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CONTROL THEORY

Finite Spectrum Assignment for Completely Regular Differential-Algebraic Systems with Aftereffect

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Abstract—For linear autonomous completely regular differential-algebraic systems with commensurable delays in the state and control, we study the problem of constructing a state feedback that ensures a finite spectrum for the closed-loop system. We propose criteria for spectral reducibility and weak spectral reducibility whose proofs contain the synthesis schemes of appropriate controllers. Several illustrative examples are given.

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INTRODUCTION

The problem of design of controllers that ensure prescribed properties of a system is one of the key problems in automatic control theory. In this connection, various feedback control synthesis problems have been studied in a vast literature. For systems with delays, these are the first stabilization problems [1–6], finite spectrum assignment [7–10], modal controllability [11–13], 0-controllability by a feedback controller [14–16], spectral reducibility [17–19], and various problems of observer design [20]. (Historical surveys can be obtained from the references in these papers.)

In the present paper, we consider the spectral reducibility problem, i.e., the problem of closing a plant with a delay by a feedback such that its spectrum becomes finite [17–19]. First of all, this problem attracts attention because the finite spectrum systems are finite-dimensional in their nature, which significantly simplifies their study and permits realizing the control in various function classes. Moreover, it plays an important role [6] in studying the problem of stabilization of a family of plants with delays by a universal controller. For delay type systems, a spectral reducibility criterion was proved in [17], but as a result of implementation of the controller synthesis scheme proposed there, the closed-loop system may become a system of neutral or advanced type. This drawback was eliminated in [18]. The spectral reducibility criteria in two classes of controllers were proposed for systems of neutral type in [19], and for each of these classes, the closed-loop system remains a system of neutral type or its special version.

In the present paper, the spectral reducibility problem is studied for linear autonomous completely regular differential-algebraic systems with commensurable delays in the state and control. Two approaches to solving this problem are considered. The first is traditional and consists in closing the plant under study by a feedback controller that ensures a finite spectrum of the closed-loop system. The second approach consists in constructing the feedback so as to separate, in the closed-loop system, an independent finite-dimensional subsystem whose solution contains all components of the original (open-loop) system, namely, so that a certain subset of the set of variables of the closed-loop system could be completely determined by the finite-dimensional system. In the applied aspect, this permits reducing the study of the original infinite-dimensional system to the study of a chosen finite-dimensional system, which seems a significantly simpler problem. In this paper, we propose solvability criteria for such problems, and their proofs contain the synthesis schemes of the corresponding controllers that do not take closed-loop systems outside the class of completely regular differential-algebraic systems with commensurable delays.

1. STATEMENT OF THE PROBLEM

Consider the linear autonomous completely regular [16] differential-algebraic system with commensurable delays in the state and control

$$\frac{d}{dt}(\widetilde{A}_0 z(t)) = \widetilde{A}(\lambda)z(t) + \widetilde{B}(\lambda)u(t), \qquad t > 0, \tag{1}$$

where $z \in \mathbb{R}^n$ is the solution vector of the original system, $u \in \mathbb{R}^r$ is the vector of piecewise continuous control, $\widetilde{A}_0 \in \mathbb{R}^{n\times n}$, $\widetilde{A}(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$, $\widetilde{B}(\lambda) \in \mathbb{R}^{n\times r}[\lambda]$ ($\mathbb{R}^{i\times j}[\lambda]$ is the set of polynomial $i \times j$ matrices), and λ is the shift operator ($\lambda f(t) = f(t-h)$ for an arbitrary function f, where $h = \text{const} > 0$ is a constant delay). We write $\operatorname{rank}\widetilde{A}_0 = n_1$. A completely regular differential-algebraic system is defined by the condition [16] $\deg|p\widetilde{A}_0 - \widetilde{A}(0)| = n_1$; here and below, deg denotes the degree of a polynomial, and $|\cdot|$ denotes the determinant of a matrix. In this paper, the systems with $|\widetilde{A}_0| \neq 0$ are also placed in the class of completely regular systems.

By $\widetilde{\mathcal{W}}_0(p, e^{-ph}) = p\widetilde{A}_0 - \widetilde{A}(e^{-ph})$ we denote the characteristic matrix of the open-loop ($u \equiv 0$) system (1). The characteristic quasipolynomial of system (1) has the form

$$|\widetilde{\mathcal{W}}_0(p, e^{-ph})| = \sum_{i=0}^{n_1} p^i \chi_i(e^{-ph}), \tag{2}$$

where the $\chi_i(p)$, $i = 1, \ldots, n$, are polynomials and $\chi_{n_1}(0) \neq 0$. It follows from (2) that the spectrum of systems of the form (1) is infinite in the general case, which significantly complicates the study of such plants as compared with finite-dimensional linear systems. In this connection, we pose the following problem.

Problem. It is required to close system (1) with a linear state feedback

$$u(t) = u(\{\widetilde{A}_0 z(t), z(t-ih), i = 1, \ldots, \varepsilon\}) \tag{3}$$

so that the following conditions are satisfied: (i) the closed-loop system remains a linear autonomous completely regular differential-algebraic system with commensurable delays; (ii) the component z of the solution of the closed-loop system satisfies a linear autonomous completely regular differential-algebraic system with a finite spectrum.

To solve this problem, we use a controller of the form

$$u(t) = R_u(\lambda)Z(t), \qquad \frac{d}{dt}(R_0 z_1(t)) = R(\lambda)Z(t), \qquad t > 0, \tag{4}$$

where $z_1 \in \mathbb{R}^{\bar{n}}$ is an auxiliary variable, $Z = \operatorname{col}[z, z_1]$, $R_u(\lambda) \in \mathbb{R}^{r\times(n+\bar{n})}[\lambda]$, $R_0 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $R(\lambda) \in \mathbb{R}^{\bar{n}\times(n+\bar{n})}[\lambda]$. The matrix $R_u(\lambda)$ satisfies the condition $R_u(0) = [R_u^1\widetilde{A}_0, R_u^2]$, where $R_u^1 \in \mathbb{R}^{r\times n}$ and $R_u^2 \in \mathbb{R}^{r\times\bar{n}}$, which ensures the formation of a control u by using the information about the vector z shown on the right-hand side in the expression (3). For the closed-loop system (1), (4) to remain a completely regular differential-algebraic system, it is required that $\deg|pR_0 - R(0)| = \operatorname{rank} R_0$.

To determine the degree of solvability of the problem stated above (see Definitions 1 and 2 below), we introduce the following characteristics of an arbitrary matrix $\Omega(p,\lambda) = p\Omega_0 + \widehat{\Omega}(\lambda)$, where $\Omega_0 \in \mathbb{R}^{m\times m}$, $\widehat{\Omega}(\lambda) \in \mathbb{R}^{m\times m}[\lambda]$, $m \in \mathbb{N}$. We say that the matrix $\Omega(p,\lambda)$ has a CR-structure if the condition $\deg|p\Omega_0 + \widehat{\Omega}(0)| = \operatorname{rank}\Omega_0$ is satisfied. In particular, the characteristic matrix $\widetilde{\mathcal{W}}_0(p,\lambda)$ of the open-loop system (1) has a CR-structure. More precisely, any matrix $\Omega(p,\lambda) = p\Omega_0 + \widehat{\Omega}(\lambda)$ can be associated with a linear autonomous completely regular differential-algebraic system with commensurable delays whose characteristic matrix is the matrix $\Omega(p, e^{-ph})$. The converse is also true.

We close system (1) with the controller (4) and arrange the variables of the closed-loop system (1), (4) as the vector $\mathrm{col}\,[z, z_1]$. By $0_{m\times n} \in \mathbb{R}^{m\times n}$ we denote the zero matrix, and by

$$\widetilde{\mathcal{W}}(p, e^{-ph}) = p \begin{bmatrix} \widetilde{A}_0 & 0_{n\times\bar{n}} \\ 0_{\bar{n}\times n} & R_0 \end{bmatrix} - \begin{bmatrix} \widetilde{A}(e^{-ph}) & 0_{n\times\bar{n}} \\ 0_{\bar{n}\times n} & 0_{\bar{n}\times\bar{n}} \end{bmatrix} + \begin{bmatrix} \widetilde{B}(e^{-ph})R_u(e^{-ph}) \\ R(e^{-ph}) \end{bmatrix}$$

we denote the characteristic matrix of the closed-loop system (1), (4). We will study this problem from the viewpoint of the following properties.

Definition 1. System (1) is said to be *spectrally reducible* if there exists a controller of the form (4) such that (i) the matrix $\widetilde{\mathcal{W}}(p, \lambda)$ has a CR-structure; (ii) the determinant $|\widetilde{\mathcal{W}}(p, e^{-ph})|$ is a polynomial.

Definition 2. System (1) is said to be *weakly spectrally reducible* if there exists a controller of the form (4) such that (i) the matrix $\widetilde{\mathcal{W}}(p, \lambda)$ has a CR-structure; (ii) there exists a unimodular matrix $\widetilde{\mathcal{P}}(\lambda) \in \mathbb{R}^{(n+\bar{n})\times(n+\bar{n})}[\lambda]$ (i.e., a matrix for which $|\widetilde{\mathcal{P}}(\lambda)| \equiv \mathrm{const} \neq 0$) and a number $n_* \leq \bar{n}$ such that

$$\widetilde{\mathcal{P}}(\lambda)\widetilde{\mathcal{W}}(p, \lambda) = \begin{bmatrix} \widetilde{\mathcal{W}}_{11}(p, \lambda) & 0_{(n+\bar{n}-n_*)\times n_*} \\ \widetilde{\mathcal{W}}_{21}(\lambda) & \widetilde{\mathcal{W}}_{22}(\lambda) \end{bmatrix}, \tag{5}$$

where the matrix $\widetilde{\mathcal{W}}_{11}(p, \lambda)$ has a CR-structure, $\widetilde{\mathcal{W}}_{21}(\lambda) \in \mathbb{R}^{n_*\times(n+\bar{n}-n_*)}[\lambda]$ and $\widetilde{\mathcal{W}}_{22}(\lambda) \in \mathbb{R}^{n_*\times n_*}[\lambda]$; (iii) the determinant $|\widetilde{\mathcal{W}}_{11}(p, e^{-ph})|$ is a polynomial.

Remark 1. Let us explain the meaning of the weak spectral reducibility property. Relation (5) means that there exists a time instant $\bar{t} > 0$ such that for $t > \bar{t}$ in system (1) closed with the controller (4) one can single out a completely regular differential-algebraic subsystem with characteristic matrix $\widetilde{\mathcal{W}}_{11}(p, \lambda)$ that has a finite spectrum and uniquely determines the variable z. In this case, the existence of a unimodular matrix $\widetilde{\mathcal{P}}(\lambda)$ permits obtaining such a subsystem by elementary transformations of the equation in system (1), (4) which do not contain the differentiation operation. Further, we show (see the proof of necessity of the conditions in Lemma 4) that the set of components of the solution of the subsystem corresponding to the characteristic matrix $\widetilde{\mathcal{W}}_{11}(p, e^{-ph})$ contains not only the variable z but also some components of the function z_1 which correspond to the basis columns of the matrix R_0 and possibly some other components of the vector z_1. We also show that the lower blocks of the matrix (5) are independent of the variable p. Moreover, we prove the estimate $n_* \leq \bar{n} - \mathrm{rank} R_0$. If $n_* = 0$, then $\widetilde{\mathcal{W}}_{11}(p, \lambda) = \widetilde{\mathcal{W}}(p, \lambda)$; i.e., the weak spectral reducibility property coincides in this case with the spectral reducibility property.

Let us illustrate the weak spectral reducibility properties.

Example 1. Consider system (1) of the form

$$\dot{z}^1(t) = -z^2(t-h), \qquad z^2(t) = z^2(t-h) + u(t) - u(t-h), \tag{6}$$

which is a special case of system (1) ($n = 2$, $z = \mathrm{col}\,[z^1, z^2]$) and has an infinite spectrum. A simple analysis shows that the traditional approach to the spectral reducibility problem cannot be used in this case. (Owing to the form of the second equation in system (6), the characteristic quasipolynomial contains the factor $1 - e^{-ph}$.) At the same time, if we close Eq. (6) with the controller

$$u(t) = z_1(t), \qquad z_1(t) = z_1(t-h) - z^2(t-h), \qquad t > 0, \tag{7}$$

then for $t > h$ we have the equation

$$z^2(t) = z^2(t-h) + z_1(t) - z_1(t-h) = 0. \tag{8}$$

We see that for $t > h$ the vector $z = \mathrm{col}\,[z^1,\, z^2]$ is completely determined by the first equation in system (6) and Eq. (8); namely, they form a system that does not contain the variable z_1 and has the characteristic matrix

$$\begin{bmatrix} p & e^{-ph} \\ 0 & 1 \end{bmatrix} \tag{9}$$

and a finite spectrum.

Now we obtain the same results in a somewhat different way. The characteristic matrix of the closed-loop system (6), (7) has the form ($\lambda = e^{-ph}$)

$$\widetilde{\mathcal{W}}(p, e^{-ph}) = \begin{bmatrix} p & \lambda & 0 \\ 0 & 1-\lambda & -1+\lambda \\ 0 & \lambda & 1-\lambda \end{bmatrix}.$$

We multiply it by the matrix

$$\widetilde{\mathcal{P}}(\lambda) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and obtain the matrix $\mathcal{W}_{11}(p, \lambda)$ from the expression (5); for $\lambda = e^{-ph}$, this matrix coincides with the matrix (9).

2. MAIN RESULTS

To state the main results of this paper, we need some notation and well-known facts. Consider the matrix $\widetilde{B}(\lambda) = \sum_{i=0}^{m} \widetilde{B}^{(i)}\lambda^i$, $\widetilde{B}^{(i)} \in \mathbb{R}^{n\times r}$, $m \in \mathbb{N} \sqcup \{0\}$. By analogy with [13–16, 20, 21], we introduce a sequence $(q_k)_{k=0}^{\infty}$ of vectors that is a solution of the difference equation

$$\widetilde{B}^{(0)}q_k + \sum_{i=1}^{m} \widetilde{B}^{(i)}q_{k-i} = 0_{n\times 1}, \qquad k = m, m+1, \ldots, \tag{10}$$

and is generated by the initial condition $q_i = \tilde{q}_i$, $i = 0, \ldots, m-1$. The sequence q_k, $k = m, m+1, \ldots$, defined by Eq. (10) exists if and only [21] if $\tilde{q}_{m-i} = T_i c$, $i = 1, \ldots, m$, where $T_i \in \mathbb{R}^{r\times r_T}$ are some matrices and $c \in \mathbb{R}^{r_T}$ is an arbitrary constant vector (the same for all matrices T_i). The procedure for constructing the matrices T_i is described in [21], and hence we do not describe it here. Note that its realization is always possible and consists in solving a finite chain of homogenous algebraic systems. We write $T = T_m$ and introduce a matrix $S \in \mathbb{R}^{r_T\times r_T}$ as a solution of the system of equations

$$\widetilde{B}^{(0)}T_1S + \sum_{i=1}^{m} \widetilde{B}^{(i)}T_i = 0_{n\times r_T}, \qquad T_kS = T_{k-1}, \qquad k = 2, \ldots, m,$$

whose solvability follows from the definition of the matrices T_i. Note that

$$\sum_{i=0}^{m} \widetilde{B}^{(i)}TS^{m-i} = 0_{n\times r_T}.$$

Now we introduce the matrices $\widetilde{G}^{(0)} = \widetilde{B}^{(0)}T$ and $\widetilde{G}^{(i)} = \widetilde{G}^{(i-1)}S + \widetilde{B}^{(i)}T$, $i = 1, \ldots, m$. Then $\widetilde{G}^{(m)} = \sum_{i=0}^{m} \widetilde{B}^{(i)}TS^{m-i} = 0_{n\times r_T}$. We define the matrix $\widetilde{G}(\lambda) = \sum_{i=0}^{m} \widetilde{G}^{(i)}\lambda^i$. For a given polynomial matrix $\widetilde{B}(\lambda)$, the matrix $\widetilde{G}(\lambda)$ constructed by the method described above will be called the matrix of additional inputs [16].

Let Φ and Ψ be the fundamental solution matrices of the linear algebraic systems $\Phi\widetilde{A}_0 = 0_{n_2\times n}$ and $\widetilde{A}_0\Psi = 0_{n\times n_2}$, respectively, where $n_2 = n - n_1$. We introduce the sets

$$\Lambda_{\widetilde{B}} = \{p \in \mathbb{C} : \operatorname{rank}[\widetilde{\mathcal{W}}(p, e^{-ph}), \widetilde{B}(e^{-ph})] < n\},$$
$$\Lambda_{\widetilde{B},\widetilde{G}} = \{p \in \mathbb{C} : \operatorname{rank}[\widetilde{\mathcal{W}}(p, e^{-ph}), \widetilde{B}(e^{-ph}), \widetilde{G}(e^{-ph})] < n\}.$$

Let us state the main results of the paper.

Theorem 1 (criterion for spectral reducibility). *System* (1) *is spectrally reducible if and only if the following conditions are satisfied simultaneously*:

$$\text{the set } \Lambda_{\widetilde{B}} \text{ is finite}; \tag{11}$$

$$\operatorname{rank}[\Phi\widetilde{A}(\lambda)\Psi, \Phi\widetilde{B}(\lambda)] = n - \operatorname{rank}\widetilde{A}_0 \quad \text{for any} \quad \lambda \in \mathbb{C}. \tag{12}$$

Theorem 2 (criterion for weak spectral reducibility). *System* (1) *is weakly spectrally reducible if and only if the following conditions are satisfied simultaneously*:

$$\text{the set } \Lambda_{\widetilde{B},\widetilde{G}} \text{ is finite}; \tag{13}$$

$$\operatorname{rank}[\Phi\widetilde{A}(\lambda)\Psi, \Phi\widetilde{B}(\lambda), \Phi\widetilde{G}(\lambda)] = n - \operatorname{rank}\widetilde{A}_0 \quad \text{for any} \quad \lambda \in \mathbb{C}. \tag{14}$$

Theorems 1 and 2 are proved in Sections 4 and 5, respectively. First (in Section 3), we consider the problem of reducing system (1) to a simpler form.

3. REDUCTION OF THE ORIGINAL SYSTEM TO A SPECIAL FORM

Using the fact that the matrix $\widetilde{\mathcal{W}}_0(p,\lambda)$ has a CR-structure, we reduce system (1) to a form more convenient for further study [16]. To this end, we choose nonsingular matrices H and $\widehat{H}_1$ such that $\widehat{H}_1 A_0 H = \operatorname{diag}[I_{n_1}, 0_{n_2\times n_2}]$, where $I_i \in \mathbb{R}^{i\times i}$ is the identity matrix. Let

$$\widehat{H}_1\widetilde{A}(\lambda)H = \begin{bmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ \overline{A}_{21}(\lambda) & \overline{A}_{22}(\lambda) \end{bmatrix}, \qquad \widehat{H}_1\widetilde{B}(\lambda) = \begin{bmatrix} B_1(\lambda) \\ \widehat{B}_2(\lambda) \end{bmatrix},$$

$B_1(\lambda) \in \mathbb{R}^{n_1\times r}[\lambda]$, and $\widehat{B}_2(\lambda) \in \mathbb{R}^{n_2\times r}[\lambda]$. Since $\deg|p\widehat{H}_1\widetilde{A}_0 H - \widehat{H}_1\widetilde{A}(0)H| = n_1$, it follows that the determinant $|\overline{A}_{22}(0)|$ is nonzero. We set $A_{21}(\lambda) = -(\overline{A}_{22}(0))^{-1}\overline{A}_{21}(\lambda)$, $A_{22}(\lambda) = -\lambda^{-1} \times (\overline{A}_{22}(0))^{-1}(\overline{A}_{22}(\lambda) - \overline{A}_{22}(0))$, and $B_2(\lambda) = -(\overline{A}_{22}(0))^{-1}\widehat{B}_2(\lambda)$, change the variables $z = H \operatorname{col}[x, y]$, $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$ in system (1), and obtain a new system of the form

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t) + A_{12}(\lambda)y(t) + B_1(\lambda)u(t), \\ y(t) &= A_{21}(\lambda)x(t) + A_{22}(\lambda)y(t-h) + B_2(\lambda)u(t), \qquad t > 0. \end{aligned} \tag{15}$$

Let

$$\mathcal{W}_0(p, e^{-ph}) = \begin{bmatrix} pI_{n_1} - A_{11}(e^{-ph}) & -A_{12}(e^{-ph}) \\ -A_{21}(e^{-ph}) & I_{n_2} - e^{-ph}A_{22}(e^{-ph}) \end{bmatrix}$$

be the characteristic matrix of system (15). The characteristic quasipolynomial of system (15) coincides up to a constant factor with the characteristic quasipolynomial of system (1); i.e., $|\mathcal{W}_0(p, e^{-ph})| = |H_1||H||\widetilde{\mathcal{W}}(p, e^{-ph})|$, where $H_1 = \operatorname{diag}[I_{n_1}, (\overline{A}_{22}(0))^{-1}]\widehat{H}_1$.

Take nonsingular matrices H_2 and $\widehat{H}_3$ such that $\widehat{H}_3 R_0 H_2 = \operatorname{diag}[I_{\bar{n}_1}, 0_{\bar{n}_2\times\bar{n}_2}]$, where $\bar{n}_1 = \operatorname{rank} R_0$, $\bar{n}_2 = \bar{n} - \bar{n}_1$ and, after similar transformations, associate system (15) with the controller

$$u(t) = R_{01}(\lambda)X(t) + R_{02}(\lambda)Y(t), \tag{16}$$

$$\dot{x}_1(t) = R_{11}(\lambda)X(t) + R_{12}(\lambda)Y(t), \tag{17}$$

$$y_1(t) = R_{21}(\lambda)X(t) + R_{22}(\lambda)Y(t), \qquad t > 0, \tag{18}$$

where $x_1 \in \mathbb{R}^{\bar{n}_1}$, $y_1 \in \mathbb{R}^{\bar{n}_2}$ are auxiliary variables, $X = \operatorname{col}[x, x_1]$, $Y = \operatorname{col}[y, y_1]$, $R_{0j}(\lambda) \in \mathbb{R}^{r\times(n_j+\bar{n}_j)}[\lambda]$, $j = 1, 2$, and $R_{ij}(\lambda) \in \mathbb{R}^{\bar{n}_i\times(n_j+\bar{n}_j)}[\lambda]$, $i, j = 1, 2$. In this case, condition (3) for system (15) becomes

$$u(t) = u(\{x(t), x(t-ih), y(t-ih), i = 1, \dots, \varepsilon\}). \tag{19}$$

The definitions of spectral reducibility and weak spectral reducibility for system (15) are stated using the controller (16)–(18) just as in the case of system (1). Obviously, the fact that one of systems (1) or (15) has the property of spectral reducibility (weak spectral reducibility) implies that the other system has the same property. Therefore, in what follows, we first prove the main results for system (15) and then prove that they are equivalent to the conditions of Theorems 1 and 2.

4. PROOF OF THEOREM 1

According to the preceding, we first prove a criterion for the spectral reducibility of system (15). We write $B(\lambda) = \operatorname{col}[B_1(\lambda), B_2(\lambda)]$ and introduce the set

$$\Lambda_B = \{p \in \mathbb{C} : \operatorname{rank}[\mathcal{W}_0(p, e^{-ph}), B(e^{-ph})] < n\}.$$

Lemma 1. *System* (15) *is spectrally reducible is and only if the following conditions are satisfied simultaneously:*

$$\textit{the set } \Lambda_B \textit{ is finite}; \tag{20}$$

$$\operatorname{rank}[I_{n_2} - \lambda A_{22}(\lambda), B_2(\lambda)] = n_2 \quad \textit{for any} \quad \lambda \in \mathbb{C}. \tag{21}$$

Proof. Necessity. The proof of the necessity of condition (20) does not encounter substantial difficulties [18] and hence is omitted.

Let us prove the necessity of condition (21). Assume that the controller (16)–(18) ensures that the closed-loop system (15)–(18) has a finite spectrum. We write the characteristic polynomial of system (15)–(18),

$$\begin{vmatrix} pI_{n_0} - A^0_{11}(e^{-ph}) & -A^0_{12}(e^{-ph}) \\ -A^0_{21}(e^{-ph}) & I_{n_y} - A^0_{22}(e^{-ph}) \end{vmatrix} = p^{n_0} + \sum_{i=0}^{n_0-1} \nu_i p^i, \tag{22}$$

where the $A^0_{ij}(\lambda)$, $i = 1, 2$, $j = 1, 2$, are some polynomial matrices, $n_0 = n_1 + \bar{n}_1$, $n_y = n_2 + \bar{n}_2$, and $\nu_i \in \mathbb{R}$, $i = 0, \dots, n_0 - 1$. Let $|I_{n_y} - A^0_{22}(\lambda)| = d_0(\lambda)$, and let $\Pi_0(\lambda)$ be the adjoint of the matrix $(I_{n_y} - A^0_{22}(\lambda))$; i.e., $(I_{n_y} - A^0_{22}(\lambda))\Pi_0(\lambda) = d_0(\lambda)I_{n_y}$. We multiply both sides of (22) on the left by the respective parts of the relation $|\operatorname{diag}[I_{n_0}, \Pi_0(e^{-ph})]| = (d_0(e^{-ph}))^{n_y-1}$ and obtain the quasipolynomial

$$\begin{vmatrix} pI_{n_0} - A^0_{11}(e^{-ph}) & -A^0_{12}(e^{-ph})\Pi_0(e^{-ph}) \\ -A^0_{21}(e^{-ph}) & d_0(e^{-ph})I_{n_y} \end{vmatrix} = \left(p^{n_0} + \sum_{i=0}^{n_0-1} \nu_i p^i\right)(d_0(e^{-ph}))^{n_y-1}. \tag{23}$$

On the other hand, it follows from the form of the determinant on the left-hand side in (23) that it can be represented as a quasipolynomial of the form

$$\begin{vmatrix} pI_{n_0} - A^0_{11}(e^{-ph}) & -A^0_{12}(e^{-ph})\Pi_0(e^{-ph}) \\ -A^0_{21}(e^{-ph}) & d_0(e^{-ph})I_{n_y} \end{vmatrix} = p^{n_0}(d_0(e^{-ph}))^{n_y} + \hat{d}_0(e^{-ph}), \tag{24}$$

where $\hat{d}_0(\lambda)$ is a polynomial. Matching the coefficients of like powers of the variable p on the right-hand sides in (23) and (24), we conclude that $d_0(\lambda) \equiv 1$. But this is impossible if condition (21) is violated [20]. The necessity has thus been proved.

Sufficiency. It follows from condition (21) that there exist polynomial matrices $L_{ij}(\lambda)$, $i,j=1,2$, of appropriate dimensions such that the following identity holds [15, 20]:

$$\begin{vmatrix} I_{n_2}-\lambda A_{22}(\lambda)-\lambda B_2(\lambda)L_{11}(\lambda) & -\lambda B_2(\lambda)L_{12}(\lambda) \\ -\lambda L_{21}(\lambda) & I_{r^*}-\lambda L_{22}(\lambda) \end{vmatrix} \equiv 1 \tag{25}$$

(where $r^* \in \mathbb{N} \sqcup \{0\}$). We close system (1) with the controller

$$\begin{aligned} u(t) &= L_{11}(\lambda)y(t-h)+L_{12}(\lambda)y_1(t-h)+w_1(t), \\ y_1(t) &= L_{21}(\lambda)y(t-h)+L_{22}(\lambda)y_1(t-h)+w_2(t), \end{aligned} \tag{26}$$

where $y_1 \in \mathbb{R}^{r^*}$ is an auxiliary variable and $w = \operatorname{col}[w_1,\, w_2]$ is the new control. We introduce the matrices

$$\begin{gathered} \overline{A}_{12}(\lambda) = [A_{12}(\lambda)+\lambda B_1(\lambda)L_{11}(\lambda), \lambda B_1(\lambda)L_{12}(\lambda)], \\ \overline{A}_{21}(\lambda) = \begin{bmatrix} A_{21}(\lambda) \\ 0_{r^*\times n_1} \end{bmatrix}, \qquad \overline{A}_{22}(\lambda) = \begin{bmatrix} A_{22}(\lambda)+B_2(\lambda)L_{11}(\lambda) & B_2(\lambda)L_{12}(\lambda) \\ L_{21}(\lambda) & L_{22}(\lambda) \end{bmatrix}, \\ \overline{B}_1(\lambda) = [B_1(\lambda), 0_{n_1\times r^*}], \qquad \overline{B}_2(\lambda) = \begin{bmatrix} B_2(\lambda) & 0_{n_2\times r^*} \\ 0_{r^*\times r} & I_{r^*} \end{bmatrix}. \end{gathered}$$

We write system (15), (26) in the form

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t)+\overline{A}_{12}(\lambda)Y_1(t)+\overline{B}_1(\lambda)w(t), \\ Y_1(t) &= \overline{A}_{21}(\lambda)x(t)+\overline{A}_{22}(\lambda)Y_1(t-h)+\overline{B}_2(\lambda)w(t), \qquad t>0, \end{aligned} \tag{27}$$

where $Y_1 = \operatorname{col}[y, y_1]$. Then

$$\mathcal{W}_1(p, e^{ph}) = \begin{bmatrix} pI_{n_1}-A_{11}(e^{-ph}) & -\overline{A}_{12}(e^{-ph}) \\ -\overline{A}_{21}(e^{-ph}) & I_{n_2+r^*}-p\overline{A}_{22}(e^{-ph}) \end{bmatrix}$$

is the characteristic matrix of system (27). We write $\overline{\overline{B}}(\lambda) = \operatorname{col}[\overline{B}_1(\lambda), \overline{B}_2(\lambda)]$ and consider the set

$$\Lambda^1_{\overline{\overline{B}}} = \{p \in \mathbb{C}:\ \operatorname{rank}[\mathcal{W}_1(p, e^{-ph}), \overline{\overline{B}}(e^{-ph})] < n\}.$$

One can readily show that

$$\Lambda_B = \Lambda^1_{\overline{\overline{B}}}. \tag{28}$$

By identity (25), the matrix $M(\lambda) = (I_{n_2+r^*}-\lambda\overline{A}_{22}(\lambda))^{-1}$ belongs to the set $\mathbb{R}^{(n_2+r^*)\times(n_2+r^*)}[\lambda]$; i.e., it is a polynomial. In system (27), we introduce the new variable $Y_1 = M(\lambda)Y_2$; then the system becomes

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t)+\widetilde{A}_{12}(\lambda)Y_2(t)+\overline{B}_1(\lambda)w(t), \\ Y_2(t) &= \overline{A}_{21}(\lambda)x(t)+\overline{B}_2(\lambda)w(t), \qquad t>0, \end{aligned} \tag{29}$$

where $\widetilde{A}_{12}(\lambda) = \overline{A}_{12}(\lambda)M(\lambda)$. By $\mathcal{W}_2(p, e^{-ph})$ we denote the characteristic matrix of system (29) and define the set

$$\Lambda^2_{\overline{\overline{B}}} = \{p \in \mathbb{C}: \operatorname{rank}[\mathcal{W}_2(p, e^{-ph}), \overline{\overline{B}}(e^{-ph})] < n\}.$$

Since $\mathcal{W}_2(p, e^{-ph}) = \mathcal{W}_1(p, e^{-ph})\operatorname{diag}[I_{n_1},\, M(\lambda)]$ and (28) holds, we also have

$$\Lambda^2_{\overline{\overline{B}}} = \Lambda_B. \tag{30}$$

It follows from the form of the equations in system (29) that x satisfies the system

$$\dot{x}(t) = K(\lambda)x(t) + F(\lambda)w(t), \qquad t > 0, \tag{31}$$

where $K(\lambda) = A_{11}(\lambda) + \widetilde{A}_{12}(\lambda)\overline{A}_{21}(\lambda)$ and $F(\lambda) = \widetilde{A}_{12}(\lambda)\overline{B}_2(\lambda) + \overline{B}_1(\lambda)$. Consider the characteristic matrix $\mathcal{W}_K(p, e^{-ph}) = I_{n_1} - K(e^{-ph})$ of system (31) and introduce the set

$$\Lambda_F = \{p \in \mathbb{C} : \operatorname{rank}[\mathcal{W}_K(p, e^{-ph}), F(e^{-ph})] < n\}.$$

Let us show that

$$\Lambda_F = \Lambda^2_{\overline{\overline{B}}}. \tag{32}$$

Let $p_0 \in \Lambda_F$. We determine a nonzero vector $g \in \mathbb{R}^{n_1}$ satisfying the algebraic system

$$g'\mathcal{W}_K(p_0, e^{-p_0 h}) = 0_{n_1 \times 1}, \qquad g'F(e^{-p_0 h}) = 0_{(r+r^*)\times 1}$$

(here and below, the prime stands for transposition) and set $g_1 = [g',\, g'\widetilde{A}_{12}(\lambda)]$. Then we have the relation

$$g_1 \mathcal{W}_2(p_0, e^{-p_0 h}) = 0_{(n+r^*)\times 1}, \qquad g_1 \overline{\overline{B}}(e^{-p_0 h}) = 0_{(r+r^*)\times 1}.$$

Therefore, $p_0 \in \Lambda^2_{\overline{\overline{B}}}$, and hence the inclusion $\Lambda_F \subseteq \Lambda^2_{\overline{\overline{B}}}$ holds. By a similar argument (with minor changes) in the backward direction, we obtain the inclusion $\Lambda_F \supseteq \Lambda^2_{\overline{\overline{B}}}$, which implies (32).

Relations (30) and (32) imply that $\Lambda_F = \Lambda_B$, and the set Λ_F is finite by condition (20). Therefore, there exists a controller of the form [19, 20]

$$w(t) = \overline{R}_{01}(\lambda)x(t) + \overline{R}_{02}(\lambda)x_1(t), \qquad \dot{x}_1(t) = \overline{R}_{11}(\lambda)x(t) + \overline{R}_{12}(\lambda)x_1(t) \tag{33}$$

ensuring that system (31), (33) has a finite spectrum; here $x_1 \in \mathbb{R}^{\bar{n}_1}$ is an auxiliary variable, $\overline{R}_{01}(\lambda) \in \mathbb{R}^{(r+r^*)\times n_1}[\lambda]$, $\overline{R}_{02}(\lambda) \in \mathbb{R}^{(r+r^*)\times \bar{n}_1}[\lambda]$, $\overline{R}_{11}(\lambda) \in \mathbb{R}^{\bar{n}_1 \times n_1}[\lambda]$, and $\overline{R}_{12}(\lambda) \in \mathbb{R}^{\bar{n}_1 \times \bar{n}_1}[\lambda]$.

We return to the controller (16)–(18), where we set

$$\begin{aligned} &R_{01}(\lambda) = [I_r, 0_{r\times r^*}][\overline{R}_{01}(\lambda), \overline{R}_{02}(\lambda)], \qquad R_{02}(\lambda) = [\lambda L_{11}(\lambda), \lambda L_{12}(\lambda)],\\ &R_{11}(\lambda) = [\overline{R}_{11}(\lambda), \overline{R}_{012}(\lambda)], \qquad R_{12}(\lambda) = 0_{\bar{n}_1 \times \bar{n}_2},\\ &R_{21}(\lambda) = [0_{r^*\times r}, I_{r^*}][\overline{R}_{01}(\lambda), \overline{R}_{02}(\lambda)], \qquad R_{22}(\lambda) = [\lambda L_{21}(\lambda), \lambda L_{22}(\lambda)]. \end{aligned} \tag{34}$$

Then the controller (16)–(18) whose matrices are defined by formulas (34) is the controller (26), (33), and system (15)–(18), (34) coincides with system (15), (26), (33).

Now let us show that the spectrum of system (15), (26), (33) is finite. Since the spectrum of system (31), (33) is finite, it follows that there exists a polynomial $\Delta_0(p)$ such that

$$\Delta_0\left(\frac{d}{dt}\right)x(t) = 0_{n\times 1}, \qquad \Delta_0\left(\frac{d}{dt}\right)x_1(t) = 0_{\bar{n}\times 1}, \qquad t > t_1, \tag{35}$$

where $t_1 > 0$ is a sufficiently large number. (We have taken into account the fact that the smoothness of system (31), (33) increases in time.) We apply the operator $\Delta_0(d/dt)$ to both sides of the second equation in system (29) and obtain

$$\Delta_0\left(\frac{d}{dt}\right)Y_2(t) = 0, \qquad t > t_2,$$

where $t_2 = t_1 + \max\{\deg\overline{A}_{21}(\lambda), \deg\overline{B}_2(\lambda)\overline{R}_{01}(\lambda), \deg\overline{B}_2(\lambda)\overline{R}_{02}(\lambda)\}$ (the maximum degree of the corresponding entries of the matrices). This implies the relation

$$\Delta_0\left(\frac{d}{dt}\right)Y_1(t) = 0, \qquad t > t_3,$$

where $t_3 = t_2 + \deg M(\lambda)$. Thus, the solution of system (15), (26), (33) satisfies relations (35) and

$$\Delta_0\left(\frac{d}{dt}\right)y(t) = 0, \qquad \Delta_0\left(\frac{d}{dt}\right)y_1(t) = 0, \qquad t > t_3;$$

i.e., the spectrum of this system is finite. The proof of the lemma is complete.

Lemma 2. *For conditions* (11) *and* (12) *to be satisfied for system* (1), *it is necessary and sufficient that conditions* (20) *and* (21) *be satisfied for system* (15).

Proof. The equivalence of conditions (12) and (21) was proved in [16], and the equivalence of conditions (11) and (20) is obvious.

Now the proof of Theorem 1 follows from Lemmas 1 and 2. The proof of Theorem 1 is complete.

5. PROOF OF THEOREM 2

Let $G(\lambda)$ be the matrix of additional inputs constructed for the matrix $B(\lambda)$ (see Section 2). We write it as $G(\lambda) = \operatorname{col}[G_1(\lambda), G_2(\lambda)]$, where $G_1(\lambda) \in \mathbb{R}^{n_1\times r_T}[\lambda]$ and $G_2(\lambda) \in \mathbb{R}^{n_2\times r_T}[\lambda]$. Recall that $H_1\widetilde{B}(\lambda) = B(\lambda)$. By setting $B^{(i)} = H_1\widetilde{B}^{(i)}$ and $G^{(i)} = H_1\widetilde{G}^{(i)}$, we obtain $H_1\widetilde{G}(\lambda) = G(\lambda)$. The matrix S defined in Section 2 and the matrix $G(\lambda)$ have the following property (which also holds for the matrix $\widetilde{G}(\lambda)$).

Lemma 3. *The relation*

$$\begin{bmatrix} I_n & G(\lambda) \\ 0_{r_T\times n} & I_{r_T} \end{bmatrix}\begin{bmatrix} \mathcal{W}_0(p,\lambda) & -B(\lambda)T \\ \Gamma(\lambda) & I_{r_T} - \lambda S \end{bmatrix} = \begin{bmatrix} \mathcal{W}_0(p,\lambda) + G(\lambda)\Gamma(\lambda) & 0_{n\times r_T} \\ \Gamma(\lambda) & I_{r_T} - \lambda S \end{bmatrix} \tag{36}$$

holds, where $\Gamma(\lambda) \in \mathbb{R}^{r_T\times n}[\lambda]$ *is an arbitrary matrix.*

Proof. This assertion follows from the chain of relations [13, 15]

$$\begin{aligned} B(\lambda)T &= B^{(0)}T + \sum_{i=1}^{m} B^{(i)}\lambda^i T = G^{(0)} + \sum_{i=1}^{m}(G^{(i)} - G^{(i-1)}S)\lambda^i \\ &= \sum_{i=0}^{m} G^{(i)}\lambda^i - \sum_{i=1}^{m} G^{(i-1)}S\lambda^i = \sum_{i=0}^{m-1} G^{(i)}\lambda^i(I_{r_T} - \lambda S) = G(\lambda)(I_{r_T} - \lambda S). \end{aligned}$$

The proof of the lemma is complete.

Remark 2. Lemma 3 can be stated differently [16]. To this end, we use the parameters of system (15) to construct the auxiliary system

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t) + A_{12}(\lambda)y(t) + B_1(\lambda)U_1(t) + G_1(\lambda)U_2(t), \\ y(t) &= A_{21}(\lambda)x(t) + A_{22}(\lambda)y(t-h) + B_2(\lambda)U_1(t) + G_2(\lambda)U_2(t), \qquad t > t_0, \end{aligned} \tag{37}$$

where $U = \operatorname{col}[U_1, U_2]$, $U_1 \in \mathbb{R}^r$, $U_2 \in \mathbb{R}^{r_T}$, is a new control and $t_0 > 0$ is a time instant.

Lemma 3′. *Assume that the control* $u(t)$, $t > 0$, *for system* (15) *is determined by the relations*

$$u(t) = T\psi(t) + U_1(t), \qquad \psi(t) = S\psi(t-h) + U_2(t), \qquad t > 0,$$

where $U_i(t)$, $t > 0$, $i = 1, 2$, *and* $\psi(t)$, $t \le 0$, *are arbitrary piecewise continuous functions. Then for* $t > t_0 = mh$ *system* (15) *has the form of system* (37).

We introduce the set

$$\Lambda_{B,G} = \{p \in \mathbb{C} : \operatorname{rank}[\mathcal{W}_0(p, e^{-ph}), B(e^{-ph}), G(e^{-ph})] < n\}.$$

As in the proof of Theorem 1, we first prove a criterion for the weak spectral reducibility of system (15).

Lemma 4. *For system* (15) *be weakly spectrally reducible*, *it is necessary and sufficient that the following conditions be satisfied simultaneously*:

$$\text{the set } \Lambda_{B,G} \text{ is finite}; \tag{38}$$

$$\operatorname{rank}[I_{n_2} - \lambda A_{22}(\lambda), B_2(\lambda), G_2(\lambda)] = n_2 \quad \text{for any } \lambda \in \mathbb{C}. \tag{39}$$

Proof. Necessity. We close system (15) with the controller (16)–(18). For system (15)–(18), we write (5) as

$$\mathcal{P}(\lambda)\mathcal{W}(p,\lambda) = \begin{bmatrix} \mathcal{W}_{11}(p,\lambda) & 0_{(n+\bar{n}-n_*)\times n_*} \\ \mathcal{W}_{21}(\lambda) & \mathcal{W}_{22}(\lambda) \end{bmatrix}, \tag{40}$$

where $\mathcal{P}(\lambda) \in \mathbb{R}^{(n+\bar{n})\times(n+\bar{n})}[\lambda]$ is a unimodular matrix, the matrix $\mathcal{W}_{11}(p,\lambda)$ has a CR-structure, $\mathcal{W}_{21}(\lambda) \in \mathbb{R}^{n_*\times(n+\bar{n}-n_*)}[\lambda]$, and $\mathcal{W}_{22}(\lambda) \in \mathbb{R}^{n_*\times n_*}[\lambda]$. We assume that the determinant $|\mathcal{W}_{11}(p, e^{-ph})|$ is a polynomial.

Let us determine which of the variables of the closed-loop system is associated with the zero block in the matrix on the right-hand side in (40). To this end, we arrange the variables of system (15)–(18) in the same order as the components of the vector $\operatorname{col}[x, y, x_1, y_1]$ and write the characteristic matrix of this system,

$$\begin{aligned} &\mathcal{W}(p,\lambda) \\ &= \begin{bmatrix} pI_{n_1} - A_{11}(\lambda) - B_1(\lambda)R^1_{01}(\lambda) & -A_{12}(\lambda) - B_1(\lambda)R^1_{02}(\lambda) & -B_1(\lambda)R^2_{01}(\lambda) & -B_1(\lambda)R^2_{02}(\lambda) \\ -A_{21}(\lambda) - B_2(\lambda)R^1_{01}(\lambda) & I_{n_2} - \lambda A_{22}(\lambda) - B_2(\lambda)R^1_{02}(\lambda) & -B_2(\lambda)R^2_{01}(\lambda) & -B_2(\lambda)R^2_{02}(\lambda) \\ -R^1_{11}(\lambda) & -R^1_{12}(\lambda) & pI_{\bar{n}_1} - R^2_{11}(\lambda) & -R^2_{12}(\lambda) \\ -R^1_{21}(\lambda) & -R^1_{22}(\lambda) & -R^2_{21}(\lambda) & I_{\bar{n}_2} - R^2_{22}(\lambda) \end{bmatrix}, \end{aligned}$$

where $\lambda = e^{-ph}$ and the matrices $R^1_{ij}(\lambda)$ and $R^2_{ij}(\lambda)$ $(i = 0, \dots, 2,\ j = 1, 2)$, respectively, consist of the first n_j columns of the matrix $R_{ij}(\lambda)$ and the remaining $\bar{n}_j$ columns of the same matrix taken so that their sequence order remains unchanged. The structure of the matrix $\mathcal{W}(p,\lambda)$ shows that if, by elementary transformations, one can single out the linear autonomous completely regular subsystem corresponding to the matrix $\mathcal{W}_{11}(p, e^{-ph})$ and containing the variables x and y as components of the solution, then this is possible only by multiplying certain rows in the lower right block and adding them to the other rows, i.e., by "nullifying the elements" in some columns of the right blocks of the matrix $\mathcal{W}(p,\lambda)$. Indeed, the application of the same argument to the third columns of blocks of the matrix $\mathcal{W}(p,\lambda)$ necessarily results in the operation of differentiation, and the same argument applied to the second or first blocks eliminates the components of the vector $\operatorname{col}[x, y]$. Thus, we conclude that to the zero block of the matrix (40) there correspond some components of the vector y_1. Obviously, we can assume that these components are the last n_* components of the vector y_1. Moreover, if necessary, we can interchange the rows of the matrices contained in Eq. (18) so as to obtain the zero block according to (40). This argument also shows that the last n_* rows of the matrix (40) are independent of p.

The above analysis permits concluding that the matrix $\mathcal{P}(\lambda)$ has the form (up to elementary transformations of columns)

$$\mathcal{P}(\lambda) = \begin{bmatrix} I_{n+\bar{n}-n_*} & \mathcal{P}_{12}(\lambda) \\ 0_{n_*\times(n+\bar{n}-n_*)} & I_{n_*} \end{bmatrix},$$

where $\mathcal{P}_{12}(\lambda) \in \mathbb{R}^{(n+\bar{n}-n_*)\times n_*}[\lambda]$ is a certain matrix.

Now let us determine the form of the first n rows of the matrix $\mathcal{P}(\lambda)\mathcal{W}(p,\lambda)$, or, which is the same, the matrix $\mathcal{W}_{11}(p,\lambda)$. We represent the matrices $R^2_{i2}(\lambda)$, $i=0,2$, in the form $R^2_{i2}(\lambda)=[R^{21}_{i2}(\lambda),R^{22}_{i2}(\lambda)]$, where the matrices $R^{21}_{i2}(\lambda)$ are composed of the first $\bar{n}_2-n_*$ columns of the matrix $R^2_{i2}(\lambda)$ preserving their sequence order, and the matrices $R^{22}_{i2}(\lambda)$ are composed of the remaining n_* columns of the matrix $R^2_{i2}(\lambda)$, also preserving their sequence order. As in the proof of the necessity in Lemma 5 in [13], we conclude the following: if the subsystem corresponding to the characteristic matrix $\mathcal{W}_{11}(p,e^{-ph})$ does not contain the last n_* components of the vector y_1, then there exists a matrix $\widehat{R}(\lambda)\in\mathbb{R}^{(r+r_T)\times n_*}[\lambda]$ such that

$$B(\lambda)R^{22}_{02}(\lambda)=[B(\lambda),G(\lambda)]\widehat{R}(\lambda)(I_{n_*}-R^{22}_{22}(\lambda)) \tag{41}$$

and the first n_* rows of the matrix $\mathcal{P}_{12}(\lambda)$ have the form (up to elementary transformations of the columns) $[B(\lambda),G(\lambda)]\widehat{R}(\lambda)$. We write

$$\widehat{\widehat{R}}_i(\lambda)=[R(\lambda)^1_{i1},R(\lambda)^1_{i2},R(\lambda)^2_{i1},R(\lambda)^{21}_{i2}],\qquad i=0,2.$$

The structure of the matrix $\mathcal{W}(p,\lambda)$, the form of the matrix $\mathcal{P}(\lambda)$, and the representation (41) show that the first n rows of the matrix $\mathcal{W}_{11}(p,\lambda)$ have the form

$$\begin{aligned}[][I_n,\,0_{n\times(\bar{n}-n_*)}]\mathcal{W}_{11}(p,\lambda)&=[\mathcal{W}_0(p,\lambda),\,0_{n\times(\bar{n}-n_*)}]\\ &\quad-[B(\lambda),\,G(\lambda)](\operatorname{col}[I_r,0_{r_T\times r}]\widehat{\widehat{R}}_0(\lambda)+\widehat{R}(\lambda)[0_{n_*\times(\bar{n}_2-n_*)},\,I_{n_*}]\widehat{\widehat{R}}_2(\lambda)).\end{aligned} \tag{42}$$

It follows from (42) that system (37) is spectrally reducible. By Lemma 1, conditions (38), (39) are necessary for the spectral reducibility of system (37). The necessity is proved.

Sufficiency. Consider system (37). By the assumptions of Lemma 4, this system is spectrally reducible; i.e., there exists a controller of the form (16)–(18) ensuring that the closed-loop system has a finite spectrum. To be definite, we write this controller in the form

$$\begin{aligned}U(t)&=\overline{K}_{01}(\lambda)X^*(t)+\overline{K}_{02}(\lambda)Y^*(t),\\ \dot{x}^*_1(t)&=K_{11}(\lambda)X^*(t)+K_{12}(\lambda)Y^*(t),\\ y^*_1(t)&=K_{21}(\lambda)X^*(t)+K_{22}(\lambda)Y^*(t),\end{aligned} \tag{43}$$

where $x^*_1\in\mathbb{R}^{\bar{n}^*_1}$ and $y^*_1\in\mathbb{R}^{\bar{n}^*_2}$ are auxiliary variables, $X^*=\operatorname{col}[x,x^*_1]$, $Y^*=\operatorname{col}[y,y^*_1]$, $\overline{K}_{0j}(\lambda)\in\mathbb{R}^{(r+r_T)\times(n_j+\bar{n}^*_j)}[\lambda]$, $j=1,2$, and $K_{ij}(\lambda)\in\mathbb{R}^{\bar{n}^*_i\times(n_j+\bar{n}^*_j)}[\lambda]$, $i,j=1,2$. Let $\mathcal{W}^*(p,e^{-ph})$ be the characteristic matrix of system (37), (43), and let $|\mathcal{W}^*(p,e^{-ph})|=d^*(p)$, where $d^*(p)$ is a polynomial. We assume that $\overline{K}_{0j}(\lambda)=\operatorname{col}[K_{0j}(\lambda),K_{3j}(\lambda)]$, $j=1,2$, where $K_{0j}(\lambda)\in\mathbb{R}^{r\times(n_j+\bar{n}^*_j)}[\lambda]$ and $K_{3j}(\lambda)\in\mathbb{R}^{r_T\times(n_j+\bar{n}^*_j)}$, $j=1,2$. We use the parameters of the controller (43) to construct the new controller

$$\begin{aligned}u(t)&=K_{01}(\lambda)X(t)+[K_{02}(\lambda),T]Y(t),\\ \dot{x}_1(t)&=K_{11}(\lambda)X(t)+[K_{12}(\lambda),0_{\bar{n}^*_1\times r_T}]Y(t),\\ y_{11}(t)&=K_{21}(\lambda)X(t)+[K_{22}(\lambda),0_{\bar{n}^*_2\times r_T}]Y(t),\\ y_{12}(t)&=K_{31}(\lambda)X(t)+[K_{32}(\lambda),\lambda S]Y(t),\end{aligned} \tag{44}$$

where x_1,y_{11}, and y_{12} are auxiliary variables, $X=\operatorname{col}[x,x_1]$, $Y=\operatorname{col}[y,y_{11},y_{21}]$, and the other notation is described above. The controller (44) is the controller (16)–(18), where $y_1=\operatorname{col}[y_{11},y_{12}]$, $\bar{n}_1=\bar{n}^*_1$, and $\bar{n}_2=\bar{n}^*_2+r_T$.

Let us show that the characteristic matrix of the closed-loop system (15), (44) has a CR-structure and the controller (44) ensures a finite spectrum for some completely regular subsystem that uniquely determines the original variables x, y (i.e., all assumptions of Definition 2 are satisfied). We arrange the components of the solution vector of the closed-loop system (15), (44) in accordance with the components of the vector $\operatorname{col}[x,y,x_1,y_{11},y_{12}]$ and write the characteristic matrix of system (15), (44) in the form

$$\mathcal{W}(p,\lambda)=\begin{bmatrix}\mathcal{W}_0(p,\lambda)-B(\lambda)\Gamma_{01}(\lambda) & -B(\lambda)\Gamma_{02}(\lambda) & -B(\lambda)T\\ -\Gamma_{11}(\lambda) & E(p)-\Gamma_{12}(\lambda) & 0_{(\bar n_1+\bar n_2-r_T)\times r_T}\\ -\Gamma_{21}(\lambda) & -\Gamma_{22}(\lambda) & I_{r_T}-\lambda S\end{bmatrix}, \tag{45}$$

where the first column of the blocks corresponds to the vector $\operatorname{col}[x,y]$, the second column corresponds to the vector $\operatorname{col}[x_1,y_{11}]$, and the third, to the vectors y_{12}, the polynomial matrices $\Gamma_{ij}(\lambda)$, $i=0,\dots,2$, $j=1,2$, can readily be determined from the closed-loop system (15), (44) and need not be described in detail, and $E(p)=\operatorname{diag}[pI_{\bar n_1},I_{\bar n_2-r_T}]$. Consider relation (40), where we set

$$\mathcal{P}(\lambda)=\begin{bmatrix}I_n & 0_{n\times(\bar n_1+\bar n_2-r_T)} & G(\lambda)\\ 0_{(\bar n_1+\bar n_2)\times n} & I_{\bar n_1+\bar n_2-r_T} & 0_{(\bar n_1+\bar n_2)\times r_T}\\ 0_{r_T\times n} & 0_{r_T\times(\bar n_1+\bar n_2-r_T)} & I_{r_T}\end{bmatrix}. \tag{46}$$

We write $\overline{\Omega}(p,\lambda)=\mathcal{P}(\lambda)\mathcal{W}(p,\lambda)$. Obviously, we have $\deg|\overline{\Omega}(p,0)|=\deg|\mathcal{W}(p,0)|$. On the other hand, $\deg|\overline{\Omega}(p,0)|=\deg|\mathcal{W}^*(p,0)|$ by Lemma 3. Hence $\deg|\mathcal{W}(p,0)|=\deg|\mathcal{W}^*(p,0)|$. Since the matrix $\mathcal{W}^*(p,\lambda)$ has a CR-structure and the additional variables x_1 in system (15), (44) and x_1^* in system (37), (43) are determined by the same equations, we see that the matrix $\mathcal{W}(p,\lambda)$ also has a CR-structure. Condition (i) in Definition 2 is satisfied.

Conditions (ii) and (iii) in Definition 2 are satisfied as well, because the relation $\mathcal{W}_{11}(p,\lambda)=\mathcal{W}^*(p,\lambda)$ holds if the matrix $\mathcal{P}(\lambda)$ is chosen in the form (46). The proof of the lemma is complete.

Let us return to the proof of Theorem 2. Now its assertion follows from Lemmas 4 and 2. The proof of Theorem 2 is complete.

Remark 3. Assume that, in the weak spectral reducibility problem, the controller is constructed according to the proof of sufficiency in Lemma 4. In this case, in (5) one can set

$$\widetilde{\mathcal{P}}(\lambda)=\begin{bmatrix}H_1^{-1}(\lambda) & 0_{n\times\bar n}\\ 0_{\bar n\times n} & I_{\bar n}\end{bmatrix}\mathcal{P}(\lambda)\begin{bmatrix}H_1(\lambda) & 0_{n\times\bar n}\\ 0_{\bar n\times n} & I_{\bar n}\end{bmatrix}, \tag{47}$$

and then one obtains $\widetilde{\mathcal{W}}_{11}(p,\lambda)=H_1^{-1}(\lambda)\mathcal{W}_{11}(p,\lambda)H^{-1}(\lambda)$, $\widetilde{\mathcal{W}}_{21}(\lambda)=\mathcal{W}_{21}(p,\lambda)H^{-1}(\lambda)$, $\widetilde{\mathcal{W}}_{22}(\lambda)=\mathcal{W}_{22}(\lambda)$, and $|\widetilde{W}_{11}(p,\lambda)|=|H_1^{-1}||H^{-1}|d^*(p)$.

Remark 4. Consider system (1) under the assumption that $\det\widetilde{A}_0\neq 0$. In this case, we have a retarded type differential-difference system. Following the argument in [19] (see Remark 2), one can state that, for such a system, the violation of condition (11) in Theorem 1 implies the violation of condition (13) in Theorem 2.

Remark 5. It follows from the proof of sufficiency of the assumptions of Lemma 2 that the verification of condition (11) can be replaced by the verification of a similar condition for the set Λ_F, and the latter, in turn, can be replaced by the calculation of the rank of the polynomial matrix [22]. A similar conclusion holds for condition (13).

6. CONCLUSIONS. EXAMPLE

We have studied the problem of reducing a linear autonomous completely regular differential-algebraic system with commensurable delays to a system with finite spectrum. Two approaches to solving this problem were proposed. The first approach is based on the assumption that there is a feedback that ensures a finite spectrum for the closed-loop system. The other approach imposes weaker conditions of the problem solvability, but the feedback ensures a finite spectrum only for a subsystem of the closed-loop system which determines the variables of the original system. The proofs of the main results are constructive; i.e., for each of the problems under study, they permit constructing a synthesis scheme for the corresponding controller. The stages of such a synthesis are illustrated by the following example.

Example 2. Consider system (1) of the form

$$\frac{d}{dt}\left(\begin{bmatrix}1 & 0 & 1\\ 0 & -1 & 0\\ 1 & -1 & 1\end{bmatrix} z(t)\right) = \begin{bmatrix}-1+\lambda & 1+\lambda & 0\\ 0 & -1 & 0\\ 0 & -1+\lambda & 0\end{bmatrix} z(t) + \begin{bmatrix}2\lambda^2-\lambda & \lambda\\ \lambda-\lambda^2 & 0\\ \lambda & \lambda\end{bmatrix} u(t). \tag{48}$$

Its characteristic quasipolynomial has the form $|\widetilde{\mathcal{W}}_0(p, e^{-ph})| = p^2(-1+e^{-ph}) - pe^{-ph} + p$, and hence the spectrum of system (48) is infinite. The problem is to reduce this system to a system with finite spectrum.

1. The set $\Lambda_{\widetilde{B}}$ is infinite in this case, and hence system (48) is not spectrally reducible. Let us verify the conditions of Theorem 2. Following the scheme proposed in [21], we obtain

$$T = [1, -1]', \qquad S = 1, \qquad \widetilde{G}(\lambda) = [-2\lambda, \lambda, 0]'.$$

One can see that the set $\Lambda_{\widetilde{B},\widetilde{G}}$ is finite; i.e., condition (13) is satisfied. An easy verification shows that (14) is true as well. By Theorem 2, system (48) is weakly spectrally reducible. Let us give an example of synthesis of the corresponding controller.

2. First, we reduce the original system (48) to the form of system (15). Take the matrices

$$\widehat{H}_1 = \begin{bmatrix}0 & 0 & 1\\ 1 & 2 & -1\\ 1 & 1 & -1\end{bmatrix}, \qquad H = \begin{bmatrix}0 & 0 & 1\\ 0 & -1 & 0\\ 1 & -1 & -1\end{bmatrix}$$

and, changing the variable $z = H\,\mathrm{col}\,[x, y]$ in system (48), obtain the new system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 1-\lambda\\ 0 & 0\end{bmatrix} x(t) + \begin{bmatrix}0\\ -1+\lambda\end{bmatrix} y(t) + \begin{bmatrix}\lambda & \lambda\\ 0 & 0\end{bmatrix} u(t),\\ y(t) &= [0, -1]x(t) + y(t-1) + [\lambda-\lambda^2, 0]u(t), \qquad t > 0.\end{aligned} \tag{49}$$

By the proof of sufficiency of the assumptions of Lemma 4, to construct such a controller, we first must construct a controller ensuring a finite spectrum for system (37). To this end, we use the proof of sufficiency of the conditions of Lemma 1, where instead of the matrix $B(\lambda)$ we must take the matrix $[B(\lambda), G(\lambda)]$. Therefore, at stage 2, the matrix $[B(\lambda), G(\lambda)]$ is interpreted as the matrix $B(\lambda)$, and the function U, as u. Before this, we must calculate the matrix $G(\lambda) = H_1\widetilde{G}(\lambda) = \mathrm{col}\,[0, 0, -\lambda]$ (in this example, $H_1 = \widehat{H}_1$).

First, we construct the controller (26). To this end, we use [19],

$$u(t) = [0, 0, 1]' y_1(t-h) + w_1(t), \qquad y_1(t) = y(t-1) - y_1(t-1) - y_1(t-2h) + w_2(t), \tag{50}$$

where $w_1 \in \mathbb{R}^3$ and $w_2 \in \mathbb{R}$. We close system (48) with the controller (50) and, after several elementary operations, obtain system (31),

$$\dot{x}(t) = \begin{bmatrix}0 & 1-\lambda\\ 0 & 1-\lambda^3\end{bmatrix} x(t) + \begin{bmatrix}\lambda & \lambda & 0 & 0\\ -\lambda^5+\lambda^4+\lambda^2-\lambda & 0 & \lambda-\lambda^4 & \lambda^2-\lambda^3\end{bmatrix} w(t), \tag{51}$$

where $w = \mathrm{col}\,[w_1, w_2]$. Further, following [18, 19], we construct the controller

$$w(t) = \begin{bmatrix}0 & \lambda^2 & \lambda^2 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}' x(t) \tag{52}$$

(the variable x_1 turned out to be unnecessary), which ensures that system (51), (52) has a finite spectrum (in this case, the characteristic polynomial is $p(p-1)$). Further, we use formulas (34) to obtain a controller that ensures that system (37) has a finite spectrum. For convenience, we write it in the form of relations (43),

$$U(t) = \begin{bmatrix} 0 & \lambda^2 & \lambda^2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}' X^*(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{bmatrix}' Y^*(t), \qquad y_1^*(t) = [\lambda, -\lambda-\lambda^2]Y^*(t), \tag{53}$$

where $X^* = x$ and $Y^* = \operatorname{col}[y, y_1^*]$.

3. We use the parameters of the controller (53) and formulas (44) to write the controller (16)–(18) as follows:

$$\begin{aligned} u(t) &= \begin{bmatrix} 0 & 0 \\ \lambda^2 & 1 \end{bmatrix} X(t) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} Y(t), \\ y_{11}(t) &= [\lambda, -\lambda-\lambda^2, 0]Y(t), \\ y_{12}(t) &= [\lambda^2, 0]X(t) + [0, \lambda, \lambda]Y(t), \end{aligned} \tag{54}$$

where $X = x$, $Y = \operatorname{col}[y, y_{11}, y_{12}]$, and $y_{1i} \in \mathbb{R}$, $i = 1, 2$. Passing to the variable z in Eqs. (53), we obtain the controller (4),

$$u(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ \lambda^2 & -\lambda^2-1 & \lambda^2 & 0 & -1 \end{bmatrix} Z(t), \quad \begin{bmatrix} -\lambda & 0 & 0 & \lambda^2+\lambda+1 & 0 \\ -\lambda^2 & \lambda^2 & -\lambda^2 & -\lambda & 1-\lambda \end{bmatrix} Z(t) = 0_{2\times 1}. \tag{55}$$

Note that the control u in formula (55) is constructed using the information about the vector z according to the right-hand side of the expression (3), because, as is easy to see,

$$R_u^1(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The characteristic matrix of system (48), (55) has the form

$$\widetilde{\mathcal{W}}(p,\lambda) = \begin{bmatrix} p+1-\lambda-\lambda^3 & -1+\lambda^3 & p-\lambda^3 & 0 & 2\lambda-2\lambda^2 \\ 0 & -p+1 & 0 & 0 & -\lambda+\lambda^2 \\ p-\lambda^3 & -p+1+\lambda^3 & p-\lambda^3 & 0 & 0 \\ -\lambda & 0 & 0 & \lambda^2+\lambda+1 & 0 \\ -\lambda^2 & \lambda^2 & -\lambda^2 & -\lambda & 1-\lambda \end{bmatrix},$$

which implies that system (48), (55) is a completely regular differential-algebraic system with commensurable delays. Further, by formula (47), we take the matrix

$$\widetilde{\mathcal{P}}(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 & -2\lambda \\ 0 & 1 & 0 & 0 & \lambda \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix};$$

then (see formula (5)), as is easy to calculate,

$$\widetilde{\mathcal{W}}_{11}(p,\lambda) = \begin{bmatrix} p+1-\lambda+\lambda^3 & -1-\lambda^3 & p+\lambda^3 & 2\lambda^2 \\ -\lambda^3 & -p+1+\lambda^3 & -\lambda^3 & -\lambda^2 \\ p-\lambda^3 & -p+1+\lambda^3 & p-\lambda^3 & 0 \\ -\lambda & 0 & 0 & 1+\lambda+\lambda^2 \end{bmatrix} \tag{56}$$

and $|\widetilde{\mathcal{W}}_{11}(p,\lambda)| = p - p^2$; i.e., the solution vector z of the original system (48) satisfies the finite-dimensional system with characteristic matrix (56).

Let us show how else the result can be verified. To this end, we eliminate the variable y_{12} from the closed-loop system (48), (55). This can be done as follows. The second relation in (55) (see the last row) implies

$$\begin{bmatrix} 0 & -2\lambda + 2\lambda^2 \\ 0 & \lambda - \lambda^2 \end{bmatrix} z_1(t) = \begin{bmatrix} -2\lambda^3 & 2\lambda^3 & -2\lambda^3 \\ \lambda^3 & -\lambda^3 & \lambda^3 \end{bmatrix} z(t) + \begin{bmatrix} -2\lambda^2 & 0 \\ \lambda^2 & 0 \end{bmatrix} z_1(t). \tag{57}$$

Substituting the function u defined by the first equation in (55) into the expression $\widetilde{B}(\lambda)u(t)$ and using relations (57), we obtain

$$\widetilde{B}(\lambda)u(t) = \begin{bmatrix} -\lambda^3 & \lambda^3 - \lambda & -\lambda^3 & -2\lambda^2 \\ \lambda^3 & -\lambda^3 & \lambda^3 & \lambda^2 \\ \lambda^3 & -\lambda^3 - \lambda & \lambda^3 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ z_{11}(t) \end{bmatrix}, \tag{58}$$

where z_{11} is the first component of the vector z_1. Now we replace $B(\lambda)u(t)$ in Eq. (48) by formula (58) and supplement the resulting expressions with the relation

$$z_{11}(t) = [\lambda, 0, 0, -\lambda - \lambda^2] \begin{bmatrix} z(t) \\ z_{11}(t) \end{bmatrix}$$

(see formulas (55)). As a result, we obtain the system with characteristic matrix (56) whose solution vector is $\operatorname{col}[z, z_{11}]$.

Remark 6. There is one more method for obtaining the matrix $\widetilde{W}_{11}(p,\lambda)$. To this end, to system (49) and the controller (54), we must apply Lemma $3'$ (with $\psi = y_{12}$) and then pass to the variable z. The characteristic matrix of the resulting system is $\widetilde{W}_{11}(p, e^{-ph})$.

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