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Stability and L_∞ performance analysis of positive systems with bounded time-varying delays on time scales[☆,☆☆]

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ABSTRACT

This paper investigates the stability and L_∞-gain problems of positive linear systems with bounded time-varying delays on time scales. For such systems on time scales, the positivity conditions are firstly established. Then, in virtue of the monotonic and asymptotic properties of a positive linear system with the specified time delay, the stability condition is presented for positive systems with bounded time-varying delays on time scales, which is independent of the delay magnitude and time scale. Furthermore, we prove that the L_∞-gain of such systems is also independent of both the magnitude of delays and time scale, which can be fully determined by system matrices. Finally, two numerical examples are presented to show the validity of the obtained results.

1. Introduction

In recent years, positive systems have attracted increasing interest due to its widely application in economy, biology, chemistry, and engineering fields. Since the state and output variables of positive system are confined to the positive orthant rather than the whole space $\mathbb{R}^n$, many interesting and distinctive dynamical properties arise, for example, its stability condition and Bounded Real Lemma in terms of linear matrix inequality (LMI) admit the existence of diagonal positive definite solution (see [1–3]), the linear copositive Lyapunov function can be a candidate for its stability analysis (see [4,5]), its stability (see [6–10]), L_1- and L_∞-gain performances (see [11–15]) are independent of the size of time-delays and so on. Therefore, the analysis and synthesis of positive systems have become a challenging and interesting topic [16–20].

On the other hand, in order to unify differential equations and difference equations, Stefan Hilger proposed the time scales and built the relevant basic theories [21]. Since a time scale may be an arbitrary closed subset of the real number set, dynamic systems on time scales not only integrate continuous-time systems and discrete-time systems, but also they can describe more complicated phenomena. Recently, the theory of time scales has been applied into neural networks [22–24], social networks [25], as well as a variety of control systems [26,27].

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In contrast with abundant works on continuous or discrete positive systems, the study of positive systems on time scales is a new trend in recent years. Some pioneering researches on positive realization, controllability and reachability of positive systems on time scales can be found in [28–30]. Stability analysis is known to be a fundamental problem in dynamic system theory. The stable radii of positive system on general time scales were studied in [31]. The condition of asymptotic stability was established for positive systems with time delays on time scales in [32]. It turned out that stability condition for positive systems on time scales was independent of time-delay and time scale.

In addition, the copositive storage function plays a key role in the performance analysis of positive systems due to the nonnegative nature of its state variables. In virtue of the copositive storage function, the L_1- and L_∞-gain are more suitable to take for performance index rather than the traditional L_2-gain. It is well known that the L_1- and L_∞-gain of positive systems with or without time-delays on time scales $\mathbb{T}=\mathbb{R}$ or $\mathbb{T}=\mathbb{Z}$ are all independent of magnitudes of time-delays and can be fully determined by system matrices. Therefore, it is natural to consider whether the L_∞-gain or L_1-gain of positive systems on general time scales is dependent on time-delays.

In this paper, we will investigate the stability and L_∞-gain problems of positive systems with bounded time-varying delays on time scales. Firstly, we will present positivity conditions for system with bounded time-varying delays on time scales. Then, the stability condition will be reproved for positive systems with bounded time-varying delays on general time scales only based on the comparison principle rather than using Lyapunov method adopted in [32]. Furthermore, we will discuss the L_∞-gain problem for positive systems with bounded time-varying delays on time scales.

Notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}$ denotes the set of real numbers. For a vector $x\in\mathbb{R}^n$, x_i denotes the ith entry and $x\geq 0$ $(x>0)$ means that $x_i\geq 0$ $(x_i>0)$ for any $i=1,\ldots,n$, respectively. In relation to this notation, we define the sets $\mathbb{R}^n_+ := \{x\in\mathbb{R}^n:\ x\geq 0\}$ and $\mathbb{R}^n_{++} := \{x\in\mathbb{R}^n:\ x>0\}$. Similarly, a_{ij} stands the (i,j)th entry of matrix A, and $A\geq 0$ means that $a_{ij}\geq 0$ for all (i,j), we also define the set $\mathbb{R}^{n\times m}_+ := \{A\in\mathbb{R}^{n\times m}:\ A\geq 0\}$. For given two matrices A and B with same dimensions, $A\geq B$ means that $A-B\geq 0$. A matrix $A\in\mathbb{R}^{n\times n}$ is said to be Metzler if $a_{ij}\geq 0, i\neq j$. $\mathbf{1}_n\in\mathbb{R}^n$ denotes the column vector with all entries equal to 1. And $I\in\mathbb{R}^{n\times n}$ denotes the n-dimensions unit matrix.

In addition, the ∞-norm of a vector x is defined by $\|x\|_\infty := \max_{1\leq i\leq n}|x_i|$. Given a vector-valued function $\omega:\mathbb{T}\mapsto\mathbb{R}^p$, the L_∞-norm $\|\omega\|_{L_\infty}$ is defined by $\|\omega\|_{L_\infty} := ess\sup_{t\in\mathbb{T}}\|\omega(t)\|_\infty$. The space of functions $\omega:\mathbb{T}\mapsto\mathbb{R}^p$ having finite L_∞-norm is denoted by $L_\infty(\mathbb{T},\mathbb{R}^p)$. Throughout this paper, we define $[a,b]_\mathbb{T} := [a,b]\bigcap\mathbb{T}$. $C([a,b]_\mathbb{T},\mathbb{R}^n)$ represents the set of all vector-valued continuous functions defined on the interval $[a,b]_\mathbb{T}$.

2. Preliminaries

In this section, we will firstly introduce some concepts and results concerning time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. It is a topological space with the topology induced from $\mathbb{R}$. Throughout this paper, we assume that $\mathbb{T}$ is unbounded, i.e. $\sup\mathbb{T}=+\infty$ and contains 0.

The forward jump operator and backward jump operator are, respectively, defined by

$$\sigma(t) := \inf\{s\in\mathbb{T}: s>t\},\quad \rho(t) := \sup\{s\in\mathbb{T}: s<t\}.$$

If $t=\sup\mathbb{T}$, then $\sigma(\sup\mathbb{T})=\sup\mathbb{T}$ when $\sup\mathbb{T}$ is finite; If $t=\inf\mathbb{T}$, then $\rho(\inf\mathbb{T})=\inf\mathbb{T}$ when $\inf\mathbb{T}$ is finite.

The forward graininess function $\mu:\mathbb{T}\to[0,+\infty)$ is defined by $\mu(t):=\sigma(t)-t$, while the backward graininess function $\nu:\mathbb{T}\to[0,+\infty)$ is defined by $\nu(t):=t-\rho(t)$. The time scale $\mathbb{T}$ is homogeneous, if μ and ν are constant.

If $\sigma(t)>t$, we say t is right-scattered; while if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, we say that t is right-dense. If $\rho(t)<t$, t is called left-scattered; while if $t>\inf\mathbb{T}$ and $\rho(t)=t$, t is called left-dense. Moreover, if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}^k=\mathbb{T}$.

Definition 2.1 (*[21]*)**.** Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$. Then the delta derivative of f at t, denoted by $f^\Delta(t)$ or $\frac{\Delta f}{\Delta t}(t)$, is a real number with the property that for any $\varepsilon>0$ there is a neighborhood U of t (i.e., $U=(t-\delta,t+\delta)\cap\mathbb{T}$ for some $\delta>0$) such that $\|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)\|\leq\varepsilon\|\sigma(t)-s\|$ for all $s\in U$. If $f^\Delta(t)$ exists, then we say that f is delta differentiable at t.

Definition 2.2 (*[33]*)**.** A function $f:\mathbb{T}\to\mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.3 (*[33]*)**.** A function $f:\mathbb{T}\to\mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$.

Definition 2.4 (*[33]*)**.** We say that a function $f:\mathbb{T}\to\mathbb{R}$ is regressive if

$$1+\mu(t)f(t)\neq 0 \ \text{ for all } t\in\mathbb{T}$$

holds; while a function $f(t)$ is positively regressive if $f(t)$ is regressive and $1+\mu(t)f(t)>0$ holds for all $t\in\mathbb{T}$. The set of all regressive and rd-continuous functions $f:\mathbb{T}\to\mathbb{R}$ is denoted by $\mathfrak{R}(\mathbb{T})$ or $\mathfrak{R}$, while the set of positively regressive functions is denoted by $\mathfrak{R}^+(\mathbb{T})$ or $\mathfrak{R}^+$.

Definition 2.5 (*[33]*)**.** Assume that function $f:\mathbb{T}\to\mathbb{R}$ is regulated. A function $F:\mathbb{T}\to\mathbb{R}$ is called an antiderivative of f if it satisfies $F^{\Delta}(t)=f(t)$ for all $t\in\mathbb{T}$. Then, the delta integral of function $f(t)$ is defined by

$$\int_{t_1}^{t_2} f(s)\Delta s = F(t_2)-F(t_1)\ \ for\ all\ \ s\in[t_1,t_2]_{\mathbb{T}}.$$

Definition 2.6 (*[33]*)**.** If $f\in\mathfrak{R}$, the generalized exponential function is defined by

$$e_f(t,s)=exp\left(\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau\right)\quad \mathrm{s},\ t\in\mathbb{T}. \tag{1}$$

where $\xi_{\mu(\tau)}(f(\tau))=\frac{1}{\mu(\tau)}Log(1+\mu(\tau)f(\tau))$ and Log is the principal logarithm function.

Definition 2.7 (*[34]*)**.** Assume that function $x:\mathbb{T}\to\mathbb{C}$ is regulated. Then, the Laplace transform of function x is defined by

$$\mathscr{L}\{x\}(z)=\int_0^{+\infty} x(t)e^{\sigma}_{\ominus z}(t,0)\Delta t,$$

where $z\in\mathfrak{R}$ and $\ominus z=-\frac{z}{1+\mu z}$, $e^{\sigma}_{\ominus z}(t,0)=e_{\ominus z}(\sigma(t),0)$.

Lemma 2.1 (*[34]*)**.** *Assume that function $x:\mathbb{T}\to\mathbb{C}$ is such that x^{Δ} is regulated. Then*

$$\mathscr{L}\{x^{\Delta}\}(z)=z\mathscr{L}\{x\}(z)-x(0)$$

for those regressive $z\in\mathbb{C}$ satisfying

$$\lim_{t\to+\infty}[x(t)e_{\ominus z}(t,0)]=0.$$

Lemma 2.2 (*Final Value Theorem*)**.** *Supposed that functions $x:\mathbb{T}\to\mathbb{R}$ and x^{Δ} possess Laplace transform , $\lim_{t\to+\infty}x(t)$ exists. Then,*

$$\lim_{t\to+\infty}x(t)=\lim_{z\to 0}(zX(z)),$$

where $\mathscr{L}\{x\}(z)=X(z)$.

Proof. According to the Definition 2.7 and differential properties of Laplace transform presented in Lemma 2.1, we can obtain

$$\mathscr{L}\{x^{\Delta}\}(z)=\int_0^{+\infty}x^{\Delta}(t)e^{\sigma}_{\ominus z}(t,0)\Delta t=zX(z)-x(0).$$

Let $z\to 0$ and take the limit on both sides of above equality, we have

$$\lim_{z\to 0}\int_0^{+\infty}x^{\Delta}(t)e^{\sigma}_{\ominus z}(t,0)\Delta t=\lim_{z\to 0}[zX(z)-x(0)]. \tag{2}$$

Note that the left side of the equality (2) is

$$\begin{aligned}\lim_{z\to 0}\int_0^{+\infty}x^{\Delta}(t)e^{\sigma}_{\ominus z}(t,0)\Delta t &= \int_0^{+\infty}x^{\Delta}(t)\lim_{z\to 0}e^{\sigma}_{\ominus z}(t,0)\Delta t\\ &= \int_0^{+\infty}x^{\Delta}(t)\Delta t\\ &= \lim_{t\to+\infty}\int_0^{t}x^{\Delta}(\tau)\Delta\tau\\ &= \lim_{t\to+\infty}[x(t)-x(0)]\end{aligned}$$

Then, we have $\lim_{t\to+\infty}x(t)=\lim_{z\to 0}(zX(z))$.

Lemma 2.3 (*[9]*)**.** *Let $A\in\mathbb{R}^{n\times n}$ be Metzler. The following conditions are equivalent:*

(i) *A is Hurwitz;*

(ii) *A is nonsingular and $A^{-1}\le 0$;*

(iii) *there exists a vector $\lambda\in\mathbb{R}^n_{++}$ such that $\lambda^TA<0$.*

Lemma 2.4 (*[33] Induction Principle*)**.** *Let $t_0 \in \mathbb{T}$ and assume that $\{S(t) : t \in [t_0, +\infty)\}$ is a family of statements satisfying:*

(i) *The statement $S(t_0)$ is true.*

(ii) *If $t \in [t_0, +\infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.*

(iii) *If $t \in [t_0, +\infty)$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t, +\infty)$.*

(iv) *If $t \in (t_0, +\infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0, t)$, then $S(t)$ is true.*

Then, $S(t)$ is true for all $t \in [t_0, +\infty)$.

For $A \in \mathbb{R}^{n\times n}$, consider the following linear system on time scale $\mathbb{T}$

$$x^{\Delta}(t) = Ax(t), \quad x(t_0) = x_0 \tag{3}$$

If $\mathbb{T} = \mathbb{R}$, system (3) is a continuous system $\dot{x}(t) = Ax(t)$ $(t \in \mathbb{R})$, while if $\mathbb{T} = \mathbb{Z}$, system (3) is a discrete system $x(k+1) = (I + A)x(k)$ $(k \in \mathbb{N})$.

Function $e_A : \{(t, \tau) \in \mathbb{T} \times \mathbb{T} : t \geq \tau\} \to \mathbb{R}^{n\times n}$ denotes the transition matrix of system (3), that is, $x(t) = e_A(t, t_0)x_0$ represents the solution of system (3) with initial condition $x(t_0) = x_0$, where $t \in [t_0, +\infty)_{\mathbb{T}}$.

In this paper, we assume that $\mathbb{T}$ contains right-scattered points and is with bounded graininess and define $\eta = \eta(\mathbb{T}) := \frac{1}{\sup\{\mu(t), t\in\mathbb{T}\}}$.

Definition 2.8 (*[31]*)**.** System (3) is said to be positive if its state trajectory $x(t) \in \mathbb{R}^n_+$, $t \in [t_0, +\infty)_{\mathbb{T}}$, for any initial condition $x_0 \in \mathbb{R}^n_+$.

In the following, the relevant definitions and Lemmas will be given for the subsequent discussion.

Lemma 2.5 (*[31]*)**.** *System (3) is positive if and only if $A + \eta I_n \geq 0$.*

Lemma 2.6 (*[31]*)**.** *The transition matrix $e_A(t, t_0) \in \mathbb{R}^{n\times n}_+$ for every $t \in [t_0, +\infty)_{\mathbb{T}}$ if $A + \eta I_n \geq 0$.*

Remark 2.1. In fact, if time scale $\mathbb{T}$ contains no right-scattered points, i.e. $\mathbb{T} = [a, +\infty)$, $a \in \mathbb{R}$, we denote $\eta = +\infty$. Now, $A + \eta I_n \geq 0$ implies that A is Metzler.

Definition 2.9 (*[35]*)**.** The equilibrium $x_e = 0$ of system (3) is stable if for every $\epsilon > 0$, there exists a $\delta(\epsilon,\ t_0) > 0$ such that when $\|x(t_0)\| < \delta(\epsilon,\ t_0)$, $\|x(t)\| < \epsilon$ for $t \in [t_0, +\infty)_{\mathbb{T}}$. The equilibrium $x_e = 0$ of system (3) is uniformly stable if it is stable and δ can be chosen independently of t_0.

Definition 2.10 (*[35]*)**.** The equilibrium $x_e = 0$ of system (3) is uniformly asymptotically stable if it is uniformly stable and for any $\delta > 0$, there exists a constant $T > 0$ such that for any t_0 and $x(t_0)$, the corresponding solution satisfies

$$\|x(t)\| \leq \delta\|x(t_0)\|, \quad \forall\, t \geq t_0 + T,\ \ t, t_0 \in \mathbb{T}.$$

Definition 2.11 (*[36]*)**.** The equilibrium $x_e = 0$ of system (3) is exponentially stable if for every t_0, there exists $K(t_0) \geq 1$ and $\alpha > 0$, and an open neighborhood V of 0 in $\mathbb{R}^n$ such that for every $t_0,\ t \in \mathbb{T}$ with $t > t_0$ and every $x_0 \in \mathbb{R}^n \cap V$, the forward trajectory x of the system corresponding to the initial condition $x(t_0) = x_0$ satisfies

$$\|x(t)\| \leq K(t_0)exp(-\alpha(t - t_0))\|x_0\|. \tag{4}$$

The equilibrium $x_e = 0$ of system (3) is uniformly exponentially stable if it is exponentially stable and K in (4) can be chosen independently of t_0.

Throughout this paper, we abuse semantics and say that system is stable rather than saying that the equilibrium is stable.

Lemma 2.7 (*[36]*)**.** *System (3) is uniformly exponentially stable if and only if it is uniformly asymptotically stable.*

Remark 2.2. Refs. [36,37] show that if system (3) is uniformly exponentially stable, then the eigenvalues of its system matrix A are all contained in the open left half complex plane. And for linear time-invariant system on time scale, stability and uniform stability above addressed are equivalent.

3. Positivity and stability analysis

In this section, we will deduce the positivity and stability conditions of linear systems with bounded time-varying delays on time scales. Consider the following system with bounded time-varying delays on time scale $\mathbb{T}$

$$\begin{cases} x^{\Delta}(t) &= Ax(t) + A_d x(\varrho(t-d(t))) + B\omega(t) \\ y(t) &= Cx(t) + C_d x(\varrho(t-\tau(t))) + D\omega(t) \\ x(s) &= \phi(s),\ s \in [-\widetilde{d},\ 0]_{\mathbb{T}} \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^q$ denote system state and output trajectories, respectively. $\omega(t) \in \mathbb{R}^p$ is disturbance input, $\phi(\cdot) \in C([-\widetilde{d}, 0]_{\mathbb{T}}, \mathbb{R}^n)$ represents the initial condition. A, A_d, B, C, C_d and D are constant matrices with compatible dimensions. Time delay functions $d(t)$ and $\tau(t)$ satisfy $0 \le d(t) \le d$ and $0 \le \tau(t) \le \tau$ for all $t \in \mathbb{T}$, respectively, where d and τ are positive constants. $\widetilde{d} = \max\{d, \tau\}$. In most cases, $t - d(t)$ and $t - \tau(t)$ may be not in time scale $\mathbb{T}$ we consider, so define function $\varrho(t) := \sup\{s : s \le t, s \in \mathbb{T}\}$ such that $x(\varrho(t-d(t)))$ and $x(\varrho(t-\tau(t)))$ are well defined. The state trajectory starting from initial condition $\phi(\cdot)$ is always denoted by $x(t; \phi)$.

Lemma 3.1 ([33]). *Let $A \in \mathbb{R}^{n\times n}$ be a matrix-valued function on $\mathbb{T}$ and suppose that function $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Then, for given $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$, the initial value problem*

$$x^{\Delta} = A(t)x + f(t),\ \ x(t_0) = x_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$, which is given by

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

The above lemma provides a condition for the existence and uniqueness of solution of the differential equation on time scale $\mathbb{T}$, we always assume that the solutions of systems addressed in this paper are existent and unique.

In the following, we will give the positivity definition of system (5) and investigate the sufficient and necessary conditions ensuring the solution of system (5) to be positive.

Definition 3.1. System (5) is said to be positive if for all initial condition $\phi(s) \ge 0$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$) and all input $\omega(t) \ge 0$, the state $x(t) \ge 0$ and output $y(t) \ge 0$ hold for all $t \in [0, +\infty)_{\mathbb{T}}$.

Now, followed the proof of Lemma 3 in reference [32], a necessary condition of positive system is given as follows.

Theorem 3.1. *If system (5) is positive, then A is Metzler, $A_d \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times p}$, $C \in \mathbb{R}_+^{q\times n}$, $C_d \in \mathbb{R}_+^{q\times n}$ and $D \in \mathbb{R}_+^{q\times p}$.*

Proof. Assume that system (5) is positive, that is, for any initial condition $x(s) = \phi(s) \ge 0$ ($s \in [-\tilde{d}, 0]_{\mathbb{T}}$) and input $\omega(t) \ge 0$, then $x(t) \ge 0$ and $y(t) \ge 0$ hold for all $t \in [0, +\infty)_{\mathbb{T}}$.

Firstly, let $x(0) = 0$ and $\phi(\varrho(-d(0))) = 0$, then $x^{\Delta}(0) = B\omega(0)$, the positivity of system (5) implies that $B \in \mathbb{R}_+^{n\times p}$ since $\omega(0) \in \mathbb{R}_+^p$ and $x^{\Delta}(0) \in \mathbb{R}_+^n$.

Secondly, let $\phi(\varrho(-d(0))) = 0$, $\omega(0) = 0$ and $x(0) = e_j$, where e_j represents the unit vector whose jth element equals to 1 and other elements equal to 0. Then, for any $x(0) \in \mathbb{R}_+^n$, $x^{\Delta}(0) = Ax(0) = Ae_j = [a_{1j}, a_{2j}, \ldots, a_{nj}]^T$ (the jth column of matrix A). Based on the fact that system (5) is positive and $x(0) = e_j$, we can conclude that $x_i^{\Delta}(0) \ge 0$ and $a_{ij} \ge 0$ ($i,\ j = 1, \ldots, n,\ i \ne j$), which implies that A is Metzler.

Thirdly, let $x(0) = 0$ and $\omega(0) = 0$ and $\phi(\varrho(-d(0))) = e_j \ge 0$, we obtain $x^{\Delta}(0) = A_d\phi(\varrho(-d(0))) = [\bar{a}_{1j}, \bar{a}_{2j}, \ldots, \bar{a}_{nj}]^T$, where $\bar{a}_{ij}$ represents the element of matrix A_d. Note that $x(0) = 0$ and system (5) is positive, we can deduce that $x^{\Delta}(0) \ge 0$, which implies that $A_d \in \mathbb{R}_+^{n\times n}$ since $\phi(\varrho(-d(0))) \ge 0$.

Finally, by the similar way, we can prove that $C \in \mathbb{R}_+^{q\times n}$, $C_d \in \mathbb{R}_+^{q\times n}$ and $D \in \mathbb{R}_+^{q\times p}$. We completes the proof of necessary condition.

Next, we will give a sufficient condition on positivity of system (5) with time-varying delays on time scale $\mathbb{T}$.

Theorem 3.2. *System (5) is positive if $A + \eta I \in \mathbb{R}_+^{n\times n}$, $A_d \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times p}$, $C \in \mathbb{R}_+^{q\times n}$, $C_d \in \mathbb{R}_+^{q\times n}$ and $D \in \mathbb{R}_+^{q\times p}$.*

Proof. Firstly, for any given initial condition $x(s) = \phi(s) \ge 0$ ($s \in [-\tilde{d}, 0]_{\mathbb{T}}$) and input $\omega(t) \ge 0$, we will prove that $x(t) \ge 0$ holds for all $t \in [0, +\infty)_{\mathbb{T}}$. It follows from Lemma 3.1 that the solution of system (5) has the form of

$$x(t) = e_A(t, 0)x(0) + \int_0^t e_A(t, \sigma(r))(A_d x(\varrho(r - d(r))) + B\omega(r))\Delta r.$$

Now, we will show that $\Omega \triangleq \{t \in [0, +\infty)_{\mathbb{T}} | x(t) \notin \mathbb{R}_+^n\}$ is an empty set. Applying contradiction approach, suppose that $\Omega \ne \emptyset$ and define the infimum of Ω as $t^* \triangleq \inf \Omega$. Since $A + \eta I_n \in \mathbb{R}_+^{n\times n}$, it follows from Lemma 2.6 that $e_A(t, 0) \in \mathbb{R}_+^{n\times n}$

and $e_A(t, \sigma(r)) \in \mathbb{R}_+^{n\times n}$ for $t \in [0, +\infty)_{\mathbb{T}}$ and $0 \le r \le t$. And note that $A_d \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times p}$, $x(\varrho(r - d(r))) \in \mathbb{R}_+^n$, $x(r) \in \mathbb{R}_+^n$ and $\omega(r) \in \mathbb{R}_+^p$ for all $0 \le r < t^*$ and $r \in \mathbb{T}$. Then, we can deduce that $x(t^*) \in \mathbb{R}_+^n$, which contradicts with $x(t^*) \notin \mathbb{R}_+^n$, that is, $\Omega = \emptyset$. Thus, we can conclude that for any initial condition $x(s) = \phi(s) \ge 0$ ($s \in [-\tilde{d}, 0]_{\mathbb{T}}$), $x(t) \in \mathbb{R}_+^n$ holds for all $t \in [0, +\infty)_{\mathbb{T}}$.

Furthermore, since $x(t) \in \mathbb{R}_+^n$ ($t \in [0, +\infty)_{\mathbb{T}}$) and matrices C, C_d, D are all nonnegative, one can easily deduce that $y(t) \in \mathbb{R}_+^q$ holds for all $t \in [0, +\infty)_{\mathbb{T}}$.

Remark 3.1. A sufficient condition on positivity of system (5) is also given in [32], which needs $A \ge 0$, but we only need $A + \eta I_n \ge 0$ in Theorem 3.2. For example, in the case of $\mathbb{T} = \mathbb{R}$, the fact that A is just Metzler can ensure the positivity of (5) rather than $A \ge 0$. However, it follows from the definition of η that $\eta = +\infty$ when $\mathbb{T} = \mathbb{R}$, thus $A + \eta I_n \ge 0$ is equivalent that A is Metzler. So the condition in Theorem 3.2 is less conservative than one in [32].

Based on Theorem 3.2, one can easily obtain the Comparison Lemma as follows.

Lemma 3.2. *Assume that system (5) with $\omega(t) = 0$ is positive, $x(t; \phi_1)$ and $x(t; \phi_2)$ are its state trajectories under initial conditions $\phi_1(s)$ and $\phi_2(s)$, $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$, respectively. Then, $\phi_1(s) \le \phi_2(s)$ for $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$ implies that $x(t, \phi_1) \le x(t, \phi_2)$ for all $t \in [0, +\infty)_{\mathbb{T}}$.*

Proof. This Lemma directly follows from the linearity and positivity of system (5).

Now, let us consider the asymptotical stability of the following delta derivative system with bounded time-varying delays on time scale $\mathbb{T}$.

$$\begin{cases} x^{\Delta}(t) &= Ax(t) + A_d x(\varrho(t - d(t))) \\ x(s) &= \phi(s),\ s \in [-\widetilde{d},\ 0]_{\mathbb{T}} \end{cases} \tag{6}$$

Before analyzing the stability of positive system (6), we firstly investigate the monotonic property of positive system (6) with special time-delay.

Lemma 3.3. *Given a constant $d \ge 0$, assume that $A + \eta I_n \in \mathbb{R}_+^{n\times n}$, $A_d \in \mathbb{R}_+^{n\times n}$ and $A + A_d$ is Hurwitz, consider the following delta derivative system on time scale $\mathbb{T}$*

$$\bar{x}^{\Delta}(t) = A\bar{x}(t) + A_d\bar{x}(\varrho(t - d)) \tag{7}$$

with initial condition $\phi(s) = \xi$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$), where ξ satisfies $(A + A_d)\xi < 0$. Then, the following Statements hold:

(a) *$\bar{x}(t) \le \xi$ for all $t \in [0, +\infty)_{\mathbb{T}}$;*
(b) *$\bar{x}(s) \ge \bar{x}(t)$ for all $s \le t$ and $s, t \in \mathbb{T}$;*
(c) *$\lim_{t\to+\infty} \bar{x}(t) = 0$;*
(d) *$x(t) \le \bar{x}(t)$ for all $t \in [0, +\infty)_{\mathbb{T}}$, where $x(t)$ is the state trajectory of system (6) under initial condition $\phi(s) = \xi$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$.*

Proof. (a) Define error variable $e(t) \triangleq \xi - \bar{x}(t)$ for all $t \in [0, +\infty)_{\mathbb{T}}$, then we have error system

$$e^{\Delta}(t) = Ae(t) + A_d e(\varrho(t - d)) - (A + A_d)\xi, \tag{8}$$

its initial condition satisfies that $e(s) = 0$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$. Since $A + A_d$ is Hurwitz and satisfies $(A + A_d)\xi < 0$, by regarding $-(A + A_d)\xi$ as a nonnegative input of error system (8), and $A + \eta I_n \in \mathbb{R}_+^{n\times n}$, $A_d \in \mathbb{R}_+^{n\times n}$, it follows from Theorem 3.2 that $e(t) \ge 0$ for all $t \in [0, +\infty)_{\mathbb{T}}$, which naturally implies that $\bar{x}(t) \le \xi$ for all $t \in [0, +\infty)_{\mathbb{T}}$.

(b) We will use the Induction Principle to prove Statement (b) holds, and the family of statement $S(t)$ refers to that $\bar{x}(s_1) \ge \bar{x}(s_2)$ for all $s_1, s_2 \in \mathbb{T}$, $s_1 \le s_2 \le t$.

(i) It follows from Statement (a) that $\xi = \bar{x}(s_1) \ge \bar{x}(s_2) = \xi$ for all $s_1 \le s_2$, $s_1, s_2 \in [-\widetilde{d},\ 0]_{\mathbb{T}}$, thus statement $S(0)$ holds.

(ii) Let $t \in [0, +\infty)_{\mathbb{T}}$ be right-scattered and assume that Statement $S(t)$ be true, i.e. $\bar{x}(s_1) \ge \bar{x}(s_2)$ for all $s_1, s_2 \in \mathbb{T}$, $s_1 \le s_2 \le t$. Then, define error variable $e(t) = \bar{x}(t) - \bar{x}(\sigma(t))$, we have error system

$$e^{\Delta}(t) = Ae(t) + A_d(\bar{x}(\varrho(t - d)) - \bar{x}(\varrho(\sigma(t) - d))). \tag{9}$$

Note that the initial condition of error system (9) satisfies that $e(s) = \bar{x}(s) - \bar{x}(\sigma(s)) \ge \xi - \xi = 0$ since $\bar{x}(s) = \xi$ and $\bar{x}(\sigma(s)) \le \xi$ from Statement (a). And by the fact that $\varrho(t - d) \le \varrho(\sigma(t) - d) \le t$ and assumption $S(t)$ is true, it is easy to deduce that $e(t) \ge 0$ by regarding $\bar{x}(\varrho(t - d)) - \bar{x}(\varrho(\sigma(t) - d))$ as a nonnegative input, which implies that $\bar{x}(t) \ge \bar{x}(\sigma(t))$, i.e. $S(\sigma(t))$ is true.

(iii) Let $t \in [0, +\infty)_{\mathbb{T}}$ be right-dense and assume that Statement $S(t)$ is true. We will show that there exists a neighborhood $U = (t, t + d)$ of t such that $S(\tilde{t})$ is true for all $\tilde{t} \in U \cap [0, +\infty)_{\mathbb{T}}$. Given an arbitrary constant c

such that $0 < c < d$ and $\tilde{t} + c \in U \cap [0, +\infty)_{\mathbb{T}}$, define error variable $e(\tilde{t}) \triangleq \bar{x}(\tilde{t}) - \bar{x}(\tilde{t} + c)$, then we have error system

$$e^{\Delta}(\tilde{t}) = Ae(\tilde{t}) + A_d(\bar{x}(\varrho(\tilde{t} - d)) - \bar{x}(\varrho(\tilde{t} + c - d))). \tag{10}$$

Note that the initial condition of error system (10) satisfies that $e(s) = \bar{x}(s) - \bar{x}(s + c) = \xi - \bar{x}(s + c) \geq 0$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$ by Statement (a). Since $\varrho(\tilde{t} - d) \leq \varrho(\tilde{t} + c - d) \leq t$ and due to the assumption $S(t)$ is true, we conclude that $e(\tilde{t}) \geq 0$ for all $\tilde{t} \in U \cap [0, +\infty)_{\mathbb{T}}$ by regarding $\bar{x}(\varrho(\tilde{t} - d)) - \bar{x}(\varrho(\tilde{t} + c - d))$ as a nonnegative input, this implies that $\bar{x}(\tilde{t}) \geq \bar{x}(\tilde{t} + c)$ for any $0 \leq c \leq d$ and $\tilde{t},\ \tilde{t} + c \in U \cap [0, +\infty)_{\mathbb{T}}$, that is, $S(\tilde{t})$ is true for all $\tilde{t} \in U \cap [0, +\infty)_{\mathbb{T}}$.

(iv) Let $t \in (0, +\infty)_{\mathbb{T}}$ be left-dense and assume that $S(s)$ is true for all $s \in [0, t)_{\mathbb{T}}$. We have to show that $S(t)$ is true, i.e. $\bar{x}(s_1) \geq \bar{x}(t)$ for all $s_1 \leq t$ and $s_1, t \in \mathbb{T}$. Then, for given any constant $h > 0$ such that $t - h \in [0, t)_{\mathbb{T}}$, define error variable $e(t) = \bar{x}(t - h) - \bar{x}(t)$, we have error system

$$e^{\Delta}(t) = Ae(t) + A_d(\bar{x}(\varrho(t - h - d)) - \bar{x}(\varrho(t - d))) \tag{11}$$

Note that the initial condition of error system (11) satisfies that $e(s) = \bar{x}(s-h) - \bar{x}(s) = \xi - \xi = 0$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$ by Statement (a). Since $\varrho(t - h - d) \leq \varrho(t - d) < t$ and due to the assumption $S(s)$ is true for all $s \in [0, t)$, it is easy to see that $e(t) \geq 0$ by regarding $\bar{x}(\varrho(t - h - d)) - \bar{x}(\varrho(t - d))$ as a nonnegative input, which implies that $\bar{x}(t - h) \geq \bar{x}(t)$ for any $h > 0$ and $t - h \in [0, t)_{\mathbb{T}}$, i.e. $S(t)$ is true.

Therefore, $S(t)$ is true for all $t \in [0, +\infty)_{\mathbb{T}}$.

(c) From Statement (b), we can conclude that the state trajectory $\bar{x}(t)$ of system (7) is monotonically nonincreasing about t and $\bar{x}(t) \geq 0$. Thus $\lim_{t\to+\infty} \bar{x}(t)$ exists. Since $\bar{x}(t)$ is bounded, i.e. $\bar{x}(t) \leq \xi$ for all $t \in [0, +\infty)_{\mathbb{T}}$ and by the linearity of system (7), we can deduce that $\bar{x}^{\Delta}(t)$ is also bounded. Letting $\lim_{t\to+\infty} \bar{x}(t) = l$ and applying Final Value Theorem, we have $\lim_{s\to 0}(s\mathscr{L}(\bar{x})) = \lim_{s\to 0}(sX(s)) = \lim_{t\to+\infty} \bar{x}(t) = l$ and $\lim_{s\to 0}(s\mathscr{L}(\bar{x}^{\Delta})) = \lim_{t\to+\infty} \bar{x}^{\Delta}(t) = (A + A_d)l$, where $\mathscr{L}$ denotes the Laplace transformation operator. Furthermore, we have

$$\begin{aligned}(A + A_d)l &= \lim_{s\to 0}(s\mathscr{L}(\bar{x}^{\Delta})) \\ &= \lim_{s\to 0}(s(s\mathscr{L}(\bar{x}) - \bar{x}(0))) \\ &= \lim_{s\to 0}(s(s\mathscr{L}(\bar{x}) - \xi)) \\ &= \lim_{s\to 0}(s(sX(s) - \xi)) = (l - \xi)\lim_{s\to 0} s = 0.\end{aligned}$$

Then, we obtain $(A + A_d)l = 0$. Note that $A + A_d$ is Hurwitz, it follows from Lemma 2.3 that matrix $A + A_d$ is nonsingular. Thus, we can easily deduce that $l = 0$.

(d) Define error variable $e(t) = \bar{x}(t) - x(t)$, then we have error system

$$e^{\Delta}(t) = Ae(t) + A_d e(\varrho(t - d(t))) + A_d(\bar{x}(\varrho(t - d)) - \bar{x}(\varrho(t - d(t)))), \tag{12}$$

with initial condition $e(s) = 0$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$). By Statement (b), we obtain that $\bar{x}(\varrho(t - d)) - \bar{x}(\varrho(t - d(t))) \geq 0$ for all $t \in [0, +\infty)_{\mathbb{T}}$. Note that $A + \eta I_n \in \mathbb{R}_+^{n\times n}$, $A_d \in \mathbb{R}_+^{n\times n}$, it follows from Theorem 3.2 that $e(t) \geq 0$ for all $t \in [0, +\infty)_{\mathbb{T}}$ by regarding $\bar{x}(\varrho(t - d)) - \bar{x}(\varrho(t - d(t)))$ as a nonnegative input. Thus, we obtain that $x(t) \leq \bar{x}(t)$ for all $t \in [0, +\infty)_{\mathbb{T}}$. This completes the whole proof.

Remark 3.2. In Lemma 3.3, since we assume that $A + A_d$ is Hurwitz, it follows form Lemma 2.3 that there always exists a vector $\xi \in \mathbb{R}_{++}^n$ such that $(A + A_d)\xi < 0$.

Theorem 3.3. *Assume that $A + \eta I_n$ is positive, A_d is nonnegative. Then, system (6) is asymptotically stable for any time-varying delay $d(t)$ satisfying $0 \leq d(t) \leq d$ and any initial condition $\phi(s) \geq 0$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$) if and only if $A + A_d$ is Hurwitz.*

Proof. (Sufficiency) Assume that $A + A_d$ is Hurwitz, we need to prove that system (6) is asymptotically stable for any time-varying delay $d(t)$ satisfying $0 \leq d(t) \leq d$.

Firstly, we will show the attraction of the state trajectory for system (6). Based on Lemma 2.3, for any initial condition $\phi(s) \geq 0$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$), one can always find a vector $\xi \in \mathbb{R}_{++}^n$ with sufficiently large magnitude such that $(A + A_d)\xi < 0$ and $0 \leq \phi(s) \leq \xi$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$. From Lemma 3.2, we have $0 \leq x(t; \phi) \leq x(t; \xi)$ for all $t \in [0, +\infty)_{\mathbb{T}}$. By Statement (d) in Lemma 3.3, one has $x(t; \xi) \leq \bar{x}(t; \xi)$. Thus, we obtain $0 \leq x(t; \phi) \leq \bar{x}(t; \xi)$, and note that $\lim_{t\to+\infty} \bar{x}(t; \xi) = 0$, we can conclude that $\lim_{t\to+\infty} x(t; \phi) = 0$.

In the following, we will prove the Lyapunov stability of system (6). Based on Lemma 2.3, for any constant $\epsilon > 0$, one can always find a vector $\xi \in \mathbb{R}_{++}^n$ with sufficiently small magnitude such that $(A + A_d)\xi < 0$ and $\xi \leq \epsilon\mathbf{1}_n$. It follows from Statements (a) and (d) in Lemma 3.3 that $x(t; \xi) \leq \bar{x}(t; \xi) \leq \xi \leq \epsilon\mathbf{1}_n$. Meanwhile, one can always find a sufficiently small $\delta > 0$, such that $\xi \geq \delta\mathbf{1}_n$. Thus, for any initial condition $\phi(s)$ satisfying $\phi(s) \in \mathbb{R}_+^n$ and $\|\phi(s)\|_\infty \leq \delta$ for all $s \in [-\widetilde{d}, 0]_{\mathbb{T}}$, we can deduce that $0 \leq \phi(s) \leq \delta\mathbf{1}_n \leq \xi$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$. Therefore, from Lemmas 3.2 and 3.3, we can conclude that $0 \leq x(t; \phi) \leq x(t; \xi) \leq \xi \leq \epsilon 1_n$, this implies that $\|x(t; \phi)\|_\infty \leq \epsilon$. Which completes the sufficiency proof.

(Necessity) Now, we should mention that $A+A_d$ is Metzler implies that its eigenvalue with the largest real part must be real. It follows from Remark 2.2 that the asymptotical stability of system (6) with $d(t)=0$ implies that the rightmost real eigenvalue of $A+A_d$ is strictly negative, therefore, $A+A_d$ is Hurwitz. This completes the necessity proof.

Remark 3.3. The asymptotically stability condition of system (6) has been established in [32], that is, there exists a vector $\lambda \in \mathbb{R}^n_{++}$ such that $(A+A_d)\lambda < 0$. In fact, this condition is equivalent to that in Theorem 3.3. Since when A is Metzler, $A_d \in \mathbb{R}^{n\times n}_+$, the condition $(A+A_d)\lambda < 0$ for $\lambda \in \mathbb{R}^n_{++}$ is equivalent to that $A+A_d$ is Hurwitz. But the proof techniques adopted in [32] is different from those in Theorem 3.3. The stability condition is established by mainly using the linear copositive Lyapunov method and in virtue of integration properties on time scales in [32]. However, we mainly resort to the Comparison Lemma of positive system on time scales and the positivity property of system. In addition, the idea of Induction Principle to prove Statement (b) of Lemma 3.2 which is the important lemma to establish the stable condition also apply to analyze L_∞-gain of positive system on time scale in the following section.

Remark 3.4. From Theorem 3.3, one can know that the stability of positive system is not only independent of the magnitude of time-delays, but also is not dependent on time scales.

4. L_∞-Gain analysis

In this section, we will study the L_∞-gain of positive system (5) with bounded time-varying delays. Throughout this section, we always make the following Assumption.

Assumption 4.1. We assume that system (5) satisfies the following conditions:

(i) Matrices $A+\eta I_n \in \mathbb{R}^{n\times n}_+$, $A_d \in \mathbb{R}^{n\times n}_+$, $B \in \mathbb{R}^{n\times p}_+$, $C \in \mathbb{R}^{q\times n}_+$, $C_d \in \mathbb{R}^{q\times n}_+$ and $D \in \mathbb{R}^{q\times p}_+$;
(ii) $A+A_d$ is Hurwitz;
(iii) the initial condition $\phi(s)=0$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$.

Definition 4.1 (*[38]*)**.** Given a p-dimensions input and q-dimensions output linear system Σ:

$$\begin{aligned} \Sigma : L^p_\infty &\ \mapsto\ L^q_\infty \\ \omega(t) &\ \mapsto\ z(t) \end{aligned}$$

with input $\omega \in L_\infty(\mathbb{T},\ \mathbb{R}^p)$, output $y \in L_\infty(\mathbb{T},\ \mathbb{R}^q)$. The L_∞-gain of system Σ is defined as:

$$\|\Sigma\|_{L_\infty} := \sup_{\|\omega\|_{L_\infty}=1} \|y\|_{L_\infty} = \sup_{\|\omega\|_{L_\infty}\neq 0} \frac{\|y\|_{L_\infty}}{\|\omega\|_{L_\infty}}.$$

From Definition 4.1, one can know that the L_∞-gain of system (5) can be defined as $\sup_{\|\omega\|_{L_\infty}=1}\|y\|_{L_\infty}$, which is the smallest γ such that $\|y\|_{L_\infty} \le \gamma\|\omega\|_{L_\infty}$ for all $\omega \in L_\infty(\mathbb{T},\ \mathbb{R}^p)$. The objective of this section is to give a characterization on L_∞-gain of positive system (5). In order to facilitate the L_∞-gain analysis, we need to prepare some lemmas.

Lemma 4.1. *Under Assumption 4.1, suppose that $0 \le \omega_1(t) \le \omega_2(t)$ ($t \in [0,+\infty)_{\mathbb{T}}$) and let $x(t;\omega)$ ($y(t;\omega)$, respectively) denote the state trajectory (output trajectory, respectively) of positive system (5) under input $\omega(t)$. Then, $x(t;\omega_1) \le x(t;\omega_2)$ and $y(t;\omega_1) \le y(t;\omega_2)$ hold for all $t \in [0,+\infty)_{\mathbb{T}}$.*

Proof. This proof is obvious by virtue of linearity and positivity of system (5).

From Lemma 4.1 and based on the definition of L_∞-gain of system (5), it is enough to study the L_∞-gain of system (5) with a constant input $\omega_0 = \mathbf{1}_p$ rather than all nonnegative input $\omega(t)$ satisfying $\|\omega\|_{L_\infty}=1$.

Before moving on, the following monotonic and asymptotic properties are needed.

Lemma 4.2. *Under Assumption 4.1 and given a constant $d \ge 0$, the state trajectory of system $x^\Delta(t) = Ax(t)+A_dx(\varrho(t-d))+B\omega_0$ is monotonically nondecreasing, that is, $x(t) \ge x(s)$ for any $t \ge s \ge 0$ and $t,\ s \in \mathbb{T}$.*

Proof. The proof of this Lemma can follow the proof way of Statement (b) in Lemma 3.3 and is omitted here.

Lemma 4.3. *Under Assumption 4.1 and given a constant $d \ge 0$, the state trajectory of system $x^\Delta(t) = Ax(t)+A_dx(\varrho(t-d))+B\omega_0$ with zero initial condition satisfies that $\lim_{t\to+\infty}x(t) = -(A+A_d)^{-1}B\omega_0$.*

Proof. Under Assumption 4.1, note that matrix $A+A_d$ is Hurwitz, it follows from Lemma 2.3 that $(A+A_d)^{-1} \le 0$. Define $x_0 \triangleq -(A+A_d)^{-1}B\omega_0$, one can deduce that $x_0 \ge 0$. Define $e(t) = x_0 - x(t)$, then we have error system $e^\Delta(t) = Ae(t)+A_de(\varrho(t-d))$ with initial condition $e(s) = x_0 \ge 0$ ($s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$). By Theorem 3.3 and $A+A_d$ is Hurwitz, we can deduce that $\lim_{t\to+\infty}e(t)=0$, which means $\lim_{t\to+\infty}x(t) = -(A+A_d)^{-1}B\omega_0$.

In order to investigate the L_∞-gain of system (5) with a constant input $\omega_0 = \mathbf{1}_p$, we need to consider the following two systems with constant input ω_0, respectively:

$$\begin{cases} \bar{x}^{\Delta}(t) = (A + A_d)\bar{x}(t) + B\omega_0 \\ \bar{y}(t) = (C + C_d)\bar{x}(t) + D\omega_0 \end{cases} \tag{13}$$

and

$$\begin{cases} \underline{x}^{\Delta}(t) = A\underline{x}(t) + A_d\underline{x}(\varrho(t-d)) + B\omega_0 \\ \underline{y}(t) = C\underline{x}(t) + C_d\underline{x}(\varrho(t-\tau)) + D\omega_0 \end{cases} \tag{14}$$

The following lemma will give the relationship among the state and output trajectories of system (5) and those of systems (13) and (14) with same constant input ω_0.

Lemma 4.4. *Under Assumption 4.1, let $\bar{x}(t)$, $\underline{x}(t)$ and $\bar{y}(t)$, $\underline{y}(t)$ be state trajectories and outputs of system (13) and system (14), respectively. Then, the trajectories $x(t)$ and $y(t)$ of system (5) satisfy $\underline{x}(t) \leq x(t) \leq \bar{x}(t)$ and $\underline{y}(t) \leq y(t) \leq \bar{y}(t)$ for all $t \in [0, +\infty)_{\mathbb{T}}$, respectively.*

Proof. Define error variable $e(t) = x(t) - \underline{x}(t)$, then we can obtain

$$\begin{aligned} e^{\Delta}(t) = {} & Ae(t) + A_d e(\varrho(t - d(t))) \\ & + A_d(\underline{x}(\varrho(t-d(t))) - \underline{x}(\varrho(t-d))) \end{aligned} \tag{15}$$

with initial condition $e(s) = 0$ for all $s \in [-\widetilde{d},\ 0]_{\mathbb{T}}$. Based on Lemma 4.2, we obtain $\underline{x}(\varrho(t-d(t))) - \underline{x}(\varrho(t-d)) \geq 0$ since $\varrho(t-d(t)) \geq \varrho(t-d) \geq 0$. Regarding $\underline{x}(\varrho(t-d(t))) - \underline{x}(\varrho(t-d))$ as an nonnegative input of the error system (15), it follows from Theorem 3.2 that $e(t) \geq 0$ for all $t \in [0, +\infty)_{\mathbb{T}}$, which implies that $x(t) \geq \underline{x}(t)$ for all $t \in [0, +\infty)_{\mathbb{T}}$.

In addition, note that matrices $C \geq 0$, $C_d \geq 0$ and $D \geq 0$, we can conclude that

$$\begin{aligned} y(t) &= Cx(t) + C_d x(\varrho(t-\tau(t))) + D\omega_0 \\ &\geq C\underline{x}(t) + C_d\underline{x}(\varrho(t-\tau(t))) + D\omega_0 \\ &\geq C\underline{x}(t) + C_d\underline{x}(\varrho(t-\tau)) + D\omega_0 = \underline{y}(t). \end{aligned}$$

Following a similar manner, one can prove that $x(t) \leq \bar{x}(t)$ and $y(t) \leq \bar{y}(t)$. This completes the proof.

Now, we are ready to give a characterization on L_∞-gain of system (5).

Theorem 4.1. *Under Assumption 4.1, the L_∞-gain of system (5) is $\|(D - (C + C_d)(A + A_d)^{-1}B)\omega_0\|_\infty$.*

Proof. Given any input $\omega(t)$ satisfying $\omega(t) \geq 0$ and $\|\omega\|_{L_\infty} = 1$, we have $\omega(t) \leq \omega_0 = \mathbf{1}_p$. From Lemma 4.3, we have $\lim_{t\to+\infty}\underline{x}(t) = -(A + A_d)^{-1}B\omega_0$. Then, $\lim_{t\to+\infty}\underline{y}(t;\omega_0) = (D - (C + C_d)(A + A_d)^{-1}B)\omega_0$. From Lemma 4.4, one has $y(t;\omega_0) \geq \underline{y}(t;\omega_0)$. Thus, $\sup_{t\in\mathbb{T}}\|y(t;\omega)\|_{L_\infty} \geq \|(D - (C + C_d)(A + A_d)^{-1}B)\omega_0\|_\infty$.

Similarly, from Lemmas 4.3 and 4.4, we can also obtain that $\lim_{t\to+\infty}\bar{y}(t;\omega_0) = (D - (C + C_d)(A + A_d)^{-1}B)\omega_0$ and $y(t;\omega_0) \leq \bar{y}(t;\omega_0)$. Then, we obtain $\sup_{t\in\mathbb{T}}\|y(t;\omega)\|_{L_\infty} \leq \|\left(D - (C + C_d)(A + A_d)^{-1}B\right)\omega_0\|_\infty$ for all $\omega(t)$ satisfying $\|\omega\|_{L_\infty} = 1$. Thus, we conclude that $\|y\|_{L_\infty} = \|(D - (C + C_d)(A + A_d)^{-1}B)\omega_0\|_\infty$ for all $\omega(t)$ satisfying $\|\omega\|_{L_\infty} = 1$. This completes the proof.

Remark 4.1. From Theorem 4.1, we can know that the L_∞-gain of positive system is independent of both magnitude of time-delays and time scale, which can be fully determined by system matrices.

5. Numerical example

Example 5.1. Let us consider the linear system (5) with following system matrices:

$$A = \begin{bmatrix} -1.0 & 0.04 & 0.05 \\ 0.03 & -1.0 & 0.08 \\ 0.01 & 0.05 & -1.0 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0.3 & 0.1 \\ 0.03 & 0.05 & 0.1 \\ 0.06 & 0.03 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \\ 0 & 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 1.0 & 0.3 & 0.1 \end{bmatrix}, C_d = \begin{bmatrix} 0.2 & 0.7 & 0.4 \\ 0 & 0.1 & 0.5 \end{bmatrix}, \ D = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}.$$

Given time scale $\mathbb{T} = \{t_n\} \cup [-\frac{3}{2}, 0]$, where $t_0 := 0$, $t_n := \sum_{k=1}^{n}\frac{1}{k}$, $n \geq 1$. Since the graininess at t_n is $\mu(t_n) = \frac{1}{n+1}$, then $\eta = 1$. It is easily checked that $A + \eta I_n \geq 0$, $A_d \geq 0$, $B \geq 0$, $C \geq 0$, $C_d \geq 0$ and $D \geq 0$. Thus, it follows from Theorem 3.2 that this system is positive.

Note that $A + A_d$ is Hurwitz, it follows from Theorem 3.3 that system (6) is asymptotically stable. Let $d(t_n) = \frac{1}{n}$ and $\tau(t_n) = \frac{1}{n}$, under initial condition $\phi(s) = [1.2|\cos(10s)|,\ 1.2|\cos(5s)|,\ 0.5|\cos(7s)|]$ $(s \in [-\frac{3}{2}, 0])$, the trajectory of system

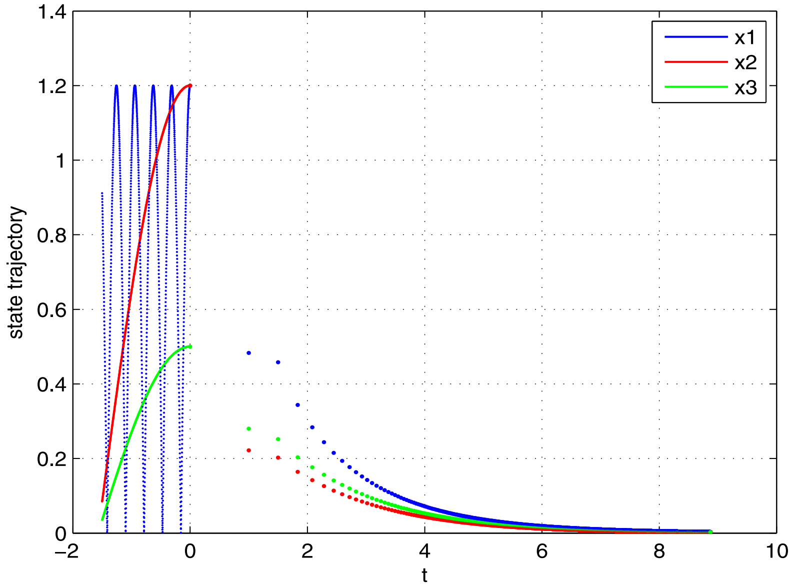

Fig. 1. State trajectory of system (6).

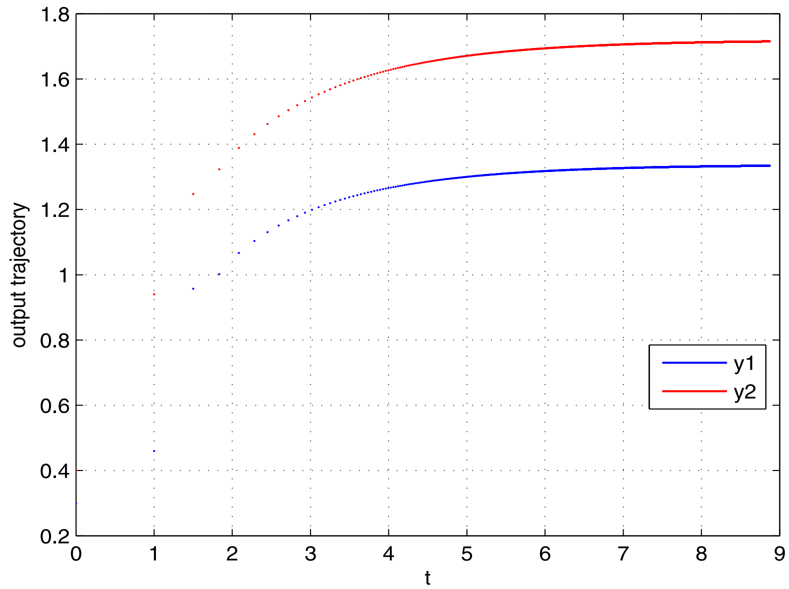

Fig. 2. Output trajectory of system (5) with zero initial conditions and input $\omega = \mathbf{1}_2$.

(6) is depicted in Fig. 1, which demonstrates that system (6) is asymptotically stable and coincides with the conclusion of Theorem 3.3.

In addition, under zero initial conditions, the output trajectories of system (5) with input $\omega(t) = \mathbf{1}_2$ are depicted in Fig. 2. From the proof process of Theorem 4.1, we can know that the limit of the output trajectory is $(D - (C + C_d)(A + A_d)^{-1}B)\mathbf{1}_2 = [1.3371, 1.7191]^T$ as $t \to +\infty$, which is consistent with the output trajectory depicted in Fig. 2.

Example 5.2. Let us consider the linear system (5) with following system matrices:

$$A = \begin{bmatrix} -2.0 & 0.4 & 0.5 \\ 0.3 & -2.0 & 0.8 \\ 0.1 & 0.5 & -2.0 \end{bmatrix}, A_d = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 \\ 0.2 & 0.1 & 0.2 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.1 \\ 0 & 0.5 \end{bmatrix},$$

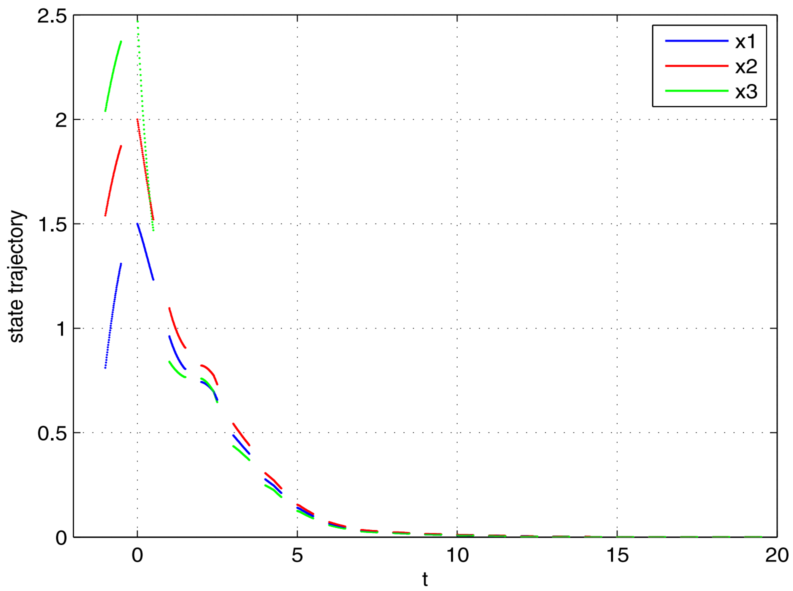

Fig. 3. State trajectory of system (6).

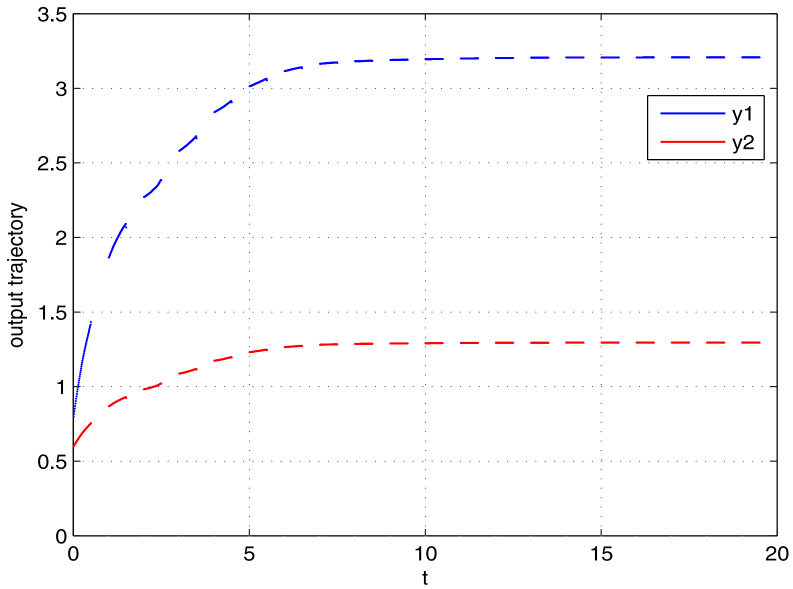

Fig. 4. Output trajectory of system (5) with zero initial conditions and input $\omega = \mathbf{1}_2$.

$$C = \begin{bmatrix} 2.0 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}, C_d = \begin{bmatrix} 0.3 & 0.1 & 0.2 \\ 0 & 0.1 & 0.1 \end{bmatrix}, \ D = \begin{bmatrix} 0.5 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}.$$

Given time scale $\mathbb{T} = \bigcup_{k\in\mathbb{N}_0}[k, k+\frac{1}{2}] \cup [-1, 0]$, we can obtain $\eta = 2$. It is easily checked that $A + \eta I_n \geq 0$, $A_d \geq 0$, $B \geq 0$, $C \geq 0$, $C_d \geq 0$ and $D \geq 0$. Thus, it follows from Theorem 3.2 that this system is positive.

Note that $A+A_d$ is Hurwitz, it follows from Theorem 3.3 that system (6) is asymptotically stable. Let $d(t) = \sin(0.5\pi t)+1$ and $\tau(t) = \sin(0.1\pi t) + 1$, under initial condition $\phi(s) = [1.5|\cos(s)|,\ |\cos(s)| + 1,\ |\cos(s)| + 1.5]$ $(s \in [-1, 0] \cap \mathbb{T})$, the trajectory of system (6) is depicted in Fig. 3, which demonstrates that system (6) is asymptotically stable and coincides with the conclusion of Theorem 3.3.

In addition, under zero initial conditions, the output trajectories of system (5) with input $\omega(t) = \mathbf{1}_2$ are depicted in Fig. 4. From the proof process of Theorem 4.1, we can know that the limit of the output trajectory is $(D - (C + C_d)(A + A_d)^{-1}B)\mathbf{1}_2 = [3.2073,\ 1.2952]^T$ as $t \to +\infty$, which is consistent with the output trajectory depicted in Fig. 4.

6. Conclusions

In this paper, the asymptotic stability and L_∞-gain analysis of positive systems with bounded time-varying delays on general time scale have been studied. Firstly, a sufficient condition and a necessary condition are given to ensure the positivity of systems with bounded time-varying delays on general time scale. Then, based on comparison principle, we reproved the asymptotical stability of positive systems with bounded time-varying delays on general time scales. In addition, the L_∞-gain of positive systems on general time scales was characterized only by the system matrices. Overall, throughout this paper, one can know that the positivity, asymptotical stability and L_∞-gain of positive systems on time scales are all independent of both the magnitudes of delays and time scales, which can be fully determined by system matrices, and that is no difference from the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Finally, two examples with simulations were also given to show the validity of the provided results.

References

[1] L. Farina, S. Rinaldi, Positive Linear Systems: Theory and Applications, John Wiley & Sons, 2011.
[2] T. Kaczorek, Positive 1D and 2D Systems, Springer Science & Business Media, 2012.
[3] T. Tanaka, C. Langbort, The Bounded Real Lemma for internally positive systems and H_∞ structured static state feedback, IEEE Trans. Automat. Control 56 (9) (2011) 2218–2223.
[4] O. Mason, R. Shorten, On linear copositive Lyapunov functions and the stability of switched positive linear systems, IEEE Trans. Automat. Control 52 (7) (2007) 1346–1349.
[5] X. Liu, Stability analysis of switched positive systems: A switched linear copositive Lyapunov function method, IEEE Trans. Circuits Syst. II 56 (5) (2009) 414–418.
[6] X. Liu, W. Yu, L. Wang, Stability analysis for continuous-time positive systems with time-varying delays, IEEE Trans. Automat. Control 55 (4) (2010) 1024–1028.
[7] X. Zhao, L. Zhang, P. Shi, Stability of a class of switched positive linear time-delay systems, Internat. J. Robust Nonlinear Control 23 (5) (2012) 578–589.
[8] H.R. Feyzmahdavian, T. Charalambous, M. Johansson, Asymptotic stability and decay rates of homogeneous positive systems with bounded and unbounded delays, SIAM J. Control Optim. 52 (4) (2014) 2623–2650.
[9] Y. Ebihara, D. Peaucelle, D. Arzelier, Analysis and synthesis of interconnected positive systems, IEEE Trans. Automat. Control 62 (2) (2017) 652–667.
[10] Y. Cui, J. Shen, Z. Feng, Y. Chen, Stability analysis for positive singular systems with time-varying delays, IEEE Trans. Automat. Control 63 (5) (2018) 1487–1494.
[11] C. Briat, Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization, Internat. J. Robust Nonlinear Control 23 (17) (2013) 1932–1954.
[12] X. Chen, J. Lam, P. Li, Positive filtering for continuous-time positive systems under L_1 performance, Internat. J. Control 87 (9) (2014) 1906–1913.
[13] J. Shen, J. Lam, On l_∞ and L_∞-gains for positive systems with bounded time-varying delays, Internat. J. Systems Sci. 46 (11) (2015) 1953–1960.
[14] J. Shen, J. Lam, Stability and performance analysis for positive fractional-order systems with time-varying delays, IEEE Trans. Automat. Control 61 (9) (2016) 2676–2681.
[15] S. Li, Z. Xiang, Stochastic stability analysis and L_∞-gain controller design for positive Markov jump systems with time-varying delays, Nonlinear Anal. Hybrid Syst. 22 (2016) 31–42.
[16] H. Gao, J. Lam, C. Wang, S. Xu, Control for stability and positivity: equivalent conditions and computation, IEEE Trans. Circuits Syst. II 52 (9) (2005) 540–544.
[17] W.M. Haddad, V. Chellaboina, Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physiological systems, Nonlinear Anal. RWA 6 (1) (2005) 35–65.
[18] X. Zhao, L. Zhang, P. Shi, M. Liu, Stability of switched positive linear systems with average dwell time switching, Automatica 48 (6) (2012) 1132–1137.
[19] P.H.A. Ngoc, Stability of positive differential systems with delay, IEEE Trans. Automat. Control 58 (1) (2013) 203–209.
[20] J. Shen, J. Lam, H_∞ model reduction for discrete-time positive systems with inhomogeneous initial conditions, Internat. J. Robust Nonlinear Control 25 (1) (2015) 88–102.
[21] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, Results Math. 18 (1) (1990) 18–56.
[22] Y. Li, T. Zhang, Global exponential stability of fuzzy interval delayed neural networks with impulses on time scales, Int. J. Neural Syst. 19 (6) (2009) 449–456.
[23] A. Chen, D. Du, Global exponential stability of delayed bam network on time scale, Neurocomputing 71 (16) (2008) 3582–3588.
[24] Q. Song, Z. Zhao, Stability criterion of complex-valued neural networks with both leakage delay and time-varying delays on time scales, Neurocomputing 171 (2016) 179–184.
[25] A. Ogulenko, Asymptotical properties of social network dynamics on time scales, J. Comput. Appl. Math. 319 (1) (2017) 413–422.
[26] E. Pawluszewicz, D.F.M. Torres, Avoidance control on time scales, J. Optim. Theory Appl. 145 (3) (2010) 527–542.
[27] M.S. Ali, J. Yogambigai, Synchronization of complex dynamical networks with hybrid coupling delays on time scales by handling multitude Kronecker product terms, Appl. Math. Comput. 291 (2016) 244–258.
[28] Z. Bartosiewicz, On positive reachability of time-variant linear systems on time scales, Bull. Pol. Acad. Sci.: Tech. Sci. 61 (4) (2013) 905–910.
[29] Z. Bartosiewicz, Positive realizations on time scales, Control Cybernet. 42 (3) (2013) 315–327.
[30] Z. Bartosiewicz, Linear positive control systems on time scales; controllability, Math. Control Signals Systems 25 (3) (2012) 327–343.
[31] T.S. Doan, A. Kalauch, S. Siegmund, F. Wirth, Stability radii for positive linear time-invariant systems on time scales, Systems Control Lett. 59 (3) (2010) 173–179.
[32] C. Zhang, Y. Sun, Stability analysis of linear positive systems with time delays on time scales, Adv. Difference Equ. 56 (2012) 1–11.
[33] M. Bohner, A. Peterson, Dynamic equations on time scales, in: An Introduction with Applications, Birkhauser, Boston, 2001.
[34] M. Bohner, A. Peterson, Laplace transform and Z-transform: unification and extension, Methods Appl. Anal. 9 (1) (2002) 151–158.
[35] J.J. DaCunha, Stability for time-varying linear dynamic systems on time scales, J. Comput. Appl. Math. 176 (2) (2005) 381–410.
[36] T.S. Doan, A. Kalauch. S. Siegmund, Exponential stability of linear time-invariant system on time scales, Nonlinear Dyn. Syst. Theory 9 (1) (2009) 37–50.
[37] Z. Bartosiewicz, Exponential stability of nonlinear positive systems on time scales, Nonlinear Anal. Hybrid Syst. 33 (2019) 143–150.
[38] C. Briat, Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization, Internat. J. Robust Nonlinear Control 23 (17) (2013) 1932–1954.