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A numerical method based on finite difference for solving fractional delay differential equations

Behrouz P. Moghaddam*, Zeynab S. Mostaghim
Department of Mathematics, Lahijan Branch, Islamic Azad University, P. O. Box 1616, Lahijan, Iran

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Abstract

Fractional delay differential equations (FDDEs) are widely used in ecology, physiology, physical Sciences and many other areas of applied science. Fractional Delay differential equations usually do not have analytic solutions and can only be solved by some numeric methods. In this paper, a new method, which is generalized from finite difference method, has been provided to solve the delay differential equations. It has been used for numerical solution of such models and used for solving a number of problems such as the problem of the impact of food on the population changes of an area [1], the problem of the fluctuations of the population of adults in an area per time [2], the problem of the number of blood cells in humans [3] and the problem of the effect of noise on light which is reflected from laser to mirror [4]. The proposed method, besides being simple, is so exact and sensible in the solved problems.

Keywords: Finite difference method; Caputo functional derivative; Fractional integral; Fractional delay differential equations

1. Introduction

A functional differential equation is one in which the rate of change of $y(x)$ depends not only on the values of y for the same time value but also on time values less than x. In the simplest case, this has the form

$$y'(x) = f(x, y(x), y(x - \delta)) \tag{1.1}$$

where δ is a constant delay. Throughout this article they will be referred to as delay differential equations (DDEs) or difference differential equations.

The general theory of DDEs is widely developed and we refer the readers to the classical books by Bellman and Cooke [5], Hale [6], Driver [7], El'sgol'ts and Norkin [8] and to the more recent books by Hale and Verduyn Lunel [9], Kolmanovskii and Myshkis [10], Kolmanovskii and Nosov [11], Diekmann et al. [12] and Kuang [13,14],which also include many real-life examples of DDEs and more general retarded functional differential equations.

The Caputo fractional derivative of order α of a function f is defined as follows

$$ {}_0^C D_x^{\alpha} y(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-\tau)^{n-\alpha-1} \dfrac{d^n}{d\tau^n} y(\tau)\, d\tau, \\ \quad n-1 < \alpha < n, \\ y^{(n)}(x), \quad \alpha = n. \end{cases}$$

where n is a positive integer.

* Corresponding author. Tel.: +98 9113359210.
E-mail address: Parsa.math@gmail.com (B.P. Moghaddam).
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For Riemann–Liouville fractional integration and the Caputo fractional derivative, we have the following properties [15]:

(1) For real values of $\alpha > 0$, the Caputo fractional derivative reverses the operation to the Riemann–Liouville integration from the left

$$ {}_{0}^{C}D_{x}^{\alpha}\,{}_{0}I_{x}^{\alpha}y(x) = y(x), \quad \alpha > 0, \qquad y(x) \in C[0, 1]. $$

(2) If $y(x) \in C^{[\alpha]}[0, 1]$, then

$$ {}_{0}I_{x}^{\alpha}\,{}_{0}^{C}D_{x}^{\alpha}y(x) = y(x) - \sum_{j=0}^{[\alpha]-1} \frac{x^j}{j!}\left(\frac{d^j}{dx^j}y\right)(0), \qquad n-1 < \alpha \le n, $$

where $C^{[\alpha]}[0, 1]$ is the space of $[\alpha]$ times, continuously differentiable functions.

In this article we consider the fractional structure of relation (1.1) as follows:

$$ {}_{0}D_{x}^{\alpha}y(x) = f(x, y(x), y(x-\delta)) \tag{1.2} $$

where ${}_{0}D_{x}^{\alpha}$ is the standard Caputo fractional derivative. Throughout this paper, they will be referred to as fractional order functional differential equations with infinite delay (FDDEs) where the existence of solutions for Eq. (1.2) has been considered by Benchohra et al. [16]. In this regard, the authors in sources [17–19] have investigated the existence and uniqueness of the solutions for fractional delay differential equations (FDDEs) and fractional order integro-differential equations. Also in these works the authors present a large amount of abstract of works which have been published so far.

Bhalekar and Daftardar [20] recently solved DDEs of fractional order as follows

$$ {}_{0}D_{x}^{\alpha}y(x) = f(x, y(x), y(x-\delta)), \quad x \in [0, \delta],\ 0 < \alpha \le 1 $$
$$ y^{(k)}(0) = y_0^{(k)}, \quad k = 0, 1, \ldots, m-1, \quad \alpha \in (m-1, m], $$

by extending the Adams–Bashforth–Moulton scheme where f is in general a nonlinear function of its arguments. Besides, Wang [21] lately approximated the delayed fractional order differential equation by combining the general Adams–Bashforth–Moulton method with the linear interpolation method as:

$$ {}_{0}D_{x}^{\alpha}y(x) = f(x, y(x), y(x-\delta)), \quad x \ge 0, \qquad m-1 < \alpha \le m $$
$$ y(x) = \varphi(x), \quad x \le 0, $$

where α is the order of the differential equations, $\varphi(x)$ is the initial value, and m is an integer.

In addition, Wang et al. [22], based on Grunwald–Letnikov definition, introduced a numerical method for nonlinear functional order differential equations with constant time varying delay:

$$ {}_{0}D_{x}^{\alpha}y(x) = f(x, y(x), y(x-\delta)), \quad x \in [a, b], \qquad m-1 < \alpha \le m $$
$$ y(x) = \varphi(x), \quad x \le a, $$

where α is the order of the differential equations, $\varphi(x)$ is the initial value, and m is an integer.

Finally, Morgado et al. [23] concentrated on the following initial value problem for a linear fractional differential equation with finite delay $\delta > 0$

$$ {}_{0}D_{x}^{\alpha}y(x) = ay(x-\delta) + by(x) + f(x), \qquad x > 0, \quad 0 < \alpha \le 1, $$
$$ y(x) = \varphi(x), \quad x \in [-\delta, 0] $$

and was solved by using adaptation of a fractional backward difference method where a and b are constant, f is a continuous function on $[0, \delta]$, $T > 0$, the initial function $\varphi(x)$ is continuous on $[-\delta, 0]$.

Throughout the article, the finite difference method is generalized by the researchers and used for FDDEs. This numerical method includes finite differences without preserving the delay index. To this aim, the paper is divided into three sections. The first part describes the problem. Then, the numerical method for solving the boundary problem is discussed. The Last part, but not the least important one, deals with the analysis of errors by the proposed method. In addition, the adaptation of a variety of differential equations in the mathematical modeling process of difference applications will be considered; for example, the problem of the impact of food on the population changes of an area [1], the problem of the fluctuations of the population of adults in an area per time [2], the problem of the number of blood cells in humans [3] and the problem of the effect of noise on light which is reflected from laser to mirror [4]; the point is that these models are so similar to real phenomena.

2. Description of the problem

In general, we will consider a modeling problem for boundary problems in delay differential equations as follows:

$$ {}_0D_x^{\alpha} y(x) = f(x, y(x), y(x-\delta)) \tag{2.1}$$

where f is in general a nonlinear function of its arguments in a way that if $a<x<b$ and $0<\alpha\leq 1$, subject to the interval and initial condition we have:

$$ y(x) = \phi(x) \qquad -\delta \leq x \leq a \tag{2.2}$$

In which $\phi(x)$ and coefficients $y(x)$ and $y(x-\delta)$ are the smooth function and δ is the amount of delay. If $y(x)$ is to have smooth answer in problem (2.1), it should be verified in the boundary problems of (2.1) and (2.2); they also should be consistent in the interval$[a,b]$ and differential consistent in the interval(a,b) .Finally, the researchers will present a number of examples based on the delayed differential equations of fractional order.

3. Numerical method

In this section we will present a numerical method for solving the boundary problem (2.1) and (2.2) based on finite differences. However; the researchers first introduce $y(b)$, because two boundary conditions for finite differences method are needed.

$$ y(b) = {}_0I_x^{\alpha}\, {}_0^C D_x^{\alpha} y(x)|_{x=b}$$

This numerical method includes the finite difference operator on a specific consistent mesh. In order to save the delay sentences, a mesh parameter can be chosen such as $h=(\delta/m)$ in which $m=pq$ and p is an positive integer and q is the δ Mantissa. This difference method for boundary problem (2.1) and (2.2) is based on the following relations:

$$ for \quad i=1,\ldots,N \qquad y(x)\to y_i \tag{3.1}$$

$$ for \quad i=1,\ldots,N \qquad y(x-\delta)\to y_{i-m} \tag{3.2}$$

$$ for \quad i=-m,-m+1,\ldots,0 \qquad y_i=\phi_i \tag{3.3}$$

$$ {}_0D_x^{\alpha} y(x) = \frac{h^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}]\, D_+ y_{i-j} \tag{3.4}$$

In which $D_+ y_{i-j} = ((y_{i-j+1} - y_{i-j})/h)$.

4. Numerical results

For $\delta=0$ the solution of the boundary problems (2.1) and (2.2) will show the behavior of layers at the end of boundaries. In the following part, the examples are considered and solved them by the use of the proposed method. Then, the graphs will be shown according to different amounts of α. With the use of principle of the binary mesh in [24], the maximum amount of absolute error for the considered problems has been calculated because there is no real answer for the following examples.

$$ E^N = \max_{0\leq i\leq N} |y_i^N - y_{2i}^{2N}| \tag{4.1}$$

Example 1. In food-limited model which is related to the nutrition range of an area, the impact of food on the population of that area is surveyed, and its equation is as:

$$ {}_0D_x^{\alpha} y(x) = r y(x) \left(\frac{k - y(x-\delta)}{k + rcy(x-\delta)} \right) \tag{4.2}$$

under the initial condition:

$$ y(x) = 0.5 \qquad -\delta \leq x \leq 0$$

To obtain $y(b)$, by applying Rieman–Liouvile fractional integration with respect x from the left on both sides of Eq. (4.2), the researchers yield

$$ y(b) = 0.5r \left[\frac{k-0.5}{k+0.5rc} \right] \frac{\Gamma(1)}{\Gamma(\alpha+1)} b^{\alpha}.$$

The finite difference method algorithm for this boundary problem is:

$$ L^N y_i = \begin{cases} for\ i=1,\ldots,N \\ \left(\dfrac{h^{1-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{j=0}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}]\, D_+ y_{i-j} \right) \\ \quad - \dfrac{r k y_i}{k + rcy_{i-m}} + \dfrac{r y_i y_{i-m}}{k + rcy_{i-m}} = 0 \end{cases}$$

$$ for \quad i=-m,-m+1,\ldots,0 \qquad y_i = 0.5$$

Based on the relations (3.1)–(3.4) we have:

$$ for \quad i=1,\ldots,N$$

$$ y_{i+1} = y_i - \sum_{j=1}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] y_{i+1-j} + \sum_{j=1}^{i} [(j+1)^{1-\alpha} - j^{1-\alpha}] y_{i-j} + \frac{rkh^{\alpha}\Gamma(2-\alpha) y_i}{k + rcy_{i-m}} - \frac{rh^{\alpha}\Gamma(2-\alpha) y_i y_{i-m}}{k + rcy_{i-m}}$$

Table 1
The maximum amount of absolute error based on different amounts of δ.

$\delta\downarrow N\rightarrow$	100	150	200	250	300	350
$\delta=8$	1.04E−5	8.17E−7	1.52E−8	0	0	0
$\delta=13$	2.73E−5	4.82E−6	1.35E−7	1.06E−8	0	0

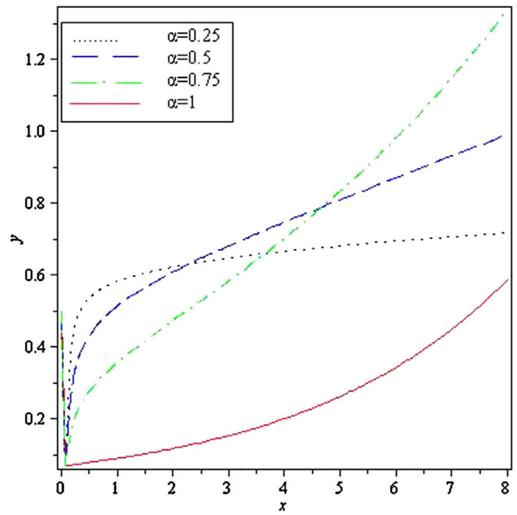

Fig. 1. The numerical solution of Example 1 ($\delta=8$, $r=0.15$, $k=100$, $c=1$).

$$for \quad i=-m,-m+1,\ldots,0 \quad y_i=0.5$$

The results are shown in Table 1 and Figs. 1 and 2.

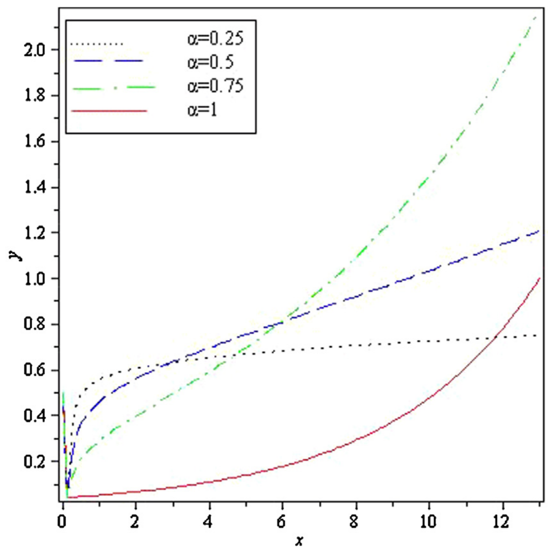

Fig. 2. The numerical solution of Example 1 ($\delta=13$, $r=0.15$, $k=100$, $c=1$).

Table 2
The maximum amount of absolute error based on different amounts of δ.

$\delta\downarrow N\rightarrow$	100	150	200	250	300	350
$\delta=6$	9.61E−7	2.58E−7	1.3E−8	0	0	0
$\delta=20$	7.7E−5	3.76E−7	6.03E−8	1.54E−8	0	0

Example 2. In Mackey–Glass model, the number of blood cells in human body which is dependent on the density of blood is surveyed whose equation is live:

$$_0D_x^{\alpha}y(x)=\frac{\lambda\, a^k y(x-\delta)}{a^k+y^k(x-\delta)}-gy(x) \tag{4.3}$$

Under the initial condition:

$$y(x)=0.02 \quad -\delta\le x\le 0$$

Similar to Example 1, we have

$$y(b)=\frac{\Gamma(1)}{\Gamma(\alpha+1)}\left(\frac{0.02\lambda a^k}{a^k+(0.02)^k}\right)(-0.02g)b^{\alpha},$$

The finite difference method algorithm for this boundary problem is:

$$L^N y_i=\begin{cases} for \quad i=1,\ldots,N \\ \left(\dfrac{h^{1-\alpha}}{\Gamma(2-\alpha)}\displaystyle\sum_{j=0}^{i}[(j+1)^{1-\alpha}-j^{1-\alpha}]\,D_+y_{i-j}\right) \\ \quad +g\,y_i-\dfrac{\lambda\,a^k y_{i-m}}{a^k+(y_{i-m})^k}=0 \end{cases}$$

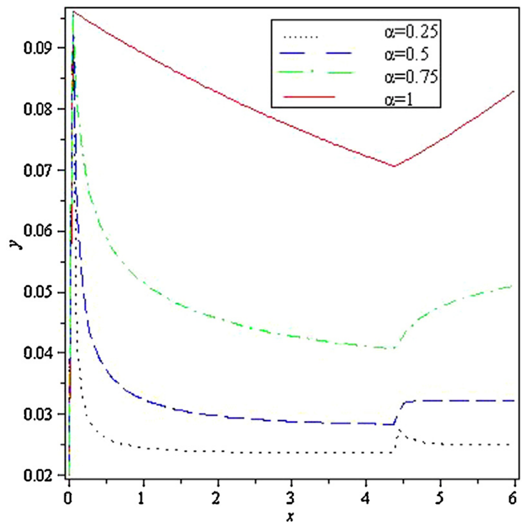

Fig. 3. The numerical solution of Example 2 ($\delta=6$, $\lambda=0.2$, $a=0.1$, $g=0.1$, $k=10$).

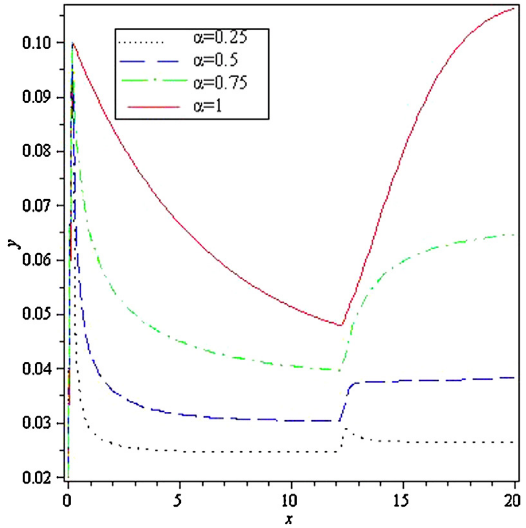

Fig. 4. The numerical solution of Example 2 ($\delta = 20$, $\lambda = 0.2$, $a = 0.1$, $g = 0.1$, $k = 10$).

$$for \quad i = -m, -m+1, \ldots, 0 \quad y_i = 0.02$$

Based on the relations (3.1)–(3.4), we have:

$$for \quad i = 1, \ldots, N$$

$$\begin{aligned} y_{i+1} = {} & (1 - gh^{\alpha}\Gamma(2-\alpha)y_i) \\ & - \sum_{j=1}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]y_{i+1-j} \\ & + \sum_{j=1}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]y_{i-j} \\ & + \frac{\lambda a^m h^{\alpha}\Gamma(2-\alpha)y_{i-m}}{a^m + (y_{i-m})^k} \end{aligned}$$

$$for \ i = -m, -m+1, \ldots, 0 \quad y_i = 0.02$$

The results are shown in Table 2 and Figs. 3 and 4.

Example 3. In Houseflies model, the fluctuation of the adults' population in one area per time is surveyed by the following equation:

$$_0D_x^{\alpha}y(x) = -dy(x) + cy(x-\delta)(k - czy(x-\delta)) \quad (4.4)$$

Under the initial condition:

$$y(x) = 160 \quad -\delta \le x \le 0$$

and

$$y(b) = \frac{160\Gamma(1)}{\Gamma(\alpha+1)}[c(k - 160bz) - d)]b^{\alpha},$$

Table 3
The maximum amount of absolute error based on different amounts of δ.

$\delta\downarrow N\rightarrow$	100	150	200	250	300	350	
$\delta = 3$	4.92E−5	3.08E−6	1.6E−7	1E−8	0	0	
$\delta = 5$	6.32E−5	5.9E−6	4.25E−7	2	41E−7	3.52E−8	0

The finite difference method algorithm for this boundary problem is:

$$L^N y_i = \begin{cases} for \quad i = 1, \ldots, N \\ \left(\dfrac{h^{1-\alpha}}{\Gamma(2-\alpha)} \displaystyle\sum_{j=0}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]\, D_+ y_{i-j} \right) \\ \quad + d\, y_i - cky_{i-m} + c^2 z (y_{i-m})^2 = 0 \end{cases}$$

$$y_i = 160 \quad i = -m, -m+1, \ldots, 0$$

Based on the relations (3.1)–(3.4), we have:

$$for \ i = 1, \ldots, N$$

$$\begin{aligned} y_{i+1} = {} & (1 - dh^{\alpha}\Gamma(2-\alpha)y_i) - \sum_{j=1}^{i}[(j+1)^{1-\alpha} \\ & - j^{1-\alpha}]y_{i+1-j} + \sum_{j=1}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]y_{i-j} \\ & + ck\, h^{\alpha}\Gamma(2-\alpha)y_{i-m} \\ & + c^2 z\, h^{\alpha}\Gamma(2-\alpha)(y_{i-m})^2 \end{aligned}$$

$$for \quad i = -m, -m+1, \ldots, 0 \quad y_i = 160$$

The results are shown in Table 3 and Figs. 5 and 6.

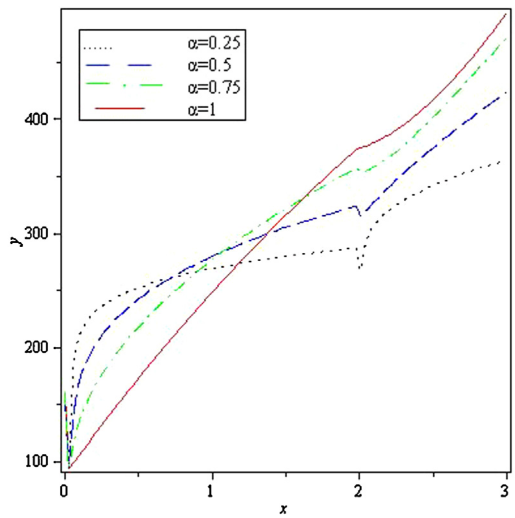

Fig. 5. The numerical solution of Example 3 ($\delta = 3$, $c = 1.81$, $k = 0.5107$, $d = 0.147$, $z = 0.000226$).

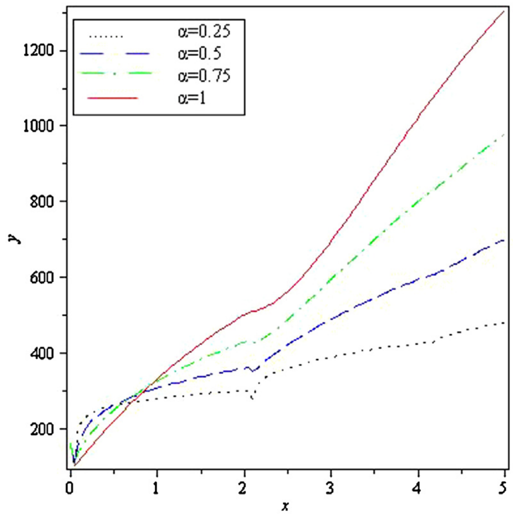

Fig. 6. The numerical solution of Example 3 ($\delta=5$, $c=1.81$, $k=0.5107$, $d=0.147$, $z=0.000226$).

Example 4. This model which is based on the effect of noise on light which is reflected from laser to mirror has been introduced by Pieroux [25]. The equation is as:

$$ {}_0D_x^{\alpha}y(x) = \frac{-1}{\varepsilon}\,y(x) + \frac{1}{\varepsilon}y(x)y(x-\delta) $$

Under the interval and boundary condition:

$$ y(x) = 0.9 \qquad -\delta \le x \le 0 $$

and

$$ y(b) = \left[\frac{-0.1}{\varepsilon} + \frac{(0.1)^2}{\varepsilon}\right]\frac{\Gamma(1)}{\Gamma(\alpha+1)}b^{\alpha} $$

The finite difference method algorithm for this boundary problem is:

$$ L^N y_i = \begin{cases} for \quad i = 1, \ldots, N \\ \left(\dfrac{h^{1-\alpha}}{\Gamma(2-\alpha)}\displaystyle\sum_{j=0}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]\,D_+ y_{i-j}\right) \\ +\dfrac{1}{\varepsilon}\,y_i - \dfrac{1}{\varepsilon}\,y_i y_{i-m} = 0 \end{cases} $$

$$ for \quad i = -m, -m+1, \ldots, 0 \qquad y_i = 0.9 $$

Table 4
The maximum absolute error based on different amounts of δ in example 4.

$\delta\downarrow N$	100	150	200	250	300	350
$\delta=1$	1.28E−6	4.12E−8	0	0	0	0
$\delta=3$	9.57E−6	6.73E−7	5.18E−8	0	0	0

Based on the relations (3.1)–(3.4), we have:

$$ \begin{aligned} &for\, i = 1, \ldots, N \\ &y_{i+j} = \left(1 - \frac{h^{\alpha}}{\varepsilon}\Gamma(2-\alpha)[1 - y_{i-m}]\right)y_i \\ &\qquad - \sum_{j=1}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]\,y_{i+1-j} \\ &\qquad + \sum_{j=1}^{i}[(j+1)^{1-\alpha} - j^{1-\alpha}]\,y_{i-j} \end{aligned} $$

$$ for\, i = -m, -m+1, \ldots, 0 \quad y_i = 0.9 $$

The results are shown in Table 4 and Figs. 7–9.

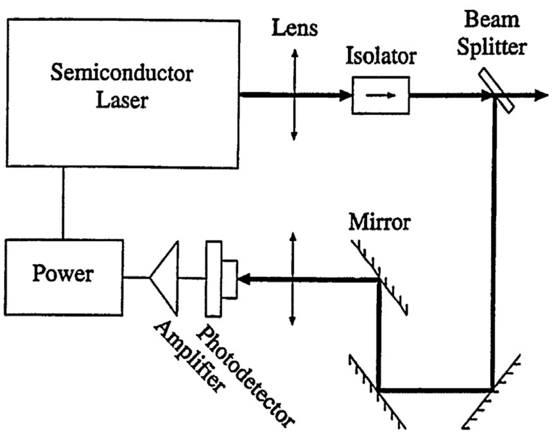

Fig. 7. Semiconductor laser subject to an optoelectronic feedback. The figure illustrates the optoelectronic device used by Saboureau et al. [26]. The feedback operates on the pump of the laser by using part of the output light which is injected into a photodetector connected to the pump. The delay of the feedback is controlled by changing the length of the optical path.

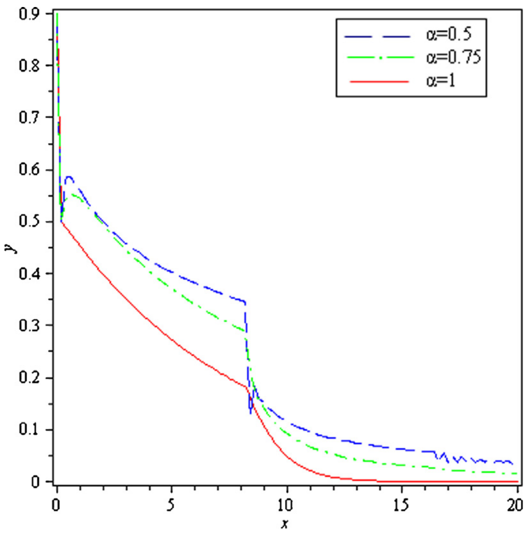

Fig. 8. The numerical solution of Example 4 ($\delta = 1$, $\varepsilon = 0.1$).

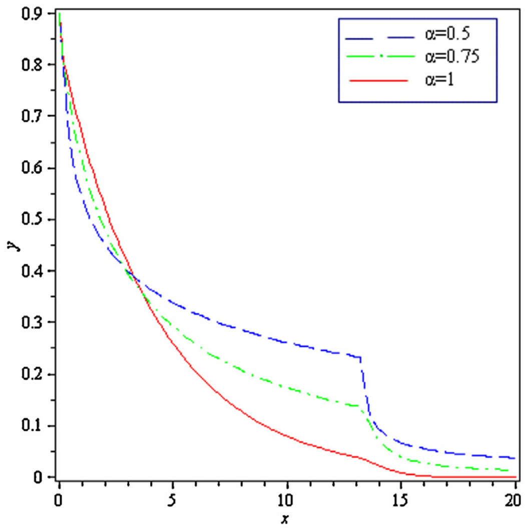

Fig. 9. The numerical solution of Example 4 ($\delta = 3$, $\varepsilon = 0.1$).

5. Conclusion

We consider a boundary problem for differential equation of fractional order as nonlinear. By this method, which is based on finite differences, an estimated solution for solving different kinds of boundary problems with fractional order is obtained. In tables, the maximum amount of error which obeys the prediction theory has been listed, and among them the impact of delay on the amount of error has been studied. It can be seen that with the increase in delay the amount of error has been increased. In Figs. 1–9, the graphs of the mentioned examples for different amounts of α and δ have been illustrated and among them the impact of delay on producing fluctuations has been presented. The merit and prominence of the suggested method is that it is not limited to a specific form or problem. In fact, any differential equation can be solved with it. It, also, provides some results with high accuracy and the least amount of error. However; the proposal is not efficient for solving system of fractional delay differential equation.

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Behrouz Parsa Moghaddam received his M.s and PhD in applied mathematics from the University of Guilan (2005–2009) under the supervision of Professor Arman Aghili. He is currently an assistant professor of applied mathematics in Islamic Azad University, Lahijan Branch. His research interests include integral transforms, orthogonal function and Boundary value problems.