Exponential Stability of Impulsive Delayed Linear Differential Equations

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***Abstract*—This brief considers the global exponential stability of impulsive delayed linear differential equations. By utilizing the Lyapunov function methods combined with the Razumikhin technique, several criteria on exponential stability are analytically derived, which are substantially an extension and a generalization of the corresponding results in recent literature. Compared with some existing works, a distinctive feature of this brief is to address exponential stability problems for any fixed-time delays. It is shown that the delayed linear differential equations can be globally exponentially stabilized by impulses even if it may be unstable itself. An example and its simulation are also given to illustrate the theoretic results.**

***Index Terms*—Global exponential stability, impulsive delayed linear differential equations, time delays.**

I. Introduction

IMPULSIVE delayed dynamical systems have attracted increasing interest in recent years from various fields of science and engineering, including sociology, biology, physics, and chemistry, as well as control technology, industrial robotics, communication engineering, and so on. In particular, special attention has been focused on the stability of impulsive differential equations [1]–[8]. One of the most investigated problems in the stability analysis of such systems is exponential stability, since it has played an important role in many areas, such as designs and applications of neural networks, synchronizations of coupled oscillators, consensus problems of networked control systems, etc. (see [10]–[16], [20]–[23] and relevant references therein).

Several research works appeared in the literature on impulsive delayed linear differential equations [17]–[19]. In [17], Zhang and Sun have proved two criteria on (*local*) uniform stability for a class of impulsive delayed linear differential equations based on the Lyapunov function technique. In this brief, we shall consider exponential stability problems for impulsive delayed linear differential equations. The main objective of this brief is to extend the results of [17] to the case of (*global*) exponential stability for more general impulsive delayed linear differential equations. As a result, several criteria on the exponential stability of such equations are established under the same conditions by utilizing the Lyapunov function methods combined with the Razumikhin technique, which improved and generalized some of the known results found in the literature [17]. It should also be noticed that the analysis technique we use here is applicable to deal with exponential stability problems *for any fixed time delays*, which is different from that used in the recent literature [13]–[15]. The same as the stability results for impulsive-free delayed linear differential equations [9], our results will show that if the impulsive delayed linear differential equation is (*local*) stable, then it must be (*global*) exponentially stable. Furthermore, the results imply that impulses can contribute to global exponential stability for delayed linear differential equations even if it may be unstable itself, which can be used as an effective control strategy to stabilize an unstable delayed linear dynamical system in practical applications. In addition, our main results can be extended to studying exponential stability problems for more general impulsive delayed nonlinear dynamical systems, which will be applied to many engineering areas, such as neural networks, coupled oscillator systems, and networks of multiagent, among many others [20]–[23].

The outline of this brief is as follows. Some basic definitions, notations, and preliminaries of impulsive stability theory on dynamical systems are presented in Section II. In Section III, several criteria on the exponential stability of impulsive delayed differential equations are analytically derived. Following that, Section IV gives an example and its simulations, and finally, some conclusions are drawn in Section V.

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II. Preliminaries

Throughout this brief, the following notations and definitions will be used. Let $R=(-\infty,+\infty)$ be the set of real numbers, $R^+=[0,+\infty)$ be the set of nonnegative real numbers, and $N=\{1,2,\ldots\}$ be the set of positive integer numbers. For the vector $u\in R^n$, $u^\top$ denotes its transpose. The norm of the vector u is defined as $\|u\|=(u^\top u)^{1/2}$. $R^{n\times n}$ stands for the set of $n\times n$ real matrices. E is the identity matrix of order n. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations.

Consider the following impulsive delayed linear differential equations:

$$\begin{cases} \dot{x}(t)=Ax(t)+Bx(t-\tau), & t\neq t_k, \quad t\geq t_0 \\ \Delta x(t)=C_k x(t^-), & t=t_k, \quad k\in N \end{cases} \tag{1}$$

where $x\in R^n$, A, B, and $C_k\in R^{n\times n}$, the time sequence $\{t_k\}_{k=1}^{+\infty}$ satisfies $0=t_0<t_1<t_2<\cdots<t_k<\cdots$, $\lim_{k\to\infty} t_k=+\infty$, the time delay $\tau>0$, and $\Delta x(t)=x(t_k^+)-x(t_k^-)$, in which

$x(t^+) = \lim_{s\to 0^+} x(t+s)$, and $x(t^-) = \lim_{s\to 0^-} x(t+s)$. Obviously, $x(t) = 0$ is a solution of (1), which we call the zero solution.

Let $PC([a,b], S) = \{\phi : [a,b] \to S | \phi(t) = \phi(t^+)\ \forall t \in [a,b)$; $\phi(t^-)$ exists in $S\ \forall t \in (a,b]$ and $\phi(t^-) = \phi(t)$ for all but at most a finite number of points $t \in (a,b]\}$. For $\psi \in PC([-\tau,0], R^n)$, the norm of ψ is defined by $\|\psi\|_\tau = \sup_{-\tau\le s\le 0}\|\psi(s)\|$. x_t, $x_{t^-} \in PC([-\tau,0], R^n)$ are defined by $x_t(s) = x(t+s)$ and $x_{t^-}(s) = x(t^- + s)$for $s \in [-\tau, 0]$, respectively.

For a given $t > t_0$ and $\varphi \in PC([-\tau,0], R^n)$, the initial value problem of (1) is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \neq t_k, \quad t \geq t_0 \\ \Delta x(t) = C_k x(t^-), & t = t_k, \quad k \in N \\ x_{t_0} = \varphi. \end{cases} \tag{2}$$

We always assume that (1) has a unique solution with respect to initial conditions. Obviously, (1) is a generalization of the equations discussed in [17]. Denote by $x(t) = x(t, t_0, \varphi)$ the solution of (1) such that $x_{t_0} = \varphi$. Now, we have following definitions.

Definition 1: The zero solution of (1) is said to be globally exponentially stable if there exist some constants $\lambda > 0$ and $M \geq 1$ such that for any initial data $x_{t_0} = \varphi$

$$\|x(t, t_0, \varphi)\| \leq M\|\varphi\|_\tau e^{-\lambda(t-t_0)}, \qquad t \geq t_0$$

where $(t_0, \varphi) \in R^+ \times PC([-\tau,0], R^n)$.

Definition 2: The function $V : R^+ \times R^n \to R^+$ is said to belong to the class ν_0 if we have the following.

1) V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$, and for each $x \in R^n$, $t \in [t_{k-1}, t_k), k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t,y) = V(t_k^-, x)$ exists.
2) $V(t,x)$ is locally Lipschitzian in all $x \in R^n$, and for all $t \geq t_0, V(t, 0) \equiv 0$.

Definition 3: Given a function $V : R^+ \times R^n \to R^+$, the upper right-hand derivative of V with respect to system (1) is defined by

$$D^+V(t, x(t)) = \limsup_{\delta\to 0^+} \frac{1}{\delta}\left[V(t+\delta, x(t+\delta)) - V(t, x(t))\right].$$

III. Main Results

In the following, we shall address the exponential stability problems for impulsive delayed linear differential equations [see (1)]. By combining the Lyapunov–Razumikhin methods and some analysis techniques, we shall present several criteria on global exponential stability for the delayed linear differential equations [see (1)]. Our results show that impulses play an important role in making the delayed linear differential equations globally exponentially stable even if it may be unstable itself.

Theorem 1: Let the $n \times n$ matrix P be symmetric and positive definite, λ_3 be the largest eigenvalue of $P^{-1}(A^TP + PA + PP)$, λ_4 be the largest eigenvalue of $P^{-1}B^TB$, and $\lambda_{5_k} (0 < \lambda_{5_k} \leq 1, k \in N)$ be the largest eigenvalue of $P^{-1}(E + C_k)^TP(E + C_k)$. Assume that there exist constants $\lambda > 0$ and $\sigma > 0$ such that, for all $k \in N$, the following conditions are satisfied:

(i) $F_{k-1}(\lambda) = \sigma - \lambda - (\lambda_3 + (\lambda_4/\lambda_{5_{k-1}})e^{\lambda\tau}) \geq 0$, where $0 < \lambda_{5_0} \leq 1$.
(ii) $\ln \lambda_{5_{k-1}} < -(\sigma + \lambda)(t_k - t_{k-1})$.

Then, the zero solution of the impulsive delayed linear differential equations [see (1)] is globally exponentially stable with convergence rate $\lambda/2$ for any fixed delays $\tau \in (0, \infty)$.

Proof: Let $x(t) = x(t, t_0, \varphi)$ be any solution of the impulsive delayed linear differential equations [see (1)] with $x_{t_0} = \varphi$ and construct a Lyapunov function as

$$V(t, x(t)) = x^T(t)Px(t). \tag{3}$$

Letting $\lambda_1 > 0$ and $\lambda_2 > 0$ be the smallest and largest eigenvalues of P, respectively, we have

$$\lambda_1\|x(t)\|^2 \leq V(t, x(t)) \leq \lambda_2\|x(t)\|^2. \tag{4}$$

We shall show that

$$V(t, x(t)) \leq \lambda_2 M\|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \qquad t \in [t_{k-1}, t_k), k \in N. \tag{5}$$

For $t \neq t_k (k \in N)$, by calculating the upper right-hand derivative of $V(t, x(t))$ along the solution of (1) and using the inequality $2x^\top y \leq x^\top x + y^\top y$, we have

$$\begin{aligned} D^+V(t, x(t)) &= x^T(t)(A^TP + PA)x(t) + 2x^T(t-\tau)B^TPx(t) \\ &\leq x^T(t)(A^TP + PA + PP)x(t) \\ &\quad + x^T(t-\tau)B^TBx(t-\tau) \\ &\leq \lambda_3 V(t, x(t)) + \lambda_4 V(t-\tau, x(t-\tau)). \end{aligned} \tag{6}$$

Letting $\gamma = \inf_{k\in N}\{1/\lambda_{5_{k-1}}\} \geq 1$ in condition (ii), we can get

$$\ln\gamma + \lambda\tau - (\sigma + \lambda)(t_k - t_{k-1}) > 0. \tag{7}$$

Therefore, from the inequality in (7), we can choose $M \geq 1$ such that

$$1 < e^{(\sigma+\lambda)(t_1-t_0)} \leq M \leq \gamma e^{\lambda\tau - (\sigma+\lambda)(t_1-t_0)} e^{(\sigma+\lambda)(t_1-t_0)}. \tag{8}$$

It then follows that

$$\|\varphi\|_\tau^2 < \|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \leq M\|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)}. \tag{9}$$

We first prove that

$$V(t, x(t)) \leq \lambda_2 M\|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \qquad t \in [t_0, t_1). \tag{10}$$

To do this, we only need to prove that

$$V(t, x(t)) \leq \lambda_2 M\|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)}, \qquad t \in [t_0, t_1). \tag{11}$$

If (11) is not true, by (4) and (9), then there exists some $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} V(\bar{t}, x(\bar{t})) &> \lambda_2 M\|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)} \geq \lambda_2\|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \\ &> \lambda_2\|\varphi\|_\tau^2 \geq V(t_0+s, x(t_0+s)), \qquad s \in [-\tau, 0] \end{aligned}$$

which implies that there exists some $t^* \in (t_0, \bar{t})$ such that

$$\begin{aligned} &V(t^*, x(t^*)) = \lambda_2 M\|\varphi)\|_\tau^2 e^{-\lambda(t_1-t_0)} \\ &V(t, x(t)) \leq V(t^*, x(t^*)), \qquad t \in [t_0 - \tau, t^*] \end{aligned} \tag{12}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$\begin{aligned} &V(t^{**}, x(t^{**})) = \lambda_2\|\varphi\|_\tau^2 \\ &V(t^{**}, x(t^{**})) \leq V(t, x(t)), \qquad t \in [t^{**}, t^*]. \end{aligned} \tag{13}$$

Hence, for any $s \in [-\tau, 0]$, we can get

$$\begin{aligned} V(t+s, x(t+s)) &\leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)} \\ &\leq \lambda_2 \gamma e^{\lambda\tau-(\sigma+\lambda)(t_1-t_0)} e^{(\sigma+\lambda)(t_1-t_0)} \|\varphi\|_\tau^2 \\ &= \gamma e^{\lambda\tau} \lambda_2 \|\varphi\|_\tau^2 = \gamma e^{\lambda\tau} V(t^{**}, x(t^{**})) \\ &\leq \gamma e^{\lambda\tau} V(t, x(t)), \qquad t \in [t^{**}, t^*] \end{aligned} \tag{14}$$

and, thus, by condition (i), (6), and (14), we get

$$\begin{aligned} D^+ V(t, x(t)) &\leq (\lambda_3 + \gamma\lambda_4 e^{\lambda\tau}) V(t, x(t)) \\ &\leq \left(\lambda_3 + \frac{\lambda_4}{\lambda_{5_0}} e^{\lambda\tau}\right) V(t, x(t)) \\ &\leq (\sigma - \lambda) V(t, x(t)), \qquad t \in [t^{**}, t^*]. \end{aligned} \tag{15}$$

It follows from (9), (12), and (13) that

$$\begin{aligned} V(t^*, x(t^*)) &\leq V(t^{**}, x(t^{**})) e^{(\sigma-\lambda)(t^*-t^{**})} \\ &= \lambda_2 \|\varphi\|_\tau^2 e^{(\sigma-\lambda)(t^*-t^{**})} \\ &< \lambda_2 \|\varphi\|_\tau^2 e^{\sigma(t_1-t_0)} \\ &\leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t_1-t_0)} \\ &= V(t^*, x(t^*)) \end{aligned}$$

which is a contradiction. Hence, (10) holds, and then (5) is true for $k = 1$.

Now, we assume that (5) holds for $k = 1, 2, \ldots, m$ ($m \in N, m \geq 1$), i.e.,

$$V(t, x(t)) \leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \qquad t \in [t_{k-1}, t_k), \quad k = 1, \ldots, m. \tag{16}$$

Next, we shall show that (5) holds for $k = m+1$, i.e.,

$$V(t, x(t)) \leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)}, \qquad t \in [t_m, t_{m+1}). \tag{17}$$

For the sake of contradiction, suppose (17) is not true. Then, we define

$$\bar{t} = \inf \left\{ t \in [t_m, t_{m+1}) | V(t, x(t)) > \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)} \right\}.$$

From condition (ii) and (16), we can get

$$\begin{aligned} V(t_m, x(t_m)) &= x^T(t_m^-)(E + C_m)^T P (E + C_m) x(t_m^-) \\ &\leq \lambda_{5_m} V(t_m^-, x(t_m^-)) \\ &\leq \lambda_{5_m} \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t_m-t_0)} \\ &= \lambda_{5_m} \lambda_2 M \|\varphi\|_\tau^2 e^{\lambda(\bar{t}-t_m)} e^{-\lambda(\bar{t}-t_0)} \\ &< \lambda_{5_m} \lambda_2 e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ &< \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \end{aligned} \tag{18}$$

and so $\bar{t} \neq t_m$. By employing the continuity of $V(t, x(t))$ in the interval $[t_m, t_{m+1})$, we have

$$\begin{aligned} V(\bar{t}, x(\bar{t})) &= \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ V(t, x(t)) &\leq V(\bar{t}, x(\bar{t})), \qquad t \in [t_m, \bar{t}]. \end{aligned} \tag{19}$$

From (18), we know that there exists some $t^* \in (t_m, \bar{t})$ such that

$$\begin{aligned} V(t^*, x(t^*)) &= \lambda_{5_m} \lambda_2 e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ V(t^*, x(t^*)) &\leq V(t, x(t)) \leq V(\bar{t}, x(\bar{t})), \qquad t \in [t^*, \bar{t}]. \end{aligned} \tag{20}$$

On the other hand, for any $t \in [t^*, \bar{t}]$, $s \in [-\tau, 0]$, then either $t + s \in [t_0 - \tau, t_m)$ or $t + s \in [t_m, \bar{t}]$. Two cases will be discussed as follows.

If $t + s \in [t_0 - \tau, t_m)$, from (16), we obtain

$$\begin{aligned} V(t+s, x(t+s)) &\leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(t-t_0)} e^{-\lambda s} \\ &\leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} e^{\lambda(\bar{t}-t)} e^{\lambda\tau} \\ &\leq \lambda_2 e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \end{aligned} \tag{21}$$

whereas if $t + s \in [t_m, \bar{t}]$, from (19), then

$$\begin{aligned} V(t+s, x(t+s)) &\leq \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ &\leq \lambda_2 e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)}. \end{aligned} \tag{22}$$

In any case, from (20)–(22), we all have for any $s \in [-\tau, 0]$

$$\begin{aligned} V(t+s, x(t+s)) &= \frac{e^{\lambda\tau}}{\lambda_{5_m}} V(t^*, x(t^*)) \\ &\leq \frac{e^{\lambda\tau}}{\lambda_{5_m}} V(t, x(t)), \qquad t \in [t^*, \bar{t}]. \end{aligned} \tag{23}$$

Finally, by condition (i), (6), and (23), we have

$$\begin{aligned} D^+ V(t, x(t)) &\leq \left(\lambda_3 + \frac{\lambda_4}{\lambda_{5_m}} e^{\lambda\tau}\right) V(t, x(t)) \\ &\leq (\sigma - \lambda) V(t, x(t)). \end{aligned} \tag{24}$$

Thus, in view of condition (ii), we have

$$\begin{aligned} &V(\bar{t}, x(\bar{t})) \\ &\quad \leq V(t^*, x(t^*)) e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &\quad = \lambda_{5_m} \lambda_2 e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &\quad < \lambda_2 e^{-(\sigma+\lambda)(t_{m+1}-t_m)} e^{\lambda(t_{m+1}-t_m)} M \|\varphi\|_\tau^2 \\ &\qquad \times e^{-\lambda(\bar{t}-t_0)} e^{(\sigma-\lambda)(\bar{t}-t^*)} \\ &\quad = \lambda_2 M \|\varphi\|_\tau^2 e^{-\sigma(t_{m+1}-t_m)} e^{(\sigma-\lambda)(\bar{t}-t^*)} e^{-\lambda(\bar{t}-t_0)} \\ &\quad < \lambda_2 M \|\varphi\|_\tau^2 e^{-\lambda(\bar{t}-t_0)} \\ &\quad = V(\bar{t}, x(\bar{t})) \end{aligned}$$

which is a contradiction. This implies that the assumption is not true, and, hence, (5) holds for $k = m+1$. Therefore, by some simple mathematical induction, we can obtain that (5) holds for any $k \in N$. It immediately follows from (4) that

$$\|x(t)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} M^{\frac{1}{2}} \|\varphi\|_\tau e^{-\frac{\lambda}{2}(t-t_0)}, \qquad t \geq t_0$$

which implies that the zero solution of the impulsive delayed system [see (1)] is globally exponentially stable with convergence rate $\lambda/2$ for any fixed delays $\tau \in (0, \infty)$. This completes the proof of Theorem 1. ■

Letting the impulsive control matrices $C_k = C$, the corresponding $\lambda_{5_k} = \lambda_5$, then (1) can be reduced to the following impulsive delayed linear differential equations discussed in [17]:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \neq t_k, \quad t \geq t_0 \\ \Delta x(t) = Cx(t^-), & t = t_k, \quad k \in N. \end{cases} \tag{25}$$

We have the following results, which is an immediate consequence of Theorem 1.

Corollary 1: Let λ_3, λ_4, and λ_5 be precisely the same as those of Theorem 1, and assume that, for all $k \in N$, we have the following condition:

(iii) $(\lambda_3 + (\lambda_4/\lambda_5))(t_k - t_{k-1}) < -\ln \lambda_5$.

Then, the zero solution of the impulsive delayed linear differential equations [see (25)] is globally exponentially stable for any fixed delays $\tau \in (0, \infty)$.

Proof: Let $\gamma = 1/\lambda_5$ in Theorem 1. In this case, it is easy to verify from the proof of Theorem 1 that condition (i) can be reduced to the following condition:

(i)′ $F_{k-1}(\lambda) = \sigma - \lambda - (\lambda_3 + (\lambda_4/\lambda_5)e^{\lambda\tau}) \geq 0$.

Moreover, conditions (i)′ and (ii) are equivalent to the following inequality:

$$\left[2\lambda + \left(\lambda_3 + \frac{\lambda_4}{\lambda_5}e^{\lambda\tau}\right)\right](t_k - t_{k-1}) < -\ln \lambda_5, \qquad k \in N \tag{26}$$

which is an immediate consequence of condition (iii) for a small-enough real number $\lambda > 0$. In fact, we introduce a function for all $k \in N$ as

$$H(\lambda) = \left[2\lambda + \left(\lambda_3 + \frac{\lambda_4}{\lambda_5}e^{\lambda\tau}\right)\right](t_k - t_{k-1}) + \ln \lambda_5$$

which yields, from condition (iii), that $H(0) < 0$. Then, we can conclude by the continuity of the function $H(\lambda)$ that there exists a small enough real number $\lambda > 0$ such that $H(\lambda) < 0$, and so inequality (26) holds. By Theorem 1, this completes the proof of Corollary 1. ■

Remark 1: Obviously, Corollary 1 has extended the main results of [17, Theorem 1] concerning the (*local*) uniform stability to the (*global*) exponential stability for impulsive delayed linear differential equations [see (25)] under the same conditions. Therefore, Theorem 1 is an important improvement and generalization of the main results in [17]. As a result, an interesting conclusion on stability problems for impulsive delayed linear differential equations is presented, i.e., just the same as the stability results for impulsive-free delayed linear differential equations [9]. If the impulsive delayed linear differential equations is (*local*) uniform stable, then it must be (*global*) exponentially stable. Furthermore, our results show that impulses do contribute to the global exponential stability of delayed linear differential equations even if it may be unstable itself, which can usually be used as an effective control strategy to stabilize the underlying delayed linear dynamical systems in some practical applications.

Remark 2: It is important to emphasize that, in contrast with some existing exponential stability results for impulsive delayed differential equations in the literature [13]–[15], Theorem 1 and Corollary 1 are also valid for any fixed delays $\tau \in (0, +\infty)$. Therefore, our new results are more practically applicable than those in the literature, since the restrictive condition that the time delays are less than the length of all the impulsive intervals is actually removed here (see, for example, [13, Theorem 3.1] and [14, Theorem 3.1]).

Remark 3: The results of Theorem 1 can further be extended to dealing with the exponential stability problems for the following more general impulsive delayed nonlinear dynamical system:

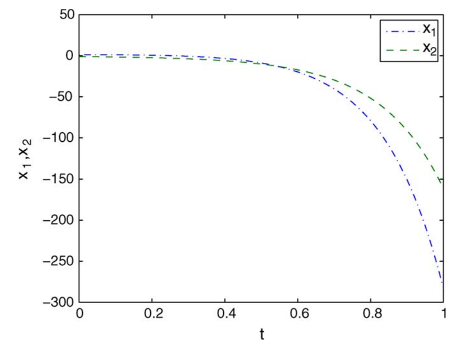

Fig. 1. System without impulses.

$$\begin{cases} \dot{x}(t) = Ax(t) + f\left(t; x(t), x(t-\tau)\right), & t \neq t_k, \quad t \geq t_0 \\ \Delta x(t) = C_k x(t^-), & t = t_k, \quad k \in N \end{cases} \tag{27}$$

in which $f : R^+ \times R^n \times R^n \to R^n$ is a continuously vector-valued function, and $f(t; 0, 0) = 0$. We further assume that the nonlinear vector-valued function f possesses some of the following properties with respect to t, i.e.,

(iv) There exist two positive numbers $l_1 > 0$ and $l_2 > 0$ such that

$$\left\| f\left(t; x(t), x(t-\tau)\right)\right\|^2 \leq l_1 \left\|x(t)\right\|^2 + l_2 \left\|x(t-\tau)\right\|^2.$$

In general, both (27) and condition (iv) can formulate some representative nonlinear system models, such as neural networks, coupled oscillator systems, networks of multiagent, etc. [20]–[23].

Theorem 2: Let the $n \times n$ matrix P be symmetric and positive definite, λ_3 be the largest eigenvalue of $P^{-1}(A^TP + PA + PP) + l_1P^{-1}$, λ_4 be the largest eigenvalue of l_2P^{-1}, and λ_{5_k} $(0 < \lambda_{5_k} \leq 1, k \in N)$ be the largest eigenvalue of $P^{-1}(E + C_k)^TP(E + C_k)$. Assume that both conditions (i) and (ii) are satisfied in Theorem 1. Then, the zero solution of the impulsive delayed nonlinear differential equations [see (27)] is globally exponentially stable with convergence rate $\lambda/2$ for any fixed delays $\tau \in (0, \infty)$.

Proof: Similar to the proof of Theorem 1, by employing the same Lyapunov function (3), we have for $t \neq t_k$ $(k \in N)$

$$\begin{aligned} D^+V(t) & \\ &= x^T(t)(A^TP + PA)x(t) + 2x^T(t)Pf\left(x(t), x(t-\tau)\right) \\ &\leq x^T(t)(A^TP + PA)x(t) + x^T(t)PPx(t) \\ &\quad + f^T\left(x(t), x(t-\tau)\right)f\left(x(t), x(t-\tau)\right) \\ &\leq x^T(t)(A^TP + PA + PP)x(t) + l_1P^{-1}x^T(t)Px(t) \\ &\quad + l_2P^{-1}x^T(t-\tau)Px(t-\tau) \\ &\leq \lambda_3 V\left(t, x(t)\right) + \lambda_4 V\left(t-\tau, x(t-\tau)\right). \end{aligned}$$

The rest of the proof is precisely the same as that for Theorem 1, and hence, it is omitted. Theorem 2 is proven. ■

IV. Application Examples

In this section, an application example given in [17] and its simulation are presented here again to illustrate our main results.

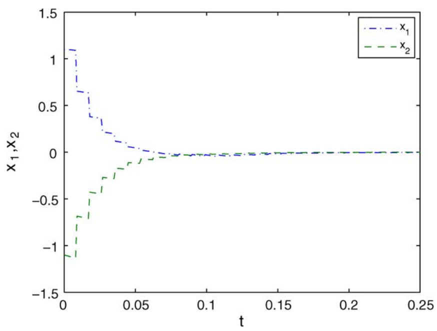

Fig. 2. Impulsively stabilized system.

Example 1: Consider the following impulsive delayed linear differential equations:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 2 & \frac{5}{3} \\ \frac{1}{2} & 4 \end{pmatrix} x(t) + \begin{pmatrix} 3 & 5 \\ \frac{1}{3} & 1 \end{pmatrix} x(t-\tau), & t \neq t_k,\ t \geq t_0 \\ x(t_k) = \begin{pmatrix} \frac{3}{5} & 0 \\ 0 & \frac{3}{5} \end{pmatrix} x\left(t_k^-\right), & t = t_k,\ k \in N \end{cases} \tag{28}$$

where $x(t) = (x_1(t), x_2(t)) \in R^2$.

It is easy to know that, for the time delay $\tau = 0.1$, the corresponding system without impulses is unstable, and the numerical simulation of this delayed differential equation with respect to initial functions

$$\varphi_1(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ 1.1, & t = 0; \end{cases}$$
$$\varphi_2(t) = \begin{cases} 0, & t \in [-0.1, 0), \\ -1.1, & t = 0 \end{cases}$$

is given in Fig. 1.

However, if we choose $P = E$, then $\lambda_3 = 9.9486$, $\lambda_4 = 35.0604$, and $\lambda_4 = 0.3600$. It is easy to verify that if the following condition holds:

$$t_k - t_{k-1} < -\frac{\ln \lambda_5}{\lambda_3 + \frac{\lambda_4}{\lambda_5}} = 0.0095$$

then condition (iii) of Corollary 1 is satisfied, which means that the zero solution of the impulsive delayed linear differential equations [see (28)] is globally exponentially stable for any fixed delays $\tau \in (0, \infty)$. This conclusion cannot be derived by applying similar exponential stability results for impulsive delayed differential equations given in the literature [13]–[15] here since the length of the impulsive intervals is excessively less than the fixed time delays, i.e., $t_k - t_{k-1} < 0.0095 \ll \tau = 0.1$. For simplicity, let the equidistant impulsive interval be taken as $t_k - t_{k-1} = 0.009$. Fig. 2 visualizes the change process of the state variables of the delayed differential system [see (28)] in time interval [0, 0.25]. It can be seen that impulses do contribute to the global exponential stability of the delayed differential system even if the corresponding system without impulses is unstable itself.

V. Conclusion

In this brief, we have investigated the global exponential stability of impulsive delayed linear differential equations. Some explicit and conclusive results on exponential stability have been derived, which are the substantial extension and generalization of some known results existing in the literature. Numerical examples are also given to demonstrate the effectiveness of the theoretical results. It should be mentioned that our results may allow us to develop an effective impulsive control strategy to stabilize an underlying delayed dynamical system even if it may be unstable in practice, which is particularly meaningful for applications in engineering and technology.

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