On the stability analysis of arbitrarily high-index singular systems with multiple delays

Ha Phi[a,*]

[a]*Faculty of Mathematics, Mechanics, and Informatics, VNU University of Science, Vietnam National University, Hanoi, Vietnam.*

Abstract

This paper is devoted to the stability analysis for the class of arbitrarily high-index (continuous-time) singular linear systems with multiple delays. By transforming the originally given system to an equivalent regular, impulse-free system, the global exponential stability problem is addressed by both approaches: spectral and Lyanpunov-Krasovskii. Characterizations for the stability are developed in both the spectral condition and the linear matrix inequality (LMI) setting. Numerical examples are presented to illustrate the advantages of the proposed results.

Keywords: Singular systems, Delay, LMIs, Spectral, Stabilization, Feedback.
2000 MSC: 34D20, 93D05, 93D20

1. Introduction

Consider the linear singular time-delay system of the form

$$E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \tag{1}$$

{delay-descriptor}

$$x(t) = \phi(t), \ \text{for all } t_0 - \tau_m \leqslant t \leqslant t_0, \tag{2}$$

where $E \in \mathbb{R}^{n,n}$ is allowed to be singular. Here the state is $x : [t_0 - \tau_m, \infty) \to \mathbb{R}^n$, and the (constant) time-delays satisfy $0 < \tau_1 < \tau_2 < \ldots < \tau_m$. The capital letters are real-valued matrices of appropriate dimensions. The system is called *free (or DDAE)* if we let $u \equiv 0$, i.e., the system reads

$$E\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i)\ . \tag{3}$$

{free system}

*Corresponding author
Email address: haphi.hus@vnu.edu.vn (Ha Phi)

The motivation for the system description 1 in the context of designing controllers lies in its generality in modelling interconnected systems.

The rest of the paper is organized as follows. In Section 2, some definitions concerning about the solution and the system classification are stated. Auxiliary Lemmas about the solution's presentation and the non-advanced test are also recalled. In Section 3, our first main results about the stability of arbitrarily high-index system are given, making use of both approaches above. Finally, in Section 4, numerical examples and the conclusion are given.

2. Preliminaries

To keep the brevity of this research, we refer the interested readers to [1, 2, 3, 4, 5] for the solvability analysis of the IVP (1).

Definition 1. *The null solution $x = 0$ of the free system* (3) *is called* exponentially stable *if there exist positive constants δ and γ such that for any consistent initial function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, the solution $x = x(t, \varphi)$ of the corresponding IVP to* (3) *satisfies*

$$\|x(t)\| \leqslant \delta e^{-\gamma t} \|\varphi\|_\infty, \ \textit{for every} \ \ t \geqslant 0.$$

Definition 2. *i) Consider the DDAE* (1)*. The matrix pair (E, A_0) is called* regular *if the* polynomial $\det(\lambda E - A_0)$ *is not identically zero.*
ii) The sets $\sigma(E, A_0, ..., A_m) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A_0 - e^{-\lambda \tau_i} A_i) = 0\}$ is called the spectrum *of* (1).

Provided that the pair (E, A_0) is regular, we can transform them to the Kronecker-Weierstraß canonical form as follows.

Lemma 3. *([6, 7]) Provided that the matrix pair (E, A_0) is regular, then there exist regular matrices W, $T \in \mathbb{R}^{n,n}$ such that*

$$(WET, WA_0T) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{4}$$

{KW form}

where N is a nilpotent, upper triangular matrix of nilpotency index ν. We also say that the pair (E, A_0) has an index *ν, i.e.,* $\operatorname{ind}(E, A_0) = \nu$*. Furthermore, the system* (1) *is called* impulse-free (index 1, or strangeness-free) *if $N = 0$.*

Remark 1. In general, the two concepts index and stability are independent. In fact, Examples 5 in [8] has illustrated that there exist systems with arbitrarily high-index (and hence, not impulse-free) which are stable.

Lemma 4. *For a nilpotent, upper triangular matrix N of nilpotency index ν, the matrix $I - \lambda N$ is invertible for all $\lambda \in \mathbb{C}$, and* $\det(I - \lambda N) = 1$. *Furthermore, the following identity holds true.*

$$(I - \lambda N)^{-1} = I + \sum_{i=1}^{\nu} (\lambda N)^i.$$

PROOF. The proof is simple and can be found in classical matrix theory textbooks, for example [9].

2.1. System classification

It is well-known (see e.g. [10, 11]) that in general, time-delayed systems has been classified into three different types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0 \dot{x}(t) + a_1 \dot{x}(t - \tau) + b_0 x(t) + b_1 x(t - \tau) = f(t)$$

is retarded if $a_0 \neq 0$ and $a_1 = 0$; is neutral if $a_0 \neq 0$, $a_1 \neq 0$; is advanced if $a_0 = 0, a_1 \neq 0, b_0 \neq 0$. This classification is based on the smoothness comparison between $x(t)$ and $x(t - \tau)$. In literature, not only the theoretical but also the numerical solution has been studied mainly for retarded and neutral systems, due to their appearance in various applications. For this reason, in [4, 5, 12] the authors proposed a concept of *non-advancedness* for the free system (see Definition 5 below). We also notice, that even though not clearly proposed, due to the author's knowledge, so far results for delay-descriptor are only obtained for certain classes of non-advanced systems, e.g. [1, 3, 13, 14, 15, 16, 17, 18, 19].

Definition 5. *A regular delay-descriptor system* (1) *is called* non-advanced *if for any consistent and continuous initial function φ, there exists a continuous, piecewise differentiable solution $x(t)$.*

Making use of Lemma 3, we change the variable $x = Ty$ and scale the whole system (3) with W to obtain the transformed system

$$\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \dot{y}(t) = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} y(t) + \sum_{i=1}^{m} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix} y(t - \tau_i), \tag{5}$$ {eq9}

where $WA_iT = \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}$ for all $i = 1, ..., m$. The following lemma gives us the necessary and sufficient condition for the non-advancedness of system (3).

Lemma 6. *i) System* (3) *is non-advanced if and only if the matrix coefficients of the transformed system* (5) *satisfy*

$$N\begin{bmatrix}\tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix} \text{ for all } i = 1, \dots, m. \tag{6}$$

{non-advanced cond.}

ii) Consequently, system (5) *has exactly the same solution as the so-called* index-reduced system

$$\tilde{E}\dot{y}(t) = \tilde{A}_0 y(t) + \sum_{i=1}^{m} \tilde{A}_i y(t-\tau_i), \tag{7}$$

{index reduced system

where

$$\tilde{E} := \begin{bmatrix} I & 0 \\ 0 & \mathbf{0} \end{bmatrix},\ \tilde{A}_0 := \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix},\ \tilde{A}_i := \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix},\ i = 1, ..., m.$$

PROOF. Partitioning $y := \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ conformably, we can rewrite system (5) as follows

$$\begin{aligned} \dot{y}_1 &= Jy_1 + \sum_{i=1}^{m} \begin{bmatrix}\tilde{A}_{i,1} & \tilde{A}_{i,2}\end{bmatrix} y(t-\tau_i), \\ N\dot{y}_2 &= y_2 + \sum_{i=1}^{m} \begin{bmatrix}\tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix} y(t-\tau_i), \end{aligned} \tag{8}$$

{eq14.2}

The second equation has a unique solution

$$y_2(t) = -\begin{bmatrix}\tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix} y(t-\tau_i) - \sum_{j=1}^{\nu}\sum_{i=1}^{m} N^i \begin{bmatrix}\tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix} y^{(j)}(t-\tau_i). \tag{9}$$

{eq10}

Since the system (3) is non-advanced, then so is system (5). Consequently, $y(t)$ must not depend on $y^{(j)}(t-\tau_i)$ for all $j \geqslant 1$ and $i = 1, ..., m$. This implies the identity (6). Then, equation (9) becomes

$$y_2(t) = -\begin{bmatrix}\tilde{A}_{i,3} & \tilde{A}_{i,4}\end{bmatrix} y(t-\tau_i),$$

and hence, the second claim is trivially followed.

Remark 2. From Lemma 6 ii), we see that if system (3) is non-advanced, then there is a linear, bijective mapping $x \mapsto y = T^{-1}x$ (where T is the matrix given in the Kronecker-Weierstraß form (4)) between the solution set of the high-index system (3) and the impulse-free system (7). This will play the key role in the stability analysis in Section 3.

Remark 3. Since the numerical computation of the Kronecker-Weierstraß form (4) is quite complicated and unstable (see [20]), Lemma 6 has more theoretical than numerical meaning for checking the non-advancedness of (3). Below we will construct another test, which is more practical.

Assume that the pair (E, A_0) is regular with index $\operatorname{ind}(E, A_0) = \nu$. We want to give a simple check whether the system (3) is non-advanced or not. In analoguous to the case of DAEs, see e.g. [21, 7], we aim to extract the so-called *underlying delay equation* of the form

$$\dot{x}(t) = \mathbf{A}_0 x(t) + \sum_{i=1}^{m} \mathbf{A}_i x(t-\tau_i) + \sum_{i=1}^{m} \mathbf{F}_i \dot{x}(t-\tau_i), \tag{10}$$

{underlying DDEs}

from an augmented system consisting of system (3) and its derivatives, which read in details

$$\frac{d^j}{dt^j}\left(E\dot{x}(t) - A_0 x(t) - \sum_{i=1}^{m} A_i x(t-\tau_i)\right) = 0, \text{ for all } j = 0, 1, \ldots, \nu.$$

We rewrite these equations into the so-called *inflated system*

$$\underbrace{\begin{bmatrix} E & & & \\ -A_0 & E & & \\ & \ddots & \ddots & \\ & & -A_0 & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{\mathcal{A}_0} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix}$$
$$+ \sum_{i=1}^{m} \underbrace{\begin{bmatrix} A_i & & & \\ & A_i & & \\ & & \ddots & \\ & & & A_i \end{bmatrix}}_{\mathcal{A}_i} \begin{bmatrix} x(t-\tau_i) \\ \dot{x}(t-\tau_i) \\ \vdots \\ x^{(\nu)}(t-\tau_i) \end{bmatrix}. \tag{11}$$

{inflated}

Here the matrix coefficients are $\mathcal{E}, \mathcal{A}_0, \mathcal{A}_i \in \mathbb{R}^{(\nu+1)n,(\nu+1)n}$ for all $i = 1, ..., m$. For the reader's convenience, below we will use MATLAB notations. An underlying delay system (10) can be extracted from (11) if and only if there exists a matrix $P = \begin{bmatrix} P_0 & P_1 & \ldots & P_\nu \end{bmatrix}^T$ in $\mathbb{R}^{(\nu+1)n,n}$ such that

$$P^T \mathcal{E} = \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix},$$
$$P^T \mathcal{A}_i = \begin{bmatrix} * & * & 0_{n,(\nu-1)n} \end{bmatrix}, \text{ for all } i = 1, ..., m,$$

where $*$ stands for an arbitrary matrix. Consequently, P is the solution to the following linear systems

$$\begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:,2n+1:end)^T \\ \vdots \\ \mathcal{A}_m(:,2n+1:end)^T \end{bmatrix} P = \begin{bmatrix} \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}^T \\ 0_{(\nu-1)n,n} \\ \vdots \\ 0_{(\nu-1)n,n} \end{bmatrix}.$$

Therefore, making use of Crammer's rule we directly obtain the simple check for the non-advancedness of system (3) in the following theorem.

Theorem 7. *Consider the zero-input descriptor system* (3) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = \nu$. *Then, this system is non-advanced if and only if the following rank condition is satisfied*

$$\operatorname{rank} \begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_1(:,2n{+}1:end)^T \\ \vdots \\ \mathcal{A}_m(:,2n{+}1:end)^T \end{bmatrix} = \operatorname{rank} \left[\begin{array}{c|c} \mathcal{E}^T & \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix}^T \\ \mathcal{A}_1(:,2n{+}1:end)^T & 0_{(\nu-1)n,n} \\ \vdots & \vdots \\ \mathcal{A}_m(:,2n{+}1:end)^T & 0_{(\nu-1)n,n} \end{array}\right]. \tag{12}$$

{adv. check eq.}

Theorem 7 applied to the index two case straightly gives us the following corollary.

Corollary 8. *Consider the zero-input descriptor system* (3) *and assume that the pair* (E, A_0) *is regular with index* $\operatorname{ind}(E, A_0) = 2$. *Then, system* (3) *is non-advanced if and only if the following identity hold true.*

$$\operatorname{rank} \left[\begin{array}{ccc} E^T & -A_0^T & 0 \\ 0 & E^T & -A_0^T \\ 0 & 0 & E^T \\ \hline 0 & 0 & A_1^T \\ \vdots & \vdots & \vdots \\ 0 & 0 & A_m^T \end{array}\right] = n + \operatorname{rank} \left[\begin{array}{cc} E^T & -A_0^T \\ 0 & E^T \\ \hline 0 & A_1^T \\ \vdots & \vdots \\ 0 & A_m^T \end{array}\right]. \tag{13}$$

{check advanced}

3. Stability

3.1. Spectral method

The stability analysis of the null solution of (1) in this work is based on a spectrum determined growth property of the solutions, which allows us to infer stability information from the location of the characteristic roots. For instance, exponential stability will be related to a strictly negative spectral abscissa (the supremum of the real parts of the characteristic roots). As we shall see, the spectral abscissa of (1) may not be a continuous function of the delays. Moreover, this may lead to a situation where infinitesimal delay perturbations destabilise an exponentially stable system. These properties are very similar to the spectral properties of neutral equations (see, e.g. [2, Section 2]), which are known to be closely related to DDAEs [3].

Proposition 9. *([15, 22]) Consider the linear, homogeneous DDAE* (3)*. Furthermore, assume that it is regular, impulse-free. Then it is stable if and only if the corresponding spectrum of this system lies entirely on the left half plane and it is bounded away from the imaginary axis.*

The following lemma plays the key role in the proof of the main Theorem 11 below.

Lemma 10. *Consider the linear, homogeneous DDAE* (3)*. Furthermore, assume that it is non-advanced. Then system* (3) *has the same spectrum (without counting multiplicity) as the index-reduced system* (7)*.*

PROOF. We will show that both systems (3) and (7) have the same spectrum (without counting multiplicity) as the system (5). Due to the variable transformation $x = Ty$ and the identity

$$W\left(\lambda E - A_0 - e^{-\lambda\tau_i} A_i\right) T = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - e^{-\lambda\tau_i} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix},$$

it is straightforward that

$$\sigma(E, A_0, ..., A_m) = \sigma\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} \\ \tilde{A}_{1,3} & \tilde{A}_{1,4} \end{bmatrix}, ..., \begin{bmatrix} \tilde{A}_{m,1} & \tilde{A}_{m,2} \\ \tilde{A}_{m,3} & \tilde{A}_{m,4} \end{bmatrix}\right). \tag{14}$$

{eq11}

Now let us consider the right hand side of (14), due to Lemma 4 we see that for an arbitrary $\lambda \in \mathbb{C}$

$$\begin{aligned} &\det\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} - \sum_{i=1}^{m} e^{-\lambda\tau_i} \begin{bmatrix} \tilde{A}_{i,1} & \tilde{A}_{i,2} \\ \tilde{A}_{i,3} & \tilde{A}_{i,4} \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} I & 0 \\ 0 & (I-\lambda N)^{-1} \end{bmatrix} \cdot \begin{bmatrix} I - J - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,1} & -\sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,2} \\ -\sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,3} & \lambda N - I - \sum_{i=1}^{m} e^{-\lambda\tau_i} \tilde{A}_{i,4} \end{bmatrix}\right). \end{aligned}$$

Due to Lemma 4 and the identity (6), we have

$$(I+\sum_{i=1}^{\nu}(\lambda N)^i)\cdot\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,3}=\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,3},$$
$$(I+\sum_{i=1}^{\nu}(\lambda N)^i)\cdot\left(\lambda N-I-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,4}\right)=-I-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,4}.$$

Hence, it follows that for any $\lambda\in\mathbb{C}$

$$\det\left(\begin{bmatrix}I&0\\0&N\end{bmatrix}-\begin{bmatrix}J&0\\0&I\end{bmatrix}-\sum_{i=1}^{m}e^{-\lambda\tau_i}\begin{bmatrix}\tilde{A}_{i,1}&\tilde{A}_{i,2}\\\tilde{A}_{i,3}&\tilde{A}_{i,4}\end{bmatrix}\right)$$
$$=\det\left(\begin{bmatrix}I-J-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,1}&-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,2}\\-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,3}&-I-\sum_{i=1}^{m}e^{-\lambda\tau_i}\tilde{A}_{i,4}\end{bmatrix}\right),$$

which yields that

$$\sigma\left(\begin{bmatrix}I&0\\0&N\end{bmatrix},\begin{bmatrix}J&0\\0&I\end{bmatrix},\begin{bmatrix}\tilde{A}_{1,1}&\tilde{A}_{1,2}\\\tilde{A}_{1,3}&\tilde{A}_{1,4}\end{bmatrix},...,\begin{bmatrix}\tilde{A}_{m,1}&\tilde{A}_{m,2}\\\tilde{A}_{m,3}&\tilde{A}_{m,4}\end{bmatrix}\right)=\sigma(\tilde{E},\tilde{A}_0,...,\tilde{A}_m). \tag{15}$$

{eq12}

From (14) and (15) we have $\sigma(E,A_0,...,A_m)=\sigma(\tilde{E},\tilde{A}_0,...,\tilde{A}_m)$. □

Theorem 11. *Consider the free system* (3)*. Furthermore, we assume that the matrix pair* (E,A_0) *is regular. Then,* (3) *is exponentially stable if and only if the following assertions hold.*

i) System (3) *is non-advanced.*

ii) The spectrum $\sigma(E,A_0,\ldots,A_m)$ *lies entirely on the left half plane and it is bounded away from the imaginary axis.*

PROOF. "$\Rightarrow$" Assume that system (3) is exponentially stable. Clearly, it is non-advanced, so we only need to prove ii). Furthermore, due to Lemma 6ii), system (3) is stable if and only if the index-reduced system (7) is also stable. Thus, the spectrum $\sigma(\tilde{E},\tilde{A}_0,\ldots,\tilde{A}_m)$ lies entirely on the left half plane and it is bounded away from the imaginary axis, and hence, due to Lemma 10 we obtain the desired claim.

"$\Leftarrow$" Since the index-reduced system (7) is impulse-free, Proposition 9 applied to it implies that the index-reduced system (7) is exponentially stable, and so is system (3). This completes the proof. □

Remark 4. Again, we notice that due to the numerical instabilities in computing the Kronecker-Weierstraß form (4), we will not compute the spectrum $\sigma(E, A_0, \ldots, A_m)$ based on (4). Instead, we refer the reader to the spectral discretisation approach in [15]. Nevertheless, since this method has only been developed for impulse-free (or index-1) system, we need the pre-processing step as in Lemma 12 below.

Let us consider the (reordered) QZ-decomposition ([23]) of the matrix pair (E, A_0) as follows

$$QEZ^T = \begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & N_E \end{bmatrix},\ QA_0Z^T = \begin{bmatrix} J_A & \hat{A}_2 \\ 0 & \Sigma_A \end{bmatrix},\ QA_iZ^T = \begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix}, \quad (16)$$

{eq15}

where Q and Z are orthogonal matrices, Σ_E and Σ_A are nonsingular, upper triangular matrices, N_E is a nilpotent, upper triangular matrix.

Using the same argument as in Lemma 6, we have the following lemma.

Lemma 12. *Consider the free system* (3) *and the QZ-decomposition* (16)*. Then, the following assertions hold true.*

i) System (3) *is non-advanced if and only if* $N_E\Sigma_A^{-1}\begin{bmatrix} \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix} = 0$ *for all* $i = 1, ..., m$.

ii) If this is the case, then there is a linear, bijective mapping $x \mapsto y = Zx$ *(where* Z *is the matrix given in* (16)*) between the solution set of the high-index system* (3) *and the following index-reduced system*

$$\begin{bmatrix} \Sigma_E & \hat{E}_2 \\ 0 & \mathbf{0} \end{bmatrix} \dot{y}(t) = \begin{bmatrix} J_A & \hat{A}_2 \\ 0 & \Sigma_A \end{bmatrix} y(t) + \sum_{i=1}^{m} \begin{bmatrix} \hat{A}_{i,1} & \hat{A}_{i,2} \\ \hat{A}_{i,3} & \hat{A}_{i,4} \end{bmatrix} y(t-\tau_i)\ . \quad (17)$$

{impulse free system}

PROOF. The proof is essentially the same as the proof of Lemma 6 and will be omitted to keep the brevity of this research.

Example 13. *To illustrate the advantage of the proposed method, we consider the following system, motivated from [24].*

$$\begin{bmatrix} -1 & 2 & 0.2648 \\ -2 & 4 & 0.8476 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 4.7 & 0.4 & 0.1192 \\ -4.9 & 0.8 & 1.1783 \\ 0 & 0 & 0.6473 \end{bmatrix} x(t) + \begin{bmatrix} 0.7 & -0.95 & 0.6456 \\ 1.1 & -1.75 & 1.7706 \\ 0 & 0 & 0 \end{bmatrix} x(t-0.2)$$
$$+ \begin{bmatrix} 1 & -0.8 & 0.6393 \\ 1.4 & -1.3 & 1.8234 \\ 0 & 0 & 0 \end{bmatrix} x(t-2). \quad (18)$$

{eq17}

We notice that the matrix pair (E, A_0) *in system* (18) *has index* $\nu = 2$*, and hence the system is not impulse-free. Using the MATLAB Toolbox TDS_STABIL ([25, 15]) we obtain the dominant eigenvalues of the original system* (18) *and that of the index-reduced system* (17)*. The result is presented in Figure 1. Clearly, we see that without the index-reduced step, the spectrum is not properly computed and hence, is not reliable to determine the stability of system* (18).

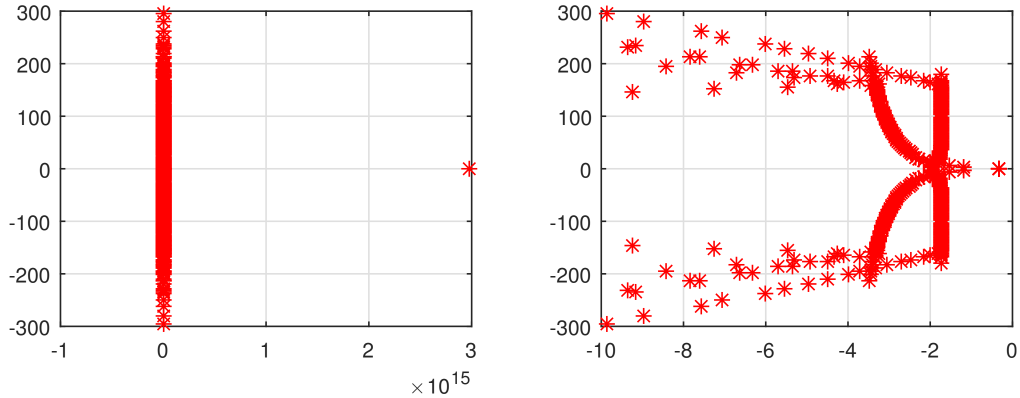

Figure 1: Spectrum of the system (18) (left) and the index-reduced system (17) (right), using the MATLAB Toolbox TDS_STABIL ([25]).

3.2. *Lyapunov-Krasovskii functional method*

Adopting the Lyapunov-Krasovskii approach, (sufficient) stability conditions for many classes of singular systems with different types of delays (single, multiple, time-varying, etc.) have been proposed, see for example, [24, 26, 27, 28, 29, 30, 31, 32, 33]. We, again, notice that all the conditions on the references mentioned above are only valid for impulse-free system. We recall one important result in the following proposition.

Proposition 14. *([30, 28]) Consider the linear, homogeneous DDAE* (3)*. Furthermore, assume that it is regular, impulse-free. Then it is stable if there exist matrices* $Q_i \succ 0$ *and matrices* P_i*,* $i = 1, ..., m$ *such that following LMI are satisfied*

$$M := \left[\begin{array}{c|ccc} AP^T + PA^T + Q & A_1 P_1^T & \dots & A_m P_m^T \\ \hline P_1 A_1^T & -Q_1 & & \\ \vdots & & \ddots & \\ P_m A_d^T & & & -Q_m \end{array}\right] \prec 0 \,. \qquad (19)$$

{LMI}

Similarly, in order to apply these results for arbitrarily-high index system, first we transform system (3) to the index-reduced form (17). We illustrate the advantage of this strategy in the following example.

Example 15. *Motivated from [18], let us consider the following system whose matrix coefficients are*

$$E = \begin{bmatrix} -11 & 1 & 0 \\ 0 & 0 & 0.127 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0.2 & 0.61 & 0.1891 \\ -1 & 0.6 & 0.5607 \\ 0 & 0 & 0.2998 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & -0.2 & -1.597 \\ -0.8 & -0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (20)$$

{eq18}

The system is not impulse-free and having an index $\nu(E, A) = 2$. If we directly apply the MATLAB LMI-Toolbox or the package CVX [34, 35] to the system (18) *then the obtained matrix M (defined by* (19)*) is not negative definite. Nevertheless, by transforming the system to the index-reduced form* (17) *which reads*

$$\begin{bmatrix} -4.802 & -9.9469 & -0.7885 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0.626 & -0.1423 & -0.1891 \\ 0 & 1.1662 & 0.5607 \\ 0 & 0 & 0.2998 \end{bmatrix} x(t) + \begin{bmatrix} -0.686 & -0.7546 & 1.597 \\ 0.4202 & 0.6808 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t-\tau) \tag{21}$$

{eq19}

then both the MATLAB LMI-Toolbox or the package CVX work properly. The matrices P, Q are

$$P = \begin{bmatrix} -15.3413 & 2.2457 & -2.6746 \\ 8.5630 & -4.1706 & 0.4629 \\ 0.1705 & -0.0140 & -0.8608 \end{bmatrix}, \; Q = \begin{bmatrix} 5.9597 & -1.1767 & 0.2239 \\ -1.1767 & 2.9958 & 0.3278 \\ 0.2239 & 0.3278 & 0.3995 \end{bmatrix}.$$

Remark 5. In comparison to the stability result obtained in [18], we do not make use of the Drazin inverse, and hence, the computation is stable and more reliable.

4. Conclusion and Outlook

Acknowledgment The author would like to thank the anonymous referee for his suggestions to improve this paper.

References

[1] U. M. Ascher, L. R. Petzold, The numerical solution of delay-differential algebraic equations of retarded and neutral type, SIAM J. Numer. Anal. 32 (1995) 1635–1657.

[2] S. L. Campbell, Nonregular 2D descriptor delay systems, IMA J. Math. Control Appl. 12 (1995) 57–67.

[3] L. F. Shampine, P. Gahinet, Delay-differential-algebraic equations in control theory, Appl. Numer. Math. 56 (3-4) (2006) 574–588.

[4] P. Ha, Analysis and numerical solutions of delay differential-algebraic equations, Dissertation, Institut für Mathematik, TU Berlin, Berlin, Germany (2015).

[5] P. Ha, V. Mehrmann, Analysis and numerical solution of linear delay differential-algebraic equations, BIT 56 (2016) 633 – 657.

[6] L. Dai, Singular Control Systems, Springer-Verlag, Berlin, Germany, 1989.

[7] P. Kunkel, V. Mehrmann, Differential-Algebraic Equations – Analysis and Numerical Solution, EMS Publishing House, Zürich, Switzerland, 2006.

[8] P. Ha, P. T. Nam, Stability analysis of arbitrarily high-index, positive delay-descriptor systems, in preparation (July 2021).

[9] R. Horn, C. Johnson, Matrix Analysis, Cambridge University Press, 1990.

[10] R. Bellman, K. L. Cooke, Differential-difference equations, Mathematics in Science and Engineering, Elsevier Science, 1963.

[11] J. Hale, S. Lunel, Introduction to Functional Differential Equations, Springer, 1993.

[12] B. Unger, Discontinuity propagation in delay differential-algebraic equations, The Electronic Journal of Linear Algebra 34 (2018) 582–601.

[13] W. Zhu, L. R. Petzold, Asymptotic stability of linear delay differential-algebraic equations and numerical methods, Appl. Numer. Math. 24 (1997) 247 – 264.

[14] W. Zhu, L. R. Petzold, Asymptotic stability of Hessenberg delay differential-algebraic equations of retarded or neutral type, Appl. Numer. Math. 27 (3) (1998) 309 – 325.

[15] W. Michiels, Spectrum-based stability analysis and stabilisation of systems described by delay differential algebraic equations, IET Control Theory Appl. 5 (16) (2011) 1829–1842.

[16] V. Phat, N. Sau, On exponential stability of linear singular positive delayed systems, Applied Mathematics Letters 38 (2014) 67–72.

[17] N. H. Sau, P. Niamsup, V. N. Phat, Positivity and stability analysis for linear implicit difference delay equations, Linear Algebra and its Applications 510 (2016) 25–41.

[18] Y. Cui, J. Shen, Z. Feng, Y. Chen, Stability analysis for positive singular systems with time-varying delays, IEEE Transactions on Automatic Control 63 (5) (2018) 1487–1494.

[19] P. H. A. Ngoc, Exponential stability of coupled linear delay time-varying differential-difference equations, IEEE Transactions on Automatic Control 63 (3) (2018) 843–848.

[20] P. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, Lin. Alg. Appl. 27 (1979) 103–141.

[21] K. E. Brenan, S. L. Campbell, L. R. Petzold, Numerical Solution of Initial-Value Problems in Differential Algebraic Equations, 2nd Edition, SIAM Publications, Philadelphia, PA, 1996.

[22] N. H. Du, V. H. Linh, V. Mehrmann, D. D. Thuan, Stability and robust stability of linear time-invariant delay differential-algebraic equations., SIAM J. Matr. Anal. Appl. 34 (4) (2013) 1631–1654.

[23] G. H. Golub, C. F. Van Loan, Matrix Computations, 3rd Edition, The Johns Hopkins University Press, Baltimore, MD, 1996.

[24] A. Haidar, E. Boukas, Exponential stability of singular systems with multiple time-varying delays, Automatica 45 (2) (2009) 539–545.

[25] W. Michiels, TDS-STABIL: A MATLAB tool for designing stabilizing fixed-order controllers for time-delay systems, available at http://twr.cs.kuleuven.be/research/software/delay-control/stab/, 2010.

[26] C. Noreddine, B. A. Sadek, T. El Houssaine, B. Boukili, Improved results on stability criteria for singular systems with interval time-varying delays, Journal of Circuits, Systems and Computers 30 (01) (2021) 2130001.

[27] H. Chen, P. Hu, New result on exponential stability for singular systems with two interval time-varying delays, IET Control Theory & Applications 7 (15) (2013) 1941–1949.

[28] S. Xu, P. Van Dooren, R. Ştefan, J. Lam, Robust stability and stabilization for singular systems with state delay and parameter uncertainty, IEEE Trans. Automat. Control 47 (7) (2002) 1122–1128.

[29] A. A. Sawoor, M. Sadkane, Lyapunov-based stability of delayed linear differential algebraic systems, Applied Mathematics Letters 118 (2021) 107–185.

[30] E. Fridman, Stability of linear descriptor systems with delay: a Lyapunov-based approach, J. Math. Anal. Appl. 273 (1) (2002) 24 – 44.

[31] Y.-J. Sun, Exponential Stability for Continuous-Time Singular Systems With Multiple Time Delays , Journal of Dynamic Systems, Measurement, and Control 125 (2) (2003) 262–264.

[32] P. H. A. Ngoc, Stability of coupled functional differential-difference equations, International Journal of Control 93 (8) (2020) 1920–1930.

[33] W. bin Chen, F. Gao, New results on stability analysis for a kind of neutral singular systems with mixed delays, European Journal of Control 53 (2020) 59–67.

[34] I. CVX Research, CVX: Matlab software for disciplined convex programming, version 2.0, `http://cvxr.com/cvx` (Aug. 2012).

[35] M. Grant, S. Boyd, Graph implementations for nonsmooth convex programs, in: V. Blondel, S. Boyd, H. Kimura (Eds.), Recent Advances in Learning and Control, Lecture Notes in Control and Information Sciences, Springer-Verlag Limited, 2008, pp. 95–110, `http://stanford.edu/~boyd/graph_dcp.html`.