Asian Journal of Control, Vol. 19, No. 6, pp. 1–8, November 2017
Published online in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.1510

–Brief Paper–

MINIMUM ENERGY CONTROL OF FRACTIONAL DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS WITH BOUNDED INPUTS USING THE DRAZIN INVERSE

Tadeusz Kaczorek

ABSTRACT

The Drazin inverse of matrices is applied to solve the minimum energy control problem of fractional descriptor discrete-time linear systems with bounded inputs. Necessary and sufficient conditions for the reachability of fractional descriptor linear systems are established. The minimum energy control problem for the fractional descriptor systems with bounded inputs is formulated and solved. A procedure for the computation of the optimal inputs sequence and the minimal value of the performance index is proposed.

Key Words: Minimum energy control, Drazin inverse, Descriptor, Fractional, Discrete-time, Linear system.

I. INTRODUCTION

Fractional linear systems have been investigated in [1,2]. Mathematical fundamentals of fractional calculus are given in monographs [3–5]. Some recent interesting results in fractional systems theory and its applications can be found in [6–9]. Minimum energy control of fractional systems has been addressed in [1,10,11].

Descriptor (singular) linear systems have been considered in many papers and books [1,12–35]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [19,25,26] and the minimum energy control of descriptor linear systems has been addressed in [27–29]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [34]. Positive linear systems with different fractional orders have been addressed in [36]. A survey on the controllability of dynamical systems has been presented in [37].

Descriptor and standard positive linear systems using the Drazin inverse has been addressed in [12–15, 21,23,24, 26]. The shuffle algorithm has been applied for checking the positivity of descriptor linear systems in [22]. The stability of positive descriptor systems has been investigated in [35]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [30]. A new class of descriptor fractional linear discrete-time system has been introduced in [31].

The Drazin inverse for solving the state equation of fractional continuous-time linear systems has been applied in [23] and the controllability, reachability, and minimum energy control of fractional discrete-time linear systems with delays in state have been investigated in [10]. A comparison of three different methods for solving the descriptor fractional discrete-time linear system has been made in [33].

Manuscript received April 3, 2016; revised October 14, 2016; accepted February 8, 2017.

T. Kaczorek (corresponding author, email: kaczorek@isep.pw.edu.pl) is with Faculty of Electrical Engineering, Białystok University of Technology, Wiejska 45D, 15-351 Białystok.

This work was supported by the National Science Centre in Poland work No. 2014/13/B/ST7/03467.

A solution to the state equation of fractional descriptor discrete-time linear systems using the Drazin inverse of matrices has been derived in [24].

In this paper, the Drazin inverse of matrices will be applied to solve the minimum energy control problem of fractional descriptor discrete-time linear systems with bounded inputs.

The paper is organized as follows. In section II some definitions and theorems concerning the Drazin inverse of matrices are recalled. The solution to the state equation using the Drazin inverse is given in section III. Necessary and sufficient conditions for the reachability of the fractional descriptor discrete-time linear systems are established in section IV. The minimum energy control problem of fractional descriptor linear systems with bounded inputs is formulated and solved in section V. Concluding remarks are given in section VI. The procedure for computation of Drazin inverse matrices is given in the appendix.

The following notation will be used: $\mathfrak{R}$ – the set of real numbers; $\mathfrak{R}^{n\times m}$ – the set of $n\times m$ real matrices and $\mathfrak{R}^n=\mathfrak{R}^{n\times 1}$; Z_+– the set of nonnegative integers; I_n– the $n\times n$ identity matrix; and ker A (im A) – the kernel (image) of the matrix.

II. FRACTIONAL DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS

Consider the fractional descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1}=Ax_i+Bu_i, i\in Z_+=\{0,\ 1,\ ...\}, \tag{2.1}$$

where $x_i\in\mathfrak{R}^n$ is the state vector, $u_i\in\mathfrak{R}^m$ is the input vector, E, $A\in\mathfrak{R}^{n\times n}$, $B\in\mathfrak{R}^{n\times m}$ and

$$\Delta^\alpha x_i=\sum_{j=0}^{i}(-1)j\binom{\alpha}{j}x_{i-j} \tag{2.2a}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j=0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,\ldots \end{cases} \tag{2.2b}$$

is the fractional $\alpha \in \Re$ order difference of x_i.

Substituting (2.2) into (2.1) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} cjEx_{i-j+1} + Bu_i \tag{2.3a}$$

where

$$A_\alpha = A + E\alpha,\ c_j = (-1)^j \binom{\alpha}{j}. \tag{2.3b}$$

It is assumed that det $E = 0$ but

$$\det[Ez - A_\alpha] \neq 0 \text{ for some } z \in \mathrm{C}, \tag{2.4}$$

where C is the field of complex numbers.

Assuming that for some chosen $c \in \mathrm{C}$, $\det[Ec - A_\alpha] \neq 0$ and premultiplying (2.3a) by $[Ec - A_\alpha]^{-1}$ we obtain

$$\overline{E}x_{i+1} = \overline{A}_\alpha x_i + \sum_{j=2}^{i+1} c_j\overline{E}x_{i-j+1} + \overline{B}u_i, \tag{2.5a}$$

where

$$\overline{E} = [Ec - A_\alpha]^{-1}E, \overline{A}_\alpha = [Ec - A_\alpha]^{-1}A_\alpha, \overline{B} = [Ec - A_\alpha]^{-1}B. \tag{2.5b}$$

Note that (2.3a) and (2.5a) have the same solution x_i, $i \in Z_+$.

Definition 2.1. [15,26] The smallest nonnegative integer q is called the index of the matrix $\overline{E} \in \Re^{n\times n}$ if

$$\text{rank } \overline{E}^q = \text{rank } \overline{E}^{q+1}. \tag{2.6}$$

Definition 2.2. [15,26] A matrix $\overline{E}^D$ is called the Drazin inverse of $\overline{E} \in \Re^{n\times n}$ if it satisfies the conditions

$$\overline{E}\overline{E}^D = \overline{E}^D\overline{E}, \tag{2.7a}$$

$$\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D, \tag{2.7b}$$

$$\overline{E}^D\overline{E}^{q+1} = \overline{E}^q, \tag{2.7c}$$

where q is the index of $\overline{E}$ defined by (2.6).

The Drazin inverse $\overline{E}^D$ of a square matrix $\overline{E}$ always exists and is unique [15,26]. If $\det \overline{E} \neq 0$ then $\overline{E}^D = \overline{E}^{-1}$. Some methods for computation of the Drazin inverse are given in [8,26] and in the appendix.

Theorem 2.1. The matrices $\overline{E}$ and $\overline{A}_\alpha$ defined by (2.5b) satisfy the following equalities

$$1.\ \overline{A}_\alpha\overline{E} = \overline{E}\overline{A}_\alpha \text{ and } \overline{A}_\alpha^D\overline{E} = \overline{E}\overline{A}_\alpha^D, \overline{E}^D\overline{A}_\alpha = \overline{A}_\alpha\overline{E}^D, \overline{A}_\alpha^D\overline{E}^D = \overline{E}^D\overline{A}_\alpha^D, \tag{2.8a}$$

$$2.\ \ker\overline{A}_\alpha \cap \ker\overline{E} = \{0\}, \tag{2.8b}$$

$$3.\ \overline{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}, \overline{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \tag{2.8c}$$

$\det T \neq 0$, $J \in \Re^{n_1\times n_1}$, is nonsingular, $N \in \Re^{n_2\times n_2}$ is nilpotent, $n_1 + n_2 = n$,

$$4.\ \left(I_n - \overline{E}\overline{E}^D\right)\overline{A}_\alpha\overline{A}_\alpha^D = I_n - \overline{E}\overline{E}^D \text{ and } \left(I_n - \overline{E}\overline{E}^D\right)\left(\overline{E}\overline{A}_\alpha^D\right)^q = 0. \tag{2.8d}$$

Proof. Using (2.5b) we obtain

$$\overline{E}c - \overline{A}_\alpha = [Ec - A_\alpha]^{-1}[Ec - A_\alpha] = I_n \tag{2.9}$$

and

$$\overline{A}_\alpha = \overline{E}c - I_n. \tag{2.10}$$

Therefore

$$\overline{E}\overline{A}_\alpha = \overline{E}\left[\overline{E}c - I_n\right] = \left[\overline{E}c - I_n\right]\overline{E} = \overline{A}_\alpha\overline{E}. \tag{2.11}$$

The proof of the remaining equalities (2.8) are similar.

Theorem 2.2. Let

$$P = \overline{E}\overline{E}^D, \tag{2.12a}$$

and

$$Q = \overline{E}^D\overline{A}_\alpha. \tag{2.12b}$$

Then

$$5.\ P^k = P \text{ for } k = 2, 3, \ldots \tag{2.13}$$

$$6.\ PQ = QP = Q, \tag{2.14}$$

$$7.\ P\overline{E}^D = \overline{E}^D. \tag{2.15}$$

Proof. Using (2.12a) we obtain

$$P^2 = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = \overline{E}\overline{E}^D = P \tag{2.16}$$

since by (2.7b) $\overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D$ and by induction

$$P^k = P^{k-1}P = \overline{E}\overline{E}^D\overline{E}\overline{E}^D = P^2 = P \text{ for } k = 2, 3, \ldots. \tag{2.17}$$

Using (2.12) we obtain

$$PQ = \overline{E}\overline{E}^D\overline{E}^D\overline{A}_\alpha = \overline{E}^D\overline{E}\overline{E}^D\overline{A}_\alpha = \overline{E}^D\overline{A}_\alpha = Q \tag{2.18}$$

and

$$\begin{aligned} QP &= \overline{E}^D\overline{A}_\alpha\overline{E}\overline{E}^D = \overline{E}^D\overline{E}\overline{A}_\alpha\overline{E}^D = \overline{E}^D\overline{E}\overline{E}^D\overline{A}_\alpha \\ &= \overline{E}^D\overline{A}_\alpha = Q \end{aligned} \tag{2.19}$$

Using (2.12a), (2.7a), and (2.7b) we obtain

$$P\overline{E}^D = \overline{E}\overline{E}^D\overline{E}^D = \overline{E}^D\overline{E}\overline{E}^D = \overline{E}^D. \tag{2.20}$$

III. FRACTIONAL DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS

In this section the solution to (2.1) will be derived using the Drazin inverses of the matrices $\overline{E}$ and $\overline{A}_\alpha$.

Theorem 3.1. The solution to (2.5a) is given by

$$\begin{aligned} x_i &= Q^iPv + c_2Q^{i-2}Pv + c_3Q^{i-3}Pv + \ldots + 2c_{i-1}QPv + c_iPv \\ &+ \sum_{k=0}^{i-1}\overline{E}^DQ^{i-k-1}\overline{B}u_k + (P - I_n)\sum_{k=0}^{q-1}Q^k\overline{A}_\alpha^D\overline{B}u_{i+k} \end{aligned} \tag{3.1}$$

where Q and P are defined by (2.12), coefficients c_j can be computed using (2.3b), and $v \in \Re^n$ is arbitrary.

Proof. Proof is given in [24].
From (3.1) for $i = 0$ we have

$$x_0 = Pv + (P - I_n)\sum_{k=0}^{q-1}Q^k\overline{A}_\alpha^D\overline{B}u_k. \tag{3.2}$$

The equality (3.2) defines the set of consistent initial conditions X_0, $x_0 \in X_0$ for a given set of admissible inputs U_{ad}, $u_k \in U_{ad}$, $k = 0,1,\ldots,q-1$.
If $u_k = 0$, $k = 0,1,\ldots,q-1$ then from (3.2) we obtain

$$x_0 = Pv \text{ and } x_0 \in \mathrm{Im}P \tag{3.3}$$

where ImP denotes the image of P.

Remark 3.1. The solution to (2.5a) for $u_i = 0$, $i \in Z_+$ can be computed recurrently using the formula

$$x_i = Qx_{i-1} + \sum_{k=2}^{i}c_kPx_{i-k}. \tag{3.4}$$

Theorem 3.2. Let

$$\Phi_0(i) = Q^i + \sum_{k=2}^{i}c_k\overline{A}_\alpha P, \tag{3.5}$$

$$\Phi(i) = \sum_{k=0}^{i-1}\overline{E}^DQ^{i-k-1}\overline{B}, \tag{3.6}$$

where Q and P are defined by (2.12).

Then.

$$P\Phi_0(i) = \Phi_0(i), \tag{3.7}$$

$$P\Phi(i) = \Phi(i). \tag{3.8}$$

Proof. Using (2.12) and (3.5) we obtain

$$P\Phi_0(i) = P\left[Q^i + \sum_{k=2}^{i}c_k\overline{A}_\alpha P\right] = Q^i + \sum_{k=2}^{i}c_k\overline{A}_\alpha P \tag{3.9}$$

since (2.14) and (2.13) hold.

Similarly, using (3.6), (2.14), and (2.15) we obtain

$$P\Phi(i) = \sum_{k=0}^{i-1}P\overline{E}^DQ^{i-k-1}\overline{B} = \sum_{k=0}^{i-1}\overline{E}^DQ^{i-k-1}\overline{B} = \Phi(i). \tag{3.10}$$

IV. REACHABILITY OF THE FRACTIONAL DESCRIPTOR LINEAR SYSTEMS

Consider the fractional descriptor linear system (2.1) or (2.5).

Definition 4.1. The fractional descriptor linear system (2.1) is called reachable in $\overline{h}$ steps if for a given final state $x_f \in \Re^n$ there exists an input sequence for u_i for $i = 0,1,\ldots,\overline{h}-1$ that steers the state of the system from $x_0 = 0$ to x_f, *i.e.* $x_{\overline{h}} = x_f$.

Theorem 4.1. The fractional descriptor system (2.1) is reachable in $\overline{h}$ steps if and only if

$$\mathrm{rank}\, R_{\overline{h}} = \mathrm{rank}\left[\overline{E}^DQ^{h-1}\overline{B} \quad \cdots \quad \overline{E}^DQ\overline{B} \quad \overline{E}^D\overline{B} \quad \overline{A}_\alpha\overline{B} \quad \cdots \quad Q^{q-1}\overline{A}_\alpha\overline{B}\right] = n \tag{4.1}$$

where $\overline{E}^D$ and Q are defined by (2.7) and (2.12b), respectively.

Proof. Using (3.1) for $v=0$ and $i=\overline{h}$ we obtain

$$x_f = x_{\overline{h}} = \left[\overline{E}^D Q^{h-1}\overline{B} \quad \cdots \quad \overline{E}^D Q\overline{B} \quad \overline{E}^D\overline{B} \quad \overline{A}_\alpha\overline{B} \quad \cdots \quad Q^{q-1}\overline{A}_\alpha\overline{B}\right]\begin{bmatrix} u_0 \\ u_0 \\ u_1 \\ \vdots \\ u_{h-1} \\ u_h \\ \vdots \\ u_{h+q-1}\end{bmatrix} = R_{\overline{h}}\overline{u}_{\overline{h}}, (4.2) \tag{4.2}$$

where $\overline{u}_{\overline{h}} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{h-1} \\ u_h \\ \vdots \\ u_{h+q-1}\end{bmatrix}$.

According to the Kronecker-Capelli theorem there exists an input sequence u_i, $i=0,1,...,\overline{h}-1$ for any given final state x_f only if the condition (4.1) is satisfied. In this case from (4.2) we have

$$\overline{u}_{\overline{h}} = \left\{R_{\overline{h}}^T\left[R_{\overline{h}}R_{\overline{h}}^T\right]^{-1} + \left[I_n - R_{\overline{h}}^T\left[R_{\overline{h}}R_{\overline{h}}^T\right]^{-1}R_{\overline{h}}\right]K\right\}x_f \tag{4.3}$$

for any given matrix $K\in\Re^{n\times n}$, where $R_{\overline{h}}^T$ denotes the transpose of $R_{\overline{h}}$.

By substitution it is easy to verify that (4.3) satisfies (4.2) for any matrix K.

Example 4.1. Consider (2.1) with the matrices

$$E = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},\ A = \begin{bmatrix}0 & 1 & 0\\ -2 & -3 & 0\\ 1 & 2 & -1\end{bmatrix},\ B = \begin{bmatrix}1\\0\\2\end{bmatrix}\ \text{for } \alpha = 0.5. \tag{4.4}$$

In this case we have

$$A_\alpha = A + E\alpha = \begin{bmatrix}0.5 & 1 & 0\\ -2 & -2.5 & 0\\ 1 & 2 & -1\end{bmatrix}. \tag{4.5}$$

The pencil of (4.4) is regular since

$$\det[Ez - A_\alpha] = \begin{vmatrix} z-0.5 & -1 & 0\\ 2 & z+2.5 & 0\\ -1 & -2 & 1\end{vmatrix} \tag{4.6}$$

$$= (z-0.5)(z+2.5)+2\neq 0.$$

We choose $c = 0$ and the matrices (2.5b) take the form

$$\overline{E} = [Ec - A_\alpha]^{-1}E = [-A_\alpha]^{-1}E = \begin{bmatrix}-0.5 & -1 & 0\\ 2 & 2.5 & 0\\ -1 & -2 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= \frac{1}{0.75}\begin{bmatrix}2.5 & 1 & 0\\ -2 & -0.5 & 0\\ -1.5 & 0 & 0\end{bmatrix},$$

$$\overline{A}_\alpha = [-A_\alpha]^{-1}A_\alpha = \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}, \tag{4.7}$$

$$\overline{B} = [-A_\alpha]^{-1}B = \begin{bmatrix}-0.5 & -1 & 0\\ 2 & 2.5 & 0\\ -1 & -2 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\0\\2\end{bmatrix} = \frac{1}{0.75}\begin{bmatrix}2.5\\-2\\0\end{bmatrix}$$

and by (2.6) $q=1$ since

$$\text{rank}\begin{bmatrix}3.333 & 1.333 & 0\\ -2.667 & 0.667 & 0\\ -2 & 0 & 0\end{bmatrix} = \text{rank}\begin{bmatrix}3.333 & 1.333 & 0\\ -2.667 & 0.667 & 0\\ -2 & 0 & 0\end{bmatrix}^2$$

$$= \text{rank}\begin{bmatrix}7.556 & 5.333 & 0\\ -10.667 & -3.111 & 0\\ -6.667 & -2.667 & 0\end{bmatrix}.$$

To compute the Drazin inverse of the matrix $\overline{E}$ we use the procedure given in the appendix and we obtain

$$\overline{E} = VW,\ V = \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 \\ -2 & -0.5 \\ -1.5 & 0 \end{bmatrix},\ W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\overline{E}^D = V\left[W\overline{E}V\right]^{-1}W = \frac{1}{0.75}\begin{bmatrix} 2.5 & 1 \\ -2 & -0.5 \\ -1.5 & 0 \end{bmatrix}\begin{bmatrix} 7.556 & 3.556 \\ -7.111 & -3.111 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix}. \tag{4.8}$$

and

$$P = \overline{E}^D\overline{E} = \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix}\frac{1}{0.75}\begin{bmatrix} 2.5 & 1 & 0 \\ -2 & -0.5 & 0 \\ -1.5 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \tag{4.9}$$

$$Q = \overline{E}^D\overline{A}_\alpha = \begin{bmatrix} -0.5 & -1 & 0 \\ 2 & 2.5 & 0 \\ 3.5 & 4 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0 \\ -2 & -2.5 & 0 \\ -3.5 & -4 & 0 \end{bmatrix}. \tag{4.10}$$

Using (4.7) and (4.1) for $\overline{h} = 3$ and $q = 1$ we obtain

$$R_3 = \begin{bmatrix} \overline{E}^D Q\overline{B} & \overline{E}^D\overline{B} & \overline{A}_\alpha\overline{B} \end{bmatrix} = \begin{bmatrix} 0.5 & 1 & -3.333 \\ -2 & 0 & 2.667 \\ -3.5 & 1 & 0 \end{bmatrix}. \tag{4.11}$$

The fractional descriptor system with (4.4) is reachable in three steps since rank $R_3 = 3$ and the input sequence which steers the system from $x_0 = 0$ to $x_f = [1 \quad 1 \quad 1]^T$ has the form

$$\begin{aligned}\overline{u}_3 &= \left\{R_3^T\left[R_3R_3^T\right]^{-1} + \left[I_3 - R_3^T\left[R_3R_3^T\right]^{-1}R_3\right]K\right\}x_f \qquad (4.12)\\ &= R_3^T\left[R_3R_3^T\right]^{-1}x_f = \begin{bmatrix} 0.833 \\ 3.917 \\ 1 \end{bmatrix}.\end{aligned}$$

V. MINIMUM ENERGY CONTROL PROBLEM

Consider the fractional descriptor positive system (2.1) reduced to the form (2.5). In section IV it was shown that if the system is reachable then there exist many input sequences that steer the state of the system from $x_0 = 0$ to the given final state $x_f \in \Re^n$. Among these input sequences we are looking for sequence $u_k \in \Re^m$ for $k = 0, 1, ..., \overline{h} - 1$ that minimizes the performance index

$$I(u) = \sum_{i=0}^{\overline{h}-1} u_i^T Q u_i, \tag{5.1}$$

where $Q \in \Re^{m\times m}$ is a symmetric defined matrix, which steers the state of the system from $x_0 = 0$ to the given final state $x_f \in \Re^n$.

The minimum energy control problem for the fractional descriptor positive discrete-time linear systems (2.1) with bounded inputs can be stated as follows.

Given the matrices E, A, B of the descriptor positive system (2.1), α, the final state $x_f \in \Re^n$ and the matrix $Q \in \Re^{m\times m}$ of the performance index (5.1) find a sequence of inputs $u_k \in \Re^m$ for $k = 0, 1, ..., \overline{h} - 1$ satisfying

$$u_k \le U\ (U \in \Re^m \textit{is given})\ \textit{for}\ k = 0, 1, ..., \overline{h} - 1 \tag{5.2}$$

that steers the state of the system from $x_0 = 0$ to $x_f \in \Re^n$ and minimizes the performance index (5.1).
To solve the problem, we define the matrix

$$W_{\overline{h}} = R_{\overline{h}}Q_{\overline{h}}^{-1}R_{\overline{h}}^T \in \Re^{n\times n} \tag{5.3}$$

where $R_{\overline{h}} \in \Re^{n\times \overline{h}m}$ is defined by (4.1) and

$$Q_{\overline{h}}^{-1} = \text{blockdiag}\left[Q^{-1}, ..., Q^{-1}\right] \in \Re^{\overline{h}m\times\overline{h}m}. \tag{5.4}$$

The matrix (5.3) is non-singular if the system is reachable in $\overline{h}$ steps.

For a given $x_f \in \Re^n$ we may define the input sequence

$$\hat{u}_{\overline{h}} = \begin{bmatrix} \hat{u}_{\overline{h}-1} \\ \hat{u}_{\overline{h}-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} = Q_{\overline{h}}^{-1}R_{\overline{h}}^T W_{\overline{h}}^{-1}x_f \tag{5.5}$$

where $Q_{\overline{h}}^{-1}$, $W_{\overline{h}}$ and $R_{\overline{h}}$ are defined by (5.4), (5.3), and (4.1), respectively.

Theorem 5.1. Let the fractional descriptor positive system (2.1) be reachable in $\overline{h}$ steps. Moreover, let

$$\overline{u}_{\overline{h}} = \begin{bmatrix} \overline{u}_{\overline{h}-1} \\ \overline{u}_{\overline{h}-2} \\ \vdots \\ \overline{u}_0 \end{bmatrix} \in \mathfrak{R}^{\overline{h}m} \tag{5.6}$$

be an input sequence satisfying (5.2) that steers the state of the system from $x_0 = 0$ to $x_f \in \mathfrak{R}^n$. Then the input sequence (5.5) satisfying (5.2) also steers the state of the system from $x_0 = 0$ to $x_f \in \mathfrak{R}^n$ and minimizes the performance index (5.1), *i.e.*

$$I(\hat{u}) \le I(\overline{u}). \tag{5.7}$$

The minimal value of the performance index (5.1) is given by

$$I(\hat{u}) = x_f^T W_{\overline{h}}^{-1} x_f. \tag{5.8}$$

Proof. If the system is reachable in $\overline{h}$ steps then the input sequence (5.5) is well defined and $\hat{u}_{\overline{h}} \in \mathfrak{R}^{\overline{h}m}$. We shall show that the input sequence (5.5) satisfying (5.2) steers the state of the system from $x_0 = 0$ to $x_f \in \mathfrak{R}^n$. Using (4.2) and (5.5) we obtain

$$x_{\overline{h}} = R_{\overline{h}} \hat{u}_{\overline{h}} = R_{\overline{h}} Q_{\overline{h}}^{-1} R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f = x_f \tag{5.9}$$

since by (5.3) $R_{\overline{h}} Q_{\overline{h}}^{-1} R_{\overline{h}}^T = W_{\overline{h}}$. Hence $x_f = R_{\overline{h}} \hat{u}_{\overline{h}} = R_h \overline{u}_{\overline{h}}$ or

$$R_{\overline{h}} \left[\hat{u}_{\overline{h}} - \overline{u}_{\overline{h}} \right] = 0. \tag{5.10}$$

The transposition of (5.10) yields

$$\left[\hat{u}_{\overline{h}} - \overline{u}_{\overline{h}} \right]^T R_{\overline{h}}^T = 0. \tag{5.11}$$

Postmultiplying the equality (5.11) by $W_{\overline{h}}^{-1} x_f$ we obtain

$$\left[\hat{u}_{\overline{h}} - \overline{u}_{\overline{h}} \right]^T R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f = 0. \tag{5.12}$$

From (5.5) we have $Q_{\overline{h}} \hat{u}_{\overline{h}} = R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f$. Substitution of this equality into (5.12) yields

$$\left[\hat{u}_{\overline{h}} - \overline{u}_{\overline{h}} \right]^T Q_{\overline{h}} \hat{u}_{\overline{h}} = 0 \tag{5.13}$$

where $Q_{\overline{h}} = \text{blockdiag}\,[Q, \ldots, Q] \in \mathfrak{R}^{\overline{h}m \times \overline{h}m}$.
From (5.13) it follows that

$$\overline{u}_{\overline{h}}^T Q_{\overline{h}} \overline{u}_{\overline{h}} = \hat{u}_{\overline{h}}^T Q_{\overline{h}} \hat{u}_{\overline{h}} + \left[\overline{u}_{\overline{h}} - \hat{u}_{\overline{h}} \right]^T Q_{\overline{h}} \left[\overline{u}_{\overline{h}} - \hat{u}_{\overline{h}} \right] \tag{5.14}$$

since by (5.13) $\overline{u}_{\overline{h}}^T Q_{\overline{h}} \hat{u}_{\overline{h}} = \hat{u}_{\overline{h}}^T Q_{\overline{h}} \hat{u}_{\overline{h}} = \hat{u}_{\overline{h}}^T Q_{\overline{h}} \overline{u}_{\overline{h}}$.

From (5.14) it follows that (5.7) holds since $\left[\overline{u}_{\overline{h}} - \hat{u}_{\overline{h}} \right]^T Q_{\overline{h}} \left[\overline{u}_{\overline{h}} - \hat{u}_{\overline{h}} \right] \ge 0$. To find the minimal value of the performance index (5.1) we substitute (5.5) into (5.1) and we obtain

$$\begin{aligned} I(\hat{u}) &= \sum_{i=0}^{\overline{h}-1} \hat{u}_i^T Q \hat{u}_i = \hat{u}_{\overline{h}}^T Q_{\overline{h}} \hat{u}_{\overline{h}} = \left[Q_{\overline{h}}^{-1} R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f \right]^T Q_{\overline{h}} \left[Q_{\overline{h}}^{-1} R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f \right] \\ &= x_f^T W_{\overline{h}}^{-1} R_{\overline{h}} Q_{\overline{h}}^{-1} R_{\overline{h}}^T W_{\overline{h}}^{-1} x_f = x_f^T W_{\overline{h}}^{-1} x_f \end{aligned} \tag{5.15}$$

since by (5.3) $W_{\overline{h}}^{-1} R_{\overline{h}} Q_{\overline{h}}^{-1} R_{\overline{h}}^T = I_n$.

The optimal input sequence (5.5) and the minimal value of the performance index (5.8) can be computed using the following procedure.

Procedure 5.1

Step 1. Knowing the matrices $\overline{E}$, $\overline{E}^D$, $\overline{A}_\alpha$, $\overline{B}$, Q and using (4.1) and (5.3) compute the matrices $R_{\overline{h}}$ and $W_{\overline{h}}$ for a chosen $\overline{h}$ such that the matrix $R_{\overline{h}}$ contains at least n linearly independent columns.

Step 2. Using (5.5) find the input sequence $u_k \in \mathfrak{R}^m$, $k = 0, 1, \ldots, \overline{h} - 1$ satisfying the condition (5.2). If the condition (5.2) is not satisfied increase $\overline{h}$ by one and repeat the computation for $i + 1$.

Step 3. Using (5.8) compute the minimal value of the performance index $I(\hat{u})$.

VI. CONCLUDING REMARKS

Necessary and sufficient conditions for the reachability of fractional descriptor discrete-time linear systems have been established (Theorem 4.1). The minimum energy control problem for the fractional descriptor systems has been formulated and solved (Theorem 4.1). A procedure for the computation of optimal input sequences and minimal value of the performance index has been proposed (Procedure). The procedure has been demonstrated in numerical examples. An open problem is an extension of these considerations to fractional positive descriptor 1D discrete-time and 2D continuous-discrete linear systems.

REFERENCES

1. Kaczorek, T., *Selected Problems of Fractional Systems Theory*, Springer-Verlag, Berlin (2011a).
2. Ostalczyk, P., *Discrete Fractional Calculus*, World Scientific Publ. Co., New Jersey (2016).
3. Oldham, K. B. and J. Spanier, *The Fractional Calculus*, Academic Press, New York (1974).
4. Ostalczyk, P., *Epitome of the Fractional Calculus: Theory and its Applications in Automatics*, Wydawnictwo Politechniki Łódzkiej, Łódź (2008) (in Polish).
5. Podlubny, I., *Fractional Differential Equations*, Academic Press, San Diego (1999).
6. Dzieliński, A., D. Sierociuk, and G. Sarwas, "Ultracapacitor parameters identification based on fractional order model," *Proc. ECC'09*, Budapest (2009).

7. Radwan, A. G., A. M. Soliman, A. S. Elwakil, and A. Sedeek, “On the stability of linear systems with fractional-order elements,” *Chaos Solitons Fractals*, Vol. 40, No. 5, pp. 2317–2328 (2009).
8. Tenreiro Machado, J. A. and S. Ramiro Barbosa, “Functional dynamics in genetic algorithms,” *Workshop on Fractional Differentiation and Its Application*, Vol. 1, pp. 439–444 (2006).
9. Vinagre, B. M., C. A. Monje, and A. J. Calderon, “Fractional order systems and fractional order control actions,” *Proc. 41st IEEE Conference on Decision and Control*, Las Vegas, USA (2002).
10. Busłowicz, M., “Controllability, reachability and minimum energy control of fractional discrete-time linear systems with multiple delays in state,” *Bull. Pol. Acad. Sci. Tech.*, Vol. 62, No. 2, pp. 233–239 (2014).
11. Klamka, J., “Controllability and minimum energy control problem of fractional discrete-time systems,” In D. Baleanu, Z. B. Guvenc, J. A. Tenreiro Machado (Eds) *New Trends in Nanotechology and Fractional Calculus*, Springer-Verlag, New York, pp. 503–509 (2010).
12. Bru, R., C. Coll, S. Romero-Vivo, and E. Sanchez, *Some problems about structural properties of positive descriptor systems*, Lecture Notes in Control and Inform. Sci 294, Springer, Berlin, pp. 233–240 (2003).
13. Bru, R., C. Coll, and E. Sanchez, “About positively discrete-time singular systems,” In *System and Control: Theory and Applications, Electr. Comput. Eng. Ser., World Sci. Eng. Soc. Press*, CORE, Athens, pp. 44–48 (2000).
14. Bru, R., C. Coll, and E. Sanchez, “Structural properties of positive linear time-invariant difference-algebraic equations,” *Linear Alg. Appl.*, Vol. 349, pp. 1–10 (2002).
15. Campbell, S. L., C. D. Meyer, and N. J. Rose, “Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients,” *SIAMJ Appl. Math.*, Vol. 31, No. 3, pp. 411–425 (1976).
16. Dai, L., *Singular Control Systems*, Lectures Notes in Control and Information Sciences, Springer-Verlag, Berlin (1989).
17. Dodig, M. and M. Stosic, “Singular systems state feedbacks problems,” *Linear Alg. Appl.*, Vol. 431, No. 8, pp. 1267–1292 (2009).
18. Fahmy, M. M., and J. O’Reill, “Matrix pencil of closed-loop descriptor systems: Infinite-eigenvalues assignment,” *Int. J. Control*, Vol. 49, No. 4, pp. 1421–1431 (1989).
19. Gantmacher, F. R., *The Theory of Matrices*, Chelsea Publishing Co., New York (1960).
20. Guang-Ren, D., *Analysis and Design of Descriptor Linear Systems*, Springer, New York (2010).
21. Kaczorek, T., “Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils,” *Int. J. Appl. Math. Comput. Sci.*, Vol. 23, No. 1, pp. 29–34 (2013a).
22. Kaczorek, T., “Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm,” *Archives of Control Sciences*, Vol. 21, No. 3, pp. 287–298 (2011b).
23. Kaczorek, T., “Drazin inverse matrix method for fractional descriptor continuous-time linear systems,” *Bull. Pol. Acad. Sci. Tech.*, Vol. 62, No. 3, pp. 233–239 (2014a).
24. Kaczorek, T., “Drazin inverse matrix method for fractional descriptor discrete-time linear systems,” *Bull. Pol. Acad. Sci. Tech.*, Vol. 64, No. 2, pp. 395–399 (2016).
25. Kaczorek, T., “Infinite eigenvalue assignment by output-feedbacks for singular systems,” *Int. J. Appl. Math. Comput. Sci.*, Vol. 14, No. 1, pp. 19–23 (2004).
26. Kaczorek, T., *Linear Control Systems* 1, Research Studies Press, J. Wiley, New York (1992).
27. Kaczorek, T., “Minimum energy control of descriptor positive discrete-time linear systems,” *COMPEL*, Vol. 33, No. 3, pp. 976–988 (2014b).
28. Kaczorek, T., “Minimum energy control of fractional descriptor positive discrete-time linear systems with bounded inputs,” *Proc. of the 19th IFAC World Congress*, Vol. 47, No. 3, pp. 2909–2914 (2014c).
29. Kaczorek, T., “Minimum energy control of positive fractional descriptor continuous-time linear systems,” *IET Contr. Theory Appl.*, Vol. 8, No. 4, pp. 219–225 (2013b).
30. Kaczorek, T., “Reduction and decomposition of singular fractional discrete-time linear systems,” *Acta Mechanica et Automatica*, Vol. 5, No. 4, pp. 62–66 (2011c).
31. Kaczorek, T., “Singular fractional discrete-time linear systems,” *Control and Cybernetics*, Vol. 40, No. 3, pp. 753–761 (2011d).
32. Kucera, V. and P. Zagalak, “Fundamental theorem of state feedback for singular systems,” *Automatica*, Vol. 24, No. 5, pp. 653–658 (1988).
33. Sajewski, Ł., “Descriptor fractional discrete-time linear system and its solution – comparison of three different methods,” In R. Szewczyk, C. Zieliński, M. Kaliczyńska (Eds) *Challenges in Automation, Robotics and Measurement Techniques, Advances in Intelligent Systems and Computing* 440, Springer, Switzerland, pp. 37–50 (2016).
34. Van Dooren, P., “The computation of Kronecker’s canonical form of a singular pencil,” *Linear Alg. Appl.*, Vol. 27, pp. 103–140 (1979).
35. Virnik, E., “Stability analysis of positive descriptor systems,” *Linear Algebra and Its Applications*, Vol. 429, No. 10, pp. 2640–2659 (2008).

36. Kaczorek, T., "Positive linear systems with different fractional orders," *Bull. Pol. Acad. Sci. Tech.*, Vol. 58, No. 3, pp. 453–458 (2010).
37. Klamka, J., "Controllability of dynamical systems. A survey," *Bull. Pol. Acad. Sci. Tech.*, Vol. 61, No. 2, pp. 221–229 (2013).

A. PROCEDURE FOR COMPUTATION OF DRAZIN INVERSE MATRICES

To compute the Drazin inverse $\overline{E}^D$ of the matrix $\overline{E} \in \Re^{n\times n}$ defined by (2.7b) the following procedure is recommended.

Procedure A.1

Step 1. Find the pair of matrices $V \in \Re^{n\times r}$, $W \in \Re^{r\times n}$ such that

$$\overline{E} = VW, \text{rank } V = \text{rank } W = \text{rank } \overline{E} = r. \tag{A.1}$$

As the r columns (rows) of the matrix V (W) the r linearly independent columns (rows) of the matrix $\overline{E}$ can be chosen.

Step 2. Compute the nonsingular matrix

$$W\overline{E}V \in \Re^{r\times r}. \tag{A.2}$$

Step 3. The desired Drazin inverse matrix is given by

$$\overline{E}^D = V\left[W\overline{E}V\right]^{-1}W. \tag{A.3}$$

Proof. It will be shown that the matrix (A.3) satisfies the three conditions (2.7) of Definition . Taking into account that $\det WV \neq 0$ and (A.1) we obtain

$$\left[W\overline{E}V\right]^{-1} = [WVWV]^{-1} = [WV]^{-1}[WV]^{-1}. \tag{A.4}$$

Using (2.7a), (A.1), and (A.4) we obtain

$$\begin{aligned}\overline{E}\overline{E}^D &= VWV\left[W\overline{E}V\right]^{-1}W = VWV[WV]^{-1}[WV]^{-1}W\\ &= V[WV]^{-1}W\end{aligned} \tag{A.5a}$$

and

$$\begin{aligned}\overline{E}^D\overline{E} &= V\left[W\overline{E}V\right]^{-1}WVW\\ &= V[WV]^{-1}[WV]^{-1}WVW = V[WV]^{-1}W.\end{aligned} \tag{A.5b}$$

Therefore, the condition 2.7a is satisfied.
To check the condition 2.7b we compute

$$\begin{aligned}\overline{E}^D\overline{E}\overline{E}^D &= V\left[W\overline{E}V\right]^{-1}WVWV\left[W\overline{E}V\right]^{-1}W\\ &= V[WVWV]^{-1}WVWV\left[W\overline{E}V\right]^{-1}W\\ &= V\left[W\overline{E}V\right]^{-1}W = \overline{E}^D.\end{aligned} \tag{A.6}$$

Therefore, the condition 2.7b is also satisfied.
Using (2.7c), (A.1), (A.3), and (A.4) we obtain

$$\begin{aligned}\overline{E}^D\overline{E}^{q+1} &= V\left[W\overline{E}V\right]^{-1}W(VW)^{q+1} = V[WV]^{-1}[WV]^{-1}WVW(VW)^q\\ &= V[WV]^{-1}W(VW)^q = VW(VW)^{q-1} = (VW)^q = \overline{E}^q\end{aligned} \tag{A.7}$$

where q is the index of $\overline{E}$.
Therefore, the condition 2.7c is also satisfied.