



Integral manifolds for partial functional differential equations in admissible spaces on a half-line[☆]



Nguyen Thieu Huy^{a,*}, Trinh Viet Duoc^b

^a School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, 1 Dai Co Viet, Hanoi, Viet Nam

^b Faculty of Mathematics, Mechanics, and Informatics, Hanoi University of Science, 334 Nguyen Trai, Hanoi, Viet Nam

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ABSTRACT

In this paper we investigate the existence of stable and center-stable manifolds for solutions to partial functional differential equations of the form $\dot{u}(t) = A(t)u(t) + f(t, u_t)$, $t \geq 0$, when its linear part, the family of operators $(A(t))_{t \geq 0}$, generates the evolution family $(U(t, s))_{t \geq s \geq 0}$ having an exponential dichotomy or trichotomy on the half-line and the nonlinear forcing term f satisfies the φ -Lipschitz condition, i.e., $\|f(t, u_t) - f(t, v_t)\| \leq \varphi(t)\|u_t - v_t\|_C$ where $u_t, v_t \in C := C([-r, 0], X)$, and $\varphi(t)$ belongs to some admissible function space on the half-line. Our main methods invoke Lyapunov-Perron methods and the use of admissible function spaces.

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1. Introduction

Consider the partial functional differential equation

$$\frac{du}{dt} = A(t)u(t) + f(t, u_t), \quad t \in [0, +\infty), \quad (1.1)$$

where $A(t)$ is a (possibly unbounded) linear operator on a Banach space X for every fixed t ; $f : \mathbb{R}_+ \times C \rightarrow X$ is a continuous nonlinear operator with $C := C([-r, 0], X)$, and u_t is the history function defined by $u_t(\theta) := u(t + \theta)$ for $\theta \in [-r, 0]$. When the family of operators $(A(t))_{t \geq 0}$ generates the evolution family having an exponential dichotomy (or trichotomy), one tries to find conditions on the nonlinear forcing term f such that Eq. (1.1) has an integral manifold (e.g., a stable, unstable or center manifold). The most popular condition imposed on f is its uniform Lipschitz continuity with a sufficiently small Lipschitz constant, i.e., $\|f(t, \phi) - f(t, \psi)\| \leq q\|\phi - \psi\|_C$ for q small enough (see [1,3,14] and references therein). However, for equations arising in complicated reaction-diffusion processes, the function f represents the source of material (or population) which, in many contexts, depends on time in diversified manners (see [5, Chapt. 11], [6,15]). Therefore, sometimes one may not hope to have the uniform Lipschitz continuity of f . Recently, for the case of partial differential equations without delay, we have obtained exciting results in [9], where we have used the Lyapunov-Perron method and the characterization of the exponential dichotomy (obtained in [8]) of evolution equations in admissible function spaces to construct the structures of solutions in mild forms, which belong to some certain classes of admissible spaces on which we could employ some well-known principles in mathematical analysis such as the contraction mapping principle, the implicit function theorem, etc. The use of admissible spaces has helped us to construct the invariant manifolds without using the smallness of Lipschitz constants of nonlinear forcing terms in classical sense (see [9,10]).

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* Corresponding author.

E-mail addresses: huy.nguyenthieu@hust.vn (T.H. Nguyen), tvduoc@gmail.com (V.D. Trinh).

Another point we would like to mention is that in some applications the partial differential operator $A(t)$ is defined only for $t \geq 0$ (see e.g., [4,11,13] and references therein). Therefore, the evolution family generated by $(A(t))_{t \geq 0}$ is defined only on a half-line.

The purpose of the present paper is to prove the existence of stable and center-stable manifolds for Eq. (1.1) when its linear part $(A(t))_{t \geq 0}$ generates the evolution family having an exponential dichotomy or trichotomy on the half-line under more general conditions on the nonlinear forcing term f , that is the φ -Lipschitz continuity of f , i.e., $\|f(t, \phi) - f(t, \psi)\| \leq \varphi(t)\|\phi - \psi\|_C$ where $\phi, \psi \in \mathcal{C}$, and $\varphi(t)$ is a real and positive function which belongs to admissible function spaces defined in Definition 2.3 below. We will extend the methods in [9] to the case of partial functional differential equations (PFDE). The main difficulties that we face when passing to the case of PFDE are the following two features: Firstly, since the nonlinear operator f is φ -Lipschitz, the existence and uniqueness theorem for solutions to (1.1) is not available. Secondly, the evolution family generated by $(A(t))_{t \geq 0}$ is defined only on a half-line \mathbb{R}_+ and doesn't act on the same Banach space as that the surfaces of the integral manifold belong to (in fact, the former acts on X , and the latter belongs to \mathcal{C}). Therefore, the standard methods of nonlinear perturbations of an evolutionary process using graph transforms as formulated in [1,3] cannot be applied here.

To overcome such difficulties, we reformulate the definition of invariant manifolds such that it contains the existence and uniqueness theorem as a property of the manifold (see Definition 3.3 below). Furthermore, we construct the structure of the mild solutions to (1.1) using the Lyapunov–Perron equation (see Eq. (3.5)) in such a way that it allows to combine the exponential dichotomy of the linear part of Eq. (1.1) with the existence and uniqueness of its bounded solutions in the case of φ -Lipschitz nonlinear forcing terms. Then, we use the admissible spaces to construct the integral manifolds for Eq. (1.1) in the case of dichotomic linear part without using the smallness of Lipschitz constants of the nonlinear terms in classical sense. Instead, the “smallness” is now understood as the sufficient smallness of $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau$. Consequently, we obtain the existence of invariant stable manifolds for the case of dichotomic linear parts under very general conditions on the nonlinear term $f(t, u_t)$. Moreover, using these results and rescaling procedures we prove the existence of center-stable manifolds for the mild solutions to Eq. (1.1) in the case of trichotomic linear parts under the same conditions on the nonlinear term f as in the dichotomic case. Our main results are contained in Theorems 3.7, 4.2.

We now recall some notions.

For a Banach space X (with a norm $\|\cdot\|$) and a given $r > 0$ we denote by $\mathcal{C} := \mathcal{C}([-r, 0], X)$ the Banach space of all continuous functions from $[-r, 0]$ into X , equipped with the norm $\|\phi\|_C = \sup_{\theta \in [-r, 0]} \|\phi(\theta)\|$ for $\phi \in \mathcal{C}$. For a continuous function $v : [-r, \infty) \rightarrow X$ the *history function* $v_t \in \mathcal{C}$ is defined by $v_t(\theta) = v(t + \theta)$ for all $\theta \in [-r, 0]$.

Definition 1.1. A family of bounded linear operators $\{U(t, s)\}_{t \geq s \geq 0}$ on a Banach space X is a (*strongly continuous, exponential bounded*) *evolution family* if

- (i) $U(t, t) = Id$ and $U(t, r)U(r, s) = U(t, s)$ for all $t \geq r \geq s \geq 0$,
- (ii) the map $(t, s) \mapsto U(t, s)x$ is continuous for every $x \in X$,
- (iii) there are constants $K, c \geq 0$ such that $\|U(t, s)x\| \leq K e^{c(t-s)} \|x\|$ for all $t \geq s \geq 0$ and $x \in X$.

The notion of an evolution family arises naturally from the theory of non-autonomous evolution equations which are well-posed. Meanwhile, if the abstract Cauchy problem

$$\begin{cases} \frac{du}{dt} = A(t)u(t), & t \geq s \geq 0, \\ u(s) = x_s \in X \end{cases} \quad (1.2)$$

is well-posed, there exists an evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ such that the solution of the Cauchy problem (1.2) is given by $u(t) = U(t, s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [11] (see also Nagel and Nickel [7] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line \mathbb{R}).

2. Function spaces and admissibility

We recall some notions on function spaces and refer to Massera and Schäffer [2], Räbiger and Schnaubelt [12] for concrete applications.

Denote by \mathcal{B} the Borel algebra and by λ the Lebesgue measure on \mathbb{R}_+ . The space $L_{1,loc}(\mathbb{R}_+)$ of real-valued locally integrable functions on \mathbb{R}_+ (modulo λ -nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)| dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{N}$ (see [2, Chapt. 2, §20]).

We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on \mathbb{R}_+ (modulo λ -nullfunctions) is called a *Banach function space* (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if

- 1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$, λ -a.e., then $\psi \in E$ and $\|\psi\|_E \leq \|\varphi\|_E$,
- 2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t \geq 0} \|\chi_{[t, t+1]}\|_E < \infty$ and $\inf_{t \geq 0} \|\chi_{[t, t+1]}\|_E > 0$,
- 3) $E \hookrightarrow L_{1,loc}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R}_+)$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leq \beta_{p_n} \|f\|_E$ for all $f \in E$.

We then define Banach spaces of vector-valued functions corresponding to Banach function spaces as follows.

Definition 2.2. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) = \{f : \mathbb{R}_+ \rightarrow X : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

(modulo λ -nullfunctions) endowed with the norm $\|f\|_{\mathcal{E}} = \|\|f(\cdot)\|\|_E$. One can easily see that \mathcal{E} is a Banach space. We call it the Banach space corresponding to the Banach function space E .

We now introduce the notion of admissibility in the following definition.

Definition 2.3. The Banach function space E is called *admissible* if

- (i) there is a constant $M \geq 1$ such that for every compact interval $[a, b] \in \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E,$$

- (ii) for $\varphi \in E$ the function $\Lambda_1 \varphi$ defined by $\Lambda_1 \varphi(t) = \int_t^{t+1} \varphi(\tau) d\tau$ belongs to E ,
- (iii) E is T_τ^+ -invariant and T_τ^- -invariant, where T_τ^+ and T_τ^- are defined for $\tau \in \mathbb{R}_+$ by

$$T_\tau^+ \varphi(t) = \begin{cases} \varphi(t - \tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases}$$

$$T_\tau^- \varphi(t) = \varphi(t + \tau) \quad \text{for } t \geq 0.$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leq N_1$, $\|T_\tau^-\| \leq N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.4. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+1} |f(\tau)| d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t \geq 0} \int_t^{t+1} |f(\tau)| d\tau$, many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty$ are admissible.

Remark 2.5. One can easily see that if E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [9, Proposition 2.6]).

Proposition 2.6. Let E be an admissible Banach function space. Then the following assertions hold.

- (a) Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1 \varphi \in E$, where Λ_1 is defined as in Definition 2.3(ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ by

$$\Lambda'_\sigma \varphi(t) = \int_0^t e^{-\sigma(t-s)} \varphi(s) ds,$$

$$\Lambda''_\sigma \varphi(t) = \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds.$$

Then, $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ belong to E . In particular, if $\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.5)) then $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for ess sup-norm, we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty.$$

- (b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geq 0$ and any fixed constant $\alpha > 0$.
- (c) E does not contain exponentially growing functions $f(t) = e^{bt}$ for $t \geq 0$ and any constant $b > 0$.

3. Exponential dichotomy and stable manifolds

In this section, we will find condition for the existence of stable manifolds. Throughout this section we assume that the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ has an exponential dichotomy on \mathbb{R}_+ . We now make precisely the notion of exponential dichotomies in the following definition.

Definition 3.1. An evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ on the Banach space X is said to have an *exponential dichotomy* on $[0, \infty)$ if there exist bounded linear projections $P(t)$, $t \geq 0$, on X and positive constants N, ν such that

- (a) $U(t, s)P(s) = P(t)U(t, s)$, $t \geq s \geq 0$,
- (b) the restriction $U(t, s)_| : \text{Ker } P(s) \rightarrow \text{Ker } P(t)$, $t \geq s \geq 0$, is an isomorphism, and we denote its inverse by $U(s, t)_| := (U(t, s)_|)^{-1}$, $0 \leq s \leq t$,
- (c) $\|U(t, s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geq s \geq 0$,
- (d) $\|U(s, t)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \text{Ker } P(t)$, $t \geq s \geq 0$.

The projections $P(t)$, $t \geq 0$, are called the *dichotomy projections*, and the constants N, ν – the *dichotomy constants*.

Using the projections $(P(t))_{t \geq 0}$ on X , we can define the family of operators $(\tilde{P}(t))_{t \geq 0}$ on \mathcal{C} as follows.

$$\begin{aligned} \tilde{P}(t) : \mathcal{C} &\rightarrow \mathcal{C}, \\ (\tilde{P}(t)\phi)(\theta) &= U(t - \theta, t)P(t)\phi(0) \quad \text{for all } \theta \in [-r, 0]. \end{aligned} \tag{3.1}$$

Then, we have that $(\tilde{P}(t))^2 = \tilde{P}(t)$, and therefore the operators $\tilde{P}(t)$, $t \geq 0$, are projections on \mathcal{C} . Moreover, $\text{Im } \tilde{P}(t) = \{\phi \in \mathcal{C} : \phi(\theta) = U(t - \theta, t)v_0 \forall \theta \in [-r, 0] \text{ for some } v_0 \in \text{Im } P(t)\}$.

To obtain the existence of stable manifolds we need the following notion of the φ -Lipschitz of the nonlinear term f .

Definition 3.2. Let E be an admissible Banach function space and φ be a positive function belonging to E . A function $f : [0, \infty) \times \mathcal{C} \rightarrow X$ is said to be φ -Lipschitz if f satisfies

- (i) $\|f(t, 0)\| \leq \varphi(t)$ for all $t \in \mathbb{R}_+$,
- (ii) $\|f(t, \phi_1) - f(t, \phi_2)\| \leq \varphi(t)\|\phi_1 - \phi_2\|_{\mathcal{C}}$ for all $t \in \mathbb{R}_+$ and all $\phi_1, \phi_2 \in \mathcal{C}$.

Note that if $f(t, \phi)$ is φ -Lipschitz then $\|f(t, \phi)\| \leq \varphi(t)(1 + \|\phi\|_{\mathcal{C}})$ for all $\phi \in \mathcal{C}$ and $t \geq 0$.

To prove the existence of a stable manifold, instead of Eq. (1.1) we consider the following integral equation

$$\begin{cases} u(t) = U(t, s)u(s) + \int_s^t U(t, \xi)f(\xi, u_\xi)d\xi & \text{for } t \geq s \geq 0, \\ u_s = \phi \in \mathcal{C}. \end{cases} \tag{3.2}$$

We note that, if the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ arises from the well-posed Cauchy problem (1.2), then the function $u : [s - r, \infty) \rightarrow X$, which satisfies (3.2) for some given function f , is called a mild solution of the semilinear problem

$$\begin{cases} \frac{du}{dt} = A(t)u(t) + f(t, u_t), & t \geq s \geq 0, \\ u_s = \phi \in \mathcal{C}. \end{cases}$$

We refer the reader to J. Wu [14] for more detailed treatments on the relations between classical and mild solutions of functional evolution equations.

We now give the definition of a stable manifold for the solutions of the integral equation (3.2).

Definition 3.3. A set $S \subset \mathbb{R}_+ \times \mathcal{C}$ is said to be an *invariant stable manifold* for the solutions to Eq. (3.2) if for every $t \in \mathbb{R}_+$ the phase space \mathcal{C} splits into a direct sum $\mathcal{C} = \tilde{X}_0(t) \oplus \tilde{X}_1(t)$ with corresponding projections $\tilde{P}(t)$ (i.e., $\tilde{X}_0(t) = \text{Im } \tilde{P}(t)$ and $\tilde{X}_1(t) = \text{Ker } \tilde{P}(t)$) such that

$$\sup_{t \geq 0} \|\tilde{P}(t)\| < \infty,$$

and there exists a family of Lipschitz continuous mappings

$$\Phi_t : \tilde{X}_0(t) \rightarrow \tilde{X}_1(t), \quad t \in \mathbb{R}_+$$

with the Lipschitz constants independent of t such that

(i) $S = \{(t, \psi + \Phi_t(\psi)) \in \mathbb{R}_+ \times (\tilde{X}_0(t) \oplus \tilde{X}_1(t)) \mid t \in \mathbb{R}_+, \psi \in \tilde{X}_0(t)\}$, and we denote

$$S_t := \{\psi + \Phi_t(\psi) : (t, \psi + \Phi_t(\psi)) \in S\},$$

(ii) S_t is homeomorphic to $\tilde{X}_0(t)$ for all $t \geq 0$,

(iii) to each $\phi \in S_s$ there corresponds one and only one solution $u(t)$ to Eq. (3.2) on $[s - r, \infty)$ satisfying the conditions that $u_s = \phi$ and $\sup_{t \geq s} \|u_t\|_{\mathcal{C}} < \infty$. Moreover, any two solutions $u(t)$ and $v(t)$ of Eq. (3.2) corresponding to different functions $\phi_1, \phi_2 \in S_s$ attract each other exponentially in the sense that, there exist positive constants μ and C_μ independent of $s \geq 0$ such that

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{-\mu(t-s)} \|(\tilde{P}(s)\phi_1)(0) - (\tilde{P}(s)\phi_2)(0)\| \quad \text{for } t \geq s, \quad (3.3)$$

(iv) S is positively invariant under Eq. (3.2), i.e., if $u(t)$, $t \geq s - r$, is a solution to Eq. (3.2) satisfying conditions that $u_s \in S_s$ and $\sup_{t \geq s} \|u_t\|_{\mathcal{C}} < \infty$, then we have $u_t \in S_t$ for all $t \geq s$.

Note that if we identify $\tilde{X}_0(t) \oplus \tilde{X}_1(t)$ with $\tilde{X}_0(t) \times \tilde{X}_1(t)$, then we can write $S_t = \text{graph}(\Phi_t)$.

Let $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N, \nu > 0$. Note that the exponential dichotomy of $\{U(t, s)\}_{t \geq s \geq 0}$ implies that $H := \sup_{t \geq 0} \|P(t)\| < \infty$ and the map $t \mapsto P(t)$ is strongly continuous (see [4, Lemma 4.2]). We can then define the Green function on the half-line as follows

$$\mathcal{G}(t, \tau) = \begin{cases} P(t)U(t, \tau) & \text{for } t > \tau \geq 0, \\ -U(t, \tau)(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases} \quad (3.4)$$

It follows from the exponential dichotomy of $\{U(t, s)\}_{t \geq s \geq 0}$ that

$$\|\mathcal{G}(t, \tau)\| \leq N(1 + H)e^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau.$$

The following lemma gives the form of bounded solutions to Eq. (3.2).

Lemma 3.4. Let the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N, \nu > 0$. Suppose that φ is a positive function which belongs to the admissible space E . Let $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ be φ -Lipschitz and $u(t)$ be a solution to Eq. (3.2) such that $\sup_{t \geq s-r} \|u(t)\| < \infty$ for fixed $s \geq 0$. Then, for $t \geq s$ we can rewrite $u(t)$ in the form

$$\begin{cases} u(t) = U(t, s)v_0 + \int_s^\infty \mathcal{G}(t, \tau)f(\tau, u_\tau)d\tau, \\ u_s = \phi \in \mathcal{C} \end{cases} \quad (3.5)$$

for some $v_0 \in X_0(s) = P(s)X$, where $\mathcal{G}(t, \tau)$ is the Green function defined as in (3.4).

Proof. Put $y(t) = \int_s^\infty \mathcal{G}(t, \tau)f(\tau, u_\tau)d\tau$. We have

$$\begin{aligned} \|y(t)\| &\leq \int_s^\infty N(1 + H)e^{-\nu|t-\tau|}(1 + \|u_\tau\|_{\mathcal{C}})\varphi(\tau)d\tau \\ &\leq N(1 + H)\left(1 + \sup_{\xi \geq s-r} \|u(\xi)\|\right) \int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)d\tau. \end{aligned}$$

Using Proposition 2.6 we obtain

$$\|y(t)\| \leq N(1 + H)\left(1 + \sup_{\xi \geq s-r} \|u(\xi)\|\right) \frac{(N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty)}{1 - e^{-\nu}} \quad \text{for all } t \geq 0. \quad (3.6)$$

On the other hand,

$$\begin{aligned} U(t, s)y(s) &= - \int_s^t U(t, s)U(s, \tau)|I - P(\tau)\|f(\tau, u_\tau)d\tau \\ &\quad - \int_t^\infty U(t, s)U(s, \tau)|I - P(\tau)\|f(\tau, u_\tau)d\tau \\ &= - \int_s^t U(t, \tau)(I - P(\tau))f(\tau, u_\tau)d\tau - \int_t^\infty U(t, \tau)(I - P(\tau))f(\tau, u_\tau)d\tau. \end{aligned}$$

Therefore,

$$y(t) = U(t, s)y(s) + \int_s^t U(t, \tau)f(\tau, u_\tau)d\tau.$$

Since $u(t)$ is a solution of Eq. (3.2) we obtain that $u(t) - y(t) = U(t, s)(u(s) - y(s))$. Put now $v_0 = u(s) - y(s)$. The boundedness of $u(t)$ and $y(t)$ on $[s - r, \infty)$ implies that $v_0 \in X_0(s)$ and $P(s)u(s) = P(s)\phi(0) = v_0$. Therefore, $u(t) = U(t, s)v_0 + y(t)$ for $t \geq s$. \square

Remark 3.5. Eq. (3.5) is called the *Lyapunov–Perron equation*. By computing directly, we can see that the converse of Lemma 3.4 is also true. This means that, all solutions of the integral equation (3.5) satisfy Eq. (3.2) for $t \geq s$.

Theorem 3.6. Let the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N, \nu > 0$. Suppose that φ is a positive function which belongs to E . Let $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ be φ -Lipschitz, and let

$$k := \frac{e^{\nu r}(1 + H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1 - e^{-\nu}}. \quad (3.7)$$

Then, if $k < 1$, there corresponds to each $\phi \in \text{Im } \tilde{P}(s)$ one and only one solution $u(t)$ of Eq. (3.5) on $[s - r, \infty)$ satisfying the conditions that $\tilde{P}(s)u_s = \phi$ and $\sup_{t \geq s} \|u_t\|_{\mathcal{C}} < \infty$. Moreover, the following estimate is valid for any two solutions $u(t), v(t)$ corresponding to different initial functions $\phi_1, \phi_2 \in \text{Im } \tilde{P}(s)$:

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{-\mu(t-s)} \|\phi_1(0) - \phi_2(0)\| \quad \text{for all } t \geq s \geq 0$$

where μ is a positive number satisfying

$$\begin{aligned} 0 < \mu < \nu + \ln(1 - N(1 + H)e^{\nu r}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)), \quad \text{and} \\ C_\mu := \frac{Ne^{\nu r}}{1 - e^{-(\nu - \mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty). \end{aligned}$$

Proof. Denote by $C_b([s - r, \infty), X)$ the Banach space of bounded, continuous and X -valued functions defined on $[s - r, \infty)$, which is endowed with the sup-norm $\|\cdot\|_\infty$. Setting $v_0 := \phi(0)$ we consider the transformation T defined by

$$(Tu)(t) = \begin{cases} U(2s - t, s)v_0 + \int_s^\infty \mathcal{G}(2s - t, \tau)f(\tau, u_\tau)d\tau & \text{for } s - r \leq t \leq s, \\ U(t, s)v_0 + \int_s^\infty \mathcal{G}(t, \tau)f(\tau, u_\tau)d\tau & \text{for } t \geq s. \end{cases}$$

Since $v_0 \in P(0)X$, using the inequality (3.6) we can easily see that T acts from $C_b([s - r, \infty), X)$ into itself. We next prove that, if $k < 1$, then T is a contraction mapping. To do this, we estimate

$$\begin{aligned} \|(Tu)(t) - (Tv)(t)\| &\leq \int_s^\infty \|\mathcal{G}(t, \tau)(f(\tau, u_\tau) - f(\tau, v_\tau))\| d\tau \\ &\leq N(1 + H) \int_s^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \\ &\leq ke^{-\nu r} \sup_{t \geq s-r} \|u(t) - v(t)\| \quad \text{for } t \geq s, \end{aligned}$$

and

$$\begin{aligned}
\|(Tu)(t) - (Tv)(t)\| &\leq \int_s^\infty \|\mathcal{G}(2s-t, \tau)(f(\tau, u_\tau) - f(\tau, v_\tau))\| d\tau \\
&\leq N(1+H) \int_s^\infty e^{-\nu|2s-t-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \\
&\leq N(1+H) e^{\nu r} \int_s^\infty e^{-\nu|s-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}} d\tau \\
&\leq k \sup_{t \geq s-r} \|u(t) - v(t)\| \quad \text{for } -r+s \leq t \leq s.
\end{aligned}$$

Therefore, $\sup_{t \geq s-r} \|(Tu)(t) - (Tv)(t)\| \leq k \sup_{t \geq s-r} \|u(t) - v(t)\|$.

Hence, for $k < 1$ the transformation $T : C_b([s-r, \infty), X) \rightarrow C_b([s-r, \infty), X)$ is a contraction mapping. Thus, there exists a unique $u(\cdot) \in C([s-r, \infty), X)$ such that $Tu = u$. This yields that $u(t)$, $t \geq s-r$, is the unique solution of Eq. (3.5) with $u_s(\theta) = U(s-\theta, s)v_0 + \int_s^\infty \mathcal{G}(s-\theta, \tau)f(\tau, u_\tau)d\tau$ for all $\theta \in \mathcal{C}$, and $P(s)u(s) = v_0 = \phi(0)$. Therefore, $\tilde{P}(s)u_s = \phi$ by definition of $\tilde{P}(s)$ (see equality (3.1)).

Let $u(t), v(t)$ be the two solutions to Eq. (3.5) corresponding to different initial functions $\phi_1, \phi_2 \in \text{Im } \tilde{P}(s)$, respectively. Putting $v_1 := \phi_1(0)$, $v_2 := \phi_2(0)$ we have that

$$\|u(t) - v(t)\| \leq \begin{cases} Ne^{-\nu(t-s)}\|v_1 - v_2\| + N(1+H) \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}} d\tau & \text{if } t \geq s, \\ Ne^{-\nu(s-t)}\|v_1 - v_2\| + N(1+H) \int_s^\infty e^{-\nu|2s-t-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}} d\tau & \text{if } s-r \leq t \leq s. \end{cases}$$

Since $t+\theta \in [-r+t, t]$ for fixed $t \in [s, \infty)$ and $\theta \in [-r, 0]$, we obtain

$$\|u_t - v_t\|_{\mathcal{C}} \leq Ne^{\nu r} e^{-\nu(t-s)}\|v_1 - v_2\| + N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) \|u_\tau - v_\tau\|_{\mathcal{C}} d\tau, \quad t \geq s.$$

Put $h(t) = \|u_t - v_t\|_{\mathcal{C}}$. Then, $\sup_{t \geq s} h(t) < \infty$ and

$$h(t) \leq Ne^{\nu r} e^{-\nu(t-s)}\|v_1 - v_2\| + N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) h(\tau) d\tau, \quad t \geq s. \quad (3.8)$$

We will use the cone inequality theorem (see [9, Theorem 2.8]) applying to Banach space $L_\infty[s, \infty)$ which is the space of real-valued functions defined and essentially bounded on $[s, \infty)$ (endowed with the sup-norm denoted by $\|\cdot\|_\infty$) with the cone \mathcal{K} being the set of all nonnegative functions. We then consider the linear operator A defined for $g \in L_\infty[s, \infty)$ by

$$(Ag)(t) = N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) g(\tau) d\tau, \quad t \geq s.$$

By Proposition 2.6 we have that

$$\sup_{t \geq s} (Ag)(t) = \sup_{t \geq s} N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) g(\tau) d\tau \leq k \|g\|_\infty.$$

Therefore, $A \in \mathcal{L}(L_\infty[s, \infty))$ and $\|A\| \leq k < 1$. Obviously, the cone \mathcal{K} is invariant under the operator A . The inequality (3.8) can now be rewritten by

$$h \leq Ah + z \quad \text{for } z(t) = Ne^{\nu r} e^{-\nu(t-s)}\|v_1 - v_2\|.$$

By the cone inequality theorem [9, Theorem 2.8] we obtain that $h \leq g$, where g is a solution in $L_\infty[s, \infty)$ of the equation $g = Ag + z$ which can be rewritten as

$$g(t) = Ne^{\nu r} e^{-\nu(t-s)}\|v_1 - v_2\| + N(1+H)e^{\nu r} \int_s^\infty e^{-\nu|t-\tau|} \varphi(\tau) g(\tau) d\tau, \quad t \geq s \geq 0.$$

To estimate g , we put $w(t) = e^{\mu(t-s)}g(t)$ for $t \geq s \geq 0$. Then, we obtain that

$$w(t) = Ne^{\nu r}e^{-(\nu-\mu)(t-s)}\|\nu_1 - \nu_2\| + N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)w(\tau)d\tau. \quad (3.9)$$

We next consider the linear operator D defined on $L_\infty[s, \infty)$ as

$$(D\phi)(t) = N(1+H)e^{\nu r}\int_s^\infty e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)\phi(\tau)d\tau \quad \text{for all } t \geq s.$$

One can easily see that $D \in \mathcal{L}(L_\infty[s, \infty))$ and $\|D\| \leq \frac{N(1+H)e^{\nu r}}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+ \varphi\|_\infty + N_2\|\Lambda_1 \varphi\|_\infty)$. Eq. (3.9) can be rewritten as

$$w = Dw + \tilde{z} \quad \text{for } \tilde{z}(t) = Ne^{\nu r}e^{-(\nu-\mu)(t-s)}\|\nu_1 - \nu_2\|.$$

We have $\|D\| < 1$ if $0 < \mu < \nu + \ln(1 - N(1+H)e^{\nu r}(N_1\|\Lambda_1 T_1^+ \varphi\|_\infty + N_2\|\Lambda_1 \varphi\|_\infty))$. Under this condition, the equation $w = Dw + \tilde{z}$ has the unique solution $w \in L_\infty[s, \infty)$ and $w = (I - D)^{-1}\tilde{z}$. Hence, we obtain that

$$\begin{aligned} \|w\|_\infty &= \|(I - D)^{-1}\tilde{z}\|_\infty \leq \frac{Ne^{\nu r}}{1 - \|D\|}\|\nu_1 - \nu_2\| \\ &\leq \frac{Ne^{\nu r}\|\nu_1 - \nu_2\|}{1 - \frac{N(1+H)e^{\nu r}}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+ \varphi\|_\infty + N_2\|\Lambda_1 \varphi\|_\infty)} := C_\mu\|\nu_1 - \nu_2\|. \end{aligned}$$

This yields that $w(t) \leq C_\mu\|\nu_1 - \nu_2\|$ for $t \geq s$. Hence,

$$h(t) = \|u_t - v_t\|_C \leq g(t) = e^{-\mu(t-s)}w(t) \leq C_\mu e^{-\mu(t-s)}\|\nu_1 - \nu_2\| \quad \text{for } t \geq s. \quad \square$$

We now prove our main result of this section.

Theorem 3.7. *Let the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N, \nu > 0$. Suppose that φ is a positive function which belongs to the admissible space E . Let $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ be φ -Lipschitz satisfying $k < \frac{1}{1+Ne^{\nu r}}$ where k is defined by (3.7). Then, there exists an invariant stable manifold S for the solutions to Eq. (3.2).*

Proof. Since $\{U(t, s)\}_{t \geq s \geq 0}$ has an exponential dichotomy, we have that for each $t \geq 0$ the phase space \mathcal{C} splits into the direct sum $\mathcal{C} = \text{Im } \tilde{P}(t) \oplus \text{Ker } \tilde{P}(t)$ where the projections $\tilde{P}(t)$, $t \geq 0$, are defined as in equality (3.1). Clearly, $\sup_{t \geq 0} \|\tilde{P}(t)\| < \infty$. We now construct a stable manifold $S = \{(t, S_t)\}_{t \geq 0}$ for the solutions to Eq. (3.2). To do this, we determine the surface S_t for $t \geq 0$ by the formula

$$S_t := \{\phi + \Phi_t(\phi) : \phi \in \text{Im } \tilde{P}(t)\} \subset \mathcal{C}$$

where the operator Φ_{t_0} is defined for each $t_0 \geq 0$ by

$$\Phi_{t_0}(\phi)(\theta) = \int_{t_0}^\infty \mathcal{G}(t_0 - \theta, \tau) f(\tau, u_\tau) d\tau \quad \text{for all } \theta \in [-r, 0],$$

here $u(\cdot)$ is the unique solution of Eq. (3.2) on $[-r + t_0, \infty)$ satisfying $\tilde{P}(t_0)u_{t_0} = \phi$ (note that the existence and uniqueness of $u(\cdot)$ is guaranteed by Theorem 3.6). On the other hand, by the definition of the Green function \mathcal{G} (see Eq. (3.4)) we have that $\Phi_{t_0}(\phi) \in \text{Ker } \tilde{P}(t_0)$. We next show that the stable manifold S satisfies the conditions of Definition 3.3.

Firstly, we prove that Φ_{t_0} is Lipschitz continuity with Lipschitz constant independent of t_0 . Indeed, for ϕ_1 and ϕ_2 belonging to $\text{Im } \tilde{P}(t_0)$ we have

$$\begin{aligned} \|\Phi_{t_0}(\phi_1)(\theta) - \Phi_{t_0}(\phi_2)(\theta)\| &\leq N(1+H)\int_{t_0}^\infty e^{-\nu|t_0-\theta-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_C d\tau \\ &\leq N(1+H)e^{\nu r}\int_{t_0}^\infty e^{-\nu|t_0-\tau|}\varphi(\tau)\|u_\tau - v_\tau\|_C d\tau \\ &\leq N(1+H)e^{\nu r}\sup_{\tau \geq t_0}\|u_\tau - v_\tau\|_C \int_{t_0}^\infty e^{-\nu|t_0-\tau|}\varphi(\tau) d\tau \\ &\leq \frac{N(1+H)e^{\nu r}}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+ \varphi\|_\infty + N_2\|\Lambda_1 \varphi\|_\infty) \sup_{\tau \geq t_0}\|u_\tau - v_\tau\|_C. \end{aligned} \quad (3.10)$$

Moreover, by the Lyapunov–Perron equation for $u(\cdot)$ and $v(\cdot)$ (see Eq. (3.5)) we have

$$\sup_{\tau \geq t_0} \|u_\tau - v_\tau\|_{\mathcal{C}} \leq Ne^{\nu r} \|\phi_1 - \phi_2\|_{\mathcal{C}} + \frac{N(1+H)e^{\nu r}}{1-e^{-\nu}} (N_1 \|\Lambda_1 T_1^+ \varphi\|_\infty + N_2 \|\Lambda_1 \varphi\|_\infty) \sup_{\tau \geq t_0} \|u_\tau - v_\tau\|_{\mathcal{C}}.$$

It follows that

$$\sup_{\tau \geq t_0} \|u_\tau - v_\tau\|_{\mathcal{C}} \leq \frac{Ne^{\nu r}}{1-k} \|\phi_1 - \phi_2\|_{\mathcal{C}}.$$

Substituting this inequality into (3.10) we obtain that

$$\|\Phi_{t_0}(\phi_1) - \Phi_{t_0}(\phi_2)\|_{\mathcal{C}} = \sup_{\theta \in [-r, 0]} \|\Phi_{t_0}(\phi_1)(\theta) - \Phi_{t_0}(\phi_2)(\theta)\| \leq \frac{Nke^{\nu r}}{1-k} \|\phi_1 - \phi_2\|_{\mathcal{C}}.$$

Therefore, Φ_{t_0} is Lipschitz continuous with the Lipschitz constant $\frac{Nke^{\nu r}}{1-k}$ independent of t_0 .

To show that S_{t_0} is homeomorphic to $\text{Im } \tilde{P}(t_0)$. We define the transformation $\mathbf{D} : \text{Im } \tilde{P}(t_0) \rightarrow S_{t_0}$ by $\mathbf{D}\phi := \phi + \Phi_{t_0}(\phi)$ for all $\phi \in \text{Im } \tilde{P}(t_0)$. Then, applying the implicit function theorem for Lipschitz continuous mappings (see [3, Lemma 2.7]) we have that if the Lipschitz constant $\frac{Nke^{\nu r}}{1-k} < 1$ (equivalently $k < \frac{1}{1+Ne^{\nu r}}$) then \mathbf{D} is a homeomorphism. Therefore, the condition (ii) in Definition 3.3 is satisfied.

The condition (iii) in Definition 3.3 now follows from Theorem 3.6. We now prove that the condition (iv) of Definition 3.3 is satisfied. Indeed, let $u(\cdot)$ be solution of Eq. (3.2) such that the function $u_s(\theta) \in S_s$. Then, by Lemma 3.4, the solution $u(t)$ for $t \in [s, \infty)$ can be rewritten in the form

$$u(t) = U(t, s)v_0 + \int_s^\infty \mathcal{G}(t, \tau)f(\tau, u_\tau)d\tau \quad \text{for some } v_0 \in \text{Im } P(s).$$

Thus, for $t \geq s$ and $\theta \in [-r, 0]$ we have

$$\begin{aligned} u(t-\theta) &= U(t-\theta, s)v_0 + \int_s^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau \\ &= U(t-\theta, s)v_0 + \int_s^t \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau + \int_t^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau \\ &= U(t-\theta, s)v_0 + \int_s^t U(t-\theta, \tau)P(\tau)f(\tau, u_\tau)d\tau + \int_t^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau \\ &= U(t-\theta, t) \left[U(t, s)v_0 + \int_s^t U(t, \tau)P(\tau)f(\tau, u_\tau)d\tau \right] + \int_t^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau. \end{aligned}$$

Put $\mu_0 = U(t, s)v_0 + \int_s^t U(t, \tau)P(\tau)f(\tau, u_\tau)d\tau$. We have $P(t)\mu_0 = \mu_0$, hence $\mu_0 \in \text{Im } P(t)$. We thus obtain that $U(t-\theta, t)\mu_0$ belongs to $\text{Im } \tilde{P}(t)$ and

$$u(t-\theta) = U(t-\theta, t)\mu_0 + \int_t^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau.$$

By the uniqueness of $u(\cdot)$ on $[s-r, \infty)$ as in the proof of Theorem 3.6 we have that Eq. (3.2) has a unique solution $u(\cdot)$ on $[-r+t, \infty)$ satisfying $(\tilde{P}(t)u_t)(\theta) = U(t-\theta, t)\mu_0$ and

$$u(\xi) = U(2t-\xi, t)\mu_0 + \int_t^\infty \mathcal{G}(2\xi-t, \tau)f(\tau, u_\tau)d\tau$$

for $\xi \in [-r+t, t]$. Therefore, the history function u_t can be viewed as

$$u_t(\theta) = u(t+\theta) = U(t-\theta, t)\mu_0 + \int_t^\infty \mathcal{G}(t-\theta, \tau)f(\tau, u_\tau)d\tau = \phi(\theta) + \Phi_t(\phi)(\theta).$$

Hence, $u_t \in S_t$ for $t \geq s$. \square

4. Exponential trichotomy and center-stable manifolds

In this section, we will generalize [Theorem 3.7](#) to the case that the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ has an exponential trichotomy on \mathbb{R}_+ and the nonlinear forcing term f is φ -Lipschitz. In this case, under similar conditions as in the above section we will prove that there exists a center-stable manifold for the solutions to Eq. [\(3.2\)](#). We now recall the definition of an exponential trichotomy.

Definition 4.1. A given evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ is said to have an exponential trichotomy on the half-line if there are three families of projections $\{P_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, and positive constants N, α, β with $\alpha < \beta$ such that the following conditions are fulfilled:

- (i) $\sup_{t \geq 0} \|P_j(t)\| < \infty$, $j = 1, 2, 3$.
- (ii) $P_1(t) + P_2(t) + P_3(t) = Id$ for $t \geq 0$ and $P_j(t)P_i(t) = 0$ for all $j \neq i$.
- (iii) $P_j(t)U(t, s) = U(t, s)P_j(s)$ for $t \geq s \geq 0$ and $j = 1, 2, 3$.
- (iv) $U(t, s)|_{\text{Im } P_j(s)}$ are homeomorphisms from $\text{Im } P_j(s)$ onto $\text{Im } P_j(t)$, for all $t \geq s \geq 0$ and $j = 2, 3$, respectively; we also denote the inverse of $U(t, s)|_{\text{Im } P_j(s)}$ by $U(s, t)|_t$, $0 \leq s \leq t$.
- (v) For all $t \geq s \geq 0$ and $x \in X$, the following estimates hold:

$$\begin{aligned} \|U(t, s)P_1(s)x\| &\leq Ne^{-\beta(t-s)}\|P_1(s)x\|, \\ \|U(s, t)|_t P_2(t)x\| &\leq Ne^{-\beta(t-s)}\|P_2(t)x\|, \\ \|U(t, s)P_3(s)x\| &\leq Ne^{\alpha(t-s)}\|P_3(s)x\|. \end{aligned}$$

The projections $\{P_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, are called the *trichotomy projections*, and the constants N, α, β – the *trichotomy constants*.

Using the trichotomy projections we can now construct three families of projections $\{\tilde{P}_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, on \mathcal{C} as follows:

$$(\tilde{P}_j(t)\phi)(\theta) = U(t - \theta, t)P_j(t)\phi(0) \quad \text{for all } \theta \in [-r, 0] \text{ and } \phi \in \mathcal{C}. \quad (4.1)$$

We come to our second main result. It proves the existence of a center-stable manifold for solutions of Eq. [\(3.2\)](#).

Theorem 4.2. Let the evolution family $\{U(t, s)\}_{t \geq s \geq 0}$ have an exponential trichotomy with the trichotomy projections $\{P_j(t)\}_{t \geq 0}$, $j = 1, 2, 3$, and constants N, α, β given as in [Definition 4.1](#). Suppose that $f : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ is φ -Lipschitz, where φ is a positive function which belongs to the admissible space E . Set $q := \sup\{\|P_j(t)\| : t \geq 0, j = 1, 3\}$, $N_0 := \max\{N, 2Nq\}$, and

$$k := \frac{(1+H)e^{vr}N_0}{1-e^{-v}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty). \quad (4.2)$$

Then, if $k < \frac{1}{1+N_0e^{vr}}$, for each fixed $\delta > \alpha$ there exists a center-stable manifold $S = \{(t, S_t)\}_{t \geq 0} \subset \mathbb{R}_+ \times \mathcal{C}$ for the solutions to Eq. [\(3.2\)](#), which is represented by a family of Lipschitz continuous mappings

$$\Phi_t : \text{Im}(\tilde{P}_1(t) + \tilde{P}_3(t)) \rightarrow \text{Im} \tilde{P}_2(t)$$

with Lipschitz constants being independent of t such that $S_t = \text{graph}(\Phi_t)$ has the following properties:

- (i) S_t is homeomorphic to $\text{Im}(\tilde{P}_1(t) + \tilde{P}_3(t))$ for all $t \geq 0$.
- (ii) To each $\phi \in S_s$ there corresponds one and only one solution $u(t)$ to Eq. [\(3.2\)](#) on $[s-r, \infty)$ satisfying $e^{-\gamma(s+\theta)}u_s(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$ and $\sup_{t \geq s} \|e^{-\gamma(t+\cdot)}u_t(\cdot)\|_{\mathcal{C}} < \infty$, where $\gamma = \frac{\delta+\alpha}{2}$. Moreover, for any two solutions $u(t)$ and $v(t)$ to Eq. [\(3.2\)](#) corresponding to different functions $\phi_1, \phi_2 \in S_s$ we have the estimate

$$\|u_t - v_t\|_{\mathcal{C}} \leq C_\mu e^{(\gamma-\mu)(t-s)} \|(\tilde{P}(s)\phi_1)(0) - (\tilde{P}(s)\phi_2)(0)\| \quad \text{for } t \geq s \quad (4.3)$$

where μ and C_μ are positive constants independent of s , $u(\cdot)$, and $v(\cdot)$.

- (iii) S is positively invariant under Eq. [\(3.2\)](#) in the sense that, if $u(t)$, $t \geq s-r$, is the solution to Eq. [\(3.2\)](#) satisfying the conditions that the function $e^{-\gamma(s+\cdot)}u_s(\cdot) \in S_s$ and $\sup_{t \geq s} \|e^{-\gamma(t+\cdot)}u_t(\cdot)\|_{\mathcal{C}} < \infty$, then the function $e^{-\gamma(t+\cdot)}u_t(\cdot) \in S_t$ for all $t \geq s$.

Proof. Set $P(t) := P_1(t) + P_3(t)$ and $Q(t) := P_2(t) = Id - P(t)$ for $t \geq 0$. We have that $P(t)$ and $Q(t)$ are projections complemented to each other on X . We then define the families of projections $\{\tilde{P}_j(t)\}$, $t \geq 0$, $j = 1, 2, 3$, on \mathcal{C} as in equality [\(4.1\)](#). Setting $\tilde{P}(t) = \tilde{P}_1(t) + \tilde{P}_3(t)$ and $\tilde{Q}(t) = \tilde{P}_2(t)$, $t \geq 0$, we obtain that $\tilde{P}(t)$ and $\tilde{Q}(t)$ are complemented projections on \mathcal{C} for each $t \geq 0$. We consider the following rescaling evolution family

$$\tilde{U}(t, s) = e^{-\gamma(t-s)}U(t, s) \quad \text{for all } t \geq s \geq 0.$$

We now prove that the evolution family $\tilde{U}(t, s)$ has an exponential dichotomy with dichotomy projections $P(t)$, $t \geq 0$. Indeed,

$$\begin{aligned} P(t)\tilde{U}(t, s) &= e^{-\gamma(t-s)}(P_1(t) + P_3(t))U(t, s) \\ &= e^{-\gamma(t-s)}U(t, s)(P_1(s) + P_3(s)) = \tilde{U}(t, s)P(s). \end{aligned}$$

Since $U(t, s)|_{\text{Im } P_2(s)}$ is a homeomorphism from $\text{Im } P_2(s)$ onto $\text{Im } P_2(t)$ and $\text{Im } P_2(t) = \text{Ker } P(t)$ for all $t \geq 0$, we have that $\tilde{U}(t, s)|_{\text{Ker } P(s)}$ is also a homeomorphism from $\text{Ker } P(s)$ onto $\text{Ker } P(t)$, and we denote $\tilde{U}(s, t)| := (\tilde{U}(t, s)|_{\text{Ker } P(s)})^{-1}$ for $0 \leq s \leq t$. By the definition of exponential trichotomy we have

$$\|\tilde{U}(s, t)|Q(t)x\| \leq e^{-(\beta+\gamma)(t-s)}\|Q(t)x\| \quad \text{for all } t \geq s \geq 0.$$

On the other hand,

$$\begin{aligned} \|\tilde{U}(t, s)P(s)x\| &= e^{-\gamma(t-s)}\|U(t, s)(P_1(s) + P_3(s))x\| \\ &\leq Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)x\| + e^{\alpha(t-s)}\|P_3(s)x\|) \\ &= Ne^{-\gamma(t-s)}(e^{-\beta(t-s)}\|P_1(s)P(s)x\| + e^{\alpha(t-s)}\|P_3(s)P(s)x\|) \end{aligned}$$

for all $t \geq s \geq 0$ and $x \in X$.

Putting $q := \sup\{\|P_j(t)\|, t \geq 0, j = 1, 3\}$, we finally get the following estimate

$$\|\tilde{U}(t, s)P(s)x\| \leq 2Nqe^{-\frac{(\delta-\alpha)}{2}(t-s)}\|P(s)x\|.$$

Therefore, $\tilde{U}(t, s)$ has an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants $N_0 := \max\{N, 2Nq\}$, $v := \frac{\delta-\alpha}{2}$.

Put $\tilde{x}(t) = e^{-\gamma t}x(t)$, and define the mapping $F : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ as follows

$$F(t, \phi) = e^{-\gamma t}f(t, e^{\gamma(t+)}\phi(\cdot)) \quad \text{for } (t, \phi) \in \mathbb{R}_+ \times \mathcal{C}.$$

Obviously, F is also φ -Lipschitz. Thus, we can rewrite Eq. (3.2) in the new form

$$\begin{cases} \tilde{x}(t) = \tilde{U}(t, s)\tilde{x}(s) + \int_s^t \tilde{U}(t, \xi)F(\xi, \tilde{x}_\xi)d\xi \quad \text{for all } t \geq s \geq 0, \\ \tilde{x}_s(\cdot) = e^{-\gamma(s+)}\phi(\cdot) \in \mathcal{C}. \end{cases} \quad (4.4)$$

Hence, by Theorem 3.7, we obtain that, if $k < \frac{1}{1+N_0e^{\gamma r}}$, then there exists an invariant stable manifold S for the solutions to Eq. (4.4). Returning to Eq. (3.2) by using the relation $x(t) := e^{\gamma t}\tilde{x}(t)$ and Theorems 3.6, 3.7, we can easily verify the properties of S which are stated in (i), (ii), (iii) and (iv). Thus, S is a center-stable manifold for the solutions of Eq. (3.2). \square

Example 4.3. We consider the problem

$$\begin{cases} \frac{\partial}{\partial t}u(t, x) = \sum_{k,l=1}^n \frac{\partial}{\partial x_k} \left(a_{kl}(t, x) \frac{\partial}{\partial x_l}u(t, x) \right) + \delta u(t, x) + bte^{-\alpha t} \int_{-r}^0 \ln(1 + |u(t+\theta, x)|)d\theta \quad \text{for } t \geq s \geq 0, x \in \Omega, \\ \sum_{k,l=1}^n n_k(x)a_{kl}(t, x) \frac{\partial}{\partial x_l}u(t, x) = 0, \quad x \in \partial\Omega, \\ u_s(\theta, x) = u(s+\theta, x) = \phi(\theta, x), \quad \theta \in [-r, 0], x \in \Omega. \end{cases} \quad (4.5)$$

Here, Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$ oriented by outer unit normal vector $n(x)$. The coefficients $a_{kl}(t, x) \in C_b^\mu(\mathbb{R}_+, C(\overline{\Omega})) \cap C_b(\mathbb{R}_+, C^1(\overline{\Omega}))$, $\mu > \frac{1}{2}$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^n a_{kl}(t, x)v_kv_l \geq \eta|v|^2, \quad \text{for all } x \in \Omega \text{ and for some constant } \eta > 0.$$

Finally, the constants $\alpha > 0$, $b \neq 0$ and $\delta > 0$ is large enough. We now choose the Hilbert space $X = L_2(\Omega)$ and define the differential operator

$$A(t, x, D) = \sum_{k,l=1}^n \frac{\partial}{\partial x_k} \left(a_{kl}(t, x) \frac{\partial}{\partial x_l} \right) + \delta$$

with domain

$$D(A(t)) = \left\{ f \in W^{2,2}(\Omega) : \sum_{k,l=1}^n n_k(x) a_{kl}(t, x) \frac{\partial}{\partial x_l} f(x) = 0, x \in \partial\Omega \right\}.$$

Therefore, this problem can rewrite as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt} u(t, \cdot) = A(t)u(t, \cdot) + F(t, u_t(\theta, \cdot)) & \text{for } t \geq s \geq 0, \\ u_s(\theta, \cdot) = \phi(\theta, \cdot) \in \mathcal{C} & \text{for } \theta \in [-r, 0] \end{cases}$$

where $F : \mathbb{R}_+ \times \mathcal{C} \rightarrow X$ is defined by

$$F(t, \phi)(x) = bte^{-\alpha t} \int_{-r}^0 \ln(1 + |(\phi(\theta))(x)|) d\theta, \quad x \in \Omega.$$

We have $F(t, \phi)(\cdot) \in X$ because by Minkowski's inequality it follows that

$$\begin{aligned} \left(\int_{\Omega} |F(t, \phi)(x)|^2 dx \right)^{\frac{1}{2}} &= |b|te^{-\alpha t} \left(\int_{\Omega} \left(\int_{-r}^0 \ln(1 + |(\phi(\theta))(x)|) d\theta \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq |b|te^{-\alpha t} \int_{-r}^0 \left(\int_{\Omega} \ln^2(1 + |(\phi(\theta))(x)|) dx \right)^{\frac{1}{2}} d\theta \\ &\leq |b|te^{-\alpha t} \int_{-r}^0 \left(\int_{\Omega} |(\phi(\theta))(x)|^2 dx \right)^{\frac{1}{2}} d\theta \\ &= |b|te^{-\alpha t} \int_{-r}^0 \|\phi(\theta)\|_2 d\theta < \infty. \end{aligned}$$

By Schnaubelt [13, Theorem 3.3, Example 4.2], the family of operators $(A(t))_{t \geq 0}$ generates an evolution family having an exponential dichotomy with the dichotomy constants N, ν provided that the Hölder constants of a_{kl} are sufficiently small. Also, the dichotomy projections $P(t)$, $t \geq 0$, satisfy $\sup_{t \geq 0} \|P(t)\| \leq N$.

We now check that F is φ -Lipschitz with $\varphi(t) = |b|rte^{-\alpha t} \in E = L_p(\mathbb{R}_+)$, $p \geq 1$. Indeed, the condition (i) is evident. To verify the condition (ii) we use Minkowski's inequality and the fact that $\ln(1 + h) \leq h$ for all $h \geq 0$. Then,

$$\begin{aligned} \|F(t, \phi_1)(x) - F(t, \phi_2)(x)\|_2 &= |b|te^{-\alpha t} \left(\int_{\Omega} \left(\int_{-r}^0 \ln \frac{1 + |(\phi_1(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} d\theta \right)^2 dx \right)^{\frac{1}{2}} \\ &\leq |b|te^{-\alpha t} \int_{-r}^0 \left(\int_{\Omega} \ln^2 \frac{1 + |(\phi_1(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} dx \right)^{\frac{1}{2}} d\theta \\ &= |b|te^{-\alpha t} \int_{-r}^0 \left(\int_{\Omega} \ln^2 \left(1 + \frac{|(\phi_1(\theta))(x)| - |(\phi_2(\theta))(x)|}{1 + |(\phi_2(\theta))(x)|} \right) dx \right)^{\frac{1}{2}} d\theta \\ &\leq |b|te^{-\alpha t} \int_{-r}^0 \left(\int_{\Omega} |(\phi_1(\theta))(x) - (\phi_2(\theta))(x)|^2 dx \right)^{\frac{1}{2}} d\theta \\ &= |b|te^{-\alpha t} \int_{-r}^0 \|\phi_1(\theta) - \phi_2(\theta)\|_2 d\theta \\ &\leq |b|rte^{-\alpha t} \sup_{\theta \in [-r, 0]} \|\phi_1(\theta) - \phi_2(\theta)\|_2. \end{aligned}$$

Hence, F is φ -Lipschitz. In the space $L_p(\mathbb{R}_+)$, the constants N_1, N_2 in [Definition 2.3](#) are defined by $N_1 = N_2 = 1$. Also, we have

$$\Lambda_1 \varphi(t) = \int_t^{t+1} \varphi(\tau) d\tau \quad \text{and} \quad \Lambda_1 T_1^+ \varphi(t) = \int_{(t-1)_+}^t \varphi(\tau) d\tau$$

where $(t-1)_+ = \max\{0, t-1\}$. Thus,

$$\max\{\|\Lambda_1 \varphi\|_\infty, \|\Lambda_1 T_1^+ \varphi\|_\infty\} < \frac{|b|r(1 + e^{-1} - e^{-\alpha})}{\alpha^2}.$$

By [Theorem 3.7](#) we obtain that if

$$\frac{|b|r(1 + e^{-1} - e^{-\alpha})}{\alpha^2} \leq \frac{e^{-\nu r}(1 - e^{-\nu})}{2N(1 + N)(1 + Ne^{\nu r})}$$

then there is an invariant stable manifold S for the mild solutions to the problem [\(4.5\)](#).

References

- [1] B. Aulbach, N.V. Minh, Nonlinear semigroups and the existence and stability of semilinear nonautonomous evolution equations, *Abstr. Appl. Anal.* 1 (1996) 351–380.
- [2] J.J. Massera, J.J. Schäffer, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.
- [3] N.V. Minh, J. Wu, Invariant manifolds of partial functional differential equations, *J. Differential Equations* 198 (2004) 381–421.
- [4] N.V. Minh, F. Räbiger, R. Schnaubelt, Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half line, *Integral Equations Operator Theory* 32 (1998) 332–353.
- [5] J.D. Murray, *Mathematical Biology I: An Introduction*, Springer-Verlag, Berlin, 2002.
- [6] J.D. Murray, *Mathematical Biology II: Spatial Models and Biomedical Applications*, Springer-Verlag, Berlin, 2003.
- [7] R. Nagel, G. Nickel, Well-posedness for non-autonomous abstract Cauchy problems, *Progr. Nonlinear Differential Equations Appl.* 50 (2002) 279–293.
- [8] Nguyen Thieu Huy, Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line, *J. Funct. Anal.* 235 (2006) 330–354.
- [9] Nguyen Thieu Huy, Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line, *J. Math. Anal. Appl.* 354 (2009) 372–386.
- [10] Nguyen Thieu Huy, Trinh Viet Duoc, Integral manifolds and their attraction property for evolution equations in admissible function spaces, *Taiwanese J. Math.* 16 (2012) 963–985.
- [11] A. Pazy, *Semigroup of Linear Operators and Application to Partial Differential Equations*, Springer-Verlag, Berlin, 1983.
- [12] F. Räbiger, R. Schnaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, *Semigroup Forum* 48 (1996) 225–239.
- [13] R. Schnaubelt, Asymptotically autonomous parabolic evolution equations, *J. Evol. Equ.* 1 (2001) 19–37.
- [14] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer Verlag, 1996.
- [15] A. Yagi, *Abstract Parabolic Evolution Equations and Their Applications*, Springer Verlag, 2009.