

Preservation of Controllability of Single-Input Time-Varying Linear Systems Under Sampling

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Abstract—We show that a time-varying controllable continuous-time system with real analytic coefficients, when sampled in almost any rate, yields a completely controllable discrete-time system.

Index Terms—Analytically varying systems, linear control systems, preservation of controllability, sampling.

I. INTRODUCTION AND MAIN RESULT

In this note, we study preservation of controllability under zero-hold uniform sampling for time-varying single-input linear systems with real analytic coefficients.

Sampling and controllability are key notions in modern control-theory (see, e.g., [1]). In particular, the problem of preservation of controllability under sampling attracted attention since the pioneering work of Kalman *et al.* [5]. Extensions to multi-input systems [3], [6] and to nonlinear systems [8] were established. Different sampling models, such as multirate-sampling [7] and generalized sampled-data hold functions [4] were also considered. This work contributes to this line of research by investigating the problem of preservation of controllability for time-varying systems with real analytic coefficients under zero-hold uniform sampling.

A formal description of the main result follows.

Consider an n -dimensional single-input time-varying control system with real analytic coefficients

$$\dot{x}(t) = A(t)x(t) + b(t)u(t) \quad (1)$$

where $A(t)$ and $b(t)$ are time-varying $n \times n$ matrix and n -dimensional column vector, respectively. All the coefficients in these matrices are assumed to be real analytic in t .

The sampling control strategy that we consider is to sample the state at prescribed equidistributed times and to hold the control constant in periods between samplings. Let δ be the length of the sampling interval and let $u_\delta(k)$ be the control value at the k th time step. For the discrete set of times where sampling occur, say $t_k = k\delta$, $k \in \mathbb{Z}$, we get a discrete-time system of the form

$$x_{k+1} = A_\delta(k)x_k + b_\delta(k)u_\delta(k) \quad (2)$$

where

$$\begin{aligned} A_\delta(k) &= \Phi(t_{k+1}, t_k), \\ b_\delta(k) &= \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, s)b(s) ds \end{aligned} \quad (3)$$

and Φ is the fundamental matrix solution associated to $A(t)$.

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Generally speaking, a system is said to be controllable if, at all times, there exists an input that drives any initial state to any final state. The following definitions give the precise notions for continuous-time and discrete-time systems that we use in this note.

Definition I.1: *The continuous-time control system (1) is said to be controllable if for every $t_i \in \mathbb{R}$ and every pair of states, $\xi_i, \xi_f \in \mathbb{R}^n$, there is a time t_f and a control $u : [t_i, t_f] \rightarrow \mathbb{R}$ such that if $x(t_i) = \xi_i$ then $x(t_f) = \xi_f$.*

Definition I.2: *The discrete-time control system (2) is said to be completely controllable (in n time steps) if for every $k \in \mathbb{Z}$ and every pair of states, $\xi_i, \xi_f \in \mathbb{R}^n$, there are control values $u_\delta(k), \dots, u_\delta(k+n-1)$ such that if $x_k = \xi_i$ then $x_{k+n} = \xi_f$.*

In the statement of the main result that follows, the term “almost every sampling period” means except for a countable set of sampling periods.

Theorem I.3: *If the system (1) is controllable then, for almost every sampling period $\delta > 0$, the sampled-data system (2) is completely controllable.*

II. PROOF OF THE MAIN RESULT

The proof of Theorem I.3 is presented in two steps: First, a statement about real analytic curves is given as Proposition II.1 and proved using some intermediate claims. Then, the proof of the theorem is derived as a corollary of that proposition.

Proposition II.1: *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a real analytic curve. Assume that there is an uncountable set $\Delta \subseteq \mathbb{R}_{>0}$ such that, for every $\delta \in \Delta$, there exists $k_\delta \in \mathbb{Z}$ such that the vectors*

$$\int_{k_\delta \delta}^{(k_\delta+1)\delta} \psi(t) dt, \dots, \int_{(k_\delta+n-1)\delta}^{(k_\delta+n)\delta} \psi(t) dt$$

are linearly dependent. Then there exists a proper linear subspace $V \subset \mathbb{R}^n$ such that $\psi(\mathbb{R}) \subseteq V$.

Claim II.2: *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a curve satisfying the conditions of Proposition II.1. Then there exists $k \in \mathbb{Z}$ such that the function*

$$f(\delta) = \det \left[\int_{k\delta}^{(k+1)\delta} \psi(t) dt, \dots, \int_{(k+n-1)\delta}^{(k+n)\delta} \psi(t) dt \right]$$

vanishes for every $\delta > 0$.

Proof: For a curve satisfying the conditions of Proposition II.1, we have a map $\delta \mapsto k_\delta$ from an uncountable set to a countable set. Since the union of a countable number of countable sets is a countable set, there must be $k \in \mathbb{Z}$ whose preimage is uncountable. For this k , the real analytic function $f(\delta)$ has an uncountable zero set, therefore it is identically zero. \square

Claim II.3: *Let $\psi : \mathbb{R} \rightarrow \mathbb{R}^n$ be a real analytic curve. The m th derivative at zero, $(d^m/d\delta^m)f(0)$, of the function defined in the preceding claim is given by*

$$\sum_{\substack{m_1 + \dots + m_n = m \\ 0 < m_1 < \dots < m_n}} C(m_1, \dots, m_n) \det \left[\psi^{(m_1-1)}(0), \dots, \psi^{(m_n-1)}(0) \right]$$

where $C(m_1, \dots, m_n)$ is

$$\frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} \det \left((k+i)^{m_j} - (k+i-1)^{m_j} \right)_{i,j=1}^n$$

and $\psi^{(i)}(0)$ denotes the i th derivative of ψ at zero.

Proof: Recall that the m th derivative, d^m/dt^m , of the determinant of a time-varying matrix, $M(t)$, is given by

$$\sum_{m_1+\dots+m_n=m} \frac{m!}{m_1! \dots m_n!} \det \left[\frac{d^{m_1}}{dt^{m_1}} M_1(t), \dots, \frac{d^{m_n}}{dt^{m_n}} M_n(t) \right] \quad (4)$$

where, for $i = 1, \dots, n$, the vector $M_i(t)$ denotes the i th column of $M(t)$.

To shorten notation, we will use the shortcuts

$$c(i, m) = (k+i)^m - (k+i-1)^m$$

and

$$y_m = \psi^{m-1}(0).$$

For $i = 1, \dots, n$, consider the functions $g_i(\delta) = \int_{(k+i-1)\delta}^{(k+i)\delta} \psi(t) dt$. The i th function, $g_i(\delta)$, is the i th column of the matrix inside the determinant in $f(\delta)$. It is easy to verify that

$$\frac{d^m}{d\delta^m} g_i(0) = \begin{cases} c(i, m) y_m, & m > 0 \\ 0, & m = 0. \end{cases}$$

In particular, by formula (4), the m th derivative of f at 0 is

$$\sum_{\substack{m_1+\dots+m_n=m \\ m_1+\dots+m_n \neq 0}} \frac{m!}{m_1! \dots m_n!} \det[c(1, m_1)y_{m_1}, \dots, c(n, m_n)y_{m_n}].$$

Factoring out scalars from the columns yields

$$\sum_{\substack{m_1+\dots+m_n=m \\ m_1+\dots+m_n \neq 0}} \frac{m!}{m_1! \dots m_n!} \prod_{i=1}^n c(i, m_i) \det[y_{m_1}, \dots, y_{m_n}].$$

Collecting together terms that correspond to permutations of the same partition gives

$$\sum_{\substack{m_1+\dots+m_n=m \\ 0 < m_1 < \dots < m_n}} \frac{m!}{m_1! \dots m_n!} D(m_1, \dots, m_n) \det[y_{m_1}, \dots, y_{m_n}]$$

where

$$D(m_1, \dots, m_n) = \sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^n c(i, m_{\pi(i)}) = \det(c(i, m_j))_{i,j=1}^n.$$

□

We proceed with the proof of Proposition II.1. The proof scheme is to use the fact that f is identically zero in order to prove that the range of ψ is confined within a proper linear subspace. We will do this by proving that the derivatives of ψ at the origin are all in a proper linear space using the fact that all the derivatives of f at zero vanish. The main tool for this proof scheme is provided by the following lemma.

Lemma II.4: Let x_1, x_2, \dots be a sequence of n -dimensional vectors. Assume that there is a function $C : \mathbb{N}^n \rightarrow \mathbb{R} \setminus \{0\}$ such that

$$\sum_{\substack{m_1+\dots+m_n=m \\ 0 < m_1 < \dots < m_n}} C(m_1, \dots, m_n) \det[x_{m_1}, \dots, x_{m_n}] = 0 \quad (5)$$

for every $m \in \mathbb{N}$. Then the sequence is contained in a proper linear subspace.

Proof: We begin by introducing a linear order over the set of ordered n -tuples. Say that $(m_1, \dots, m_n) \prec (\hat{m}_1, \dots, \hat{m}_n)$ if $(\sum_{i=1}^n m_i, \hat{m}_n, \dots, \hat{m}_1)$ precedes $(\sum_{i=1}^n \hat{m}_i, m_n, \dots, m_1)$ lexicographically. Note that this is a well-founded order (see [2, p. 27]).

We argue by contradiction and let (m_1, \dots, m_n) be the minimal (according to the previous order) ordered tuple for which

$$\det[x_{m_1}, \dots, x_{m_n}] \neq 0. \quad (6)$$

Take another ordered tuple, $(\hat{m}_1, \dots, \hat{m}_n)$, such that $\sum_{i=1}^n m_i = \sum_{i=1}^n \hat{m}_i$. Toward a contradiction to the existence of a tuple satisfying (6), we will show that $\det[x_{\hat{m}_1}, \dots, x_{\hat{m}_n}] = 0$, i.e., that all the terms in (5) vanish, except the one that corresponds to (m_1, \dots, m_n) .

Consider first the case where (m_n, \dots, m_1) precedes $(\hat{m}_n, \dots, \hat{m}_1)$ lexicographically. Thus, $(\hat{m}_n, \dots, \hat{m}_1)$ precedes (m_n, \dots, m_1) in our order. Since (m_1, \dots, m_n) is the first tuple for which the determinant is not zero, we have $\det[x_{\hat{m}_1}, \dots, x_{\hat{m}_n}] = 0$.

For the other case, assume that $(\hat{m}_n, \dots, \hat{m}_1)$ precedes (m_n, \dots, m_1) lexicographically. Let $i \in \{0, \dots, n-1\}$ be the first index such that $m_{n-i} \neq \hat{m}_{n-i}$ (more specifically, $m_{n-i} > \hat{m}_{n-i}$). Then, $\hat{m}_{n-j} = m_{n-j}$ for $j = 0, \dots, i-1$ and $\hat{m}_{n-j} < m_{n-i}$ for $j = i, \dots, n-1$. Therefore

$$\det[x_{m_1}, \dots, x_{m_{n-i-1}}, x_{\hat{m}_j}, x_{m_{n-i+1}}, \dots, x_{m_n}] = 0$$

for all $j = 1, \dots, n$ (some because of repeated columns and the others because $\hat{m}_j = m_{n-i} + \sum_{k=1}^n m_k < \sum_{k=1}^n m_k$). Since the vectors x_{m_1}, \dots, x_{m_n} are linearly independent [by (6)], we get that

$$\{x_{\hat{m}_1}, \dots, x_{\hat{m}_n}\} \subset \text{span}(\{x_{m_1}, \dots, x_{m_n}\} \setminus \{x_{m_{n-i}}\}).$$

In particular, $\det[x_{\hat{m}_1}, \dots, x_{\hat{m}_n}] = 0$.

We get that, all the terms in (5), except the term that corresponds to (m_1, \dots, m_n) , vanish. This yields a contradiction with inequality (6). □

Note the similarity of (5) and the expression for the m th derivative of f given in Claim II.3. To apply Lemma II.4, we need to verify that the coefficients are not zero. In the following claim we show that they are all positive.

Claim II.5: For every $k > 0$ and integers $0 < m_1 < \dots < m_n$, the matrix

$$M = \left((k+i)^{m_j} - (k+i-1)^{m_j} \right)_{i,j=1}^n$$

has a positive determinant.

Proof: Define

$$f(k; m_1, \dots, m_n) = \det M.$$

Note that this function vanishes when there is a repeating parameter, i.e., $m_i = m_j$ for some $i \neq j \in \{0, 1, \dots, n\}$ where $m_0 = 0$.

By adding the rows of M and deleting telescopic terms, it is easy to see that

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{pmatrix} M = \left((k+i)^{m_j} - k^{m_j} \right)_{i,j=1}^n.$$

When $k = 0$, this is a generalized Vandermonde matrix (see [9] for more information about generalized Vandermonde matrices). Since the determinant of a generalized Vandermonde matrix is positive, we get that $f(0; m_1, \dots, m_n) > 0$ for all integers $0 < m_1 < \dots < m_n$.

By formula (4), the derivative of $f(k; m_1, \dots, m_n)$ with respect to k is

$$\frac{d}{dk} f(k; m_1, \dots, m_n) = \sum_{i=1}^n m_i f(k; m_1, \dots, m_i - 1, \dots, m_n). \quad (7)$$

We now prove, by induction on $m = m_1 + \dots + m_n$, that $f(k; m_1, \dots, m_n) > 0$ for all $k > 0$ and integers $0 < m_1 < \dots < m_n$.

By (7), $(d/dk)f(k; 1, \dots, n) = 0$ for every k (because all the terms have repeating parameter: $m_{i-1} = m_i - 1$). Therefore, $f(k; 1, \dots, n) = f(0; 1, \dots, n) > 0$. This establishes the case $m = n(n+1)/2$, which is the base of the induction.

If $m_1 + \dots + m_n > n(n+1)/2$ then each of the summands on the right hand side of (7) is nonnegative (by the induction hypothesis) so the derivative $(d/dk)f(k; m_1, \dots, m_n)$ is positive. In particular, for every $k > 0$, $f(k; m_1, \dots, m_n) \geq f(0; m_1, \dots, m_n) > 0$. \square

The proof of the proposition follows from the preceding claims.

Proof of Proposition II.1: By Claim II.2 and Claim II.3, the sequence $x_1 = \psi(0), x_2 = \psi^{(1)}(0), \dots$ of the derivatives of ψ at zero satisfies, for every $m \in \mathbb{N}$

$$\sum_{\substack{m_1+\dots+m_n=m \\ 0 < m_1 < \dots < m_n}} C(m_1, \dots, m_n) \det[x_{m_1}, \dots, x_{m_n}] = 0$$

where

$$C(m_1, \dots, m_n) = \frac{(m_1 + \dots + m_n)!}{m_1! \cdots m_n!} \det(c(i, m_j))_{i,j=1}^n.$$

By Claim II.5, $C(m_1, \dots, m_n) > 0$ for any tuple $0 < m_1 < \dots < m_n$. Hence, by Lemma II.4, all the derivatives of ψ at the origin lie in a proper linear subspace. Since ψ is real analytic, its whole image is contained in that subspace. \square

The proof of the theorem follows as a corollary of the preceding proposition.

Proof of Theorem I.3: Consider the curve $\psi(t) = \Phi(0, t)b(t)$, where Φ is the fundamental matrix solution associated to $A(t)$. Let Δ be the set of δ values for which the sampled data system (2) is *not* completely controllable.

By Definition I.2, for every $\delta \in \Delta$ there exists $k_\delta \in \mathbb{Z}$ and a pair of states, $\xi_i, \xi_f \in \mathbb{R}^n$, for which there are no control values $u_\delta(k_\delta), \dots, u_\delta(k_\delta + n - 1)$ such that if $x_{k_\delta} = \xi_i$ then $x_{k_\delta+n} = \xi_f$.

For notational convenience, we define

$$P_i = A_\delta(k_\delta + n - 1) \cdots A_\delta(k_\delta + n - i)$$

for $i = 0, \dots, n$. Especially, P_0 is the n -dimensional identity matrix.

By (2)

$$x_{k_\delta+n} = P_n x_{k_\delta} + \sum_{i=0}^{n-1} P_{n-i-1} b_\delta(k_\delta + i) u_\delta(k_\delta + i). \quad (8)$$

If we plug ξ_i and ξ_f instead of x_{k_δ} and $x_{k_\delta+n}$ respectively, we get that there are no control values $u_\delta(k_\delta), \dots, u_\delta(k_\delta + n - 1)$ such that the second term of the right-hand side of (8) is $\xi_f - P_n^{-1}\xi_i$ (P_n is regular, being a product of regular matrices). In particular, the matrix

$$[P_{n-1} b_\delta(k_\delta), P_{n-2} b_\delta(k_\delta + 1), \dots, P_0 b_\delta(k_\delta + n - 1)]$$

is singular. By (3), for every $i = 0, 1, \dots, n - 1$

$$\begin{aligned} P_{n-1-i} b_\delta(k_\delta + i) &= \int_{(k_\delta+i)\delta}^{(k_\delta+i+1)\delta} \Phi((k_\delta+n)\delta, s) b(s) ds \\ &= \Phi((k_\delta+n)\delta, 0) \int_{(k_\delta+i)\delta}^{(k_\delta+i+1)\delta} \psi(s) ds. \end{aligned}$$

Therefore, the matrix

$$\Phi((k_\delta+n)\delta, 0) \left[\int_{k_\delta\delta}^{(k_\delta+1)\delta} \psi(s) ds, \dots, \int_{(k_\delta+n-1)\delta}^{(k_\delta+n)\delta} \psi(s) ds \right]$$

is singular. Because $\Phi((k_\delta+n)\delta, 0)$ is regular, we conclude that

$$\left[\int_{k_\delta\delta}^{(k_\delta+1)\delta} \psi(s) ds, \dots, \int_{(k_\delta+n-1)\delta}^{(k_\delta+n)\delta} \psi(s) ds \right]$$

is singular.

If Δ is not countable, the conditions of Proposition II.1 are met, so the image of ψ is contained in a proper subspace. In particular, the system (1) is not controllable [8, p. 109, Th. 5]. \square

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