CONTROLLABILITY OF SECOND ORDER DISCRETE-TIME DESCRIPTOR SYSTEMS VERSION June 14, 2021

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Abstract. This paper is mainly devoted to controllability of second order discrete-time descriptor systems. Characterizations for controllability different concepts are derived and feedback designs are investigated by transforming the system into an appropriate form. Some observability conditions are also studied for these descriptor systems. It shows how the classical conditions for first order discrete-time systems can be generalized to second order discrete-time descriptor systems. We will develop the algebraic approach to establish concise and stably computed condensed forms, which play a key role in our controllability analysis. This work completes the researches about controllability/observability of higher order descriptor systems.
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1. Introduction.

In this paper we study the second order descriptor system in discrete-time

$$\begin{aligned} &Mx(n+2) + Dx(n+1) + Kx(n) = Bu(n) \quad \text{for all } n \geqslant n_0, \\ &y(k) = Cx(k), \\ &x(n_0) = x_0,\ x(n_0+1) = x_1, \end{aligned} \tag{1.1}$$

{descriptor 2nd order discrete}

where M, D, $K \in \mathbb{R}^{d,d}$, $B \in \mathbb{R}^{d,p}$, $C \in \mathbb{R}^{q,d}$ are real, constant coefficient matrices. Here $x = \{x(n)\}_{n \geqslant n_0}$, $u = \{u(n)\}_{n \geqslant n_0}$ are real-valued vector sequences. System (1.1) is concerned with the singular difference equations (SiDE)

$$Mx(n+2) + Dx(n+1) + Kx(n) = f(n) \quad \text{for all } n \geqslant n_0. \tag{1.2}$$

{SiDE 2nd ord}

They arise as mathematical models in various fields such as population dynamics, economics, the discretization of some differential- algebraic equations (DAEs) or partial differential equations (PDEs), from sampling in dynamical systems; e.g., see [6, 12, 21, 22, 27]. Recently, solvability and stability of SiDEs of second order has been investigated in [23, 24, 29]. However, controllability for these systems has not been reached althought it has been well-studied for both DAEs and SiDEs of first order [5, 11, 19].

In classical approach [4, 14, 20, 30, 31], usually new variables are introduced such that a high order system can be reformulated as a first order one. As will be seen later in Examples 2.6 and 2.7, this method, however, is not only non-unique but also has presented some substantial disadvantages from both theoretical and numerical viewpoints. These drawbacks include (1) give a wrong prediction on the index and hence, increase the complexity of a numerical solution method, (2) increase the computational effort due to the bigger size of a reformulated system, (3) affect the controllability/observability of the system itself, i.e. a first order resulting system is uncontrollable, even though the original one is.

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To overcome these obstacles, the *algebraic approach*, which treats the system directly without reformulating it, has been studied in [25, 28, 34, 35]. Nevertheless, the proposed method therein has also presented some additional difficulties as follows. Firstly, important condensed forms numbered (2.3)-(2.5) are big and complicated, which is really hard to be generalized for higher order systems. More importantly, the system transformations are not unitary, and hence, condensed forms and characteristic values could not be stably computed. Secondly, even though characterizations for the impulse controllability are given, a feedback strategy to obtain gain matrices is still missing. Finally, since feedbacks are involved in the system transformations, they may destroy desired properties, in particular the system observability, see [25, Sec.4].

From the observation above, the motivation of this work includes: Firstly, we want to develop and modify the algebraic method suggested in [25] to make it more convenient to study different controllability concepts for second order discrete-time descriptor systems. Secondly, we want to fill in missing gaps in previous researches that we have mentioned above for causal controllability. In particular, motivated by recent researches on the control properties of multi-body systems (e.g. [1, 2, 3, 17, 36]), we will study another types of feedback, namely acceleration, beside the classical displacement/velocity feedbacks. After that, a comparable framework for controllability of discrete-time systems is set up by using the algebraic approach. Finally, based on controllability, we derive some characterization for observability of second order discrete-time descriptor systems.

It should be noted, that all results in this paper also carry over to descriptor systems with time-variable, complex-valued coefficients or higher order descriptor systems. However, for notational convenience, and because that this is the most important case in practice, we restrict ourself to time-invariant, real-valued systems of second order.

The outline of this paper is as follows. After recalling some preliminary concepts and some auxiliary lemmas, in Section 3 we present the the condensed forms (3.4), (3.11) for (1.1). Based on these, we discuss the causal controllability of (1.1) via different types of feedbacks and their characterization. Here we also discuss the advantage of an acceleration feedback to the causal controllability of the system, while the other feedbacks fail. In Section 4, making use of (3.4), we analyze other controllability concepts for system (1.1). There, we also highlight a new feature of second order systems compare to first order ones, as well as the difference between continuous-time and discrete-time systems. In Section 5, observability for (1.1) is investigated. Finally, we finish with some conclusion.

2. Preliminaries and auxiliary lemmas.

First let us briefly recall some important concepts for a first order descriptor system

$$E\xi(n+1) - A\xi(n) = B_1 u(n) \quad \text{for all } n \geqslant n_0, \tag{2.1}$$

{SiDE 1st ord}

where $E,\ A \in \mathbb{R}^{\tilde{d},\tilde{d}}$, $B_1 \in \mathbb{R}^{\tilde{d},p}$ for some $\tilde{d} \in \mathbb{N}$. Here we notice that the matrix E may be rank deficient, and the matrix pair (E, A) is regular, i.e., $\det(\lambda E - A) \not\equiv 0$ in the polynomial sense. It is well-known, that the regularity of the pair (E, A) is the necessary and sufficient condition for the existence and uniqueness of a solution to (2.1), see, e.g. [11]. Moreover, the regular pair (E, A) can be transformed to Kronecker-Weierstraß canonical form (see, e.g. [29]), i.e., there exist nonsingular matrices U, V such that

$$UEV = \begin{bmatrix} I_{\tilde{d}_1} & 0 \\ 0 & N \end{bmatrix},\ UAV = \begin{bmatrix} J & 0 \\ 0 & I_{\tilde{d}_2} \end{bmatrix},\ \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = UB_1\ , \tag{2.2}$$

{Kronecker}

where N is a nilpotent matrix of nilpotency index ν, i.e., $N^\nu = 0$ and $N^i \neq 0$ for $i = 1, 2 \ldots \nu - 1$. The index ν is called the index of the pair (E, A) which doesn't depend on U, V and we write $ind(E, A) = \nu$. Consequently, the explicit solution of (2.1) is of the form $\xi(n) = V \begin{bmatrix} \xi_1(n) \\ \xi_2(n) \end{bmatrix}$ with

$$\begin{aligned} \xi_1(n+1) &= J^{n-n_0+1}\ x(n_0) + \sum_{i=0}^{n-n_0} J^i\ B_{11}\ u(n-i), \\ \xi_2(n) &= -\sum_{i=0}^{\nu-1} N^i\ B_{12}\ u(n+i) \end{aligned} \tag{2.3}$$

{solution}

for all $n \geqslant n_0$.

Clearly, the initial condition $\xi(n_0)$ could not be arbitrarily taken. System (2.1) is called *causal* if the state $\xi(n)$ is determined completely by the initial condition $\xi(n_0)$ and former inputs $u(i)$ with $i = n_0, n_0+1 \ldots, n$. It is easy to see that if $\mathrm{ind}(E, A) = 1$ then system (2.1) is causal. For a given input sequence $u = \{u(n)\}_{n \geqslant n_0}$, the set of consistent initial condition is given by

$$\mathcal{S}_0 = \left\{ V \begin{bmatrix} \xi_1(n) \\ \xi_2(n) \end{bmatrix} \;\middle|\; \xi_1(n_0) \in \mathbb{R}^{\tilde{d}_1},\ \xi_2(n_0) = -\sum_{i=0}^{\nu-1} N^i\ B_{12}\ u(n+i) \right\} .$$

The set $\mathcal{R}$ of *reachable states* or *reachable set* of (2.1) is the set of all vector that can be reached from some consistent initial vector $\xi(n_0)$ and some input sequence $\{u(n)\}_{n \geqslant n_0}$. In fact, for (2.1), it is well-known (e.g. [33]) that

$$\mathcal{R} = \mathbb{R}^{\tilde{d}_1} \oplus \mathrm{Im}\mathcal{K}(N, B_{12}),$$

where $\mathcal{K}(N, B_{12}) := \begin{bmatrix} B_{12}, & NB_{12}, & \ldots, & N^{\nu-1}B_{12} \end{bmatrix}$. In particular, if $N = 0$, the following corollary is directly followed.

Corollary 2.1. *Assume that the first order, discrete-time descriptor system of the form*

$$\begin{bmatrix} \mathbf{E}_1 \\ 0 \end{bmatrix} \xi(n+1) - \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix} \xi(n) = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} u(n) \quad \textit{for all } n \geqslant 0,$$

where $\begin{bmatrix} \mathbf{E}_1 \\ \mathbf{A}_2 \end{bmatrix}$ *is nonsingular, and* $\mathbf{B}_2$ *has full row rank. Then the reachable subspace* $\mathcal{R}$ *is the whole space* $\mathbb{R}^d$.

Definition 2.2. *The first order descriptor system* (2.1) *is called*

i) completely controllable or C-controllable *if for any* $x_0 \in \mathbb{R}^n$ *and any* $x_0^f \in \mathbb{R}^n$ *there exist a finite time* n_f *and an input sequence* u *such that* $x(n_f) = x_0^f$.

ii) controllable on a reachable set or R-controllable *if for any* $x_0 \in \mathbb{R}^n$ *and any* $x_0^f \in \mathbb{R}^n$ *there exist a finite time* n_f *and an input sequence* u *such that* $x(n_f) = x_0^f$.

iii) causal controllable or Y-controllable *if if there exists a feedback* $u(k) = Fx(k)$ *such that its closed-loop system* $Ex(k+1) = (A + B_1F)x(k)$ *is causal.*

iv) normalizable *if there exists a feedback* $u(k) = Fx(k+1)$ *such that its closed-loop system* $(E + B_1F)x(k+1) = Ax(k)$ *is an explicit difference equation, i.e.,* $E + B_1F$ *is nonsingular.*

For most classical control design aim, typically, one or more of the following rank conditions are required

$$\begin{aligned}
&\mathbf{C0}: \quad \operatorname{rank}\left[\alpha E-\beta A,\ B_1\right]=\tilde{d} \ \text{ for all } (\alpha,\beta)\in\mathbb{C}^2\backslash\{(0,0)\},\\
&\mathbf{C1}: \quad \operatorname{rank}\left[\lambda E-A,\ B_1\right]=\tilde{d} \ \text{ for all } \lambda\in\mathbb{C},\\
&\mathbf{C2}: \quad \operatorname{rank}\left[E,\ AS_\infty(E),\ B_1\right]=\tilde{d},\\
&\mathbf{C3}: \quad \operatorname{rank}\left[E,\ B_1\right]=\tilde{d},
\end{aligned} \tag{2.4}$$

{rank 1st ord}

where $S_\infty(E)$ is a matrix whose columns span an orthogonal basis of $\ker(E)$. Furthermore, it should be noted that $\mathbf{C0} = \mathbf{C1}+\mathbf{C3}$. From characterizations of controllability in [5, 11, 19] and by Kronecker-Weierstraß canonical form we can deduce

Proposition 2.3. *Consider the first order descriptor system* (2.1)*, whose the matrix pair* (E, A) *is regular. Then* (2.1) *is*

*i) C-controllable if and only if **C0** holds.*

*ii) R-controllable if and only if **C1** holds.*

*iii) Y-controllable if and only if **C2** holds.*

*iv) normalizable if and only if **C3** holds.*

For the physical meanings of these controllability concepts and their properties, we refer the interested readers to classical textbooks [7, 16, 32, 37].

Definition 2.4. *i) System* (1.1) *is called* regular *if there exists an input sequence* $u = \{u(n)\}_{n\geqslant n_0}$ *such that the corresponding IVP* (1.1) *is uniquely solvable. In this situation, we also say that the input u and the initial vectors x_0, x_1 are* consistent.
ii) In addition, a regular system (1.1) *is called* causal *if for each* $n \geqslant n_0$, $x(n)$ *does not depend on an input u at future time, i.e.,* $u(n+1)$, $u(n+2)$,.... *but only at present and past time, i.e.,* $u(n)$, $u(n-1), ..., u(n_0)$.

Definition 2.5. *([23]) System* (1.2) *is called* strangeness-free *if there exists a nonsingular matrix* $P \in \mathbb{R}^{n,n}$ *such that by scaling* (1.2) *with* P*, we obtain a new system of the form*

$$\begin{array}{c}\hat{r}_2\\ \hat{r}_1\\ \hat{r}_0\\ \hat{v}\end{array}\begin{bmatrix}\hat{M}_1\\0\\0\\0\end{bmatrix}x(n+2)+\begin{bmatrix}\hat{D}_1\\\hat{D}_2\\0\\0\end{bmatrix}x(n+1)+\begin{bmatrix}\hat{K}_1\\\hat{K}_2\\\hat{K}_3\\0\end{bmatrix}x(n)=\begin{bmatrix}\hat{f}_{n,1}\\\hat{f}_{n,2}\\\hat{f}_{n,3}\\0\end{bmatrix} \text{ for all } n\geqslant n_0, \tag{2.5}$$

{SiDE 2nd order sfree}

where the matrix $\begin{bmatrix}\hat{M}_1^T & \hat{D}_2^T & \hat{K}_3^T\end{bmatrix}^T$ *has full row rank. Notice that, restricted to the case that* $M = 0$*, we obtain exactly the well-known concept* strangeness-free *for the first order DAEs in [21].*

To study control properties of second order descriptor systems, the classical approach is to reformulate (1.1) in the form of (2.1). In the following example we demonstrate some critical difficulties that may arise while performing this approach for SiDEs.

Example 2.6. *Consider* (1.1)*, where the matrix coefficients are*

$$M=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ D=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ K=\begin{bmatrix}1&1\\1&0\end{bmatrix},\ B=\begin{bmatrix}1\\0\end{bmatrix}. \tag{2.6}$$

{eq1.4}

In fact, we have at least four ways to reformulate (1.1) *as follows*

$$\begin{aligned}
\textbf{\textit{companion form}}:\ &\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n+1) \\ x(n+2) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}\begin{bmatrix} x(n) \\ x(n+1) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(n), \\
\textbf{\textit{2nd form}}:\ &\begin{bmatrix} D & M \\ M & 0 \end{bmatrix}\begin{bmatrix} x(n+1) \\ x(n+2) \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n) \\ x(n+1) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(n), \\
\textbf{\textit{3rd form}}:\ &\begin{bmatrix} D & M \\ -M & 0 \end{bmatrix}\begin{bmatrix} x(n+1) \\ x(n+2) \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & -M \end{bmatrix}\begin{bmatrix} x(n) \\ x(n+1) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(n), \\
\textbf{\textit{4th form}}:\ &\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} x(n+1) \\ x(n+2) \end{bmatrix} = \begin{bmatrix} 0 & -K \\ -K & -D \end{bmatrix}\begin{bmatrix} x(n) \\ x(n+1) \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} u(n).
\end{aligned} \tag{2.7}$$

{first order companion form}

Each form above has its advantage, especially in case that M, K, D has a symmetric or skew-symmetric structure. Now let us check the controllability of these systems by verifying the rank conditions (2.4)*. Direct computations turns out that only in the fourth form, the index of the matrix pair (E, A) is three, while in the others, the index is four, which suggests a wrong prediction, that $x(n)$ depends also on $u(n+3)$, instead of only $u(n)$, $u(n+1)$, $u(n+2)$.*

In control theory, classical design approaches usually require that the system is at least Y-controllable. Nevertheless, this is not always fulfilled as shown in Example 2.7 below.

Example 2.7. *Consider the artificial descriptor system* (1.1) *with*

$$M = 0,\ D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ K = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This is in fact a first order system, since $M = 0$. We can directly check that this system is Y-controllable. Nevertheless, all the first order formulations in (2.7) *are not. Furthermore, for another input matrix $B = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ direct computations yield that* (1.1) *is C-controllable, while all the formulations in* (2.7) *are not.*

In view of all these difficulties, it is natural to seek for a suitable first order reformulation that is Y-controllable and be beneficial to study other controllability properties of (1.1). This task will be done in the next section. Two auxiliaries lemmata below will be very useful for our analysis later.

Lemma 2.8. *([24, Lemma 4.1]) Given four matrices $\breve{A}$, $\breve{B}$, $\breve{C}$ in $\mathbb{R}^{m,d}$ and $\breve{D}$ in $\mathbb{R}^{m,p}$. Then there exists an orthogonal matrix $\breve{U} \in \mathbb{R}^{m,m}$ such that*

$$\breve{U}\begin{bmatrix} \breve{A} & \breve{B} & \breve{C} & | & \breve{D} \end{bmatrix} = \left[\begin{array}{ccc|c} \breve{A}_1 & \breve{B}_1 & \breve{C}_1 & \breve{D}_1 \\ 0 & \breve{B}_2 & \breve{C}_2 & 0 \\ 0 & 0 & \breve{C}_3 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \breve{B}_4 & \breve{C}_4 & \breve{D}_4 \\ 0 & 0 & \breve{C}_5 & \breve{D}_5 \end{array}\right], \tag{2.8}$$

{eq1.6}

where the matrices $\breve{A}_1$, $\breve{B}_2$, $\breve{B}_4$, $\breve{C}_3$, $\left[\begin{smallmatrix} \breve{D}_4 \\ \breve{D}_5 \end{smallmatrix}\right]$ have full row rank.

Lemma 2.9. *Let $P = \left[\begin{smallmatrix} P_1 \\ P_2 \end{smallmatrix}\right] \in \mathbb{R}^{p,d}$, $Q = \left[\begin{smallmatrix} 0 \\ Q_2 \end{smallmatrix}\right] \in \mathbb{R}^{q,d}$ be two matrices. Furthermore, assume that Q_2 has full row rank. Then there exist a matrix $F \in \mathbb{R}^{d,d}$ such that $P + QF$ has full row rank if and only if P_1 also has full row rank.*

Proof. The necessary part is followed directly from the observation that

$$P + QF = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \begin{bmatrix} 0 \\ Q_2 \end{bmatrix} F = \begin{bmatrix} P_1 \\ P_2 + Q_2 F \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ P_2 & Q_2 \end{bmatrix} \begin{bmatrix} I \\ F \end{bmatrix}.$$

For the sufficient part, see [24, Lemma 2.8]. □

3. Condensed forms and causal controllability. In this section, we will modify an *algebraic method* presented in [25] to study the causal controllability (Y-controllability) of system (1.1). The main idea is to transform (1.1) directly, but not reformulate it as a first order one, into so-called *condensed forms*. Moreover, in comparison to [25], the main advantage of our method is two folds. First, the condensed form is much more concise, and can be computed in a stable way. Second, it is helpful to design a suitable feedback that make the closed-loop system to be causal (resp., impulse-free) in the discrete (resp., continuous) time case. Now let us introduce some rank conditions, which generalize the ones in (2.4).

$$\begin{aligned} &\mathbf{C21}: \quad \operatorname{rank}\left[\lambda^2 M + \lambda D + K,\ B\right] = d \ \text{ for all } \lambda \in \mathbb{C}, \\ &\mathbf{C22}: \quad \operatorname{rank}\left[M,\ DS_\infty^1,\ KS_\infty^2,\ B\right] = d, \\ &\mathbf{C23}: \quad \operatorname{rank}\left[M,\ D,\ B\right] = d, \\ &\mathbf{C24}: \quad \operatorname{rank}\left[M,\ B\right] = d, \end{aligned} \tag{3.1}$$

{rank 2nd ord}

where columns of S_∞^1 form a basis of kernel M, and columns of S_∞^2 form the basis of

$$\operatorname{kernel}\begin{bmatrix} M \\ Z_1^T D \end{bmatrix} \setminus \operatorname{kernel}\begin{bmatrix} M \\ Z_1^T D \\ Z_3^T K \end{bmatrix},$$

and columns of Z_1 and of Z_3 span the left null spaces of M and $[M\ D]$, respectively.

Definition 3.1. *Two second order descriptor systems of the form* (1.1) *with system matrices* (M, D, K, B), *and* $(\tilde{M}, \tilde{D}, \tilde{K}, \tilde{B})$ *are called* strongly (left) equivalent *if there exist nonsingular matrices* $U \in \mathbb{R}^{d,d}$ *and* $V \in \mathbb{R}^{m,m}$ *such that*

$$\tilde{M} = UM,\ \tilde{D} = UD,\ \tilde{K} = UK,\ \tilde{B} = UBV,$$

We write $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D}, \tilde{K}, \tilde{B})$.

It should be noted that, in contrast to [25, 28, 35], we avoid to perform variable transformations, i.e. $x(n) = W(n)y(n)$ for some nonsingular matrix $W(n)$. This approach will make our analysis more concise and clearer. More importantly, we aim at stably computable condensed forms, which is not available by the approach presented in the references above. Recently, using condensed forms under strongly left equivalence transformation, solvability analysis for second order discrete-time systems has been discussed in [24]. Furthermore, we also incorporate another class of equivalent transformations as follows.

Definition 3.2. *Two systems* $Mx(n+2) + Dx(n+1) + Kx(n) = Bu(n)$ *and* $\tilde{M}x(n+2) + \tilde{D}x(n+1) + \tilde{K}x(n) = \tilde{B}u(n)$ *are called* equivalent under

i) displacement/position feedback *if there exists a matrix* $F_d \in \mathbb{R}^{m,d}$ *such that* $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D}, \tilde{K} + F_d\tilde{B}, \tilde{B})$.

ii) velocity feedback *if there exists a matrix* $F_v \in \mathbb{R}^{m,d}$ *such that* $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M}, \tilde{D} + F_v\tilde{B}, \tilde{K}, \tilde{B})$.

iii) acceleration feedback *if there exists a matrix* $F_a \in \mathbb{R}^{m,d}$ *such that* $(M, D, K, B) \overset{\ell}{\sim} (\tilde{M} + F_a \widetilde{B}, \widetilde{D}, \widetilde{K}, \widetilde{B})$.

Here F_d, F_v, F_a *are called displacement, velocity, acceleration gain matrices.*

We notice that this concept is equivalent to classical feedback concepts as in mechanics for continuous-time descriptor systems [26, 27]. Furthermore, in general, a chosen feedback may contain all acceleration part $F_a x(n+2)$, velocity part $F_v x(n+1)$ and displacement/position part $F_d x(n)$, i.e.,

$$u(n) = -F_a x(n+2) - F_v x(n+1) - F_d x(n). \tag{3.2}$$

{feedback}

Consequently, the resulting closed-loop system is

$$(M + BF_a)x(n+2) + (D + BF_v)x(n+1) + (K + BF_d)x(n) = 0. \tag{3.3}$$

{close-loop}

Now let us recall the concept of Y-controllability for system (1.1).

Definition 3.3. *The descriptor system* (1.1) *is called* Y-controllable via displacement-velocity-acceleration feedback *if there exists a feedback of the form* (3.2) *such that the closed-loop system* (3.3) *is regular and strangeness-free.*

Lemma 3.4. *The Y-controllability is invariant under left equivalent transformations.*

Proof. Due to Definition 3.1, by choosing

$$u(n) = -V^{-1}F_a x(n+2) - V^{-1}F_v x(n+1) - V^{-1}F_d x(n)$$

the proof is straightforward. □

In the following theorem, we present the first condensed form of system (1.1).

Theorem 3.5. *Consider the descriptor system* (1.1)*. Then there exist two orthogonal matrices* U, V *such that the following identities hold.*

$$U\begin{bmatrix} M, & D, & K \end{bmatrix} = \left[\begin{array}{ccc} M_1 & D_1 & K_1 \\ 0 & D_2 & K_2 \\ 0 & 0 & K_3 \\ \hline 0 & D_4 & K_4 \\ 0 & 0 & K_5 \\ 0 & 0 & 0 \end{array}\right], \; UBV = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right], \quad \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.4}$$

{condensed form 1}

where sizes of the block rows are r_2, r_1, r_0, φ_1, φ_0, v, *the matrices* M_1, $\left[\begin{smallmatrix} D_2 \\ D_4 \end{smallmatrix}\right]$, K_3 *are of full row rank, and the matrices* Σ_1, Σ_0 *are nonsingular and diagonal.*

Proof. The proof is followed directly from Lemma 2.8 by consecutively partitioning two matrices $\breve{D}_5$ and $\breve{D}_4$ in (2.8) via Singular Value Decompositions. □

Theorem 3.5 has one direct corollary below.

Corollary 3.6. *In the condensed form* (3.4)*, the condition* $r_0 = v_0 = 0$ *holds true if and only if condition* ***C23*** *holds true, i.e. the matrix* $\begin{bmatrix} M, & D, & B \end{bmatrix}$ *has full row rank* d.

Remark 3.7. *The orthogonality of* U *and* V *guarantees that the condensed form* (3.4) *can be numerically stably computed. This is an important advantage, in comparison to the condensed form in Theorem 2.4, [25]. Furthermore, we refer the interested reader to Remark 2.7 in the same article.*

3.1. Causal controllability via displacement and velocity feedbacks. Now we are ready to present our first main result about the Y-controllability of (1.1) in Theorem 3.8 below. We emphasize, that due to different roles of feedback types, the characteristic condition for Y-controllability via displacement feedback is more strict than the corresponding one for velocity feedback.

THEOREM 3.8. *Consider the second order descriptor system* (1.1) *and the condensed form* (3.4)*. Then we have that:*
i) System (1.1) *is Y-controllable via displacement-velocity feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank.*
ii) System (1.1) *is Y-controllable via displacement feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T & D_4^T \end{bmatrix}^T$ *has full row rank.*
iii) System (1.1) *is Y-controllable via velocity feedback if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank.*

Proof. Since the proofs of these three claims are essentially the same, for the sake of brevity we will present only the detailed arguments for part i).
Necessity: Due to (3.4) we see that

$$\begin{bmatrix} M & D & K & \mid & B \end{bmatrix} \overset{\ell}{\sim} \begin{array}{c} \left[\begin{array}{ccc|ccc} M_1 & D_1 & K_1 & B_{11} & B_{12} & B_{13} \\ 0 & D_2 & K_2 & 0 & 0 & B_{23} \\ 0 & 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & D_4 & K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \varphi_1 \\ \varphi_0 \\ v \end{array}.$$

Thus, by using Gaussian elimination, we obtain

$$\begin{bmatrix} M & D & K & \mid & B \end{bmatrix} \overset{\ell}{\sim} \left[\begin{array}{ccc|ccc} M_1 & D_1^{new} & K_1^{new} & B_{11} & 0 & 0 \\ 0 & D_2 & K_2^{new} & 0 & 0 & 0 \\ 0 & 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & D_4 & K_4^{new} & 0 & \Sigma_1 & 0 \\ 0 & 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad (3.5)$$ {eq3.1}

where by the super script *new* we indicate a (possibly) new matrix at the same block position. This form implies that no matter what feedback has been applied, it will not affect the strangeness property of the upper part of the corresponding system, and hence, system (1.1) is Y-controllable only if the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ has full row rank. Finally, notice that system (1.1) is of square size, so it is regular only if $v = 0$. This completes the necessity part.
Sufficiency: By applying Lemma 2.9 for the matrices $P = \begin{bmatrix} M_1^T \; D_2^T \; K_3^T \end{bmatrix}^T$, $Q = \left[\begin{smallmatrix} 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_0 \end{smallmatrix}\right]$ and $G = \begin{bmatrix} D_4^T & K_5^T \end{bmatrix}^T$, we see that there exist two matrices F_d, F_v such that the matrix

$$\begin{bmatrix} M_1 \\ D_2 \\ K_3 \\ D_4 + \begin{bmatrix} 0 & \Sigma_1 & B_{43} \end{bmatrix} F_v \\ K_5 + \begin{bmatrix} 0 & 0 & \Sigma_0 \end{bmatrix} F_d \end{bmatrix}$$

has full row rank. Consequently, for the displacement-velocity feedback

$$u(n) = -F_v x(n+1)(t) - F_d x(n) \text{ for all } n \geqslant n_0, \tag{3.6}$$ {eq5.5}

the closed loop system

$$Mx(n+2) + (D + BF_v)x(n+1) + (K + BF_d)x(n) = 0 \tag{3.7}$$ {eq5.6}

is strangeness-free. Furthermore, due to the fact that in (3.4) $v = 0$, the closed-loop system (3.7) is regular, and hence, this finishes the proof. □

Making use of (3.4), we can rewrite our system (1.1) as follows

$$\begin{bmatrix} M_1 & D_1 & K_1 \\ 0 & D_2 & K_2 \\ 0 & 0 & K_3 \\ \hline 0 & D_4 & K_4 \\ 0 & 0 & K_5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{bmatrix} v(n), \qquad \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.8}$$ {system condensed form 1}

where $u(n) = Vv(n)$ for all $n \geqslant n_0$. Let $z(n) := M_1 x(n+1)$ we can then introduce a new variable $\xi(n) = \left[\begin{smallmatrix} z(n) \\ x(n) \end{smallmatrix}\right] \in \mathbb{R}^{r_2+d}$ and rewrite system (3.8) in the so-called *minimal extension form*

$$\underbrace{\begin{bmatrix} I_{r_2} & D_1 \\ 0 & M_1 \\ 0 & D_2 \\ 0 & 0 \\ \hline 0 & D_4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\tilde{E}} \xi(n+1) + \underbrace{\begin{bmatrix} 0 & K_1 \\ -I_{r_2} & 0 \\ 0 & K_2 \\ 0 & K_3 \\ \hline 0 & K_4 \\ 0 & K_5 \\ 0 & 0 \end{bmatrix}}_{-\tilde{A}} \xi(n) = \underbrace{\begin{bmatrix} B_{11} & B_{12} & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & B_{23} \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & B_{43} \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{bmatrix}}_{\tilde{B}} v(n). \qquad \begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{3.9}$$ {descriptor minimal extension}

THEOREM 3.9. *Consider the descriptor system* (1.1) *and the condensed form* (3.4). *Furthermore, assume that* $v = 0$ *and the matrix* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank. Then the minimal extension form* (3.9) *is also Y-controllable.*

Proof. In order to prove the desired claim we will verify the rank condition (2.4). Let $S_\infty(\tilde{E})$ be a full column rank matrix whose columns form an orthogonal basis of the vector space $\ker(\tilde{E})$. Partition $S_\infty(\tilde{E}) = \left[\begin{smallmatrix} U_1 \\ V_1 \end{smallmatrix}\right] \in \mathbb{R}^{r_2+d, r_2+d}$ correspondingly to (3.9), we see that

$$D_2 V_1 = 0, \; M_1 V_1 = 0.$$

Now we will prove that $K_3 V_1$ has full row rank. To do it first we perform an SVD for the matrix $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$, and due to the fact that the matrix $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$ has full row rank, it follows that

$$U_2^T \left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right] V_2 = \begin{bmatrix} \Sigma & 0 \end{bmatrix},$$

where Σ is a nonsingular, diagonal matrix. Hence, $V_1 = V_2 \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$. Partitioning $U_2^T K_3 V_2$ correspondingly, we have $U_2^T K_3 V_2 = \begin{bmatrix} K_{31} & K_{32} \end{bmatrix}$. Notice that since the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ has full row rank, K_{32} has full row rank. Thus,

$$K_3 V_1 = U_2 \; \begin{bmatrix} K_{31} & K_{32} \end{bmatrix} V_2^T \; V_2 \begin{bmatrix} 0 \\ I \end{bmatrix} = U_2 \; K_{32},$$

which has full row rank. Therefore, we see that

$$\begin{bmatrix}\tilde{E} & \tilde{A}S_\infty(\tilde{E}) & \tilde{B}\end{bmatrix} = \left[\begin{array}{cc|c|ccc} I & D_1 & K_1V_1 & B_{11} & B_{12} & B_{13}\\ 0 & M_1 & U_1 & 0 & 0 & 0\\ 0 & D_2 & K_2V_1 & 0 & 0 & B_{23}\\ 0 & 0 & K_3V_1 & 0 & 0 & 0\\ \hline 0 & D_4 & K_5V_1 & 0 & \Sigma_1 & B_{43}\\ 0 & 0 & 0 & 0 & 0 & \Sigma_0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{c} r_2\\ r_2\\ r_1\\ r_0\\ \hline \varphi_1\\ \varphi_0\\ v \end{array}$$

has full row rank if and only if $v = 0$. This completes the proof. □

REMARK 3.10. *From Theorems 3.8, 3.9 above, we see that one can interpret the upper part of system* (3.8) *as a* causal uncontrollable part*, while the lower part is the* causal controllable part*. Furthermore, the key point for constructing a suitable first order reformulation to* (1.1) *(and also for feedback design strategies) is to bring system* (1.1) *to the form* (3.4)*, where the upper part must be* strangeness-free*, i.e.,* $\begin{bmatrix} M_1^T & D_2^T & K_3^T \end{bmatrix}^T$ *has full row rank. In other words, the index reduction procedure has been performed only for the casual uncontrollable part. Recently, this task has been finished in both theoretical and numerical ways. To keep the brevity of this paper, we will omit the details and refer the interested readers to [24, Section 4]. Below we recall one important result taken from this research.*

PROPOSITION 3.11. *([24, Theorem 4.7]) Consider the descriptor system* (1.1)*. Then it has exactly the same solution set as the so-called* strangeness-free descriptor system

$$\underbrace{\left[\begin{array}{c} \hat{M}_1\\ 0\\ 0\\ \hline 0\\ 0\\ 0 \end{array}\right]}_{\hat{M}} x(n+2) + \underbrace{\left[\begin{array}{c} \hat{D}_1\\ \hat{D}_2\\ 0\\ \hline \hat{D}_4\\ 0\\ 0 \end{array}\right]}_{\hat{D}} x(n+1) + \underbrace{\left[\begin{array}{c} \hat{K}_1\\ \hat{K}_2\\ \hat{K}_3\\ \hline \hat{K}_4\\ \hat{K}_5\\ 0 \end{array}\right]}_{\hat{K}} x(n) = \underbrace{\left[\begin{array}{ccc} \hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13}\\ 0 & 0 & \hat{B}_{23}\\ 0 & 0 & 0\\ \hline 0 & \hat{\Sigma}_1 & \hat{B}_{43}\\ 0 & 0 & \hat{\Sigma}_0\\ 0 & 0 & 0 \end{array}\right]}_{\hat{B}} v(n), \quad \begin{array}{c} \hat{r}_2\\ \hat{r}_1\\ \hat{r}_0\\ \hline \hat{\varphi}_1\\ \hat{\varphi}_0\\ \hat{v} \end{array} \quad (3.10)$$

{descriptor 2nd order sfree}

for all $t \geqslant t_0$*, where* $\begin{bmatrix} \hat{M}_1^T & \hat{D}_2^T & \hat{K}_3^T \end{bmatrix}^T$ *has full row rank,* $\hat{\Sigma}_1$ *and* $\hat{\Sigma}_0$ *are nonsingular and diagonal, and* $u(n) = Vv(n)$ *for all* $n \geqslant n_0$*, where* V *is nonsingular. Furthermore, if system* (1.1) *is regular then* $\hat{v} = 0$*.*

Therefore, making use of Theorems 3.8, 3.9 and Proposition 3.11, we can completely analyze the Y-controllability and feedback design of (1.1). We, furthermore, can deduce from these theorems other conditions that help us directly verify the Y-controllability of (1.1) (without any feedback design strategy) as below.

COROLLARY 3.12. *Consider the second order descriptor system* (1.1) *and the condensed form* (3.4)*. Then system* (1.1) *is Y-controllable via displacement-velocity feedback if and only if condition* **C21** *is satisfied.*

REMARK 3.13. *In comparison to the continuous-time case, we see that Corollary 3.12 is similar to Theorem 3.14 i) ([25]). Nevertheless, if one wants to use only one type of feedback (displacement or velocity), then it could lead to extra difficulties, since the condensed form (2.3) ([25]) could not be stably-computed. Therefore, we suggest the reader to use Theorem 3.8.*

3.2. Causal controllability via acceleration feedback. For second order systems, one can consider different types of feedback (acceleration/velocity/displacement) separately, or mimic them together. In the pioneering work [25], Loose and Mehrmann considered three feedback types: position, velocity, and position-velocity; while recently Abdelaziz ([1]) considered displacement-accerleration feedback, and Zhu and Zhang ([36]) considered the most general form (3.2). In this section, we will not limit ourself to velocity/displacement feedback as in previous section, but study also the effectiveness of acceleration feedback. Clearly, to in-cooperate another feedback type, we need a new condensed form, instead of using (3.4). This is given in the following theorem.

THEOREM 3.14. *Consider the descriptor system* (1.1). *Then, there exist two orthogonal matrices* U, V *such that the following identities hold.*

$$U\left[M,\ D,\ K\right] = \left[\begin{array}{ccc} \tilde{M}_1 & \tilde{D}_1 & \tilde{K}_1 \\ 0 & \tilde{D}_2 & \tilde{K}_2 \\ 0 & 0 & \tilde{K}_3 \\ \hline \tilde{M}_4 & \tilde{D}_4 & \tilde{K}_4 \\ 0 & \tilde{D}_5 & \tilde{K}_5 \\ 0 & 0 & \tilde{K}_6 \\ \hline 0 & 0 & 0 \end{array}\right],\ UBV = \left[\begin{array}{cccc} 0 & 0 & \tilde{B}_{13} & \tilde{B}_{14} \\ 0 & 0 & 0 & \tilde{B}_{24} \\ 0 & 0 & 0 & 0 \\ \hline 0 & \tilde{\Sigma}_2 & \tilde{B}_{43} & \tilde{B}_{44} \\ 0 & 0 & \tilde{\Sigma}_1 & \tilde{B}_{54} \\ 0 & 0 & 0 & \tilde{\Sigma}_0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],\ \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_2 \\ \varphi_1 \\ \varphi_0 \\ \hline v \end{array} \qquad (3.11)$$

{condensed form 2}

where sizes of the block rows are $r_2,\ r_1,\ r_0,\ \varphi_2,\ \varphi_1,\ \varphi_0,\ v$, *the matrices* $\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_4 \end{bmatrix}$, $\begin{bmatrix} \tilde{D}_2 \\ \tilde{D}_5 \end{bmatrix}$, $\tilde{K}_3$ *are of full row rank, and the matrices* $\tilde{\Sigma}_2$, $\tilde{\Sigma}_1$, $\tilde{\Sigma}_0$ *are nonsingular and diagonal.*

Proof. The proof can be obtained directly by using Theorem 3.5. To keep the brevity of this paper we will omit the detail. ◻

The following corollaries are direct consequences of Theorem 3.14 and Lemma 2.8.

COROLLARY 3.15. *Consider the descriptor system* (1.1) *and the factorization* (3.11). *Then, the following assertions hold true.*
i) System (1.1) *is Y-controllable via only displacement feedback if and only if in* (3.4), *we have* $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T & \tilde{M}_4^T & \tilde{D}_5^T \end{bmatrix}^T$ *is of full row rank.*
ii) System (1.1) *is Y-controllable via displacement-velocity feedback (or velocity feedback) if and only if in* (3.4), $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T & \tilde{M}_4^T \end{bmatrix}^T$ *is of full row rank.*

COROLLARY 3.16. *Consider the descriptor system* (1.1) *and the factorization* (3.11). *Then, for any kind of feedback that involves acceleration (d-v-a, d-a, v-a, a), system* (1.1) *is Y-controllable via that feedback type if and only if* $v = 0$ *and the matrix* $\begin{bmatrix} \tilde{M}_1^T & \tilde{D}_2^T & \tilde{K}_3^T \end{bmatrix}^T$ *is of full row rank.*

EXAMPLE 3.17. *To illustrate the effectiveness of an acceleration feedback, we consider the discrete-time version of a non-gyroscopic system (e.g. [18])*

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(n)\ . \qquad (3.12)$$

Here we have that $\tilde{M}_4 = \tilde{K}_3 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\tilde{M}_1 = \tilde{D}_2 = \tilde{D}_4 = \tilde{D}_5 = \tilde{K}_6 = [\ ]$. *Due to Corollary 3.16i) this system is Y-controllable by acceleration feedback. Furthermore, it* *is not possible to eliminate the causal behavior by using only displacement and velocity feedbacks, since all the rank conditions in Corollary 3.15 fail.*

EXAMPLE 3.18. *Similarly, using Corollaries 3.15, 3.16 we see that one could not use only displacement-velocity feedback to eliminate the causal behavior of the system*

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(n) \ .$$

We notice, that we can construct any of four feedback types (d-v-a, d-a, v-a, a) to regularize this system.

REMARK 3.19. *We also notice, that even though different feedback types can be applied to achieve the causality of the closed-loop systems, two condensed forms* (3.4) *and* (3.11) *are still useful to achieve a desired rank for the system, i.e., there is a desired number of zero-, first- and second-order equations. For more details on this issue, we refer the readers to [8, 9, 10].*

4. Other controllability concepts and their characterizations. In this section, using the condensed forms (3.4), (3.9) proposed above, we will discuss other controllability concepts for second order systems. We will also point out the difference between a discrete and continuous time cases and discuss a new feature of second order system as well.

DEFINITION 4.1. *Consider the descriptor system* (1.1).
i) A set $\mathcal{R} \subseteq \mathbb{R}^n$ is called reachable from *the pair (x_0, x_1) if for every $x_0^f \in \mathcal{R}$ there exists an input sequence u that transfers the system in finite time from $x(n_0) = x_0$ to x_f.*
ii) A set $\mathcal{R}_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is called reachable from *the pair (x_0, x_1) if for every $(x_0^f, x_1^f) \in \mathcal{R}_2$ there exists an input sequence u that transfers the system in finite time from $x(n_0) = x_0$, $x(n_1) = x_1$ to x_0^f, x_1^f.*
iii) The system is called C-controllable *if for any pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ and any $x_0^f \in \mathbb{R}^n$ there exist a finite time n_f and an input sequence u such that $x(n_f) = x_0^f$.*
iv) The system is called strongly C2-controllable *if for any pair $(x_0, x_1) \in \mathbb{R}^n \times \mathbb{R}^n$ and any pair $(x_0^f, x_1^f) \in \mathbb{R}^n \times \mathbb{R}^n$ there exist a finite time n_f and an input sequence u such that $x(n_f) = x_0^f$, $x(n_f + 1) = x_1^f$.*
v) The system is called R-controllable *if any state $x_0^f \in \mathcal{R}$ can be reached from some pair (x_0, x_1) in finite time.*
vi) The system is called R2-controllable *if any pair $(x_0^f, x_1^f) \in \mathcal{R}_2$ can be reached from some pair (x_0, x_1) in finite time.*

It is straightforward to see that all these controllability concepts are invariant under left equivalent transformation. In the following theorem, we give a characterization for the strongly C2- and R2-controllability.

THEOREM 4.2. *Consider the descriptor system* (1.1) *and its first order companion form* (2.7). *Then the following assertions hold true.*
i) System (1.1) *is R2-controllable if and only if the system matrix coefficients satisfy condition* **C21**.
ii) Besides that, system (1.1) *is strongly C2-controllable if and only if the system matrix coefficients satisfy both conditions* **C21** *and* **C24**.

Proof. Following directly from Definition 4.1, we see that system (1.1) is strongly C2-controllable (resp., R2-controllable) if and only if its first order companion form

(2.7) is C-controllable (resp., R-controllable). Thus, the proof is directly followed by checking the rank criteria in Proposition 2.3. ∎

Now let us come back to the strangeness-free form (3.10). Clearly, we see that it is reasonable to control $x(n)$ and only the part $M_1x(n+1)$ but not the whole $x(n+1)$. This fact motivates another concept below, which is more suitable for singular descriptor systems.

DEFINITION 4.3. *Consider the descriptor system* (1.1) *and assume that it is already in the strangeness-free form* (3.10)*. Then system* (1.1) *is called* C2-controllable *if the minimal extension form* (3.9) *is C-controllable.*

LEMMA 4.4. *Consider the descriptor system* (1.1) *and its the strangeness-free from* (3.10) *and the minimal extension form* (3.9)*. Then we have that:*
i) System (3.9) *is R-controllable if and only if system* (3.10) *satisfies condition* **C21***.*
ii) System (3.9) *is C-controllable if and only if system* (3.10) *satisfies both conditions* **C21** *and* **C23***.*
iii) The constant rank condition **C21** *is preserved under the strangeness-free formulation, which transform* (1.1) *to* (3.10)*.*

Proof. For notational convenience, within this proof, we will omit the superscript $\hat{}$ on all matrices in the strangeness-free form (3.10). Due to Definition 2.3, system (3.9) is R-controllable (resp. C-controllable) if and only if the matrix coefficients $\tilde{E}$, $\tilde{A}$, $\tilde{B}$ satisfy the constant rank **C1** (resp., **C0**).
i) Condition **C1** applied to system (3.9) reads

$$\operatorname{rank}\left[\begin{array}{cc|ccc} \lambda I_{r_2} & \lambda D_1+K_1 & B_{11} & B_{12} & B_{13} \\ -I_{r_2} & \lambda M_1 & 0 & 0 & 0 \\ 0 & \lambda D_2+K_2 & 0 & 0 & B_{23} \\ 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & \lambda D_4+K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = d + r_2 \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.1}$$ {a1}

By using matrix row manipulation in order to eliminate λI_{r_2} in the first row, we see that (4.1) is equivalent to the condition

$$\operatorname{rank}\left[\begin{array}{cc|ccc} 0 & \lambda^2 M_1+\lambda D_1+K_1 & B_{11} & B_{12} & B_{13} \\ -I_{r_2} & \lambda M_1 & 0 & 0 & 0 \\ 0 & \lambda D_2+K_2 & 0 & 0 & B_{23} \\ 0 & K_3 & 0 & 0 & 0 \\ \hline 0 & \lambda D_4+K_4 & 0 & \Sigma_1 & B_{43} \\ 0 & K_5 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = d + r_2 \quad \text{for all } \lambda \in \mathbb{C}. \tag{4.2}$$ {a2}

Clearly, this holds true if and only if $\operatorname{rank}\left[\lambda^2 M+\lambda D+K,\ B\right] = d$, which is exactly the rank condition **C21**.
ii) Due to Definition 2.3, we see that **C0** = **C1** + **C3**, and hence we need to prove that condition **C3** is equivalent to condition **C23**. Now let us look at condition **C3**,

which means that the matrix

$$\begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{cc|ccc} I_{r_2} & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & M_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has full row rank $(d + r_2)$. Recall that in the strangeness-free form (3.10) the matrix $\begin{bmatrix} \hat{M}_1 \\ \hat{D}_2 \end{bmatrix}$ has full row rank. Therefore, condition **C3** holds true if and only if $r_0 = v = 0$. Moreover, condition **C23**, which means that the matrix

$$\begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{cc|ccc} M_1 & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

has full row rank, is fulfilled also only when $r_0 = v = 0$. Thus, two conditions **C3** and **C23** are equivalent, and hence, this completes the proof of this part.

iii) In order to prove that condition **C21** is preserved under the strangeness-free formulation we only need to prove that it is preserved under one index reduction step. First we notice that for any two strongly equivalent tuples (M, D, K, B) and $(\hat{M}, \hat{D}, \hat{K}, \hat{B})$ we have that

$$\left[\lambda^2 M + \lambda D + K,\ B\right] = U\ \left[\lambda^2 M + \lambda D + K,\ B\right] \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix}.$$

Thus, rank $\left[\lambda^2 M + \lambda D + K,\ B\right]$ is invariant under strongly equivalent relation. Consequently, we may assume that (M, D, K, B) takes the form as in the right hand side of (3.5). For notational convenience, we will omit the super script *new* and rewrite our system as follows.

$$\left[\begin{array}{c} M_1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right] x(n+2) + \left[\begin{array}{c} D_1 \\ D_2 \\ 0 \\ \hline D_4 \\ 0 \\ 0 \end{array}\right] x(n+1) + \left[\begin{array}{c} K_1 \\ K_2 \\ K_3 \\ \hline K_4 \\ K_5 \\ 0 \end{array}\right] x(n) = \left[\begin{array}{ccc} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right] v(n), \qquad \begin{array}{c} r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \tag{4.3}$$

{a3}

where M_1, D_2, K_3 have full row rank, and the matrices Σ_0, Σ_1 are digonal and nonsingular. We recall, that due to [24, Lemma 4.4], one step index reduction in the strangeness-free formulation is indeed transforming (4.3) into the new form which reads

$$\underbrace{\left[\begin{array}{c} S^{(2)}M_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]}_{\tilde{M}} x(n+2) + \underbrace{\left[\begin{array}{c} S^{(2)}D_1 \\ Z^{(2)}D_1 + Z^{(4)}K_2 \\ S^{(1)}D_2 \\ 0 \\ 0 \\ \hline D_4 \\ 0 \\ 0 \end{array}\right]}_{\widetilde{D}} x(n+1) + \underbrace{\left[\begin{array}{c} S^{(2)}K_1 \\ Z^{(2)}K_1 \\ S^{(1)}K_2 \\ Z^{(1)}K_2 \\ K_3 \\ \hline K_4 \\ K_5 \\ 0 \end{array}\right]}_{\widetilde{K}} x(n) = \underbrace{\left[\begin{array}{ccc} S^{(2)}B_{11} & 0 & 0 \\ Z^{(2)}B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_1 & 0 \\ 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 \end{array}\right]}_{\tilde{B}} v(n)\ . \quad \begin{array}{c} d_2 \\ s_2 \\ d_1 \\ s_1 \\ r_0 \\ \hline \varphi_2 \\ \varphi_1 \\ v \end{array} \tag{4.4}$$

{a4}

Here, the matrices $S^{(i)}$, $i = 1, 2$, and $Z^{(j)}$, $j = 1, ..., 5$ satisfy the following conditions.

i) For $i = 1, 2$, the matrices $\begin{bmatrix} S^{(i)} \\ Z^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$ are orthogonal, and $r_i = d_i + s_i$.

ii) The following identities hold true.

$$\begin{aligned} Z^{(1)}D_2 + Z^{(3)}K_3 &= 0, \\ Z^{(2)}M_1 + Z^{(4)}D_2 + Z^{(5)}K_3 &= 0. \end{aligned}$$

Consider the matrix $\left[\lambda^2 \tilde{M} + \lambda \widetilde{D} + \widetilde{K},\ \widetilde{B}\right]$, we directly see that

$$\left[\lambda^2 \tilde{M} + \lambda \widetilde{D} + \widetilde{K},\ \widetilde{B}\right] = U_\lambda \left[\lambda^2 M + \lambda D + K,\ B\right],$$

where the matrix U_λ is defined as

$$U_\lambda := \begin{bmatrix} \begin{bmatrix} S^{(2)} \\ Z^{(2)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda Z^{(4)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda^2 Z^{(5)} \end{bmatrix} & 0 & 0 & 0 \\ 0 & \begin{bmatrix} S^{(1)} \\ Z^{(1)} \end{bmatrix} & \begin{bmatrix} 0 \\ \lambda Z^{(3)} \end{bmatrix} & 0 & 0 & 0 \\ 0 & 0 & I_{r_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\varphi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\varphi_0} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_v \end{bmatrix}.$$

Since all matrices on the main diagonal are orthogonal, we see that U_λ is nonsingular for all $\lambda \in \mathbb{C}$. Therefore,

$$\operatorname{rank}\left[\lambda^2 \tilde{M} + \lambda \widetilde{D} + \widetilde{K},\ \widetilde{B}\right] = \operatorname{rank}\left[\lambda^2 M + \lambda D + K,\ B\right] \quad \text{for all } \lambda \in \mathbb{C},$$

and hence, condition **C21** is preserved under one index reduction step. This finishes our proof. ∎

In comparison to Theorem 3.9, the advantage of the minimal extension form (3.9) will be proven in the following theorem.

Theorem 4.5. *Consider the descriptor system* (1.1)*, its the strangeness-free from* (3.10) *and the minimal extension form* (3.9)*. If system* (1.1) *is R2-controllable then so is system* (3.10)*. Furthermore, if this is the case, then system* (3.9) *is R-controllable.*

Proof. Making use of Theorem 4.2 i) and Lemma 4.4 ii) we see that the constant rank condition **C21** holds for the coefficients of system (3.9). As in the proof of Lemma 4.4, due to simple matrix row manipulations, from system (3.9) we see that

$$\operatorname{rank}\left[\lambda \tilde{E} - \tilde{A},\ \widetilde{B}\right] = \operatorname{rank}\left[\lambda^2 M + \lambda D + K,\ B\right] + r_2\ ,$$

and hence, $\operatorname{rank}\left[\lambda \tilde{E} - \tilde{A}, \widetilde{B}\right] = d + r_2$. This implies that system (3.9) is R-controllable.

∎

Theorem 4.6. *Consider the descriptor system* (1.1) *and its the strangeness-free from* (3.10). *Then system* (1.1) *is C2-controllable if and only if the following conditions are satisfied.*
i) The matrix coefficients of system (1.1) *satisfies condition* **C21**.
ii) The matrix coefficients of the strangeness-free system (3.10) *satisfies condition* **C23**.

Proof. The proof is followed directly from Definition 4.3 and Lemma 4.4. □

The following example shows that condition **C23** is not invariant under the strangeness-free formulation.

Example 4.7. *Consider the following system*

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{M} x(n+2) + \underbrace{\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{D} x(n+1) + \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{K} x(n) = \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{B} u(n)\ . \tag{4.6}$$ {eq4.1}

Due to the strangeness-free formulation in [24], we can shift the second row equation forward to obtain

$$\begin{bmatrix}1&0&0\end{bmatrix} x(n+2) + \begin{bmatrix}0&1&0\end{bmatrix} x(n+1) = 0\ .$$

By removing this from the first equation, we obtain that $\begin{bmatrix}1&0&0\end{bmatrix} x(n) = 0$. *Therefore, we obtain the system*

$$\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix} x(n+1) + \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} x(n) = \begin{bmatrix}0\\0\\1\end{bmatrix} u(n)\ .$$

Analogously, by subtracting the shifted version of the first row equation from the second equation, we obtain the strangeness-free formulation (2.5) *that reads*

$$\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\hat{M}} x(n+2) + \underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{\hat{D}} x(n+1) + \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{\hat{K}} x(n) = \underbrace{\begin{bmatrix}0\\0\\1\end{bmatrix}}_{\hat{B}} u(n)\ . \tag{4.7}$$ {eq4.2}

Clearly, $\operatorname{rank}\begin{bmatrix}M, & D, & B\end{bmatrix} = 3 > 1 = \operatorname{rank}\begin{bmatrix}\hat{M}, & \hat{D}, & \hat{B}\end{bmatrix}$. *This means that condition* **C23** *is not invariant under the strangeness-free formulation.*
Furthermore, by verifying condition **C21**, *we directly see that system* (4.6) *is* $R2$-*controllable. Indeed, we have that*

$$\operatorname{rank}\begin{bmatrix}\lambda^2 M + \lambda D + K & | & B\end{bmatrix} = \operatorname{rank}\left[\begin{array}{ccc|c}\lambda^2+1 & \lambda & 0 & 0\\ \lambda & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right] = 3\ .$$

As obtained above, since $\operatorname{rank}\begin{bmatrix}\hat{M}, & \hat{D}, & \hat{B}\end{bmatrix} = 1 < 3$, *system* (4.6) *is not* $C2$-*controllable. In fact, from* (4.7), *it is straightforward that system* (4.6) *is not* C-*controllable.*

Remark 4.8. *As stated in Theorem 4.6, condition* **C23** *must be required for the strangeness-free system* (3.10) *instead of for the original system* (1.1). *This is the main difference between discrete and continuous time descriptor systems. In details, [25, Corollary 3.11 ii) and Theorem 3.18 iv)] imply that the continuous-time version of system* (4.6) *is* $C2$-*controllable (resp.* C-*controllable).*

Naturally, one may ask whether one can verify the $C2$-controllability of system (1.1) without performing an index reduction procedure (i.e., without determining the strangeness-free form (3.10)). In fact, the positive answer is given in the following theorem.

Theorem 4.9. *Consider the descriptor system* (1.1) *and its condensed form* (3.4)*. Then, system* (1.1) *is $C2$-controllable if and only if two following conditions are satisfied.*
i) The matrix coefficients of system (1.1) *satisfies condition* **C21**.
ii) In the upper part of system (3.4)*, $r_0 = v_0 = 0$ and the matrix $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$ has full row rank.*
Finally, condition ii) is equivalent to the requirement that $\operatorname{rank}\begin{bmatrix} M, & D, & B \end{bmatrix} = d$ *and the matrix $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$ has full row rank.*

Proof. Due to Definition 4.3 system (1.1) is $C2$-controllable if and only if the minimal extension form (3.9) is C-controllable. From Definition 2.3 and Lemma 4.4 iii, we see that $\mathbf{C0} = \mathbf{C1} + \mathbf{C3}$ and $\mathbf{C1}$ is equivalent to condition **C21**.
Hence, we only need to prove that condition **C3** is equivalent to the claim ii). Now let us look at condition **C3**, which means that the matrix

$$\begin{array}{c} r_2 \\ r_2 \\ r_1 \\ r_0 \\ \hline \varphi_1 \\ \varphi_0 \\ v \end{array} \left[\begin{array}{cc|ccc} I_{r_2} & D_1 & B_{11} & B_{12} & B_{13} \\ 0 & M_1 & 0 & 0 & 0 \\ 0 & D_2 & 0 & 0 & B_{23} \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & D_4 & 0 & \Sigma_1 & B_{43} \\ 0 & 0 & 0 & 0 & \Sigma_0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has full row rank, is fulfilled if and only if $\left[\begin{smallmatrix} M_1 \\ D_2 \end{smallmatrix}\right]$ has full row rank and $r_0 = v = 0$, which is nothing else than the claim ii). Finally, the last claim is directly followed from Corollary 3.6. This completes the proof. □

We summarize the relation between the controllability of the systems discussed above in Figure 4.1. Now let us discuss the C-controllability of system (1.1). In the following example we illustrate that for second order systems, C-controllability does not always imply Y-controllability.

Example 4.10. *Consider the following system*

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}}_{M} x(n+2) + \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{D} x(n+1) + \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}}_{K} x(n) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(n) \ . \tag{4.8}$$ {eq3.6}

Clearly, the structure of the pair $(M{,}D)$ *implies that system* (4.8) *is not Y-controllable. By adding the shifted version of the second row equation to the first row, we can transform* (4.8) *to the first order system*

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(n+1) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(n) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(n) \ ,$$

which can be directly verified that is C-controllable. Thus, C-controllability does not imply Y-controllability. The same observation can be made for continuous-time second order descriptor systems by considering the following system

$$\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \ .$$

Example (4.10) suggests, that we should discuss the C-controllability of the strangeness-free formulation (3.10) instead of the original system (1.1). The characterizations of C-controllability for system (1.1) are given in the following theorem.

THEOREM 4.11. *Consider the system* (1.1) *and assume that it is already in the strangeness-free form* (3.10)*. Let $\mathcal{R}_{ext}$ be the reachable set of the minimal extension form* (3.9)*. Let $E_0 = \mathrm{diag}(0_{r_2}, I_d)$. Then the following assertions are equivalent.*
i) System (1.1) *is C-controllable.*
ii) System (1.1) *is R-controllable and* $\mathrm{Im} E_0 \subseteq \mathcal{R}_{ext}$.
iii) System (1.1) *is R-controllable and* $\mathrm{rank}\begin{bmatrix} M, & D, & B\end{bmatrix} = d$.

Proof. Notice that in system (3.9) $\xi_n = \left[\begin{smallmatrix} z_n \\ x_n \end{smallmatrix}\right] \in \mathbb{R}^{r_2+d}$, so the equivalence between i) and ii) is straightforward. From the definition of C-controllability and the fact that system (1.1) is square, we have $r_0 = v_0 = 0$. Corollary 3.6, therefore, implies that $\mathrm{rank}\begin{bmatrix} M, & D, & B\end{bmatrix} = d$. Hence, we have proved that $i) \Rightarrow iii)$. Now we prove that $iii) \Rightarrow ii)$.
Due to Corollary 3.6, we see that $r_0 = v_0 = 0$, and hence the 3rd and 6th rows are not present in the form (3.9). Applying Theorem 3.8 i), in analogous to the sufficiency part, we see that there exist two matrices $F_d,\ F_v$ such that the matrix $\begin{bmatrix} M_1^T & D_2^T & K_3^T & \widetilde{D}_4^T & \widetilde{K}_5^T \end{bmatrix}^T$ has full row rank, where

$$\widetilde{D}_4 := D_4 + \begin{bmatrix} 0 & \Sigma_1 & B_{43}\end{bmatrix} F_v,\ \widetilde{K}_5 := K_5 + \begin{bmatrix} 0 & 0 & \Sigma_0\end{bmatrix} F_d\ .$$

Consequently, by introducing a new input function $w = \{w(n)\}$ such that

$$u(n) = -F_v x(n+1)(t) - F_d x(n) + w(n) \quad \text{for all } n \geqslant n_0,$$

we can transform the minimal extension form (3.9) to the closed loop system

$$\left[\begin{array}{cc} I_{r_2} & D_1 \\ 0 & M_1 \\ 0 & D_2 \\ 0 & \widetilde{D}_4 \\ \hline 0 & 0 \end{array}\right] \xi(n+1) + \left[\begin{array}{cc} 0 & K_1 \\ -I_{r_2} & 0 \\ 0 & K_2 \\ 0 & K_4 \\ \hline 0 & \widetilde{K}_5 \end{array}\right] \xi(n) = \left[\begin{array}{ccc} B_{11} & B_{12} & B_{13} \\ 0 & 0 & 0 \\ 0 & 0 & B_{23} \\ 0 & \Sigma_1 & B_{43} \\ \hline 0 & 0 & \Sigma_0 \end{array}\right] w(n), \quad \begin{array}{c} r_2 \\ r_2 \\ r_1 \\ \varphi_1 \\ \varphi_0 \end{array}\ . \tag{4.9}$$

{eq4.3}

Notice that, since $w(n)$ can be freely chosen like $u(n)$, we neither change the R-controllability or change the reachable set $\mathcal{R}$ of system (1.1). Since the matrix $\begin{bmatrix} M_1^T & D_2^T & \widetilde{D}_4^T & \widetilde{K}_5^T \end{bmatrix}^T$ has full row rank, the matrix

$$\left[\begin{array}{cc} I_{r_2} & D_1 \\ 0 & M_1 \\ 0 & D_2 \\ 0 & \widetilde{D}_4 \\ \hline 0 & \widetilde{K}_5 \end{array}\right]$$

also has full row rank, and hence, system (4.9) is regular and strangeness-free. Corollary 2.1 applied to system (4.9) implies that the reachable subspace of (4.9) is $\mathcal{R}_{ext} = \mathbb{R}^{r_2+d}$ and hence, $\mathrm{Im} E_0 \subseteq \mathcal{R}_{ext}$. This completes the proof. □

By following [11], we can determine the reachable set $\mathcal{R}$ of system (4.9) based on the Kronecker-Weierstraßcanonical form of (1.1)

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \tilde{\xi}(n+1) = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{\varphi_0} \end{bmatrix} \tilde{\xi}(n) + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} v(n), \tag{4.10}$$

{eq4.4}

where $n_1 = 2r_2 + r_1 + \varphi_1$. Now we are ready to discuss the R-controllability of the strangeness-free system (1.1).

THEOREM 4.12. *Consider the system* (1.1) *and assume that it is already in the strangeness-free form* (3.10)*. Let us also consider the system* (4.10)*. Then, system* (1.1) *is R-controllable if and only if for the corresponding first order system* (4.10) *the matrix product* $\begin{bmatrix} 0 & I_{n_1-r_2} \end{bmatrix} \mathcal{K}(\bar{A}_1, \bar{B}_1)$ *has full row rank, where*

$$\mathcal{K}(\bar{A}_1, \bar{B}_1) := \begin{bmatrix} \bar{B}_1, & \bar{A}_1\bar{B}_1, & \ldots, & \bar{A}_1^{n_1-1}\bar{B}_1 \end{bmatrix}, \tag{4.11}$$ {eq4.5}

Here the matrix $\begin{bmatrix} 0 & I_{n_1-r_2} \end{bmatrix} \in \mathbb{R}^{n_1-r_2,n_1}$.

Proof. From [11, Chap. 2] we see that the first order system (4.10) has the reachable set $\mathcal{R} = \mathbb{R}^{n_1} \oplus \mathrm{Im}(B_2)$, and (4.10) is R-controllable if and only if $\mathrm{Im}\mathcal{K}(\bar{A}_1, \bar{B}_1) = \mathbb{R}^{n_1}$. Furthermore, notice that the first r_2 variables of (4.9) come from the transformation of second order system (3.10) to the first order system (4.9) and are not relevant to consider for R-controllability. Therefore, the proof is straightly followed. ∎

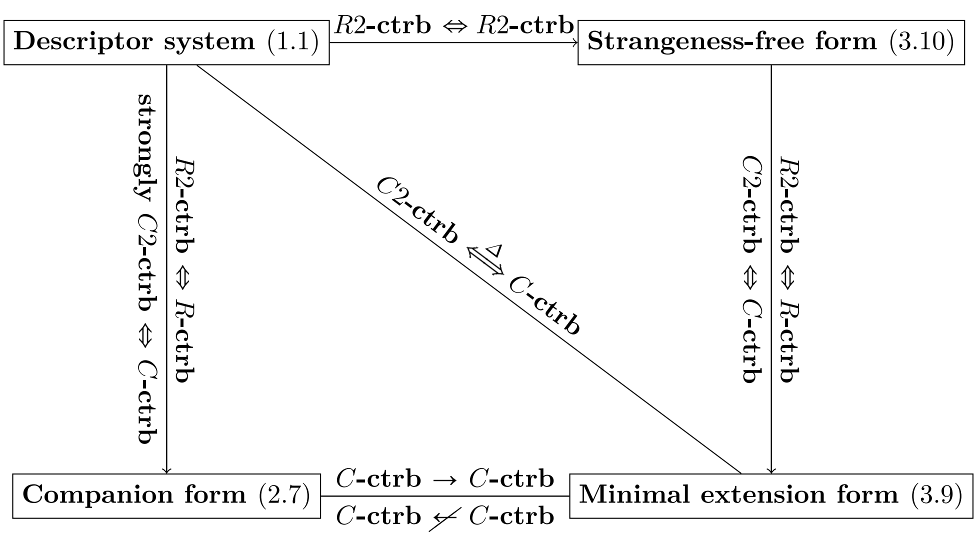

FIG. 4.1. *Controllability diagrams of system* (1.1) *and its reformulations*

5. Observability of second order descriptor systems. In this section we give a few result about the corresponding observability of system (1.1). For this let us denote by $\mathcal{P}_{r,2}$ the projection onto the right finite eigenspace corresponding to the finite eigenvalues of the matrix polynomial $\lambda^2 M + \lambda D + K$, [15]. First we recall three important concepts.

DEFINITION 5.1. *i) System* (1.1) *is called* C-observable *if from a response* $y = 0$ *for the input* $u = 0$ *it follows that system* (1.1) *has only one trivial solution* $x = 0$.
ii) It is called R-observable *if from a response* $y = 0$ *for the input* $u = 0$ *it follows that* $\mathcal{P}_{r,2}x = 0$.
iii) It is called causal observable (Y-observable) *if its state* $x(k)$ *at any time point* k *is uniquely determined by initial condition* $(x(0), x(1))$ *and the former (k included) inputs* $u(i)$*, together with former outputs* $y(i)$*,* $i = 0, \ldots, k$.

REMARK 5.2. *Due to linear property of system* (1.1)*, C-observability also means that for any unknown initial condition* $(x(0), x(1))$*, there exists a finite integer* $k > 0$*, such that the knowledge about former (k included) inputs* $u(i)$*, together with former output* $y(i)$*,* $i = 0, \ldots, k$ *suffices to determine uniquely the initial condition* $(x(0), x(1))$.

It is straightforward to see that all three observability concepts above are invariant under left equivalent transformation. On the other hand, since the index reduction procedure, which transforms system (1.1) to the form (3.10), does not alter the solution set of system (1.1), the C- and R-observability are preserved. Furthermore, due to Remark 3.10, the index reduction procedure has been performed only on the causal uncontrollable part, which implies that the Y-observability is also preserved. The following lemma plays the key role in our study about the observability of (1.1).

LEMMA 5.3. *Consider system* (1.1)*, the the strangeness-free from* (3.10) *and the minimal extension form* (3.9)*. Then, system* (3.10) *is Y-observable (resp., R-observable) if and only if system* (3.9) *is also Y-observable (resp., R-observable).*

Proof. Concerning about the Y-observability, the proof is straightforward, since the transformation from (3.10) to (3.9) keeps both the input and output, while the second block equation of (3.9) is nothing else than $z(n) = Mx(n+1)$, which does not have any impact on the causality of the system. About the R-observability, the proof is essentially the same as the proof of [25, Thm 4.3], so we will omit it to keep the brevity of this paper. □

Making use of Lemma 5.3, we see that the first order duality of controllability and observability [11, 13] can be directly extended to the second order case for system (1.1) and the dual system

$$\begin{aligned} &M^T x(n+2) + D^T x(n+1) + K^T x(n) = C^T u(n) \ \text{ for all } n \geqslant n_0,\\ &y(k) = Bx(k), \\ &x(n_0) = x_0,\ x(n_0+1) = x_1. \end{aligned} \tag{5.1}$$

{dual system}

THEOREM 5.4. *Consider the second order descriptor system* (1.1) *and the dual system* (5.1)*. Then the following assertions hold true.*
i) System (1.1) *is C-observable if and only if the dual system* (5.1) *is $C2$-controllable.*
ii) System (1.1) *is R-observable if and only if the dual system* (5.1) *is $R2$-controllable.*
iii) System (1.1) *is Y-observable if and only if the dual system* (5.1) *is Y-controllable via displacement-velocity feedback.*

Proof. Due to Lemma 5.3, the proof is directly obtained by checking rank conditions for the first order system (3.9), so it will be omitted to keep the brevity of this paper. □

Corollary 5.5. *Consider the second order descriptor system* (1.1)*. Then, it is*
i) R-observable if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda^2 M + \lambda D + K \\ C\end{bmatrix} = d;$$

ii) C-observable if and only if it is R-observable and the matrix coefficients in the strangeness-free form of the dual system (5.1) *satisfy*

$$\operatorname{rank}\begin{bmatrix}\hat{M}^T & \hat{D}^T & \hat{C}^T\end{bmatrix} = d; \tag{5.2}$$

{eq5.1}

iii) Y-observable if and only if

$$\operatorname{rank}\begin{bmatrix} M \\ T_\infty^1 D \\ T_\infty^2 K \\ C\end{bmatrix} = d,$$

where rows of T_∞^1 form a basis of $\operatorname{cokernel} M$*, and rows of T_∞^2 form the basis of*

$$\operatorname{cokernel}\begin{bmatrix} M \\ DZ_1\end{bmatrix} \setminus \operatorname{cokernel}\begin{bmatrix} M \\ DZ_1 \\ KZ_3\end{bmatrix},$$

and rows of Z_1 and of Z_3 form a basis of $\operatorname{kernel} M$ *and* $\operatorname{kernel}\begin{bmatrix} M \\ D\end{bmatrix}$*, respectively.*

In analogous to the controllability case, see Example 4.7, here we notice that the rank condition $\operatorname{rank}\begin{bmatrix}M^T & D^T & C^T\end{bmatrix} = d$ implies (5.2), but the converse is not true. Consequently, system (1.1) may not be Y-observable, even if $\operatorname{rank}\begin{bmatrix}M^T & D^T & C^T\end{bmatrix} = d$, as illustrated in the following example.

Example 5.6. *Consider the system* (1.1) *which reads*

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{M} x(n+2) + \underbrace{\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{D} x(n+1) + \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}}_{K} x(n) = \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{B} u(n), \tag{5.3}$$

$$y(n) = \underbrace{\begin{bmatrix}0 & 0 & 1\end{bmatrix}}_{C} x(n)\ .$$

{eq5.3}

Since the matrices M, D, K are symmetric and $C^T = B$, we see that the dual system of (5.3) *is nothing else than itself. As in Example 4.7, the strangeness-free formulation of this dual system reads*

$$\underbrace{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{M}^T} x(n+2) + \underbrace{\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{D}^T} x(n+1) + \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\hat{K}^T} x(n) = \underbrace{\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}}_{\hat{C}^T} u(n)\ . \tag{5.4}$$

{eq5.4}

Consequently, the dual system is not C-controllable (and hence not C2-controllable). Theorem 5.4 i) applied to system (5.3) *implies that this system is not C-observable, despite the fact that* $\operatorname{rank}\begin{bmatrix}M^T & D^T & C^T\end{bmatrix} = 3$*. This agrees with Corollary 5.5 ii), since* $\operatorname{rank}\begin{bmatrix}\hat{M}^T & \hat{D}^T & \hat{C}^T\end{bmatrix} = 1 < 3$*. Besides that, by direct computation, we see that system* (5.3) *is R-observable but not Y-observable.*

6. Conclusion and Outlook. In this paper we have presented the theoretical analysis for the controllability of linear, second order descriptor systems in discrete-time. We have modified an algebraic method proposed in [25, 28] to make it more convenient and reliable to apply, in order to study second order descriptor systems. We have given several necessary and sufficient conditions, which are numerically verifiable, in order to characterize all the fundamental controllability concepts for the considered systems. We have pointed out that C-controllable does not imply Y-controllable, and have also presented suitable feedback design strategy in order to eliminate the causal behavior of the considered systems. Future research includes the generalization of this approach to higher order descriptor systems, and also a comparable framework for the observability concepts.

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