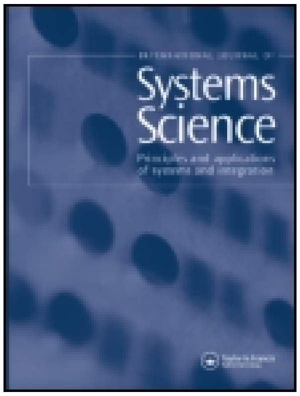

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Admissibility for positive continuous-time descriptor systems

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Positive descriptor systems are a new research branch in descriptor systems. This article is devoted to the study of the admissibility property for positive continuous-time descriptor systems. Based on Lyapunov inequality existed for stability of positive descriptor systems, we provide a necessary and sufficient condition to guarantee the admissibility via linear matrix inequalities (LMIs). Furthermore, a necessary and sufficient condition to guarantee the admissibility is established by means of generalised Lyapunov equation if c-monomial decomposition is applied to positive descriptor systems. Finally, examples are given to illustrate the validity of the results obtained.

Keywords: positive systems; descriptor systems; impulse-free; admissibility; c-monomial decomposition

1. Introduction

Positive systems are systems whose variables are always in the nonnegative orthant for any nonnegative initial state and any nonnegative input. These systems are frequently encountered in many practical applications such as population models (Farina and Rinaldi 2000), Leontief economic models (Cantó, Coll, and Sánchez 2008), biology systems (Klipp, Herwig, Kowald, Wierlig, and Lehrach 2005; Liu, Zhang, Feng, and Yang 2009), pharmacokinetics (Macheras and Iliadis 2006; Liu, Zhang, Liu, and Zhang 2009) and chemical model (Niu, Zhang, and Zhang 2010), where the variables represent quantities of goods, concentrations, population levels and concentrations of chemical substances, that have no meaning with negative values. Since, positive systems are not defined on linear spaces but on cones, their analysis and synthesis are more complicated and more challenging. During the past decades, inasmuch as their wide applications, much attention has been paid to positive standard systems (i.e. state space systems). Many theoretic problems on positive standard systems have been well studied. Controllability and reachability properties were investigated in Coxson and Shapiro (1987), Caccetta and Rumchev (2000), Commault and Alamir (2007) and Valcher (2008). Positive realisation problem was considered in Kaczorek (2001). Pole assignment was studied in James, Kostova, and Rumchev (2001). In Leenheer and Aeyels (2001), Ait Rami, Tadeo, and Benzaouia (2007) and Feng, Lam, Li, and Shu (2011), positive stabilisation was studied. A survey about positive linear observer can be found in Back and Astolfi (2006). Complete introductions for positive standard systems were given in Farina and Rinaldi (2000) and Kaczorek (2001, 2007).

However, some models in electrical circuit network, aerospace engineering, economic systems and chemical processes can be accurately described by descriptor systems. Naturally, in some descriptor systems, the nonnegativity of the variables should be guaranteed. Therefore, positive descriptor systems are also important in practical applications, worth our concern and study. Positive descriptor systems have only been investigated in recent years. In addition, because of the complexity of derivative matrix and the nonnegativity of variables in positive descriptor systems, much of the developed theory for such systems is still not up to a quantitative level. Reachability and controllability properties for positive discrete-time descriptor systems were studied in Bru, Coll, and Sánchez (2002) and Bru, Coll, Romero-Vivo, and Sánchez (2003). Stability, generalised Lyapunov equations and positivity preserving model reduction of positive descriptor systems were analysed by Virnik (2006, 2008a, b). The conditions to assure the existence of a state-feedback such that the closed-loop system is nonnegative and stable were given by Herrero, Ramírez, and Thome (2006). Positive normalisation problem and positive solution

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for discrete-time descriptor systems were solved in Cantó, Coll, and Sánchez (2006) and Cantó et al. (2008). Positive descriptor systems were introduced and analysed in Kaczorek (2001).

In general, whether a descriptor system is in normal operation, the role of impulse is a critical key. For a regular descriptor system, the internal stability contains not only asymptotic stability, but also impulse-free. This kind of internal stability is called admissibility. To the best of author's knowledge, there has been little research results on admissibility of positive descriptor systems.

In this article, we are interested in admissibility property for positive continuous-time descriptor systems. According to Lyapunov inequality for stability of positive descriptor systems, which has been proposed already, a necessary and sufficient condition in terms of linear matrix inequalities (LMIs) to guarantee the admissibility is established. On the other hand, we point out that, if c-monomial decomposition is applied to a positive descriptor system, the nonnegativity of the system with c-monomial form is equivalent to the origin one. Furthermore, a necessary and sufficient condition for admissibility of such class of positive descriptor systems by means of generalised Lyapunov equation is presented. Finally, numerical examples are given to illustrate the obtained results.

2. Preliminaries

This section provides some notations, definitions and statements which will be essentially used for our main admissibility results.

The following notations will be used throughout this article.

$\mathbb{R}^n$: the set of n-dimensional real vectors. $\mathbb{R}^n_+$: the set of n-dimensional real vectors with nonnegative elements. $\mathbb{R}^{n\times m}$: the set of $n\times m$ real matrices. $\mathbb{R}^{n\times m}_+$: the set of $n\times m$ matrices with nonnegative elements. $\mathbb{C}$: the set of complex numbers, A^T: transpose of matrix A, A^{-1}: inverse of matrix A, rank(A): rank of matrix A and ind(A): matrix index of matrix A, which is defined as the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$. $x\geq 0{:}x\in\mathbb{R}^n_+$. $A\geq 0$: $A\in\mathbb{R}^{n\times n}_+$. A_{ij}: the (i,j)-th entry of matrix A. $A\succ 0$ ($A\succeq 0$): positive definite (semi-definite) matrix. $A\prec 0$ ($A\preceq 0$): negative definite (semi-definite) matrix. A^D: Drazin inverse of matrix A, which satisfies the following properties: (i) $A^DAA^D = A^D$, (ii) $AA^D = A^DA$ and (iii) $A^{k+1}A^D = A^k$, where $k\geq \mathrm{ind}(A)$. Metzler matrix: $A\in\mathbb{R}^{n\times n}$ with nonnegative off-diagonal elements. Monomial matrix: its every row and column has only one positive element and the remaining elements are equal to zero.

Now consider a continuous-time descriptor system of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t)\in\mathbb{R}^n, u(t)\in\mathbb{R}^m$ are the state and input vectors, respectively. E, A and B are real matrices of compatible dimensions, and $\mathrm{rank}(E) = r\leq n$. System (1) is called standard system if $E = I$.

If there exists a scalar $\lambda\in\mathbb{C}$ such that $\det(\lambda E - A)\neq 0$, the matrix pair (E, A) is said to be regular. In this case, there exist two nonsingular matrices $P, Q\in\mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_r & 0\\ 0 & N\end{bmatrix},\quad PAQ = \begin{bmatrix} J & 0\\ 0 & I_{n-r}\end{bmatrix},$$

where J and N are matrices in Jordan canonical forms, moreover, N is a nilpotent matrix. This transformation is called Weierstrass canonical form (Gantmacher 1959). Index of matrix pair (E, A) is defined as the index of nilpotency of the nilpotent matrix N in Weierstrass canonical form (Kunkel and Mehrmann 2006) and denoted by ind(E, A). In particular, $\mathrm{ind}(E, A) = 0$ when E is nonsingular, and $\mathrm{ind}(E, A) = 1$ if $N = 0$ when E is singular.

If (E, A) is regular, $\mathrm{ind}(E, A) = \nu$, then we can obtain an explicit solution to (1) which can be represented in terms of Drazin inverses (Kaczorek 1992)

$$x(t) = e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}x(0) + \int_0^t e^{\hat{E}^D\hat{A}(t-\tau)}\hat{E}^D\hat{B}u(\tau)\mathrm{d}\tau - (I - \hat{E}^D\hat{E})\sum_{i=0}^{\nu-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B}u^{(i)}(t),$$

where $\hat{E} = (\lambda E - A)^{-1}E$, $\hat{A} = (\lambda E - A)^{-1}A$, $\hat{B} = (\lambda E - A)^{-1}B$, $x(0)$ is an admissible initial condition, $u^{(i)}, i = 0, 1, \ldots, \nu - 1$ is the i-th derivative of u. Note that matrices $\hat{E}$ and $\hat{A}$ are commutative, that is, $\hat{E}\hat{A} = \hat{A}\hat{E}$.

We rewrite the matrices $\hat{E}$ and $\hat{A}$ by transforming them into Weierstrass canonical form

$$\begin{aligned}\hat{E} &= (\hat{\lambda}E - A)^{-1}E\\ &= \left(\hat{\lambda}P^{-1}\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}Q^{-1} - P^{-1}\begin{bmatrix} J & 0\\ 0 & I\end{bmatrix}Q^{-1}\right)^{-1}\\ &\quad\times P^{-1}\begin{bmatrix} I & 0\\ 0 & N\end{bmatrix}Q^{-1}\\ &= Q\begin{bmatrix} (\hat{\lambda}I - J)^{-1} & 0\\ 0 & (\hat{\lambda}N - I)^{-1}N\end{bmatrix}Q^{-1}\\ \hat{A} &= Q\begin{bmatrix} (\hat{\lambda}I - J)^{-1}J & 0\\ 0 & (\hat{\lambda}N - I)^{-1}\end{bmatrix}Q^{-1}.\end{aligned}$$

Then we can obtain the Drazin inverses of $\hat{E}$ and $\hat{A}$ (Kunkel and Mehrmann 2006) by

$$\hat{E}^D = Q\begin{bmatrix} \hat{\lambda}I - J & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$
$$\hat{A}^D = Q\begin{bmatrix} J^D(\hat{\lambda}I - J) & 0 \\ 0 & \hat{\lambda}N - I \end{bmatrix}Q^{-1}.$$

Therefore, we have the following results by computation

$$\hat{E}^D\hat{E} = Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$
$$\hat{E}^D\hat{A} = \hat{A}\hat{E}^D = Q\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$
$$\hat{E}\hat{A}^D = \hat{A}^D\hat{E} = Q\begin{bmatrix} J^D & 0 \\ 0 & N \end{bmatrix}Q^{-1}.$$

For convenience, system (1) in absence of input is denoted by (E, A). (E, A) is said to be stable if all finite roots of $\det(sE - A) = 0$ have negative real parts. (E, A) is said to be impulse-free if $N = 0$. (E, A) is said to be admissible if it is regular, impulse-free and stable (Dai 1989). It is obvious that the stability of (E, A) is related to the sub-matrix J in $\hat{E}^D\hat{A}$ and the impulse of (E, A) is relevant to the sub-matrix N in $\hat{E}\hat{A}^D$.

The following definition and lemmas will be used later for our main results.

Definition 1 (Virnik 2008b)**:** System (1) in absence of input is called positive if for any admissible initial state $x(0) \geq 0$, the state trajectory is nonnegative, that is, $x(t) \geq 0, \quad \forall t \in \mathbb{R}_+$.

Lemma 1 (Kaczorek 2001)**:** *Let* $A \in \mathbb{R}_+^{n\times n}$, *then* $A^{-1} \geq 0$ *if and only if* A *is a monomial matrix. Furthermore,* A^{-1} *is also a monomial matrix.*

Lemma 2 (Virnik 2008b)**:** *Suppose that* (E, A) *is regular and* $\hat{E}^D\hat{E} \geq 0$, *then* (E, A) *is positive if and only if there exists a scalar* $\alpha \geq 0$ *such that the matrix*

$$\bar{M} := -\alpha I + \hat{E}^D\hat{A} + \alpha\hat{E}^D\hat{E}$$

is a Metzler matrix.

Lemma 3 (Farina and Rinaldi 2000)**:** *Let* $E = I$ *and* $B = 0$, *then the following statements hold*:

(a) *System* (1) *is positive if and only if* A *is a Metzler matrix*;
(b) *Suppose that system* (1) *is positive, then it is stable if and only if there exists a positive definite diagonal matrix* X *such that*

$$A^TX + XA \prec 0. \qquad (2)$$

Remark 1: It is worth pointing out that positive standard systems possess many elegant properties. For a positive continuous-time standard system, one of these elegant properties is diagonal stability, namely, it is stable if and only if there exists a positive definite diagonal matrix such that Lyapunov inequality (2) holds.

The following lemma can be interpreted as an extension of the elegant result on positive continuous-time standard systems, and this lemma offers an easy test for checking the stability of positive continuous-time descriptor systems.

Lemma 4 (Virnik 2008b)**:** *Suppose that* (E, A) *is regular and* $\hat{E}^D\hat{E} \geq 0$. *If* (E, A) *is positive, then* (E, A) *is stable if and only if there exists a positive definite diagonal matrix* X *such that*

$$(\hat{E}^D\hat{A})^TX + X(\hat{E}^D\hat{A}) \preceq 0.$$

3. Main results

In this section, the admissibility of positive continuous-time descriptor systems is studied. The obtained relevant results involve necessary and sufficient conditions.

In the following, we present a necessary and sufficient condition for positive descriptor systems to guarantee admissibility, which is obviously different from that of descriptor systems without nonnegativity restriction.

Theorem 1: *Suppose that* (E, A) *is regular and* $\hat{E}^D\hat{E} \geq 0$. *If* (E, A) *is positive, then* (E, A) *is admissible if and only if there exists a positive definite diagonal matrix* X *such that*

$$(\hat{E}^D\hat{A})^TX + X(\hat{E}^D\hat{A}) \preceq 0 \qquad (3)$$

$$(\hat{E}\hat{A}^D)^TX + X(\hat{E}\hat{A}^D) \preceq 0. \qquad (4)$$

Proof: Condition (3) has already been given to guarantee the stability of (E, A) in Lemma 4. Therefore, we only present condition (4) to ensure that (E, A) is impulse-free.

Sufficiency: Let

$$\tilde{X} = Q^TXQ = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}.$$

Substituting matrices $\hat{E}$ and $\hat{A}^D$ with Weierstrass canonical form into inequality (4), and premultiplying by Q^T and postmultiplying by Q on both sides of (4), we obtain

$$\begin{bmatrix} (J^D)^T X_1 + X_1 J^D & (J^D)^T X_2 + X_2 N \\ X_2^T J^D + N^T X_2^T & N^T X_3 + X_3 N \end{bmatrix} \preceq 0. \tag{5}$$

Inequality (5) holds only if

$$N^T X_3 + X_3 N \preceq 0 \tag{6}$$

holds. We have known that matrix N is in Jordan canonical form. Therefore, without loss of generality, suppose that

$$N = \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \tag{7}$$

where * represents the entries that are not relevant in the following discussion. In addition,

$$\begin{aligned} \tilde{X} &= Q^T X Q \\ &= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \\ &= \begin{bmatrix} q_{11} & q_{21} & \cdots & q_{n1} \\ q_{12} & q_{22} & \cdots & q_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ q_{1n} & q_{2n} & \cdots & q_{nn} \end{bmatrix} \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix} \\ &\quad \times \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} \end{bmatrix}. \end{aligned}$$

A computation allows to show that

$$x_{ii} = \sum_{j=1}^{n} q_{ji}^2 x_j > 0, i = 1, 2, \ldots, n$$

since X is positive definite diagonal matrix and matrix Q is nonsingular. Then, rewriting (6) with the hypothesis in (7) on matrix N, we observe that

$$N^T X_3 + X_3 N = \begin{bmatrix} 0 & X_{3_{11}} & * & * \\ X_{3_{11}} & 2X_{3_{12}} & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \preceq 0.$$

Taking into account that $X_{3_{11}} > 0$, we can get

$$\det \begin{bmatrix} 0 & X_{3_{11}} \\ X_{3_{11}} & 2X_{3_{12}} \end{bmatrix} < 0$$

which contradicts inequality (6). Therefore, (6) holds only if $N = 0$, that is, (E, A) is impulse-free.

Necessity: Suppose that (E, A) is admissible. According to Lemma 4, condition (3) holds. Write $\tilde{X}$ as

$$\tilde{X} = Q^T X Q = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}.$$

Then transforming matrices $\hat{E}^D$ and $\hat{A}$ into Weierstrass canonical form, and premultiplying by Q^T and postmultiplying by Q on both sides of (3), it can be rewritten as

$$\begin{bmatrix} J^T X_1 + X_1 J & J^T X_2 \\ X_2^T J & 0 \end{bmatrix} \preceq 0. \tag{8}$$

(E, A) is stable if and only if the diagonal elements of J have negative real parts, since stability of (E, A) is related to the sub-matrix J which is in Jordan canonical form, where J is the sub-matrix in Weierstrass canonical form. Thus, J is invertible. On the other hand, X_1 is also invertible, then, from inequality (8), we obtain

$$J^T X_1 + X_1 J \prec 0 \tag{9}$$

and (8) holds only if $X_2 = 0$. After premultiplying by J^{-T} and postmultiplying by J^{-1} on both sides of (9), we obtain

$$(J^{-1})^T X_1 + X_1 J^{-1} \prec 0.$$

Then, we obtain the following inequality

$$Q^{-T} \begin{bmatrix} (J^{-1})^T X_1 + X_1 J^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \preceq 0.$$

Inasmuch as (E, A) is impulse-free, we have

$$\hat{E}\hat{A}^D = Q \begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}.$$

Therefore, it is easy to see that

$$\begin{aligned} &(\hat{E}\hat{A}^D)^T X + X(\hat{E}\hat{A}^D) \\ &= (\hat{E}\hat{A}^D)^T X Q Q^{-1} + Q^{-T} Q^T X (\hat{E}\hat{A}^D) \\ &= Q^{-T} \begin{bmatrix} (J^{-1})^T X_1 + X_1 J^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} \preceq 0 \end{aligned}$$

holds. □

Next, we generalise the diagonal property for admissibility of positive continuous-time descriptor systems in terms of generalised Lyapunov equation.

This result is only related to original matrices E, A and offers a simple and convenient test for checking the admissibility for a class of positive continuous-time descriptor systems.

Definition 2: Let (E, A) be regular. If there exist two monomial matrices M and W such that E and A have the following decomposition

$$E = M\begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}W, \quad A = M\begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}W, \tag{10}$$

where $A_1 \in \mathbb{R}^{r\times r}$ and $N \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent matrix. Then we call this decomposition c-monomial decomposition. Moreover, under such decomposition, the system with the following parameter matrices

$$\bar{E} = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} \tag{11}$$

is called c-monomial form.

It is easy to verify that the nonnegativity of the system with c-monomial form is equivalent to the origin one, since M and W are both monomial matrices.

Theorem 2: *Suppose that (E, A) is regular and has c-monomial decomposition. If (E, A) is positive, then (E, A) is admissible if and only if there exists a positive definite diagonal matrix X and a positive definite matrix Y such that*

$$A^TXE + E^TXA + E^TYE = 0. \tag{12}$$

Proof: Since the matrix W is a monomial matrix, then from Lemma 1, the nonnegativity of (E, A) is equivalent to the nonnegativity of the system with c-monomial form (11). It is easy to verify that $\bar{E}\bar{A} = \bar{A}\bar{E}$. Inasmuch as $(\bar{E}, \bar{A})$ is positive, then there exists a scalar $\alpha \geq 0$ such that

$$\bar{M} := -\alpha I + \bar{E}^D\bar{A} + \alpha\bar{E}^D\bar{E}$$

is a Metzler matrix. In addition,

$$\bar{M} = \begin{bmatrix} A_1 & 0 \\ 0 & -\alpha I \end{bmatrix}$$

is a Metzler matrix if and only if A_1 is a Metzler matrix.

Sufficiency: From the hypothesis of this theorem, substituting (10) into Equation (12), it follows that

$$W^T\begin{bmatrix} A_1^T & 0 \\ 0 & I \end{bmatrix}M^TXM\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}W + W^T\begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix} M^TXM\begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}W + W^T\begin{bmatrix} I & 0 \\ 0 & N^T \end{bmatrix}M^TYM\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}W = 0. \tag{13}$$

Write

$$\bar{X} = M^TXM, \quad \bar{Y} = M^TYM = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix},$$

where $Y_1 \in \mathbb{R}^{r\times r}$. M is a monomial matrix and X is a positive definite diagonal matrix, we can define

$$M^T = \begin{bmatrix} m_1e_{i_1} & m_2e_{i_2} & \cdots & m_ne_{i_n} \end{bmatrix}^T$$
$$X = \begin{bmatrix} x_1e_1 & x_2e_2 & \cdots & x_ne_n \end{bmatrix},$$

where $e_i, i = 1, 2, \ldots, n$ denote the i-th vector of the canonical basis in $\mathbb{R}^n$. $i_1, i_2, \ldots, i_n$ is a permutation of $1, 2, \ldots, n$, $m_i > 0$, $x_i > 0$, $i = 1, 2, \ldots, n$, then, we have

$$\begin{aligned}\bar{X} &= M^TXM \\ &= \begin{bmatrix} m_1e_{i_1} & m_2e_{i_2} & \cdots & m_ne_{i_n} \end{bmatrix}^T\begin{bmatrix} x_1e_1 & x_2e_2 & \cdots & x_ne_n \end{bmatrix} \begin{bmatrix} m_1e_{i_1} & m_2e_{i_2} & \cdots & m_ne_{i_n} \end{bmatrix} \\ &= \begin{bmatrix} m_1x_1e_{i_1} & m_2x_2e_{i_2} & \cdots & m_nx_ne_{i_n} \end{bmatrix}^T\begin{bmatrix} m_1e_{i_1} & m_2e_{i_2} & \cdots & m_ne_{i_n} \end{bmatrix} \\ &= \begin{bmatrix} m_1^2x_1 & 0 & \cdots & 0 \\ 0 & m_2^2x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n^2x_n \end{bmatrix} \\ &= \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix},\end{aligned}$$

where $X_1 \in \mathbb{R}^{r\times r}$, $X_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ are positive definite diagonal matrices. Therefore, (13) can be derived as

$$W^T\begin{bmatrix} A_1^TX_1 + X_1A_1 + Y_1 & Y_2N \\ N^TY_2^T & X_2N + N^TX_2 + N^TY_3N \end{bmatrix}W = 0,$$

that is,

$$\begin{bmatrix} A_1^TX_1 + X_1A_1 + Y_1 & Y_2N \\ N^TY_2^T & X_2N + N^TX_2 + N^TY_3N \end{bmatrix} = 0.$$

Furthermore, we have

$$A_1^TX_1 + X_1A_1 = -Y_1 \prec 0$$

which implies that (E, A) is stable, according to Lemma 3. In addition, we have

$$N^TX_2 + X_2N + N^TY_3N = 0. \tag{14}$$

Suppose that the nilpotent index of N is $\nu > 1$, that is, $N^{\nu-1} \neq 0$, $N^\nu = 0$. Premultiplying (14) by $(N^T)^{\nu-1}$, it yields that

$$(N^T)^{\nu-1}X_2N = 0 \Rightarrow (N^T)^{\nu-1}X_2(N)^{\nu-1} = 0.$$

Then, we obtain $N^{\nu-1} = 0$ on the account of $X_2 \succ 0$. This contradicts the nilpotent index ν.

So (14) holds only if $N = 0$. Therefore, (E, A) is admissible.

Necessity: Suppose that (E, A) is admissible. Then $N = 0$ and all roots of $\det(sI - A_1) = 0$ have negative real parts. According to Lemma 3, there exists a positive definite diagonal matrix X_1 and a positive definite matrix Y_1 such that

$$A_1^T X_1 + X_1 A_1 + Y_1 = 0.$$

Thus, take

$$\bar{X} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \bar{Y} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix},$$

where $X_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ is a positive definite diagonal matrix and $\bar{Y} \succ 0$, we get

$$\begin{bmatrix} A_1^T & 0 \\ 0 & I \end{bmatrix} \bar{X} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \bar{X} \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \bar{Y} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = 0.$$

Therefore, from the above equation, it follows that

$$\begin{aligned} & W^T \begin{bmatrix} A_1^T & 0 \\ 0 & I \end{bmatrix} M^T M^{-T} \bar{X} M^{-1} M \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W \\ & + W^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} M^T M^{-T} \bar{X} M^{-1} M \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} W \\ & + W^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} M^T M^{-T} \bar{Y} M^{-1} M \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W \\ & = A^T X E + E^T X A + E^T Y E \\ & = 0, \end{aligned}$$

where $X = M^{-T} \bar{X} M^{-1}, Y = M^{-T} \bar{Y} M^{-1}$, in a similar way as for the proof in sufficiency, it is easy to see that X is a positive definite diagonal matrix. □

Remark 2: If $E = I$, then condition (12) is equivalent to condition (2). Furthermore, if E is nonsingular, then condition (12) is restated as

$$(\hat{E}^{-1}\hat{A})^T X + X(\hat{E}^{-1}\hat{A}) + Y = 0.$$

On the other hand, if positive definite matrix substitutes for positive definite diagonal matrix, Theorem 2 is a sufficient and necessary condition for admissibility of descriptor systems without nonnegativity restriction.

4. Numerical examples

In this section, we give two numerical examples to illustrate the effectiveness of the results obtained.

Example 1: Consider a continuous-time descriptor system with the following parameter matrices

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 5 & 0 \\ 2 & 0 & -8 \\ -1 & 0 & 3 \end{bmatrix}.$$

By calculating, we obtain

$$\hat{E}^D \hat{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{E}^D \hat{A} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 0 & 0 \\ 1 & 0 & -4 \end{bmatrix},$$

$$\hat{E}\hat{A}^D = \begin{bmatrix} -4 & 0 & -3 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}.$$

According to Lemma 2, choose $\alpha = 0$, then, we have that the system is positive since $\hat{E}^D \hat{A}$ is a Metzler matrix. Therefore, we solved the LMIs from Theorem 1 and obtain one feasible solution

$$X = \begin{bmatrix} 1.1608 & 0 & 0 \\ 0 & 3.9576 & 0 \\ 0 & 0 & 2.4852 \end{bmatrix}.$$

Furthermore, there exist two monomial matrices

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}$$

such that

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 3 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 0 \end{bmatrix}.$$

We solved the generalised Lyapunov equation from Theorem 2, one feasible solution is obtained as

$$X = \begin{bmatrix} 4.0671 & 0 & 0 \\ 0 & 0.6958 & 0 \\ 0 & 0 & 1.6557 \end{bmatrix},$$

$$Y = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0 & 5.5667 & -3.8753 \\ 0 & -3.8753 & 3.3115 \end{bmatrix}.$$

Example 2: Consider a continuous-time descriptor system with the following parameter matrices

$$E=\begin{bmatrix}0&0&0&0\\0&0&2&0\\0&0&0&5\\1&0&0&0\end{bmatrix},\quad A=\begin{bmatrix}0&0&0&5\\2&0&-8&0\\0&1&0&0\\-1&0&3&0\end{bmatrix}.$$

By computation, we obtain

$$\hat{E}^D\hat{E}=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad \hat{E}^D\hat{A}=\begin{bmatrix}-1&0&3&0\\0&0&0&0\\1&0&-4&0\\0&0&0&0\end{bmatrix},$$

$$\hat{E}\hat{A}^D=\begin{bmatrix}-4&0&-3&0\\0&0&0&5\\-1&0&-1&0\\0&0&0&0\end{bmatrix}.$$

According to Lemma 2, choose $\alpha=0$, then, we conclude that the system is positive since $\hat{E}^D\hat{A}$ is a Metzler matrix. Then, we solved the LMIs in Theorem 1 and we can observe that there exists a solution for inequality (4), but no solution for inequalities (4) and (5). Therefore, the system is stable, but has impulse.

Moreover, there exist two monomial matrices

$$M=\begin{bmatrix}0&0&0&1\\0&2&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix},\quad W=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&5\end{bmatrix}$$

such that

$$E=\begin{bmatrix}0&0&0&1\\0&2&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&5\end{bmatrix}$$

$$A=\begin{bmatrix}0&0&0&1\\0&2&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}-1&3&0&0\\1&-4&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&5\end{bmatrix}.$$

We solved the generalised Lyapunov equation from Theorem 2 and there is also no feasible solution for Equation (12).

To sum up, the positive descriptor system is not admissible.

5. Conclusion

The admissibility property for positive continuous-time descriptor systems is studied. A sufficient and necessary condition is presented in terms of LMIs to guarantee the admissibility based on Lyapunov inequality for stability of positive descriptor systems. Furthermore, a sufficient and necessary condition is proposed by means of generalised Lyapunov equation for admissibility if c-monomial decomposition is applied to a positive descriptor system. Finally, numerical examples are given to illustrate the obtained results.

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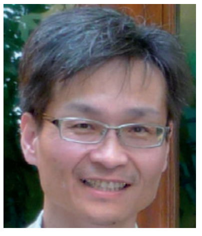

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