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Stability analysis of arbitrarily high-index positive delay-descriptor systems

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Abstract This paper deals with the stability analysis of positive delay-descriptor systems with arbitrarily high index. First we discuss the solvability problem (i.e., about the existence and uniqueness of a solution), which is followed by the study on characterizations of the (internal) positivity. Finally, we discuss the stability analysis. Numerically verifiable conditions in terms of matrix inequality for the system's coefficients are proposed, and are examined in several examples.

Keywords Positivity · Delay · Descriptor systems · Strangeness-index .

Nomenclature

$\mathbb{N}$ ($\mathbb{N}_0$)	the set of natural numbers (including 0)
$\mathbb{R}$ ($\mathbb{C}$)	the set of real (complex) numbers
$\mathbb{C}_-$	the set $\{\lambda \in \mathbb{C} \mid Re\lambda < 0\}$
I (I_n)	the identity matrix (of size $n \times n$)
$x^{(j)}$	the j-th derivative of a function x
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of p-times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$ (for $0 \leqslant p \leqslant \infty$)
$\lVert \cdot \rVert_\infty$	the norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$
$\mathrm{im}_+ W$	the space $\{Ww_1 \text{ for all } w_1 \in \mathbb{R}^n_+\}$
$\mathcal{K}(U, W)$	the matrix $\mathcal{K}(U, W) := \begin{bmatrix} W, & UW, & \ldots, & U^{\nu-1}W \end{bmatrix}$.

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1 Introduction

{sec1}

Our focus in the present paper is on the positivity and stability analysis of linear, constant coefficients *delay-descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), \quad \text{for all } t \in [t_0, t_f), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

{delay-descriptor}

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, $C \in \mathbb{R}^{q,n}$, $x : [t_0 - \tau, t_f) \to \mathbb{R}^n$, $f : [t_0, t_f) \to \mathbb{R}^n$, and $\tau > 0$ is a constant delay. Together with (1), we are also concern with the associated *zero-input/free system*

$$E\dot{x}(t) = Ax(t) + A_d x(t-\tau), \quad \text{for all } t \in [t_0, t_f). \tag{2}$$

{free system}

Systems of the form (1) can be considered as a general combination of two important classes of dynamical systems, namely *differential-algebraic equations (descriptor systems)* (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

{eq1.2}

where the matrix E is allowed to be singular ($\det E = 0$), and *delay-differential equations* (DDEs)

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t). \tag{4}$$

{eq1.3}

Delay-descriptor systems of the form (1) have been arisen in various applications, see Ascher and Petzold [1995], Campbell [1980], Hale and Lunel [1993], Shampine and Gahinet [2006], Zhu and Petzold [1997] and the references there in. From the theoretical viewpoint, the study for such systems is much more complicated than that for standard DDEs or DAEs. The dynamics of DDAEs has been strongly enriched, and many interesting properties, which occur neither for DAEs nor for DDEs, have been observed for DDAEs Campbell [1995], Du et al. [2013], Ha and Mehrmann [2012, 2016]. Due to these reasons, recently more and more attention has been devoted to DDAEs, Campbell and Linh [2009], Fridman [2002], Ha and Mehrmann [2012, 2016], Michiels [2011], Shampine and Gahinet [2006], Tian et al. [2014], Linh and Thuan [2015].

[....]

The short outline of this work is as follows. Firstly, in Section 2, we briefly recall the solvability analysis to system (1), followed by a result about solution comparison for the free system (2) (Theorem 3). Based on the explicit solution representation in Section 2, we present a characterization for the positivity of system (1) in Section 3. Algebraic, numerically verifiable conditions in terms of the system matrix coefficients are established there. To follow, in Section 4 we discuss further about the free system (2) under biconditional requirements: stability and positivity. Finally, we conclude this research with some discussion and open questions.

2 Preliminaries

{sec2}

In this section we discuss the solvability analysis, including the solution representation and the comparison principal for the corresponding IVP to system (1), which consists of (1) together with an initial condition

$$x|_{[t_0-\tau,t_0]} = \varphi : [t_0 - \tau, t_0] \to \mathbb{R}^n. \tag{5}$$ {initial condition}

Here, φ is a prescribed initial trajectory (preshape function), which is necessary to achieve uniqueness of solutions. Without loss of generality, we assume that $t_0 = 0$ and $t_f = n_f \tau$, where $n_f \in \mathbb{N}$.

2.1 Existence, uniqueness and explicit solution formula

It is well-known (e.g. Du et al. [2013]) that we may consider different solution concepts for system (1). The reason is, that $E(0)\dot{x}(0^+)$ which arises from the right hand side in (1) at 0 may not be equal to $E(0)\dot{\varphi}(0^-)$. Moreover, it has been observed in Baker et al. [2002], Campbell [1980], Guglielmi and Hairer [2008] that a discontinuity of $\dot{x}$ at $t = 0$ may propagate with time, and typically $\dot{x}$ is discontinuous at every point $j\tau,\ j \in \mathbb{N}_0$ or it may not even exist. To deal with this property of DDAEs, we use the following solution concept.

{solution}

Definition 1 Let us consider a fixed input function $u(t)$.
i) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *piecewise differentiable solution* of (1), if Ex is piecewise continuously differentiable, x is continuous and satisfies (1) at every $t \in [t_0, t_f) \setminus \bigcup_{j \in \mathbb{N}_0} \{j\tau\}$.
ii) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *classical solution* of (1) if it is at least continuous and satisfies (1) at every $t \in [t_0, t_f)$.

Throughout this paper whenever we speak of a solution, we mean a piecewise differentiable solution. Notice that, like DAEs, DDAEs are not solvable for arbitrary initial conditions, but they have to obey certain consistency conditions.

{consistency}

Definition 2 An initial function φ is called *consistent* with (1) if the associated initial value problem (IVP) (1), (5) has at least one solution. System (1) is called *solvable* (resp. *regular*) if for every consistent initial function φ, the IVP (1), (5) has a solution (resp. has a unique solution).

Introducing sequences of matrix-valued and vector-valued functions f_j, u_j, x_j for each $j \in \mathbb{N}$, on the time interval $[0, \tau]$ via

$$f_j(t) = f(t + (j-1)\tau),\ u_j(t) = u(t + (j-1)\tau),$$
$$x_j(t) = x(t + (j-1)\tau),\ x_0(t) := \varphi(t - \tau),$$

we can rewrite the IVP (1)-(5) as a sequence of non-delayed descriptor systems

$$E\dot{x}_j(t) = Ax_j(t) + A_d x_{j-1}(t) + Bu_j(t), \tag{6}$$ {j-th DAE}

for all $t \in (0, \tau)$ and for all $j = 1, 2, ..., n_f$. We notice, that for each j, the initial condition $x_j(0)$ is given due to the continuity of the solution $x(t)$ at the point $(j-1)\tau$, i.e.,

$$x_j(0) = x_{j-1}(\tau) \ . \tag{7}$$

{continuity condition}

In particular, $x_1(0) = \phi(0)$ and the function x_0 is given.

It is well-known (see e.g. Bellman and Cooke [1963], Hale and Lunel [1993]) that in general, time-delayed systems has been classified into three different types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0 \dot{x}(t) + a_1 \dot{x}(t-\tau) + b_0 x(t) + b_1 x(t-\tau) = f(t)$$

is retarded if $a_0 \neq 0$ and $a_1 = 0$; is neutral if $a_0 \neq 0$, $a_1 \neq 0$; is advanced if $a_0 = 0$, $a_1 \neq 0$, $b_0 \neq 0$. Obviously, this classification is based on the smoothness comparison between $x(t)$ and $x(t-\tau)$. In literature, not only the theoretical but also numerical solution has been studied mainly for non-advanced systems (i.e., retarded or neutral), due to their apperance in various applications. For this reason, in Ha [2015], Ha and Mehrmann [2016], Unger [2018] the authors poposed a concept of *non-advancedness* for (1) (see Definition 3 below). We also notice, that even though not clearly proposed, due to the author's knowledge, so far results for delay-descriptor are only obtained for certain classes of non-advanced systems, e.g. Ascher and Petzold [1995], Shampine and Gahinet [2006], Zhu and Petzold [1997, 1998], Michiels [2011].

{def2}

Definition 3 A regular delay-descriptor system (1) is called *non-advanced* if for any consistent and continuous initial function φ, there exists a piecewise differentiable solution $x(t)$ to the IVP (1), (5).

{regularity}

Definition 4 Consider the DDAE (1). The matrix triple (E, A, B) is called *regular* if the (two variable) *characteristic polynomial* $\det(\lambda E - A - \omega B)$ is not identically zero. If, in addition, $B = 0$ we say that the matrix pair (E, A) (or the pencil $\lambda E - A$) is regular. The sets $\sigma(E, A, B) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A - e^{-\lambda\tau} B) = 0\}$ and $\rho(E, A, B) = \mathbb{C} \setminus \sigma(E, A, B)$ are called the *spectrum* and the *resolvent set* of (1), respectively.

Provided that the pair (E, A) is regular, we can transform them to the Kronecker-Weierstraßcanonical form (see e.g. Dai [1989], Kunkel and Mehrmann [2006]). That is, there exist regular matrices W, $T \in \mathbb{R}^{n,n}$ such that

$$(E, A) = \left(W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, \ W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right) \ , \tag{8}$$

{KW form}

where N is a nilpotent matrix of nilpotency index ν. We also say that the pair (E, A) has a *differentiation index* ν, i.e., $\operatorname{ind}(E, A) = \nu$.

Remark 1 Two concepts non-advancedness and differentiation index are independent. In details, a non-advanced system can have arbitrarily high index, as can be seen in the following example.

{example 1}

Example 1 Consider the following systems with the parameters ε_1, ε_2.

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E} \dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 & \varepsilon_1 \\ 0 & \varepsilon_2 \end{bmatrix}}_{A_d} x(t-\tau) \ . \tag{9}$$ {eq11}

It is well-known that in this example $\mathrm{ind}(E,A) = 2$. Furthermore, depending on the value of ε_2, the system will be advanced (if $\varepsilon_2 \neq 0$) and be non-advanced (if $\varepsilon_2 \neq 0$). Analogously, one can construct a non-advanced system which has an arbitrarily high index.

Let E have index $\tilde{\nu}$, i.e., $\mathrm{ind}(E, I_n) = \tilde{\nu}$, the Drazin inverse E^D of E is uniquely defined by the properties

$$E^D E = E E^D, \ E^D E E^D = E^D, \ E^D E^{\tilde{\nu}+1} = E^{\tilde{\nu}}. \tag{10}$$ {Drazin property}

{lem1}

Lemma 1 *Kunkel and Mehrmann [2006] Let (E,A) be a regular matrix pair. Then for any $\lambda \in \rho(E,A)$, two following matrices commute.*

$$\hat{E} := (\lambda E - A)^{-1} E, \ \hat{A} := (\lambda E - A)^{-1} A. \tag{11}$$ {eq20}

Furthermore, the following commutative identities hold true.

$$\hat{E}\hat{A}^D = \hat{A}^D \hat{E}, \ \hat{E}^D \hat{A} = \hat{A}\hat{E}^D, \ \hat{E}^D \hat{A}^D = \hat{A}^D \hat{E}^D \ . \tag{12}$$ {eq12}

We notice that the matrix products $\hat{E}^D\hat{E}$, $\hat{E}^D\hat{A}$, $\hat{E}\hat{A}^D$, $\hat{E}^D\hat{B}$, $\hat{A}^D\hat{B}$ do not depend on the choice of λ (see e.g. Dai [1989]). Furthermore, they can be numerically computed by transforming the pair (E,A) to their Weierstrass canonical form (8) (see e.g. Varga [2019], Virnik [2008]).

For any $\lambda \in \rho(E,A)$, we denote

$$\hat{A}_d := (\lambda E - A)^{-1} A_d, \ \hat{B} := (\lambda E - A)^{-1} B \ . \tag{13}$$ {eq21}

Making use of the Drazin inverse, in the following theorem we present the explicit solution representation of system (1).

{sol. rep. DAE}

Theorem 1 *Consider the delay-descriptor system* (1)*. Assume that* (E,A) *is a regular matrix pair with a differentiation index* $ind(E,A) = \nu$*. Let* $\hat{E}$, $\hat{A}$, $\hat{A}_d$, $\hat{B}$ *be defined as in* (11), (13)*. Furthermore, assume that* u *is sufficiently smooth. Then, every solution* x_j *of the DAE* (6) *has the form*

$$\begin{aligned} x_j(t) = {} & e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_j + \int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D \ \left(\hat{A}_d x_{j-1}(s) + \hat{B}u_j(s)\right) ds \\ & + (\hat{E}^D\hat{E} - I) \ \sum_{i=0}^{\nu-1} (\hat{E}^D\hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(t) + \hat{B}u_j^{(i)}(t)\right), \end{aligned} \tag{14}$$ {j-th solution}

for some vector $v_j \in \mathbb{R}^n$.

Proof. The proof is straightly followed from the explicit solution of DAEs, see [Kunkel and Mehrmann, 2006, Chap. 2]. □

Making use of (7), we directly obtain the following corollary.

Corollary 1 *The solution $x(t)$ of system* (1) *is continuous at the point* $(j-1)\tau$ *if and only if the following condition holds.*

$$(\hat{E}^D\hat{E}-I)\ x_{j-1}(\tau) = (\hat{E}^D\hat{E}-I)\ \sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\left(\hat{A}_d x_{j-1}^{(i)}(0)+\hat{B}u_j^{(i)}(0)\right)\ .$$

In particular, for the preshape function $\varphi(t)$, we must require

$$(\hat{E}^D\hat{E}-I)\ \left(\varphi(0)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\left(\hat{A}_d\varphi^{(i)}(-\tau)+\hat{B}u^{(i)}(0)\right)\right)=0\ .$$

Following from (14), we directly obtain a simpler form in case of non-advanced system as follows.

Corollary 2 *Consider system* (1) *and assume that it is regular and non-advanced. Then, we have*

$$\begin{aligned} x_j(t) &= e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_j+\int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\left(\hat{A}_d x_{j-1}(s)+\hat{B}u_j(s)\right)ds \\ &+(\hat{E}^D\hat{E}-I)\ \left(\hat{A}^D\hat{A}_d x_{j-1}(t)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u_j^{(i)}(t)\right), \end{aligned} \qquad (15)$$ {sol. formula non-advanced}

Furthermore, the consistency condition at $t=0$ reads

$$(\hat{E}^D\hat{E}-I)\left(\varphi(0)+\hat{A}^D\hat{A}_d\varphi(-\tau)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u^{(i)}(0)\right)=0\ . \qquad (16)$$ {consistency}

2.2 A simple check for the non-advancedness

Assume that the pair (E,A) is regular with index $\mathrm{ind}(E,A)=\nu$. We want to give a simple check whether the free system (2) is non-advanced or not. In analoguous to the case of DAEs Brenan et al. [1996], Kunkel and Mehrmann [2006], we aim to extract the so-called *underlying delay equation* of the form

$$\dot{x}(t)=\mathbf{A}x(t)+\mathbf{A}_{d0}x(t-h)+\mathbf{A}_{d1}\dot{x}(t-h), \qquad (17)$$ {underlying DDEs}

from an augmented system consisting of system (2) and its derivatives, which read in details

$$\frac{d^i}{dt^i}\left(E\dot{x}(t)-Ax(t)-A_d x(t-\tau)\right)=0,\ \text{for all } i=0,1,\ldots,\nu.$$

We rewrite these equations into the so-called *inflated system*

$$\underbrace{\begin{bmatrix} E & & & \\ -A & E & & \\ & \ddots & \ddots & \\ & & -A & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} A_d & & & \\ & A_d & & \\ & & \ddots & \\ & & & A_d \end{bmatrix}}_{\mathcal{A}_d} \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \\ \vdots \\ x^{(\nu)}(t-h) \end{bmatrix} . \tag{18}$$ {inflated}

Here the matrix coefficients are $\mathcal{E}, \mathcal{A}, \mathcal{A}_d \in \mathbb{R}^{(\nu+1)n,(\nu+1)n}$. For the reader's convenience, below we will use MATLAB notations. An underlying delay system (17) can be extracted from (18) if and only if there exists a matrix $P = \begin{bmatrix} P_0 \ P_1 \ \dots \ P_\nu \end{bmatrix}^T$ in $\mathbb{R}^{(\nu+1)n,n}$ such that

$$P^T \mathcal{E} = \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix} ,$$
$$P^T \mathcal{A}_d = \begin{bmatrix} * & * & 0_{n,(\nu-1)n} \end{bmatrix} ,$$

where $*$ stands for an arbitrary matrix. Consequently, P is the solution to the following linear systems

$$\begin{bmatrix} \mathcal{E} \ \mathcal{A}_d(:, 2n+1 : end) \end{bmatrix}^T P = \begin{bmatrix} I_n & 0_{n,\nu n} & 0_{n,(\nu-1)n} \end{bmatrix}^T .$$

Therefore, making use of Crammer's rule we directly obtain the simple check for the non-advancedness of system (2) in the following theorem.

{thm check advancedness}

Theorem 2 *Consider the zero-input descriptor system* (2) *and assume that the pair* (E, A) *is regular with index* $\mathrm{ind}(E, A) = \nu$*. Then, this system is non-advanced if and only if the following rank condition is satisfied*

$$\mathrm{rank} \begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_d(:, 2n+1 : end)^T \end{bmatrix} = \mathrm{rank} \left[\begin{array}{c|c} \mathcal{E}^T & I_n \\ \mathcal{A}_d(:, 2n+1 : end)^T & 0_{(2\nu-1)n,n} \end{array} \right] \tag{19}$$ {adv. check eq.}

Theorem 2 applied to the index two case straightly gives us the following corollary.

Corollary 3 *Consider the zero-input descriptor system* (2) *and assume that the pair* (E, A) *is regular with index* $\mathrm{ind}(E, A) = 2$*. Then, system* (2) *is non-advanced if and only if the following identity hold true.*

$$\mathrm{rank} \begin{bmatrix} E^T & -A^T & 0 \\ 0 & E^T & -A^T \\ 0 & 0 & A_d^T \end{bmatrix} = n + \mathrm{rank} \begin{bmatrix} E^T & -A^T \\ 0 & E^T \\ 0 & A_d^T \end{bmatrix} . \tag{20}$$ {check advanced}

Example 2 Let us reconsider system (9) in Example 1. Numerical verification of non-advancedness via condition (20) completely agrees with theoretical observation.

2.3 Comparison principal

{sec2b}

In this part of Section 2, we will show how to generalize our result to delay-descriptor systems with time-varying delay of the following form

$$E\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) + Bu(t), \quad \text{for all } t \in [t_0, t_f), \tag{21}$$ {ltv delay-descriptor}

where the delay function $\tau(t)$ is preassumed continuous and bounded, i.e. $0 < \underline{\tau} \leqslant \tau(t) \leqslant \bar{\tau}$ for all $t \geq 0$. Here $\underline{\tau}$, $\bar{\tau}$ are two positive constants. Following Ha and Mehrmann [2016], it can be shown that the solution to system (21) exists, unique and totally determined by any consistent initial function φ such that $x(t) = \varphi(t)$ for all $-\bar{\tau} \leqslant t \leqslant 0$. Indeed, also making use of the method of steps, the solution x is constructively built on consecutive interval $[t_{i-1}, t_i]$, $i \in \mathbb{N}$ such that $0 = t_0 < t_1 < t_2 < ....$ and

$$t_i - \tau(t_i) = t_{i-1} \; .$$

As shown in Theorems 3, 4 below, we can directly generalize our result to systems with bounded, time varying delay.

{solution comparison 1}

Theorem 3 *Consider system* (21) *and assume that the corresponding constant delay system* (1) *is positive and non-advanced. For a fixed input u, let $x(t)$ (resp. $\tilde{x}(t)$) be a state function corresponds to a preshape function $\varphi(t)$ (resp. $\tilde{\varphi}(t)$). Furthermore, assume that $\varphi(t) \leqslant \tilde{\varphi}(t)$ for all $t \in [-\bar{\tau}, 0]$. Then, we have $x(t) \leqslant \tilde{x}(t)$ for all $t \geq 0$.*

Proof. Based on the linearity of system (1), $\tilde{x}(t) - x(t)$ satisfies the free system (2). Furthermore, since this system is non-advanced and positive the non-negativity of $\tilde{\varphi}(t) - \varphi(t)$ implies that $\tilde{x}(t) - x(t) \geq 0$ for all t. □

{solution comparison 2}

Theorem 4 *Consider system* (21) *and assume that the corresponding constant delay system* (1) *is positive. Furthermore, assume that*

$$(\hat{E}^D \hat{E} - I) \; (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$$

for all $i = 0, \ldots, \nu - 1$. Let $x(t)$ (resp. $\tilde{x}(t)$) be a state function corresponds to a reference input $u(t)$ (resp. $\tilde{u}(t)$) and a preshape function $\varphi(t)$ (resp. $\tilde{\varphi}(t)$). Then we have $x(t) \leqslant \tilde{x}(t)$ for all $t \geq 0$, provided that the following conditions are fulfilled.
i) $\varphi(t) \leqslant \tilde{\varphi}(t)$ for all $t \in [-\tau, 0]$,
ii) $u^{(i)}(t) \leqslant \tilde{u}^{(i)}(t)$ for all $t \geq 0$ and for all $i \leqslant (\nu - 1) \; \lfloor t/\tau \rfloor$.

Proof. The proof is also straightforward from the solution's representation (14). □

From Theorems 3, 4 above, we see that the time varying delay will affect neither the positivity nor the stability of system (1).

3 Characterizations of positive delay-descriptor system

{sec3}

Since most systems occur in application are non-advanced, in this section we focus on the chracterization for positivity of non-advanced delay descriptor systems. We, furthermore, notice that the non-advancedness is a necessary condition for the stability (in the Lyapunov sense) of any time-delayed system, see e.g. Hale and Lunel [1993], Du et al. [2013].

Definition 5 Consider the delay-descriptor system (1) and assume that it is non-advanced, and that the pair (E, A) is regular with $\operatorname{ind}(E, A) = \nu$. We call (1) positive if for all $t \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for any input function u and any consistent initial function $\varphi(t)$ that satisfy two following conditions.
i) $\varphi(t) \geq 0$ for all $t \in [-\tau, 0]$,
ii) $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and all $i \leqslant (\nu - 1)\ \lfloor t/\tau \rfloor$.

For nontiaonal convenience, let us denote by

$$\begin{aligned} &P := \hat{E}^D \hat{E},\ \bar{\mathbf{A}} := \hat{E}^D \hat{A},\ \bar{\mathbf{A}}_d := \hat{E}^D \hat{A}_d,\ \bar{\mathbf{B}} := \hat{E}^D \hat{B}, \\ &\mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D \hat{B}) := \left[\hat{A}^D \hat{B}, \bar{\mathbf{A}}\ \hat{A}^D \hat{B}, \ldots, \bar{\mathbf{A}}^{\nu-1} \hat{A}^D \hat{B}\right]\ . \end{aligned} \tag{22}$$ {can. proj}

Since our systems is linear, time invariant coefficients, it would be sufficient to study the positivity on the first time interval $[0, \tau]$. Making use of (15), and let $j = 1$, we can split the solution $x_1 = x|_{[0,\tau]}$ as follows

$$\begin{aligned} x_1(t) = &\underbrace{e^{\bar{\mathbf{A}}t} P x_0(\tau) + (P - I)\hat{A}^D \hat{A}_d x_0(t) + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d x_0(s) ds}_{x_{zi}(t)} \\ &+ \underbrace{\int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{B}} u_j(s) ds + (P - I) \sum_{i=0}^{\nu-1} \bar{\mathbf{A}}^i \hat{A}^D \hat{B} u_j^{(i)}(t)}_{x_{zs}(t)}\ . \end{aligned} \tag{23}$$ {eq16}

In the theory of linear systems, $x_{zi}(t)$ (resp. $x_{zs}(t)$) is often called the *zero input/free* (resp. *zero state*) solution.

{Castelan'93}

Lemma 2 *Let $F \in \mathbb{R}^{p,n}$, $M \in \mathbb{R}^{n,n}$ and the system $\dot{z}(t) = Mz(t)$. Then, the implication $[Fz(0) \geq 0] \Rightarrow [Fz(t) \geq 0$ for all $t \geq 0]$ holds true if and only if $FM = HF$ for some Metzler matrix H.*

The characterization for the positivity of the free solution x_{zi} is given in Rami and Napp [2012] as follows.

Proposition 1 *Rami and Napp [2012] The following statements are equivalent.* {Rami12}
i) The non-delayed free system $E\dot{x}(t) = Ax(t)$ is positive.
ii) There exists a Metzler matrix H such that $\bar{\mathbf{A}} = HP$, where P is defined via (22)*.*
iii) There exists a matrix D such that $H := \bar{\mathbf{A}} + D\ (I - P)$ is Metzler.

{zero input lemma}

Lemma 3 *Consider the delay-descriptor system* (1) *and assume that it is non-advanced, and the pair* (E, A) *is regular with index* $\mathrm{ind}(E, A) = \nu$. *Then, the free system* (2) *has a non-negative solution* $x_{zi}(t) \geq 0$ *for all* $t \geq 0$ *and for all consistent initial function* $\varphi(t) \geq 0$ *if and only if the following conditions are satisfied.*
i) There exists a Metzler matrix H *such that* $\bar{\mathbf{A}} = HP$.
ii) $\bar{\mathbf{A}}_d \geq 0$, $(P - I)\hat{A}^D\hat{A}_d \geq 0$.

Proof. "⇒" For any fixed $t \in (0, \tau)$, since the integral part $\int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds$ can be arbitrarily small chosen, independent of the two boundary points 0 and t, we see that the sum $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$ must be non-negative for any non-negative vectors $x_0(\tau)$ and $x_0(t)$. The independence of these two vectors leads to the fact that the sum $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$ is non-negative if and only if both terms are non-negative. Thus, due to Proposition 1, the non-negativity of the term $e^{\bar{\mathbf{A}}t}Px_0(\tau)$ is equivalent to the claim i). On the other hand, the non-negativity of the term $(P - I)\hat{A}^D\hat{A}_d x_0(t)$ implies that $(P - I)\hat{A}^D\hat{A}_d \geq 0$.

To prove that $\bar{\mathbf{A}}_d \geq 0$, we assume the contrary, i.e. there exist some indices i, j with $[\bar{\mathbf{A}}_d]_{ij} < 0$. Thus, for the jth unit vector e_j, we have $[\bar{\mathbf{A}}_d e_j]_i < 0$. For a sufficiently small $\varepsilon > 0$, let us choose the initial function x_0 as follows

$$x_0(s) = \begin{cases} \left(1 - \frac{1}{\varepsilon}|t - \varepsilon - s|\right) e_j & \text{for all} \quad |t - \varepsilon - s| \leqslant \varepsilon, \\ 0 \quad \text{otherwise.} \ . \end{cases} \tag{24}$$ {x0 function}

The graph of the magnitude of $x_0(s)$ is given in Figure 1. Since $u \equiv 0$,

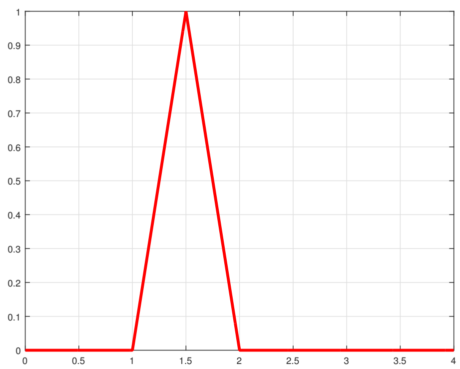

Fig. 1 The function x_0 in (24) with $\tau = 4$, $t = 2$, $\varepsilon = 0.5$. {fig1}

$x_0(0) = x_0(\tau) = 0$, the consistency condition (16) is trivially satisfied. Then, we have that

$$x_1(t) = \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds = \int_{t-2\varepsilon}^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds,$$
$$= \int_{t-2\varepsilon}^t \left(I + \bar{\mathbf{A}}(t-s) + \mathcal{O}((t-s)^2)\right)\left(1 - \frac{1}{\varepsilon}|t-\varepsilon-s|\right)\bar{\mathbf{A}}_d e_j ds \ .$$

Thus, for sufficiently small ε, the coordinate $(x_1(t))_i$ have exactly the same sign as $[\bar{\mathbf{A}}_d e_j]_i$, which is strictly negative. This is contradicted to the non-negativity of the solution $x(t)$, and hence, we conclude that $\bar{\mathbf{A}}_d \geq 0$.
"$\Leftarrow$" It is directly followed from i) and ii) that all three summands of $x_{zi}(t)$ are non-negative, □

{Thm positivity}

Theorem 5 *Consider the delay-descriptor system* (1) *and assume that it is non-advanced, and the pair* (E,A) *is regular with index* $\operatorname{ind}(E,A) = \nu$. *Furthermore, assume that* $(P-I)\ \bar{\mathbf{A}}^i \hat{A}^D \hat{B} \geq 0$ *for all* $i = 0,\dots,\nu-1$. *Then, system* (1) *is positive if and only if the following conditions hold.*
i) $\bar{\mathbf{A}} = H\ P$ *for some Metzler matrix* H.
ii) $\bar{\mathbf{A}}_d \geq 0,\ \ \bar{\mathbf{B}} \geq 0,\ \ (P-I)\hat{A}^D\hat{A}_d \geq 0,$
iii) C *is non-negative on the subspace*

$$\mathcal{X} := \operatorname{im}_+ \left[P,\ (P-I)\hat{A}^D\hat{A}_d,\ (P-I)\ \mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D\hat{B})\right] \ . \tag{25}$$

{reachable subspace}

Proof. "$\Rightarrow$" By consecutively choosing $u \equiv 0$ and $\phi \equiv 0$, we see that both the free solution $x_{zi}(t)$ and the zero-state solution x_{zs} are non-negative for all $t \geq 0$. Analogous to the proof of the necessity part in Lemma 3, from the non-negativity of the integral $\int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{B}}u_j(s)ds$, we obtain $\bar{\mathbf{B}} \geq 0$. Thus, only the claim iii) needs to be proven. We notice that due to Lemma 1 and the property (10) of the Drazin inverse, we have that P and $\bar{\mathbf{A}}$ commute, and $P\hat{E}^D = \hat{E}^D$, and hence,

$$e^{\bar{\mathbf{A}}}\hat{E}^D = \hat{E}^D e^{\bar{\mathbf{A}}} = \hat{E}^D\hat{E}\hat{E}^D e^{\bar{\mathbf{A}}} = Pe^{\bar{\mathbf{A}}}\hat{E}^D \ .$$

Therefore, we see that

$$e^{\bar{\mathbf{A}}t}Px_0(\tau) + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{B}}u_j(s)ds \subseteq \operatorname{im}_+(P),$$
$$(P-I)\hat{A}^D\hat{A}_d x_0(t) + (P-I)\sum_{i=0}^{\nu-1}\bar{\mathbf{A}}^i\hat{A}^D\hat{B}u_j^{(i)}(t)$$
$$\subseteq \operatorname{im}_+\left[(P-I)\hat{A}^D\hat{A}_d,\ (P-I)\ \mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D\hat{B})\right] \ .$$

Thus, the claim iii) is directly followed.
"$\Leftarrow$" It is straightforward that from i) and ii) we obtain the non-negativity of $x(t)$, and due to iii) we obtain the non-negativity of $y(t)$. This completes the proof. □

If we restrict ourself to the non-delayed case (i.e. $A_d = 0$), the direct corollary of Theorem 5 is straightforward. We, moreover, notice that this corollary has slightly improved the result [Virnik, 2008, Thm. 3.4].

{Thm positivity - DAE version}

Corollary 4 *Consider the descriptor system* (3) *and assume that the pair* (E, A) *is regular with index* $\mathrm{ind}(E, A) = \nu$. *Furthermore, assume that the inequalities* $(P - I)\ \bar{\mathbf{A}}^i \hat{A}^D \hat{B} \geq 0$ *hold true for* $i = 0, \ldots, \nu - 1$.
Then, system (3) *is positive if and only if the following conditions hold.*
i) $\bar{\mathbf{A}} = H\ P$ *for some Metzler matrix* H.
ii) $\bar{\mathbf{B}} \geq 0$,
iii) C *is non-negative on the subspace* $\mathcal{X}$ *defined in* (25).

4 Stability of positive delay-descriptor system

{sec4}

Remark 2 Remark 3.6: We stress out that in previous results on positivity of autonomous descriptor systems (the case when .....) it is assumed that .... , which is an unnecessary condition, see for instance [11], [14]. In contrast, our result in Theorem 3.5 provides necessary and sufficient conditions for the positivity of (1) without any a priori assumptions on the projector .... .

In light of Remark 3.6, we illustrate how Theorem 3.5 applies to general situations by presenting an example where ... is not positive and .... is not Metzler, but the system is nevertheless positive.

{exam 3}

Example 3 Let us consider system (1) whose the matrix coefficients are

$$E = \begin{bmatrix} -8.5025 & 0.9037 & -6.1960 \\ -4.8967 & 0.7359 & -3.5750 \\ -0.2285 & 0.1870 & -0.1715 \end{bmatrix},\ A = \begin{bmatrix} 0.1628 & 0.7510 & 0.3814 \\ -0.2259 & 1.0891 & 0.1289 \\ -0.1859 & 0.5633 & -0.0226 \end{bmatrix},$$
$$Ad = \begin{bmatrix} -0.6120 & 0.1289 & -0.5673 \\ -0.7736 & 0.1510 & -0.6626 \\ -0.2798 & 0.1117 & -0.2308 \end{bmatrix}.$$

Direct computation yields that the matrix polynomial $\det(sE - A)$ is

$$\det(sE - A) = 0.0688184\ s + 0.00897097,$$

and hence the system is not impulse-free, since $\mathrm{rank}(E) = 2$. Nevertheless, Theorem 2 implies that the system is non-advanced. Furthermore, by verifying Theorem 5 we see that the system is both positive and stable.

5 Conclusion

{conclusion}

In this paper, we have discussed the positivity of strangeness-free descriptor systems in continuous time. Beside that, the characterization of positive delay-descriptor systems has been treated as well. The theoretical results are obtained mainly via an algebraic approach and a projection approach. The projection approach investigates the positivity of a given descriptor system by the positivity of an inherent ODE obtained by projecting the given system onto a subspace. On the other hand, the algebraic approach derives an underlying ODE without changing the state, input and output. Then, studying these hidden ODEs is the key point. The main difficulty here is that the derivative of the input u may occur in the new system. Despite their disadvantages, these methods can provide both necessary conditions and sufficient conditions. Beside these theoretical methods, the behaviour approach, which leads to some feasible conditions, is also implemented.

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