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Condensed Forms for linear Port-Hamiltonian Descriptor Systems

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Abstract

Motivated by the structure which arises in the port-Hamiltonian formulation of constraint dynamical systems, we derive structure preserving condensed forms for skew-adjoint differential-algebraic equations (DAEs). Moreover, structure preserving condensed forms under constant rank assumptions for linear port-Hamiltonian differential-algebraic equations are developed. These condensed forms allow us to further analyze the properties of port-Hamiltonian DAEs and to study e.g. existence and uniqueness of solutions. As examples the equations of motion of linear multibody systems and of linear electrical circuit equations are considered.

Keywords: Port-Hamiltonian system; descriptor system; differential-algebraic equation; system transformation; strangeness index; skew-adjoint pair of matrix functions; condensed form.

AMS subject classification: 34H05, 93A30, 93B11, 93B17, 93C05, 93C15.

1 Introduction

In this paper we study linear variable coefficient descriptor systems of the form

$$E\dot{x} = [(J-R)Q - EK]x + (B-P)u \tag{1.1a}$$

$$y = (B+P)^T Qx + (S+N)u \tag{1.1b}$$

where $J, R, K \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $Q, E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $B, P \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, and $S, N \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$ on a real time interval $\mathbb{I} = [t_0, t_f]$ with $S(t) = S(t)^T$, $N(t) = -N(t)^T$ for all $t \in \mathbb{I}$. Here, $C^\ell(\mathbb{I}, \mathbb{R}^{n,m})$ denotes the ℓ-times continuously differentiable functions from $\mathbb{I}$ to the real $n \times m$ matrices. Moreover, $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ (or from an appropriate subspace) denotes the *state* of the system, $u \in C^0(\mathbb{I}, \mathbb{R}^m)$ denotes the m-dimensional *input* of the system and $y \in C^0(\mathbb{I}, \mathbb{R}^m)$ denotes the m-dimensional *output* of the system. Note that for simplicity we omit the argument t in all matrix and vector valued functions. Systems of the form (1.1) have been investigated in [2] as a new modeling framework of port-Hamiltonian systems with constrained dynamics. In [2] also the term *linear port-Hamiltonian differential-algebraic equations* was established.

Definition 1.1 *[2] A linear descriptor system of the form* (1.1) *is called* linear time-varying port-Hamiltonian descriptor system *or* linear port-Hamiltonian differential-algebraic equations (pHDAE) *if*

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(i) the differential-algebraic operator $\mathcal{L} : D \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)$ *defined by*

$$\mathcal{L}(x) = Q^T E \frac{\mathrm{d}}{\mathrm{d}t} x - (Q^T J Q - Q^T E K) x \tag{1.2}$$

is skew-adjoint, i. e., for all $t \in \mathbb{I}$ *it holds that* $Q^T(t)E(t) = E^T(t)Q(t)$, *and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(Q^T E) = Q^T[EK - JQ] + [EK - JQ]^T Q; \tag{1.3}$$

(ii) the matrix function $Q^T E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *is bounded from below by a constant symmetric matrix* $H_0 \in \mathbb{R}^{n,n}$, *i. e.,* $Q^T(t)E(t) - H_0 \geq 0$ *for all* $t \in \mathbb{I}$;

(iii) the matrix function

$$W := \begin{bmatrix} Q^T R Q & Q^T P \\ P^T Q & S \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{n+m,n+m})$$

is symmetric positive semi-definite, i. e., $W(t) = W^T(t) \geq 0$ *for all* $t \in \mathbb{I}$.

The associated Hamiltonian *is defined as*

$$\mathcal{H}(x) := \frac{1}{2} x^T Q^T E x. \tag{1.4}$$

Descriptor systems of the form (1.1) arise in *energy based modeling* where underlying physical properties (such as conservation laws) are directly encoded into the structure of the system model. With this respect, statement (iii) in Definition 1.1 is related to the stability and passivity of the system, while the Hamiltonian (1.4) describes the total energy of the system, cf. [2]. The assumption that $Q^T E$ is bounded by a constant matrix from below implies that the Hamiltonian $\mathcal{H}$ is bounded from below by a constant in order to guarantee that the Hamiltonian can be interpreted as energy. In most of the cases assumption (ii) can be replaced by the stronger condition that $Q^T E$ is positive semi-definite on $\mathbb{I}$.

Theorem 1.2 *[2] A linear time-varying pHDAE* (1.1) *has the following properties:*

1. *If* $W \equiv 0$, *then* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = u^T y$. *In particular, if* $u \equiv 0$ *and* $W \equiv 0$, *then* $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = 0$ *(*conservation of energy*).*

2. *The system satisfies the* dissipation inequality

$$\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^T u(t)\, dt. \tag{1.5}$$

Linear port-Hamiltonian DAEs of the described form can be seen as generalization of linear port-Hamiltonian and gyroscopic systems, see e. g. [1, 9, 11, 20, 21], where $E = Q = I_n$ and $K = 0$ such that we get

$$\dot{x} = (J - R)x + (B - P)u, \tag{1.6a}$$

$$y = (B + P)^T x + (S + N)u, \tag{1.6b}$$

and (1.3) reduces to the condition that J has to be (pointwise) skew-symmetric. In this case, J is referred to as structure matrix describing energy flux among energy storage elements within the system, $R = R^T$ is the dissipation matrix describing energy dissipation/loss in the system, $B \pm P$ are port matrices, describing the manner in which energy enters and exits the system, and $S + N$ describes the direct feed-through from input to output. In general, port-Hamiltonian systems generalize Hamiltonian systems in the sense that the conservation of energy for Hamiltonian systems is replaced by the dissipation inequality (1.5) that shows that the dynamical system is passive, see also [5].

The presented definition of a pHDAE is based on the concept of skew-adjoint differential-algebraic operators. In this paper, we will derive condensed forms for skew-adjoint pairs of matrix function as well as for linear port-Hamiltonian DAEs that will serve as theoretical basis and main tool for the further analysis of port-Hamiltonian DAEs. We will see that the derived condensed forms require certain constant rank assumptions that are often required in the theory of general DAEs, see [13]. Condensed forms for structured DAE systems have also been considered in [24]. In [14] we have considered condensed forms for linear self-adjoint DAE systems that arise e.g. in the necessary optimality conditions for linear optimal control problems. Based on the condensed forms a further analysis of the system properties as e.g. existence and uniqueness of solution or the index of the DAE is possible.
The remainder of this paper is organized as follows. After introducing some preliminary results in Section 2, we develop condensed forms for skew-adjoint pairs of matrix functions under orthogonal and general congruence transformations using certain constant rank assumptions in Section 3. Next, we derive condensed forms for linear port-Hamiltonian DAEs in Section 4. In Section 5, we apply the obtained results to two major classes of applications, namely linear mechanical multibody systems and linear electrical circuit equations. We close with some concluding remarks in Section 6.

2 Preliminaries

We consider linear differential-algebraic equations (DAEs) of the form

$$\mathcal{E}\dot{x} = \mathcal{A}x + f, \tag{2.1}$$

where $\mathcal{E}, \mathcal{A} : \mathbb{I} \to \mathbb{R}^{n,n}$ are continuous matrix-valued functions, $\mathbb{I} = [t_0, t_f] \subset \mathbb{R}$, $x : \mathbb{I} \to \mathbb{R}^n$ is a continuously differentiable unknown function, and $f : \mathbb{I} \to \mathbb{R}^n$ is a given continuous function. For a differentiable time depending function x, the i-th derivative of x with respect to t is denoted by $x^{(i)}(t) = \mathrm{d}^i x(t)/\mathrm{d}t^i$ for $i \in \mathbb{N}$, using the convention $x^{(1)}(t) = \dot{x}(t)$, and $x^{(2)}(t) = \ddot{x}(t)$. The same notation is used for the derivatives of matrix-valued functions. For a matrix $A \in \mathbb{R}^{n,n}$, A^T denotes the transposed of A, $\operatorname{rank} A$ denotes the rank of A, and a real symmetric matrix A that is positive definite or positive semi-definite is denoted by $A > 0$ or $A \geq 0$, respectively. Moreover, for a differential operator $\mathcal{L} : D \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n)$ we denote by $\mathcal{L}^*$ the (unique) conjugate operator.

At first, we gather some facts about linear skew-adjoint differential-algebraic operators. For a more detailed discussion we refer to [2].

Definition 2.1 *A differential-algebraic operator*

$$\mathcal{L} = \mathcal{E}\frac{\mathrm{d}}{\mathrm{d}t} - \mathcal{A} : D \subset C^1(\mathbb{I}, \mathbb{R}^n) \to C^0(\mathbb{I}, \mathbb{R}^n) \tag{2.2}$$

with coefficient functions $\mathcal{E} \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$*,* $\mathcal{A} \in C(\mathbb{I}, \mathbb{R}^{n,n})$ *is called* skew-adjoint *if*

$$\mathcal{E}(t)^T = \mathcal{E}(t), \quad \dot{\mathcal{E}}(t) = -(\mathcal{A}(t) + \mathcal{A}(t)^T) \quad \textit{for all } t \in \mathbb{I}. \tag{2.3}$$

The motivation for the above definition is the fact that in the L_2-inner product we have for all $x \in D$ and all $z \in C^0(\mathbb{I}, \mathbb{R}^n)$ that

$$\langle z, \mathcal{L}(x)\rangle = \langle z, \mathcal{E}\frac{\mathrm{d}}{\mathrm{d}t}x - \mathcal{A}x\rangle = z^T\mathcal{E}x|_{t_0}^{t_f} + \langle -\mathcal{E}^T\dot{z} - (\mathcal{A}^T + \dot{\mathcal{E}}^T)z, x\rangle = \langle \mathcal{L}^*(z), x\rangle$$

using partial integration and assuming zero boundary conditions at the end points of the interval $\mathbb{I} = [t_0, t_f]$. Thus, for the unique conjugate operator $\mathcal{L}^* = -\mathcal{E}^T\frac{\mathrm{d}}{\mathrm{d}t} - (\mathcal{A}^T + \dot{\mathcal{E}}^T)$ of $\mathcal{L}$ in (2.2) and with the above condition (2.3) we have that $\mathcal{L}^* = -\mathcal{L}$.

Lemma 2.2 *[2] Consider a skew-adjoint differential-algebraic operator* (2.2)*. Let* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$*, then the operator* $\mathcal{L}_V$ *defined by*

$$\mathcal{L}_V(x) := V^T\mathcal{E}V\frac{\mathrm{d}}{\mathrm{d}t}x - (V^T\mathcal{A}V - V^T\mathcal{E}\dot{V})x$$

is again skew-adjoint, i. e., $\mathcal{L}_V^* = -\mathcal{L}_V$*.*

It should be noted that we have $\mathcal{L}_V(x(t)) = V^T(t)\mathcal{L}(V(t)x(t))$ for all $x \in D$ and all $t \in \mathbb{I}$, which corresponds to an equivalence transformation of the underlying homogeneous DAE system $\mathcal{E}\dot{x} = \mathcal{A}x$ if V is pointwise invertible on $\mathbb{I}$.

Next, we recall some of the basic settings used in the strangeness index concept for general linear DAEs of the form (2.1). A throughout discussion of the theory can be found in [13].
Since the solution of (2.1) may depend on derivatives of the coefficient functions $\mathcal{E}, \mathcal{A}$ and f, these functions usually will have to satisfy some further smoothness requirements, see [3, 10, 13]. Generally, it is difficult or even impossible to differentiate data that is numerically computed, thus, an idea due to [6] is to differentiate (2.1) and consider the equation together with its derivatives. In this way, we get so-called *derivative arrays*

$$M_\ell\dot{\zeta}_\ell = N_\ell\zeta_\ell + g_\ell, \tag{2.4}$$

where the coefficient functions form an inflated pair of block matrix functions

$$\begin{aligned}
(M_\ell)_{i,j} &= \tbinom{i}{j}\mathcal{E}^{(i-j)} - \tbinom{i}{j+1}\mathcal{A}^{(i-j-1)}, \ i, j = 0, \dots, \ell, \\
(N_\ell)_{i,j} &= \begin{cases} \mathcal{A}^{(i)} & \text{for } i = 0, \dots, \ell, \ j = 0, \\ 0 & \text{otherwise,} \end{cases} \\
(\zeta_\ell)_j &= x^{(j)}, \ j = 0, \dots, \ell, \\
(g_\ell)_i &= f^{(i)}, \ i = 0, \dots, \ell.
\end{aligned} \tag{2.5}$$

Here we have used the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$. It is then known, [13], that the following hypothesis is sufficient to characterize the solution behavior of (2.1).

Hypothesis 2.3 *Consider the system of differential-algebraic equations (2.4). There exist integers μ, a, d, v such that the following properties hold.*

1. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} M_\mu(t) = (\mu+1)n - a - v$. This implies the existence of a smooth matrix function Z with orthonormal columns and size $((\mu+1)n, a+v)$ satisfying $Z^T M_\mu = 0$.*

2. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} Z(t)^T N_\mu(t)[I_n\, 0 \cdots 0]^T = a$, and without loss of generality Z can be partitioned as $[Z_2, Z_3]$, with Z_2 of size $((\mu+1)n, a)$ and Z_3 of size $((\mu+1)n, v)$, such that $\hat{\mathcal{A}}_2 = Z_2^T N_\mu [I_n\ 0\ \cdots\ 0]^T$ has full row rank a and $Z_3^T N_\mu [I_n\ 0\ \cdots\ 0]^T = 0$. Furthermore, there exists a smooth matrix function T_2 with orthonormal columns and size $(n, n-a)$, satisfying $\hat{\mathcal{A}}_2 T_2 = 0$.*

3. *For all $t \in \mathbb{I}$ we have that $\operatorname{rank} \mathcal{E}(t) T_2(t) = d$ with $d = n - a - \hat{v}$ and*

$$\hat{v} = n - \operatorname{rank}[M_\mu\, N_\mu] + \operatorname{rank}[M_{\mu-1}\, N_{\mu-1}]$$

with the convention that $\operatorname{rank}[M_{-1}\, N_{-1}] = 0$. This implies the existence of a smooth matrix function Z_1 with orthonormal columns and size (n, d) so that $\hat{\mathcal{E}}_1 = Z_1^T \mathcal{E}$ has constant rank d.

If Hypothesis 2.3 holds, then system (2.1) has the same solution set as the so-called *reduced system*

$$\begin{bmatrix} \hat{\mathcal{E}}_1 \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \hat{\mathcal{A}}_1 \\ \hat{\mathcal{A}}_2 \\ 0 \end{bmatrix} x + \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{bmatrix}, \tag{2.6}$$

where $\hat{\mathcal{E}}_1 = Z_1^T \mathcal{E}$, $\hat{\mathcal{A}}_1 = Z_1^T \mathcal{A}$, $\hat{\mathcal{A}}_2 = Z_2^T N_\mu [I_n\, 0 \cdots 0]^T$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_i = Z_i^T g_\mu$ for $i = 2, 3$. The block rows in (2.6) have dimensions d, a and v, respectively.
If $v > 0$ and $\hat{f}_3 \neq 0$, then the system (2.6) (and thus also (2.1)) has no solution. Note that v is in general larger that $\hat{v}$, where $n = d + a + \hat{v}$. If $v = \hat{v} = 0$, then every consistent initial condition fixes a unique solution. In the latter case we call the system *regular*. If the system is regular, then from (2.6) we see that an initial condition $x(t_0) = x_0$ for (2.1) is consistent if and only if

$$\hat{\mathcal{A}}_2(t_0) x_0 + \hat{f}_2(t_0) = 0$$

holds, or the second block row in (2.6) is void. The quantity μ in Hypothesis 2.3 is called the *strangeness index* of the DAE system. Note that the reduced system (2.6) is strangeness-free in the sense that it satisfies Hypothesis 2.3 with $\mu = 0$. It is well known, [13], that a system that satisfies Hypothesis 2.3 with $v = 0$ has a well-defined *differentiation index*, [3]. The differentiation index is commonly used to classify regular DAEs, and except for the case of ordinary differential equations it is one more than the strangeness index.

3 Condensed forms for skew-adjoint pairs of matrix functions

Motivated by the observation that a linear pHDAE is related to a skew-adjoint DAE operator (see Definition 1.1), we start by considering a general linear DAE

$$\mathcal{E}\dot{x} = \mathcal{A}x + f \tag{3.1}$$

with skew-adjoint pair $(\mathcal{E}, \mathcal{A})$ of matrix functions that satisfy condition (2.3). For the analysis of such systems we will derive condensed forms for skew-adjoint pairs of matrix functions in this section.
Typically, we can scale equation (3.1) with a pointwise nonsingular matrix function $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, and perform a change of variables $x = V\tilde{x}$ with a pointwise nonsingular matrix function $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, which gives

$$U\mathcal{E}V\dot{\tilde{x}} = U\mathcal{A}V\tilde{x} - U\mathcal{E}\dot{V}\tilde{x} + Uf. \tag{3.2}$$

In view of Lemma 2.2, we know that we have to restrict to *congruence transformations* with $U = V^T$ in order to preserve the skew-adjointness of the pair of matrix functions. For matrix pairs $(\mathcal{E}, \mathcal{A})$, with $\mathcal{E}, \mathcal{A} \in \mathbb{R}^{n,n}$, $\mathcal{E} = \mathcal{E}^T$ and $\mathcal{A} = -\mathcal{A}^T$, the canonical form under congruence, i. e., $(V^T\mathcal{E}V, V^T\mathcal{A}V)$ with nonsingular V is well known, see e. g. [18, 19]. If the transformation matrices are restricted to be real orthogonal matrices, then the resulting staircase form has been developed in [4], modifying the staircase form of [23]. For self-adjoint pairs of matrix functions $(\mathcal{E}, \mathcal{A})$ where $\mathcal{E} = -\mathcal{E}^T$ and $\dot{\mathcal{E}} = \mathcal{A}^T - \mathcal{A}$ that arise e.g. in the necessary optimality conditions for linear quadratic optimal control problems, a condensed form under congruence transformations as well as global condensed forms have been derived in [14]. In this paper, we will extend these results to skew-adjoint pairs of matrix functions.
We will make use of the following theorem which is an extended real and structured version of Theorem 3.9 in [13] originating from [8].

Theorem 3.1 *Let $\mathcal{E} \in C^\ell(\mathbb{I}, \mathbb{R}^{m,n})$, $\ell \in \mathbb{N}_0 \cup \{\infty\}$, with* rank $\mathcal{E}(t) = r$ *for all* $t \in \mathbb{I}$. *Then there exist pointwise real orthogonal matrix functions $U \in C^\ell(\mathbb{I}, \mathbb{R}^{m,m})$ and $V \in C^\ell(\mathbb{I}, \mathbb{R}^{n,n})$, such that*

$$U^T\mathcal{E}V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

with pointwise nonsingular $\Sigma \in C^\ell(\mathbb{I}, \mathbb{R}^{r,r})$.
If $\mathcal{E} \in C^\ell(\mathbb{I}, \mathbb{R}^{n,n})$ is symmetric (skew-symmetric), with $\operatorname{rank} \mathcal{E}(t) = r$ *for all $t \in \mathbb{I}$, then there exists a pointwise real orthogonal matrix function $U \in C^\ell(\mathbb{I}, \mathbb{R}^{n,n})$ such that*

$$U^T\mathcal{E}U = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix} \tag{3.4}$$

with pointwise nonsingular and symmetric (skew-symmetric) $\Delta \in C^\ell(\mathbb{I}, \mathbb{R}^{r,r})$.

Thus, to achieve a condensed form using Theorem 3.1, we have to assume that certain matrix functions have constant rank in the given interval $\mathbb{I}$. If this is not the case, then one can restrict the interval $\mathbb{I}$ under consideration to a smaller interval where this condition holds and consider the problem piecewise, cp. [13, Theorem 3.25] where it is shown that the constant rank assumptions hold on dense subsets. Then, based on sequences of factorizations as in Theorem 3.1 we can construct a staircase form for a skew-adjoint pair $(\mathcal{E}, \mathcal{A})$ via the following recursive procedure, starting with $\mathcal{E}^{[1]} := \mathcal{E} = \mathcal{E}^T$, $\mathcal{A}^{[1]} := \mathcal{A} = -\mathcal{A}^T - \dot{\mathcal{E}}$, $n_1 := n$, $i = 1$.

Procedure 3.2

1. (a) *Let* $\operatorname{rank} \mathcal{E}^{[i]}(t) = r_i$ *for all* $t \in \mathbb{I}$.

(b) *Determine a pointwise orthogonal matrix function* $U_1 \in C^1(\mathbb{I}, \mathbb{R}^{n_i,n_i})$ *such that*

$$U_1^T \mathcal{E}^{[i]} U_1 = \begin{bmatrix} \Delta_i & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.5}$$

with pointwise nonsingular $\Delta_i = \Delta_i^T \in C^1(\mathbb{I}, \mathbb{R}^{r_i,r_i})$.

(c) *Perform a congruence transformation with* U_1*:*

$$U_1^T \mathcal{E}^{[i]} U_1 = \begin{bmatrix} \Delta_i & 0 \\ 0 & 0 \end{bmatrix} =: \mathcal{E}_1^{[i]}, \tag{3.6}$$

$$U_1^T \mathcal{A}^{[i]} U_1 - U_1^T \mathcal{E}^{[i]} \dot{U}_1 = \begin{bmatrix} \hat{\mathcal{A}}_{11} & \hat{\mathcal{A}}_{12} \\ \hat{\mathcal{A}}_{21} & \hat{\mathcal{A}}_{22} \end{bmatrix} =: \mathcal{A}_1^{[i]} \tag{3.7}$$

with $\hat{\mathcal{A}}_{21} = -\hat{\mathcal{A}}_{12}^T$ *and* $\hat{\mathcal{A}}_{22} = -\hat{\mathcal{A}}_{22}^T$ *on* $\mathbb{I}$. *If* $r_i = n_i$, *stop the procedure.*

2. (a) *Let* $\operatorname{rank} \hat{\mathcal{A}}_{22}(t) = a_i$ *for all* $t \in \mathbb{I}$.

   (b) *Determine a pointwise orthogonal matrix function* $\hat{U}_2 \in C^0(\mathbb{I}, \mathbb{R}^{n_i-r_i,n_i-r_i})$ *such that*

$$\hat{U}_2^T \hat{\mathcal{A}}_{22} \hat{U}_2 = \begin{bmatrix} \Sigma_i & 0 \\ 0 & 0 \end{bmatrix},$$

   with pointwise nonsingular $\Sigma_i = -\Sigma_i^T \in C^0(\mathbb{I}, \mathbb{R}^{a_i,a_i})$.

   (c) *Perform a congruence transformation with* $U_2 := \begin{bmatrix} I_{r_i} & 0 \\ 0 & \hat{U}_2 \end{bmatrix}$*:*

$$U_2^T \mathcal{E}_1^{[i]} U_2 = \begin{bmatrix} \Delta_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =: \mathcal{E}_2^{[i]}, \tag{3.8}$$

$$U_2^T \mathcal{A}_1^{[i]} U_2 - U_2^T \mathcal{E}_1^{[i]} \dot{U}_2 = \begin{bmatrix} \tilde{\mathcal{A}}_{11} & \tilde{\mathcal{A}}_{12} & \tilde{\mathcal{A}}_{13} \\ \tilde{\mathcal{A}}_{21} & \Sigma_i & 0 \\ \tilde{\mathcal{A}}_{31} & 0 & 0 \end{bmatrix} =: \mathcal{A}_2^{[i]}, \tag{3.9}$$

   with $\tilde{\mathcal{A}}_{21} = -\tilde{\mathcal{A}}_{12}^T$ *and* $\tilde{\mathcal{A}}_{31} = -\tilde{\mathcal{A}}_{13}^T$. *If* $a_i = n_i - r_i$, *i. e.,* $\tilde{\mathcal{A}}_{13} = [.]$ *is an empty matrix of size* $r_i \times 0$, *then stop the procedure.*

3. (a) *Let* $\operatorname{rank} \tilde{\mathcal{A}}_{13}(t) = s_i$ *for all* $t \in \mathbb{I}$. *Then,* $s_i \leq q_i := n_i - r_i - a_i$.

   (b) *Determine pointwise orthogonal matrix functions* $\tilde{U}_3 \in C^1(\mathbb{I}, \mathbb{R}^{r_i,r_i})$ *and* $\tilde{V}_3 \in C^0(\mathbb{I}, \mathbb{R}^{q_i,q_i})$ *such that*

$$\tilde{U}_3^T \tilde{\mathcal{A}}_{13} \tilde{V}_3 = \begin{bmatrix} \Gamma_i & 0 \\ 0 & 0 \end{bmatrix}$$

   with pointwise nonsingular $\Gamma_i \in C^0(\mathbb{I}, \mathbb{R}^{s_i,s_i})$.

(c) Perform a congruence transformation with $U_3 := \begin{bmatrix} \tilde{U}_3 & 0 & 0 \\ 0 & I_{a_i} & 0 \\ 0 & 0 & \tilde{V}_3 \end{bmatrix}$

$$U_3^T \mathcal{E}_2^{[i]} U_3 = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & 0 & 0 & 0 \\ \mathcal{E}_{12}^T & \mathcal{E}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.10}$$

$$U_3^T \mathcal{A}_2^{[i]} U_3 - U_3^T \mathcal{E}_2^{[i]} \dot{U}_3 = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & \Gamma_i & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} & 0 & 0 \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \Sigma_i & 0 & 0 \\ -\Gamma_i^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.11}$$

where $\tilde{U}_3^T \Delta_i \tilde{U}_3 = \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \mathcal{E}_{12}^T & \mathcal{E}_{22} \end{bmatrix}$ *is symmetric,* $\mathcal{A}_{31} = -\mathcal{A}_{13}^T$, $\mathcal{A}_{32} = -\mathcal{A}_{23}^T$.

4. *Set* $\mathcal{E}^{[i+1]} := \begin{bmatrix} \mathcal{E}_{22} & 0 \\ 0 & 0 \end{bmatrix}$, $\mathcal{A}^{[i+1]} := \begin{bmatrix} \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{32} & \Sigma_i \end{bmatrix}$ *and* $n_{i+1} = r_i - s_i + a_i$. *Continue with step 1. setting* $i \mapsto i+1$.

To carry out Procedure 3.2 until termination, it is assumed that all necessary derivatives exist, i. e., that the coefficient functions $\mathcal{E}, \mathcal{A}$ are sufficiently smooth. Note that under the discussed constant rank assumptions Procedure 3.2 terminates after finitely many steps.

Theorem 3.3 *Consider a skew-adjoint pair* $(\mathcal{E}, \mathcal{A})$ *of matrix functions and assume that Procedure 3.2 can be carried out until termination, i. e., the ranks* r_i, a_i, s_i *are constant on* $\mathbb{I}$ *in each iteration. Then, there exists a congruence transformation with a pointwise orthogonal* $U \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$*, leading to a congruent matrix pair given by*

$$U^T \mathcal{E} U =$$

$$\begin{array}{cc} \left[\begin{array}{cccc|c|cccc} \mathcal{E}_{11} & \cdots & \cdots & \mathcal{E}_{1,\omega} & \mathcal{E}_{1,\omega+1} & \mathcal{E}_{1,\omega+2} & \cdots & \mathcal{E}_{1,2\omega} & 0 \\ \vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\ \vdots & & \ddots & \vdots & \vdots & \mathcal{E}_{\omega-1,\omega+2} & \iddots & & \\ \mathcal{E}_{1,\omega}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega} & \mathcal{E}_{\omega,\omega+1} & 0 & & & \\ \hline \mathcal{E}_{1,\omega+1}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega+1}^T & \mathcal{E}_{\omega+1,\omega+1} & & & & \\ \hline \mathcal{E}_{1,\omega+2}^T & \cdots & \mathcal{E}_{\omega-1,\omega+2}^T & 0 & & & & & \\ \vdots & \iddots & \iddots & & & & & & \\ \mathcal{E}_{1,2\omega}^T & \iddots & & & & & & & \\ 0 & & & & & & & & \end{array}\right] & \begin{array}{c} s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ q_2 \\ q_1 \end{array} \end{array}$$

$U^T\mathcal{A}U - U^T\mathcal{E}\dot{U} =$

$$
\left[\begin{array}{cccc|c|cccc}
\mathcal{A}_{11} & \cdots & \cdots & \mathcal{A}_{1,\omega} & \mathcal{A}_{1,\omega+1} & \mathcal{A}_{1,\omega+2} & \cdots & \cdots & \mathcal{A}_{1,2\omega+1} \\
\vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\
\mathcal{A}_{\omega,1} & \cdots & \cdots & \mathcal{A}_{\omega,\omega} & \mathcal{A}_{\omega,\omega+1} & \mathcal{A}_{\omega,\omega+2} & & & \\
\hline
\mathcal{A}_{\omega+1,1} & \cdots & \cdots & \mathcal{A}_{\omega+1,\omega} & \mathcal{A}_{\omega+1,\omega+1} & & & & \\
\hline
\mathcal{A}_{\omega+2,1} & \cdots & \cdots & \mathcal{A}_{\omega+2,\omega} & & & & & \\
\vdots & & \iddots & & & & & & \\
\vdots & \iddots & & & & & & & \\
\mathcal{A}_{2\omega+1,1} & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ \vdots \\ q_1
\end{array}, \tag{3.12}
$$

where $q_1 \geq s_1 \geq q_2 \geq s_2 \geq \ldots \geq q_\omega \geq s_\omega$, $b := r_{\omega+1} + a_{\omega+1}$,

$$
\begin{array}{rcl}
\mathcal{E}_{j,2\omega+1-j} & \in & C^1(\mathbb{I}, \mathbb{R}^{s_j,q_{j+1}}), \qquad 1 \leq j \leq \omega - 1, \\
\mathcal{E}_{\omega+1,\omega+1} & = & \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \quad \Delta = \Delta^T \in C^1(\mathbb{I}, \mathbb{R}^{r_{\omega+1},r_{\omega+1}}), \\
\mathcal{E}_{j,j} & = & \mathcal{E}_{j,j}^T, \qquad j = 1, \ldots, \omega, \\
\mathcal{A}_{j,2\omega+2-j} & = & -\mathcal{A}_{2\omega+2-j,j}^T = \begin{bmatrix} \Gamma_j & 0 \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_j}),\ \Gamma_j \in C^0(\mathbb{I}, \mathbb{R}^{s_j,s_j}),\ 1 \leq j \leq \omega, \\
\mathcal{A}_{\omega+1,\omega+1} & = & \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ -\Sigma_{12}^T & \Sigma_{22} \end{bmatrix},\ \Sigma_{11} = -\Sigma_{11}^T - \dot{\Delta} \in C^0(\mathbb{I}, \mathbb{R}^{r_{\omega+1},r_{\omega+1}}), \\
\Sigma_{22} & = & -\Sigma_{22}^T \in C^0(\mathbb{I}, \mathbb{R}^{a_{\omega+1},a_{\omega+1}}),
\end{array}
$$

and the blocks Σ_{22}, Δ *and* Γ_j, $j = 1, \ldots, \omega$ *are pointwise nonsingular, implying that* Σ_{22} *has even dimension* $a_{\omega+1} = 2p$. *Furthermore, each of the first* ω *block columns (block rows) of the matrix* $U^T\mathcal{E}U$ *has full column rank (full row rank).*

Proof. The proof is analogous to the proof of Theorem 4.4 in [14] for self-adjoint matrix pairs. $\square$

With nonsingular congruence transformations it is possible to reduce the system even further.

Corollary 3.4 *Consider a skew-adjoint pair* $(\mathcal{E}, \mathcal{A})$ *of matrix functions and assume that Procedure 3.2 can be carried out until termination, i. e., the ranks* r_i, a_i, s_i *are constant on* $\mathbb{I}$ *in each iteration. Then, there exists a congruence transformation with a pointwise nonsingular* $T \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, *leading to the congruent matrix pair*

$T^T\mathcal{E}T =$

$$
\left[\begin{array}{cccc|c|cccc}
\mathcal{E}_{11} & \cdots & \cdots & \mathcal{E}_{1,\omega} & \mathcal{E}_{1,\omega+1} & \mathcal{E}_{1,\omega+2} & \cdots & \mathcal{E}_{1,2\omega} & 0 \\
\vdots & \ddots & & \vdots & \vdots & \vdots & \iddots & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \mathcal{E}_{\omega-1,\omega+2} & \iddots & & \\
\mathcal{E}_{1,\omega}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega} & \mathcal{E}_{\omega,\omega+1} & 0 & & & \\
\hline
\mathcal{E}_{1,\omega+1}^T & \cdots & \cdots & \mathcal{E}_{\omega,\omega+1}^T & \mathcal{E}_{\omega+1,\omega+1} & & & & \\
\hline
\mathcal{E}_{1,\omega+2}^T & \cdots & \mathcal{E}_{\omega-1,\omega+2}^T & 0 & & & & & \\
\vdots & \iddots & \iddots & & & & & & \\
\mathcal{E}_{1,2\omega}^T & \iddots & & & & & & & \\
0 & & & & & & & &
\end{array}\right]
\begin{array}{c}
s_1 \\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots \\ q_2 \\ q_1
\end{array}
$$

$$
T^T\mathcal{A}T - T^T\mathcal{E}\dot{T} =
\left[\begin{array}{cccc|c|cccc}
\mathcal{A}_{1,1} & \cdots & \cdots & \mathcal{A}_{1,\omega} & \mathcal{A}_{1,\omega+1} & \mathcal{A}_{1,\omega+2} & \cdots & \cdots & \mathcal{A}_{1,2\omega+1}\\
\vdots & \ddots & & \vdots & \vdots & \vdots & & \iddots & \\
\vdots & & \ddots & \vdots & \vdots & \vdots & \iddots & & \\
\mathcal{A}_{\omega,1} & \cdots & \cdots & \mathcal{A}_{\omega,\omega} & \mathcal{A}_{\omega,\omega+1} & \mathcal{A}_{\omega,\omega+2} & & & \\ \hline
\mathcal{A}_{\omega+1,1} & \cdots & \cdots & \mathcal{A}_{\omega+1,\omega} & \mathcal{A}_{\omega+1,\omega+1} & & & & \\ \hline
0 & \cdots & 0 & \mathcal{A}_{\omega+2,\omega} & & & & & \\
\vdots & & \iddots & & & & & & \\
0 & \iddots & & & & & & & \\
\mathcal{A}_{2\omega+1,1} & & & & & & & &
\end{array}\right]
\begin{array}{c} s_1\\ \vdots \\ \vdots \\ s_\omega \\ \hline b \\ \hline q_\omega \\ \vdots\\ \vdots \\ q_1\end{array}, \tag{3.13}
$$

where $q_1 \geq s_1 \geq q_2 \geq s_2 \geq \ldots \geq q_\omega \geq s_\omega$, $b := r_{\omega+1} + a_{\omega+1}$,

$$\mathcal{E}_{j,2\omega+1-j} \in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_{j+1}}), \qquad 1 \leq j \leq \omega - 1,$$

$$\mathcal{E}_{\omega+1,\omega+1} = \begin{bmatrix} I_k & 0 & 0 \\ 0 & -I_{r_{\omega+1}-k} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{A}_{j,2\omega+2-j} = -\mathcal{A}_{2\omega+2-j,j}^T = \begin{bmatrix} I_{s_j} & 0 \end{bmatrix} \in C^0(\mathbb{I}, \mathbb{R}^{s_j,q_j}), \qquad 1 \leq j \leq \omega,$$

$$\mathcal{A}_{i,j} = -\dot{\mathcal{E}}_{i,j}, \quad i = 1, \ldots, \omega - 1,\ j = \omega + 2, \ldots, 2\omega + 1 - i,$$

$$\mathcal{A}_{\omega+1,\omega+1} = \begin{bmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{bmatrix}, \quad \Sigma_{11} = -\Sigma_{11}^T \in C^0(\mathbb{I}, \mathbb{R}^{r_{\omega+1},r_{\omega+1}}),\ \Sigma_{22} = -\Sigma_{22}^T \in C^0(\mathbb{I}, \mathbb{R}^{2p,2p}),$$

and the block Σ_{22} *is pointwise nonsingular. Furthermore, each of the first* ω *block columns (block rows) of the matrix* $T^T\mathcal{E}T$ *has full column rank (full row rank).*

Proof. The proof is similar to the proof of Corollary 4.6 in [14]. Starting from the staircase form (3.12) we can first perform a congruence transformation

$$(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}) = (T_1^T U^T \mathcal{E} U T_1, T_1^T U^T \mathcal{A} U T_1 - T_1^T U^T \mathcal{E} \tfrac{\mathrm{d}}{\mathrm{d}t}(UT_1))$$

with $T_1^T = \operatorname{diag}(\Gamma_1^{-1}, \ldots, \Gamma_\omega^{-1}, L, I_{q_\omega}, \ldots, I_{q_1})$ where

$$L = \begin{bmatrix} I_{r_{\omega+1}} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{r_{\omega+1}} \end{bmatrix}.$$

Then, with block-Gaussian congruence transformations, we can eliminate all elements above the block anti-diagonal of $\tilde{\mathcal{A}}$ in block-columns $1, \ldots, \omega$. Finally, we perform a congruence transformation to the nonsingular first diagonal block Δ in $\mathcal{E}_{\omega+1,\omega+1}$ using a nonsingular matrix V such that

$$V^T \Delta V = \begin{bmatrix} I_k & 0 \\ 0 & -I_{r_{\omega+1}-k} \end{bmatrix}$$

by Sylvester's law of inertia. $\square$

Note that neither the orthogonal staircase form (3.12) nor the condensed form (3.13) is a normal form in the algebraic sense, since there is still further refinement possible using congruence transformations. For the purpose of analyzing systems of differential-algebraic equations, however, these condensed forms are sufficient.

Corollary 3.5 *Consider a skew-adjoint pair $(\mathcal{E}, \mathcal{A})$ of matrix functions and suppose that the assumptions of Theorem 3.3 hold so that there exists a congruence transformation with a pointwise orthogonal $U \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ to the staircase form (3.12).*

i) The differential-algebraic equation (3.1) is regular if and only if $s_j = q_j$ for all $j = 1, \ldots, \omega$ in the staircase form (3.12).

ii) If $\omega = 0$, i. e., if Procedure 3.2 stops in the first iteration, then the DAE (3.1) is regular and strangeness-free.

iii) If $\omega > 0$, then $\mu \leq 2\omega - 1$ differentiations will be necessary to solve the system if $a_{\omega+1} = 0$, and $\mu \leq 2\omega$ differentiations will be necessary otherwise.

Proof.

i) If $s_j = q_j$ for $j = 1, \ldots, \omega$, then we can successively solve the equation by backward substitution in a unique way, thus the system is regular. Conversely, if $q_1 > s_1$ the DAE is non-regular, because then it has a zero row and hence the problem is not solvable for every smooth right hand side. If $s_j = q_j$ for $j = 1, \ldots, \ell - 1$ but $q_\ell > s_\ell$, then we can successively solve the equation from the bottom up in a unique way, until we reach the remaining system with a non-square block $\mathcal{A}_{2\omega+2-\ell,\ell} = -\mathcal{A}^T_{\ell,2\omega+2-\ell} = \begin{bmatrix} -\Gamma_\ell & 0 \end{bmatrix}^T$. Then again, the last $q_\ell - s_\ell$ equations associated with this block are not solvable for every smooth right hand side and, hence, the problem is not regular.

ii) If $\omega = 0$, then the associated staircase form has the form (3.7) with $\hat{\mathcal{A}}_{22}$ pointwise nonsingular and it is well known already from the unstructured case, see [12, 13], that the associated DAE is regular and strangeness-free.

iii) Using the condensed form (3.13), we can apply backward substitution starting with the last block row. Then we have to differentiate the right hand side at most ω times until we reach the middle block. If after backward substitution the middle block contains an algebraic part, then we continue with at most ω further differentiations. If the middle block has no algebraic part, then at most $\omega - 1$ further differentiations are necessary.

□

Example 3.6 Consider the DAE

$$\left[\begin{array}{c|cc|c|c} 0 & 0 & 1 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \left[\begin{array}{c|cc|c|c} 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix},$$

which is given in the condensed form (3.12) with $\omega = 2$, $q_1 = s_1 = 1$, $q_2 = 1$, $s_2 = 0$, $b = 2$, $a_3 = 0$. Since $q_2 \neq s_2$, the system is non-regular. To solve the system, we first get $x_1 = f_5$, and by substituting $\dot{x}_1 = \dot{f}_5$ the solution for x_2, x_3 can be determined from

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_2 \\ f_3 - \dot{f}_5 \end{bmatrix}.$$

Next, we can solve the differential system

$$\dot{x}_4 = x_3 + x_5 + f_1 + f_2$$

to obtain a solution for x_4. The component x_5 is undetermined and we have the consistency condition $f_4 - \dot{f}_5 = 0$ for the inhomogeneity. Thus, $\mu = 1 < 3$ differentiations are necessary to solve the system. $\triangleleft$

If the pair $(\mathcal{E}, \mathcal{A})$ is in the condensed form (3.13) and the associated DAE (3.1) is regular, then we can permute and re-arrange the condensed form to

$$\left(\begin{bmatrix} \tilde{\mathcal{E}}_{11} & \tilde{\mathcal{E}}_{12} & \tilde{\mathcal{E}}_{13} & \tilde{\mathcal{E}}_{14} \\ \tilde{\mathcal{E}}_{12}^T & \tilde{\mathcal{E}}_{22} & 0 & 0 \\ \tilde{\mathcal{E}}_{13}^T & 0 & 0 & 0 \\ \tilde{\mathcal{E}}_{14}^T & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{\mathcal{A}}_{11} & \tilde{\mathcal{A}}_{12} - \dot{\tilde{\mathcal{E}}}_{12} & \tilde{\mathcal{A}}_{13} - \dot{\tilde{\mathcal{E}}}_{13} & I_s - \dot{\tilde{\mathcal{E}}}_{14} \\ -\tilde{\mathcal{A}}_{12}^T & \tilde{\mathcal{A}}_{22} & 0 & 0 \\ -\tilde{\mathcal{A}}_{13}^T & 0 & \tilde{\mathcal{A}}_{33} & 0 \\ -I_s & 0 & 0 & 0 \end{bmatrix} \right), \tag{3.14}$$

where $s = \sum_{i=1}^{\omega} s_i$, $\tilde{\mathcal{E}}_{22} = \begin{bmatrix} I_k & 0 \\ 0 & -I_{r_{\omega+1}-k} \end{bmatrix}$ and $\tilde{\mathcal{A}}_{33} = \Sigma_{22}$ are invertible, and $\tilde{\mathcal{E}}_{14}$ is block upper-triangular with square diagonal blocks, which are zero matrices. Performing some further block-Gaussian elimination congruence transformations, we can eliminate the block to the left of $\tilde{\mathcal{A}}_{33}$. Then, due to the skew-adjoint structure, the part $\tilde{\mathcal{A}}_{13}$ above the block $\tilde{\mathcal{A}}_{33}$ is eliminated as well, while the part $\dot{\tilde{\mathcal{E}}}_{13}$ remains. In the same way, the block above and to the left of $\tilde{\mathcal{E}}_{22}$ can be eliminated. One further block permutation (exchanging the first two block rows and columns), partitioning the blocks further, and renaming the blocks, finally leads to the form

$$\left(\begin{bmatrix} I_k & 0 & 0 & 0 & 0 \\ 0 & -I_{r_{\omega+1}-k} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{E}_{33} & \mathcal{E}_{34} & \mathcal{E}_{35} \\ 0 & 0 & \mathcal{E}_{34}^T & 0 & 0 \\ 0 & 0 & \mathcal{E}_{35}^T & 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} & 0 & 0 \\ -\mathcal{A}_{12}^T & \mathcal{A}_{22} & \mathcal{A}_{23} & 0 & 0 \\ -\mathcal{A}_{13}^T & -\mathcal{A}_{23}^T & \mathcal{A}_{33} & -\dot{\mathcal{E}}_{34} & I_s - \dot{\mathcal{E}}_{35} \\ 0 & 0 & 0 & \mathcal{A}_{44} & 0 \\ 0 & 0 & -I_s & 0 & 0 \end{bmatrix} \right), \tag{3.15}$$

with $\mathcal{A}_{44} = -\mathcal{A}_{44}^T$ invertible (and of even dimension), and $\mathcal{E}_{35}$ block upper-triangular with square diagonal blocks, which are zero matrices.

In our original motivation the leading matrix $\mathcal{E}$ of the skew-adjoint differential-algebraic system is given by $\mathcal{E} = Q^T E$, where Q and E are coefficient functions of a pHDAE (1.1) (see Definition 1.1). We will now assume that the matrix function $\mathcal{E} = Q^T E$ is positive semi-definite on $\mathbb{I}$, a somewhat stronger assumption than (ii) in Definition 1.1, but often satisfied in physical applications (for examples we refer to Section 5), and furthermore that the DAE (3.1) is regular. Under these assumptions we can transform the pair $(\mathcal{E}, \mathcal{A})$ to the condensed form (3.15), where due to the positive semi-definiteness of $\mathcal{E}$ we have that $r_{\omega+1} - k = 0$ as well as $\mathcal{E}_{34} \equiv 0$ and $\mathcal{E}_{35} \equiv 0$. Thus, the condensed form (3.15) reduces to

$$\begin{array}{c} \left(\begin{bmatrix} I_{r_{\omega+1}} & 0 & 0 & 0 \\ 0 & \mathcal{E}_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{13} & 0 & 0 \\ -\mathcal{A}_{13}^T & \mathcal{A}_{33} & 0 & I_s \\ 0 & 0 & \mathcal{A}_{44} & 0 \\ 0 & -I_s & 0 & 0 \end{bmatrix} \right) \end{array} \begin{array}{c} r_{\omega+1} \\ s \\ 2p \\ s \end{array} \tag{3.16}$$

with $\mathcal{A}_{11}(t) = -\mathcal{A}_{11}^T(t)$, $\mathcal{A}_{44}(t) = -\mathcal{A}_{44}^T(t)$ for all $t \in \mathbb{I}$ and $\mathcal{A}_{44}$ of even dimension $2p$ is pointwise nonsingular, as well as $\dot{\mathcal{E}}_{33} = -(\mathcal{A}_{33} + \mathcal{A}_{33}^T)$ on $\mathbb{I}$. Moreover, we know that $\mathcal{E}_{33}$ of size $s \times s$ is pointwise nonsingular due to the full column rank condition in Theorem 3.4. The corresponding DAE takes the form

$$\begin{array}{rcl} \dot{x}_1 & = & \mathcal{A}_{11}x_1 + \mathcal{A}_{13}x_2 + f_1, \\ \mathcal{E}_{33}\dot{x}_2 & = & -\mathcal{A}_{13}^T x_1 + \mathcal{A}_{33}x_2 + x_4 + f_2, \\ 0 & = & \mathcal{A}_{44}x_3 + f_3, \\ 0 & = & -x_2 + f_4, \end{array} \tag{3.17}$$

for a sufficiently smooth inhomogeneity $f = [f_1, f_2, f_3, f_4]^T$. The last two equations in (3.17) can be solved for x_2 and x_3 giving $x_2 = f_4$ and $x_3 = -\mathcal{A}_{44}^{-1} f_3$. Differentiating the relation for x_2 and inserting it into the second equation of (3.17) gives

$$\begin{array}{rcl} \dot{x}_1 & = & \mathcal{A}_{11}x_1 + \mathcal{A}_{13}f_4 + f_1, \\ x_4 & = & \mathcal{E}_{33}\dot{f}_4 + \mathcal{A}_{13}^T x_1 - \mathcal{A}_{33}f_4 - f_2, \end{array}$$

i. e., an ordinary differential equation for x_1 and subsequently the solution for x_4. Thus, we require at most one differentiation of equations to obtain the unique solution of the system. Also we see that x_1 is the only differential component in the system which is related to the dynamics, while x_2, x_3 and x_4 are algebraic components related to algebraic constraints on the dynamics. The algebraic component x_2 and its coupling to the second equation in (3.17) results in an index greater than 0. We formulate this result in the following Theorem.

Theorem 3.7 *Consider a regular DAE* (3.1) *with coefficient functions* $(\mathcal{E}, \mathcal{A})$ *that form a skew-adjoint pair. Suppose that the assumptions of Theorem 3.3 hold and that* $\mathcal{E}$ *is positive semi-definite for all* $t \in \mathbb{I}$*. Then the DAE* (3.1) *has strangeness index* $\mu \leq 1$*. In particular, the DAE* (3.1) *has strangeness index* $\mu = 1$ *if and only if* $s > 0$ *in the reduced condensed form (3.16). If* $s = p = 0$ *in the reduced condensed form (3.16), then the system* (3.1) *is an ODE.*

Regularity of the DAE (3.1) is an assumption that should always be satisfied in reasonable models. Non-regular DAE systems usually result from modeling or discretization errors and in this case the system should be regularized in a preprocessing step (e.g., by using feedback regularization [7]). In the context of port-Hamiltonian DAEs we will see that undetermined components can always be reinterpreted as port variables, see Lemma 4.8.

4 Condensed forms for linear port-Hamiltonian DAEs

In this section we derive condensed forms for linear pHDAEs (1.1) under equivalence transformations. The development of the condensed forms is based on the following result.

Theorem 4.1 *[2] Consider a linear pHDAE system* (1.1) *with Hamiltonian* (1.4)*. Let* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *be pointwise invertible in* $\mathbb{I}$*. Then the transformed system*

$$\tilde{E}\dot{\tilde{x}} = [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \tag{4.1a}$$

$$y = (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + (S + N)u, \tag{4.1b}$$

with

$$\begin{aligned}
&\tilde{E} = U^T E V, \quad \tilde{Q} = U^{-1} Q V, \quad \tilde{J} = U^T J U, \quad \tilde{R} = U^T R U, \\
&\tilde{B} = U^T B, \quad \tilde{P} = U^T P, \quad \tilde{K} = V^{-1} K V + V^{-1}\dot{V}, \quad \tilde{x} = V^{-1} x
\end{aligned}$$

is again a pHDAE with the same Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{Q}^T \tilde{E}\tilde{x} = \mathcal{H}(x)$.

Thus, we can define equivalence of linear port-Hamiltonian DAEs in the following way.

Definition 4.2 (Equivalence of pHDAEs) *Two pHDAE systems of the form* (1.1) *defined by tuples of matrix functions* $(E, J, R, Q, K, B, P, S, N)$ *and* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, \tilde{S}, \tilde{N})$ *are called* equivalent *if there exist pointwise invertible matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that*

$$\begin{aligned}
&\tilde{E} = U^T E V, \quad \tilde{Q} = U^{-1} Q V, \quad \tilde{J} = U^T J U, \quad \tilde{R} = U^T R U, \\
&\tilde{B} = U^T B, \quad \tilde{P} = U^T P, \quad \tilde{K} = V^{-1} K V + V^{-1}\dot{V}, \quad \tilde{S} = S, \quad \tilde{N} = N.
\end{aligned} \tag{4.2}$$

It should be noted that the matrix K is required to describe equivalence transformations in the time-varying setting. However, we can always find an equivalence transformation that eliminates K in the pHDAE (1.1) due to the following result.

Lemma 4.3 *[2] Consider a pHDAE*

$$\begin{aligned}
\tilde{E}\dot{\tilde{x}} &= [(\tilde{J} - \tilde{R})\tilde{Q} - \tilde{E}\tilde{K}]\tilde{x} + (\tilde{B} - \tilde{P})u, \\
y &= (\tilde{B} + \tilde{P})^T \tilde{Q}\tilde{x} + (S + N)u,
\end{aligned}$$

with Hamiltonian $\tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{Q}^T \tilde{E}\tilde{x}$, *where* $\tilde{K} \in C(\mathbb{I}, \mathbb{R}^{n,n})$. *If* $V_{\tilde{K}}$ *is a pointwise invertible solution of the matrix differential equation* $\dot{V} = -\tilde{K}V$ *with initial condition* $V(t_0) = I_n$, *then defining* $E = \tilde{E}V_{\tilde{K}}^{-1}$, $Q = \tilde{Q}V_{\tilde{K}}^{-1}$, $x = V_{\tilde{K}}\tilde{x}$, $J = \tilde{J}$, $R = \tilde{R}$, $B = \tilde{B}$, $P = \tilde{P}$, *the system*

$$\begin{aligned}
E\dot{x} &= (J - R)Qx + (B - P)u, \\
y &= (B + P)^T Q x + (S + N)u,
\end{aligned}$$

is again a pHDAE with $\mathcal{H}(x) = \frac{1}{2}x^T Q^T E x = \tilde{\mathcal{H}}(\tilde{x})$.

Remark 4.4 The matrix differential equation for $V_{\tilde{K}}$ in Lemma 4.3 can be solved numerically, and if $\tilde{K} = -\tilde{K}^T$, then $V_{\tilde{K}}$ from Lemma 4.3 can be assumed to be pointwise orthogonal. On the other hand, if we restrict to equivalence transformations using only pointwise orthogonal matrices U and V in Definition 4.2, then we will always get a skew-symmetric matrix function $\tilde{K}$.

We start our investigations by restricting to equivalence transformations with orthogonal matrix functions U and V such that $U^{-1} = U^T$ and $V^{-1} = V^T$ in (4.2). In this case, we can assume that $K = -K^T$ (see Remark 4.4). As before, in order to derive the condensed form using Theorem 3.1, we have to assume constant rank of certain matrix functions.
Assuming that $\operatorname{rank} E(t) = r$ for all $t \in \mathbb{I}$ we can at first compute a smooth SVD of E, i. e., there exist pointwise orthogonal matrix functions U_1 and V_1 of size $n \times n$ such that

$$\tilde{E}_1 = U_1^T E V_1 = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix},$$

with Σ_r of size $r \times r$ pointwise nonsingular. The other matrix functions are transformed according to (4.2) into

$$\tilde{J}_1 := U_1^T J U_1 = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \; \tilde{R}_1 := U_1^T R U_1 = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \; \tilde{Q}_1 := U_1^T Q V_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$
$$\tilde{K}_1 := V_1^T K V_1 + V_1^T \dot{V}_1 = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad \tilde{B}_1 := U_1^T B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \tilde{P}_1 := U_1^T P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}.$$

Since the transformed system is again a pHDAE, we have that $\tilde{Q}_1^T \tilde{E}_1 = \tilde{E}_1^T \tilde{Q}_1$, such that

$$\begin{bmatrix} Q_{11}^T \Sigma_r & 0 \\ Q_{12}^T \Sigma_r & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_r^T Q_{11} & \Sigma_r^T Q_{12} \\ 0 & 0 \end{bmatrix},$$

giving $Q_{12} \equiv 0$, as well as $Q_{11}^T \Sigma_r = \Sigma_r^T Q_{11}$ or $Q_{11} = \Sigma_r^{-T} Q_{11}^T \Sigma_r$ on $\mathbb{I}$. Thus, we have

$$\tilde{Q}_1 = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}. \tag{4.3}$$

Assuming that

$$\operatorname{rank} Q_{22}(t) = q \text{ for all } t \in \mathbb{I}, \tag{R1}$$

we can then perform a smooth SVD of Q_{22}, i. e., there exist pointwise orthogonal matrix functions U_{22} and V_{22}, both of size $(n-r) \times (n-r)$, such that

$$U_{22}^T Q_{22} V_{22} = \begin{bmatrix} \Sigma_q & 0 \\ 0 & 0 \end{bmatrix}$$

where Σ_q of size $q \times q$ is pointwise nonsingular. Using $U_2 = \begin{bmatrix} I_r & 0 \\ 0 & U_{22} \end{bmatrix}$ and $V_2 = \begin{bmatrix} I_r & 0 \\ 0 & V_{22} \end{bmatrix}$ we get the transformed matrix functions

$$\tilde{E}_2 := U_2^T \tilde{E}_1 V_2 = \begin{bmatrix} \Sigma_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{J}_2 := U_2^T \tilde{J}_1 U_2 = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix},$$
$$\tilde{Q}_2 := U_2^T \tilde{Q}_1 V_2 = \begin{bmatrix} Q_{11} & 0 & 0 \\ Q_{21} & \Sigma_q & 0 \\ Q_{31} & 0 & 0 \end{bmatrix},$$

as well as $\tilde{R}_2 := U_2^T \tilde{R}_1 U_2$, $\tilde{K}_2 := V_2^T \tilde{K}_1 V_2 + V_2^T \dot{V}_2$, $\tilde{B}_2 := U_2^T \tilde{B}_1$, $\tilde{P}_2 := U_2^T \tilde{P}_1$ partitioned accordingly. The skew-adjointness condition (1.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{Q}_2^T \tilde{E}_2) = \tilde{Q}_2^T \tilde{E}_2 \tilde{K}_2 + \tilde{K}_2^T \tilde{E}_2^T \tilde{Q}_2 - \tilde{Q}_2^T (\tilde{J}_2 + \tilde{J}_2^T) \tilde{Q}_2,$$

which takes the form

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}(Q_{11}^T \Sigma_r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} * & * & Q_{11}^T \Sigma_r K_{13} \\ * & -\Sigma_q^T (J_{22} + J_{22}^T) \Sigma_q & 0 \\ K_{13}^T \Sigma_r^T Q_{11} & 0 & 0 \end{bmatrix},$$

and, thus, gives $J_{22}(t) = -J_{22}^T(t)$ for all $t \in \mathbb{I}$, as well as

$$Q_{11}^T \Sigma_r K_{13} \equiv 0 \quad \text{on } \mathbb{I}.$$

Assuming that

$$\operatorname{rank} Q_{31}(t) = w \text{ for all } t \in \mathbb{I}, \tag{R2}$$

we can proceed with a smooth SVD of Q_{31}, i. e., there exist pointwise orthogonal matrix functions U_{31} of size $(n-r-q) \times (n-r-q)$ and V_{31} of size $r \times r$ such that

$$U_{31}^T Q_{31} V_{31} = \begin{bmatrix} \Sigma_w & 0 \\ 0 & 0 \end{bmatrix}$$

where Σ_w of size $w \times w$ is pointwise nonsingular for all $t \in \mathbb{I}$. Using $U_3 = \begin{bmatrix} I_r & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & U_{31} \end{bmatrix}$ and $V_3 = \begin{bmatrix} V_{31} & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & I_{n-r-q} \end{bmatrix}$, we get

$$\tilde{E}_3 := U_3^T \tilde{E}_2 V_3 = \left[\begin{array}{c|cc} \Sigma_r V_{31} & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|ccc} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$\tilde{J}_3 := U_3^T \tilde{J}_2 U_3 = \left[\begin{array}{cc|ccc} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ \hline J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} \end{array}\right], \quad \tilde{Q}_3 := U_3^T \tilde{Q}_2 V_3 = \left[\begin{array}{cc|ccc} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ \hline Q_{31} & Q_{32} & \Sigma_q & 0 & 0 \\ \Sigma_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

as well as $\tilde{R}_3 := U_3^T \tilde{R}_2 U_3$, $\tilde{K}_3 := V_3^T \tilde{K}_2 V_3 + V_3^T \dot{V}_3$, $\tilde{B}_3 := U_3^T \tilde{B}_2$, $\tilde{P}_3 := U_3^T \tilde{P}_2$ partitioned accordingly, where $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$ is pointwise nonsingular and

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

on $\mathbb{I}$. Furthermore, we have $J_{33}(t) = -J_{33}^T(t)$ for all $t \in \mathbb{I}$ by the same arguments as above.

Theorem 4.5 *Consider a linear pHDAE* (1.1) *that is defined by the tuple of matrix functions* $(E, J, R, Q, K, B, P, S, N)$ *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$. *Under the constant rank assumptions* (R1) *and* (R2) *there exist pointwise orthogonal matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by*

$$U^T E V = \begin{array}{c} \left[\begin{array}{cc|ccc} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} w \\ d \\ q \\ w \\ v \end{array} \\ \begin{array}{ccccc} w & d & q & w & v \end{array} \end{array}, \quad U^T Q V = \begin{array}{c} \left[\begin{array}{cc|ccc} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ \hline Q_{31} & Q_{32} & \Sigma_q & 0 & 0 \\ \Sigma_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} w \\ d \\ q \\ w \\ v \end{array} \\ \begin{array}{ccccc} w & d & q & w & v \end{array} \end{array}, \tag{4.4}$$

where $d = r - w$, $v = n - r - q - w$, with pointwise nonsingular blocks Σ_q, Σ_w and $\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$, and

$$U^T J U = \left[\begin{array}{cc|ccc} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ \hline J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} \end{array}\right], \quad U^T R U = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix},$$

$$U^T B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}, \quad U^T P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix}, \quad \tilde{K} := V^T K V + V^T \dot{V} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix},$$

partitioned accordingly, with $J_{33}(t) = -J_{33}^T(t)$, $\tilde{K}^T(t) = -\tilde{K}(t)$ for all $t \in \mathbb{I}$, as well as

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}^T \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \begin{bmatrix} K_{14} & K_{15} \\ K_{24} & K_{25} \end{bmatrix} \equiv 0$$

on $\mathbb{I}$.

Proof. The proof follows directly from the previous discussion. □

If we also allow non-orthogonal transformations, then we can further simplify the matrices in (4.4). At first, we can transform the nonsingular blocks in $\tilde{Q}_3$ into identity blocks by using pointwise nonsingular matrix functions $U_4 = I_n$ and $V_4 = \begin{bmatrix} \Sigma_w^{-1} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \Sigma_q^{-1} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$ giving

$$\tilde{E}_4 := \tilde{E}_3 V_4 = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_4 := \tilde{Q}_3 V_4 = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ Q_{31} & Q_{32} & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\tilde{J}_4 := \tilde{J}_3$, $\tilde{R}_4 := \tilde{R}_3$, $\tilde{K}_4 := V_4^{-1}\tilde{K}_3 V_4 + V_4^{-1}\dot{V}_4$, $\tilde{B}_4 := \tilde{B}_3$, $\tilde{P}_4 := \tilde{P}_3$. Next, using the identity block I_q in $\tilde{Q}_4$, we can eliminate Q_{31} and Q_{32} by defining the pointwise nonsingular matrix functions $U_5 = I$ and $V_5 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ -Q_{31} & -Q_{32} & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$ giving

$$\tilde{E}_5 := \tilde{E}_4 V_5 = \begin{bmatrix} E_{11} & E_{12} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_5 := \tilde{Q}_4 V_5 = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\tilde{J}_5 := \tilde{J}_4$, $\tilde{R}_5 := \tilde{R}_4$, $\tilde{K}_5 := V_5^{-1}\tilde{K}_4 V_5 + V_5^{-1}\dot{V}_5$, $\tilde{B}_5 := \tilde{B}_4$, $\tilde{P}_5 := \tilde{P}_4$. Now, we can restore the identity in $\tilde{E}_5$ by using $U_6^T = \begin{bmatrix} X_{11} & X_{12} & 0 & 0 & 0 \\ X_{21} & X_{22} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$, where $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ is the pointwise inverse of the nonsingular upper left 2-by-2 block in $\tilde{E}_5$ and $V_6 = I_n$. We get

$$\tilde{E}_6 := U_6^T \tilde{E}_5 V_6 = \begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_6 := U_6^{-1}\tilde{Q}_5 V_6 = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where $d = r - w$, and $\tilde{J}_6 := U_6^T \tilde{J}_5 U_6$, $\tilde{R}_6 := U_6^T \tilde{R}_5 U_6$, $\tilde{K}_6 := \tilde{K}_5$, $\tilde{B}_6 := U_6^T \tilde{B}_5$, $\tilde{P}_6 := U_6^T \tilde{P}_5$ partitioned accordingly. Now, we can eliminate the blocks Q_{11} and Q_{21} using $V_7 = I_n$ and

$$U_7^{-1} = \begin{bmatrix} I & 0 & 0 & -Q_{11} & 0 \\ 0 & I & 0 & -Q_{21} & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

to get

$$\tilde{E}_7 := U_7^T \tilde{E}_6 V_7 = \begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_7 := U_7^{-1}\tilde{Q}_6 V_7 = \begin{bmatrix} 0 & Q_{12} & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and $\tilde{J}_7 := U_7^T \tilde{J}_6 U_7$, $\tilde{R}_7 := U_7^T \tilde{R}_6 U_7$, $\tilde{K}_7 := \tilde{K}_6$, $\tilde{B}_7 := U_7^T \tilde{B}_6$, $\tilde{P}_7 := U_7^T \tilde{P}_6$ partitioned accordingly. From $\tilde{Q}_7^T \tilde{E}_7 = \tilde{E}_7^T \tilde{Q}_7$ we have that

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ Q_{12}^T & Q_{22}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q_{12} & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

such that $Q_{12} \equiv 0$ as well as $Q_{22}^T(t) = Q_{22}(t)$ for all $t \in \mathbb{I}$. Finally, we can determine pointwise nonsingular matrix functions V_{ii}, $i = 1, \ldots, 5$, such that

$$\dot{V}_{ii} = -K_{ii}V_{ii}, \quad V_{ii}(t_0) = I, \quad i = 1, \ldots, 5.$$

Then, with

$$U_8 = \begin{bmatrix} V_{11}^{-T} & 0 & 0 & 0 & 0 \\ 0 & V_{22}^{-T} & 0 & 0 & 0 \\ 0 & 0 & V_{33} & 0 & 0 \\ 0 & 0 & 0 & V_{11} & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad V_8 = \begin{bmatrix} V_{11} & 0 & 0 & 0 & 0 \\ 0 & V_{22} & 0 & 0 & 0 \\ 0 & 0 & V_{33} & 0 & 0 \\ 0 & 0 & 0 & V_{44} & 0 \\ 0 & 0 & 0 & 0 & V_{55} \end{bmatrix},$$

we get that

$$\tilde{E}_8 := U_8^T \tilde{E}_7 V_8 = \begin{bmatrix} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{Q}_8 := U_8^{-1} \tilde{Q}_7 V_8 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{K}_8 := V_8^{-1} \tilde{K}_7 V_8 + V_8^{-1}\dot{V}_8 = \begin{bmatrix} 0 & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & 0 & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & 0 & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & 0 & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & 0 \end{bmatrix},$$

and $\tilde{J}_8 := U_8^T \tilde{J}_7 U_8$, $\tilde{R}_8 := U_8^T \tilde{R}_7 U_8$, $\tilde{B}_8 := U_8^T \tilde{B}_7$, $\tilde{P}_8 := U_8^T \tilde{P}_7$ partitioned accordingly. In summary, we have the following result.

Theorem 4.6 *Consider a linear pHDAE* (1.1) *that is defined by the tuple of matrix functions* $(E, J, R, Q, K, B, P, S, N)$ *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$. *Under the assumptions of Theorem 4.5 there exist pointwise nonsingular matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by the tuple of matrix functions* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, S, N)$ *given by*

$$\tilde{E} := U^T E V = \begin{array}{c} \left[\begin{array}{cc|ccc} I_w & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} w \\ d \\ q \\ w \\ v \end{array} \\ \begin{array}{ccccc} w & d & q & w & v \end{array} \end{array}, \quad \tilde{Q} := U^{-1} Q V = \begin{array}{c} \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ \hline 0 & 0 & I_q & 0 & 0 \\ I_w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} w \\ d \\ q \\ w \\ v \end{array} \\ \begin{array}{ccccc} w & d & q & w & v \end{array} \end{array}, \tag{4.5a}$$

where $d = r - w$, $v = n - r - q - w$, and $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{d,d}$ for all $t \in \mathbb{I}$, and

$$
\begin{aligned}
\tilde{J} &:= U^T J U = \left[\begin{array}{cc|ccc} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} \\ \hline J_{31} & J_{32} & J_{33} & J_{34} & J_{35} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} \end{array}\right], \quad \tilde{R} := U^T R U = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} \end{bmatrix}, \\
\tilde{K} &:= V^{-1} K V + V^{-1} \dot{V} = \begin{bmatrix} 0 & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & 0 & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & 0 & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & 0 & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & 0 \end{bmatrix}, \qquad (4.5b) \\
\tilde{B} &:= U^T B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}, \quad \tilde{P} := U^T P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{bmatrix},
\end{aligned}
$$

partitioned accordingly, with

$$
\begin{aligned}
& J_{44} = -J_{44}^T, \quad J_{33} = -J_{33}^T, \quad J_{43} = -J_{34}^T, \\
& R_{33} = R_{33}^T, \quad R_{44} = R_{44}^T, \quad R_{32} = R_{23}^T, \quad R_{42} = R_{24}^T, \quad R_{43} = R_{34}^T, \\
& Q_{22} K_{24} \equiv 0, \quad Q_{22} K_{25} \equiv 0, \quad Q_{22}(J_{23} - K_{23}) = -Q_{22} J_{32}^T, \quad Q_{22}(J_{24} - K_{21}) = -Q_{22} J_{42}^T, \\
& \dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T) Q_{22},
\end{aligned}
$$

as well as

$$
\begin{bmatrix} R_{44} & R_{24}^T Q_{22} & R_{34}^T \\ Q_{22} R_{24} & Q_{22} R_{22} Q_{22} & Q_{22} R_{23} \\ R_{34} & R_{23}^T Q_{22} & R_{33} \end{bmatrix} \geq 0
$$

on $\mathbb{I}$.

Proof. The structure of the condensed form (4.5) follows from the previous discussion. Moreover, the matrices of the corresponding skew-adjoint operator (1.2) takes the form

$$
\left(\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} J_{44} & J_{42} Q_{22} & J_{43} & 0 & 0 \\ Q_{22}(J_{24} - K_{21}) & Q_{22} J_{22} Q_{22} & Q_{22}(J_{23} - K_{23}) & -Q_{22} K_{24} & -Q_{22} K_{25} \\ J_{34} & J_{32} Q_{22} & J_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right),
$$

and

$$
\tilde{W} := \begin{bmatrix} \tilde{Q}^T \tilde{R} \tilde{Q} & \tilde{Q}^T \tilde{P} \\ \tilde{P}^T \tilde{Q} & S \end{bmatrix},
$$

such that the skew-adjointness condition (1.3) and the condition that $\tilde{W} = \tilde{W}^T \geq 0$ due to (iii) in Definition 1.1 yield the remaining conditions for the block matrices in (4.5). □

Remark 4.7 It should be noted that the total energy of the pHDAE (1.1) is given by

$$\mathcal{H}(x) = \tilde{\mathcal{H}}(\tilde{x}) = \frac{1}{2}\tilde{x}^T \tilde{E}^T \tilde{Q}\tilde{x} = \frac{1}{2}\tilde{x}_2^T Q_{22}\tilde{x}_2$$

for the transformed state vector $\tilde{x} = V^{-1}x$ partitioned according to the condensed form (4.5). Thus, the only contribution to the total energy comes from the component $\tilde{x}_2$ and the matrix function Q_{22}. The remaining components of the state vector belong to algebraic constraints that have no energy contribution to the system or to undetermined components that should have been considered as port variables in the modeling (see also Lemma 4.8).

If there are undetermined components of the state vector in a pHDAE (1.1) this usually means that a modeling error has occurred. Such undetermined components should be reinterpreted as port variables. For a pHDAE given in condensed form (4.5) such a reinterpretation can be easily performed.

Lemma 4.8 *Consider a linear pHDAE* (1.1) *given in condensed form* (4.5) *and let the state vector* $x = [x_1^T, x_2^T, x_3^T, x_4^T, x_5^T]^T$ *be partitioned according to the block structure of* (4.5)*. If* $v > 0$ *there are undetermined components of the state vector that can be reinterpreted as port variables of a pHDAE*

$$\begin{aligned} \hat{E}\dot{\hat{x}} &= [(\hat{J} - \hat{R})\hat{Q} - \hat{E}\hat{K}]\hat{x} + (\hat{B} - \hat{P})\hat{u} \\ \hat{y} &= (\hat{B} + \hat{P})^T \hat{Q}\hat{x} + (\hat{S} + \hat{N})\hat{u} \end{aligned} \tag{4.6}$$

where

$$\hat{E} := \begin{bmatrix} I_w & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{Q} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ I_w & 0 & 0 & 0 \end{bmatrix}, \quad \hat{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad \hat{u} := \begin{bmatrix} u \\ x_5 \end{bmatrix}, \quad \hat{y} := \begin{bmatrix} y \\ 0 \end{bmatrix}$$

$$\hat{J} := \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \\ J_{41} & J_{42} & J_{43} & J_{44} \end{bmatrix}, \hat{R} := \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix}, \hat{K} := \begin{bmatrix} 0 & K_{12} & K_{13} & K_{14} \\ K_{21} & 0 & K_{23} & K_{24} \\ K_{31} & K_{32} & 0 & K_{34} \\ K_{41} & K_{42} & K_{43} & 0 \end{bmatrix}$$

$$\hat{B} := \begin{bmatrix} B_1 & \frac{1}{2}(J_{51}^T - R_{51}^T - K_{15}) \\ B_2 & \frac{1}{2}(J_{52}^T - R_{52}^T - K_{25}) \\ B_3 & \frac{1}{2}(J_{53}^T - R_{53}^T) \\ B_4 & \frac{1}{2}(J_{54}^T - R_{54}^T) \end{bmatrix}, \quad \hat{P} := \begin{bmatrix} P_1 & \frac{1}{2}(J_{51}^T - R_{51}^T + K_{15}) \\ P_2 & \frac{1}{2}(J_{52}^T - R_{52}^T + K_{25}) \\ P_3 & \frac{1}{2}(J_{53}^T - R_{53}^T) \\ P_4 & \frac{1}{2}(J_{54}^T - R_{54}^T) \end{bmatrix}$$

$$\hat{S} := \begin{bmatrix} S & \frac{1}{2}(B_5^T - P_5^T) \\ \frac{1}{2}(B_5 - P_5) & 0 \end{bmatrix}, \quad \hat{N} := \begin{bmatrix} N & \frac{1}{2}(P_5^T - B_5^T) \\ \frac{1}{2}(B_5 - P_5) & 0 \end{bmatrix}.$$

Proof. In a pHDAE in condensed form (4.5), the components x_5 of dimension v are undetermined components of the state vector x. These components can be reinterpreted as port variables by rewriting the system as

$$\begin{bmatrix} I_w & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} L_{14} & L_{12}Q_{22} - K_{12} & L_{13} - K_{13} & -K_{14} \\ L_{24} - K_{21} & L_{22}Q_{22} & L_{23} - K_{23} & -K_{24} \\ L_{34} & L_{32}Q_{22} & L_{33} & 0 \\ L_{44} & L_{42}Q_{22} & L_{43} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_1 - P_1 & -K_{15} \\ B_2 - P_2 & -K_{25} \\ B_3 - P_3 & 0 \\ B_4 - P_4 & 0 \end{bmatrix} \begin{bmatrix} u \\ x_5 \end{bmatrix},$$

$$\begin{bmatrix} y \\ 0 \end{bmatrix} = \begin{bmatrix} (B_1 + P_1)^T & (B_2 + P_2)^T & (B_3 + P_3)^T & (B_4 + P_4)^T \\ L_{51} & L_{52} & L_{53} & L_{54} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ I_w & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} (S+N) & 0 \\ B_5 - P_5 & 0 \end{bmatrix} \begin{bmatrix} u \\ x_5 \end{bmatrix},$$

where we define $L_{ij} := J_{ij} - R_{ij}$ for $i, j = 1, \ldots, 5$. By using the above definitions it can be easily checked that all properties of a phDAE are satisfied for the system (4.6). $\square$

In many practical examples and applications not only $Q^T E$ but also the product EQ^T is symmetric, i. e.,

$$E(t)Q^T(t) = Q(t)E^T(t) \quad \text{for all } t \in \mathbb{I}$$

(cf. again Section 5). In this case, the condensed form (4.5) simplifies as follows.

Corollary 4.9 *Consider a linear pHDAE (1.1) that is defined by the tuple of matrix functions $(E, J, R, Q, K, B, P, S, N)$ with $r = \operatorname{rank} E(t)$ for all $t \in \mathbb{I}$, which satisfies $E(t)Q^T(t) = Q(t)E^T(t)$ for all $t \in \mathbb{I}$. Under the assumption of Theorem 4.5 there exist pointwise nonsingular matrix functions $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ and $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ such that the system is equivalent to a pHDAE described by the tuple $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, S, N)$ given by*

$$\tilde{E} := U^T E V = \begin{array}{c} \left[\begin{array}{c|cc} I_r & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} r \\ q \\ v \end{array} \\ \begin{array}{ccc} r & q & v \end{array} \end{array}, \quad \tilde{Q} := U^{-1} Q V = \begin{array}{c} \left[\begin{array}{c|cc} Q_{22} & 0 & 0 \\ \hline 0 & I_q & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{array}{c} r \\ q \\ v \end{array} \\ \begin{array}{ccc} r & q & v \end{array} \end{array}, \tag{4.7a}$$

where $v = n - r - q$, with $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{r,r}$ for all $t \in \mathbb{I}$, and

$$\tilde{J} := U^T J U = \begin{bmatrix} J_{22} & J_{23} & J_{25} \\ J_{32} & J_{33} & J_{35} \\ J_{52} & J_{53} & J_{55} \end{bmatrix}, \quad \tilde{R} := U^T R U = \begin{bmatrix} R_{22} & R_{23} & R_{25} \\ R_{32} & R_{33} & R_{35} \\ R_{52} & R_{53} & R_{55} \end{bmatrix}, \tag{4.7b}$$

$$\tilde{K} := V^{-1} K V + V^{-1} \dot{V} = \begin{bmatrix} 0 & K_{23} & K_{25} \\ K_{32} & 0 & K_{35} \\ K_{52} & K_{53} & 0 \end{bmatrix}, \tilde{B} := U^T B = \begin{bmatrix} B_2 \\ B_3 \\ B_5 \end{bmatrix}, \quad \tilde{P} := U^T P = \begin{bmatrix} P_2 \\ P_3 \\ P_5 \end{bmatrix},$$

partitioned accordingly, with

$$\begin{aligned} & J_{33} = -J_{33}^T, \quad R_{33} = R_{33}^T, \quad R_{32} = R_{23}^T, \\ & Q_{22} K_{25} \equiv 0, \quad Q_{22}(J_{23} - K_{23}) = -Q_{22} J_{32}^T, , \\ & \dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T) Q_{22}, \end{aligned}$$

as well as

$$\begin{bmatrix} Q_{22} R_{22} Q_{22} & Q_{22} R_{23} \\ R_{23}^T Q_{22} & R_{33} \end{bmatrix} \geq 0$$

on $\mathbb{I}$.

Proof. From the symmetry condition $E(t)Q^T(t) = Q(t)E^T(t)$ for all $t \in \mathbb{I}$ we get that $Q_{21} \equiv 0$ in (4.3), and consequently $w = 0$. The rest follows from Theorem 4.6. □

Another additional property that occurs frequently in practical applications is the case that the matrix function Q is pointwise invertible (cf. Section 5). This assumption can possibly be met after reinterpretation of state variables as port variables (as in Lemma 4.8).

Corollary 4.10 *Consider a linear pHDAE* (1.1) *that is defined by the tuple of matrix functions* $(E, J, R, Q, K, B, P, S, N)$ *with* $r = \operatorname{rank} E(t)$ *for all* $t \in \mathbb{I}$ *and pointwise invertible matrix function* Q*. Under the assumption of Theorem 4.5 there exist pointwise nonsingular matrix functions* $U \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$ *and* $V \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$ *such that the system is equivalent to a pHDAE described by the tuple* $(\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q}, \tilde{K}, \tilde{B}, \tilde{P}, S, N)$ *given by*

$$\tilde{E} := U^T E V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{Q} := U^{-1} Q V = \begin{bmatrix} Q_{22} & 0 \\ 0 & I_{n-r} \end{bmatrix}, \tag{4.8a}$$

with $Q_{22}(t) = Q_{22}^T(t) > 0$ *for all* $t \in \mathbb{I}$ *and*

$$\tilde{J} := U^T J U = \begin{bmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{bmatrix}, \quad \tilde{R} := U^T R U = \begin{bmatrix} R_{22} & R_{23} \\ R_{23}^T & R_{33} \end{bmatrix}, \tag{4.8b}$$

$$\tilde{K} := V^{-1} K V + V^{-1} \dot{V} = \begin{bmatrix} 0 & K_{23} \\ K_{32} & 0 \end{bmatrix}, \tilde{B} := U^T B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \quad \tilde{P} := U^T P = \begin{bmatrix} P_2 \\ P_3 \end{bmatrix},$$

partitioned accordingly, with

$$\begin{aligned} &J_{33} = -J_{33}^T, \quad K_{23} = J_{32}^T + J_{23}, \quad R_{22} = R_{22}^T, \quad R_{33} = R_{33}^T, \\ &\dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T) Q_{22}, \end{aligned}$$

as well as $\tilde{R} \geq 0$ *on* $\mathbb{I}$.

Proof. From the condition that Q is pointwise nonsingular we get that Q_{22} in (4.3) is pointwise nonsingular and, thus, $q = n - r$. It follows that $d = r$, and $w = 0$ as well as $v = 0$ in the condensed form (4.5). □

If the matrix function Q in (1.1) is pointwise nonsingular, the results from Theorem 3.7 can be applied.

Corollary 4.11 *Consider a linear pHDAE* (1.1) *with pointwise invertible matrix function* Q*. Suppose that the assumptions of Theorem 3.3 hold for the skew-adjoint pair of matrix functions* $(Q^T E, Q^T J Q - Q^T E K)$*. Then, the undamped and uncontrolled DAE system* (1.1a) *with* $R \equiv 0$ *and* $u \equiv 0$ *has strangeness index* $\mu \leq 1$*. In particular, the undamped and uncontrolled system is strangeness-free if and only if the matrix function* J_{33} *of size* $(n - r) \times (n - r)$ *in the condensed form* (4.8) *is pointwise invertible for all* $t \in \mathbb{I}$ *(and, thus, is of even dimension).*

Proof. If Q is pointwise invertible, then the index of the DAE (1.1a) with $R \equiv 0$ and $u \equiv 0$ is the same as the index of the skew-adjoint DAE

$$Q^T E \dot{x} = Q^T J Q x - Q^T E K x.$$

Moreover, using the condensed form (4.8) we see that the matrix function $Q^T E$ is symmetric positive semi-definite for all $t \in \mathbb{I}$ such that the result follows from Theorem 3.7. $\square$

The condensed forms (4.5),(4.7) and (4.8) are very useful if we want to determine the strangeness index of a pHDAE (1.1a). To simplify the notation we write a pHDAE (1.1a) in condensed form (4.5) as

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \end{bmatrix} x + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{4.9}$$

with

$$\begin{aligned}
A_{11} &:= \begin{bmatrix} L_{14} & L_{12}Q_{22} - K_{12} \\ L_{24} - K_{21} & L_{22}Q_{22} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,r}), \quad A_{12} := \begin{bmatrix} L_{13} - K_{13} \\ L_{23} - K_{23} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,q}), \\
A_{13} &:= \begin{bmatrix} -K_{14} & -K_{15} \\ -K_{24} & -K_{25} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{r,w+v}) \\
A_{21} &:= \begin{bmatrix} L_{34} & L_{32}Q_{22} \\ L_{44} & L_{42}Q_{22} \\ L_{54} & L_{52}Q_{22} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{q+w+v,r}), \quad A_{22} := \begin{bmatrix} L_{33} \\ L_{43} \\ L_{53} \end{bmatrix} \in C(\mathbb{I}, \mathbb{R}^{q+w+v,q}),
\end{aligned}$$

where we use the definition $L_{ij} := J_{ij} - R_{ij}$ for $i, j = 1, \ldots, 5$ and

$$f_1 := \begin{bmatrix} B_1 - P_1 \\ B_2 - P_2 \end{bmatrix} u, \quad f_2 := \begin{bmatrix} B_3 - P_3 \\ B_4 - P_4 \\ B_5 - P_5 \end{bmatrix}$$

are seen as given input functions (or inhomogeneity). Then the pair of matrix functions corresponding to the DAE system (4.9) can be transformed into an equivalent pair of matrix functions in a similar manner as before. We get

$$\begin{aligned}
\left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & 0 \end{bmatrix} \right) &\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \Sigma_a & 0 \\ \tilde{A}_{31} & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} E_{11} & E_{21} & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ \hat{A}_{31} & \hat{A}_{32} & \Sigma_a & 0 \\ \Sigma_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_{r-s} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & \hat{A}_{14} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ \hat{A}_{31} & \hat{A}_{32} & \Sigma_a & 0 \\ \Sigma_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)
\end{aligned}$$

with pointwise nonsingular matrix functions Σ_a of size $a \times a$ and Σ_s of size $s \times s$, assuming that $\operatorname{rank} A_{22}(t) = a$ for all $t \in \mathbb{I}$, as well as $\operatorname{rank} \tilde{A}_{31}(t) = s$ for all $t \in \mathbb{I}$. Finally, we obtain the following result.

Theorem 4.12 *Consider a linear pHDAE* (1.1) *given in condensed form* (4.5) *and assume that*

$$\operatorname{rank}\begin{bmatrix} J_{33}(t)-R_{33}(t) \\ J_{43}(t)-R_{43}(t) \\ J_{53}(t)-R_{53}(t) \end{bmatrix} = a \leq q \quad \textit{for all } t \in \mathbb{I}.$$

Furthermore, let Z be a matrix function of size $(q+w+v)\times(q-a)$ and pointwise maximal rank such that

$$Z(t)^T \begin{bmatrix} J_{33}(t)-R_{33}(t) \\ J_{43}(t)-R_{43}(t) \\ J_{53}(t)-R_{53}(t) \end{bmatrix} = 0$$

for all $t \in \mathbb{I}$ and assume that

$$\operatorname{rank} Z^T(t) \begin{bmatrix} J_{34}(t)-R_{34}(t) & (J_{32}(t)-R_{32}(t))Q_{22}(t) \\ J_{44}(t)-R_{44}(t) & (J_{42}(t)-R_{42}(t))Q_{22}(t) \\ J_{54}(t)-R_{54}(t) & (J_{52}(t)-R_{52}(t))Q_{22}(t) \end{bmatrix} = s \quad \textit{for all } t \in \mathbb{I}.$$

Then, the following hold.

1. *The pHDAE* (1.1) *is strangeness-free if and only if $s = 0$.*
2. *The pHDAE* (1.1) *is regular and strangeness-free if and only if $s = w = v = 0$ and $q = a$. In this case, the pHDAE in condensed form* (4.5) *reduces to*

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}\dot{\tilde{x}} = \left(\begin{bmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{bmatrix} - \begin{bmatrix} R_{22} & R_{23} \\ R_{23}^T & R_{33} \end{bmatrix}\right)\begin{bmatrix} Q_{22} & 0 \\ 0 & I_a \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} - \begin{bmatrix} 0 & K_{23} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + \begin{bmatrix} B_2 - P_2 \\ B_3 - P_3 \end{bmatrix}u,$$

$$y = \begin{bmatrix} B_2 + P_2 \\ B_3 + P_3 \end{bmatrix}^T \begin{bmatrix} Q_{22} & 0 \\ 0 & I_a \end{bmatrix}\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + (S+N)u,$$

with $J_{33}(t) = -J_{33}^T(t)$, $R_{33}(t) = R_{33}^T(t)$, $Q_{22}(t) = Q_{22}^T(t) \geq Q_0 \in \mathbb{R}^{r,r}$ for all $t \in \mathbb{I}$, where the matrix function $J_{33} - R_{33}$ is pointwise invertible on $\mathbb{I}$ and

$$0 = Q_{22}\left[J_{32}^T + J_{23} - K_{23}\right], \quad \dot{Q}_{22} = -Q_{22}(J_{22} + J_{22}^T)Q_{22},$$

as well as

$$\begin{bmatrix} Q_{22}R_{22}Q_{22} & Q_{22}R_{23} \\ R_{23}^T Q_{22} & R_{33} \end{bmatrix} \geq 0$$

on $\mathbb{I}$.

Proof. The proof follows directly from the previous discussion. □

For linear pHDAEs that are not strangeness-free an index reduction is necessary. Usually, the differentiation and elimination step used in the index reduction procedure proposed in [13] will destroy the port-Hamiltonian structure of the system. However, a modification of the regularization procedure has been presented in [2] that preserve the pHDAE structure and, under some (local) constant rank assumption, allows us to reformulated any linear pHDAE as an implicitly defined standard port-Hamiltonian system plus an algebraic constraint.
For pHDAEs that fulfill further structural assumptions we get the corresponding results.

Corollary 4.13 *Consider a linear pHDAE* (1.1) *which satisfies* $E(t)Q^T(t) = Q(t)E^T(t)$ *for all* $t \in \mathbb{I}$ *given in condensed form* (4.7) *and assume that*

$$\operatorname{rank} \begin{bmatrix} J_{33}(t) - R_{33}(t) \\ J_{53}(t) - R_{53}(t) \end{bmatrix} = a \leq q \quad \textit{for all } t \in \mathbb{I}.$$

Furthermore, let Z *be a matrix function of size* $(q+v) \times (q-a)$ *and pointwise maximal rank such that*

$$Z(t)^T \begin{bmatrix} J_{33}(t) - R_{33}(t) \\ J_{53}(t) - R_{53}(t) \end{bmatrix} = 0$$

for all $t \in \mathbb{I}$ *and assume that*

$$\operatorname{rank} Z^T \begin{bmatrix} (J_{32}(t) - R_{32}(t))Q_{22}(t) \\ (J_{52}(t) - R_{52}(t))Q_{22}(t) \end{bmatrix} = s \quad \textit{for all } t \in \mathbb{I}.$$

Then the pHDAE (1.1) *is regular and strangeness-free if and only if* $s = v = 0$ *and* $q = a$.

Corollary 4.14 *Consider a linear pHDAE* (1.1) *with pointwise nonsingular matrix function* Q *given in condensed form* (4.8) *and assume that*

$$\operatorname{rank} \begin{bmatrix} J_{33}(t) - R_{33}(t) \end{bmatrix} = a \leq q \quad \textit{for all } t \in \mathbb{I}.$$

Furthermore, let Z *be a matrix function of size* $q \times (q-a)$ *and pointwise maximal rank such that*

$$Z(t)^T \begin{bmatrix} J_{33}(t) - R_{33}(t) \end{bmatrix} = 0$$

for all $t \in \mathbb{I}$ *and assume that*

$$\operatorname{rank} Z^T(t) \begin{bmatrix} (J_{32}(t) - R_{32}(t))Q_{22}(t) \end{bmatrix} = s \quad \textit{for all } t \in \mathbb{I}.$$

Then the pHDAE (1.1) *is regular and strangeness-free if and only if* $s = 0$ *and* $J_{33} - R_{33}$ *is pointwise nonsingular on* $\mathbb{I}$.

Comparing this last result with Corollary 4.11, we see that if a regular pHDAE is of strangeness index larger than 1, then the matrix function R that describes the energy dissipation in the system is responsible for the higher index. A regular port-Hamiltonian DAE with pointwise nonsingular matrix function Q and no dissipation terms will always be of strangeness index less than or equal to 1.

5 Applications

In this section, we consider two important application classes of linear port-Hamiltonian DAEs, namely linear mechanical multibody systems and linear electrical RLC circuits. We will see how the previous results conform with these examples.

5.1 Linear Multibody Systems

We consider linear mechanical systems under holonomic and nonholonomic constraints [15] given in the form

$$M\ddot{q} + (D_0 + D_G)\dot{q} + Sq = G^T\lambda + H^T\eta + B_1 u, \tag{5.1a}$$
$$0 = Gq, \tag{5.1b}$$
$$0 = H\dot{q}, \tag{5.1c}$$

with constant mass matrix $M = M^T > 0$, damping matrix $D_0 = D_0^T \geq 0$, gyroscopic terms $D_G = -D_G^T$ and stiffness matrix $S = S^T > 0$, for $M, D_0, D_G, S \in \mathbb{R}^{n_q,n_q}$. The holonomic constraint matrix $G \in \mathbb{R}^{n_\lambda,n_q}$ and the nonholonomic constraint matrix $H \in \mathbb{R}^{n_\eta,n_q}$ are assumed to be of full row rank (i. e., we assume non-redundant constraints). Here, q denotes the n_q-dimensional vector of generalized position coordinates, whereas λ and η denote the n_λ- and n_η-dimensional vectors of Lagrange multipliers corresponding to the holonomic and nonholonomic constraints, respectively. Introducing the velocity vector $v = \dot{q}$ the corresponding first order system takes the form

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{q} \\ \dot{\lambda} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -(D_0 + D_G) & -S & G^T & H^T \\ I & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ H & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \\ \lambda \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u. \tag{5.2}$$

It is well-known that the DAE system (5.2) with $f = Bu$ seen as inhomogeneity is regular and of strangeness index $\mu = 2$. Given in the representation above, system (5.2) cannot be written as pHDAE since the symmetric/anti-symmetric structure cannot be met. However, if the system contains only nonholonomic constraints, i. e., $n_\lambda = 0$, then the system (5.2) reduces to

$$\begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{q} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -(D_0 + D_G) & -S & H^T \\ I & 0 & 0 \\ H & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u, \tag{5.3}$$

which can be written as linear pHDAE by setting

$$E = \begin{bmatrix} M & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; Q = \begin{bmatrix} I & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & I \end{bmatrix}, \; B = \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, \; J = \begin{bmatrix} -D_G & -I & H^T \\ I & 0 & 0 \\ -H & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} D_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and $K = 0$ as well as $P = 0$. We can easily see that (1.3) is satisfied since $J = -J^T$ and

$$Q^T E = E^T Q = \begin{bmatrix} M & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \geq 0,$$

as well as $EQ^T = QE^T \geq 0$. Moreover, we have the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \begin{bmatrix} v \\ q \\ \eta \end{bmatrix}^T \begin{bmatrix} M & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ q \\ \eta \end{bmatrix} = \frac{1}{2} v^T M v + \frac{1}{2} q^T S q$$

representing the total energy of the system. The condensed form (4.8) for (5.3) takes the form

$$\left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\begin{bmatrix} M\dot{v} \\ \dot{q} \\ \dot{\eta} \end{bmatrix} = \left(\left[\begin{array}{cc|c} -D_G & -I & H^T \\ I & 0 & 0 \\ \hline -H & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|c} D_0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]\right)\left[\begin{array}{cc|c} M^{-1} & 0 & 0 \\ 0 & S & 0 \\ \hline 0 & 0 & I \end{array}\right]\begin{bmatrix} Mv \\ q \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix} u$$

with $r = 2n_q$. Since Q is pointwise nonsingular, Corollary 4.11 applies and the undamped and uncontrolled DAE system (5.3) has strangeness index $\mu = 1$. Note that in this case even the damped system has strangeness index $\mu = 1$ (as a system in Hessenberg form, cf. e.g. [13]), since the only contribution D_0 from R does not lead to an increase in the index.
In order to formulate system (5.2) as a port-Hamiltonian DAE we perform an index reduction at first. A well-known strategy to reduce the index of (5.2) is to add the time derivative of the holonomic constraints to the system equations and to introduce another Lagrange multiplier ϑ to couple these holonomic constraints on velocity level to the original equations. The resulting system is given by

$$\begin{bmatrix} M & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{q} \\ \dot{\lambda} \\ \dot{\eta} \\ \dot{\vartheta} \end{bmatrix} = \begin{bmatrix} -(D_0 + D_G) & -S & G^T & H^T & 0 \\ I & 0 & 0 & 0 & G^T \\ 0 & G & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 0 \\ G & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ q \\ \lambda \\ \eta \\ \vartheta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u. \tag{5.4}$$

This formulation is also known as the *Gear-Gupta-Leimkuhler formulation* of the equations of motion and it can be shown that the DAE (5.4) is of strangeness index $\mu = 1$, see [13]. Reordering variables and equations yields a system of the form

$$\begin{bmatrix} M & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{q} \\ \dot{\lambda} \\ \dot{\vartheta} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -(D_0 + D_G) & -S & G^T & 0 & H^T \\ I & 0 & 0 & G^T & 0 \\ G & 0 & 0 & 0 & 0 \\ 0 & G & 0 & 0 & 0 \\ H & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ q \\ \lambda \\ \vartheta \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \tag{5.5}$$

which has the form of a linear pHDAE in the same manner as before. Again, the only contribution D_0 of R does not lead to an increase in the index compared to the undamped system formulation.

5.2 Linear RLC Circuits

The model equations of electrical RLC circuits arise from Kirchhoff's laws together with the constitutive element relations. For details see e.g. [16]. Assuming linear element relations for inductors, capacitors, resistors, current and voltage sources, we get a DAE system of the form

$$\begin{aligned}
C\frac{\mathrm{d}}{\mathrm{d}t}\nu_C &= \imath_C,\\
L\frac{\mathrm{d}}{\mathrm{d}t}\imath_L &= {A_L}^T\eta,\\
0 &= -\nu_C + {A_C}^T\eta\\
0 &= -\nu_R + {A_R}^T\eta\\
0 &= {A_V}^T\eta - V_s(t)\\
0 &= \imath_R - G\nu_R,\\
0 &= -A_C\imath_C - A_L\imath_L - A_R\imath_R - A_V\imath_V - A_I I_s(t),\\
0 &= -\nu_I + {A_I}^T\eta,
\end{aligned} \tag{5.6}$$

where the *conductance matrix* $G \in \mathbb{R}^{n_R \times n_R}$, the *capacitance matrix* $C \in \mathbb{R}^{n_C \times n_C}$, and the *inductance matrix* $L \in \mathbb{R}^{n_L \times n_L}$ are assumed to be symmetric and positive definite, and

$$A = \begin{bmatrix} A_C & A_L & A_R & A_V & A_I \end{bmatrix}, \tag{5.7}$$

denotes the (reduced) incidence matrix of the circuit graph with $A_C \in \mathbb{R}^{n_\eta, n_C}$, $A_L \in \mathbb{R}^{n_\eta, n_L}$, $A_R \in \mathbb{R}^{n_\eta, n_R}$, $A_V \in \mathbb{R}^{n_\eta, n_V}$, and $A_I \in \mathbb{R}^{n_\eta, n_I}$. Here n_η denotes the number of nodes in the circuit (without the reference node), n_V denotes the number of voltage sources, n_I the number of current sources, n_C the number of capacitors, n_L the number of inductors, and n_R the number of resistors in the circuit, respectively. We restrict to independent current and voltage sources described by the source functions $I_s(t)$ and $V_s(t)$, respectively. Moreover, η denotes the vector of all node potentials, ν_C, ν_R, ν_I denotes the vectors of all branch voltages through capacitive branches, resistive branches and branches corresponding to current sources, respectively, whereas $\imath_C$, $\imath_L$, $\imath_R$, $\imath_V$ denote the vectors of all branch currents for capacitive, inductive, resistive branches and branches corresponding to voltages sources, respectively.
If we consider the last equation in (5.6) as an output equation, we obtain a linear pHDAE of the form (1.1) with

$$E = \left[\begin{array}{cc|ccccc} C & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & L & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad R = \left[\begin{array}{cc|ccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$J = \left[\begin{array}{cc|ccccc} 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & A_L^T\\ \hline -I & 0 & 0 & 0 & 0 & 0 & A_C^T\\ 0 & 0 & 0 & 0 & 0 & -I & A_R^T\\ 0 & 0 & 0 & 0 & 0 & 0 & A_V^T\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & -A_L & -A_C & -A_R & -A_V & 0 & 0 \end{array}\right], \quad B = \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\\ -I & 0\\ 0 & 0\\ 0 & -A_I \end{bmatrix}, \quad Q = I,$$

as well as $K = 0$, $P = 0$ for the state variables $x = [\nu_C^T, \imath_L^T, \imath_C^T, \imath_R^T, \imath_V^T, \nu_R^T, \eta^T]^T$, input variables $u = \begin{bmatrix} V_s \\ I_s \end{bmatrix}$, and output variables $y = \begin{bmatrix} -\imath_V \\ -\nu_I \end{bmatrix}$. The Hamiltonian is given by

$$\mathcal{H}(x) = \frac{1}{2} x^T E^T Q x = \frac{1}{2} \nu_C^T C \nu_C + \frac{1}{2} \imath_L^T L \imath_L$$

describing the total energy of the system. We have the following result on the index of the circuit equations (5.6).

Theorem 5.1 *[17] Consider an electrical circuit with circuit equations* (5.6)*. Assume that A_V has full column rank, $[A_C \ A_L \ A_R \ A_V]$ has full row rank, and that C, L and G are symmetric and positive definite. Then the port-Hamiltonian circuit equations* (5.6) *are of strangeness index $\mu = 0$ if and only if* $\operatorname{rank}[A_R, A_C, A_V] = n_\eta - 1$ *and* $\ker[A_C, A_V] = \{0\}$*. The port-Hamiltonian circuit equations* (5.6) *are of strangeness index $\mu = 1$ if* $\operatorname{rank}[A_R, A_C, A_V] < n_\eta - 1$ *or* $\ker[A_C, A_V] \neq \{0\}$*.*

In the port-Hamiltonian formulation of the circuit equations (5.6) we have again a nonsingular matrix Q and symmetric products $Q^T E \geq 0$ and $EQ^T \geq 0$. Thus, the results of Corollary 4.14 and for non-resistive circuits also the results of Corollary 4.11 apply.

6 Conclusion

We have considered linear port-Hamiltonian differential-algebraic equations that arise in the energy-based modeling of constrained dynamical systems and the corresponding skew-adjoint differential-algebraic operator that is related to this structure. For skew-adjoint differential-algebraic equations we have developed structure preserving condensed forms under orthogonal and non-orthogonal congruence transformations in Theorem 3.3 and Corollary 3.4. These condensed forms require some constant rank assumptions that are usually satisfied in the common applications and are trivially satisfied for systems with constant coefficients. The constant rank restriction can also be removed by considering the system in a piecewise manner, see [13]. Based on the derived condensed forms an analysis of existence and uniqueness of solutions and of the index of skew-adjoint DAEs is possible (Corollary 3.5). In particular, for regular skew-adjoint DAEs with positive semi-definite leading matrix we have shown that the strangeness index is always less than or equal to 1 (Theorem 3.7).
In the second part of the paper we have derived condensed forms for linear port-Hamiltonian DAEs under orthogonal and non-orthogonal equivalence transformations (Theorem 4.5 and Theorem 4.6). Under additional structural properties (that are often satisfied in applications) these condensed forms can be further simplified (Corollary 4.9 and Corollary 4.10). As a result we get that a regular linear port-Hamiltonian DAE with pointwise nonsingular matrix function Q and no dissipation terms always has a strangeness index less than or equal to 1 (Corollary 4.11). Again, the derived condensed forms allow us to analyze existence and uniqueness of solutions as well as the index of linear port-Hamiltonian DAEs (Theorem 4.12, Corollary 4.13 and Corollary 4.14). If a linear port-Hamiltonian DAE is of strangeness index larger than 1, then the matrix function R that describes the energy dissipation in the system is responsible for the higher index. It was stated in [22] that port-Hamiltonian DAEs are of differentiation index at most one (i.e. strangeness-free). That this is not the case can be seen in the presented examples. In case of higher-index pHDAEs a regularization procedure is necessary. We refer

to [2], where a structure preserving regularization procedure for port-Hamiltonian DAEs has been presented.

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