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On the Stability of Linear Delay-Differential Algebraic Systems: Exact Conditions via Matrix Pencil Solutions

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Abstract— **In this paper we study the stability properties of a class of linear systems expressed by semi-explicit delay differential algebraic equations, that is a system of functional differential equations coupled with a system of (continuous-time) difference equations. We show that the stability analysis (delay-independent, delay-dependent, crossing characterization) in the commensurate delay case can be performed by computing the generalized eigenvalues of certain matrix pencils, which can be executed efficiently and with high precision. The results extend previously known work on retarded, neutral, and lossless propagation systems, and demonstrate that similar stability tests can be derived for such systems.**

Index Terms— **delay-differential algebraic equations, stability, switches, matrix pencil.**

I. INTRODUCTION

In certain control problems, we encounter partial differential equations of hyperbolic type with mixed initial, and derivative boundary conditions in feedback interconnection, see, for instance, processes including lossless transmission lines, steam and/or water pipes. As seen in [1], [4], such models can be easily described by semi-explicit delay differential algebraic equations, that is dynamical systems of coupled differential, and (continuous-time) difference equation, where the elements of "interconnection" are represented by the delay terms, constant or distributed. Several examples in this sense can be found in [13], [18]. As pointed out by [17], the theoretical, and the numerical analysis of such systems described by delay-differential algebraic equations is far from complete even for the linear (relatively simple) case (see also [2] for further discussions). Some discussions on the use of such models in control theory can be found in [25].

The aim of this paper is to give a complete characterization of exponential stability of linear propagation systems in state-space representation, under the assumption of commensurate delays. The considered approach is based on the computation of the generalized eigenvalues of some appropriate matrix pencils, and their distribution with respect to the unit circle of the complex plane will give the stability type: delay-independent/delay-dependent, and in the second case, the existence or not of several (stability) delay intervals, that is the characterization of the crossing direction of the roots of the characteristic equation as a function of the delay parameter.

The derived conditions are *necessary, and sufficient*, and to the best of the authors' knowledge, there does not exist any similar results in the literature. Although the proposed method follows the lines of previous works of the authors devoted to retarded, and/or neutral systems (see, for instance, [3], [9], [19]), however the construction is quite distinct, and more general, and it makes use of some appropriate *transformation* of the original characteristic equation. It is important to note that the approach considered in the paper allows to handle *unitarily* both *retarded*, *neutral* systems, and opens the perspective to give complete solutions for the stability characterization of the *singular* delay systems in terms of matrix pencils (both retarded, and neutral cases). Such singular cases are not treated, since they are out of the scope of the paper, but the way to solve them is briefly outlined. Furthermore, the solutions in the particular case of *lossless propagation* systems will be also derived.

As in the retarded, and neutral cases, the advantage of the method lies in its *simplicity*, and in the fact that the corresponding matrix pencils are *finite-dimensional*. Finally, the approach considered here allows some new insights for the known retarded, and neutral cases.

The remaining paper is organized as follows: Section 2 includes some preliminary results on the stability of propagation systems. Section 3 is devoted to the main results, including delay-independent, and delay-dependent characterizations using the distribution of the generalized eigenvalues of some appropriate finite-dimensional matrix pencils. Various comments and interpretations in the retarded, and neutral cases are considered in Section 4. A (simple) illustrative example is presented in Section 5, and some concluding remarks end the paper. For the brevity of the paper, the proofs of the results are omitted, but the underlying (main) ideas are presented.

II. PROBLEM FORMULATION, AND PRELIMINARY RESULTS

We begin with a brief description of our notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{C}$ the set of complex numbers, and $\mathbb{R}_+$ the set of nonnegative real numbers. Denote the open right half plane by $\mathbb{C}_+ := \{s : \Re(s) > 0\}$, the closed right half plane by $\overline{\mathbb{C}}_+$, and the imaginary axis by $\partial\mathbb{C}_+$. Similarly, denote the open unit disc by $\mathbb{D}$, the unit circle by $\partial\mathbb{D}$, and the closed exterior of the unit disc by $\mathbb{D}^c$. For any $z \in \mathbb{C}$, we denote its complex conjugate by $\bar{z}$. For a matrix

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A, denote its spectrum by $\sigma(A)$, and its spectral radius by $\rho(A)$. Let $A \oplus B$ denote the Kronecker sum, and $A \otimes B$ the Kronecker product, of the matrices A and B (see, e.g., [11]).

Let P_1, P_2 be projections on $\mathbb{R}^n$, such that $P_1 + P_2 = I_n$, $P_1 P_2 = 0$, $P_2 P_1 = 0$. Define now the following system:

$$P_1 \dot{z}(t) + P_2 z(t) \quad = \quad g(t, z_t), \tag{1}$$

where $z_t(\theta) = z(t+\theta)$, $\theta \in [-\bar{\tau}, 0]$, and $\bar{\tau} \in \mathbb{R}_+$, and g an appropriate continuous function. Following the terminology in [14], such a system is described by a set of *"hybrid" functional differential equations*. It is easy to see that, by an appropriate change of coordinates, (1) can be rewritten as a system of functional differential equations coupled with a a system of functional equations. In fact, if the dimension of the range of the operator P_1 is n_1, then there exists a nonsingular matrix S such that $SP_1S^{-1} = diag(I_{n_1}, 0)$, with the zero block of appropriate dimension. Since $P_1 + P_2 = I_n$, it follows that: $SP_2S^{-1} = diag(0, I_{n_2})$, with $n_2 = n - n_1$. Thus, if we take $x(t) = Sz(t)$, the system (1) rewrites as follows:

$$\begin{cases} \dot{x}_1(t) = f_1(t, x_t) \\ x_2(t) = f_2(t, x_t), \end{cases} \tag{2}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$, $x_t(\theta) = x(t+\theta)$, for all $\theta \in [-\tau, 0]$ and $f(t, \phi) = Sg(t, S^{-1}\phi)$, and f_1, f_2 denotes an appropriate partition of f (see, e.g. [14] for further details).

In the sequel, we shall focus on the following linearization of the system (2) in the presence of multiple delays, supposed commensurate:

$$\begin{cases} \dot{x}_1(t) = Ax_1(t) + \displaystyle\sum_{k=1}^{m} (A_k x_1(t-k\tau) + B_k x_2(t-k\tau)) \\ x_2(t) = Cx_1(t) + \displaystyle\sum_{k=1}^{m} (C_k x_1(t-k\tau) + D_k x_2(t-k\tau)), \end{cases} \tag{3}$$

where $x_i \in \mathbb{R}^{n_i}$, for $i = 1, 2$, and under some appropriate initial conditions on $(x_{1t}(\cdot), x_{2t}(\cdot)) \in \mathcal{C}([-m\tau, 0], \mathbb{R}^{n_1}) \times \mathcal{C}([-m\tau, 0], \mathbb{R}^{n_2})$, where $x_{it}(\cdot)$ denotes the restriction to the interval $[t - m\tau, t]$, translated to $[-m\tau, 0]$. In the case when $A_k = 0$, $C_k = 0$, for all $k = 1, \ldots, m$, the system (3) becomes a standard *lossless propagation system* in the sense specified by [12], [13], [24]. Further remarks, and examples concerning this last case when the matrix C is *not invertible*, and when $n_1 \neq n_2$ can be found in [22]. The fact that we have the same number of delays in both differential, and (continuous-time) difference equations does not represent a restriction since we may consider the corresponding matrices $B_m = B_{m-1} = \ldots = B_{m_1+1}$, or $D_m = D_{m-1} = \ldots = D_{m_2+1}$ matrices equal to 0, where the corresponding positive integers $m_1, m_2 < m$ satisfy $m_1 \neq m_2$ *etc.*

Remark 1 (Retarded/neutral cases): It is easy to see that if $n_1 = n_2 = \nu$, $A_k = C_k = D_k = 0$, for all $k = 1, \ldots, \nu$, and $C = I_n$, the system (3) rewrites as a standard retarded (delay) system. Similarly, if $n_1 = n_2 = \nu$, $C = I_\nu$, and there exist at least one positive integer k_0, such that $D_{k_0} \neq 0$, (3) rewrites as a linear neutral system.

Notice that there is no any rank constraint on the matrix C, fact which opens the possibility to treat also some (particular) singular cases for neutral systems. □

In the stability analysis of (3), as seen in [15], [16], [18], one needs explicitly the stability of the *difference operator* $\mathcal{D} : \mathcal{C}([-m\tau, 0], \mathbb{R}^{n_2}) \mapsto \mathbb{R}^{n_2}$, and defined by

$$\mathcal{D}(\phi) = \phi(0) - \sum_{k=1}^{m} D_k \phi(-k\tau). \tag{4}$$

The stability condition is given by:

Lemma 1: The difference operator $\mathcal{D}$ defined by (4) is stable if and only if:

$$\rho(D) < 1, \tag{5}$$

where

$$D := \begin{bmatrix} D_1 & \cdots & D_{m-1} & D_m \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}. \tag{6}$$

Furthermore, the stability is guaranteed for all positive delay values τ.

Let us consider now the case free of delays ($\tau = 0$). We shall say that (3) with $\tau = 0$ is *exponentially stable* if the corresponding characteristic equation defined by $\mathcal{P}(s; 0)$ has all its roots in $\mathbb{C}^-$. With the assumption (5), it follows that the exponential (or asymptotic) stability of the system free of delays is reduced to the *Hurwitz stability* of the matrix

$$A + \sum_{k=1}^{m} A_k + \sum_{k=1}^{m} B_k \left(I_{n_2} - \sum_{k=1}^{m} D_k \right)^{-1} \left(C + \sum_{k=1}^{m} C_k \right).$$

III. MAIN RESULTS

Introduce now the following matrix pencils:

$$\Lambda_i(z) \quad = \quad zU_i + V_i, \qquad i = 1, 2 \tag{7}$$

where U_i, V_i are given by:

$$U_1 = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & H_m \end{bmatrix}, \tag{8}$$

$$V_1 = \begin{bmatrix} 0 & -I & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & -I \\ H_{-m} & H_{-m+1} & \cdots & H_0 & \cdots & H_{m-1} \end{bmatrix}, \tag{9}$$

$$U_2 = \begin{bmatrix} -\displaystyle\sum_{k=1}^{m} A_k & -\displaystyle\sum_{k=1}^{m} B_k \\ -\displaystyle\sum_{k=1}^{m} C_k & -\displaystyle\sum_{k=1}^{m} D_k \end{bmatrix}, \tag{10}$$

$$V_2 = \begin{bmatrix} -A & 0 \\ -C & I_{n_2} \end{bmatrix}, \tag{11}$$

where the identity, and zero matrices have appropriate dimensions, and the matrices H_0 and $H_{\pm k}$ $(k = 1, \ldots, m)$ are given as follows:

$$H_0 = \begin{bmatrix} A \oplus A^T & 0 & I_{n_1} \otimes C^T \\ C \otimes I_{n_1} & I_{n_1 n_2} & 0 \\ 0 & 0 & I_{n_1 n_2} \end{bmatrix}, \quad (12)$$

$$H_k = \begin{bmatrix} A_k \otimes I_{n_1} & B_k \otimes I_{n_1} & 0 \\ C_k \otimes I_{n_1} & -D_k \otimes I_{n_1} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{for all } 1 \le k \le m, \quad (13)$$

$$H_{-k} = \begin{bmatrix} I_{n_1} \otimes A_k^T & 0 & I_{n_1} \otimes C_k^T \\ 0 & 0 & 0 \\ I_{n_1} \otimes B_k^T & 0 & -I_{n_1} \otimes D_k^T \end{bmatrix}. \quad (14)$$

The construction of the matrix pencils Λ_i, $i = 1, 2$ leads to the following result, similar to the one developed in [19] in the context of retarded delay systems:

Lemma 2: Assume that $\rho(D) < 1$. If the original system free of delay is asymptotically stable, then the matrix pencils Λ_1, and Λ_2 are both regular.

A. Delay-independent stability

Using the preliminary results, and the notations above, we have the following:

Theorem 1: Consider the system (3), and assume that $\rho(D) < 1$. Then the delay-differential algebraic system is delay-independent asymptotically stable if and only if:

(i) the system free of delay is asymptotically stable, and

(ii) the matrix pencil Λ_1 has no generalized eigenvalues on the unit circle, or if it has, all its generalized eigenvalues z_0 on the unit circle are also generalized eigenvalues of the matrix pencil Λ_2 and

$$m_{\Lambda_1}(z_0) = m_{\Lambda_2}(z_0)^2, \quad (15)$$

where $m_{\Lambda_i}(z_0)$ $(i = 1, 2)$ denotes the (algebraic) multiplicity of the generalized eigenvalue z_0 of the corresponding matrix pencil Λ_i $(i = 1, 2)$.

Proof idea : Some tedious algebraic manipulations prove that the condition (ii) above is equivalent to the fact that the characteristic equation associated with (3) has no roots on the imaginary axis for all positive delays τ. Using the continuity type property of the roots with respect to the delay parameter (in the sense of Datko [7]) and the arguments presented in [10], [18] for the retarded case, the equivalence between the delay-independent stability and the conditions (i)–(ii) above follows[1]. □

Theorem 1 simply states that the delay-independent stability problem for lossless propagation systems is reduced to the computation of the generalized eigenvalues of two appropriate finite-dimensional matrix pencils. The distribution of these eigenvalues with respect to the unit circle of the complex plane gives the type of stability: *delay-independent* (no eigenvalues, or eigenvalues exist, but satisfying some particular properties), or *delay-dependent* stability.

[1]See the full version of the paper [21] for a complete proof.

Remark 2: As seen in Remark 1, the result above allows to handle *unitarily* the delay-independent stability of retarded and neutral systems. To the best of the authors' knowledge, there does not exist any similar approach in the open literature. □

Remark 3 (Propagation systems: Single delay): In the case of a single delay propagation system, that is $m = 1$ $A_1 = C_1 = 0$, some simple algebraic manipulation allows to rewrite the system (3) as a single delay propagation model defined as follows:

$$\begin{cases} \dot{x}_1(t) = Ax_1(t) + [B_1 \ldots B_m]\, y(t-\tau) \\ y(t) = \begin{bmatrix} C \\ 0 \end{bmatrix} x_1(t) + \begin{bmatrix} D_1 & \cdots & D_{m-1} & D_m \\ I & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix} y(t-\tau), \end{cases}$$

where $y(t)^T = [x_2(t)^T \ \ldots \ x_2(t-(k-1)\tau)^T]^T$, and the zero blocks have appropriate dimensions. Theorem 1 applied to the single delay propagation model above will lead to a different form for the matrix pencil Λ_1, which is equivalent to the form (7). □

Remark 4 (Singular retarded case. Discussions): Consider the single delay case in (3), but with $D_1 = 0$. It is easy to see that the corresponding retarded system simply rewrites as a *singular* delay system (in the state-variable x_t) of the form:

$$E\dot{x}(t) = Fx(t) + Gx(t-\tau),$$

where the state-vector x is given by $x^T = [x_1^T \ x_2^T]^T$, and E, F, G are appropriate real matrices in $\mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$, with $E = diag(I_{n_1}, 0)$, and the pair (E, F) *regular* in the sense proposed by Dai [6].

In conclusion, the approach above can be applied to handle also this case, and to the best of the authors' knowledge, there does not exist any complete solution for the corresponding *delay-independent* stability problem. □

B. Delay-dependent stability

Under the assumption that the system free of delay is asymptotically stable, we can explicitly compute the *delay margin*, that is the *first delay-interval* guaranteeing asymptotic stability if the conditions of the Theorem 1 do not hold. Introduce now the following complex matrix:

$$\mathcal{A}(z_0) = A + \sum_{k=1}^{m} A_k z_0 + \sum_{k=1}^{m} B_k z_0^k \left(I_{n_2} - \sum_{k=1}^{m} D_k z_0^k \right)^{-1} \left(C + \sum_{k=1}^{m} C_k z_0^k \right) \quad (16)$$

for some $z_0 \in \mathbb{C}$ We have the following result:

Theorem 2: Consider the system (3), and assume that $\rho(D) < 1$. Then the delay-differential algebraic system is delay-dependent asymptotically stable if and only if:

(i) the system free of delay is asymptotically stable, and

(ii) the matrix pencil Λ_1 has at least one generalized eigenvalue on the unit circle z_0, which is not a generalized eigenvalue of the matrix pencil Λ_2, or if it is, then the following condition is satisfied:

$$m_{\Lambda_1}(z_0) \quad > \quad m_{\Lambda_2}(z_0)^2, \tag{17}$$

where $m_{\Lambda_i}(z_0)$ $(i = 1, 2)$ denotes the (algebraic) multiplicity of the generalized eigenvalue z_0 of the corresponding matrix pencil Λ_i $(i = 1, 2)$.

Furthermore, the delay-differential algebraic system is asymptotically stable for all delays $\tau \in [0, \overline{\tau})$, where:

$$\overline{\tau} \quad = \quad \min_{1\leq k\leq mn_1(n_1+2n_2)} \min_{1\leq i\leq n_1} \frac{\alpha_k}{\omega_{k_i}}, \tag{18}$$

where $\alpha_k \in [0, 2\pi)$, $e^{-j\alpha_k} \in \sigma(\Lambda_1) - \sigma(\Lambda_2)$, and $j\omega_{k_i} \in \sigma(\mathcal{A}(z_0))$.

Proof idea : Using the continuity argument of the roots with respect to the delay parameter [7], delay-dependent stability is equivalent to the existence of roots crossing the imaginary axis for some delay value, fact which is further equivalent to the conditions (i)–(ii) in the Theorem statement[2]. Some tedious, but straightforward algebraic manipulations lead to the upper bound (18) for the corresponding delay margin. □

Notice that the ideas proposed in the remarks 2–4 above are still valid for the *delay-dependent* stability analysis.

C. Crossing direction characterization

The remaining problem in the delay-dependent stability case consists in analyzing the following two cases:

a) the characterization (if any) of other delay intervals guaranteeing (asymptotic) stability under the assumption that the system free of delay is stable;
b) the same analysis, but by relaxing the assumption on the system free of delays.

As mentioned in [18] (see also [5]), the solution of the cases above can be constructed if, for each root of the characteristic equation associated with (3) on the imaginary axis, the *crossing direction* is explicitly computed. Without any loss of generality, we assume in the sequel that all the roots of the characteristic equation are *simple*.

From the Theorems 1, and 2, it follows that the roots crossing the imaginary axis can be easily detected by computing the generalized eigenvalues of the matrix pencils Λ_i, i=1,2. Thus, the remaining problem consists in computing the *sensitivity* of the root with respect to the delay parameter when crossing the imaginary axis. If the problem of crossing direction in the retarded, and neutral cases was largely treated in the literature since the 80s (see, for instance, [5], [26], [23]), explicit formulae for such quantities for delay systems in *state-space representation* were reported in the literature only recently [20], but only for the *retarded* case.

Theorem 3 (crossing characterization): Assume that the matrix pencil Λ_1 is regular. Then the characteristic equation associated to (3) has a crossing root on the imaginary axis for some positive delay value τ_0 if and only if the following conditions are satisfied simultaneously:

(i) The matrix pencil Λ_1 has generalized eigenvalues on the unit circle;
(ii) There exists some $z_0 \in \sigma(\Lambda_1) \cap \partial\mathbb{D}$, such that:

$$\sigma\left(\mathcal{A}((z_0)) \cap \partial\mathbb{C}_+^* \quad \neq \quad \emptyset, \tag{19}$$

where $\partial\mathbb{C}_+^* = \partial\mathbb{C}_+ - \{0\}$[3].

Furthermore, for some z_0 satisfying the condition (ii) above, the set of "crossing" delays is given by:

$$\mathcal{T}(z_0) \quad = \quad \left\{ \frac{Log(\overline{z_0})}{j\omega_0} + \frac{2\pi\ell}{\omega_0} > 0 : \right. \\ \left. j\omega_0 \in \sigma(\mathcal{A}(z_0)) - \{0\}, \ell \in \mathbb{Z} \right\} \tag{20}$$

where $Log(\cdot)$ denotes the principal value of the logarithm.

Definition 1: A complex number z_0 satisfying the condition (ii) in Theorem 3 will be called a crossing generator, and denote σ_g the set of all such crossing generators. Then

$$\mathcal{T} \quad = \quad \bigcup_{z\in\sigma_g} \mathcal{T}(z) \tag{21}$$

will be called the delay crossing generator set.

Using the definition above, Theorem 3 simply says that *the existence of crossing roots is equivalent to the property that the delay crossing generator set is not empty*. Next, we have the following result:

Proposition 1 (switches characterization: simple roots): Assume that the crossing roots are simple, and let $z_0 \in \sigma_g$ be a crossing generator of some root $j\omega_0 \neq 0$ of the delay-differential algebraic system (3). Then, we have a root crossing the imaginary axis towards instability (stability) if and only if:

$$Re\left\{ -\frac{j\omega_0}{u_1^* u_1} \sum_{k=1}^{m} k z_0^k \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^* \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} \\ > 0 (< 0), \tag{22}$$

where $u_1^* \in \mathbb{C}^{n_1}$ $(v_1 \in \mathbb{C}^{n_1})$, and $u_2^* \in \mathbb{C}^{n_2}$ $(v_2 \in \mathbb{C}^{n_2})$ represent a partition of the corresponding left (right) eigenvectors u^* (v) of the eigenvalue $j\omega_0$ in the spectrum of $\mathcal{A}(z_0)$, where $\mathcal{A}(z_0)$ is given by (16).

The proof can be found in the full version of the paper [21], and it generalizes the results proposed by [20] for the retarded case. It is based on the so-called Jacobi's formula for computing the differential of the determinant of some square matrix M:

$$d\mathbf{det}(M) \quad = \quad \mathbf{Tr}\left(\mathbf{Adj}(M) dM\right),$$

where dM, and $d\mathbf{det}(M)$ define the differentials of A, and of its determinant, respectively.

Remark 5: The quantity to be evaluated in (23) is well-defined. Indeed, a simple algebraic argument proves that if the roots on the imaginary axis $\partial\mathbb{C}_+$ are *simple*, than the corresponding ratio always exists, is different from 0, and it is finite. Finally, it is important to note that the *crossing*

[2] See the full version of the paper [21] for a complete proof.

[3] $\partial\mathbb{C}_+^*$ denotes the imaginary axis without the origin.

direction is *independent* of the *delay* value (see also [23] for a different approach). □

Remark 6 (Lossless propagation systems): In the case of lossless propagation systems, that is $A_k = 0$, $C_k = 0$, for all $k = 1, \dots, m$, the crossing condition in the Proposition above rewrites as follows:

$$Re\left\{-\frac{j\omega_0}{u_1^* u_1}\sum_{k=1}^{m} k z_0^k \left(u_1^* B_k v_2 + u_2^* D_k v_2\right)\right\} > 0 (< 0) \quad (23)$$

where $u_1^* \in \mathbb{C}^{n_1}$ ($v_1 \in \mathbb{C}^{n_1}$), and $u_2^* \in \mathbb{C}^{n_2}$ ($v_2 \in \mathbb{C}^{n_2}$) represent a partition of the corresponding left (right) eigenvectors u^* (v) of the eigenvalue $j\omega_0$ in the spectrum of $\mathcal{A}(z_0)$, where $\mathcal{A}(z_0)$ is given by the corresponding complex matrix (16).

IV. Comments, and interpretations

For the sake of simplicity, we shall consider in the sequel only the single delay case ($m = 1$). However, the proposed comparisons, discussions, and comments are also valid for the general case. Our interest is to see how the results above rewrites in the particular retarded, and neutral cases, and to establish the existing connections between the corresponding matrix pencils with the ones proposed in the literature [3], [19], [9] for handling the corresponding cases.

For the brevity of the paper, we further consider only the *delay-independent* stability analysis, but it is clear that the same arguments apply to the delay-dependent, and crossing root characterization cases.

A. Retarded case

It is easy to see that we completely recover the *delay-independent* conditions for *retarded* linear systems including a single delay, but with a different definition of the matrix pencil Λ_1 [3], [19].

In fact, the corresponding matrix pencil in the above references is derived using a different *strong* linearization of the corresponding matrix polynomial (see also [18]):

$$\mathcal{P}(z) = B \otimes I_{n_1} z^2 + A \oplus A^T z + I_{n_1} \otimes B^T,$$

leading to:

$$\widetilde{\Lambda_1}(z) = z\begin{bmatrix} I_{n_1^2} & 0 \\ 0 & B \otimes I_{n_1} \end{bmatrix} + \begin{bmatrix} 0 & -I_{n_1^2} \\ I_{n_1} \otimes B^T & A \oplus A^T \end{bmatrix}.$$

Here Λ_1 rewrites as follows:

$$\Lambda_1(z) = z\begin{bmatrix} 0 & B \otimes I_{n_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -I_{n_1^2} \end{bmatrix} + \begin{bmatrix} A \oplus A^T & 0 & I_{n_1^2} \\ I_{n_1^2} & -I_{n_1^2} & 0 \\ I_{n_1} \otimes B^T & 0 & 0 \end{bmatrix}.$$

Simple computations of $\mathbf{det}(\Lambda_1(z))$, and $\mathbf{det}(\widetilde{\Lambda_1}(z))$ show that:

$$\mathbf{det}(\Lambda_1(z)) = \mathbf{det}(\widetilde{\Lambda_1}(z)),$$

and, in conclusion:

Proposition 2: The matrix pencils Λ_1, and $\widetilde{\Lambda}_1$ have the same generalized eigenvalues on the unit circle of complex plane.

The approach considered in this paper takes advantage on the form of Λ_1 instead of $\widetilde{\Lambda}_1$, which cannot be used directly in the lossless propagation case. Furthermore, the form of the pencil Λ_1 is particularly adapted to handle (regular) *singular* systems with delays (see, for instance, Remark 4).

B. Neutral case

Consider now the neutral system:

$$\dot{x}(t) - A_{-1}x(t-\tau) = A_0 x(t) + A_1 x(t-\tau). \quad (24)$$

As shown in [9], the delay-independent stability can be expressed in terms of generalized eigenvalues of the matrix pencil $\widehat{\Lambda_1}$ defined as follows:

$$\widehat{\Lambda_1}(z) = z\begin{bmatrix} I & 0 \\ 0 & A_1 \otimes I - A_{-1} \otimes A_0^T \end{bmatrix} + \begin{bmatrix} 0 & -I \\ I \otimes A_1^T - A_0 \otimes A_{-1}^T & A_0 \oplus A_0^T - A_1 \otimes A_{-1}^T - A_{-1} \otimes A_1^T \end{bmatrix}.$$

Let us rewrite (24) in the form (3). Simple computations prove that:

$$A = A_0, \qquad B = A_0 A_{-1} + A_1$$
$$C = I_n, \qquad D = A_{-1}.$$

Some simple, but tedious computations of $\mathbf{det}(\Lambda_1(z))$, and $\mathbf{det}(\widehat{\Lambda_1}(z))$ show that:

$$\mathbf{det}(\Lambda_1(z)) = \mathbf{det}\left[\left((I_n - Dz) \otimes I_n\right)^{-1}\right] \cdot \mathbf{det}(\widehat{\Lambda_1}(z)) \cdot \mathbf{det}\left[\left(I_n \otimes (I_n - D^T\bar{z})\right)^{-1}\right].$$

Next, since $\rho(D) < 1$, it follows that:

Proposition 3: The matrix pencils Λ_1, and $\widehat{\Lambda_1}$ have the same generalized eigenvalues on the unit circle of complex plane.

As in the retarded case, the matrix pencil $\widehat{\Lambda_1}$ cannot be directly adapted to the lossless propagation case. It is important to note that Λ_1 gives also a simple stability characterization to some of the *descriptor representations* considered by Fridman [8], and opens the perspective to give complete solutions to the *singular* neutral systems (a remark similar to Remark 4 holds also in the neutral case).

V. Illustrative example

For the brevity of the paper, we consider below only the scalar (propagation) system ($n_1 = n_2 = 1$), and we shall explicitly give the corresponding stability conditions[4].

The system (3) rewrites as:

$$\begin{cases} \dot{x}_1(t) = a x_1(t) + b x_2(t-\tau) \\ x_2(t) = c x_1(t) + d x_2(t-\tau), \end{cases} \quad (25)$$

[4] See, for instance, the full version of the paper [21] for further examples

where $a, b, c, d \in \mathbb{R}$. The matrix pencils Λ_1, and Λ_2 are given by:

$$\Lambda_1(z) = z \begin{bmatrix} 0 & b & 0 \\ 0 & d & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 2a & 0 & c \\ c & -1 & 0 \\ b & 0 & d \end{bmatrix}, \quad (26)$$

$$\Lambda_2(z) = z \begin{bmatrix} 0 & -b \\ 0 & -d \end{bmatrix} + \begin{bmatrix} -a & 0 \\ -c & 1 \end{bmatrix}. \quad (27)$$

Applying Theorem 1 to (25) leads to the following result:

Proposition 4: The system (25) with $| d |< 1$ is delay-independent asymptotically stable if and only if:

$$- | a | (1 + d) \leq bc <| a | (1 - d). \quad (28)$$

By similarity, we have the following delay-dependent stability result:

Proposition 5: The system (25) with $| d |< 1$ is delay-dependent asymptotically stable if and only if:

$$a(1 - d) + bc < 0, \qquad a(1 + d) - bc > 0. \quad (29)$$

Furthermore, the lossless propagation system (25) is asymptotically stable for all the delays $\tau \in [0, \overline{\tau})$, where:

$$\overline{\tau} = \sqrt{\frac{1 - d^2}{(ad - bc)^2 - a^2}} \cdot arcctg\left(\frac{a(1 + d^2) - bcd}{\sqrt{((bc - ad)^2 - a^2)(1 - d^2)}}\right). \quad (30)$$

Finally, applying Theorem 3 it follows that:

Proposition 6: Under the assumption $| d |< 1$, and (29), there does not exist any delay-interval other than $[0, \overline{\tau})$ guaranteeing asymptotic stability of the scalar system (25).

VI. Concluding remarks

This paper addressed the asymptotic stability analysis of some class of linear delay-differential algebraic systems including multiple commensurate delays in their representation. More explicitly, *necessary, and sufficient stability* conditions have been derived for characterizing the delay-independent, or delay-dependent stability, and in the second case, the crossing root behavior. The corresponding conditions have been expressed in terms of generalized eigenvalue distribution of some appropriate finite-dimensional matrix pencils. Furthermore, the construction has given a *unitary* treatment of both *retarded*, and *neutral* systems, and opens the perspective to handle also the *singular* systems. To complete the presentation, an illustrative example has been detailed.

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