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Observability and Controllability of Fractional Linear Dynamical Systems

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Abstract: In this paper we study the observability and controllability of fractional linear dynamical systems in finite dimensional spaces. Examples are included to illustrate the theoretical results proved in this manuscript.

Keywords: Controllability; Observability; Fractional Differential Equations; Mittag Leffler Matrix Function.

1. INTRODUCTION

Fractional differential equations (FDEs) are considered as a emergent branch of applied mathematics with many applications in the field of physical and engineering to model the dynamics of different processes through anomalous media, but also introduce more efficient model in fields as signal processing or control theory (see, for example, the survey [29], the books [5, 10, 17, 21] and the following special issues [6, 20, 24, 25, 28]). FDEs capture non local relations in space and time with power-law memory kernels, due to this fact, research in this topic has grown significantly all around the world. Fractional differentials and integrals provide more accurate models of systems under consideration. Few examples of how many authors have demonstrated the application of fractional calculus are the following: in electrochemistry [9], thermal systems and heat conduction [3], viscoelastic materials [1], fractal electrical networks [22] and many others areas. Differential equations with fractional order have recently proved to be valuable tools to the modelling of many physical phenomena [12, 23].

In particular, an increasing interest in issues related to fractional dynamical systems oriented towards the field of control theory can be observed in the literature. The study of the observability and controllability of the fractional dynamical systems are two important issues for many applied problems. It is well known that the problem of controllability of dynamical systems are widely used in analysis and the design of control system. Any system is said to be controllable if every state corresponding to this process can be affected or controlled in respective time by some controller. Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs. The observability and controllability of a system are mathematical duals. The concept of observability and controllability were introduced by R.E. Kalman for linear dynamical systems. Several authors [2, 8, 27, 30] studied the controllability results for linear and nonlinear dynamical systems in finite dimensional spaces. This is not the case of fractional linear systems. In fact there are few works reporting the study of observability and controllability of fractional linear systems (see, for example, [4, 7, 15, 16, 26]).

In this paper we study the observability and controllability of the linear fractional dynamical system in finite dimensional spaces. The observability and controllability Grammian matrices are obtained by using Mittag-Leffler matrix function. Examples are constructed to verify the results proved in this paper.

2. THE FRACTIONAL DERIVATIVE

To work in a general context we will use the general formulation of the incremental ratio derivative valid for any order, real or complex. In the following we will consider the real case.

Definition 1. Similarly to the classic case, we define fractional derivative by the limit of the fractional incremental ratio ([18])

$$D^{\alpha}_{\theta} f(t) = e^{-j\alpha\theta} \lim_{|h|\to 0} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^{\alpha}}, \quad (1)$$

where $h = e^{j\theta}$ is a complex number, with $\theta \in (-\pi, \pi]$.

The above defined derivative is a general incremental ratio based derivative that generalizes classical Grünwald-Letnikov fractional derivative. To understand and give an interpretation to the above formula, assume that $t \in \mathbb{R}$ is a time and that h is real, $\theta = 0$ or $\theta = \pi$. If $\theta = 0$, only the present and past values are being used (2), while, if $\theta = \pi$, only the present and future values are used. This means that if we look at (1) as a linear system, the first case is causal, while the second is anti-causal [21].

In general, if $\theta = 0$, we call (1) the forward Grünwald-Letnikov [1] derivative, which is well known; however its properties are not so well studied:

$$D_f^\alpha f(t) = \lim_{|h|\to 0+} \frac{\sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha}. \qquad (2)$$

In particular, if we assume that $f(t) = 0$ for $t < a$ and let $[\cdot]$ be the "integer part of the argument", we obtain from (2).

$$D_f^\alpha f(t) = \lim_{|h|\to 0+} \frac{\sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} f(t-kh)}{h^\alpha}, \qquad (3)$$

that is, the formulation we find frequently [23]. With $\theta = \pi$ we would obtain the corresponding called backward derivative $D_b^\alpha f(t)$.

In the following, we will use mainly the forward derivative. So we will remove, in such case, the subscript "f ".

2.1 Simple examples

(1) **The exponential**

Applying the above definitions to the function $f(t) = e^{st}$, $s \in \mathbb{C}$. we obtain see [18]

$$D^\alpha f(t) = \lim_{h\to 0+} \frac{\left(1-e^{-sh}\right)^\alpha}{h^\alpha} e^{st} = |\, h \,|^\alpha e^{j\theta\alpha} e^{st} \qquad (4)$$

iff $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ which corresponds to be working with the principal branch of $(\cdot)^\alpha$ and assuming a branch cut line in the left hand complex half plane.

This result can be used to generalize a well known property of the Laplace transform. If we return back to equation (2) and apply the bilateral Laplace transform

$$F(s) = \int_{-\infty}^{+\infty} f(t) e^{-st} dt \qquad (5)$$

to both sides we conclude that:

$$LT\left[D^\alpha f(t)\right] = s^\alpha F(s) \quad (\Re e(s) > 0) \qquad (6)$$

where in s^α we assume the principal branch and a cut line in the left half plane. This result is valid for all functions that have Laplace transform with region of convergence on the right half complex plane.

(2) **The causal power function**

We start by computing the fractional derivative of the Heaviside unit step function, $\epsilon(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$. From (2) we have:

$$D^\alpha \epsilon(t) = \lim_{h\to 0+} \frac{\sum_{k=0}^{\left[\frac{t}{h}\right]} (-1)^k \binom{\alpha}{k}}{h^\alpha}.$$

where we assumed $h > 0$. On the other hand we can put with $k = t/h$, with $t > 0$

$$(-1)^k \binom{\alpha}{k} = \frac{(-\alpha)_k}{k!} = \frac{\Gamma(-\alpha+k)}{\Gamma(-\alpha)\Gamma(k+1)} = \frac{\Gamma(-\alpha+t/h)}{\Gamma(-\alpha)\Gamma(t/h+1)}$$

Using a well-known property of the gamma function, when $h \to 0+$ we can write

$$(-1)^k \binom{\alpha}{k} = \frac{(-\alpha)_k}{k!} \approx \frac{(t/h)^{-\alpha-1}}{\Gamma(-\alpha)}$$

Inserting this expression above we obtain

$$D^\alpha \epsilon(t) = \lim_{h\to 0+} \frac{t}{h} \frac{(t/h)^{-\alpha-1}}{\Gamma(-\alpha) h^\alpha}.$$

allowing us to write:

$$D^\alpha \epsilon(t) = \frac{t^{-\alpha}\epsilon(t)}{\Gamma(-\alpha)} \qquad (7)$$

In the following we will be interested in negative values of the order, meaning powers of the type $\frac{t^\alpha}{\Gamma(\alpha)}\epsilon(t)$. Its Laplace transform is equal to $\frac{1}{s^{\alpha+1}}$ [21]

3. THE LINEAR SYSTEMS AND THEIR INITIAL CONDITIONS

The linear systems we will consider assume the general format

$$D^\alpha x(t) = Ax(t) + f(t), \qquad (8)$$

with the derivative defined in (2) and where $0 < \alpha < 1$, $t \in \mathbb{R}$, $x, f \in \mathbb{R}^n$, A is a $n \times n$ constant matrix and f is a continuous function on $J = [0, T] \in \mathbb{R}$.
The above equation could be consider on $\mathbb{R}$. However, we shall be concerned with the system behaviour after the application of $f(t)$. This means that our observation window is in $\mathbb{R}^+$. This leads us to the need for considering a initial condition of the system. This can be done by introducing a change in the above equation making the initial condition appear explicitly - see [19]. As it was shown there the definition of the system to account for an initial condition $x_0 = x(t_0)$ is

$$D^\alpha x(t) = Ax(t) + x(t_0)\delta^{(\alpha-1)}(t-t_0) + f(t), \qquad (9)$$

where $\delta(t)$ is the Dirac delta function. The fractional derivative of the delta is given by the first order derivative of (7), accordingly to the theory presented in [18].
Applying the Laplace transform to this equation, we obtain:

$$s^\alpha X(s) = AX(s) + s^{\alpha-1}x(t_0) + F(s),$$

and

$$X(s) = \left[s^\alpha I - A\right]^{-1}\left[s^{\alpha-1}x(t_0) + F(s)\right] \qquad (10)$$

To obtain a general expression to the output $x(t)$ we must compute the inverse of $\left[s^\alpha I - A\right]^{-1}$. To do it, proceed formally to get the series expansion

$$\left[s^\alpha I - A\right]^{-1} = \sum_{n=1}^{\infty} A^{n-1} s^{-n\alpha}$$

[1] The terms forward and backward are used here in agreement to the way the time flows, from past to future or the reverse.

that inverted gives

$$LT^{-1}\left\{[s^{\alpha}I-A]^{-1}\right\}=\sum_{n=1}^{\infty}A^{n-1}\frac{t^{n\alpha-1}}{\Gamma(n\alpha)}\epsilon(t)$$

This function is called $\alpha-$exponential [11]. However it is commonly expressed in terms of the Mittag-Leffler function.

Definition 2. The Mittag-Leffler function $E_{\alpha,\beta}(z)$ is a complex function which depends on two complex parameter. It is defined by (see, for example, [11, 14])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ \alpha,\beta\geq 0. \tag{11}$$

When $\beta=1, E_{\alpha,1}(z)=E_{\alpha}(z)$ converges for all values of the argument z. Thus the Mittag-Leffler function is an entire function. For a $n\times n$ matrix A, the matrix extension of the above Mittag-Leffler function is

$$E_{\alpha,\beta}(A)=\sum_{k=0}^{\infty}\frac{A^k}{\Gamma(\alpha k+\beta)}.$$

Finally, using the Mittag-Leffler function the solution of the system (9) is given by

$$\begin{aligned}x(t)&=E_{\alpha}(At^{\alpha})x_0\\&+\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})f(s)\mathrm{d}s.\end{aligned}\tag{12}$$

with $x_0=x(t_0)$.

4. OBSERVABILITY RESULT

Consider the fractional order linear time invariant system

$$D^{\alpha}x(t)=Ax(t), \tag{13}$$

where again $x\in\mathbb{R}^n$ and A is a $n\times n$ constant matrix. Along with (13) we have a linear observation in the interval $t\in[t_0,t_1]$,

$$y(t)=Hx(t), \tag{14}$$

where $y\in\mathbb{R}^m$ and H is an $m\times n$ constant matrix.

Definition 3. The system (13), (14) is *observable* on an interval $[t_0,t_1]$ if

$$y(t)=Hx(t)=0,\quad t\in[t_0,t_1],$$

implies

$$x(t)=0,\qquad t\in[t_0,t_1].$$

Theorem 1. The observed linear system (13), (14) is observable on $[t_0,t_1]$ if and only if the observability Grammian matrix

$$W=\int_{t_0}^{t_1}E_{\alpha}(A^T(t-t_0)^{\alpha})H^THE_{\alpha}(A(t-t_0)^{\alpha})\mathrm{d}t \tag{15}$$

is positive definite, where the T denotes the transpose matrix.

Proof. The solution $x(t)$ of (13) corresponding to the initial condition $x(t_0)=x_0$ is given by

$$x(t)=E_{\alpha}(A(t-t_0)^{\alpha})x_0$$

and we have, for $y(t)=Hx(t)=HE_{\alpha}(A(t-t_0)^{\alpha})x_0$

$$\|y\|^2=\int_{t_0}^{t_1}y^T(t)y(t)\mathrm{d}t$$

$$=x_0^T\int_{t_0}^{t_1}E_{\alpha}(A^T(t-t_0)^{\alpha})H^THE_{\alpha}(A(t-t_0)^{\alpha}\mathrm{d}t\ x_0$$

$$=x_0^T\,W\,x_0,$$

a quadratic form in x_0. Clearly W is an $n\times n$ symmetric matrix. If W is positive definite, then $y=0$ implies $x_0^T\,W\,x_0=0$. Therefore $x_0=0$. Hence (13), (14) is observable on $[t_0,t_1]$. If W is not positive definite, then there is some $x_0\neq 0$ such that $x_0^T\,W\,x_0=0$. Then $x(t)=E_{\alpha}(A(t-t_0)^{\alpha})x_0\neq 0$, for $t\in[t_0,t_1]$ but $\|y\|^2=0$, so $y=0$ and we conclude that (13), (14) is not observable on $[t_0,t_1]$.

5. CONTROLLABILITY RESULT

Consider the fractional order linear time invariant system

$$\begin{aligned}D^{\alpha}x(t)&=Ax(t)+Bu(t)\\x(t_0)&=x_0,\end{aligned}\tag{16}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ and A, B are $n\times n$, $n\times m$ matrices respectively. Let $u(t)\in L^2([t_0,t_1],\mathbb{R}^m)$, the space of all square integrable $\mathbb{R}^m$ valued measurable functions defined on $[t_0,t_1]$.

Definition 4. The system (16) is *controllable* on $[t_0,t_1]$ if for every pair of vectors $x_0,x_1\in\mathbb{R}^n$, there is a control $u(t)\in L^2([t_0,t_1],\mathbb{R}^m)$ such that the solution $x(t)$ of (16) which satisfies

$$x(t_0)=x_0, \tag{17}$$

also satisfies

$$x(t_1)=x_1. \tag{18}$$

We say that u steers the system form x_0 to x_1 during the interval $[t_0,t_1]$.

Lemma 2. The system (16) is controllable on $[t_0,t_1]$ if and only if for each vector $x_1\in\mathbb{R}^n$ there is a control $u\in L^2([t_0,t_1],\mathbb{R}^m)$ which steers 0 to x_1 during $[t_0,t_1]$.

Proof. Suppose the system (16) is controllable on $[t_0,t_1]$, then by taking $x_0=0$ we see that u steers form 0 to x_1 during $[t_0,t_1]$. To prove the sufficiency it is only necessary to choose two vectors $x_0,x_1\in\mathbb{R}^n$ and put

$$\bar{x}_1=x_1-E_{\alpha}(A(t_1-t_0)^{\alpha})x_0.$$

If u steers 0 to $\bar{x}_1$ during $[t_0,t_1]$, then

$$\bar{x}_1=\int_{t_0}^{t_1}(t_1-t)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-t)^{\alpha})Bu(t)\mathrm{d}t.$$

By the above we have,

$$x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0 = \int_{t_0}^{t_1}(t_1-t)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-t)^\alpha)Bu(t)\mathrm{d}t$$

$$x_1 = E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1}(t_1-t)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-t)^\alpha)Bu(t)\mathrm{d}t = x(t_1),$$

this implies that the solution of (16) for this control u satisfies both (17) and (18), and steers the system form x_0 to x_1. Hence the given system (16) is controllable on $[t_0, t_1]$.

Theorem 3. The linear control system (16) is controllable on $[t_0, t_1]$ if and only if the controllability Grammian matrix

$$M = \int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)BB^T \times E_{\alpha,\alpha}(A^T(t_1-\tau)^\alpha)\mathrm{d}\tau, \tag{19}$$

is positive definite, for some $t_1 > t_0$.

Proof. Since M is positive definite, that is, it is non-singular and therefore its inverse is well-defined. Define the input as,

$$u(t) = B^T E_{\alpha,\alpha}(A^T(t_1-t)^\alpha) \times M^{-1}[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0]. \tag{20}$$

The solution of (16) with initial condition $x(t_0) = x_0$ is

$$x(t) = E_\alpha(A(t-t_0)^\alpha)x_0 + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t-\tau)^\alpha)Bu(\tau)\mathrm{d}\tau. \tag{21}$$

From (20), we have

$$\begin{aligned} x(t_1) &= E_\alpha(A(t_1-t_0)^\alpha)x_0 \\ &\quad + \int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)BB^T \\ &\quad \times E_{\alpha,\alpha}(A^T(t_1-\tau)^\alpha) \\ &\quad \times M^{-1}[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0]\mathrm{d}\tau \\ &= E_\alpha(A(t_1-t_0)^\alpha)x_0 \\ &\quad + MM^{-1}[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0] = x_1. \end{aligned}$$

Thus (16) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero y such that

$$y^T M y = 0$$

that is,

$$y^T\int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)BB^T \times E_{\alpha,\alpha}(A^T(t_1-\tau)^\alpha)y\ \mathrm{d}\tau = 0,$$

which implies that $y^T E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)B = 0$ on $[t_0, t_1]$. Let $x_0 = [E_\alpha(A(t_1-t_0)^\alpha)]^{-1}y$. By the assumption, there exists a control u such that it steers x_0 to the origin in the interval $[t_0, t_1]$. It follows that

$$x(t_1) = 0 = E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)Bu(\tau)\mathrm{d}\tau.$$

Then,

$$0 = y^T y + \int_{t_0}^{t_1}(t_1-\tau)^{\alpha-1}y^T E_{\alpha,\alpha}(A(t_1-\tau)^\alpha)Bu(\tau)\mathrm{d}\tau.$$

But the second term is zero leading to the conclusion that $y^T y = 0$. This is a contradiction to $y \neq 0$. Hence M is positive definite.

Theorem 4. The system (16) is controllable on $[t_0, t_1]$ if and only if the adjoint linear observed system

$$D_b^\alpha y(t) = A^T y \tag{22}$$

$$w(t) = B^T y \tag{23}$$

is observable on $[t_0, t_1]$.

Proof. Define a linear subspace $\mathbb{R}(t_0, t_1) \subset \mathbb{R}^n$ by

$$\mathbb{R}(t_0, t_1) = \{x_1 \in \mathbb{R}^n\,;\, x_1 = I(t_0, t_1)\}, \tag{24}$$

with

$$I(t_0, t_1) = \int_{t_0}^{t_1}(t_1-t)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-t)^\alpha)Bu(t)\mathrm{d}t,$$

and the control $u \in L^2([t_0, t_1], \mathbb{R}^m)$. Thus $\mathbb{R}(t_0, t_1)$ is the subspace of states reachable from the origin using the control $u \in L^2([t_0, t_1], \mathbb{R}^m)$. Suppose $y_1 \in \mathbb{R}^n$ has the property

$$y_1^T x_1 = 0, \text{ for } x_1 \in \mathbb{R}(t_0, t_1). \tag{25}$$

Therefore, using (24)

$$y_1^T\int_{t_0}^{t_1}(t_1-t)^{\alpha-1}E_{\alpha,\alpha}(A(t_1-t)^\alpha)Bu(t)\mathrm{d}t = 0,$$

and since $u(t)$ is an arbitrary element of $L^2([t_0, t_1], \mathbb{R}^m)$. So, conclude that

$$y_1^T E_{\alpha,\alpha}(A(t_1-t)^\alpha)B = 0,$$

or

$$w(t) = B^T E_{\alpha,\alpha}(A^T(t_1-t)^\alpha)y_1 = 0,\ \ t \in [t_0, t_1]. \tag{26}$$

Now $y(t) = E_{\alpha,\alpha}(A^T(t_1-t)^\alpha)y_1$ is a solution of (22) on $[t_0, t_1]$ and $w(t)$ is the associated observation (23). If (22), (23) is observable, then (26) implies $y(0) = 0$, which gives $y_1 = 0$. Then (25) shows that $y_1 = 0$ and we conclude that $\mathbb{R}(t_0, t_1) = \mathbb{R}^n$. Hence the system (16) is controllable on $[t_0, t_1]$. If (22), (23) is not observable on $[t_0, t_1]$, there is some $y_1 = y(0) \neq 0$ such that (26) holds. Then we conclude that (25) holds for this nonzero y_1 and $\mathbb{R}(t_0, t_1) \neq \mathbb{R}^n$. So (16) is not controllable on $[t_0, t_1]$. Hence the pair (22), (23) is observable, whenever (16) is controllable.

6. EXAMPLES

The examples consider in this section involve the general derivative defined above.

Example 1. Consider the sequential linear fractional dynamical equation of order 2α and $0 < \alpha < 1$

$$D^{2\alpha}x(t) + x(t) = 0 \quad (t \in [0, t_1]) \tag{27}$$

$$x(0) = x_0,$$

$$D^{\alpha}x(0) = \lim_{t\to 0} x^{1-\alpha}x'(t) = x'_0, \tag{28}$$

with the linear observation $y = x'_0$.

Let us introduce the auxiliary variables $x_1(t) = x(t)$ and $x_2(t) = D^{\alpha}x_1(t)$. Then

$$D^{\alpha}x_1(t) = D^{\alpha}x(t) = x_2(t)$$

$$D^{\alpha}x_2(t) = D^{2\alpha}x(t) = -x_1(t),$$

and therefore the problem (27) can be expressed as $D^{\alpha}\bar{x}(t) = A\bar{x}(t)$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.
From the linear observation

$$y = x_1 = [\,1\ 0\,]\,\bar{x}(t), \quad H = [\,1\ 0\,].$$

The Mittag-Leffler matrix function of the given matrix A is

$$E_{\alpha}(At^{\alpha}) = \begin{pmatrix} L_1 & L_2 \\ -L_2 & L_1 \end{pmatrix}$$

where

$$L_1 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n\alpha}}{\Gamma(2\alpha n + 1)}$$

$$L_2 = \sum_{n=0}^{\infty} \frac{(-1)^n t^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha + 1)}.$$

The observability Grammian of this system is

$$W = \int_0^T E_{\alpha}(A^T t^{\alpha}) H^T H E_{\alpha}(At^{\alpha})\mathrm{d}t$$

$$= \int_0^T \begin{pmatrix} L_1^2 & L_1L_2 \\ L_1L_2 & L_2^2 \end{pmatrix} \mathrm{d}t,$$

which is non-singular if $t_1 > 0$. Hence the given system is observable.

Example 2. Consider the sequential linear control fractional dynamical equation of order 2α and $0 < \alpha < 1$

$$D^{2\alpha}x(t) - x(t) = u(t) \tag{29}$$

observed in the interval $[0, t_1]$. Let us introduce the following auxiliary variables $x_1(t) = x(t)$ and $x_2(t) = D^{\alpha}x_1(t)$. Then

$$D^{\alpha}x_1(t) = D^{\alpha}x(t) = x_2(t)$$

$$D^{\alpha}x_2(t) = D^{2\alpha}x(t) = x_1(t) + u(t).$$

Therefore the problem (29) can be expressed as $D^{\alpha}\bar{x}(t) = A\bar{x}(t) + Bu(t)$,

where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\bar{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

We will assume the following boundary conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} x_1(T) \\ x_2(T) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The Mittag-Leffler matrix function of the given matrix A is

$$E_{\alpha,\alpha}(A(T-\tau)^{\alpha}) = \begin{pmatrix} N_1 & N_2 \\ N_2 & N_1 \end{pmatrix}$$

where

$$N_1 = \sum_{k=0}^{\infty} \frac{(T-\tau)^{2k\alpha}}{\Gamma(2k\alpha + \alpha)}$$

$$N_2 = \sum_{k=0}^{\infty} \frac{(T-\tau)^{(2k+1)\alpha}}{\Gamma(2k\alpha + 2\alpha)}.$$

The controllability Grammian of this system is

$$M = \int_0^{t_1} (t_1-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(T-\tau)^{\alpha})BB^T \times E_{\alpha,\alpha}(A^T(t_1-\tau)^{\alpha})\mathrm{d}\tau$$

$$= \int_0^T (t_1-\tau)^{\alpha-1} \begin{pmatrix} N_2^2 & N_1N_2 \\ N_1N_2 & N_1^2 \end{pmatrix} \mathrm{d}\tau.$$

Therefore M is non-singular if $T > 0$, and the control defined by

$$u(t) = B^T E_{\alpha,\alpha}(A^T(t_1-t)^{\alpha})M^{-1} \times [x_1 - E_{\alpha}(A(T)^{\alpha})x_0]$$

steers the given system from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

This example is interesting, because the system as defined in (29) is unstable.

7. CONCLUSIONS

Observability is a measure for how well internal states of a system can be inferred by knowledge of its external outputs, while the controlability informs us about the ability to change the state of a system in order to assume a pre-specified value in a given time interval. The study of both the observability and controllability of the fractional dynamical systems are important issues for many practical daily applications. In this paper we have proved two new main results connected with the observability and controllability of fractional linear system by the use of so called Grammian matrix. The results are formally very simple and open a new way into interesting possibilities for applications. Also, we included two application examples illustrating the presented theory. In the second example we can see a formal structure of the controller.

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