IET Control Theory & Applications

Research Article

Periodicity and stability analysis of periodic switched positive systems

ISSN 1751-8644
Received on 27th March 2019
Revised 7th July 2019
Accepted on 27th September 2019
E-First on 22nd October 2019
doi: 10.1049/iet-cta.2019.0336
www.ietdl.org

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Abstract: Addressed in this study are the minimal period and stability issues of periodic switched positive systems. With the assumption that all the system matrices, delays and switching signal are periodic, it is shown that the considered switched system is also periodic with the minimal period being a divisor of the least common multiple of the periods of subsystems and switching signal, and an algorithm is presented to determine the minimal period. Then some necessary and sufficient exponential stability conditions are proposed for periodic switched positive systems. These conditions are also extended to a more general class of periodic switched systems. Finally, a numerical example with three cases is provided to demonstrate the effectiveness of the theoretical results and reveals two interesting facts: The system matrices of periodic positive systems are unnecessarily non-negative and system delays influence system stability. These facts imply that there exist some remarkable differences between general switched positive systems and periodic switched positive systems.

1 Introduction

Positive systems are a special kind of dynamical systems whose system states remain non-negative provided their initial states are non-negative. This kind of systems have wide applications in many fields, such as bioengineering [1], environmentology [2], finance [3] and electronics [4]. As stability is one of the most important properties in dynamical systems, many stability results of positive systems have been presented. Farina and Rinaldi [5] presented an equivalent relation between asymptotic stability and diagonal square stability of positive systems [5], which plays a fundamental role in the analysis and synthesis of positive systems. As for delayed positive systems, it was found that the system stability is completely determined by system matrices and is not affected by delay size if the delays are bounded [6–8]. This property was then extended to the case where the delays are unbounded [9–11]. Similar topics are investigated in [12–18].

As a type of complex positive systems, switched positive systems have attracted considerable attention [19–21]. The analysis and synthesis for such systems are challenging due to the fact that they possess both characteristics of switched systems and positive ones. Several kinds of switching signals are widely considered in the literature on switched positive systems, e.g. arbitrary switching [22, 23], average dwell-time switching [24, 25] and Markovian switching [26, 27]. The authors of these references show that switching signals have significant influence on the properties of switched systems. In this paper, we consider a special type of switching signal called periodic switching signal [28, 29]. Actually, many engineering systems in the real world display periodicity, such as clocks, automatic assembly lines and traffic lights. Such systems are said to be periodic [30–34]. In 2010, the concept of periodic positive systems was proposed by Bougatef *et al.* [35]. Then, positivity and stability of periodic positive systems were discussed in [36]. The same topics of periodic positive systems with constant delays and periodic delays were, respectively, studied in [37, 38], each of the two references concludes that the least common multiple of minimal periods of system matrices and delays is the minimal period of the system. In this paper, a special kind of switched systems called periodic switched positive systems with delays are considered and three properties (periodicity, positivity and stability) are discussed.

For the considered system, it is assumed that all the minimal periods of subsystems and switching signal are known. One of the main tasks is to determine the minimal period of the overall switched system. Clearly, the minimal period of a periodic switched system is a key parameter of the system. It seems that, generally, there is nothing to study for this parameter because it is known in many situations. However, in some complicated cases (say, the context of this paper), determining the minimal period is not an easy work because the minimal period of the switched system not only depends on the given minimal periods but also on the specific configurations of system matrices and the value of delays. Ignoring this fact, the results about system periodicity in [37, 38] are actually questionable.

The contribution in this paper lies in three points. (i) It is shown that the minimal period of a periodic switched system is a divisor of the least common multiple of all the periods of subsystems and switching signal. A reduced case (non-switched system) is that the minimal period of a periodic system is a divisor of the least common multiple of periods of the system matrices and delays, which refines the results in [37, 38]. (ii) With the minimal period known, a positivity condition is proposed to guarantee the system to be positive. Then, two equivalent necessary and sufficient exponential stability conditions of periodic switched positive systems are presented. (iii) It is found that the two stability conditions aforementioned are necessary and sufficient to determine the stability of a more general class of periodic switched systems.

In addition, it is interesting to see the following two aspects: first, the system matrices of periodic positive systems are unnecessarily non-negative, and second, the delays influence the stability of periodic switched positive systems.

The remainder of this paper is organised as follows: In Section 2, some necessary preliminaries are presented, and some definitions and lemmas are provided. Section 3 presents a method to determine to minimal period of systems, and then proposes a positivity condition and two sufficient and necessary stability conditions. Numerical examples are provided in Section 4 to demonstrate the effectiveness of the presented results, and Section 5 concludes this paper.

2 Preliminaries

Some basic notations, lemmas and definitions are listed below for later use.

$A \succeq 0(\succ 0, \preceq 0, \prec 0)$: All elements of matrix A are non-negative (positive, non-positive, negative).

$A > 0(<0)$: Matrix A is positive (negative) definite.

A^{T}: Transpose of matrix A.

$\mathbb{R}^{i\times j}(\mathbb{R}_{0,+}^{i\times j}, \mathbb{R}_{+}^{i\times j})$: The set of $i\times j$-dimensional real (non-negative, positive) matrices, and let $\mathbb{R}^{i\times 1} = \mathbb{R}^{i}$ and $\mathbb{R}^{1\times 1} = \mathbb{R}$.

$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.

$\mathbb{N}_0$: $\{0\}\bigcup\mathbb{N}$.

$\underline{q}$: $\{1, 2, \ldots, q\}$ with $q\in\mathbb{N}$.

$\mathrm{lcm}(a, b)$: The least common multiple of a and b with $a, b\in\mathbb{N}$.

$a \bmod b$: The reminder of a divided by b with $a, b\in\mathbb{N}$.

$\rho(A)$: The spectral radius of matrix A.

$\|\boldsymbol{x}\|$: Infinite norm of vector $\boldsymbol{x}$, i.e. $\|\boldsymbol{x}\| = \max_{i\in\{1,\ldots,m\}}\{|x_i|\}$, where $\boldsymbol{x} = [x_1, x_2, \ldots, x_m]^{\mathrm{T}}$.

$\|A\|$: Induced infinite norm of matrix A, i.e. $\|A\| = \max\{\|A\boldsymbol{x}\| : \|\boldsymbol{x}\| = 1\}$.

Throughout this paper, we define $\boldsymbol{x}(k)\in\mathbb{R}^n$, i.e. the dimension of vector $\boldsymbol{x}(k)$ is n. Let $0_{j,k}$ denote the $jn\times kn$-dimensional zero matrix, and $I_{j,k}$ the $jn\times kn$-dimensional unit matrix, especially, $I_{1,1} = I$.

A function $\alpha: [0, a)\to[0,\infty)$ is called class $\mathscr{K}$ function if it is strictly increasing and $\alpha(0) = 0$, and a function $\beta: [0,\infty)\times\mathbb{N}_0\to[0,\infty)$ is called class $\mathscr{KL}$ function if $\beta(\cdot, k)$ is of class $\mathscr{K}$ for every fixed $k\geq 0$ and $\beta(r, k)\to 0$ as $k\to\infty$ for every fixed $r\geq 0$.

Consider the following system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{x}(k-\tau(k)), \quad k\in\mathbb{N}_0 \\ \boldsymbol{x}(k) &= \boldsymbol{\varphi}(k), \quad k\in\{-\tau, -\tau+1, \ldots, 0\} \end{aligned} \tag{1}$$

where $\boldsymbol{x}(k)\in\mathbb{R}^n$ is the state variable and $A(k), B(k)\in\mathbb{R}^{n\times n}$ are system matrices, $\boldsymbol{\varphi}(\cdot)$ is the initial condition, $0\leq\tau(k)\leq\tau$. Let $\|\boldsymbol{\varphi}\| = \sup_{-\tau\leq k\leq 0}\|\boldsymbol{\varphi}(k)\|$.

Definition 1: System (1) is positive if $\boldsymbol{x}(k)\succeq 0$ for all $k\in\mathbb{N}$ whenever $\boldsymbol{\varphi}(\cdot)\succeq 0$.

Definition 2: System (1) is asymptotically stable if there exists a class $\mathscr{KL}$ function β satisfying $\|\boldsymbol{x}(k)\|\leq\beta(\|\boldsymbol{\varphi}\|, k)$ for all $k\in\mathbb{N}$, and is exponentially stable if there exist two scalars $\alpha > 0$, $r > 1$ satisfying $\|\boldsymbol{x}(k)\|\leq\alpha r^{-k}\|\boldsymbol{\varphi}\|$ for all $k\in\mathbb{N}$.

Lemma 1: System

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k), \quad k\in\mathbb{N}_0 \tag{2}$$

is positive if and only if $A\succeq 0$.

Since system (2) is linear time-invariant, its asymptotic stability and exponential stability are equivalent.

The following lemma shows that positive systems possess some particular stability properties.

Lemma 2: Positive system (2) is asymptotically stable if and only if one of the following statements hold:

(i) There exists a vector $\boldsymbol{\lambda}\in\mathbb{R}_+^n$ satisfying $(A-I)\boldsymbol{\lambda}\prec 0$.
(ii) There exists a diagonal positive definite matrix P satisfying $A^{\mathrm{T}}PA - P < 0$.

Now consider the following time-varying system:

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k), \quad k\in\mathbb{N}_0 \tag{3}$$

where $A(k)\in\mathbb{R}^{n\times n}$ is the time-varying system matrix.

Definition 3: If there exists $p\in\mathbb{N}$ satisfying $A(k+p) = A(k)$ for all $k\in\mathbb{N}_0$, then the system (3) is said to be periodic and has a period p; if p is the minimal one possessing this property, then we say (3) has minimal period p, or briefly, system (3) is p-periodic.

Remark 1: The following two points about period should be noted:

(i) In general literature, minimal period is briefly called period. However, in this paper, the difference between period and minimal period needs to be distinguished.
(ii) If $p = 1$, then $A(k) = A(0)$ holds for all $k\in\mathbb{N}$, thus system (3) is constant, which is a special case of periodic systems.

Lemma 3: Suppose system (3) is p-periodic, it is positive if and only if

$$\prod_{i=0}^{l} A(i) = A(l)A(l-1)\cdots A(0)\succeq 0, \quad \forall l\in\{0, 1, \ldots, p-1\}$$

Lemmas 1 and 2 can be found in [5] and Lemma 3 in [36].

3 Main results

In this section, we first discuss the minimal period of switched systems with delays and then investigate their positivity and stability.

3.1 Periodicity analysis of systems with delays

Applying the augmented method [39], one has the following equivalent system of (1):

$$\boldsymbol{y}(k+1) = C(k)\boldsymbol{y}(k), \quad k\in\mathbb{N}_0 \tag{4}$$

where $\boldsymbol{y}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \ldots, \boldsymbol{x}^{\mathrm{T}}(k-\tau)]^{\mathrm{T}}\in\mathbb{R}^{n(\tau+1)}$, and $C(k)\in\mathbb{R}^{n(\tau+1)\times n(\tau+1)}$, for $\tau(k) > 0$

$$C(k) = \begin{bmatrix} A(k) & 0_{1,\tau(k)-1} & B(k) & 0_{1,\tau-\tau(k)} \\ I & & & 0_{1,\tau} \\ 0_{1,1} & I & & 0_{1,\tau-1} \\ \vdots & & \ddots & \vdots \\ 0_{1,\tau-1} & & I & 0_{1,1} \end{bmatrix} \tag{5}$$

and for $\tau(k) = 0$

$$C(k) = \begin{bmatrix} A(k)+B(k) & & 0_{1,\tau} \\ I & & 0_{1,\tau} \\ 0_{1,1} & I & 0_{1,\tau-1} \\ \vdots & \ddots & \vdots \\ 0_{1,\tau-1} & I & 0_{1,1} \end{bmatrix}$$

Note that $C(k)$ in (5) is defined by block submatrices in the following manner: The first n rows of $C(k)$ is partitioned into four parts: $A(k), 0_{1,\tau(k)-1}, B(k), 0_{1,\tau-\tau(k)}$. Similar, the second n rows of $C(k)$ consists of two submatrices: $I, 0_{1,\tau}$. The remaining part of $C(k)$ can be understood in the same way.

Theorem 1: Suppose $A(k)$, $B(k)$ and $\tau(k)$ are p_a-, p_b- and p_τ-periodic, respectively, where p_a, p_b, $p_\tau\in\mathbb{N}$. Let $\overline{p} = \mathrm{lcm}(p_a, p_b, p_\tau)$. The minimal period of system (1) (or equivalently (4)) is a divisor of $\overline{p}$ and can be calculated by Algorithm 1 (see Fig. 1).

Proof: Note that the following fact holds:

Fact 1: If p is the minimal period of system (3), then the set of all the periods of system (3) is $\{ip: i\in\mathbb{N}\}$.

Indeed, given $i\in\mathbb{N}$, ip is a period since

$$A(k+ip) = A(k+(i-1)p) = \cdots = A(k), \quad k\in\mathbb{N}_0$$

Moreover, suppose that there exists a period γ not in the set $\{ip : i \in \mathbb{N}\}$. Thus, there exist $c \in \mathbb{N}, m \in \mathbb{N}_0$ with $c < p$ satisfying $\gamma = mp + c$. Since p is the minimal period, it yields that

$$\begin{aligned} A(k) &= A(k+\gamma) \\ &= A(k+mp+c) \\ &= A(k+(m-1)p+c) \\ &\cdots \\ &= A(k+c), \quad \forall k \in \mathbb{N}_0 \end{aligned} \tag{6}$$

That is, $A(k) = A(k+c)$ for all $k \in \mathbb{N}_0$, which implies that c is also a period of system (3). This conclusion contradicts the assumption that p is the minimal period.

Now consider system (1). Since $\overline{p} = \mathrm{lcm}(p_a, p_b, p_\tau)$, there exists a scalar $d \in \mathbb{N}$ such that $\overline{p} = p_a d$, and hence $A(k+\overline{p}) = A(k+p_a d) = A(k)$. Similarly, $B(k+\overline{p}) = B(k)$, $\tau(k+\overline{p}) = \tau(k)$. By the definition of $C(k)$, it follows that $C(k+\overline{p}) = C(k)$ for all $k \in \mathbb{N}_0$, which means $\overline{p}$ is a period of matrix $C(k)$. By Fact 1, there exists some $i \in \mathbb{N}$ such that $ip = \overline{p}$, i.e. p is a divisor of $\overline{p}$. □

Remark 2: Two points should be pointed out:

(i) The following two special cases should be noted for Algorithm 1 (Fig. 1):

a. If $C(\jmath) = C(\ell)$ for any $\jmath, \ell \in \{0, 1, \ldots, \overline{p}-1\}, \jmath \neq \ell$, then the minimal period of the system is 1.
b. If $C(\jmath) \neq C(\ell)$ for any $\jmath, \ell \in \{0, 1, \ldots, \overline{p}-1\}, \jmath \neq \ell$, then the minimal period of the system is $\overline{p}$.

(ii) The minimal period p of system (1) depends on not only the periods of $A(k), B(k)$ and $\tau(k)$ but also their specific configurations.

Now consider the following switched system:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A_{\sigma(k)}(k)\boldsymbol{x}(k) + B_{\sigma(k)}(k)\boldsymbol{x}(k-\tau_{\sigma(k)}(k)),\ k \in \mathbb{N}_0 \\ \boldsymbol{x}(k) &= \boldsymbol{\varphi}(k), \quad k \in \{-\tau, -\tau+1, \ldots, 0\} \end{aligned} \tag{7}$$

where $\sigma : \mathbb{N}_0 \to \underline{q}$ is a switching signal with q being the number of subsystems, $A_i(k), B_i(k) (i \in \underline{q})$ are system matrices of the ith subsystem, $\tau \in \mathbb{N}$ is the maximal delay and $\tau_i(k) \in \{0, 1, \ldots, \tau\}$.

Similar to (4), (7) can be recast as

$$\boldsymbol{y}(k+1) = C_{\sigma(k)}(k)\boldsymbol{y}(k), \quad k \in \mathbb{N}_0 \tag{8}$$

where $\boldsymbol{y}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \ldots, \boldsymbol{x}^{\mathrm{T}}(k-\tau)]^{\mathrm{T}} \in \mathbb{R}^{n(\tau+1)}$, and $C_{\sigma(k)}(k) \in \mathbb{R}^{n(\tau+1)\times n(\tau+1)}$, for $\tau_{\sigma(k)}(k) > 0$

$$C_{\sigma(k)}(k) = \begin{bmatrix} A_{\sigma(k)}(k) & 0_{1,\tau_{\sigma(k)}(k)-1} & B_{\sigma(k)}(k) & 0_{1,\tau-\tau_{\sigma(k)}(k)} \\ I & & & 0_{1,\tau} \\ 0_{1,1} & I & & 0_{1,\tau-1} \\ \vdots & \ddots & & \vdots \\ 0_{1,\tau-1} & & I & 0_{1,1} \end{bmatrix}$$

and for $\tau_{\sigma(k)}(k) = 0$

$$C_{\sigma(k)}(k) = \begin{bmatrix} A_{\sigma(k)}(k) + B_{\sigma(k)}(k) & & 0_{1,\tau} \\ I & & 0_{1,\tau} \\ 0_{1,1} & I & 0_{1,\tau-1} \\ \vdots & \ddots & \vdots \\ 0_{1,\tau-1} & I & 0_{1,1} \end{bmatrix}$$

Assumption 1: System (7) (or (8)) satisfying the following two conditions:

Input: $\overline{p}$ and all augmented system matrices $C(0), C(1), \ldots, C(\overline{p}-1)$
Output: p being the minimal period
1: Compute all the divisors of $\overline{p}$ and sort them in an increasing order: $q_1, q_2, \ldots, q_\xi$ %$q_1 = 1, q_\xi = \overline{p}$
2: $v \leftarrow 1$
3: $u = q_\xi / q_v, k \leftarrow 0$
4: **if** $C(k) = C(k+q_v) = \ldots = C(k+(u-1)q_v)$ **then**
5: Let $k \leftarrow k+1$
6: **if** $k < q_v$ **then**
7: Return to 4
8: **else**
9: Break to 12
10: **else**
11: Let $v \leftarrow v+1$ and return to 3
12: $p \leftarrow q_v$
13: **return** p

Fig. 1 *Algorithm 1: Calculate minimal period of (1)*

Input: $p_{i_j}, C_{i_j}(\pi_{jl})$ for $l \in \{0, 1, \ldots, r_j\}, j \in \{1, 2, \ldots, \theta\}, p_\sigma$ and $\sigma(k) = i_j$ for $k \in \{0, 1, \ldots, p_\sigma - 1\}$
Output: p being the minimal period
1: Compute $\overline{p}$ =lcm(p_{i_1},p_{i_2},...,p_{i_θ},p_σ) and all the divisors of $\overline{p}$, sort divisors in increasing order: $q_1, q_2, \ldots, q_\xi$ % $q_1 = 1, q_\xi = \overline{p}$.
2: **for** $k \leftarrow 0 : \overline{p}-1$ **do**
3: $i_j \leftarrow \sigma(k \bmod p_\sigma), l \leftarrow k \bmod p_{i_j}, D(k) \leftarrow C_{i_j}(\pi_{jl})$
4: **end for**
5: Let $v \leftarrow 1$
6: $u \leftarrow q_\xi / q_v, k \leftarrow 0$
7: **if** $D(k) = D(k+q_v) = \ldots = D(k+(u-1)q_v)$ **then**
8: Let $k \leftarrow k+1$
9: **if** $k < q_v$ **then**
10: Return to 6
11: **else**
12: Break to 14
13: **else**
14: Let $v \leftarrow v+1$ and return to 5
15: $p \leftarrow q_v$
16: **return** p

Fig. 2 *Algorithm 2: Calculate minimal period of (7)*

(i) The minimal period of the ith subsystem is $p_i (i \in \underline{q})$ and the minimal period of the switching signal σ is p_σ.
(ii) There are θ subsystems being activated during a minimal switching period, i.e. there exists a subset $\{i_1, i_2, \ldots, i_\theta\}$ of $\underline{q}$ having the property that for any given $k \in \{0, 1, \ldots, p_\sigma - 1\}$, there exists a unique $j \in \{1, 2, \ldots, \theta\}$ for which $\sigma(k) = i_j$. Let $\overline{p} = \mathrm{lcm}(p_{i_1}, p_{i_2}, \ldots, p_{i_\theta}, p_\sigma)$.

Subsystems activated during a switching period are all from $\underline{q}$, and one subsystem may be activated more than once and may not be activated in a switching period. Therefore, $\theta \leq q$ and $\theta \leq p_\sigma$.

Theorem 2: Assume that Assumption 1 holds. Define

$$\begin{aligned} K_j &= \{k : \sigma(k) = i_j, k \in \{0, 1, \ldots, \overline{p}-1\}\} \\ &\triangleq \{k_1^j, k_2^j, \ldots, k_{\eta_j}^j\}, \quad j \in \{1, 2, \ldots, \theta\} \\ \pi_j &= \{k_r^j \, mod \, p_{i_j} : r \in \{1, 2, \ldots, \eta_j\}\} \\ &= \{\pi_{j0}, \pi_{j1}, \ldots, \pi_{jr_j}\}, \quad r_j < p_{i_j} \end{aligned} \tag{9}$$

The minimal period of system (7) (or equivalently (8)) is a divisor of $\overline{p}$, and can be calculated by Algorithm 2 (see Fig. 2).

Proof: By the definition of $\overline{p}$, there exists a set of positive integers $d_1, \ldots, d_\theta, d_\sigma$ such that

$$p_{i_j}d_j = \overline{p}, \quad j \in \{1, 2, \ldots, \theta\}$$
$$p_\sigma d_\sigma = \overline{p}$$

It holds that

$$\begin{aligned} C_{i_j}(k + \overline{p}) &= C_{i_j}(k + p_{i_j}d_j) \\ &= C_{i_j}(k), \quad k \in \mathbb{N}_0, j \in \{1, 2, \ldots, \theta\} \\ \sigma(k + \overline{p}) &= \sigma(k + p_\sigma d_\sigma) = \sigma(k), \quad k \in \mathbb{N}_0 \end{aligned} \tag{10}$$

For any $k \in \mathbb{N}_0$, there exists a unique $j \in \{1, 2, \ldots, \theta\}$ such that $\sigma(k) = i_j$. It follows from (10) that

$$\begin{aligned} C_{\sigma(k+\overline{p})}(k + \overline{p}) &= C_{\sigma(k + p_\sigma d_\sigma)}(k + \overline{p}) \\ &= C_{\sigma(k)}(k + \overline{p}) \\ &= C_{i_j}(k + p_{i_j}d_j) \\ &= C_{i_j}(k) \\ &= C_{\sigma(k)}(k) \end{aligned}$$

in other words, $\overline{p}$ is a period of system (8) (or (7)). By Theorem 1, it implies that if p is the minimal period of system (7), then p is a divisor of $\overline{p}$.

Define

$$\begin{aligned} \pi_{j,l} &= \{k_r^j : k_r^j \bmod p_{i_j} = \pi_{j_l}, r \in \{1, 2, \ldots, \eta_j\}\} \\ & l \in \{0, 1, \ldots, r_j\}, \quad r_j < p_{i_j}, \quad j \in \{1, 2, \ldots, \theta\} \end{aligned}$$

where k_r^j, π_{j_l} are in (9). Clearly

$$\bigcup_{l=0}^{r_j} \pi_{j,l} = K_j, \quad \bigcup_{j=1}^{\theta} \bigcup_{l=0}^{r_j} \pi_{j,l} = \{0, 1, \ldots, \overline{p} - 1\}$$

Given $k \in \pi_{j,l}$, there exists a scalar $\alpha_j \in \mathbb{N}_0$ such that $k = \alpha_j p_{i_j} + \pi_{j_l}$ and hence

$$\begin{aligned} C_{\sigma(k)}(k) &= C_{i_j}(k) \\ &= C_{i_j}(\alpha_j p_{i_j} + \pi_{j_l}) \\ &= C_{i_j}(\pi_{j_l}), \quad k \in \{0, 1, \ldots, \overline{p} - 1\} \end{aligned} \tag{11}$$

Let $D(k) = C_{\sigma(k)}(k)$. Equation (11) shows for all $k \in \{0, 1, \ldots, \overline{p} - 1\}$, $D(k) \in \bigcup_{j=1}^{\theta} \bigcup_{l=0}^{r_j} C_{i_j}(\pi_{j_l})$. Moreover, since $r_j < p_{i_j}$ and $\overline{p} = \mathrm{lcm}(p_{i_1}, p_{i_2}, \ldots, p_{i_\theta}, p_\sigma)$, the number of the elements in the set $\{C_{i_j}(\pi_{j_l}) : l \in \{0, 1, \ldots, r_j\}, j \in \{1, 2, \ldots, \theta\}\}$ may less than the length of the sequence $\overline{p}$. Based on the above results, the number of considered matrices may be remarkably reduced. □

Remark 3: The minimal period of system (7) depends on not only the switching sequence $\{\sigma(k)\}_{k=0}^{p_\sigma - 1}$, but also the specific configurations of $A_{i_j}(k), B_{i_j}(k)$ and $\tau_{i_j}(k)$ of i_jth subsystem. See Case 1 in Example section for details.

3.2 Positivity and stability analysis of periodic switched positive systems

The purpose of this section is to study the properties of periodic switched positive systems. A positivity condition is presented first, and two exponential stability conditions are proposed.

Theorem 3: Consider system (7) and suppose it is p-periodic. The following statements hold:

(i) The system is positive if and only if

$$\prod_{k=0}^{l} C_{\sigma(k)}(k) \geq 0, \quad l \in \{0, 1, 2, \ldots, p - 1\} \tag{12}$$

(ii) Suppose the system is positive. Then the system is exponentially stable if and only if one of the following conditions holds:

a. There exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^{(\tau+1)n}$ satisfying

$$\left(\prod_{k=0}^{p-1} C_{\sigma(k)}(k) - I_{\tau+1,\tau+1}\right)\boldsymbol{\lambda} \prec 0 \tag{13}$$

b. There exists a diagonal positive definite matrix $P \in \mathbb{R}^{(\tau+1)n \times (\tau+1)n}$ satisfying

$$\left(\prod_{k=0}^{p-1} C_{\sigma(k)}(k)\right)^{\mathrm{T}} P \left(\prod_{k=0}^{p-1} C_{\sigma(k)}(k)\right) - P < 0 \tag{14}$$

Proof: By Lemma 3, together with (8), item (i) follows immediately. It is worth pointing out that the positivity of switched systems is depended on switching signal. Indeed, a system being positive under a switching signal may not be positive under another.

Now, we show (a) in item (ii). The corresponding proof is divided into two parts: sufficiency and necessity.

Sufficiency. Since (7) is positive and periodic, (12) holds clearly. Hence

$$\prod_{k=0}^{p-1} C_{\sigma(k)}(k) \succeq 0 \tag{15}$$

Let $\bar{C} = \prod_{k=0}^{p-1} C_{\sigma(k)}(k)$, then by (ii) in Lemma 2, it follows that the following linear positive time-invariant system:

$$\boldsymbol{z}(t+1) = \bar{C}\boldsymbol{z}(t), \quad t \in \mathbb{N}_0 \tag{16}$$

is exponentially stable, i.e. there exist $\alpha > 0, r > 1$ satisfying

$$\| \boldsymbol{z}(t) \| \leq \alpha r^{-t} \| \boldsymbol{z}(0) \|, \quad t \in \mathbb{N}_0 \tag{17}$$

Setting $\boldsymbol{z}(0) = \boldsymbol{y}(0)$ for system (8) and (16), it holds that

$$\boldsymbol{y}(pt) = \boldsymbol{z}(t), \quad t \in \mathbb{N}_0 \tag{18}$$

therefore

$$\| \boldsymbol{y}(pt) \| \leq \alpha r^{-t} \| \boldsymbol{y}(0) \|, \quad t \in \mathbb{N}_0 \tag{19}$$

On the other hand, for any $l \in \{1, 2, \ldots, p - 1\}$, it follows that

$$\begin{aligned} \boldsymbol{y}(pt + l) &= C_{\sigma(pt+l-1)}(pt + l - 1)\boldsymbol{y}(pt + l - 1) \\ &= C_{\sigma(pt+l-1)}(pt + l - 1)C_{\sigma(pt+l-2)}(pt + l - 2) \\ &\quad \cdots C_{\sigma(pt)}(pt)\boldsymbol{y}(pt) \\ &= C_{\sigma(l-1)}(l - 1)C_{\sigma(l-2)}(l - 2) \\ &\quad \cdots C_{\sigma(0)}(0)\boldsymbol{y}(pt), \quad t \in \mathbb{N}_0 \end{aligned}$$

Set $M = \max_{l \in \{0,1,\ldots,p-1\}} \left\{ \| \prod_{t=0}^{l} C_{\sigma(t)}(t) \| \right\}$ and then we have

$$\begin{aligned} & \| \boldsymbol{y}\ (pt + l) \| \\ &= \| C_{\sigma(l-1)}(l - 1)C_{\sigma(l-2)}(l - 2)\cdots C_{\sigma(0)}(0)\boldsymbol{y}(pt) \| \\ &\leq \| C_{\sigma(l-1)}(l - 1)C_{\sigma(l-2)}(l - 2)\cdots C_{\sigma(0)}(0) \| \ \| \boldsymbol{y}(pt) \| \\ &\leq M \| \boldsymbol{y}(pt) \|, \quad t \in \mathbb{N}_0 \end{aligned} \tag{20}$$

Let $\bar{M} = \max\{1, M\}$. By (19) and (20), it follows that

$$\| \boldsymbol{y}(pt + l) \| \leq \bar{M}\alpha r^{-t} \| \boldsymbol{y}(0) \|, \quad t \in \mathbb{N}_0, l \in \{0, 1, 2, \ldots, p - 1\} \tag{21}$$

Given $k \in \mathbb{N}$, there exist $t \in \mathbb{N}_0, l \in \{0, 1, 2, \ldots, p-1\}$ such that $k = pt + l$, thus, $k \le pt + p$ and $t \ge kp^{-1} - 1$, this inequality, together with (21), indicates that

$$\begin{aligned} \| \boldsymbol{y}(k) \| &\le \bar{M}\alpha r^{-t} \| \boldsymbol{y}(0) \| \\ &\le \bar{M}\alpha r^{-(kp^{-1}-1)} \| \boldsymbol{y}(0) \| \\ &= \bar{M}\alpha r (r^{p^{-1}})^{-k} \| \boldsymbol{y}(0) \|, \quad k \in \mathbb{N}_0 \end{aligned}$$

Let $\alpha_1 = \bar{M}\alpha r$, $r_1 = r^{p^{-1}}$, clearly $\alpha_1 > 0, r_1 > 1$, and the last inequality means

$$\| \boldsymbol{y}(k) \| \le \alpha_1 r_1^{-k} \| \boldsymbol{y}(0) \|, \quad k \in \mathbb{N}_0$$

thus system (8) is exponentially stable. By the formation of $\boldsymbol{y}(k)$ and definition of infinite norm, it yields that

$$\| \boldsymbol{x}(k) \| \le \| \boldsymbol{y}(k) \| \le \alpha_1 r_1^{-k} \| \boldsymbol{y}(0) \|, \quad k \in \mathbb{N}_0 \tag{22}$$

Since $\boldsymbol{y}(0) = [\boldsymbol{\varphi}^{\mathrm{T}}(0), \boldsymbol{\varphi}^{\mathrm{T}}(-1), \ldots, \boldsymbol{\varphi}^{\mathrm{T}}(-\tau)]^{\mathrm{T}}$, $\| \boldsymbol{\varphi} \| = \sup_{-\tau \le k \le 0} \| \boldsymbol{\varphi}(k) \|$, it holds that

$$\| \boldsymbol{x}(k) \| \le \alpha_1 r_1^{-k} \| \boldsymbol{\varphi} \|, \quad k \in \mathbb{N}_0$$

therefore, system (7) is exponentially stable, and the sufficiency part is finished.

Necessity. We prove this part by contradiction. Setting $\boldsymbol{z}(0) = \boldsymbol{y}(0) \ne \boldsymbol{0} \in \mathbb{R}^{(\tau+1)n}$ and assume that the condition (a) in item (ii) does not hold. By Lemma 2, system (16) is not exponentially stable. In other words, for any $\alpha > 0, r > 1$, there must exist an infinite sequence $\{k_i\}_{i=1}^{\infty}$, $k_i \in \mathbb{N}_0$, $k_i \ne k_j$ if $i \ne j$, $k_i \to \infty$ if $i \to \infty$ such that $\| z(k_i) \| > \alpha r^{-k_i} \| z(0) \|$. Indeed, if the sequence $\{k_i\}$ is finite, by scaling, there must exist a scalar b satisfying

$$\| z(k_i) \| \le b\alpha r^{-k_i} \| z(0) \|$$

which implies the system is stable. Therefore, we can obtain from (18) that there exists a infinite subsequence $\boldsymbol{y}(pk_i)$ from $\boldsymbol{y}(k)$ satisfying

$$\| \boldsymbol{y}(pk_i) \| > \alpha r^{-k_i} \| \boldsymbol{y}(0) \|$$

By the formation of $\boldsymbol{y}(k)$ and definition of infinite norm, there exists $l_i \in \{0, 1, \ldots, \tau\}$ and $pk_i \ge \tau$ for each $i \in \mathbb{N}$ such that

$$\| \boldsymbol{x}(pk_i - l_i) \| = \| \boldsymbol{y}(pk_i) \| > \alpha r^{-k_i} \| \boldsymbol{y}(0) \|$$

Since $\boldsymbol{y}(0) = [\boldsymbol{\varphi}^{\mathrm{T}}(0), \boldsymbol{\varphi}^{\mathrm{T}}(-1), \ldots, \boldsymbol{\varphi}^{\mathrm{T}}(-\tau)]^{\mathrm{T}}$, $\| \boldsymbol{\varphi} \| = \sup_{-\tau \le k \le 0} \| \boldsymbol{\varphi}(k) \|$, it holds that

$$\| \boldsymbol{x}(pk_i - l_i) \| > \alpha r^{-k_i} \| \boldsymbol{\varphi} \| = \alpha r^{-(k_i - l_i) - l_i} \| \boldsymbol{\varphi} \| \tag{23}$$

Set $t_i = pk_i - l_i$. Because $\{k_i\}_{i=1}^{\infty}$ is an infinite sequence as well as l_i is bounded, we have $\{t_i\}_{i=1}^{\infty}$ is also an infinite sequence and $t_i \to \infty$ if $i \to \infty$. Since $pk_i \ge \tau$, $l_i \in \{0, 1, \ldots, \tau\}$, we have $t_i \ge 0$. Due to the fact that for $i, j \in \mathbb{N}$, $i \ne j$, the equation $\| \boldsymbol{x}(t_i) \| = \| \boldsymbol{x}(t_j) \|$ may hold, we define a new infinite subsequence $\{\bar{t}_i\}_{i=1}^{\infty}$ of $\{t_i\}_{i=1}^{\infty}$ with the property that $\bar{t}_i \ne \bar{t}_j$ if $i \ne j$, $\bar{t}_i \to \infty$ if $i \to \infty$. Then it follows from (23) that

$$\| \boldsymbol{x}(\bar{t}_i) \| > \alpha r^{-l_i} r^{-\bar{t}_i} \| \boldsymbol{\varphi} \| > \alpha r^{-\tau} r^{-\bar{t}_i} \| \boldsymbol{\varphi} \| \tag{24}$$

Set $\alpha_1 = \alpha r^{-\tau}$, by (24), it holds that

$$\| \boldsymbol{x}(\bar{t}_i) \| > \alpha_1 r^{-\bar{t}_i} \| \boldsymbol{\varphi} \|$$

so (7) is not exponentially stable and a contradiction occurs.

Based on the proof of (a) in item (ii), (b) in item (ii) can be proved directly by item (ii) of Lemma 2. □

Remark 4: The specific structures of matrices $C_{\sigma(k)}(k)$ depend on the parameter values of $A_{\sigma(k)}(k)$, $B_{\sigma(k)}(k)$, $\tau_{\sigma(k)}(k)$ and $\sigma(k)$, to be specific, the parameter values $A_{\sigma(k)}(k)$, $B_{\sigma(k)}(k)$ and $\sigma(k)$ influence the parameter value of $C_{\sigma(k)}(k)$, $\tau_{\sigma(k)}(k)$ influence the position of matrix $B_{\sigma(k)}(k)$ in $C_{\sigma(k)}(k)$, thus any change of parameter values may influence the positivity and stability of the considered system.

In fact, the results above can be used for a more general class (may be not positive) of systems.

Corollary 1: Consider the system (7) and suppose that it is p-periodic and that $\prod_{k=\iota}^{p-1+\iota} C_{\sigma(k)}(k) \ge 0$ for some $\iota \in \{0, \ldots, p-1\}$. The system is exponentially stable if and only if one of the following conditions hold:

(i) There exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_{+}^{(\tau+1)n}$ satisfying

$$\left(\prod_{k=\iota}^{p-1+\iota} C_{\sigma(k)}(k) - I_{\tau+1, \tau+1} \right) \boldsymbol{\lambda} \prec 0 \tag{25}$$

(ii) There exists a diagonal positive definite matrix $P \in \mathbb{R}_{+}^{(\tau+1)n \times (\tau+1)n}$ satisfying

$$\left(\prod_{k=\iota}^{p-1+\iota} C_{\sigma(k)}(k) \right)^{\mathrm{T}} P \left(\prod_{k=\iota}^{p-1+\iota} C_{\sigma(k)}(k) \right) - P < 0 \tag{26}$$

Proof: It suffices to prove item (i) in Corollary 1 since items (i) and (ii) are equivalent. We only prove the sufficiency part of item (i) because the necessity part is very similar to that part of Theorem 3.

If $\iota = 0$, then the proof is identical with that of Theorem 3. Therefore, suppose that $\iota \in \{1, \ldots, p-1\}$. Fix initial function $\boldsymbol{\varphi}(\cdot)$. It follows from (7) that

$$\begin{aligned} \| \boldsymbol{x}(1) \| &\le \| A_{\sigma(0)}(0)\boldsymbol{x}(0) \| + \| B_{\sigma(0)}(0)\boldsymbol{x}(-\tau(0)) \| \\ &\le \| A_{\sigma(0)}(0) \| \| \boldsymbol{x}(0) \| + \| B_{\sigma(0)}(0) \| \| \boldsymbol{\varphi} \| \\ &\le \| A_{\sigma(0)}(0) \| \| \boldsymbol{\varphi} \| + \| B_{\sigma(0)}(0) \| \| \boldsymbol{\varphi} \| \\ &= g_1 \| \boldsymbol{\varphi} \| \end{aligned}$$

with $g_1 = \| A_{\sigma(0)}(0) \| + \| B_{\sigma(0)}(0) \|$.

$$\begin{aligned} \| \boldsymbol{x}(2) \| &\le \| A_{\sigma(1)}(1)\boldsymbol{x}(1) \| + \| B_{\sigma(1)}(1)\boldsymbol{x}(1-\tau(1)) \| \\ &\le \| A_{\sigma(1)}(1) \| \| \boldsymbol{x}(1) \| + \| B_{\sigma(1)}(1) \| \| \boldsymbol{x}(1-\tau(1)) \| \\ &\le \| A_{\sigma(1)}(1) \| g_1 \| \boldsymbol{\varphi} \| + \| B_{\sigma(1)}(1) \| \max\{g_1, 1\} \| \boldsymbol{\varphi} \| \\ &= g_2 \| \boldsymbol{\varphi} \| \end{aligned}$$

where $g_2 = \| A_{\sigma(1)}(1) \| g_1 + \| B_{\sigma(1)}(1) \| \max\{g_1, 1\}$. Inductively, for any $i \in \{1, 2, \ldots, \iota\}$, there must exist a scalar g_i satisfying

$$\| \boldsymbol{x}(i) \| \le g_i \| \boldsymbol{\varphi} \|$$

with g_i being a function in $g_1, \ldots, g_{i-1}, A_{\sigma(i-1)}(i-1), B_{\sigma(i-1)}(i-1)$ and $\| \boldsymbol{\varphi} \|$. Note that ι is given and all $A_{\sigma(i)}(i), B_{\sigma(i)}(i)$ and $\sigma(i), \forall i \in \{1, 2, \ldots, \iota\}$, are known, $g = \sup_{i \in \{1, 2, \ldots, \iota\}} \{g_i\}$, a finite number, necessarily exists. As a result,

$$\| \boldsymbol{x}(i) \| \le g \| \boldsymbol{\varphi} \|, \quad \forall i \in \{1, 2, \ldots, \iota\} \tag{27}$$

Let $\boldsymbol{y}(\iota) = \left[\boldsymbol{x}^{\mathrm{T}}(\iota) \cdots \boldsymbol{x}^{\mathrm{T}}(\iota - \tau)\right]^{\mathrm{T}}$. Since $\prod_{k=\iota}^{p-1+\iota} C_{\sigma(k)}(k) \ge 0$ holds, following a process similar to that from (15) to (22), one claims that there exist two scalars $a_2 > 0, r_2 > 1$ such that

$$\begin{aligned} \| \boldsymbol{x}(k) \| &\le \alpha_2 r_2^{-(k-\iota)} \| \boldsymbol{y}(\iota) \| \\ &= \alpha_2 r_2^{-(k-\iota)} \max_{0 \le m \le \tau} \| \boldsymbol{x}(\iota - m) \|, \quad \forall k \ge \iota \end{aligned}$$

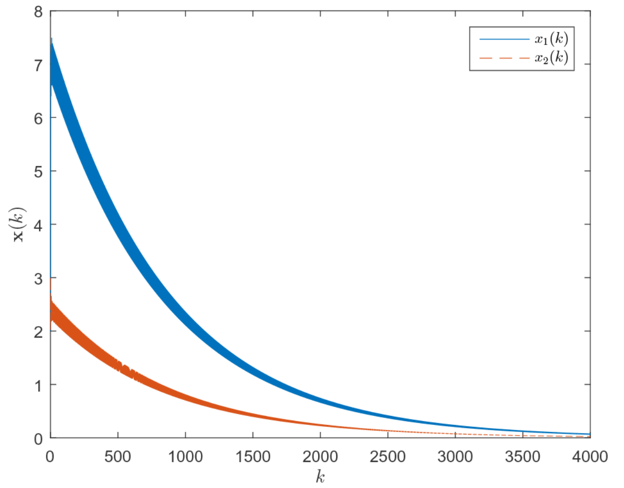

Fig. 3 *Evolution of Case 1 with* $\boldsymbol{\varphi} = [2, 3]^{\mathrm{T}}$

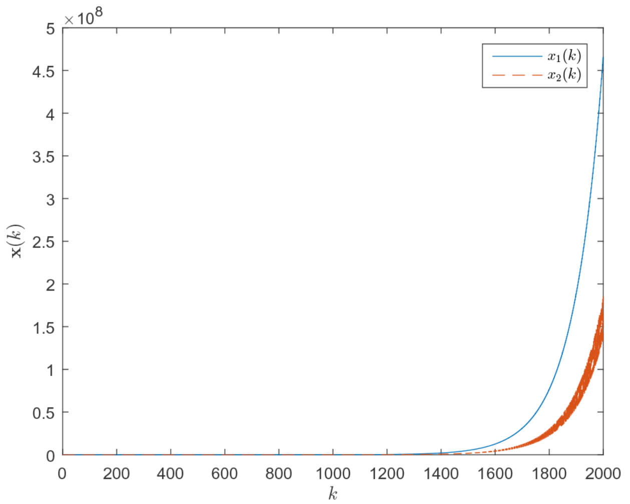

Fig. 4 *Evolution of Case 2 with* $\boldsymbol{\varphi} = [2, 3]^{\mathrm{T}}$

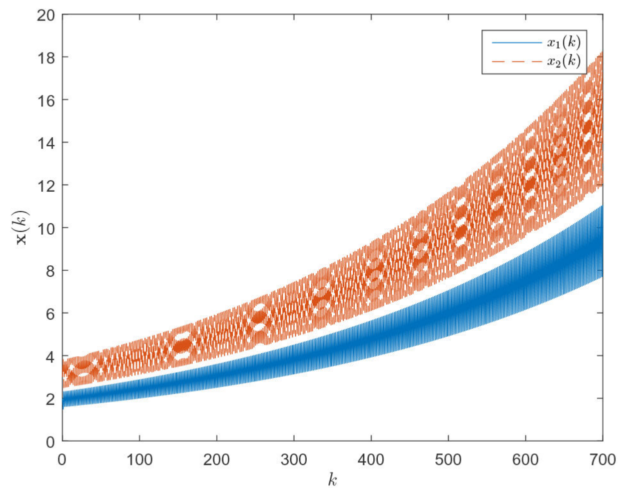

Fig. 5 *Evolution of Case 3 with* $\boldsymbol{\varphi} = [2, 3]^{\mathrm{T}}$

which combining (27) produces

$$\| \boldsymbol{x}(k) \| \le g\alpha_2 r_2^l r_2^{-k} \| \boldsymbol{\varphi} \| = \alpha_3 r_2^{-k} \| \boldsymbol{\varphi} \|, \quad \forall k \ge 0$$

with $\alpha_3 = g\alpha_2 r_2^l$. □

4 Example

We will study an example in this section to illustrate the theoretical results in this paper. Three cases are considered. Case 1 is concerned with the periodicity result, Cases 2 and 3 show how delays and switching signals influence the stability of periodic switched positive systems.

Example 1: Consider

$$\boldsymbol{x}(k+1) = A_{\sigma(k)}(k)\boldsymbol{x}(k) + B_{\sigma(k)}(k)\boldsymbol{x}(k - \tau_{\sigma(k)}(k)) \tag{28}$$

where $\boldsymbol{x}(k) = [x_1(k), x_2(k)]^{\mathrm{T}} \in \mathbb{R}^2$, $\sigma(k) \in \{1, 2\}$.

In the first subsystem, all $A_1(k)$, $B_1(k)$ are 4-periodic with

$$A_1(0) = \begin{bmatrix} 0.48 & 0.33 \\ 0.1 & 0.188 \end{bmatrix}, \quad A_1(1) = \begin{bmatrix} 0.15 & 0.1 \\ 0.33 & 0.347 \end{bmatrix}$$
$$A_1(2) = \begin{bmatrix} 0.15 & 0.1 \\ 0.33 & 0.347 \end{bmatrix}, \quad A_1(3) = \begin{bmatrix} 0.48 & 0.33 \\ 0.1 & 0.188 \end{bmatrix}$$
$$B_1(0) = \begin{bmatrix} 0.12 & 1 \\ 0 & 0.412 \end{bmatrix}, \quad B_1(1) = \begin{bmatrix} 0.1 & 0.4 \\ 0.1 & 0.1 \end{bmatrix}$$
$$B_1(2) = \begin{bmatrix} 0.43 & 0.01 \\ 0.6 & 0.3 \end{bmatrix}, \quad B_1(3) = \begin{bmatrix} -0.01 & 1.1 \\ 0.01 & 0.6 \end{bmatrix}$$

In the second subsystem, all $A_2(k)$, $B_2(k)$ are 4-periodic with

$$A_2(0) = \begin{bmatrix} 0.15 & 0.1 \\ 0.33 & 0.347 \end{bmatrix}, \quad A_2(1) = \begin{bmatrix} 0.48 & 0.33 \\ 0.1 & 0.188 \end{bmatrix}$$
$$A_2(2) = \begin{bmatrix} 0.48 & 0.33 \\ 0.1 & 0.188 \end{bmatrix}, \quad A_2(3) = \begin{bmatrix} 0.15 & 0.1 \\ 0.33 & 0.347 \end{bmatrix}$$
$$B_2(0) = \begin{bmatrix} 0.43 & 0.01 \\ 0.6 & 0.3 \end{bmatrix}, \quad B_2(1) = \begin{bmatrix} -0.01 & 1.1 \\ 0.01 & 0.6 \end{bmatrix}$$
$$B_2(2) = \begin{bmatrix} 0.12 & 1 \\ 0 & 0.412 \end{bmatrix}, \quad B_2(3) = \begin{bmatrix} 0.1 & 0.4 \\ 0.1 & 0.1 \end{bmatrix}$$

Three cases are considered with different delays or different switching signals as follows:

Case 1: $\sigma(0) = 1, \sigma(1) = 2, \sigma(2) = 2, \sigma(3) = 1$, $\tau_1 = \tau_2 = 2$. By Algorithm 2 (Fig. 2), we have Case 1 is 2-periodic and it is not difficult to obtain $C_{\sigma(1)}(1)C_{\sigma(0)}(0) \ge 0$. Using the linprog toolbox in MATLAB, we can get a vector $\boldsymbol{\lambda} = [1.0203\ 0.3944\ 1.3048\ 0.3389\ 1.0388\ 0.3946]^{\mathrm{T}} \in \mathbb{R}_+^6$ satisfying $(\prod_{k=0}^{1} C_{\sigma(k)}(k) - I)\boldsymbol{\lambda} \prec 0$. By Theorem 3, system (28) is positive and stable, as shown in Fig. 3.

.

It is seen that there exist some negative elements in the system matrices of the periodic switched positive system. Therefore, condition (12) is less conservative than that in [40].

Case 2: $\sigma(0) = 1, \sigma(1) = 2, \sigma(2) = 2, \sigma(3) = 1$, $\tau_1 = \tau_2 = 1$. The only difference between Cases 1 and 2 is delay, which causes the system 4-periodic and unstable. The simulation result is plotted in Fig. 4.

Case 3: $\sigma(0) = 2, \sigma(1) = 1, \sigma(2) = 1, \sigma(3) = 2$, $\tau_1 = \tau_2 = 2$. The only difference between Cases 1 and 3 is switching signal, which makes system (28) 4-periodic and unstable. The simulation figure is plotted in Fig. 5.

Comparing Case 1 with Case 2 shows that the stability of system (28) is delay-dependent, while for a non-switched positive system with delays, its stability is not affected by the size of delays. Comparing Case 1 with Case 3, one can see that switching signal may influence the stability of system (28). Observe that there are some negative elements in system matrices, and this fact shows an important difference from the conventional situation of switched positive systems where all system matrices need to be non-negative.

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5 Conclusions

We have addressed three properties (periodicity, positivity and stability) of a class of switched systems with delays. First, a property about the minimal period of periodic switched systems has been revealed, and an algorithm has proposed to determine the minimal period of this kind of systems. Based on this result, a set of necessary and sufficient conditions have been presented to verify the positivity and exponential stability of the considered switched system. A corollary has been put forward to extend the stability results to more general (not only positive) systems. Future studies will be focused on the dynamics of non-linear periodic switched systems.

6 Acknowledgments

This work was partially supported by the National Nature Science Foundation (61673016, 61703353), Innovative Research Team of the Education Department of Sichuan Province (15TD0050), Sichuan Youth Science and Technology Innovation Research Team (2017TD0028), and the Fundamental Research Funds for the Central Universities, Southwest Minzu University (2018YXXS27).

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