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A NOTE ON THE CONTROLLABILITY AND THE OBSERVABILITY OF FRACTIONAL DYNAMICAL SYSTEMS

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Abstract: In this paper, we give some new results on the controllability and the observability of linear dynamical systems with a fractional derivative of order α, where α is a non integer number. We show that the observability and the controllability Gramians, recently introduced for a fractional order system, are solutions of fractional differential Lyapunov equations, thus generalizing the classical result for the integer case ($\alpha = 1$). Our results can be considered as a generalization of the known corresponding results in the integer order case to the fractional order one since for $\alpha = 1$, the results for the integer case are recovered.

Keywords: Fractional order systems, Controllability, Observability, Lyapunov equation.

1. INTRODUCTION

Over the last years, the concept of fractional derivative has been used increasingly to model the behavior of real systems in various fields of science and engineering such as electrochemistry (Ichise *et al*, 1971), electromagnetism and electrical machines (Lin *et al*, 2001), thermal systems and heat conduction (Battaglia *et al*, 2001), (Cois, 2002), transmission and acoustic (Matignon, 1994), (Matignon *et al*, 1994), viscoelastic material (Hanyga, 2003), electrical fractals networks (Arena *et al*, 2000), (Petras, 2000), robotics (Valerio, 2004), (Fonseca *et al*, 2004) and in many other areas. These systems exhibit hereditary properties and long memory transients. From a mathematical point of view, the fractional derivative dates back two centuries. Well later, theoretical research of the fractional derivative and integral has been developed. Recent books, (Oldham and Spanier, 1974), (Oustaloup, 1983), (Samko *et al*, 1993), (Miller and Ross, 1993), (Oustaloup, 1995), (Podlubny, 1999), (Gorenflo and Mainardi, 2000), provide a rich source of references on fractional-order calculus. In control theory, several authors have been interested by this aspect from the sixties. The first contributions, (Manabe, 1963), (Oustaloup, 1983), (Axtell and Bise, 1990), give the generalization of classical analysis methods for fractional order systems (transfer function definition, frequency response, pole and zero analysis,...).

The state space representation of fractional order systems is also introduced in (Raynaud and Zergainoh, 2000), (Hotzel and Fliess, 1998), (Sabatier *et al*, 2002), (Dorcak, 2000), (Vinagre *et al*, 2002). It has been pointed out that for fractional order systems, two different cases can be taken into account that yield two kinds of interesting systems, the commensurate order and the rational order (Vinagre *et al*, 2004), (Matignon and d'Andréa Novel, 1997). The system is of commensurate order if all the orders of derivative are multiple integer of a base α, where α denotes the order of the derivative. The state space representation has been exploited in order to analysis the performances of the system. In fact, the solution of the state space equation has been derived by using the Mittag-Leffler function (Mittag-Leffler, 1904). The stability of the fractional order system has been investigated (Matignon, 1996). A condition based on the argument principle has been established to guarantee the asymptotic stability of the fractional order system. Further, the controllability and the observability properties have been defined and some algebraic criteria of these two properties have been derived (Matignon and d'Andréa Novel, 1996). Recently, some new interesting results concerning the robust stability and the robust controllability of uncertain fractional order linear time invariant systems are derived in (Yan Quan Chen *et al*, 2005a) and in (Yan Quan Chen *et al*, 2005b), respectively.

The aim of the present paper is to give a new contribution for the analysis of the controllability and the observability of commensurate fractional order systems modeled by fractional state space equations. We show that the controllability and observability Gramians are solutions of fractional order differential Lyapunov equations.

The rest of this paper is organized as follows. In Section 2, we recall some fundamental definitions on fractional derivative and fractional order systems. Section 3 addresses the controllability and the observability properties.

2. THE FRACTIONAL ORDER DYNAMICAL SYSTEM

There are different definitions of the fractional derivative, (Oldham and Spanier, 1974), (Samko *et al*, 1993), (Miller and Ross, 1993). The first one is due to K. Grunwald:

$$ {}_a^G D_t^\alpha f(t) = \lim_{h\to 0} \frac{{}_a\Delta_h^\alpha f(t)}{h^\alpha} \tag{1}$$

where the real number α denotes the order of the derivative, a is the initial time. The difference operator Δ is given by:

$$ {}_a\Delta_h^\alpha f(t) = \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{2}$$

Introducing the positive integer number m such that $m-1<\alpha<m$, we obtain a similar definition given by A.V. Letnikov:

$$ {}_a^G D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha+1)} \int_a^t (t-\tau)^{m-\alpha} f^{(m+1)}(\tau)d\tau + \sum_{k=0}^{m} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} \tag{3}$$

where Γ is the Euler gamma function:

$$ \Gamma(\beta) = \int_0^\infty z^{\beta-1} e^{-z} dz \tag{4}$$

The definition given by (3) assumes that the function f is sufficiently differentiable and that $f^{(k)}(a) < \infty, k = 0,1,...,m$. A new version is the Letnikov-Reimann-Liouville definition (LRL) given by:

$$ {}_a^L D_t^\alpha f(t) = \frac{d^m}{dt^m} \{ \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \} \tag{5}$$

Naturally as the physical systems are modeled by differential equations containing eventually fractional derivatives, it is necessary to give for, these equations, initial conditions which must be physically interpretable. Unfortunately, the LRL definition leads to initial conditions containing the value of the fractional derivative at the initial conditions. To overcome this difficulty, Caputo proposed another definition given by:

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \tag{6}$$

Note the remarkable fact that the LRL fractional derivative of a constant function $f(t) = C$ is not zero:

$$ {}_a^L D_t^\alpha C = \frac{Ct^{-\alpha}}{\Gamma(1-\alpha)} \tag{7}$$

while the Caputo derivative of a constant is identically zero.

Thereafter, we assume that $a = 0$ and we denote by $d^\alpha f(t)$ and by $D^\alpha f(t)$ the Caputo and Reimann-Liouville fractional derivative of $f(t)$, respectively. Furthermore, we assume that $0<\alpha<1$ without loss of generality.

A single input, single output linear continuous-time dynamical system of fractional order is described by the following input-output fractional differential equation (Vinagre *et al*, 2002):

$$ a_n d^{\alpha_n} y(t) + a_{n-1} d^{\alpha_{n-1}} y(t) + \ldots a_0 d^{\alpha_0} y(t) = b_m d^{\beta_m} u(t) + b_{m-1} d^{\beta_{m-1}} u(t) + \ldots b_0 d^{\beta_0} u(t) \tag{8}$$

where the α_i and β_i are real numbers. By applying the Laplace transform to this equation and assuming zero initial conditions, we obtain the input-output description by the following fractional order transfer function:

$$ G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \ldots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \ldots + a_0 s^{\alpha_0}} \tag{9}$$

In the above, the orders of differentiation α_i and β_j are different real numbers. A particular case which yields to an interesting system namely the commensurate order system is when all these orders are multiple integers of a base α, that is:

$$ \alpha_k \quad , \quad \beta_k = k\alpha, \quad k \in \mathcal{N}^+, \quad 0<\alpha<1 \tag{10}$$

The state space representation of a multivariable commensurate order continuous time system takes the form:

$$ d^\alpha x(t) = Ax(t) + Bu(t) \quad x(t_0) = x_0 \tag{11a}$$

$$ y(t) = Cx(t) + Du(t) \tag{11b}$$

Where $u(t) \in \Re^m$ is the input vector, $y(t) \in \Re^q$ is the output vector and $x(t) \in \Re^n$ is the state vector. As in the classical integer case, a relation between the state space matrices A, B, C and D and the transfer function $G(s)$ can be easy obtained by using the Laplace transform in the case of zero initial conditions:

$$G(s) = C(s^\alpha I - A)^{-1}B + D \qquad (12)$$

The response of the state space equation $(11a)$ is given by (Vinagre *et al*, 2002)

$$x(t) = \Phi(t - t_0)x_o + \int_{t_0}^{t} \Phi(t - \tau)Bu(\tau)d\tau \qquad (13)$$

where the $\Phi(t)$ is the transition matrix of (11) given by the Mittag-Leffler function, (Mittag-Leffler, 1904):

$$\Phi(t) \triangleq E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(1 + k\alpha)} \qquad (14)$$

The Mittag-Leffler function is an extension of the exponential matrix to the fractional case. Indeed, for the integer case, i.e. $\alpha = 1$, $E_1(At^1) = exp(At)$. The Mittag-Leffler function satisfies the following relation:

$$d^\alpha E_\alpha(At^\alpha) = AE_\alpha(At^\alpha), \quad E(0) = I_n \qquad (15)$$

3. CONTROLLABILITY AND OBSERVABILITY

The controllability and the observability of linear fractional order systems have not received a lot of attention. In our knowledge, only some preliminary results are given in (Matignon and d'Andréa Novel, 1996), (Vinagre *et al*, 2002). The definition of the controllability given below is the same as for the integer order case (Chen, 1984).

Definition 1. The system (11) is controllable if for a given time t_0 there exists a finite time $t_1 > t_0$ such that from any $x(t_0) = x_0$ and any $x(t_1) = x_1$ in the state space, there exists an input $u(t), t \in [t_0 \quad t_1]$ that transfers the state $x(t)$ from x_0 to x_1 at time t_1.

Theorem 1. System (11) is controllable if and only if the controllability Gramian

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} E_\alpha(A(t_0 - \tau)^\alpha)BB^T E_\alpha(A^T(t_0 - \tau)^\alpha)d\tau \qquad (17)$$

is a positive definite matrix. Furthermore, the following input

$$u(t) = -B^T E_\alpha(A^T(t_0 - t)^\alpha)W_c^{-1}(t_0, t_1) [x_0 - E_\alpha(A(t_0 - t_1)^\alpha)x_1] \qquad (18)$$

transfers the state vector $x(t)$ from x_0 at t_0 to x_1 at t_1

Proof : The above theorem can be proofed as in the integer order case (Chen, 1984). Indeed, substituting the input control (18) in the solution (13) of the state equation (11), we obtain:

$$x(t_1) = E_\alpha(A(t_1 - t_0)^\alpha)x_0$$

$$- \int_{t_0}^{t_1} E_\alpha(A(t_1 - \tau)^\alpha BB^T E_\alpha(A^T(t_0 - \tau)^\alpha)d\tau$$

$$W_c^{-1}(t_0, t_1)[x_0 - E_\alpha(A(t_0 - t_1)^\alpha)]x_1$$

Using the following properties of the Mittag-Leffler function (Vinagre *et al*, 2002)

$$\Phi(t_1 - t_0)^{-1} = \Phi(t_0 - t_1)$$

$$\Phi(t_2 - t_1)\Phi(t_1 - t_0) = \Phi(t_2 - t_0)$$

it follows that

$$x(t_1) = E_\alpha(A(t_1 - t_0)^\alpha)$$

$$\{x_0 - \int_{t_0}^{t_1} E_\alpha(A(t_0 - \tau)^\alpha)BB^T E_\alpha(A^T(t_0 - \tau)^\alpha)d\tau$$

$$W_c^{-1}(t_0, t_1)[x_0 - E_\alpha(A(t_0 - t_1)^\alpha)x_1]\}$$

$$= E_\alpha(A(t_1 - t_0)^\alpha)\{x_0 - W_c(t_0, t_1)W_c^{-1}(t_0, t_1)$$

$$[x_0 - E_\alpha(A(t_0 - t_1)^\alpha)x_1]\}$$

$$= E_\alpha(A(t_1 - t_0)^\alpha)E_\alpha(A(t_0 - t_1)^\alpha)x_1 = x_1 \qquad \square$$

Remark 1. Matignon, (Matignon and d'Andréa Novel, 1996), introduced another definition of the controllability Gramian given by:

$$W_c(t) = \int_0^t (t-\tau)^{2(1-\alpha)} \mathcal{E}_\alpha(A\tau^\alpha)BB^T \mathcal{E}_\alpha(A^T\tau^\alpha)d\tau \qquad (19)$$

where

$$\mathcal{E}_\alpha(At^\alpha) = t^{\alpha-1}\Sigma_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma((1 + k)\alpha)} \qquad (20)$$

The term $(t - \tau)^{2(1-\alpha)}$ is added in order to ensure the convergence of the integral at $\tau = t$.

It has been shown, (Matignon and d'Andréa Novel, 1996), that system (11) is controllable if the rank of the controllability matrix:

$$\mathcal{C} = [B \quad AB \quad A^2B \quad ...A^{n-1}B] \qquad (21)$$

is equal to n. In other words, the controllability condition of the state space representation of a commensurate fractional order system is the same as for the integer case.

Same results are established with respect to the observability.

Definition 2. System (11) is observable on $[t_0 \quad t_1]$, $t_1 > 0$, if $x(t_0)$ can be deduced from the knowledge of the output $y(t)$ and the input $u(t)$ for $t \in [t_0 \quad t_1]$.

System (11) is observable if the rank of the observability matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{22}$$

is equal to n. The observabilty Gramian is defined as follows

Definition 3. (Matignon and d'Andréa Novel, 1996) The observability Gramian is the positive symmetric matrix:

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} E_\alpha(A^T(\tau - t_0)^\alpha) C^T C E_\alpha(A(\tau - t_0)^\alpha) d\tau \tag{23}$$

Theorem 2. (Matignon and d'Andréa Novel, 1996) System (11) is observable on $[t_0 \quad t_1]$ if and only either $W_o(t_0, t) = \int_{t_0}^{t} E_\alpha(A^T\tau^\alpha) C^T C E_\alpha(A\tau^\alpha) d\tau$ is positive definite for $t \in [t_0 \quad t_1]$ or $rank(\mathcal{O}) = n$.

In the integer case, the controlability and observability Gramians are solutions of Lyapunov matrix equations. In the following, we show that for the fractional order case, the controllability and observability Gramians are solutions of fractional order differential Lyapunov equations. In the sequel, we need the following rule for fractional differentiation of an integral depending on a parameter.

Lemma 1.

$$D^\alpha \int_0^t f(t-\tau) d\tau = \int_0^t D^\alpha f(\tau) d\tau + \lim_{\tau \to 0} D^{\alpha-1} f(\tau) \tag{24}$$

Proof : Following of the Letnikov-Reiman-Liouville's definition of the fractional derivative we have:

$D^\alpha \int_0^t f(t - \tau) d\tau =$

$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{d\tau}{(t-\tau)^\alpha} \int_0^\tau f(\tau - \xi) d\xi =$

$$\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t d\xi \int_\xi^t \frac{f(\tau - \xi)}{(t-\tau)^\alpha} d\tau = \frac{d}{dt} \int_0^t \tilde{f}(t - \xi) d\xi \tag{25}$$

where

$$\tilde{f}(t - \xi) = \frac{1}{\Gamma(1-\alpha)} \int_\xi^t \frac{f(\tau - \xi)}{(t-\tau)^\alpha} d\tau$$

Using the Leibniz formulae in (25), i.e.:

$$\frac{d}{dt} \int_0^t \tilde{f}(t - \xi) d\xi = \lim_{\xi \to t} \tilde{f}(t - \xi) + \int_0^t \frac{d\tilde{f}(t - \xi)}{dt} d\xi$$

it follows

$$D^\alpha \int_0^t f(t - \tau) d\tau = \lim_{\xi \to t} \frac{1}{\Gamma(1-\alpha)} \int_\xi^t \frac{f(\tau - \xi)}{(t-\tau)^\alpha} d\tau$$

$$+ \int_0^t \frac{d}{dt} \int_\xi^t \frac{f(\tau - \xi)}{(t-\tau)^\alpha} d\tau d\xi$$

Since $f(\tau - \xi) = 0$ for $\tau < \xi$ and using the fact that

$$D^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f(\tau - \xi)}{(t-\tau)^\alpha} d\tau$$

the relation (24) is obtained. $\square$

If we consider $t_0 = 0$, the controllability and the observability Gramians for each instant $t \in [0 \quad t_1]$ are given by:

$$W_o(t) = \int_0^t E_\alpha(A^T\tau^\alpha) C^T C E_\alpha(A\tau^\alpha) d\tau \tag{26}$$

$$W_c(t) = \int_0^t E_\alpha(-A\tau^\alpha) B B^T E_\alpha(-A^T\tau^\alpha) d\tau \tag{27}$$

Theorem 3. Let $W_c(t)$ and $W_o(t)$ be the controllability and observability Gramians of system (11) assumed to be asymptotically stable, controllable and observable, then $W_c(t)$ and $W_o(t)$ are solutions of the fractional order differential Lyapunov equations:

$$D^\alpha W_o(t) = A^T W_o(t) + W_o(t) A + D^{(\alpha-1)}(C^T C) \tag{28}$$

and

$$D^\alpha W_c(t) = A W_c(t) + W_c(t) A^T + D^{(\alpha-1)}(B B^T) \tag{29}$$

respectively.

Proof : Carrying out the change of variable $\tau = t - \tilde{\tau}$, the observability Gramian (26) is rewritten as

$$W_o(t) = \int_0^t E_\alpha(A^T(t - \tilde{\tau})^\alpha) C^T C E_\alpha(A(t - \tilde{\tau})^\alpha) d\tilde{\tau}$$

Using Lemma 1, with

$$f(t - \tilde{\tau}) = E_\alpha(A^T(t - \tilde{\tau})^\alpha) C^T C E_\alpha(A(t - \tilde{\tau})^\alpha)$$

then

$$D^\alpha W_o(t) = D^{(\alpha-1)}\{C^T C\} + \int_0^t D^\alpha\{E_\alpha(A^T\tau^\alpha) C^T C E_\alpha(A\tau^\alpha) d\tau$$

According to the following relation between the Caputo derivative and the Reimann-Liouville derivative:

$$D^\alpha f(t) = d^\alpha f(t) + \frac{t^{-\alpha}}{\Gamma(1-\alpha)} f(0^+), \quad 0 < \alpha < 1$$

and using the relation (15) of the Mittag-Leffler function, we obtain

$D^\alpha W_o(t) = D^{\alpha-1}\{C^T C\}$

$+ \int_0^t \{A^T E_\alpha(A^T\tau^\alpha) C^T C E_\alpha(A\tau^\alpha)$

$+ E_\alpha(A^T\tau^\alpha) C^T C E_\alpha(A\tau^\alpha) A\} d\tau$

which implies that

$$D^\alpha W_o(t) = D^{(\alpha-1)}(C^T C) + A^T W_o(t) + W_o(t) A$$

(29) is proved in the same manner. $\square$

Remark 2. The results of the integer case are recovered. Setting $\alpha = 1$ in both (28) and (29), we obtain the well known algebraic Lyapunov equations for the controllability and observability constant matrices Gramians, that is

$$AW_c(t)+W_c(t)A^T+BB^T = exp(At)BB^Texp(A^Tt) = \frac{dW_c(t)}{dt} \quad (30)$$

and

$$A^TW_o(t)+W_o(t)A+C^TC = exp(A^Tt)C^TCexp(At) = \frac{dW_o(t)}{dt} \quad (31)$$

Remark 3. The controllability and observability properties are both time-invariant as shown by Theorem 3. It is clear that rank $(W_c(t))$ and rank $(W_o(t))$ are also time-invariant. However, the fact that the controllability and observability Gramians are given by fractional differential Lyapunov equations means that the Gramians include the memory effect and the past history. It is known that the singular values of W_c measure the amount of control energy required to reach the state at a given position and the singular values of W_o are the amount of the output energy generated by an initial state x_0. It seems that for the fractional order case, these singular values take into account the past history of the system. This dependency on the past history of the singular values reflects the accumulated impact of the past of the dynamical process on its state (long memory behavior) which is the main particularity of fractional dynamical systems.

4. CONCLUSION

In this paper, we have introduced a fractional differential Lyapunov equation as a new tool for analyzing the controllability and the observability of fractional order linear dynamical systems. The main interpretation of the preliminary results given in this paper is the past history dependency of the fractional order dynamical linear systems modeled by constant state space matrices. For example, the dependence on the past history of the singular values of the controllability and observability Gramians may be explored for future development. Furthermore, these contributions can be also used in the derivation of new results of optimal control and filtering of fractional linear dynamical systems.

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