DEPARTAMENTO DE ANÁLISE
MATEMÁTICA, ESTATÍSTICA
E OPTIMIZACIÓN

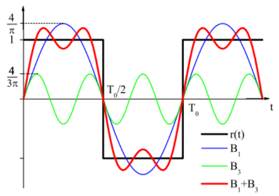
Fractional derivatives and periodic functions

Jorge Losada Rodríguez
jorge.losada@usc.es

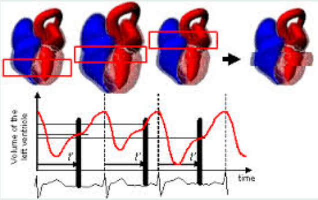
Summer School on Fractional and Other Nonlocal Models
Basque Center for Applied Mathematics

Periodic functions

They play a major role in mathematics...



...and also in our lives



- If f is a T-periodic function $\Longrightarrow$ f' is a T-periodic function
- If f is a T-periodic function $\Longrightarrow$ Is $D^{\alpha}f$ a T-periodic function??

[1] M.S. Tavazoei, **A note on fractional order derivatives of periodic functions**.
Automatica 46 (5) (2010) 945–948.

Quasi-periodic functions

Definition (Asymptotically periodicity)

A function f defined in $\mathbb{R}^+$ is said to be *asymptotically T-periodic* if we can write

$$f = f_1 + f_2,$$

with f_1 T-periodic and $\lim_{t\to+\infty} f_2(t) = 0$.

Definition (S-Asymptotically periodicity)

A bounded function f is called *S-asymptotically T-periodic* if there exists $T > 0$ such that

$$\lim_{t\to+\infty} [f(t+T) - f(t)] = 0.$$

We say that T is an asymptotic period of f.

Quasi-periodic functions

Definition (Almost periodicity)

A function f is *almost periodic* if for every $\varepsilon > 0$ there exists a relatively dense subset $H(\varepsilon, f)$ of $\mathbb{R}$ such that

$$|f(t+\eta) - f(t)| < \varepsilon$$

for every $t \in \mathbb{R}$ and $\eta \in H(\varepsilon, f)$.

A set $E \subset \mathbb{R}$ is said to be *relatively dense* if it exists a number $\ell > 0$ such that any interval of lenght ℓ contains at least on number of E.

Fractional Calculus: Basic definitions

Let $\alpha > 0$ and $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$.

Fractional integral

The fractional integral of order α of a given function f is defined as

$$I^{\alpha} f\,(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0.$$

Riemann-Liouville fractional derivative

The R-L fractional derivative of order α of a given function f is defined for $t > 0$ by

$$\begin{aligned} {}^{\mathrm{RL}}D^{\alpha} f\,(t) &= D^n I^{n-\alpha} f\,(t) \\ &= \frac{1}{\Gamma(\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds. \end{aligned}$$

Fractional Calculus: Basic definitions

Let $\alpha > 0$ and $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$.

Fractional integral

The fractional integral of order α of a given function f is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0.$$

Caputo fractional derivative

Caputo introduced in 1967 another definition of fractional derivative

$$\begin{aligned} {}^{\mathrm{C}}D^{\alpha} f(t) &= I^{n-\alpha} D^{n} f(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds. \end{aligned}$$

Fractional Calculus: Basic definitions

Let $\alpha > 0$ and $n \in \mathbb{N}$ such that $n = \lceil \alpha \rceil$.

Fractional integral

The fractional integral of order α of a given function f is defined as

$$I^{\alpha} f\,(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds, \quad t > 0.$$

Riemann-Liouville & Caputo

$$g(s) = f(s) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} s^{k-\alpha}$$

$${}^{\mathrm{C}}D^{\alpha} f\,(t) = {}^{\mathrm{RL}}D^{\alpha} g\,(t)$$

In summary

Let f be a T-periodic function and $0 < \alpha < 1$.
Consider then: $I^{\alpha} f$, ${}^{\mathrm{C}}D^{\alpha} f$ and ${}^{\mathrm{RL}}D^{\alpha} f$.

1.- Are these functions **periodic**?

2.- Are these functions S-**asymptotically periodic**?

3.- Are these functions **asymptotically periodic**?

4.- Are these functions **almost periodic**?

5.- When these functions are **bounded**?

In summary

Let f be a T-periodic function and $0 < \alpha < 1$.
Consider then: $I^{\alpha} f$, ${}^{\mathrm{C}}D^{\alpha} f$ and ${}^{\mathrm{RL}}D^{\alpha} f$.

1.- Are these functions **periodic**? ✗
2.- Are these functions S-**asymptotically periodic**? ✓
3.- Are these functions **asymptotically periodic**? ✓
4.- Are these functions **almost periodic**? ✗
5.- When these functions are **bounded**?

In summary

Let f be a T-periodic function and $0 < \alpha < 1$.
Consider then: $I^{\alpha} f$, ${}^{\mathrm{C}}D^{\alpha} f$ and ${}^{\mathrm{RL}}D^{\alpha} f$.

1.- Are these functions **periodic**? ✗

2.- Are these functions S-**asymptotically periodic**? ✓

3.- Are these functions **asymptotically periodic**? ✓

4.- Are these functions **almost periodic**? ✗

5.- When these functions are **bounded**?

The basic hypotheses

$$\frac{1}{T} \int_0^T f(s)\, ds = 0 \qquad (*)$$

In summary

Let f be a T-periodic function and $0 < \alpha < 1$.
Consider then: $I^\alpha f$, ${}^{\mathrm{C}}D^\alpha f$ and ${}^{\mathrm{RL}}D^\alpha f$.

1.- Are these functions **periodic**? ✗

2.- Are these functions S-**asymptotically periodic**? ✓

3.- Are these functions **asymptotically periodic**? ✓

4.- Are these functions **almost periodic**? ✗

5.- When these functions are **bounded**?

The basic tool: Weyl fractional integral

$$ {}^{\mathrm{W}}I^\alpha f\,(t) = \frac{1}{2\pi}\int_0^{2\pi} f(t-s)g(s)\,ds $$

$$ g(s) = \lim_{n\to\infty}\left[\frac{2\pi}{\Gamma(\alpha)}\sum_{m=1}^{n}(s+2\pi m)^{\alpha-1} - \frac{(2\pi n)^\alpha}{\alpha\Gamma(\alpha)}\right] + \frac{2\pi}{\Gamma(\alpha)}s^{\alpha-1}. $$

In summary

Let f be a T-periodic function and $0 < \alpha < 1$.
Consider then: $I^\alpha f$, ${}^{\mathrm{C}}D^\alpha f$ and ${}^{\mathrm{RL}}D^\alpha f$.

1.- Are these functions **periodic**? ✗

2.- Are these functions S-**asymptotically periodic**? ✓

3.- Are these functions **asymptotically periodic**? ✓

4.- Are these functions **almost periodic**? ✗

5.- When these functions are **bounded**?

The basic tool: Weyl fractional integral

$$ {}^{\mathrm{W}}I^\alpha f\,(t) = \frac{1}{\Gamma(\alpha)} \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \int_{t-nT}^{t} (t-s)^{\alpha-1} f(s)\,ds. $$

Results about asymptotically periodicity

Theorem

Let f be a T-periodic and continuous function and $\alpha \in (0,1)$. Assume that $I^\alpha f$ is a bounded function, then $I^\alpha f$ is asymptoically T-periodic.

Proof: Let us denote for the sake of brevity

$$\varphi(t) = I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad t > 0.$$

Given $n \in \mathbb{N}$, consider the function $\varphi_n(t) = \varphi(t + nT)$. Define

$$\Phi_n(t) = \sup_{k \geq n} \varphi_k(t) \quad \text{and} \quad \Psi_n(t) = \inf_{k \geq n} \varphi_k(t).$$

We have that φ_n, Φ_n and Ψ_n are bounded and continuous. Moreover, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$,

$$\Phi_{n+1}(t) \leq \Phi_n(t) \quad \text{and} \quad \Psi_{n+1}(t) \geq \Psi_n(t).$$

Results about asymptotically periodicity

Theorem

Let f be a T-periodic and continuous function and $\alpha \in (0,1)$. Assume that $I^{\alpha} f$ is a bounded function, then $I^{\alpha} f$ is asymptoically T-periodic.

Proof: Using sequences Φ_n and Ψ_n we prove that

$$\varphi_n \rightrightarrows \Phi \quad \text{in } \mathbb{R}^+, \ \Phi \text{ is } T\text{-periodic}$$

$$\lim_{t\to\infty} (\varphi(t) - \Phi(t)) = 0.$$

Conclusion will be clear if we write

$$\varphi(t) = \Phi(t) + (\varphi(t) - \Phi(t)).$$

Pointwise convergence of Φ_n and Ψ_n

Since f is a T-periodic function,

$$\lim_{n\to\infty}\Phi_n(t) = \lim_{n\to\infty}\left[\sup_{k\geq n}\varphi(t+kT)\right] = \limsup_{n\to\infty}\varphi(t+nT)$$
$$= \limsup_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_{t-nT}^{t}(t-s)^{\alpha-1}f(s)\,ds.$$

Therefore, we can define

$$\lim_{n\to\infty}\Phi_n(t) = \Phi(t)\in\mathbb{R}, \quad \text{for all } t\geq 0.$$

and we have

$$\Phi(t)\equiv {}^{\mathrm{W}}I^{\alpha}f\,(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}f(s)\,ds, \qquad t\geq 0.$$

Moreover all hypothesis of Dini's Theorem are satisfied and we deduce that Φ_n uniformly converges to Φ on any given closed subinterval of $\mathbb{R}^+$.

Φ is a T-periodic function

$$\begin{aligned}
\Phi(t+T)-\Phi(t) &= \lim_{n\to\infty}\left(\Phi_n(t+T)-\Phi_n(t)\right) \\
&= \lim_{n\to\infty}\left(\sup_{k\geq n}\varphi(t+T+kT)-\sup_{k\geq n}\varphi(t+kT)\right) \\
&= \limsup_{n\to\infty}\left(\varphi(t+T+nT)-\varphi(t+nT)\right) \\
&= \lim_{n\to\infty}\left(\varphi(t+T+nT)-\varphi(t+nT)\right)=0.
\end{aligned}$$

$\varphi_n \rightrightarrows \Phi$ uniformly in $\mathbb{R}^+$

Given $m, n \in \mathbb{N}$, define

$$I_n = [n\,T, (n+1)\,T] \quad \text{and} \quad A_{m,n} = \sup_{t \in I_m} |\Phi_n(t) - \Phi(t)|\,.$$

We have:

$$\Phi \text{ is } T\text{-periodic} \implies A_{m,\,n} = A_{0,m+n},$$
$$I^\alpha f = \varphi \text{ is bounded} \implies \lim_{s\to\infty} A_{0,\,s} = 0.$$

Therefore, for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $k \geq N$

$$A_{0,\,k} = \sup_{t \in I_0} |\Phi_k(t) - \Phi(t)| < \varepsilon.$$

Given $t \in I_m$ arbitrary, when $n \geq N$,

$$A_{m,\,n} = A_{0,\,m+n} = \sup_{t \in I_m} |\Phi_n(t) - \Phi(t)| < \varepsilon.$$

That is to say, we have that for $n \geq N$

$$\sup_{t \in \mathbb{R}^+} |\Phi_n(t) - \Phi(t)| < \varepsilon \iff (\Phi_n)_{n\in\mathbb{N}} \rightrightarrows \Phi \text{ uniformly in } \mathbb{R}^+$$
$$\sup_{t \in \mathbb{R}^+} |\Psi_n(t) - \Phi(t)| < \varepsilon \iff (\Psi_n)_{n\in\mathbb{N}} \rightrightarrows \Phi \text{ uniformly in } \mathbb{R}^+$$

We have that $(\varphi_n)_{n\in\mathbb{N}} \rightrightarrows \Phi$ uniformly in $\mathbb{R}^+$. In fact, since

$$\Psi_n(t) = \inf_{k\geq n} \varphi_k(t) \leq \varphi_n(t) \leq \sup_{k\geq n} \varphi_k(t) = \Phi_n(t).$$

$$\lim_{t\to\infty} (\varphi(t) - \Phi(t)) = 0$$

Moreover,

$$|\varphi(t) - \Phi(t)| \leq |\varphi(t) - \varphi_n(t)| + |\varphi_n(t) - \Phi(t)| < |\varphi(t) - \varphi_n(t)| + \frac{\varepsilon}{2}.$$

For the above $\varepsilon > 0$ there exists $S > 0$ such that if $t > S$ then

$$|\varphi(t+T) - \varphi(t)| < \frac{\varepsilon}{2n}. \qquad \text{(S-asymptotically periodicity)}$$

Using this, for $t > S$, it follows that

$$\begin{aligned}
|\varphi(t) - \varphi_n(t)| &= |\varphi(t) - \varphi(t+nT)| \\
&\leq |\varphi(t) - \varphi(t+T)| + |\varphi(t+T) - \varphi(t+2T)| \\
&\quad + \cdots + |\varphi(t+nT) - \varphi(t+(n-1)T)| \\
&< \frac{\varepsilon}{2n}\, n = \frac{\varepsilon}{2}.
\end{aligned}$$

Results about boundedness

Theorem: Boundedness of the fractional integral

Let f be a continuous and T-periodic function and $\alpha \in (0,1)$. Then, $I^\alpha f$ is a bounded function if, and only if,

$$\int_0^T f(t)\,dt = 0, \qquad \left[c = \int_0^T f^+(t)\,dt = \int_0^T f^-(t)\,dt > 0\right]. \qquad (*)$$

Laplace transform

If both limits exist and are finite

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0^+} s\,\mathcal{L}[f](s)$$

$$\lim_{t\to 0^+} f(t) = \lim_{s\to+\infty} s\,\mathcal{L}[f](s)$$

for asymptotically periodic functions

If f is an (asymptotically) T-periodic function then

$$\lim_{s\to 0^+} s\,\mathcal{L}[f](s) = \frac{1}{T}\int_0^T f(s)\,ds.$$

[2] E. Gluskin, **Let us teach this generalization of the final value-theorem**. European Journal of of Physics 24 (2003) 591–597.

Results about boundedness

Theorem: Boundedness of the fractional integral

Let f be a continuous and T-periodic function and $\alpha \in (0,1)$. Then, $I^\alpha f$ is a bounded function if, and only if,

$$\int_0^T f(t)\,dt = 0, \qquad \left[c = \int_0^T f^+(t)\,dt = \int_0^T f^-(t)\,dt > 0 \right]. \qquad (*)$$

Proof: Assume that $I^\alpha f$ is a bounded function. Since f is a T-periodic function, using properties of Laplace transform, we obtain that

$$s\,\mathcal{L}\left[I^\alpha f\right](s) = \frac{s}{s^\alpha}\mathcal{L}[f](s),$$

Therefore, using Generalized Final Value Theorem, we have that

$$\lim_{s\to 0^+} s\,\mathcal{L}\left(I^\alpha f\right)(s) = \frac{1}{T}\int_0^T f(t)\,dt = \lim_{s\to 0^+} \frac{1}{s^\alpha}\frac{1}{T}\int_0^T f(t)\,dt.$$

Hence, it is clear that if $I^\alpha f$ is bounded, then $(*)$ holds.

Proof: Assume now that $(*)$ holds.

$$I^{\alpha} f\,(t) = {}^{\mathrm{W}}I^{\alpha} f\,(t) - \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \int_{-nT}^{0} (t-s)^{\alpha-1} f(s)\,ds.$$

The first summand of the right hand side is bounded. Moreover,

$$\int_{-nT}^{0} (t-s)^{\alpha-1} f(s)\,ds = \sum_{j=1}^{n} \int_{-jT}^{(1-j)T} (t-s)^{\alpha-1} f(s)\,ds$$

$$= \sum_{j=1}^{n} \int_{0}^{T} (t-r+jT)^{\alpha-1} f^{+}(r)\,dr - \sum_{j=1}^{n} \int_{0}^{T} (t-r+jT)^{\alpha-1} f^{-}(r)\,dr$$

$$\leq c\Big(\sum_{j=1}^{n} (t+(j-1)T)^{\alpha-1} - \sum_{j=1}^{n} (t-jT)^{\alpha-1} \Big) = c\, t^{\alpha-1},$$

Hence,

$$-c\, t^{\alpha-1} \leq \int_{-nT}^{0} (t-s)^{\alpha-1} f(s)\,ds \leq c\, t^{\alpha-1}$$

and then,

$$\lim_{t\to\infty} \lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} \int_{-nT}^{0} (t-s)^{\alpha-1} f(s)\,ds = 0.$$

Example and more results

Example

Consider $f(t) = \sin(t)$, then

$$I^{\alpha} f\,(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \sum_{j=0}^{\infty} \frac{4^{-j}\left(-t^2\right)^j}{\left(\frac{\alpha}{2}+1\right)_j \left(\frac{\alpha+3}{2}\right)_j}$$

$$= \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\, {}_1F_2\left(1; \frac{\alpha}{2}+1, \frac{\alpha}{2}+\frac{3}{2}; -\frac{t^2}{4}\right).$$

Series expansion for $\alpha = 1/2$

$$I^{1/2} f\,(t) \sim \frac{\sqrt{t^{-1}}}{\sqrt{\pi}} + \frac{\sin(t) - \cos(t)}{\sqrt{2}}$$

Series expansion for $\alpha = 3/2$

$$I^{3/2} f\,(t) \sim \frac{2\sqrt{t}}{\sqrt{\pi}} - \frac{\sin(t) + \cos(t)}{\sqrt{2}}.$$

Example and more results

What happens if $\alpha > 1$?

Consider now $f(t) = \sin(t)$ and $g(t) = \cos(t)$ and $\alpha \in (1,2)$ then

$$I^{\alpha} \sin(t) = I^{\alpha-1} I^{1} \sin(t) = I^{\alpha-1}(-\cos s + 1)(t),$$
$$I^{\alpha} \cos(t) = I^{\alpha-1} I^{1} \cos(t) = I^{\alpha-1} \sin(t).$$

Discrete fractional calculus We can obtain similar results using discrete fractional operators:

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s),$$

$${}^{\mathrm{RL}}\Delta_a^{\alpha} f(t) = \Delta^{n} \Delta_a^{-(n-\alpha)} f(t),$$

$${}^{\mathrm{C}}\Delta_a^{\alpha} f(t) = \Delta_a^{-(n-\alpha)} \Delta^{n} f(t).$$

References

[1] S.H.S. Nicola, M. Pierri, **A note on S-asymptotically periodic functions**.
Nonlinear Analysis Real world Applications 10 (2009) 2937–2938.

[2] E. Gluskin, **Let us teach this generalization of the final value-theorem**.
European Journal of of Physics 24 (2003) 591–597.

[3] I. Area, J. Losada, J. J. Nieto, **On quasi-periodicity properties of fractional integrals and fractional derivatives of periodic functions**.
Integral Transforms and Special Functions 27:1 (2016) 1–16.

[4] I. Area, J. Losada, J. J. Nieto, **On quasi-periodic properties of fractional sums and fractional differences of periodic functions**.
Applied Mathematics and Computation 273 (2016) 190–200.

Gracias

Eskerrik asko

Thank you

USC
UNIVERSIDADE DE SANTIAGO DE COMPOSTELA
DEPARTAMENTO DE ANÁLISE MATEMÁTICA, ESTATÍSTICA E OPTIMIZACIÓN

Fractional derivatives and periodic functions

Jorge Losada Rodríguez
jorge.losada@usc.es

Summer School on Fractional and Other Nonlocal Models
Basque Center for Applied Mathematics