State Space for Time Varying Delay

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Abstract. The construction of a state space for systems with time variant delay is analyzed. We show that under causality and consistency constraints a state space can be derived, but fails if the conditions are not satisfied. We rederive a known result on spectral reachability using an discretization approach followed by taking limits. It is also shown that when a system with fixed delay is modeled as one in a class with larger delay, reachability can no longer be preserved. This has repercussions in modeling systems with bounded time varying delay by embedding them in the class of delay systems with fixed delay, equal to the maximum of the delay function $\tau(t)$, or by using lossless causalization.

1 Introduction

One problem with mathematical modeling is that it is easy to loose oneself doing mathematics and forget about the physical reality one tried to model in the first place. For instance connecting two charged capacitors over an Ohmless wire leads to a mathematical model in the realm of impulsive currents. But is this physical reality? A closer inspection would reveal that when a charge packet moves between the capacitors, a magnetic field is created, and inductive effects should not be ignored especially with large $\mathrm{d}i/\mathrm{d}t$. It becomes appropriate to consider the closed circuit as an LC circuit. This illustrates that mathematical equations may be 'cheap', but insufficient to capture reality. This is especially so if the mathematical equations turn out to lead to inconsistencies. It is easy to write down the equations $x = 1$ and $2x = 3$, but they have no solution. Hadamard's whole idea of well-posedness is an attempt not to stray off-line.

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Recently, there has been an increased interest in systems with time varying delays. With the condition $\dot{\tau} < 1$, the Lyapunov-Krasovskii method yields simple sufficient conditions for stability [4]. It was emphasized that this is not merely a technical condition to make the proof work, but has some deeper causality meaning associated with it. In particular, we believe that one should be very cautious when models are used for systems where the derivative of the time delay can exceed one [1]. It is not clear what the state space should be in such a case. The same holds for systems where the delay may depend on the solution itself. The term "state-dependent delay" has been used in these cases, but we want to avoid this until the notion of state has been made clear.

One way of avoiding problems with the conceptual definition of state is to envision a system with time varying delay as the limit of one with distributed delay over a fixed interval [5]. In this limit the kernel must be singular, which may already cause a problem. Obviously this embedding into a larger delay can only work if the time variant delay is bounded [1, 2].

2 State

In order to be able to speak of trajectories (the 'state' as function of time) of a dynamical system, it is necessary that what one considers to be a state space is a stationary construct. The space itself cannot depend on time. For a delay systems of the form

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)) + bu(t), \tag{1}$$

the 'obvious' choices $C([-\tau(t),0]$, or $L_2([-\tau(t),0]$ cannot work. We have shown in [7, 8, 9] that if $\mathrm{d}\tau(t)/\mathrm{d}t < 1$, then a diffeomeorphism $h : \mathbb{R} \to \mathbb{R}$ may be found such that in the time variable $h(t)$, the delay system is transformed to a timevarying coefficient delay system with fixed time delay. Moreover linearity can be preserved. Hence in the case $\dot{\tau} < 1$, a state space can be well-defined. Also, the limit case, $\mathrm{d}\tau(t)/\mathrm{d}t = 1$, poses no problem. In this case, the system is equivalent to a finite dimensional one

$$\dot{x}(t) = Ax(t) + Bx(t_0) + bu(t), \tag{2}$$

where t_0 is some fixed time. Causality requires that one should only consider such an equation for $t \geq t_0$. If x is a scalar variable, and $B \neq 0$, the state space is $\mathbb{R}^2$ with the true state at time t equal to (x_0, x). Omitting $x(t_0)$ no longer leaves a sufficient statistic. However, at once we see that this state cannot be reachable. If $\dim x = n$, the state space has dimension $n + \mathrm{rk}B$, as only $Bx(t_0)$ and not all of x_0 is required.

If $\dot{\tau} > 1$, x in the time interval $(t - \tau(t), t)$ no longer is a sufficient statistic to determine future behavior uniquely, and therefore fails to be a state in the sense conceived by Nerode. No diffeomorphism, h, transforming the system into one with constant delay exists. In [7] we introduced *lossless causalization* as an attempt to interpret the state equation with intervals where $\dot{\tau} > 1$ occurs as a causal system. This pertains to embedding the delay, $\tau(t)$, in a bigger delay, i.e., $\widehat{\tau}(t)$, such that the graph of $t - \widehat{\tau}(t)$ is nondecreasing, but in some minimal sense. See Figure 1.

It was shown in [7, 8] that this only moves the causality problem because of the necessity of an anticipatory information structure. The precise form of $\tau(\cdot)$ must be known at each time, since $t - \widehat{\tau}(t) = \inf_{s>t}(s - \tau(s))$. Surely, if the delay depends on system outputs, this cannot be assumed.

We caution that a general time varying system in $\mathbb{R}^n$ does not fall into this class: at time t only the parameters at that time, and not their future values determine $\dot{x}(t)$. Perhaps the information structure is such that only an upper bound to the delay is known [2]. Of course one could be super cautious, and sweep all problems away by letting $\tau = \infty$ from the beginning and cast all systems in a nonparsimonious way as *Volterra systems with infinite aftereffect*[3].

$$\dot{x}(t) = f\left(t, \int_{-\infty}^{t} K(t, \theta, x(\theta))\, \mathrm{d}\theta\right).$$

But there another potential emerges, not related to the information structure: The well-defined state space may not be minimal, just as it wasn't in the border case $\dot{\tau} = 1$. In particular, it may not be possible to reach any arbitrary preassigned state with this system.

We illustrate the difficulty in defining state spaces for some evolutions defined by functional equations. Even if a state space can be defined, we show that reachability may not hold, and thus the quest for a *minimal* state space may remain unsolved.

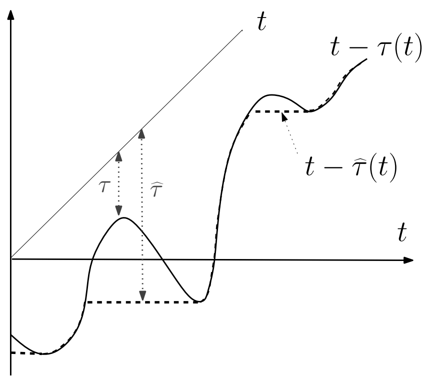

Fig. 1 Lossless Causalization

3 Functional Differential Equation

In this section we consider the scalar evolution system, where the derivative at some future time $t_f(t)$, called *forward time*, is determined by the present state. Thus,

$$\dot{x}(t_f) = f(x(t), u(t)). \tag{3}$$

First, one carefully needs to interpret the left hand side, as it may be ambiguous. Let's introduce an evaluation functional σ_t, mapping functions from a suitable function space to $\mathbb{R}$. For any function f, the number $\sigma_t f = f(t)$. Likewise we have the differentiation operator $\mathbf{D} : x \to \dot{x}$. The left hand side of (3) means $\sigma_{t_f}\mathbf{D}x$, which differs from $\frac{\mathrm{d}}{\mathrm{d}t}x(t_f(t)) = \sigma_{t_f}(\mathbf{D}x)\,\sigma_t(\mathbf{D}t_f)$. With $t_f(t) = t+\tau$, this corresponds to a system with delay τ. We favor this representation as causality is built in when $t_f \geq t$ can be guaranteed. Suppose now that at two different times, t_1 ad t_2, it holds that $t_f(t_1) = t_f(t_2)$. This would imply that $x(t_f)$ may have two different values, which is inconsistent. Hence, for consistency, we impose in addition that the forward time function, $t_f(t)$ be monotonically increasing. If $t_f(t)$ is anticipated, i.e., a known function of t, or at least computable at t [8], this system is causal as long as $t_f(t)$ dominates t.

Thus the forward time, $t_f(t)$, has a *causality* constraint

$$\forall t : \quad t_f(t) \geq t. \tag{4}$$

and a *consistency* constraint[1]

$$\forall t : \quad \dot{t}_f(t) > 0. \tag{5}$$

In some systems, the forward time may depend on the actual (partial) state at time t, and therefore only implicitly be time varying [6]. Let thus

$$t_f(t) = t + F(x(t)). \tag{6}$$

Causality imposes $F(x) \geq 0$, and consistency $\frac{\mathrm{d}}{\mathrm{d}t}[t + F(x(t))] > 0$, or

$$\frac{\mathrm{d}F}{\mathrm{d}x}\dot{x} > -1. \tag{7}$$

Thus consider a delay system $\dot{x}(t_f) = f(x(t), u(t))$, with a delay dependent on the magnitude of the state, say of the form $F(x) = 1 - \mathrm{e}^{-|x|}$. Then the causality and consistency constraints mandate that such a system restricts $\dot{x}$ to $\dot{x} < \mathrm{e}^{|x|}$ if $x < 0$ and $\dot{x} > -\mathrm{e}^{x}$ if $x > 0$. When violated such a model cannot accurately represent some physical phenomenon. Clearly the system (3) with $t_f(\cdot)$ as above and with $|f(x,u)| < 1$ will satisfy these conditions.

Likewise, the system with unbounded delay $F(x) = |x|$ imposes the consistency constraint $\dot{x}\,\mathrm{sgn}\,x > -1$. This holds also for a system satisfying $|f(x,u)| < 1$. For a linear system, $f(x,u) = ax + bu$, with $b > 0$, it imposes the control constraints $-(1+ax)/b < u < -(ax-1)/b$.

In fact, let us look at the autonomous system

$$\dot{x}(t + |x(t)|) = ax(t) \tag{8}$$

[1] $\dot{t_f} = 0$ implies the impossibility of a jump in x at $t_f(t_0)$.

with initial condition $x(0) = x_0$. The first time that something can be discovered about its evolution is at time $t_1 = |x_0|$. Does this mean that $\{\phi(t)|t \in [0,t_1]\}$ is the necessary initial condition? Obviously not all of $C([0,|x_0|],\mathbb{R})$ can be allowed. Causality is obvious, consistency requires that $\dot{\phi}\,\mathrm{sgn}\,\phi > -1$. Strangely enough, the interval length in this space of initial conditions is determined by $\phi(0)$. With this information all subsequent states in the interval (t_1,t_2) are determined, where $t_1 = \phi(0)$ and $t_2 = \phi(t_1)$. More generally, an adaptation of the method of steps allows the computation of $x(t)$ in the k-th step in the interval (t_k,t_{k+1}), where $t_{k+1} = t_k + |x(t_k)|$.

Clearly, if $a > 0$, then $x(t) > 0$, with $\dot{x}(t) > -1$ for all $t < t_1 = x_0$ implies $\dot{x}(t + x(t)) = ax(t) > 0$, and consequently the conditions $x(t) > 0$ and $\dot{x}(t) > -1$ are preserved. The system remains causal and consistent and the solution will diverge, but slower than exponentially. However, the case $a < 0$ can only give consistent solutions for the initial condition $x(0) = x_0 > 0$ if $\dot{x}(t) > -1$ and $x(t) < 1/|a|$ in the interval $[0,x_0]$, or, if $x_0 < 0$, then $\dot{x}(t) < 1$ and $x(t) > -1/|a|$ in the same interval $[0,x_0]$. In these cases the solution converges to zero and is faster than $\exp(at)$. Note that the border line data $\phi(t) = 1/|a| - t$ in the interval $[0,1/|a|]$ is self consistent, except at $t = 1/|a|$, since

$$\dot{x}(t + \phi(t)) = a\phi(t)$$

yields for the left hand side

$$\dot{x}(t + 1/|a| - t) = \dot{x}(1/|a|).$$

and for the right hand side

$$a\phi(t) = a/|a| - at = at - 1.$$

The derivative at $t = 1/|a|$, when x is zero, is multivalued!

If at some time, say t_0, $x(t_0) = 0$, then the evolution equation yields $\dot{x}(t_0) = ax(t_0) = 0$, so that $x(t)$ remains zero. The equilibrium state is $x = 0$. Consequently a trajectory of the autonomous system cannot cross zero. Thus, the above border line case has a deadbeat character and remains zero after $t = 1/|a|$ for the given initial data. In Figure 2 some trajectories are displayed for constant initial data, $\phi(\cdot) = \phi_0$ in the interval $(0,\phi_0)$. Consistency requires that $\phi_0 < 1/|a| = 1$. We plot the nondelay equation solution starting with initial condition ϕ_0 at ϕ_0 as well (exponentials).

The long term behavior of the equation in the asymptotically stable case can be approximated for small $x(t)$. Indeed, using a Taylor expansion, we get ($x_0 > 0$)

$$\dot{x}(t) + \ddot{x}(t)x(t) = ax(t).$$

Upon dividing by $x(t)$, this yields, assuming $x_0 > 0$, (hence $x(t) \geq 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\dot{x}(t) + \log x(t) - at\right] = 0.$$

Setting $x(t) = \exp(\psi(t))$ gives, introducing the integration constant, b,

$$\dot{\psi} = (at + b - \psi)\mathrm{e}^{-\psi},$$

which has the solution converging asymptotically to

$$\widehat{\psi}(t) = (at + b) - a\exp(at + b).$$

Finally

$$x(t) \approx \exp(at + b - a\exp(at + b)).$$

Note that if $\exp(at + b) < \varepsilon$, then up to first order in ε:

$$\frac{\widehat{\psi}(t)\exp\widehat{\psi}(t)}{at + b - \widehat{\psi}(t)} \approx 1 - 2a\varepsilon.$$

Equation (8) has the remarkable property that first of all the initial data must be restricted for the causality and consistency to hold. In addition, we have seen that the minimal sufficient statistic for the equation is a function in an interval whose length depends on the initial value in that interval. It seems hopeless to try to define an invariant structure as state space in this case. However an invariant structure for the state space is readily constructed. Restricting the solution set of (8) to the set of positive continuous functions for which causality and consistency hold, let us first single out $x(t_0)$, as a partial state at time t_0. Next consider the function

$$\hat{x}_{t_0}(\theta) = \frac{x(t_0 + x(t_0)\theta)}{x(t_0)}\Pi_{[0,1)}(\theta), \tag{9}$$

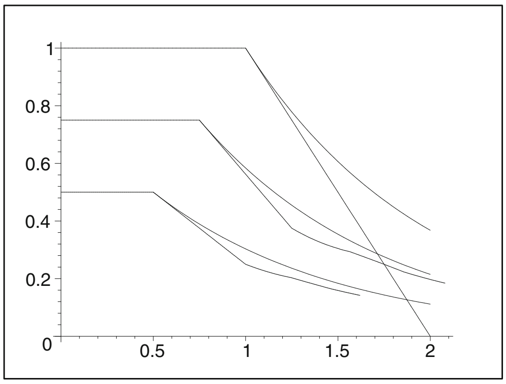

Fig. 2 Solutions to $\dot{x}(t + x(t)) = ax(t)$ for positive initial conditions with $a = -1$, and this in 4 consecutive couplets (not counting the ID). The initial data (minimal sufficient statistic) is chosen to be constant The solutions to this state dependent delay equation decay faster than in the non delay equation. The border case $\phi_0 = 1$ has deadbeat behavior.

where $\Pi_{[0,1)}$ is the indicator function of the interval $[0,1)$. We note that $\hat{x}_{t_0}(0) = 1$ and $\frac{\mathrm{d}\hat{x}_{t_0}(\theta)}{\mathrm{d}\theta} > -1$ in the interval $[0,1)$. This interval is the same for all t_0. Consistency requires that this function has a derivative larger than -1, and since it starts at 1, its values must be lower bound by $1-\theta$ in $[0,1)$. A simple characterization of this sufficient statistic is then by the pair $(x(t_0), \psi(\cdot))$ where $\psi = \mathbf{D}\hat{x}_{t_0}$. Hence a suitable state space for (8) (restricted to positive solutions) is the set $\mathbb{R}_+ \times X$, where $X = \{x | x(\theta) = 1 + \int_0^\theta \psi(s)\mathrm{d}s,\ \theta \in [0,1), \psi \in L_2([0,1), \mathbb{R}_+ - 1\}$ where $\mathbb{R}_+ - 1 = \{r | r > -1\}$. Then, knowing the state (x, ψ) at time t, readily reproduces the partial state $x(t')$ for $t' \geq t$ as $x(t') = x[1 + \int_0^{(t'-t)/x} \psi(\theta)\mathrm{d}\theta]$ in the interval $(t, t+x)$. In the next section we shall explore a somewhat simpler equation with similar behavior.

4 Non-differential Functional Equation

Consider now the simpler functional equation of the form

$$x(t + |x(t)|) = ax(t). \tag{10}$$

One may think of this as a continuous form of a discrete system, prescribing the value of the state at some future time t_f in terms of the value of x at the present time, t. In addition, we let this future time depend on the present value of x. Here, the forward time jump is precisely equal to $|x(t)|$.

What are equilibria for this equation? Clearly if it has a constant solution, say x_∞, then it must hold that $x_\infty = ax_\infty$. Hence, if $a = 1$, all constant solutions are possible, while if $a \neq 1$, the only equilibrium solution is $x(t) \equiv 0$. However, note that if at some time t_0, we have $x(t_0) = 0$, then it follows that the ‘next‘ step is also t_0, so that the solution does not evolve any further. Thus also, the initial condition $x(0) = 0$ does not lead to a propagating solution if $a \neq 1$.

Taking $V(x) = |x|$, then between successive jumps, we get the forward increment

$$\Delta V(t) = |x(t + |x(t)|)| - |x(t)| = (|a| - 1)|x|(t). \tag{11}$$

Clearly, then $\Delta V(t)$ is decreasing if $|a| < 1$, but this does not yet imply asymptotic stability as the time sequence $t_0 = t, t_1 = t_0 + |x(t_0)|, \cdots, t_{k+1} = t_k + |x(t_k)|$, may be clustering. However, we find

$$\frac{\Delta V(t)}{\Delta t} = (|a| - 1). \tag{12}$$

Not only does this imply asymptotic stability, but also that the equilibrium must be found in *finite time*. Necessarily, this autonomous system has time jumps that cluster. Consider thus the behavior for small differentiable initial data. Thus $x(t_1) = \varepsilon > 0$ implies $x(t_1 + \varepsilon) = a\varepsilon$. The left hand side can be approximated by

$$x(t_1 + \varepsilon) = x(t_1) + \dot{x}(t_1)\varepsilon.$$

Hence, $\dot{x}(t_1) = (a-1)$ approximately, and $x(t)$ must decrease for $t > t_1$ if $a < 1$. So, let us check the decay towards zero: If t_F is the finite time when the differentiable solution hits 0, then $x(t) = (1-a)(t_F - t)$. It is now readily verified that this solution is consistent for all $t < t_F$. (It seems also to imply that this would hold even if $|a| > 1$, but in this case the assumption of a finite t_F is false.

What is in this case a minimal statistic that allows to evolve a solution to either t_F (in the asymptotically stable case) or indefinitely (if unstable)? Let $x_0 > 0$ (we exclude zero for there is no propagation, and the case $x_0 < 0$ can be dealt with in a symmetrical fashion.) Just as in the case of the differential functional equation, causality imposes the constraint that the forward time exceeds present time, $t_f(t) > t$, and the consistency constraint imposes that as function of time t, the forward time cannot trace it steps back, i.e., $\dot{t}_f(t) > 0$. In the example this means that the initial data, $\phi(t)$, must be restricted to the class of functions satisfying $\dot{\phi}(t) > -1$, for $\phi(t) > 0$. But what is the minimal interval? With $t_0 = 0$, we get $t_1 = t_f(t_0) = x_0$, the initial condition. Let $x_1 = ax_0$. Is any differentiable curve from $(0, x_0)$ to (x_0, x_1), satisfying the causality and consistency constraint allowed?

Suppose that at time θ, it holds that $x(\theta) = \theta$. Define the iterates as $t_{k+1} = t_k + x(t_k)$, and $x_{k+1} = ax_k$ with $x_0 = t_0 = \theta$. It is easily shown by induction that, if $|a| < 1$, then $t_k = (1 + \frac{a^k - 1}{a-1})\theta$ and $x_k = x(t_k) = a^k\theta$. It follows from the above that t_k and x_k respectively converge to $t_k \to t_F = \frac{(a-2)}{(a-1)}\theta$ and $x_k \to x_F = 0$. If $a = 1$, $t_k = (k+1)\theta$ and $x_k = \theta$, and the solution periodically revisits its initial value θ. It should not go unnoticed that $x(t)$ in each interval $(t_k, t_k + 1)$ determines a minimal statistic for evolving the solution. These intervals have in general a nonuniform length. Moreover, this length is dictated by the data itself.

5 Reachability of Systems with Fixed Point Delay

We give in this section a simple proof for the spectral reachability condition for sytems with fixed delay, and use it to illustrate the loss of reachability when the system is embedded in one with a larger delay. Consider the system (1) with constant delay. Assume that u is a locally integrable function, and that the initial data, ϕ, is integrable. Then the solution $x(t)$ is continuous for $t > 0$. We are interested in conditions that guarantee that any arbitrary continuous function x over $(T - \tau, T)$ is attainable, by proper choice of $u(t)$ for $0 < t < T$.

5.1 PBH Test for Delay Systems

As a first approach of the problem, we discretize the delay model by letting $N\Delta = \tau$, and setting

$$\dot{x}(t) \approx \frac{1}{\Delta}(x(t+\Delta) - x(t))$$

from which a forward difference model follows

$$x(t+\Delta) = (I+\Delta A)x(t) + \Delta Bx(t-N\Delta) + \Delta bu(t).$$

In turn, this gives the approximate dynamics for the delay system in terms of the approximating state $\xi_N(t)^\top = [x(t)^\top, x(t-\Delta)^\top, \ldots, x(t-N\Delta)^\top]$.

$$\xi_N(t+\Delta) = \begin{bmatrix} I+\Delta A & 0 & \cdots & 0 & \Delta B \\ I & & & & 0 \\ 0 & I & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & I & 0 \end{bmatrix} \xi_N(t) + \begin{bmatrix} \Delta b \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t). \tag{13}$$

or, $\xi_N(t+\Delta) = \Phi_N(A,B)\xi_N(t) + \beta_N u(t)$. By the PBH-test, this discrete time system is reachable if $\operatorname{rank}[zI - \Phi_N : \beta_N] = (N+1)n$. Using elementary column and row operations, this reduces to

$$\begin{aligned} &\operatorname{rank} \begin{bmatrix} [(z-1)I-\Delta A] & 0 & \cdots & 0 & -\Delta B & -\Delta b \\ -I & zI & & & & \\ & -I & zI & & & \\ & & \ddots & & & \\ & & & -I & zI & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} 0 & & & 0 & [z^N[(z-1)I-\Delta A]-\Delta B] & -\Delta b \\ I & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & I & 0 & 0 \end{bmatrix} \\ &= Nn + \operatorname{rank}[z^N[(z-1)I-\Delta A] - \Delta B :\ \Delta b] \end{aligned}$$

It follows that the discretized model is reachable iff

$$\operatorname{rank}[z^N[(z-1)I-\Delta A] - \Delta B :\ \Delta b] = n, \tag{14}$$

In the limit for $N \to \infty$ and $\Delta \to 0$ such that $N\Delta = \tau$, we get for $z \neq 0$ the reachability condition

$$\operatorname{rank}\lim \left[\frac{(z-1)}{\Delta} I - A - z^{-N}B :\ b\right] = n, \tag{15}$$

When $z = 0$, an additional condition is obtained:

$$\operatorname{rank}[B :\ b] = n. \tag{16}$$

Set now $z = \mathrm{e}^{s\Delta}$, to get

$$\operatorname{rank}[sI - A - \mathrm{e}^{-s\tau}B :\ b] = n, \tag{17}$$

which nicely generalizes the PBH test for finite dimensional LTI systems.

Also note that the condition (16) is easily understood. If it is desired to obtain the state ψ at time t, then it follows from the dynamics that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi - A\psi = Bx(T-\tau) - bu(T)$$

Hence since $\frac{\mathrm{d}}{\mathrm{d}t}\psi - A\psi$ can be arbitrary, it follows that the range space of $[B:\ b]$ must be the $\mathbb{R}^n$. This corresponds to the limit $\Re s \to -\infty$ in (17).

For a reachable finite dimensional discrete time system of order n, it is known that at most n steps are required to accomplish the requisite state transfer. Hence the approximating discretized system with state $\xi_N(t)$ will require at most $(N+1)n$ steps. Since with each step a time Δ elapses, the total elapsed time is $n(N+1)\Delta$. In the limit for $N \to \infty$ and $\Delta \to 0$ with constraint $N\Delta = \tau$, we obtain a lower bound on the requisite time for the reachability problem for the delay system. This lower limit is $n\tau$.

This gives an alternative proof to the spectral reachability theorem.

Theorem 1. *The state of the constant delay system (1) can be made arbitrary iff*

$$\mathrm{rank}[sI - A - \mathrm{e}^{-s\tau}B:\ b] = n, \quad \forall s,$$

provided that $n\tau$ time units are available to achieve this state transfer.

5.2 State Augmentation

It has been argued that for systems with time variant delay, the state should be taken as the fixed interval corresponding to the maximum of the delay. Likewise, lossless causalization embeds the system into one with larger delay. We show that this necessarily compromises reachability. In order to streamline the ideas we analyze the case for a linear time invariant delay system with delay τ. It will be shown that if one embeds this system into the class of delay systems with state space $C^1([-(\tau+\varepsilon),0),\mathbb{R}^n)$, where $\varepsilon > 0$, then reachability will not hold in this bigger space.

Theorem 2. Failure of reachability by embedding.
Let the LTI delay system (1) be reachable. Then in general, it is not possible to make $\{x(t+\theta)|\theta \in (-\tau-\varepsilon,0])\}$ for any $t>0$ coincide with an arbitrary preassigned continuous function, no matter what value of t is chosen, for any particular choice of the input $u(\cdot)$.

Proof. Proceed with the discretization as in the reachability problem. Assume first that $\tau = N\Delta$ and $\varepsilon = p\Delta$. We now get, letting again $\xi_{N+p}(t)^\top = [x(t)^\top,$ $x(t-\Delta)^\top,\ldots,x(t-N\Delta)^\top,\ldots,x(t-(N+p)\Delta))^\top]$.

$$\xi_{N+p}(t+\Delta) = \begin{bmatrix} I+\Delta A & 0 & \cdots \Delta B \cdots & 0 \\ I & & & \\ & \ddots & \ddots & \\ & 0 & I & 0 \\ & & \ddots & \ddots \\ & & & I \quad 0 \end{bmatrix} \xi_{N+p}(t) + \begin{bmatrix} \Delta b \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t). \qquad (18)$$

The PBH test applied to this approximation with $(N+p+1)$ steps yields now

$$\begin{aligned}
&\operatorname{rank} \begin{bmatrix} [(z-1)I-\Delta A] & 0 & \cdots & -\Delta B & 0 & \cdots & 0 & -\Delta b \\ -I & zI & & & & & & \\ & \ddots & \ddots & & & & & \\ & & -I & zI & & & & \\ & & & \ddots & \ddots & & & \\ & & & & & -I & zI & \end{bmatrix} \\
&= \operatorname{rank} \begin{bmatrix} 0 & 0 & \cdots & 0 & z^p[z^N[(z-1)I-\Delta A]-\Delta B] & -\Delta b \\ I & & & & & \\ & I & & & & \\ & & \ddots & & & \\ & & & I & 0 & \end{bmatrix} \\
&= (N+p)n + \operatorname{rank}[z^p[z^N[(z-1)I-\Delta A]-\Delta B] : \ \Delta b]
\end{aligned}$$

Clearly, for $z=0$, $\operatorname{rank}[z^p[z^N[(z-1)I-\Delta A]-\Delta B] : \ \Delta b] = \operatorname{rank}[b]$. Unless $n=1$, and $b \neq 0$, the embedded delay system cannot be reachable. $\square$

If the embedding does not yield a reachable realization, the lossless causalized systems cannot be minimal. Consequently, pole placement based on the maximal delay may not be possible.

Remark: The reachability condition is of the form $\operatorname{rank}[\Delta(s), b] = n$. where $\Delta(s) = sI - A - Be^{-s\tau}$. The one-sided Laplace transform of $x(t)$ satisfies $\Delta(s)X(s) = E(s)$ for some entire function $E(s)$. Hence $\Delta(s)$ is the denominator matrix in a left fraction description of the system. See [10] for further generalizations.

6 Conclusions

We discussed some issues regarding the existence of an invariant state space for systems with time varying delay. It was shown that the condition $\dot{\tau} < 1$ is fundamental for its existence. We have given a simple derivation of the spectral reachability condition for LTI delay systems. We also discussed embedding the system into one with larger delay when the delay derivative condition is not met. In this case we have shown that reachability may fail. Physical reality should constrain the mathematical models. [11]

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