

## REGULARIZATION OF DESCRIPTOR SYSTEMS BY DERIVATIVE AND PROPORTIONAL STATE FEEDBACK\*

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*Dedicated to Gene Golub on the occasion of his 60th birthday*

**Abstract.** For linear multivariable time-invariant continuous or discrete-time singular systems it is customary to use a proportional feedback control in order to achieve a desired closed loop behaviour. Derivative feedback is rarely considered. This paper examines how derivative feedback in descriptor systems can be used to alter the structure of the system pencil under various controllability conditions. It is shown that derivative and proportional feedback controls can be constructed such that the closed loop system has a given form and is also regular and has index at most 1. This property ensures the solvability of the resulting system of dynamic-algebraic equations. The construction procedures used to establish the theory are based only on orthogonal matrix decompositions and can therefore be implemented in a numerically stable way. The problem of pole placement with derivative feedback alone and in combination with proportional state feedback is also investigated. A computational algorithm for improving the “conditioning” of the regularized closed loop system is derived.

**Key words.** differential-algebraic systems, singular systems, controllability, regularizability, numerical stability, optimal conditioning

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**1. Introduction.** We consider linear time-invariant continuous or discrete-time dynamical systems of the form

$$(1) \quad E\dot{x} := E dx/dt = Ax(t) + Bu(t), \quad x(t_0) = x_0,$$

$$(2) \quad y(t) = Cx(t),$$

or

$$(3) \quad Ex_{k+1} = Ax_k + Bu_k, \quad x_0 \text{ given},$$

$$(4) \quad y_k = Cx_k,$$

where  $E, A \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,m}$ ,  $C \in \mathbb{R}^{p,n}$ , and  $\text{rank}(B) = m \leq n$ ,  $\text{rank}(C) = p \leq n$ . Here  $x(t)$  or  $x_k \in \mathbb{R}^n$  is the state,  $y(t)$  or  $y_k \in \mathbb{R}^p$  is the output, and  $u(t)$  or  $u_k \in \mathbb{R}^m$  is the input or control of the system. Such systems are called *descriptor* or *generalized state-space* systems. In the case  $E = I$ , the identity matrix, we refer to (1), (2) and (3), (4) as *standard* systems.

Descriptor systems arise naturally in a variety of circumstances [19], [13] and have recently been investigated in a number of papers [18], [4]–[7], [10]–[12], [14], [16], [17], [20]–[26]. The response of a descriptor system can be described in terms of the eigenstructure of the matrix pencil

$$(5) \quad \alpha E - \beta A.$$

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In order to alter the behaviour of the system, it is customary to use proportional state or output feedback to modify the matrix  $A$ . The closed loop system pencil then becomes

$$(6) \quad \alpha E - \beta(A + BFC),$$

where the control is taken to be  $u = Fy + v$  or  $u_k = Fy_k + v_k$ . In the theory of matrix pencils, the roles of  $E$  and  $A$  are interchangeable, but the analogous use of derivative state or output feedback in multivariable systems has not been investigated much in the literature. Derivative feedback modifies the matrix  $E$ , and the closed loop system pencil then becomes

$$(7) \quad \alpha(E + BGC) - \beta A,$$

where the control is taken to be  $u = -G\dot{y} + v$  or  $u_k = -Gy_{k+1} + v_k$ .

Derivative information has long been used in the practical design of PD controllers. Recently it has been applied in the construction of a discrete-time observer using both current and past output data in the current state estimation [18]. This leads to a system for the error with a matrix pencil of the form

$$(8) \quad \alpha(E + GC) - \beta(A + FC).$$

Even for nonsingular  $E$  the use of the output derivative information is valuable, and it is shown in [18] that choosing  $G$  such that the condition number of  $E + GC$  is small gives improved state estimates.

Theoretical aspects of derivative feedback for descriptor systems are studied in a few recent papers [4], [16], [21], [26]. A control of the restricted form  $u = F(\alpha x - \dot{x}) + v$  is discussed in [4], [21], [26]. In [16] a full state feedback of the form  $u = -G\dot{x} + Fx + v$  is studied for the pole placement problem. In these papers the main task of the derivative feedback is to transform  $E$  into a nonsingular matrix  $E + BG$ . Complete controllability and regularity of the system pencil (5) is assumed.

In this paper we investigate both derivative and proportional state feedback and examine the properties that can be achieved with these types of feedback under various controllability conditions. Applications to pole placement are also considered. Detailed proofs of results previously presented in [2] are given and new results on strongly controllable systems are derived.

The principal aim of this paper is to provide *numerically stable* methods for constructing the feedback controllers based on orthogonal matrix decompositions [9]. Parts of the mathematical theory developed here have been derived concurrently by Dai [6]. Additional assumptions are required in [6], however, and the techniques used for constructing the feedback matrices in [6] are not suitable for numerical computation. It is assumed in [6] that the matrix pencil (5) associated with the system (1), (2) or (3), (4) is regular. This assumption is not required to establish the results presented here. Furthermore, in [6] it is necessary to transform the system into separate “fast” and “slow” subsystems in order to obtain the feedback controls. This transformation is well known to be computationally unreliable [22]. The proofs given here do not require this transformation; and it is shown specifically how to select a feedback in a numerically stable way so as to ensure that the closed loop system is regular and that the controllability (observability) properties of the system are preserved.

In the next section of the paper we introduce notation and examine how the response of the system depends on the eigenstructure of the associated matrix pencil. Definitions of *complete* and *strong* controllability are given and the significance of these conditions is discussed.

In § 3 we summarize the system properties that can be achieved by derivative and proportional *state* feedback under the different controllability conditions. It is shown

that a system that is *completely* controllable can be transformed into a *standard* system by derivative feedback. It is shown, furthermore, that a system that is *strongly* controllable can be transformed into a regular system of index at most 1 (that is, a system in which impulses are excluded) by either proportional or derivative state feedback. Derivative feedback can be used, however, to increase the explicit degrees of freedom defining the solution space (*reachable subspace*) of the system. The construction of the required feedback matrices is obtained by reducing the system pencil to an equivalent “canonical” form using orthogonal transformations that are numerically stable [9]. Most but not all of the conclusions of this section can also be achieved by *output* derivative and proportional feedback. Preliminary results are presented in [1] and [2].

In § 4 applications to the pole placement problem are discussed. The extent to which the poles can be assigned by derivative and/or proportional state feedback whilst retaining regularity is examined under the different controllability conditions.

In the final section we discuss a numerical technique for regularizing the dynamical part of a descriptor system by a derivative feedback which optimizes the conditioning of  $E + BG$ . The results of the paper are then summarized, and concluding remarks are given.

**2. Definitions and properties.** The system equations (1) and (3) are said to be *solvable* if and only if the system pencil (5) is *regular*, that is,

$$(9) \quad \det(\alpha E - \beta A) \neq 0 \quad \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{0, 0\}.$$

For solvable systems there exist unique solutions for any *sufficiently smooth* input and any admissible initial conditions corresponding to an admissible input [3], [25]. The behaviour of the system response is then governed by the eigenstructure of the system pencil. In the next section we examine the eigenstructure of generalized state-space systems and in the following section we define conditions that ensure the controllability (observability) of the system.

**2.1. Eigenstructure of descriptor systems.** For a regular pencil *generalized eigenvalues* are defined to be pairs  $(\alpha_j, \beta_j) \in \mathbb{C}^2$  such that

$$(10) \quad \det(\alpha_j E - \beta_j A) = 0, \quad j = 1, 2, \dots, n.$$

Observe that pairs  $(\alpha_j, \beta_j)$  and  $(t\alpha_j, t\beta_j)$ ,  $t \in \mathbb{C} \setminus \{0\}$  are identified. Eigenvalue pairs  $(\alpha_j, \beta_j)$  where  $\beta_j \neq 0$  are said to be *finite* and, without loss of generality, can be taken to have the “value”  $\lambda_j = \alpha_j/\beta_j$ . Pairs where  $\beta_j = 0$  are said to be *infinite* eigenvalues. The maximum number of finite eigenvalues that a pencil can have is less than or equal to the rank of  $E$ . (For a pencil that is *not* regular, the generalized eigenvalues are similarly defined as pairs  $(\alpha_j, \beta_j)$  such that the pencil loses rank.)

For regular pencils the solution of the system equations can be characterized in terms of the Kronecker canonical form (KCF) [8]. In this case there exist nonsingular matrices  $X$  and  $Y$  (representing the right and left generalized eigenvectors and principal vectors of the system pencil, respectively) which transform  $E$  and  $A$  into the KCF:

$$(11) \quad Y^T E X = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad Y^T A X = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}.$$

Here  $J$  is a Jordan matrix corresponding to the finite eigenvalues of the pencil and  $N$  is a nilpotent Jordan matrix such that  $N^m = 0$ ,  $N^{m-1} \neq 0$ , corresponding to the infinite eigenvalues. The *index* of the system, denoted by  $\text{ind}_\infty(E, A)$ , is defined to be equal to the degree  $m$  of nilpotency. (For pencils that are not regular, the KCF can also be defined and the index is then given similarly by the dimension of the largest nilpotent block in the KCF. See [1] and [8].)

We observe that if a descriptor system is regular then it is of index 0 if and only if  $E$  is nonsingular. In this case the system can be reformulated as a standard system and the usual theory applies. In practice the reduction to standard form can be numerically unstable, however, if  $E$  is ill conditioned with respect to inversion. Hence, even for index 0 systems, it may be preferable to work directly with the generalized state-space form.

We observe that if a descriptor system is regular then it is of index at most 1 if and only if it has exactly  $q = \text{rank}(E)$  finite eigenvalues. Conditions for the system to be regular and of index less than or equal to 1 are given in the following lemma [10]. (Here and in the following we denote the nullspace of a matrix  $M$  by  $\mathcal{N}(M)$ .)

LEMMA 1. Let  $E, A \in \mathbb{R}^{n,n}$ . Let  $S_\infty$  and  $T_\infty$  be full rank matrices whose columns span the null spaces  $\mathcal{N}(E)$  and  $\mathcal{N}(E^T)$ , respectively. Then the following are equivalent:

- (i)  $\alpha E - \beta A$  is regular and  $\text{ind}_\infty(E, A) \leq 1$ ,
- (ii)  $\text{rank}([E, AS_\infty]) = n$ ,

$$(iii) \quad \text{rank} \left( \begin{bmatrix} E \\ T_\infty^H A \end{bmatrix} \right) = n.$$

For systems that are regular and of index at most 1, there exists a unique solution for all admissible controls with consistent initial conditions. Such systems separate into purely dynamical and purely algebraic parts, and in theory the algebraic part can be eliminated to give a reduced-order *standard* system. The reduction process, however, may not be numerically stable [15].

For higher-index systems, if the control is not sufficiently smooth, impulses can arise in the response of the system and the system can lose causality [23], [1]. It is desirable, therefore, to use a feedback control that ensures that the closed loop system is regular and of index less than or equal to 1, if possible. In the next sections we show that this can be achieved under certain “controllability” (“observability”) conditions.

**2.2. Controllability and observability of descriptor systems.** The definitions of controllability and observability for standard control systems can be extended to descriptor systems. However, various types of controllability/observability can be identified [25]. Here we investigate the properties of the generalized state-space system (1), (2) and (3), (4) under the following conditions:

$$(12) \quad \begin{aligned} &C0: \text{rank}([\alpha E - \beta A, B]) = n \quad \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}; \\ &C1: \text{rank}([\lambda E - A, B]) = n \quad \forall \lambda \in \mathbb{C}; \\ &C2: \text{rank}([E, AS_\infty, B]) = n \quad \text{where the columns of } S_\infty \text{ span } \mathcal{N}(E). \end{aligned}$$

For systems that are *regular*, these conditions characterize the controllability of the system. We have the following definition.

DEFINITION 2. Let  $\alpha E - \beta A$  be a regular pencil. Then the triple  $(E, A, B)$  and the corresponding descriptor system are said to be *completely controllable* (C-controllable) if and only if condition C0 holds.

We remark that a descriptor system satisfies C0, i.e., is completely controllable, only if

$$(13) \quad \text{rank}([E, B]) = n.$$

Complete controllability ensures that for any given initial and final states  $x_0, x_f \in \mathbb{R}^n$  of the system, there exists an admissible control that transfers the system from  $x_0$  to  $x_f$  in finite time [25]. Hence, descriptor systems that are completely controllable can be expected to have properties similar to those of standard systems.

A weaker definition of controllability is given by the following.

DEFINITION 3. Let  $\alpha E - \beta A$  be a regular pencil. Then the triple  $(E, A, B)$  and the corresponding descriptor system are said to be strongly controllable (S-controllable) if and only if C1 and C2 hold.

We remark that C-controllability implies S-controllability. Clearly C1 follows from C0 for  $\beta \neq 0$  and  $\lambda = \alpha/\beta$ . Condition C2 follows from (13), but is weaker. In the literature, regular systems that satisfy C2 are often described as “controllable at infinity” or “impulse controllable” [5], [10], [23]. For these systems “impulsive modes” can be excluded. A descriptor system that has a regular pencil of index less than or equal to 1 is always controllable at infinity, since by Lemma 1 we have  $\text{rank}([E, AS_\infty]) = n$ .

The controllability conditions are preserved under certain transformations of the system. Specifically, C0, C1, and C2, are all preserved under nonsingular “equivalence” transformations of the pencil and under proportional state feedback. With the exception of C2, these same conditions are also preserved under derivative state feedback. The following lemma summarizes these results.

LEMMA 4. Let  $(E, A, B)$  satisfy C0 or C1 or C2. Then for any nonsingular  $P$  and  $Q \in \mathbb{R}^{n,n}$  and for any  $F \in \mathbb{R}^{m,n}$ , the system  $(\tilde{E}, \tilde{A}, \tilde{B})$ , where

$$(14) \quad \tilde{E} = PEQ, \quad \tilde{A} = PAQ, \quad \tilde{B} = PB$$

or

$$(15) \quad \tilde{E} = E, \quad \tilde{A} = A + BF, \quad \tilde{B} = B,$$

also satisfies these conditions.

Furthermore, for any matrix  $G \in \mathbb{R}^{m,n}$ , the system  $(\tilde{E}, \tilde{A}, \tilde{B})$ , where

$$(16) \quad \tilde{E} = E + BG, \quad \tilde{A} = A, \quad \tilde{B} = B,$$

also satisfies these conditions with the exception of C2.

*Proof.* In case (14), for all  $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$  we have

$$(17) \quad \begin{aligned} \text{rank}([\alpha E - \beta A, B]) &= \text{rank}\left(P[\alpha E - \beta A, B] \begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}\right) \\ &= \text{rank}([\alpha \tilde{E} - \beta \tilde{A}, \tilde{B}]) \end{aligned}$$

and

$$(18) \quad \begin{aligned} \text{rank}([E, AS_\infty, B]) &= \text{rank}\left(P[E, AQQ^{-1}S_\infty, B] \begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\right) \\ &= \text{rank}([\tilde{E}, \tilde{A}\tilde{S}_\infty, \tilde{B}]), \end{aligned}$$

where  $\tilde{S}_\infty = Q^{-1}S_\infty$  spans  $\mathcal{N}(\tilde{E})$ . Therefore, C0, C1, and C2 are preserved under the transformation (14).

In case (15) we have

$$(19) \quad \begin{aligned} \text{rank}([\alpha E - \beta A, B]) &= \text{rank}\left([\alpha E - \beta A, B] \begin{bmatrix} I & 0 \\ -\beta F & I \end{bmatrix}\right) \\ &= \text{rank}([\alpha \tilde{E} - \beta \tilde{A}, \tilde{B}]) \end{aligned}$$

and

$$(20) \quad \begin{aligned} \text{rank}([E, AS_\infty, B]) &= \text{rank} \left( [E, AS_\infty, B] \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & FS_\infty & I \end{bmatrix} \right) \\ &= \text{rank}([\tilde{E}, \tilde{A}\tilde{S}_\infty, \tilde{B}]), \end{aligned}$$

where  $\tilde{S}_\infty = S_\infty$ , since  $\tilde{E} = E$ . Therefore, C0, C1, and C2 are all retained.

In case (16) the proof that C0 and C1 are preserved is shown analogously to case (15). Condition C2 is not necessarily preserved, however, since in the case (16), the nullspace  $S_\infty$  is altered by the feedback and  $\tilde{S}_\infty \neq S_\infty$ .  $\square$

An example is given in [1] demonstrating that C2 is not necessarily preserved under derivative feedback. If derivative feedback is used to change the system dynamics, it is therefore necessary to be careful not to lose controllability at infinity. In the next section we investigate the use of derivative feedback to make the system regular and of index at most 1. Thus, the resulting system is always controllable at infinity. Regularity of the original system is not needed to achieve this result.

Observability conditions for the time-invariant systems (1), (2) and (3), (4) can be defined as the dual of the controllability conditions. Specifically, a system represented by the triple  $(E, A, C)$  is said to satisfy conditions O0, O1, O2 if and only if the dual system, represented by the triple  $(E^T, A^T, C^T)$ , satisfies C0, C1, C2, respectively. A regular system is defined to be *completely observable* (C-observable) if and only if O0 is satisfied and *strongly observable* (S-observable) if and only if O1 and O2 hold.

In the following sections we derive numerically stable techniques for constructing feedback controllers to achieve particular objectives. By duality these techniques can also be used in the construction of state estimators and observer-based controllers.

**3. Derivative and proportional feedback for descriptor systems.** In this section we discuss conditions under which we can alter the structure of the system pencil (5) by the use of derivative and/or proportional state feedback. We show that if the triple  $(E, A, B)$  satisfies C0, i.e., is C-controllable, then the system (1) or (3) can be transformed into a *completely controllable standard* system by derivative feedback [1]. We show also that if a system satisfies C1 and C2, then a closed loop system that is *strongly controllable, regular, and of index at most 1* can be obtained by derivative or proportional feedback. With derivative feedback, however, the explicit degrees of freedom describing the reachable subspace of the system (corresponding to the number of finite poles of the closed loop system) can be increased to a maximum equal to  $\text{rank}([E, B])$ . Previously it has been shown that proportional state feedback can be used to obtain a regular closed loop system of index at most 1 and simultaneously to place  $q = \text{rank}(E)$  poles [10]. Here we describe a simpler numerical procedure for constructing a regular closed loop system of index at most 1 by proportional state feedback. This procedure does not guarantee that the closed loop poles take specified values. In § 4 techniques for pole placement are discussed.

In the first part of this section we give basic theorems that form the core of the numerical construction techniques. Subsequently, the C-controllable and S-controllable cases are each examined, and finally, the combined use of both derivative and proportional feedback is discussed. Throughout the development we make extensive use of the singular value decomposition (SVD) of a matrix  $M \in \mathbb{R}^{m,n}$ , e.g., [9]. In the usual notation the SVD is given by

$$(21) \quad M = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T,$$

where  $U$  and  $V$  are  $m \times m$  and  $n \times n$  orthogonal matrices, respectively, and  $\Sigma$  is a  $\text{rank}(M) \times \text{rank}(M)$  diagonal matrix with positive diagonal entries. Here we also refer to the orthogonal reduction of  $M$  to diagonal form

$$(22) \quad U^T M V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$

as an SVD of  $M$ , because we always need it in this form.

**3.1. Preliminary theory.** The first lemma serves as a basic tool and provides a “canonical” form for the system (1) or (3), which can be obtained in a numerically stable way.

**LEMMA 5.** *Let  $E \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,m}$ , and  $\text{rank}(B) = m \leq n$ . There exist orthogonal matrices  $Q$ ,  $U$ , and  $V$  such that*

$$(23) \quad QEU = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad QBV = \begin{bmatrix} 0 \\ \Sigma_B \\ 0 \end{bmatrix},$$

where  $\Sigma_1$  and  $\Sigma_B$  are  $l \times l$  and  $m \times m$  diagonal matrices, respectively, with positive diagonal entries, and  $E_{22}$  is an  $m \times s$  matrix with full column rank. The partitioning in  $QEU$  and  $QBV$  is conformable.

*Proof.* Let

$$(24) \quad \tilde{P}BV = \begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix}$$

be an SVD of  $B$ . Let

$$(25) \quad P = \begin{bmatrix} 0 & I_{n-m} \\ I_m & 0 \end{bmatrix} \tilde{P}.$$

Then we obtain

$$(26) \quad PBV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}, \quad PE = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix},$$

with a compatible partitioning. Let

$$(27) \quad WE_1Z_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

be an SVD of  $E_1$ , where  $\Sigma_1$  is an  $l \times l$  diagonal matrix with positive diagonal entries. Then

$$(28) \quad \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} PEZ_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \\ E_{21} & \tilde{E}_{22} \end{bmatrix},$$

where  $[E_{21}, \tilde{E}_{22}]$  is a compatible partitioning of  $E_2Z_1$ . Let  $Z_2$  be an orthogonal matrix that does a “column compression”

$$(29) \quad \tilde{E}_{22}Z_2 = [E_{22}, 0]$$

on  $\tilde{E}_{22}$  such that  $E_{22}$  has full column rank. The matrix  $Z_2$  could, for example, be derived from an RQ-decomposition of  $\tilde{E}_{22}$  (e.g., [9]),

$$(30) \quad \tilde{E}_{22} = [R, 0]Z_2^T.$$

Then from (26), (28), and (29) we get the desired transformation as

$$(31) \quad \begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} PEZ_1 \begin{bmatrix} I_l & 0 \\ 0 & Z_2 \end{bmatrix} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$(32) \quad \begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-m-l} & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} PBV = \begin{bmatrix} 0 \\ \Sigma_B \\ 0 \end{bmatrix}. \quad \square$$

In the next theorem we establish conditions that guarantee that there exist matrices  $F$  and  $G$  such that the matrix pencil  $\alpha(E + BG) - \beta(A + BF)$  is regular and of index at most 1 and such that  $\text{rank}(E + BG) = r$ , where  $r$  can be chosen to be any integer satisfying  $q - m \leq r \leq q = \text{rank}([E, B])$ .

**THEOREM 6.** *Let  $E, A \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,m}$  with  $\text{rank}(B) = m \leq n$ , and let  $S_\infty$  be a full rank matrix whose columns span  $\mathcal{N}(E)$ . If  $\text{rank}([E, AS_\infty, B]) = n$  and  $r \in \mathbb{N}$  such that  $0 \leq l = q - m \leq r \leq q = \text{rank}([E, B])$ , then there exist matrices  $F, G \in \mathbb{R}^{m,n}$  such that the matrix pencil  $\alpha(E + BG) - \beta(A + BF)$  is regular,  $\text{ind}_\infty(E + BG, A + BF) \leq 1$ , and  $\text{rank}(E + BG) = r$ .*

*Proof.* By Lemma 5 there exist orthogonal matrices  $Q, U$ , and  $V$  such that (23) holds and we may choose

$$(33) \quad S_\infty = U \begin{bmatrix} 0 \\ 0 \\ I_{n-l-s} \end{bmatrix}.$$

Partitioning  $QAU$  compatibly with  $QEU$  we have

$$(34) \quad QAU = \begin{bmatrix} A_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ A_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ A_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}$$

and then

$$(35) \quad n = \text{rank}([E, AS_\infty, B]) = \text{rank} \left( \begin{bmatrix} QEU, \begin{bmatrix} \tilde{A}_{13} \\ \tilde{A}_{23} \\ \tilde{A}_{33} \end{bmatrix}, QBV \end{bmatrix} \right)$$

implies that  $\tilde{A}_{33}$  must have full row rank, that is,  $\text{rank}(\tilde{A}_{33}) = n - l - m$ .

Without loss of generality we may assume that the last  $n - m - l$  columns of  $\tilde{A}_{33}$  are linearly independent. If this is not the case, we can achieve this property by a ‘‘column compression’’ of  $\tilde{A}_{33}$  to the right using an RQ-decomposition of  $\tilde{A}_{33}$  or with an SVD.

From (23) we see that  $\text{rank}(E) = l + s$  and  $q = \text{rank}([E, B]) = l + m$ . Let

$$(36) \quad \tilde{G} = [G_1, G_2, G_3]$$

and choose  $G_1 = -\Sigma_B^{-1} E_{21}$  and  $G_2, G_3$  such that

$$(37) \quad [E_{22} + \Sigma_B G_2, \Sigma_B G_3] = [\mathcal{E}_2, 0],$$

where  $\mathcal{E}_2$  is an  $m \times (r - l)$  matrix of full column rank. For instance, if  $r > l + s$ , we may select  $G_2 = 0$  and

$$(38) \quad G_3 = \begin{bmatrix} \Sigma_B^{-1} \hat{E}_{22} \begin{bmatrix} I_{r-l-s} \\ 0 \end{bmatrix}, 0 \end{bmatrix},$$

where  $\hat{E}_{22}$  forms a basis for the orthogonal complement of  $E_{22}$ ; if  $r < l + s$ , then we may choose  $G_3 = 0$  and

$$(39) \quad G_2 = \begin{bmatrix} 0, -\Sigma_B^{-1} E_{22} \begin{bmatrix} 0 \\ I_{l+s-r} \end{bmatrix} \end{bmatrix};$$

and if  $r = l + s = \text{rank}(E)$  then we may choose  $G_3, G_2 = 0$ . (The matrix  $\hat{E}_{22}$  can be obtained in practice from the RQ-decomposition (30) used in the reduction of Lemma 5.) Then  $QEU + QBV\tilde{G}$  has rank equal to  $r$  precisely and its nullspace is spanned by

$$(40) \quad \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}.$$

Now let  $\tilde{Z}$  be an orthogonal matrix that gives a column compression of the last  $n - r$  columns of  $[\tilde{A}_{32}, \tilde{A}_{33}]$  to the right; that is, such that

$$(41) \quad [\tilde{A}_{32}, \tilde{A}_{33}] \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \tilde{Z} = [0, A_{34}],$$

where  $A_{34}$  is a nonsingular  $(n - l - m) \times (n - l - m)$  matrix. This is achievable by our assumption that the last  $n - l - m$  columns of  $\tilde{A}_{33}$  are linearly independent. Then with

$$(42) \quad Z = \begin{bmatrix} I_r & 0 \\ 0 & \tilde{Z} \end{bmatrix}$$

we obtain

$$(43) \quad (QEUZ + QBV\tilde{G}) = \begin{bmatrix} \Sigma_1 & 0 & 0 & 0 \\ 0 & \mathcal{E}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$(44) \quad QAUZ = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \end{bmatrix}.$$

Now let

$$(45) \quad \tilde{F} = [F_1, F_2, F_3, F_4],$$

partitioned conformably with  $QAUZ$ , and choose  $F_3$  such that the  $m \times m$  matrix  $[\mathcal{E}_2, A_{23} + \Sigma_B F_3]$  is of full rank. For instance, if  $r < l + m$ , we may select  $F_3 = \Sigma_B^{-1}(\mathcal{E}_2 - A_{23})$ , where  $\mathcal{E}_2$  spans the orthogonal complement of  $\mathcal{E}_2$ . (If  $\tilde{G}$  is as previously suggested,  $\mathcal{E}_2$  is easily constructed from  $E_{22}$  and  $\hat{E}_{22}$ .)

If  $r = l + m = \text{rank}([E, B])$ , then we may select  $\tilde{F} = 0$ . Finally, with  $G = V\tilde{G}Z^T U^T$ ,  $F = V\tilde{F}Z^T U^T$ , we find that the nullspace of  $E + BG = E + BV\tilde{G}Z^T U^T$  is spanned by

$$(46) \quad \hat{S}_\infty = UZ \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$$

and it follows that

$$(47) \quad \begin{aligned} & \text{rank}([E + BG, (A + BF)\hat{S}_\infty]) \\ &= \text{rank}\left(\begin{bmatrix} QEUZ + QBV\tilde{G}, (QAUZ + QBV\tilde{F}) \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \end{bmatrix}\right) \\ &= \text{rank}\left(\begin{bmatrix} \Sigma_1 & 0 & A_{13} & A_{14} \\ 0 & \mathcal{E}_2 & A_{23} + \Sigma_B F_3 & A_{24} + \Sigma_B F_4 \\ 0 & 0 & 0 & A_{34} \end{bmatrix}\right) = n. \end{aligned}$$

By Lemma 1, the pencil  $\alpha(E + BG) - \beta(A + BF)$  is therefore regular and has index less than or equal to 1.  $\square$

An immediate consequence of Theorem 6 is the following corollary.

**COROLLARY 7.** *Let  $E, A \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,m}$  with  $\text{rank}(B) = m \leq n$ , and let  $S_\infty$  be a full rank matrix whose columns span  $\mathcal{N}(E)$ . If  $\text{rank}([E, AS_\infty, B]) = n$ , then the following hold:*

(i) *There exists a matrix  $G \in \mathbb{R}^{m,n}$  such that the matrix pencil  $\alpha(E + BG) - \beta A$  is regular, has index at most 1, and  $\text{rank}(E + BG) = \text{rank}([E, B])$ ;*

(ii) *There exists a matrix  $F \in \mathbb{R}^{m,n}$  such that the matrix pencil  $\alpha E - \beta(A + BF)$  is regular and has index at most 1.*

*Proof.* The first result follows directly from the construction of  $F$  and  $G$  in Theorem 6 in the case where  $r = \text{rank}([E, B])$ . The second result, where  $r = \text{rank}(E)$ , also follows as in Theorem 6, with the exception that  $\hat{G} = 0$  is selected and  $F_3$  is constructed such that the  $r \times r$  matrix

$$(48) \quad \begin{bmatrix} \Sigma_1 & 0 & A_{13} \\ E_{21} & E_{22} & A_{23} + \Sigma_B F_3 \end{bmatrix}$$

is of full rank. The feedback  $F_3$  could, for instance, be taken as

$$(49) \quad F_3 = \Sigma_B^{-1}(\hat{E}_{22} - A_{23} + E_{21}\Sigma_1^{-1}A_{13}),$$

where  $\hat{E}_{22}$  gives a basis for the orthogonal complement of  $E_{22}$ .  $\square$

We remark that the decomposition (43) of Theorem 6 reveals the extent to which the structure of  $E + BG$  can be controlled by a derivative feedback  $G$ . In a later section we discuss techniques for selecting  $G$  to give a “well-conditioned” regularization of the descriptor system. Lemma 5, Theorem 6, and Corollary 7 provide the key steps in the proofs of the following theorems. We note that these results can also be achieved using the generalized singular value decomposition (see [9]), but the full reduction to this decomposition is not needed here and it is preferable to use the decomposition (23), which requires only orthogonal transformations, for numerical stability.

**3.2. C-controllable systems: Derivative feedback.** We now show that if the triple  $(E, A, B)$  satisfies condition C0, then systems (1) and (3) can be transformed into completely controllable *standard* systems by derivative state feedback. These results have also been established in [16] and [6], but here numerically stable techniques for constructing the feedback are provided. Regularity of the system  $(E, A, B)$  is not required.

The main theorem is given as follows.

**THEOREM 8.** *There exists a real feedback control  $u = -G\dot{x} + v$  or  $u_k = -Gx_{k+1} + v_k$  such that the system defined by the triple  $(E + BG, A, B)$  is C-controllable and the matrix  $E + BG$  is nonsingular if and only if the triple  $(E, A, B)$  satisfies C0, that is,  $\text{rank}([\alpha E - \beta A, B]) = n$  for all  $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ .*

*Proof.* Condition C0 implies that  $\text{rank}([E, B]) = n$ , and hence  $\text{rank}([E, AS_\infty, B]) = n$ . Therefore, by Corollary 7, there exists  $G \in \mathbb{R}^{m,n}$  such that  $\text{rank}(E + BG) = \text{rank}([E, B]) = n$ . Now by Lemma 4, the condition C0 is preserved under derivative state feedback and the theorem follows immediately from the definition of C-controllability.  $\square$

We remark that the condition  $\text{rank}([E, B]) = n$  is both necessary and sufficient to find  $G$  such that  $E + BG$  is nonsingular. Sufficiency follows from Corollary 7 and necessity from the observation that

$$(50) \quad E + BG = [E, B] \begin{bmatrix} I \\ G \end{bmatrix}.$$

From Theorem 8 we conclude that systems (1) or (3), which satisfy C0, can be transformed into standard systems by derivative feedback. If the system matrix  $S = E + BG$  is nonsingular, then the corresponding closed loop system is equivalent to the standard system

$$(51) \quad \dot{x} = \tilde{A}x + \tilde{B}v$$

or

$$(52) \quad x_{k+1} = \tilde{A}x_k + \tilde{B}v_k,$$

where  $\tilde{A} = S^{-1}A$ ,  $\tilde{B} = S^{-1}B$ . Transformation to this standard form may not be numerically reliable, however, if  $S = E + BG$  is ill conditioned with respect to inversion. The decomposition (43) of Theorem 6 reveals the extent to which the conditioning of  $S$  can be controlled by an appropriate choice of  $G$ . In a later section of this paper we discuss techniques for selecting  $G$  to provide an optimally conditioned "regularization" of the system.

The following example illustrates the regularization of a very simple system (given in [23]).

*Example 1.* Let

$$(53) \quad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then (1) gives the equation of a simple electrical circuit, where  $x_1$  is the current and  $x_2$  is the potential of the capacitor. The system is C-controllable. Let  $G = [g_1, g_2]$  with  $g_1 \neq 0$ ; then

$$(54) \quad E + BG = \begin{bmatrix} 0 & 1 \\ g_1 & g_2 \end{bmatrix}$$

is nonsingular, and choosing  $u = -G\dot{x} + v$  transforms the system into the system

$$(55) \quad \begin{aligned} \dot{x}_2 &= x_1, \\ g_1\dot{x}_1 + g_2\dot{x}_2 &= x_2 + v, \end{aligned}$$

which is equivalent to the completely controllable standard system

$$(56) \quad \begin{aligned} \dot{x}_1 &= -\frac{g_2}{g_1}x_1 + \frac{1}{g_1}x_2 + \frac{1}{g_1}v, \\ \dot{x}_2 &= x_1. \end{aligned}$$

If  $g_1$  is taken to be very small, the conditioning of  $E + BG$  is very poor and the standard system (56) may be very sensitive to perturbations. Selecting  $g_1 = 1$ ,  $g_2 = 0$  optimizes the conditioning of  $E + BG$  and ensures that the system (56) is robust.

We have established here that a C-controllable descriptor system can be transformed by derivative state feedback into a completely controllable standard system of full order. By duality, the analogous results hold for C-observable systems.

We remark that the transformation to standard form cannot be achieved with *proportional* state feedback alone. In the next sections we show that under the weaker S-controllability condition, a closed loop system that is regular and of index at most 1 can be achieved by either derivative or proportional state feedback. Such systems are equivalent to reduced-order standard systems and are completely controllable within a subspace of less than full dimension.

**3.3. S-controllable systems: Derivative feedback.** We now show that a system  $(E, A, B)$  that satisfies C1 and C2 can be transformed by derivative state feedback into a system  $(E + BG, A, B)$ , which is regular and has index at most 1, has system matrix  $E + BG$  of maximal rank equal to  $\text{rank}([E, B])$ , and is S-controllable. The main theorem is given as follows.

**THEOREM 9.** *There exists a real feedback control  $u = -G\dot{x} + v$  or  $u_k = -Gx_{k+1} + v_k$  such that the continuous or discrete closed loop system defined by the triple  $(E + BG, A, B)$  is S-controllable and the system pencil  $\alpha(E + BG) - \beta A$  is regular,  $\text{ind}_\infty(E + BG, A) \leq 1$ , and  $\text{rank}(E + BG) = \text{rank}([E, B])$  if the triple  $(E, A, B)$  satisfies C1 and C2, that is,  $\text{rank}([\lambda E - A, B]) = n$  for all  $\lambda \in \mathbb{C}$  and  $\text{rank}([E, AS_\infty, B]) = n$ , where  $S_\infty$  forms a basis for  $\mathcal{N}(E)$ .*

*Proof.* By Corollary 7, C2 ensures the existence of a matrix  $G$  such that  $\alpha(E + BG) - \beta A$  is regular,  $\text{ind}_\infty(E + BG, A) \leq 1$ , and  $\text{rank}(E + BG) = \text{rank}([E, B])$ . The triple  $(E + BG, A, B)$  therefore also satisfies C2 and by Lemma 4, C1 is preserved under derivative feedback. Thus, the closed loop system is S-controllable.  $\square$

We remark that the converse of Theorem 9 does not hold. The condition  $\text{rank}([E, AS_\infty, B]) = n$  is not necessarily preserved under derivative feedback, since  $S_\infty$  is altered. The condition  $\text{rank}([E, AS_\infty, B]) = n$  is therefore sufficient, but *not* necessary to obtain a regular pencil  $\alpha(E + BG) - \beta A$  of index at most 1 with  $\text{rank}(E + BG) = \text{rank}([E, B])$ . An example is given in [1].

From Theorem 9 we conclude that systems that satisfy C1 and C2, or are S-controllable, can be transformed by derivative feedback into completely controllable, reduced-order, standard systems with maximal dimension equal to the dimension of the reachable subspace of the original system. By Lemma 5 and Theorem 6, C2 ensures that there exist orthogonal matrices  $Q, U, V$ , and  $Z$  such that (23), (43), and (44) hold, where  $r = \text{rank}([E, B]) = m + l$ ,  $A_{34}$  is nonsingular, and

$$(57) \quad E_R := \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \mathcal{E}_2 \end{bmatrix}$$

is also nonsingular. The last  $(n - l - m)$ -block of algebraic equations of the equivalent system  $(Q(E + BG)UZ, QAUZ, QBV)$  can thus be solved and the corresponding variables can be eliminated from the first  $(l + m)$ -block of equations, leaving a purely dynamical descriptor system of the form  $(E_R, A_R, B_R)$ , with  $E_R$  nonsingular. This reduced-order system has dimension  $l + m = \text{rank}([E, B])$  and is equivalent to the completely controllable, standard system

$$(58) \quad \dot{z} = E_R^{-1} A_R z + E_R^{-1} B_R v$$

or

$$(59) \quad z_{k+1} = E_R^{-1} A_R z_k + E_R^{-1} B_R v_k.$$

The reachable subspace of this system is thus of dimension  $l + m$  and the degrees of freedom are all explicit in the initial conditions. To illustrate this result, consider the following example.

*Example 2.* Let

$$(60) \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

This system is S-controllable but not C-controllable, and is already in decomposed form (37) and (43). The solutions to this system are given by  $x_1 = -u$ ,  $x_2 = -\dot{u}$ , and

$x_3 = 0$ . For a specific choice of control, there are no degrees of freedom in the initial state of the system. The reachable subspace over all possible choices of the control has dimension 2 and is given by

$$\text{span} \left( \begin{bmatrix} I_2 \\ 0 \end{bmatrix} \right).$$

If we now let  $G = [g_1, g_2, 0]$  with  $g_2 \neq 0$ , then the feedback  $u = -G\dot{x} + v$  transforms the system into

$$\begin{aligned} \dot{x}_1 &= x_2, \\ (61) \quad g_1 \dot{x}_1 + g_2 \dot{x}_2 &= x_1 + v, \\ 0 &= x_3. \end{aligned}$$

The last variable can be eliminated and the remaining dynamical system can be transformed into standard form by inverting the matrix

$$(62) \quad \begin{bmatrix} 1 & 0 \\ g_1 & g_2 \end{bmatrix}$$

to obtain the completely controllable system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ (63) \quad \dot{x}_2 &= \frac{1}{g_2} x_1 - \frac{g_1}{g_2} x_2 + \frac{1}{g_2} v. \end{aligned}$$

These equations can be initiated from any state with any control and the dimension of the reachable subspace is precisely 2.

We remark that the reduction to standard form of systems that are regular and have index at most 1 may not be numerically reliable if the matrices  $A_{34}$  and  $E_R$  obtained from the decompositions (43) and (44) are not well conditioned for inversion. The conditioning of  $E_R$  is influenced by the selection of the derivative feedback  $G$ ; the matrix  $A_{34}$  is not affected by the feedback  $G$ .

We have now established that an S-controllable descriptor system can be transformed by derivative state feedback into a reduced-order, completely controllable standard system with explicit degrees of freedom in the initial conditions equal to the dimension of the reachable subspace. By duality, the analogous results hold for S-observable systems. In the next sections we examine what can be achieved with proportional feedback alone and in combination with derivative state feedback.

**3.4. S-controllable systems: Proportional feedback and other results.** We next show that a system  $(E, A, B)$  that satisfies C1 and C2 can be transformed by proportional state feedback into a system  $(E, A + BF, B)$  that is regular, has index at most 1, and is S-controllable. This result has been established (implicitly) in [5], [6], [7], and [10] using various approaches. Here we give another proof, based on the decomposition of Lemma 5, which allows for the construction of the required feedback in a numerically stable manner.

The main theorem is given as follows.

**THEOREM 10.** *There exists a real feedback control  $u = Fx + v$  or  $u_k = Fx_k + v_k$  such that the continuous or discrete system defined by the triple  $(E, A + BF, B)$  is S-controllable and the system pencil  $\alpha E - \beta(A + BF)$  is regular and  $\text{ind}_\infty(E, A + BF) \leq 1$  if and only if the triple  $(E, A, B)$  satisfies C1 and C2, that is,  $\text{rank}([\lambda E - A, B]) = n$  for all  $\lambda \in \mathbb{C}$  and  $\text{rank}([E, AS_\infty, B]) = n$ , where  $S_\infty$  forms a basis for  $\mathcal{N}(E)$ .*

*Proof.* By Corollary 7, C2 ensures the existence of a matrix  $F$  such that  $\alpha E - \beta(A + BF)$  is regular and  $\text{ind}_\infty(E, A + BF) \leq 1$ . The remainder of the theorem is established by applying Lemma 4, which ensures that C1 and C2 are both preserved under proportional state feedback.  $\square$

We remark that the condition  $\text{rank}([E, AS_\infty, B]) = n$  is both necessary and sufficient to find  $F$  such that  $\alpha E - \beta(A + BF)$  is regular and of index at most 1. This follows because  $E$  and therefore  $S_\infty$  are not changed by proportional state feedback.

From Theorem 10 we see that systems that satisfy C1 and C2 can also be transformed by proportional state feedback into regular systems of index at most 1 and hence into completely controllable reduced-order *standard* systems. The order of the standard system is minimal, however, being equal to  $\text{rank}(E)$ . The degrees of freedom in the reduced-order dynamical system thus do not reflect the dimension of the solution space (i.e., reachable subspace) of the original descriptor system.

As an illustration consider again Example 2.

*Example 3.* Let  $(E, A, B)$  be given as in Example 2 by (60). Let  $F = [f_1, f_2, 0]$ . The feedback  $u = Fx + v$  transforms the system into

$$\begin{aligned} \dot{x}_1 &= x_2, \\ (64) \quad 0 &= (1 + f_1)x_1 + f_2x_2 + v, \\ 0 &= x_3, \end{aligned}$$

which is regular and of index 1, provided  $f_2 \neq 0$ . The last two equations of (64) can then be solved explicitly and eliminated from the first to give

$$(65) \quad \dot{x}_1 = \frac{1 + f_1}{f_2}x_1 - \frac{1}{f_2}v,$$

a standard system of order 1. The solution space of the system (64) is in fact of dimension 2, and hence the degrees of freedom in the reduced-order system (65) do not explicitly describe the solution space of the original system.

We conclude that although proportional state feedback can be used to eliminate impulses by controlling poles at infinity, it cannot be used to regularize a descriptor system completely. Using proportional feedback in combination with derivative feedback, on the other hand, can provide good design techniques that are computationally reliable. A suitable strategy is to use derivative feedback to obtain a well-conditioned regularization of the dynamic-algebraic equations and then to apply proportional feedback to achieve further objectives, such as pole assignment or stable reduction to a reduced-order system. This approach is particularly attractive, since proportional feedback cannot make the system lose regularity, once the rank of  $E$  has been maximized so that  $\text{rank}(E) = \text{rank}([E, B])$ . We have the following theorem.

**THEOREM 11.** *If  $\text{rank}(E) = \text{rank}([E, B])$  and  $\text{rank}([E, AS_\infty]) = n$ , then for any  $F \in \mathbb{R}^{m,n}$ ,  $\text{rank}([E, (A + BF)S_\infty]) = n$ . Here  $S_\infty$  defines a basis for  $\mathcal{N}(E)$ .*

*Proof.* Suppose there exists  $z \neq 0$  such that  $z^T[E, (A + BF)S_\infty] = 0$ . Then  $z^TE = 0$  and  $z^TAS_\infty = -z^TBF S_\infty$ . But since  $\text{rank}(E) = \text{rank}([E, B])$ , it follows that  $z^TB = 0$ , and thus  $z^TAS_\infty = 0$ . But then  $z^T[E, AS_\infty] = 0$ , which contradicts the assumption that  $\text{rank}([E, AS_\infty]) = n$ .  $\square$

If a system  $(E, A, B)$  satisfies C1 and C2, it follows that there exists a derivative feedback  $G$  such that the system  $(E + BG, A, B)$  satisfies  $\text{rank}([E + BG, B]) = \text{rank}([E, B]) = \text{rank}(E + BG)$ , is regular and of index at most 1, and is S-controllable. By Lemma 1, then,  $\text{rank}([E + BG, A\hat{S}_\infty]) = n$ , where  $\hat{S}_\infty$  gives a basis for  $\mathcal{N}(E + BG)$ . Theorem 11 then guarantees that for any choice of  $F$  the system triple

$(E + BG, A + BF, B)$  satisfies  $\text{rank}([E + BG, (A + BF)\hat{S}_\infty]) = n$ , and hence the system remains regular with index at most 1.

We have shown here that an S-controllable system can be transformed by proportional state feedback into a regular system of index at most 1, and hence into a reduced-order, controllable standard system. By duality, analogous results hold for S-observable systems. Of more practical significance, however, we have established that if a system has already been transformed into a regular system of index at most 1 by a derivative feedback that maximizes the dimension of the dynamic part of the system, that is, the system has been fully “regularized” by derivative feedback, then no proportional feedback can cause the system to lose regularity.

We complete this part of the paper by examining the results that can be obtained in general with a combination of derivative and proportional feedback.

**3.5. Combined derivative and proportional feedback.** We now summarize the results that can be achieved by using both derivative and proportional state feedback together. We show that for a system  $(E, A, B)$  that satisfies C1 and C2, a closed loop system can be obtained such that the system pencil  $\alpha(E + BG) - \beta(A + BF)$  is regular and of index at most 1, and such that  $\text{rank}(E + BG) = r$ , where  $r$  is any integer between  $l = q - m$  and  $q = \text{rank}([E, B])$ . (Here  $m = \text{rank}(B)$ .) We have the following theorem, which follows directly from Theorem 6.

**THEOREM 12.** *There exists a real feedback control  $u = Fx - G\dot{x} + v$  or  $u_k = Fx_k - Gx_{k+1} + v_k$  such that the continuous or discrete time system defined by the triple  $(E + BG, A + BF, B)$  is S-controllable and the system pencil  $\alpha(E + BG) - \beta(A + BF)$  is regular,  $\text{ind}_\infty(E + BG, A + BF) \leq 1$ , and  $\text{rank}(E + BG) = r$  with  $l \leq r \leq q$ , where  $q = \text{rank}([E, B])$ ,  $m = \text{rank}(B)$ , and  $l = q - m$ , if the triple  $(E, A, B)$  satisfies C1 and C2, that is,  $\text{rank}([\lambda E - A, B]) = n$  for all  $\lambda \in \mathbb{C}$  and  $\text{rank}([E, AS_\infty, B]) = n$ , where  $S_\infty$  forms a basis for  $\mathcal{N}(E)$ .*

*Proof.* The existence of  $F$  and  $G$  such that  $\alpha(E + BG) - \beta(A + BF)$  is regular and of index at most 1, and  $\text{rank}(E + BG) = r$  follows from C2 and Theorem 6. Then the transformed system given by  $(E + BG, A + BF, B)$  must also satisfy C2, and by Lemma 4 C1 is preserved under both derivative and proportional state feedback, which establishes the theorem.  $\square$

We include Theorem 12 here primarily for completeness. It essentially shows that if C1 and C2 hold, then we can transform the system (1) or (3) by derivative and proportional state feedback into a regular system of index at most 1 with precisely  $r$  finite poles, where  $r$  is between  $\text{rank}([E, B])$  and  $\text{rank}([E, B]) - \text{rank}(B)$ . We emphasize that regularity of the original system is *not* required. Moreover, the feedback matrices  $F$  and  $G$  that achieve the result can be constructed in a *numerically stable* manner, using only orthogonal transformations.

Since the transformed system is regular and of index at most 1, it can be further transformed into a completely controllable, reduced-order, standard system of precise order  $r$ . For this reduction, however, the feedback matrices  $F$  and  $G$  must be selected with care.

In the next section we examine how derivative and proportional state feedback can be used to place the poles of the system in prescribed locations. In the final section we derive a computational algorithm for optimizing the conditioning of regularized dynamical systems obtained by derivative and proportional state feedback.

**4. Eigenvalue assignment in descriptor systems.** We now examine the consequences of the theory of § 3 for the problem of eigenvalue assignment. The conclusions follow directly from the “regularizability” results of Theorems 8 and 9. We begin by stating the pole assignment problem.

**PROBLEM 1.** Given a triple of real matrices  $(E, A, B)$  and a set  $\mathcal{L} = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n)\}$ , where  $(\alpha_j, \beta_j) \in \mathbb{C}^2$  and  $(\alpha_j, \beta_j) \in \mathcal{L}$  implies  $(\bar{\alpha}_j, \bar{\beta}_j) \in \mathcal{L}$  for  $j = 1, \dots, n$ , find  $F, G \in \mathbb{R}^{m,n}$  such that all pairs in  $\mathcal{L}$  are generalized eigenvalues of the matrix pencil  $\alpha(E + BG) - \beta(A + BF)$  and such that

$$(66) \quad \det(\alpha(E + BG) - \beta(A + BF)) \neq 0 \quad \text{for some } (\alpha, \beta) \notin \mathcal{L} \cup \{(0, 0)\}.$$

The condition (66) ensures that the closed loop system obtained by the feedback  $u = Fx - G\dot{x}$  or  $u_k = Fx_k - Gx_{k+1}$  in system (1) or (3), respectively, is *regular*. In assigning a set of eigenpairs by feedback, it is always possible for the closed loop system to lose regularity, even if the original system is regular. It is important, therefore, in assigning eigenpairs, to ensure that (66) holds.

The problem of pole assignment by proportional feedback alone has been treated in [7] and [10]. In this case for systems that satisfy C1 and C2, at most  $r = \text{rank}(E)$  finite generalized eigenvalues  $(\alpha_j, \beta_j)$ ,  $\beta_j \neq 0$ ,  $j = 1, 2, \dots, r$ , can be assigned such that the closed loop pencil is regular. The remaining  $n - r$  infinite eigenvalues  $(\alpha_j, 0)$ ,  $j = n - r + 1, \dots, n$  cannot be reassigned. By exchanging the role of  $E$  and  $A$  in the system pencil, it can be seen that under analogous conditions, at most  $s = \text{rank}(A)$  nonzero eigenvalues  $(\alpha_j, \beta_j)$ ,  $\alpha_j \neq 0$ ,  $j = 1, 2, \dots, s$  (including infinite eigenvalues) can be assigned with derivative feedback alone. It might, therefore, be expected that with both derivative and proportional feedback, a full set of  $n$  eigenpairs could be assigned. This is, in fact, the case if and only if the system satisfies C0. We note that no assumptions are needed about the regularity of the system. We have the following theorem.

**THEOREM 13.** For any arbitrary set  $\mathcal{L}$  of  $n$  self-conjugate poles there exists a pair of real matrices  $F$  and  $G$  solving the pole placement problem, Problem 1, if and only if the triple of real matrices  $(E, A, B)$  satisfies C0, that is,

$$(67) \quad \text{rank}([\alpha E - \beta A, B]) = n \quad \forall (\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}.$$

*Proof.* Since the triple  $(E, A, B)$  satisfies C0, the triple  $(A, E, B)$  also satisfies this condition. Therefore, by Theorem 8 there exists a feedback matrix  $F_1 \in \mathbb{R}^{m,n}$  such that  $A + BF_1$  is nonsingular and the standard system  $(I, (A + BF_1)^{-1}E, (A + BF_1)^{-1}B)$  is completely controllable. It follows that there exists  $G \in \mathbb{R}^{m,n}$  such that  $G$  assigns  $k$ ,  $1 \leq k \leq n$ , zero poles to this standard system, and, therefore, such that the pencil  $\alpha(E + BG) - \beta(A + BF_1)$  has  $k$  infinite eigenvalues  $(\alpha_j, 0)$ ,  $j = 1, 2, \dots, k$ . Let  $P, Q \in \mathbb{C}^{n,n}$  be nonsingular matrices that transform this pencil into Kronecker canonical form:

$$(68) \quad P(\alpha(E + BG) - \beta(A + BF_1))Q = \alpha \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \beta \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}.$$

Partition  $PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$  analogously. The new triple

$$(69) \quad \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$$

still satisfies C0, by Lemma 4, and hence the triple  $(I, J, B_1)$  is completely controllable. Thus, there exists  $F_2 \in \mathbb{R}^{m,(n-k)}$  such that the eigenvalues of  $J + B_1F_2$  are the finite eigenvalues  $(\alpha_j, \beta_j)$ ,  $\beta_j \neq 0$ ,  $j = n - k + 1, \dots, n$ , belonging to  $\mathcal{L}$ . Let  $F = [F_2, 0]Q^{-1} + F_1$ . Then the pencil

$$(70) \quad \alpha(E + BG) - \beta(A + BF) = P^{-1} \left( \alpha \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \beta \begin{bmatrix} J + B_1F_2 & 0 \\ B_2F_2 & I \end{bmatrix} \right) Q^{-1}$$

has the required eigenvalues.

Conversely, if there exist  $F, G \in \mathbb{R}^{m,n}$  such that the pencil  $\alpha(E + BG) - \beta(A + BF)$  has arbitrary generalized eigenvalues, then there exist  $F$  and  $G$  such that the pencil has arbitrary *finite* eigenvalues, that is, such that  $E + BG$  is nonsingular and the eigenvalues of  $(E + BG)^{-1}(A + BF)$  are arbitrary. The standard system  $(I, (E + BG)^{-1}(A + BF), (E + BG)^{-1}B)$  must therefore be controllable. The triple  $(E + BG, A + BF, B)$  must then satisfy C0, and by Lemma 4 the triple  $(E, A, B)$  also satisfies this condition.  $\square$

The construction of feedback matrices  $F, G$  in the proof of Theorem 13 requires a reduction to Kronecker canonical form, which in general is not a numerically reliable technique. Furthermore, the poles of the closed loop pencil obtained by this construction are not in general robust with respect to perturbations in the system matrices. In order to assign an arbitrary number of infinite poles to the closed loop system, the pencil must be allowed to have index greater than 1. Such systems are necessarily less robust than systems of index less than or equal to 1. Moreover, due to the Jordan form of the nilpotent part of the system, ill-conditioned transformations cannot be avoided.

In practice, it is not generally desirable to assign finite poles to infinite positions. If the number of infinite poles to be prescribed is limited, then the feedback matrices  $F$  and  $G$  can be constructed such that the closed loop pencil is not only regular and has the required finite poles, but also has index at most 1. Up to  $n$  finite eigenvalues can be assigned if and only if the triple  $(E, A, B)$  satisfies C0. Under the weaker assumptions C1 and C2, up to  $q = \text{rank}([E, B])$ , finite poles can be prescribed. These results follow directly from Theorem 12. We have the following general result.

**THEOREM 14.** *For any arbitrary set  $\mathcal{L}$  of  $r$  self-conjugate finite poles  $(\alpha_j, \beta_j)$ ,  $\beta_j \neq 0$ ,  $j = 1, \dots, r$ , and  $n - r$  infinite poles  $(\alpha_j, 0)$ ,  $j = r + 1, \dots, n$ , where  $q = \text{rank}([E, B]) \geq r \geq q - \text{rank}(B)$ , there exists a pair of real matrices  $F$  and  $G$  solving the pole placement problem, Problem 1, such that the pencil  $\alpha(E + BG) - \beta(A + BF)$  is regular and  $\text{ind}_\infty(E + BG, A + BF) \leq 1$  if the triple of real matrices  $(E, A, B)$  satisfies C1 and C2, that is,  $\text{rank}([\lambda E - A, B]) = n$  for all  $\lambda \in \mathbb{C}$  and  $\text{rank}([E, AS_\infty, B]) = n$ , where  $S_\infty$  forms a basis for  $\mathcal{N}(E)$ .*

*Proof.* By Theorem 12, there exist matrices  $G$  and  $F_1$  such that the pencil  $\alpha(E + BG) - \beta(A + BF_1)$  is regular and of index at most 1 and  $\text{rank}(E + BG) = r$ , where  $l = q - m \leq r \leq q = \text{rank}([E, B])$ ,  $m = \text{rank}(B)$ . The system  $(E + BG, A + BF_1, B)$  is, moreover, S-controllable. It follows that there exists  $F_2$  which assigns to this system up to  $r = \text{rank}(E + BG)$  finite poles and such that the closed loop system  $(E + BG, A + BF, B)$ , with  $F = F_1 + F_2$ , is regular and of index at most 1. (See [5], [7], [10].) By definition, this system has precisely  $n - r$  infinite poles, which establishes the theorem.  $\square$

Conditions C1 and C2 are sufficient but not necessary for the results of Theorem 14 to hold. If it is required to assign precisely  $n$  finite poles, then C0 is both necessary and sufficient. Sufficiency follows directly from Theorem 14, since C0 implies C1 and C2 and  $\text{rank}([E, B]) = n$ . Necessity follows from Theorem 13.

In order to assign precisely  $n$  finite eigenvalues (assuming C0 holds) we may select  $G$  such that  $E + BG$  is nonsingular, by Theorem 8, and then select  $F$  to assign the prescribed poles to the equivalent standard system  $(I, (E + BG)^{-1}A, (E + BG)^{-1}B)$ . For this strategy to be computationally reliable, it is important to ensure that  $E + BG$  is well conditioned for inversion. In the next section we describe a technique for selecting  $G$  to optimize the conditioning of  $E + BG$ . (In practice, it may not be possible to ensure that  $E + BG$  is nicely conditioned; in this case the techniques of [10] can be applied to the generalized state-space system  $(E + BG, A, B)$  to assign the  $n$  prescribed finite poles as robustly as possible.)

If the weaker conditions C1 and C2 hold, but C0 does not, then it is possible to assign a maximum of precisely  $q = \text{rank}([E, B]) < n$  finite poles. In this case, by

Theorem 9 we may select  $G$  such that  $\text{rank}(E + BG) = \text{rank}([E, B]) = q$  and the pencil is regular and of index 1. As demonstrated in § 3.3, the corresponding closed loop system can then be transformed into a reduced-order, completely controllable system  $(E_R, A_R, B_R)$  of dimension  $q$ , where  $E_R$  is nonsingular. It is then possible to choose  $F_R$  to assign the required finite poles to the standard system  $(I, E_R^{-1}A_R, E_R^{-1}B_R)$ , and hence to construct  $F$  such that the pencil  $\alpha(E + BG) - \beta(A + BF)$  has the required finite eigenvalues. By Theorem 11, this pencil is regular and of index 1. For this strategy to be numerically stable, it is necessary for  $E_R$  to be well conditioned, and also for  $A_{34}$  (defined in § 3.1) to be well conditioned in order for the reduction to the lower-order system to be computationally reliable.

A similar approach can be used for constructing the solution to the general problem of assigning  $r$  finite poles, where  $q - m \leq r \leq q$ , first applying Theorem 12 to obtain a regular S-controllable system  $(E + BG, A + BF, B)$ , where  $\text{rank}(E + BG) = r$ , and then using a reduction to a lower-order standard form. In practice, however, this “reduced-order” approach may not be as efficient or as reliable as applying a direct procedure such as that of [10] to the “regularized” descriptor system in order to assign the poles.

In the next section, we develop techniques for “regularizing” the descriptor system so as to ensure that the dynamic part of the closed loop system is as well conditioned as possible.

**5. Algorithm for regularizing a descriptor system.** In previous sections we have examined conditions under which the descriptor systems (1) and (3) can be “regularized” by derivative and proportional state feedback, that is, conditions that ensure that a closed loop system can be constructed which is regular and of index at most 1, and is S-controllable. Regularity of the original system is not required, and the construction procedures are based on numerically stable techniques.

It has been shown in general that it is desirable in constructing a closed loop system of the form  $(E + BG, A + BF, B)$  to ensure that  $E + BG$  is “well conditioned” in some sense. In this final section of the paper we present a computational technique for generating a feedback  $G$  in such a way as to control the conditioning of the system matrix  $E + BG$ . In addition it is desirable to ensure that  $A + BF$  is chosen such that the transformed descriptor system can be reduced to a standard system in a numerically stable way. A technique is also described for achieving this result. It is assumed that the system  $(E, A, B)$  satisfies C1 and C2.

In order for the matrix  $E + BG$  to be well conditioned (with respect to inversion of the nonsingular part), it is necessary for the ratio  $\sigma_{\max}/\sigma_{\min}$  of the largest singular value  $\sigma_{\max}$ , to the smallest *nonzero* singular value  $\sigma_{\min}$  of  $E + BG$ , to be minimal. Now by Theorem 6, there exist orthogonal transformations  $Q, U, V$ , and  $Z$  and a feedback  $G$  such that  $Q(E + BG)UZ$  is of form (43); moreover,  $G$  can be chosen such that  $\mathcal{E}_2$ , defined in (37), is of the form

$$(71) \quad \mathcal{E}_2 = \begin{bmatrix} \Sigma_2 \\ 0 \end{bmatrix},$$

where  $\Sigma_2$  is an  $r \times r$  diagonal matrix with positive diagonal components and  $q - \text{rank}(B) \leq r \leq q = \text{rank}([E, B])$ . It follows that the singular values of  $E + BG$  are given by the diagonal components of  $\Sigma_1$  and  $\Sigma_2$ . Since  $\Sigma_1$  arises from the decomposition (23) of  $E$  and cannot be altered by feedback, we find that the minimal possible condition number is  $\sigma_{\max}/\sigma_{\min} = \|\Sigma_1\|_2 \|\Sigma_1^{-1}\|_2$ . This value is attained provided the diagonal components of  $\Sigma_2$  are selected to lie between the smallest and largest diagonal components of  $\Sigma_1$ .

In the case  $r = q = \text{rank}([E, B])$ , the system generated by this procedure is regular and of index at most 1. In the case  $r < q$ , in order to obtain a system that is guaranteed to have these properties, it is necessary to use both derivative and proportional feedback. The proportional feedback matrix  $F$  must be selected, by Theorem 6, such that (47) holds. It is desirable also to select  $F$  such that the last  $(n - r) \times (n - r)$  principal submatrix of  $Q(A + BF)UZ$  is well conditioned with respect to inversion. As indicated in previous sections, the reduction of the descriptor system  $(E + BG, A + BF, B)$  to a lower-order *standard* system is then expected to be computationally reliable.

From Theorem 6 it can be seen that if  $\mathcal{E}_2$  is of the form (71), then (47) holds if we select

$$(72) \quad F_3 = \Sigma_B^{-1} \left( \begin{bmatrix} 0 \\ \Sigma_3 \end{bmatrix} - A_{23} \right), \quad F_4 = -\Sigma_B^{-1} A_{24},$$

where  $\Sigma_3$  is an  $(m + l - r) \times (m + l - r)$  diagonal matrix with positive diagonal elements. Then

$$(73) \quad Q(A + BF)UZ \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & A_{34} \end{bmatrix}$$

has singular values given by the singular values of  $A_{34}$  and the diagonal components of  $\Sigma_3$ . To optimize the conditioning of (73), we must therefore select the components of  $\Sigma_3$  to lie between  $\|A_{34}^{-1}\|_2^{-1}$  and  $\|A_{34}\|_2$ .

If we let

$$(74) \quad W_2 A_{34} Z_3 = \Sigma_4$$

be an SVD of  $A_{34}$  and define

$$(75) \quad \tilde{Q} = \begin{bmatrix} I_{l+m} & 0 \\ 0 & W_2 \end{bmatrix} Q, \quad \tilde{U} = UZ \begin{bmatrix} I_{l+m} & 0 \\ 0 & Z_3 \end{bmatrix},$$

then the pencil  $\alpha(E + BG) - \beta(A + BF)$  constructed in this way is orthogonally equivalent to the pencil

$$(76) \quad \tilde{Q}[\alpha(E + BG) - \beta(A + BF)]\tilde{U} =: \alpha \begin{bmatrix} \Sigma_R & 0 \\ 0 & 0 \end{bmatrix} - \beta \begin{bmatrix} A_1 & A_2 \\ A_3 & \Sigma_A \end{bmatrix},$$

where

$$(77) \quad \Sigma_R = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}, \quad \Sigma_A = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & \Sigma_4 \end{bmatrix},$$

and  $\Sigma_R, \Sigma_A$  are as well conditioned as possible. The transformed descriptor system given by the triple  $(E + BG, A + BF, B)$  can therefore be reduced to the standard system

$$(78) \quad \dot{z} = \tilde{A}z + \tilde{B}v$$

or

$$(79) \quad z_{k+1} = \tilde{A}z_k + \tilde{B}v_k,$$

where the system matrix  $\tilde{A}$  is given by

$$(80) \quad \tilde{A} = \Sigma_R^{-1}(A_1 - A_2 \Sigma_A^{-1} A_3).$$

The sensitivity of this computation to round-off errors then depends on the conditioning of  $\Sigma_1$  and  $\Sigma_4$ , which are determined by  $E, A$ , and  $B$ .

We have established here a stable numerical technique for constructing a “regularized” descriptor system  $(E + BG, A + BF, B)$  that is as well conditioned as possible. It is assumed that  $(E, A, B)$  satisfies conditions that correspond to S-controllability, but regularity of the original system pencil  $\alpha E - \beta A$  is *not* needed. The computational algorithm for determining the required derivative and proportional state feedback matrices  $G$  and  $F$  is summarized in full in the Appendix. This procedure can also be extended to the problem of regularizing the systems (1), (2) and (3), (4) by *output* feedback. This topic is currently under investigation. Preliminary results are given in [1].

**6. Conclusions.** We investigate here the use of derivative and proportional feedback in descriptor, or generalized state-space, systems. We define various conditions for controllability (observability) and demonstrate to what extent the system can be altered by derivative and/or proportional state feedback under these conditions.

It is established that systems that satisfy conditions ensuring *complete* controllability can be transformed into *standard* systems (of full dimension) by a combination of derivative and proportional state feedback. It is shown, furthermore, that in this case, with state feedback, all of the poles of the system can be assigned to prescribed positions.

It is also established that systems that satisfy conditions ensuring *strong* controllability can be transformed by derivative and proportional state feedback into systems that are regular and of index at most 1 and have precisely  $r$  finite poles, where  $r$  lies between  $q = \text{rank}([E, B])$  and  $q - \text{rank}(B)$ . Moreover, it is shown that these  $r$  poles can be assigned to arbitrary (finite) locations. Such systems are “impulse controllable” and can be transformed into reduced-order *standard* systems of precise dimension  $r$ .

The proofs of these results do not require regularity of the original system. Furthermore, the procedure for constructing the feedback matrices which regularize the closed loop system are based on orthogonal matrix decompositions and are numerically stable. In practice it is desirable not only that the closed loop descriptor system is regular, but also “well conditioned,” in the sense that the reduction to standard form is computationally reliable. We show here that the feedback matrices that regularize the system can also be chosen to optimize the “conditioning” of the closed loop system, and a computational algorithm for achieving this result is presented.

## 7. Appendix: Algorithm for regularizing a descriptor system.

*Step 1.* Find orthogonal matrices  $\tilde{P}$ ,  $V$  such that

$$\tilde{P}BV = \begin{bmatrix} \Sigma_B \\ 0 \end{bmatrix},$$

using the singular value decomposition of  $B$ .

*Step 2.* Let

$$P = \begin{bmatrix} 0 & I_{n-m} \\ I_m & 0 \end{bmatrix}$$

and partition

$$P\tilde{P}E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

compatibly with

$$P\tilde{P}BV = \begin{bmatrix} 0 \\ \Sigma_B \end{bmatrix}.$$

Step 3. Find orthogonal matrices  $W, Z_1$  such that

$$(81) \quad WE_1Z_1 = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_l),$$

by the singular value decomposition of  $E_1$ .

Step 4. Partition  $E_2Z_1 = [E_{21}, \tilde{E}_{22}]$  compatibly with  $WE_1Z_1$  and find an orthogonal matrix  $Z_2$  such that  $\tilde{E}_{22}Z_2 = [E_{22}, 0]$ , where  $E_{22}$  is of full column rank. This can, for example, be achieved by an RQ-decomposition of  $E_{22}$ .

Step 5. Let

$$(82) \quad Q = \begin{bmatrix} I_l & 0 & 0 \\ 0 & 0 & I_m \\ 0 & I_{n-l-m} & 0 \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I_m \end{bmatrix} P\tilde{P}, \quad U = Z_1 \begin{bmatrix} I_l & 0 \\ 0 & Z_2 \end{bmatrix}.$$

Step 6. Select  $r$  such that  $q = \text{rank}([E, B]) \geq r \geq q - \text{rank}(B)$ . Find orthogonal matrices  $\tilde{W}, \tilde{Z}$  such that

$$(83) \quad \tilde{W}[0, I_{n-l-m}]QAU\hat{U} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix} \tilde{Z} = [0, \Sigma_4], \quad \Sigma_4 = \text{diag}(\sigma_{l+m+1}, \dots, \sigma_n)$$

by the singular value decomposition of  $[0, I_{n-l-m}]QAU\hat{U} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$ , where  $\hat{U}$  is chosen such that the lower right  $(n-l-m) \times (n-l-m)$  block of  $QAU\hat{U}$  is nonsingular. The matrix  $\hat{U}$  can, for example, be found by an RQ-decomposition of the lower right  $(n-l-s) \times (n-l-m)$  block of  $QAU$ , which is of full rank.

Step 7. Let

$$(84) \quad \tilde{Q} = \begin{bmatrix} I_{l+m} & 0 \\ 0 & \tilde{W} \end{bmatrix} Q, \quad \tilde{U} = U\hat{U} \begin{bmatrix} I_r & 0 \\ 0 & \tilde{Z} \end{bmatrix}.$$

Step 8. Select

$$(85) \quad \Sigma_2 = \text{diag}(\sigma_{l+1}, \dots, \sigma_r), \quad \Sigma_3 = \text{diag}(\sigma_{r+1}, \dots, \sigma_{l+m}),$$

where

$$(86) \quad \begin{aligned} \|\Sigma_1^{-1}\|_2^{-1} &\leq \sigma_j \leq \|\Sigma_1\|_2, & j &= l+1, \dots, r, \\ \|\Sigma_4^{-1}\|_2^{-1} &\leq \sigma_j \leq \|\Sigma_4\|_2, & j &= r+1, \dots, l+m. \end{aligned}$$

Step 9. Select

$$(87) \quad G = V[G_1, G_2, G_3, 0]\tilde{U}^T, \quad F = V[0, 0, F_3, F_4]\tilde{U}^T,$$

where

$$(88) \quad \begin{aligned} G_1 &= -\Sigma_B^{-1}E_{21}, & [G_2, G_3] &= \Sigma_B^{-1} \left( \begin{bmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{bmatrix} - [E_{22}, 0] \right), \\ F_3 &= \Sigma_B^{-1} \left( \begin{bmatrix} 0 \\ \Sigma_3 \end{bmatrix} - A_{23} \right), & F_4 &= -\Sigma_B^{-1}A_{24}, \end{aligned}$$

with

$$(89) \quad [A_{23}, A_{24}] = [0, I_m, 0]\tilde{Q}\tilde{A}\tilde{U} \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}.$$

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