiently small $h$, and so

$$|\epsilon_{k+1}| \le \mathrm{e}^{-h\lambda_1}|\epsilon_k| + C_1 \int_0^h |u(kh+s) - u_\bullet|\,\mathrm{d}s + 2C_0\lambda_N h^2.$$

Now all that is needed is a bound on the integrand,

$$|u(kh+s)-u_\bullet| \le |\epsilon_\bullet| + |u(kh+s)-u(\bullet h)|,$$

so that the change in $u(t)$ between $t=kh$ and $t=(k+1)h$ must be taken into account.

Now note that $|u(t)| \le K_1(u(0))$ for all $t \ge 0$:

$$\begin{aligned} |u(t)| &\le \mathrm{e}^{-\lambda_1 t}|u(0)| + \int_0^\infty \mathrm{e}^{-\lambda_1 s} C_0 \,\mathrm{d}s \\ &\le |A^\alpha u(0)| + C_0/\lambda_1 \\ &\equiv K_1(X) \end{aligned}$$

(using the variation of constants formula). It follows that

$$|-Au(t) - f(u(t))| \le K_1\lambda_N + C_0 \equiv K_2,$$

and so certainly $|u(t)-u(s)| \le K_2|t-s|$.

Thus

$$|\epsilon_{k+1}| \le \mathrm{e}^{-h\lambda_1}|\epsilon_k| + C_1\int_0^h |\epsilon_\bullet| + K_2 h \,\mathrm{d}s + 2C_0\lambda_N h^2,$$

which gives

$$|\epsilon_{k+1}| \le C_\bullet(h)|\epsilon_k| + K_3 h^2$$

with $K_3 = C_1K_2 + 2C_0\lambda_N$ and $C_\bullet(h) = \mathrm{e}^{-h\lambda_1} + C_1 h$ for the explicit scheme, and $\mathrm{e}^{-h\lambda_1}/(1-C_1h)$ for the implicit scheme. In both cases $C_\bullet(h) \sim 1 + (C_1-\lambda_1)h$ as $h \to 0$.

Since $\epsilon_0 = 0$ it follows that

$$|\epsilon_k| \le K_3(X)\, h^2\, \frac{C^k-1}{C-1}.$$

Writing $\Lambda = 2(C_1-\lambda_1)/3$ and using the asymptotics of $C_\bullet$ as $h \to 0$,

$$1 + \Lambda h \le C_\bullet \le 1 + 2\Lambda h$$

for $h$ small enough. Thus, for $kh \in [0,T]$,

$$\begin{aligned} |\epsilon_k| &\le K_3 h^2\, \frac{(1+2\Lambda h)^{T/h}}{\Lambda h} \\ &\le 2K_3\Lambda^{-1}\mathrm{e}^{2\Lambda T}\, h \end{aligned}$$

where the final algebraic step follows since $(1+x/k)^k \to \mathrm{e}^x$ as $k \to \infty$, and so $(1+x/k)^k \le 2\mathrm{e}^x$ for $k$ large enough, i.e. $h$ small enough. □

3.2. **Interpolation.** One way in which this paper differs significantly from previous works is that in seeking a computational method it is necessary to take into account that it is only possible to deal with a discrete set of gridpoints over which the "manifold function" $\phi$ is defined. So issues of interpolating $\phi$ from these points will be important.

For some given set of gridpoints $G \subset \mathbb{R}^n$ we introduce the discrete analogue of the space $\mathcal{F}_l^n$, the space $\mathcal{G}_l^G$ consisting of functions $\psi : G \to \mathbb{R}^{N-n}$ that satisfy

$$|\psi(g)| < \infty \;\forall\; g \in G$$

$$|\psi(g_1) - \psi(g_2)| \le l|g_1 - g_2| \;\forall\; g_1, g_2 \in G,$$

and define a norm on $\mathcal{G}$ by

$$\|\psi\|_G \equiv \sup_{g\in G} |\psi(g)|.$$

The interpolation scheme used in the analysis here is in fact obtained from an extension theorem due to McShane [17]. Although easy to calculate, the interpolation coarsens the Lipschitz constant of the discrete function. Denoting the interpolation by $\phi \mapsto \tilde{\phi}$, it will be shown that

$$\mathrm{Lip}(\tilde{\phi}) \leq \sigma\ \mathrm{Lip}_{\mathcal{G}}(\phi), \tag{3.5}$$

where $\mathrm{Lip}(\psi)$ is the Lipschitz constant of $\psi \in \mathcal{F}_l$ and $\mathrm{Lip}_{\mathcal{G}}(\chi)$ is the "discrete Lipschitz constant" for $\chi \in \mathcal{G}$ (i.e. $l$ for $\chi \in \mathcal{G}_l$). In other words,

$$\phi \mapsto \tilde{\phi} : \mathcal{G}_l^G \to \mathcal{F}_{\sigma l}.$$

Ideally the inequality would be an equality with $\sigma = 1$ (that such an "ideal" extension *does* exist is shown by Federer [5]).

In the statement of the proposition,

$$\|f\|_K = \sup_{x\in K} |f(x)|.$$

**Proposition 3.2.** *Let $G$ be a discrete set of points contained within a compact subset $K$ of $\mathbb{R}^n$, and $\phi : G \to \mathbb{R}^m$ a Lipschitz continuous function such that*

$$|\phi(x) - \phi(y)| \leq C|x-y| \qquad x, y \in G. \tag{3.6}$$

*Then the function $\tilde{\phi}$, with components*

$$\tilde{\phi}_j(x) = \sup_{y\in G}\big[\phi_j(y) - C|x-y|\big],$$

*is an interpolant of $\phi$ and (3.5) holds with $\sigma = \sqrt{m}$. Furthermore the interpolation scheme converges as the grid is refined, in that if*

$$s(G) \equiv \sup_{\phi\in\mathcal{F}} \|\widetilde{[\phi]_G} - \phi\|_K,$$

*where $[f]_G$ denotes the restriction of the function $f$ to $G$, and*

$$\delta(G) = \sup_{x\in K} \inf_{g\in G} |x-g|$$

*then*

$$s(G) \leq 2\sigma\delta(G).$$

*Proof.* Each component $\phi_j(x)$ of $\phi(x)$ satisfies (3.6). It is straightforward to show (see [17]) that $\tilde{\phi}_j$ is an extension (and hence interpolant in this case) of $\phi_j$ which also satisfies (3.6). That $\sigma = \sqrt{m}$ is almost immediate since

$$\begin{aligned} |\tilde{\phi}(x) - \tilde{\phi}(y)|^2 &= \sum_{j=1}^m |\tilde{\phi}_j(x) - \tilde{\phi}_j(y)|^2 \\ &\leq \sum_{j=1}^m C^2|x-y|^2 \\ &= mC^2|x-y|^2. \end{aligned}$$

The convergence result follows since $\phi - \widetilde{[\phi]_G}$ has Lipschitz constant at most $1 + \sigma \leq 2\sigma$, and any point in $K$ lies within $\delta(G)$ of one of the points in $G$ on which $\phi - \widetilde{[\phi]_G} = 0$. $\square$

The following simple observation will be useful in what follows. If $f$ and $g$ have Lipschitz constants $L_f$ and $L_g$ respectively, then

$$\|f-g\| \leq \|f-g\|_G + (L_f+L_g)\delta(G). \tag{3.7}$$

4. **An approximation scheme based on the Lyapunov-Perron method.** We first use the Lyapunov-Perron fixed point method to provide a candidate approximation scheme. In what follows $P=P_n$ and $Q=Q_n$.

In the continuous case the method involves finding the fixed point of an integral operator, and this corresponds to an invariant manifold for the equation (Chow *et al.* [1]; Foias *et al.* [9]; Henry [13]; Miklavčič [19]; Rodriguez-Bernal [28]; Temam [30]). For a given Lipschitz function $\phi \in \mathcal{F}^n$ and point $p_0 \in \mathbb{R}^n$ denote by $p(t)$ the solution of the equation

$$\mathrm{d}p/\mathrm{d}t + Ap + Pf(p+\phi(p)) = 0 \qquad p(0)=p_0;$$

the integral operator $T$, which maps $\phi$ into another function, is defined by

$$[T\phi](p_0) = -\int_{-\infty}^{0} \mathrm{e}^{\tau AQ} Qf(p+\phi(p))\,\mathrm{d}\tau.$$

Under the gap condition (2.3) $T$ is a contraction mapping on $\mathcal{F}_1$,

$$\|T\phi - T\psi\|_{\leq}\kappa_T\|\phi-\psi\| \tag{4.1}$$

for some $\kappa_T < 1$.

For the discrete time method used here, $T$ has to be replaced by the sum

$$[\hat{T}\phi](p_0) = -h\sum_{k=0}^{\infty} \mathrm{e}^{-Akh} Qf(p_{-k}+\phi(p_{-k})), \tag{4.2}$$

where the $p_k$ satisfy the implicit relation from (3.4) with $q_k=\phi(p_k)$ enforced for all $k$:

$$p_{k+1} = \mathrm{e}^{-Ah}p_k - hPf(p_{k+1}+\phi(p_{k+1})), \tag{4.3}$$

with $p_0$ specified. Fixed points of $\hat{T}$ in $\mathcal{F}^n$ will be invariant manifolds for the discrete implicit scheme

$$u_{k+1} = \mathrm{e}^{-Ah}u_k - hf(u_{k+1}). \tag{4.4}$$

Indeed, if $\phi = \hat{T}\phi$ then

$$\begin{aligned}
\phi(p_1) &= -h\sum_{k=0}^{\infty} \mathrm{e}^{-Akh} Qf(p_{1-k}+\phi(p_{1-k})) \\
&= -h\mathrm{e}^{-Ah}\sum_{k=0}^{\infty} \mathrm{e}^{-Akh} Qf(p_{-k}+\phi(p_{-k})) \\
&\qquad\qquad\qquad -h\,Qf(p_1+\phi(p_1)) \\
&= \mathrm{e}^{-Ah}\phi(p_1) - h\,Qf(p_1+\phi(p_1)),
\end{aligned}$$

the appropriate $Q$ component of the solution on the invariant manifold.

The contraction mapping argument applied to $\hat{T}$ will show the existence of an invariant manifold. Note that this approach is similar to that of Demengel & Ghidaglia [4], except that the form of $\hat{T}$ is simplified by careful choice of the scheme (4.4).

The following simple lemma, on the growth of deviations in the successive backward iterates of the finite-dimensional $p$ equation, is necessary in the proof of the proposition.

**Lemma 4.1.** *The separation* $\delta_k = |p_{-k} - \bar{p}_{-k}|$ *of the solutions of*

$$p_{-(k+1)} = \mathrm{e}^{Ah}[p_{-k} + h\,Pf(p_{-k} + \phi(p_{-k}))], \qquad p_0 \text{ given,}$$

*and*

$$\bar{p}_{-(k+1)} = \mathrm{e}^{Ah}[\bar{p}_{-k} + h\,Pf(\bar{p}_{-k} + \bar{\phi}(\bar{p}_{-k}))], \qquad \bar{p}_0 \text{ given,}$$

*satisfies*

$$\delta_k \leq \mathrm{e}^{kh\gamma_h}[\delta_0 + khC_1\Delta\phi], \tag{4.5}$$

*where*

$$\Delta\phi = \sup_{p\in\mathbb{R}^n} |\phi(p) - \bar{\phi}(p)|$$

*and* $\gamma_h(l) \to \lambda_n + (1+l)C_1$ *as* $h \to 0$.

*Proof.* We have

$$p_{-(k+1)} - \bar{p}_{-(k+1)} = \mathrm{e}^{Ah}[p_{-k} - \bar{p}_{-k} + h\{Pf(p_{-k} + \phi(p_{-k})) - Pf(\bar{p}_{-k} + \bar{\phi}(\bar{p}_{-k}))\}],$$

and so

$$\begin{aligned} \delta_{k+1} &\leq \mathrm{e}^{\lambda_n h}[\delta_k + hC_1(\delta_k + |\phi(p_{-k}) - \bar{\phi}(\bar{p}_{-k})|)] \\ &\leq \mathrm{e}^{\lambda_n h}[\delta_k + hC_1\{(1+l)\delta_k + \Delta\phi\}] \\ &= \mathrm{e}^{\lambda_n h}[1 + (1+l)hC_1]\delta_k + hC_1\mathrm{e}^{\lambda_n h}\Delta\phi. \end{aligned}$$

Thus

$$\delta_k \leq \mathcal{E}^k\delta_0 + \frac{\mathcal{E}^k - 1}{\mathcal{E} - 1}\,hC_1\Delta\phi, \tag{4.6}$$

with

$$\mathcal{E} = \mathrm{e}^{\lambda_n h}[1 + (1+l)hC_1] \equiv \mathrm{e}^{h\gamma_h},$$

where $\gamma_h$ is defined by

$$\gamma_h(l) = \lambda_n + \frac{1}{h}\ln\{1 + (1+l)hC_1\} \tag{4.7}$$

and clearly has the behaviour given in the lemma. The estimate (4.5) now follows, since

$$\frac{x^k - 1}{x - 1} = \sum_{j=0}^{k-1} x^j.$$

□

4.1. **The operator $\hat{T}$ as a contraction mapping.** Lemma 4.1 is now applied in the analysis of $\hat{T}$.

**Proposition 4.2.** *Provided that* $\lambda_{n+1} - \gamma_h(l) > 0$ *then* $\hat{T}$ *maps* $\mathcal{F}_l$ *into* $\mathcal{F}_{\tilde{l}}$, *where*

$$\tilde{l} = \frac{(1+l)C_1}{\lambda_{n+1} - \gamma_h(l)}. \tag{4.8}$$

*Proof.* Clearly $|\hat{T}\phi|$ is bounded by

$$\begin{aligned} |\hat{T}\phi| &\leq h\sum_{k=0}^{\infty} e^{-\lambda_{n+1}kh}C_0 \\ &\leq C_0\int_0^{\infty} \mathrm{e}^{-\lambda_{n+1}t}\,\mathrm{d}t \\ &= C_0/\lambda_{n+1}, \end{aligned}$$

where the sum is bounded by the integral since the terms are decreasing.

To show that $\hat{T}\phi$ is Lipschitz, consider

$$\begin{aligned} |\hat{T}\phi(p) - \hat{T}\phi(\bar{p})| &\leq h\sum_{k=0}^{\infty} e^{-\lambda_{n+1}kh}|Qf(p_{-k}+\phi(p_{-k})) - Qf(\bar{p}_{-k}+\phi(\bar{p}_{-k}))| \\ &\leq h\sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_{n+1}kh}(1+l)C_1|p_{-k}-\bar{p}_{-k}|. \end{aligned}$$

Using the estimate (4.5) with $\Delta\phi = 0$ gives

$$\leq (1+l)C_1 h\sum_{k=0}^{\infty} \mathrm{e}^{-[\lambda_{n+1}-\gamma_h]kh}|p-\bar{p}|,$$

and provided $\lambda_{n+1} > \gamma_h$ the terms are decreasing and the sum can be bounded, as above, by an integral

$$\leq (1+l)C_1|p-\bar{p}|\int_0^{\infty} \mathrm{e}^{-[\lambda_{n+1}-\gamma_h]t}\,\mathrm{d}t$$

to obtain

$$|\hat{T}\phi(p) - \hat{T}\phi(\bar{p})| \leq \frac{(1+l)C_1}{\lambda_{n+1}-\gamma_h}|p-\bar{p}|.$$

□

**Proposition 4.3.** *Provided that*

$$\lambda_{n+1} - \gamma_h(l) > (1+l^{-1})C_1 \tag{4.9}$$

*holds, $\hat{T}$ is a contraction mapping on $\mathcal{F}_l$,*

$$\|\hat{T}\phi - \hat{T}\bar{\phi}\| \leq \kappa_{\hat{T}}\|\phi - \bar{\phi}\| \tag{4.10}$$

*for some $\kappa_{\hat{T}} < 1$. It follows that the implicit scheme (3.4) has an invariant manifold given as the graph of some function $\phi \in \mathcal{F}_l$.*

Note that, using (4.7), the condition (4.9) is in fact a spectral gap condition once again, namely

$$\lambda_{n+1} - \lambda_n > (1+l)^{-1}C_1 + \frac{1}{h}\ln(1+(1+l)C_1).$$

With $l = 1$ in the limit as $h \to 0$ this reduces to the standard gap condition (2.3)

$$\lambda_{n+1} - \lambda_n > 4C_1.$$

*Proof.* First, it is clear from (4.8) that (4.9) implies that $\hat{T}$ maps $\mathcal{F}_l$ into itself. To show that $\hat{T}$ is a contraction write

$$|\hat{T}\phi(p) - \hat{T}\bar{\phi}(p)| = \left|h\sum_{k=0}^{\infty} \mathrm{e}^{-Akh}\{Qf(p_{-k}+\phi(p_{-k})) - Qf(\bar{p}_{-k}+\bar{\phi}(\bar{p}_{-k}))\}\right|$$

$$\begin{aligned}
&\leq \left| h \sum_{k=0}^{\infty} e^{-\lambda_{n+1}kh}[Qf(p_{-k}+\phi(p_{-k})) - Qf(p_{-k}+\bar{\phi}(p_{-k})) \right. \\
&\qquad \left. + Qf(p_{-k}+\bar{\phi}(p_{-k})) - Qf(\bar{p}_{-k}+\bar{\phi}(\bar{p}_{-k}))] \right| \\
&\leq h \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_{n+1}kh}[C_1\Delta\phi + |Qf(p_{-k}+\bar{\phi}(p_{-k})) - Qf(\bar{p}_{-k}+\bar{\phi}(\bar{p}_{-k}))|] \\
&\leq h \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda_{n+1}kh}[C_1\Delta\phi + (1+l)C_1|p_{-k}-\bar{p}_{-k}|].
\end{aligned}$$

Now using the estimate from (4.5) this gives

$$\begin{aligned}
|\hat{T}\phi(p) - \hat{T}\bar{\phi}(p)| &\leq \frac{C_1\Delta\phi}{\lambda_{n+1}} + (1+l)(hC_1)^2\Delta\phi \sum_{k=0}^{\infty} k\mathrm{e}^{-[\lambda_{n+1}-\gamma_h]kh} \\
&\leq \left[\frac{C_1}{\lambda_{n+1}} + (1+l)hC_1^2 \int_0^\infty t\mathrm{e}^{-[\lambda_{n+1}-\gamma_h]t}\,\mathrm{d}t\right]\Delta\phi.
\end{aligned}$$

Thus

$$\|T\phi - T\bar{\phi}\| \leq \left[\frac{C_1}{\lambda_{n+1}} + \frac{(1+l)C_1^2h}{(\lambda_{n+1}-\gamma_h)^2}\right]\|\phi-\bar{\phi}\|,$$

and so for $\hat{T}$ to be a contraction the condition is

$$\frac{C_1}{\lambda_{n+1}} + \frac{(1+l)C_1^2h}{(\lambda_{n+1}-\gamma_h)^2} < 1.$$

Since the spectral gap condition (4.9) implies in particular that

$$\lambda_{n+1} - \gamma_h > (1+l)C_1$$

for $h$ small enough, $\hat{T}$ is a contraction provided that

$$\frac{C_1}{\lambda_{n+1}} + \frac{h}{1+l} < 1.$$

The important term here is clearly the first, and it follows from the spectral gap condition that $\lambda_{n+1} > (1+l^{-1})C_1$: thus $\hat{T}$ is a contraction provided that $h$ is small enough ($h < (1+l)/(1+l^{-1})$ certainly suffices). □

Since $\hat{T}$ has a fixed point, the implicit scheme (3.4) has an invariant manifold given as the graph of some function $\phi_h \in \mathcal{F}_1$. It is simple to show that $\phi_h$ converges to $\phi_N$ as $h \to 0$.

**Proposition 4.4.** *Provided that*

$$\lambda_{n+1} - \gamma_h(1) > 2C_1$$

*we have* $\phi_h \to \phi_N$ *as* $h \to 0$.

*Proof.* Given $\epsilon > 0$, first choose $T$ so large that

$$\begin{aligned}\left|T\phi(p) + \int_{-T}^{0} \mathrm{e}^{sAQ} f(p(s) + \phi(p(s)))\,\mathrm{d}s\right| &= \left|\int_{-\infty}^{-T} \mathrm{e}^{sAQ} f(p(s) + \phi(p(s)))\,\mathrm{d}s\right| \\ &\leq \int_{-\infty}^{-T} \mathrm{e}^{s\lambda_{n+1}} C_0\,\mathrm{d}s \\ &= C_0 \mathrm{e}^{-T\lambda_{n+1}} \\ &\leq \epsilon/3\end{aligned}$$

and (arguing similarly) that

$$\left\|\hat{T}\phi + h\sum_{k=0}^{T/h} \mathrm{e}^{-kAQ} f(p_{-k} + \phi(p_{-k}))\right\| \leq \frac{\epsilon}{3}.$$

Then, since the simple integration scheme

$$\int_0^a f(x)\,dx \simeq h\sum_{k=0}^{a/h} f(kh)$$

is $O(h)$ for each fixed $a$ provided that $f$ is Lipschitz, and likewise the implicit scheme is $O(h)$ on bounded time intervals, for any fixed $T$ we have

$$\left|\int_{-T}^{0} \mathrm{e}^{sAQ} f(p(s) + \phi(p(s))\,\mathrm{d}s - h\sum_{k=0}^{T/h} \mathrm{e}^{-kAQ} f(p_{-k} + \phi(p_{-k}))\right| = O(h),$$

so choosing $h$ small enough it follows that

$$\|T\phi - \hat{T}\phi\| \leq \epsilon.$$

Thus

$$\begin{aligned}\|\phi_h - \phi_N\| &= \|\hat{T}\phi_h - T\phi_N\| \\ &\leq \|\hat{T}\phi_h - T\phi_h\| + \|T\phi_h - T\phi_N\| \\ &\leq \epsilon + \kappa_T\|\phi_h - \phi_N\|,\end{aligned}$$

so that

$$\|\phi_h - \phi_N\| \leq \frac{\epsilon}{1-\kappa_T}\|\phi_h\|,$$

showing convergence as $h \to 0$. □

We note here that other more realistic schemes could be considered in a similar way - for example, a standard fourth order Runge-Kutta method is analysed in a similar way to section 3 in Robinson [22].

4.2. **Truncating the summation operator.** The approximation to $\hat{T}$ given by the finite sum $\hat{T}_\tau$ ($\tau = Mh$),

$$\hat{T}_\tau\phi = -h\sum_{k=0}^{M} \mathrm{e}^{-khAQ} Qf(p_{-k} + \phi(p_{-k})),$$

can be computed numerically. It is clear that the finite sum $\hat{T}_\tau$ approaches the infinite sum as the number of terms (i.e. $\tau$) increases. The existence of fixed points for $\hat{T}_\tau$ and the convergence of these fixed points follows swiftly from the results of the previous subsection.

**Corollary 4.5.** *Under condition (4.9) each of the operators $\hat{T}_\tau$ possesses a unique fixed point $\hat{\phi}_{h,\tau} \in \mathcal{F}_l$. These fixed points tend to $\hat{\phi}_h$ as $\tau \to \infty$.*

*Proof.* The argument of proposition 4.3, which showed that $\hat{T}$ is a contraction, can easily be adapted to treat $\hat{T}_\tau$ and deduce the existence of fixed points $\hat{\phi}_{h,\tau}$ for each operator. Furthermore, the truncations of $\hat{T}$, $\hat{T}_\tau$, converge uniformly to $\hat{T}$ on $\mathcal{F}$: following the argument of proposition 4.4 we can easily obtain

$$\|\hat{T}\phi - \hat{T}_\tau\phi\| \leq C_0 \mathrm{e}^{-\lambda_{n+1}\tau}.$$

Now the convergence of the fixed points of $\hat{T}_\tau$ to that of $\hat{T}$ as $\tau \to \infty$ follows simply (as above), since

$$\begin{aligned} \|\hat{\phi} - \hat{\phi}_\tau\| &= \|\hat{T}\hat{\phi} - \hat{T}_\tau\hat{\phi}_\tau\| \\ &\leq \|\hat{T}\hat{\phi} - \hat{T}\hat{\phi}_\tau\| + \|\hat{T}\hat{\phi}_\tau - \hat{T}_\tau\hat{\phi}_\tau\| \\ &\leq \kappa_{\hat{T}}\|\hat{\phi} - \hat{\phi}_\tau\| + C_0\mathrm{e}^{-\lambda_{n+1}\tau}, \end{aligned}$$

where $\kappa_{\hat{T}}$ is the contraction constant of $\hat{T}$ from (4.10). Thus

$$\|\hat{\phi} - \hat{\phi}_\tau\| \leq \frac{C_0\mathrm{e}^{-\lambda_{n+1}\tau}}{1-\kappa_{\hat{T}}}. \tag{4.11}$$

□

Thus $\hat{T}_\tau$ is indeed a reasonable starting point for a numerical attempt to approximate $\phi$.

4.3. **Including interpolation.** We now have to take the problems arising from interpolation into account. Rather than having a function $\phi$ defined over the whole of $PH$, in fact we will have a function $\psi$ defined only over a fixed grid $G \subset P\Omega_\rho$,

$$\psi \in \mathcal{G}_1.$$

Since the calculation of the backwards trajectory of $p_k$ will require values of $\phi$ at other points, we first have to replace $\psi$ by its interpolation $\tilde{\psi}$,

$$\tilde{\psi} \in \mathcal{F}_\sigma$$

(see (3.5)). Once this is done we can apply the truncated summation operator $\hat{T}_\tau$ introduced above: in theory this produces a new function $\hat{T}_\tau\tilde{\psi}$ with

$$\hat{T}_\tau\tilde{\psi} \in \mathcal{F}_{\tilde{\sigma}}$$

(see (4.8)). Of course, in fact we will only compute the values of $\hat{T}_\tau\tilde{\psi}$ over the particular gridpoints $G$, thus obtaining the somewhat notation-heavy expression

$$[\hat{T}_\tau\tilde{\psi}]_G$$

(where $[f]_G$ denotes the restriction of the function $f$ to $G$ as above) for the "fully numeric" version of $T$.

We will denote this operator by $T^*$,

$$T^*\psi = [T_\tau\tilde{\psi}]_G.$$

We expect that $T^*$ will map $\mathcal{G}_1$ into $\mathcal{G}_l$. It remains to determine conditions that ensure that we can take $l = 1$ and to show that iterates of $T^*$ will produce an approximation to the fixed point of $\hat{T}_\tau$. This requires a slightly strengthened version of the spectral gap condition.

**Proposition 4.6.** *Provided that*

$$\lambda_{n+1} - \gamma_h(\sigma) > (1 + \sigma^{-1})C_1 \tag{4.12}$$

*then $T^*$ maps $\mathcal{G}_1$ into itself. Furthermore, for any $\psi \in \mathcal{G}_1$, there exists an $j_0(\psi)$ such that*

$$\|(T^*)^j \psi - \hat{\phi}_{h,\tau}\|_G \leq \frac{2\kappa(1+\sigma)}{1-\kappa}\,\delta(G) \tag{4.13}$$

*for all $j \geq j_0$, where $\kappa$ is the contraction constant of $\hat{T}_\tau$.*

Note that as $h \to 0$, (4.12) becomes

$$\lambda_{n+1} - \lambda_n > (1 + \sigma)(1 + \sigma^{-1})C_1. \tag{4.14}$$

*Proof.* To begin, note that if $\psi \in \mathcal{G}_1$ then, from (3.5), $\tilde{\psi} \in \mathcal{G}_\sigma$. Now, we showed in proposition 4.2 that if $\phi \in \mathcal{F}_\sigma$ and the strengthened condition (4.12) holds then $\hat{T}\phi$ (and also $\hat{T}_\tau\phi$) is an element of $\mathcal{F}_1$. Thus $T^*$ maps $\mathcal{G}_1$ into itself.

To obtain the approximation result (4.13), observe (with $\kappa = \kappa_{\hat{T}_\tau}$ and $\phi = \hat{\phi}_{h,\tau}$) that

$$\begin{aligned} \|T^*\psi - \phi\|_{\mathcal{G}} &= \|\hat{T}_\tau\tilde{\psi} - \hat{T}_\tau\phi\|_{\mathcal{G}} \\ &\leq \|\hat{T}_\tau\tilde{\psi} - \hat{T}_\tau\phi\| \\ &\leq \kappa\|\tilde{\psi} - \phi\|, \end{aligned}$$

since $\hat{T}_\tau$ is a contraction on $\mathcal{F}_\sigma$ when (4.12) holds. Thus

$$\|T^*\psi - \phi\|_{\mathcal{G}} \leq \kappa\|\psi - \phi\|_G + \kappa(1+\sigma)\delta(G)$$

and the iterates of $\psi$ under $T^*$ satisfy

$$\|(T^*)^j\psi - \hat{\phi}_\tau\|_{\mathcal{G}} \leq \kappa^j\|\psi - \phi\| + \frac{\kappa(1+\sigma)}{1-\kappa}\,\delta(G).$$

After a sufficient number of iterations one has

$$\|(T^*)^n\psi - \hat{\phi}_\tau\|_{\mathcal{G}} \leq \frac{2\kappa(1+\sigma)}{1-\kappa}\,\delta(G).$$

□

## 5. An approximation scheme based on the graph transform method.

It is also possible to produce an approximation scheme based on the more geometric "graph transform method" – this is the aim of this section.

### 5.1. A discrete cone condition.

The Hadamard or "graph transform" existence proof of Mallet-Paret & Sell [16] is based on following a particular set of initial conditions under the flow. Fundamental to this "evolution" method is the cone condition, which states that once the difference of two solutions lies in a certain cone in $\mathbb{R}^N$ it remains there in the future (see Robinson [21] for a fuller discussion). This section gives a proof of a discrete version of this property for the one-step method.

We begin by investigating the evolution of Lipschitz graphs under the flow.

**Proposition 5.1.** *Suppose that*

$$|Q(u_0 - \bar{u}_0)| \leq l|P(u_0 - \bar{u}_0)|. \tag{5.1}$$

*Then*

$$|Q(u_1 - \bar{u}_1)| \leq \tilde{l}|P(u_1 - \bar{u}_1)|$$

*where*

$$\tilde{l} = \left[\frac{\mathrm{e}^{-\lambda_{n+1}h} + (1+l^{-1})C_1 h}{\mathrm{e}^{-\lambda_n h} - (1+l)C_1 h}\right] l \tag{5.2}$$

*Furthermore, if*

$$|Q(u_0 - \bar{u}_0)| \geq l|P(u_0 - \bar{u}_0)|$$

*then*

$$|Q(u_1 - \bar{u}_1)| \leq [\mathrm{e}^{-\lambda_{n+1}h} + hC_1(1 + l^{-1})]|Q(u_0 - \bar{u}_0)|. \tag{5.3}$$

*Proof.* Write $w_k = u_k - \bar{u}_k$. We bound $|Qw_1|$ above in terms of $|Pw_0|$, and then bound $|Pw_1|$ below in terms of $|Pw_0|$.

For the $Q$-part,

$$Qw_1 = \mathrm{e}^{-Ah}Qw_0 - h(Qf(u_0) - Qf(\bar{u}_0)),$$

and so

$$\begin{aligned} |Qw_1| &\leq \mathrm{e}^{-\lambda_{n+1}h}|Qw_0| + hC_1|w_0| \\ &\leq (l\mathrm{e}^{-\lambda_{n+1}h} + (1+l)C_1 h)|Pw_0| \end{aligned} \tag{5.4}$$

since $|Qw_0| \leq l|Pw_0|$. Now,

$$Pw_1 = \mathrm{e}^{-Ah}Pw_0 - h(Pf(u_0) - Pf(\bar{u}_0)),$$

and so

$$\begin{aligned} |Pw_1| &\geq \mathrm{e}^{-\lambda_n h}|Pw_0| - hC_1|w_0|, \\ &\geq \mathrm{e}^{-\lambda_n h}|Pw_0| - hC_1|w_0| \\ &\geq (\mathrm{e}^{-\lambda_n h} - (1+l)C_1 h)|Pw_0|, \end{aligned} \tag{5.5}$$

so that

$$|Qw_1| \leq \frac{(l\mathrm{e}^{-\lambda_{n+1}h} + (1+l)C_1 h)}{(\mathrm{e}^{-\lambda_n h} - (1+l)C_1 h)}|Pw_1|$$

from which (5.2) follows. If $|Qw_0| \geq l|Pw_0|$ then (5.3) follows directly from (5.4). $\square$

It is a consequence of this proposition that the image under the discrete evolution $S_h$ of a manifold given as the graph of an $l$-Lipschitz function $\phi$ is another Lipschitz manifold provided that $\mathrm{e}^{-\lambda_n h} > (1+l)C_1 h$. In this case we define an evolution $\mathcal{S}_h$ on the space of Lipschitz functions via

$$S_h\mathcal{G}[\phi] = \mathcal{G}[\mathcal{S}_h\phi].$$

From above, $\mathcal{S}_h : \mathcal{F}_l \to \mathcal{F}_{\tilde{l}}$.

**Proposition 5.2.** *If the condition*

$$\mathrm{e}^{-\lambda_{n+1}h} + 2C_1 h \leq \mathrm{e}^{-\lambda_n h} - 2C_1 h \tag{5.6}$$

*holds then $\mathcal{S}_h$ maps $\mathcal{F}_1$ into itself, and there is an inertial manifold given as the graph of*

$$\phi_\infty = \lim_{k\to\infty} \mathcal{S}_h^k[0].$$

*Furthermore*

$$\|\mathcal{S}_h\psi - \phi_\infty\| \leq \chi\|\psi - \phi_\infty\|. \tag{5.7}$$

Note that (5.6) reduces to the spectral gap condition (2.3) as $h \to 0$.

*Proof.* It is immediate from proposition 5.1 that if (5.6) holds then whenever

$$|Qu_0 - Q\bar{u}_0| \le |Pu_0 - P\bar{u}_0| \tag{5.8}$$

we must have

$$|Qu_k - Q\bar{u}_k| \le |Pu_k - P\bar{u}_k| \tag{5.9}$$

for all $k \ge 0$, and in particular it follows that $\mathcal{S}_h$ maps $\mathcal{F}_1$ into itself.

As a first step to constructing an inertial manifold we note that

$$|Qu_{k+1}| \le \mathrm{e}^{-\lambda_{n+1}h}|Qu_k| + hC_0,$$

so in particular if $Qu_0 = 0$ we have

$$|Qu_k| \le \frac{hC_0}{1-\mathrm{e}^{-\lambda_{n+1}h}} \qquad \text{for all} \qquad k \in \mathbb{Z}_+.$$

Now let

$$\phi_k = \mathcal{S}_h^k[0].$$

Consider for $r > s$ the two points $u = p + \phi_r(p)$ and $\bar{u} = p + \phi_s(p)$. Then if $\phi_r(p) \ne \phi_s(p)$

$$|Q(u-\bar{u})| = |\phi_r(p) - \phi_s(p)| > 0 = |P(u-\bar{u})|,$$

and so it follows from the "cone invariance" in (5.8) and (5.9) that the backwards iterates $u_{-k}$ and $\bar{u}_{-k}$ of $u$ and $\bar{u}$ must satisfy

$$|Q(u_{-k} - \bar{u}_{-k})| > |P(u_{-k} - \bar{u}_{-k})|.$$

Therefore

$$\begin{aligned}
|\phi_r(p) - \phi_s(p)| &= |Q(u-\bar{u})| \\
&\le [\mathrm{e}^{-\lambda_{n+1}h} + hC_1(1+l^{-1})]^s |Qu_{r-s}| \\
&\le [\mathrm{e}^{-\lambda_n h} - hC_1(1+l)]^s \frac{hC_0}{1-\mathrm{e}^{-\lambda_{n+1}h}},
\end{aligned}$$

and so $\{\phi_k\}_{k=1}^{\infty}$ is a Cauchy sequence with limit $\phi_\infty \in \mathcal{F}_1$. Since

$$\mathcal{S}_h\phi_k = \phi_{k+1}$$

it is clear that $\phi_\infty$ satisfies

$$\mathcal{S}_h\phi_\infty = \phi_\infty,$$

and so $\mathcal{M} = \mathcal{G}[\phi_\infty]$ is an invariant manifold.

That $\mathcal{M}$ attracts exponentially can either be shown following almost exactly the above argument as above, or more directly by noting that the invariance of $\mathcal{G}[\phi_\infty]$ implies that

$$\phi_\infty(p_{k+1}) = \mathrm{e}^{-Ah}\phi_\infty(p_k) - h\,Qf(p_k + \phi_\infty(p_k)).$$

Now,

$$q_{k+1} = \mathrm{e}^{-Ah}q_k - h\,Qf(p_k + q_k),$$

and thus, writing $\delta_k = q_k - \phi_\infty(p_k)$,

$$\delta_{k+1} = \mathrm{e}^{-Ah}\delta_k + h\,(Qf(p_k+q_k) - Qf(p_k + \phi_\infty(p_k))).$$

It is immediate that

$$|\delta_{k+1}| \le (\mathrm{e}^{-\lambda_{n+1}h} + C_1 h)|\delta_k|.$$

□

If this discrete scheme possesses inertial manifolds, they will converge to those of the continuous time system. This follows using the convergence result from Robinson [25] reproduced below (the notation is specialised to the case under consideration here).

**Theorem 5.3.** *Suppose that a semigroup $S(t) : \mathbb{R}^N \to \mathbb{R}^N$ is approximated by a family of schemes $\{S_h\}_{h>0}$, and that $S_h \to S$ uniformly on bounded intervals of time and for initial conditions in bounded subsets of $H$. Then if each $S_h$ has an inertial manifold $\mathcal{M}_h$ given as the graph of $\phi_h \in \mathcal{F}_l^n$, and the rate of attraction is uniform, in that*

$$\mathrm{dist}(S_h^k u_0, \mathcal{M}_h) \leq C\mathrm{e}^{-\gamma kh} \qquad \textit{for all} \qquad k \in \mathbb{Z}_+$$

*then $\phi_h \to \phi$ uniformly and $\phi \in \mathcal{F}_l^n$. Furthermore $\mathcal{M} = \mathcal{G}[\phi]$ is an inertial manifold for $S(t)$ that attracts at the same rate:*

$$\mathrm{dist}(S(t)u_0, \mathcal{M}) \leq C\mathrm{e}^{-\gamma t}.$$

(This theorem assumes nothing about the properties of the limiting equation; when one assumes that the limiting equation has an inertial manifold, Jones *et al.* [15] and Jones & Stuart [14] show that the manifolds for various approximation schemes converge to the "true" manifold in a $C^1$ fashion.)

We now use the attraction property from (5.7) along with knowledge of how Lipschitz graphs behave under $\mathcal{S}_h$ (5.2) to examine an approximation method based on the graph transform property.

5.2. **Including interpolation.** If initially one considers only the graph of $\psi \in \mathcal{G}_1$ over the discrete set of gridpoints in $G$, under the evolution this will become the set

$$G_h \equiv \{S_h(g + \psi(g)) : \ g \in G.\}$$

In this case, in something of an abuse of notation, we will also define

$$\mathcal{S}_h G = PG_h,$$

i.e. the "evolved" grid.

The numerical scheme begins with a 1-Lipschitz function $\psi$ defined over the discrete grid $G$,

$$\psi \in \mathcal{G}_1$$

There is no need to interpolate at this stage. The collection of initial points

$$\{g + \psi(g)\}$$

are now evolved $d$ times ($d$ will be specified below) to give a new set of points

$$\{S_h^d(g + \psi(g)) : \ g \in G\} \tag{5.10}$$

that lie on $\mathcal{G}[\mathcal{S}_h^d\tilde{\psi}]$. These points give values of the function $\mathcal{S}_h^d\tilde{\psi}$, defined over the new grid

$$G' = P\{S_h^d(g + \psi(g)) : \ g \in G\}.$$

Note that

$$|Pw_{k+1}| \leq (\mathrm{e}^{-\lambda_1 h} + 2hC_1)|Pw_k|,$$

and so

$$\delta(G') \leq (\mathrm{e}^{-\lambda_1 h} + 2hC_1)^d \delta(G). \tag{5.11}$$

We now want to ensure that (5.6) holds for every value of $l$ with $\sigma^{-1} \leq l \leq 1$, so we assume that

$$\mathrm{e}^{-\lambda_{n+1}h} + (1+\sigma)C_1 h < \mathrm{e}^{-\lambda_n h} - 2C_1 h.$$

In particular this means that for every $\sigma^{-1} \leq l \leq 1$ we have $\tilde{l} \leq \theta l$ for some fixed $\theta < 1$. Thus if $\phi \in \mathcal{F}_1$ we have

$$\mathrm{Lip}(\mathcal{S}_h^k \phi) \leq \theta^k.$$

In particular

$$\mathrm{Lip}(\mathcal{S}_h^d \phi) \leq \sigma^{-1} \qquad \text{for all} \qquad \phi \in \mathcal{F}_1$$

if we take $d$ sufficiently large.

We will write the interpolant of $\mathcal{S}_h^d \tilde{\psi}$ as $E(\mathcal{S}_h^d \tilde{\psi})$ in order to emphasise that the interpolation is *not* over the regular grid $G$ but over $G'$. This gives a 1-Lipschitz function again, which we then restrict back to $G$, thus preventing the points that we are calculating with from separating too far.

We now show that if there is an invariant Lipschitz manifold that attracts exponentially then, provided that the grid is fine enough, this scheme will produce a good approximation.

The scheme can now be thought of as an operator $\mathcal{T}$ acting on 1-Lipschitz functions $\psi \in \mathcal{G}_1$, given by

$$\mathcal{T}\phi = [E(\mathcal{S}_h^d \psi)]_G. \tag{5.12}$$

(Note that there is some slight abuse of notation in the above definition, since strictly speaking $\mathcal{S}_h$ acts on functions in $\mathcal{F}_l$ rather than functions in $\mathcal{G}_1$. However, the procedure outlined above shows that in order to find the values of $\mathcal{S}_h^d \tilde{\phi}$ over the transformed grid "$\mathcal{S}_h^d G$" it is not actually necessary to interpolate over the initial gridpoints, but only over the transformed gridpoints.)

**Proposition 5.4.** *Suppose that*

$$\mathrm{e}^{-\lambda_{n+1}h} + (1+\sigma)C_1 h < \mathrm{e}^{-\lambda_n h} - 2C_1 h \tag{5.13}$$

*holds. Then given $\psi \in \mathcal{G}_\sigma$, there exists an $k_0(\psi)$ such that,*

$$\|\mathcal{T}^k \psi - \phi_\infty\|_G \leq 2\theta\delta(G)$$

*for all $k \geq k_0$, where $\phi_\infty$ is the function from proposition 5.2 and $\theta$ is defined in (5.14).*

Note that (5.13) reduces to

$$\lambda_{n+1} - \lambda_n > (3+\sigma)C_1$$

as $h \to 0$.

*Proof.* From the definition (6.10),

$$\begin{aligned}
\|\mathcal{T}\psi - \phi\|_G &= \|E(\mathcal{S}_h^d \psi) - \phi\|_G \\
&\leq \|E(\mathcal{S}_h^d \psi) - \phi\| \\
&\leq \|\mathcal{S}_h^d \psi - \phi\|_{G'} + (1+L)\delta(G') \\
&= \|\mathcal{S}_h^d \tilde{\psi} - \phi\|_{G'} + (1+L)\delta(G') \\
&\leq \chi^d \|\tilde{\psi} - \phi\| + (1+L)\delta(G') \\
&\leq \chi^d \big[\|\psi - \phi\|_G + (\sigma + L)\delta(G)\big] + (1+L)\delta(G');
\end{aligned}$$

and so, using (5.11),

$$\|\mathcal{T}\psi - \phi\|_G \leq \chi^d \|\psi - \phi\|_G + \theta\delta(G),$$

where

$$\theta = \chi^d(\sigma + L) + (1 + L)(\mathrm{e}^{-\lambda_1 h} + 2hC_1)^d. \tag{5.14}$$

The result holds as stated for $k$ sufficiently large. □

6. **Conclusion.** Two possible methods of computing inertial manifolds have been suggested, the main advance here being that we have included those issues that arise from having to interpolate the manifold from a discrete set of gridpoints. Surmounting these problems, it has been shown that both methods converge to the true manifold for the continuous problem as the approximation parameters are refined. Which of the methods would be the more convenient is practice remains to be seen in a full numerical study.

In order to study the behaviour of inertial manifolds away from the parameter range in which the spectral gap condition holds it may be possible to make use of one method of proof that we have not turned into a numerical method here. This is the "Cauchy" construction of Constantin *et al.* introduced in [2] and treated in a more elementary way in Robinson [23]: here the manifold is constructed as the forward image of $\Gamma = \partial P_n \Omega_\rho$,

$$\mathcal{M} = \overline{\cup_{t \geq 0} S(t)\Gamma},$$

where $S(t)\Gamma = \{u(t) : \ u_0 \in \Gamma\}$. Although a spectral gap condition is required to show that $\mathcal{M}$ is an inertial manifold, there are cases in which the same construction produces an invariant exponentially attracting set [24] or alternatively (see Robinson [26]) the "multi-valued inertial manifold" introduced by Debussche & Temam [3]. There is thus some hope that investigations that follow the evolution of $\Gamma$ under the discrete time evolution will produce interesting results away from the restrictions of the spectral gap condition.

**Acknowledgments.** This work has had many versions, so thanks for support are due to: the SERC (as they then were); Trinity College, Cambridge; Lincoln College, Oxford; and the Royal Society of London. I am currently a Royal Society University Research Fellow.

## REFERENCES

[1] S.-N. Chow, K. Lu and G.R. Sell, *Smoothness of inertial manifolds*, J. Math. Anal. Appl. 169 (1992), 283–312.

[2] P. Constantin, C. Foias, B. Nicolaenko and R. Temam, "Integral manifolds and inertial manifolds for dissipative partial differential equations" (Springer AMS 70, 1988).

[3] A. Debussche and R. Temam, *Some new generalizations of inertial manifolds*, Discrete & Continuous Dynamical Systems, 2 (1996), 543–558.

[4] F. Demengel and J.M. Ghidaglia, *Inertial manifolds for partial differential evolution equations under time discretization: existence, convergence, and applications*, J. Math. Anal. Appl. 155 (1991), 177–225.

[5] H. Federer, "Geometric measure theory" (Springer Classics in Mathematics, 1991, reprint of 1969 edition).

[6] C. Foias, G.R. Sell and R. Temam, *Variétés inertielles des équations différentielles dissipatives*, C. R. Acad. Sci. Paris I 301 (1985), 139–141.

[7] C. Foias, M.S. Jolly, I.G. Kevrekidis, G.R. Sell and E.S. Titi, *On the computation of inertial manifolds*, Phys. Lett. A 131 (1988), 433–436.

[8] C. Foias, B. Nicolaenko, G.R. Sell and R. Temam, *Inertial manifolds for the Kuramoto Sivashinsky equation and an estimate of their lowest dimension*, J. Math. Pures Appl. 67 (1988), 197–226.
[9] C. Foias, G.R. Sell and R. Temam, *Inertial Manifolds for nonlinear evolution equations*, J. Diff. Eq. 73 (1988), 309–353.
[10] C. Foias, O. Manley and R. Temam, *Modelling of the interaction of small and large eddies in two dimensional turbulent flows*, Math. Mod. Num. Anal. 22 (1988), 93–114.
[11] C. Foias, G.R. Sell and E.S. Titi, *Exponential Tracking and approximation of inertial manifolds for dissipative nonlinear equations*, J. Dyn. Diff. Eq. 1 (1989), 199–244.
[12] H. Haken, "Synergetics" (Springer, Berlin, 1978).
[13] D. Henry, "Geometric Theory of Semilinear Parabolic Equations" (Lecture Notes in Math. Vol 840, Springer-Verlag, New York, 1981).
[14] D.A. Jones and A.M. Stuart, *Attractive invariant manifolds under approximation - inertial manifolds*, J. Diff. Eq. 123 (1995), 588–637.
[15] D.A. Jones, A.M. Stuart and E.S. Titi, *Persistence of invariant sets for partial differential equations*, J. Math. Anal. Appl. 219 (1998) 479–502.
[16] J. Mallet-Paret and G.R. Sell, *Inertial manifolds for reaction diffusion equations in higher space dimensions*, J. Amer. Math. Soc. 1 (1988), 805–866.
[17] E.J. McShane, *Extension of range of functions*, Bull. Amer. Math. Soc. 40 (1934), 837–842.
[18] M. Miklavčič, *Stability for semilinear equations with noninvertible linear operator*, Pacific J. Math. 118 (1985), 199-214.
[19] M. Miklavčič, *A sharp condition for the existence of an inertial manifold*, J. Dyn. Diff. Eq. 3 (1991), 437–456.
[20] M. Renardy and R.C. Rogers, "An Introduction to Partial Differential Equations" (Springer-Verlag Texts in Applied Maths. Vol. 13, New York, 1992).
[21] J.C. Robinson, *Inertial manifolds and the cone condition*, Dyn. Sys. Appl. 2 (1993), 311-330.
[22] J.C. Robinson, "Inertial manifolds" (Ph.D. Thesis, University of Cambridge, 1995).
[23] J.C. Robinson, *A concise proof of the "geometric" construction of inertial manifolds*, Phys. Lett. A 200 (1995), 415–417.
[24] J.C. Robinson, *Some closure results for inertial manifolds*, J. Dyn. Diff. Eq. 9 (1997) 373–400.
[25] J.C. Robinson, *Convergent Families of Inertial Manifolds for Convergent Approximations*, Numerical Algorithms 14 (1997) 179–188.
[26] J.C. Robinson, *Inertial manifolds with and without delay*, Disc. Cont. Dyn. Sys. 5 (1999) 813–824.
[27] J.C. Robinson, *Infinite-dimensional dynamical systems.* Cambridge University Press, Cambridge (2001).
[28] A. Rodriguez Bernal, *Inertial manifolds for dissipative semiflows in Banach spaces*, Appl. Anal. 37 (1990), 95–141.
[29] G.R. Sell, *An optimality condition for approximate inertial manifolds*, in G.R. Sell, C. Foias and R. Temam, R. (Eds.), "Turbulence in Fluid Flows: A dynamical systems approach" (IMA Volumes in Mathematics and its Applications 55, Springer, New York, 1993).
[30] R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics" (Springer AMS 68, 1988).

Received March 2001; revised September 2001; final version March 2002.

*E-mail address*: jcr@maths.warwick.ac.uk