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Drazin inverse conditions for stability of positive singular systems

Xiuyong Ding[a,b,*], Guisheng Zhai[c]

[a] School of Mathematics, Southwest Jiaotong University, Chengdu, Sichuan 610031, PR China

[b] National Engineering Laboratory of Integrated Transportation Big Data Application Technology, Southwest Jiaotong University, Chengdu, Sichuan 610031, PR China

[c] Department of Mathematical Sciences, Shibaura Institute of Technology, Saitama 337-8570, Japan

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Abstract

In this paper, positive singular systems in both continuous and discrete cases are addressed, and a complete characterization for stability is provided. First, it is shown that positive singular systems can be stable for a non-negative initial condition. The presented stability criteria are necessary and sufficient, and can be checked by means of linear matrix inequality (LMI) or linear programming (LP). Further, we generalize the Lyapunov stability theory for positive singular systems.

1. Introduction

In many cases, the dynamical behavior of physical processes is modeled via differential equations. However if the states of the physical system are in some ways constrained, for example by conservation laws such as Kirchhoff's laws in electrical networks, or by position constraints such as the movement of mass points on a surface, then the mathematical model also contains algebraic equations to describe these constraints. Such systems, consisting of both differential and algebraic equations, are called singular systems (also known as differential-algebraic systems, implicit equations, or descriptor systems). The applications for

* Corresponding author.

E-mail address: dingxingzhi@hotmail.com (X. Ding).

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singular systems are well known in economics, demography, interconnected systems, electrical networks, robotics, aircraft dynamics, and so on, see [1–4]. Over the past few years, many fundamental issues have been well-investigated for singular systems, see for instance [2,3,5–9] and some references therein.

On the other hand, in dynamical systems an additional frequent and inherent constraint is the nonnegativity of the states. Many physical systems in the real world involve variables that have nonnegative sign: population levels, absolute temperature, density of gas, concentration of substances, level of liquid in tanks, and so on [10–14]. Singular systems, if the non-negativeness of initial condition implies that the state and output are non-negative, are called positive singular systems [7,15]. There is an increased interest in positive singular systems during recent years. Previous work on positive singular systems has been made on the positivity, existence of solutions, regularization, and some structural properties, see e.g., [7,8,15–19]. and some references therein.

Stability is a basic requirement in control problems. The stability of standard singular systems (non-positive) has been well-studied, see e.g., [2–4,6,20,21]. On the other hand, for positive singular systems the stability issue was considered in [7,8,19,22–24]. Specially, in [7] a Lyapunov-type stability condition was derived under a special assumption that the projector matrix on the set of admissible initial conditions is non-negative. In [8], the stability is treated for a specific conic set, that is, the positive singular systems can be stable under an assumption of positive admissible initial conditions. This result is extended to discrete positive singular systems in [19]. It should be emphasized, when the projector matrix is non-positive and the admissible initial conditions are not strict positive, that the existing results on stability do not work, thus leading to deeper insights into the stability problem of positive singular systems must be endowed with.

In this paper we treat the stability of positive singular systems for a non-negative initial condition. We provide some checkable necessary and sufficient conditions for stability, and these conditions are formulated as some LP problems and LMI problems. In addition, we generalize the Lyapunov stability theory for positive singular systems from the standard case (non-positive) to the singular case. The layout of the paper is as follows: Section 2 states some preliminaries. In Section 3, we concentrate our attention on continuous positive singular systems. We show that the positivity and stability of singular systems is equivalent to a Metzler Hurwitz condition on the system matrices. Based on such a condition, we introduce three different Lyapunov functions for positive singular systems and show that such Lyapunov functions are equivalent to each other. We then extend these results to discrete positive singular systems in Section 4. Section 5 presents some brief concluding remarks.

2. Notation and Preliminaries

Throughout this paper, $\mathfrak{R}^n$ ($\mathfrak{R}^{n\times n}$) stands for the n-dimensional ($n \times n$) real vector space. For a matrix M, $M \succeq 0(\succ 0, \preceq 0, \prec 0)$ means that all the elements of M are non-negative (positive, non-positive, negative). $M > 0(\geq 0, < 0, \leq 0)$ means that M is symmetric positive definite (positive semidefinite, negative definite, negative semidefinite). M^T and $rank(M)$ represent its transpose and rank, respectively. We denote the image and kernel of M by $\mathrm{im}(M)$ and $\ker(M)$, respectively. $[M]_i$ and $[M]_{ij}$ denote the ith row and the (i, j)th entry of M, respectively. For a vector $\mathbf{v}$, $[\mathbf{v}]_i$ denotes its ith entry. We use $\mathrm{diag}(\mathbf{v})$ to denote the diagonal matrix of the vector $\mathbf{v}$. The vector $\mathbf{e}_i$ is the ith element of the canonical basis of $\mathfrak{R}^n$. $\vec{\mathbf{1}}$ is the vector $(1, \ldots, 1)^T$. We denote the **vec** operation by $\mathrm{vec}(Y) = [(Y\mathbf{e}_1)^T, \ldots, (Y\mathbf{e}_n)^T]^T$.

For any appropriate matrices A, B, X, $\otimes$ denotes the Kronecker product which is defined as $\text{vec}(AXB) = [B^T \otimes A]\text{vec}(X)$. When referring to stability of a linear system, we always mean global asymptotic stability.

The Drazin inverse [25] of a matrix M is denoted by M^D, which is uniquely defined by the properties $M^D M = MM^D$, $M^D MM^D = M^D$, and $M^D M^{\nu+1} = M^\nu$ where ν is the smallest nonnegative integer such that $rank(M^\nu) = rank(M^{\nu+1})$. M's Drazin inverse can be computed by $M^D = T\begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix}T^{-1}$(see, e.g., [4,26]), where matrices T, J are given by the Jordan canonical form $M = T\begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix}T^{-1}$ with invertible J and nilpotent N.

For a matrix pair (E, A), it said to be regular if $\lambda E - A$ is invertible for some complex numbers λ. It said to be impulse-free if $deg(det(\lambda E - A)) = rank(E)$.

For a matrix M, it is said to be Metzler if its off-diagonal entries are non-negative. M is said to be non-negative if $M \succeq 0$. Obviously, M is Metzler if and only if $M + \alpha I \succeq 0$ for some scalars $\alpha > 0$. If all eigenvalues of M are in the open left half plane, it said to be Huwitz. If the spectral radius of M is less than one, it said to be Schur. The following well-known equivalent conditions [10,27] will be used in this paper.

Lemma 1. *Let $M \in \Re^{n\times n}$ be a Metzler matrix. Then the following are equivalent.*

(i) *The matrix M is Hurwitz.*
(ii) *There exists a vector $\mathbf{v} \succ 0$ such that $M^T\mathbf{v} \prec 0$.*
(iii) *There exists a diagonal matrix $D > 0$ with $M^T D + DM < 0$.*

Lemma 2. *Let $M \succeq 0$ be in $\Re^{n\times n}$. Then the following are equivalent.*

(i) *The matrix M is Schur.*
(ii) *There exists a vector $\mathbf{v} \succ 0$ such that $(M - I)^T\mathbf{v} \prec 0$.*
(iii) *There exists a diagonal matrix $D > 0$ with $M^T DM - D < 0$.*

3. Continuous case

This section is concerned with the stability of continuous linear time-invariant (LTI) singular systems in the form

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t) \in \Re^n$ is the state, $E, A \in \Re^{n\times n}$, E may be singular.

Lemma 3 [1]**.** *The singular system (1) admits a unique solution for each admissible initial condition if and only if (E, A) is regular.*

As a consequence of Lemma 3, we in the sequel steadily make the following assumption which has been frequently introduced in many references, see e.g., [6,8].

Assumption 1. The matrix pair (E, A) is regular and impulse-free.

The set of initial conditions for which (1) has at least one solution is called the consistency space [28] (namely, the set of admissible initial conditions) and it is denoted by χ^0. Under Assumption 1, set $\hat{E} = (\hat{\lambda}E - A)^{-1}E, \hat{A} = (\hat{\lambda}E - A)^{-1}A$, where $\hat{\lambda}$ is a complex number such that $(\hat{\lambda}E - A)^{-1}$ exists. Then, the consistency space is given by $\chi^0 = \mathrm{im}(\hat{E}^D\hat{E})$ and the solutions of (1) have the form $\mathbf{x}(t) = e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}\mathbf{w}$ [1], where $\mathbf{w} \in \mathfrak{R}^n$ is an arbitrary vector, and $\hat{E}^D$ is the Drazin inverse of $\hat{E}$. This implies that (1) admits the same solution as the following system [3,8]:

$$\dot{\mathbf{x}}(t) = \mathbb{A}\mathbf{x}(t), \ \ \mathbf{x}(0) \in \mathrm{im}(\mathbb{E}), \tag{2}$$

where $\mathbb{A} = \hat{E}^D\hat{A}$ and $\mathbb{E} = \hat{E}^D\hat{E}$ (note that $\mathbb{E}, \mathbb{A}$ depend on each other, irrespective of the value of $\hat{\lambda}$, see [1,4] for details). Hence in the sequel of this section we only need to study the equivalent system (2). For the matrices $\mathbb{E}, \mathbb{A}$ the following properties will be extensively used later.

Lemma 4 [8,29]**.** *Let $\mathbb{E}, \mathbb{A}$ be defined above. Then, $\mathbb{E}\mathbb{A} = \mathbb{A}\mathbb{E} = \mathbb{A}$, $\mathbb{E}^2 = \mathbb{E}$ (or equivalently, $\mathbb{E}$ is a projector), and $\mathbb{E}\mathbf{x}(t) = \mathbf{x}(t)$ for any solution $\mathbf{x}(t)$ of (2).*

The system (2) is positive if for any admissible initial condition $\mathbf{x}(0) \succeq 0$, the state $\mathbf{x}(t) \succeq 0$ for all $t \geq 0$. The following lemma can be easily concluded from [7,8].

Lemma 5. *The singular system (2) is positive if and only if there exists a matrix M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler. Specially, if $\mathbb{E} \succeq 0$, then the system (2) is positive if and only if there exists a scalar $c > 0$ such that $\mathbb{A} + c(\mathbb{E} - I)$ is Metzler.*

The following stability condition is obtained from [8].

Lemma 6. *Assume that there exists $\mathbf{w}_0 \in \mathfrak{R}^n$ such that $\mathbb{E}\mathbf{w}_0 \succ 0$. Then the system (2) is positive and stable for $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succ 0$ if and only if there exists a matrix M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler and Hurwitz.*

We should emphasize that the above result is based on the assumption $\mathbb{E}\mathbf{w}_0 \succ 0$. However, such an assumption does not work for some common cases. For example, consider the system (1) with $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. We can directly compute $\mathbb{E} = E$, hence there is no vector $\mathbf{w}_0$ such that $\mathbb{E}\mathbf{w}_0 \succ 0$ (In fact, we can only guarantee $\mathbb{E}\mathbf{w}_0 \succeq 0$). However, it is clear that the singular system is stable as its equivalent system is $[\dot{\mathbf{x}}]_1 = -[\mathbf{x}]_1$. To solve this problem, we shall provide an extension to Lemma 6.

Theorem 1. *Assume that there exists a vector $\mathbf{w}_0 \succ 0$ such that $\mathbb{E}\mathbf{w}_0 \succeq 0$. Then the following statements are equivalent:*

(i) *The singular system (2) is positive and stable for all initial conditions such that $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succeq 0$.*
(ii) *There exists a matrix M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler and Hurwitz.*
(iii) *There exist a matrix W and diagonal matrices D_1, D_2 with $D_1 > 0$ such that*

$$D_1\mathbb{A} + W(\mathbb{E} - I) + D_2 \succeq 0, \tag{3}$$

$$\mathbb{A}^T D_1 + D_1\mathbb{A} + (\mathbb{E} - I)^T W^T + W(\mathbb{E} - I) < 0. \tag{4}$$

(iv) *There exist a scalar c, a vector $\mathbf{v} \succ 0$, and a matrix Y such that*

$$\sum_{i=1}^{n} \mathbb{A}^T \mathbf{e}_i \mathbf{e}_i^T \mathbf{v} \mathbf{e}_i^T + (\mathbb{E} - I)^T Y + cI \succeq 0, \tag{5}$$

$$\mathbb{A}^T \mathbf{v} + (\mathbb{E} - I)^T Y \vec{\mathbf{1}} \prec 0. \tag{6}$$

Proof. The proof will be completed by proving (i) $\Leftrightarrow$ (ii), (ii) $\Leftrightarrow$ (iii), and (ii) $\Leftrightarrow$ (iv).

(i) $\Rightarrow$ (ii): If $\mathbb{E}\mathbf{w}_0 \succ 0$, Lemma 6 has shown (ii). Now consider the case $\mathbb{E}\mathbf{w}_0 \succeq 0$ and $\mathbb{E}\mathbf{w}_0 \nsucc 0$. Without loss of generality, assume that there exists $\mathbf{w}_0 \succ 0$ such that $[\mathbb{E}\mathbf{w}_0]_n = 0$, $[\mathbb{E}\mathbf{w}_0]_i > 0$ for $1 \le i \le n-1$ (note that for the other cases that there exist two or more zero entries of $\mathbb{E}\mathbf{w}_0$, the proof is similar), then there are the following two cases:

Case (1): There exist some nonzero entries for $[\mathbb{E}]_n$.

Case (2): The nth row $[\mathbb{E}]_n = 0$.

For Case (1) there must be some positive entries for $[\mathbb{E}]_n$ as $[\mathbb{E}\mathbf{w}_0]_n = 0$ with $\mathbf{w}_0 \succ 0$. Without loss of generality assume $[\mathbb{E}]_{nn} > 0$, and note that the assumption $[\mathbb{E}\mathbf{w}_0]_i > 0$ for $1 \le i \le n-1$. Now we choose a positive scalar ϵ such that

$$0 < \epsilon < \min_{1 \le i \le n-1, [\mathbb{E}]_{in} \ne 0} \frac{\sum_{k=1}^{n} [\mathbb{E}]_{ik} [\mathbf{w}_0]_k}{|[\mathbb{E}]_{in}|},$$

where $|[\mathbb{E}]_{in}|$ denotes its absolute value. By setting $\bar{\mathbf{w}}_0 = \mathbf{w}_0 + [0, \ldots, 0, \epsilon]^T$, it is easy to check that $\mathbb{E}\bar{\mathbf{w}}_0 \succ 0$, which is the case of Lemma 6.

Now in Case (2) $[\mathbb{E}]_n = 0$, we can denote

$$\mathbb{E} = \begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix}, \ \widetilde{\mathbb{E}} \in \mathfrak{R}^{(n-1)\times(n-1)}, \mathbf{q} \in \mathfrak{R}^{n-1}.$$

Note that $\mathbb{E}^2 = \mathbb{E}$ shown by Lemma 4, i.e.,

$$\begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix},$$

this leads to

$$\widetilde{\mathbb{E}}^2 = \widetilde{\mathbb{E}}, \ \widetilde{\mathbb{E}}\mathbf{q} = \mathbf{q}. \tag{7}$$

Also by Lemma 4, that is, $\mathbb{E}\mathbf{x}(t) = \mathbf{x}(t)$ for any solution $\mathbf{x}(t)$ to (2), denoting $\mathbf{x}(t) = (\widetilde{\mathbf{x}}^T, [\mathbf{x}]_n)^T$ gives

$$\begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\mathbf{x}} \\ [\mathbf{x}]_n \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{x}} \\ [\mathbf{x}]_n \end{pmatrix},$$

which implies that

$$\mathbf{x}(t) = \begin{pmatrix} \widetilde{\mathbf{x}} \\ 0 \end{pmatrix}, \ \widetilde{\mathbb{E}}\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}, \tag{8}$$

i.e., $\widetilde{\mathbb{E}}$ is a projector for $\widetilde{\mathbf{x}}$.

Again, noting that $\mathbb{E}\mathbb{A} = \mathbb{A}\mathbb{E} = \mathbb{A}$ and denoting $\mathbb{A} = \begin{pmatrix} \widetilde{\mathbb{A}} & \mathbf{g} \\ \mathbf{h} & c \end{pmatrix}$ we get

$$\begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \widetilde{\mathbb{A}} & \mathbf{g} \\ \mathbf{h} & c \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbb{A}} & \mathbf{g} \\ \mathbf{h} & c \end{pmatrix}\begin{pmatrix} \widetilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbb{A}} & \mathbf{g} \\ \mathbf{h} & c \end{pmatrix}.$$

It follows that

$$\mathbb{A} = \begin{pmatrix} \widetilde{\mathbb{A}} & \mathbf{g} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{\mathbb{E}}\widetilde{\mathbb{A}} = \widetilde{\mathbb{A}}\widetilde{\mathbb{E}} = \widetilde{\mathbb{A}}. \tag{9}$$

Now consider the system

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbb{A}}\tilde{\mathbf{x}}(t). \tag{10}$$

Under the assumption $[\mathbb{E}\mathbf{w}_0]_n = 0,\ [\mathbb{E}\mathbf{w}_0]_i > 0\ (1 \le i \le n-1)$, one can easily check from (7) and (8) that there exists $\tilde{\mathbf{w}}_0 \succ 0$ in $\mathfrak{R}^{n-1}$ such that $\tilde{\mathbf{x}}(0) = \tilde{\mathbb{E}}\tilde{\mathbf{w}}_0 \succ 0$ as $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succeq 0$. Further from (8) the system (10) is positive and stable. Hence by using Lemma 6 there exists a matrix $\tilde{M}$ such that

$$\tilde{\mathbb{A}} + \tilde{M}(\tilde{\mathbb{E}} - I_{n-1}) \ \text{ is Metzler and Hurwitz.} \tag{11}$$

Now setting $M = \begin{pmatrix} \tilde{M} & \tilde{M}\mathbf{q} + \mathbf{g} \\ 0 & c \end{pmatrix}$ with $c > 0$, we obtain

$$\begin{aligned}
&\mathbb{A} + M(\mathbb{E} - I) \\
&= \begin{pmatrix} \tilde{\mathbb{A}} & \mathbf{g} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \tilde{M} & \tilde{M}\mathbf{q} + \mathbf{g} \\ 0 & c \end{pmatrix}\left(\begin{pmatrix} \tilde{\mathbb{E}} & \mathbf{q} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} \tilde{\mathbb{A}} + \tilde{M}(\tilde{\mathbb{E}} - I) & 0 \\ 0 & -c \end{pmatrix},
\end{aligned}$$

therefore from (11) $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler and Hurwitz.

(ii) $\Rightarrow$ (i): Firstly, if $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler, then by Lemma 5 the system (2) is positive. In addition, if $\mathbb{A} + M(\mathbb{E} - I)$ is Hurwitz, note that $\mathbb{E}\mathbf{x}(t) = \mathbf{x}(t)$ shown by Lemma 4, the stability of (2) follows directly as $\dot{\mathbf{x}}(t) = \mathbb{A}\mathbf{x}(t) = (\mathbb{A} + M(\mathbb{E} - I))\mathbf{x}(t)$.

(ii) $\Leftrightarrow$ (iii): Firstly, pre-multiplying (3) by D_1^{-1} gives that

$$\mathbb{A} + D_1^{-1}W(\mathbb{E} - I) + D_1^{-1}D_2 \succeq 0. \tag{12}$$

Further, note that $D_1^{-1}D_2$ is diagonal, (12) holds if and only if there exists a diagonal matrix $D_1 > 0$ and W such that

$$\mathbb{A} + D_1^{-1}W(\mathbb{E} - I) \text{ is Metzler.} \tag{13}$$

By denoting $M = D_1^{-1}W$ (or, $W = D_1 M$), one can see that (13) is equivalent to the existence of M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler.

On the other hand, as $W = D_1 M$, (4) can be rewritten as

$$\begin{aligned}
&\mathbb{A}^T D_1 + D_1\mathbb{A} + (\mathbb{E} - I)^T M^T D_1 + D_1 M(\mathbb{E} - I) \\
&= [\mathbb{A} + M(\mathbb{E} - I)]^T D_1 + D_1[\mathbb{A} + M(\mathbb{E} - I)] < 0.
\end{aligned} \tag{14}$$

Then (14) holds if and only if, by Lemma 1, $\mathbb{A} + M(\mathbb{E} - I)$ is Hurwitz.

(ii) $\Leftrightarrow$ (iv): Using the fact $\mathrm{diag}(\mathbf{v}) = \sum_{i=1}^n \mathbf{e}_i\mathbf{e}_i^T\mathbf{v}\mathbf{e}_i^T$ and post-multiplying (5) by $\mathrm{diag}(\mathbf{v})^{-1}$, we can express (5) and (6) as

$$\begin{aligned}
&\mathbb{A}^T + (\mathbb{E} - I)^T Y \mathrm{diag}(\mathbf{v})^{-1} + c\,\mathrm{diag}(\mathbf{v})^{-1} \succeq 0, \\
&\mathbb{A}^T\mathbf{v} + (\mathbb{E} - I)^T Y\vec{\mathbf{1}} \prec 0,
\end{aligned}$$

which is equivalent, as $c\,\mathrm{diag}(\mathbf{v})^{-1}$ is diagonal, to that

$$\begin{aligned}
&\mathbb{A}^T + (\mathbb{E} - I)^T Y \mathrm{diag}(\mathbf{v})^{-1} \text{ is Metzler}, \\
&\mathbb{A}^T\mathbf{v} + (\mathbb{E} - I)^T Y\vec{\mathbf{1}} \prec 0.
\end{aligned} \tag{15}$$

By denoting $M^T = Y\mathrm{diag}(\mathbf{v})^{-1}$ (or equivalently, $Y = M^T\mathrm{diag}(\mathbf{v})$), one can see that (15) is true if and only if there exists M such that

$$\begin{aligned} &\mathbb{A}^T + (\mathbb{E}-I)^T M^T \text{ is } \quad Metzler, \\ &[\mathbb{A} + M(\mathbb{E}-I)]^T \mathbf{v} \quad \prec 0. \end{aligned}$$

That is, by Lemma 1, $\mathbb{A} + M(\mathbb{E}-I)$ is Metzler and Hurwitz. □

Remark 1. We should emphasize that Theorem 1 is applicable for the most special case $\mathbb{E}\mathbf{w}_0 = 0$ (i.e., the initial condition is zero).

Remark 2. In Theorem 1, if $[\mathbb{E}\mathbf{w}_0]_i = 0$ always implies $[\mathbb{E}]_i = 0$, then the restriction $\mathbf{w}_0 \succ 0$, according to the proof process, is not required. In other words, in this case Theorem 1 works for any nonnegative admissible condition $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0$ with $\mathbf{w}_0 \in \Re^n$.

Remark 3. The inequalities (3) and (4) can be transformed into the following LMI problem: $\forall\ i, j = 1, \cdots n$,

$$\begin{aligned} \mathbf{e}_i^T D_1 \mathbf{e}_i \mathbf{e}_i^T \mathbb{A}\mathbf{e}_j + \mathbf{e}_i^T W(\mathbb{E}-I)\mathbf{e}_j + \mathbf{e}_i^T D_2 \mathbf{e}_j &\geq 0, \\ D_1 &> 0, \\ \mathbb{A}^T D_1 + D_1\mathbb{A} + (\mathbb{E}-I)^T W^T + W(\mathbb{E}-I) &< 0. \end{aligned} \tag{16}$$

Also, it is easy to check that the inequalities (5) and (6) can be transformed into the following LP problem:

$$\begin{aligned} &\begin{bmatrix} \Phi & I \otimes (\mathbb{E}-I)^T & \mathrm{vec}(I) \end{bmatrix} \mathbf{y} \succeq 0, \\ &\begin{bmatrix} -I & 0 & 0 \\ \mathbb{A}^T & \vec{\mathbf{1}}^T \otimes (\mathbb{E}-I)^T & 0 \end{bmatrix} \mathbf{y} \prec 0, \end{aligned} \tag{17}$$

where the new variables are $\mathbf{y} = \begin{bmatrix} \mathbf{v}^T & \mathrm{vec}(Y)^T & c \end{bmatrix}^T$, $\Phi = \sum_{i=1}^n \mathbf{e}_i \otimes \mathbb{A}^T \mathbf{e}_i \mathbf{e}_i^T$.

For Theorem 1, if we make a stronger assumption $\mathbb{E} \succeq 0$, the matrix M in (ii) can be strengthened as a positive scalar.

Corollary 1. *Assume that* $\mathbb{E} \succeq 0$. *Then the following statements are equivalent.*

(i) *The singular system (2) is positive and stable.*
(ii) *There exists a scalar* $c > 0$ *such that* $\mathbb{A} + c(\mathbb{E}-I)$ *is Metzler and Hurwitz.*

Proof. The sufficiency has been shown by Lemma 4.7 in [7]. Now if $\mathbb{A} + c(\mathbb{E}-I)$ with $c > 0$ is Metzler and Hurwitz, then denoting $M = cI$ gives that $\mathbb{A} + M(\mathbb{E}-I)$ is Metzler and Hurwitz. By Theorem 1 this implies that the system (2) is positive and stable. □

Next we will alternatively provide a generalized Lyapunov stability theory for the positive singular system (2). We need to define generalized Lyapunov functions for (2).

Definition 1. The function $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T R\mathbb{E}\mathbf{x}(t)$ with $R > 0$ is called a quadratic Lyapunov function (QLF) of (2), if along the solution of (2) its derivative $\dot{V}(t)$ is negative definite on the consistency space $\mathrm{im}(\mathbb{E})$ (i.e., $\dot{V}(t) \leq 0$, and $\dot{V}(t) = 0$ only for $\mathbf{x}(t) = 0$). Specially, if R is a diagonal matrix, $V(\mathbf{x}(t))$ is called a diagonal QLF.

Let (2) be positive, $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ with a vector $\mathbf{v} \succ 0$ is called a linear co-positive Lyapunov function (LCLF) if along the solution of (2) $\dot{V}(t)$ is negative definite on $\mathrm{im}(\mathbb{E})$.

Remark 4. Obviously, the system's solution $\mathbf{x}(t) \in \mathrm{im}(\mathbb{E})$ ensures that $V(\mathbf{x}(t))$ is positive definite and $\dot{V}(\mathbf{x}(t))$ is negative definite on the consistency space $\mathrm{im}(\mathbb{E})$, respectively.

For a QLF, one can see that $\dot{V}(\mathbf{x}(t))$ is negative definite on $\mathrm{im}(\mathbb{E})$ if and only if there exist a matrix $R > 0$ and a matrix Q which is positive definite on $\mathrm{im}(\mathbb{E})$ such that $(\mathbb{E}\mathbb{A})^T R + R(\mathbb{E}\mathbb{A}) = \mathbb{A}^T R + R\mathbb{A} = -Q$.

When (1) reduces to the standard system, i.e., E is nonsingular, in this case, $\mathbb{E} = I$, and then $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T R\mathbb{E}\mathbf{x}(t) = \mathbf{x}^T(t)R\mathbf{x}(t)$ is the well known (diagonal) QLF of a standard LTI system [30]. Also, $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v} = \mathbf{x}^T(t)\mathbf{v}$ is the well known LCLF of a standard positive LTI system, see e.g., [31].

Theorem 2. *Assume that there exists $\mathbf{w}_0 \succ 0$ such that $\mathbb{E}\mathbf{w}_0 \succeq 0$. If (2) is positive, then the following statements are equivalent:*

(i) *The system (2) is stable for all initial conditions such that $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succeq 0$.*
(ii) *There exists a vector $\mathbf{v} \succ 0$ such that $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ is an LCLF for (2).*
(iii) *There exists a diagonal matrix $D > 0$ such that $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T D\mathbb{E}\mathbf{x}(t)$ is a diagonal QLF for (2).*
(iv) *There exists a matrix $R > 0$ such that $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T R\mathbb{E}\mathbf{x}(t)$ is a QLF for (2).*

Proof. The proof will be completed by proving (i) $\Leftrightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): If (2) is positive and stable, then by Theorem 1 there exists a matrix M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler and Hurwitz. This implies by Lemma 1 that there exists a vector $\mathbf{v} \succ 0$ such that $[\mathbb{A} + M(\mathbb{E} - I)]^T\mathbf{v} \prec 0$. Now define $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ which is obvious definite positive on $\mathrm{im}(\mathbb{E})$. Along (2) we can derive that

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbb{A}^T\mathbf{v} = \mathbf{x}^T(t)[\mathbb{A} + M(\mathbb{E} - I)]^T\mathbf{v} \le 0,$$

and $\dot{V}(\mathbf{x}(t)) = 0$ only for $\mathbf{x}(t) = 0$ as $\mathbf{x}(t) \in \mathrm{im}(\mathbb{E})$. This is to say, $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ is an LCLF.

(ii) $\Rightarrow$ (i): Let $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ with $\mathbf{v} \succ 0$ be an LCLF of (2), then along the system solution we have $\dot{V}(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbb{A}^T\mathbf{v}$. As $V(\mathbf{x}(t))$ and $\dot{V}(\mathbf{x}(t))$ is positive definite and negative definite on $\mathrm{im}(\mathbb{E})$, respectively, we can set $\delta = \min_{\mathbf{x}\in\mathrm{im}(\mathbb{E})/\{0\}} \frac{-\mathbf{x}^T\mathbb{E}^T\mathbb{A}^T\mathbf{v}}{V(\mathbf{x})} > 0$ and obtain that $\dot{V}(\mathbf{x}(t)) \le -\delta V(\mathbf{x}(t))$. It therefore follows that $V(\mathbf{x}(t)) \le V(\mathbf{x}(0))\exp(-\delta t) \to 0$ as $t \to \infty$, which implies $\mathbb{E}\mathbf{x}(t) = \mathbf{x}(t) \to 0$ as $t \to \infty$.

(i) $\Rightarrow$ (iii): As the system (2) is positive and stable, then by Theorem 1 there exists a matrix M such that $\mathbb{A} + M(\mathbb{E} - I)$ is Metzler and Hurwitz. That is, by Lemma 1, there exists a diagonal matrix $D > 0$ such that $[\mathbb{A} + M(\mathbb{A} - I)]^T D + D[\mathbb{A} + M(\mathbb{E} - I)] < 0$. This implies that, for $\mathbf{x}(t) \in \mathrm{im}(\mathbb{E})/\{0\}$,

$$\begin{aligned} &\mathbf{x}^T(t)[\mathbb{A}^T D + D\mathbb{A}]\mathbf{x}(t) \\ &= \mathbf{x}^T(t)[\mathbb{A} + M(\mathbb{E} - I)]^T D + D[\mathbb{A} + M(\mathbb{E} - I)]\mathbf{x}(t) < 0 \end{aligned} \tag{18}$$

and for $\mathbf{x}(t) \in \ker(\mathbb{E})$, as $\mathbb{A}\mathbf{x}(t) = \mathbb{A}\mathbb{E}\mathbf{x}(t) = 0$, then

$$\mathbf{x}^T(t)[\mathbb{A}^T D + D\mathbb{A}]\mathbf{x}(t) = 0. \tag{19}$$

Now define the function $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T D\mathbb{E}\mathbf{x}(t)$, from (18) and (19) it shows that $V(\mathbf{x}(t))$ is a diagonal QLF for (2).

(iii) $\Rightarrow$ (iv): It is straightforward by setting $R = D$.

Table 1
Feasibility analysis for different methods (Example 1).

(a,b,c)	$\mathbb{E}$	Thm 4.8 [7]	Thm 4.5 [8]	Thm 1
(0,0,0)	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Feasible	Infeasible	Feasible
(-3,1.9854,0.5162)	$\begin{pmatrix} 1 & 0 & 0 \\ 0.2450 & 0.3356 & 0.3356 \\ -0.2450 & 0.6644 & 0.6644 \end{pmatrix}$	Infeasible	Feasible	Feasible
(0,-0.6868,0.3783)	$\begin{pmatrix} 1 & 0 & 0.2872 \\ 0 & 1 & -0.4928 \\ 0 & 0 & 0 \end{pmatrix}$	Infeasible	Infeasible	Feasible

(iv) $\Rightarrow$ (i): Let $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T R\mathbb{E}\mathbf{x}(t)$ with $R > 0$ be a QLF of (2), then along the system solution we have $\dot{V}(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T(\mathbb{A}^T R - R\mathbb{A})\mathbb{E}\mathbf{x}(t) = -\mathbf{x}^T(t)\mathbb{E}^T Q\mathbb{E}\mathbf{x}(t)$, which is negative definite on $\mathrm{im}(\mathbb{E})$. we can set $\delta = \min_{\mathbf{x}\in \mathrm{im}(\mathbb{E})/\{0\}} \frac{\mathbf{x}^T(t)\mathbb{E}^T Q\mathbb{E}\mathbf{x}(t)}{V(\mathbf{x})} > 0$ and obtain that $\dot{V}(\mathbf{x}(t)) \le -\delta V(\mathbf{x}(t))$. It therefore follows that $V(\mathbf{x}(t)) \le V(\mathbf{x}(0))\exp(-\delta t) \to 0$ as $t \to \infty$, which implies $\mathbb{E}\mathbf{x}(t) = \mathbf{x}(t) \to 0$ as $t \to \infty$. $\square$

Remark 5. Note that the existence of a diagonal QLF and an LCLF for (2) is equivalent to the solvability of $[\mathbb{A} + M(\mathbb{E} - I)]^T D + D[\mathbb{A} + M(\mathbb{E} - I)] < 0$ and $[\mathbb{A} + M(\mathbb{E} - I)]^T \mathbf{v} \prec 0$, respectively, that is to say, $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T D\mathbb{E}\mathbf{x}(t)$ and $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v}$ can be located by solving the LMI (16) in D_1 and LP (17) in $\mathbf{v}$, respectively.

Example 1. As an illustrative example, consider the system (1) given by

$$E = \begin{pmatrix} 2.2031 & 0.5162 & c \\ 1.0152 & 1.9854 & b \\ 0 & a & a \end{pmatrix}, A = \begin{pmatrix} -4.1236 & 0.6387 & 0.3089 \\ 0.5567 & -6.3671 & 0.5003 \\ 0 & 0 & 5.1629 \end{pmatrix}.$$

In Table 1, by considering different E, we do a feasibility study and make a comparison between Theorem 1 and the existing results (Theorem 4.8 in [7] and Theorem 4.5 in [8]). It is easy to see that our work provides a less conservative criterion for stability of (1). For example, setting $a = 0, b = -0.6868, c = 0.3783$, we calculate the corresponding

$$\mathbb{E} = \begin{pmatrix} 1 & 0 & 0.2872 \\ 0 & 1 & -0.4928 \\ 0 & 0 & 0 \end{pmatrix}, \mathbb{A} = \begin{pmatrix} -2.2011 & 1.1831 & -1.2151 \\ 1.4059 & -3.8119 & 2.2821 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, it is clear that $\mathbb{E} \succeq 0$ is not true, in addition, we can not find a vector $\mathbf{w}_0$ such that $\mathbb{E}\mathbf{w}_0 \succ 0$. Hence the existing results (see e.g., [7,8]) are not available. However one can easily pick up some vectors such as $\mathbf{w}_0 = (1, 1, 1)^T$ such that $\mathbb{E}\mathbf{w}_0 \succeq 0$. This implies Theorem 1 can work. To see this, for the case $a = 0, b = -0.6868, c = 0.3783$ we can solve the LMI (16) and obtain a feasible solution

$$D_1 = \begin{pmatrix} 0.0986 & 0 & 0 \\ 0 & 0.0827 & 0 \\ 0 & 0 & 0.8791 \end{pmatrix}, W = \begin{pmatrix} 86.3117 & 27.0136 & 11.3550 \\ -24.5368 & 42.5026 & -27.8012 \\ -1.7818 & 0.0074 & 0.0149 \end{pmatrix},$$

which gives from Remark 5, a diagonal QLF as $V(\mathbf{x}) = \mathbf{x}^T\mathbb{E}^T D_1\mathbb{E}\mathbf{x} = 0.0986[\mathbf{x}]_1^2 + 0.0827[\mathbf{x}]_2^2 + 0.0282[\mathbf{x}]_3^2 + 0.0566[\mathbf{x}]_1[\mathbf{x}]_3 - 0.0816[\mathbf{x}]_2[\mathbf{x}]_3$. Moreover, noting that $M =$

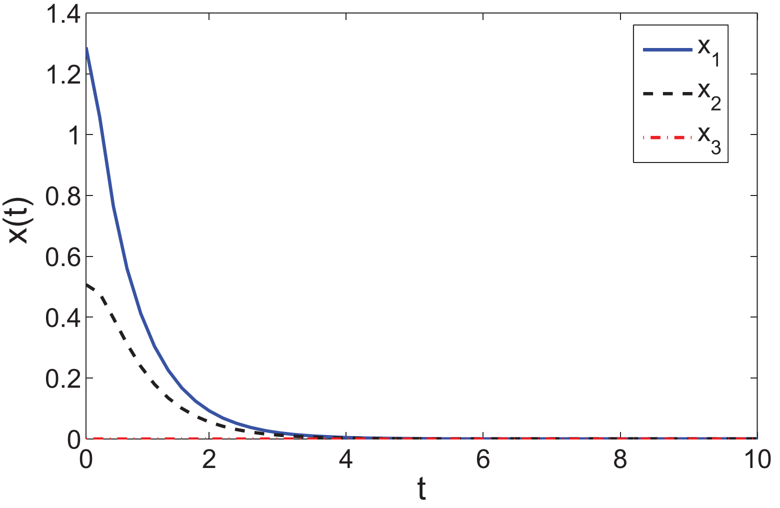

Fig. 1. State response of the system, where the initial condition $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 = (1.2872; 0.5072; 0)^T$ with $\mathbf{w}_0 = (1, 1, 1)^T$.

$D_1^{-1}W$ we have

$$\mathbb{A} + M(\mathbb{E} - I) = \begin{pmatrix} -2.2011 & 1.1831 & 0.0000 \\ 1.4059 & -3.8119 & 0.0000 \\ 0.0000 & 0.0000 & -0.6032 \end{pmatrix},$$

which is Metzler and its spectrum is $(-1.4860, -4.5270, -0.6032)$. Hence from the statement (iii) in Theorem 1 the system (1) is positive and stable. The state response is shown in Figure 1.

Alternatively, we can conclude the positivity and stability of (1) by solving the LP (17). A feasible solution is

$$\mathbf{v} = \begin{pmatrix} 8.5460 \\ 8.8559 \\ 6.0091 \end{pmatrix}, Y = \begin{pmatrix} -580.5077 & -601.7011 & -660.2949 \\ -220.0559 & -192.1517 & -128.7204 \\ -93.2345 & -75.3774 & -29.7718 \end{pmatrix},$$

which defines an LCLF as $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v} = 8.5460[\mathbf{x}]_1 + 8.8559[\mathbf{x}]_2 - 1.9097[\mathbf{x}]_3$. Moreover, note that $M = (Y\mathrm{diag}^{-1}(\mathbf{v}))^T$ we obtain

$$\mathbb{A} + M(\mathbb{E} - I) = \begin{pmatrix} -2.2011 & 1.1831 & 2.8764 \\ 1.4059 & -3.8119 & 1.9741 \\ 0.0000 & 0.0000 & -16.0452 \end{pmatrix},$$

which is Metzler and with a stable spectrum $(-1.4860, -4.5270, -16.0452)$, i.e., the system (1) is positive and stable.

4. Discrete case

In this section, we restrict our attention on discrete LTI singular systems in the form

$$E\mathbf{x}(k+1) = A\mathbf{x}(k). \tag{20}$$

Lemma 7 [1]**.** *The singular system (20) admits a unique solution for each admissible initial condition if and only if* (E, A) *is regular.*

Consequently, we introduce the following assumption for (20), see e.g., [6,19].

Assumption 2. The matrix pair (E, A) is regular and impulse-free.

As in the continuous case, set $\hat{E} = (\hat{\lambda}E - A)^{-1}E, \hat{A} = (\hat{\lambda}E - A)^{-1}A$, then the consistency space is given by $\chi^0 = \mathrm{im}(\hat{E}^D\hat{E})$ and the solutions of (20) have the form $\mathbf{x}(k) = (\hat{E}^D\hat{A})^k\hat{E}^D\hat{E}\mathbf{w}$ with any $\mathbf{w} \in \mathfrak{R}^n$ [1]. Moreover, denote $\mathbb{A} = \hat{E}^D\hat{A}$ and $\mathbb{E} = \hat{E}^D\hat{E}$, the equivalent system (in the sense of a same solution) of (20) is given by [1,19]:

$$\mathbf{x}(k+1) = \mathbb{A}\mathbf{x}(k), \ \ \mathbf{x}(0) \in \mathrm{im}(\mathbb{E}). \tag{21}$$

Lemma 8 [19]**.** *For* $\mathbb{E}$, $\mathbb{A}$, *we have the following properties* $\mathbb{E}\mathbb{A} = \mathbb{A}\mathbb{E} = \mathbb{A}$, $\mathbb{E}^2 = \mathbb{E}$, $\mathbb{E}\mathbf{x}(k) = \mathbf{x}(k)$ *for any solution* $\mathbf{x}(k)$ *of (21).*

The system (21) is positive if for any admissible initial condition $\mathbf{x}(0) \succeq 0$, the state $\mathbf{x}(k) \succeq 0$ for all $k \geq 0$.

Lemma 9 [7,19]**.** *The singular system (21) is positive if and only if there exists a matrix* M *such that* $\mathbb{A} + M(\mathbb{E} - I) \succeq 0$. *Specially, if* $\mathbb{E} \succeq 0$, *then the system (21) is positive if and only if* $\mathbb{A} \succeq 0$.

The following stability result is presented by [19].

Lemma 10. *Assume that there exists* $\mathbf{w}_0 \in \mathfrak{R}^n$ *such that* $\mathbb{E}\mathbf{w}_0 \succ 0$. *Then the system (21) is positive and stable for* $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succ 0$ *if and only if there exists a matrix* M *such that* $\mathbb{A} + M(\mathbb{E} - I)$ *is nonnegative and Schur.*

Similarly, the above result is just applicable for the case of strict positive initial conditions, i.e., $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succ 0$. Next we will provide an extended result to deal with a more general case.

Theorem 3. *Assume that there exists a vector* $\mathbf{w}_0 \succ 0$ *such that* $\mathbb{E}\mathbf{w}_0 \succeq 0$. *Then the following statements are equivalent:*

(i) *The singular system (21) is positive and stable for all initial conditions such that* $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 \succeq 0$.
(ii) *There exists a matrix* M *such that* $\mathbb{A} + M(\mathbb{E} - I)$ *is nonnegative and Schur.*
(iii) *The matrix* $\mathbb{A}$ *is Schur and there exists a matrix* M *such that* $\mathbb{A} + M(\mathbb{E} - I)$ *is nonnegative.*
(iv) *There exist a vector* $\mathbf{v} \succ 0$, *and a matrix* Y *such that*

$$\sum_{i=1}^{n} \mathbb{A}^T \mathbf{e}_i \mathbf{e}_i^T \mathbf{v} \mathbf{e}_i^T + (\mathbb{E} - I)^T Y \succeq 0, \tag{22}$$

$$(\mathbb{A} - I)^T \mathbf{v} + (\mathbb{E} - I)^T Y \vec{\mathbf{1}} \prec 0. \tag{23}$$

(v) *There exist a vector* $\mathbf{v} \succ 0$, *and a matrix* M *such that*

$$\mathbb{A} + M(\mathbb{E} - I) \succeq 0, \tag{24}$$

$$(\mathbb{A}-I)^T\mathbf{v} \prec 0. \tag{25}$$

(vi) *There exist a matrix M and a diagonal matrix $D>0$ such that*

$$\mathbb{A}+M(\mathbb{E}-I) \succeq 0, \tag{26}$$
$$\mathbb{A}^T D\mathbb{A}-D < 0. \tag{27}$$

Proof. (i) $\Leftrightarrow$ (ii): Based on Lemma 10, the proof is essentially same as that of (i) $\Leftrightarrow$ (ii) in Theorem 1.

(i) $\Leftrightarrow$ (iii): The positivity follows from Lemma 9. We just need to consider the stability. Firstly, it is obvious that the system (21) is stable for all initial conditions such that $\mathbf{x}(0)=\mathbb{E}\mathbf{w}_0 \succeq 0$ if $\mathbb{A}$ is Schur. Now it suffices to prove the stability of (21) implies the Schur property of $\mathbb{A}$. The proof will be achieved by using a well-known result on the matrices $\mathbb{E}, \mathbb{A}$, see e.g., [4,7]. That is, for $\mathbb{E}, \mathbb{A}$ there exist invertible matrices T, J such that $(\mathbb{E}, \mathbb{A}) = T^{-1}\left(\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & 0\end{bmatrix}\right)T$. In this case, one can see that the matrix J must be Schur if the system (21) is stable, this gives $\mathbb{A} = T^{-1}\begin{pmatrix} J & 0 \\ 0 & 0\end{pmatrix}T$ is Schur.

(ii) $\Leftrightarrow$ (iv): The proof is very similar to that of (ii) $\Leftrightarrow$ (iv) in Theorem 1.

(iii) $\Leftrightarrow$ (v) and (iii) $\Leftrightarrow$ (vi): The proofs are straightforward from Lemma 2, $\square$

Remark 6. The equivalence between (i) and (iii) shows that the stability of (20) is equivalent to that $\mathbb{A}$ is stable (i.e., Schur). This result is very different from the stability result presented in Theorem 1 on the continuous system (1), for which the associated matrix $\mathbb{A}$ is not necessarily stable (i.e., Hurwitz).

Remark 7. The inequalities (22) and (23) can be transformed into the following LP problem:

$$\begin{aligned}&\begin{bmatrix}\Phi & I\otimes(\mathbb{E}-I)^T\end{bmatrix}\mathbf{y} \succeq 0,\\ &\begin{bmatrix} -I & 0 \\ (\mathbb{A}-I)^T & \vec{\mathbf{1}}^T\otimes(\mathbb{E}-I)^T\end{bmatrix}\mathbf{y} \prec 0,\end{aligned} \tag{28}$$

where $\mathbf{y} = \begin{bmatrix}\mathbf{v}^T & \mathrm{vec}(Y)^T\end{bmatrix}^T$ and $\Phi = \sum_{i=1}^n \mathbf{e}_i \otimes \mathbb{A}^T\mathbf{e}_i\mathbf{e}_i^T$.

Similarly, the inequalities (24) and (25) can be transformed into the LP problem:

$$\begin{aligned}[(\mathbb{E}-I)^T\otimes I]\ \mathrm{vec}(M) &\succeq -\mathbb{A},\\ \mathbf{v} &\succ 0,\\ (\mathbb{A}-I)^T\mathbf{v} &\prec 0.\end{aligned} \tag{29}$$

The inequalities (26) and (27) can be transformed into the following LMI problem:

$$\begin{aligned}\mathbf{e}_i^T\mathbb{A}\mathbf{e}_j+\mathbf{e}_i^T M(\mathbb{E}-I)\mathbf{e}_j &\geq 0,\\ D &> 0,\\ \mathbb{A}^T D\mathbb{A}-D &< 0.\end{aligned} \tag{30}$$

For the special case of $\mathbb{E}\succeq 0$, we can derive the following result.

Corollary 2. *Let $\mathbb{E}\succeq 0$. Then the following statements are equivalent:*

(i) *The singular system (21) is positive and stable.*
(ii) *The matrix $\mathbb{A}$ is nonnegative and Schur.*
(iii) *The matrix $\mathbb{A}$ is nonnegative and $\mathbb{A}+c(\mathbb{E}-I)$ is Schur for any $|c|<1$.*

Proof. (i) $\Leftrightarrow$ (ii): The positivity follows from Lemma 9. The stability follows from the equivalence between (i) and (iii) of Theorem 3.

(ii) $\Leftrightarrow$ (iii): The necessity is trivial. Now consider the sufficiency. Since there exist invertible matrices T, J such that $(\mathbb{E},\mathbb{A}) = T^{-1}\left(\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & 0\end{bmatrix}\right)T$, see [4,7]. If $\mathbb{A}$ is Schur, i.e., $\mathbb{A} = T^{-1}\begin{pmatrix} J & 0 \\ 0 & 0\end{pmatrix}T$ is Schur, then for any $|c|<1$ the matrix $T^{-1}\begin{pmatrix} J & 0 \\ 0 & -cI\end{pmatrix}T$ is also Schur, or equivalently, $T^{-1}\begin{pmatrix} J & 0 \\ 0 & 0\end{pmatrix}T + cT^{-1}\begin{pmatrix} 0 & 0 \\ 0 & -I\end{pmatrix}T$ is Schur. That is, $\mathbb{A}+c(\mathbb{E}-I)$ is Schur as $\mathbb{E} = T^{-1}\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}T$. $\square$

As in the continuous case, a generalized Lyapunov stability theory for the positive discrete singular system (21) shall be presented.

Definition 2. The function $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T R\mathbb{E}\mathbf{x}(k)$ with $R>0$ is called a QLF of (21), if along the solution of (21) its difference $\triangle V(k)$ is negative definite on the consistency space $\mathrm{im}(\mathbb{E})$ (i.e., $\triangle V(k) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \le 0$, and $\triangle V(k) = 0$ only for $\mathbf{x}(k) = 0$). Specially, if R is diagonal, $V(\mathbf{x}(k))$ is called a diagonal QLF.

Let (21) be positive, $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ with a vector $\mathbf{v}\succ 0$ is called an LCLF if along the solution of (21), $\triangle V(k)$ is negative definite on $\mathrm{im}(\mathbb{E})$.

Remark 8. The solution $\mathbf{x}(k)\in \mathrm{im}(\mathbb{E})$ ensures that $V(\mathbf{x}(k))$ is positive definite and $\triangle V(\mathbf{x}(k))$ is negative definite on $\mathrm{im}(\mathbb{E})$, respectively.

For a QLF, one can see that $\triangle V(k)$ is negative definite on $\mathrm{im}(\mathbb{E})$ if and only if there exist $R>0$ and Q which is positive definite on $\mathrm{im}(\mathbb{E})$ such that $(\mathbb{E}\mathbb{A})^T R(\mathbb{E}\mathbb{A}) - R = \mathbb{A}^T R\mathbb{A} - R = -Q$.

When (1) reduces to the standard system, in this case $V(\mathbf{x}(k)) = \mathbf{x}^T(t)R\mathbf{x}(t)$ and $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{v}$, which are the well known (diagonal) QLF and LCLF of a standard discrete LTI system, respectively. See e.g., [30,32].

Theorem 4. *Assume that there exists a vector $\mathbf{w}_0\succ 0$ such that $\mathbb{E}\mathbf{w}_0\succeq 0$. If the system (21) is positive, then the following statements are equivalent:*

(i) *The system (21) is stable for all initial conditions such that $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0\succeq 0$.*
(ii) *There exists a vector $\mathbf{v}\succ 0$ such that $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ is an LCLF for (21).*
(iii) *There exists a diagonal matrix $D>0$ such that $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T D\mathbb{E}\mathbf{x}(k)$ is a diagonal QLF for (21).*
(iv) *There exists a matrix $R>0$ such that $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T R\mathbb{E}\mathbf{x}(k)$ is an QLF for (21).*

Proof. The proof will be completed by proving (i) $\Leftrightarrow$ (ii), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv)$\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): If (21) is positive and stable, then by Theorem 3 there exists a matrix M such that $\mathbb{A}+M(\mathbb{E}-I)$ is nonnegative and Schur. This implies by Lemma 2 that there exists a vector $\mathbf{v}\succ 0$ such that $[\mathbb{A}+M(\mathbb{E}-I)-I]^T\mathbf{v}\prec 0$. Now define $V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ which is obvious definite positive on $\mathrm{im}(\mathbb{E})$. Along the solution of the system (21) we can derive that

$$\begin{aligned}\triangle V(k) &= \mathbf{x}^T(k)\mathbb{E}^T(\mathbb{A}-I)^T\mathbf{v}\\ &= \mathbf{x}^T(k)[\mathbb{A}+M(\mathbb{E}-I)-I]^T\mathbf{v} \le 0\end{aligned}$$

and $\triangle V(k)=0$ only for $\mathbf{x}(k)=0$ as $\mathbf{x}(k)\in \mathrm{im}(\mathbb{E})$. That is to say, $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ is an LCLF.

(ii) $\Rightarrow$ (i): Let $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ with $\mathbf{v}\succ 0$ be an LCLF of (21), then along the system solution we have

$$\triangle V(k) = \mathbf{x}^T(k)\mathbb{E}^T(\mathbb{A}-I)^T\mathbf{v}.$$

Since $V(\mathbf{x}(k))$ and $\triangle V(k)$ is positive and negative definite on $\mathrm{im}(\mathbb{E})$, respectively, we can choose δ small enough such that $0<\delta<\{1,\min_{\mathbf{x}\in \mathrm{im}(\mathbb{E})/\{0\}}\frac{-\mathbf{x}^T\mathbb{E}^T(\mathbb{A}-I)^T\mathbf{v}}{V(\mathbf{x})}\}$, then we can get

$$\triangle V(k) = \mathbf{x}^T(k)(\mathbb{A}-I)^T\mathbf{v} \le -\delta V(\mathbf{x}(k)).$$

It further follows that $V(\mathbf{x}(k+1)) \le (1-\delta)V(\mathbf{x}(k)) \le \cdots \le (1-\delta)^{k+1}V(\mathbf{x}(0)) \to 0$ as $k\to\infty$. This implies $\mathbb{E}\mathbf{x}(k+1)=x(k+1)\to 0$ as $k\to\infty$, i.e., the system (21) is stable.

(i) $\Rightarrow$ (iii): As the system (21) is positive and stable, then by Theorem 3 there exists a matrix M such that $\mathbb{A}+M(\mathbb{E}-I)$ is nonnegative and Schur. That is, by Lemma 2, there exists a diagonal matrix $D>0$ such that $[\mathbb{A}+M(\mathbb{A}-I)]^TD[\mathbb{A}+M(\mathbb{E}-I)]-D<0$. This implies that, $\forall\ \mathbf{x}(k)\in \mathrm{im}(\mathbb{E})/\{0\}$,

$$\begin{aligned}&\mathbf{x}^T(k)[\mathbb{A}^TD\mathbb{A}-D]\mathbf{x}(k)\\ &= \mathbf{x}^T(k)([\mathbb{A}+M(\mathbb{E}-I)]^TD[\mathbb{A}+M(\mathbb{E}-I)]-D)\mathbf{x}(k)\\ &< 0\end{aligned} \tag{31}$$

and $\forall\ \mathbf{x}(k)\in \ker(\mathbb{E})$, as $\mathbb{A}\mathbf{x}(k)=\mathbb{A}\mathbb{E}\mathbf{x}(k)=0$, then

$$\mathbf{x}^T(k)[\mathbb{A}^TD\mathbb{A}-D]\mathbf{x}(k)=0. \tag{32}$$

Now define the function $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^TD\mathbb{E}\mathbf{x}(k)$, together (31) with (32) it shows that $V(\mathbf{x}(k))$ is a diagonal QLF for (21).

(iii) $\Rightarrow$ (iv): Trivial.

(iv) $\Rightarrow$ (i): Let $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^TR\mathbb{E}\mathbf{x}(k)$ with $R>0$ is the QLF of (21), then along the system solution we have

$$\begin{aligned}\triangle V(k) &= \mathbf{x}^T(k)\mathbb{E}^T(\mathbb{A}^TR\mathbb{A}-R)\mathbb{E}\mathbf{x}(k)\\ &= \mathbf{x}^T(k)\mathbb{E}^T(\mathbb{A}^TR\mathbb{A}-R)\mathbb{E}\mathbf{x}(k)\\ &= -\mathbf{x}^T(k)\mathbb{E}^TQ\mathbb{E}\mathbf{x}(k),\end{aligned}$$

where Q is positive definite on $\mathrm{im}(\mathbb{E})$. Now choose δ small enough such that $0<\delta<\{1,\frac{\lambda_{\min}(Q)}{\lambda_{\max}(R)}\}$, where $\lambda_{\min}$ and $\lambda_{\max}$ denotes the minimum and maximum eigenvalue, respectively. Then we can get

$$\triangle V(k) = -\mathbf{x}^T(k)\mathbb{E}^TQ\mathbb{E}\mathbf{x}(k) \le -\delta V(\mathbf{x}(k)).$$

The rest of proof is similar to that in (ii) $\Rightarrow$ (i). □

Remark 9. $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^T\mathbf{v}$ can be located by solving the LP (28) or (29) in $\mathbf{v}$. A diagonal QLF $V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbb{E}^TD\mathbb{E}\mathbf{x}(k)$ can be obtained by solving the LMI (30) in D.

Table 2
Feasibility analysis for different methods (Example 2)

(a,b,c)	$\mathbb{E}$	Thm 4.8 [7]	Thm 4.4 [19]	Thm 3
(0,0,0)	$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$	Feasible	Infeasible	Feasible
(0.5,1.2854,0.1168)	$\begin{pmatrix}1 & 0 & 0\\-0.1748 & 0.4470 & 0.4470\\0.1748 & 0.5530 & 0.5530\end{pmatrix}$	Infeasible	Feasible	Feasible
(0,0.2116,0.1592)	$\begin{pmatrix}1 & 0 & -0.1380\\0 & 1 & 0.1782\\0 & 0 & 0\end{pmatrix}$	Infeasible	Infeasible	Feasible

Example 2. As an illustrative example, consider the system (20) given by

$$E=\begin{pmatrix}-1.0031 & 0.1168 & c\\0.1269 & 1.2854 & b\\0 & a & a\end{pmatrix},\quad A=\begin{pmatrix}-0.3123 & -0.1256 & 0.0581\\0.1189 & 0.2671 & 0.2132\\0 & 0 & 0.1629\end{pmatrix}.$$

In table 2, by considering different E, we do a feasibility study and make a comparison between Theorem 3 and the existing results (Theorem 4.8 in [7] and Theorem 4.4 in [19]). It is clear that our work provides a less conservative criterion for stability of (2). For example, set $a=0, b=0.2116, c=0.1592,$ we calculate the corresponding

$$\mathbb{E}=\begin{pmatrix}1 & 0 & -0.1380\\0 & 1 & 0.1782\\0 & 0 & 0\end{pmatrix},\quad \mathbb{A}=\begin{pmatrix}0.3184 & 0.1477 & -0.0176\\0.0611 & 0.1932 & 0.0260\\0 & 0 & 0\end{pmatrix}.$$

One can see that $\mathbb{E}\succeq 0$ is not true and there dose not exist a vector $\mathbf{w}_0$ such that $\mathbb{E}\mathbf{w}_0\succ 0$ as $[\mathbb{E}]_3=[0,0,0]$. Hence the existing results [7,19] are not available. However one can easily find some vectors such as $\mathbf{w}_0=(1,1,1)^T$ such that $\mathbb{E}\mathbf{w}_0\succeq 0$. This means that Theorem 3 can work. To see this, for the case $a=0, b=0.2116, c=0.1592$ we can solve the LP (28) obtain a feasible solution

$$\mathbf{v}=\begin{pmatrix}4.8244\\4.8118\\4.7661\end{pmatrix},\quad Y=\begin{pmatrix}-0.1479 & -0.1051 & -0.1306\\0.2273 & 0.1745 & 0.2059\\-1.3624 & -1.1762 & -1.2868\end{pmatrix},$$

which defines an LCLF $V(\mathbf{x})=\mathbf{x}^T\mathbb{E}^T\mathbf{v}=4.8244[\mathbf{x}]_1+4.8118[\mathbf{x}]_2+0.1921[\mathbf{x}]_3$. Moreover, noting that $M=(Y\mathrm{diag}(\mathbf{v})^{-1})^T$ we have

$$\mathbb{A}+M(\mathbb{E}-I)=\begin{pmatrix}0.3184 & 0.1477 & 0.2774\\0.0611 & 0.1932 & 0.2799\\0 & 0 & 0.2815\end{pmatrix},$$

which is nonnegative and Schur as its spectrum is (0.1421, 0.3696, 0.2815). This means from the statement (ii) in Theorem 3 that the system (21) is positive and stable. This is shown in Figure 2.

Alternatively, we can check the positivity and stability by using the statement (iii) in Theorem 3. By solving the LP (29), a feasible solution can be given as

$$\mathbf{v}=\begin{pmatrix}4.2445\\4.4342\\4.7690\end{pmatrix},\quad M=\begin{pmatrix}-1.3547 & 1.7129 & -5.7840\\-1.3483 & 1.7060 & -5.7741\\-1.3521 & 1.7101 & -5.7800\end{pmatrix},$$

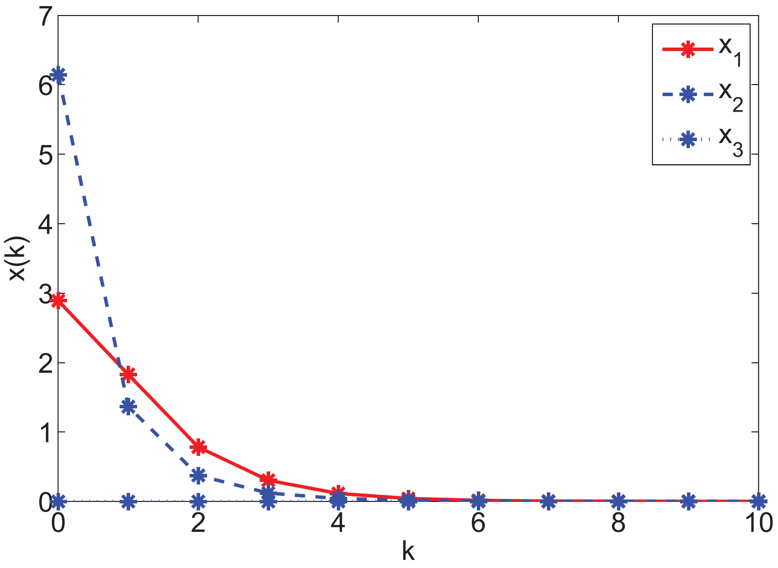

Fig. 2. State response of the system, where the initial condition $\mathbf{x}(0) = \mathbb{E}\mathbf{w}_0 = (2.8896; 6.1426; 0)^T$ with $\mathbf{w}_0 = (3, 6, 0.8)^T$.

which gives an LCLF as $V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbb{E}^T\mathbf{v} = 4.2445[\mathbf{x}]_1 + 4.4342[\mathbf{x}]_2 + 4.7690[\mathbf{x}]_3$ and

$$\mathbb{A} + M(\mathbb{E} - I) = \begin{pmatrix} 0.3184 & 0.1477 & 6.2586 \\ 0.0611 & 0.1932 & 6.2902 \\ 0 & 0 & 6.2713 \end{pmatrix}.$$

One can see that it is nonnegative (note that its spectrum is (0.3696, 0.1421, 6.2713), i.e., not Schur). In addition, since $\mathbb{A}$ with spectrum (0.3696, 0.1421, 0) is Schur, we can conclude that the system (21) is positive and stable.

Effectively, we can solve the LMI (30) to check the positivity and stability of (21). A feasible solution is

$$D = \begin{pmatrix} 1.2946 & 0 & 0 \\ 0 & 1.2705 & 0 \\ 0 & 0 & 1.2351 \end{pmatrix}, \quad M = \begin{pmatrix} 0.7605 & 1.2877 & 0 \\ 1.0179 & 1.2422 & 0 \\ 1.0789 & 1.4354 & 0 \end{pmatrix},$$

which give a diagonal QLF $V(\mathbf{x}) = \mathbf{x}^T\mathbb{E}^T D\mathbb{E}\mathbf{x} = 1.2946[\mathbf{x}]_1^2 + 1.2705[\mathbf{x}]_2^2 + 0.0650[\mathbf{x}]_3^2 + -0.3572[\mathbf{x}]_1[\mathbf{x}]_3 + 0.4530[\mathbf{x}]_2[\mathbf{x}]_3$ and

$$\mathbb{A} + M(\mathbb{E} - I) = \begin{pmatrix} 0.3184 & 0.1477 & 0.1070 \\ 0.0611 & 0.1932 & 0.1070 \\ 0 & 0 & 0.1070 \end{pmatrix},$$

which is nonnegative. Together this with Schur property of $\mathbb{A}$, the positivity and stability of (21) thus follow by applying the statement (iii) in Theorem 3.

5. Concluding remarks

In this paper, we have investigated the stability of positive singular systems in the continuous as well as in the discrete case. Based on Drazin inverse, we have provided some checkable necessary and sufficient conditions for stability. These conditions are less conservative than the existing results. We have also provided a generalized Lyapunov stability theory for positive singular systems. Our future work will involve the stability issue of positive singular systems with delay. We suspect that the results presented here will be of great value in this future study.

Statement of Contribution

In this paper, we provide a complete characterization for the stability of positive singular systems. Some Drazin-inverse-based necessary and sufficient conditions to check the stability are presented. These conditions take a simpler form than the existing results, and are easily solved by LP and LMI toolboxes in Matlab. We also generalize the Lyapunov stability theory for positive singular systems. Our work will provide a key idea for the future work on positive singular systems, such as the control problems, properties of switched singular systems. We believe that our work is meaningful, and some readers of your journal will benefit from it.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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