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Stability radii of differential–algebraic equations with respect to stochastic perturbations

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ABSTRACT

In this paper, we investigate differential–algebraic equations (DAEs) subject to stochastic perturbations. We introduce the index-ν concept and establish a formula of solution for these equations. After that the stability is studied by using the method of Lyapunov functions. Finally, the robust stability of DAEs with respect to stochastic perturbations is considered. Formulas of the stability radii are derived. An example is given to illustrate the obtained results.

1. Introduction

This paper is concerned with differential–algebraic equations (DAEs) subject to stochastic perturbations of the form

$$\begin{cases} Edx(t) = (Ax(t) + g(t))dt + f(t, x(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{1.1}$$

where $E, A \in \mathbb{K}^{n\times n}$, the leading coefficient E is allowed to be a singular matrix and $w(t)$ is an m-dimensional Wiener process.

While standard differential–algebraic equations (DAEs) without random noise are today standard mathematical models for dynamical systems in many application areas, such as multibody systems, electrical circuit simulation, control theory, fluid dynamics, and chemical engineering (see, e.g., [1–4]), the stochastic version is typically needed to model effects that do not arise deterministically (see, e.g., [5,6]). In fact, an accurate mathematical model of a dynamic system in electrical, mechanical, or control engineering often requires the consideration of stochastic elements. Electronic circuit systems or multibody mechanism systems with random noise are often modeled by stochastic differential–algebraic equations (SDAEs), or sometimes called stochastic implicit dynamic systems. These models have been studied recently in [7–10]. It is well known that, due to the fact that the dynamics of (1.1) are constrained, some extra difficulties appear in the analysis of stability as well numerical treatments of SDAEs. These difficulties are typically characterized by index concepts, see [1–3]. Note that in [7–10], authors consider SDAEs only in the case of index-1.

On the other hand, in a lot of applications there is a frequently arising question, namely, how robust is a characteristic qualitative property of a system (e.g. stability) when the system comes under the effect of uncertain perturbations. The aspect of developing measures of stability robustness for linear uncertain systems with state space description has received significant attention in system and control theory. These measures can be characterized by stability radius. The problem of evaluating and calculating this stability radius is of great importance, from both theoretical and practical points of view and has attracted a lot of attention from researchers (see, e.g., [11–15] and the references given therein). For a systematic introduction to the topic, the interested readers are referred to the earlier work due to Hinrichsen and Pritchard [16] and their more recent monograph [17], which contains, along with rigorous theoretical developments, also an extensive literature review on the subject. It is remarkable that the similar problems have been considered for many other types of linear dynamical systems, including time-varying and time-delay systems, implicit systems, positive systems, linear systems in infinite-dimensional spaces as well as linear systems with respect to stochastic perturbations (see, e.g., [14,18–22]).

On the basis of the above discussion, there arises a natural question whether one can define measures of stability robustness

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for DAEs with respect to stochastic perturbations and, moreover, how to calculate these measures. To the best of our knowledge, such kind of questions has not been addressed so far in the literature, although different aspects of robust analysis for stability of DAEs with respect to deterministic perturbations have been studied already (see, e.g., [11,23–25]). The purpose of the present paper is to fill this gap. Firstly, we will study the consistency condition of random noise and define the index-ν concept for SDAEs. By using this index notion, we can establish the explicit expression of solution and the variation of constants formula. Secondly, we shall establish the necessary and sufficient conditions for the exponential L^2-stability of SIDEs by using the method of Lyapunov functions which is well known for the stability theory of dynamic systems. Finally, we will establish formula of the stability radius of DAEs with respect to stochastic perturbations. A problem, however, occurs in the case that the equation may not be solvable under stochastic perturbations, because then consistency conditions arise. To deal with this problem either a reformulation of the system has to be performed which characterizes the consistency conditions or the perturbations have to be further restricted.

The paper is organized as follows: In Section 2 we summarize some preliminary results on matrix analysis and stochastic analysis. In Section 3, solvability and stability for SDAEs are presented and the formula of solution is derived. In Section 4, we study robust stability for DAEs with respect to stochastic perturbations. We close the paper with a conclusion where some comments and the further work will be given.

2. Preliminary

For the reader's convenience, we recall some notations as well as some known results on matrix pair which will be used in the sequence; see, e.g., [2]. Let $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$, the set of complex or real numbers, $\mathbb{R}_+=\{\lambda\in\mathbb{R}:\lambda\geq 0\}$, $\mathbb{C}_-=\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$, n,m,k,l,q be positive integers. $\mathbb{K}^{n\times m}$ will stand for the set of all $n\times m$-matrices, $A^*\in\mathbb{K}^{m\times n}$ denotes the adjoint matrix of $A\in\mathbb{K}^{n\times m}$ and $I_r\in\mathbb{K}^{r\times r}$ denotes the identity matrix. The space $\mathbb{K}^{n\times m}$ is equipped by the inner product $\langle u,v\rangle=\operatorname{Trace}(v^*u)$ for all $u,v\in\mathbb{K}^{n\times m}$.

Definition 2.1. A matrix pair (E,A), $E,A\in\mathbb{K}^{n\times n}$ is called *regular* if there exists $s\in\mathbb{C}$ such that $\det(sE-A)$ is different from zero. Otherwise, if $\det(sE-A)=0$ for all $s\in\mathbb{C}$, then we say that (E,A) is *singular*.

If (E,A) is regular, then a complex number λ is called a *(generalized finite) eigenvalue* of (E,A) if $\det(\lambda E-A)=0$. The set of all (finite) eigenvalues of (E,A) is called the *(finite) spectrum of the pencil* (E,A) and denoted by $\sigma(E,A)$. If E is singular and the pair is regular, then we say that (E,A) has the eigenvalue ∞.

Regular pairs (E,A) can be transformed to *Weierstraß-Kronecker canonical form*, see [1–3], i.e., there exist nonsingular matrices $W,\ T\in\mathbb{K}^{n\times n}$ such that

$$E=W\begin{bmatrix}I_r & 0\\ 0 & N\end{bmatrix}T^{-1},\ \ A=W\begin{bmatrix}J & 0\\ 0 & I_{n-r}\end{bmatrix}T^{-1}, \tag{2.1}$$

where I_r, I_{n-r} are identity matrices of indicated size, $J\in\mathbb{K}^{r\times r}$, and $N\in\mathbb{K}^{(n-r)\times(n-r)}$ are matrices in Jordan canonical form and N is nilpotent. If E is invertible, then $r=n$, i.e., the second diagonal block does not occur.

Definition 2.2. Consider a regular pair (E,A) with $E,A\in\mathbb{K}^{n\times n}$ in Weierstraß–Kronecker form (2.1). If $r<n$ and N has nilpotency index $\nu\in\{1,2,\ldots\}$, i.e., $N^\nu=0$, $N^i\neq 0$ for $i=1,2,\ldots,\nu-1$, then ν is called the *index of the pair* (E,A) and we write $\operatorname{ind}(E,A)=\nu$. If $r=n$ then the pair has index $\nu=0$.

We note that $\nu=\operatorname{ind}(E,A)$ does not depend on the special transformation to canonical form. If $E\in\mathbb{K}^{n\times n}$ then the quantity $\nu=\operatorname{ind}(E,I)$ is called index (of nilpotency) of E and is denoted by $\nu=\operatorname{ind}(E)$.

Definition 2.3. Let $E\in\mathbb{K}^{n\times n}$ have $\nu=\operatorname{ind}E$. A matrix $X\in\mathbb{K}^{n\times n}$ satisfying

$$EX=XE, \tag{2.2a}$$

$$XEX=X, \tag{2.2b}$$

$$XE^{\nu+1}=E^\nu, \tag{2.2c}$$

is called a Drazin inverse of E.

Theorem 2.4. *Every $E\in\mathbb{K}^{n\times n}$ has one and only one Drazin inverse E^D. Moreover, if $E\in\mathbb{K}^{n\times n}$ is nonsingular then*

$$E^D=E^{-1}, \tag{2.3}$$

and for arbitrary nonsingular $T\in\mathbb{K}^{n\times n}$, we have

$$(T^{-1}ET)^D=T^{-1}E^DT. \tag{2.4}$$

Theorem 2.5. *Let $E\in\mathbb{K}^{n\times n}$ with $\nu=\operatorname{ind}E$. There is one and only one decomposition*

$$E=\widetilde{C}+\widetilde{N} \tag{2.5}$$

with the properties

$$\widetilde{C}\widetilde{N}=\widetilde{N}\widetilde{C}=0,\ \widetilde{N}^\nu=0,\ \widetilde{N}^{\nu-1}\neq 0,\ \operatorname{ind}\widetilde{C}\leq 1. \tag{2.6}$$

In particular, the following statements hold:

$$\widetilde{C}^D\widetilde{N}=0,\qquad \widetilde{N}\widetilde{C}^D=0, \tag{2.7a}$$

$$E^D=\widetilde{C}^D, \tag{2.7b}$$

$$\widetilde{C}\widetilde{C}^D\widetilde{C}=\widetilde{C}, \tag{2.7c}$$

$$\widetilde{C}^D\widetilde{C}=E^DE, \tag{2.7d}$$

$$\widetilde{C}=EE^DE,\quad \widetilde{N}=E(I-E^DE). \tag{2.7e}$$

Theorem 2.6. *Let $E,A\in\mathbb{K}^{n\times n}$ satisfy $AE=EA$. Then we have*

$$EA^D=A^DE,\ \ E^DA=AE^D,\ \ E^DA^D=A^DE^D. \tag{2.8}$$

Moreover, if

$$\ker E\cap\ker A=\{0\} \tag{2.9}$$

then we have

$$(I-E^DE)A^DA=I-E^DE. \tag{2.10}$$

Note that for the commuting matrices E and A, condition (2.9) is equivalent to the regularity of (E,A) and in formula (2.1) we can choose $T=W$.

Now, we recall some notations as well as some known results on stochastic analysis; see, e.g. [26,27]. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and $\mathbb{E}$ be the mathematical expectation. $L_2(\Omega,\mathbb{K}^{n\times m})$ denotes the set of all square integrable random variables, i.e. $\mathbb{E}(\|X\|^2)<\infty$. Let $w(t)$ be an m-dimensional Wiener process given on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. For $0\leq t_0<T<\infty$, denote $\mathbb{F}=(\mathcal{F}_t)_{t\geq t_0}$ by the canonical filtration generated by $\{w(s),t_0\leq s\leq t\}$. Assume that $h:[t_0,T]\times\Omega\to\mathbb{K}^{n\times m}$ is a measurable $\mathbb{F}$-adapted function such that $\mathbb{E}\int_{t_0}^T\|h(t)\|^2dt<\infty$, where $\|h(t)\|=\sqrt{\operatorname{Trace}(h(t)^*h(t))}$. Put $I(t)=\int_{t_0}^t h(s)dw(s)$ and $J(t)=\int_{t_0}^t h(s)ds$ for $t_0\leq t\leq T$. Then $I=\{I(t)\}_{t_0\leq t\leq T}$ is a square-integrable continuous martingale, $J=\{J(t)\}_{t_0\leq t\leq T}$ is a square-integrable stochastic process and their quadratic variations are given by

$$\langle I, I\rangle_t = \int_{t_0}^{t} \|h(s)\|^2 ds, \quad \langle J, J\rangle_t = 0 \quad \text{for } t_0 \le t \le T.$$

Moreover, for $t_0 \le t \le \tau \le T$,

$$\mathbb{E}\int_t^{\tau} h(s)dw(s) = 0, \quad \mathbb{E}\left\|\int_t^{\tau} h(s)dw(s)\right\|^2 = \mathbb{E}\int_t^{\tau} \|h(s)\|^2 ds.$$

Finally, assume that $\Delta : U \to V$ is Lipschitz continuous. Then, the Lipschitz norm of Δ is defined by

$$\|\Delta\| = \inf\{\gamma \ge 0,\ \forall y, \hat{y} \in U : \|\Delta(y) - \Delta(\hat{y})\| \le \gamma\|y - \hat{y}\|\},$$

where U, V are the normed vector spaces.

3. Stochastic differential–algebraic equations of index-ν

In this section, we consider the linear stochastic differential–algebraic equations with constant coefficients of the form

$$\begin{cases} Edx(t) = (Ax(t) + g(t))dt + f(t, x(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices, $g : [t_0, \infty) \to \mathbb{K}^n$ is a $(\nu - 1)$-times continuously differentiable vector-valued function, $w(t)$ is an m-dimensional Wiener process, $f : [t_0, \infty) \times \mathbb{K}^n \to \mathbb{K}^{n\times m}$ plays the role of a perturbation such that it is Lipschitz continuous in x, $f(t, x(t))$ is $\mathbb{F}$-adapted and $f(t, 0)$ is square integrable on $[t_0, T]$.

Definition 3.1. A function $x : [t_0, \infty) \times \Omega \to \mathbb{K}^n$ is a called solution of the initial value problem (3.1) if x is continuous and $\mathbb{F}$-adapted, $\int_{t_0}^{T} \|x(t)\|dt < \infty$ a.s., $\int_{t_0}^{T} \|f(t, x(t))\|^2 dt < \infty$ a.s. for $T > t_0$ and

$$Ex(t) = Ex_0 + \int_{t_0}^{t} (Ax(s) + g(s))ds + \int_{t_0}^{t} f(s, x(s))dw(s)$$

a.s. for all $t \in [t_0, \infty)$. The functions f, g and the initial condition x_0 are consistent with (3.1) if the associated initial value problem has at least one solution. Eq. (3.1) is called solvable if for every consistent f, g and x_0, the associated initial value problem has a solution.

Remark 3.2. If $E = I_n$ then Eq. (3.1) becomes a stochastic differential equation (SDE) and with the above assumption it has a unique solution (see, e.g., [28]). Moreover, the well-posedness of the SDAE (3.1) in the case of index-1 also has been studied in [7–10].

3.1. Solvability

We first treat the special case where E and A commute, i.e.

$$EA = AE. \tag{3.2}$$

According to Theorem 2.5, we have decomposition $E = \widetilde{C} + \widetilde{N}$ with the properties of $\widetilde{C}$ and $\widetilde{N}$ as given there. We get the following lemma.

Lemma 3.3. *Eq. (3.1) with property (3.2) is equivalent to the system*

$$\widetilde{C}dx_1(t) = Ax_1(t)dt + E^D Eg(t)dt + E^D Ef(t, x(t))dw(t), \tag{3.3a}$$

$$\widetilde{N}dx_2(t) = Ax_2(t)dt + (I - E^D E)g(t)dt + (I - E^D E)f(t, x(t))dw(t), \tag{3.3b}$$

where

$$x_1(t) = E^D Ex(t), \quad x_2(t) = (I - E^D E)x(t). \tag{3.4}$$

Moreover, Eq. (3.3a) is equivalent to the stochastic differential equation

$$dx_1(t) = E^D Ax_1(t)dt + E^D g(t)dt + E^D f(t, x(t))dw(t). \tag{3.5}$$

Proof. Assume that (3.1) holds. Multiplying (3.1) by $\widetilde{C}^D\widetilde{C}$, we have

$$\widetilde{C}^D\widetilde{C}Edx(t) = \widetilde{C}^D\widetilde{C}Ax(t)dt + \widetilde{C}^D\widetilde{C}g(t)dt + \widetilde{C}^D\widetilde{C}f(t, x(t))dw(t)$$

By using $\widetilde{C}^D\widetilde{C}E = \widetilde{C}\widetilde{C}^D E = \widetilde{C}E^D E$, $\widetilde{C}^D\widetilde{C} = E^D E$ and $E^D A = AE^D$, it implies that

$$\widetilde{C}E^D Edx(t) = AE^D Ex(t)dt + E^D Eg(t)dt + E^D Ef(t, x(t))dw(t),$$

or equivalently,

$$\widetilde{C}dx_1(t) = Ax_1(t)dt + E^D Eg(t)dt + E^D Ef(t, x(t))dw(t).$$

Note that by definition,

$$\begin{aligned}(I - E^D E)(I - E^D E) &= I - 2E^D E + E^D EE^D E \\ &= I - 2E^D E + E^D E = I - E^D E.\end{aligned}$$

Therefore, multiplying (3.1) by $I - E^D E$, we get

$$\begin{aligned}&(I - E^D E)Edx(t) = (I - E^D E)Ax(t)dt + \\ &\quad (I - E^D E)g(t)dt + (I - E^D E)f(t, x(t))dw(t) \\ &\Leftrightarrow (I - E^D E)(I - E^D E)Edx(t) = A(I - E^D E)x(t)dt + \\ &\quad (I - E^D E)g(t)dt + (I - E^D E)f(t, x(t))dw(t) \\ &\Leftrightarrow E(I - E^D E)(I - E^D E)dx(t) = A(I - E^D E)x(t)dt + \\ &\quad (I - E^D E)g(t)dt + (I - E^D E)f(t, x(t))dw(t) \\ &\Leftrightarrow \widetilde{N}dx_2(t) \\ &= Ax_2(t)dt + (I - E^D E)g(t)dt + (I - E^D E)f(t, x(t))dw(t).\end{aligned}$$

Conversely, assume that (3.3a) and (3.3b) hold, we have

$$\begin{aligned}\widetilde{C}dx_1(t) + \widetilde{N}dx_2(t) &= \widetilde{C}E^D Edx(t) + \widetilde{N}(I - E^D E)dx(t) \\ &= EE^D Edx(t) + E(I - E^D E)dx(t) = Edx(t).\end{aligned}$$

Therefore

$$\begin{aligned}Edx(t) &= \widetilde{C}dx_1(t) + \widetilde{N}dx_2(t) \\ &= A(x_1(t) + x_2(t))dt + (E^D E + I - E^D E)g(t)dt \\ &\qquad + (E^D E + I - E^D E)f(t, x(t))dw(t) \\ &= Ax(t)dt + g(t)dt + f(t, x(t))dw(t).\end{aligned}$$

Now, assume that (3.3a) holds. Multiplying (3.3a) by $\widetilde{C}^D = E^D$, we have

$$\begin{aligned}&\widetilde{C}^D\widetilde{C}dx_1(t) = \widetilde{C}^D Ax_1(t)dt + \widetilde{C}^D E^D Eg(t)dt + \widetilde{C}^D E^D Ef(t, x(t))dw(t) \\ &\Leftrightarrow E^D Edx_1(t) = E^D Ax_1(t)dt + E^D E^D Eg(t)dt + E^D E^D Ef(t, x(t))dw(t) \\ &\Leftrightarrow E^D Edx_1(t) = E^D Ax_1(t)dt + E^D g(t)dt + E^D f(t, x(t))dw(t).\end{aligned}$$

Since $E^D Edx_1(t) = E^D EE^D Edx(t) = E^D Edx(t) = dx_1(t)$, it implies that

$$dx_1(t) = E^D Ax_1(t)dt + E^D g(t)dt + E^D f(t, x(t))dw(t).$$

Conversely, assume that (3.5) holds. Multiplying (3.5) by $\widetilde{C}$ yields

$$\begin{aligned}\widetilde{C}dx_1(t) &= \widetilde{C}E^D Ax_1 dt + \widetilde{C}E^D g(t)dt + \widetilde{C}E^D f(t, x(t))dw(t) \\ &= \widetilde{C}\widetilde{C}^D Ax_1 dt + \widetilde{C}\widetilde{C}^D g(t)dt + \widetilde{C}\widetilde{C}^D f(t, x(t))dw(t) \\ &= \widetilde{C}^D\widetilde{C}Ax_1 dt + \widetilde{C}^D\widetilde{C}g(t)dt + \widetilde{C}^D\widetilde{C}f(t, x(t))dw(t) \\ &= E^D EAx_1 dt + E^D Eg(t)dt + E^D Ef(t, x(t))dw(t).\end{aligned}$$

Since

$$E^D EAx_1(t) = AE^D Ex_1(t) = AE^D E\big(E^D Ex(t)\big) = AE^D Ex(t) = Ax_1(t),$$

it implies that

$$\widetilde{C}dx_1(t) = Ax_1(t)dt + E^DEg(t)dt + E^DEf(t,x(t))dw(t).$$

The proof is complete. □

Remark 3.4. By the above lemma, a solution of the SDAE (3.1) can be expressed by sum of a solution of the classical SDE (3.5) and a solution of the SDAE (3.3b) with the nilpotent leading matrix.

Proposition 3.5. *Let $E, A \in \mathbb{K}^{n\times n}$ satisfy (3.2) and (2.9). Then the consistent condition of the perturbation f for solvability of (3.1) is*

$$(I - E^DE)f = 0. \tag{3.6}$$

Moreover, the solution of Eq. (3.3b) has only the form

$$x_2(t) = -(I - E^DE)\sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t), \quad \text{a.s. } \forall t \ge t_0, \tag{3.7}$$

and the consistent condition of g, x_0 for solvability of (3.1) is

$$(I - E^DE)\left(x_0 + \sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t_0)\right) = 0.$$

Proof. By using Theorem 2.5, we obtain

$$\begin{aligned} A\widetilde{N} &= AE(I - E^DE) = E(I - E^DE)A = \widetilde{N}A, \\ \widetilde{N}(I - E^DE) &= E(I - E^DE)(I - E^DE) = E(I - E^DE) = \widetilde{N}. \end{aligned} \tag{3.8}$$

Since $\widetilde{N}$ is a nilpotent matrix of degree ν, from (3.3b) it follows that

$$\begin{aligned} 0 &= \widetilde{N}^\nu dx_2(t) = \widetilde{N}^{\nu-1}\widetilde{N}dx_2(t) \\ &= \widetilde{N}^{\nu-1}\left(Ax_2(t)dt + (I - E^DE)g(t)dt + (I - E^DE)f(t,x(t))dw(t)\right) \\ &= \widetilde{N}^{\nu-1}Ax_2(t)dt + \widetilde{N}^{\nu-1}(I - E^DE)g(t)dt \\ &\qquad + \widetilde{N}^{\nu-1}(I - E^DE)f(t,x(t))dw(t) \\ &= \left(\widetilde{N}^{\nu-1}Ax_2(t) + \widetilde{N}^{\nu-1}g(t)\right)dt + \widetilde{N}^{\nu-1}f(t,x(t))dw(t). \end{aligned}$$

This implies that

$$\int_{t_0}^t \left(\widetilde{N}^{\nu-1}Ax_2(s) + \widetilde{N}^{\nu-1}g(s)\right)ds + \int_{t_0}^t \widetilde{N}^{\nu-1}f(s,x(s))dw(s) = 0.$$

Define the stochastic processes

$$X(t) = \int_{t_0}^t \left(\widetilde{N}^{\nu-1}Ax_2(s) + \widetilde{N}^{\nu-1}g(s)\right)ds = \int_{t_0}^t -\widetilde{N}^{\nu-1}f(s,x(s))dw(s).$$

Taking quadratic variation of X, we obtain

$$0 = \langle X, X\rangle_t = \int_{t_0}^t \left(\widetilde{N}^{\nu-1}f(s,x(s))\right)^2 ds.$$

This implies that $\widetilde{N}^{\nu-1}f(t,x(t)) = 0$, a.s. for all $t \ge t_0$ and hence

$$A\widetilde{N}^{\nu-1}x_2(t) + \widetilde{N}^{\nu-1}g(t) = 0, \ \text{a.s. } \forall t \ge t_0.$$

Multiplying this equation by $(I - E^DE)A^D$, we obtain

$$(I - E^DE)A^DA\widetilde{N}^{\nu-1}x_2(t) + (I - E^DE)A^D\widetilde{N}^{\nu-1}g(t) = 0,$$

a.s. for all $t \ge 0$. By (2.10), (3.2) and (3.8) we get

$$\widetilde{N}^{\nu-1}x_2(t) + A^D\widetilde{N}^{\nu-1}g(t) = 0, \ \text{a.s. } \forall t \ge t_0.$$

Differentiating this equation gives

$$\widetilde{N}^{\nu-1}dx_2(t) + A^D\widetilde{N}^{\nu-1}g'(t)dt = 0, \ \text{a.s. } \forall t \ge t_0.$$

This implies that

$$\begin{aligned} 0 &= \widetilde{N}^{\nu-2}\widetilde{N}dx_2(t) + A^D\widetilde{N}^{\nu-1}g'(t)dt \\ &= \widetilde{N}^{\nu-2}\left(Ax_2(t)dt + (I - E^DE)g(t)dt + (I - E^DE)f(t,x(t))dw(t)\right) \\ &\qquad + A^D\widetilde{N}^{\nu-1}g'(t)dt \\ &= \left(\widetilde{N}^{\nu-2}Ax_2(t) + \widetilde{N}^{\nu-2}(I - E^DE)g(t) + A^D\widetilde{N}^{\nu-1}g'(t)\right)dt \\ &\qquad + \widetilde{N}^{\nu-2}(I - E^DE)f(t,x(t))dw(t) \\ &= \left(\widetilde{N}^{\nu-2}Ax_2(t) + \widetilde{N}^{\nu-2}g(t) + A^D\widetilde{N}^{\nu-1}g'(t)\right)dt \\ &\qquad + \widetilde{N}^{\nu-2}f(t,x(t))dw(t). \end{aligned}$$

By taking quadratic variation, similar with the above argument, we obtain $\widetilde{N}^{\nu-2}f(t,x(t)) = 0$ and

$$A\widetilde{N}^{\nu-2}x_2(t) + \widetilde{N}^{\nu-2}g(t) + A^D\widetilde{N}^{\nu-1}g'(t) = 0, \ \text{a.s. } \forall t \ge t_0.$$

Multiplying this equation by $(I - E^DE)A^D$ we get

$$\widetilde{N}^{\nu-2}x_2(t) + A^D\widetilde{N}^{\nu-2}g(t) + (A^D)^2\widetilde{N}^{\nu-1}g'(t) = 0 \text{ a.s. } \forall t \ge t_0.$$

Applying this procedure continuously yields

$$\widetilde{N}x_2(t) + A^D\widetilde{N}g(t) + (A^D\widetilde{N})^2g'(t) + \cdots + (A^D\widetilde{N})^{\nu-1}g^{(\nu-2)}(t) = 0.$$

Differentiating this equation we get

$$\widetilde{N}dx_2(t) + A^D\widetilde{N}g'(t)dt + \cdots + (A^D\widetilde{N})^{\nu-1}g^{(\nu-1)}(t)dt = 0,$$

or equivalently,

$$\begin{aligned} &\left(Ax_2(t) + (I - E^DE)g(t) + A^D\widetilde{N}g'(t) + \cdots + (A^D\widetilde{N})^{\nu-1}g^{(\nu-1)}(t)\right)dt \\ &+ (I - E^DE)f(t,x(t))dw(t) = 0. \end{aligned}$$

By taking quadratic variation, it implies that $(I - E^DE)f(t,x(t)) = 0$ and

$$Ax_2(t) + (I - E^DE)g(t) + A^D\widetilde{N}g'(t) + \cdots + (A^D\widetilde{N})^{\nu-1}g^{(\nu-1)}(t) = 0,$$

a.s. for all $t \ge t_0$. Similarly, multiplying this equation by $(I - E^DE)A^D$ we obtain

$$\begin{aligned} (I - E^DE)A^D\Big(&Ax_2(t) + (I - E^DE)g(t) + A^D\widetilde{N}g'(t) + (A^D\widetilde{N})^2g''(t) \\ &+ (A^D\widetilde{N})^3g^{(3)}(t) + \cdots + (A^D\widetilde{N})^{\nu-1}g^{(\nu-1)}(t)\Big) = 0. \end{aligned}$$

By (2.10) and $x_2(t) = (I - E^DE)x_2(t)$, it implies that

$$\begin{aligned} &x_2(t) + (I - E^DE)\Big[A^Dg(t) + A^D(A^D\widetilde{N})g'(t) + A^D(A^D\widetilde{N})^2g''(t) \\ &+ A^D(A^D\widetilde{N})^3g^{(3)}(t) + \cdots + A^D(A^D\widetilde{N})^{\nu-1}g^{(\nu-1)}(t)\Big] = 0, \end{aligned}$$

or equivalently,

$$x_2(t) = -(I - E^DE)\sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t),$$

a.s. for all $t \ge t_0$. For $t = t_0$, $x_2(t_0) = (I - E^DE)x_0$ and we obtain the consistent condition of g, x_0,

$$(I - E^DE)\left(x_0 + \sum_{i=0}^{\nu-1} A^D(A^D\widetilde{N})^i g^{(i)}(t_0)\right) = 0.$$

Finally, since $f(t,x)$ is Lipschitz continuous in x, $E^Df(t, z + x_2(t))$ is so in z and there exists a constant $K > 0$ such that

$$\|E^Df(t,x_2(t))\| \le \|E^D\|\|f(t,0)\| + K\|E^D\|\|x_2(t)\|, \forall t \in [t_0, T].$$

This implies that $E^Df(t,x_2(t))$ is square integrable on $[t_0, T]$. Thus, by Remark 3.2, Eq. (3.5) has a unique solution $x_1(t)$ on $[t_0, T]$, and hence $x(t) = x_1(t) + x_2(t)$ solves only (3.1). The proof is complete. □

Motivated by the consistent condition of the perturbation f, we now derive the following notion.

Definition 3.6. The SDAE (3.1) is called tractable with index-ν (or for short, of index-ν) if

(i) $\operatorname{ind}(E, A) = \nu$,
(ii) $(I - E^D E)f = 0$.

Remark 3.7. In the case $\nu = 1$ the condition $(I - E^D E)f = 0$ is equivalent to $\operatorname{Im} f \subset \operatorname{Im} E$ and we return the notation of index-1 in [7–10]. The natural restriction (*ii*) is the so called condition that the noise sources do not appear in the constraints, or equivalently a requirement that the constraint part of solution process is not directly affected by random noise.

Theorem 3.8. *Assume that the SDAE (3.1) has index-ν and satisfies (3.2). Then, the solution of (3.1) is given by the formula*

$$\begin{aligned} x(t) =& e^{E^D A(t-t_0)} E^D E x_0 - (I - E^D E) \sum_{i=0}^{\nu-1} A^D (A^D \widetilde{N})^i g^{(i)}(t) \\ &+ \int_{t_0}^{t} e^{E^D A(t-s)} g(s) ds + \int_{t_0}^{t} e^{E^D A(t-s)} f(s, x(s)) dw. \end{aligned} \tag{3.9}$$

Proof. We have decomposition $x(t) = x_1(t) + x_2(t)$ for all $t \ge 0$. Since $x_1(t)$ solves Eq. (3.5), by variation of constants formula (see, e.g. [27]) we get

$$\begin{aligned} x_1(t) = e^{E^D A(t-t_0)} E^D E x_0 &+ \int_{t_0}^{t} e^{E^D A(t-s)} g(s) ds \\ &+ \int_{t_0}^{t} e^{E^D A(t-s)} f(s, x(s)) dw. \end{aligned}$$

Since the SDAE (3.1) has index-ν and satisfies (3.2), by Proposition 3.5 we have

$$x_2(t) = -(I - E^D E) \sum_{i=0}^{\nu-1} A^D (A^D \widetilde{N})^i g^{(i)}(t).$$

Thus we obtain (3.9). The proof is complete. □

In the general case, E and A may not be commutative. Since (E, A) is regular, there exists $\lambda_0 \in \mathbb{C}$ with $\det(\lambda_0 E - A) \neq 0$. Put

$$\widetilde{E} = (\lambda_0 E - A)^{-1} E,$$
$$\widetilde{A} = (\lambda_0 E - A)^{-1} A,$$
$$\widetilde{f} = (\lambda_0 E - A)^{-1} f,$$
$$\widetilde{g} = (\lambda_0 E - A)^{-1} g.$$

Then it is easy to see that $\widetilde{E}\widetilde{A} = \widetilde{A}\widetilde{E}$ and Eq. (3.1) is equivalent to

$$\widetilde{E} dx(t) = (\widetilde{A} x(t) + \widetilde{g}(t)) dt + \widetilde{f}(t, x(t)) dw(t). \tag{3.10}$$

Thus, by applying Proposition 3.5 and Theorem 3.8 for Eq. (3.10), we obtain the consistent condition of the perturbation and the formula of solution for the SDAE (3.1).

In what folows, without lost of generality, we will assume that E and A are commutative.

3.2. Stability

In this subsection, we study the L^2-stability and the exponential L^2-stability for SDAEs by using the method of Lyapunov functions. For defining the stability of zero solution, in Eq. (3.1) we assume that $g(t) = 0$ and $f(0, x) = 0$. Moreover, assume that there exist $a_1, a_2 > 0$ and a function $\gamma(t)$ such that $\|f(t, x)\| \le \gamma(t)\|x\|$ and $\int_{t_0}^{t} \gamma^2(s) ds \le a_1(t - t_0) + a_2$ for all $t \ge t_0$. Let us consider the equation

$$\begin{cases} E dx(t) = A x(t) dt + f(t, x(t)) dw(t), \\ x(t_0) = x_0, \end{cases} \tag{3.11}$$

where $E, A \in \mathbb{K}^{n \times n}$ are constant matrices and $w(t)$ be an m-dimensional Wiener process. For solvability of (3.11), by Proposition 3.5, the initial condition x_0 needs to satisfy the consistent condition $(I - E^D E) x_0 = x_2(t_0) = 0$, or equivalently, $x_0 \in \operatorname{Im}(E^D E)$.

Definition 3.9. Eq. (3.11) is said to be L^2-stable if $\int_{t_0}^{\infty} \mathbb{E} (\|x(t, t_0, x_0)\|^2) dt < \infty$ for all $x_0 \in \operatorname{Im}(E^D E)$. Eq. (3.11) is said to be exponentially L^2-stable if there exist $\alpha, \beta > 0$ such that

$$\mathbb{E}\|x(t, t_0, x_0)\|^2 \le \beta e^{-\alpha(t-t_0)} \|x_0\|^2, \tag{3.12}$$

for all $t \ge t_0 \ge 0$ and $x_0 \in \operatorname{Im}(E^D E)$.

Theorem 3.10. *Assume that the SDAE (3.11) has index-ν. Then Eq. (3.11) is exponentially L^2-stable if and only if there exists $\eta > 0$ such that*

$$\int_{t_0}^{\infty} \mathbb{E}(\|x(t, t_0, x_0)\|^2) dt \le \eta \|x_0\|^2, \tag{3.13}$$

for all $t \ge t_0 \ge 0$ and $x_0 \in \operatorname{Im}(E^D E)$.

Proof (*Necessity*)**.** Suppose that (3.11) is exponentially L^2-stable, i.e., inequality (3.12) holds with some constants $\alpha, \beta > 0$. Then we obtain

$$\int_{t_0}^{\infty} \mathbb{E}(\|x(t, t_0, x_0)\|^2) dt \le \int_{t_0}^{\infty} \beta e^{-\alpha(t-t_0)} \|x_0\|^2 dt = \frac{\beta}{\alpha} \|x_0\|^2.$$

Sufficiency: Suppose that (3.13) holds. By Lemma 3.3 and Proposition 3.5, it implies that $x(t)$ satisfies the equation

$$dx(t) = E^D A x(t) dt + E^D f(t, x(t)) dw(t) \tag{3.14}$$

with the constrained condition $(I - E^D E) x(t) = x_2(t) = 0$. Let L be the parabolic differential operator associated with system (3.14), i.e.

$$\begin{aligned} (Lv)(t, x) = & \frac{\partial v}{\partial t}(t, x) + (E^D A x)^* \frac{\partial v}{\partial x}(t, x) \\ &+ \frac{1}{2} \operatorname{Trace} \left[(E^D f(t, x))^* \frac{\partial^2 v}{\partial x \partial x}(t, x) (E^D f(t, x)) \right], \end{aligned}$$

for all $t \ge t_0, x \in \operatorname{Im}(E^D E)$. Taking $v(t, x) = \|x\|^2$, $x = x(t, t_0, x_0)$, $x_0 \in \operatorname{Im}(E^D E)$, we get easily that

$$\begin{aligned} |(Lv)(t, x)| &= \left| (E^D A x)^* 2x + \frac{1}{2} \operatorname{Trace} \left(\left(E^D f(t, x)\right)^* 2 \left(E^D f(t, x)\right) \right) \right| \\ &\le 2 \left| (E^D A x)^* x \right| + \|E^D\|^2 \|f(t, x)\|^2 \\ &\le 2 \|E^D A\| \|x\|^2 + \|E^D\|^2 \gamma^2(t) \|x\|^2 \\ &\le \left(2 \|E^D A\| + \gamma^2(t) \|E^D\|^2 \right) \|x\|^2 = a(t) \|x\|^2, \end{aligned}$$

where $a(t) = 2\|E^D A\| + \gamma^2(t) \|E^D\|^2$, $\|E^D\|^2 = \operatorname{Trace}\left(E^{D*} E^D\right)$ and $\|f(t, x)\|^2 = \operatorname{Trace}(f(t, x)^* f(t, x))$. Hence, $(Lv)(t, x) \ge -a(t)\|x\|^2$ for all $t \ge t_0$, $x \in \operatorname{Im}(E^D E)$. By using the Ito's formula (see, e.g., [26,27]), we have

$$\begin{aligned} \mathbb{E}\|x(t, t_0, x_0)\|^2 - \mathbb{E}\|x(t_0, t_0, x_0)\|^2 &= \mathbb{E}\|x(t, t_0, x_0)\|^2 - \|x_0\|^2 \\ &= \int_{t_0}^{t} \mathbb{E}[(Lv)(s, x(s, t_0, x_0))] ds. \end{aligned}$$

This implies that

$$\frac{d\mathbb{E}\|x(t, t_0, x_0)\|^2}{dt} = \mathbb{E}[(Lv)(t, x(t, t_0, x_0))] \ge -a(t) \mathbb{E}\|x(t, t_0, x_0)\|^2,$$

for all $t \geq t_0$, $x_0 \in \mathrm{Im}(E^D E)$. It is equivalent to $\frac{d}{dt}\left(e^{\int_{t_0}^t a(s)ds}\mathbb{E}\|x(t,t_0,x_0)\|^2\right) \geq 0$. Therefore, $e^{\int_{t_0}^t a(s)ds}\mathbb{E}\|x(t,t_0,x_0)\|^2 \geq \mathbb{E}\|x(t_0,t_0,x_0)\|^2$, or equivalently, $\mathbb{E}\|x(t,t_0,x_0)\|^2 \geq e^{-\int_{t_0}^t a(s)ds}\|x_0\|^2$, for all $t_0 \geq 0$ and $x_0 \in \mathrm{Im}(E^D E)$. By the assumption of $\gamma(t)$, this implies that

$$\mathbb{E}\|x(t,t_0,x_0)\|^2 \geq Me^{-\theta(t-t_0)}\|x_0\|^2, \tag{3.15}$$

where $M = e^{-a_2\|E^D\|^2}$ and $\theta = 2\|E^D A\| + a_1\|E^D\|^2$.

We define $V : \mathbb{R}_+ \times \mathrm{Im}(E^D E) \to \mathbb{R}_+$ by $V(t,y) = \int_t^\infty \mathbb{E}\|x(s,t,y)\|^2 ds$. It is obviously that $V(t,y) \leq \eta\|y\|^2, t \geq t_0, y \in \mathrm{Im}(E^D E)$. By (3.15), it implies that $V(t,y) \geq \frac{M}{\theta}\|y\|^2, t \geq t_0, y \in \mathrm{Im}(E^D E)$. Now, for a stochastic variable $\xi : \Omega \to \mathrm{Im}(E^D E)$, define $\mathbb{V}(t,\xi) = \mathbb{E}V(t,\xi)$. Then, for $y \in \mathrm{Im}(E^D E)$, we have $\frac{M}{\theta}\|y\|^2 \leq V(t,y) = \mathbb{V}(t,y) \leq \eta\|y\|^2$. On the other hand, since $x(s,t,x(t,t_0,x_0)) = x(s,t_0,x_0)$ a.s., we imply that (similarly, see Lemma 1 in [22]),

$$\mathbb{V}(t,x(t,t_0,x_0)) = \int_t^\infty \mathbb{E}\|x(s,t,x(t,t_0,x_0))\|^2 ds = \int_t^\infty \mathbb{E}\|x(s,t_0,x_0)\|^2 ds.$$

Therefore, $\frac{d\mathbb{V}(t,x(t,t_0,x_0))}{dt} = -\mathbb{E}\|x(t,t_0,x_0)\|^2$. On the other hand, we have

$$\frac{M}{\theta}\|x(t,t_0,x_0)\|^2 \leq V(t,x(t,t_0,x_0)) \leq \eta\|x(t,t_0,x_0)\|^2, \quad \text{a.s. } \forall t \geq t_0.$$

Taking the expectation on both sides, we get

$$\frac{M}{\theta}\mathbb{E}\|x(t,t_0,x_0)\|^2 \leq \mathbb{V}(t,x(t,t_0,x_0)) \leq \eta\mathbb{E}\|x(t,t_0,x_0)\|^2.$$

This implies that

$$\frac{d\mathbb{V}(t,x(t,t_0,x_0))}{dt} \leq -\frac{1}{\eta}\mathbb{V}(t,x(t,t_0,x_0)), \quad \forall t \geq t_0.$$

It is equivalent to $\frac{d}{dt}\left(\mathbb{V}(t,x(t,t_0,x_0))e^{\frac{1}{\eta}(t-t_0)}\right) \leq 0$, and hence

$$\mathbb{V}(t,x(t,t_0,x_0)) \leq \mathbb{V}(t_0,x(t_0,t_0,x_0))e^{\frac{-1}{\eta}(t-t_0)} \leq \eta\|x_0\|^2 e^{\frac{-1}{\eta}(t-t_0)}.$$

Thus

$$\mathbb{E}\|x(t,t_0,x_0)\|^2 \leq \frac{\theta}{M}\mathbb{V}(t,x(t,t_0,x_0)) \leq \frac{\theta\eta}{M}\|x_0\|^2 e^{\frac{-1}{\eta}(t-t_0)},$$

for all $t \geq t_0 \geq 0, x_0 \in \mathrm{Im}(E^D E)$. The proof is complete. □

4. Stability radii for stochastic differential–algebraic equations with respect to stochastic perturbations

In this section, we will develop approach in [19] to investigate the robust stability of DAEs subject to stochastic perturbations. Consider the regular SDAEs

$$\begin{cases} Edx(t) = Ax(t)dt + C\Delta(Bx(t))dw(t), \\ x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $E, A \in \mathbb{K}^{n\times n}$ are constant matrices, $B \in \mathbb{K}^{q\times n}$, $C \in \mathbb{K}^{n\times l}$ are structure matrices of perturbations, $w(t) \in \mathbb{R}^m, t \geq t_0 \geq 0$ are an m-dimensional Wiener process and x_0 is independent of $w(t), t \geq t_0 \geq 0$ and the disturbance operator $\Delta : \mathbb{K}^q \to \mathbb{K}^l$ is Lipschitz continuous with $\Delta(0) = 0$. The equation

$$Edx(t) = Ax(t)dt \tag{4.2}$$

is called the deterministic part of (4.1). Assume that $\sigma(E,A) \subset \mathbb{C}_-$, or equivalently, Eq. (4.2) is exponentially stable (see, e.g., [11]).

It is already known for the case of perturbed DAEs (see, e.g., [24,25,29]), that it is necessary to restrict the perturbations in order to get a meaningful concept of the structured stability radius, since a DAE system may lose its regularity, solvability and/or stability under infinitesimal perturbations. We therefore introduce the allowable stochastic perturbations in which the consistency condition (3.6) is satisfied, i.e.,

$$(I - E^D E)C = 0. \tag{4.3}$$

Definition 4.1. Assume that condition (4.3) holds. Then, the L^2-stability radius and the exponential L^2-stability radius of the exponentially stable equation (4.2) with respect to the stochastic perturbation in the form of (4.1) are defined by

$$r^s_{\mathbb{K}}(E,A;C,B) = \inf\{\|\Delta\|; (4.1) \text{ is not } L^2-\text{stable}\},$$
$$r^{es}_{\mathbb{K}}(E,A;C,B) = \inf\{\|\Delta\|; (4.1) \text{ is not exponentially } L^2-\text{stable}\}.$$

By Theorem 3.8 and the consistent initial condition $E^D Ex_0 = x_0$, the solution of (4.1) satisfies the equation

$$x(t) = e^{E^D A(t-t_0)}x_0 + \int_{t_0}^t e^{E^D A(t-s)}E^D C\Delta(Bx(s))dw(s). \tag{4.4}$$

Let

$\mathcal{V} = L^2[[t_0,\infty), L_2(\Omega,\mathbb{K}^{l\times m})]$, $\mathcal{H}_0 = L^2[[t_0,\infty), L_2(\Omega,\mathbb{K}^{n\times n})]$,
$\mathcal{H} = L^2[[t_0,\infty), L_2(\Omega,\mathbb{K}^{q\times n})]$.

The spaces $\mathcal{V}, \mathcal{H}_0, \mathcal{H}$ are equipped by the inner product $\langle\cdot,\cdot\rangle$ as follows

$$\langle u(\cdot), v(\cdot)\rangle = \int_{t_0}^\infty \mathbb{E}\langle u(t), v(t)\rangle dt = \int_{t_0}^\infty \mathbb{E}\,\mathrm{Trace}(v(t)^* u(t))dt.$$

With this inner product, $\mathcal{V}, \mathcal{H}_0, \mathcal{H}$ become the Hilbert spaces. We now define the operators $\mathbb{M} : \mathcal{V} \longrightarrow \mathcal{H}_0$ by

$$(\mathbb{M}v)(t) = \int_{t_0}^t e^{E^D A(t-s)}E^D Cv(s)dw(s), \tag{4.5}$$

and $\mathbb{L} : \mathcal{V} \longrightarrow \mathcal{H}$ by

$$(\mathbb{L}v)(t) = B(\mathbb{M}v)(t). \tag{4.6}$$

Using Weierstraß–Kronecker canonical form for commutative matrix pair, we have

$$e^{E^D A(t-s)}E^D = T\begin{bmatrix} e^{J(t-s)} & 0 \\ 0 & 0 \end{bmatrix}T^{-1},$$

where $J \in \mathbb{K}^{r\times r}$ with $\sigma(J) = \sigma(E,A) \subset \mathbb{C}_-$. Therefore there exist $K, \alpha > 0$ such that

$$\left\|e^{E^D A(t-s)}E^D\right\| \leq Ke^{-\alpha(t-s)}.$$

This implies that the operators $\mathbb{M}$ and $\mathbb{L}$ are bounded. Now, we derive an upper bound for the perturbation such that Eq. (4.1) preserves the exponential stability.

Theorem 4.2. *Assume that (4.3) holds and $\|\Delta\| < \|\mathbb{L}\|^{-1}$ where $\mathbb{L}$ is defined by (4.6). Then Eq. (4.1) is exponentially L^2-stable.*

Proof. From (4.4),

$$Bx(t) = Be^{E^D A(t-t_0)}x_0 + \int_{t_0}^t Be^{E^D A(t-s)}E^D C\Delta(Bx(s))dw(s). \tag{4.7}$$

Let $y(t) = Bx(t)$, $y_0(t) = Be^{E^D A(t-t_0)}x_0$, then (4.7) can be rewritten as

$$y(\cdot) = y_0(\cdot) + \mathbb{L}(\Delta(y))(\cdot). \tag{4.8}$$

For $y, \hat{y} \in H$,

$$\begin{aligned}\|\mathbb{L}(\Delta(y))(\cdot) - \mathbb{L}(\Delta(\hat{y}))(\cdot)\|_{\mathcal{H}} &\le \|\mathbb{L}\| \|\Delta(y)(\cdot) - \Delta(\hat{y})(\cdot)\|_{\mathcal{V}} \\ &\le \|\mathbb{L}\| \|\Delta\| \|y(\cdot) - \hat{y}(\cdot)\|_{\mathcal{H}}.\end{aligned}$$

Therefore (4.8) has a unique solution $y(\cdot)$ in $\mathcal{H}$ by the contraction theorem. On the other hand,

$$\begin{aligned}\|y(\cdot) - y_0(\cdot)\|_{\mathcal{H}} &= \|\mathbb{L}(\Delta(y))(\cdot)\|_{\mathcal{H}} \\ &\le \|\mathbb{L}(\Delta(y))(\cdot) - \mathbb{L}(\Delta(y_0))(\cdot)\|_{\mathcal{H}} + \|\mathbb{L}(\Delta(y_0))(\cdot)\|_{\mathcal{H}} \\ &\le \|\mathbb{L}\| \|\Delta(y)(\cdot) - \Delta(y_0)(\cdot)\|_{\mathcal{V}} + \|\mathbb{L}\| \|\Delta(y)(\cdot)\|_{\mathcal{V}} \\ &\le \|\mathbb{L}\| \|\Delta\| \|y(\cdot) - y_0(\cdot)\|_{\mathcal{H}} + \|\mathbb{L}\| \|\Delta\| \|y_0(\cdot)\|_{\mathcal{H}}.\end{aligned}$$

This implies that $\|y(\cdot) - y_0(\cdot)\|_{\mathcal{H}} \le \dfrac{\|\mathbb{L}\| \|\Delta\|}{1 - \|\mathbb{L}\| \|\Delta\|} \|y_0(\cdot)\|_{\mathcal{H}}$ and

$$\begin{aligned}\|y(\cdot)\|_{\mathcal{H}} &\le \|y(\cdot) - y_0(\cdot)\|_{\mathcal{H}} + \|y_0(\cdot)\|_{\mathcal{H}} \\ &\le \frac{\|\mathbb{L}\| \|\Delta\|}{1 - \|\mathbb{L}\| \|\Delta\|} \|y_0(\cdot)\|_{\mathcal{H}} + \|y_0(\cdot)\|_{\mathcal{H}} \\ &= \frac{1}{1 - \|\mathbb{L}\| \|\Delta\|} \|y_0(\cdot)\|_{\mathcal{H}} \\ &= \frac{1}{1 - \|\mathbb{L}\| \|\Delta\|} \left(\int_{t_0}^{\infty} \left\| B e^{E^D A(t-t_0)} E^D E x_0 \right\|^2 dt \right)^{1/2} \\ &\le \frac{1}{1 - \|\mathbb{L}\| \|\Delta\|} \\ &\quad \times \left(\int_{t_0}^{\infty} \|B\|^2 \left\| e^{E^D A(t-t_0)} E^D \right\|^2 \|E\|^2 \|x_0\|^2 dt \right)^{1/2} \\ &\le \frac{\|B\| \|E\|}{1 - \|\mathbb{L}\| \|\Delta\|} \|x_0\| \left(\int_{t_0}^{\infty} K^2 e^{-2\alpha(t-t_0)} dt \right)^{1/2} = K_0 \|x_0\|,\end{aligned}$$

where $K_0 = \dfrac{\|B\| \|E\| K}{(1 - \|\mathbb{L}\| \|\Delta\|)\sqrt{2\alpha}}$. Now, we define

$$\begin{aligned}x(t) &= e^{E^D A(t-t_0)} x_0 + \mathbb{M}(\Delta(y))(t) \\ &= e^{E^D A(t-t_0)} x_0 + \int_{t_0}^{t} e^{E^D A(t-s)} E^D C \Delta(y(s)) dw(s), \ \ t \ge t_0.\end{aligned}$$

Since Eq. (4.8) has a unique solution $y(\cdot)$ in $\mathcal{H}$, $x(\cdot)$ uniquely solves (4.1). Moreover,

$$\mathbb{E}\|x(t)\|^2 \le \left\| e^{E^D A(t-t_0)} x_0 \right\|^2 + \mathbb{E}\|\mathbb{M}(\Delta(y))(t)\|^2$$

and hence

$$\begin{aligned}\int_{t_0}^{\infty} \mathbb{E}\|x(t)\|^2 dt &= \int_{t_0}^{\infty} \left\| e^{E^D A(t-t_0)} x_0 \right\|^2 dt + \int_{t_0}^{\infty} \mathbb{E}\|\mathbb{M}(\Delta(y)(t))\|^2 dt \\ &= \int_{t_0}^{\infty} \left\| e^{E^D A(t-t_0)} E^D E x_0 \right\|^2 dt + \|\mathbb{M}(\Delta(y))(\cdot)\|^2_{\mathcal{H}_0} \\ &\le \frac{K^2 \|E\|^2}{2\alpha} \|x_0\|^2 + \|\mathbb{M}\|^2 \|\Delta\|^2 \|y(\cdot)\|^2_{\mathcal{H}} \\ &\le \left[\frac{K^2 \|E\|^2}{2\alpha} + \|\mathbb{M}\|^2 \|\Delta\|^2 K_0^2 \right] \|x_0\|^2,\end{aligned}$$

for all $x_0 \in \mathrm{Im}(E^D E)$. By Theorem 3.10, Eq. (4.1) is exponentially L^2-stable. □

Remark 4.3. In the case $E = I_n$, the above theorem shows that if $\|\Delta\| < \|\mathbb{L}\|^{-1}$ then the equation preserves the exponential L^2-stability while Theorem 2.1 in [19] has concluded that it preserves only the L^2-stability.

Define the matrix

$$P_\rho = P_\rho^* = \rho^2 \int_{t_0}^{\infty} E^{D*} e^{(E^D A)^*(t-t_0)} B^* B e^{E^D A(t-t_0)} E^D dt. \tag{4.9}$$

This is well-defined because $\|e^{E^D At} E^D\| \le K e^{-\alpha t}$. Since E and A commute, it implies that $e^{E^D At} E^D = E^D e^{E^D At}$. Therefore P_ρ is a solution of the Lyapunov equation

$$P_\rho E^D A + (E^D A)^* P_\rho + \rho^2 (B E^D)^* B E^D = 0, \tag{4.10}$$

and satisfies $P_\rho(I - E^D E) = 0$. We now derive a computable formula for $\|\mathbb{L}\|$ based on the matrix P_ρ.

Proposition 4.4. *Let P_ρ be defined in Eq. (4.9). Then we have*

$$\|\mathbb{L}\|^{-1} = \sup\{\rho > 0 : I_l - C^* P_\rho C \ge 0\}. \tag{4.11}$$

Proof. We have

$$\|v(\cdot)\|^2_{\mathcal{V}} - \rho^2 \|(\mathbb{L}v)(\cdot)\|^2_{\mathcal{H}} = \|v(\cdot)\|^2_{\mathcal{V}} - \rho^2 \int_{t_0}^{\infty} \mathbb{E}(\|(\mathbb{L}v)(t)\|^2) dt.$$

By Fubini's theorem, we have

$$\begin{aligned}&\rho^2 \int_{t_0}^{\infty} \mathbb{E}(\|(\mathbb{L}v)(t)\|^2) dt \\ =&\rho^2 \int_{t_0}^{\infty} \left(\mathbb{E} \left\| \int_{t_0}^{t} B e^{E^D A(t-s)} E^D C v(s) dw(s) \right\|^2 \right) dt \\ =& \int_{t_0}^{\infty} \rho^2 \mathbb{E} \int_{t_0}^{t} \left\| B e^{E^D A(t-s)} E^D C v(s) \right\|^2 ds dt \\ =& \int_{t_0}^{\infty} \mathbb{E} \int_{s}^{\infty} \rho^2 \left\| B e^{E^D A(t-s)} E^D C v(s) \right\|^2 dt ds \\ =& \int_{t_0}^{\infty} \mathbb{E} \int_{s}^{\infty} \rho^2 \left\langle B e^{E^D A(t-s)} E^D C v(s), B e^{E^D A(t-s)} E^D C v(s) \right\rangle dt ds \\ =& \int_{t_0}^{\infty} \mathbb{E} \left\langle \rho^2 \int_{s}^{\infty} E^{D*} e^{(E^D A)^*(t-s)} B^* B e^{E^D A(t-s)} E^D dt C v(s), C v(s) \right\rangle ds \\ =& \int_{0}^{\infty} \mathbb{E} \left\langle P_\rho C v(s), C v(s) \right\rangle ds.\end{aligned}$$

Hence

$$\begin{aligned}&\|v(\cdot)\|^2_{\mathcal{V}} - \rho^2 \|(\mathbb{L}v)(\cdot)\|^2_{\mathcal{H}} = \|v(\cdot)\|^2_{\mathcal{V}} - \rho^2 \int_{0}^{\infty} \mathbb{E}(\|(\mathbb{L}v)(t)\|^2) dt \\ &= \|v(\cdot)\|^2_{\mathcal{V}} - \langle P_\rho C v(\cdot), C v(\cdot) \rangle = \langle v(\cdot), v(\cdot) \rangle - \langle C^* P_\rho C v(\cdot), v(\cdot) \rangle \\ &= \langle (I_l - C^* P_\rho C) v(\cdot), v(\cdot) \rangle,\end{aligned}$$

for all $v \in \mathcal{V}$. This implies that $I_l - C^* P_\rho C \ge 0$ if and only if $\rho \le \|\mathbb{L}\|^{-1}$. Thus, we obtain

$$\|\mathbb{L}\|^{-1} = \sup\{\rho > 0 : I_l - C^* P_\rho C \ge 0\}.$$

The proof is complete. □

By using construction of the stochastic perturbation destroying stability in [19], we will construct a stochastic perturbation preserving stability with the norm near the stability radius to get the formula of these radii.

Theorem 4.5. *Assume that condition (4.3) holds. Then, the stability radii of the exponentially stable equation (4.2) with respect to the stochastic perturbation in the form of (4.1) is given by the formula*

$$r^s_{\mathbb{K}}(E, A; C, B) = r^{es}_{\mathbb{K}}(E, A; C, B) = \|\mathbb{L}\|^{-1}. \tag{4.12}$$

Proof. By Theorem 4.2, we have $r^{es}_{\mathbb{K}}(E, A; C, B) \ge \|\mathbb{L}\|^{-1}$. It follows from the definition of the stability radius that $r^s_{\mathbb{K}}(E, A; C, B) \ge r^{es}_{\mathbb{K}}(E, A; C, B)$. Now we will show that $r^s_{\mathbb{K}}(E, A; C, B) \le \|\mathbb{L}\|^{-1}$. Let $\rho = r^s_{\mathbb{K}}(E, A; C, B) - \epsilon > 0$ and $z \in \mathbb{K}^{l \times m}$, $\|z\| = 1$. Define $\Delta(y) = \rho\|y\| z$ then $\|\Delta\| = \rho < r^s_{\mathbb{K}}(E, A; C, B)$. By Definition 4.1 and Theorem 3.8, Eq. (4.1) is L^2-stable and has a solution $x(t)$ satisfying

$$Bx(t) = B e^{E^D A(t-t_0)} x_0 + \rho \int_{t_0}^{t} B e^{E^D A(t-s)} E^D C \|Bx(s)\| z dw(s).$$

This implies that

$$\begin{aligned}
\mathbb{E}\ \|Bx(t)\ \|^2 &= \mathbb{E}\left\|Be^{E^DA(t-t_0)}x_0 + \rho\int_{t_0}^t Be^{E^DA(t-s)}E^DC\|Bx(s)\|zdw(s)\right\|^2 \\
&= \|Be^{E^DA(t-t_0)}x_0\|^2 \\
&\quad + \mathbb{E}\Big(\rho^2\Big\|\int_{t_0}^t Be^{E^DA(t-s)}E^DC\|Bx(s)\|zdw(s)\Big\|^2\Big) \\
&\quad + 2\Big\langle Be^{E^DA(t-t_0)}x_0, \rho\mathbb{E}\Big(\int_{t_0}^t Be^{E^DA(t-s)}E^DC\|Bx(s)\|zdw(s)\Big)\Big\rangle \\
&= \|Be^{E^DA(t-t_0)}x_0\|^2 \\
&\quad + \rho^2\mathbb{E}\Big(\Big\|\int_{t_0}^t Be^{E^DA(t-s)}E^DCz\|Bx(s)\|dw(s)\Big\|^2\Big) \\
&= \|Be^{E^DA(t-t_0)}x_0\|^2 + \rho^2\mathbb{E}\Big(\int_{t_0}^t\Big\|Be^{E^DA(t-s)}E^DCz\|Bx(s)\|\Big\|^2ds\Big) \\
&= \|Be^{E^DA(t-t_0)}x_0\|^2 \\
&\quad + \rho^2\int_{t_0}^t \|\ Be^{E^DA(t-s)}E^DCz\ \|^2\ \ \mathbb{E}\big(\|Bx(s)\|^2\big)ds.
\end{aligned}$$

Therefore

$$\int_{t_0}^\infty \mathbb{E}\big(\|Bx(t)\|^2\big)dt = \int_{t_0}^\infty \big(\|Be^{E^DA(t-t_0)}x_0\|^2\big)dt + \rho^2\int_{t_0}^\infty\int_{t_0}^t \|Be^{E^DA(t-s)}E^DCz\|^2\ \mathbb{E}\big(\|Bx(s)\|^2\big)dsdt.$$

On the other hand,

$$\begin{aligned}
&\int_{t_0}^\infty\int_{t_0}^t \rho^2\|Be^{E^DA(t-s)}E^DCz\|^2\ \mathbb{E}\big(\|Bx(s)\|^2\big)dsdt \\
&= \int_{t_0}^\infty\int_s^\infty \rho^2\|Be^{E^DA(t-s)}E^DCz\|^2dt\mathbb{E}\big(\|Bx(s)\|^2\big)ds \\
&= \int_{t_0}^\infty \rho^2\int_s^\infty \Big\langle Be^{E^DA(t-s)}E^DCz, Be^{E^DA(t-s)}E^DCz\Big\rangle dt\ \mathbb{E}\big(\|Bx(s)\|^2\big)ds \\
&= \int_{t_0}^\infty \Big\langle \rho^2\int_s^\infty E^{D*}e^{(E^DA)^*(t-s)}B^*Be^{E^DA(t-s)}E^DdtCz, Cz\Big\rangle\mathbb{E}\big(\|Bx(s)\|^2\big)ds \\
&= \langle P_\rho Cz, Cz\rangle\int_{t_0}^\infty \mathbb{E}\big(\|Bx(s)\|^2\big)ds = \langle C^*P_\rho Cz, z\rangle\int_{t_0}^\infty \mathbb{E}\big(\|Bx(s)\|^2\big)ds.
\end{aligned}$$

This implies that

$$(1-\langle C^*P_\rho Cz, z\rangle)\int_{t_0}^\infty \mathbb{E}(\|Bx(t)\|^2)dt = \int_{t_0}^\infty \mathbb{E}(\|Be^{E^DA(t-t_0)}x_0\|^2)dt,$$

and hence $1-\langle C^*P_\rho Cz, z\rangle \geq 0$ for all $z\in\mathbb{K}^{l\times m}$, $\|z\|=1$. This is equivalent to $I_l - C^*P_\rho C \geq 0$. By Proposition 4.4, we imply that $r^s_{\mathbb{K}}(E,A;C,B)-\epsilon = \rho \leq \|\mathbb{L}\|^{-1}$ for arbitrary $\epsilon > 0$. Let $\epsilon\to 0$ we get $r^s_{\mathbb{K}}(E,A;C,B) \leq \|\mathbb{L}\|^{-1}$. The proof is complete. □

Remark 4.6. The above theorem shows that the L^2-stability radius equal the exponential L^2-stability radius. In the case $E = I_n$, by letting $w = (w_i)$, where w_i is one dimensional Wiener process, we will obtain the formula of stability radii in [19,22].

Example 4.7. Consider a DAE subject to stochastic perturbation:

$$EdX = AXdt + C\Delta(BX)dw, \tag{4.13}$$

where

$$E = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}, A = \begin{bmatrix}-2 & 2\\ 2 & -3\end{bmatrix}, C = \begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix},$$
$$B = \begin{bmatrix}1 & -1\end{bmatrix}.$$

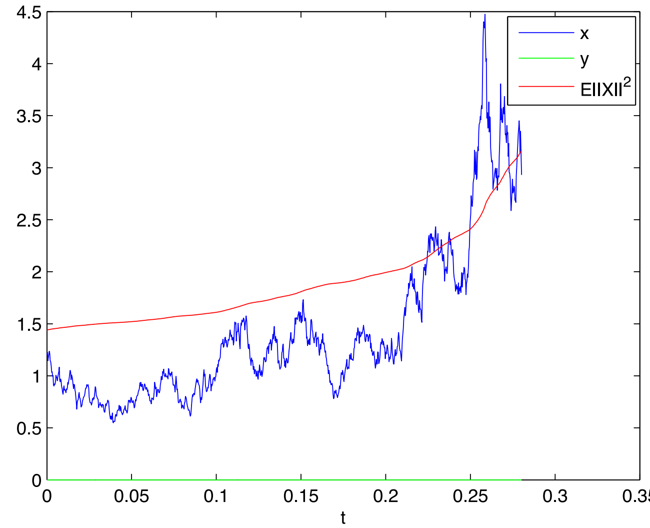

Fig. 1. The unstable solution $X(t) = \binom{x(t)}{0}$.

Because of $EA \neq AE$, put

$$E_1 = (-3E-A)^{-1}E = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix},$$
$$A_1 = (-3E-A)^{-1}A = \begin{bmatrix}2 & -3\\ 0 & -1\end{bmatrix},$$
$$C_1 = (-3E-A)^{-1}C = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}.$$

Then, Eq. (4.13) is equivalent to

$$E_1dX = A_1Xdt + C_1\Delta(BX)dw. \tag{4.14}$$

Since $\sigma(E_1, A_1) = -2$, the homogeneous equation of (4.14) is exponentially stable. By the direct computation, $E_1^D = E_1 = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix}$. Therefore, solving the Lyapunov equation

$$P_\rho E_1^DA_1 + (E_1^DA_1)^*P_\rho + \rho^2(BE_1^D)^*BE_1^D = 0$$

with $P_\rho(I-E_1^DE_1) = 0$ we get $P_\rho = \begin{bmatrix}\rho^2/4 & -\rho^2/4\\ -\rho^2/4 & \rho^2/4\end{bmatrix}$ and $C_1^*P_\rho C_1 = \begin{bmatrix}\rho^2/4 & \rho^2/4\\ \rho^2/4 & \rho^2/4\end{bmatrix}$. By Proposition 4.4, we have

$$\begin{aligned}
\|\mathbb{L}\|^{-1} &= \sup\{\rho > 0 : I - C_1^*P_\rho C_1 \geq 0\} \\
&= \sup\left\{\rho > 0 : \begin{bmatrix}1-\rho^2/4 & -\rho^2/4\\ -\rho^2/4 & 1-\rho^2/4\end{bmatrix} \geq 0\right\} \\
&= \sup\left\{\rho > 0 : \rho^2/4 \leq \frac{1}{2}\right\} = \sqrt{2}.
\end{aligned}$$

By Theorem 4.5, we obtain

$$\begin{aligned}
r^s_{\mathbb{K}}(E,A;C,B) &= r^{es}_{\mathbb{K}}(E,A;C,B) \\
&= r^s_{\mathbb{K}}(E_1,A_1;C_1,B) = r^{es}_{\mathbb{K}}(E_1,A_1;C_1,B) = \sqrt{2}.
\end{aligned}$$

Now, we construct two perturbations

$$\Delta_1(y) = \rho_1\|y\|z_1,\ \Delta_2(y) = \rho_2\|y\|z_2,$$

where $\rho_1 = 1.5392$, $z_1 = \begin{bmatrix}0.866\\ 0.5\end{bmatrix}$, $\rho_2 = 1,412$, $z_2 = \begin{bmatrix}0.866\\ -0.5\end{bmatrix}$. Then, it is easy to see that $\|\Delta_1\| = \rho_1 > \|\mathbb{L}\|^{-1} > \|\Delta_2\| = \rho_2$. With the perturbation Δ_1, Eq. (4.1) is exponentially L^2-unstable, see Fig. 1. With the perturbation Δ_2, Eq. (4.1) is exponentially L^2-stable, see Fig. 2.

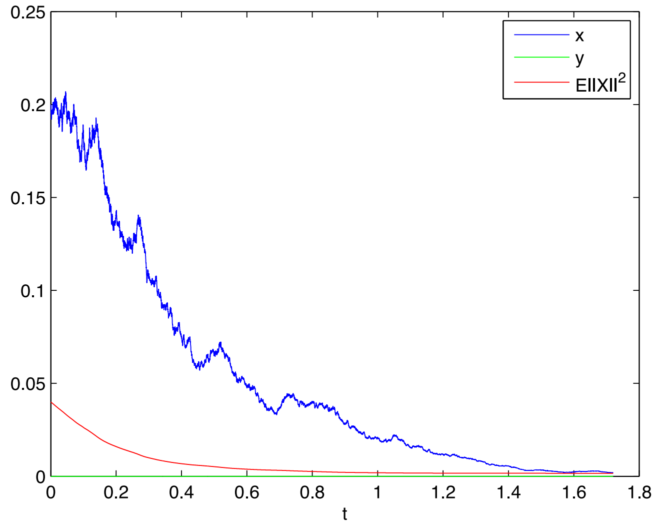

Fig. 2. The stable solution $X(t) = \binom{x(t)}{0}$.

5. Conclusion

In this paper, we have studied DAEs subject to stochastic perturbations. We derive the index-ν concept and establish formula of solution for these equations. The exponential L^2-stability is investigated by using the method of Lyapunov functions. Formulas of the stability radii are derived. However, numerical algorithms must be elaborated further to solve the optimization problems which are involved in calculation of the stability radii we have established in the previous section.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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