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Positive solutions for boundary value problems of second order difference equations and their computation

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ABSTRACT

We consider the following two classes of second order boundary value problems for difference equation:

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leqslant i \leqslant n, \qquad y_0 - \tau y_1 = y_{n+1} - \delta y_n = 0$$

with $\delta, \tau \in [0,1]$ and

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leqslant i \leqslant n, \qquad y_0 = \alpha y_n, \qquad y_{n+1} = \beta y_1$$

with $\alpha, \beta \in [0,1]$. We establish the existence of positive solutions to both problems. A solver with linear computational complexity for almost tridiagonal linear systems is developed by exploring the special structure of linear system of equations. Based on fast solvers for linear systems, effective algorithms for the computation of positive solutions will be proposed.

1. Introduction

In this paper, we consider the following two classes of second order boundary value problems for difference equations:

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leqslant i \leqslant n, \tag{1.1a}$$

$$y_0 - \tau y_1 = y_{n+1} - \delta y_n = 0, \tag{1.1b}$$

and

$$\Delta(r_{i-1}\Delta y_{i-1}) - b_i y_i + \lambda a_i y_i = 0, \quad 1 \leqslant i \leqslant n, \tag{1.2a}$$

$$y_0 = \alpha y_n, \qquad y_{n+1} = \beta y_1, \tag{1.2b}$$

where the forward difference operator Δ is defined as $\Delta y_i = y_{i+1} - y_i$.

Throughout we assume that $n \geqslant 3$ is a fixed integer and the following condition holds:

(H1) The $\{r_i\}_{i=0}^n$ and $\{b_i\}_{i=1}^n$ are finite sequences of real numbers such that $r_i > 0$ for $0 \leqslant i \leqslant n$ and $b_i \geqslant 0$ for $1 \leqslant i \leqslant n$. The α, β, τ, and δ are constants in $[0,1]$.

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The boundary conditions in (1.1b) and (1.2b) include many important cases. Eq. (1.1b) with $\tau = \delta = 0$ (also (1.2b) with $\alpha = \beta = 0$) is the Dirichlet boundary conditions. If $\tau = \delta = 1$, then (1.1b) becomes the Neumann boundary conditions. If $\alpha = \beta = 1$, then (1.2b) becomes the periodic boundary conditions.

If λ is a number (maybe complex) such that a boundary value problem, (1.1) or (1.2), has a nontrivial solution $\{y_i\}_{i=0}^{n+1}$, then λ is said to be an eigenvalue of the problem, and the corresponding nontrivial solution $\{y_i\}_{i=0}^{n+1}$ is called an eigenvector of the problem corresponding to λ.

The structures of the eigenvalues and the monotonicity of all eigenvalues of these problems as the coefficients $\{a_i\}$, $\{b_i\}$, and $\{r_i\}$ change have recently been studied in [2–5]. For problem (1.2), only the case when $r_0 = r_n$ and $\alpha = \beta$ has been analyzed in [2]. In addition, the existence of positive solution to the problem (1.1) was established in [5] for $\tau = 0$ and nonzero non-negative sequences $\{a_i\}$ and $\{b_i\}$. In this paper, we will focus on the existence and computation of positive solutions and establish the existence of positive solutions to both problems for a wider range of the parameters. Algorithms for the computation of positive solutions will be proposed. In the design of our proposed algorithms, we will fully explore the structures of the problems. The Crout factorization algorithm for tridiagonal systems is utilized in the algorithm for problem (1.1). A solver for almost tridiagonal linear systems will be developed following the steps of Crout. Due to the linear computational complexities for our linear solvers, the algorithms proposed are very effective for large scale problems.

2. Existence of positive solutions

Define

$$A = \mathrm{diag}(a_1, a_2, \dots, a_{n-1}, a_n),$$
$$B = \mathrm{diag}(b_1, b_2, \dots, b_{n-1}, b_n),$$
$$y = (y_1, y_2, \dots, y_{n-1}, y_n)^T.$$

Then the problem (1.1) is equivalent to the equation

$$(-G_1 + \lambda A)y = 0, \tag{2.3}$$

where $G_1 = D_1 + B$ and D_1 is the $n \times n$ matrix given by

$$D_1 = \begin{pmatrix} d_{11} & -r_1 & 0 & \cdots & 0 & 0 & 0 \\ -r_1 & r_1 + r_2 & -r_2 & \cdots & 0 & 0 & 0 \\ 0 & -r_2 & r_2 + r_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & r_{n-3} + r_{n-2} & -r_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -r_{n-2} & r_{n-2} + r_{n-1} & -r_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -r_{n-1} & d_{nn} \end{pmatrix},$$

$$d_{11} = (1 - \tau)r_0 + r_1,$$
$$d_{nn} = r_{n-1} + (1 - \delta)r_n.$$

The problem (1.2) is equivalent to the equation

$$(-G_2 + \lambda A)y = 0, \tag{2.4}$$

where $G_2 = D_2 + B$ and D_2 is the $n \times n$ matrix given by

$$D_2 = \begin{pmatrix} r_0 + r_1 & -r_1 & 0 & \cdots & 0 & 0 & -\alpha r_0 \\ -r_1 & r_1 + r_2 & -r_2 & \cdots & 0 & 0 & 0 \\ 0 & -r_2 & r_2 + r_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & r_{n-3} + r_{n-2} & -r_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & -r_{n-2} & r_{n-2} + r_{n-1} & -r_{n-1} \\ -\beta r_n & 0 & 0 & \cdots & 0 & -r_{n-1} & r_{n-1} + r_n \end{pmatrix}.$$

Obviously, from a solution $\{y_0, y_1, \dots, y_n, y_{n+1}\}$ to the problem (1.1) corresponding to λ, a solution $y = (y_1, y_2, \dots, y_n)^T$ to the problem (2.3) corresponding to λ can be obtained. On the other hand, if $y = (y_1, y_2, \dots, y_n)^T$ is a solution to the problem (2.3) corresponding to λ, then $\{\tau y_1, y_1, \dots, y_n, \delta y_n\}$ is a solution to the problem (1.1) corresponding to λ. Similarly, from a solution $\{y_0, y_1, \dots, y_n, y_{n+1}\}$ to the problem (1.2) corresponding to λ, a solution $y = (y_1, y_2, \dots, y_n)^T$ to the problem (2.4) corresponding to λ can be obtained. On the other hand, if $y = (y_1, y_2, \dots, y_n)^T$ is a solution to the problem (2.4) corresponding to λ, then $\{\alpha y_n, y_1, \dots, y_n, \beta y_1\}$ is a solution to the problem (1.2) corresponding to λ. In this sense, problems (1.1) and (1.2) are equivalent to (2.3) and (2.4), respectively. In this paper we will work with Eqs. (2.3) and (2.4).

Next, we will focus on the positivity of the elements of the inverse matrices of G_1 and G_2. To this end, in what follows we will write $X=(x_{ij}) \geqslant Y=(y_{ij})$ if $x_{ij} \geqslant y_{ij}$, for all i, j, and write $X=(x_{ij}) > Y=(y_{ij})$ if $x_{ij} > y_{ij}$, for all i, j. A matrix is said to be positive if each element of the matrix is positive.

Lemma 2.1. *Under the hypotheses of* (H1), *we have*:

(1) *If* $\tau+\delta \neq 2$ *or*

$$\tau=\delta=1, \qquad B \neq 0,$$

then G_1^{-1} *exists and is positive.*

(2) *If* $\alpha+\beta \neq 2$ *or*

$$\alpha=\beta=1, \qquad B \neq 0,$$

then G_2^{-1} *exists and is positive.*

Moreover, the elements of both G_1^{-1} *and* G_2^{-1} *are non-increasing when any member of*

$$\{b_1, b_2, \ldots, b_n\}$$

increases.

Proof. We only prove the results on G_2 and the ones on G_1 can be proved similarly. Let K_2 be the diagonal matrix whose diagonal elements are the same as those of G_2. Obviously, K_2 and K_2^{-1} have positive diagonal elements. Define $J_2 = I - K_2^{-1}G_2 \geqslant 0$, the Jacobian matrix of G_2. It is easily seen that the matrix G_2 is irreducibly diagonally dominant under the condition (2). Thus, G_2 is non-singular and $\rho(J_2) < 1$ which further imply

$$G_2^{-1} = (I-J_2)^{-1}K_2^{-1} = \left(\sum_{j=0}^{\infty} J_2^j\right)K_2^{-1}. \tag{2.5}$$

Define

$$\xi_1 = \frac{\alpha r_0}{r_0+r_1+b_1}, \qquad \xi_i = \frac{r_{i-1}}{r_{i-1}+r_i+b_i}, \quad \text{for } 2 \leqslant i \leqslant n,$$
$$\zeta_n = \frac{\beta r_n}{r_{n-1}+r_n+b_n}, \qquad \zeta_i = \frac{r_i}{r_{i-1}+r_i+b_i}, \quad \text{for } 1 \leqslant i \leqslant n-1,$$

and

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$J_2 = I - K_2^{-1}G_2 = \begin{pmatrix} 0 & \zeta_1 & 0 & \cdots & 0 & 0 & \xi_1 \\ \xi_2 & 0 & \zeta_2 & \cdots & 0 & 0 & 0 \\ 0 & \xi_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \zeta_{n-2} & 0 \\ 0 & 0 & 0 & \cdots & \xi_{n-1} & 0 & \zeta_{n-1} \\ \zeta_n & 0 & 0 & \cdots & 0 & \xi_n & 0 \end{pmatrix} \geqslant \eta\big(J+J^T\big) \geqslant 0, \tag{2.6}$$

where $\eta = \min\{\xi_2, \xi_3, \ldots, \xi_n, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}\} > 0$. It is seen from (2.5)–(2.6), together with the facts that $0 < \eta < 1$, $\xi_1 \geqslant 0$, and $\zeta_n \geqslant 0$, that

$$G_2^{-1} \geqslant \left(\sum_{j=0}^{n-1} J_2^j\right) K_2^{-1} \geqslant \left(I + \sum_{j=1}^{n-1} \eta^j \left(J^j + \left(J^T\right)^j\right)\right) K_2^{-1}$$
$$\geqslant \eta^{n-1}\left(I + \sum_{j=1}^{n-1} \left(J^j + \left(J^T\right)^j\right)\right) K_2^{-1} = \eta^{n-1} ee^T K_2^{-1} > 0,$$

where e is a column vector of all ones.

If any of the diagonal elements of B increases, then the ξ_i, ζ_i $(i = 1, 2, \ldots, n)$ and the diagonal elements of K_2^{-1} will decrease. Therefore, the monotonicity of elements of G_2^{-1} comes from the monotonicity of elements of J_2 and K_2^{-1}, then Eq. (2.5), and the non-negativity of J_2 and K_2^{-1}. □

Lemma 2.2. *Under the conditions of Lemma* 2.1, *each eigenvalue of the problems* (2.3) *and* (2.4) *is nonzero.*

Proof. Let λ be any eigenvalue of the problem (2.3) and y be an eigenvector corresponding to λ. If $\lambda = 0$, then $G_1 y = \lambda A y = 0$ implying $y = 0$ in view of Lemma 2.1, contracting the fact that $y \neq 0$. The result for problem (2.4) can be established in the same way. Thus, the proof is complete. □

Next we study the existence of positive eigenvector corresponding to the eigenvalue of problems (2.3) and (2.4) with minimum module and the monotonic behavior of the eigenvalue with minimum module as the sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ change. To this end, we further assume the following condition:

(H2) The $\{a_i\}_{i=1}^n$ is a finite sequence of real numbers such that $a_i \geqslant 0$ for $1 \leqslant i \leqslant n$. And there is at least one nonzero member of the sequence $\{a_i\}_{i=1}^n$.

If every member of the sequence $\{a_i\}_{i=1}^n$ is zero, then (2.3) reduces to

$$G_1 y = 0,$$

which is no longer an eigenvalue problem since the only solution to $G_1 y = 0$ is $y = 0$. This justifies our requirement of having at least one nonzero a_{i_0} $(1 \leqslant i_0 \leqslant n)$ in the assumption (H2).

As a tool, we will employ the following well-known Perron–Frobenius theorem. For a proof, please refer to [6, p. 30].

Theorem 2.3 *(Perron–Frobenius). Let $C \geqslant 0$ be an irreducible square matrix. Then*:

1. *C has a positive eigenvalue equal to its spectral radius $\rho(C)$.*
2. *To $\rho(C)$ there corresponds an eigenvector $x > 0$.*
3. *$\rho(C)$ increases when any entry of C increases.*
4. *$\rho(C)$ is a simple eigenvalue of C.*

Theorem 2.4. *Assume the hypotheses of* (H1)–(H2) *and the conditions of Lemma* 2.1. *If λ_1 is the eigenvalue of problem* (2.3) *or problem* (2.4) *with minimum module, then*:

1. *λ_1 is simple and positive, and there exists an eigenvector $y > 0$ corresponding to λ_1.*
2. *For a fixed sequence $\{r_i\}_{i=0}^n$, λ_1 is monotonically decreasing when any of the positive entries of A increases while it is monotonically non-decreasing when any entry of $\{b_1, b_2, \ldots, b_n\}$ increases.*

Proof. Let G stand for either G_1 or G_2. We note that $\lambda_1 \neq 0$ in view of Lemma 2.2 and

$$G^{-1} A y = \frac{1}{\lambda_1} y. \tag{2.7}$$

Thus, $1/\lambda_1$ is the eigenvalue of $G^{-1}A$ with maximum module and y is a corresponding eigenvector. There exists a permutation matrix P such that

$$P^T A P = \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix},$$

where $Z = \operatorname{diag}(\bar{a}_1, \ldots, \bar{a}_t)$, and $\bar{a}_1, \ldots, \bar{a}_t$ are the positive elements in the set

$$\{a_1, a_2, \ldots, a_n\}.$$

Then, $(-G+\lambda A)y=0$ becomes

$$\frac{1}{\lambda}P^Ty=P^TG^{-1}P\begin{pmatrix} Z & 0\\ 0 & 0\end{pmatrix}P^Ty. \tag{2.8}$$

Note

$$P^TG^{-1}P\begin{pmatrix} Z & 0\\ 0 & 0\end{pmatrix}$$

is in the form

$$\begin{pmatrix} W & 0\\ V & 0\end{pmatrix},$$

where W is non-singular and both W and V are positive matrices in view of Lemma 2.1. Thus, $1/\lambda_1$ is the eigenvalue of W with maximum module. Therefore, Theorem 2.3 indicates that $1/\lambda_1$ is simple and positive, and that there exists a positive eigenvector u_1 of W corresponding to $1/\lambda_1$. Finally, we see that

$$\hat{y}\equiv P\begin{pmatrix} u_1\\ \lambda_1 V u_1\end{pmatrix}$$

is a positive eigenvector of $(-G+\lambda A)y=0$ corresponding to λ_1. Therefore, part (1) follows immediately.

When any of the positive entries of A increases, the entries of Z increase and the same P in (2.8) can be employed in the analysis. Consequently, the entries of W increase as well due to the fact that $P^TG^{-1}P$ is positive in view of Lemma 2.1. Therefore, $1/\lambda_1$ increases in this case in view of Theorem 2.3. Similarly, when any diagonal entry of B increases, it is seen from Lemma 2.1 that entries of G^{-1} are positive and monotonically non-increasing and so are entries of W. Again, Lemma 2.3 indicates that $1/\lambda_1$ is monotonically non-increasing. Thus, part (2) follows immediately for either case. □

3. Computation of positive solutions

In this section we will focus on the computation of a positive solution to problems (2.3) and (2.4) corresponding to the eigenvalue λ_1 with minimum module. Let G be either G_1 or G_2. In view of Theorem 2.4, any positive eigenvector of $G^{-1}A$ corresponding to the eigenvalue $1/\lambda_1$ is such a solution for the least eigenvalue λ_1. Due to the fact that $1/\lambda_1$ is simple and dominant, a positive solution can be found by using Power method with any initial vector $y^{(0)}$ if its representation in terms of the eigenvectors of the matrix $G^{-1}A$ contains a nonzero contribution from the eigenvector associated with the dominant eigenvalue. Obviously, $y^{(0)}=e$, a vector of all ones, guarantees non-orthogonality with any positive vector. Therefore, the Power method with the initial vector e

$$y^{(0)}=e,\qquad z^{(k)}=G^{-1}Ay^{(k-1)},\qquad \lambda^{(k)}=\left\|z^{(k)}\right\|_\infty,\qquad y^{(k)}=z^{(k)}/\lambda^{(k)} \tag{3.9}$$

is certainly convergent to a positive solution of the problem. Note that $z^{(k)}=G^{-1}Ay^{(k-1)}$ is equivalent to the system of linear equations:

$$Gz^{(k)}=Ay^{(k-1)}. \tag{3.10}$$

When $G=G_1$, (3.10) is a tridiagonal linear system which can be solved very effectively with Crout factorization algorithm [1, p. 408]. It is easily seen that G_1 is positive definite under the conditions of Lemma 2.1 (also see [3,4]). Therefore, the Crout factorization algorithm is always successful. We mention that the tridiagonal linear system $G_1z=d$ can be solved in only $(5n-4)$ multiplications or divisions. Consequently, this has a considerable computational advantage over methods that do not explore the tridiagonality of the coefficient matrix. Next, we present an algorithm by combining the Power method in (3.9) with the Crout factorization algorithm.

Algorithm 1. The Power method for problem (1.1).

Step 0 Input $\tau,\delta,\{r_i\},\{a_i\},\{b_i\}$ and set $\epsilon=10^{-6}$, $y=e$, a vector of all ones.
Step 1 Repeat
  Step 1.1 Solve $G_1z=Ay$ with Crout factorization algorithm.
  Step 1.2 Set $\lambda=\|z\|_\infty$ and $y^+=z/\lambda$.
  Step 1.3 If $\|y^+-y\|_2<\epsilon$, then stop and goto Step 2
    else set $y=y^+$ and goto Step 1.1.
Step 2 Output a positive solution y^+ and its associated eigenvalue $1/\lambda$.

Now, we turn our attention to the problem (2.4). If $G = G_2$, then (3.10) becomes

$$G_2 z^{(k)} = A y^{(k-1)}. \tag{3.11}$$

Note that G_2 is no longer a tridiagonal matrix but it does have a special LU decomposition. In this special LU decomposition of G_2, L is a lower triangular matrix and U is an upper triangular matrix with 1's along the diagonal. For the matrix G_2, it is easily seen that the leading principal minor $G_2[k,k]$ of order $k \leqslant n-1$ is positive definite and $\det(G_2[k,k]) = l_{11} l_{22} \cdots l_{kk}$. Thus, we have $l_{11} = \det(G_2[1,1]) = g_{11} = r_0 + r_1 + b_1 > 0$ and deductively, we have $l_{kk} = \det(G_2[k,k])/\det(G_2[k-1,k-1]) > 0$ for $2 \leqslant k \leqslant n-1$. Finally, it is seen from Lemma 2.1 that $l_{nn} \neq 0$. Therefore, G_2 has a unique LU decomposition. Performing the LU decomposition on G_2, we find that all elements in L are zeros except for those in (i,i), (n,i), $i = 1,2,\dots,n$ and $(i+1,i)$, $i = 1,2,\dots,n-1$; while all elements in U are zeros except for those in (i,i), (i,n), $i = 1,2,\dots,n$ and $(i,i+1)$, $i = 1,2,\dots,n-1$. For simplicity, let us denote l_{ii} by l_i. Then L and U can be found in the form

$$L = \begin{pmatrix} l_1 & 0 & \cdots & 0 & 0 \\ l_{21} & l_2 & 0 & 0 & 0 \\ 0 & l_{32} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & l_{n-1} & 0 \\ l_{n1} & l_{n2} & \cdots & l_{n,n-1} & l_n \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} 1 & u_{12} & 0 & \cdots & 0 & u_{1n} \\ 0 & 1 & u_{23} & \cdots & 0 & u_{2n} \\ 0 & 0 & 1 & \cdots & 0 & u_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & u_{n-1,n} \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$

The multiplication involved with $(g_{ij}) = G_2 = LU$ gives the following equations for $(5n-6)$ nonzero entries in L and U:

$$g_{11} = l_1, \qquad g_{n1} = l_{n1}, \tag{3.12}$$

$$g_{i,i-1} = l_{i,i-1}, \quad \text{for } i = 2,3,\dots,n-1, \tag{3.13}$$

$$g_{ii} = l_{i,i-1} u_{i-1,i} + l_i, \quad \text{for } i = 2,3,\dots,n-1, \tag{3.14}$$

$$g_{i,i+1} = l_i u_{i,i+1}, \quad \text{for } i = 1,2,\dots,n-2, \tag{3.15}$$

$$0 = l_{n,i-1} u_{i-1,i} + l_{ni}, \quad \text{for } i = 2,3,\dots,n-2, \tag{3.16}$$

$$g_{n,n-1} = l_{n,n-2} u_{n-2,n-1} + l_{n,n-1}, \tag{3.17}$$

$$g_{1n} = l_1 u_{1n}, \tag{3.18}$$

$$0 = l_{i,i-1} u_{i-1,n} + l_i u_{in}, \quad \text{for } i = 2,3,\dots,n-2, \tag{3.19}$$

$$g_{n-1,n} = l_{n-1,n-2} u_{n-2,n} + l_{n-1} u_{n-1,n}, \tag{3.20}$$

$$g_{nn} = \sum_{i=1}^{n-1} l_{ni} u_{in} + l_n. \tag{3.21}$$

A solution to this system is found by first using (3.13) to obtain the nonzero entries $l_{i,i-1}$, $i = 2,3,\dots,n-1$ in L and then using the other equations to alternatively obtain entries l_i, l_{ni} in L and entries $u_{i,i+1}$, u_{in} in U. After factorizing G_2 into specially structured L and U, we solve two triangular systems of linear equations $Lx = d$ and $Uz = x$ to get the solution of $G_2 z = d$. Such a method for the solution to $G_2 z = d$ is called Crout-like factorization algorithm in what follows.

Procedure 1. Crout-like factorization algorithm for $G_2 z = d$.

Step 0 Input the dimension n and the entries of d and $G_2 = (g_{ij})$.
Step 1 Set $l_1 = g_{11}$, $l_{21} = g_{21}$, $l_{n1} = g_{n1}$, $u_{12} = g_{12}/l_1$, $u_{1n} = g_{1n}/l_1$, $x_1 = d_1/l_1$.
Step 2 For $i = 2,\dots,n-2$ set

$$l_i = g_{ii} - l_{i,i-1} u_{i-1,i}, \qquad l_{i+1,i} = g_{i+1,i}, \qquad l_{ni} = -l_{n,i-1} u_{i-1,i},$$

$$u_{i,i+1} = g_{i,i+1}/l_i, \qquad u_{in} = -l_{i,i-1} u_{i-1,n}/l_i,$$

$$x_i = (d_i - l_{i,i-1} x_{i-1})/l_i.$$

Step 3 Set

$$l_{n-1} = g_{n-1,n-1} - l_{n-1,n-2} u_{n-2,n-1},$$

$$l_{n,n-1} = g_{n,n-1} - l_{n,n-2} u_{n-2,n-1},$$

$$u_{n-1,n} = (g_{n-1,n} - l_{n-1,n-2} u_{n-2,n})/l_{n-1},$$

$$l_n = g_{nn} - \sum_{i=1}^{n-1} l_{ni} u_{in},$$
$$x_{n-1} = (d_{n-1} - l_{n-1,n-2} x_{n-2})/l_{n-1},$$
$$x_n = \left(d_n - \sum_{k=1}^{n-1} l_{nk} x_k\right)/l_n.$$

Step 4 Set $z_n = x_n$, $z_{n-1} = x_{n-1} - u_{n-1,n} z_n$.
Step 5 For $i = n-2, \dots, 1$ set $z_i = x_i - u_{i,i+1} z_{i+1} - u_{in} z_n$.
Step 6 Output the solution z to $G_2 z = d$.

Like Crout factorization algorithm for tridiagonal linear systems, this method is very effective by avoiding computing many zeros in the regular LU factorization method. To see this, let us count the total number of multiplications and divisions by ignoring the additions and subtractions. There are three divisions in Step 1, $7(n-3)$ multiplications or divisions in Step 2, $(2n+3)$ multiplications or divisions in Step 3, one multiplication in Step 4, and $2(n-2)$ multiplications in Step 5. Thus, the linear system $G_2 z = d$ can be solved in only $(11n - 18)$ multiplications or divisions.

Combining the Power method in (3.9) with the Crout-like procedure described above, we have the following algorithm for a positive solution to problem (1.2).

Algorithm 2. The Power method for problem (1.2).

Step 0 Input $\alpha, \beta, \{r_i\}, \{a_i\}, \{b_i\}$ and set $\epsilon = 10^{-6}$, $y = e$, a vector of all ones.
Step 1 Repeat
  Step 1.1 Solve $G_2 z = Ay$ with Procedure 1.
  Step 1.2 Set $\lambda = \|z\|_\infty$ and $y^+ = z/\lambda$.
  Step 1.3 If $\|y^+ - y\|_2 < \epsilon$, then stop and goto Step 2
    else set $y = y^+$ and goto Step 1.1.
Step 2 Output a positive solution y^+ and its associated eigenvalue $1/\lambda$.

Both Algorithms 1 and 2 are iterative in nature. At each iteration, a structured linear system is solved. For problem (1.1), the linear system is tridiagonal while it is almost tridiagonal for problem (1.2). Due to the linear computational complexities of our solvers for linear system of equations, Algorithms 1 and 2 are very fast. Moreover, these algorithms are extremely suitable for large scale problems since only a few vector variables are needed in the implementations of both algorithms.

Both algorithms were implemented in Matlab6.1 and executed in HP Pavilion dv3-1075us Entertainment Notebook PC which is equipped with AMD Turion X2 Dual-Core Mobile processor RM-72 (2.1 GHz, 1MB L2 Cache) and 4 GB DDR2 SDRAM. A few problems were tested and all were solved successfully. The speed of convergence depends on the dimension n and the other data of the problems. Theoretically, the computational complexity of Procedure 1, the solver for $G_2 z = d$, is more than twice of that of the Crout factorization algorithm for tridiagonal systems. It is observed in our test runs that finding a positive solution to an instance of problem (1.2) requires twice the time for finding a positive solution to an instance of problem (1.1) if the numbers of calling its corresponding linear solver are almost the same. For example, it takes 13.16 minutes for Algorithm 1 to solve the instance of problem (1.1) with data $n = 4000$, $\tau = 0.5$, $\sigma = 0$, $r(0) = 1$, $r(i) = i$ $(1 \leqslant i \leqslant n)$, $b(i) = 1$ for all i and $a(100) = a(200) = 0$, $a(i) = 1$ for $i \neq 100$, or 200 by calling its linear system solver 3869 times and it takes 27.66 minutes for Algorithm 2 to solve the instance of problem (1.2) with data $n = 4000$, $\alpha = 0.5$, $\beta = 0$, $r(0) = 1$, $r(i) = i$ $(1 \leqslant i \leqslant n)$, $b(i) = 1$ for all i and $a(100) = a(200) = 0$, $a(i) = 1$ for $i \neq 100$, or 200 with 3866 calls to its linear system solver.

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