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Brief paper

Stability and L_1-gain analysis of linear periodic piecewise positive systems☆

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ABSTRACT

This paper investigates the stability, stabilization and L_1-gain of linear periodic piecewise positive systems. The monotonicity of linear periodic piecewise positive systems is first studied. Then a time-varying co-positive Lyapunov function for periodic piecewise positive systems is employed and a sufficient condition for the asymptotic stability of the system is established. Based on the provided co-positive Lyapunov function and the sufficient stability condition, a state-feedback periodic piecewise controller to stabilize the system is formed and an upper bound of L_1-gain of the system is given. Finally, numerical examples are given to illustrate the theoretical results.

1. Introduction

Positive system, whose trajectory is always in the nonnegative orthant under any nonnegative initial conditions, has drawn considerable attention in recent years due to its wide application in real world physical processes involving nonnegative variables. Examples can be found in viral infection (Eisele & Siliciano, 2012), disease transmission (Ait Rami, Bokharaie, Mason, & Wirth, 2014) and networked fluid flow (Blanchini, Colaneri, & Valcher, 2015). Over the past decades, there have been many investigations on positive systems (Li, Yu, Gao, 2015; Shu, Lam, Gao, Du, & Wu, 2008). Some important properties of positive systems, like the stability and input–output gain performance, have been extensively studied. For stability problems, the system matrix and related LMI problems have been studied. In Farina and Rinaldi (2000), Frobenius eigenvalues are used to prove the asymptotic stability of positive systems and several necessary and sufficient stability conditions are presented. In Ebihara, Peaucelle, and Arzelier (2014), Ebihara, Peaucelle and Arzelier give some LMIs to verify the stability of linear continuous-time and discrete-time positive systems by a duality based argument. For time-delay positive systems, Shen and Chen point out that the stability criteria for positive systems is not affected by the augmentation of time delay (Shen & Chen, 2017). Furthermore, the L_1- and L_∞-gain for positive systems have been investigated. In Briat (2013), L_1- and L_∞- gain for positive systems are defined and the relationship between the two gains is revealed by introducing the transposed positive system. In Chen, Lam, Li, and Shu (2013), Chen et al. give a necessary and sufficient condition for the existence of a state-feedback controller to ensure the asymptotic stability of closed-loop positive systems with a prescribed L_1-induced performance. In Shen and Lam (2016), the L_1- and L_∞-optimal controller synthesis problems are solved via linear programming by adding a one-dimensional search. The L_1- and L_∞-gain problems for positive systems with time delay are also investigated in Shen and Lam (2014, 2015).

Recently, switched positive system theory has drawn considerable attention due to the fact that the model of a switched positive system can better describe many physical situations and the switched controller is more effective than time-invariant. By combining the analytical methods for switched systems (Xiang, 2015; Xiang, Xiao, & Iqbal, 2012; Zhao, Zhang, Shi, & Liu, 2012) and the property of positive systems, the stability and input–output gain performance for switched positive systems have been intensively studied (Lian & Liu, 2013; Liu, 2015; Liu & Dang, 2011; Shi, Tian, Zhao, & Zheng, 2015; Wang, Liang, & Wang, 2018; Zhang, Han, & Zhu, 2015; Zheng, Ge, & Wang, 2018; Zhu, Wang, & Zhang, 2017). In Lian and Liu (2013), Lian and Liu apply the average dwell-time approach to investigate a class of linear switched positive systems composed of both stable and unstable subsystems, and a sufficient stability condition is proposed. The switched positive systems with time delay are also taken into consideration in Liu (2015) and Liu and Dang (2011). Liu applies a comparison method to analyse the asymptotic stability of linear and nonlinear switched

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positive systems with bounded time delay and all subsystems stable under arbitrary switching signals. For the input–output gain of switched positive systems, discrete-time switched positive systems with time-varying delay and possibly unstable subsystems are investigated in Shi et al. (2015). The L_1-gain characterization is investigated by applying the continuous-time co-positive Lyapunov function. In Zhang et al. (2015), the continuous-time case is taken into consideration. By determining the average dwell-time of the positive system and constructing a co-positive Lyapunov function, a sufficient condition for bounded L_1-gain of switched positive systems is proposed.

However, most of the aforementioned research focuses on the switched positive systems under arbitrary switching protocols. When investigating the stability and input–output gain of the switched systems under arbitrary switching protocols, the conventional method is to establish a monotonically decreasing Lyapunov function over the whole time period. This method not only imposes a major restriction to the switching interval but also makes the stability condition of the switched systems conservative. As a special class of switched systems, periodic piecewise systems not only possess the property of general switched systems, but also have their own properties and applications (Li, Lam, Cheung, 2015; Li, Lam, Kwok, & Lu, 2018; Xie, Lam, & Li, 2017). In Li, Lam et al. (2015), Li, Lam and Cheung analyse the stability and L_2-gain of the linear periodic piecewise systems with possibly unstable subsystems. By employing a discontinuous Lyapunov function, the L_2-gain characterization of the linear periodic piecewise systems is obtained. In Xie et al. (2017), Xie, Lam and Li apply the time-varying Lyapunov-like function to analyse the finite-time stability, L_2-gain and H_∞ control problems. Exponential stability of continuous-time periodic piecewise systems is discussed in Li et al. (2018). A new time-varying Lyapunov function with a continuous matrix polynomial dependent on time is introduced. In this paper, motivated by the above works, a discontinuous time-varying copositive Lyapunov function is exploited. The Lyapunov function can be monotonically increasing over a time interval in some subsystems. By increasing the convergent speed of the Lyapunov function in the other subsystems and introducing discontinuity at the switching instants between each subsystems, we can compensate for the increase of the Lyapunov function over some subsystems and guarantee the Lyapunov function finally converges to zero. Based on the possibly nonmonotonicity decreasing co-positive Lyapunov function, the asymptotic stability of linear periodic piecewise positive systems is first analysed. Then a theorem for state-feedback controller synthesis problem is obtained, and a convergent algorithm for implementing the synthesis is provided. Furthermore, the L_1-gain characterization with co-positive Lyapunov function is provided.

The structure of the paper is shown as follows. In Section 2, a time-varying co-positive Lyapunov function is first constructed. Then several sufficient conditions for asymptotic stability of a periodic piecewise positive system are derived. Based on those conditions, the stabilization and L_1-gain of the system are analysed. In Section 3, numerical examples are given to illustrate the theoretical results. Finally, the conclusion is provided in Section 4.

Notation: $\mathbb{R}^n$ denotes the n-dimensional real vector space, $\mathbb{R}^{m\times n}$ denotes the set of all $m \times n$ real matrices, A^{T} denotes the transpose of matrix A, 1_n denotes an n-dimensional column vector with each entry equals to 1, $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{R}_+ = \{x \mid x > 0,\ x \in \mathbb{R}\}$, and $\mathbb{R}_{0,+} = \{x \mid x \geq 0,\ x \in \mathbb{R}\}$. $\rho(A)$ represents the spectral radius of matrix A. The product of n matrices $M_{j_1}, M_{j_2}, \ldots, M_{j_n}$ is denoted by $\prod_{j=j_1}^{j_n} M_j = M_{j_1}M_{j_2}\cdots M_{j_n}$. In addition, $\|v\|_1 = \sum_{i=1}^{n} |v_i|$ stands for the 1-norm of a vector v, $\|\omega\|_1 = \int_0^\infty \|\omega(t)\|_1\, dt$ stands for the L_1-norm of a function ω. We say $\omega \in L_1$, if $\|\omega\|_1 < \infty$.

Furthermore, some basic notations for positive systems are recalled (Berman & Plemmons, 1994). $v \succeq (\succ) 0$ or $v \in \mathbb{R}^n_{0,+}$ $\left(\mathbb{R}^n_+\right)$ means a real vector v is a nonnegative (positive) vector whose entries are all nonnegative (positive). $A \succeq (\succ) 0$ or $A \in \mathbb{R}^{m\times n}_{0,+}$ $\left(\mathbb{R}^{m\times n}_+\right)$ means a real matrix $A \in \mathbb{R}^{m\times n}$ is a nonnegative (positive) matrix. For two nonnegative (positive) matrix A and $B \in \mathbb{R}^{m\times n}_{0,+}$ $\left(\mathbb{R}^{m\times n}_+\right)$, $A \succeq (\succ) B$ means $A - B$ is a nonnegative (positive) matrix. $\mathbb{M}^{n\times n}$ denotes the set of $n\times n$ Metzler matrices whose off-diagonal entries are nonnegative.

2. Main results

Considering a linear continuous-time periodic piecewise system given as

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_\omega(t)\omega(t) + B_u(t)u(t), \\ z(t) &= C(t)x(t) + D_\omega(t)\omega(t) + D_u(t)u(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$, $\omega(t) \in \mathbb{R}^{n_\omega}$, $u(t) \in \mathbb{R}^{n_u}$, and $z(t) \in \mathbb{R}^{n_z}$ are the state, disturbance, control input and output, respectively. $A(t) = A(t + T_p)$, $B_\omega(t) = B_\omega(t + T_p)$, $B_u(t) = B_u(t + T_p)$, $C(t) = C(t + T_p)$, $D_\omega(t) = D_\omega(t + T_p)$, and $D_u(t) = D_u(t + T_p)$ for all $t \geq 0$. $T_p > 0$ is the fundamental period. Furthermore, the time-varying matrices $A(t)$, $B_\omega(t)$, $B_u(t)$, $C(t)$, $D_\omega(t)$ and $D_u(t)$ are given as follows:

$$A(t) = A_{\sigma(i)}, \quad B_\omega(t) = B_{\omega,\sigma(i)}, \quad B_u(t) = B_{u,\sigma(i)},$$

$$C(t) = C_{\sigma(i)}, \quad D_\omega(t) = D_{\omega,\sigma(i)}, \quad D_u(t) = D_{u,\sigma(i)},$$

when $t \in \left[t_{i-1,\sigma(i)-1}, t_{i,\sigma(i)}\right)$, where $i \in \{1, 2, \ldots, m\}$, $(\sigma(1), \sigma(2), \ldots, \sigma(m))$ is the cyclic permutation of $(1, 2, \ldots, m)$ with $t_{0,\sigma(1)-1} = 0$ and $t_{m,\sigma(m)} = T_p$. We also define the time interval $T_{\sigma(i)} = t_{i,\sigma(i)} - t_{i-1,\sigma(i)-1}$. When $u(t) \equiv 0$, definitions of positivity, asymptotic stability and L_1-gain performance of a periodic piecewise system are given.

Definition 1. A periodic piecewise system (1) is said to be positive if for any initial condition $x(0) \succeq 0$, disturbance $\omega(t) \succeq 0$ and cyclic permutation $(\sigma(1), \sigma(2), \ldots, \sigma(m))$, its state $x(t)$ and output $z(t)$ are in the nonnegative orthant for all $t \geq 0$.

Definition 2. A periodic piecewise system (1) with $\omega(t) \equiv 0$ is said to be asymptotically stable if system (1) is Lyapunov stable and, for any positive initial state $x(0)$, the state trajectory $x(t)$ asymptotically converges to zero.

Definition 3. A periodic piecewise system (1) is said to be asymptotically stable with an L_1-gain performance level β, if the system (1) is asymptotically stable when $\omega(t) \equiv 0$, and under zero initial condition, the following condition

$$\int_0^\infty \|z(t)\|_1\, dt \leq \beta \int_0^\infty \|\omega(t)\|_1\, dt, \tag{2}$$

holds for all $\omega \in L_1$.

Based on Definitions 1 and 2, several positivity and asymptotic stability conditions of the periodic piecewise positive system are given as follows.

Lemma 1. *System (1) is positive if and only if A_i is Metzler matrix, $B_{\omega,i}$, C_i and $D_{\omega,i}$ are nonnegative matrices for all $i \in \{1, 2, \ldots, m\}$.*

Proof. Sufficiency is proved first. When $u(t) \equiv 0$, the state $x(t)$ and output $z(t)$ satisfies

$$x(t) = \phi_{l,i}(t)x\left(lT_p + t_{i-1,\sigma(i)-1}\right) + \int_{lT_p+t_{i-1,\sigma(i)-1}}^{t} \varphi_i(t,\tau)\,\omega(\tau)d\tau, \tag{3}$$

$$z(t)=C_{\sigma(i)}x(t)+D_{\omega,\sigma(i)}\omega(t), \tag{4}$$

where $\phi_{l,i}(t) = e^{A_{\sigma(i)}(t-lT_p-t_{i-1,\sigma(i)-1})}$, $\varphi_i(t,\tau) = e^{A_{\sigma(i)}(t-\tau)}B_{\sigma(i)}$ and $t \in \left[lT_p+t_{i-1,\sigma(i)-1}, lT_p+t_{i,\sigma(i)}\right)$, for all $l \in \mathbb{N}_0$ and $i \in \{1,2,\ldots,m\}$. Since $A_{\sigma(i)}$ is Metzler, $B_{\sigma(i)}$ is nonnegative for all $i = 1,2,\ldots,m$, and the exponential of a Metzler matrix is a nonnegative matrix, $\phi_{l,i}(t)$ and $\varphi(t,\tau)$ are nonnegative matrix functions for any $l \in \mathbb{N}_0$ and $i \in \{1,2,\ldots,m\}$. When initial state $x(0) \succeq 0$ and disturbance $\omega(t) \succeq 0$, $x(t) \succeq 0$ for all $t \geq 0$. Since $x(t) \succeq 0$, $\omega(t) \succeq 0$ and matrices $C_{\sigma(i)}$, $D_{\omega,\sigma(i)}$ are nonnegative, $z(t) \succeq 0$ for all $t \geq 0$, and system (1) is positive.

The necessity is proved by contradiction. Definition 1 shows that the initial state $x(0)$ in system (1) can start from any subsystem. According to Theorem II.2 in Ngoc (2013), when any matrices A_i are not Metzler or matrices $B_{\omega,i}$, C_i or $D_{\omega,i}$ are not nonnegative, one can find a cyclic permutation $(\sigma(1),\sigma(2),\ldots,\sigma(m))$, a nonnegative initial condition $x(0)=x'$ and a nonnegative disturbance $\omega(t)$ such that the state $x(t)$ or output $z(t)$ leave the nonnegative orthant within a time interval $t \in (0,\varepsilon]$, where $\varepsilon > 0$ is an arbitrary scalar. The necessity is proved. □

In the sequel, 'system (1)' means a periodic piecewise system with the state matrix satisfying the condition in Lemma 1. Based on the stability condition for general periodic piecewise systems, a necessary and sufficient asymptotic stability condition is given.

Theorem 1. *Considering a periodic piecewise positive system (1) with $\omega(t) \equiv 0$ and $u(t) \equiv 0$. The system is asymptotically stable if and only if matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}} = e^{A_mT_m}e^{A_{m-1}T_{m-1}}\cdots e^{A_1T_1}$ is a Schur matrix.*

Proof. Since $\rho\left(\prod_{j=0}^{m-1} e^{A_{\sigma(m-j)}T_{\sigma(m-j)}}\right) = \rho\left(\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}\right)$ holds for any cyclic permutation of $(\sigma(1),\sigma(2),\ldots,\sigma(m))$, without loss of generality, we assume $\sigma(i) = i$ for all $i = 1,2,\ldots,m$. For sufficiency, when matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}$ is a Schur matrix, there exists a scalar $\lambda > 0$ satisfying

$$\rho\left(\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}\right) < e^{-\lambda T_p}. \tag{5}$$

According to condition (i) of Theorem 1 in Li, Lam et al. (2015), the system is λ-exponentially stable. Therefore, when matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}$ is a Schur matrix, system (1) is asymptotically stable.

The necessity is proved by contradiction. Assume that system (1) is asymptotically stable and matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}$ is not a Schur matrix. For the nonnegative matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}$, there exists a Perron–Frobenius eigenvector $\upsilon_{PF} \in \mathbb{R}^{n_x}_{0,+}$ satisfying $\left(\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}\right)\upsilon_{PF} = \delta\upsilon_{PF}$, where $\delta \geq 1$. Choose initial state $x(0)=\upsilon_{PF}$, we have $x(kT_p) = \left[\prod_{j=0}^{m-1}\left(e^{A_{m-j}T_{m-j}}\right)\right]^k = \delta^k\upsilon_{PF}$. When $k\to\infty$, the equation

$$\lim_{k\to\infty}\left(\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}\right)^k \upsilon_{PF} \neq 0 \tag{6}$$

holds. It contradicts the assumption that system (1) is asymptotically stable. Theorem 1 is proved. □

Remark 1. Matrix $\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}$ is a nonnegative Schur matrix. According to Liu, Yu, and Wang (2009) and Ngoc (2013), another necessary and sufficient asymptotic stability condition for system (1) is that there exists a vector $\lambda' \succ 0$ satisfying

$$\left(I-\prod_{j=0}^{m-1} e^{A_{m-j}T_{m-j}}\right)\lambda' \prec 0. \tag{7}$$

Theorem 1 presents a necessary and sufficient asymptotic stability condition for the periodic piecewise positive systems based on the Schur stability of the transition matrix. It is easy to judge the asymptotic stability of system (1) by Theorem 1. However, when introducing the periodic piecewise state-feedback controller

$$u(t)=K(t)x(t), \tag{8}$$

where $K(t)=K(t+T_p)$, and $K(t)=K_{\sigma(i)}$ when $t \in \left[t_{i-1,\sigma(i)-1}, t_{i,\sigma(i)}\right)$, the closed-loop state-feedback periodic piecewise positive system is given as

$$\begin{aligned}\dot{x}(t) &= (A(t)+B_u(t)K(t))\,x(t)+B_\omega(t)\omega(t),\\ z(t) &= (C(t)+D_u(t)K(t))\,x(t)+D_\omega(t)\omega(t),\end{aligned} \tag{9}$$

and the state transition matrix changes into

$$\prod_{j=0}^{m-1} e^{\left(A_{\sigma(m-j)}+B_{u,\sigma(m-j)}K_{\sigma(m-j)}\right)T_{\sigma(m-j)}}.$$

The control synthesis for the closed-loop system (9) is to design a set of matrices K_i such that the spectral radius of the closed-loop state transition matrix is less than 1. This problem is not a convex problem and cannot be directly solved via Theorem 1. Furthermore, the L_1-gain of system (1) is not intuitive to derive from Theorem 1. Based on the time-varying co-positive Lyapunov function, a sufficient asymptotic stability criterion is given for further investigation of stabilization and L_1-gain characterization.

Theorem 2. *Consider a periodic piecewise positive system (1) with $u(t) \equiv 0$ and $\omega(t) \equiv 0$. The system is asymptotically stable if there exist scalars $\lambda_i \in \mathbb{R}$ and $\mu_i \geq 1$ and vectors $p_{i,i-1} \succ 0$ and $p_{i,i+1} \succ 0$, $i=1,2,\ldots,m$, satisfying*

$$A_i^{\mathrm{T}}p_{i,i-1}+\frac{p_{i,i+1}-p_{i,i-1}}{T_i}+\lambda_ip_{i,i-1} \preceq 0, \tag{10}$$

$$A_i^{\mathrm{T}}p_{i,i+1}+\frac{p_{i,i+1}-p_{i,i-1}}{T_i}+\lambda_ip_{i,i+1} \preceq 0, \tag{11}$$

$$p_{i+1,i} \preceq \mu_ip_{i,i+1}, \qquad i=1,2,\ldots,m-1, \tag{12}$$

$$p_{1,0} \preceq \mu_mp_{m,m+1}, \tag{13}$$

$$\sum_{i=1}^{m}(\ln\mu_i-\lambda_iT_i) < 0. \tag{14}$$

Proof. First, a time-varying co-positive Lyapunov function candidate for the periodic piecewise positive system is employed:

$$V(t)=\sum_{i=1}^{m}\zeta_i(t)V_i(t), \tag{15}$$

where

$$\zeta_i(t)=\begin{cases}1, & t\in\left[lT_p+t_{i-1}, lT_p+t_i\right)\\ 0, & t\notin\left[lT_p+t_{i-1}, lT_p+t_i\right)\end{cases}, \quad l\in\mathbb{N}_0,$$

$V_i(t) = x^{\mathrm{T}}(t)p_i(t)$, $p_i(t) = \alpha_i(t)p_{i,i-1}+(1-\alpha_i(t))p_{i,i+1}$, $\alpha_i(t) = \frac{lT_p+t_i-t}{T_i}$. For $t \in \left[lT_p+t_{i-1}, lT_p+t_i\right)$, the derivative of the time-varying co-positive Lyapunov function is derived as

$$\begin{aligned}\dot{V}(t) =& x^{\mathrm{T}}(t)\left[\alpha_i(t)\left(A_i^{\mathrm{T}}p_{i,i-1}+\frac{p_{i,i+1}-p_{i,i-1}}{T_i}\right)\right.\\ &\left.+\ (1-\alpha_i(t))\left(A_i^{\mathrm{T}}p_{i,i+1}+\frac{p_{i,i+1}-p_{i,i-1}}{T_i}\right)\right].\end{aligned} \tag{16}$$

Combining inequality (16) with conditions (10) and (11), we have

$$\dot{V}_i(t)+\lambda_iV_i(t) \leq 0. \tag{17}$$

Inequality (17) implies that

$$V_i(t) \leq e^{-\lambda_i(t-lT_p-t_{i-1})}V_i(lT_p+t_{i-1}). \tag{18}$$

Then the inequality for time-varying co-positive Lyapunov function at switching instant $lT_p + t_i$ is taken into consideration. The time-varying co-positive Lyapunov function at time $\left(lT_p + t_i\right)^-$, where $\left(lT_p + t_i\right)^-$ is the left-hand limit at time $lT_p + t_i$, is

$$V(\left(lT_p + t_i\right)^-) = V_i(\left(lT_p + t_i\right)^-) = x^{\mathrm{T}}(\left(lT_p + t_i\right)^-)p_{i,i+1}. \tag{19}$$

The co-positive Lyapunov function at time $lT_p + t_i$ is

$$V(lT_p + t_i) = V_{i+1}(lT_p + t_i) = x^{\mathrm{T}}(lT_p + t_i)p_{i+1,i}. \tag{20}$$

Due to the facts that the state $x(t)$ is continuous, and there exists a μ_i satisfies conditions (12) and (13), inequality

$$V_{i+1}(lT_p + t_i) \le \mu_i V_i\left(\left(lT_p + t_i\right)^-\right) \tag{21}$$

holds. By combining the inequality (18) and inequality (21), the copositive Lyapunov function satisfies the inequality

$$V\left(lT_p + t_i\right) \le \mu_i e^{-\lambda_i T_i} V\left(lT_p + t_{i-1}\right), \tag{22}$$

for all $l \in \mathbb{N}_0$ and $i \in \{1, 2, \dots, m\}$. According to inequality (22), $V((l+1)T_p + t_0)$ satisfies

$$V((l+1)T_p + t_0) = \left(\prod_{j=1}^{m} \mu_j\right) e^{-\sum_{j=1}^{m} \lambda_j T_j} V(lT_p + t_0) \tag{23}$$

for all $l \in \mathbb{N}_0$. By combining inequalities (18), (22) and (23), the following inequality holds:

$$V(t) \le \left[\left(\prod_{j=1}^{m} \mu_j\right) e^{-\sum_{j=1}^{m} \lambda_j T_j}\right]^l \left(\prod_{j=1}^{i-1} \mu_j\right) e^{-\left(\sum_{j=1}^{i-1} \lambda_j T_j + \lambda_i (t - lT_p - t_{i-1})\right)} V(0), \tag{24}$$

when $t \in \left[lT_p + t_{i-1}, lT_p + t_i\right)$. Based on condition (14), when $t \to \infty$, $V(t) \to 0$ and system (1) is asymptotically stable. Theorem 2 is proved. □

In the following, stabilization problem for a periodic piecewise positive system is taken into consideration. Based on Theorem 2, by introducing a periodic piecewise feedback controller (8), a sufficient stability condition for closed-loop periodic piecewise positive system (9) is given in Theorem 3.

Theorem 3. *Consider a closed-loop state-feedback periodic piecewise system (9) with $\omega(t) \equiv 0$. The system is positive and asymptotically stable if there exist scalars $\lambda_i \in \mathbb{R}$ and $\mu_i \ge 1$, vectors $p_{i,i-1} \succ 0$ and $p_{i,i+1} \succ 0$ and matrices K_i, $i = 1, 2, \dots, m$, satisfying*

$$\left(A_i^{\mathrm{T}} + K_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}}\right) p_{i,i-1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i-1} \preceq 0, \tag{25}$$

$$\left(A_i^{\mathrm{T}} + K_i^{\mathrm{T}} B_{u,i}^{\mathrm{T}}\right) p_{i,i+1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i+1} \preceq 0, \tag{26}$$

$$p_{i+1,i} \preceq \mu_i p_{i,i+1}, \qquad i = 1, 2, \dots, m-1, \tag{27}$$

$$p_{1,0} \preceq \mu_m p_{m,m+1}, \tag{28}$$

$$\sum_{i=1}^{m} (\ln \mu_i - \lambda_i T_i) < 0, \tag{29}$$

$$A_i + B_{u,i} K_i \in \mathbb{M}^{n_x \times n_x}. \tag{30}$$

Due to the fact that the conditions in Theorem 3 are nonconvex, convex optimization approach cannot be directly employed to design the state-feedback controller (8). An iterative scheme is applied to solve the nonconvex problems. Algorithm Periodic Piecewise Positive System State-feedback Controller Design (Algorithm PPPSSCD) is designed to find K_i with feasible λ_i, μ_i, $p_{i,i+1}$ and $p_{i,i-1}$ in Theorem 3.

Algorithm PPPSSCD:

- Step 1. Set initial iteration step $k = 1$ and maximum iteration number k_{max}. Select initial scalars $\mu_{k,i} \ge 1$ and $\lambda_{k,i} \in \mathbb{R}$ for all $i = 1, 2, \dots, m$. Set $K_{k,i} = 0$ for all $i = 1, 2, \dots, m$.
- Step 2. For fixed $\mu_{k,i}$, $\lambda_{k,i}$ and $K_{k,i}$, solve the following feasibility problem for $p_{i,i-1}$ and $p_{i,i+1}$.
  FP: Find a set of $p_{i,i-1}$ and $p_{i,i+1}$, $i = 1, 2, \dots, m$, satisfying the following convex constraints:

$$\left(A_i^{\mathrm{T}} + K_{k,i}^{\mathrm{T}} B_{u,i}^{\mathrm{T}}\right) p_{i,i-1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_{k,i} p_{i,i-1} \preceq 0, \tag{31}$$

$$\left(A_i^{\mathrm{T}} + K_{k,i}^{\mathrm{T}} B_{u,i}^{\mathrm{T}}\right) p_{i,i+1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_{k,i} p_{i,i+1} \preceq 0, \tag{32}$$

$$0 \prec p_{i+1,i} \preceq \mu_{k,i} p_{i,i+1}, \qquad i = 1, 2, \dots, m-1, \tag{33}$$

$$0 \prec p_{1,0} \preceq \mu_{k,m} p_{m,m+1}. \tag{34}$$

- Step 3. For fixed $p_{i,i+1}$ and $p_{i+1,i}$, update the value of $\mu_{k+1,i}$ as follows:

$$\mu_{k+1,i} = \max\left\{1, \max_{\forall j}\left[\frac{(p_{i+1,i})_{[j]}}{(p_{i,i+1})_{[j]}}\right]\right\}, \tag{35}$$

  where $\left(p_{i+1,i}\right)_{[j]}$ and $\left(p_{i,i+1}\right)_{[j]}$ denote the jth element of vectors $p_{i+1,i}$ and $p_{i,i+1}$, respectively.
- Step 4. For fixed $p_{i,i-1}$ and $p_{i,i+1}$, update the value of $\lambda_{k+1,i}$ and $K_{k+1,i}$ by solving the following convex optimization problem.
  OP: Minimize $-\sum_{i=1}^{m} \lambda_{k+1,i} T_i$ subject to (31)–(32) and $A_i + B_{u,i} K_{k+1,i} \in \mathbb{M}^{n_x \times n_x}$.
- Step 5. If $\sum_{i=1}^{m} (\ln \mu_{k+1,i} - \lambda_{k+1,i} T_i) < 0$, then $\lambda_{k+1,i}$, $\mu_{k+1,i}$, $p_{i,i+1}$, $p_{i,i-1}$ and $K_{k+1,i}$ are the solutions, otherwise go to Step 6.
- Step 6. If $k = k_{max}$, a solution is not found; else set $k = k + 1$ and go to Step 2.

Remark 2. The Algorithm PPPSSCD is a convergent algorithm. In Step 3, for fixed $p_{i,i+1}$ and $p_{i+1,i}$, the inequality $\mu_{k+1,i} \preceq \mu_{k,i}$ holds for all $i = 1, 2, \dots, m$. In Step 4, the function $-\sum_{i=1}^{m} \lambda_{k+1,i} T_i$ is minimized with given vectors $p_{i,i+1}$ and $p_{i+1,i}$. Therefore, the value of function $\ln \mu_{k+1,i} - \lambda_{k+1,i} T_i$ is always less than the value of function $\ln \mu_{k,i} - \lambda_{k,i} T_i$ for each iteration. Finally, $\sum_{i=1}^{m} (\ln \mu_i - \lambda_i T_i)$ will decrease to a certain value.

In the following, the disturbance $\omega(t)$ in system (1) is taken into consideration. Theorem 4 is derived to characterize an upper bound of L_1-gain of the periodic piecewise positive systems.

Theorem 4. *Consider a periodic piecewise positive system (1) with $u(t) \equiv 0$. If there exist scalars $\lambda_i \in \mathbb{R}$, $\gamma > 0$ and $\mu_i \ge 1$ and vectors $p_{i,i-1} \succ 0$ and $p_{i,i+1} \succ 0$, $i = 1, 2, \dots, m$, satisfying*

$$A_i^{\mathrm{T}} p_{i,i-1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i-1} + C_i^{\mathrm{T}} 1_{n_z} \preceq 0, \tag{36}$$

$$A_i^{\mathrm{T}} p_{i,i+1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i+1} + C_i^{\mathrm{T}} 1_{n_z} \preceq 0, \tag{37}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i,i-1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega} \preceq 0, \tag{38}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i,i+1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega} \preceq 0, \tag{39}$$

$$p_{i+1,i} \preceq \mu_i p_{i,i+1}, \qquad i = 1, 2, \dots, m-1, \tag{40}$$

$$p_{1,0} \preceq \mu_m p_{m,m+1}, \tag{41}$$

$$\sum_{i=1}^{m} (\ln \mu_i - \lambda_i T_i) < 0, \tag{42}$$

the system is asymptotically stable and, under zero initial condition, an upper bound of the L_1-gain is

$$\frac{\gamma\mu_{max}\lambda_{max}\Lambda^2 e^{\sum_{i=1}^{m^+}\lambda_i^+ T_i^+}}{\lambda^*}, \tag{43}$$

where $\lambda_{max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, $\lambda_{min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, $T_{max} = \max\{T_1, T_2, \ldots, T_m\}$, $\lambda^* = \frac{\sum_{i=1}^m(\lambda_i T_i - \ln\mu_i)}{T_p}$, $\Lambda = \max\left\{e^{(-\lambda_{min}+\lambda^*)T_{max}}, 1\right\}$, $\mu_{max} = \max\{\mu_1, \mu_2, \ldots, \mu_m\}$, λ_i^+, T_i^+ *and* m^+ *denote the* $\lambda_{i'}$ *such that* $\lambda_{i'} > 0$, *the time interval* $T_{i'}$ *and the total number of* i', *respectively.*

Proof. According to inequalities (36) and (37) in Theorem 4, the following two inequalities hold:

$$A_i^{\mathrm{T}} p_{i,i-1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i-1} \preceq 0, \tag{44}$$

$$A_i^{\mathrm{T}} p_{i,i+1} + \frac{p_{i,i+1} - p_{i,i-1}}{T_i} + \lambda_i p_{i,i+1} \preceq 0. \tag{45}$$

Combining inequalities (44), (45) with inequalities (40)–(42) in Theorem 4, the asymptotic stability of periodic piecewise positive system is guaranteed according to Theorem 2. By employing the time-varying co-positive Lyapunov function candidate (15), when $t \in \left[lT_p + t_{i-1}, lT_p + t_i\right)$, the derivative of the co-positive Lyapunov function can be written as

$$\begin{aligned}\dot{V}_i(t) \le{}& \alpha_i(t)\left(\gamma\omega^{\mathrm{T}}(t)1_{n_\omega} - \omega^{\mathrm{T}}(t)D_{\omega,i}^{\mathrm{T}}1_{n_z} - \lambda_i x^{\mathrm{T}}(t)p_{i,i-1}\right.\\ &\left.-x^{\mathrm{T}}(t)C_i^{\mathrm{T}}1_{n_z}\right) + (1-\alpha_i(t))\left(\gamma\omega^{\mathrm{T}}(t)1_{n_\omega}\right.\\ &\left.-\omega^{\mathrm{T}}(t)D_{\omega,i}^{\mathrm{T}}1_{n_z} - \lambda_i x^{\mathrm{T}}(t)p_{i,i+1} - x^{\mathrm{T}}(t)C_i^{\mathrm{T}}1_{n_z}\right)\\ ={}& -\lambda_i V_i(t) - F(t),\end{aligned} \tag{46}$$

where $F(t) = z^{\mathrm{T}}(t)1_{n_z} - \gamma\omega^{\mathrm{T}}(t)1_{n_\omega}$. According to the inequality of derivative of $V(t)$ in (46), we have

$$\begin{aligned}V_i(t) \le{}& e^{-\lambda_i(t-lT_p-t_{i-1})}V_i(lT_p + t_{i-1})\\ &-\int_{lT_p+t_{i-1}}^{t} e^{-\lambda_i(t-\tau)}F(\tau)d\tau,\end{aligned} \tag{47}$$

when $t \in \left[lT_p + t_{i-1}, lT_p + t_i\right)$. When $t = \left(lT_p + t_i\right)^-$, inequality (47) indicates that

$$\begin{aligned}V_i\left(\left(lT_p + t_i\right)^-\right) \le{}& e^{-\lambda_i T_i}V_i(lT_p + t_{i-1})\\ &-\int_{lT_p+t_{i-1}}^{lT_p+t_i} e^{-\lambda_i(lT_p+t_i-\tau)}F(\tau)d\tau.\end{aligned} \tag{48}$$

Combining inequality (48) with (21), $V(lT_p + t_i)$ satisfies the following inequality:

$$\begin{aligned}V(lT_p + t_i) &= V_{i+1}(lT_p + t_i)\\ &\le \mu_i V_i(lT_p + t_i^-)\\ &\le \mu_i e^{-\lambda_i T_i}V_i(lT_p + t_{i-1})\\ &\quad - \mu_i\int_{lT_p+t_{i-1}}^{lT_p+t_i} e^{-\lambda_i(lT_p+t_i-\tau)}F(\tau)d\tau.\end{aligned} \tag{49}$$

Inequality (49) gives the relation of the co-positive Lyapunov function at time $t = lT_p + t_i$ and $t = lT_p + t_{i-1}$ for all $l \in \mathbb{N}_0$ and $i \in \{1, 2, \ldots, m\}$. Based on inequality (49), the relation of the co-positive Lyapunov function at time $t = lT_p + t_i$ and $t = 0$ can be derived as follows:

$$\begin{aligned}V(lT_p + t_i) \le{}& \left(\prod_{k=1}^{lm+i}\mu_k e^{-\lambda_k T_k}\right)V(0)\\ &-\sum_{j=1}^{lm+i-1}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\right.\\ &\left.\times\int_{t_{j-1}}^{t_j} e^{-\lambda_j(t_j-\tau)}F(\tau)d\tau\right]\\ &-\mu_{lm+i}\int_{lT_p+t_{i-1}}^{lT_p+t_i} e^{-\lambda_i(lT_p+t_i-\tau)}F(\tau)d\tau,\end{aligned} \tag{50}$$

where $T_j = t_j - t_{j-1}$, $T_j = T_{j-m}$, $\lambda_j = \lambda_{j-m}$ and $\mu_j = \mu_{j-m}$. Combining inequality (50) with inequality (47), the inequality for the Lyapunov function at time t, where $t \in \left[lT_p + t_{i-1}, lT_p + t_i\right)$, is given as follows:

$$\begin{aligned}V(t) \le{}& e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\left(\prod_{k=1}^{lm+i-1}\mu_k e^{-\lambda_k T_k}\right)V(0)\right.\\ &-\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\int_{t_{j-1}}^{t_j} e^{-\lambda_j(t_j-\tau)}F(\tau)d\tau\right]\\ &\left.-\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}} e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}F(\tau)d\tau\right\}\\ &-\int_{lT_p+t_{i-1}}^{t} e^{-\lambda_i(t-\tau)}F(\tau)d\tau.\end{aligned} \tag{51}$$

Since $V(t) \ge 0$ for all $t \in \mathbb{R}_{0,+}$ and $V(0) = 0$, use $z^{\mathrm{T}}(t)1_{n_z} - \gamma\omega^{\mathrm{T}}(t)1_{n_\omega}$ to replace $F(t)$ in inequality (51), inequality (51) can be rewritten as inequality (52), given in Box I. Since $\lambda_{max} \ge \lambda_i$ and $\mu_i \ge 1$ holds for all $i \in \{1, 2, \ldots, m\}$ and inequality (42) holds, $\lambda_{max} > 0$ always holds. The left-hand side of inequality (52) satisfies the following inequality:

$$\begin{aligned}&e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\right.\right.\\ &\quad\left.\times\int_{t_{j-1}}^{t_j} e^{-\lambda_j(t_j-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\right]\\ &\quad\left.+\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}} e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\right\}\\ &\quad+\int_{lT_p+t_{i-1}}^{t} e^{-\lambda_i(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\\ &\ge \sum_{j=1}^{lm+i-1}\left(\int_{t_{j-1}}^{t_j} e^{-\lambda_{max}(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\right)\\ &\quad+\int_{lT_p+t_{i-1}}^{t} e^{-\lambda_{max}(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\\ &=\int_0^t e^{-\lambda_{max}(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau.\end{aligned} \tag{53}$$

Then the right-hand side of inequality (52) is taken into consideration. Since $\lambda^* T_p = \sum_{i=1}^m(\lambda_i T_i - \ln\mu_i) > 0$ and $\mu_i \ge 1$ for all $i = 1, 2, \ldots, m$, the following inequality holds:

$$\begin{aligned}\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right) &= \frac{\left(\prod_{k_j=j+1}^{\left\lceil\frac{lm+i-j-1}{m}\right\rceil m+j}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)}{\left(\prod_{k_j=lm+i}^{\left\lceil\frac{lm+i-j-1}{m}\right\rceil m+j}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)}\\ &\le \frac{e^{-\left\lceil\frac{lm+i-j-1}{m}\right\rceil\lambda^* T_p}}{e^{-\sum_{k_j=lm+i}^{\left\lceil\frac{lm+i-j-1}{m}\right\rceil m+j}\lambda_{k_j}T_{k_j}}}.\end{aligned} \tag{54}$$

$$
\begin{aligned}
& e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\int_{t_{j-1}}^{t_j}e^{-\lambda_j(t_j-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\right]\right. \\
& \left.\quad+\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}}e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau\right\}+\int_{lT_p+t_{i-1}}^{t}e^{-\lambda_i(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau \\
& \le \gamma e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\int_{t_{j-1}}^{t_j}e^{-\lambda_j(t_j-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right]\right. \\
& \left.\quad+\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}}e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right\}+\gamma\int_{lT_p+t_{i-1}}^{t}e^{-\lambda_i(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau \qquad (52)
\end{aligned}
$$

Box I.

Let λ_i^+, T_i^+ and m^+ denote the $\lambda_{i'}$ such that $\lambda_{i'} > 0$, the time interval $T_{i'}$ and the total number of i', respectively, the inequality

$$\sum_{k_j=lm+i}^{\left\lceil\frac{lm+i-j-1}{m}\right\rceil m+j}\lambda_{k_j}T_{k_j}\le\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+ \qquad (55)$$

holds. Then the left-hand side of inequality (54) satisfies the following inequality,

$$\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\le e^{-\left\lceil\frac{lm+i-j-1}{m}\right\rceil\lambda^*T_p}e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}. \qquad (56)$$

Since $\lambda^* > 0$ and $\left\lceil\frac{lm+i-j-1}{m}\right\rceil T_p \ge \sum_{k_j=j+1}^{lm+i-1}T_{k_j}$, inequality (56) can be written as

$$\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\le e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}e^{-\lambda^*\sum_{k_j=j+1}^{lm+i-1}T_{k_j}}. \qquad (57)$$

According to inequality (57), the right-hand side of the inequality (52) satisfies the following inequality:

$$
\begin{aligned}
& \gamma e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\right.\right. \\
& \quad\left.\times\int_{t_{j-1}}^{t_j}e^{-\lambda_j(t_j-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right] \\
& \quad\left.+\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}}e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right\} \\
& \quad+\gamma\int_{lT_p+t_{i-1}}^{t}e^{-\lambda_i(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau \\
& \le\gamma\mu_{max}e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}\left\{e^{-\lambda_i(t-lT_p-t_{i-1})}\right. \\
& \quad\times\left[\sum_{j=1}^{lm+i-2}\left(\prod_{k_j=j+1}^{lm+i-1}e^{-\lambda^*T_{k_j}}\right)\int_{t_{j-1}}^{t_j}e^{-\lambda_j(t_j-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right. \\
& \quad\left.+\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}}e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right] \\
& \quad\left.+\int_{lT_p+t_{i-1}}^{t}e^{-\lambda_i(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right\}. \qquad (58)
\end{aligned}
$$

Since $\varLambda = \max\left\{e^{(-\lambda_{min}+\lambda^*)T_{max}}, 1\right\}$, $\varLambda^2 \ge \varLambda$, and the following inequality holds:

$$e^{-\lambda_j(t_j-\tau)}\le e^{-(\lambda_{min}-\lambda^*+\lambda^*)(t_j-\tau)}\le\varLambda e^{-\lambda^*(t_j-\tau)}, \qquad (59)$$

for all $\tau \le t_j$ and $j = 1, 2, \ldots, m$. Combining inequality (59) with inequality (58), the right-hand side of the inequality (52) satisfies the following inequality:

$$
\begin{aligned}
& \gamma e^{-\lambda_i(t-lT_p-t_{i-1})}\left\{\sum_{j=1}^{lm+i-2}\left[\mu_j\left(\prod_{k_j=j+1}^{lm+i-1}\mu_{k_j}e^{-\lambda_{k_j}T_{k_j}}\right)\right.\right. \\
& \quad\left.\times\int_{t_{j-1}}^{t_j}e^{-\lambda_j(t_j-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right] \\
& \quad\left.+\mu_{i-1}\int_{lT_p+t_{i-2}}^{lT_p+t_{i-1}}e^{-\lambda_{i-1}(lT_p+t_{i-1}-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau\right\} \\
& \quad+\gamma\int_{lT_p+t_{i-1}}^{t}e^{-\lambda_i(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau \\
& \le\gamma\mu_{max}\varLambda^2e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}\int_0^t e^{-\lambda^*(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau. \qquad (60)
\end{aligned}
$$

By putting inequalities (53) and (60) into (52), the following inequality holds:

$$
\begin{aligned}
& \int_0^t e^{-\lambda_{max}(t-\tau)}z^{\mathrm{T}}(\tau)1_{n_z}d\tau \\
& \le\gamma\mu_{max}\varLambda^2e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}\int_0^t e^{-\lambda^*(t-\tau)}\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau. \qquad (61)
\end{aligned}
$$

By integrating inequalities (61) from $t = 0$ to ∞, we have

$$
\begin{aligned}
& \frac{1}{\lambda_{max}}\int_0^\infty z^{\mathrm{T}}(\tau)1_{n_z}d\tau \\
& \le\frac{\gamma\mu_{max}\varLambda^2e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}}{\lambda^*}\int_0^\infty\omega^{\mathrm{T}}(\tau)1_{n_\omega}d\tau. \qquad (62)
\end{aligned}
$$

Finally, an upper bound of the L_1-gain of the periodic piecewise positive system is derived, and the form of the bound is

$$\frac{\gamma\mu_{max}\lambda_{max}\varLambda^2e^{\sum_{k_+=1}^{m^+}\lambda_{k_+}^+T_{k_+}^+}}{\lambda^*}. \quad\Box \qquad (63)$$

The above L_1-gain criterion is proposed based on a discontinuous time-varying co-positive Lyapunov function. With different choices of $p(t)$ in the co-positive Lyapunov function, different corollaries are proposed.

Case I: Continuous time-varying $p(t)$. Consider a continuous time Lyapunov function $V(t) = x^{\mathrm{T}}(t)p(t)$ with continuous function $p(t)$ as

$$p(t) = p_{i-1} + (t - lT_p - t_{i-1})\frac{p_i - p_{i-1}}{T_i}, \quad t \in \left[lT_p + t_{i-1}, lT_p + t_i\right),$$

where $l \in \mathbb{N}_0$, $p_i \in \mathbb{R}_+^{n_x}$ for all $i \in \{1, 2, \ldots, m\}$, and $p_m = p_0$. By letting $p_{i,i-1} = p_{i-1}$ and $p_{i,i+1} = p_i$ in Theorem 4, the corollary is obtained as follows.

Corollary 1. *Consider a periodic piecewise positive system (1) with $u(t) \equiv 0$. If there exist scalars $\lambda_i \in \mathbb{R}$, $\gamma > 0$ and vectors $p_i \succ 0$, $i = 1, 2, \ldots, m$, satisfying*

$$A_i^{\mathrm{T}} p_{i-1} + \frac{p_i - p_{i-1}}{T_i} + \lambda_i p_{i-1} + C_i^{\mathrm{T}} 1_{n_z} \preceq 0, \tag{64}$$

$$A_i^{\mathrm{T}} p_i + \frac{p_i - p_{i-1}}{T_i} + \lambda_i p_i + C_i^{\mathrm{T}} 1_{n_z} \preceq 0, \tag{65}$$

$$B_{\omega,i}^{\mathrm{T}} p_{i-1} + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega} \preceq 0, \tag{66}$$

$$B_{\omega,i}^{\mathrm{T}} p_i + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega} \preceq 0, \tag{67}$$

$$p_m = p_0, \tag{68}$$

$$\sum_{i=1}^{m} \lambda_i T_i > 0, \tag{69}$$

the system is asymptotically stable and an upper bound of the L_1-gain is

$$\frac{\gamma \lambda_{max} \Lambda^2 e^{\sum_{i=1}^{m^+} \lambda_i^+ T_i^+}}{\lambda^*}, \tag{70}$$

where $\lambda_{max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, $\lambda_{min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, $T_{max} = \max\{T_1, T_2, \ldots, T_m\}$, $\lambda^* = \frac{\sum_{i=1}^m \lambda_i T_i}{T_p}$, $\Lambda = e^{(-\lambda_{min} + \lambda^*)T_{max}}$, λ_i^+, T_i^+ and m^+ denote the $\lambda_{i'}$ such that $\lambda_{i'} > 0$, the time interval $T_{i'}$ and the total number of i', respectively.

Case II: Piecewise constant $p(t)$. Consider a Lyapunov function $V(t) = x^{\mathrm{T}}(t)p(t)$ with piecewise constant function $p(t)$ as

$$p(t) = p_i, \quad t \in \left[lT_p + t_{i-1}, lT_p + t_i\right),$$

where $l \in \mathbb{N}_0$, $p_i \in \mathbb{R}_+^{n_x}$ for all $i \in \{1, 2, \ldots, m\}$, and $p_m = p_0$. By letting $p_{i,i-1} = p_i$ and $p_{i,i+1} = p_i$ in Theorem 4, the corollary is obtained as follows.

Corollary 2. *Consider a periodic piecewise positive system (1) with $u(t) \equiv 0$. If there exist scalars $\lambda_i \in \mathbb{R}$, $\gamma > 0$, $\mu_i \geq 1$ and vectors $p_i \succ 0$, $i = 1, 2, \ldots, m$, satisfying*

$$A_i^{\mathrm{T}} p_i + \lambda_i p_i + C_i^{\mathrm{T}} 1_{n_z} \preceq 0, \tag{71}$$

$$B_{\omega,i}^{\mathrm{T}} p_i + D_{\omega,i}^{\mathrm{T}} 1_{n_z} - \gamma 1_{n_\omega} \preceq 0, \tag{72}$$

$$p_{i+1} \preceq \mu_i p_i, \qquad i = 1, 2, \ldots, m-1, \tag{73}$$

$$p_1 \preceq \mu_m p_m, \tag{74}$$

$$\sum_{i=1}^{m} (\ln \mu_i - \lambda_i T_i) < 0, \tag{75}$$

the system is asymptotically stable and an upper bound of the L_1-gain is (43).

3. Illustrative examples

A periodic piecewise positive system is taken into consideration as follows:

$$\dot{x}(t) = A(t)x(t) + B_u(t)u(t), \tag{76}$$

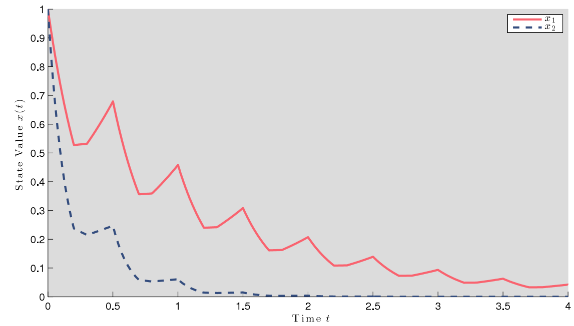

Fig. 1. Evolution of each state in periodic piecewise positive systems.

where $\sigma(i) = i$, $m = 3$, $T_1 = 0.2$, $T_2 = 0.1$, $T_3 = 0.2$ and

$$A_1 = \begin{bmatrix} -3 & 1.1 \\ 1.2 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix},$$

$$B_{u,1} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad B_{u,2} = \begin{bmatrix} 1.2 \\ 1.3 \end{bmatrix}, \quad B_{u,3} = \begin{bmatrix} 1.1 \\ 0.7 \end{bmatrix}.$$

The eigenvalues of $e^{A_3T_3}e^{A_2T_2}e^{A_1T_1}$ are 0.557 and 1.204, thus the system is unstable according to Theorem 1. Then a state-feedback controller $u(t) = K(t)x(t)$ is introduced. According to Algorithm PPPSSCD, we select the initial scalars $\mu_{1,1} = 5$, $\mu_{1,2} = 5$, $\mu_{1,3} = 5$, $\lambda_{1,1} = -6$, $\lambda_{1,2} = -6$, and $\lambda_{1,3} = -6$. The scalars are chosen to guarantee the feasibility of the FP in Step 2. By applying Algorithm PPPSSCD, the value of $\sum_{i=1}^m \left(\ln \mu_{k,i} - \lambda_{k,i}T_i\right)$ decreases with iteration. When $k = 4$, $\sum_{i=1}^m \left(\ln \mu_{k,i} - \lambda_{k,i}T_i\right) = -0.0967 < 0$, and the closed-loop system is asymptotically stable. The state-feedback controller $K_{k,i}$ converges to K_i, which are given as follows:

$$K_1 = \begin{bmatrix} -1.2 & -5.1936 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.7692 & 0.0127 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.7142 & -0.4353 \end{bmatrix}.$$

The eigenvalues of state transition matrix $e^{(A_3+B_{u,3}K_3)T_3}\, e^{(A_2+B_{u,2}K_2)T_2}\, e^{(A_1+B_{u,1}K_1)T_1}$ of the closed-loop system (76) are 0.672 and 0.2471. When initial state $x(0) = [1\ 1]^{\mathrm{T}}$, the evolution of the state components in the closed-loop periodic piecewise positive system is shown in Fig. 1. One can find that the closed-loop periodic piecewise positive system is asymptotically stable.

In what follows, an example of calculating an upper bound of L_1-gain of a periodic piecewise positive system is taken into consideration. An asymptotically stable periodic piecewise positive system is given as follows:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B_\omega(t)\omega(t), \\ z(t) &= C(t)x(t) + D_\omega(t)\omega(t), \end{aligned} \tag{77}$$

where $\sigma(i), m, T_1, T_2, T_3$ are the same as the ones in system (76). A_i in system (77) is equal to $A_i + B_{u,i}K_i$ in system (76) for all $i \in \{1, 2, 3\}$. Furthermore, the other matrices are given as follows:

$$B_{\omega,1} = \begin{bmatrix} 0.8 \\ 1.1 \end{bmatrix}, \quad B_{\omega,2} = \begin{bmatrix} 1.2 \\ 2.2 \end{bmatrix}, \quad B_{\omega,3} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.2 & 0 \\ 0.3 & 0.1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$

$$D_{\omega,1} = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \quad D_{\omega,2} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad D_{\omega,3} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix}.$$

There exist parameters satisfying conditions (36)–(42) in Theorem 4 and are given as follows:

$$p_{1,0} = \begin{bmatrix} 0.8009 \\ 0.0948 \end{bmatrix}, \quad p_{1,2} = \begin{bmatrix} 1.2198 \\ 0.2013 \end{bmatrix}, \quad p_{2,1} = \begin{bmatrix} 1.2197 \\ 0.2012 \end{bmatrix},$$

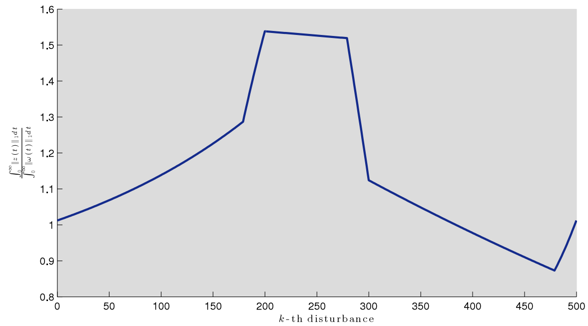

Fig. 2. $\frac{\int_0^\infty \|z(t)\|_1 dt}{\int_0^\infty \|\omega(t)\|_1 dt}$ of system (77) with disturbances (78).

$$p_{2,3} = \begin{bmatrix} 0.1902 \\ 0.2091 \end{bmatrix}, \quad p_{3,2} = \begin{bmatrix} 1.1901 \\ 0.2090 \end{bmatrix}, \quad p_{3,4} = \begin{bmatrix} 0.8010 \\ 0.0949 \end{bmatrix},$$

$\mu_1 = \mu_2 = \mu_3 = 1$, $\lambda_1 = 2.199 \times 10^{-4}$, $\lambda_2 = 2.193 \times 10^{-4}$, $\lambda_3 = 2.2 \times 10^{-4}$, $\gamma = 2.30626$. According to Theorem 4, an upper bound of the L_1-gain of the linear periodic piecewise positive system is 2.3082. In order to test the effectiveness of the theorem, different disturbances are given as follows:

$$\omega(t) = \omega_k(t) = \begin{cases} 1, & t \in [0.001k, 0.001k + 0.02] \\ 0, & t \notin [0.001k, 0.001k + 0.02] \end{cases}, \tag{78}$$

where $k \in \mathbb{N}_0 \cap [0, 500]$. The ratio $\frac{\int_0^\infty \|z(t)\|_1 dt}{\int_0^\infty \|\omega_k(t)\|_1 dt}$ of the system with different disturbances $\omega_k(t)$ is shown in Fig. 2. It shows that the supremum of ratio $\frac{\int_0^\infty \|z(t)\|_1 dt}{\int_0^\infty \|\omega_k(t)\|_1 dt}$ is less than the calculated upper bound from Theorem 4.

4. Conclusion

In this paper, the stability, stabilization and L_1-gain analysis problems of linear periodic piecewise positive systems are studied. First, a sufficient asymptotic stability condition for linear periodic piecewise positive systems is proposed based on the time-varying co-positive Lyapunov function. Then based on the sufficient condition, a state-feedback periodic piecewise controller to stabilize the system is proposed and an algorithm to design the controller is given. Furthermore, by using the obtained asymptotic stability condition, a sufficient condition to characterize an upper bound of L_1-gain of linear periodic piecewise positive systems is obtained. Finally, some numerical examples are given to illustrate the theoretical results.

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