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Criterions for detecting the existence of the exponential dichotomies in the asymptotic behavior of the solutions of variational equations

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Abstract

We prove that the admissibility of any pair of vector-valued Schäffer function spaces (satisfying a very general technical condition) implies the existence of a "no past" exponential dichotomy for an exponentially bounded, strongly continuous cocycle (over a semiflow). Roughly speaking the class of Schäffer function spaces consists in all function spaces which are invariant under the right-shift and therefore our approach addresses most of the possible pairs of admissible spaces. Complete characterizations for the exponential dichotomy of cocycles are also obtained. Moreover, we involve a concept of a "no past" exponential dichotomy for cocycles weaker than the classical concept defined by Sacker and Sell (1994) in [23]. Our definition of exponential dichotomy follows partially the definition given by Chow and Leiva (1996) in [4] in the sense that we allow the unstable subspace to have infinite dimension. The main difference is that we do not assume *a priori* that the cocycle is invertible on the unstable space (actually we do not even assume that the unstable space is invariant under the cocycle). Thus we generalize some known results due to O. Perron (1930) [14], J. Daleckij and M. Krein (1974) [7], J.L. Massera and J.J. Schäffer (1966) [11], N. van Minh, F. Räbiger and R. Schnaubelt (1998) [26].
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1. Introduction

The study of linear systems seems to be more than classic by now but it still plays a central role in the qualitative theory of dynamical systems. This is mainly due to the fact that a comprehensive analysis of nonlinear systems using perturbation techniques requires a linear machinery, since, in most cases, the stability of solutions can be obtained from the linearization along the solution, the so-called variational equation. It is known for instance that the qualitative theory of (nonlinear) (semi)flows on (locally) compact spaces or (σ-)finite measure spaces relies heavily on notions like stability or exponential dichotomy for the associated linear skew-product (semi)flow. It is worth to note that all truly infinite-dimensional situations, e.g. flows originating from partial differential equations and functional differential equations, only yield linear skew-product (semi)flows. For instance, it is known by now that well-known equations like Navier–Stokes, Bubnov–Galerkin, Taylor–Couette can be modeled asymptotically by associating a linear skew-product (semi)flow (for details we refer the reader to [16]).

In this paper we investigate the existence of exponential dichotomies for linear skew-product semiflows (LSPS). An exponential dichotomy is one of the most basic concepts arising in the theory of dynamical systems. This topic, for example, plays a central role in the Hadamard–Perron theory of invariant manifolds for dynamical systems, and in many aspects of the theory of stability. Even in the context of bifurcation theory, the exponential dichotomy has a role. However in this context, the exponential dichotomy is represented by its younger sibling, the exponential trichotomy. In particular, topics such as the reduction principle and the center manifold theorem, the robustness of periodic solutions and invariant manifolds, as seen in the Poincaré–Melnikov scenario, are based on the theory of exponential trichotomies.

The notion of exponential dichotomy of linear differential equations was introduced by Perron [14], which approaches the problem of conditional stability of a system $\dot{x}(t) = A(t)x$ and its connection with the existence of bounded solutions of the equation $\dot{x}(t) = A(t)x + f(x,t)$, where the state space is a Banach space X and the operator-valued function $A(\cdot)$ is bounded, continuous in the strong operator topology. An important contribution to these problems is the work done by Massera and Schäffer [11], Daleckij and Krein [7], Coppel [6], Sacker and Sell [22]. The need for a new approach came from the observation that for a time dependent linear differential equation with unbounded operator $A(t)$, the solutions, generally speaking, either cannot be extended in the direction of the negative times, or can be extended, but not uniquely. All the problems above can be treated in the unified setting of a linear skew-product semiflow (LSPS). In [23] Sacker and Sell employ a notion of exponential dichotomy for skew-product semiflow with the restriction that the unstable subspace has finite dimension, and they point out a sufficient condition for the existence of exponential dichotomy for skew-product semiflow. In this work we use a concept of a **no past** exponential dichotomy for skew-product semiflow weaker than the concept used by Sacker and Sell. Our definition follows partially the definition (of exponential dichotomy) introduced by Chow and Leiva in [4] in the sense that we allow the unstable subspace to have infinite dimension. We go even more general and we do not assume *a priori* that the cocycle is invertible on the unstable space (actually we do not even assume that the unstable space is invariant under the cocycle). We continue the approach initiated by Perron (the so-called “admissibility condition” or “test function method”) and we prove that the admissibility of any pair of vector-valued Schäffer function spaces (satisfying a very general technical condition) implies the existence of a (no past) exponential dichotomy. Roughly speaking the class of Schäffer function spaces consists in all function spaces which are invariant under the right-shift (see Definition 2.1) and therefore our approach addresses most of the possible pairs of admissible spaces (see Theorem 5.4). Thus we generalize some known results due to O. Perron [14], J. Daleckij and M. Krein [7], J.L. Massera and J.J. Schäffer [11], N. van Minh, F. Räbiger and R. Schnaubelt [26].

2. Preliminaries

We now recall some preliminaries.

We will use the symbol $\mathbb{R}_+$ to denote the set $\{t \in \mathbb{R}\colon\ t \geqslant 0\}$. Also, let X be real or complex Banach space and X^* its dual space. By $\mathcal{M}(\mathbb{R}_+, X)$ we will denote the space of all strongly measurable functions from $\mathbb{R}_+$ to X. Furthermore $B(X)$ denotes the Banach algebra of all bounded linear operators acting on the Banach space X. The norms on X, X^*, $B(X)$ shall be denoted by the symbol $\|\cdot\|$.

As usual, we put

$$C_b(\mathbb{R}, X) = \{f : \mathbb{R} \to X\colon\ f \text{ is continuous and bounded}\};$$

$$L^1_{loc}(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X)\colon \int_K \|f(t)\|\,dt < \infty,\ \text{for each compact } K \text{ from } \mathbb{R}_+ \right\};$$

$$L^p(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X)\colon \int_{\mathbb{R}_+} \|f(t)\|^p\,dt < \infty \right\}, \quad \text{where } p \in [1, \infty);$$

$$L^\infty(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X)\colon \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \|f(t)\| < \infty \right\};$$

$$M^p(\mathbb{R}_+, X) = \left\{ f \in \mathcal{M}(\mathbb{R}_+, X)\colon \sup_{t \in \mathbb{R}_+} \int_t^{t+1} \|f(s)\|^p\,ds < \infty \right\}, \quad \text{where } p \in [1, \infty);$$

$T(\mathbb{R}_+, X)$ is the space of all functions $f \in L^1_{loc}(\mathbb{R}_+, X)$ with the property that there exist $(\tau_n)_{n \in \mathbb{N}}$ and $(a_n)_{n \in \mathbb{N}}$ two sequences of positive real numbers such that

$$\sum_{n=0}^{\infty} a_n < \infty \quad \text{and} \quad \|f(t)\| \leqslant \sum_{n=0}^{\infty} a_n \chi_{[\tau_n, \tau_n + 1]}(t) \quad \text{a.e.}$$

We recall that $C_b(\mathbb{R}, X)$, $L^p(\mathbb{R}_+, X)$, $L^\infty(\mathbb{R}_+, X)$, $M^p(\mathbb{R}_+, X)$, $T(\mathbb{R}_+, X)$ are Banach spaces endowed with the respectively norms:

$$|||f||| = \sup_{t \in \mathbb{R}} |f(t)|;$$

$$\|f\|_p = \left(\int_{\mathbb{R}_+} \|f(t)\|^p\,dt \right)^{\frac{1}{p}};$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t \in \mathbb{R}_+} \|f(t)\|;$$

$$\|f\|_{M^p(\mathbb{R}_+,X)} = \sup_{t\in\mathbb{R}_+}\left(\int_t^{t+1}\|f(s)\|^p\,ds\right)^{\frac{1}{p}};$$

$$\|f\|_{T(\mathbb{R}_+,X)} = \inf\left\{\sum_{n=0}^{\infty} a_n\colon \text{ where } (a_n)_{n\in\mathbb{N}} \text{ satisfy the above inequality}\right\}.$$

Definition 2.1. A Banach space $(E(\mathbb{R}_+,\mathbb{R}), \|\cdot\|_{E(\mathbb{R}_+,\mathbb{R})})$ is said to be a scalar-valued Schäffer function space if the following conditions hold:

(s_1) $E \subset L^1_{loc}(\mathbb{R}_+,\mathbb{R})$ and for each compact $K\subset\mathbb{R}_+$ there is $\alpha_K > 0$ such that

$$\int_K \left|f(t)\right|dt \leqslant \alpha_K\|f\|_{E(\mathbb{R}_+,\mathbb{R})}, \quad \text{for all } f\in E(\mathbb{R}_+,\mathbb{R}).$$

(s_2) $\phi_{[0,t]}\in E(\mathbb{R}_+,\mathbb{R})$, for each $t\geqslant 0$, where $\phi_{[0,t]}$ denotes the characteristic function (indicator) of the interval $[0,t]$.

(s_3) If $f\in E(\mathbb{R}_+,\mathbb{R})$ and $h\in\mathcal{M}(\mathbb{R}_+,\mathbb{R})$ with $|h|\leqslant|f|$, then $h\in E(\mathbb{R}_+,\mathbb{R})$ and $\|h\|_{E(\mathbb{R}_+,\mathbb{R})}\leqslant \|f\|_{E(\mathbb{R}_+,\mathbb{R})}$.

(s_4) If $f\in E(\mathbb{R}_+,\mathbb{R})$, $t\geqslant 0$, $g_t:\mathbb{R}_+\to\mathbb{R}$,

$$g_t(s)=\begin{cases}0, & s\in[0,t),\\ f(s-t), & s\in[t,\infty),\end{cases}$$

then $g_t\in E(\mathbb{R}_+,\mathbb{R})$ and $\|g_t\|_{E(\mathbb{R}_+,\mathbb{R})} = \|f\|_{E(\mathbb{R}_+,\mathbb{R})}$.

Example 2.1. It is a routine to verify that $M^p(\mathbb{R}_+,\mathbb{R})$, $L^p(\mathbb{R}_+,\mathbb{R})$, $L^\infty(\mathbb{R}_+,\mathbb{R})$ and $T(\mathbb{R}_+,\mathbb{R})$, the spaces mentioned above are particular examples of scalar-valued Schäffer function spaces. One can easily remark that $T(\mathbb{R}_+,\mathbb{R})\subset E(\mathbb{R}_+,\mathbb{R})\subset M^1(\mathbb{R}_+,\mathbb{R})$, for any scalar-valued Schäffer function space $E(\mathbb{R}_+,\mathbb{R})$. For details we refer the reader to [11, 23.G, p. 60].

Example 2.2. The class of scalar-valued Schäffer function spaces contains also the well-known class of scalar-valued Orlicz function spaces. For convenience we recall briefly to the reader the notion of a scalar-valued Orlicz function space. Let $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ be a function which is non-decreasing, left-continuous, $\varphi(t)>0$, for all $t>0$. Define

$$\Phi(t)=\int_0^t\varphi(s)\,ds.$$

A function Φ of this form is called a Young function. For $f:\mathbb{R}_+\to\mathbb{R}$ a measurable function and Φ a Young function we define

$$M^\Phi(f)=\int_0^\infty \Phi\left(\left|f(s)\right|\right)ds.$$

The set $L^{\Phi}(\mathbb{R}_+,\mathbb{R})$ of all f for which there exists a $k>0$ that $M^{\Phi}(kf)<\infty$ is clearly a vector space. Using the Luxemburg norm,

$$\rho^{\Phi}(f)=\inf\left\{k>0\colon\ M^{\Phi}\left(\frac{1}{k}f\right)\leqslant 1\right\},$$

we get that $(L^{\Phi}(\mathbb{R}_+,\mathbb{R}),\rho^{\Phi})$ is a Banach space. It is trivial to see that $(L^{\Phi}(\mathbb{R}_+,\mathbb{R}),\rho^{\Phi})$ verifies the conditions (s_2), (s_3), (s_4). For checking (s_1) we consider $f\in L^{\Phi}(\mathbb{R}_+,\mathbb{R})$, $t>0$, $k>0$ such that $M^{\Phi}(\frac{1}{k}f)\leqslant 1$. It follows that

$$\Phi\left(\frac{1}{kt}\int_0^t\left|f(s)\right|ds\right)\leqslant\frac{1}{t}\int_0^t\Phi\left(\frac{1}{k}\left|f(s)\right|\right)ds\leqslant\frac{1}{t},$$

and so

$$\int_0^t\left|f(s)\right|ds\leqslant t\Phi^{-1}\left(\frac{1}{t}\right)k$$

which implies that

$$\int_0^t\left|f(s)\right|ds\leqslant t\Phi^{-1}\left(\frac{1}{t}\right)\rho^{\Phi}(f),$$

for all $f\in L^{\Phi}(\mathbb{R}_+,\mathbb{R})$, $t>0$, and hence the condition (s_1) is also verified.

Remark 2.1. Let now $L^{\Phi}(\mathbb{R}_+,\mathbb{R})$ be a scalar-valued Orlicz function space. We denote by

$$L^{\Phi}(\mathbb{R}_+,X)=\left\{f\in\mathcal{M}(\mathbb{R}_+,X):t\mapsto\left\|f(t)\right\|\colon\ \mathbb{R}_+\to\mathbb{R}_+\text{ is in }L^{\Phi}(\mathbb{R}_+,\mathbb{R})\right\}.$$

It is easy to check that $L^{\Phi}(\mathbb{R}_+,X)$ is a Banach space endowed with the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}_+,X)}=\left\|\left\|f(\cdot)\right\|\right\|_{\rho^{\Phi}}.$$

We will call $L^{\Phi}(\mathbb{R}_+,X)$ as a *vector-valued Orlicz function space.*

Remark 2.2. $L^{\Phi}(\mathbb{R}_+,\mathbb{R})=L^{p}(\mathbb{R}_+,\mathbb{R})$ if and only if $\Phi(t)=t^p$, for all $t\geqslant 0$. For details we refer the reader to [20].

Remark 2.3. Take now $E(\mathbb{R}_+,\mathbb{R})$ to be a scalar-valued Schäffer function space. We define

$$E(\mathbb{R}_+,X)=\left\{f\in\mathcal{M}(\mathbb{R}_+,X):t\mapsto\left\|f(t)\right\|\colon\ \mathbb{R}_+\to\mathbb{R}_+\text{ is in }E(\mathbb{R}_+,\mathbb{R})\right\}.$$

Obviously that $E(\mathbb{R}_+,X)$ will be called as a vector-valued Schäffer function space.

Remark 2.4. $E(\mathbb{R}_+, X)$ is a Banach space endowed with the norm

$$\|f\|_{E(\mathbb{R}_+,X)} = \big\|\|f(\cdot)\|\big\|_{E(\mathbb{R}_+,\mathbb{R})}.$$

For details we refer the reader to [19, Remark 2.1, p. 196].

Remark 2.5. If $\{f_n\}_{n\in\mathbb{N}} \subset E(\mathbb{R}_+, X)$, $f \in E(\mathbb{R}_+, X)$, $f_n \to f$ in $E(\mathbb{R}_+, X)$ when $n \to \infty$, then there exists $\{f_{n_k}\}_{k\in\mathbb{N}}$ a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ such that

$$f_{n_k} \to f \quad \text{a.e.}$$

For the proof of this fact see [19, Remark 2.2, p. 197].

For a scalar-valued Schäffer function space $E(\mathbb{R}_+, \mathbb{R})$, we denote by $\alpha(\cdot, E), \beta(\cdot, E) : \mathbb{R}_+ \to \mathbb{R}_+$ the following functions:

$$\alpha(t, E) = \inf\left\{\alpha > 0\colon \int_0^t \big|f(s)\big|\,ds \leqslant \alpha\|f\|_{E(\mathbb{R}_+,\mathbb{R})},\ \text{for all } f \in E\right\},$$
$$\beta(t, E) = \|\chi_{[0,t]}\|_{E(\mathbb{R}_+,\mathbb{R})}.$$

Remark 2.6. It is known (see for instance [11, (23.1) from p. 61, and (23.K) from p. 62]) that $\alpha(\cdot, E), \beta(\cdot, E)$ are non-decreasing functions and

$$t \leqslant \alpha(t, E)\beta(t, E) \leqslant 2t, \quad \text{for all } t \geqslant 0. \tag{1}$$

Remark 2.7. It is easy to compute the above numbers for $L^p(\mathbb{R}_+, \mathbb{R})$ and $M^p(\mathbb{R}_+, \mathbb{R})$:

$$\alpha\big(t, L^p\big) = \begin{cases} t^{1-\frac{1}{p}}, & p \in [1, \infty), \\ t, & p = \infty, \end{cases} \quad t \geqslant 0,$$
$$\beta\big(t, L^p\big) = \begin{cases} t^{\frac{1}{p}}, & p \in [1, \infty), \\ 1, & p = \infty, \end{cases} \quad t \geqslant 0.$$

Also we can see that $t \leqslant \alpha(t, M^p) \leqslant [t] + \{t\}^{1-\frac{1}{p}}$, for each $(p, t) \in [1, \infty) \times \mathbb{R}_+$. Here $[t]$ denotes the largest integer less than or equal t and $\{t\} = t - [t]$:

$$\beta\big(t, M^p\big) = \begin{cases} t^{\frac{1}{p}}, & t \in [0, 1), \\ 1, & t \geqslant 1. \end{cases}$$

Furthermore $\alpha(t, L^\Phi) = t\Phi^{-1}(\frac{1}{t})$ and $\beta(t, L^\Phi) = (\Phi^{-1}(\frac{1}{t}))^{-1}$.

3. Linear skew-product semiflows (LSPS)

Consider now the trivial Banach bundle $\mathcal{E} = X \times \Theta$, where X is a fixed Banach space (the *state space*) and Θ is a metric space (the *base space*). For details of Banach bundles we refer to [24, Chapter 4].

Definition 3.1. A (nonlinear) semiflow $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ is defined by the properties:

(i) $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$;
(ii) $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$, for all $\theta \in \Theta$ and $t, s \in \mathbb{R}_+$.

If in addition $(\theta, t) \to \sigma(\theta, t)$ is continuous, then σ is called a *continuous* (nonlinear) semiflow on Θ.

Also, if (ii) holds for any $t, s \in \mathbb{R}$ then σ is said to be a (nonlinear) *flow* on Θ.

Definition 3.2. A family $\{T(t)\}_{t\geqslant 0}$ of linear and bounded operators acting on X, is said to be a C_0-semigroup on X if the following conditions hold:

(i) $T(0) = I$;
(ii) $T(t+s) = T(t)T(s)$, for all $t, s \geqslant 0$;
(iii) there exists $\lim_{t\to 0_+} T(t)x = x$, for all $x \in X$.

If the second property holds for any $t, s \in \mathbb{R}$ then $\{T(t)\}_{t\in\mathbb{R}}$ is called a C_0-group.

For a general presentation of the theory of C_0-semigroups we refer the reader to [13].

Remark 3.1. It is known the connection between (nonlinear) (semi)flows, first order differential operators, and (linear) one-parameter (semi)groups. For instance, consider a continuously differentiable vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with $\sup_{\theta\in\mathbb{R}^n} \|DF(\theta)\| < \infty$, for the derivative $DF(\theta)$ of F and $\theta \in \mathbb{R}^n$. Take the first order differential operator on

$$X := C_0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n\colon\ f \text{ is continuous and vanishes at infinity}\}$$

corresponding to the vector field F,

$$Af(\theta) = \langle \operatorname{grad} f(\theta), F(\theta)\rangle = \sum_{i=1}^{n} F_i(\theta)\frac{\partial f}{\partial\theta_i}(\theta),$$

for $f \in C_c^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n\colon\ f \text{ continuously differentiable, with compact support}\}$, and $\theta \in \mathbb{R}^n$.

For $0 \neq f \in X$, the duality set $\mathcal{J}(f) = \{x^* \in X^*\colon\ x^*(f) = \|f\|^2 = \|x^*\|^2\}$ contains all point measures supported by those points $\theta_0 \in \mathbb{R}^n$ where $|f|$ reaches its maximum. More precisely,

$$\{\overline{f(\theta_0)}\delta_{\theta_0}\colon\ \theta_0 \in \mathbb{R}^n,\ |f(\theta_0)| = \|f\|\} \subset \mathcal{J}(f).$$

Since $\frac{\partial f(\theta_0)}{\partial\theta_i} = 0$ while $|f(\theta_0)| = \|f\|$, it follows that A is dissipative (i.e. there exists $j(f) \in \mathcal{J}(f)$ such that $\operatorname{Re} j(f)(Af) \leqslant 0$). Since F is globally Lipschitz it follows from standard arguments that there exists a continuous flow $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ which solves the differential equation

$$\frac{\partial}{\partial t}\sigma(\theta, t) = F\big(\sigma(\theta, t)\big), \quad \text{for all } t \in \mathbb{R} \text{ and } \theta \in \mathbb{R}^n \ \big(\text{see [1, Thm. 10.3]}\big).$$

To such a flow we associate a one-parameter group of linear operators on $C_0(\mathbb{R}^n)$ given by

$$\big(T(t)f\big)(\theta) := f\big(\sigma(\theta, t)\big), \quad \text{for } \theta \in \mathbb{R}^n,\ t \in \mathbb{R},$$

the so-called group induced by the flow σ. It can be proved that the generator of the above group is the closure of the first order differential operator A. The domain of the generator will be $D(A) = C_c^1(\mathbb{R}^n)$. For details we refer the reader to [8].

The general relation between (nonlinear) semiflows and linear semigroups is given in the example below.

Example 3.1. Let Θ be a compact metric space and take $X = C(\Theta)$, where

$$C(\Theta) = \{f : \Theta \to \mathbb{C}:\ f \text{ continuous on } \Theta\}.$$

(i) The (nonlinear) semiflow σ is continuous if and only if it induces a strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on X by the formula:

$$\big(T(t)f\big)(\theta) := f\big(\sigma(\theta, t)\big), \quad \text{for } \theta \in \Theta,\ t \geqslant 0,\ f \in X. \tag{2}$$

(ii) The generator $(A, D(A))$ of $\{T(t)\}_{t\geqslant 0}$ is a derivation.

(iii) Every strongly continuous semigroup $\{T(t)\}_{t\geqslant 0}$ on X that consists of algebra homomorphisms originates, via (2), from a continuous (nonlinear) semiflow on Θ (see [12, B-II, Thm. 3.4]).

We will state in the next the basic definitions concerning cocycles, linear skew-product (semi)flows, and dichotomy. Before to state the formal definitions, let us recall the prototypical example of a linear skew-product flow; namely, the skew-product flow associated with the solutions of a nonautonomous differential equation $\dot{u} = A(t)u$ on a Banach space X. For this case, consider the translation flow $\sigma(\theta, t) = \theta + t$ on $\mathbb{R}$ and the trivial bundle $X \times \mathbb{R}$ over $\mathbb{R}$. A linear skew-product flow on $X \times \mathbb{R}$ is defined by $(u_0, \theta, t) \mapsto (u(u_0, \theta, t), \theta + t)$ where $t \mapsto u(u_0, \theta, t)$ is the solution of the differential equation with the initial condition $u(u_0, \theta, \theta) = u_0$. To avoid technical complications for the general case, we will define the notion of a cocycle and a linear skew-product (semi)flow in the setting of a trivial vector bundle. It is worth to mention that the theory is valid for general vector bundles, but the topology of nontrivial bundles plays no role in the analysis. In fact, the constructions of this section are local. They can always be carried out in a natural vector bundle chart.

Definition 3.3. Let σ be a (nonlinear) continuous semiflow on Θ. A strongly continuous cocycle over the continuous semiflow σ is an operator-valued function

$$\Phi : \Theta \times \mathbb{R}_+ \to B(X), \quad (\theta, t) \mapsto \Phi(\theta, t),$$

that satisfies the following properties:

(i) $\Phi(\theta, 0) = I$ (I – the identity operator in X), for all $\theta \in \Theta$, $t \in \mathbb{R}_+$;
(ii) $\Phi(\theta, \cdot)x$ is continuous for each $\theta \in \Theta$, $x \in X$;
(iii) $\Phi(\theta, t+s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $t, s \geqslant 0$ and $\theta \in \Theta$ (the cocycle identity).

If, in addition,

(iv) there exist constants M, ω such that

$$\left\|\Phi(\theta, t)\right\| \leqslant Me^{\omega t}, \quad \text{for } t \geqslant 0,\ \theta \in \Theta,$$

then the strongly continuous cocycle is exponentially bounded.

The linear skew-product semiflow (LSPS), associated with the above cocycle, is the dynamical system $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ defined by

$$\pi : X \times \Theta \times \mathbb{R}_+ \to X \times \Theta, \quad (x, \theta, t) \mapsto \pi(x, \theta, t) = \big(\Phi(\theta, t)x, \sigma(\theta, t)\big).$$

Remark 3.2. Note that the operators in a strongly continuous cocycle are not assumed to be invertible. For this reason, the cocycle is parameterized by $t \geqslant 0$, but *not* by $t \in \mathbb{R}$. By the Uniform Boundedness Principle, if the base space Θ is compact, then a strongly continuous cocycle is exponentially bounded.

Example 3.2. The classic example of a cocycle arises as the solution operator for a variational equation. Take σ to be a continuous flow on the locally compact metric space Θ, and $\{A(\theta): \theta \in \Theta\}$ be a family of (possibly unbounded) densely defined closed operators on the Banach space X. A strongly continuous cocycle $\Phi(\cdot, t)x$ is said to solve the variational equation

$$\dot{u}(t) = A\big(\sigma(\theta, t)\big)u(t), \quad \theta \in \Theta,\ t \in \mathbb{R}, \tag{3}$$

if, for every $\theta \in \Theta$, we can find a dense subset $Z_\theta \subset D(A(\theta))$ such that, for every $u_\theta \in Z_\theta \subset D(A(\theta))$, the function

$$t \mapsto \Phi(\theta, t)u_\theta$$

is differentiable (for $t \geqslant 0$) and the values $u(t) \in D(A(\sigma(\theta, t)))$, and $t \mapsto u(t)$ verifies the above differential equation. More restrictive definition can be given if we impose that $Z_\theta = D(A(\theta))$ or even $Z_\theta = D(A(\theta)) = D$; that is, Z_θ is independent of θ. Characterizations of (exponential, discrete, pointwise) dichotomy for the solutions of the above variational systems were obtained through various techniques. For a complete presentation of these results we refer the reader to [2, Chapter 7].

Differential equations of type (3) arise from two reach (and essential) sources that we describe below. First, consider a nonlinear differential equation on X:

$$\dot{x} = F(x), \tag{4}$$

with $F : X \to X$ being Fréchet differentiable. Assume that (4) has a compact invariant set $\Theta \subset X$ (i.e. the solution $t \mapsto x(\theta, t)$, with $x(\theta, 0) = \theta$, has its values in Θ, for each $t \in \mathbb{R}$, whenever the initial point $\theta \in \Theta$). Then the family of functions $\{\sigma(\cdot, t) : \theta \mapsto x(\theta, t),\ t \in \mathbb{R}\}$ describes a flow on Θ. If $\theta \in \Theta$ then for any other initial condition $u_0 \in X$, the difference $u(t) = x(u_0, t) - x(\theta, t)$ such that

$$\dot{u}(t) = DF\big(x(\theta, t)\big)u(t) + \eta(u, x), \qquad \big|\eta(u, x)\big| = o(u), \quad |u| \to 0.$$

Then the differential equation (3) with $A(\theta) = DF(\theta)$, called the *variational equation*, determines the linearized flow of $\dot{x} = F(x)$. It is worth to note that, in an infinite-dimensional context, the operators $A(\theta)$ could be unbounded.

Second, define the *hull* of a continuous function $f : \mathbb{R} \to B(X)$ to be the set of operator-valued functions, given below:

$$\mathit{Hull}(f) = \mathit{closure}\big\{f(\cdot + \tau)\colon\ \tau \in \mathbb{R}\big\}.$$

Under appropriate assumptions, the set $\Theta := \mathit{Hull}(f)$ may be a compact set of operator-valued functions on $\mathbb{R}$. For example, if $f : \mathbb{R} \to B(\mathbb{R}^n)$ is almost-periodic and the closure is taken in the topology of uniform convergence on compact subsets of $\mathbb{R}$, then, by Bochner's Theorem, Θ is compact in the space of continuous matrix-valued functions. Consider now the flow (on Θ) given by $\sigma(\theta, t)(s) = \theta(s + t)$, with $t, s \in \mathbb{R}$. If we put also $A(\theta) = \theta(0) \in B(X)$, then we get in (3) all differential equations of the form $\dot{u} = \theta(t)u$, where the function θ is in the hull of f.

The next example shows how a cocycle arises from the linearization of a nonlinear partial differential equation. We will sketch extremely briefly the case of the linearized Navier–Stokes equation.

Example 3.3. Consider the Navier–Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} = \nu \Delta \mathbf{v} - \langle \mathbf{v}, \nabla \rangle \mathbf{v} - \operatorname{grad} \mathbf{p} + g, \qquad \operatorname{div} \mathbf{v} = 0$$

on a bounded domain $\Omega \subset \mathbb{R}^2$ with zero boundary conditions. As usual, $\mathbf{v} : \Omega \to \mathbb{R}^2$ denotes the velocity of an incompressible fluid, ν measures the viscosity of the fluid, $\mathbf{p} : \Omega \to \mathbb{R}$ represents the pressure, and the function $g : \mathbb{R} \times \Omega \to \mathbb{R}^2$ is a time-dependent forcing term. Take the orthogonal decomposition $L^2(\Omega; \mathbb{R}^2) = X \oplus H_\pi$, with X being the closure in $L^2(\Omega; \mathbb{R}^2)$ of the C^∞ divergence-free ($\nabla \cdot \mathbf{v} = 0$) vector fields with compact support in Ω, and H_π being the closure in $L^2(\Omega; \mathbb{R}^2)$ of the gradients $\nabla \mathbf{p}$ of all $\mathbf{p} \in C^1(\Omega; \mathbb{R})$, see for instance Constantin and Foias [5]. Let $P : L^2(\Omega; \mathbb{R}^2) \to X$ be the corresponding orthogonal projection, and define $A = P\Delta$, $B(\mathbf{v}, \mathbf{u}) = -P\langle \mathbf{v}, \nabla \rangle \mathbf{u}$, and $f = Pg$. Thus, see for instance Temam [25], the Navier–Stokes equation can be rewritten as an abstract equation on the Hilbert space X:

$$\frac{d\mathbf{v}}{dt} = \nu A\mathbf{v} + B(\mathbf{v}, \mathbf{v}) + f, \qquad \mathbf{v}(0) = \mathbf{v}_0. \tag{5}$$

The operator A with $D(A) = X \cap H^2(\Omega; \mathbb{R}^2)$ is a negative operator, and thus it generates an analytic semigroup on X (see for instance [25]). Define now $V = D((-A)^{\frac{1}{2}})$. Suppose that the function F, defined by $F(t) = f(t, \cdot)$, $t \in \mathbb{R}$, is in $C_b(\mathbb{R}; X)$. Furthermore, suppose that the positive hull of F,

$$H^{+}(F) = closure_{C_b(\mathbb{R};X)}\{F_\tau = F(\cdot + \tau):\ \tau \in \mathbb{R}_+\}$$

is a compact subset of $C_b(\mathbb{R}; X)$. Therefore, we have that the omega-limit set $\omega(F) = \bigcap_{\tau \geqslant 0} H^{+}(F_\tau)$ is nonempty and compact. Moreover, we can find a global compact attractor $\Theta \subset D(A) \times \omega(F)$ for the semiflow generated by the strong solutions of the abstract equation (5). For details we refer the reader to Raugel and Sell [21, Sections 2.11–2.12] and the references therein. This attractor is invariant under the flow σ defined by

$$(\mathbf{v}, f) \mapsto \theta_\tau = (\mathbf{v}_\tau, f_\tau), \quad \tau \in \mathbb{R},$$

where $f_\tau(t, \cdot) = f(t + \tau, \cdot)$ and $\mathbf{v}_\tau(t, \cdot) = \mathbf{v}(t + \tau, \cdot)$ for the strong solution $\mathbf{v}(t, \cdot)$ of Eq. (5). If $\theta = (\mathbf{v}_0, f) \in \Theta$ and $\mathbf{v}(t) = \mathbf{v}(t; f, \mathbf{v}_0)$, $t \geqslant 0$, is the corresponding strong solution of Eq. (5), then

$$\mathbf{v}(\cdot; f, \mathbf{v}_0) \in C\big([0, \infty); V\big) \cap L^\infty\big((0, \infty); V\big) \cap L^\infty_{loc}\big((0, \infty); D(A)\big).$$

For details we refer the reader to [5]. The linearized Navier–Stokes equation along the solution $\mathbf{v}$ is given by

$$\frac{dx}{dt} = \nu Ax + B\big(\mathbf{v}(t), x\big) + B\big(x, \mathbf{v}(t)\big), \quad x(0) = x_0 \in X. \tag{6}$$

Accordingly to [23], we have that if $x_0 \in V$, then there is a unique strong solution $x(t) = \Phi(\theta, t)x_0$ of the linearized equation (6) such that

$$x(\cdot) \in C\big([0, \infty); V\big) \cap L^\infty_{loc}\big((0, \infty); D(A)\big), \quad x_t(\cdot) \in L^2_{loc}\big((0, \infty); X\big),$$

where $\Phi(\theta, t)$ is the solution operator of (6). Clearly Φ is a cocycle over the flow σ on Θ. Also, for the study of exponential dichotomies for the Navier–Stokes equations we refer the reader to [15].

Example 3.4. Let X be a Banach space and Θ a compact topological Hausdorff space. Consider the following linear dependent system

$$\dot{x}(t) = A\big(\sigma(\theta, t)\big)x(t), \quad t > 0, \tag{7}$$

where $A(\sigma(\theta, t))x(t) = A + B(\sigma(\theta, t))$, A is the infinitesimal generator of the strongly continuous semigroup $\{T(t)\}_{t \geqslant 0}$ and σ is a flow on Θ and $B(\theta) \in B(X)$, $\theta \in \Theta$. To be precise in which sense the above equation generates a linear skew-product semiflow, we shall consider the following family of integral differential equations:

$$x(t) = T(t)x_0 + \int_0^t T(t - s)B\big(\sigma(\theta, s)\big)x(s)\, ds, \quad t \geqslant 0,\ \theta \in \Theta. \tag{8}$$

A solution $x(t) = x(t, \theta)$ of Eq. (8) is called a mild solution of (7).

Chow and Leiva establish in [3] that if A is the infinitesimal generator of the C_0-semigroup $\{T(t)\}_{t\geqslant 0}$ on X and if $B:\Theta\to B(X)$ is also strongly continuous, then for each $\theta\in\Theta$ and $x_0\in X$ the problem

$$\dot{x}(t)=A\big(\sigma(\theta,t)\big)x(t)=\big(A+B\big(\sigma(\theta,t)\big)\big)x(t);\qquad x(0)=x_0$$

has a unique mild solution given by

$$\Phi(\theta,t)x_0=T(t)x_0+\int_0^t T(t-s)B\big(\sigma(\theta,s)\big)\Phi(\theta,s)x_0\,ds.$$

Moreover if A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_0(t)\}_{t\geqslant 0}$, and the mapping $\theta\mapsto A(\theta)-A_0:\Theta\to B(X)$ is strongly continuous and the equation $\dot{x}(t)=A(\sigma(\theta,t))x(t)$ has an exponential dichotomy over Θ then there exists $\epsilon>0$ such that for any mapping $\theta\mapsto B(\theta):\Theta\to B(X)$ strongly continuous and $\|B(\theta)\|<\epsilon$, $\theta\in\Theta$, the equation

$$\dot{x}(t)=\big(A\big(\sigma(\theta,t)\big)+B\big(\sigma(\theta,t)\big)\big)x(t)$$

has also exponential dichotomy. For details we refer the reader to [3].

Definition 3.4. A family of linear and bounded operators $\{U(t,t_0)\}_{t\geqslant t_0\geqslant 0}$ is said to be a two-parameter evolution family if the following conditions hold:

(i) $U(t,t)=I$, for all $t\geqslant 0$;
(ii) $U(t,s)U(s,t_0)=U(t,t_0)$, for all $t\geqslant s\geqslant t_0\geqslant 0$;
(iii) $U(\cdot,t_0)x$ is continuous on $[t_0,\infty)$, for all $t_0\geqslant 0$, $x\in X$;
$U(t,\cdot)x$ is continuous on $[0,t]$, for all $t\geqslant 0$, $x\in X$;
(iv) there exist $M,\omega>0$ such that

$$\big\|U(t,t_0)\big\|\leqslant Me^{\omega(t-t_0)},\quad \text{for all } t\geqslant t_0\geqslant 0.$$

For a general presentation of the theory of two-parameter evolution families we refer the reader to [2] or [7].

Example 3.5. Let $\Theta=\mathbb{R}_+$, $\sigma(\theta,t)=\theta+t$ and let $\{U(t,s)\}_{t\geqslant s}$ be an evolution family on the Banach space X. We define

$$\Phi_U(\theta,t)=U(t+\theta,\theta),\quad \text{for all } (\theta,t)\in\mathbb{R}_+\times\mathbb{R}_+.$$

Then $\{\Phi_U(\theta,t)\}_{\theta\in\Theta,\,t\geqslant 0}$ is an exponentially bounded, strongly continuous cocycle (over the above semiflow σ). Therefore, we can say that the notion of a cocycle generalizes the classic notion of a two-parameter evolution family. An account of the results concerning the analysis of the exponential dichotomy for evolution families is given in [2, Chapter 4].

4. (No past) exponential dichotomy and admissibility

Let now $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ be a pair of vector-valued Schäffer function spaces and take $\{\Phi(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$ as an exponentially bounded, strongly continuous cocycle (over a semiflow $\{\sigma(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$) and $\pi = (\Phi, \sigma)$ as the associated linear skew-product semiflow on $\mathcal{E}$. Since $\mathcal{E} = X \times \Theta$ is a trivial Banach bundle (see for instance [3, Remark 2.1], and [24, Chapter 4]), we define for any subset $\mathcal{X} \subset \mathcal{E}$ the fibers

$$\mathcal{X}(\theta) = \bigl\{x \in X\colon\ (x, \theta) \in \mathcal{X}\bigr\}, \quad \theta \in \Theta.$$

In particular $\mathcal{E}(\theta) = X$.

Now we denote

$$\mathcal{X}_{1,F} = \bigl\{(x, \theta) \in \mathcal{E}\colon\ \Phi(\theta, \cdot)x \in F(\mathbb{R}_+, X)\bigr\}.$$

The corresponding fiber is $\mathcal{X}_{1,F}(\theta) = \{x \in X\colon\ (x, \theta) \in \mathcal{X}_{1,F}\}$, $\theta \in \Theta$.

It can be seen that $\mathcal{X}_{1,F}(\theta)$ is a vector subspace of X. In what follows $\mathcal{X}_{1,F}(\theta)$ will be assumed complemented (i.e. $\mathcal{X}_{1,F}(\theta)$ is closed and there exists $\mathcal{X}_{2,F}(\theta)$ a closed subspace such that $X = \mathcal{X}_{1,F}(\theta) \oplus \mathcal{X}_{2,F}(\theta)$). Also we denote by $P_F(\theta)$ a projection onto $\mathcal{X}_{1,F}(\theta)$ along $\mathcal{X}_{2,F}(\theta)$ (that is $P_F(\theta) \in B(X)$, $P_F(\theta)^2 = P_F(\theta)$ and $\mathrm{Ker}(P_F(\theta)) = \mathcal{X}_{2,F}(\theta)$) and by $Q_F(\theta) = I - P_F(\theta)$.

Remark 4.1. If $x \in \mathcal{X}_{2,F}(\theta)$, $x \neq 0$ then $\Phi(\theta, t)x \neq 0$, for all $t \geqslant 0, \theta \in \Theta$.

Proof. Assume for a contradiction that there exist $t_0 \geqslant 0$ and $\theta_0 \in \Theta$ such that $\Phi(\theta_0, t_0)x = 0$. Then

$$\Phi(\theta_0, t_0 + s)x = \Phi\bigl(\sigma(\theta_0, s), s\bigr)\Phi(\theta_0, t_0)x = 0,$$

for each $s \geqslant 0$. It follows that $x \in \mathcal{X}_{1,F}(\theta_0)$ and thus $x = 0$. $\square$

Definition 4.1. The pair $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ is said to be admissible to an exponentially bounded, strongly continuous cocycle $\{\Phi(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$) if for each $f \in E(\mathbb{R}_+, X)$ and $\theta \in \Theta$, there exists $x \in X$ such that

$$u(\cdot; \theta, x, f) : \mathbb{R}_+ \to X, \quad u(t; \theta, x, f) = \Phi(\theta, t)x + \int_0^t \Phi\bigl(\sigma(\theta, s), t - s\bigr) f(s)\, ds$$

belongs to $F(\mathbb{R}_+, X)$.

Definition 4.2. An exponentially bounded, strongly continuous cocycle $\{\Phi(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$) has an **exponential dichotomy** if there exists a family of projectors $\{P(\theta)\}_{\theta\in\Theta}$ (i.e. $P(\theta) \in B(X)$, and $P^2(\theta) = P(\theta)$, for each $\theta \in \Theta$) such that

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$ (the invariance property).
(ii) $\Phi(\theta, t) : \mathrm{Ker}\, P(\theta) \to \mathrm{Ker}\, P(\sigma(\theta, t))$ is an isomorphism, for each $(\theta, t) \in \Theta \times \mathbb{R}_+$.

(iii) There exist constants $N, \nu > 0$, such that
- $\|\Phi(\theta,t)P(\theta)x\| \leqslant Ne^{-\nu t}\|P(\theta)x\|$, $(\theta,t) \in \Theta \times \mathbb{R}_+$ and $x \in X$;
- $\|\Phi^{-1}(\sigma(\theta,t))Q(\theta)x\| \leqslant \frac{1}{N}e^{-\nu t}\|Q(\theta)x\|$, $(\theta,t) \in \Theta \times \mathbb{R}_+$ and $x \in X$.

The following definition of exponential dichotomy for an exponentially bounded, strongly continuous cocycle (over a semiflow) is weaker than Definition 4.2.

Definition 4.3. An exponentially bounded, strongly continuous cocycle $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$) has a **no past exponential dichotomy** if there exist a family of projectors $\{P(\theta)\}_{\theta\in\Theta}$ and the constants $N_1, N_2, \nu_1, \nu_2 > 0$ such that the following conditions hold:

(i) $\|\Phi(\theta,t)x\| \leqslant N_1 e^{-\nu_1 t}\|x\|$, for all $t \geqslant 0$, $\theta \in \Theta$ and $x \in \operatorname{Im} P(\theta)$;
(ii) $\|\Phi(\theta,t)x\| \geqslant N_2 e^{\nu_2 t}\|x\|$, for all $t \geqslant 0$, $\theta \in \Theta$ and $x \in \operatorname{Ker} P(\theta)$.

Remark 4.2. As it is known, **exponential dichotomy** means that X can be decomposed, at every $\theta \in \Theta$, as a direct sum between two subspaces such that solutions (of the variational equation (3)) starting in the first subspace (respectively, in the second one) decay exponentially in forward time (respectively, in backward time). Assuming the existence of an **exponential dichotomy** we practically force the solutions that starts in the second subspace to exist for negative time. However there are situations which require to drop off this requirement and to replace the exponential decay in negative time for the solutions starting in the second subspace with an exponential blow-up in positive time (we called *ad hoc* this behavior as a **no past exponential dichotomy**).

Remark 4.3. It is obvious that the existence of an **exponential dichotomy** implies the existence of a **no past exponential dichotomy**. However, for infinite-dimensional subspaces $\operatorname{Im} Q(\theta)$, the inequality (ii) of Definition 4.3 does not imply the second inequality in condition (iii) of Definition 4.2. Assuming $\dim \operatorname{Im} Q(\theta) < \infty$ (and condition (i) in Definition 4.2) we get an equivalence between the two definitions.

Lemma 4.1. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a function with the property that there exist $H > 0$, $\delta > 0$, $\eta > 1$ such that*:

(i) *$h(t) \geqslant Hh(t_0)$, for all $t \in [t_0, t_0+\delta]$, $t_0 \geqslant 0$;*
(ii) *$h(t_0+\delta) \geqslant \eta h(t_0)$, for all $t_0 \geqslant 0$.*

Then there exist two constants $N, \nu > 0$ such that

$$h(t) \geqslant Ne^{\nu(t-t_0)}h(t_0), \quad \textit{for all } t \geqslant t_0 \geqslant 0.$$

Proof. See [11, 20C, p. 39]. □

Lemma 4.2. *If $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following conditions*:

(i) *$h_1(t) \leqslant h_1(s)h_2(t-s)$ for all $t \geqslant s \geqslant 0$;*
(ii) *$\sup_{t\in[0,a]} h_2(s) < \infty$, for all $a > 0$;*
(iii) *$\inf_{t\geqslant 0} h_2(t) < 1$,*

then there exist two constants $N, \nu > 0$ *such that*

$$h_1(t) \leqslant N e^{-\nu(t-s)} h_1(s) \quad \textit{for all } t \geqslant s \geqslant 0.$$

Proof. See [19, Lemma 3.4, p. 202]. □

5. Results

Proposition 5.1. *If* $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ *is admissible to an exponentially bounded, strongly continuous cocycle* $\{\Phi(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$ *(over a semiflow* $\{\sigma(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$*) then for every* $f \in E(\mathbb{R}_+, X)$ *and* $\theta \in \Theta$ *there is a unique* $x_2 \in \mathcal{X}_{2,F}(\theta)$ *such that* $u(\cdot;\theta, x_2, f) \in F(\mathbb{R}_+, X)$.

Proof. Let $f \in E(\mathbb{R}_+, X)$. Since $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ is admissible to $\{\Phi(\theta, t)\}_{\theta\in\Theta,\, t\geqslant 0}$ it follows that there exists $x \in X$ such that

$$u(\cdot;\theta, x, f) : \mathbb{R}_+ \to X, \quad u(t;\theta, x, f) = \Phi(\theta, t)x + \int_0^t \Phi\bigl(\sigma(\theta, s), t-s\bigr) f(s)\, ds$$

belongs to $F(\mathbb{R}_+, X)$, for each $\theta \in \Theta$. Denoting by $v(t;\theta, x) = \Phi(\theta, t)P_F(\theta)x$ and $z(t;\theta, x, f) = u(t;\theta, x, f) - v(t;\theta, x)$ we have that $z(t;\theta, x, f) \in F(\mathbb{R}_+, X)$ with

$$z(t;\theta, x, f) = \Phi(\theta, t)Q_F(\theta)x + \int_0^t \Phi\bigl(\sigma(\theta, s), t-s\bigr) f(s)\, ds.$$

The uniqueness follows easily using a simple proof by contradiction. □

Given $f \in E(\mathbb{R}_+, X)$ *we will denote, throughout of this paper, the unique vector* $x_2 \in \mathcal{X}_{2,F}(\theta)$ *by* x_f.

Proposition 5.2. *If* $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ *is admissible to an exponentially bounded, strongly continuous cocycle* $\{\Phi(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$ *(over a semiflow* $\{\sigma(\theta, t)\}_{\theta\in\Theta, t\geqslant 0}$*) then for each* $\theta \in \Theta$ *there exists* $K(\theta) > 0$ *such that*

$$\bigl\| u(\cdot;\theta, x_f, f)\bigr\|_{F(\mathbb{R}_+,X)} \leqslant K(\theta)\|f\|_{E(\mathbb{R}_+,X)}$$

and

$$\|x_f\| \leqslant K(\theta)\|f\|_{E(\mathbb{R}_+,X)}.$$

Proof. Let $\theta \in \Theta$. We define

$$U_\theta : E(\mathbb{R}_+, X) \to \mathcal{X}_{2,F}(\theta) \oplus F(\mathbb{R}_+, X), \quad U_\theta(f) = \bigl(x_f, u(\cdot;\theta, x_f, f)\bigr).$$

It is obvious that U_θ is a linear operator. We will show that U_θ is also closed. Let $\{f_n\}_{n\in\mathbb{N}} \subset E(\mathbb{R}_+, X)$, $f \in E(\mathbb{R}_+, X)$, $g \in F(\mathbb{R}_+, X)$ such that

$$f_n \xrightarrow{E(\mathbb{R}_+,X)} f, \quad \big(U_\theta(f_n)\big)_n \xrightarrow{\mathcal{X}_{2,F}(\theta)\oplus F(\mathbb{R}_+,X)} (y, g).$$

Then

$$\lim_{n\to\infty} \|f_n - f\|_{E(\mathbb{R}_+,X)} = \lim_{n\to\infty} \|x_{f_n} - y\| = \lim_{n\to\infty} \big\|u(\cdot;\theta, x_{f_n}, f_n) - g\big\|_{F(\mathbb{R}_+,X)} = 0.$$

Taking into account that

$$\begin{aligned}\left\|\int_0^t \Phi\big(\sigma(\theta,s), t-s\big)\big(f_n(s) - f(s)\big)\,ds\right\| &\leqslant Me^{\omega t}\int_0^t \|f_n(s) - f(s)\|\,ds\\ &\leqslant Me^{\omega t}\alpha(t,E)\|f_n - f\|_{E(\mathbb{R}_+,X)},\end{aligned}$$

we have that

$$\lim_{n\to\infty}\int_0^t \Phi\big(\sigma(\theta,s), t-s\big) f_n(s)\,ds = \int_0^t \Phi\big(\sigma(\theta,s), t-s\big) f(s)\,ds.$$

By Remark 2.5 we have that there exists a subsequence $(f_{n_k}) \subset (f_n)$, $f_{n_k} \to f$ a.e. Since

$$u(\cdot;\theta, x_{f_{n_k}}, f_{n_k}) = \Phi(\theta,t)x_{f_{n_k}} + \int_0^t \Phi\big(\sigma(\theta,s), t-s\big) f_{n_k}(s)\,ds,$$

we have that

$$g(t) = \Phi(\theta,t)y + \int_0^t \Phi\big(\sigma(\theta,s), t-s\big) f(s)\,ds.$$

This proves that $u(\cdot;\theta, x_f, f) = g$ and $x_f = y$. Thus U_θ is a closed linear operator and by the Closed-Graph Theorem it is also bounded. It follows that there exists $K(\theta) > 0$ such that

$$\big\|u(\cdot;\theta, x_f, f)\big\|_{F(\mathbb{R}_+,X)} \leqslant K(\theta)\|f\|_{E(\mathbb{R}_+,X)}$$

and

$$\|x_f\| \leqslant K(\theta)\|f\|_{E(\mathbb{R}_+,X)}. \qquad \square$$

Definition 5.1. $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ is said to be uniformly admissible to the exponentially bounded, strongly continuous cocycle $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$) if $\sup_{\theta\in\Theta} K(\theta) = K < \infty$.

Theorem 5.1. *If* $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ *is uniformly admissible to an exponentially bounded, strongly continuous cocycle* $\{\Phi(\theta, t)\}_{\theta\in\Theta,\, t\geqslant 0}$ *(over a semiflow* $\{\sigma(\theta, t)\}_{\theta\in\Theta,\, t\geqslant 0}$*) and*

$$\lim_{t\to\infty} \alpha(t, E)\beta(t, F) = \infty,$$

then $\{\Phi(\theta, t)\}_{\theta\in\Theta,\, t\geqslant 0}$ *has a* ***no past exponential dichotomy***.

Proof. Let $x \in X$ with $Q_F(\theta)x \neq 0$, $t_0 \geqslant 0$, and

$$f_\theta(t) = \varphi_{[t_0,t_0+1]}(t)\frac{\Phi(\theta, t)Q_F(\theta)x}{\|\Phi(\theta, t)Q_F(\theta)x\|}.$$

By Remark 4.1 it follows that the above function is well defined.

We can see that $\|f_\theta(t)\| \leqslant \varphi_{[t_0,t_0+1]}(t)$, for any $t \geqslant 0$. This shows that $f_\theta \in E(\mathbb{R}_+, X)$ and $\|f_\theta\|_{E(\mathbb{R}_+,X)} \leqslant \beta(1, E)$. But

$$\begin{aligned}
v(t;\theta,x) &= -\int_t^\infty \varphi_{[t_0,t_0+1]}(s)\frac{ds}{\|\Phi(\theta,s)Q_F(\theta)x\|}\Phi(\theta,t)Q_F(\theta)x \\
&= -\int_0^\infty \varphi_{[t_0,t_0+1]}(s)\frac{ds}{\|\Phi(\theta,s)Q_F(\theta)x\|}\Phi(\theta,t)Q_F(\theta)x \\
&\quad + \int_0^t \Phi\big(\sigma(\theta,s), t-s\big)f_\theta(s)\,ds \\
&= -\Phi(\theta,t)\left(\int_0^\infty \varphi_{[t_0,t_0+1]}(s)\frac{ds}{\|\Phi(\theta,s)Q_F(\theta)x\|}Q_F(\theta)x\right) \\
&\quad + \int_0^t \Phi\big(\sigma(\theta,s), t-s\big)f_\theta(s)\,ds \\
&= \begin{cases} 0, & t \geqslant t_0+1, \\ -\int_t^{t_0+1} \frac{ds}{\|\Phi(\theta,s)Q_F(\theta)x\|}\Phi(\theta,t)Q_F(\theta)x, & t_0 < t < t_0+1, \\ -\int_{t_0}^{t_0+1} \frac{ds}{\|\Phi(\theta,s)Q_F(\theta)x\|}\Phi(\theta,t)Q_F(\theta)x, & t \leqslant t_0. \end{cases}
\end{aligned}$$

It follows that $v(\cdot;\theta,x) \in F(\mathbb{R}_+, X)$ and by using that $v(0;\theta,x) \in \mathcal{X}_{2,F}(\theta)$ we get that $v(t;\theta,x) = u(t;\theta,x_{f_\theta}, f_\theta)$. Thus

$$\big\|u(t;\theta,x_{f_\theta}, f_\theta)\big\|_{F(\mathbb{R}_+,X)} \leqslant K\beta(1,E) \quad \text{and} \quad \|x_{f_\theta}\| \leqslant K\beta(1,E).$$

But

$$\begin{aligned}
u(t;\theta,x_{f_\theta},f_\theta) &= \Phi(\theta,t)x_{f_\theta} + \int_0^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau \\
&= \Phi\big(\sigma(\theta,s),t-s\big)\Phi(\theta,s)x_{f_\theta} \\
&\quad + \int_0^s \Phi\big(\sigma\big(\sigma(\theta,\tau),s-\tau\big),t-s\big)\Phi\big(\sigma(\theta,s),t-s\big)f_\theta(\tau)\,d\tau \\
&\quad + \int_s^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau \\
&= \Phi\big(\sigma(\theta,s),t-s\big)\Phi(\theta,s)x_{f_\theta} \\
&\quad + \int_0^s \Phi\big(\sigma(\theta,s),t-s\big)\Phi\big(\sigma(\theta,\tau),s-\tau\big)f_\theta(\tau)\,d\tau \\
&\quad + \int_s^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau \\
&= \Phi\big(\sigma(\theta,s),t-s\big)u(s;\theta,x_{f_\theta},f_\theta) + \int_s^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau,
\end{aligned}$$

for each $0 \leqslant s \leqslant t$. If $t \geqslant 1$ and $s \in [t-1,t]$ we have that

$$\big\|u(t;\theta,x_{f_\theta},f_\theta)\big\| \leqslant Me^{\omega}\big\|u(s;\theta,x_{f_\theta},f_\theta)\big\| + \int_s^t Me^{\omega}\big\|f_\theta(\tau)\big\|\,d\tau.$$

Thus, for each $t \geqslant 1$ we have that:

$$\begin{aligned}
\big\|u(t;\theta,x_{f_\theta},f_\theta)\big\| &\leqslant Me^{\omega}\int_{t-1}^t \big\|u(s;\theta,x_{f_\theta},f_\theta)\big\|\,ds + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)} \\
&\leqslant Me^{\omega}\alpha(1,F)\big\|u(\cdot;\theta,x_{f_\theta},f_\theta)\big\|_{F(\mathbb{R}_+,X)} + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}.
\end{aligned}$$

Taking now $t \in [0,1]$ we have that

$$\begin{aligned}
\big\|u(t;\theta,x_{f_\theta},f_\theta)\big\| &\leqslant Me^{\omega}\int_0^1 \big\|u(s;\theta,x_{f_\theta},f_\theta)\big\|\,ds + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)} \\
&\leqslant Me^{\omega}\alpha(1,F)\big\|u(\cdot;\theta,x_{f_\theta},f_\theta)\big\|_{F(\mathbb{R}_+,X)} + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}.
\end{aligned}$$

Thus, for any $t \geqslant 0$ we have that:

$$\begin{aligned}\big\|u(t;\theta,x_{f_\theta},f_\theta)\big\| &\leqslant Me^{\omega}\big(\alpha(1,F)\big\|u(\cdot;\theta,x_{f_\theta},f_\theta)\big\|_{F(\mathbb{R}_+,X)}+\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}\big)\\ &\leqslant L=Me^{\omega}\big(K\alpha(1,F)+\alpha(1,E)\big)\beta(1,E).\end{aligned}$$

Thus

$$\int_{t_0}^{t_0+1}\frac{d\tau}{\|\Phi(\theta,\tau)Q_F(\theta)x\|}\big\|\Phi(\theta,t)Q_F(\theta)x\big\|\leqslant L,$$

for any $t\leqslant t_0$. But

$$\big\|\Phi(\theta,\tau)Q_F(\theta)x\big\|=\big\|\Phi\big(\sigma(\theta,t_0),\tau-t_0\big)\Phi(\theta,t_0)Q_F(\theta)x\big\|\leqslant Me^{\omega}\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|,$$

for any $t\in[t_0,t_0+1]$. It follows that

$$\frac{1}{\|\Phi(\theta,t_o)Q_F(\theta)x\|}\leqslant Me^{\omega}\int_{t_0}^{t_0+1}\frac{d\tau}{\|\Phi(\theta,\tau)Q_F(\theta)x\|}$$

and from here we have that

$$\big\|\Phi(\theta,t)Q_F(\theta)x\big\|\leqslant Me^{\omega}L\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|,$$

for any $0\leqslant t\leqslant t_0$, or equivalent

$$\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|\leqslant Me^{\omega}L\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|,$$

for any $0\leqslant t_0\leqslant t$.

Let now $t_0\geqslant 0$, $\delta>0$ and

$$g_\delta(t)=\varphi_{[t_0,t_0+\delta]}(t)\frac{\Phi(\theta,t)Q_F(\theta)x}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|}.$$

We have that $\|g_\delta(t)\|\leqslant Me^{\omega}L\varphi_{[t_0,t_0+\delta]}(t)$, for any $t\geqslant 0$, which shows that $g_\delta\in E(\mathbb{R}_+,X)$ and $\|g_\delta\|_{E(\mathbb{R}_+,X)}\leqslant Me^{\omega}L\beta(\delta,E)$. Consider now

$$\begin{aligned}z(t;\theta,x)&=-\int_t^{\infty}\varphi_{[t_0,t_0+\delta]}(s)\frac{ds}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|}\Phi(\theta,t)Q_F(\theta)x\\ &=-\Phi(\theta,t)\left(\int_0^{\infty}\varphi_{[t_0,t_0+\delta]}(s)\frac{ds}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|}Q_F(\theta)x\right)\end{aligned}$$

$$+\int_0^t \Phi\big(\sigma(\theta,s),t-s\big)g_\theta(s)\,ds$$

$$=\begin{cases} 0, & t \geqslant t_0+\delta, \\ -(t_0+\delta-t)\frac{\Phi(\theta,t)Q_F(\theta)x}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|}, & t_0<t<t_0+\delta, \\ -\delta\frac{\Phi(\theta,t)Q_F(\theta)x}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|}, & t\leqslant t_0. \end{cases}$$

Then $z(\cdot;\theta,x)\in F(\mathbb{R}_+,X)$ and $z(0;\theta,x)\in\mathcal{X}_{2,F}(\theta)$. Thus it follows that

$$\big\|z(\cdot;\theta,x)\big\|_{F(\mathbb{R}_+,X)}\leqslant KMe^{\omega}L\beta(\delta,E).$$

Integrating on $[t_0,t_0+\delta]$ and using (1), we have that

$$\begin{aligned}\frac{\delta^2}{2}\frac{\|\Phi(\theta,t)Q_F(\theta)x\|}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|} &= \int_{t_0}^{t_0+\delta}(t_0+\delta-s)\,ds\,\frac{\|\Phi(\theta,t)Q_F(\theta)x\|}{\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\|} \\ &\leqslant MLe^{\omega}\int_{t_0}^{t_0+\delta}\big\|z(s;\theta,x)\big\|\,ds \\ &\leqslant KLMe^{\omega}\alpha(\delta,F)\big\|z(\cdot;\theta,x)\big\|_{F(\mathbb{R}_+,X)} \\ &\leqslant \alpha(\delta,F)K^2Me^{2\omega}L^2\beta(\delta,E) \\ &\leqslant \frac{2\delta}{\beta(\delta,F)}K^2Me^{2\omega}L^2\frac{2\delta}{\alpha(\delta,E)}=\frac{4K^2M^2e^{2\omega}L^2\delta^2}{\alpha(\delta,E)\beta(\delta,F)}.\end{aligned}$$

Thus we have that

$$\big\|\Phi(\theta,t_0+\delta)Q_F(\theta)x\big\|\geqslant\frac{1}{8K^2M^2L^2e^{2\omega}}\alpha(\delta,E)\beta(\delta,F)\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|.$$

Therefore we can choose $\delta_0>0$ such that

$$\big\|\Phi(\theta,t_0+\delta_0)Q_F(\theta)x\big\|\geqslant 2\big\|\Phi(\theta,t_0)Q_F(\theta)x\big\|,$$

for each $t_0\geqslant 0$. Using Lemma 4.1 we have that there exist $N_2,\nu_2>0$ such that

$$\big\|\Phi(\theta,t)Q_F(\theta)x\big\|\geqslant N_2e^{\nu_2 t}\big\|Q_F(\theta)x\big\|,$$

for all $t\geqslant 0$ and $x\in X$.

Let $x\in X$, $t_0\geqslant 0$ and $\theta\in\Theta$. We define the map

$$f_\theta(t)=\varphi_{[t_0,t_0+1]}(t)\Phi(\theta,t)P_F(\theta)x.$$

Thus, we have

$$\begin{aligned}\|f_\theta(t)\| &= \varphi_{[t_0,t_0+1]}(t)\big\|\Phi\big(\sigma(\theta,t_0),t-t_0\big)\Phi(\theta,t_0)P_F(\theta)x\big\| \\ &\leqslant \varphi_{[t_0,t_0+1]}(t)Me^{\omega}\big\|\Phi(\theta,t_0)P_F(\theta)x\big\|, \quad \text{for all } t\geqslant 0.\end{aligned}$$

Then $f_\theta \in E(\mathbb{R}_+,X)$ and $\|f_\theta\|_{E(\mathbb{R}_+,X)} \leqslant Me^{\omega}\|\Phi(\theta,t_0)P_F(\theta)x\|\beta(1,E)$.

Taking the function

$$\begin{aligned} v(t;\theta) &= \int_0^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau \\ &= \int_0^t \varphi_{[t_0,t_0+1]}(\tau)\Phi(\theta,t)P_F(\theta)x\,d\tau \\ &= \begin{cases} 0, & t<t_0, \\ (t-t_0)\Phi(\theta,t)P_F(\theta)x, & t_0\leqslant t\leqslant t_0+1, \\ \Phi(\theta,t)P_F(\theta)x, & t>t_0+1, \end{cases}\end{aligned}$$

it follows that $v(\cdot;\theta)\in F(\mathbb{R}_+,X)$. Taking into account that $v(0;\theta)\in\mathcal{X}_{2,F}(\theta)$ we have that

$$\big\|v(\cdot;\theta)\big\|_{F(\mathbb{R}_+,X)} \leqslant K\|f_\theta\|_{E(\mathbb{R}_+,X)} \leqslant KMe^{\omega}\beta(1,E)\big\|\Phi(\theta,t_0)P_F(\theta)x\big\|.$$

We can see that

$$\begin{aligned} v(t;\theta) &= \int_0^s \Phi\big(\sigma\big(\sigma(\theta,\tau),s-\tau\big),t-s\big)\Phi\big(\sigma(\theta,\tau),s-\tau\big)f_\theta(\tau)\,d\tau \\ &\quad + \int_s^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau \\ &= \Phi\big(\sigma(\theta,s),t-s\big)\int_0^s \Phi\big(\sigma(\theta,s),s-\tau\big)f_\theta(\tau)\,d\tau \\ &\quad + \int_s^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau,\end{aligned}$$

for all $t\geqslant s\geqslant 0$. If we choose $t\geqslant 1$ and $s\in[t-1,t]$, we have that

$$\begin{aligned}\big\|v(t;\theta)\big\| &\leqslant Me^{\omega}\big\|v(s;\theta)\big\| + Me^{\omega}\int_s^t \big\|f_\theta(\tau)\big\|\,d\tau \\ &\leqslant Me^{\omega}\big\|v(s;\theta)\big\| + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}.\end{aligned}$$

Integrating on $[t-1,t]$ we have that

$$\begin{aligned}\|v(t;\theta)\| &\leqslant Me^{\omega}\int_{t-1}^{t}\|v(s;\theta)\|\,ds + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}\\ &\leqslant Me^{\omega}\alpha(1,F)\|v(\cdot;\theta)\|_{F(\mathbb{R}_+,X)} + Me^{\omega}\alpha(1,E)\|f_\theta\|_{E(\mathbb{R}_+,X)}\\ &\leqslant Me^{\omega}\big(\alpha(1,F)K+\alpha(1,E)\big)\|f_\theta\|_{E(\mathbb{R}_+,X)}.\end{aligned}$$

If we choose now $t \geqslant t_0+1$ we obtain that

$$\|\Phi(\theta,t)P_F(\theta)x\| \leqslant Me^{\omega}\big(\alpha(1,F)K+\alpha(1,E)\big)Me^{\omega}\|\Phi(\theta,t_0)P_F(\theta)x\|.$$

If we let $t\in[t_0,t_0+1)$ we have that

$$\|\Phi(\theta,t)P_F(\theta)x\| = \|\Phi\big(\sigma(\theta,t_0),t-t_0\big)\Phi(\theta,t_0)P_F(\theta)x\| \leqslant Me^{\omega}\|\Phi(\theta,t_0)P_F(\theta)x\|.$$

Denoting by $L' = \max\{M^2e^{2\omega}(\alpha(1,F)K+\alpha(1,E)), Me^{\omega}\}$ we have that

$$\|\Phi(\theta,t)P_F(\theta)x\| \leqslant L'\|\Phi(\theta,t_0)P_F(\theta)x\|,$$

for all $t\geqslant t_0\geqslant 0$ and $\theta\in\Theta$.

If $\delta>0$ and we set $g_\theta(t)=\varphi_{[t_0,t_0+\delta]}(t)\Phi(\theta,t)P_F(\theta)x$, then we have

$$\|g_\theta(t)\| = \varphi_{[t_0,t_0+\delta]}(t)L'\|\Phi(\theta,t_0)P_F(\theta)x\|.$$

Thus we have $g_\theta\in E(\mathbb{R}_+,X)$ and $\|g_\theta\|_{E(\mathbb{R}_+,X)}\leqslant\beta(\delta,E)L'\|\Phi(\theta,t_0)P_F(\theta)x\|$. We set

$$\begin{aligned}z(t;\theta) &= \int_0^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)g_\theta(\tau)\,d\tau\\ &= \begin{cases}0, & t<t_0,\\ (t-t_0)\Phi(\theta,t)P_F(\theta)x, & t_0\leqslant t\leqslant t_0+\delta,\\ \delta\Phi(\theta,t)P_F(\theta)x, & t>t_0+\delta.\end{cases}\end{aligned}$$

It follows that $z(\cdot;\theta)\in F(\mathbb{R}_+,X)$. Taking into account now that $z(0;\theta)=0\in\mathcal{X}_{2,F}(\theta)$, we obtain that

$$\|z(\cdot;\theta)\|_{F(\mathbb{R}_+,X)} \leqslant KL\beta(\delta,E)\|\Phi(\theta,t_0)P_F(\theta)x\|.$$

Using again (1) we have that

$$
\begin{aligned}
\frac{\delta^2}{2}\left\|\Phi(\theta, t_0+\delta)P_F(\theta)x\right\| &= \int_{t_0}^{t_0+\delta}(s-t_0)\left\|\Phi(\theta, t_0+\delta)P_F(\theta)x\right\| ds \\
&\leqslant \int_{t_0}^{t_0+\delta}(s-t_0)L'\left\|\Phi(\theta, s)P_F(\theta)x\right\| ds \\
&= L'\int_{t_0}^{t_0+\delta}\left\|z(s;\theta)\right\| ds \leqslant L\alpha(\delta, F)\left\|z(\cdot;\theta)\right\|_{F(\mathbb{R}_+,X)} \\
&\leqslant KL\alpha(\delta, F)\beta(\delta, E)\left\|\Phi(\theta, t_0)P_F(\theta)x\right\| \\
&\leqslant KL\frac{4\delta^2}{\beta(\delta, F)\alpha(\delta, E)}\left\|\Phi(\theta, t_0)P_F(\theta)x\right\|.
\end{aligned}
$$

It follows that

$$\left\|\Phi(\theta, t_0+\delta)P_F(\theta)x\right\| \leqslant \frac{8KL}{\alpha(\delta, E)\beta(\delta, F)}\left\|\Phi(\theta, t_0)P_F(\theta)x\right\|,$$

for all $t_0 \geqslant 0$, $\theta \in \Theta$ and $\delta > 0$. Thus there exists $\delta_0 > 0$ such that

$$\left\|\Phi(\theta, t_0+\delta_0)P_F(\theta)x\right\| \leqslant \frac{1}{2}\left\|\Phi(\theta, t_0)P_F(\theta)x\right\|,$$

for each $t_0 \geqslant 0$ and $\theta \in \Theta$. Using now Lemma 4.2 we have that there exist $N_1, \nu_1 > 0$ such that

$$\left\|\Phi(\theta, t)P_F(\theta)x\right\| \leqslant N_1 e^{-\nu_1 t}\left\|P_F(\theta)x\right\|,$$

for each $t \geqslant 0$, $\theta \in \Theta$ and $x \in X$. $\square$

Remark 5.1. The reader will find an impressive list of papers by screening the literature regarding the connection between the *admissibility* of some function spaces, and the existence of an *exponential dichotomy* for dynamical systems. The milestone of this subject is the paper by O. Perron (see [14]) from 30's, where he establishes for the first time an equivalence between the condition that the non-homogeneous equation has some bounded solution for every bounded "second member" on the one hand and a certain form of conditional stability of the solutions of the homogeneous equation on the other.

This concept was called "admissibility" (or the "test function method" or "Perron's method") and it was extended in the more general framework of infinite-dimensional Banach spaces by J.L. Daleckij and M.G. Krein [7], J.L. Massera and J.J. Schäffer [11], and more recently by C. Chicone and Y. Latushkin [2], Nguyen van Minh, F. Räbiger and R. Schnaubelt [26], Nguyen Thieu Huy [9]. For more details, we also refer the reader to [17–20] and the references therein.

Latest, it is known the equivalence between the admissibility of the pair $(L^p(\mathbb{R}_+, X), L^p(\mathbb{R}_+, X))$, $1 \leqslant p \leqslant q \leqslant \infty$ and $(p,q) \neq (1,\infty)$ and the *exponential dichotomy* of a two-parameter evolution family $\{U(t,t_0)\}_{t \geqslant t_0 \geqslant 0}$ when we assume *a priori* that there exists a family of projectors $\{P(t)\}_{t\in\mathbb{R}_+}$ such that

- $U(t,t_0)P(t_0) = P(t)U(t,t_0)$, for all $t \geqslant t_0 \geqslant 0$;
- $U(t,t_0) : \operatorname{Ker} P(t_0) \to \operatorname{Ker} P(t)$ is an isomorphism, for all $t \geqslant t_0 \geqslant 0$.

The above equivalence has been proved by Nguyen van Minh, F. Räbiger and R. Schnaubelt in [26] by associating the evolution semigroup on $(L^p(\mathbb{R}_+, X)$. Also, a direct proof (i.e. by choosing the appropriate "test functions") was obtained in [18, Theorem 3.9]. For related results we refer the reader to [19,20].

Theorem 5.1 extends the above results in few directions. First, we analyze the case of an exponentially bounded, strongly continuous cocycle (over a semiflow) which extends the classical notion of a two-parameter family (see Example 3.5). Also, most important is that we do **not** assume *a priori* that the family of projectors $\{P(\theta)\}_{\theta\in\Theta}$ satisfy the restrictive requirements:

- $\Phi(\theta,t)P(\theta) = P(\sigma(\theta,t))\Phi(\theta,t)$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$;
- $\Phi(\theta,t) : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta,t))$ is an isomorphism, for every $(\theta,t) \in \Theta \times \mathbb{R}_+$;

and still we succeed to prove that the admissibility of any pair of vector-valued Schäffer function spaces (satisfying a certain technical condition) implies the existence of a (no past) exponential dichotomy. Also it is worth to note that the class of vector-valued Schäffer function spaces is extremely large (see for instance Examples 2.1, 2.2) and this fact allows the reader to choose the "test functions" in various ways and in the same time it does not force the *output* (i.e. the solution of the inhomogeneous problem) to stay in $L^p(\mathbb{R}_+, X)$, as before. Moreover, this approach can provide "small" *input spaces* (i.e. the spaces consisting in "test functions") which are obviously more convenient in the admissibility condition. Taking for instance $\Phi(t) = e^t - 1$ in Example 2.2, we observe that the corresponding scalar-valued Orlicz function space $L^\Phi(\mathbb{R}_+, \mathbb{R}) \subset L^p(\mathbb{R}_+, \mathbb{R})$, for all $p \in [1,\infty)$. Moreover, there is no $p \in [1,\infty)$, such that $L^\Phi(\mathbb{R}_+, \mathbb{R}) = L^p(\mathbb{R}_+, \mathbb{R})$.

We also prove that if there exists a pair of vector-valued Schäffer function spaces, $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$, which is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ and with the property that $\lim_{t\to\infty} \alpha(t,E)\beta(t,F) = \infty$ then the fiber $\mathcal{X}_{1,F}(\theta)$ (which induces a (no past) exponential dichotomy) is always the same fiber $\mathcal{X}_{1,L^\infty}(\theta)$ (see Remark 5.2 below). More interesting is the result from Theorem 5.2 below, that is if we assume in addition the invariance property (i.e. condition (i) in Definition 4.2) then the above admissibility condition implies the invertibility of the operators $\{\Phi(\theta,t)\}$ on the unstable fiber (i.e. the complement of $\mathcal{X}_{1,L^\infty}(\theta)$). Thus we can conclude with the following schema:

- uniform admissibility $\Rightarrow$ no past exponential dichotomy,
- uniform admissibility + invariance property $\Rightarrow$ exponential dichotomy.

Equivalences are also established in Theorems 5.3 and 5.4. below.

Let now $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ be a pair of vector-valued Schäffer function spaces.

Remark 5.2. If $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ is uniformly admissible to an exponentially bounded, strongly continuous cocycle $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$) and $\lim_{t\to\infty}\alpha(t,E)\beta(t,F)=\infty$, then

$$\mathcal{X}_{1,F}(\theta)=\mathcal{X}_{1,L^\infty}(\theta)=\bigl\{x\in X\colon\ \Phi(\theta,\cdot)\in L^\infty(\mathbb{R}_+,X)\bigr\}.$$

Proof. Indeed if we choose $x\in\mathcal{X}_{1,F}(\theta)$, we have (from the above theorem) that there exist $N_1,\nu_1>0$ such that

$$\bigl\|\Phi(\theta,t)x\bigr\|\leqslant N_1e^{-\nu_1 t}\|x\|,$$

for each $t\geqslant 0$ and $\theta\in\Theta$. It follows that $x\in\mathcal{X}_{1,L^\infty}(\theta)$.

Take now $x\in\mathcal{X}_{1,L^\infty}(\theta)$ and assume for a contradiction that $Q_F(\theta)x\neq 0$. Then there exist $N_2,\nu_2>0$ such that

$$\bigl\|\Phi(\theta,t)Q_F(\theta)x\bigr\|\geqslant N_2e^{\nu_2 t}\bigl\|Q_F(\theta)x\bigr\|.$$

It follows that

$$\begin{aligned}\bigl\|\Phi(\theta,t)x\bigr\|&\geqslant\bigl\|\Phi(\theta,t)Q_F(\theta)x\bigr\|-\bigl\|\Phi(\theta,t)P_F(\theta)x\bigr\|\\&\geqslant N_2e^{\nu_2 t}\bigl\|Q_F(\theta)x\bigr\|-N_1e^{-\nu_1 t}\bigl\|P_F(\theta)x\bigr\|.\end{aligned}$$

This shows that $\Phi(\theta,\cdot)\notin L^\infty(\mathbb{R}_+,X)$ and thus we get the contradiction. Then $Q_F(\theta)x=0$ which implies that $x\in\mathcal{X}_{1,F}(\theta)$. This ends the proof. □

Let again $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ be a pair of vector-valued Schäffer function spaces.

Theorem 5.2. *Assume that $(E(\mathbb{R}_+, X), F(\mathbb{R}_+, X))$ is uniformly admissible to an exponentially bounded, strongly continuous cocycle $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$) with $\lim_{t\to\infty}\alpha(t,E)\beta(t,F)=\infty$. If $\Phi(\theta,t)P_{L^\infty}(\theta)=P_{L^\infty}(\sigma(\theta,t))\Phi(\theta,t)$ then*

$$\Phi(\theta,t):\operatorname{Ker}P_{L^\infty}(\theta)\to\operatorname{Ker}P_{L^\infty}\bigl(\sigma(\theta,t)\bigr),$$

is invertible for all $(\theta,t)\in\Theta\times\mathbb{R}_+$.

Proof. Let $(\theta,t_0)\in\Theta\times\mathbb{R}_+$ and $x\in\operatorname{Ker}P_{L^\infty}(\theta)$ such that $\Phi(\theta,t_0)x=0$. From the above theorem it follows that there exist $N_2,\nu_2>0$ such that $\|\Phi(\theta,t_0)x\|\geqslant N_2e^{\nu_2 t_0}\|x\|$. Thus we obtain that $x=0$ and from here it follows that $\Phi(\theta,t_0)$ is one-to-one when the domain is restricted to $\operatorname{Ker}P_{L^\infty}(\theta)$.

Take now $y\in\operatorname{Ker}P_{L^\infty}(\sigma(\theta,t_0))$, and set

$$f_\theta(t)=\begin{cases}0, & t\in[0,t_0],\\-\Phi(\sigma(\theta,t_0),t-t_0)y, & t\in(t_0,t_0+1],\\0, & t>t_0+1.\end{cases}$$

Since $\|f_\theta(t)\|\leqslant Me^\omega\|y\|\varphi_{[t_0,t_0+1]}(t)$, for each $t\geqslant 0$, it follows that $f_\theta\in E(\mathbb{R}_+,X)$ and $\|f_\theta\|_{E(\mathbb{R}_+,X)}\leqslant Me^\omega\|y\|\beta(1,E)$. Thus there exists a unique $x\in\operatorname{Ker}P_{L^\infty}(\theta)$ such that

$$u(t;\theta,x,f_\theta)=\Phi(\theta,t)x+\int_0^t \Phi\big(\sigma(\theta,\tau),t-\tau\big)f_\theta(\tau)\,d\tau$$
$$=\Phi(\theta,t)x+\begin{cases}0, & t\in[0,t_0],\\ -(t-t_0)\Phi(\sigma(\theta,t_0),t-t_0)y, & t\in(t_0,t_0+1],\\ -\Phi(\sigma(\theta,t_0),t-t_0)y, & t>t_0+1\end{cases}$$

belongs to $F(\mathbb{R}_+,X)$. Using a similar argument with the one from the proof of the above theorem we have that $u(\cdot;\theta,x,f_\theta)\in L^\infty(\mathbb{R}_+,X)$. If we choose $t\geqslant t_0$ we have that

$$u(t;\theta,x,f_\theta)=\Phi(\theta,t)x-\Phi\big(\sigma(\theta,t_0),t-t_0\big)y.$$

Thus it follows that

$$\begin{aligned}\big\|u(t;\theta,x,f_\theta)\big\|&=\big\|\Phi(\theta,t)x-\Phi\big(\sigma(\theta,t_0),t-t_0\big)y\big\|\\&=\big\|\Phi\big(\sigma(\theta,t_0),t-t_0\big)\Phi(\theta,t_0)x-\Phi\big(\sigma(\theta,t_0),t-t_0\big)y\big\|\\&=\big\|\Phi\big(\sigma(\theta,t_0),t-t_0\big)\big(\Phi(\theta,t_0)x-y\big)\big\|\\&\geqslant N_2e^{\nu_2(t-t_0)}\big\|\Phi(\theta,t_0)x-y\big\|,\end{aligned}$$

for all $t\geqslant t_0+1$. Since $u(\cdot;\theta,x,f_\theta)$ is bounded, we have that $\Phi(\theta,t_0)x=y$. Thus $\Phi(\theta,t):\operatorname{Ker}P_{L^\infty}(\theta)\to\operatorname{Ker}P_{L^\infty}(\sigma(\theta,t))$ is also onto. □

Proposition 5.3. *If the exponentially bounded, strongly continuous cocycle $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant0}$ (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta,t\geqslant0}$) has a* ***no past exponential dichotomy*** *then* $\operatorname{Im}P(\theta)=\mathcal{X}_{1,L^\infty}(\theta)$ (*where $\{P(\theta)\}_{\theta\in\Theta}$ is a family of projectors provided by Definition* 4.2). *Moreover* $\sup_{\theta\in\Theta}\|P(\theta)\|<\infty$.

Proof. The first claim is obvious. For proving the second part we take $x_1\in\operatorname{Im}P(\theta)$ and $x_2\in\operatorname{Ker}P(\theta)$ with $\|x_1\|=\|x_2\|=1$. Recall that the angular distance between $\operatorname{Im}P(\theta)$ and $\operatorname{Ker}P(\theta)$ is defined by

$$\gamma\big[\operatorname{Im}P(\theta),\operatorname{Ker}P(\theta)\big]=\inf\left\{\left\|\frac{x}{\|x\|}-\frac{y}{\|y\|}\right\|:\ x\in\operatorname{Im}P(\theta),\ y\in\operatorname{Ker}P(\theta),\ x,y\neq0\right\}.$$

But

$$\|x_1-x_2\|\geqslant\frac{1}{Me^{\omega t}}\big\|\Phi(\theta,t)x_2-\Phi(\theta,t)x_1\big\|\geqslant\frac{1}{Me^{\omega t}}\left(Ne^{\nu t}-\frac{1}{N}e^{-\nu t}\right).$$

Choose $t_0>0$ such that $Ne^{\nu t_0}-\frac{1}{N}e^{-\nu t_0}=\psi_0>0$. Then

$$\|x_1-x_2\|\geqslant\psi=\frac{\psi_0}{Me^{\omega t_0}},$$

and thus $\gamma[\operatorname{Im} P(\theta), \operatorname{Ker} P(\theta)] \geqslant \psi$, for all $\theta \in \Theta$. Taking into account that

$$\frac{1}{\|P(\theta\|} \leqslant \gamma\big[\operatorname{Im} P(\theta), \operatorname{Ker} P(\theta)\big] \leqslant \frac{2}{\|P(\theta)\|} \quad \big(\text{see [11, (11.D), p. 8]}\big)$$

it follows that $\sup_{\theta\in\Theta} \|P(\theta)\| < \infty$. $\square$

Theorem 5.3. *Let $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ be an exponentially bounded, strongly continuous cocycle (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$). Assume that $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$ has an exponential dichotomy and that the corresponding family of projectors is strongly continuous (i.e. $P(\cdot)x$ is continuous for each $x \in X$).*

If $E(\mathbb{R}_+, X)$ is a vector-valued Schäffer function space then the pair $(E(\mathbb{R}_+, X), L^\infty(\mathbb{R}_+, X))$ is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta, t\geqslant 0}$.

Proof. Let $f \in E(\mathbb{R}_+, X)$ and set

$$v(t;\theta,f) = \int_0^t \Phi\big(\sigma(\theta,\tau), t-\tau\big) P\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau - \int_t^\infty \Phi^{-1}\big(\sigma(\theta,t), \tau-t\big) Q\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau.$$

Denoting by $x = v(0;\theta,f) = -\int_0^\infty \Phi^{-1}(\theta,\tau) Q(\sigma(\theta,\tau)) f(\tau)\,d\tau$ we have that

$$\begin{aligned}
&\Phi(\theta,t)x + \int_0^t \Phi\big(\sigma(\theta,\tau), t-\tau\big) f(\tau)\,d\tau \\
&= \int_0^t \Phi\big(\sigma(\theta,\tau), t-\tau\big) Q\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau \\
&\quad - \int_t^\infty \Phi^{-1}\big(\sigma(\theta,t), \tau-t\big) Q\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau + \int_0^t \Phi\big(\sigma(\theta,\tau), t-\tau\big) f(\tau)\,d\tau \\
&= \int_0^t \Phi\big(\sigma(\theta,\tau), t-\tau\big) P\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau \\
&\quad - \int_t^\infty \Phi^{-1}\big(\sigma(\theta,t), \tau-t\big) Q\big(\sigma(\theta,\tau)\big) f(\tau)\,d\tau \\
&= u(t;\theta,x,f).
\end{aligned}$$

It can bee seen that $x \in \operatorname{Ker} P(\theta)$ and by [11, (23.V), p. 69] (or alternatively [6, Lemma 1, p. 21]) it follows that $u(\cdot;\theta,x,f)$ belongs to $L^\infty(\mathbb{R}_+, X)$. $\square$

Theorem 5.4. *Let $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$ be exponentially bounded, strongly continuous cocycle (over a semiflow $\{\sigma(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$) and assume that there exists $\{P(\theta)\}_{\theta\in\Theta}$ a family of projectors with the following properties:*

- $\Phi(\theta,t)P(\theta) = P(\sigma(\theta,t))\Phi(\theta,t)$, *for all* $(\theta,t)\in\Theta\times\mathbb{R}_+$;
- $P(\cdot)x$ *is continuous for each* $x\in X$.

Then $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$ has an exponential dichotomy if and only if one of the following statements is true:

(i) *There exists $E(\mathbb{R}_+,X)$ a vector-valued Schäffer function space such that $(E(\mathbb{R}_+,X), L^\infty(\mathbb{R}_+,X))$ is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$;*
(ii) *There exist $p,q,\in[1,\infty]$, $(p,q)\neq(1,\infty)$ such that $(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$ is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$;*
(iii) *$(L^p(\mathbb{R}_+,X), L^q(\mathbb{R}_+,X))$ is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$, for any $p,q,\in[1,\infty]$, $(p,q)\neq(1,\infty)$;*
(iv) *there exists a vector-valued Orlicz function space $L^\Phi(\mathbb{R}_+,X)$ such that the pair $(L^\Phi(\mathbb{R}_+,X), L^\Phi(\mathbb{R}_+,X))$ is uniformly admissible to $\{\Phi(\theta,t)\}_{\theta\in\Theta,t\geqslant 0}$.*

Proof. It follows easily from Examples 2.1, 2.2, Remark 2.7 and above theorems. □

Remark 5.3. It is worth to note that so far, it has been extensively analyzed the asymptotic behavior of exponentially bounded, strongly continuous cocycles over *flows*. The main results in this direction are focused on the characterization of exponential dichotomy of an exponentially bounded, strongly continuous cocycles over a *flow* in terms of Sacker–Sell spectral properties [23] or the hyperbolicity of the associated evolution semigroups and their generators [10]. In particular, a characterization of exponential dichotomy for cocycles over *flows* was given in [23] assuming the dimension of the unstable manifold to be finite. Meanwhile, in [10] a characterization is given through the hyperbolicity of the associated evolution semigroup and its generator. Another characterization in [3] uses a discrete cocycle over a discretized *flow*. In this paper we made an attempt to characterize the exponential dichotomy in a more general setting and we did consider an exponentially bounded, strongly continuous cocycles over a *semiflow*, i.e., there is only a *semiflow* on the base space. This setting is particularly appropriate in the infinite-dimensional case since in this case the dynamical systems restricted to invariant manifolds are only *semiflows* in general.

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