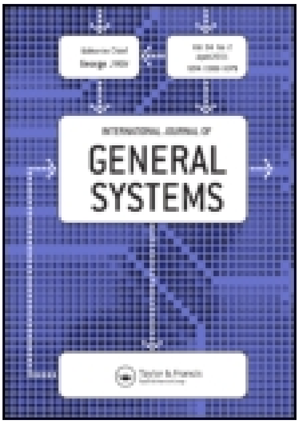

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Relationships between asymptotic stability and exponential stability of positive delay systems

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Relationships between asymptotic stability and exponential stability of positive delay systems

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This paper explores the relations between asymptotic stability and exponential stability of continuous-time and discrete-time positive systems with delay. A system is said to be positive if its state and output are non-negative whenever the initial condition and input are non-negative. Two results are obtained. First, if a positive system is asymptotically stable for all bounded (further continuous, for continuous-time systems) time-varying delays, then it is exponentially stable for all such delays. In particular, if a positive system is asymptotically stable for a given constant delay, then it is exponentially stable for all constant delays. Second, if the involved delays are unbounded, then the positive system may be not exponentially stable even if it is asymptotically stable.

Keywords: asymptotic stability; exponential stability; positive systems; time-varying delays; unbounded delays

Nomenclatures

$A \succeq 0$ ($\preceq 0$)	All elements of matrix A are non-negative (non-positive)
$A \succ 0$ ($\prec 0$)	All elements of matrix A are positive (negative)
A^{T}	Transpose of matrix A
$\mathbb{R}$ ($\mathbb{R}_+$)	The set of all real (positive) numbers
$\mathbb{R}^n$ $\left(\mathbb{R}_+^n\right)$	The set of n-dimensional real (positive) vectors
$\mathbb{R}^{n\times m}$	The set of all real matrices of $n \times m$-dimension
$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{N}_0$	$\{0\} \bigcup \mathbb{N}$
$\underline{p}$	$\{1, 2, \ldots, p\}$, where $p \in \mathbb{N}$
$\underline{p}_0$	$\{0\} \cup \underline{p}$
$\lvert a\rvert$	Absolute value of real number a
$\lVert\mathbf{x}\rVert_\infty$	l_∞ norm of $\mathbf{x} \in \mathbb{R}^n$ i.e. $\lVert\mathbf{x}\rVert_\infty = \max_{i\in\underline{n}}\{\lvert x_i\rvert\}$ with x_i being the ith element of $\mathbf{x}$
$\lceil a\rceil$	The minimum integer not less than real number a
$\mathbb{M}$	The set of $n \times n$ Metzler matrices

1. Introduction

In the real world, there are many systems in which their states are always non-negative, for example population levels, absolute temperature, concentration, and density of substances,

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and data packets flowing in a network. Such systems can be mathematically modelled as positive systems. A system is said to be positive if its state and output are non-negative whenever the initial condition and input are non-negative (Berman, Neumann, and Stern 1989; Kaczorek 2002). Because positive systems have broad applications in many areas (Farina and Rinaldi 2000; Benvenuti and Farina 2004), in recent years researchers from different fields have paid much attention to analyse and synthesize positive systems (Laffey and Šmigoc 2006).

Positive systems have many important and interesting properties (Gao, Lam, Wang, and Xu 2005), among which the most fundamental property is their stability. As an example, for a linear time-invariant positive system, its asymptotic stability and diagonal quadratic stability are equivalent (Farina and Rinaldi 2000); for positive systems with delays, asymptotic stability is completely determined by system matrices and has nothing to do with delays (Liu, Yu, and Wang 2009; Wu, Lam, Shu, and Du 2009; Liu and Zhong 2010; Liu and Dang 2011; Liu, Yu, and Wang 2011). When dealing with the stability issues of positive systems with delays, especially in the case where the delays are time varying, we frequently use the 'comparison approach' (Liu and Dang 2011) rather than the popular Lyapunov–Krasovskii or Razumikhin method (Jiao and Shen 2005). During the past few years, many researchers have also paid their attention to positive switched systems – a combination of positive systems and switched systems (Mason and Shorten 2007; Liu 2009). Recently, Liu and Dang (2011) have extended some interesting properties of positive systems to positive switched systems with delays. One should note that most of the references cited above are only concerned with the asymptotic stability of positive systems and not with their exponential stability.

There are several related concepts of stability in the sense of Lyapunov. Generally speaking, exploring conditions under which the considered systems are asymptotically stable or exponentially stable is one of the most important tasks in control theory (Wang, Lam, Xu, and Gao 2005; Xu and Lam 2008; Liang, Dong, and Lam 2011). For instance, investigations have been made on the exponential stability of various systems, and issues on asymptotic stability have been studied. It is well known that exponential stability implies asymptotic stability, but the converse is not true. In the theory of linear systems, a basic result is that for linear time-invariant systems, these two stability notions are equivalent (Kailath 1980). For a linear time-invariant system with retarded constant delays, the same conclusion also holds [see Theorem 5.1.7 of Curtain and Zwart (1995) for details]. However, for a system with time-varying delays, the relation between asymptotic stability and exponential stability is rather elusive. As a matter of fact, though many papers studied delay systems under these two stability notions, very little effort has been devoted to consider the relationship between them for systems with time-varying delays. More precisely, in this case under what conditions does asymptotic stability imply exponential stability? As far as general delay systems are concerned, a complete answer to this issue is very difficult. However, such a problem is very interesting, since exponential stability is a concept stronger than asymptotic stability, answering this question means that we can study the exponential stability of a delay system by means of its asymptotic stability, which motivates the present study.

This paper considers the relationships between asymptotic stability and exponential stability of positive delay systems. The main contribution of this paper lies in the following aspects: first, we show that if a positive system is asymptotically stable for all bounded (further continuous, for continuous-time systems) time-varying delays, then it is also exponentially stable for the same delays. Furthermore, if a positive system with a given constant delay is asymptotically stable, then it is exponentially stable for all constant delays. Second, by virtue of two counterexamples, we conclude that even if a positive system with unbounded delays is asymptotically stable, it may not be exponentially stable.

The rest of this paper is organized as follows. In Section 2, preliminaries are presented and technical lemmas are provided. Section 3 establishes the equivalence between asymptotic stability and exponential stability for positive systems with bounded delays, Section 4 provides two counterexamples and shows that such an equivalent relation does not hold for positive systems with unbounded delays, and Section 5 concludes this paper.

2. Problem statement and preliminaries

Consider a discrete-time linear system as follows:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + \sum_{i=1}^{p} A_i\mathbf{x}(k - \tau_i(k)), \quad k \in \mathbb{N}_0, \tag{1}$$
$$\mathbf{x}(k) = \boldsymbol{\varphi}(k), \quad k = -\tau, \ldots, 0,$$

where $A_i \in \mathbb{R}^{n\times n}$, $i \in \underline{p}_0$, are system matrices, $0 < \tau_i(k) \in \mathbb{N}_0$ are time-varying delays, and τ is a non-negative integer.

Definition 2.1. System (1) is said to be positive if, for any initial condition $\boldsymbol{\varphi}(k) \succeq 0$, $k \in \{-\tau, \ldots, 0\}$, the corresponding trajectory $\mathbf{x}(k) \succeq 0$ holds for all $k \in \mathbb{N}$ (Liu and Zhong 2010).

Lemma 2.2. System (1) is positive if and only if $A_i \succeq 0$, $i \in \underline{p}_0$ (Liu and Zhong 2010), Lemma 3).

Consider a continuous-time linear system with delays of the following form:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + \sum_{i=1}^{p} A_i\mathbf{x}(t - \tau_i(t)), \quad t \geq 0, \tag{2}$$
$$\mathbf{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [-\tau, 0],$$

where delays $\tau_i(t) > 0$ are continuous in t and $\boldsymbol{\varphi}(t)$ is the vector-valued initial function.

From now on, we denote by $\dot{\mathbf{x}}(t)$ the right-hand derivative of $\mathbf{x}(t)$ at time t if t is the instant from which the system evolves.

Definition 2.3. System (2) is said to be positive if, for any initial condition $\boldsymbol{\varphi}(t) \succeq 0$, $t \in [-\tau, 0]$, the corresponding trajectory $\mathbf{x}(t) \succeq 0$ holds for all $t > 0$ (Liu, Yu, and Wang 2010).

Lemma 2.4. System (2) is positive if and only if $A_0 \in \mathbb{M}$, $A_i \succeq 0$, $\forall i \in \underline{p}$ (Liu et al. 2010, Lemma 2).

Hereafter, we always assume that (1) and (2) are both positive and that the initial condition $\boldsymbol{\varphi}(\cdot) \succeq 0$. The following several definitions are also required.

Definition 2.5. A continuous function $\alpha : [0, a) \to [0, +\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$ (Khalil 2002). It is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\alpha(r) \to +\infty$ as $r \to +\infty$. A continuous function $\beta : [0, a) \times [0, +\infty) \to [0, +\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ decreases with respect to s and $\beta(r, s) \to 0$ as $s \to +\infty$.

Definition 2.6. System (1) is *asymptotically stable* if there exists a class $\mathcal{KL}$ function β such that the solution $\mathbf{x}(k)$ to (1) satisfies $\|\mathbf{x}(k)\|_\infty \leq \beta(\|\boldsymbol{\varphi}\|_d, k)$, $\forall k \in \mathbb{N}$, where $\|\boldsymbol{\varphi}\|_d = \max_{k \in \{-\tau, \dots, 0\}} \|\boldsymbol{\varphi}(k)\|_\infty$ (Jiang and Wang 2002). System (1) is *exponentially stable* if there exist two scalars $\alpha > 0$ and $\gamma > 1$ such that $\|\mathbf{x}(k)\|_\infty \leq \alpha\gamma^{-k}\|\boldsymbol{\varphi}\|_d$ holds for all $k \in \mathbb{N}$.

Definition 2.7. System (2) is *asymptotically stable* if there exists a class $\mathcal{KL}$ function β such that the solution $\mathbf{x}(t)$ to (2) satisfies $\|\mathbf{x}(t)\|_\infty \leq \beta(\|\boldsymbol{\varphi}\|_c, t)$, $\forall t \geq 0$, where $\|\boldsymbol{\varphi}\|_c = \sup_{-\tau \leq t \leq 0} \|\boldsymbol{\varphi}(t)\|_\infty$ (Liberzon 2003). System (2) is *exponentially stable* if there exist two positive scalars α and γ such that $\|\mathbf{x}(t)\|_\infty \leq \alpha \exp(-\gamma t)\|\boldsymbol{\varphi}\|_c$ holds for all $t > 0$.

3. Positive systems with bounded delays

This section will consider the case where the delays are all bounded, and will reveal the property that if a positive system is asymptotically stable for all bounded (and continuous, for continuous-time systems) time-varying delays, then it is exponentially stable for the same delays.

Consider system (1) and assume that all the involved delays are bounded, that is there exists a set of constants $\{\tau_i \in \mathbb{N}\}_{i \in \underline{p}}$ with the property that $0 \leq \tau_i(k) \leq \tau_i$, $\forall k \in \mathbb{N}$. The following lemmas will pave the way for establishing Theorem 3.4.

Lemma 3.1. Positive system (1) is asymptotically stable for all bounded delays if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (Liu et al. 2009, Theorem 1)

$$\left(\sum_{i=0}^{p} A_i - I\right)\boldsymbol{\lambda} < 0 \tag{3}$$

holds.

Lemma 3.2. Assume that system (1) is positive. Let $\mathbf{x}_a(k)$ and $\mathbf{x}_b(k)$ be the trajectories solution of (1) under the initial conditions $\boldsymbol{\varphi}_a(k)$ and $\boldsymbol{\varphi}_b(k), k \in \{-\tau, \dots, 0\}$, respectively (Liu et al. 2009, Lemma 4). Then $\boldsymbol{\varphi}_a(k) \preceq \boldsymbol{\varphi}_b(k)$, $k \in \{-\tau, \dots, 0\}$ implies that $\mathbf{x}_a(k) \preceq \mathbf{x}_b(k)$ for all $k \in \mathbb{N}$.

Let all delays $\tau_i(k)$ in Equation (1) be replaced with their upper bounds τ_i, and we obtain the following system:

$$\begin{aligned} \mathbf{y}(k+1) &= A_0\mathbf{y}(k) + \sum_{i=1}^{p} A_i\mathbf{y}(k - \tau_i), \quad k \in \mathbb{N}_0, \\ \mathbf{y}(k) &= \boldsymbol{\varphi}(k), \quad k = -\tau, \dots, 0, \end{aligned} \tag{4}$$

whose solution is denoted by $\mathbf{y}(k)$.

Lemma 3.3. Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (3) holds (Liu et al. 2009, Lemma 6). Let the initial conditions of (1) and (4) be $\boldsymbol{\varphi}(k) = \lambda$, $k \in \{-\tau, \dots, 0\}$. Then $\mathbf{y}(k) \succeq \mathbf{x}(k)$, $\forall k \in \mathbb{N}$.

Theorem 3.4. If system (1) is asymptotically stable for all bounded delays, then it is exponentially stable for all bounded delays.

Proof. Arbitrarily pick an initial condition $\boldsymbol{\varphi}$ with the property that $\boldsymbol{\varphi}(k) \neq 0$ for some $k \in \{-\tau, \dots, 0\}$, and denote the corresponding solution of (1) by $\mathbf{x}(k), \forall k \in \mathbb{N}$. Since system (1) is asymptotically stable for all bounded delays, then system (4) is also asymptotically stable. By the well-known fact that the asymptotic stability and exponential stability of system (4) are equivalent to each other, we reach the conclusion that system (4) is exponentially stable. Hence, there exist two scalars $\alpha > 0$ and $\gamma > 1$ such that $\|\mathbf{y}(k)\|_\infty \leq \alpha\gamma^{-k}\|\boldsymbol{\varphi}\|_d$ holds for each $k \in \mathbb{N}$.

Since system (1) is asymptotically stable for all bounded delays, by means of Lemma 3.1, there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ such that (3) holds.

Because $\boldsymbol{\varphi}(k) \neq 0$ for some $k \in \{-\tau, \ldots, 0\}$, there is a unique scalar $\alpha_1 > 0$ satisfying $\alpha_1\|\boldsymbol{\varphi}\|_d = \min\{\lambda_1, \ldots, \lambda_n\}$, where λ_i is the ith component of $\boldsymbol{\lambda}$. Hence $\alpha_1\boldsymbol{\varphi}(k) \preceq \boldsymbol{\lambda}, \forall k \in \{-\tau, \ldots, 0\}$. Let $\mathbf{x}_a(k)$, $\forall k \in \mathbb{N}$, be the solution under the initial condition $\alpha_1\boldsymbol{\varphi}(k), k \in \{-\tau, \ldots, 0\}$. Choose another initial condition as $\boldsymbol{\varphi}(k) = \boldsymbol{\lambda}$, $k \in \{-\tau, \ldots, 0\}$, and denote the corresponding solutions to (1) and (4) by $\mathbf{x}_b(k)$ and $\mathbf{y}_b(k), \forall k \in \mathbb{N}$, respectively. According to Lemmas 3.2 and 3.3,

$$\begin{aligned}\|\mathbf{x}(k)\|_\infty &= \frac{1}{\alpha_1}\|\mathbf{x}_a(k)\|_\infty \\ &\leq \frac{1}{\alpha_1}\|\mathbf{x}_b(k)\|_\infty \\ &\leq \frac{1}{\alpha_1}\|\mathbf{y}_b(k)\|_\infty \\ &\leq \frac{1}{\alpha_1}\alpha\gamma^{-k}\|\boldsymbol{\lambda}\|_\infty.\end{aligned}$$

For the constant vector $\boldsymbol{\lambda}$, there exists constant scalar $r = ((\max\{\lambda_1, \ldots, \lambda_n\})/(\min\{\lambda_1, \ldots, \lambda_n\}))$ such that $\|\boldsymbol{\lambda}\|_\infty = r\min\{\lambda_1, \ldots, \lambda_n\}$, which, together with the fact that $\alpha_1\|\boldsymbol{\varphi}\|_d = \min\{\lambda_1, \ldots, \lambda_n\}$, shows that

$$\begin{aligned}\|\mathbf{x}(k)\|_\infty &\leq \frac{1}{\alpha_1}\alpha\gamma^{-k}r\min\{\lambda_1, \ldots, \lambda_n\} \\ &= \frac{1}{\alpha_1}\alpha\gamma^{-k}r\alpha_1\|\boldsymbol{\varphi}\|_d \\ &= (r\alpha)\gamma^{-k}\|\boldsymbol{\varphi}\|_d.\end{aligned}$$

The proof is thus completed. □

To deal with the continuous-time system (2), we need the following lemmas.

Lemma 3.5. The positive system (2) is asymptotically stable for all continuous and bounded delays if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ such that (Liu et al. 2010, Theorem 1)

$$\sum_{i=0}^{p} A_i\boldsymbol{\lambda} \prec 0 \tag{5}$$

holds.

Lemma 3.6. Assume that system (2) is positive. Let $\mathbf{x}_a(t)$ and $\mathbf{x}_b(t)$, $t \geq 0$, be the solution trajectories of (2) under the initial conditions $\boldsymbol{\varphi}_a(t)$ and $\boldsymbol{\varphi}_b(t)$, $t \in [-\tau, 0]$, respectively

(Liu et al. 2010, Lemma 3). Then $\boldsymbol{\varphi}_a(t) \preceq \boldsymbol{\varphi}_b(t)$, $t \in [-\tau, 0]$ implies that $\mathbf{x}_a(t) \preceq \mathbf{x}_b(t)$, $t > 0$.

Suppose that there exists a set of constants $\{\tau_i \in \mathbb{R}_+\}_{i \in \underline{p}}$ satisfying $\tau_i \geq \tau_i(t) \geq 0, \forall t \geq 0$. Replacing $\tau_i(t)$ in Equation (2) with τ_i, we have

$$\dot{\mathbf{y}}(t) = A_0\mathbf{y}(t) + \sum_{i=1}^{p} A_i\mathbf{y}(t - \tau_i), \quad t \geq 0, \tag{6}$$

$$\mathbf{y}(t) = \boldsymbol{\varphi}(t), \quad t \in [-\tau, 0].$$

LEMMA 3.7. Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ such that (5) holds (Liu et al. 2010, Lemma 6). Let the initial conditions of (2) and (6) be $\boldsymbol{\varphi}(t) = \boldsymbol{\lambda}, t \in [-\tau, 0]$. Then $\mathbf{y}(t) \succeq \mathbf{x}(t), \forall t > 0$.

THEOREM 3.8. If system (2) is asymptotically stable for all continuous and bounded delays, then it is exponentially stable for all continuous and bounded delays.

Proof. Arbitrarily pick a $\boldsymbol{\varphi} \not\equiv 0$, and denote the corresponding solutions to (2) and (6) by $\mathbf{x}(t)$ and $\mathbf{y}(t), \forall t > 0$. Since system (2) is asymptotically stable for all continuous and bounded delays, system (6) is necessarily asymptotically stable. Curtain and Zwart (1995, Theorem 5.1.7) say that for continuous-time system (6), asymptotic stability and exponential stability are equivalent to each other. Hence, there exist two positive scalars α and γ such that $\|\mathbf{y}(t)\|_\infty \leq \alpha \exp(-\gamma t)\|\boldsymbol{\varphi}\|_c$ holds for all $t > 0$.

Because the system (2) is asymptotically stable for all continuous and bounded delays, there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ satisfying (5) (Lemma 3.5). Clearly, there is a unique scalar $\alpha_1 > 0$ satisfying $\alpha_1\|\boldsymbol{\varphi}\|_c = \min\{\lambda_1, \ldots, \lambda_n\}$. Denote the solution of (2) by $\mathbf{x}_a(t), \forall t > 0$, under initial condition $\alpha_1\boldsymbol{\varphi}(t), t \in [-\tau, 0]$. Choose another initial condition $\boldsymbol{\varphi}(t) = \boldsymbol{\lambda}, t \in [-\tau, 0]$ and denote the corresponding solutions to (2) and (6) by $\mathbf{x}_b(t)$ and $\mathbf{y}_b(t), \forall t \geq 0$, respectively. By Lemmas 3.6 and 3.7,

$$\begin{aligned}
\|\mathbf{x}(t)\|_\infty &= \frac{1}{\alpha_1}\|\mathbf{x}_a(t)\|_\infty \\
&\leq \frac{1}{\alpha_1}\|\mathbf{x}_b(t)\|_\infty \\
&\leq \frac{1}{\alpha_1}\|\mathbf{y}_b(t)\|_\infty \\
&\leq \frac{1}{\alpha_1}\alpha \exp(-\gamma t)\|\boldsymbol{\lambda}\|_\infty \\
&\leq \frac{1}{\alpha_1}\alpha \exp(-\gamma t) r \min\{\lambda_1, \ldots, \lambda_n\} \\
&= (r\alpha)\gamma^{-k}\|\boldsymbol{\varphi}\|_c,
\end{aligned}$$

where $r = ((\max\{\lambda_1, \ldots, \lambda_n\})/(\min\{\lambda_1, \ldots, \lambda_n\}))$. The proof is thus completed. □

Now consider the constant delay. Clearly, constant delay is also bounded. If the involved delays in Equations (1) and (2) are constant, we can obtain some results stronger than Theorems 3.4 and 3.8. Let us begin with the discrete-time case.

If all the delays in Equation (1) are constant, then we obtain the following system:

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + \sum_{i=1}^{p} A_i\mathbf{x}(k-\tau_i), \quad k \in \mathbb{N}_0, \tag{7}$$

$$\mathbf{x}(k) = \boldsymbol{\varphi}(k), \quad k = -\tau, \ldots, 0$$

for which the following lemma holds.

LEMMA 3.9. Let $\{\tau_i \in \mathbb{N}\}_{i\in \underline{p}}$ be arbitrarily given. System (7) is asymptotically stable if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (3).

Proof. Sufficiency is an immediate consequence of Theorem 1 in Liu et al. (2009). In the following, we show necessity.

Note that for any $k \in \mathbb{N}$, it follows from (7) that

$$\begin{aligned}\mathbf{x}(k)-\mathbf{x}(0) &= (\mathbf{x}(k)-\mathbf{x}(k-1))+(\mathbf{x}(k-1)-\mathbf{x}(k-2))+\cdots+(\mathbf{x}(1)-\mathbf{x}(0)) \\ &= (A_0-I)\mathbf{x}(k-1)+\sum_{i=1}^{p}A_i\mathbf{x}(k-1-\tau_i)+\cdots+(A_0-I)\mathbf{x}(0)+\sum_{i=1}^{p}A_i\mathbf{x}(-\tau_i) \\ &= (A_0-I)\sum_{l=0}^{k-1}\mathbf{x}(l)+\sum_{i=1}^{p}A_i\sum_{l=0}^{k-1}\mathbf{x}(l-\tau_i).\end{aligned} \tag{8}$$

Applying the identity $\mathbf{x}(l-\tau_i) = \mathbf{x}(l) + \mathbf{x}(l-\tau_i) - \mathbf{x}(l)$ into (8), we have

$$\begin{aligned}\mathbf{x}(k)-\mathbf{x}(0) &= (A_0-I)\sum_{l=0}^{k-1}\mathbf{x}(l)+\sum_{i=1}^{p}A_i\sum_{l=0}^{k-1}(\mathbf{x}(l)+\mathbf{x}(l-\tau_i)-\mathbf{x}(l)) \\ &= \left(\sum_{i=0}^{p}A_i - I\right)\sum_{l=0}^{k-1}\mathbf{x}(l)+\sum_{i=1}^{p}A_i\sum_{l=0}^{k-1}(\mathbf{x}(l-\tau_i)-\mathbf{x}(l)) \\ &= \left(\sum_{i=0}^{p}A_i - I\right)\sum_{l=0}^{k-1}\mathbf{x}(l)+\sum_{i=1}^{p}A_i\left(\sum_{l=1}^{\tau_i}\mathbf{x}(-l)-\sum_{l=k-\tau_i}^{k-1}\mathbf{x}(l)\right).\end{aligned}$$

That is

$$\mathbf{x}(k)+\sum_{i=1}^{p}A_i\sum_{l=k-\tau_i}^{k-1}\mathbf{x}(l) = \mathbf{x}(0)+\left(\sum_{i=0}^{p}A_i - I\right)\sum_{l=0}^{k-1}\mathbf{x}(l)+\sum_{i=1}^{p}A_i\sum_{l=1}^{\tau_i}\mathbf{x}(-l).$$

Let $\mathbf{x}(k) \succ 0, \forall k \in \{-\tau, \ldots, 0\}$. Since system (7) is positive, we have $\mathbf{x}(k) \succeq 0, \forall k \in \mathbb{N}$, its asymptotic stability implies $\lim_{k\to+\infty}\mathbf{x}(k) = 0$ and $\lim_{k\to+\infty}\sum_{l=k-\tau_i}^{k-1}\mathbf{x}(l) = 0$. On the other hand, we have $\mathbf{x}(0) \succ 0$, $\sum_{l=0}^{k-1}\mathbf{x}(l) \succ 0$, and $\sum_{l=0}^{k-1}\mathbf{x}(l) \succ 0$. Hence, we may conclude that $\left(\sum_{i=0}^{p}A_i - I\right)\sum_{l=0}^{k-1}\mathbf{x}(l) \prec 0$, which means that (3) does hold. □

Remark 1. Theorem 3.1 of Haddad and Chellaboina (2004) has in fact proved that system (7) is asymptotically stable for all constant delays if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that condition (3) is satisfied. Clearly, it is a reduced version of the above lemma.

THEOREM 3.10. If system (7) is asymptotically stable for given delays $\{\tau_i \in \mathbb{N}\}_{i\in\underline{p}}$, then it is exponentially stable for all constant delays.

Proof. Since system (7) is asymptotically stable, in the light of Lemma 3.9, there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that condition (3) holds. According to Theorem 2.1 of Ait Rami, Helmke, and Tadeo (2007b), condition (3) implies that system (7) is asymptotically stable for all constant delays. By the well-known fact that the asymptotic stability and exponential stability of system (7) are equivalent to each other, we reach the conclusion that if system (7) is asymptotically stable for given delays $\{\tau_i \in \mathbb{N}\}_{i\in\underline{p}}$, then it is exponentially stable for all constant delays. □

Now, consider the following continuous-time linear system with constant delays:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + \sum_{i=1}^{p} A_i\mathbf{x}(t-\tau_i), \quad t \geq 0, \tag{9}$$

$$\mathbf{x}(t) = \boldsymbol{\varphi}(t), \quad t \in [-\tau, 0]$$

for which the following lemma holds.

LEMMA 3.11. Let $\{\tau_i \in \mathbb{R}_+\}_{i\in\underline{p}}$ be arbitrarily given. System (9) is asymptotically stable if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (5) (Ait Rami 2009, Theorem 1).

THEOREM 3.12. If system (9) is asymptotically stable for given delays $\{\tau_i \in \mathbb{R}_+\}_{i\in\underline{p}}$, then it is exponentially stable for all constant delays.

Proof. The proof of this theorem is similar to that of Theorem 3.10. By Lemma 3.11, the asymptotic stability of system (9) indicates that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (5), which further implies that system (9) is asymptotically stable for all constant delays (Ait Rami, Helmke, and Tadeo 2007a, Theorem 2.1). Since the asymptotic stability and exponential stability of system (9) are equivalent, the conclusion follows. □

Remark 2. At the first glance, Theorems 3.4 and 3.10 are very similar. Actually, Theorem 3.10 only requires system (1) to be asymptotically stable for *any arbitrarily given* constant delay. On the contrary, Theorem 3.4 requires that system (1) to be asymptotically stable for *all* bounded delays. The reason for the difference between Theorems 3.4 and 3.10 lies in that we have proved that condition (3) is necessary to guarantee asymptotic stability of system (1) for an arbitrarily given constant delay. However, it still remains open to prove or disprove that (3) is necessary to guarantee asymptotic stability of system (1) for a given time-varying delay. A similar remark can be made for Theorems 3.8 and 3.12.

Remark 3. Note that Theorem 5.1.7 of Curtain and Zwart (1995) can only claim that if a system is asymptotically stable for some delays, then it is exponentially stable for the *same* delays. Therefore, Theorem 3.12 is stronger than Theorem 5.1.7 of Curtain and Zwart (1995). A similar remark can be made for Theorem 3.10.

Remark 4. In Zhu, Li, and Zhang (2011), sufficient conditions for systems (7) and (9) with $p = 1$ to be α-exponentially stable were established. Note that the magnitude of delays affects the α-exponential stability of systems (7) and (9), but does not affect their exponential stability.

4. Positive systems with unbounded delays

This section explores the relationship between asymptotic stability and exponential stability of positive systems with unbounded delays. Having shown that these two properties are equivalent for positive systems with bounded delays, one may conjecture that they are also equivalent to each other for positive systems with unbounded delays. However, such a conjecture is not true, which is shown by the following two counterexamples.

Generally speaking, certain constraints should be imposed on the unbounded delays in order to guarantee the asymptotic stability of such delay systems; see Remark 2 of Liu and Dang (2011) for details. In this section, we make the following assumptions.

Assumption 1. In system (1), there exists an integer $\upsilon \in \mathbb{N}$ such that

$$\sup_{k>\upsilon, i\in\underline{p}} \frac{\tau_i(k)}{k} < 1. \tag{10}$$

Remark 5. In Equation (10), the delays $\tau_i(k)$'s may be unbounded. In this case, it is also enough to define $\boldsymbol{\varphi}$ on a finite set $\{-\tau, \ldots, 0\}$. Indeed, define $\theta = \sup_{k>\upsilon, i\in\underline{p}} (\tau_i(k)/k)$, then (10) implies that $k - \tau_i(k) \geq k - \theta k > 0, \forall k > \upsilon$. Let $-\tau = \min_{i\in\underline{p}} \left\{ \min_{0\leq k\leq \upsilon} \{k - \tau_i(k)\}, \min_{k>\upsilon} \{k - \tau_i(k)\} \right\} = \min_{i\in\underline{p}} \min_{0\leq k\leq\upsilon} \{k - \tau_i(k)\}$. Of course, τ is bounded. A similar remark can be made for Assumption 2.

Assumption 2. In system (2), there exists a real number $\upsilon > 0$ such that

$$\sup_{t>\upsilon, i\in\underline{p}} \frac{\tau_i(t)}{t} < 1. \tag{11}$$

Lemma 4.1. Positive system (7) is asymptotically stable for all delays satisfying (10) if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (3) holds (Liu and Zhong 2010, Theorem 1).

Lemma 4.2. Positive system (2) is asymptotically stable for all continuous delays satisfying (11) if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (5) holds (Liu and Dang 2011, Corollary 1).

Lemma 4.3. Consider the following positive system:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + \sum_{i=1}^{p} A_i\mathbf{x}(t - \tau_i(t)), \quad t \geq t_0, \tag{12}$$

$$\mathbf{x}(t) = \boldsymbol{\varphi}(t) \succeq 0, \quad t \in [t_0 - \tau, t_0].$$

Let $\mathbf{x}_a(t)$ and $\mathbf{x}_b(t)$, $t \geq t_0$, be solution trajectories of (13) under the initial conditions $\boldsymbol{\varphi}_a(t)$ and $\boldsymbol{\varphi}_b(t)$ for $t \in [t_0 - \tau, t_0]$, respectively. Then $\boldsymbol{\varphi}_a(t) \preceq \boldsymbol{\varphi}_b(t)$ for $t \in [t_0 - \tau, t_0]$ implies that $\mathbf{x}_a(t) \preceq \mathbf{x}_b(t), t > t_0$.

Lemma 4.3 is a natural extension of Lemma 3 in Liu et al. (2010), hence its proof is omitted.

In what follows, we will show that a positive system with unbounded delays may not be exponentially stable, though it is asymptotically stable. Two examples are given. The first one is concerned with discrete-time systems, and the second one with continuous-time systems.

Example 4.4. Consider a scalar discrete-time positive delay system:

$$x(k+1) = 0.9x(k - 0.9k), \quad k \in \mathbb{N}_0. \tag{13}$$

By Lemma 4.1, system (13) is asymptotically stable. We now show that it is not exponentially stable by contradiction.

Define a map $a : \mathbb{N} \to \mathbb{N}$ such that $a(k) = \kappa$ if $k \in \left\{\sum_{l=0}^{\kappa-1} 10^l, \sum_{l=0}^{\kappa-1} 10^l + 1, \ldots, \sum_{l=1}^{\kappa} 10^l\right\}$, for $\kappa \in \mathbb{N}$. The map a is a surjection, since the set $\left\{\sum_{l=0}^{\kappa-1} 10^l, \sum_{l=0}^{\kappa-1} 10^l + 1, \ldots, \sum_{l=1}^{\kappa} 10^l\right\}$ can never be empty for $\kappa \in \mathbb{N}$. Clearly, $a(k) \to +\infty$ as $k \to +\infty$. It is not difficult to verify that

$$x(k) = 0.9^{a(k)} x(0), \quad \forall k \in \mathbb{N}. \tag{14}$$

Let $x(0) > 0$. Suppose that system (13) is exponentially stable. Thus, there exist two scalars $\alpha > 0$ and $\gamma > 1$ such that

$$\|x(k)\|_\infty \leq \alpha\gamma^{-k} x(0), \quad \forall k \in \mathbb{N}. \tag{15}$$

Equation (14), together with Equation (15), gives that

$$0.9^{a(k)} x(0) \leq \alpha\gamma^{-k} x(0), \quad \forall k \in \mathbb{N},$$

which means that

$$0.9^{a(k)} \leq \alpha\gamma^{-k}, \quad \forall k \in \mathbb{N}. \tag{16}$$

Let $c = 1/\gamma$. Clearly, $0 < c < 1$. By Equation (16), we have that

$$0.9^{a(k)} \leq \alpha c^k \leq \alpha c^{\sum_{l=0}^{a(k)-1} 10^l} \leq \alpha c^{10^{a(k)-1}}, \quad \forall a(k) \in \mathbb{N}.$$

Let $\alpha = c^d$ with $d \in \mathbb{R}$. It follows from the above inequality that

$$0.9^a \leq c^{d+10^{a-1}}, \quad \forall a \in \mathbb{N} \tag{17}$$

in which we have dropped the argument k of $a(k)$. Taking the natural logarithms on both sides of (17), we get $a \ln 0.9 \leq (d + 10^{a-1}) \ln c$, which, together with the fact that both $\ln 0.9$ and $\ln c$ are negative, means that $(\ln 0.9 / \ln c) \geq ((d + 10^{a-1})/a)$ holds for all $a \in \mathbb{N}$. Clearly, this inequality will not hold when a is sufficiently large. Therefore, system (10) is asymptotically stable but not exponentially stable.

Example 4.5. Consider a continuous-time scalar continuous-time positive delay system:

$$\dot{x}(t) = -0.4x(t) + 0.2x(t - 0.6t), \quad \forall t \geq 0. \tag{18}$$

Clearly, here $\tau(t) = 0.6t$ is unbounded on the interval $[0, +\infty)$ and satisfies condition (11). By Lemma 4.2, (18) is asymptotically stable. However, it is not exponentially stable, as shown below.

Claim 1. System (18) is strictly decreasing on the interval $[0, +\infty)$.

Define $\upsilon = \sup\{t \geq 0 : \dot{x}(s) < 0, \forall s \in [0, t)\}$. Clearly, $\upsilon > 0$ since $\dot{x}(0) < 0$. If $\upsilon = +\infty$, the claim is surely true. If $\upsilon < +\infty$, we have $\dot{x}(\upsilon) = 0$, and therefore $\dot{x}(0.4\upsilon) < 0$. By continuity of $\dot{x}(t)$, we may pick a scalar $\xi > 0$ sufficiently small so that $\theta_1 < -0.2\theta_2$ with

$$\theta_1 = \sup_{\upsilon-\xi\leq t\leq\upsilon}\{|\dot{x}(t)|\} > 0, \qquad \theta_2 = \sup_{0.4(\upsilon-\xi)\leq t\leq 0.4\upsilon}\{\dot{x}(t)\} < 0.$$

By Equation (18)

$$\begin{aligned}\dot{x}(\upsilon) - \dot{x}(\upsilon - \xi) &= -0.4x(\upsilon) + 0.2x(\upsilon - 0.6\upsilon) + 0.4x(\upsilon - \xi) - 0.2x(\upsilon - \xi - 0.6(\upsilon - \xi)) \\ &= -0.4(x(\upsilon) - x(\upsilon - \xi)) + 0.2(x(0.4\upsilon) - x(0.4(\upsilon - \xi))).\end{aligned}$$

By the mean-value theorem, there exist two scalars $\xi_1 \in [\upsilon - \xi, \upsilon]$ and $\xi_2 \in [0.4(\upsilon - \xi), 0.4\upsilon]$ such that

$$\begin{aligned}\dot{x}(\upsilon) - \dot{x}(\upsilon - \xi) &= -0.4\xi\dot{x}(\xi_1) + 0.08\xi\dot{x}(\xi_2) \\ &\leq \xi(0.4\theta_1 + 0.08\theta_2) \\ &< 0\end{aligned}$$

which, by the fact that $\dot{x}(\upsilon - \xi) < 0$, implies that $\dot{x}(\upsilon) < 0$, contradicting the fact that $\dot{x}(\upsilon) = 0$. Hence, Claim 1 holds.

Take $x(0) = 1$. Since system (18) strictly decreases on the interval $[0, +\infty)$ and converges to zero as $t \to +\infty$ (it is asymptotically stable), for any given $0 < \delta_1 < 1$, there exists a unique instant t_1 such that $x(t_1) = \delta_1$. Recursively define $t_{i+1} = t_i/0.4$, $\delta_{i+1} = 0.5\delta_i(1 + \exp(-0.4(t_{i+1} - t_i)))$, $\forall i \in \mathbb{N}$. Let $t_0 = 0$.

Consider the following scalar system:

$$\begin{aligned}&\dot{y}(t) = -0.4y(t) + 0.2y(t - 0.6t), \quad \forall t \in (t_i, t_{i+1}], \ \forall i \in \mathbb{N}, \\ &y(t) = \delta_i, \quad \forall t \in (t_{i-1}, t_i].\end{aligned} \tag{19}$$

At the first glance, (19) is somehow 'strange'. In fact, it is defined piecewisely, and one can verify that it is indeed well defined on $[t_1, +\infty)$ as shown below.

Claim 2. Suppose $y(t) = \delta_1, \forall t \in [0, t_1]$. When $t > t_1$, $y(t)$ is the solution of system (19). Then

$$x(t) \geq y(t), \quad \forall t > 0.$$

Note that for each $i \in \mathbb{N}, \forall t \in (t_i, t_{i+1}]$, it holds that

$$\begin{aligned} y(t) &= \exp(-0.4(t-t_i))\delta_i + 0.2\int_{t_i}^{t} \exp(-0.4(t-s))x(0.4s)\,\mathrm{d}s \\ &= \exp(-0.4(t-t_i))\delta_i + 0.2\delta_i\int_{t_i}^{t} \exp(-0.4(t-s))\,\mathrm{d}s \\ &= \exp(-0.4(t-t_i))\delta_i + 0.2\delta_i\exp(-0.4t)\int_{t_i}^{t} \exp(0.4s)\,\mathrm{d}s \\ &= \exp(-0.4(t-t_i))\delta_i + 0.5\delta_i\exp(-0.4t)(\exp(0.4t)-\exp(0.4t_i)) \\ &= 0.5\delta_i(1+\exp(-0.4(t-t_i))). \end{aligned} \tag{20}$$

Therefore, $y(t_{i+1}) = 0.5\delta_i(1+\exp(-0.4(t_{i+1}-t_i))) = \delta_{i+1}$. Hence, the solution of system (19) is continuous.

Of course, $y(t) \le x(t)$, $\forall t \in [0, t_1]$. Hence, when $t \in (t_1, t_2]$, applying Lemma 4.3, we have that $y(t) \le x(t)$, $\forall t \in (t_1, t_2]$. Repeating the same procedure, we can prove that for any $i \in \mathbb{N}_0$, $y(t) \le x(t)$, $\forall t \in (t_i, t_{i+1}]$ holds. Hence, Claim 2 is true.

For each $t > 0$, there exists a unique scalar $i(t) \in \mathbb{N}$ such that $t \in (t_{i(t)-1}, t_{i(t)}]$. Clearly, as $t \to +\infty$, $i(t) \to +\infty$.

Now, define a function $f(t)$ as follows:

$$f(t) = \begin{cases} \delta_1, & t \in \left[0, t_1\right], \\ 0.5^{i(t)-1}\delta_1, & t \in (t_{i(t)-1}, t_{i(t)}], \quad i(t) \in \frac{\mathbb{N}}{\{1\}}. \end{cases}$$

Claim 3. $y(t) \ge f(t)$, $\forall t > 0$.

By the definition of δ_i, we have that

$$\delta_{i(t)} = 0.5^{i(t)-1}\delta_1 \prod_{l=1}^{i(t)-1} (1+\exp(-0.4(t_{l+1}-t_l))) > 0.5^{i(t)-1}\delta_1, \quad \forall i(t) \in \frac{\mathbb{N}}{\{1\}}.$$

On the other hand, by Equation (20),

$$\dot{y}(t) = -0.2\delta_{i(t)-1}(1+\exp(-0.4(t-t_{i(t)-1}))) < 0, \quad \forall t \in (t_{i(t)-1}, t_{i(t)}], \quad i(t) \in \frac{\mathbb{N}}{\{1\}}.$$

Hence, $y(t) \ge \delta_{i(t)}$, $\forall t \in (t_{i(t)-1}, t_{i(t)}]$, $i(t) \in \mathbb{N}/\{1\}$. Thus, Claim 3 is true.

We now prove that for any $\alpha > 0$, $\gamma > 0$, the following claim holds.

Claim 4.

$$0.5^{i(t)-1}\delta_1 \le \alpha\exp(-\gamma t) \tag{21}$$

does not hold for all $t > 0$.

If (21) holds for all $t > 0$, then

$$0.5^{i(t)-1}\delta_1 \le \alpha\exp(-\gamma t_{i(t)}) = \alpha\exp(-2.5^{i(t)-1}\gamma t_1), \quad \forall i(t) \in \frac{\mathbb{N}}{\{1\}}$$

necessarily holds. Dropping the argument t of $i(t)$, we obtain

$$0.5^{i-1}\delta_1 \le \alpha\exp(-2.5^{i-1}\gamma t_1), \quad \forall i \in \frac{\mathbb{N}}{\{1\}}. \tag{22}$$

There exists a unique scalar $\nu \in \mathbb{R}$ such that $\exp(\nu) = (\alpha/\delta_1)$. Hence, (22) further implies that $(i-1)\ln 0.5 < \nu - 2.5^{i-1}\gamma t_1$, that is $\ln 0.5 < ((\nu - 2.5^{i-1}\gamma)/(i-1))$. Clearly, it will not hold if i is sufficiently large. Therefore, Claim 4 holds.

Combining Claims 1–4, we conclude that for any $\alpha > 0, \gamma > 0$, $x(t) \le \alpha \exp(-\gamma t)$ cannot hold for all $t > 0$. In other words, system (18) is not exponentially stable.

5. Conclusions

This paper has discussed the relationship between asymptotic stability and exponential stability of positive systems with delays. It has been shown that, for a positive system with bounded delays, asymptotic stability and exponential stability are equivalent. However, for positive systems with unbounded delays, asymptotic stability and exponential stability are generally not equivalent.

The significance of this paper is that it not only reveals the relationship between asymptotic stability and exponential stability of positive systems with bounded time-varying delays but also sheds light on the future study of such relationship for general dynamic systems, which is a very important and challenging task in the control field.

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