

# EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE

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**ABSTRACT.** We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line  $\mathbb{R}_+$ . Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform  $\varphi$ -Lipschitz condition, in which the Lipschitz function  $\varphi$  belongs to wide classes of admissible function spaces such as  $L_p$ ,  $1 \leq p \leq \infty$ ,  $L_{p,q}$ , etc.

## 1. INTRODUCTION AND PRELIMINARIES

The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear non-autonomous differential-algebraic equations (DAEs) of the form

$$\begin{array}{c} d \text{ rows} \\ a \text{ rows} \end{array} \underbrace{\begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}}_{E(t)} \dot{x}(t) = \underbrace{\begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}}_{A(t)} x(t) + \underbrace{\begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}}_{f(t, x(t))}, \quad t \in \mathbb{R}_+ := [0, +\infty). \quad \text{semi linDAE} \quad (1.1)$$

To do that, we start by investigating the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \quad \text{linDAE} \quad (1.2)$$

Here  $E = \begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}$  are matrix-valued functions acting on  $\mathbb{R}_+$  to  $\mathbb{R}^{n,n}$ ,  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , and  $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Furthermore, we assume that for all  $t$ , the matrices  $\mathbf{E}_1(t)$ ,  $\mathbf{A}_2(t)$  have full row rank.

DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuits, chemical engineering, and many other applications. Due to the rank-deficiency of  $E(t)$ , the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the reader to recent monographs [2, 12–14] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [12, Chapter 2]), there are only few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [15] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [17] for Lyapunov and other spectra for linear DAEs, to [4, 8] for the robustness of exponential stability and Bohl exponents.

On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is characterized, the next important question in the qualitative theory of DAEs is to study the existence of integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1) [3, 6]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps, this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1).

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Our method is based on the classical "Lyapunov-Perron method" ([6, 25]) and the admissibility of function spaces ([10, 11]).

The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for later use, including the notion of the exponential dichotomy and its properties, as well as some important features of admissible function spaces. In Section 2 we give a characterization for the existence of exponential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations, and we conclude this research by a summary and some open problems.

**1.1. Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By  $(\mathbb{R}^n, \|\cdot\|)$  we denote the  $n$ -dimensional real vector space equipped with the Euclidean norm. For any matrix  $V$ , by  $V^T$  we denote its transpose. For any  $p \in \mathbb{N}$ , by  $C^p([0, \infty), \mathbb{R}^n)$  we denote the space of  $p$ -times continuously differentiable functions acting on  $[0, \infty)$  with values in  $\mathbb{R}^n$ . By  $C_b([0, \infty), \mathbb{R}^n)$  we denote the space of continuous and bounded functions mapping from  $[0, \infty)$  into  $\mathbb{R}^n$ . This space is a Banach space with the *ess sup*-norm  $\|f\|_\infty := \sup\{\|f(t)\|, t \geq 0\}$ .

It is well-known (e.g. [3]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \geq s \geq 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{eq3} \tag{1.3}$$

is well-posed, then there exists a pointwise nonsingular matrix-valued function  $(t, s) \mapsto X(t, s) \in \mathbb{R}^{n,n}$  such that the solution of (1.3) is given by  $x(t) = X(t, s)x_s$ . This fact motivates the existence of an evolution family  $(X(t, s))_{t \geq s \geq 0}$  associated with the matrix function  $A(t)$ . This family satisfies the condition  $X(t, t) = Id$  and the so-called *semi-group property*

$$X(t, r)X(r, s) = X(t, s), \quad \text{for all } t \geq r \geq s \geq 0. \tag{semigroup prop} \tag{1.4}$$

Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), \quad \text{for all } t \geq s \geq 0,$$

also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \tau)f(\tau, x(\tau))d\tau, \quad \text{for all } t \geq s \geq 0. \tag{variational form} \tag{1.5}$$

For more details on the notion and discussion on properties and applications of evolution families we refer the readers to Pazy [21].

**Definition 1.1.** A given evolution family  $\{X(t, s)\}_{t \geq s \geq 0}$  of the ODE (1.3) is said to have an *exponential dichotomy* on the half-line if there exist a family of projection matrices  $\{P(t)\}_{t \geq 0}$  and two positive constants  $N, \nu$  such that the following conditions are satisfied.

- i)  $P(t)X(t, s) = X(t, s)P(s)$  for all  $t \geq s \geq 0$ ,
- ii) for all  $t \geq s \geq 0$ , the restriction  $X(t, s)|_{\ker P(s)} : \ker P(s) \rightarrow \ker P(t)$  is an isomorphism, and we denote its inverse by  $X(s, t)|$ ,
- iii)  $\|X(t, s)P(s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$ , for all  $t \geq s \geq 0$ ,  $x \in \mathbb{R}^n$ ,
- iv)  $\|X(t, s)|_{(I - P(s))x}\| \leq Ne^{\nu(t-s)}\|(I - P(s))x\|$ , for all  $s \geq t \geq 0$ ,  $x \in \mathbb{R}^n$ .

Here  $\{P(t)\}_{t \geq 0}$  (reps.  $N, \nu$ ) are called *dichotomy projections* (resp. *dichotomy constants*).

The concept exponential dichotomy means that the state space  $\mathbb{R}^n$  has been splitted into the (exponentially) stable subspace ( $\text{Im}(P(t))$ ) and the (exponentially) unstable subspace ( $\text{ker}(P(t))$ ).

**1.2. A short review of DAE solvability theory.** Linear DAEs of the form 1.2 have been extensively studied in the last thirty years, see [13] and the references therein. In order to understand the solution behavior and to obtain numerical solutions, the necessary information about derivatives of equations has to be utilized. This has led to the concept of the strangeness index, which allows to use the DAE and (some of) its derivatives to be reformulated as a system with the same solution that is *strangeness-free*, i.e., for which the algebraic and differential part of the system are easily separated. In this paper we restrict ourselves to regular DAEs with sufficiently smooth coefficients, i.e., we require that (1.2) (or the nonlinear DAE (1.1) locally) has a unique solution for appropriately chosen (consistent) initial conditions, see [13] for a discussion of more general nonregular DAEs. With the theory and appropriate numerical methods available, then throughout this paper, for regular DAEs we may assume that the homogeneous DAE 1.2 in consideration fulfills the following presumption.

**Assumption 1.2.** Assume that the function pair  $(E, A)$  in the DAEs (1.1), (1.2) is *strangeness-free*, i.e.,

$$\text{rank} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} = n,$$

for all  $t \geq 0$ . Furthermore, we assume that  $E \in C^1([0, \infty), \mathbb{R}^{n,n})$  and  $A \in C^0([0, \infty), \mathbb{R}^{n,n})$ .

**Definition 1.3.** The DAE

$$\tilde{E}(t)\dot{y}(t) = \tilde{A}(t)x(t) + \tilde{f}(t, y(t)) \quad (1.6)$$

is called *orthogonally equivalent* to the DAE (1.1) if there exist pointwise-orthogonal matrix-valued functions  $U \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that after changing variable  $x(t) = V(t)y(t)$ , and scaling (1.2) with  $U(t)$ , we obtain exactly (1.19). In details, this means that the following identities hold true.

$$\tilde{E} = UEV, \quad \tilde{A} = UAV - UE\dot{V}, \quad \tilde{f}(t, y(t)) = U(t)f(t, Vy(t)), \quad \text{for all } t \geq 0. \quad (1.7)$$

We denote this orthogonal equivalence by  $(E, A, f) \sim (\tilde{E}, \tilde{A}, \tilde{f})$  and omit the terms  $f, \tilde{f}$  if the homogeneous system (1.2) is considered.

Indeed, one can directly verify that this orthogonal equivalence concept is an equivalent relation, i.e., it fulfills three properties: reflexivity, symmetry and transitivity. We omit the detailed proof here in order to keep the brevity of this work. By making use of some smooth factorizations, for example QR or SVD ([7] or [13, Thm 3.9]), we can decouple the differential and algebraic parts of the DAE (1.2) in the following lemma.

**Lemma 1.4.** Consider the DAE (1.2) and assume that it satisfies Assumption 1.2. Then, there exist pointwise-orthogonal matrix-valued functions  $U \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that by changing variable  $x(t) = V(t)y(t)$ , and scaling (1.2) with  $U(t)$ , we obtain the so-called semi-explicit system

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{semi-explicit system} \quad (1.8)$$

with pointwise nonsingular matrix-valued functions  $\Sigma(t) \in \mathbb{R}^{d,d}$  and  $A_4(t) \in \mathbb{R}^{a,a}$ .

*Proof.* Applying an SVD factorization for  $\mathbf{E}_1(t)$  we can find pointwise-orthogonal matrix functions  $U_1(t) \in C^1([0, \infty), \mathbb{R}^{d,d})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$  such that  $U_1(t)\mathbf{E}_1(t)V(t) = \begin{bmatrix} \Sigma(t) & 0 \end{bmatrix}$ , where  $\Sigma(t)$  is a continuous, pointwise nonsingular function with values in  $\mathbb{R}^{d,d}$ . Changing the variable  $x(t) = V(t)y(t)$  and scaling (1.2) with  $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$ , we obtain a new system exactly of the form (1.8). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ A_3(t) & A_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} V(t),$$

then Assumption 1.2 yields that both  $\Sigma$  and  $A_4$  are nonsingular. This completes the proof.  $\square$

Let  $\hat{A}_3 := -A_4^{-1}A_3$ ,  $\hat{A}_1 := \Sigma^{-1}(A_1 - A_2A_4^{-1}A_3)$ , we rewrite system (1.8) as

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t), & \text{eq10.1} \\ y_2(t) &= \hat{A}_3(t)y_1(t). & \text{(1.9a)} \\ & & \text{eq10.2} \\ & & \text{(1.9b)} \end{aligned}$$

Since  $V(t)$  is orthogonal for all  $t \geq 0$ , we see that all important qualitative properties of  $x(t)$ , such as boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function  $y(t)$  without any difficulty. Clearly, we see that (1.9b) gives an *algebraic constraint* that the solution to (1.8) must obey, while (1.9a) gives the dynamic of (1.8). For this reason, we call it *an underlying ODE* to (1.8).

**Definition 1.5.** (i) Consider the DAE (1.2). A matrix function  $X \in C([0, \infty), \mathbb{R}^{n,k})$ ,  $d \leq k \leq n$ , is called a *fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and  $\text{rank } X(t) = d$ , for all  $t \geq 0$ .

(ii) A fundamental solution matrix is said to be *maximal* if  $k = n$  and *minimal* if  $k = d$ , respectively. A maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)(X(0) - Id) = 0. \quad \text{projected initial condition} \quad (1.10)$$

We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of full rank. Furthermore, each fundamental solution matrix has exactly  $d$ -linear independent columns, and a minimal fundamental solution matrix can be made maximal by adding  $n - d$  zero columns. This is the major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been proposed and carefully discussed in [17]. We recall it below, and notice that this concept is equivalent to the one proposed by Lentini and März in [15] within the context of the matrix chains approach and tractability index.

Let  $\{\hat{Y}_1(t, s)\}_{t \geq s \geq 0}$  be the evolution family associated with (1.9a), then we can define the corresponding evolution families for two DAEs (1.8), (1.2) consecutively as follows.

$$\hat{Y}(t, s) := \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, s) & 0 \end{bmatrix}, \quad \hat{X}(t, s) := V(t)\hat{Y}(t, s)V^T(s), \quad \text{for all } t \geq s \geq 0. \quad \text{eq11} \quad (1.11)$$

Nevertheless, since  $X(t, s)$  is not invertible, we will define the *reflexive generalized inverse matrix function* as in [17] by

$$\hat{Y}^-(t, s) := \begin{bmatrix} \hat{Y}_1^{-1}(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t, s) & 0 \end{bmatrix}, \quad \hat{X}^-(t, s) := V(s)\hat{Y}^-(t, s)V^T(t), \quad \text{for all } t \geq s \geq 0. \quad \text{eq12} \quad (1.12)$$

Then, we can directly verify the semigroup properties, i.e.

$$\begin{aligned} \hat{X}(t, r) &= \hat{X}(t, s)\hat{X}(s, r), \quad \text{for all } t \geq s \geq r \geq 0, \\ \hat{X}(t, s) &= \hat{X}(t, 0)\hat{X}^-(s, 0), \quad \text{for all } t \geq s \geq 0. \end{aligned}$$

Furthermore, Lemmas 1.6 and 1.7 below show that the family  $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$  does not depend on the choice of orthogonal transformations, and it plays the same role as the evolution family  $\{X(t, s)\}_{t \geq s \geq 0}$ , in comparison to (1.5).

**Lemma 1.6.** *The families  $\{X(t, s)\}_{t \geq s \geq 0}$ ,  $\{X^-(t, s)\}_{t \geq s \geq 0}$  defined by (1.11), (1.12) do not depend on the choice of orthogonal transformations.*

*Proof.* We will prove this claim only for the first family  $\{X(t, s)\}_{t \geq s \geq 0}$ , since for the second family the proof is essentially the same. Let us assume that we have two semi-explicit forms of system (1.2) obtained under orthogonal transformations, i.e.,

$$(E, A) \simeq \left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \quad \text{eq13.1} \quad (1.13a)$$

$$(E, A) \simeq \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right). \quad \text{eq13.2} \quad (1.13b)$$

Now we will prove that the two corresponding systems have the same evolution family  $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$ . Without loss of generality, we assume that  $(E, A)$  is already in the form of the right hand side of (1.13a), so  $\hat{X}(t, s) = \hat{Y}(t, s)$  for all  $t \geq s \geq 0$ . The corresponding system to the right hand side of (1.13b) reads

$$\dot{\tilde{y}}_1(t) = \hat{\mathcal{A}}_1(t) \tilde{y}_1(t), \quad \text{eq14.1} \quad (1.14a)$$

$$\dot{\tilde{y}}_2(t) = \hat{\mathcal{A}}_3(t) \tilde{y}_1(t). \quad \text{eq14.2} \quad (1.14b)$$

where  $\hat{\mathcal{A}}_3 := -\tilde{A}_4^{-1} \tilde{A}_3$ ,  $\hat{\mathcal{A}}_1 := \tilde{\Sigma}^{-1} (\tilde{A}_1 - \tilde{A}_2 \tilde{A}_4^{-1} \tilde{A}_3)$ .

The transitivity applied to (1.13) implies that there exist pointwise-orthogonal matrix-valued functions  $S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that  $y(t) = T(t) \tilde{y}(t)$  and

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad \text{eq15.1} \quad (1.15a)$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \left( \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T}_1 & \dot{T}_2 \\ \dot{T}_3 & \dot{T}_4 \end{bmatrix} \right). \quad \text{eq15.2} \quad (1.15b)$$

117 Let  $\{\hat{\mathcal{Y}}_1(t, s)\}_{t \geq s \geq 0}$  be the evolution family associated with (1.14a), then the evolution family associated with  
118 system (1.14) is

$$\hat{\mathcal{Y}}(t, s) = \begin{bmatrix} \hat{\mathcal{Y}}_1(t, s) & 0 \\ \hat{\mathcal{A}}_3(t) \hat{\mathcal{Y}}_1(t, s) & 0 \end{bmatrix}, \quad \hat{\mathcal{X}}(t, s) := T(t) \hat{\mathcal{Y}}(t, s) T^T(s) \text{ for all } t \geq s \geq 0. \quad \text{eq16} \quad (1.16)$$

119 Thus, we need to prove that  $\hat{\mathcal{X}}(t, s) = \hat{X}(t, s)$ .

120 From (1.15a), it implies that  $S_3 \Sigma \begin{bmatrix} T_1 & T_2 \end{bmatrix} = 0$ . Thus, we have

$$\begin{bmatrix} S_3 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

121 and hence, this follows that  $S_3 = 0$ . Thus,  $S = \begin{bmatrix} S_1 & S_2 \\ 0 & S_4 \end{bmatrix}$ , and then, due to the orthogonality of  $S$ , we see that  
122 both  $S_1$  is nonsingular and  $S_4$  is also orthogonal. Also from (1.15a), we see that  $S_1 \Sigma T_2 = 0$ , which yields  
123 that  $T_2 = 0$ . Moreover, due to the orthogonality of  $S$  and  $T$ , from (1.14a) we see that

$$\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^T & S_3^T \\ S_2^T & S_4^T \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^T & T_3^T \\ T_2^T & T_4^T \end{bmatrix}.$$

Therefore, using similar arguments as above, we can prove that  $S_2 = 0$  and  $T_3 = 0$ .

Consequently, by inserting  $S_3 = T_3 = 0$  and  $S_2 = T_2 = 0$  into (1.15a) and (1.15b) we obtain

$$\tilde{\Sigma} = S_1 \Sigma T_1, \quad (1.17a)$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 \left( A_1 T_1 - \Sigma \dot{T}_1 \right) & S_1 A_2 T_4 \\ S_4 A_3 T_1 & S_4 A_4 T_4 \end{bmatrix}, \quad (1.17b)$$

where the matrix-valued function  $S_i, T_i$  ( $i = 1, 4$ ) are pointwise orthogonal. Thus, we see that

$$\hat{A}_3 = -\tilde{A}_4^{-1}\tilde{A}_3 = -(S_4A_4T_4)^{-1}S_4A_3T_1 = T_4^{-1}\hat{A}_3T_1, \quad (1.18a)$$

$$\hat{A}_1 = \tilde{\Sigma}^{-1}(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3) = T_1^{-1}(\hat{A}_1T_1 - \dot{T}_1). \quad (1.18b)$$

Furthermore, since  $y = T\tilde{y}$ , the structure of  $T$  implies that  $y_1 = T_1\tilde{y}_1$  and  $y_4 = T_4\tilde{y}_4$ . Therefore, the underlying ODE (1.14a) is directly obtained from (1.9a) by applying the variable transformation  $\tilde{y}_1(t) = T_1^{-1}(t)y_1(t)$  and scaling the system with  $T_1^{-1}$ . So, we have that  $\hat{\mathcal{Y}}_1(t, s) = T_1^{-1}\hat{Y}_1(t, s)T_1(s)$ . Making use of (1.18), we can deduce the evolution family  $\{\hat{\mathcal{X}}(t, s)\}_{t \geq s \geq 0}$  as follows

$$\hat{\mathcal{X}}(t, s) = \begin{bmatrix} T_1(t) & 0 \\ 0 & T_4(t) \end{bmatrix} \begin{bmatrix} \hat{\mathcal{Y}}_1(t, s) & 0 \\ \hat{A}_3\hat{\mathcal{Y}}_1(t, s) & 0 \end{bmatrix} \begin{bmatrix} T_1^T(s) & 0 \\ 0 & T_4^T(s) \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3\hat{Y}_1(t, s) & 0 \end{bmatrix},$$

and hence, this completes the proof.  $\square$

**Lemma 1.7.** *Consider the DAE (1.1) and the evolution family  $(X(t, s))_{t \geq s \geq 0}$  defined by (1.11). Furthermore, we also consider the pointwise-orthogonal matrix-valued functions  $U, V$  defined in Lemma 1.7. Then, the solution to (1.1), if exists, also satisfies the so-called mild equation*

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \hat{X}(t, s) \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t, \tau) \begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau)f(\tau, x(\tau))d\tau \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t)), \end{aligned}$$

for all  $t \geq s \geq 0$ .

*Proof.* The proof can be obtained directly by using Lemma 1.4. Thus, in order to keep the brevity we will omit the details here.  $\square$

In the following, for ease of notation, we will use the abbreviation  $\hat{X}(t) := \hat{X}(t, 0)$ ,  $\hat{X}^-(t) := \hat{X}^-(t, 0)$ ,  $\hat{Y}(t) := \hat{Y}(t, 0)$  and  $\hat{Y}^-(t) := \hat{Y}^-(t, 0)$ . The concept of exponential dichotomy for the DAE (1.8) is given as below.

**Definition 1.8.** ([17]) The DAE (1.8) is said to have an *exponential dichotomy* if there exist a family of projection matrices  $\{P_y(t)\}_{t \geq 0}$  in  $\mathbb{R}^{d,d}$  and positive constants  $N, \nu$  such that

$$\begin{aligned} \left\| \hat{Y}(t) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq Ne^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0, \\ \left\| \hat{Y}(t) \begin{bmatrix} I_d - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq Ne^{\nu(t-s)}, \text{ for all } s \geq t \geq 0, \end{aligned} \quad (1.19)$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.11)-(1.19) we see that

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq Ne^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0.$$

and

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq Ne^{\nu(t-s)}, \text{ for all } s \geq t \geq 0.$$

In addition, since  $V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$  is also a projection matrix for any  $t \geq 0$ , we can interpret the exponential dichotomy of (1.2) as the one of (1.8).

**1.3. Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces that play a fundamental role in the study of differential equations and refer to Nguyen [10], Massera and Schäffer [18, Chap. 2] and Răbiger and Schnaubelt [22, §1] for various applications.

Let  $E$  (endowed with the norm  $\|\cdot\|_E$ ) be Banach function space of real-valued functions defined as in [10]. We then recall the Banach space corresponding to the space  $E$  as follows.

**Definition 1.9** ([10]). Consider the Banach space  $(\mathbb{R}^n, \|\cdot\|)$ . For a Banach function space  $E$  we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

endowed with the norm  $\|f\|_{\mathcal{E}} := \| \|f(\cdot)\| \|_E$ . Thus, one can directly see that  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  is a Banach space. We call it *the Banach space corresponding to  $E$* .

We now introduce the notion of admissibility in the following definition.

**Definition 1.10** ([10]). The Banach function space  $E$  is called *admissible* if for any  $\varphi \in E$  the following conditions hold.

- (i) There exists a constant  $M \geq 1$  such that for every compact interval  $[a, b] \subset \mathbb{R}_+$  we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \quad (1.20)$$

where  $\chi_{[a,b]}$  is the indicator function of  $[a, b]$ .

- (ii) The function  $\Lambda_1 \varphi$  defined by  $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$  belongs to  $E$ .

- (iii) For any  $\tau \geq 0$ , the space  $E$  is  $T_\tau^+$ -invariant and  $T_\tau^-$ -invariant, where  $T_\tau^+$  and  $T_\tau^-$  are defined as

$$\begin{aligned} T_\tau^+ \varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \\ T_\tau^- \varphi(t) &:= \varphi(t+\tau) \text{ for } t \geq 0. \end{aligned} \quad (1.21)$$

Furthermore, there exist constants  $N_1, N_2$  such that  $\|T_\tau^+\|_E \leq N_1$ ,  $\|T_\tau^-\|_E \leq N_2$  for all  $\tau \in \mathbb{R}_+$ .

**Example 1.11.** Besides the spaces  $L_p(\mathbb{R}_+)$ ,  $1 \leq p \leq \infty$ , and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau < \infty\},$$

(for any fixed  $\alpha > 0$ ), endowed with the norm  $\|h\|_{\mathbf{M}_\alpha} := \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau$ , many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces  $L_{p,q}$ ,  $1 < p < \infty, 1 \leq q < \infty$  (see [3], [24]) and, more general, the class of rearrangement invariant function spaces (see [16]) are admissible.

*Remark 1.12.* Following directly from Definition 1.10 we have that

$$\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau \leq \frac{M}{\inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

and hence,  $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$ . Furthermore,  $C_b(\mathbb{R}^+)$  is dense in  $\mathbf{M}_1$ .

We present here some important features of admissible spaces in the following proposition (see [10, Proposition 2.6] and originally in [18, 23.V.(1)]).

**Proposition 1.13** ([10]). Let  $E$  be an admissible Banach function space. Then the following assertions hold.

a) Let  $\varphi \in L_{1,loc}(\mathbb{R}_+)$  such that  $\varphi \geq 0$  and  $\Lambda_1 \varphi \in E$ , where,  $\Lambda_1$  is defined as in definition 1.10 (ii). For  $\sigma > 0$  we define functions  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  by

$$\begin{aligned}\Lambda'_\sigma \varphi(t) &:= \int_0^t e^{-\sigma(t-s)} \varphi(s) ds, \\ \Lambda''_\sigma \varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds.\end{aligned}$$

161 Then,  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  belong to  $E$ . In particular, if  $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau < \infty$  (this will be satisfied if  $\varphi \in E$  (see  
162 remark 1.12)) then  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  are bounded. Moreover, denoted by  $\|\cdot\|_\infty$  for *ess sup*-norm, we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty \quad (\text{eq2.2})$$

163 for operator  $T_1^+$  and constants  $N_1, N_2$  defined as in Definition 1.10.

164 b)  $E$  contains exponentially decaying functions  $\psi(t) = e^{-\alpha t}$  for any constant  $\alpha > 0$ .

165 c)  $E$  does not contain exponentially growing functions  $f(t) := e^{bt}$  for any constant  $b > 0$ .

## 166 2. EXPONENTIAL DICHOTOMY FOR LINEAR DAEs

167 In the qualitative analysis of ODEs, one of the central topics is to find necessary and sufficient conditions  
168 such that the considered system admits an exponential dichotomy. Many researches have been devoted to  
169 this topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces  
170 (e.g. [6, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

171 **Proposition 2.1.** ([17]) The DAE (1.8) has an exponential dichotomy if and only if the corresponding  
172 underlying ODE (1.9a) also has exponential dichotomy and the matrix function  $\hat{A}_3(t)$  is bounded. Moreover,  
173 the existence of an exponential dichotomy implies that  $\sup_{t \geq 0} \|P_Y(t)\| < \infty$ .

174 Notice that, Proposition 2.1 is only valid for finite-dimensional but it is very hard to generalize for infinite  
175 dimensional DAE systems. For this reason, we aim at another approach, motivated from one classical result  
176 stated below.

177 **Proposition 2.2.** ([5, Chap. 3]) The ODE (1.3) has an exponential dichotomy if and only if one of the  
178 following conditions is satisfied.

179 i) For any function  $g \in \mathbf{M}_1(\mathbb{R}_+)$  there exists a continuous, bounded solution  $x(t)$  to the inhomogeneous  
180 system

$$\dot{x}(t) = A(t)x(t) + g(t). \quad (\text{inho. ODE } (2.1))$$

181 ii) For any function  $g \in C_\infty(\mathbb{R}_+)$ , there exists a continuous, bounded solution  $x(t)$  to the inhomogeneous  
182 system (2.1), provided that the ODE (1.3) has bounded growth.

183 Comparable results to Proposition 2.2 have not been achieved for DAEs, and hence, this will be our main  
184 aim in this section. Together with (1.2), let us consider the following system

$$E(t)\dot{x}(t) = A(t)x(t) + g(t). \quad (\text{eq2.1 } (2.2))$$

185 The following example shows that Proposition 2.2 could not be directly applied to the DAE (2.2).

186 **Example 2.3.** Consider the system (2.2) with  $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} -1 & 0 \\ 0 & e^{-t} \end{bmatrix}$ ,  $f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Clearly,  $f$  is bounded.  
187 The explicit solution  $x(t) = \begin{bmatrix} e^{-t} x_1(0) \\ e^t \end{bmatrix}$  is clearly unbounded no matter how the initial condition  $x(0)$  is  
188 chosen. On the other hand, the homogeneous system clearly has an exponential dichotomy.



**Definition 2.4.** Consider the matrix functions  $E, A$  in system (1.2). Then, a function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  satisfying the condition

$$\sup_{t \geq 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} g(t) \right\| \right\} < +\infty, \quad \text{eq2.2} \quad (2.3)$$

is called  $(E, A)$ -bounded. We denote the set of all continuous and  $(E, A)$ -bounded functions by  $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$ .

**Lemma 2.5.** The  $(E, A)$ -boundedness is invariant with respect to orthogonal transformations.

*Proof.* Let us consider two orthogonally equivalent systems

$$\left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \simeq \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued function  $S_i, T_i$  ( $i = 1, 4$ ). Thus, we see that

$$\begin{bmatrix} \tilde{\Sigma}^{-1}(t) & -\tilde{\Sigma}^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & \tilde{A}_4^{-1}(t) \end{bmatrix} g(t) = \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} \begin{bmatrix} S_1^T & 0 \\ 0 & S_4^T \end{bmatrix} g(t).$$

Since Euclidean norm is preserved under orthogonal transformation, this identity completes the proof.  $\square$

The main result of this section is to prove a characterization of the exponential dichotomy for DAEs. Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping  $\mathcal{L} := E \frac{d}{dt} - A$  is surjective on the space  $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$ . We formulate our main result in this section as follows.

**Theorem 2.6.** Consider the linear, strangeness-free DAE (1.2) and the associated inhomogeneous DAE (2.2). Then the following assertions hold.

- (i) If the DAE (1.2) admits an exponential dichotomy then for any continuous,  $(E, A)$ -bounded function  $g(t)$  on  $[0, \infty)$ , there exists a continuous, bounded solution  $x(t)$  to the DAE (2.2).
- (ii) If the matrix function  $\hat{A}_3(t)$  is bounded, then the converse of assertion (i) holds true.

*Proof.* Firstly, we notice that, since  $\hat{g} = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t)g(t)$ , where  $U$  is orthogonal, so the  $(E, A)$ -boundedness of  $g$  is equivalent to the boundedness of  $\hat{g}$ . Recall that the semi-explicit system (1.8) reads

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t) + \hat{g}_1(t), & \text{eq3.10a} & (2.4a) \\ y_2(t) &= \hat{A}_3(t)y_1(t) + \hat{g}_2(t). & \text{eq3.10b} & (2.4b) \end{aligned}$$

(i) Assuming that the DAE (1.2) admits an exponential dichotomy, then (1.8) also has an exponential dichotomy. Proposition 2.1 implies that equation (2.4a) has an exponential dichotomy, and the function  $\hat{A}_3$  is bounded. Therefore, Proposition 2.2 implies that  $y_1$  is bounded, and consequently,  $y_2$  is also bounded.

(ii) Notice that the mapping

$$\begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) : \mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C_\infty(\mathbb{R}_+, \mathbb{R}^n)$$

is surjective, so  $g_1(t)$  can be freely chosen in the space  $C_\infty(\mathbb{R}_+, \mathbb{R}^n)$ . Proposition 2.2 applied to system (2.4a) follows that (2.4a) has exponential dichotomy. On the other hand, the boundedness of  $\hat{A}_3$  implies that (1.2) admits exponential dichotomy.  $\square$

### 3. LOCAL STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout this section we assume that the evolution family  $(X(t, s))_{t \geq s \geq 0}$  associated with the linear, homogeneous DAE (1.2) admits an exponential dichotomy on  $\mathbb{R}_+$ .

From Lemma 1.4, by using orthogonal transformation  $x(t) = V(t)y(t)$ , where  $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$  we can transform (1.1) to the coupled system

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), \quad \text{eq4.1a} \quad (3.1)$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), \quad \text{eq4.1b} \quad (3.2)$$

where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) \begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}. \quad \text{eq4.2} \quad (3.3)$$

Notice that, unlike the DAEs (1.2) and (2.2), equation (3.2) only gives an implicit algebraic constraint in terms of  $y_1$  and  $y_2$ . In order to guarantee the strangeness-free of system (1.1), we need the following assumption.

**Assumption 3.1.** Assume that for some  $\rho > 0$ , the function  $A_4^{-1}(t)f_2(t, x)$  is a contraction mapping in the ball  $B_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$  (uniformly in time), i.e.,

$$\|A_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

for a.e.  $t \in \mathbb{R}_+$ , and for all  $x, \tilde{x} \in B_\rho$  where the Lipschitz constant  $L$  satisfies that  $L < 1$ .

**Lemma 3.2.** Under Assumption 3.1 and given  $y_1 \in B_\rho$ , there exists a unique function  $y_2 \in B_\rho$  satisfying (3.2).

*Proof.* Firstly, notice that Assumption 3.1 implies that  $\hat{f}_2(t, y)$  is also Lipschitz in  $y$  with the same constant  $L$ . Then, the desired claim is obtained directly by making use of [19, Lem. 2.7].  $\square$

**Remark 3.3.** Lemma 3.2 leads to one important fact, that under Assumption 3.1, the coupled system (3.1)-(3.2) is still strangeness-free, as defined in [13, Chap. 4]. Therefore, in analogue to the linear case, (3.2) is called an *algebraic constraint*, whereas (3.1) is called an *underlying ODE*.

To obtain the existence of a stable manifold we need the following property of the nonlinear part  $f_1$  defined as follows.

**Definition 3.4.** Let  $\varphi$  be a positive function belonging to an admissible Banach function space  $E$ . A function  $h : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to belong to the class  $(M, \varphi, \rho)$  for some positive constant  $M$ ,  $\rho$  if  $h$  satisfies

- (i)  $\|h(t, x)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$  and for all  $x \in B_\rho$ ,
- (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$ , for all  $x, \tilde{x} \in B_\rho$ .

**Assumption 3.5.** Assume that the function  $t \mapsto \Sigma^{-1}(t)[I_d - A_2(t)A_4^{-1}(t)]f(t, x(t))$  belongs to class  $(M, \varphi, \rho)$  for some positive constants  $M$ ,  $\rho$  and a positive function  $\varphi \in E$ .

The following lemma shows that Assumptions 3.1, 3.5 are invariant with respect to orthogonal transformations.

**Lemma 3.6.** Assumptions 3.1, 3.5 are also invariant with respect to orthogonal transformations.

240 *Proof.* Let us consider two orthogonally equivalent systems

$$\left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \simeq \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued function  $S_i$ ,  $T_i$  ( $i = 1, 4$ ). Therefore, we have that

$$\|\tilde{A}_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| = \|T_4^{-1}(t)A_4^{-1}(t)S_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

241 Then, the desired claim is directly followed.  $\square$

242 The following proposition gives one sufficient condition for examining Assumptions 3.1, 3.5.

243 **Proposition 3.7.** Consider the semi-linear DAE (1.1). Furthermore, assume that all three functions  $\Sigma^{-1}$ ,  
244  $A_4^{-1}$ ,  $\Sigma^{-1}A_2A_4^{-1}$  are bounded. If the function  $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$  belongs to the class  $(M, \varphi, \rho)$  then the following  
245 claims hold true.

- 246 i)  $\hat{f}_1$  belongs to the class  $(M, \varphi, \rho)$ , and  
247 ii)  $f_2$  is Lipschitz with the Lipschitz constant  $\varphi \sup_{t \geq 0} \|A_4^{-1}\|$ .

248 We notice that a sufficient condition for Assumption (3.1) is that

$$\|f_2(t, x) - f_2(t, \tilde{x})\| \leq \frac{L}{\|A_4^{-1}(t)\|} \|x - \tilde{x}\|. \quad \text{Lipschitz (3.4)}$$

249 For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.1)-  
250 (3.2). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad \text{mild equation (3.5)}$$

251 for all  $t \geq s \geq 0$ .

252 **Lemma 3.8.** Let Assumptions 3.1 and 3.5 hold true. Then, for all  $y, \tilde{y} \in B_\rho$  the following assertions hold.

- 253 (i)  $\|\hat{f}_1(t, y)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$ ,  
254 (ii)  $\|\hat{f}_1(t, y) - \hat{f}_1(t, \tilde{y})\| \leq \varphi(t)\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ ,  
255 (iii)  $\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| \leq L\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ .

256 *Proof.* The proof is trivially followed from Assumptions 3.1 and 3.5 due to the fact that  $\|y\| = \|Qy\|$  for any  
257 orthogonal matrix  $V$ .  $\square$

258 Let  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  has an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$   
259 and the dichotomy constants  $N, \nu > 0$  as in Definition 1.8. Furthermore, as in Proposition 2.1, let us denote  
260 by  $H_1 := \sup_{t \geq 0} \|\hat{A}_3(t)\|$  and  $H_2 := \sup_{t \geq 0} \|P_y(t)\|$ . Then, we can define the Green function on the half-line as  
261 follows

$$G(t, \tau) := \begin{cases} \hat{Y}(t, \tau) \begin{bmatrix} P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)P_y(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, \tau)P_y(\tau) & 0 \end{bmatrix}, & \text{for all } t \geq \tau \geq 0, \\ -\hat{Y}(t, \tau) \begin{bmatrix} I_d - P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \end{bmatrix}, & \text{for all } 0 \leq t < \tau. \end{cases} \quad (4.2) \quad (3.6)$$

262 Then, we have

$$\|G(t, \tau)\| \leq (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geq 0. \quad (4.3) \quad (3.7)$$

263 In the following lemma, we give an explicit form for bounded solutions to system (3.5).

**Lemma 3.9.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy with the corresponding projection matrices  $\{P_Y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumptions 3.1, 3.5 hold true. Let  $y(t)$  be any solution to (3.5) such that  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$  for fixed  $t_0 \geq 0$  and some  $\rho > 0$ . Then, for  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^t G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (3.8)$$

for some  $v_0 \in \text{Im}P_Y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.6).

*Proof.* Put

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^t G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}.$$

By direct computation, we can verify that  $z$  satisfies the integral equation

$$z(t) = \hat{Y}(t, t_0) \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix},$$

for all  $t \geq t_0$ . Now let us estimate  $\|z(t)\|$ . Making use of Lemma 3.8 and (3.7), we see that

$$\|z(t)\| \leq \int_{t_0}^t (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} M \varphi(\tau) d\tau + L\rho,$$

and then, from (1.22) it follows that

$$\|z(t)\| \leq M (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\| \Lambda_1 T_1^+ \varphi \|_\infty + \| \Lambda_1 \varphi \|_\infty) + L\rho,$$

for all  $t \geq t_0$ . Thus,  $z(t) - y(t)$  is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t, t_0) (z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t) \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

we see that  $v_0 := z_1(t_0) - y_1(t_0) \in \text{Im}P_Y(t_0)$ . Finally, since  $z(t) = y(t) + \hat{Y}(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$  for all  $t \geq t_0$ , equality (3.8) follows.  $\square$

**Remark 3.10.** By computing directly, we can see that the converse of Lemma 3.9 is also true. It means, that all solutions to (3.8) also satisfy equation (3.5) for all  $t \geq t_0$ .

Let us denote by

$$H_3 := (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\| \Lambda_1 T_1^+ \varphi \|_\infty + \| \Lambda_1 \varphi \|_\infty) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)} \rho. \quad (3.9)$$

**Lemma 3.11.** *Under the assumptions of Lemma 3.9, let  $y(t)$ ,  $\tilde{y}(t)$  be any two functions lying in the ball  $B_\rho$  and satisfy (3.8) for  $v_0, \tilde{v}_0 \in \text{Im}P_Y(t_0)$ . If  $H_3$  defined as in (3.9) satisfies  $H_3 + L < 1$  then the following estimate holds true:*

$$\|y - \tilde{y}\|_\infty \leq \frac{N}{1 - H_3 - L} \|v_0 - \tilde{v}_0\|. \quad (3.10)$$

*Proof.* Using the same arguments as in the proof of Lemma 3.8, we see that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq N \|v_0 - \tilde{v}_0\| + \int_{t_0}^t (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq N \|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\| \Lambda_1 T_1^+ \varphi \|_\infty + \| \Lambda_1 \varphi \|_\infty) \|y - \tilde{y}\|_\infty + L \|y(t) - \tilde{y}(t)\|, \\ &\leq N \|v_0 - \tilde{v}_0\| + (H_3 + L) \|y - \tilde{y}\|_\infty, \end{aligned}$$

281 which directly implies (3.10). □

282 In the following theorem, we exploit the local structure of bounded solutions to (3.5).

283 **Theorem 3.12.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy*  
 284 *with the corresponding projection matrices  $\{P_Y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore,*  
 285 *assume that Assumptions 3.1, 3.5 hold true, and constant  $H_3$  defined as in (3.9). Then, the following*  
 286 *assertions hold true.*

287 (i) If

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)\rho}{2M} \right\}, \quad \text{eq4.7} \quad (3.11)$$

288 then there corresponds to each  $v_0 \in B_{\tilde{\rho}} \cap \text{Im} P_Y(t_0)$  one and only one solution  $y(t)$  to (3.5) on  $[t_0, \infty)$  satisfying  
 289  $P_Y(t_0)y_1(t_0) = v_0$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ .

290 (ii) Moreover, any two solutions  $y(t), \tilde{y}(t)$  corresponding to different  $v_0, \tilde{v}_0$  in  $B_{\tilde{\rho}} \cap \text{Im} P_Y(t_0)$  attract each  
 291 other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0, \quad \text{eq4.8} \quad (3.12)$$

292 for some positive constants  $H_4, \mu$ .

293 *Proof.* (i) Consider in the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  the ball  $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) : \|y(\cdot)\|_\infty := \text{ess sup}_{t \geq 0} \|y(t)\| \leq \rho\}$ .

294 For each fixed  $v_0 \in B_{\tilde{\rho}}$  we will prove the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix} & \text{for all } t \geq t_0, \\ 0 & \text{for all } t < t_0, \end{cases} \quad (3.13)$$

295 is a contraction mapping from  $\mathcal{B}_\rho$  to itself. Using the same argument as in the proof of Lemma 3.8, we see  
 296 that

$$\begin{aligned} \|(Ty)(t)\| &\leq (1 + H_1)N e^{-\nu(t-t_0)} \|v_0\| + M(1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho, \\ &\leq (1 + H_1)N \|v_0\| + MH_3 + L\rho \quad \text{for all } t \geq 0, \end{aligned}$$

297 and by (3.11) we see that

$$\|(Ty)(t)\| \leq (1 + H_1)N\tilde{\rho} + \frac{(1 - L)\rho}{2} + L\rho = \rho \quad \text{for all } t \geq 0.$$

Therefore,  $T$  is a mapping from  $\mathcal{B}_\rho$  to itself. Now we prove its contraction property. Indeed, making use of (3.7), we obtain the following estimate:

$$\begin{aligned} \|Ty(t) - T\tilde{y}(t)\| &\leq \int_{t_0}^\infty \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq \int_{t_0}^\infty (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq (H_3 + L) \|y(\cdot) - \tilde{y}(\cdot)\|_\infty \quad \text{for all } t \geq 0. \end{aligned}$$

298 Consequently, due to (3.11), we see that  $T$  is a contraction mapping with the contraction constant  $H_3 + L$ .  
 299 Thus, there exist a unique function  $y \in \mathcal{B}_\rho$  such that  $y = Ty$ , and hence, due to the definition of  $T$ ,  $y$  is the  
 300 solution to the mild/integral-algebraic system (3.5).

(ii) The proof of the estimate (3.12) can be done in a similar way as in [11, Thm 3.7]. We present here for seek of completeness. Let  $y(t)$  and  $\tilde{y}(t)$  be two essentially bounded solutions of (3.5) corresponding to different values  $v_0, \tilde{v}_0 \in B_{\tilde{p}} \cap \text{Im}P_Y(t_0)$ . Then, we have that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq Y(t, t_0)\|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq (1 + H_1)Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \end{aligned}$$

and hence,

$$\|y(t) - \tilde{y}(t)\| \leq \frac{1 + H_1}{1 - L} Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} \frac{(1 + H_1)(1 + H_2)}{1 - L} Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau.$$

Then, due to the Cone Inequality, [6, Theorem 1.9.3], in analogue to [20, Theorem 3.7], we obtain the estimation (3.12) with  $H_4, \mu$  are given by

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu-\nu}}}.$$

Furthermore, notice that from (3.11) it follows that  $\mu < \nu$  implying the positivity of  $H_4$ . This completes the proof.  $\square$

Under Assumption 3.1, we then define the so-called *constrained manifold*, which all solutions to (3.1)-(3.2) must belong to

$$\mathbb{L}(t, y) := \{(t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t, y_1, y_2)\} \quad \text{constraint manifold} \quad (3.14)$$

We further notice that this manifold is of dimension  $d$ , which is the degree of freedom to the DAE (3.5). Now, we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic system (3.5).

**Definition 3.13.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *local stable manifold* for solutions to (3.5) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf \{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exist positive constants  $\rho, \rho_1, \rho_2$  and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t), \quad t \in \mathbb{R}_+,$$

with a common Lipschitz constant independent of  $t$  such that

- (i)  $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$ , and we denote by  $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$ ,
- (ii)  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$  for all  $t \geq 0$ ,
- (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.5) satisfying  $y_1(t_0) = \tilde{w}$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ .

We now state and prove our main result on the existence of a local stable manifold for DAEs.

**Theorem 3.14.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumptions 3.1, 3.5 hold true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)\rho}{2M}, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a local stable manifold for the solutions of (3.5). Moreover, every two solutions  $y(t), \tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)y_2(t_0)\|, \quad \text{for all } t \geq t_0. \quad \text{eq4.12 (3.15)}$$

*Proof.* First we notice that the phase subspace  $\mathbb{R}^d$  splits into the direct sum  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel } P_y(t)$  for all  $t \geq 0$ . We set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{kernel } P_y(t)$ , then due to Proposition 2.1, we see that  $\sup_{t \geq 0} \|P_y(t)\| < \infty$ , and hence,  $\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$ .

For any  $\rho > 0$  defined as in Assumptions 3.1, 3.5, let  $\rho_1 := \tilde{\rho} = \frac{1 - L}{2N(1 + H_1)}\rho$  and  $\rho_2 := \frac{(1 - L)\rho}{2}$ . For each  $t \geq 0$  we define the mapping  $g_t$  acting on  $B_{\rho_1} \cap W_1(t)$  as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function  $y(t)$  is uniquely defined via Theorem 3.12 i). Clearly,  $g_t(w_1) \in \text{ker } P_y(t) = W_2(t)$ .

Now, we prove that  $\|g_t(w_1)\| \leq \rho_2$ . Due to Theorem 3.12 (i) and Lemma 3.8 (i), we have that  $\|y(t)\| \leq \rho$  and  $\|f_1(\tau, y(\tau))\| \leq M\varphi(\tau)$  for a.e.  $t \geq 0$ . Therefore,

$$\begin{aligned} \|g_t(w_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} M\varphi(\tau) d\tau, \\ &\leq M(1 + H_2) \frac{N}{1 - e^{-\nu}} (\| \Lambda_1 T_1^+ \varphi \|_\infty + \| \Lambda_1 \varphi \|_\infty) = \frac{MH_3}{1 + H_1} \leq \frac{(1 - L)\rho}{2}, \end{aligned}$$

and hence,  $g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t)$ .

Notice that both part (iii) in Definition 3.13 and estimation (3.15) are followed directly from Theorem 3.12. We now only need to prove that  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$ . We first prove that  $g_t$  is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1 - e^{-\nu}} (\| \Lambda_1 T_1^+ \varphi \|_\infty + \| \Lambda_1 \varphi \|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1 + H_1)(1 + H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1 + H_1)(1 + H_2)(1 - H_3 - L)} \|w_1 - \tilde{w}_1\|.$$

Finally,  $H_3 < \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)}$  yields that  $\frac{NH_3}{(1 + H_1)(1 + H_2)(1 - H_3 - L)} < 1$ , and hence,  $g_t$  is a contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous mappings ([19, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow B_{\rho_1} \cap W_1(t)$  is a homeomorphism. This implies the condition (ii) of Definition 3.13 finishing the proof.  $\square$

## 4. GLOBAL INVARIANT STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1). We begin with the concept of  $\varphi$ -Lipschitz functions.

**Definition 4.1.** Let  $E$  be an admissible Banach function space and  $\varphi \in E$  be a positive function. A function  $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^d$  is said to be  $\varphi$ -Lipschitz if the following conditions hold true.

- (i)  $\|h(t, 0)\| = 0$  for a.e.  $t \in \mathbb{R}_+$ ,
- (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$  and all  $x, \tilde{x} \in \mathbb{R}^n$ .

In comparability to Assumptions 3.1, 3.5, we also need some global properties of the nonlinear term  $f$ .

**Assumption 4.2.** Assume that the following hypotheses hold true.

- (i) The function  $\Sigma^{-1}(t) f_1(t, x(t)) - \Sigma^{-1}(t) A_2(t) A_4^{-1}(t) f_2(t, x(t))$  is  $\varphi$ -Lipschitz.
- (ii) The function  $A_4^{-1}(t) f_2(t, x(t))$  is a contraction mapping with the Lipschitz constant  $L < 1$  for all  $(t, x(t))$  lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid A_2(t)x + f_2(t, x) = 0\}.$$

We can directly verify that orthogonal transformations of the form  $x = Vy$  preserves the  $\varphi$ -Lipschitz property, and hence, function  $\hat{f}_1$  in (3.1) is also  $\varphi$ -Lipschitz. Besides that, function  $\hat{f}_2$  in (3.2) is also a contraction mapping with the Lipschitz constant  $L < 1$ . For notational simplicity, now we will study the transformed system (1.8) and the integral-algebraic system (3.5).

**Definition 4.3.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *global, invariant stable manifold* for solutions to (3.5) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \rightarrow W_2(t), \quad t \in \mathbb{R}_+,$$

with the Lipschitz constants independent of  $t$  such that

- (i)  $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$ , and we denote by  $\mathbb{M}_t := \{(y_1, y_2) \mid (t, y_1, y_2) \in \mathbb{M}\}$ ,
- (ii)  $\mathbb{M}_t$  is homeomorphic to  $W_1(t)$  for all  $t \geq 0$ ,
- (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.5) satisfying  $y_1(t_0) = \tilde{w}$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ ,
- (iv)  $\mathbb{M}$  is invariant under system (3.5), i.e., if  $y$  is a solution to (3.5), and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ , then  $y(s) \in \mathbb{M}_s$  for all  $s \geq t_0$ .

Analogously to Lemma 3.9, we give the explicit form of bounded solutions to system (3.5) as below.

**Lemma 4.4.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumption 4.2 holds true. Let  $y(t)$  be any solution to (3.5) such that  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$  for a fixed  $t_0 \geq 0$ . Then, for all  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^t G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (4.1)$$

for some  $v_0 \in \text{Im} P_y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.6).



364 *Proof.* The proof can be done by using similar arguments as in the proof of Lemma 3.2.  $\square$

365 In the following two theorems, we present the global versions of Theorems 3.12 and 3.14, where we  
 366 construct the structure of bounded solutions to (3.5) and prove the existence of a global, stable manifold,  
 367 respectively.

368 **Theorem 4.5.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy with the  
 369 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
 370 that Assumption 4.2 holds true.*

- 371 (i) *For any fixed  $t_0 \geq 0$ , if  $H_3 < 1 - L$  then there corresponds to each  $v_0 \in \text{Im}P_y(t_0)$  one and only one  
 372 solution  $y(t)$  to (3.5) on  $[t_0, \infty)$  satisfying  $P_y(t_0)y_1(t_0) = v_0$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ .*  
 (ii) *Any two solutions  $y(t), \tilde{y}(t)$  corresponding to different initial conditions  $v_0, \tilde{v}_0$  in  $\text{Im}P_y(t_0)$ , are expo-  
 nentially attracted to each other, i.e.,*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0,$$

with some positive constants  $H_4, \mu$  satisfying

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu-\nu}}}.$$

*Proof.* The proof of this theorem is essentially the same as the proof of Theorem 3.12. The only change is, that instead of considering the ball  $B_\rho$  we will work with the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  itself. Then, we can prove (without any difficulty) that for each fixed  $v_0 \in \text{Im}P_y(t_0)$ , the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geq t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

373 is a contraction mapping, and therefore, all the assertions of the theorem follows.  $\square$

**Theorem 4.6.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.8) have an exponential dichotomy with the  
 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
 that Assumption 4.2 holds true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a global invariant stable manifold for the solutions of (3.5). Moreover, every two solutions  $y(t), \tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)y_2(t_0)\| \quad \text{for all } t \geq t_0.$$

374 *Proof.* Analogous to the proof of Theorem 3.14, we consider the decomposition  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel } P_y(t)$   
 375 and set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{kernel } P_y(t)$ . Thus, we see that  $\inf_{t \in \mathbb{R}_+} \text{Sn}(W_1(t), W_2(t)) > 0$ .  
 376 Now we define the family of mappings  $(g_t)_{t \geq 0}$  acting on  $W_1$  as

$$g_t(w_1) := \int_t^{\infty} \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function  $y(t)$  is bounded and be uniquely defined via Theorem 4.5 i). Clearly,  $g_t(w_1) \in \ker P_y(t) = W_2(t)$ . To verify the Lipschitz property of  $g_t$ , let us consider two arbitrary elements  $w_1$  and  $\tilde{w}_1$  in  $W_1$  and

let  $y$  and  $\tilde{y}$  be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

Finally,  $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N + (1+H_1)(1+H_2)}$  yields that  $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$ , and hence,  $g_t$  is a contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous mapping ([19, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow W_1(t)$  is a homeomorphism. This implies the condition ii) of Definition 3.13, and hence, the proof is finished.  $\square$

Now let us illustrate our results by the following examples.

**Example 4.7.** The dynamical behavior of a system in fluid mechanics and turbulence modeling is often described by the incompressible Navier-Stokes equation on an open, bounded domain  $\Omega \subset \mathbb{R}^k$ ,  $k = 2$  or  $3$ , of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p - (u \cdot \nabla)u + f(t, u, p), \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned}$$

where  $\nu > 0$  is the viscosity,  $u = u(t, \xi)$  is the velocity field which is a function of the time  $t$  and the position  $\xi$ ,  $p$  is the pressure,  $f$  is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [9], one obtains a differential-algebraic system of the following form.

$$\begin{aligned} M\dot{U} &= (K + N(U)) U - CP + F(t, U, P), \\ C^T U &= 0, \end{aligned}$$

where  $U(t)$ ,  $P(t)$  approximate the velocity  $u(t, \xi)$  and the pressure  $p(t, \xi)$ , respectively. Here the leading matrix  $M$  is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix  $C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$  is nonsingular.

We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free, so Assumption 1.2 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned} M\dot{U} &= -(K + N(U)) U - CP + F(t, U, P), \\ 0 &= C^T M^{-1} C P - C^T M^{-1} (F - (K + N(U)) U). \end{aligned} \tag{4.2} \text{eq5.3}$$

Clearly, we still need to linearize (4.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.8)

$$\begin{aligned} M\dot{U} &= A_1(t)U + A_2(t)P + g_1(t, U, P), \\ 0 &= C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right) P - C^T M^{-1} \left( \frac{\partial F}{\partial U} - K \right) U + C^T M^{-1} g_2(t, U, P). \end{aligned} \tag{4.3} \text{eq5.4}$$

386 We further notice that since  $C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$  is nonsingular, from the second equation we can uniquely  
 387 determine  $P$  in term of  $U$ , and hence, system (4.2) is indeed strangeness-free. Let

$$A_3(t) := -C^T M^{-1} \left( \frac{\partial F}{\partial U} - K \right), \quad A_4(t) := C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$$

388 Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

389 admits an exponential dichotomy, and  $g_1$  satisfies the  $\varphi$ -Lipschitz condition, and  $g_2$  is a contraction mapping  
 390 (uniformly in time), then there exists a stable manifold for the solution to (4.2).

**Example 4.8.** Consider the nonlinear electrical circuit with Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation  $i_2 = I_0 \sin(k\phi_2)$ , where  $I_0$  and  $k$  are positive constants depend on the device itself. Moreover, the resistance  $R$ , inductance  $L$  and conductance  $G$  are positive. Furthermore,  $i_1$  is the current going through the inductance,  $v_1$  and  $v_2$  are voltage drops across the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead

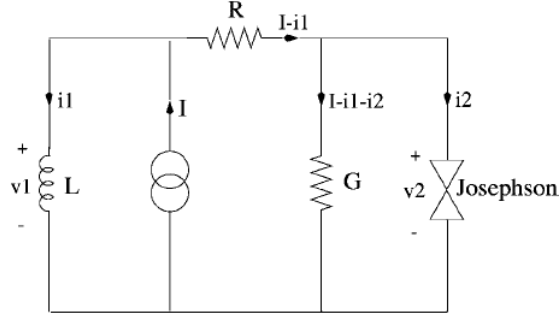


FIGURE 1. Electric circuit with Josephson junction, [23]

of linear resistance, inductance and conductance as in [23], and hence, we see that for the inductance  $i_1 = i_L(L, \phi_1)$ , for the resistance  $v_R = v_R(R, i_1)$ , and for the conductance  $i_G = i_G(G, v_2)$ . Therefore, we obtain the following system, which completely describes the behavior of this circuit.

$$\begin{aligned} \dot{\phi}_1 &= v_1, & (4.4a) \\ \dot{\phi}_2 &= v_2, & (4.4b) \\ i_1 &= i_L(L, \phi_1), & (4.4c) \\ i_2 &= I_0 \sin(k\phi_2), & (4.4d) \\ 0 &= v_1 - v_R(R, i_1) + v_2, & (4.4e) \\ 0 &= -i_G(G, v_2) + I - i_1 - i_2. & (4.4f) \end{aligned}$$

From (4.4c)-(4.4f) we obtain an explicit form of  $v_1$  in terms of  $\phi_1$ ,  $i_1$  and  $v_2$ , so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \quad (4.5a)$$

$$\dot{\phi}_2 = v_2, \quad (4.5b)$$

$$i_1 = i_L(L, \phi_1), \quad (4.5c)$$

$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \quad (4.5d)$$

The linearized version of this system along equilibrium points defined by  $v_2 = 0$ ,  $i_1 = I$ ,  $\phi_1 = LI$ ,  $\phi_2 = n\pi/k$ , reads

$$\dot{\phi}_1 = RI - (R/L)\phi_1 + v_2,$$

$$\dot{\phi}_2 = v_2,$$

$$i_1 = \phi_1/L,$$

$$0 = -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2),$$

will have a positive eigenvalue and a negative one (e.g. [23]). Hence, it admits exponential dichotomy for any odd number  $n$ . Thus, for  $\varphi$ -Lipschitz function  $v_R$  and contraction mapping  $i_G$ , we obtain a stable manifold for (4.5).

#### REFERENCES

- [1] R. Altmann and J. Heiland. *Continuous, Semi-discrete, and Fully Discretised Navier-Stokes Equations*, pages 277–312. Springer International Publishing, Cham, 2019.
- [2] Lorenz T. Biegler, Stephen L. Campbell, and Volker Mehrmann, editors. *Control and Optimization with Differential-Algebraic Constraints*. SIAM Publications, Philadelphia, PA, 2012.
- [3] C. Chicone. *Ordinary Differential Equations with Applications*. Texts in Applied Mathematics. Springer New York, 2013.
- [4] Chuan-Jen Chyan, Nguyen Huu Du, and Vu Hoang Linh. On data-dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems. *Journal of Differential Equations*, 245(8):2078 – 2102, 2008.
- [5] W. A. Coppel. *Dichotomies in Stability Theory*. Springer-Verlag, New York, NY, 1978.
- [6] J.L. Daleckiĭ and M.G. Krein. *Stability of Solutions of Differential Equations in Banach Space*. Translations of mathematical monographs. American Mathematical Society, 2002.
- [7] L. Dieci and T. Eirola. On smooth decompositions of matrices. *SIAM J. Matr. Anal. Appl.*, 20:800–819, 1999.
- [8] Nguyen Huu Du and Vu Hoang Linh. Stability radii for linear time-varying differential-algebraic equations with respect to dynamic perturbations. *Journal of Differential Equations*, 230(2):579 – 599, 2006.
- [9] C. Grossmann, H.G. Roos, and M. Stynes. *Numerical Treatment of Partial Differential Equations*. Springer-Verlag Berlin Heidelberg, 2007.
- [10] Nguyen Thieu Huy. Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line. *Journal of Functional Analysis*, 235(1):330 – 354, 2006.
- [11] Nguyen Thieu Huy. Invariant manifolds of admissible classes for semi-linear evolution equations. *Journal of Differential Equations*, 246(5):1820 – 1844, 2009.
- [12] A. Ilchmann and T. Reis. *Surveys in Differential-Algebraic Equations I*. Differential-Algebraic Equations Forum. Springer, 2013.
- [13] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations – Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

- [14] R. Lamour, R. März, and C. Tischendorf. *Differential-algebraic equations: A projector based analysis*. Differential-Algebraic Equations Forum 1. Berlin: Springer, 2013.
- [15] M. Lentini and R. März. Conditioning and dichotomy in differential algebraic equations. *SIAM Journal on Numerical Analysis*, 27(6):1519–1526, 1990.
- [16] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces II: Function Spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Berlin Heidelberg, 2013.
- [17] Vu Hoang Linh and Volker Mehrmann. Lyapunov, Bohl and Sacker-Sell spectral intervals for differential-algebraic equations. *Journal of Dynamics and Differential Equations*, 21(1):153–194, Mar 2009.
- [18] J.L. Massera and J.J. Schäffer. *Linear differential equations and function spaces*. Pure and applied mathematics. Academic Press, 1966.
- [19] Nguyen Van Minh and Jianhong Wu. Invariant manifolds of partial functional differential equations. *Journal of Differential Equations*, 198(2):381 – 421, 2004.
- [20] Thieu Huy Nguyen. Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line. *Journal of Mathematical Analysis and Applications*, 354(1):372 – 386, 2009.
- [21] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer New York, 2012.
- [22] Frank Rübiger and Roland Schnaubelt. The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions. *Semigroup Forum*, 52(1):225–239, Dec 1996.
- [23] R. Riaza. A matrix pencil approach to the local stability analysis of non-linear circuits. *Internat. J. Circ. Theor. Appl.*, 32:23–46, 2004.
- [24] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. Carnegie-Rochester Conference Series on Public Policy. North-Holland Publishing Company, 1978.
- [25] V.V. Zhikov. On the theory of admissibility of pairs of function spaces. *Soviet Math. Dokl.*, 13(4):1108–1111, 1972.

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