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Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability

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A B S T R A C T

Stability of fractional-order nonlinear dynamic systems is studied using Lyapunov direct method with the introductions of Mittag–Leffler stability and generalized Mittag–Leffler stability notions. With the definitions of Mittag–Leffler stability and generalized Mittag–Leffler stability proposed, the decaying speed of the Lyapunov function can be more generally characterized which include the exponential stability and power-law stability as special cases. Finally, four worked out examples are provided to illustrate the concepts.

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1. Introduction

In nonlinear systems, Lyapunov's direct method (also called the second method of Lyapunov) provides a way to analyze the stability of a system without explicitly solving the differential equations. The method generalizes the idea which shows that the system is stable if there are some Lyapunov function candidates for the system. The Lyapunov direct method is a sufficient condition to show the stability of systems, which means the system may still be stable even one cannot find a Lyapunov function candidate to conclude the system stability property.

Recently, fractional calculus was introduced to the stability analysis of nonlinear systems, for example, [1–3]. The integer-order methods of stabilization were used in these works. Motivated by the application of fractional calculus in nonlinear systems, we propose the (generalized) Mittag–Leffler stability and the (generalized) fractional Lyapunov direct method with a hope to enrich the knowledge of both system theory and fractional calculus. Meanwhile, the fact that computation becomes faster and memory becomes cheaper makes the application of fractional calculus in reality possible and affordable [4]. To demonstrate the advantage of fractional calculus in characterizing system behavior, let us consider the following illustrative example.

Example. Compare the following two systems with initial condition $x(0)$ for $0 < \nu < 1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \nu t^{\nu-1}, \tag{1}$$

$$ {}_0^C D_t^{\alpha} x(t) = \nu t^{\nu-1}, \quad 0 < \alpha < 1. \tag{2}$$

The analytical solutions of (1) and (2) are $t^\nu + x(0)$ and $\frac{\nu\Gamma(\nu)t^{\nu+\alpha-1}}{\Gamma(\nu+\alpha)} + x(0)$, respectively. Obviously, the integer-order system (1) is unstable for any $\nu \in (0, 1)$. However, the fractional dynamic system (2) is stable as $0 < \nu \le 1-\alpha$, which implies that the fractional-order system may have additional attractive feature over the integer-order system.

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This work is motivated by the simple fact, as also indicated in [4], that the generalized energy of a system does not have to decay exponentially for the system to be stable in the sense of Lyapunov. Moreover, these three papers [5–7] are strongly related to various stability problems of fractional systems. Lastly, an early version of this paper is [8].

Our contributions of this paper include:

- The study of the (generalized) fractional Lyapunov direct method and the (generalized) Mittag–Leffler stability of nonautonomous systems.
- The extension of the application of Riemann–Liouville fractional-order systems by using Caputo fractional-order systems.
- The fractional comparison principle and several other fractional inequalities extend the applications of fractional calculus.

The paper is organized as follows. In Section 2, we recall some basic definitions. In Section 3, we discuss the definition of fractional-order systems and the relationships between Lipschitz condition and fractional-order systems. In Section 4, we propose the definition of the (generalized) Mittag–Leffler stability. In Section 5, we prove the (generalized) fractional Lyapunov direct method of nonautonomous systems. In Section 6, we introduce the class-$\mathcal{K}$ functions to the fractional Lyapunov direct method and provide the fractional comparison principle. In Section 7, four illustrative examples are given as a proof of concept. Conclusion is given in Section 8.

2. Fractional calculus

2.1. Caputo and Riemann–Liouville fractional derivatives

Fractional calculus plays an important role in modern science [3,9–12]. In this paper, we use both Riemann–Liouville and Caputo fractional operators as our main tools. The uniform formula of a fractional integral with $\alpha \in (0, 1)$ is defined as

$$ {}_a\mathcal{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \tag{3} $$

where $f(t)$ is an arbitrary integrable function, ${}_a\mathcal{D}_t^{-\alpha}$ is the fractional integral of order α on $[a, t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number p, the Riemann–Liouville and Caputo fractional derivatives are defined respectively as

$$ {}_aD_t^p f(t) = \frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} [{}_aD_t^{-([p]-p+1)} f(t)] \tag{4} $$

and

$$ {}_a^CD_t^p f(t) = {}_aD_t^{-([p]-p+1)} \left[\frac{\mathrm{d}^{[p]+1}}{\mathrm{d}t^{[p]+1}} f(t) \right], \tag{5} $$

where $[p]$ stands for the integer part of p, D and ${}^CD$ are Riemann–Liouville and Caputo fractional derivatives, respectively.

Some of the properties of the Riemann–Liouville fractional operator are recalled below [10,11,13]:

Property 1.

$$ {}_aD_t^p (t-a)^\nu = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-p)} (t-a)^{\nu-p}, $$

where $p \in \mathbb{R}$ and $\nu > -1$.

Property 2.

$$ {}_0D_t^p H(t) = \frac{t^{-p}}{\Gamma(1-p)}, \tag{6} $$

where $H(t)$ is the Heaviside unit step function.

Property 3.

$$ {}_aD_t^p ({}_aD_t^q f(t)) = {}_aD_t^{p+q} f(t) - \sum_{j=1}^{n} [{}_aD_t^{q-j} f(t)]_{t=a} \frac{(t-a)^{-p-j}}{\Gamma(1-p-j)}, \tag{7} $$

where $p, q \in \mathbb{R}$, $n \in \mathbb{Z}$ and $n - 1 \le q < n$.

Property 4.

$$ {}_0D_t^\gamma (t^{k\alpha+\beta-1} E_{\alpha,\beta}^{(k)}(\lambda t^\alpha)) = t^{k\alpha+\beta-\gamma-1} E_{\alpha,\beta-\gamma}^{(k)}(\lambda t^\alpha), $$

where $\gamma \in \mathbb{R}$, $k \in \mathbb{Z} \setminus \mathbb{Z}^-$ and $E^{(k)}(y) = \frac{\mathrm{d}^k}{\mathrm{d}y^k} E(y)$.

2.2. Mittag–Leffler function

Similar to the exponential function frequently used in the solutions of integer-order systems, a function frequently used in the solutions of fractional-order systems is the Mittag–Leffler function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \tag{8}$$

where $\alpha > 0$ and $z \in \mathbb{C}$. The Mittag–Leffler function with two parameters appears most frequently and has the following form:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \tag{9}$$

where $\alpha > 0$, $\beta > 0$ and $z \in \mathbb{C}$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$. Also, $E_{1,1}(z) = e^x$.

Moreover, the Laplace transform of Mittag–Leffler function in two parameters is

$$\mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \quad (\Re(s) > |\lambda|^{\frac{1}{\alpha}}), \tag{10}$$

where $t \geq 0$, s is the variable in Laplace domain, $\Re(s)$ denotes the real part of s, $\lambda \in \mathbb{R}$ and $\mathscr{L}\{\cdot\}$ stands for the Laplace transform.

3. Fractional nonautonomous systems

Consider the Caputo fractional nonautonomous system [3,10]

$${}_{t_0}^{C}D_t^{\alpha}x(t) = f(t,x) \tag{11}$$

with initial condition $x(t_0)$, where $\alpha \in (0,1)$, $f : [t_0,\infty] \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0,\infty] \times \Omega$, and $\Omega \in \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The equilibrium point of (11) is defined as follows:

Definition 3.1. The constant x_0 is an equilibrium point of Caputo fractional dynamic system (11), if and only if $f(t, x_0) = 0$.

Remark 3.2. When $\alpha \in (0, 1)$, it follows form (5) that the Caputo fractional-order system (11) has the same equilibrium points as the integer-order system $\dot{x}(t) = f(t, x)$.

Remark 3.3. For convenience, we state all definitions and theorems for the case when the equilibrium point is the origin of $\mathbb{R}^n$; i.e. $x_0 = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose the equilibrium point for (11) is $\bar{x} \neq 0$ and consider the change of variable $y = x - \bar{x}$. The αth order derivative of y is given by

$${}_{t_0}^{C}D_t^{\alpha}y = {}_{t_0}^{C}D_t^{\alpha}(x-\bar{x}) = f(t,x) = f(t,y+\bar{x}) = g(t,y),$$

where $g(t, 0) = 0$ and in the new variable y, the system has equilibrium at the origin.

Consider the Riemann–Liouville fractional-order system

$${}_{t_0}D_t^{\alpha}x(t) = f(t,x) \tag{12}$$

with initial condition $x(t_0)$, where $\alpha \in (0,1)$, $f : [t_0,\infty] \times \Omega \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on $[t_0,\infty] \times \Omega$, and $\Omega \in \mathbb{R}^n$ is a domain that contains the equilibrium point $x = 0$. The equilibrium point of (12) is defined as follows.

Definition 3.4. The constant x_0 is an equilibrium point of the Riemann–Liouville fractional dynamic system (12), if and only if ${}_{t_0}D_t^{\alpha}x_0 = f(t, x_0)$.

Remark 3.5. For convenience, we state all definitions and theorems for the case when the equilibrium point is the origin of $\mathbb{R}^n$; i.e. $x_0 = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose the equilibrium point for (12) is $\bar{x} \neq 0$ and consider the change of variable $y = x - \bar{x}$. The αth order derivative of y is given by

$$\begin{aligned}{}_{t_0}D_t^{\alpha}y &= {}_{t_0}D_t^{\alpha}(x-\bar{x}) = f(t,x) - \frac{\bar{x}t^{-\alpha}}{\Gamma(1-\alpha)} \\ &= f(t,y+\bar{x}) - \frac{\bar{x}t^{-\alpha}}{\Gamma(1-\alpha)} = \bar{g}(t,y),\end{aligned}$$

where $\bar{g}(t, 0) = 0$ and in the new variable y, the system has equilibrium at the origin.

Remark 3.6. For the system

$$_0\mathcal{D}_t^{\alpha}x(t) = f(t, x), \tag{13}$$

where $\alpha \in (0, 1]$, and the symbol $\mathcal{D}$ in (13) can denote both the Caputo and Riemann–Liouville fractional operators. If $x = 0$ is the equilibrium point of the fractional-order system (13), and there exists t_1 satisfying $x(t_1) = 0$, then $x(t) = 0$ for $t \geq t_1$.

The existence and uniqueness of the solution to system (13) can be stated by the following theorem.

Theorem 3.7 (*Existence and Uniqueness Theorem [11]*). *Let $f(t, x)$ be a real-valued continuous function, defined in the domain G, satisfying in G the Lipschitz condition with respect to x, i.e.*

$$|f(t, x_1) - f(t, x_2)| \leq l|x_1 - x_2|,$$

where l is a positive constant, such that

$$|f(t, x)| \leq M < \infty \quad \text{for all } (t, x) \in G.$$

Let also

$$K \geq \frac{Mh^{\sigma_n - \sigma_1 + 1}}{\Gamma(1 + \sigma_n)}.$$

Then there exists in a region R(h,K) a unique and continuous solution $y(t)$ of the following initial-value problem,

$$_0\mathcal{D}_t^{\sigma_n}x(t) = f(t, x), \tag{14}$$

$$[_0\mathcal{D}_t^{\sigma_k - 1}x(t)]_{t=0} = b_k, \quad k = 1, 2, \ldots, n, \tag{15}$$

where

$$_a\mathcal{D}_t^{\sigma_k} \equiv {_a\mathcal{D}_t^{\alpha_k}}\, {_a\mathcal{D}_t^{\alpha_{k-1}}} \ldots {_a\mathcal{D}_t^{\alpha_1}};$$
$$_a\mathcal{D}_t^{\sigma_k - 1} \equiv {_a\mathcal{D}_t^{\alpha_k - 1}}\, {_a\mathcal{D}_t^{\alpha_{k-1}}} \ldots {_a\mathcal{D}_t^{\alpha_1}};$$
$$\sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \ldots, n);$$
$$0 < \alpha_j \leq 1, \quad (j = 1, 2, \ldots, n).$$

Remark 3.8. Obviously, if $f(t, x)$ in (13) satisfies the locally Lipschitz condition with respect to x, then there exists a unique solution of (13) on $[t_0, \infty] \times \Omega$.

3.1. Lipschitz condition and Caputo fractional nonautonomous systems

The fact that $f(t, x)$ is locally bounded and is locally Lipschitz in x implies the existence and uniqueness of the solution to the Caputo fractional-order system (11) [11]. In the following of this subsection, we study the relationship between the Lipschitz condition and the Caputo fractional nonautonomous system (11).

Lemma 3.1. *For the real-valued continuous $f(t, x)$ in (11), we have*

$$\|{_{t_0}\mathcal{D}_t^{-\alpha}}f(t, x(t))\| \leq {_{t_0}\mathcal{D}_t^{-\alpha}}\|f(t, x(t))\|,$$

where $\alpha \geq 0$ and $\|\cdot\|$ denotes an arbitrary norm. It follows from (3) and (5) that

$$_{t_0}\mathcal{D}_t^{-\alpha}f(t, x(t)) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau, x(\tau))}{(t - \tau)^{1-\alpha}} \mathrm{d}\tau,$$

which implies

$$\begin{aligned}
\|{_{t_0}\mathcal{D}_t^{-\alpha}}f(t, x(t))\| &= \left\| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau, x(\tau))}{(t - \tau)^{1-\alpha}} \mathrm{d}\tau \right\| \\
&= \frac{1}{\Gamma(\alpha)} \left\| \lim_{\Delta t \to 0} \sum_{n=0}^{\frac{t-t_0}{\Delta t}} \frac{f(t_0 + n\Delta t, x(t_0 + n\Delta t))}{(t - t_0 - n\Delta t)^{1-\alpha}} \Delta t \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \lim_{\Delta t \to 0} \sum_{n=0}^{\frac{t-t_0}{\Delta t}} \left\| \frac{f(t_0 + n\Delta t, x(t_0 + n\Delta t))}{(t - t_0 - n\Delta t)^{1-\alpha}} \Delta t \right\|
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)} \lim_{\Delta t \to 0} \sum_{n=0}^{\frac{t-t_0}{\Delta t}} \frac{\|f(t_0 + n\Delta t, x(t_0 + n\Delta t))\|}{(t - t_0 - n\Delta t)^{1-\alpha}} \Delta t$$

$$= {}_{t_0}\mathcal{D}_t^{-\alpha} \|f(t, x(t))\|. \tag{16}$$

Here we have used the triangle inequality to derive (16).

Theorem 3.9. *If $x = 0$ is an equilibrium point of system (11), f is Lipschitz on x with Lipschitz constant l and is piecewise continuous with respect to t, then the solution of (11) satisfies*

$$\|x(t)\| \le \|x(t_0)\| E_\alpha(l(t - t_0)^\alpha), \tag{17}$$

where $\alpha \in (0, 1)$.

Proof. By applying the fractional integral operator ${}_{t_0}\mathcal{D}_t^{-\alpha}$ to both sides of (11), it follows from (7), Lemma 3.1 and the Lipschitz condition that

$$\begin{aligned} |\|x(t)\| - \|x(t_0)\|| &\le \|x(t) - x(t_0)\| = \|{}_{t_0}\mathcal{D}_t^{-\alpha} f(t, x(t))\| \\ &\le {}_{t_0}\mathcal{D}_t^{-\alpha} \|f(t, x(t))\| \le l {}_{t_0}\mathcal{D}_t^{-\alpha} \|x(t)\|, \end{aligned}$$

where $\int_{t_0}^{t} x(\tau)\mathrm{d}\tau|_{t=t_0} = 0$.[1] There exists a nonnegative function $M(t)$ satisfying

$$\|x(t)\| - \|x(t_0)\| = l {}_{t_0}\mathcal{D}_t^{-\alpha} \|x(t)\| - M(t). \tag{18}$$

By applying the Laplace transform($\mathcal{L}\{\cdot\}$) to (18), it follows that

$$\|x(s)\| = \frac{\|x(t_0)\| s^{\alpha-1} - s^{\alpha} M(t)}{s^{\alpha} - l}, \tag{19}$$

where $\|x(s)\| = \mathcal{L}\{\|x(t)\|\}$. Applying the inverse Laplace transform to (19) gives

$$\|x(t)\| = \|x(t_0)\| E_\alpha(l(t - t_0)^\alpha) - M(t) * [t^{-1} E_{\alpha,0}(l(t - t_0)^\alpha)],$$

where $*$ denotes the convolution operator and $t^{-1} E_{\alpha,0}(l(t - t_0)^\alpha) = \frac{\mathrm{d}E_\alpha(l(t-t_0)^\alpha)}{\mathrm{d}t} \ge 0$.[2] It then follows that

$$\|x(t)\| \le \|x(t_0)\| E_\alpha(l(t - t_0)^\alpha). \quad \blacksquare$$

Remark 3.10. In Theorem 3.9, if $\alpha = 1$, it follows that [14]

$$\|x(t)\| \le \|x(t_0)\| \mathrm{e}^{l(t-t_0)}.$$

Remark 3.11. In the proof of Theorem 3.9, we cannot establish $\|x(t_0)\| E_\alpha(-l(t - t_0)^\alpha) \le \|x(t)\|$ because the relationship between $\|{}_{t_0}^{C}D_t^{\alpha} x(t)\|$ and $|{}_{t_0}^{C}D_t^{\alpha} \|x(t)\||$ cannot be ascertained as in the integer-order case. To show this, let us use the Grünwald–Letnikov form of the definition of fractional integral, so we have

$$\|{}_{t_0}^{C}D_t^{\alpha} x(t)\| = \left\| \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \left\{ \sum_{r=0}^{n} \begin{bmatrix} -\alpha \\ r \end{bmatrix} x(t + h - rh) - \begin{bmatrix} \alpha \\ n \end{bmatrix} x(0) \right\} \right\|$$

and

$$|{}_{t_0}^{C}D_t^{\alpha} \|x(t)\|| = \left| \lim_{\substack{h \to 0 \\ nh = t - t_0}} h^{-\alpha} \left\{ \sum_{r=0}^{n} \begin{bmatrix} -\alpha \\ r \end{bmatrix} \|x(t + h - rh)\| - \begin{bmatrix} \alpha \\ n \end{bmatrix} \|x(0)\| \right\} \right|,$$

where

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} = \frac{\zeta(\zeta + 1) \cdots (\zeta + \eta - 1)}{\eta!}.$$

Clearly, the relationship between $\|{}_{t_0}^{C}D_t^{\alpha} x(t)\|$ and $|{}_{t_0}^{C}D_t^{\alpha} \|x(t)\||$ cannot be established as in integer-order case.

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1 We used the fact that $\|x(t_0)\|$ is a finite constant.

2 This inequality can be derived directly from the definition of Mittag–Leffler function (8).

4. Generalized Mittag–Leffler stability

Lyapunov stability provides an important tool for stability analysis in nonlinear systems. In fact, stability issues have been extensively covered by Lyapunov and there are several tests associated with this name. We primarily consider what is often called, Lyapunov's direct method which involves finding a Lyapunov function candidate for a given nonlinear system. If such a function exists, the system is stable. Applying Lyapunov's direct method is to search for an appropriate function. Note that Lyapunov direct method is a sufficient condition which means if one cannot find a Lyapunov function candidate to conclude the system stability property, the system may still be stable and one cannot claiming the system is not stable. In this paper, we extend Lyapunov direct method by considering incorporating fractional-order operators. That is, the nonlinear dynamic systems itself could be fractional order as well as the evolution of the Lyapunov function could be time-fractional order. Let us first define the stability in sense of Mittag–Leffler.

Definition 4.1 (*Mittag–Leffler Stability*). The solution of (13) is said to be Mittag–Leffler stable if

$$\|x(t)\| \le \{m[x(t_0)]E_\alpha(-\lambda(t-t_0)^\alpha)\}^b, \tag{20}$$

where t_0 is the initial time, $\alpha \in (0, 1)$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant m_0.

Definition 4.2 (*Generalized Mittag–Leffler Stability*). The solution of (13) is said to be Generalized Mittag–Leffler stable if

$$\|x(t)\| \le \left\{m[x(t_0)](t-t_0)^{-\gamma}E_{\alpha,1-\gamma}(-\lambda(t-t_0)^\alpha)\right\}^b, \tag{21}$$

where t_0 is the initial time, $\alpha \in (0, 1)$, $-\alpha < \gamma \le 1-\alpha$, $\lambda \ge 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on $x \in \mathbb{B} \in \mathbb{R}^n$ with Lipschitz constant m_0.

Remark 4.3. It follows from the properties of Mittag–Leffler function and completely monotonic function that

$${}_0\mathcal{D}_t^\gamma E_\alpha(-\lambda t^\alpha) = {}_0D_t^\gamma E_\alpha(-\lambda t^\alpha) = {}_0^C D_t^\gamma E_\alpha(-\lambda t^\alpha) = t^{-\gamma}E_{\alpha,1-\gamma}(-\lambda t^\alpha)$$

is a completely monotonic function for $\alpha \in (0, 1)$, $\lambda \ge 0$ and $\gamma \in [0, 1-\alpha]$ [15,16].

Remark 4.4. Mittag–Leffler stability and Generalized Mittag–Leffler stability imply asymptotic stability.

Remark 4.5. Let $\lambda = 0$, it follows from (21) that

$$\|x(t)\| \le \left[\frac{m(x(t_0))}{\Gamma(1-\gamma)}\right]^b (t-t_0)^{-\gamma b},$$

which implies that the power-law stability is a special case of the Mittag–Leffler stability.

Remark 4.6. The following two statements are equivalent:

(a) $m(x)$ is Lipschitz with respect to x.
(b) There exist a Lipschitz constant m_0 satisfying

$$\|m(x_1) - m(x_2)\| \le m_0\|x_1 - x_2\|.$$

As a special case, when $x_2 = 0$, it follows from $m(0) = 0$ that

$$\|m(x_1)\| \le m_0\|x_1\|.$$

Without loss of generality, the initial time can be taken as $t_0 = 0$.

5. Fractional-order extension of Lyapunov direct method

By using the Lyapunov direct method, we can get the asymptotic stability of the corresponding systems. In this section, we extend the Lyapunov direct method to the case of fractional-order systems, which leads to the Mittag–Leffler stability.

Theorem 5.1. *Let $x = 0$ be an equilibrium point for the system (13) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0,\infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1\|x\|^a \le V(t, x(t)) \le \alpha_2\|x\|^{ab}, \tag{22}$$

$${}_0^C D_t^\beta V(t, x(t)) \le -\alpha_3\|x\|^{ab}, \tag{23}$$

where $t \ge 0$, $x \in \mathbb{D}$, $\beta \in (0, 1)$, α_1, α_2, α_3, a and b are arbitrary positive constants. Then $x = 0$ is Mittag–Leffler stable. If the assumptions hold globally on $\mathbb{R}^n$, then $x = 0$ is globally Mittag–Leffler stable.

Proof. It follows from equations (22) and (23) that

$${}_{0}^{C}D_{t}^{\beta}V(t,x(t)) \leq -\frac{\alpha_3}{\alpha_2}V(t,x(t)).$$

There exists a nonnegative function $M(t)$ satisfying

$${}_{0}^{C}D_{t}^{\beta}V(t,x(t)) + M(t) = -\frac{\alpha_3}{\alpha_2}V(t,x(t)). \tag{24}$$

Taking the Laplace transform of (24) gives

$$s^{\beta}V(s) - V(0)s^{\beta-1} + M(s) = -\frac{\alpha_3}{\alpha_2}V(s), \tag{25}$$

where nonnegative constant $V(0) = V(0, x(0))$ and $V(s) = \mathcal{L}\{V(t, x(t))\}$. It then follows that

$$V(s) = \frac{V(0)s^{\beta-1} - M(s)}{s^{\beta} + \frac{\alpha_3}{\alpha_2}}.$$

If $x(0) = 0$, namely $V(0) = 0$, the solution to (13) is $x = 0$.

If $x(0) \neq 0$, $V(0) > 0$. Because $V(t, x)$ is locally Lipschitz with respect to x, it follows from Theorem 3.7 and the inverse Laplace transform that the unique solution of (24) is

$$V(t) = V(0)E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right) - M(t) * \left[t^{\beta-1}E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right].$$

Since both $t^{\beta-1}$ and $E_{\beta,\beta}(-\frac{\alpha_3}{\alpha_2}t^{\beta})$ are nonnegative functions, it follows that

$$V(t) \leq V(0)E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right). \tag{26}$$

Substituting (26) into (22) yields

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1}E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]^{\frac{1}{a}},$$

where $\frac{V(0)}{\alpha_1} > 0$ for $x(0) \neq 0$.

Let $m = \frac{V(0)}{\alpha_1} = \frac{V(0,x(0))}{\alpha_1} \geq 0$, then we have

$$\|x(t)\| \leq \left[mE_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]^{\frac{1}{a}},$$

where $m = 0$ holds if and only if $x(0) = 0$. Because $V(t, x)$ is locally Lipschitz with respect to x and $V(0, x(0)) = 0$ if and only if $x(0) = 0$, it follows that $m = \frac{V(0,x(0))}{\alpha_1}$ is also Lipschitz with respect to $x(0)$ and $m(0) = 0$, which imply the Mittag–Leffler stability of system (13).

Theorem 5.2. *In Theorem 5.1, there exists a constant t_1 such that the equilibrium point $x = 0$ is generalized Mittag–Leffler stability for $t \geq t_1$.*

Proof. It follows from the proof of Theorem 5.1 that there exists a $M(s)$ satisfying $s^{\beta}V(s) - V(0)s^{\beta-1} + M(s) = -\frac{\alpha_3}{\alpha_2}V(s)$. Let $\mathcal{L}\{\tilde{M}(t)\} = \tilde{M}(s) = M(s) - V(0)s^{\beta-1} + V(0)s^{\beta-\tilde{\beta}}$, where $\tilde{\beta} \in [\beta, 1+\beta)$, we have

$$V(s) = \frac{V(0)s^{\beta-\tilde{\beta}} - \tilde{M}(s)}{s^{\beta} + \frac{\alpha_3}{\alpha_2}}. \tag{27}$$

In time domain

$$V(t) = V(0)t^{\tilde{\beta}-1}E_{\beta,\tilde{\beta}}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right) - \tilde{M}(t) * \left[t^{\beta-1}E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]. \tag{28}$$

It follows from $\beta \in (0, 1)$ that there exist $\varepsilon > 0$ and $t_1 > 0$ such that

$$\left[\frac{t^{\tilde{\beta}-\beta-1}}{\Gamma(\tilde{\beta}-\beta)} - \frac{t^{-\beta}}{\Gamma(1-\beta)}\right] * \left[t^{\beta-1}E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right] \geq 0$$

for all $t \geq t_1$ and $\tilde{\beta} \in (1+\beta-\varepsilon, 1+\beta)$. Therefore, $\tilde{M}(t) * \left[t^{\beta-1}E_{\beta,\beta}(-\frac{\alpha_3}{\alpha_2}t^{\beta})\right] \geq 0$ for all $t \geq t_1$ and $\tilde{\beta} \in (1+\beta-\varepsilon, 1+\beta)$. Substituting this inequality into (28) yields

$$\|x(t)\| \leq [V(t)]^{\frac{1}{a}} \leq \left[V(0)t^{\tilde{\beta}-1}E_{\beta,\tilde{\beta}}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]^{\frac{1}{a}}$$

for all $t \geq t_1$ and $\tilde{\beta} \in (1+\beta-\varepsilon, 1+\beta)$. ■

Lemma 5.1. *Let $\beta \in (0, 1)$ and $M(0)$ be an arbitrary nonnegative constant, then*

$${}_0^C D_t^{\beta} M(t) \leq {}_0D_t^{\beta} M(t),$$

where D and ${}^C D$ are the Riemann–Liouville and the Caputo fractional operators, respectively.

Proof. By using (7), we have

$${}_0^C D_t^{\beta} M(t) = {}_0D_t^{\beta-1}\frac{\mathrm{d}}{\mathrm{d}t}M(t) = {}_0D_t^{\beta}M(t) - \frac{M(0)t^{-\beta}}{\Gamma(1-\beta)}.$$

Because $\beta \in (0, 1)$ and $M(0) \geq 0$,

$${}_0^C D_t^{\beta} M(t) \leq {}_0D_t^{\beta} M(t). \quad ■$$

Theorem 5.3. *Assume that the assumptions in Theorem 5.1 are satisfied except replacing ${}_0^C D_t^{\beta}$ by ${}_0D_t^{\beta}$, then we have*

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1}E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]^{\frac{1}{a}}.$$

Proof. It follows from Lemma 5.1 and $V(t, x) \geq 0$ that

$${}_0^C D_t^{\beta} V(t, x(t)) \leq {}_0D_t^{\beta} V(t, x(t)),$$

which implies

$${}_0^C D_t^{\beta} V(t, x(t)) \leq {}_0D_t^{\beta} V(t, x(t)) \leq -\alpha_3\|x\|^{ab}.$$

Following the same proof in Theorem 5.1 yields

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1}E_{\beta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta}\right)\right]^{\frac{1}{a}}. \quad ■$$

We pay special attention to the Mittag–Leffler stability for the following reasons. First, as shown in Remark 4.4, the Mittag–Leffler stability implies asymptotic stability. Second, the convergence speed of the corresponding system is an important character when evaluating the system. The Mittag–Leffler stability shows a faster convergence speed than the exponential stability near the origin, which can be illustrated by the following two derivatives.

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{-\lambda t}]\right\}_{t=0} = -\lambda \mathrm{e}^{-\lambda t}|_{t=0} = -\lambda,$$

$$\left\{\frac{\mathrm{d}}{\mathrm{d}t}[E_{\beta}(-\lambda t^{\beta})]\right\}_{t=0} = -\infty,$$

where $\beta \in (0, 1)$ and $\lambda > 0$. Because $\mathrm{e}^0 = E_{\beta}(0) = 1$, it follows from the comparison principle that $E_{\beta}(-\lambda t^{\beta})$ decreases much faster than $\mathrm{e}^{-\lambda t}$ near the origin.

At the end of the section, a generalized fractional Lyapunov direct method is proposed.

Theorem 5.4. *Let $x = 0$ be an equilibrium point for the system (13) and $\mathbb{D} \subset \mathbb{R}^n$ be a domain containing the origin. Let $V(t, x(t)) : [0, \infty) \times \mathbb{D} \to \mathbb{R}$ be a continuously differentiable function and locally Lipschitz with respect to x such that*

$$\alpha_1\|x\|^a \leq V(t, x(t)) \leq \alpha_2 {}_0^C D_t^{-\eta}\|x\|^{ab}, \tag{29}$$

$${}_0^C D_t^{\beta} V(t, x(t)) \leq -\alpha_3\|x\|^{ab}, \tag{30}$$

where $t \geq 0$, $x \in \mathbb{D}$, $\beta \in (0, 1)$, $\eta \neq \beta$, $\eta > 0$, $|\beta - \eta| < 1$, α_1, α_2, α_3, a and b are arbitrary positive constants. Then the equilibrium point $x = 0$ is asymptotic stable.

Proof. It follows from (29) and (30) that there exists a nonnegative function $M(t)$ satisfying

$$ {}_0^C D_t^\beta V(t, x(t)) + \frac{\alpha_3}{\alpha_2} {}_0^C D_t^\eta V(t, x(t)) + M(t) = 0. $$

Therefore

$$ V(s) = \frac{V(0)s^{\beta-1} + \frac{\alpha_3}{\alpha_2} V(0)s^{\eta-1} - M(s)}{s^\beta + \frac{\alpha_3}{\alpha_2} s^\eta}. $$

In time domain

$$ V(t) = \begin{cases} V(0)E_{\beta-\eta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta-\eta}\right) + \frac{\alpha_3}{\alpha_2}V(0)t^{\beta-\eta}E_{\beta-\eta,\beta-\eta+1}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta-\eta}\right) \\ \quad -M(t) * \left[t^{\beta-1}E_{\beta-\eta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)\right], \quad \beta > \eta, \\ V(0)E_{\eta-\beta}\left(-\frac{\alpha_2}{\alpha_3}t^{\eta-\beta}\right) + \frac{\alpha_2}{\alpha_3}V(0)t^{\eta-\beta}E_{\eta-\beta,\eta-\beta+1}\left(-\frac{\alpha_2}{\alpha_3}t^{\eta-\beta}\right) \\ \quad -\frac{\alpha_2}{\alpha_3}M(t) * \left[t^{\eta-1}E_{\eta-\beta,\eta}\left(-\frac{\alpha_2}{\alpha_3}t^{\eta-\beta}\right)\right], \quad \beta < \eta. \end{cases} $$

Therefore,

$$ \|x\|^a \le \alpha_1^{-1}V(t) \le \begin{cases} V(0)t^{-\eta}E_{\beta-\eta,1-\eta}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta-\eta}\right) + \frac{\alpha_3}{\alpha_2}V(0)t^{\beta-2\eta}E_{\beta-\eta,\beta-2\eta+1}\left(-\frac{\alpha_3}{\alpha_2}t^{\beta-\eta}\right), \quad \beta > \eta, \\ V(0)t^{-\eta}E_{\eta-\beta,1-\eta}\left(-\frac{\alpha_2}{\alpha_3}t^{\eta-\beta}\right) + \frac{\alpha_2}{\alpha_3}V(0)t^{-\beta}E_{\eta-\beta,1-\beta}\left(-\frac{\alpha_2}{\alpha_3}t^{\eta-\beta}\right), \quad \beta < \eta. \end{cases} $$

Lastly, it follows from a is an arbitrary positive constant that the equilibrium point $x = 0$ is asymptotic stable. ■

6. Fractional Lyapunov direct method by using the class-$\mathcal{K}$ functions

In this section, the class-$\mathcal{K}$ functions are applied to the analysis of fractional Lyapunov direct method.

Definition 6.1. A continuous function $\alpha : [0, t) \to [0, \infty)$ is said to belong to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$ [17].

Lemma 6.1 (*Fractional Comparison Principle*). *Let ${}_0^C D_t^\beta x(t) \ge {}_0^C D_t^\beta y(t)$ and $x(0) = y(0)$, where $\beta \in (0, 1)$. Then $x(t) \ge y(t)$.*

Proof. It follows from ${}_0^C D_t^\beta x(t) \ge {}_0^C D_t^\beta y(t)$ that there exists a nonnegative function $m(t)$ satisfying

$$ {}_0^C D_t^\beta x(t) = m(t) + {}_0^C D_t^\beta y(t). \tag{31} $$

Taking the Laplace transform of equation (31) yields

$$ s^\beta X(s) - s^{\beta-1}x(0) = M(s) + s^\beta Y(s) - s^{\beta-1}y(0). $$

It follows from $x(0) = y(0)$ that

$$ X(s) = s^{-\beta}M(s) + Y(s). \tag{32} $$

Applying the inverse Laplace transform to (32) gives

$$ x(t) = {}_0\mathcal{D}_t^{-\beta}m(t) + y(t). $$

It follows from $m(t) \ge 0$ and (3) that

$$ x(t) \ge y(t). \quad ■ $$

Theorem 6.2. *Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system (13). Assume that there exists a Lyapunov function $V(t, x(t))$ and class-$\mathcal{K}$ functions α_i $(i = 1, 2, 3)$ satisfying*

$$ \alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|) \tag{33} $$

and

$$ {}_0^C D_t^\beta V(t, x(t)) \le -\alpha_3(\|x\|) \tag{34} $$

where $\beta \in (0, 1)$. Then the system (13) is asymptotically stable.

Proof. It follows from (33) and (34) that

$${}_0^C D_t^\beta V \leq -\alpha_3(\alpha_2^{-1}(V)).$$

As shown in Lemma 6.1 that $V(t, x(t))$ is bounded by the unique nonnegative solution of the scalar differential equation

$${}_0^C D_t^\beta g(t) = -\alpha_3(\alpha_2^{-1}(g(t))), \quad g(0) = V(0, x(0)), \tag{35}$$

it follows from Definition 3.1 that $g(t) = 0$ for $t \geq 0$ if $g(0) = 0$, because $\alpha_3\alpha_2^{-1}$ is a class-$\mathcal{K}$ function.

Otherwise, $g(t) \geq 0$ on $t \in [0, +\infty)$, it then follows from (35) that ${}_0^C D_t^\beta g(t) \leq 0$.

Following the same proof in Lemma 6.1 gives

$$g(t) \leq g(0) \tag{36}$$

for $t \in (0, +\infty)$. Then the asymptotic stability of (35) is proved by contradiction.

Case 1: Suppose there exists a constant $t_1 \geq 0$ satisfying

$${}_0^C D_{t_1}^\beta g(t) = -\alpha_3(\alpha_2^{-1}(g(t_1))) = 0,$$

which implies that

$${}_0^C D_t^\beta g(t) = {}_{t_1}^C D_t^\beta g(t) = -\alpha_3(\alpha_2^{-1}(g(t)))$$

for any $t \geq t_1$. From Definition 3.1, $x = 0$ is the equilibrium point of ${}_{t_1}^C D_t^\beta g(t) = -\alpha_3(\alpha_2^{-1}(g(t)))$. Then $g(t) = 0$ for $t \geq t_1$ if $g(t_1) = 0$.

Case 2: Assume that there exists a positive constant ε such that $g(t) \geq \varepsilon$ for $t \geq 0$. Then it follows from (36) that

$$0 < \varepsilon \leq g(t) \leq g(0), \quad t \geq 0. \tag{37}$$

Substituting (37) to (35) gives

$$\begin{aligned} -\alpha_3(\alpha_2^{-1}(g(t))) &\leq -\alpha_3(\alpha_2^{-1}(\varepsilon)) \\ &= -\frac{\alpha_3(\alpha_2^{-1}(\varepsilon))}{g(0)} g(0) \leq -lg(t), \end{aligned}$$

where $0 < l = \frac{\alpha_3(\alpha_2^{-1}(\varepsilon))}{g(0)}$. It then follows that

$${}_0^C D_t^\beta g(t) = -\alpha_3(\alpha_2^{-1}(g(t))) \leq -lg(t).$$

Following the same proof in Theorem 5.1 gives

$$g(t) \leq g(0) E_\beta(-lt^\beta),$$

which contradicts the assumption that $g(t) \geq \varepsilon$.

Based on the discussions in both **Case 1** and **Case 2**, we have $g(t)$ tends to zero as $t \to \infty$. Because $V(t, x(t))$ is bounded by $g(t)$, it follows from (33) that $\lim_{t\to\infty} x(t) = 0$. ■

Theorem 6.3. *If the assumptions in Theorem 6.2 are satisfied except replacing ${}_0^C D_t^\beta$ by ${}_0 D_t^\beta$, then we have $\lim_{t\to\infty} x(t) = 0$.*

Proof. It follows from Lemma 5.1 and $V(t, x) \geq 0$ that

$${}_0^C D_t^\beta V(t, x(t)) \leq {}_0 D_t^\beta V(t, x(t)),$$

which implies

$${}_0^C D_t^\beta V(t, x(t)) \leq {}_0 D_t^\beta V(t, x(t)) \leq -\alpha_3(\|x\|).$$

Following the same proof in Theorem 6.2 gives $\lim_{t\to\infty} x(t) = 0$. ■

Remark 6.4. For the corresponding integer-order cases, please see [17].

7. Four illustrative examples

The following illustrative examples are used as proofs of concept.

Example 7.1. For the fractional-order system

$$ {}_0D_t^{\alpha}|x(t)| = -|x(t)|, \tag{38}$$

where $\alpha \in (0, 1)$. Consider the Lipschitz function candidate $V(t, x) = |x|$, it follows from Lemma 5.1 that

$$ {}_0^CD_t^{\alpha}V = {}_0^CD_t^{\alpha}|x| \le {}_0D_t^{\alpha}|x| = {}_0D_t^{\alpha}V \le -|x|.$$

Let $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = -1$, applying them in Theorem 5.1 gives

$$|x(t)| \le |x(0)|E_{\alpha}(-t^{\alpha}).$$

Remark 7.2. If we use the Laplace transform directly on ${}_0D_t^{\alpha}|x(t)| = -|x(t)|$, it follows from (2.248) in [11] that

$$ s^{\alpha}\mathcal{L}\{|x(t)|\} - \left[{}_0\mathcal{D}_t^{\alpha-1}|x(t)|\right]_{t=0} = -\mathcal{L}\{|x(t)|\}. \tag{39}$$

Applying the inverse Laplace transform to (39) gives

$$|x(t)| = \left[{}_0\mathcal{D}_t^{\alpha-1}|x(t)|\right]_{t=0} E_{\alpha}(-t^{\alpha}),$$

where $\left[{}_0\mathcal{D}_t^{\alpha-1}|x(t)|\right]_{t=0} = 0$ for any finite $x(0)$, which implies that the stabilization of system ${}_0D_t^{\alpha}|x(t)| = -|x(t)|$ cannot be derived directly from solving (38). However, in Example 7.1, we can not only prove the asymptotic but also the Mittag–Leffler stability of Riemann–Liouville system (38).

Example 7.3. For the fractional-order system

$$ {}_0^CD_t^{\alpha}x(t) = f(t, x), \tag{40}$$

where $\alpha \in (0, 1)$, $x = 0$ is the equilibrium point of system (40), $x(0) = x_0$ and $f(t, x)$ satisfies Lipschitz condition with Lipschitz constant $l > 0$. Suppose that there exists a Lyapunov candidate $V(t, x)$ satisfying

$$\alpha_1\|x\| \le V(t, x) \le \alpha_2\|x\|, \tag{41}$$

$$\dot{V}(t, x) \le -\alpha_3\|x\|, \tag{42}$$

where $\alpha_1, \alpha_2, \alpha_3$ are positive constants and $\dot{V}(t, x) = \frac{dV(t,x)}{dt}$. We have

$$\|x(t)\| \le \frac{V(0, x(0))}{\alpha_1}E_{1-\alpha}\left(-\frac{\alpha_3}{\alpha_2 l}t^{1-\alpha}\right).$$

Proof. It follows from (40)–(42), Property 3 and Lemma 3.1 that

$$\begin{aligned} {}_0^CD_t^{1-\alpha}V(t, x) &= {}_0D_t^{-\alpha}\dot{V}(t, x) \le -\alpha_3 {}_0D_t^{-\alpha}\|x\| \\ &= -\alpha_3 l^{-1} {}_0D_t^{-\alpha}\|f(t, x)\| \\ &\le -\alpha_3 l^{-1}\|{}_0D_t^{-\alpha}f(t, x)\| \\ &= -\alpha_3 l^{-1}\|x\|, \end{aligned}$$

where $\left[{}_0^CD_t^{\alpha-1}x(t)\right]_{t=0} = 0$. Therefore, the conclusion can be obtained by using Theorem 5.1. ■

Example 7.4. For the system

$$ {}_0^CD_t^{\alpha}x(t) = f(x), \tag{43}$$

where $\alpha \in (0, 1]$, $x(0) = x_0$, $x = 0$ is the equilibrium point of system (43), $\|x\|_2 \le \tilde{l}\|f(x)\|_2$ ($\tilde{l} > 0$ and $\|\cdot\|_2$ denotes the 2-norm.), and $f(x)\frac{df(x)}{dx}\dot{x} \le 0$. We have the equilibrium point $x = 0$ is stable.

Proof. Let the Lyapunov candidate be $V(x) = f^2(x)$, it follows from $f(x)\frac{df(x)}{dx}\dot{x} \le 0$ that

$$\frac{dV}{dt} = \frac{dV}{dx}\dot{x}(t) = 2f(x)\frac{df(x)}{dx}\dot{x} \le 0. \tag{44}$$

It then follows from $\|x\|_2 \le \tilde{l}\|f(x)\|_2$ and $x = 0$ is the equilibrium point that $\|x\|_2^2 \le \tilde{l}^2\|f(x)\|_2^2 \le \tilde{l}^2V(x_0)$. Therefore, the equilibrium point $x = 0$ is stable. ■

Example 7.5. For the fractional-order system

$$ {}_0^C D_t^\alpha x(t) = f(t, x), \tag{45} $$

where $\alpha \in (0, 1]$, $x(0) = x_0 \geq 0$, $x = 0$ is the equilibrium point of system (45) and $f(t, x) < 0$. We have the equilibrium point $x = 0$ is stable.

Proof. It follows from the fractional comparison principle and ${}_0^C D_t^\alpha x_0 = 0 > f(t, x) = {}_0^C D_t^\alpha x$ that $x \leq x_0$. It then follows from $x = 0$ is the equilibrium point of system (45) that $0 \leq x \leq x_0$. Therefore, the equilibrium point $x = 0$ is stable.

8. Conclusion and future works

In this paper, we studied the stabilization of nonlinear fractional-order dynamic systems. We discussed fractional nonautonomous systems and the application of the Lipschitz condition to fractional-order systems. We proposed the definition of (generalized) Mittag–Leffler stability and the (generalized) fractional Lyapunov direct method, which enriches the knowledge of both the system theory and the fractional calculus. We introduced the fractional comparison principle. We partly extended the application of Riemann–Liouville fractional-order systems by using fractional comparison principle and Caputo fractional-order systems. Four illustrative examples were provided to demonstrate the applicability of the proposed approach.

Our future works include the Mittag–Leffler stability of multi-variables fractional-order systems and the searching to Lyapunov functions of fractional-order systems.

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