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DESCRIPTOR SYSTEMS: FUNDAMENTAL MATRIX, REACHABILITY AND OBSERVABILITY MATRICES, SUBSPACES*

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Abstract

This paper uses the fundamental matrix of a regular discrete descriptor system to derive expressions for descriptor reachability and observability matrices. Reachable and unobservable subspaces and a subspace of admissible boundary conditions are defined. It is shown that the natural space for analyzing descriptor system properties seems to be R^{2n} (where n is the dimension of the system), not R^n as is the case for state-space systems. Solutions are provided for the descriptor open-loop control and estimation problems.

I. INTRODUCTION

There have been many approaches to analysis of descriptor or generalized state systems. These may basically be divided into time domain methods [1-6,15,22] and polynomial matrix/matrix pencil methods [7-9,19,20,23,25]. Various approaches have used the Drazin inverse [3,9,15,22], generalized eigenstructure [13,25,27-30], Weierstrass form [4,6,7,17,18, 24,25,27,31], numerical methods [6], and the concept of the output-zeroing problem [32]. The above approaches are not mutually exclusive and indeed their interdependencies are what make the study of descriptor systems so fascinating. These systems provide a focus which highlights some new relations between many different techniques.

In this paper we use a time-domain point of view and the descriptor fundamental matrix to define reachability and observability matrices. Our approach allows us to define reachable and unobservable subspaces in terms of the descriptor fundamental matrix. We are also able to solve the descriptor open-loop control and estimation problems.

A distinguishing feature of our work is that, in consonance with the noncausal or "nonoriented" nature of descriptor systems, we work with

$$\begin{vmatrix} x_0 \\ x_N \end{vmatrix} \varepsilon R^{2n}, \tag{1.1}$$

where 0, N represent initial and final times. This allows us to help clarify the duality between reachability and observability in the time domain (see also [17] which treats continuous systems). All of our results depend on the fundamental matrix; so no transformation to special form is required.

The focus here is on discrete systems, though the descriptor Cayley-Hamilton theorem [11] should allow a generalization to continuous systems.

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At each step it is shown how the familiar state space results are recovered as a special case of the results presented herein.

II. FUNDAMENTAL MATRIX AND CAYLEY-HAMILTON THEOREM

Consider the linear time invariant system over the real numbers R

$$Ex_{k+1} = Fx_k + Gu_k \tag{2.1a}$$

$$y_k = Hx_k \ ; \quad k = 0,\ 1,\ \ldots,\ N-1 \tag{2.1b}$$

where $u_k \varepsilon R^m$, $x_k \varepsilon R^n$, $y_k \varepsilon R^p$, and N specifies the time interval of interest. If E is singular then x_k should not be considered the state of (2.1), and following Luenberger [1] we call x_k the descriptor variable. Note that if E is nonsingular it is possible to solve for x_k by forward iteration given x_o. On the other hand, if F is nonsingular it is possible to solve for x_k by backward iteration given x_N. In general, however, it is not possible to solve (2.1) by simple iteration in one direction [2-6].

We assume throughout that (2.1) is regular, i.e. $|zE-F| \neq 0$ [1,2,7,8]. In this case we can write the unique Laurent expansion for the resolvent matrix for large values of z as

$$(zE-F)^{-1} = z^{-1} \sum_{k=-\mu}^{\infty} \phi_k z^{-k} , \tag{2.2}$$

where μ is the index of nilpotency of the pencil (zE-F) [9-11]. We call $\{\phi_k\}$ the <u>fundamental matrix</u> for (2.1). Note that ϕ_k satisfies the well-known equalities

$$E\phi_k - F\phi_{k-1} = \delta_{ok} I , \tag{2.3a}$$

$$\phi_k E - \phi_{k-1} F = \delta_{ok} I , \tag{2.3b}$$

where δ_{ok} is the Kronecker delta [10,12].

The importance of the descriptor fundamental matrix has been discussed, but is not generally realized. In [12] it is shown how to compute $\{\phi_k\}$ given ϕ_0 and ϕ_{-1}, and in [9] it is shown how to compute $\{\phi_k\}$ from E and F by using the Drazin inverse. It is known [10,12] that $\phi_0 E$ is the projection on H_I along H_N, and that $-\phi_{-1}F$ is the projection on H_N along H_I, where the finite and infinite eigenspaces H_I and H_N are defined in [12-14]. H_I may also be interpreted as the subspace of admissible initial conditions with zero input [1,2,13].

In this paper we focus on the properties and uses of the descriptor fundamental matrix. The first result is the following.

Theorem 2.1

Let

$$\Delta(z) = |zE-F| = p_0 z^n - p_1 z^{n-1} - \dots - p_n \quad (2.4)$$

Then

$$p_0\phi_k - p_1\phi_{k-1} - \dots - p_n\phi_{k-n} = 0 \quad (2.5)$$

for $k > n$ and $k < -1$.

proof: see [11]. ●

If $E=I$, then $\mu = 0$ and $\phi_k = F^k$; so that (2.5) with $k=n$ becomes simply $\Delta(F) = 0$. We therefore call Theorem 2.1 the descriptor Cayley-Hamilton theorem.

We shall require the following notation. Write (2.1) in expanded form (c.f. [1,2]) as

$$\begin{vmatrix} E & -F & 0 & \cdots & 0 & 0 \\ 0 & E & -F & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ 0 & & & \cdots & -F & 0 \\ 0 & \cdots & & \cdots & E & -F \end{vmatrix} \begin{vmatrix} x_N \\ \vdots \\ x_1 \\ x_o \end{vmatrix} = \begin{vmatrix} G & & 0 \\ & \ddots & \\ 0 & & G \end{vmatrix} \begin{vmatrix} u_{N-1} \\ \vdots \\ u_1 \\ u_o \end{vmatrix} \quad (2.6a)$$

$$\begin{vmatrix} y_{N-1} \\ \vdots \\ y_1 \\ y_o \end{vmatrix} = \begin{vmatrix} H & & 0 \\ & \ddots & \\ 0 & & H \end{vmatrix} \begin{vmatrix} x_{N-1} \\ \vdots \\ x_1 \\ x_o \end{vmatrix}, \quad (2.6b)$$

or by appropriate definition of the coefficient matrices A_N, B_N, C_N and the input, descriptor, and output sequence vectors $\overline{u}_{o,N}$, $\overline{x}_{o,N}$, $\overline{y}_{o,N}$ as

$$A_N\,\overline{x}_{o,N+1} = B_N\,\overline{u}_{o,N} \quad (2.7a)$$

$$\overline{y}_{o,N} = C_N\,\overline{x}_{o,N}\,. \quad (2.7b)$$

Define also the auxiliary matrix

$$a_N = \begin{vmatrix} -F & 0 & \cdots & 0 & 0 \\ E & -F & & \vdots & \vdots \\ 0 & E & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & & \\ 0 & 0 & \cdots & E & -F \\ 0 & 0 & \cdots & 0 & E \end{vmatrix}_{nN \times n(N-1)} \quad (2.8)$$

which is just A_N with the first and last block columns deleted.

The next results will subsequently be required.

Lemma 2.2

Suppose (2.1) is regular. Then a right inverse of A_N for any $N > 0$ is given by

$$A_N^r = \begin{vmatrix} \phi_o & \phi_1 & \phi_2 & \cdots & \phi_{N-1} \\ \phi_{-1} & \phi_o & \phi_1 & \cdots & \phi_{N-2} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \phi_o \\ \phi_{-N} & \cdots & \cdots & \cdots & \phi_{-1} \end{vmatrix}. \quad (2.9)$$

proof: Show $A_N A_N^r = I$ by using (2.3a). ●

Lemma 2.3

Suppose (2.1) is regular. Then a left inverse of a_N for any $N > 0$ is given by

$$a_N^\ell = \begin{vmatrix} \phi_{-1} & \phi_0 & \phi_1 & \cdots & \phi_{N-2} \\ \phi_{-2} & \phi_{-1} & \phi_0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \\ \phi_{-N+1} & \cdots & & \phi_{-1} & \phi_0 \end{vmatrix}. \quad (2.10)$$

proof: Show $a_N^\ell a_N = I$ by using (2.3b). ●

It is well known [1,2] that regularity of (2.1) is equivalent to the full rank of A_N and a_N, and hence of A_N^r and a_N^ℓ, for all $N > 1$.

III. REACHABILITY

Given $z_1, z_2 \in R^n$, define the pair (z_1,z_2) to be reachable if there exists a control $\overline{u}_{0,N}$ for some $N > 0$ such that $\overline{x}_{0,N+1}$ is a solution to (2.1a) with $x_0 = z_1$, $x_N = z_2$. We will loosely speak of (x_0,x_N) as reachable. Define system (2.1) to be reachable if all pairs (x_0,x_N) are reachable. This is consistent with definitions in [4,10,15].

It is well known [4] that (2.1) is reachable if and only if

$$\text{rank}[zE-F \quad G] = n \text{ all } z\,, \quad (3.1a)$$

and

$$\text{rank}[E \quad G] = n\,. \quad (3.1b)$$

The next result presents one possible condition for reachability in terms of the fundamental matrix.

Theorem 3.1

If (2.1) is regular, it is reachable if and only if

$$\text{rank}[\phi_{-\mu}G \dots \phi_{-1}G \quad \phi_0 G \dots \phi_{n-1}G] = n\,. \quad (3.2)$$

proof:

Reachability is equivalent [10] to the condition

$$q^T(zE-F)^{-1}G = 0 \text{ for } q \in R^n, \text{ all } z \text{ implies } q=0\,. \quad (3.3)$$

By using (2.2) and (2.5), the theorem follows readily. ●

If $E=I$, then condition (3.2) becomes

$$\text{rank } [G \quad FG \dots F^{n-1}G] = n\,.$$

Theorem (3.1) is not useful for our objectives, which include the computation of the open-loop control required to make the solution to (2.1) have desired values of x_0 and x_N. Accordingly the next result is presented.

Theorem 3.2

Let (2.1) be regular and define descriptor reachability matrix

$$U_N = \begin{vmatrix} \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{vmatrix}. \quad (3.4)$$

Then over any interval $[0,N]$, the control sequence and the initial and final values of the descriptor variable are related by

$$\begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} = U_N \bar{u}_{0,N} . \qquad (3.5)$$

Furthermore, reachability for (2.1) can be studied in terms of (3.5).

proof:

Rewrite (2.6a)/(2.7a)

$$\begin{vmatrix} -Ex_N \\ 0 \\ \vdots \\ 0 \\ Fx_o \end{vmatrix} = [a_N \; B_N] \begin{vmatrix} \bar{x}_{1,N} \\ -\bar{u}_{o,N} \end{vmatrix} . \qquad (3.6)$$

Premultiply both sides of (3.6) by A_N^r as given by (2.9) to obtain

$$\begin{vmatrix} -\phi_0 E & \phi_{N-1}F \\ \hline -\phi_{-1}E & \phi_{N-2}F \\ \vdots & \\ -\phi_{-N+1}E & \phi_0 F \\ \hline -\phi_{-N}E & \phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & \phi_0 G & \phi_1 G & \cdots & \phi_{N-1}G \\ \hline & \phi_{-1}G & \phi_0 G & \cdots & \phi_{N-2}G \\ I & \vdots & & & \vdots \\ & \phi_{-N+1}G & & & \phi_0 G \\ \hline 0 & \phi_{-N}G & \cdots & \phi_{-2}G & \phi_{-1}G \end{vmatrix} \begin{vmatrix} \bar{x}_{1,N} \\ -\bar{u}_{0,N} \end{vmatrix} . \qquad (3.7)$$

Since A_N^r has full column rank (3.6) has a solution if and only if (3.7) does. The form of (3.7) guarantees that we can solve for $\bar{x}_{1,N}$ given any x_0, x_N, $\bar{u}_{0,N}$ which are otherwise consistent with (3.7). Therefore (3.6) is equivalent to (3.5) •

In (3.4), $\phi_k = 0$ for $k < -\mu$. For ease of notation this is not xplicitly indicated.

From this theorem there follow several results whose proofs are quite trivial. R(•) denotes range of linear operator, and superscripts "-1" and "+" denote inverse image of a linear operator and Moore-Penrose Matrix inverse respectively. The first result provides another reachability test.

Corollary 3.3

A regular system (2.1) is reachable if and only if, for some N > 0,

$$R\left(\begin{vmatrix} \phi_0 E & \phi_{N-1}F \\ \phi_{-N}E & \phi_{-1}F \end{vmatrix}\right) \subset R(U_N) . \qquad (3.8)\bullet$$

Next, we characterize the reachable subspace of (2.1).

Corollary 3.4

Let (2.1) be regular. Then (x_0, x_N) is reachable if and only if

$$\begin{vmatrix} x_N \\ x_o \end{vmatrix} \varepsilon \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix}^{-1} R(U_N) . \qquad (3.9)\bullet$$

Equation (3.9) should be compared to the equivalent characterization by deflating subspaces in [25] and by Weierstrass form in [4].

If E=I then with N=n (3.5) becomes

$$\begin{vmatrix} I & -F^n \\ 0 & 0 \end{vmatrix} \begin{vmatrix} x_n \\ x_o \end{vmatrix} \equiv \begin{vmatrix} G & FG & \cdots & F^{n-1}G \\ 0 & 0 & \cdots & 0 \end{vmatrix} \bar{u}_{0,n} \quad (3.10)$$

which is equivalent to $(x_n - F^n x_o) \; \varepsilon \; R[G \;\; FG \; \ldots \; F^{n-1}G]$, the familiar state system result. Note that the usual definitions of reachability and controllability for discrete state systems both derive from our single definition for reachability of the pair (x_o, x_n). Hence, in the usual terminology, a state system is reachable if $x_n \; \varepsilon \; R[G \;\; FG \; \ldots \; F^{n-1}G]$ for all $x_n \; \varepsilon \; R^n$, and controllable if $F^n x_o \; \varepsilon \; R[G \;\; FG \; \ldots \; F^{n-1}G]$ for all $x_o \; \varepsilon \; R^n$.

The open-loop control problem for descriptor systems is solved next.

Corollary 3.5

The minimum-norm control which makes the descriptor variable of the regular system (2.1) take on prescribed values x_0 and x_N exists if and only if (3.9) holds, and then it is given by

$$\bar{u}_{0,N} = U_N^+ \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix} \begin{vmatrix} x_N \\ x_0 \end{vmatrix} . \qquad (3.11)\bullet$$

This reduces to the well-known state space result in the case E=I.

It is worthwhile to compare the notions of reachability and solvability [1,2,16]. (2.1) is solvable if a solution $\bar{x}_{0,N+1}$ exists for all $\bar{u}_{0,N}$. In general this occurs if and only if $R(B_N) \subset R(A_N)$, or equivalently rank [zE-F] = rank[zE-F G] for almost every z [16]. Compare this condition to (3.1). Regularity implies solvability. (In [1,2], solvability and regularity are defined to be equivalent, though in fact they should be thought of as distinct properties. See [16].) Solvability is defined in terms of (2.7a) while reachability is defined in terms of (3.7), or equivalently (3.5).

IV. OBSERVABILITY

Observability for descriptor systems seems to be a more difficult problem conceptually than reachability, as is attested to by the dearth of references on the subject. We deal with a form of observability which gives dual results in the time domain to those of section III, and which reduces if E=I to the state space results. Our definition is similar to that in [17,20] and should be contrasted to the definition in [4].

Given z_1, $z_2 \; \varepsilon \; R^n\_$, define the pair (z_1, z_2) to be observable if for $\bar{u}_{0,N+1} = 0$ and some N > 0, knowledge of the output $\bar{y}_{1,N+1}$ resulting when $x_0 = z_1$ and $x_{N+1} = z_2$ is sufficient to uniquely determine Fx_0, Ex_{N+1}. We shall loosely speak of (x_0, x_{N+1}) as observable. Define system (2.1) to be observable if all pairs (x_0, x_{N+1}) are observable. It will become clear that the choice of final time (i.e. N+1 not N) will result in a duality with the results of section III. (Note: the pair (x_0, x_{N+1}) must be an admissible set of boundary values for (2.1).)

Theorem 4.1

Let (2.1) be regular and define descriptor observability matrix

$$V_N = \begin{vmatrix} H\phi_{-1} & H\phi_{N-1} \\ H\phi_{-2} & \vdots \\ \vdots & H\phi_1 \\ H\phi_{-N} & H\phi_0 \end{vmatrix} . \qquad (4.1)$$

Then over any interval $[0, N+1]$, the output and the initial and final values of the descriptor variable are related by

$$\begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \\ \hline \multicolumn{2}{c}{V_N} \end{vmatrix} \begin{vmatrix} w_N \\ w_0 \end{vmatrix} = \begin{vmatrix} Ex_{N+1} \\ Fx_0 \\ \hline \bar{y}_{1,N+1} \end{vmatrix} \qquad (4.2a)$$

$$\begin{vmatrix} H & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} x_{N+1} \\ x_0 \end{vmatrix} = \begin{vmatrix} y_{N+1} \\ y_0 \end{vmatrix} , \qquad (4.2b)$$

where w_0, w_N are intermediate variables. Furthermore, the observability of (2.1) can be studied in terms of (4.2).

proof: Let $\bar{u}_{0,N+1} = 0$ and write (2.6)/(2.7) as

$$\begin{vmatrix} A_{N+1} \\ C_{N+2} \end{vmatrix} \bar{x}_{0,N+2} = \begin{vmatrix} 0 \\ \bar{y}_{0,N+2} \end{vmatrix} ,$$

which can be rewritten as

$$\begin{vmatrix} a_{N+1} \\ C_N \end{vmatrix} \bar{x}_{1,N+1} = \begin{vmatrix} -Ex_{N+1} \\ 0 \\ 0 \\ Fx_0 \\ \hline \bar{y}_{1,N+1} \end{vmatrix} \qquad (4.3a)$$

and

$$\begin{vmatrix} y_{N+1} \\ y_0 \end{vmatrix} = \begin{vmatrix} H & 0 \\ 0 & H \end{vmatrix} \begin{vmatrix} x_{N+1} \\ x_0 \end{vmatrix} . \qquad (4.3b)$$

Define intermediate variable $\bar{w}_{0,N+1}$ by $\bar{x}_{1,N+1} = a^{\ell}_{N+1}\bar{w}_{0,N+1}$ and write (4.3a) as

$$\begin{vmatrix} L \\ \hline M \end{vmatrix} \bar{w}_{0,N+1} \triangleq \left|\begin{array}{c|ccc|c} -F\phi_{-1} & -F\phi_0 & \cdots & -F\phi_{N-2} & -F\phi_{N-1} \\ \hline 0 & & I & & 0 \\ \hline E\phi_{-N} & E\phi_{-N+1} & \cdots & E\phi_{-1} & E\phi_0 \\ \hline\hline H\phi_{-1} & H\phi_0 & \cdots & E\phi_{N-2} & H\phi_{N-1} \\ H\phi_{-2} & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & H\phi_1 \\ H\phi_{-N} & H\phi_{-N+1} & \cdots & H\phi_{-1} & H\phi_0 \end{array}\right|$$

$$\bar{w}_{0,N+1} = \begin{vmatrix} -Ex_{N+1} \\ \hline 0 \\ \hline Fx_0 \\ \hline\hline \bar{y}_{1,N+1} \end{vmatrix} \qquad (4.4)$$

Now, given $\bar{y}_{1,N+1}$, (4.3a) has a unique solution with respect to Fx_0, Ex_{N+1} (i.e. possibly different $\bar{x}_{1,N+1}$ give rise to the same Fx_0, Ex_{N+1}) if and only if $N(C_N) \subset N(a_{N+1})$. Since a^{ℓ}_{N+1} has full row rank, this is equivalent to $N(M) \subset N(L)$. Thus (4.3a) and (4.4) are equivalent with respect to uniqueness of solution. Due to the structure of L, $w_{1,N} = 0$ and (4.4) is effectively equivalent to (4.2a). ●

At this point the next results are immediate.

Corollary 4.2

A regular system (2.1) is observable if and only if, for some $N > 0$,

$$N(V_N) \subset N \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} . \qquad (4.5)$$

proof:

For a given $\bar{y}_{1,N+1}$, the solution to (4.2a) is unique with respect to Fx_0, Ex_{N+1} if and only if (4.5) obtains. ●

The unobservable subspace is characterized next. Superscript "⊥" denotes orthogonal complement of a subspace.

Corollary 4.3

Let (2.1) be regular. Then an admissible pair (x_0, x_{N+1}) makes a zero contribution to the output iff

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} \in \left(\begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} N(V_N) \right) . \qquad (4.6)$$

proof: Any $\begin{vmatrix} w_N \\ w_0 \end{vmatrix} \in N(V_N)$ results in $\bar{y}_{1,N+1} = 0$.

Such "unobservable intermediate variables" contribute a component to the solution given by

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} = \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} \begin{vmatrix} w_N \\ w_0 \end{vmatrix} .$$

●

If E=I then (4.2a) becomes, with $N = n-1$,

$$\begin{vmatrix} 0 & F^{n-1} \\ 0 & I \\ \hline 0 & HF^{n-2} \\ \vdots & \vdots \\ 0 & HF \\ 0 & H \end{vmatrix} \begin{vmatrix} w_{n-1} \\ w_0 \end{vmatrix} = \begin{vmatrix} x_n \\ Fx_0 \\ \hline \bar{y}_{1,n} \end{vmatrix} ;$$

so that $w_0 = Fx_0$. Taking into account this and (4.2b), write the above as

$$\begin{vmatrix} F^n \\ I \\ \hline V^s_n \end{vmatrix} \triangleq \begin{vmatrix} F^n \\ I \\ \hline HF^{n-1} \\ \vdots \\ HF \\ H \end{vmatrix} x_0 = \begin{vmatrix} x_n \\ x_0 \\ \hline \bar{y}_{0,n} \end{vmatrix} , \qquad (4.7)$$

i.e. the familiar state space result. Note that the usual definitions of observability and reconstructibility for discrete state systems both derive from our single definition for observability of the pair (x_0,x_n). Hence, in the usual terminology, x_0 is observable if (4.7) has a unique solution with respect to x_0, i.e. $N(V_n^s) = 0$. On the other hand, x_n is reconstructible if (4.7) has a unique solution with respect to x_n, i.e. $N(V_n^s) \subset N(F^n)$.

Finally, the descriptor variable boundary value reconstruction problem is solved.

Corollary 4.4

Let (2.1) be regular and $\bar{u}_{0,N+1} = 0$. Then the values of Fx_0 and Ex_{N+1} can be uniquely determined from $\bar{y}_{1,N+1}$ if and only if (4.6) holds. In this case they are given by

$$\begin{vmatrix} Ex_{N+1} \\ Fx_0 \end{vmatrix} = \begin{vmatrix} F\phi_{-1} & F\phi_{N-1} \\ E\phi_{-N} & E\phi_0 \end{vmatrix} V_N^+ \bar{y}_{1,N+1} . \qquad (4.8)\bullet$$

If E=I this reduces to the known state-space result.

It appears in general to be impossible, or at least too complex notationally, to solve for x_0, x_{N+1} themselves. Work on this is in progress. It should also be possible to show the relation between (4.5) and the duals of conditions (3.1) and (3.3) [17].

It is worthwhile to compare the notions of observability and conditionability [1,2,16]. (2.1) is conditionable if there exists a unique $\bar{y}_{0,N+1}$ for all $\bar{x}_{0,N+1}$ satisfying $x_0 \in N(E)$, $x_N \in N(F)$ [16]. This occurs if and only if rank $[zE-F]$ = rank $\begin{vmatrix} zE-F \\ H \end{vmatrix}$ for almost every z, and is implied by regularity. Conditionability is defined in terms of uniqueness of solution $\bar{y}_{0,N+1}$ of (2.7), while observability is defined in terms of (4.2).

V. ADMISSIBLE BOUNDARY CONDITIONS

Define the pair (x_0,x_N) to be an admissible boundary condition if, given (x_0,x_N), there exists a solution $\bar{x}_{0,N+1}$ to (2.1) for some $\bar{u}_{0,N}$. This is a generalization of the notion of admissible initial condition x_0 discussed in [1-6,9,10,12-14,15,17-23]. Such a generalization seems appropriate because (2.1) is inherently a noncasual (i.e. non-oriented) system; so that a symmetric treatment of initial and final conditions is more natural. This approach was also successful in the above treatment of reachability and observability.

To characterize the set of admissible boundary conditions as a subspace of R^{2n}, write the general least-squares solution to (2.7a) as

$$\bar{x}_{0,N+1} = A_N^r B_N \bar{u}_{0,N} + (I - A_N^r A_N)\bar{w}_{0,N+1} \qquad (5.1)$$

for arbitrary $\bar{w}_{0,N+1}$. Since (2.1) is regular, (5.1) is an exact solution (i.e. $R(B_N) \subset R(A_N)$). Now we have

Theorem 5.1

Let (2.1) be regular. Then (x_0,x_N) is an admissible boundary condition if and only if

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \in R \begin{vmatrix} \phi_{-1}F & \phi_{N-1}F \\ \phi_{-N}E & \phi_0 E \end{vmatrix} + \begin{vmatrix} \phi_0 E & -\phi_{N-1}F \\ \phi_{-N}E & -\phi_{-1}F \end{vmatrix}^{-1} R(U_N) \qquad (5.2)$$

where superscript "-1" represents inverse image.

proof:

Note that in (5.1) the first term of RHS is the input-dependent portion and the second term of RHS is the zero-input portion of the solution. Let (2.1) be regular. Then the zero-input solution supplies n additional conditions which, together with $\bar{u}_{0,N}$, specify a unique solution $\bar{x}_{0,N}$. The zero-input solution can be written as

$$\begin{vmatrix} x_N \\ \vdots \\ x_1 \\ x_0 \end{vmatrix} = (I - A_N^r A_N)\bar{w}_{0,N+1} = \begin{vmatrix} I-\phi_0 E \\ -\phi_{-1}E \\ \vdots \\ -\phi_{-N}E \end{vmatrix} \; 0 \; \begin{vmatrix} \phi_{N-1}F \\ \vdots \\ \phi_0 F \\ I+\phi_{-1}F \end{vmatrix} \bar{w}_{0,N+1} \qquad (5.3)$$

since a_N^ℓ is a submatrix of A_N^r. Hence the zero-input solution satisfies

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \in R \begin{vmatrix} I- {}_0E & {}_{N-1}F \\ - {}_{-N}E & I+ {}_{-1}F \end{vmatrix}$$

$$= R \begin{vmatrix} {}_{-1}F & {}_{N-1}F \\ {}_{-N}E & {}_0E \end{vmatrix} . \qquad (5.4)$$

The input-dependent solution was found to satisfy (3.9), which represents pairs (x_0,x_N) which are "admissible for some $u_{0,N}$".

The characterization (5.4) of the "zero-input admissible boundary conditions" is consistent with [5,10,12,13], where it is shown that ${}_0E$ is the projection onto the subspace of zero-input admissible initial conditions. It is shown in [5,12] that $- {}_{-1}F$ is the projection onto the "subspace of admissible final conditions," which is again consistent with (5.4).

If E=I then (5.2) becomes

$$\begin{vmatrix} x_N \\ x_0 \end{vmatrix} \quad R \begin{vmatrix} F^N \\ I \end{vmatrix} + [I \quad -F^N]^{-1} R[G \quad FG \quad \ldots \quad F^{N-1}G] . \qquad (5.5)$$

Thus for state systems with zero input any x_0 is admissible while any x_N $R(F^N)$ is admissible. The second term of this equation was discussed in connection with (3.10).

Equation (5.2) should be compared to the characterizations of the subspace of admissible x_0 by Drazin inverse in [3,9,15,22], by Weierstrass form in [4,6,17,18], by deflating subspaces in [5], by recursion relations in [12,13,23], by matrix transformations in [1,2,21], and by eigenstructure in [14].

VI. CONCLUSION

Our results show that the fundamental matrix is more important than previously realized in the study of discrete descriptor systems, and that the properties of these systems should be studied in terms of (1.1) where x_0 and x_N are the boundary values of the descriptor variable. Thus, reachable, observable, and admissible boundary value subspaces should be thought of as residing in R^{2n}, not R^n.

The descriptor open-loop control and estimation problems were solved.

Work is in progress on extending these results to continuous systems using the descriptor Cayley-Hamilton Theorem [11]

REFERENCES

[1] D. G. Luenberger, "Dynamic equations in descriptor form," IEEE Trans. Automat. Contr., pp. 312-321, June 1977.

[2] D. G. Luenberger, "Time-invariant descriptor systems," Automatica, vol. 14, pp. 473-480, 1978.

[3] S. L. Campbell, C. D. Meyer, Jr., and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," SIAM J. Appl. Math., vol. 31, no. 3, pp. 411-425, Nov. 1976.

[4] E. L. Yip and R. F. Sincovec, "Solvability, Controllability, and Observability of Continuous Descriptor Systems," IEEE Trans. Automat. Contr., pp. 702-707, June 1981.

[5] F. L. Lewis, "Descriptor systems: decomposition into forward and backward subsystems," IEEE Trans. Automat. Contr., vol. AC-29, no. 1, Jan. 1984.

[6] R. F. Sincovec, A. M. Erisman, E. L. Yip, and M. A. Epton, "Analysis of descriptor systems using numerical algorithms," IEEE Trans. Automat. Contr., pp. 139-147, Feb. 1981.

[7] F. R. Gantmacher, The Theory of Matrices, New York: Chelsea, 1974.

[8] H. H. Rosenbrock, State-Space and Multivariable Theory, New York: Wiley, 1970.

[9] N. J. Rose, "The Laurent expansion of a generalized resolvent with some applications," SIAM J. Math. Analy., vol. 9, no. 4, pp. 751-758, Aug. 1978.

[10] C. E. Langenhop, "Controllability and stabilization of regular singular linear systems with constant coefficients," Dept. of Mathematics, S. Illinois Univ., Dec. 6, 1979.

[11] F. L. Lewis, "Adjoint matrix, Cayley-Hamilton Theorem, and Fadeev's method for the matrix pencil (sE-A)," Proc. 22nd IEEE Conf. Dec. and Contr., San Antonio, TX, pp. 1282-1288, 1983.

[12] C. E. Langenhop, "The Laurent expansion for a nearly singular matrix," Linear Algebra and its Applic., 4, pp. 329-340, 1971.

[13] K-T Wong, "The eigenvalue problem Tx + Sx," J. Diff. Equations, vol. 16, pp. 270-281, 1974.

[14] F. L. Lewis and K. Ozcaldiran, "The relative eigenstructure problem and descriptor systems," SIAM National Meeting, Denver, CO, June 1983.

[15] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," JOTA, vol. 30, no. 4, pp. 601-620, April 1980.

[16] F. L. Lewis, "Descriptor systems: expanded descriptor equation and Markov parameters," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, May 1983.

[17] D. Cobb, "Controllability, observability, and duality in singular systems," submitted.

[18] D. Cobb, "Feedback and pole placement in descriptor variable systems," Int. J. Control, vol. 33, no. 6, pp. 1135-1146, 1981.

[19] G. C. Verghese, B. C. Levy, and T. Kailath, "A generalized state-space for singular systems," IEEE Trans. Automat. Contr., vol. AC-26, no. 4, pp. 811-831, August 1981.

[20] G. C. Verghese and T. Kailath, "Impulsive behavior in dynamical systems: structure and significance," Proc. 4th Int. Symp. Math. Theory, Networks Syst., Delft, The Netherlands, pp. 162-168, July 1979.

[21] F. L. Lewis, "Inversion of descriptor systems," Proc. ACC, San Francisco, CA, pp. 1153-1158, June 1983.

[22] B. Dziurla and R. Newcomb, "The Drazin inverse and semi-state equations," Proc. 4th Int. Symp. Mathematical Theory of Networks and Systems, Delft, The Netherlands, pp. 283-289, July 1979.

[23] G. C. Verghese, "Further Notes on Singular Descriptions," Proc. JACC, paper TA-4, June 1981.

[24] J. H. Wilkinson, "Linear differential equations and Kronecker's Canonical Form," in Recent Advances in Numerical Analysis, C. de Boor and G. H. Golub, ed., New York: Academic, 1978.

[25] P. M. Van Dooren, "The generalized eigenstructure problem in linear system theory," IEEE Trans. Automat. Contr., vol. AC-26, no. 1, pp. 111-129, Feb. 1981.

[26] H. H. Rosenbrock, "Structural properties of linear dynamical systems," Int. J. Control, vol. 20, no. 2, pp. 191-202, 1974.

[27] P. M. Van Dooren, "The computation of Kronecker's canonical form of a singular pencil," Linear Algebra and its Applications, 27, pp. 103-140, 1979.

[28] G. Peters and J. H. Wilkinson, "Ax = Bx and the generalized eigenproblem," SIAM J. Numer. Anal., vol. 7, no. 4, pp. 479-492, Dec. 1970.

[29] P. Lancaster, "A fundamental theorem on Lambda-matrices with applications," Linear Algebra and its Applications, vol. 18, pp. 189-211, 1977.

[30] I. Gohberg and L. Rodman, "On spectral analysis of non-monic matrix and operator polynomials," Israel J. Mathematics, vol. 30, nos. 1-2, pp. 133-151, 1978.

[31] D. Cobb, "Descriptor variable systems and optimal state regulation," IEEE Trans. Automat. Contr., vol. AC-28, no. 5, pp. 601-611, May 1983.

[32] N. Karcanias and G. E. Hayton, "State-space and transfer function invariant infinite zeros: a unified approach," Proc. JACC, vol. 1, paper TA-4C, 1981.