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Stability radius of implicit dynamic equations with constant coefficients on time scales[☆]

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ABSTRACT

This paper deals with the stability radii of implicit dynamic equations on time scales when the structured perturbations act on both the coefficient of derivative and the right-hand side. Formulas of the stability radii are derived as a unification and generalization of some previous results. A special case where the real stability radius and the complex stability radius are equal is studied. Examples are derived to illustrate results.

1. Introduction

In the past decades, there have been extensive works on studying of robust measures, where one of the most powerful ideas is the concept of the stability radii, introduced by Hinrichsen and Pritchard [1]. The stability radius is defined as the smallest (in norm) complex or real perturbations destabilizing the equation. In [2], the authors consider the equation $x' = Bx$ and assume that the perturbed equation can be represented in the form

$$x' = (B + D\Sigma E)x, \tag{1.1}$$

where Σ is an unknown disturbance matrix and D, E are known scaling matrices defining the "structure" of the perturbation. The complex stability radius is then given by

$$\left[\max_{t\in i\mathbb{R}} \|E(tI - B)^{-1}D\|\right]^{-1}. \tag{1.2}$$

If the nominal equation is the difference equation $x_{n+1} = Bx_n$ with a structured perturbation of the form

$$x_{n+1} = (B + D\Sigma E)x_n, \tag{1.3}$$

then we have a formula in [3] for computing the complex stability radius

$$\left[\max_{\omega\in\mathbb{C}:|\omega|=1} \|E(\omega I - B)^{-1}D\|\right]^{-1}. \tag{1.4}$$

Moreover, in recent years, several technical problems in electronic circuit theory and robotic designs lead to the problem of investigating the differential–algebraic equation $f(x'(t), x(t)) = 0$, where the leading term x' cannot be explicitly solved from $x(t)$. The linear form of this equation is

$$Ax'(t) = Bx(t), \tag{1.5}$$

with A and B denoting two constant matrices. Assume that Eq. (1.5) is subjected to perturbations of the form

$$Ax'(t) = (B + D\Sigma E)x(t). \tag{1.6}$$

Then the formula of the complex stability radius is given by (see [4])

$$\left[\max_{t\in i\mathbb{R}} \|E(tA - B)^{-1}D\|\right]^{-1}. \tag{1.7}$$

When the nominal equation is the difference equation $Ax_{n+1} = Bx_n$ with the structured perturbation of the form

$$Ax_{n+1} = (B + D\Sigma E)x_n, \tag{1.8}$$

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we obtained an expression in [5] for the complex stability radius given by

$$\left[\max_{\omega\in\mathbb{C}:|\omega|=1} \|E(\omega A - B)^{-1}D\|\right]^{-1}. \tag{1.9}$$

Earlier results of stability radii for time-varying equations can be found, e.g., in [6,7]. The most successful attempt for finding a formula for the stability radius was an elegant result given by Jacob [7]. Using this result, the notion and formula of the stability radius were extended to linear time-invariant differential–algebraic equations [8,9,4]; and to linear time-varying differential and difference–algebraic equations [10,5].

On the other hand, in order to unify the continuous and discrete analysis, the theory of the analysis on time scales was introduced by Stefan Hilger in his Ph.D thesis in 1988 (supervised by Bernd Aulbach) [11] and has received a lot of attention. By using the notation of the analysis on time scales, Eqs. (1.1) and (1.3) can be rewritten under the unified form

$$x^{\Delta} = (B + D\Sigma E)x, \tag{1.10}$$

and also Eqs. (1.6) and (1.8) become

$$Ax^{\Delta} = (B + D\Sigma E)x.$$

A formula of the stability radius for (1.10) is derived recently in [12] and it is given by

$$\left[\max_{t\in\Gamma_{us}} \|E(tI - B)^{-1}D\|\right]^{-1}, \tag{1.11}$$

where Γ_{us} is the stability domain of the time scale $\mathbb{T}$.

The purpose of this paper is to present a unified formula for (1.2), (1.4), (1.7), (1.9) and (1.11) and to generalize them by studying the stability radius of the implicit dynamic equations on time scales

$$Ax^{\Delta}(t) = Bx(t), \tag{1.12}$$

under the general structured perturbations of the form

$$[A, B] \rightsquigarrow [\tilde{A}, \tilde{B}] = [A, B] + D\Sigma E. \tag{1.13}$$

When $\mathbb{T} = \mathbb{R}$ (resp. $\mathbb{T} = \mathbb{N}$), we consider it as a generalization of Eqs. (1.1) and (1.6) (resp. Eqs. (1.3) and (1.8)).

The difficulty we are faced when dealing with this problem is that although A, B, D, E are constant matrices, the structure of time scale (also the stability domain) is rather complicated and it can make Eq. (1.12) become a time-varying equation. Moreover, the disturbances affect not only the term on the right, but also the coefficient of the derivative on the left-hand side and it seems that we are working with an ill-posed problem.

This paper is organized as follows. In Section 2, we summarize some preliminary results on time scales. In Section 3, by defining the so-called domain of the uniformly exponential stability of time scales, we give the formulas of the stability radii of Eq. (1.12), where the general structured perturbations are considered. Section 4 is concerned with special classes of $\{A, B\}$ where the complex and real stability radii are equal.

2. Preliminaries

A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$ and we usually denote it by the symbol $\mathbb{T}$. The most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We assume throughout that a time scale $\mathbb{T}$ inherits the topology from the standard topology of the real numbers. We define the *forward jump operator* $\varsigma : \mathbb{T} \to \mathbb{T}$ by $\varsigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ (supplemented by $\inf\emptyset = \sup\mathbb{T}$) and the *backward jump operator* $\varrho : \mathbb{T} \to \mathbb{T}$ by $\varrho(t) = \sup\{s \in \mathbb{T} : s < t\}$ (supplemented by $\sup\emptyset = \inf\mathbb{T}$). The *positively graininess function* $\mu : \mathbb{T} \to \mathbb{R}^{+} \cup \{0\}$ is given by $\mu(t) = \varsigma(t) - t$. For our purpose, we will assume that the time scale $\mathbb{T}$ is unbounded above, i.e., $\sup\mathbb{T} = \infty$.

Definition 2.1 (*Delta Derivative*)**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called *delta differentiable* at t if there exists a scalar $f^{\Delta}(t)$ such that for all $\epsilon > 0$

$$|f(\varsigma(t)) - f(s) - f^{\Delta}(t)(\varsigma(t) - s)| \leqslant \epsilon|\varsigma(t) - s|$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The scalar $f^{\Delta}(t)$ is called the *delta derivative* of f at t.

If $\mathbb{T} = \mathbb{R}$ then delta derivative is $f'(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the delta derivative is the forward difference, Δf, from discrete calculus.

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\varsigma(t) = t$, *right-scattered* if $\varsigma(t) > t$, *left-dense* if $\varrho(t) = t$ and *left-scattered* if $\varrho(t) < t$. A function f defined on $\mathbb{T}$ is *rd-continuous* if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. For any rd-continuous functions $p(\cdot)$ from $\mathbb{T}$ to $\mathbb{R}$, the solution of the dynamic equation $x^{\Delta} = p(t)x$, with the initial condition $x(s) = 1$, defines a so-called exponential function. We denote this exponential function by $e_p(t, s)$. For the properties of exponential function $e_p(t, s)$ the interested reader can see [13–15].

Denote $\mathbb{T}^{+} = [t_0, \infty) \cap \mathbb{T}$. We consider the dynamic equation on the time scale $\mathbb{T}$

$$x^{\Delta} = f(t, x), \tag{2.14}$$

where $f : \mathbb{T}\times\mathbb{R}^d \to \mathbb{R}^d$ is a rd-continuous function and $f(t, 0) = 0$.

For the existence, uniqueness and extensibility of solution of Eq. (2.14) we refer to [14]. A function f from $\mathbb{T}$ to $\mathbb{R}$ is *positively regressive* if $1 + \mu(t)f(t) > 0$ for every $t \in \mathbb{T}$. We denote $\mathcal{R}^{+}$ the set of positively regressive functions from $\mathbb{T}$ to $\mathbb{R}$. For any $\tau \in \mathbb{T}^{+}$, let $x(t) = x(t, \tau, x_0)$ be a solution of (2.14) with the initial condition $x(\tau, \tau) = x_0 \in \mathbb{R}^d$. On the exponential stability of dynamic equations on time scales, we use the following definition, see, e.g. [16,11,17]:

Definition 2.2 (*Exponential Stability*)**.** The dynamic equation (2.14) is called exponentially stable if the condition

- for every $\tau \in \mathbb{T}^{+}$ there exists an $N = N(\tau) \geqslant 1$ satisfying
$$\|x(t, \tau, x_0)\| \leqslant N(\tau)\|x_0\|e_{-\alpha}(t, \tau) \tag{2.15}$$
for all $t \geqslant \tau, t \in \mathbb{T}^{+}$ and $x_0 \in \mathbb{R}^d$, where $x(t, \tau, x_0)$ is the solution of (2.14) with the initial condition $x(\tau, \tau) = x_0$

holds for some $\alpha > 0$ such that $-\alpha \in \mathcal{R}^{+}$. If the constant N can be chosen independent of $\tau \in \mathbb{T}^{+}$ then the dynamic equation (2.14) is called uniformly exponentially stable.

Note that the condition $-\alpha \in \mathcal{R}^{+}$ is equivalent to $\mu(t) \leqslant \frac{1}{\alpha}$. This means that we are working on time scales with bounded graininess. Beside this definition one can find other definitions of exponential stability in [18–20] where instead of using the exponential function $e_{-\alpha}(t, \tau)$ on time scale, one uses the classical exponential function $\exp\{-\alpha(t - \tau)\}$ in (2.15). However, it is easy to prove that these definitions are equivalent.

We now consider the condition of exponential stability for linear time-invariant equations

$$x^{\Delta} = Ax, \tag{2.16}$$

where $A \in \mathbb{K}^{d\times d}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). We denote the set of the eigenvalues of A by $\sigma(A)$.

The following theorem can be proved by a similar way as in [19], although we use the exponential function on time scales to define exponential stability.

Theorem 2.3 (*See [19, Lemma 6.1]*)**.** *The linear equation (2.16) is uniformly exponentially stable if and only if for every $\lambda \in \sigma(A)$, the scalar equation $x^{\Delta} = \lambda x$ is uniformly exponentially stable.*

It is easy to give an example where on the time scale $\mathbb{T}$, the scalar dynamic equation $x^\Delta = \lambda x$ is exponentially stable but it is not uniformly exponentially stable. Indeed, denote $((a, b)) = \{n \in \mathbb{N} : a < n < b\}$. Consider the time scale

$$\mathbb{T} = \bigcup_n [2^{2n}, 2^{2n+1}] \bigcup_n ((2^{2n+1}, 2^{2n+2})).$$

Let $\lambda = -2$ and $\tau \in \mathbb{T}$, say $2^m \leqslant \tau < 2^{m+1}$. We can choose $\alpha = -1$ and $N = 2^{m+1}$ to obtain $|e_\lambda(t, \tau)| \leqslant Ne_{-1}(t, \tau)$. However, it is not possible to choose N independent of τ.

Now, we denote

$$S = \{\lambda \in \mathbb{C},\ \text{the scalar equation } x^\Delta = \lambda x \text{ is uniformly exponentially stable}\}.$$

The set S is called the domain of the uniform exponential stability of the time scale $\mathbb{T}$. By the definition, if $\lambda \in S$, there exist $\alpha > 0$ and $N \geqslant 1$ satisfying $-\alpha \in \mathcal{R}^+$ and $|e_\lambda(t, \tau)| \leqslant Ne_{-\alpha}(t, \tau)$ for all $t \geqslant \tau$. As a corollary of proposition 3.1 in [21], we have the following result.

Theorem 2.4. *S is an open set in $\mathbb{C}$.*

For illustrating the domain of the uniform exponential stability S of the time scale $\mathbb{T}$, we consider some simple cases.

- When $\mathbb{T} = \mathbb{R}$ then $S = \{\lambda \in \mathbb{C}, \operatorname{Re}\lambda < 0\}$.
- When $\mathbb{T} = h\mathbb{Z}(h > 0)$ then $S = \{\lambda \in \mathbb{C}, |1 + \lambda h| < 1\}$.
- When $\mathbb{T} = \bigcup_{k=0}^\infty [2k, 2k + 1]$ then $S = \{\lambda \in \mathbb{C}, \operatorname{Re}\lambda + \ln|1 + \lambda| < 0\}$.

3. Stability radii of implicit dynamic equations on time scales

Consider the implicit dynamic equation on time scale $\mathbb{T}$

$$Ax^\Delta(t) = Bx(t), \tag{3.1}$$

where $x(t) \in \mathbb{K}^m$, and $\{A, B\} \in \mathbb{K}^{m\times m}$ are constant matrices; underlying field $\mathbb{K}$ is either real or complex. We assume that the pencil of matrices $\{A, B\}$ is regular (that is, $\det(\lambda A - B) \not\equiv 0$) and the index of $\{A, B\}$ is $k \geqslant 1$. The Kronecker decomposition of the pencil of matrices $\{A, B\}$ indicates that there exists a pair of nonsingular matrices W, T such that

$$A = W\operatorname{diag}(I_r, U)T^{-1}, \qquad B = W\operatorname{diag}(B_1, I_{m-r})T^{-1}, \tag{3.2}$$

where I_r is the unit matrix in $\mathbb{K}^{r\times r}$ and B_1 is a matrix in $\mathbb{K}^{r\times r}$. Further, $U \in \mathbb{K}^{(m-r)\times(m-r)}$ is a nilpotent matrix whose nilpotency degree is exactly k. Denote

$$\widehat{Q} = T\operatorname{diag}(0_r, I_{m-r})T^{-1},$$
$$\widehat{P} = I_m - \widehat{Q} = T\operatorname{diag}(I_r, 0_{m-r})T^{-1}. \tag{3.3}$$

It is known that for any $\alpha \in \mathbb{K}$ such that $\alpha A + B$ is nonsingular, one has

$$\mathbb{K}^m = \ker[(\alpha A + B)^{-1}A]^k \oplus \operatorname{im}[(\alpha A + B)^{-1}A]^k,$$

and $\widehat{Q}$ is the projection onto $\ker[(\alpha A + B)^{-1}A]^k$ along the space $\operatorname{im}[(\alpha A + B)^{-1}A]^k$. In particular, $\widehat{Q}$ does not depend on the choice of W and T. Let

$$A_1 = A - B\widehat{Q} = W\operatorname{diag}(I_r, U - I_{m-r})T^{-1}. \tag{3.4}$$

Since U is a nilpotent matrix, it is clear that A_1 is invertible. Further, by using (3.2) and (3.3) it follows that $\widehat{P}A_1^{-1}A = A_1^{-1}A\widehat{P} = \widehat{P}$ and $\widehat{P}A_1^{-1}B = A_1^{-1}B\widehat{P} = \widehat{P}A_1^{-1}B\widehat{P}$. Multiplying both sides of (3.1) by $\widehat{P}A_1^{-1}$ and $\widehat{Q}A_1^{-1}$ respectively we obtain

$$\begin{cases} (\widehat{P}x)^\Delta(t) = \widehat{P}A_1^{-1}B(\widehat{P}x)(t),\\ (\widehat{Q}A_1^{-1}Ax)^\Delta(t) = \widehat{Q}A_1^{-1}Bx(t). \end{cases} \tag{3.5}$$

Also by the decomposition (3.2) and the definition (3.3) of $\widehat{Q}$ we see that

$$\widehat{Q}A_1^{-1}A = T\operatorname{diag}(0, (U - I_{m-r})^{-1}U)T^{-1}$$

is a nilpotent matrix and

$$\widehat{Q}A_1^{-1}B = T\operatorname{diag}(0, (U - I_{m-r})^{-1})T^{-1}.$$

Denoting $T^{-1}x(t) = \binom{y(t)}{z(t)}$ where $z(t) \in \mathbb{K}^{m-r}$, we obtain

$$Uz^\Delta(t) = z(t). \tag{3.6}$$

It is easy to see that this equation has a unique solution $z(t) \equiv 0$. Therefore, the solution $x(t)$ of (3.1) with the initial condition $\widehat{P}(x(t_0) - x_0) = 0$ exists in $\mathbb{T}^+ = [t_0, \infty) \cap \mathbb{T}$, and it satisfies

$$\widehat{Q}x(t) = T\operatorname{diag}(0, I_{m-r})T^{-1}x(t) = T\operatorname{diag}(0, I_{m-r})\begin{pmatrix} y(t)\\ 0\end{pmatrix} = 0, \quad \text{for all } t \in \mathbb{T}^+. \tag{3.7}$$

In particular, the initial condition $x(t_0) = \widehat{P}x_0$ must hold. Let $x(t, \tau, \widehat{P}x_0)$ be the solution of (3.1) with the initial value $x(\tau, \tau) = \widehat{P}x_0$. According to Definition 2.2, we get the following definition of exponential stability:

Definition 3.1. The implicit dynamic equation (3.1) is called exponentially stable if the condition

- for every $\tau \in \mathbb{T}^+$ and $x_0 \in \mathbb{R}^m$ there exists an $N = N(\tau) \geqslant 1$ satisfying
$$\|x(t, \tau, \widehat{P}x_0)\| \leqslant N(\tau)\|\widehat{P}x_0\|e_{-\alpha}(t, \tau) \tag{3.8}$$
for all $t \geqslant \tau, t \in \mathbb{T}^+$ where $x(t, \tau, \widehat{P}x_0)$ is the solution of (3.1) with the initial value $x(\tau, \tau) = \widehat{P}x_0$

holds for some $\alpha > 0$ such that $-\alpha \in \mathcal{R}^+$. If the constant N can be chosen independent of τ then the implicit dynamic equation (3.1) is called uniformly exponentially stable.

We denote by $\sigma(C, D)$ the spectrum of the pencil $\{C, D\}$, i.e., the set of all solutions of the equation $\det(\lambda C - D) = 0$. When $C = I$, we write simply $\sigma(D)$ for $\sigma(I, D)$.

Theorem 3.2. *The implicit dynamic equation (3.1) is uniformly exponentially stable if and only if $\sigma(A, B) \subset S$, where S is the domain of the uniformly exponential stability of the time scale $\mathbb{T}$.*

Proof. Let $x(t) = T\binom{y(t)}{z(t)}$. Then, we have $\widehat{P}x(t) = T\binom{y(t)}{0}$. By decomposition (3.2) and (3.3) we get

$$\widehat{P}A_1^{-1}B = T\operatorname{diag}(B_1, 0_{m-r})T^{-1}.$$

From (3.5) and (3.6) it follows that Eq. (3.1) is equivalent to

$$\begin{cases} y^\Delta(t) = B_1y(t),\\ z(t) \equiv 0. \end{cases} \tag{3.9}$$

Therefore, Eq. (3.1) is uniformly exponentially stable if and only if the linear equation $y^\Delta(t) = B_1y(t)$ is so. By Theorem 2.3, this is equivalent to $\sigma(B_1) \subset S$. On the other hand,

$$\lambda A - B = W\operatorname{diag}(\lambda I_r - B_1, \lambda U - I_{m-r})T^{-1}.$$

This implies that

$$\det(\lambda A - B) = 0 \Longleftrightarrow \det(\lambda I_r - B_1) = 0.$$

Thus, $\sigma(A, B) = \sigma(B_1)$ and the uniformly exponential stability of Eq. (3.1) is equivalent to $\sigma(A, B) \subset S$. The proof is complete. □

Now, we consider Eq. (3.1) subjected to general structured perturbations of the form

$$\widetilde{A}x^{\Delta}(t) = \widetilde{B}x(t), \tag{3.10}$$

with

$$[\widetilde{A}, \widetilde{B}] = [A, B] + D\Sigma E, \tag{3.11}$$

where $D \in \mathbb{K}^{m\times l}$, $E \in \mathbb{K}^{q\times 2m}$, the perturbation $\Sigma \in \mathbb{K}^{l\times q}$. The matrix $D\Sigma E$ is called a structured perturbation of the Eq. (3.1). If we let $E = [E_1, E_2]$ with $E_1, E_2 \in \mathbb{K}^{q\times m}$ then (3.11) is equivalent to

$$\widetilde{A} = A + D\Sigma E_1, \qquad \widetilde{B} = B + D\Sigma E_2.$$

It is easy to see that the perturbed model of the form

$$A \rightsquigarrow A + D_A \Sigma_A E_A, \qquad B \rightsquigarrow B + D_B \Sigma_B E_B,$$

where $E_A \in \mathbb{C}^{q_1\times m}$, $E_B \in \mathbb{C}^{q_2\times m}$, $D_A = D_B \in \mathbb{C}^{m\times l}$, can be rewritten in the form (3.11) with $D = D_A = D_B$, $\Sigma = [\Sigma_A, \Sigma_B]$, $E = \operatorname{diag}(E_A, E_B)$.

We define

$$\Xi_{\mathbb{K}} = \{\Sigma \in \mathbb{K}^{l\times q} : \text{Eq. (3.10) is either irregular or not uniformly exponentially stable}\}.$$

Definition 3.3. The stability radius of Eq. (3.1) under structured perturbations of the form (3.11) is defined by

$$r_{\mathbb{K}}(A, B; D, E) = \inf\{\|\Sigma\| : \Sigma \in \Xi_{\mathbb{K}}\},$$

where $\|\cdot\|$ can be any vector-induced matrix norm.

Let us use the notation $E_\lambda = E\begin{bmatrix}\lambda I_m \\ -I_m\end{bmatrix}$. We have the following theorem.

Theorem 3.4. *The complex stability radius of Eq. (3.1) under structured perturbations of the form (3.11) is given by the formula*

$$r_{\mathbb{C}}(A, B; D, E) = \left(\sup_{\lambda\in\infty\cup\partial S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1}. \tag{3.12}$$

Proof. Let $\Sigma \in \mathbb{C}^{l\times q}$ be such that the perturbed equation (3.10) is irregular or it is regular but not uniformly exponentially stable. In both cases, we can always choose an eigenvalue $\lambda_0 \in \sigma(\widetilde{A}, \widetilde{B}) \bigcap (\mathbb{C}\setminus S)$ and an eigenvector $x \neq 0$ corresponding to λ_0, i.e., $(\lambda_0\widetilde{A} - \widetilde{B})x = 0$. From (3.11) this yields

$$\lambda_0\widetilde{A} - \widetilde{B} = [\widetilde{A}, \widetilde{B}]\begin{bmatrix}\lambda_0 I_m \\ -I_m\end{bmatrix} = ([A, B] + D\Sigma E)\begin{bmatrix}\lambda_0 I_m \\ -I_m\end{bmatrix} = \lambda_0 A - B + D\Sigma E_{\lambda_0}.$$

Therefore,

$$(\lambda_0 A - B)x = -D\Sigma E_{\lambda_0}x.$$

This relation implies

$$E_{\lambda_0}x = -E_{\lambda_0}(\lambda_0 A - B)^{-1}D\Sigma E_{\lambda_0}x.$$

Since $E_{\lambda_0}x \neq 0$,

$$\|\Sigma\| \geqslant \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1} \geqslant \left(\sup_{\lambda\in\mathbb{C}\setminus S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1}.$$

Thus,

$$r_{\mathbb{C}}(A, B; D, E) \geqslant \left(\sup_{\lambda\in\mathbb{C}\setminus S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1}.$$

Conversely, take $\epsilon > 0$ and a $\lambda_0 \in \mathbb{C}\setminus S$ satisfying

$$\left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1} \leqslant \left(\sup_{\lambda\in\mathbb{C}\setminus S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1} + \epsilon.$$

Following the same argument as in [15], we find a vector $u \in \mathbb{C}^l$ satisfying $\|u\| = 1$ and

$$\|E_{\lambda_0}(\lambda_0 A - B)^{-1}Du\| = \|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|.$$

Let y^* be a linear functional defined on $\mathbb{C}^l$ such that $\|y^*\| = 1$ and

$$y^*(E_{\lambda_0}(\lambda_0 A - B)^{-1}Du) = \|E_{\lambda_0}(\lambda_0 A - B)^{-1}Du\| = \|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|.$$

Consider

$$\Sigma = -\left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1}uy^*, \quad x = (\lambda_0 A - B)^{-1}Du. \tag{3.13}$$

It is clear that

$$\|\Sigma\| \leqslant \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1}\|u\|\|y^*\| = \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1},$$

and

$$\Sigma E_{\lambda_0}(\lambda_0 A - B)^{-1}Du = \frac{-u}{\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|}\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\| = -u. \tag{3.14}$$

Since $u \neq 0$,

$$\|\Sigma\| \geqslant \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1}.$$

Combining these inequalities we obtain

$$\|\Sigma\| = \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1}.$$

Furthermore, from (3.13) and (3.14) it follows that $(\lambda_0 A - B + D\Sigma E_{\lambda_0})x = 0$, i.e., $\lambda_0 \in \sigma(\widetilde{A}, \widetilde{B})$, with $[\widetilde{A}, \widetilde{B}] = [A, B] + D\Sigma E$, which implies that the equation

$$\widetilde{A}x^{\Delta}(t) = \widetilde{B}x(t)$$

is either irregular or not uniformly exponentially stable. This means that $\Sigma \in \Xi_{\mathbb{C}}$ which implies

$$r_{\mathbb{C}}(A, B; D, E) \leqslant \|\Sigma\| = \left(\|E_{\lambda_0}(\lambda_0 A - B)^{-1}D\|\right)^{-1} \leqslant \left(\sup_{\lambda\in\mathbb{C}\setminus S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1} + \epsilon. \tag{3.15}$$

Since ϵ is arbitrary,

$$r_{\mathbb{C}}(A, B; D, E) = \left(\sup_{\lambda\in\mathbb{C}\setminus S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1}.$$

Note that the function $G(\lambda) = E_\lambda(\lambda A - B)^{-1}D$ is analytic on $\mathbb{C}\setminus S$. By the maximum principle, $\|G(\cdot)\|$ either reaches its maximum value on the boundary ∂S of S or $\sup_{\lambda\in\mathbb{C}\setminus S}\|G(\lambda)\| = \lim_{\lambda\to\infty}\|G(\lambda)\|$. Thus, we obtain

$$r_{\mathbb{C}}(A, B; D, E) = \left(\sup_{\lambda\in\infty\cup\partial S} \|E_\lambda(\lambda A - B)^{-1}D\|\right)^{-1}.$$

The proof is complete. □

Corollary 3.5. *The complex stability radius of Eq. (3.1) under the structured perturbation of the form*

$$Ax^{\Delta}(t) = (B + D_B \Sigma E_B)x(t), \tag{3.16}$$

is given by

$$r_{\mathbb{C}}(B; D_B, E_B) = \left(\sup_{\lambda \in \infty \cup \partial S} \|E_B(\lambda A - B)^{-1} D_B\| \right)^{-1}, \tag{3.17}$$

and under the structured perturbation of the form

$$(A + D_A \Sigma E_A)x^{\Delta}(t) = Bx(t), \tag{3.18}$$

is given by

$$r_{\mathbb{C}}(A; D_A, E_A) = \left(\sup_{\lambda \in \infty \cup \partial S} \|\lambda E_A(\lambda A - B)^{-1} D_A\| \right)^{-1}. \tag{3.19}$$

Proof. With $D = D_B$ and $E = [E_1, E_2] = [0, E_B]$, the perturbation (3.16), we can write

$[\widetilde{A}, \widetilde{B}] = [A, B] + D\Sigma E,$

Further, $E_\lambda = E \begin{bmatrix} \lambda I_m \\ -I_m \end{bmatrix} = -E_B$ and by Theorem 3.4, we get (3.17).

For the perturbation (3.18), we choose $D = D_A$ and $E = [E_1, E_2] = [E_A, 0]$. By seeing that $E_\lambda = E \begin{bmatrix} \lambda I_m \\ -I_m \end{bmatrix} = \lambda E_A$ we get (3.19). □

Theorem 3.6. (a) $r_{\mathbb{C}}(B; D_B, E_B) > 0$ *if and only if the polynomial* $p(\lambda) = E_B\widehat{Q}(\lambda A - B)^{-1} D_B$ *is constant.*
(b) $r_{\mathbb{C}}(A; D_A, E_A) > 0$ *if and only if the polynomial* $q(\lambda) = \lambda E_A \widehat{Q}(\lambda A - B)^{-1} D_A$ *is constant.*
(c) *Let* $E = [E_1, E_2]$. *Then* $r_{\mathbb{C}}(A, B; D, E) > 0$ *if and only if the polynomial* $s(\lambda) = (\lambda E_1 - E_2)\widehat{Q}(\lambda A - B)^{-1} D$ *is constant.*

Proof. (a) We have

$$E_B(\lambda A - B)^{-1} D_B = E_B\widehat{P}(\lambda A - B)^{-1} D_B + E_B\widehat{Q}(\lambda A - B)^{-1} D_B.$$

It is easy to prove that $\lim_{\lambda\to\infty} \|E_B\widehat{P}(\lambda A - B)^{-1} D_B\| = \lim_{\lambda\to\infty} \|E_B T \operatorname{diag}\left((\lambda I - B_1)^{-1}, 0_{m-r}\right) W^{-1} D_B\| = 0$. Moreover, since $U^k = 0$,

$$\begin{aligned} p(\lambda) &= E_B\widehat{Q}(\lambda A - B)^{-1} D_B \\ &= E_B T \operatorname{diag}\left(0_r, (\lambda U - I_{m-r})^{-1}\right) W^{-1} D_B \\ &= E_B T \operatorname{diag}\left(0_r, -\sum_{i=0}^{k-1} (\lambda U)^i\right) W^{-1} D_B, \end{aligned}$$

where T, W and U as mentioned in (3.2) and $\widehat{Q} = T\operatorname{diag}(0_r, I_{m-r}) T^{-1}$, $\widehat{P} = I_m - \widehat{Q} = T\operatorname{diag}(I_r, 0_{m-r})T^{-1}$. Hence, $\lim_{\lambda\to\infty} \|E_B(\lambda A - B)^{-1} D_B\|$ exists and it equals ∞ if $p(\lambda)$ is not constant. Thus, we get (a).
A similar argument can be applied to prove (b) and (c).

The proof is complete. □

Corollary 3.7. *Let* $\operatorname{ind}(A, B) = 1$. *Then,* $r_{\mathbb{C}}(A, B; D, E) > 0$ *if and only if* $r_{\mathbb{C}}(A; D, E_1) > 0$ *where* $E = [E_1, E_2]$.

When $A = I$, (3.17) has been proved in [12] in order to unify the continuous and discrete stability radii of the linear dynamic equations. In case $\mathbb{T} = \mathbb{R}$, it is seen that $S = \mathbb{C}_-$ and we get (1.7). If $\mathbb{T} = \mathbb{N}$, $S = \{\omega : |1 + \omega| < 1\}$ and (1.9) is deduced.

Example 3.8. Let us calculate the stability radius of the equation $Ax^{\Delta}(t) = Bx(t)$ under the structured perturbation of the form

$$[A, B] \rightsquigarrow [\tilde{A}, \tilde{B}] = [A, B] + D\Sigma E,$$

where $\mathbb{T} = \bigcup_{k=0}^{\infty} [2k, 2k + 1]$,

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 3 & 3 & 3 \end{bmatrix},$$

$$E = [E_1, E_2] = \begin{bmatrix} 1 & 0 & 1 & 0 & -3 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{bmatrix}.$$

Since $\mathbb{T} = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, $S = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda + \ln|\lambda + 1| < 0\}$. It is easy to see that $\operatorname{ind}(A, B) = 2$ and $\sigma(A, B) = -\frac{1}{3}$. Therefore, the pencil $\{A, B\}$ is exponentially stable. When $\lambda \in \partial S$, by the direct computations, we obtain

$$(\lambda A - B)^{-1} = \frac{1}{3\lambda + 1} \begin{bmatrix} \lambda + 1 & -1 & -\lambda^2 \\ \lambda - 1 & 2 & -\lambda^2 - \lambda \\ -\lambda - 1 & 1 & \lambda^2 + 3\lambda + 1 \end{bmatrix},$$

$$\widehat{Q} = \frac{1}{9} \begin{bmatrix} 7 & 3 & 1 \\ 4 & 3 & -2 \\ 2 & -3 & 8 \end{bmatrix}, \quad E_\lambda = \lambda E_1 - E_2 = \begin{bmatrix} \lambda & 3 & \lambda \\ \lambda & -\lambda & -1 \\ 2 & \lambda & \lambda \end{bmatrix},$$

and therefore,

$$E_\lambda(\lambda A - B)^{-1} D = \frac{1}{3\lambda + 1} \begin{bmatrix} -12 & -12 & -12 \\ 0 & 0 & 0 \\ 6 & 6 & 6 \end{bmatrix}.$$

Let $\|\cdot\|_\infty^v$ be the maximum norm of $\mathbb{C}^3$. We have

$$\|E_\lambda(\lambda A - B)^{-1} D\|_\infty = \frac{36}{|3\lambda + 1|},$$

where $\|\cdot\|_\infty$ is the operator's norm induced by $\|\cdot\|_\infty^v$. This implies that

$$\sup_{\lambda \in \infty \cup \partial S} \|E_\lambda(\lambda A - B)^{-1} D\|_\infty = \|E_0(-B)^{-1} D\|_\infty = 36.$$

Thus, we get

$$r_{\mathbb{C}}(A, B; D, E) = \frac{1}{36}.$$

Moreover, we see that

$$[A, B] \rightsquigarrow [\tilde{A}, \tilde{B}] = [A, B] + D\Sigma E \iff \begin{cases} A \rightsquigarrow \tilde{A} = A + D\Sigma E_1, \\ B \rightsquigarrow \tilde{B} = B + D\Sigma E_2, \end{cases}$$

and the polynomials

$$q(\lambda) = \lambda E_1 \widehat{Q}(\lambda A - B)^{-1} D = \begin{bmatrix} 3\lambda & 3\lambda & 3\lambda \\ \lambda & \lambda & \lambda \\ 2\lambda & 2\lambda & 2\lambda \end{bmatrix} \neq \text{constant},$$

$$\begin{aligned} p(\lambda) &= E_2 \widehat{Q}(\lambda A - B)^{-1} D \\ &= \begin{bmatrix} 3\lambda & 3\lambda & 3\lambda \\ \lambda + 2 & \lambda + 2 & \lambda + 2 \\ 2\lambda - 2 & 2\lambda - 2 & 2\lambda - 2 \end{bmatrix} \neq \text{constant}. \end{aligned}$$

Therefore

$$r_{\mathbb{C}}(A; D, E_1) = r_{\mathbb{C}}(B; D, E_2) = 0.$$

4. The equality of real and complex radii

Now, we consider the problem when the real and complex stability radii are equal. It seems that this is a difficult problem in the implicit dynamic equations on time scales because in this case, the positive cone $\mathbb{R}_+^m$ is no longer invariant under the action of the pencil of matrices $\{A, B\}$; even when both A and B are positive. Moreover, the domain of uniformly exponential stability S has the property that although $\lambda \in \partial S$, $\operatorname{Re}\lambda \in S$ which means we cannot use the approach in [4]. However, we are able to answer this question under some assumptions.

Let us consider Eq. (3.1) subjected to structured perturbations (3.10). Firstly, we prove the following result which provides a difference between ordinary dynamic equations and implicit dynamic equations. Denoting $G(\lambda) = E_\lambda(\lambda A - B)^{-1}D$, we have:

Theorem 4.1. *If $\|G(\lambda)\|$ does not reach its maximum at a finite point on ∂S then*

$$r_{\mathbb{C}}(A, B; D, E) = r_{\mathbb{R}}(A, B; D, E).$$

Proof. It is clear that $r_{\mathbb{C}}(A, B; D, E) \leqslant r_{\mathbb{R}}(A, B; D, E)$. Therefore, it is sufficient to prove that there exists a sequence of disturbances $\{\Sigma_n\} \subset \Xi_{\mathbb{R}}$ such that

$$\lim_{n\to\infty} \|\Sigma_n\| \leqslant r_{\mathbb{C}}(A, B; D, E).$$

It is seen

$$G(\lambda) = E_\lambda \widehat{P}(\lambda A - B)^{-1}D + E_\lambda \widehat{Q}(\lambda A - B)^{-1}D.$$

By using the Kronecker decomposition (3.2), we get

$$E_\lambda \widehat{Q}(\lambda A - B)^{-1}D = E_\lambda T \operatorname{diag}\left(0_r, -\sum_{i=0}^{k-1}(\lambda U)^i\right) W^{-1}D,$$

and

$$E_\lambda \widehat{P}(\lambda A - B)^{-1}D = E_\lambda T \operatorname{diag}\left((\lambda I_r - B_1)^{-1}, 0_{m-r}\right) W^{-1}D.$$

Assume that $E = [E_1, E_2]$ with $E_1, E_2 \in \mathbb{R}^{m\times m}$. Then, $E_\lambda = \lambda E_1 - E_2$, and it follows that

$$\begin{aligned}\lim_{\lambda\to\infty} E_\lambda \widehat{P}(\lambda A - B)^{-1}D &= E_1 T \operatorname{diag}(I_r, 0_{m-r})\, W^{-1}D \\ &= E_1 \widehat{P} A_1^{-1} D,\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}&E_\lambda T \operatorname{diag}\left(0_r, -\sum_{i=0}^{k-1}(\lambda U)^i\right) W^{-1}D \\ &\quad = (\lambda E_1 - E_2) T \operatorname{diag}\left(0_r, -\sum_{i=0}^{k-1}(\lambda U)^i\right) W^{-1}D\end{aligned}$$

is a polynomial in λ. Therefore the limit $\lim_{\lambda\to\infty}\|G(\lambda)\|$ exists (possibly $+\infty$). Since $\|G(\lambda)\|$ does not reach its maximum at a finite point on ∂S, it follows that

$$r_{\mathbb{C}}(A, B; D, E)^{-1} = \sup_{\lambda\in\infty\cup\partial S} \|G(\lambda)\| = \lim_{\lambda\to\infty}\|G(\lambda)\|.$$

This implies that $r_{\mathbb{C}}(A, B; D, E)^{-1} = \lim_{n\in\mathbb{N}; n\to\infty}\|G(n)\|$. For any $n \in \mathbb{N}$, let $u_n \in \mathbb{R}^l$ be a vector with $\|u_n\| = 1$: $\|G(n)u_n\| = \|G(n)\|$; let y_n^* be a linear functional defined on $\mathbb{R}^q$ with $\|y_n^*\| = 1$ and $y_n^*(G(n)u_n) = \|G(n)u_n\|$ as in Theorem 3.4. By denoting $\Sigma_n = \|G(n)\|^{-1}u_n y_n^*$ we see that n is an eigenvalue of the pencil $\{\widetilde{A}, \widetilde{B}\}$ with $[\widetilde{A}, \widetilde{B}] = [A, B] + D\Sigma_n E$ and the corresponding eigenvector $x_n = (nA - B)^{-1}Du_n$. Note that Σ_n is indeed a real perturbation. Therefore, $\Sigma_n \in \Xi_{\mathbb{R}}$. Further, $\|\Sigma_n\| = \|G(n)\|^{-1}\|u_n y_n^*\| \leqslant \|G(n)\|^{-1}$ and $\|\Sigma_n(G(n)u_n)\| = \|G(n)\|^{-1}\|u_n y_n^*(G(n)u_n)\| = \|u_n\| = 1$ which implies that $\|\Sigma_n\| = \|G(n)\|^{-1}$ and $\lim_{n\to\infty}\|\Sigma_n\| = \lim_{n\to\infty}\|G(n)\|^{-1} = r_{\mathbb{C}}(A, B; D, E)$. This relation says that $r_{\mathbb{R}}(A, B; D, E) \leqslant r_{\mathbb{C}}(A, B; D, E)$. The proof is complete. □

When $\|G(\lambda)\|$ attains its maximum value at a finite point in ∂S, we need further assumptions. A matrix $M = (m_{ij}) \in \mathbb{R}^{k\times p}$ is said to be positive, written as $M \geqslant 0$, if $m_{ij} \geqslant 0$ for any i, j. We define a partial order relation in $\mathbb{R}^{k\times p}$ by $M \geqslant N \Leftrightarrow M - N \geqslant 0$. We define the absolute value of a matrix $M = (m_{ij})$ as the matrix $|M| = (|m_{ij}|)$; similarly for a vector x we use the notation $|x| = (|x_1|, |x_2|, \ldots, |x_p|)$. Let $\rho(C, D)$ be the spectral radius of the pencil of matrices $\{C, D\}$, i.e., $\rho(C, D) := \max\{|\lambda| : \lambda \in \sigma(C, D)\}$.

Consider the implicit dynamic equation on time scale with structured perturbations of the form

$$Ax^{\Delta}(t) = (B + D_B\Sigma E_B)x(t), \tag{4.2}$$

where $A, B \in \mathbb{R}^{m\times m}$, $D_B \in \mathbb{R}^{m\times l}$, and $E_B \in \mathbb{R}^{q\times m}$. Suppose that $\operatorname{ind}(A, B) = 1$. Then, $\widehat{Q}$ is the projection onto $\ker A$ along the space

$$S = \{y \in \mathbb{R}^m : By \in \operatorname{im} A\}.$$

Recall that $A_1 = A - B\widehat{Q}$ is nonsingular, $A\widehat{Q} = 0$ and $\widehat{P} = A_1^{-1}A$. Moreover, from Perron–Frobenius extension theorem (see [22]) we have $\rho(A, B) = \rho(A_1^{-1}B\widehat{P})$ and if $A_1^{-1}B\widehat{P} \geqslant 0$ then $\rho(A, B)$ is an eigenvalue of the pencil of matrices $\{A, B\}$.

Let $\widehat{B} = A_1^{-1}B\widehat{P} = \widehat{P}A_1^{-1}B$. From (3.2)–(3.4), it follows that

$$\begin{aligned}(\lambda A - B)^{-1} &= T\operatorname{diag}\left((\lambda I_r - B_1)^{-1}, -I_{m-r}\right) W^{-1} \\ &= (\lambda I - \widehat{B})^{-1}\widehat{P}A_1^{-1} + \widehat{Q}A_1^{-1}.\end{aligned} \tag{4.3}$$

With the perturbed equation (4.2) then $G(\lambda) = E_B(\lambda A - B)^{-1}D_B$. The relation (4.3) implies that

$$G(\infty) = \lim_{\lambda\to\infty} G(\lambda) = E_B\widehat{Q}A_1^{-1}D_B. \tag{4.4}$$

For $\alpha \geqslant 0$, we define the ball $B_\alpha(-\alpha) = \{z \in \mathbb{C} : |z + \alpha| < \alpha\}$. For a time scale with bounded graininess several essential features are captured by an associated characteristic ball. The analysis of positive linear equations show that $B_\eta(-\eta) \subset S$, with S is the domain of uniform exponential stability of the time scale $\mathbb{T}$, and η is defined by

$$\eta = \frac{1}{\sup\{\mu(t) : t \in \mathbb{T}\}}, \tag{4.5}$$

see, e.g. [12].

Hypotheses 4.2. (i) $A_1^{-1}D_B \geqslant 0$, $E_B\widehat{P} \geqslant 0$ and $E_B\widehat{Q}A_1^{-1}D_B \geqslant 0$.
(ii) There exists $\alpha \geqslant 0$ with $B_\alpha(-\alpha) \subset S$ such that $\widehat{B} + \alpha\widehat{P} \geqslant 0$.

We need the following simple lemma.

Lemma 4.3. *Suppose that the bounded linear operator triplet: $\mathbb{M} : X \to Y, \mathbb{N} : Y \to Z, \mathbb{P} : Z \to X$ is given, where X, Y, Z are Banach spaces. Then the operator $I - \mathbb{MNP}$ is invertible if and only if $I - \mathbb{NPM}$ is invertible, moreover,*

$$(I - \mathbb{NPM})^{-1} = I + \mathbb{NP}(I - \mathbb{MNP})^{-1}\mathbb{M}.$$

Proof. Suppose that $I - \mathbb{MNP}$ is invertible. By direct calculation, it is easy to verify that the inverse of $I - \mathbb{NPM}$ is

$$(I - \mathbb{NPM})^{-1} = I + \mathbb{NP}(I - \mathbb{MNP})^{-1}\mathbb{M}. \tag{4.6}$$

Furthermore, if $(I - \mathbb{MNP})^{-1}$ is bounded then so is $(I - \mathbb{NPM})^{-1}$. The converse is proved similarly. □

Theorem 4.4. *Assume that the pencil of matrices $\{A, B\}$ has $\operatorname{ind}(A, B) = 1$ and satisfies Hypotheses 4.2. Then, we have*

$$r_{\mathbb{C}}(B; D_B, E_B) = r_{\mathbb{R}}(B; D_B, E_B). \tag{4.7}$$

Proof. Clearly, it is sufficient to prove that $r_{\mathbb{C}}(B; D_B, E_B) \geqslant r_{\mathbb{R}}(B; D_B, E_B)$. If $\|G(\lambda)\|$ does not reach its maximum value at a finite point on ∂S then from Theorem 4.1 it follows that $r_{\mathbb{C}}(B; D_B, E_B) = r_{\mathbb{R}}(B; D_B, E_B)$. Else, let $\lambda_0 \in \mathbb{C}$ satisfy $\|G(\lambda_0)\| = \sup_{\partial S} \|G(\lambda)\|$ and the perturbation Σ, given by (3.13), destroy stability. We aim to show

- Σ can be the complex perturbation, but $|\Sigma|$ is the real perturbation making the dynamic equation unstable,
- furthermore $\||\Sigma|\| = \|\Sigma\|$.

It is seen that $\|\Sigma\| = \|G(\lambda_0)\|^{-1} < \|G(\infty)\|^{-1}$. Moreover, Σ has rank one which implies that $\||\Sigma|\| = \|\Sigma\|$.

Since $B_\alpha(-\alpha) \subset S$ and the pencil of matrices $\{A, B + D_B\Sigma E_B\}$ is unstable, it follows that $\sigma(A, B + D_B\Sigma E_B) \not\subset S$ and

$$\rho(A, B + \alpha A + D_B\Sigma E_B) = \max\{|z + \alpha| : z \in \sigma(A, B + D_B\Sigma E_B)\} \geqslant \alpha. \tag{4.8}$$

Further, the inequality $\|\Sigma\|\|E_B\widehat{Q}A_1^{-1}D_B\| = \|\Sigma\|\|G(\infty)\| < 1$ implies that the matrix $I - E_B\widehat{Q}A_1^{-1}D_B\Sigma$ is invertible. Define

$$A_{1,\Sigma} = A - (B + D_B\Sigma E_B)\widehat{Q}.$$

From (3.4) it follows that

$$A_{1,\Sigma} = A_1(I - A_1^{-1}D_B\Sigma E_B\widehat{Q}). \tag{4.9}$$

Applying Lemma 4.3 with $\mathbb{M} = E_B\widehat{Q}, \mathbb{N} = I, \mathbb{P} = A_1^{-1}D_B\Sigma$ we get $I - A_1^{-1}D_B\Sigma E_B\widehat{Q}$ is invertible. This implies that $A_{1,\Sigma}$ is invertible as well. Now, we will prove that

$$\sigma(A, B + \alpha A + D_B\Sigma E_B) \cup \{0\} = \sigma(A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}). \tag{4.10}$$

Indeed, for any $\lambda \neq 0$, because of the properties $A\widehat{Q} = \widehat{P}\widehat{Q} = 0, \widehat{Q}\widehat{Q} = \widehat{Q}, \widehat{P} + \widehat{Q} = I$, we have

$$\begin{aligned}
&\det(\lambda I + A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}) = 0\\
&\iff \det(\lambda A_{1,\Sigma} + (B + \alpha A + D_B\Sigma E_B)\widehat{P}) = 0\\
&\iff \det\left[(\lambda A_{1,\Sigma} + (B + \alpha A + D_B\Sigma E_B)\widehat{P})\left(\widehat{P} - \frac{\widehat{Q}}{\lambda}\right)\right] = 0\\
&\iff \det[(A - (B + D_B\Sigma E_B)\widehat{Q})(\lambda\widehat{P} - \widehat{Q})\\
&\qquad + (B + \alpha A + D_B\Sigma E_B)\widehat{P}] = 0\\
&\iff \det[A(\lambda\widehat{P} - \widehat{Q}) + (B + D_B\Sigma E_B)\widehat{Q}\\
&\qquad + (B + \alpha A + D_B\Sigma E_B)\widehat{P}] = 0\\
&\iff \det[\lambda A(\widehat{P} + \widehat{Q}) + (B + \alpha A + D_B\Sigma E_B)(\widehat{P} + \widehat{Q})\\
&\qquad - (1 + \alpha + \lambda)A\widehat{Q}] = 0\\
&\iff \det[\lambda A + B + \alpha A + D_B\Sigma E_B] = 0.
\end{aligned}$$

This implies that the spectral equality (4.10) holds. In particular,

$$\rho(A, B + \alpha A + D_B\Sigma E_B) = \rho(A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}). \tag{4.11}$$

Similarly, since $\||\Sigma|\| = \|\Sigma\| < \|G(\infty)\|^{-1}$,

$$\sigma(A, B + \alpha A + D_B|\Sigma|E_B) \cup \{0\} = \sigma(A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}), \tag{4.12}$$

and

$$\rho(A, B + \alpha A + D_B|\Sigma|E_B) = \rho(A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}). \tag{4.13}$$

From (4.9) it follows that

$$\begin{aligned}
&A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}\\
&= (I - A_1^{-1}D_B\Sigma E_B\widehat{Q})^{-1}(A_1^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P})\\
&= (I - A_1^{-1}D_B\Sigma E_B\widehat{Q})^{-1}(\widehat{B} + \alpha\widehat{P} + A_1^{-1}D_B\Sigma E_B\widehat{P}).
\end{aligned} \tag{4.14}$$

Above we use the identities $\widehat{B} = A_1^{-1}B\widehat{P}$ and $A_1^{-1}A = \widehat{P}$. Using (4.6) with $\mathbb{M} = E_B\widehat{Q}, \mathbb{N} = I, \mathbb{P} = A_1^{-1}D_B\Sigma$, we get

$$\begin{aligned}
&(I - A_1^{-1}D_B\Sigma E_B\widehat{Q})^{-1}\\
&= I + A_1^{-1}D_B\Sigma(I - E_B\widehat{Q}A_1^{-1}D_B\Sigma)^{-1}E_B\widehat{Q}\\
&= I + A_1^{-1}D_B\Sigma(I - G(\infty)\Sigma)^{-1}E_B\widehat{Q}.
\end{aligned} \tag{4.15}$$

Define $\Phi(\Sigma) = A_1^{-1}D_B\Sigma(I - G(\infty)\Sigma)^{-1}G(\infty)\Sigma E_B\widehat{P}$. Then, from (4.14) and (4.15), because of the property $\widehat{Q}\widehat{B} = \widehat{Q}\widehat{P} = 0$, we obtain

$$\begin{aligned}
&A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}\\
&= \widehat{B} + \alpha\widehat{P} + A_1^{-1}D_B\Sigma E_B\widehat{P} + \Phi(\Sigma).
\end{aligned}$$

Similarly, we also have

$$\begin{aligned}
&A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}\\
&= \widehat{B} + \alpha\widehat{P} + A_1^{-1}D_B|\Sigma|E_B\widehat{P} + \Phi(|\Sigma|).
\end{aligned}$$

Moreover, by Hypotheses 4.2 and the inequality $\|G(\infty)\|\|\Sigma\| = \|G(\infty)\|\||\Sigma|\| < 1$, it follows that

$$\begin{aligned}
|\Phi(\Sigma)| &= A_1^{-1}D_B|\Sigma(I - G(\infty)\Sigma)^{-1}G(\infty)\Sigma|E_B\widehat{P}\\
&= A_1^{-1}D_B\left|\Sigma\sum_{i=0}^{\infty}(G(\infty)\Sigma)^{i+1}\right|E_B\widehat{P}\\
&\leqslant A_1^{-1}D_B\left(|\Sigma|\sum_{i=0}^{\infty}(G(\infty)|\Sigma|)^{i+1}\right)E_B\widehat{P}\\
&= A_1^{-1}D_B|\Sigma|(I - G(\infty)|\Sigma|)^{-1}G(\infty)|\Sigma|E_B\widehat{P}\\
&= \Phi(|\Sigma|),
\end{aligned}$$

and therefore,

$$\begin{aligned}
&|A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P}|\\
&\leqslant |\widehat{B} + \alpha\widehat{P}| + |A_1^{-1}D_B\Sigma E_B\widehat{P}| + |\Phi(\Sigma)|\\
&\leqslant \widehat{B} + \alpha\widehat{P} + A_1^{-1}D_B|\Sigma|E_B\widehat{P} + \Phi(|\Sigma|)\\
&= A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}.
\end{aligned} \tag{4.16}$$

From theory of nonnegative matrices, see, e.g. [23], it follows that

$$\begin{aligned}
&\rho(A_{1,\Sigma}^{-1}(B + \alpha A + D_B\Sigma E_B)\widehat{P})\\
&\leqslant \rho(A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}) := \beta.
\end{aligned} \tag{4.17}$$

Since $A_{1,|\Sigma|}^{-1}(B+\alpha A+D_B|\Sigma|E_B)\widehat{P} \geqslant 0$, by Perron–Frobenius theorem, β is an eigenvalue of the matrix $A_{1,|\Sigma|}^{-1}(B + \alpha A + D_B|\Sigma|E_B)\widehat{P}$ with maximum module and by (4.12) it follows that $\beta \in \sigma(A, B + \alpha A + D_B|\Sigma|E_B)$. Therefore, by (4.11), (4.8) and (4.17), we obtain $0 \leqslant \beta - \alpha \in \sigma(A, B + D_B|\Sigma|E_B)$. Thus the perturbation $|\Sigma| \in \mathbb{R}^{l\times q}$, with $\||\Sigma|\| = \|\Sigma\|$, destroys stability which implies $r_{\mathbb{C}}(B; D_B, E_B) \geqslant r_{\mathbb{R}}(B; D_B, E_B)$. The proof is complete. □

We consider the case where Eq. (3.1) is positive, i.e., for any $x_0 \in \mathbb{R}_+^m$, the solution $x(t)$ of Eq. (3.1) with $\widehat{P}(x(t_0) - x_0) = 0$ satisfies the condition $x(t) \geqslant 0$ for all $t \in \mathbb{T}, t \geqslant t_0$. It is known that if $x(t)$ is a solution of (3.1) then $\widehat{Q}x(t) = 0$ which implies $\widehat{P}x(t) = x(t)$ for all $t \in \mathbb{T}, t \geqslant t_0$. Therefore, (3.1) can be rewritten

$$\begin{cases} x^{\Delta} = \widehat{B}x,\\ x(t_0) = \widehat{P}x_0, \end{cases} \tag{4.18}$$

where $\widehat{B} = (A - B\widehat{Q})^{-1}B\widehat{P} = \widehat{P}(A - B\widehat{Q})^{-1}B$. It is easy to see that Eq. (4.18) has a general solution

$$x(t) = x(t, t_0, x_0) = \left(\prod_{t_0 \leqslant s < t} (\widehat{P} + \mu(s)\widehat{B}) \exp\{\text{mes}\,[t_0, t)\widehat{B}\}\right) \widehat{P}x_0,$$

where mes(C) is the Lebesgue measure of the set C. With $t = t_0$, we have $x(t_0) = \widehat{P}x_0 \geqslant 0$. If $\widehat{B}$ is a $\widehat{P}$-Metzler matrix, i.e., there exists $\alpha \in \mathbb{R}$ such that $\widehat{B} + \alpha\widehat{P} \geqslant 0$, then by paying attention that $\widehat{P}$ and $\widehat{B}$ are commutative, we see that $\exp\{\text{mes}\,[t_0, t)\widehat{B}\} = \exp\{\text{mes}\,[t_0, t)(\widehat{B} + \alpha\widehat{P})\} \exp\{-\text{mes}\,[t_0, t)\alpha\widehat{P}\}$. Therefore $\exp\{\text{mes}\,[t_0, t)\widehat{B}\}\widehat{P}x_0 = \exp\{\text{mes}\,[t_0, t)(\widehat{B} + \alpha\widehat{P})\} \exp\{-\alpha\,\text{mes}\,[t_0, t)\}\widehat{P}x_0 \geqslant 0$. Further, we need $\widehat{P} + \mu(t)\widehat{B} \geqslant 0$ for all $t \in \mathbb{T}$ which implies that $\eta\widehat{P} + \widehat{B} \geqslant 0$. Thus, the positiveness condition of Eq. (3.1) is equivalent to $\eta\widehat{P} + \widehat{B} \geqslant 0$. This means that the condition (ii) of Hypotheses 4.2 holds.

Corollary 4.5. *Assume that the solution of the linear equation $x^{\Delta} = Bx$ is positive and this equation is subjected to perturbations of the form $x^{\Delta} = (B + D_B\Sigma E_B)x$ with $D_B \geqslant 0$, $E_B \geqslant 0$. Then, we have*

$$r_{\mathbb{C}}(B; D_B, E_B) = r_{\mathbb{R}}(B; D_B, E_B). \tag{4.19}$$

Example 4.6. Let us consider the stability radius of the perturbed equation

$$Ax^{\Delta}(t) - (B + D_B\Sigma E_B)x(t) = 0, \tag{4.20}$$

with $\mathbb{T} = \mathbb{Z}$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & 1/2 & 0 \\ 1/2 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$D_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad E_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is seen that ind$(A, B) = 1$ and

$$\sigma(A, B) = \{-1/2; -3/2\} \subset S = \{\lambda \in \mathbb{C} : |1 + \lambda| < 1\}.$$

Moreover,

$$\widehat{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \widehat{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_1^{-1} = (A - B\widehat{Q})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, for $\alpha = 1$, we have $B_1(-1) = S$ and

$$\widehat{B} + \widehat{P} = A_1^{-1}B\widehat{P} + \widehat{P} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \geqslant 0.$$

It is easy to see that $A_1^{-1}D_B \geqslant 0$, $E_B\widehat{P} \geqslant 0$ and $G(\infty) = E_B\widehat{Q}A_1^{-1}D_B \geqslant 0$. From Theorem 4.4, we have $r_{\mathbb{C}}(B; D_B, E_B) = r_{\mathbb{R}}(B; D_B, E_B)$. By the computations, we get

$$(\lambda A - B)^{-1} = \frac{1}{(\lambda+1)^2 - 1/4} \times \begin{bmatrix} \lambda + 1 & 1/2 & 1/2 \\ 1/2 & \lambda + 1 & \lambda + 1 \\ 0 & 0 & (\lambda+1)^2 - 1/4 \end{bmatrix},$$

$$G(\lambda) = E_B(\lambda A - B)^{-1}D_B = \frac{1}{(\lambda+1)^2 - 1/4} \begin{bmatrix} \lambda + 1 & 1/2 & \lambda + 1 \\ \lambda + 3/2 & \lambda + 3/2 & \lambda + 3/2 \\ \lambda + 3/2 & \lambda + 3/2 & (\lambda + 3/2)^2 \end{bmatrix}.$$

Let $\|\cdot\|_3$ be the maximum norm of $\mathbb{C}^3$, it follows that

$$\sup_{\lambda \in \infty \cup \partial S} \|G(\lambda)\|_{\infty} = \|G(0)\|_{\infty} = 7.$$

Thus, we obtain

$$r_{\mathbb{C}}(B; D_B, E_B) = r_{\mathbb{R}}(B; D_B, E_B) = \left(\sup_{\lambda \in \infty \cup \partial S} \|G(\lambda)\|_{\infty}\right)^{-1} = \frac{1}{7}.$$

5. Conclusion

In this paper we have considered the uniformly exponential stability and given the formulas for the stability radius of implicit dynamic equations with general structured perturbations on time scales and obtain the characterizations of these formulas. We also provide some sufficient conditions for which the complex stability radius and the real stability radius are the same. So far we do not know whether the positive condition of the implicit dynamic equations under general structured perturbations implies the equality of the complex and real stability radii. An answer to this problem would be of great interest.

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