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Partial eigenstructure assignment for undamped vibration systems using acceleration and displacement feedback

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A B S T R A C T

A new method for partial eigenstructure assignment using acceleration and displacement feedback for undamped vibration systems is presented in this paper. Firstly, a necessary and sufficient condition is proposed for the incremental mass and stiffness matrices that modify some eigenpairs while keeping other eigenpairs unchanged. Secondly, based on this condition, an algorithm for determining the required control gain matrices of acceleration and displacement feedback, which assign the desired eigenstructure, is developed. This algorithm is easy to implement, and works directly on the second-order system model. More importantly, the algorithm allows the control matrix to be specified beforehand and also leads naturally to a small norm solution of the feedback gain matrices. Finally, some numerical examples are given to demonstrate the effectiveness and accuracy of the proposed algorithm.

1. Introduction

Vibration in engineering is usually undesirable and must be controlled to prevent damage to or premature fatigue failure of structures or machines. There are many ways of controlling vibration. Shifting natural frequencies away from excitation frequencies or adding damping is a common control strategy. Both can be realised by means of pole or eigenvalue assignment. It is also very useful to be able to force a structure or machine to possess modes (eigenvectors) in certain forms. Assignment of both eigenvalues and eigenvectors is called eigenstructure assignment. The eigenvalue or eigenstructure assignment with robustness against uncertainty in the system matrices and/or to measurement noise is also an important problem in active control design.

Eigenvalue assignment and eigenstructure assignment working directly on second-order dynamic system models has attracted much attention over the past ten years, partly because of the demands in general control and vibration control applications in engineering, and partly because of the advantage of those peculiar properties afforded by the second-order system models. For these two kinds of eigendata assignment problems, the undamped vibration models lead to an inverse generalised eigenvalue problem and the damped vibration models lead to an inverse quadratic eigenvalue problem.

Juang and Maghami [1] adapted the established first-order approach to second-order systems, and presented a robust full pole assignment algorithm. Datta and his colleagues pioneered research into inverse eigenvalue problems. Chu and Datta [2] proposed a modification to the foregoing algorithm that produced well-conditioned closed-loop eigenvectors, and they also gave a numerically robust algorithm by minimising the condition numbers of the eigenvectors. Furthermore, Datta

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et al. [3] established three bi-orthogonality relations for eigenvectors of symmetric definite quadratic pencils. One of these relations was used to derive an explicit solution to the *partial pole assignment* of symmetric definite second-order systems by state feedback with a single-input, where some desired eigenvalues were relocated to prescribed positions, while all the other unassigned eigenvalues of the open-loop system remained unchanged in the closed-loop system, i.e., possessing "*no spill-over*" property. Ram and Elhay [4] generalised partial assignment of poles to multi-input control, and they demonstrated significant reductions in the magnitude of the control forces by using multi-input control. Moreover, the method does not require knowledge of the unassigned eigenvalues and their corresponding eigenvectors of the open-loop system, and was generalised to multi-input and asymmetric quadratic pencils by Zhang [5] and Chu [6]. In addition, Ramadan and EI-Sayed [7] introduced an explicit solution to the partial eigenvalue assignment problem for high order systems via a single input. They used the well-known criteria of orthogonality relations between the eigenvectors of the matrix polynomial and their solution can be implemented with only partial knowledge of the spectrum and the corresponding left eigenvectors of the matrix polynomial.

Chan et al. [8] examined robust eigenvalue assignment by combined derivative and proportional state feedback, and an alternative objective function was defined and minimised by utilising the gradient flow technique based on the condition number of the closed-loop eigenvectors. Xu and Qian [9,10] put forward some robust partial pole assignment algorithms, where eigenvectors were chosen in certain subspaces such that some measure of the distance between the eigenvectors and some orthogonal bases of a certain subspace was minimised. Recently, Brahma and Datta [11], and Bai et al. [12] solved robust partial quadratic eigenvalue assignment problems for vibrating structures based on minimisation of a relevant norm. Additionally, robust partial pole assignment problem for high order control systems was studied in [13,14]. It is worthwhile to note that, for large scale systems and structures, usually only a small part of eigenvalues that significantly affects the stability and other performances of the system need to be relocated, and meanwhile it is required to keep other eigenvalues not to be affected by the assignment. This is one main reason why partial eigenvalue or eigenstructure assignment plays an important role.

The eigenstructure assignment problem of vibration systems is to assign both the eigenvalues and their corresponding eigenvectors. Datta et al. [15] developed a method for *partial eigenstructure assignment*, where only a small part of the eigenstructure was assigned and the rest remained unchanged. Nichols and Kautsky [16] derived new sensitivity measures, or condition numbers, for the eigenvalues of the quadratic matrix polynomial and defined a measure of the robustness of the corresponding system. They showed that the robustness of the quadratic inverse eigenvalue problem could be achieved by solving a generalised linear eigenvalue assignment problem subject to structured perturbations. For eigenstructure assignment in second-order systems, a parametric expression of the feedback gain matrices was explicitly established in terms of the eigenstructure and system matrices by Schulz and Inman [17], whereby the closed-loop system was optimised by minimising the norms related to the feedback gains. Duan [18,19] developed another parametric method. Additionally, other eigenstructure assignment methods were developed by Triller and Kammer [20], and Chu et al. [21]. It should be pointed out that for the eigenstructure assignment of vibration systems not just any eigenvector can be assigned to a given system, that is, the assigned or desired eigenvectors cannot be chosen arbitrarily, as discussed in [22,16,9]; and it is believed that the "achievable" eigenvectors and then the required feedback gain matrices generally should be determined from the original system matrices, the given control matrix, and the desired assigned eigenvectors, as shown in this paper.

Eigenstructure assignment can also be made by means of structural modifications. Kypriano et al. [23] assigned frequencies and zeros to lumped mass-spring systems with a mass and springs using the Groebner bases. They did the same for a continuous structure of an L-shaped beam and validated the theoretical results by experiment [24]. Ouyang et al. [25,26] developed an optimisation approach to assign the eigenstructure to a lumped mass-spring system and implemented the approach on an experimental rig. Interestingly, the continuous optimisation approach and integer optimisation approach were shown to yield fairly different modifications.

In addition to the eigendata assignment methods mentioned above, which can be called the model-based approach, a new approach to eigenvalue assignment in structural vibration systems was introduced by Ram and Mottershead [27], and extended by them and their colleagues [28–31] based on measured receptances and without the need to know or evaluate the system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. Ouyang [29] presented a receptance-based inverse method for assigning complex poles to second-order asymmetric systems through state-feedback control using a combination of active stiffness, active damping and active mass. Tehrani et al. [30] studied robust pole placement to symmetric systems. Tehrani and Ouyang [31] developed partial pole assignment to asymmetric systems.

As the focus of this paper is on eigenstructure assignment using the second-order framework, works using the first-order formulation are largely omitted from the paper. It should also be mentioned that some works on frequency placement, for example in [32], are also very useful but are not identical to frequency assignment, and are thus not discussed.

From the open literature, it may be concluded that most researchers used velocity and displacement feedback to assign eigendata of vibration systems. This paper instead uses acceleration and displacement feedback for partial eigenstructure assignment of undamped vibration systems. This is even more interesting because of the frequent use of accelerometers in practice. Section 2 presents a necessary and sufficient condition that the modification of the mass and stiffness matrices should satisfy so that spill-over does not occur when undamped vibration systems are modified for a desirable eigenstructure. Based on this condition, a partial eigenstructure assignment algorithm is proposed to determine the acceleration and displacement feedback matrices in Section 3. Several numerical examples are used to demonstrate advantages of the algorithm in Section 4. Finally, some conclusions are drawn in Section 5.

2. Partial eigenstructure modification

2.1. The problem description

Consider an n-degree-of-freedom undamped vibration system that is modelled by the following set of second-order ordinary differential equations:

$$\mathbf{M}_0\ddot{\mathbf{q}}(t)+\mathbf{K}_0\mathbf{q}(t)=\mathbf{0} \tag{1}$$

where $\mathbf{q}(t)\in\mathbf{R}^n$ is displacement vector, $\mathbf{M}_0$, $\mathbf{K}_0\in\mathbf{R}^{n\times n}$ are mass and stiffness matrices, respectively. In general, $\mathbf{M}_0$ is symmetric and positive definite, and $\mathbf{K}_0$ is symmetric and positive semi-definite, i.e. $\mathbf{M}_0=\mathbf{M}_0^{\mathrm{T}}>\mathbf{0}$, $\mathbf{K}_0=\mathbf{K}_0^{\mathrm{T}}\geq\mathbf{0}$. For the sake of convenience, the model is simply denoted by $\{\mathbf{M}_0,\mathbf{K}_0\}$.

It is well known that if $\mathbf{q}(t)=\mathbf{x}e^{i\omega t}$ is a fundamental solution of Eq. (1), then the natural frequency ω and the mode shape vector $\mathbf{x}$ must satisfy the following generalised eigenvalue equation:

$$(\mathbf{K}_0-\lambda_i\mathbf{M}_0)\mathbf{x}_i=0,\quad i=1,2,\ldots,n \tag{2}$$

where $\lambda_i=\omega_i^2$ is the square of the ith natural frequency ω_i, called the ith eigenvalue, and $\mathbf{x}_i$ is the corresponding ith eigenvector. Eq. (2) can be written in a compact representation as follows:

$$\mathbf{K}_0\mathbf{X}=\mathbf{M}_0\mathbf{X}\boldsymbol{\Lambda} \tag{3}$$

where $\boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\ \lambda_2,\ldots,\lambda_n)$ and $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ make up the complete eigenstructure.

Suppose that the system described by Eq. (1) is modified by the incremental matrices $\boldsymbol{\Delta}\mathbf{M}\in\mathbf{R}^{n\times n}$ and $\boldsymbol{\Delta}\mathbf{K}\in\mathbf{R}^{n\times n}$. Then the motion of the modified system is governed by

$$(\mathbf{M}_0+\boldsymbol{\Delta}\mathbf{M})\ddot{\mathbf{q}}(t)+(\mathbf{K}_0+\boldsymbol{\Delta}\mathbf{K})\mathbf{q}(t)=\mathbf{0} \tag{4}$$

Let $\mathbf{M}_c=\mathbf{M}_0+\boldsymbol{\Delta}\mathbf{M}$, $\mathbf{K}_c=\mathbf{K}_0+\boldsymbol{\Delta}\mathbf{K}$. The modified system is denoted simply by $\{\mathbf{M}_c,\mathbf{K}_c\}$ which satisfy the following equation:

$$\mathbf{K}_c\mathbf{Y}=\mathbf{M}_c\mathbf{Y}\boldsymbol{\Sigma} \tag{5}$$

where $\boldsymbol{\Sigma}=\mathrm{diag}(\mu_1,\ \mu_2,\ldots,\mu_n)$ and $\mathbf{Y}=(\mathbf{y}_1,\ \mathbf{y}_2,\ldots,\mathbf{y}_n)$ are the complete eigenstructure of $\{\mathbf{M}_c,\mathbf{K}_c\}$. Mathematically, the partial eigenstructure modification (PESM) problem may be formulated as follows.

PESM Problem: Given a system model $\{\mathbf{M}_0,\mathbf{K}_0\}$, a set of its associated eigenpairs $(\lambda_i,\ \mathbf{x}_i)(i=1,\ 2,\ldots,m)$ with $m<n$, and another set of modified (or desired) eigenpairs $(\mu_i,\ \mathbf{y}_i)(i=1,\ 2,\ldots,m)$, find the incremental matrices $\boldsymbol{\Delta}\mathbf{M}$ and $\boldsymbol{\Delta}\mathbf{K}$ such that:

(1) The subset $(\lambda_i,\ \mathbf{x}_i)(i=1,\ 2,\ldots,m)$ of $\{\mathbf{M}_0,\mathbf{K}_0\}$ is replaced by $(\mu_i,\ \mathbf{y}_i)(i=1,\ 2,\ldots,m)$ as m eigenpairs of the modified model $\{\mathbf{M}_c,\mathbf{K}_c\}$, where $\mathbf{M}_c=\mathbf{M}_0+\boldsymbol{\Delta}\mathbf{M}$, $\mathbf{K}_c=\mathbf{K}_0+\boldsymbol{\Delta}\mathbf{K}$.
(2) The remaining (unknown) n–m eigenpairs of the modified model $\{\mathbf{M}_c,\mathbf{K}_c\}$ are the same as those of the original model $\{\mathbf{M}_0,\mathbf{K}_0\}$.This is the "*no spill-over*" property.

It is necessary to partition the complete eigenstructure into two parts: the assigned (part 1) and the unassigned (part 2). Throughout this paper, the following notation is used. Let $\boldsymbol{\Lambda}_1=\mathrm{diag}(\lambda_1,\ \lambda_2,\ldots,\lambda_m)$, $\boldsymbol{\Lambda}_2=\mathrm{diag}(\lambda_{m+1},\ \lambda_{m+2},\ldots,\lambda_n)$, so that $\boldsymbol{\Lambda}=\mathrm{diag}(\boldsymbol{\Lambda}_1,\ \boldsymbol{\Lambda}_2)$.

$$\mathbf{X}_1=(\mathbf{x}_1,\ \mathbf{x}_2,\ldots,\mathbf{x}_m),\ \ \mathbf{X}_2=(\mathbf{x}_{m+1},\ \mathbf{x}_{m+2},\ldots,\mathbf{x}_n),\mathbf{X}=(\mathbf{X}_1,\ \mathbf{X}_2)$$

and

$$\boldsymbol{\Sigma}_1=\mathrm{diag}(\mu_1,\ \mu_2,\ldots,\mu_m),\mathbf{Y}_1=(\mathbf{y}_1,\ \mathbf{y}_2,\ldots,\mathbf{y}_m).$$

In addition, for any real matrix $\mathbf{A}$, $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$, $\mathbf{A}^{+}$ the Moore–Penrose inverse of $\mathbf{A}$, $r(\mathbf{A})$ the rank of $\mathbf{A}$, $N(\mathbf{A})$ the null space of $\mathbf{A}$, and $\mathbf{I}_n$ represents the identity matrix of size n. The symbol $\|\bullet\|_F$ stands for the Frobenius norm.

The following assumptions, which are quite reasonable in practice, are made:

$$\text{(A1)}\quad \mathbf{M}_0=\mathbf{M}_0^{\mathrm{T}}>\mathbf{0},\mathbf{K}_0=\mathbf{K}_0^{\mathrm{T}}\geq\mathbf{0},\mathbf{X}^{\mathrm{T}}\mathbf{M}_0\mathbf{X}=\mathbf{I}_n;$$

$$\text{(A2)}\quad \{\lambda_1,\lambda_2,\ldots,\lambda_m\}\cap\{\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_n\}=\emptyset\ \text{(an empty set)};$$

$$\text{(A3)}\quad \{\mu_1,\mu_2,\ldots,\mu_m\}\cap\{\lambda_{m+1},\lambda_{m+2},\ldots,\lambda_n\}=\emptyset;$$

$$\text{(A4)}\quad \{\mathbf{M}_0,\ \mathbf{K}_0\}\ \text{are non-defective systems.}$$

It should be pointed out that non-defective matrices always possess the same number of independent modes as the dimension of the system matrix. In such a system, all modes of the distinct eigenvalues are orthogonal to one another; all modes corresponding to the same (identical) eigenvalue can be made orthogonal through a proper linear transformation. Therefore, it can be said that all modes of a non-defective system are orthogonal to one another. This essential property is used in the mathematical proof of Theorem 2.1.

2.2. PESM: A necessary and sufficient condition for $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$

A necessary and sufficient condition for $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$ to jointly satisfy in the PESM to prevent *spill-over* is presented in the following Theorem 2.1. Before that, two lemmas have to be introduced below.

Lemma 2.1. (*Sylvester's law of nullity* [33]). *For any two matrices*, $\mathbf{A}\in\mathbf{R}^{n\times p}$ *and* $\mathbf{B}\in\mathbf{R}^{p\times s}$, if $\mathbf{AB}=\mathbf{0}$, *then* $r(\mathbf{A})+r(\mathbf{B})\le p$.

Lemma 2.2. *If the rank of a matrix* $\mathbf{A}$ *is zero,* i.e., $r(\mathbf{A})=0$, *then matrix* $\mathbf{A}$ *is a null matrix*, i.e., $\mathbf{A}=\mathbf{0}$.
With the above two lemmas, Theorem 2.1 can be stated below.

Theorem 2.1. (*Eigenstructure preserving modification*). *Under assumptions* (A1), (A2) *and* (A3), *the PESM can avoid spill-over, i.e. scenario* (2) *of the PESM is satisfied, if and only if* $\Delta\mathbf{M}$ and $\Delta\mathbf{M}$ *jointly satisfy the following matrix equations:*

$$\Delta\mathbf{K}(\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}})-\Delta\mathbf{M}(\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}})=\mathbf{0} \tag{6a}$$

$$\Delta\mathbf{K}(\mathbf{I}_n-\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0)-\Delta\mathbf{M}(\mathbf{M}_0^{-1}\mathbf{K}_0-\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0)=\mathbf{0} \tag{6b}$$

Proof. Necessity: Using the normalisation of modes in (A1), it is easy to show that $\mathbf{X}^{-1}=\mathbf{X}^{\mathrm{T}}\mathbf{M}_0$ and $\mathbf{X}\mathbf{X}^{\mathrm{T}}=\mathbf{M}_0^{-1}$. Partition of the left-hand side of the previous matrix equation, that is, $\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}+\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}=\mathbf{M}_0^{-1}$, leads to

$$\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}=\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} \tag{7}$$

As $\mathbf{K}_0\mathbf{X}=\mathbf{M}_0\mathbf{X}\mathbf{\Lambda}$, one can get $\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{X}=\mathbf{X}\mathbf{\Lambda}$, $\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{X}\mathbf{X}^{\mathrm{T}}=\mathbf{X}\mathbf{\Lambda}\mathbf{X}^{\mathrm{T}}$ and $\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1}=\mathbf{X}_1\mathbf{\Lambda}\mathbf{X}_1^{\mathrm{T}}+\mathbf{X}_2\mathbf{\Lambda}\mathbf{X}_2^{\mathrm{T}}$. So it follows that

$$\mathbf{X}_2\mathbf{\Lambda}\mathbf{X}_2^{\mathrm{T}}=\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{\Lambda}\mathbf{X}_1^{\mathrm{T}} \tag{8}$$

From Eqs. (7) and (8), one can derive

$$\Delta\mathbf{K}\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}=\Delta\mathbf{K}(\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}) \tag{7a}$$

$$\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}\mathbf{X}_2^{\mathrm{T}}=\Delta\mathbf{M}(\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{\Lambda}\mathbf{X}_1^{\mathrm{T}}) \tag{8a}$$

It follows from Eqs. (7a)–(8a) that

$$(\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda})\mathbf{X}_2^{\mathrm{T}}=\Delta\mathbf{K}(\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}})-\Delta\mathbf{M}(\mathbf{M}_0^{-1}\mathbf{K}_0\mathbf{M}_0^{-1}-\mathbf{X}_1\mathbf{\Lambda}\mathbf{X}_1^{\mathrm{T}}) \tag{9}$$

Moreover, partial eigenstructure assignment of $\{\mathbf{M}_c,\mathbf{K}_c\}$ means $(\mathbf{K}_0+\Delta\mathbf{K})\mathbf{X}_2=(\mathbf{M}_0+\Delta\mathbf{M})\mathbf{X}_2\mathbf{\Lambda}_2$. Subtracting the eigenstructure equation $\mathbf{K}_0\mathbf{X}_2=\mathbf{M}_0\mathbf{X}_2\mathbf{\Lambda}_2$ of $\{\mathbf{M}_0,\mathbf{K}_0\}$ from the afore-mentioned matrix equation yields

$$\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2=\mathbf{0} \tag{10}$$

When Eq. (10) is used in (9), Eq. (6a) results (end of proof of the necessity).

In addition, multiplying matrix Eq. (6a) on the right by $\mathbf{M}_0$, matrix Eq. (6b) is obviously obtained.

Sufficiency: Suppose matrix Eq. (6a) holds. It follows from Eq. (9) that $(\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda})\mathbf{X}_2^{\mathrm{T}}=\mathbf{0}$. Let $\mathbf{A}=(\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2)\in\mathbf{R}^{n\times(n-m)}$ and $\mathbf{B}=\mathbf{X}_2^{\mathrm{T}}\in\mathbf{R}^{(n-m)\times n}$. Because all modes of $\{\mathbf{M}_0,\mathbf{K}_0\}$ are orthogonal to one another, that is, $\mathbf{X}_2$ contains $n-m$ orthogonal modes, obviously $r(\mathbf{B})=r(\mathbf{X}_2^{\mathrm{T}})=n-m$. Lemma 2.1 dictates that $r(\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2)+r(\mathbf{X}_2^{\mathrm{T}})\le n-m$. Therefore one can conclude that $r(\Delta\mathbf{K}\mathbf{X}_2-\Delta\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2)=0$. It can immediately be inferred from lemma 2.2 that the matrix formulation (10) is valid. Then it is easy to verify that

$$(\mathbf{K}_0+\Delta\mathbf{K})\mathbf{X}_2=(\mathbf{M}_0+\Delta\mathbf{M})\mathbf{X}_2\mathbf{\Lambda}_2$$

This completes the proof. □

In the next section, an algorithm for solving the partial eigenstructure assignment of undamped vibration systems is developed using acceleration and displacement feedback.

3. Partial eigenstructure assignment of undamped vibration systems

3.1. The problem description

Partial eigenstructure assignment of an undamped vibration system by applying a control force is presented in this section. Eq. (1) now becomes

$$\mathbf{M}_0\ddot{\mathbf{q}}(t)+\mathbf{K}_0\mathbf{q}(t)=\mathbf{B}\mathbf{u}(t) \tag{11}$$

where $\mathbf{B}\in\mathbf{R}^{n\times p}$ $(p<n)$ is known as the control matrix, and without loss of generality, $\mathbf{B}$ is assumed to have a full column rank, that is, $r(\mathbf{B})=p$. Suppose that system (11) is controllable. The control force vector $\mathbf{u}(t)\in\mathbf{R}^p$ is a time-dependent real vector and the following special form is taken:

$$\mathbf{u}(t)=-\mathbf{F}\ddot{\mathbf{q}}(t)-\mathbf{G}\mathbf{q}(t) \tag{12}$$

where $\mathbf{F}$ and $\mathbf{G} \in \mathbf{R}^{p \times n}$ are acceleration and displacement feedback gain matrices, respectively. Thus Eq. (11) may be written in the form of

$$(\mathbf{M}_0 + \mathbf{BF})\ddot{\mathbf{q}}(t) + (\mathbf{K}_0 + \mathbf{BG})\mathbf{q}(t) = \mathbf{0} \tag{13}$$

with the corresponding eigenstructure equation of the closed-loop system (13) as follows:

$$(\mathbf{K}_0 + \mathbf{BG})\mathbf{Y} = (\mathbf{M}_0 + \mathbf{BF})\mathbf{Y}\boldsymbol{\Sigma} \tag{14}$$

where $\boldsymbol{\Sigma} = \mathrm{diag}(\mu_1,\ \mu_2, \ldots, \mu_n)$ and $\mathbf{Y} = (\mathbf{y}_1,\ \mathbf{y}_2, \ldots, \mathbf{y}_n)$ are eigenvalues and the corresponding eigenvector matrix of the closed-loop system (13), respectively.

Let $\mathbf{M}_c = \mathbf{M}_0 + \mathbf{BF}$ and $\mathbf{K}_c = \mathbf{K}_0 + \mathbf{BG}$. The partial eigenstructure assignment (PESA) problem is to find matrices $\mathbf{F}$ and $\mathbf{G}$ such that the partial eigenstructure $\boldsymbol{\Lambda}_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mathbf{X}_1 = (\mathbf{x}_1,\ \mathbf{x}_2, \ldots, \mathbf{x}_m)$ of the open-loop system $\{\mathbf{M}_0, \mathbf{K}_0\}$ is replaced by the partial eigenstructure $\boldsymbol{\Sigma}_1 = \mathrm{diag}(\mu_1,\ \mu_2, \ldots, \mu_m)$ and $\mathbf{Y}_1 = (\mathbf{y}_1,\ \mathbf{y}_2, \ldots, \mathbf{y}_m)$ of the closed-loop system $\{\mathbf{M}_c, \mathbf{K}_c\}$, while the remaining n–m eigenpairs of $\{\mathbf{M}_0, \mathbf{K}_0\}$ are also eigenpairs of $\{\mathbf{M}_c, \mathbf{K}_c\}$. Let $\Delta\mathbf{M} = \mathbf{BF}$ and $\Delta\mathbf{K} = \mathbf{BG}$. Then it is easy to see that solving the PESA problem of undamped vibration systems includes solving the PESM problem in Section 2. In what follows the procedure to determine $\mathbf{F}$ and $\mathbf{G}$ solving the PESA problem is presented.

Substituting $\mathbf{F}$ and $\mathbf{G}$ into Eq. (6b) with $\Delta\mathbf{M} = \mathbf{BF}$ and $\Delta\mathbf{K} = \mathbf{BG}$ gives

$$\mathbf{BG}(\mathbf{I}_n - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) - \mathbf{BF}(\mathbf{M}_0^{-1}\mathbf{K}_0 - \mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) = \mathbf{0} \tag{15a}$$

Because $\mathbf{B}$ is assumed to be of full column rank, Eq. (15a) implies that

$$\mathbf{G}(\mathbf{I}_n - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) - \mathbf{F}(\mathbf{M}_0^{-1}\mathbf{K}_0 - \mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) = \mathbf{0} \tag{15b}$$

Let $\mathbf{F} = \mathbf{F}_1\mathbf{M}_0$ and $\mathbf{G} = \mathbf{G}_1\mathbf{M}_0$. Substituting them into Eq. (15b) gives

$$\mathbf{G}_1(\mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) - \mathbf{F}_1(\mathbf{K}_0 - \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0) = \mathbf{0} \tag{16}$$

Thus it can be deduced that $\mathbf{F} = \mathbf{F}_1\mathbf{M}_0$ and $\mathbf{G} = \mathbf{G}_1\mathbf{M}_0$ are a pair of solutions to the PESA problem without *spill-over* if and only if $\mathbf{F}_1$ and $\mathbf{G}_1$ satisfy Eq. (16).

Now the partial eigenstructure equation of $\{\mathbf{M}_c, \mathbf{K}_c\}$ that $\mathbf{F}$ and $\mathbf{G}$ must also satisfy is investigated. By definition,

$$(\mathbf{K}_0 + \mathbf{BG})\mathbf{Y}_1 = (\mathbf{M}_0 + \mathbf{BF})\mathbf{Y}_1\boldsymbol{\Sigma}_1 \tag{17}$$

Substituting $\mathbf{F} = \mathbf{F}_1\mathbf{M}_0$ and $\mathbf{G} = \mathbf{G}_1\mathbf{M}_0$ into Eq. (17) and rearranging it gives

$$\mathbf{BG}_1\mathbf{M}_0\mathbf{Y}_1 - \mathbf{BF}_1\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1 = \mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1 - \mathbf{K}_0\mathbf{Y}_1 \tag{18}$$

Eqs. (16) and (18) are two key matrix equations used to solve the PESA problem.

3.2. Assignment algorithm

In what follows, the general solutions of Eq. (16) for $\mathbf{F}_1$ and $\mathbf{G}_1$ are explored. Eq. (16) is rewritten as

$$(\mathbf{G}_1,\ \mathbf{F}_1)\begin{pmatrix} \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \\ -\mathbf{K}_0 + \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \end{pmatrix} = \mathbf{0} \tag{19}$$

To this end, ranks of several matrices must be known. It can be proved that

$$r\begin{pmatrix} \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \\ -\mathbf{K}_0 + \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \end{pmatrix}_{2n \times n} = n - m, \text{ when } \mathbf{K}_0 > \mathbf{0} \tag{20}$$

With $\mathbf{K}_0\mathbf{X}_1 = \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1$, one gets

$$\begin{pmatrix} \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \\ -\mathbf{K}_0 + \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \\ -\mathbf{K}_0 + \mathbf{K}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_0 \end{pmatrix}\begin{pmatrix} \mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} \\ -(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}) \end{pmatrix}\mathbf{M}_0.$$

Note that $\mathbf{K}_0 > \mathbf{0}$, then $\begin{pmatrix} \mathbf{M}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_0 \end{pmatrix}$ is invertible, and therefore

$$r\begin{pmatrix} \mathbf{M}_0 - \mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \\ -\mathbf{K}_0 + \mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0 \end{pmatrix} = r\begin{pmatrix} \mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} \\ -(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}) \end{pmatrix},$$

Note also that $\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} = \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}$, and $r(\mathbf{X}_2) = r(\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}) = n - m$ [33]. Thus $r(\mathbf{M}_0^{-1} - \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}) = n - m$. Together with lemma 3.1 below, it is easy to see that Eq. (20) holds.

Lemma 3.1. *If* $r(\mathbf{A}) = s,\ \ \mathbf{A} \in \mathbf{R}^{n \times n}$, *then* $r\begin{pmatrix} \mathbf{A} \\ -\mathbf{A} \end{pmatrix} = s$.

Proof.

$$r\begin{pmatrix}\mathbf{A}\\-\mathbf{A}\end{pmatrix}=r\begin{pmatrix}\mathbf{PAQ}\\-\mathbf{PAQ}\end{pmatrix}=r\begin{pmatrix}\mathbf{I}_s&\mathbf{0}\\\mathbf{0}&\mathbf{0}\\-\mathbf{I}_s&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}=r\left(\begin{matrix}\mathbf{I}_s\\\begin{pmatrix}-\mathbf{I}_s\\\mathbf{0}\\\mathbf{0}\end{pmatrix}\end{matrix}\quad\begin{matrix}\mathbf{0}\\\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}\end{matrix}\right),$$

where $\mathbf{P}$ and $\mathbf{Q}$ are invertible matrices. Note that the rank does not change if rows of a matrix are interchanged. Moreover, it is well known that [33]

$$r\begin{pmatrix}\mathbf{S}&\mathbf{U}\\\mathbf{V}&\mathbf{0}\end{pmatrix}=r(\mathbf{S})+r(\mathbf{VS}^{-1}\mathbf{U})$$

if $\mathbf{S}$ is square and nonsingular. Thus

$$r\begin{pmatrix}\mathbf{A}\\-\mathbf{A}\end{pmatrix}=r\left(\begin{matrix}\mathbf{I}_s\\\begin{pmatrix}-\mathbf{I}_s\\\mathbf{0}\\\mathbf{0}\end{pmatrix}\end{matrix}\quad\begin{matrix}\mathbf{0}\\\begin{pmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{pmatrix}\end{matrix}\right)=r(\mathbf{I}_s)=s,$$

which completes the proof. □

Now it is time to attempt to solve Eq. (19). Using QR decomposition, one gets

$$\begin{pmatrix}\mathbf{M}_0-\mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0\\-\mathbf{K}_0+\mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0\end{pmatrix}=\mathbf{Q}\begin{pmatrix}\mathbf{P}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}=(\mathbf{Q}_1,\ \mathbf{Q}_2)\begin{pmatrix}\mathbf{P}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}\tag{21}$$

where $(\mathbf{Q}_1,\mathbf{Q}_2)$ is an orthogonal matrix with $\mathbf{Q}_1\in\mathbf{R}^{2n\times(n-m)}$ and $\mathbf{Q}_2\in\mathbf{R}^{2n\times(n+m)}$, and $\mathbf{P}\in\mathbf{R}^{(n-m)\times(n-m)}$ is upper triangular and nonsingular. Let

$$\mathbf{U}=(\mathbf{U}_1,\mathbf{U}_2)=(\mathbf{G}_1,\ \mathbf{F}_1)\mathbf{Q}\tag{22}$$

where $\mathbf{U}_1\in\mathbf{R}^{p\times(n-m)}$, $\mathbf{U}_2\in\mathbf{R}^{p\times(n+m)}$. It follows from Eq. (19) that

$$\begin{aligned}(\mathbf{G}_1,\mathbf{F}_1)\begin{pmatrix}\mathbf{M}_0-\mathbf{M}_0\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0\\-\mathbf{K}_0+\mathbf{M}_0\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{X}_1^{\mathrm{T}}\mathbf{M}_0\end{pmatrix}&=(\mathbf{G}_1,\mathbf{F}_1)\mathbf{Q}\begin{pmatrix}\mathbf{P}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}\\&=(\mathbf{U}_1,\mathbf{U}_2)\begin{pmatrix}\mathbf{P}&\mathbf{0}\\\mathbf{0}&\mathbf{0}\end{pmatrix}=\mathbf{0}\end{aligned}\tag{23}$$

Eq. (23) indicates that $\mathbf{U}_1=\mathbf{0}$. Furthermore,

$$(\mathbf{G}_1,\mathbf{F}_1)=(\mathbf{U}_1,\mathbf{U}_2)\mathbf{Q}^{\mathrm{T}}=(\mathbf{U}_1,\mathbf{U}_2)\begin{pmatrix}\mathbf{Q}_1^{\mathrm{T}}\\\mathbf{Q}_2^{\mathrm{T}}\end{pmatrix}=\mathbf{U}_2\mathbf{Q}_2^{\mathrm{T}}\tag{24}$$

where $\mathbf{U}_2$ is to be determined below.

At this point, the general solutions of Eqs. (19) or (16) are obtained. It should be noted that when $\mathbf{K}_0=0$, and the rank determination of the matrix in (20) is not straightforward, then the more expensive singular value decompositions (SVD) could be used in (21) instead of the QR decomposition, and the general solutions (24) stay unchanged.

Secondly, the solutions of Eq. (18) for $\mathbf{F_1}$ and $\mathbf{G_1}$ are sought below. Eq. (18) is rewritten as

$$\mathbf{B}(\mathbf{G}_1,\mathbf{F}_1)\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\-\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1\end{pmatrix}=\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1\tag{25}$$

Substituting the QR decomposition of $\mathbf{B}$, i.e. $\mathbf{B}=(\mathbf{V}_0,\mathbf{V}_1)\begin{pmatrix}\mathbf{Z}\\\mathbf{0}\end{pmatrix}$, where $(\mathbf{V}_0,\mathbf{V}_1)\in\mathbf{R}^{n\times n}$ is an orthogonal matrix with $\mathbf{V}_0\in\mathbf{R}^{n\times p}$ and $\mathbf{V}_1\in\mathbf{R}^{n\times(n-p)}$, and $\mathbf{Z}\in\mathbf{R}^{p\times p}$ is upper triangular and nonsingular, into Eq. (25) gives

$$\begin{pmatrix}\mathbf{Z}\\\mathbf{0}\end{pmatrix}(\mathbf{G}_1,\mathbf{F}_1)\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\-\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1\end{pmatrix}=\begin{pmatrix}\mathbf{V}_0^{\mathrm{T}}\\\mathbf{V}_1^{\mathrm{T}}\end{pmatrix}(\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1),$$

that is

$$\mathbf{Z}(\mathbf{G}_1,\ \mathbf{F}_1)\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\-\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1\end{pmatrix}=\mathbf{V}_0^{\mathrm{T}}(\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1),\tag{26}$$

$$\mathbf{0}=\mathbf{V}_1^{\mathrm{T}}(\mathbf{M}_0\mathbf{Y}_1\boldsymbol{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1).\tag{27}$$

As is known, for the given assigned eigenvalues $\mathbf{\Sigma}_1$ of $\{\mathbf{M}_c,\mathbf{K}_c\}$, not any given corresponding eigenvectors $\mathbf{Y}_1$ can be assigned to $\{\mathbf{M}_c,\mathbf{K}_c\}$ such that there exist $\mathbf{F_1}$ and $\mathbf{G_1}$ satisfying Eq. (25) of the PESA problem. From Eq. (27), the following Theorem 3.1 indicates which eigenvectors $\mathbf{Y}_1$ corresponding to $\mathbf{\Sigma}_1$ would be "achievable" for the PESA problem.

Theorem 3.1. *Given eigenvalues* $\mathbf{\Sigma}_1=\mathrm{diag}(\mu_1,\ \mu_2,\ldots,\mu_m)$, *there exist* $\mathbf{F}_1$ *and* $\mathbf{G}_1$ *satisfying Eq.* (25) *of the PESA problem, if and only if the given corresponding eigenvectors* $\mathbf{Y}_1=(\mathbf{y}_1,\ \mathbf{y}_2,\ldots,\mathbf{y}_m)$ *satisfy Eq.* (27), *or in other words,* $\mathbf{y}_i, i=1,\ 2,\ldots,m$ *must belongs to the right null space of matrix* $\mathbf{V}_1^{\mathrm{T}}(\mu_i\mathbf{M}_0-\mathbf{K}_0)$, i.e. $\mathbf{y}_i\in N\{\mathbf{V}_1^{\mathrm{T}}(\mu_i\mathbf{M}_0-\mathbf{K}_0)\}$.

It should be noted that the similar requirements on $\mathbf{Y}_1$ were given in [34,16] for the eigenstructure assignment of linear descriptor systems and time-invariant second-order control systems, respectively.

In order to ensure the accuracy of partial eigenstructure assignment, it is necessary to 'condition' the given $\mathbf{Y}_1$ according to Theorem 3.1. In what follows the conditioning Algorithm 3.1 is presented.

Conditioning algorithm 3.1.

(i) Solve the orthogonal basis vectors of $N\{\mathbf{V}_1^{\mathrm{T}}(\mu_i\mathbf{M}_0-\mathbf{K}_0)\}$, which make up a full column rank matrix $\mathbf{S}_i, i=1,\ 2,\ldots,m$.
(ii) Solve the linear system of equations $\mathbf{S}_i\mathbf{v}_i=\mathbf{y}_i$, which is usually over-determined. So $\mathbf{v}_i^*=\mathbf{S}_i^+\mathbf{y}_i=(\mathbf{S}_i^{\mathrm{T}}\mathbf{S}_i)^{-1}\mathbf{S}_i^{\mathrm{T}}\mathbf{y}_i$ gives a unique least square approximation of $\mathbf{y}_\mathrm{i}$ for $\mathbf{y}_i^*=\mathbf{S}_i\mathbf{v}_i^*, i=1,\ 2,\ldots,m$. Then $\mathbf{Y}_1^*=(\mathbf{y}_1^*,\ \mathbf{y}_2^*,\ldots,\mathbf{y}_m^*)$ is taken to replace $\mathbf{Y}_1$ in the partial eigenstructure assignment of Eq. (25) with the assigned eigenvalues $\mathbf{\Sigma}_1=\mathrm{diag}(\mu_1,\ \mu_2,\ldots,\mu_m)$, i.e., as if $\mathbf{\Sigma}_1$ and $\mathbf{Y}_1^*$ are the assigned partial eigenstructure.

Before the final solutions of $\mathbf{F}_1$ and $\mathbf{G}_1$ from Eq. (26) are presented, the solutions of the following matrix equation are discussed, as they are used subsequently:

$$\mathbf{AXC}=\mathbf{D} \tag{28}$$

where $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{D}$ are known matrices. Eq. (28) has a special solution as follows [35]:

$$\mathbf{X}=\mathbf{A}^+\mathbf{DC}^+ \tag{29}$$

When Eq. (28) is consistent, the solution is a unique minimal norm solution of Eq. (28); when Eq.(28) is inconsistent, the solution is a unique minimal norm least square solution.

Now the solution of Eq. (26) can be sought. Substituting the general solutions of Eq. (19), i.e. Eq. (24), into (26) gives

$$\mathbf{ZU}_2\mathbf{Q}_2^{\mathrm{T}}\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\ -\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1\end{pmatrix}=\mathbf{V}_0^{\mathrm{T}}(\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1). \tag{30}$$

A solution of Eq. (30) for matrix $\mathbf{U}_2$ is obtained using Eq. (29) as follows:

$$\mathbf{U}_2=\mathbf{Z}^{-1}\mathbf{V}_0^{\mathrm{T}}(\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1-\mathbf{K}_0\mathbf{Y}_1)\left(\mathbf{Q}_2^{\mathrm{T}}\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\ -\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1\end{pmatrix}\right)^+ \tag{31}$$

Then substituting the obtained $\mathbf{U}_2$ back into Eq. (24), feedback matrices $\mathbf{F}$ and $\mathbf{G}$ are eventually determined, which solve the PESA problem as:

$$(\mathbf{G},\ \mathbf{F})=\mathbf{U}_2\mathbf{Q}_2^{\mathrm{T}}\mathbf{M}_0 \tag{32}$$

Note that matrix $\mathbf{Q}_2^{\mathrm{T}}\begin{pmatrix}\mathbf{M}_0\mathbf{Y}_1\\ -\mathbf{M}_0\mathbf{Y}_1\mathbf{\Sigma}_1\end{pmatrix}$ is usually of full column rank.

Based on the above discussion, the following algorithm is developed for solving the PESA problem.

Algorithm 3.2.

Inputs: The analytical mass and stiffness matrices $\mathbf{M}_0$ and $\mathbf{K}_0$, and the eigenpairs $\mathbf{\Lambda}_1$ and $\mathbf{X}_1$ to be assigned to $\{\mathbf{M}_0,\mathbf{K}_0\}$, the assigned eigenpairs $\mathbf{\Sigma}_1$ and $\mathbf{Y}_1$ of $\{\mathbf{M}_c,\mathbf{K}_c\}$.
Output: Feedback matrices $\mathbf{F}$ and $\mathbf{G}$.
(1) Solve the QR decomposition of Eq. (21) and obtain $\mathbf{Q}_2^{\mathrm{T}}$.
(2) Solve the QR decomposition of the control matrix $\mathbf{B}=(\mathbf{V}_0,\mathbf{V}_1)\begin{pmatrix}\mathbf{Z}\\ \mathbf{0}\end{pmatrix}$.
(3) Use Algorithm 3.1 to condition $\mathbf{Y}_1$ and obtain the corresponding $\mathbf{Y}_1^*$ that is substituted into Eq. (31).
(4) Solve $\mathbf{U}_2$ from Eq. (31), and then substitute the obtained $\mathbf{U}_2$ into (32).

Finally, some important features of Algorithm 3.2 are summarised below.

(a) Algorithm 3.2 gives the acceleration and displacement feedback matrices to the PESA problem of undamped vibration systems, which is rarely seen in the published literature. Accelerometers have some good properties, such as being easy to install, wide frequency range and high sensitivity, and are thus widely used in practice.

(b) Algorithm 3.2 is easy to implement, and its steps are simple and clear. It works directly on the second-order system model, and can be implemented with the knowledge of only those few eigenvalues and the corresponding eigenvectors of $\{\mathbf{M}_0,\mathbf{K}_0\}$ to be assigned and $\{\mathbf{M}_c,\mathbf{K}_c\}$. More importantly, this algorithm can accurately solve the PESA problem, and can also be used to solve the partial eigenvalue assignment with a sparse control matrix **B**.

(c) Because the transformations (or decompositions) in steps (1) and (2) of Algorithm 3.2 are all of orthogonal type, and step (4) gives the minimal norm solution of $\mathbf{U}_2$, the presented algorithm leads naturally to a small norm solutions for **F** and **G**. The results shown in the numerical examples in the next section are comparable in the magnitude of the norms of the control gain matrices with those obtained by an expensive optimisation procedure.

(d) Control matrix **B** in [15] cannot be prescribed beforehand but must be determined during the process of solving the velocity and displacement feedback gain matrices, and **B** thus obtained is usually a dense matrix. This kind of **B** would be difficult to realise in practice. In contrast, **B** in this paper can be prescribed in a simple form (see for example, **B** used in Example 4.1 in Section 4). Additionally, its solution in [15] is not unique, and the norms of the gain matrices incorporating **B** are often much bigger. However, the presented algorithm does not have these problems when solving the PESA problem.

4. Numerical examples

To demonstrate the performance of the present algorithm, three numerical examples are analysed in this section, using MATLAB 7.11.

Example 4.1. In this example, $n=6$, $m=p=3$, and

$$\mathbf{M}_0=\begin{pmatrix} 1.56 & 0.66 & 0.54 & -0.39 & 0 & 0\\ 0.66 & 0.36 & 0.39 & -0.27 & 0 & 0\\ 0.54 & 0.39 & 3.12 & 0 & 0.54 & -0.39\\ -0.39 & -0.27 & 0 & 0.72 & 0.39 & -0.27\\ 0 & 0 & 0.54 & 0.39 & 3.12 & 0\\ 0 & 0 & -0.39 & -0.27 & 0 & 0.72 \end{pmatrix},$$

$$\mathbf{K}_0=\begin{pmatrix} 12 & 18 & -12 & 18 & 0 & 0\\ 18 & 36 & -18 & 18 & 0 & 0\\ -12 & -18 & 24 & 0 & -12 & 18\\ 18 & 18 & 0 & 72 & -18 & 18\\ 0 & 0 & -12 & -18 & 24 & 0\\ 0 & 0 & 18 & 18 & 0 & 72 \end{pmatrix},\quad \mathbf{B}=\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$

The open-loop eigenvalues are $\mathbf{\Lambda}=\mathrm{diag}(0.036346,\ 1.4365,\ 11.4697,\ 58.1668,\ 206.023,\ 818.8383)$ with $\mathbf{\Lambda}_1$

$$=\mathrm{diag}(0.036346, 1.4365,\ 11.4697),\quad \mathbf{\Sigma}_1=\mathrm{diag}(0.05, 1.80, 12.0).$$

Eigenvector matrices $\mathbf{X}$, $\mathbf{Y}_1$ and $\mathbf{Y}_1^*$ are listed in Table 1.
The assignment results are provided as follows (Tables 2 and 3).

Table 1
Eigenvector matrices $\mathbf{X}$, $\mathbf{Y}_1$ and $\mathbf{Y}_1^*$.

X					
1.0000	1.0000	1.0000	**0.7039**	**0.4525**	**0.3001**
−0.1529	−0.5317	−0.8832	**−1.0000**	**−1.0000**	**−1.0000**
0.5469	−0.4235	−0.6561	**−0.0819**	**0.2105**	**0.0565**
−0.1454	−0.3288	0.1410	**0.7738**	**0.4879**	**−0.3469**
0.1655	−0.5899	0.7440	**−0.1773**	**−0.1195**	**0.0361**
−0.1005	0.1960	0.1847	**−0.7552**	**0.7418**	**−0.1234**
$\mathbf{Y}_1$			$\mathbf{Y}_1^*$		
1.0000	1.0000	1.0000	**1.0000**	**1.0000**	**1.0000**
−0.0152	−0.1317	−0.3832	**−0.0312**	**−0.2149**	**−0.7661**
0.6469	−0.3235	−0.5561	**0.6878**	**−0.2187**	**−0.7466**
−0.2454	−0.4288	0.2410	**−0.1563**	**−0.4360**	**0.0829**
0.2655	−0.3899	0.5440	**0.2342**	**−0.6176**	**0.8050**
−0.2005	0.2960	0.2847	**−0.1103**	**0.2460**	**0.3105**

Table 2
The closed-loop eigenvalues and eigenvectors.

μ_i	$\mathbf{y}_i$					
0.05	**1.0000**	**1.0000**	**1.0000**	**0.7039**	**0.4525**	**0.3001**
1.8	**−0.0312**	**−0.2149**	**−0.7661**	**−1.0000**	**−1.0000**	**−1.0000**
12	**0.6878**	**−0.2187**	**−0.7466**	**−0.0819**	**0.2105**	**0.0565**
58.1668	**−0.1563**	**−0.4360**	**0.0829**	**0.7738**	**0.4879**	**−0.3469**
206.023	**0.2342**	**−0.6176**	**0.8050**	**−0.1773**	**−0.1195**	**0.0361**
818.8383	**−0.1103**	**0.2460**	**0.3105**	**−0.7552**	**0.7418**	**−0.1234**

Table 3
Feedback matrices **G** and **F** and their norms.

G						$\Vert\mathbf{G}\Vert_F$
−0.1506	−0.0752	−0.1767	0.0504	0.0043	0.0108	1.9973
−0.0218	−0.0138	−0.1173	−0.0156	−0.1147	0.0178	
−1.2870	−0.6198	−0.7930	0.6082	0.9264	−0.0348	
F						$\Vert\mathbf{F}\Vert_F$
0.0144	−0.0043	−0.1448	−0.0126	−0.0333	0.0294	2.2831
−0.0347	−0.0166	0.0195	0.0402	0.1566	−0.0080	
0.3923	0.0754	−1.5978	−0.4168	−1.4662	0.3539	

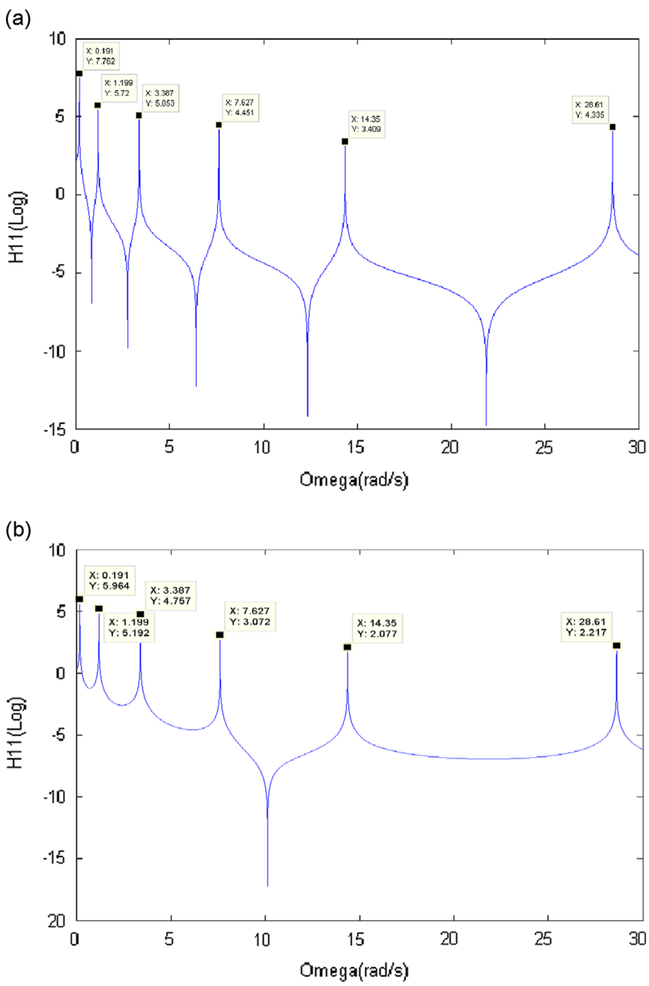

Fig. 1. Receptances of the open-loop system. (a) H_{11} and (b) H_{15}.

The F-norms of the closed-loop eigenstructure equations are $\|\mathbf{M}_c\mathbf{Y}_1^*\mathbf{\Sigma}_1 - \mathbf{K}_c\mathbf{Y}_1^*\|_F = 3.0257\text{e}-014$, $\|\mathbf{M}_c\mathbf{X}_2\mathbf{\Lambda}_2 - \mathbf{K}_c\mathbf{X}_2\|_F = 5.5639\text{e}-013$.

For this numerical example, the open-loop and closed-loop natural frequencies are (0.1906, 1.1985, 3.3867, 7.6267, 14.3535, 28.6154) and (0.2236, 1.3416, 3.4641, 7.6267, 14.3535, 28.6154) rad/s, respectively. Figs. 1 and 2 depict the frequency response functions H_{11} and H_{15} of the open-loop and closed-loop systems, which visually confirm the excellent realisation of the eigenvalue assignment. The X-values of the peak points in the figures correspond to the natural frequencies.

Example 4.2. (P 5.2 in [12]). In [12], the minimal norm solutions of gain matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ for the velocity and displacement feedback were determined. In this example, $n=3$, $m=p=2$, and

$$\mathbf{M}_0 = 10\mathbf{I}_3\,,\mathbf{K}_0 = \begin{pmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{pmatrix},\quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix}$$

The open-loop eigenvalues are $\mathbf{\Lambda} = \text{diag}(0.79225,\ 6.2198,\ 12.9879)$. $\mathbf{\Lambda}_1 = \text{diag}(0.79225,\ 6.2198)$ and $\mathbf{\Sigma}_1 = \text{diag}(1.0,\ 2.0)$ are chosen, which are the same as those in [12]. The eigenvector matrices $\mathbf{X}$, $\mathbf{Y}_1$ and $\mathbf{Y}_1^*$, and some other results are not listed for the sake of saving space. The present algorithm also accurately implements the partial eigenstructure assignment here, as can be seen in Table 4 for feedback matrices $\mathbf{G}$ and $\mathbf{F}$, and their norms. The minimal norms of feedback matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ in [12] are also given for comparison.

$$\|\mathbf{M}_c\mathbf{Y}_1^*\mathbf{\Sigma}_1 - \mathbf{K}_c\mathbf{Y}_1^*\|_F = 6.9078\text{e}-014,\quad \|\mathbf{M}_c\mathbf{X}_2\mathbf{\Lambda}_2 - \mathbf{K}_c\mathbf{X}_2\|_F = 8.3167\text{e}-014.$$

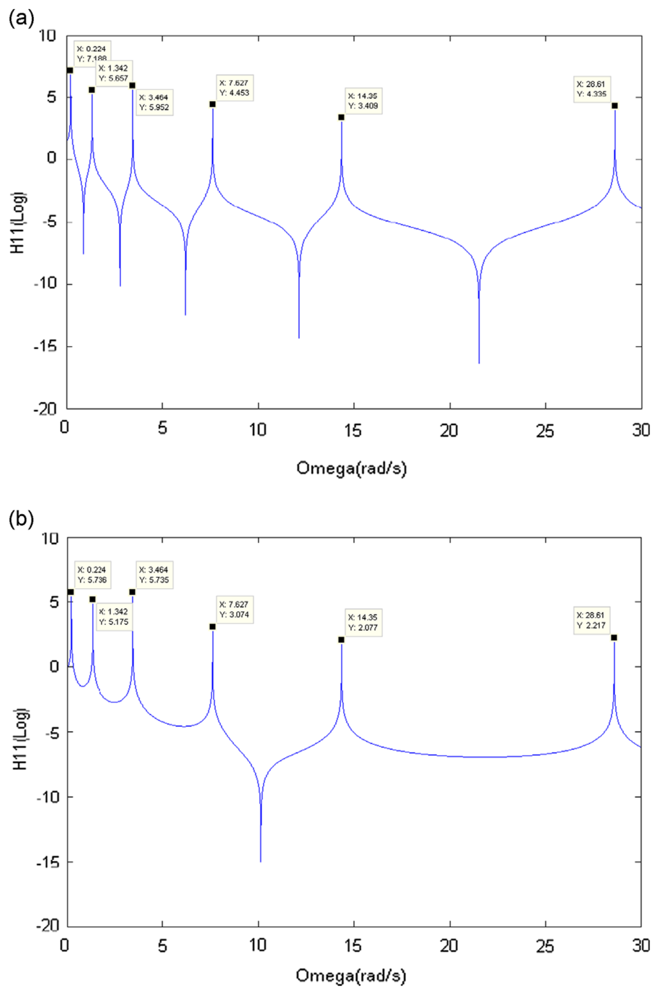

Fig. 2. Receptances of the closed-loop system. (a) H_{11} and (b) H_{15}.

Table 4
Feedback matrices **G** and **F** and its norm.

G			$\|G\|_F$	$\|\mathbf{F}_2\|_F$[12]
−4.6075 4.3028	−12.5875 11.1973	−4.9483 4.3457	19.1556	70.89
F			$\|\mathbf{F}\|_F$	$\|\mathbf{F}_1\|_F$ [12]
0.8254 −1.3951	9.2954 −8.2487	11.7638 −10.0683	19.9207	19.36

Example 4.3. (P 5.5 in [12]). In this example, $n=20$, $m=2$, $p=3$, and

$$\mathbf{M}_0=\mathbf{I}_{20}\,,\mathbf{K}_0=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}, \quad \mathbf{B}=\begin{pmatrix} \mathbf{I}_3 \\ 0 \end{pmatrix}$$

For the sake of comparison with P 5.5 in [12], $\mathbf{\Sigma}_1=\mathrm{diag}(\sqrt{10},\ \sqrt{20})$ is chosen to replace $\mathbf{\Lambda}_1=\mathrm{diag}(0.0058684,\ 0.052609)$, which is the same as P 5.5 in [12] as far as assigned undamped natural frequencies are concerned. In the same way, the present algorithm accurately finds the partial eigenstructure assignment needed, while keeping the other frequencies of the open-loop system unchanged with $\|\mathbf{M}_c\mathbf{Y}_1^*\mathbf{\Sigma}_1-\mathbf{K}_c\mathbf{Y}_1^*\|_F=4.3414\text{e}{-015}$ and $\|\mathbf{M}_c\mathbf{X}_2\mathbf{\Lambda}_2-\mathbf{K}_c\mathbf{X}_2\|_F=2.1161\text{e}{-014}$. Additionally, the present algorithm allows selective assignment of some eigenvalues without changing the remainder of the frequency spectrum in this example. The F-norms of feedback matrices **G** and **F** are $\|\mathbf{G}\|_F=2.4257$ and $\|\mathbf{F}\|_F=1.8188$, respectively. In [12], the velocity feedback is used so as to increase the system damping and to improve system response. It turned out that the F-norms of feedback matrices $\mathbf{F}_1$ and $\mathbf{F}_2$ are very high: $\|\mathbf{F}_1\|_F=1602$ and $\|\mathbf{F}_2\|_F=8838$.

From all these three examples, it can be seen that the resulting assigned frequencies and modes are quite accurate.

5. Conclusions

A partial eigenstructure modification formulation for the incremental mass and stiffness matrices to be satisfied is proposed. The formulation is successfully used to develop a partial eigenstructure assignment algorithm for undamped vibration systems. This algorithm can accurately assign prescribed eigenpairs while keeping other unassigned eigenpairs unchanged using acceleration and displacement feedback. The algorithm mainly involves numerically stable matrix computations, such as QR decomposition, and only need those few eigenpairs to be assigned and the analytical mass and stiffness matrices of the open-loop vibration system. The norms of feedback gain matrices obtained are found to be small.

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