

¹ **Stability analysis of arbitrarily high-index positive
² delay-descriptor systems**

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⁶ **Abstract** This paper deals with the stability analysis of positive delay-descrip-
⁷ tor systems with arbitrarily high index. First we discuss the solvability problem
⁸ (i.e., about the existence and uniqueness of a solution), which is followed by
⁹ the study on characterizations of the (internal) positivity. Finally, we discuss
¹⁰ the stability analysis. Numerically verifiable conditions in terms of matrix in-
¹¹ equality for the system's coefficients are proposed, and are examined in several
¹² examples.

¹³ **Keywords** Positivity · Delay · Descriptor systems · Strangeness-index .

¹⁴ **Nomenclature**

$\mathbb{N} (\mathbb{N}_0)$	the set of natural numbers (including 0)
$\mathbb{R} (\mathbb{C})$	the set of real (complex) numbers
\mathbb{C}_-	the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$
$I (I_n)$	the identity matrix (of size $n \times n$)
$x^{(j)}$	the j -th derivative of a function x
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of p -times continuously differentiable functions from $[-\tau, 0]$ to \mathbb{R}^n (for $0 \leq p \leq \infty$)
$\ \cdot\ _\infty$	the norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$.
$\operatorname{im}_+ W$	the space $\{Ww_1 \text{ for all } w_1 \in \mathbb{R}_+^n\}$.

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16 1 Introduction

{sec1}

Our focus in the present paper is on the positivity and stability analysis of linear, constant coefficients *delay-descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_dx(t - \tau) + Bu(t), \quad \text{for all } t \in [t_0, t_f], \\ y(t) &= Cx(t), \end{aligned} \quad (1) \quad \{\text{delay-descriptor}\}$$

¹⁷ where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, $C \in \mathbb{R}^{q,n}$, $x : [t_0 - \tau, t_f] \rightarrow \mathbb{R}^n$, $f : [t_0, t_f] \rightarrow \mathbb{R}^n$,
¹⁸ and $\tau > 0$ is a constant delay. Together with (1), we are also concern with
¹⁹ the associated *zero-input system*

$$E\dot{x}(t) = Ax(t) + A_dx(t - \tau), \quad \text{for all } t \in [t_0, t_f]. \quad (2) \quad \{\text{zero-input system}\}$$

²⁰ Systems of the form (1) can be considered as a general combination of two
²¹ important classes of dynamical systems, namely *differential-algebraic equations*
²² (*descriptor systems*) (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (3) \quad \{\text{eq1.2}\}$$

²³ where the matrix E is allowed to be singular ($\det E = 0$), and *delay-differential*
²⁴ *equations* (DDEs)

$$\dot{x}(t) = Ax(t) + A_dx(t - \tau) + Bu(t). \quad (4) \quad \{\text{eq1.3}\}$$

²⁵ delay-descriptor systems of the form (1) have been arisen in various applica-
²⁶ tions, see Ascher and Petzold [1995], Campbell [1980], Hale and Lunel [1993],
²⁷ Shampine and Gahinet [2006], Zhu and Petzold [1997] and the references there
²⁸ in. From the theoretical viewpoint, the study for such systems is much more
²⁹ complicated than that for standard DDEs or DAEs. The dynamics of DDAEs
³⁰ has been strongly enriched, and many interesting properties, which occur nei-
³¹ ther for DAEs nor for DDEs, have been observed for DDAEs Campbell [1995],
³² Du et al. [2013], Ha and Mehrmann [2012, 2016]. Due to these reasons, re-
³³ cently more and more attention has been devoted to DDAEs, Campbell and
³⁴ Linh [2009], Fridman [2002], Ha and Mehrmann [2012, 2016], Michiels [2011],
³⁵ Shampine and Gahinet [2006], Tian et al. [2014], Linh and Thuan [2015].

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³⁷ [...]
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³⁹ The short outline of this work is as follows. Firstly, in Section 2, we briefly
⁴⁰ recall the solvability analysis to system (1), which is followed by an imporant
⁴¹ result about solution comparison for system (2) (Theorem 2). Based on the
⁴² explicit solution representation in Section 2, we characterize the positivity of
⁴³ system (1) in Section 3. We establish there algebraic, numerically verifiable
⁴⁴ conditions in terms of the system matrix coefficients. To follow, in Section
⁴⁵ 4 we discuss further about the zero-input system (2) under biconditional re-
⁴⁶ quirements: stability and positivity.

47 2 Preliminaries

In this section we discuss the solvability analysis, including the solution representation and the comparison principal for the corresponding IVP to system (1), which reads in details

$$E\dot{x}(t) = Ax(t) + A_dx(t - \tau) + Bu(t), \quad \text{for all } t \in [t_0, t_f], \\ x|_{[t_0 - \tau, t_0]} = \varphi : [t_0 - \tau, t_0] \rightarrow \mathbb{R}^n. \quad (5)$$

{initial condition}

- 48 Here, φ is a prescribed initial trajectory (preshape function), which is necessary
 49 to achieve uniqueness of solutions. Without loss of generality, we assume that
 50 $t_0 = 0$ and $t_f = n_f\tau$, where $n_f \in \mathbb{N}$.

51 **2.1 Existence, uniqueness and explicit solution formula**

52 It is well-known (e.g. Du et al. [2013]) that we may consider different solution
 53 concepts for system (1). The reason is, that $E(0)\dot{x}(0^+)$ which arises from the
 54 right hand side in (1) at 0 may not be equal to $E(0)\dot{\varphi}(0^-)$. Moreover, it has
 55 been observed in Baker et al. [2002], Campbell [1980], Guglielmi and Hairer
 56 [2008] that a discontinuity of \dot{x} at $t = 0$ may propagate with time, and typically
 57 \dot{x} is discontinuous at every point $j\tau$, $j \in \mathbb{N}_0$ or it may not even exist. To deal
 58 with this property of DDAEs, we use the following solution concept.

59 **Definition 1** Let us consider a fixed input function $u(t)$.

- 60 i) A function $x : [-\tau, \infty) \rightarrow \mathbb{R}^n$ is called a *piecewise differentiable solution* of
 61 (1), if Ex is piecewise continuously differentiable, x is continuous and satisfies
 62 (1) at every $t \in [t_0, t_f) \setminus \bigcup_{j \in \mathbb{N}_0} \{j\tau\}$.
 63 ii) A function $x : [-\tau, \infty) \rightarrow \mathbb{R}^n$ is called a *classical solution* of (1) if it is at
 64 least continuous and satisfies (1) at every $t \in [t_0, t_f]$.

65 Throughout this paper whenever we speak of a solution, we mean a piece-
 66 wise differentiable solution. Notice that, like DAEs, DDAEs are not solvable
 67 for arbitrary initial conditions, but they have to obey certain consistency con-
 68 ditions.

69 **Definition 2** An initial function φ is called *consistent* with (1) if the associ-
 70 ated initial value problem (IVP) (1), (5) has at least one solution. System (1)
 71 is called *solvable* (resp. *regular*) if for every consistent initial function φ , the
 72 IVP (1), (5) has a solution (resp. has a unique solution).

Introducing sequences of matrix-valued and vector-valued functions $f_j, u_j,$
 x_j for each $j \in \mathbb{N}$, on the time interval $[0, \tau]$ via

$$f_j(t) = f(t + (j - 1)\tau), \quad u_j(t) = u(t + (j - 1)\tau), \\ x_j(t) = x(t + (j - 1)\tau), \quad x_0(t) := \varphi(t - \tau),$$

- 73 we can rewrite the IVP (1)-(5) as a sequence of non-delayed descriptor systems

$$E\dot{x}_j(t) = Ax_j(t) + A_dx_{j-1}(t) + Bu_j(t), \quad (6)$$

{j-th DAE}

74 for all $t \in (0, \tau)$ and for all $j = 1, 2, \dots, n_f$. We notice, that for each j , the
 75 initial condition $x_j(0)$ is given due to the continuity of the solution $x(t)$ at the
 76 point $(j-1)\tau$, i.e.,

$$x_j(0) = x_{j-1}(\tau). \quad (7) \quad \{\text{continuity condition}\}$$

77 In particular, $x_1(0) = \phi(0)$ and the function x_0 is given. Inherited from the the-
 78 ory of delay-different equations (Hale and Lunel [1993]), we recall the concept
 79 of *non-advancedness* as follow.

80 **Definition 3** A regular delay-descriptor system (1) is called *non-advanced* if
 81 for any consistent and continuous initial function φ , there exists a piecewise
 82 differentiable solution $x(t)$ to the IVP (1), (5).

83 Obviously, the non-advancedness of system (1) is equivalent to the fact
 84 that the function x_j is at least as smooth as x_{j-1} for all $j \in \mathbb{N}$. In deed,
 85 most of systems that we have encountered in applications are non-advanced,
 86 Ascher and Petzold [1995], Shampine and Gahinet [2006], Ha [2015]. For more
 87 detailed discussions about the types of the DDAE (2), we refer the readers to
 88 Ha [2015], Ha and Mehrmann [2016], Unger [2018].

89 **Definition 4** Consider the DDAE (1). The matrix triple (E, A, B) is called
 90 *regular* if the (two variable) *characteristic polynomial* $\mathfrak{P}(\lambda, \omega) := \det(\lambda E -$
 91 $A - \omega B)$ is not identically zero. If, in addition, $B = 0$ we say that the matrix
 92 pair (E, A) (or the pencil $\lambda E - A$) is regular. The sets $\sigma(E, A, B) := \{\lambda \in$
 93 $\mathbb{C} \mid \det(\lambda E - A - e^{-\lambda\tau} B) = 0\}$ and $\rho(E, A, B) = \mathbb{C} \setminus \sigma(E, A, B)$ are called the
 94 *spectrum* and the *resolvent set* of (1), respectively.

{regularity}

95 Provided that the pair (E, A) is regular, we can transform them to the
 96 Kronecker-Weierstraß canonical form (see e.g. Dai [1989], Kunkel and Mehrmann
 97 [2006]). That is, there exist regular matrices $W, T \in \mathbb{R}^{n,n}$ such that

$$(E, A) = \left(W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right), \quad (8) \quad \{\text{KW form}\}$$

98 where N is a nilpotent matrix of nilpotency index ν . We also say that the pair
 99 (E, A) has a *differentiation index* ν , i.e., $\text{ind}(E, A) = \nu$.

100 *Remark 1* Two concepts non-advancedness and differentiation index are inde-
 101 pendent. In details, a non-advanced system can have arbitrarily high index, as
 102 can be seen in the following example.

{example 1}

103 *Example 1* Consider the following systems with the parameters $\varepsilon_1, \varepsilon_2$.

$$\underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_E \dot{x}(t) = \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_A x(t) + \begin{bmatrix} 0 & & \varepsilon_1 \\ & \ddots & 0 \\ & & \varepsilon_2 \end{bmatrix} x(t-h), \quad (9) \quad \{\text{eq11}\}$$

¹⁰⁴ It is well-known that in this example $\text{ind}(E, A) = n$. Furthermore, depending
¹⁰⁵ on the value of ε_2 , the system will be advanced (if $\varepsilon_2 \neq 0$) and be non-advanced
¹⁰⁶ (if $\varepsilon_2 = 0$).

¹⁰⁷ Let E have index $\tilde{\nu}$, i.e., $\text{ind}(E, I_n) = \tilde{\nu}$, the Drazin inverse E^D of E is
¹⁰⁸ uniquely defined by the properties

$$E^D E = E E^D, \quad E^D E E^D = E^D, \quad E^D E^{\tilde{\nu}+1} = E^{\tilde{\nu}}. \quad (10)$$

¹⁰⁹ **Lemma 1** *Kunkel and Mehrmann [2006]* Let (E, A) be a regular matrix pair.
¹¹⁰ Then for any $\lambda \in \rho(E, A)$, two following matrices commute.

$$\hat{E} := (\lambda E - A)^{-1} E, \quad \hat{A} := (\lambda E - A)^{-1} A. \quad (11) \quad \{\text{eq20}\}$$

¹¹¹ Furthermore, the following commutative identities hold true.

$$\hat{E} \hat{A}^D = \hat{A}^D \hat{E}, \quad \hat{E}^D \hat{A} = \hat{A} \hat{E}^D, \quad \hat{E}^D \hat{A}^D = \hat{A}^D \hat{E}^D.$$

¹¹² We notice that the matrix products $\hat{E}^D \hat{E}$, $\hat{E}^D \hat{A}$, $\hat{E} \hat{A}^D$, $\hat{E}^D \hat{B}$, $\hat{A}^D \hat{B}$ do
¹¹³ not depend on the choice of λ (see e.g. Dai [1989]). Furthermore, they can
¹¹⁴ be numerically computed by transforming the pair (E, A) to their Weierstrass
¹¹⁵ canonical form (8) (see e.g. Gerdts [2005], Virnik [2008]).

¹¹⁶ For any $\lambda \in \rho(E, A)$, we denote

$$\hat{A}_d := (\lambda E - A)^{-1} A_d, \quad \hat{B} := (\lambda E - A)^{-1} B. \quad (12) \quad \{\text{eq21}\}$$

¹¹⁷ Making use of the Drazin inverse, in the following theorem we present the
¹¹⁸ explicit solution representation of system (1).

Theorem 1 Consider the delay-descriptor system (1). Assume that (E, A) is
¹¹⁹ a regular matrix pair with a differentiation index $\text{ind}(E, A) = \nu$. Let \hat{E} , \hat{A} ,
 \hat{A}_d , \hat{B} be defined as in (11), (12). Furthermore, assume that u is sufficiently
¹²⁰ smooth. Then, every solution x_j of the DAE (6) has the form

$$\begin{aligned} x_j(t) &= e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left(\hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds \\ &+ (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(t) + \hat{B} u_j^{(i)}(t) \right), \end{aligned} \quad (13) \quad \{\text{j-th solution}\}$$

¹²¹ for some vector $v_j \in \mathbb{R}^n$.

¹²² *Proof.* The proof is straightly followed from the explicit solution of DAEs, see
¹²³ [Kunkel and Mehrmann, 2006, Chap. 2]. \square

¹²⁴ Making use of (7), we directly obtain the following corollary.

123 **Corollary 1** *The solution $x(t)$ of system (1) is continuous at the point $(j-1)\tau$*
124 *if and only if the following condition holds.*

$$(\hat{E}^D \hat{E} - I) x_{j-1}(\tau) = (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(0) + \hat{B} u_j^{(i)}(0) \right).$$

125 *In particular, for the preshape function $\varphi(t)$, we must require*

$$(\hat{E}^D \hat{E} - I) \left(\varphi(0) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d \varphi^{(i)}(-\tau) + \hat{B} u^{(i)}(0) \right) \right) = 0.$$

126 Following from (13), we directly obtain a simpler form in case of non-
127 advanced system as follows.

Corollary 2 *Consider system (1) and assume that it is regular and non-advanced. Then, we have*

$$\begin{aligned} x_j(t) &= e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left(\hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds \\ &+ (\hat{E}^D \hat{E} - I) \left(\hat{A}^D \hat{A}_d x_{j-1}(t) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u_j^{(i)}(t) \right), \end{aligned} \quad (14) \quad \{\text{sol. formula non-advanced}\}$$

128 Furthermore, the consistency condition at $t = 0$ reads

$$(\hat{E}^D \hat{E} - I) \left(\varphi(0) + \hat{A}^D \hat{A}_d \varphi(-\tau) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u^{(i)}(0) \right) = 0. \quad (15) \quad \{\text{consistency}\}$$

129 2.2 Comparison principal

130 **Lemma 2** *It suffices to prove that if $u_j(t) \leq \tilde{u}_j(t)$ and $x_{j-1}(t) \leq \tilde{x}_{j-1}(t)$ for*
131 *all $t \in [0, \tau]$ then it follows that $x_j(t) \leq \tilde{x}_j(t)$ for all $t \in [0, \tau]$.*

132 By simple induction, making use of Lemma 2, we obtain the solution com-
133 parison for system (1).

134 **Theorem 2** *Consider system (1) and assume that it is positive. Let $x(t)$*
135 *(resp. $\tilde{x}(t)$) be a state function corresponds to a reference input $u(t)$ (resp.*
136 *$\tilde{u}(t)$) and a preshape function $\varphi(t)$ (resp. $\tilde{\varphi}(t)$). Furthermore, assume that the*
137 *following conditions hold.*

- 138 i) $\varphi(t) \leq \tilde{\varphi}(t)$ for all $t \in [-\tau, 0]$,
- 139 ii) $u^{(i)}(t) \leq \tilde{u}^{(i)}(t)$ for all $t \geq 0$ and all $i \leq (\nu - 1) \lfloor t/\tau \rfloor$. Then we have
- 140 $x(t) \leq \tilde{x}(t)$ for all $t \geq 0$.

141 *Proof.*

{lem2b}}

{solution comparison 1}

142 **Theorem 3** *Time-dependent delay will affect neither the positivity nor the*
143 *stability of system (1).*

144 3 Characterizations of positive delay-descriptor system

145 Since most systems occur in application are non-advanced, in this section we
 146 focus on the characterization for positivity of non-advanced delay descriptor
 147 systems. We, furthermore, notice that the non-advancedness is a necessary
 148 condition for the stability (in the Lyapunov sense) of any time-delayed system,
 149 see e.g. Hale and Lunel [1993], Du et al. [2013].

150 **Definition 5** Consider the delay-descriptor system (1) and assume that it is
 151 non-advanced, and that the pair (E, A) is regular with $\text{ind}(E, A) = \nu$. We call
 152 (1) positive if for all $t \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for any input function
 153 u and any consistent initial function $\varphi(t)$ that satisfy two following conditions.
 154 i) $\varphi(t) \geq 0$ for all $t \in [-\tau, 0]$,
 155 ii) $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and all $i \leq (\nu - 1) \lfloor t/\tau \rfloor$.

156 Let us denote

$$\mathcal{K}_\nu(\hat{E}\hat{A}^D, \hat{A}^D\hat{B}) := [\hat{A}^D\hat{B}, \hat{E}\hat{A}^D\hat{A}^D\hat{B}, \dots, (\hat{E}\hat{A}^D)^{\nu-1}\hat{A}^D\hat{B}] .$$

Since our systems is linear, time invariant coefficients, it would be sufficient to study the positivity on the first time interval $[0, \tau]$. Making use of (14), and let $j = 1$, we can split the solution $x_1 = x|_{[0, \tau]}$ as follows

$$\begin{aligned} x_1(t) &= e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_1 + \underbrace{\int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{A}_d x_0(s) + (\hat{E}^D\hat{E} - I)\hat{A}^D\hat{A}_d x_0(t)}_{x_{zi}(t)} \\ &\quad + \underbrace{\int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{B}u_j(s) + (\hat{E}^D\hat{E} - I)\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u_j^{(i)}(t)}_{x_{zs}(t)}, \end{aligned} \tag{16} \quad \{\text{eq16}\}$$

157 where $x_{zi}(t)$ (resp. $x_{zs}(t)$) is often called (in the theory of linear systems) the
 158 zero input (resp. zero state) solution.

159 **Lemma 3** Let $F \in \mathbb{R}^{p,n}$ and $M \in \mathbb{R}^{n,n}$ and consider the linear system $\dot{z}(t) =$
 160 $Mz(t)$. Then, the following implication holds true:

$$[Fz(0) \geq 0] \Rightarrow [Fz(t) \geq 0 \text{ for all } t \geq 0]$$

161 if and only if there exists a Metzler matrix H such that $FM = HF$.

162 **Proposition 1** Rami and Napp [2012] The following statements are equivalent. {Rami12}

- 163 i) The differential-algebraic equation $E\dot{x}(t) = Ax(t)$ is positive.
- 164 ii) There exists a Metzler matrix H such that $\hat{E}^D\hat{A} = H\hat{E}^D\hat{E}$.
- 165 iii) There exists a matrix D such that $H := \bar{A} + D(I - P)$ is Metzler.

167 **Lemma 4** Consider the delay-descriptor system (1) and assume that it is
 168 non-advanced, and the pair (E, A) is regular with index $\text{ind}(E, A) = \nu$. Let the
 169 input $u = 0$. Then, system (1) has a solution $x(t) \geq 0$ for all $t \geq 0$ and all
 170 consistent initial function $\varphi(t) \geq 0$ if and only if the following conditions are
 171 satisfied.

- 172 i) There exists a Metzler matrix H s.t. $\hat{E}^D \hat{A} = H \hat{E}^D \hat{E}$.
 173 ii) $\hat{E}^D \hat{A}_d \geq 0$, $(\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d \geq 0$.

174 **Theorem 4** Consider the delay-descriptor system (1) and assume that it is
 175 non-advanced, and the pair (E, A) is regular with index $\text{ind}(E, A) = \nu$. Fur-
 176 thermore, assume that $(\hat{E}^D \hat{E} - I) (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$ for all $i = 0, \dots, \nu - 1$.
 177 Then, system (1) is positive if and only if the following conditions hold.

- 178 i) $\hat{E}^D \hat{A} = H \hat{E}^D \hat{E}$ for some Metzler matrix H .
 179 ii) $\hat{E}^D \hat{A}_d \geq 0$, $(\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d \geq 0$, $\hat{E}^D \hat{B} \geq 0$,
 180 iii) C is non-negative on the subspace

$$\mathcal{X} := \text{im}_+ [\hat{E}^D \hat{E}, (\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d, (\hat{E}^D \hat{E} - I) \mathcal{K}_\nu(\hat{E} \hat{A}^D, \hat{A}^D \hat{B})]. \quad (17) \quad \{\text{reachable subspace}\}$$

181 *Proof.* \Rightarrow Due to Lemma 4, we only need to prove part 3.

182 \Leftarrow Quite simple. \square

183 If we restrict ourself to the non-delayed case (i.e. $A_d = 0$), the direct corol-
 184 lary of Theorem 4 is straightforward. We, moreover, notice that this corollary
 185 has slightly improved the result [Virnik, 2008, Thm. 3.4].

186 **Corollary 3** Consider the descriptor system (3) and assume that the pair
 187 (E, A) is regular with index $\text{ind}(E, A) = \nu$. Furthermore, assume that the
 188 inequalities $(\hat{E}^D \hat{E} - I) (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$ hold true for $i = 0, \dots, \nu - 1$.
 189 Then, system (3) is positive if and only if the following conditions hold.
 190 i) $\hat{E}^D \hat{A} = H \hat{E}^D \hat{E}$ for some Metzler matrix H .
 191 ii) $\hat{E}^D \hat{B} \geq 0$,
 192 iii) C is non-negative on the subspace \mathcal{X} .

193 4 Stability of positive delay-descriptor system

194 5 Conclusion

195 In this paper, we have discussed the positivity of strangeness-free descrip-
 196 tor systems in continuous time. Beside that, the characterization of positive
 197 delay-descriptor systems has been treated as well. The theoretical results are
 198 obtained mainly via an algebraic approach and a projection approach. The
 199 projection approach investigates the positivity of a given descriptor system
 200 by the positivity of an inherent ODE obtained by projecting the given sys-
 201 tem onto a subspace. On the other hand, the algebraic approach derives an
 202 underlying ODE without changing the state, input and output. Then, study-
 203 ing these hidden ODEs is the key point. The main difficulty here is that the

{zero input lemma}

{Thm positivity}

{Thm positivity - DAE version}

{sec4}

{conclusion}

204 derivative of the input u may occur in the new system. Despite their disadvantages,
 205 these methods can provide both necessary conditions and sufficient
 206 conditions. Beside these theoretical methods, the behaviour approach, which
 207 leads to some feasible conditions, is also implemented.

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