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RESEARCH PAPER

CAUCHY FORMULA FOR THE TIME-VARYING LINEAR SYSTEMS WITH CAPUTO DERIVATIVE

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Abstract

In the paper, existence, uniqueness and a Cauchy formula for the solution to a time-varying linear system containing fractional Caputo derivative is obtained. This formula shows that nonnegativity of the data of the system implies nonnegativity of the solution. In the context of a strenghthening of this result, an example illustrating the absence (in the case of Caputo derivative) of the standard relation "monotonicity of function - sign of derivative".

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1. Introduction

In this paper, a Cauchy formula for the solution to a fractional time-varying linear system containing Caputo derivative will be derived and some result concerning the positivity of the system will be obtained with the aid of this formula.

A dynamic system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. An overview of the state of art in positive systems theory is given in monographs [2], [6]. Positivity of fractional time-varying discrete-time linear

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systems have been addressed in [7], [8], [9]. The fractional positive linear autonomous systems have been analyzed in [10], [11].

The paper is organized as follows. In Section 2 some preliminaries are recalled. In Section 3 existence, uniqueness and Cauchy formula for the solution to a fractional time-varying linear system containing Caputo derivative are obtained. An example illustrating this formula is also given. The positivity of the system is analyzed in Section 4.

2. Preliminaries

The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{n\times m}$ - the set of $n \times m$ real matrices, $\mathbb{R}_+^{n\times m}$ - the set of $n \times m$ real matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$, M_n - the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n \times n$ identity matrix, superscript T denotes the transposition of matrix (vector).

Let us recall that a function $x : [a, b] \to \mathbb{R}^n$ is named absolutely continuous if there exist a constant $c \in \mathbb{R}^n$ and a function $\varphi \in L^1 = L^1([a, b], \mathbb{R}^n)$ such that

$$x(t) = c + \int_a^t \varphi(s)ds$$

for $t \in [a, b]$. In such a case x has a.e. on $[a, b]$ (with respect to the Lebesgue measure) derivative $x'(t)$ and

$$x'(t) = \varphi(t),\ t \in [a, b] \text{ a.e.}$$

Moreover,

$$x(a) = c.$$

The set of all absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ will be denoted by AC. This set with the metric

$$\chi(x, y) = |x(a) - y(a)| + \int_a^b \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt$$

is a complete metric space. By AC_c we denote the set $\{x \in AC;\ x(a) = c\}$. Of course, AC_c is closed in AC. So, AC_c with the metric $\varkappa = \chi\ |_{AC_c \times AC_c}$ is complete, too.

It is clear that ρ_k given by

$$\rho_k(x, y) = \int_a^b e^{-kt} \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt,\ x, y \in AC_c,$$

where $k > 0$ is any fixed positive integer, is an equivalent metric to the metric $\varkappa$ in AC_c. In consequence, AC_c with ρ_k is complete.

By the left-sided Caputo fractional derivative ${}^C D_{a+}^{\alpha} x$ of order $\alpha \in (0, 1)$ on the interval $[a, b]$ of a function $x \in AC$ we mean the Riemann-Liouville derivative of the function $x(\cdot) - x(a)$, i.e.

$$({}^C D_{a+}^{\alpha} x)(t) := D_{a+}^{\alpha}(x(\cdot) - x(a))(t),\ t \in [a, b] \text{ a.e.}$$

So,

$$({}^C D^{\alpha}_{a+}x)(t) = (D^{\alpha}_{a+}x)(t) - \frac{1}{\Gamma(1-\alpha)}\frac{x(a)}{(t-a)^{\alpha}}. \tag{2.1}$$

One can show that

$$({}^C D^{\alpha}_{a+}x)(t) = I^{1-\alpha}_{a+}(\frac{d}{dt}x)(t)$$

for $t \in [a,b]$ a.e.

3. Existence of a solution and Cauchy formula

Now, let us consider the following time-varying linear Cauchy problem

$$\begin{cases} ({}^C D^{\alpha}_{a+}x)(t) = A(t)x(t) + v(t), \quad t \in [a,b] \text{ a.e.} \\ x(a) = c, \end{cases} \tag{3.1}$$

where $\alpha \in (0,1)$, $A : [a,b] \to \mathbb{R}^{n\times n}$ is absolutely continuous with essentially bounded derivative $\frac{d}{dt}A$ ($\frac{d}{dt}A \in L^{\infty} = L^{\infty}([a,b],\mathbb{R}^n)$) and $v \in I^{1-\alpha}_{a+}(L^1)$. By a solution of this problem we mean an absolutely continuous function $x : [a,b] \to \mathbb{R}^n$ which satisfies the equation a.e. on $[a,b]$ and the initial condition.

It is easy to see that the existence of a unique, in AC, solution to problem (3.1) is equivalent to the existence of a unique, in AC_c, solution to the integral equation

$$x(t) = c + I^{\alpha}_{a+}(A(\cdot)x(\cdot))(t) + I^{\alpha}_{a+}v(t).$$

In other words, the existence of a unique, in AC, solution to problem (3.1) is equivalent to the existence of a unique fixed point of the operator

$$\Phi : AC_c \ni x(\cdot) \longmapsto c + I^{\alpha}_{a+}(A(\cdot)x(\cdot))(t) + I^{\alpha}_{a+}v(t) \in AC_c,$$

(well-possedness of Φ follows from the results of [4] and [1]).

Let us observe that

$$\begin{aligned}\rho_k(\Phi(x),\Phi(y)) &= \int_a^b e^{-kt}\left|\frac{d}{dt}I^{\alpha}_{a+}(A(\cdot)(x(\cdot)-y(\cdot)))(t)\right|dt \\ &= \int_a^b e^{-kt}\left|I^{\alpha}_{a+}(\frac{d}{dt}(A(\cdot)(x(\cdot)-y(\cdot))))(t)\right|dt \\ &\le \int_a^b e^{-kt}I^{\alpha}_{a+}\left|\frac{d}{dt}A(\cdot)(x(\cdot)-y(\cdot)) + A(\cdot)(\frac{d}{dt}x(\cdot)-\frac{d}{dt}y(\cdot))\right|(t)dt.\end{aligned}$$

Integrating by parts (fractionally) and using formula for the fractional integral of exponent function we obtain

$$\int_a^b e^{-kt} I_{a+}^{\alpha} \left| \frac{d}{dt}A(\cdot)(x(\cdot)-y(\cdot)) + A(\cdot)(\frac{d}{dt}x(\cdot) - \frac{d}{dt}y(\cdot)) \right| (t)dt$$
$$= \int_a^b I_{b-}^{\alpha}(e^{-k\cdot})(t) \left| \frac{d}{dt}A(\cdot)(x(\cdot)-y(\cdot)) + A(t)(\frac{d}{dt}x(t) - \frac{d}{dt}y(t)) \right| dt$$
$$\leq \int_a^b I_{b-}^{\alpha}(e^{-k\cdot})(t) \left| \frac{d}{dt}A(t) \right| |x(t)-y(t)|\, dt$$
$$+ \int_a^b I_{b-}^{\alpha}(e^{-k\cdot})(t) \, |A(t)| \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt$$
$$\leq \frac{1}{k^{\alpha}} \int_a^b e^{-kt} \left| \frac{d}{dt}A(t) \right| |x(t)-y(t)|\, dt + \frac{1}{k^{\alpha}} \int_a^b e^{-kt} \, |A(t)| \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt$$
$$\leq \frac{1}{k^{\alpha}} ess\sup \left| \frac{d}{dt}A(\cdot) \right| \int_a^b e^{-kt} \, |x(t)-y(t)|\, dt$$
$$+ \frac{1}{k^{\alpha}} \max |A(\cdot)| \int_a^b e^{-kt} \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt$$
$$= \frac{1}{k^{\alpha}} ess\sup \left| \frac{d}{dt}A(\cdot) \right| \int_a^b e^{-kt} \left| I_{a+}^{1} \frac{d}{dt}x(t) - I_{a+}^{1} \frac{d}{dt}y(t) \right| dt$$
$$+ \frac{1}{k^{\alpha}} \max |A(\cdot)| \, \rho_k(x,y)$$

for $x,\, y \in AC_c$, where $ess\sup \left|\frac{d}{dt}A(\cdot)\right|$ is the smallest constant δ such that $\left|\frac{d}{dt}A(t)\right| \leq \delta$ for $t \in [a,b]$ a.e. But integrating by parts (classically) we can write

$$\int_a^b e^{-kt} \left| I_{a+}^{1} \frac{d}{dt}x(t) - I_{a+}^{1} \frac{d}{dt}y(t) \right| dt \leq \int_a^b e^{-kt} \int_a^t \left| \frac{d}{dt}x(s) - \frac{d}{dt}y(s) \right| dsdt$$
$$= (-\frac{1}{k} e^{-kb} \int_a^b \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt + \frac{1}{k} \int_a^b e^{-kt} \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt$$
$$\leq \frac{1}{k} \int_a^b e^{-kt} \left| \frac{d}{dt}x(t) - \frac{d}{dt}y(t) \right| dt = \frac{1}{k} \rho_k(x,y).$$

So,

$$\rho_k(\Phi(x), \Phi(y)) \leq (\frac{1}{k^{\alpha+1}} ess\sup \left| \frac{d}{dt}A(\cdot) \right| + \frac{1}{k^{\alpha}} \max |A(\cdot)|) \rho_k(x,y)$$

for $x,\, y \in AC_c$. Since $(\frac{1}{k^{\alpha+1}} \max \left|\frac{d}{dt}A(\cdot)\right| + \frac{1}{k^{\alpha}} \max |A(\cdot)|) \to 0$ as $k \to \infty$, therefore from the Banach contraction principle it follows that there exists a unique point $x_* \in AC_c$ such that

$$x_*(t) = c + I_{a+}^{\alpha}(A(\cdot)x_*(\cdot))(t) + I_{a+}^{\alpha} v(t)$$

for $t \in [a,b]$ a.e.

From the Banach principle it also follows that x_* is the limit in AC_c (equivalently, in AC) of the sequence

$$\Phi(0) = c + I^{\alpha}_{a+}v(t),$$

$$\begin{aligned}\Phi^2(0) &= c + I^{\alpha}_{a+}(A(\cdot))(t)c + I^{\alpha}_{a+}(A(\cdot)I^{\alpha}_{a+}v(\cdot))(t) + I^{\alpha}_{a+}v(t)\\ &= (I_n + I^{\alpha}_{a+}(A(\cdot))(t))c + I^{\alpha}_{a+}v(t) + I^{\alpha}_{a+}(A(\cdot)I^{\alpha}_{a+}v(\cdot))(t)\\ &= (I_n + I^{\alpha,A}_{a+}(I_n)(t))c + I^{\alpha}_{a+}v(t) + I^{\alpha,A}_{a+}(I^{\alpha}_{a+}v(\cdot))(t),\end{aligned}$$

$$\begin{aligned}\Phi^3(0) &= c + I^{\alpha}_{a+}(A(\cdot)(c + I^{\alpha}_{a+}(A(:))(\cdot)c + I^{\alpha}_{a+}(A(:)I^{\alpha}_{a+}v(:))(\cdot) + I^{\alpha}_{a+}v(\cdot))(t)\\ &\quad + I^{\alpha}_{a+}v(t)\\ &= (I_n + I^{\alpha,A}_{a+}(I_n)(t) + I^{\alpha,A}_{a+}(I^{\alpha,A}_{a+}(I_n))(t))c\\ &\quad + I^{\alpha}_{a+}v(t) + I^{\alpha,A}_{a+}(I^{\alpha}_{a+}v(\cdot))(t) + I^{\alpha,A}_{a+}(I^{\alpha,A}_{a+}(I^{\alpha}_{a+}v(:))(\cdot))(t)\end{aligned}$$

$$\vdots$$

where

$$I^{\alpha,A}_{a+}(B)(t) = I^{\alpha}_{a+}(A(\cdot)B(\cdot))(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{A(s)B(s)}{(t-s)^{1-\alpha}}ds$$

for a time-varying matrix $B(s) \in \mathbb{R}^{n\times n}$ or vector $B(s) \in \mathbb{R}^n$.

Using the induction principle one can show that

$$\Phi^l(0) = (\sum_{i=0}^{l-1}(I^{\alpha,A}_{a+})^i(I_n)(t))c + \sum_{i=0}^{l-1}(I^{\alpha,A}_{a+})^i(I^{\alpha}_{a+}v)(t)$$

for $l \in \mathbb{N}$ (we put $(I^{\alpha,A}_{a+})^0(I_n)(t) = I_n$). It means that

$$x_*(t) = (\sum_{i=0}^{\infty}(I^{\alpha,A}_{a+})^i(I_n)(t))c + \sum_{i=0}^{\infty}(I^{\alpha,A}_{a+})^i(I^{\alpha}_{a+}v)(t)$$

in AC_c (convergence of the both series follows from the case $v = 0$). Consequently, the above formula holds true pointwise on the interval $[a, b]$ (even uniformly pointwise).

Finally, we proved the following theorem.

THEOREM 3.1. *If $A \in AC$ and $\frac{d}{dt}A \in L^{\infty}$, $v \in I^{1-\alpha}_{a+}(L^1)$, then problem (3.1) has a unique solution $x \in AC$ given by the following Cauchy formula*

$$x(t) = (\sum_{i=0}^{\infty}(I^{\alpha,A}_{a+})^i(I_n)(t))c + \sum_{i=0}^{\infty}(I^{\alpha,A}_{a+})^i(I^{\alpha}_{a+}v)(t) \tag{3.2}$$

for $t \in [a, b]$.

Remark 3.1. The assumption "$A \in AC$ and $\frac{d}{dt}A \in L^{\infty}$" can be replaced by the following (equivalent) one: there exists a constant $L > 0$ such that

$$|A_{i,j}(t_1) - A_{i,j}(t_2)| \leq L\,|t_1 - t_2|$$

for $t_1,\, t_2 \in [a,b]$ and $i,\, j = 1, ..., n$ (we say in such a case that $A_{i,j}$ satisfies Lipschitz condition).

Remark 3.2. If we consider (3.1) with the function $v = 0$ and initial points

$$c_i = (0, ..., 0, 1, 0, ..., 0),\ i = 1, ..., n,$$

we see that the matrix linear Cauchy problem

$$\begin{cases} {}^{C}D^{\alpha}_{a+}X(t) = A(t)X(t), \quad t \in [a,b] \text{ a.e.} \\ X(a) = I_n \end{cases}$$

where $A(t) \in \mathbb{R}^{n\times n}$ satisfies assumptions of Theorem 3.1 and $X(t) \in \mathbb{R}^{n\times n}$, has a unique solution $X(t)$ in AC, given by

$$X(t) = \sum_{i=0}^{\infty} (I^{\alpha,A}_{a+})^i (I_n)(t)$$

for $t \in [a,b]$. The above matrix-valued function is a counterpart of the well known fundamental matrix from the theory of linear differential equations of order one.

It is easy to see that if A does not depend on t, then

$$(I^{\alpha,A}_{a+})^i (I_n)(t) = A^i I^{i\alpha}_{a+}(1)(t) = A^i \frac{1}{\Gamma(i\alpha)} \int_a^t \frac{ds}{(t-s)^{1-i\alpha}}$$

$$= A^i \frac{1}{\Gamma(i\alpha)} \frac{1}{i\alpha}(t-a)^{i\alpha} = A^i \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}$$

and, consequently,

$$X(t) = \sum_{i=0}^{\infty} (I^{\alpha,A}_{a+})^i (I_n)(t) = \sum_{i=0}^{\infty} \frac{A^i (t-a)^{i\alpha}}{\Gamma(i\alpha+1)} = E_{\alpha}(A(t-a)^{\alpha}),$$

where $E_{\alpha} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ defined by

$$E_{\alpha}(B) = \sum_{i=0}^{\infty} \frac{B^i}{\Gamma(i\alpha+1)}$$

for $B \in \mathbb{R}^{n\times n}$, is the "matrix" (one-parameter) Mittag-Leffler function (cf. [12] for the case $E_{\alpha} : \mathbb{R} \to \mathbb{R}$). Similarly (cf. [3]),

$$\sum_{i=0}^{\infty} (I^{\alpha,A}_{a+})^i (I^{\alpha}_{a+}v)(t) = \sum_{i=0}^{\infty} A^i I^{(i+1)\alpha}_{a+} v(t)$$

$$= \int_a^t \left(\sum_{i=0}^{\infty} \frac{A^i (t-s)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} \right) v(s)ds = \int_a^t \left((t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \right) v(s)ds$$

with $E_{\alpha,\beta} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, $\alpha, \beta > 0$, defined by

$$E_{\alpha,\beta}(B) = \sum_{i=0}^{\infty} \frac{B^i}{\Gamma(i\alpha + \beta)}$$

for $B \in \mathbb{R}^{n\times n}$, is the "matrix" (two-parameter) Mittag-Leffler type function (cf. [12] for the case $E_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$).

If A does not depend on t and $\alpha = 1$, then

$$X(t) = e^{A(t-a)} \quad \text{and} \quad (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) = e^{A(t-s)}.$$

Example 3.1. Let us consider system (3.1) with $n = 2$ and $A(t) = \begin{bmatrix} 0 & f(t) \\ 0 & 0 \end{bmatrix}$, i.e.

$$\begin{cases} {}^{C}D_{0+}^{\alpha} x_1(t) = f(t)x_2(t) + v_1(t) \\ {}^{C}D_{0+}^{\alpha} x_2(t) = v_2(t) \end{cases}$$

for $t \in [a, b]$ a.e., with initial condition

$$(x_1(a), x_2(a)) = (c_1, c_2).$$

Assume that f, v_1, v_2 satisfy assumptions of Theorem 3.1. We have

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= (I_2 + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{A(s)I_2 ds}{(t-s)^{1-\alpha}} \\ &\quad + \frac{1}{\Gamma(\alpha)^2} \int_a^t \frac{A(s)}{(t-s)^{1-\alpha}} (\int_a^s \frac{A(\tau)I_2}{(t-s)^{1-\alpha}} d\tau) ds + ...) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &\quad + I_{a+}^{\alpha} v(t) + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{A(s) I_{a+}^{\alpha} v(s) ds}{(t-s)^{1-\alpha}} \\ &\quad + \frac{1}{\Gamma(\alpha)^2} \int_a^t \frac{A(s)}{(t-s)^{1-\alpha}} (\int_a^s \frac{A(\tau) I_{a+}^{\alpha} v(\tau)}{(t-s)^{1-\alpha}} d\tau) ds + ... \end{aligned}$$

$$\begin{aligned} &= (I_2 + \frac{1}{\Gamma(\alpha)} \begin{bmatrix} 0 & \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} \\ 0 & 0 \end{bmatrix} \\ &\quad + \frac{1}{\Gamma(\alpha)^2} \int_a^t \begin{bmatrix} 0 & \frac{f(s)}{(t-s)^{1-\alpha}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \int_a^s \frac{f(\tau)d\tau}{(s-\tau)^{1-\alpha}} \\ 0 & 0 \end{bmatrix} ds + ...) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} I_{a+}^{\alpha}v_1(t) \\ I_{a+}^{\alpha}v_2(t) \end{bmatrix} + \frac{1}{\Gamma(\alpha)}\int_a^t \begin{bmatrix} f(s)I_{a+}^{\alpha}v_2(s) \\ 0 \end{bmatrix} \frac{ds}{(t-s)^{1-\alpha}}$$

$$+\frac{1}{\Gamma(\alpha)^2}\int_a^t \begin{bmatrix} 0 & \frac{f(s)}{(t-s)^{1-\alpha}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \int_a^s \frac{f(\tau)I_{a+}^{\alpha}v_2(\tau)d\tau}{(s-\tau)^{1-\alpha}} \\ 0 \end{bmatrix} ds + \ldots$$

$$= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}} c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} I_{a+}^{\alpha}v_1(t) \\ I_{a+}^{\alpha}v_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)I_{a+}^{\alpha}v_2(s)}{(t-s)^{1-\alpha}} ds \\ 0 \end{bmatrix}.$$

It is known that

$$I_{a+}^{\alpha}(\cdot - a)^{\nu}(t) = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}(t-a)^{\nu+\alpha}$$

for $\alpha > 0$, $\nu > -1$. So, if we put

$$f(t) = (t-a)^2,$$

$$v_1(t) = I_{a+}^{1-\alpha}(\cos\cdot)(t),$$

$$v_2(t) = I_{a+}^{1-\alpha}(2)(t),$$

then

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{\Gamma(2+1)}{\Gamma(\alpha+2+1)}(t-a)^{2+\alpha}c_2 \\ 0 \end{bmatrix}$$

$$+\begin{bmatrix} \sin t - \sin a \\ 2(t-a) \end{bmatrix} + \begin{bmatrix} 2I_{a+}^{\alpha}(\cdot - a)^3(t) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + \frac{\Gamma(2+1)}{\Gamma(\alpha+2+1)}(t-a)^{2+\alpha}c_2 + \sin t - \sin a + 2\frac{\Gamma(3+1)}{\Gamma(\alpha+3+1)}(t-a)^{3+\alpha} \\ c_2 + 2(t-a) \end{bmatrix}.$$

4. **Positivity**

Now, let us consider the Cauchy problem

$$\begin{cases} ({}^{C}D_{0+}^{\alpha}x)(t) = A(t)x(t) + v(t), \quad t \in [0,\infty) \text{ a.e.} \\ x(0) = c, \end{cases} \tag{4.1}$$

where $A : [0,\infty) \to \mathbb{R}^{n\times n}$ is absolutely continuous with essentially bounded derivative $\frac{d}{dt}A$ on $[0,b]$ and $v : [0,\infty) \to \mathbb{R}^n$ is such that $v \in I_{a+}^{1-\alpha}(L^1([0,b],\mathbb{R}^n)$ for any $b \in (0,\infty)$. By a solution of this problem we mean a function $x : [0,\infty) \to \mathbb{R}^n$ which is absolutely continuous on each interval $[0,b]$, satisfies the equation a.e. on $[0,\infty]$ and the initial condition.

From Theorem 3.1 it follows that problem (4.1) possesses a unique solution x and x is such that $x|_{[0,b]}$ is the unique solution to problem (3.1) on the interval $[0,b]$ for any $b \in (0,\infty)$. This solution is given by the Cauchy formula (3.2) for $t \in [0,\infty)$, i.e.

$$x(t) = (\sum_{i=0}^{\infty} (I_{a+}^{\alpha,A})^i (I_n)(t))c + \sum_{i=0}^{\infty} (I_{a+}^{\alpha,A})^i (I_{a+}^{\alpha} v)(t),\ t \in [0,\infty). \tag{4.2}$$

We say that system (4.1) is positive if for any initial vector c with nonnegative entries and for any function $v : [0,\infty) \to \mathbb{R}^n$ satisfying the above assumptions and such that the entries of the vector $v(t)$ are nonnegative for $t \in [0,\infty)$ a.e. the state vector $x(t)$ has nonnegative entries for any $t \in [0,\infty)$.

It is known that in the case of the classical control system (with $\alpha = 1$) with matrix A which does not depend on t the necessary and sufficient condition for the positivity is $A \in M_n$. The proof of this fact can be found in monograph [2]. It is based on the dependence of the monotonicity of a function on the sign of its (classical) derivative. In the case of the fractional Caputo derivative we have not such a clear dependence and, consequently the method of the proof used in [2] can not be applied in the fractional case.

Example 4.1. Let $e \in (1,2)$ be such a number that

$$t^{1-\alpha} - \frac{3}{2}(t-1)^{1-\alpha} > 0$$

for $t \in [1,e]$ (such an e exists because the function $t \longmapsto t^{1-\alpha} - \frac{3}{2}(t-1)^{1-\alpha}$ is continuous on $[0,\infty)$ and its value at $t = 1$ is 1) and consider the function

$$z(t) = \begin{cases} t & ;\ t \in [0,1] \\ -\frac{1}{2}t + \frac{3}{2} & ;\ t \in (1,e] \\ -\frac{1}{2}e + \frac{3}{2} & ;\ t \in (e,\infty) \end{cases} .$$

Of course, z is absolutely continuous on each interval $[0,b]$ and its derivative (of order one) is as follows:

$$z'(t) = \begin{cases} 1 & ;\ t \in (0,1) \\ -\frac{1}{2} & ;\ t \in (1,e) \\ 0 & ;\ t \in (e,\infty) \end{cases} .$$

So,

$$z'(t) = x_1(t) + x_2(t),$$

where

$$x_1(t) = \begin{cases} 1 & ;\ t \in (0,1) \\ 0 & ;\ t \in (1,\infty) \end{cases} ,$$

$$x_2(t) = \begin{cases} 0 & ;\ t \in (0,1) \\ -\frac{1}{2} & ;\ t \in (1,e) \\ 0 & ;\ t \in (e,\infty) \end{cases} ,$$

and, consequently,

$$I_{0+}^{1-\alpha} z' = I_{0+}^{1-\alpha} x_1 + I_{0+}^{1-\alpha} x_2.$$

In an elementary way, we calculate

$$I_{0+}^{1-\alpha} x_1(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} t^{1-\alpha} & ; \quad t \in (0,1) \\ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} (t^{1-\alpha} - (t-1)^{1-\alpha}) & ; \quad t \in (1,\infty) \end{cases}$$

and

$$I_{0+}^{1-\alpha} x_2(t) = \begin{cases} 0 & ; \quad t \in (0,1) \\ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} (-\frac{1}{2})(t-1)^{1-\alpha} & ; \quad t \in (1,e) \\ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} (-\frac{1}{2})((t-1)^{1-\alpha} - (t-e)^{1-\alpha}) & ; \quad t \in (e,\infty) \end{cases}.$$

In consequence,

$$\begin{aligned} {}^C D_{0+}^{\alpha} z(t) &= I_{0+}^{1-\alpha} z'(t) = I_{0+}^{1-\alpha} x_1(t) + I_{0+}^{1-\alpha} x_2(t) \\ &= \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} t^{1-\alpha} & ; \quad t \in (0,1) \\ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} (t^{1-\alpha} - \frac{3}{2}(t-1)^{1-\alpha}) & ; \quad t \in (1,e) \\ \frac{1}{\Gamma(1-\alpha)} \frac{1}{1-\alpha} (t^{1-\alpha} - \frac{3}{2}(t-1)^{1-\alpha} + \frac{1}{2}(t-e)^{1-\alpha}) & ; \quad t \in (e,\infty) \end{cases}. \end{aligned}$$

Thus,

$${}^C D_{0+}^{\alpha} z(t) > 0$$

for $t \in (0,\infty) \setminus \{1,e\}$, but z is not monotone on $[0,\infty)$.

It is worth to point out that the derivative ${}^C D_{0+}^{\alpha} z(t)$ is positive (even increasing) on the interval (e,∞) whereas the function z is constant on this interval.

From the Cauchy formula (4.2) we immediately obtain

THEOREM 4.1. *If the matrix $A(t)$ has nonnegative entries for any $t \in [0,\infty)$, then system (4.1) is positive.*

5. **Concluding remarks**

The existence and uniqueness of a solution to fractional time-varying Cauchy problem with Caputo derivative have been derived. The Cauchy formula for the solution has been obtained, too. Sufficient condition for the positivity of the system under consideration has been established. The example illustrating these results is given. An open problem is the following: is it possible to replace in Theorem 4.1 nonnegative matrix A by the Metzler matrix.

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