



## Brief Paper

# Stability analysis of interconnected Hamiltonian systems under time delays

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**Abstract:** Sufficient conditions are derived for checking whether interconnected port-Hamiltonian systems are stable in the presence of time delays. It is assumed that the time delay parameters are unknown time-varying functions for which the only available information is about the upper bounds on their magnitude and/or variation. The stability conditions proposed here are established by constructing Lyapunov–Krasovskii stability certificates, based on the Hamiltonians of the individual port-Hamiltonian systems. The forms of the Lyapunov–Krasovskii functionals vary according to the information available on the delay parameter. It is shown how different informations on the delay are utilised to construct stability certificates.

## 1 Introduction

Port-Hamiltonian models are natural candidates to describe many physical systems [1]. These classes of systems are basically defined with respect to a power conserving geometric structure capturing the basic interconnection laws and an Hamiltonian function given by the total stored energy of the system. A key feature of port-Hamiltonian systems is that a power conserving interconnection of a number of port-Hamiltonian systems is again a port-Hamiltonian system. This concept of interconnection is important from a control point of view, since implementing a control law or controlling a system is usually done with an external device via external port variables. An immediate example is the control by interconnection of port-Hamiltonian systems [1, 2], where the plant port-Hamiltonian system is connected to a controller port-Hamiltonian system via a feedback loop such that the closed-loop system has desired stability properties, by using various energy shaping techniques. Since the interconnection preserves the port-Hamiltonian structure, the control by interconnection method has some inherent robustness properties largely because of the structure of the port-Hamiltonian systems. Now assume that there is some time delay in the communication between the plant and the controller. The presence of time delays may often result in a closed-loop system which is not exactly in the port-Hamiltonian form. In other words the ‘ $J - R$ ’ structure is actually destroyed and hence does not reveal any information on the stability of the system. This motivates us to ask the following question: When is a time delay Hamiltonian system stable?

In the case of linear time delay systems, the stability analysis is either based on generating Lyapunov–Krasovskii

or Lyapunov–Razumikin functions (time domain approach [3–11]) or using tools such as the small gain theorem or integral quadratic constraints (frequency domain approach [12–14]). For a comprehensive review of various stability criterion we refer to [15]. Few delay-dependent and delay-independent criteria for general non-linear systems have been reported in [16–18], wherein the delay parameter is assumed to be constant.

In this paper we analyse stability of interconnected port-Hamiltonian systems in the presence of time delays for which we have information on the upper bounds of the magnitude of the delay and its variation. Based on these bounds, we derive sufficient conditions for the interconnected system with time-varying delays to be stable. These conditions are derived based on the construction of Lyapunov–Krasovskii functionals by making use of the Hamiltonians of the individual subsystems. This is an endeavour continuing from our previous work ([19]) where we consider a similar interconnected port-Hamiltonian structure. In [19], we consider the case where the time delay is slowly varying (i.e. rate of variation of the delay parameter is less than 1). In this paper we consider more scenarios where different information on the delay is assumed to be available. It is shown that the forms of Lyapunov–Krasovskii stability certificates vary according to the information available on the delay parameter. Furthermore, we provide more insights on how different information on the delay is utilised to construct stability certificates.

**Notation:** We use the symbols  $\mathbb{R}$  to denote the set of real numbers,  $\mathbb{R}^+$  to denote non-negative real numbers,  $\mathbb{R}^n$  to denote  $n \times 1$  real vectors and  $\mathbb{R}^{n \times m}$  to denote  $n \times m$  real matrices. For  $a < b$ , the Banach space of

continuous functions mapping the interval  $[a, b]$  into  $\mathbb{R}^n$ , with the topology of uniform convergence, is denoted by  $c([a, b], \mathbb{R}^n)$ . For  $\psi \in c([a, b], \mathbb{R}^n)$ , the norm is defined as  $\|\psi\|_c = \sup_{a \leq \theta \leq b} |\psi(\theta)|$ , where  $|\cdot|$  is the standard Euclidean norm of  $\mathbb{R}^n$ . Finally,  $c^r([a, b], \mathbb{R}^n) := \{\psi \in c([a, b], \mathbb{R}^n) : \|\psi\|_c < r\}$ .

For a given continuous function  $x : [\beta, \infty) \rightarrow \mathbb{R}^n$ , the notation  $\dot{x}$  denotes the mapping from  $[\beta - a, \infty)$  to  $c([a, b], \mathbb{R}^n)$  defined as  $\dot{x}[t](\theta) := x(t + \theta)$ ,  $a \leq \theta \leq b$ .

Given a function  $H(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ , the notations  $\nabla H(x)$  and  $\nabla^2 H(x)$  are used to denote the gradient vector and the Hessian matrix of  $H(x)$  at  $x$ , respectively. Likewise, given a vector-valued function  $G(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , the notation  $\nabla G(x)$  denotes the matrix-valued function

$$\begin{bmatrix} \frac{\partial G_1}{\partial x_1}(x) & \frac{\partial G_1}{\partial x_2}(x) & \cdots & \frac{\partial G_1}{\partial x_n}(x) \\ \frac{\partial G_2}{\partial x_1}(x) & \frac{\partial G_2}{\partial x_2}(x) & \cdots & \frac{\partial G_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial x_1}(x) & \frac{\partial G_m}{\partial x_2}(x) & \cdots & \frac{\partial G_m}{\partial x_n}(x) \end{bmatrix}$$

In the paper, often the argument of a function is a function of time; for example  $H(x(t))$ . For the sake of simplifying the notation, we will drop the time dependency and simply write  $H(x)$  for  $H(x(t))$  when it is clear from the context. Likewise, time dependency of a quantity will be dropped when there is no risk of confusion.

The  $n \times m$ -dimensional identity and zero matrices are denoted by  $I_{n \times m}$  and  $O_{n \times m}$ , respectively. When  $n = m$ , we will simply write  $I_n$  and  $O_n$ . For a given matrix  $A$ , we use the notation  $\mathbb{S}(A)$  to denote  $A + A'$ , where  $A'$  denotes the transposition of  $A$ . When  $A$  is symmetric, ' $A > 0$ ' (' $A < 0$ ') is used to denote positive definiteness (negative definiteness).

## 2 Preliminaries and problem formulation

Consider a functional differential equation of retarded type

$$\dot{x}(t) = f(\dot{x}[t]) \quad (1)$$

defined on the positive time interval  $[0, \infty)$ , where  $f : \Omega \times \mathbb{R}^p \rightarrow \mathbb{R}^n$ ,  $\Omega \subseteq c([- \xi, 0], \mathbb{R}^n)$  and  $\dot{x}[t] \in \Omega$ . The initial condition of (1) is denoted by  $\dot{x}[0]$ , and it is also a function taking values in  $\Omega$ . We assume that (1) has a unique solution for any initial condition  $\dot{x}[0] \in \Omega$  and that  $x(t) \equiv 0$  (the zero function) is a solution of (1). The concept of internal stability of the solution  $x(t) \equiv 0$  of (1) is given below.

**Definition 1:** The solution  $x(t) \equiv 0$  of (1) is called stable if for any  $\epsilon > 0$  there exists a  $\delta(\epsilon) > 0$  such that for any initial condition  $\dot{x}[0]$  which satisfies  $\|\dot{x}[0]\|_c < \delta(\epsilon)$ , the corresponding solution  $x(t)$  satisfies  $|x(t)| < \epsilon$  for all  $t \geq 0$ . It is called asymptotically stable if it is stable and  $\delta(\epsilon)$  can be chosen such that  $\|\dot{x}[0]\|_c < \delta(\epsilon)$  implies that  $\lim_{t \rightarrow \infty} x(t) \rightarrow 0$ . It is called globally asymptotically stable if it is asymptotically stable, and for any initial condition  $\dot{x}[0]$  the corresponding solution  $x(t)$  approaches 0 as  $t \rightarrow \infty$ , no matter how large  $\|\dot{x}[0]\|_c$  is.

A corner stone for studying internal stability of (1) is the Lyapunov–Krasovskii theorem. The theorem is briefly summarised below. The readers are referred to [5, 20] for details. Let  $V : \Omega \rightarrow \mathbb{R}$ . The derivative of  $V$  with respect to time along a solution  $x$  of (1) is defined as

$$\dot{V}(\dot{x}[t]) = \limsup_{\Delta t \rightarrow 0} \frac{1}{\Delta t} (V(\dot{x}[t + \Delta t]) - V(\dot{x}[t]))$$

We have the following theorem for stability of the autonomous system (1).

**Theorem 2 (Lyapunov–Krasovskii):** Suppose that for any given  $r > 0$ ,  $f$  defined in (1) maps  $c^r([- \xi, 0], \mathbb{R}^n)$  into a bounded set in  $\mathbb{R}^n$ . Let  $w_1, w_2, w_3$  be continuous non-negative non-decreasing functions which satisfy  $w_1(0) = w_2(0) = 0$ , and  $w_1(s) > 0, w_2(s) > 0$  for  $s \neq 0$ . If there exists  $V : \Omega \rightarrow \mathbb{R}$  such that  $V$  and the time derivative of  $V$  along the solution of (1) satisfy

$$w_1(|\phi(0)|) \leq V(\phi) \leq w_2(\|\phi\|_c) \text{ and } \dot{V}(\phi) \leq -w_3(|\phi(0)|)$$

for all  $\phi \in \Omega$ , then the zero solution  $x(t) \equiv 0$  of (1) is stable. If, in addition,  $w_3(s) > 0$  for any  $s \neq 0$ , then the solution  $x(t) \equiv 0$  of (1) is asymptotically stable. Finally, if  $\lim_{s \rightarrow \infty} w_1(s) = \infty$ , then the stability is global.

As usual, stability of any non-zero equilibrium  $x_*$  can be studied as the stability of the zero solution  $x(t) \equiv 0$ , by 'shifting the equilibrium' to the origin.

### 2.1 Port-Hamiltonian systems

In this paper, we are interested in stability of interconnection of non-linear systems, called port-Hamiltonian systems, with a specific structure which is of the form

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \nabla H(x) + g(x)u \\ y &= g(x)' \nabla H(x) \end{aligned} \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  is the state vector at time  $t$  and  $u(t) \in \mathbb{R}^p$  is the control action.  $H : \mathbb{R}^n \rightarrow \mathbb{R}^+$ , which is sometimes termed as 'total stored energy', satisfies  $H(0) = 0, \nabla H(0) = 0$  and  $H(x) > 0$  for all  $x \neq 0$ .  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  is called 'the port matrix'.  $J : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  and  $R : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  satisfy  $J(x) = -J(x)'$  for all  $x$  and  $R(x) = R(x)' \geq 0$  for all  $x \neq 0$ .  $J(x)$  and  $R(x)$  are often referred to as 'the natural interconnection matrix' and 'the damping matrix', respectively.

Owing to the skew symmetry of  $J(x)$  and semi-positive definiteness of  $R(x)$ , the time derivative of  $H$  along the solution of (2) satisfies

$$\dot{H} = -\nabla H(x)' R(x) \nabla H(x) + u'y \leq u'y \quad (3)$$

which shows that (2) is passive, with the total energy  $H(x)$  acting as storage function. Physically, (3) corresponds to the fact that the internal interconnection structure of the system is power conserving, whereas  $u$  and  $y$  are the power variables of the ports defined by  $g$  and thus  $u'y$  is the externally supplied power and  $-\nabla H(x)' R(x) \nabla H(x)$  the dissipated power. An important corollary of (3) is that, in the absence of input  $u$ , the energy of the autonomous system decreases with time because of dissipation (when  $R(x) > 0$ , or  $R(x) \geq 0$  and the invariance principle of LaSalle holds).

Since the energy function  $H$  is bounded from below, the system will eventually stop at a point of minimum energy.

Now consider two port-Hamiltonian systems of the form (2), and denote the input, output and stored energy function of system  $i$ ,  $i = 1, 2$ , as  $u_i$ ,  $y_i$  and  $H_i$ , respectively. One of the two systems could be thought as a plant to be controlled and the other as the controller. Interconnecting the two systems via the standard (power preserving) feedback interconnection

$$u_2(t) = y_1(t) + v_2(t), \quad u_1(t) = -y_2(t) + v_1(t) \quad (4)$$

where  $v_1$  and  $v_2$  are external signals injected at input ports of the first and the second Hamiltonian systems, respectively, one can readily verify that the composed system is still of port-Hamiltonian form with the total Hamiltonian being  $H = H_1 + H_2$ . Assume that the total Hamiltonian  $H$  is a non-negative function. Since the port-Hamiltonian structure is preserved under interconnection, the interconnected system remains passive with  $H$  serving as storage function and hence, when  $v_1$  and  $v_2$  are zero, stable in the sense of Lyapunov.

## 2.2 Interconnected Hamiltonian systems with communication delays

The feedback interconnection (4) assumes ideal signal transmission between the two port-Hamiltonian systems, so that signals  $y_1$  and  $y_2$  reach their respective destinations instantaneously, or with delays that are not significant enough to be taken into account. This assumption would not be realistic if the two systems were far apart from each other, or communicating with each other through channels that have high traffic. In those cases, the signal transmission delays must be taken into account, and the feedback interconnection relationship should be modelled as

$$\begin{aligned} u_2(t) &= y_1(t - \tau_2(t)) + v_2(t) \\ u_1(t) &= -y_2(t - \tau_1(t)) + v_1(t) \end{aligned} \quad (5)$$

where  $u_i, y_i$ ,  $i = 1, 2$ , are the respective inputs and outputs of the individual subsystems.  $v_1, v_2$  are the external signals injected at the input ports of the first and second Hamiltonian systems, respectively, and  $\tau_2$  and  $\tau_1$  model the forward and backward signal transmission delays, respectively. The closed-loop system now takes the following form

$$\begin{aligned} \dot{x} &= (J(x) - R(x))\nabla H(x) \\ &+ \sum_{i=1,2} T_i(x)[g(x)'\nabla H(x)]_{\tau_i} + g(x)v \end{aligned} \quad (6)$$

where  $H(x) := H_1(x) + H_2(x)$ ,  $J(x) = \text{diag}(J_1(x), J_2(x))$ ,  $R(x) = \text{diag}(R_1(x), R_2(x))$ ,  $g(x) = \text{diag}(g_1(x), g_2(x))$

$$T_1(x) = \begin{bmatrix} 0 & -g_1(x) \\ 0 & 0 \end{bmatrix}, \quad T_2(x) = \begin{bmatrix} 0 & 0 \\ g_2(x) & 0 \end{bmatrix}$$

and  $[g(x)'\nabla H(x)]_{\tau_i} := g(x(t - \tau_i(t)))'\nabla H(x(t - \tau_i(t)))$ . The above equation no longer preserves the port-Hamiltonian structure. The energy balance equation now takes the form

$$\begin{aligned} \dot{H} &= -\nabla H(x)'R(x)\nabla H(x) \\ &+ \sum_{i=1,2} \nabla H(x)'T_i(x)[g(x)'\nabla H(x)]_{\tau_i} + y'v \end{aligned} \quad (7)$$

Since the second term on the right-hand side of (7) may not be negative semi-definite, the total Hamiltonian

$H := H_1 + H_2$  does not help in revealing any information about the passivity/stability property of the interconnected system (6). To deduce stability, one has to seek for Lyapunov–Krasovskii functionals other than the total Hamiltonian. Furthermore, in certain cases the presence of time delays may actually destroy these properties of the system – the feedback interconnection may no longer preserve stability.

## 2.3 Problem formulation

The discussion in the previous session motivates us to consider the following port-Hamiltonian system with time delays

$$\dot{x} = (J(x) - R(x))\nabla H(x) + \sum_{i=1}^m T_i(x)[g(x)'\nabla H(x)]_{\tau_i} \quad (8)$$

Note that the zero function  $x(t) \equiv 0$  is a solution of (8) since we assume  $\nabla H(0) = 0$  [cf. (2)]. We are interested in verifying stability properties of systems in the form of (8) under the following alternative assumptions on the delay parameters  $\tau_i$ :

- A1 Each  $\tau_i$  satisfies  $0 \leq \tau_i(t) \leq h_i$  and  $\dot{\tau}_i(t) \leq d_i$  for all  $t$ .
- A2 Each  $\tau_i$  satisfies  $0 \leq \tau_i(t) \leq h_i$  for all  $t$ , but the variation  $\dot{\tau}_i(t)$  is arbitrary.
- A3 Each  $\tau_i$  satisfies  $\dot{\tau}_i(t) \leq d_i < 1$  and  $\tau_i(t)$  is upper bounded for all  $t$ , but the bound of  $\tau_i$  is unknown to us.

Note that the initial condition of (8) is a function taking values in  $\Omega \subseteq c([-h_{\max}, 0], \mathbb{R}^n)$ , where  $h_{\max} := \max_i \{h_i\}$  for cases A1 and A2. For case A3, we do not know the exact value of  $h_{\max}$  but only that it exists.

## 3 Results

In this section, criteria for verifying dissipativity property of (8) under the assumptions A1–A3 are presented, followed by a series of remarks commenting on various aspects of the results. To facilitate the development, we denote  $J(x) - R(x) + \sum_{i=1}^m T_i(x)g(x)'$  as  $\mathcal{R}(x)$ ,  $g(x)'\nabla H(x)$  as  $\mathcal{G}(x)$ , and define matrix-valued function  $F: \mathbb{R}^n \rightarrow \mathbb{R}^{(n+pm) \times (n+pm)}$  as

$$F(x) = \begin{bmatrix} I_n & O_{n \times p} & O_{n \times p} & \cdots & O_{n \times p} \\ g(x)' & -I_p & O_p & \cdots & O_p \\ g(x)' & O_p & -I_p & \cdots & O_p \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g(x)' & O_p & O_p & \cdots & -I_p \end{bmatrix}$$

**Proposition 3:** Under assumption A1, the system (8) is stable w.r.t.  $x(t) \equiv 0$  if there exist symmetric and positive – definite matrices  $P$ ,  $\Psi_i$  and  $\Phi_i$ ,  $i = 1, \dots, m$  such that

$$\begin{aligned} M(x) &:= F(x)'(M_H(x) + M_P(x) + M_\Phi(x))F(x) \\ &+ M_\Psi(x) < 0, \quad \forall x \in \mathbb{R}^n \end{aligned} \quad (9)$$

where  $M_H(x)$ ,  $M_P(x)$ ,  $M_\Psi(x)$  and  $M_\Phi(x)$  are described in (12) to (15).

**Proposition 4:** Under assumption A2, the system (8) is stable w.r.t.  $x(t) \equiv 0$  if there exist symmetric and positive

definite matrices  $P$  and  $\Psi_i$ ,  $i = 1, \dots, m$  such that

$$M(x) := F(x)'(M_H(x) + M_P(x))F(x) + M_\Psi(x) < 0, \quad \forall x \in \mathbb{R}^n \quad (10)$$

where  $M_H(x)$ ,  $M_P(x)$ ,  $M_\Psi(x)$  are described in (12), (13) and (15).

**Proposition 5:** Under assumption A3, the system (8) is stable w.r.t.  $x(t) \equiv 0$  if there exist symmetric and positive – definite matrices  $P$  and  $\Phi_i$ ,  $i = 1, \dots, m$  such that

$$M(x) := M_H(x) + M_P(x) + M_\Phi(x) < 0, \quad \forall x \in \mathbb{R}^n \quad (11)$$

where  $M_H(x)$ ,  $M_P(x)$ ,  $M_\Phi(x)$  are described in (12)–(14).

**Remark 6:** Note that inequalities (10) and (11) are special cases of inequality (9). More precisely, one recovers (10) and (11) from (9) by setting  $\Phi_i$  and  $\Psi_i$  to zero, respectively. Thus, criteria stated in Propositions 4 and 5 are more conservative than the criterion in Proposition 3. This is expected since under assumptions A2 and A3, less information is available to us. Without information on the upper bounds of  $\dot{\tau}_i(t)$  and  $\tau_i(t)$ , we lose the terms  $M_\Phi$  and  $M_\Psi$ , respectively. Moreover, note that (2,2) to  $(m+1, m+1)$  diagonal blocks of  $M_H(x) + M_P(x) + M_\Phi(x)$  are  $(d_i - 1)\Phi_i$ ,  $i = 1, \dots, m$ , respectively. Thus,  $d_i < 1$  is necessary for stability condition (11) to hold.

$$M_H = \mathbb{S} \left( \begin{bmatrix} I_n \\ O_{p \times n} \\ \vdots \\ O_{p \times n} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}R & \frac{1}{2}T_1 & \cdots & \frac{1}{2}T_m \end{bmatrix} \right) \quad (12)$$

$$M_P = \mathbb{S} \left( \begin{bmatrix} I_n \\ O_{p \times n} \\ \vdots \\ O_{p \times n} \end{bmatrix} P(\nabla^2 H) \begin{bmatrix} (J - R) & T_1 & \cdots & T_m \end{bmatrix} \right) \quad (13)$$

$$M_\Phi = \begin{bmatrix} g(x) \left( \sum_{i=1}^m \Phi_i \right) g(x)' & O_{n \times p} & \cdots & O_{n \times p} \\ O_{p \times n} & (d_1 - 1)\Phi_1 & \cdots & O_p \\ \vdots & \vdots & \ddots & \vdots \\ O_{p \times n} & O_p & \cdots & (d_m - 1)\Phi_m \end{bmatrix} \quad (14)$$

$$M_\Psi = \begin{bmatrix} -\mathcal{R}' \\ T_1' \\ \vdots \\ T_m' \end{bmatrix} (\nabla \mathcal{G})' \left( \sum_{i=1}^m h_i \Psi_i \right) (\nabla \mathcal{G}) \begin{bmatrix} -\mathcal{R} & T_1 & \cdots & T_m \end{bmatrix} - \begin{bmatrix} O_n & O_{n \times p} & \cdots & O_{n \times p} \\ O_{p \times n} & \frac{1}{h_1} \Psi_1 & \cdots & O_p \\ \vdots & \vdots & \ddots & \vdots \\ O_{p \times n} & O_p & \cdots & \frac{1}{h_m} \Psi_m \end{bmatrix} \quad (15)$$

## 4 Proofs

In this section, we prove the results presented in Section 3. It should be clear that the proofs of Propositions 3–5 follow

the same arguments. The only variation between them is the Lyapunov–Krasovskii functional candidate used. Hence, we will only present the proof of Proposition 3, and then remark on what Lyapunov–Krasovskii functional candidates one should use to obtain criteria stated in Propositions 4 and 5. To prove Proposition 3, we need the following technical lemma.

**Lemma 7:** Suppose that  $\tau : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $0 \leq \tau(t) \leq h$   $\forall t$ , where  $h$  is a given positive constant. Furthermore, let  $v$  be an  $\mathbb{R}^n$ -valued function which is integrable and square-integrable over any interval  $[t-h, t]$ . Finally, let

$$z(t) = \int_{t-\tau(t)}^t v(\theta) d\theta \quad (t \in \mathbb{R}) \quad (16)$$

Then for any given positive-definite matrix  $\Phi$ ,  $z(t)$  and  $v(t)$  satisfy the inequality

$$z(t)' \Phi z(t) \leq h \int_{t-h}^t v(\theta)' \Phi v(\theta) d\theta \quad (17)$$

**Proof:** Inequality (17) is obtained by exploiting the fact that the left-hand side of (17) is a convex function of  $z(t)$ . One version of the proof can be found in [13].  $\square$

The proof of Proposition 3 relies on the following transformation of (8)

$$\begin{aligned} \dot{x} &= \left( J(x) - R(x) + \sum_{i=1}^m T_i(x) g(x)' \right) \nabla H(x) - \sum_{i=1}^m T_i(x) \zeta_i \\ &= \mathcal{R}(x) \nabla H(x) - \sum_{i=1}^m T_i(x) \zeta_i \end{aligned} \quad (18)$$

where  $\zeta_i := g(x)' \nabla H(x) - [g(x)' \nabla H(x)]_{\tau_i}$  can also be expressed as

$$\begin{aligned} &\int_{t-\tau_i}^t \frac{d}{d\theta} \{ g(x(\theta))' \nabla H(x(\theta)) \} d\theta \\ &= \int_{t-\tau_i}^t \nabla \{ g(x)' \nabla H(x) \} |_{x \leftarrow x(\theta)} \dot{x}(\theta) d\theta \\ &= \int_{t-\tau_i}^t \nabla \mathcal{G}(x(\theta)) \left\{ (J(x(\theta)) - R(x(\theta)) \nabla H(x(\theta))) \right. \\ &\quad \left. + \sum_{j=1}^m T_j(x(\theta)) \mathcal{G}(x(\theta - \tau_j(\theta))) \right\} d\theta \end{aligned} \quad (19)$$

(Recall:  $\mathcal{G}(x) := g(x)' \nabla H(x)$ ) It can be shown that the transformation introduces additional dynamics; see [21] and [5]. The two systems are not equivalent; however, the behaviour of (18) is richer than that of (8) in the sense that each solution of (8) is also a solution of (18) but not vice versa. Therefore to show that (8) is stable, it is sufficient to prove that the same property holds for (18). In the following, we will show that condition (9) implies that system (18) is dissipative. To facilitate the development, let us denote  $\nabla \mathcal{G}(x(t)) \dot{x}(t)$  by  $v(t)$ .

Consider system (18) and the Lyapunov–Krasovskii functional candidate  $V_1 : \Omega \rightarrow \mathbb{R} := V_H + V_P + V_\Psi + V_\Phi$ ,

where

$$\begin{aligned} V_H(x(t)) &= H(x(t)) \\ V_P(x(t)) &= \nabla H(x(t))' P \nabla H(x(t)) \\ V_\Psi(\dot{x}[t]) &= \sum_{i=1}^m \int_{-h_i}^0 \int_{\theta}^0 \left( \nabla \mathcal{G}(\dot{x}[t](s)) \frac{d\dot{x}[t](s)}{ds} \right)' \\ &\quad \times \Psi_i \left( \nabla \mathcal{G}(\dot{x}[t](s)) \frac{d\dot{x}[t](s)}{ds} \right) ds d\theta \\ V_\Phi(\dot{x}[t]) &= \sum_{i=1}^m \int_{-\tau_i}^0 \mathcal{G}(\dot{x}[t](s))' \Phi_i \mathcal{G}(\dot{x}[t](s)) ds \end{aligned}$$

Note that  $d\dot{x}[t](s)/ds = \dot{x}(t+s)$ . Thus, by the change of variables  $\lambda = t+s$ ,  $\xi = t+\theta$  one can verify that

$$\begin{aligned} V_\Psi(\dot{x}[t]) &= \sum_{i=1}^m \int_{t-h_i}^t \int_{\xi}^t v(\lambda)' \Psi_i v(\lambda) d\lambda d\xi \\ V_\Phi(\dot{x}[t]) &= \sum_{i=1}^m \int_{t-\tau_i}^t \mathcal{G}(x(\lambda))' \Phi_i \mathcal{G}(x(\lambda)) d\lambda \end{aligned}$$

It can also be verified that the derivatives of  $V_H$ ,  $V_P$ ,  $V_\Psi$ , and  $V_\Phi$  along the solution of (18) have the following expressions

$$\dot{V}_H = \nabla H(x)' \left( \mathcal{R}(x) \nabla H(x) - \sum_{i=1}^m T_i(x) \zeta_i \right) \quad (20)$$

$$\dot{V}_P = \nabla H(x)' (2P \nabla^2 H(x)) \left( \mathcal{R}(x) \nabla H(x) - \sum_{i=1}^m T_i(x) \zeta_i \right) \quad (21)$$

$$\dot{V}_\Psi = \sum_{i=1}^m h_i v(t)' \Psi_i v(t) - \int_{t-h_i}^t v(\lambda)' \Psi_i v(\lambda) d\lambda \quad (22)$$

$$\begin{aligned} \dot{V}_\Phi &= \sum_{i=1}^m (g(x)' \nabla H(x))' \Phi_i (g(x)' \nabla H(x)) - (1 - \dot{\tau}_i) \\ &\quad \times [g(x)' \nabla H(x)]'_{\tau_i} \Phi_i [g(x)' \nabla H(x)]_{\tau_i} \end{aligned} \quad (23)$$

Furthermore,  $\dot{V}_\Psi$  satisfies the following inequality

$$\dot{V}_\Psi \leq \sum_{i=1}^m h_i v(t)' \Psi_i v(t) - \frac{1}{h_i} \zeta_i(t)' \Psi_i \zeta_i(t) \quad (24)$$

which follows Lemma 7 and the fact that  $\zeta_i(t)$  and  $v(t)$  satisfy (16) [with  $\zeta_i(t)$  and  $v(t)$  playing the roles of  $z(t)$  and  $v(t)$ , respectively]. Moreover, since  $1 - \dot{\tau}_i \geq 1 - d_i$  and  $\Phi_i$  is positive definite, we have

$$\begin{aligned} \dot{V}_\Phi &\leq \sum_{i=1}^m (g(x)' \nabla H(x))' \Phi_i (g(x)' \nabla H(x)) - (1 - d_i) \\ &\quad \times [g(x)' \nabla H(x)]'_{\tau_i} \Phi_i [g(x)' \nabla H(x)]_{\tau_i} \\ &= \eta(t)' F(x(t))' M_\Phi(x(t)) F(x(t)) \eta(t) \end{aligned} \quad (25)$$

where  $\eta(t) = [\nabla H(x(t))' \quad \zeta_1(t)' \quad \cdots \quad \zeta_m(t)']'$ , and  $M_\Phi(x)$  is described in (14). Now, adding (20)–(22) together and

utilising (24), we obtain

$$\begin{aligned} \dot{V}_H + \dot{V}_P + \dot{V}_\Psi &\leq \nabla H(x)' (I_n + 2P \nabla^2 H(x)) \\ &\quad \times \left( \mathcal{R}(x) \nabla H(x) - \sum_{i=1}^m T_i(x) \zeta_i \right) \\ &\quad + \sum_{i=1}^m h_i v' \Psi_i v - \frac{1}{h_i} \zeta_i' \Psi_i \zeta_i \end{aligned}$$

By replacing  $v(t)$  by  $\nabla \mathcal{G}(x(t)) \dot{x}(t)$  and rearranging terms, we obtain

$$\begin{aligned} \dot{V}_H + \dot{V}_P + \dot{V}_\Psi &\leq \eta(t)' F(x(t))' (M_H(x(t)) + M_P(x(t))) \\ &\quad \times F(x(t)) \eta(t) + \eta(t)' M_\Psi(x(t)) \eta(t) \end{aligned} \quad (26)$$

where  $M_H$ ,  $M_P$  and  $M_\Psi$  are described in (12), (13) and (15).

Finally, by adding (25) and (26), we obtain

$$\dot{V}_1(\dot{x}[t]) \leq \eta(t)' M(x(t)) \eta(t)$$

Thus, if the condition  $M(x) < 0$  for all  $x \in \mathbb{R}^n$  holds, system (18) is stable with stability certificate  $V_1$ , and so is system (8).

*Remark 8:* To prove Propositions 4 and 5, one takes Lyapunov–Krasovskii functional candidates  $V_2 := V_H + V_P + V_\Psi$  and  $V_3 := V_H + V_P + V_\Phi$ , respectively. The proofs follow exactly the same arguments. Also note that, to prove Propositions 5, transforming (8) into (18) is not required. In this case, one works with system (8) directly.

*Remark 9:* Regarding functions  $w_1$ ,  $w_2$  and  $w_3$  as required in Theorem 2, it is difficult to give explicit forms for them; however, we can easily argue that such functions exist. The reasonings are as follows.

In our assumptions, we assume each  $\tau_i$  is lower bounded by zero. Therefore stability of the ‘zero delay’ Hamiltonian system is ‘necessary’ for verifying stability for all delay  $\tau_i \in [0, h_i]$ , and it is natural to assume that  $H(x)$  serves as the Lyapunov function for proving stability. Thus, we know there exist two continuous non-negative non-decreasing functions  $\kappa_1$  and  $\kappa_2$ ,  $\kappa_1(s) > 0$ ,  $\kappa_2(s) > 0$  for  $s \neq 0$ , such that  $\kappa_1(s) \leq H(s) \leq \kappa_2(s)$  for all  $s \in \mathbb{R}^n$ . Since  $P$ ,  $\Psi_i$  and  $\Phi_i$  are all positive definite, we clearly have  $\kappa_1(x(t)) \leq V(\dot{x}[t])$  and  $\kappa_1(\cdot)$  may serve as  $w_1(\cdot)$  in Theorem 2. Furthermore, note that

$$\begin{aligned} V_P(x(t)) &= \nabla H(x(t))' P \nabla H(x(t)) \leq \bar{\lambda}_P \|\nabla H(x(t))\|^2 \\ V_\Psi(\dot{x}[t]) &= \sum_{i=1}^m \int_{-h_i}^0 \int_{\theta}^0 \left( \nabla \mathcal{G}(\dot{x}[t](s)) \frac{d\dot{x}[t](s)}{ds} \right)' \\ &\quad \times \Psi_i \left( \nabla \mathcal{G}(\dot{x}[t](s)) \frac{d\dot{x}[t](s)}{ds} \right) ds d\theta \\ &\leq \sum_{i=1}^m (h_i)^2 \bar{\lambda}_{\Psi_i} \left( \sup_{s \in [-h_i, 0]} \|\nabla \mathcal{G}(x(t+s)) \dot{x}(t+s)\|^2 \right) \\ V_\Phi(\dot{x}[t]) &= \sum_{i=1}^m \int_{-\tau_i}^0 \mathcal{G}(\dot{x}[t](s))' \Phi_i \mathcal{G}(\dot{x}[t](s)) ds \\ &\leq \sum_{i=1}^m h_i \bar{\lambda}_{\Phi_i} \left( \sup_{s \in [-h_i, 0]} \|\mathcal{G}(x(t+s))\|^2 \right) \end{aligned}$$

where  $\bar{\lambda}_P$ ,  $\bar{\lambda}_{\Psi_i}$  and  $\bar{\lambda}_{\Phi_i}$  denote the maximum eigenvalue of  $P$ ,  $\Psi_i$  and  $\Phi_i$ , respectively. Thus, for any given  $\dot{x}[t]$ , we may

choose a continuous non-negative non-decreasing function  $w_2$  satisfying  $w_2(s) > 0$  for  $s \neq 0$ , such that

$$w_2(\|\dot{x}[t]\|_c) \geq \kappa_2(x(t)) + \bar{\lambda}_p \|\nabla H(x(t))\|^2 + \sum_{i=1}^m h_i \bar{\lambda}_{\Phi_i} \left( \sup_{s \in [-h_i, 0]} \|\mathcal{G}(x(t+s))\|^2 \right) + \sum_{i=1}^m (h_i)^2 \bar{\lambda}_{\Psi_i} \left( \sup_{s \in [-h_i, 0]} \|\nabla \mathcal{G}(x(t+s))\dot{x}(t+s)\|^2 \right)$$

Then  $w_2$  satisfies  $V(\dot{x}[t]) \leq w_2(\|\dot{x}[t]\|_c)$ , fulfilling its role in Theorem 2. Finally, if the stability condition ' $M(x) < 0$  for all  $x \in \mathbb{R}^n$ ' holds, we clearly have  $\dot{V}(\dot{x}[t]) \leq \eta(t)'M(x(t))\eta(t) \leq 0$ . Thus, we can simply choose the zero function as  $w_3$ .

## 5 Example

To illustrate our results, let us consider the equation of a normalised undamped pendulum

$$\ddot{x}_1 + \sin x_1 = u, \quad y = \dot{x}_1 := x_2$$

where  $u$  is the actuation torque and the output  $y$  is the angular velocity. The pendulum system can be expressed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \nabla H_p + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = [0 \quad 1] \nabla H_p$$

where the Hamiltonian  $H_p(x_1, x_2) := \frac{1}{2}x_2^2 + (1 - \cos x_1)$  is the total energy. It can be readily verified that the origin is an equilibrium and the pendulum system is stable, but not asymptotically stable around it. To asymptotically stabilise the system and to improve the tracking performance, a feedback controller of the following form

$$\dot{x}_c = -10(1 + e^{-x_c^2})x_c + u_c, \quad y_c = x_c + u_c$$

is interconnected with the pendulum. The interconnection constraints  $u = -y_c + r$ ,  $u_c = y$  lead to the following closed-loop system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & -1 \\ 0 & 1 & -10(1 + e^{-x_c^2}) \end{bmatrix} \begin{bmatrix} \sin x_1 \\ x_2 \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r \quad (27)$$

Let  $x = [x_1 \ x_2 \ x_c]'$ ,  $H_c(x_c) = \frac{1}{2}x_c^2$  and  $H(x) = H_p(x_1, x_2) + H_c(x_c)$ . One can verify that  $H$  is the total Hamiltonian of the closed-loop system (27). Note that  $H$  is positive definite and can thus serve as the stability and passivity certificate of (27). Furthermore, one can also verify that asymptotic tracking for a range of constant  $x_1$  can be achieved. In particular, if  $x_1^* \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ , setting  $r$  equal to  $\sin x_1^*$  would asymptotically bring the state to the equilibrium  $(x_1^*, 0, 0)$ .

Introducing the communication delays between the plant and the controller leads to the following form of the closed-loop system:

$$\dot{x} = (J - R(x)) \nabla H + T_1[g' \nabla H]_{\tau_1} + T_2[g' \nabla H]_{\tau_2} + T_3[g' \nabla H]_{\tau_3} + \tilde{g}r \quad (28)$$

where  $\tau_1(t)$  is the forward delay from the plant to the controller,  $\tau_2(t)$  is the backward delay from the controller

to the plant,  $\tau_3(t) := \tau_1(t) + \tau_2(t)$  is the loop delay, and

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 10(1 + e^{-x_c^2}) \end{bmatrix},$$

$$\tilde{g} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Suppose  $\tau_i(t) \in [0, h]$ ,  $\dot{\tau}_i(t) \leq d$ ,  $i = 1, 2$ . We first note that under the assumption A3 system (28) is not stable. As we will show later, numerical simulations reveal that the system becomes unstable as the delays become sufficiently large. Using Propositions 3 and 4, we obtain the margins shown in Table 1 for the autonomous version of system (28) to be asymptotically stable w.r.t. the origin. Furthermore, using the same storage functionals, one can also show that the autonomous version of (28) is asymptotically stable w.r.t. the origin if the delay parameters are within the margins displayed in Table 1.

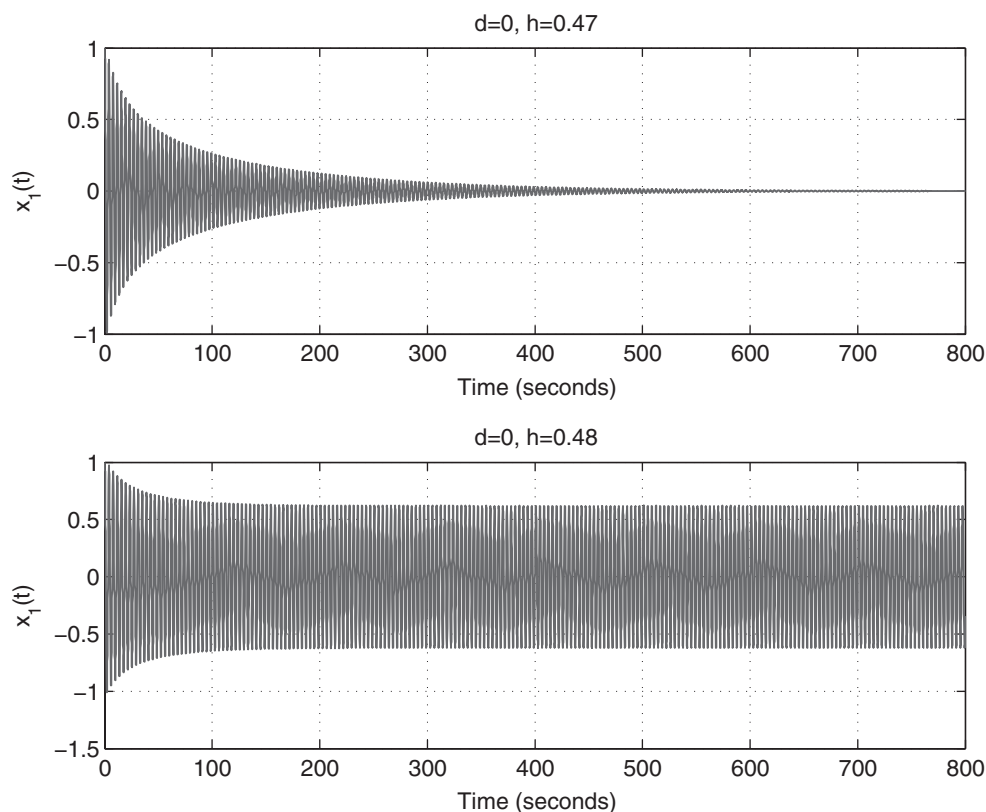
To investigate how tight these stability margins are, we performed several numerical simulations. For  $d = 0$ , simulations indicate that the system remains asymptotically stable for  $h$  up to 0.47. The margin predicted by Proposition 3 is about 84% of the margin revealed by simulations. Since the criteria proposed in Propositions 3–5 are only sufficient, some conservatism is expected. The simulation results are shown in Fig. 1.

For time-varying delay ( $d \neq 0$ ), we tested two cases where  $\tau_1(t) = \tau_2(t) = \frac{1}{4}(1 + \sin t)$  in one case and  $\tau_1(t) = \tau_2(t) = t - \frac{1}{2}[2t]$  in another, where  $[2t]$  denotes the largest integer among those smaller than  $2t$ . Note that in both cases,  $\tau_i(t) \in [0, \frac{1}{2}]$  and  $\sup_t \dot{\tau}_i(t) = 1$ . The upper left and the upper right figures in Fig. 2 illustrate these two functions, respectively, whereas the lower left and the lower right figures illustrate the corresponding  $x_1(t)$ . Apparently, the system remains asymptotically stable under the influence of the two time-varying delays. Note that the maximum delay length in these two cases is 0.5, which exceeds the stability margin for the constant delay scenario. Thus, in terms of destabilisation, the two particular time-varying delay functions we select are actually more benign than the time-invariant delay  $\tau_i \equiv 0.5$ ,  $i = 1, 2$ . This is a good indication that, for time-varying delays, there is more useful information than bounds on the magnitude and the variation one could exploit.

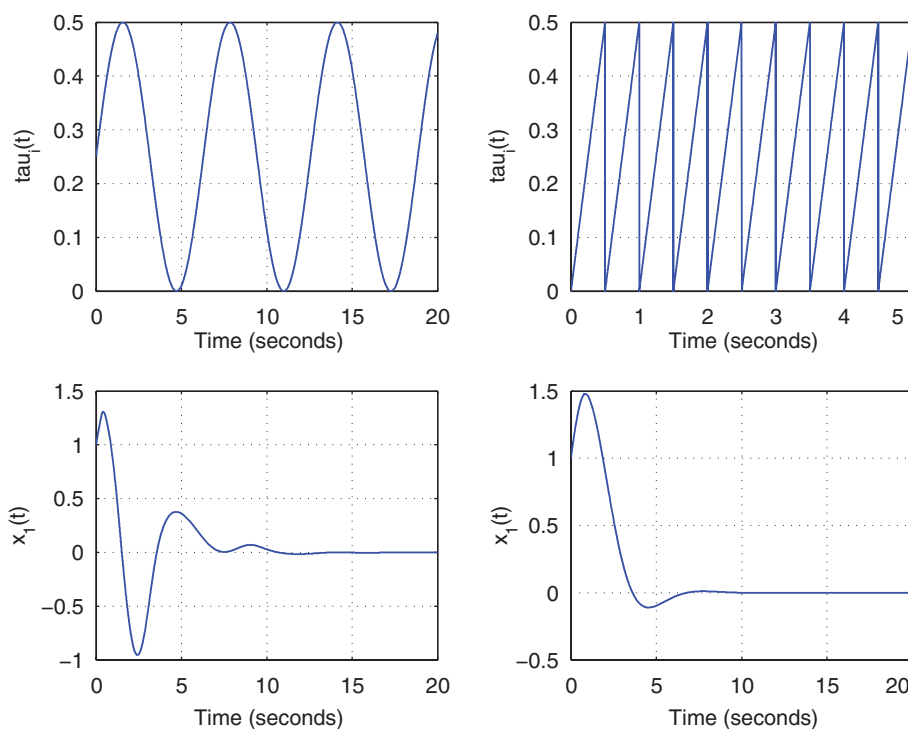
We conclude this section with a remark on the numerical computation. Inequalities (9)–(11) seem to suggest that, to numerically solve the matrix inequalities, one has to partition  $\mathbb{R}^n$  and evaluate  $M(x)$  at a dense grid of states. This may not be the case and sometimes it may be more efficient to partition the range of  $M(x)$ . For instance, in this example the only two non-constant elements involved in  $M(x)$  are  $\cos x_1$  and  $e^{-x_c^2}$ . Thus, instead of partitioning  $\mathbb{R}^3$ , we partition the

**Table 1** Margins for dissipativity (stability) of system (28)

$d$	0.0	0.2	0.4	0.6	0.8	0.99	$\infty$
$h$	0.396	0.393	0.390	0.385	0.376	0.338	0.332



**Fig. 1** Upper figure: time history of  $x_1$  when  $\tau_1 = \tau_2 = 0.47$ . Lower figure: time history of  $x_1$  when  $\tau_1 = \tau_2 = 0.48$



**Fig. 2** Upper left: time history of  $\tau_1$  and  $\tau_2$ , which are both equal to  $\frac{1}{4}(1 + \sin t)$ . Lower left: time history of  $x_1$  corresponding to  $\tau_1$  and  $\tau_2$  shown in the upper left figure; Upper right: time history of  $\tau_1$  and  $\tau_2$ , which are both equal to  $t - \frac{1}{2}[2t]$ . Lower right: time history of  $x_1$  corresponding to  $\tau_1$  and  $\tau_2$  shown in the upper right figure

range space  $\{(\cos x_1, e^{-x_c^2}) | x_1 \in \mathbb{R}, x_c \in \mathbb{R}\}$ , which is  $[0, 1] \times [0, 1]$ , a bounded subset of  $\mathbb{R}^2$ . In this way, we are able to obtain fairly accurate results by taking only 10 000 ( $100 \times 100$ ) grid points.

## 6 Conclusions

In this paper we presented a methodology to test the stability of non-linear time delay systems in the port-Hamiltonian

framework. Sufficient conditions for stability are derived by construction of appropriate stability certificates. To illustrate use of the main results, an example is presented in detail with a set of margins for stability found and simulations to verify these margins. A few issues, however, remain open. One issue is how conservative the proposed method is and how to improve it. Simulation results indicate that utilising only the bounds on  $\tau(t)$  and  $\hat{\tau}(t)$  could be conservative at times. Another important issue is how to solve the state-parameterised matrix inequalities in a sound and efficient fashion. Besides approximating the entire state space by a dense grid of state vectors, a solution to this problem could lie in the so-called ‘sum-of-square’ approach ([22]), the applicability of which is yet to be investigated.

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