

# Stability Analysis of Discrete-Dime Positive Switched Linear Delay Systems

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**Abstract**—This paper is concerned with the stability problem of discrete-time positive switched linear systems with delays. The states of the systems under consideration are confined in the positive orthant, and the delays can be time-varying and not necessarily bounded. Based on a novel approach, a delay-independent stability criterion is established for such systems. The result coincides with some well-known properties of positive system when the considered switched system has only one mode, and also improves an earlier result. A numerical example is provided to demonstrate the obtained result.

## I. INTRODUCTION

Many systems in the real world can be modelled by dynamic systems with combined continuous and discrete states [1], [2], [3]. Such systems are called switched systems. A switched system consists of a family of (finitely or infinitely many) dynamic subsystems (sometimes called modes) with a rule, called a switching signal, that determines the switching behavior amongst the subsystems. Switched systems have attracted a lot of attention from the control and system theory communities due to their abilities to capture the dynamics of various physical systems [4], [5], [6], [7], [8], [9]. There are still many open and challenging issues remained to be tackled, despite great successes have been reported during the last several decades [10], [11].

A special class of switched systems is positive switched system. Positive switched linear systems (PSLSs) have attracted many researchers from the community of control. The importance of PSLSs has been highlighted due to their broad applications in formation flying [12] and systems theory [13], [14]. A positive system is one that its state and output are nonnegative whenever the initial condition

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and input are nonnegative [15], [16], [17]. A PSLS corresponds to a switched linear system in which each subsystem is itself a positive system. Positive systems have numerous applications in areas such as economics, biology, sociology and communication [18], [19]. It is well known that positive systems have many special and interesting properties. For example, their stability is not affected by delays [20], [21], [22], [23], [24]. It should be pointed out that studying the dynamics of positive switched systems is more challenging than that of general switched systems because, in order to obtain some “elegant” results, one has to combine the features of positive systems and switched systems [25].

A wealth of literature is concerned with the issue of stability of PSLSs [26], [27], [28]. It has been proven in [26] that for a PSLS consisting of two 2nd order subsystems, it is uniformly asymptotically stable if and only if the system has a common copositive quadratic Lyapunov function. A necessary and sufficient condition has been established in [28] for the existence of a common linear copositive Lyapunov function, where the switched system consists of two  $n$ th order subsystems. A switched copositive linear Lyapunov function has been proposed in [27], which yields a less conservative stability condition for PSLSs.

Delays are universal in physical processes and have very significant impacts on their dynamics. This fact makes the task of studying the dynamics of positive switched linear delay systems (PSLDSs) important. However, to our knowledge, up to now, there has been little results obtained for the stability problem of PSLDSs. Since PSLDSs form a special class of switched linear systems, the relevant methods applicable to switched linear delay systems are also suitable for PSLDSs. These methods, such as the popular Lyapunov-Krasovskii functional method, however, generally fail to capture the inherent characteristics of PSLDSs. In order to obtain the desirable results, it is necessary to develop new approaches and techniques. The stability issue of PSLDSs is studied in [29], where it is assumed that there exists a common positive vector  $\lambda$  such that  $(\sum_{i=0}^p A_{il} - I)\lambda \prec 0$  with  $\sum_{i=0}^p A_{il}$  being the sum of

all the system matrices of the  $l$ th subsystem. The main idea in [29] is to construct a “cover” that covers all the solution trajectories starting from a specified region.

This paper focuses on the stability problem of discrete-time PSLDSs. Similar to the idea of [29], the “covering” approach is also used here. The main difference between this paper and [29] lies in that the cover in [29] is constructed using a common positive vector  $\lambda$ , while the cover in this paper is constructed using  $m$  different positive vectors  $\lambda_l, l \in \{1, 2, \dots, m\}$ , where  $m$  is the number of subsystems contained in the considered switched system.

The rest of this paper is organized as follows. In Section II, preliminaries are presented and the problem to be treated is stated. Section III proposes a stability criterion for discrete-time PSLDSs. An example is given in Section IV, and Section V concludes this paper.

## II. PROBLEM STATEMENTS AND PRELIMINARIES

### Nomenclatures

$A \succeq 0 (\succ 0)$	Matrix $A$ with nonnegative (positive) elements
$A \preceq 0 (\prec 0)$	$-A \succeq 0 (\succ 0)$
$A^T$	Transpose of matrix $A$
$\mathbb{R}^n (\mathbb{R}_+^n)$	The set of $n$ -dimensional real (positive) vectors
$\mathbb{R} (\mathbb{R}_+)$	$\mathbb{R}^1 (\mathbb{R}_+^1)$
$\mathbb{R}^{n \times m}$	The set of all real matrices of $n \times m$ -dimension
$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{N}_0$	$\{0\} \cup \mathbb{N}$
$\underline{p}$	$\{1, 2, \dots, p\}$ , where $p \in \mathbb{N}$
$\underline{p}_0$	$\{0\} \cup \underline{p}$
$\ \mathbf{x}\ _\infty$	$l_\infty$ norm of vector $\mathbf{x} \in \mathbb{R}^n$ , that is, $\ \mathbf{x}\ _\infty = \max \{ x_1 ,  x_2 , \dots,  x_n \}$
$[a]$	Minimum integer not less than real number $a$

Consider a discrete-time switched linear system as follows:

$$\begin{aligned} \mathbf{x}(k+1) &= A_{0\sigma(k)} \mathbf{x}(k) + \sum_{i=1}^p A_{i\sigma(k)} \mathbf{x}(k - \tau_{i\sigma(k)}(k)) \\ k &\in \mathbb{N}_0 \\ \mathbf{x}(k) &= \varphi(k), \quad k = -\tau, \dots, 0 \end{aligned} \tag{1}$$

where the map  $\sigma : \mathbb{N}_0 \rightarrow \underline{m}$  is an arbitrary switching signal with  $m$  being the number of subsystems,  $A_{il} \in \mathbb{R}^{n \times n}, i \in \underline{p}_0, l \in \underline{m}$ , are the system matrices,  $\tau_{il}(k) \geq 0$  are time-varying delays, and  $\tau \in \mathbb{N}_0$  is a constant so that system (1) is well defined. Of

course, for each  $i \in \underline{p}, l \in \underline{m}$ , it is assumed that  $\tau_{il}(k) \not\equiv 0, k \in \mathbb{N}_0$ . A subsystem, say the  $l$ th one, is activated if and only if  $\sigma(k) = l$ . Hence, system (1) evolves according to the equation  $\mathbf{x}(k+1) = A_{0l} \mathbf{x}(k) + \sum_{i=1}^p A_{il} \mathbf{x}(k - \tau_{il}(k))$  if  $\sigma(k) = l$ .

*Definition 1 ([29]):* System (1) is said to be positive if, for any initial condition  $\varphi(k) \succeq 0, k \in \{-\tau, \dots, 0\}$ , and any switching signal, the corresponding trajectory  $\mathbf{x}(k) \succeq 0$  holds for all  $k \in \mathbb{N}$ .

*Lemma 1 ([29]):* System (1) is positive if and only if  $A_{il} \succeq 0, i \in \underline{p}_0, l \in \underline{m}$ .

*Lemma 2 ([29]):* Assume that system (1) is positive and arbitrarily fix a switching signal  $\sigma$ . Let  $\mathbf{x}_a(k)$  and  $\mathbf{x}_b(k), \forall k \in \mathbb{N}$ , be the solution trajectories of (1) under initial conditions  $\varphi_a(k)$  and  $\varphi_b(k), k \in \{-\tau, \dots, 0\}$ , respectively. Then  $0 \preceq \varphi_a(k) \preceq \varphi_b(k) (k \in \{-\tau, \dots, 0\})$  implies that  $\mathbf{x}_a(k) \preceq \mathbf{x}_b(k), \forall k \in \mathbb{N}$ .

In the sequel, we always assume that the following assumption holds for system (1).

*Assumption 1:* In system (1), there exist  $T \in \mathbb{N}$  and a set of scalars  $0 \leq \theta_{il} < 1$  such that

$$\theta_{il} = \sup_{k \geq T} \frac{\tau_{il}(k)}{k}, \quad i \in \underline{p}, l \in \underline{m}. \tag{2}$$

*Remark 1:* Constraint (2) on the delays in system (1) is very mild. In fact, all the bounded delays satisfy (2), whether time-varying or not. If  $\tau_{il}(k) \leq \tau^*$ , then  $T$  can take any integral value such that  $T \geq \tau^* + 1$  (hence  $T \geq 2$ ). Obviously, delays satisfying (2) may be unbounded. In this case,  $T$  can always be assumed to be greater than or equal to 2.

It should be pointed out that even when delays  $\tau_{il}(k)$  satisfying (2) are unbounded, the initial condition can also be defined on a finite set so that (1) is well defined, see [29, Remark 2] for details.

The aim of this paper is to present a condition which enables us to determine whether or not the positive system (1) is asymptotically stable under arbitrary switching signal.

## III. MAIN RESULTS

This section establishes a stability condition for discrete-time PSLDSs, which begins with the following lemma.

*Lemma 3:* Consider PSLDS (1) under Assumption 1. Suppose that there exist  $m$  vectors  $\lambda_l \in \mathbb{R}_+^n$  such that

$$\sum_{i=0}^p A_{il} \lambda_{l(i)} \prec \lambda_l \tag{3}$$

hold for all  $l \in \underline{m}$  and all choices of  $l(i) \in \underline{m}$ .

Let  $T \geq 2$  be given in (2) and  $\theta = \max_{i \in \underline{p}, l \in \underline{m}} \{\theta_{il}\} <$

1. Define  $Q_0 = 0, Q_1 = T, Q_{q+1} = \left\lceil \frac{Q_q + 1}{1 - \theta} \right\rceil, q \in \mathbb{N}$ .

Suppose that the initial condition is  $\phi(k) \equiv \lambda, k \in \{-\tau, \dots, 0\}$ , with  $\lambda$  being a constant positive vector satisfying  $\lambda \preceq \lambda_l, \forall l \in \underline{m}$ . Then there exists a scalar  $0 < \mu < 1$  such that

$$\mathbf{x}(k) \prec \mu^{q+1} \lambda_{\sigma(k-1)}, \quad \forall k \in \mathcal{R}_q \quad (4)$$

hold for all  $q \in \mathbb{N}_0$ , where  $\mathcal{R}_q = \{Q_q + 1, Q_q + 2, \dots, Q_{q+1}\}$ .

*Proof:* By condition (3), there exists a set of sufficiently small constants  $\alpha_l > 0$  such that

$$\sum_{i=0}^p A_{il} \lambda_{l(i)} \prec \lambda_l - \alpha_l \lambda_l, \quad \forall l \in \underline{m}. \quad (5)$$

Take  $\alpha = \min_{l \in \underline{m}} \{\alpha_l\}$ . Then

$$\sum_{i=0}^p A_{il} \lambda_{l(i)} \prec (1 - \alpha) \lambda_l, \quad \forall l \in \underline{m}.$$

Let  $\mu = 1 - \alpha$  and we have

$$\sum_{i=0}^p A_{il} \lambda_{l(i)} \prec \mu \lambda_l, \quad \forall l \in \underline{m}. \quad (6)$$

Since  $\alpha > 0$  is a sufficiently small constant, one can ensure that  $0 < \mu < 1$ . The rest of the proof will be completed by induction.

First, we prove that (4) holds for  $q = 0$ . Since  $\phi(k) \equiv \lambda, k \in \{-\tau, \dots, 0\}$ ,

$$\begin{aligned} \mathbf{x}(1) &= A_{0\sigma(0)} \mathbf{x}(0) + \sum_{i=1}^p A_{i\sigma(0)} \mathbf{x}(0 - \tau_{i\sigma(0)}(0)) \\ &= \sum_{i=0}^p A_{i\sigma(0)} \lambda. \end{aligned}$$

By the facts that  $\lambda \preceq \lambda_l, \forall l \in \underline{m}$  and that  $A_{il} \succeq 0, i \in \underline{p}_0, l \in \underline{m}$ , inequality (6) implies that

$$\mathbf{x}(1) = \sum_{i=0}^p A_{i\sigma(0)} \lambda \preceq \mu \lambda_{\sigma(0)}.$$

Suppose that  $\mathbf{x}(k) \prec \mu \lambda_{\sigma(k-1)}$  hold for all  $k$  satisfying

$$Q_0 + 1 \leq k \leq s$$

for some

$$s \in \{Q_0 + 1, Q_0 + 2, \dots, Q_1 - 1\}.$$

Clearly, such an  $s$  exists and  $\mathbf{x}(s) \prec \mu \lambda_{\sigma(s-1)}$ . Let  $S_{0,1}(s) = \{i : s - \tau_{i\sigma(s)}(s) \leq Q_0\}$  and  $S_{0,2}(s) = \{i : s - \tau_{i\sigma(s)}(s) > Q_0\}$ . Then,

$$\begin{aligned} \mathbf{x}(s+1) &= A_{0\sigma(s)} \mathbf{x}(s) + \sum_{i=1}^p A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\ &= A_{0\sigma(s)} \mathbf{x}(s) + \sum_{i \in S_{0,1}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\ &\quad + \sum_{i \in S_{0,2}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)). \end{aligned} \quad (7)$$

If  $i \in S_{0,1}(s)$ , then  $\mathbf{x}(s - \tau_{i\sigma(s)}(s)) = \lambda$ ; if  $i \in S_{0,2}(s)$ , then by assumption,  $\mathbf{x}(s - \tau_{i\sigma(s)}(s)) \prec \mu \lambda_{\sigma(s - \tau_{i\sigma(s)}(s) - 1)}$ . It follows from (7) and (3) that

$$\begin{aligned} \mathbf{x}(s+1) &\preceq \mu A_{0\sigma(s)} \lambda_{\sigma(s-1)} + \sum_{i \in S_{0,1}(s)} A_{i\sigma(s)} \lambda \\ &\quad + \sum_{i \in S_{0,2}(s)} \mu A_{i\sigma(s)} \lambda_{\sigma(s - \tau_{i\sigma(s)}(s) - 1)} \\ &\prec \mu \lambda_{\sigma(s)}. \end{aligned}$$

Therefore, (4) holds for  $q = 0$ .

Second, assume that (4) holds for  $q = j \in \mathbb{N}_0$ , that is,

$$\begin{aligned} \mathbf{x}(k) &\prec \mu^{j+1} \lambda_{\sigma(k-1)} \\ \forall k \in \mathcal{R}_j &= \{Q_j + 1, Q_j + 2, \dots, Q_{j+1}\}. \end{aligned} \quad (8)$$

Finally, we show that (4) holds for  $q = j + 1$ , that is,

$$\begin{aligned} \mathbf{x}(k) &\prec \mu^{j+2} \lambda_{\sigma(k-1)} \\ \forall k \in \mathcal{R}_{j+1} &= \{Q_{j+1} + 1, Q_{j+1} + 2, \dots, Q_{j+2}\}. \end{aligned} \quad (9)$$

By the definitions of  $\theta$  and  $Q_q, q \in \mathbb{N}$ , one obtains that  $0 \leq \tau_{i\sigma(s)}(s) \leq \theta_{i\sigma(s)} s \leq \theta s$  holds for all  $s \geq T = Q_1$ . So, if  $s \geq Q_{j+1}, j \in \mathbb{N}_0$ , then

$$\begin{aligned} s - \tau_{i\sigma(s)}(s) &\geq s - \theta s = (1 - \theta)s \\ &\geq (1 - \theta)Q_{j+1} \\ &\geq Q_j + 1. \end{aligned} \quad (10)$$

Combining (8), (10), and (3) yields that

$$\begin{aligned} &\mathbf{x}(Q_{j+1} + 1) \\ &= A_{0\sigma(Q_{j+1})} \mathbf{x}(Q_{j+1}) + \\ &\quad \sum_{i=1}^p A_{i\sigma(Q_{j+1})} \mathbf{x}(Q_{j+1} - \tau_{i\sigma(Q_{j+1})}(Q_{j+1})) \\ &\preceq \mu^{j+1} A_{0\sigma(Q_{j+1})} \lambda_{\sigma(Q_{j+1}-1)} + \\ &\quad \sum_{i=1}^p \mu^{j+1} A_{i\sigma(Q_{j+1})} \lambda_{\sigma(Q_{j+1} - \tau_{i\sigma(Q_{j+1})}(Q_{j+1}) - 1)} \\ &\prec \mu^{j+2} \lambda_{\sigma(Q_{j+1})}. \end{aligned}$$

Having shown that (9) holds for all  $k$  satisfying  $Q_{j+1} + 1 \leq k \leq s$  for some  $s \in \{Q_{j+1} + 1, Q_{j+1} + 2, \dots, Q_{j+2} - 1\}$ , now we prove that it holds for  $k = s + 1$ .

Define  $S_{j+1,1}(s) = \{i : s - \tau_{i\sigma(s)}(s) \leq Q_{j+1}\}$  and  $S_{j+1,2}(s) = \{i : s - \tau_{i\sigma(s)}(s) > Q_{j+1}\}$ . It follows

from (10) that

$$\begin{aligned}
& \mathbf{x}(s+1) \\
&= A_{0\sigma(s)} \mathbf{x}(s) + \sum_{i=1}^p A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\
&\preceq \mu^{j+2} A_{0\sigma(s)} \lambda_{\sigma(s-1)} \\
&\quad + \sum_{i \in S_{j+1,1}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)) \\
&\quad + \sum_{i \in S_{j+1,2}(s)} A_{i\sigma(s)} \mathbf{x}(s - \tau_{i\sigma(s)}(s)).
\end{aligned}$$

Note that if  $i \in S_{j+1,1}(s)$  then  $\mathbf{x}(s - \tau_{i\sigma(s)}(s)) \preceq \mu^{j+1} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)}$ , otherwise  $\mathbf{x}(s - \tau_{i\sigma(s)}(s)) \preceq \mu^{j+2} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)}$ . Hence,

$$\begin{aligned}
& \mathbf{x}(s+1) \\
&\preceq \mu^{j+2} A_{0\sigma(s)} \lambda_{\sigma(s-1)} \\
&\quad + \sum_{i \in S_{j+1,1}(s)} \mu^{j+1} A_{i\sigma(s)} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)} \\
&\quad + \sum_{i \in S_{j+1,2}(s)} \mu^{j+2} A_{i\sigma(s)} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)} \\
&\preceq \mu^{j+1} A_{0\sigma(s)} \lambda_{\sigma(s-1)} \\
&\quad + \sum_{i \in S_{j+1,1}(s)} \mu^{j+1} A_{i\sigma(s)} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)} \\
&\quad + \sum_{i \in S_{j+1,2}(s)} \mu^{j+1} A_{i\sigma(s)} \lambda_{\sigma(s-\tau_{i\sigma(s)}(s)-1)} \\
&\prec \mu^{j+2} \lambda_{\sigma(s)}
\end{aligned}$$

which implies that (9) holds for all  $k \in \mathcal{R}_{j+1}$ . Therefore, if (4) holds for  $q = j \in \mathbb{N}_0$ , then so does for  $q = j + 1$ . By the principle of mathematical induction, the lemma follows immediately. ■

*Remark 2:* Suppose that in system (1),  $m = 2$  and  $p = 1$ . Let  $l = 1$ , and it follows from (3) that

$$A_{01}\lambda_{1(0)} + A_{11}\lambda_{1(1)} \prec \lambda_1.$$

Since both  $1(0)$  and  $1(1)$  run over the set  $\{1, 2\}$ , we further obtain from the above inequality that

$$\begin{aligned}
A_{01}\lambda_1 + A_{11}\lambda_1 &\prec \lambda_1, \\
A_{01}\lambda_2 + A_{11}\lambda_1 &\prec \lambda_1, \\
A_{01}\lambda_2 + A_{11}\lambda_2 &\prec \lambda_1.
\end{aligned}$$

It can be easily shown that if system (1) consists of  $m$  subsystems and each subsystem has  $p$  delays, then condition (3) involves a total  $m^{p+2}$  such inequalities.

Now we present our main result.

*Theorem 1:* Positive system (1) under Assumption 1 is asymptotically stable if there exist  $m$  vectors  $\lambda_l \in \mathbb{R}_+^n$  such that (3) holds.

*Proof:* Suppose that there exist  $m$  vectors  $\lambda_l \in \mathbb{R}_+^n$  such that (3) holds. Pick a vector  $\lambda$  satisfying  $\lambda \preceq \lambda_l, \forall l \in \underline{m}$ .

For an arbitrary given scalar  $\varepsilon > 0$ , there always exists a scalar  $v > 0$  such that  $\|v\lambda_l\|_\infty < \varepsilon, \forall l \in \underline{m}$ . By Lemma 3, under the initial condition  $\tilde{\varphi}(k) = v\lambda, k \in \{-\tau, \dots, 0\}$ , the corresponding solution  $\tilde{\mathbf{x}}(k)$  of (1) satisfies

$$\tilde{\mathbf{x}}(k) \prec v\mu^{q+1} \lambda_{\sigma(k-1)}, \forall k \in \mathcal{R}_q, \forall q \in \mathbb{N}_0$$

for some scalar  $0 < \mu < 1$ .

Arbitrarily choose another initial condition  $\varphi(k)$  such that  $\|\varphi(k)\|_\infty < \delta = \min\{v\lambda_1, \dots, v\lambda_n\}, k \in \{-\tau, \dots, 0\}$ , with  $\lambda_i$  being the  $i$ th element of  $\lambda$  so that  $\|\varphi(k)\|_\infty < \|\tilde{\varphi}(k)\|_\infty, \forall k \in \{-\tau, \dots, 0\}$ , and denote the corresponding solution to system (1) by  $\mathbf{x}(k)$ . By Lemma 2,

$$\begin{aligned}
\mathbf{x}(k) &\preceq \tilde{\mathbf{x}}(k) \\
&\prec v\mu^{q+1} \lambda_{\sigma(k-1)}, \forall k \in \mathcal{R}_q, q \in \mathbb{N}_0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathbf{x}(k)\|_\infty &\leq \|\tilde{\mathbf{x}}(k)\|_\infty \\
&\leq \|v\mu^{q+1} \lambda_{\sigma(k-1)}\|_\infty \\
&= \mu^{q+1} \|v\lambda_{\sigma(k-1)}\|_\infty \\
&< \mu^{q+1} \varepsilon, \forall k \in \mathcal{R}_q, q \in \mathbb{N}_0.
\end{aligned} \tag{11}$$

Since  $\bigcup_{q=0}^\infty \mathcal{R}_q = \mathbb{N}$ , (11) indicates that  $\|\mathbf{x}(k)\|_\infty < \varepsilon, \forall k \in \mathbb{N}$ .

Moreover, as  $\mathcal{R}_q \cap \mathcal{R}_{q+1} = \emptyset$ , for each  $k \in \mathbb{N}$ , there exists a unique  $q(k) \in \mathbb{N}_0$  such that  $k \in \mathcal{R}_{q(k)}$ , and  $q(k) \rightarrow +\infty$  as  $k \rightarrow +\infty$ . By Lemma 3, one has

$$\begin{aligned}
\lim_{k \rightarrow +\infty} \|\mathbf{x}(k)\|_\infty &\leq \lim_{k \rightarrow +\infty} \left\| \mu^{q(k)+1} v\lambda_{\sigma(k-1)} \right\|_\infty \\
&\leq \lim_{k \rightarrow +\infty} \mu^{q(k)+1} \varepsilon = 0.
\end{aligned}$$

Hence, system (1) is asymptotically stable. ■

*Remark 3:* Some notes can be made with respect to Theorem 1.

- 1) In [29], a simpler stability condition was established, which says that system (1) is asymptotically stable if there exists a vector  $\lambda \in \mathbb{R}_+^n$  satisfying  $(\sum_{i=0}^p A_{il} - I) \lambda \prec 0, \forall l \in \underline{m}$ , which can be obtained from (3) by replacing all  $\lambda_{l(i)}$  and  $\lambda_l$  with  $\lambda$ . Therefore, that result is a corollary of Theorem 1 in this paper.
- 2) Condition (3) is delay size-independent and delay rate-independent, and the considered switching signal is arbitrary.
- 3) Condition (3) in Theorem 1 is a linear programming problem in a set of vectors  $\lambda_l, l \in \underline{m}$ , and can be numerically solved by standard linear programming package (see [30]). However, when the number of subsystems or delays is large, computation can still be heavy.

- 4) If system (1) consists of only one mode, that is,  $l = 1$ , then it reduces to a standard positive system. In this case, condition (3) becomes  $(\sum_{i=0}^p A_{il}/\lambda - I) \prec 0$ , which coincides with the condition in [29, Corollary 1]. And if system (1) has no delay, that is,  $p = 0$ , then condition (3) leads to a result in [27, Theorem 1, Item 2)].

#### IV. NUMERICAL EXAMPLE

Consider the following system:

$$\mathbf{x}(k+1) = A_{0\sigma(k)}\mathbf{x}(k) + A_{1\sigma(k)}\mathbf{x}(k - \tau_{\sigma(k)}(k)) \quad (12)$$

where  $\mathbf{x}(k) = [x_1(k), x_2(k)]^T \in \mathbb{R}^2$ ,  $\sigma : \mathbb{N}_0 \rightarrow \{1, 2\}$ , and

$$A_{01} = \begin{bmatrix} 0.3a & 0.33 \\ 0.04 & 0.1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.7a & 0.33 \\ 0.06 & 0.15 \end{bmatrix}$$

$$A_{02} = \begin{bmatrix} 0.0733 & 0.06 \\ 0.32 & 0.12 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.66 & 0.14 \\ 0.48 & 0.28 \end{bmatrix}$$

with  $a > 0$  is an adjustable parameter. Clearly, system (12) is a positive switched system.

Using the routine `linprog.m` in MATLAB, we can easily check that the stability domain of  $a$  is the interval  $[0, 0.1198]$  according to Theorem 1 in [29], while the stability domain of  $a$  is the interval  $[0, 0.3109]$  according to Theorem 1 in this paper. Of course, our result is less conservative.

To see that  $[0, 0.3109]$  is indeed a stability region, let us choose  $a = 0.3109$ , and the simulations with different delays are shown in Figures 1–4, respectively, where the initial conditions and switching signals are randomly generated. From these figures, one can see that system (12) is indeed asymptotically stable. Comparing Figures 1 and 3, one may find that, in general, the decay rate of the trajectory becomes small when the ratio of  $\tau_i(k)/k$  increases.

#### V. CONCLUSION

The stability issue of PSLDSs has been addressed and a stability condition has been proposed. Compared with previous results, the condition in this paper is less conservative. A numerical example has been carried out to demonstrate the obtained theoretical result.

The key of the method used in this paper is that several different constant vector  $\lambda_l$  are used to construct a “cover” which covers all the solution trajectories of the system starting from a specified domain. In spite of its low conservatism, this result is only a *sufficient* stability condition. How to further reduce its conservatism is a challenging work in the future.

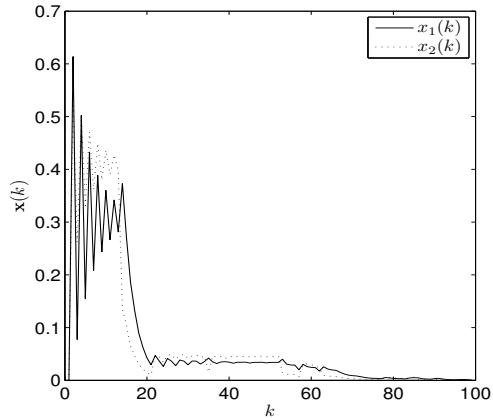


Fig. 1. Evolution of system (12) with delays  $\tau_1 = [0.05k]$ ,  $\tau_2 = [0.06k]$ .

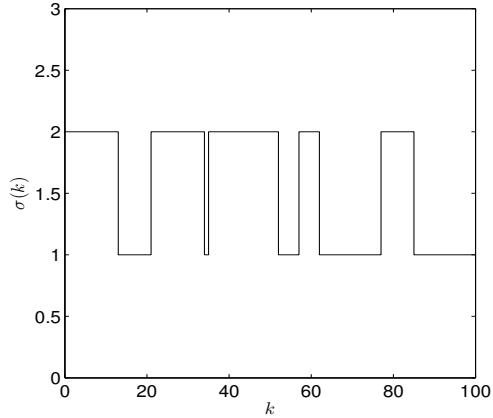


Fig. 2. Switching signal  $\sigma(k)$  corresponding to Figure 1

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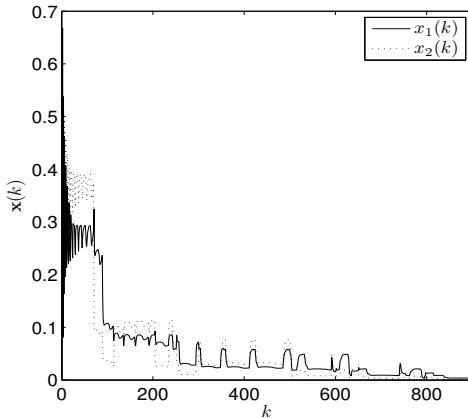


Fig. 3. Evolution of system (12) with delays  $\tau_1 = \lceil 0.2k \rceil$ ,  $\tau_2 = \lceil 0.15k \rceil$ .

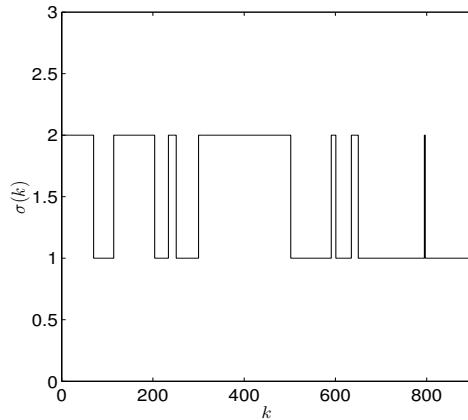


Fig. 4. Switching signal  $\sigma(k)$  corresponding to Figure 3

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