Applied Mathematics and Computation 208 (2009) 397–415

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions

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ARTICLE INFO

Keywords:
Delay differential-algebraic equation
Multiple delays
Asymptotic stability
Regular pencil
Numerical solution

ABSTRACT

This paper is concerned with the asymptotic stability of differential-algebraic equations with multiple delays and their numerical solutions. First, we give a sufficient condition for delay-independent stability. After characterizing the coefficient matrices that satisfy this stability condition, we propose some practical checkable criteria for asymptotic stability. Then we investigate the stability of numerical solutions obtained by θ-methods and BDF methods. Finally, solvability and stability of a class of weakly regular delay differential-algebraic equations are analyzed.

1. Introduction

In this paper we consider the linear differential-algebraic equation with multiple delays

$$A\dot{x}(t) + Bx(t) + \sum_{i=1}^{M} C_i\dot{x}(t-\tau_i) + \sum_{i=1}^{M} D_i x(t-\tau_i) = 0, \tag{1.1}$$

where A, B, C_i, D_i ($i = 1,2,\ldots,M$), are real (or complex) constant matrices of size $m \times m$. The time-delays are ordered increasingly, $0 < \tau_1 < \tau_2 < \cdots < \tau_M$. Matrix A is assumed to be singular with rank $A = d < m$. We are also interested in a special subclass of (1.1) in the form,

$$A\dot{x}(t) + Bx(t) + \sum_{i=1}^{M} C_i\dot{x}(t-i\tau) + \sum_{i=1}^{M} D_i x(t-i\tau) = 0. \tag{1.2}$$

That is by $\tau_i = i\tau$ ($i = 1,2,\ldots,M$), where $\tau > 0$ is given. From now on, if the unknown functions appear without argument and no confusion arises, we mean that they are evaluated at the actual time t. For example, we write x instead of $x(t)$ and $\dot{x}$ instead of $\dot{x}(t)$.

Differential-algebraic equations (DAEs) play important roles in mathematical modeling of real-life problems arising in a wide range of applications, for example, multibody mechanics, prescribed path control, electrical design, chemically reacting systems, biology and biomedicine. See [3,16] and references therein. In many problems, the systems in consideration contain time-delays, see [2,5–7,10,19,20,22,24–26]. While the theory and the numerics for delay ordinary differential equations (DODEs) have been well known and discussed for decades in a wide range of literature, see [12] and references therein, there are very few results for the theory of delay differential-algebraic equations (DDAEs). The main reason is that even for linear DDAEs, their dynamics have not been well understood yet, in particular when the pencil $\{A,B\}$ in (1.1) is not regular. The

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2008.12.008

most difficult problem is that there exists no compressed form into which a tuple of more than two matrices can be simultaneously transformed. Most of the existing results are only for linear time-invariant regular DDAEs [10,24] or DDAEs of special form [2,19,25,26]. Until now there have been only two papers concerning nonregular DAEs [7,20]. A general result for DDAEs' solvability and stability is still missing. The following examples illustrate some significant differences between delay ODEs, DAEs without delays, and delay DAEs.

Example 1. Consider the system

$$\begin{cases} \dot{x}_1(t) + x_1(t) - x_1(t-1) - x_2(t-1) = 0 \\ \quad 2x_2(t) + x_1(t-1) + x_2(t-1) = 0 \end{cases} \quad (t \geqslant 0),$$

where x_1 and x_2 are given by continuous functions on the initial interval $(-1,0]$. The dynamics of x_1 is governed by a differential operator and continuity of x_1 is expected. The dynamics of x_2 is determined by a difference operator and unlike x_1, this component is piecewise continuous, in general.

Example 2 [7]. Consider the following inhomogenous system:

$$\begin{cases} \dot{x}_1(t) = f(t) \\ x_1(t) - x_2(t-1) = g(t) \end{cases} \quad (t \geqslant 0).$$

The solution is given by

$$x_1(t) = \int_0^t f(s)ds + c, \quad x_2(t) = -g(t+1) + \int_0^{t+1} f(s)ds + c \quad (t \geqslant 0),$$

where c is a constant. The system dynamics is not causal. Not only is x_2 specified on $(-1,0]$, but the solution depends on future integrals of the input $f(t)$. This interesting phenomenon should be noted in addition to the well-known fact in the DAE theory that the solution may depend on derivatives of the input.

A sufficient condition for the delay-independent asymptotic stability of DAEs with single delay is proposed in [25]. Under this condition, the asymptotic stability of θ-methods, BDF methods, general linear multistep methods, as well as implicit Runge–Kutta methods are analyzed. Unfortunately, it is very difficult to verify this condition in practice. The main aim of the present paper is to give a complement to this result in the stability theory for DDAEs. Namely, we intend to derive delay-independent stability criteria for DDAEs of the form (1.1) and (1.2). We focus on practical stability criteria that are easily checkable. Our results extend those obtained for neutral DODEs [13,14] to neutral DDAEs. Under these criteria, we will show that numerical solutions obtained by the θ-methods and BDF methods preserve the asymptotic stability of the DDAE. This result includes the single delay DAEs result of [25] as a special case. Further, we also investigate the solvability and the stability of a special class of nonregular delay DAEs.

The paper is organized as follows. In the next section we review basic notions and results from the theory of DAEs and regular delay DAEs. The main results of the paper lie in Section 3, where we formulate sufficient conditions to provide the asymptotic stability of regular DDAEs. We give a characterization of those coefficient matrices that satisfy the sufficient conditions. We also propose some practical criteria for the asymptotic stability of DDAEs with multiple delays. In Section 4, we analyze the stability of numerical solutions to (1.1) and (1.2) using θ-methods and BDF methods. Finally, in the last section, we discuss solvability and stability issues of a special class of weakly regular DDAEs.

2. Preliminary

In this section, we give a brief summary of needed results on linear constant coefficient and delay DAEs. We assume the reader is familiar with the basic theory of linear time invariant DAEs [3,11,16], such as

$$A\dot{x} + Bx = 0. \tag{2.1}$$

The matrix pencil $\{A,B\}$ is said to be regular if there exists $\lambda \in \mathbb{C}$ such that the determinant of $\lambda A + B$, denoted by $\det(\lambda A + B)$, is nonzero. The system (2.1) is solvable if and only if $\{A,B\}$ is regular. If $\det(\lambda A + B) = 0 \quad \forall \lambda \in \mathbb{C}$, we say that $\{A,B\}$ is irregular or non-regular. If $\{A,B\}$ is regular, then λ is a (generalized finite) eigenvalue of $\{A,B\}$ if $\det(\lambda A + B) = 0$. The set of all eigenvalues is called the spectrum of the pencil $\{A,B\}$ and denoted by $\sigma\{A,B\}$. The maximum of the absolute values of the finite eigenvalues is called the spectral radius of the pencil $\{A,B\}$ and denoted by $\rho(A,B)$. These concepts are also extended to the case of a given tuple of matrices $\{A_i\}_{i=0}^n$ (the generalized polynomial eigenvalue problem) by defining $\sigma(\{A_i\}_{i=0}^n) = \{\lambda \in \mathbb{C} : \det(\sum_{i=0}^n \lambda^{n-i} A_i) = 0\}$, and $\rho(\{A_i\}_{i=0}^n) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{and} \det(\sum_{i=0}^n \lambda^{n-i} A_i) = 0\}$. Thus, for a given matrix $A \in \mathbb{C}^{m\times m}$, the well-known spectrum $\sigma(A)$ and the spectral radius $\rho(A)$ are $\sigma(-I,A)$ and $\rho(-I,A)$, respectively.

Suppose that A is singular and pencil $\{A,B\}$ is regular. Then there exist nonsingular matrices W, T such that

$$WAT = \begin{pmatrix} I_d & 0 \\ 0 & N \end{pmatrix}, \quad WBT = \begin{pmatrix} B_1 & 0 \\ 0 & I_{m-d} \end{pmatrix}, \tag{2.2}$$

where N is nilpotent of index k [3,11,16]. If N is a zero matrix, then $k = 1$. Furthermore, we may assume without loss of generality, that N and B_1 are upper triangular. If $\{A, B\}$ is regular, the nilpotency index of N in (2.2) is called the index of matrix pencil $\{A, B\}$ and we write index $\{A, B\} = k$. If A is nonsingular, we set index $\{A, B\} = 0$.

Definition 1. Suppose that $\{A, B\}$ is regular. Let Q be a projector onto the subspace of consistent initial conditions. Let $P = I - Q$. We say that the zero solution of (2.1) is stable if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for an arbitrary vector $x_0 \in \mathbb{C}^m$ satisfying $\|x_0\| < \delta$, the solution of the initial value problem

$$\begin{cases} A\dot{x} + Bx = 0, \quad t \in [0, \infty), \\ P(x(0) - x_0) = 0 \end{cases}$$

exists uniquely and the estimate $\|x(t)\| < \varepsilon$ holds for all $t \geqslant 0$. The zero solution is said to be asymptotically stable if it is stable and $\lim_{t\to\infty}\|x(t)\| = 0$ for solutions x of (2.1). If the zero solution of (2.1) is asymptotically stable, we say that system (2.1) is asymptotically stable.

If index $\{A, B\} = 1$ one may choose Q as the projector onto (A) [11]. A difference between ODE-s and DAE-s is that the equality $x(0) = x_0$ is not expected here, in general. That is, for DAEs, we need *consistent* initial value x_0 such that (2.1) with the initial condition $x(0) = x_0$ holds for a smooth solution. We do not consider impulsive solutions in this paper and for that reason will frequently make an index one assumption. For linear time-invariant systems, the concepts of asymptotic stability and exponential stability are equivalent. The system (2.1) is asymptotically stable if and only if the matrix pencil $\{A, B\}$ is (asymptotically) stable, i.e., $\sigma(A, B) \subset \mathbb{C}^-$, where $\mathbb{C}^-$ denotes the *open* left half complex plane [23]. Clearly $\sigma(RAS, RBS) = \sigma(A, B)$ for nonsingular R, S.

2.1. Solvability of regular delay DAEs

The theory of delay ordinary differential equations (DODEs), when the leading matrix A in (2.3) is the identity matrix, has been widely discussed [12]. These systems are classified by their type. For a scalar DODE $a\dot{x} + bx + c\dot{x}(t-\tau) + dx(t-\tau) = f(t)$, the system is of retarded type if $a \neq 0, c = 0$, of neutral type if $a \neq 0, c \neq 0$, and of advanced type if $a = 0, b \neq 0$, and $c \neq 0$. One important attribute of the type is that it classifies how DODEs propagate discontinuities to future delay intervals (assuming an initial value problem). Discontinuities in retarded systems become smoother in each successive interval, whereas discontinuities in advanced systems become less smooth in each successive interval. Discontinuities in neutral systems are carried into successive delay intervals with the same degree of smoothness. Hence, we wish to study separately DDAEs which include retarded and neutral DODEs, but to avoid altogether those which lead to DODEs of advanced type. For some interesting examples of DDAEs and some DDAEs which "look like" they should be of retarded type but are actually neutral or advanced type, see [6,7].

In this section we consider DAEs with single delay

$$A\dot{x} + Bx + Dx(t-\tau) = 0. \tag{2.3}$$

The delay DAE (2.3) is called regular [7] if the pencil $\{A, B\}$ is regular and weakly regular if there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $\det(\alpha A + \beta B + \gamma D) \neq 0$, i.e., the triplet $\{A, B, D\}$ is regular. We suppose initially that $\{A, B\}$ is regular and has index k. Note that neutral DAEs with single delay

$$A\dot{x} + Bx + C\dot{x}(t-\tau) + Dx(t-\tau) = 0 \tag{2.4}$$

can always be transformed to the form (2.3). Indeed, by defining a new variable y by $y(t) = x(t-\tau)$, we obtain a new delay DAE

$$\widetilde{A}\dot{\tilde{x}} + \widetilde{B}\tilde{x} + \widetilde{D}\tilde{x}(t-\tau) = 0 \tag{2.5}$$

with

$$\tilde{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A & C \\ 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} B & D \\ 0 & I \end{pmatrix}, \quad \widetilde{D} = \begin{pmatrix} 0 & 0 \\ -I & 0 \end{pmatrix}.$$

However, this transformation can increase the index of the DAE system. Further, the dimension of the new transformed system becomes $2m$, which is less advantageous in practical computation.

Proposition 1. *The pencil $\{\widetilde{A}, \widetilde{B}\}$ is regular if and only if $\{A, B\}$ is regular. However, the index of $\{\widetilde{A}, \widetilde{B}\}$ is equal either k or $k+1$, where k is the index of $\{A, B\}$.*

Proof. The equivalence between the regularity of the two pencils is clear. We verify the statement on the index of $\{\widetilde{A}, \widetilde{B}\}$. Without loss of generality, we assume that the pencil $\{A, B\}$ is given in the Kronecker normal form (2.2) Correspondingly, C and D are given in block form

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}. \tag{2.6}$$

Thus $\{\widetilde{A}, \widetilde{B}\}$ can be assumed to be

$$\widetilde{A} = \begin{pmatrix} I & 0 & C_1 & C_2 \\ 0 & N & C_3 & C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} B_1 & 0 & D_1 & D_2 \\ 0 & I & D_3 & D_4 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

It is not difficult to verify that

$$\mathrm{index}\{\widetilde{A}, \widetilde{B}\} = \mathrm{index}\left\{ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & N & C_3 & C_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} B_1 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \right\}.$$

Then $\mathrm{index}\{\widetilde{A}, \widetilde{B}\} = \mathrm{index}\widetilde{N}$ where

$$\widetilde{N} = \begin{pmatrix} N & C_3 & C_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$\widetilde{N}^k = \begin{pmatrix} N^k & N^{k-1}C_3 & N^{k-1}C_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\mathrm{index}\widetilde{N} = k$ if $N^{k-1}C_3 = 0$ and $N^{k-1}C_4 = 0$. Otherwise $\mathrm{index}\widetilde{N} = k + 1$. $\square$

Corollary 1. *Suppose that the pencil $\{A, B\}$ is regular and has index-1. Further, suppose the matrices are in block form as in the proof of Proposition 1. Then the new pencil $\{\widetilde{A}, \widetilde{B}\}$ has index-1 if and only if $C_3 = 0$, $C_4 = 0$.*

Corollary 1 means that the transformed system (2.5) has index-1 if and only if the pencil $\{A, B\}$ has index-1 and the derivative of $x(t-\tau)$ does not appear in the "algebraic part".

Now, we turn back to the regular delay DAE (2.3) with an initial condition $x(t) = \varphi(t)$, $t \in [-\tau, 0]$, where φ is a continuous function defined on $[-\tau, 0]$. The solvability of regular delay DAEs was discussed in detail in [5,6]. Using appropriate constant coordinate changes, first we transform the matrix triplet A,B,D into the block form (2.2), (2.6). Then system (2.3) is decomposed as follows:

$$\begin{aligned} z' + B_1 z + D_1 z(t-\tau) + D_2 w(t-\tau) &= 0, \\ Nw' + w + D_3 z(t-\tau) + D_4 w(t-\tau) &= 0, \end{aligned} \tag{2.7}$$

where x is decomposed into "differential" variables z and "algebraic" variables w. Using the nilpotency of N,

$$w(t) = -\sum_{i=0}^{k-1} (-N)^i [D_3 z^{(i)}(t-\tau) + D_4 w^{(i)}(t-\tau)]. \tag{2.8}$$

Setting $t = 0$ in (2.8), we obtain the consistency condition for the initial condition. The initial value problem for (2.3) with a consistent initial condition admits a unique solution, see [5,6,10] which can be obtained by solving the system (2.7) for z,w recursively on each interval $((l-1)\tau, l\tau]$, $l = 1, 2, \ldots$ The definition of the asymptotic stability for DDAEs of the form (2.3) is similar to that for DODEs.

Definition 2. [10,24,25] The trivial solution of the DDAE (2.3) is said to be stable if for any $\varepsilon > 0$ theres exists $\delta = \delta(\varepsilon)$ such that for all contiuous functions φ satisfying the consistent condition and $\sup_{t\in[\tau,0]}\|\varphi(t)\| < \delta$, the solution $x = x(t, \varphi)$ of the initial value problem for (2.3) satifies $\|x(t, \varphi)\| < \varepsilon$ for all $t \geqslant 0$. The trivial solution of the DDAE (2.3) is said to be asymptotically stable if it is stable and furthermore $\lim_{t\to\infty} \|x(t, \varphi)\| = 0$.

For higher-index problems ($k > 1$), the formula for w involves derivatives of the solution taken in the past. It was shown in [5] that solutions for (2.3) can be continuous on only finite intervals even if the initial function φ (or the input, if there is an input function) is infinitely differentiable. Further, discontinuities do not necessarily get smoothed out as with the nonsingular problem.

Example 3 [6]. Consider a two dimensional delay DAE system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x'(t) + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x(t-1) = 0,$$

with $x = (x_1, x_2)^T$. This system has index-2. It is easy to see that x_1 satisfies an advanced type equation $x_1(t) = x_1'(t-1)$, so that $x_1(t) = x_1^{(m)}(t-m)$. That is, x_1 (and x_2, too) becomes progressively less smooth. The system behaves like those of advanced type.

For the simplest case $k = 1$, the situation is somewhat better. The evolution of z is given by a delay differential equation, meanwhile a difference operator defines the dynamics of w. If a continuous initial function φ is given, then z is continuous and w is piecewise continuous in general. Furthermore, z is differentiable and w is continuous except possibly at integer multiples of τ. The system (2.3) behaves like a neutral delay system.

Extending all the results in this section to multiple-delay DAEs of the form (1.1) or (1.2) is straightforward. We note that the smoothness of solutions now may be even worse. Even in index-1 problems, the distance between the jump (or break) points can become arbitrarily small as t is increasing except for the case when all the ratios τ_i/τ_j, $i \neq j$ are rational numbers. This fact gives rise to practical difficulties for numerical methods.

2.2. Delay DAEs of Hessenberg form

Delay DAEs arising in applications frequently have special structure. One of the most important class of systems is that of Hessenberg forms which generalizes non-delay DAEs of Hessenberg form [3].

Definition 3. Linear delay DAEs of the form

$$\dot{x}_1 + B_1x_1 + B_2x_2 + D_1x_1(t-\tau) + D_2x_2(t-\tau) = 0, \tag{2.9}$$
$$B_3x_1 + B_4x_2 + D_3x_1(t-\tau) = 0, \tag{2.10}$$

where B_4 is nonsingular, is called semi-explicit index-1 linear DDAEs or index-1 linear DDAEs in Hessenberg form. Note $\{A, B\}$ is an index one pencil and $D_4 = 0$.

Linear delay DAEs of the form

$$\dot{x}_1 + B_1x_1 + B_2x_2 + D_1x_1(t-\tau) = 0, \tag{2.11}$$
$$B_3x_1 = 0, \tag{2.12}$$

where B_3B_2 is nonsingular, is called semi-explicit index-2 linear DDAEs or index-2 linear DDAEs in Hessenberg form. Here $\{A, B\}$ is an index two Hessenberg pencil and $D_2 = 0$, $D_3 = 0$, and $D_4 = 0$.

Delay DAEs of the form (2.9)-(2.10) come from the linearization of index-1 nonlinear DDAEs in Hessenberg form

$$f(t, \dot{x}_1(t), x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau)) = 0, \tag{2.13}$$
$$g(t, x_1(t), x_1(t-\tau), x_2(t)) = 0 \tag{2.14}$$

along a particular solution, where the Jacobian g_{x_2} is assumed nonsingular. Similarly, by linearizing index-2 nonlinear DDAEs

$$f(t, \dot{x}_1(t), x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau)) = 0, \tag{2.15}$$
$$g(t, x_1(t)) = 0, \tag{2.16}$$

where $g_{x_1}f_{x_2}$ is assumed nonsingular, one obtains DDAEs of the form (2.11) and (2.12), see [2,26].

The derivative of the unknown function at a delayed time may appear in (2.9), (2.10) and (2.11), (2.12), as well. Namely, DDAEs of the form

$$\dot{x}_1 + B_1x_1 + B_2x_2 + C_1\dot{x}_1(t-\tau) + C_2\dot{x}_2(t-\tau) + D_1x_1(t-\tau) + D_2x_2(t-\tau) = 0, \tag{2.17}$$
$$B_3x_1 + B_4x_2 + D_3x_1(t-\tau) = 0, \tag{2.18}$$

where B_4 is nonsingular, are called index-1 linear neutral DDAEs in Hessenberg form. Further, DDAEs of the form

$$\dot{x}_1 + B_1x_1 + B_2x_2 + C_1\dot{x}_1(t-\tau) + D_1x_1(t-\tau) = 0, \tag{2.19}$$
$$B_3x_1 = 0, \tag{2.20}$$

where B_3B_2 is nonsingular, are called index-2 linear neutral DDAEs in Hessenberg form.

Neutral delay DAEs of the forms (2.17), (2.18) and (2.19), (2.20) can be transformed to delay DAEs of the forms (2.9), (2.10) and (2.11), (2.12) by introducing new auxiliary variables as discussed in the previous section. Proposition 1 shows the index of the transformed delay DAEs have the same index as the original neutral delay DAEs.

One of the most important features of delay DAEs in Hessenberg form is that one can easily get the so-called underlying DODEs. For example, for the system 2.9,2.10, one can solve x_2 from (2.10), then insert into (2.9), and get the underlying DODE

$$\dot{x}_1 + (B_1 - B_2B_4^{-1}B_3)x_1 + (D_1 - D_2B_4^{-1}B_3 - B_2B_4^{-1}D_3)x_1(t-\tau) - D_2B_4^{-1}D_3x_1(t-2\tau) = 0. \tag{2.21}$$

Note that the UDODE (2.21) now has double delays while the original DDAE (2.9) and (2.10) has a single delay. Further, it is easy to see that the index-1 DDAE (2.9) and (2.10) is asymptotically stable if and only if its UDODE (2.21) is asymptotically stable. Similarly, we can derive the underlying neutral DODE for the index-1 neutral DDAE of the form (2.17) and (2.18).

Obtaining the underlying DODE for semi-explicit index-2 DDAE of the form (2.11) and (2.12) is a little bit more complicated than the index-1 case. First, observe that by differentiating (2.12) and inserting the result into (2.11), we obtain a hidden constraint

$$B_3B_1x_1 + B_3B_2x_2 + B_3D_1x_1(t-\tau) = 0. \tag{2.22}$$

Since B_3B_2 is invertible, one can calculate the index-2 algebraic variable x_2 from x_1. Next, we proceed as follows (see [1,8]). Denote the row number and the column number of B_2 by m_1 and m_2, respectively. Take a matrix $R \in \mathbb{R}^{(m_1-m_2)\times m_1}$ whose linearly independent normalized rows form a basis for the null space of B_2^T. Then $RB_2 = 0$ and the matrix $\begin{pmatrix} R \\ B_3 \end{pmatrix}$ is invertible.

Defining new variables $u = Rx_1$, we can calculate x_1 from u by

$$x_1 = \begin{pmatrix} R \\ B_3 \end{pmatrix}^{-1} \begin{pmatrix} u \\ 0 \end{pmatrix} = Su, \tag{2.23}$$

where S is defined by $RS = I$, $B_3S = 0$. The underlying DODE is

$$\dot{u} + RB_1Su + RD_1Su(t-\tau) = 0. \tag{2.24}$$

From (2.22) and (2.23), it is clearly seen that the semi-explicit index-2 DDAE (2.11) and (2.12) is asymptotically stable if and only if the UDODE (2.24) is. We obtain analogously the underlying neutral DODE for the index-2 neutral DDAE of the form (2.19) and (2.20).

From the above introduction of DDAE in Hessenberg form, we conclude that if one wants to investigate the stability of DDAEs in Hessenberg form, it makes sense to consider their underlying DODEs.

3. Stability criteria

Now consider the DAE of multiple delays of the form (1.1) or (1.2). The characteristic equation for (1.1) is defined by

$$P(s) = \det\left(sA + B + s\sum_{i=1}^{M} C_i e^{-s\tau_i} + \sum_{i=1}^{M} D_i e^{-s\tau_i}\right) = 0. \tag{3.1}$$

For a given $s \in \mathbb{C}$, we denote its real and imaginary parts by $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$, respectively. It is well known, see [25], that the system (1.1) is asymptotically stable if all the roots of (3.1) have negative real part and they are bounded away from the imaginary axis, i.e., for all root λ_i of (3.1) ($i = 1,2,\ldots$) and for some positive μ, the inequalities

$$\mathrm{Re}(\lambda_i) \leqslant -\mu < 0 \tag{3.2}$$

hold. Note that (3.1) may have infinitely roots and they may accumulate at a finite point on the complex plane or at infinity. In this section, we will derive some sufficient conditions for (3.2). We will need the following definition and an auxiliary result, which are well known in the theory of nonnegative matrices [17].

Definition 4. Let $W \in \mathbb{C}^{n\times n}$ with elements w_{ij} and $|W|$ denote the nonnegative matrix in $\mathbb{R}^{n\times n}$ with element $|w_{ij}|$. For two matrices $U, V \in \mathbb{R}^{n\times n}$, we write $U \leqslant V$ if and only if $u_{ij} \leqslant v_{ij}$ for each $i,j \in \{1,2,\ldots,n\}$. In particular, $\rho(W) \leqslant \rho(|W|)$.

Lemma 1. *Let $W \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{R}^{n\times n}$. If $|W| \leqslant V$, then $\rho(W) \leqslant \rho(V)$.*

3.1. Delay-independent asymptotic stability

We introduce the following two-variable polynomials:

$$Q(s,z) = \det\left(sA + B + \sum_{i=1}^{M}(sC_i + D_i)z\right), \tag{3.3}$$

$$R(s,z) = \det\left(sA + B + \sum_{i=1}^{M}(sC_i + D_i)z^i\right). \tag{3.4}$$

Lemma 2. *Suppose that*

$$\text{(i)}\quad \sigma(A,B) \in \mathbb{C}^-, \tag{3.5}$$

$$\text{(ii)}\quad \sup_{\mathrm{Re}(s)\geqslant 0} \rho\left(\sum_{i=1}^{M} |(sA+B)^{-1}(sC_i + D_i)|\right) < 1. \tag{3.6}$$

Then $Q(s,z) \neq 0$ for all $s, z \in \mathbb{C}$ such that $\mathrm{Re}(s) \geqslant 0$, $|z| \leqslant 1$.

Proof. Suppose that the contrary happens, i.e., there exist s, $\mathrm{Re}(s) \geqslant 0$ and z, $|z| \leqslant 1$ such that $Q(s,z) = 0$. This implies that there exist a vector $v \neq 0$ such that

$$\left[sA + B + \sum_{i=1}^{M}(sC_i + D_i)z\right] v = 0$$

or equivalently (since $(sA + B)$ is invertible)

$$(sA + B)^{-1}\sum_{i=1}^{M}(sC_i + D_i)zv = -v.$$

This means that -1 is an eigenvalue of $(sA + B)^{-1}\sum_{i=1}^{M}(sC_i + D_i)z$, which implies

$$\rho\left((sA + B)^{-1}\sum_{i=1}^{M}(sC_i + D_i)z\right) \geqslant 1.$$

But

$$\rho\left(\sum_{i=1}^{M}|(sA + B)^{-1}(sC_i + D_i)|\right) \geqslant \rho\left((sA + B)^{-1}\sum_{i=1}^{M}(sC_i + D_i)z\right),$$

which contradicts (3.6). □

Note that in the single delay case, i.e. $M = 1$, the statement holds without the need of taking the absolute value in (3.6), see also [25]. Further, due to the maximum principle in complex analysis, it suffices to take the supremum on the imaginary axis $\mathrm{Re}(s) = 0$ in the assumption (3.6).

Theorem 1. *Suppose that the assumptions (3.5) and (3.6) in Lemma 2 hold. Then the system (1.1) is asymptotically stable for all sets of the delays $\{\tau_i\}_{i=1}^{M}$, i.e., the asymptotic stability of (1.1) is delay-independent.*

Proof. Similarly to the proof of Lemma 2, it is not difficult to show that the equation $P(s) = 0$ has only roots with negative real part. Next, we prove that the real parts of the roots are bounded away from 0. Suppose that this statement is not true. Then, there exists a sequence $\{s_n\}$ such that $\lim_{n\to\infty}\mathrm{Re}(s_n) = -0$ meanwhile $P(s_n) = 0$. Choose a positive number ε such that

$$\varepsilon < 1 - \sup_{\mathrm{Re}(s)=0}\rho\left(\sum_{i=1}^{M}|(sA + B)^{-1}(sC_i + D_i)|\right).$$

It is obvious that there exists a sufficiently large N_0 such that for $n \geqslant N_0$, we have $\mathrm{Re}(s_n) \geqslant \frac{1}{2}\bar{\mu}$, where $\bar{\mu} = \max\{\mathrm{Re}(\lambda), \lambda \in \sigma(A,B)\} < 0$ is the spectral abscissa of the pencil $\{A,B\}$, and $|e^{-\tau_M s_n}| \leqslant (1 - \varepsilon/2)^{-1}$. For each s_n, there exists a vector $v_n \neq 0$ such that

$$\left[s_nA + B + \sum_{i=1}^{M}(s_nC_i + D_i)e^{-\tau_i s_n}\right] v_n = 0,$$

which implies

$$\rho\left(\sum_{i=1}^{M}|(s_nA + B)^{-1}(s_nC_i + D_i)|\right) \geqslant 1 - \varepsilon/2.$$

Now we observe that the entries of the matrix functions

$$(sA + B)^{-1}(sC_i + D_i), \quad i = 1, 2, \ldots, M$$

are rational function of s. Only the finite eigenvalues of $\{A,B\}$ may be poles of these functions. Thus, each element of the (nonnegative) matrix function

$$\sum_{i=1}^{M}|(sA + B)^{-1}(sC_i + D_i)|$$

has the form $|s|^{\alpha_{pq}}(a_{pq} + O(1/|s|))$, where α_{pq} are some integers and a_{pq} are nonnegative numbers $(1 \leqslant p,q \leqslant m)$. Hence, for an arbitrarily small $\epsilon > 0$, there exists a bound $s_\infty > 0$ such that

$$\mathscr{A}(s)(1 - \epsilon) \leqslant \sum_{i=1}^{M}|(sA + B)^{-1}(sC_i + D_i)| \leqslant \mathscr{A}(s)(1 + \epsilon) \tag{3.7}$$

for all $|s| \geqslant s_\infty$, where the elements of the matrix function $\mathscr{A}(s)$ are defined by

$$\mathscr{A}_{pq}(s) = |s|^{\alpha_{pq}} a_{pq}.$$

Let $\alpha = \max_{p,q} \alpha_{pq}$ and the matrix function $\mathscr{A}(s)$ be decomposed such that

$$\mathscr{A}(s) = |s|^{\alpha}\left[\mathscr{A}^{(0)} + \mathscr{A}^{(1)}(s)\right],$$

where $\mathscr{A}^{(0)}$ is a nonnegative constant matrix and each entry of $\mathscr{A}^{(1)}(s)$ is either zero or negative power of $|s|$. Next, we investigate the asymptotic behavior of the spectral radius of $\mathscr{A}(s)$ as $|s|$ tends to infinity. The following cases are possible:

- If $\alpha \leqslant 0$, then $\rho(\mathscr{A}(s))$ has a finite limit as $|s| \to \infty$;
- If $\alpha > 0$ and $\rho(\mathscr{A}^{(0)}) > 0$, then $\rho(\mathscr{A}(s))$ tends to infinity as $|s| \to \infty$;
- If $\alpha > 0$ and $\rho(\mathscr{A}^{(0)}) = 0$, then due to the Puisseux series of the eigenvalues of $(\mathscr{A}^{(0)} + \mathscr{A}^{(1)}(s))$, see [15], we have

$$\rho(\mathscr{A}^{(0)} + \mathscr{A}^{(1)}(s)) = |s|^{\beta}(c + o(1)),$$

where $c > 0$ is a constant and β is a negative fractional number. In other words, we use the fact that the eigenvalues can be expanded into fractional power series of $1/|s|$. Depending on the sign of $\alpha + \beta$, the spectral radius of $\mathscr{A}(s)$ either converges to a finite number or tends to infinity as $|s| \to \infty$.

Summarizing the above cases, the spectral radius of $\mathscr{A}(s)$ either converges to a finite number or tends to infinity as $|s| \to \infty$. Since ϵ in (3.7) is arbitrarily chosen, the same statement holds for the spectral radius of

$$\sum_{i=1}^{M} |(sA + B)^{-1}(sC_i + D_i)|,$$

which is a function of s. The assumption (3.6) implies that the latter function must converge to a finite limit as $|s| \to \infty$. On the other hand, this function is continuous in the domain $\{s \in \mathbb{C},\ \mathrm{Re}(s) \geqslant \frac{1}{2}\bar{\mu}\}$. Consequently, it is uniformly continuous in the considered domain.

Finally, due to the verified uniform continuity, there exists s_n sufficiently close to the imaginary axis such that

$$\left|\rho\left(\sum_{i=1}^{M} |(s_nA + B)^{-1}(s_nC_i + D_i)|\right) - \rho\left(\sum_{i=1}^{M} |(\mathrm{Im}(s_n)A + B)^{-1}(\mathrm{Im}(s_n)C_i + D_i)|\right)\right| \leqslant \varepsilon/2.$$

We obtain

$$\begin{aligned}\rho\left(\sum_{i=1}^{M} |(s_nA + B)^{-1}(s_nC_i + D_i)|\right) &\leqslant \rho\left(\sum_{i=1}^{M} |(\mathrm{Im}(s_n)A + B)^{-1}(\mathrm{Im}(s_n)C_i + D_i)|\right) + \varepsilon/2 \\ &\leqslant \sup_{\mathrm{Re}(s)=0} \rho\left(\sum_{i=1}^{M} |(sA + B)^{-1}(sC_i + D_i)|\right) + \varepsilon/2 < 1 - \varepsilon/2,\end{aligned}$$

which yields contradiction. The proof is complete. □

Assumptions (3.5) and (3.6) come from the straightforward generalization of the corresponding stability conditions for neutral DAEs with single delay given in [25]. In that paper, a third assumption $|u^TAu| > |u^TCu|, \forall u \in \mathbb{C}^m$, was needed to ensure that all the roots of the characteristic equations are bounded away from the imaginary axis. From the proof of Theorem 1, we see that such an assumption is redundant and can be ignored.

Sometimes it is more convenient to check the assumptions by using an operator norm instead of the spectral radius.

Corollary 2. *Suppose that the assumption (3.5) holds and*

$$\sup_{\mathrm{Re}(s)=0} \left\|\sum_{i=1}^{M} |(sA + B)^{-1}(sC_i + D_i)|\right\| < 1.$$

Then the system (1.1) is delay-independently asymptotically stable.

We have similar statements for the system (1.2).

Lemma 3. *Suppose that*

$$\text{(i)}\ \sigma(A, B) \in \mathbb{C}^-, \tag{3.8}$$

$$\text{(ii)}\ \sup_{\mathrm{Re}(s)\geqslant 0} \rho\left(\{sC_i + D_i\}_{i=0}^{M}\right) < 1, \tag{3.9}$$

where $C_0 := A$, $D_0 := B$. Then $R(s,z) \neq 0$ for all $s, z \in \mathbb{C}$ such that $Re(s) \geqslant 0$, $|z| \leqslant 1$.

Proof. Recall that

$$\rho\left(\{sC_i + D_i\}_{i=0}^{M}\right) = \max\left\{|\lambda|, \det\left(\sum_{i=0}^{M}(sC_i + D_i)\lambda^{M-i}\right) = 0\right\} \quad \text{for a given fixed } s.$$

Hence, for a given s, $\mathrm{Re}(s) \geqslant 0$, if $z \in \mathbb{C}$ is such that $R(s,z) = 0$, then $1/z$ is an eigenvalue of the polynomial eigenvalue problem with data $\{sC_i + D_i\}_{i=0}^{M}$. Note that z cannot be zero because $\det(sA + B) \neq 0$ for all s, $\mathrm{Re}(s) \geqslant 0$. Therefore, assumption (3.9) implies that for a given s, $\mathrm{Re}(s) \geqslant 0$, if $R(s,z) = 0$, then $|z| > 1$. □

Theorem 2. *Suppose that the assumptions (3.8) and (3.9) in Lemma 3 hold. Then the system (1.2) is asymptotically stable for all $\tau \geqslant 0$, i.e., the asymptotic stability of (1.2) is delay-independent.*

Proof. Similar to the proof of Theorem 1. □

Next, we attempt to characterize the set of admissible coefficient matrices which satisfy the assumptions of Theorem 1 and analyze the effect of the index of the pencil $\{A,B\}$ with second assumption (3.6). For sake of simplicity, and due to Proposition 1, we consider the single delay Eq. (2.3). The assumption (3.6) now becomes $\sup_{\mathrm{Re}(s)=0}\rho((sA + B)^{-1}D) < 1$.

Assume the coefficient matrices are transformed again in block form. We have

$$(sA + B)^{-1}D = \begin{pmatrix} (sI + B_1)^{-1}D_1 & (sI + B_1)^{-1}D_2 \\ (sN + I)^{-1}D_3 & (sN + I)^{-1}D_4 \end{pmatrix}.$$

Using the nilpotency of N, it is obvious that the spectral radius of the matrix

$$\begin{pmatrix} (sI + B_1)^{-1}D_1 & (sI + B_1)^{-1}D_2 \\ \sum_{i=0}^{k-1}(-N)^i s^i D_3 & \sum_{i=0}^{k-1}(-N)^i s^i D_4 \end{pmatrix}$$

is necessarily bounded for s, $\mathrm{Re}(s) = 0$. Since all the entries of the first block row tend to 0 as $|s|$ tends to infinity, we get some consequences on D_4. Namely, if $k = 1$, then $\rho(D_4) < 1$ must be satisfied. For higher index cases, we have $\rho(D_4) < 1$ and $\rho(N^i D_4) = 0$ for $i = 1,2,\ldots,k-1$, otherwise the spectral radius in question is unbounded. That is, for higher index pencil $\{A,B\}$, the block D_4 (and D_3 as well) must be of special structure. Taking into account the result of Proposition 1, the same statement holds for higher index neutral delay DAEs (2.4) and for higher index DAEs systems with multiple delays of the form (1.1) and (1.2). Note that these necessary conditions on D_4 are trivially satisfied by the delay and neutral delay DAEs of Hessenberg forms.

With the assumption (3.5), the problem of finding sufficient condition for asymptotic stability for delay DAEs is closely related to the robust stability question of DAE, see [4,21,9,18]. We have a nominal DAE system without delays which is assumed to be asymptotically stable. The delay terms can be considered uncertain perturbations. From this point of view, a somewhat simpler condition can be given instead of (3.6).

Proposition 2. *Consider the delay DAE of the form (1.1). Suppose that $C_i = 0$, for $i = 1,2,\ldots,M$ and assumption (3.5) holds. Then if*

$$\|(D_1 \quad D_2 \quad \cdots \quad D_M)\| < \left(\sup_{\mathrm{Re}(s)=0}\|(sA + B)^{-1}\|\right)^{-1}, \tag{3.10}$$

the delay DAE system is asymptotically stable.

Proof. Eq. (3.10) implies (3.6). See also [18]. Note that in (3.10) we can take any matrix norm induced by a vector norm. □

Unfortunately, if the index of $\{A,B\}$ is greater than 1, then the right hand side of (3.10) is simply zero, and the proposition does not apply. This once again confirms that for higher-index problems, the coefficient matrices C_i, D_i, $(i = 1,2,\ldots,M)$ must be highly structured so that the asymptotic stability would be preserved.

3.2. Practical algebraic stability criteria

Theorems 1 and 2 give us sufficient conditions for the asymptotic stability of delay DAEs of the form (1.1) and (1.2), respectively. Unfortunately, checking the conditions (3.6) or (3.9) is rather difficult because computing the supremum of the spectral radius of a matrix function or a polynomial matrix function over an unbounded domain is very costly. In this section, we propose some checkable algebraic criteria for the asymptotic stability. Our results extend some recent results for neutral delay ODEs, see [13,14], to neutral delay DAEs.

(a) The index-1 case. We first restrict the investigation to index-1 problems. Since the matrices A and B can easily be transformed to the upper block-triangular form using QZ or QR decompositions, we assume that

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ 0 & B_4 \end{pmatrix}.$$

The index-1 assumption on the pencil $\{A,B\}$ implies that the submatrix B_4 is invertible. Furthermore, due to Corollary 1, we assume that

$$C_i = \begin{pmatrix} C_{i1} & C_{i2} \\ 0 & 0 \end{pmatrix}, \quad D_i = \begin{pmatrix} D_{i1} & D_{i2} \\ D_{i3} & D_{i4} \end{pmatrix}, \quad i = 1,2,\ldots,M.$$

We introduce some auxiliary matrix sequences

$$L_i = (A+B)^{-1}(D_i + C_i), \quad M_i = (A+B)^{-1}(D_i - C_i), \quad E = (A+B)^{-1}(A-B) \tag{3.11}$$

for $i = 1,2,\ldots,M$.

Lemma 4. *Let the assumption (3.5) hold. Then*

$$(sA+B)^{-1}(sC_i + D_i) = (I - zE)^{-1}(zM_i + L_i)$$

for all $Re(s) \geqslant 0$, where $z = \frac{1-s}{1+s}$ (which implies $|z| \leqslant 1$, $z \neq -1$).

Proof. The proof is similar to the proof of Theorem 2.2 [13]. It is easy to derive

$$\begin{aligned} I - zE &= \left[I - \frac{1-s}{1+s}(A+B)^{-1}(A-B) \right] \\ &= (A+B)^{-1}[(A+B)(1+s) - (1-s)(A-B)](1+s)^{-1} \\ &= 2(A+B)^{-1}(sA+B)(1+s)^{-1}. \end{aligned}$$

In the same way, we get

$$zM_i + L_i = 2(A+B)^{-1}(sC_i + D_i)(1+s)^{-1} \quad \text{for all } i = 1,2,\ldots,$$

which yields the statement. □

Now, let $S = (A_1 + B_1)^{-1}$. By direct calculations, we have

$$E = \begin{pmatrix} S(A_1 - B_1) & 2SA_2 \\ 0 & -I \end{pmatrix} =: \begin{pmatrix} E_1 & E_2 \\ 0 & -I \end{pmatrix},$$

$$L_i = \begin{pmatrix} S[D_{i1} + C_{i1} - (A_2 + B_2)B_4^{-1}D_{i3}] & S[D_{i2} + C_{i2} - (A_2 + B_2)B_4^{-1}D_{i4}] \\ B_4^{-1}D_{i3} & B_4^{-1}D_{i4} \end{pmatrix} =: \begin{pmatrix} L_{i1} & L_{i2} \\ L_{i3} & L_{i4} \end{pmatrix},$$

$$M_i = \begin{pmatrix} S[D_{i1} - C_{i1} - (A_2 + B_2)B_4^{-1}D_{i3}] & S[D_{i2} - C_{i2} - (A_2 + B_2)B_4^{-1}D_{i4}] \\ B_4^{-1}D_{i3} & B_4^{-1}D_{i4} \end{pmatrix} =: \begin{pmatrix} M_{i1} & M_{i2} \\ M_{i3} & M_{i4} \end{pmatrix}$$

Matrix E always has an eigenvalue $\lambda = -1$, which makes the straightforward extension of the results in [13,14] impossible since $\rho(|E|) < 1$ would be required. We can still give estimates for the left hand-side of (3.6) and (3.9) by estimating separately the "differential" part and the "algebraic" one. Furthermore, in order to ease the matrix calculations, we may transform A_1, B_1, and B_4 into upper triangular form prior to the calculations.

We have

$$I - zE = \begin{pmatrix} I - zE_1 & -zE_2 \\ 0 & (1+z)I \end{pmatrix}, \quad zM_i + L_i = \begin{pmatrix} zM_{i1} + L_{i1} & zM_{i2} + L_{i2} \\ (1+z)L_{i3} & (1+z)L_{i4} \end{pmatrix}. \tag{3.12}$$

Note that $L_{i3} = M_{i3}$, $L_{i4} = M_{i4}$.

We introduce the following auxiliary matrices.

Definition 5. Assume $\rho(|E_1|) < 1$. For an integer $l \geqslant 0$, and for $i = 1,\ldots,M$; $j = 1,2$, let

$$G_{ij}(l) = \sum_{\nu=0}^{l} \{|E_1^\nu L_{ij}| + |E_1^\nu \widetilde{M}_{ij}|\} + (I - |E_1|)^{-1}(|E_1^{l+1}L_{ij}| + |E_1^{l+1}\widetilde{M}_{ij}|), \tag{3.13}$$

where $\widetilde{M}_{i1} = M_{i1} + E_2 L_{i3}$, $\widetilde{M}_{i2} = M_{i2} + E_2 L_{i4}$, $i = 1,2,\ldots,M$. Further, let

$$G_i(l) = \begin{pmatrix} G_{i1}(l) & G_{i2}(l) \\ |L_{i3}| & |L_{i4}| \end{pmatrix}. \tag{3.14}$$

The following estimate will be very useful (see also [13, Theorem 3.1]).

Proposition 3. *Assume that the assumption (3.5) holds and the pencil $\{A,B\}$ has index 1. Further, assume $\rho(|E_1|) < 1$. Then for any z satisfying $|z| \leqslant 1$, we have*

$$|(I - zE_1)^{-1}(L_{ij} + z\widetilde{M}_{ij})| \leqslant G_{ij}(l) \leqslant G_{ij}(0).$$

Furthermore $G_{ij}(l) \leqslant G_{ij}(l-1)$ for all $l \geqslant 1$ and $i = 1,2,\dots,M$; $j = 1,2$.

Proof. The required inequalities can be verified by the same arguments as in the proof of [13, Theorem 3.1]. By defining $T = zE_1$, for $|z| \leqslant 1$, we have

$$(I - zE_1)^{-1}(L_{ij} + z\widetilde{M}_{ij}) = (I + T + T^2 + \cdots)(L_{ij} + z\widetilde{M}_{ij}) = \sum_{v=0}^{l}\{T^v L_{ij} + zT^v \widetilde{M}_{ij}\} + (I + T + T^2 + \cdots)(T^{l+1}L_{ij} + zT^{l+1}\widetilde{M}_{ij}).$$

Since $|z| \leqslant 1$ and $|T| \leqslant |E_1|$, the inequality

$$|(I - zE_1)^{-1}(L_{ij} + z\widetilde{M}_{ij})| \leqslant \sum_{v=0}^{l}\{|E_1^v L_{ij}| + |E_1^v \widetilde{M}_{ij}|\} + (I + |E_1| + |E_1|^2 + \cdots)(|E_1^{l+1}L_{ij}| + |E_1^{l+1}\widetilde{M}_{ij}|)$$

holds, from which the estimate $|(I - zE_1)^{-1}(L_{ij} + z\widetilde{M}_{ij})| \leqslant G_{ij}(l)$ comes immediately. Next, we show $G_{ij}(l) \leqslant G_{ij}(l-1)$ for $l \geqslant 1$. Indeed, we have

$$\begin{aligned} G_{ij}(l) &= \sum_{v=0}^{l}\{|E_1^v L_{ij}| + |E_1^v \widetilde{M}_{ij}|\} + (I + |E_1| + |E_1|^2 + \cdots)(|E_1^{l+1}L_{ij}| + |E_1^{l+1}\widetilde{M}_{ij}|) \\ &\leqslant \sum_{v=0}^{l}\{|E_1^v L_{ij}| + |E_1^v \widetilde{M}_{ij}|\} + (|E_1| + |E_1|^2 + \cdots)(|E_1^{l}L_{ij}| + |E_1^{l}\widetilde{M}_{ij}|) \\ &= \sum_{v=0}^{l-1}\{|E_1^v L_{ij}| + |E_1^v \widetilde{M}_{ij}|\} + (I + |E_1| + |E_1|^2 + \cdots)(|E_1^{l}L_{ij}| + |E_1^{l}\widetilde{M}_{ij}|) = G_{ij}(l-1). \end{aligned}$$

As a consequence, the inequality $G_{ij}(l) \leqslant G_{ij}(0)$ holds for any positive integer l. □

Note that transforming A_1 and B_1 into upper triangular form has an additional advantage. If A_1, B_1 are upper triangular, then E_1 is of upper triangular form, too. In this case the eigenvalues of E_1 are $(1-\lambda_i)/(1+\lambda_i)$, where λ_i, $i = 1,2,\dots$ are finite eigenvalues of the pencil $\{A,B\}$. Thus (3.5) implies that $\rho(E_1) < 1$ and $\rho(|E_1|) < 1$ as well. Hence, the condition $\rho(|E_1|) < 1$ is obviously fulfilled and it does not mean an extra assumption for the existence of the inverse of $(I - |E_1|)$ is required.

Using this result, it is easy to get estimates for the left-hand side of (3.5).

Proposition 4. *Assume that the assumptions of Proposition 3 hold. Then*

$$\rho\left(\sum_{i=1}^{M}|(I - zE)^{-1}(L_i + zM_i)|\right) \leqslant \rho\left(\sum_{i=1}^{M}G_i(l)\right) \leqslant \rho\left(\sum_{i=1}^{M}G_i(0)\right)$$

holds for any z satisfying $|z| \leqslant 1$, $z \neq -1$, where the parametrized matrices $G_i(\cdot)$ are defined in (3.14).

Proof. Using the formulas in (3.12), it is easy to verify that

$$(I - zE)^{-1}(L_i + zM_i) = \begin{pmatrix} (I - zE_1)^{-1}(L_{i1} + \widetilde{M}_{i1}) & (I - zE_1)^{-1}(L_{i2} + \widetilde{M}_{i2}) \\ L_{i3} & L_{i4} \end{pmatrix}.$$

Then, using Proposition 3 and the definition of matrices $G_i(l)$ given in (3.14), we have $|(I - zE)^{-1}(L_i + zM_i)| \leqslant G_i(l) \leqslant G_i(0)$, $i = 1,\dots,M$, for any positive integer l. Summing up the inequalities and then invoking Lemma 1, the proof is complete. □

Now, we are in position to state a practical algebraic criterion for the asymptotic stability of the delay DAE system of the form (1.1).

Theorem 3. *Assume the assumptions of Proposition 3 hold. If there exists an integer $l \geqslant 0$ such that*

$$\rho\left(\sum_{i=1}^{M}G_i(l)\right) < 1 \tag{3.15}$$

then (1.1) is asymptotically stable.

Proof. Using the result of Propostion 4, it is easy to see that (3.15) implies (3.6). Invoking Theorem 1 completes the proof. □

By either simply setting $l = 0$ or estimating the limit as $l \to \infty$ in (3.15), we obtain simple stability criteria for index-1 DDAEs of the form (1.1).

Corollary 3. *Assume that the assumptions of Proposition 3 hold. If*

$$\rho\left(\sum_{i=1}^{M}G_i(0)\right) < 1 \tag{3.16}$$

or

$$\rho\left(\sum_{i=1}^{M} H_i\right) < 1, \tag{3.17}$$

where

$$H_i = \begin{pmatrix} (1-|E_1|)^{-1}(|L_{i1}|+|\widetilde{M}_{i1}|) & (1-|E_1|)^{-1}(|L_{i2}|+|\widetilde{M}_{i2}|) \\ |L_{i3}| & |L_{i4}| \end{pmatrix}$$

($i = 1,\ldots,M$), then (1.1) is asymptotically stable.

Note that there are practical criteria for checking whether a given matrix is a Schur matrix or not. Furthermore, by a similar argument as in the proof of Proposition 4, it is not difficult to show that $G_i(0) \leqslant H_i$, $i = 1,\ldots,M$. Hence

$$\rho\left(\sum_{i=1}^{M} G_i(0)\right) \leqslant \rho\left(\sum_{i=1}^{M} H_i\right).$$

Example 4. Consider Eq. (1.1) with the following data:

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 1 & 0.5 & -1 \\ 0 & 5 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad C_1 = \alpha\begin{pmatrix} -2 & -1 & -1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_2 = \alpha\begin{pmatrix} -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_1 = \beta\begin{pmatrix} -2 & 1 & -1 & 1 \\ -2 & 0 & 2 & -1 \\ -2 & 1 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \quad D_2 = \beta\begin{pmatrix} -1 & 1 & -1 & 1 \\ -2 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

where α and β are real parameters. The number of delays $M = 2$. The pencil $\{A,B\}$) has index 1 and $\sigma(A,B) = \{-3,-5\}$. Further

$$E_1 = \begin{pmatrix} -\frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{2}{3} \end{pmatrix}.$$

The values of $\rho(\sum_{i=1}^{M} G(l))$ are calculated with different values of α and β. First, we fix $\alpha = 0.125$. The numerical results for $l = 0,1,2,3,4$ and $\beta = 0.089, 0.079, 0.069, 0.059$ are displayed in Table 1. By Theorem 3, we conclude that Eq. (1.1) in this example with the above chosen parameters is delay-independently asymptotically stable. Further, the monotonicity of $\rho(\sum_{i=1}^{M} G(l))$ with respect to l is illustrated well.

Next we plot $\rho(\sum_{i=1}^{M} G(0))$ as a function of $\beta \in [-0.1, 0.1]$ at $\alpha = 0.05$ on Fig. 1 and as a function of $\alpha \in [-0.1, 0.2]$ at $\beta = 0.05$ on Fig. 2, respectively. Finally, a 3D plot of $\rho(\sum_{i=1}^{M} G(0))$ as a two-variable function of α and β is shown on Fig. 3.

Theorem 3 and Corollary 3 include the stability criteria for neutral ODEs with multiple delays in [14] as a special case (when the algebraic part vanishes).

By analogue, we obtain the stability criteria for the class of delay DAEs of the form (1.2).

Theorem 4. *Assume assumption (3.8) holds and A,B are given in upper triangular form. If there exists an integer $l \geqslant 0$ such that*

$$\rho\left(\{G_i(l)\}_{i=0}^{M}\right) < 1, \tag{3.18}$$

where $G_0(l) := I$, then (1.2) is asymptotically stable.

Table 1
The spectral radius of $\sum_{i=1}^{M} G(l)$ with $\alpha = 0.125$.

l	$\beta = 0.089$	$\beta = 0.079$	$\beta = 0.069$	$\beta = 0.059$
0	1.0082	0.9252	0.8537	0.8155
1	1.0020	0.9214	0.8475	0.8108
2	0.9979	0.9188	0.8467	0.8077
3	0.9976	0.9171	0.8462	0.8058
4	0.9975	0.9162	0.8460	0.8054

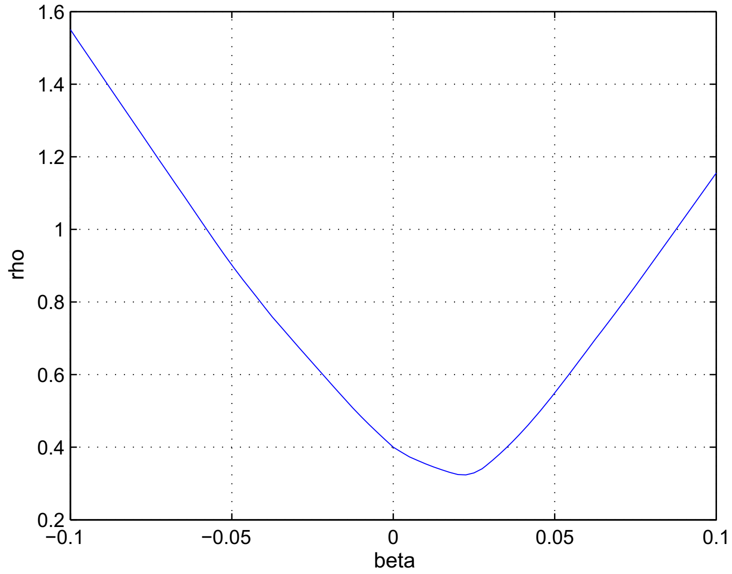

Fig. 1. The graph of $\rho(\sum_{i=1}^{M} G(0))$ as a function of β ($\alpha = 0.05$).

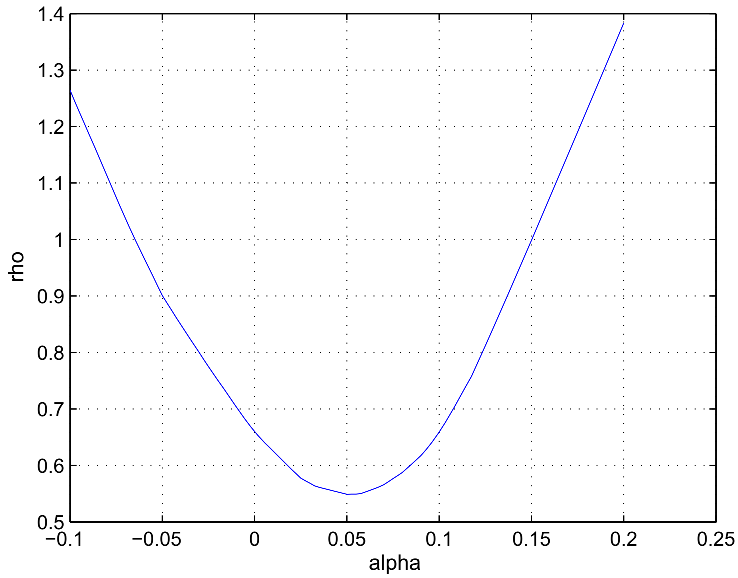

Fig. 2. The graph of $\rho(\sum_{i=1}^{M} G(0))$ as a function of α ($\beta = 0.05$).

Corollary 4. *Assume assumption (3.8) holds and A,B are given in upper triangular form. If*

$$\rho\left(\{G_i(0)\}_{i=0}^{M}\right) < 1, \tag{3.19}$$

or

$$\rho\left(\{H_i\}_{i=0}^{M}\right) < 1, \tag{3.20}$$

where H_0: = I, then (1.2) is asymptotically stable.

Example 5. Consider the Eq. (1.2) with the following data:

$$A = \begin{pmatrix} 1 & 2 & 1 & -3 \\ -1 & 1 & 1 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 1 & 1 & -2 \\ -1 & 5 & 2 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad C_1 = \alpha \begin{pmatrix} -1 & -3 & -2 & 1 \\ 4 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

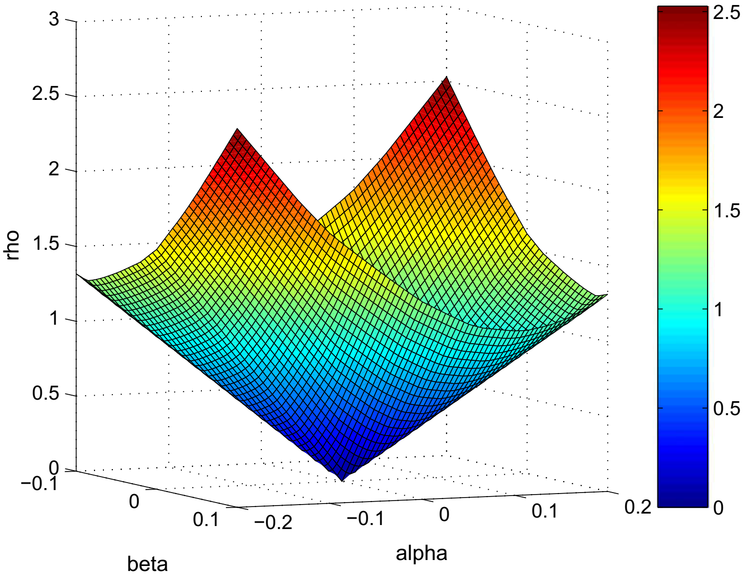

Fig. 3. The 3D plot of $\rho(\sum_{i=1}^{M} G(0))$ as a function of α and β.

$$C_2 = \alpha\begin{pmatrix} -2 & 1 & 1 & -2 \\ -3 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad D_1 = \beta\begin{pmatrix} -1 & 2 & 1 & -1 \\ -1 & 3 & 3 & 1 \\ -3 & -1 & -1 & 2 \\ 2 & -1 & 1 & 3 \end{pmatrix}, \quad D_2 = \beta\begin{pmatrix} -2 & -1 & -3 & 1 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 2 & -1 \\ 1 & 2 & -3 & 1 \end{pmatrix},$$

where α and β are real parameters. The number of delays $M = 2$. The pencil $\{A, B\}$ has index 1 and $\sigma(A, B) = \{-0.2857 \pm 0.2474i\}$. Since A_1 and B_1 are not upper triangular, we need to check the condition $\rho(|E_1|) < 1$. We have

$$E_1 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{pmatrix}$$

from which $\rho(|E_1|) = 0.7887$ is easily calculated. Now by solving auxiliary polynomial eigenvalue problems, the values of $\rho(\{G(l)\}_{i=0}^{M})$ are calculated with different values of α and β. Here, we fix $\beta = 0.05$. The numerical results for $l = 0, 1, 2, 3, 4$ and $\alpha = 0.1, 0.08, 0.06, 0.04$ are displayed in Table 2. Based on the obtained results, by Theorem 4, we conclude that the Eq. (1.2) in this example with the above chosen parameters is delay-independently asymptotically stable. The monotonicity of $\rho(\{G(l)\}_{i=0}^{M})$ with respect to l can also be observed.

(b) The higher-index case. Higher index DDAEs are more complicated. As we noted previously, the matrices D_i must be of appropriate structure so that DDAEs of the form (1.1) and (1.2) are delay-independently asymptotically stable. Now we suppose that the pencil $\{A, B\}$ has index $k \geqslant 2$ and it is given in Kronecker form (2.2).

Matrices C_i, D_i are in block form as in the index-1 case. We assume further that

$$ND_{3i} = 0, \quad ND_{4i} = 0, \; i = 1, 2, \ldots, M. \tag{3.21}$$

We define again the auxiliary matrices L_i, M_i, E as in (3.11). Using the nilpotency of N as well as the extra assumptions (3.21), it is easy to calculate

$$E = \begin{pmatrix} (I + B_1)^{-1}(I - B_1) & 0 \\ 0 & (I + N)^{-1}(N - I) \end{pmatrix} =: \begin{pmatrix} E_1 & 0 \\ 0 & (I + N)^{-1}(N - I) \end{pmatrix},$$

Table 2
The spectral radius of $\{G(l)\}_{i=0}^{M}$ with $\beta = 0.05$.

l	$\alpha = 0.1$	$\alpha = 0.08$	$\alpha = 0.06$	$\alpha = 0.04$
0	1.2347	1.1271	1.0181	0.9076
1	1.1020	1.0096	0.9235	0.8398
2	1.0321	0.9504	0.8705	0.7901
3	1.0005	0.9201	0.8409	0.7617
4	0.9674	0.8911	0.8178	0.7452

$$L_i = \begin{pmatrix} (I+B_1)^{-1}(D_{i1}+C_{i1}) & (I+B_1)^{-1}(D_{i2}+C_{i2}) \\ D_{i3} & D_{i4} \end{pmatrix} =: \begin{pmatrix} L_{i1} & L_{i2} \\ L_{i3} & L_{i4} \end{pmatrix},$$
$$M_i = \begin{pmatrix} (I+B_1)^{-1}(D_{i1}-C_{i1}) & (I+B_1)^{-1}(D_{i2}-C_{i2}) \\ D_{i3} & D_{i4} \end{pmatrix} =: \begin{pmatrix} M_{i1} & M_{i2} \\ M_{i3} & M_{i4} \end{pmatrix}.$$

Once again using the nilpotency of N as well as the extra assumptions (3.21), we get

$$(I-zE)^{-1}(zM_i+L_i) = \begin{pmatrix} (I-zE_1)^{-1}(zM_{i1}+L_{i1}) & (I-zE_1)^{-1}(zM_{i2}+L_{i2}) \\ L_{i3} & L_{i4} \end{pmatrix},$$

which is exactly the same as in the index-1 case (that is, the trouble-causing matrix N has been eliminated). From now on the formulation of stability criteria for higher index DDAEs is quite similar to the index-1 case. The matrices $G_i(l)$ are defined as in Definition 5. We obtain the analogue of Theorem 3.

Theorem 5. *Assume assumption (3.5) holds and A, B are given in Kronecker form. Further we suppose that D_i's have special structure satisfying (3.21) and $\rho(|E_1|) < 1$. If there exists an integer $l \geqslant 0$ such that*

$$\rho\left(\sum_{i=1}^{M} G_i(l)\right) < 1, \tag{3.22}$$

then (1.1) is asymptotically stable.

In the index-1 case we have $N = 0$, so assumption (3.21) is trivially satisfied. Thus, Theorem 5 can apply to index-1 DDAEs without any restriction on D_{i3} and D_{i4}, that is, it includes Theorem 3 as a special case.

Under the same assumptions as in Theorem 5, analogous statements of Corollary 3, Theorem 4, and Corollary 4 hold true.

Now we look at index-1 and index-2 neutral DDAEs of Hessenberg form. In the index-1 case, it is easy to see that the neutral DDAE (2.17) and (2.18) is asymptotically stable if and only if the transformed system

$$\begin{aligned} &\dot{\tilde{x}}_1 + (B_1 - B_2B_4^{-1}B_3)\tilde{x}_1 + B_2\tilde{x}_2 + C_1\dot{\tilde{x}}_1(t-\tau) + C_2\dot{\tilde{x}}_2(t-\tau) \\ &\quad + (D_1 - D_2B_4^{-1}B_3)\tilde{x}_1(t-\tau) + D_2\tilde{x}_2(t-\tau) = 0, \\ &B_4\tilde{x}_2 + D_3\tilde{x}_1(t-\tau) = 0. \end{aligned} \tag{3.23}$$

Here, the new variables $\tilde{x}_i$ are defined by $\tilde{x}_1 = x_1, \tilde{x}_2 = B_4^{-1}(B_3x_1 + x_2)$. Then, Theorem 3 and Corollary 3 can be applied to the transformed system (3.23) to obtain algebraic stability criteria. The results can be extended to the multiple delay case without any difficulty. An alternative way is to derive the underlying NDODE and then apply one of the well-known stability criteria for NDODEs. In particular, this latter approach can be useful for index-2 NDDAEs of Hessenberg form, too. In this case, it is enough to consider the underlying NDODE

$$\dot{u} + RB_1Su + RC_1S\dot{u}(t-\tau) + RD_1Su(t-\tau) = 0,$$

where R, S are defined as in Eq. (2.24).

We illustrate the stability analysis of index-2 NDDAEs of Hessenberg form by an example.

Example 6. Consider the index-2 NDDAEs of Hessenberg form 2.19,2.20 with the following data:

$$B_1 = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 1 & -1 \\ 0 & -1 & 5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad B_3 = (-1 \quad 2 \quad 0).$$

Here, we consider a single delay with

$$C_1 = \alpha \begin{pmatrix} -2 & 1 & -2 \\ -1 & -2 & -1 \\ 1 & -1 & -3 \end{pmatrix}, \quad D_1 = \beta \begin{pmatrix} 1 & 2 & -1 \\ -1 & 3 & -2 \\ -2 & 4 & 1 \end{pmatrix}.$$

Following the construction of the underlying neutral delay ODE (2.24), first we have

$$R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\widetilde{B}_1 = RB_1S, \widetilde{C}_1 = RC_1S, \widetilde{D}_1 = RD_1S$, and introduce a new variable $u = Rx_1$. We obtain a neutral delay ODE

$$\dot{u} + \widetilde{B}_1u + \widetilde{C}_1\dot{u}(t-\tau) + \widetilde{D}_1u(t-\tau) = 0. \tag{3.24}$$

Table 3
The spectral radius of $G(l)$ with $\alpha = 0.2$.

l	$\beta = 0.19$	$\beta = 0.16$	$\beta = 0.13$	$\alpha = 0.10$
0	1.0349	1.0136	0.9973	0.9970
1	1.0178	0.9969	0.9811	0.9694
2	1.0078	0.9872	0.9715	0.9600
3	1.0019	0.9814	0.9658	0.9544
4	0.9984	0.9779	0.9624	0.9511

Then

$$\widetilde{B}_1 = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}, \quad \widetilde{C}_1 = \begin{pmatrix} 4 & 1 \\ -1 & 3 \end{pmatrix}, \quad \widetilde{D}_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}.$$

By applying one of the stability criteria given in [13,14], the asymptotic stability of (3.24) can be established. This index-0 problem can also be considered as a special case of the index-1 problem (the dimension of the algebraic part is zero, i.e., the algebraic variables disappear). The values of $\rho(G(l))$ are calculated with different values of α and β. Here, we fix $\alpha = 0.2$. The numerical results for $l = 0,1,2,3,4$ and $\beta = 0.19, 0.16, 0.13, 0.1$ are displayed in Table 3. Based on the obtained results, we conclude that the index-2 NDDAEs of Hessenberg form 2.19,2.20 in this example with the above chosen parameters is delay-independently asymptotically stable.

4. Stability of numerical solutions

In this section we investigate the stability of numerical methods applied to DAEs with multiple delays. For simplicity, we consider only the form (1.2) and also assume that the constant stepsize h is chosen such that $l = \tau/h$ is a positive integer. We will prove that under the delay-independent criteria, numerical solutions by θ-methods and BDF methods are asymptotically stable. The result extends some previous results for single-delay DAEs in [25] to the multiple delay case. However, our proof is simplified and shortened. We assume throughout this section that the assumptions (3.8) and (3.9) are fulfilled.

Definition 6. Given a constant stepsize $h > 0$, a numerical solution $\{x_n\}$ for a DDAE of the form (1.2) is called asymptotically stable if $lim_{n\to\infty}\|x_n\| = 0$.

4.1. The θ-methods

A θ-method [3] applied to the delay DAE of the form (1.2) yields a difference system

$$A\frac{x_n - x_{n-1}}{h} + \theta B x_n + (1-\theta)Bx_{n-1} + \sum_{i=1}^{M} C_i \frac{x_{n-il} - x_{n-il-1}}{h} + \sum_{i=1}^{M}[\theta D_i x_{n-il} + (1-\theta)D_i x_{n-il-1}] = 0, \tag{4.1}$$

where $\theta \in (0,1]$ is the method parameter, x_n is the approximate value of $x(nh)$, $n = -Ml, -Ml+1, \ldots, 0, 1, 2, \ldots$ The "past" values of the solution, i.e., x_n's with negative index are given. Rewriting (4.1), we have

$$\left(\frac{A}{h} + \theta B\right)x_n + \left(-\frac{A}{h} + (1-\theta)B\right)x_{n-1} + \sum_{i=1}^{M}\left(\frac{C_i}{h} + \theta D_i\right)x_{n-il} + \sum_{i=1}^{M}\left(-\frac{C_i}{h} + (1-\theta)D_i\right)x_{n-il-1} = 0.$$

Note that under assumption (3.8), the leading matrix of the difference system is nonsingular. Thus, the difference system is solvable. The characteristic equation for this difference system is

$$\det\left[\left(\frac{A}{h} + \theta B\right)\lambda^{Ml+1} + \left(-\frac{A}{h} + (1-\theta)B\right)\lambda^{Ml} + \sum_{i=1}^{M}\left(\frac{C_i}{h} + \theta D_i\right)\lambda^{(M-i)l+1} + \sum_{i=1}^{M}\left(-\frac{C_i}{h} + (1-\theta)D_i\right)\lambda^{(M-i)l}\right] = 0. \tag{4.2}$$

Theorem 6. *Suppose that assumptions (3.8) and (3.9) hold for (1.2). Then, for $\theta \in (1/2, 1]$, the numerical solution $\{x_n\}$ by the θ-method is asymptotically stable.*

Proof. We need only prove that all the roots of the characteristic Eq. (4.2) are inside the unit circle of the complex plane. Suppose that (4.2) has a root λ satisfying $|\lambda| \geqslant 1$. First, since $\theta\lambda + (1-\theta) \neq 0$ for $\theta \in (1/2, 1]$, we reformulate the characteristic equation as

$$\det\left[\left(A\frac{\lambda - 1}{h(\theta\lambda + (1-\theta))} + B\right)\lambda^{Ml} + \sum_{i=1}^{M}\left(C_i\frac{\lambda - 1}{h(\theta\lambda + (1-\theta))} + D_i\right)\lambda^{(M-i)l}\right] = 0.$$

Define

$$s := \frac{\lambda - 1}{h(\theta\lambda + (1-\theta))}, \text{or equivalently } \lambda = \frac{1 + hs(1-\theta)}{1 - hs\theta}.$$

It is easy to see that for $\theta \in (1/2, 1]$, if $\mathrm{Re}(s) < 0$, then $|\lambda| < 1$. That is, $|\lambda| \geqslant 1$ implies $\mathrm{Re}(s) \geqslant 0$. Now, by letting $w := \lambda^l$, we conclude that $\det[(As + B)w^M + \sum_{i=1}^{M}(C_i s + D_i)w^{M-i}] = 0$ has solution s,w satisfying $\mathrm{Re}(s) \geqslant 0$, $|w| \geqslant 1$. This contradicts the assumption (3.9). □

4.2. The BDF methods

Applying a k-step BDF method [3] to a DDAE of the form (1.2), we have the difference system

$$A\frac{1}{h}\sum_{j=0}^{k}\alpha_j x_{n-j} + Bx_n + \sum_{i=1}^{M}\left[C_i\frac{1}{h}\sum_{j=0}^{k}\alpha_j x_{n-il-j} + D_i x_{n-il}\right] = 0. \tag{4.3}$$

Since $\alpha_0 > 0$, the leading term is nonsingular and the difference system is solvable. Its characteristic equation is

$$\det\left[A\frac{1}{h}\sum_{j=0}^{k}\alpha_j\lambda^{Ml-j} + B\lambda^{Ml} + \sum_{i=1}^{M}\left(C_i\frac{1}{h}\sum_{j=0}^{k}\alpha_j\lambda^{(M-i)l-j} + D_i\lambda^{(M-i)l}\right)\right] = 0$$

or equivalently

$$\det\left[\left(A\frac{1}{h}\sum_{j=0}^{k}\alpha_j\lambda^{-j} + B\right)\lambda^{Ml} + \sum_{i=1}^{M}\left(C_i\frac{1}{h}\sum_{j=0}^{k}\alpha_j\lambda^{-j} + D_i\right)\lambda^{(M-i)l}\right] = 0. \tag{4.4}$$

Theorem 7. *Suppose assumptions (3.8) and (3.9) hold for (1.2). Then, for an A-stable BDF method, the numerical solution $\{x_n\}$ is asymptotically stable.*

Proof. We will show that the characteristic Eq. (4.4) has no root satisfying $|\lambda| \geqslant 1$. Suppose the opposite holds true, that is, there exists a root λ of (4.4) such that $|\lambda| \geqslant 1$. Let

$$s := \frac{1}{h}\sum_{j=0}^{k}\alpha_j\lambda^{-j} \quad \text{and} \quad w := \lambda^l.$$

As in the proof of Theorem 3.6 [25], it can be verified that for an A-stable BDF method, if $|\lambda| \geqslant 1$ then $\mathrm{Re}(s) \geqslant 0$. It implies that the equation $[As + B]w^M + \sum_{i=1}^{M}[C_i s + D_i]w^{M-i} = 0$ has solution s,w satisfying $\mathrm{Re}(s) \geqslant 0, -w- \geqslant 1$. This latter statement contradicts the assumption (3.9). □

It is not difficult to prove an analogous result for numerical solutions obtained by A-stable linear multistep methods.A similar analysis of A-stable implicit Runge–Kutta methods without the restrictive assumptions on the structure of A, B, C_i, D_i that were needed in [25] is an open interesting problem.

5. Weakly regular delay DAEs

We are only aware of two papers [7,20] that investigate the case when $\{A, B\}$ is nonregular but weakly regular. We will show that for certain special weakly regular DDAEs, although these systems are noncausal, solvability (existence and uniqueness of solution) of initial value problems can be established. Further, stability analysis can be done similarly to the index-1 regular DDAEs discussed in Section 3.

For simplicity, we consider single delay DAEs of the form (2.3). The result can be extended to multiple-delay case without any difficulty. Without loss of generality, we can suppose that the matrices A, B, D are transformed into block form as follows:

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}. \tag{5.1}$$

It is easy to see the following characterization.

Lemma 5. *If the pencil $\{A, B\}$ is not regular, but the pencil $\{B_4, D_4\}$ is regular, then the triplet $\{A, B, D\}$ is weakly regular.*

However, a weakly regular triplet $\{A, B, D\}$ need not have regular $\{B_4, D_4\}$.

Example 7. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $\{A,B\}$ is not regular, but $\{A,B,D\}$ is weakly regular. However, the pencil $\{B_4,D_4\}$ is not regular. Consider the system

$$A\dot{x}(t)+Bx(t)+Dx(t-1)=f(t),\quad x(t)=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix},\quad f(t)=\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix},$$

with the above data. Assume that the input f is sufficiently smooth. Then the system has the solution $x_1(t)=f_2(t+1),\ x_2(t)=f_1(t+1)-\dot{f}_2(t+2)$, which depends on the derivative of the input at a future time.

From now on, we assume that the assumption of Lemma 5 holds. First, we consider the case index$\{B_4,D_4\}$ = 1.

Proposition 5. *Suppose that index$\{B_4,D_4\}$ = 1. Then, there exist nonsingular matrices P and Q such that*

$$A=P\begin{pmatrix}I&0&0\\0&0&0\\0&0&0\end{pmatrix}Q,\quad B=P\begin{pmatrix}\overline{B}_1&0&\overline{B}_3\\0&I&0\\\overline{B}_7&0&0\end{pmatrix}Q,\quad D=P\begin{pmatrix}\overline{D}_1&\overline{D}_2&0\\\overline{D}_4&\overline{D}_5&0\\0&0&I\end{pmatrix}Q.\tag{5.2}$$

Proof. The proof is based on elementary matrix manipulations, so it is omitted. □

Thus, by multiplying with P from the left and using a variable change, we obtain a new DDAE system which has the same stability property as the original DDAE system (2.3)

$$\begin{aligned}&\dot{\bar{x}}_1+\overline{B}_1\bar{x}_1+\overline{B}_3\bar{x}_3+\overline{D}_1\bar{x}_1(t-\tau)+\bar{D}_2\bar{x}_2(t-\tau)=0,\\&\bar{x}_2+\overline{D}_4\bar{x}_1(t-\tau)+\overline{D}_5\bar{x}_2(t-\tau)=0,\\&\overline{B}_7\bar{x}_1+\bar{x}_3(t-\tau)=0\end{aligned}\tag{5.3}$$

is. Here the new variable is defined by $\bar{x}=Qx$. From the last equation, we get $\bar{x}_3(t)=-\overline{B}_7\bar{x}_1(t+\tau)$. Inserting this into the first equation, the obtained equation is of advanced type if $\overline{B}_3\overline{B}_7\neq 0$. Otherwise, the system (5.3) is well-solvable. Further, it is not necessary to assign initial values for $\bar{x}_3$ on interval $[-\tau,0]$ (or they must be consistent).

Theorem 8. *Suppose that $\overline{B}_3\overline{B}_7=0$. Then the initial value problem for the DDAE (5.3) is uniquely solvable.*

Proof. Since $\overline{B}_3\overline{B}_7=0$, we can eliminate x_3 in the first equation. Then the first two equations form an index-1 regular DDAE for which the dynamics is well known. Note that $\bar{x}_3$ can be determined from the future values of $\bar{x}_1$, i.e. the system is noncausal. □

Note that in this case, the stability analysis of (5.3) can be reduced to that of index-1 DDAE of the form (2.3). However, the stability of (5.3), if it holds, is less robust than the stability of an index-1 regular DDAE of the form (2.3). The reason is that an arbitrary small perturbation in the data may make the system (5.3) become a DDAE of advanced type for which the stability cannot be expected in general.

By analogue, we can deal with the case index $\{B_4,D_4\}$ > 1. Then, there exist nonsingular matrices P and Q such that

$$A=P\begin{pmatrix}I&0&0\\0&0&0\\0&0&0\end{pmatrix}Q,\quad B=P\begin{pmatrix}\overline{B}_1&0&\overline{B}_3\\0&I&0\\\overline{B}_7&0&N\end{pmatrix}Q,\quad D=P\begin{pmatrix}\overline{D}_1&\overline{D}_2&0\\\overline{D}_4&\overline{D}_5&0\\0&0&I\end{pmatrix}Q,\tag{5.4}$$

where N is a matrix of nilpotency index k.

The third equation of the corresponding system now reads $\overline{B}_7\bar{x}_1+N\bar{x}_3+\bar{x}_3(t-\tau)=0$. It is easy to derive

$$\bar{x}_3(t)=-\sum_{i=0}^{k-1}N^i\overline{B}_3\bar{x}_1(t+(i+1)\tau).$$

Note that now $\bar{x}_3$ depends on more future terms of $\bar{x}_1$. Hence, it is not necessary to assign initial values for $\bar{x}_3$ on interval $[-\tau,0]$ (or they must be consistent). Similarly, we have the following result.

Theorem 9. *Suppose that $\overline{B}_3N^i\overline{B}_7=0$ for all $i=0,1,\ldots,k-1$. Then the initial value problem for the DDAE (5.3) is uniquely solvable.*

It is not difficult to show that under the assumption $\overline{B}_3\overline{B}_7=0$, the pencil $\{A,B\}$ is not regular. So the DDAE in question is weakly regular, but still well-solvable and can be analyzed similarly to a regular DDAE system. Its dynamical behaviour is like a mixture of DODEs and higher index singular difference equations. Unfortunately, the characterization of DDAEs when $\{B_4,D_4\}$ is nonregular is still an open question. There exist examples showing that in this case, the DDAE system may be weakly regular or not regular, as well. Further, everything can happen concerning its dynamics: the system may behave like system of advanced type, noncausality may appear, and (or) some solution components may depend on derivatives of other components and inputs.

6. Conclusion

The stability of delay DAEs has been investigated. A number of checkable criteria for delay independent stability are presented. These results are then applied to the stability of A-stable numerical integrators.

Deriving other practical delay-independent stability conditions and comparing them with those given in this paper would be of great interest. Further, a complete analysis of weakly regular DDAEs is still an open problem.

Acknowledgements

This research was carried out during Vu Hoang Linh's visit to the University of California, Santa Barbara, USA, and the North Carolina State University, Raleigh, USA. This author appreciates the financial support for the visit from the Vietnam National University, Hanoi, and the working facilities provided by the two US institutions. The authors thank Linda Petzold for useful discussions on the topic of DDAEs that initiated the research. The research of S. Campbell was supported in part by the National Science Foundation under ECS-0620986.

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