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Reachability of Discrete Time ARMA Representations

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A new formula for the solution of discrete time Auto Regressive Moving Average (ARMA) representations is provided. Then, the concept of reachability (also known as controllability-from-the-origin) is examined by showcasing the reachable subspace of the system and the consistent inputs that drive the system to a desired state, taking into account the admissible initial conditions. The theory is illustrated by an example and the study of the special cases of ARMA representations i.e. state space systems, descriptor systems and causal systems.

Keywords: discrete-time systems; reachability; ARMA representations; control systems; singular systems; descriptor systems; causality; linear systems; multivariable systems; matrix theory.

1. Introduction

Let \mathbb{R} be the field of reals, $\mathbb{R}[\sigma]$ the ring of polynomials with coefficients from \mathbb{R} and $\mathbb{R}(\sigma)$ the field of rational functions. By $\mathbb{R}[\sigma]^{p \times m}$, $\mathbb{R}(\sigma)^{p \times m}$, $\mathbb{R}_{pr}(\sigma)^{p \times m}$ we denote the sets of $p \times m$ polynomial, rational and proper rational matrices with real coefficients. We are going to study the behavior of non-homogenous systems of linear algebraic and difference equations, described by the matrix equation

$$A(\sigma)\beta(k) = B(\sigma)u(k) \quad (1.1)$$

where $k \in \mathbb{N}$ and

$$\begin{aligned} A(\sigma) &= A_0 + A_1\sigma + \dots + A_q\sigma^q \\ B(\sigma) &= B_0 + B_1\sigma + \dots + B_q\sigma^q \end{aligned} \quad (1.2)$$

with $A_i \in \mathbb{R}^{r \times r}$, $B_i \in \mathbb{R}^{r \times m}$, at least one of A_q, B_q is a non zero matrix and $\det A(\sigma) \neq 0$. The discrete time functions $u(k) : \mathbb{N} \rightarrow \mathbb{R}^m$ and $\beta(k) : \mathbb{N} \rightarrow \mathbb{R}^r$ define the input and output vectors of the system respectively, whereas σ denotes the forward shift operator, i.e. $\sigma^i \beta(k) = \beta(k+i)$. Systems described by (1.1) are called *ARMA (Auto-Regressive Moving Average)* representations.

A notable feature is that there is no distinction between states and outputs, in contrast to the state-space case. Such systems find applications in electrical circuit networks, mechanical, social, biological and economic systems, see for example Dai (1989); Rao (2011). It is easily seen that such representations can be considered as a generalisation of the known descriptor systems.

A usual method of approach for the analysis of ARMA representations (1.1) is through the transformation of the system into an equivalent state-space or descriptor system, as in Antsaklis and Michel (2006); Karampetakis and Vologianidis (2009); Mahmood et al. (1998); Ogata (2010). Descriptor systems have been extensively studied in Brull (2009); Dai (1989); Duan (2010); Lewis and Mertzios

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(1990); Yip and Sincovec (1981) and references therein and results have been extended to positive systems in Kaczorek (2002). Yet, as Antsaklis and Michel (2006) comments, the transformation of the system (1.1) into an equivalent representation may not always be desirable, since it involves a change of the internal variables. This may be inconvenient, since it can lead to the loss of the physical meaning of the original variables. This paper utilises a more direct approach, through the use of the Jordan Pairs of the polynomial matrix $A(\sigma)$ and the Laurent expansion of its inverse matrix. The algebraic structure of polynomial matrices and the theory of Jordan pairs has been studied in the early works of Cohen (1983); Gohberg et al. (2009) and later in Bernstein (2009); Kaczorek (2007b); Vardulakis (1991) and references therein. Results regarding the general solution of (1.1) have been previously provided in Fragulis and Vardulakis (1995); Hou et al. (2000); Karampetakis (1997); Vardulakis (1991) for the continuous time case and in Antoniou et al. (1998); Jones et al. (2003); Karampetakis et al. (2001); Karampetakis and Vologiannidis (2009) for the discrete time case.

The concept of reachability has been thoroughly examined by numerous authors, initially for state space systems in Antsaklis and Michel (2006); Hespanha (2009); Kailath (1980). These results have been extended to continuous and discrete time descriptor systems in Berger and Reis (2013); Coll et al. (2002); Dai (1989); Duan (2010); Karampetakis and Gregoriadou (2014); Koumboulis and Mertzios (1999); Lewis and Mertzios (1990); Malabre et al. (1990); Yip and Sincovec (1981), in Laub and Arnold (1984); Losse and Mehrmann (2008) for second order descriptor systems, in Ishihara and Terra (2001); Mishra and Tomar (2015); Mishra et al. (2016) for rectangular descriptor systems, in Commault and Alamir (2007); Kaczorek (2007a); Valcher (1996) for positive systems and in Fragulis and Vardulakis (1995); Karampetakis (1997); Mahmood et al. (1998) for continuous time ARMA systems. In contrast to the state-space case though, the following paradox is encountered. Despite the numerous papers studying the concepts of reachability and controllability, there is no in depth analysis regarding the input that drives the system to a desired state. This question goes in pair with the problem of consistent initial conditions, i.e. the equations that the initial values of the state and input need to satisfy, in order for the system to be impulse free for the continuous time case, and causal for the discrete time case. Although consistent initial conditions have been studied by many authors in Brull (2009); Dai (1989); Karampetakis et al. (2001); Lewis and Mertzios (1990); Vardulakis et al. (1999), their effect on the choice of the appropriate input that drives the system to the desired state has been left out of question, up until recently in Karampetakis and Gregoriadou (2014). There, reachability criteria for discrete time descriptor systems have been derived in a constructive way, by providing a consistent input that drives the system to a desired state, taking into account the consistency of initial conditions. As an example, for the descriptor system

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}}_E x(k+1) = \underbrace{\begin{pmatrix} -4 & -2 & 0 \\ 3 & 1 & 1 \\ 6 & 2 & 1 \end{pmatrix}}_A x(k) + \underbrace{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}_B u(k) \quad (1.3)$$

the authors of Karampetakis and Gregoriadou (2014) proposed the input sequence

$$u(0) = 0, u(1) = 0, u(2) = 2z_1 + z_2, u(3) = \frac{1}{4}z_3, u(4) = \frac{3}{4}z_1 + \frac{1}{4}z_2 + \frac{3}{16}z_3 \quad (1.4)$$

that drives the state of the system from the origin to any arbitrary state $z = (z_1 \ z_2 \ z_3)^T \in \mathbb{R}^3$ in three time steps, that is $x(3) = z$ and in addition, it satisfies the initial conditions of consistency proposed in Karampetakis et al. (2001).

The novelty of the present work lies in the extension of these results to the general case of discrete time ARMA representations.

Hence, the main aim of this work is twofold. Firstly, to present the solution of (1.1) in terms of the finite and infinite Jordan Pairs of $A(\sigma)$ and secondly, to construct the reachable subspace from the zero state, i.e. $R(\mathbf{0})$ of (1.1) by proposing the *consistent* inputs that drive the system from the origin to any vector in $R(\mathbf{0})$. This constructive method will yield a simple criterion for the reachability of the system, which is a generalised version of the already known results for state space and descriptor systems. In addition, these results are examined for the special case where the system (1.1) is causal.

This paper is organized as follows: In Section 2, some preliminary concepts from the theory of polynomial matrices are presented. A new expression for the solution of (1.1) is given, in terms of the finite and infinite Jordan Pairs of $A(\sigma)$. Section 3 describes the reachable subspace of (1.1) and proposes a criterion for the reachability of the system. In Section 4, causal systems are studied and in the last section, the established theory is illustrated by an example.

2. Preliminaries

In this section we provide some background from the theory of polynomial matrices. Results regarding the finite and infinite zero structure of a polynomial matrix shall be used extensively in the following sections. The general solution and admissible initial conditions of (1.1) are also presented.

2.1 Matrix Polynomials

Consider a polynomial matrix

$$A(\sigma) = A_q \sigma^q + A_{q-1} \sigma^{q-1} + \dots + A_1 \sigma + A_0 \quad (2.1)$$

with $A_i \in \mathbb{R}^{r \times r}$, $A_q \neq 0$ and $\det(A(\sigma)) \neq 0$.

DEFINITION 2.1 [Vardulakis \(1991\)](#) A square polynomial matrix $A(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$ is called *unimodular* if $\det A(\sigma) = c \in \mathbb{R}$, $c \neq 0$. A rational matrix $A(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r \times r}$ is called *biproper* if $\lim_{\sigma \rightarrow \infty} A(\sigma) = E \in \mathbb{R}^{r \times r}$ with $\text{rank} E = r$.

THEOREM 2.1 [Vardulakis \(1991\)](#) Let $A(\sigma)$ as in (2.1). There exist unimodular matrices $U_L(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$, $U_R(\sigma) \in \mathbb{R}[\sigma]^{r \times r}$ such that

$$S_{A(\sigma)}^{\mathbb{C}}(\sigma) = U_L(\sigma) A(\sigma) U_R(\sigma) = \text{diag}(1, \dots, 1, f_z(\sigma), f_{z+1}(\sigma), \dots, f_r(\sigma)) \quad (2.2)$$

with $1 \leq z \leq r$ and $f_j(\sigma)/f_{j+1}(\sigma)$, for $j = z, z+1, \dots, r$. $S_{A(\sigma)}^{\mathbb{C}}(\sigma)$ is called the *Smith form* of $A(\sigma)$, where $f_j(\sigma) \in \mathbb{R}[\sigma]$ are the *invariant polynomials* of $A(\sigma)$. The zeros $\lambda_i \in \mathbb{C}$ of $f_j(\sigma)$, $j = z, z+1, \dots, r$ are called *finite zeros* of $A(\sigma)$. Assume that $A(\sigma)$ has ℓ finite, distinct zeros. The partial multiplicities $n_{i,j}$ of each zero $\lambda_i \in \mathbb{C}$, $i = 1, \dots, \ell$ satisfy

$$0 \leq n_{i,z} \leq n_{i,z+1} \leq \dots \leq n_{i,r} \quad (2.3)$$

with

$$f_j(\sigma) = (\sigma - \lambda_i)^{n_{i,j}} \hat{f}_j(\sigma) \quad (2.4)$$

$j = z, \dots, r$ and $\hat{f}_j(\lambda_i) \neq 0$. The terms $(\sigma - \lambda_i)^{n_{i,j}}$ are called *finite elementary divisors* of $A(\sigma)$ at λ_i .

Denote by n the sum of the degrees of the finite elementary divisors of $A(\sigma)$, i.e.

$$n := \deg \det(A(\sigma)) = \deg \left[\prod_{j=z}^r f_j(\sigma) \right] = \sum_{i=1}^{\ell} \sum_{j=z}^r n_{i,j} \quad (2.5)$$

Similarly, we can find $U_L(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$, $U_R(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$ having no poles and zeros at $\sigma = \lambda_0$ such that

$$S_{A(\sigma)}^{\lambda_0}(\sigma) = U_L(\sigma)A(\sigma)U_R(\sigma) = \text{diag}(1, \dots, 1, (\sigma - \lambda_0)^{n_z}, \dots, (\sigma - \lambda_0)^{n_r}) \quad (2.6)$$

$S_{A(\sigma)}^{\lambda_0}(\sigma)$ is called the Smith form of $A(\sigma)$ at the local point λ_0 .

THEOREM 2.2 [Vardulakis \(1991\)](#); [Vardulakis et al. \(1982\)](#) Let $A(\sigma)$ as in (2.1). There exist biproper matrices $U_L(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r \times r}$, $U_R(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r \times r}$ such that

$$U_L(\sigma)A(\sigma)U_R(\sigma) = S_{A(\sigma)}^{\infty}(\sigma) = \text{diag} \left(\underbrace{\sigma^{q_1}, \sigma^{q_2}, \dots, \sigma^{q_u}}_u, \overbrace{\frac{1}{\sigma^{\hat{q}_{u+1}}}, \frac{1}{\sigma^{\hat{q}_{u+2}}}, \dots, \frac{1}{\sigma^{\hat{q}_r}}}_{r-u} \right) \quad (2.7)$$

with

$$q_1 \geq \dots \geq q_u \geq 0 \quad (2.8)$$

$$\hat{q}_r \geq \hat{q}_{r-1} \geq \dots \hat{q}_{u+1} > 0 \quad (2.9)$$

and $1 \leq u \leq r$. $S_{A(\sigma)}^{\infty}(\sigma)$ is called the *Smith form of $A(\sigma)$ at infinity*. If p_{∞} is the number of q_i 's in (2.8) with $q_i > 0$, then we say that $A(\sigma)$ has p_{∞} *poles at infinity*, each one of order $q_i > 0$. Also, if z_{∞} is the number of \hat{q}_i 's in (2.9), then we say that $A(\sigma)$ has z_{∞} *zeros at infinity*, each one of order $\hat{q}_i > 0$. It is proved in [Vardulakis \(1991\)](#) that $q_1 = q$.

DEFINITION 2.2 [Vardulakis \(1991\)](#) The dual polynomial matrix of $A(\sigma)$ is defined as

$$\tilde{A}(\sigma) := \sigma^q A\left(\frac{1}{\sigma}\right) = A_0 \sigma^q + A_1 \sigma^{q-1} + \dots + A_q \quad (2.10)$$

THEOREM 2.3 [Vardulakis \(1991\)](#) Let $\tilde{A}(\sigma)$ as in (2.10). There exist matrices $\tilde{U}_L(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$, $\tilde{U}_R(\sigma) \in \mathbb{R}(\sigma)^{r \times r}$ having no poles or zeros at $\sigma = 0$, such that

$$S_{\tilde{A}(\sigma)}^0(\sigma) = \tilde{U}_L(\sigma)\tilde{A}(\sigma)\tilde{U}_R(\sigma) = \text{diag}[\sigma^{\mu_1}, \dots, \sigma^{\mu_r}] \quad (2.11)$$

$S_{\tilde{A}(\sigma)}^0(\sigma)$ is the local Smith form of $\tilde{A}(\sigma)$ at $\sigma = 0$. The terms σ^{μ_j} are the finite elementary divisors of $\tilde{A}(\sigma)$ at zero and are called the *infinite elementary divisors (i.e.d.)* of $A(\sigma)$.

The connection between the Smith form at infinity of $A(\sigma)$ and the Smith form at zero of the dual matrix is given in [Hayton et al. \(1988\)](#); [Vardulakis \(1991\)](#):

$$S_{\tilde{A}(\sigma)}^0(\sigma) = \text{diag} \left[1, \underbrace{\sigma^{q-q_2}, \dots, \sigma^{q-q_u}}_{i.p.e.d.}, \underbrace{\sigma^{q+\hat{q}_{u+1}}, \dots, \sigma^{q+\hat{q}_r}}_{i.z.e.d.} \right] = \text{diag}[\sigma^{\mu_1}, \sigma^{\mu_2}, \dots, \sigma^{\mu_r}] \quad (2.12)$$

where by i.p.e.d. and i.z.e.d. we denote the infinite pole and infinite zero elementary divisors respectively. From the above formula it is seen that the the orders of the infinite elementary divisors of $A(\sigma)$ are given by

$$\begin{aligned}\mu_1 &= q - q_1 \stackrel{q=q_1}{=} 0 \\ \mu_j &= q - q_j \quad j = 2, 3, \dots, u \\ \mu_j &= q + \hat{q}_j \quad j = u + 1, \dots, r\end{aligned}\tag{2.13}$$

We denote by μ the sum of the degrees of the infinite elementary divisors of $A(\sigma)$ i.e.

$$\mu := \sum_{j=1}^r \mu_j\tag{2.14}$$

THEOREM 2.4 [Cohen \(1983\)](#); [Gohberg and Rodman \(1978\)](#); [Gohberg et al. \(2009\)](#) Let $(C \ J)$ and $(\bar{C}_\infty \ \bar{J}_\infty)$ be the finite and infinite Jordan Pairs of $A(\sigma)$, with $C \in \mathbb{R}^{r \times n}$, $J \in \mathbb{R}^{n \times n}$, $\bar{C}_\infty \in \mathbb{R}^{r \times \mu}$, $\bar{J}_\infty \in \mathbb{R}^{\mu \times \mu}$. The Jordan Pairs determine $A(\sigma)$ uniquely and satisfy the following properties:

- $\text{rank} \begin{pmatrix} C & CJ & \dots & CJ^{q-1} \end{pmatrix}^T = n$
- $\sum_{i=0}^q A_i C J^i = 0$

and

- $\det \tilde{A}(s)$ has a zero at $\lambda = 0$ of multiplicity μ .
- $\text{rank} \begin{pmatrix} \bar{C}_\infty & \bar{C}_\infty \bar{J}_\infty & \dots & \bar{C}_\infty \bar{J}_\infty^{q-1} \end{pmatrix}^T = \mu$
- $\sum_{i=0}^q A_i \bar{C}_\infty \bar{J}_\infty^{q-i} = 0$

THEOREM 2.5 [Vardulakis \(1991\)](#) Let $A(\sigma)^{-1}$ be the inverse of $A(\sigma)$. Then $A(\sigma)^{-1}$ has the representation

$$A(\sigma)^{-1} = C(\sigma I_n - J)^{-1} B_F + C_\infty (I_{\hat{\mu}} - \sigma J_\infty)^{-1} B_\infty\tag{2.15}$$

where the matrix triples (C, J, B_F) and $(C_\infty, J_\infty, B_\infty)$ are the *minimal realizations* of the strictly proper and polynomial parts of $A(\sigma)^{-1}$ respectively, with $C \in \mathbb{R}^{r \times n}$, $J \in \mathbb{R}^{n \times n}$, $B_F \in \mathbb{R}^{n \times r}$, $C_\infty \in \mathbb{R}^{r \times \hat{\mu}}$, $J_\infty \in \mathbb{R}^{\hat{\mu} \times \hat{\mu}}$, $B_\infty \in \mathbb{R}^{\hat{\mu} \times r}$, $\hat{\mu} = \sum_{i=u+1}^r (\hat{q}_i + 1) \leq \mu$ and

$$J_\infty = \text{blockdiag}(J_{\infty, r}, \dots, J_{\infty, u+1})\tag{2.16}$$

$$J_{\infty, i} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{(\hat{q}_i+1) \times (\hat{q}_i+1)}\tag{2.17}$$

for $i = u + 1, \dots, r$. It holds that

$$(\sigma I_n - J)^{-1} = \sigma^{-1} I_n + \sigma^{-2} J + \sigma^{-3} J^2 + \dots\tag{2.18}$$

$$(I_{\hat{\mu}} - \sigma J_\infty)^{-1} = I_{\hat{\mu}} + \sigma J_\infty + \sigma^2 J_\infty^2 + \dots + \sigma^{\hat{q}_r} J_\infty^{\hat{q}_r}\tag{2.19}$$

LEMMA 2.1 [Vardoulakis \(1991\)](#) The Laurent expansion of $A(\sigma)^{-1}$ at infinity is given by

$$A(\sigma)^{-1} = H_{\hat{q}_r} \sigma^{\hat{q}_r} + H_{\hat{q}_r-1} \sigma^{\hat{q}_r-1} + \dots + H_1 \sigma + H_0 + H_{-1} \sigma^{-1} + H_{-2} \sigma^{-2} + \dots \quad (2.20)$$

By equating the coefficients of the powers of σ^i of the last two expressions for $A(\sigma)^{-1}$ we get

$$H_i = C_\infty J_\infty^i B_\infty \quad i = 0, 1, 2, \dots, \hat{q}_r \quad (2.21a)$$

$$H_{-i} = C J^{i-1} B_F \quad i = 1, 2, \dots \quad (2.21b)$$

2.2 The Solution Space of $A(\sigma)B(k) = B(\sigma)u(k)$

The aim of this section is to present the solution of (1.1) in terms of the finite and infinite Jordan pairs.

THEOREM 2.6 The solution of $A(\sigma)\beta(k) = B(\sigma)u(k)$ is given by

$$\beta(k) = \begin{pmatrix} C & C_\infty \end{pmatrix} \begin{pmatrix} \sum_{i=1}^k J^{i-1} \Omega u(k-i) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \tilde{\Omega} u(k+q+i) \end{pmatrix} + \sum_{i=0}^{q-1} (C \Phi_{i+1} + C_\infty Z_i) u(k+i) - C J^k \sum_{i=0}^{q-1} \Phi_{i+1} u(i) + C J^k \beta_f(0) \quad (2.22)$$

where

$$\beta_f(0) = \begin{pmatrix} J^{q-1} B_F & \dots & B_F \end{pmatrix} \begin{pmatrix} A_q & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \dots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} \quad (2.23)$$

$$\Omega = J^q B_F B_q + J^{q-1} B_F B_{q-1} + \dots + J B_F B_1 + B_F B_0 \quad (2.24)$$

$$\tilde{\Omega} = B_\infty B_q + J_\infty B_\infty B_{q-1} + \dots + J_\infty^q B_\infty B_0 \quad (2.25)$$

$$\Phi_j = \sum_{i=0}^{q-j} J^i B_F B_{i+j} \quad Z_{q-j} = \sum_{i=0}^q J_\infty^i B_\infty B_{q-j-i} \quad (2.26)$$

with $j = 1, 2, \dots, q$ and $B_k \equiv 0$ for $k < 0$.

Proof. The general solution of the system, given in [Jones et al. \(2003\)](#); [Karampetakis et al. \(2001\)](#), is

$$\begin{aligned} \beta(k) = & \begin{pmatrix} H_{-k-q} & \dots & H_{-k-1} \end{pmatrix} \begin{pmatrix} A_q & \dots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \dots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} + \\ & + \begin{pmatrix} H_{-k} & \dots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} B_0 & \dots & B_q & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & B_0 & \dots & B_q \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ u(k+q+\hat{q}_r) \end{pmatrix} \end{aligned} \quad (2.27)$$

By replacing in (2.27) the fundamental matrix sequences H_i given in and (2.21a), (2.21b) and taking into account that

$$\Omega = \begin{pmatrix} J^q B_F & \dots & J B_F & B_F \end{pmatrix} \begin{pmatrix} B_q \\ \vdots \\ B_0 \end{pmatrix} \quad (2.28)$$

$$\begin{pmatrix} \tilde{\Omega} & J_\infty \tilde{\Omega} & \cdots & J_\infty^{\hat{q}_r} \tilde{\Omega} \end{pmatrix} = \begin{pmatrix} B_\infty & \cdots & J_\infty^{\hat{q}_r} B_\infty \end{pmatrix} \begin{pmatrix} B_q \\ \vdots \\ B_0 \end{pmatrix}, \quad q \leq \hat{q}_r \quad (2.29)$$

$$\begin{pmatrix} \tilde{\Omega} & J_\infty \tilde{\Omega} & \cdots & J_\infty^{\hat{q}_r} \tilde{\Omega} \end{pmatrix} = \begin{pmatrix} B_\infty & \cdots & J_\infty^{\hat{q}_r} B_\infty \end{pmatrix} \begin{pmatrix} B_q \\ \vdots \\ B_{q-\hat{q}_r} \end{pmatrix}, \quad q > \hat{q}_r \quad (2.30)$$

$$\begin{pmatrix} \Phi_1 & \cdots & \Phi_q \end{pmatrix} = \begin{pmatrix} J^{q-1} B_F & \cdots & B_F \end{pmatrix} \begin{pmatrix} B_q \\ \vdots \\ B_1 \end{pmatrix} \quad (2.31)$$

$$\begin{pmatrix} Z_0 & \cdots & Z_{q-1} \end{pmatrix} = \begin{pmatrix} B_\infty & \cdots & J_\infty^{q-1} B_\infty \end{pmatrix} \begin{pmatrix} B_0 & \cdots & B_{q-1} \\ \vdots \\ B_0 \end{pmatrix} \quad (2.32)$$

we obtain after a number of calculations the formula given in (2.22). \square

The last formula for the general solution of the system, offers a great convenience. It showcases which parts of the input are connected with the finite or infinite Jordan Pairs. So we can see that for an arbitrary k , the first k inputs, from $u(0)$ to $u(k-1)$ are connected to the finite Jordan Pairs, the next $u(k)$ to $u(k+q-1)$ are connected to both finite and infinite Jordan Pairs and the last $u(k+q)$ to $u(k+q+\hat{q}_r)$ are connected only to the infinite Jordan Pairs. This separation will be of utmost importance in the creation of the reachable subspace.

In order for the above formula to be a solution of (1.1), it needs to satisfy (1.1) for $k = 0, 1, \dots, q-1$. This brings us to the following definition and theorem.

DEFINITION 2.3 The set of all $\{u(k), \beta(k)\}$ that satisfy (1.1) for $k = 0, 1, \dots, q-1$ is called the set of *Admissible Initial Conditions* of the system and is denoted by H_{ad} .

THEOREM 2.7 Karampetakis et al. (2001) The set of all H_{ad} is given by

$$\begin{aligned} H_{ad} &:= \left\{ \beta(k), k = 0, \dots, q-1 \text{ and } u(k), k = 0, \dots, 2q + \hat{q}_r - 1 : \right. \\ &\quad \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_0 & \cdots & H_{q-1} \\ \vdots & & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \underbrace{\begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix}}_{\beta_{in}} = \\ &= \left. \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_0 & \cdots & H_{\hat{q}_r} & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ H_{-q+1} & \cdots & H_{\hat{q}_r-q+1} & \cdots & H_{\hat{q}_r} \end{pmatrix} \begin{pmatrix} B_0 & \cdots & B_q & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & B_0 & \cdots & B_q \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ u(2q + \hat{q}_r - 1) \end{pmatrix} \right\} \quad (2.33) \end{aligned}$$

3. Reachable Subspace

In this section the Reacability of system (1.1) is considered. We present a constructive proof to showcase the reachable subspace of the system and provide a criterion to test when a system is reachable. The results are illustrated by an example.

DEFINITION 3.1 We say that a vector $z \in \mathbb{R}^r$ is *reachable (controllable from the origin)* if there exists an input $u(k)$ that transfers $\beta(k)$ from the origin $\beta_{in} = \mathbf{0}$ to z in some finite time $k_0 \in \mathbb{N}$ i.e.

$$\beta(k_0) = z \quad (3.1)$$

The set of all reachable outputs from $\mathbf{0} \in H_{ad}$ is denoted by $R(\mathbf{0})$.

DEFINITION 3.2 The system (1.1) is called *reachable (controllable from the origin)* if every point $z \in \mathbb{R}^r$ is reachable. In mathematical terms, a system is reachable iff

$$R(\mathbf{0}) = \mathbb{R}^r \quad (3.2)$$

To describe $R(\mathbf{0})$, we first need to introduce the following formulas

$$C \langle J / \text{Im } \Omega \rangle := C \text{Im } \Omega + C J \text{Im } \Omega + \dots + C J^{n-1} \text{Im } \Omega \quad (3.3)$$

$$C_\infty \langle J_\infty / \text{Im } \tilde{\Omega} \rangle := C_\infty \text{Im } \tilde{\Omega} + C_\infty J_\infty \text{Im } \tilde{\Omega} + \dots + C_\infty J_\infty^{\hat{q}_r} \text{Im } \tilde{\Omega} \quad (3.4)$$

$$\text{Im } \Omega := \{x/x \in \mathbb{R}^n, \exists u \in \mathbb{R}^m : x = \Omega u\} \subset \mathbb{R}^n \quad (3.5)$$

$$\text{Im } \tilde{\Omega} := \{x/x \in \mathbb{R}^{\hat{\mu}}, \exists u \in \mathbb{R}^m : x = \tilde{\Omega} u\} \subset \mathbb{R}^{\hat{\mu}} \quad (3.6)$$

where $n = \deg \det(A(\sigma))$ is the sum of the finite elementary divisors of $A(\sigma)$ and $\hat{\mu} = \sum_{i=u+1}^r (\hat{q}_i + 1)$.

LEMMA 3.1 Define $W(1, n) := \sum_{i=1}^n C J^{i-1} \Omega \Omega^T (J^T)^{i-1} C^T$. We shall prove that

$$\text{Im } W(1, n) = C \langle J / \text{Im } \Omega \rangle \quad (3.7)$$

Proof. It is clear that $C \langle J / \text{Im } \Omega \rangle$ is produced by the linearly independent columns of

$$Q = (C \Omega \quad C J \Omega \quad \dots \quad C J^{n-1} \Omega) \quad (3.8)$$

It is sufficient to show that

$$\ker W^T(1, n) = \ker Q^T \Leftrightarrow \ker W(1, n) = \bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T \quad (3.9)$$

since $W(1, n)$ is symmetric, i.e. $W^T(1, n) = W(1, n)$.

First, we prove that $\ker W(1, n) \subseteq \bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T$. Let $x \in \ker W(1, n)$, then

$$\begin{aligned} W(1, n)x &= 0 \Rightarrow x^T W(1, n)x = 0 \Rightarrow \\ \sum_{i=1}^n x^T C J^{i-1} \Omega \Omega^T (J^T)^{i-1} C^T x &= 0 \Rightarrow \sum_{i=1}^n \left\| \Omega^T (J^T)^{i-1} C^T x \right\|^2 = 0 \Rightarrow \\ \Omega^T C^T x &= 0, \quad \Omega^T J^T C^T x = 0, \quad \dots, \quad \Omega^T (J^T)^{n-1} C^T x = 0 \end{aligned}$$

So $x \in \bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T$, therefore

$$\ker W(1, n) \subseteq \bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T \quad (3.10)$$

Now we shall prove that $\bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T \subseteq \ker W(1, n)$. Let $x \in \bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T$, then

$$\begin{aligned} W(1, n)x &= \sum_{i=1}^n C J^{i-1} \Omega \Omega^T (J^T)^{i-1} C^T x = \\ &= \underbrace{\Omega \Omega^T C^T x}_0 + J \underbrace{\Omega \Omega^T J^T C^T x}_0 + \dots + J^{n-1} \underbrace{\Omega \Omega^T (J^T)^{n-1} C^T x}_0 = 0 \Rightarrow x \in \ker W(1, n) \end{aligned}$$

Thus $\bigcap_{i=0}^{n-1} \ker \Omega^T (J^T)^i C^T \subseteq \ker W(1, n)$. \square

LEMMA 3.2 $\text{Im } W(1, n) = \text{Im } W(1, k)$ for $k \geq n$.

Proof. We know according to the Cayley-Hamilton theorem that each matrix satisfies its characteristic polynomial. So the matrix J , which as we know is in block diagonal form, has the characteristic polynomial $p(\lambda) = \det(\lambda I_n - J) = \lambda^n + \sum_{i=0}^{n-1} p_i \lambda^i$. Hence, J satisfies its characteristic equation i.e.

$$p(J) = J^n + p_{n-1} J^{n-1} + \dots + p_0 I_n = 0 \quad (3.11)$$

This means that the matrix $J^n = -p_{n-1} J^{n-1} - \dots - p_0 I_n$ is linearly dependent to the matrices $J^{n-1}, J^{n-2}, \dots, J, I_n$. Multiplying the above equation from the right by J and replacing J^n , we get that the same holds for J^{n+1} . Following this procedure, we get that all $J^k, k \geq n$ are linearly dependent to $J^{n-1}, J^{n-2}, \dots, J, I_n$. As a result

$$C \text{Im} \begin{pmatrix} \Omega & J\Omega & \dots & J^{n-1}\Omega \end{pmatrix} = C \text{Im} \begin{pmatrix} \Omega & J\Omega & \dots & J^{k-1}\Omega \end{pmatrix} \Leftrightarrow \quad (3.12)$$

$$\text{Im } W(1, n) = \text{Im } W(1, k) \quad (3.13)$$

for $k \geq n$. \square

LEMMA 3.3 Define $\tilde{W}(0, \hat{q}_r) := \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} \tilde{\Omega}^T (J_\infty^T)^i C_\infty^T$. Then

$$\text{Im } \tilde{W}(0, \hat{q}_r) = C_\infty \langle J_\infty / \text{Im } \tilde{\Omega} \rangle \quad (3.14)$$

Proof. The proof follows the same procedure as in Lemma 3.1 and therefore shall be omitted. \square

Now we proceed to the main theorem of this paper.

THEOREM 3.1 The reachable subspace $R(\mathbf{0})$ of (1.1) is

$$R(\mathbf{0}) = C \langle J / \text{Im } \Omega \rangle + C_\infty \langle J_\infty / \text{Im } \tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im} (C \Phi_{i+1} + C_\infty Z_i) \quad (3.15)$$

Proof. Firstly, we prove that

$$R(\mathbf{0}) \subseteq C\langle J/\text{Im}\Omega \rangle + C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.16)$$

Let $\beta(k) \in R(\mathbf{0})$. This means that the initial conditions vector is

$$\beta_{in} = \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (3.17)$$

and the above vector, along with the first $2q + \hat{q}_r$ input values

$$\begin{pmatrix} u(0) \\ \vdots \\ u(2q + \hat{q}_r - 1) \end{pmatrix} \quad (3.18)$$

need to satisfy the equation of consistency (2.33). The simplest choice of initial input values, for (2.33) to hold true is

$$u(i) = 0 \quad i = 0, \dots, 2q + \hat{q}_r - 1 \quad (3.19)$$

For an arbitrary $k_0 > q$ the output of the system becomes

$$\beta(k_0) = \begin{pmatrix} C & C_\infty \end{pmatrix} \begin{pmatrix} \sum_{i=1}^{k_0} J^{i-1} \Omega u(k_0 - i) \\ \sum_{i=0}^{\hat{q}_r} J_\infty^i \tilde{\Omega} u(k_0 + q + i) \end{pmatrix} + \sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) u(k_0 + i) \quad (3.20)$$

and it is easy to see that

$$C \sum_{i=1}^{k_0} J^{i-1} \Omega u(k_0 - i) \in \text{Im}W(1, k_0) \subseteq C\langle J/\text{Im}\Omega \rangle \quad (3.21)$$

$$C_\infty \sum_{i=0}^{\hat{q}_r} J_\infty^i \tilde{\Omega} u(k_0 + q + i) \in \text{Im}\tilde{W}(0, \hat{q}_r) = C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle \quad (3.22)$$

$$\sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) u(k_0 + i) \in \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.23)$$

and thus

$$R(\mathbf{0}) \subseteq C\langle J/\text{Im}\Omega \rangle + C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.24)$$

Now we shall prove the opposite, that

$$R(\mathbf{0}) \supseteq C\langle J/\text{Im}\Omega \rangle + C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.25)$$

Let

$$z \in C\langle J/\text{Im}\Omega \rangle + C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.26)$$

i.e.

$$z = \beta_1 + \beta_2 + \beta_3 \quad (3.27)$$

where $\beta_1 \in C\langle J/\text{Im}\Omega \rangle$, $\beta_2 \in C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle$ and $\beta_3 \in \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i)$. We need to construct a consistent input $u(k)$, so that for a certain k , we have

$$\beta(k) = z \quad (3.28)$$

We shall show that this control sequence can be constructed for $k = 2q + \hat{q}_r + n$. For this choice of k the output is

$$\begin{aligned} \beta(2q + \hat{q}_r + n) = & \underbrace{\sum_{i=1}^{2q+\hat{q}_r+n} C J^{i-1} \Omega u(2q + \hat{q}_r + n - i)}_A + \\ & + \underbrace{\sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) u(2q + \hat{q}_r + n + i)}_B + \underbrace{\sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} u(3q + \hat{q}_r + n + i)}_G \end{aligned} \quad (3.29)$$

where

$$A = \sum_{i=1}^{2q+\hat{q}_r+n} C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) = \sum_{i=1}^n C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) + \sum_{i=n+1}^{2q+\hat{q}_r+n} C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) \quad (3.30)$$

corresponds to the input sequence $u(0), \dots, u(2q + \hat{q}_r + n - 1)$.

$$B = \sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) u(2q + \hat{q}_r + n + i) \quad (3.31)$$

corresponds to the input sequence $u(2q + \hat{q}_r + n), \dots, u(3q + \hat{q}_r + n - 1)$.

$$G = \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} u(3q + \hat{q}_r + n + i) \quad (3.32)$$

corresponds to the input sequence $u(3q + \hat{q}_r + n), \dots, u(3q + 2\hat{q}_r + n)$.

Since $\beta_1 \in C\langle J/\text{Im}\Omega \rangle \equiv \text{Im}W(1, n)$, there exists a vector $w \in \mathbb{R}^r$ such that

$$\beta_1 = \sum_{i=1}^n C J^{i-1} \Omega \Omega^T (J^T)^{i-1} C^T w = C \Omega \Omega^T C^T w + \dots + C J^{n-1} \Omega \Omega^T (J^T)^{n-1} C^T w \quad (3.33)$$

and since

$$\sum_{i=1}^n C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) = C \Omega u(2q + \hat{q}_r + n - 1) + \dots + C J^{n-1} \Omega u(2q + \hat{q}_r) \quad (3.34)$$

by choosing $u(i) = \Omega^T (J^T)^{2q+\hat{q}_r+n-1-i} C^T w$ for $i = 2q + \hat{q}_r, \dots, 2q + \hat{q}_r + n - 1$ and $u(i) = 0$ for $i = 0, \dots, 2q + \hat{q}_r - 1$ we get

$$A = \sum_{i=1}^{2q+\hat{q}_r+n} C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) = \sum_{i=1}^n C J^{i-1} \Omega u(2q + \hat{q}_r + n - i) = \beta_1 \quad (3.35)$$

Since $\beta_3 \in \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i)$, there exist vectors $\beta_{3i} \in \mathbb{R}^m$ such that

$$\beta_3 = \sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) \beta_{3i} \quad (3.36)$$

By choosing

$$\begin{aligned} u(2q + \hat{q}_r + n) &= \beta_{3,0} \\ u(2q + \hat{q}_r + n + 1) &= \beta_{3,1} \\ &\vdots \\ u(3q + \hat{q}_r + n - 1) &= \beta_{3,q-1} \end{aligned} \quad (3.37)$$

we get

$$B = \sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) u(2q + \hat{q}_r + n + i) = \sum_{i=0}^{q-1} (C\Phi_{i+1} + C_\infty Z_i) \beta_{3i} = \beta_3 \quad (3.38)$$

Since $\beta_2 \in C_\infty \langle J_\infty / \text{Im } \tilde{\Omega} \rangle \equiv \text{Im } \tilde{W}(0, q_r)$, there exists a vector $v \in \mathbb{R}^r$ such that

$$\beta_2 = \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} \tilde{\Omega}^T (J_\infty^T)^i C_\infty^T v = C_\infty \tilde{\Omega} \tilde{\Omega}^T C_\infty^T v + \dots + C_\infty J_\infty^{\hat{q}_r} \tilde{\Omega} \tilde{\Omega}^T (J_\infty^T)^{\hat{q}_r} C_\infty^T v \quad (3.39)$$

and since

$$\sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} u(3q + \hat{q}_r + n + i) = C_\infty \tilde{\Omega} u(3q + \hat{q}_r + n) + \dots + C_\infty J_\infty^{\hat{q}_r} \tilde{\Omega} u(3q + 2\hat{q}_r + n) \quad (3.40)$$

by choosing $u(i) = \tilde{\Omega}^T (J_\infty^T)^{i-3q-\hat{q}_r-n} C_\infty^T v$ for $i = 3q + \hat{q}_r + n, \dots, 3q + 2\hat{q}_r + n$ we get

$$C = \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} u(3q + \hat{q}_r + n + i) = \sum_{i=0}^{\hat{q}_r} C_\infty J_\infty^i \tilde{\Omega} \tilde{\Omega}^T (J_\infty^T)^i C_\infty^T v = \beta_2 \quad (3.41)$$

Overall, by choosing as an input

$$u(k) = \begin{cases} 0 & k = 0, \dots, 2q + \hat{q}_r - 1 \\ \Omega^T (J^T)^{2q+\hat{q}_r+n-1-k} C^T w & k = 2q + \hat{q}_r, \dots, 2q + \hat{q}_r + n - 1 \\ \beta_{3,k-2q-\hat{q}_r-n} & k = 2q + \hat{q}_r + n, \dots, 3q + \hat{q}_r + n - 1 \\ \tilde{\Omega}^T (J_\infty^T)^{k-3q-\hat{q}_r-n} C_\infty^T v & k = 3q + \hat{q}_r + n, \dots, 3q + 2\hat{q}_r + n \end{cases} \quad (3.42)$$

we end up with

$$\beta(2q + \hat{q}_r + n) = \beta_1 + \beta_2 + \beta_3 = z \quad (3.43)$$

which proves that

$$R(\mathbf{0}) \supseteq C\langle J/\text{Im}\Omega \rangle + C_\infty\langle J_\infty/\text{Im}\tilde{\Omega} \rangle + \sum_{i=0}^{q-1} \text{Im}(C\Phi_{i+1} + C_\infty Z_i) \quad (3.44)$$

□

In the above proof, it was shown that any desired output can be reached within $2q + \hat{q}_r + n$ time steps. However, an admissible input sequence could possibly be constructed in less steps, as will be shown in the illustrative example of the last section.

So far we have managed to describe the reachable subspace $R(\mathbf{0})$ by the use of the finite and infinite Jordan Pairs of $A(\sigma)$. Since the system is reachable when $R(\mathbf{0}) = \mathbb{R}^r$ we can easily conclude to the following theorem that provides a reachability test for the system.

THEOREM 3.2 The system (1.1) is reachable from the origin, iff

$$\text{rank} \begin{pmatrix} Q_1 & Q_2 & Q_3 \end{pmatrix} = r \quad (3.45)$$

where

$$Q_1 = (CJ^{n-1}\Omega \quad \cdots \quad C\Omega) \quad (3.46a)$$

$$Q_2 = (C\Phi_1 + C_\infty Z_0 \quad \cdots \quad C\Phi_q + C_\infty Z_{q-1}) \quad (3.46b)$$

$$Q_3 = (C_\infty \tilde{\Omega} \quad \cdots \quad C_\infty J_\infty^{\hat{q}_r} \tilde{\Omega}) \quad (3.46c)$$

By expanding the matrices in (3.46) and rewriting their coefficients in terms of the Laurent series expansion using (2.21a),(2.21b), bearing in mind that $J_\infty^i = 0$ for $i \geq \hat{q}_r + 1$, since $\hat{q}_r + 1$ is the *index of nilpotency* of J_∞ we get

$$Q_1 = (H_{-q-n} \quad \cdots \quad H_{-1}) \begin{pmatrix} B_q & & \\ \vdots & \ddots & \\ B_0 & & B_q \\ & \ddots & \vdots \\ & & B_0 \end{pmatrix} \quad Q_2 = (H_{-q} \quad \cdots \quad H_q) \begin{pmatrix} B_q & & \\ \vdots & \ddots & \\ B_0 & & B_q \\ & \ddots & \vdots \\ & & B_0 \end{pmatrix} \quad (3.47)$$

$$Q_3 = (H_0 \quad \cdots \quad H_{\hat{q}_r}) \begin{pmatrix} B_q & & & \\ \vdots & \ddots & & \\ B_0 & & \ddots & \\ & \ddots & & \ddots & \\ & & B_0 & \cdots & B_q \end{pmatrix} \quad (3.48)$$

By replacing (3.47),(3.48) in (3.45) we obtain an equivalent reachability test:

$$\text{rank} \begin{pmatrix} H_{-q-n} & \cdots & H_{\hat{q}_r} \\ \vdots & \ddots & \\ B_0 & & \ddots & \\ & \ddots & & \ddots & \\ & & B_0 & \cdots & B_q \end{pmatrix} = r \quad (3.49)$$

(3.49) indicates that the matrix $Q = (Q_1 \ Q_2 \ Q_3)$ which contains the first $q + n + \hat{q}_r + 1$ terms of the Laurent expansion of $A(\sigma)^{-1}B(\sigma)$ does not lose rank if and only if the system (1.1) is reachable. This result is comparable to the one presented in Karampetakis and Vologiannidis (2009) where a different approximation was used. The advantage of the approximation used here is that we find the consistent input that drives the system from the zero initial conditions to the desired output z .

REMARK 3.1 In the special case of state space systems

$$\beta(k+1) = A\beta(k) + Bu(k) \quad (3.50)$$

or equivalently

$$(\sigma I_r - A)\beta(k) = Bu(k) \quad (3.51)$$

the system's matrices are $A_0 = -A$, $A_1 = I_r$, $B_0 = B$, with $q = 1$ and $n = r$. Now, by using the formula

$$(\sigma I_r - A)^{-1} = \sigma^{-1}I_r + \sigma^{-2}A + \sigma^{-3}A^2 + \dots \quad (3.52)$$

we get

$$H_{-1} = I_r \quad H_{-2} = A \quad H_{-3} = A^2 \quad \dots \quad (3.53)$$

and the reachability criterion takes the form

$$\text{rank} \begin{pmatrix} A^{n-1}B & \dots & B \end{pmatrix} = r \quad (3.54)$$

which is the reachability criterion for discrete time state space systems provided in Antsaklis and Michel (2006); Kailath (1980).

REMARK 3.2 In the special case of generalised state space or descriptor systems

$$E\beta(k+1) = A\beta(k) + Bu(k) \quad (3.55)$$

or equivalently

$$(E\sigma - A)\beta(k) = Bu(k) \quad (3.56)$$

with $\det E = 0$, the system's matrices are $A_0 = -A$, $A_1 = E$, $B_0 = B$, with $q = 1$. The reachability criterion takes the form

$$\text{rank} \begin{pmatrix} H_{-n-1}B & \dots & H_{\hat{q}_r}B \end{pmatrix} = r \quad (3.57)$$

This result is comparable to the ones presented in Lewis and Mertzios (1990) and later in Karampetakis and Gregoriadou (2014):

$$\text{rank}(U_n) = \text{rank} \begin{pmatrix} \phi_{-\mu}B & \dots & \phi_{-1}B & \phi_0B & \dots & \phi_{n-1}B \end{pmatrix} = r \quad (3.58)$$

where the coefficients ϕ_i are computed from the Laurent expansion of $(\sigma E - A)^{-1}$:

$$(\sigma E - A)^{-1} = \phi_{-\mu}\sigma^{\mu-1} + \dots + \phi_{-1}\sigma^0 + \phi_0\sigma^{-1} + \dots \quad (3.59)$$

4. Causal Systems

In this section, the solution and reachable subspace of *causal systems* is presented.

DEFINITION 4.1 The ARMA representation $A(\sigma)\beta(k) = B(\sigma)u(k)$ is called *causal* iff for every admissible input $u(k)$ such that $u(k) = 0, k < 0$ the solution $\beta(k)$ of $A(\sigma)\beta(k) = B(\sigma)u(k)$ depends only on the initial conditions of $\beta(k)$ and the present and past values of $u(k)$.

Following a procedure similar to [Vardulakis et al. \(1999\)](#) it can be shown that the system (1.1) is causal iff:

1. $A(\sigma)^{-1} \in \mathbb{R}_{pr}(\sigma)^{r \times r}$
2. $A(\sigma)^{-1}B(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r \times r}$

Under the condition that $A(\sigma)^{-1}$ is proper, the Laurent expansion of $A(\sigma)^{-1}$ becomes

$$A(\sigma)^{-1} = H_0 + H_{-1}\sigma^{-1} + H_{-2}\sigma^{-2} + \dots \quad (4.1)$$

or equivalently

$$A(\sigma)^{-1} = C(\sigma I_n - J)^{-1}B_F + H_0 \quad (4.2)$$

so $H_0 = \dots = H_{\hat{q}_r} = 0$, and the matrix $A(\sigma)$ has no zeros at infinity. By equating coefficients in (4.1) and (4.2) we get

$$H_{-i} = CJ^{i-1}B_F \quad i = 1, 2, \dots \quad (4.3)$$

In addition, from $A(\sigma)^{-1}A(\sigma) = I_r$, we obtain the relations

$$H_{i-q}A_q + \dots + H_iA_0 = \delta_i I_r, \quad (4.4)$$

where δ_i denotes the Kronecker delta.

In this special case the admissible initial conditions of (1.1) are the following.

COROLLARY 4.1 If the system (1.1) is causal, the input/output initial conditions become

$$\begin{aligned} & \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} = \\ & = \begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix} \begin{pmatrix} H_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ H_{-q+1} & \cdots & H_0 \end{pmatrix} \begin{pmatrix} B_0 & \cdots & B_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_0 \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ u(q-1) \end{pmatrix} \quad (4.5) \end{aligned}$$

COROLLARY 4.2 From (2.22), under the conditions that $A(\sigma)^{-1} \in \mathbb{R}_{pr}(\sigma)^{r \times r}$ and $A(\sigma)^{-1}B(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r \times r}$, the solution of the causal system (1.1) is

$$\beta(k) = CJ^k\beta_f(0) + (CJ^{k-1}B_F \quad \cdots \quad CB_F \quad H_0) \begin{pmatrix} B_0 & \cdots & B_q & \cdots \\ & \ddots & & \ddots \\ & & \ddots & B_q \\ & & & \ddots & B_0 \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ u(k) \end{pmatrix} \quad (4.6)$$

Using the same method as in the proof of Theorem 3.1 we derive the reachable subspace of the causal system.

THEOREM 4.1 The reachable subspace $R(\mathbf{0})$ of the causal system is

$$R(\mathbf{0}) = \text{Im} \left(\underbrace{\begin{pmatrix} C J^{q+n-1} B_F & \dots & C B_F & H_0 \end{pmatrix}}_{\Lambda} \begin{pmatrix} B_q \\ \vdots \\ B_0 \end{pmatrix} \right) \quad (4.7)$$

REMARK 4.1 Similar to the proof of Theorem 3.1, we find that an admissible input can be constructed that drives the system from the origin to an arbitrary $z \in R(\mathbf{0})$ in $q+n$ steps, that is $\beta(q+n) = z$. Writing z as $z = \beta_1 + \beta_2$, with $\beta_1 \in C \langle J/\text{Im} \Omega \rangle$ and $\beta_2 \in \text{Im} M$, with

$$\Omega = J^q B B_q + J^{q-1} B B_{q-1} + \dots + J B B_1 + B B_0 \quad (4.8)$$

$$M = C J^{q-1} B B_q + \dots + C B B_1 + H_0 B_0 \quad (4.9)$$

we find that the admissible input is

$$u(k) = \begin{cases} 0 & k = 0, \dots, q-1 \\ \Omega^T (J^T)^{q+n-1-k} C^T w & k = q, \dots, q+n-1 \\ M^T v & k = q+n \end{cases} \quad (4.10)$$

where $w, v \in \mathbb{R}^r$, such that

$$\beta_1 = \sum_{i=1}^n C J^{i-1} \Omega \Omega^T (J^T)^{i-1} C^T w, \quad \beta_2 = M M^T v \quad (4.11)$$

and so the output for $k = q+n$ is

$$\beta(q+n) = \Lambda \begin{pmatrix} u(q) \\ \vdots \\ u(q+n) \end{pmatrix} = \sum_{i=0}^{n-1} C J^i \Omega u(q+n-1-i) + M u(q+n) = \beta_1 + \beta_2 = z \quad (4.12)$$

The above results lead to the following reachability criterion.

THEOREM 4.2 The causal system (1.1) is reachable from the origin iff

$$\text{rank} \left(\begin{pmatrix} C J^{q+n-1} B_F & \dots & C B_F & H_0 \end{pmatrix} \begin{pmatrix} B_q \\ \vdots \\ B_0 \end{pmatrix} \right) = r \quad (4.13)$$

or equivalently

$$\text{rank} \left(\begin{pmatrix} H_{-q-n} & \cdots & H_{-1} & H_0 \end{pmatrix} \begin{pmatrix} B_q & & \\ \vdots & \ddots & \\ B_0 & & B_q \\ & \ddots & \vdots \\ & & B_0 \end{pmatrix} \right) = r \quad (4.14)$$

5. An Illustrative Example

EXAMPLE 5.1 Let

$$\underbrace{\begin{pmatrix} 7-3\sigma & -3\sigma+\sigma^2 & 2+\sigma \\ -5+2\sigma & 2 & 1-2\sigma \\ -5+5\sigma-\sigma^2 & -\sigma & -3\sigma+\sigma^2 \end{pmatrix}}_{A(\sigma)} \beta(k) = \underbrace{\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & 1 & 1 \\ \sigma & 0 & 1 \end{pmatrix}}_{B(\sigma)} u(k) \quad (5.1)$$

For the matrix $A(\sigma)$ we have

$$S_{A(\sigma)}^{\mathbb{C}}(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (\sigma-2)^2 \end{pmatrix}, \quad S_{\tilde{A}(\sigma)}^0(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma^4 \end{pmatrix} \quad (5.2)$$

and so $q=2, n=2, \mu=4$, with $\mu = q + \hat{q}_3 = 4 \Rightarrow \hat{q}_3 = 4 - q = 2$. So $\hat{\mu} = \hat{q}_3 + 1 = 3$ and a realization of $A(s)^{-1}$ is given by

$$C = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad B_F = \begin{pmatrix} 0 & \frac{2}{5} & 0 \\ -1 & -\frac{6}{5} & -\frac{1}{5} \end{pmatrix} \quad (5.3)$$

$$C_{\infty} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad J_{\infty} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_{\infty} = \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & -\frac{1}{5} & 0 \end{pmatrix} \quad (5.4)$$

The matrices $\Omega, \tilde{\Omega}$ are

$$\Omega = J^2 B_F B_2 + J B_F B_1 + B_F B_0 = \begin{pmatrix} -\frac{21}{5} & \frac{2}{5} & \frac{2}{5} \\ -\frac{22}{5} & -\frac{6}{5} & -\frac{7}{5} \end{pmatrix} \quad (5.5)$$

with $\text{rank}(\Omega) = 2$ and

$$\tilde{\Omega} = B_{\infty} B_2 + J_{\infty} B_{\infty} B_1 + J_{\infty}^2 B_{\infty} B_0 = \begin{pmatrix} -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (5.6)$$

with $\text{rank}(\tilde{\Omega}) = 1$. Computing Φ_1, Φ_2 and Z_0, Z_1 in the same fashion, we get

$$\Phi_1 = J B_F B_2 + B_F B_1 = \begin{pmatrix} -1 & 0 & 0 \\ -\frac{11}{5} & 0 & 0 \end{pmatrix}, \quad \Phi_2 = B_F B_2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (5.7)$$

$$Z_0 = B_\infty B_0 = \begin{pmatrix} 0 & 0 & \frac{1}{5} \\ 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & -\frac{1}{5} & -\frac{1}{5} \end{pmatrix}, \quad Z_1 = B_\infty B_1 + J_\infty B_\infty B_0 = \begin{pmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 \end{pmatrix} \quad (5.8)$$

$$Q_1 = (C\Omega \quad CJ\Omega) = \left(\begin{array}{ccc|ccc} \frac{-64}{5} & \frac{-2}{5} & \frac{-3}{5} & \frac{-172}{5} & \frac{-16}{5} & -4 \\ \frac{-21}{5} & \frac{2}{5} & \frac{2}{5} & \frac{-64}{5} & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-22}{5} & \frac{-6}{5} & \frac{-7}{5} & \frac{-44}{5} & \frac{-12}{5} & \frac{-14}{5} \end{array} \right) \quad (5.9)$$

$$Q_2 = (C\Phi_1 + C_\infty Z_0 \quad C\Phi_2 + C_\infty Z_1) = \left(\begin{array}{ccc|ccc} -\frac{21}{5} & \frac{2}{5} & \frac{1}{5} & -\frac{6}{5} & \frac{2}{5} & 0 \\ -1 & 1 & \frac{1}{5} & -\frac{4}{5} & -\frac{2}{5} & -\frac{2}{5} \\ -\frac{11}{5} & -\frac{3}{5} & 0 & -\frac{2}{5} & \frac{4}{5} & \frac{2}{5} \end{array} \right) \quad (5.10)$$

$$Q_3 = (C_\infty \tilde{\Omega} \quad C_\infty J_\infty \tilde{\Omega} \quad C_\infty J_\infty^2 \tilde{\Omega}) = \left(\begin{array}{ccc|ccc|ccc} -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \quad (5.11)$$

Overall

$$\text{rank}(Q_1 \quad Q_2 \quad Q_3) = 3 \quad (5.12)$$

So the above system is reachable. That means that every vector in \mathbb{R}^3 can be reached from the origin within a finite number of time steps. We will now construct such an input using the methodology presented in the Proof of Theorem 3.1.

First of all, we need to choose the appropriate admissible initial conditions for the input, so that (2.33) is satisfied. Thus, select the inputs

$$\begin{pmatrix} u(0) \\ \vdots \\ u(2q + \hat{q}_3 - 1) \end{pmatrix} = \begin{pmatrix} u(0) \\ \vdots \\ u(5) \end{pmatrix} \quad (5.13)$$

to belong to the kernel of:

$$\begin{pmatrix} A_0 & A_1 \\ 0 & A_0 \end{pmatrix} \begin{pmatrix} H_0 & H_1 & H_2 & 0 \\ H_{-1} & H_0 & H_1 & H_2 \end{pmatrix} \begin{pmatrix} B_0 & B_1 & B_2 & 0 & 0 & 0 \\ 0 & B_0 & B_1 & B_2 & 0 & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & 0 \\ 0 & 0 & 0 & B_0 & B_1 & B_2 \end{pmatrix} \quad (5.14)$$

The simplest choice is

$$\begin{pmatrix} u(0) \\ \vdots \\ u(5) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad (5.15)$$

Now, since $\text{rank} Q_2 = 3$, every vector in \mathbb{R}^3 can be written as a linear combination of the columns of Q_2 . So we only require the input time sequences corresponding to Q_2 to reach any output in \mathbb{R}^3 . Let an arbitrary vector

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \in \mathbb{R}^3 \quad (5.16)$$

The above vector can be written as a linear combination of $C\Phi_1 + C_\infty Z_0$ and $C\Phi_2 + C_\infty Z_1$. That is, $\exists c_1, c_2 \in \mathbb{R}^3$ such that

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (C\Phi_1 + C_\infty Z_0)c_1 + (C\Phi_2 + C_\infty Z_1)c_2 \quad (5.17)$$

Solving the system we get

$$c_1 = \begin{pmatrix} c_{11} \\ c_{12} \\ c_{13} \end{pmatrix} = \begin{pmatrix} -\beta_1 + \beta_2 + \beta_3 \\ c_{12} \\ c_{13} \end{pmatrix} \quad (5.18)$$

$$c_2 = \begin{pmatrix} c_{21} \\ c_{22} \\ c_{23} \end{pmatrix} = \begin{pmatrix} c_{21} \\ -8\beta_1 + \frac{21\beta_2}{2} + \frac{21\beta_3}{2} - c_{12} - \frac{c_{13}}{2} + 3c_{21} \\ \frac{21\beta_1}{2} - \frac{31\beta_2}{2} - 13\beta_3 + \frac{7c_{12}}{2} + c_{13} - 5c_{21} \end{pmatrix} \quad (5.19)$$

where $c_{12}, c_{13}, c_{21} \in \mathbb{R}$ are arbitrary. To reach the desired output, we need to set $k = 2q + \hat{q}_r = 6$. Choosing as an input

$$u(k) = \begin{cases} 0_{3 \times 1} & k = 0, \dots, 5 \\ c_1 & k = 6 \\ c_2 & k = 7 \\ 0_{3 \times 1} & k = 8, \dots \end{cases} \quad (5.20)$$

we get the output of the system (under zero initial conditions)

$$\begin{aligned} \beta(6) &= [C \quad C_\infty] \begin{pmatrix} \sum_{i=0}^6 J^{i-1} \tilde{\Omega} u(6-i) \\ \sum_{i=0}^6 J_\infty^i \tilde{\Omega} u(8+i) \end{pmatrix} + \sum_{i=0}^1 (C\Phi_{i+1} + C_\infty Z_i) u(6+i) - C J^6 \sum_{i=0}^{2-1} \Phi_{i+1} u(i) \Rightarrow \\ \beta(6) &= (C\Phi_1 + C_\infty Z_0) u(6) + (C\Phi_2 + C_\infty Z_1) u(7) \Rightarrow \\ \beta(6) &= \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \end{aligned} \quad (5.21)$$

So any arbitrary output can be reached within 6 time steps.

Of course, the above input sequence is by no means unique, since it depends on c_{12}, c_{13}, c_{21} that can be chosen arbitrarily. One can find multiple consistent inputs that can drive the system to a desired output. The advantage of our methodology lies in the fact that we take into account the admissible initial conditions (2.33).

Another approach for solving this problem might be to use the formula proposed by Karampetakis et al. (2001), Jones et al. (2003). Let the general solution of the system, under zero initial conditions $\beta(0) = \beta(1) = 0_3$:

$$\beta(k) = (H_{-k} \quad \dots \quad H_2) \begin{pmatrix} B_0 & B_1 & B_2 & \dots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \dots & B_0 & B_1 & B_2 \end{pmatrix} \begin{pmatrix} u(0) \\ \vdots \\ u(k+4) \end{pmatrix} \quad (5.22)$$

It is easy to check that for $k = 2$

$$\text{rank} \begin{pmatrix} H_{-2} & H_{-1} & H_0 & H_1 & H_2 \end{pmatrix} \begin{pmatrix} B_0 & B_1 & B_2 & 0 & 0 & 0 & 0 \\ 0 & B_0 & B_1 & B_2 & 0 & 0 & 0 \\ 0 & 0 & B_0 & B_1 & B_2 & 0 & 0 \\ 0 & 0 & 0 & B_0 & B_1 & B_2 & 0 \\ 0 & 0 & 0 & 0 & B_0 & B_1 & B_2 \end{pmatrix} = 3 \quad (5.23)$$

So for an arbitrary vector in \mathbb{R}^3 , $\exists c_1, c_2, \dots, c_7 \in \mathbb{R}^3$ such that:

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = (H_{-2} \quad \dots \quad H_2) \begin{pmatrix} B_0 & B_1 & B_2 & \dots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ 0 & \dots & B_0 & B_1 & B_2 \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_7 \end{pmatrix} \quad (5.24)$$

Solving the above system, we can find a plethora of possible choices for the input. However we need to be careful, since the admissible conditions (2.33) need to be taken into account. For example, by choosing

$$\begin{aligned} u(0) &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & u(1) &= \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix} & u(2) &= \begin{pmatrix} -\frac{17}{5} - \beta_1 + \beta_2 + \beta_3 \\ \frac{227}{15} + \frac{11}{3}\beta_1 - \frac{11}{3}\beta_2 - \frac{16}{3}\beta_3 \\ -\frac{209}{3} - \frac{70\beta_1}{3} + \frac{85\beta_2}{3} + \frac{95\beta_3}{3} \end{pmatrix} \\ u(3) &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & u(k) &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & k &= 4, \dots \end{aligned}$$

as an input, we get that

$$\beta(2) = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \quad (5.25)$$

Yet, although we managed to reach the desired output in 2 time steps, the choice of initial conditions is wrong, since they do not satisfy (2.33). So the above input sequence is not feasible. Moreover, if we try to solve the system of equations (5.14), (5.24), we find that they have no solutions in common, so for $k = 2$ we cannot find an input sequence that will drive the system to any arbitrary output.

6. Conclusions

A formula for the general solution of discrete time ARMA representations was provided in terms of the finite and infinite Jordan Pairs of $A(\sigma)$. Furthermore, the reachability subspace of discrete time ARMA systems has been defined and criteria regarding the reachability of a system have been proposed. The special case of causal systems has also been studied. The results presented in this work constitute an extension of the results presented in Karampetakis and Gregoriadou (2014) for descriptor systems. The advantage of the method used in this work is that it provides us with the form of the consistent input that drives the system from the zero vector to the desired output. Although the input and output of the system are defined as real vector valued functions, these results can be generalized to an arbitrary field \mathbb{F} . Further research still remains for the derivation of a criterion connecting the reachability of (1.1) to the absence of decoupling zeros of the compound matrix $[A(\sigma) \quad B(\sigma)]$, the controllability to the origin and the extension of these results to positive systems or fractional order systems.

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References

- E.N. Antoniou, A.I.G. Vardulakis, and N.P. Karampetakis. A spectral characterization of the behavior of discrete time AR-representations over a finite time interval. *Kybernetika*, 34(5):555–564, 1998. ISSN 0023-5954.
- Panos J. Antsaklis and Anthony N. Michel. *Linear systems*. Boston: Birkhäuser, 2nd corrected printing edition, 2006. ISBN 0-8176-4434-2/hbk.
- Thomas Berger and Timo Reis. Controllability of linear differential-algebraic systems - a survey. In *Surveys in Differential-Algebraic Equations I*, pages 1–61. Springer, 2013.
- D. S. Bernstein. *Matrix mathematics. Theory, facts, and formulas*. Princeton, University Press, Second edition, 2009. ISBN 978-0-691-14039-1/pbk; 978-0-691-13287-7/hbk.
- Tobias Brull. Explicit solutions of regular linear discrete-time descriptor systems with constant coefficients. *ELA. The Electronic Journal of Linear Algebra [electronic only]*, 18:317–338, 2009. URL <http://eudml.org/doc/226958>.
- Nir Cohen. Spectral analysis of regular matrix polynomials. *Integral Equations Oper. Theory*, 6:161–183, 1983. ISSN 0378-620X; 1420-8989/e. doi: 10.1007/BF01691894.
- Carmen Coll, Màrius J. Fullana, and Elena Sánchez. Some invariants of discrete-time descriptor systems. *Appl. Math. Comput.*, 127(2-3):277–287, 2002. ISSN 0096-3003. doi: 10.1016/S0096-3003(01)00005-4.
- Christian Commault and Mazen Alamir. On the reachability in any fixed time for positive continuous-time linear systems. *Syst. Control Lett.*, 56(4):272–276, 2007. ISSN 0167-6911. doi: 10.1016/j.sysconle.2006.10.021.
- Liyi Dai. *Singular control systems*. Berlin etc.: Springer-Verlag, 1989. ISBN 3-540-50724-8.
- Guang-Ren Duan. *Analysis and design of descriptor linear systems*. Dordrecht: Springer, 2010. ISBN 978-1-4419-6396-3/hbk; 978-1-4614-2684-4/pbk; 978-1-4419-6397-0/ebook. doi: 10.1007/978-1-4419-6397-0.
- G.F. Fragulis and A.I.G. Vardulakis. Reachability of polynomial matrix descriptions (PMDs). *Circuits, Systems and Signal Processing*, 14(6):787–815, 1995. ISSN 0278-081X. doi: 10.1007/BF01204685. URL <http://dx.doi.org/10.1007/BF01204685>.

- I. Gohberg and L. Rodman. On spectral analysis of non-monic matrix and operator polynomials. I: reduction to monic polynomials. *Isr. J. Math.*, 30:133–151, 1978. ISSN 0021-2172; 1565-8511/e. doi: 10.1007/BF02760835.
- Israel Gohberg, Peter Lancaster, and Leiba Rodman. *Matrix polynomials*. Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), reprint of the 1982 original edition, 2009. ISBN 978-0-898716-81-8/pbk; 978-0-89871-902-4/ebook. doi: 10.1137/1.9780898719024.
- G.E. Hayton, A.C. Pugh, and P. Fretwell. Infinite elementary divisors of a matrix polynomial and implications. *Int. J. Control*, 47(1):53–64, 1988. ISSN 0020-7179; 1366-5820/e. doi: 10.1080/00207178808905995.
- João P. Hespanha. *Linear systems theory*. Princeton, NJ: Princeton University Press, 2009. ISBN 978-0-691-14021-6/hbk.
- M. Hou, A.C. Pugh, and G.E. Hayton. General solution to systems in polynomial matrix form. *Int. J. Control*, 73(9):733–743, 2000. ISSN 0020-7179; 1366-5820/e. doi: 10.1080/00207170050029241.
- Joao Yoshiyuki Ishihara and Marco Henrique Terra. Impulse controllability and observability of rectangular descriptor systems. *Automatic Control, IEEE Transactions on*, 46(6):991–994, 2001.
- J. Jones, N.P. Karampetakis, and A.C. Pugh. Solution of discrete ARMA-representations via MAPLE. *Appl. Math. Comput.*, 139(2-3):437–489, 2003. ISSN 0096-3003. doi: 10.1016/S0096-3003(02)00210-2.
- Tadeusz Kaczorek. *Positive 1D and 2D systems*. London: Springer, 2002. ISBN 1-85233-508-4.
- Tadeusz Kaczorek. New reachability and observability tests for positive linear discrete-time systems. *Bull. Pol. Acad. Sci., Tech. Sci.*, 55(1):19–21, 2007a. ISSN 0239-7528.
- Tadeusz Kaczorek. *Polynomial and rational matrices. Applications in dynamical systems theory*. Dordrecht: Springer, 2007b. ISBN 1-84628-604-2/hbk.
- T. Kailath. *Linear Systems*. Information and System Sciences Series. Prentice-Hall, 1980. ISBN 9780135369616. URL <http://books.google.gr/books?id=ggYqAQAAMAAJ>.
- N. Karampetakis, J. Jones, and E.N. Antoniou. Forward, backward, and symmetric solutions of discrete ARMA representations. *Circuits Syst. Signal Process.*, 20(1):89–109, 2001. ISSN 0278-081X; 1531-5878/e. doi: 10.1007/BF01204924.
- N.P. Karampetakis. Comments on reachability of polynomial matrix descriptions (PMDs) by Fragulis, G.F. and Vardoulakis, A.I.G. *Circuits, Systems and Signal Processing*, 16(5):559–568, 1997. ISSN 0278-081X. doi: 10.1007/BF01185005. URL <http://dx.doi.org/10.1007/BF01185005>.
- N.P. Karampetakis and A. Gregoriadou. Reachability and controllability of discrete-time descriptor systems. *International Journal of Control*, 87(2):235–248, 2014. doi: 10.1080/00207179.2013.827798. URL <http://www.tandfonline.com/doi/abs/10.1080/00207179.2013.827798>.
- N.P. Karampetakis and S. Vologiannidis. On the fundamental matrix of the inverse of a polynomial matrix and applications to ARMA representations. *Linear Algebra Appl.*, 431(11):2261–2276, 2009. ISSN 0024-3795. doi: 10.1016/j.laa.2009.07.017.

- F.N. Koumboulis and B.G. Mertzios. On Kalman's controllability and observability criteria for singular systems. *Circuits Syst. Signal Process.*, 18(3):269–290, 1999. ISSN 0278-081X; 1531-5878/e. doi: 10.1007/BF01225698.
- Alan J. Laub and W.F. Arnold. Controllability and observability criteria for multivariable linear second-order models. *IEEE Trans. Autom. Control*, 29:163–165, 1984. ISSN 0018-9286. doi: 10.1109/TAC.1984.1103470.
- F.L. Lewis and B.G. Mertzios. On the analysis of discrete linear time-invariant singular systems. *IEEE Trans. Autom. Control*, 35(4):506–511, 1990. ISSN 0018-9286. doi: 10.1109/9.52316.
- Philip Losse and Volker Mehrmann. Controllability and observability of second order descriptor systems. *SIAM J. Control Optim.*, 47(3):1351–1379, 2008. ISSN 0363-0129; 1095-7138/e. doi: 10.1137/060673977.
- S. Mahmood, N.P. Karampetakis, and A.C. Pugh. Solvability, reachability, controllability and observability of regular PMDs. *International Journal of Control*, 70(4):617–630, 1998. doi: 10.1080/002071798222235. URL <http://www.tandfonline.com/doi/abs/10.1080/002071798222235>.
- M. Malabre, V. Kučera, and P. Zagalak. Reachability and controllability indices for linear descriptor systems. *Syst. Control Lett.*, 15(2):119–123, 1990. ISSN 0167-6911. doi: 10.1016/0167-6911(90)90005-F.
- Vikas Kumar Mishra and Nutan Kumar Tomar. On complete and strong controllability for rectangular descriptor systems. *Circuits, Systems, and Signal Processing*, pages 1–12, 2015.
- Vikas Kumar Mishra, Nutan Kumar Tomar, and Mahendra Kumar Gupta. On controllability and normalizability for linear descriptor systems. *Journal of Control, Automation and Electrical Systems*, 27(1):19–28, 2016.
- K. Ogata. *Modern Control Engineering*. Instrumentation and controls series. Prentice Hall, 2010. ISBN 9780136156734.
- S.S. Rao. *Mechanical Vibrations*. Number v. 978, nos. 0-212813 in Mechanical Vibrations. Prentice Hall, 5th edition, 2011. ISBN 9780132128193.
- Maria Elena Valcher. Controllability and reachability criteria for discrete time positive systems. *Int. J. Control*, 65(3):511–536, 1996. ISSN 0020-7179; 1366-5820/e. doi: 10.1080/00207179608921708.
- A.I.G. Vardulakis. *Linear multivariable control. Algebraic analysis and synthesis methods*. Chichester etc.: John Wiley &— Sons, 1991. ISBN 0-471-92859-3.
- A.I.G. Vardulakis, D.N.J. Limebeer, and N. Karcianas. Structure and Smith-MacMillan form of a rational matrix at infinity. *Int. J. Control*, 35:701–725, 1982. ISSN 0020-7179; 1366-5820/e. doi: 10.1080/00207178208922649.
- A.I.G. Vardulakis, E.N. Antoniou, and N.P. Karampetakis. On the solution and impulsive behaviour of polynomial matrix descriptions of free linear multivariable systems. *International Journal of Control*, 72(3):215–228, 1999. doi: 10.1080/002071799221208. URL <http://www.tandfonline.com/doi/abs/10.1080/002071799221208>.

Elizabeth L. Yip and Richard F. Sincovec. Solvability, controllability, and observability of continuous descriptor systems. *IEEE Trans. Autom. Control*, 26:702–707, 1981. ISSN 0018-9286. doi: 10.1109/TAC.1981.1102699.