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Stability of Linear Delay Differential Systems with Matrices Having Common Eigenvectors

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We are concerned with the asymptotic stability of linear delay systems. Under the condition that parameter matrices of the systems have common eigenvectors, the stability analysis may be reduced to that of lower dimensional systems. A method for the analysis is presented by means of eigenpairs of the parameter matrices. Our results are illustrated by two examples.

Key words: stability analysis, delay system, simultaneous triangularizability, commuting family, common eigenvectors

1. Introduction

We are concerned with the asymptotic stability of systems of linear delay differential equations (DDEs)

$$\dot{x}(t) = A_1 x(t) + A_2 x(t-\tau) \tag{1.1}$$

where A_1 and A_2 are constant complex-valued matrices, and τ is a positive constant. The present paper starts with the case that the parameter matrices of the system (1.1) are commutative, *i.e.* the equation

$$A_1 A_2 = A_2 A_1 \tag{1.2}$$

holds. Then we can apply many interesting results in linear algebra to derive much insight into the stability analysis of DDEs.

For instance, the asymptotic stability analysis of linear delay system

$$\dot{x}(t) = Bx(t-\tau) \tag{1.3}$$

is studied in [2, 3, 4, 6] and reduced to that of n scalar equations. The system (1.3) is a special case of the system (1.1) with commuting matrices. On the other hand, the asymptotic stability analysis of linear delay differential system

$$\dot{x}(t) = -x(t) + \beta Bx(t-\tau) \qquad (\beta \text{ is a constant}) \tag{1.4}$$

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is studied in [1] and again reduced to that of n scalar equations. Since this is a special case of (1.1) with $A_1 = -I$ and $A_2 = \beta B$, the commuting condition (1.2) obviously holds. Generalizing the results of [1, 2, 3, 4, 6], we shall establish criteria and algorithms on linear stability analysis for DDEs and neutral delay differential equations (NDDEs).

As a matter of fact, a commuting family of matrices is characterized by all of them being simultaneously triangularizable through unitary transformation (see Lemma 2.2 below). The condition of simultaneous triangularizability of a pair of matrices can be further relaxed so that the rank of their commutator is less than or equal to one. Therefore we shall treat the stability analysis for the system (1.1) with matrices having several common eigenvectors, for the existence of such common eigenvectors yields a partial simultaneous triangularization of the matrices.

2. Lemmata from Linear Algebra

The following lemmata are already known in linear algebra, however some are not so popular. In conjunction with our algorithms given later, their proofs will be included.

LEMMA 2.1 ([5], p.51). *Let $\{A_j\}$ $(j = 1, 2, \ldots, m)$ be a commuting family (i.e. $A_i A_j = A_j A_i$ for every i and j) of matrices in $C^{n\times n}$. Then there exists a common eigenvector in C^n of the family.*

This serves for the proof of the following Lemma.

LEMMA 2.2 ([5], p.81). *Let $\{A_j\}$ $(j = 1, 2, \ldots, m)$ be a commuting family. Then, there exists a unitary matrix $U \in C^{n\times n}$ such that $U^{-1}A_jU$ is upper triangular for every j.*

The outline of the proof can be given as follows. Exploiting Lemma 2.1 at each stage in which a choice of an eigenvector (and unitary matrix) is made, the same eigenvector may be chosen for every A_j $(j = 1, 2, \ldots, m)$. Since the unitary equivalence preserves commutativity, we can get a commuting family whose matrices are reduced by one in their dimension. Repeating the process, we complete the composition of U.

This means that a commuting family of matrices may be simultaneously upper-triangularized. The upper triangular form of matrices will be fully utilized in the stability analysis of DDEs.

The following lemma gives another sufficient condition for simultaneous triangularization.

LEMMA 2.3 ([7], p.175). *Let a pair of matrices A_1 and A_2 in $C^{n\times n}$ be satisfying the condition* $\operatorname{rank}(A_1A_2 - A_2A_1) \le 1$. *Then A_1 and A_2 are simultaneously triangularizable.*

Once this lemma is established, then Lemma 2.2 of the case dealing only with two matrices, can be regarded as its special case.

In connection with Lemma 2.2, we obtain the following as a special case.

LEMMA 2.4 ([5], p.82). *Let $\{A_j\}$ $(j = 1, 2, \dots, m)$ be a commuting family of matrices in $\boldsymbol{R}^{n \times n}$. Then there is a real orthogonal matrix Q such that $Q^T A_j Q$ is of the following block upper triangular form for every j :*

$$Q^{-1}A_jQ = \begin{bmatrix} A_j^{(1)} & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ \underline{0} & & & A_j^{(k)} \end{bmatrix} \qquad (1 \le k \le n)$$

where each diagonal block $A_j^{(i)}$ is a real 1-by-1 matrix, or a real 2-by-2 matrix with a nonreal pair of complex conjugate eigenvalues. The diagonal blocks may be arranged in any prescribed order.

3. Stability Criteria for DDEs and NDDEs

Taking advantage of the upper triangularization for the commuting matrices, we can reduce the condition of asymptotic stability of a system of DDEs into that of a scalar one. As a matter of fact, we obtain the following

THEOREM 3.1. *The system of DDEs (1.1) with the commuting pair of matrices is asymptotically stable if and only if the following scalar equations are asymptotically stable*:

$$\dot{y}(t) = \beta_1^{(k)} y(t) + \beta_2^{(k)} y(t - \tau) \quad \text{for} \quad k = 1, 2, \dots, n \tag{3.1}$$

where $\beta_1^{(k)}$ and $\beta_2^{(k)}$ stand for eigenvalue of A_1 and A_2, respectively. The orders of $\beta_1^{(k)}$ and $\beta_2^{(k)}$ are those which can be taken from the proof of Lemma 2.2.

Proof. Owing to Lemma 2.2 and the condition (1.2), there exists a unitary matrix U satisfying the identities

$$U^{-1}A_1U = T_1 \quad \text{and} \quad U^{-1}A_2U = T_2$$

where T_1 and T_2 are upper triangular matrices whose diagonal elements are $\beta_1^{(k)}$ and $\beta_2^{(k)}$, respectively.

The characteristic equation of the system (1.1) is given by

$$\det[sI - A_1 - A_2 \exp(-s\tau)] = 0.$$

Since the equation

$$\begin{aligned} \det[sI - A_1 - A_2 \exp(-s\tau)] &= \det[sI - T_1 - T_2 \exp(-s\tau)] \\ &= \prod_{k=1}^{n} (s - \beta_1^{(k)} - \beta_2^{(k)} \exp(-s\tau)) = 0 \end{aligned}$$

holds, the characteristic root s should satisfy the equation

$$s - \beta_1^{(k)} - \beta_2^{(k)} \exp(-s\tau) = 0 \qquad (k = 1, 2, \ldots, n),$$

which is the characteristic equation of (3.1). $\square$

REMARK. Even when the condition (1.2) holds, A_1 and A_2 may not be simultaneously diagonalizable. Actually the following lemma is known.

LEMMA 3.1 ([5], p.50). *Let $A_1, A_2 \in \boldsymbol{C}^{n\times n}$ be diagonalizable. Then A_1 and A_2 commute if and only if they are simultaneously diagonalizable.*

Hence the simultaneous diagonalizability is a special case of the simultaneous upper-triangularizability in commutative matrices.

We now consider the asymptotic stability of the system (1.1) with a non-commutative pair of matrices. If the condition $\mathrm{rank}(A_1A_2 - A_2A_1) \le 1$ holds, due to Lemma 2.3 the stability criteria of the system can be reduced to that of n scalar equations the same as in Theorem 3.1. Furthermore we can obtain a broader criterion for non-commutative cases.

THEOREM 3.2. *Suppose A_1 and A_2 have ℓ $(1 \le \ell \le n)$ common eigenvectors. The system (1.1) is asymptotically stable if and only if the following two conditions are satisfied.*

(i) *The ℓ scalar equations*

$$\dot{y}(t) = \beta_1^{(k)} y(t) + \beta_2^{(k)} y(t-\tau) \qquad (k = 1, 2, \ldots, \ell)$$

are asymptotically stable. Here $\beta_1^{(k)}$ and $\beta_2^{(k)}$ are eigenvalues of A_1 and A_2, respectively, corresponding to their ℓ common eigenvectors.

(ii) *The following system is asymptotically stable.*

$$\dot{z}(t) = A_1^{(n-\ell)} z(t) + A_2^{(n-\ell)} z(t-\tau)$$

where $z(t) \in \boldsymbol{C}^{n-\ell}$ and the $(n-\ell)$-square matrices $A_1^{(n-\ell)}$ and $A_2^{(n-\ell)}$ are the restriction of A_1 and A_2, respectively, onto the complementary subspace of $\boldsymbol{C}^n$ of that spanned by the common eigenvectors.

This shows that the stability criteria of the system (1.1) may be reduced to that of lower dimension systems under the existence of eigenvectors common to A_1 and A_2.

Proof. Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_\ell$ be the common eigenvectors of A_1 and A_2. There exists a nonsingular matrix

$$U = [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_\ell, \boldsymbol{w}_{\ell+1}, \ldots, \boldsymbol{w}_n]$$

whose similarity transformation makes

$$U^{-1}A_1U = \begin{bmatrix} \beta_1^{(1)} & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \beta_1^{(\ell)} & * \\ \underline{0} & & & A_1^{(n-\ell)} \end{bmatrix} \quad \text{and} \quad U^{-1}A_2U = \begin{bmatrix} \beta_2^{(1)} & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \beta_2^{(\ell)} & * \\ \underline{0} & & & A_2^{(n-\ell)} \end{bmatrix}.$$

Hence the characteristic equation of the system (1.1) is factorized as

$$\det[sI - A_1 - A_2 \exp(-s\tau)]$$
$$= \left\{ \prod_{k=1}^{\ell} (s - \beta_1^{(k)} - \beta_2^{(k)} \exp -s\tau) \right\} \det[sI^{(n-\ell)} - A_1^{(n-\ell)} - A_2^{(n-\ell)} \exp(-s\tau)] = 0.$$

$I^{(n-\ell)}$ is the $(n-\ell)$-square unit matrix. Thus the desired result follows. □

Theorems 3.1 and 3.2 can be readily extended to the following linear systems of neutral delay differential equations (NDDEs) including several positive constant delays.

$$\dot{x}(t) = A_1 x(t) + \sum_{j=2}^{m} A_j x(t - \tau_j) + \sum_{j=m+1}^{N} A_j \dot{x}(t - \tau_j). \tag{3.2}$$

We shall impose the commuting condition for matrices, *i.e.*

$$A_i A_j = A_j A_i \quad \text{for} \quad i, j = 1, 2, \ldots, N. \tag{3.3}$$

Denote by $\beta_j^{(k)}$ the k-th eigenvalue of the matrix A_j.

THEOREM 3.3. *Suppose that the system* (3.2) *has a commuting family of matrices* $\{A_j\}$. *It is asymptotically stable if and only if the scalar equations*

$$\dot{y}(t) = \beta_1^{(k)} y(t) + \sum_{j=2}^{m} \beta_j^{(k)} y(t - \tau_j) + \sum_{j=m+1}^{N} \beta_j^{(k)} \dot{y}(t - \tau_j) \quad \textit{for} \quad k = 1, 2, \ldots, n$$

are asymptotically stable. The order of $\beta_j^{(k)}$ $(k = 1, 2, \ldots, n)$ *is that stated in Lemma* 2.2.

Proof is parallel with that of Theorem 3.1.

The key for application of Theorem 3.3 is the determination of the order of $\beta_j^{(k)}$ when the commuting condition (3.3) is satisfied. We give an algorithm to determine the order.

Step 1. Seek a common eigenvector $\boldsymbol{z}$ of $\{A_j\}$ $(j = 1, 2, \ldots, N)$, and determine $\beta_j^{(1)}$.

Step 2. Find a nonsingular matrix U_1 so as

$$U_1 = [\boldsymbol{z}, \boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-1}]$$

and

$$U_1^{-1} A_j U_1 = \begin{bmatrix} \beta_j^1 & * \\ 0 & A_j^{(n-1)} \end{bmatrix} \quad \text{for} \quad j = 1, 2, \ldots, N$$

to obtain $(n-1)$-dimensional matrices $\{A_j^{(n-1)}\}$ $(j = 1, 2, \ldots, N)$.

Step 3. Because $\{A_j^{(n-1)}\}$ are again a commuting family, repeat Steps 1 and 2 to obtain $\beta_j^{(2)}$ and $\{A_j^{(n-2)}\}$ $(j = 1, 2, \ldots, N)$. Theoretically we can continue the above process and determine the order of $\beta_j^{(k)}$ $(j = 1, 2, \ldots, N; k = 1, 2, \ldots, n)$.

REMARK. According to Lemma 2.4, if the family $\{A_j\}$ is real and commuting, the stability analysis of (3.2) can be carried out within the real number arithmetic. Even when the condition (3.3) fails, as in Theorem 3.2, the existence of several common eigenvectors for A_j $(j = 1, 2, \ldots, N)$ leads us to the condition for asymptotic stability.

4. Two Examples

We illustrate our method through two numerical examples given below.

Example 1.

$$\dot{x}(t) = A_1 x(t) + A_2 x(t - \tau) \tag{4.1}$$

with

$$A_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since

$$A_1 A_2 = A_2 A_1 = \begin{bmatrix} -2 & 0 & 0 \\ -2 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

A_1 and A_2 are commutative and simultaneously upper-triangularizable. Let $\lambda_i(F)$ represent i-th eigenvalue of a matrix $F \in \boldsymbol{C}^{n \times n}$ for $i = 1, \ldots, n$.

$$\lambda_1(A_1) = \lambda_2(A_1) = \lambda_3(A_1) = -2,$$

$$\lambda_1(A_2) = 1, \ \lambda_2(A_2) = \lambda_3(A_2) = 0.$$

The common eigenvector of A_1 and A_2 corresponding to $\lambda_1(A_1)$ and $\lambda_2(A_2)$, respectively, is $[0, 1, 0]^T$. We can obtain nonsingular U_1 whose first column is $[0, 1, 0]^T$, and have the identities

$$U_1 = U_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$U_1^{-1}A_1U_1 = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \; U_1^{-1}A_2U_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that $\beta_1^{(1)} = \lambda_1(A_1)$ and $\beta_2^{(1)} = \lambda_2(A_2)$. Owing to Theorem 2.1, the system (4.1) is asymptotically stable if and only if the following three scalar equations are asymptotically stable.

$$\dot{y}_1(t) = -2y_1(t), \quad \dot{y}_2(t) = -2y_2(t) + y_2(t-\tau), \quad \dot{y}_3(t) = -2y_3(t)$$

where, y_1, y_2 and $y_3 \in \boldsymbol{R}$. Because all the above three systems are asymptotically stable, the system (4.1) is asymptotically stable.

Example 2.

$$\dot{x}(t) = A_1x(t) + A_2x(t-\tau) \tag{4.2}$$

with

$$A_1 = \begin{bmatrix} 0 & 4 & -4 \\ 3 & 1 & -3 \\ -1 & 1 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}.$$

In this case $A_1A_2 \neq A_2A_1$, and $\operatorname{rank}(A_1A_2 - A_2A_1) = 2$. We can not determine whether A_1 and A_2 are simultaneously triangularizable by Lemma 2.2. Because $[1,1,0]^T$ is the common eigenvector of A_1 and A_2 corresponding to $\lambda_1(A_1) = 4$ and $\lambda_1(A_2) = 2$, respectively ($\beta_1^{(1)} = \lambda_1(A_1), \beta_2^{(1)} = \lambda_1(A_2)$), by Theorem 3.2, we can obtain nonsingular U_1 whose first column is $[1,1,0]^T$ and derive

$$U_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad U_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$U_1^{-1}A_1U_1 = \begin{bmatrix} 4 & 3 & -3 \\ 0 & -3 & -1 \\ 0 & -1 & -3 \end{bmatrix}, \quad U_1^{-1}A_2U_1 = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} &\det[sI - A_1 - A_2\exp(-s\tau)] \\ &\quad = (s - 4 - 2\exp(-s\tau))\det[sI - A_1^{(2)} - A_2^{(2)}\exp(-s\tau)] = 0, \end{aligned}$$

where

$$A_1^{(2)} = \begin{bmatrix} -3 & -1 \\ -1 & -3 \end{bmatrix}, \quad A_2^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

The system (4.2) is asymptotically stable if and only if the following two systems are asymptotically stable.

$$\dot{y}(t) = 4y(t) + 2y(t-\tau) \qquad (y(t) \in \boldsymbol{R}) \tag{4.3}$$

$$\dot{z}(t) = A_1^{(2)} z(t) + A_2^{(2)} z(t-\tau) \qquad (z(t) \in \boldsymbol{R}^2) \tag{4.4}$$

The system (4.2) is unstable for (4.3) and (4.4) are both unstable.

5. Conclusions

Under the condition that parameter matrices of the linear delay differential system have some common eigenvectors, its stability analysis may be reduced to that of lower dimensional systems. For stability analysis of the system we present a method which needs only the evaluation of eigenpairs of the parameter matrices.

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