

1 **EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR**
 2 **DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE**

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ABSTRACT. We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line \mathbb{R}_+ . Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform φ -Lipschitz condition, in which the Lipschitz function φ belongs to wide classes of admissible function spaces such as L_p , $1 \leq p \leq \infty$, $L_{p,q}$, etc.

4 1. INTRODUCTION AND PRELIMINARIES

5 The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear
 6 non-autonomous differential-algebraic equations (DAEs) of the form

$$\begin{array}{ll} d \text{ rows} & \left[\begin{array}{c} \mathbf{E}_1(t) \\ 0 \end{array} \right] \dot{x}(t) = \underbrace{\left[\begin{array}{c} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{array} \right]}_{A(t)} x(t) + \underbrace{\left[\begin{array}{c} f_1(t, x(t)) \\ f_2(t, x(t)) \end{array} \right]}_{f(t, x(t))}, & t \in \mathbb{R}_+ := [0, +\infty). \end{array} \quad \begin{matrix} \text{semi linDAE} \\ (1.1) \end{matrix}$$

7 To do that, we start by investigating the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \quad \begin{matrix} \text{linDAE} \\ (1.2) \end{matrix}$$

8 Here $E = \begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}$ are assumed to be matrix-valued functions acting on \mathbb{R}_+ to $\mathbb{R}^{n,n}$, $x : \mathbb{R}_+ \rightarrow$

9 \mathbb{R}^n , $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Furthermore, we assume that for all t , the matrices $\mathbf{E}_1(t)$, $\mathbf{A}_2(t)$ have full row rank.
 10 DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuits, chemical engineering, and many other applications. Due to the rank-deficiency of $E(t)$, the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the reader to recent monographs [2, 12–14] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [12, Chapter 2]), there are only few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [15] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [17] for Lyapunov and other spectra for linear DAEs, to [4, 7] for the robustness of exponential stability and Bohl exponents.

19 On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is
 20 characterized, the next important question in the qualitative theory of DAEs is to study the existence of
 21 integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1)
 22 [3, 5]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps,
 23 this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1).

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24 Our method is based on the classical "Lyapunov-Perron method" ([5, 25]) and the admissibility of function
 25 spaces ([10, 11]).

26 The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for
 27 later use, including the notion of the exponential dichotomy and its properties, as well as some important
 28 features of admissible function spaces. In Section 2 we give a characterization for the existence of expo-
 29 nential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of
 30 local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in
 31 Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations,
 32 and we conclude this research by a summary and some open problems.
 33

34 **1.1. Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By $(\mathbb{R}^n,$
 35 $\|\cdot\|)$ we denote the n -dimensional real vector space equipped with the Euclidean norm. For any matrix
 36 V , by V^T we denote its transpose. For any $p \in \mathbb{N}$, by $C^p([0, \infty), \mathbb{R}^n)$ we denote the space of p -times
 37 continuously differentiable functions acting on $[0, \infty)$ with values in \mathbb{R}^n . By $C_b([0, \infty), \mathbb{R}^n)$ we denote the
 38 space of continuous and bounded functions mapping from $[0, \infty)$ into \mathbb{R}^n . This space is a Banach space with
 39 the *ess sup*-norm $\|f\|_\infty := \sup\{\|f(t)\|, t \geq 0\}$.

It is well-known (e.g. [3]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \geq s \geq 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

40 is well-posed, then there exists a pointwise nonsingular matrix-valued function $(t, s) \mapsto X(t, s) \in \mathbb{R}^{n,n}$ such
 41 that the solution of (1.3) is given by $x(t) = X(t, s)x_s$. This fact motivates the existence of an evolution family
 42 $(X(t, s))_{t \geq s \geq 0}$ associated with the matrix function $A(t)$. This family satisfies the condition $X(t, t) = Id$ and
 43 the so-called *semi-group property*

$$X(t, r)X(r, s) = X(t, s), \quad \text{for all } t \geq r \geq s \geq 0. \tag{1.4}$$

44 Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), \quad \text{for all } t \geq s \geq 0,$$

45 also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \tau)f(\tau, x(\tau))d\tau, \quad \text{for all } t \geq s \geq 0. \tag{1.5}$$

46 For more details on the notion and discussion on properties and applications of evolution families we refer
 47 the readers to Pazy [21].

48 **Definition 1.1.** A given evolution family $\{X(t, s)\}_{t \geq s \geq 0}$ of the ODE (1.3) is said to have an *exponential*
 49 *dichotomy* on the half-line if there exist a family of projection matrices $\{P(t)\}_{t \geq 0}$ and two positive constants
 50 N, ν such that the following conditions are satisfied.

- 51 i) $P(t)X(t, s) = X(t, s)P(s)$ for all $t \geq s \geq 0$,
 - 52 ii) for all $t \geq s \geq 0$, the restriction $X(t, s)| : \ker P(s) \rightarrow \ker P(t)$ is an isomorphism, and we denote its
 53 inverse by $X(s, t)|$,
 - 54 iii) $\|X(t, s)P(s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$, for all $t \geq s \geq 0$, $x \in \mathbb{R}^n$,
 - 55 iv) $\|X(t, s)|(I - P(s))x\| \leq Ne^{\nu(t-s)}\|(I - P(s))x\|$, for all $s \geq t \geq 0$, $x \in \mathbb{R}^n$.
- 56 Here $\{P(t)\}_{t \geq 0}$ (reps. N, ν) are called *dichotomy projections* (resp. *dichotomy constants*).

57 The concept exponential dichotomy means that the state space \mathbb{R}^n has been splitted into the (exponentially) stable subspace ($\text{Im}(P(t))$) and the (exponentially) unstable subspace ($\text{ker}(P(t))$).

59 **1.2. A short review of DAE solvability theory.**

60 **Definition 1.2.** (i) Consider the DAE (1.2). A matrix function $X \in C([0, \infty), \mathbb{R}^{n,k})$, $d \leq k \leq n$, is called a fundamental solution matrix of (1.2) if each of its columns is a solution to (1.2) and $\text{rank } X(t) = d$, for all $t \geq 0$.

63 (ii) A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively. A maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)(X(0) - Id) = 0. \quad \text{projected intial condition} \quad (1.6)$$

65 We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of full rank. Furthermore, each fundamental solution matrix has exactly d -linear independent columns, and a minimal fundamental solution matrix can be made maximal by adding $n - d$ zero columns. This is the major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been proposed and carefully discussed in [17]. We recall it below, and notice that this concept is equivalent to the one proposed by Lentini and März in [15] within the context of the matrix chains approach and tractability index.

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74 Therefore, throughout this paper, we will assume the following.

75 **Assumption 1.3.** Assume that the function pair (E, A) in the DAEs (1.1), (1.2) is *strangeness-free*, i.e.,

$$\text{rank} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} = n,$$

76 for all $t \geq 0$. Furthermore, we assume that $E \in C^1([0, \infty), \mathbb{R}^{n,n})$ and $A \in C^0([0, \infty), \mathbb{R}^{n,n})$.

77 **Definition 1.4.** The DAE

$$\tilde{E}(t)\dot{y}(t) = \tilde{A}(t)x(t) + \tilde{f}(t, y(t)) \quad \text{eq7} \quad (1.7)$$

78 is called *orthogonally equivalent* to the DAE (1.1) if there exist pointwise-orthogonal matrix-valued functions $U \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that after changing variable $x(t) = V(t)y(t)$, and scaling (1.2) with $U(t)$, we obtain exactly (1.19). In details, this means that the following identities hold true.

$$\tilde{E} = UEV, \quad \tilde{A} = UAV - UEV, \quad \tilde{f}(t, y(t)) = U(t)f(t, Vy(t)), \quad \text{for all } t \geq 0. \quad (1.8)$$

81 We denote this orthogonal equivalence by $(E, A, f) \sim (\tilde{E}, \tilde{A}, \tilde{f})$ and omit the terms f, \tilde{f} if the homogeneous system (1.2) is considered.

83 Indeed, one can directly verify that this orthogonal equivalence concept is an equivalent relation, i.e., it fulfills three properties: reflexivity, symmetry and transitivity. We omit the detailed proof here in order to keep the brevity of this work. By making use of some smooth factorizations, for example QR or SVD ([6] or [13, Thm 3.9]), we can decouple and then exploit the structure of the DAE (1.2) in the following lemma.

87 **Lemma 1.5.** Consider the DAE (1.2) and assume that it satisfies Assumption 1.3. Then, there exist 88 pointwise-orthogonal matrix-valued functions $U \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that by 89 changing variable $x(t) = V(t)y(t)$, and scaling (1.2) with $U(t)$, we obtain the so-called semi-explicit system

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{semi-explicit system} \quad (1.9)$$

90 with pointwise nonsingular matrix-valued functions $\Sigma(t) \in \mathbb{R}^{d,d}$ and $A_4(t) \in \mathbb{R}^{a,a}$.

91 *Proof.* Applying an SVD factorization for $\mathbf{E}_1(t)$ we can find pointwise-orthogonal matrix functions $U_1(t) \in$
92 $C^1([0, \infty), \mathbb{R}^{d,d})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ such that $U_1(t)\mathbf{E}_1(t)V(t) = [\Sigma(t) \ 0]$, where $\Sigma(t)$ is a continuous,
93 pointwise nonsingular function with values in $\mathbb{R}^{d,d}$. Changing the variable $x(t) = V(t)y(t)$ and scaling (1.2)
94 with $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$, we obtain a new system exactly of the form (1.9). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ A_3(t) & A_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} V(t),$$

95 then Assumption 1.3 yields that both Σ and A_4 are nonsingular. This completes the proof. \square

Let $\hat{A}_3 := -A_4^{-1}A_3$, $\hat{A}_1 := \Sigma^{-1}(A_1 - A_2A_4^{-1}A_3)$, we rewrite system (1.9) as

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t), \quad \text{eq10.1}$$

$$y_2(t) = \hat{A}_3(t)y_1(t). \quad \text{eq10.2}$$

96 Since $V(t)$ is orthogonal for all $t \geq 0$, we see that all important qualitative properties of $x(t)$, such as
97 boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function $y(t)$
98 without any difficulty. Clearly, we see that (1.10b) gives an *algebraic constraint* that the solution to (1.9)
99 must obey, while (1.10a) gives the dynamic of (1.9). For this reason, we call it *an underlying ODE* to (1.9).

100 Let $\{\hat{Y}_1(t, s)\}_{t \geq s \geq 0}$ be the evolution family associated with (1.10a), then we can define the corresponding
101 evolution families for two DAEs (1.9), (1.2) consecutively as follows.

$$\hat{Y}(t, s) := \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, s) & 0 \end{bmatrix}, \quad \hat{X}(t, s) := V(t)\hat{Y}(t, s)V^T(s), \quad \text{for all } t \geq s \geq 0. \quad \text{eq11}$$

103 Nevertheless, since $X(t, s)$ is not invertible, we will define the *reflexive generalized inverse matrix function*
104 as in [17] by

$$\hat{Y}^-(t, s) := \begin{bmatrix} \hat{Y}_1^{-1}(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t, s) & 0 \end{bmatrix}, \quad \hat{X}^-(t, s) := V(s)\hat{Y}^-(t, s)V^T(t), \quad \text{for all } t \geq s \geq 0. \quad \text{eq12}$$

Then, we can directly verify the semigroup properties, i.e.

$$\hat{X}(t, r) = \hat{X}(t, s)\hat{X}(s, r), \quad \text{for all } t \geq s \geq r \geq 0,$$

$$\hat{X}(t, s) = \hat{X}(t, 0)\hat{X}^-(s, 0), \quad \text{for all } t \geq s \geq 0.$$

105 Furthermore, Lemmas 1.6 and 1.7 below show that the family $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$ does not depend on the
106 choice of orthogonal transformations, and it plays the same role as the evolution family $\{X(t, s)\}_{t \geq s \geq 0}$, in
107 comparison to (1.5).

108 **Lemma 1.6.** *The families $\{X(t, s)\}_{t \geq s \geq 0}$, $\{X^-(t, s)\}_{t \geq s \geq 0}$ defined by (1.11), (1.12) do not depend on the
109 choice of orthogonal transformations.*

Proof. We will prove this claim only for the first family $\{X(t, s)\}_{t \geq s \geq 0}$, since for the second family the proof
is essentially the same. Let us assume that we have two semi-explicit forms of system (1.2) obtained under
orthogonal transformations, i.e.,

$$(E, A) \sim \left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \quad \text{eq13.1}$$

$$(E, A) \sim \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right). \quad \text{eq13.2}$$

Now we will prove that the two corresponding systems have the same evolution family $\{X(t, s)\}_{t \geq s \geq 0}$. Without loss of generality, we assume that (E, A) is already in the form of the right hand side of (1.13a). The corresponding system to the right hand side of (1.13b) reads

$$\dot{\tilde{y}}_1(t) = \hat{\mathcal{A}}_1(t)\tilde{y}_1(t), \quad (\text{eq17.1})$$

$$\dot{\tilde{y}}_2(t) = \hat{\mathcal{A}}_3(t)\tilde{y}_1(t). \quad (\text{eq17.2})$$

where $\hat{\mathcal{A}}_3 := -\tilde{A}_4^{-1}\tilde{A}_3$, $\hat{\mathcal{A}}_1 := \tilde{\Sigma}^{-1}(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3)$.

The transitivity applied to (1.13) implies that there exist pointwise-orthogonal matrix-valued functions $S = [S_1 \ S_2 \ S_3 \ S_4] \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $T = [T_1 \ T_2 \ T_3 \ T_4] \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that $y(t) = T(t)\tilde{y}(t)$ and

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad (1.15)$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T}_1 & \dot{T}_2 \\ \dot{T}_3 & \dot{T}_4 \end{bmatrix} \right). \quad (1.16)$$

Let $\{\hat{\mathcal{Y}}_1(t, s)\}_{t \geq s \geq 0}$ be the evolution family associated with (1.14a), then the evolution family associated with system (1.14) is

$$\hat{\mathcal{Y}}(t, s) = \begin{bmatrix} \hat{\mathcal{Y}}_1(t, s) & 0 \\ \hat{\mathcal{A}}_3(t)\hat{\mathcal{Y}}_1(t, s) & 0 \end{bmatrix}, \quad \hat{\mathcal{X}}(t, s) := T(t)\hat{\mathcal{Y}}(t, s)T^T(s) \text{ for all } t \geq s \geq 0. \quad (\text{eq18})$$

Thus, we need to prove that $\hat{\mathcal{X}}(t, s) = \hat{X}(t, s)$.

From (1.15), it implies that $S_3\Sigma[T_1 \ T_2] = 0$. Thus, we have

$$[S_3 \ 0] \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = [0 \ 0],$$

and hence, this follows that $S_3 = 0$. Thus, $S = [S_1 \ S_2 \ 0 \ S_4]$, and hence, due to the orthogonality of S , we see that both S_1 is nonsingular and S_4 is also orthogonal. Also from (1.15), we see that $S_1\Sigma T_2 = 0$, and hence $T_2 = 0$. Consequently, by inserting $S_3 = 0$ and $T_2 = 0$ into (1.15) and (1.16) we obtain

$$\tilde{\Sigma} = S_1\Sigma T_1, \quad (1.18a)$$

$$\tilde{A}_1 = [S_1 \ S_2] \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} - \begin{bmatrix} \Sigma \dot{T}_1 \\ 0 \end{bmatrix} \right), \quad (1.18b)$$

$$\tilde{A}_2 = [S_1 \ S_2] \begin{bmatrix} A_2 \\ A_4 \end{bmatrix} T_4, \quad (1.18c)$$

$$\tilde{A}_3 = S_4 [A_3 \ A_4] \begin{bmatrix} T_1 \\ T_3 \end{bmatrix}, \quad (1.18d)$$

$$\tilde{A}_4 = S_4 A_4 T_4. \quad (1.18e)$$

Furthermore, making use of (1.18) we see that

$$\hat{\mathcal{A}}_3 = -\tilde{A}_4^{-1}\tilde{A}_3 = -(S_4 A_4 T_4)^{-1} S_4 [A_3 \ A_4] \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} = -T_4^{-1} [-\hat{A}_3 \ I] \begin{bmatrix} T_1 \\ T_3 \end{bmatrix},$$

and

$$\begin{aligned}
\tilde{\Sigma}\hat{A}_1 &= \tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3 \\
&= [S_1 \ S_2] \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} - \begin{bmatrix} \Sigma \dot{T}_1 \\ 0 \end{bmatrix} \right) - [S_1 \ S_2] \begin{bmatrix} A_2 \\ A_4 \end{bmatrix} T_4 (S_4 A_4 T_4)^{-1} S_4 [A_3 \ A_4] \begin{bmatrix} T_1 \\ T_3 \end{bmatrix} \\
&= S_1 \left((A_1 + A_2 A_4^{-1} A_3) T_1 - \Sigma \dot{T}_1 \right) \\
&= S_1 \Sigma \left(\hat{A}_1 T_1 - \dot{T}_1 \right).
\end{aligned}$$

¹¹⁸ Hence, we have

$$\hat{A}_1 = (S_1 \Sigma T_1)^{-1} S_1 \Sigma \left(\hat{A}_1 T_1 - \dot{T}_1 \right) = T_1^{-1} \left(\hat{A}_1 T_1 - \dot{T}_1 \right).$$

¹¹⁹ Therefore, the underlying ODE (1.14a) is directly obtained from (1.10a) by the variable transformation
¹²⁰ $\tilde{y}_1(t) = T_1(t)y_1(t)$. \square

Lemma 1.7. Consider the DAE (1.1) and the evolution family $(X(t, s))_{t \geq s \geq 0}$ defined by (1.11). Furthermore, we also consider the pointwise-orthogonal matrix-valued functions U, V defined in Lemma 1.7. Then, the solution to (1.1), if exists, also satisfies the so-called mild equation

$$\begin{aligned}
\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \hat{X}(t, s) \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t, \tau) \begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau)f(\tau, x(\tau))d\tau \\
&\quad + \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t)),
\end{aligned}$$

¹²¹ for all $t \geq s \geq 0$.

¹²² *Proof.* The proof can be obtained directly by using Lemma 1.5. Thus, in order to keep the brevity we will
¹²³ omit the details here. \square

¹²⁴ In the following, for ease of notation, we will use the abbreviation $\hat{X}(t) := \hat{X}(t, 0)$, $\hat{X}^-(t) := \hat{X}^-(t, 0)$,
¹²⁵ $\hat{Y}(t) := \hat{Y}(t, 0)$ and $\hat{Y}^-(t) := \hat{Y}^-(t, 0)$. The concept of exponential dichotomy for the DAE (1.9) is given as
¹²⁶ below.

Definition 1.8. ([17]) The DAE (1.9) is said to have an *exponential dichotomy* if there exist a family of projection matrices $\{P_y(t)\}_{t \geq 0}$ in $\mathbb{R}^{d,d}$ and positive constants N, ν such that

$$\begin{aligned}
\left\| \hat{Y}(t) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0, \\
\left\| \hat{Y}(t) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0,
\end{aligned} \tag{1.19}$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.11)-(1.19) we see that

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0.$$

and

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0.$$

¹²⁷ In addition, since $V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$ is also a projection matrix for any $t \geq 0$, we can interpret the
¹²⁸ exponential dichotomy of (1.2) as the one of (1.9). eq7

129 **1.3. Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces
 130 that play a fundamental role in the study of differential equations and refer to Nguyen [10], Massera and
 131 Schäffer [18, Chap. 2] and Räbiger and Schnaubelt [22, §1] for various applications.

132 Let E (endowed with the norm $\|\cdot\|_E$) be Banach function space of real-valued functions defined as in [10].
 133 We then recall the Banach space corresponding to the space E as follows.

134 **Definition 1.9** ([10]). Consider the Banach space $(\mathbb{R}^n, \|\cdot\|)$. For a Banach function space E we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

135 endowed with the norm $\|f\|_{\mathcal{E}} := \|f(\cdot)\|_E$. Thus, one can directly see that $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space. We
 136 call it *the Banach space corresponding to E* .

137 We now introduce the notion of admissibility in the following definition.

138 **Definition 1.10** ([10]). The Banach function space E is called *admissible* if for any $\varphi \in E$ the following
 139 conditions hold.

140 (i) There exists a constant $M \geq 1$ such that for every compact interval $[a, b] \subset \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \quad (1.20)$$

141 where $\chi_{[a,b]}$ is the indicator function of $[a, b]$.

142 (ii) The function $\Lambda_1 \varphi$ defined by $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$ belongs to E .

143 (iii) For any $\tau \geq 0$, the space E is T_τ^+ -invariant and T_τ^- -invariant, where T_τ^+ and T_τ^- are defined as

$$T_\tau^+ \varphi(t) := \begin{cases} \varphi(t - \tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \quad (1.21)$$

$$T_\tau^- \varphi(t) := \varphi(t + \tau) \text{ for } t \geq 0.$$

144 Furthermore, there exist constants N_1, N_2 such that $\|T_\tau^+\|_E \leq N_1, \|T_\tau^-\|_E \leq N_2$ for all $\tau \in \mathbb{R}_+$.

145 **Example 1.11.** Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau < \infty\},$$

146 (for any fixed $\alpha > 0$), endowed with the norm $\|h\|_{\mathbf{M}_\alpha} := \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau$, many other function spaces
 147 occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty, 1 \leq q < \infty$ (see [3], [24]) and,
 148 more general, the class of rearrangement invariant function spaces (see [16]) are admissible.

149 *Remark 1.12.* Following directly from Definition 1.10 we have that

$$\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau \leq \frac{M}{\inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

150 and hence, $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$. Furthermore, $C_b(\mathbb{R}^+)$ is dense in \mathbf{M}_1 .

151 We present here some important features of admissible spaces in the following proposition (see [10, Propo-
 152 sition 2.6] and originally in [18, 23.V.(1)]).

Proposition 1.13 ([10]). Let E be an admissible Banach function space. Then the following assertions hold.

a) Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1 \varphi \in E$, where, Λ_1 is defined as in definition 1.10 (ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ by

$$\begin{aligned}\Lambda'_\sigma \varphi(t) &:= \int_0^t e^{-\sigma(t-s)} \varphi(s) ds, \\ \Lambda''_\sigma \varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds.\end{aligned}$$

153 Then, $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ belong to E . In particular, if $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see
154 remark 1.12)) then $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for *ess sup*-norm, we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty \quad (1.22)$$

155 for operator T_1^+ and constants N_1, N_2 defined as in Definition 1.10.

156 b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for any constant $\alpha > 0$.

157 c) E does not contain exponentially growing functions $f(t) := e^{bt}$ for any constant $b > 0$.

158 2. EXPONENTIAL DICHOTOMY FOR LINEAR DAEs

159 In the qualitative analysis of ODEs, one of the central topic is to find sufficient and necessary conditions
160 for the considered systems to admit exponential dichotomy. Many researches have been devoted to this
161 topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces (e.g.
162 [5, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

163 **Proposition 2.1.** ([17]) The DAE (1.9) has an exponential dichotomy if and only if the corresponding un-
164 derlying ODE (1.10a) also has exponential dichotomy, and the matrix function $\hat{A}_3(t)$ is bounded. Moreover,
165 the existence of exponential dichotomy implies that $\sup_{t \geq 0} \|P_y(t)\| < \infty$.

166 Together with (1.2), let us consider the following system

$$E(t) \dot{x}(t) = A(t)x(t) + g(t). \quad (2.1)$$

167 Notice that, even for ODEs, Proposition 2.1 is only valid for finite-dimensional systems but not for infinite
168 dimensional systems, [5]. For this reason, we recall a classical result by Perron below.

169 **Proposition 2.2.** ([5]) The ODE (1.3) has an exponential dichotomy if and only if for any continuous,
170 bounded function $g(t)$ on $[0, \infty)$, there exists a continuous, bounded solution $x(t)$ to the system

$$\dot{x}(t) = A(t)x(t) + g(t). \quad (2.2)$$

171 In view of Proposition 2.2, comparable result has not been achieved for DAEs, and hence, this will be our
172 main aim in this section.

173 **Definition 2.3.** Consider the matrix functions E, A in system (1.2). Then, a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}^n$
174 satisfying the condition

$$\sup_{t \geq 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} g(t) \right\| \right\} < +\infty,$$

175 is called (E, A) -bounded. We denote the set of all continuous and (E, A) -bounded functions by $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$.

176 The main result of this section is to prove a characterization of the exponential dichotomy for DAEs.

177 Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping $\mathcal{L} := E \frac{d}{dt} - A$
178 is surjective on the space $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$. We formulate our main result in this section as follows.

¹⁷⁹ **Theorem 2.4.** Consider the linear, strangeness-free DAE (1.2) and the associated inhomogeneous DAE
¹⁸⁰ (2.1). Then the following assertions hold.

- ¹⁸¹ (i) If the DAE (1.2) admits exponential dichotomy then for any continuous, (E, A) -bounded function $g(t)$
¹⁸² on $[0, \infty)$, there exists a continuous, bounded solution $x(t)$ to the DAE (2.1).
- ¹⁸³ (ii) If the matrix function $\hat{A}_3(t)$ is bounded, then the converse of assertion (i) holds true.

Proof. Firstly, we notice that, since $\hat{g} = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t)g(t)$, the (E, A) -boundedness of f is equivalent to the boundedness of \hat{f} . Recall that the semi-explicit system (1.9) reads

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{g}_1(t), \quad \text{eq3.10a} \quad (2.3)$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{g}_2(t). \quad \text{eq3.10b} \quad (2.4)$$

¹⁸⁴ (i) Assuming that the DAE (1.2) admits exponential dichotomy, then (1.9) also has an exponential di-
¹⁸⁵ chotomy. Proposition 2.1 implies that equation (2.3) has an exponential dichotomy, and the function \hat{A}_3 is
¹⁸⁶ bounded. Therefore, Proposition 2.2 implies that y_1 is bounded, and consequently, y_2 is also bounded.

¹⁸⁷ (ii) From Proposition 2.2, it follows that (2.3) has exponential dichotomy **May be wrong, since the**
¹⁸⁸ **domains of g and hg are not the same.** On the other hand, the boundedness of \hat{A}_3 implies that (1.2)
¹⁸⁹ admits exponential dichotomy. \square

190 3. LOCAL STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

¹⁹¹ In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout
¹⁹² this section we assume that the evolution family $(X(t, s))_{t \geq s \geq 0}$ associated with the linear, homogeneous DAE
¹⁹³ (1.2) admits an exponential dichotomy on \mathbb{R}_+ .

From Lemma 1.5, by using orthogonal transformation $x(t) = V(t)y(t)$, where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$ we can transform (1.1) to the coupled system

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), \quad \text{eq4.1a} \quad (3.1)$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), \quad \text{eq4.1b} \quad (3.2)$$

¹⁹⁴ where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) \begin{bmatrix} f_1(t, y(t)) \\ f_2(t, y(t)) \end{bmatrix}. \quad \text{eq4.2}$$

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t))$$

¹⁹⁵ Notice that, unlike the DAEs (1.2) and (2.1), equation (3.2) only gives an implicit algebraic constraint
¹⁹⁶ in terms of y_1 and y_2 . In order to guarantee the strangeness-free of system (1.1), we need the following
¹⁹⁷ assumption.

¹⁹⁸ **Assumption 3.1.** Assume that for some $\rho > 0$, the function $A_4^{-1}(t)f_2(t, x)$ is a contraction mapping in the
¹⁹⁹ ball $B_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$ (uniformly in time), i.e.,

$$\|A_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

²⁰⁰ for a.e. $t \in \mathbb{R}_+$, and for all $x, \tilde{x} \in B_\rho$ where the Lipschitz constant L satisfies that $L < 1$.

²⁰¹ **Lemma 3.2.** Under Assumption 3.1 and given $y_1 \in B_\rho$, there exists a unique function $y_2 \in \mathcal{B}_\rho$ satisfying
²⁰² (3.2).

203 *Proof.* Firstly, notice that Assumption 3.1 implies that $\hat{f}_2(t, y)$ is also Lipschitz in y with the same constant
 204 L . Then, the desired claim is obtained directly by making use of [19, Lem. 2.7]. \square

205 *Remark 3.3.* Lemma 3.2 leads to one important fact, that under Assumption 3.1, the coupled system (3.1)-
 206 (3.2) is still strangeness-free, as defined in [13, Chap. 4]. Therefore, in analogue to the linear case, (3.2) is
 207 called *an algebraic constraint*, whereas (3.1) is called *an underlying ODE*.

208 To obtain the existence of a stable manifold we need the following property of the nonlinear part f_1
 209 defined as follows.

210 **Definition 3.4.** Let φ be a positive function belonging to an admissible Banach function space E . A
 211 function $h : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to belong to the class (M, φ, ρ) for some positive constant M , ρ if h
 212 satisfies

- 213 (i) $\|h(t, x)\| \leq M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $x \in B_\rho$,
 214 (ii) $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$, for all $x, \tilde{x} \in B_\rho$.

215 **Assumption 3.5.** Assume that the function $t \mapsto \Sigma^{-1}(t) [I_d \ -A_2(t)A_4^{-1}(t)] f(t, x(t))$ belongs to class
 216 (M, φ, ρ) for some positive constants M, ρ and a positive function $\varphi \in E$.

217 The following proposition gives one sufficient condition for examining Assumptions 3.1, 3.5.

218 **Proposition 3.6.** Consider the semi-linear DAE (1.1). Furthermore, assume that all three functions Σ^{-1} ,
 219 A_4^{-1} , $\Sigma^{-1}A_2A_4^{-1}$ are bounded. If the function $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ belongs to the class (M, φ, ρ) then the following
 220 claims hold true.

- 221 i) \hat{f}_1 belongs to the class (M, φ, ρ) , and
 222 ii) f_2 is Lipschitz with the Lipschitz constant $\varphi \sup_{t \geq 0} \|A_4^{-1}\|$.

223 We notice that a sufficient condition for Assumption (3.1) is that

$$\|f_2(t, x) - f_2(t, \tilde{x})\| \leq \frac{L}{\|A_4^{-1}(t)\|} \|x - \tilde{x}\|. \quad \text{Lipschitz (3.4)}$$

224 For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.1)-
 225 (3.2). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad \text{mild equation (3.5)}$$

226 for all $t \geq s \geq 0$.

227 **Lemma 3.7.** Let Assumptions 3.1 and 3.5 hold true. Then, for all $y, \tilde{y} \in B_\rho$ the following assertions hold.

- 228 (i) $\|\hat{f}_1(t, y)\| \leq M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$,
 229 (ii) $\|\hat{f}_1(t, y) - \hat{f}_1(t, \tilde{y})\| \leq \varphi(t)\|y - \tilde{y}\|$ for a.e. $t \in \mathbb{R}_+$,
 230 (iii) $\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| \leq L\|y - \tilde{y}\|$ for a.e. $t \in \mathbb{R}_+$.

231 *Proof.* The proof is trivially followed from Assumptions 3.1 and 3.5 due to the fact that $\|y\| = \|Qy\|$ for any
 232 orthogonal matrix V . \square

233 Let $(\hat{Y}(t, s))_{t \geq s \geq 0}$ has an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$
 234 and the dichotomy constants $N, \nu > 0$ as in Definition 1.8. Furthermore, as in Proposition 2.1, let us denote
 235 by $H_1 := \sup_{t \geq 0} \|\hat{A}_3(t)\|$ and $H_2 := \sup_{t \geq 0} \|P_y(t)\|$. Then, we can define the Green function on the half-line as

²³⁶ follows

$$G(t, \tau) := \begin{cases} \hat{Y}(t, \tau) \begin{bmatrix} P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)P_y(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, \tau)P_y(\tau) & 0 \end{bmatrix}, & \text{for all } t \geq \tau \geq 0, \\ -\hat{Y}(t, \tau) \begin{bmatrix} I_d - P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \end{bmatrix}, & \text{for all } 0 \leq t < \tau. \end{cases} \quad (3.6)$$

²³⁷ Then, we have

$$\|G(t, \tau)\| \leq (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geq 0. \quad (3.7)$$

²³⁸ In the following lemma, we give an explicit form for bounded solutions to system (3.5).

²³⁹ **Lemma 3.8.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumptions 3.1, 3.5 hold true. Let $y(t)$ be any solution to (3.5) such that $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ for fixed $t_0 \geq 0$ and some $\rho > 0$. Then, for $t \geq t_0 \geq 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (3.8)$$

²⁴³ for some $v_0 \in \text{Im}P_y(t_0)$, where $G(t, \tau)$ is the Green function defined by (3.6).

Proof. Put

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}.$$

²⁴⁴ By direct computation, we can verify that z satisfies the integral equation

$$z(t) = \hat{Y}(t, t_0) \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix},$$

²⁴⁵ for all $t \geq t_0$. Now let us estimate $\|z(t)\|$. Making use of Lemma 3.7 and (3.7), we see that

$$\|z(t)\| \leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} M\varphi(\tau) d\tau + L\rho,$$

²⁴⁶ and then, from (1.22) it follows that

$$\|z(t)\| \leq M (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_{\infty} + \|\Lambda_1 \varphi\|_{\infty}) + L\rho,$$

²⁴⁷ for all $t \geq t_0$. Thus, $z(t) - y(t)$ is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t, t_0) (z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t)\hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

²⁴⁸ we see that $v_0 := z_1(t_0) - y_1(t_0) \in \text{Im}P_y(t_0)$. Finally, since $z(t) = y(t) + \hat{Y}(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$ for all $t \geq t_0$, equality

²⁴⁹ (3.8) follows. \square

²⁵⁰ *Remark 3.9.* By computing directly, we can see that the converse of Lemma 3.8 is also true. It means, that all solutions to (3.8) also satisfy equation (3.5) for all $t \geq t_0$.

²⁵² Let us denote by

$$H_3 := (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_{\infty} + \|\Lambda_1 \varphi\|_{\infty}) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)} \rho. \quad (3.9)$$

²⁵³ **Lemma 3.10.** *Under the assumptions of Lemma 3.8, let $y(t)$, $\tilde{y}(t)$ be any two functions lying in the ball
²⁵⁴ B_ρ and satisfy (3.8) for $v_0, \tilde{v}_0 \in \text{Im}P_y(t_0)$. If H_3 defined as in (3.9) satisfies $H_3 + L < 1$ then the following
²⁵⁵ estimate holds true:*

$$\|y - \tilde{y}\|_\infty \leq \frac{N}{1 - H_3 - L} \|v_0 - \tilde{v}_0\|. \quad (\text{eq4.9})$$

Proof. Using the same arguments as in the proof of Lemma 3.7, we see that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq N\|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (H_3 + L) \|y - \tilde{y}\|_\infty, \end{aligned}$$

²⁵⁶ which directly implies (3.10). \square

²⁵⁷ In the following theorem, we exploit the local structure of bounded solutions to (3.5).

²⁵⁸ **Theorem 3.11.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy
²⁵⁹ with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore,
²⁶⁰ assume that Assumptions 3.1, 3.5 hold true, and constant H_3 defined as in (3.9). Then, the following
²⁶¹ assertions hold true.*

²⁶² (i) *If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)\rho}{2M} \right\}, \quad (\text{eq4.7})$$

²⁶³ *then there corresponds to each $v_0 \in B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$ one and only one solution $y(t)$ to (3.5) on $[t_0, \infty)$ satisfying
²⁶⁴ $P_y(t_0)y_1(t_0) = v_0$ and $\text{esssup}_{t \geq t_0} \|y(t)\| \leq \rho$.*
²⁶⁵ (ii) *Moreover, any two solutions $y(t), \tilde{y}(t)$ corresponding to different v_0, \tilde{v}_0 in $B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$ attract each
²⁶⁶ other exponentially, i.e.,*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0, \quad (\text{eq4.8})$$

²⁶⁷ *for some positive constants H_4, μ .*

²⁶⁸ *Proof.* (i) Consider in the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ the ball $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) : \|y(\cdot)\|_\infty := \text{esssup}_{t \geq 0} \|y(t)\| \leq \rho\}$.

²⁶⁹ For each fixed $v_0 \in B_{\tilde{\rho}}$ we will prove the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix} & \text{for all } t \geq t_0, \\ 0 & \text{for all } t < t_0, \end{cases} \quad (3.13)$$

²⁷⁰ is a contraction mapping from \mathcal{B}_ρ to itself. Using the same argument as in the proof of Lemma 3.7, we see
²⁷¹ that

$$\begin{aligned} \|(Ty)(t)\| &\leq (1 + H_1)Ne^{-\nu(t-t_0)} \|v_0\| + M(1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho, \\ &\leq (1 + H_1)N\|v_0\| + MH_3 + L\rho \quad \text{for all } t \geq 0, \end{aligned}$$

²⁷² and by (3.11) we see that

$$\|(Ty)(t)\| \leq (1 + H_1)N\tilde{\rho} + \frac{(1 - L)\rho}{2} + L\rho = \rho \quad \text{for all } t \geq 0.$$

Therefore, T is a mapping from \mathcal{B}_ρ to itself. Now we prove its contraction property. Indeed, making use of (3.7), we obtain the following estimate:

$$\begin{aligned} \|Ty(t) - T\tilde{y}(t)\| &\leq \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq (H_3 + L) \|y(\cdot) - \tilde{y}(\cdot)\|_\infty \text{ for all } t \geq 0. \end{aligned}$$

Consequently, due to (3.11), we see that T is a contraction mapping with the contraction constant $H_3 + L$. Thus, there exist a unique function $y \in \mathcal{B}_\rho$ such that $y = Ty$, and hence, due to the definition of T , y is the solution to the mild/integral-algebraic system (3.5).

(ii) The proof of the estimate (3.12) can be done in a similar way as in [11, Thm 3.7]. We present here for seek of completeness. Let $y(t)$ and $\tilde{y}(t)$ be two essentially bounded solutions of (3.5) corresponding to different values $v_0, \tilde{v}_0 \in B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$. Then, we have that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq Y(t, t_0) \|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq (1 + H_1) Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \end{aligned}$$

and hence,

$$\|y(t) - \tilde{y}(t)\| \leq \frac{1 + H_1}{1 - L} Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} \frac{(1 + H_1)(1 + H_2)}{1 - L} Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau.$$

Then, due to the Cone Inequality, [5, Theorem 1.9.3], in analogue to [20, Theorem 3.7], we obtain the estimation (3.12) with H_4, μ are given by

$$0 < \mu < \nu + \ln \left(1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu-\nu}}}.$$

Furthermore, notice that from (3.11) it follows that $\mu < \nu$ implying the positivity of H_4 . This completes the proof. \square

Under Assumption 3.1, we then define the so-called *constrained manifold*, which all solutions to (3.1)-(3.2) must belong to

$$\mathbb{L}(t, y) := \{(t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t, y_1, y_2)\}. \quad \text{constraint manifold} \quad (3.14)$$

We further notice that this manifold is of dimension d , which is the degree of freedom to the DAE (3.5). Now, we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic system (3.5).

Definition 3.12. A subset \mathbb{M} of the constrained manifold $\mathbb{L}(t, y)$ is said to be a *local stable manifold* for solutions to (3.5) if for every $t \in \mathbb{R}_+$ the phase subspace \mathbb{R}^d splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf \{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exist positive constants ρ, ρ_1, ρ_2 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t), \quad t \in \mathbb{R}_+,$$

with a common Lipschitz constant independent of t such that

- (i) $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$, and we denote by
 $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$,
(ii) \mathbb{M}_t is homeomorphic to $B_{\rho_1} \cap W_1(t)$ for all $t \geq 0$,
(iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and
 $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$.

We now state and prove our main result on the existence of a local stable manifold for DAEs.

Theorem 3.13. *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumptions 3.1, 3.5 hold true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)\rho}{2M}, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a local stable manifold for the solutions of (3.5). Moreover, every two solutions $y(t), \tilde{y}(t)$ on the manifold \mathbb{M} attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0 \geq 0$ such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\|, \quad \text{for all } t \geq t_0. \quad (\text{eq4.12})$$

Proof. First we notice that the phase subspace \mathbb{R}^d splits into the direct sum $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel}P_y(t)$ for all $t \geq 0$. We set $W_1(t) := \text{Im}P_y(t)$ and $W_2(t) := \text{kernel}P_y(t)$, then due to Proposition 2.1, we see that $\sup_{t \geq 0} \|P_y(t)\| < \infty$, and hence, $\inf_{t \in \mathbb{R}_+} S_n(W_1(t), W_2(t)) > 0$.

For any $\rho > 0$ defined as in Assumptions 3.1, 3.5, let $\rho_1 := \tilde{\rho} = \frac{1 - L}{2N(1 + H_1)}\rho$ and $\rho_2 := \frac{(1 - L)\rho}{2}$. For each $t \geq 0$ we define the mapping g_t acting on $B_{\rho_1} \cap W_1(t)$ as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is uniquely defined via Theorem 3.11 i). Clearly, $g_t(w_1) \in \text{ker}P_y(t) = W_2(t)$.

Now, we prove that $\|g_t(w_1)\| \leq \rho_2$. Due to Theorem 3.11 (i) and Lemma 3.7 (i), we have that $\|y(t)\| \leq \rho$ and $\|f_1(\tau, y(\tau))\| \leq M\varphi(\tau)$ for a.e. $t \geq 0$. Therefore,

$$\begin{aligned} \|g_t(w_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} M\varphi(\tau) d\tau, \\ &\leq M(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) = \frac{MH_3}{1 + H_1} \leq \frac{(1 - L)\rho}{2}, \end{aligned}$$

and hence, $g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t)$.

Notice that both part (iii) in Definition 3.12 and estimation (3.15) are followed directly from Theorem 3.11. We now only need to prove that \mathbb{M}_t is homeomorphic to $B_{\rho_1} \cap W_1(t)$. We first prove that g_t is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1 + H_1)(1 + H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

306 and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

307 Finally, $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a
308 contraction mapping for all $t \geq 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous
309 mappings ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \rightarrow B_{\rho_1} \cap W_1(t)$ is a homeomorphism.
310 This implies the condition (ii) of Definition 3.12 finishing the proof. \square

311 4. GLOBAL INVARIANT STABLE MANIFOLDS FOR SEMI-LINEAR DAES

312 In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1).
313 We begin with the concept of φ -Lipschitz functions.

314 **Definition 4.1.** Let E be an admissible Banach function space and $\varphi \in E$ be a positive function. A function
315 $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^d$ is said to be φ -Lipschitz if the following conditions hold true.

- 316 (i) $\|h(t, 0)\| = 0$ for a.e. $t \in \mathbb{R}_+$,
- 317 (ii) $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$ and all $x, \tilde{x} \in \mathbb{R}^n$.

318 In comparability to Assumptions 3.1, 3.5, we also need some global properties of the nonlinear term f .

319 **Assumption 4.2.** Assume that the following hypotheses hold true.

- 320 (i) The function $\Sigma^{-1}(t)f_1(t, x(t)) - \Sigma^{-1}(t)A_2(t)A_4^{-1}(t)f_2(t, x(t))$ is φ -Lipschitz.
- 321 (ii) The function $A_4^{-1}(t)f_2(t, x(t))$ is a contraction mapping with the Lipschitz constant $L < 1$ for all
322 $(t, x(t))$ lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid A_2(t)x + f_2(t, x) = 0\}.$$

323 We can directly verify that orthogonal transformations of the form $x = Vy$ preserves the φ -Lipschitz
324 property, and hence, function \hat{f}_1 in (3.1) is also φ -Lipschitz. Besides that, function \hat{f}_2 in (3.2) is also a
325 contraction mapping with the Lipschitz constant $L < 1$. For notational simplicity, now we will study the
326 transformed system (1.9) and the integral-algebraic system (3.5).

Definition 4.3. A subset \mathbb{M} of the constrained manifold $\mathbb{L}(t, y)$ is said to be a *global, invariant stable manifold* for solutions to (3.5) if for every $t \in \mathbb{R}_+$ the phase subspace \mathbb{R}^d splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \rightarrow W_2(t), \quad t \in \mathbb{R}_+,$$

325 with the Lipschitz constants independent of t such that

- 326 (i) $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$, and we denote by
327 $\mathbb{M}_t := \{(y_1, y_2) \mid (t, y_1, y_2) \in \mathbb{M}\}$,
- 328 (ii) \mathbb{M}_t is homeomorphic to $W_1(t)$ for all $t \geq 0$,
- 329 (iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and
330 $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$,
- 331 (iv) \mathbb{M} is invariant under system (3.5), i.e., if y is a solution to (3.5), and $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$, then
332 $y(s) \in \mathbb{M}_s$ for all $s \geq t_0$.

333 Analogously to Lemma 3.8, we give the explicit form of bounded solutions to system (3.5) as below.

³³⁴ **Lemma 4.4.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy with the
³³⁵ corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
³³⁶ that Assumption 4.2 holds true. Let $y(t)$ be any solution to (3.5) such that $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ for a fixed
³³⁷ $t_0 \geq 0$. Then, for all $t \geq t_0 \geq 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (4.1)$$

³³⁸ for some $v_0 \in \text{Im}P_y(t_0)$, where $G(t, \tau)$ is the Green function defined by (3.6).

³³⁹ *Proof.* The proof can be done by using similar arguments as in the proof of Lemma 3.2. \square

³⁴⁰ In the following two theorems, we present the global versions of Theorems 3.11 and 3.13, where we
³⁴¹ construct the structure of bounded solutions to (3.5) and prove the existence of a global, stable manifold,
³⁴² respectively.

³⁴³ **Theorem 4.5.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy with the
³⁴⁴ corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
³⁴⁵ that Assumption 4.2 holds true.*

- ³⁴⁶ (i) *For any fixed $t_0 \geq 0$, if $H_3 < 1 - L$ then there corresponds to each $v_0 \in \text{Im}P_y(t_0)$ one and only one
³⁴⁷ solution $y(t)$ to (3.5) on $[t_0, \infty)$ satisfying $P_y(t_0)y_1(t_0) = v_0$ and $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$.*
(ii) *Any two solutions $y(t), \tilde{y}(t)$ corresponding to different initial conditions v_0, \tilde{v}_0 in $\text{Im}P_y(t_0)$, are exponentially attracted to each other, i.e.,*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0,$$

with some positive constants H_4, μ satisfying

$$0 < \mu < \nu + \ln \left(1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu - \nu}} }.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 3.11. The only change is, that instead of considering the ball B_ρ we will work with the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ itself. Then, we can prove (without any difficulty) that for each fixed $v_0 \in \text{Im}P_y(t_0)$, the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geq t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

³⁴⁸ is a contraction mapping, and therefore, all the assertions of the theorem follows. \square

Theorem 4.6. *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.9) have an exponential dichotomy with the
corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
that Assumption 4.2 holds true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a global invariant stable manifold for the solutions of (3.5). Moreover, every two solutions $y(t), \tilde{y}(t)$ on the manifold \mathbb{M} attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0 \geq 0$ such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\| \quad \text{for all } t \geq t_0.$$

³⁴⁹ *Proof.* Analogous to the proof of Theorem 3.13, we consider the decomposition $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel } P_y(t)$
³⁵⁰ and set $W_1(t) := \text{Im}P_y(t)$ and $W_2(t) := \text{kernel } P_y(t)$. Thus, we see that $\inf_{t \in \mathbb{R}_+} S_n(W_1(t), W_2(t)) > 0$.
³⁵¹ Now we define the family of mappings $(g_t)_{t \geq 0}$ acting on W_1 as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is bounded and be uniquely defined via Theorem 4.5 i). Clearly, $g_t(w_1) \in \ker P_y(t) = W_2(t)$. To verify the Lipschitz property of g_t , let us consider two arbitrary elements w_1 and \tilde{w}_1 in W_1 and let y and \tilde{y} be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1 + H_1)(1 + H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

³⁵² and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1 + H_1)(1 + H_2)(1 - H_3 - L)} \|w_1 - \tilde{w}_1\|.$$

³⁵³ Finally, $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a
³⁵⁴ contraction mapping for all $t \geq 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous
³⁵⁵ mapping ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \rightarrow W_1(t)$ is a homeomorphism. This
³⁵⁶ implies the condition ii) of Definition 3.12, and hence, the proof is finished. \square

³⁵⁷ Now let us illustrate our results by the following examples.

³⁵⁸ **Example 4.7.** The dynamical behavior of a system in fluid mechanics and turbulence modeling is often
³⁵⁹ described by the incompressible Navier-Stokes equation on an open, bounded domain $\Omega \subset \mathbb{R}^k$, $k = 2$ or 3 ,
³⁶⁰ of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p - (u \cdot \nabla) u + f(t, u, p), \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned}$$

where $\nu > 0$ is the viscosity, $u = u(t, \xi)$ is the velocity field which is a function of the time t and the position ξ , p is the pressure, f is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [9], one obtains a differential-algebraic system of the following form.

$$\begin{aligned} M \dot{U} &= (K + N(U)) U - CP + F(t, U, P), \\ C^T U &= 0, \end{aligned}$$

where $U(t)$, $P(t)$ approximate the velocity $u(t, \xi)$ and the pressure $p(t, \xi)$, respectively. Here the leading matrix M is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix $C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$ is nonsingular. We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free,

so Assumption 1.3 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned} M\dot{U} &= -(K + N(U)) U - CP + F(t, U, P), \\ 0 &= C^T M^{-1} C P - C^T M^{-1} (F - (K + N(U)) U) . \end{aligned} \quad \text{eq5.3} \quad (4.2)$$

Clearly, we still need to linearize (4.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.9)

$$\begin{aligned} M\dot{U} &= A_1(t)U + A_2(t)P + g_1(t, U, P), \\ 0 &= C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right) P - C^T M^{-1} \left(\frac{\partial F}{\partial U} - K \right) U + C^T M^{-1} g_2(t, U, P) . \end{aligned} \quad \text{eq5.4} \quad (4.3)$$

- 361 We further notice that since $C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$ is nonsingular, from the second equation we can uniquely
 362 determine P in term of U , and hence, system (4.2) is indeed strangeness-free. Let

$$A_3(t) := -C^T M^{-1} \left(\frac{\partial F}{\partial U} - K \right), \quad A_4(t) := C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$$

- 363 Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

- 364 admits an exponential dichotomy, and g_1 satisfies the φ -Lipschitz condition, and g_2 is a contraction mapping
 365 (uniformly in time), then there exists a stable manifold for the solution to (4.2).

Example 4.8. Consider the nonlinear electrical circuit with Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation $i_2 = I_0 \sin(k\phi_2)$, where I_0 and k are positive constants depend on the device itself. Moreover, the resistance R , inductance L and conductance G are positive. Furthermore, i_1 is the current going through the inductance, v_1 and v_2 are voltage drops across the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead

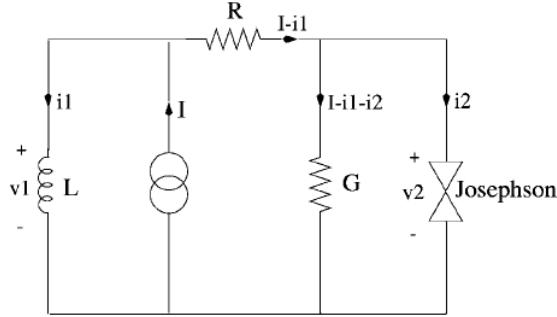


FIGURE 1. Electric circuit with Josephson junction, [23]

of linear resistance, inductance and conductance as in [23], and hence, we see that for the inductance $i_1 = i_L(L, \phi_1)$, for the resistance $v_R = v_R(R, i_1)$, and for the conductance $i_G = i_G(G, v_2)$. Therefore, we

obtain the following system, which completely describes the behavior of this circuit.

$$\begin{aligned}\dot{\phi}_1 &= v_1, & (4.4a) \\ \dot{\phi}_2 &= v_2, & (4.4b) \\ i_1 &= i_L(L, \phi_1), & (4.4c) \\ i_2 &= I_0 \sin(k\phi_2), & (4.4d) \\ 0 &= v_1 - v_R(R, i_1) + v_2, & (4.4e) \\ 0 &= -i_G(G, v_2) + I - i_1 - i_2. & (4.4f)\end{aligned}$$

From (4.4c)-(4.4f) we obtain an explicit form of v_1 in terms of ϕ_1 , i_1 and v_2 , so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \quad (4.5a)$$

$$\dot{\phi}_2 = v_2, \quad (4.5b)$$

$$i_1 = i_L(L, \phi_1), \quad (4.5c)$$

$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \quad (4.5d)$$

The linearized version of this system along equilibrium points defined by $v_2 = 0$, $i_1 = I$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, reads

$$\begin{aligned}\dot{\phi}_1 &= RI - (R/L)\phi_1 + v_2, \\ \dot{\phi}_2 &= v_2, \\ i_1 &= \phi_1/L, \\ 0 &= -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2),\end{aligned}$$

will have a positive eigenvalue and a negative one (e.g. [23]). Hence, it admits exponential dichotomy for any odd number n . Thus, for φ -Lipschitz function v_R and contraction mapping i_G , we obtain a stable manifold for (4.5).

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