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Partial differential-algebraic systems of second order with symmetric convection

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Abstract

This paper deals with initial boundary value problems (IBVPs) of linear and some semilinear partial differential algebraic equations (PDAEs) with symmetric first order (convection) terms which are semidiscretized with respect to the space variables by means of a standard conform finite element method. The aim is to give L^2-convergence results for the semidiscretized systems when the finite element mesh parameter h goes to zero. In general, without the assumption of symmetry (and some further conditions) it is difficult to get such results. According to many practical applications, the PDAEs may have also hyperbolic parts. These are described by means of Friedrichs' theory for symmetric positive systems of differential equations. The PDAEs are assumed to be of time index 1.

Keywords: Partial differential algebraic equations; Finite element method; Numerical solution

1. Introduction

In this paper the approximate numerical solution of linear or semilinear PDAEs for $u = u(t,x),\ t \in (0,t_e),\ t_e > 0,\ x \in \Omega \subset \mathbb{R}^d,\ d \geqslant 1$, of the form

$$Au_t + B(t,x)\Delta u + \sum_{j=1}^{d} D_j(t,x)\partial_j u + E(t,x)u + F(t,x,u)u = f(t,x), \tag{1}$$

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is considered. Δ is the Laplace operator, and ∂_j is the partial derivative operator with respect to the spatial variable x_j. For $d > 1$ Ω is assumed to be a bounded open set with smooth boundary. u, f are mappings $u, f : [0, t_e] \times \overline{\Omega} \to \mathbb{R}^{n+m}$, $n + m > 1$, where f (supposed to be sufficiently smooth) is given. A, B, D_j, E and F are assumed to be real $(n + m, n + m)$-matrices, and B, D_j, E, F may depend on t and x. If F is present in the system ($F \neq 0$), it is a function of components of u (for details see Section 5.2). A and B are supposed to be of block form,

$$A = \begin{pmatrix} I_n & \\ & 0_{m,m} \end{pmatrix}, \qquad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad B_{11} \in \mathbb{R}^{n,n}, \quad B_{22} \in \mathbb{R}^{m,m}, \quad \text{etc.} \tag{2}$$

I_n and $0_{m,m}$ denote the unit matrix of order n and a square zero matrix of order m, respectively. B_{11} is assumed to satisfy

$$-\big(B_{11}(t, x)z, z\big) \geqslant b_0 \sum_{i=1}^{n} z_i^2, \quad b_0 \geqslant 0,\ z \in \mathbb{R}^n. \tag{3}$$

According to the block form of A, we also split the vector u as

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1 \in \mathbb{R}^n,\ u_2 \in \mathbb{R}^m. \tag{4}$$

Then we consider IBVPs with initial value

$$u_1(0, x) = u^{10}(x), \quad x \in \overline{\Omega}, \tag{5}$$

where u^{10} is a given sufficiently smooth function. Initial values for u_2 (if needed) has to determined consistently by means of Eq. (1) and u^{10}. Further specifications of the matrices used in Eq. (1) and the form of the boundary values are given in Section 4.

This paper deals with the estimate of the L^2- norm $\|u - u_h\|$ in terms of the mesh parameter h. u_h denotes the solution of the semidiscretized system of the PDAE (1) whose convection terms are assumed to be symmetric. In general, it is not easy (and sometimes seems impossible) to obtain convergence results in the case that the matrices D_j are not symmetric (see, e.g., [8] where finite difference methods are used). However, if $D_j^{\mathrm{T}} = D_j$ and some further assumptions are valid, then convergence can be proved. In this paper, for the semidiscretization a standard FEM is used. An example of a PDAE with symmetric convection is given in Section 2. If the PDAEs have such terms, they are described by means of Friedrichs' corresponding theory recalled in Section 3. In Section 4 we state the problems in weak form with the corresponding assumptions for linear systems, and estimates of $\|u - u_h\|$ are derived in Sections 5.1 and 5.2 for linear and semilinear PDAEs, respectively.

2. Motivation

In order to motivate the investigation of "PDAEs with symmetric convection" we cite a known example from magnetohydrodynamics (MHD) which shows that the first order terms of this model can be described by matrices which are symmetric. This is the basic assumption here which together with some further assumptions (e.g., according to Friedrichs' theory of symmetric positive differential equations systems [2,3]) can be used to obtain certain convergence results.

Example 1. The example is the system of MHD equations in three space dimensions. Using standard notation, the corresponding model system for determining the electric field E, the magnetic field H, the velocity vector v, the density ρ and the pressure p is a nonlinear coupled system of Maxwells equations and the equations of fluid motion which is explicitly

$$\begin{aligned}
&\operatorname{curl} E = -\partial_t(\mu H),\\
&\operatorname{curl} H = J := \sigma\big(E + v \times \mu H\big),\\
&\operatorname{div}(\mu H) = 0,\\
&\frac{D}{Dt}\rho + \rho \operatorname{div} v = 0,\\
&\rho\frac{D}{Dt}v - \eta\Delta v = -\nabla p + J \times \mu H,\\
&\varphi(\rho) - p = 0.
\end{aligned}$$

μ, σ and η denote the (given) magnetic permeability, the electric conductivity and the viscosity of the fluid, respectively. The first three equations are Maxwells equations in the quasistationary case, and the last three equations are the equations of motion of the fluid. The last equation is the equation of state.

The system can be linearized in several ways. Here we linearize the expressions on the left side of the equations of fluid motion (i.e., we linearize the left side of the last three equations). To do this we introduce for $\rho(t,x)$, $v(t,x)$, $p(t,x)$ some reference states $R(t,x)$, $V(t,x)$, $P(t,x) = \varphi(R(t,x))$, $R(t,x) \neq 0$, and write $\rho = R + \rho'$, $v = V + v'$, $p = P + p'$. Defining a new vector $u = (E^{\mathrm T}, H^{\mathrm T}, v'^{\mathrm T}, \tilde\rho)^{\mathrm T}$ where $\tilde\rho = \frac{c}{R}\rho'$, $c^2 = \frac{\mathrm d\varphi(R)}{\mathrm d\rho}$, the MHD equations can be written with block diagonal matrices approximately (here 0_6 is a zero vector with 6 components)

$$\begin{aligned}
&\begin{pmatrix} A_M & \\ & I_4 \end{pmatrix}\partial_t u + \sum_{j=1}^{3}\begin{pmatrix} D_{Mj} & \\ & D_{Fj}\end{pmatrix}\partial_j u + \begin{pmatrix} E_M & \\ & E_F\end{pmatrix} u\\
&\qquad - \begin{pmatrix} 0_{6,6} & \\ & \begin{pmatrix} I_3 & \\ & 0\end{pmatrix}\end{pmatrix}\Delta u = \begin{pmatrix} 0_6 \\ \begin{pmatrix}\psi_a\\ \psi_b\end{pmatrix}\end{pmatrix}, \qquad (6)\\
&\operatorname{div}(\mu H) = 0. \qquad (7)
\end{aligned}$$

The coefficient matrices are given by

$$A_M = \begin{pmatrix} 0 &&&&& \\ & 0 &&&& \\ && 0 &&& \\ &&& \mu && \\ &&&& \mu & \\ &&&&& \mu \end{pmatrix}, \qquad D_{M1} = \begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&-1&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&1&0&0&0&0 \end{pmatrix},$$

$$D_{M2} = \begin{pmatrix} 0&0&0&0&0&-1\\ 0&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ -1&0&0&0&0&0 \end{pmatrix}, \qquad D_{M3} = \begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0 \end{pmatrix},$$

$$D_{F1} = \begin{pmatrix} V_1 & 0 & 0 & C \\ 0 & V_1 & 0 & 0 \\ 0 & 0 & V_1 & 0 \\ C & 0 & 0 & V_1 \end{pmatrix}, \qquad D_{F2} = \begin{pmatrix} V_2 & 0 & 0 & 0 \\ 0 & V_2 & 0 & C \\ 0 & 0 & V_2 & 0 \\ 0 & C & 0 & V_2 \end{pmatrix},$$

$$D_{F3} = \begin{pmatrix} V_3 & 0 & 0 & 0 \\ 0 & V_3 & 0 & 0 \\ 0 & 0 & V_2 & C \\ 0 & 0 & C & V_2 \end{pmatrix}, \qquad E_M = \sigma \begin{pmatrix} 1 & 0 & 0 & 0 & -\mu v_3 & \mu v_2 \\ 0 & 1 & 0 & \mu v_3 & 0 & -\mu v_1 \\ 0 & 0 & 1 & -\mu v_2 & \mu v_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{v=V+v'},$$

$$E_F = \begin{pmatrix} \partial_1 V_1 & \partial_2 V_1 & \partial_3 V_1 & Z_{a1} \\ \partial_1 V_2 & \partial_2 V_2 & \partial_3 V_2 & Z_{a2} \\ \partial_1 V_3 & \partial_2 V_3 & \partial_3 V_3 & Z_{a3} \\ \frac{C}{R}\partial_1 R & \frac{C}{R}\partial_2 R & \frac{C}{R}\partial_3 R & Z_b \end{pmatrix},$$

$$Z_a = \frac{1}{C}\left(\frac{D}{Dt}V + \nabla C^2\right) + \frac{C^2}{R}\nabla\frac{R}{C}, \qquad Z_b = \frac{C}{R}\frac{D}{Dt}\frac{R}{C} + \operatorname{div} V.$$

The right vector $\left(\begin{smallmatrix} 0_6 \\ \left(\begin{smallmatrix}\psi_a \\ \psi_b\end{smallmatrix}\right)\end{smallmatrix}\right)$ is easily obtained from the right side of the original MHD system but it is not needed here (and therefore it is not given explicitly).

What is essential is the fact that in Eq. (6) the matrices in front of the first spatial derivatives are symmetric, and the matrices A and B are diagonal. The coupling of the electromagnetic and mechanical quantities on the left side is given by the matrix $E = \left(\begin{smallmatrix} E_M & \\ & E_F \end{smallmatrix}\right)$. System (6) can be seen to consist of a parabolic and a first order part and a part which does not have time derivatives. If a weak formulation is used, the constraint (7) could be incorporated into a suitably chosen space (below a FEM is considered). In this way the MHD equations are a system with "symmetric convection".

In Eq. (6), the matrix in front of the time derivative ∂u_t is not of the structure of the matrix A as given in (2). Since in this paper A is assumed to have that form, we reformulate the MHD system appropriately. It is easy to see that by re-ordering the components of the vector u and after division of the electromagnetic equations by $\mu > 0$, the MHD system (6) can be written in the following block form (0_{ij} means a zero matrix with i rows and j columns)

$$\begin{aligned} &\begin{pmatrix} I_4 & & \\ & I_3 & \\ & & 0_{33} \end{pmatrix} v_t + \begin{pmatrix} -\begin{pmatrix} I_3 & \\ & 0 \end{pmatrix} & & \\ & 0_{33} & \\ & & 0_{33} \end{pmatrix} \Delta v \\ &\quad + \sum_{j=1}^{3} \begin{pmatrix} D_{Fj} & 0_{43} & 0_{43} \\ 0_{34} & 0_{33} & \frac{1}{\mu} D_{Mj21} \\ 0_{34} & \frac{1}{\mu} D_{Mj12} & 0_{33} \end{pmatrix} \partial_j v \\ &\quad + \begin{pmatrix} E_F & 0_{43} & 0_{43} \\ 0_{34} & 0_{33} & 0_{33} \\ 0_{34} & \frac{1}{\mu} E_{M12} & \frac{1}{\mu} E_{M11} \end{pmatrix} v = \begin{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} \\ 0_3 \\ 0_3 \end{pmatrix}. \end{aligned} \tag{8}$$

This form follows from the fact that the matrices D_{Mj} and E_M can be written as block matrices,

$$D_{Mj} = \begin{pmatrix} 0_{33} & D_{Mj12} \\ D_{Mj21} & 0_{33} \end{pmatrix} \quad \text{with } D_{Mj21}^{\mathrm{T}} = D_{Mj12},\ j = 1, 2, 3,$$

and

$$E_M = \begin{pmatrix} E_{M11} & E_{M12} \\ 0_{33} & 0_{33} \end{pmatrix},$$

respectively. If we write (according to Eq. (4)) $u = \binom{u_1}{u_2}$ and $u_1 = (\binom{u_{11}}{u_{12}})$ where $u_1 \in \mathbb{R}^6$, u_{11}, $u_{12} \in \mathbb{R}^3$, $u_2 \in \mathbb{R}^4$, then we obtain the system (6) in the form given in (8) with

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \qquad v_1 = u_2, \qquad v_2 = u_{12}, \qquad v_3 = u_{11}.$$

Of course, the matrix A in system (8) has now the required structure, and it is again a system with symmetric convection (the matrix in front of $\partial_j v$ is symmetric). Furthermore, the matrix B has the form as assumed in (2) and (19).

Further systems with symmetric first order terms are the system of incompressible Navier–Stokes equations and a linearization of the system of compressible Euler equations of gas dynamics (see, e.g., [6]).

3. Symmetric first order systems

In this section we recall K.O. Friedrichs theory for symmetric first order systems [2,3], see also [7]. Let $t \in (0, t_e)$, $x \in \Omega \subset \mathbb{R}^d$, $f, u, u^0 \in \mathbb{R}^k$, $k \geqslant 1$, d_j, $e \in \mathbb{R}^{k,k}$, and

$$u_t + Lu = f(t,x), \qquad L = \sum_{j=1}^{d} d_j(t,x)\partial_j + e(t,x), \tag{9}$$

$$u(0,x) = u^0(x), \quad x \in \overline{\Omega}. \tag{10}$$

Suppose d_j, e are sufficiently smooth and

(1) $d_j^{\mathrm{T}} = d_j$,
(2) F1: there is a square matrix $M = M(t,x)$ with

$$\frac{1}{2}M \geqslant \sigma_M I_k \quad \text{on } \partial\Omega,\ \sigma_M > 0, \tag{11}$$

$$\operatorname{Ker}(G - M) \oplus \operatorname{Ker}(G + M) = \mathbb{R}^k \quad \text{on } \partial\Omega \tag{12}$$

where $G = \sum_{j=1}^{d} \bar{n}_j d_j$, $\quad \bar{n} = (\bar{n}_1, \ldots, \bar{n}_d)^{\mathrm{T}}$ is the outward unit normal on $\partial\Omega$,

$$(M - G)u = 0 \quad \text{on } \partial\Omega, \tag{13}$$

F2: $$\frac{1}{2}\left[e + e^{\mathrm{T}} - \sum_{j=1}^{d} (\partial_j d_j)\right] \geqslant \sigma I_k, \quad \sigma > 0. \tag{14}$$

For matrices P and Q, the relation $P \geqslant Q$ means that $P - Q$ is positive semi definite.

Then the boundary values of the hyperbolic system are defined by (13), and the following Theorem is valid.

Theorem 2 (For zero boundary values). *Let $f \in (L^2((0, t_e) \times \Omega))^k$ and $u^0 \in (L^2(\Omega))^k$. Then the IBVP* (9), (10), (13) *admits a unique weak solution.*

For completeness the weak form of (9), (10) with some boundary value g,

$$(M - G)(u - g) = 0 \quad \text{on } \partial\Omega,$$

used below is given here. We impose the boundary conditions weakly (see, e.g., [5, Chapter 9]), i.e., the following variational problem is considered:

Problem 1. Find $u(t) \in V = (H^1)^k$, $t \in (0, t_e)$, such that

$$\big(u'(t), w\big) + b\big(u(t), w\big) = l(w), \quad \forall w \in V, \tag{15}$$

$$u(0) = \tilde{u}^0, \tag{16}$$

where the bilinear form $b(\cdot\,,\cdot)$ and the linear form $l(\cdot)$ are defined by

$$b(v, w) = (Lv, w) + \frac{1}{2}\big\langle (M - G)v, w\big\rangle, \tag{17}$$

$$l(w) = (f, w) + \frac{1}{2}\big\langle (M - G)g, w\big\rangle. \tag{18}$$

H^1 is the usual Sobolew space $H^1(\Omega)$, and $\tilde{u}^0 \in V$ is a suitable approximation of u^0 in V.

$(v, w) = \int_\Omega v^{\mathrm{T}} w \,\mathrm{d}x$ and $\langle v, w\rangle = \int_{\partial\Omega} v^{\mathrm{T}} w \,\mathrm{d}s$ are the L^2-scalar products on Ω and $\partial\Omega$, respectively. The corresponding norms are denoted by $\|v\| = (v, v)^{1/2}$ and $|v|_{\partial\Omega} = \langle v, v\rangle^{1/2}$.

4. The linear PDAE and its weak form

In this section we set $F = 0$ in (1) and deal with the case of weak coupling of the two functions u_1 and u_2 (see (4)) which means that they are coupled by the matrices E_{12} and E_{21} only. Guided by the examples in Section 2 we specify the PDAE (1) with A as defined in Eq. (2) in detail. First we suppose the matrix B_{11} to be of block diagonal form,

$$B_{11} = \begin{pmatrix} B_1 & \\ & 0_{n_2,n_2} \end{pmatrix}, \quad B_1 \in \mathbb{R}^{n_1,n_1},\ n_1 + n_2 = n. \tag{19}$$

According to this structure of B_{11} we write analogously $u_1 = \binom{u_{11}}{u_{12}}$, $u_{1k} \in \mathbb{R}^{n_k}$. If we decompose all matrices as shown in (2), then the PDAE (1) studied here is

$$u_{1t} + B_{11}\Delta u_1 + \sum_{j=1}^{d} D_{j11}\partial_j u_1 + E_{11}u_1 + E_{12}u_2 = f_1, \tag{20}$$

$$B_{22}\Delta u_2 + \sum_{j=1}^{d} D_{j22}\partial_j u_2 + E_{22}u_2 + E_{21}u_1 = f_2. \tag{21}$$

The first equation, (20), can be written obviously[1]

$$\begin{aligned} &\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix}_t + \begin{pmatrix} B_1 & \\ & 0_{n_2,n_2} \end{pmatrix} \Delta \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} + \sum_{j=1}^{d} \begin{pmatrix} D_{j11}^{11} & D_{j11}^{12} \\ D_{j11}^{21} & D_{j11}^{22} \end{pmatrix} \partial_j \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \\ &\quad + \begin{pmatrix} E_{11}^{11} & E_{11}^{12} \\ E_{11}^{21} & E_{11}^{22} \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} + E_{12}u_2 = f_1. \end{aligned} \tag{22}$$

The system (20), (21) is studied now under the following:

General assumptions:

(1) All data are sufficiently smooth,
(2) $D_j^{\mathrm{T}} = D_j$,
(3) the matrix B_1 is negative definite. Then (according to Eq. (22)) the subsystem for u_{11},

$$u_{11t} + \left(B_1\Delta + \sum_{j=1}^{d} D_{j11}^{11}\partial_j + E_{11}^{11}\right)u_{11} = f_{11} - \left(\sum_{j=1}^{d} D_{j11}^{12}\partial_j + E_{11}^{12}\right)u_{12} - (E_{12}u_2)_1,$$

and the subsystem for u_{12},

$$u_{12t} + \left(\sum_{j=1}^{d} D_{j11}^{22}\partial_j + E_{11}^{22}\right)u_{12} = f_{12} - \left(\sum_{j=1}^{d} D_{j11}^{21}\partial_j + E_{11}^{21}\right)u_{11} - (E_{12}u_2)_2,$$

are considered to be the parabolic and the hyperbolic part (in the $(t-x)$-domain), respectively (here, $f_1 = (f_{11}^{\mathrm{T}}, f_{12}^{\mathrm{T}})^{\mathrm{T}}$ and $E_{12}u_2 = ((E_{12}u_2)_1^{\mathrm{T}}, (E_{12}u_2)_2^{\mathrm{T}})^{\mathrm{T}}$),
(4) Friedrichs assumptions F1, F2 are valid for the first order part, and F2 holds for the parabolic part.

For u_1 initial values are needed, see (5). The boundary values (BVs) are as follows.

Boundary values:

(1) Neumann BVs are chosen for the parabolic component, i.e., $\frac{\partial u_{11}}{\partial \bar{n}}(t,x) = u_{11N}(t,x)$, $t \in [0, t_e]$, $x \in \partial\Omega$, u_{11N} given,

[1] Mixed hyperbolic-parabolic systems of the form (22) (but with $E_{12} = 0$ and with C^{∞}-smooth coefficients which are 1-periodic in x) under periodic boundary conditions are considered by Kreiss and Lorenz [6].

(2) BVs for u_{12} are constructed by means of the M-matrix which is based now on D^{22}_{j11} and E^{22}_{11}, i.e., $d_j = D^{22}_{j11}$, $e = E^{22}_{11}$ in Eqs. (9), and (13) is replaced by

$$\big(M^{22}_{11} - G^{22}_{11}\big)(u_{12} - g_{12}) = 0 \quad \text{on } \partial\Omega, \qquad G^{22}_{11} = \sum_{j=1}^{d} \bar{n}_j D^{22}_{j11}. \tag{23}$$

The (n_2, n_2)-matrix M^{22}_{11} is the matrix M defined in relations (11), (12). The function g_{12} is considered to be given on that parts of $\partial\Omega$ where BVs are defined by means of (23).

(3) BVs for u_2 are defined by inspection of Eq. (21).

With these preliminaries a weak form of the problem can be given:

Problem 2. Find $(u_1(t), u_2(t)) \in V_1 \times V_2$, $V_1 = (H^1)^n$ and, e.g., $V_2 \subseteq (H^1)^m$ such that

$$(u_{1t}, v_1) + a_1(u_1, v_1) + (E_{12}u_2, v_1) = f_1(v_1) \quad v_1 \in V_1, \tag{24}$$

$$a_2(u_2, v_2) + (E_{21}u_1, v_2) = f_2(v_2) \quad v_2 \in V_2, \tag{25}$$

$$u_1(0) = \tilde{u}^{10}, \tag{26}$$

$\tilde{u}^{10} \in V_1$ is a suitable approximation of u^{10}, and

$$a_1(u_1, v_1) = (Q_{11}u_1, v_1) + \frac{1}{2}\langle M_s u_1, v_1\rangle, \qquad a_2(u_2, v_2) = (Q_{22}u_2, v_2), \tag{27}$$

$$(Q_{ii}u_i, v_i) = -\sum_{j=1}^{d}\big[\big((\partial_j B_{ii})\partial_j u_i, v_i\big) + (B_{ii}\partial_j u_i, \partial_j v_i)\big] + \sum_{j=1}^{d}(D_{jii}\partial_j u_i, v_i) + (E_{ii}u_i, v_i), \tag{28}$$

$$f_1(v_1) = (f_1, v_1) - \langle B_{11}u_{1N}, v_1\rangle + \frac{1}{2}\langle M_s g_1, v_1\rangle, \qquad f_2(v_2) = (f_2, v_2) + \cdots, \tag{29}$$

$$u_{1N} = \begin{pmatrix} u_{11N} \\ 0_{n_2} \end{pmatrix}, \qquad g_1 = \begin{pmatrix} 0_{n_1} \\ g_{12} \end{pmatrix}, \qquad M_s = \begin{pmatrix} 0_{n_1,n_1} & \\ & (M^{22}_{11} - G^{22}_{11}) \end{pmatrix}. \tag{30}$$

"$+\cdots$" means that there may be further terms depending on the boundary conditions for u_2 (to be specified in each case).

5. Convergence

In this section L^2-error estimates in the linear case (Section 5.1) and the semilinear case (Section 5.2) are derived. The starting point are the following two basic Lemmata, and for completeness we repeat here also essential assumptions stated already before.

Lemma 3. *Suppose*

(1) B_1 =*const.* (*for simplicity*), B_1 *is negative definite,*
(2) $D_j^{\mathrm{T}} = D_j$ (*this implies, of course,* $D^{\mathrm{T}}_{j11} = D_{j11}$),

(3) $\frac{1}{2}M_{11}^{22} \geqslant \sigma_M I_{n_2}$ *on* $\partial\Omega$, $\sigma_M \geqslant 0$,
(4) $\frac{1}{2}[E_{11} + E_{11}^{\mathrm{T}} - \sum_{j=1}^{d}(\partial_j D_{j11})] \geqslant \sigma_1 I_n$, *in* Ω, $\sigma_1 > 0$,
(5) $\frac{1}{2}\sum_{j=1}^{d}\bar{n}_j D_{j11}^{11} \geqslant \sigma_{11} I_{n_1}$ *on* $\partial\Omega$, $\sigma_{11} \geqslant 0$,
(6) $\sum_{j=1}^{d}\bar{n}_j(D_{j11}^{12} + D_{j11}^{21\mathrm{T}}) = 0$ *on* $\partial\Omega$.

Then $a_1(v_1, v_1) \geqslant \sigma_1\|v_1\|^2 + \sigma_0|v_1|_{\partial\Omega}^2$, $\forall v_1 \in V_1$, *where* $\sigma_0 = \min\{\sigma_{11}, \sigma_M\}$.

Remark 4. If one introduces spaces $V_{11} = (H^1)^{n_1}$ and $V_{12} = (H^1)^{n_2}$, the last two assumptions can be replaced by

(5′) $\frac{1}{2}\sum_{j=1}^{d} < v, \bar{n}_j D_{j11}^{11} v > \geqslant \sigma_{11}|v|_{\partial\Omega}^2$, $v \in V_{11}$, $\sigma_{11} \geqslant 0$,
(6′) $\sum_{j=1}^{d} < v, \bar{n}_j(D_{j11}^{12} + D_{j11}^{21\mathrm{T}})w > = 0$, $v \in V_{11}$, $w \in V_{12}$.

These assumptions may be less restrictive. From a physical point of view the sixth assumption means that the parabolic and the hyperbolic part of the system do interact in the interior Ω, but not on the surface $\partial\Omega$.

Proof. Because the bilinear form a_1 is connected with $(1,1)$-components of the matrices only, we simplify the notation in this proof. Some vector and matrix indices not needed here are suppressed, e.g., the bilinear form $a_1(u_1, v_1) = -\sum_{j=1}^{d}(B_{11}\partial_j u_1, \partial_j v_1) + \sum_{j=1}^{d}(D_{j11}\partial_j u_1, v_1) + (E_{11}u_1, v_1) + \frac{1}{2}\langle M_s u_1, v_1\rangle$, $u_1, v_1 \in V_1$, is written shortly

$$a_1(u, v) = -\sum_{j=1}^{d}(B\partial_j u, \partial_j v) + \sum_{j=1}^{d}(D_j\partial_j u, v) + (Eu, v) + \frac{1}{2}\langle M_s u, v\rangle, \quad u, v \in V_1,$$

where $M_s = \begin{pmatrix} 0_{n_1,n_1} & \\ & (M^{22} - G^{22}) \end{pmatrix}$, $G^{22} = \sum_{j=1}^{d}\bar{n}_j D_j^{22}$.

Let $u, v \in V_1$. The proof of the Lemma starts from the identity $(D_j\partial_j u, v) = (\partial_j u, D_j^{\mathrm{T}} v) = (\partial_j u, D_j v)$ where the last equality comes from the second assumption. Using Green's identity $\int_\Omega(\partial_j w)\bar{w}\,\mathrm{d}x = \int_{\partial\Omega} w\bar{w}\bar{n}_j\,\mathrm{d}s - \int_\Omega w\partial_j\bar{w}\,\mathrm{d}x$, $w, \bar{w} \in \mathbb{R}$, $j = 1, \ldots, d$, one finds $(D_j\partial_j u, v) = \langle u, \bar{n}_j D_j v\rangle - (u, \partial_j(D_j v))$ $= \langle u, \bar{n}_j D_j v\rangle - (u, (\partial_j D_j)v) - (u, D_j\partial_j v)$. The special case $u = v$ gives $(D_j\partial_j v, v) = \frac{1}{2}\langle v, \bar{n}_j D_j v\rangle - \frac{1}{2}(v, (\partial_j D_j)v)$. With $(Ev, v) = \frac{1}{2}(v, E^{\mathrm{T}} v) + \frac{1}{2}(Ev, v) = \frac{1}{2}(v, (E + E^{\mathrm{T}})v)$ and the fourth assumption, one obtains

$$\sum_{j=1}^{d}(D_j\partial_j v, v) + (Ev, v) = \frac{1}{2}\sum_{j=1}^{d}\langle v, \bar{n}_j D_j v\rangle + \left(v, \frac{1}{2}\left[E + E^{\mathrm{T}} - \sum_{j=1}^{d}(\partial_j D_j)\right]v\right)$$
$$\geqslant \frac{1}{2}\sum_{j=1}^{d}\langle v, \bar{n}_j D_j v\rangle + \sigma_1\|v\|^2.$$

This with the first assumption and with $v = (v_1^{\mathrm{T}}, v_2^{\mathrm{T}})^{\mathrm{T}}$ implies

$$a_1(v, v) \geqslant \frac{1}{2}\sum_{j=1}^{d} \langle v, \bar{n}_j D_j v\rangle + \sigma_1 \|v\|^2 + \frac{1}{2}\langle M^{22} v_2, v_2\rangle - \frac{1}{2}\langle G^{22} v_2, v_2\rangle. \tag{31}$$

The first and the last term on the right side can be combined as follows:

$$\begin{aligned}\sum_{j=1}^{d} \langle v, \bar{n}_j D_j v\rangle - \langle G^{22} v_2, v_2\rangle &= \sum_{j=1}^{d} \big[\langle v_1, \bar{n}_j D_j^{11} v_1\rangle + \langle v_1, \bar{n}_j D_j^{12} v_2\rangle\big] \\ &\quad + \sum_{j=1}^{d} \big[\langle v_2, \bar{n}_j D_j^{21} v_1\rangle + \langle v_2, \bar{n}_j D_j^{22} v_2\rangle\big] - \sum_{j=1}^{d} \langle \bar{n}_j D_j^{22} v_2, v_2\rangle \\ &= \sum_{j=1}^{d} \big[\langle v_1, \bar{n}_j D_j^{11} v_1\rangle + \langle v_1, \bar{n}_j \big[D_j^{12} + D_j^{21T}\big] v_2\rangle\big] \geqslant 2\sigma_{11} |v_1|_{\partial\Omega}^2.\end{aligned}$$

The last estimate follows from the two last assumptions of the lemma. Using this estimate and the third assumption, the lower bound in inequality (31) can be estimated further to give

$$a_1(v, v) \geqslant \sigma_1 \|v\|^2 + \sigma_{11} |v_1|_{\partial\Omega}^2 + \sigma_M |v_2|_{\partial\Omega}^2 \geqslant \sigma_1 \|v\|^2 + \min\{\sigma_{11}, \sigma_M\} |v|_{\partial\Omega}^2,$$

which proves the lemma. □

Lemma 5. *Suppose*

(1) $B_{22} = \mathit{const.}$ (*for simplicity*), B_{22} *is negative semi definite,*
(2) $D_{j22}^{\mathrm{T}} = D_{j22}$,
(3) $\frac{1}{2}[E_{22} + E_{22}^{\mathrm{T}} - \sum_{j=1}^{d} (\partial_j D_{j22})] \geqslant \sigma_2 I_m$, *in* Ω, $\quad \sigma_2 > 0$,
(4) $\frac{1}{2}\sum_{j=1}^{d} \bar{n}_j D_{j22} \geqslant \sigma_{22} I_m$ *on* $\partial\Omega$, $\sigma_{22} \geqslant 0$.

Then $a_2(v_2, v_2) \geqslant \sigma_2 \|v_2\|^2 + \sigma_{22} |v_2|_{\partial\Omega}^2$, $\forall v_2 \in V_2$.

Remark 6. Again, the last assumption can be replaced by $\frac{1}{2}\sum_{j=1}^{d} \langle v, \bar{n}_j D_{j22} v\rangle \geqslant \sigma_{22} |v|_{\partial\Omega}^2$, $v \in V_2$, $\sigma_{22} \geqslant 0$.

Proof. The proof of this lemma is similar to the proof of Lemma 3, and therefore it is suppressed here. In the proof of Lemma 3 only the parts concerning the matrix M_s must be omitted. □

To construct a numerical solution of Problem 2 we consider a standard conforming finite element method:

Problem 3. Find $(u_{1h}(t), u_{2h}(t)) \in V_{1h} \times V_{2h}$, $V_{1h} \subset V_1$, $V_{2h} \subset V_2$ such that

$$(u_{1ht}, v_{1h}) + a_1(u_{1h}, v_{1h}) + (E_{12} u_{2h}, v_{1h}) = f_1(v_{1h}) \quad v_{1h} \in V_{1h}, \tag{32}$$

$$a_2(u_{2h}, v_{2h}) + (E_{21} u_{1h}, v_{2h}) = f_2(v_{2h}) \quad v_{2h} \in V_{2h}, \tag{33}$$

$$u_{1h}(0) = \tilde{u}_h^{10}. \tag{34}$$

$\tilde{u}_h^{10} \in V_{1h}$ is a projection of u^{10} onto V_{1h} (see below).

Let e be the error $u - u_h = \binom{u_1 - u_{1h}}{u_2 - u_{2h}} = \binom{e_1}{e_2}$ which is splitted in the usual way,

$$e = u - u_h = (u - \tilde{u}_h) + \eta, \quad \eta = \tilde{u}_h - u_h, \tag{35}$$

where $\tilde{u}_h = \binom{\tilde{u}_{1h}}{\tilde{u}_{2h}}$, and $\tilde{u}_{jh}$ is the Ritz projection of u_j onto V_{jh}, $j = 1, 2$,

$$a_j(\tilde{u}_{jh}, v_{jh}) = a_j(u_j, v_{jh}), \quad \forall v_{jh} \in V_{jh}. \tag{36}$$

$\|u - \tilde{u}_h\|$ can be estimated by standard interpolation theory, see, e.g., [1,4]. A typical result for $\|u - \tilde{u}_h\|$ is ($r \in \mathbb{N}$, $r \geqslant 1$, h is the finite element mesh parameter, C is a positive constant independent of h, and $|\cdot|$ means the semi norm)

$$\|u - \tilde{u}_h\| \leqslant Ch^{r+1}|u|_{(H^{r+1})^{n+m}}, \qquad \|(u - \tilde{u}_h)_t\| \leqslant Ch^{r+1}|u_t|_{(H^{r+1})^{n+m}}, \tag{37}$$

provided u and u_t have the regularity required on the right side of the inequalities. Therefore, in order to obtain an upper bound of $\|e\|^2$ we have to estimate $\|\eta\|^2 = \|\eta_1\|^2 + \|\eta_2\|^2$ only.

5.1. Linear system

First we consider the case that the PDAE (1) is linear ($F = 0$) and of time index $\nu_t = 1$, i.e., Eq. (25) can be solved for u_2 in terms of u_1. By $P_r(K)$ we denote the space of polynomials of degree $\leqslant r$ in the variables $x_1, \dots, x_d$ on the element K of the finite element decomposition $\mathcal{T}_h$ of Ω. Under standard assumptions of a conform and quasiuniform (or regular, see, e.g., [1, p. 124]) FEM we now prove

Theorem 7 (For Problems 2 and 3). *Suppose*

(1) $\nu_t = 1$,
(2) *the assumptions of Lemma* 3 *are valid,*
(3) $u_1 \in V_1 \cap (H^{r_1+1})^n$, $u_2 \in V_2 \cap (H^{r_2+1})^m$, $r_1, r_2 \geqslant 1$,
(4) $V_{1h} = \{v \in (H^1)^n,\ v|_K \in (P_{r_1}(K))^n\ \forall K \in \mathcal{T}_h\}$, *and, e.g.,* $V_{2h} = \{v \in (H_0^1)^m,\ v|_K \in (P_{r_2}(K))^m\ \forall K \in \mathcal{T}_h\}$,
(5) *either* $a_2(\cdot,\cdot)$ *is* V_2*−elliptic, i.e.,* $a_2(v_2, v_2) \geqslant \alpha\|v_2\|_{V_2}^2$, $v_2 \in V_2$, $\alpha > 0$, *or Lemma* 5 *is valid, i.e.,* $a_2(v_2, v_2) \geqslant \alpha\|v_2\|^2$, *where now* $\alpha = \sigma_2$.

Then

$$\|u_1(t) - u_{1h}(t)\| \leqslant \|u_{1h}(0) - \tilde{u}_{1h}(0)\| e^{-\gamma t} + \mathrm{O}(h^{r_1+1}) + \mathrm{O}(h^{r_2+1}), \tag{38}$$

$$\|u_2(t) - u_{2h}(t)\| \leqslant \frac{1}{\alpha}\|E_{21}(t)\| \cdot \|u_1(t) - u_{1h}(t)\| + \mathrm{O}(h^{r_2+1}), \tag{39}$$

where the constant $\gamma = \sigma_1 - \beta_1$, *see* (46), *is independent of the mesh parameter* h, *and*

$$\mathrm{O}(h^{r_j+1}) \sim h^{r_j+1} \max_{\tau \in [0,t_e]} |u_j(\tau)|_{(H^{r_j+1})^{k_j}}, \quad j = 1, 2,\ k_1 = n,\ k_2 = m. \tag{40}$$

(As usual, in any estimate or inequality, the quantity C or $C(t)$ denotes a generic positive constant always independent of h and need not be the same constant in different occurrences.)

Proof. $\|\eta_1\|$ and $\|\eta_2\|$ are estimated separately. To obtain an equation for $\|\eta_1(t)\|$, consider the identity $(\eta_{1t}, v_{1h}) + a_1(\eta_1, v_{1h}) = (\tilde{u}_{1ht}, v_{1h}) - (u_{1ht}, v_{1h}) + a_1(\tilde{u}_{1h}, v_{1h}) - a_1(u_{1h}, v_{1h})$ and note that $a_1(\tilde{u}_{1h}, v_{1h}) = a_1(u_1, v_{1h})$ and that (by Eq. (32)) $-(u_{1ht}, v_{1h}) - a_1(u_{1h}, v_{1h}) = -f_1(v_{1h}) + (E_{12}u_{2h}, v_{1h})$. Therefore, the right side of the identity can be written $(\tilde{u}_{1ht}, v_{1h}) - f_1(v_{1h}) + (E_{12}u_{2h}, v_{1h}) + a_1(u_1, v_{1h})$ which (by Eq. (24)) reduces to $(\tilde{u}_{1ht}, v_{1h}) - (u_{1t}, v_{1h}) - (E_{12}u_2, v_{1h}) + (E_{12}u_{2h}, v_{1h})$, and η_1 satisfies

$$(\eta_{1t}, v_{1h}) + a_1(\eta_1, v_{1h}) = (\tilde{u}_{1ht} - u_{1t}, v_{1h}) + \bigl(E_{12}(u_{2h} - u_2), v_{1h}\bigr). \tag{41}$$

Since $\eta_1 \in V_{1h}$, $v_{1h} = \eta_1$ can be chosen, and one gets from the last equation and Lemma 3 $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1\|^2 + \sigma_1\|\eta_1\|^2 \leqslant \|\tilde{u}_{1ht} - u_{1t}\|\|\eta_1\| + \|E_{12}e_2\|\|\eta_1\|$ or (because $\|E_{12}\| < \infty$)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1\| + \sigma_1\|\eta_1\| &\leqslant \bigl\|(u_1 - \tilde{u}_{1h})_t\bigr\| + \|E_{12}\|\|e_2\| \\ &\leqslant \bigl\|(u_1 - \tilde{u}_{1h})_t\bigr\| + \|E_{12}\|\bigl(\|u_2 - \tilde{u}_{2h}\| + \|\eta_2\|\bigr).\end{aligned} \tag{42}$$

The next step is to estimate $\|\eta_2\|$. Set $v_2 = v_{2h}$ in Eq. (25), and then subtract Eq. (33) from Eq. (25). The result is $a_2(e_2, v_{2h}) = -(E_{21}e_1, v_{2h})\ \forall v_{2h} \in V_{2h}$. Thus, $a_2(e_2, v_{2h}) = a_2(u_2 - \tilde{u}_{2h}, v_{2h}) + a_2(\eta_2, v_{2h}) = -(E_{21}e_1, v_{2h})$, and with $\eta_2 \in V_{2h}$ we obtain $a_2(\eta_2, \eta_2) = -(E_{21}e_1, \eta_2) - a_2(u_2 - \tilde{u}_{2h}, \eta_2)$. This together with Eq. (36) and the fifth assumption implies $\alpha\|\eta_2\|^2 \leqslant a_2(\eta_2, \eta_2) \leqslant \|E_{21}e_1\|\|\eta_2\|$ or

$$\|\eta_2\| \leqslant \frac{\|E_{21}\|}{\alpha}\|e_1\| \leqslant \frac{\|E_{21}\|}{\alpha}\bigl(\|u_1 - \tilde{u}_{1h}\| + \|\eta_1\|\bigr). \tag{43}$$

If this is inserted into inequality (42), one finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1\| + \gamma\|\eta_1\| \leqslant \bigl\|(u_1 - \tilde{u}_{1h})_t\bigr\| + \beta_1\|u_1 - \tilde{u}_{1h}\| + \beta_2\|u_2 - \tilde{u}_{2h}\|, \tag{44}$$

or by integration

$$\begin{aligned}\|\eta_1(t)\| &\leqslant \bigl\|\eta_1(0)\bigr\|e^{-\gamma t} + \int_0^t e^{-\gamma(t-\tau)}\bigl\|(u_1 - \tilde{u}_{1h})_\tau(\tau)\bigr\|\,\mathrm{d}\tau \\ &\quad + \beta_1\int_0^t e^{-\gamma(t-\tau)}\bigl\|(u_1 - \tilde{u}_{1h})(\tau)\bigr\|\,\mathrm{d}\tau + \beta_2\int_0^t e^{-\gamma(t-\tau)}\bigl\|(u_2 - \tilde{u}_{2h})(\tau)\bigr\|\,\mathrm{d}\tau\end{aligned} \tag{45}$$

where

$$\beta_1 = \max_{t\in[0,t_e]}\frac{\|E_{12}(t)\|\|E_{21}(t)\|}{\alpha}, \qquad \beta_2 = \max_{t\in[0,t_e]}\bigl\|E_{12}(t)\bigr\|. \tag{46}$$

Applying now the interpolation result (37) ($u - \tilde{u}_h$ replaced by $u_1 - \tilde{u}_{1h}$ etc.), the first estimate (38) is obtained by means of (35). Then the second one, in Eq. (39), follows from (35), (37), (38) and (43).

Example 8. If a linear finite element method is used for both V_{1h} and V_{2h}, then $r_1 = r_2 = 1$, and it is necessary that $(u_1, u_2) \in (H^2)^n \times (H^2)^m$.

5.2. Semilinear system

In this subsection the approximate numerical solution of semilinear PDAEs of the form

$$u_{1t} + B_{11}\Delta u_1 + \sum_{j=1}^{d} D_{j11}\partial_j u_1 + E_{11}u_1 + E_{12}u_2 + F_{11}(u_1)u_1 + F_{12}(u_2)u_2 = f_1, \tag{47}$$

$$B_{22}\Delta u_2 + \sum_{j=1}^{d} D_{j22}\partial_j u_2 + E_{22}u_2 + E_{21}u_1 + F_{21}(u_1)u_1 + F_{22}(u_2)u_2 = f_2, \tag{48}$$

is considered.

Example 9. A simple example for systems of this type is given when the second equation reduces to a nonlinear “equation of state”. In this case, Eq. (48) is simply

$$F_{21}(u_1)u_1 + F_{22}(u_2)u_2 = f_2,$$

i.e., u_2 may be expressed in terms of u_1.

The basic assumption we use here is that the matrices F_{ij} fulfill a Lipschitz condition

$$\left\| F_{ij}(v_j)v_j - F_{ij}(w_j)w_j \right\| \leqslant L_{ij}\|v_j - w_j\|, \quad v_j, w_j \in V_j,\ j = 1, 2, \tag{49}$$

where the Lipschitz constant L_{ij} is assumed to be independent of $t \in [0, t_e]$.

A weak form of the system (47), (48) can be obtained as an obvious generalization of Problem 2:

Problem 4. Find $(u_1(t), u_2(t)) \in V_1 \times V_2$, $V_1 = (H^1)^n$ and, e.g., $V_2 \subseteq (H^1)^m$ such that

$$(u_{1t}, v_1) + a_1(u_1, v_1) + (E_{12}u_2, v_1) + \big(F_{11}(u_1)u_1, v_1\big) + \big(F_{12}(u_2)u_2, v_1\big) = f_1(v_1), \quad \forall v_1 \in V_1, \tag{50}$$

$$a_2(u_2, v_2) + (E_{21}u_1, v_2) + \big(F_{21}(u_1)u_1, v_2\big) + \big(F_{22}(u_2)u_2, v_2\big) = f_2(v_2), \quad \forall v_2 \in V_2, \tag{51}$$

$$u_1(0) = \tilde{u}^{10}, \quad \tilde{u}^{10} \in V_1. \tag{52}$$

The bilinear forms and the linear forms are defined as in Eqs. (27)–(30). The corresponding finite element approximations u_{1h}, u_{2h} are assumed to be a solution of

Problem 5. Find $(u_{1h}(t), u_{2h}(t)) \in V_{1h} \times V_{2h}$, $V_{1h} \subset V_1$, $V_{2h} \subset V_2$ such that

$$\begin{aligned}(u_{1ht}, v_{1h}) &+ a_1(u_{1h}, v_{1h}) + (E_{12}u_{2h}, v_{1h}) + \big(F_{11}(u_{1h})u_{1h}, v_{1h}\big)\\ &+ \big(F_{12}(u_{2h})u_{2h}, v_{1h}\big) = f_1(v_{1h}), \quad \forall v_{1h} \in V_{1h},\end{aligned} \tag{53}$$

$$\begin{aligned}a_2(u_{2h}, v_{2h}) &+ (E_{21}u_{1h}, v_{2h}) + (F_{21}(u_{1h})u_{1h}, v_{2h}) + \big(F_{22}(u_{2h})u_{2h}, v_{2h}\big) = f_2(v_{2h}),\\ &\forall v_{2h} \in V_{2h},\end{aligned} \tag{54}$$

$$u_{1h}(0) = \tilde{u}_h^{10}, \quad \tilde{u}_h^{10} \in V_{1h}. \tag{55}$$

Then the following theorem is valid:

Theorem 10 (For Problems 4 and 5). *Suppose*

(1) $\nu_t = 1$,
(2) *the Lipschitz condition* (49) *holds with time independent constants* $L_{i,j} < \infty$, $i, j = 1, 2$,
(3) *the assumptions of Lemma* 3 *are valid,*
(4) $u_1 \in V_1 \cap (H^{r_1+1})^n$, $u_2 \in V_2 \cap (H^{r_2+1})^m$, $r_1, r_2 \geqslant 1$,
(5) $V_{1h} = \{v \in (H^1)^n,\ v|_K \in (P_{r_1}(K))^n\ \forall K \in \mathcal{T}_h\}$, *and, e.g.*, $V_{2h} = \{v \in (H_0^1)^m,\ v|_K \in (P_{r_2}(K))^m\ \forall K \in \mathcal{T}_h\}$,
(6) *either* $a_2(\cdot\,,\cdot)$ *is* V_2*-elliptic, i.e.,* $a_2(v_2, v_2) \geqslant \alpha\|v_2\|_{V_2}^2$, $v_2 \in V_2$, $\alpha > 0$, *or Lemma* 5 *is valid, i.e.,* $a_2(v_2, v_2) \geqslant \alpha\|v_2\|^2$ *with* $\alpha = \sigma_2$,
(7) $\alpha - L_{22} > 0$.

Then

$$\big\|u_1(t) - u_{1h}(t)\big\| \leqslant \big\|u_{1h}(0) - \tilde{u}_{1h}(0)\big\|e^{-\bar{\gamma}t} + \mathrm{O}\big(h^{r_1+1}\big) + \mathrm{O}\big(h^{r_2+1}\big), \tag{56}$$

$$\big\|u_2(t) - u_{2h}(t)\big\| \leqslant \left(\frac{\|E_{21}(t)\| + L_{21}}{\alpha - L_{22}}\right)\big\|u_1(t) - u_{1h}(t)\big\| + \mathrm{O}\big(h^{r_2+1}\big), \tag{57}$$

where the constant $\bar{\gamma} = \sigma_1 - \bar{\beta}_1$ *is independent of the mesh parameter* h, *and* $\bar{\beta}_1$ *is given in Eq.* (62).

Proof. The proof is similar to the proof of Theorem 7. An equation for η_1 is found by means of Eqs. (36) and (50) as follows: $(\eta_{1t}, v_{1h}) + a_1(\eta_1, v_{1h}) = (\tilde{u}_{1ht}, v_{1h}) - (u_{1ht}, v_{1h}) + a_1(\tilde{u}_{1h}, v_{1h}) - a_1(u_{1h}, v_{1h}) = (\tilde{u}_{1ht}, v_{1h}) - f_1(v_{1h}) + (E_{12}u_{2h}, v_{1h}) + (F_{11}(u_{1h})u_{1h}, v_{1h}) + (F_{12}(u_{2h})u_{2h}, v_{1h}) + a_1(u_1, v_{1h}) = (\tilde{u}_{1ht}, v_{1h}) - (u_{1t}, v_{1h}) - (E_{12}u_2, v_{1h}) - (F_{11}(u_{1h})u_{1h}, v_{1h}) - (F_{12}(u_{2h})u_{2h}, v_{1h}) + (E_{12}u_{2h}, v_{1h}) + (F_{11}(u_{1h})u_{1h}, v_{1h}) + (F_{12}(u_{2h})u_{2h}, v_{1h})$. Therefore, η_1 satisfies

$$\begin{aligned}(\eta_{1t}, v_{1h}) + a_1(\eta_1, v_{1h}) &= (\tilde{u}_{1ht} - u_{1t}, v_{1h}) + \big(E_{12}(u_{2h} - u_2), v_{1h}\big) + \big(F_{11}(u_{1h})u_{1h} - F_{11}(u_1)u_1, v_{1h}\big)\\ &\quad + \big(F_{12}(u_{2h})u_{2h} - F_{12}(u_2)u_2, v_{1h}\big),\end{aligned} \tag{58}$$

and (with $v_{1h} = \eta_1$, Lemma 3, (35) and (49))

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1\| + \sigma_1\|\eta_1\| &\leqslant \big\|(u_1 - \tilde{u}_{1h})_t\big\| + \|E_{12}\|\big(\|u_2 - \tilde{u}_{2h}\| + \|\eta_2\|\big)\\ &\quad + L_{11}\big(\|u_1 - \tilde{u}_{1h}\| + \|\eta_1\|\big) + L_{12}\big(\|u_2 - \tilde{u}_{2h}\| + \|\eta_2\|\big).\end{aligned} \tag{59}$$

To estimate $\|\eta_2\|$, we set in Eq. (51) $v_2 = v_{2h}$ and then subtract Eq. (54) from Eq. (51), $a_2(e_2, v_{2h}) = -(E_{21}e_1, v_{2h}) - (F_{21}(u_1)u_1 - F_{21}(u_{1h})u_{1h}, v_{2h}) - (F_{22}(u_2)u_2 - F_{22}(u_{2h})u_{2h}, v_{2h})$, $\forall v_{2h} \in V_{2h}$. Using, on the other hand, $a_2(e_2, v_{2h}) = a_2(u_2 - \tilde{u}_{2h}, v_{2h}) + a_2(\eta_2, v_{2h})$, we obtain by means of $v_{2h} = \eta_2 \in V_{2h}$ the equation $a_2(\eta_2, \eta_2) = -(E_{21}e_1, \eta_2) - (F_{21}(u_1)u_1 - F_{21}(u_{1h})u_{1h}, \eta_2) - (F_{22}(u_2)u_2 - F_{22}(u_{2h})u_{2h}), \eta_2) - a_2(u_2 - \tilde{u}_{2h}, \eta_2)$. This together with Eq. (36) and with the second and seventh assumption gives

$$\begin{aligned}&\alpha\|\eta_2\|^2 \leqslant a_2(\eta_2, \eta_2) \leqslant \|E_{21}\|\|e_1\|\|\eta_2\| + L_{21}\|e_1\|\|\eta_2\| + L_{22}\|e_2\|\|\eta_2\| \quad \text{or}\\ &\|\eta_2\| \leqslant \frac{\|E_{21}\| + L_{21}}{\alpha - L_{22}}\|e_1\| + \frac{L_{22}}{\alpha - L_{22}}\|u_2 - \tilde{u}_{2h}\|,\end{aligned} \tag{60}$$

provided $\alpha - L_{22} > 0$. If this is inserted into in Eq. (59), we obtain the analogue to in Eq. (44), $\frac{\mathrm{d}}{\mathrm{d}t}\|\eta_1\| + \bar{\gamma}\|\eta_1\| \leqslant \|(u_1 - \tilde{u}_{1h})_t\| + \bar{\beta}_1\|u_1 - \tilde{u}_{1h}\| + \bar{\beta}_2\|u_2 - \tilde{u}_{2h}\|$, and

$$\|\eta_1(t)\| \leqslant \|\eta_1(0)\| e^{-\bar{\gamma}t} + \int_0^t e^{-\bar{\gamma}(t-\tau)} \|(u_1 - \tilde{u}_{1h})_\tau(\tau)\| \, d\tau$$
$$+ \bar{\beta}_1 \int_0^t e^{-\bar{\gamma}(t-\tau)} \|(u_1 - \tilde{u}_{1h})(\tau)\| \, d\tau + \bar{\beta}_2 \int_0^t e^{-\bar{\gamma}(t-\tau)} \|(u_2 - \tilde{u}_{2h})(\tau)\| \, d\tau, \tag{61}$$

where now

$$\bar{\beta}_1 = \max_{t\in[0,t_e]} \left\{ L_{11} + \frac{(\|E_{12}(t)\| + L_{12})(\|E_{21}(t)\| + L_{21})}{\alpha - L_{22}} \right\},$$
$$\bar{\beta}_2 = \max_{t\in[0,t_e]} \left\{ \|E_{12}(t)\| + L_{12} + \frac{L_{22}(\|E_{12}(t)\| + L_{12})}{\alpha - L_{22}} \right\}. \tag{62}$$

The first estimate, (56), and the second estimate, (57), is obtained by means of (35), (37), (45) and (35), (37), (56), (60), respectively. □

6. Conclusion

It was shown that for weakly coupled PDAEs of second order with first order terms of time index one L^2-estimates of $\|u - u_h\|$ in terms of the finite element mesh parameter h can be derived under the symmetry condition $D_j^{\mathrm{T}} = D_j$, $j = 1, \ldots, d$, and some further assumptions given in Theorems 7 and 10 for linear and semilinear systems, respectively. An essential basic assumption of this paper is that the matrix A has the block diagonal structure defined in (2). In applications, this is often fulfilled as illustrated in Section 2. This block form of A together with the block structure of B, Eq. (19), with the requirement that the matrix B_1 is negative definite implies that the system (1) has a certain "normal form", e.g., it may consist of a parabolic system, a first order system (e.g., of hyperbolic type in $(t - x)$-space) and a time independent system (which may be, for example, an elliptic system or a hyperbolic system in x-space). The MHD equations mentioned in Section 2 are an example for PDAEs of this structure. The "normal form" enables one to specify (general) boundary conditions needed for the weak formulation.

As expected, the results obtained show that the regularity of $u = (u_1^{\mathrm{T}}, u_2^{\mathrm{T}})^{\mathrm{T}}$ plays again a fundamental role. The order of convergence with respect to h is given by $r + 1$, $r = \min\{r_1, r_2\}$, where r_j characterizes the regularity of u_j, i.e., $u_j \in (H^{r_j+1})^{k_j}$, $j = 1, 2$, $k_1 = n$, $k_2 = m$.

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