Relationship between controllability and observability of standard and fractional different orders discrete-time linear system

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Abstract—**The controllability and observability of fractional different orders discrete-time linear systems are analyzed. The relationship between controllability and observability of standard and fractional discrete-time linear systems are investigated. The influence of the fractional order on the controllability and observability is shown. Investigations are illustrated on numerical examples.**

I. INTRODUCTION

The notion of controllability and observability and the decomposition of linear systems have been introduced by Kalman [1-2]. These notions are the basic concepts of the modern control theory [3-7]. They have been also extended to positive linear systems [8, 9]. The decomposition of the pair (A,B) and (A,C) of the positive discrete-time linear system has been addressed in [10]. The reachability of linear systems is closely related to the controllability of the systems. Especially for positive linear systems, the conditions for the controllability are much stronger than for the reachability [9, 11]. Tests for the reachability and controllability of standard and positive linear systems are given in [9, 12-15].

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [16, 17] and another on was proposed in 20th century by Caputo [18]. This idea has been used by engineers for modeling different processes [19-21]. Mathematical fundamentals of fractional calculus are given in the monographs [16-18, 22]. The positive fractional linear systems have been investigated in [23, 24]. The positive linear systems with different fractional orders have been addressed in [25, 26]. Stability of fractional continuous-time linear systems consisting of n subsystem with different fractional orders has been given in [27]. Reachability and minimum energy control problem for systems with two different fractional orders has been considered in [28]. Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional orders has been introduced in [29]. Comparison of three different methods for finding the solution of the descriptor fractional discrete-time linear system has been given in [30].

In this paper the relationship between controllability and observability of standard and fractional different orders discrete-time linear systems will be established.

The paper is organized as follows. In section 2 the fractional different orders discrete-time linear systems are introduced. The controllability and observability of standard discrete-time linear systems are recalled in section 3. The controllability and observability of fractional different orders discrete-time linear systems are discussed in section 4. The main result of the paper is presented in section 5 where the relationship between controllability and observability of standard and fractional is analyzed. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, Z_+ - the set of nonnegative integers, I_n - the $n\times n$ identity matrix, A^T - the transpose matrix A.

II. FRACTIONAL DIFFERENT ORDERS DISCRETE-TIME LINEAR SYSTEMS

Consider the fractional discrete-time linear system with two different fractional orders α and β of the form

$$\begin{bmatrix} \Delta^{\alpha} x_1(k+1) \\ \Delta^{\beta} x_2(k+1) \end{bmatrix} = A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Bu(k), \tag{1a}$$

$$y(k) = C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \tag{1b}$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [C_1 \quad C_2]$, $k \in Z_+$, $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector, $y(k) \in \Re^p$ is the output vector and $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{p \times n_i}$; $i, j = 1,2$, $n = n_1 + n_2$.

The studies have been carried out in the framework of work No. S/WE/1/2016 and financed from the funds for science by the Polish Ministry of Science and Higher Education.

The fractional difference of α (β) order is defined by [24]

$$\Delta^{\alpha} x(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) = \sum_{j=0}^{k} c_{\alpha}(j) x(k-j),$$
$$c_{\alpha}(j) = (-1)^j \binom{\alpha}{j} = (-1)^j \frac{\alpha(\alpha-1)...(\alpha-j+1)}{j!}, \tag{2}$$
$$c_{\alpha}(0) = 1, \quad j = 1,2,...$$

Using (2) we can write the equation (1a) in the matrix form

$$\begin{aligned}\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} &= \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\ &- \sum_{j=2}^{k+1} \begin{bmatrix} c_{\alpha}(j) I_{n_1} & 0 \\ 0 & c_{\beta}(j) I_{n_2} \end{bmatrix} \begin{bmatrix} x_1(k-j+1) \\ x_2(k-j+1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(k),\end{aligned} \tag{3}$$

where $A_{1\alpha} = A_{11} + I_{n_1}\alpha$, $A_{2\beta} = A_{22} + I_{n_2}\beta$.

Theorem 1. The solution to the fractional system described by equation (1a) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{k-1} \Phi_{k-j-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(j), \; k \in Z_+, \tag{4}$$

where Φ_k is defined by

$$\Phi_i = \begin{cases} I_{n_1+n_2} & \text{for} \quad i = 0 \\ \tilde{A}\Phi_{i-1} - D_1\Phi_{i-2} - ... - D_{i-1}\Phi_0 & \text{for} \quad i = 1,2,...,j \\ \tilde{A}\Phi_{i-1} - D_1\Phi_{i-2} - ... - D_i\Phi_{i-j-1} & \text{for} \quad i = j+1, j+2,... \end{cases} \tag{5a}$$

and

$$\tilde{A} = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} = A + D_0,$$
$$D_k = \begin{bmatrix} c_{\alpha}(k+1) I_{n_1} & 0 \\ 0 & c_{\beta}(k+1) I_{n_2} \end{bmatrix}. \tag{5b}$$

Proof is given in [24-26].

III. STANDARD DISCRETE-TIME LINEAR SYSTEMS

Consider the standard discrete-time linear system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Bu(k), \tag{6a}$$

$$y(k) = C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \tag{6b}$$

where $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [C_1 \quad C_2]$, $k \in Z_+$, $x_1(k) \in \Re^{n_1}$ and $x_2(k) \in \Re^{n_2}$ are the state vectors, $u(k) \in \Re^m$ is the input vector, $y(k) \in \Re^p$ is the output vector and $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $C_i \in \Re^{p \times n_i}$; $i, j = 1,2$, $n = n_1 + n_2$.

Theorem 2. The solution to the standard system described by equation (6a) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = A^k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j), \; k \in Z_+. \tag{7}$$

Proof. Substituting (7) into (6a) we obtain

$$\begin{aligned} & A^{k+1} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{k} A^{k-j} Bu(j) \\ &= A \left[A^k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{k-1} A^{k-j-1} Bu(j) \right] + Bu(k) \\ &= A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Bu(k). \end{aligned} \tag{8}$$

Therefore, (7) satisfies the equation (6a). □

Definition 1. The system (6) is called controllable in q steps ($q \le n$) if for any initial conditions $[x_1^T(0) \quad x_2^T(0)]^T$ and given final state $x_f = [x_{f1}^T \quad x_{f2}^T]^T \in \Re^n$ there exists an input sequence $u(i) \in \Re^m$, $i = 0,1,...,q-1$ such that $x(q) = x_f$.

Theorem 3. The system (6a) is controllable in q steps if and only if

$$\text{rank}[B \quad AB \quad \cdots \quad A^{q-1}B] = n. \tag{9}$$

Proof. From (7) for $k = q$ we obtain

$$\begin{aligned} x_f &= \begin{bmatrix} x_1(q) \\ x_2(q) \end{bmatrix} = A^q \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &+ [B \quad AB \quad \cdots \quad A^{q-1}B] \begin{bmatrix} u(q-1) \\ u(q-2) \\ \vdots \\ u(0) \end{bmatrix}. \end{aligned} \tag{10}$$

By Kronecker-Cappelli theorem [3, 4] the equation (10) has a solution $u(0), u(1),..., u(q-1)$ if and only if the condition (9) is satisfied. □

Definition 2. The system (6) is called observable in q steps $(q \le n)$ if knowing the input sequence $u(i) \in \Re^m$, $i = 0,1,...,q-1$ and the corresponding output sequence $y(i) \in \Re^p$, $i = 0,1,...,q-1$ it is possible to find its unique initial condition $[x_1^T(0) \quad x_2^T(0)]^T$.

Theorem 4. The system (6) is observable in q steps if and only if

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} = n \,. \tag{11}$$

Proof. Without loss of generality we may assume that in (6) the matrix $B = 0$. In this case from (6b) and (7) we have

$$y(k) = CA^k x(0) \tag{12}$$

for $k \in Z_+$, $C = [C_1 \quad C_2]$, $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$.

Using (12) for $k = 0, i, ..., q-1$ we obtain

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} x(0) \,. \tag{13}$$

By Kronecker-Cappelli theorem [3, 4] the equation (13) has the unique solution $x(0)$ if and only if the condition (11) is satisfied. □

Example 1. Check the controllability and observability of the system (6) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 0 \quad 0 \quad 1] \,. \tag{14}$$

Using (9), (11) for $q = 4$ and (14) we obtain

$$\text{rank}[B \quad AB \quad A^2B \quad A^3B] = \text{rank}\begin{bmatrix} 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 5 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 6 \end{bmatrix} = 4 = n \tag{15}$$

and

$$\text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{q-1} \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix} = 4 = n \,. \tag{16}$$

Therefore, the system (6) with (14) is controllable and observable.

IV. Controllability and Observability of Fractional Different Orders Systems

The controllability of the fractional systems (1) is defined in a similar way as for the system (6) (Definition 1).

Theorem 5. The fractional system (1) is controllable in q steps if and only if

$$\text{rank}[\Phi_0 B \quad \Phi_1 B \quad \cdots \quad \Phi_{q-1} B] = n \,, \tag{17}$$

where Φ_i for $i = 0,1,...,q-1$ is defined by (5a) and (5b).

Proof. From (4) for $k = q$ we obtain

$$\begin{aligned} x_f &= \begin{bmatrix} x_1(q) \\ x_2(q) \end{bmatrix} = \Phi_q \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &+ [\Phi_0 B \quad \Phi_1 B \quad \cdots \quad \Phi_{q-1} B] \begin{bmatrix} u(q-1) \\ u(q-2) \\ \vdots \\ u(0) \end{bmatrix}. \end{aligned} \tag{18}$$

By Kronecker-Cappelli theorem [3, 4] the equation (18) has a solution $u(0), u(1), ..., u(q-1)$ if and only if the condition (17) is satisfied. □

The observability of the fractional systems (1) is defined in a similar way as for the system (6) (Definition 2).

Theorem 6. The fractional system (1) is observable in q steps if and only if

$$\text{rank}\begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ \vdots \\ C\Phi_{q-1} \end{bmatrix} = n \,. \tag{19}$$

Proof. Without loss of generality we may assume that in (1) the matrix $B = 0$. In this case from (1b) and (4) we have

$$y(k) = C\Phi_k x(0) \tag{20}$$

for $k \in Z_+$, $C = [C_1 \quad C_2]$, $x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$.

Using (20) for $k = 0,1,...,q-1$ we obtain

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(q-1) \end{bmatrix} = \begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ \vdots \\ C\Phi_{q-1} \end{bmatrix} x(0) . \qquad (21)$$

By Kronecker-Cappelli theorem [3, 4] the equation (21) has the unique solution $x(0)$ if and only if the condition (19) is satisfied. □

Example 2. (Continuation of Example 1) Check the controllability and observability of the fractional system (1) with the matrices (14).

Using (17), (19), (5a) and (14) we obtain

$$\begin{aligned} &\text{rank}[\Phi_0 B \quad \Phi_1 B \quad \Phi_2 B \quad \Phi_3 B] \\ &= \text{rank}\begin{bmatrix} 0 & 1 & 2.4 & 7.38 \\ 1 & 1.2 & 3.82 & 8.287 \\ 0 & 1 & 3 & 6.7 \\ 1 & 2.5 & 5.075 & 11.683 \end{bmatrix} = 4 = n \end{aligned} \qquad (22)$$

and

$$\text{rank}\begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ C\Phi_2 \\ C\Phi_3 \end{bmatrix} = \text{rank}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1.5 \\ 1 & 1.7 & 0 & 3.375 \\ 1.9 & 4.795 & 1 & 6.888 \end{bmatrix} = 4 = n . \qquad (23)$$

Therefore, by Theorem 5 and 6 the fractional system (1) with (14) is also controllable and observable in $q = 4$ steps.

V. Relationship between controllability and observability of standard and fractional systems

In this section the relationship between controllability and observability of the standard system (6) and the fractional system (1) will be discussed and it will be shown that if the standard system is controllable (observable) then the fractional system can be uncontrollable (unobservable).

On the following simple example it will be shown that the standard system (6) can be controllable but the fractional system (1) is uncontrollable for some special choice of its parameters.

Example 3. Consider the fractional system (1) with

$$A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \alpha = \frac{3}{5}, \quad \beta = \frac{2}{3} . \qquad (24)$$

The standard system with $\alpha = \beta = 1$ and A, B given by (24) is controllable since

$$\text{rank}[B \quad AB] = \text{rank}\begin{bmatrix} 2 & \frac{4}{5} \\ 3 & 1 \end{bmatrix} = 2 = n , \qquad (25)$$

but the fractional system (1) with (24) is uncontrollable, since

$$\begin{aligned} &\text{rank}[B \quad \Phi_1 B] = \text{rank}[B \quad AB + D_0 B] \\ &= \text{rank}\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} = 1 < n. \end{aligned} \qquad (26)$$

If we assume that $\alpha = \beta = \frac{3}{5}$, then we have

$$\text{rank}[B \quad \Phi_1 B] = \text{rank}\begin{bmatrix} 2 & 2 \\ 3 & 2.8 \end{bmatrix} = 2 = n . \qquad (27a)$$

If $\alpha = \beta = \frac{2}{3}$, then we have

$$\text{rank}[B \quad \Phi_1 B] = \text{rank}\begin{bmatrix} 2 & 2.13 \\ 3 & 3 \end{bmatrix} = 2 = n \qquad (27b)$$

and the fractional system for $\alpha = \beta$ is controllable.

Theorem 7. If $\alpha = \beta \neq 0$, then the fractional system (1) is always controllable if and only if the standard system (6) is controllable.

Proof. If $\alpha \neq \beta$ then we can find the combination of α and β for which first two columns of the controllability matrix are linearly depended (the controllability matrix is not full rank) and the fractional system (1) is uncontrollable. If $\alpha = \beta \neq 0$ the controllability matrix is always full rank if the controllability matrix of the standard system (6) is of full rank. □

Theorem 8. The fractional system (1) with $\alpha \neq \beta$ is controllable in q steps if and only if

$$\text{rank}[\Phi_0 B \quad \Phi_1 B \quad \cdots \quad \Phi_{q-1} B] = n . \qquad (28)$$

Proof is similar to the proof of theorem 7.

Example 4. (Continuation of Example 3) The fractional dual system (1) with

$$A^T = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}, \quad C = B^T = [2 \quad 3], \quad \alpha = \frac{3}{5}, \quad \beta = \frac{2}{3} \qquad (29)$$

is not observable since

$$\operatorname{rank}\begin{bmatrix} C \\ C\Phi_1 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = 1 < n \,. \qquad (30)$$

Similar as previously, if we assume that $\alpha = \beta = \frac{3}{5}$ or $\alpha = \beta = \frac{2}{3}$ the fractional dual system (1) is observable, since observability matrix is of the full rank.

Theorem 9. If $\alpha = \beta \neq 0$, then the fractional system (1) is always observable if and only if the standard system (6) is observable.

Proof is similar (dual) to the proof of theorem 7.

Theorem 10. The fractional system (1) with $\alpha \neq \beta$ is controllable in q steps if and only if

$$\operatorname{rank}\begin{bmatrix} C\Phi_0 \\ C\Phi_1 \\ \vdots \\ C\Phi_{q-1} \end{bmatrix} = n \,. \qquad (31)$$

Proof is similar (dual) to the proof of theorem 7.

VI. Concluding Remarks

The relationship between controllability and observability of standard and fractional discrete-time linear systems have been investigated. The necessary and sufficient conditions for the controllability and observability of the standard and fractional different order discrete-time linear systems have been established. The influence of the fractional order on the controllability and observability has been shown. Investigations have been illustrated on numerical examples. The considerations can be extended to n different orders fractional discrete-time linear systems.

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