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Inertial manifolds for semi-linear parabolic equations in admissible spaces ☆

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ABSTRACT

We prove the existence of inertial manifolds for the solutions to the semi-linear parabolic equation $\frac{du(t)}{dt} + Au(t) = f(t,u)$ when the partial differential operator A is positive definite and self-adjoint with a discrete spectrum having a sufficiently large distance between some two successive points of the spectrum, and the nonlinear forcing term f satisfies the φ-Lipschitz conditions on the domain $D(A^\theta)$, $0 \leqslant \theta < 1$, i.e., $\|f(t,x) - f(t,y)\| \leqslant \varphi(t)\|A^\theta(x-y)\|$ and $\|f(t,x)\| \leqslant \varphi(t)(1 + \|A^\theta x\|)$ where $\varphi(t)$ belongs to one of admissible function spaces containing wide classes of function spaces like L_p-spaces, the Lorentz spaces $L_{p,q}$ and many other function spaces occurring in interpolation theory.

1. Introduction

Consider the semi-linear parabolic equation of the form

$$\frac{dx}{dt} + Ax(t) = f\big(t, x(t)\big), \quad t > s,\ x(s) = x_s,\ s \in \mathbb{R}, \tag{1.1}$$

where A is in general an unbounded linear operator on a Hilbert space X and $f : \mathbb{R} \times X_\theta \to X$ is a nonlinear operator with $X_\theta := D(A^\theta)$ being the domain of the fractional power A^θ for $0 \leqslant \theta < 1$.

One of important directions of research regarding the asymptotic behavior of solutions to Eq. (1.1) is to find conditions for this equation to have an integral manifold (e.g., a stable, unstable, or center manifold). Such early results can be traced back to Hadamard [11], Perron [23,24], Bogoliubov and Mitropolsky [2,3] for the case of matrix coefficients $A(t)$, to Daleckii and Krein [9] for the case of bounded coefficients acting on Banach spaces, and to Henry [12] for the case of unbounded coefficients (see also [1,5,6,15,21,26] and references therein for more information on the matter).

The methods and results on invariant manifolds have been used to derive the notion of inertial manifolds and to obtain their existence and properties (see [7,8,18,26,27] and references therein). The importance of the discovery of inertial manifolds is that such manifolds are of finite dimensions and exponentially attract all solutions of the evolution equation under consideration. This allows to apply the reduction principles to consider the asymptotic behavior of the partial differential equation by determining the structures of its induced solutions belonging to these inertial manifolds, which turn out to be solutions to some induced ordinary differential equations.

To our best knowledge, the most popular conditions for the existence of inertial manifolds are the spectral gap condition of the linear operator A and the uniform Lipschitz continuity of the nonlinear term $f(t,x)$ (i.e., $\|f(t,x) - f(t,y)\| \leqslant q\|x-y\|$

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for a Lipschitz constant q independent of t) (see [7,8,18,20,26]). However, for equations arising in complicated reaction–diffusion processes, the Lipschitz coefficients may depend on time and the restricted spectral gap condition may not be fulfilled. Therefore, one tries to extend the conditions on the operator A and the nonlinear term so that they describe more exactly such processes. Recently, we have obtained exciting results in [14], where we have used the Lyapunov–Perron method and the characterization of the exponential dichotomy (obtained in [13]) of evolution equations in admissible function spaces to construct the structures of solutions of Eq. (1.1) in a mild form, which belong to some certain classes of admissible spaces on which we could implement some well-known procedures in functional analysis such as: constructing of contraction mapping; using of Implicit Function Theorem, etc. The use of admissible spaces has helped us to construct the invariant manifolds for Eq. (1.1) in the case of dichotomic linear parts without using the smallness of Lipschitz constants of nonlinear forcing terms in classical sense. Consequently, we have obtained the existence of invariant stable manifolds for the case of dichotomic linear parts under very general conditions on the nonlinear term $f(t,x)$ (see [14]).

The purpose of the present paper is to establish the existence of inertial manifolds under two conditions. Firstly, the linear operator A is positive definite and self-adjoint with a discrete spectrum having a sufficiently large distance between some two successive points of the spectrum, which can be considered as a generalization of the restricted spectral gap conditions as in [7,8,20,26,27]; and secondly, the nonlinear term $f(t,x)$ is non-uniformly Lipschitz continuous on some interpolation space, i.e., $\|f(t,x)-f(t,y)\| \leqslant \varphi(t)\|A^\theta(x-y)\|$ for φ being a real and positive function which belongs to an admissible function space defined in Definition 2.4 below and $0 \leqslant \theta < 1$. Under some conditions on φ, we will prove the existence of inertial manifolds for Eq. (1.1) provided that the linear partial differential operator A has a generalized spectral gap condition. Moreover, in case $\theta = 0$, we obtain the existence of such manifolds without the generalized spectral gap condition (in fact, what we need is a gap in the spectrum and the smallness of the norm of φ in some admissible spaces, e.g., the smallness of $\sup_{t\in\mathbb{R}} \int_{t-1}^t \varphi(\tau)\,d\tau$).

Our method is to use some weighted (or rescaling) function spaces to obtain some dichotomy estimates, and then to apply our techniques in [14] (see also [13,15]) of using admissibility of function spaces to construct the solutions of Lyapunov–Perron equation which will be used to derive the existence of inertial manifolds. Our main results are contained in Lemma 3.4, Theorem 3.5. We also illustrate our results in Example 3.7.

2. Preliminaries

We now recall some notions. Let X be a separable Hilbert space and suppose that A is a closed linear operator on X satisfying the following standing hypothesis.

Standing Hypothesis 2.1. We suppose that A is a positive definite, self-adjoint operator with a discrete spectrum, say

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \text{each with finite multiplicity} \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and assume that $\{e_k\}_{k=1}^\infty$ is the orthonormal basis in X consisting of the corresponding eigenfunctions of the operator A (i.e., $Ae_k = \lambda_k e_k$). Let now λ_N and λ_{N+1} be two successive and different eigenvalues with $\lambda_N < \lambda_{N+1}$, let further P be the orthogonal projection onto the first N eigenvectors of the operator A.

Denote by $(e^{-tA})_{t\geqslant 0}$ the semigroup generated by $-A$. Since $\operatorname{Im} P$ is of finite dimension, we have that the restriction $(e^{-tA}P)_{t\geqslant 0}$ of the semigroup $(e^{-tA})_{t\geqslant 0}$ to $\operatorname{Im} P$ can be extended to the whole line $\mathbb{R}$.

For $\theta > 0$ we then recall the following dichotomy estimates (see [26]):

$$\begin{aligned}
&\|e^{-tA}P\| \leqslant Me^{\lambda_N|t|}, \quad t\in\mathbb{R} \text{ for some constant } M \geqslant 1,\\
&\|A^\theta e^{-tA}P\| \leqslant \lambda_N^\theta Me^{\lambda_N|t|}, \quad t\in\mathbb{R},\\
&\|e^{-tA}(I-P)\| \leqslant Me^{-\lambda_{N+1}t}, \quad t\geqslant 0,\\
&\|A^\theta e^{-tA}(I-P)\| \leqslant M\left[\left(\frac{\theta}{t}\right)^\theta + \lambda_{N+1}^\theta\right]e^{-\lambda_{N+1}t}, \quad t>0.
\end{aligned} \tag{2.1}$$

We next recall some notions on function spaces and refer to Massera and Schäffer [19], Räbiger and Schnaubelt [25] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\,dt$, where $J_n = [n, n+1]$ for each $n\in\mathbb{Z}$ (see [19, Chapter 2, §20]).

We can now define Banach function spaces as follows.

Definition 2.2. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called *a Banach function space* (over $(\mathbb{R},\mathcal{B},\lambda)$) if

(1) E is a Banach lattice with respect to the norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,
(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0$,
(3) $E \hookrightarrow L_{1,loc}(\mathbb{R})$.

We remark that condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}$ there exists a number $\beta_J \geqslant 0$ such that $\int_J |f(t)|\,dt \leqslant \beta_J \|f\|_E$ for all $f \in E$.

We state the following trivial lemma which will be frequently used in our strategy.

Lemma 2.3. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}$ such that they coincide outside a compact interval and they are essentially bounded* (*in particular, continuous*) *on this compact interval. Then $\varphi \in E$ if and only if $\psi \in E$.*

We now introduce the notion of admissibility in the following definition.

Definition 2.4. The Banach function space E is called *admissible* if it satisfies

(i) there is a constant $M \geqslant 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\int_a^b \left|\varphi(t)\right| dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{2.2}$$

(ii) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_{t-1}^t \varphi(\tau)\,d\tau$ belongs to E.
(iii) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined, for $\tau \in \mathbb{R}$, by

$$\begin{aligned} &T_\tau^+\varphi(t) := \varphi(t-\tau) \quad \text{for } t \in \mathbb{R}, \\ &T_\tau^-\varphi(t) := \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{2.3}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leqslant N_1$, $\|T_\tau^-\| \leqslant N_2$ for all $\tau \in \mathbb{R}$.

Example 2.5. Besides the spaces $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ (see [19, Chapter 2, Thm. 23.V]), and the space

$$\mathbf{M}(\mathbb{R}) := \left\{ f \in L_{1,loc}(\mathbb{R}):\ \sup_{t\in\mathbb{R}} \int_{t-1}^t \left|f(\tau)\right| d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\in\mathbb{R}} \int_{t-1}^t |f(\tau)|\,d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty$ (see [4, Thm. 3 and p. 284], [28, 1.18.6, 1.19.3]) and, more generally, the class of rearrangement invariant function spaces over $(\mathbb{R}, \mathcal{B}, \lambda)$ (see [16, 2.a]) are admissible.

For the reader's convenience we now prove that the property (ii) in Definition 2.4 holds true for the space $L_p(\mathbb{R})$ with $1 < p < \infty$. Indeed, for $\varphi \in L_p(\mathbb{R})$ we now prove that $\int_{\mathbb{R}} |\int_{t-1}^t \varphi(\tau)\,d\tau|^p\,dt < \infty$. To do this, we will estimate the sum

$$\sum_{n=-\infty}^{\infty} \int_n^{n+1} \left| \int_{t-1}^t \varphi(\tau)\,d\tau \right|^p dt.$$

Using Hölder Inequality we obtain $|\int_{t-1}^t \varphi(\tau)\,d\tau|^p \leqslant \int_{t-1}^t |\varphi(\tau)|^p\,d\tau$. For each $n \in \mathbb{Z}$, it follows from Fubini Theorem that

$$\begin{aligned} \int_n^{n+1} \int_{t-1}^t \left|\varphi(\tau)\right|^p d\tau\,dt &= \int_{n-1}^n \int_n^{\tau+1} \left|\varphi(\tau)\right|^p dt\,d\tau + \int_n^{n+1} \int_\tau^{n+1} \left|\varphi(\tau)\right|^p dt\,d\tau \\ &= \int_{n-1}^n (\tau+1-n)\left|\varphi(\tau)\right|^p d\tau + \int_n^{n+1} (n+1-\tau)\left|\varphi(\tau)\right|^p d\tau \\ &\leqslant \int_{n-1}^n \left|\varphi(\tau)\right|^p d\tau + \int_n^{n+1} \left|\varphi(\tau)\right|^p d\tau. \end{aligned}$$

Therefore, we obtain

$$\sum_{n=-\infty}^{\infty}\int_{n}^{n+1}\left|\int_{t-1}^{t}\varphi(\tau)\,d\tau\right|^{p}dt \leqslant \sum_{n=-\infty}^{\infty}\int_{n-1}^{n}\left|\varphi(\tau)\right|^{p}d\tau + \sum_{n=-\infty}^{\infty}\int_{n}^{n+1}\left|\varphi(\tau)\right|^{p}d\tau$$
$$= 2\int_{\mathbb{R}}\left|\varphi(\tau)\right|^{p}d\tau < \infty.$$

This follows that $\int_{\mathbb{R}}|\int_{t-1}^{t}\varphi(\tau)\,d\tau|^{p}\,dt < \infty$.

Remark 2.6. If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R})$. Indeed, put $\beta := \inf_{t\in\mathbb{R}}\|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.2(2)). Then, from Definition 2.4(i) we derive

$$\int_{t-1}^{t}\left|\varphi(\tau)\right|d\tau \leqslant \frac{M}{\beta}\|\varphi\|_E \quad \text{for all } t\in\mathbb{R} \text{ and } \varphi\in E. \tag{2.4}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R})$ and $\|\varphi\|_{\mathbf{M}} \leqslant \frac{M}{\beta}\|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R})$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [13, Proposition 2.6] and originally in [19, 23.V(1)]).

Proposition 2.7. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,loc}(\mathbb{R})$ be such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is defined as in Definition 2.4(ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)\,ds,$$
$$\Lambda''_\sigma\varphi(t) := \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\in\mathbb{R}}\int_{t-1}^{t}\varphi(\tau)\,d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.6)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, the following estimates hold:

$$\left\|\Lambda'_\sigma\varphi\right\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \quad \text{and} \quad \left\|\Lambda''_\sigma\varphi\right\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \tag{2.5}$$

where constants N_1, N_2 are defined in Definition 2.4.

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) = e^{b|t|}$ for $t \in \mathbb{R}$ and any fixed constant $b > 0$.*

Remark 2.8. If we replace the whole line $\mathbb{R}$ by a half-infinite interval $(-\infty, t_0]$ for any fixed $t_0 \in \mathbb{R}$, then we have the similar notions of admissible spaces of functions defined on $(-\infty, t_0]$ with slight changes as follow:

(1) In Definition 2.4, the translation semigroups T^+_τ and T^-_τ for $\tau \in \mathbb{R}$ should be replaced by T^+_τ and T^-_τ defined for $\tau \leqslant t_0$ and $t \leqslant t_0$ as

$$T^+_\tau\varphi(t) := \begin{cases}\varphi(t-\tau) & \text{for } t \leqslant \tau + t_0,\\ 0 & \text{for } t > t_0 + \tau,\end{cases}$$
$$T^-_\tau\varphi(t) := \begin{cases}\varphi(t+\tau) & \text{for } t \leqslant t_0 - \tau,\\ 0 & \text{for } t > t_0 - \tau.\end{cases} \tag{2.6}$$

(2) In Proposition 2.7(a), the functions Λ'_σ and Λ''_σ should be replaced by

$$\Lambda_\sigma^{t_0}\varphi(t) \quad := \int_t^{t_0} e^{-\sigma(s-t)}\varphi(s)\,ds,$$

$$\Lambda_\sigma^{-\infty}\varphi(t) := \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)\,ds$$

for $t \leqslant t_0$.

(3) In Proposition 2.7(b) and (c) the functions $\psi(t) = e^{-\alpha|t|}$ ($t \in \mathbb{R}$, and fixed $\alpha > 0$) should be replaced by $\psi(t) = e^{\alpha t}$, $t \leqslant t_0$, and fixed $\alpha > 0$; and the functions $f(t) := e^{b|t|}$ ($t \in \mathbb{R}$, and any fixed constant $b > 0$) should be replaced by $f(t) := e^{-bt}$, $t \leqslant t_0$ and fixed $b > 0$.

We denote the admissible function space of the functions defined on $(-\infty, t_0]$ by $E_{(-\infty,t_0]}$. For a function φ defined on the whole line we denote the restriction of φ on $(-\infty, t_0]$ by $\varphi|_{(-\infty,t_0]}$. It is obvious that, if the function $\varphi \in E$, then $\varphi|_{(-\infty,t_0]} \in E_{(-\infty,t_0]}$.

In the case of infinite-dimensional phase spaces, instead of Eq. (1.1), we consider the integral equation

$$u(t) = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\xi)A} f\bigl(\xi, u(\xi)\bigr)\,d\xi \quad \text{for a.e. } t \geqslant s. \tag{2.7}$$

By a *solution* of (2.7) we mean a *strongly measurable* function $u(t)$ defined on an interval J with the values in X_θ that satisfies Eq. (2.7) for $t, s \in J$. We note that the solution u to Eq. (2.7) is called a mild solution of Eq. (1.1).

We refer the reader to Pazy [22] for more detailed treatment on the relations between classical and mild solutions of evolution equations (see also [7,10,17,26]).

To obtain the existence of an inertial manifold for Eq. (2.7), besides the assumptions on the operator A, we also need the φ-Lipschitz property of the nonlinear term f in the following definition.

Definition 2.9 *(φ-Lipschitz functions).* Let E be an admissible Banach function space on $\mathbb{R}$ and φ be a positive function belonging to E. Put $X_\theta := D(A^\theta)$ for $\theta \in [0, 1)$. Then, a function $f : \mathbb{R} \times X_\theta \to X$ is said to be φ-Lipschitz if f satisfies

(i) $\|f(t, x)\| \leqslant \varphi(t)(1 + \|A^\theta x\|)$ for a.e. $t \in \mathbb{R}$ and all $x \in X_\theta$;
(ii) $\|f(t, x_1) - f(t, x_2)\| \leqslant \varphi(t)\|A^\theta(x_1 - x_2)\|$ for a.e. $t \in \mathbb{R}$ and all $x_1, x_2 \in X_\theta$.

3. Inertial manifolds

In this section we will prove the existence of the inertial manifolds for solutions to Eq. (2.7). We suppose that A satisfies Standing Hypothesis 2.1 and recall that P is the orthogonal projection onto the first N orthonormal eigenvectors of A. We then make precisely the notion of inertial manifolds in the following definition.

Definition 3.1. An *inertial manifold* of Eq. (2.7) is a collection of Lipschitz surfaces $(\mathbb{M}_t)_{t\in\mathbb{R}}$ in X such that each $\mathbb{M}_t$ is the graph of a Lipschitz function $\Phi_t : \operatorname{Im} P \to (I - P)X_\theta$, i.e.,

$$\mathbb{M}_t = \{x + \Phi_t x \mid x \in \operatorname{Im} P\} \quad \text{for } t \in \mathbb{R},$$

and the following conditions are satisfied:

(i) The Lipschitz constants of Φ_t are independent of t, i.e., there exists a constant C independent of t such that

$$\bigl\|A^\theta(\Phi_t x_1 - \Phi_t x_2)\bigr\| \leqslant C\bigl\|A^\theta(x_1 - x_2)\bigr\|.$$

(ii) There exists $\gamma > 0$ such that to each $x_0 \in \mathbb{M}_{t_0}$ there corresponds one and only one solution $u(t)$ to Eq. (2.7) on $(-\infty, t_0]$ satisfying that $u(t_0) = x_0$ and

$$\operatorname*{esssup}_{t\leqslant t_0}\bigl\|e^{-\gamma(t_0-t)}A^\theta u(t)\bigr\| < \infty. \tag{3.1}$$

(iii) $(\mathbb{M}_t)_{t\in\mathbb{R}}$ is positively invariant under Eq. (2.7), i.e., if a solution $x(t)$, $t \geqslant s$, of Eq. (2.7) satisfies $x(s) \in \mathbb{M}_s$, then we have that $x(t) \in \mathbb{M}_t$ for $t \geqslant s$.

(iv) $(\mathbb{M}_t)_{t\in\mathbb{R}}$ exponentially attracts all the solutions to Eq. (2.7), i.e., for any solution $u(\cdot)$ of Eq. (2.7) and any fixed $s \in \mathbb{R}$, there is a positive constant H such that

$$\operatorname{dist}_{X_\theta}\bigl(u(t), \mathbb{M}_t\bigr) \leqslant He^{-\gamma(t-s)} \quad \text{for } t \geqslant s,$$

where γ is the same constant as the one in item (ii), and dist_{X_θ} denotes the Hausdorff semi-distance generated by the norm in X_θ.

Let A satisfy Standing Hypothesis 2.1. Then, we can define the Green function as follows:

$$G(t,\tau) := \begin{cases} e^{-(t-\tau)A}[I-P] & \text{for } t > \tau, \\ -e^{-(t-\tau)A}P & \text{for } t \leqslant \tau. \end{cases} \tag{3.2}$$

Then, one can see that $G(t,s)$ maps X into X_θ. Also, by the dichotomy estimates (2.1) and for $\gamma = (\lambda_N + \lambda_{N+1})/2$ we have

$$\left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \leqslant K(t,\tau) e^{-\alpha|t-\tau|} \quad \text{for all } t,\tau \in \mathbb{R} \tag{3.3}$$

where $\alpha = (\lambda_{N+1} - \lambda_N)/2$ and

$$K(t,\tau) = \begin{cases} M\left(\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right) & \text{if } t > \tau, \\ M\lambda_N^\theta & \text{if } t \leqslant \tau. \end{cases}$$

We can now construct the form of the solutions of Eq. (2.7) which are rescaledly bounded on the half-line $(-\infty, t_0]$ in the following lemma.

Lemma 3.2. *Let the operator A satisfy Standing Hypothesis 2.1 and $f : \mathbb{R} \times X_\theta \to X$ be φ-Lipschitz for a positive function φ belonging to an admissible space E such that*

$$R(\varphi,\theta) := \sup_{t\in\mathbb{R}} \left(\int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}} \, d\tau \right)^{\frac{2\theta}{1+\theta}} < \infty. \tag{3.4}$$

For a fixed $t_0 \in \mathbb{R}$ let $x(t)$, $t \leqslant t_0$, be a solution of (2.7) such that $x(t) \in X_\theta$ for $t \leqslant t_0$ and

$$\operatorname*{esssup}_{t \leqslant t_0} \left\| e^{-\gamma(t_0-t)} A^\theta x(t) \right\| < \infty, \quad \text{where } \gamma \text{ is defined as in (3.3).}$$

Then, this solution $x(t)$ satisfies

$$x(t) = e^{-(t-t_0)A} v_1 + \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big) \, d\tau \quad \text{for a.e. } t \leqslant t_0, \tag{3.5}$$

where $v_1 \in PX$, and $G(t,\tau)$ is the Green's function defined as in (3.2).

Proof. Put

$$y(t) := \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big) \, d\tau \quad \text{for } t \leqslant t_0. \tag{3.6}$$

By the definition of $G(t,\tau)$, we have that $y(t) \in X_\theta$ for $t \leqslant t_0$. We then estimate $\|A^\theta e^{-\gamma(t_0-t)} y(t)\|$.

Indeed, since f is φ-Lipschitz, using estimate (3.3) we obtain

$$\begin{aligned} \left\| A^\theta e^{-\gamma(t_0-t)} y(t) \right\| &\leqslant \int_{-\infty}^{t_0} \left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \varphi(\tau) e^{-\gamma(t_0-\tau)} \big(1 + \left\| A^\theta x(\tau) \right\|\big) \, d\tau \\ &\leqslant \int_{-\infty}^{t_0} \left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \varphi(\tau) \, d\tau \left(1 + \operatorname*{esssup}_{t \leqslant t_0} \left\| e^{-\gamma(t_0-t)} A^\theta x(t) \right\| \right) \quad \text{for } t \leqslant t_0. \end{aligned} \tag{3.7}$$

Using (2.5) and (3.3) we estimate the integral

$$\begin{aligned} \int_{-\infty}^{t_0} \left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \varphi(\tau) \, d\tau &\leqslant \int_{-\infty}^{t} M\left(\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta \right) e^{-\alpha(t-\tau)} \varphi(\tau) \, d\tau + \int_{t}^{t_0} M\lambda_N^\theta e^{-\alpha(\tau-t)} \varphi(\tau) \, d\tau \\ &\leqslant \int_{-\infty}^{t} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)} \varphi(\tau) \, d\tau + \frac{M(\lambda_{N+1}^\theta N_1 + \lambda_N^\theta N_2)}{1 - e^{-\alpha}} \|\Lambda_1 \varphi\|_\infty \end{aligned} \tag{3.8}$$

where α is as in (3.3).

The first integral on the right-hand side is now estimated for $0 < \theta < 1$ as follows

$$\begin{aligned}
&\int_{-\infty}^{t} M\left(\frac{\theta}{t-\tau}\right)^{\theta} e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau \\
&\quad = \int_{-\infty}^{t-1} M\left(\frac{\theta}{t-\tau}\right)^{\theta} e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau + \int_{t-1}^{t} M\left(\frac{\theta}{t-\tau}\right)^{\theta} e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau \\
&\quad \leqslant \int_{-\infty}^{t-1} M\theta^{\theta} e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau + M\theta^{\theta}\left(\int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau\right)^{\frac{2\theta}{1+\theta}} \left(\int_{t-1}^{t} e^{-\alpha\frac{1+\theta}{1-\theta}(t-\tau)}\,d\tau\right)^{\frac{1-\theta}{1+\theta}} \\
&\quad \leqslant \frac{M\theta^{\theta}N_1}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty + M\theta^{\theta}R(\varphi,\theta)\left(\frac{1-\theta}{(1+\theta)\alpha}\right)^{\frac{1-\theta}{1+\theta}}
\end{aligned}$$

(here we have used the Hölder inequality for the second term on the right-hand side), also for $\theta = 0$ we have

$$\int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^{\theta}G(t,\tau)\right\|\varphi(\tau)\,d\tau = \int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}G(t,\tau)\right\|\varphi(\tau)\,d\tau \leqslant \frac{M(N_1+N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty.$$

Substituting above inequalities into (3.8) we obtain

$$\int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^{\theta}G(t,\tau)\right\|\varphi(\tau)\,d\tau \leqslant k \quad \text{for } t \leqslant t_0 \tag{3.9}$$

where

$$k = \begin{cases} \frac{M(\theta^{\theta}N_1+\lambda_{N+1}^{\theta}N_1+\lambda_N^{\theta}N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty + M\theta^{\theta}R(\varphi,\theta)(\frac{1-\theta}{(1+\theta)\alpha})^{\frac{1-\theta}{1+\theta}} & \text{for } 0<\theta<1, \\ \frac{M(N_1+N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty & \text{for } \theta = 0. \end{cases} \tag{3.10}$$

Now, substituting this estimate to (3.7) we have that

$$\operatorname*{esssup}_{t\leqslant t_0}\left\|A^{\theta}e^{-\gamma(t_0-t)}y(t)\right\| \leqslant k\left(1+\operatorname*{esssup}_{t\leqslant t_0}\left\|e^{-\gamma(t_0-t)}A^{\theta}x(t)\right\|\right) < \infty.$$

Next, by computing directly we will verify that $y(\cdot)$ satisfies the integral equation

$$y(t_0) = e^{-(t_0-t)A}y(t) + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau \quad \text{for } t \leqslant t_0. \tag{3.11}$$

Indeed, substituting y from (3.6) to the right-hand side of (3.11) we obtain

$$\begin{aligned}
&e^{-(t_0-t)A}y(t) + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau \\
&\quad = e^{-(t_0-t)A}\int_{-\infty}^{t_0} G(t,\tau)f\big(\tau,x(\tau)\big)\,d\tau + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau \\
&\quad = e^{-(t_0-t)A}\int_{-\infty}^{t} e^{-(t-\tau)A}(I-P)f\big(\tau,x(\tau)\big)\,d\tau - e^{-(t_0-t)A}\int_{t}^{t_0} e^{-(t-\tau)A}Pf\big(\tau,x(\tau)\big)\,d\tau + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau \\
&\quad = \int_{-\infty}^{t} e^{-(t_0-\tau)A}(I-P)f\big(\tau,x(\tau)\big)\,d\tau - \int_{t}^{t_0} e^{-(t_0-t)A}e^{-(t-\tau)A}Pf\big(\tau,x(\tau)\big)\,d\tau + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau
\end{aligned}$$

$$= \int_{-\infty}^{t_0} e^{-(t_0-\tau)A}(I-P) f\big(\tau, x(\tau)\big)\,d\tau = \int_{-\infty}^{t_0} G(t_0,\tau) f\big(\tau, x(\tau)\big) = y(t_0),$$

here we use the fact that $e^{-(t_0-t)A}e^{-(t-\tau)A}P = e^{-(t_0-\tau)A}P$ for all $t \leqslant \tau \leqslant t_0$.

Thus, we have that (3.11) is satisfied.

On the other hand,

$$x(t_0) = e^{-(t_0-t)A}x(t) + \int_t^{t_0} e^{-(t_0-\tau)A} f\big(\tau, x(\tau)\big)\,d\tau.$$

Then $x(t_0) - y(t_0) = e^{-(t_0-t)A}[x(t) - y(t)]$. We need to prove that $x(t_0) - y(t_0) \in PX$.

Applying the operator $A^\theta(I-P)$ to the expression $x(t_0) - y(t_0) = e^{-(t_0-t)A}[x(t) - y(t)]$, we have

$$\begin{aligned}\big\|A^\theta(I-P)\big[x(t_0) - y(t_0)\big]\big\| &= \big\|e^{-(t_0-t)A}A^\theta(I-P)\big[x(t) - y(t)\big]\big\| \\ &\leqslant M e^{-(\lambda_{N+1}-\gamma)(t_0-t)}\|I-P\|\big\|e^{-\gamma(t_0-t)}A^\theta\big(x(t) - y(t)\big)\big\|.\end{aligned}$$

Since $\|e^{-\gamma(t_0-t)}A^\theta(x(t) - y(t))\| < \infty$, letting $t \to -\infty$ we obtain

$$\big\|A^\theta(I-P)\big[x(t_0) - y(t_0)\big]\big\| = 0, \quad \text{hence } A^\theta(I-P)\big[x(t_0) - y(t_0)\big] = 0.$$

Since A^θ is injective, it follows that $(I-P)[x(t_0) - y(t_0)] = 0$. Thus, $v_1 := x(t_0) - y(t_0) \in PX$. Using the fact that the restriction of $e^{-(t_0-t)A}$ on PX is invertible with the inverse $e^{-(t-t_0)A}$ we obtain

$$x(t) = e^{-(t-t_0)A}v_1 + y(t) = e^{-(t-t_0)A}v_1 + \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\,d\tau \quad \text{for } t \leqslant t_0,$$

finishing the proof. □

Remark 3.3.

(i) Eq. (3.5) is called Lyapunov–Perron equation which will be used to determine the inertial manifold for Eq. (2.7). By computing directly, we can see that the converse of Lemma 3.2 is also true. This means that all solutions of Eq. (3.5) satisfy Eq. (2.7) for $t \leqslant t_0$.

(ii) We note that any positive and strongly measurable function φ with

$$H := \operatorname*{ess\,sup}_{t\in\mathbb{R}} \operatorname*{ess\,sup}_{\tau\in[t-1,t]} \varphi(\tau) < \infty$$

(e.g., $\varphi \in L_\infty(\mathbb{R})$) will satisfy condition (3.4) since

$$\sup_{t\in\mathbb{R}} \int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau \leqslant \frac{2H^{\frac{1+\theta}{2\theta}}}{1-\theta}.$$

We now introduce another function $\varphi(t)$ defined for a constant $c > 1$ by

$$\varphi(t) = \begin{cases} |n|^{\frac{2\theta}{1+\theta}} & \text{if } t \in [\frac{6n+1}{2} - \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}, \frac{6n+1}{2} + \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}] \quad \text{for } n = 0, \pm1, \pm2, \ldots, \\ 0 & \text{otherwise.}\end{cases}$$

Here we note that values of φ are arbitrarily large but we still have

$$\sup_{t\in\mathbb{R}} \int_{t-1}^{t} \varphi(\tau)\,d\tau \leqslant \sup_{n\in\mathbb{Z}} \int_{\frac{6n+1}{2} - \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}}^{\frac{6n+1}{2} + \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}} |n|^{\frac{2\theta}{1+\theta}}\,dt = \sup_{n\in\mathbb{Z}} \frac{|n|^{\frac{2\theta}{1+\theta}}}{2^{\frac{2|n|}{1-\theta}+c-1}} \leqslant \frac{2}{2^{c-1}(1-\theta)}.$$

Therefore, $\varphi \in \mathbf{M}(\mathbb{R})$ which is an admissible space. We next check the condition (3.4). Indeed, since the interval $I_n = [\frac{6n+1}{2} - \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}, \frac{6n+1}{2} + \frac{1}{2^{\frac{2|n|}{1-\theta}+c}}]$ has length $\frac{1}{2^{\frac{2|n|}{1-\theta}+c-1}} < 1$, it follows that I_n can contain only one of the points t or $t-1$. Moreover, since the distance between the two centers of two successive intervals I_n is equal to 3, the interval $[t-1,t]$ cannot overlap two intervals I_n. Therefore, for each $t \in \mathbb{R}$ there are three possibilities: $t \in I_n$ or $t-1 \in I_n$ or $I_n \subset [t-1,t]$. Hence, we obtain

$$\int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau$$

$$\leqslant \max\left\{ \int_{\frac{6n+1}{2}-\frac{1}{2^{\frac{2|n|}{1-\theta}+c}}}^{t} \frac{|n|}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau,\ \int_{t-1}^{\frac{6n+1}{2}+\frac{1}{2^{\frac{2|n|}{1-\theta}+c}}} \frac{|n|}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau,\ \int_{\frac{6n+1}{2}-\frac{1}{2^{\frac{2|n|}{1-\theta}+c}}}^{\frac{6n+1}{2}+\frac{1}{2^{\frac{2|n|}{1-\theta}+c}}} \frac{|n|}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau \right\}$$

$$\leqslant \frac{2|n|}{(1-\theta)2^{|n|}2^{\frac{(1-\theta)(c-1)}{2}}} \leqslant \frac{2}{(1-\theta)2^{\frac{(1-\theta)(c-1)}{2}}}.$$

We thus obtain

$$\sup_{t\in\mathbb{R}} \int_{t-1}^{t} \frac{\varphi(\tau)^{\frac{1+\theta}{2\theta}}}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau \leqslant \frac{2}{(1-\theta)2^{\frac{(1-\theta)(c-1)}{2}}}.$$

Therefore, the function φ satisfies condition (3.4).

We then have the following lemma which describes the existence and uniqueness of certain solutions belonging to weighted spaces.

Lemma 3.4. *Let the operator A satisfy Standing Hypothesis 2.1. Let $f : \mathbb{R} \times X_\theta \to X$ be φ-Lipschitz with the positive function φ satisfying the condition (3.4). Put*

$$k := \begin{cases} \frac{M(\theta^\theta N_1+\lambda_{N+1}^\theta N_1+\lambda_N^\theta N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty + M\theta^\theta R(\varphi,\theta)\left(\frac{1-\theta}{(1+\theta)\alpha}\right)^{\frac{1-\theta}{1+\theta}} & \text{for } 0<\theta<1, \\ \frac{M(N_1+N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty & \text{for } \theta=0, \end{cases} \tag{3.12}$$

where $\lambda_N < \lambda_{N+1}$ are two successive eigenvalues of A, $\alpha = (\lambda_{N+1}-\lambda_N)/2$, and $R(\varphi,\theta)$ is defined as in (3.4). Then, if $k<1$, there corresponds to each $v \in PX$ one and only one solution $x(t)$ of Eq. (3.5) on $(-\infty, t_0]$ satisfying the condition $Px(t_0) = v$ and $\operatorname{esssup}_{t\leqslant t_0} e^{-\gamma(t_0-t)}\|A^\theta x(t)\| < \infty$.

Proof. Denote

$$\begin{aligned} L_\infty^{\gamma,t_0,\theta} &= L_\infty^\gamma\big((-\infty,t_0],X_\theta\big) \\ &:= \Big\{h:(-\infty,t_0]\to X_\theta \;\Big|\; h \text{ is strongly measurable and } \operatorname*{esssup}_{t\leqslant t_0} e^{-\gamma(t_0-t)}\big\|A^\theta h(t)\big\| < \infty\Big\} \end{aligned}$$

endowed with the norm $\|h\|_{\gamma,\theta,\infty} := \operatorname{esssup}_{t\leqslant t_0} e^{-\gamma(t_0-t)}\|A^\theta h(t)\|$.

For each $t_0 \in \mathbb{R}$ and $v \in PX$ we will prove that the transformation T defined by

$$(Tx)(t) = e^{-(t-t_0)A}v + \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\,d\tau \quad \text{for } t\leqslant t_0$$

acts from $L_\infty^{\gamma,t_0,\theta}$ into itself and is a contraction.

In fact, for $x(\cdot) \in L_\infty^{\gamma,t_0,\theta}$, we have that $\|f(t,x(t))\| \leqslant \varphi(t)(1+\|A^\theta x(t)\|)$. Therefore, putting

$$y(t) = e^{-(t-t_0)A}v + \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\,d\tau \quad \text{for } t\leqslant t_0$$

we derive that

$$\begin{aligned} \big\|A^\theta e^{-\gamma(t_0-t)}y(t)\big\| &\leqslant \lambda_N^\theta M e^{-(\gamma-\lambda_N)(t_0-t)}\|v\| \\ &\quad + \int_{-\infty}^{t_0} \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)e^{-\gamma(t_0-\tau)}\big(1+\big\|A^\theta x(\tau)\big\|\big)\,d\tau \quad \text{for all } t\leqslant t_0. \end{aligned}$$

Using the estimate (3.9) we obtain

$$\left\| A^\theta e^{-\gamma(t_0-t)} y(t) \right\| \leqslant \lambda_N^\theta M\|v\| + k\Big(1 + \operatorname*{esssup}_{\tau\leqslant t_0}\left\| e^{-\gamma(t_0-\tau)} A^\theta x(\tau)\right\|\Big) \quad \text{for all } t \leqslant t_0.$$

It follows that $y(\cdot) \in L_\infty^{\gamma,t_0,\theta}$ and

$$\left\| y(\cdot)\right\|_{\gamma,\theta,\infty} \leqslant \lambda_N^\theta M\|v\| + k\Big(1 + \operatorname*{esssup}_{t\leqslant t_0}\left\| e^{-\gamma(t_0-t)} A^\theta x(t)\right\|\Big).$$

Therefore, the transformation T acts from $L_\infty^{\gamma,t_0,\theta}$ to $L_\infty^{\gamma,t_0,\theta}$.

For $x, z \in L_\infty^{\gamma,t_0,\theta}$ we now estimate

$$\begin{aligned}\left\| e^{-\gamma(t_0-t)} A^\theta \big(Tx(t) - Tz(t)\big)\right\| &\leqslant \int_{-\infty}^{t_0} \left\| e^{-\gamma(t_0-t)} A^\theta G(t,\tau)\right\| \left\| f\big(\tau, x(\tau)\big) - f\big(\tau, z(\tau)\big)\right\| d\tau \\ &\leqslant \int_{-\infty}^{t_0} \left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau)\right\| \varphi(\tau) e^{-\gamma(t_0-\tau)} \left\| A^\theta \big(x(\tau)\big) - z(\tau))\right\| d\tau.\end{aligned}$$

Again, using (3.9) we derive

$$\left\| Tx(\cdot) - Tz(\cdot)\right\|_{\gamma,\theta,\infty} \leqslant k\left\| x(\cdot) - z(\cdot)\right\|_{\gamma,\theta,\infty}$$

where k is defined as in (3.12).

Hence, since $k < 1$, we obtain that $T : L_\infty^{\gamma,t_0,\theta} \to L_\infty^{\gamma,t_0,\theta}$ is a contraction. Thus, there exists a unique $u(\cdot) \in L_\infty^{\gamma,t_0,\theta}$ such that $Tu = u$. By definition of T we have that $u(\cdot)$ is the unique solution in $L_\infty^{\gamma,t_0,\theta}$ of Eq. (3.5) for $t \leqslant t_0$. By Lemma 3.2 and Remark 3.3 we have that $u(\cdot)$ is the unique solution in $L_\infty^{\gamma,t_0,\theta}$ of Eq. (2.7) for $t \leqslant t_0$. □

Theorem 3.5. *Let the operator A satisfy Standing Hypothesis* 2.1 *and φ belong to some admissible space E. Let f be φ-Lipschitz satisfying condition* (3.4)*. Suppose that there are two successive eigenvalues $\lambda_N < \lambda_{N+1}$ of A satisfying*

$$k < 1 \quad \textit{and} \quad \frac{kM^3\lambda_N^{2\theta} N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1 \tag{3.13}$$

where the constant k is defined by (3.12) *and $\alpha = (\lambda_{N+1} - \lambda_N)/2$.*

Then, Eq. (2.7) *has an inertial manifold.*

Proof. Lemma 3.4 allows us to define a collection of surfaces $(\mathbb{M}_{t_0})_{t_0\in\mathbb{R}}$ by

$$\mathbb{M}_{t_0} := \{y + \Phi_{t_0} y \mid y \in PX\}$$

here $\Phi_{t_0} : PX \to (I-P)X_\theta$ is defined by

$$\Phi_{t_0}(y) = \int_{-\infty}^{t_0} e^{-(t_0-\tau)A}(I-P) f\big(\tau, x(\tau)\big)\, d\tau = (I-P)x(t_0) \tag{3.14}$$

where $x(t)$ is the unique solution in $L_\infty^{\gamma,t_0,\theta}$ of Eq. (3.5) satisfying that $Px(t_0) = y$ (note that the existence and uniqueness of $x(t)$ is proved in Lemma 3.4).

We then prove that Φ_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for y_1 and y_2 belonging to PX we have

$$\begin{aligned}\left\| A^\theta\big(\Phi_{t_0}(y_1) - \Phi_{t_0}(y_2)\big)\right\| &\leqslant \int_{-\infty}^{t_0} \left\| A^\theta e^{-(t_0-s)A}(I-P)\right\| \left\| f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\right\| ds \\ &= \int_{-\infty}^{t_0} e^{\gamma(t_0-s)} \left\| A^\theta G(t_0,s)\right\| \left\| e^{-\gamma(t_0-s)}\big(f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big)\right\| ds \\ &\leqslant \int_{-\infty}^{t_0} e^{\gamma(t_0-s)} \varphi(s) \left\| A^\theta G(t_0,s)\right\| \left\| e^{-\gamma(t_0-s)} A^\theta \big(x_1(s) - x_2(s)\big)\right\| ds \\ &\leqslant k\left\| x_1(\cdot) - x_2(\cdot)\right\|_{\gamma,\theta,\infty} \quad \big(\text{here we use the estimate (3.9)}\big).\end{aligned} \tag{3.15}$$

We now estimate $\|x_1(\cdot)-x_2(\cdot)\|_{\gamma,\theta,\infty}$. Since $x_i(\cdot)$ is the unique solution in $L_\infty^{\gamma,t_0,\theta}$ of Eq. (3.5) on $(-\infty,t_0]$ satisfying $Px_i(t_0)=y_i$, $i=1,2$, respectively, we have that

$$\begin{aligned}
&\left\|e^{-\gamma(t_0-t)}A^\theta\big(x_1(t)-x_2(t)\big)\right\| \\
&\quad=\left\|e^{-\gamma(t_0-t)}A^\theta\left(e^{-(t-t_0)A}(y_1-y_2)+\int_{-\infty}^{t_0}G(t,\tau)\big(f\big(\tau,x_1(\tau)\big)-f\big(\tau,x_2(\tau)\big)\big)\,d\tau\right)\right\| \\
&\quad\leqslant M\lambda_N^\theta\left\|A^\theta(y_1-y_2)\right\|+k\left\|x_1(\cdot)-x_2(\cdot)\right\|_{\gamma,\theta,\infty}\quad\text{for all } t\leqslant t_0.
\end{aligned}$$

Hence, we obtain

$$\left\|x_1(\cdot)-x_2(\cdot)\right\|_{\gamma,\theta,\infty}\leqslant M\lambda_N^\theta\left\|A^\theta(y_1-y_2)\right\|+k\left\|x_1(\cdot)-x_2(\cdot)\right\|_{\gamma,\theta,\infty}.$$

Therefore, since $k<1$, we get

$$\left\|x_1(\cdot)-x_2(\cdot)\right\|_{\gamma,\theta,\infty}\leqslant\frac{M\lambda_N^\theta}{1-k}\left\|A^\theta(y_1-y_2)\right\|.$$

Substituting this inequality to (3.15) we obtain

$$\left\|A^\theta\big(\Phi_{t_0}(y_1)-\Phi_{t_0}(y_2)\big)\right\|\leqslant\frac{M\lambda_N^\theta k}{1-k}\left\|A^\theta(y_1-y_2)\right\|$$

yielding that Φ_{t_0} is Lipschitz continuous with the Lipschitz constant $C:=\frac{M\lambda_N^\theta k}{1-k}$ independent of t_0. We thus obtain the property (i) in Definition 3.1 of the Inertial Manifold.

The property (ii) follows from Lemma 3.4, Lemma 3.2 and Remark 3.3(i).

We now prove the property (iii). To do this, let $x(\cdot)$ be a solution to Eq. (2.7) satisfying $x(s)=x_0\in\mathbb{M}_s$, i.e., $x(s)=Px(s)+\Phi_s(Px(s))$. Then, we fix an arbitrary number $t_0\in[s,\infty)$ and define a function $w(t)$ on $(-\infty,t_0]$ by

$$w(t)=\begin{cases}x(t) & \text{if } t\in[s,t_0],\\ u(t) & \text{if } t\in(-\infty,s]\end{cases}$$

where $u(t)$ is the unique solution in $L_\infty^{\gamma,s,\theta}$ of Eq. (2.7) satisfying $u(s)=x(s)\in\mathbb{M}_s$. Then, using Eqs. (2.7) and (3.14) we obtain

$$\begin{aligned}
w(t)&=e^{-(t-s)A}\big(Px(s)+\Phi_s\big(Px(s)\big)\big)+\int_s^t e^{-(t-\tau)A}f\big(\tau,w(\tau)\big)\,d\tau \\
&=e^{-(t-s)A}\big(Px(s)\big)+\int_{-\infty}^t e^{-(t-\tau)A}(I-P)f\big(\tau,w(\tau)\big)\,d\tau \\
&\quad+\int_s^t e^{-(t-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau\quad\text{for } s\leqslant t\leqslant t_0.
\end{aligned}\tag{3.16}$$

Obviously, Eq. (3.16) also remains true for $t\in(-\infty,s]$. Now, in Eq. (3.16) setting $t=t_0$ and applying the projection P we obtain

$$Pw(t_0)=e^{-(t_0-s)A}\big(Px(s)\big)+\int_s^{t_0}e^{-(t_0-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau\quad\text{for } s\leqslant t_0.$$

Since the restriction of the semigroup $(e^{-tA})_{t\geqslant0}$ on $\operatorname{Im}P$ can be extended to the group $(e^{-tA}P)_{t\in\mathbb{R}}$ and using the fact that $w(t_0)=x(t_0)$, it follows from the above equation that

$$\begin{aligned}
Px(s)&=e^{(t_0-s)A}\big(Px(t_0)\big)-\int_s^{t_0}e^{(t_0-s)A}e^{-(t_0-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau \\
&=e^{-(s-t_0)A}\big(Px(t_0)\big)-\int_s^{t_0}e^{-(s-\tau)A}Pf\big(\tau,w(\tau)\big)\quad\text{for } s\leqslant t_0.
\end{aligned}$$

Substituting this form of $Px(s)$ to Eq. (3.16) we obtain

$$w(t) = e^{-(t-t_0)A}Px(t_0) + \int_{t_0}^{t} e^{-(t-\tau)A}Pf\big(\tau, w(\tau)\big)\,d\tau + \int_{-\infty}^{t} e^{-(t-\tau)A}(I-P)f\big(\tau, w(\tau)\big)\,d\tau$$

$$= e^{-(t-t_0)A}Px(t_0) + \int_{-\infty}^{t_0} G(t,\tau)f\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \leqslant t_0. \tag{3.17}$$

Therefore, $x(t_0) = w(t_0) = Px(t_0) + \Phi_{t_0}(Px(t_0)) \in \mathbb{M}_{t_0}$ for all $t_0 \geqslant s$.

Lastly, we prove the property (iv) of Definition 3.1. To do this, we will prove that for any solution $u(\cdot)$ to Eq. (2.7) and any $s \in \mathbb{R}$ there is a solution $u^*(\cdot)$ of (2.7) such that $u^*(t) \in \mathbb{M}_t$ for $t \geqslant s$ and

$$\big\|A^\theta\big(u(t) - u^*(t)\big)\big\| \leqslant \frac{M\eta}{1-L}e^{-\gamma(t-s)} \quad \text{for all } t \geqslant s \text{ and some constant } \eta \tag{3.18}$$

where $L = \frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1$ given as in (3.13). Note that this solution $u^*(\cdot)$ is called an *induced trajectory*.

To this purpose, we will find the induced trajectory in the form $u^*(t) = u(t) + w(t)$ such that

$$\|w\|_{s,+} = \operatorname*{esssup}_{t\geqslant s}\big\|e^{\gamma(t-s)}A^\theta\big(w(t)\big)\big\| < \infty. \tag{3.19}$$

Substituting $u^*(\cdot)$ to Eq. (2.7) we obtain that $u^*(\cdot)$ is a solution to (2.7) for $t \geqslant s$ if and only if $w(\cdot)$ is a solution to the equation

$$w(t) = e^{-(t-s)A}w(s) + \int_s^t e^{-(t-\xi)A}\big[f\big(\xi, u(\xi) + w(\xi)\big) - f\big(\xi, u(\xi)\big)\big]\,d\xi. \tag{3.20}$$

For the sake of simplicity in the presentation we put $F(t,w) = f(t, u+w) - f(t,u)$ and set

$$L_\infty^{s,+} = \Big\{v : [s,\infty) \to X_\theta \;\Big|\; v \text{ is strongly measurable and } \operatorname*{esssup}_{t\geqslant s}\big\|e^{\gamma(t-s)}A^\theta v(t)\big\| < \infty\Big\}$$

endowed with the norm $\|\cdot\|_{s,+}$ defined as in (3.19).

Then, by the same way as in Lemma 3.2 and Remark 3.3 we can prove that a function $w(\cdot) \in L_\infty^{s,+}$ is a solution to (3.20) if and only if it satisfies

$$w(t) = e^{-(t-s)A}x_0 + \int_s^\infty G(t,\tau)F\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \geqslant s \text{ and some } x_0 \in (I-P)X_\theta. \tag{3.21}$$

Here the value $x_0 \in (I-P)X_\theta$ is chosen such that $u^*(s) = u(s) + w(s) \in \mathbb{M}_s$, i.e.,

$$(I-P)\big(u(s) + w(s)\big) = \Phi_s\big(P\big(u(s) + w(s)\big)\big).$$

From (3.21) it follows that

$$w(s) = x_0 - \int_s^\infty e^{-(s-\tau)A}PF\big(\tau, w(\tau)\big)\,d\tau. \tag{3.22}$$

Hence $P(u(s) + w(s)) = Pu(s) - \int_s^\infty e^{-(s-\tau)A}PF(\tau, w(\tau))\,d\tau$, and therefore

$$x_0 = (I-P)w(s) = -(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A}PF\big(\tau, w(\tau)\big)\,d\tau\right). \tag{3.23}$$

Substituting this form of x_0 into (3.21) we obtain

$$w(t) = e^{-(t-s)A}\left[-(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A}PF\big(\tau, w(\tau)\big)\,d\tau\right)\right]$$

$$+ \int_s^\infty G(t,\tau)F\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \geqslant s. \tag{3.24}$$

What we have to do now to prove the existence of u^* satisfying (3.18) is to prove that Eq. (3.24) has a solution $w(\cdot) \in L_\infty^{s,+}$. To do this we will prove that the transformation T defined by

$$(Tx)(t) = e^{-(t-s)A}\left[-(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, x(\tau)\big)\,d\tau\right)\right] + \int_s^\infty G(t,\tau)F\big(\tau, x(\tau)\big)\,d\tau \quad \text{for } t \geqslant s$$

acts from $L_\infty^{s,+}$ into itself and is a contraction.

Indeed, for $x(\cdot) \in L_\infty^{s,+}$, we have that $\|F(t, x(t))\| \leqslant \varphi(t)\|A^\theta x(t)\|$, therefore, putting

$$q(x) := -(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, x(\tau)\big)\,d\tau\right)$$

we can estimate

$$\begin{aligned}\big\|e^{\gamma(t-s)}A^\theta (Tx)(t)\big\| &\leqslant \big\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\big\| + \int_s^\infty \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)e^{\gamma(\tau-s)}\big\|A^\theta x(\tau)\big\|\,d\tau \\ &\leqslant \big\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\big\| + k\big\|x(\cdot)\big\|_{s,+}. \end{aligned} \tag{3.25}$$

Using Lipschitz property of Φ_s and for $t \geqslant s$ we now estimate the first term in the right-hand side of the above formula. In fact,

$$\begin{aligned}&\big\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\big\| \\ &\quad\leqslant \big\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A}\big(-(I-P)u(s) + \Phi_s\big(Pu(s)\big)\big)\big\| + \big\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A}\big(q(x) + (I-P)u(s) - \Phi_s\big(Pu(s)\big)\big)\big\| \\ &\quad\leqslant Me^{-(\lambda_{N+1}-\gamma)(t-s)}\big(\big\|A^\theta\big(-(I-P)u(s) + \Phi_s\big(Pu(s)\big)\big)\big\| + \big\|A^\theta\big(q(x) + (I-P)u(s) - \Phi_s\big(Pu(s)\big)\big)\big\|\big) \\ &\quad\leqslant M\eta + M\big\|A^\theta\big(q(x) + (I-P)u(s) - \Phi_s\big(Pu(s)\big)\big)\big\| \quad \big(\text{here } \eta := \big\|A^\theta\big(-(I-P)u(s) + \Phi_s\big(Pu(s)\big)\big)\big\|\big) \\ &\quad= M\eta + M\left\|A^\theta\left[\Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A}PF\big(\tau, x(\tau)\big)\,d\tau\right) - \Phi_s\big(Pu(s)\big)\right]\right\| \\ &\quad\leqslant M\eta + \frac{M^2\lambda_N^\theta k}{1-k}\left\|\int_s^\infty A^\theta e^{-(s-\tau)A}PF\big(\tau, x(\tau)\big)\,d\tau\right\| \\ &\quad\leqslant M\eta + \frac{kM^3\lambda_N^{2\theta}}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\big\|e^{\gamma(\tau-s)}A^\theta x(\tau)\big\|\,d\tau \\ &\quad\leqslant M\eta + \frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty\big\|x(\cdot)\big\|_{s,+}\end{aligned}$$

where k is defined as in (3.12).

Substituting these estimates to (3.25) we obtain $Tx \in L_\infty^{s,+}$ and

$$\|Tx\|_{s,+} \leqslant M\eta + \left[\frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\big\|x(\cdot)\big\|_{s,+}. \tag{3.26}$$

Therefore, the transformation T acts from $L_\infty^{s,+}$ to $L_\infty^{s,+}$.

Using the fact that $\|F(t, w_1) - F(t, w_2)\| \leqslant \varphi(t)\|A^\theta(w_1 - w_2)\|$ and for $x, z \in L_\infty^{s,+}$ we now estimate

$$\begin{aligned}\big\|e^{\gamma(t-s)}A^\theta\big(Tx(t) - Tz(t)\big)\big\| &\leqslant \frac{M^2\lambda_N^\theta k}{1-k}\left\|\int_s^\infty A^\theta e^{-(s-\tau)A}P\big(F\big(\tau, x(\tau)\big) - F\big(\tau, z(\tau)\big)\big)\,d\tau\right\| \\ &\quad + \int_s^\infty \big\|e^{\gamma(t-s)}A^\theta G(t,\tau)\big\|\,\big\|F\big(\tau, x(\tau)\big) - F\big(\tau, z(\tau)\big)\big\|\,d\tau\end{aligned}$$

$$\leqslant \frac{kM^3\lambda_N^{2\theta}}{1-k}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\left\|e^{\gamma(\tau-s)}A^\theta\big(x(\tau)-z(\tau)\big)\right\|d\tau$$

$$+\int_s^\infty \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\|\varphi(\tau)e^{\gamma(\tau-s)}\left\|A^\theta\big(x(\tau)-z(\tau)\big)\right\|d\tau$$

$$\leqslant \left[\frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\left\|x(\cdot)-z(\cdot)\right\|_{s,+}$$

for all $t \geqslant s$.

Therefore,

$$\left\|Tx(\cdot)-Tz(\cdot)\right\|_{s,+} \leqslant \left[\frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k\right]\left\|x(\cdot)-z(\cdot)\right\|_{s,+}$$

where k is defined as in (3.12).

Hence, if $\frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k < 1$, then we obtain that $T : L_\infty^{s,+} \to L_\infty^{s,+}$ is a contraction. Thus, there exists a unique $w(\cdot) \in L_\infty^{s,+}$ such that $Tw = w$. By the definition of T we have that $w(\cdot)$ is the unique solution in $L_\infty^{s,+}$ of Eq. (3.24) for $t \geqslant s$. Also, using (3.26) we have the estimate for $\|w(\cdot)\|_{s,+}$ as

$$\left\|w(\cdot)\right\|_{s,+} \leqslant \frac{M\eta}{1-L}$$

where $\eta = \|A^\theta(-(I-P)u(s) + \Phi_s(Pu(s)))\|$ and $L = \frac{kM^3\lambda_N^{2\theta}N_2}{(1-k)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty + k$.

Furthermore, by determination of w we obtain the existence of the solution $u^* = u + w$ to Eq. (2.7) such that $u^*(t) \in \mathbb{M}_t$ for $t \geqslant s$, and u^* satisfies the inequality (3.18) yielding that

$$\left\|A^\theta\big(u^*(t)-u(t)\big)\right\| = \left\|A^\theta w(t)\right\| \leqslant \frac{M\eta}{1-L}e^{-\gamma(t-s)} \quad \text{for all } t \geqslant s.$$

Putting $H := \frac{M\eta}{1-L}$ it follows from this inequality that

$$\operatorname{dist}_{X_\theta}\big(u(t), \mathbb{M}_t\big) \leqslant He^{-\gamma(t-s)} \quad \text{for all } t \geqslant s.$$

Therefore, $(\mathbb{M}_t)_{t\in\mathbb{R}}$ exponentially attracts every solution u of (2.7). $\square$

Remark 3.6. By the definition of the constant k (see (3.12)) we have that, for $0 < \theta < 1$, the condition (3.13) is fulfilled if the following two conditions:

(i) the difference $\lambda_{N+1} - \lambda_N$ is sufficiently large, and
(ii) the norm $\|\Lambda_1\varphi\|_\infty = \sup_{t\in\mathbb{R}}\int_{t-1}^t \varphi(\tau)\,d\tau$ is sufficiently small

hold true.

On the other hand, if $\theta = 0$, then for the fulfillment of the condition (3.13) we need only the fact that the norm $\|\Lambda_1\varphi\|_\infty$ is sufficiently small. This is meaningful, e.g., in case that X is finite dimensional since in this case the spectrum of A has a finite number of eigenvalues and hence we don't have the condition $\lim_{k\to\infty}\lambda_k = \infty$.

We illustrate our result in the following example.

Example 3.7. Consider the reaction–diffusion problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = g\left(x, u, \dfrac{\partial u}{\partial x}, t\right), & 0 < x < l,\ t > s, \\ u|_{x=0} = u|_{x=l} = 0; \quad u|_{t=s} = u_s(x) \end{cases} \tag{3.27}$$

where $s \in \mathbb{R}$ and $g(x, u, \xi, t)$ is a continuous function such that

$$\left|g(x, u_1, \xi_1, t) - g(x, u_1, \xi_2, t)\right| \leqslant \psi(t)\big(L_1|u_1-u_2| + L_2|\xi_1-\xi_2|\big) \quad \text{for all } x \in (0,l),\ t \in \mathbb{R},\ u_i, \xi_i \in \mathbb{R},$$

$$\left|g(x, 0, 0, t)\right| \leqslant L_3\psi(t) \quad \text{for all } x \in (0,l),\ t \in \mathbb{R}$$

here L_i, $i = 1, 2, 3$, are positive numbers, and ψ is a positive function belonging to an admissible function space E and satisfying (3.4) with $\theta = \frac{1}{2}$. We choose the Hilbert space $X = L^2(0, l)$ and consider the operators

$$A = -\frac{d^2}{dx^2} \quad \text{with } D(A) = H_0^1(0, l) \cap H^2(0, l)$$

and

$$f : \mathbb{R} \times D\big(A^{\frac{1}{2}}\big) \to X \quad \text{defined by } f(t, u)(x) = g\left(x, u, \frac{\partial u}{\partial x}, t\right).$$

Then, we know that A satisfies Standing Hypothesis 2.1 with the discrete point spectrum being

$$\left(\frac{\pi}{l}\right)^2, \left(\frac{\pi}{l}\right)^2 4, \ldots, \left(\frac{\pi}{l}\right)^2 n^2, \ldots.$$

Obviously,

$$\big\| f(t, u) - f(t, v)\big\| \leqslant \psi(t)\left(L_1\|u - v\| + L_2\left\|\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right\|\right).$$

Using now Poincaré inequality

$$\left\|\frac{\partial u}{\partial x}\right\|^2 \geqslant \left(\frac{\pi}{l}\right)^2 \|u\|^2 \quad \text{for } u \in H_0^1(0, l),$$

we obtain

$$\big\| f(t, u) - f(t, v)\big\| \leqslant \psi(t)\frac{(lL_1 + \pi L_2)}{\pi}\big\|A^{\frac{1}{2}}(u - v)\big\| \quad \text{for all } t \in \mathbb{R} \text{ and } u, v \in X_{\frac{1}{2}}.$$

Moreover, since $\| f(t, 0)\| \leqslant L_3\psi(t)$, we obtain

$$\big\| f(t, u)\big\| \leqslant \big\| f(t, u) - f(t, 0)\big\| + \big\| f(t, 0)\big\| \leqslant \psi(t)\frac{(lL_1 + \pi L_2)}{\pi}\big\|A^{\frac{1}{2}}u\big\| + L_3\psi(t).$$

Therefore, f is φ-Lipschitz with $\varphi = \max\{L_3\psi(t), \frac{(lL_1+\pi L_2)}{\pi}\psi(t)\}$. Applying Theorem 3.5 we see that, if N is sufficiently large (i.e., the difference $(\frac{\pi}{l})^2(N+1)^2 - (\frac{\pi}{l})^2N^2$ is large enough) and the norm $\|\Lambda_1\varphi\|_\infty = \sup_{t\in\mathbb{R}}\int_{t-1}^{t}\varphi(t)\,dt$ is sufficiently small, then Eq. (3.27) has an inertial manifold.

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