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LYAPUNOV METHOD FOR STABILITY OF DESCRIPTOR SECOND-ORDER AND HIGH-ORDER SYSTEMS

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Abstract. In this study, the stability problem of descriptor second-order systems is considered. Lyapunov equations for stability of second-order systems are established by using Lyapunov method. The existence of solutions for Lyapunov equations are discussed and linear matrix inequality condition for stability of second-order systems are given. Then, based on the feasible solutions of the linear matrix inequality, all parametric solutions of Lyapunov equations are derived. Furthermore, the results of Lyapunov equations and linear matrix inequality condition for stability of second-order systems are extended to high-order systems. Finally, illustrating examples are provided to show the effectiveness of the proposed method.

1. **Introduction.** The second-order systems are an extension of descriptor systems in form, and can be treated as polynomial matrix systems with degree 2 [10, 17]. There are wide applications for second-order systems theory in last twenty years, such as in mechanics and spacecraft control field [17, 4, 11, 16]. Some mechanical systems, for instance, vibration systems [16, 21] and linear gyroscopic systems [13, 14] can be described by descriptor second-order systems. The current development of second-order systems is far from prefect for theory and practical applications [11, 14, 15], and most results of descriptor systems cannot be extended directly to second-order systems or high-order systems.

There are many methods to investigate stability of standard state space systems and descriptor systems [22, 6]. A direct criterion for stability of standard state space systems ($\dot{x} = Ax$) is whether all the eigenvalues of system matrix (A) have negative real parts or not [22]. The stability can also be determined by solving a Lyapunov equation ($A^T X + X A = -Q$), which is the known Lyapunov stability criterion of the standard state space systems [6]. There is also corresponding Lyapunov equation ($E^T X A + A^T X E = -P_r^T Y P_r$) for stability of descriptor systems ($E\dot{x} = Ax$) [6]. In descriptor systems, the singularity of E may lead to impulse solutions in the state response that are not expected to exist. Hence, stability and impulse-freeness which are termed as admissibility are usually required to meet in descriptor systems [6, 8]. Admissibility has been one of the important research topics of descriptor systems for the past few years [6, 7, 20]. There is also Lyapunov equation for determining admissibility in [6] ($E^T X A + A^T X E = -E^T Y E$).

The stability of second-order systems has been investigated since 1970s by many researchers, and there are many methods to involve the stability of second-order systems [1, 19, 18, 2, 5]. Some sufficient conditions for stability of second-order systems are given by positive definiteness of system matrices in [1], and several conditions are given via linear matrix equation in [19]. The stability conditions for matrix polynomials based on algebraic properties of the matrix coefficients are presented in [18]. Lyapunov stability and semistability conditions of matrix second-order systems are given by the rank of system matrices for some special cases in [2]. The generalized Hurwitz criteria are used to analyze stability for a class of second-order systems in [5]. Some stability results of fractional-order systems are presented by using Lyapunov method in [9, 3]. Up to now, there are no relevant results in studying stability of second-order systems by general Lyapunov method. Inspired by these existing stability results, we will study the stability of descriptor (singular) second-order systems by using Lyapunov method. This study will extend Lyapunov stability criterion of descriptor systems to second-order systems, and obtained results will weaken the existing stability conditions (for instance, [1, 19]) of second-order systems.

The main work of this paper is to establish Lyapunov equations and linear matrix inequality (LMI) condition for the stability of descriptor second-order systems. By selecting a Lyapunov candidate function, we will derive Lyapunov equations to determine the stability of descriptor second-order systems. Then based on Lyapunov function, we will also establish LMI condition for the stability of descriptor second-order systems. More precisely, based on the feasible solutions of LMI, we will derive all the parametric solutions of Lyapunov equations. Furthermore, the results will be generalized to high-order systems.

The rest of this paper is organized as follows. In Section 2, we give preliminaries and some lemmas. Section 3 establishes Lyapunov equations for stability of second-order systems. In Section 4, we derive LMI condition for stability of second-order systems, and give the relationship of Lyapunov equations and LMI. Section 5 generalizes stability results of descriptor second-order to descriptor high-order systems. Section 6 gives some conclusions and describes possible future work.

2. **Preliminaries and lemmas.** The purpose of this section is to introduce known results about stability which are essential for deriving the main results.

Consider following descriptor second-order systems

$$A_2\ddot{x}(t) + A_1\dot{x}(t) + A_0x(t) = Bu(t)$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^r$ are the state vector and input vector, respectively; $A_i \in \mathbb{R}^{n \times n}, i = 0, 1, 2$ and $B \in \mathbb{R}^{n \times r}$ are system matrices and input matrix, respectively. Like the standard state space systems and descriptor systems case, when studying stability of systems, we only need to consider the following homogeneous equation

$$A_2\ddot{x} + A_1\dot{x} + A_0x = 0 \tag{1}$$

In this study, we assume that the system (1) is regular, that is, A_2 is nonzero matrix and the polynomial $det(A_2s^2 + A_1s + A_0)$ is not identically zero.

Lemma 2.1 ([12]). *Consider the autonomous system*

$$\dot{x} = f(x) \tag{2}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Let $x = 0$ be an equilibrium point for (2) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \longrightarrow \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0 \text{ in } x \in D - \{0\}.$$

Then we say, the system is stable at the equilibrium if $\dot{V}(x) \leq 0$; the system is asymptotically stable at the equilibrium if $\dot{V}(x) < 0$ or $\dot{V}(x) \leq 0$, but for the arbitrary initial state $x(t_0) \neq 0$, $\dot{V}(x) \not\equiv 0, \forall x \neq 0$. Furthermore, the system is globally asymptotically stable at the equilibrium if $V(x)$ is radially unbounded.

The Schur complement lemma is a powerful tool to deal with LMI. The following result is well known.

Lemma 2.2. *Let S be a real square matrix with $S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$, where $S_{11} \in \mathbb{R}^{r \times r}$. Then the following statements are equivalent:*
  a) $S < 0$;
  b) $S_{11} < 0, \ S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
  c) $S_{22} < 0, \ S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

The following lemma can be obtained directly.

Lemma 2.3. *Let S be a real square matrix with $S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \geq 0$, $S_{11} \in \mathbb{R}^{r \times r}$,*
  a) If S_{22} is nonsingular, then $S_{11} - S_{12} S_{22}^{-1} S_{12}^T \geq 0$;
  b) If S_{11} is nonsingular, then $S_{22} - S_{12}^T S_{11}^{-1} S_{12} \geq 0$.

3. **Lyapunov equations for stability.** We are now in a position to present and prove our main results.

Theorem 3.1. *The system (1) is stable, if there exist two positive definite matrices $P_1, P_2 \in \mathbb{R}^{n \times n}$ and nonzero matrices $C_0, C_1 \in \mathbb{R}^{m \times n}$ such that*

$$A_0^T P_1 A_1 + A_1^T P_1 A_0 = C_0^T C_0 \tag{3a}$$

$$A_1^T P_2 A_2 + A_2^T P_2 A_1 = C_1^T C_1 \tag{3b}$$

$$A_0^T P_1 A_2 + A_0^T P_2 A_2 = C_0^T C_1 \tag{3c}$$

Furthermore, the system (1) is globally asymptotically stable if $C_1 \dot{x} + C_0 x \neq 0$ for arbitrary $x \in \mathbb{R}^n - \{0\}$ or $C_1 \dot{x} + C_0 x = 0$ and (1) has one and only one common solution $x = 0$.

Proof. Let $V(x) = (A_2 \dot{x})^T P_2 (A_2 \dot{x}) + (A_2 \dot{x} + A_1 x)^T P_1 (A_2 \dot{x} + A_1 x)$. Then

$$\begin{aligned} \dot{V}(x) &= -(A_1 \dot{x} + A_0 x)^T P_2 (A_2 \dot{x}) - (A_2 \dot{x})^T P_2 (A_1 \dot{x} + A_0 x) \\ &\quad -(A_0 x)^T P_1 (A_2 \dot{x} + A_1 x) - (A_2 \dot{x} + A_1 x)^T P_1 A_0 x \\ &= -x^T (A_0^T P_1 A_1 + A_1^T P_1 A_0) x - \dot{x}^T (A_1^T P_2 A_2 + A_2^T P_2 A_1) \dot{x} \\ &\quad -x^T (A_0^T P_1 A_2 + A_0^T P_2 A_2) \dot{x} - \dot{x}^T (A_2^T P_1 A_0 + A_2^T P_2 A_0) x \\ &= -(C_1 \dot{x} + C_0 x)^T (C_1 \dot{x} + C_0 x) \leq 0 \end{aligned}$$

$\dot{V}(x)$ is negative semidefinite. Hence, by Lemma 2.1, the system (1) is stable. Furthermore, if $C_1 \dot{x} + C_0 x \neq 0$ for $\forall x \in \mathbb{R}^n - \{0\}$, then $\dot{V}(x) < 0$, thus the system (1) is asymptotically stable. Besides, if $C_1 \dot{x} + C_0 x = 0$ and (1) have one and only one common solution $x = 0$, then $\dot{V}(x) \leq 0$, but $\dot{V}(x) \not\equiv 0$, $\forall x \neq 0$,

by LaSalle's invariance principle, the system (1) is asymptotically stable. $V(x)$ is radially unbounded, therefore the system (1) is globally asymptotically stable. $\square$

Remark 1. In Theorem 3.1, if $C_1\dot{x} + C_0x = 0$ and (1) have one and only one common solution $x = 0$, then the system (1) is asymptotically stable. The condition is a direct application of LaSalle's invariance principle in second-order systems. Especially, the condition has two special cases. There exist $C_0, C_1 \in \mathbb{R}^{n \times n}$ such that (3) and one of the following conditions hold:

    i) matrix C_1 and $A_0 - A_1C_1^{-1}C_0 + A_2C_1^{-1}C_0C_1^{-1}C_0$ is nonsingular;

    ii) matrix C_0 and $A_2 - A_1C_0^{-1}C_1 + A_0C_0^{-1}C_1C_0^{-1}C_1$ is nonsingular.

(3a)-(3c) are called Lyapunov equations of the system (1). In fact, the existence of C_0, C_1, P_1, P_2 are constrained by each other. In general, P_1, P_2 are given first, then C_0, C_1 are determined by (3). One can know that there exist second-order systems with the form (1) and matrices $P_i, C_i, i = 1, 2$ satisfying (3). For example, if $A_2 = A_1 = A_0 = I$, then select $P_1 = P_2 = P > 0$, and $C_0 = C_1 = \sqrt{2}\bar{P}$ by (3), where $\bar{P}^T\bar{P} = P$. On the other hand, one can observe that the solutions are asymptotically stable when $A_2 = A_1 = A_0 = I$.

Remark 2. We know that Lyapunov equation of standard state space systems is a special case of Lyapunov equation of descriptor systems. In Theorem 3.1, if $A_2 = 0$, then the system (1) is a descriptor system and (3) degenerates a Lyapunov equation of descriptor systems. Therefore, (3) is consistent with Lyapunov equation of descriptor systems, and Lyapunov stability criterion of descriptor systems is a special case of Theorem 3.1.

Theorem 3.1 is a direct and efficient stability criterion. However, it is not easy to solve P_1, P_2, C_0, C_1 from (3). In fact, if there exist solutions for equations (3), then the system (1) is stable. The following theorem shows an equivalent condition for existence of solution of (3).

Theorem 3.2. *For given $P_1 > 0, P_2 > 0$, matrices $A_0^T P_1 A_1 + A_1^T P_1 A_0$ and $A_1^T P_2 A_2 + A_2^T P_2 A_1$ are positive definite. Then, there exist solutions for Lyapunov equations (3) if and only if*

$$(A_0^T P_1 A_1 + A_1^T P_1 A_0)^{-\frac{1}{2}} A_0^T (P_1 + P_2) A_2 (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}$$

can be decomposed into product of a row orthogonal matrix and a column orthogonal matrix.

Proof. Sufficiency. Let $\Phi \in \mathbb{R}^{n \times m}, \Psi \in \mathbb{R}^{m \times n}$ be row orthogonal matrix (i.e. $\Phi\Phi^T = I$) and column orthogonal matrix (i.e. $\Psi^T\Psi = I$), respectively, and satisfy

$$(A_0^T P_1 A_1 + A_1^T P_1 A_0)^{-\frac{1}{2}} [A_0^T (P_1 + P_2) A_2] (A_2^T P_1 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}} = \Phi\Psi$$

Namely,

$$A_0^T (P_1 + P_2) A_2 = (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \Phi\Psi (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}}$$

Let

$$C_0 = \Phi^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}}, C_1 = \Psi (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}},$$

then $C_i \in \mathbb{R}^{m \times n}, i = 1, 2$, and

$$A_0^T (P_1 + P_2) A_2 = C_0^T C_1$$

and then one can obtain

$$
\begin{aligned}
A_0^T P_1 A_1 + A_1^T P_1 A_0 &= (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \\
&= (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \Phi \Phi^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \\
&= [\Phi^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}}]^T \Phi^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \\
&= C_0^T C_0
\end{aligned}
$$

$$
\begin{aligned}
A_1^T P_2 A_2 + A_2^T P_2 A_1 &= (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \\
&= (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \Psi^T \Psi (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \\
&= [\Psi (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}}]^T \Psi (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \\
&= C_1^T C_1.
\end{aligned}
$$

Therefore, the sufficiency is true.

Necessity. Since

$$
\begin{aligned}
A_0^T P_1 A_1 + A_1^T P_1 A_0 &= (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \\
&= (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \alpha \alpha^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
A_1^T P_2 A_2 + A_2^T P_2 A_1 &= (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \\
&= (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \beta^T \beta (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}}.
\end{aligned}
$$

where, α, β are row orthogonal matrix and column orthogonal matrix, respectively. Then all the solutions C_0 of (3a) and all the solutions C_1 of (3b) are in the following two sets, respectively,

$$
\mathbb{C}_0 = \{\alpha^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \mid \alpha \in \Theta_1\},
$$

$$
\mathbb{C}_1 = \{\beta (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \mid \beta \in \Theta_2\},
$$

where, Θ_1, Θ_2 are set of all row orthogonal matrices and set of all column orthogonal matrices, respectively. (3a), (3b) and (3c) imply that there exist $\alpha_0 \in \Theta_1$ and $\beta_0 \in \Theta_2$. That is, there are two matrices C_0, C_1 with

$$
C_0 = \alpha_0^T (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}}, \ C_1 = \beta_0 (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}}
$$

such that (3c) holds. Thus,

$$
(A_0^T P_1 A_1 + A_1^T P_1 A_0)^{-\frac{1}{2}} [A_0^T (P_1 + P_2) A_2](A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}} = \alpha_0 \beta_0,
$$

Note that α_0 with $\alpha_0 \alpha_0^T = I$ is a row orthogonal matrix and β_0 with $\beta_0^T \beta_0 = I$ is a column orthogonal matrix. So, necessity is true. The proof is therefore complete. □

As a special case, one can derive following result when the solutions of (3) are square matrices.

Corollary 1. *For given $P_1 > 0, P_2 > 0$ matrices $A_0^T P_1 A_1 + A_1^T P_1 A_0$ and $A_1^T P_2 A_2 + A_2^T P_2 A_1$ are positive definite. Then, there exist $C_0, C_1 \in \mathbb{R}^{n \times n}$ satisfying Lyapunov equations (3) if and only if*

$$
(A_0^T P_1 A_1 + A_1^T P_1 A_0)^{-\frac{1}{2}} A_0^T (P_1 + P_2) A_2 (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}
$$

is an orthogonal matrix.

Theorem 3.2 and Corollary 1 provide a approach to determine the existence of solutions in (3). The following section will give an applicable approach to determine the stability of second-order systems. We will derive LMI condition for stability of the system (1), and give the parametric solutions of (3) by using the feasible solutions of LMI.

4. LMI conditions for stability.

Theorem 4.1. *The system* (1) *is stable if there exist two positive definite matrices* P_1, P_2 *such that the following LMI holds:*

$$\begin{bmatrix} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{bmatrix} \geq 0 \tag{4}$$

The system (1) *is globally asymptotically stable if LMI* (4) *is strictly true.*

Proof. Consider the Lyapunov function

$$V(x) = (A_2 \dot{x})^T P_2 (A_2 \dot{x}) + (A_2 \dot{x} + A_1 x)^T P_1 (A_2 \dot{x} + A_1 x)$$

and substitute (1) into derivation of V, one can deduce that,

$$\begin{aligned} \dot{V}(x) &= -x^T (A_0^T P_1 A_1 + A_1^T P_1 A_0) x - \dot{x}^T (A_1^T P_2 A_2 + A_2^T P_2 A_1) \dot{x} \\ &\quad -x^T (A_0^T P_1 A_2 + A_0^T P_2 A_2) \dot{x} - \dot{x}^T (A_2^T P_1 A_0 + A_2^T P_2 A_0) x \\ &= -\begin{bmatrix} x^T, \dot{x}^T \end{bmatrix} \begin{bmatrix} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \end{aligned}$$

This implies that $\dot{V}(x) \leq 0$ (*or* < 0) *if and only if*

$$\begin{bmatrix} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{bmatrix} \geq 0 \ (or > 0).$$

The proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

By Theorem 4.1, if LMI (4) has feasible solutions, then the system (1) is stable. The system may even be asymptotically stable if LMI (4) is not strictly true. This will be presented later in this section (see Theorem 4.4).

Remark 3. Using LMI toolbox in MATLAB, feasible solutions can be obtained. Therefore, LMI condition is more easily derived and applied to engineering applications. Many vibration systems and linear gyroscopic systems can be described by descriptor second-order systems. Thus, the results of Theorem 4.1 provide a feasible approach to solve stability problem of mechanical systems.

In the following several theorems, we will study the relation of Lyapunov equations (3) and LMI (4) from the view of the algebra.

Theorem 4.2. *For given positive definite matrices* $P_1, P_2 \in \mathbb{R}^{n \times n}$, *let* $A_1^T P_2 A_2 + A_2^T P_2 A_1$ *be a nonsingular matrix, then, there exist* $C_0, C_1 \in \mathbb{R}^{m \times n}$ *such that*

$$\begin{aligned} A_0^T P_1 A_1 + A_1^T P_1 A_0 &= C_0^T C_0 \\ A_1^T P_2 A_2 + A_2^T P_2 A_1 &= C_1^T C_1 \\ A_0^T P_1 A_2 + A_0^T P_2 A_2 &= C_0^T C_1 \end{aligned}$$

if and only if P_1, P_2 *are feasible solutions of LMI*

$$\begin{bmatrix} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{bmatrix} \geq 0.$$

Proof. Necessity. Consider quadratic form

$$\begin{aligned} &\begin{bmatrix} x_1^T, x_2^T \end{bmatrix} \begin{bmatrix} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1^T (A_0^T P_1 A_1 + A_1^T P_1 A_0) x_1 + x_2^T (A_1^T P_2 A_2 + A_2^T P_2 A_1) x_2 \\ &\quad + x_1^T (A_0^T P_1 A_2 + A_0^T P_2 A_2) x_2 + x_2^T (A_2^T P_1 A_0 + A_2^T P_2 A_0) x_1 \\ &= (C_1 x_2 + C_0 x_1)^T (C_1 x_2 + C_0 x_1) \geq 0 \end{aligned}$$

Thus,

$$\left[\begin{array}{cc} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{array} \right] \geq 0.$$

Sufficiency. Since $A_1^T P_2 A_2 + A_2^T P_2 A_1$ is nonsingular, by a) of Lemma 2.3, we have

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$

$$-(A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0) \geq 0.$$

This implies that there exists $L \in \mathbb{R}^{k \times n}$ such that

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$
$$-(A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0) = L^T L \tag{5}$$

Rewriting (5) in a matrix form

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$
$$= L^T L + (A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0)$$
$$= W^T W$$

where

$$W = \left[\begin{array}{c} L \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}(A_2^T P_1 A_0 + A_2^T P_2 A_0) \end{array} \right]$$

Note that

$$A_1^T P_2 A_2 + A_2^T P_2 A_1 = \left((A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \right)^T \left(A_1^T P_2 A_2 + A_2^T P_2 A_1 \right)^{\frac{1}{2}}$$

Let

$$C_0 = W,$$
$$C_1 = \left[\begin{array}{c} 0 \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \end{array} \right],$$

Then

$$C_0^T C_0 = A_0^T P_1 A_1 + A_1^T P_1 A_0,$$
$$C_0^T C_1 = A_0^T P_1 A_2 + A_0^T P_2 A_2,$$
$$C_1^T C_1 = A_1^T P_2 A_2 + A_2^T P_2 A_1.$$

The proof is complete. □

Remark 4. If the condition of Theorem 4.2 is substituted for which $A_0^T P_1 A_1 + A_1^T P_1 A_0$ is nonsingular, then the conclusion still holds and can be proved by using b) of Lemma 2.3. Theorem 4.2 provides an approach to construct all the parametric solutions of (3) based on the feasible solutions of LMI (4).

Theorem 4.3. *Assume that LMI (4) is true, and $A_1^T P_2 A_2 + A_2^T P_2 A_1$ is nonsingular. Then all the parametric solutions of (3) are expressed by*

$$C_0 = Q \left[\begin{array}{c} L_0 \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}(A_2^T P_1 A_0 + A_2^T P_2 A_0) \end{array} \right],$$

$$C_1 = Q \left[\begin{array}{c} 0 \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \end{array} \right],$$

where, $L_0 \in \mathbb{R}^{l \times n}$ has minimum number of rows. $Q \in \mathbb{R}^{m \times (l+n)}$ with $Q^T Q = I$ is column orthogonal matrix (has also minimum number of columns).

Proof. LMI (4) implies that there exists $\bar{C} \in \mathbb{R}^{m \times 2n}$ such that

$$\left[\begin{array}{cc} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{array} \right] = \bar{C}^T \bar{C} \tag{6}$$

Hence, all the matrices $\bar{C}$ in (6) can be expressed by

$$\bar{C} = QM, \tag{7}$$

where, $M \in \mathbb{R}^{q \times 2n}$ has minimum number of rows. $Q \in \mathbb{R}^{m \times q}$ satisfies $Q^T Q = I_q$, that is, Q is a column orthogonal matrix. Partition M as

$$M := [M_1, M_2] \tag{8}$$

where $M_i \in \mathbb{R}^{q \times n}$. It is seen that (3) can be equivalently rewritten a matrix equation form

$$\left[\begin{array}{cc} C_0^T C_0 & C_0^T C_1 \\ C_1^T C_0 & C_1^T C_1 \end{array} \right] = \left[\begin{array}{cc} A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_2 A_2 + A_2^T P_2 A_1 \end{array} \right] \tag{9}$$

To complete proof, it is sufficient to construct matrix M. By (5), if

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$

$$\neq (A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0)$$

then there exists a row full rank matrix $L_0 \in \mathbb{R}^{l \times n}$, such that

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$
$$= L_0^T L_0 + (A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0)$$

Let

$$M_1 = \left[\begin{array}{c} L_0 \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}(A_2^T P_1 A_0 + A_2^T P_2 A_0) \end{array} \right],$$

$$M_2 = \left[\begin{array}{c} 0 \\ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \end{array} \right].$$

Note that q is minimum number of rows. This implies $q = l + n$. Thus, C_1, C_2 are obtained. If

$$A_0^T P_1 A_1 + A_1^T P_1 A_0$$

$$= (A_0^T P_1 A_2 + A_0^T P_2 A_2)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0),$$

then $L_0 = 0$ and $M_i \in \mathbb{R}^{n \times n}, C_i \in \mathbb{R}^{m \times n}, i = 1, 2$. The parametric solutions can be expressed as the following degenerated forms

$$\begin{array}{l} C_0 = Q \ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-\frac{1}{2}}(A_2^T P_1 A_0 + A_2^T P_2 A_0) \\ C_1 = Q \ (A_1^T P_2 A_2 + A_2^T P_2 A_1)^{\frac{1}{2}} \end{array} \tag{10}$$

where, $Q \in \mathbb{R}^{m \times n}$ with $Q^T Q = I$. The proof is complete. $\qquad \square$

If LMI (4) is not strictly true, but C_0, C_1 satisfy i) or ii) of Remark 1, then the system (1) is also asymptotically stable. In this case, C_0, C_1 are square matrices with forms (10) and $m = n$. Therefore, Theorem 4.1 can be rewritten as following Theorem 4.4.

Theorem 4.4. *The system (1) is stable if there exist $P_1 > 0, P_2 > 0$ such that the LMI (4) holds; The system (1) is globally asymptotically stable if LMI (4) is strictly true or LMI (4) is not strictly true but one of the following conditions holds.*
  i) $A_1^T P_2 A_2 + A_2^T P_2 A_1$ *and*

$$A_0 - A_1(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0) + A_2(A_1^T P_2 A_2 +$$
$$A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0)(A_1^T P_2 A_2 + A_2^T P_2 A_1)^{-1}(A_2^T P_1 A_0 + A_2^T P_2 A_0)$$

are nonsingular.
  ii) $A_1^T P_2 A_2 + A_2^T P_2 A_1$, $A_2^T P_1 A_0 + A_2^T P_2 A_0$ *and*

$$A_2 - A_1(A_2^T P_1 A_0 + A_2^T P_2 A_0)^{-1}(A_1^T P_2 A_2 + A_2^T P_2 A_1) + A_0(A_2^T P_1 A_0 +$$
$$A_2^T P_2 A_0)^{-1}(A_1^T P_2 A_2 + A_2^T P_2 A_1)(A_2^T P_1 A_0 + A_2^T P_2 A_0)^{-1}(A_1^T P_2 A_2 + A_2^T P_2 A_1)$$

are nonsingular.

The same argument with Theorem 4.3, the following results can be obtained.

Theorem 4.5. *Assume that $A_0^T P_1 A_1 + A_1^T P_1 A_0$ is nonsingular matrix, and LMI (4) is true, then all the parametric solutions of (3) are expressed by*

$$C_0 = \begin{bmatrix} 0 \\ (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{\frac{1}{2}} \end{bmatrix}$$
$$C_1 = Q \begin{bmatrix} L_0 \\ (A_0^T P_1 A_1 + A_1^T P_1 A_0)^{-\frac{1}{2}}(A_0^T P_1 A_2 + A_0^T P_2 A_2) \end{bmatrix}$$

where, $L_0 \in \mathbb{R}^{l \times n}$ has minimum number of rows. $Q \in \mathbb{R}^{m \times (l+n)}$ with $Q^T Q = I$ has minimum number of columns.

Remark 5. The stability of second-order systems can be determined by the positive (negative) definiteness of system matrices in existing results (see [1, 19, 18]). This study weakened these stability conditions. In other words, by the results of Theorem 3.1 and Theorem 4.1, some descriptor second-order systems are stable, but the systems matrices are not positive (negative) definite. This can also be shown by the following examples.

Example 1. Consider the stability of the following second-order singular system

$$\begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix} \ddot{x} + \begin{bmatrix} 5 & 0 \\ 2 & 6 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad (11)$$

Using LMI Toolbox in MATLAB, we obtain the feasible solutions that LMI (4) is strictly true.

$$P_1 = \begin{bmatrix} 582.9031 & -82.5770 \\ -82.5770 & 44.6823 \end{bmatrix}$$

$$P_2 = 10^3 \times \begin{bmatrix} 1.3245 & -0.6686 \\ -0.6686 & 0.3738 \end{bmatrix}$$

Obviously, P_1, P_2 are positive definite matrices, thus the system is asymptotically stable. The impulse input response curves of state solutions are shown in Figure 1 (a), where, $x = [x_1^T, x_2^T]^T$.

Example 2. Concern with the second-order nonsingular system

$$\begin{bmatrix} 0.1 & 0 \\ 0.1 & 1 \end{bmatrix} \ddot{x} + \begin{bmatrix} 0.5 & 0 \\ 0.2 & 1 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x = 0 \qquad (12)$$

Since

$$P_1 = 10^3 \times \left[\begin{array}{cc} 1.4272 & -0.0991 \\ -0.0991 & 0.0743 \end{array} \right]$$

$$P_2 = 10^3 \times \left[\begin{array}{cc} 1.9855 & -0.0347 \\ -0.0347 & 0.0743 \end{array} \right]$$

are feasible solutions of LMI (4), the system is global asymptotically stable. The zero input response curves of state solutions with $x(0) = [-2, 5]^T, \dot{x}(0) = [-1, 1]^T$ are shown in Figure 1 (b), where, $x = [x_1^T, x_2^T]^T$.
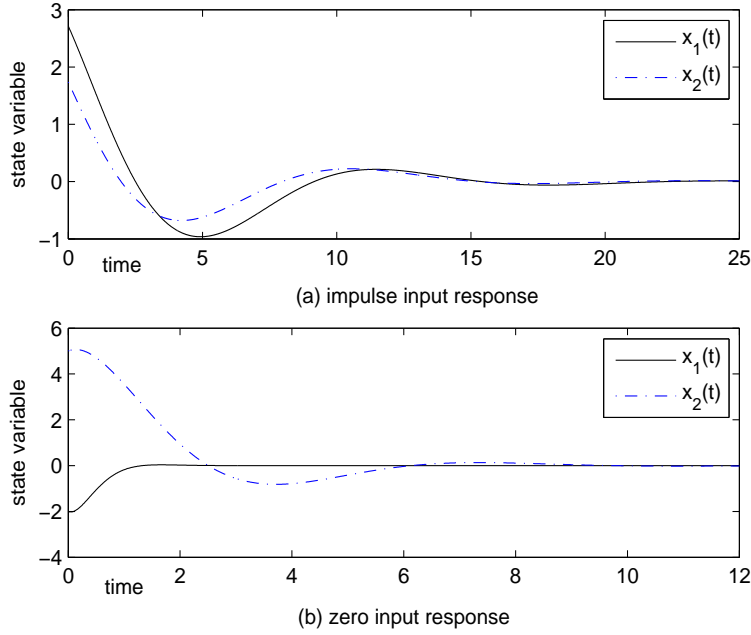

FIGURE 1. State responses of systems

5. **Generalize to high-order systems.** In this section, Lyaunov equations and LMI condition of second-order systems are generalized to mth-order descriptor systems. Now, we consider stability of the following system

$$A_m x^{(m)} + A_{m-1} x^{(m-1)} + \cdots + A_1 \dot{x} + A_0 x = Bu(t)$$

Similarly, we are only to consider the following system without input term.

$$A_m x^{(m)} + A_{m-1} x^{(m-1)} + \cdots + A_1 \dot{x} + A_0 x = 0 \tag{13}$$

Theorem 5.1. *Consider the system (13). If there exist $P_i > 0, C_{i-1}, i = 1, \cdots, m$, such that*

$$A_{i-1}^T P_i A_i + A_i^T P_i A_{i-1} = C_{i-1}^T C_{i-1}, \quad i = 1, \cdots, m,$$
$$A_{i-1}^T \left(\sum_{k=i}^{j} P_k \right) A_j = C_{i-1}^T C_{j-1}, \quad i < j, 2 \le j \le m, \tag{14}$$

then the system (13) is stable. Further, if $\sum\limits_{i=1}^{m} C_{i-1} x^{(i-1)} \ne 0$ for arbitrary $x \in \mathbb{R}^n - \{0\}$, then the system (13) is global asymptotically stable.

Proof. Select Lyapunov function

$$
\begin{aligned}
V(x) = &(A_m x^{(m-1)})^T P_m A_m x^{(m-1)} \\
&+ (A_m x^{(m-1)} + A_{m-1} x^{(m-2)})^T P_{m-1}(A_m x^{(m-1)} + A_{m-1} x^{(m-2)}) \\
&+ \cdots + (A_m x^{(m-1)} + \cdots + A_1 x)^T P_1 (A_m x^{(m-1)} + \cdots + A_1 x),
\end{aligned}
$$

where $P_1, P_2, \cdots, P_m$ are positive definite matrices.

$$
\begin{aligned}
\dot{V}(x) = \ &- (A_{m-1} x^{(m-1)} + \cdots + A_0 x)^T P_m A_m x^{(m-1)} \\
&- (A_m x^{(m-1)})^T P_m (A_{m-1} x^{(m-1)} + \cdots + A_0 x) \\
&- (A_{m-2} x^{(m-2)} + \cdots + A_0 x)^T P_{m-1}(A_m x^{(m-1)} + A_{m-1} x^{(m-2)}) \\
&- (A_m x^{(m-1)} + A_{m-1} x^{(m-2)})^T P_{m-1}(A_{m-2} x^{(m-2)} + \cdots + A_0 x) - \cdots \\
&- (A_0 x)^T P_1 (A_m x^{(m-1)} + \cdots + A_1 x) - (A_m x^{(m-1)} + \cdots + A_1 x)^T P_1 A_0 x \\
= \ &- \sum_{i=1}^{m} (x^{(i-1)})^T (A_{i-1}^T P_i A_i + A_i^T P_i A_{i-1}) x^{(i-1)} \\
&- \sum_{i<j} (x^{(i-1)})^T \left(A_{i-1}^T \left(\sum_{k=i}^{j} P_k \right) A_j \right) x^{(j-1)} \\
&- \sum_{i<j} (x^{(j-1)})^T \left(A_j^T \left(\sum_{k=i}^{j} P_k \right) A_{i-1} \right) x^{(i-1)}
\end{aligned}
$$

By (14), one can deduce that

$$
\dot{V}(x) = - \left(\sum_{i=1}^{m} C_{i-1} x^{(i-1)} \right)^T \left(\sum_{i=1}^{m} C_{i-1} x^{(i-1)} \right) \le 0.
$$

Further, $\dot{V}(x)$ is negative definite if $\sum_{i=1}^{m} C_{i-1} x^{(i-1)} \ne 0$. $\square$

(14) is called Lyapunov equations of system (13). The LMI conditions for stability can be given as follows.

Theorem 5.2. *Consider system (13). If LMIs*

$$
\begin{bmatrix}
A_0^T P_1 A_1 + A_1^T P_1 A_0 & A_0^T (P_1 + P_2) A_2 & \cdots & A_0^T (\sum_{i=1}^{m} P_i) A_m \\
* & A_1^T P_2 A_2 + A_2^T P_2 A_1 & \cdots & A_1^T (\sum_{i=2}^{m} P_i) A_m \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & A_{m-1}^T P_m A_m + A_m^T P_m A_{m-1}
\end{bmatrix} \ge 0
$$

(15a)

$$
P_i > 0, i = 1, 2, \cdots, m
$$

(15b)

hold, then system (13) is stable. Furthermore, if LMI (15a) is strictly true, then system (13) is global asymptotically stable, where $*$ *represent a term that is induced by symmetry.*

Proof. Select the same Lyapunov function as Theorem 5.1. It is seen that

$$
\dot{V}(x) = - \sum_{i=1}^{m} (x^{(i-1)})^T (A_{i-1}^T P_i A_i + A_i^T P_i A_{i-1}) x^{(i-1)}
$$

$$- \sum_{i<j} (x^{(i-1)})^T \left(A_{i-1}^T \left(\sum_{k=i}^{j} P_k \right) A_j \right) x^{(j-1)}$$

$$- \sum_{i<j} (x^{(j-1)})^T \left(A_j^T \left(\sum_{k=i}^{j} P_k \right) A_{i-1} \right) x^{(i-1)}$$

$$= \begin{bmatrix} x^T, \dot{x}^T, \cdots, (x^{(m-1)})^T \end{bmatrix} (a_{ij})_{m \times m} \begin{bmatrix} x \\ \dot{x} \\ \cdots \\ x^{(m-1)} \end{bmatrix}$$

where $(a_{ij})_{m \times m}$ is symmetric matrix with

$$(a_{ij})_{m \times m} = \begin{cases} a_{ii} = A_{i-1}^T P_i A_i + A_i^T P_i A_{i-1}, \ 1 \le i \le m \\ a_{ij} = A_{i-1}^T \left(\sum_{k=i}^{j} P_k \right) A_j, \ i < j, 1 \le i, j \le m \end{cases}$$

which is the matrix in LMI (15a). The proof is complete. $\qquad \square$

Example 3. Consider the following third-order matrix descriptor system

$$\begin{bmatrix} 0.1 & -0.2 \\ 0 & 0.3 \end{bmatrix} x^{(3)} + \begin{bmatrix} 1 & 3 \\ 0 & 8 \end{bmatrix} \ddot{x} + \begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix} \dot{x} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix} u \quad (16)$$

It follows that

$$P_1 = \begin{bmatrix} 0.3118 & -0.0722 \\ -0.0722 & 0.0705 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.2998 & -0.1268 \\ -0.1268 & 0.0859 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0.6185 & -0.0626 \\ -0.0626 & 0.6666 \end{bmatrix}$$

are the feasible solutions of strict LMIs (15) with $m = 3$. Hence, the system is global asymptotically stable by Theorem 5.2. The zero input response curves and impulse input response curves of state solutions are shown in (a) and (b) of Figure 2, respectively, where, $x(0) = [1, 0]^T, \dot{x}(0) = [0, -2]^T, \ddot{x}(0) = [1, 1]^T, x = [x_1^T, x_2^T]^T$.

6. **Conclusions.** In this paper, we studied stability issue of descriptor second-order systems. Lyapunov equations for the stability of second-order systems were derived by using Lyapunov method. We also presented LMI condition for stability of second-order systems, and derived the relationship of Lyapunov equations and LMI. Then, based on the solutions of LMI, the parametric solutions of Lyapunov equations were constructed. Furthermore, the stability results of second-order systems were extended to high order systems. Many mechanical systems can be described by descriptor second-order systems. Therefore, the results of this study will be useful for analyzing stability of second-order mechanical models. On the basic of these stability results, the problems of stabilization and robustness for second-order systems can be investigated in future work.

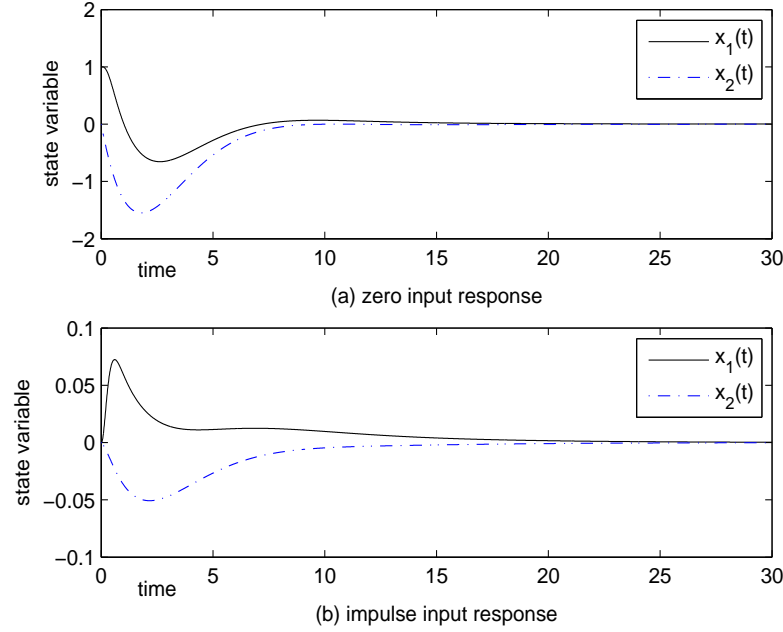FIGURE 2. State responses of system (16)

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