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Existence and stability results for nonlinear fractional order Riemann–Liouville Volterra–Stieltjes quadratic integral equations ☆

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ABSTRACT

Our aim in this paper is to study the existence and the stability of solutions for Riemann–Liouville Volterra–Stieltjes quadratic integral equations of fractional order. Our results are obtained by using some fixed point theorems. Some examples are provided to illustrate the main results.

1. Introduction

Fractional differential and integral equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering and other applied sciences. There has been a significant development in ordinary and partial fractional differential and integral equations in recent years; see the monographs of Abbas et al. [5], Baleanu et al. [7], Diethelm [15], Hilfer [17], Kilbas et al. [18], Lakshmikantham et al. [19], Podlubny [20] and Tarasov [28], and the papers by Abbas et al. [1–3,6], Qian et al. [21–23], Vityuk and Golushkov [29]. Recently interesting results of the stability of the solutions of various classes of integral equations of fractional order have obtained by Abbas et al. [4,5], Banaś et al. [8–10], Darwish et al. [12], Dhage [13,14] and the references therein.

In [8,10], Banaś et al. proved some existence results for the following nonlinear Volterra–Stieltjes quadratic integral equation

$$x(t) = a(t) + f(t, x(t)) \int_0^t u(t, \tau, x(\tau)) d_\tau g(t, \tau); \quad t \geqslant 0, \tag{1}$$

where $a : [0, \infty) \to [0, \infty)$ and $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

In this paper we establish some sufficient conditions for the existence of solutions of the following fractional order Riemann–Liouville Volterra–Stieltjes quadratic integral equations of the form

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$$u(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(t,x,s,u(s,x)) d_s g(t,s), \tag{2}$$

for $(t,x) \in J := [0,a] \times [0,b], a,b > 0,\ r \in (0,\infty),\ \mu : J \to \mathbb{R},\ g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ f : J \times \mathbb{R} \to \mathbb{R},\ h : J_1 \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $\mathbb{R}_+ = [0,\infty),\ J_1 = \{(t,x,s) \in J \times [0,a] : s \leqslant t\}$, and $\Gamma(\cdot)$ is the Euler's Gamma function.

We present two results for the existence of solutions of the Eq. (2). The first one is based on Schauder's fixed point theorem (Theorem 3.2) and the second one on the nonlinear alternative of Leray–Schauder type (Theorem 3.5).

Next, we establish a sufficient condition for the existence and the stability of solutions of the following fractional order Riemann–Liouville Volterra–Stieltjes quadratic integral equations of the form

$$u(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(t,x,s,u(s,x)) d_s g(t,s), \tag{3}$$

where $(t,x) \in J' := \mathbb{R}_+ \times [0,b],\ b > 0,\ r \in (0,\infty),\ \mu : J' \to \mathbb{R},\ g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R},\ f : J' \times \mathbb{R} \to \mathbb{R},\ h : J'_1 \times \mathbb{R} \to \mathbb{R}$ are given continuous functions and $J'_1 = \{(t,x,s) \in J' \times \mathbb{R}_+ : s \leqslant t\}$. We use the Schauder fixed point theorem for the existence of solutions of the Eq. (3), and we prove that all solutions are locally asymptotically stable (Theorem 4.3).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1(J)$, we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}$ with the norm

$$\|u\|_1 = \int_0^a \int_0^b |u(t,x)| dx dt.$$

Let $\mathscr{C} := \mathscr{C}(J)$ be the Banach space of all continuous functions from J into $\mathbb{R}$ endowed with the norm

$$\|u\|_{\mathscr{C}} = \sup_{(t,x)\in J} |u(t,x)|.$$

By $BC := BC(J')$ we denote the Banach space of all bounded and continuous functions from J' into $\mathbb{R}$ equipped with the standard norm

$$\|u\|_{BC} = \sup_{(t,x)\in J'} |u(t,x)|.$$

For $u_0 \in \mathscr{C}$ (or $u_0 \in BC$) and $\eta \in (0,\infty)$, we denote by $B(u_0,\eta)$, the closed ball in $\mathscr{C}$ (or BC) centered at u_0 with radius η.

Definition 2.1 [27]. Let $r \in (0,\infty)$ and $u \in L^1(J)$. The partial Riemann–Liouville integral of order r of $u(t,x)$ with respect to t is defined by the expression

$$I^r_{0,t} u(t,x) = \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} u(s,x) ds; \quad \text{for almost all } (t,x) \in J.$$

Analogously, we define the integral

$$I^r_{0,x} u(t,x) = \frac{1}{\Gamma(r)} \int_0^x (x-s)^{r-1} u(t,s) ds; \quad \text{for almost all } (t,x) \in J.$$

Example 2.2. Let $\lambda, \omega \in (-1,\infty)$ and $r \in (0,\infty)$, then

$$I^r_{0,t} t^\lambda x^\omega = \frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+r)} t^{\lambda+r} x^\omega; \quad \text{for almost all } (t,x) \in J.$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w : [a,b] \times [c,b] \to \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t \to w(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [24].

If u and φ are two real functions defined on the interval $[a,b]$, then under some conditions (see [24]) we can define the Stieltjes integral (in the Riemann–Stieltjes sense)

$$\int_a^b u(t) d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a,b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [24,26]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a,b]$.

In what follows we will use a few properties of the Stieltjes integral contained in the below given lemma ([25], Section 8.13):

Lemma 2.3 [25]. *If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation then*

$$\left|\int_a^b u(t)d\varphi(t)\right| \leqslant \int_a^b |u(t)|d\left(\bigvee_a^t \varphi\right).$$

In the sequel we will also consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t,s),$$

and Riemann–Liouville Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s)d_s g(t,s),$$

where $g:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R},\ r\in(0,\infty)$ and the symbol d_s indicates the integration with respect to s.

Let $\emptyset \neq \Omega \subset BC$, and let $G:\Omega\to\Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{4}$$

Inspired by the definition of the attractivity of solutions of integral equations (for instance [9]), we introduce the following concept of attractivity of solutions for Eq. (4).

Definition 2.4. Solutions of Eq. (4) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space BC such that, for arbitrary solutions $v=v(t,x)$ and $w=w(t,x)$ of Eq. (4) belonging to $B(u_0,\eta)\cap\Omega$, we have that, for each $x\in[0,b]$,

$$\lim_{t\to\infty}(v(t,x)-w(t,x)) = 0. \tag{5}$$

When the limit (5) is uniform with respect to $B(u_0,\eta)\cap\Omega$, solutions of Eq. (4) are said to be uniformly locally attractive (or equivalently that solutions of (4) are locally asymptotically stable).

Lemma 2.5 [11]. *Let $D\subset BC$. Then D is relatively compact in BC if the following conditions hold:*

(a) D is uniformly bounded in BC,
(b) The functions belonging to D are almost equicontinuous on $\mathbb{R}_+\times[0,b]$, i.e. equicontinuous on every compact of $\mathbb{R}_+\times[0,b]$,
(c) The functions from D are equiconvergent, that is, given $\epsilon>0,\ x\in[0,b]$ there corresponds $T(\epsilon,x)>0$ such that $|u(t,x)-\lim_{t\to\infty}u(t,x)|<\epsilon$ for any $t\geqslant T(\epsilon,x)$ and $u\in D$.

3. Local existence results

Let us start in this section by defining what we mean by a solution of the Eq. (2).

Definition 3.1. We mean by a solution of Eq. (2), every function $u\in\mathscr{C}$ such that u satisfies Eq. (2) on J.

The following hypotheses will be used in the sequel:

(H_1) The function f is a continuous and there exists a positive function $p\in\mathscr{C}$ such that

$$|f(t,x,u)-f(t,x,v)| \leqslant p(t,x)|u-v|;\quad (t,x)\in J,\quad u,v\in\mathbb{R}.$$

(H_2) For all $t_1,t_2\in[0,a]$ such that $t_1<t_2$ the function $s\mapsto g(t_2,s)-g(t_1,s)$ is nondecreasing on $[0,a]$.
(H_3) The function $s\mapsto g(0,s)$ is nondecreasing on $[0,a]0$.
(H_4) The functions $s\mapsto g(t,s)$ and $t\mapsto g(t,s)$ are continuous on $[0,a]$ for each fixed $t\in[0,a]$ or $s\in[0,a]$, respectively.
(H_5) The function h is continuous and there exist continuous functions $q=q(t,x,s):J_1\to\mathbb{R}_+;\ \Phi:\mathbb{R}_+\to\mathbb{R}_+$ such that Φ is nondecreasing and

$$|h(t,x,s,u)| \leqslant q(t,x,s)\Phi(|u|);\quad (t,x,s)\in J_1,\quad u\in\mathbb{R}.$$

Set

$$\mu^* := \sup_{(t,x)\in J}\mu(t,x),\quad f^* := \sup_{(t,x)\in J} f(t,x,0),\quad g^* = \sup_{t\in[0,a]}\bigvee_{k=0}^{t} g(t,k),$$

$$p^* := \sup_{(t,x)\in J} p(t,x) \quad \text{and} \quad q^* := \sup_{(t,x,s)\in J_1} \frac{(t-s)^{r-1}q(t,x,s)}{\Gamma(r)}.$$

Now, we shall prove the following theorem concerning the existence of a solution of the Eq. (2), based on Schauder's fixed point theorem [16].

Theorem 3.2. *Assume that the hypotheses* $(H_1)-(H_5)$ *and the following hypothesis hold*

(H_6) *There exists a constant* $\eta > 0$, *such that* $\mu^* + q^*g^*\Phi(\eta)(f^* + p^*\eta) \leqslant \eta$.

Then the Eq. (2) has at least one solution in the space $\mathscr{C}$.

Proof 3.3. Let us define the operator $N : \mathscr{C} \to \mathscr{C}$ such that, for any $u \in \mathscr{C}$,

$$(Nu)(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(t,x,s,u(s,x)) d_s g(t,s); \quad (t,x) \in J. \tag{6}$$

It is clear that the operator N maps $\mathscr{C}$ into $\mathscr{C}$. The problem of finding the solutions of the Eq. (2) is reduced to finding the solutions of the operator equation $N(u) = u$. Hypothesis (H_6) implies that N transforms the ball $B_\eta := B(0,\eta)$ into itself. Indeed, for any $u \in B_\eta$, and for each $(t,x) \in J$ we have

$$\begin{aligned}
|(Nu)(t,x)| &\leqslant |\mu(t,x)| + \frac{|f(t,x,0)|}{\Gamma(r)} \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))| |d_s g(t,s)| + \frac{|f(t,x,u(t,x)) - f(t,x,0)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))| d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant \mu^* + \frac{f^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s) \Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) + \frac{p(t,x)|u(t,x)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} q(t,x,s) \Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant \mu^* + q^* g^* \Phi(\|u\|_{\mathscr{C}})(f^* + p^*\|u\|_{\mathscr{C}}) \leqslant \mu^* + q^* g^* \Phi(\eta)(f^* + p^*\eta) \leqslant \eta.
\end{aligned}$$

Hence, $N(u) \in B_\eta$. We shall show that $N : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [16]. The proof will be given in several steps.

Step 1: *N is continuous.*

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in B_η. Then, for each $(t,x) \in J$, we have

$$\begin{aligned}
|(Nu_n)(t,x) - (Nu)(t,x)| &\leqslant |f(t,x,u_n(t,x)) - f(t,x,u(t,x.))| \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} |h(t,x,s,u_n(s,x))| d_s g(t,s)| \\
&\quad + |f(t,x,u(t,x))| \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} |h(t,x,s,u_n(s,x)) - h(t,x,s,u(s,x))| |d_s g(t,s)| \\
&\leqslant p^* q^* g^* \Phi(\|u_n\|_{\mathscr{C}}) |u_n(t,x) - u(t,x)| + \frac{f^* + p^*\|u\|_{\mathscr{C}}}{\Gamma(r)} \int_0^t (t-s)^{r-1} |h(t,x,s,u_n(s,x)) \\
&\quad - h(t,x,s,u(s,x))| d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant p^* q^* g^* \Phi(\eta) |u_n(t,x) - u(t,x)| + \frac{f^* + p^*\eta}{\Gamma(r)} \int_0^t (t-s)^{r-1} |h(t,x,s,u_n(s,x)) \\
&\quad - h(t,x,s,u(s,x))| d_s \left(\bigvee_{k=0}^{s} g(t,k) \right).
\end{aligned} \tag{7}$$

Since $u_n \to u$ as $n \to \infty$ and h is continuous, (7) gives

$\|N(u_n) - N(u)\|_{\mathscr{C}} \to 0 \quad as\ n \to \infty.$

Step 2: $N(B_\eta)$ *is bounded.*

This is clear since $N(B_\eta) \subset B_\eta$ and B_η is bounded.

Step 3: $N(B_\eta)$ *is equicontinuous.*

Let $(t_1,x_1), (t_2,x_2) \in J,\ t_1 < t_2,\ x_1 < x_2$ and let $u \in B_\eta$. Thus we have

$$|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \leqslant |\mu(t_2,x_2)-\mu(t_1,x_1)| + |f(t_2,x_2,u(t_2,x_2))-f(t_1,x_1,u(t_1,x_1))|$$
$$\times \int_0^{t_2} \frac{(t_2-s)^{r-1}}{\Gamma(r)} |h(t_2,x_2,s,u(s,x_2))| d_s\left(\bigvee_{k=0}^{s} g(t_2,k)\right)$$
$$+ |f(t_1,x_1,u(t_1,x_1))| \left| \int_0^{t_2} \frac{(t_2-s)^{r-1}}{\Gamma(r)} h(t_2,x_2,s,u(s,x_2)) d_s g(t_2,s) - \int_0^{t_1} \frac{(t_1-s)^{r-1}}{\Gamma(r)} h(t_1,x_1,s,u(s,x_1)) d_s g(t_1,s) \right|.$$

Thus

$$|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \leqslant |\mu(t_2,x_2)-\mu(t_1,x_1)| + q^*g^*\Phi(\eta)|f(t_2,x_2,u(t_2,x_2))-f(t_1,x_1,u(t_1,x_1))|$$
$$+ p^*\eta \left| \int_0^{t_2} \frac{(t_2-s)^{r-1}}{\Gamma(r)} h(t_2,x_2,s,u(s,x_2)) d_s g(t_2,s) - \int_0^{t_1} \frac{(t_1-s)^{r-1}}{\Gamma(r)} h(t_1,x_1,s,u(s,x_1)) d_s g(t_1,s) \right|.$$

Hence

$$|(Nu)(t_2,x_2)-(Nu)(t_1,x_1)| \leqslant |\mu(t_2,x_2)-\mu(t_1,x_1)| + p^*q^*g^*\Phi(\eta)|u(t_2,x_2)-u(t_1,x_1)| + \frac{p^*\eta}{\Gamma(r)}$$
$$\times \int_0^{t_1} \left|(t_2-s)^{r-1} h(t_2,x_2,s,u(s,x_2))(t_1-s)^{r-1} h(t_1,x_1,s,u(s,x_1))\right| d_s\left(\bigvee_{k=0}^{s} g(t_1,k)\right) + \frac{p^*\eta}{\Gamma(r)}$$
$$\times \int_0^{t_1} |(t_2-s)^{r-1} h(t_2,x_2,s,u(s,x_2))||d_s(g(t_2,s)-g(t_1,s))| + p^*q^*\eta\Phi(\eta)\bigvee_{k=t_1}^{t_2} g(t_2,k).$$

From continuity of μ, g, h and as $t_1 \to t_2$ and $x_1 \to x_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $N : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [16], we deduce that N has a fixed point u which is a solution of the Eq. (2). $\square$

In the sequel, we need the following theorem.

Theorem 3.4 ([16] Nonlinear alternative of Leray–Schauder type). *By $\overline{U}$ and ∂U we denote the closure of U and the boundary of U respectively. Let X be a Banach space and C a nonempty convex subset of X. Let U be a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to C$ completely continuous operator. Then either*

(a) T has fixed points. Or
(b) There exist $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda T(u)$.

Now, we present another result based on the nonlinear alternative of Leray–Schauder type (Theorem 3.4).

Theorem 3.5. *Assume that the hypotheses $(H_1) - (H_4)$ and the following hypothesis holds*

(H_7) *The function h is continuous and there exist continuous functions $k_1, k_2 : J_1 \to \mathbb{R}_+$ such that*

$$|h(t,x,s,u)| \leqslant k_1(t,x,s) + \frac{k_2(t,x,s)}{1+|u|}; \quad (t,x,s) \in J_1, \quad u \in \mathbb{R},$$

with

$$k_i^* := \sup_{(t,x,s)\in J_1} \frac{1}{\Gamma(r)} (t-s)^{r-1} k_i(t,x,s); \quad i = 1,2.$$

If

$$p^*g^*k_1^* < 1, \tag{8}$$

then the Eq. (2) has at least one solution in the space $\mathscr{C}$.

Proof 3.6. We shall show that the operator N defined in (6) satisfies all the conditions of Theorem 3.4. As in Theorem 3.2, we can show that N is completely continuous.

A priori bounds.

We shall show there exists an open set $U \subseteq \mathscr{C}$ with $u \neq \lambda N(u)$, for $\lambda \in (0,1)$ and $u \in \partial U$. Let $u \in \mathscr{C}$ be such that $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus for each $(x,y) \in J$, we have

$$
\begin{aligned}
|u(t,x)| &\leqslant |\lambda\mu(t,x)| + \frac{|\lambda f(t,x,0)|}{\Gamma(r)}\int_0^t (t-s)^{r-1}|h(t,x,s,u(s,x))||d_s g(t,s)| \\
&\quad + \frac{|\lambda||f(t,x,u(t,x)) - f(t,x,0)|}{\Gamma(r)}\int_0^t (t-s)^{r-1}|h(t,x,s,u(s,x))|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leqslant \mu^* + \frac{f^*}{\Gamma(r)}\int_0^t (t-s)^{r-1}\left(k_1(t,x,s) + \frac{k_2(t,x,s)}{1+|u(s,x)|}\right)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\quad + \frac{p(t,x)|u(t,x)|}{\Gamma(r)}\int_0^t (t-s)^{r-1}\left(k_1(t,x,s) + \frac{k_2(t,x,s)}{1+|u(s,x)|}\right)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \leqslant \mu^* + f^*g^*(k_1^* + k_2^*) + p^*g^*k_2^* + p^*g^*k_1^*\|u\|_{\mathscr{C}}.
\end{aligned}
$$

Then,

$$\|u\|_{\mathscr{C}} \leqslant \mu^* + f^*g^*(k_1^* + k_2^*) + p^*g^*k_2^* + p^*g^*k_1^*\|u\|_{\mathscr{C}}.$$

Thus, by (8) we get

$$\|u\|_{\mathscr{C}} \leqslant \frac{\mu^* + f^*g^*(k_1^* + k_2^*) + p^*g^*k_2^*}{1 - p^*g^*k_1^*} := M.$$

Set

$$U = \{u \in \mathscr{C} : \|u\|_{\mathscr{C}} < M + 1\}.$$

By our choice of U, there is no $u \in \partial U$ such that $u = \lambda N(u)$, for $\lambda \in (0,1)$.

From Theorem 3.4, we deduce that N has a fixed point u in $\overline{U}$ which is a solution to the Eq. (2). □

4. Global existence and stability of solutions

Definition 4.1. We mean by a solution of Eq. (3), every function $u \in BC$ such that u satisfies Eq. (3) on J'.

The following hypotheses will be used in the sequel:

(H_1') The function μ is in BC. Moreover, assume that $\lim_{t\to\infty}\mu(t,x) = 0;\ x \in [0,b]$.

(H_2') The function f is a continuous and there exists a positive function $p \in BC$ such that

$$|f(t,x,u) - f(t,x,v)| \leqslant p(t,x)|u - v|; \quad (t,x) \in J, \quad u,v \in \mathbb{R}.$$

Moreover, assume that $\lim_{t\to\infty} f(t,x,0) = 0;\ x \in [0,b]$.

(H_3') For all $t_1, t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \mapsto g(t_2,s) - g(t_1,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_4') The function $s \mapsto g(0,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_5') The functions $s \mapsto g(t,s)$ and $t \mapsto g(t,s)$ are continuous on $\mathbb{R}_+$ for each fixed $t \in \mathbb{R}_+$ or $s \in \mathbb{R}_+$, respectively.

(H_6') The function h is continuous and there exist continuous functions $q : J_1' \to \mathbb{R}_+,\ \Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ is nondecreasing and

$$|h(t,x,s,u)| \leqslant q(t,x,s)\Phi(|u|); \quad (t,x,s) \in J_1', \quad u \in \mathbb{R}.$$

Moreover, assume that $\lim_{t\to\infty} \int_0^t (t-s)^{r-1} q(t,x,s) d_s g(t,s) = 0$.

Remark 4.2. Set

$$\mu^* := \sup_{(t,x)\in J'} \mu(t,x), \quad f^* := \sup_{(t,x)\in J'} f(t,x,0),$$

$$p^* := \sup_{(t,x)\in J'} p(t,x) \quad \textit{and} \quad q^* := \sup_{(t,x)\in J'} \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1} q(t,x,s) d_s\left(\bigvee_{k=0}^{s} g(t,k)\right).$$

From hypotheses, we infer that $\mu^*,\ f^*,\ p^*$ and q^* are finite.

Now, we shall prove the following theorem concerning the existence and the stability of a solution of the Eq. (3).

Theorem 4.3. *Assume that the hypotheses* $(H_1') - (H_6')$ *and the following hypothesis hold*

(H_7') *There exists a constant* $\eta > 0$, *such that* $\mu^* + q^*\Phi(\eta)(f^* + p^*\eta) \leqslant \eta$.

Then the Eq. (3) has at least one solution in the space BC. Moreover, if there exists a constant $\eta^* > 0$, such that

$$q^*(f^* + p^*\eta)\Phi(\eta) + [q^*f^* + p^*q^*(\eta + \eta^*)]\Phi(\eta + \eta^*) \leqslant \eta^*,$$

then solutions of the Eq. (3) are locally asymptotically stable.

Proof 4.4. Let us define the operator N' such that, for any $u \in BC$,

$$(N'u)(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(t,x,s,u(s,x)) d_s g(t,s); \quad (t,x) \in J'. \tag{9}$$

The operator N' maps BC into BC. Indeed the map $N'(u)$ is continuous on J' for any $u \in BC$, and for each $(t,x) \in J'$ we have

$$\begin{aligned}
|(N'u)(t,x)| &\leqslant |\mu(t,x)| + \frac{|f(t,x,0)|}{\Gamma(r)} \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))||d_s g(t,s)| + \frac{|f(t,x,u(t,x)) - f(t,x,0)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))| d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant \mu^* + \frac{f^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s)\Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) + \frac{p(t,x)|u(t,x)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} q(t,x,s)\Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant \mu^* + \frac{f^*\Phi(\|u\|_{BC})}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) + \frac{p^*\|u\|_{BC}\Phi(\|u\|_{BC})}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} q(t,x,s) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant \mu^* + q^*\Phi(\|u\|_{BC})(f^* + p^*\|u\|_{BC}).
\end{aligned}$$

Thus,

$$\|N'(u)\|_{BC} \leqslant \mu^* + q^*\Phi(\|u\|_{BC})(f^* + p^*\|u\|_{BC}). \tag{10}$$

Hence, $N'(u) \in BC$. This proves that the operator N' maps BC into itself.

The problem of finding the solutions of the Eq. (3) is reduced to finding the solutions of the operator equation $N'(u) = u$. Hypothesis (H_7') implies that N' transforms the ball $B_\eta := B(0,\eta)$ into itself. We shall show that $N' : B_\eta \to B_\eta$ satisfies the assumptions of Schauder's fixed point theorem [16]. As in the proof of Theorem 1, we can show that N' is continuous, $N'(B_\eta)$ is uniformly bounded, and equicontinuous on every compact subset $[0,a] \times [0,b]$ of J', $a > 0$.

Step 1: *$N'(B_\eta)$ is equiconvergent.*
Let $(t,x) \in \mathbb{R}_+ \times [0,b]$ and $u \in B_\eta$, then we have

$$\begin{aligned}
|(Nu)(t,x)| &\leqslant |\mu(t,x)| + \frac{|f(t,x,0)|}{\Gamma(r)} \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))||d_s g(t,s)| + \frac{|f(t,x,u(t,x)) - f(t,x,0)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} |h(t,x,s,u(s,x))| d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant |\mu(t,x)| + \frac{f^*}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s)\Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) + \frac{p(t,x)|u(t,x)|}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1} q(t,x,s)\Phi(|u(s,x)|) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right) \\
&\leqslant |\mu(t,x)| + \frac{\Phi(\eta)(f^* + p^*\eta)}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s) d_s \left(\bigvee_{k=0}^{s} g(t,k) \right).
\end{aligned}$$

Thus, for each $x \in [0,b]$, we get

$$|(Nu)(t,x)| \to 0, \quad as \quad t \to +\infty.$$

Hence,

$$|(Nu)(t,x) - (Nu)(+\infty,x)| \to 0, \quad as \quad t \to +\infty.$$

As a consequence of Step 1 together with Lemma 2.5, we can conclude that $N' : B_\eta \to B_\eta$ is continuous and compact. From an application of Schauder's theorem [16], we deduce that N' has a fixed point u which is a solution of the Eq. (3).

Step 2: *The local asymptotic stability of solutions.*

Now we investigate the local asymptotic stability of solutions of the Eq. (3). Let us assume that u_0 is a solution of the Eq. (3) with the conditions of this theorem. Taking $u \in B(u_0, \eta^*)$, we have

$$\begin{aligned}
|(N'u)(t,x) - u_0(t,x)| &= |(N'u)(t,x) - (N'u_0)(t,x)| \\
&\leqslant p^*q^*\Phi(\eta+\eta^*)|u(t,x)-u_0(t,x)| + \frac{f^*+p^*\eta}{\Gamma(r)}\int_0^t (t-s)^{r-1}|h(t,x,s,u(s,x)) \\
&\quad - h(t,x,s,u_0(s,x))|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leqslant p^*q^*\Phi(\eta+\eta^*)|u(t,x)-u_0(t,x)| + \frac{f^*+p^*\eta}{\Gamma(r)}\int_0^t (t-s)^{r-1}q(t,x,s)(\Phi(|u(s,x)|) \\
&\quad + \Phi(|u_0(s,x)|))d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leqslant p^*q^*\eta^*\Phi(\eta+\eta^*) + q^*(f^*+p^*\eta)[\Phi(\eta)+\Phi(\eta+\eta^*)] \leqslant \eta^*.
\end{aligned}$$

Thus we observe that N' is a continuous function such that

$$N'(B(u_0,\eta^*)) \subset B(u_0,\eta^*).$$

Moreover, if u is a solution of the Eq. (3), then

$$\begin{aligned}
|u(t,x) - u_0(t,x)| &= |(N'u)(t,x) - (N'u_0)(t,x)| \\
&\leqslant p^*\Phi(\eta+\eta^*)|u(t,x)-u_0(t,x)| \times \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}q(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) + \frac{f^*+p^*\eta}{\Gamma(r)} \\
&\quad \times \int_0^t (t-s)^{r-1}|h(t,x,s,u(s,x)) - h(t,x,s,u_0(s,x))|d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leqslant \frac{p^*\eta^*\Phi(\eta+\eta^*)}{\Gamma(r)}\int_0^t (t-s)^{r-1}q(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) + \frac{f^*+p^*\eta}{\Gamma(r)}\int_0^t (t-s)^{r-1}q(t,x,s)(\Phi(|u(s,x)|) \\
&\quad + \Phi(|u_0(s,x)|))d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\
&\leqslant (p^*\eta^*\Phi(\eta+\eta^*) + (f^*+p^*\eta)[\Phi(\eta)+\Phi(\eta+\eta^*)]) \times \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}q(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right).
\end{aligned}$$

Thus

$$|u(t,x) - u_0(t,x)| \leqslant (p^*\eta^*\Phi(\eta+\eta^*) + (f^*+p^*\eta)[\Phi(\eta)+\Phi(\eta+\eta^*)]) \times \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)}q(t,x,s)d_s\left(\bigvee_{k=0}^{s} g(t,k)\right). \tag{11}$$

By using (11) we deduce that

$$\lim_{t\to\infty} |u(t,x) - u_0(t,x)| = 0.$$

Consequently, all solutions of the Eq. (3) are locally asymptotically stable. □

5. Examples

As applications and to illustrate our results, we present two examples.

Example 1. Consider the following fractional order Riemann–Liouville Volterra–Stieltjes quadratic integral equations

$$u(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(t,x,s,u(s,x))d_s g(t,s); \quad \text{for} \quad (t,x) \in J := [0,1]\times[0,1], \tag{12}$$

where $r = \frac{1}{4}$, $\mu(t,x) = \frac{1}{2+t^2}$; $t,x \in [0,1]$,

$$f(t,x,u) = \frac{e^{x-t}|u|}{1+|u|}; \quad t, \quad x \in [0,1], \; u \in \mathbb{R},$$

$$g(t,s) = s; \quad t, \quad s \in [0,1],$$

$$h(t,x,s,u) = \frac{cx|u|}{1+|u|}(t-s)^{\frac{3}{4}} \sin t \sin s; \quad (t,x,s) \in J_1,$$

$$c = \frac{\Gamma(\frac{1}{4})}{8e} \quad \textit{and} \quad J_1 = \{(t,x,s) \in [0,1]^3 : s \leqslant t\}.$$

The function f is a continuous, $f^* = 0$, and

$$|f(t,x,u) - f(t,x,v)| \leqslant e^{x-t}|u-v|; \quad t, \quad x \in [0,1], \; u, \; v \in \mathbb{R}.$$

Then, the assumption (H_1) is satisfies with $p(t,x) = e^{x-t}; \; t,x \in [0,1]$, and then $p^* = e$. Also, we can easily see that the function g satisfies the hypotheses $(H_2) - (H_4)$.

The function h satisfies the assumption (H_5). Indeed, h is continuous and

$$|h(t,x,s,u)| \leqslant q(t,x,s)\Phi(|u|); \quad (t,x,s) \in J_1, \quad u \in \mathbb{R},$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+ : \; \Phi(w) = w$, and

$$q(t,x,s) = cx(t-s)^{\frac{3}{4}} \sin t \sin s; \quad (t,x,s) \in J_1.$$

Then,

$$q^* = \sup_{(t,x,s)\in J_1} \frac{(t-s)^{\frac{-3}{4}} q(t,x,s)}{\Gamma(\frac{1}{4})} = \frac{1}{8e}.$$

Finally, we can see that the hypothesis (H_6) is satisfies with $\eta = 1$. Indeed, we have $\mu^* = \frac{1}{2}, \; g^* = 1$ and the inequality

$$\mu^* + q^* g^* \Phi(\eta)(f^* + p^* \eta) \leqslant \eta$$

implies

$$\frac{1}{2} + \frac{1}{8}\eta^2 \leqslant \eta,$$

which is satisfies for $\eta = 1$. Consequently, by Theorem 3.2, the Eq. (12) has a solution defined on $[0,1] \times [0,1]$.

Example 2. Consider now the following fractional order Riemann–Liouville Volterra–Stieltjes quadratic integral equations

$$u(t,x) = \mu(t,x) + \frac{f(t,x,u(t,x))}{\Gamma(r)} \int_0^t (t-s)^{r-1} h(t,x,s,u(s,x)) d_s g(t,s); \quad \text{for } (t,x) \in J' := \mathbb{R}_+ \times [0,1], \tag{13}$$

where $r = \frac{1}{4}, \; \mu(t,x) = \frac{1}{2+t^2}; \; (t,x) \in J'$,

$$f(t,x,u) = \frac{e^{x-t}|u|}{1+|u|}; \quad (t,x) \in J', \quad u \in \mathbb{R}, \; g(t,s) = s; \; (t,s) \in \mathbb{R}_+^2,$$

$$h(t,x,s,u) = \frac{cxs^{\frac{-3}{4}}|u| \sin \sqrt{t} \sin s}{(1+t^2)(2+|u|)}; \quad (t,x,s) \in J_1', \quad s \neq 0 \quad \textit{and} \quad u \in \mathbb{R},$$

$$h(t,x,0,u) = 0; \quad (t,x) \in J \quad \textit{and} \quad u \in \mathbb{R},$$

$$c = \frac{\sqrt{\pi}}{8e\Gamma(\frac{1}{4})} \quad \textit{and} \quad J_1' = \{(t,x,s) \in J \times \mathbb{R}_+ : s \leqslant t, \quad x \in [0,1]\}.$$

First, we can see that $\lim_{t\to+\infty} \frac{1}{2+t^2} = 0$, then the assumption (H_1') is satisfies and $\mu^* = \frac{1}{2}$. Next, the function f is a continuous, $f^* = 0$, and

$$|f(t,x,u) - f(t,x,v)| \leqslant e^{x-t}|u-v|; \quad (t,x) \in J', \quad u,v \in \mathbb{R}.$$

Then, the assumption (H_2') is satisfies with $p(t,x) = e^{x-t}; \; (t,x) \in J$, and then $p^* = e$. Also, we can easily see that the function g satisfies the hypotheses $(H_3') - (H_5')$.

The function h satisfies the assumption (H_6'). Indeed, h is continuous and

$$|h(t,x,s,u)| \leqslant q(t,x,s)\Phi(|u|); \quad (t,x,s) \in J_1', \quad u \in \mathbb{R},$$

where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+ : \ \Phi(w) = w$, and

$$q(t,x,s) = \frac{cxs^{\frac{-3}{4}} \sin\sqrt{t} \sin s}{1+t^2}; \quad (t,x,s) \in J_1', \quad s \neq 0,$$

$$q(t,x,0) = 0; \quad (t,x) \in J'.$$

Then,

$$\left|\int_0^t (t-s)^{r-1} q(t,x,s) d_s g(t,s)\right| \leqslant \int_0^t (t-s)^{\frac{-3}{4}} cxs^{\frac{-3}{4}} |\sin\sqrt{t} \sin s| d_s \left(\bigvee_{k=0}^{s} g(t,k)\right) \leqslant cx|\sin\sqrt{t}| \int_0^t (t-s)^{\frac{-3}{4}} s^{\frac{-3}{4}} ds$$
$$\leqslant \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi}} \left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| \leqslant \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi t}} \longrightarrow 0 \ as \ t \to \infty,$$

and

$$q^* := \sup_{(t,x)\in J} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} q(t,x,s) d_s \left(\bigvee_{k=0}^{s} g(t,k)\right) \leqslant \sup_{(t,x)\in J} \frac{cx\Gamma(\frac{1}{4})}{\sqrt{\pi}} \left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right| = \frac{c\Gamma(\frac{1}{4})}{\sqrt{\pi}} = \frac{1}{8e}.$$

Finally, we can see that hypothesis (H_7') is satisfies with $\eta = 1$. Indeed, the inequality

$$\mu^* + q^*\Phi(\eta)(f^* + p^*\eta) \leqslant \eta$$

implies $\frac{1}{2} + \frac{1}{8}\eta^2 \leqslant \eta$, which is satisfies for $\eta = 1$. Moreover, the inequality

$$q^*(f^* + p^*\eta)\Phi(\eta) + [q^*f^* + p^*q^*(\eta + \eta^*)]\Phi(\eta + \eta^*) \leqslant \eta^*$$

is satisfies with $\eta = \eta^* = 1$ and gives $\frac{5}{8} \leqslant 1$. Consequently, by Theorem 4.3, Eq. (13) has a solution defined on $\mathbb{R}_+ \times [0,1]$ and all solutions are locally asymptotically stable on $\mathbb{R}_+ \times [0,1]$.

References

[1] S. Abbas, M. Benchohra, Darboux problem for implicit impulsive partial hyperbolic differential equations, Electron. J. Differ. Equ. 2011 (2011) 15.
[2] S. Abbas, M. Benchohra, On the set of solutions of fractional order Riemann–Liouville integral inclusions, Demonstratio Math. 46 (2013) 271–281.
[3] S. Abbas, M. Benchohra, Fractional order Riemann–Liouville integral equations with multiple time delay, Appl. Math. E-Notes 12 (2012) 79–87.
[4] S. Abbas, M. Benchohra, J. Henderson, On global asymptotic stability of solutions of nonlinear quadratic Volterra integral equations of fractional order, Commun. Appl. Nonlinear Anal. 19 (2012) 79–89.
[5] S. Abbas, M. Benchohra, G.M. N'Guérékata, Topics in Fractional Differential Equations, Springer, New York, 2012.
[6] S. Abbas, M. Benchohra, A.N. Vityuk, On fractional order derivatives and Darboux problem for implicit differential equations, Fract. Calc. Appl. Anal. 15 (2) (2012) 168–182.
[7] D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, Fractional Calculus Models and Numerical Methods, World Scientific Publishing, New York, 2012.
[8] J. Banaś, Existence results for Volterra–Stieltjes quadratic integral equations on an unbounded interval, Math. Scand. 98 (2006) 143–160.
[9] J. Banaś, B.C. Dhage, Global asymptotic stability of solutions of a functional integral equation, Nonlinear Anal. Theory Methods Appl. 69 (7) (2008) 1945–1952.
[10] J. Banaś, T. Zaja¸c, A new approach to the theory of functional integral equations of fractional order, J. Math. Anal. Appl. 375 (2011) 375–387.
[11] C. Corduneanu, Integral Equations and Stability of Feedback Systems, Academic Press, New York, 1973.
[12] M.A. Darwish, J. Henderson, D. O'Regan, Existence and asymptotic stability of solutions of a perturbed fractional functional integral equations with linear modification of the argument, Bull. Korean Math. Soc. 48 (3) (2011) 539–553.
[13] B.C. Dhage, Global attractivity results for nonlinear functional integral equations via a Krasnoselskii type fixed point theorem, Nonlinear Anal. 70 (2009) 2485–2493.
[14] B.C. Dhage, Attractivity and positivity results for nonlinear functional integral equations via measure of noncompactness, Differ. Equ. Appl. 2 (3) (2010) 299–318.
[15] K. Diethelm, The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics, Springer, Berlin, 2010.
[16] A. Granas, J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.
[17] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, New Jersey, 2000.
[18] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam, 2006.
[19] V. Lakshmikantham, S. Leela, J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.
[20] I. Podlubny, Fractional Differential Equation, Academic Press, San Diego, 1999.
[21] C. Qian, Global attractivity in nonlinear delay differential equations, J. Math. Anal. Appl. 197 (1996) 529–547.
[22] C. Qian, Global attractivity in a delay differential equation with application in a commodity model, Appl. Math. Lett. 24 (2011) 116–121.
[23] C. Qian, Y. Sun, Global attractivity of solutions of nonlinear delay differential equations with a forcing term, Nonlinear Anal. 66 (2007) 689–703.
[24] I.P. Natanson, Theory of Functions of a Real Variable, Ungar, New York, 1960.
[25] W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1970.
[26] R. Sikorski, Real Functions, PWN, Warsaw, 1958 (in Polish).
[27] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordon and Breach, Yverdon, 1993.
[28] V.E. Tarasov, Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media, Springer, Heidelberg; Higher Education Press, Beijing, 2010.
[29] A.N. Vityuk, A.V. Golushkov, Existence of solutions of systems of partial differential equations of fractional order, Nonlinear Oscil. 7 (2004) 318–325.