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Exponential Trichotomy of Differential Systems

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In [7] Sacker and Sell introduced a notion of trichotomy for linear differential systems. A stronger notion of trichotomy has been recently introduced in [4] by Elaydi and Hajek. This paper is concerned with the latter notion of trichotomy which will be called here exponential trichotomy.

Section 1 contains the required definitions, preliminary results, and examples. In Section 2 we give a characterization of exponential trichotomy via a certain Liapunov function (Theorem 2.3). In Section 3, we show that an upper triangular system possesses an exponential trichotomy iff the system corresponding to its diagonal possesses one (Theorem 3.1). The limiting equations are used in Section 4 to give several criteria for the existence of exponential trichotomy of the given system. In Section 5, we study the roughness of exponential trichotomy. We show here that if a linear system possesses an exponential trichotomy, then under certain non-linear perturbations, the perturbed system exhibits a qualitative behavior which is similar to that of the nonperturbed system.

1. PRELIMINARIES

Consider the system

$$x' = A(t)\,x, \tag{1}$$

where $x \in C^n$ and $A(t)$ is a bounded and continuous $n \times n$ matrix on the

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whole real line $\mathbf{R}$. Let $X(t)$ be the fundamental matrix of (1) with $X(0)=I$. Let $F(A)=\{A_\tau \mid A(\tau)=A(\tau+t),\ \tau\in\mathbf{R}\}$ be the set of translations of A and $\mathbf{H}(A)=\operatorname{cl} F(A)$ be the hull of A, where the closure is taken in the topology of uniform convergence on compact intervals. The ω-limit set of A is defined as $\omega(A)=\{\tilde{A} \mid \lim_{h\to\infty} A(t+h)=\tilde{A}\}$ and the α-limit set of A is defined as $\alpha(A)=\{\tilde{A} \mid \lim_{h\to-\infty} A(t+h)=\tilde{A}\}$.

DEFINITION 1.1 [4]. The system (1) is said to have an exponential trichotomy (on $\mathbf{R}$) if there exist linear projections P, Q such that

$$PQ=QP,\ P+Q-PQ=I, \tag{2}$$

and constants $K\geqslant 1$, $\alpha>0$ such that

$$\begin{aligned} |X(t)\,PX^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } 0\leqslant s\leqslant t\\ |X(t)(I-P)\,X^{-1}(s)| &\leqslant Ke^{-\alpha(s-t)} && \text{for } t\leqslant s,\ s\geqslant 0\\ |X(t)\,QX^{-1}(s)| &\leqslant Ke^{-\alpha(s-t)} && \text{for } t\leqslant s\leqslant 0\\ |X(t)(I-Q)\,X^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } s\leqslant t,\ s\leqslant 0. \end{aligned} \tag{3}$$

If in the above definition we put $Q=I-P$, then (3) becomes

$$\begin{aligned} |X(t)\,PX^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } s\leqslant t\\ |X(t)(I-P)\,X^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } t\leqslant s \end{aligned} \tag{4}$$

and we then have an exponential dichotomy on $\mathbf{R}$ [2–8].

LEMMA 1.2. *The following statements are pairwise equivalent*:

(i) *The system* (1) *has an exponential trichotomy*,

(ii) *The system* (1) *has an exponential dichotomy on* $\mathbf{R}^{\pm}$ (*that is, exponential dichotomy on* $\mathbf{R}^+$ *and on* $\mathbf{R}^-$) *with projections* P_+ *and* P_-, *respectively, such that* $P_+P_-=P_-P_+=P_-$,

(iii) *There are three mutually orthogonal projections* P_1, P_2, P_3 *with sum* I *and such that*

$$\begin{aligned} |X(t)\,P_1X^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } s\leqslant t\\ |X(t)\,P_2X^{-1}(s)| &\leqslant Ke^{-\alpha(s-t)} && \text{for } t\leqslant s\\ |X(t)\,P_3X^{-1}(s)| &\leqslant Ke^{-\alpha(t-s)} && \text{for } 0\leqslant s\leqslant t\\ &\leqslant Ke^{-\alpha(s-t)} && \text{for } t\leqslant s\leqslant 0. \end{aligned} \tag{5}$$

Proof. (i) implies (ii). It is clear that the first two inequalities in (3) imply exponential dichotomy of (1) on $\mathbf{R}^+$ with the projection $P_+ = P$ and the last two inequalities of (3) imply exponential dichotomy of (1) on $\mathbf{R}^-$ with the projection $P_- = I - Q$. Then from (2) it follows that $P_+ P_- = P - PQ = I - Q = P_- = P_- P_+$.

(ii) implies (iii). Let $P_1 = P_-$, $P_2 = I - P_+$, $P_3 = P_+ - P_-$. Then P_1, P_2, and P_3 are mutually orthogonal projections with sum I. We now verify the inequalities in (5):

$$\begin{aligned}
|X(t)\, P_1 X^{-1}(s)| &= |X(t)\, P_+ P_- X^{-1}(s)| \\
&\leqslant |X(t)\, P_+ X^{-1}(0)|\; |X(0)\, P_- X^{-1}(s)| \\
&\leqslant Le^{-\alpha t} \cdot Le^{\alpha s} \\
&\leqslant L^2 e^{-\alpha(t-s)} = Ke^{-\alpha(t-s)} \qquad \text{for } t \geqslant s; \\
|X(t)\, P_2 X^{-1}(s)| &= |X(t)(I - P_+)\, X^{-1}(s)| \\
&\leqslant |X(t)(I - P_-)\, X^{-1}(0)|\; |X(0)(I - P_+)\, X^{-1}(s)| \\
&\leqslant Le^{-\alpha(0-t)} \cdot Le^{-\alpha(s-0)} \\
&= Ke^{-\alpha(t-s)} \qquad \text{for } t \geqslant s.
\end{aligned}$$

The last two inequalities may be established by noting that $P_+ - P_- = P_+(I - P_-)$ and range of $P_3 =$ range of $(P_+ - P_-) =$ range of $P_+ \cap$ range of $(I - P_-)$.

(iii) implies (ii). Let $P = I - P_3$, $Q = I - P_1$. Then, clearly, (2) and (3) hold. This completes the proof of the lemma.

Remark 1.3. Statement (ii) in the above lemma may be rephrased as follows:

(ii)′ The system (1) has an exponential dichotomy on $\mathbf{R}^\pm$ and every solution of (1) is the sum of two solutions, one is bounded on $\mathbf{R}^+$ and the other is bounded on $\mathbf{R}^-$.

DEFINITION 1.4. The system (1) is said to have an SS-trichotomy (in the sense of Sacker and Sell [7]) if there are three mutually orthogonal projections P_1, P_2, P_3 with sum I and such that

$$\begin{aligned}
&|X(t)\, P_1 X^{-1}(s)| \leqslant Ke^{-\alpha(t-s)} && \text{for } t \geqslant s \\
&|X(t)\, P_2 X^{-1}(s)| \leqslant Ke^{-\alpha(s-t)} && \text{for } t \leqslant s \\
&|X(t)\, P_3 X^{-1}(s)| \leqslant K && \text{for all } s,\ t \in \mathbf{R}.
\end{aligned} \tag{6}$$

It follows from (5) and (6) that exponential trichotomy implies the SS-trichotomy. However, the converse is clearly false as it may be shown by simple examples.

Remark 1.4. (a) For time-variant systems exponential dichotomy on $\mathbf{R}$ is equivalent to exponential trichotomy. This is due to the fact that in this case, exponential dichotomy on $\mathbf{R}^+$ or $\mathbf{R}^-$ implies exponential dichotomy on $\mathbf{R}$. Furthermore, it is always true that exponential dichotomy on $\mathbf{R}$ implies exponential trichotomy.

(b) The conclusion of part (a) holds also for almost periodic systems [2, 8].

(c) If $A(t)$ in Eq. (1) is almost periodic and Eq. (1) has an ordinary dichotomy on $\mathbf{R}^+$ or $\mathbf{R}^-$ [2], then it follows from [2; 9.4] that Eq. (1) has an SS-trichotomy.

We now give an example which is typical in the sense that the general theory of trichotomy follows closely the conclusions obtained in this example.

EXAMPLE 1.5. Consider the scalar equation

$$x' = (-\tanh t)\, x \tag{7}$$

and its adjoint equation

$$y' = (\tanh t)\, y. \tag{8}$$

Then $x(t) = \operatorname{sech}(t)\, x_0$ is the solution of (7) with $x(0) = x_0$ and $y(t) = \cosh(t)\, y_0$ is the solution of (8) with $y(0) = y_0$. Hence Eq. (8) has an exponential dichotomy on $\mathbf{R}^\pm$ with projections $\tilde{P}_+ = 0$, $\tilde{P}_- = I$, respectively. Thus $\tilde{P}_+ \tilde{P}_- = \tilde{P}_- \tilde{P}_+ = \tilde{P}_+$ and, consequently, Eq. (8) has no nontrivial solution which is bounded on $\mathbf{R}$. On the other hand, Eq. (7) has an exponential dichotomy on $\mathbf{R}^+$ with projections $P_+ = I - \tilde{P}_+ = I$ and $P_- = I - \tilde{P}_- = 0$, respectively. Thus $P_+ P_- = P_- P_+ = P_-$. Hence, according to Lemma 1.2, Eq. (7) has an exponential trichotomy. Since $H(\tanh t) = \{\tanh(t+h) \mid h \in \mathbf{R}\} \cup \{1, -1\}$, it follows that no equation in the hull of (8) has a nontrivial solution bounded on $\mathbf{R}$. Furthermore, since $H(-\tanh t) = \{-\tanh(t+h) \mid h \in \mathbf{R}\} \cup \{-1, 1\}$, it follows that the solutions of every equation in $F(-\tanh t)$ are bounded on $\mathbf{R}$ and no nontrivial solution in the α or ω-limit set of $(-\tanh t)$ is bounded on $\mathbf{R}$.

2. TRICHOTOMIES AND LIAPUNOV FUNCTIONS

We first recall the following theorem from Kulik and Kulik [5].

THEOREM 2.1 [5]. *For Eq.* (1) *to have an exponential dichotomy on* $\mathbf{R}$, *it* *is necessary and sufficient that there be a nonsingular bounded continuously differentiable Hermitian matrix $H(t)$ such that*

$$H'(t)+H(t)\,A(t)+A^*(t)\,H(t)\leqslant -\gamma I \qquad \text{for all } t\in\mathbf{R}, \tag{9}$$

where $\gamma>0$ is a constant, and $A^*(t)$ is the adjoint of $A(t)$.

Before giving the main results of this section we need the following lemma.

LEMMA 2.2. *Equation* (1) *has an exponential trichotomy iff its adjoint equation*

$$y'=-A^*(t)\,y \tag{10}$$

has an exponential dichotomy on $\mathbf{R}^{\pm}$ and has no nontrivial solution bounded on $\mathbf{R}$.

Proof. Assume that Eq. (1) has an exponential trichotomy. Then by Lemma 1.2, Eq. (1) has an exponential dichotomy on $\mathbf{R}^{\pm}$ with projections P_+, P_-, respectively, such that $P_+P_-=P_-P_+=P_-$. Let $\tilde{P}_+=I-P_+^*$, $\tilde{P}_-=I-P_-^*$. Then the adjoint equation (10) has an exponential dichotomy on $\mathbf{R}^{\pm}$ with projections $\tilde{P}_+$, $\tilde{P}_-$, respectively. Furthermore, since $\tilde{P}_+\tilde{P}_-=\tilde{P}_-\tilde{P}_+=\tilde{P}_+$, it follows that Eq. (10) has no nontrivial solutions bounded on $\mathbf{R}$. The converse may be proved by reversing the steps in the above argument.

THEOREM 2.3. *Equation* (1) *has an exponential trichotomy iff there exists a bounded continuously differentiable Hermitian matrix $H(t)$ and $\gamma>0$ such that*

$$H'(t)-H(t)\,A^*(t)-A(t)\,H(t)\leqslant -\gamma I \qquad \text{for all } t\in\mathbf{R}. \tag{11}$$

Proof. Assume that (1) has an exponential trichotomy. Then it follows from Lemma 2.2 that the adjoint equation (10) has an exponential dichotomy on $\mathbf{R}^{\pm}$ and it has no nontrivial solutions bounded on $\mathbf{R}$. Hence by [5; 2] there exists a Liapunov transformation $y=T(t)\,z$ which reduces Eq. (10) to the form

$$z'=B(t)\,z, \tag{12}$$

where $z=\operatorname{col}(z_1, z_2, z_3)$ and

$$\begin{aligned} B(t)&=\begin{pmatrix} B_{11}(t) & 0 & B_{13}(t)\\ 0 & B_{22}(t) & B_{23}(t)\\ 0 & 0 & B_{33}(t)\end{pmatrix}\\ &=-T^{-1}A^*T-T^{-1}T'. \end{aligned}$$

We may assume that the norms of $B_{13}(t)$ and $B_{23}(t)$ are sufficiently small, which may be obtained by replacing z_3 by εz_3 with a sufficiently small $\varepsilon > 0$. Furthermore, if $Z_i(t)$ is the fundamental matrix of $z_i' = B_{ii}(t) z_i$, $1 \leqslant i \leqslant 3$, with $Z_i(0) = I$, then

$$\begin{aligned} |Z_1(t) Z_1^{-1}(s)| &\leqslant K e^{-\alpha(t-s)} && \text{for } t \geqslant s \\ |Z_2(t) Z_2^{-1}(s)| &\leqslant K e^{-\alpha(s-t)} && \text{for } t \leqslant s \\ |Z_3(t) Z_3^{-1}(s)| &\leqslant K e^{-\alpha(t-s)} && \text{for } 0 \leqslant s \leqslant t \\ &\leqslant K e^{-\alpha(s-t)} && \text{for } t \leqslant s \leqslant 0. \end{aligned} \tag{13}$$

For brevity, let $Z_i(t, s) = Z_i(t) Z_i^{-1}(s)$, $1 \leqslant i \leqslant 3$. Let $\tilde{H}(t) = \operatorname{diag}(\tilde{H}_1(t), \tilde{H}_2(t), \tilde{H}_3(t))$, where

$$\tilde{H}_1(t) = \int_t^{\infty} Z_1(t, s) Z_1^*(t, s)\, ds$$

$$\tilde{H}_2(t) = -\int_{-\infty}^{t} Z_2(t, s) Z_2^*(t, s)\, ds$$

$$\tilde{H}_3(t) = \int_t^{0} Z_3(t, s) Z_3^*(t, s)\, ds.$$

Then it follows from (13) that $\tilde{H}(t)$ is a bounded continuously differentiable Hermitian matrix. Moreover, $\tilde{H}'(t) = \operatorname{diag}(\tilde{H}_1 B_{11}^* + B_{11} \tilde{H}_1, \tilde{H}_2 B_{22}^* + B_{22} \tilde{H}_2, \tilde{H}_3 B_{33}^* + B_{33} \tilde{H}_3) - 3I$. Since $B_{13}(t)$ and $B_{23}(t)$ are sufficiently small in norm, it follows that

$$\tilde{H}'(t) - \tilde{H}(t) B(t) - B(t) \tilde{H}(t) \leqslant -2I. \tag{14}$$

Going back to the variables of (10) we obtain a Hermitian matrix $H(t)$ which is bounded, and continuously differentiable and satisfies (11).

Conversely, assume that (11) holds. Then it follows from [2; 7.1] that Eq. (10) has an exponential dichotomy on $\mathbf{R}^{\pm}$ with projections P_+ and P_-, respectively. We claim that $P_+ P_- = P_- P_+ = P_+$. This is equivalent to saying that Eq. (10) has no nontrivial solutions bounded on $\mathbf{R}$. To prove the claim let $V(t, y) = y^*(t) H(t) y(t)$. Then from (11) it follows that $V(t, y)$ is strictly decreasing. Let $\hat{y}(t)$ be a nontrivial solution of (10) bounded on $\mathbf{R}$. Then $V(t, \hat{y}(t))$ is bounded on $\mathbf{R}$. Furthermore, $V(t, \hat{y}(t)) > 0$ for all $t \in \mathbf{R}^+$. For otherwise, if $V(t_1, \hat{y}(t_1)) \leqslant 0$ for some $t_1 \in \mathbf{R}^+$, then $V(t, y(t)) < 0$ for all $t \in (t_1, \infty)$. Therefore

$$V'(t, \hat{y}(t)) \leqslant -|\hat{y}(t)|^2 \leqslant \alpha V(t, \hat{y}(t)) \qquad \text{for all } t \in (t_1, \infty), \text{ for some } \alpha > 0.$$

Hence $V(t, \hat{y}(t)) \leqslant V(s, \hat{y}(s))\, e^{\alpha(t-s)}$ for $t_1 < s \leqslant t < \infty$, and $-V(t, \hat{y}(t)) \geqslant$

$-V(s, \hat{y}(s))\, e^{\alpha(t-s)}$ for $t_1 < s \leqslant t < \infty$. This contradicts the boundedness of $V(t, \hat{y})$ on $\mathbf{R}$. Thus $V(t, \hat{y}(t)) > 0$ for all $t \in \mathbf{R}^+$.

By a similar argument, one can show that $V(t, \hat{y}(t)) > 0$ for all $t \in \mathbf{R}^-$. This implies that $V(t, \hat{y}(t))$ is unbounded on $\mathbf{R}$ and we then have a contradiction. Consequently $V(t, \hat{y}(t)) = 0$ and again we have a contradiction. This completes the proof of the claim. Now let $Q_+ = I - P_+^*$ and $Q_- = I - P_-^*$. Then Eq. (1) has an exponential dichotomy on $\mathbf{R}^+$ with projections Q_+, Q_-, respectively, with $Q_+ Q_- = Q_- Q_+ = Q_-$. It follows from Lemma 1.2 that Eq. (1) has an exponential trichotomy. The proof of the theorem is now complete.

3. Triangular Systems

In [6] Palmer has shown that if $A(t)$ in (1) is upper triangular, then Eq. (1) has an exponential dichotomy on $\mathbf{R}^+$ iff each scalar equation $x_i' = a_{ii} x_i$ has. Sacker and Sell [9] gave an example which shows that the necessity of the above result of Palmer does not hold for exponential dichotomy on $\mathbf{R}$. We now give a result which shows that Palmer's theorem extends for exponential trichotomy.

Theorem 3.1. *Let $A(t) = (a_{ij})$ be a bounded, continuous, and upper triangular matrix function defined on* $\mathbf{R}$. *Then* Eq. (1) *has an exponential trichotomy iff each scalar equation*

$$x_i' = a_{ii}(t)\, x_i \tag{15}$$

has.

Proof. Assume that Eq. (1) has an exponential trichotomy. Then by Lemma 2.2 the adjoint equation (10) has an exponential dichotomy on $\mathbf{R}^\pm$ and has no nontrivial solution bounded on $\mathbf{R}$. Then as in the proof of Theorem 1 in Palmer [6] each scalar equation

$$y_i' = -\overline{a_{ii}}\, y_i \tag{16}$$

has an exponential dichotomy on $\mathbf{R}^\pm$. Claim that no equation in (16) has a nontrivial solution bounded on $\mathbf{R}$. For otherwise the whole diagonal system

$$y_i' = -\overline{a_{ii}}\, y_i, \qquad 1 \leqslant i \leqslant n, \tag{17}$$

has a nontrivial solution bounded on $\mathbf{R}$. Then a sufficiently small perturbation of (17) also has a nontrivial solution bounded on $\mathbf{R}$. Bylov [1] has shown that the adjoint upper triangular system (10) is kinematically similar

[2] to an arbitrary small perturbation of the diagonal system (17) by using the so-called β-transformation, where $\beta > 0$ is sufficiently small. Hence Eq. (10) must have a nontrivial bounded solution on $\mathbf{R}$, which is a contradiction. It follows from Lemma 2.2 that each scalar equation (15) has an exponential trichotomy. The converse may be established using a similar argument to that used by Palmer [6] in the proof of Theorem 1.

4. The Limiting Equations

In this section we assume that $A(t)$ is bounded and uniformly continuous $\mathbf{R}$.

Theorem 3.1. *The following statements are pairwise equivalent:*

(i) *Equation* (1) *has an exponential trichotomy.*

(ii) *No equation in the hull of the adjoint equation* (10) *has a nontrivial solution bounded on* $\mathbf{R}$.

(iii) *No equation in the* α*- or* ω*-limit set of* (1) *has a nontrivial solution bounded on* $\mathbf{R}$ *and the adjoint equation* (11) *has no nontrivial solution bounded on* $\mathbf{R}$.

Proof. (i) implies (ii). If Eq. (1) has an exponential trichotomy, then according to Lemma 1.2, it has an exponential dichotomy on $\mathbf{R}^{\pm}$ with projections P_+ and P_-, respectively, such that $P_+P_- = P_-P_+ = P_-$. Hence, it follows from Lemma 2.2 that the adjoint equation (10) has an exponential dichotomy on $\mathbf{R}^{\pm}$ and has no nontrivial solution bounded on $\mathbf{R}$. This implies that no equation in the hull of (10) has a nontrivial solution bounded on $\mathbf{R}$.

(ii) implies (iii). Assume that no equation in the hull of the adjoint system (10) has a nontrivial solution bounded on $\mathbf{R}$. Then it follows from [8; 3] that each equation in the α- or ω-limit set of (10) has an exponential dichotomy on $\mathbf{R}$. This implies that each equation in the α- or ω-limit set of (1) has an exponential dichotomy on $\mathbf{R}$. The conclusion of (iii) now follows.

(iii) implies (i). This follows easily from the previous argument.

5. Roughness of Exponential Trichotomy

Throughout this section we will assume that $A(t)$ is a continuous matrix on $\mathbf{R}$. Let C denote the Banach space of all bounded continuous vector functions $f(t)$ with the norm $\|f\|_C = \sup_{t \in R} |f(t)|$. Let M denote the

Banach space of all locally integrable vector functions $f(t)$ (i.e., $f(t)$ is measurable and $\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$) with $\int_t^{t+1} |f(s)|\, ds < \infty$ and norm $\|f\|_M = \int_{-\infty}^{\infty} |f(t)|\, dt$.

The following lemma extends [2; 3.1].

LEMMA 5.1. *Let $\gamma(t)$ be a nonnegative locally integrably function such that*

$$\int_t^{t+1} \gamma(s)\, ds \leqslant b \qquad \textit{for all } t \in \mathbf{R}.$$

If $\alpha > 0$, then for all $t \in \mathbf{R}$

$$\int_{-\infty}^{\infty} e^{-\alpha|t-s|} \gamma(s)\, ds \leqslant \frac{2b}{1-e^{-\alpha}}$$

Proof.

$$\int_{-\infty}^{\infty} e^{-\alpha|t-s|} \gamma(s)\, ds = \int_{-\infty}^{t} e^{-\alpha(t-s)} \gamma(s)\, ds + \int_t^{\infty} e^{-\alpha(s-t)} \gamma(s)\, ds. \quad (18)$$

Now

$$\begin{aligned} \int_{t-m-1}^{t-m} e^{-\alpha(t-s)} \gamma(s)\, ds &= e^{-\alpha t} \int_{t-m-1}^{t-m} e^{\alpha s}\, \gamma\,(s)\, ds \\ &\leqslant b e^{-\alpha t} \sup e^{\alpha s} \int_{t-m-1}^{t-m} \\ &\leqslant b e^{-\alpha t} e^{\alpha(t-m)} = b e^{-\alpha m}. \end{aligned}$$

Similarly, $\int_{t+m}^{t+m+1} e^{-\alpha(s-t)} \gamma(s)\, ds \leqslant b e^{-\alpha m}$. Hence

$$\int_{-\infty}^{t} e^{-\alpha(t-s)} \gamma(s)\, ds = \sum_{m=0}^{\infty} \int_{t-m-1}^{t-m} e^{-\alpha(t-s)} \gamma(s)\, ds \leqslant b \sum_{m=0}^{\infty} e^{-\alpha m} = \frac{b}{1-e^{-\alpha}}, \quad (19)$$

and

$$\int_t^{\infty} e^{-\alpha(s-t)} \gamma(s)\, ds = \sum_{m=0}^{\infty} \int_{t+m}^{t+m+1} e^{-\alpha(s-t)} \gamma(s)\, ds \leqslant b \sum_{m=0}^{\infty} e^{-\alpha m} = \frac{b}{1-e^{-\alpha}}. \quad (20)$$

Substituting from (19) and (20) into (18) one obtains the conclusion of the lemma.

Consider the inhomogeneous differential equation

$$y' = A(t)\, y + f(t). \tag{21}$$

THEOREM 5.2. *The inhomogeneous equation* (21) *has at least one bounded solution on* $\mathbf{R}$ *for every* $f \in M$ *iff Eq.* (1) *has an exponential trichotomy.*

Proof. The necessity follows from [2; 3.2]. To prove the sufficiency, assume that Eq. (1) has an exponential trichotomy with projections P_+ and P_- (Lemma 1.2). Put

$$G(t, s) = \begin{cases} X(t)\, P_+ X^{-1}(s) & \text{for } 0 < s \leqslant \max(t, 0) \\ -X(t)(I - P_+)\, X^{-1}(s) & \text{for } \max(t, 0) < s \\ X(t)\, P_- X^{-1}(s) & \text{for } s \leqslant \min(t, 0) \\ -X(t)(I - P_-)\, X^{-1}(s) & \text{for } \min(t, 0) < s \leqslant 0. \end{cases} \tag{22}$$

Then

$$y(t) = \int_{-\infty}^{\infty} G(t, s)\, f(s)\, ds$$

is a solution of Eq. (21). Furthermore $|G(t, s)| \leqslant K e^{-\alpha|t-s|}$. It follows from Lemma 5.1 that

$$|y(t)| \leqslant \frac{2b}{1 - e^{-\alpha}} \qquad \text{for all } t \in \mathbf{R}.$$

The proof of the theorem is now complete.

According to [2] Eq. (1) is said to have a bounded growth on an interval J if, for some fixed $h > 0$, there exists a constant $b \geqslant 1$ such that every solution $x(t)$ of (1) satisfies

$$|X(t)| \leqslant b\, |x(s)| \qquad \text{for } s, t \in J,\ s \leqslant t \leqslant s + h.$$

Equivalently, Eq. (1) has a bounded growth on J if

$$|X(t)\, X^{-1}(s)| \leqslant K e^{\alpha(t-s)} \qquad \text{for } t \geqslant s,\ t, s \in J,$$

for some real constants K and α. It is clear that Eq. (1) has bounded growth if its coefficient matrix $A(t)$ is bounded.

THEOREM 5.3. *Suppose that Eq.* (1) *has a bounded growth on* $\mathbf{R}$. *Then the inhomogeneous equation* (21) *has at least one bounded solution for every function* $f \in C$ *iff Eq.* (1) *has an exponential trichotomy.*

Proof. The necessity follows from [2; 3.2]. The proof of the sufficiency is similar to that of Theorem 5.2.

Consider now the nonlinear differential system

$$y' = A(t)\, y + g(t, y), \tag{23}$$

where $|g(t, y)| \leqslant \varepsilon$ and $|g(t, y_1) - g(t, y_2)| \leqslant \varepsilon\, |y_1 - y_2|$ for all $t \in \mathbf{R}$; $y_1, y_2 \in \mathbf{R}^n$; and $2K\varepsilon < 1$.

THEOREM 5.4. *Suppose that Eq.* (1) *has an exponential trichotomy with projections* P_1, P_2, P_3 *as defined in Lemma* 1.2. *Then for any* $b \in \mathbf{R}^n$, *there is a unique solution to the integral equation*

$$y_i(t) = X(t)\, P_i b + \int_{-\infty}^{\infty} G(t, s)\, g(s, y_i(s))\, ds, \qquad 1 \leqslant i \leqslant 3. \tag{24}$$

Furthermore, $y_i(t)$, $1 \leqslant i \leqslant 3$, is a solution of (1) on $\mathbf{R}$, where $y_1(t)$ is bounded on $\mathbf{R}^+$, $y_2(t)$ is bounded on $\mathbf{R}^-$, and $y_3(t)$ is bounded on $\mathbf{R}$.

Proof. For each $\theta \in \mathbf{R}$, put

$$y_{1\theta}(t) = X(t)\, P_1 b + \int_{\theta}^{\infty} G(t, s)\, g(s, y_1(s))\, ds. \tag{25}$$

Consider the operator

$$Fy(t) = X(t)\, P_1 b + \int_{\theta} G(t, s)\, g(s, y(s)), \qquad t \geqslant \theta.$$

Then F is a mapping from the space of continuous bounded functions on $[\theta, \infty)$ into itself. Furthermore,

$$\begin{aligned} |Fy(t) - Fz(t)| &\leqslant \int_{\theta}^{\infty} |G(t, s)|\, |g(s, y(s)) - g(s, z(s))|\, ds \\ &\leqslant K\varepsilon \int_{\theta}^{\infty} e^{-\alpha |t-s|}\, |y(s) - z(s)|\, ds. \end{aligned}$$

This implies that $\|Fy - Fz\| \leqslant 2K\varepsilon\, \|x - y\|$, where $\|\ \|$ is the supremum norm. Since $2K\varepsilon < 1$, F is a contraction map. Consequently, F has a unique fixed point $y = y_{1\theta}$, where

$$y_{1\theta} = X(t)\, P_1 b + \int_{\theta}^{\infty} G(t, s)\, g(s, y_{1\theta}(s))\, ds, \qquad t \geqslant \theta.$$

Consider any two such fixed points $y_{1\theta}$, $y_{1\lambda}$, with $\lambda \geqslant \theta$. Then for any $t \geqslant \lambda$,

$$y_{1\theta}(t) - y_{1\lambda}(t) = \int_\theta^\lambda G(t, s)\, g(s, y_{1\theta}(s))\, ds$$

$$+ \int_\lambda^\infty G(t, s)[g(s, y_{1\theta}(s)) - g(s, y_{1\lambda}(s))]\, ds$$

$$|y_{1\theta}(t) - y_{1\lambda}(t)| \leqslant \varepsilon K e^{-\alpha(t-\lambda)} + \varepsilon K \int_\lambda^\infty e^{-\alpha|t-s|}\, |y_{1\theta}(s) - y_{1\lambda}(s)|\, ds.$$

This implies by [2; 4.1] that

$$|y_{1\theta}(t) - y_{1\lambda}(t)| \leqslant L e^{-\beta(t-\lambda)}, \qquad t \geqslant \lambda \geqslant \theta,$$

where L, β are certain positive constants. Hence the net $\{y_{1\theta}(t)\}$ converges to $y_1(t)$ as $\theta \to -\infty$. Moreover,

$$y_1(t) - \left[X(t)\, P_1 b + \int_{-\infty}^\infty G(t, s)\, g(s, y_1(s))\, ds\right]$$

$$= y_1(t) - y_{1\theta}(t) + \int_\lambda^\infty G(t, s)[g(s, y_{1\theta}(s)) - g(s, y_1(s))]\, ds$$

$$+ \left[\int_\theta^\lambda G(t, s)\, g(s, y_{1\theta}(s))\, ds - \int_{-\infty}^\lambda G(t, s)\, g(s, y_1(s))\, ds\right].$$

Thus

$$\left|y_1(t) - \left[X(t)\, P_1 b + \int_{-\infty}^\infty G(t, s)\, g(s, y_1(s))\, ds\right]\right|$$

$$\leqslant |y_1(t) - y_{1\theta}(t)| + 2K\varepsilon\, \|y_1 - y_{1\theta}\|_{[\lambda, \infty)} + 2K\varepsilon e^{-\alpha(t-\lambda)}.$$

Keeping t fixed, for any $\delta > 0$ there exists λ such that $2K\varepsilon e^{-\alpha(t-\lambda)} < \delta/3$. For this λ, $y_{1\theta}(t) \to y_1(t)$ uniformly on $[\lambda, \infty)$. We may assume that $|y_1(t) - y_{1\theta}(t)| < \delta/3$, $2K\varepsilon\, \|y_\theta - y_{1\theta}\|_{[\lambda, \infty)} < \delta/3$ for large $-\theta$. Hence

$$y_1(t) = X(t)\, P_1 b + \int_{-\infty}^\infty G(t, s)\, g(s, y_1(s))\, ds.$$

Similarly, one may establish similar conclusions for $y_2(t)$ and $y_3(t)$. Furthermore, $y_1(t)$ is bounded on $\mathbf{R}^+$, $y_2(t)$ is bounded on $\mathbf{R}^-$, and $y_3(t)$ is bounded on $\mathbf{R}$. This completes the proof of the theorem.

COROLLARY 5.5. *Suppose that Eq.* (1) *has an exponential trichotomy. Then for each solution* $x(t)$ *of* (1) *there exists a solution* $y(t)$ *of* (23) *such that for all* $t \in \mathbf{R}$, $|x(t) - y(t)| \leqslant L$ *for some positive constant L. Furthermore, if* $x(t)$ *is bounded on* $\{\mathbf{R}^+\}\{\mathbf{R}^-\}\{\mathbf{R}\}$, *then* $y(t)$ *possesses the same boundedness property.*

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