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Existence and uniqueness of $C^{(n)}$-almost periodic solutions to some ordinary differential equations

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Abstract

In this paper we prove the existence and uniqueness of $C^{(n)}$-almost periodic solutions to the ordinary differential equation $x'(t) = A(t)x(t) + f(t),\quad t \in \mathbb{R}$, where the matrix $A(t) : \mathbb{R} \to \mathcal{M}_k(\mathbb{C})$ is τ-periodic and $f : \mathbb{R} \to \mathbb{C}^k$ is $C^{(n)}$-almost periodic. We also prove the existence and uniqueness of an ultra-weak $C^{(n)}$-almost periodic solution in the case when $A(t) = A$ is independent of t. Finally we prove also the existence and uniqueness of a mild $C^{(n)}$-almost periodic solution of the semilinear hyperbolic equation $x'(t) = Ax(t) + f(t, x)$ considered in a Banach space, assuming $f(t, x)$ is $C^{(n)}$-almost periodic in t for each $x \in X$, satisfies a global Lipschitz condition and takes values in an extrapolation space $F_{A_{-1}}$ associated to A.

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1. Introduction

Harald Bohr's interest in which functions could be represented by a Dirichlet series, i.e. of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$, where $a_n, z \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ is a monotone increasing

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sequence of real numbers (series which play an important role in complex analysis and analytic number theory), led him to devise a theory of almost periodic real (and complex) functions, founding this theory between the years 1923 and 1926. Several generalizations and classes of almost periodic functions have been introduced in the literature, including pseudo-almost periodic functions, almost automorphic functions [13,14], p-almost automorphic functions, etc.

The concept of $C^{(n)}$-almost periodic functions has been introduced in [1] and [2] for functions $\mathbb{R} \to \mathbb{R}$. These are functions which are almost periodic up to their nth derivatives. In [6], Bugajewski and N'Guérékata have extended the study to functions $\mathbb{R} \to X$, where X is a Banach space. They also introduced the concept of $C^{(n)}$-asymptotically almost periodic functions and discussed some applications to ordinary and partial differential equations, as well as some results in [5].

In the present paper we would like to pursue this study. The work is organized as follows. In Section 2, we recall some definitions and preliminary facts about extrapolation spaces. In Section 3, we present the so-called notion of $C^{(n)}$-almost periodic functions and elementary properties of such functions in Banach spaces. We also prove an equivalent of Kadets' theorem for $C^{(n)}$-almost periodic functions at the end of this section. In Section 4, we use the method of reduction (see [10,13]) to prove a Massera type theorem for $C^{(n)}$-almost periodic functions in finite dimensional spaces. Then we prove the existence and uniqueness of $C^{(n)}$-almost periodic ultra-weak solutions (in Lions' sense) of linear inhomogeneous differential equations in a separable Hilbert space. Finally we establish the existence and uniqueness of $C^{(n)}$-almost periodic mild solutions of semilinear hyperbolic differential equations in a Banach space. The main result of the paper is Theorem 4.7. Throughout this work X will denote a complex Banach space with norm $\|\cdot\|$.

2. Preliminaries

We begin in this section, by fixing some notations and recalling a few basic results on extrapolation spaces of generators. For more details, we refer the reader to [8] and [12]. Let $(A, D(A))$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X.

Define on X a new norm by

$$\|x\|_{-1} = \left\|(\lambda - A)^{-1}x\right\|, \quad x \in X,\ \lambda \in \rho(A).$$

The completion of $(X, \|\cdot\|_{-1})$ is called the *extrapolation space* of X associated to A and will be denoted by X_{-1}. By the resolvent equation, the space X_{-1} does not depend on λ.

Since $T(t)$ commutes with the operator resolvent $R(\lambda, A) := (\lambda I - A)^{-1}$, the extension of $T(t)$ to X_{-1} exists and defines a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ which is generated by A_{-1} with $D(A_{-1}) = X$.

We recall that the Favard class associated to a generator A (or $T(\cdot)$) is the Banach space

$$F_A := \left\{x \in X : \sup_{t>0} \frac{1}{t} \left\|e^{-\omega t} T(t)x - x\right\| < \infty\right\}$$

endowed with the norm

$$\|x\|_{F_A} := \sup_{t>0} \frac{1}{t} \left\|e^{-\omega t} T(t)x - x\right\|.$$

Here $\omega > \omega_0(T(\cdot))$, the growth bound of $T(\cdot)$. We note that F_A is independent of the choice of ω, contains the domain of A, $F_A \hookrightarrow X \hookrightarrow F_{A_{-1}} \hookrightarrow X_{-1}$, and

$$(\lambda - A_{-1}) : F_A \longrightarrow F_{A_{-1}} \tag{1}$$

is an isomorphism for every $\lambda \in \rho(A)$. In the case when X is a reflexive Banach space, the Favard class associated to $T(\cdot)$ is exactly the domain of its generator (see, e.g., [8, Section. II.5.b] for more properties).

A C_0-semigroup $(T(t))_{t\geq 0}$ is said to be hyperbolic if it satisfies the following properties:

(i) there exist two subspaces X^S (the stable space) and X^U (the unstable space) of X such that $X = X^S \oplus X^U$;

(ii) $T(t)$ is defined on X^U, $T(t)X^U \subset X^U$, and $T(t)X^S \subset X^S$ for all $t \geq 0$;

(iii) there exist constants $M,\ \delta > 0$ such that

$$\|T(t)P_S\| \leq M\mathrm{e}^{-\delta t}, \quad t \geq 0, \qquad \|T(t)P_U\| \leq M\mathrm{e}^{\delta t}, \quad t \leq 0, \tag{2}$$

where P_S and P_U are, respectively, the projections onto X^S and X^U.

In the following we need the following main proposition.

Proposition 2.1 (*[3]*). *Assume that the semigroup $T(\cdot)$ is hyperbolic. Then,*

(i) *$P_S T(t) = T(t)P_S$ and $P_U T(t) = T(t)P_U$, for all $t \geq 0$.*

(ii) *$(T(t))_{t\geq 0}$ is a stable C_0-semigroup on $X^S = P_S X$ with generator $P_S A$, and $(T(t))_{t\in\mathbb{R}}$ is a C_0-group on $X^U = P_U X$ with generator $P_U A$.*

(iii) *P_S and P_U can be extended on X_{-1} to two unique bounded operators $P_{S,-1}$ and $P_{U,-1}$.*

We need also the following fundamental lemma, see [3].

Theorem 2.2. *Let $f : \mathbb{R} \longrightarrow F_{A_{-1}}$ be a bounded function. Then, the following assertions hold.*

$$\int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s)\mathrm{d}s,\ \int_{t}^{\infty} T_{-1}(t-s)P_{U,-1}f(s)\mathrm{d}s \in X \quad \text{for all } t \in \mathbb{R},$$
$$\left\| \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s)\mathrm{d}s \right\| \leq C\mathrm{e}^{-\delta t} \int_{-\infty}^{t} \mathrm{e}^{\delta s}\, \|f(s)\|_{F_{A_{-1}}}\, \mathrm{d}s,$$
$$\left\| \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}f(s)\mathrm{d}s \right\| \leq C\mathrm{e}^{\delta t} \int_{t}^{+\infty} \mathrm{e}^{-\delta s}\, \|f(s)\|_{F_{A_{-1}}}\, \mathrm{d}s \quad \text{for all } t \in \mathbb{R}.$$

3. $C^{(n)}$-almost periodic functions

Let $X = (X, \|\bullet\|)$ be a (complex) Banach space and $f_\tau(x) := f(x+\tau)$, where $f : \mathbb{R} \to X$, and $x,\ \tau \in \mathbb{R}$.

Denote by $C^{(n)}(\mathbb{R}, X)$ (briefly $C^{(n)}(X)$) the space of all functions $\mathbb{R} \to X$ which have a continuous nth derivative on $\mathbb{R}$. Let $C_b^{(n)}(\mathbb{R}, X)$ (briefly $C_b^{(n)}(X)$) be the subspace of $C^{(n)}(\mathbb{R}, X)$ consisting of such functions satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|f^{(i)}(t)\| < \infty$$

where $f^{(i)}$ denote the ith derivative of f and $f^{(0)} := f$. Clearly $C_b^{(n)}(X)$ turns out to be a Banach space with the norm

$$\|f\|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{n} \|f^{(i)}(t)\|.$$

Definition 3.1. Let $\epsilon > 0$. A number $\tau \in \mathbb{R}$ is said to be a $(\| \bullet \|_n, \epsilon)$-almost period of a function $f \in C^{(n)}(X)$, if $\|f_\tau - f\|_n < \epsilon$.

The set of all $(\| \bullet \|_n, \epsilon)$-almost periods of a function f will be denoted by $E^{(n)}(\epsilon, f)$.

Definition 3.2. A function $f \in C^{(n)}(X)$ is said to be $C^{(n)}$-almost periodic (briefly $C^{(n)}$-a.p.) if for every $\epsilon > 0$, the set $E^{(n)}(\epsilon, f)$ is relatively dense in $\mathbb{R}$. The set of all $C^{(n)}$-a.p functions $f : \mathbb{R} \to X$ will be denoted by $AP^{(n)}(\mathbb{R}, X)$ (or briefly $AP^{(n)}(X)$). $AP^{(0)}(X) = AP(X)$, the classical Banach space of all almost periodic functions in Bohr's sense.

Equipped with the $\| \bullet \|_n$ norm above, $AP^{(n)}(X)$ turns out to be a Banach space (cf. [6, Corollary 2.12]).

Example 3.3. Let $g(t) = \cos(\alpha t) + \cos(\beta t),\ t \in \mathbb{R}$ where α and β are incommensurate real numbers. Then the function $f(t) = \mathrm{e}^{g(t)}$ is $C^{(n)}$-almost periodic for any $n = 1, 2, \ldots$. The proof is straightforward from [6, Theorem 4.3].

We recall that $AP^{(n+1)}(X) \subset AP^{(n)}(X) \subset C_b^{(n)}(X)$, for all $n = 0, 1, 2, \ldots$. All the inclusions are strict (cf. [6, Example 4.5]).

One can find more examples of $C^{(n)}$-almost periodic functions in [1] and [6].

The uniform limit of $C^{(n)}$-almost periodic functions in $AP^n(X)$ is also in $AP^n(X)$ (see [6, Theorem 2.11]).

We also have the following (cf. [6, Theorem 3.4]).

Theorem 3.4. *Let $F(t) := \int_0^t f(s)\mathrm{d}s$ where $f \in AP^{(n)}(X)$, $t \in \mathbb{R}$. Then $F \in AP^{(n+1)}(X)$ if $\mathcal{R}_F$, the range of F, is relatively compact in X.*

Let us recall that for $f \in AP(X)$ where X is a uniformly convex Banach space, the primitive $F(t) = \int_0^t f(s)\mathrm{d}s$ is a.p. iff $\mathcal{R}_F$ is bounded in X. This is known as the Bohl–Bohr theorem (see for instance [7, Theorem 6.20]).

This result can be extended to $AP^{(n)}(X)$ as follows.

Theorem 3.5. *Let X be a uniformly convex Banach space and $f \in AP^{(n)}(X)$. Then the function $F(t) = \int_0^t f(s)\mathrm{d}s \in AP^{(n+1)}(X)$ iff $\mathcal{R}_F$ is bounded in X.*

Proof. In view of Bohl–Bohr's theorem, $F \in AP(X)$ if $\mathcal{R}_F$ is bounded in X. Since $F' = f \in AP^{(n)}(X)$, we deduce that $F \in AP^{(n+1)}(X)$ by the Bohl–Bohr theorem above. The converse is obviously true since any almost periodic function is bounded in norm (see for instance [7], or [13]). $\square$

We recall that uniformly convex Banach spaces include also finite dimensional spaces.

In fact using Kadets' theorem one can prove that Theorem 3.5 holds true in any Banach space which does not contain a subspace isomorphic to c_0. Such spaces are sometimes called perfect Banach spaces (see [13]).

We recall also the following definition from [6].

Definition 3.6. A function $f : \mathbb{R} \times X \to X, (t, x) \longmapsto f(t, x)$ is said to be $C^{(n)}$-almost periodic in t for each $x \in X$ if $D^{(i)} f(t, x) := \frac{\partial^i f}{\partial t^i}(t, x)$ is almost periodic in t for each $x \in X$, and for $i = 0, 1, \ldots, n$. Here $\frac{\partial^0 f}{\partial t^0} := f$.

Theorem 3.7 *([6, Theorem 4.9]). Let $f : \mathbb{R} \times X \to X$ be such that $D^{(i)} f(t, x)$ are continuous for $i = 0, 1, \ldots, n$. Then f is $C^{(n)}$-almost periodic if and only if for every sequence of real numbers (s'_n), there exists a subsequence (s_n) such that $(D^{(i)} f(t + s_n, x))$ converges uniformly in $t \in \mathbb{R}$ and $x \in X$, $i = 0, 1, \ldots, n$.*

We end this section with two results which are easy to establish. See [9] for the definition and properties of the Carleman spectrum of a bounded function.

Lemma 3.8. *Let $f \in AP^{(n)}(X)$ and $\phi \in L^1(\mathbb{R})$ whose Fourier transform has compact support. Then $g := \phi * f \in AP^{(n)}(X)$ and $Sp(g) \subset Sp(f) \cap Sp(\phi)$, where $Sp(h)$ denotes the Carlman spectrum of $h \in BC(\mathbb{R}, X)$.*

Theorem 3.9. *Let $\nu \in AP^{(n)}(\mathbb{R}, \mathcal{L}_s(X, Y))$ and $f \in AP^{(n)}(\mathbb{R}, X)$. Then $\nu f \in AP^{(n)}(\mathbb{R}, Y)$ for two Banach spaces X and Y.*

Proof. It suffices to observe that $\nu^{(i)} f^{(n-i)} : \mathbb{R} \to Y$ is almost periodic, for each $i = 0, 1, \ldots, n$. $\square$

4. Main results

4.1. Linear equations

We begin this section by a study of linear ordinary differential equations with $C^{(n)}$-almost periodic forced terms. Here we denote the set $\Pi := \{z \in \mathbb{C} : \Re z \neq 0\}$.

Lemma 4.1. *Consider in $\mathbb{C}$ the differential equation*

$$x'(t) = \lambda x(t) + f(t), \quad t \in \mathbb{R}, \tag{3}$$

where $\lambda \in \mathbb{C}$ and $f \in AP^{(n)}(\mathbb{C})$. Then every bounded solution x of Eq. (3) satisfies $x \in AP^{(n+1)}(\mathbb{C})$, if $\lambda \notin \Pi$ and $x \in AP^{(n)}(\mathbb{C})$ if $\lambda \in \Pi$.

Proof. First suppose $\lambda = i\theta$ for some $\theta \in \mathbb{R}$. Then the unique solution of Eq. (3) is given by

$$x(t) = \mathrm{e}^{\mathrm{i}\theta} c + \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\theta} f(s)\mathrm{d}s,$$

for some $c \in \mathbb{C}$. It is clear that the functions $s \mapsto \mathrm{e}^{\pm \mathrm{i}s\theta} c$ are $C^{(n+1)}$-a.p., thus $C^{(n)}$-a.p. Now by [6, Proposition 2.10] the function $s \mapsto \mathrm{e}^{-\mathrm{i}s\theta} f(s)$ is also $C^{(n)}$-a.p. Since $\int_0^t \mathrm{e}^{-\mathrm{i}s\theta} f(s)\mathrm{d}s$ is bounded, it is then $C^{(n+1)}$-a.p. by Theorem 3.5 above. So is $\mathrm{e}^{\mathrm{i}t\theta} \int_0^t \mathrm{e}^{-\mathrm{i}s\theta} f(s)\mathrm{d}s = \int_0^t \mathrm{e}^{\mathrm{i}(t-s)\theta} f(s)\mathrm{d}s$, by Theorem 3.9. Finally $x(t) \in AP^{(n+1)}(X)$.

Now, if $\lambda \in \Pi$ then the homogeneous equation associated with Eq. (3) has an exponential dichotomy, so it has a unique bounded solution which we denote by $x_{f,\lambda}(\cdot)$. Moreover, from the theory of ordinary differential equations, it follows that for every fixed $\xi \in \mathbb{R}$,

$$x_{f,\lambda}(\xi) := \begin{cases} \displaystyle\int_{-\infty}^{\xi} e^{\lambda(\xi - t)} f(t)dt & (\text{if } \Re\lambda < 0) \\ \displaystyle -\int_{\xi}^{\infty} e^{\lambda(\xi - t)} f(t)dt & (\text{if } \Re\lambda > 0). \end{cases} \tag{4}$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{0} e^{-\lambda\eta} f(\xi + \eta)d\eta & (\text{if } \Re\lambda < 0) \\ \displaystyle -\int_{0}^{\infty} e^{-\lambda\eta} f(\xi + \eta)d\eta & (\text{if } \Re\lambda > 0). \end{cases} \tag{5}$$

It is clear from the above that $x_{f,\lambda} \in C^n(\mathbb{C})$ and

$$x_{f,\lambda}^{(i)}(\xi) = \begin{cases} \displaystyle\int_{-\infty}^{0} e^{-\lambda\eta} f^{(i)}(\xi + \eta)d\eta & (\text{if } \Re\lambda < 0) \\ \displaystyle -\int_{0}^{\infty} e^{-\lambda\eta} f^{(i)}(\xi + \eta)d\eta & (\text{if } \Re\lambda > 0). \end{cases}$$

Now if τ is an $(\| \bullet \|_n, \epsilon)$-a.p of f, then for λ with $\Re\lambda < 0$ we have

$$\sum_{i=0}^{n} |x_{f,\lambda}^{(i)}(t + \tau) - x_{f,\lambda}^{(i)}(t)| \le \int_{-\infty}^{0} e^{-(\Re\lambda)\eta} \sum_{i=0}^{n} |f^{(i)}(\xi + \eta + \tau) - f^{(i)}(\xi + \eta)|d\eta,$$

which gives

$$\|x_{f,\lambda}(\cdot + \tau) - x_{f,\lambda}(\cdot)\|_n \le \frac{1}{|\Re\lambda|} \|f_\tau - f\|_n < \frac{\epsilon}{|\Re\lambda|}.$$

Therefore $\tau \in E^{(n)}(\frac{\epsilon}{|\Re\lambda|}, x_{f.\lambda})$.

A similar inequality holds for the case $\Re\lambda > 0$. Thus $x_{f,\lambda} \in AP^{(n)}(\mathbb{C})$. The proof is now complete. □

We now extend Lemma 4.1 to n-dimensional spaces. To achieve the result, we use the method of reduction to reduce the equation to a one dimensional space case as presented in [10] and [13]. We recall that for the almost periodic case, this result is known. There are two types of proofs: one is Favard's proof, the other is based on discretization. Both proofs are long and complicated. The method of reduction below is rather elementary and short.

Theorem 4.2. *Let $f \in AP^{(n)}(\mathbb{C}^k)$. Then every bounded solution of the differential equation*

$$x'(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R}, \tag{6}$$

where $A(t) : \mathbb{R} \to \mathbb{C}^k$ is τ-periodic, is in $AP^{(n)}(\mathbb{C}^k)$.

Proof. By Floquet's theory and without loss of generality we may assume that $A(t) = A$ is independent of t. Next we will show that the problem can be reduced to the one dimensional case. In fact, if A is independent of t, by a change of variable if necessary, we may assume that

A is of Jordan normal form. In this direction we can go further with the assumption that A has only one Jordan box. That is, we have to prove the theorem for equations of the form

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_k'(t) \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_k(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_k(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

The last equation

$$x_k'(t) = \lambda x_k(t) + f_k(t), \quad t \in \mathbb{R},\ f_k \in AP^{(n)}(\mathbb{C})$$

stands in $\mathbb{C}$. By the previous lemma, $x_k \in AP^{(n)}(\mathbb{C})$.

Next we have the equation

$$x_{k-1}'(t) = \lambda x_{k-1}(t) + x_k(t) + f_{k-1}(t), \quad t \in \mathbb{R},\ f_{k-1} \in AP^{(n)}(\mathbb{C}).$$

Since $x_k \in AP^{(n)}(\mathbb{C})$, we deduce as above that $x_{k-1} \in AP^{(n)}(\mathbb{C})$. We continue the process until x_1 and obtain the result. $\square$

4.2. Ultra-weak solutions

Now let H be a separable Hilbert space with inner product $(\bullet, \bullet)_H$ and orthonormal base $(e_j)_{j=1}^{\infty}$ and consider in H the differential equation

$$x'(t) = Ax(t) + f(t), \quad t \in \mathbb{R}. \tag{7}$$

Definition 4.3. A linear operator $A : H \to H$ is said to be diagonal if the domain of A is given by

$$D(A) = \left\{ \sum_{j=1}^{k} \alpha_j e_j | k \in \mathbb{N}, \alpha_j \in \mathbb{C} \right\};$$

and if $x \in D(A)$, with $x = \sum_{j=1}^{k} \alpha_j e_j$, then we write $Ax = \sum_{j=1}^{k} \alpha_j \lambda_j e_j$ for some $\lambda_j \in \mathbb{C}$.

Observe that $Ae_j = \lambda_j e_j$ and $\overline{D(A)} = H$.

Let $K_{A^*} := \{\phi : \mathbb{R} \to D(A^*) : \phi \in C_0^1(\mathbb{R}, H)$ and $A^*\phi \in C(\mathbb{R}, H)\}$ be a class of test functions.

Definition 4.4. A function $x \in C(\mathbb{R}, H)$ is said to be an ultra-weak solution to Eq. (7) (in the sense of Lions [11]) if

$$\int_{\mathbb{R}} (x(t), \phi'(t) + (A^*\phi)(t))_H \mathrm{d}t = -\int_{\mathbb{R}} (f(t), \phi(t))_H \mathrm{d}t \tag{8}$$

for any $\phi \in K_{A^*}$.

It is clear that any classical solution, that is a function $x \in C^1(\mathbb{R}, X)$ which satisfies Eq. (7) is also an ultra-weak solution. The converse is not true in general.

Theorem 4.5. *Let A be a diagonal operator in H and assume $Ae_j = \lambda_j e_j$ and $\Re\lambda_j \neq 0$ for all $j = 1, 2, \ldots$. Let $f \in AP^{(n)}(H)$ such that*

$$\sum_{j=1}^{\infty} \frac{1}{|\Re\lambda_j|^2} |(f, e_j)_H|_n^2 < \infty. \tag{9}$$

Then Eq. (7) possesses a unique $C^{(n)}$-almost periodic ultra-weak solution.

Proof. Let $f_j : \mathbb{R} \to \mathbb{C}$ be defined by

$$f_j(t) = (f(t), e_j)_H, \quad j = 1, 2, \ldots.$$

Then obviously each $f_j \in AP^{(n)}(\mathbb{C})$. Moreover for each $i = 0, 1, \ldots, n$, we have

$$f^{(i)}(t) = \lim_{k\to\infty} \sum_{j=1}^{k} f_j^{(i)}(t) e_j$$

uniformly on $\mathbb{R}$ (cf. [15] or [16, pages 98–99].

Now consider $u_j : \mathbb{R} \to \mathbb{C}$ be defined by

$$u_j(t) = \int_{-\infty}^{t} \mathrm{e}^{\lambda_j(t-\xi)} f_j(\xi) \mathrm{d}\xi, \quad \Re\lambda_j < 0$$

and

$$u_j(t) = -\int_{t}^{\infty} \mathrm{e}^{\lambda_j(t-\xi)} f_j(\xi) \mathrm{d}\xi, \quad \Re\lambda_j > 0,$$

$j = 1, 2, \ldots$. By Lemma 4.1, each $u_j \in AP^{(n)}(\mathbb{C})$ and satisfies the inequality

$$\sum_{i=0}^{n} |u_j^{(i)}(t)|^2 \leq \frac{1}{|\Re\lambda_j|^2} |f_j|_n^2 \quad \text{for all } t \in \mathbb{R}. \tag{10}$$

Let the function $u(t) = \sum_{j=1}^{\infty} u_j(t) e_j$ and $v_k(t) = \sum_{j=1}^{k} u_j(t) e_j$, $k \geq 1$. The estimate Eq. (10) yields the following

$$\sum_{i=0}^{n} \|v_k^{(i)}(t) - v_m^{(i)}(t)\|_H^2 \leq \sum_{j=k+1}^{m} \frac{1}{|\Re\lambda_j|^2} |f_j|_n^2$$

for all $t \in \mathbb{R}, m \geq k \geq 1$. Hence, the inequality (9) implies that the sequence of functions v_k converges to u in $C_b^{(n)}(H)$, and then $u \in AP^{(n)}(H)$, since $v_k \in AP^{(n)}(H)$ for each $k \geq 1$.

The function u is then a $C^{(n)}$-almost periodic ultra-weak solution of Eq. (7). Let us show the uniqueness. Suppose $x_1,\ x_2$ are two bounded ultra-weak solutions of Eq. (7) and consider $x = x_1 - x_2$. Then x satisfies the equation

$$\int_{\mathbb{R}} (x(t), \phi'(t) + (A^*\phi)(t))_H \mathrm{d}t = 0, \quad \forall \phi \in K_{A^*}.$$

Let $y \in C_0^1(\mathbb{R})$ and consider the sequence $\phi_j(t) := y(t)e_j,\quad j = 1, 2, \ldots$. Obviously $\phi_j \in C_0^1(\mathbb{R}, H),\ j = 1, 2, \ldots$. So we may replace ϕ with ϕ_j in the above equation to get

$$\int_{\mathbb{R}} (x(t), y'(t)e_j + \bar{\lambda}_j y(t) e_j)_H \mathrm{d}t = 0, \quad j = 1, 2, \ldots,$$

or

$$\int_{\mathbb{R}} y'(t)(x(t), e_j)_H + y(t)(x(t), \bar{\lambda}_j e_j)_H \mathrm{d}t = 0, \quad j = 1, 2, \ldots.$$

Now let $x_j(t) := (x(t), e_j)_H,\ j = 1, 2, \ldots$.

Then it comes

$$\int_{\mathbb{R}} y'(t)x_j(t) + y(t)x_j'(t)\mathrm{d}t = 0, \quad j = 1, 2, \ldots.$$

That means

$$\frac{\mathrm{d}}{\mathrm{d}t}x_j(t) = \lambda_j x_j(t), \quad j = 1, 2, \ldots$$

in the sense of $\mathcal{D}'(\mathbb{R})$. Thus $x_j(t) = \mathrm{e}^{\lambda_j t}x_j(0),\ j = 1, 2, \ldots$. Note that each $x_j(t) \in BC(\mathbb{R}, H)$. Since each $\lambda_j \in \Pi$, then necessarily $x_j(t) = 0, \forall t \in \mathbb{R}$, and consequently $x(t) = 0, \forall t \in \mathbb{R}$. So $x_1(t) = x_2(t), \forall t \in \mathbb{R}$.

The proof is now complete. $\square$

4.3. Semilinear equations

Consider a generator A of a hyperbolic C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X over the field $\mathcal{K}$, and the semilinear evolution equation

$$x'(t) = A_{-1}x(t) + f(t, x(t)), \quad t \in \mathbb{R}. \tag{SEE}$$

The function $f : \mathbb{R} \times X \longrightarrow F_{A_{-1}}$ is assumed to be of the form

$$f(t, x) = a(t) + b(t)x \quad \text{for all } t \in \mathbb{R},\ x \in X, \tag{11}$$

where $a(\cdot) \in AP^{(n)}(F_{A_{-1}})$ and $b(t) \in \mathcal{L}(X, F_{A_{-1}}), t \in \mathbb{R}$, such that $b(\cdot)x \in AP^{(n)}(F_{A_{-1}})$ for each $x \in X$. This provides that $f(t, u)$ is $C^{(n)}$-almost periodic in t for each $u \in X$. Observe that since $b^{(i)}(\cdot)x$ is bounded for each $x \in X$, we deduce by the Uniform Boundedness Principle that $b^{(i)}(\cdot)$ is bounded uniformly on $\mathbb{R}$ for every $0 \leq i \leq n$. Therefore, for each $x, y \in C_b^{(n)}(X)$, there is $k > 0$ such that we have the following Lipschitz condition

$$\|f(\cdot, x(\cdot)) - f(\cdot, y(\cdot))\|_{F_{A_{-1}},n} \leq k \|x(\cdot) - y(\cdot)\|_n. \tag{12}$$

Now by a mild solution of (SEE) we will understand a continuous function $x : \mathbb{R} \longrightarrow X$, which satisfies the following variation of constants formula

$$x(t) = T(t-s)x(s) + \int_s^t T_{-1}(t-\tau)f(\tau, x(\tau))\mathrm{d}\tau \quad \text{for all } t \geq s,\ t, s \in \mathbb{R}. \tag{13}$$

We study first the existence of $C^{(n)}$-almost periodic mild solutions for the inhomogeneous evolution equation

$$x'(t) = A_{-1}x(t) + g(t), \quad t \in \mathbb{R}. \tag{IEE}$$

We have the following main result.

Theorem 4.6. *Let $g \in AP^{(n)}(F_{A_{-1}})$. Then, the equation (IEE) admits a unique mild solution $x \in AP^{(n)}(X)$ given by*

$$x(t) = \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}g(s)\mathrm{d}s - \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}g(s)\mathrm{d}s, \quad t \in \mathbb{R}. \tag{14}$$

Proof. Actually by [4, Theorem 3.1]

$$x(t) := \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}g(s)\mathrm{d}s - \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}g(s)\mathrm{d}s$$

is the unique bounded mild solution to (IEE).

Obviously $x(t) \in C^{(n)}(X)$. It remains to prove that it is $C^{(n)}$-almost periodic.

Let τ be an $(\| \bullet \|_n, \epsilon)$-almost period of g. Then we have

$$\begin{aligned}
&\left\| \int_{-\infty}^{\cdot+\tau} T_{-1}(\cdot+\tau-s)P_{S,-1}g(s)\mathrm{d}s - \int_{-\infty}^{\cdot} T_{-1}(\cdot-s)P_{S,-1}g(s)\mathrm{d}s \right\|_n \\
&\quad \le C \int_{-\infty}^{0} \mathrm{e}^{\delta s} \|g_\tau - g\|_{F_{A_{-1}},n}\mathrm{d}s \\
&\quad < \frac{C\epsilon}{\delta}
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_{\cdot+\tau}^{+\infty} T_{-1}(\cdot+\tau-s)P_{U,-1}g(s)\mathrm{d}s - \int_{\cdot}^{+\infty} T_{-1}(\cdot-s)P_{U,-1}g(s)\mathrm{d}s \right\|_n \\
&\quad \le C \int_{0}^{+\infty} \mathrm{e}^{-\delta s} \|g_\tau - g\|_{F_{A_{-1}},n}\mathrm{d}s \\
&\quad < \frac{C\epsilon}{\delta},
\end{aligned}$$

where C, δ are the constants given in Theorem 2.2. Therefore, this gives

$$\|x_\tau - x\|_n < 2\frac{C\epsilon}{\delta}.$$

So $\tau \in E^{(n)}(\frac{2C\epsilon}{\delta}, x)$. This establishes the result. □

Theorem 4.7. *Under the above conditions, we assume that $kC < \frac{\delta}{2}$. Then (SEE) has a unique $C^{(n)}$-a.p. mild solution.*

Proof. Consider $y \in AP^{(n)}(X)$. Then, by Theorem 3.9, the function $g(\cdot) := f(\cdot, y(\cdot)) \in AP^{(n)}(F_{A_{-1}})$, and from Theorem 4.6, the inhomogeneous evolution equation

$$x'(t) = Ax(t) + g(t), \quad t \in \mathbb{R},$$

admits a unique mild solution $x \in AP^{(n)}(X)$ given by

$$x(t) = \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s, y(s))\mathrm{d}s - \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}f(s, y(s))\mathrm{d}s.$$

It follows that the operator $F : AP^{(n)}(X) \longrightarrow AP^{(n)}(X)$ where

$$(Fy)(t) : \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s,y(s))\mathrm{d}s - \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}f(s,y(s))\mathrm{d}s, \quad t \in \mathbb{R}$$

is well-defined. By Theorem 2.2 and (12), we have for any $x, y \in AP^{(n)}(X)$

$$\begin{aligned}\|Fx - Fy\|_n &\le C\int_{-\infty}^{0} \mathrm{e}^{\delta s}\|f(\cdot,x(\cdot)) - f(\cdot,y(\cdot))\|_{F_{A_{-1}},n}\mathrm{d}s \\ &\quad + C\int_{0}^{+\infty} \mathrm{e}^{-\delta s}\|f(\cdot,x(\cdot)) - f(\cdot,y(\cdot))\|_{F_{A_{-1}},n}\mathrm{d}s \\ &\le \frac{2kC}{\delta}\|x-y\|_n.\end{aligned}$$

This shows that F has a unique fixed point $\bar{x} \in AP^{(n)}(X)$, and

$$\bar{x}(t) = F\bar{x}(t) = \int_{-\infty}^{t} T_{-1}(t-s)P_{S,-1}f(s,\bar{x}(s))\mathrm{d}s - \int_{t}^{+\infty} T_{-1}(t-s)P_{U,-1}f(s,\bar{x}(s))\mathrm{d}s$$

which means that $\bar{x}(t)$ is a mild solution of (SEE). The proof is now complete. □

Remark 4.8. By the same proof, the result of the above theorem remains true for general functions $f : \mathbb{R} \times X \longrightarrow F_{A_{-1}}$ satisfying the Lipschitz condition Eq. (12) and such that the function $g(\cdot) := f(\cdot, y(\cdot)) \in AP^{(n)}(F_{A_{-1}})$ for each $y \in AP^{(n)}(X)$.

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