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Robust Finite-time H_∞ Control of Linear Time-varying Delay Systems with Bounded Control via Riccati Equations

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Abstract: In this paper, we will present new results on robust finite-time H_∞ control for linear time-varying systems with both time-varying delay and bounded control. Delay-dependent sufficient conditions for robust finite-time stabilization and H_∞ control are first established to guarantee finite-time stability of the closed-loop system via solving Riccati differential equations. Applications to finite-time H_∞ control to a class of linear autonomous time-delay systems with bounded control are also discussed in this paper. Numerical examples are given to illustrate the effectiveness of the proposed method.

Keywords: Finite-time stability, H_∞ control, bounded control, time-varying delay, Riccati equation.

1 Introduction

Finite-time stability is one of the fundamental concepts in mathematical control theory, which has been studied by different approaches and for different kind of systems (see, e.g., [1−6] and the references therein). In general, the finite-time stability (FTS) introduced in [7] means that the state of a system does not exceed some bound during a fixed time interval. FTS focuses its attention on the transient behavior of a system response. It is worth pointing out that finite-time stability and Lyapunov asymptotic stability are different concepts, and they are independent of each other. A system is finite-time stable if its state retains certain pre-specified bound in the fixed time interval in the case that the initial bound is given. Often Lyapunov asymptotic stability is enough for practical applications, but there are some cases where large values of the state are not acceptable, for instance in the presence of saturations. In these cases, we need to check that these unacceptable values are not attained by the state. Moreover, finite-time stability analysis for linear time-varying delay systems is more difficult, because time-varying delay systems have more complicated dynamic behaviors than the systems without delays or with constant delays. On the other hand, the H_∞ control problem of dynamical control systems has attracted much attention due to its both practical and theoretical importance[8]. Traditionally, Lyapunov theory has served as a powerful tool for H_∞ control design. The idea of a Lyapunov function was applied in the context of control design to yield control Lyapunov functions (CLFs). For continuous linear time-invariant systems, there exists a well known method to construct CLFs, which essentially involves finding a positive definite solution of a Riccati equation. Various approach have been developed and a great number of results on finite-time H_∞ control for continuous systems as well as discrete systems have been reported in the literatures (see, e.g., [9−11]). However, most of the results in this field relate to stability and performance criteria defined over an infinite-time interval. The finite-time H_∞ control concerns with the design of a feedback controller which ensures the FTS of the closed-loop system and guarantees a maximum H_∞ performance bound[12−17]. On the other hand, the problem of stabilization of systems with control constraints arises not only in mathematical control theory, but also in many applied areas[18−20]. It is clear that control constraints on the structure of the feedbacks and the neglect of geometric constraints on the control are hardly in accord with present-day requirements for control systems. Input constraints are ubiquitous in control and operation of all control systems. These constraints usually arise due to the physical limitation of control actuators such as pumps or valves. It is well established that neglecting these constraints while designing controllers can lead to significant performance deterioration and even closed-loop instability[21]. Existing attempts for control analysis of non-autonomous continuous systems are mere extensions of the approaches for autonomous continuous case. In such cases, there exists a well known method to design feedback controllers, which essentially involve finding a positive definite solution of Riccati differential equations (RDEs) or

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differential linear matrix inequalities (DLMIs)[22−24]. Note that, in general, numerically solving RDEs or DLMIs can be realiable. This approach also applies to the finite-time control design for the system with bounded controls[25−28]. On the other hand, the procedures are derived under the assumption of unconstrained control action. For linear systems with bounded control, under appropriate assumptions on the spectral and controllability property, papers[18, 19, 23] proposed a nonlinear feedback control to stabilize (in the Lyapunov sense) the system without delays. It is worth noting that the approach in these works cannot be readily applied to the systems with time-varying delays. The main difficulty is that the investigation of the spectrum of the time-varying delay matrices is still complicated and there are no appropriate properties available as in the un-delayed case. To the best of our knowledge, the issue of robust finite-time H_∞ control for linear time-delay systems with bounded control has not been investigated. Due to the fact that the existence of constraints on the control may have great influence on the finite-time stability of the closed-loop system[17−19], which increases difficulties for us to discuss this topic. Consequently, the problem of the finite-time control of linear non-autonomous systems with both time-varying delay and bounded control is of interest in its own right.

In this paper, unlike the previous reported results on finite-time control of linear non-autonomous systems, problem of finite-time control is fully investigated without any spectral and controllability assumption and the derived conditions involve solving a Riccati differential equation. By exploring an auxiliary control system with time-varying delay, novel delay-dependent sufficient conditions for robust finite-time stabilization are proposed via a newly constructed Riccati differential equation. Applications to finite-time H_∞ control to a class of linear autonomous time-delay systems with bounded control are also discussed in this paper. The proposed approach is numerically appealing for checking finite-time H_∞ control conditions of a given linear time-varying delay system with bounded controls.

The structure of the paper is as follows. Section 2 gives the necessary background on linear non-autonomous delay systems with bounded control and some technical propositions. In Section 3, the nonlinear feedback controller design for robust finite-time stabilization and solution to H_∞ control problem are presented with some applications to linear autonomous time-delay systems with bounded control. Numerical examples illustrated the obtained results are given in Section 4. Section 5 ends with some conclusions.

2 Problem formulation and preliminaries

In this section, we introduce some notations, definitions and technical propositions. $\mathbf{R}^+$ denotes the set of all real non-negative numbers. $\mathbf{R}^n$ denotes the n-dimensional space with the scalar product $x^{\mathrm{T}}y$. $M^{n\times r}$ denotes the space of all matrices of $(n\times r)$-dimensions with the spectral norm $||A|| = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$. A^{T} denotes the transpose of matrix A. I denotes the identity matrix. $\lambda(A)$ denotes the set of all eigenvalues of A. $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda, \lambda\in\lambda(A)\}$. $x_t = \{x(t+s) : s\in[-h,0]\}$, $\| x_t \| = \sup_{s\in[-h,0]} \| x(t+s) \|$, $C([0,t],\mathbf{R}^n)$ denotes the set of all $\mathbf{R}^n-$valued continuous functions on $[0,t]$. $L_2([0,t],\mathbf{R}^m)$ denotes the set of all the $\mathbf{R}^m$-valued square integrable functions on $[0,t]$. Matrix A is called semi-positive definite ($A\ge 0$) if $x^{\mathrm{T}}Ax\ge 0$, for all $x\in\mathbf{R}^n$. A is positive definite ($A>0$) if $x^{\mathrm{T}}Ax>0$ for all $x\ne 0$. $A>B$ means $A-B>0$.

Consider a linear non-autonomous time-varying delay system with bounded control of the form:

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + D(t)x(t-h(t)) + \\ &\quad B(t)u(t) + B_1(t)w(t)\\ z(t) &= C_1(t)x(t) + C_2(t)x(t-h(t))\\ x(t) &= \varphi(t),\quad t\in[-h,0]\end{aligned}\tag{1}$$

where $x(t)\in\mathbf{R}^n, u(t)\in\mathbf{R}^m, w\in\mathbf{R}^{m_1}, z(t)\in\mathbf{R}^{m_2}$ are the state, the control, the disturbance and the observation vector, respectively. $A(t), D(t)\in\mathbf{R}^{n\times n}, B(t)\in\mathbf{R}^{n\times m}, B_1(t)\in\mathbf{R}^{n\times m_1}, C_1(t), C_2(t)\in\mathbf{R}^{n\times m_2}$ are given continuous matrix functions. The initial function $\varphi(t)\in C([-h,0],\mathbf{R}^n)$. The delay function $h(t)$ is continuous and satisfies

$$0\le h(t)\le h,\quad \dot{h}(t)\le\delta<1,\quad \forall t\ge 0.\tag{2}$$

The control $u\in L_2([0,T],\ \mathbf{R}^m)$ satisfies

$$\exists r>0:\quad ||u(t)||\le r,\quad \forall t\in[0,T].\tag{3}$$

The disturbance $w(t)\in L_2([0,T],\ \mathbf{R}^{m_1})$ satisfies

$$\exists d>0:\quad \int_0^T w^{\mathrm{T}}(t)w(t)\mathrm{d}t\le d.\tag{4}$$

Once the above assumption on the initial function $\varphi(\cdot)$ and the matrix data are given, the solution of system (1) is well defined (see, e.g., [29]).

Definition 1. Robust finite-time stabilization

For given positive numbers T, c_1, c_2 and a symmetric positive definite matrix R, the control system (1) is robustly finite-time stabilizable w.r.t (c_1, c_2, T, R) if there exists a state feedback controller $u(t) = g(x(t))$ satisfying (3) such that

$$\sup_{-h\le s\le 0}\varphi^{\mathrm{T}}(s)R\varphi(s)\le c_1\Rightarrow x^{\mathrm{T}}(t)Rx(t)<c_2, \forall t\in[0,\ T]$$

for all disturbances $w(t)$ satisfying (4).

In the above definition, and differently from the definition of Lyapunov stability, we focus on the input-output behavior of linear systems over a finite time interval. Roughly speaking, and consistently with the definition of FTS given in [7], a system is defined bounded if, given a class of norm bounded input signals over a specified time interval $[0,\ T]$, the outputs of the system do not exceed an assigned threshold during $[0,\ T]$.

The objective of this paper is to design a state feedback controller $u(t) = g(x(t))$ satisfying (3) such that the resulting closed-loop system

$$\dot{x} = A(t)x + B(t)g(x) + D(t)x(t-h(t)) + B_1(t)w \tag{5}$$

is finite-time stable for all disturbances $w(t)$ satisfying (4) and guarantees a maximum H_∞ performance. In the following, we define the robust finite-time H_∞ controller for a class of linear control time-varying delay system such that the corresponding closed-loop system is finite-time stable with H_∞ disturbance attenuation level γ.

Definition 2. Finite-time H_∞ control

Given $\gamma > 0$, the finite-time H_∞ control problem for the systems (1) has a solution if

1) The system (1) is robustly finite-time stabilizable w.r.t. (c_1, c_2, T, R).

2) There exists a number $c_0 > 0$ such that

$$\sup \frac{\int_0^{\rm T} \|z(t)\|^2 {\rm d}t}{c_0\|\varphi\|^2 + \int_0^{\rm T} \|w(t)\|^2 {\rm d}t} \le \gamma \tag{6}$$

where the supremum is taken over all $\varphi \in C([-h,\ 0], R^n)$ and non-zero disturbances $w(\cdot)$ satisfying (4).

Propositions 1 to 4 play an important role in our later development.

Proposition 1. Cauchy matrix inequality[29]

For any matrices $P, Q \in \mathbf{R}^{n\times n}, Q > 0$ is symmetric, the following inequality holds

$$2y^{\rm T}Px - y^{\rm T}Qy \le x^{\rm T}P^{\rm T}Q^{-1}Px, \quad \forall x, y \in \mathbf{R}^n.$$

Proposition 2. Let

$$f(t,x) = r\frac{B(t)B^{\rm T}(t)P(t)x}{1 + \|B^{\rm T}(t)P(t)x\|}$$
$$b = \sup_{t\in[0,\,T]} ||B(t)||, \quad p = \sup_{t\in[0,\,T]} ||P(t)||.$$

Then we have the following assertions:

1) $f(t,x)$ is globally Lipschitz in $\mathbf{R}^n$.

2) $||f(t,x)||^2 \le r^2b^2x^{\rm T}P^2(t)x, \quad \forall x \in \mathbf{R}^n,\ t \in [0,\ T]$.

Proof. Let us denote $x_1, x_2 \in \mathbf{R}^n$ and

$$y_1 = B^{\rm T}(t)P(t)x_1, \quad y_2 = B^{\rm T}(t)P(t)x_2.$$

We have

$$\begin{aligned}
&\|f(t,x_1) - f(t,x_2)\| = \\
&\quad r\Big\|B(t)\Big[\frac{y_1}{1+\|y_1\|} - \frac{y_2}{1+\|y_2\|}\Big]\Big\| \le \\
&\quad rb\Big\|\frac{y_1}{1+\|y_1\|} - \frac{y_2}{1+\|y_2\|}\Big\| \le \\
&\quad rb\frac{\|y_1 - y_2\| + \|y_2\|y_1 - \|y_1\|y_2}{(1+\|y_1\|)(1+\|y_2\|)}.
\end{aligned}$$

Since

$$\begin{aligned}
y_1\|y_2\| - y_2\|y_1\| &= y_1(\|y_2\| - \|y_1\|) + \|y_1\|(y_1 - y_2) \le \\
&\|y_1\|(\|y_1 - y_2\|) + \|y_1\|(\|y_1 - y_2\|) = \\
&2\|y_1\|(\|y_1 - y_2\|)
\end{aligned}$$

and

$$\frac{\|y_1 - y_2\|}{(1+\|y_1\|)(1+\|y_2\|)} \le \|y_1 - y_2\|$$
$$\frac{\|y_1\|}{(1+\|y_1\|)(1+\|y_2\|)} \le 1$$

we have

$$||f(t,x_1) - f(t,x_2)|| \le 3rb||y_1 - y_2|| \le\ 3rpb^2||x_1 - x_2||$$

where $p = \sup_{t\in[0,\,T]} ||P(t)||$.

3) We have

$$\begin{aligned}
&f^{\rm T}(t,x)f(t,x) = \\
&\quad \frac{r^2}{(1+\|B^{\rm T}(t)P(t)x\|)^2}\|B(t)B^{\rm T}(t)P(t)x\|^2 \le \\
&\quad \frac{r^2b^4}{(1+\|B^{\rm T}(t)P(t)x\|)^2}x^{\rm T}P^2(t)x \le \\
&\quad r^2b^4x^{\rm T}P^2(t)x
\end{aligned}$$

because of $1 + ||B^{\rm T}(t)P(t)x|| \ge 1$. □

Proposition 3. Schur complement Lemma[29]

Given matrices X, Y, Z, where $Y = Y^{\rm T} > 0$, we have

$$\begin{pmatrix} X & Z \\ Z^{\rm T} & -Y \end{pmatrix} < 0 \Longleftrightarrow X < 0,\ X + ZY^{-1}Z^{\rm T} < 0.$$

Proposition 4. Let $P \in M^{n\times n}, R \in M^{n\times n}$ be symmetric positive definite matrices. We have

1) $\lambda_{\min}(P)(R) > 0, \lambda_{\max}(P)(R) > 0$ and

$$\lambda_{\min}(P)x^{\rm T}x \le x^{\rm T}Px \le \lambda_{\max}(P)x^{\rm T}x, \quad \forall x \in \mathbf{R}^n.$$

2) $x^{\rm T}x \le \lambda_{\max}(R^{-1})x^{\rm T}Rx, \quad \forall x \in \mathbf{R}^n.$

3) $x^{\rm T}Px \le \lambda_{\max}(P)\lambda_{\max}(R^{-1})x^{\rm T}Rx, \quad \forall x \in \mathbf{R}^n.$

Proof. 1) is obvious. To prove 2), we use the assertion 1) as

$$x^{\rm T}x = x^{\rm T}R^{\frac{1}{2}}R^{-1}R^{\frac{1}{2}}x \le \lambda_{\max}(R^{-1})x^{\rm T}Rx.$$

The assertion 3) is easily followed from 1) and 2). □

3 Main result

In this section, we start by designing the state feedback controllers for finite-time stabilization and H_∞ control. The approach we use here is the Lyapunov-like function method in the context of stabilization of linear non-autonomous systems subject to control constraint. Let us consider the following Riccati differential equation (RDE)

$$\begin{aligned}
&\dot{P}_R(t) + A^{\rm T}(t)P_R(t) + P_R(t)A(t) + \\
&\quad (2 + r^2b^4 + \frac{b_1}{\gamma\beta_2})P_R^2(t) + (\beta_2 + \zeta\xi)I = 0
\end{aligned} \tag{7}$$

where

$$P_R(t) = P(t) + R,\ \zeta = \frac{1}{1-\delta}$$
$$b = \sup_{t\in[0,T]} ||(B(t))||, \quad b_1 = \sup_{t\in[0,T]} \lambda_{\max}(B_1^{\mathrm{T}}(t)B_1(t))$$
$$\sigma = \sup_{t\in[0,\,T]} \lambda_{\max}(D^{\mathrm{T}}(t)D(t))$$
$$\bar{c}_i = 2 \sup_{t\in[0,\,T]} \lambda_{\max}(C_i^{\mathrm{T}}(t)C_i(t)),\ i=1,2$$
$$\beta_2 = \max\{\bar{c}_1, \bar{c}_2\}, \quad \xi = \sigma + \beta_2.$$

Theorem 1. The finite-time H_∞ control problem for system (1) has a solution if there exist a symmetric matrix $P(t) \geq 0$ of RDE (7) and a positive number $\eta > 0$ satisfying

$$\alpha_1 c_1 + \beta_2 \gamma d \leq c_2 \mathrm{e}^{-\eta \mathrm{T}} \tag{8}$$

where

$$\alpha_1 = \lambda_{\max}(P_R(0))\lambda_{\max}(R^{-1}) + \zeta h \xi \lambda_{\max}(R^{-1}).$$

Moreover, the state feedback controller is

$$u(t) = \frac{r}{1+||B^{\mathrm{T}}(t)P(t)x(t)||} B^{\mathrm{T}}(t)P(t)x(t). \tag{9}$$

Proof. Let us consider the bounded feedback control (9). By Proposition 2, the function

$$f(t,x) = r\frac{B(t)B^{\mathrm{T}}(t)P(t)x}{1+\|B^{\mathrm{T}}(t)P(t)x\|}$$

is globally Lipschitz, hence, the closed-loop system

$$\dot{x} = A(t)x + D(t)x(t-h(t)) + B_1(t)w) + f(t,x) \tag{10}$$

has a unique solution. Consider the following non-negative quadratic function

$$V(t,x_t) = V_1(t,x_t) + V_2(t,x_t)$$

where

$$V_1(\cdot) = \mathrm{e}^{\eta t} x^{\mathrm{T}} P_R(t) x(t)$$
$$V_2(\cdot) = \mathrm{e}^{\eta t} \zeta\xi \int_{t-h(t)}^{t} x^{\mathrm{T}}(s)x(s)\mathrm{d}s.$$

Since the integral item of $V(\cdot)$ is non-negative and $P(t) \geq 0$, we have

$$\begin{aligned} V(t,x_t) \geq & \mathrm{e}^{\eta t} x^{\mathrm{T}}(P(t)+R)x(t) \geq \\ & x^{\mathrm{T}}(P(t)+R)x(t) \geq \\ & x^{\mathrm{T}}(t)Rx(t). \end{aligned} \tag{11}$$

Using 2) of Proposition 4 for the following estimations

$$\begin{aligned} x^{\mathrm{T}}(0)x(0) = & x^{\mathrm{T}}(0)R^{\frac{1}{2}}R^{-1}R^{\frac{1}{2}}x(0) \leq \\ & \lambda_{\max}(R^{-1})x^{\mathrm{T}}(0)Rx(0) \end{aligned}$$
$$\int_{-h(0)}^{0} x^{\mathrm{T}}(s)x(s)\mathrm{d}s \leq h\lambda_{\max}(R^{-1}) \sup_{-h\leq s\leq 0} \varphi^{\mathrm{T}}(s)R\varphi(s)$$

we estimate the value of $V(\cdot)$ at $t=0$ to have

$$\begin{aligned} V(0,x_0) = & x^{\mathrm{T}}(0)P_R(0)x(0) + \zeta\xi \int_{-h(0)}^{0} x^{\mathrm{T}}(s)x(s)\mathrm{d}s \leq \\ & \lambda_{\max}(P_R(0))\lambda_{\max}(R^{-1})x^{\mathrm{T}}(0)x(0) + \\ & \zeta h\xi\lambda_{\max}(R^{-1})\phi^{\mathrm{T}}(t)R\phi(t) \leq \\ & \alpha_1 \sup_{-h\leq s\leq 0}\{\varphi^{\mathrm{T}}(s)R\varphi(s)\} \leq \alpha_1 c_1. \end{aligned} \tag{12}$$

Moreover, using 3) of Proposition 4, we get the following estimation

$$V(0,x_0) \leq \alpha_2 ||\varphi||^2 \tag{13}$$

where

$$\alpha_2 = \lambda_{\max}(P_R(0)) + h\zeta\xi,\ \|\varphi\| = \max_{s\in[-h,0]} \|\varphi(s)\|.$$

Taking the derivative of $V_1(\cdot)$ along the solution of the system (10), we have

$$\begin{aligned} \dot{V}_1(t,x_t) = & \eta \mathrm{e}^{\eta t} x^{\mathrm{T}}(t)P_R(t)x(t) + \\ & \mathrm{e}^{\eta t}[x^{\mathrm{T}}(t)\dot{P}_R(t)x(t) + 2x^{\mathrm{T}}(t)P_R(t)\dot{x}(t)] = \\ & \eta \mathrm{e}^{\eta t} x^{\mathrm{T}}(t)P_R(t)x(t) + \\ & \mathrm{e}^{\eta t}\Big[x^{\mathrm{T}}[\dot{P}_R(t) + A^{\mathrm{T}}(t)P_R(t) + P_R(t)A(t)]x + \\ & 2x^{\mathrm{T}}(t)P_R(t)D(t)x(t-h(t)) + 2x^{\mathrm{T}}(t)P_R(t)f(\cdot) + \\ & 2x^{\mathrm{T}}(t)P_R(t)B_1(t)w(t)\Big]. \end{aligned}$$

To get the derivative of $V_2(\cdot)$, we apply the differentiation rule

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} f(s)\mathrm{d}s = \dot{b}(t)f(b(t)) - \dot{a}(t)f(a(t))$$

and have

$$\begin{aligned} \dot{V}_2(.) = \mathrm{e}^{\eta t}\zeta\xi\Big[& x^{\mathrm{T}}(t)x(t) - \\ & [1-\dot{h}(t)]x^{\mathrm{T}}(t-h(t))x(t-h(t))\Big] + \eta V_2(.). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \dot{V}(\cdot) = & \mathrm{e}^{\eta t}\Big[x^{\mathrm{T}}(t)[\dot{P}_R(t) + A^{\mathrm{T}}(t)P_R(t) + P_R(t)A(t) + \\ & \zeta\xi)I]x(t) + 2x^{\mathrm{T}}P_R(t)D(t)x(t-h(t)) + \\ & 2x^{\mathrm{T}}(t)P_R(t)f(\cdot) + 2x^{\mathrm{T}}(t)P_R(t)B_1(t)w(t) - \\ & (1-\dot{h}(t))\zeta\xi x^{\mathrm{T}}(t-h(t)x(t-h(t))\Big] + \eta V(\cdot). \end{aligned}$$

Since

$$-(1-\dot{h}(t)) < -(1-\delta) = -\frac{1}{\zeta}$$

we have

$$\begin{aligned} \dot{V}(\cdot) \leq \mathrm{e}^{\eta t}\Big[& [x^{\mathrm{T}}(t)\dot{P}_R(t) + A^{\mathrm{T}}(t)P_R(t) + P_R(t)A(t) + \\ & \zeta\xi I]x(t) + 2x^{\mathrm{T}}P_R(t)D(t)x(t-h(t)) - \\ & \xi x^{\mathrm{T}}(t-h(t))x(t-h(t)) + 2x^{\mathrm{T}}(t)P_R(t)f(\cdot) + \\ & 2x^{\mathrm{T}}(t)P_R(t)B_1(t)w(t)\Big] + \eta V(\cdot). \end{aligned}$$

Applying Cauchy matrix inequality (Proposition 1) for the following estimations

$$2x^{\mathrm T}P_R(t)D(t)x_h \le \sigma x_h^{\mathrm T}x_h + x^{\mathrm T}P_R^2(t)x$$
$$2x^{\mathrm T}P_RB_1w \le \frac{b_1}{\gamma\beta_2}x^{\mathrm T}P_R^2x + \frac{\gamma\beta_2}{b_1}w^{\mathrm T}B_1^{\mathrm T}B_1w \le \frac{b_1}{\gamma\beta_2}x^{\mathrm T}P_R^2x + \gamma\beta_2 w^{\mathrm T}(t)w.$$

To get estimation of the value $2x^{\mathrm T}P_Rf(\cdot)$, we use 2) of Proposition 2:

$$2x^{\mathrm T}P_R(t)f(\cdot) \le x^{\mathrm T}P_R^2(t)x + f^{\mathrm T}(\cdot)f(\cdot) \le (1+r^2b^4)x^{\mathrm T}P_R^2x.$$

We now apply the Cauchy matrix inequality (Proposition 1) to get the value $z^{\mathrm T}(t)z(t)$ defined from the second equation of system (1) as

$$\begin{aligned}z^{\mathrm T}(t)z(t) =& [C_1(t)x(t)+C_2(t)x(t-h(t))]^{\mathrm T}[C_1(t)x(t)+ C_2(t)x(t-h(t))] =\\ & x^{\mathrm T}(t)C_1^{\mathrm T}(t)C_1(t)x(t)+2x^{\mathrm T}(t-h(t))\times C_2^{\mathrm T}(t)C_1(t)x(t)+x^{\mathrm T}(t-h(t))\times C_2^{\mathrm T}(t)C_2(t)x(t-h(t)) \le\\ & 2x^{\mathrm T}(t)C_1^{\mathrm T}(t)C_1(t)x(t)+2x^{\mathrm T}(t-h(t))\times C_2^{\mathrm T}(t)C_2(t)x(t-h(t)) \le\\ & \beta_2x^{\mathrm T}(t)x(t)+\beta_2x^{\mathrm T}(t-h(t))x(t-h(t))\end{aligned}$$

hence, we obtain

$$\begin{aligned}\dot V(\cdot) \le & \mathrm{e}^{\eta t}\Big[x^{\mathrm T}(t)[\dot P_R(t)+A^{\mathrm T}(t)P_R(t)+P_R(t)A(t)+ (2+r^2b^4+\frac{b_1}{\gamma\beta_2})P_R^2(t)+(\beta_2+\zeta\xi)I]x(t)\Big]+\\ & \mathrm{e}^{\eta t}\gamma\beta_2w^{\mathrm T}(t)w(t)+\eta V(\cdot)-\beta_2\mathrm{e}^{\eta t}z^{\mathrm T}(t)z(t). \qquad (14)\end{aligned}$$

Therefore, from the conditions (7) and (14), it follows that

$$\dot V(\cdot) \le \eta V(\cdot)+\mathrm{e}^{\eta t}\beta_2\gamma w^{\mathrm T}(t)w(t)-\beta_2\mathrm{e}^{\eta t}z^{\mathrm T}(t)z(t). \qquad (15)$$

Multiplying both sides of (15) with $\mathrm{e}^{-\eta t}$, we have

$$\mathrm{e}^{-\eta t}\dot V(\cdot)-\eta\mathrm{e}^{-\eta t}V(\cdot) < \beta_2\gamma w^{\mathrm T}(t)w(t)-\beta_2z^{\mathrm T}(t)z(t) \qquad (16)$$

and hence

$$\mathrm{e}^{-\eta t}\dot V(\cdot)-\eta\mathrm{e}^{-\eta t}V(\cdot) < \beta_2\gamma w^{\mathrm T}(t)w(t),\ \forall t\in[0,\,T] \qquad (17)$$

because of $z^{\mathrm T}(t)z(t)\ge 0$. Note that

$$\frac{\mathrm d}{\mathrm dt}\Big[\mathrm{e}^{-\eta t}V(\cdot)\Big] = \mathrm{e}^{-\eta t}\dot V(\cdot)-\eta\mathrm{e}^{-\eta t}V(\cdot)$$

thus integrating both sides of (17) from 0 to t gives

$$\mathrm{e}^{-\eta t}V(t,x_t) < V(0,x_0)+\beta_2\gamma\int_0^t w^{\mathrm T}(s)w(s)\mathrm ds \quad \forall t\in[0,\,T].$$

Therefore, from the conditions (8), (11) and (12), it follows that

$$\mathrm{e}^{-\eta t}x(t)^{\mathrm T}Rx(t) < V(t,x_t) \le \alpha_1c_1+\beta_2\gamma d,\ \forall t\in[0,T]$$

and hence from (8), it follows that

$$x(t)^{\mathrm T}Rx(t) < (\alpha_1c_1+\beta_2\gamma d)\mathrm{e}^{\eta\mathrm T} \le c_2$$

which implies that the closed-loop system is robustly finite-time stable w.r.t. (c_1,c_2,T,R).

To complete the proof of the theorem, it remains to show the $\gamma-$optimal level condition (6). For this, we consider the following relation

$$\begin{aligned}\int_0^T\left[\beta_2\|z(t)\|^2-\gamma\beta_2\|w(t)\|^2\right]\mathrm dt = & \int_0^T[\beta_2\|z(t)\|^2- \beta_2\gamma\|w(t)\|^2+\frac{\mathrm d}{\mathrm dt}\left(\mathrm{e}^{-\eta t}V(t,x_t)\right)]\mathrm dt-\\ & \int_0^T\frac{\mathrm d}{\mathrm dt}\left(\mathrm{e}^{-\eta t}V(t,x_t)\right)\mathrm dt.\end{aligned}$$

Since $V(t,x_t)\ge 0$ and from (13), it follows that

$$-\int_0^T\frac{\mathrm d}{\mathrm dt}\left(\mathrm{e}^{-\eta t}V(t,x_t)\right)\mathrm dt = -\mathrm{e}^{-\eta\mathrm T}V(T,x_T)+V(0,x_0) \le \alpha_2\|\varphi\|^2.$$

On the other hand, from (16), we have

$$\beta_2\|z(t)\|^2-\beta_2\gamma\|w(t)\|^2+\frac{\mathrm d}{\mathrm dt}\left(\mathrm{e}^{-\eta t}V(t,x_t)\right) < 0$$

therefore,

$$\int_0^T\left[\beta_2\|z(t)\|^2-\beta_2\gamma\|w(t)\|^2\right]\mathrm dt \le \alpha_2\|\varphi\|^2.$$

Setting $c_0=\frac{\alpha_2}{\beta_2\gamma}$, the above inequality yields

$$\sup\frac{\int_0^T\|z(t)\|^2\mathrm dt}{c_0\|\varphi\|^2+\int_0^T\|w(t)\|^2\mathrm dt} \le \gamma.$$

This inequality holds for all non-zero disturbances $w\in L_2([0,T],\mathbf R^{m_1})$, all $\varphi\in C([-h,0],\mathbf R^n)$, and hence the condition (6) holds. □

Remark 1. Theorem 1 gives sufficient conditions for H_∞ control problem via solving some RDEs. Note that, the problem of solving RDEs is, in general, still complicated and some effective methods for solving RDEs can be found, for instance, in [30–32]. However, there is another useful approach used in [27] to solve RDEs via DLMIs and LMIs. We first reduce RDE (7) to the following DLMI by using the Schur complement lemma (Proposition 3):

$$\begin{pmatrix} W(t) & P_R(t)\\ P_R(t) & -\dfrac{\gamma\beta_2}{2\gamma\beta_2+r^2b^4\gamma\beta_2+b_1}I\end{pmatrix} < 0$$

where $W(t)=\dot P_R(t)+A^{\mathrm T}(t)P_R(t)+P_R(t)A(t)+\zeta\xi I$, and then we recast the DLMI in terms of LMIs which can be solved by the approach adopted as in [27]. For example, to recast the above DLMI conditions in terms of LMIs, the matrix-valued function $P(t)$ can be assumed piecewise affine, i.e.,

$$\begin{aligned}&P(0)=L_1^0\\&P(t)=L_k^0+L_k^s(t-(k-1)T_s)\\&\qquad k\in\mathbf{N},\ \ k\le\bar{k},\ \ t\in[(k-1)T_s,kT_s]\\&P(t)=L_{\bar{k}}^0+L_{\bar{k}}^s(t-\bar{k}T_s),\quad t\in[\bar{k}T_s,T]\end{aligned}$$

where $\bar{k}=\max\{k\in\mathbf{N}:k<\frac{T}{T_s}\}$, and

$$L_k^0=L_{k-1}^0+L_{k-1}^sT_s,k=2,3,\cdots,\bar{k}.$$

Since in the k-th time interval it is $\dot{P}(t)=L_k^s$, it readily follows that this approximation permits to recast the proposed DLMI problem into an LMIs feasibility problem with $2\bar{k}$ optimization variables L_k^0,L_k^s. Furthermore, such a piecewise function can approximate a generic continuously differentiable $P(\cdot)$ with adequate accuracy, provided that the length of T is sufficiently small. It should be noticed that a similar approach to solve DLMIs has been adopted in [33].

Remark 2. In Theorem 1, the condition (8) is an exponential inequality with respect to η, since η includes only in an exponent term, this inequality always has solutions. Moreover, since the unknown $\eta>0$ is not included in RDE (7), we first find the solutions of (7) and then check the condition (8).

As an application of Theorem 1, we derive sufficient conditions for finite-time H_∞ control of the following linear autonomous time-delay system with bounded control via Riccati algebraic equations (RAEs):

$$\begin{cases}\dot{x}(t)=Ax(t)+Dx(t-h(t))+Bu(t)+B_1w(t)\\z(t)=C_1x(t)+C_2x(t-h(t))\\x(t)=\varphi(t),\quad t\in[-h,0]\end{cases}\tag{18}$$

where $A,D\in\mathbf{R}^{n\times n},B\in\mathbf{R}^{n\times m},B_1\in\mathbf{R}^{n\times m_1},C_1,C_2\in\mathbf{R}^{n\times m_2}$ are given constant matrices. The control $u(t)$, the disturbance $w(t)$ satisfy the following conditions

$$\begin{aligned}&\exists r>0:\quad ||u(t)||\le r,\quad\forall t\in[0,T]\\&\exists d>0:\quad\int_0^T w^{\mathrm{T}}(t)w(t)\mathrm{d}t\le d.\end{aligned}$$

The following result on solving finite-time H_∞ control problem for linear autonomous time-delay system with bounded control (18) is derived from Theorem 1.

Corollary 1. The finite-time H_∞ control problem for system (18) has a solution if there exist a symmetric positive definite matrix P and a positive number $\eta>0$ satisfying the following conditions:

$$A^{\mathrm{T}}P_R+P_RA+\Big(2+r^2b^4+\frac{b_1}{\gamma\beta_2}\Big)P_R^2+(\beta_2+\zeta\xi)I=0\tag{19}$$

$$\alpha_1c_1+\beta_2\gamma d\le c_2\mathrm{e}^{-\eta\mathrm{T}}.\tag{20}$$

Moreover, the static feedback controller is

$$u(t)=\frac{r}{1+||B^{\top}Px(t)||}B^{\mathrm{T}}Px(t),\quad t\in[0,T].$$

For the case of considered system (1) with constant delay: $h(t)=h$, we choose simple Lyapunov functional

$$V(t,x_t)=\mathrm{e}^{\eta t}x^{\mathrm{T}}P_R(t)x(t)+\mathrm{e}^{\eta t}\xi\int_{t-h}^t x^{\mathrm{T}}(s)x(s)\mathrm{d}s.$$

The Riccati equation (19) is then reduced to

$$A^{\mathrm{T}}P_R+P_RA+(2+r^2b^4+\frac{b_1}{\gamma\beta_2})P_R^2+(\beta_2+\xi)I=0$$

which is, by the Schur complement lemma, Proposition 3, converted to an LMI w.r.t the solution $P>0$ as

$$\begin{pmatrix}A^{\mathrm{T}}P+PA+2\bar{d}PR+W(A,R)\ +(\beta_2+\xi)I & P\\ P & -\frac{1}{\bar{d}}I\end{pmatrix}<0\tag{21}$$

where $\bar{d}=2+r^2b^4+\frac{b_1}{\gamma\beta_2}$ and

$$W(A,R)=A^{\mathrm{T}}R+RA+\bar{d}R^2.$$

Therefore, we get Corollary 2.

Corollary 2. The finite-time H_∞ control problem for system (18), where $h(t)=h$, has a solution if there exist a symmetric positive definite matrix P and a positive number $\eta>0$ satisfying the LMI (21) and the condition (20), where

$$\alpha_1=\lambda_{\max}(P_{\mathbf{R}}(0))\lambda_{\max}(R^{-1})+h\xi\lambda_{\max}(R^{-1}).$$

Moreover, the static feedback controller is

$$u(t)=\frac{r}{1+||B^{\mathrm{T}}Px(t)||}B^{\mathrm{T}}Px(t),\quad t\in[0,T].$$

4 Numerical examples

In this section, numerical examples are provided to show the effectiveness of the method developed in this paper.

Example 1. Consider a linear time-varying system with bounded control described by

$$\begin{cases}\dot{x}_1(t)=(-0.5-1.625\mathrm{e}^t-3\mathrm{e}^{-t})x_1(t)+x_2(t)+\\\qquad x_2(t-h(t))\cos t+\dfrac{1}{t+1}u_1(t)+\mathrm{e}^{-t}w_1(t)\\\dot{x}_2(t)=-x_1(t)+(-0.5-1.625\mathrm{e}^t-3\mathrm{e}^{-t})x_2(t)+\\\qquad x_1(t-h(t))\sin t+u_2(t)+0.5w_2(t)\\z_1(t)=x_1(t)+x_1(t-h(t))\\z_2(t)=\mathrm{e}^{-t}x_2(t)+\mathrm{e}^{-t}x_2(t-h(t))\\x(t)=[\varphi_1(t),\varphi_2(t)],\quad t\in[-0.25,0]\end{cases}$$

where

$$h(t)=0.25\sin^2t,\quad r=1,\quad d=2,\quad\gamma=1.$$

We have

$$A=\begin{pmatrix}-0.5-1.625\mathrm{e}^t-3\mathrm{e}^{-t} & 1\\-1 & -0.5-1.625\mathrm{e}^t-3\mathrm{e}^{-t}\end{pmatrix}$$

$$D=\begin{pmatrix}0 & 1\\\sin t & 0\end{pmatrix},\quad B=\begin{pmatrix}\frac{1}{t+1} & 0\\0 & 1\end{pmatrix}$$

$$B_1 = \begin{pmatrix} e^{-t} & 0 \\ 0 & 0.5 \end{pmatrix},\ C_1 = C_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

We can find that RDE (7) has a solution

$$P(t) = \begin{pmatrix} e^t - 1 & 0 \\ 0 & e^t - 1 \end{pmatrix} \geq 0$$

and the condition (8) is satisfied with

$$\eta = \frac{1}{40}, \quad T = 80, \quad c_1 = 1, \quad c_2 = 50, \quad R = I.$$

By Theorem 1 the system is robustly finite-time stabilizable w.r.t $(1, 50, 100, I)$ and the state feedback controller is

$$\begin{cases} u_1(t) = \dfrac{-e^t x_1(t)}{(1+t)\left(1 + \sqrt{\dfrac{e^{2t}}{(91+t)^4} x_1^2(t) + x_2^2(t)}\right)} \\ u_2(t) = \dfrac{-x_2(t)}{1 + \sqrt{\dfrac{e^{2t}}{(91+t)^4} x_1^2(t) + x_2^2(t)}}, \quad t \in [0, 80]. \end{cases}$$

Fig. 1 shows the trajectories of $x(t)^{\mathrm{T}} R x(t)$ of the closed-loop system with the initial conditions $\varphi(t) = [0.9, 0.4]$.

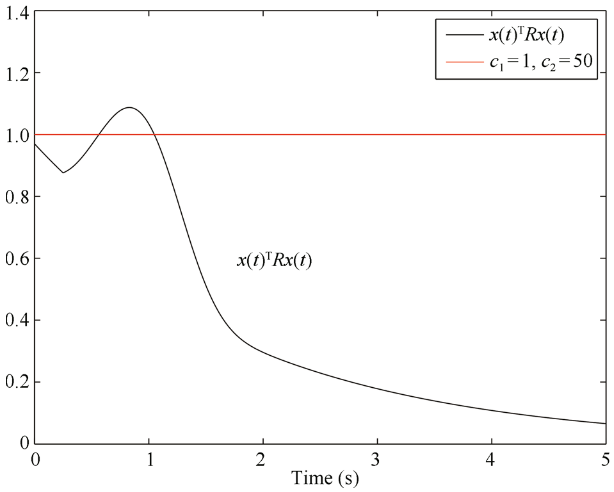

Fig. 1 Trajectories of $x(t)^{\mathrm{T}} Q x(t)$ of system (1)

Example 2. Consider a linear autonomous system with bounded control described by

$$\begin{cases} \dot{x}_1(t) = -5x_1(t) + x_2(t - h(t)) + w_1(t) \\ \dot{x}_2(t) = -5.875x_2(t) + x_1(t - h(t)) + u(t) + w_2(t) \\ z_1(t) = 0.5x_1(t) + 0.5x_1(t - h(t)) \\ z_2(t) = 0.5x_2(t) + 0.5x_2(t - h(t)) \\ x(t) = [\varphi_1(t), \varphi_2(t)], \quad t \in [-0.25, 0] \end{cases}$$

where

$$h(t) = 0.25 \sin^2 t, \quad r = d = \gamma = 1.$$

We have

$$A = \begin{pmatrix} -5 & 0 \\ 0 & -5 \end{pmatrix},\ D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_1 = C_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}.$$

Note that the control system $[A, B]$ is uncontrollable, but we can find that the algebraic Riccati equations (ARE) (19) has a solution

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} > 0$$

and the condition (20) is satisfied with

$$T = 100, \quad \eta = \frac{1}{50}, \quad c_1 = 1, \quad c_2 = 58, \quad R = I.$$

By Corollary 1, the system is robustly finite-time stabilizable w.r.t $(1, 58, 100, I)$, and the state feedback controller is

$$u(t) = \frac{2x_2(t)}{1 + 2|x_2(t)|}, \quad t \in [0, 100].$$

Fig. 2 shows the trajectories of $x(t)^{\mathrm{T}} R x(t)$ of the closed-loop system with the initial conditions $\varphi(t) = [0.7, -0.7]$.

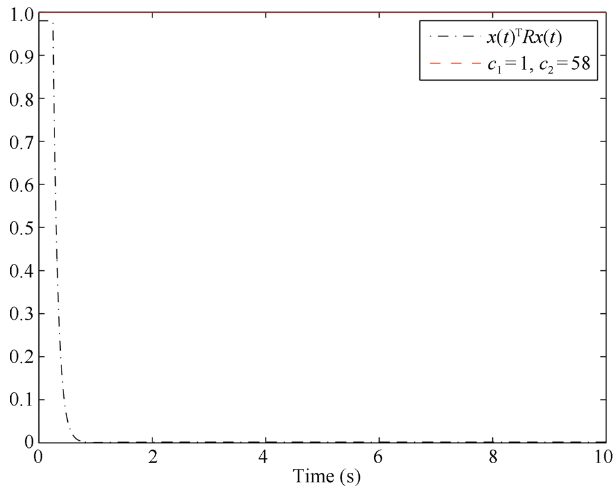

Fig. 2 Trajectories of $x(t)^{\mathrm{T}} Q x(t)$ of system (12)

Example 3. Consider the following linear system with constant delay:

$$\begin{cases} \dot{x}_1(t) = 2x_1(t) + x_2(t) + 0.5x_1(t-2) + u_1(t) + w_1(t) \\ \dot{x}_2(t) = -x_1(t) + x_2(t) + x_2(t-2) + 0.5u_2(t) + w_2(t) \\ z_1(t) = 0.2x_1(t) + x_1(t-2) \\ z_2(t) = 0.5x_2(t) + x_2(t-2) \\ x(t) = [\varphi_1(t), \varphi_2(t)], \quad t \in [-2, 0] \end{cases}$$

where $h = 2, d = 2, r = 1, \gamma = 2$. We have

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix},\ B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.5 \end{pmatrix},\ C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We can find that the LMI (20) has a solution

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} > 0$$

and the condition (20) is satisfied with

$$T = 80, \quad \eta = \frac{1}{80}, \quad c_1 = 1, \quad c_2 = 50, \quad R = I.$$

By Corollary 2, the system is robustly finite-time stabilizable w.r.t $(1, 50, 80, I)$ and the state feedback controller is

$$\begin{cases} u_1(t) = \dfrac{x_1^2(t)}{1 + x_1^2(t)}, \quad t \in [0, 80] \\ u_2(t) = 0. \end{cases}$$

5 Conclusions

In this paper, the problem of finite-time stabilization and H_∞ control for linear time-varying delay systems with bounded control has been studied. Different from the existing results, we have obtained new sufficient conditions for the finite-time stabilization and finite-time H_∞ control problem via nonlinear feedback controller $u(t) = g(x(t))$. The solution of the H_∞ control problem can be efficiently solved by means of Riccati differential equations. Applications to finite-time H_∞ control to a class of linear autonomous systems with delays have been presented in this paper. Our paper raises certain questions that are important from the point of view of the bounded stabilization theory, in particular the construction of output feedback controllers $u(t) = g(z(t))$ satisfying condition (3).

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