Math. Control Signals Syst.
DOI 10.1007/s00498-013-0123-5

ORIGINAL ARTICLE

Robustness of stability of time-varying index-1 DAEs

Thomas Berger

Received: 12 December 2012 / Accepted: 16 December 2013

Abstract We study exponential stability and its robustness for time-varying linear index-1 differential-algebraic equations. The effect of perturbations on the leading coefficient matrix is investigated. An appropriate class of allowable perturbations is introduced. Robustness of exponential stability with respect to a certain class of perturbations is proved in terms of the Bohl exponent and perturbation operator. Finally, a stability radius involving these perturbations is introduced and investigated. In particular, a lower bound for the stability radius is derived. The results are presented by means of illustrative examples.

Keywords Time-varying linear differential-algebraic equations · Exponential stability · Robustness · Bohl exponent · Perturbation operator · Stability radius

List of symbols

$\mathbb{N},\ \mathbb{N}_0,\ \mathbb{R}_+$	The set of natural numbers $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = [0, \infty)$
$\operatorname{im} A$, $\ker A$	The image and kernel of the matrix $A \in \mathbb{R}^{m \times n}$, resp.
$\mathbf{Gl}_n(\mathbb{R})$	The set of all invertible $n \times n$ matrices over $\mathbb{R}$
$\|x\|$	$:= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\|A\|$	$:= \sup\{\, \|Ax\| \mid \|x\| = 1 \,\}$, induced matrix norm of $A \in \mathbb{R}^{n \times m}$
$\mathcal{C}^k(\mathcal{I}; \mathcal{S})$	The set of k-times continuously differentiable functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$, $k \in \mathbb{N}_0$
$\mathcal{B}(\mathcal{I}; \mathcal{S})$	The set of continuous and bounded functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$

T. Berger (✉)
Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany
e-mail: thomas.berger@uni-hamburg.de

Published online: 01 January 2014

$\mathbb{1}_{\mathcal{M}}(t)$	$= \begin{cases} 1, & \text{if } t \in \mathcal{M}, \\ 0, & \text{otherwise,} \end{cases}$ for $t \in \mathbb{R}_+$ and $\mathcal{M} \subseteq \mathbb{R}_+$
dom f	The domain of the function f
$\\|f\\|_\infty$	$:= \sup\{\\|f(t)\\| \mid t \in \text{dom } f\}$ the infinity norm of the function f
$f\mid_{\mathcal{M}}$	The restriction of the function f on a set $\mathcal{M} \subseteq \text{dom } f$
$L^2(\mathcal{I}; \mathcal{S})$	The set of measurable and square integrable functions $f : \mathcal{I} \to \mathcal{S}$ from a set $\mathcal{I} \subseteq \mathbb{R}$ to a vector space $\mathcal{S}$
$\\|f\\|_{L^2[t_0,\infty)}$	$:= \left(\int_{t_0}^{\infty} \\|f(t)\\|^2 \, \mathrm{d}t\right)^{1/2}$ the L^2-norm of the function $f \in L^2([t_0, \infty); \mathcal{S})$, $t_0 \in \mathbb{R}$

1 Introduction

Differential-algebraic equations (DAEs) are a combination of differential equations along with (implicitly hidden) algebraic constraints. They have been discovered as an appropriate tool for modeling a vast variety of problems, e.g. in mechanical multibody dynamics [17], electrical networks [38] and chemical engineering [30], which often cannot be modeled by standard ordinary differential equations (ODEs). These problems indeed have in common that the dynamics are algebraically constrained, for instance by tracks, Kirchhoff laws or conservation laws. A nice example can be found in [27]: a mobile manipulator is modeled as a linear time-varying differential-algebraic control problem. In particular, the power in application is responsible for DAEs being nowadays an established field in applied mathematics and subject of various monographs and textbooks [8,12,31,32]. In this work we study questions related to the robustness of stability of linear time-varying DAEs: the concepts of index-1 DAEs, exponential stability, Bohl exponent, perturbation operator and stability radius. Due to the algebraic constraints in DAEs, most of the classical concepts of the qualitative theory have to be modified and the analysis becomes more involved.

Although differential-algebraic control systems $E(t)\dot{x} = A(t)x + B(t)u(t)$ are not considered explicitly, the methods to obtain the robustness results are borrowed from systems theory: the perturbation operator considered in Sect. 5 is essentially the input–output operator of a control system and is the main tool to investigate the stability radius in Sect. 6. The robustness results obtained in this paper are important to deal with problems in adaptive control [1] and robust control [40].

We study exponential stability and its robustness for time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x}(t) = A(t)x(t), \tag{1.1}$$

where $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$, $n \in \mathbb{N}$. For brevity, we identify the tuple (E, A) with the DAE (1.1). For the analysis, it is also important to consider the inhomogeneous system

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \tag{1.2}$$

where $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$.

In this work we concentrate on linear time-varying index-1 DAEs, which are, roughly speaking, those DAEs which are decomposable into a differential and an algebraic part and no derivatives of the algebraic variables appear in the decomposed system. The consideration of index-1 DAEs is relevant, as in a lot of applications the occurring DAEs are naturally of index-1. For instance, it is shown in [18] that any passive electrical circuit containing nonlinear and possibly time-varying elements has index less than or equal to two and the index-2 case is exceptional. Furthermore, the so-called hybrid models of electrical circuits are always index-1 [28,39]. Therefore, our approach to index-1 DAEs has a wide area of applications, e.g. in electrical engineering, linear DAEs (E, A) arise as linearizations of nonlinear DAEs $F(t, x, \dot{x}) = 0$ along trajectories [10].

Among all the available index concepts for DAEs [8,20,21,31,35], the tractability index as introduced in [34] turned out to be the most suitable for dealing with perturbations in the leading coefficient matrix E of (1.1). This is because the way it allows for the decoupling of the DAE in a differential and an algebraic part via certain projectors enables us to reuse the same projectors for the perturbed system under some appropriate assumptions. This allows for a proper analysis of the perturbation problem. Moreover, in this approach it is not necessary to carry out any state space transformations.

The present paper is concerned with perturbations in the leading coefficient matrix E. In perturbation theory of DAEs, it is usually assumed that the leading coefficient E is not perturbed at all; see e.g. [11,15,16,19,37]. Even in the time-invariant setting, only very few authors have investigated the effects of perturbations in the leading coefficient; see [7,9,14]. For time-varying DAEs, the only work where also perturbations in the leading coefficient are allowed is [33]. The main reason why perturbations in the leading term are usually not considered in the DAE community is that even in the time-invariant index-1 case, exponential stability is very sensitive with respect to such perturbations; see [9]. Byers and Nichols [9] gave the first systematic approach to this problem by introducing a class of "allowable perturbations". In the present article, we will generalize their results to time-varying systems in a certain sense; see Sect. 6. Bracke [7] also generalized the approach of [9] within the setting of time-invariant DAEs to obtain a better treatment of higher index DAEs.

The paper is organized as follows. In Sect. 2 we briefly introduce the class of DAEs with tractability index-1. The perturbation problem is outlined in Sect. 3 and the class of allowable perturbations is defined. The notion of Bohl exponent for DAEs is recapitulated in Sect. 4 and it is shown in Theorem 4.6 that the Bohl exponent is robust with respect to perturbations introduced in Sect. 3. In Sect. 5 we introduce the perturbation operator for the DAE (1.1) and, after recapitulating some of its properties, we show in Theorem 5.3 that its norm can be used to determine a bound ρ such that, roughly speaking, exponential stability is preserved for any perturbation with norm less than ρ. We also prove another robustness result which incorporates the norm of the perturbation operator in Theorem 5.6. In Sect. 6 we introduce a stability radius for index-1 DAEs and prove the essential properties. The main theorem of this section is Theorem 6.11 which provides a lower bound for the stability radius. This lower bound then enables us to prove a statement about certain subsets of exponentially stable index-1 DAEs being open in the respective supersets. Note that the results obtained in this article are new even for time-invariant systems.

2 Index-1 DAEs

In this section, we briefly recall the concept of tractability index-1 and state some crucial results for DAEs with this property. The results can be found in the relevant literature [32,34]; see also [2,3,11,16,20,35]. For a discussion of the tractability index concept in relation to other index concepts, such as the differentiation index [8] or the strangeness index [31], see [32, Secs. 2.10 & 3.10] and [36].

Definition 2.1 (*Index-1 DAE*) The DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ is called *index-1* if, and only if, there exists $Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ such that $Q^2 = Q$, $\ker E = \operatorname{im} Q$ and $E + (E\dot{Q} - A)Q \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$.

Note that by Definition 2.1 the set of index-1 DAEs includes all implicit ODEs, i.e., any system (1.2), where $E \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$, even though such systems are often referred to as index-0 in the literature. In Lemma 2.4 we show that any index-1 DAE is decomposable into a differential and an algebraic part, which then justifies this notion.

It is important to observe that the projector Q in Definition 2.1 can always be chosen to be bounded[1]: If $Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is not bounded, then its pointwise Moore–Penrose inverse $Q^+ \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ is well defined (cf. also [31, Thm. 3.9]) and we have that $QQ^+ \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ and, for all $t \in \mathbb{R}_+$,

$$\big(Q(t)Q^+(t)\big)^2 = Q(t)Q^+(t), \quad \operatorname{im} Q(t)Q^+(t) = \operatorname{im} Q(t) = \ker E(t), \quad \|Q(t)Q^+(t)\| = 1.$$

Finally, a pointwise application of [20, Thm. A.13] yields that $E + (E\frac{\mathrm{d}}{\mathrm{d}t}(QQ^+) - A)QQ^+ \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$.

To get a better understanding of the structure of index-1 DAEs, we may now introduce, for $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$, the following set of projector functions:

$$\mathfrak{Q}_{E,A} := \left\{ Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n}) \cap \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ : \; Q(t)^2 = Q(t) \,\wedge\, \ker E(t) = \operatorname{im} Q(t), \\ E + (E\dot{Q} - A)Q \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R})) \end{array} \right\}.$$

It is immediate that $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ is index-1 if, and only if, $\mathfrak{Q}_{E,A} \neq \emptyset$. With any $Q \in \mathfrak{Q}_{E,A}$ and $P := I - Q$, we may immediately rewrite (1.2) as

$$E\frac{\mathrm{d}}{\mathrm{d}t}(Px) = (A - E\dot{Q})x + f. \tag{2.1}$$

It can be shown that the solutions of (2.1) are independent of the choice of Q; thus the following set of solutions is well defined.

[1] I thank the Associate Editor who handled this paper for pointing this out.

Definition 2.2 (*Solution space*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$ and $Q \in \mathfrak{Q}_{E,A}$. We call a function $x : \mathbb{R}_+ \to \mathbb{R}^n$ a *solution* of (1.2) if, and only if,

$$x \in \mathcal{C}^1_{E,A,f} := \left\{ x \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n) \,\middle|\, (I-Q)x \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n) \text{ and } x \text{ solves (1.2) for all } t \in \mathbb{R}_+ \right\}.$$

Note that this solution concept only incorporates global solutions and does not account for possible local solutions, which however must be expected for time-varying DAEs; see e.g. [4,5]. This is reasonable since it can be shown that any local solution can be uniquely extended to a global solution for the class of index-1 DAEs.

The following proposition is important for later purposes and gives more insight into the set $\mathfrak{Q}_{E,A}$. Its proof follows from a pointwise application of [20, Thm. A.13].

Proposition 2.3 (Index-1 and projectors on ker E) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. *If* $\mathfrak{Q}_{E,A} \neq \emptyset$, *then*

$$\mathfrak{Q}_{E,A} = \Big\{ Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n}) \cap \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \\ \Big|\, \forall\, t \in \mathbb{R}_+ :\ Q(t)^2 = Q(t) \ \wedge\ \ker E(t) = \operatorname{im} Q(t) \Big\}.$$

The opposite implication of Proposition 2.3, in general, does not hold true: considering the simple example $E = A = 0$ shows that it is possible that $\mathfrak{Q}_{E,A} = \emptyset$, but at the same time there are projectors onto ker E, i.e., only the index-1 property guarantees that all these projectors are contained in $\mathfrak{Q}_{E,A}$.

Lemma 2.4 (Decomposition) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$ *and* $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. *Set*

$$P := I - Q, \quad \bar{A} := A - E\dot{Q}, \quad G := E + (E\dot{Q} - A)Q = E - \bar{A}Q. \tag{2.2}$$

Then, for $x \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$, *we have* $x \in \mathcal{C}^1_{E,A,f}$ *if, and only if,* $Px \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ *and* x *solves the following system for all* $t \in \mathbb{R}_+$*:*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)x(t)) = (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)x(t) + P(t)G(t)^{-1}f(t), \\ \qquad Q(t)x(t) = Q(t)G(t)^{-1}\bar{A}(t)P(t)x(t) + Q(t)G(t)^{-1}f(t). \end{cases} \tag{2.3}$$

It can be seen from Lemma 2.4 that, roughly speaking, the solutions of the index-1 DAE (E, A) can be calculated by solving an ODE for Px and then Qx (and therefore x) is given in terms of Px. Therefore, all solutions of the DAE (1.2) are fully determined by the solutions of the ODE (first equation) in (2.3). It is also important to note that no derivatives of the so-called "algebraic variables" Qx are involved in (2.3), which justifies the use of the notion "index-1", cf. [34].

To define a transition matrix, we need to consider the initial value conditions of the form

$$E(t_0)(x(t_0) - x^0) = 0 \tag{2.4}$$

for $t_0 \in \mathbb{R}_+$ and $x^0 \in \mathbb{R}^n$. It is crucial that an initial value problem (1.2), (2.4) can be considered for arbitrary $x^0 \in \mathbb{R}^n$ and that this problem has a unique solution.

Proposition 2.5 (Solution mapping) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.2) and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. Then, for every $t_0 \in \mathbb{R}_+$, the map*

$$\varphi_{t_0} : \mathbb{R}^n \to \mathcal{C}^1_{E,A,f},\ x^0 \mapsto x,\ \textit{where } E(t_0)(x(t_0) - x^0) = 0,$$

is well defined and surjective.

Definition 2.6 (Transition matrix) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and φ_{t_0} be the solution map given by Proposition 2.5 for (1.1). Then the *transition matrix* $\Phi(\cdot,\cdot) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ of (E, A) is defined by

$$\Phi(t, t_0) := \left[\left(\varphi_{t_0}(e_1)\right)(t), \ldots, \left(\varphi_{t_0}(e_n)\right)(t)\right], \quad t, t_0 \in \mathbb{R}_+,$$

where e_i is the ith unit vector.

In general, we have that $\Phi(\cdot, t_0)$ is continuous, but not continuously differentiable, whilst $P(\cdot)\Phi(\cdot, t_0)$ is always continuously differentiable. For inhomogeneous problems (1.2), using the transition matrix $\Phi(\cdot,\cdot)$, a variation of constants formula can be derived.

Proposition 2.7 (Variation of constants) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.2) and $f \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$. Furthermore, let $t_0 \in \mathbb{R}_+$, $\Phi(\cdot,\cdot)$ be the transition matrix of (E, A), and φ_{t_0} be as in Proposition 2.5. Then, for all $x^0 \in \mathbb{R}^n$,*

$$\begin{aligned} \forall\, t \in \mathbb{R}_+ :\ \left(\varphi_{t_0}(x^0)\right)(t) &= \Phi(t, t_0)P(t_0)x^0 \\ &\quad + \int_{t_0}^{t} \Phi(t, s)P(s)G(s)^{-1} f(s)\, \mathrm{d}s\ + Q(t)G(t)^{-1} f(t). \end{aligned} \tag{2.5}$$

3 The perturbation problem

In this section, we introduce the class of allowable perturbations considered in this article. For given $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ and perturbation $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, we consider the perturbed system

$$(E(t) + \Delta_E(t))\, \dot{x}(t) = A(t)x(t), \tag{3.1}$$

i.e., perturbations of the matrix-valued function E. Since exponential stability is very sensitive with respect to arbitrary perturbations in the leading term [9], we do not allow for general perturbations $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$, but restrict ourselves to the class of perturbations defined in the following.

Definition 3.1 (*Allowable perturbations*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. Then the *set of allowable perturbations* (*in the leading coefficient*) is defined by

$$\mathcal{P}_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ \ker E(t) = \ker(E(t) + \Delta_E(t)), \\ \text{and } (E + \Delta_E, A) \text{ is index-1} \end{array} \right\}.$$

Remark 3.2 (Allowable perturbations) The matrix $\dot{Q}Q$ is nilpotent and the index of nilpotency is at most 2 everywhere. As $\dot{Q} = \frac{\mathrm{d}}{\mathrm{d}t} Q^2 = \dot{Q}Q + Q\dot{Q}$, we obtain $Q\dot{Q}Q = 0$ and hence $(\dot{Q}Q)^2 = 0$. Therefore, $I + \dot{Q}Q$ is invertible everywhere with $(I + \dot{Q}Q)^{-1} = I - \dot{Q}Q$.

Remark 3.3 (Kernel assumption) The definition of the set $\mathcal{P}_{E,A}$ may seem restrictive, in particular the claim for the kernel of E to be preserved. But on the one hand, as shown later in this section, perturbations of the algebraic part are still possible. On the other hand, in the perturbation theory of DAEs, it is usually assumed that the leading coefficient E is not perturbed at all; see e.g., [11,15,16,19,37]. Moreover, the condition on perturbations of the leading term to preserve some kernel is not uncommon, as in [9], where time-invariant systems are considered, it is assumed that the left kernel of E is preserved under the perturbation (see proof of [9, Lem. 3.2]). Furthermore, the singularly perturbed systems considered in [22] to regularize index-2 DAEs belong to $\mathcal{P}_{E,A}$, provided they are applied to index-1 systems.

As argued in [9], in practical applications the set of allowable perturbations is limited anyway, restricted by the physical structure of the considered system. Therefore, as it is widely believed, if the algebraic part of the DAE represents path constraints, then the zero blocks in E are structural and are not subject to disturbances or uncertainties. Perturbations which preserve the (physical) structure of an index-1 DAE are exactly those which preserve the rank of E and the index-1 property of (E, A). However, exponential stability is, in general, not robust with respect to perturbations in this class as Example 4.5 shows. If we restrict ourselves to those perturbations which preserve the kernel of E, then robustness of exponential stability can be shown; see Theorem 4.6.

Lemma 3.4 (Sufficient condition for preserved index) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$*,* G *as in* (2.2) *and* $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$*. Then the following holds true:*

(i) $E - AQ \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$,

(ii) $\left.\begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ \ker E(t) = \ker(E(t) + \Delta_E(t)), \\ \forall\, t \in \mathbb{R}_+ : \ \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\| < 1 \end{array}\right\} \Rightarrow Q \in \mathfrak{Q}_{E+\Delta_E,A} \ \wedge \ \Delta_E \in \mathcal{P}_{E,A}$.

Proof (i) Since $Q\dot{Q}Q = 0$, we have $(E - AQ)(I + \dot{Q}Q) = E + (E\dot{Q} - A)Q = G$ and hence, invoking that $I + \dot{Q}Q$ is invertible by Remark 3.2,

$$(E - AQ)^{-1} = (I + \dot{Q}Q)G^{-1}. \tag{3.2}$$

(ii) As Δ_E preserves the kernel of E, it is clear that Q is a bounded projector on $\ker(E+\Delta_E)$. Hence, it only remains to prove that $E+\Delta_E+((E+\Delta_E)\dot{Q}-A)Q =$

$G + \Delta_E(I + \dot{Q}Q) \in \mathcal{C}(\mathbb{R}_+; \mathbf{Gl}_n(\mathbb{R}))$. Since $G + \Delta_E(I + \dot{Q}Q) = (I + \Delta_E(E - AQ)^{-1})G$, the invertibility immediately follows from the assumption. □

Lemma 3.4 gives rise to the following definition of subsets of $\mathcal{P}_{E,A}$.

Definition 3.5 Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then we define

$$\mathcal{P}^Q_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ker E(t) = \ker(E(t) + \Delta_E(t)) \text{ and} \\ \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\| < 1 \end{array} \right\}.$$

Note that, if $E = 0$, then $I \in \mathfrak{Q}_{E,A}$ and we have $\mathcal{P}^I_{E,A} = \{0\} = \mathcal{P}_{E,A}$. In general, we have $\mathcal{P}^Q_{E,A} \subseteq \mathcal{P}_{E,A}$. For perturbations in $\mathcal{P}^Q_{E,A}$, we may also reformulate the perturbed system (3.1) in a form similar to (2.3).

Lemma 3.6 (Decomposition of perturbed system) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$, $P, \bar{A}, G$ as in (2.2) and $\Delta_E \in \mathcal{P}^Q_{E,A}$. Then $x \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$ is a solution of (3.1) if, and only if, $Px \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ and x solves the following system for all $t \in \mathbb{R}_+$:*

$$\left\{ \begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(P(t)x(t)) &= (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)x(t) + P(t)G(t)^{-1}\Delta(t)P(t)x(t), \\ Q(t)x(t) &= Q(t)G(t)^{-1}\bar{A}(t)P(t)x(t) + Q(t)G(t)^{-1}\Delta(t)P(t)x(t), \end{aligned} \right. \tag{3.3}$$

where

$$\Delta := -(I + \Lambda)^{-1}\Lambda A(I - Q\dot{Q}), \quad \Lambda = \Delta_E(E - AQ)^{-1}. \tag{3.4}$$

Proof Using Lemma 2.4 and the fact that $Q \in \mathfrak{Q}_{E+\Delta_E,A}$ by Lemma 3.4, and defining $\tilde{A} := \bar{A} - \Delta_E\dot{Q}$, $\tilde{G} := E + \Delta_E - \tilde{A}Q$, it is immediate that x is a solution of (3.1) if, and only if, $Px \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ and x solves

$$\left\{ \begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(P(t)x(t)) &= (\dot{P}(t) + P(t)\tilde{G}(t)^{-1}\tilde{A}(t))P(t)x(t), \\ Q(t)x(t) &= Q(t)\tilde{G}(t)^{-1}\tilde{A}(t)P(t)x(t). \end{aligned} \right. \tag{3.5}$$

Now observe that by (3.2) we have $\Lambda G = \Delta_E(I + \dot{Q}Q)$; thus, $\tilde{G} = G + \Lambda G$ and hence under the assumption that $\|\Lambda(t)\| = \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\| < 1$ for all $t \in \mathbb{R}_+$, it is immediate that $\tilde{G}^{-1} = G^{-1}(I + \Lambda)^{-1} = G^{-1}\left(I - \Lambda(I + \Lambda)^{-1}\right)$. By some simple calculation, we then obtain that $\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} - G^{-1}\big((I + \Lambda)^{-1}\Delta_E\dot{Q} + \Lambda(I + \Lambda)^{-1}\bar{A}\big)$. Using that $\Delta_E\dot{Q} = \Lambda(E - AQ)\dot{Q}$ and $(I + \Lambda)^{-1}\Lambda = \Lambda - \Lambda(I + \Lambda)^{-1}\Lambda = \Lambda(I + \Lambda)^{-1}$, we get $\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} - G^{-1}(I + \Lambda)^{-1}\Lambda\left((E - AQ)\dot{Q} + \bar{A}\right)$. Now, $(E - AQ)\dot{Q} + \bar{A} = (E - AQ)\dot{Q} + A - E\dot{Q} = A(I - Q\dot{Q})$, thus $\tilde{G}^{-1}\tilde{A} = G^{-1}\bar{A} + G^{-1}\Delta$, which yields that (3.5) is equivalent to (3.3). □

In the subsequent sections we will also need the following lemma, the proof of which is straightforward.

Lemma 3.7 (Bound on Δ) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and $\Delta_E \in \mathcal{P}^Q_{E,A}$. Then, for Δ as in (3.4) and all $t \in \mathbb{R}_+$, we have*

$$\|\Delta(t)\| \leq \frac{\|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}A(t)(I - Q(t)\dot{Q}(t))\|}{1 - \|\Delta_E(t)(E(t) - A(t)Q(t))^{-1}\|}.$$

From (3.3), it can be seen that the perturbation does not only affect the differential part, but also the algebraic part of the DAE. To make this more clear, consider the following example which will serve as a running example in the following.

Example 3.8 Consider the system (1.1) with constant $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The solutions of this system are given by $x_1(t) = c_1 e^{-t}$, $x_2(t) = c_2 e^{-t}$, $x_3(t) = 0$ for $c_1, c_2 \in \mathbb{R}$. Now, let $\Delta_E = \begin{bmatrix} 0 & \delta & 0 \\ 0 & \delta & 0 \\ \delta & 0 & 0 \end{bmatrix}$, $\delta \in \mathbb{R}$ and observe that $\ker E = \ker(E + \Delta_E)$ for all $\delta \in \mathbb{R}$. Furthermore, choosing $Q = I - E \in \mathfrak{Q}_{E,A}$, we have that, for G as in (2.2),

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and hence} \quad G + \Delta_E(I + \dot{Q}Q) = \begin{bmatrix} 1 & \delta & 0 \\ 0 & 1+\delta & 0 \\ \delta & 0 & -1 \end{bmatrix},$$

which is invertible for all $\delta \neq -1$. Hence, $\Delta_E \in \mathcal{P}_{E,A}$ for $\delta \neq -1$. As it is easy to calculate that $\|\Delta_E\| = \sqrt{2}|\delta|$, we have $\Delta_E \in \mathcal{P}^Q_{E,A}$ if, and only if, $|\delta| < \sqrt{2}/2$. The perturbed system (3.1) reads, after some rearrangement,

$$\dot{x}_1 = -x_1 + \delta(1+\delta)^{-1}x_2, \quad \dot{x}_2 = -(1+\delta)^{-1}x_2, \quad x_3 = -\delta x_1 + \delta^2(1+\delta)^{-1}x_2,$$

Therefore, the solutions are

$$\begin{aligned} x_1(t) &= (c_1 - c_2)e^{-t} + c_2 e^{-(1+\delta)^{-1}t}, \quad x_2(t) = c_2 e^{-(1+\delta)^{-1}t}, \\ x_3(t) &= -\delta(c_1 - c_2)e^{-t} - \delta c_2(1+\delta)^{-1}e^{-(1+\delta)^{-1}t}, \end{aligned}$$

for $c_1, c_2 \in \mathbb{R}$, and it is clear that both the differential and the algebraic part of the DAE have been perturbed as all components of the solution have changed. Furthermore, we see that for $\delta > -1$ the perturbed system is exponentially stable (cf. Definition 4.3), whilst it is unstable for $\delta < -1$. For $\delta = -1$ we have $\Delta_E \notin \mathcal{P}_{E,A}$; however, the system is still exponentially stable as the equations read, after some rearrangement,

$\dot{x}_1 = -x_1, x_2 = 0, x_3 = x_1$, but this is beyond the scope of this approach because the index of the system did change (it is index-2 tractable in the sense of [34] for $\delta = -1$).

Remark 3.9 Note that, as shown in Example 3.8, the perturbations may change the algebraic equations, but not the algebraic *structure* of the system as it was pointed out in Remark 3.3.

4 Bohl exponent

The Bohl exponent, introduced by Piers Bohl [6], describes the uniform exponential growth of the solutions of a system. For ODEs, the Bohl exponent has been successfully used to characterize exponential stability and to derive robustness results; see e.g. [13, 23]. In this section we give the definition for the Bohl exponent as stated in [4] for general DAE systems and derive some formulae for it which hold in the index-1 setting. We also state the equivalence between a negative Bohl exponent and exponential stability and derive a robustness result, namely Theorem 4.6, for the Bohl exponent under the class of perturbations introduced in Sect. 3. In particular, this shows that exponential stability is robust under these perturbations.

Definition 4.1 (*Bohl exponent*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1. The *Bohl exponent of* (E, A) is defined as

$$k_{\mathrm{B}}(E, A) := \inf\left\{\rho \in \mathbb{R} \,\middle|\, \exists\, N_\rho > 0\ \forall\, x \in \mathcal{C}^1_{E,A,0}\ \forall\, t \geq s \geq 0 : \|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|\right\}.$$

Note that we use the usual convention $\inf \emptyset := +\infty$.

Lemma 4.2 (Representation of the Bohl exponent) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1 with transition matrix* $\Phi(\cdot, \cdot)$*. Then we have*

$$k_{\mathrm{B}}(E, A) = \inf\left\{\rho \in \mathbb{R} \,\middle|\, \exists\, N_\rho > 0\ \forall\, t \geq s \geq 0 :\ \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)}\right\}$$

and $k_{\mathrm{B}}(E, A) < \infty$ *if, and only if,* $\sup_{0\leq t-s\leq 1} \|\Phi(t, s)\| < \infty$*. Furthermore, if* $k_{\mathrm{B}}(E, A) < \infty$*, then it holds that*

$$k_{\mathrm{B}}(E, A) = \limsup_{s, t-s\to\infty} \frac{\ln \|\Phi(t, s)\|}{t - s}, \quad \textit{where } \ln 0 := -\infty.$$

Proof The first statement is immediate from the definition of the Bohl exponent and the second is a special case of [4, Prop. 3.7]. For the last statement, see [11, Prop. 4.4]. Note that in the second and last statement, a Bohl exponent $k_{\mathrm{B}}(E, A) = -\infty$ is explicitly allowed. □

We stress that the equivalent condition for a Bohl exponent $k_{\mathrm{B}}(E, A) < \infty$ is also valid in the case $k_{\mathrm{B}}(E, A) = -\infty$. Moreover, the formula for the calculation of the Bohl exponent does also hold true in this case. The Bohl exponent can become $-\infty$ if all solutions of (1.1) vanish identically; hence, $\Phi(t, s) = 0$ for all $t, s \in \mathbb{R}_+$.

However, it is possible that the Bohl exponent is $-\infty$ even in the ODE case, as e.g., $k_B(1, -2t) = -\infty$. Compared to a Bohl exponent of $+\infty$, it is of a more “good-natured” kind, as a system with Bohl exponent $-\infty$ is in particular exponentially stable (see Definition 4.3). Therefore, we will usually consider the cases of finite Bohl exponent and Bohl exponent $-\infty$ together, i.e., the latter is not excluded when $k_B(E, A) < \infty$ is required, if not stated otherwise.

Next, we state the definition of exponential stability of DAEs (E, A). The definition for general DAEs can be found, e.g., in [4,5]. Here, we state the already simplified version derived from [4, Prop. 5.2].

Definition 4.3 (*Exponential stability*) A linear index-1 DAE $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ with transition matrix $\Phi(\cdot, \cdot)$ is called *exponentially stable* if, and only if,

$$\exists\, \mu, M > 0\, \forall\, t \geq t_0 \geq 0 :\ \|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}. \tag{4.1}$$

As shown in [4, Cor. 5.3], we have the following result.

Lemma 4.4 (Bohl exponent and exponential stability) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 with transition matrix $\Phi(\cdot, \cdot)$, let $Q \in \mathfrak{Q}_{E,A}$ and suppose that $k_B(E, A) < \infty$. Then the following statements are equivalent:*

(i) $k_B(E, A) < 0$.
(ii) (E, A) *is exponentially stable.*
(iii) $\forall\, p > 0\, \exists\, c > 0\, \forall\, t_0 \in \mathbb{R}_+ :\ \int_{t_0}^{\infty} \|\Phi(t, t_0)\|^p \,\mathrm{d}t \leq c.$

In the following, we derive robustness results for exponential stability. That the assumption of preserved kernel of E, in general, cannot be relaxed to preserved rank of E is shown by the following counterexample.

Example 4.5 Consider the system (1.1) with

$$E_{\varepsilon,\mu}(t) = \begin{bmatrix} 1 & \varepsilon\alpha_\mu(t) \\ \varepsilon\beta_\mu(t) & \varepsilon^2\alpha_\mu(t)\beta_\mu(t) \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

where

$$\alpha_\mu(t) = \cos(\mu t), \quad \beta_\mu(t) = \sin(\mu t), \quad \mu, t \in \mathbb{R}.$$

Then $(E_{0,0}, A)$ is exponentially stable. By choosing $\varepsilon > 0$ small enough, it is possible to achieve that for the perturbation term $\Delta_E^{\varepsilon,\mu}(t) = \begin{bmatrix} 0 & \varepsilon\alpha_\mu(t) \\ \varepsilon\beta_\mu(t) & \varepsilon^2\alpha_\mu(t)\beta_\mu(t) \end{bmatrix}$ the norm $\|\Delta_E^{\varepsilon,\mu}\|_\infty$ gets as small as desired. Furthermore,

$$\forall\, t, \varepsilon, \mu \in \mathbb{R} :\ \mathrm{rk}\, E_{\varepsilon,\mu}(t) = \mathrm{rk}\, E_{0,0},$$

i.e., the rank of $E_{0,0}$ is preserved under the perturbation $\Delta_E^{\varepsilon,\mu}$. If $1 - \varepsilon^2\alpha_\mu(t)\beta_\mu(t) \neq 0$ for all $t \in \mathbb{R}$, then the perturbed system is again index-1; this can be achieved by choosing $0 < \varepsilon < 1$. However, for any $\varepsilon \in (0, 1)$ we may choose $\mu > 0$ large enough so that the perturbed system $(E_{\varepsilon,\mu}, A)$ is not exponentially stable: To this end observe that (1.1) can be written as

$$\dot{x}_1(t) = \frac{\varepsilon^2\alpha_\mu(t)\dot{\beta}_\mu(t) - 1}{1 - \varepsilon^2\alpha_\mu(t)\beta_\mu(t)} x_1(t), \quad x_2(t) = -\varepsilon\beta_\mu(t)x_1(t).$$

The solution x_1 of the first equation together with the initial condition $x_1(0) = x_1^0$, $x_1^0 \in \mathbb{R}$, is given by

$$x_1(t) = \mathrm{e}^{\int_0^t \frac{\varepsilon^2\mu\cos(\mu\tau)^2 - 1}{1-\varepsilon^2\sin(\mu\tau)\cos(\mu\tau)}\,\mathrm{d}\tau} x_1^0.$$

Now, consider the sequence $t_k := \frac{2k\pi}{\mu}$, $k \in \mathbb{N}_0$, and calculate that $x_1(t_k) = \mathrm{e}^{\frac{2k\pi(\varepsilon^2\mu-2)}{\mu\sqrt{4-\varepsilon^4}}} x_1^0$ for $k \in \mathbb{N}_0$. We may conclude that for any $\varepsilon \in (0, 1)$ we may choose $\mu > 0$ large enough so that $x_1(t_k) \to \infty$ for $k \to \infty$. Therefore, exponential stability of $(E_{0,0}, A)$ is not robust with respect to perturbations which preserve the rank of $E_{0,0}$ and the index-1 property of $(E_{0,0}, A)$.

In view of Lemma 4.4, the next result shows the robustness result of exponential stability.

Theorem 4.6 (Robustness of Bohl exponent) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1, $Q \in \mathfrak{Q}_{E,A}$ and suppose that $k_\mathrm{B}(E, A) > -\infty$. Further, let P and G be as in (2.2). Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\Delta_E \in \mathcal{P}^Q_{E,A}$ which satisfy, for Δ as in (3.4), the condition*

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\,\mathrm{d}\tau < \delta \tag{4.2}$$

it holds that

$$k_\mathrm{B}(E + \Delta_E, A) \le k_\mathrm{B}(E, A) + \varepsilon.$$

Proof Since $\Delta_E \in \mathcal{P}^Q_{E,A}$, we may apply Lemma 3.6 and obtain the reformulated perturbed system (3.3). This system can be interpreted as a system where A has been perturbed to $A + \Delta P$ and hence can be treated within the framework of [11]. The assertion of the theorem can then be inferred from [11, Thm. 5.2] by observing that Assumptions A1 and A3* of [11, Thm. 5.2] can be relaxed by assuming the boundedness of Q if the perturbation has the form $F = \Delta P$. The latter follows from the observation that in the proof of [11, Thm. 5.2] the reformulation carried out in the third equation is not necessary and it suffices to choose $h(t) = \|P(t)G(t)^{-1}\Delta(t)P(t)\|$. $\square$

In the case of bounded perturbations, the statement of Theorem 4.6 can, under some further assumptions, be simplified.

Corollary 4.7 (Robustness of Bohl exponent) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$ *and suppose that* $k_{\mathrm{B}}(E, A) > -\infty$*. Further let* P, G *be as in* (2.2) *and suppose that* G^{-1}, $P(E - AQ)^{-1}$ *and* $P(E - AQ)^{-1}A(P - \dot{Q}P)$ *are bounded. Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for all* $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})$ *which satisfy* $\ker E(t) = \ker(E(t) + \Delta_E(t))$*,* $t \in \mathbb{R}_+$*, and* $\|\Delta_E\|_\infty < \delta$*, it holds that*

$$k_{\mathrm{B}}(E + \Delta_E, A) \le k_{\mathrm{B}}(E, A) + \varepsilon.$$

Proof First, observe that by choosing δ sufficiently small we may assure $\Delta_E \in \mathcal{P}^Q_{E,A}$ and $\|\Delta_E\|_\infty \|P(E - AQ)^{-1}\|_\infty < 1$. Furthermore, for Δ as in (3.4), ΔP is bounded, as from Lemma 3.7

$$\|\Delta P\|_\infty \le \frac{\|P(E - AQ)^{-1}A(P - \dot{Q}P)\|_\infty \|\Delta_E\|_\infty}{1 - \|P(E - AQ)^{-1}\|_\infty \|\Delta_E\|_\infty}, \tag{4.3}$$

where it was used that $\Delta_E = \Delta_E P$ and $(I - Q\dot{Q})P = (I - \dot{Q} + \dot{Q}Q)P = P - \dot{Q}P$. It follows that

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, \mathrm{d}\tau \le \|PG^{-1}\|_\infty \|\Delta P\|_\infty.$$

Now, $\|\Delta_E\|_\infty$ can be chosen sufficiently small so that Theorem 4.6 can be applied. □

In Theorem 4.6, the case $k_{\mathrm{B}}(E, A) = -\infty$ is excluded. Together with the case $k_{\mathrm{B}}(E, A) = +\infty$, this is treated in the following proposition, which provides a condition under which the Bohl exponent is invariant.

Proposition 4.8 (Equal Bohl exponents) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1 and* $Q \in \mathfrak{Q}_{E,A}$*. Further, let* P *and* G *be as in* (2.2)*. If* $\Delta_E \in \mathcal{P}^Q_{E,A}$ *and* Δ *as in* (3.4) *satisfies*

$$\limsup_{t,s\to\infty} \frac{1}{s} \int_t^{t+s} \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, \mathrm{d}\tau = 0, \tag{4.4}$$

then $k_{\mathrm{B}}(E + \Delta_E, A) = k_{\mathrm{B}}(E, A)$*. This means, in particular, if*

$$\lim_{t\to\infty} \|P(\tau)G(\tau)^{-1}\Delta(t)P(t)\| = 0 \quad or \quad \int_0^\infty \|P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\|\, \mathrm{d}\tau < \infty,$$

then $k_{\mathrm{B}}(E + \Delta_E, A) = k_{\mathrm{B}}(E, A)$.

Proof Let P, $\bar{A}$, G be as in (2.2). If $k_{\mathrm{B}}(E, A) = -\infty$, then it is easy to observe that choosing sequences $\mu_k \to -\infty$ and $\delta_k \searrow 0$ in the proof of Theorem 4.6 shows that $k_{\mathrm{B}}(E + \Delta_E, A) = -\infty$. Suppose now $k_{\mathrm{B}}(E, A) \ne -\infty$. Observe that Theorem 4.6

implies $k_{\mathrm{B}}(E+\Delta_E, A) \leq k_{\mathrm{B}}(E, A)$. We now show that it may be applied to $(E+\Delta_E, A)$ with perturbation $-\Delta_E$ as well. To this end, note that, by Lemma 3.4, $Q \in \mathfrak{Q}_{E+\Delta_E,A}$ and $-\Delta_E \in \mathcal{P}^Q_{E+\Delta_E,A}$. It remains to prove that $\tilde{G} := E+\Delta_E + ((E+\Delta_E)\dot{Q} - A)Q$ and $\tilde{\Delta} := -(I+\tilde{\Lambda})^{-1}\tilde{\Lambda}A(I-Q\dot{Q})$, where $\Lambda = -\Delta_E(E+\Delta_E - AQ)^{-1}$, satisfy (4.4) as well. This follows from observing that $E+\Delta_E - AQ$ is invertible everywhere and calculating $\tilde{G}^{-1}\tilde{\Delta} = -G^{-1}\Delta$. $\square$

Remark 4.9 (Invariance of Bohl exponent $\pm\infty$) Note that condition (4.4) is a very strong condition on the perturbation for the Bohl exponent of $\pm\infty$ to be preserved. The invariance of Bohl exponent $\pm\infty$ under an appropriately large class of perturbations is an open problem. If we assume

$$\exists\, f \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}) \text{ s.t. } \lim_{t\to\infty} \dot{f}(t) = \infty\ \exists\, M>0\ \forall\, t \geq s \geq 0: \|\Phi(t,s)\| \leq Me^{-(f(t)-f(s))}, \tag{4.5}$$

it is straightforward to prove (using the mean value theorem) the following:

Let $(E, A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Further, let P and G be as in (2.2). If (4.5) holds, $\Delta_E \in \mathcal{P}^Q_{E,A}$ and Δ as in (3.4) satisfies (4.4) with "$< \infty$" instead of "$= 0$", then $k_{\mathrm{B}}(E+\Delta_E, A) = k_{\mathrm{B}}(E, A) = -\infty$.

The author conjectures that Condition (4.5) is equivalent to $k_{\mathrm{B}}(E, A) = -\infty$; however it is only clear that (4.5) implies $k_{\mathrm{B}}(E, A) = -\infty$.

We close this section by illustrating the main result by means of our running example.

Example 4.10 (Example 3.8 revisited) It can be immediately seen from the representation of the solutions in Example 3.8 that

$$k_{\mathrm{B}}(E, A) = -1 \quad \text{and} \quad k_{\mathrm{B}}(E+\Delta_E, A) = \max\left\{-1, -(1+\delta)^{-1}\right\}$$

for all $\delta \neq -1$. Therefore, given $\varepsilon > 0$ we have that for all $\delta \in \mathbb{R}$ which satisfy

$$\varepsilon < 1: \ \delta \in \left(-1, \varepsilon(1-\varepsilon)^{-1}\right), \quad \varepsilon = 1: \ \delta \in (-1, \infty),$$
$$\varepsilon > 1: \ \delta \in \left(-\infty, \varepsilon(1-\varepsilon)^{-1}\right] \cup (-1, \infty),$$

the Bohl exponents satisfy

$$k_{\mathrm{B}}(E+\Delta_E, A) \leq k_{\mathrm{B}}(E, A) + \varepsilon.$$

5 Perturbation operator

In this section we investigate the robustness of exponential stability (1.1) in terms of the perturbation operator. As a system (E, A) is exponentially stable if, and only if, its Bohl exponent is negative by Lemma 4.4, Theorem 4.6 states in particular that exponential stability of index-1 DAEs is robust with respect to perturbations in $\mathcal{P}^{Q}_{E,A}$ for any $Q \in \mathfrak{Q}_{E,A}$. However, Theorem 4.6 only states that the perturbation has to be sufficiently small to preserve exponential stability. In this section we provide a calculable upper bound on the perturbation such that exponential stability is preserved by using the perturbation operator. In [23], it was shown that the perturbation operator is an appropriate tool for investigating perturbations and robustness for ODEs; see also [11,16] for index-1 DAEs.

Motivated by the variation of constants formula (2.5), the perturbation operator is defined as follows.

Definition 5.1 (*Perturbation operator*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Further, let $\Phi(\cdot,\cdot)$ be the transition matrix of (E, A), let P and G be as in (2.2) and suppose that G^{-1} is bounded. Then the *perturbation operator* of (E, A) is defined by $L_{t_0} : L^2([t_0,\infty); \mathbb{R}^n) \to L^2([t_0,\infty); \mathbb{R}^n)$,

$$(L_{t_0} f)(t) = \int_{t_0}^{t} \Phi(t,s)P(s)G(s)^{-1} f(s)\, \mathrm{d}s \; + Q(t)G(t)^{-1} f(t).$$

Lemma 5.2 (Properties of the perturbation operator) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable such that (4.1) holds. Let $Q \in \mathfrak{Q}_{E,A}$, $\Phi(\cdot,\cdot)$ be the transition matrix of (E, A) and let P and G be as in (2.2) and suppose that G^{-1} is bounded. Then we have:*

(i) *For any $t_0 \in \mathbb{R}_+$: L_{t_0} is independent of the choice of Q and well defined, i.e., $L_{t_0}(f) \in L^2([t_0,\infty); \mathbb{R}^n)$ for all $f \in L^2([t_0,\infty); \mathbb{R}^n)$.*

(ii) *For all $t_0 \in \mathbb{R}_+$, the operator L_{t_0} is bounded by*

$$\|L_{t_0}\| \le \frac{M}{\mu} \left\| PG^{-1}\big|_{[t_0,\infty)} \right\|_\infty + \left\| QG^{-1}\big|_{[t_0,\infty)} \right\|_\infty .$$

(iii) *$t_0 \mapsto \|L_{t_0}\|$ is monotonically nonincreasing on $\mathbb{R}_+$, i.e., $\|L_{t_0}\| \ge \|L_{t_1}\|$ for all $0 \le t_0 \le t_1$.*

Proof See [11,16]. □

As mentioned before, the perturbation operator is motivated by the variation of constants formula (2.5), but since an introduction of a solution theory for (1.1) involving L^2-inhomogeneities and therefore Sobolev spaces for the solutions would be very technical and not provide any more insight, we restricted ourselves to the class of continuous solutions as introduced in Definition 2.2. Nevertheless, Lemma 5.2 shows that the perturbation operator is well defined.

We show now that the robustness of exponential stability can be related to the inverse norm of the perturbation operator. In fact, we prove that the latter provides a calculable bound on the perturbation such that exponential stability is preserved. This result is a DAE version of [23, Cor. 4.3] and to this end we also introduce the notation

$$\ell(E, A) := \lim_{t_0\to\infty} \|L_{t_0}\|^{-1} \overset{\text{Lemma 5.2}}{=} \sup_{t_0\ge 0} \|L_{t_0}\|^{-1}$$

for an index-1 $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. Note that $\ell(E, A) = \infty$ is explicitly allowed. The next theorem states that if the perturbation term Δ as in (3.4) is sufficiently small, then exponential stability is preserved.

Theorem 5.3 (Exponential stability and perturbation operator) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in (2.2) and suppose that G^{-1} is bounded. Furthermore, let $\Delta_E \in \mathcal{P}^Q_{E,A}$ and suppose that for Δ as in (3.4) the matrix ΔP is bounded. If*

$$\lim_{t_0\to\infty} \left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \min\left\{\ell(E, A), \|QG^{-1}\|_\infty^{-1}\right\}, & \text{if } Q \neq 0, \\ \ell(E, A), & \text{if } Q = 0, \end{cases}$$

then the perturbed system (3.1) is exponentially stable.

Proof **Case 1:** $Q \neq 0$. First, note that $t_0 \mapsto \|(\Delta P)|_{[t_0,\infty)}\|_\infty$ is monotonically decreasing on $\mathbb{R}_+$ and hence the limit always exists since ΔP is bounded. Then, the assumption and the fact that $t_0 \mapsto \|L_{t_0}\|^{-1}$ is monotonically nondecreasing imply that there exists $\hat{t} \in \mathbb{R}_+$ such that

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \min\left\{\|L_{t_0}\|^{-1}, \|QG^{-1}\|_\infty^{-1}\right\}, \quad t_0 \geq \hat{t}. \tag{5.1}$$

By exponential stability of (E, A) we have (4.1), where $\Phi(\cdot,\cdot)$ is the transition matrix of (E, A). To show that (3.1) is exponentially stable, we will show in Step 1 that $k_{\mathrm{B}}(E + \Delta_E, A) < \infty$ using Lemma 4.2 and then in Step 2 using Lemma 4.4 that $k_{\mathrm{B}}(E + \Delta_E, A) < 0$. This means to show that there exist $c_1, c_2 > 0$ such that for the transition matrix $\tilde{\Phi}(\cdot,\cdot)$ of $(E + \Delta_E, A)$, it holds that

$$\sup_{0\leq t-t_0\leq 1} \|\tilde{\Phi}(t, t_0)\| \leq c_1 \quad \text{and} \quad \forall\, t_0 \in \mathbb{R}_+ : \int_{t_0}^{\infty} \|\tilde{\Phi}(t, t_0)\|^2 \,\mathrm{d}t \leq c_2.$$

Fix $s \geq \hat{t}$ and let $\bar{A}$ be as in (2.2). Then $\tilde{\Phi}(\cdot,\cdot)$ satisfies (3.3) as a matrix equation, i.e., for all $t \geq s$,

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(P(t)\tilde{\Phi}(t, s)) = (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)\tilde{\Phi}(t, s) \\ \qquad\qquad\qquad\qquad + P(t)G(t)^{-1}\Delta(t)P(t)\tilde{\Phi}(t, s), \\ \quad Q(t)\tilde{\Phi}(t, s) = Q(t)G(t)^{-1}\bar{A}(t)P(t)\tilde{\Phi}(t, s) + Q(t)G(t)^{-1}\Delta(t)P(t)\tilde{\Phi}(t, s). \end{cases} \tag{5.2}$$

By $P(s)\tilde{\Phi}(s, s)x^0 = P(s)x^0$ for $x^0 \in \mathbb{R}^n$, we find that the variation of constants formula (2.5) yields

$$\begin{aligned} \tilde{\Phi}(t, s)x^0 = \Phi(t, s)P(s)x^0 &+ \int_s^t \Phi(t, \tau)P(\tau)G(\tau)^{-1}\Delta(\tau)P(\tau)\tilde{\Phi}(\tau, s)x^0 \,\mathrm{d}\tau \\ &+ Q(t)G(t)^{-1}\Delta(t)P(t)\tilde{\Phi}(t, s)x^0. \end{aligned} \tag{5.3}$$

Step 1: We show that $\sup_{0\le t-t_0\le 1}\|\tilde\Phi(t,t_0)\|\le c_1$. Let $t_0\ge\hat t$ and observe that by (5.1)

$$\left\|(QG^{-1}\Delta P)\Big|_{[t_0,\infty)}\right\|_\infty \le \|QG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty < 1.$$

Therefore,

$$e^{\mu t}\|\tilde\Phi(t,t_0)x^0\| \le \left(1-\|QG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty\right)^{-1} Me^{\mu t_0}\|P(t_0)x^0\| \\ +\|PG^{-1}\|_\infty\left\|(\Delta P)|_{[t_0,\infty)}\right\|_\infty\int_{t_0}^{t} Me^{\mu\tau}\|\tilde\Phi(\tau,t_0)x^0\|\,\mathrm{d}\tau$$

for all $x^0\in\mathbb{R}^n$ and an application of Gronwall's inequality (see e.g., [26, Lem. 2.1.18]) yields

$$\|\tilde\Phi(t,t_0)x^0\|\le\kappa_1\|P(t_0)\|e^{-\mu(t-t_0)}\|x^0\|e^{\kappa_2(t-t_0)},$$

where $\kappa_1=(1-\|QG^{-1}\|_\infty\|\Delta P|_{[\hat t,\infty)}\|_\infty)^{-1}M$ and $\kappa_2=\|PG^{-1}\|_\infty\|\Delta P\|_\infty M$. This implies that

$$\|\tilde\Phi(t,t_0)x^0\|\le c_1\|x^0\|,\quad c_1=\kappa_1\|P\|_\infty e^{\kappa_2},\ t\in[t_0,t_0+1],$$

and c_1 is independent of $t_0\ge\hat t$. It remains to prove that $\sup_{t\in[t_0,t_0+1]}\|\tilde\Phi(t,t_0)\|\le\tilde c_1$ for all $0\le t_0\le\hat t$ and some $\tilde c_1>0$. However, this is clear since the mapping $t_0\mapsto\sup_{t\in[t_0,t_0+1]}\|\tilde\Phi(t,t_0)\|$ is uniformly continuous on $[0,\hat t]$.

Step 2: We show that $\int_{t_0}^\infty\|\tilde\Phi(t,t_0)\|^2\,\mathrm{d}t\le c_2$ for all $t_0\in\mathbb{R}_+$. To this end, consider, for $\hat t\le s\le T$, the operator

$$M_{s,T}:\mathbb{R}^n\to L^2([\hat t,\infty);\mathbb{R}^n),\ x^0\mapsto x_{s,T}(\cdot):=\mathbb{1}_{[s,T)}(\cdot)\tilde\Phi(\cdot,s)x^0.$$

Let, for $x^0\in\mathbb{R}^n$, $x_{0,s,T}(\cdot):=\mathbb{1}_{[s,T)}(\cdot)\Phi(\cdot,s)P(s)x^0$ and define the operator

$$L_{s,T}:L^2([s,\infty);\mathbb{R}^n)\to L^2([s,\infty);\mathbb{R}^n),\ f\mapsto\mathbb{1}_{[s,T)}L_s(f).$$

Then we have

$$x_{s,T}(t)=x_{0,s,T}(t)+L_{s,T}(\Delta Px_{s,T})(t),\quad t\ge s. \tag{5.4}$$

Note that $x_{0,s,T}\big|_{[s,\infty)}, x_{s,T}\big|_{[s,\infty)}\in L^2([s,\infty);\mathbb{R}^n)$. By (5.1), we find that the operator

$$J:L^2([s,\infty);\mathbb{R}^n)\to L^2([s,\infty);\mathbb{R}^n),\ f\mapsto x_{0,s,T}\big|_{[s,\infty)}+L_{s,T}(\Delta Pf)$$

is a contraction and hence the Banach fixed-point theorem yields that $x_{s,T}$ is the unique solution of (5.4) and

$$\begin{aligned}\|x_{s,T}\|_{L^2[s,\infty)} &\leq \|(I - L_{s,T}\Delta P)^{-1}\| \cdot \|x_{0,s,T}\|_{L^2[s,\infty)} \\ &\leq \underbrace{\left(1 - \|L_{s,T}\| \left\| (\Delta P)|_{[s,\infty)} \right\|_{\infty}\right)^{-1}}_{=:\kappa_{s,T}} \|x_{0,s,T}\|_{L^2[s,\infty)},\end{aligned}$$

and by exponential stability of (E, A),

$$\|x_{s,T}\|_{L^2[\hat{t},\infty)} = \|x_{s,T}\|_{L^2[s,\infty)} \leq \frac{\kappa_{s,T} M}{\sqrt{2\mu}} \sqrt{1 - e^{-2\mu(T-s)}} \|x^0\|.$$

Now, we have $\|L_{s,T}\| \leq \|L_s\| \leq \|L_{\hat{t}}\|$ and $\left\| (\Delta P)|_{[s,\infty)} \right\|_{\infty} \leq \left\| (\Delta P)|_{[\hat{t},\infty)} \right\|_{\infty}$; thus

$$\kappa_{s,T} \leq \left(1 - \|L_{\hat{t}}\| \left\| \Delta P|_{[\hat{t},\infty)} \right\|_{\infty}\right)^{-1}, \quad \hat{t} \leq s \leq T.$$

Therefore, we find that for all $x^0 \in \mathbb{R}^n$,

$$\sup \left\{ \|M_{s,T} x^0\|_{L^2[\hat{t},\infty)} \;\middle|\; (s, T) \in \mathbb{R}^2 \text{ and } \hat{t} \leq s \leq T \right\} < \infty,$$

and hence the uniform boundedness principle yields the existence of $K > 0$ such that $\|M_{s,T}\|_{L^2[\hat{t},\infty)} \leq K$ for all $\hat{t} \leq s \leq T$. This implies that, for all $x^0 \in \mathbb{R}^n$ and $s \geq \hat{t}$, we have

$$\int_s^{\infty} \|\tilde{\Phi}(t, s)x^0\|^2 \, \mathrm{d}t = \lim_{T\to\infty} \int_s^{T} \|(M_{s,T} x^0)(t)\|^2 \, \mathrm{d}t \leq K^2 \|x^0\|^2;$$

thus $\int_s^{\infty} \|\tilde{\Phi}(t, s)\|^2 \, \mathrm{d}t \leq K^2$ and K is independent of s. Since we had fixed $s \geq \hat{t}$, it remains to prove the assertion for $t_0 \leq \hat{t}$. The latter follows from

$$\begin{aligned}\int_{t_0}^{\infty} \|\tilde{\Phi}(t, t_0)\|^2 \, \mathrm{d}t &\leq \int_0^{\hat{t}} \|\tilde{\Phi}(t, 0)\|^2 \, \mathrm{d}t \sup_{t_0[0,\hat{t}]} \|\tilde{\Phi}(0, t_0)\|^2 \\ &\quad + \int_{\hat{t}}^{\infty} \|\tilde{\Phi}(t, \hat{t})\|^2 \, \mathrm{d}t \sup_{t_0\in[0,\hat{t}]} \|\tilde{\Phi}(\hat{t}, t_0)\|^2 < \infty,\end{aligned}$$

which holds by continuity of $\tilde{\Phi}(\cdot, \cdot)$.

Case 2: $Q = 0$. The proof of this case is established along similar lines. □

The boundedness of G^{-1} and ΔP in Theorem 5.3 is guaranteed if $\dot{Q}$, $(E-AQ)^{-1}$, A and Δ_E are bounded and $\|\Delta_E(E-AQ)^{-1}\|_\infty < 1$. Note also that the case $k_{\mathrm{B}}(E,A) = -\infty$ is explicitly allowed in Theorem 5.3.

The following corollary gives a bound directly on the perturbation Δ_E such that exponential stability is preserved for all perturbations within the so defined set.

Corollary 5.4 *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in (2.2) and suppose that G^{-1}, $P(E-AQ)^{-1}$ and $P(E-AQ)^{-1}A(P-\dot{Q}P)$ are bounded. Furthermore, let $\Delta_E \in \mathcal{P}^{Q}_{E,A}$ be bounded and suppose that $\Delta_E \neq 0$, which readily implies $P \neq 0$. Set $\kappa_1 := \|P(E-AQ)^{-1}A(P-\dot{Q}P)\|_\infty \geq 0$ and $\kappa_2 := \|P(E-AQ)^{-1}\|_\infty > 0$. If*

$$\lim_{t_0\to\infty} \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \frac{\min\{\ell(E,A), \|QG^{-1}\|_\infty^{-1}\}}{\kappa_1+\kappa_2 \min\{\ell(E,A), \|QG^{-1}\|_\infty^{-1}\}}, & \text{if } Q \neq 0, \\ \frac{\ell(E,A)}{\|E^{-1}A\|_\infty + \|E^{-1}\|_\infty \ell(E,A)}, & \text{if } Q = 0 \wedge \ell(E,A) < \infty, \\ \|E^{-1}\|_\infty^{-1}, & \text{if } Q = 0 \wedge \ell(E,A) = \infty, \end{cases}$$

then the perturbed system (3.1) is exponentially stable.

Proof **Case 1:** $Q \neq 0$. First note that by assumption $\left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty < \kappa_2^{-1} = \|P(E-AQ)^{-1}\|_\infty^{-1}$ for t_0 large enough. Furthermore, Lemma 3.7 yields (cf. also (4.3)), for Δ as in (3.4) and t_0 large enough,

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty \leq \frac{\|P(E-AQ)^{-1}A(P-\dot{Q}P)\|_\infty \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty}{1 - \|P(E-AQ)^{-1}\|_\infty \left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty};$$

thus the statement follows from Theorem 5.3.

Case 2: $Q = 0$. In this case, observe that $G = E$ and $P = I$; thus the proof is similar to Case 1. □

Remark 5.5 (Sharp perturbation bound) Consider the perturbed ODE

$$(1+\delta)\dot{x}(t) = -x(t) \quad \Longleftrightarrow \quad \dot{x}(t) = -\frac{1}{1+\delta}x(t) = \left(-1 + \frac{\delta}{1+\delta}\right)x(t),$$

which is exponentially stable for $|\delta| < 1$. After the reformulation of the *multiplicative* perturbation on the right hand side as an *additive* perturbation of A we obtain the perturbation term $\frac{\delta}{1+\delta}$, the absolute value of which must be bounded by 1. However, this reduces the range of possible perturbation values to $|\delta| < 1/2$. Therefore, the drawback of the reformulated perturbed system (3.3) is that, in general, it does not yield sharp perturbation bounds.

The next theorem is a version of [23, Prop. 4.5] and [11, Thm. 5.8] for perturbations of the leading coefficient E of an index-1 DAE (E, A). It is a further robustness result under perturbations within the class $\mathcal{P}^{Q}_{E,A}$, as it shows, for perturbations which converge to zero, that the norm of the difference of the two perturbation operators corresponding to the nominal system and the perturbed system gets arbitrary small for sufficiently large t_0.

Theorem 5.6 (Perturbation operator under perturbations) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1 and exponentially stable and let* $Q \in \mathfrak{Q}_{E,A}$. *Let* P *and* G *be as in* (2.2) *and suppose that* G^{-1}, $P(E - AQ)^{-1}$ *and* $P(E - AQ)^{-1}A(P - \dot{Q}P)$ *are bounded. Furthermore, let* $\Delta_E \in \mathcal{P}^Q_{E,A}$ *be such that* $\|\Delta_E(E - AQ)^{-1}\|_\infty < 1$. *If*

$$\lim_{t\to\infty} \|\Delta_E(t)\| = 0,$$

then for the perturbation operator L_{t_0} *of* (E, A) *and the perturbation operator* $\tilde{L}_{t_0}$ *of the perturbed system* $(E + \Delta_E, A)$, *it holds that*

$$\lim_{t_0\to\infty} \|L_{t_0} - \tilde{L}_{t_0}\| = 0.$$

In particular,

$$\ell(E, A) = \ell(E + \Delta_E, A).$$

Proof Except for slight modifications, the proof follows the lines of the proof of [11, Thm. 5.8] applied to the reformulated perturbed system (3.3) with perturbation term $F = \Delta P$ and $h = \|\Delta P\|$. It is only necessary to observe that by [11, Lem. 4.3] and $k_{\mathrm{B}}(E, A) < \infty$, the matrix $QG^{-1}\bar{A}$ is bounded and $\lim_{t\to\infty} \|\Delta_E(t)\| = 0$ implies, using the assumptions of the theorem and Lemma 3.7, that $\lim_{t\to\infty} \|\Delta(t)P(t)\| = 0$. □

We illustrate some of the results by means of our running example.

Example 5.7 (Examples 3.8 and 4.10 revisited) First, we calculate $\ell(E, A)$ for the system (E, A) and projector Q given in Example 3.8. Simple calculations yield that the transition matrix of (E, A) is given by $\Phi(t, s) = \mathrm{diag}\,(e^{-(t-s)}, e^{-(t-s)}, 1)$ for $t, s \in \mathbb{R}_+$, and the perturbation operator by

$$(L_{t_0} f)(t) = \left(\int_{t_0}^{t} e^{-(t-s)} f_1(s)\, \mathrm{d}s\,,\ \int_{t_0}^{t} e^{-(t-s)} f_2(s)\, \mathrm{d}s\,, -f_3(t) \right),$$
$$f \in L^2([t_0, \infty); \mathbb{R}^3),\ t \geq t_0.$$

We may now calculate that, for any $t_0 \in \mathbb{R}_+$ and $f \in L^2([t_0, \infty); \mathbb{R}^3)$,

$$\|L_{t_0} f\|^2_{L^2[t_0,\infty)} = \int_0^\infty \left(\int_0^t e^{-(t-s)} f_1(s + t_0)\, \mathrm{d}s \right)^2 \mathrm{d}t$$

$$+\int_0^\infty \left(\int_0^t e^{-(t-s)} f_2(s+t_0)\, \mathrm{d}s \right)^2 \mathrm{d}t$$
$$+\int_{t_0}^\infty f_3(t)^2\, \mathrm{d}t \le \left(\int_0^\infty e^{-t}\, \mathrm{d}t \right)^2 (\|f_1\|^2_{L^2[t_0,\infty)} + \|f_2\|^2_{L^2[t_0,\infty)})$$
$$+\|f_3\|^2_{L^2[t_0,\infty)} = \|f\|^2_{L^2[t_0,\infty)},$$

which gives $\|L_{t_0}\| \le 1$. On the other hand, for $f = \big(t \mapsto (0, 0, e^{-(t-t_0)})\big) \in L^2([t_0,\infty); \mathbb{R}^3)$ we obtain

$$\|L_{t_0} f\|^2_{L^2[t_0,\infty)} = \|f\|^2_{L^2[t_0,\infty)} = 1/2;$$

thus it holds $\|L_{t_0}\| = 1$ for all $t_0 \in \mathbb{R}_+$ and hence $\ell(E, A) = 1$. For the constants in Corollary 5.4 we find that $\kappa_1 = 1$, $\kappa_2 = 1$ and $\min\{\ell(E, A), \|QG^{-1}\|_\infty^{-1}\} = 1$ as it can easily be calculated. Now Corollary 5.4 states that, for the perturbations Δ_E in Example 3.8, if $\|\Delta_E\| < \frac{1}{2}$, then the perturbed system $(E + \Delta_E, A)$ is exponentially stable. As $\|\Delta_E\| = \sqrt{2}|\delta|$ this is satisfied if

$$|\delta| < \sqrt{2}/4.$$

Indeed, as seen in Example 3.8, the perturbed system is exponentially stable for all $\delta > -1$, so the above statement is true but not very sharp; however, a sharp bound could not have been expected in view of Remark 5.5.

6 Stability radius

In Theorem 5.3 and Corollary 5.4, we have derived a bound on the perturbation such that exponential stability is preserved. This rises the question for the distance to instability of an index-1 DAE (E, A). For ODEs this question has been successfully treated by Hinrichsen and Pritchard, who introduced the stability radius as an appropriate measure for robustness [24,25]. Roughly speaking, the stability radius is the largest bound ρ such that exponential stability and the "algebraic structure" (which is important for DAEs) of the nominal system are preserved for all perturbations of norm less than ρ. After the investigation by Hinrichsen and Pritchard [24,25] for time-invariant ODEs, the stability radius was generalized to time-varying ODEs; see e.g. [23,29]. For time-invariant DAEs a stability radius has been defined and investigated in [7,9,15,37], the most general version (in the sense that the set of allowable perturbations is large) is given in [7], and for time-varying DAEs in [11,16]. In contrast to the definition of the stability radius for time-varying DAEs given in [11,16], we define the stability radius by also allowing for perturbations in the leading coefficient matrix E.

For time-invariant DAEs, the first stability radius was introduced by Byers and Nichols [9] who also introduced a set of allowable perturbations, that is perturbations which preserve regularity and the so called nilpotent part of the matrix pencil $sE - A$. As shown in the proof of [9, Lem. 3.2], the assumption of preserved nilpotent part is, provided that the perturbation preserves the index-1 property, equivalent to a common *left* kernel of the leading coefficient matrices of the perturbed and the nominal matrix pencil. Therefore, it differs from our approach as we require the *right* kernel of E to be preserved. In this sense, our definition of the stability radius can be viewed as both a generalization of the definition given in [9] to time-varying systems and as a generalization of the definition given in [11,16] to a larger set of allowable perturbations with respect to the leading coefficient.

Compared to [9], where the Frobenius norm $\|[\Delta_E, \Delta_A]\|_F$ is considered, and [7], where the norm of the block matrix $\left\| \begin{bmatrix} \Delta_E & 0 \\ 0 & \Delta_A \end{bmatrix} \right\|$ is used, we use the infinity norm of the time-varying perturbation pair $\|[\Delta_E, \Delta_A]\|_\infty$ as a measure of the distance to the nominal DAE.

Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. We introduce the following sets:

$$\begin{aligned} \mathcal{K}(E, A) &:= \left\{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \;\middle|\; \forall\, t \in \mathbb{R}_+ : \ker E(t) = \ker(E(t) + \Delta_E(t)) \right\}, \\ \mathcal{I} &:= \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2 \;\middle|\; (E, A) \text{ is index-1} \right\}, \\ \mathcal{S} &:= \left\{ (E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2 \;\middle|\; (E, A) \text{ is exponentially stable} \right\}. \end{aligned}$$

$\mathcal{K}(E, A)$ is the set of *allowable perturbations*.

Definition 6.1 (*Stability radius*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$. Then the *stability radius* $r(E, A) \in [0, \infty]$ of (E, A) is defined as

$$\begin{aligned} &r(E, A) \\ &\quad := \inf\{ \|[\Delta_E, \Delta_A]\|_\infty \,|\, [\Delta_E, \Delta_A] \in \mathcal{K}(E, A) \;\wedge\; ((E + \Delta_E, A + \Delta_A) \notin \mathcal{I} \\ &\qquad \vee (E + \Delta_E, A + \Delta_A) \notin \mathcal{S})\}. \end{aligned}$$

Remark 6.2 (Stability radius)

(i) It is immediate that for exponentially stable index-1 $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ and any perturbation $[\Delta_E, \Delta_A] \in \mathcal{K}(E, A)$ with $\|[\Delta_E, \Delta_A]\|_\infty < r(E, A)$ the perturbed system $(E + \Delta_E, A + \Delta_A)$, which corresponds to the equation

$$(E(t) + \Delta_E(t))\, \dot{x}(t) = (A(t) + \Delta_A(t))\, x(t), \tag{6.1}$$

is exponentially stable and index-1.

(ii) $r(E, A)$ is the measure of the distance to the nearest allowable system that is not exponentially stable. Note that the infimum is taken over the set $\mathcal{K}(E, A)$. If we had taken a larger set, or all of $\mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n})$, the infimum would in most cases be zero. This is due to the fact that arbitrarily small perturbations in E can cause the system to become unstable if no further structure of the perturbations is claimed. This is true even in the time-invariant case; see e.g. [9]. Nevertheless, it is still possible that there are exponentially stable systems with stability radius zero, because arbitrary small perturbations can also change the structure of the system, i.e., destroy the index-1 property; this is illustrated in Example 6.3. However, as shown in Lemma 6.4, under some boundedness assumption this cannot happen anymore.

(iii) Note that for time-invariant DAEs the definition of stability radius given in [7] is more general than ours in the sense that the set of allowable perturbations is larger, as it is only required that the index and the degree of the characteristic polynomial are preserved. However, for time-varying DAEs we have no notion like the characteristic polynomial. Concerning the higher index case, see the following item.

(iv) It may be possible to define sets of allowable perturbations and the stability radius for higher index DAEs in the following way. If (E, A) is index-μ tractable in the sense of [34], then assume that the perturbation Δ_E is such that in the chain of matrix functions [34, (2.23)] the kernel of A_i (in the notation of [34]) is preserved for $i = 0, \dots, \mu - 1$; note that $A_0 = E$. This might be a proper generalization of the set $\mathcal{K}(E, A)$. The set $\mathcal{I}$ might be generalized in a straightforward manner to the set of all index-μ systems (E, A). Then it is also an interesting question in what way the so generalized stability radius is related to the one defined in [9] in the case of time-invariant DAEs.

(v) For time-varying ODEs (I, A), the stability radius $r(I, A)$ is, in general, much smaller than the stability radius $r(A)$ defined in [23]. In fact, it may even be that $r(A) = \infty$ and $r(I, A) < \infty$: consider the system $\dot{x}(t) = -tx(t)$. It is easy to see that for any bounded perturbation $\Delta \in \mathcal{B}(\mathbb{R}_+; \mathbb{R})$ the system $\dot{x}(t) = (-t + \Delta(t))x(t)$ is still exponentially stable, thus $r(-t) = \infty$. On the other hand, let $[\Delta_E, \Delta_A] \in \mathcal{K}(1, -t)$, that is $1 + \Delta_E(t) \neq 0$ for all $t \in \mathbb{R}_+$. Hence, the perturbed system (6.1) can be rewritten as

$$\dot{x}(t) = \frac{-t + \Delta_A(t)}{1 + \Delta_E(t)}\, x(t)$$

and by choosing $\Delta_A \equiv 0$ and, for any $\varepsilon > 0$, $\Delta_E \equiv -1-\varepsilon$, the perturbed system gets unstable, as it reads $\dot{x} = \frac{t}{\varepsilon}x$. Thus $r(1, -t) \leq \|[-1 - \varepsilon, 0]\| = 1 + \varepsilon$ and, as $\varepsilon > 0$ was arbitrary, we get $r(1, -t) \leq 1 < \infty = r(-t)$.

Example 6.3 Consider system (1.1) with $E = 0$ and $A(t) = \frac{1}{t+1}$. Now, let $\Delta_E \equiv 0$ and $\Delta_A \equiv -\delta$ for any $\delta > 0$. Then $[\Delta_E, \Delta_A] \in \mathcal{K}(E, A)$. However, there exists some $t > 0$ such that $A(t) + \Delta_A(t) = \frac{1}{t+1} - \delta = 0$ and hence $(E + \Delta_E, A + \Delta_A) \notin \mathcal{I}$. This means $r(E, A) \leq \|[0, -\delta]\| = \delta$ for all $\delta > 0$, i.e., $r(E, A) = 0$. However, the nominal system (E, A) is exponentially stable, as any solution x satisfies $x(t) = 0$ for all $t \in \mathbb{R}_+$. This shows that $r(E, A) = 0$ and $(E, A) \in \mathcal{S}$, but the vanishing stability radius is only due to the structural index-1 property getting weaker and weaker with increasing time t, which may be compensated by appropriate boundedness conditions; see Lemma 6.4.

As stressed in the preceding example, for an index-1 DAE (E, A) the properties $r(E, A) = 0$ and $(E, A) \in \mathcal{S}$ are not equivalent. If, however, some boundedness assumptions are satisfied, then this equivalence becomes valid. This and other properties of the stability radius are derived in the following. Note that the stability radius does not have any invariance properties, as we consider an unstructured stability radius.

As shown in [23], the unstructured stability radius is not invariant with respect to Bohl transformations (see also [4] for the latter).

Lemma 6.4 (Properties of the stability radius)

(i) *If $Q \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}^{n\times n})$ is such that Q and $\dot{Q}$ are bounded and $(E,A) \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ is such that $Q \in \mathfrak{Q}_{E,A}$ and $(E-AQ)^{-1}$ is bounded, then it holds that*

$$r(E,A)=0 \iff (E,A)\notin\mathcal{S}.$$

(ii) *For all $(E,A) \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}^{n\times n})^2$ and all $\alpha \geq 0$, we have $r(\alpha(E,A)) = r(\alpha E,\alpha A) = \alpha\, r(E,A)$.*

(iii) *Let $\mathcal{V}(t) \subseteq \mathbb{R}^n$ be a time-varying subspace of $\mathbb{R}^n$ with constant dimension, and define*

$$\mathcal{K}_{\mathcal{V}} := \left\{[E,A]\in\mathcal{B}(\mathbb{R}_+;\mathbb{R}^{n\times 2n}) \,\middle|\, (E,A) \text{ is index-1 and } \ker E(t)=\mathcal{V}(t) \text{ for all } t\in\mathbb{R}_+\right\}.$$

Then the map $\mathcal{K}_{\mathcal{V}} \ni [E,A] \mapsto r(E,A)$ is continuous.

Proof (i) "$\Leftarrow$" is clear. To show "$\Rightarrow$" we use the result of Theorem 6.11 which will be proved later. So assume that $r(E,A)=0$ and $(E,A)\in\mathcal{S}$. Observe that, for G as in (2.2), we have $G^{-1} = (I-\dot{Q}Q)(I+\dot{Q}Q)G^{-1}$ and hence the boundedness of $(E-AQ)^{-1}$, Q and $\dot{Q}$ implies, invoking (3.2), boundedness of G^{-1}. This guarantees $\ell(E,A)\in(0,\infty]$. Together with boundedness of E and A, it also follows that κ_1 and κ_2 as in Theorem 6.11 are finite. Now Theorem 6.11 implies $r(E,A)>0$, a contradiction.

(ii) Follows directly from the definition of the stability radius.

(iii) Let $\varepsilon>0$ and $[E_1,A_1]\in\mathcal{K}_{\mathcal{V}}$. Choose $\delta=\varepsilon$ and $[E_2,A_2]\in\mathcal{K}_{\mathcal{V}}$ such that $\|[E_1-E_2,A_1-A_2]\|_\infty<\delta$. Since $[E_1,A_1]$ is bounded we have $r(E_1,A_1)<\infty$, because $[-E_1,-A_1]\in\mathcal{K}(E_1,A_1)$ but $(E_1-E_1,A_1-A_1)=(0,0)\notin\mathcal{I}$, thus $r(E_1,A_1)\leq\|[E_1,A_1]\|_\infty$. Let $[\Delta_E,\Delta_A]\in\mathcal{K}(E_1,A_1)$ be such that $(E_1+\Delta_E,A_1+\Delta_A)\notin\mathcal{I}$ or $(E_1+\Delta_E,A_1+\Delta_A)\notin\mathcal{S}$, that is $r(E_1,A_1)\leq\|[\Delta_E,\Delta_A]\|_\infty$. Since

$$(E_1+\Delta_E,A_1+\Delta_A) = (E_2+(E_1-E_2)+\Delta_E,\, A_2+(A_1-A_2)+\Delta_A),$$

it follows that $r(E_2,A_2)\leq\|[E_1-E_2,A_1-A_2]\|_\infty+\|[\Delta_E,\Delta_A]\|_\infty$. Now, taking the infimum over all such $[\Delta_E,\Delta_A]$, we obtain that $r(E_2,A_2)\leq\|[E_1-E_2,A_1-A_2]\|_\infty+r(E_1,A_1)$, thus having $|r(E_2,A_2)-r(E_1,A_1)|<\delta=\varepsilon$. This proves the continuity. $\square$

Note that in Lemma 6.4(iii) we consider the set of bounded functions to get a proper notion of distance between two pairs of matrix functions. Moreover, as can be deduced from the proof, boundedness is essential to get a finite stability radius, which is in turn crucial for continuity. Furthermore, the constant dimension of $\mathcal{V}$ is not restrictive since if (E,A) is index-1, then E has constant rank, and hence the kernel is of constant dimension. Therefore, it is shown that the stability radius is continuous on every set of bounded pairs of index-1 matrix functions where the leading coefficients share a common kernel. In fact, this is no longer true on sets where the kernel may change, as the following example illustrates.

Example 6.5 Let $\varepsilon \geq 0$ and consider the system (1.1) with $E = \varepsilon$ and $A = -1$. First, we consider the case $\varepsilon > 0$. Let $[\Delta_E, \Delta_A] \in \mathcal{K}(\varepsilon, -1)$. Note that $\varepsilon + \Delta_E(t)$ must be always invertible to preserve the kernel and hence (6.1) can be rewritten as

$$\dot{x}(t) = \frac{-1 + \Delta_A(t)}{\varepsilon + \Delta_E(t)}\, x(t).$$

Now, for any $\gamma > 0$, $\Delta_E \equiv -\varepsilon - \gamma$ and $\Delta_A \equiv 0$ are allowable perturbations and make the system unstable, as it reads $\dot{x} = \frac{1}{\gamma}x$. Hence, $r(\varepsilon, -1) \leq \|[-\varepsilon - \gamma, 0]\| = \varepsilon + \gamma$ for all $\gamma > 0$; thus $0 \leq r(\varepsilon, -1) \leq \varepsilon$. In particular, this gives

$$\lim_{\varepsilon \to 0} r(\varepsilon, -1) = 0.$$

Now, for $\varepsilon = 0$ and any $[\Delta_E, \Delta_A] \in \mathcal{K}(0, -1)$, the system (6.1) reads $0 = (-1 + \Delta_A(t))x$. First, observe that $[\Delta_E, \Delta_A] \equiv [0, 1] \in \mathcal{K}(0, -1)$ and the resulting perturbed system reads $0 = 0$ which is not index-1 anymore and has any function as a solution. Therefore, it is in particular not exponentially stable, which gives $r(0, -1) \leq 1$. On the other hand, for any Δ_A with $\|\Delta_A\|_\infty < 1$ the perturbed system stays exponentially stable, so we obtain $r(0, -1) = 1$. Finally, we may conclude that

$$\lim_{\varepsilon \to 0} r(\varepsilon, -1) = 0 \neq 1 = r(0, -1).$$

In the following, we derive a lower bound for the stability radius. To do this, we further investigate the perturbation structure. Similar to Sect. 3, we introduce the following.

Definition 6.6 (*Pairs of perturbations*) Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then

$$\widehat{\mathcal{P}}^Q_{E,A} := \left\{ [\Delta_E, \Delta_A] \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times 2n}) \;\middle|\; \begin{array}{l} \forall\, t \in \mathbb{R}_+ : \ker E(t) = \ker\,(E(t) + \Delta_E(t)) \text{ and} \\ \left\| [\Delta_E(t), \Delta_A(t)] \begin{bmatrix} P(t)(E(t) - A(t)Q(t))^{-1} \\ -Q(t)(E(t) - A(t)Q(t))^{-1} \end{bmatrix} \right\| < 1 \end{array} \right\}.$$

It is crucial that perturbations in $\widehat{\mathcal{P}}^Q_{E,A}$ preserve the index-1 property of the nominal system. This is stated in the next lemma.

Lemma 6.7 (Condition for preserved index) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n \times n})^2$ be index-1 and $Q \in \mathfrak{Q}_{E,A}$. Then we have*

$$[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A} \Longrightarrow Q \in \mathfrak{Q}_{E + \Delta_E, A + \Delta_A}.$$

Proof As we only have to show that $E + \Delta_E + ((E + \Delta_E)\dot{Q} - (A + \Delta_A))Q = G + [\Delta_E, \Delta_A]\begin{bmatrix} I+\dot{Q}Q \\ -Q \end{bmatrix}$ is invertible everywhere, the statement follows immediately from the assumptions and the observations $\Delta_E(I + \dot{Q}Q)G^{-1} \overset{(3.2)}{=} \Delta_E P(E - AQ)^{-1}$ and $QG^{-1} = Q(I + \dot{Q}Q)G^{-1} = Q(E - AQ)^{-1}$. □

We may also reformulate the perturbed system (6.1) in a decomposition as in (3.3).

Lemma 6.8 (Decomposition of perturbed system) *Let* $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ *be index-1,* $Q \in \mathfrak{Q}_{E,A}$*,* P*,* $\bar{A}$*,* G *as in* (2.2) *and* $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$*. Then* $x \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^n)$ *is a solution of* (6.1) *if, and only if,* $Px \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^n)$ *and* x *solves* (3.3) *with*

$$\Delta := (I + \Lambda)^{-1}(\Delta_A - \Lambda A)(I - Q\dot{Q}), \quad \Lambda = [\Delta_E, \Delta_A]\begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix}. \tag{6.2}$$

Proof The proof is a straightforward modification of the proof of Lemma 3.6. It is only important to use that $[\Delta_E, \Delta_A]\begin{bmatrix} -\dot{Q} \\ I \end{bmatrix} = -\Lambda G(I - \dot{Q}Q)\dot{Q} + \Delta_A(I - Q\dot{Q})$. □

In fact, with the new Δ in (6.2), it is easy to generalize *all* of the results of Sects. 4 and 5 to perturbations $[\Delta_E, \Delta_A]$ in E and A. We state this in the following theorem.

Theorem 6.9 (Results for perturbations in E *and* A) *The statements of Theorem* 4.6*, Corollary* 4.7*, Proposition* 4.8*, Theorem* 5.3*, Corollary* 5.4 *and Theorem* 5.6 *remain the same for perturbations in E and A, that is they are true if the following substitutions are applied where possible:*

- $\Delta_E \in \mathcal{P}^Q_{E,A} \;\mapsto\; [\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$,
- $\Delta_E \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n}) \;\mapsto\; (\Delta_E, \Delta_A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$,
- Δ *as in* (3.4) $\;\mapsto\;$ Δ *as in* (6.2),
- $k_{\mathrm{B}}(E + \Delta_E, A) \;\mapsto\; k_{\mathrm{B}}(E + \Delta_E, A + \Delta_A)$,
- $\|\Delta_E\|_\infty \;\mapsto\; \|[\Delta_E, \Delta_A]\|_\infty$,
- *perturbed system* (3.1) $\;\mapsto\;$ *perturbed system* (6.1),
- $\left\| \Delta_E|_{[t_0,\infty)} \right\|_\infty \;\mapsto\; \left\| [\Delta_E, \Delta_A]|_{[t_0,\infty)} \right\|_\infty$,
- $\lim_{t\to\infty} \|\Delta_E(t)\| = 0 \;\mapsto\; \lim_{t\to\infty} \|[\Delta_E(t), \Delta_A(t)]\| = 0$,
- *perturbed system* $(E + \Delta_E, A) \;\mapsto\;$ *perturbed system* $(E + \Delta_E, A + \Delta_A)$,
- $\|\Delta_E(E - AQ)^{-1}\|_\infty < 1 \;\mapsto\; \left\| [\Delta_E, \Delta_A]\begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix} \right\|_\infty < 1$,
- $\ell(E + \Delta_E, A) \;\mapsto\; \ell(E + \Delta_E, A + \Delta_A)$.

Furthermore, in Corollary 4.7*, Corollary* 5.4*, and Theorem* 5.6 *the assumption of boundedness of* $(I - QG^{-1}A)(P - \dot{Q}P)$ *has to be added and in Corollary* 5.4 *the constants* κ_1 *and* κ_2 *have to substituted with the ones defined in Theorem* 6.11 *and in the second case* $\|E^{-1}A\|_\infty$ *has to be substituted with* $\left\| \begin{bmatrix} -E^{-1}A \\ I \end{bmatrix} \right\|_\infty$.

Proof Except for slight but obvious modifications, the proofs of the results need not to be changed if it is remembered that Δ is another matrix. At some instances Lemma 6.7 must be used instead of Lemma 3.4. Furthermore, in Corollary 4.7, Eq. (4.3) has to be changed to the inequality presented in Step 2 of the proof of Theorem 6.11 and in Corollary 5.4 the inequality in Case 1 has to be changed in the same manner. $\square$

Nevertheless, we separately state the following generalized version of Theorem 5.3 which is important in due course.

Proposition 6.10 (Exponential stability and perturbation operator anew) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in (2.2) and suppose that G^{-1} is bounded. Furthermore, let $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^{Q}_{E,A}$ and suppose that for Δ as in (6.2) the matrix ΔP is bounded. If*

$$\lim_{t_0\to\infty} \left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty < \begin{cases} \min\left\{\ell(E, A), \|QG^{-1}\|_\infty^{-1}\right\}, & \text{if } Q \neq 0,\\ \ell(E, A), & \text{if } Q = 0,\end{cases}$$

then the perturbed system (6.1) is exponentially stable.

The main theorem of this section essentially relies on the preceding proposition. It gives a lower bound for the stability radius in terms of the norm of the perturbation operator, more precisely the number $\ell(E, A)$ introduced in Sect. 5.

Theorem 6.11 (Lower bound for the stability radius) *Let $(E, A) \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}^{n\times n})^2$ be index-1 and exponentially stable and let $Q \in \mathfrak{Q}_{E,A}$. Let P and G be as in (2.2) and suppose that G^{-1} is bounded. Suppose further that $\kappa_1 := \left\| \begin{bmatrix} -P(E - AQ)^{-1}A(P - \dot{Q}P) \\ (I - Q(E - AQ)^{-1}A)(P - \dot{Q}P) \end{bmatrix} \right\|_\infty < \infty$ and $\kappa_2 := \left\| \begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix} \right\|_\infty < \infty$, i.e., the corresponding matrices are bounded. Then $\kappa_2 > 0$ and*

$$r(E, A) \geq \begin{cases} \frac{\min\{\ell(E,A), \|QG^{-1}\|_\infty^{-1}\}}{\kappa_1+\kappa_2 \min\{\ell(E,A), \|QG^{-1}\|_\infty^{-1}\}}, & \text{if } Q \neq 0,\\ \frac{\ell(E,A)}{\kappa_1+\kappa_2 \ell(E,A)}, & \text{if } Q = 0 \wedge \ell(E, A) < \infty,\\ \frac{1}{\kappa_2}, & \text{if } Q = 0 \wedge \ell(E, A) = \infty.\end{cases}$$

Proof If $P \neq 0$, then $\kappa_2 > 0$ is obvious. If $P = 0$, then $Q = I$ and hence $\kappa_2 > 0$ as well.

Case 1: $Q \neq 0$. We show that for any $[\Delta_E, \Delta_A] \in \mathcal{K}(E, A)$ with $\|[\Delta_E, \Delta_A]\|_\infty < \frac{\alpha}{\kappa_1+\kappa_2\alpha}$, where $\alpha := \min\left\{\ell(E, A), \|QG^{-1}\|_\infty^{-1}\right\} < \infty$, we have $(E+\Delta_E, A+\Delta_A) \in \mathcal{I}$ and $(E + \Delta_E, A + \Delta_A) \in \mathcal{S}$.

Step 1: We show $(E + \Delta_E, A + \Delta_A) \in \mathcal{I}$. To this end observe that it follows from the assumption that $\kappa_2\|[\Delta_E, \Delta_A]\|_\infty < \frac{\kappa_2\alpha}{\kappa_1+\kappa_2\alpha} \leq 1$ and hence

$$\|[\Delta_E, \Delta_A]\|_\infty < \left\| \begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1} \end{bmatrix} \right\|_\infty^{-1},$$

which yields $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$, thus, invoking Lemma 6.7, $(E + \Delta_E, A + \Delta_A)$ is index-1.

Step 2: We show $(E + \Delta_E, A + \Delta_A) \in \mathcal{S}$. By Step 1 we have $[\Delta_E, \Delta_A] \in \widehat{\mathcal{P}}^Q_{E,A}$. Further invoking that for Δ and Λ as in (6.2)

$$\begin{aligned}\Delta P &= (I + \Lambda)^{-1}(\Delta_A - \Lambda A)(I - Q\dot{Q})P \\ &= (I + \Lambda)^{-1}[\Delta_E, \Delta_A]\begin{bmatrix} -P(E - AQ)^{-1}A(P - \dot{Q}P) \\ (I - Q(E - AQ)^{-1}A)(P - \dot{Q}P)\end{bmatrix},\end{aligned}$$

we obtain

$$\left\| (\Delta P)|_{[t_0,\infty)} \right\|_\infty \le \frac{\kappa_1 \, \|[\Delta_E, \Delta_A]\|_\infty}{1 - \kappa_2 \, \|[\Delta_E, \Delta_A]\|_\infty} < \alpha$$

for all $t_0 \in \mathbb{R}_+$, hence we may apply Proposition 6.10 to conclude the exponential stability.

Case 2: $Q = 0$ and $\ell(E, A) < \infty$. With $\alpha := \ell(E, A)$ the proof is similar to Case 1.

Case 3: $Q = 0$ and $\ell(E, A) = \infty$. We may observe that, as in Case 1, $\|[\Delta_E, \Delta_A]\|_\infty < \kappa_2^{-1}$ implies $(E+\Delta_E, A+\Delta_A) \in \mathcal{I}$. Then $(E+\Delta_E, A+\Delta_A) \in \mathcal{S}$ follows immediately from Proposition 6.10. □

Note that in Theorem 6.11 the boundedness of G^{-1} is still important to guarantee that $\ell(E, A) \in (0, \infty]$ exists.

Remark 6.12 (Special cases) We consider Theorem 6.11 for two special cases.

Case 1: $E = I$. In this case we have $Q = 0$, thus $P = I$ and hence $\kappa_1 = \left\| \begin{bmatrix} -A \\ I \end{bmatrix} \right\|_\infty$ and $\kappa_2 = 1$. Suppose that $\ell(I, A) < \infty$. Then we obtain from Theorem 6.11 that

$$\frac{\ell(I, A)}{1 + \|A\|_\infty + \ell(I, A)} \le \frac{\ell(I, A)}{\kappa_1 + \ell(I, A)} \le r(I, A).$$

Note that this does not coincide with any bounds known for the stability radius of an ODE, as still perturbations of the identity and therefore multiplicative perturbations of A are possible. More precisely, A may be perturbed to $(I + \Delta_E)^{-1}(A + \Delta_A)$.

If we considered only perturbations in A, then in κ_1 and κ_2 we neglect the first rows (because these correspond to Δ_E) and thus obtain $\kappa_1 = 1$ and $\kappa_2 = 0$, i.e., $\ell(I, A) \le r(I, A)$, which is just the bound obtained in [23, Prop. 4.1] for ODEs.

Case 2: $E = 0$. In the case of a purely algebraic equation, we have $Q = I$. This gives $\kappa_1 = 0$ and, as A must be invertible everywhere, $\kappa_2 = \|A^{-1}\|_\infty$. Now, Theorem 6.11 gives

$$\|A^{-1}\|_\infty^{-1} \le r(0, A).$$

This bound is sharp: Any allowable perturbation $[\Delta_E, \Delta_A]$ with $\|[\Delta_E, \Delta_A]\|_\infty < \|A^{-1}\|_\infty^{-1}$ has $\Delta_E = 0$ and the perturbed system (6.1) reads $0 = (A(t) + \Delta_A(t))x$, or, equivalently, $0 = (I + A(t)^{-1}\Delta_A(t))x$. Then

$$\|A(t)^{-1}\Delta_A(t)\| \leq \|\Delta_A\|_\infty \|A^{-1}\|_\infty < 1$$

for all $t \in \mathbb{R}_+$ and the resulting invertibility of $I + A(t)^{-1}\Delta_A(t)$ yields that the perturbed system (6.1) is exponentially stable (as it only has the trivial solution). Therefore, $r(I, A) = \|A^{-1}\|_\infty^{-1}$.

In fact, $\|A^{-1}\|_\infty^{-1}$ also coincides with the stability radius as defined in [11,16], see [16, Sec. 5.2], which is reasonable as in this case no perturbations of E are involved.

An important consequence of Theorem 6.11 is that, roughly speaking, the set of all exponentially stable index-1 DAEs where the E matrices share the same kernel is open in the respective superset where exponential stability is not required.

Corollary 6.13 (Set of stable DAEs is open) *Let $Q \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}^{n\times n})$ be such that Q and $\dot{Q}$ are bounded and $Q(t)^2 = Q(t)$ for all $t \in \mathbb{R}_+$. Define*

$$\begin{aligned}
\mathcal{K}_Q &:= \left\{[E, A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \,\middle|\, \forall\, t \in \mathbb{R}_+ : \ker E(t) = \operatorname{im} Q(t)\right\},\\
\mathcal{S}_Q &:= \left\{[E, A] \in \mathcal{B}(\mathbb{R}_+; \mathbb{R}^{n\times 2n}) \,\middle|\, Q \in \mathfrak{Q}_{E,A} \wedge (E, A) \in \mathcal{S} \wedge (E\text{-}AQ)^{-1} \textit{is bounded}\right\}.
\end{aligned}$$

Then $\mathcal{S}_Q$ is open in $\mathcal{K}_Q$.

Proof Observe that clearly $\mathcal{S}_Q \subseteq \mathcal{K}_Q$ and let $[E, A] \in \mathcal{S}_Q$. For G as in (2.2), $G^{-1} = (I - \dot{Q}Q)(I + \dot{Q}Q)G^{-1}$ the boundedness of $(E - AQ)^{-1}$, Q and $\dot{Q}$ implies, invoking (3.2), the boundedness of G^{-1}. Together with the boundedness of E and A, it then follows that κ_1 and κ_2 as in Theorem 6.11 are finite. Set

$$\begin{aligned}
\varepsilon &:= \frac{\alpha}{\kappa_1 + \kappa_2\alpha}, \text{ where}\\
\alpha &:= \begin{cases} \min\left\{\ell(E, A), \|QG^{-1}\|_\infty^{-1}\right\}, & \text{if } Q \neq 0,\\ \ell(E, A), & \text{if } Q = 0 \wedge \ell(E, A) < \infty. \end{cases}
\end{aligned}$$

If $Q = 0$ and $\ell(E, A) = \infty$, set $\varepsilon = \kappa_2^{-1}$. If now $[\tilde{E}, \tilde{A}] \in \mathcal{K}_Q$ with $\|[E - \tilde{E}, A - \tilde{A}]\|_\infty < \varepsilon$, then it follows that $[\Delta_E, \Delta_A] := [\tilde{E} - E, \tilde{A} - A] \in \mathcal{K}(E, A)$ and, hence, applying Theorem 6.11, we may conclude that $(\tilde{E}, \tilde{A}) \in \mathcal{I} \cap \mathcal{S}$. It remains to prove $Q \in \mathfrak{Q}_{\tilde{E},\tilde{A}}$ and the boundedness of $(\tilde{E} - \tilde{A}Q)^{-1}$.

To this end observe that $(\tilde{E}, \tilde{A}) \in \mathcal{I} \cap \mathcal{K}_Q$ and Proposition 2.3 imply that $Q \in \mathfrak{Q}_{\tilde{E},\tilde{A}}$. We also calculate that

$$\begin{aligned}
(\tilde{E} - \tilde{A}Q)^{-1} &= ((E - AQ) + (\Delta_E - \Delta_A Q))^{-1}\\
&= (E - AQ)^{-1}\left(I + (\Delta_E - \Delta_A Q)(E - AQ)^{-1}\right)^{-1}
\end{aligned}$$

and since

$$\left\|(\Delta_E - \Delta_A Q)(E - AQ)^{-1}\right\|_\infty = \left\|[\Delta_E, \Delta_A]\begin{bmatrix} P(E - AQ)^{-1} \\ -Q(E - AQ)^{-1}\end{bmatrix}\right\|_\infty$$
$$\leq \|[\Delta_E, \Delta_A]\|_\infty \kappa_2 < \varepsilon\kappa_2 \leq 1$$

we find

$$\|(\tilde{E} - \tilde{A}Q)^{-1}\|_\infty \leq \frac{\|(E - AQ)^{-1}\|_\infty}{1 - \left\|(\Delta_E - \Delta_A Q)(E - AQ)^{-1}\right\|_\infty} < \infty.$$

□

Acknowledgments I am indebted to Achim Ilchmann (TU Ilmenau), Stephan Trenn (TU Kaiserslautern), and Volker Mehrmann (TU Berlin) for several constructive discussions. I also thank the three anonymous referees for their valuable comments which helped to improve the paper. This work was supported by the DFG grant Il25/9 and partially supported by the DAAD.

References

1. Åström KJ, Wittenmark B (1994) Adaptive control, 2nd edn. Addison-Wesley, Boston
2. Balla K, Kurina GA, März R (2006) Index criteria for differential algebraic equations arising from linear-quadratic optimal control problems. J Dyn Control Syst 12(3):289–311
3. Balla K, März R (2002) A unified approach to linear differential algebraic equations and their adjoints. Z Anal Anwend 21:783–802
4. Berger T (2012) Bohl exponent for time-varying linear differential-algebraic equations. Int J Control 85(10):1433–1451
5. Berger T, Ilchmann A (2013) On stability of time-varying linear differential-algebraic equations. Int J Control 86(6):1060–1076
6. Bohl P (1913) Über Differentialgleichungen. J für Reine und Angewandte Mathematik 144:284–313
7. Bracke M (2000) On stability radii of parametrized, linear differential-algebraic systems. PhD Thesis, Fachbereich Mathematik, Universität Kaiserslautern, Kaiserslautern
8. Brenan KE, Campbell SL, Petzold LR (1989) Numerical solution of initial-value problems in differential-algebraic equations. North-Holland, Amsterdam
9. Byers R, Nichols NK (1993) On the stability radius of a generalized state-space system. Linear Algebra Appl 188–189:113–134
10. Campbell SL (1995) Linearization of DAEs along trajectories. Z Angew Math Phys 46:70–84
11. Chyan C-J, Du NH, Linh VH (2008) On data-dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems. J Differ Equ 245:2078–2102
12. Dai L (1989) Singular control systems. Number 118 in Lecture Notes in Control and Information Sciences. Springer, Berlin
13. Daleckiĭ JL, Kreĭn MG (1974) Stability of solutions of differential equations in banach spaces. Number 43 in Translations of Mathematical Monographs. American Mathematical Society, Providence
14. De Terán F, Dopico FM, Moro J (2008) First order spectral perturbation theory of square singular matrix pencils. Linear Algebra Appl 429:548–576
15. Du NH, Linh VH (2006) On the robust stability of implicit linear systems containing a small parameter in the leading term. IMA J Math Control Inf 23:67–84 (published online July 27, 2005)
16. Du NH, Linh VH (2006) Stability radii for linear time-varying differential-algebraic equations with respect to dynamic perturbations. J Differ Equ 230(2):579–599
17. Eich-Soellner E, Führer C (1998) Numerical methods in multibody dynamics. Teubner, Stuttgart
18. Estévez Schwarz D, Tischendorf C (2000) Structural analysis for electric circuits and consequences for MNA. Int J Circuit Theory Appl 28(2):131–162

19. Fang C-H, Chang F-R (1993) Analysis of stability robustness for generalized state-space systems with structured perturbations. Syst Control Lett 21:109–114
20. Griepentrog E, März R (1986) Differential-algebraic equations and their numerical treatment. Number 88 in Teubner-Texte zur Mathematik. Teubner, Leipzig
21. Hairer E, Wanner G (1991) Solving ordinary differential equations II: Stiff and differential-algebraic problems. Springer, Berlin
22. Hanke M (1990) On the regularization of index 2 differential-algebraic equations. J Math Anal Appl 151:236–253
23. Hinrichsen D, Ilchmann A, Pritchard AJ (1989) Robustness of stability of time-varying linear systems. J Differ Equ 82(2):219–250
24. Hinrichsen D, Pritchard AJ (1986) Stability radii of linear systems. Syst Control Lett 7:1–10
25. Hinrichsen D, Pritchard AJ (1986) Stability radius for structured perturbations and the algebraic Riccati equation. Syst Control Lett 8:105–113
26. Hinrichsen D, Pritchard AJ (2005) Mathematical systems theory I. Modelling, state space analysis, stability and robustness, volume 48 of Texts in Applied Mathematics. Springer, Berlin
27. Ilchmann A, Mehrmann V (2005) A behavioural approach to time-varying linear systems. Part 1: general theory. SIAM J Control Optim 44(5):1725–1747
28. Iwata S, Takamatsu M, Tischendorf C (2012) Tractability index of hybrid equations for circuit simulation. Math Comput 81(278):923–939
29. Jacob B (1998) A formula for the stability radius of time-varying systems. J Differ Equ 142:167–187
30. Kumar A, Daoutidis P (1999) Control of nonlinear differential algebraic equation systems with applications to chemical processes, volume 397 of Chapman and Hall/CRC Research Notes in Mathematics. Chapman and Hall, Boca Raton
31. Kunkel P, Mehrmann V (2006) Differential-algebraic equations. Analysis and numerical solution. EMS Publishing House, Zürich
32. Lamour R, März R, Tischendorf C (2013) Differential algebraic equations: a projector based analysis, volume 1 of Differential-Algebraic Equations Forum. Springer, Heidelberg
33. Linh VH, Mehrmann V (2009) Lyapunov, Bohl and Sacker-Sell spectral intervals for differential-algebraic equations. J. Dyn Differ Equ 21:153–194
34. März R (1991) Numerical methods for differential algebraic equations. Acta Numer, 141–198
35. März R (2002) The index of linear differential algebraic equations with properly stated leading terms. Results Math. 42:308–338
36. Mehrmann V (2012) Index concepts for differential-algebraic equations. Institute of Mathematics, TU Berlin (preprint)
37. Qiu L, Davison EJ (1992) The stability robustness of generalized eigenvalues. IEEE Trans Autom Control 37(6):886–891
38. Riaza R (2008) Differential-algebraic systems. Analytical aspects and circuit applications. World Scientific Publishing, Basel
39. Takamatsu M, Iwata S (2010) Index characterization of differential-algebraic equations in hybrid analysis for circuit simulation. Int J Circuit Theory Appl 38:419–440
40. Zhou K, Doyle JC, Glover K (1996) Robust and optimal control. Prentice Hall, Upper Saddle River