Initial-Value Methods for Discrete Boundary Value Problems

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1. INTRODUCTION

Motivated by the work of Angel and Kalaba [5] and Golberg [8] on two-point boundary value problems for difference equations, in this paper we shall study several initial-value methods for multipoint boundary value problems for linear and nonlinear difference systems. We generalize method of adjoints (Goodman and Lance [9]), method of complementary functions (Tifford [17]) and method of particular solutions (Miele [11]) proposed for two-point continuous problems to linear multipoint problems for difference equations. Next, we show that method of adjoint and method of complementary functions can be used in an iterative way to solve nonlinear problems. To justify the results obtained here several examples are also given.

The methods discussed here are similar to our earlier work [1–3] on continuous problems. We will take up the initial adjusting method of Ojika, Kasue, and Welsh [12–15, 18] for the discrete case later.

Throughout the paper we shall assume the existence, uniqueness, and stability properties of solutions of the problems discussed and refer to [4] for some of these questions.

2. LINEAR SYSTEMS

We shall consider the set of n linear difference equations with variable coefficients

$$y(t + 1) = A(t)y(t) + f(t), \qquad t \in I_\infty \tag{2.1}$$

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together with the boundary conditions

$$\sum_{i=0}^{N} L_i y(t_i) = l, \tag{2.2}$$

where $A(t)$ is an $n \times n$ matrix with elements $a_{ij}(t)$, $1 \leqslant i, j \leqslant n$; $y(t)$ is an $n \times 1$ vector with components $y_i(t)$, $1 \leqslant i \leqslant n$; $f(t)$ is an $n \times 1$ vector with components $f_i(t)$, $1 \leqslant i \leqslant n$; I is the set of discrete points $\{0, 1, ..., \}$; L_i, $0 \leqslant i \leqslant N$ are the given $n \times n$ matrices with elements c_{pq}^i, $1 \leqslant p, q \leqslant n$; l is a given $n \times 1$ vector with components l_i, $1 \leqslant i \leqslant n$; t_i are ordered so that $t_i < t_{i+1}$, with $t_0 = 0$.

In what follows we shall assume that $A(t)$ and $f(t)$ are defined for all $t \in I_\infty$. Further, we shall denote $U(t)$ as the fundamental matrix solution of the homogeneous difference system

$$u(t+1) = A(t) u(t), \qquad t \in I_\infty \tag{2.3}$$

such that $U(0) = E$(unit matrix). The matrix G is defined as

$$G = \sum_{i=0}^{N} L_i U(t_i) \tag{2.4}$$

which we shall assume nonsingular throughout our discussion.

2.1. Method of Complementary Functions

Any solution of (2.1) in component form can be expressed as

$$y_j(t) = \sum_{i=1}^{n} u_j^{(i)}(t) y_i(0) + w_j(t), \qquad 1 \leqslant j \leqslant n \tag{2.5}$$

where $u^{(i)}(t)$ is the ith column of $U(t)$ and $w(t)$ is a particular solution of (2.1) with the initial conditions

$$w_j(0) = 0, \qquad 1 \leqslant j \leqslant n. \tag{2.6}$$

Substituting (2.5) in (2.2) leads to n linear algebraic equations in n unknowns $y_i(0)$, $1 \leqslant i \leqslant n$. Since the matrix G is nonsingular, these algebraic equations can be solved uniquely for $y_i(0)$, $1 \leqslant i \leqslant n$. Substituting these values of $y_i(0)$, $1 \leqslant i \leqslant n$ in (2.5) we find the solution of (2.1), (2.2).

Thus, to obtain the solution of (2.1), (2.2) we need n solutions of (2.3) with the initial conditions

$$u_j^{(i)}(0) = \begin{cases} 1, & i = j \\ 0, & i \neq j, \ 1 \leqslant i, j \leqslant n \end{cases} \tag{2.7}$$

and a particular solution $w(t)$ of (2.1) with initial conditions (2.6). Hence, the total $(n + 1)$ solutions.

2.2. *Method of Adjoints*

The adjoint system of (2.1) is defined as

$$x(t) = A^T(t) x(t + 1), \qquad t \in I_\infty, \tag{2.8}$$

where $A^T(t)$ is an $n \times n$ matrix with elements $a_{ji}(t)$, $1 \leqslant j$, $i \leqslant n$ and $x(t)$ is an $n \times 1$ vector with components $x_i(t)$, $1 \leqslant i \leqslant n$.

We multiply the ith equation of (2.1) by $x_i(t + 1)$ and sum over all n equations to obtain

$$\sum_{i=1}^{n} x_i(t + 1) y_i(t + 1)$$

$$= \sum_{i=1}^{n} x_i(t + 1) \sum_{j=1}^{n} a_{ij}(t) y_j(t) + \sum_{i=1}^{n} f_i(t) x_i(t + 1). \tag{2.9}$$

Next, we multiply the ith equation of (2.8) by $y_i(t)$, and summing over all n equations, we find

$$\sum_{i=1}^{n} x_i(t) y_i(t) = \sum_{i=1}^{n} y_i(t) \sum_{j=1}^{n} a_{ji}(t) x_j(t + 1). \tag{2.10}$$

On substracting (2.10) in (2.9), we get

$$\sum_{i=1}^{n} [x_i(t + 1) y_i(t + 1) - x_i(t) y_i(t)] = \sum_{i=1}^{n} f_i(t) x_i(t + 1).$$

Summing up the above identity from $t = 0$ to $t = t - 1$, we get

$$\sum_{i=1}^{n} [x_i(t) y_i(t) - x_i(0) y_i(0)] = \sum_{s=0}^{t-1} \sum_{i=1}^{n} f_i(t) x_i(t + 1). \tag{2.11}$$

From the identity (2.11), if $X(t)$ is the fundamental solution of (2.8) satisfying $X(t_N) = E$, then any solution of (2.1) can be written as

$$y(t) = [X^T(t)]^{-1} X^T(0) y(0) + [X^T(t)]^{-1} \sum_{s=0}^{t-1} X^T(s + 1) f(s). \tag{2.12}$$

Substituting (2.12) in (2.2) once again leads to n linear algebraic equations in n unknowns $y_i(0)$, $1 \leqslant i \leqslant n$. These equations will have a unique solution since the matrix

$$G_1 = \sum_{i=0}^{N} L_i [X^T(t_i)]^{-1} X^T(0)$$

is equivalent to G.

The second term in (2.12) is again a particular solution of (2.1) and hence to find the solution of (2.1), (2.2) we need $(n + 1)$ solutions.

2.3. Method of Particular Solutions

In order to find the solution of (2.1), (2.2) we solve (2.1) with $(n + 1)$ different sets of initial conditions and obtain $y^{(i)}(t)$, $1 \leqslant i \leqslant n + 1$; $(n + 1)$ particular solutions of (2.1). Next, we introduce $(n + 1)$ constants c_i, $1 \leqslant i \leqslant n + 1$ and demand that the linear combination

$$y(t) = \sum_{i=1}^{n+1} c_i y^{(i)}(t) \tag{2.13}$$

to be a solution of (2.1), for this we must have

$$\sum_{i=1}^{n+1} c_i = 1. \tag{2.14}$$

Substituting (2.13) in (2.2), we obtain another n linear algebraic equations in $(n + 1)$ unknowns c_i, $1 \leqslant i \leqslant n + 1$. These equations with (2.14) form $(n + 1)$ algebraic equations in $(n + 1)$ unknowns c_i, $1 \leqslant i \leqslant n + 1$, and will have a unique solution.

2.4. Discussion

(1)   All three methods discussed above require same number $(n + 1)$ solutions and are theoretically the same. Method of complementary functions and method of particular solutions both are forward process, whereas method of adjoints is backward process. In practical computation all three methods have an advantage or disadvantage over the other methods.

(2)   The fundamental matrix $U(t)$ has the representation $U(t) = \prod_{i=0}^{t-1} A(t - 1 - i)$, and the particular solution $w(t) = \sum_{s=1}^{t} (\prod_{i=0}^{t-1-s} A(t - 1 - i))f(s - 1)$, $\prod_{i=0}^{-1} A(i) = 1$. Thus, in terms of $A(t)$ and $f(t)$, Eq. (2.5) takes the form

$$y(t) = \prod_{i=0}^{t-1} A(t - 1 - i) y(0) + \sum_{s=1}^{t} \left(\prod_{i=0}^{t-1-s} A(t - 1 - i) \right) f(s - 1).$$

In the process of calculating $U(t)$, we store all $\prod_{i=0}^{t-1-s} A(t - 1 - i)$, $1 \leqslant s \leqslant t - 1$, and use it to calculate $w(t)$.

(3)   Between $U(t)$ and $X(t)$ the relation $U(t) = [X^T(t)]^{-1} x^T(0)$ can be easily verified.

(4)   In the method of particular solutions let $y^{(n+1)}(t) = w(t)$, then (2.13) is the same as (2.5) where now $u^{(i)}(t) = y^{(i)}(t) - w(t)$, $1 \leqslant i \leqslant n$.

3. Separated Boundary Conditions

In Section 2, we noticed that to solve problem (2.1), (2.2) we need $(n + 1)$ solutions, however for some particular cases of (2.3) we actually need fewer solutions. Here, we shall consider the following boundary conditions:

$$y_{i_{k_i}}(t_i) = \alpha_{i_{k_i}}, \tag{3.1}$$

where $0 \leqslant i \leqslant N \leqslant n - 1$; $k_0 = 1, 2,..., k_0$; $k_1 = 1, 2,..., k_1$;...; $k_N = 1, 2,..., k_N$; $\sum_{j=0}^{N} k_j = n$.

The subscripts on the specified boundary points are written i_{k_i} to allow the possibility that the set of variables specified at the boundary points may not be disjoint. For example, let $n = 7$, $N = 3$, $y_1(t_0)$, $y_3(t_0)$, $y_2(t_1)$, $y_3(t_2)$, $y_1(t_3)$, $y_6(t_3)$, $y_7(t_3)$ so y_1 is fixed at t_0 and t_3, also y_3 is fixed at t_0 and t_2, whereas no condition is prescribed at y_4 and y_5. The indexing for the boundary conditions is $0_1 = 1$, $0_2 = 3$, $1_1 = 2$, $2_1 = 3$, $3_1 = 1$, $3_2 = 6$, $3_3 = 7$.

We shall assume that k_0 is greater than all other k_j, $1 \leqslant j \leqslant N$, otherwise the role of t_0 and the boundary point where the greatest k_j is defined can be interchanged.

3.1. Method of Complementary Functions for (2.1), (3.1)

Rewriting (2.5) as

$$y_j(t) = \sum_{i=1, i \neq 0_{k_0}}^{n} u_j^{(i)}(t) \, y_i(0) + v_j(t) \tag{3.2}$$

where the expression

$$v_j(t) = \sum_{i=0_{k_0}} u_j^{(i)}(t) \, y_i(0) + w_j(t)$$

is the jth component of the solution of (2.1) with the initial conditions

$$v_j(0) = \begin{cases} y_j(0), & j = 0_{k_0} \\ 0, & j \neq 0_{k_0} \end{cases}. \tag{3.3}$$

Thus, we solve (2.3), $(n - k_0)$ times with the initial conditions

$$u_j^{(i)}(0) = \begin{cases} 1, & i = j \neq 0_{k_0} \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

and need a particular solution of (2.1) satisfying (3.3).

Now, using the boundary conditions (3.1) other than at the point t_0, we find from (3.2) on arranging the terms

$$\sum_{i=1, i \neq 0_{k_0}}^{n} u_{j_{k_j}}^{(i)}(t_j) \, y_i(0) = \alpha_{j_{k_j}} - v_{j_{k_j}}(t_j), \quad 1 \leqslant j \leqslant N, \tag{3.5}$$

which is a sum of $(n - k_0)$ algebraic equations in $(n - k_0)$ unknowns $y_i(0)$; $1 \leqslant i \leqslant n, \, i \neq 0_{k_0}$.

3.2. Method of Adjoints for (2.1), (3.1)

To utilize the identity (2.11) we solve the adjoint equations (2.8) k_j times from the point t_j, $1 \leqslant j \leqslant N$, and hence the total $(n - k_0)$ times with the conditions

$$x_i^{m(k_j)}(t_j) = \begin{cases} 1, & i = j_{k_j} \\ 0, & i \neq j_{k_j} \end{cases}, \tag{3.6}$$

where the superscript $m(k_j)$ refers to the mth solution of (2.8) from the point t_j and j_{k_j} refers to the subscripts of the boundary condition $y_{j_{k_j}}(t_j)$ in (3.1).

Using (3.6) in (2.11) and arranging the terms, we obtain

$$\sum_{i=1, i \neq 0_{k_0}}^{n} x_i^{m(k_j)}(0) \, y_i(0)$$

$$= y_{j_{k_j}}(t_j) - \sum_{i=0_{k_0}} x_i^{m(k_j)}(0) \, y_i(0)$$

$$- \sum_{s=0}^{t_j - 1} \sum_{i=1}^{n} x_i^{m(k_j)}(s+1) f_i(s), \quad 1 \leqslant j \leqslant N. \tag{3.7}$$

In (3.7) the right part is known from (3.1), (2.8), (3.6); hence the generation of (3.7) yields a set of $(n - k_0)$ linear algebraic equations in $(n - k_0)$ unknowns $y_i(0)$; $1 \leqslant i \leqslant n, \, i \neq 0_{k_0}$.

3.3. Method of Particular Solutions for (2.1), (3.1)

We solve (2.1) with $(n - k_0 + 1)$ different sets of initial conditions

$$y_j^{(i)}(0) = \begin{cases} \alpha_{0_{k_0}}, & j = 0_{k_0} \\ \delta_{i,j}, & 1 \leqslant i \leqslant n + 1, \, 1 \leqslant j \leqslant n; \, i, j \neq 0_{k_0} \end{cases} \tag{3.8}$$

so that the linear combination

$$y(t) = \sum_{i=1, i \neq 0_{k_0}}^{n+1} c_i \, y^{(i)}(t) \tag{3.9}$$

is a solution of (2.1). For this, (3.9) reduces to

$$\sum_{i=1, i \neq 0_{k_0}}^{n+1} c_i = 1.$$
(3.10)

Naturally (3.9) satisfies (3.1) at the point t_0 as long as (3.10) holds, also at other points from (3.9), we find

$$\sum_{i=1, i \neq 0_{k_0}}^{n+1} c_i y_{j_{k_j}}^{(i)}(t_j) = a_{j_{k_j}}, \qquad 1 \leqslant j \leqslant N.$$
(3.11)

Thus, (3.11) together with (3.10) are $(n - k_0 + 1)$ equations in $(n - k_0 + 1)$ unknowns c_i, $1 \leqslant i \leqslant n + 1$, $i \neq 0_{k_0}$.

3.4. *Some Examples*

(1) Consider the problem

$$\left| \begin{matrix} y_1(t+1) \\ y_2(t+1) \end{matrix} \right| = \left| \begin{matrix} 0 & 1 \\ -1 & 33/16 \end{matrix} \right| \left| \begin{matrix} y_1(t) \\ y_2(t) \end{matrix} \right| + \left| \begin{matrix} 0 \\ (t+1)/64 \end{matrix} \right|, t = 0, 1, 2, 3,$$

$$y_1(0) = 0, \qquad y_1(4) = 1$$
(3.12)

which is the discrete analogue of $y'' = \frac{1}{16} y + \frac{1}{64} t$; $y(0) = 0$, $y(4) = 1$ discussed previously by Fox [7], Roberts and Shipman [16], and several others.

For (3.12) we get from (3.5) only one equation

$$u_1^{(2)}(4) y_2(0) = 1 - v_1(4).$$
(3.13)

The system

$$\left| \begin{matrix} u_1(t+1) \\ u_2(t+1) \end{matrix} \right| = \left| \begin{matrix} 0 & 1 \\ -1 & 33/16 \end{matrix} \right| \left| \begin{matrix} u_1(t) \\ u_2(t) \end{matrix} \right|$$

is solved with the conditions $u_1^{(2)}(0) = 0$, $u_2^{(2)}(0) = 1$, to obtain $u_1^{(2)}(4) = 4.6486817$. Next, (3.12) is solved with the conditions $y_1(0) = 0$, $y_2(0) = 0$ to find $v_1(4) = 0.16217042$. These values are used in (3.13) to find $y_2(0) = 0.1802295$.

Similarly for (3.12) we get from (3.7) only one equation

$$x_2^{1(1)}(0) y_2(0) = 1 - \frac{1}{64} x_2^{1(1)}(1) - \frac{2}{64} x_2^{1(1)}(2)$$

$$- \frac{3}{64} x_2^{1(1)}(3).$$
(3.14)

The system

$$\begin{vmatrix} x_1(t) \\ x_2(t) \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 33/16 \end{vmatrix} \begin{vmatrix} x_1(t+1) \\ x_2(t+1) \end{vmatrix}$$

is solved with the conditions $x_1^{1(1)}(4) = 1$, $x_2^{1(1)}(4) = 0$ to obtain $x_2^{1(1)}(0) = 4.6486817$, $x_2^{1(1)}(1) = 3.2539063$, $x_2^{1(1)}(2) = 2.0625$, $x_2^{1(1)}(3) = 1$. Substituting these values in (3.14) we obtain the same value for $y_2(0)$ as obtained earlier from (3.13).

Next, to use method of particular solutions we need to solve

$$c_2 y_1^{(2)}(4) + c_3 y_1^{(3)}(4) = 1, \qquad c_2 + c_3 = 1, \tag{3.15}$$

where $y^{(2)}(t)$ and $y^{(3)}(t)$ are the solutions of (3.12) satisfying $y_1^{(2)}(0) = 0$, $y_2^{(2)}(0) = 1$; and $y_1^{(3)}(0) = 0$, $y_2^{(3)}(0) = 0$. The required values are $y_1^{(2)}(4) = 4.81085212$, $y_1^{(3)}(4) = 0.16217042$ and the value of c_2 obtained from (3.15) is same as $y_2(0)$ found earlier by other two methods.

(2)   Consider the problem

$$\begin{vmatrix} y_1(t+1) \\ y_2(t+1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 2 + \dfrac{2}{(21+t)^2} \end{vmatrix} \begin{vmatrix} y_1(t) \\ y_2(t) \end{vmatrix} + \begin{vmatrix} 0 \\ -\dfrac{1}{10(21+t)} \end{vmatrix}, \tag{3.16}$$

$$y_1(0) = y_1(10) = 0$$

which is the discrete analogue of $y'' - (2/(20+t)^2)\,y(t) + (1/10(20+t)) = 0$ discussed previously by Collatz [6].

For (3.16), all the three methods are used to obtain the same missing value $y_2(0) = 0.01888$.

These two simple examples justify our statement that all three methods are theoretically the same. However, in several situations where numerical difficulties cannot be cured on a computer with the forward process, a backward process may provide good results. Hence, it is expected that all the three methods will work differently when used on the computer. We shall take up more details of this matter later.

4. NONLINEAR PROBLEMS

Here, we shall use the method of complementary functions and method of adjoints given in Section 2 in an iterative way to solve the nonlinear problem

$$y_i(t+1) = f_i(t, y_1(t), y_2(t),..., y_n(t)); \; t \in I_\infty, \; 1 \leqslant i \leqslant n, \tag{4.1}$$

$$q_i[y(t_0), y(t_1),..., y(t_N)] = 0; \qquad 1 \leqslant i \leqslant n. \tag{4.2}$$

Assume trial values of $y_i(t_0)$, $1 \leqslant i \leqslant n$, and find the solution $y_i(t)$ of (4.1). Let us consider a nearby solution $y_i(t) + \delta y_i(t)$, $1 \leqslant i \leqslant n$, where $\delta y_i(t)$ is the first-order correction to $y_i(t)$ to produce the actual solution of (4.1), (4.2). The equations of the nearby solutions are

$$y_i(t + 1) + \delta y_i(t + 1)$$
$$= f_i(t, y_1(t) + \delta y_1(t), ..., y_n(t) + \delta y_n(t)) \quad t \in I_\infty, 1 \leqslant i \leqslant n. \quad (4.3)$$

Expanding the right side of (4.3) in a Taylor's series up to and including first-order terms, we obtain the variational equations

$$\delta y_i(t + 1) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} \delta y_j(t), \qquad 1 \leqslant i \leqslant n. \quad (4.4)$$

In a similar way, the boundary conditions for the variational equations are obtained and appear as

$$\sum_{j=1}^{n} \frac{\partial q_i}{\partial y_j(0)} \delta y_j(0) + \cdots + \sum_{j=1}^{n} \frac{\partial q_i}{\partial y_j(t_N)} \delta y_j(t_N)$$
$$= -q_{i(\text{cal})}, \qquad 1 \leqslant i \leqslant n, \quad (4.5)$$

where $q_{i(\text{cal})}$ is the value of (4.2) calculated from the assumed initial conditions for (4.1) and $y(t_i)$, $1 \leqslant i \leqslant N$, is found from the solution of (4.1). In a similar manner, the partial derivatives are evaluated. Equations (4.4) and (4.5) form a linear system and play the role of (2.1), (2.2).

Note that we have interpreted the variation $\delta y_i(t)$ to be the difference between the true (but unknown) and the calculated solution i.e.,

$$\delta y_i(t) = y_{i(\text{true})}(t) - y_{i(\text{cal})}(t), \qquad t_0 \leqslant t \leqslant t_N, 1 \leqslant i \leqslant n. \quad (4.6)$$

For the boundary conditions (3.1), we always take the given $y_i(0)$ so that we need to assume trial values of $y_i(0)$; $1 \leqslant i \leqslant n$, $i \neq 0_{k_0}$. From (4.6), the boundary conditions for (4.4) appear as

$$\delta y_i(0) = 0, \qquad i = 0_{k_0}$$
$$\delta y_{i_{k_i}}(t_i) = \alpha_{i_{k_i}} - [y_{i_{k_i}(\text{cal})}(t_i)], \qquad 1 \leqslant i \leqslant n, i \neq 0_{k_0}. \quad (4.7)$$

Equations (4.7) play the role of (3.1).

4.1. *Method of Section* 2.1

We solve the homogeneous system associated with (4.4)

$$u(t+1) = A(t)\, u(t),\, A(t) = \left[\frac{\partial f_i}{\partial y_j}\right] \tag{4.8}$$

n times with the initial conditions (2.7).

The missing $\delta y_i(t_0)$, $1 \leqslant i \leqslant n$, for the problem (4.4), (4.5) are obtained from the equations

$$\sum_{s=1}^{n} \left[\frac{\partial q_i}{\partial y_s(0)} + \sum_{j=1}^{n} \sum_{k=1}^{N} \frac{\partial q_i}{\partial y_j(t_k)} u_j^{(s)}(t_k)\right] \delta y_s(t_0)$$
$$= -q_{i(\text{cal})}, \qquad 1 \leqslant i \leqslant n. \tag{4.9}$$

Since Eqs. (4.4)–(4.6) are only approximate equations, the process of finding the true solution is iterative and terminates when $\delta y_i(t)$, $t_0 \leqslant t \leqslant t_N$, $1 \leqslant i \leqslant n$, are sufficiently small (less than a preassigned tolerance). Equations (4.6), (4.8), and (4.9) for the mth iteration are written as

$$[\delta y_i(t)]^{(m)} = y_{i(\text{true})}(t) - [y_{i(\text{cal})}(t)]^{(m)}, \qquad t_0 \leqslant t \leqslant t_N, 1 \leqslant i \leqslant n \tag{4.10}$$

$$[u(t+1)]^{(m)} = [A(t)]^{(m)} [u(t)]^{(m)}, \quad [A(t)]^{(m)} = \left.\left[\frac{\partial f_i}{\partial y_j}\right]\right.^{(m)}, \tag{4.11}$$

$$\sum_{s=1}^{n} \left|\left[\frac{\partial q_i}{\partial y_s(0)}\right]^{(m)} + \sum_{j=1}^{n} \sum_{k=1}^{N} \left|\frac{\partial q_i}{\partial y_j(t_k)}\right|^{(m)} [u_j^{(s)}(t_k)]^{(m)}\right|$$
$$\times [\delta y_s(t_0)]^{(m)} = -[q_{i(\text{cal})}]^{(m)}, \quad 1 \leqslant i \leqslant n. \tag{4.12}$$

For the next iteration, the new conditions are found from

$$[y_i(t_0)]^{(m+1)} = [y_i(t_0)]^{(m)} + [\delta y_i(t_0)]^{(m)},$$
$$1 \leqslant i \leqslant n, \quad m = 1, 2, ... \tag{4.13}$$

The calculations will terminate whenever $\max\{|q_{i(\text{cal})}|, 1 \leqslant i \leqslant n\}$ is less than a preassigned tolerance.

4.2. *Method of Section* 2.2

We solve the adjoint system associated with (4.4),

$$x(t) = A^T(t)\, x(t+1), \qquad A^T(t) = \left[\frac{\partial f_j}{\partial y_i}\right] \tag{4.14}$$

satisfying $x(t_N) = E$. If we denote $\mathbf{X}(t) = [X^T(t)]^{-1} x^T(0)$ and $\mathbf{x}^{(i)}(t)$ as the ith column of $\mathbf{X}(t)$, then the missing $\delta y_i(t_0)$, $1 \leqslant i \leqslant n$ for problem (4.4), (4.5) are obtained from the same equation (4.9) replacing $u_j^{(s)}(t_k)$ by $\mathbf{x}_j^{(s)}(t_k)$, $1 \leqslant j$, $s \leqslant n$, $1 \leqslant k \leqslant N$. Thus, for the mth iteration Eqs. (4.10) and (4.13) remain the same, whereas (4.11) takes the form

$$[x(t)]^{(m)} = [A^T(t)]^{(m)} [x(t+1)]^{(m)}, \quad [A^T(t)]^{(m)} = \left[\frac{\partial f_j}{\partial y_i}\right]^{(m)} \quad (4.15)$$

and in Eq. (4.12), $|u_j^{(s)}(t_k)|^{(m)}$ are replaced by $[\mathbf{x}^{(s)}(t_k)]^{(m)}$, $1 \leqslant j$, $s \leqslant n$, $1 \leqslant k \leqslant N$, $m = 1, 2, \dots$.

4.3. Method of Section 3.1

For the mth iteration Eqs. (4.10) and (4.11) remain the same, whereas Eq. (4.12) is simplified to

$$\sum_{i=1, i \neq 0_{k_k}}^{n} [u_{j_{k_j}}^{(i)}(t_j)]^{(m)} [y_i(0)]^{(m)}$$

$$= [\delta y_{j_{k_j}}(t_j)]^{(m)}, \qquad 1 \leqslant j \leqslant N, m = 1, 2, \dots. \qquad (4.16)$$

and Eq. (4.13) is replaced by

$$[y_i(t_0)]^{(m+1)} = \begin{cases} \alpha_{0_{k_0}}, & i = 0_{k_0} \\ [y_i(t_0)]^{(m)} + [\delta y_i(t_0)]^{(m)}, & 1 \leqslant i \leqslant n, i \neq 0_{k_k} \end{cases}. \quad (4.17)$$

4.4. Method of Section 3.2

The mth iteration is given by Eqs. (4.10) and (4.14) and

$$\sum_{i=1, i \neq 0_{k_0}}^{n} [x_i^{m(k_j)}(0)]^{(m)} [\delta y_i(0)]^{(m)} = [\delta y_{j_{k_j}}(t_j)]^{1(m)},$$

$$1 \leqslant j \leqslant N, m = 1, 2, \dots, \qquad (4.18)$$

also by Eq. (4.17).

5. CONVERGENCE AND ERROR ANALYSIS

For problem (4.1), (4.2) the missing conditions at t_0 are obtained by an iterative process. We solve (4.12) to find $[y_i(0)]^{(m)}$, $1 \leqslant i \leqslant n$, and the next $(m+1)$th iteration is obtained by (4.13).

We shall denote the solution of (4.1) as $y_i[y_1(0), \dots, y_n(0), t]$ or in short

$y_i[\cdot, t]$, $1 \leqslant i \leqslant n$, which is continuously dependent on given and assumed $y_i(0)$, $1 \leqslant i \leqslant n$. If we denote

$$\phi_i = q_i[\, y(t_0), y[\cdot, t_1], y[\cdot, t_2],..., y[\cdot, t_N]\,] = 0, \qquad 1 \leqslant i \leqslant n, \quad (4.19)$$

then ϕ_i is also continuously dependent on $y_i(t_0)$, $1 \leqslant i \leqslant n$. Thus, solving (4.1), (4.2) is equivalent to finding $y_i(t_0)$, $1 \leqslant i \leqslant n$, for which $\phi_i = 0$, $1 \leqslant i \leqslant n$.

Assume that the mth approximation to the initial conditions $[\, y_i(t_0)]^{(m)}$, $1 \leqslant i \leqslant n$, has been found. Newton's method gives the $(m+1)$th approximation by the equations

$$[\phi_i]^{(m)} + \sum_{s=1}^{n} \left| \frac{\partial \phi_i}{\partial y_s(t_0)} \right|^{(m)}$$
$$\times \{[\, y_s(t_0)]^{(m+1)} - [\, y_s(t_0)]^{(m)}\} = 0, \qquad 1 \leqslant i \leqslant n. \quad (4.20)$$

Using (4.13), (4.19) in (4.20), we find

$$[q_{i(\text{cal})}]^{(m)} + \sum_{s=1}^{n} \left[\left| \frac{\delta q_i}{\partial y_s(0)} \right|^{(m)} \sum_{j=1}^{n} \sum_{k=1}^{N} \left| \frac{\partial q_i}{\partial y_j[., t_k]} \right|^{(m)} \right.$$
$$\left. \times \left| \frac{\partial y_j[\cdot, t_k]}{\partial y_s(t_0)} \right|^{(m)} \right] [\delta y_s(t_0)]^{(m)} = 0, \qquad 1 \leqslant i \leqslant n. \quad (4.21)$$

The total variation in $y_i[\cdot, t_k]$ can be expressed as

$$\delta y_i[\cdot, t_k] = \sum_{s=1}^{n} \frac{\partial y_i[\cdot, t_k]}{\partial y_s(t_0)} \delta y_s(t_0),$$
$$1 \leqslant i \leqslant n, 1 \leqslant k \leqslant N; \qquad (4.22)$$

also, the solution of (4.4) at the point t_k is represented by virtue of (2.5) by

$$\delta y_i[\cdot, t_k] = \sum_{s=1}^{n} u_i^{(s)}(t_k) \, \delta y_s(t_0), \qquad 1 \leqslant i \leqslant n, 1 \leqslant k \leqslant N. \quad (4.23)$$

Comparing the coefficients of $\delta y_s(t_0)$ in (4.22) and (4.23), it follows that

$$\frac{\partial y_i[\cdot, t_k]}{\partial y_s(t_0)} = u_i^{(s)}(t_k), \qquad 1 \leqslant i, s \leqslant n, 1 \leqslant k \leqslant N. \quad (4.24)$$

Using (4.24) in (4.21) we obtain the same equation as (4.12). Thus, the method used to find the missing conditions at t_0 is equivalent to Newton's method to solve the system of equations. Therefore, the Kantorovich sufficient theorem [10, p. 367] can be applied which furnishes a theoretical

basis for the convergence of the process and an estimate of the rate of convergence. For other cases discussed in Sections 4.2–4.4 it is shown analogously.

6. Some Examples

(3) Consider the problem

$$y_1(t + 1) = y_2(t),$$

$$y_2(t + 1) = 2y_2(t) - y_1(t) + \frac{1}{N^2} e^{-y_2(t)}, \quad 0 \leqslant t \leqslant N - 1, \tag{6.1}$$

$$y_1(0) = y_1(N) = 0$$

which is the discrete analogue of $y''(x) = e^{-y(x)}, y(0) = y(1) = 0$ having exactly two solutions

$$^iy(x) = 2 \left\{ \log \left[\cosh \left(\frac{^ic}{2} \left(x - \frac{1}{2} \right) \right) \right] - \log \left[\cosh \left(\frac{^ic}{4} \right) \right] \right\},$$

where ^ic $(i = 1, 2)$ are the solutions of $c = \sqrt{2} \cosh (\frac{1}{4}c)$. The numerical values for ^ic up to three decimal places are $^1c = 1.517, {}^2c = 10.939$. With these values of ^ic, $^1y(x)$ drops below to -0.14050941 and $^2y(x)$ up to -4.0916146.

The method of section 4.4 is used for (6.1); the results are given in Table I up to $N = 50$ with the assumed starting value $y_2(0) = -2$. The results agree with sufficient number of places with the exact values of $^2y(x)$. For $N = 100$ the results are given in Table II with different starting values of $y_2(0)$; the results agree with the exact values of $^1y(x)$ as expected to happen for large N.

TABLE I

k th iteration	N	$[y_2(0)]^{(k)}$	$[y_1(0.2)]^{(k)}$	$[y_1(1)]^{(k)}$
3	5	−2.03690085	−2.03690085	−1.42928911E−07
5	10	−1.06290478	−2.09686189	−4.5838533E−09
6	50	−0.216605421	−2.12140715	3.46576599E−08

TABLE II

kth iteration	$y_2(0)$	$[y_2(0)]^{(k)}$	$[y_1(.2)]^{(k)}$	$[y_1(1)]^{(k)}$
6	-2	$-5.44348668E-03$	-0.0891908076	$4.9512181E-09$
4	-0.001	$-5.4434866E-03$	-0.0891908085	$-4.40110171E-09$
4	0.001	$-5.44348663E-03$	-0.0891908079	$4.2962256E-10$

(4) Consider the problem

$$y_1(t+1) = y_2(t),$$

$$y_2(t+1) = 2y_2(t) - y_1(t) + \frac{1}{N^2} e^{y_2(t)}, \qquad 0 \leqslant t \leqslant N-1, \qquad (6.2)$$

$$y_1(0) = y_1(N) = 0$$

which is the discrete analogue of $y''(x) = e^{y(x)}$, $y(0) = y(1) = 0$ having exactly one solution

$$y(x) = -\log 2 + 2 \log \left\{ \frac{c}{\cos(\frac{1}{2}c(x - \frac{1}{2}))} \right\},$$

where c is the solution of $c = \sqrt{2} \cos(\frac{1}{4}c)$.

The method of Section 4.4 is used for (6.2) for different N with the starting value $y_2(0) = 0.001$; the results for the fourth iteration are presented in Table III.

(5) Consider the problem

$$y_1(t+1) = y_2(t),$$

$$y_2(t+1) = 2y_2(t) - y_1(t) + \frac{2}{N^2} y_2^3(t), \quad 0 \leqslant t \leqslant N-1, \qquad (6.3)$$

$$y_1(0) = 1, \qquad y_2(N) - y_1(N-1) + \frac{2}{N} y_1^2(N) = 0$$

which is the discrete analogue of $y''(x) = 2y^3(x)$, $y(1) = 1$, $y'(2) + y^2(2) = 0$ discussed previously by Fox [7]. The continuous problem has only one solution, $y(x) = 1/x$.

The method of Section 4.1 is used for (6.3) for different N with the starting value $y_2(0) = 1$. The results are presented in Table IV for the nth iteration.

TABLE III

x	Calculated $y(x)$					Exact $y(X)$
	$N = 5$	$N = 10$	$N = 50$	$N = 100$	$N = 200$	
0.2	−0.0730539075	−0.0732143313	−0.0732662139	−0.0732678385	−0.0732682461	−0.0732683791
0.4	−0.108925786	−0.109159129	−0.109234569	−0.109236932	−0.109237524	−0.10923772
0.6	−0.108925786	−0.109159129	−0.10923457	−0.109236932	−0.109237525	−0.10923772
0.8	−0.0730539075	−0.0732143314	−0.0732662144	−0.0732678385	−0.0732682478	0.0732683791
1.0	6.18456397E−11	−1.35514711E−10	−7.43057172E−10	5.90340221E−10	−1.48980917E−09	0.0

TABLE IV

	Calculated $y(x)$				
x	$N = 5$	$N = 10$	$N = 50$	$N = 100$	ext $y(x)$
0.2	0.834892169	0.833735968	0.833349665	0.833337485	0.83333333
0.4	0.716340926	0.714813225	0.714307103	0.714291215	0.71428511
0.6	0.627196585	0.625561175	0.625022777	0.625005951	0.625
0.8	0.557790149	0.556124286	0.555578689	0.555561713	0.5555555
1.0	0.502267329	0.500575173	0.500023466	0.500006385	0.5

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REFERENCES

1. R. P. AGARWAL, The numerical solutions of multipoint boundary value problems, *J. Comput. Appl. Math.* **5** (1979), 17–24.
2. R. P. AGARWAL, On the periodic solutions of nonlinear second order differential systems, *J. Comput. Appl. Math.* **5** (1979), 117–123.
3. R. P. AGARWAL, On the method of complementary functions for nonlinear boundary value problems, *J. Optim. Theory Appl.* **36** (1982) 139–144.
4. R. P. AGARWAL, On multi-point boundary value problems for discrete equations, *J. Math. Anal. Appl.* **96** (1983), 520–534.
5. E. ANGEL AND R. KALABA, A one-sweep numerical method for vector-matrix difference equations with two-point boundary conditions, *J. Optim. Theory Appl.* **6** (1970), 345–355.
6. L. COLLATZ, "The Numerical Treatment of Differential Equations, Springer-Verlag," New York, 1966.
7. L. FOX, The Numerical Solution of Two-Point Boundary Value Problems in Ordinary Differential Equations," Oxford Univ. Press, London, 1957.
8. M. A. GOLBERG, Derivation and validation of initial-value methods for boundary-value problems for difference equations, *J. Optim. Theory Appl.* **7** (1971), 411–419.
9. T. R. GOODMAN AND G. N. LANCE, The numerical solution of two point boundary value problems, *MTAC*, **10** (1956), 82–86.
10. P. HENRICI, "Discrete Variable Methods in Ordinary Differential Equations, Wiley, New York, 1962.
11. A. MIELE, Method of particular solutions for linear two-point boundary value problems, *J. Optim. Theory Appl.* **2** (1968), 260–273.
12. T. OJIKA AND Y. KASUE, Initial-value adjusting method for the nonlinear multipoint boundary-value problems, *J. Math. Anal. Appl.* **69** (1979), 359–371.

13. T. OJIKA AND W. WELSH, A numerical method for the solution of multi-point problems for ordinary differential equations with integral constraints, *J. Math. Anal. Appl.* **72** (1979), 500–511.

14. T. OJIKA, On the quadratic convergence of the initial value adjusting method for nonlinear multipoint boundary-value problems. *J. Math. Anal. Appl.* **73** (1980), 192–203.

15. T. OJIKA AND W. WELSH, A numerical method for multipoit boundary value problems with application to a restricted three body problem, *Internat. J. Comput. Math.* **8** (1980), 329–344.

16. S. M. ROBERTS AND J. S. SHIPMAN, "Two-Point Boundary Value Problems: Shooting Methods," American Elsvier, New York, 1972.

17. A. N. TIFFORD, On the solution of total differential, boundary value problems, *J. Aeronaut. Sci.* **18** (1951), 65–66.

18. W. WELSH AND T. OJIKA, Multipoint boundary value problems with discontinuities. I. Algorithms and applications, *J. Comput. Appl. Math.* **6** (1980), 133–143.