



Partial eigenvalue assignment with time delay in high order system using the receptance



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ABSTRACT

An explicit solution to the partial eigenvalue assignment problem of high order control system is presented by the method of receptance. Conventional methods, e.g. finite elements, are known to contain inaccuracies and assumptions that may hinder the calculations. An alternative approach was given by Ram and Mottershead [1] in the form of receptances, typically available from a modal test. This paper generalizes the earlier work on partial assignment that is applicable to multi-input delayed system without use of the Sherman–Morrison formula. The results of our numerical experiments support the validity of our proposed method.

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1. Introduction

Consider the high order dynamical system

$$P_K(D)v(t) = f(t), \quad (1.1)$$

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where $P_K(D) = \sum_{k=0}^K M_k D^k$, $D^k = \frac{d^k}{dt^k}$ is a differential operator, M_k is real constant $n \times n$ matrices, $k = 0, 1, \dots, K$, and M_K is nonsingular. Using separation of variables of the form $v(t) = xe^{\lambda t}$ in $P_K(D)v(t) = 0$, where x is a constant vector, we obtain the high order eigenvalue problem

$$P_K(\lambda)x = 0, \quad (1.2)$$

where

$$P_K(\lambda) = \sum_{k=0}^K M_k \lambda^k. \quad (1.3)$$

The characteristic roots $\lambda_1, \dots, \lambda_{Kn}$ of the polynomial Eq. (1.3), namely $\det [P_K(\lambda_i)] = 0$, $i = 1, 2, \dots, Kn$, are known as eigenvalues or natural frequencies. The nonzero corresponding vectors x_1, \dots, x_{Kn} are corresponding right eigenvectors or mode shapes which satisfy respectively

$$P_K(\lambda_i)x_i = 0, \quad i = 1, 2, \dots, Kn. \quad (1.4)$$

From the knowledge of these, the system responses as well as stability can be determined. It is well known that if the Kn eigenvalues $\{\lambda_i\}_{i=1}^{Kn}$ of Eq. (1.3) satisfy $\text{Re}(\lambda_i) \leq 0$ for all $i = 1, 2, \dots, Kn$, then the response of Eq. (1.1) is bounded for arbitrary initial conditions. The response of the system to initial conditions is required in some applications to diminish rapidly. This objective can be achieved by assigning eigenvalues further to the left-hand side of the complex plane.

In most practical applications only a small number of eigenvalues, which are responsible for instability and other undesirable phenomena, need to be assigned. Active control strategy is an effective way to control dangerous poles in a structure. Implementation of this strategy requires real-time computations of feedback control matrices such that a small amount of eigenvalues of the associated matrix pencil are replaced by suitably chosen ones while the remaining large number of eigenvalues and eigenvectors remain unchanged ensuring the no spill-over. This consideration gives rise to partial eigenvalue assignment problem (PEAP).

Orthogonality conditions given by Fawzy and Bishop [2] were used to derive an explicit solution to the partial eigenvalue assignment problem. The solution shed new light on the stabilization and control of large flexible space structures. Since the introduction of PEAP in the literature by Datta, Elhay and Ram [3] showed an elegant mathematical solution to the single input case, much research has been done in recent years. These works have extended the single input solutions to the multi-input case [4–6] and also to the solution of eigenstructure assignment problem [7]. Furthermore, the aspects of robust feedback stabilization and the norm minimization have been considered and solved by the sophisticated techniques of numerical linear algebra and numerical optimization [8,9].

It is necessary to take into account of the practical aspects of the problem; namely the time delay in control. Receptance methods were used in solving time-delayed PEAP [10] and a formal procedure was formulated by Ram and Mottershead [11]. In [12,13] the partial pole assignment with time delay for the single input case has been solved. In [14] an explicit solution to the partial eigenvalue assignment problem of high order control systems with time delay was presented. Bai, Datta and Wang [15] further developed the hybrid approach [13] to include multiple inputs with time delay using a multi-step technique. Recently, multiple-input active vibration control by the method of receptance based on the Sherman–Morrison formula was explained by Ram and Mottershead [1]. The method allows for complete multiple-input control in a single application of the routine and is superior to the sequential application of single-input control.

Motivated by the recent work of Ram and Mottershead [1], we provide an algorithm, readily achievable, for the time-delayed partial eigenvalue assignment problem. In [16] Li and Chu pointed out “The generalization of the receptance method for multi-input systems, with or without time-delay, is still an open problem”. We solve this open problem with respect to time delay. The paper is organized as follows. In Section 2 we generalize Ram’s work on partial assignment to constant time-delay high-order systems. We then consider the partial assignment of eigenvalues by multi-input control. In Section 3, we introduce an explicit solution to the partial eigenvalue assignment problem for time-delay system in multi-input case such that $\{M_k\}_{k=0}^K$ are non-symmetric matrices and M_K is nonsingular. Numerical examples are presented in Section 4. Conclusions are finally drawn in Section 5.

2. Partial eigenvalue assignment for time-delayed system by single-input control

In this section, we develop the method of receptance for solving time-delayed partial eigenvalue assignment problem. For single-input control application, the system modeled by Eq. (1.1) is modified by applying a controlling force $bu(t - \tau)$ as

$$P_K(D)v(t) = bu(t - \tau), \quad (2.1)$$

where $b \in \mathbb{R}^n$ is a constant vector, and the location of control and associated control force $u(t)$ is defined as

$$u(t) = \sum_{j=0}^{K-1} f_{K-j}^T D^j v(t), \quad (2.2)$$

with constant vectors $f_1, \dots, f_K \in \mathbb{R}^n$. Substituting $u(t)$ into Eq. (2.1) we get

$$P_K(D)v(t) - b \sum_{j=0}^{K-1} f_{K-j}^T D^j v(t - \tau) = 0. \quad (2.3)$$

Applying the separation of variables $v(t) = ye^{\mu t}$ to Eq. (2.3), we get the closed-loop system

$$F_{Kc}(\mu)y = 0, \quad (2.4)$$

where

$$F_{Kc}(\mu) = P_K(\mu) - e^{-\mu\tau}bh^T(\mu), \quad h(\mu) = \sum_{j=0}^{K-1} \mu^j f_{K-j}. \quad (2.5)$$

We assume that the eigenvalues in $\{\mu_k\}_{k=1}^p$ of Eq. (2.5) are distinct from eigenvalues in $\{\lambda_k\}_{k=1}^{Kn}$ of the open-loop system, while

$$\mu_k = \lambda_k, \quad k = p+1, p+2, \dots, Kn. \quad (2.6)$$

Substituting Eq. (2.6) in Eq. (2.4) gives

$$P_K(\mu_k)y_k = e^{-\mu_k\tau}bh^T(\mu)y_k, \quad k = p+1, p+2, \dots, Kn. \quad (2.7)$$

A non-trivial solution to Eq. (2.7) is

$$y_k = x_k, \quad k = p+1, p+2, \dots, Kn. \quad (2.8)$$

By virtue of Eq. (1.2), we have

$$bh^T(\lambda_k)x_k = 0, \quad k = p+1, p+2, \dots, Kn. \quad (2.9)$$

Since $b \neq 0$, this implies that

$$h^T(\lambda_k)x_k = 0, \quad k = p+1, p+2, \dots, Kn. \quad (2.10)$$

Note that $P_K(\mu)$ is invertible since we assume that the eigenvalues in $\{\mu_k\}_{k=1}^p$ of Eq. (2.5) are distinct from eigenvalues in $\{\lambda_k\}_{k=1}^{Kn}$. We define the receptance matrix

$$H(\mu) = [P_K(\mu)]^{-1}. \quad (2.11)$$

Then the p equations similar to Eq. (2.7) give

$$y_k = e^{-\mu_k\tau}H(\mu_k)bh^T(\mu_k)y_k, \quad k = 1, 2, \dots, p. \quad (2.12)$$

Since y_k may be scaled arbitrarily, we choose y_k such that

$$h^T(\mu_k)y_k = e^{\mu_k\tau}, \quad k = 1, 2, \dots, p. \quad (2.13)$$

By letting

$$r_{\mu_k} = H(\mu_k)b, \quad k = 1, 2, \dots, p, \quad (2.14)$$

we can obtain from Eq. (2.13)

$$y_k = r_{\mu_k}, \quad k = 1, 2, \dots, p. \quad (2.15)$$

Substituting Eq. (2.15) into Eq. (2.13) gives

$$r_{\mu_k}^T h(\mu_k) = e^{\mu_k \tau}, \quad k = 1, \dots, p. \quad (2.16)$$

We denote

$$S = \sum_{i=1}^p \sum_{j=1}^K E_{ij}^{(p \times K)} \otimes \mu_i^{K-j} r_{\mu_i}^T, \quad Q = \sum_{i=1}^{Kn-p} \sum_{j=1}^K E_{ij}^{((kn-p) \times K)} \otimes \mu_{p+i}^{K-j} x_{p+i}^T, \quad (2.17)$$

where \otimes is Kronecker product and $E_{ij}^{(h \times l)}$ is the $h \times l$ matrix with 1 at its entry (i, j) and zeros elsewhere. Write the Kn equations in Eq. (2.13) and Eq. (2.9) in matrix form

$$\begin{pmatrix} S \\ Q \end{pmatrix} (f_1^T, f_2^T, \dots, f_K^T)^T = \begin{pmatrix} \zeta \\ 0 \end{pmatrix}, \quad (2.18)$$

where

$$\zeta = (e^{\mu_1 \tau}, e^{\mu_2 \tau}, \dots, e^{\mu_p \tau})^T. \quad (2.19)$$

By $\{r_{\mu_i}\}_{i=1}^p$, $\{y_i\}_{i=1}^p$ and a complete set of assigned eigenvalues $\{\mu_i\}_{i=1}^p$, the control vector $\{f_i\}_{i=1}^K$ can be found from Eq. (2.18). The linear system of Eq. (2.18) defines the method of time-delayed eigenvalue assignment by receptances in the case where some eigenvalues in the open and closed loop systems are common. Note that the open-loop eigenvectors $\{x_i\}_{i=p+1}^{Kn}$ in Q may be expressed in terms of measured receptances so in practice there is no need to know the system matrices.

3. Partial eigenvalue assignment for time-delayed system by multi-input control

Assume that we wish to alter the location of the eigenvalues by applying the control force $f(t) = Bu(t - \tau)$, where B is an $n \times m$ ($m \leq n$) real matrix and $u(t)$ is a time-dependent $m \times 1$ real vector. Matrix B is known as the control matrix, and without loss of generality, we assume that B has full column rank, that is, $\text{rank}(B) = m$. The special choice

$$u(t - \tau) = \sum_{j=0}^{K-1} F_{K-j}^T D^j v(t - \tau), \quad (3.1)$$

where $F_1, \dots, F_K \in \mathbb{R}^{n \times m}$ and it leads to the closed-loop eigenvalue problem

$$F_{Kc}(\mu)y = (P_K(\mu) - e^{-\mu\tau}Bh^T(\mu))y, \quad h(\mu) = \sum_{j=0}^{K-1} \mu^j F_{K-j}. \quad (3.2)$$

Eq. (3.2) can be written equivalently in the form

$$P_K(\mu)y = e^{-\mu\tau} \left(\sum_{j=1}^m b_j \left(\sum_{i=1}^K \mu^{K-i} f_{i,j}^T \right) \right) y, \quad (3.3)$$

where $b_j, f_{i,j}$, are the j -th columns of B and F_i respectively. For the p eigenvalues that are intended to be changed we have

$$y_k = e^{-\mu\tau} \sum_{j=1}^m \sum_{i=1}^K \mu^{K-i} f_{i,j}^T y_k H(\mu_k) b_j. \quad (3.4)$$

We denote

$$r_{\mu_k,j} = H(\mu_k) b_j \quad (3.5)$$

and

$$\alpha_{\mu_k,j} = e^{-\mu\tau} \left(\sum_{i=1}^K \mu^{K-i} f_{i,j}^T \right) y_k, \quad k = 1, 2, \dots, p, \quad j = 1, 2, \dots, m. \quad (3.6)$$

Then Eq. (3.4) may be written in the form

$$y_k = \sum_{j=1}^m \alpha_{\mu_k,j} r_{\mu_k,j}, \quad (3.7)$$

i.e., the eigenvector y_k of the closed loop system is a linear combination of $\{r_{\mu_k,j}\}_{j=1}^m$. Therefore, we can obtain $\{y_k\}_{k=1}^p$ by choosing arbitrary $\{\alpha_{\mu_k,j}\}_{k=1}^p$ from Eq. (3.7). The equations in Eq. (3.6) may be casted in matrix form

$$S_k \tilde{f} = e^{\mu_k \tau} \alpha_k, \quad (3.8)$$

where

$$\begin{aligned} S_k &= (Y_{\mu_k, K-1}, Y_{\mu_k, K-2}, \dots, Y_{\mu_k, 0}), \quad Y_{\mu_k, i} = I_m \otimes \mu_k^i y_k^T, \quad i = 0, 1, \dots, K-1, \\ \tilde{f} &= (\tilde{f}_1^T, \tilde{f}_2^T, \dots, \tilde{f}_K^T)^T, \quad \tilde{f}_j = (f_{j,1}^T, f_{j,2}^T, \dots, f_{j,m}^T)^T, \quad j = 1, 2, \dots, K, \\ \alpha_k &= (\alpha_{\mu_k,1}, \alpha_{\mu_k,2}, \dots, \alpha_{\mu_k,m})^T, \end{aligned}$$

and I_m is the $m \times m$ identity matrix. For the invariant eigenvalues we have

$$\left(\sum_{j=1}^m b_j \left(\sum_{i=1}^K \mu^{K-i} f_{i,j}^T \right) \right) x_k = 0, \quad k = p+1, p+2, \dots, Kn. \quad (3.9)$$

The Kn equations in Eq. (3.9) are satisfied whenever

$$Q_k \tilde{f} = 0, \quad (3.10)$$

where

$$Q_k = (X_{\mu_k, K-1}, X_{\mu_k, K-2}, \dots, X_{\mu_k, 0}), \quad X_{\mu_k, i} = I_m \otimes \mu_k^i x_k^T, \quad i = 0, 1, \dots, K-1. \quad (3.11)$$

From combining Eq. (3.8) with Eq. (3.10), eigenvalues assignment for multi-input system by the method of receptance can be done by solving the system of linear equations

$$\begin{pmatrix} S_1 \\ \vdots \\ S_p \\ Q_{p+1} \\ \vdots \\ Q_{Kn} \end{pmatrix} \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_m \end{pmatrix} = \begin{pmatrix} e^{\mu_1 \tau} \alpha_1 \\ \vdots \\ e^{\mu_p \tau} \alpha_p \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (3.12)$$

It is seen that the gains $\{\tilde{f}_i\}_{i=1}^K$ obtained by the solution of Eq. (3.12) may be determined from measured receptance $H(s)$ at the desired eigenvalues $\{\mu_i\}_{i=1}^p$ without knowing the system matrices.

Remark. The solution to the problem when $m > 1$ is not unique. Each choice of appropriate parameters $\alpha_{\mu_k, i}$ will lead to different solution. This provides us with some freedom in the design of the controller. Therefore, we may set different $\alpha_{\mu_k, i}$ and obtain different feedback matrices from the following Algorithm 3.1. Moreover if $\{F_i\}_{i=1}^K$ are real and $\{\mu, y_\mu\}$ is an eigenpair of Eq. (3.2) then it can be shown by direct substitution that $\{\bar{\mu}, y_{\bar{\mu}}\}$ is an eigenpair of Eq. (3.2) as well, where bars denote complex conjugation. It thus follows by Eq. (3.6) that we have to choose $\alpha_{\mu_k, j} = \bar{\alpha}_{\bar{\mu}_k, j}$ so that $\{F_i\}_{i=1}^K$ are real.

Now we give summary of the algorithm.

Algorithm 3.1.

Input:

- The system matrices $\{M_i\}_{i=0}^K \in \mathbb{R}^{n \times n}$, M_K nonsingular.
- The control matrix $B \in \mathbb{R}^{n \times m}$, $1 \leq m \leq n$.

- A self-conjugate subset $\{\lambda_i\}_{i=1}^p$ of the spectrum of $P_K(\lambda)$ and the associated eigenvectors $\{x_i\}_{i=1}^p$.
- A suitably chosen self-conjugate set $\{\mu_i\}_{i=1}^p$ and the measured receptances $\{H(\mu_i)\}_{i=1}^p$.
- The time delay τ .

Procedure:

- Set b_i is the i -th column of B .
- For $k = 1, 2, \dots, p$.
 - Calculate $\{r_{\mu_k,j}\}_{j=1}^m$ by Eq. (3.5).
 - Determine $\{\alpha_{\mu_k,j}\}_{j=1}^m$ and calculate y_k by Eq. (3.7).
 - Compute P_k and Q_k .
- Evaluate $\{\tilde{f}_j\}_{j=1}^m$ by Eq. (3.12).

Output: $\{F_i\}_{i=1}^K = \{f_{1,i}, f_{2,i}, \dots, f_{m,i}\}_{i=1}^K$, which make $\{\mu_i\}_{i=1}^p$ eigenvalues of Eq. (3.2) other eigenvalues unchanged.

4. Numerical example

Example 4.1. Consider the control of the quadratic model with time delay ($\tau = 0$) in [1] described by

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 5 & -5 \\ -5 & 5 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 10 & -5 \\ -5 & 15 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}.$$

The eigenpairs of this system are

$$\begin{aligned} \left\{ \lambda_{1,2} = \pm\sqrt{5}, x_{1,2} = (1, 1)^T \right\}, \quad & \left\{ \lambda_3 = -2.5, x_3 = (2, -1)^T \right\}, \\ \left\{ \lambda_4 = -5, x_4 = (2, -1)^T \right\}. \end{aligned}$$

We wish to change $\lambda_{1,2}$ to $\mu_{1,2} = -1 \pm i$ and to keep $\lambda_{3,4}$ and $x_{3,4}$ unchanged. We choose $\alpha_{\mu_1,1} = \alpha_{\mu_2,1} = 1$ and $\alpha_{\mu_1,2} = \alpha_{\mu_2,2} = 0.5$ arbitrarily. By using Algorithm 3.1 we obtained

$$F_1 = \begin{pmatrix} -4 & -2 \\ -8 & -4 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 6 & 3 \\ 12 & 6 \end{pmatrix},$$

which are the same as the result in [1]. It can be checked that $\sum_{p=1}^2 \|F_{Kc}(\mu_p)y_{\mu_p}\| = 6.75322301446426 \times 10^{-16}$, which validates the desired partial eigenvalue assignment.

Example 4.2. Consider the three-order system with time delay ($\tau = 0.1$). We generate the randomly matrices M_3 , M_2 , M_1 , M_0 and B as follows

Table 1Comparison of eigenvalues of $P_3(\lambda)$ with eigenvalues of $F_{3c}(\lambda)$.

Eigenvalues of matrix polynomial $P_3(\lambda)$	Eigenvalues of matrix polynomial $F_{3c}(\lambda)$
0.48796902370407 ± 0.30962754205854i	-1.000000000000000 ± 1.000000000000000i
-1.00849513650939 ± 1.61127166760996i	-1.00849513650939 ± 1.61127166760996i
-0.54242173254294 ± 0.73319251823676i	-0.54242173254294 ± 0.73319251823676i
-0.23706733951077 ± 0.80969058643599i	-0.23706733951077 ± 0.80969058643599i
-1.03782016257817	-1.03782016257817

$$M_3 = \begin{pmatrix} 0.6813 & 0.5028 & 0.3046 \\ 0.3795 & 0.7095 & 0.1897 \\ 0.8318 & 0.4289 & 0.1934 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0.6822 & 0.1509 & 0.8600 \\ 0.3028 & 0.6979 & 0.8537 \\ 0.5417 & 0.3784 & 0.5936 \end{pmatrix},$$

$$M_1 = \begin{pmatrix} 0.4966 & 0.6449 & 0.3420 \\ 0.8998 & 0.8180 & 0.2897 \\ 0.8216 & 0.6602 & 0.3412 \end{pmatrix}, \quad M_0 = \begin{pmatrix} 0.5341 & 0.8385 & 0.7027 \\ 0.7271 & 0.5681 & 0.5466 \\ 0.3093 & 0.3704 & 0.4449 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.6946 & 0.9568 \\ 0.6213 & 0.5226 \\ 0.7948 & 0.8801 \end{pmatrix}.$$

The polynomial $P_3(\lambda) = M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0$ has eigenvalues shown in [Table 1](#). Now we wish to replace $p = 2$ eigenvalues $\lambda_{1,2}$ with $\eta_{1,2} = -1 \pm i$ without altering other eigenpairs. We choose $\alpha_{\mu_1,1} = \alpha_{\mu_2,1} = 1$ and $\alpha_{\mu_1,2} = \alpha_{\mu_2,2} = 0.5$ arbitrarily. By using [Algorithm 3.1](#) we obtained

$$F_1 = \begin{pmatrix} -3.3057 & -1.6528 \\ 2.5098 & 1.2549 \\ -0.4047 & -0.2023 \end{pmatrix}, \quad F_2 = \begin{pmatrix} -6.4133 & -3.2067 \\ 5.2990 & 2.6495 \\ 1.4788 & 0.7394 \end{pmatrix},$$

$$F_3 = \begin{pmatrix} -2.1524 & -1.0762 \\ 3.3628 & 1.6814 \\ 0.7837 & 0.3919 \end{pmatrix}.$$

With these matrices, the eigenvalues of the closed-loop system Eq. (3.2) are shown in [Table 1](#). It can be checked that $\sum_{p=1}^2 \|F_{Kc}(\mu_p)y_{\mu_p}\| = 3.84472477288761 \times 10^{-14}$, which validates the desired partial eigenvalue assignment.

Remark. It is important to note that the characteristic equation of a controlled system with delay is a quasi-polynomial and has generally infinite number of roots over the complex field. [13] has shown that for pole assignment in delayed system a posterior analysis has to be done to identify and assign the primary eigenvalues ensuring that such assignment will avoid instability. In light of their ideas, we check the stability of [Example 4.2](#). In this case, the pole assignment is achieved and the approximate eigenvalues are

$\eta_{1,2} = 1.00030892209161 \pm 0.99904864623100i$ without positive real eigenvalue around them which means the primary poles have been assigned and the controlled system is stable.

5. Conclusion

An algorithm is developed for high order delayed systems by eigenvalue assignment using the receptance method that is applicable to multi-input delayed system without use of the Sherman–Morrison formula. This method is applicable to both single-input and multi-input control without turning high order systems into first order form. In particular, when $K = 2$ and $\tau = 0$, we know that [1] is a special case of our results.

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