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Stability Analysis of Positive Switched Linear Systems With Delays

Xingwen Liu and Chuangyin Dang

Abstract—This technical note addresses the stability problem of delayed positive switched linear systems whose subsystems are all positive. Both discrete-time systems and continuous-time systems are studied. In our analysis, the delays in systems can be unbounded. Under certain conditions, several stability results are established by constructing a sequence of functions that are positive, monotonically decreasing, and convergent to zero as time tends to infinity (additionally continuous for continuous-time systems). It turns out that these functions can serve as an upper bound of the systems' trajectories starting from a particular region. Finally, a numerical example is presented to illustrate the obtained results.

Index Terms—Delays, dual systems, positive switched linear system (PSLS), stability.

I. INTRODUCTION

A switched system is a type of hybrid dynamical system that combines discrete states and continuous states. Informally, it consists of a family of dynamical subsystems and a rule, called a switching signal, that determines the switching manner between the subsystems. Many dynamical systems can be modeled as such switched systems [1]–[3]. Switched systems possess rich dynamics due to the multiple subsystems and various possible switching signals [4], [5]. Many interesting and challenging issues in switched systems have attracted a lot of attention [6]–[9].

Recently, the importance of positive switched linear systems (PSLSs) has been highlighted by many researchers due to their broad applications in communication systems [10], formation flying [11], and systems theory [12], [13]. A positive system implies that its states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative [14], [15]. A PSLS means a switched linear system in which each subsystem is itself a positive system. Positive systems have numerous applications in areas such as economics, biology, sociology and communications [16], [17], [30]. It is well known that positive systems have many special and interesting properties. For example, their stability is not affected by delays [18]–[20]. It should be pointed out that studying the dynamics of positive switched systems is more challenging than that of general switched system because, in order to obtain some "elegant" results, one has to combine the features of positive systems and switched systems [21].

A mass of literature is concerned with the issue of stability of PSLSs [22]–[25]. When stability of positive systems is considered, it is natural

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X. Liu is with the College of Electrical and Information Engineering, Southwest University for Nationalities of China, Chengdu, Sichuan, 610041, China (e-mail: xingwenliu@gmail.com).

C. Dang is with the Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong, China (e-mail: mecdang@cityu.edu.hk).

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to apply a linear copositive Lyapunov function [26], [27]. A necessary and sufficient condition has been established in [25] for the existence of common linear copositive Lyapunov function, where the switched system consists of two subsystems of n -dimension. A switched linear copositive Lyapunov function has been proposed in [24], which yields a less conservative stability condition for PSLs.

Because positive linear systems have many elegant properties, one naturally expects that PSLs also possess such properties. This results in a number of conjectures in the field of PSLs. Some of these conjectures are shown correct [23], [25], whereas some of them are shown incorrect by counterexamples [22], [28], [29]. Note that the existence of a common linear copositive Lyapunov function ensures the asymptotic stability of a PSL and that a positive system shares the same stability with its dual system (a pair of dual systems are ones with mutually transposed system matrices). Thus it is reasonable to conjecture that a PSL is asymptotically stable if its dual system has a common linear copositive Lyapunov function. And if this conjecture does hold, one may expect to extend it to the delayed case due to the fact that delays do not affect the stability of positive linear systems. Foregoing statements are the driving force behind this research.

Delays are universal in real engineering processes and have very complex impacts on system dynamics, which make the task of studying dynamics of PSLs with delays necessary. However, to our knowledge, up to now, there has been little literature studying the stability problem of delayed PSLs. Since PSLs are a special class of switched linear systems, the relevant methods applicable to delayed switched linear systems are also suitable for delayed PSLs. These methods, such as the popular Lyapunov-Krasovskii functional method, however, generally fail to capture the nature of PSLs. So, in order to obtain desirable results, it is necessary to develop new methods.

This technical note focuses on the stability problem of PSLs with delays, both discrete- and continuous-time systems are considered. We neither adopt the Lyapunov-Krasovskii functional method widely used to analyze switched linear systems, nor the approaches used in [19], [20], since [19], [20] transformed the stability issue of a positive system with bounded delays into that of positive system with constant delays, and the delays considered in this technical note may be unbounded. The idea of this technical note mainly lies in: under some specified conditions, the continuous-time system's trajectory solution starting from a certain region can be "covered" by a family of functions on the whole interval $[0, \infty)$, and piecing together these functions forms a function which is monotonically decreasing on interval $[0, \infty)$, and approaches zero as $t \rightarrow \infty$, thus establishing the system's asymptotic stability. A similar idea is applied to the discrete-time systems.

The rest of this technical note is organized as follows. In Section II, necessary preliminaries are presented and the problems to be treated are stated. Sections III and IV propose some stability criteria for discrete- and continuous-time PSLs with delays, respectively. Section V gives some remarks on the results in Sections III and IV. An example is given in Section VI, and Section VII concludes this technical note.

Notations: $A \succeq 0$ ($\preceq 0$, $\succ 0$, $\prec 0$) means that all elements of matrix A are nonnegative (nonpositive, positive, negative), $\mathbb{R}(\mathbb{R}_{0,+})$ stands for the set of all real (nonnegative) numbers, \mathbb{R}^n is an n -dimensional real vector space, \mathbb{R}_+^n is the set of positive vectors, and $\mathbb{R}^{n \times n}$ is the set of real $n \times n$ -dimensional matrices. A^T denotes the transpose of matrix A and \mathbf{M} the set of Metzler matrices whose off diagonal entries are nonnegative. $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. $\underline{\mathbf{p}} = \{1, \dots, p\}$ and $\underline{\mathbf{p}}_0 = \{0\} \cup \underline{\mathbf{p}}$ for $p \in \mathbb{N}$. $\lceil a \rceil$ is the smallest integer greater than or equal to real number a . $\|\mathbf{x}\|$ denotes the l_∞ norm of vector $\mathbf{x} \in \mathbb{R}^n$, i.e., $\|\mathbf{x}\| = \max\{|x_1|, \dots, |x_n|\}$.

II. PROBLEM STATEMENTS AND PRELIMINARIES

A discrete-time switched linear system can be stated as

$$\begin{aligned} \mathbf{x}(k+1) &= A_{0\sigma(k)}\mathbf{x}(k) \\ &+ \sum_{i=1}^p A_{i\sigma(k)}\mathbf{x}(k - \tau_{i\sigma(k)}(k)), \quad k \in \mathbb{N}_0 \\ \mathbf{x}(k) &= \boldsymbol{\varphi}(k), \quad k = -\tau, \dots, 0 \end{aligned} \quad (1)$$

where the map $\sigma : \mathbb{N}_0 \rightarrow \underline{\mathbf{m}}$ is an arbitrary switching signal with m being the number of subsystems, $A_{il} \in \mathbb{R}^{n \times n}$, $i \in \underline{\mathbf{p}}_0$, $l \in \underline{\mathbf{m}}$, are system matrices, $\tau_{il}(k) \geq 0$ are delays.

Definition 1: System (1) is said to be positive if, for any initial condition $\boldsymbol{\varphi}(\cdot) \succeq 0$ and any switching signal, the corresponding trajectory $\mathbf{x}(k) \succeq 0$ holds for all $k \in \mathbb{N}$.

The following two lemmas are direct extensions of Lemmas 3 and 4 in [19], respectively.

Lemma 1: System (1) is positive if and only if $A_{il} \succeq 0$, $i \in \underline{\mathbf{p}}_0$, $l \in \underline{\mathbf{m}}$.

Lemma 2: Assume that system (1) is positive and arbitrarily fix a switching signal σ . Let $\mathbf{x}_a(\cdot)$ and $\mathbf{x}_b(\cdot)$ be solution trajectories of (1) under the initial conditions $\boldsymbol{\varphi}_a(\cdot)$ and $\boldsymbol{\varphi}_b(\cdot)$, respectively. Then $0 \preceq \boldsymbol{\varphi}_a(\cdot) \preceq \boldsymbol{\varphi}_b(\cdot)$ implies that $\mathbf{x}_a(k) \preceq \mathbf{x}_b(k)$, $\forall k \in \mathbb{N}$.

A continuous-time switched linear system with delays is as follows:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A_{0\varrho(t)}\mathbf{x}(t) + \sum_{i=1}^p A_{i\varrho(t)}\mathbf{x}(t - \tau_{i\varrho(t)}(t)), \quad t \geq 0 \\ \mathbf{x}(t) &= \boldsymbol{\phi}(t), \quad t \in [-\tau, 0] \end{aligned} \quad (2)$$

where $\varrho : \mathbb{R}_{0,+} \rightarrow \underline{\mathbf{m}}$ is a piecewise constant function, referred to as switching signal, with m being the number of subsystems, delays $\tau_{il}(t) \geq 0$ are continuous in t , and $\boldsymbol{\phi}(t)$ is the vector-valued initial function.

Remark 1: "Piecewise constant" means that the switching signals are a kind of "slow" switchings in the sense that the set of time-lapses between any consecutive switchings has a positive lower bound, i.e., if $\{t_k; k \in \mathbb{N}_0\}$ is the sequence of switching instants, then there exists a scalar t^* satisfying

$$t^* = \inf_{k \in \mathbb{N}_0} \{t_{k+1} - t_k\} > 0. \quad (3)$$

Definition 2: System (2) is said to be positive if, for any initial condition $\boldsymbol{\phi}(\cdot) \succeq 0$ and for any switching signal, the corresponding trajectory $\mathbf{x}(t) \succeq 0$ holds for all $t > 0$.

Lemma 3: System (2) is positive if and only if $A_{0l} \in \mathbf{M}$, $A_{il} \succeq 0$, $i \in \underline{\mathbf{p}}$, $l \in \underline{\mathbf{m}}$.

Proof: Combining Lemma 2.1 in [31] and Lemma 2 in [20], one can obtain the remaining proof similar to that of Lemma 1. ■

Hereafter, we always assume that in system (1) $A_{il} \succeq 0$, $i \in \underline{\mathbf{p}}_0$, $l \in \underline{\mathbf{m}}$, and that in system (2) $A_{0l} \in \mathbf{M}$, $A_{il} \succeq 0$, $i \in \underline{\mathbf{p}}$, $l \in \underline{\mathbf{m}}$. By Lemmas 1 and 3, systems (1) and (2) are both positive.

Lemma 4: Assume that system (2) is positive and arbitrarily fix a switching signal ϱ . Let $\mathbf{x}_a(\cdot)$ and $\mathbf{x}_b(\cdot)$ be solution trajectories of (2) under the initial conditions $\boldsymbol{\phi}_a(\cdot)$ and $\boldsymbol{\phi}_b(\cdot)$ on $[-\tau, t_s]$ with $t_s \geq 0$, respectively. Then $0 \preceq \boldsymbol{\phi}_a(\cdot) \preceq \boldsymbol{\phi}_b(\cdot)$ implies that $\mathbf{x}_a(t) \preceq \mathbf{x}_b(t)$, $\forall t \geq t_s$.

Proof: The proof of this lemma is similar to that of Lemma 2 and thus omitted here. ■

In this technical note, we will establish several stability conditions for positive systems (1) and (2). For simplicity, when speaking of the asymptotic stability of a switched linear system, we mean the global uniform asymptotic stability for arbitrary switching signals [32].

To accomplish these tasks, we make the following two necessary assumptions on (1) and (2).

Assumption 1: In system (1), there exist $T \in \mathbb{N}$ and a scalar $0 \leq \theta < 1$ such that

$$\theta = \sup_{k > T, i \in \underline{p}, l \in \underline{m}} \tau_{il}(k)/k. \quad (4)$$

Assumption 2: In system (2), there exist $T > 0$ and a scalar $0 \leq \vartheta < 1$ such that

$$\vartheta = \sup_{t > T, i \in \underline{p}, l \in \underline{m}} \tau_{il}(t)/t. \quad (5)$$

Remark 2: Constraint (4) on the delays in system (1) is very mild. It is easy to see that all the bounded delays satisfy (4), including the constant delays and time-varying and bounded delays. If all $\tau_{il}(k)$ are bounded by τ^* , then T can take any value equal to or greater than $\tau^* + 1$ and we set $\tau = \tau^*$ in (1); if all $\tau_{il}(k) \equiv 0$, i.e., system (1) is free of delays, then T can take any positive integer and $\theta = 0$ and $\tau = 0$. Delays satisfying (4) may be unbounded. In this case, it suffices to define $\varphi(\cdot)$ on a finite set $\{-\tau, \dots, 0\}$. Note that (4) implies that $k - \tau_{il}(k) \geq k - \theta k \geq 0, \forall k > T$. Let $-\tau = \min_{i \in \underline{p}, l \in \underline{m}} \{\inf_{0 \leq k \leq T} \{k - \tau_{il}(k)\}, \inf_{k > T} \{k - \tau_{il}(k)\}\} = \min_{i \in \underline{p}, l \in \underline{m}} \inf_{0 \leq k \leq T} \{k - \tau_{il}(k)\}$. Clearly, τ is bounded.

Intuitively, if constraint (4) is not satisfied, it may occur that even if for a very large k , evolution of $\mathbf{x}(k)$ may still involve the initial conditions $\varphi(\cdot)$ and those values near them, which may result in that $\mathbf{x}(k)$ does not approach zero as $k \rightarrow \infty$. Let us consider scalar equation $x(k+1) = 0.6x(k - \tau(k))$. One can easily see that it is not asymptotically stable if $\tau(k) = \lceil k \sin k \rceil$ (such a $\tau(k)$ does not satisfy constraint (4)). However, as indicated in Theorem 1 below, once $\tau(k)$ satisfies constraint (4), this equation is indeed asymptotically stable. Similar remark can be made for (5).

Hereafter, it is assumed that Assumption 1 holds for system (1) and Assumption 2 holds for (2).

III. STABILITY ANALYSIS OF DISCRETE-TIME PSLSS

This section establishes a stability condition for discrete-time PSLSS, beginning with the following lemma.

Lemma 5: Consider the positive system (1). Suppose that there exists a vector $\lambda \in \mathbb{R}_+^n$ satisfying

$$\left(\sum_{i=0}^p A_{il} - I \right) \lambda < 0, \forall l \in \underline{m}. \quad (6)$$

Let the initial condition be $\varphi(\cdot) \equiv \lambda$. Then there exists a scalar $0 < \mu < 1$ such that

$$\mathbf{x}(k) < \mu^{q+1} \lambda, \forall k \in \mathcal{Q}_q, q \in \mathbb{N}_0 \quad (7)$$

where $\mathcal{Q}_q = \{Q_q + 1, \dots, Q_{q+1}\}$ with $Q_0 = 0, Q_1 = T \geq 2$ (given in (4)), and $Q_{q+1} = \lceil (Q_q + 1)/(1 - \theta) \rceil, q \in \mathbb{N}$.

Proof: By (6), there exists a scalar $0 < \nu < 1$ satisfying $(\sum_{i=0}^p A_{il} - I) \lambda < -\nu \lambda, \forall l \in \underline{m}$, which implies that $\sum_{i=0}^p A_{il} \lambda < \mu \lambda$ with $0 < \mu = 1 - \nu < 1$. The remainder of the proof is completed by induction.

(I). We show that (7) holds for $q = 0$. Let $k = 1$. Then, $\mathbf{x}(1) = \sum_{i=0}^p A_{i\sigma(0)} \lambda < \mu \lambda$.

Suppose that $\mathbf{x}(k) < \mu \lambda$ holds for all k with $1 \leq k \leq s \in \{1, \dots, Q_1 - 1\}$. Let $S_{0,1}(s) = \{i : s - \tau_{i\sigma(s)}(s) > 0\}$ and $S_{0,2}(s) = \underline{p} \setminus S_{0,1}(s)$. Then

$$\mathbf{x}(s+1) = A_{0\sigma(s)} \mathbf{x}(s) + \sum_{i \in S_{0,1}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s))$$

$$\begin{aligned} & + \sum_{i \in S_{0,2}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\ & \preceq A_{0\sigma(s)} \mu \lambda + \sum_{i \in S_{0,1}(s)} A_{i\sigma(s)} \mu \lambda + \sum_{i \in S_{0,2}(s)} A_{i\sigma(s)} \lambda \\ & \preceq A_{0\sigma(s)} \lambda + \sum_{i \in S_{0,1}(s)} A_{i\sigma(s)} \lambda + \sum_{i \in S_{0,2}(s)} A_{i\sigma(s)} \lambda \\ & < \mu \lambda. \end{aligned}$$

Therefore, (7) holds for $q = 0$.

(II). Assume that (7) holds for $q = j \in \mathbb{N}_0$, i.e.

$$\mathbf{x}(k) < \mu^{j+1} \lambda, \forall k \in \mathcal{Q}_j. \quad (8)$$

(III). We prove that (7) holds for $q = j + 1$, i.e.

$$\mathbf{x}(k) < \mu^{j+2} \lambda, \forall k \in \mathcal{Q}_{j+1}. \quad (9)$$

By the definitions of θ and $\mathcal{Q}_q, q \in \mathbb{N}_0$, we have that $0 \leq \tau_{i\sigma(s)}(s) \leq \theta s$ for all $s \geq T = Q_1$. Thus

$$s - \tau_{i\sigma(s)}(s) \geq s - \theta s \geq (1 - \theta)Q_{q+1} \geq Q_q + 1 \quad (10)$$

holds for all $s \geq Q_{q+1}, q \in \mathbb{N}_0$. Combining (8) and (10) yields that

$$\begin{aligned} \mathbf{x}(Q_{j+1} + 1) & = A_{0\sigma(Q_{j+1})} \mathbf{x}(Q_{j+1}) + \sum_{i=1}^p A_{i\sigma(Q_{j+1})} \mathbf{x} \\ & \quad \times (Q_{j+1} - \tau_{i\sigma(Q_{j+1})}(Q_{j+1})) \\ & \preceq A_{0\sigma(Q_{j+1})} \mu^{j+1} \lambda + \sum_{i=1}^p A_{i\sigma(Q_{j+1})} \mu^{j+1} \lambda \\ & = \mu^{j+1} \sum_{i=0}^p A_{i\sigma(Q_{j+1})} \lambda < \mu^{j+2} \lambda. \end{aligned}$$

Having shown that (9) holds for all k with $Q_{j+1} + 1 \leq k \leq s \in \{Q_{j+1} + 1, \dots, Q_{j+2} - 1\}$, now we prove that it holds for $k = s + 1$.

Define $S_{j,1}(s) = \{i : s - \tau_{i\sigma(s)}(s) > Q_j\}$ and $S_{j,2}(s) = \underline{p} \setminus S_{j,1}(s)$. It follows from (10) that

$$\begin{aligned} \mathbf{x}(s+1) & = A_{0\sigma(s)} \mathbf{x}(s) + \sum_{i \in S_{j+1,1}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\ & \quad + \sum_{i \in S_{j+1,2}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\ & \preceq A_{0\sigma(s)} \mu^{j+2} \lambda + \sum_{i \in S_{j+1,1}(s)} A_{i\sigma(s)} \mu^{j+2} \lambda \\ & \quad + \sum_{i \in S_{j+1,2}(s)} A_{i\sigma(s)} \mu^{j+1} \lambda \\ & \preceq A_{0\sigma(s)} \mu^{j+1} \lambda + \sum_{i \in S_{j+1,1}(s)} A_{i\sigma(s)} \mu^{j+1} \lambda \\ & \quad + \sum_{i \in S_{j+1,2}(s)} A_{i\sigma(s)} \mu^{j+1} \lambda < \mu^{j+2} \lambda \end{aligned}$$

which implies that (9) holds for all $k \in \mathcal{Q}_{j+1}$. ■

Theorem 1: The positive system (1) is asymptotically stable if there exists a vector $\lambda \in \mathbb{R}_+^n$ such that (6) holds.

Proof: Suppose that there exists a vector $\lambda \in \mathbb{R}_+^n$ satisfying (6). For an arbitrary given scalar $\epsilon > 0$, there exists a scalar $\nu > 0$ such that $\|\nu \lambda\| < \epsilon$. By Lemma 5, under the initial condition $\varphi(\cdot) = \nu \lambda$,

there exists a scalar $0 < \mu < 1$ such that the corresponding solution $\mathbf{x}_1(k)$ to (1) satisfies $\mathbf{x}_1(k) \prec \nu \mu^{q+1} \boldsymbol{\lambda}$, $\forall k \in \mathcal{Q}_q$, $\forall q \in \mathbb{N}_0$.

Choose $\boldsymbol{\varphi}(\cdot)$ such that $\|\boldsymbol{\varphi}(\cdot)\| < \min\{\nu \lambda_1, \dots, \nu \lambda_n\}$ with λ_i being the i th element of $\boldsymbol{\lambda}$. By Lemma 2, the corresponding solution $\mathbf{x}(k)$ to (1) satisfies $\mathbf{x}(k) \preceq \mathbf{x}_1(k) \prec \nu \mu^{q+1} \boldsymbol{\lambda}$, $\forall k \in \mathcal{Q}_q$, $q \in \mathbb{N}_0$. Therefore, $\|\mathbf{x}(k)\| \leq \|\nu \mu^{q+1} \boldsymbol{\lambda}\| = \mu^{q+1} \|\nu \boldsymbol{\lambda}\| < \mu^{q+1} \epsilon$, $\forall k \in \mathcal{Q}_q$, $q \in \mathbb{N}_0$, which, by the fact that $\bigcup_{q=0}^{\infty} \mathcal{Q}_q = \mathbb{N}$, indicates that $\|\mathbf{x}(k)\| < \epsilon$, $\forall k \in \mathbb{N}$.

For each $k \in \mathbb{N}$, there exists a unique $q(k) \in \mathbb{N}_0$ such that $k \in \mathcal{Q}_{q(k)}$, and $k \rightarrow \infty$ implies $q(k) \rightarrow \infty$. By Lemma 5, one has $\lim_{k \rightarrow \infty} \|\mathbf{x}(k)\| \leq \lim_{k \rightarrow \infty} \|\mu^{q(k)+1} \nu \boldsymbol{\lambda}\| \leq \lim_{k \rightarrow \infty} \mu^{q(k)+1} \epsilon = 0$. Hence, system (1) is asymptotically stable. ■

Remark 3: Condition (6) in Theorem 1 is a linear programming problem in $\boldsymbol{\lambda}$, and thus can be numerically solved by linear programming optimal toolbox [33] with slight computational effort.

IV. STABILITY ANALYSIS OF CONTINUOUS-TIME PSLSS

In this section, we analyze stability of continuous-time PSLSSs. We start with exploring the boundary of the trajectory of system (2) under certain conditions. It is a remark that, for system (2), sometimes we may need to modify the initial condition by defining $\boldsymbol{\phi}(\cdot)$ on $[t_{s_1}, t_{s_2}]$ with $t_{s_1} \geq -\tau$ and $t_{s_2} \geq 0$ so that system (2) will evolve from t_{s_2} on. In the following discussion, we denote by $\dot{\mathbf{x}}(t_k)$ the right-hand derivative of $\mathbf{x}(t)$ at time t_k if t_k is a switching instant or an instant from which the system evolves on.

Lemma 6: Consider system (2). Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying

$$\sum_{i=0}^p A_{il} \boldsymbol{\lambda} \prec 0, \quad \forall l \in \underline{\mathbf{m}}. \quad (11)$$

Let $\boldsymbol{\phi}(t) \equiv \boldsymbol{\lambda}$, $t \in [-\tau, t_s]$ with $t_s \geq 0$. Then it holds that the solution $\mathbf{x}(t) \preceq \boldsymbol{\lambda}$, $\forall t \in [t_s, \infty)$.

Proof: Condition (11) implies that $\dot{\mathbf{x}}(t_s) \prec 0$, and therefore $\mathbf{x}(t) \prec \boldsymbol{\lambda}$, $\forall t \in (t_s, \nu)$ for some $\nu > t_s$.

Suppose that this lemma is false. Then there exist two scalars $\delta' > \delta > t_s$ such that $\mathbf{x}(t) \preceq \boldsymbol{\lambda}$, $\forall t \in [t_s, \delta]$ and that some elements of $\mathbf{x}(t)$ are greater than the corresponding ones of $\boldsymbol{\lambda}$ on $(\delta, \delta']$.

Define two initial conditions $\boldsymbol{\phi}_a(t) = \boldsymbol{\lambda}$, $\forall t \in [-\tau, \delta]$ and

$$\boldsymbol{\phi}_b(t) = \begin{cases} \boldsymbol{\lambda}, & \forall t \in [-\tau, 0] \\ \mathbf{x}(t) & \forall t \in (0, \delta]. \end{cases}$$

Since (11) holds, $\mathbf{x}_a(t) \prec \boldsymbol{\lambda}$ on a certain interval $(\delta, \delta'']$ with $\delta'' > \delta$. Hence, $\mathbf{x}(t) = \mathbf{x}_b(t) \preceq \mathbf{x}_a(t) \prec \boldsymbol{\lambda}$, i.e., $\mathbf{x}(t) \prec \boldsymbol{\lambda}$, $\forall t \in (\delta, \delta'']$. A contradiction occurs. ■

Lemma 7: Consider system (2). Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (11). For any given scalar $t_s > T$ (defined in (5)) and switching signal $\varrho(\cdot)$ with switching instants $\{t_i; i \in \mathbb{N}_0\}$, let $k \in \mathbb{N}_0$ and $l \in \underline{\mathbf{m}}$ satisfy that $\varrho(t_s) = \varrho(t_k) = l$ and $t_k \leq t_s < t_{k+1}$. If $\boldsymbol{\phi}(t) = \boldsymbol{\lambda}$, $t \in [-\tau, t_s]$, then, on interval $[t_s, t_{k+1})$, the corresponding solution $\mathbf{x}(t)$ satisfies (i). $e^{A_{0l}(t-t_s)} \boldsymbol{\lambda} \preceq \mathbf{x}(t) \preceq \boldsymbol{\lambda}$ and (ii). $e^{A_{0l}(t-t_s)} \boldsymbol{\lambda} \preceq \mathbf{x}(t - \tau_{il}(t)) \preceq \boldsymbol{\lambda}$, $\forall i \in \underline{\mathbf{p}}$.

Proof: By Lemma 6, $\mathbf{x}(t) \preceq \boldsymbol{\lambda}$ in (i) and $\mathbf{x}(t - \tau_{il}(t)) \preceq \boldsymbol{\lambda}$ in (ii) hold for all $t \in [t_s, t_{k+1})$.

Under the condition $\boldsymbol{\phi}(t) = \boldsymbol{\lambda}$, $t \in [-\tau, t_s]$, the local solution ($\forall t \in [t_s, t_{k+1})$) to (2) is $\mathbf{x}(t) = e^{A_{0l}(t-t_s)} \boldsymbol{\lambda} + \int_{t_s}^t e^{A_{0l}(t-s)} (\sum_{i=1}^p A_{il} \mathbf{x}(s - \tau_{il}(s))) ds$. Since system (2) is positive, $\mathbf{x}(t) \succeq 0$ for all $t \geq t_s$. Moreover, $A_{0l} \in \mathbf{M}$ implies that $e^{A_{0l}(t-s)} \succeq 0$ holds for all $s \leq t$.¹ Hence, $\int_{t_s}^t e^{A_{0l}(t-s)} (\sum_{i=1}^p A_{il} \mathbf{x}(s - \tau_{il}(s))) ds \succeq 0$. Claim (i) holds.

¹Theorem 1.18 in Chapter 1, [14]: Let $A \in \mathbb{R}^{n \times n}$. Then $e^{At} \succeq 0$, $\forall t \geq 0$, if and only if $A \in \mathbf{M}$.

By (11), $A_{0l} \boldsymbol{\lambda} \prec 0$. Clearly, $(d/dt)e^{A_{0l}(t-t_s)} \boldsymbol{\lambda} = e^{A_{0l}(t-t_s)} A_{0l} \boldsymbol{\lambda}$. So, $(d/dt)e^{A_{0l}(t-t_s)} \boldsymbol{\lambda} \preceq 0$. Hence, $e^{A_{0l}(t-t_s)} \boldsymbol{\lambda}$ monotonically decreases. If $t - \tau_{il}(t) \leq t_s$, then $\mathbf{x}(t - \tau_{il}(t)) = \boldsymbol{\lambda}$, and (ii) holds. Otherwise, $t - \tau_{il}(t) > t_s$, which together with (i) implies $\mathbf{x}(t - \tau_{il}(t)) \succeq e^{A_{0l}(t-\tau_{il}(t)-t_s)} \boldsymbol{\lambda} \succeq e^{A_{0l}(t-t_s)} \boldsymbol{\lambda}$, thus claim (ii) also holds. ■

Lemma 8: Consider system (2). Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (11). Given $t_a \geq T$ with t_a being one of switching instants of switching signal $\varrho(\cdot)$, define $t_s = t_a - \vartheta t_a$, where T and ϑ are defined by (5), and set the initial condition as $\boldsymbol{\phi}_1(t) = \boldsymbol{\lambda}$, $t \in [t_s, t_a]$. Then, there exist two positive scalars α_1, α_2 , both independent of t_a , such that the corresponding solution $\mathbf{x}_1(t)$ satisfies

$$\mathbf{x}_1(t) \prec (1 - \alpha_1(t - t_a)) \boldsymbol{\lambda}, \quad \forall t \in (t_a, t_a + \alpha_2]. \quad (12)$$

Furthermore, if $t_b > t_a + \alpha_2$ and, for any given $t_c \in [t_a + \alpha_2, t_b)$, we define

$$\boldsymbol{\phi}_2(t) = \begin{cases} \boldsymbol{\lambda}, & t \in (t_s, t_a] \\ (1 - \alpha_1(t - t_a)) \boldsymbol{\lambda}, & t \in (t_a, t_a + \alpha_2] \\ (1 - \alpha_1 \alpha_2) \boldsymbol{\lambda}, & t \in (t_a + \alpha_2, t_c] \end{cases} \quad (13)$$

whose corresponding solution is denoted by $\mathbf{x}_2(t)$, then there exists a scalar $\alpha_3 > 0$, independent of t_c , such that

$$\mathbf{x}_2(t) \prec (1 - \alpha_1 \alpha_2) \boldsymbol{\lambda}, \quad \forall t \in (t_c, t_c + \alpha_3]. \quad (14)$$

Proof: Idea of the proof: By the strict negativity of $\sum_{i=0}^p A_{il} \boldsymbol{\lambda}$ in condition (11), we show that scalars α_1 and α_2 satisfying (12) do exist; using uniform continuity of $\mathbf{x}_2(t)$ over $[t_c, \infty)$ (which will be shown below) and that of $\tau_{il}(t)$ on some closed interval, we prove the existence of α_3 satisfying (14).

Define $\mathbf{f}_i(\mathbf{z}_0, \dots, \mathbf{z}_p) = \sum_{i=0}^p A_{il} \mathbf{z}_i$ with $\mathbf{z}_i \in \mathbb{R}^n$. Then, it follows from (11) that there exists a largest scalar $\alpha_1 > 0$, independent of t_a , such that $\mathbf{f}_i(\boldsymbol{\lambda}, \dots, \boldsymbol{\lambda}) \preceq -2\alpha_1 \boldsymbol{\lambda}$, $\forall l \in \underline{\mathbf{m}}$.

By continuity of $\mathbf{f}_i(\mathbf{z}_0, \dots, \mathbf{z}_p)$, there exists a scalar $\delta > 0$ such that $\|\mathbf{f}_i(\mathbf{z}_0, \dots, \mathbf{z}_p) - \mathbf{f}_i(\boldsymbol{\lambda}, \dots, \boldsymbol{\lambda})\| < \epsilon$, $\forall l \in \underline{\mathbf{m}}$ if $\|\mathbf{z}_i - \boldsymbol{\lambda}\| \leq \delta$, $i \in \underline{\mathbf{p}}$, where $\epsilon = \alpha_1 \min_{i \in \underline{\mathbf{m}}} \{\lambda_i\}$, which immediately indicates that $\mathbf{f}_i(\mathbf{z}_0, \dots, \mathbf{z}_p) \prec -\alpha_1 \boldsymbol{\lambda}$.

Clearly, there exists a sufficiently small scalar α_2 satisfying $0 < \alpha_2 < t^*/2$ and $0 < \alpha_1 \alpha_2 < 1$ and independent of t_a , such that $\|e^{A_{0l}(t-t_a)} \boldsymbol{\lambda} - \boldsymbol{\lambda}\| < \delta$, $t \in [t_a, t_a + \alpha_2]$, $l \in \underline{\mathbf{m}}$, where t^* is given by (3).

By Lemma 7, $\|e^{A_{0l}(t-t_a)} \boldsymbol{\lambda} - \boldsymbol{\lambda}\| < \delta$ means that $\|\mathbf{x}_1(t) - \boldsymbol{\lambda}\| < \delta$ and $\|\mathbf{x}_1(t - \tau_{il}(t)) - \boldsymbol{\lambda}\| < \delta$, $\forall l \in \underline{\mathbf{m}}$, $t \in [t_a, t_a + \alpha_2]$. Moreover, by (2), $\dot{\mathbf{x}}_1(t) \prec -\alpha_1 \boldsymbol{\lambda}$, $t \in [t_a, t_a + \alpha_2]$. Thus (12) holds.

By Lemma 6, we have that $\mathbf{x}_2(t) \preceq \boldsymbol{\lambda}$, $\forall t \geq t_c$. Since system (2) is positive, $\mathbf{x}_2(t) \succeq 0$, $\forall t \geq t_c$. Thus, $\dot{\mathbf{x}}_2(t)$ is bounded on $[t_c, \infty)$. Define $\kappa = \sup_{t \geq t_c} \|\dot{\mathbf{x}}_2(t)\|$. Now prove that $\mathbf{x}_2(t)$ is uniformly continuous in t over $[t_c, \infty)$. Arbitrarily fix $\epsilon_a > 0$ and let $\epsilon_b = (\epsilon_a / \kappa) > 0$. For given $\varrho(\cdot)$ and positive scalars \bar{t}_a, \bar{t}_b satisfying $\bar{t}_b - \bar{t}_a < \epsilon_b$, if $\mathbf{x}_2(t)$ is differentiable on $[\bar{t}_a, \bar{t}_b]$, then $\|\mathbf{x}_2(\bar{t}_b) - \mathbf{x}_2(\bar{t}_a)\| = \|\int_{\bar{t}_a}^{\bar{t}_b} \dot{\mathbf{x}}_2(s) ds\| \leq (\bar{t}_b - \bar{t}_a) \kappa < \epsilon_a$; otherwise by Assumption (3), there exist finite instants $\bar{t}_a \leq \bar{t}_1 < \dots < \bar{t}_q \leq \bar{t}_b$ at which $\mathbf{x}_2(t)$ is non-differentiable, then $\|\mathbf{x}_2(\bar{t}_b) - \mathbf{x}_2(\bar{t}_a)\| = \|\lim_{t \rightarrow \bar{t}_1} \int_{\bar{t}_a}^t \dot{\mathbf{x}}_2(s) ds + \lim_{t \rightarrow \bar{t}_2} \int_{\bar{t}_1}^t \dot{\mathbf{x}}_2(s) ds + \dots + \int_{\bar{t}_q}^{\bar{t}_b} \dot{\mathbf{x}}_2(s) ds\| \leq \lim_{t \rightarrow \bar{t}_1} \int_{\bar{t}_a}^t \kappa ds + \lim_{t \rightarrow \bar{t}_2} \int_{\bar{t}_1}^t \kappa ds + \dots + \int_{\bar{t}_q}^{\bar{t}_b} \kappa ds < \epsilon_a$, where $\lim_{t \rightarrow \bar{t}_1} \int_{\bar{t}_a}^t \dot{\mathbf{x}}_2(s) ds = 0$ and $\lim_{t \rightarrow \bar{t}_1} \int_{\bar{t}_a}^t \kappa ds = 0$ if $\bar{t}_a = \bar{t}_1$.

It follows from (13) and $\|\mathbf{x}_1(t) - \boldsymbol{\lambda}\| < \delta$, $\forall t \in [t_a, t_a + \alpha_2]$ that $\|\mathbf{x}_2(t_c) - \boldsymbol{\lambda}\| = \|(1 - \alpha_1 \alpha_2) \boldsymbol{\lambda} - \boldsymbol{\lambda}\| \leq \|\mathbf{x}_1(t_a + \alpha_2) - \boldsymbol{\lambda}\| < \delta$ and similarly that $\|\mathbf{x}_2(t_c - \tau_{il}(t_c)) - \boldsymbol{\lambda}\| < \delta$, $\forall i \in \underline{\mathbf{p}}$. Thus, $\dot{\mathbf{x}}_2(t_c) \preceq -\alpha_1 \boldsymbol{\lambda}$.

Define $\tilde{\epsilon} = (1/2) \min_{l \in \underline{\mathbf{m}}, j \in \underline{\mathbf{n}}} \{-f_{lj}(\mathbf{x}_2(t_c), \dots, \mathbf{x}_2(t_c - \tau_{pl}(t_c)))\} > 0$ with f_{lj} being the j th element of \mathbf{f}_l . By continuity of \mathbf{f}_l , there exists

$\tilde{\delta}$ such that $\|\mathbf{x}_2(t_c) - \mathbf{z}_0\| < \tilde{\delta}$, $\|\mathbf{x}_2(t_c - \tau_{il}(t_c)) - \mathbf{z}_i\| < \tilde{\delta}$, $i \in \mathbf{p}$, imply that $\|\mathbf{f}_i(\mathbf{x}_2(t_c), \dots, \mathbf{x}_2(t_c - \tau_{pl}(t_c))) - \mathbf{f}_i(\mathbf{z}_0, \dots, \mathbf{z}_p)\| < \tilde{\epsilon}$. Since $\tau_{il}(t)$ is continuous on $[0, \infty)$, it is uniformly continuous on $[t_a + \alpha_2, t_b + \beta]$ with $\beta > 0$ being constant, and, so is $\mathbf{x}_2(t)$. Therefore, there exists a sufficiently small positive scalar $\alpha_3 < \beta$, independent of t_c , such that, for all $t \in (t_c, t_c + \alpha_3]$, $\|\mathbf{x}_2(t_c) - \mathbf{x}_2(t)\| < \tilde{\delta}$ and $\|\mathbf{x}_2(t_c - \tau_{il}(t_c)) - \mathbf{x}_2(t - \tau_{il}(t))\| < \tilde{\delta}$, $\forall i \in \mathbf{p}$ and $l \in \mathbf{m}$. Briefly, there exists a scalar $\alpha_3 > 0$, independent of t_c , such that for all $t \in (t_c, t_c + \alpha_3]$ and all $l \in \mathbf{m}$, $\|\mathbf{f}_i(\mathbf{x}_2(t_c), \dots, \mathbf{x}_2(t_c - \tau_{pl}(t_c))) - \mathbf{f}_i(\mathbf{x}_2(t), \dots, \mathbf{x}_2(t - \tau_{pl}(t)))\| < \tilde{\epsilon}$ holds. Clearly, this implies that $\dot{\mathbf{x}}_2(t) \prec 0$, $\forall t \in (t_c, t_c + \alpha_3]$, which immediately indicates (14). ■

Lemma 9: Consider system (2). Suppose that there exists a vector $\lambda \in \mathbb{R}_+^n$ satisfying (11). Let ϱ be an arbitrarily given switching signal with switching instants $\{t_k; k \in \mathbb{N}_0\}$. Let $\phi(t) = \lambda$, $t \in [-\tau, 0]$, and denote $\mathbf{x}(t)$ the corresponding solution. Choose a subsequence $\{t_{k_i}; i \in \mathbb{N}_0\}$ from $\{t_k; k \in \mathbb{N}_0\}$ in such a way: $t_{k_0} = \min\{t_k : t_k \geq T\}$ with T defined by (5), and $t_{k_{i+1}} = \min\{t_k : t_k - \vartheta t_{k_i} \geq t_{k_i} + \alpha_2\}$, $\forall i \in \mathbb{N}_0$, where ϑ is given by (5) and α_2 by (12). For notational simplicity, let $\alpha = 1 - \alpha_1\alpha_2$ and $\gamma_{i,l} = t_{k_i} + \alpha_2 + l\alpha_{3,i}$, where α_1 is given by (12) and $\alpha_{3,i} > 0$ will be determined later.

Define $\mathcal{I}_{0,-1} = [0, t_{k_0}]$, $\mathcal{I}_{i,-1} = (\gamma_{i-1,0}, t_{k_i}]$, $i \in \mathbb{N}$; $\mathcal{I}_{i,0} = (t_{k_i}, \gamma_{i,0}]$, $\mathcal{I}_i = \mathcal{I}_{i,-1} \cup \mathcal{I}_{i,0}$, $i \in \mathbb{N}_0$ and

$$\phi_i(t) = \begin{cases} \alpha^i \lambda, & t \in \mathcal{I}_{i,-1} \\ (1 - \alpha_1(t - t_{k_i})) \alpha^i \lambda, & t \in \mathcal{I}_{i,0} \forall i \in \mathbb{N}_0. \end{cases} \quad (15)$$

Then, for any $i \in \mathbb{N}_0$, it holds that

$$\mathbf{x}(t) \preceq \phi_i(t), \forall t \in \mathcal{I}_i. \quad (16)$$

Proof: Strategy of the proof: We first divide $[0, \infty)$ into infinitely many disjoint intervals \mathcal{I}_i , $i \in \mathbb{N}_0$ and then prove (16) on each interval by induction.

Divide the interval $\mathcal{I}_{i+1,-1} = (\gamma_{i,0}, t_{k_{i+1}}]$, $i \in \mathbb{N}_0$, into ρ_i subintervals in the following manner:

$$\begin{aligned} \mathcal{I}_{i,l} &= (\gamma_{i,l-1}, \gamma_{i,l}], \forall 1 \leq l \leq \rho_i - 1; \\ \mathcal{I}_{i,\rho_i} &= (\gamma_{i,\rho_i-1}, t_{k_{i+1}}] \subseteq (\gamma_{i,\rho_i-1}, \gamma_{i,\rho_i}] \end{aligned} \quad (17)$$

where $\rho_i \in \mathbb{N}$ will be determined later. For $i \in \mathbb{N}_0$, let $\phi_{i,-1}(t) = \alpha^i \lambda(t \in \mathcal{I}_{i,-1})$, $\phi_{i,0}(t) = \phi_i(t)(t \in \mathcal{I}_i)$

$$\phi_{i,l}(t) = \begin{cases} \phi_{i,0}(t) & t \in \mathcal{I}_i, \\ \alpha^{i+1} \lambda & t \in \bigcup_{r=1}^l \mathcal{I}_{i,r}, \end{cases} \quad 1 \leq l \leq \rho_i.$$

For all $\phi_{i,l}$, $l \in \{-1, 0, \dots, \rho_i\}$, denote the corresponding solution as $\mathbf{x}_{i,l}(t)$.

Now determine $\alpha_{3,i}$ and ρ_i , $i \in \mathbb{N}_0$. Note that $\alpha_2 < (t^*/2)$. By Lemma 8, there exists a scalar $\alpha_{3,i} > 0$ such that $\mathbf{x}_{i,0}(t) \prec \alpha^{i+1} \lambda$ for all $i \in \mathbb{N}_0$, $t \in \mathcal{I}_{i,1}$. Then, $\rho_i = \lceil (t_{k_{i+1}} - t_{k_i} - \alpha_2) / \alpha_{3,i} \rceil$, which satisfies (17).

The remaining proof is completed by induction.

By Lemma 6, $\mathbf{x}(t) \preceq \lambda = \alpha^0 \lambda$, $\forall t \geq 0$. An application of Lemma 4 yields that $\mathbf{x}(t) \preceq \mathbf{x}_{0,-1}(t)$, $\forall t \geq t_{k_0}$. By Lemma 8, one has that $\mathbf{x}_{0,-1}(t) \prec (1 - \alpha_1(t - t_{k_0})) \lambda$, $\forall t \in \mathcal{I}_{0,0}$. Therefore, $\mathbf{x}(t) \preceq (1 - \alpha_1(t - t_{k_0})) \lambda$, $\forall t \in \mathcal{I}_{0,0}$, which implies that (16) holds for $i = 0$.

Having shown that (16) holds for $i = j \in \mathbb{N}_0$, that is, $\mathbf{x}(t) \preceq \phi_j(t)$, $\forall t \in \mathcal{I}_j$, we need to prove that (16) holds for $i = j + 1$, i.e., $\mathbf{x}(t) \preceq \phi_{j+1}(t)$, $\forall t \in \mathcal{I}_{j+1}$.

By Lemma 4 and the assumption that $\mathbf{x}(t) \preceq \phi_j(t)$, $\forall t \in \mathcal{I}_j$, it follows that $\mathbf{x}(t) \preceq \mathbf{x}_{j,0}(t)$, $\forall t \geq \gamma_{j,0}$. Applying Lemma 8 yields that $\mathbf{x}_{j,0}(t) \prec \alpha^{j+1} \lambda$, $\forall t \in \mathcal{I}_{j,1}$.

According to Lemma 4, it holds that

$$\mathbf{x}_{j,0}(t) \preceq \mathbf{x}_{j,1}(t), \forall t \geq \gamma_{j,1}. \quad (18)$$

Similarly, we obtain that

$$\begin{aligned} \mathbf{x}_{j,l}(t) &\preceq \mathbf{x}_{j,l+1}(t), t \geq \gamma_{j,l+1}, \forall l \in \{1, \dots, \rho_j - 2\} \\ \mathbf{x}_{j,\rho_j-1}(t) &\preceq \mathbf{x}_{j,\rho_j}(t), t \geq t_{k_{j+1}}. \end{aligned} \quad (19)$$

From (18), (19), and the fact that $\mathbf{x}(t) \preceq \mathbf{x}_{j,0}(t)$, $\forall t \geq \gamma_{j,0}$, the following relations are obtained:

$$\mathbf{x}(t) \preceq \alpha^{j+1} \lambda, \forall t \in \bigcup_{r=1}^{\rho_j} \mathcal{I}_{j,r} = \mathcal{I}_{j+1,-1} \quad (20)$$

$$\mathbf{x}(t) \preceq \mathbf{x}_{j,\rho_j}(t) = \mathbf{x}_{j+1,-1}(t), \forall t > t_{k_{j+1}}. \quad (21)$$

By the choice of t_{k_i} , it holds that $t_{k_{j+1}} - \vartheta t_{k_{j+1}} \geq t_{k_j} + \alpha_2$. By Lemma 8, it follows from (2) that for all $t \in \mathcal{I}_{j+1,0}$

$$\mathbf{x}_{j,\rho_j}(t) = \mathbf{x}_{j+1,-1}(t) \prec (1 - \alpha_1(t - t_{k_{j+1}})) \alpha^{j+1} \lambda. \quad (22)$$

It follows from (21) and (22) that for all $t \in \mathcal{I}_{j+1,0}$, $\mathbf{x}(t) \prec (1 - \alpha_1(t - t_{k_{j+1}})) \alpha^{j+1} \lambda$. This together with (20) yields that $\mathbf{x}(t) \preceq \phi_{j+1}(t)$, $\forall t \in \mathcal{I}_{j+1}$. By induction, (16) holds for any $i \in \mathbb{N}_0$. ■

Theorem 2: Consider system (2). Suppose that there exists a vector $\lambda \in \mathbb{R}_+^n$ satisfying (11). Then, system (2) is asymptotically stable.

Proof: Suppose that there exists a vector $\lambda \in \mathbb{R}_+^n$ satisfying (11). For an arbitrary given scalar $\epsilon > 0$, there always exists a scalar $\nu > 0$ such that $\|\nu \lambda\| < \epsilon$. In the remaining proof, we use the notations in Lemma 9, but $\phi_i(t)$ are modified by replacing λ with $\nu \lambda$. Define $\Phi(t)$ on $[0, \infty)$ as: $\Phi(t) = \phi_i(t)$ if $t \in \mathcal{I}_i$. Since $\mathcal{I}_i \cap \mathcal{I}_j = \emptyset$ if $i \neq j$ and $\bigcup_{i=0}^{\infty} \mathcal{I}_i = [0, \infty)$, $\Phi(t)$ is well-defined on $[0, \infty)$.

For every $t \geq 0$, there exists a unique $i(t) \in \mathbb{N}$ such that $t \in \mathcal{I}_{i(t)}$. Obviously, $i(t) \rightarrow \infty$ as $t \rightarrow \infty$. By (15), $\Phi(t) = \phi_{i(t)}(t) \preceq \nu \alpha^{i(t)} \lambda$, $\forall t \geq 0$. Moreover, $\lim_{t \rightarrow \infty} \Phi(t) = \lim_{t \rightarrow \infty} \phi_{i(t)}(t) \preceq \lim_{t \rightarrow \infty} \nu \alpha^{i(t)} \lambda = 0$. Therefore, $\Phi(t)$ is continuous in t , monotonically decreasing, and $\lim_{t \rightarrow \infty} \Phi(t) = 0$.

First let $\phi(\cdot) = \nu \lambda$, $\forall t \in [-\tau, 0]$ and the corresponding solution be $\mathbf{x}_1(t)$. By Lemma 9, $\mathbf{x}_1(t) \preceq \Phi(t)$ for all $t > 0$. Choose another initial condition $\phi(\cdot)$ such that $\|\phi(\cdot)\| < \min\{\nu \lambda_1, \dots, \nu \lambda_n\}$ and denote the corresponding solution by $\mathbf{x}(t)$. By Lemma 4, $\mathbf{x}(t) \preceq \mathbf{x}_1(t)$, $\forall t \geq 0$. Thus, $\mathbf{x}(t) \preceq \Phi(t)$, $\forall t \geq 0$. Clearly, $\|\mathbf{x}(t)\| < \epsilon$, $\forall t \geq 0$, and $\lim_{t \rightarrow \infty} \|\mathbf{x}(t)\| = 0$. The proof is completed. ■

V. RELEVANT REMARKS

This section discusses some reduced cases of systems (1) and (2).

If systems (1) and (2) consist of just one subsystem, then we get the delayed positive systems

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + \sum_{i=1}^p A_i \mathbf{x}(k - \tau_i(k)), \quad k \in \mathbb{N}_0 \quad (23)$$

$$\dot{\mathbf{x}}(t) = A_0 \mathbf{x}(t) + \sum_{i=1}^p A_i \mathbf{x}(t - \tau_i(t)), \quad t \geq 0 \quad (24)$$

respectively. We assume that.

Assumption 3: For system (23), there exist $T \in \mathbb{N}$ and a scalar $0 \leq \theta < 1$ such that $\theta = \sup_{k > T} (\tau_i(k)/k)$, $i \in \mathbf{p}$.

Assumption 4: For system (24), there exist $T > 0$ and a scalar $0 \leq \vartheta < 1$ such that $\vartheta = \sup_{t > T} (\tau_i(t)/t)$, $i \in \mathbf{p}$.

As a result of Theorems 1 and 2, we obtain.

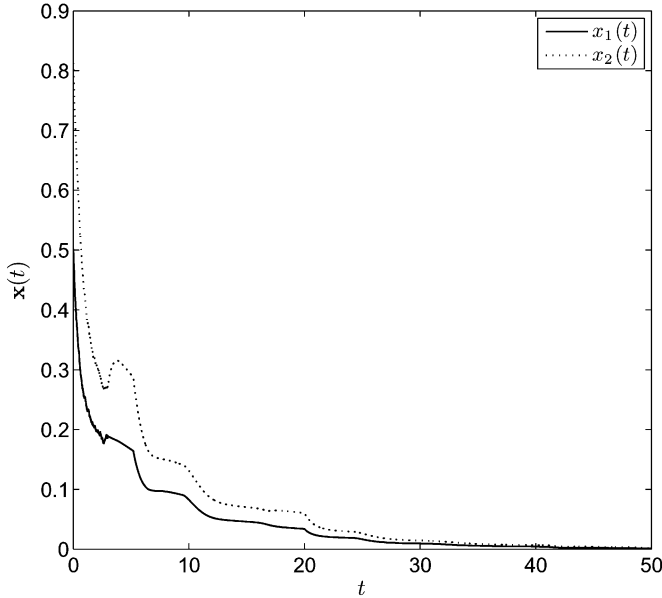


Fig. 1. Evolution of system (25), $\phi(\cdot)$ and $\sigma(\cdot)$ are randomly generated.

Corollary 1: The positive system (23) is asymptotically stable for all delays satisfying Assumption 3 if and only if there exists a vector $\lambda \in \mathbb{R}_+^n$ such that $(\sum_{i=0}^p A_i - I)\lambda \prec 0$. The positive system (24) is asymptotically stable for all delays satisfying Assumption 4 if and only if there exists a vector $\lambda \in \mathbb{R}_+^n$ such that $\sum_{i=0}^p A_i \lambda \prec 0$.

Proof: Sufficiency is obvious by Theorems 1 and 2. To see necessity, let delays in (23) and (24) be constant, and the conclusions follow from the known results [19], [20]. ■

When the systems are one-dimensional, Theorems 1 and 2 together lead to the following corollary.

Corollary 2: Suppose system (2) is positive and $x \in \mathbb{R}$. Then system (2) is asymptotically stable for all delays satisfying Assumption 4 if and only if so are all the subsystems.

A similar result also holds for system (1).

Proof: Only the first conclusion need to be proven, the second one is analogous. Necessity is obvious.

To show sufficiency, suppose all the subsystems are asymptotically stable. By Corollary 1, for each subsystem, say, the l th one, there exists a positive scalar λ_l such that $(\sum_{i=0}^p A_{il} - I)\lambda_l < 0, \forall l \in \underline{m}$. Since these inequalities are all scalar, there must exist a positive scalar λ satisfying $(\sum_{i=0}^p A_{il} - I)\lambda < 0$ for all $l \in \underline{m}$. By Theorem 2, system (2) is asymptotically stable. ■

Remark 4: References [19], [20] have established two necessary and sufficient stability conditions for systems (23) and (24), but required that the delays be bounded.

VI. EXAMPLE

To illustrate the theoretical results, we give a numerical example of Theorem 2, omitting that of Theorem 1 due to their similarity.

Example 1: Consider the following delayed system:

$$\begin{aligned} \dot{x}(t) &= A_{0\sigma(t)}x(t) + A_{1\sigma(t)}x(t - \tau_{1\sigma(t)}(t)), \quad t \geq 0 \\ x(t) &= \phi(t), \quad t \in [-\tau, 0] \end{aligned} \quad (25)$$

where $x(t) \in \mathbb{R}^2$, $\sigma: \mathbb{N}_0 \rightarrow \{1, 2\}$, and

$$\begin{aligned} A_{01} &= \begin{bmatrix} -2.8 & 0.6 \\ 0.1 & -1.25 \end{bmatrix}, & A_{11} &= \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 0.5 \end{bmatrix} \\ A_{02} &= \begin{bmatrix} -0.6 & 0.2 \\ 1.366 & -1.9 \end{bmatrix}, & A_{12} &= \begin{bmatrix} 0.1 & 0.1 \\ 0.8 & 0.6 \end{bmatrix}. \end{aligned}$$

Clearly, system (25) is a PSLS. It is easy to verify that there exists a vector $\lambda = [66.5536, 110.8886]^T$ such that $\sum_{i=0}^1 A_{il}\lambda \prec 0$ for $l = 1, 2$. By Theorem 2, system (25) is asymptotically stable for any delays satisfying condition (4). For example, we take $\tau_{11}(t) = 0.3t + 3$, $\tau_{12}(t) = 0.4t + 2$ and the simulation result is shown in Fig. 1, from which one can see that system (25) is indeed asymptotically stable.

VII. CONCLUSION

This technical note has addressed the stability problem of positive switched linear systems with delays, where both discrete- and continuous-time cases are considered. By means of a novel method, some new stability conditions have been established. An example has been given to illustrate the obtained results. There still exist many appealing and challenging problems to be solved. It is expected that the idea and technique in this technical note will be helpful for the future research in this field.

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Global Disturbance Rejection of Lower Triangular Systems With an Unknown Linear Exosystem

Lu Liu, Zhiyong Chen, and Jie Huang

Abstract—In this technical note, we study the global disturbance rejection problem of nonlinear systems in lower triangular form with unknown exosystem via state feedback control. The problem is dealt with in two steps. In the first step, an augmented system composed of the given plant and an internal model is constructed. Owing to the presence of the unknown parameter in the exosystem, the augmented system contains both nonlinearly and linearly parameterized uncertainties. In the second step, the stabilization of the augmented system is solved by an approach integrating both robust and adaptive techniques. The solution of the stabilization problem of the augmented system in turn leads to the solution of the global disturbance rejection problem of the original system. Further, the convergence issue of an estimated unknown parameter vector is also discussed.

Index Terms—Adaptive control, disturbance rejection, nonlinear systems, robust control.

I. INTRODUCTION

Consider the class of nonlinear systems in the lower triangular form as follows:

$$\begin{aligned}\dot{z} &= f(z, x_1, w) \\ \dot{x}_1 &= f_1(z, x_1, w) + x_2 \\ &\vdots \\ \dot{x}_r &= f_r(z, x_1, \dots, x_r, v(t), w) + u \\ e &= x_1\end{aligned}\quad (1)$$

where $z \in \mathbb{R}^n$, $x_i \in \mathbb{R}$, $i = 1, \dots, r$, are the states, $e \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ the input, $w \in \mathbb{R}^{\ell_1}$ an unknown constant parameter vector. It is assumed that all the functions in system (1) are sufficiently smooth with $f(0, 0, w) = 0$ and $f_i(0, \dots, 0, w) = 0$. The function $v(t) \in \mathbb{R}^q$ represents an unknown time-varying disturbance which enters the system in a non-matching way, and $v(t)$ is generated by the exosystem

$$\dot{v} = A_1(\sigma)v \quad (2)$$

where $\sigma \in \mathbb{R}^{\ell_2}$ represents the uncertainty in the exosystem. Assume that the eigenvalues of $A_1(\sigma)$, are distinct with zero real parts for all $\sigma \in \mathbb{S}$.

The global disturbance rejection problem is described as follows: Given W , V and S , which are known compact subsets of \mathbb{R}^{ℓ_1} , \mathbb{R}^q , and \mathbb{R}^{ℓ_2} containing the origins, respectively, design a partial state feedback

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L. Liu is with the Department of Mechanical, Materials and Manufacturing, The University of Nottingham, Nottingham NG7 2RD, U.K. (e-mail: lu.liu@nottingham.ac.uk).

Z. Chen is with the School of Electrical Engineering and Computer Science, The University of Newcastle, Newcastle, Australia (e-mail: zhiyong.chen@newcastle.edu.au).

J. Huang is with the Department of Mechanical and Automation Engineering, The Chinese University of Hong Kong, China (e-mail: jhuang@mae.cuhk.edu.hk).

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