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Controllability analysis of linear time-varying systems with multiple time delays and impulsive effects

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ABSTRACT

In this paper, the issue of controllability for linear time-varying systems with multiple time delays in the control and impulsive effects is addressed. The solution of such systems based on the variation of parameters is derived. Several sufficient and necessary algebraic conditions for two kinds of controllability, i.e., controllability to the origin and controllability, are derived. The relation among these conditions are established. A numerical example is provided to illustrate the effectiveness of the proposed methods.

1. Introduction

Many real systems in physics, chemistry, biology, engineering, and information science, may experience abrupt changes at certain instants. These systems exhibit impulsive dynamical behaviors during the continuous dynamical processes. Hence, the study on the control of dynamical systems with impulsive effects is of great importance and has received increasing interest in the recent years; see [1–5]. It is well-known that the controllability and observability play a significant role in the modern control theory and engineering since they are closely related to pole assignment, structural decomposition, quadratic optimal control and observer design. Different techniques were developed to investigate the controllability of various systems, such as geometric analysis [6,7], Lie algebraic approach [8], functional analysis [9] and algebraic method [10,11].

In particular, for the controllability of time-varying impulsive hybrid systems, the algebraic analysis method has proved to be effective. Sufficient and necessary conditions in terms of matrix rank for state controllability and observability of different kinds of time-varying impulsive systems are established in [12–14]. Moreover, this method was extended to consider the controllability and observability of time-varying switched impulsive systems in [15,16].

As we know, time delay is one of the inevitable problems in practical engineering applications. Considering time delay in the controller design process will be very important to the system stability and performance. The systems with multiple time delays in the control have strong practical backgrounds in various chemical process systems, hydraulically actuated systems and combustion systems [17–21]. Hence, it is necessary to investigate the control theory of systems with multiple input delays. Some research efforts have been focused on the controllability of linear continuous systems with multiple input delays [17–19]. However, to the best of our knowledge, there is no result concerning the controllability of time-varying impulsive systems with multiple delays. Due to the co-existence of impulses and time-delay, the algebraic analysis of the controllability becomes much more complex. This motivates our current work. In this paper, we study the controllability problem for time-varying impulsive systems with multiple delays. By explicitly characterizing solutions of such systems, we derive the controllability criteria expressed in terms of matrix rank conditions which are easy to check. Specializing

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to the cases of linear time-varying impulsive systems or linear delay systems, the obtained criteria include some existing results [12–14,22,23].

The rest of this paper is organized as follows. In Section 2, the linear time-varying impulsive systems with multiple delays is formulated and the solution of such systems is presented by the variation of parameters. In Section 3, algebraic conditions for the controllability of linear time-varying impulsive delayed systems are established. When reduced to the impulsive systems without delay or time-invariant delay systems, the corresponding results are derived. A numerical example is given to show the effectiveness of the proposed methods in Section 4. Finally, some concluding remarks are drawn in Section 5.

2. Preliminaries

We consider the time-varying impulsive system with multiple delays as follows,

$$\begin{aligned}
&\dot{x}(t) = A(t)x(t) + \sum_{i=1}^{N} B_i(t)u(t-h_i), \quad t \neq t_k,\\
&\Delta x = E_k x(t_k) + F_k u_k, \quad t = t_k,\\
&x(t_0^+) = x_0,\\
&u(t) = u_0(t), \quad t \in [t_0 - h_N, t_0],
\end{aligned} \tag{1}$$

where $k = 1, 2, \dots, A(t) \in \mathbb{R}^{n\times n}$, $B_i(t) \in \mathbb{R}^{n\times d}$ are continuous functions of $\mathbb{R}^+$, and $E_k \in \mathbb{R}^{n\times n}$, $F_k \in \mathbb{R}^{n\times d}$ are constant matrices, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^d$ is a piecewise continuous input, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k) = x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$ with discontinuity points $t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$, which implies that the solution of (1) is left-continuous at t_k. $u_k = u(t_k)$, $k = 1, 2, \dots$. $d_i = t_i - t_{i-1}$, $d_i > h_N \geq h_{N-1} \geq \cdots \geq h_1 \geq 0$, $i = 1, 2, \dots$, where $h_1, \dots, h_N$ are time delays in the control input. $u_0(t) \in \mathbb{R}^d$ is an initial input mapping $t \in [t_0 - h_N, t_0]$. Let $A^\top$ be the transpose of the matrix A. $\prod_{i=k}^{1} A_i$ stands for the matrix product $A_k A_{k-1} \cdots A_1$.

Corresponding to impulsive system (1), consider the differential equation

$$\dot{x}(t) = A(t)x(t). \tag{2}$$

Suppose that $X(t)$ is the fundamental solution matrix of system (2). Then $X(t, s) := X(t)X^{-1}(s)$ is the transition matrix associated with the matrix $A(t)$. It is clear that $X(t, t) = I$, where I is the identity matrix of order n, $X(t, z)X(z, s) = X(t, s)$ and $X(t, s) = X^{-1}(s, t)$ for $s, t, z \in \mathbb{R}^+$.

Lemma 2.1. *For any $t \in (t_{k-1} + h_N, t_k]$, $k = 2, 3, \dots$ the solution of system (1) is given as follows:*

$$\begin{aligned}
x(t) = {} & X(t, t_k) \prod_{j=k}^{1} X(t_j, t_{j-1})(I + E_{j-1}) \left\{ (x_0 + a_0) + \sum_{i=1}^{N} \int_{t_0}^{t_1 - h_i} X(t_0, s + h_i) B_i(s + h_i) u(s) \mathrm{d}s \right\} \\
& + X(t, t_k) \left\{ \sum_{m=2}^{k-1} \prod_{j=k}^{m+1} X(t_j, t_{j-1})(I + E_{j-1}) X(t_m, t_{m-1}) \right. \\
& \times \left[\sum_{i=1}^{N} \int_{t_{m-1} - h_i}^{t_m - h_i} X(t_{m-1}, s + h_i) B_i(s + h_i) u(s) \mathrm{d}s + F_{m-1} u_{m-1} \right] \\
& \left. + X(t_k, t_{k-1}) \left[\sum_{i=1}^{N} \int_{t_{k-1} - h_i}^{t - h_i} X(t_{k-1}, s + h_i) B_i(s + h_i) u(s) \mathrm{d}s + F_{k-1} u_{k-1} \right] \right\},
\end{aligned} \tag{3}$$

where

$$a_0 := \sum_{i=1}^{N} \int_{t_0 - h_i}^{t_0} X(t_0, s + h_i) B_i(s + h_i) u_0(s) \mathrm{d}s, \quad E_0 = 0. \tag{4}$$

Proof. It follows from the initial condition of system (1) and the ordinary variation of parameters that

$$\begin{aligned}
x(t) &= X(t, t_0) x_0 + \int_{t_0}^{t} X(t, s) \sum_{i=1}^{N} B_i(s) u(s - h_i) \mathrm{d}s \\
&= X(t, t_0) x_0 + \sum_{i=1}^{N} \int_{t_0 - h_i}^{t_0} X(t, s + h_i) B_i(s + h_i) u_0(s) \mathrm{d}s + \sum_{i=1}^{N} \int_{t_0}^{t - h_i} X(t, s + h_i) B_i(s + h_i) u(s) \mathrm{d}s,
\end{aligned}$$

for $t \in [t_0 + h_N, t_1]$, which leads to

$$x(t_1) = X(t_1, t_0)(x_0 + a_0) + \sum_{i=1}^{N} \int_{t_0}^{t_1 - h_i} X(t_1, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s. \tag{5}$$

Since $\Delta x = E_k x(t_k) + F_k u_k$, $t = t_k$, we have $x(t_1^+) = (I + E_1)x(t_1) + F_1 u_1$. Moreover, for $t \in (t_1 + h_N, t_2]$, the solution can be written as

$$x(t) = X(t, t_1)x(t_1^+) + \sum_{i=1}^{N} \int_{t_1 - h_i}^{t - h_i} X(t, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s.$$

By the expression of $x(t_1^+)$, we have

$$\begin{aligned} x(t_2) = {} & X(t_2, t_1)(I + E_1)X(t_1, t_0)(x_0 + a_0) + X(t_2, t_1)(I + E_1) \sum_{i=1}^{N} \int_{t_0}^{t_1 - h_i} X(t_1, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s \\ & + \sum_{i=1}^{N} \int_{t_1 - h_i}^{t_2 - h_i} X(t_2, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s + X(t_2, t_1)F_1 u_1. \end{aligned}$$

$x(t_2^+) = (I + E_2)x(t_2) + F_2 u_2$. Hence, for any $t \in (t_2 + h_N, t_3]$,

$$\begin{aligned} x(t) = {} & X(t, t_3) \prod_{j=3}^{1} X(t_j, t_{j-1})(I + E_{j-1}) \left\{ (x_0 + a_0) + \sum_{i=1}^{N} \int_{t_0}^{t_1 - h_i} X(t_0, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s \right\} \\ & + X(t, t_3) \left\{ \sum_{m=2}^{2} \prod_{j=3}^{m+1} X(t_j, t_{j-1})(I + E_{j-1})X(t_m, t_{m-1}) \right. \\ & \times \left[\sum_{i=1}^{N} \int_{t_{m-1} - h_i}^{t_m - h_i} X(t_{m-1}, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s + F_{m-1}u_{m-1} \right] \\ & \left. + X(t_3, t_2) \left[\sum_{i=1}^{N} \int_{t_2 - h_i}^{t - h_i} X(t_2, s + h_i)B_i(s + h_i)u(s)\mathrm{d}s + F_2 u_2 \right] \right\}. \end{aligned}$$

By repeating the same procedure, we can easily deduce the general result (3) for $k = 2, \ldots$. This completes the proof. □

3. Controllability

In this section, we proceed to investigate two kinds of controllability for system (1) with multiple input delays based on the solution expression. Explicitly sufficient conditions and necessary condition for the controllability to the origin are derived, respectively. The sufficient and necessary condition of the controllability is obtained by algebraic analysis method. The relation among these conditions are discussed. Moreover, when reduced to some special cases, the corresponding criteria are also obtained.

The definitions of two kinds of controllability are introduced first.

Definition 3.1. The time-varying impulsive system (1) with multiple time delays is called controllable to the origin on $[t_0, t_f]$ $(t_0 < t_f)$, if given any initial state $x_0 \in \mathbb{R}^n$ and initial control mapping $u_0(t)\colon [t_0 - h_N, t_0] \to \mathbb{R}^m$, there exists a piecewise continuous control input $u(t)$ such that the corresponding solution of (1) satisfies $x(t_f) = 0$.

Definition 3.2. The time-varying impulsive system (1) with multiple time delays is called controllable on $[t_0, t_f]$ $(t_0 < t_f)$ if for all $x_0, x_f \in \mathbb{R}^n$ and initial control mapping $u_0(t)\colon [t_0 - h_N, t_0] \to \mathbb{R}^m$, there exists a piecewise continuous control input $u(t)$ such that the corresponding solution of (1) on $[t_0, t_f]$ satisfies $x(t_0) = x_0$ and $x(t_f) = x_f$.

Remark 3.3. From the definitions, it is easy to see that when the system is controllable, it must be controllable to the origin. However, it was shown in [3,15] that the controllability of impulsive systems is not equivalent to the controllability to the origin.

3.1. Sufficient conditions for the controllability to the origin

For convenience, some notations are introduced and the solution expression will be described in another form. We denote

$$
\begin{aligned}
& t_f = t_{M+1}, \qquad \Theta_i := \prod_{j=M+1}^{i+1} X(t_j, t_{j-1})(I+E_{j-1})X(t_i, t_{i-1}), \quad i = 1, \dots, M; \\
& \Theta_{M+1} := X(t_{M+1}, t_M), \qquad \Theta_{M+2} = 0, \qquad E_{M+1} = 0; \\
& G(l, k) = X(t_l, s+h_k)B_k(s+h_k), \quad l = 0, \dots, M, k = 1, \dots, N-1.
\end{aligned} \tag{6}
$$

From Lemma 2.1, we have

$$
x_f = \Theta_1 \left\{ (x_0 + a_0) + \sum_{i=1}^{N} \int_{t_0}^{t_1 - h_i} G(0, i)u(s)\mathrm{d}s \right\} + \sum_{m=2}^{M+1} \Theta_m \left[\sum_{i=1}^{N} \int_{t_{m-1}-h_i}^{t_m - h_i} G(m-1, i)u(s)\mathrm{d}s + F_{m-1}u_{m-1} \right]. \tag{7}
$$

We rewrite the third term in (7) as follows:

$$
\begin{aligned}
& \Theta_m \left[\sum_{i=1}^{N} \int_{t_{m-1}-h_i}^{t_m-h_i} G(m-1, i)u(s)\mathrm{d}s \right] \\
&= \Theta_m \left[\int_{t_{m-1}-h_N}^{t_{m-1}-h_{N-1}} G(m-1, N)u(s)\mathrm{d}s + \int_{t_{m-1}-h_{N-1}}^{t_{m-1}-h_{N-2}} \sum_{i=N-1}^{N} G(m-1, i)u(s)\mathrm{d}s \right. \\
&\quad + \cdots + \int_{t_{m-1}-h_2}^{t_{m-1}-h_1} \sum_{i=2}^{N} G(m-1, i)u(s)\mathrm{d}s + \int_{t_{m-1}-h_1}^{t_m-h_N} \sum_{i=1}^{N} G(m-1, i)u(s)\mathrm{d}s \\
&\quad + \int_{t_m-h_N}^{t_m-h_{N-1}} \sum_{i=1}^{N-1} G(m-1, i)u(s)\mathrm{d}s + \int_{t_m-h_{N-1}}^{t_m-h_{N-2}} \sum_{i=1}^{N-2} G(m-1, i)u(s)\mathrm{d}s \\
&\quad \left. + \cdots + \int_{t_m-h_3}^{t_m-h_2} \sum_{i=1}^{2} G(m-1, 1)u(s)\mathrm{d}s + \int_{t_m-h_2}^{t_m-h_1} G(m-1, 1)u(s)\mathrm{d}s \right] \\
&= \sum_{j=1}^{N-1} \int_{t_{m-1}-h_{j+1}}^{t_{m-1}-h_j} \Theta_m \sum_{i=j+1}^{N} G(m-1, i)u(s)\mathrm{d}s + \int_{t_{m-1}-h_1}^{t_m-h_N} \Theta_m \sum_{i=1}^{N} G(m-1, i)u(s)\mathrm{d}s \\
&\quad + \sum_{j=1}^{N-1} \int_{t_m-h_{j+1}}^{t_m-h_j} \Theta_m \sum_{i=1}^{j} G(m-1, i)u(s)\mathrm{d}s.
\end{aligned} \tag{8}
$$

Similarly,

$$
\Theta_1 \left[\sum_{i=1}^{N} \int_{t_0}^{t_1-h_i} G(0, i)u(s)\mathrm{d}s \right] = \int_{t_0}^{t_1-h_N} \Theta_1 \sum_{i=1}^{N} G(0, i)u(s)\mathrm{d}s + \sum_{j=1}^{N-1} \int_{t_1-h_{j+1}}^{t_1-h_j} \Theta_1 \sum_{i=1}^{j} G(0, i)u(s)\mathrm{d}s. \tag{9}
$$

Using (8) and (9), (10) can be expressed as follows:

$$
\begin{aligned}
x_f &= \Theta_1(x_0 + a_0) + \int_{t_0}^{t_1-h_N} \Theta_1 \sum_{i=1}^{N} G(0, i)u(s)\mathrm{d}s + \sum_{m=1}^{M+1} \left\{ \sum_{j=1}^{N-1} \int_{t_m-h_{j+1}}^{t_m-h_j} \left[\Theta_{m+1} \sum_{i=j+1}^{N} G(m, i) \right.\right. \\
&\quad \left.\left. + \Theta_m \sum_{i=1}^{j} G(m-1, i) \right] u(s)\mathrm{d}s \right\} + \sum_{m=1}^{M} \left\{ \int_{t_m-h_1}^{t_{m+1}-h_N} \Theta_{m+1} \sum_{i=1}^{N} G(m, i)u(s)\mathrm{d}s + \Theta_{m+1}F_m u_m \right\}.
\end{aligned} \tag{10}
$$

For $j = 1, \dots, N-1$, denote

$$
H_{m,j} = \sum_{i=j+1}^{N} G(m, i) + (I + E_m)X(t_m, t_{m-1}) \sum_{i=1}^{j} G(m-1, i), \quad m = 1, 2, \dots, M. \tag{11}
$$

Then we can rewrite (10) as follows:

$$
\begin{aligned}
x_f &= \Theta_1(x_0 + a_0) + \int_{t_0}^{t_1-h_N} \Theta_1 \sum_{i=1}^{N} G(0, i)u(s)\mathrm{d}s + \sum_{m=1}^{M} \left\{ \sum_{j=1}^{N-1} \int_{t_m-h_{j+1}}^{t_m-h_j} \Theta_{m+1} H_{m,j} u(s)\mathrm{d}s \right. \\
&\quad \left. + \int_{t_m-h_1}^{t_{m+1}-h_N} \Theta_{m+1} \sum_{i=1}^{N} G(m, i)u(s)\mathrm{d}s + \Theta_{m+1}F_m u_m \right\} + \sum_{j=1}^{N-1} \int_{t_{M+1}-h_{j+1}}^{t_{M+1}-h_j} \Theta_{M+1} \sum_{i=1}^{j} G(M, i)u(s)\mathrm{d}s.
\end{aligned} \tag{12}
$$

Furthermore, we denote

$$
\begin{aligned}
W_{l,j} &:= \int_{t_l-h_{j+1}}^{t_l-h_j} H_{l,j}H_{l,j}^{\top}\mathrm{d}s, \quad j=1,2,\dots,N-1, l=1,2,\dots,M,\\
W_{M+1,j} &:= \int_{t_{M+1}-h_{j+1}}^{t_{M+1}-h_j} \left[\Theta_{M+1}\sum_{i=1}^{j}G(M,i)\right]\left[\Theta_{M+1}\sum_{i=1}^{j}G(M,i)\right]^{\top}\mathrm{d}s,\\
V_1 &:= \int_{t_0}^{t_1-h_N}\sum_{i=1}^{N}G(0,i)\left[\sum_{i=1}^{N}G(0,i)\right]^{\top}\mathrm{d}s,\\
V_{l+1} &:= \int_{t_l-h_1}^{t_{l+1}-h_N}\sum_{i=1}^{N}G(l,i)\left[\sum_{i=1}^{N}G(l,i)\right]^{\top}\mathrm{d}s, \quad l=1,\dots,M,
\end{aligned}
\tag{13}
$$

where $G(m,i)$ is given by (6).

It should be noticed that some known results on the controllability of (switched) impulsive systems need the assumption that matrix $(I+E_k)$ is nonsingular; see [15,24]. Based on (12), we obtain the following sufficient conditions on the controllability without this assumption. While the necessary condition still needs this assumption.

Theorem 3.4. *If one of the following conditions holds, then system (1) is controllable to the origin on $[t_0,t_f]$ with $t_{M+1}=t_f$*

(i) *if there exists $W_{m,j}$ or V_m, $m\in\{1,2,\dots,M+1\}$, $j\in\{1,\dots,N-1\}$ such that*

$$rank(W_{m,j})=n \quad or \quad rank(V_m)=n;$$

(ii) *if there exist $i\in\{1,2,\dots,M\}$ and $d\times n$ matrix F_i' such that $F_iF_i'=I$.*

Proof. Without loss of generality, suppose that there exists $W_{m,j}$, $m\in\{1,2,\dots,M\}$, $j\in\{1,\dots,N-1\}$ such that $rank(W_{m,j})=n$, namely, $W_{m,j}$ is invertible. Then for any $n\times 1$ initial vector x_0, choose

$$
u(s)=\begin{cases}-H_{m,j}^{\top}W_{m,j}^{-1}\prod_{l=m}^{1}(I+E_l)X(t_l,t_{l-1})(x_0+a_0), & s\in(t_m-h_{j+1},t_m-h_j];\\ 0, & s\in[t_0,t_f]\setminus(t_m-h_{j+1},t_m-h_j].\end{cases}
$$

By (12), x_f can be written as

$$x_f=\Theta_1(x_0+a_0)-\Theta_{m+1}\int_{t_m-h_{j+1}}^{t_m-h_j}H_{m,j}H_{m,j}^{\top}W_{m,j}^{-1}\prod_{l=m}^{1}(I+E_l)X(t_l,t_{l-1})(x_0+a_0)\mathrm{d}s=0.$$

Thus, the impulsive system (1) is controllable to the origin on $[t_0,t_f]$ by Definition 3.1.

If $W_{M+1,j}$ is invertible for some $j\in\{1,\dots,N-1\}$, the control is chosen as

$$
u(s)=\begin{cases}-\left[\Theta_{M+1}\sum_{i=1}^{j}G(M,i)\right]^{\top}W_{M+1,j}^{-1}\Theta_1(x_0+a_0), & s\in(t_{M+1}-h_{j+1},t_{M+1}-h_j];\\ 0, & s\in[t_0,t_f]\setminus(t_{M+1}-h_{j+1},t_{M+1}-h_j].\end{cases}
$$

Then x_f can be written as

$$x_f=\Theta_1(x_0+a_0)-\int_{t_{M+1}-h_{j+1}}^{t_{M+1}-h_j}\left[\Theta_{M+1}\sum_{i=1}^{j}G(M,i)\right]\left[\Theta_{M+1}\sum_{i=1}^{j}G(M,i)\right]^{\top}W_{M+1,j}^{-1}\Theta_1(x_0+a_0)\mathrm{d}s=0.$$

Hence, system (1) is controllable to the origin on $[t_0,t_f]$.

If there exists $m\in\{1,\dots,M\}$ such that $rank(V_{m+1})=n$, then for any $n\times 1$ initial vector x_0, choose

$$
u(s)=\begin{cases}-\left[\sum_{i=1}^{N}G(m,i)\right]^{\top}V_m^{-1}\prod_{j=m}^{1}(I+E_j)X(t_j,t_{j-1})(x_0+a_0), & s\in(t_m-h_1,t_{m+1}-h_N]\setminus\{t_m\};\\ 0, & s\in[t_0,t_f]\setminus(t_m-h_1,t_{m+1}-h_N]\cup\{t_m\}.\end{cases}
$$

x_f can be written as

$$x_f=\Theta_1(x_0+a_0)-\int_{t_m-h_1}^{t_{m+1}-h_N}\Theta_{m+1}\sum_{i=1}^{N}G(m,i)\left[\sum_{i=1}^{N}G(m,i)\right]^{\top}V_m^{-1}\times\prod_{j=m}^{1}(I+E_j)X(t_j,t_{j-1})(x_0+a_0)\mathrm{d}s=0.$$

Thus, the impulsive system (1) is controllable to the origin on $[t_0,t_f]$.

Similarly, if $rank(V_1) = n$, then for any $n \times 1$ initial vector x_0, choose

$$u(s) = \begin{cases} -\left[\sum_{i=1}^{N} G(0,i)\right]^\top V_1^{-1}(x_0 + a_0), & s \in (t_0, t_1 - h_N]; \\ 0, & s \in [t_0, t_f] \setminus (t_0, t_1 - h_N]. \end{cases}$$

Then x_f can be written as

$$x_f = \Theta_1(x_0 + a_0) - \int_{t_0}^{t_1 - h_N} \Theta_1 \sum_{i=1}^{N} G(0,i) \left[\sum_{i=1}^{N} G(0,i)\right]^\top V_1^{-1}(x_0 + a_0)\mathrm{d}s = 0.$$

Thus, the impulsive system (1) is also controllable to the origin on $[t_0, t_f]$.

Next, we consider case (ii). Suppose that there exist $i \in \{1, \ldots, M\}$ and $d \times n$ matrix F_i' such that $F_i F_i' = I$. Then for any $n \times 1$ initial vector x_0, choose

$$u(s) = \begin{cases} -F_i' \prod_{j=i}^{1} (I + E_j) X(t_j, t_{j-1})(x_0 + a_0), & s = t_i; \\ 0, & t \in [t_0, t_f] \setminus \{t_i\}. \end{cases}$$

x_f can be written as

$$x_f = \Theta_1(x_0 + a_0) - \Theta_{i+1} F_i F_i' \prod_{j=i}^{1} (I + E_j) X(t_j, t_{j-1})(x_0 + a_0) = 0. \tag{14}$$

Thus, the impulsive system (1) is controllable to the origin on $[t_0, t_f]$ by Definition 3.1. The proof is completed. □

Remark 3.5. Different from [15,24], the controllability conditions are derived without assuming that $(I + E_k)$ is nonsingular, $k = 1, 2, \ldots$, which implies that the proposed methods can be applied to more general cases. Moreover, from condition (ii), the impulsive matrices in system (1) can be designed to achieve the controllability of system (1).

3.2. Necessary condition for the controllability to the origin

In this subsection, the necessary condition for the controllability of system (1) is derived. For convenience, the following notations are presented first. We denote

$$\begin{aligned} &Z_{m-1} := \Theta_1^{-1}\Theta_m = \prod_{j=1}^{m-1} X(t_{j-1}, t_j)(I + E_j)^{-1}, \\ &U_l := Z_l F_l, \qquad W_{l,j}^* := Z_l W_{l,j} Z_l^\top, \quad l = 1, \ldots, M, j = 1, \ldots, N-1, \\ &W_{M+1,j}^* := Z_M W_{M+1,j} Z_M^\top, \qquad W_m^* = (W_{m,1}^* \cdots W_{m,N-1}^*), \\ &V_m^* = Z_{m-1} V_m Z_{m-1}^\top, \quad m = 1, \ldots, M+1. \end{aligned} \tag{15}$$

Theorem 3.6. *Assume that all the matrices $(I + E_k)$ are nonsingular, $k = 1, 2, \ldots, M$. If system (1) is controllable to the origin, then*

$$rank(\Upsilon) = n, \tag{16}$$

where $\Upsilon = (W_1^* \cdots W_{M+1}^*\ V_1^* \cdots V_{M+1}^*\ U_1 \cdots U_M)$.

Proof. We proceed to prove the theorem by contradiction. Suppose that system (1) is controllable to the origin on $[t_0, t_{M+1}]$, but $rank(\Upsilon) < n$. Then there exists a nonzero $n \times 1$ vector x_α, such that the following equalities hold:

$$\begin{aligned} &x_\alpha^\top W_{m,j}^* x_\alpha = x_\alpha^\top V_m^* x_\alpha = 0, \quad m = 1, \ldots, M+1, j = 1, \ldots, N-1, \\ &x_\alpha^\top U_l = 0, \quad l = 1, \ldots, M. \end{aligned}$$

Since $W_{m,j}^*$ and V_m^* are nonnegative definite matrices, we can get the following equalities for $l = 1, \ldots, M, m = 1, \ldots, M+1, j = 1, \ldots, N-1$.

$$
\begin{aligned}
& x_\alpha^\top Z_0 \sum_{i=1}^{N} G(0,i) = 0, \quad s \in (t_0, t_1 - h_N], \\
& x_\alpha^\top Z_l \sum_{i=1}^{N} G(l,i) = 0, \quad s \in (t_l - h_1, t_{l+1} - h_N] \setminus \{t_l\}, \\
& x_\alpha^\top Z_l H_{l,j} = 0, \quad s \in (t_l - h_{j+1}, t_l - h_j], \\
& x_\alpha^\top Z_M \sum_{i=1}^{j} G(M,i) = 0, \quad s \in (t_{M+1} - h_{j+1}, t_{M+1} - h_j], \\
& x_\alpha^\top Z_l F_l = 0, \quad s = t_l.
\end{aligned} \tag{17}
$$

Since Θ_1 is non-singular, choose the initial state $x_0 = x_\alpha - a_0$. By the controllability assumption of system (1) on $[t_0, t_f]$, there exists a piecewise continuous control input $u(t)$ satisfying

$$
\begin{aligned}
0 = {} & \Theta_1 x_\alpha + \int_{t_0}^{t_1 - h_N} \Theta_1 \sum_{i=1}^{N} G(0,i) u(s) \mathrm{d}s + \sum_{m=1}^{M} \left\{ \sum_{j=1}^{N-1} \int_{t_m - h_{j+1}}^{t_m - h_j} \Theta_{m+1} H_{m,j} u(s) \mathrm{d}s \right. \\
& \left. + \int_{t_m - h_1}^{t_{m+1} - h_N} \Theta_{m+1} \sum_{i=1}^{N} G(m,i) u(s) \mathrm{d}s + \Theta_{m+1} F_m u_m \right\} + \sum_{j=1}^{N-1} \int_{t_{M+1} - h_{j+1}}^{t_{M+1} - h_j} \Theta_{M+1} \sum_{i=1}^{j} G(M,i) u(s) \mathrm{d}s.
\end{aligned} \tag{18}
$$

Multiplying (18) by $x_\alpha^\top \Theta_1^{-1}$, and from (17), we obtain

$$
\begin{aligned}
-x_\alpha^\top x_\alpha = {} & x_\alpha^\top \int_{t_0}^{t_1 - h_N} \sum_{i=1}^{N} G(0,i) u(s) \mathrm{d}s + \sum_{m=1}^{M} \left\{ \sum_{j=1}^{N-1} \int_{t_m - h_{j+1}}^{t_m - h_j} Z_m H_{m,j} u(s) \mathrm{d}s \right. \\
& \left. + \int_{t_m - h_1}^{t_{m+1} - h_N} Z_m \sum_{i=1}^{N} G(m,i) u(s) \mathrm{d}s + Z_m F_m u_m \right\} + \sum_{j=1}^{N-1} \int_{t_{M+1} - h_{j+1}}^{t_{M+1} - h_j} Z_M \sum_{i=1}^{j} G(M,i) u(s) \mathrm{d}s = 0,
\end{aligned} \tag{19}
$$

which contradicts with the assumption that $x_\alpha \neq 0$ and therefore, we can conclude that (16) holds if system (1) is controllable to the origin on $[t_0, t_{M+1}]$. This completes the proof. □

Remark 3.7. Especially, when $h_N = \cdots = h_1 = 0$ for system (1), $W_{m,j}$ in (13) vanishes, $m = 1, \ldots, M+1, j = 1, \ldots, N-1$ and $V_{l+1} = \int_{t_l}^{t_{l+1}} \sum_{i=1}^{N} G(l,i)[\sum_{i=1}^{N} G(l,i)]^\top \mathrm{d}s$, $l = 0, \ldots, M$. Then Theorems 3.4 and 3.6 derive the sufficient and necessary conditions of controllability to the origin for time-varying impulsive systems, which include the results in [12,13]. On the other hand, if there is no impulse in system (1) (i.e., $E_k = F_k = 0, k = 1, 2, \ldots$), then $x(t_f) = X(t_f, t_0)(x_0 + a_0) + \sum_{i=1}^{N} \int_{t_0}^{t_f - h_i} X(t_f, s + h_i) B_i(s + h_i) u(s) \mathrm{d}s$. Simple calculations imply that Theorems 3.4 and 3.6 include the results in [22,23,25]. Therefore, our results generalize the existing results to more general cases.

3.3. A sufficient and necessary condition for the controllability

In this subsection, we aim to investigate the controllability of the linear time-varying system (1) with multiple input delays and impulsive effects based on Definition 3.2. A sufficient and necessary condition is derived for system (1) by algebraic analysis. When reduced to systems without impulsive effects or delays, the conditions are obtained accordingly. Also, the relation among the resulting conditions is discussed. First, we denote the following $n \times n$ matrices for $j = 1, \ldots, N-1$:

$$
\begin{aligned}
& \Phi_{l,j} := \Theta_{l+1} W_{l,j} \Theta_{l+1}^\top, \qquad \Phi_{M+1,j} := W_{M+1,j}, \\
& \Phi_m = (\Phi_{m,1} \cdots \Phi_{m,N-1}), \\
& \Psi_m := \Theta_m V_m \Theta_m^\top, \quad m = 1, \ldots, M+1, \\
& \Xi_l := \Theta_1 U_l U_l^\top \Theta_1^\top, \quad l = 1, \ldots, M,
\end{aligned} \tag{20}
$$

where $G(m,i)$ and Θ_m are given by (6).

Theorem 3.8. *If the impulsive control matrices $(I + E_k)$ are non-singular, $k = 1, \ldots, M$, then system (1) is controllable on $[t_0, t_f]$ with $t_{M+1} = t_f$ if and only if*

$$rank(\mathcal{C}) = n,$$

where $\mathcal{C} = (\Phi_1 \cdots \Phi_{M+1}\ \Psi_1 \cdots \Psi_{M+1}\ \Xi_1 \cdots \Xi_M)$.

Proof. Since $\Phi_{m,j}, \Psi_m, \Xi_k, \ (m = 1, \dots, M+1, j = 1, \dots, N-1, k = 1, \dots, M)$ are all nonnegative definite matrices. We denote $\tilde{\Phi}_m = \sum_{j=1}^{N} \Phi_{m,j}, m = 1, \dots, M+1$; $\tilde{\mathcal{C}} = \sum_{m=1}^{M+1}(\tilde{\Phi}_m + \Psi_m) + \sum_{i=1}^{M} \Xi_i$. Then it is easy to see that $rank(\tilde{\mathcal{C}}) = rank(\mathcal{C})$.

Sufficiency: If $rank(\mathcal{C}) = rank(\tilde{\mathcal{C}}) = n$, we will show that system (1) is controllable on $[t_0, t_f]$. From Definition 3.1, we only need to find a piecewise continuous control $u(t)$ such that the corresponding solution of (1) on $[t_0, t_f]$ satisfies $x(t_0) = x_0$ and $x(t_f) = x_f$.

Let

$$u(s) = \begin{cases} \left(\Theta_1 \sum_{i=1}^{N} G(0,i)\right)^\top \alpha, \quad s \in (t_0, t_1 - h_N], \\ \left(\Theta_{l+1} \sum_{i=1}^{N} G(l,i)\right)^\top \alpha, \quad s \in (t_l - h_1, t_{l+1} - h_N] \setminus \{t_l\}, \\ \left(\Theta_{m+1} \sum_{i=j+1}^{N} G(m,i) + \Theta_m \sum_{i=1}^{j} G(m-1,i)\right)^\top \alpha, \quad s \in (t_m - h_{j+1}, t_m - h_j], \\ \qquad m = 1, \dots, M+1, j = 1, \dots, N-1, \\ F_l^\top \Theta_{l+1}^\top \alpha, \quad s = t_l, l = 1, \dots, M \end{cases} \tag{21}$$

where $\alpha \in \mathbb{R}^n$ is a constant vector to be determined. Then from (20), (10) can be rewritten by

$$x_f - \Theta_1(x_0 + a_0) = \sum_{m=1}^{M+1}(\tilde{\Phi}_m + \Psi_m)\alpha + \sum_{i=1}^{M} \Xi_i \alpha = \tilde{\mathcal{C}}\alpha. \tag{22}$$

Since $\tilde{\mathcal{C}}$ has full rank, there exists a solution α satisfying (22). Therefore, system (1) is controllable on $[t_0, t_f]$.

Necessity: Now we are in the position to prove the necessity by contradiction. Suppose that system (1) is controllable on $[t_0, t_f]$, but $rank(\mathcal{C}) < n$. Then there exists a nonzero $n \times 1$ vector x_α such that

$$x_\alpha^\top \Phi_m x_\alpha = x_\alpha^\top \Psi_m x_\alpha = x_\alpha^\top \Xi_l x_\alpha = 0, \quad m = 1, \dots, M+1, l = 1, \dots, M.$$

From (20), since $\Phi_{m,j}, \Psi_m, \Xi_l, \ (m = 1, \dots, M+1, j = 1, \dots, N-1, l = 1, \dots, M)$ are all nonnegative definite matrices, we can get

$$\begin{aligned} &\left(\Theta_1 \sum_{i=1}^{N} G(0,i)\right)^\top x_\alpha = 0, \quad s \in (t_0, t_1 - h_N], \\ &\left(\Theta_{l+1} \sum_{i=1}^{N} G(l,i)\right)^\top x_\alpha = 0, \quad s \in (t_l - h_1, t_{l+1} - h_N] \setminus \{t_l\}, \\ &\left(\Theta_{m+1} \sum_{i=j+1}^{N} G(m,i) + \Theta_m \sum_{i=1}^{j} G(m-1,i)\right)^\top x_\alpha = 0, \quad s \in (t_m - h_{j+1}, t_m - h_j], \\ &\quad m = 1, \dots, M+1, j = 1, \dots, N-1; \\ &F_l^\top \Theta_{l+1}^\top x_\alpha = 0, \quad s = t_l, \ l = 1, \dots, M. \end{aligned} \tag{23}$$

Since Θ_1 is non-singular, choose the initial state $x_0 = \Theta_1^{-1} x_\alpha - a_0$ and $x_f = 0$. By the controllability assumption of system (1) on $[t_0, t_f]$, there exists a piecewise continuous control input $u(t)$ such that the right side of (10) equals to zero. Left-multiplying $x_\alpha^\top$ to (10), and from (23), we obtain

$$\begin{aligned} -x_\alpha^\top x_\alpha &= x_\alpha^\top \int_{t_0}^{t_1 - h_N} \Theta_1 \sum_{i=1}^{N} G(0,i) u(s) \mathrm{d}s \\ &\quad + x_\alpha^\top \sum_{m=1}^{M+1} \left\{ \sum_{j=1}^{N-1} \int_{t_m - h_{j+1}}^{t_m - h_j} \left[\Theta_{m+1} \sum_{i=j+1}^{N} G(m,i) + \Theta_m \sum_{i=1}^{j} G(m-1,i) \right] u(s) \mathrm{d}s \right\} \\ &\quad + x_\alpha^\top \sum_{m=1}^{M} \left\{ \int_{t_m - h_1}^{t_{m+1} - h_N} \Theta_{m+1} \sum_{i=1}^{N} G(m,i) u(s) \mathrm{d}s + \Theta_{m+1} F_m u_m \right\} = 0 \end{aligned} \tag{24}$$

which contradicts with the assumption $x_\alpha \neq 0$. Therefore, we can conclude that $rank(\mathcal{C}) = n$. This completes the proof. □

In some special cases, we obtain the following results.

Case 1: $\Delta x = 0$, i.e., there is no impulses in system (1), then $\Theta_i = X(t_{M+1}, t_{i-1})$, $i = 1, \ldots, M$. From Theorem 3.8, we have

Corollary 3.9. *System (1) without impulsive effects is controllable on $[t_0, t_f]$ with $t_{M+1} = t_f$ if and only if*

$$rank(\mathcal{C}_1) = n,$$

where $\mathcal{C}_1 = (\Phi_1 \cdots \Phi_{M+1}\ \Psi_1 \cdots \Psi_{M+1})$.

Case 2: $N = 2$.

Corollary 3.10. *If the impulsive control matrices $I + E_k$, $k = 1, \ldots, M$ are non-singular, then system (1) is controllable on $[t_0, t_f]$ with $t_{M+1} = t_f$ if and only if*

$$rank(\mathcal{C}_2) = n,$$

where $\mathcal{C}_2 = (\Phi_{1,1} \cdots \Phi_{M+1,1}\ \Psi_1 \cdots \Psi_{M+1}\ \Xi_1 \cdots \Xi_M)$.

Case 3: There is no delay in the control, i.e., $h_1 = \cdots = h_N = 0$.

Corollary 3.11. *If the impulsive control matrices $I + E_k$, $k = 1, \ldots, M$ are non-singular, then system (1) is controllable on $[t_0, t_f]$ with $t_{M+1} = t_f$ if and only if*

$$rank(\mathcal{C}_3) = n,$$

where $\mathcal{C}_3 = (\Psi_1' \cdots \Psi_{M+1}'\ \Xi_1 \cdots \Xi_M)$,

where

$$\Psi_{l+1}' := \int_{t_l}^{t_{l+1}} \left[\Theta_{l+1} \sum_{i=1}^{N} G(l, i)\right] \left[\Theta_{l+1} \sum_{i=1}^{N} G(l, i)\right]^\top ds, \quad l = 0, \ldots, M. \tag{25}$$

It is natural to consider the relation among the conclusions in the above theorems.

Remark 3.12. For the necessity part of Theorem 3.8, to prove the full rank of $\mathcal{C}$, the controllability to the origin condition for system (1) is needed. If we multiple Θ_1^{-1} and its transpose to the matrix $\mathcal{C}$ from the left and right sides, respectively, it yields that $\bar{\Psi} = (W_1^* \cdots W_{M+1}^* V_1^* \cdots V_{M+1}^*\ U_1 U_1^\top \cdots U_M U_M^\top)$. It is easy to get that $rank\ (\bar{\Psi}) = rank\ (\Upsilon)$. Hence, the necessity part of Theorem 3.8 is equivalent to Theorem 3.6.

If the impulsive control matrices $I + E_k$ are non-singular, $k = 1, 2, \ldots, M$, then the non-singularity of $W_{m,j}$ and V_m implies the non-singularity of $\Phi_{m,j}$ and Ψ_m, $m = 1, \ldots, M + 1, j = 1, \ldots, N - 1$, respectively, subsequently the non-singularity of $\mathcal{C}$. But the inverse may be not hold because the full rank of $\mathcal{C}$ cannot guarantee the full rank of any of $W_{m,j}$ and V_m, which is required by Theorem 3.4. Hence, we can conclude that the rank condition in Theorem 3.8 is less conservative than that in Theorem 3.4 to check the controllability of system (1) if the non-singularity of impulsive control matrices is assumed.

4. A numerical example

In this section, we give an example to illustrate the effectiveness of the obtained results. For system (1), consider the two-dimensional system with two delays ($N = 2$) as follows:

$$A(t) = \begin{bmatrix} -1 & -e^{2t} \\ 0 & -4 \end{bmatrix}, \qquad B_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad B_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$E_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad F_1(t) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad E_2(t) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad F_2(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $M = 2$ and $t_0 = 0$, $t_1 = 0.1$, $t_2 = 0.2$, $t_3 = 0.4$, $h_1 = 0.05$, $h_2 = 0.06$, $u_0(s) = 0$, $s \in [-0.06, 0]$. By calculation, the associated transition matrix is

$$X(t, s) = \begin{bmatrix} e^{-t+s} & -e^{-t+3s} + e^{-2t+4s} \\ 0 & e^{-4t+4s} \end{bmatrix}.$$

From (20), it yields that

$$\begin{aligned}
\Phi_{1,1} &:= \int_{t_1-h_2}^{t_1-h_1} [\Theta_2 G(1,2) + \Theta_1 G(0,1)][\Theta_2 G(1,2) + \Theta_1 G(0,1)]^\top \mathrm{d}s, \\
\Phi_{2,1} &:= \int_{t_2-h_2}^{t_2-h_1} [\Theta_3 G(2,2) + \Theta_2 G(1,1)][\Theta_3 G(2,2) + \Theta_2 G(1,1)]^\top \mathrm{d}s, \\
\Phi_{3,1} &:= \int_{t_3-h_2}^{t_3-h_1} [\Theta_3 G(2,1)][\Theta_3 G(2,1)]^\top \mathrm{d}s; \\
\Psi_1 &:= \int_{t_0}^{t_1-h_2} \left[\Theta_1 \sum_{i=1}^{2} G(0,i)\right]\left[\Theta_1 \sum_{i=1}^{2} G(0,i)\right]^\top \mathrm{d}s, \\
\Psi_2 &:= \int_{t_1-h_1}^{t_2-h_2} \left[\Theta_2 \sum_{i=1}^{2} G(1,i)\right]\left[\Theta_2 \sum_{i=1}^{2} G(1,i)\right]^\top \mathrm{d}s, \\
\Psi_3 &:= \int_{t_2-h_1}^{t_3-h_2} \left[\Theta_3 \sum_{i=1}^{2} G(2,i)\right]\left[\Theta_3 \sum_{i=1}^{2} G(2,i)\right]^\top \mathrm{d}s; \\
\Xi_1 &:= \Theta_2 F_1 F_1^\top \Theta_2^\top, \qquad \Xi_2 := \Theta_3 F_2 F_2^\top \Theta_3^\top.
\end{aligned} \tag{26}$$

Then we have

$$\Phi_{1,1} = \begin{bmatrix} 2.0472 & 0.0879 \\ 0.0879 & 0.0038 \end{bmatrix}, \qquad \Phi_{2,1} = \begin{bmatrix} 0.4073 & 0.0293 \\ 0.0293 & 0.0021 \end{bmatrix}, \qquad \Phi_{3,1} = \begin{bmatrix} 0.0099 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Psi_1 = \begin{bmatrix} 57.4983 & 2.5353 \\ 2.5353 & 0.1118 \end{bmatrix}, \qquad \Psi_2 = \begin{bmatrix} 21.4793 & 1.0549 \\ 1.0549 & 0.0518 \end{bmatrix}, \qquad \Psi_3 = \begin{bmatrix} 12.4807 & 1.1036 \\ 1.1036 & 0.0977 \end{bmatrix},$$

$$\Xi_1 = \begin{bmatrix} 8.7810 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Xi_2 = \begin{bmatrix} 4.9346 & 0.9981 \\ 0.9981 & 0.2019 \end{bmatrix}.$$

Obviously, $rank(\mathcal{C}) = 2$. It follows from Theorem 3.8 that the system is controllable on $[t_0, t_f]$. However, results in papers [12–19,24,26–28] are limited to check the controllability of this linear time-varying impulsive system with multiple input delays.

5. Conclusion

In this paper, two kinds of controllability of time-varying impulsive systems with multiple delays in the control have been investigated. Several explicit sufficient and necessary conditions for the controllability of such systems have been established and the relation among these conditions has also been discussed. Compared with some existing results, it can be found that the algebraic approach has been extended to consider the controllability of more general systems.

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