Impulsive Mode Elimination for Descriptor Systems by a Structured P-D Feedback

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***Abstract*—The problem of impulsive mode elimination for descriptor systems via a structured output proportional and derivative (P-D) feedback is investigated in this technical note. The motivation of solving this problem arises from an observation that the impulse behavior for some descriptor systems can not be eliminated completely by a parameterized output P-D feedback, but it can be removed by a structured P-D feedback elegantly designed. In order to explain this phenomenon explicitly, a concept of the structured P-D feedback is first introduced, then an explicit necessary and sufficient condition is constructively derived for the closed loop systems to be regular and impulse-free by the structured P-D feedback. The main result is represented in terms of the original system parameters instead of decomposed subsystem parameters. Finally, an example is provided to illustrate the effectiveness of the proposed result.**

***Index Terms*—Descriptor systems, impulsive mode elimination, regularization, structured proportional-derivative (P-D) feedback.**

I. Introduction

The descriptor systems have attracted much attention of researchers in last three decades and many important results have been achieved. For the research of the descriptor systems, the output proportional and derivative (P-D) feedback is a significant type of feedback, in which the derivative feedback term can be used to eliminate the impulse behavior as well as to change the dynamical order. By this type of feedback, the regularization problem has been investigated extensively and some necessary and sufficient conditions have been obtained, see [1]–[3] and the references therein. The results of [1] and [2] are presented via decomposed subsystem parameters and the results of [3] are presented via orthogonal matrix transformations. In [4], a decentralized P-D feedback is explored for the impulsive elimination problem where the centralized P-D feedback is regarded as a special case. All the existing results are given by a parameterized feedback or presented via decomposed subsystem parameters or orthogonal matrix transformations.

In this technical note we consider the regularization problem (i.e., regular and impulse free) with a structured P-D feedback. The motivation of our work is based on a question: if the impulsive modes of descriptor systems can not be eliminated by a P-feedback (output feedback) or a parameterized output P-D feedback, can we remove them by using a sort of new feedback structure? We will explore the new feedback control structure, which is called a structured P-D feedback control, i.e., the derivative (D-) feedback gain is a structured matrix [5]. Using this type of feedback, an explicit necessary and sufficient condition for the systems to be regular and impulse-free is derived in terms of the original system parameters. The novelty of main result in this technical note is in three folds: i) Compared to the conventional parameterized P-D feedback, it is a new result as well as a significant extension of the result with an output P-D feedback [4]. ii) The structured P-D feedback includes all possible cases of the P-D feedback for eliminating impulsive modes and permitting nonzero parameter perturbation. iii) The derived necessary and sufficient condition with the original system parameters for this problem indicates that a structured P-D feedback control is a more general type of controller and satisfies with practical requirements.

The layout of this technical note is as follows. Some preliminary results and problem formulation are presented in Section II. The problem for elimination of impulsive modes by a structured P-D feedback is investigated in Section III where the main result is presented. A numerical example is presented to illustrate the main result in Section IV. Section V gives the conclusion.

Manuscript received October 19, 2009; revised April 18, 2010, November 28, 2010, and June 07, 2011; accepted June 17, 2011. Date of publication June 23, 2011; date of current version December 07, 2011. This work was supported by the National Natural Science Foundation of China under Grants 60674019 and 61074088. Recommended by Associate Editor T. Zhou.

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Digital Object Identifier 10.1109/TAC.2011.2160597

II. Preliminaries and Problem Formulation

A. Preliminaries

Consider a following linear time-invariant descriptor system:

$$\begin{cases} E\dot{x} = Ax + Bu, \\ y = Cx. \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$ is system state, $u \in \mathbf{R}^m$ is system input and $y \in \mathbf{R}^p$ is system output. $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are constant matrices. Assume that E is *singular* with $0 < \operatorname{rank}(E) < n$. Usually system (1) is represented by (E, A, B, C) for simplicity. The system $E\dot{x} = Ax$, represented as (E, A), is said to be *regular*, if $\det(sE - A)$ is not identically zero. If $\det(sE - A) = 0$, i.e., (E, A) is singular, we define that $\deg\det(sE - A) = -\infty$. It is well known that (E, A) is regular and impulse-free if and only if $\deg\det(sE - A) = \operatorname{rank}(E)$, or equivalently [7]

$$\operatorname{rank}\begin{bmatrix} 0 & E \\ E & A \end{bmatrix} = n + \operatorname{rank}(E).$$

Next we consider an output P-D feedback

$$u = Ky - L\dot{y} \tag{2}$$

then the resulted closed-loop system is

$$(E + BLC)\dot{x} = (A + BKC)x. \tag{3}$$

It can be directly proved that system (3) is regular and impulse-free if and only if

$$\deg\det[s(E + BLC) - (A + BKC)] = \operatorname{rank}(E + BLC)$$

or equivalently

$$\operatorname{rank}\begin{bmatrix} 0 & E + BLC \\ E + BLC & A + BKC \end{bmatrix} = n + \operatorname{rank}(E + BLC). \tag{4}$$

B. Problem Formulation

In the existing literatures, K and L in (2)–(4) are generally assumed to be parameterized matrices, i.e., K and L can vary arbitrarily in a robust subset of $\mathbf{R}^{m\times p}$ (see [4] and the references therein), which can also be interpreted as an open and dense subset of $\mathbf{R}^{m\times p}$. It can be observed that if there exist controller gains $K, L \in \mathbf{R}^{m\times p}$ such that system (3) is regular and impulse-free, then this will be true for almost all $K \in \mathbf{R}^{m\times p}$, but not necessary true for almost all $L \in \mathbf{R}^{m\times p}$. This motivates us to consider L as a structured matrix instead of a parameterized matrix such that the rank of $E + BLC$ is required to be an appropriate integer satisfying (4). Actually, a structured matrix, which is introduced in [5], is a matrix in which some elements are zeros and others vary in robust subsets of $\mathbf{R}$. The structured matrix is adopted because it is consistent with some physical system requirements in the sense that the system elements are never known precisely except zeros. Bear in mind that some results in [2] are the necessary and sufficient conditions only for the existence of the feedback gains and the derived feedback gains may not be allowed for any perturbations there. What we are considering in this technical note is to derive a necessary and sufficient condition for the regularization problem by using a structure derivative feedback gain L and a parameterized matrix K. This new type controller is different from the controllers used in the existing literatures.

In order to formulate the problem precisely, we give the following definitions.

Definition 1: A matrix $L \in \mathbf{R}^{m\times p}$, denoted as $L = [l_{ij}]_{m\times p}$, is said to be *a structured matrix*, if for any i, j, either $l_{ij} = 0$ or $l_{ij} \neq 0$ is a variable parameter in an open and dense subset of $\mathbf{R}$.

Definition 2: An output P-D feedback (2) is said to be *a structured P-D feedback* if L is a structured matrix and K is a parameterized matrix.

The problem studied in this technical note: Find a necessary and sufficient condition, under which, there exists a structured P-D feedback (2) such that the closed-loop system (3) is regular and impulse-free.

C. Some Notations and Lemmas

In order to solve the proposed problem, we give some basic notations and lemmas. Without loss of generality, let B be a full column rank matrix and C be a full row rank matrix with notations

$$B = [b_1 \quad b_2 \quad \dots \quad b_m], \quad C^{\mathrm{T}} = \left[c_1^{\mathrm{T}} \quad c_2^{\mathrm{T}} \quad \dots \quad c_p^{\mathrm{T}}\right],$$
$$\underline{m} := \{1, 2, \dots, m\}, \quad \underline{p} := \{1, 2, \dots, p\}.$$

Let Φ, Ψ be subsets of $\underline{m}, \underline{p}$ respectively, denoted as

$$\Phi = \{i_1, i_2, \dots, i_k\}, \quad \Psi = \{j_1, j_2, \dots, j_t\}$$

and accordingly define

$$B_\Phi := \{b_{i_1}, b_{i_2}, \dots, b_{i_k}\}, \quad C_\Psi^{\mathrm{T}} := \left\{c_{j_1}^{\mathrm{T}}, c_{j_2}^{\mathrm{T}}, \dots, c_{j_t}^{\mathrm{T}}\right\}.$$

Also the complements of sets Φ, Ψ are denoted as $\overline{\Phi}, \overline{\Psi}$, respectively, i.e.

$$\begin{aligned}\overline{\Phi} &:= \underline{m} - \Phi = \{1, 2, \dots, m\} - \{i_1, i_2, \dots, i_k\}\\ &= \{j : j \in \underline{m}, j \notin \Phi\},\\ \overline{\Psi} &:= \underline{p} - \Psi = \{1, 2, \dots, p\} - \{j_1, j_2, \dots, j_t\}\\ &= \{j : j \in \underline{p}, j \notin \Psi\}.\end{aligned}$$

The following lemmas are useful in proving the main result of this technical note.

Lemma 1: [6] For the given matrices E, B, C, and a parameterized matrix L with compatible dimension, then

$$\underset{L}{\text{g.r.}}[E + BLC] = \min\left\{\text{rank}\,[E \quad B], \text{rank}\begin{bmatrix}E\\ C\end{bmatrix}\right\} \tag{5}$$

where, "g.r." denotes the generic rank, i.e., $\underset{L}{\text{g.r.}}[E + BLC]$ denotes the rank of $E + BLC$ for almost all $L \in \mathbf{R}^{m\times p}$.

Using Lemma 1 repeatedly, we can derive following Lemma 2 and Lemma 3.

Lemma 2: Let $A \in \mathbf{R}^{m\times n}$, $B_i \in \mathbf{R}^{m\times l_i}$, and $C_i \in \mathbf{R}^{p_i\times n}$ be given matrices, and $K_i \in \mathbf{R}^{l_i\times p_i}$ be a variable matrix, $i = 1, 2$. Then

$$\begin{aligned}&\underset{K_1,K_2}{\text{g.r.}}\ [A + B_1K_1C_1 + B_2K_2C_2]\\ &\quad = \min\left\{\text{rank}\,[A \quad B_1 \quad B_2], \text{rank}\left[A^{\mathrm{T}} \quad C_1^{\mathrm{T}} \quad C_2^{\mathrm{T}}\right],\right.\\ &\qquad\qquad \left.\text{rank}\begin{bmatrix}A & B_1\\ C_2 & 0\end{bmatrix}, \text{rank}\begin{bmatrix}A & B_2\\ C_1 & 0\end{bmatrix}\right\}.\end{aligned}$$

Lemma 3: Let $A \in \mathbf{R}^{m\times n}$, $b_i \in \mathbf{R}^m$, and $c_i^{\mathrm{T}} \in \mathbf{R}^n$, be given matrix and vectors, and $l_i \in \mathbf{R}$ be a variable number, $i \in \underline{N} := \{1, 2, \dots, N\}$. Then

$$\begin{aligned}&\underset{l_i, i\in\underline{N}}{\text{g.r.}}\left[A + \sum_{i=1}^{N} b_i l_i c_i\right]\\ &= \min\left\{\text{rank}\begin{bmatrix}A & b_{i_1} & \dots & b_{i_p}\\ c_{i_{p+1}} & 0 & \dots & 0\\ \vdots & & \dots & \\ c_{i_N} & 0 & \dots & 0\end{bmatrix}, \underline{N} = \{i_1, i_2, \dots, i_N\}\right\}\end{aligned}$$

where $\{i_1, i_2, \dots, i_N\}$ is a permutation of $\{1, 2, \dots, N\}$, and any $i_p \in \underline{N}$.

In order to make discussion more precise and easy to follow, we introduce some new notations and definitions.

Definition 3: Let $L = [l_{ij}]_{m\times p}$ be a structured matrix. Θ is said to be *an induced set of* L, if

$$\Theta := \left\{(i, j) : l_{ij} \neq 0, L = [l_{ij}]_{m\times p}, i \in \underline{m}, j \in \underline{p}\right\}.$$

Apparently, Θ is a subset of $\underline{m} \times \underline{p}$, and mapping $L \to \Theta(l_{ij} \to (i, j))$ is a bijection.

Definition 4: A set of pairs (Φ, Ψ) is said to be *a feasible subset pairs associated with* Θ, if there exists a pair of disjoint subsets Θ_l and Θ_r of Θ, satisfying the following four conditions:

$$\begin{aligned}&\Phi = \{i : (i, j) \in \Theta_l\}, \quad \Psi = \{j : (i, j) \in \Theta_r\},\\ &\Theta_l \bigcup \Theta_r = \Theta, \quad \Theta_l \bigcap \Theta_r = \emptyset.\end{aligned} \tag{6}$$

The set of feasible subset pairs associated with Θ is denoted as Θ_F, i.e.

$$\Theta_F := \{(\Phi, \Psi) : (\Phi, \Psi) \text{ is subject to } (6)\}.$$

Now we consider the variable l_{ij}, an element of L with double-subscripts, Lemma 3 can be extended to a more general case with above new notations.

Lemma 4: For any structured matrix $L \in \mathbf{R}^{m\times p}$, its induced set Θ, and the set Θ_F of feasible subset pairs associated with Θ, the following equality holds:

$$\underset{l_{ij},(i,j)\in\Theta}{\text{g.r.}}[E + BLC] = \min_{(\Phi,\Psi)\in\Theta_F}\left\{\text{rank}\begin{bmatrix}E & B_\Phi\\ C_\Psi & 0\end{bmatrix}\right\}. \tag{7}$$

Proof: Let b_i and c_j represent the ith column of B and jth row of C, then

$$\text{rank}[E + BLC] = \text{rank}\left[E + \sum_{(i,j)\in\Theta} b_i l_{ij} c_j\right].$$

It is noted that this form is the same structure as that in Lemma 3. By the definition of Θ, Θ_F and Φ, Ψ in (6), using Lemma 3, one can directly obtain the equality (7). This completes the proof. ∎

The following lemma is novel and plays a key role in proving the main result of this technical note.

Lemma 5: Let $E \in \mathbf{R}^{m\times n}$, $B_1 \in \mathbf{R}^{m\times p}$, $B_2 \in \mathbf{R}^{m\times q}$, $C_1 \in \mathbf{R}^{s\times n}$, $C_2 \in \mathbf{R}^{t\times n}$ with the following rank conditions:

$$\operatorname{rank}[E \quad B_1 \quad B_2] = r_B,\ \operatorname{rank}\begin{bmatrix}E^{\mathrm{T}} & C_1^{\mathrm{T}} & C_2^{\mathrm{T}}\end{bmatrix} = r_C.$$

If $\operatorname{rank}\begin{bmatrix}E & B_1\\ C_1 & 0\end{bmatrix} \le \min\{r_B, r_C\}$, then $\operatorname{rank}\begin{bmatrix}E & B_2\\ C_2 & 0\end{bmatrix} \ge \max\{r_B, r_C\}$.

Proof: Let $\operatorname{rank}(E) = r$. First by using elementary operations on the first m rows and first n columns of the following matrix, we can obtain

$$\operatorname{rank}\begin{bmatrix}E & B_1 & B_2\\ C_1 & 0 & 0\\ C_2 & 0 & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}I_r & 0 & B_{11} & B_{12}\\ 0 & 0 & B_{21} & B_{22}\\ C_{11} & C_{12} & 0 & 0\\ C_{21} & C_{22} & 0 & 0\end{bmatrix}. \quad (8)$$

Then,

$$\operatorname{rank}[B_{21} \quad B_{22}] = r_B - r,\ \operatorname{rank}\begin{bmatrix}C_{12}\\ C_{22}\end{bmatrix} = r_C - r. \quad (9)$$

According to (8) and given conditions, we have the following inequality:

$$\operatorname{rank}\begin{bmatrix}E & B_1\\ C_1 & 0\end{bmatrix} = r + \operatorname{rank}\begin{bmatrix}0 & B_{21}\\ C_{12} & -C_{11}B_{11}\end{bmatrix} \le \min\{r_B, r_C\}.$$

Then

$$\operatorname{rank}\begin{bmatrix}0 & B_{21}\\ C_{12} & 0\end{bmatrix} \le \operatorname{rank}\begin{bmatrix}0 & B_{21}\\ C_{12} & -C_{11}B_{11}\end{bmatrix} \le \min\{r_B, r_C\} - r. \quad (10)$$

On the other hand, we can further give the following derivation of rank inequality by using (8), (9) and (10):

$$\begin{aligned}\operatorname{rank}\begin{bmatrix}E & B_2\\ C_2 & 0\end{bmatrix} &= r + \operatorname{rank}\begin{bmatrix}0 & B_{22}\\ C_{22} & -C_{21}B_{12}\end{bmatrix}\\ &\ge r + \operatorname{rank}\begin{bmatrix}0 & B_{22}\\ C_{22} & 0\end{bmatrix}\\ &\quad + \left(r + \operatorname{rank}\begin{bmatrix}0 & B_{21}\\ C_{12} & 0\end{bmatrix}\right.\\ &\quad \left. - \min\{r_B, r_C\}\right)\\ &\ge \operatorname{rank}\begin{bmatrix}C_{12}\\ C_{22}\end{bmatrix} + \operatorname{rank}[B_{21} \quad B_{22}]\\ &\quad + 2r - \min\{r_B, r_C\}\\ &= r_B + r_C - \min\{r_B, r_C\} = \max\{r_B, r_C\}.\end{aligned}$$

This completes the proof. ■

It is reminded that Lemma 5 has its independent algebraic significance.

III. Main Results

In this section we will derive an explicit necessary and sufficient condition for the closed-loop system (3) to be regular and impulse-free in terms of the original system parameters.

Theorem 1: There exist a structured P-D feedback (2) such that the closed-loop system (3) is regular and impulse-free if and only if there exist Φ and Ψ that are subsets of $\underline{m}$ and $\underline{p}$, respectively, such that

$$\min\left\{\operatorname{rank}\begin{bmatrix}0 & E & 0 & B_\Phi\\ E & A & B & 0\\ 0 & C_\Psi & 0 & 0\end{bmatrix}, \operatorname{rank}\begin{bmatrix}0 & E & 0\\ E & A & B_\Phi\\ 0 & C & 0\\ C_\Psi & 0 & 0\end{bmatrix}\right\} = n + \operatorname{rank}\begin{bmatrix}E & B_\Phi\\ C_\Psi & 0\end{bmatrix} \quad (11)$$

and

$$n + \operatorname{rank}\begin{bmatrix}E & B_\Phi\\ C_\Psi & 0\end{bmatrix} \le \operatorname{rank}\begin{bmatrix}0 & E & 0\\ E & A & B\\ 0 & C & 0\end{bmatrix}. \quad (12)$$

Proof: For a structured matrix L and a parameterized matrix K, the closed-loop system (3) is regular and impulse-free if and only if

$$\underset{K, l_{ij}, (i,j)\in\Theta}{\text{g.r.}}\begin{bmatrix}0 & E+BLC\\ E+BLC & A+BKC\end{bmatrix} = n + \underset{l_{ij}, (i,j)\in\Theta}{\text{g.r.}}[E+BLC]. \quad (13)$$

From Lemma 1, we can directly obtain that for any L

$$\begin{aligned}\underset{K}{\text{g.r.}}\begin{bmatrix}0 & E+BLC\\ E+BLC & A+BKC\end{bmatrix} &= \min\left\{\operatorname{rank}\begin{bmatrix}0 & E+BLC & 0\\ E & A & B\end{bmatrix},\right.\\ &\qquad \left.\operatorname{rank}\begin{bmatrix}0 & E\\ E+BLC & A\\ 0 & C\end{bmatrix}\right\}.\end{aligned} \quad (14)$$

Next we will prove the necessity and the sufficiency respectively.

Necessity part: We prove that (13) implies (11) and (12). Let Θ be the induced set of the structured matrix L and Θ_F be the set of feasible subset pairs associated with Θ. Note that L is a structured matrix in (14), according to Lemma 4, there exist two feasible subset pairs $(\Phi_1, \Psi_1) \in \Theta_F$ and $(\Phi_2, \Psi_2) \in \Theta_F$ such that

$$\underset{l_{ij}, (i,j)\in\Theta}{\text{g.r.}}\begin{bmatrix}0 & E+BLC & 0\\ E & A & B\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & E & 0 & B_{\Phi_1}\\ E & A & B & 0\\ 0 & C_{\Psi_1} & 0 & 0\end{bmatrix} =: p(\Phi_1, \Psi_1). \quad (15)$$

$$\underset{l_{ij}, (i,j)\in\Theta}{\text{g.r.}}\begin{bmatrix}0 & E\\ E+BLC & A\\ 0 & C\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & E & 0\\ E & A & B_{\Phi_2}\\ 0 & C & 0\\ C_{\Psi_2} & 0 & 0\end{bmatrix} =: q(\Phi_2, \Psi_2). \quad (16)$$

Then from (13)–(16), it can be obtained that

$$\underset{K, l_{ij}, (i,j)\in\Theta}{\text{g.r.}}\begin{bmatrix}0 & E+BLC\\ E+BLC & A+BKC\end{bmatrix} = \min\{p(\Phi_1, \Psi_1),\ q(\Phi_2, \Psi_2)\}. \quad (17)$$

On the other hand, from (7), there exists a feasible subset pairs $(\Phi, \Psi) \in \Theta_F$, such that

$$\underset{l_{ij}, (i,j)\in\Theta}{\text{g.r.}}[E+BLC] = \operatorname{rank}\begin{bmatrix}E & B_\Phi\\ C_\Psi & 0\end{bmatrix}. \quad (18)$$

Then from (13), (17) and (18), we obtain

$$\min\{p(\Phi_1,\Psi_1),\ q(\Phi_2,\Psi_2)\} = n + \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix}. \tag{19}$$

But from Lemma 4, (15) and (16), $p(\Phi_1,\Psi_1)$ and $p(\Phi_2,\Psi_2)$ are minimum values in set $\{p(\Phi,\Psi),(\Phi,\Psi) \in \Theta_F\}$ and $\{q(\Phi,\Psi),(\Phi,\Psi)\in\Theta_F\}$, respectively, i.e.

$$p(\Phi_1,\Psi_1) \le p(\Phi,\Psi),\ q(\Phi_2,\Psi_2) \le q(\Phi,\Psi). \tag{20}$$

Then it can be obtained from (19) and (20) that

$$n + \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} \le \min\{p(\Phi,\Psi),\ q(\Phi,\Psi)\}. \tag{21}$$

But the left side of (21) is never less than the right side, then (21) implies that its equality holds, i.e., (11) holds.

On the other hand, from (13), (14) and (18), it can be directly derived that

$$\begin{aligned} n &+ \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} \\ &= \min\left\{ \underset{l_{ij},(i,j)\in\Theta}{\text{g.r.}} \begin{bmatrix} 0 & E+BLC & 0 \\ E & A & B \end{bmatrix}, \right. \\ &\qquad \left. \underset{l_{ij},(i,j)\in\Theta}{\text{g.r.}} \begin{bmatrix} 0 & E \\ E+BLC & A \\ 0 & C \end{bmatrix} \right\} \\ &\le \operatorname{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} \end{aligned}$$

i.e., (12) holds. The necessity is proved.

Sufficiency part: We will prove that (11) and (12) implies (13). A constructive method is presented for proving this part. Logically we need to discuss all the possible choice of feasible subset pairs (Φ,Ψ).

Case 1: If there exist two nonempty proper subsets Φ of $\underline{m}$, and Ψ of $\underline{p}$, such that (11) and (12) hold. Let

$$BLC = B_\Phi L_1 C_{\overline{\Psi}} + B_{\overline{\Phi}} L_2 C_\Psi \tag{22}$$

where L_1 and L_2 are parameterized matrices. Then $\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} =: \operatorname{diag}\{L_1, L_2\}$ is a structured matrix. Apparently, $[B_\Phi\ B_{\overline{\Phi}}]$ and $\left[\begin{smallmatrix} C_{\overline{\Psi}} \\ C_\Psi \end{smallmatrix}\right]$ can be obtained by permuting the columns of B and the rows of C, i.e., there exist permutation matrices U and V, such that

$$[B_\Phi \quad B_{\overline{\Phi}}] = BU,\ \begin{bmatrix} C_{\overline{\Psi}} \\ C_\Psi \end{bmatrix} = VC.$$

Corresponding to (22),

$$L = U\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} V \tag{23}$$

is also a structured matrix since it is obtained via only permutation operations on another structured matrix.

Now we prove (13) from the structured matrix L in (23). For simplicity, let

$$\begin{aligned} p_1 &:= p(\Phi,\Psi),\ q_1 := q(\Phi,\Psi), \\ p_2 &:= \operatorname{rank}\begin{bmatrix} 0 & E & 0 & B \\ E & A & B & 0 \end{bmatrix}, p_3 := \operatorname{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix}, \\ p_4 &:= p(\overline{\Phi},\overline{\Psi}). \end{aligned}$$

We first prove that $p_1 \le p_i$, $i = 2, 3, 4$. From (11) and (12), it can be directly obtained that $p_1 \le p_3$, and furthermore, from (12)

$$\operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} \le \min\left\{\operatorname{rank}[E \quad B],\ \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix}\right\}. \tag{24}$$

Let

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} &= \operatorname{rank}[E \quad B_\Phi] + r_1 \\ \operatorname{rank}[E \quad B] &= \operatorname{rank}[E \quad B_\Phi \quad B_{\overline{\Phi}}] \\ &= \operatorname{rank}[E \quad B_\Phi] + r_2. \end{aligned}$$

From (24), $r_1 \le r_2$, with some algebraic manipulations, one can derive the following inequalities:

$$\begin{aligned} p_1 &\le \operatorname{rank}\begin{bmatrix} 0 & E & 0 & B_\Phi \\ E & A & B & 0 \end{bmatrix} + r_1 \\ &\le \operatorname{rank}\begin{bmatrix} 0 & E & 0 & B_\Phi \\ E & A & B & 0 \end{bmatrix} + r_2 \\ &\le \operatorname{rank}\begin{bmatrix} 0 & E & 0 & B_\Phi & B_{\overline{\Phi}} \\ E & A & B & 0 & 0 \end{bmatrix} = p_2. \end{aligned}$$

Therefore $p_1 \le \min\{p_2, p_3\}$. By using Lemma 5, one can obtain that $\max\{p_2,p_3\} \le p_4$. With Lemma 2 and (22), one can derive the following equality:

$$\underset{L_1,L_2}{\text{g.r.}} \begin{bmatrix} 0 & E+BLC & 0 \\ E & A & B \end{bmatrix} = \min\{p_1,p_2,p_3,p_4\} = p_1. \tag{25}$$

Similarly

$$\underset{L_1,L_2}{\text{g.r.}} \begin{bmatrix} 0 & E \\ E+BLC & A \\ 0 & C \end{bmatrix} = q_1. \tag{26}$$

Then from (14), (25) and (26), we can conclude that

$$\underset{K,L_1,L_2}{\text{g.r.}} \begin{bmatrix} 0 & E+BLC \\ E+BLC & A+BKC \end{bmatrix} = \min\{p_1,q_1\}. \tag{27}$$

On the other hand, from (24) and Lemma 5, one can see that

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} &\le \min\left\{\operatorname{rank}[E \quad B], \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix}\right\} \\ &\le \operatorname{rank}\begin{bmatrix} E & B_{\overline{\Phi}} \\ C_{\overline{\Psi}} & 0 \end{bmatrix}. \end{aligned} \tag{28}$$

Based on (28) and Lemma 2

$$\begin{aligned} \underset{L_1,L_2}{\text{g.r.}} [E+BLC] &= \underset{L_1,L_2}{\text{g.r.}} [E + B_\Phi L_1 C_{\overline{\Psi}} + B_{\overline{\Phi}} L_2 C_\Psi] \\ &= \operatorname{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix}. \end{aligned} \tag{29}$$

Finally combining (11), (27) and (29), we can derive that (13) holds.

Case 2: If $B_\Phi = B$, then from (11) and (12), one can see that C_Ψ does not play any role in (11) and (12), so we can let $\Psi = \emptyset$. In this case, L turns to be a parameterized matrix, i.e., $BLC = B_\Phi L C_{\overline{\Psi}}$. Then (11) and (12) can be equivalent to the following equalities:

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} 0 & E & 0 & B \\ E & A & B & 0 \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} \\ &= n + \operatorname{rank}[E \quad B]. \end{aligned} \tag{30}$$

Now we need to prove that (30) implies (13) from L being a parameterized matrix. From (30), it can be directly obtained that

$$\operatorname{rank}[E \quad B] \le \operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix}. \tag{31}$$

Let

$$\mathrm{rank}\,[E \quad B] = \mathrm{rank}(E) + h_1,$$
$$\mathrm{rank}[E^{\mathrm{T}} \quad C^{\mathrm{T}}] = \mathrm{rank}(E) + h_2. \tag{32}$$

Then $h_1 \leq h_2$ and moreover, by some algebraic manipulations, we have

$$\mathrm{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} \leq \mathrm{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \end{bmatrix} + h_1 \tag{33}$$

and

$$\mathrm{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \\ C & 0 \end{bmatrix} \geq \mathrm{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \end{bmatrix} + h_2. \tag{34}$$

Comparing (33) with (34), and $h_1 \leq h_2$, we obtain that

$$\mathrm{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} \leq \mathrm{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \\ C & 0 \end{bmatrix}. \tag{35}$$

So from (14), (30), (31) and (35), the following two equalities hold:

$$\underset{K,L}{\mathrm{g.r.}}\begin{bmatrix} 0 & E+BLC \\ E+BLC & A+BKC \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix},$$
$$\underset{L}{\mathrm{g.r.}}[E+BLC] = \mathrm{rank}\,[E \quad B].$$

This implies that (13) holds.

Case 3: If $C_\Psi = C$, then we can prove the sufficiency part of the theorem in the same way as Case 2. Meanwhile, (11) and (12) are equivalent to the following equalities:

$$\mathrm{rank}\begin{bmatrix} 0 & E \\ E & A \\ 0 & C \\ C & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} = n + \mathrm{rank}\begin{bmatrix} E \\ C \end{bmatrix}. \tag{36}$$

Case 4: If Φ (or Ψ) is a nonempty proper subsets of $\underline{m}$ (or $\underline{p}$), and $\Psi = \emptyset$ (or $\Phi = \emptyset$), such that (11) and (12) hold. Then we can derive $BLC = B_\Phi L_1 C$ (or $BLC = BL_2 C_\Psi$), where L_1 (or L_2) is a parameterized matrix. The remaining part of the proof is the same as in the Case 2.

Concluding all these cases, the theorem has been proved completely. ■

Remark 1: The proof of sufficiency part in Theorem 1 provides a constructive approach for obtaining the structured matrix L if B_Φ and C_Ψ have been found satisfying (11) and (12) [see from (22) to (23)].

Remark 2: (11) and (12) contain all possible cases that the closed-loop system is regular and impulse-free when the nonzero parameters of the structured P-D feedback are permitted with perturbation. Two special cases are the P-feedback ($\Phi = \emptyset$ and $\Psi = \emptyset$) and the parameterized P-D feedback ($\Phi = \underline{m}$ and $\Psi = \emptyset$, or $\Phi = \emptyset$ and $\Psi = \underline{p}$). For the first case, (11) implies that the system (1) is impulse-controllable and impulse-observable, this is consistent with the known results. For the second case, (11) is equivalent to (30) or (36), and it is a more detailed description than the Corollary 3.1 in [4].

If we consider designing a structured P-D feedback controller (2) such that the closed-loop system (3) is regular and impulse-free and has a given dynamical order, this is so-called dynamical order assignment problem. We have the following corollary to solve the problem.

Corollary 1: There exist a structured P-D feedback (2) such that the closed-loop system (3) is regular and impulse-free for a given dynamical order l if and only if there exist Φ and Ψ that are subsets of $\underline{m}$ and $\underline{p}$, respectively, such that (11) and (12) hold, and $\mathrm{rank}\begin{bmatrix} E & B_\Phi \\ C_\Psi & 0 \end{bmatrix} = l$. Furthermore, l must be in the range of

$$\mathrm{rank}(E) \leq l \leq \mathrm{rank}\begin{bmatrix} 0 & E & 0 \\ E & A & B \\ 0 & C & 0 \end{bmatrix} - n. \tag{37}$$

Remark 3: We can give an intuitive way to check (11) and (12). Under the rank constraint (12) or (37), we can check (11) for a given dynamical order l or taking l from small value to large value as a checking prioritization, and this way will reduce the computation. A fast algorithm for checking (11) and (12) still needs further investigation.

IV. Illustrative Example

In this section we will consider an illustrative example to show the effectiveness of the structured P-D feedback control proposed in this technical note. Let the descriptor system parameters be

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & * & * \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} * & * & 0 & 0 & * & 0 \\ * & * & 0 & 0 & * & 0 \\ * & * & * & 0 & * & * \\ * & * & * & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Here "$*$" represents any entries. It is easy to verify that the impulsive modes of the system can not be eliminated by P-feedback and the parameterized P-D feedback by checking (4). In order to find a structured matrix L such that the closed-loop system is regular and impulse-free, we need to test the conditions (11) and (12) in Theorem 1. According to (37), we only need to verify the dynamic order with $l = 4$. By verification, one can find that the unique solution of (11) and (12) is $(B_\Phi, C_\Psi) = (\emptyset, c_1)$. This corresponds to Case 4 in the proof of sufficiency part of Theorem 1. Then the corresponding structured matrix L can be selected as

$$L = \begin{bmatrix} l_{11} & 0 \\ 0 & 0 \end{bmatrix} \text{ or } L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & 0 \end{bmatrix},\ (l_{11} \neq 0, l_{21} \neq 0).$$

We can directly verify the two structured matrices Ls satisfying (4). This shows that Theorem 1 is correct and effective.

V. Conclusion

In this technical note we have investigated the problem of impulsive modes elimination for descriptor systems via a structured P-D feedback. An explicit necessary and sufficient condition has been derived with the original system parameters. The main result shows that the conditions contain all possible cases that the closed-loop system is regular and impulse-free when nonzero terms of the feedback gain matrix are permitted with perturbation. This accords with some physical system requirements. So this type of P-D feedback controller is a more practical than the parameterized controllers. The presentation of the main result using the original system parameters shows the possibility of a new approach to investigate different problems for descriptor systems since many previous results for descriptor systems are based on decomposed subsystem parameters or subspace constraints. Moreover, the techniques used in this technical note are algebraically graceful and some results have their own independent interests, which may have applications in other control problems.

ACKNOWLEDGMENT

The authors would like to thank the reviewers and the Associate Editor for their constructive comments and suggestions.

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Infinite-Dimensional Negative Imaginary Systems

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***Abstract*—We consider second order infinite-dimensional systems with force control and collocated position measurement interconnected with finite-dimensional controllers of the same type. We show that under assumptions that generalize those in the finite-dimensional case (the theory of negative imaginary systems), asymptotic stability of the closed-loop system can be concluded, but that the closed-loop system may not be exponentially stable nor input-output stable.**

***Index Terms*—Infinite-dimensional systems, linear systems, negative imaginary systems, stability.**

I. INTRODUCTION

Flexible systems with force control and collocated velocity measurement lead to positive real transfer functions. This class of systems is very well studied. The case of collocated position measurement has received far less attention. An important recent contribution to the latter situation was the introduction of the concept of a negative imaginary transfer function [1]–[3]. Many flexible systems (such as beams, strings and plates) are best described by partial differential equations and are therefore infinite-dimensional systems. These are not covered by the current theory on negative imaginary functions. In this article we investigate in how far the existing theory on negative imaginary functions generalizes to infinite-dimensional systems. The for applications most relevant case is that of second order systems with force control and collocated position measurement interconnected with finite-dimensional controllers of the same form, and we restrict ourselves to this case. We make extensive use of existing theory of second order infinite-dimensional systems, see, e.g., [4] for an extensive discussion and bibliography of such systems.

Manuscript received October 26, 2010; revised February 11, 2011; accepted July 08, 2011. Date of publication July 29, 2011; date of current version December 07, 2011. Recommended by Associate Editor K. Morris.

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Digital Object Identifier 10.1109/TAC.2011.2162886

The paradigmatic example in this article is the following wave equation with boundary force control and collocated position measurement (see, e.g., [5, Chapter 9] and see [6] for the similar situation involving a beam equation)

$$\begin{aligned} &w_{tt}(x,t) = w_{xx}(x,t),\ w_x(0,t) = u(t),\ w(1,t) = 0, \\ &\quad y(t) = -w(0,t). \end{aligned}$$

Here $w(x,t)$ denotes the displacement at position $x \in [0,1]$ and time $t \geq 0$. This can be written in an abstract second order operator-theoretic form as (the details are in Section IV)

$$\ddot{w}_p(t) + K_p w_p(t) = G_p u(t),\ y(t) = G_p^* w_p(t) \tag{1}$$

where the state space is infinite-dimensional. Systems of this form, but with a finite-dimensional state space, are shown in the theory of negative imaginary systems to be stabilized by a finite-dimensional controller of the form

$$\ddot{w}_c(t) + D_c \dot{w}_c(t) + K_c w_c(t) = G_c y(t),\ u(t) = G_c^* w_c(t)$$

under some assumptions on the controller parameters [7, Theorem 1], [8]. Stabilization means here that the closed-loop A-matrix which appears in the first order form of the closed-loop system is Hurwitz. We show that -under essentially the same conditions as in the finite-dimensional case- in the infinite-dimensional case the closed-loop system is asymptotically stable, but not necessarily exponentially stable (the eigenvalues are in the left half-plane, but can converge to the imaginary axis). If we add an input disturbance to the plant (i.e., replace u in (1) by $u+d$) and consider the transfer function from this input disturbance to the output y of the plant, then contrary to the finite-dimensional case, this transfer function is generally not stable.

II. ABSTRACT SECOND ORDER SYSTEMS

We review the set-up of abstract second order infinite-dimensional systems in the special case of bounded damping operators (a more general case can be found in, e.g., [4]). How the above wave equation fits into this framework is explained in Section IV.

The second order equation that we consider is

$$\ddot{w}(t) + D\dot{w}(t) + Kw(t) = Gu(t),\ y(t) = Hw(t). \tag{2}$$

The stiffness operator K is assumed to be a densely defined nonnegative self-adjoint operator on the Hilbert space $\mathscr{H}$ with a bounded inverse. We denote the domain of K by $\mathscr{H}_1$ and that of its (nonnegative self-adjoint) square root by $\mathscr{H}_{1/2}$, we equip these spaces with their graph norm, thus making them into Hilbert spaces. The space $\mathscr{H}_{-1/2}$ is defined as the completion of $\mathscr{H}$ under the norm $\|x\|_{-1/2} := \|(I + K^{1/2})^{-1}x\|_{\mathscr{H}}$, which makes it into a Hilbert space. The operator K extends to a bounded operator from $\mathscr{H}_{1/2}$ to $\mathscr{H}_{-1/2}$.

We assume that the damping operator D is a bounded operator on $\mathscr{H}$. The second order control operator G is assumed to be a bounded