NORTH-HOLLAND

Derivative and Proportional State Feedback for Linear Descriptor Systems With Variable Coefficients*

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ABSTRACT

We study linear descriptor systems with rectangular variable coefficient matrices. Using local and global equivalence transformations, we introduce normal and condensed forms and get sets of characteristic quantities. These quantities allow us to decide whether a linear descriptor system with variable coefficients is regularizable by derivative and/or proportional state feedback or not. Regularizable by feedback means for us that there exists a feedback which makes the closed loop system uniquely solvable for every consistent initial vector. © Elsevier Science Inc., 1997

1. INTRODUCTION

In this paper, we study descriptor systems with linear variable coefficients

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

in the interval $[t_1, t_2] \subset \mathbb{R}$ together with an initial condition

$$x(t_0) = x_0. \tag{2}$$

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Let $C^r([t_1, t_2], \mathbb{C}^{n,l})$ denote the set of r times continuously differentiable functions from the interval $[t_1, t_2]$ to the vector space $\mathbb{C}^{n,l}$ of complex $n \times l$ matrices. We assume that

$$\begin{aligned} E(t),\, A(t) &\in C([t_1, t_2], \mathbb{C}^{n,l}), \\ B(t) &\in C([t_1, t_2], \mathbb{C}^{n,m}), \\ x(t) &\in C([t_1, t_2], \mathbb{C}^{l}), \\ u(t) &\in C([t_1, t_2], \mathbb{C}^{m}), \end{aligned} \tag{3}$$

and $B(t)$ has full column rank for all $t \in [t_1, t_2]$. Here $x(t)$ is called the state and $u(t)$ the control of the system.

Descriptor systems of the form (1) arise naturally in a variety of circumstances; e.g., they are used in modeling of mechanical multibody systems [30, 31] and electrical circuits [19].

For a square constant coefficient system $(n = l)$

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

it is well known that the behavior of the system (4), (2) [and the corresponding differential-algebraic equation (DAE)] depends upon the properties of the matrix pencil

$$\alpha E - \beta A. \tag{5}$$

The system (4) and the corresponding pencil (5) are called regular if

$$\det(\alpha E - \beta A) \neq 0 \qquad \text{for some} \quad (\alpha, \beta) \in \mathbb{C}^2. \tag{6}$$

While regularity of the system (4) guarantees the existence and uniqueness of classical solutions [1, 6], this is not true for the system (1) with variable coefficients [18, 23].

The constant coefficient system (4) and the corresponding pencil (5) are said to have index at most one if the dimension of the largest nilpotent block in the Kronecker canonical form of the pencil (5) is less than or equal to one (see e.g. [1, 14, 32]). For higher index descriptor systems (4), impulses can arise if the control is not sufficiently smooth, or the system can even lose causality (see [16, 17, 33]). Therefore, one is interested in a proportional and/or derivative feedback for which the closed loop system is regular and at most of index one in order to guarantee the existence and uniqueness of the solution and to avoid impulsive modes [2, 4].

The main difficulty in understanding the DAE that corresponds to the descriptor system (1) is that different generalizations of the concepts of solvability, index, etc. from constant DAEs to variable coefficient DAEs are possible and have been discussed in the literature [1, 18, 20, 23]. These different concepts can be used as a basis for different results for linear descriptor systems with variable coefficients. So far only a few results have been achieved in this direction. The results in [11, 12], for example, use the solvability concepts for (DAEs as described in [1, 7–9]).

In a series of articles, Kunkel and Mehrmann discussed a more general solvability concept and presented new canonical forms for linear DAEs with variable coefficients [23, 24]. Furthermore, they presented new numerical methods based on an index reduction process [22]. The solvability concept in [22–24] is based on the so-called *strangeness index*, which generalizes the differentiation index [1] for systems with undetermined components. Rabier and Rheinboldt used a different approach to derive a coordinate-free reduction procedure [27], and in [28] they showed that, as in the constant coefficient case, impulse modes can only occur for higher index systems.

In Sections 2 and 3 we show that methods analogous to those in [23, 24] can be used to study linear descriptor control systems with variable coefficients. First, we obtain local characteristic quantities and local canonical forms for the system (1) in Section 2. Then, in Section 3, we show that these local quantities can be used to study the global properties of the system, and we end up with global canonical forms from which we can read off system properties.

Finally, in Section 4 we study under which conditions a linear descriptor system with variable coefficients is regularizable. That means we give necessary and sufficient conditions for the existence of derivative and/or proportional state feedback so that the closed loop system is uniquely solvable for all consistent initial values.

For example, take the descriptor system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ 0 &= u(t), \end{aligned} \tag{7}$$

which does not allow any control $u(t)$. The system (7) is regularizable, since the feedback $u(t) = x_2(t) + w(t)$ yields the closed loop system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ 0 &= x_2(t) + w(t), \end{aligned} \tag{8}$$

which is uniquely solvable for every consistent initial condition and any control $w(t)$.

Furthermore, Section 4 shows how we can get in theory a closed loop system of index at most one.

2. LOCAL CANONICAL FORMS

In this section we will generalize the local canonical form for linear DAEs with variable coefficients of [23, 24] for the descriptor system (1). For constant coefficient systems, canonical and condensed forms have been studied for unitary transformations in [2–4] and for general transformations in [26].

Note that for a linear descriptor system with variable coefficients (1) we cannot apply directly the results of [23, 24], since usually we cannot assume that the control $u(t)$ is sufficiently differentiable.

EXAMPLE 1. Choosing the descriptor system

$$\begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ -1 & t \end{bmatrix}x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \tag{9}$$

and applying the results of [23, 24] for a given control $u(t) \in C^2([t_1, t_2])$, we get the unique solution

$$x_1(t) = u(t) - t\dot{u}(t),$$

$$x_2(t) = -\dot{u}(t).$$

Note that the system (9) has strangeness index 1, i.e. differentiation index 2, and therefore we need a twice differentiable control $u(t)$ to guarantee the existence of a (classical) solution (see [23, 24]).

In principle we can apply differentiation of components only in the uncontrollable subspace, i.e., the part of the system operating in the left nullspace of $B(t)$. Recently, a condensed form for unitary transformations has been studied in [5]. In the approach of [11, 12] it is assumed that the control is sufficiently smooth, which is a major difference to our approach.

The standard variable coefficient transformations that can be applied to the linear descriptor system (1) are premultiplication of (1) by a nonsingular matrix $P(t)$ and changes of the bases for the state $x(t)$ and control $u(t)$ of the system. Therefore, we use the following global transformations for a triple of matrix functions $(E(t), A(t), B(t))$.

DEFINITION 2. Two triples of matrix functions $(E_i(t), A_i(t), B_i(t))$, $B_i(t) \in C([t_1, t_2], \mathbb{C}^{n,m})$, $E_i(t), A_i(t) \in C([t_1, t_2], \mathbb{C}^{n,l})$, $i = 1, 2$, are called equivalent if there are $P(t) \in C([t_1, t_2], \mathbb{C}^{n,n})$, $Q(t) \in C^1([t_1, t_2], \mathbb{C}^{l,l})$, $S(t) \in C([t_1, t_2], \mathbb{C}^{m,m})$ with $P(t), Q(t), S(t)$ nonsingular for all $t \in [t_1, t_2]$ such that

$$\begin{aligned}(E_2(t), A_2(t), B_2(t))& \\ &= P(t)(E_1(t), A_1(t), B_1(t))\begin{bmatrix} Q(t) & -\dot{Q}(t) & 0 \\ 0 & Q(t) & 0 \\ 0 & 0 & S(t) \end{bmatrix}. \quad (10)\end{aligned}$$

Standard rules for differentiation show that this is indeed an equivalence relation.

Taking into account that at a fixed point $t \in [t_1, t_2]$ we can choose $Q(t)$ and $\dot{Q}(t)$ independently [15, 23], we obtain the following definition of a local equivalence.

DEFINITION 3. Two triples of matrices (E_i, A_i, B_i), $E_i, A_i \in \mathbb{C}^{n,l}$, $B_i \in \mathbb{C}^{n,m}$, $i = 1, 2$, are called equivalent if there are matrices $P \in \mathbb{C}^{n,n}$, $Q, R \in \mathbb{C}^{l,l}$, $S \in \mathbb{C}^{m,m}$ with P, Q, S nonsingular such that

$$(E_2, A_2, B_2) = P(E_1, A_1, B_1)\begin{bmatrix} Q & -R & 0 \\ 0 & Q & 0 \\ 0 & 0 & S \end{bmatrix}. \quad (11)$$

Again, it is easily checked that the local transformations describe an equivalence transformation.

Using local equivalence transformations, we obtain the following canonical form for a triple of matrices (E, A, B).

THEOREM 4. *Let* $E, A \in \mathbb{C}^{n,l}$, $B \in \mathbb{C}^{n,m}$, *and*

T *be the basis of* kernel E, (12a)

Z *be the basis of* corange E = kernel E^*, (12b)

$$T' \textit{ be the basis of } \operatorname{cokernel} E = \operatorname{range} E^*, \tag{12c}$$

$$K \textit{ be the basis of } \operatorname{corange}(Z^* B), \tag{12d}$$

$$L \textit{ be the basis of } \operatorname{kernel}(Z^* B), \tag{12e}$$

$$V \textit{ be the basis of } \operatorname{corange}(K^* Z^* AT), \tag{12f}$$

$$Y \textit{ be the basis of } \operatorname{kernel}(V^* K^* Z^* AT'), \tag{12g}$$

$$Y' \textit{ be the basis of } \operatorname{cokernel}(V^* K^* Z^* AT'), \tag{12h}$$

$$N \textit{ be the basis of } \operatorname{kernel}\left([I_s \;\; 0][Y' \;\; Y]^{-1}(Z'^* ET')^{-1} Z'^* BL\right). \tag{12i}$$

Then the quantities

$$r = \operatorname{rank} E \qquad (\textit{rank}), \tag{13a}$$

$$f = \operatorname{rank}(Z^* B) \qquad (\textit{feedback part}), \tag{13b}$$

$$a = \operatorname{rank}(K^* Z^* AT) \qquad (\textit{algebraic part}), \tag{13c}$$

$$s = \operatorname{rank}(V^* K^* Z^* AT') \qquad (\textit{strangeness}), \tag{13d}$$

$$d = r - s \qquad (\textit{differential part}), \tag{13e}$$

$$u^l = n - r - a - s - f \qquad (\textit{left undermined part}), \tag{13f}$$

$$u^r = l - r - a \qquad (\textit{right undermined part}), \tag{13g}$$

$$v = m - f, \tag{13h}$$

$$s^c = \operatorname{rank}\left\{[I_s \;\; 0][Y' \;\; Y]^{-1}(Z'^* ET')^{-1} Z'^* BL\right\}, \tag{13i}$$

$$s^u = s - s^c, \tag{13j}$$

$$d^c = \operatorname{rank}\left\{[0 \;\; I_d][Y' \;\; Y]^{-1}(Z'^* ET')^{-1} Z'^* BLN\right\}, \tag{13k}$$

$$d^u = d - d^c \tag{13l}$$

are invariant under (11), *and* $(E,\ A,\ B)$ *is equivalent to the canonical form*

$$\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d^c} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^u} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_a & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d^c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) \begin{array}{l} s^c \\ s^u \\ d^c \\ d^u \\ a \\ s^c \\ s^u \\ f \\ u^l \end{array} \qquad (14)$$

and the last column in the first and second matrix has width u^r.

Proof. Let $(E_i,\ A_i,\ B_i)$, $i = 1, 2$, be equivalent. Since

$$\operatorname{rank} E_2 = \operatorname{rank}(PE_1Q) = \operatorname{rank} E_1,$$

r is invariant. For f, a, s, s^c, and d^c we must first show that they are well defined with respect to the choice of the bases. Each change of bases can be represented by

$$\tilde{Z} = ZM_Z \qquad \tilde{T}' = T'M_{T'}, \qquad \tilde{Z}' = Z'M_{Z'}, \qquad \tilde{L} = LM_L,$$

$$\tilde{Y}' = M_{T'}^{-1}Y'M_{Y'}, \qquad \tilde{Y} = M_{T'}^{-1}YM_Y$$

with nonsingular matrices M_Z, $M_{T'}$, $M_{Z'}$, M_L, $M_{Y'}$, and M_Y. The well-definedness follows from

$$\operatorname{rank}(\tilde{Z}^* B) = \operatorname{rank}(M_Z^* Z^* B) = \operatorname{rank}(Z^* B),$$

$$\begin{aligned}
&\operatorname{rank}\left\{[I_s\ \ 0]\left[\tilde{Y}'\ \ \tilde{Y}\right]^{-1}(\tilde{Z}'^* E\tilde{T}')^{-1}\tilde{Z}'^* B\tilde{L}\right\} \\
&\quad = \operatorname{rank}\left\{[I_s\ \ 0]\left[M_{T'}^{-1}Y'M_{Y'}\, M_{T'}^{-1}YM_Y\right]^{-1}\right. \\
&\qquad\qquad \left.\times(M_{Z'}^* Z'^* ET'M_{T'})^{-1}M_{Z'}^* Z'^* BLM_L\right\} \\
&\quad = \operatorname{rank}\left([I_s\ \ 0]\left\{\operatorname{diag}(M_{Y'}^{-1}, M_Y^{-1})[Y'\ \ Y]^{-1}M_{T'}\right\}\right. \\
&\qquad\qquad \left.\times\left\{M_{T'}^{-1}(Z'^* ET')^{-1}M_{Z'}^{-*}\right\}M_{Z'}^* Z'^* BLM_L\right) \\
&\quad = \operatorname{rank}\left\{M_{Y'}^{-1}[I_s\ \ 0][Y'\ \ Y]^{-1}(Z'^* ET')^{-1}Z'^* BLM_L\right\} \\
&\quad = \operatorname{rank}\left\{[I_s\ \ 0][Y'\ \ Y]^{-1}(Z'^* ET')^{-1}Z'^* BL\right\}
\end{aligned}$$

and similar calculations for the other values.

Let now bases $Z_2, Z_2', T_2', L_2, Y_2', Y_2$ be given for (E_2, A_2, B_2), e.g.

$$\operatorname{rank}(E_2 T_2) = 0, \quad T_2^* T_2 \text{ nonsingular}, \qquad \operatorname{rank}(T_2^* T_2) = n - r.$$

Using (11) and setting

$$\begin{aligned}
Z_1^* &= Z_2^* P, & T_1' &= QT_2', & Z_1'^* &= Z_2'^* P, \\
L_1 &= SL_2, & Y_1 &= Y_2, & Y_1' &= Y_2',
\end{aligned}$$

the above $Z_1, T_1', Z_1', L_1, Y_1, Y_1'$, form bases according to (12). Since

$$\begin{aligned}
f_2 &= \operatorname{rank}(Z_2^* B_2) \\
&= \operatorname{rank}(Z_2^* PB_1 S) \\
&= \operatorname{rank}(Z_1^* B_1) = f_1,
\end{aligned}$$

we get the invariance of f. Now s^c is invariant, since

$$\begin{aligned} s_2^c &= \operatorname{rank}\{[I_s\ \ 0][Y_2'\ \ Y_2]^{-1}(Z_2'^*E_2T_2')^{-1}Z_2'^*B_2L_2\} \\ &= \operatorname{rank}\{[I_s\ \ 0][Y_1'\ \ Y_1]^{-1}(Z_2'^*PE_1QT_2')^{-1}Z_2'^*PB_1SL_2\} \\ &= \operatorname{rank}\{[I_s\ \ 0][Y_1'\ \ Y_1]^{-1}(Z_1'^*E_1T_1')^{-1}Z_1'^*B_1L_1\} = s_1^c. \end{aligned}$$

With the same technique, the invariance of a, s, and d^c can be shown. The invariance of the other values in (13) follows immediately.

For the derivation of the canonical form (14) we always use nonsingular transformation matrices, i.e., in the first step we take a basis Z' of range E and set $Q = [Z'\ Z]$, etc. As result we obtain the following sequence of equivalent ($\sim$) matrix pairs:

$$(E, A, B) \sim \left(\begin{bmatrix} Z'^*ET' & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} Z'^*AT' & Z'^*AT \\ Z^*AT' & Z^*AT \end{bmatrix}, \begin{bmatrix} Z'^*B \\ Z^*B \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} Z'^*ET' & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \begin{bmatrix} Z'^*BL' & Z'^*BL \\ K'^*Z^*BL' & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \begin{bmatrix} (Z'^*ET')^{-1}Z'^*BL' & (Z'^*ET')^{-1}Z'^*BL \\ K'^*Z^*BL' & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ K'^*Z^*AT' & K'^*Z^*AT \\ K^*Z^*AT' & K^*Z^*AT \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ V'^*K^*Z^*AT' & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & (Z'^*ET')^{-1}Z'^*BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & * \\ 0 & I_a & 0 \\ V^*K^*Z^*AT' & 0 & 0 \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} 0 & (Z'^{*}ET')^{-1}Z'^{*}BL \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right)$$

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & * \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

(where $B_{12} = [I_s \ 0][Y' \ Y]^{-1}(Z'^{*}ET')^{-1}Z'^{*}BL$ and $B_{22} = [0 \ I_d] \times [Y' \ Y]^{-1}(Z'^{*}ET')^{-1}Z'^{*}BL$)

$$\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_a & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 \\ 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right),$$

where $B_{33} = [0 \ I_d](Z'^{*}ET')^{-1}Z'^{*}BLN$, which at last leads to (14) by a similar final transformation step. ∎

If we do not split the d and s blocks of $B(t)$ in the proof of Theorem 4, we get the following condensed form.

COROLLARY 5. *Let E, $A \in \mathbb{C}^{n,l}$, $B \in \mathbb{C}^{n,m}$. Then (E, A, B) is equivalent to the form*

$$
\left(
\begin{bmatrix} I_s&0&0&0\\0&I_d&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},
\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&I_a&0\\I_s&0&0&0\\0&*&0&*\\0&0&0&0\end{bmatrix},
\begin{bmatrix}0&*\\0&*\\0&0\\0&0\\I_f&0\\0&0\end{bmatrix}
\right)
\begin{array}{l} s\\d\\a\\s\\f\\u^l\end{array} \tag{15}
$$

where the last block column in the first and second matrices has width u^r, and the last block column of the last matrix has width v. The quantities s, d, a, f, u^l, u^r, and v are defined as in Theorem 4 and invariant under (11).

3. GLOBAL CANONICAL FORMS

We can apply the results for the local canonical form (14) to Equation (1), and one obtains functions $r, f, a, s, s^c\ d^c : [t_1, t_2] \to \mathbb{N}_0$. Note that the other values depend only on these invariants. Currently we do not know in general how to characterize points where these quantities change their values with t. A general classification of these points is under investigation. For DAEs partial results have been obtained in [10, 25, 27]. Here, we exclude such phenomena by assuming

$$
\begin{aligned}
r(t) &\equiv r, \qquad & f(t) &\equiv f, \qquad & a(t) &\equiv a,\\
s(t) &\equiv s, \qquad & s^c(t) &\equiv s^c, \qquad & d^c(t) &\equiv d^c.
\end{aligned} \tag{16}
$$

Applying the transformation (10) to (1), we get the following canonical form:

THEOREM 6. *Let E, A, B in* (1) *be sufficiently smooth, and let* (16) *hold. Then the triple $(E(t), A(t), B(t))$ is equivalent to a triple of matrix* *functions of the form (without arguments)*

$$
\left(
\begin{bmatrix}
I_{s^c} & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 \\
0 & 0 & I_{d^c} & 0 & 0 & 0 \\
0 & 0 & 0 & I_{d^u} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & A_{13} & A_{14} & 0 & A_{16} \\
0 & 0 & A_{23} & A_{24} & 0 & A_{26} \\
0 & 0 & 0 & A_{34} & 0 & A_{36} \\
0 & 0 & A_{43} & 0 & 0 & A_{46} \\
0 & 0 & 0 & 0 & I_a & 0 \\
I_{s^c} & 0 & 0 & 0 & 0 & 0 \\
0 & I_{s^u} & 0 & 0 & 0 & 0 \\
0 & 0 & A_{83} & A_{84} & 0 & A_{86} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\right.
$$

$$
\left.
\begin{bmatrix}
0 & I_{s^c} & 0 \\
0 & 0 & 0 \\
0 & 0 & I_{d^c} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
I_f & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\right)
\begin{array}{l}
s^c \\ s^u \\ d^c \\ d^u \\ a \\ s^c \\ s^u \\ f \\ u^l
\end{array}
\qquad (17)
$$

The proof of Theorem 6 is given in Appendix A.

Again, as in Section 2, we get a condensed form if we do not split the d and s blocks of $B(t)$.

COROLLARY 7. *Let E, A, B in* (1) *be sufficiently smooth, and let*

$$
r(t) \equiv r, \qquad f(t) \equiv f, \qquad a(t) \equiv a, \qquad s(t) \equiv s
$$

hold. Then $(E(t), A(t), B(t))$ is equivalent to a triple of matrix functions of *the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0\\ 0 & I_d & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t)\\ 0 & 0 & 0 & A_{24}(t)\\ 0 & 0 & I_a & 0\\ I_s & 0 & 0 & 0\\ 0 & A_{52}(t) & 0 & A_{54}(t)\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t)\\ 0 & B_{22}(t)\\ 0 & 0\\ 0 & 0\\ I_f & 0\\ 0 & 0 \end{bmatrix}\right) \begin{array}{l} s\\ d\\ a\\ s\\ f\\ u^l \end{array} \tag{18}$$

From the analysis of linear DAEs with variable coefficients we know that higher index problems, i.e., of index greater than one, are indicated by a nonvanishing strangeness s (see [24]).

Our main goal is to study the regularization of the descriptor system (1) by feedback. As the next lemma shows, Corollary 7 is a first step in this direction.

LEMMA 8. *Let a square descriptor system* (1), *i.e.* $n = l$, *be in the form* (18), *and assume that* $s = 0$. *If* $u^l = 0$, *then there exists a state feedback* $u(t) = F(t)x(t) + w(t)$ *such that the closed loop system*

$$E(t)\dot{x}(t) = [A(t) + B(t)F(t)]x(t) + B(t)w(t), \qquad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 *and any given control* $w(t)$.

Proof. The descriptor system is of the form

$$\begin{bmatrix} I_d & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} 0 & 0 & A_{13}(t)\\ 0 & I_a & 0\\ A_{31}(t) & 0 & A_{33}(t) \end{bmatrix}x(t) + \begin{bmatrix} B_{12}(t) & 0\\ 0 & 0\\ 0 & I_f \end{bmatrix}w(t).$$

Choosing $F(t) = \begin{bmatrix} 0 & 0 & 0 \\ -A_{31}(t) & 0 & I_f - A_{33}(t) \end{bmatrix}$, we get the closed loop system

$$\begin{bmatrix} I_d & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_a & 0 \\ 0 & 0 & I_f \end{bmatrix} x(t) + \begin{bmatrix} B_{12}(t) & 0 \\ 0 & 0 \\ 0 & I_f \end{bmatrix} w(t). \quad (19)$$

For any given control $w(t)$, (19) is a strangeness-free DAE, i.e., it has differentiation index one, with d differential and $a + f$ algebraic equations. Therefore, (19) is uniquely solvable for every consistent initial value x_0. ∎

EXAMPLE 9. Lemma 8 is applicable for the descriptor system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad (20)$$

of the introduction. The system (20) does not allow any control $u(t)$ at all, i.e., is not even solvable for any given control. But choosing the feedback $u(t) = x_2(t) + w(t)$, the closed loop system (8) is uniquely solvable for every control $w(t)$.

Lemma 8 shows that under certain assumptions the condensed form (18) allows us to construct a feedback which makes the closed loop system uniquely solvable. Even more, in Section 4 we will show that it is sufficient to study a closely related condensed form to answer the question whether there exist a state and/or derivative feedback which makes the closed loop uniquely solvable or not.

From now on we will focus our analysis on the generalization of the remaining results from [23, 24] for the condensed form (18) of Corollary 7.

Writing down the descriptor system equations that belong to the matrix triple from Corollary 7, we get

$$\dot{x}_1(t) = A_{12}(t) x_2(t) + A_{14}(t) x_4(t) + B_{12}(t) u_3(t), \quad (21a)$$

$$\dot{x}_2(t) = A_{24}(t) x_4(t) + B_{22}(t) u_3(t), \quad (21b)$$

$$0 = x_3(t), \quad (21c)$$

$$0 = x_1(t), \tag{21d}$$

$$0 = A_{52}(t)x_2(t) + A_{54}(t)x_4(t) + u_1(t), \tag{21e}$$

$$0 = 0. \tag{21f}$$

From Equation (21d) we see that $x_1(t) \equiv 0$. This implies $\dot{x}_1(t) \equiv 0$, by inserting $\dot{x}_1(t) \equiv 0$ in (21a) we get an algebraic equation. This corresponds to passing from the form (18) to

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & A_{52}(t) & 0 & A_{54}(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t) \\ 0 & B_{22}(t) \\ 0 & 0 \\ 0 & 0 \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{array}{l} s \\ d \\ a \\ s \\ f \\ u^l \end{array} \tag{22}$$

for which we again compute characteristic values r, f, a, s, d, u^l, u^r, and v.

This procedure leads to an inductive definition of a sequence $(E_i(t), A_i(t), B_i(t))$, $i \in \mathbb{N}_0$, of matrix function triples, where $(E_0(t), A_0(t), B_0(t)) = (E(t), A(t), B(t))$ and $(E_{i+1}(t), A_{i+1}(t), B_{i+1}(t))$ is derived from $(E_i(t), A_i(t), B_i(t))$ by bringing it into the form (18) and passing to the form above. Here we must assume that $r(t) \equiv r$, $f(t) \equiv f$, $a(t) \equiv a$, $s(t) \equiv s$ for every occurring triple of matrices. Connected with this sequence, we then have sequences $r_i, f_i, a_i, s_i, u_i^l, u_i^r, v_i$, $i \in \mathbb{N}_0$, of nonnegative integers.

The next theorem shows that these sequences are indeed characteristic for a given triple $(E(t), A(t), B(t))$, i.e., they do not depend on the specific way they are obtained.

THEOREM 10. *Let* $(E(t), A(t), B(t))$, $(\bar{E}(t), \bar{A}(t), \bar{B}(t))$ *be equivalent and of the form* (18). *Then the modified triples* $(E_{\mathrm{m}}(t), A_{\mathrm{m}}(t), B_{\mathrm{m}}(t))$, $(\bar{E}_{\mathrm{m}}(t), \bar{A}_{\mathrm{m}}(t), \bar{B}_{\mathrm{m}}(t))$ *obtained by passing to* (22) *are also equivalent.*

Proof. Assume that $(E(t), A(t), B(t))$, $(\bar{E}(t), \bar{A}(t), \bar{B}(t))$ are equivalent and of the form (18). Omitting arguments, we get

$$P\bar{E} = EQ, \qquad P\bar{A} = AQ - E\dot{Q}, \qquad P\bar{B} = BS,$$

where P, Q, and S are smooth, pointwise nonsingular matrix functions. From the first relation we get

$$\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 \\ P_{31} & P_{32} & 0 & 0 \\ P_{41} & P_{42} & 0 & 0 \\ P_{51} & P_{52} & 0 & 0 \\ P_{61} & P_{62} & 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{21} & Q_{22} & Q_{23} & Q_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

if we partition P and Q according to Corollary 7.

With this we obtain for the third, fourth, and sixth block rows of the second relation

$$\begin{bmatrix} P_{34} & P_{35}\bar{A}_{52} & P_{33} & P_{35}\bar{A}_{54} \\ P_{44} & P_{45}\bar{A}_{52} & P_{43} & P_{45}\bar{A}_{54} \\ P_{64} & P_{65}\bar{A}_{52} & P_{63} & P_{65}\bar{A}_{54} \end{bmatrix} = \begin{bmatrix} Q_{31} & Q_{32} & Q_{33} & Q_{34} \\ Q_{11} & Q_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the third to sixth block rows of the third relation we then deduce

$$\begin{bmatrix} P_{35} & 0 \\ P_{45} & 0 \\ P_{55} & 0 \\ P_{65} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ S_{11} & S_{12} \\ 0 & 0 \end{bmatrix},$$

where we partition $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ according to Corollary 7.

In terms of the matrices Q and S we therefore have

$$P = \begin{bmatrix} Q_{11} & 0 & P_{13} & P_{14} & P_{15} & P_{16} \\ Q_{21} & Q_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ 0 & 0 & Q_{33} & Q_{31} & 0 & P_{36} \\ 0 & 0 & 0 & Q_{11} & 0 & P_{46} \\ 0 & 0 & P_{53} & P_{54} & S_{11} & P_{56} \\ 0 & 0 & 0 & 0 & 0 & P_{66} \end{bmatrix},$$

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 \\ Q_{31} & 0 & Q_{33} & 0 \\ Q_{41} & Q_{42} & Q_{43} & Q_{44} \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{bmatrix},$$

and $Q_{11}, Q_{22}, Q_{33}, Q_{44}, S_{11}, S_{22}, P_{66}$ must be nonsingular. From the first two and the fifth block rows of the second relation, we then get

$$\begin{bmatrix} Q_{11} & 0 & P_{15} \\ Q_{21} & Q_{22} & P_{25} \\ 0 & 0 & S_{11} \end{bmatrix} \begin{bmatrix} \bar{A}_{12} & \bar{A}_{14} \\ 0 & \bar{A}_{24} \\ \bar{A}_{52} & \bar{A}_{54} \end{bmatrix} = \begin{bmatrix} A_{12} & A_{14} \\ 0 & A_{24} \\ A_{52} & A_{54} \end{bmatrix} \begin{bmatrix} Q_{22} & 0 \\ Q_{42} & Q_{44} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \dot{Q}_{22} & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, from the same block rows of the third equation, we deduce

$$\begin{bmatrix} Q_{11} & 0 & P_{15} \\ Q_{21} & Q_{22} & P_{25} \\ 0 & 0 & S_{11} \end{bmatrix} \begin{bmatrix} 0 & \bar{B}_{12} \\ 0 & \bar{B}_{22} \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ I & 0 \end{bmatrix} S.$$

Let $(\bar{E}_m, \bar{A}_m, \bar{B}_m)$ be the modified triple (22) which we obtained from the triple $(\bar{E}, \bar{A}, \bar{B})$. Setting

$$P_m = \begin{bmatrix} Q_{11} & & & & P_{15} & \\ Q_{21} & Q_{22} & & & P_{25} & \\ & & I & & & \\ & & & I & & \\ & & & & S_{11} & \\ & & & & & I \end{bmatrix}, \qquad Q_m = \begin{bmatrix} I & & & \\ & Q_{22} & & \\ & & I & \\ & Q_{42} & & Q_{44} \end{bmatrix},$$

and $S_m = S$, we immediately get

$$P_m\left(\bar{E}_m, \bar{A}_m, \bar{B}_m\right) \begin{bmatrix} Q_m^{-1} & -\dot{\widehat{Q_m^{-1}}} & 0 \\ 0 & Q_m^{-1} & 0 \\ 0 & 0 & S_m^{-1} \end{bmatrix}$$

$$= \left(E_m Q_m, A_m Q_m - E_m \dot{Q}_m, B_m S_m\right) \begin{bmatrix} Q_m^{-1} & -\dot{\widehat{Q_m^{-1}}} & 0 \\ 0 & Q_m^{-1} & 0 \\ 0 & 0 & S_m^{-1} \end{bmatrix}$$

$$= \left(E_m, A_m - E_m\left(\dot{Q}_m Q_m^{-1} - Q_m \dot{\widehat{Q_m^{-1}}}\right), B_m\right),$$

where $\dot{Q}_m Q_m^{-1} - Q_m \dot{\widehat{Q_m^{-1}}} = \dot{\widehat{(Q_m Q_m^{-1})}} = \dot{I} = 0$. ∎

Now we can state some basic properties of these quantities:

LEMMA 11. *Let $E(t)$, $A(t)$, and $B(t)$ in (1) be sufficiently smooth and such that the sequences $(E_i(t), A_i(t), B_i(t))$, $i \in \mathbb{N}_0$, and $r_i, f_i, a_i, s_i, d_i, u_i^l, u_i^r, v_i$, $i \in \mathbb{N}_0$, are well defined by the above process. Let furthermore*

$$(E_i, A_i, B_i) \sim \left(\begin{bmatrix} I_{s_i} & 0 & 0 & 0 \\ 0 & I_{d_i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}^{(i)}(t) & 0 & A_{14}^{(i)}(t) \\ 0 & 0 & 0 & A_{24}^{(i)}(t) \\ 0 & 0 & I_{a_i} & 0 \\ I_{s_i} & 0 & 0 & 0 \\ 0 & A_{52}^{(i)}(t) & 0 & A_{54}^{(i)}(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & B_{12}^{(i)}(t) \\ 0 & B_{22}^{(i)}(t) \\ 0 & 0 \\ 0 & 0 \\ I_{f_i} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{array}{l} s_i \\ d_i \\ a_i \\ s_i \\ f_i \\ u_i^l \end{array} \quad (23)$$

Then we have (for all $t \in [t_1, t_2]$, $i \in \mathbb{N}$)

$$r_{i+1} = r_i - s_i, \tag{24a}$$

$$f_{i+1} = f_i + \operatorname{rank} B_{12}^{(i)}(t), \tag{24b}$$

$$a_{i+1} = a_i + s_i + \operatorname{rank}\big(R_i(t)^* A_{14}^{(i)}(t)\big), \tag{24c}$$

$$s_{i+1} = \operatorname{rank}\big(W_i(t)^* R_i(t)^* A_{12}^{(i)}(t)\big), \tag{24d}$$

$$d_{i+1} = d_i - \operatorname{rank}\big(W_i(t)^* R_i(t)^* A_{12}^{(i)}(t)\big), \tag{24e}$$

$$u_{i+1}^l = u_i^l + \big\{s_i - \operatorname{rank}\big[A_{12}^{(i)}(t)\, A_{14}^{(i)}(t)\, B_{12}^{(i)}(t)\big]\big\}, \tag{24f}$$

$$u_{i+1}^r = u_i^r + \big\{s_i - \operatorname{rank}\big(R_i(t)^* A_{14}^{(i)}(t)\big)\big\}, \tag{24g}$$

$$v_{i+1} = v_i - \operatorname{rank} B_{12}^{(i)}(t) \tag{24h}$$

with $R_i(t) = \operatorname{corange} B_{12}^{(i)}(t)$ *and* $W_i(t) = \operatorname{corange}(R_i(t)^* A_{14}^{(i)}(t))$.
There exists a number $\nu \in \mathbb{N}_0$ defined by

$$\nu = \min\{i \in \mathbb{N}_0 \,|\, s_i = 0\}, \tag{25}$$

and the above sequences have the properties

$$r_i > r_{i+1} \quad \text{for } i < \nu, \qquad r_i = r_\nu \quad \text{for } i \geqslant \nu, \tag{26a}$$

$$f_i \leqslant f_{i+1} \quad \text{for } i < \nu, \qquad f_i = f_\nu \quad \text{for } i \geqslant \nu, \tag{26b}$$

$$a_i < a_{i+1} \quad \text{for } i < \nu, \qquad a_i = a_\nu \quad \text{for } i \geqslant \nu, \tag{26c}$$

$$s_i \geqslant s_{i+1} \quad \text{for } i < \nu, \qquad s_i = 0 \quad \text{for } i \geqslant \nu, \tag{26d}$$

$$d_i \geqslant d_{i+1} \quad \text{for } i < \nu, \qquad d_i = d_\nu \quad \text{for } i \geqslant \nu, \tag{26e}$$

$$u_i^l \leqslant u_{i+1}^l \quad \text{for } i < \nu, \qquad u_i^l = u_\nu^l \quad \text{for } i \geqslant \nu, \tag{26f}$$

$$u_i^r \leqslant u_{i+1}^r \quad for\ i < \nu, \qquad u_i^r = u_\nu^r \quad for\ i \geqslant \nu, \tag{26g}$$

$$v_i \geqslant v_{i+1} \quad for\ i < \nu, \qquad v_i = v_\nu \quad for\ i \geqslant \nu, \tag{26h}$$

Proof. Replacing I_{s_i} by 0 in $E_i(t)$, we get (24a) from $r_{i+1} = \operatorname{rank} E_{i+1}(t)$. (24b) is then a consequence of $f_{i+1} = \operatorname{rank}(Z_{i+1}(t)^* B_{i+1}(t))$, where $Z_{i+1}(t)$ is a basis of corange $E_{i+1}(t)$. Since

$$a_{i+1} = \operatorname{rank}\big(K_{i+1}(t)^* Z_{i+1}(t)^* A_{i+1}(t) T_{i+1}(t)\big),$$

where $K_{i+1}(t)$ is a basis of corange$(Z_{i+1}(t)^* B_{i+1}(t))$ and $T_{i+1}(t)$ is a basis of kernel $E_{i+1}(t)$, we get (24c). (24d) follows now immediately from the definition (12) of s_{i+1}. By direct application of (12) we now get (24e–h).

$A_{12}^{(i)}(t)$ is an (s_i, d_i) matrix, so that $s_i \geqslant s_{i+1}$ and s_i must become zero after a finite number of steps. Thus, (26) is a direct consequence of (24). ∎

The quantities ν and r_i, f_i, a_i, s_i, $i \in \{0, \ldots, \nu\}$, are characteristic for a given descriptor system, and the hope is that they are sufficient to describe the possible phenomena for (1). We now get a condensed form which reflects the above quantities, similar to the condensed form of [5].

THEOREM 12. *Let ν from Lemma 11 be well defined for a triple $(E(t), A(t), B(t))$ of smooth matrix functions. Let $r_i, f_i, a_i, s_i, d_i, u_i^l, u_i^r, v_i$, $i \in 0, \ldots, \nu$, be the related characteristic values as above. Furthermore define (in the notation of Lemma 11)*

$$b_0 = a_0, \qquad b_i = \operatorname{rank}\big(R_{i-1}(t)^* A_{14}^{(i-1)}(t)\big), \tag{27a}$$

$$g_0 = f_0, \qquad g_i = \operatorname{rank} B_{12}^{(i-1)}(t), \tag{27b}$$

$$c_0 = a_0 + s_0, \qquad c_i = \operatorname{rank}\big\{R_{i-1}(t)^* \big[\, A_{12}^{(i-1)}(t)\, A_{14}^{(i-1)}(t)\big]\big\}, \tag{27c}$$

$$w_0 = u_0^l, \qquad w_i = u_i^l - u_{i-1}^l. \tag{27d}$$

We then have

$$c_i = b_i + s_i, \qquad i = 0, \ldots, \nu, \tag{28a}$$

$$w_i = s_{i-1} - c_i - g_i, \qquad i = 1, \ldots, \nu, \tag{28b}$$

and the triple $(E(t), A(t), B(t))$ is equivalent to a triple of matrix functions of the form twithout arguments)

$$\left(\left[\begin{array}{c|c|cccc} I & 0 & 0 & * & \cdots & * \\ \hline 0 & 0 & 0 & F_\nu & & * \\ \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & & \ddots & F_1 \\ 0 & 0 & & & & 0 \\ \hline 0 & 0 & 0 & E_\nu & & * \\ \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & & & \ddots & E_1 \\ 0 & 0 & & & & 0 \\ \hline 0 & 0 & 0 & * & \cdots & * \end{array}\right], \left[\begin{array}{c|c|cccc} * & * & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & I & & & \\ \vdots & & & \ddots & & \\ \vdots & & & & \ddots & \\ 0 & 0 & & & & I \\ \hline * & * & 0 & \cdots & \cdots & 0 \end{array}\right],\right.$$

$$\left.\begin{array}{l} \left[\begin{array}{cc} 0 & * \\ \hline 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ \hline I & 0 \end{array}\right] \end{array}\right) \begin{array}{l} d_\nu \\ w_\nu \\ \vdots \\ \vdots \\ w_0 \\ c_\nu \\ \vdots \\ \vdots \\ c_0 \\ f_\nu \end{array} \tag{29}$$

where

$$\operatorname{rank}\begin{bmatrix} F_i \\ E_i \end{bmatrix} = c_i + w_i = s_{i-1} - g_i \leqslant c_{i-1}. \tag{30}$$

The second block column in the first and second matrices has width u_ν^r.

The proof of Theorem 12 is given in Appendix B.

To complete the picture, we will conclude this section with some remarks about the generalization of the above process for the canonical form (17) of Theorem 6.

Generalizing the above process, we again get an inductive definition of a sequence of matrix function triples $(E_i(t), A_i(t), B_i(t))$, $i \in \mathbb{N}_0$, and sequences of corresponding characteristic values. In this case we must assume additionally that $d^c(t) \equiv d^c$ and $s^c(t) \equiv s^c$ for every occurring pair of matrices. Finally, we get the following canonical form.

THEOREM 13. *Let ν from Lemma 11 be well-defined for a triple $(E(t), A(t), B(t))$ of smooth matrix functions, and let the values c_i, w_i, $i = 0, \ldots, \nu$, be defined as in Theorem 12. The triple $(E(t), A(t), B(t))$ is then equivalent to a triple of matrix functions of the form (without arguments)*

$$\left(\left[\begin{array}{cc|c|cccc} I & 0 & 0 & 0 & * & \cdots & * \\ 0 & I & 0 & 0 & * & \cdots & * \\ \hline 0 & 0 & 0 & 0 & F_\nu & & * \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \ddots & F_1 \\ 0 & 0 & 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & E_\nu & & * \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & & \ddots & E_1 \\ 0 & 0 & 0 & & & & 0 \\ \hline 0 & 0 & 0 & 0 & * & \cdots & * \end{array}\right], \left[\begin{array}{cc|c|cccc} * & * & * & 0 & \cdots & \cdots & 0 \\ * & * & * & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \hline 0 & 0 & 0 & I & & & \\ \vdots & \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & \vdots & & & \ddots & \\ 0 & 0 & 0 & & & & I \\ \hline * & * & * & 0 & \cdots & \cdots & 0 \end{array}\right], \right.$$

$$\left. \left[\begin{array}{cc} 0 & I \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \\ \hline I & 0 \end{array}\right] \right) \begin{array}{l} d_\nu^c \\ d_\nu^u \\ w_\nu \\ \vdots \\ \vdots \\ w_0 \\ c_\nu \\ \vdots \\ \vdots \\ c_0 \\ f_\nu \end{array} \qquad (31)$$

and (28) *and* (30) *are still valid.*

Note that we have assumed additionally that $d^c(t) \equiv d^c$ and $s^c(t) \equiv s^c$ for every occurring pair of matrices, but in the normal form (31) only the d_i^c's occur. Therefore, one can use weaker assumptions to prove Theorem 13.

REMARK 14. Until now we have studied only equivalence transformations of the form (10) and (11). For constant coefficient systems feedback canonical forms have been studied in [26]. The canonical form (31) gives us the possibility to generalize these results for the variable coefficient case.

4. REGULARIZATION BY FEEDBACK

In this final section we answer the question whether the system is regularizable by proportional and/or derivative feedbacks, i.e. whether the closed loop system is uniquely solvable for all consistent initial vectors.

For constant coefficient systems regularizability has been studied by several authors, for example [2, 3]. An approach similar to the one we present in this paper for linear descriptor systems with variable coefficients has been studied in [5].

Using the results of Section 3, we can transform (1) to an equivalent descriptor system of a very special structure. Note that equivalence here means that there is a one-to-one correspondence of the solutions, that is, we get a descriptor system which has the same solutions as the original system (1) for every consistent initial value and any given control.

THEOREM 15. *Let ν from (25) be well defined for the triple $(E(t), A(t), B(t))$ in (1). Then (1) is equivalent to a descriptor system of the form*

$$\dot{x}_1(t) = A_{13}(t)x_3(t) + B_{12}(t)u_2(t), \tag{32a}$$

$$0 = x_2(t), \tag{32b}$$

$$0 = A_{31}(t)x_1(t) + A_{33}(t)x_3(t) + u_1(t), \tag{32c}$$

$$0 = 0; \tag{32d}$$

d_ν, a_ν, and u_ν^r are the numbers of the differential, algebraic, and undetermined components of the unknown x in (32); and f_ν and u_ν^l are the numbers of equations in (32c) and (32d).

Proof. We transform the triple $(E(t), A(t), B(t))$ to the form (18) and pass to (22). From Lemma 11 we know that we can repeat this process ν times until $s_\nu = 0$. This yields a triple of matrices of the form

$$\left(\begin{bmatrix} I_{d\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ A_{31}(t) & 0 & A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{matrix} d_\nu \\ a_\nu \\ f_\nu \\ u_\nu^l \end{matrix} \qquad (33)$$

where the last block columns of the first and second matrices have width u_ν^r and all these steps are reversible. ■

Note that some solution components of (32b) which are constrained to zero come from uncontrollable higher index components of (1). The other uncontrollable higher index components are fulfilled trivially and can be found in (32d).

Before we can answer the question posed in the beginning of this section, we have to define what we mean by regularizability.

DEFINITION 16 (See [5, Definition 7]).

(a) The descriptor system (1) is called regularizable by proportional feedback if there exists a (proportional state) feedback $u(t) = F(t)x(t) + w(t)$ such that the closed loop system

$$E(t)\dot{x}(t) = [A(t) + B(t)F(t)]x(t) + B(t)w(t), \qquad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

(b) The descriptor system (1) is called regularizable by derivative feedback if there exists a (derivative) feedback $u(t) = -G(t)\dot{x}(t) + w(t)$ such that the closed loop system

$$[E(t) + B(t)G(t)]\dot{x}(t) = A(t)x(t) + B(t)w(t), \qquad x(t_0) = x_0$$

is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

(c) The descriptor system (1) is called regularizable by combined derivative and proportional state feedback if there esists a feedback $u(t) = -G(t)\dot{x}(t) + F(t)x(t) + w(t)$ such that the closed loop system

$$\begin{aligned}&[E(t) + B(t)G(t)]\dot{x}(t) \\ &\qquad = [A(t) + B(t)F(t)]x(t) + B(t)w(t), \qquad x(t_0) = x_0\end{aligned}$$

is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

Now we can formulate the main theorem of this section. It gives necessary and sufficient conditions for when the descriptor system (1) is regularizable by proportional or derivative feedback.

THEOREM 17. *Let* ν *from* (25) *be well defined for the triple* $(E(t), A(t), B(t))$ *in* (1).

(a) *The descriptor system* (1) *can be regularized by proportional state feedback if and only if* $u_\nu^r = f_\nu$.

(b) *The descriptor system* (1) *can be regularized by derivative feedback if and only if* $u_\nu^r = f_\nu$.

(c) *The descriptor system* (1) *can be regularized by combined derivative and proportional state feedback if and only if* $u_\nu^r = f_\nu$.

Proof. From Theorem 15 we know that it is sufficient to analyze the descriptor system (32). Therefore, we assume that (1) is of the form (32).

In order to show that the condition $u_\nu^r = f_\nu$ is necessary, observe that the last block row (32d) is fulfilled trivially and we can leave these equations off altogether. If $u_\nu^r > f_\nu$, the remaining system (32a–c) has more columns than rows, i.e., we can choose components of x arbitrarily and the solution will not be unique. If $u_\nu^r < f_\nu$, the system (32a–c) has more rows than columns, and we cannot apply arbitrary controls.

Assume now that $u_\nu^r = f_\nu$. We can choose the proportional feedback

$$u(t) = \begin{bmatrix} -A_{31}(t) & 0 & I_{f_\nu} - A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix} x(t) + w(t),$$

and the closed loop system is then of the form

$$\begin{bmatrix} I_{d_\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} w(t).$$

The corresponding DAE has the characteristic values $d_{\mathrm{DAE}} = d_\nu$, $a_{\mathrm{DAE}} = a_\nu + f_\nu$, $s_{\mathrm{DAE}} = 0$, $u_{\mathrm{DAE}}^r = 0$. Since $u_{\mathrm{DAE}}^l = u_\nu^l$ and since the last block row of

$B(t)$ of the closed loop system is zero, we get from [24, Corollary 20] that the closed loop system is uniquely solvable for every consistent initial value x_0 and any given control $w(t)$.

In the case of derivative feedback we choose

$$u(t) = \begin{bmatrix} 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + w(t)$$

and get the closed loop system

$$\begin{bmatrix} I_{d_\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{f_\nu} \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 & A_{13}(t) \\ 0 & I_{a_\nu} & 0 \\ A_{31}(t) & 0 & A_{33}(t) \\ 0 & 0 & 0 \end{bmatrix} x(t)$$

$$+ \begin{bmatrix} 0 & B_{12}(t) \\ 0 & 0 \\ I_{f_\nu} & 0 \\ 0 & 0 \end{bmatrix} w(t),$$

which is as required. (c) follows now immediately. ■

COROLLARY 18. *Let ν from (25) be well defined for the triple $(E(t), A(t), B(t))$ of a square system (1), i.e., $n = l$. The system (1) can be regularized by a proportional state, a derivative, or a combined derivative proportional feedback if and only if $u_\nu^l = 0$.*

EXAMPLE 19. Treating the problem of Example 1, we get

$$\left(\begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & t \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & t \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\sim \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right).$$

Therefore, $\nu = 0$, and the descriptor system (9) is equivalent to the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

Finally, choosing

$$u(t) = [1 \quad 1] x(t) + w(t),$$

we get the closed loop system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t),$$

which has unique solution

$$x_1(t) = \int x_2(t)\, dt,$$

$$x_2(t) = -w_2(t).$$

REMARK 20. Theorem 17 and Corollary 18 show that it is sufficient to study the reduction process based on the condensed form (18). Note that there is still a lot of freedom in the choice of the feedback, and the canonical form (17) can possibly be used to improve the robustness of the system or guarantee controllability of the regularized system. For constant coefficient systems this is done in [2, 3, 13], but so far it is not really clear what robustness or controllability means for linear coefficient systems with variable coefficients.

REMARK 21. We studied the descriptor system (1) under the assumption that $x(t)$ ($u(t)$) is the given state (control) of the system, and we try to construct a feedback which makes the closed loop system uniquely solvable for any control.

On the other hand, a different point of view is possible (see also [5]). In Theorem 15 the solution component x_3 can be chosen arbitrarily; hence it can be viewed as a control. Choosing u_2 and x_3, we get x_1 from Equation (32a). u_1 is now fixed by Equation (32c), i.e., u_1 is a state variable. Omitting the trivially fulfilled last equation, we can replace (32) by the square descriptor system

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\overbrace{\begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix}}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ A_{31} & 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \end{bmatrix} + \begin{bmatrix} A_{13} & B_{12} \\ 0 & 0 \\ A_{33} & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ u_2 \end{bmatrix}, \quad (34)$$

which is uniquely solvable for any control $\begin{bmatrix} x_3 \\ u_2 \end{bmatrix}$. Therefore, the feedbacks we choose in Theorem 17 change the state variable u_1 of (34) to a control variable in the closed loop systems, and vice versa for x_3.

5. CONCLUSION

We have presented local and global equivalences and corresponding canonical forms for linear descriptor systems with variable coefficients. The global canonical forms and the global condensed forms, which are not as far reduced as the canonical forms, are powerful tools in the analysis of this type of descriptor systems. Based on a condensed form we found under what conditions a linear descriptor system is regularizable, i.e., there exists a derivative and/or proportional state feedback such that the closed loop system is uniquely solvable for all consistent initial vectors. These conditions are necessary and sufficient.

While the global forms are not suitable for numerical computations, the numerical accessibility of local quantities which give essential information on the global solution behavior are of great importance in the development of numerical methods.

We assumed that sequences of characteristic values are constant. For differential algebraic equations weaker assumptions such as jumps at isolated points connected with a weak solvability concept can be considered.

APPENDIX A. PROOF OF THEOREM 6

To prove Theorem 6 we make use of the following property [21, 29]:

LEMMA 22. *Let* $E \in C^l([t_1, t_2], \mathbb{C}^{n,n})$, $l \in \mathbb{N}_0$, rank $E(t) = r$ *for all* $t \in [t_1, t_2]$. *Then there exist* $U, V \in C^l([t_1, t_2], \mathbb{C}^{n,n})$ *with* $U(t), V(t)$ *nonsin-*

gular (unitary) for every $t \in [t_1, t_2]$ *such that*

$$U(t)^* E(t)V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}, \qquad t \in [t_1, t_2],$$

where $\Sigma \in C^l([t_1, t_2], \mathbb{C}^{r,r})$.

Proof of Theorem 6. From now on, we will omit the argument t in the proofs. Applying Lemma 22 to the matrix E and setting $P = U^* \Sigma^{-1}$, $Q = V$, and $S = I$, we find that (E, A, B) is equivalent to

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \right).$$

Now, we apply Lemma 22 to B_{21} and set $P = \operatorname{diag}(I, U^* \Sigma^{-1})$, $Q = I$, and $S = V$, which yields

$$\left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ I_f & 0 \\ 0 & 0 \end{bmatrix} \right),$$

and we use the identity in the matrix B to eliminate B_{11}. As in the DAE case, we can proceed with the last block row and get

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ I_f & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right).$$

Applying Lemma 22 to B_{12} and setting $S = \mathrm{diag}(I, V)$, $P = \mathrm{diag}(U^*\Sigma^{-1}, I_{n-s})$, $Q = \mathrm{diag}(\Sigma U^{-*}, I_{l-s})$ we obtain

$$\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 \\ 0 & 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & I_a & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & B_{32} & B_{33} \\ I_f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

Choosing

$$P = \begin{bmatrix} I & & & & & & & \\ & I & & & & & & \\ -B_{32} & & I & & & & & \\ & & & I & & & & \\ & & & & I & & & \\ & & & & & I & & \\ & & & & & & I & \\ & & & & & & & I \end{bmatrix},$$

$$Q = \begin{bmatrix} I & & & & \\ & I & & & \\ B_{32} & & I & & \\ & & -A_{53} & I & \\ & & & & I \end{bmatrix}, \qquad S = I,$$

we eliminate B_{32}, A_{53} in the matrix E, B, respectively. Furthermore, applying Lemma 22 to B_{33} and using a block permutation which moves the fourth block row to the seventh block row, we obtain

$$\left(\begin{bmatrix} I_{s^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d^c} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{d^u} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ A_{51} & A_{52} & 0 & 0 & I_a & 0 \\ I_{s^c} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{s^u} & 0 & 0 & 0 & 0 \\ A_{81} & A_{82} & A_{83} & A_{84} & A_{85} & A_{86} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_{s^c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{d^c} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I_f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right).$$

Next, we use the identities in A to eliminate the remaining entries in the first, second, and fifth block columns of A. Finally, set $P = I$, $Q = \operatorname{diag}(I_s, Q_3, Q_4, I_{l-r})$, $S = I$, where Q_i, $i = 3, 4$, is chosen to be the solution of the initial problems

$$\dot{Q}_i - A_{ii}Q_i, \quad Q_i(t_0) = I, \qquad i = 3, 4,$$

which is nonsingular at every point $t \in [t_1, t_2]$. This yields the required canonical form. ∎

APPENDIX B. PROOF OF THEOREM 12

From

$$\operatorname{rank}\left\{R_{i-1}(t)^* \left[A_{12}^{(i-1)}(t)\, A_{14}^{(i-1)}(t)\right]\right\}$$
$$= \operatorname{rank}\left[R_{i-1}(t)^* A_{14}^{(i-1)}(t)\right] + \operatorname{rank}\left[W_{i-1}(t)^* R_{i-1}(t)^* A_{12}^{(i-1)}(t)\right]$$

we obtain (28a), while (28b) is a consequence of Lemma 11 (24f).

Starting from Corollary 7 in the permuted form (if we do not perform the last elimination step, i.e. do not zero out A_{22}) we obtain

$$
\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ A_{61} & A_{62} & A_{63} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}\right) \begin{array}{l} d_0 \\ s_0 \\ w_0 \\ s_0 \\ b_0 \\ f_0 \end{array}
$$

Defining

$$
X^i = \begin{bmatrix} X_i & & & * \\ 0 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & X_1 \\ & & & 0 \end{bmatrix} \qquad \text{for} \quad X = F, E
$$

and $w^i = \sum_{j=1}^{i} w_i$, $c^i = \sum_{j=1}^{i} c_i$, we get in the ith step (omitting subscripts i and denoting by $[X \; X]$ a block entry $X = F^i, I$ which extends over two block columns)

$$
\left(\left[\begin{array}{cc|c|cc|cc}
I & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & I & 0 & * & * \\ \hline
 & & & & & F^i & F^i \\ \hline
 & & & & & W_{11} & * \\
 & & & & & W_{21} & * \\ \hline
 & & & & & 0 & E^{i-1} \\ \hline
 & & & & & * & *
\end{array}\right], \left[\begin{array}{cc|c|cc|cc}
A_{11} & A_{12} & A_{13} & & & & \\
A_{21} & A_{22} & A_{23} & & & & \\ \hline
 & & & & & & \\ \hline
 & & & I & & & \\
 & & & & I & & \\ \hline
 & & & & & I & I \\ \hline
* & * & * & & & &
\end{array}\right], \right.
$$

$$
\left.\left[\begin{array}{cc}
0 & B_{12} \\
0 & B_{22} \\ \hline
 & \\ \hline
 & \\
 & \\ \hline
 & \\ \hline
I & 0
\end{array}\right]\right) \begin{array}{l} d_i \\ s_i \\ w^i \\ s_i \\ b_i \\ c^{i-1} \\ f_i \end{array}
$$

Applying Lemma 22 to B_{22} and setting $P = \operatorname{diag}(U^*, I, I, U^*, I, I)$, $Q = \operatorname{diag}(I, I, I, U^{-*}, I, I, I)$, $S = \operatorname{diag}(I, \Sigma^{-1}V)$, we obtain

$$
\left(\left[\begin{array}{cc|c|ccc|cc}
I & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & I & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & I & 0 & * & * \\
\hline
 & & & & & & F^i & F^i \\
\hline
 & & & & & & W_{11} & * \\
 & & & & & & W_{21} & * \\
 & & & & & & W_{31} & * \\
\hline
 & & & & & & 0 & E^{i-1} \\
\hline
 & & & & & & * & *
\end{array}\right],\right.
$$

$$
\left[\begin{array}{cc|c|ccc|cc}
A_{11} & A_{12} & A_{13} & & & & & \\
A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & & & \\
A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & & & \\
\hline
 & & & & & & & \\
\hline
 & & & I & & & & \\
 & & & & I & & & \\
 & & & & & I & & \\
\hline
 & & & & & & I & I \\
\hline
* & * & * & & & & &
\end{array}\right],
$$

$$
\left.\left[\begin{array}{ccc}
0 & B_{12} & B_{13} \\
0 & I & 0 \\
0 & 0 & 0 \\
\hline
 & & \\
\hline
 & & \\
 & & \\
 & & \\
\hline
 & & \\
\hline
I & 0 & 0
\end{array}\right]\right)
\begin{array}{l}
d_i \\
g_{i+1} \\
s_i - g_{i+1} \\
w^i \\
g_{i+1} \\
s_i - g_{i+1} \\
b_i \\
c^{i-1} \\
f_i
\end{array}
$$

Using block row operations to eliminate the blocks A_{25}, A_{26}, A_{35}, A_{36}, and B_{12} combined with a block permutation which moves the second block row to the end yields

$$
\left(\left[\begin{array}{cc|c|ccc|cc}
I & 0 & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & 0 & I & 0 & * & * \\
\hline
 & & & & & & F^i & F^i \\
\hline
 & & & & & & W_{11} & * \\
 & & & & & & W_{21} & * \\
 & & & & & & W_{31} & * \\
\hline
 & & & & & & 0 & E^{i-1} \\
\hline
 & & & & & & * & *
\end{array}\right],\right.
$$

$$
\left[\begin{array}{cc|c|ccc|cc}
A_{11} & A_{12} & A_{13} & & & & & \\
A_{21} & A_{22} & A_{23} & & & & & \\
\hline
 & & & & & & & \\
\hline
 & & & I & & & & \\
 & & & & I & & & \\
 & & & & & I & & \\
\hline
 & & & & & & I & I \\
\hline
* & * & * & & & & &
\end{array}\right],
$$

$$
\left.\left[\begin{array}{cc}
0 & B_{12} \\
0 & 0 \\
\hline
 & \\
\hline
 & \\
 & \\
\hline
 & \\
\hline
I & 0
\end{array}\right]\right)
\begin{array}{l}
d_i \\
s_i - g_{i+1} \\
w^i \\
g_{i+1} \\
s_i - g_{i+1} \\
b_i \\
c^{i-1} \\
f_{i+1}
\end{array}
$$

Finally, from a simple modification of the DAE case in [24] we get

$$
\left(
\left[\begin{array}{cc|c|cc|cc}
I & 0 & 0 & 0 & 0 & * & * \\
0 & 0 & 0 & I & 0 & * & * \\
\hline
 & & & & & F^{i+1} & F^{i+1} \\
\hline
 & & & & & W_{11} & * \\
 & & & & & W_{21} & * \\
\hline
 & & & & & 0 & E^{i} \\
 & & & & & * & *
\end{array}\right],
\left[\begin{array}{cc|c|c|c}
A_{11} & A_{12} & A_{13} & & \\
A_{21} & A_{22} & A_{23} & & \\
\hline
 & & & & \\
\hline
 & & & I & \\
 & & & I & \\
\hline
 & & & & I \quad I \\
\hline
* & * & * & &
\end{array}\right],
\left[\begin{array}{cc}
0 & B_{12} \\
0 & B_{22} \\
\hline
 & \\
\hline
 & \\
\hline
I & 0
\end{array}\right]
\right)
\begin{array}{l}
d_{i+1} \\ s_{i+1} \\ w^{i+1} \\ s_{i+1} \\ b_{i+1} \\ c^{i} \\ f_{i+1}
\end{array}
$$

Thus, (29) follows by induction, and (30) holds because $\begin{bmatrix} F_{i+1} \\ E_{i+1} \end{bmatrix}$ is obtained by nonsingular transformations applied to $[0 \;\; I \;\; 0]$.

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