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DYNAMIC ANALYSIS ON AN ALMOST PERIODIC PREDATOR-PREY SYSTEM WITH IMPULSIVE EFFECTS AND TIME DELAYS

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Abstract. This article is concerned with a generalized almost periodic predator-prey model with impulsive effects and time delays. By utilizing comparison theorem and constructing a feasible Lyapunov functional, we obtain sufficient conditions to guarantee the permanence and global asymptotic stability of the system. By applying Arzelà-Ascoli theorem, we establish the existence and uniqueness of almost-periodic positive solutions. A feasible numerical simulation is provided to explain the suitability of our main criteria.

1. **Introduction.** In 1924-1926, Bohr [6, 7, 8] established the theory about almost periodic functions (APFs) systematically. During the immediate decades, following Bohr's researches, numerous significant works were finished on APFs, we refer researchers to Fink [12], van Kampen [34], Bochner [4, 5] and von Neumann [35]. Almost periodic differential equations (APDEs) can be applied to various fields to characterize some phenomena such as celestial mechanics, mechanical vibration, electric or ecology system, engineering technology and so on. In view of the extensive applications from science to engineering, APDEs have been developed rapidly during the past three decades. Despite a lot of works devoted to the qualitative properties of periodic solutions (see [24]), the study of almost periodic solutions can obtain a more general and extensive application in real world because of the different time-dependent coefficients in time period. As we know, the traditional tools of solving the qualitative problems of periodic model cannot be used to solve the same problems of almost periodic issues due to the compactness of operators. Furthermore, some results were obtained in recent decades, but there have still many unresolved problems, some of them were not even mentioned in literatures. Therefore, we claim that it will be significative to further study almost periodic differential equations. In the field of biological dynamic, several useful researches on APDEs

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have been published such as hematopoiesis system [9, 10, 11, 19, 25, 42, 43, 44], cellular neural networks [14, 17, 18, 20, 32, 41], systems on time scales [13, 38, 39, 46] and so on.

Biological dynamic systems with time delays have been investigated by many researchers based on their real world applications. The predator-prey system is one of the hot and interesting research subjects, which illustrates the predation between two or more species. What we mainly concern in this article is the qualitative properties of a Holling type II predator-prey model with impulsive effects and time delays

$$\begin{cases} \dot{x}(t) = x(t)\left[a_1(t) - b_1(t)x(t-\tau_1(t))\right] - \dfrac{c_1(t)x(t)y^m(t)}{d + x(t)}, \\ \dot{y}(t) = y(t)\left[-a_2(t) - b_2(t)y(t-\tau_2(t))\right] + \dfrac{c_2(t)x(t)y^m(t)}{d + x(t)}, \\ \qquad\qquad\qquad 0 < m < 1,\ t \in \displaystyle\bigcup_{i=0}^{\infty}(t_i, t_{i+1}), \\ x(t_k^+) = (1+h_{1k})x(t_k), \quad y(t_k^+) = (1+h_{2k})y(t_k), \quad k \in \mathbb{Z}^+, \end{cases} \tag{1}$$

under the initial condition

$$\begin{aligned} &x(\alpha) = \varphi(\alpha), \quad \alpha \in (-\infty, 0], \quad \varphi(\alpha) \in C((-\infty,0],(0,+\infty)), \\ &y(\alpha) = \psi(\alpha), \quad \alpha \in (-\infty, 0], \quad \psi(\alpha) \in C((-\infty,0],(0,+\infty)). \end{aligned} \tag{2}$$

Here $x(t)$ and $y(t)$, respectively, denote the population sizes of prey and predator at time t; a_1 stands for the growth rate of prey, in the absence of preys, a_2 represents the decay of predator population; b_1 (b_2) describes the prey (predator) population decay in the competition among individuals of prey (predator); c_1 is the prey fed upon by the predators, c_2 represents the parameter of transformation from preys to predators; $k \in \mathbb{Z}^+$ stands for nonnegative integers and $0 = t_0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots$, denote impulse points with $\lim\limits_{k\to+\infty} t_k = \infty$.

The investigation of the nonlinear dynamics about biomathematics model is one of the most active and important subjects in mathematics, ecology and biological world. They described many of the relationships between two or more species, such as competitive mechanism [21, 30], predator-prey mechanism [26, 47], mutualism [40] and so on. To investigate the predator-prey effects between two or more natural species, several biological models have been introduced and investigated by many ecologists, biologists and mathematicians [16, 27, 29, 36, 37, 45]. In 2009, Lin and Chen [22] investigated the properties of almost periodic solutions for a Beddington-DeAngelis predator-prey model

$$\begin{cases} \dot{x}(t) = x(t)\left[a_1(t) - b_1(t)x(t) - \dfrac{c_1(t)y^m(t)}{d(t)+e(t)x(t)+f(t)y(t)}\right], \\ \dot{y}(t) = y(t)\left[-a_2(t) - b_2(t)y(t) + \dfrac{c_2(t)x(t)y^{m-1}(t)}{d(t)+e(t)x(t)+f(t)y(t)}\right], \ 0 < m < 1, \end{cases}$$

where $x(t)$ and $y(t)$ stand for the population density of the prey and predator at time t, respectively. However, this model did not describe the impacts of the past states of species. Recently, Lin et al [23] investigated the global asymptotic stability of a

Holling type III multispecies competition-predator model with multi-time delays

$$\begin{cases} \dot{u}_i(t) = u_i(t)\left[a_i(t) - \sum_{k=1}^{n} b_{ik}(t)u_k(t-\tau_k(t)) - \sum_{k=1}^{m} \frac{c_{ik}u_i(t)v_k(t)}{u_i^2(t)+f_{ik}(t)}\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \cdots, n, \\ \dot{v}_j(t) = v_i(t)\left[-d_j(t) + \sum_{k=1}^{n} \frac{e_{kj}u_k^2(t-\delta_k(t))}{u_k^2(t-\delta_k(t))+f_{kj}(t)} - \sum_{k=1}^{m} g_{jk}(t)v_k(t-\sigma_k(t))\right], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \cdots, m. \end{cases} \tag{3}$$

System (3) takes into account the dependence on the past states of other species j.

Impulsive mechanism can be viewed as a transient changeable phenomena and can be described as a system with impulse. Impulsive system, of course, features impacts of transient change on model status as its most distinguishing characteristic. Meanwhile, it can also deeply reflect the law of thing's change. During the past two decades, impulsive differential equation developed rapidly and widely used in many fields of science such as biotechnology, chemical technology, economics, physics and population dynamics. In addition, the equation with impulsive effects has been and will continue to be investigated by many researchers too. Impulsive differential equation was first studied by Mil'man and Myshkis [28]. We refer to scholar to the outstanding monographs of Bainov and Simeonov [1, 2, 31]. The investigation of the fundamental theory about impulsive differential systems with time delay began comparatively late, especially on nonlinear differential models. The primary difficulty is that impulsive effects and time delays appear at the same time. Hence, we argue that it is useful and interesting to investigate system (1). Therefore, motivated by above, we are firmly convinced that this modified system can illustrate the impulsive effects and the time delays of species. We also claim that it will be significant, interesting and beneficial to investigate the qualitative theory of system (1) as it extends previous theories and admits biological value.

The organization of the rest part of this article is as follows: Section 2 contains some lemmas and the existence theorem of almost periodic solutions of (1). In Section 3, two parameter conditions are proposed to guarantee the permanence of system (1) by utilizing comparison theorem. In Section 4, three sufficient conditions are introduced to guarantee the global asymptotical stability of model (1) by constructing a feasible Lyapunov functional. In Section 5, by applying Arzelà-Ascoli theorem we establish the existence and uniqueness of almost periodic positive solutions of (1). In Section 6, we offer an example to describe the applicability of our main results. The conclusion is made in Section 7.

2. **Preliminaries.**

We restate some notations, lemmas and definition which will be applied in the proofs of our main theorem. For any bounded function $f(t)$ defined on $(0, +\infty)$, we define

$$f^u = \sup_{t\in(0,+\infty)} f(t), \quad f^l = \inf_{t\in(0,+\infty)} f(t).$$

Throughout the entire article, we suppose that the following conditions hold:

(H_1) For $i = 1, 2$, the biological coefficients $a_i(t)$, $b_i(t)$ and $c_i(t)$ are all positive and bounded almost periodic continuous on $(0, +\infty)$ with $a_i^l(t), b_i^l(t), c_i^l(t) > 0$;

(H_2) For $i=1,2,\ k\in\mathbb{Z}^+,\ h_{ik}>-1,\ H_i(t)=\prod_{0<t_k<t}(1+h_{ik})$ is almost periodic and there exist two positive constants H_i^l and H_i^u such that $H_i^l\le H_i(t)\le H_i^u$;

(H_3) For $i=1,2$, the time delay $\tau_i(t)$ is continuously differentiable and nonnegative almost periodic on $[0,+\infty)$ with $\dot{\tau}_i(t)<1$ and $\tau_i(0)=0$. That is, $\phi_i(t)=t-\tau_i(t)$ possess the inverse function $\phi_i^{-1}(t)$. In addition, $\phi_i(t)<\phi_i^{-1}(t)$ for $t\ge 0$.

Definition 2.1 (Definition 1.1, [15]). A function $f(t)$ is termed to be almost periodic (Bohr) if for any given $\varepsilon>0$, the set

$$\mathcal{T}(f,\varepsilon)=\{\tau;|f(t+\tau)-f(t)|<\varepsilon,\forall t\in\mathbb{R}\}$$

is relatively dense, i.e., it is possible to discover an ε-translation constant $l=l(\varepsilon)>0$ such that there possesses an ε-translation number $\tau=\tau(\varepsilon)\in\mathcal{T}(f,\varepsilon)$ in any interval with length $l(\varepsilon)$ such that

$$|f(t+\tau)-f(t)|<\varepsilon$$

is fulfilled for any $t\in\mathbb{R}$.

Consider the following auxiliary equations

$$\begin{cases}\dot{u}(t)=u(t)\left[a_1(t)-B_1(t)u(t-\tau_1(t))-\dfrac{C_1(t)v^m(t)}{d+H_1(t)u(t)}\right],\\[2mm] \dot{v}(t)=v(t)\left[-a_2(t)-B_2(t)v(t-\tau_2(t))+\dfrac{C_2(t)u(t)v^{m-1}(t)}{d+H_1(t)u(t)}\right],\ 0<m<1,\end{cases}\tag{4}$$

under the initial condition

$$\begin{aligned}u(s)&=\varphi(s),\quad s\in(-\infty,0],\quad \varphi(s)\in C((-\infty,0],(0,+\infty)),\\ v(s)&=\psi(s),\quad s\in(-\infty,0],\quad \psi(s)\in C((-\infty,0],(0,+\infty)),\end{aligned}\tag{5}$$

and

$$\begin{aligned}B_i(t)&=\prod_{0<t_k<t}(1+h_{ik})\,b_i(t),\quad i=1,2,\\ C_1(t)&=\prod_{0<t_k<t}(1+h_{2k})^m\,c_1(t),\\ C_2(t)&=\prod_{0<t_k<t}(1+h_{1k})\,(1+h_{2k})^{m-1}\,c_2(t).\end{aligned}$$

Definition 2.2 ([24], Positive solutions). If $(u(t),v(t))$ is a solution to model (4) under initial condition $u(0)>0,v(0)>0$, then $u(t),v(t)>0$ for all $t>0$. Such solutions will be called positive.

Lemma 2.3. *Under the initial condition (2), all solutions to system (1) are positive.*

Proof. Based on $x(t_0)>0$ and $y(t_0)>0$, we obtain

$$\begin{aligned}x(t)&=x(t_0)\exp\left\{\int_{t_0}^t\left(a_1(s)-b_1(s)x(s-\tau_1(s))-\frac{c_1(s)y^m(s)}{d+x(s)}\right)ds\right\},\\ y(t)&=y(t_0)\exp\left\{\int_{t_0}^t\left(-a_2(s)-b_2(s)y(s-\tau_2(s))+\frac{c_2(s)x(s)y^{m-1}(s)}{d+x(s)}\right)ds\right\}.\end{aligned}$$

This ends the proof of Lemma 2.3. □

Lemma 2.4 ([33]). *Suppose that $a>0$, $b>0$, $c>0$ and $\dot{x} \geq (\leq) x(a-bx^c)$, when $t \geq 0$ and $x(0)>0$, we deduce*

$$\liminf_{t\to\infty} x(t) \geq \left(\frac{a}{b}\right)^{\frac{1}{c}} \left(\limsup_{t\to\infty} x(t) \leq \left(\frac{a}{b}\right)^{\frac{1}{c}}\right).$$

Lemma 2.5 (Brouwer fixed-point theorem). *Assume that the continuous operator A maps the closed and bounded convex set $Q \subset \mathbb{R}^n$ onto itself; then the operator A has at least one fixed point in set Q.*

Lemma 2.6. *The following two results hold:*

(i) *If $(u(t), v(t))^T$ stands for a positive solution of system (4), then*

$$(x(t), y(t))^T = \left(\prod_{0<t_k<t} (1+h_{1k})u(t), \prod_{0<t_k<t} (1+h_{2k})v(t)\right)^T$$

represents a positive solution of system (1);

(ii) *If $(x(t), y(t))^T$ stands for a positive solution of system (1), then*

$$(u(t), v(t))^T = \left(\prod_{0<t_k<t} (1+h_{1k})^{-1}x(t), \prod_{0<t_k<t} (1+h_{2k})^{-1}y(t)\right)^T$$

represents a positive solution of system (4).

Proof. (i) Assume that $(u(t), v(t))^T$ stands for a positive solution of model (4). Set

$$x(t) = \prod_{0<t_k<t} (1+h_{1k})u(t), \ y(t) = \prod_{0<t_k<t} (1+h_{2k})v(t),$$

then for any $t \neq t_k$, by applying

$$u(t) = \prod_{0<t_k<t} (1+h_{1k})^{-1}x(t), \ v(t) = \prod_{0<t_k<t} (1+h_{2k})^{-1}y(t)$$

to model (4), we can check that the first two equations of model (1) hold.

If $t = t_k$, $t \in \mathbb{Z}^+$, we obtain

$$\begin{aligned} x(t_k^+) &= \lim_{t\to t_k^+} x(t) = \lim_{t\to t_k^+} \prod_{0<t_k<t} (1+h_{1k})u(t) = \prod_{0<t_s\leq t_k} (1+h_{1s})u(t_k) \\ &=(1+h_{1k}) \prod_{0<t_s<t_k} u(t_k) = (1+h_{1k})x(t_k), \\ y(t_k^+) &= \lim_{t\to t_k^+} y(t) = \lim_{t\to t_k^+} \prod_{0<t_k<t} (1+h_{2k})v(t) = \prod_{0<t_s\leq t_k} (1+h_{2s})v(t_k) \\ &=(1+h_{2k}) \prod_{0<t_s<t_k} v(t_k) = (1+h_{2k})y(t_k). \end{aligned} \tag{6}$$

Therefore, (6) can be applied to all the equations of system (1). Hence, $(x(t), y(t))^T$ stands for a positive solution of system (1).

(ii) Now, we prove the second part of this theorem. We see that $x(t)$ and $y(t)$ are continuous. Then, $x(t)$ and $y(t)$ are continuous on each interval $(t_k, t_{k+1}]$. It is effortless to verify the continuity of $u(t)$ and $v(t)$ at t_k (impulse point), $t \in \mathbb{Z}^+$.

Thanks to $u(t)=\prod_{0<t_k<t}(1+h_{1k})^{-1}x(t)$ and $v(t)=\prod_{0<t_k<t}(1+h_{2k})^{-1}y(t)$, we get

$$\begin{aligned}
u(t_k^+)&=\prod_{0<t_s\le t_k}(1+h_{1s})^{-1}x(t_k^+)=\prod_{0<t_s<t_k}(1+h_{1s})^{-1}x(t_k)=u(t_k),\\
u(t_k^-)&=\prod_{0<t_s\le t_k}(1+h_{1s})^{-1}x(t_k^-)=\prod_{0<t_s<t_k}(1+h_{1s})^{-1}x(t_k)=u(t_k),\\
v(t_k^+)&=\prod_{0<t_s\le t_k}(1+h_{2s})^{-1}y(t_k^+)=\prod_{0<t_s<t_k}(1+h_{2s})^{-1}y(t_k)=v(t_k),\\
v(t_k^-)&=\prod_{0<t_s\le t_k}(1+h_{2s})^{-1}y(t_k^-)=\prod_{0<t_s<t_k}(1+h_{2s})^{-1}y(t_k)=v(t_k).
\end{aligned}$$

Hence, $u(t)$ and $v(t)$ are continuous on $[0,+\infty)$. It is effortless to verify that $(u(t),v(t))^T$ is a positive solution of system (4).

This ends the proof of Lemma 2.6. □

Lemma 2.7 ([3], Barbalar Lemma)**.** *If $f:[0,+\infty)\to\mathbb{R}$ is uniformly continuous, $f(t)\ge 0$ and $\lim\limits_{t\to\infty}\int_0^t f(s)ds<+\infty$, then $\lim\limits_{t\to\infty}f(t)=0$.*

Lemma 2.8. *Under the initial condition (5), all solutions to system (4) are positive.*

Proof. Together with Lemmas 2.3 and 2.6, we finish the proof of Lemma 2.8. □

3. **Permanence.** In this part, by utilizing comparison theorem (Lemma 2.4), we establish two parameter conditions to guarantee the permanence of system (1).

Theorem 3.1. *Assume that system (1) satisfies the following conditions:*
(C_1) $a_1^l>\frac{c_1^uX^u(Y^u)^m}{d}$,
(C_2) $\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}>a_2^u$.
Then model (1) possesses permanence, that is, any positive solution $(x(t),y(t))$ to the model (1) fulfills

$$0<X^l\le\liminf_{t\to\infty}x(t)\le\limsup_{t\to\infty}x(t)\le X^u,$$
$$0<Y^l\le\liminf_{t\to\infty}y(t)\le\limsup_{t\to\infty}y(t)\le Y^u,$$

where

$$\tau:=\max_{i=1,2}\left\{\sup_{t\in[0,+\infty)}\tau_i(t)\right\},\quad X^u=\frac{a_1^u}{b_1^l}\exp\{a_1^u\tau\},\quad Y^u=\left(\frac{a_2^l(d+X^u)}{c_2^uX^u}\right)^{\frac{1}{m-1}},$$
$$X^l=\frac{a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}}{b_1^u}\exp\left\{\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^uX^u\right]\tau\right\},$$
$$Y^l=\frac{\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u}{b_2^u}\exp\left\{\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uY^u\right]\tau\right\}.$$

Proof. Based on the first equation of model (1), we deduce that

$$\dot{x}(t)\le x(t)\left[a_1(t)-b_1(t)x(t-\tau_1(t))\right],\ t>\tau. \tag{7}$$

Here and subsequently, $x(\bar{t})$ denotes any local maximal value of $x(t)$. Thanks to (7), we obtain that

$$0=\dot{x}(\bar{t})\leq x(\bar{t})\left[a_1^u-b_1^l x(\bar{t}-\tau_1(\bar{t}))\right]. \tag{8}$$

Thanks to (8), we get

$$x(\bar{t}-\tau_1(\bar{t}))\leq \frac{a_1^u}{b_1^l},\ t>\tau. \tag{9}$$

Integrating both sides of (7) on interval $[\bar{t}-\tau_1(\bar{t}),\bar{t}]$, we get that

$$\ln\frac{x(\bar{t})}{x(\bar{t}-\tau_1(\bar{t}))}\leq\int_{\bar{t}-\tau_1(\bar{t})}^{\bar{t}}\left[a_1^u-b_1^l x(t-\tau_1(t))\right]dt\leq a_1^u\tau. \tag{10}$$

Together with (9) and (10), one has

$$x(\bar{t})\leq\frac{a_1^u}{b_1^l}\exp\{a_1^u\tau\}\overset{def}{=}X^u.$$

We need to notice that $x(\bar{t})$ represents any local maximal value of $x(t)$, hence there possesses a $T_1>\tau$, for $t>T_1$, we have

$$x(t)\leq X^u. \tag{11}$$

We denote $y(t)=\frac{1}{w(t)}>0$. Thus, $y(t-\tau_2(t))=\frac{1}{w(t-\tau_2(t))}>0$. Applying the second equation of the system (1) and (11) leads to

$$\begin{aligned}\dot{w}(t)=&w(t)\left(a_2(t)+\frac{b_2(t)}{w(t-\tau_2(t))}-\frac{c_2(t)x(t)w^{1-m}(t)}{d+x(t)}\right)\\ \geq&w(t)\left(a_2(t)-\frac{c_2(t)x(t)w^{1-m}(t)}{d+x(t)}\right)\\ \geq&w(t)\left(a_2^l-\frac{c_2^uX^u}{d+X^u}w^{1-m}(t)\right),\ 0<m<1.\end{aligned}$$

Thanks to Lemma 2.4, one has

$$\liminf_{t\to+\infty}w(t)\geq\left(\frac{a_2^l\,(d+X^u)}{c_2^uX^u}\right)^{\frac{1}{1-m}}\overset{def}{=}W^l.$$

Thanks to the last inequality, one has

$$\limsup_{t\to+\infty}y(t)\leq\frac{1}{W^l}=\left(\frac{a_2^l\,(d+X^u)}{c_2^uX^u}\right)^{\frac{1}{m-1}}\overset{def}{=}Y^u. \tag{12}$$

Together with $0<m<1$ and the first equation of model (1), it follows that

$$\dot{x}(t)\geq x(t)\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^ux(t-\tau_1(t))\right],\ t>\tau. \tag{13}$$

Here and subsequently, $x(\tilde{t})$ denotes any local minimal value of $x(t)$. Thanks to (13), we obtain that

$$0=\dot{x}(\tilde{t})\geq x(\tilde{t})\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^ux(\tilde{t}-\tau_1(\tilde{t}))\right]. \tag{14}$$

Thanks to (14), we obtain

$$x(\tilde{t}-\tau_1(\tilde{t}))\geq\frac{a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}}{b_1^u}. \tag{15}$$

Integrating both sides of (14) on interval $[\tilde{t}-\tau_1(\tilde{t}),\tilde{t}]$, noticing that

$$\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^ux(\tilde{t}-\tau_1(\tilde{t}))\le 0,$$

we obtain

$$\begin{aligned}\ln\frac{x(\tilde{t})}{x(\tilde{t}-\tau_1(\tilde{t}))}&\ge\int_{\tilde{t}-\tau_1(\tilde{t})}^{\tilde{t}}\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^ux(\tilde{t}-\tau_1(\tilde{t}))\right]dt\\&\ge\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^uX^u\right]\tau.\end{aligned}\tag{16}$$

Together with (15) and (16), we deduce that

$$x(\tilde{t})\ge\frac{a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}}{b_1^u}\exp\left\{\left[\left(a_1^l-\frac{c_1^uX^u(Y^u)^m}{d}\right)-b_1^uX^u\right]\tau\right\}\overset{def}{=}X^l.$$

We need to notice that $x(\tilde{t})$ represents any local minimal value of $x(t)$, hence there possesses a $T_2>\tau$, for $t>T_2$, we have

$$x(t)\ge X^l.\tag{17}$$

Thus, by setting the estimates (12) and (17) into the second equation of (1), one has

$$\begin{aligned}\dot{y}(t)=&y(t)\left[-a_2(t)-b_2(t)y(t-\tau_2(t))+\frac{c_2(t)x(t)y^{m-1}(t)}{d+x(t)}\right]\\\ge&y(t)\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uy(t-\tau_2(t))\right],\ t>\tau.\end{aligned}\tag{18}$$

Here and subsequently, $y(\tilde{t})$ denotes any local minimal value of $y(t)$. Thanks to (18), we obtain that

$$0=\dot{y}(\tilde{t})\ge y(\tilde{t})\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uy(\tilde{t}-\tau_2(\tilde{t}))\right],\ t>\tau.\tag{19}$$

Thanks to (19), we obtain

$$y(\tilde{t}-\tau_2(\tilde{t}))\ge\frac{\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u}{b_2^u}.\tag{20}$$

Integrating both sides of (20) on interval $[\tilde{t}-\tau_2(\tilde{t}),\tilde{t}]$, noticing that

$$\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uy(\tilde{t}-\tau_2(\tilde{t}))\le 0,$$

we obtain

$$\begin{aligned}\ln\frac{y(\tilde{t})}{y(\tilde{t}-\tau_2(\tilde{t}))}&\ge\int_{\tilde{t}-\tau_2(\tilde{t})}^{\tilde{t}}\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uy(\tilde{t}-\tau_2(\tilde{t}))\right]dt\\&\ge\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uY^u\right]\tau.\end{aligned}\tag{21}$$

Together with (20) and (21), we deduce that

$$y(\tilde{t})\ge\frac{\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u}{b_2^u}\exp\left\{\left[\left(\frac{c_2^lX^l(Y^u)^{m-1}}{d+X^u}-a_2^u\right)-b_2^uY^u\right]\tau\right\}\overset{def}{=}Y^l.$$

We need to notice that $y(\tilde{t})$ represents any local minimal value of $y(t)$, hence there possesses a $T_3 > \tau$, for $t > T_3$, we have

$$y(t) \geq Y^l.$$

The proof is complete. $\square$

We can easily get the following corollary for system (4).

Corollary 1. *Suppose that (C_1)-(C_2) hold. Then system (4) possesses permanence. That is to say, any almost periodic positive solution $(u(t), v(t))^T$ of system (4) fulfills*

$$\frac{X^l}{H_1^u} \leq \liminf_{t\to\infty} u(t) \leq \limsup_{t\to\infty} u(t) \leq \frac{X^u}{H_1^l},$$
$$\frac{Y^l}{H_2^u} \leq \liminf_{t\to\infty} v(t) \leq \limsup_{t\to\infty} v(t) \leq \frac{Y^u}{H_2^l}.$$

Proof. Suppose that $(u(t), v(t))^T$ stands for an almost periodic positive solution of model (4). By utilizing Lemma 2.6, we can effortlessly know that $(x(t), y(t))^T = (H_1(t)u(t), H_2(t)v(t))^T$ also represents an almost periodic positive solution of model (1). Thanks to Theorem 3.1, one has

$$X^l \leq \liminf_{t\to\infty} H_1(t)u(t) \leq \limsup_{t\to\infty} H_1(t)u(t) \leq X^u,$$
$$Y^l \leq \liminf_{t\to\infty} H_2(t)v(t) \leq \limsup_{t\to\infty} H_2(t)v(t) \leq Y^u,$$

which means that

$$\frac{X^l}{H_1^u} \leq \liminf_{t\to\infty} u(t) \leq \limsup_{t\to\infty} u(t) \leq \frac{X^u}{H_1^l},$$
$$\frac{Y^l}{H_2^u} \leq \liminf_{t\to\infty} v(t) \leq \limsup_{t\to\infty} v(t) \leq \frac{Y^u}{H_2^l}.$$

Thus, the proof of Corollary 1 is complete now. $\square$

Theorem 3.2. *Suppose that $\mathcal{S}$ represents the set of all solutions $(u(t), v(t))^T$ of (4) satisfying*

$$\frac{X^l}{H_1^u} \leq u(t) \leq \frac{X^u}{H_1^l}, \quad \frac{Y^l}{H_2^u} \leq v(t) \leq \frac{Y^u}{H_2^l}, \ t \in (0, +\infty).$$

Then $\mathcal{S} \neq \varnothing$.

Proof. According to the theory of almost periodic functions, there possesses a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$a_i(t+t_n) \to a_i(t), \ B_i(t+t_n) \to B_i(t), \ C_i(t+t_n) \to C_i(t),$$
$$H_i(t+t_n) \to H_i(t), \ \tau_i(t+t_n) \to \tau_i(t), \ i = 1, 2,$$

uniformly on $(0, +\infty)$ as $n \to \infty$. Based on the Brouwer fixed-point theorem, we can propose a hypothesis that $(u(t), v(t))^T$ represents a positive (almost periodic) solution of system (4) satisfying

$$\frac{X^l}{H_1^u} \leq u(t) \leq \frac{X^u}{H_1^l}, \quad \frac{Y^l}{H_2^u} \leq v(t) \leq \frac{Y^u}{H_2^l}$$

for $t > T \geq 0$. It is obvious that the sequence $(u(t+t_n), v(t+t_n))^T$ is equicontinuous and uniformly bounded on each bounded subset of $(0, +\infty)$. Applying Arzelà-Ascoli theorem leads to

$$\lim_{k\to\infty} \mathbf{w}(t+t_k) = \lim_{k\to\infty} (u(t+t_k), v(t+t_k))^T = \mathbf{z}(t) = (z_1(t), z_2(t))^T,$$

where $\mathbf{w}(t+t_k)$ represents a subsequence of $\mathbf{w}(t+t_n)$ uniformly on each bounded subset, $\mathbf{z}(t)$ stands for a continuous function. Let $T_1 \in (0, +\infty)$ be given. For all positive integer k, we can suppose that $t_k + T_1 \geq T$. If $t \geq 0$, we obtain

$$\begin{aligned}
&u(t+t_k+T_1) - u(t_k+T_1)\\
=&\int_{T_1}^{t+T_1} \Big[u(s+t_k)\left(a_1(s+t_k) - B_1(s+t_k)u(s+t_k-\tau_1(s+t_k))\right)\\
&-\frac{C_1(s+t_k)u(s+t_k)v^m(s+t_k)}{d+H_1(s+t_k)u(s+t_k)}\Bigg]\,ds,\\
&v(t+t_k+T_1) - v(t_k+T_1)\\
=&\int_{T_1}^{t+T_1} \Big[v(s+t_k)\left(-a_2(s+t_k) - B_2(s+t_k)u(s+t_k-\tau_2(s+t_k))\right)\\
&+\frac{C_2(s+t_k)u(s+t_k)v^m(s+t_k)}{d+H_1(s+t_k)u(s+t_k)}\Bigg]\,ds.
\end{aligned} \tag{22}$$

Letting $n \to \infty$ in (22) and together with Lebesgue's dominated convergence theorem, we derive that for all $t \geq 0$,

$$\begin{aligned}
&u(t+T_1) - u(T_1)\\
=&\int_{T_1}^{t+T_1} \left[u(s)\left(a_1(s) - B_1(s)u(s-\tau_1(s))\right) - \frac{C_1(s)u(s)v^m(s)}{d+H_1(s)u(s)}\right] ds,\\
&v(t+T_1) - v(T_1)\\
=&\int_{T_1}^{t+T_1} \left[v(s)\left(-a_2(s) - B_2(s)u(s-\tau_2(s))\right) + \frac{C_2(s)u(s)v^m(s)}{d+H_1(s)u(s)}\right] ds.
\end{aligned}$$

Since $T_1 \in (0, +\infty)$ is given arbitrarily, system (4) possesses a positive (almost periodic) solution $\mathbf{z}(t) = (z_1(t), z_2(t))^T$ on $(0, +\infty)$. It is obvious that $\frac{X^l}{H_1^u} \leq z_1(t) \leq \frac{X^u}{H_1^l}$, $\frac{Y^l}{H_2^u} \leq z_2(t) \leq \frac{Y^u}{H_2^l}$ for $t \in (0, +\infty)$. Therefore, $\mathbf{z}(t) \in \mathcal{S}$. This ends the proof of Theorem 3.2. □

4. **Global asymptotic stability.** In this section, by constructing a feasible Lyapunov function, we present the global asymptotic stability.

Definition 4.1 (Globally asymptotically stable, [40])**.** If $(u(t), v(t))^T$ represents a positive solution of models (4) and (5), $(\tilde{u}(t), \tilde{v}(t))^T$ stands for any positive solution of models (4) and (5) fulfilling

$$\lim_{t\to\infty} (|u(t) - \tilde{u}(t)| + |v(t) - \tilde{v}(t)|) = 0.$$

Then we term $(u(t), v(t))^T$ is globally asymptotically stable.

Theorem 4.2. *Assume that all conditions of Theorem 3.1 are satisfied and further that the parameters of (4) fulfill the following conditions:*

(C_3) m represents a rational constant;
(C_4) $\liminf_{t\to+\infty} L_i(t) > 0$, $i = 1, 2$,

where

$$\begin{aligned}
L_1(t) =& B_1(t) - \left(\frac{H_1^u \left(C_1(t) \left(\frac{Y^u}{H_2^l}\right)^m + C_2(t) \frac{X^u}{H_1^l} \left(\frac{Y^l}{H_2^u}\right)^{m-1} \right)}{(d+X^l)^2} \right. \\
& \left. + \frac{(Y^l)^{m-1}}{(H_2^u)^{m-1}(d+X^l)} + \frac{X_u B_1(\phi_1^{-1}(t))}{H_1^l \dot{\phi}_1(\phi_1^{-1}(t))} 2B_1^u \tau \right) \\
& - \left(a_1(t) + \frac{X^u}{H_1^l} B_1(t) + \frac{(Y^u)^m C_1(t)}{(H_2^l)^m \,(d+X^l)} + \frac{X^u H_1^u (Y^u)^m C_1(t)}{H_1^l (H_2^l)^m (d+X^l)^2} \right) B_1^u \tau \\
& - \left(\frac{C_2(t) H_1^u X^u Y^u (Y^l)^{m-1}}{H_1^l H_2^l (H_2^u)^{m-1} \,(d+X^l)^2} + \frac{C_2(t) Y^u (Y^l)^{m-1}}{H_2^l (d+X^l)} \right) B_2^u \tau
\end{aligned}$$

and

$$\begin{aligned}
L_2(t) = & \left(B_2(t) - \frac{Y_u B_2(\phi_2^{-1}(t))}{H_2^l \dot{\phi}_2(\phi_2^{-1}(t))} \cdot 2B_2^u \tau \right) K_p \\
& - \left(a_2(t) + \frac{Y^u}{H_2^l} B_2(t) + \frac{C_2(t) X^u (Y^l)^{m-1}}{H_1^l (H_2^u)^{m-1} (d+X^l)} \right) B_2^u \tau K_p \\
& - \left(\frac{C_2(t) X^u Y^u (Y^l)^{2m-2}}{H_1^l H_2^l (d+X^l)} B_2^u \tau + \frac{X^u (Y^l)^{2m-2}}{H_1^l (d+X^l)} \right) K_{p-q} \\
& - \left(\frac{C_1(t)}{d+X^l} + \frac{X^u C_1(t)}{H_1^l (d+X^l)} B_1^u \tau \right) K_q,
\end{aligned}$$

in which ϕ_i^{-1} stands for the inverse function of $\phi_i(t) = t - \tau_i(t)(i=1,2)$, respectively. Thus, the positive solution of model (4) possesses global asymptotic stability.

Proof. By Theorem 3.2, we see that $\mathcal{S} \neq \varnothing$. For any two positive solutions $(u_1(t), v_1(t))^T$ and $(u_2(t), v_2(t))^T$ of system (4), it follows from Corollary 1 that there exists a positive constant T such that

$$\frac{X^l}{H_1^u} \le u_i(t) \le \frac{X^u}{H_1^l}, \quad \frac{Y^l}{H_2^u} \le v_i(t) \le \frac{Y^u}{H_2^l}, \quad t \ge T, \quad i = 1, 2.$$

Set

$$V_{11}(t) = \left| \ln u_1(t) - \ln u_2(t) \right|,$$

we obtain the upper right derivative of V_{11} along system (4)

$$\begin{aligned}
D^+ V_{11} =& \operatorname{sgn}(u_1(t) - u_2(t)) \left(\frac{\dot{u}_1(t)}{u_1(t)} - \frac{\dot{u}_2(t)}{u_2(t)} \right) \\
=& \operatorname{sgn}(u_1(t) - u_2(t)) \left\{ -B_1(t) \left[u_1(t-\tau_1(t)) - u_2(t-\tau_1(t)) \right] \right. \\
& \left. -C_1(t) \left[\frac{v_1^m(t)}{d + H_1(t) u_1(t)} - \frac{v_2^m(t)}{d + H_1(t) u_2(t)} \right] \right\} \\
=& -B_1(t) \operatorname{sgn}(u_1(t) - u_2(t)) \left[u_1(t-\tau_1(t)) - u_2(t-\tau_1(t)) \right] \\
& + C_1(t) \operatorname{sgn}(u_1(t) - u_2(t)) \left(\frac{v_2^m(t)}{d + H_1(t) u_2(t)} - \frac{v_2^m(t)}{d + H_1(t) u_1(t)} \right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{v_2^m(t)}{d+H_1(t)u_1(t)}-\frac{v_1^m(t)}{d+H_1(t)u_1(t)}\Bigg)\\
\le&-B_1(t)\mathrm{sgn}(u_1(t)-u_2(t))\left[u_1(t-\tau_1(t))-u_2(t-\tau_1(t))\right]\\
&+\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)}.
\end{aligned}
$$

By utilizing inequality $-\mathrm{sgn}(f)\cdot g\le -|f|+|f-g|(f,g\in\mathbb{R})$, one gets

$$
\begin{aligned}
&D^+V_{11}\\
\le&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)}\\
&-B_1(t)\left|u_1(t)-u_2(t)\right|+B_1(t)\left|\int_{t-\tau_1(t)}^t(\dot u_1(s)-\dot u_2(s))\,ds\right|\\
=&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)}\\
&-B_1(t)\left|u_1(t)-u_2(t)\right|\\
&+B_1(t)\left|\int_{t-\tau_1(t)}^t\left\{u_1(s)\left[a_1(s)-B_1(s)u_1(s-\tau_1(s))-\frac{C_1(s)v_1^m(s)}{d+H_1(s)u_1(s)}\right]\right.\right.\\
&\left.\left.-u_2(s)\left[a_1(s)-B_1(s)u_2(s-\tau_1(s))-\frac{C_1(s)v_2^m(s)}{d+H_1(s)u_2(s)}\right]\right\}ds\right|\\
=&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)}\\
&-B_1(t)\left|u_1(t)-u_2(t)\right|-B_1(s)u_2(s)\left[u_1(s-\tau_1(s))-u_2(s-\tau_1(s))\right]\\
&+B_1(t)\left|\int_{t-\tau_1(t)}^t\{[a_1(s)-B_1(s)u_1(s-\tau_1(s))\right.\\
&-\left.\frac{C_1(s)v_1^m(s)}{d+H_1(s)u_1(s)}\right](u_1(s)-u_2(s))\\
&\left.-u_2(s)\left[\frac{C_1(s)v_1^m(s)}{d+H_1(s)u_1(s)}-\frac{C_1(s)v_2^m(s)}{d+H_1(s)u_2(s)}\right]\right\}ds\Bigg|.
\end{aligned}
\tag{23}
$$

For $t\ge T+\tau$, we can utilize (23) to obtain that

$$
\begin{aligned}
D^+V_{11}\le&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)}\\
&-B_1(t)\left|u_1(t)-u_2(t)\right|+B_1(s)u_2(s)\left|u_1(s-\tau_1(s))-u_2(s-\tau_1(s))\right|\\
&+B_1(t)\left|\int_{t-\tau_1(t)}^t\{[a_1(s)+B_1(s)u_1(s-\tau_1(s))\right.\\
&+\left.\frac{C_1(s)v_1^m(s)}{d+H_1(s)u_1(s)}\right](u_1(s)-u_2(s))\\
&+u_2(s)\left[\frac{C_1(s)H_1(s)v_2^m(s)\left|u_1(s)-u_2(s)\right|}{(d+H_1(s)u_1(s))(d+H_1(s)u_2(s))}\right.
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&+\frac{C_1(s)\left|v_2^m(s)-v_1^m(s)\right|}{d+H_1(s)u_1(s)}\Bigg]\Bigg\}ds\Bigg| \\
\leq&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)} \\
&-B_1(t)\left|u_1(t)-u_2(t)\right|+B_1(t)\int_{t-\tau_1(t)}^{t}F_1(s)ds \\
=&\frac{C_1(t)H_1(t)v_2^m(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\left|v_2^m(t)-v_1^m(t)\right|}{d+H_1(t)u_1(t)} \\
&-B_1(t)\left|u_1(t)-u_2(t)\right|+B_1(t)\left[G_1(t)-G_1(\phi_1(t))\right],
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(s)=&\left(a_1(s)+\frac{X^u}{H_1^l}B_1(s)+\frac{(Y^u)^m C_1(s)}{(H_2^l)^m\,(d+X^l)}\right)\left|u_1(s)-u_2(s)\right| \\
&+\frac{X^u}{H_1^l}B_1(s)\left|u_1(s-\tau_1(s))-u_2(s-\tau_1(s))\right| \\
&+\frac{X^u}{H_1^l}\left(\frac{H_1^u(Y^u)^m C_1(s)}{(H_2^l)^m(d+X^l)^2}\left|u_1(s)-u_2(s)\right|+\frac{C_1(s)}{d+X^l}\left|v_2^m(s)-v_1^m(s)\right|\right)
\end{aligned}
$$

and $G_1(t)$ denotes a primitive function of $F_1(t)$.

Denote

$$
V_{12}(t)=\int_t^{\phi_1^{-1}(t)}\int_{\phi_1(u)}^{t}B_1(u)F_1(s)dsdu,
$$

we have

$$
\begin{aligned}
V_{12}(t)=&\int_t^{\phi_1^{-1}(t)}B_1(u)\left[G_1(t)-G_1(\phi_1(u))\right]du \\
=&G_1(t)\int_t^{\phi_1^{-1}(t)}B_1(u)du-\int_t^{\phi_1^{-1}(t)}B_1(u)G_1(\phi_1(u))du,
\end{aligned}
$$

and for $t\geq T+\tau$,

$$
\begin{aligned}
&D^+V_{12}(t) \\
=&F_1(t)\int_t^{\phi_1^{-1}(t)}B_1(u)du+G_1(t)\left[\frac{B_1(\phi_1^{-1}(t))}{\dot{\phi}_1(t)}-B_1(t)\right] \\
&-\left[\frac{B_1(\phi_1^{-1}(t))}{\dot{\phi}_1(t)}G_1(t)-B_1(t)G_1(\phi_1(t))\right] \\
=&F_1(t)\int_t^{\phi_1^{-1}(t)}B_1(u)du-B_1(t)\left[G_1(t)-G_1(\phi_1(t))\right].
\end{aligned}
\tag{25}
$$

Let

$$
V_{13}(t)=\frac{X^u}{H_1^l}\int_{t-\tau_1(t)}^{t}\int_{\phi_1^{-1}(u)}^{\phi_1^{-1}(\phi_1^{-1}(u))}\frac{B_1(s)B_1(\phi_1^{-1}(u))}{\dot{\phi}_1(\phi_1^{-1}(u))}\left|u_1(u)-u_2(u)\right|dsdu,
$$

for $t\geq T+\tau$, we obtain

$$\begin{aligned}
&D^+V_{13}(t)\\
=&\frac{X_u B_1(\phi_1^{-1}(t))}{H_1^l\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} B_1(u)du\,|u_1(t)-u_2(t)|\\
&-\frac{X_u}{H_1^l}B_1(t)\,|u_1(t-\tau_1(t))-u_2(t-\tau_1(t))|\int_t^{\phi_1^{-1}(t)} B_1(u)du.
\end{aligned}\tag{26}$$

Letting V_1 be the sum of V_{11}, V_{12} and V_{13}, it follows from (4), (25) and (26) that for $t\geq T+\tau$,

$$\begin{aligned}
&D^+V_1\\
\leq&\frac{C_1(t)H_1(t)v_2^m(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{C_1(t)\,|v_2^m(t)-v_1^m(t)|}{d+H_1(t)u_1(t)}\\
&-B_1(t)\,|u_1(t)-u_2(t)|\\
&+\left[\left(a_1(t)+\frac{X^u}{H_1^l}B_1(t)+\frac{(Y^u)^m C_1(t)}{(H_2^l)^m\,(d+X^l)}\right)|u_1(t)-u_2(t)|\right.\\
&+\frac{X^u}{H_1^l}\left(\frac{H_1^u(Y^u)^m C_1(t)}{(H_2^l)^m(d+X^l)^2}\,|u_1(t)-u_2(t)|\right.\\
&\left.\left.+\frac{C_1(t)}{d+X^l}\,|v_2^m(t)-v_1^m(t)|\right)\right]\int_t^{\phi_1^{-1}(t)} B_1(u)du\\
&+\frac{X_u B_1(\phi_1^{-1}(t))}{H_1^l\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} B_1(u)du\,|u_1(t)-u_2(t)|\,.
\end{aligned}$$

By the same method, we define

$$V_{21}(t)=|\ln v_1(t)-\ln v_2(t)|$$

and obtain the upper right derivative of V_{21} along system (4)

$$\begin{aligned}
&D^+V_{21}\\
=&\mathrm{sgn}(v_1(t)-v_2(t))\left(\frac{\dot{v}_1(t)}{v_1(t)}-\frac{\dot{v}_2(t)}{v_2(t)}\right)\\
=&\mathrm{sgn}(v_1(t)-v_2(t))\left\{-B_2(t)(v_1(t)-v_2(t))\right.\\
&\left.+C_2(t)\left[\frac{u_1(t)v_1^{m-1}(t)}{d+H_1(t)u_1(t)}-\frac{u_2(t)v_2^{m-1}(t)}{d+H_1(t)u_2(t)}\right]\right\}\\
=&-B_2(t)\mathrm{sgn}(v_1(t)-v_2(t))\left[v_1(t-\tau_2(t))-v_2(t-\tau_2(t))\right]\\
&+C_2(t)\mathrm{sgn}(v_1(t)-v_2(t))\left(\frac{u_1(t)v_1^{m-1}(t)}{d+H_1(t)u_1(t)}-\frac{u_1(t)v_1^{m-1}(t)}{d+H_1(t)u_2(t)}\right.\\
&\left.+\frac{u_1(t)v_1^{m-1}(t)}{d+H_1(t)u_2(t)}-\frac{u_2(t)v_2^{m-1}(t)}{d+H_1(t)u_2(t)}\right)\\
\leq&-B_2(t)\mathrm{sgn}(v_1(t)-v_2(t))\left[v_1(t-\tau_2(t))-v_2(t-\tau_2(t))\right]\\
&+\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{v_1^{m-1}(t)\,|u_1(t)-u_2(t)|+u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}.
\end{aligned}\tag{27}$$

By utilizing inequality $-\mathrm{sgn}(f)\cdot g \le -|f| + |f-g| (f, g \in \mathbb{R})$, one gets

$$\begin{aligned}
&D^+V_{21}\\
\le&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{d+H_1(t)u_2(t)}\\
&+\frac{u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}-B_2(t)\,|v_1(t)-v_2(t)|\\
&+B_2(t)\left|\int_{t-\tau_2(t)}^{t}(\dot v_1(s)-\dot v_2(s))\,ds\right|\\
=&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}+\frac{v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{d+H_1(t)u_2(t)}\\
&+\frac{u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}-B_2(t)\,|v_1(t)-v_2(t)|\\
&+B_2(t)\left|\int_{t-\tau_2(t)}^{t}\{v_1(s)\,[-a_2(s)-B_2(s)v_1(s-\tau_2(s))\right.\\
&+\left.\frac{C_2(s)u_1(s)v_1^{m-1}(s)}{d+H_1(s)u_1(s)}\right]\\
&\left.-v_2(s)\left[-a_2(s)-B_2(s)v_2(s-\tau_2(s))+\frac{C_2(s)u_2(s)v_2^{m-1}(s)}{d+H_1(s)u_2(s)}\right]\right\}ds\Bigg|
\end{aligned}\tag{28}$$

$$\begin{aligned}
=&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{v_1^{m-1}(t)\,|u_1(t)-u_2(t)|+u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}\\
&-B_2(t)\,|v_1(t)-v_2(t)|+B_2(t)\left|\int_{t-\tau_2(t)}^{t}\{[-a_2(s)-B_2(s)v_1(s-\tau_2(s))\right.\\
&+\left.\frac{C_2(s)u_1(s)v_1^{m-1}(s)}{d+H_1(s)u_1(s)}\right](v_1(s)-v_2(s))\\
&-B_2(s)v_2(s)\,[v_1(s-\tau_2(s))-v_2(s-\tau_2(s))]\\
&\left.-v_2(s)\left[\frac{C_2(s)u_2(s)v_2^{m-1}(s)}{d+H_1(s)u_2(s)}-\frac{C_2(s)u_1(s)v_1^{m-1}(s)}{d+H_1(s)u_1(s)}\right]\right\}ds\Bigg|.
\end{aligned}$$

For $t \ge T+\tau$, we can utilize (28) to obtain that

$$\begin{aligned}
&D^+V_{21}\\
\le&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\,|u_1(t)-u_2(t)|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{v_1^{m-1}(t)\,|u_1(t)-u_2(t)|+u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}\\
&-B_2(t)\,|v_1(t)-v_2(t)|+B_2(t)\left|\int_{t-\tau_2(t)}^{t}\{[a_2(s)+B_2(s)v_1(s-\tau_2(s))\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C_2(s)u_1(s)v_1^{m-1}(s)}{d+H_1(s)u_1(s)}\Bigg](v_1(s)-v_2(s))\\
&+B_2(s)v_2(s)\left[v_1(s-\tau_2(s))-v_2(s-\tau_2(s))\right]\\
&+C_2(s)v_2(s)\left[\frac{H_1(s)u_2(s)v_2^{m-1}(s)\left|u_1(s)-u_2(s)\right|}{(d+H_1(s)u_1(s))\left(d+H_1(s)u_2(s)\right)}\right.\\
&\left.\left.+\frac{u_2(s)\left|v_1^{m-1}(s)-v_2^{m-1}(s)\right|+v_1^{m-1}\left|u_1(s)-u_2(s)\right|}{d+H_1(s)u_1(s)}\right]\right\}ds\Bigg|\\
\leq&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{v_1^{m-1}(t)\left|u_1(t)-u_2(t)\right|+u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}\\
&-B_2(t)\left|v_1(t)-v_2(t)\right|+B_2(t)\int_{t-\tau_1(t)}^{t}F_2(s)ds\\
\leq&\frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t)\left|u_1(t)-u_2(t)\right|}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))}\\
&+\frac{v_1^{m-1}(t)\left|u_1(t)-u_2(t)\right|+u_2(t)\left|v_1^{m-1}(t)-v_2^{m-1}(t)\right|}{d+H_1(t)u_2(t)}\\
&-B_2(t)\left|v_1(t)-v_2(t)\right|+B_2(t)\left[G_2(t)-G_2(\phi_2(t))\right],
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
F_2(s)=&\left(a_2(s)+\frac{Y^u}{H_2^l}B_2(s)+\frac{C_2(s)X^u(Y^l)^{m-1}}{H_1^l(H_2^u)^{m-1}(d+X^l)}\right)\left|v_1(s)-v_2(s)\right|\\
&+\frac{Y^u}{H_2^l}B_2(s)\left|v_1(s-\tau_2(s))-v_2(s-\tau_2(s))\right|\\
&+\frac{Y^u}{H_2^l}C_2(s)\left(\frac{H_1^uX^u(Y^l)^{m-1}}{H_1^l(H_2^u)^{m-1}\left(d+X^l\right)^2}\left|u_1(s)-u_2(s)\right|\right.\\
&+\frac{X^u}{H_1^l(d+X^l)}\left|v_1^{m-1}(s)-v_2^{m-1}(s)\right|\\
&\left.+\frac{(Y^l)^{m-1}}{(H_2^u)^{m-1}(d+X^l)}\left|u_1(s)-u_2(s)\right|\right)
\end{aligned}
$$

and $G_2(t)$ denotes a primitive function of $F_2(t)$.

Next, denote

$$
V_{22}(t)=\int_t^{\phi_2^{-1}(t)}\int_{\phi_2(u)}^{t}B_2(u)F_2(s)dsdu,
$$

we have

$$
\begin{aligned}
V_{22}(t)=&\int_t^{\phi_2^{-1}(t)}B_2(u)\left[G_2(t)-G_2(\phi_2(u))\right]du\\
=&G_2(t)\int_t^{\phi_2^{-1}(t)}B_2(u)du-\int_t^{\phi_2^{-1}(t)}B_2(u)G_2(\phi_2(u))du,
\end{aligned}
$$

and for $t \geq T + \tau$,

$$\begin{aligned} D^+V_{22}(t) =& F_2(t) \int_t^{\phi_2^{-1}(t)} B_2(u)du + G_2(t) \left[\frac{B_2(\phi_2^{-1}(t))}{\dot{\phi}_2(t)} - B_2(t) \right] \\ & - \left[\frac{B_2(\phi_2^{-1}(t))}{\dot{\phi}_2(t)} G_2(t) - B_2(t) G_2(\phi_2(t)) \right] \\ =& F_2(t) \int_t^{\phi_2^{-1}(t)} B_2(u)du - B_2(t) \left[G_2(t) - G_2(\phi_2(t)) \right]. \end{aligned} \tag{30}$$

Now, define

$$V_{23}(t) = \frac{Y^u}{H_2^l} \int_{t-\tau_2(t)}^{t} \int_{\phi_2^{-1}(u)}^{\phi_2^{-1}(\phi_2^{-1}(u))} \frac{B_2(s)B_2(\phi_2^{-1}(u))}{\dot{\phi}_2(\phi_2^{-1}(u))} \left| v_1(u) - v_2(u) \right| dsdu,$$

for $t \geq T + \tau$ we get

$$\begin{aligned} D^+V_{23}(t) =& \frac{Y_u B_2(\phi_2^{-1}(t))}{H_2^l \dot{\phi}_2(\phi_2^{-1}(t))} \int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} B_2(u)du \left| v_1(t) - v_2(t) \right| \\ & - \frac{Y_u}{H_2^l} B_2(t) \left| v_1(t-\tau_2(t)) - v_2(t-\tau_2(t)) \right| \int_t^{\phi_2^{-1}(t)} B_2(u)du. \end{aligned} \tag{31}$$

Letting V_2 be the sum of V_{21}, V_{22} and V_{23}, it follows from (29), (30) and (31) that for $t \geq T + \tau$,

$$\begin{aligned} & D^+V_2 \\ \leq & \frac{C_2(t)H_1(t)u_1(t)v_1^{m-1}(t) \left| u_1(t) - u_2(t) \right|}{(d + H_1(t)u_1(t))(d + H_1(t)u_2(t))} \\ & + \frac{v_1^{m-1}(t) \left| u_1(t) - u_2(t) \right| + u_2(t) \left| v_1^{m-1}(t) - v_2^{m-1}(t) \right|}{d + H_1(t)u_2(t)} \\ & - B_2(t) \left| v_1(t) - v_2(t) \right| \\ & + \left[\left(a_2(t) + \frac{Y^u}{H_2^l} B_2(t) + \frac{C_2(t)X^u(Y^l)^{m-1}}{H_1^l(H_2^u)^{m-1}(d + X^l)} \right) \left| v_1(t) - v_2(t) \right| \right. \\ & + \frac{Y^u}{H_2^l} C_2(t) \left(\frac{H_1^u X^u (Y^l)^{m-1} \left| u_1(t) - u_2(t) \right|}{H_1^l (H_2^u)^{m-1} (d + X^l)^2} \right. \\ & \left. \left. + \frac{X^u \left| v_1^{m-1}(t) - v_2^{m-1}(t) \right|}{H_1^l (d + X^l)} + \frac{(Y^l)^{m-1} \left| u_1(t) - u_2(t) \right|}{(H_2^u)^{m-1}(d + X^l)} \right) \right] \int_t^{\phi_2^{-1}(t)} B_2(u)du \\ & + \frac{Y_u B_2(\phi_2^{-1}(t))}{H_2^l \dot{\phi}_2(\phi_2^{-1}(t))} \int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} B_2(u)du \left| v_1(t) - v_2(t) \right|. \end{aligned} \tag{32}$$

Now, we construct the following Lyapunov function

$$V(t) = V_1(t) + V_2(t).$$

It follows from (27) and (32) that

$$
\begin{aligned}
&D^+V(t) = D^+V_1(t) + D^+V_2(t)\\
\le &\left(\frac{H_1(t)\left(C_1(t)v_2^m(t) + C_2(t)u_1(t)v_1^{m-1}(t)\right)}{(d+H_1(t)u_1(t))(d+H_1(t)u_2(t))} - B_1(t) + \frac{v_1^{m-1}(t)}{d+H_1(t)u_2(t)}\right.\\
&\left.+\frac{X_u B_1(\phi_1^{-1}(t))}{H_1^l\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} B_1(u)du\right)|u_1(t)-u_2(t)|\\
&+\left(a_1(t) + \frac{X^u}{H_1^l}B_1(t) + \frac{(Y^u)^m C_1(t)}{(H_2^l)^m\,(d+X^l)} + \frac{X^u H_1^u (Y^u)^m C_1(t)}{H_1^l (H_2^l)^m (d+X^l)^2}\right)\\
&\times\int_t^{\phi_1^{-1}(t)} B_1(u)du\,|u_1(t)-u_2(t)|\\
&+\left(\frac{C_2(t)H_1^u X^u Y^u (Y^l)^{m-1}}{H_1^l H_2^l (H_2^u)^{m-1}\left(d+X^l\right)^2}\right.\\
&\left.+\frac{C_2(t)Y^u(Y^l)^{m-1}}{H_2^l(H_2^u)^{m-1}(d+X^l)}\right)\int_t^{\phi_2^{-1}(t)} B_2(u)du\,|u_1(t)-u_2(t)|\\
&+\left(\frac{Y_u B_2(\phi_2^{-1}(t))}{H_2^l\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} B_2(u)du - B_2(t)\right)|v_1(t)-v_2(t)|\\
&+\left(a_2(t) + \frac{Y^u}{H_2^l}B_2(t) + \frac{C_2(t)X^u(Y^l)^{m-1}}{H_1^l(H_2^u)^{m-1}(d+X^l)}\right)\\
&\times\int_t^{\phi_2^{-1}(t)} B_2(u)du\,|v_1(t)-v_2(t)|\\
&+\left(\frac{C_2(t)X^uY^u}{H_1^lH_2^l(d+X^l)}\int_t^{\phi_2^{-1}(t)} B_2(u)du + \frac{u_2(t)}{d+H_1(t)u_2(t)}\right)\\
&\times\left|v_1^{m-1}(t) - v_2^{m-1}(t)\right|\\
&+\left(\frac{C_1(t)}{d+H_1(t)u_1(t)} + \frac{X^uC_1(t)}{H_1^l(d+X^l)}\int_t^{\phi_1^{-1}(t)} B_1(u)du\right)\left|v_2^m(t) - v_1^m(t)\right|.
\end{aligned}
\tag{33}
$$

Based on (C_3), there admit two positive integers p and q $(p > q)$ such that $m = \frac{q}{p}$. Hence,

$$v_1^{m-1}(t) - v_2^{m-1}(t) = \left(v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t)\right) v_1^{m-1}(t)v_2^{m-1}(t)\sum_{i=1}^{p-q} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{p-q-i}{p}}(t), \tag{34}$$

$$v_1^m(t) - v_2^m(t) = \left(v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t)\right)\sum_{i=1}^{q} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t), \tag{35}$$

$$v_1(t) - v_2(t) = \left(v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t)\right)\sum_{i=1}^{p} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t). \tag{36}$$

Substitution of (34), (35) and (36) into (33) yields

$$
\begin{aligned}
&D^+V(t) = D^+V_1(t) + D^+V_2(t)\\
\le &\left(\frac{H_1^u\left(C_1(t)\left(\frac{Y^u}{H_2^l}\right)^m + C_2(t)\frac{X^u}{H_1^l}\left(\frac{Y^l}{H_2^u}\right)^{m-1}(t)\right)}{(d+X^l)^2} - B_1(t)\right.\\
&\left.+\frac{(Y^l)^{m-1}}{(H_2^u)^{m-1}(d+X^l)} + \frac{X_uB_1(\phi_1^{-1}(t))}{H_1^l\dot{\phi}_1(\phi_1^{-1}(t))}\int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} B_1(u)du\right)\\
&\times|u_1(t)-u_2(t)|\\
&+\left(a_1(t) + \frac{X^u}{H_1^l}B_1(t) + \frac{(Y^u)^mC_1(t)}{(H_2^l)^m\,(d+X^l)} + \frac{X^uH_1^u(Y^u)^mC_1(t)}{H_1^l(H_2^l)^m(d+X^l)^2}\right)\\
&\times\int_t^{\phi_1^{-1}(t)} B_1(u)du\,|u_1(t)-u_2(t)|\\
&+\left(\frac{C_2(t)H_1^uX^uY^u(Y^l)^{m-1}}{H_1^lH_2^l(H_2^u)^{m-1}\left(d+X^l\right)^2} + \frac{C_2(t)Y^u(Y^l)^{m-1}}{H_2^l(d+X^l)}\right)\\
&\times\int_t^{\phi_2^{-1}(t)} B_2(u)du\,|u_1(t)-u_2(t)|\\
&+\left(\frac{Y_uB_2(\phi_2^{-1}(t))}{H_2^l\dot{\phi}_2(\phi_2^{-1}(t))}\int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} B_2(u)du - B_2(t)\right)\\
&\times\sum_{i=1}^{p} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t)\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|\\
&+\left(a_2(t) + \frac{Y^u}{H_2^l}B_2(t) + \frac{C_2(t)X^u(Y^l)^{m-1}}{H_1^l(H_2^u)^{m-1}(d+X^l)}\right)\\
&\times\int_t^{\phi_2^{-1}(t)} B_2(u)du\sum_{i=1}^{p} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t)\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|\\
&+\left(\frac{C_2(t)X^uY^u(Y^l)^{2m-2}}{H_1^lH_2^l(d+X^l)}\int_t^{\phi_2^{-1}(t)} B_2(u)du + \frac{X^u(Y^l)^{2m-2}}{H_1^l(d+X^l)}\right)\\
&\times\sum_{i=1}^{p-q} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{p-q-i}{p}}(t)\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|\\
&+\left(\frac{C_1(t)}{d+X^l} + \frac{X^uC_1(t)}{H_1^l(d+X^l)}\int_t^{\phi_1^{-1}(t)} B_1(u)du\right)\\
&\times\sum_{i=1}^{q} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t)\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|.
\end{aligned} \tag{37}
$$

From now on, we always denote

$$K_p \ge \sum_{i=1}^{p} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{q-i}{p}}(t),\quad K_{p-q} \ge \sum_{i=1}^{p-q} v_1^{\frac{i-1}{p}}(t)v_2^{\frac{p-q-i}{p}}(t),$$

$$\begin{aligned}
&K_q \geq \sum_{i=1}^{q} v_1^{\frac{i-1}{p}}(t) v_2^{\frac{q-i}{p}}(t), \quad B_1^u \tau \geq \int_t^{\phi_1^{-1}(t)} B_1(u)du, \\
&B_2^u \tau \geq \int_t^{\phi_2^{-1}(t)} B_2(u)du, \quad 2B_1^u \tau \geq \int_{\phi_1^{-1}(t)}^{\phi_1^{-1}(\phi_1^{-1}(t))} B_1(u)du, \\
&2B_2^u \tau \geq \int_{\phi_2^{-1}(t)}^{\phi_2^{-1}(\phi_2^{-1}(t))} B_2(u)du.
\end{aligned} \tag{38}$$

Substitution of (38) into (37) yields

$$\begin{aligned}
&D^+V(t) = D^+V_1(t) + D^+V_2(t) \\
\leq &\left(\frac{H_1^u \left(C_1(t) \left(\frac{Y^u}{H_2^l}\right)^m + C_2(t) \frac{X^u}{H_1^l} \left(\frac{Y^l}{H_2^u}\right)^{m-1} \right)}{(d+X^l)^2} \right. \\
&\left. -B_1(t) + \frac{(Y^l)^{m-1}}{(H_2^u)^{m-1}(d+X^l)} + \frac{X_u B_1(\phi_1^{-1}(t))}{H_1^l \dot{\phi}_1(\phi_1^{-1}(t))} 2B_1^u \tau \right) |u_1(t) - u_2(t)| \\
&+ \left(a_1(t) + \frac{X^u}{H_1^l} B_1(t) + \frac{(Y^u)^m C_1(t)}{(H_2^l)^m (d+X^l)} + \frac{X^u H_1^u (Y^u)^m C_1(t)}{H_1^l (H_2^l)^m (d+X^l)^2} \right) \\
&B_1^u \tau \, |u_1(t) - u_2(t)| \\
&+ \left(\frac{C_2(t) H_1^u X^u Y^u (Y^l)^{m-1}}{H_1^l H_2^l (H_2^u)^{m-1} (d+X^l)^2} + \frac{C_2(t) Y^u (Y^l)^{m-1}}{H_2^l (d+X^l)} \right) B_2^u \tau \, |u_1(t) - u_2(t)| \\
&+ \left(\frac{Y_u B_2(\phi_2^{-1}(t))}{H_2^l \dot{\phi}_2(\phi_2^{-1}(t))} \cdot 2B_2^u \tau - B_2(t) \right) K_p \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \\
&+ \left(a_2(t) + \frac{Y^u}{H_2^l} B_2(t) + \frac{C_2(t) X^u (Y^l)^{m-1}}{H_1^l (H_2^u)^{m-1} (d+X^l)} \right) B_2^u \tau K_p \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \\
&+ \left(\frac{C_2(t) X^u Y^u (Y^l)^{2m-2}}{H_1^l H_2^l (d+X^l)} B_2^u \tau + \frac{X^u (Y^l)^{2m-2}}{H_1^l (d+X^l)} \right) K_{p-q} \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \\
&+ \left(\frac{C_1(t)}{d+X^l} + \frac{X^u C_1(t)}{H_1^l (d+X^l)} B_1^u \tau \right) K_q \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \\
= &- \left(L_1(t) |u_1(t) - u_2(t)| + L_2(t) \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \right).
\end{aligned} \tag{39}$$

Based on the condition (C_4), there have positive constants α_1, α_2 and $T_0 \geq T + \tau$ such that

$$L_1 \geq \alpha_1 > 0, \ L_2 \geq \alpha_2 > 0. \tag{40}$$

Letting $\alpha^* = \min\{\alpha_1, \alpha_2\}$, we get from (39) and (40) that

$$D^+V(t) \leq -\alpha^* \left(|u_1(t) - u_2(t)| + \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \right).$$

Integrating the last inequality from T_0 to t, for $t \geq T_0$ we have

$$V(t) + \alpha^* \int_{T_0}^t \left(|u_1(t) - u_2(t)| + \left| v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t) \right| \right) du \leq V(T_0).$$

Hence, $V(t)$ represents a bounded function on interval $[T_0, +\infty)$. In addition,

$$\int_{T_0}^{+\infty}\left(|u_1(t)-u_2(t)|+\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|\right)du \le +\infty.$$

Together with system (4) and Theorem 3.1, we can deduce that $u_1(t)-u_2(t)$, $v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)$ and their derivatives are bounded on interval $[T_0, +\infty)$. In other word, $|u_1(t)-u_2(t)|+\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|$ is uniformly continuous. Thanks to Barbalar Lemma [3], we deduce that

$$\lim_{t\to+\infty}\left(|u_1(t)-u_2(t)|+\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|\right)=0.$$

That is,

$$\lim_{t\to+\infty}|u_1(t)-u_2(t)|=\lim_{t\to+\infty}\left|v_1^{\frac{1}{p}}(t)-v_2^{\frac{1}{p}}(t)\right|=0.$$

Thus, the positive solution of (4) remains globally asymptotically stable. This finishes the proof of Theorem 4.2. □

5. **Existence and uniqueness of almost periodic solutions.** In this section, by applying Arzelà-Ascoli theorem, we establish the existence and uniqueness of almost periodic positive solutions.

Theorem 5.1. *Suppose that the conditions* (H_1)*-*(H_3)*,* (C_1)*-*(C_4) *hold. Then the predator-prey system (4) possesses a unique almost periodic positive solution which admits global asymptotic stability.*

Proof. Thanks to Theorem 3.2, the predator-prey system (4) exists a positive bounded solution $(u(t), v(t))^T$ fulfilling

$$\frac{X^l}{H_1^u}\le u(t)\le\frac{X^u}{H_1^l},\quad \frac{Y^l}{H_2^u}\le v(t)\le\frac{Y^u}{H_2^l},\quad t\in(0,+\infty).$$

Hence, there possesses a sequence $\{\xi'_m\}$ with $\xi'_m\to\infty$ as $m\to\infty$ such that $(u(t+\xi'_m), v(t+\xi'_m))^T$ stands for a positive bounded solution of the following model

$$\begin{cases}\dot u(t)=u(t)\left[a_1(t+\xi'_m)-B_1(t+\xi'_m)u(t-\tau_1(t))-\dfrac{C_1(t+\xi'_m)v^m(t)}{d+H_1(t+\xi'_m)u(t)}\right],\\ \dot v(t)=v(t)\left[-a_2(t+\xi'_m)-B_2(t+\xi'_m)v(t-\tau_2(t))+\dfrac{C_2(t+\xi'_m)u(t)v^{m-1}(t)}{d+H_1(t+\xi'_m)u(t)}\right],\\ \qquad\qquad\qquad\qquad 0<m<1.\end{cases}\tag{41}$$

Together with (41) and Theorem 3.1, we can come to a conclusion that not only $(u(t+\xi'_m), v(t+\xi'_m))^T$ but also $(\dot u(t+\xi'_m), \dot v(t+\xi'_m))^T$ possesses uniformly bounded. Therefore, $(u(t+\xi'_m), v(t+\xi'_m))^T$ is equi-continuous. Thanks to Arzelà-Ascoli theorem, we can find a subsequence $\left\{(u(t+\xi_m), v(t+\xi_m))^T\right\}\subseteq\left\{(u(t+\xi'_m), v(t+\xi'_m))^T\right\}$ with uniform convergence such that for any $\varepsilon>0$, there admits a positive constant $\rho_0(\varepsilon)>0$. In addition, if $m, l>\rho_0(\varepsilon)$, we get that

$$|u(t+\xi_m)-u(t+\xi_l)|<\varepsilon,$$
$$|v(t+\xi_m)-v(t+\xi_l)|<\varepsilon,$$

which means that $(u(t+\xi_m), v(t+\xi_m))^T$ represents a positive almost periodic asymptotic function, there admit four functions $P_1(t)$, $P_2(t)$, $Q_1(t)$ and $Q_2(t)$ such that

$$u(t) = P_1(t) + Q_1(t),\ v(t) = P_2(t) + Q_2(t),\ t \in (0, +\infty),$$

where

$$\lim_{m\to+\infty} P_i(t+\xi_m) = P_i(t), \quad \lim_{m\to+\infty} Q_i(t+\xi_m) = 0, \quad i = 1, 2,$$

$P_i(t)$ stands for almost periodic function. It shows that

$$\lim_{m\to+\infty} u(t+\xi_m) = P_1(t), \ \lim_{m\to+\infty} v(t+\xi_m) = P_2(t).$$

Meanwhile,

$$\begin{aligned}
\lim_{m\to+\infty} \dot{u}(t+\xi_m) &= \lim_{m\to+\infty}\lim_{k\to0} \frac{u(t+\xi_m+k) - u(t+\xi_m)}{k} \\
&= \lim_{k\to0}\lim_{m\to+\infty} \frac{u(t+\xi_m+k) - u(t+\xi_m)}{k} = \lim_{k\to0} \frac{P_1(t+k) - P_1(t)}{k}, \\
\lim_{m\to+\infty} \dot{v}(t+\xi_m) &= \lim_{m\to+\infty}\lim_{k\to0} \frac{v(t+\xi_m+k) - v(t+\xi_m)}{k} \\
&= \lim_{k\to0}\lim_{m\to+\infty} \frac{v(t+\xi_m+k) - v(t+\xi_m)}{k} = \lim_{k\to0} \frac{P_2(t+k) - P_2(t)}{k},
\end{aligned}$$

thus the almost periodic functions $\dot{P}_1(t)$ and $\dot{P}_2(t)$ exist.

Here and subsequently, we need to verify that $P(t) = (P_1(t), P_2(t))^T$ represents an almost periodic solution to model (4). According to the theory of almost periodic functions [15], there admits a sequence $\{\xi_\rho\}$, $\xi_\rho \to \infty$ as $\rho \to +\infty$, such that

$$\begin{aligned}
&a_i(t+\xi_\rho) \to a_i(t),\ B_i(t+\xi_\rho) \to B_i(t),\ C_i(t+\xi_\rho) \to C_i(t), \\
&H_i(t+\xi_\rho) \to H_i(t),\ \tau_i(t+\xi_\rho) \to \tau_i(t),
\end{aligned}$$

as $\rho \to +\infty$ uniformly on $(0, +\infty)$, $i = 1, 2$.

It is effortless to check that

$$\lim_{\rho\to+\infty} u(t+\xi_\rho) = P_1(t), \ \lim_{\rho\to+\infty} v(t+\xi_\rho) = P_2(t).$$

Therefore, we deduce that

$$\begin{aligned}
\dot{P}_1(t) &= \lim_{\rho\to+\infty} \dot{u}(t+\xi_\rho) \\
&= \lim_{\rho\to+\infty} u(t+\xi_\rho)\left[a_1(t+\xi_\rho) - B_1(t+\xi_\rho)u(t+\xi_\rho - \tau_1(t+\xi_\rho))\right. \\
&\quad \left. - \frac{C_1(t+\xi_\rho)v^m(t+\xi_\rho)}{d + H_1(t+\xi_\rho)u(t+\xi_\rho)}\right] \\
&= P_1(t)\left[a_1(t) - B_1(t)P_1(t-\tau_1(t)) - \frac{C_1(t)v^m(t)}{d + H_1(t)u(t)}\right], \\
\dot{P}_2(t) &= \lim_{\rho\to+\infty} \dot{v}(t+\xi_\rho) \\
&= \lim_{\rho\to+\infty} v(t+\xi_\rho)\left[-a_2(t+\xi_\rho) - B_2(t+\xi_\rho)v(t+\xi_\rho - \tau_2(t+\xi_\rho))\right. \\
&\quad \left. + \frac{C_2(t+\xi_\rho)u(t+\xi_\rho)v^{m-1}(t+\xi_\rho)}{d + H_1(t+\xi_\rho)u(t+\xi_\rho)}\right] \\
&= P_2(t)\left[-a_2(t) - B_2(t)P_2(t-\tau_2(t)) + \frac{C_2(t)u(t)v^{m-1}(t)}{d + H_1(t)u(t)}\right].
\end{aligned}$$

It means that $P(t) = (P_1(t), P_2(t))$ can satisfy (4). Meanwhile, $P(t)$ represents an almost periodic solution of (4).

We proceed to show that system (4) has only one almost periodic positive solution. Suppose for the sake of contradiction that system (4) possesses two almost periodic positive solutions $P(t) = (P_1(t), P_2(t))$ and $Q(t) = (Q_1(t), Q_2(t))$. We assert that $P_i(t) = Q_i(t)$ for all $t \in (0, +\infty)$. Conversely, there exists at least a positive number $\eta \in (0, +\infty)$ such that $P_i(\eta) \neq Q_i(\eta)$ for a certain integer $i > 0$. It means that

$$\begin{aligned}\Xi_1 &= |P_1(\eta) - Q_1(\eta)| = \left|\lim_{m\to+\infty} u_1(\eta+\xi_m) - \lim_{m\to+\infty} u_2(\eta+\xi_m)\right| \\ &= \lim_{m\to+\infty} |u_1(t) - u_2(t)| > 0, \\ \Xi_2 &= \left|P_2^{\frac{1}{p}}(\eta) - Q_2^{\frac{1}{p}}(\eta)\right| = \left|\lim_{m\to+\infty} v_1^{\frac{1}{p}}(\eta+\xi_m) - \lim_{m\to+\infty} v_2^{\frac{1}{p}}(\eta+\xi_m)\right| \\ &= \lim_{m\to+\infty} \left|v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t)\right| > 0,\end{aligned}$$

which contradicts with $\lim_{m\to+\infty} |u_1(t) - u_2(t)| = 0$ and $\lim_{m\to+\infty} \left|v_1^{\frac{1}{p}}(t) - v_2^{\frac{1}{p}}(t)\right| = 0$ proposed by Definition 4.1. This finishes the proof of Theorem 5.1. $\square$

Theorem 5.2. *Suppose that the conditions (H_1)-(H_3), (C_1)-(C_4) hold. Then the predator-prey systems (1) possesses a unique almost periodic positive solution which admits global asymptotic stability.*

Proof. Thanks to Lemma 2.6, we can see that

$$(x(t), y(t))^T = \left(\prod_{0<t_k<t}(1+h_{1k})u(t), \prod_{0<t_k<t}(1+h_{2k})v(t)\right)^T$$

stands for an almost periodic solution of system (1). Because (H_2) holds, follow the proofs of Theorem 79 and Lemma 31 in [31], we can deduce that $(x(t), y(t))^T$ stands for an almost periodic solution. Thus, based on the uniqueness and global stability of $(u(t), v(t))^T$, $(x(t), y(t))^T$ represents a unique globally asymptotically stable solution to system (1). This ends the proof. $\square$

We can obtain the following corollary because the condition (C_4) has been simplified by setting $h_{ik} \equiv 0$.

Corollary 2. *We denote $h_{ik} \equiv 0$. Thanks to (H_1) and (H_2), we can suppose further that*

$$\begin{aligned}\liminf_{t\to+\infty} P_1(t) = \liminf_{t\to+\infty} \Bigg\{ b_1(t) &- \Bigg(\frac{c_1(t)\left(Y^u\right)^m + c_2(t)X^u\left(Y^l\right)^{m-1}}{(d+X^l)^2} \\ &+ \frac{(Y^l)^{m-1}}{d+X^l} + \frac{X^u b_1(\phi_1^{-1}(t))}{\dot{\phi}_1(\phi_1^{-1}(t))} 2b_1^u\tau \Bigg) \\ &- \left(a_1(t) + X^u b_1(t) + \frac{(Y^u)^m c_1(t)}{d+X^l} + \frac{X^u (Y^u)^m c_1(t)}{(d+X^l)^2} \right) b_1^u \tau \\ &- \left(\frac{c_2(t) X^u Y^u (Y^l)^{m-1}}{\left(d+X^l\right)^2} + \frac{c_2(t) Y^u (Y^l)^{m-1}}{(d+X^l)} \right) b_2^u \tau \Bigg\} > 0,\end{aligned}$$

and

$$\begin{aligned}\liminf_{t\to+\infty} P_2(t) = & \liminf_{t\to+\infty} \Bigg\{ \left(b_2(t) - \frac{Y^u b_2(\phi_2^{-1}(t))}{\dot{\phi}_2(\phi_2^{-1}(t))} \cdot 2b_2^u \tau \right) K_p \\ & - \left(a_2(t) + Y^u b_2(t) + \frac{c_2(t) X^u (Y^l)^{m-1}}{d + X^l} \right) b_2^u \tau K_p \\ & - \left(\frac{c_2(t) X^u Y^u (Y^l)^{2m-2}}{d + X^l} b_2^u \tau + \frac{X^u (Y^l)^{2m-2}}{d + X^l} \right) K_{p-q} \\ & - \left(\frac{c_1(t)}{d + X^l} + \frac{X^u c_1(t)}{d + X^l} b_1^u \tau \right) K_q \Bigg\} > 0.\end{aligned}$$

Thus, (1) possesses a unique almost periodic positive solution which has global asymptotic stability.

6. Numerical simulations.

In this section, we offer an numerical example applied to our theoretical results. Consider

$$\begin{cases} \dot{x}(t) = (0.3 - 0.05\sin\sqrt{2}t)x(t) - (0.95 - 0.01\cos t)x(t-0.01)x(t) \\ \qquad - \dfrac{0.11 x(t) y^{0.5}(t)}{3 + x(t)}, \\ \dot{y}(t) = -(0.5 + 0.05\cos\sqrt{0.2}t)y(t) - (2.5 - 1.3\sin\sqrt{0.2}t)y(t-0.02)y(t) \\ \qquad + \dfrac{3x(t) y^{0.5}(t)}{3 + x(t)}. \end{cases} \tag{42}$$

Define $X^l = m_1^l$, $Y^l = m_2^l$, $X^u = M_1^u$ and $Y^u = M_2^u$, we can obtain the following table (Table 1) by a direct computation:

TABLE 1. The biological parameters of x and y.

	a_i^l	a_i^u	b_i^l	b_i^u	c_i^l	c_i^u	m_i^l	M_i^u	P_i^l	τ_i^l	τ_i^u
x	0.25	0.35	0.94	0.96	0.11	0.11	0.2495	0.3736	0.01	0.01	0.01
y	0.45	0.55	1.20	3.80	3.00	3.00	0.0189	0.5451	0.02	0.02	0.02

It means that the conditions of $P_1(t)$ and $P_2(t)$ proposed by Corollary 2 are fulfilled. It is also shown that system (42) admits permanence and global asymptotic stability, existence and uniqueness of almost periodic positive solutions. Figure 1 supports our main results.

7. Conclusion.

In this article, we've considered a modified nonlinear impulsive differential system with time delays. By utilizing comparison theorem and constructing a feasible Lyapunov functional, we've obtained sufficient conditions to guarantee the permanence and global asymptotic stability of the system. By applying Arzelà-Ascoli theorem, we've established the existence and uniqueness of almost-periodic positive solutions. Compared with [22], we've analyzed a more general model with impulsive effects and time delays and established several more general conclusions. As a matter of fact, the example proposed in [22] can not offer an actual result by comparing theoretical derivation and experimental verification. Although [23] investigated a time-delayed multispecies competition-predator model, but it didn't consider the existence of transient changeable mechanism. All up, the

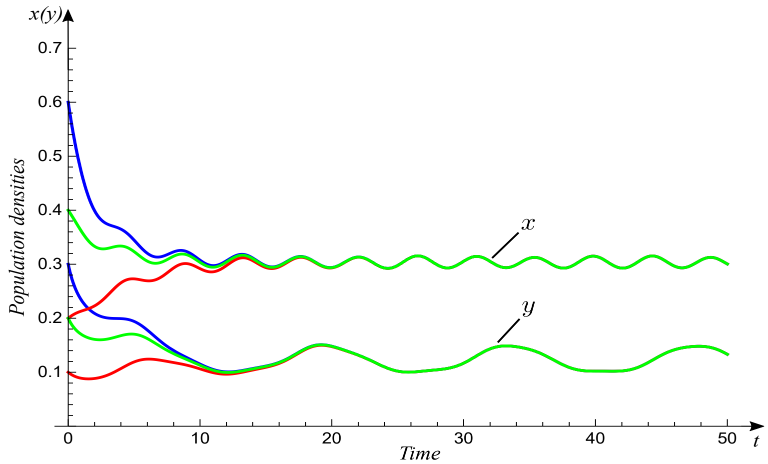

FIGURE 1. Numeric simulation of the prey $x(t)$ and the predator $y(t)$ of (42) with the initial conditions $(x(0), y(0))^T = (0.6, 0.3)^T$, $(x(0), y(0))^T = (0.2, 0.1)^T$ and $(x(0), y(0))^T = (0.4, 0.2)^T$.

system (1) makes the most sense, since it has considered many natural phenomena in its model.

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