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Nonuniform (h, k, μ, ν)-dichotomy with applications to nonautonomous dynamical systems ☆

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ABSTRACT

The paper develops a more general notion of dichotomy, referred to as "nonuniform (h, k, μ, ν)-dichotomy". The new notion unifies most versions of existing dichotomy in the literature, includes them in one comprehensive mathematics, and reveals more dichotomic behaviors of dynamical systems. Then we show that any linear nonautonomous system admits a nonuniform (h, k, μ, ν)-dichotomy if it has an (h, k) Lyapunov exponent with different signs or a Lyapunov function with different growth rates in the stable and unstable subspaces of solutions. This implies that the nonuniform (h, k, μ, ν)-dichotomy arises naturally and exists widely in the linear systems. Using the general notation, new versions of robustness, Hartman–Grobman theorem, and stable invariant manifold theorem for nonautonomous dynamical systems in Banach spaces will be presented. Especially, these new results not only generalize the corresponding ones in the literature, but also characterize the influences of different growth functions in the stable and unstable subspaces and in the uniform and nonuniform parts of a linear system for solving the above problems. This improves the application range of dichotomy in the nonautonomous systems.

1. Introduction

The well-known and established notion of exponential dichotomy (UED) in the linear analysis of nonautonomous systems, essentially originated in landmark work of Perron [58], extends the idea of hyperbolicity from autonomous systems to explicitly time-dependent ones. In the classical words of Coppel [31], "dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behavior

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Table 1
Type of dichotomies and their applications.

Dichotomy	Definition	Existence	Robustnesss	Topological conjugacy	Invariant manifolds
UED	[58,31,44,36, 23,1]	[58,31,29,57,44, 72,69,56,43,41, 37,47,23,42,49, 70,71,73,18]	[31,23,55,46,40, 48,52,60,63,62, 79]	[63,30,54,53,61, 64,68,75,76]	[25,32,33,75, 26,38,39]
(h,k)-D	[51,50,59]	[50]	[51,50]	[35]	[34]
NUED	[5,77,45,28]	[5,74,65,67,78, 80]	[78,6]	[3]	[4]
NUPD	[8]	[8]	[15]		[14,16]
N-(μ,ν)-D	[20,22]	[17]	[22,27]		[20,19]
ρ-NUED	[7,9]	[7]	[9]	[10]	[11]

Here, UED: Uniform exponential dichotomy; (h,k)-D: (h,k) dichotomy; NUED: Nonuniform exponential dichotomy; NUPD: Nonuniform polynomial dichotomy; N-(μ,ν)-D: Nonuniform (μ,ν)-dichotomy; ρ-NUED:ρ-nonuniform exponential dichotomy

for nonautonomous differential equations". Exponential dichotomies have been the subject of extensive research over (at least) the past four decades, leading to exciting new results in areas as diverse as functional differential equations [29,57,44,36], evolution equations [72,69,56,43,41,37,47,66], skew-product flows [23–25, 42], random systems or stochastic equations [1,32,33], torus [49]. We refer the reader to the references cited in the second line of Table 1 for more details.

However, dynamical systems exhibit various different kinds of dichotomic behavior and the existing notion of exponential dichotomy is too restrictive and can not well relate all those dichotomic behaviors. In 1992, Pinto [59] (also see the second line of Table 1) established (h,k)-dichotomy ((h,k)-D) for hyperbolic systems in the study of asymptotic equivalence. The (h,k)-dichotomy that possesses different functions h and k in the stable space and unstable space of every solution describes a great variety of dichotomic behaviors which are not ordinary not exponential dichotomies. The relatively recent development is that Barreira and Valls [5] introduced the concept of nonuniform exponential dichotomy (NUED) for nonuniformly hyperbolic systems (see the third line of Table 1). The nonuniform exponential dichotomy not only includes the corresponding (uniform) exponential dichotomy, but also allows to produce small destruction in the direct of the stability and instability of each solution with the increase of the initial time. In particular, any linear nonautonomous differential equation with global solutions and at least one negative Lyapunov exponent in the finite-dimensional space has a nonuniform exponential dichotomy [5]. On this basis, the notions of nonuniform polynomial dichotomy (NUPD) [8] and ρ-nonuniform exponential dichotomy (ρ-NUED) [7] were represented to characterize polynomial type or ρ-exponential type for the direct of the stability and instability of solutions. At the same time, Bento and Silva [20] gave the concept of nonuniform (μ,ν)-dichotomy (N-(μ,ν)-D) which describes the different functions for uniform part and nonuniform part of every solution. In view of the above discussion, it is important and of great interest problem is whether there is a unified framework to harmonize the existing various forms of dichotomies, to include them in one comprehensive mathematics, and to reveal more dichotomic behaviors of dynamical systems.

In this study, we introduce a new notion called nonuniform (h,k,μ,ν)-dichotomy, which is a more general framework and includes as particular cases most versions of uniform and nonuniform dichotomy usually found in the literature (see Definition 2.1). The new notion's wide scope is due to the fact that it allows different growth rates in the stable subspace and unstable subspace and in the uniform part and nonuniform part, and the comparison functions h,k,μ,ν are only assumed to be increasing and unbounded, but no specific form (e.g. exponential or polynomial) is prescribed for them. There is also evidence that the new dichotomy exhibits more widely dichotomic behavior for nonautonomous dynamical systems that can not be characterized by the existing dichotomy in the literatures (see Example 2.1).

As revealed in Section 3, where the existence of nonuniform (h,k,μ,ν)-dichotomy is characterized in terms of Lyapunov exponents and Lyapunov functions for a linear nonautonomous dynamical system in a finite-dimensional space. It is becoming clear that the notion of nonuniform (h, k, μ, ν)-dichotomy is not just a routine extension of the extensive notions of uniform or nonuniform dichotomies, but arises naturally and has a more comprehensive description for the qualitative and stable behavior of linear nonautonomous dynamical systems.

Dichotomy is of fundamental importance and plays a central role in nonautonomous dynamics. It entails a clear and simple structure of the (extended) phase space, which in turn can be utilized to address important questions regarding, for instance, robustness, topological conjugacy, and invariant manifolds of nonautonomous dynamical systems. Generally speaking, robustness or roughness of dichotomy, first proved by Massera and Schäffer [46], states that if a linear dynamical system admits a dichotomy, then all neighboring linear dynamical systems also have the same dichotomy with a similar projection. In theoretical studies it has been shown that the robustness or roughness is one of the most important and useful properties of dichotomy (see the fourth column of Table 1). There is increasing recognition that the linearization method of dynamical systems has an important position in exploring local behavior of a given nonlinear flow or nonlinear map for nonlinear dynamical systems. This is because growing evidence suggests that dynamic properties of a nonlinear dynamical system can be inferred by the corresponding linearized systems. As the most important linearization theorems, it is well established that the classical Grobman–Hartman theorem propose the topological conjugacy or topological equivalence between the nonlinear system and its corresponding linearization. It has been prove that the dichotomy is a useful method for establishing the Grobman–Hartman theorem and its generalizations (see the fifth column of Table 1). Studies suggest that the invariant manifolds are important in the geometric study of dynamics for nonlinear systems. It is worth noting that dichotomic theory has been becoming a powerful method to construct invariant manifolds of dynamical systems. We refer the reader to the sixth column of Table 1 and the references cited therein for detailed expositions of historical comments.

The another principal aim of the present paper is to provide corresponding new versions of the robustness (Section 4), Hartman–Grobman theorem (Section 5), and stable manifold theorem (Section 6) for nonautonomous continuous dynamical systems in Banach space. The study reveals that the new defined dichotomy still allows us to obtain results that generalize the ones in the literature. Moreover, our results also characterize some new features and the influences that can not be described by previous results for a linear system under linear and nonlinear perturbations when it allows different growth functions in the stable and unstable subspaces and in the uniform and nonuniform parts of each solution.

2. Definition of nonuniform (h, k, μ, ν)-dichotomy

Let $\mathcal{B}(X)$ be the space of bounded linear operators on a Banach space X. Consider the linear system

$$x' = A(t)x, \quad t \in \mathbb{R}, \tag{2.1}$$

where $A(t) \in \mathcal{B}(X)$. Let $T(t, s)$ be the evolution operator of (2.1) satisfying $T(t, s)x(s) = x(t), t, s \in \mathbb{R}$ for any solution $x(t)$ of (2.1). An increasing function $u : \mathbb{R} \to (0, +\infty)$ is said to be a growth rate if $u(0) = 1$, $\lim\limits_{t\to\infty} u(t) = \infty$ and $\lim\limits_{t\to-\infty} u(t) = 0$. In the following, we always assume that $h(t), k(t), \mu(t), \nu(t)$ are growth rates.

Definition 2.1. (2.1) is said to admit a *nonuniform* (h, k, μ, ν)*-dichotomy* on $\mathbb{R}$ if there exists a projection $P(t)$ such that $P(t)T(t, s) = T(t, s)P(s),\ t, s \in \mathbb{R}$ and there exist constants $a < 0 \le b$, $\varepsilon \ge 0$ and $K > 0$ such that

$$\begin{aligned} \|T(t,s)P(s)\| &\le K\left(\frac{h(t)}{h(s)}\right)^{a}\mu(|s|)^{\varepsilon},\ t\ge s,\\ \|T(t,s)Q(s)\| &\le K\left(\frac{k(s)}{k(t)}\right)^{-b}\nu(|s|)^{\varepsilon},\ s\ge t, \end{aligned} \tag{2.2}$$

where $Q(t) = \mathrm{id} - P(t)$ are the complementary projections of $P(t)$.

In Definition 2.1, four different functions or growth rates h, k, μ, ν are chosen to characterize the stable space and the unstable space and also the uniform part and the nonuniform part. Intuitively, on the level of generality adopted here, the parameters a, b and ε can be made part of h, k, μ and ν, at least in the non-trivial case $a\varepsilon < 0$ and this would simplify many expressions later on. For example, when $a\varepsilon < 0$, one can replace (2.2) with

$$\begin{aligned} \|T(t,s)P(s)\| &\le K\left(\frac{\bar{h}(t)}{\bar{h}(s)}\right)^{-1}\bar{\mu}(|s|),\ t\ge s,\\ \|T(t,s)Q(s)\| &\le K\left(\frac{\bar{k}(s)}{\bar{k}(t)}\right)^{-1}\bar{\nu}(|s|),\ s\ge t, \end{aligned}$$

where $\bar{h}(t), \bar{k}(t), \bar{\mu}(t), \bar{\nu}(t)$ are growth rates, or more general functions $a(t,s), b(t,s)$ (see [21], Bento and Silva consider two general bounded functions on $\mathbb{R}^+$). In the present paper, we deliberately prefer to use (2.2). The reason is that a and b play the role of Lyapunov exponents while ε measures the nonuniformity of dichotomies and the nonuniform (h, k, μ, ν)-dichotomy can be more closely connected with Lyapunov exponents, Lyapunov functions, and the theory of nonuniform hyperbolicity (see Section 3). This also implies that the nonuniform (h, k, μ, ν)-dichotomy exists widely in the theory of linear nonautonomous dynamical systems.

The nonuniform (h, k, μ, ν)-dichotomy is much more general and extends the existing notions of uniform and nonuniform dichotomies such as uniform exponential dichotomy ($h(t) = k(t) = e^t, \varepsilon = 0$) [31], (h, h)-dichotomy ($h(t) = k(t), \varepsilon = 0$) [59], (h, k)-dichotomy ($\varepsilon = 0$) [50], nonuniform exponential dichotomy ($h(t) = k(t) = e^t, \mu(t) = \nu(t) = e^{|t|}$) [5,77], nonuniform polynomial dichotomy ($h(t) = k(t) = \mu(t) = \nu(t) = t + 1$ for $t \in \mathbb{R}^+$) [8], nonuniform (μ, ν)-dichotomy ($h(t) = k(t) = \mu(t)$ and $\mu(t) = \nu(t) = \nu(t)$ for $t \in \mathbb{R}^+$) [22,20,17], ρ-nonuniform exponential dichotomy ($h(t) = k(t) = \mu(t) = \nu(t) = e^{\rho(t)}$ for $t \in \mathbb{R}^+$) [9] and so on.

The following contrived example shows the generality of the nonuniform (h, k, μ, ν)-dichotomy.

Example 2.1. Consider the differential equation in $\mathbb{R}^2$

$$z_1' = (-\eta_1\hat{h}'(t)/\hat{h}(t) + \zeta_1(t))z_1, \qquad z_2' = (\eta_3\hat{k}'(t)/\hat{k}(t) + \zeta_2(t))z_2 \tag{2.3}$$

for $t \in \mathbb{R}$, where

$$\begin{aligned} \zeta_1(t) &= \eta_2(\hat{\mu}'(t)/\hat{\mu}(t))(\log\hat{\mu}(t)\cos(\log\hat{\mu(t)}) - 1),\\ \zeta_2(t) &= \eta_2(\hat{\nu}'(t)/\hat{\nu}(t))(\log\hat{\nu}(t)\cos(\log\hat{\nu}(t)) - 1), \end{aligned}$$

$\hat{h}, \hat{k}, \hat{\mu}, \hat{\nu}$ are growth rates and η_1, η_2, η_3 are positive constants.

Set $P(t)(z_1, z_2) = z_1$ and $Q(t)(z_1, z_2) = z_2$ for $t \in \mathbb{R}$. Then we have

$$T(t,s)P(s) = \left(\hat{h}(t)/\hat{h}(s)\right)^{-\eta_1}e^{\eta_2 d_1(t)}, \quad T(t,s)Q(s) = \left(\hat{k}(t)/\hat{k}(s)\right)^{\eta_3}e^{\eta_2 d_2(t)},$$

where

$$d_1(t) = \log \hat{\mu}(t)(\sin \log \hat{\mu}(t) - 1) + \cos \log \hat{\mu}(t) - \cos \log \hat{\mu}(s) - \log \hat{\mu}(s)(\sin \log \hat{\mu}(s) - 1),$$
$$d_2(t) = \log \hat{\nu}(t)(\sin \log \hat{\nu}(t) - 1) + \cos \log \hat{\nu}(t) - \cos \log \hat{\nu}(s) - \log \hat{\nu}(s)(\sin \log \hat{\nu}(s) - 1).$$

It follows that

$$\|T(t,s)P(s)\| \le e^{2\eta_2} \left(\hat{h}(t)/\hat{h}(s)\right)^{-\eta_1} \hat{\mu}(s)^{2\eta_2} \le e^{2\eta_2} \left(\hat{h}(t)/\hat{h}(s)\right)^{-\eta_1} \hat{\mu}(|s|)^{2\eta_2}, \ t \ge s,$$
$$\|T(t,s)Q(s)\| \le e^{2\eta_2} \left(\hat{k}(s)/\hat{k}(t)\right)^{-\eta_3} \hat{\nu}(s)^{2\eta_2} \le e^{2\eta_2} \left(\hat{k}(s)/\hat{k}(t)\right)^{-\eta_3} \hat{\nu}(|s|)^{2\eta_2}, \ s \ge t.$$

This implies that (2.3) admits a nonuniform (h, k, μ, ν)-dichotomy with

$$K = e^{2\eta_2}, \quad a = -\eta_1, \quad b = \eta_3 \quad \text{and} \quad \varepsilon = 2\eta_2.$$

Particularly, when $\hat{h}, \hat{k}, \hat{\mu}, \hat{\nu}$ are chosen as different functions, we obtain new nonuniform dichotomies different to the existing ones. For example, if $\hat{h}(t) = t + 1$, $\hat{k}(t) = e^t$, $\hat{\mu}(t) = t^2 + 1$ and $\hat{\nu}(t) = e^{t^2}$ for $t \in \mathbb{R}^+$, then (2.3) admits a dichotomy that can not be covered by any known dichotomy in the literatures.

3. Existence of nonuniform (h, k, μ, ν)-dichotomy

In this section, in terms of appropriate Lyapunov exponents and Lyapunov functions, some sufficient criteria are established for linear dynamical systems in a finite-dimensional space to have a nonuniform (h, k, μ, ν)-dichotomy. Those results show that the notion of nonuniform (h, k, μ, ν)-dichotomy occurs in a very natural way for linear nonautonomous dynamical systems.

3.1. Lyapunov exponents and nonuniform (h, k, μ, ν)-dichotomy

Assume that $A(t)$ in (2.1) is a continuous $n \times n$ matrix function of block form, i.e., $A(t) = \mathrm{diag}(W_1(t), W_2(t))$ for $t \in \mathbb{R}^+$ and $\mathbb{R}^n = E \oplus F$, where $\dim E = l$ and $\dim F = n - l$. For $t \ge 0$, consider

$$x_1' = W_1(t)x_1, \tag{3.1}$$
$$x_2' = W_2(t)x_2 \tag{3.2}$$

and the corresponding adjoint systems

$$y_1' = -W_1(t)^* y_1, \tag{3.3}$$
$$y_2' = -W_2(t)^* y_2 \tag{3.4}$$

where $W_1(t)^*$ and $W_2(t)^*$ are the transpose of $W_1(t)$ and $W_2(t)$, respectively. Define $\varphi : E \to [-\infty, +\infty]$ and $\psi : F \to [-\infty, +\infty]$ by

$$\varphi(x_1^0) = \limsup_{t \to +\infty} \frac{\log \|x_1(t)\|}{\log h(t)} \quad \text{and} \quad \psi(x_2^0) = \limsup_{t \to +\infty} \frac{\log \|x_2(t)\|}{\log k(t)}, \tag{3.5}$$

where $x_1(t)$ is the solution of (3.1) with $x_1(0) = x_1^0$ and $x_2(t)$ is the solution of (3.2) with $x_2(0) = x_2^0$ (we assume that $\log 0 = -\infty$). Then, one has the following claims

(1) $\varphi(0) = -\infty$ and $\psi(0) = -\infty$;
(2) $\varphi(cx_1^0) = \varphi(x_1^0)$ and $\psi(cx_2^0) = \psi(x_2^0)$ for each $x_1^0 \in E, x_2^0 \in F$ and $c \in \mathbb{R} \setminus \{0\}$;

(3) for any $x_1', x_1'' \in E$ and $x_2', x_2'' \in F$,

$$\varphi(x_1' + x_1'') \le \max\{\varphi(x_1'), \varphi(x_1'')\}, \quad \psi(x_2' + x_2'') \le \max\{\psi(x_2'), \psi(x_2'')\};$$

(4) $x_1^1, \cdots, x_1^m$ are linearly independent if $\varphi(x_1^1), \cdots, \varphi(x_1^m)$ are distinct for $x_1^1, \cdots, x_1^m \in E \setminus \{0\}$; $x_2^1, \cdots, x_2^{m'}$ are linearly independent if $\psi(x_2^1), \cdots, \psi(x_2^{m'})$ are distinct for $x_2^1, \cdots, x_2^{m'} \in F \setminus \{0\}$;
(5) φ has at most $r \le l$ distinct values in $E \setminus \{0\}$, say $-\infty \le \lambda_1 < \cdots < \lambda_r \le +\infty$; ψ has at most $r' \le n-l$ distinct values in $F \setminus \{0\}$, say $-\infty \le \chi_1 < \cdots < \chi_{r'} \le +\infty$.

Therefore, from (1)–(3), it follows that (φ, ψ) is the so-called (h, k) Lyapunov exponent with respect to the linear equation (2.1).

Let $y_1(t)$ be the solution of (3.3) with $y_1(0) = y_1^0$ and $y_2(t)$ be the solution of (3.4) with $y_2(0) = y_2^0$. Consider $\bar\varphi : E \to [-\infty, +\infty]$ and $\bar\psi : F \to [-\infty, +\infty]$ defined by

$$\bar\varphi(y_1^0) = \limsup_{t\to+\infty} \frac{\log \|y_1(t)\|}{\log \bar h(t)} \quad \text{and} \quad \bar\psi(y_2^0) = \limsup_{t\to+\infty} \frac{\log \|y_2(t)\|}{\log \bar k(t)}, \tag{3.6}$$

where $\bar h(t), \bar k(t)$ are growth rates. Then

(6) $(\bar\varphi, \bar\psi)$ is the $(\bar h, \bar k)$ Lyapunov exponent;
(7) $\bar\varphi$ takes at most $\bar r \le l$ distinct values in $E \setminus \{0\}$, say $-\infty \le \bar\lambda_{\bar r} < \cdots < \bar\lambda_1 \le +\infty$; $\bar\psi$ takes at most $\bar r' \le n-l$ distinct values in $F \setminus \{0\}$, say $-\infty \le \bar\chi_{\bar r'} < \cdots < \bar\chi_1 \le +\infty$.

Let $\varrho_1, \cdots, \varrho_n$ and $\zeta_1, \cdots, \zeta_n$ be two bases of $\mathbb{R}^n$, they are said to be dual if $(\varrho_i, \zeta_j) = \omega_{ij}$ for every i, j, where $(\cdot,\cdot)$ is the standard inner product in $\mathbb{R}^n$ and ω_{ij} is the Kronecker symbol. In order to introduce the regularity coefficients of $\varphi, \bar\varphi$ and $\psi, \bar\psi$, $\lambda_i, \bar\lambda_i, \chi_i, \bar\chi_i$ are assumed to be finite.

Definition 3.1. The regularity coefficients of φ and $\bar\varphi$ is defined by

$$\gamma(\varphi, \bar\varphi) = \min\max\{\varphi(\delta_i) + \bar\varphi(\bar\delta_i) : 1 \le i \le l\},$$

where the minimum is taken over all dual bases $\delta_1, \cdots, \delta_l$ and $\bar\delta_1, \cdots, \bar\delta_l$ of E.

Definition 3.2. The regularity coefficients of ψ and $\bar\psi$ is defined by

$$\bar\gamma(\psi, \bar\psi) = \min\max\{\psi(\epsilon_i) + \bar\psi(\bar\epsilon_i) : 1 \le i \le n-l\},$$

where the minimum is taken over all dual bases $\epsilon_1, \cdots, \epsilon_{n-l}$ and $\bar\epsilon_1, \cdots, \bar\epsilon_{n-l}$ of F.

Theorem 3.1. *Assume that $\varphi(x_1) < 0$ for any $x_1 \in E \setminus \{0\}$ and $\psi(x_2) > 0$ for any $x_2 \in F \setminus \{0\}$ with $\lambda_r < 0 < \chi_1$. Then for any sufficiently small $\tilde\varepsilon > 0$, (2.1) with $A(t)$ admits a nonuniform (h, k, μ, ν)-dichotomy on $\mathbb{R}^+$ with*

$$a = \lambda_r + \tilde\varepsilon,\ b = \chi_1 + \tilde\varepsilon,\ \varepsilon = \max\{\gamma(\varphi, \bar\varphi),\ \bar\gamma(\psi, \bar\psi)\} + \tilde\varepsilon,\ \mu(t) = h(t)\bar h(t),\ \nu(t) = k(t)\bar k(t).$$

Proof. Let $X_1(t)$ be a fundamental solution matrix of (3.1). It is not difficult to show that $Y_1(t) = (X_1(t)^*)^{-1}$ is a fundamental solution matrix of (3.3). Let $m_j = \varphi(x_1^j(0))$ and $n_j = \bar\varphi(y_1^j(0))$ for $j = 1, \cdots, l$, where $x_1^1(t), \cdots, x_1^l(t)$ are the columns of $X_1(t)$ and $y_1^1(t), \cdots, y_1^l(t)$ are the columns of $Y_1(t)$. For any $\tilde\varepsilon > 0$ and $\bar t > 0$, it follows from (3.5) and (3.6) that there exists a constant $\bar K_1^1$ such that $\|x_1^j(t)\| \le \bar K_1^1 h(t)^{m_j+\tilde\varepsilon}$ and $\|y_1^j(t)\| \le \bar K_1^1 \bar h(t)^{n_j+\tilde\varepsilon}$ for $t \ge \bar t$ and $j = 1, \cdots, l$. On the other hand, there exists a sufficiently large

$\bar{K}_1^2$ such that $\|x_1^j(t)\| \leq \bar{K}_1^2 h(t)^{m_j+\tilde{\varepsilon}}$ and $\|y_1^j(t)\| \leq \bar{K}_1^2 \bar{h}(t)^{n_j+\tilde{\varepsilon}}$ for any $t \in [0, \bar{t}]$ and $j = 1, \cdots, l$. Let $\bar{K}_1 = \max\{\bar{K}_1^1, \bar{K}_1^2\}$, then

$$\|x_1^j(t)\| \leq \bar{K}_1 h(t)^{m_j+\tilde{\varepsilon}}, \quad \|y_1^j(t)\| \leq \bar{K}_1 \bar{h}(t)^{n_j+\tilde{\varepsilon}}, \quad t \geq 0, \quad j = 1, \cdots, l. \tag{3.7}$$

Note that $Y_1(t)^* X_1(t) = \mathrm{id}$, then $(x_1^i(t), y_1^j(t)) = \omega_{ij}$ for $i, j = 1, \cdots, l$. It is clear that, if the matrix $X_1(t)$ is appropriately select, then

$$\gamma(\varphi, \bar{\varphi}) = \max\{m_j + n_j : j = 1, \cdots, l\}.$$

Let $U(t,s) := X_1(t) X_1^{-1}(s)$ for $t \geq s$, then $U(t,s) = X_1(t) Y_1(s)^*$ and the entries of $U(t,s)$ are $u_{ik}(t,s) = \sum_{j=1}^{l} x_1^{ij}(t) y_1^{kj}(s)$. It follows from (3.7) that

$$\begin{aligned}
|u_{ik}(t,s)| &\leq \sum_{j=1}^{l} |x_1^{ij}(t)||y_1^{kj}(s)| \leq \sum_{j=1}^{l} \|x_1^j(t)\| \|y_1^j(s)\| \\
&\leq \sum_{j=1}^{l} \bar{K}_1^2 h(t)^{m_j+\tilde{\varepsilon}} \bar{h}(s)^{n_j+\tilde{\varepsilon}} \\
&\leq \sum_{j=1}^{l} \bar{K}_1^2 (h(t)/h(s))^{m_j+\tilde{\varepsilon}} h(s)^{m_j+\tilde{\varepsilon}} \bar{h}(s)^{n_j+\tilde{\varepsilon}} \\
&\leq \bar{K}_1^2 l (h(t)/h(s))^{\lambda_r+\tilde{\varepsilon}} (h(s)\bar{h}(s))^{\gamma(\varphi,\bar{\varphi})+\tilde{\varepsilon}}.
\end{aligned}$$

Let $\xi = \sum_{k=1}^{l} l_k e_k$ with $\|\xi\|^2 = \sum_{k=1}^{l} l_k^2 = 1$, where $e_1, \cdots, e_l$ are the standard orthogonal basis of E. Therefore,

$$\|U(t,s)\xi\|^2 = \left\| \sum_{i=1}^{l} \sum_{k=1}^{l} l_k u_{ik}(t,s) e_i \right\|^2 \leq \sum_{i=1}^{l} \left(\sum_{k=1}^{l} l_k^2 \sum_{k=1}^{l} u_{ik}(t,s)^2 \right) \leq \sum_{i=1}^{l} \sum_{k=1}^{l} u_{ik}(t,s)^2,$$

which implies that

$$\begin{aligned}
\|U(t,s)\| &\leq \left(\sum_{i=1}^{l} \sum_{k=1}^{l} u_{ik}(t,s)^2 \right)^{1/2} \\
&\leq \bar{K}_1^2 l^2 (h(t)/h(s))^{\lambda_r+\tilde{\varepsilon}} (h(s)\bar{h}(s))^{\gamma(\varphi,\bar{\varphi})+\tilde{\varepsilon}} \\
&\leq \bar{K}_1^2 l^2 (h(t)/h(s))^{a} \mu(s)^{\varepsilon}.
\end{aligned}$$

Let $X_2(t)$ be a fundamental solution matrix of (3.2), then $Y_2(t) = (X_2(t)^*)^{-1}$ is a fundamental solution matrix of (3.4). Let $\bar{m}_j = \psi(x_2^j(0))$ and $\bar{n}_j = \bar{\psi}(y_2^j(0))$ for $j = 1, \cdots, n-l$, where $x_2^1(t), \cdots, x_2^{n-l}(t)$ are the columns of $X_2(t)$ and $y_2^1(t), \cdots, y_2^{n-l}(t)$ are the columns of $Y_2(t)$. Proceeding similarly to the above, there exists a positive constant $\bar{K}_2$ such that

$$\|x_2^j(t)\| \leq \bar{K}_2 k(t)^{\bar{m}_j+\tilde{\varepsilon}} \quad \text{and} \quad \|y_2^j(t)\| \leq \bar{K}_2 \bar{k}(t)^{\bar{n}_j+\tilde{\varepsilon}}, \quad t \geq 0, \quad j = 1, \cdots, n-l. \tag{3.8}$$

Moreover, $(x_2^i(t), y_2^j(t)) = \omega_{ij}$ for $i, j = 1, \cdots, n-l$ since $Y_2(t)^* X_2(t) = \mathrm{id}$ and $X_2(t)$ can be appropriately selected such that

$$\bar{\gamma}(\psi, \bar{\psi}) = \max\{\bar{m}_j + \bar{n}_j : j = 1, \cdots, n-l\}.$$

Let $V(t,s) = X_2(t)X_2^{-1}(s)$ for $0 \le t \le s$. Then $V(t,s) = X_2(t)Y_2(s)^*$ and the entries of $V(t,s)$ are $v_{ik}(t,s) = \sum_{j=1}^{n-l} x_2^{ij}(t)y_2^{kj}(s)$. By (3.8), one has

$$\begin{aligned}|v_{ik}(t,s)| &\le \sum_{j=1}^{n-l} |x_2^{ij}(t)||y_2^{kj}(s)| \le \sum_{j=1}^{n-l} \|x_2^j(t)\|\|y_2^j(s)\| \\ &\le \sum_{j=1}^{n-l} \bar{K}_2^2 k(t)^{\bar{m}_j+\tilde{\varepsilon}}\bar{k}(s)^{\bar{n}_j+\tilde{\varepsilon}} \\ &\le \sum_{j=1}^{n-l} \bar{K}_2^2 (k(s)/k(t))^{-(\bar{m}_j+\tilde{\varepsilon})} k(s)^{\bar{m}_j+\tilde{\varepsilon}}\bar{k}(s)^{\bar{n}_j+\tilde{\varepsilon}} \\ &\le \bar{K}_2^2(n-l)(k(s)/k(t))^{-(\chi_1+\tilde{\varepsilon})}(k(s)\bar{k}(s))^{\bar{\gamma}(\psi,\bar{\psi})+\tilde{\varepsilon}}\end{aligned}$$

and

$$\|V(t,s)\bar{\xi}\|^2 = \left\|\sum_{i=1}^{n-l}\sum_{k=1}^{n-l} \bar{l}_k v_{ik}(t,s)\bar{e}_i\right\|^2 \le \sum_{i=1}^{n-l}\left(\sum_{k=1}^{n-l}\bar{l}_k^2\sum_{k=1}^{n-l} v_{ik}(t,s)^2\right) \le \sum_{i=1}^{n-l}\sum_{k=1}^{n-l} v_{ik}(t,s)^2,$$

where $\bar{\xi} = \sum_{k=1}^{n-l}\bar{l}_k\bar{e}_k$ and $\sum_{k=1}^{n-l}\bar{l}_k^2 = 1$ with $\bar{e}_1, \cdots, \bar{e}_{n-l}$ being the standard orthogonal basis of F. Therefore,

$$\begin{aligned}\|V(t,s)\| &\le \left(\sum_{i=1}^{n-l}\sum_{k=1}^{n-l} v_{ik}(t,s)^2\right)^{1/2} \\ &\le \bar{K}_2^2(n-l)^2(k(s)/k(t))^{-(\chi_1+\tilde{\varepsilon})}(k(s)\bar{k}(s))^{\bar{\gamma}(\psi,\bar{\psi})+\tilde{\varepsilon}} \\ &\le \bar{K}_2^2(n-l)^2(k(s)/k(t))^{-b}\nu(s)^{\varepsilon}.\end{aligned}$$

The proof is complete. □

Intuitively, it seems very restrictive that $A(t)$ is assumed to be of block form and $\lambda_i, \bar{\lambda}_i, \chi_i, \bar{\chi}_i$ are assumed to be finite. In fact, from the view point of Lyapunov's theory of regularity and ergodic theory, those assumptions are natural, typical and quite reasonable. For example, by the Oseledets–Pesin reduction theorem (see Theorem 3.5.5 in [2]), there exists a coordinate change (maintaining the values of the Lyapunov exponents) transforming the matrix $A(t)$ into a block form for μ-almost every x with respect to a time-independent decomposition. We refer the reader to [2] for a more detailed exposition.

Although the discussion is carried out for the relatively special case, in fact, one can confirm the existence of nonuniform (h,k,μ,ν)-dichotomy for more general dynamical systems. For example, consider the linear systems

$$x' = \widetilde{A}(t)x, \ t \in \mathbb{R}^+ \tag{3.9}$$

and

$$y' = \widetilde{B}(t)y = \begin{pmatrix}\widetilde{B}_1(t) & 0 \\ 0 & \widetilde{B}_2(t)\end{pmatrix} y, \ t \in \mathbb{R}^+, \tag{3.10}$$

where $\widetilde{A}(t), \widetilde{B}(t)$ are continuous $n \times n$ matrix functions, $\widetilde{B}_1(t)$ and $\widetilde{B}_2(t)$ are matrices of lower order than $\widetilde{B}(t)$. (3.9) is said to be *reducible* if there exist a continuously differentiable invertible matrix $S(t)$ and a constant $\widetilde{M} > 0$ such that

$$S' = \widetilde{A}(t)S - S\widetilde{B}(t), \quad \|S(t)\| \le \widetilde{M}, \quad \|S^{-1}(t)\| \le \widetilde{M}, \quad t \in \mathbb{R}^+.$$

Direct calculation shows that, if $y(t)$ is a solution of (3.10), then $x(t) = S(t)y(t)$ is a solution of (3.9). Therefore, if (3.9) is reducible and (3.10) admits a nonuniform $(h, k, \mu.\nu)$-dichotomy, then (3.9) also admits a nonuniform $(h, k, \mu.\nu)$-dichotomy.

3.2. Lyapunov functions and nonuniform (h, k, μ, ν)-dichotomy

Consider the function

$$H(t, x) = \langle S(t)x, x\rangle, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \tag{3.11}$$

where $S(t)$ is a given $n \times n$ matrix function and

$$\dot{H}(t, x) = \frac{d}{dh} H(t + h, T(t + h, t)x)|_{h=0}. \tag{3.12}$$

For fixed $\tau \in \mathbb{R}$, set

$$\begin{aligned} E^s_\tau &:= \{0\} \cup \{x \in \mathbb{R}^n : H(t, T(t, \tau)x) > 0, t \ge \tau\}, \\ E^u_\tau &:= \{0\} \cup \{x \in \mathbb{R}^n : H(t, T(t, \tau)x) < 0, t \ge \tau\}. \end{aligned} \tag{3.13}$$

Similar to Lemma 5 in [12], it is not difficult to show that E^s_τ and E^u_τ are both subspaces and $E^s_\tau \oplus E^u_\tau = \mathbb{R}^n$ for $\tau \in \mathbb{R}$. Let $U(t, \tau) = T(t, \tau)|_{E^s_\tau}$ and $V(t, \tau) = T(t, \tau)|_{E^u_\tau}$ for $t \ge \tau$. Then, one has

$$U(t, \tau)(E^s_\tau) = E^s_t \quad \text{and} \quad V(t, \tau)(E^u_\tau) = E^u_t. \tag{3.14}$$

Lemma 3.1 *([17]). Given continuous functions $w, f : [\tau, t] \to \mathbb{R}^+$ and $\hat{\eta} > 0$, if $w(x) - w(\tau) \ge \hat{\eta} \int_\tau^x w(z)f(z)dz$ for $x \in [\tau, t]$, then $w(x) \ge w(\tau) \exp(\hat{\eta} \int_\tau^x f(z)dz)$ for $x \in [\tau, t]$.*

Theorem 3.2. *Assume that*

(i) *there exists a symmetric invertible $n \times n$ matrices $S \in C^1(\mathbb{R}, \mathbb{R}^{n\times n})$ such that*

$$\limsup_{t\to\pm\infty} \frac{\log \|S(t)\|}{\log(\mu(|t|)^\varepsilon + \nu(|t|)^\varepsilon)} < \infty \tag{3.15}$$

and

$$S'(t) + S(t)A(t) + A(t)^* S(t) \le -\mathrm{id}, \quad t \in \mathbb{R}; \tag{3.16}$$

(ii) *for a given $\tau \in \mathbb{R}$, there exist constants $\hat{\eta}_i > 0, i = 1, 2$ such that*

$$x \in E^s_\tau, \ \dot{H}(t, U(t, \tau)x) \le -\hat{\eta}_1(h'(t)/h(t))|H(t, U(t, \tau)x)|, \tag{3.17}$$

$$x \in E^u_\tau, \ \dot{H}(t, V(t, \tau)x) \le -\hat{\eta}_2(k'(t)/k(t))|H(t, V(t, \tau)x)|; \tag{3.18}$$

(iii) *there exist constants* $\hat{d} > 0$, $\hat{k}_i, \hat{l}_i \geq 0, i = 1, 2$ *such that*

$$\|U(t,\tau)\| \leq \hat{l}_1 \mu(t)^{\hat{k}_1}, \; |t-\tau| \leq \hat{d} \tag{3.19}$$

and

$$\|V(t,\tau)\| \leq \hat{l}_2 \nu(t)^{\hat{k}_2}, \; |t-\tau| \leq \hat{d}; \tag{3.20}$$

(iv) $h(t)/h(\tau) \geq \mu(t)/\mu(\tau)$ *for* $t \geq \tau$ *when* $\hat{\eta}_1 > 2\hat{k}_1$.

Then (2.1) *has a nonuniform* (h, k, μ, ν)*-dichotomy on* $\mathbb{R}$.

Proof. First, we show that, for $\tau \in \mathbb{R}$ and $t \geq \tau$,

$$|H(\tau,x)| \geq \frac{\hat{d}}{\hat{l}_1^2} \mu(\tau)^{-2\hat{k}_1} \|x\|^2, \; x \in E_\tau^s; \quad |H(\tau,x)| \geq \frac{\hat{d}}{\hat{l}_2^2} \nu(\tau)^{-2\hat{k}_2} \|x\|^2, \; x \in E_\tau^u, \tag{3.21}$$

and

$$\begin{aligned} H(t, U(t,\tau)x) &\leq \left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1} H(\tau,x), \quad x \in E_\tau^s, \\ |H(t, V(t,\tau)x)| &\geq \left(\frac{k(t)}{k(\tau)}\right)^{\hat{\eta}_2} |H(\tau,x)|, \quad x \in E_\tau^u. \end{aligned} \tag{3.22}$$

In fact, let $x(t) = U(t,\tau)x$ for $x \in E_\tau^s$. By (3.11) and (3.16), one has

$$\begin{aligned} \frac{d}{dt} H(t, x(t)) &= \langle S'(t)x(t), x(t)\rangle + \langle S(t)x'(t), x(t)\rangle + \langle S(t)x(t), x'(t)\rangle \\ &= \langle (S'(t) + S(t)A(t) + A(t)^* S(t))x(t), x(t)\rangle \leq -\|x(t)\|^2. \end{aligned} \tag{3.23}$$

It follows from $H(\tau + \hat{d}, x(\tau + \hat{d})) \geq 0, x \in E_\tau^s$, (3.19) and (3.23) that

$$\begin{aligned} H(\tau,x) &\geq H(\tau,x) - H(\tau+\hat{d}, x(\tau+\hat{d})) \\ &= -\int_\tau^{\tau+\hat{d}} \frac{d}{dr} H(r, x(r))dr \geq \int_\tau^{\tau+\hat{d}} \|x(r)\|^2 dr \\ &= \int_\tau^{\tau+\hat{d}} \|U(r,\tau)x\|^2 dr \geq \|x\|^2 \int_\tau^{\tau+\hat{d}} \frac{dr}{\|U(\tau,r)\|^2} \\ &\geq \|x\|^2 \int_\tau^{\tau+\hat{d}} \frac{1}{\hat{l}_1^2} \mu(\tau)^{-2\hat{k}_1} dr = \frac{\hat{d}}{\hat{l}_1^2} \mu(\tau)^{-2\hat{k}_1} \|x\|^2. \end{aligned}$$

That is, (3.21) is valid.

For $x \in E_\tau^s$, define $\theta : [\tau, t] \to \mathbb{R}^+$ by $\theta(s) = H(t+\tau-s, U(t+\tau-s,\tau)x)$. (3.17) together with Lemma 3.1 gives

$$\theta(\tau) - \theta(t) = H(t, U(t,\tau)x) - H(\tau, x) = \int_\tau^t \dot{H}(v, U(v,\tau)x)dv \leq -\hat{\eta}_1 \int_\tau^t \frac{h'(v)}{h(v)} H(v, U(v,\tau)x)dv$$
$$= -\hat{\eta}_1 \int_\tau^t \frac{h'(v)}{h(v)}\theta(t+\tau-v)dv = -\hat{\eta}_1 \int_\tau^t \frac{h'(t+\tau-s)}{h(t+\tau-s)}\theta(s)ds,$$

which means

$$\theta(t) - \theta(\tau) \geq \hat{\eta}_1 \int_\tau^t \frac{h'(t+\tau-s)}{h(t+\tau-s)}\theta(s)ds$$

and

$$\theta(t) \geq \theta(\tau)(h(t)/h(\tau))^{\hat{\eta}_1}.$$

Then the first inequality of (3.22) holds.

For $x \in E_\tau^u$, note that $H(\tau - \hat{d}, x(\tau - \hat{d})) \leq 0$, by (3.20) and (3.23), one has

$$|H(\tau,x)| \geq |H(\tau,x)| - |H(\tau-\hat{d}, x(\tau-\hat{d}))| = H(\tau-\hat{d}, x(\tau-\hat{d})) - H(\tau,x)$$
$$= -\int_{\tau-\hat{d}}^{\tau} \frac{d}{dr} H(r, x(r))dr \geq \int_{\tau-\hat{d}}^{\tau} \|x(r)\|^2 dr = \int_{\tau-\hat{d}}^{\tau} \|V(r,\tau)x\|^2 dr$$
$$\geq \|x\|^2 \int_{\tau-\hat{d}}^{\tau} \frac{dr}{\|V(\tau,r)\|^2} dr \geq \frac{\hat{d}}{\hat{l}_2^2}\nu(\tau)^{-2\hat{k}_2}\|x\|^2.$$

For a given $\tau \in \mathbb{R}$ and any $x \in E_\tau^u$, it follows from (3.18) that

$$|H(t, V(t,\tau))x| - |H(\tau,x)| = H(\tau, V(\tau,\tau)x) - H(t, V(t,\tau)x)$$
$$= -\int_\tau^t \dot{H}(v, V(v,\tau)x)dv$$
$$\geq \hat{\eta}_2 \int_\tau^t \frac{k'(v)}{k(v)} |H(v, V(v,\tau)x)|\, dv, \quad t \geq \tau.$$

By Lemma 3.1, the second inequality of (3.22) holds.

By (3.15), direct calculation shows that there exist constants $\hat{a}, \hat{b}$ such that

$$\|S(t)\| \leq \hat{a}(\mu(|t|)^\varepsilon + \nu(|t|)^\varepsilon)^{\hat{b}}, \quad t \in \mathbb{R}. \tag{3.24}$$

Next we establish the norm bounds of the evolution operators $U(t,\tau)$ and $V(t,\tau)$, that is, for $\tau \in \mathbb{R}$ and $t \geq \tau$.

$$\begin{aligned} \|U(t,\tau)\|^2 &\leq \frac{\hat{a}\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1+2\hat{k}_1} (\mu(|\tau|)^\varepsilon + \nu(|\tau|)^\varepsilon)^{\hat{b}}\mu(\tau)^{2\hat{k}_1},\ x \in E_\tau^s, \\ \|V(t,\tau)^{-1}\|^2 &\leq \frac{\hat{a}\hat{l}_2^2}{\hat{d}}\left(\frac{k(t)}{k(\tau)}\right)^{-\hat{\eta}_2} (\mu(|t|)^\varepsilon + \nu(|t|)^\varepsilon)^{\hat{b}}\nu(t)^{2\hat{k}_2},\ x \in E_\tau^u. \end{aligned} \tag{3.25}$$

For any $x \in E_\tau^s$ and $t \geq \tau$, it follows from (3.21), (3.22), (3.14) and condition (iv) that

$$\begin{aligned}
\|U(t,\tau)x\|^2 &\leq \frac{\hat{l}_1^2}{\hat{d}}\mu(t)^{2\hat{k}_1}|H(t,U(t,\tau)x)| \leq \frac{\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1}\mu(t)^{2\hat{k}_1}H(\tau,x)\\
&\leq \frac{\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1}\mu(t)^{2\hat{k}_1}\|S(\tau)\|\|x\|^2\\
&\leq \frac{\hat{a}\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1}\mu(t)^{2\hat{k}_1}(\mu(|\tau|)^\varepsilon+\nu(|\tau|)^\varepsilon)^{\hat{b}}\|x\|^2\\
&= \frac{\hat{a}\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1}\left(\frac{\mu(t)}{\mu(\tau)}\right)^{2\hat{k}_1}(\mu(|\tau|)^\varepsilon+\nu(|\tau|)^\varepsilon)^{\hat{b}}\mu(\tau)^{2\hat{k}_1}\|x\|^2\\
&\leq \frac{\hat{a}\hat{l}_1^2}{\hat{d}}\left(\frac{h(t)}{h(\tau)}\right)^{-\hat{\eta}_1+2\hat{k}_1}(\mu(|\tau|)^\varepsilon+\nu(|\tau|)^\varepsilon)^{\hat{b}}\mu(\tau)^{2\hat{k}_1}\|x\|^2,
\end{aligned}$$

which implies that the first inequality of (3.25) holds.

For any $x \in E_\tau^u$, by (3.24) and (3.14), one has

$$|H(t,V(t,\tau)x)| \leq \hat{a}(\mu(|t|)^\varepsilon+\nu(|t|)^\varepsilon)^{\hat{b}}\|V(t,\tau)x\|^2.$$

Then

$$\begin{aligned}
\|V(t,\tau)x\|^2 &\geq \frac{1}{\hat{a}}(\mu(|t|)^\varepsilon+\nu(|t|)^\varepsilon)^{-\hat{b}}\left(\frac{k(t)}{k(\tau)}\right)^{\hat{\eta}_2}\frac{\hat{d}}{\hat{l}_2^2}\nu(\tau)^{-2\hat{k}_2}\|x\|^2\\
&\geq \frac{\hat{d}}{\hat{a}\hat{l}_2^2}\nu(t)^{-2\hat{k}_2}(\mu(|t|)^\varepsilon+\nu(|t|)^\varepsilon)^{-\hat{b}}\left(\frac{k(t)}{k(\tau)}\right)^{\hat{\eta}_2}\left(\frac{\nu(t)}{\nu(\tau)}\right)^{2\hat{k}_2}\|x\|^2\\
&\geq \frac{\hat{d}}{\hat{a}\hat{l}_2^2}\nu(t)^{-2\hat{k}_2}(\mu(|t|)^\varepsilon+\nu(|t|)^\varepsilon)^{-\hat{b}}\left(\frac{k(t)}{k(\tau)}\right)^{\hat{\eta}_2}\|x\|^2,
\end{aligned}$$

that is, the second inequality of (3.25) holds.

Let $P(t) : \mathbb{R}^n \to E_t^s$, $Q(t) : \mathbb{R}^n \to E_t^u$, $t \in \mathbb{R}$ be the projections. Then

$$\begin{aligned}
&\|T(t,\tau)P(\tau)\| \leq \|T(t,\tau)|_{E_\tau^t}\|\|P(\tau)\| = \|U(t,\tau)\|\|\|P(\tau)\|,\ t \geq \tau,\\
&\|T(t,\tau)Q(\tau)\| \leq \|T(t,\tau)|_{E_\tau^t}\|\|Q(\tau)\| = \|V(t,\tau)\|\|\|Q(\tau)\|,\ t \leq \tau.
\end{aligned} \tag{3.26}$$

For any $z \in \mathbb{R}^n$, we have $z = P(t)z + Q(t)z$, $P(t)z \in E_t^s$ and $Q(t)z \in E_t^u$. By (3.21), one has

$$-H(t,P(t)z)+\frac{\hat{d}}{\hat{l}_1^2}\mu(t)^{-2\hat{k}_1}\|P(t)z\|^2+H(t,Q(t)z)+\frac{\hat{d}}{\hat{l}_2^2}\nu(t)^{-2\hat{k}_2}\|Q(t)z\|^2 \leq 0.$$

Note that

$$\begin{aligned}
&\frac{\hat{d}}{\hat{l}_1^2}\mu(t)^{-2\hat{k}_1}\|P(t)z-\frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z\|^2+\frac{\hat{d}}{\hat{l}_2^2}\nu(t)^{-2\hat{k}_2}\|Q(t)z+\frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z\|^2\\
&=\frac{\hat{d}}{\hat{l}_1^2}\mu(t)^{-2\hat{k}_1}\|P(t)z\|^2+\frac{\hat{d}}{\hat{l}_2^2}\nu(t)^{-2\hat{k}_2}\|Q(t)z\|^2-H(t,P(t)z)+H(t,Q(t)z)\\
&\quad+\frac{\hat{l}_1^2}{4\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\|^2+\frac{\hat{l}_2^2}{4\hat{d}}\nu(t)^{2\hat{k}_2}\|S(t)z\|^2
\end{aligned}$$

$$\leq \frac{\hat{l}_1^2}{4\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\|^2 + \frac{\hat{l}_2^2}{4\hat{d}}\nu(t)^{2\hat{k}_2}\|S(t)z\|^2.$$

In order to complete the proof, one only needs to prove norm estimation of $P(t)$ and $Q(t)$. The discussion is divided into two cases.

Case 1. If $(\hat{d}/\hat{l}_1^2)\mu(t)^{-2\hat{k}_1} \leq \hat{d}/(\hat{l}_2^2)\nu(t)^{-2\hat{k}_2}$, then

$$\|P(t)z - \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z\|^2 + \|Q(t)z + \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z\|^2 \leq \frac{\hat{l}_1^4}{2\hat{d}^2}\mu(t)^{4\hat{k}_1}\|S(t)z\|^2.$$

Whence,

$$\begin{aligned}
\|P(t)z\| &= \|P(t)z - \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z + \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z\| \\
&\leq \|P(t)z - \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z\| + \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\| \\
&\leq \frac{\sqrt{2}\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\| + \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\| \\
&= \frac{(\sqrt{2}+1)\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\|,
\end{aligned}$$

and

$$\begin{aligned}
\|Q(t)z\| &= \|Q(t)z + \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z - \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z\| \\
&\leq \|Q(t)z + \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z\| + \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}\|S(t)z\| \\
&\leq \frac{\sqrt{2}\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\| + \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\| \\
&= \frac{(\sqrt{2}+1)\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}\|S(t)z\|.
\end{aligned}$$

Case 2. If $(\hat{d}/\hat{l}_1^2)\mu(t)^{-2\hat{k}_1} > \hat{d}/(\hat{l}_2^2)\nu(t)^{-2\hat{k}_2}$, then

$$\|P(t)z - \frac{\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}S(t)z\|^2 + \|Q(t)z + \frac{\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}S(t)z\|^2 \leq \frac{\hat{l}_2^4}{2\hat{d}^2}\nu(t)^{4\hat{k}_2}\|S(t)z\|^2.$$

Similarly, one has

$$\|P(t)z\| \leq \frac{(\sqrt{2}+1)\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}\|S(t)z\|, \ \|Q(t)z\| \leq \frac{(\sqrt{2}+1)\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}\|S(t)z\|.$$

Then

$$\begin{aligned}
\|P(t)\| &\leq \max\left\{\frac{(\sqrt{2}+1)\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}, \frac{(\sqrt{2}+1)\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}\right\}\|S(t)\|, \\
\|Q(t)\| &\leq \max\left\{\frac{(\sqrt{2}+1)\hat{l}_1^2}{2\hat{d}}\mu(t)^{2\hat{k}_1}, \frac{(\sqrt{2}+1)\hat{l}_2^2}{2\hat{d}}\nu(t)^{2\hat{k}_2}\right\}\|S(t)\|.
\end{aligned} \tag{3.27}$$

It follows from (3.24), (3.25), (3.26) and (3.27) that Theorem 3.2 is valid. □

In order to further characterize the nonuniform (h, k, μ, ν)-dichotomy, we establish a necessary condition for the existence of nonuniform (h, k, μ, ν)-dichotomies for the linear system (2.1).

Theorem 3.3. *If (2.1) admits a nonuniform (h, k, μ, ν)-dichotomy (2.2) on $\mathbb{R}$ with $h, k \in C^1(\mathbb{R}, \mathbb{R}^+)$, then there exists a symmetric invertible matrix function $S \in C^1(\mathbb{R}, \mathbb{R}^{n\times n})$ such that*

$$\limsup_{t\to\pm\infty} \frac{\log \|S(t)\|}{\log(\mu(|t|)^\varepsilon + \nu(|t|)^\varepsilon)} < \infty \tag{3.28}$$

and

$$S'(t) + S(t)A(t) + A(t)^*S(t) \le -\left(P(t)^*P(t)\frac{h'(t)}{h(t)} + Q(t)^*Q(t)\frac{k'(t)}{k(t)}\right), \quad t \in \mathbb{R}. \tag{3.29}$$

Moreover, there exist constants $\bar{k}_1$ and $\bar{k}_2$ such that

$$\begin{aligned} \dot{H}(t, T(t,\tau)x) &\le -\bar{k}_1(h'(t)/h(t))|H(t, T(t,\tau)x)|, \ x \in F_\tau^s, \\ \dot{H}(t, T(t,\tau)x) &\le -\bar{k}_2(k'(t)/k(t))|H(t, T(t,\tau)x)|, \ x \in F_\tau^u \end{aligned} \tag{3.30}$$

for given $\tau \in \mathbb{R}$, where $F_\tau^s = P(\tau)(\mathbb{R}^n)$, $F_\tau^u = Q(\tau)(\mathbb{R}^n)$ and $H(t, x)$ is as in (3.11).

Proof. For some positive constant $0 < \bar{d} < \min\{-a, b\}$, set

$$\begin{aligned} S(t) = &\int_t^\infty T(v,t)^*P(v)^*P(v)T(v,t)\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d}))}\frac{h'(v)}{h(v)}dv \\ &- \int_{-\infty}^t T(v,t)^*Q(v)^*Q(v)T(v,t)\left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})}\frac{k'(v)}{k(v)}dv. \end{aligned} \tag{3.31}$$

It is clear that $S(t)$ is symmetric for each $t \in \mathbb{R}$. Note that $\partial T(\tau, t)/\partial t = -T(\tau, t)A(t)$ and $\partial T(\tau, t)^*/\partial t = -A(t)^*T(\tau, t)^*$, then $S(t)$ is continuously differentiable. From (3.11), it follows that

$$\begin{aligned} H(t, x) = &\int_t^\infty \|T(v,t)P(t)x\|^2\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})}\frac{h'(v)}{h(v)}dv \\ &- \int_{-\infty}^t \|T(v,t)Q(t)x\|^2\left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})}\frac{k'(v)}{k(v)}dv, \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.32}$$

If (2.1) admits a nonuniform (h, k, μ, ν)-dichotomy, then $\mathbb{R}^n = F_t^s \oplus F_t^u$, where $F_t^s = P(t)(\mathbb{R}^n)$ and $F_t^u = Q(t)(\mathbb{R}^n)$. Note that $\langle S(t)x, x\rangle = H(t, x) > 0$ for $x \in F_t^s \setminus \{0\}$ and $\langle S(t)x, x\rangle = H(t, x) < 0$ for $x \in F_t^u \setminus \{0\}$, then $S(t)|_{F_t^s}$ and $S(t)|_{F_t^u}$ are both invertible. This implies that $S(t)$ is invertible for each $t \in \mathbb{R}$.

By (2.2), one has

$$\begin{aligned} L_1 =: &\int_t^\infty \|T(v,t)P(t)\|^2\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})}\frac{h'(v)}{h(v)}dv \\ \le\ &K^2\int_t^\infty \left(\frac{h(v)}{h(t)}\right)^{2a}\mu(|t|)^{2\varepsilon}\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})}\frac{h'(v)}{h(v)}dv \le (K^2/2\bar{d})\mu(|t|)^{2\varepsilon} \end{aligned}$$

and

$$L_2 =: \int_{-\infty}^{t} \|T(v,t)Q(t)\|^2 \left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})} \frac{k'(v)}{k(v)} dv$$
$$\leq K^2 \int_{-\infty}^{t} \left(\frac{k(t)}{k(v)}\right)^{-2b} v(|t|)^{2\varepsilon} \left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})} \frac{k'(v)}{k(v)} dv \leq (K^2/2\bar{d})\nu(|t|)^{2\varepsilon}.$$

Then

$$\|S(t)\| = \sup_{x\neq 0} \frac{|H(t,x)|}{\|x\|^2} \leq \sup_{x\neq 0} \frac{\|S(t)x\|\|x\|}{\|x\|^2} \leq L_1 + L_2 \leq (K^2/2\bar{d})(\mu(|t|)^{2\varepsilon} + \nu(|t|)^{2\varepsilon}), \tag{3.33}$$

which implies that

$$\limsup_{t\to\pm\infty} \frac{\log\|S(t)\|}{\log(\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon})} \leq \limsup_{t\to\pm\infty} \frac{\log(K^2/2\bar{d})(\mu(|t|)^{2\varepsilon} + \nu(|t|)^{2\varepsilon})}{\log(\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon})}$$
$$\leq \limsup_{t\to\pm\infty} \frac{\log(K^2/2\bar{d})}{\log(\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon})} + 2 < +\infty.$$

Direct calculation leads to

$$\begin{aligned} S'(t) =& - P(t)^*P(t)\frac{h'(t)}{h(t)} - Q(t)^*Q(t)\frac{k'(t)}{k(t)} - A(t)^*S(t) - S(t)A(t) \\ &+ 2(a+\bar{d})\frac{h'(t)}{h(t)} \int_{t}^{\infty} T(v,t)^*P(v)^*P(v)T(v,t)\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})} \frac{h'(v)}{h(v)} dv \\ &- 2(b-\bar{d})\frac{k'(t)}{k(t)} \int_{-\infty}^{t} T(v,t)^*Q(v)^*Q(v)T(v,t)\left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})} \frac{k'(v)}{k(v)} dv. \end{aligned} \tag{3.34}$$

Thus, (3.29) holds since $\bar{d} < \min\{-a, b\}$.

For each given $\tau \in \mathbb{R}$ and any $x \in \mathbb{R}^n$, by (3.34) and $h'(t), k'(t) \geq 0$, we have

$$\begin{aligned} \frac{dH(t,T(t,\tau)x)}{dt} =& \langle S'(t)T(t,\tau)x, T(t,\tau)x\rangle + \langle S(t)\frac{\partial T(t,\tau)}{\partial t}x, T(t,\tau)x\rangle \\ &+ \langle S(t)T(t,\tau)x, \frac{\partial T(t,\tau)}{\partial t}x\rangle \\ =& \langle (S'(t) + S(t)A(t) + A(t)^*S(t))T(t,\tau)x, T(t,\tau)x\rangle \\ \leq& -\left\langle \left(P(t)^*P(t)\frac{h'(t)}{h(t)} + Q(t)^*Q(t)\frac{k'(t)}{k(t)}\right) T(t,\tau)x, T(t,\tau)x \right\rangle \\ &+ 2(a+\bar{d})\frac{h'(t)}{h(t)} \int_{t}^{\infty} \|T(v,t)P(t)T(t,\tau)x\|^2 \left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})} \frac{h'(v)}{h(v)} dv \\ &- 2(b-\bar{d})\frac{k'(t)}{k(t)} \int_{-\infty}^{t} \|T(v,t)Q(t)T(t,\tau)x\|^2 \left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})} \frac{k'(v)}{k(v)} dv \end{aligned}$$

$$\leq 2(a+\bar{d})\frac{h'(t)}{h(t)}\int_t^\infty \|T(v,\tau)P(\tau)x\|^2\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})}\frac{h'(v)}{h(v)}dv$$

$$-2(b-\bar{d})\frac{k'(t)}{k(t)}\int_{-\infty}^t \|T(v,\tau)Q(\tau)x\|^2\left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})}\frac{k'(v)}{k(v)}dv.$$

Moreover, by (3.32), one has

$$\begin{aligned}\frac{dH(t,T(t,\tau)x)}{dt} &\leq 2(a+\bar{d})\frac{h'(t)}{h(t)}\int_t^\infty \|T(v,\tau)P(\tau)x\|^2\left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})}\frac{h'(v)}{h(v)}dv\\ &= 2(a+\bar{d})\frac{h'(t)}{h(t)}|H(t,T(t,\tau)x)|, \quad x\in F_\tau^s\end{aligned}$$

$$\begin{aligned}\frac{dH(t,T(t,\tau)x)}{dt} &\leq -2(b-\bar{d})\frac{k'(t)}{k(t)}\int_{-\infty}^t \|T(v,\tau)Q(\tau)x\|^2\left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})}\frac{k'(v)}{k(v)}dv\\ &= -2(b-\bar{d})\frac{k'(t)}{k(t)}|H(t,T(t,\tau)x)|, \quad x\in F_\tau^u.\end{aligned}$$

Therefore, (3.30) holds with $\bar{k}_1 = -2(a+\bar{d})$ and $\bar{k}_2 = 2(b-\bar{d})$. □

Corollary 3.1. *If (iii) and (iv) in Theorem 3.2 hold and*

$$P(t)^*P(t)\frac{h'(t)}{h(t)} + Q(t)^*Q(t)\frac{k'(t)}{k(t)} \geq \mathrm{id},$$

then (2.1) admits a nonuniform (h,k,μ,ν)-dichotomy as in (2.2) on $\mathbb{R}$ if and only if (i) and (ii) in Theorem 3.2 hold.

Remark 3.1. The results in Theorem 3.1 suggest that if a linear nonautonomous system has a negative (h,k) Lyapunov exponent in the stable subspace and a positive (h,k) Lyapunov exponent in the unstable subspace, then it admits a nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}^+$. In Theorems 3.2 and 3.3, we establish the sufficient condition and the necessary condition for the existence of nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$ by using Lyapunov functions with different growth rates in the stable and unstable direct of solutions. These studies show that the nonuniform (h,k,μ,ν)-dichotomic behavior arises naturally and exists widely in the linear systems and can not be characterized by the existing dichotomic behavior.

4. Robustness of nonuniform (h,k,μ,ν)-dichotomy

This section focuses on the robustness or roughness of nonuniform (h,k,μ,ν)-dichotomy, which is one of the most important properties of dichotomy. The principal aim is to show that the nonuniform (h,k,μ,ν)-dichotomy persists under sufficiently small linear perturbations of the original dynamics.

We first establish the robustness of nonuniform (h,k,μ,ν)-dichotomy in a finite-dimensional space by using Lyapunov functions. Consider the linear perturbed system

$$x' = (A(t)+B(t))x, \tag{4.1}$$

where $A, B \in C(\mathbb{R},\mathbb{R}^{n\times n})$. Let $\widehat{T}(t,\tau)$ be the evolution operator associated to (4.1) and

$$\begin{aligned}\widehat{E}^s_\tau &:= \{0\} \cup \{x \in \mathbb{R}^n : H(t, \widehat{T}(t,\tau)x) > 0, t \geq \tau\},\\ \widehat{E}^u_\tau &:= \{0\} \cup \{x \in \mathbb{R}^n : H(t, \widehat{T}(t,\tau)x) < 0, t \geq \tau\}\end{aligned} \tag{4.2}$$

for each given $\tau \in \mathbb{R}$. It is not difficult to show that $\widehat{E}^s_\tau$ and $\widehat{E}^u_\tau$ are both subspaces and $\widehat{E}^s_\tau \oplus \widehat{E}^u_\tau = \mathbb{R}^n$ for $\tau \in \mathbb{R}$.

Theorem 4.1. *Assume that (2.1) admits a nonuniform (h, k, μ, ν)-dichotomy as in (2.2) on $\mathbb{R}$ and there exist positive constants $\hat{l}$, $\hat{\delta}$ and $\hat{d}$ such that*

$$\|T(t,\tau)\| \leq \hat{l} \min\{\mu(t)^{2\varepsilon}, \nu(t)^{2\varepsilon}\}, \quad |t-\tau| \leq \hat{d}, \tag{4.3}$$

$$\|B(t)\| \leq \hat{\delta}(\mu(|t|) + \nu(|t|))^{-2\varepsilon}. \tag{4.4}$$

Moreover, if (iv) in Theorem 3.2 holds and

$$P^*(t)P(t)(h'(t)/h(t)) + Q^*(t)Q(t)(k'(t)/k(t)) - (\hat{\delta}K^2/\bar{d})\,\mathrm{id} \geq \mathrm{id}, \tag{4.5}$$

for a positive constant $\bar{d} < \min\{-a, b\}$, then (4.1) also admits a nonuniform (h, k, μ, ν)-dichotomy on $\mathbb{R}$.

Proof. We first claim that condition (iii) in Theorem 3.2 holds. In fact, by the variation of constant method, one has

$$\widehat{T}(t,\tau) = T(t,\tau) + \int_\tau^t T(t,r)B(r)\widehat{T}(r,\tau)dr$$

for each $t, \tau \in \mathbb{R}$ and $|t-\tau| \leq \bar{d}$. Then

$$\begin{aligned}\widehat{U}(t,\tau) = \widehat{T}(t,\tau)|_{\widehat{E}^s_\tau} &= T(t,\tau)|_{\widehat{E}^s_\tau} + \int_\tau^t T(t,r)B(r)\widehat{T}(r,\tau)|_{\widehat{E}^s_\tau}dr\\ &= T(t,\tau)|_{\widehat{E}^s_\tau} + \int_\tau^t T(t,r)B(r)\widehat{U}(r,\tau)dr.\end{aligned}$$

By (4.3) and (4.4), we have

$$\begin{aligned}\|\widehat{U}(t,\tau)\| &\leq \hat{l}\mu(t)^{2\varepsilon} + \hat{l}\hat{\delta}\mu(t)^{2\varepsilon}\int_\tau^t (\mu(|r|) + \nu(|r|))^{-2\varepsilon}\|\widehat{U}(r,\tau)\|dr\\ &\leq \hat{l}\mu(t)^{2\varepsilon} + \hat{l}\hat{\delta}\mu(t)^{2\varepsilon}\int_\tau^t \mu(r)^{-2\varepsilon}\|\widehat{U}(r,\tau)\|dr.\end{aligned}$$

From the Gronwall's inequality, it follows that, for $|t-\tau| \leq \hat{d}$,

$$\|\widehat{U}(t,\tau)\| \leq \hat{l}\exp\{\hat{l}\hat{\delta}\hat{d}\}\mu(t)^{2\varepsilon}, \quad \|\widehat{V}(t,\tau)\| = \|\widehat{T}(t,\tau)|_{\widehat{E}^u_\tau}\| \leq \hat{l}\exp\{\hat{l}\hat{\delta}\hat{d}\}\nu(t)^{2\varepsilon}.$$

In order to prove (i) of Theorem 3.2, consider the matrix $S(t)$ in (3.31). By (3.33), (3.29) and (4.5), we have

$$
\begin{aligned}
&S'(t) + S(t)(A(t) + B(t)) + (A^*(t) + B^*(t))S(t) \\
&\le -P(t)^*P(t)\frac{h'(t)}{h(t)} - Q(t)^*Q(t)\frac{k'(t)}{k(t)} + 2\|S(t)\|\|B^*(t)\| \,\mathrm{id} \\
&\le -P(t)^*P(t)\frac{h'(t)}{h(t)} - Q(t)^*Q(t)\frac{k'(t)}{k(t)} + (\hat{\delta}K^2/\bar{d}) \,\mathrm{id} \le -\,\mathrm{id}, \quad t \in \mathbb{R},
\end{aligned}
$$

which, together with (3.28), implies (i) of Theorem 3.2.

Direct calculation leads to

$$
\begin{aligned}
\frac{dH(t, \widehat{T}(t,\tau)x)}{dt} &= \langle (S'(t) + S(t)(A(t) + B(t)) + (A^*(t) + B^*(t))S(t))\widehat{T}(t,\tau)x, \widehat{T}(t,\tau)x\rangle \\
&\le 2(a + \bar{d})\frac{h'(t)}{h(t)} \int_t^\infty \|T(v,t)P(t)\widehat{T}(t,\tau)x\|^2 \left(\frac{h(v)}{h(t)}\right)^{-2(a+\bar{d})} \frac{h'(v)}{h(v)} dv \\
&\quad - 2(b - \bar{d})\frac{k'(t)}{k(t)} \int_{-\infty}^t \|T(v,t)Q(t)\widehat{T}(t,\tau)x\|^2 \left(\frac{k(t)}{k(v)}\right)^{2(b-\bar{d})} \frac{k'(v)}{k(v)} dv
\end{aligned}
$$

for each given $\tau \in \mathbb{R}$ and any $x \in \mathbb{R}^n$ since (3.34) holds. By (4.2) and (3.32), we get

$$
\begin{aligned}
\frac{dH(t, \widehat{T}(t,\tau)x)}{dt} &\le 2(a + \bar{d})\frac{h'(t)}{h(t)}|H(t, \widehat{T}(t,\tau)x)|, \quad x \in \widehat{E}^s_\tau, \\
\frac{dH(t, \widehat{T}(t,\tau)x)}{dt} &\le -2(b - \bar{d})\frac{h'(t)}{h(t)}|H(t, \widehat{T}(t,\tau)x)|, \quad x \in \widehat{E}^u_\tau.
\end{aligned}
$$

The proof is complete. □

Next we characterize robustness of the nonuniform (h, k, μ, ν)-dichotomy in the infinite-dimensional space.

Let X be a Banach space and $Y = (Y, |\cdot|)$ be an open subset of the parameter space (also Banach space). Consider the linear perturbed system with parameters

$$
x' = (A(t) + B(t,\lambda))x, \tag{4.6}
$$

where $B : \mathbb{R} \times Y \to \mathcal{B}(X)$.

Theorem 4.2. *Assume that*

- (a_1) *(2.1) admits a nonuniform (h, k, μ, ν)-dichotomy as in (2.2) on $\mathbb{R}$ with $\lim_{t\to\infty} k(t)^{-b}\nu(|t|)^\varepsilon = 0$ and $\lim_{t\to-\infty} h(t)^{-a}\mu(|t|)^\varepsilon = 0$;*
- (a_2) *there is a positive constant N such that*

$$
\nu(|t|)^\varepsilon \int_{-\infty}^t \mu(|\tau|)^{-\omega} d\tau + \mu(|t|)^\varepsilon \int_t^\infty \nu(|\tau|)^{-\omega} d\tau \le N, \quad t \in \mathbb{R};
$$

- (a_3) *there exist positive constants c and ω such that, for any $\lambda, \lambda_1, \lambda_2 \in Y$,*

$$
\begin{aligned}
&\|B(t,\lambda)\| \le c \min\{\mu(|t|)^{-\omega-\varepsilon}, \nu(|t|)^{-\omega-\varepsilon}\}, \\
&\|B(t,\lambda_1) - B(t,\lambda_2)\| \le c|\lambda_1 - \lambda_2| \cdot \min\{\mu(|t|)^{-\omega-\varepsilon}, \nu(|t|)^{-\omega-\varepsilon}\}.
\end{aligned}
$$

Moreover, if

$$c < [KN(2K+1)]^{-1}, \tag{4.7}$$

then (4.6) *also admits a nonuniform* (h, k, μ, ν)*-dichotomy on* $\mathbb{R}$*, i.e., for each* $\lambda \in Y$*, there exists a projection* $\widehat{P}(t, \lambda)$ *for* $t \in \mathbb{R}$ *such that*

$$\widehat{P}(t,\lambda)\widehat{T}(t,s,\lambda) = \widehat{T}(t,s,\lambda)\widehat{P}(s,\lambda) \tag{4.8}$$

and

$$\begin{aligned} \|\widehat{T}(t,s,\lambda)\widehat{P}(s,\lambda)\| &\le \frac{K\widehat{K}}{1-2K\widehat{K}cN}(h(t)/h(s))^a \mu(|s|)^\varepsilon(\mu(|s|)^\varepsilon + \nu(|s|)^\varepsilon), \quad t \ge s, \\ \|\widehat{T}(t,s,\lambda)\widehat{Q}(s,\lambda)\| &\le \frac{K\widehat{K}}{1-2K\widehat{K}cN}(k(s)/k(t))^{-b} \nu(|s|)^\varepsilon(\mu(|s|)^\varepsilon + \nu(|s|)^\varepsilon), \quad s \ge t, \end{aligned} \tag{4.9}$$

where $\widehat{Q}(t,\lambda) = \mathrm{id} - \widehat{P}(t,\lambda)$ *is the complementary projection of* $\widehat{P}(t,\lambda)$ *and* $\widehat{T}(t,s,\lambda)$ *is the evolution operator associated to* (4.6) *and*

$$\widehat{K} = K/(1-KcN). \tag{4.10}$$

Moreover, the stable subspace $\widehat{P}(t,\lambda)(X)$ *and the unstable subspace* $\widehat{Q}(t,\lambda)(X)$ *are Lipschitz continuous in* λ *if* Y *is finite-dimensional.*

The proof of Theorem 4.2 is divided into several steps. First, we characterize the existence of some bounded solutions of (4.6) (Lemma 4.1) and show that these bounded solutions are Lipschitz continuous in λ (Lemma 4.2). We also show that these bounded solutions admit a semigroup property for each $\lambda \in Y$ (Lemma 4.3). Then, with the help of the semigroup property along bounded solutions and the invertible operator $S(0,\lambda)$ defined in Lemma 4.4, we construct invariant projections $\widehat{P}(t,\lambda)$ in (4.8). Finally, we deliberately establish the estimates for the evolution operator in (4.9) (Lemmas 4.5–4.8) and show that the stable and unstable subspaces of nonuniform (h, k, μ, ν)-dichotomies for the linear perturbed system are Lipschitz continuous in λ (Lemma 4.9).

In the rest of this section, we always assume that the conditions in Theorem 4.2 are satisfied.

Step 1. Construction of bounded solutions of (4.6).

For each $s \in \mathbb{R}$, define

$$\begin{aligned} \Omega_1 &:= \{U(t,s)_{t\ge s} \in \mathcal{B}(X) : U \text{ is continuous and } \|U\|_1 < \infty, (t,s) \in \mathbb{R}\times\mathbb{R}\}, \\ \Omega_2 &:= \{V(t,s)_{t\le s} \in \mathcal{B}(X) : V \text{ is continuous and } \|V\|_2 < \infty, (t,s) \in \mathbb{R}\times\mathbb{R}\}, \end{aligned}$$

respectively with the norms

$$\begin{aligned} \|U\|_1 &= \sup\left\{\|U(t,s)\|(h(t)/h(s))^{-a}\mu(|s|)^{-\varepsilon} : t \ge s\right\}, \\ \|V\|_2 &= \sup\left\{\|V(t,s)\|(k(s)/k(t))^{b}\nu(|s|)^{-\varepsilon} : t \le s\right\}. \end{aligned}$$

It is not difficult to show that $(\Omega_1, \|\cdot\|_1)$ and $(\Omega_2, \|\cdot\|_2)$ are Banach spaces.

Lemma 4.1. *For each $\lambda \in Y$ and $s \in \mathbb{R}$,*

- *there exists a unique solution $U^\lambda \in \Omega_1$ of (4.6) satisfying*

$$U^\lambda(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)U^\lambda(\tau,s)d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)U^\lambda(\tau,s)d\tau, \quad t \geq s; \tag{4.11}$$

- *there exists a unique solution $V^\lambda \in \Omega_2$ of (4.6) satisfying*

$$V^\lambda(t,s) = T(t,s)Q(s) + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau,\lambda)V^\lambda(\tau,s)d\tau - \int_t^s T(t,\tau)Q(\tau)B(\tau,\lambda)V^\lambda(\tau,s)d\tau, \quad s \geq t. \tag{4.12}$$

Proof. It is not difficult to show that $U^\lambda(t,s)_{t\geq s}$ satisfying (4.11) and $V^\lambda(t,s)_{s\geq t}$ satisfying (4.12) are solutions of (4.6). We next show that the operator J_1^λ defined by

$$(J_1^\lambda U)(t,s) = T(t,s)P(s) + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)U(\tau,s)d\tau - \int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)U(\tau,s)d\tau$$

has a unique fixed point in Ω_1 for each $\lambda \in Y$. For $t \geq s$, by (2.2), (a_2) and (a_3), we have

$$\begin{aligned} A_1^\lambda :&= \int_s^t \|T(t,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|U(\tau,s)\|d\tau + \int_t^\infty \|T(t,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|U(\tau,s)\|d\tau \\ &\leq Kc(h(t)/h(s))^a\mu(|s|)^\varepsilon \int_s^t \mu(|\tau|)^{-\omega}d\tau\|U\|_1 \\ &\quad + Kc(h(t)/h(s))^a\mu(|s|)^\varepsilon \int_t^\infty (k(\tau)/k(t))^{-b}(h(\tau)/h(t))^a\nu(|\tau|)^{-\omega}d\tau\|U\|_1 \\ &\leq KcN(h(t)/h(s))^a\mu(|s|)^\varepsilon\|U\|_1 \end{aligned}$$

and

$$\begin{aligned} \|(J_1^\lambda U)(t,s)\| &\leq K(h(t)/h(s))^a\mu(|s|)^\varepsilon + A_1^\lambda \\ &\leq K(h(t)/h(s))^a\mu(|s|)^\varepsilon + KcN(h(t)/h(s))^a\mu(|s|)^\varepsilon\|U\|_1. \end{aligned}$$

Then

$$\|J_1^\lambda U\|_1 \le K + KcN\|U\|_1 < \infty, \tag{4.13}$$

which implies that $J_1^\lambda U$ is well-defined and $J_1^\lambda : \Omega_1 \to \Omega_1$. Moreover, for each $\lambda \in Y$, for any $U_1, U_2 \in \Omega_1$ and $t \ge s$, one has

$$\begin{aligned} A_2^\lambda :&= \int_s^t \|T(t,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|U_1(\tau,s) - U_2(\tau,s)\|d\tau \\ &\le Kc(h(t)/h(s))^a \mu(|s|)^\varepsilon \int_s^t \mu(|\tau|)^{-\omega} d\tau \|U_1 - U_2\|_1 \end{aligned}$$

and

$$\begin{aligned} A_3^\lambda :&= \int_t^\infty \|T(t,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|U_1(\tau,s) - U_2(\tau,s)\|d\tau \\ &\le Kc(h(t)/h(s))^a \mu(|s|)^\varepsilon \int_t^\infty \nu(|\tau|)^{-\omega} d\tau \|U_1 - U_2\|_1. \end{aligned}$$

Hence,

$$\|(J_1^\lambda U_1)(t,s) - (J_1^\lambda U_2)(t,s)\| \le A_2^\lambda + A_3^\lambda \le KcN(h(t)/h(s))^a \mu(|s|)^\varepsilon \|U_1 - U_2\|_1,$$

whence

$$\|J_1^\lambda U_1 - J_1^\lambda U_2\|_1 \le KcN\|U_1 - U_2\|_1.$$

If (4.7) holds, then the operator J_1^λ is a contraction and there exists a unique $U^\lambda \in \Omega_1$ such that $J_1^\lambda U^\lambda = U^\lambda$.

In addition, define an operator J_2^λ on Ω_2 by

$$\begin{aligned} (J_2^\lambda V)(t,s) = T(t,s)Q(s) &+ \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau,\lambda)V(\tau,s)d\tau \\ &- \int_t^s T(t,\tau)Q(\tau)B(\tau,\lambda)V(\tau,s)d\tau \end{aligned}$$

for each $\lambda \in Y$. It follows from (2.2), (a_2), and (a_3) that

$$\begin{aligned} A_4^\lambda :&= \int_{-\infty}^t \|T(t,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|V(\tau,s)\|d\tau + \int_t^s \|T(t,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|V(\tau,s)\|d\tau \\ &\le Kc(k(s)/k(t))^{-b}\nu(|s|)^\varepsilon \int_{-\infty}^t (h(t)/h(\tau))^a (k(t)/k(\tau))^{-b} \mu(|\tau|)^{-\omega} d\tau \|V\|_2 \\ &\quad + Kc(k(s)/k(t))^{-b}\nu(|s|)^\varepsilon \int_t^s \nu(|\tau|)^{-\omega} d\tau \|V\|_2 \end{aligned} \tag{4.14}$$

$$\leq KcN(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|V\|_2$$

and

$$\begin{aligned}\|(J_2^{\lambda}V)(t,s)\| &\leq K(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon} + A_4^{\lambda} \\ &\leq K(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon} + KcN(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|V\|_2.\end{aligned}$$

Then

$$\|J_2^{\lambda}V\|_2 \leq K + KcN\|V\|_2 < \infty \tag{4.15}$$

and $J_2^{\lambda} : \Omega_2 \to \Omega_2$ is well-defined. On the other hand, we have

$$\|J_2^{\lambda}V_1 - J_2^{\lambda}V_2\|_2 \leq KcN\|V_1 - V_2\|_2$$

for each $\lambda \in Y$ and any $V_1, V_2 \in \Omega_2$. The operator J_2^{λ} is a contraction since (4.7) holds and there exists a unique $V^{\lambda} \in \Omega_2$ such that $J_2^{\lambda}V^{\lambda} = V^{\lambda}$. The proof is complete. □

Lemma 4.2. *Both U^{λ} and V^{λ} are Lipschitz continuous in the parameter λ.*

Proof. It follows from Lemma 4.1 that, for any $\lambda_1, \lambda_2 \in Y$, there exist bounded solutions $U^{\lambda_1}, U^{\lambda_2} \in \Omega_1$ satisfying (4.11). Then, by (4.13) and (a_3),

$$\begin{aligned}A_1^{\lambda_1,\lambda_2}(\tau) :&= \|B(\tau,\lambda_1)U^{\lambda_1}(\tau,s) - B(\tau,\lambda_2)U^{\lambda_2}(\tau,s)\| \\ &\leq \|B(\tau,\lambda_1)U^{\lambda_1}(\tau,s) - B(\tau,\lambda_1)U^{\lambda_2}(\tau,s)\| \\ &\quad + \|B(\tau,\lambda_1)U^{\lambda_2}(\tau,s) - B(\tau,\lambda_2)U^{\lambda_2}(\tau,s)\| \\ &\leq c(h(\tau)/h(s))^{a}\mu(|\tau|)^{-\omega-\varepsilon}\mu(|s|)^{\varepsilon}(\|U^{\lambda_1} - U^{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|),\end{aligned}$$

which, together with (4.11), implies

$$\begin{aligned}&\|U^{\lambda_1}(t,s) - U^{\lambda_2}(t,s)\| \\ &\leq \int_s^t \|T(t,\tau)P(\tau)\|A_1^{\lambda_1,\lambda_2}(\tau)d\tau + \int_t^{\infty} \|T(t,\tau)Q(\tau)\|A_1^{\lambda_1,\lambda_2}(\tau)d\tau \\ &\leq Kc(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}\left(\int_s^t \mu(|\tau|)^{-\omega}d\tau + \int_t^{\infty} \nu(|\tau|)^{-\omega}d\tau\right)(\|U^{\lambda_1} - U^{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|) \\ &\leq KcN(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}(\|U^{\lambda_1} - U^{\lambda_2}\|_1 + \widehat{K}|\lambda_1 - \lambda_2|).\end{aligned}$$

Thus

$$\|U^{\lambda_1} - U^{\lambda_2}\|_1 \leq [\widehat{K}KcN/(1 - KcN)] \cdot |\lambda_1 - \lambda_2|.$$

Similarly, for any $\lambda_1, \lambda_2 \in Y$, there exist bounded solutions $V^{\lambda_1}, V^{\lambda_2} \in \Omega_2$ satisfying (4.12) and

$$\begin{aligned}A_2^{\lambda_1,\lambda_2}(\tau) &:= \|B(\tau,\lambda_1)V^{\lambda_1}(\tau,s) - B(\tau,\lambda_2)V^{\lambda_2}(\tau,s)\| \\ &\le \|B(\tau,\lambda_1)V^{\lambda_1}(\tau,s) - B(\tau,\lambda_1)V^{\lambda_2}(\tau,s)\| \\ &\quad + \|B(\tau,\lambda_1)V^{\lambda_2}(\tau,s) - B(\tau,\lambda_2)V^{\lambda_2}(\tau,s)\| \\ &\le c(k(\tau)/k(s))^{-b}\nu(|\tau|)^{-\omega-\varepsilon}\nu(|s|)^{\varepsilon}(\|V^{\lambda_1} - V^{\lambda_2}\|_2 + \widehat{K}|\lambda_1 - \lambda_2|).\end{aligned}$$

Then

$$\begin{aligned}\|V^{\lambda_1}(t,s) - V^{\lambda_2}(t,s)\| &\le \int_{-\infty}^{t} \|T(t,\tau)P(\tau)\|A_2^{\lambda_1,\lambda_2}(\tau)d\tau + \int_{t}^{s} \|T(t,\tau)Q(\tau)\|A_2^{\lambda_1,\lambda_2}(\tau)d\tau \\ &\le KcN(k(t)/k(s))^{a}\nu(|s|)^{\varepsilon}(\|V^{\lambda_1} - V^{\lambda_2}\|_2 + \widehat{K}|\lambda_1 - \lambda_2|).\end{aligned}$$

The proof is complete. □

Step 2. Semigroup property of the bounded solutions.

Lemma 4.3. *For each $\lambda \in Y$, one has*

$$U^{\lambda}(t,\sigma)U^{\lambda}(\sigma,s) = U^{\lambda}(t,s),\ t \ge \sigma \ge s; \quad V^{\lambda}(t,\sigma)V^{\lambda}(\sigma,s) = V^{\lambda}(t,s),\ t \le \sigma \le s.$$

Proof. It follows from (4.11) that

$$\begin{aligned}U^{\lambda}(t,\sigma)U^{\lambda}(\sigma,s) = T(t,s)P(s) &+ \int_{s}^{\sigma} T(t,\tau)P(\tau)B(\tau,\lambda)U^{\lambda}(\tau,s)d\tau \\ &+ \int_{\sigma}^{t} T(t,\tau)P(\tau)B(\tau,\lambda)U^{\lambda}(\tau,\sigma)d\tau U^{\lambda}(\sigma,s) \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)B(\tau,\lambda)U^{\lambda}(\tau,\sigma)d\tau U^{\lambda}(\sigma,s).\end{aligned}$$

Let $L^{\lambda}(t,\sigma) = U^{\lambda}(t,\sigma)U^{\lambda}(\sigma,s) - U^{\lambda}(t,s)$ for $t \ge \sigma \ge s$. Define the operator H_1^{λ} by

$$(H_1^{\lambda}l)(t,\sigma) = \int_{\sigma}^{t} T(t,\tau)P(\tau)B(\tau,\lambda)l(\tau,\sigma)d\tau - \int_{t}^{\infty} T(t,\tau)Q(\tau)B(\tau,\lambda)l(\tau,\sigma)d\tau, l \in \Omega_1^{\sigma}, t \ge \sigma,$$

where Ω_1^{σ} is obtained from Ω_1 by replacing s with σ. For any $l, l_1, l_2 \in \Omega_1^{\sigma}$, by (a$_1$)–(a$_3$), one has

$$\|(H_1^{\lambda}l)(t,\sigma)\| \le KcN(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}\|l\|_1$$

and

$$\|(H_1^{\lambda}l_1)(t,\sigma) - (H_1^{\lambda}l_2)(t,\sigma)\| \le KcN(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}\|l_1 - l_2\|_1,$$

then

$$\|H_1^{\lambda}l\|_1 \le KcN\|l\|_1 < \infty, \quad \|H_1^{\lambda}l_1 - H_1^{\lambda}l_2\|_1 \le KcN\|l_1 - l_2\|_1.$$

Hence, H_1^λ is well-defined and $H_1^\lambda(\Omega_1^\sigma) \subset \Omega_1^\sigma$. Therefore, there exists a unique $l^\lambda \in \Omega_1^\sigma$ such that $H_1^\lambda l^\lambda = l^\lambda$. Moreover, it is not difficult to show that $L^\lambda \in \Omega_1^\sigma$ and $0 \in \Omega_1^\sigma$ satisfying $H_1^\lambda 0 = 0$ and $H_1^\lambda L^\lambda = L^\lambda$, which implies that $L^\lambda = l^\lambda = 0$. Similarly, by (4.12), (a_1), (a_2), and (a_3), one has $V^\lambda(t,\sigma)V^\lambda(\sigma,s) = V^\lambda(t,s)$ for $t \le \sigma \le s$. □

Step 3. Construction of the projection $\widehat{P}(t,\lambda)$ in (4.8).
For each given $\lambda \in Y$. Define the linear operator

$$\widetilde{P}(t,\lambda) = \widehat{T}(t,0,\lambda)U^\lambda(0,0)\widehat{T}(0,t,\lambda), \quad \widetilde{Q}(t,\lambda) = \widehat{T}(t,0,\lambda)V^\lambda(0,0)\widehat{T}(0,t,\lambda),\ t \in \mathbb{R}.$$

By Lemma 4.3, $\widetilde{P}(t,\lambda)$ and $\widetilde{Q}(t,\lambda)$ are projections for each $t \in \mathbb{R}$ and

$$\widetilde{P}(t,\lambda)\widehat{T}(t,s,\lambda) = \widehat{T}(t,s,\lambda)\widetilde{P}(s,\lambda), \quad \widetilde{Q}(t,\lambda)\widehat{T}(t,s,\lambda) = \widehat{T}(t,s,\lambda)\widetilde{Q}(s,\lambda), \quad t,s \in \mathbb{R}.$$

It is obvious that $\widehat{U}^\lambda(t,0) = U^\lambda(t,0)P(0)$ satisfies (4.11) with $s = 0$ and $\widehat{V}^\lambda(t,0) = V^\lambda(t,0)Q(0)$ satisfies (4.12) with $s = 0$. By Lemma 4.1,

$$U^\lambda(t,0)P(0) = U^\lambda(t,0), \quad V^\lambda(t,0)Q(0) = V^\lambda(t,0).$$

Note that

$$\widetilde{P}(0,\lambda) = U^\lambda(0,0) = P(0) - \int_0^\infty T(0,\tau)Q(\tau)B(\tau,\lambda)U^\lambda(\tau,0)d\tau \tag{4.16}$$

and

$$\widetilde{Q}(0,\lambda) = V^\lambda(0,0) = Q(0) + \int_{-\infty}^0 T(0,\tau)P(\tau)B(\tau,\lambda)V^\lambda(\tau,0)d\tau, \tag{4.17}$$

then

$$\begin{aligned} &P(0)\widetilde{P}(0,\lambda) = P(0),\ \widetilde{P}(0,\lambda)P(0) = \widetilde{P}(0,\lambda),\ P(0)(\mathrm{id}-\widetilde{Q}(0,\lambda)) = \mathrm{id}-\widetilde{Q}(0,\lambda); \\ &Q(0)\widetilde{Q}(0,\lambda) = Q(0),\ \widetilde{Q}(0,\lambda)Q(0) = \widetilde{Q}(0,\lambda),\ Q(0)(\mathrm{id}-\widetilde{P}(0,\lambda)) = \mathrm{id}-\widetilde{P}(0,\lambda). \end{aligned} \tag{4.18}$$

To obtain the projection $\widehat{P}(t,\lambda)$, set $S(0,\lambda) = \widetilde{P}(0,\lambda) + \widetilde{Q}(0,\lambda)$.

Lemma 4.4. *For each $\lambda \in Y$, the operator $S(0,\lambda)$ is invertible.*

Proof. It follows from (4.18) that

$$\widetilde{P}(0,\lambda) + \widetilde{Q}(0,\lambda) - \mathrm{id} = Q(0)\widetilde{P}(0,\lambda) + P(0)\widetilde{Q}(0,\lambda). \tag{4.19}$$

By (4.16) and (4.17),

$$P(0)\widetilde{Q}(0,\lambda) = P(0)V^\lambda(0,0) = \int_{-\infty}^0 T(0,\tau)P(\tau)B(\tau,\lambda)V^\lambda(\tau,0)d\tau,$$

$$Q(0)\widetilde{P}(0,\lambda) = Q(0)U^\lambda(0,0) = -\int_0^\infty T(0,\tau)Q(\tau)B(\tau,\lambda)U^\lambda(\tau,0)d\tau.$$

Moreover, by (4.10), (4.13) and (4.15), one has

$$\begin{aligned} &\|U^\lambda(t,s)\| \le \widehat{K}(h(t)/h(s))^a \mu(|s|)^\varepsilon,\ t \ge s, \\ &\|V^\lambda(t,s)\| \le \widehat{K}(k(s)/k(t))^{-b} \nu(|s|)^\varepsilon,\ t \le s. \end{aligned} \tag{4.20}$$

By (4.19)–(4.20), we have

$$\begin{aligned} A_5^\lambda =: &\int_{-\infty}^{0} \|T(0,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|V^\lambda(\tau,0)\|d\tau \\ &\le K\widehat{K}c \int_{-\infty}^{0} (h(0)/h(\tau))^a (k(0)/k(\tau))^{-b} \mu(|\tau|)^{-\omega} \nu(0)^\varepsilon d\tau \\ &\le K\widehat{K}c \int_{-\infty}^{0} \mu(|\tau|)^{-\omega} d\tau \end{aligned}$$

and

$$\begin{aligned} A_6^\lambda =: &\int_{0}^{\infty} \|T(0,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|U^\lambda(\tau,0)\|d\tau \\ &\le K\widehat{K}c \int_{0}^{\infty} (k(\tau)/k(0))^{-b} (h(\tau)/h(0))^a \mu(0)^\varepsilon \nu(|\tau|)^{-\omega} d\tau \\ &\le K\widehat{K}c \int_{0}^{\infty} \nu(|\tau|)^{-\omega} d\tau. \end{aligned}$$

Then

$$\|\widetilde{P}(0,\lambda) + \widetilde{Q}(0,\lambda) - \mathrm{id}\,\| \le A_5^\lambda + A_6^\lambda \le K\widehat{K}cN.$$

Therefore, for each $\lambda \in Y$, the operator $S(0,\lambda)$ is invertible if (4.7) holds. □

For $\lambda \in Y$ and $t \in \mathbb{R}$, set

$$\begin{aligned} \widehat{P}(t,\lambda) &= \widehat{T}(t,0,\lambda)S(0,\lambda)P(0)S(0,\lambda)^{-1}\widehat{T}(0,t,\lambda), \\ \widehat{Q}(t,\lambda) &= \widehat{T}(t,0,\lambda)S(0,\lambda)Q(0)S(0,\lambda)^{-1}\widehat{T}(0,t,\lambda). \end{aligned} \tag{4.21}$$

Then $\widehat{P}(t,\lambda)$, $\widehat{Q}(t,\lambda)$ are projections for $t \in \mathbb{R}$ and $\widehat{P}(t,\lambda) + \widehat{Q}(t,\lambda) = \mathrm{id}$. Hence, (4.8) is valid.

Step 4. Norm bounds for the evolution operator.

Lemma 4.5. *For each $\lambda \in Y$, $\|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widetilde{P}(s,\lambda)\| \le \widehat{K}(h(t)/h(s))^a \mu(|s|)^\varepsilon$ for $t \ge s$.*

Proof. First, we show that, if $z^\lambda(t)_{(t\geq s)}$, $\lambda \in Y$ is a bounded solution of (4.6), then

$$\begin{aligned} z^\lambda(t) &= T(t,s)P(s)z^\lambda(s) + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau \\ &\quad - \int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau, \quad t \geq s. \end{aligned} \tag{4.22}$$

It is not difficult to show that

$$\begin{aligned} P(t)z^\lambda(t) &= T(t,s)P(s)z^\lambda(s) + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau, \\ Q(t)z^\lambda(t) &= T(t,s)Q(s)z^\lambda(s) + \int_s^t T(t,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau, \end{aligned} \tag{4.23}$$

and $z^\lambda(t) = P(t)z^\lambda(t) + Q(t)z^\lambda(t)$ for $t \in \mathbb{R}$. Then

$$Q(s)z^\lambda(s) = T(s,t)Q(t)z^\lambda(t) - \int_s^t T(s,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau. \tag{4.24}$$

On the other hand, one has

$$\|T(s,t)Q(t)\| \leq K(k(t)/k(s))^{-b}\nu(|t|)^\varepsilon$$

and

$$\begin{aligned} \int_s^\infty \|T(s,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)\|d\tau &\leq Kc\int_s^\infty \nu(|\tau|)^{-\omega}d\tau \sup_{\tau\geq s}\|z^\lambda(\tau)\| \\ &\leq KcN\sup_{\tau\geq s}\|z^\lambda(\tau)\| < \infty. \end{aligned}$$

Let $t \to \infty$ in (4.24), then

$$Q(s)z^\lambda(s) = -\int_s^\infty T(s,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau.$$

Consequently,

$$\begin{aligned} Q(t)z^\lambda(t) &= -\int_s^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau + \int_s^t T(t,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau \\ &= -\int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)z^\lambda(\tau)d\tau, \end{aligned}$$

which proves (4.22).

For each given $\xi \in X$ and $\lambda \in Y$, let $z^\lambda(t) = \widehat{T}(t,s,\lambda)\widetilde{P}(s,\lambda)\xi$ be the solution of (4.6) for $t \geq s$. Since $\widehat{T}(t,0,\lambda)U^\lambda(0,0)$ and $U^\lambda(t,0)$ are solutions of (4.6) and coincide at $t = 0$, then

$$z^\lambda(t) = \widehat{T}(t,0,\lambda)U^\lambda(0,0)\widehat{T}(0,s,\lambda)\xi = U^\lambda(t,0)\widehat{T}(0,s,\lambda)\xi.$$

Note that $U^\lambda(t,0)$ is bounded for $t \in \mathbb{R}$, then $z^\lambda(t)_{(t\geq s)}$ is a bounded solution of (4.6) with the initial value $z^\lambda(s) = \widetilde{P}(s,\lambda)\xi$. By (4.22), we have

$$\begin{aligned}\widetilde{P}(t,\lambda)\widehat{T}(t,s,\lambda)\xi = T(t,s)P(s)\widetilde{P}(s,\lambda)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau \\ - \int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau, \quad t \geq s.\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}A_7^\lambda =: &\int_s^t \|T(t,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi\|d\tau \\ &\leq Kc\int_s^t (h(t)/h(\tau))^a\mu(|\tau|)^{-\omega}\|\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\|\|\widetilde{P}(s,\lambda)\xi\|d\tau \\ &\leq Kc(h(t)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(\lambda)\widehat{T}(\lambda)\|_1\|\widetilde{P}(s,\lambda)\xi\| \int_s^t \mu(|\tau|)^{-\omega}d\tau\end{aligned}$$

and

$$\begin{aligned}A_8^\lambda =: &\int_t^\infty \|T(t,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi\|d\tau \\ &\leq Kc\int_t^\infty (k(\tau)/k(t))^{-b}\nu(|\tau|)^{-\omega}\|\widetilde{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\|\|\widetilde{P}(s,\lambda)\xi\|d\tau \\ &\leq Kc\|\widetilde{P}(\lambda)\widehat{T}(\lambda)\|_1\int_t^\infty (k(\tau)/k(t))^{-b}\nu(|\tau|)^{-\omega}(h(\tau)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(s,\lambda)\xi\|d\tau \\ &\leq Kc(h(t)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(\lambda)\widehat{T}(\lambda)\|_1\|\widetilde{P}(s,\lambda)\xi\| \int_t^\infty \nu(|\tau|)^{-\omega}d\tau.\end{aligned}$$

Then

$$\begin{aligned}\|\widetilde{P}(t,\lambda)\widehat{T}(t,s,\lambda)\xi\| &\leq K(h(t)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(s,\lambda)\xi\| + A_7^\lambda + A_8^\lambda \\ &\leq K(h(t)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(s,\lambda)\xi\| \\ &\quad + KcN(h(t)/h(s))^a\mu(|s|)^\varepsilon\|\widetilde{P}(\lambda)\widehat{T}(\lambda)\|_1\|\widetilde{P}(s,\lambda)\xi\|,\end{aligned}$$

i.e., $\|\widetilde{P}(\lambda)\widehat{T}(\lambda)\|_1 \leq \widehat{K}$. This yields the desired inequality. □

Lemma 4.6. *For each $\lambda \in Y$, $\|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widetilde{Q}(s,\lambda)\| \le \widehat{K}(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}$ for $t \le s$.*

Proof. By carrying out arguments similar to that of Lemma 4.5, we can show that, for $\lambda \in Y$, if $z^{\lambda}(t)_{(t\le s)}$ is a bounded solution of (4.6), then

$$\begin{aligned} z^{\lambda}(t) = T(t,s)Q(s)z^{\lambda}(s) + \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau,\lambda)z^{\lambda}(\tau)d\tau \\ - \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau,\lambda)z^{\lambda}(\tau)d\tau. \end{aligned} \tag{4.25}$$

Moreover,

$$z^{\lambda}(t) := \widehat{T}(t,s,\lambda)\widetilde{Q}(s,\lambda)\xi = V^{\lambda}(t,0)\widehat{T}(0,s,\lambda)\xi, \quad \xi \in X,\ t \le s$$

and $z^{\lambda}(t)_{(t\le s)}$ is a bounded solution of (4.6) with $z^{\lambda}(s) = \widetilde{Q}(s,\lambda)\xi$. From (4.25), it follows that

$$\begin{aligned} \widetilde{Q}(t,\lambda)\widehat{T}(t,s,\lambda)\xi = T(t,s)Q(s)\widetilde{Q}(s,\lambda)\xi + \int_{-\infty}^{t} T(t,\tau)P(\tau)B(\tau,\lambda)\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau \\ - \int_{t}^{s} T(t,\tau)Q(\tau)B(\tau,\lambda)\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau. \end{aligned}$$

Note that

$$\begin{aligned} A_9^{\lambda} =: &\int_{-\infty}^{t} \|T(t,\tau)P(\tau)\|\|B(\tau,\lambda)\|\|\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi\|d\tau \\ &\le Kc\int_{-\infty}^{t} (h(t)/h(\tau))^{a}\mu(\tau)^{-\omega}\|\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\|\|\widetilde{Q}(s,\lambda)\xi\|d\tau \\ &\le Kc\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2 \int_{-\infty}^{t} (h(t)/h(\tau))^{a}\mu(|\tau|)^{-\omega}(k(s)/k(\tau))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(s,\lambda)\xi\|d\tau \\ &\le Kc(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2\|\widetilde{Q}(s,\lambda)\xi\| \int_{-\infty}^{t} \mu(|\tau|)^{-\omega}d\tau \end{aligned}$$

and

$$\begin{aligned} A_{10}^{\lambda} =: &\int_{t}^{s} \|T(t,\tau)Q(\tau)\|\|B(\tau,\lambda)\|\|\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi\|d\tau \\ &\le Kc\int_{t}^{s} (k(\tau)/k(t))^{-b}\nu(|\tau|)^{-\omega}\|\widetilde{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\|\|\widetilde{Q}(s,\lambda)\xi\|d\tau \\ &\le Kc(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2\|\widetilde{Q}(s,\lambda)\xi\| \int_{t}^{s} \nu(|\tau|)^{-\omega}d\tau, \end{aligned}$$

we have

$$\begin{aligned}\|\widetilde{Q}(t,\lambda)\widehat{T}(t,s,\lambda)\xi\| &\leq K(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(s,\lambda)\xi\| + A_9^{\lambda} + A_{10}^{\lambda} \\ &\leq K(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(s,\lambda)\xi\| \\ &\quad + KcN(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2\|\widetilde{Q}(s,\lambda)\xi\|.\end{aligned}$$

Then

$$\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2 \leq K + KcN\|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2, \quad \text{i.e.,} \quad \|\widetilde{Q}(\lambda)\widehat{T}(\lambda)\|_2 \leq \widehat{K},$$

which yields the desired inequality. □

Lemma 4.7. *For each $\lambda \in Y$, one has*

$$\begin{aligned}\|\widehat{T}(t,s,\lambda)\widehat{P}(s,\lambda)\| &\leq \widehat{K}(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}\|\widehat{P}(s,\lambda)\|, \quad t \geq s, \\ \|\widehat{T}(t,s,\lambda)\widehat{Q}(s,\lambda)\| &\leq \widehat{K}(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widehat{Q}(s,\lambda)\|, \quad t \leq s.\end{aligned} \tag{4.26}$$

Proof. For $\lambda \in Y$, by (b_4), we have

$$\begin{aligned}S(0,\lambda)P(0) &= (\widetilde{P}(0,\lambda) + \widetilde{Q}(0,\lambda))P(0) = \widetilde{P}(0,\lambda), \\ S(0,\lambda)Q(0) &= (\widetilde{P}(0,\lambda) + \widetilde{Q}(0,\lambda))Q(0) = \widetilde{Q}(0,\lambda)\end{aligned}$$

Note that $S(t,\lambda) = \widehat{T}(t,0,\lambda)S(0,\lambda)\widehat{T}(0,t,\lambda)$ for $t \in \mathbb{R}$, then

$$\widehat{P}(t,\lambda)S(t,\lambda) = \widehat{T}(t,0,\lambda)S(0,\lambda)P(0)\widehat{T}(0,t,\lambda) = \widehat{T}(t,0,\lambda)\widetilde{P}(0,\lambda)\widehat{T}(0,t,\lambda) = \widetilde{P}(t,\lambda).$$

Similarly, $\widehat{Q}(t,\lambda)S(t,\lambda) = \widetilde{Q}(t,\lambda)$. Then

$$\operatorname{Im}\widehat{P}(t,\lambda) = \operatorname{Im}\widetilde{P}(t,\lambda) \quad \text{and} \quad \operatorname{Im}\widehat{Q}(t,\lambda) = \operatorname{Im}\widetilde{Q}(t,\lambda).$$

By Lemmas 4.5 and 4.6, one has

$$\begin{aligned}\|\widehat{T}(t,s,\lambda)\widehat{P}(s,\lambda)\| &\leq \|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widehat{P}(s,\lambda)\|\|\widehat{P}(s,\lambda)\| \\ &\leq \|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widetilde{P}(s,\lambda)\|\|\widehat{P}(s,\lambda)\| \\ &\leq \widehat{K}(h(t)/h(s))^{a}\mu(|s|)^{\varepsilon}\|\widehat{P}(s,\lambda)\|, \quad t \geq s\end{aligned}$$

and

$$\begin{aligned}\|\widehat{T}(t,s,\lambda)\widehat{Q}(s,\lambda)\| &\leq \|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widehat{Q}(s,\lambda)\|\|\widehat{Q}(s,\lambda)\| \\ &\leq \|\widehat{T}(t,s,\lambda)|\operatorname{Im}\widetilde{Q}(s,\lambda)\|\|\widehat{Q}(s,\lambda)\| \\ &\leq \widehat{K}(k(s)/k(t))^{-b}\nu(|s|)^{\varepsilon}\|\widehat{Q}(s,\lambda)\|, \quad t \leq s. \qquad \square\end{aligned}$$

Lemma 4.8. *For each $\lambda \in Y$, one has*

$$\begin{aligned}\|\widehat{P}(t,\lambda)\| &\leq [K/(1-2K\widehat{K}cN)](\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon}), \\ \|\widehat{Q}(t,\lambda)\| &\leq [K/(1-2K\widehat{K}cN)](\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon}).\end{aligned} \tag{4.27}$$

Proof. For $\xi \in X$ and $\lambda \in Y$, set

$$z_1^\lambda(t) = \widehat{T}(t,s,\lambda)\widehat{P}(s,\lambda)\xi,\ t \ge s; \quad z_2^\lambda(t) = \widehat{T}(t,s,\lambda)\widehat{Q}(s,\lambda)\xi,\ t \le s.$$

By Lemma 4.7, $(z_1^\lambda(t))_{t\ge s}$ and $(z_2^\lambda(t))_{t\le s}$ are bounded solutions of (4.6). By (4.22) and (4.25),

$$\widehat{P}(t,\lambda)\widehat{T}(t,s,\lambda)\xi = T(t,s)P(s)\widehat{P}(s,\lambda)\xi + \int_s^t T(t,\tau)P(\tau)B(\tau,\lambda)\widehat{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau$$
$$- \int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)\widehat{P}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau$$

and

$$\widehat{Q}(t,\lambda)\widehat{T}(t,s,\lambda)\xi = T(t,s)Q(s)\widehat{Q}(s,\lambda)\xi + \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau,\lambda)\widehat{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau$$
$$- \int_t^s T(t,\tau)Q(\tau)B(\tau,\lambda)\widehat{Q}(\tau,\lambda)\widehat{T}(\tau,s,\lambda)\xi d\tau.$$

Taking $t = s$ leads to

$$Q(t)\widehat{P}(t,\lambda)\xi = -\int_t^\infty T(t,\tau)Q(\tau)B(\tau,\lambda)\widehat{P}(\tau,\lambda)\widehat{T}(\tau,t,\lambda)\xi d\tau,$$
$$P(t)\widehat{Q}(t,\lambda)\xi = \int_{-\infty}^t T(t,\tau)P(\tau)B(\tau,\lambda)\widehat{Q}(\tau,\lambda)\widehat{T}(\tau,t,\lambda)\xi d\tau.$$

By Lemma 4.7,

$$\begin{aligned}\|Q(t)\widehat{P}(t,\lambda)\| + \|P(t)\widehat{Q}(t,\lambda)\| &\le \widehat{K}Kc\Bigg(\mu(|t|)^\varepsilon \int_t^\infty \nu(|\tau|)^{-\omega} d\tau \|\widehat{P}(t,\lambda)\| \\ &\qquad + \nu(|t|)^\varepsilon \int_{-\infty}^t \mu(|\tau|)^{-\omega} d\tau \|\widehat{Q}(t,\lambda)\|\Bigg) \\ &\le \widehat{K}KcN(\|\widehat{P}(t,\lambda)\| + \|\widehat{Q}(t,\lambda)\|).\end{aligned}$$

Since $\|P(t)\| \le K\mu(|t|)^\varepsilon$ and $\|Q(t)\| \le K\nu(|t|)^\varepsilon$, one has

$$\begin{aligned}\|\widehat{P}(t,\lambda)\| &\le \|\widehat{P}(t,\lambda) - P(t)\| + \|P(t)\| \\ &= \|\widehat{P}(t,\lambda) - P(t)\widehat{P}(t,\lambda) - P(t) + P(t)\widehat{P}(t,\lambda)\| + \|P(t)\| \\ &= \|Q(t)\widehat{P}(t,\lambda) - P(t)\widehat{Q}(t,\lambda)\| + \|P(t)\| \\ &\le \|Q(t)\widehat{P}(t,\lambda)\| + \|P(t)\widehat{Q}(t,\lambda)\| + \|P(t)\| \\ &\le \widehat{K}KcN(\|\widehat{P}(t,\lambda)\| + \|\widehat{Q}(t,\lambda)\|) + K\mu(|t|)^\varepsilon\end{aligned}$$

and

$$\|\widehat{Q}(t,\lambda)\| \le \|\widehat{Q}(t,\lambda) - Q(t)\| + \|Q(t)\| = \|\widehat{P}(t,\lambda) - P(t)\| + \|Q(t)\|$$
$$\le \widehat{K}KcN(\|\widehat{P}(t,\lambda)\| + \|\widehat{Q}(t,\lambda)\|) + K\nu(|t|)^{\varepsilon}.$$

Therefore,

$$\|\widehat{P}(t,\lambda)\| + \|\widehat{Q}(t,\lambda)\| \le 2\widehat{K}KcN(\|\widehat{P}(t,\lambda)\| + \|\widehat{Q}(t,\lambda)\|) + K(\mu(|t|)^{\varepsilon} + \nu(|t|)^{\varepsilon}).$$

The proof is complete. □

Step 5. Lipschitz continuity of $\widehat{P}(t,\lambda)(X)$, $\widehat{Q}(t,\lambda)(X)$ with respect to λ.

Lemma 4.9. *$\widehat{P}(t,\lambda)(X)$ and $\widehat{Q}(t,\lambda)(X)$ are Lipschitz continuous in λ.*

Proof. By (a_3), $\widehat{T}(t,0,\lambda)$ is Lipschitz continuous in λ. Since U^{λ} and V^{λ} are Lipschitz continuous in λ (Lemma 4.2), $\widetilde{P}(t,\lambda)$ and $\widetilde{Q}(t,\lambda)$ are Lipschitz continuous in λ. Moreover, if Y is finite-dimensional, then $S(0,\lambda)$ and $S^{-1}(0,\lambda)$ are both Lipschitz continuous in the parameter. By (4.21), Lemma 4.9 is valid. □

Remark 4.1. Theorem 4.1 includes and generalizes Theorem 4.1 in [17] (nonuniform (μ,ν)-dichotomies) and Theorem 7 in [12] (nonuniform exponential dichotomy). When (4.6) reduces to $x' = (A(t) + B(t))x$, the conclusion in Theorem 4.2 includes and extends some existing results for robustness of various dichotomies, such as, robustness of exponential dichotomy (Theorem 5.6 in [62], Theorem 3.2 in [40] and Proposition 1 of Section 4 in [31]), robustness of (h,k)-dichotomy (Theorem 6 in [50]), robustness of nonuniform exponential dichotomy (Theorem 2 in [6]), robustness of ρ-nonuniform exponential dichotomy (Theorem 2 in [9]), and robustness of nonuniform (μ,ν)-dichotomy (Theorem 4.1 in [22]).

Remark 4.2. The two robustness theorems also consider the situation when h, k, μ, ν are different functions, which can not be found in the existing literature. The studies suggest that the size of the linear perturbation is closely related to the growth functions μ, ν (see (4.4), (4.7) and (a_3)), but it has nothing to do with the growth functions h, k. This means that the nonuniform parts μ, ν of the linear system (2.1) play a more important role in the study of the robustness.

5. Existence of topological conjugacy

In this section, with the help of the nonuniform (h,k,μ,ν)-dichotomy, we explore the topological conjugacy of nonautonomous dynamical systems in Banach spaces by establishing a new version of the Grobman–Hartman theorem.

Consider the nonlinear perturbed system

$$x' = A(t)x + f(t,x), \tag{5.1}$$

where $f : \mathbb{R} \times X \to X$.

Definition 5.1 *(see [53]).* (2.1) and (5.1) are said to be topologically equivalent if there exists an function $H : \mathbb{R} \times X \to X$ having the following properties:

(i) if $\|x\| \to \infty$, then $\|H(t,x)\| \to \infty$ uniformly with respect to $t \in \mathbb{R}$;
(ii) for each fixed t, $H(t,\cdot)$ is a homeomorphism of X into X;

(iii) $L(t,\cdot) = H^{-1}(t,\cdot)$ also has property (i);
(iv) if $x(t)$ is a solution of (5.1), then $H(t,x(t))$ is a solution of (2.1).

The function H satisfying the above four properties is said to be *the equivalent function* of (2.1) and (5.1).

Theorem 5.1. *Assume that (2.1) admits a nonuniform (h,k,μ,ν)-dichotomy as in (2.2) on $\mathbb{R}$ and h,k are differentiable. If there exist positive constants α and γ such that, for any $x,x_1,x_2 \in X$,*

$$\begin{aligned}\|f(t,x)\| &\le \alpha\min\{h'(t)h(t)^{-1}\mu(|t|)^{-\varepsilon}, k'(t)k(t)^{-1}\nu(|t|)^{-\varepsilon}\},\\ \|f(t,x_1)-f(t,x_2)\| &\le \gamma\min\{h'(t)h(t)^{-1}\mu(|t|)^{-\varepsilon}, k'(t)k(t)^{-1}\nu(|t|)^{-\varepsilon}\}\|x_1-x_2\|,\end{aligned} \tag{5.2}$$

and

$$K\gamma(1/|a|+1/b) < 1, \tag{5.3}$$

then (5.1) is topologically equivalent to (2.1) and the equivalent function $H(t,x)$ satisfies

$$\|H(t,x)-x\| \le K\alpha(1/|a|+1/b), \quad t\in\mathbb{R}, \quad x\in X.$$

The proof of Theorem 5.1 is achieved in two steps. First, it is shown that either of the systems (5.4), (5.5) and (5.6) has a unique bounded solution (Lemma 5.1, 5.2, 5.3), then we construct a function $H(t,x)$ and prove that $H(t,x)$ is an equivalent function satisfying the properties (i)–(iv) in Definition 5.1 (Lemma 5.4, 5.5, 5.6, 5.7).

Let $X(t,t_0,x_0)$ be the solution of (5.1) with $X(t_0)=x_0$ and $Y(t,t_0,y_0)$ be the solution of (2.1) with $Y(t_0)=y_0$. In the rest of this section, we always assume that (5.2) and (5.3) are satisfied.

Step 1. Construction of bounded solutions.

Lemma 5.1. *For any fixed $(\bar{t},\xi)\in\mathbb{R}\times X$,*

$$z' = A(t)z - f(t, X(t,\bar{t},\xi)) \tag{5.4}$$

has a unique bounded solution $h(t,(\bar{t},\xi))$ and

$$\|h(t,(\bar{t},\xi))\| \le K\alpha(1/|a|+1/b), \quad t\in\mathbb{R}.$$

Proof. It is trivial to show that

$$h(t,(\bar{t},\xi)) = -\int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau,X(\tau,\bar{t},\xi))d\tau + \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau,X(\tau,\bar{t},\xi))d\tau$$

is a solution of (5.4). By (2.2) and (5.2), for any $t\in\mathbb{R}$, we have

$$\begin{aligned}\|h(t,(\bar{t},\xi))\| = &\int_{-\infty}^{t} \|T(t,\tau)P(\tau)\|\|f(\tau,X(\tau,\bar{t},\xi))\|d\tau\\ &+\int_{t}^{\infty} \|T(t,\tau)Q(\tau)\|\|f(\tau,X(\tau,\bar{t},\xi))\|d\tau\end{aligned}$$

$$\leq K\alpha h(t)^a \int_{-\infty}^{t} h(\tau)^{-a-1}h'(\tau)d\tau + K\alpha h(t)^b \int_{t}^{\infty} h(\tau)^{-b-1}h'(\tau)d\tau$$
$$\leq K\alpha(1/|a| + 1/b),$$

which implies that $h(t,(\bar{t},\xi))$ is the unique bounded solution of (5.4) since $z' = A(t)z$ admits a nonuniform (h,k,μ,ν)-dichotomy on $\mathbb{R}$. □

Lemma 5.2. *For any fixed $(\bar{t},\xi) \in \mathbb{R} \times X$,*

$$z' = A(t)z + f(t, Y(t,\bar{t},\xi) + z) \tag{5.5}$$

has a unique bounded solution $l(t,(\bar{t},\xi))$ and

$$\|l(t,(\bar{t},\xi))\| \leq K\alpha(1/|a| + 1/b).$$

Proof. Let

$$\Omega_3 := \{z : \mathbb{R} \to X | \|z\| \leq K\alpha(1/|a| + 1/b)\},$$

where $\|z\| := \sup_{t\in\mathbb{R}} \|z(t)\|$. Then $(\Omega_3, \|\cdot\|)$ is a Banach space. Define a mapping J on Ω_3 by

$$(Jz)(t) = \int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi) + z(\tau))d\tau$$
$$- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi) + z(\tau))d\tau.$$

It follows from (2.2) and (5.2) that

$$\|Jz\| \leq K\alpha(1/|a| + 1/b), \quad \|Jz_1 - Jz_2\| \leq K\gamma(1/|a| + 1/b)\|z_1 - z_2\|, \quad z, z_1, z_2 \in \Omega_3.$$

Then $J(\Omega_3) \subset \Omega_3$ and J is a contraction mapping. Therefore, J has a unique fixed point $l(t)$, i.e.,

$$l(t,(\bar{t},\xi)) = \int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l(\tau))d\tau$$
$$- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l(\tau))d\tau.$$

Next, we prove that $l(t,(\bar{t},\xi))$ is unique in the whole space by contradiction arguments. Otherwise, assume that there is another bounded solution $l^0(t,(\bar{t},\xi))$ of (5.5), which can be written as

$$l^0(t,(\bar{t},\xi)) = \int_{-\infty}^{t} T(t,\tau)P(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l^0(\tau))d\tau$$
$$- \int_{t}^{\infty} T(t,\tau)Q(\tau)f(\tau, Y(\tau,\bar{t},\xi) + l^0(\tau))d\tau.$$

It is trivial to show that

$$\|l - l^0\| \leq K\gamma(1/|a| + 1/b)\|l - l^0\|.$$

Then, by (5.3), one has $l \equiv l^0$. Therefore, $l(t, (\bar{t}, \xi))$ is a unique bounded solution of (5.5) and

$$\|l(t, (\bar{t}, \xi))\| \leq K\alpha(1/|a| + 1/b), \; t \in \mathbb{R}. \qquad \square$$

Lemma 5.3. *Let $x(t)$ be any solution of (5.1), then*

$$z' = A(t)z + f(t, x(t) + z) - f(t, x(t)) \tag{5.6}$$

has a unique bounded solution $z(t) \equiv 0$.

Proof. It is obvious that $z(t) \equiv 0$ is a bounded solution of (5.6). Next we show that $z(t) \equiv 0$ is the unique bounded solution. Assume that $z^0(t)$ is any bounded solution of (5.6), then $z^0(t)$ can be written in the form

$$\begin{aligned} z^0(t) = &\int_{-\infty}^{t} T(t,\tau)P(\tau)[f(\tau, x(\tau) + z(\tau)) - f(\tau, x(\tau))]d\tau \\ &- \int_{t}^{\infty} T(t,\tau)Q(\tau)[f(\tau, x(\tau) + z(\tau)) - f(\tau, x(\tau))]d\tau. \end{aligned}$$

It is easy to show that

$$\|z^0 - 0\| \leq K\gamma(1/|a| + 1/b)\|z^0 - 0\|,$$

which implies that $z^0(t) \equiv 0$. $\square$

Step 2. Construction of the topologically equivalent function.
Define

$$H(t, x) = x + h(t, (t, x)), \quad L(t, y) = y + l(t, (t, y)), \quad x, y \in X. \tag{5.7}$$

Lemma 5.4. *For any fixed $(\bar{t}, x(\bar{t})) \in \mathbb{R} \times X$, $H(t, X(t, \bar{t}, x(\bar{t})))$ is a solution of (2.1).*

Proof. By Lemma 5.1, we have

$$h(t, (t, X(t, \bar{t}, x(\bar{t})))) = h(t, (\bar{t}, x(\bar{t})))$$

and

$$H(t, X(t, \bar{t}, x(\bar{t}))) = X(t, \bar{t}, x(\bar{t})) + h(t, (t, X(t, \bar{t}, x(\bar{t})))) = X(t, \bar{t}, x(\bar{t})) + h(t, (\bar{t}, x(\bar{t}))).$$

Note that $X(t, \bar{t}, x(\bar{t}))$ and $h(t, (\bar{t}, x(\bar{t})))$ are solutions of (5.1) and (5.4), respectively, then

$$\begin{aligned} H'(t, X(t, \bar{t}, x(\bar{t}))) &= X'(t, \bar{t}, x(\bar{t})) + h'(t, (\bar{t}, x(\bar{t}))) \\ &= A(t)X(t, \bar{t}, x(\bar{t})) + f(t, X(t, \bar{t}, x(\bar{t}))) \\ &\quad + A(t)h(t, (\bar{t}, x(\bar{t}))) - f(t, X(t, \bar{t}, x(\bar{t}))) \\ &= A(t)H(t, X(t, \bar{t}, x(\bar{t}))), \end{aligned}$$

which implies that $H(t, X(t, \bar{t}, x(\bar{t})))$ is a solution of (2.1). $\square$

Lemma 5.5. *For any fixed* $(\bar{t}, y(\bar{t})) \in \mathbb{R} \times X$, $L(t, Y(t, \bar{t}, y(\bar{t})))$ *is a solution of* (5.1).

Proof. It follows from Lemma 5.2 that

$$l(t, (t, Y(t, \bar{t}, y(\bar{t})))) = l(t, (\bar{t}, y(\bar{t}))),$$

then

$$\begin{aligned} L(t, Y(t, \bar{t}, y(\bar{t}))) &= Y(t, \bar{t}, y(\bar{t})) + l(t, (t, Y(t, \bar{t}, y(\bar{t})))) \\ &= Y(t, \bar{t}, y(\bar{t})) + l(t, (\bar{t}, y(\bar{t}))). \end{aligned}$$

Since $Y(t, \bar{t}, y(\bar{t}))$ and $l(t, (\bar{t}, y(\bar{t})))$ are solutions of (2.1) and (5.5), respectively, we have

$$\begin{aligned} L'(t, Y(t, \bar{t}, y(\bar{t}))) &= Y'(t, \bar{t}, y(\bar{t})) + l'(t, (\bar{t}, y(\bar{t}))) \\ &= A(t)Y(t, \bar{t}, y(\bar{t})) + A(t)l(t, (\bar{t}, y(\bar{t}))) \\ &\quad + f(t, Y(t, \bar{t}, y(\bar{t})) + l(t, (\bar{t}, y(\bar{t})))) \\ &= A(t)L(t, Y(t, \bar{t}, y(\bar{t}))) + f(t, L(t, Y(t, \bar{t}, y(\bar{t})))). \qquad \square \end{aligned}$$

Lemma 5.6. *For any fixed* $t \in \mathbb{R}$ *and* $y \in X$, $H(t, L(t, y)) = y$ *holds.*

Proof. Let $y(t)$ be any solution of (2.1). It follows from Lemma 5.4 and Lemma 5.5 that $L(t, y(t))$ is a solution of (5.1) and $H(t, L(t, y(t)))$ is a solution of (2.1). Moreover,

$$H'(t, L(t, y(t))) - y'(t) = A(t)H(t, L(t, y(t))) - A(t)y(t) = A(t)(H(t, L(t, y(t))) - y(t))$$

and

$$\begin{aligned} \|H(t, L(t, y(t))) - y(t)\| &\le \|H(t, L(t, y(t))) - L(t, y(t))\| + \|L(t, y(t)) - y(t)\| \\ &\le 2K\alpha(1/|a| + 1/b). \end{aligned}$$

Then $H(t, L(t, y(t))) - y(t)$ is a bounded solution of (2.1) and $H(t, L(t, y(t))) - y(t) \equiv 0$. For any fixed $t \in \mathbb{R}, y \in X$, there is a solution of (2.1) with the initial value $y(t) = y$. Then $H(t, L(t, y)) = y$ holds. $\square$

Lemma 5.7. *For any fixed* $t \in \mathbb{R}$ *and* $x \in X$, $L(t, H(t, x)) = x$ *holds.*

Proof. Let $x(t)$ be any solution of (5.1). It follows from Lemma 5.4 and Lemma 5.5 that $H(t, x(t))$ is a solution of (2.1) and $L(t, H(t, x(t)))$ is a solution of (5.1). Moreover,

$$\begin{aligned} L'(t, H(t, x(t))) - x'(t) &= A(t)L(t, H(t, x(t))) + f(t, L(t, H(t, x(t)))) - A(t)x(t) - f(t, x(t)) \\ &= A(t)[L(t, H(t, x(t))) - x(t)] + f(t, L(t, H(t, x(t))) - x(t) + x(t)) - f(t, x(t)) \end{aligned}$$

and

$$\begin{aligned} \|L(t, H(t, x(t))) - x(t)\| &\le \|L(t, H(t, x(t))) - H(t, x(t))\| + \|H(t, x(t)) - x(t)\| \\ &\le 2K\alpha(1/|a| + 1/b). \end{aligned}$$

By Lemma 5.3, we conclude that $L(t, H(t, x(t))) - x(t) \equiv 0$. For any fixed $t \in \mathbb{R}, x \in X$, there exists a solution of (5.1) with the initial value $x(t) = x$. Then $L(t, H(t, x)) = x$ holds. $\square$

We are now at the right position to establish Theorem 5.1, that is, to verify that $H(t, x)$ is topologically equivalent function. From (5.7) and Lemma 5.1, it follows that, for any $t \in \mathbb{R}$,

$$\|H(t,x) - x\| = \|h(t,x)\| \le K\alpha(1/|a| + 1/b), \ x \in X.$$

Then $\|H(t,x)\| \to \infty$ as $\|x\| \to \infty$ uniformly with respect to $t \in \mathbb{R}$, i.e., Condition (i) holds. By Lemma 5.6 and Lemma 5.7, for each fixed $t \in \mathbb{R}$, $H(t,\cdot) = L^{-1}(t,\cdot)$ is homeomorphism. Then Condition (ii) holds. By (5.7) and Lemma 5.2, for any $t \in \mathbb{R}$, we have

$$\|L(t,y) - y\| = \|l(t,,y)\| \le K\alpha(1/|a| + 1/b), \ y \in X.$$

This implies that $\|L(t,y)\| \to \infty$ as $\|y\| \to \infty$ uniformly with respect to $t \in \mathbb{R}$. Hence, Condition (iii) holds. It follows from Lemma 5.4 and Lemma 5.5 that Condition (iv) holds.

Remark 5.1. Theorem 5.1 not only includes the classical Palmer's linearization theorem for hyperbolic system in [54], and also extends the idea of linearization theorems from hyperbolicity to nonuniform hyperbolicity. On the other hand, it is worth noting that the size of the nonlinear term $f(t,x)$ is related to the functions h, k, μ, ν. This implies that the Palmer's linearization theorem in the case of nonuniform hyperbolicity is closed related to the specific forms of dichotomy, which is a significant different with previous results.

6. Existence of stable invariant manifolds

We establish in this section the existence of parameter dependence of Lipschitz stable invariant manifolds for sufficiently small Lipschitz perturbations of (2.1) assuming that it admits a nonuniform (h, k, μ, ν)-dichotomy.

Consider the nonlinear perturbed system with the parameters of (2.1)

$$x' = A(t)x + f(t, x, \lambda), \tag{6.1}$$

where $f : \mathbb{R} \times X \times Y \to X$ and $f(t, 0, \lambda) = 0$ for any $t \in \mathbb{R}$ and $\lambda \in Y$. Since the problem explored here is the existence of stable invariant manifold of (6.1), one only needs to carry out the discussion on $\mathbb{R}^+$. In order to facilitate the discussion below, we make use of the following equivalent characterization of the nonuniform (h, k, μ, ν)-dichotomy

$$\|T(t,s)P(s)\| \le K\left(\frac{h(t)}{h(s)}\right)^a \mu(s)^\varepsilon, \quad \|T(t,s)^{-1}Q(t)\| \le K\left(\frac{k(t)}{k(s)}\right)^{-b} \nu(t)^\varepsilon, \quad t \ge s \ge 0. \tag{6.2}$$

Assume that there exist positive constants $\hat{c}$ and q such that

$$\begin{aligned} &\|f(t, x_1, \lambda) - f(t, x_2, \lambda)\| \le \hat{c}\|x_1 - x_2\|(\|x_1\|^q + \|x_2\|^q), \\ &\|f(t, x, \lambda_1) - f(t, x, \lambda_2)\| \le \hat{c}|\lambda_1 - \lambda_2| \cdot \|x\|^{q+1} \end{aligned} \tag{6.3}$$

for any $t \in \mathbb{R}^+$, $x, x_1, x_2 \in X$ and $\lambda, \lambda_1, \lambda_2 \in Y$. Define the stable and unstable spaces for each $t \in \mathbb{R}^+$ by

$$E(t) = P(t)(X) \quad \text{and} \quad F(t) = Q(t)(X).$$

We next establish the parameter dependence of the stable manifolds as graphs of Lipschitz functions and begin with introducing the class of functions to be considered. For each $s \ge 0$, let $B_s(\varrho) \subset E(s)$ be the open ball of radius ϱ centered at zero and set

$$\beta(t) = k(t)^{b/(\varepsilon q)} h(t)^{-a(q+1)/(\varepsilon q)} \mu(t)^{1+1/q} C(t)^{1/\varepsilon q}, \tag{6.4}$$

where

$$C(t) = \int_t^\infty h(\tau)^{aq} \max\{\mu(\tau)^\varepsilon, \nu(\tau)^\varepsilon\} d\tau.$$

Given $\eta > 0$, consider the set of initial conditions

$$Z_\beta(\eta) = \{(s,\xi) : s \geq 0,\ \xi \in B_s(\beta(s)^{-\varepsilon}/\eta)\}.$$

Let $Z_\beta = Z_\beta(1)$. Denote by $\mathcal{X}$ the space of continuous functions $\Phi\colon Z_\beta \to X$ such that

$$\Phi(s,0) = 0, \quad \Phi(s, B_s(\beta(s)^{-\varepsilon})) \subset F(s),$$

and

$$\|\Phi(s,\xi_1) - \Phi(s,\xi_2)\| \leq \|\xi_1 - \xi_2\| \tag{6.5}$$

for $s \geq 0$ and $\xi_1, \xi_2 \in B_s(\beta(s)^{-\varepsilon})$. It is not difficult to show that $\mathcal{X}$ is a complete metric space induced by

$$|\Phi|' = \sup\left\{ \frac{\|\Phi(s,\xi)\|}{\|\xi\|} : s \geq 0 \text{ and } \xi \in B_s(\beta(s)^{-\varepsilon}) \setminus \{0\} \right\}.$$

For each $\lambda \in Y$ and given $\Phi \in \mathcal{X}$, consider the graph

$$\mathcal{W}^\lambda = \{(s,\xi,\Phi(s,\xi)) : (s,\xi) \in Z_\beta\} \tag{6.6}$$

and the semiflow generated by (6.1):

$$\Psi_\kappa^\lambda(s,u(s),v(s)) = (t,u(t),v(t)), \quad \kappa = t - s \geq 0,\ (s,u(s),v(s)) \in \mathbb{R}^+ \times E(s) \times F(s) \tag{6.7}$$

where

$$\begin{aligned} u(t) &= T(t,s)u(s) + \int_s^t T(t,\tau)P(\tau) f(\tau,u(\tau),v(\tau),\lambda) d\tau, \\ v(t) &= T(t,s)v(s) + \int_s^t T(t,\tau)Q(\tau) f(\tau,u(\tau),v(\tau),\lambda) d\tau. \end{aligned} \tag{6.8}$$

We now state the existence of parameter dependence of a stable invariant manifold for (6.1).

Theorem 6.1. *Assume that*

(c_1) (2.1) *admits a nonuniform (h,k,μ,ν)-dichotomy as in* (2.2) *on* $\mathbb{R}^+$*;*
(c_2) $\lim_{t\to\infty} k(t)^{-b} h(t)^a \nu(t)^\varepsilon = 0$;
(c_3) $h(t)^a \beta(t)^\varepsilon$ *is decreasing.*

If $\hat{c}$ in (6.3) is sufficiently small, then for each $\lambda \in Y$,

(d$_1$) *there exists a unique function $\Phi = \Phi^\lambda \in \mathcal{X}$ such that $\mathcal{W}^\lambda$ is forward invariant with respect to Ψ_κ^λ in the sense that*

$$\Psi_\kappa^\lambda(s, \xi, \Phi(s, \xi)) \in \mathcal{W}^\lambda \quad \text{for any} \quad (s, \xi) \in Z_{\beta \cdot \mu}(2K),\ \kappa = t - s \geq 0; \tag{6.9}$$

(d$_2$) *there exists a constant $d > 0$ such that*

$$\|\Psi_\kappa^\lambda(s, \xi_1, \Phi(s, \xi_1)) - \Psi_\kappa^\lambda(s, \xi_2, \Phi(s, \xi_2))\| \leq d(h(t)/h(s))^a \mu(s)^\varepsilon \|\xi_1 - \xi_2\| \tag{6.10}$$

for any $\kappa = t - s \geq 0$ and $(s, \xi_1), (s, \xi_2) \in Z_{\beta \cdot \mu}(2K)$;

(d$_3$) *there exists a constant $d^* > 0$ such that*

$$\|\Psi_\kappa^{\lambda_1}(t, \xi, \Phi^{\lambda_1}(t, \xi)) - \Psi_\kappa^{\lambda_2}(t, \xi, \Phi^{\lambda_2}(t, \xi))\| \leq d(h(t)/h(s))^a \mu(s)^\varepsilon |\lambda_1 - \lambda_2| \cdot \|\xi\| \tag{6.11}$$

for any $\lambda_1, \lambda_2 \in Y$.

To obtain parameter dependence of the stable manifolds, we first introduce an auxiliary space. Let $\bar{\mathcal{X}}$ be the space of functions $\Phi\colon \mathbb{R}^+ \times X \to X$ such that $\Phi|_{Z_\beta} \in X$ and

$$\Phi(s, \xi) = \Phi\big(s, \beta(s)^{-\varepsilon} \xi / \|\xi\|\big), \quad (s, \xi) \notin Z_\beta.$$

Note that there is a one-to-one correspondence between $\mathcal{X}$ and $\bar{\mathcal{X}}$. Moreover, $\bar{\mathcal{X}}$ is a Banach space with the norm $\bar{\mathcal{X}} \ni \Phi \mapsto |\Phi|Z_\beta|'$. It is not difficult to show that, for each $\Phi \in \bar{\mathcal{X}}$, one has

$$\|\Phi(s, \xi_1) - \Phi(s, \xi_2)\| \leq 2\|\xi_1 - \xi_2\|, \quad s \geq 0,\ \xi_1, \xi_2 \in E(s). \tag{6.12}$$

The proof of Theorem 6.1 is obtained in several steps. First, we prove that, for each $(s, \xi, \Phi, \lambda) \in Z_\beta \times \mathcal{X}^* \times Y$, there exists a unique function $u^{\Phi,\lambda}(t, s, \xi)$ satisfying the first equality of (6.8) (Lemma 6.1). In order to prove that there exists a unique function $\Phi = \Phi^\lambda \in \mathcal{X}$ satisfying (6.16) for each $\lambda \in Y$, we reduce the problem to an alternative one (Lemma 6.2) and show that there exists a unique function $\Phi = \Phi^\lambda \in \mathcal{X}$ satisfying (6.17) (Lemma 6.5). The asymptotic behaviors of the unique function $u^{\Phi,\lambda}(t, s, \xi)$ defined in Lemma 6.1 are characterized by Lemma 6.3, 6.4, and 6.6. Finally, with the established lemmas, we prove Theorem 6.1 by showing that (6.9), (6.10), and (6.11) are satisfied.

Lemma 6.1. *Let $\hat{c}$ in (6.3) be sufficiently small. Then, for each $(s, \xi, \Phi, \lambda) \in Z_\beta \times \mathcal{X}^* \times Y$, there exists a unique function $u = u^{\Phi,\lambda} : \mathbb{R}^+ \to X$ with $u(s) = \xi$ such that, for any $t \geq s$, (6.8) holds and*

$$\|u(t)\| \leq 2K(h(t)/h(s))^a \mu(s)^\varepsilon \|\xi\|. \tag{6.13}$$

Proof. Let Ω_4 be the space of continuous functions $u\colon [s, \infty) \to X$ with $u(s) = \xi$ such that $u(t) \in E(t)$ for $t \geq s$ and $\|u\|_* \leq \beta(s)^{-\varepsilon}$, where

$$\|u\|_* = \frac{1}{2K} \sup\left\{ \frac{\|u(t)\|}{(h(t)/h(s))^a \mu(s)^\varepsilon} : t \geq s \right\}. \tag{6.14}$$

It is trivial to show that Ω_4 is a complete metric space induced by $\|\cdot\|_*$.

Given $(s,\xi) \in Z_\beta$ and $\Phi \in \bar{\mathcal{X}}$, for $t \geq s$ and each $\lambda \in Y$, define an operator L^λ in Ω_4 by

$$(L^\lambda u)(t) = T(t,s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau),\lambda)d\tau.$$

Then $L^\lambda u$ is continuous in $[s,\infty)$, $(L^\lambda u)(s) = \xi$ and $(L^\lambda u)(t) \in E(t)$ for $t \geq s$. It follows from (6.2) and (6.3) that

$$\begin{aligned}
B_1^\lambda(\tau) =&: \|f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)\| \\
&\leq \hat{c}\left(\|u(\tau)\| + \|\Phi(\tau,u(\tau))\|\right)\left(\|u(\tau)\| + \|\Phi(\tau,u(\tau))\|\right)^q \\
&\leq 3^{q+1}\hat{c}\|u(\tau)\|^{q+1} \\
&\leq 6^{q+1}\hat{c}K^{q+1}\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}(\|u\|_*)^{q+1},\ \tau \geq s
\end{aligned}$$

and

$$\begin{aligned}
\|(L^\lambda u)(t)\| &\leq \|T(t,s)\|\|\xi\| + \int_s^t \|T(t,\tau)P(\tau)\|B_1^\lambda(\tau)d\tau \\
&\leq K\left(\frac{\mu(t)}{\mu(s)}\right)^a \nu(s)^\varepsilon\|\xi\| + 6^{q+1}\hat{c}K^{q+2}\left(\frac{h(t)}{h(s)}\right)^a h(s)^{-aq}\mu(s)^{\varepsilon(q+1)}(\|u\|_*)^{q+1}C(s),
\end{aligned}$$

which implies that

$$\begin{aligned}
\|L^\lambda u\|_* &\leq \frac{1}{2}\left(\|\xi\| + 6^{q+1}\hat{c}K^{q+1}h(s)^{-aq}\mu(s)^{\varepsilon q}(\|u\|_*)^{q+1}C(s)\right) \\
&\leq \frac{1}{2}\left(1 + 6^{q+1}\hat{c}K^{q+1}h(s)^{-aq}\mu(s)^{\varepsilon q}\beta(s)^{-\varepsilon q}C(s)\right)\beta(s)^{-\varepsilon} \leq \frac{1}{2}(1+6^{q+1}\hat{c}K^{q+1})\beta(s)^{-\varepsilon}.
\end{aligned}$$

Since $\hat{c}$ is sufficiently small, take $\hat{c}$ such that $6^{q+1}\hat{c}K^{q+1} < 1$, then $L^\lambda(\Omega_4) \subset \Omega_4$. In addition, for any $u_1,u_2 \in \Omega_4$, one has

$$\begin{aligned}
B_2^\lambda(\tau) =&: \|f(\tau,u_1(\tau),\Phi(\tau,u_1(\tau)),\lambda) - f(\tau,u_2(\tau),\Phi(\tau,u_2(\tau)),\lambda)\| \\
&\leq 3^{q+1}\hat{c}\|u_1(\tau) - u_2(\tau)\|(\|\mu_1(\tau)\|^q + \|\mu_2(\tau)\|^q) \\
&\leq 2^{q+2}3^{q+1}\hat{c}K^{q+1}\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\|u_1-u_2\|_*
\end{aligned}$$

and

$$\begin{aligned}
\|L^\lambda u_1(t) - L^\lambda u_2(t)\| &\leq \int_s^t \|T(t,\tau)P(\tau)\|B_2^\lambda(\tau)d\tau \\
&\leq 2\cdot 6^{q+1}\hat{c}K^{q+1}\|u_1-u_2\|_*\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon.
\end{aligned}$$

Whence,

$$\|L^\lambda u_1 - L^\lambda u_2\|_* \leq 6^{q+1}\hat{c}K^{q+1}\|u_1-u_2\|_*.$$

Therefore, L^λ is a contraction in Ω_4 and there exists a unique function $u = u^{\Phi,\lambda} \in \Omega_4$ such that $L^u = u$. Moreover,

$$\|u\|_* \le \frac{1}{2}\|\xi\| + \frac{1}{2}6^{q+1}\hat{c}K^{q+1}\|u\|_*,$$

and

$$\|u(t)\| \le 2K(h(t)/h(s))^a \mu(s)^\varepsilon \|\xi\| \quad \text{for any} \quad t \ge s,$$

since $K/(1-(1/2)6^{q+1}\hat{c}K^{q+1}) < 2K$. $\square$

Let $u(t) = u^{\Phi,\lambda}(t,s,\xi)$ be the unique function defined by Lemma 6.1, that is,

$$u(t) = T(t,s)\xi + \int_s^t T(t,\tau)P(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau),\lambda)d\tau, \quad t \ge s. \tag{6.15}$$

Lemma 6.2. *Given $\hat{c} > 0$ sufficiently small and $\Phi \in \bar{\mathcal{X}}$, for each $\lambda \in Y$, the following properties hold:*

(e$_1$) *for each $(s,\xi) \in Z_\beta$ and $t \ge s$, if*

$$\Phi(t,u(t)) = T(t,s)\Phi(s,\xi) + \int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau, \tag{6.16}$$

then

$$\Phi(s,\xi) = -\int_s^\infty T(\tau,s)^{-1}Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau; \tag{6.17}$$

(e$_2$) *if (6.17) holds for $s \ge 0$ and $\xi \in B_s(\beta(s)^{-\varepsilon})$, then (6.16) holds for $(s,\xi) \in Z_{\beta\cdot\mu}(2K)$.*

Proof. By (6.2), (6.3), (6.12), and (6.13), for $\tau \ge s$, one has

$$\begin{aligned} B_3^\lambda(\tau) &=: \|T(\tau,s)^{-1}Q(\tau)\| \cdot \|f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)\| \\ &\le 3^{q+1}\hat{c}K\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^\varepsilon\|u(\tau)\|^{q+1} \\ &\le 6^{q+1}\hat{c}K^{q+2}\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^\varepsilon\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\|\xi\|^{q+1} \\ &\le 6^{q+1}\hat{c}K^{q+2}\left(\frac{k(\tau)}{k(s)}\right)^{-b}\nu(\tau)^\varepsilon\left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon(q+1)} \end{aligned}$$

and

$$\begin{aligned} \int_s^\infty B_3^\lambda(\tau)d\tau &\le 6^{q+1}\hat{c}K^{q+2}k(s)^b h(s)^{-a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon(q+1)}\left(\int_s^\infty k(\tau)^{-b}h(\tau)^{a(q+1)}\nu(\tau)^\varepsilon d\tau\right) \\ &\le 6^{q+1}\hat{c}K^{q+2}k(s)^b h(s)^{-a(q+1)}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s) < \infty, \end{aligned}$$

which imply that the right-hand side of (6.17) is well-defined.

Assume that (6.16) holds for $(s,\xi) \in Z_\beta$ and $t \geq s$, then (6.16) rewrites

$$\Phi(s,\xi) = T(t,s)^{-1}\Phi(t,u(t)) - \int_s^t T(\tau,s)^{-1}Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau. \tag{6.18}$$

From (6.2), (6.12), and (6.13), it follows that

$$\begin{aligned}\|T(t,s)^{-1}\Phi(t,u(t))\| &\leq 4K^2\left(\frac{k(t)}{k(s)}\right)^{-b}\nu(t)^\varepsilon\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon\beta(s)^{-\varepsilon} \\ &\leq 4K^2k(t)^{-b}h(t)^a\nu(t)^\varepsilon k(s)^b h(s)^{-a}\mu(s)^\varepsilon\beta(s)^{-\varepsilon}.\end{aligned}$$

Then, letting $t \to \infty$ in (6.18) yields (6.17).

If (6.17) holds for any $(s,\xi) \in Z_\beta$, then, for $(s,\xi) \in Z_{\beta\cdot\mu}(2K)$,

$$\|u(t)\| \leq 2K\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon\|\xi\| \leq \beta(t)^{-\varepsilon}\frac{h(t)^a\beta(t)^\varepsilon}{h(s)^a\beta(s)^\varepsilon} \leq \beta(t)^{-\varepsilon},$$

and hence, $(t,u(t)) \in Z_\beta$ for any $t \geq s$. By (6.17), one has

$$\begin{aligned}T(t,s)\Phi(s,\xi) &= -\int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau \\ &\quad - \int_t^\infty T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau \\ &= -\int_s^t T(t,\tau)Q(\tau)f(\tau,u(\tau),\Phi(\tau,u(\tau)),\lambda)d\tau + \Phi(t,\mu(t)),\end{aligned}$$

where, in the last equality, (6.17) is used with (s,ξ) replaced by $(t,u(t))$. □

Lemma 6.3. *If $\hat{c}$ is sufficiently small and $u_i(t) = u^{\Phi,\lambda}(t,s,\xi_i)$, $i = 1,2$, then there exists a $K_1 > 0$ such that*

$$\|u_1(t) - u_2(t)\| \leq K_1(h(t)/h(s))^a\mu(s)^\varepsilon\|\xi_1 - \xi_2\|, \quad t \geq s. \tag{6.19}$$

Proof. It follows from (6.3), (6.12) and (6.13) that

$$\begin{aligned}B_4^\lambda(\tau) &=: \|f(\tau,u_1(\tau),\Phi(\tau,u_1(\tau)),\lambda) - f(\tau,u_2(\tau),\Phi(\tau,u_2(\tau)),\lambda)\| \\ &\leq 3^{q+1}\hat{c}\|u_1(\tau) - u_2(\tau)\|(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q)\end{aligned}$$

and

$$\begin{aligned}\|u_1(t) - u_2(t)\| &\leq \|T(t,s)(\xi_1 - \xi_2)\| + \int_s^t \|T(t,\tau)P(\tau)\|B_4^\lambda(\tau)d\tau \\ &\leq K\left(\frac{h(t)}{h(s)}\right)^a\mu(s)^\varepsilon(\|\xi_1 - \xi_2\| \\ &\quad + 2\cdot 6^{q+1}\hat{c}K^{q+2}\|u_1 - u_2\|_* \times \left(\frac{h(t)}{h(s)}\right)^a\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\int_s^t h(\tau)^{aq}\nu(\tau)^\varepsilon d\tau.\end{aligned}$$

Then

$$\|u_1 - u_2\|^* \leq \frac{1}{2}\|\xi_1 - \xi_2\| + 6^{q+1}\hat{c}K^{q+1}\|u_1 - u_2\|^*,$$

which yields (6.19) with $K_1 = K/(1 - 6^{q+1}\hat{c}K^{q+1})$. $\square$

Lemma 6.4. *If $\hat{c}$ is sufficiently small and let $u_i(t) = u^{\Phi_i,\lambda}(t,s,\xi), i = 1,2$, then there exists a $K_2 > 0$ such that*

$$\|u_1(t) - u_2(t)\| \leq K_2(\mu(t)/\mu(s))^a\|\xi\| \cdot |\Phi_1 - \Phi_2|', \quad t \geq s. \tag{6.20}$$

Proof. For simplicity, write $u_i = u_\xi^{\Phi_i}$ for $i = 1,2$. A straightforward calculation shows that

$$\begin{aligned}
B_5^\lambda(\tau) =: &\|f(\tau, u_1(\tau), \Phi_1(\tau, u_1(\tau)), \lambda) - f(\tau, u_2(\tau), \Phi_2(\tau, u_2(\tau)), \lambda)\| \\
&\leq 3^q\hat{c}\,[3(\|u_1(\tau) - u_2(\tau)\|)(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q) \\
&\qquad + (\|u_1(\tau)\| \cdot |\Phi_1 - \Phi_2|')(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q)] \\
&\leq [2 \cdot 6^{q+1}\hat{c}K^{q+1}\|u_1 - u_2\|_* \\
&\qquad + 4 \cdot 6^q\hat{c}K^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|'] \times \left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)} \mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}, \ \tau \geq s
\end{aligned}$$

and

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\leq \int_s^t \|T(t,\tau)P(\tau)\|B_5^\lambda(\tau)d\tau \\
&\leq [2 \cdot 6^{q+1}\hat{c}K^{q+1}\|u_1 - u_2\|_* + 4 \cdot 6^q\hat{c}K^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|'] \\
&\qquad \times K\left(\frac{h(t)}{h(s)}\right)^a h(s)^{-aq}\mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s).
\end{aligned}$$

Then

$$\|u_1 - u_2\|_* \leq [6^{q+1}\hat{c}K^{q+1}\|u_1 - u_2\|_* + 2 \cdot 6^q\hat{c}K^{q+1}\|\xi\| \cdot |\Phi_1 - \Phi_2|']\mu(s)^{-\varepsilon}.$$

This establishes (6.20). $\square$

Lemma 6.5. *If $\hat{c}$ is sufficiently small, then, for each $\lambda \in Y$, there exists a unique function $\Phi = \Phi^\lambda \in \mathcal{X}$ such that (6.17) holds for any $(s,\xi) \in Z_\beta$.*

Proof. For each $\lambda \in Y$ and any $(s,\xi) \in Z_\beta$, define an operator J^λ by

$$(J^\lambda\Phi)(s,\xi) = -\int_s^\infty T(\tau,s)^{-1}Q(\tau)f(\tau, u(\tau), \Phi(\tau, u(\tau)), \lambda)d\tau, \quad \Phi \in \bar{\mathcal{X}}$$

where $u = u^{\Phi,\lambda}(t,s,\xi)$ is the unique function defined by Lemma 6.1. It is not difficult to show that $J^\lambda\Phi$ is continuous and $(J^\lambda\Phi)(s,0) = 0$ for $s \geq 0$. Moreover, for any $\xi_1, \xi_2 \in B_s(\beta(s)^{-\varepsilon})$, let $u_i(t) = u^{\Phi,\lambda}(t,s,\xi_i)$, $i = 1,2$. By (6.2), (6.13), and (6.19), we have

$$\begin{aligned}B_6^\lambda(\tau) &:= \|f(\tau, u_1(\tau), \Phi(\tau, u_1(\tau)), \lambda) - f(\tau, u_2(\tau), \Phi(\tau, u_2(\tau)), \lambda)\| \\ &\le 3^{q+1}\hat{c}\|u_1(\tau) - u_2(\tau)\|(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q) \\ &\le 6^{q+1}\hat{c}K^q K_1 \left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)} \mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\|\xi_1 - \xi_2\|\end{aligned}$$

and

$$\begin{aligned}\|(J^\lambda\Phi)(s,\xi_1) - (J^\lambda\Phi)(s,\xi_2)\| &\le \int_s^\infty \|T(\tau,s)^{-1}Q(\tau)\|B_6^\lambda(\tau)d\tau \\ &\le 6^{q+1}\hat{c}K^{q+1}K_1 k(s)^b h(s)^{-a(q+1)} \\ &\quad \times \mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}C(s)\|\xi_1 - \xi_2\| \\ &\le 6^{q+1}\hat{c}K^{q+1}K_1\|\xi_1 - \xi_2\|.\end{aligned}$$

If $\hat{c}$ is sufficiently small, then

$$\|(J^\lambda\Phi)(s,\xi_1) - (J^\lambda\Phi)(s,\xi_2)\| \le \|\xi_1 - \xi_2\|$$

and one can extend $J\Phi$ to $\mathbb{R}^+ \times X$ by $(J\Phi)(s,\xi) = (J\Phi)\big(s, \beta(s)^{-\varepsilon}\xi/\|\xi\|\big)$ for any $(s,\xi) \notin Z_\beta$. Hence, $J(\bar{\mathcal{X}}) \subset \bar{\mathcal{X}}$.

Now we show that J is a contraction. For any $\Phi_1, \Phi_2 \in \bar{\mathcal{X}}$, write $u_i(t) = u^{\Phi_i,\lambda}(t,s,\xi)$, $i = 1, 2$, for each $(s,\xi) \in Z_\beta$, by (6.12), (6.13), and (6.20), one has

$$\begin{aligned}B_7^\lambda(\tau) &:= \|f(\tau, u_1(\tau), \Phi_1(\tau, u_1(\tau)), \lambda) - f(\tau, u_2(\tau), \Phi_2(\tau, u_2(\tau)), \lambda)\| \\ &\le 3^q\hat{c}\big(3\|u_1(\tau) - u_2(\tau)\| + \|u_1(\tau)\| \cdot |\Phi_1 - \Phi_2|'\big)(\|u_1(\tau)\|^q + \|u_2(\tau)\|^q) \\ &\le 2 \cdot 6^q\hat{c}K^q(2K + 3K_2)\|\xi\| \cdot |\Phi_1 - \Phi_2|' \times \left(\frac{h(\tau)}{h(s)}\right)^{a(q+1)} \mu(s)^{\varepsilon(q+1)}\beta(s)^{-\varepsilon q}\end{aligned}$$

and

$$\begin{aligned}\|(J^\lambda\Phi_1)(s,\xi) - (J^\lambda\Phi_2)(s,\xi)\| &\le \int_s^\infty \|T(\tau,s)^{-1}Q(\tau)\|B_7^\lambda(\tau)d\tau \\ &\le 2 \cdot 6^q\hat{c}K^q(2K + 3K_2)\|\xi\| \cdot |\Phi_1 - \Phi_2|'.\end{aligned}$$

If $\hat{c}$ is sufficiently small, then J^λ is a contraction. Therefore, for each $\lambda \in Y$, there exists a unique function $\Phi^\lambda \in \bar{\mathcal{X}}$ such that (6.17) holds for any $(s,\xi) \in Z_\beta$. From the one-to-one correspondence between $\mathcal{X}$ and $\bar{\mathcal{X}}$, it follows that there exists a unique function $\Phi^\lambda \in \mathcal{X}$ such that (6.17) holds for each $\lambda \in Y$ and any $(s,\xi) \in Z_\beta$. $\square$

Lemma 6.6. *For each $(s,\xi) \in Z_\beta$, let $u_i(t) = u^{\Phi^{\lambda_i},\lambda_i}(t,s,\xi), i = 1, 2$, then there exists a $K_3 > 0$ such that*

$$\|u_1(t) - u_2(t)\| \le K_3(h(t)/h(s))^a\mu(s)^\varepsilon\|\lambda_1 - \lambda_2\| \cdot \|\xi\|.$$

Proof. By Lemma 6.1, 6.3, and 6.4, we have

$$B_1^{\lambda_1,\lambda_2}(\tau) := \|f(\tau, u_1(\tau), \Phi^{\lambda_1}(\tau, u_1(\tau)), \lambda_1) - f(\tau, u_2(\tau), \Phi^{\lambda_2}(\tau, u_2(\tau)), \lambda_2)\|$$

$$\begin{aligned}
&\le \|f(\tau, u_1(\tau), \Phi^{\lambda_1}(\tau, u_1(\tau)), \lambda_1) - f(\tau, u_1(\tau), \Phi^{\lambda_1}(\tau, u_1(\tau)), \lambda_2)\| \\
&\quad + \|f(\tau, u_1(\tau), \Phi^{\lambda_1}(\tau, u_1(\tau)), \lambda_2) - f(\tau, u_2(\tau), \Phi^{\lambda_2}(\tau, u_2(\tau)), \lambda_2)\| \\
&\le 6^{q+1}K^{q+1}\hat{c}(h(\tau)/h(s))^{a(q+1)}\mu(s)^{\varepsilon(q+1)} \\
&\quad \times [|\lambda_1 - \lambda_2| \cdot \|\xi\|^{q+1} + 2\|\xi\|^q\|u_1 - u_2\|_* + \frac{2}{3}|\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \cdot \|\xi\|^{q+1}]
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi^{\lambda_1}(s,\xi) - \Phi^{\lambda_2}(s,\xi)\| &\le \int_s^\infty \|T(\tau,s)^{-1}Q(\tau)\|B_1^{\lambda_1,\lambda_2}(\tau)d\tau \\
&\le h'|\lambda_1 - \lambda_2| \cdot \|\xi\| + 2h'\|u_1 - u_2\|_* + (2/3)h'|\Phi^{\lambda_1} - \Phi^{\lambda_2}|' \cdot \|\xi\|,
\end{aligned}$$

where $h' = 2 \cdot 3^{q+1}K^2\hat{c}$. If $\hat{c}$ is sufficiently small, let $H = h'/(1 - (2/3)h')$, then

$$\|\Phi^{\lambda_1}(s,\xi) - \Phi^{\lambda_2}(s,\xi)\| \le H|\lambda_1 - \lambda_2| \cdot \|\xi\| + 2H\|u_1 - u_2\|_*.$$

It follows from (6.8) that

$$\begin{aligned}
\|u_1(t) - u_2(t)\| &\le \int_s^t \|T(s,\tau)P(\tau)\|B_1^{\lambda_1\lambda_2}(\tau)d\tau \\
&\le h'\left((1 + (2/3)H)|\lambda_1 - \lambda_2| \cdot \|\xi\| + (2 + (4/3)H)\|u^{\lambda_1} - u^{\lambda_2}\|_*\right)(h(t)/h(s))^a\mu(s)^\varepsilon
\end{aligned}$$

Thus,

$$\|u_1 - u_2\|_* \le [K_3/(2K)]|\lambda_1 - \lambda_2| \cdot \|\xi\|,$$

where $K_3 = h'(1 + 2H/3)/(1 - h'(1 + 2H/3)/K)$. □

We are now at the right position to establish Theorem 6.1.

Proof of Theorem 6.1. Sum up the above claims, we have the following conclusions.

- From Lemma 6.1, it follows that, for any $(s, \xi, \Phi, \lambda) \in Z_\beta \times \bar{\mathcal{X}} \times Y$, there exists a unique function $u(t) = u^{\Phi,\lambda}(t, s, \xi) \in \Omega_4$. By Lemma 6.2, 6.5 and the one-to-one correspondence between $\mathcal{X}$ and $\bar{\mathcal{X}}$, for $s \ge 0$ and $\xi \in B_s((\beta(s) \cdot \mu(s))^{-\varepsilon}/(2K))$, there exists a unique function $\Phi \in \mathcal{X}$ such that (6.16) holds for each $\lambda \in Y$. For $(s,\xi) \in Z_{\beta\cdot\mu}(2K)$, by (6.13), one has

$$\|u(t)\| \le 2K(h(t)/h(s))^a\mu(s)^\varepsilon \frac{1}{2K}(\beta(s) \cdot \mu(s))^{-\varepsilon} \le (h(t)/h(s))^a\beta(s)^{-\varepsilon} \le \beta(s)^{-\varepsilon},$$

  which implies that $(t, u(t)) \in Z_\beta$, $t \ge s$. Therefore, (6.9) holds and $\mathcal{W}^\lambda$ is forward invariant with respect to the semiflow Ψ_κ^λ for each $\lambda \in Y$.
- For any $(s, \xi_1), (s, \xi_2) \in Z_{\beta\cdot\mu}(2K)$, $\lambda \in Y$, and $\kappa = t - s \ge 0$, by Lemma 6.3, we have

$$\begin{aligned}
&\|\Psi_\kappa^\lambda(s, \xi_1, \Phi(s,\xi_1)) - \Psi_\kappa^\lambda(s, \xi_2, \Phi(s,\xi_2))\| \\
&= \|(t, u^{\Phi,\lambda}(t,s,\xi_1), \Phi^\lambda(t, u^{\Phi,\lambda}(t,s,\xi_1))) - (t, u^{\Phi,\lambda}(t,s,\xi_2), \Phi^\lambda(t, u^{\Phi,\lambda}(t,s,\xi_2)))\| \\
&\le 3\|u^{\Phi,\lambda}(t,s,\xi_1) - u^{\Phi,\lambda}(t,s,\xi_2)\| \le 3K_1(h(t)/h(s))^a\mu(s)^\varepsilon\|\xi_1 - \xi_2\|.
\end{aligned}$$

- It follows from Lemma 6.6 that for $(s,\xi) \in Z_{\beta\cdot\mu}(2K)$, $\lambda_1, \lambda_2 \in Y$, and $\kappa = t - s \geq 0$, we have

$$
\begin{aligned}
&\|\Psi_\kappa^{\lambda_1}(s,\xi,\Phi^{\lambda_1}(s,\xi)) - \Psi_\kappa^{\lambda_2}(s,\xi,\Phi^{\lambda_2}(s,\xi))\| \\
&= \|(t,u_1(t),\Phi^{\lambda_1}(t,u_1(t)) - (t,u_2(t),\Phi^{\lambda_2}(t,u_2(t))\| \\
&\leq \|u_1(t) - u_2(t)\| + \|\Phi^{\lambda_1}(t,u_1(t)) - \Phi^{\lambda_2}(t,u_2(t))\| \\
&\leq \|u_1(t) - u_2(t)\| + |\Phi^{\lambda_1}(t,u_1(t)) - \Phi^{\lambda_1}(t,u_2(t))\| + \|\Phi^{\lambda_1}(t,u_2(t)) - \Phi^{\lambda_2}(t,u_2(t))\| \\
&\leq 3\|u_1(t) - u_2(t)\| + \|\Phi^{\lambda_1} - \Phi^{\lambda_1}|'\|u_2(t)\| \\
&\leq [3K_3 + 2KH(1+\frac{K_3}{K})]\left(\frac{h(t)}{h(s)}\right)^a \mu(s)^\varepsilon\|\lambda_1 - \lambda_2|\cdot\|\xi\|.
\end{aligned}
$$

The proof of Theorem 6.1 is complete. □

Remark 6.1. Theorem 6.1 includes and extends Theorem 1 in [13]. In particular, if the parameter λ is absent from (6.1), then Theorem 6.1 includes and extends some existing stable manifold theorems, for example, Theorem 1 in [4] (nonuniform exponential dichotomy), Theorem 2.1 in [20] (nonuniform (μ,ν)-dichotomy), and Theorem 2 in [11] (ρ-nonuniform exponential dichotomy). In addition, we note that the initial set of stable invariant manifold is related to the functions h, k, μ, ν (see (6.4)), which is a new feature. This also shows that the different forms of functions h, k, μ, ν influence the composition of stable manifolds.

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