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# Discrete-Time Inverse Dynamics Control of Flexible Joint Robots

*This paper focuses on the problem of the application of inverse dynamics control methods to robots with flexible joints and electromechanical actuators. Due to drawbacks of the continuous-time inverse dynamics, discussed in the paper, a new control strategy in discrete-time is presented. The proposed control algorithm is based on numerical methods conceived for the solution of index-three systems of differential-algebraic equations. The method is computationally efficient and admits low sampling frequencies. The results of numerical experiments confirm the advantages of the designed control algorithm.*

## 1 Introduction

In many cases the elasticity of a robot structure considerably influences the precision of operational task execution and needs to be taken into account in the robot dynamic analysis, and to be included in the control system design. For the broad class of industrial robots the elastic compliance is mainly concentrated in joints and harmonic drives (Good et al., 1985). For this reason, there is a growing interest in the area of modeling and control of flexible joint robots.

From among nonadaptive control methods developed for the rigid robot case, but those whose range of application reaches also flexible joint robots, the following methods are cited: computed torque technique (Markiewicz, 1973; Bejczy, 1974), feedback linearization (Tarn et al., 1984; Bortoff and Spong, 1987), and invariant manifold approach (Khorosani and Spong, 1985). Applying the computed torque technique or feedback linearization, the robot dynamic model nonlinearities are compensated by nonlinear feedback. In this way a number of decoupled linear subsystems corresponding to each degree of freedom in the joint space or in the task space can be obtained.

The problem to be considered in this paper is motivated by difficulties in the practical application of the classical continuous-time inverse dynamics. The principal difficulties are:

- the complexity of the control laws;
- the necessity of feedback of measured (or evaluated) values of higher order time derivatives of system coordinates;
- the need for very high sampling frequencies.

By virtue of the deficiencies of the continuous-time approach, there is a need for simplified discrete-time implementations of nonlinear system controllers. In this paper a new control method based on inverse dynamics in discrete-time is presented. The control algorithms are developed using the methods conceived for the solution of systems of differential-algebraic equations

(DAE). The DAE methods are considered to be adequate for this purpose since the inverse dynamics problem can be described by a set of differential equations governing the robot motion and by a set of algebraic equations representing the specified motion of the robot end effector in the task space. It is shown in the paper that using the solution of the developed index three DAE system in the control problem naturally leads to the predictive control scheme. In the formulation of control laws multiple differentiation of system equations is avoided. Therefore, the control laws are less complex than in the full analytical inverse dynamics control, and the application of the method does not necessitate direct feedback of higher order time derivatives of system variables. Moreover, using low sampling frequencies is possible.

## 2 Robot Dynamic Model

Physical model of a robot with flexible joints is usually represented as a system of  $2n$  interconnected rigid bodies:  $n$  links and  $n$  actuators (Spong, 1987). It is assumed that the elasticity of the  $i$ th joint is mainly concentrated in a harmonic drive reducer and in an intermediate shaft connecting the harmonic drive and a pair of gears (Potkonjak, 1988). To describe the configuration of the considered mechanical system two sets of coordinates are introduced:  $\varphi_i$ —the angle of rotation of  $i$ th actuator rotor with respect to  $(i-1)$ th link, and  $q_i$ —the joint variable defining the position of  $i$ th link with respect to  $(i-1)$ th one.

The dynamic model of a flexible joint robot can be formulated using Lagrange's or other approaches. Using usually adopted simplifications concerning omission of the cross-coupling terms due to motor mass rotation (Spong, 1987) one obtains:

$$\mathbf{A}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{B}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{n}\mathbf{K}(\mathbf{n}\mathbf{q} - \mathbf{N}^{-1}\boldsymbol{\varphi}) = \mathbf{0} \quad (1)$$

$$\mathbf{I}_r\ddot{\boldsymbol{\varphi}} + \mathbf{B}_{\boldsymbol{\varphi}}\dot{\boldsymbol{\varphi}} - \mathbf{N}^{-1}\mathbf{K}(\mathbf{n}\mathbf{q} - \mathbf{N}^{-1}\boldsymbol{\varphi}) = \mathbf{T}. \quad (2)$$

While the first equation in the above presented set describes the motion of the robot links, the second governs the motion of the actuator rotors. In Eq. (1),  $\mathbf{A}(\mathbf{q})$  is the symmetric positive definite inertia matrix corresponding to rigid degrees of free-

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dom;  $\mathbf{I}_i = \text{diag}[I_{ii}]$ —where  $I_{ii}$  is the moment of inertia of  $i$ th rotor;  $\mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})$  contains centrifugal, Coriolis, and gravitational forces;  $\mathbf{B}_\varphi = \text{diag}[B_{\varphi i}]$ —where  $B_{\varphi i}$  is the viscous friction coefficient of  $i$ th rotor;  $\mathbf{n} = \text{diag}[n_i]$ —where  $n_i$  is the gear ratio;  $\mathbf{N} = \text{diag}[N_i]$ —where  $N_i$  is the ratio of the harmonic drive;  $\mathbf{K} = \text{diag}[K_i]$ —where  $K_i$  is the elastic constant;  $\mathbf{T}$ —the vector of torques applied to the actuator rotors.

The dynamic equations of the electric circuits of the armature controlled d-c motors can be presented in the following matrix form (Tarn et al., 1988):

$$\dot{\mathbf{L}}\mathbf{T} + \mathbf{R}\mathbf{T} + \mathbf{C}_t\mathbf{C}_v\dot{\mathbf{q}} = \mathbf{C}_t\mathbf{u}. \quad (3)$$

In Eq. (3),  $\mathbf{L}$  and  $\mathbf{R}$  are diagonal matrices with elements  $L_i$  and  $R_i$  being the inductance and the resistance of armature circuit, respectively;  $\mathbf{C}_t$  and  $\mathbf{C}_v$  are diagonal matrices with  $C_{ti}$  and  $C_{vi}$  denoting the torque and voltage constants of the motor, respectively; and  $\mathbf{u}$  is the vector of the armature voltages treated as the control inputs.

Equations (1)–(3) constitute the complete dynamic model of the flexible joint robot with electromechanical actuators.

### 3 Inverse Dynamics Task Control

In many robotic applications, the manipulator end effector is required to follow a preplanned path specified in the robot task space. The relation between the vector  $\mathbf{x}$  representing the position and orientation of the end effector, and the joint variables  $\mathbf{q}$  can be written as:  $\mathbf{x} = \mathbf{f}(\mathbf{q})$ , where it is assumed that  $\mathbf{x}$  has dimension equal to  $n$ . The motion of the robot will be programmed by specifying the desired values of the end effector configuration parameters:  $\mathbf{x} = \mathbf{x}_d(t)$ , which corresponds to imposing the so-called "program" constraints (Jankowski, 1989) on the system:

$$\mathbf{F}(\mathbf{q}, t) = \mathbf{f}(\mathbf{q}) - \mathbf{x}_d(t) = \mathbf{0} \quad (4)$$

The analytical approach to solve the set of differential algebraic Eqs. (1)–(4) consists in differentiation with respect to time of those equations which do not depend on the controls  $\mathbf{u}$  and subsequent elimination of the time derivatives of angles  $\varphi$  and torques  $\mathbf{T}$  (Jankowski and Van Brussel, 1991). Therefore, Eq. (1) is solved for  $\varphi$  and the result is differentiated three times with respect to time to obtain  $\dot{\varphi}$ ,  $\ddot{\varphi}$ , and  $\dddot{\varphi}$ . Next, Eq. (2) is differentiated once with respect to time to get  $\dot{\mathbf{T}}$ . Now, using the obtained expressions and combining the Eqs. (1)–(3), the controls  $\mathbf{u}$  can be obtained in function of the time derivatives of the generalized coordinates  $\mathbf{q}$  up to the fifth order inclusive.

Moreover, in order to obtain the control laws realizing the prescribed motion of the end effector, the constraints (4) have to be differentiated successively with respect to time up to the appearance of  $\dot{\mathbf{q}}^5$ . It follows that in the resulting equation higher order time derivatives of the manipulator Jacobian  $\mathbf{J} = \partial \mathbf{f} / \partial \mathbf{q}$  appear. Following the extended computed torque approach, the vector  $\dot{\mathbf{q}}^5$  is calculated from the obtained equation modified to form the desired error equation in the robot task space:

$$\begin{matrix} (5) & (4) & (3) \\ \mathbf{F} + \mathbf{k}_4 \mathbf{F} + \mathbf{k}_3 \mathbf{F} + \mathbf{k}_2 \ddot{\mathbf{F}} + \mathbf{k}_1 \dot{\mathbf{F}} + \mathbf{k}_0 \mathbf{F} = \mathbf{0}, \end{matrix} \quad (5)$$

where  $\mathbf{k}_i (i=0, \dots, 4)$  are constant feedback gain matrices. In this way, the control law which ensures the nonlinear decoupling and feedback linearization can be obtained.

The above presented analytical approach can theoretically be implemented to the inverse dynamics control of flexible joint manipulators. However, it was observed that for the number of links greater than three difficulties with analytical derivation of the control laws can appear, even if a computer package for symbolic calculation, such as MACSYMA, is used. The complicated form of these equations is caused by the necessity of multiple differentiation with respect to time of the inertia  $\mathbf{A}(\mathbf{q})$  and Jacobian  $\mathbf{J}(\mathbf{q})$  matrices as well as the vector

of nonlinear terms  $\mathbf{B}(\mathbf{q}, \dot{\mathbf{q}})$ . For the numbers of degrees of freedom  $n > 3$ , it is only for special structure robots, for example a SCARA type, that the control laws with a relatively reasonable size and complexity can be obtained.

The complicated form of the control law creates difficulties in practical on-line implementation of the continuous-time control scheme. For a high-performance robot arm under digital computer control, the sampling rate is critical for robot dynamic control. The complexity of the robot control system results in increasing the time delay of the reconstructed error signal, so it places a maximum on the sampling rate. In the same time, it was observed that to achieve satisfactory performance, very high sampling frequencies are required for continuous-time computed torque (Uhlik, 1990).

Therefore, there is a need to develop another approach which would permit the application, at lower costs, of the methodology of inverse dynamics outlined in this section. It seems to be possible to design a nonlinear discrete-time controller with the use of numerical methods which have been developed for solving systems of differential-algebraic equations (DAE). A technique which is based on this idea is described in the following sections.

### 4 Discrete-Time Solution

The numerical solution of higher index DAE systems is not a straightforward task. In particular, forward differentiation formulae fail when applied to such systems. As it was proven by Gear et al. (1985), variable-step, variable-order backward differentiation formula (BDF) methods applied to solve a DAE system converge if the index of the system does not exceed two and the methods are convergent for an ordinary differential equation. Moreover, Gear and Petzold (1984) and Lötstedt and Petzold (1986) showed that index three systems arising in the dynamics of constrained mechanical systems can be solved with constant stepsize BDF methods.

The index of the DAE system described in Section 3, which is used to solve the control problem if the full analytical approach is adopted, equals one. The index three DAE system is obtained if two differentiations of Eqs. (1) and (4), and one differentiation of Eq. (2), are not performed.

For the numerical solution purposes, we can write the above defined index three system as a set of first order differential equations:

$$\dot{\mathbf{q}} = \mathbf{v}, \quad (6)$$

$$\dot{\mathbf{v}} = \mathbf{a}, \quad (7)$$

$$\mathbf{A}\dot{\mathbf{a}} + \dot{\mathbf{A}}\mathbf{a} + \dot{\mathbf{B}} + \mathbf{n}\mathbf{K}(\mathbf{nv} - \mathbf{N}^{-1}\omega) = \mathbf{0}, \quad (8)$$

$$\mathbf{I}_r\dot{\omega} + \mathbf{B}_\varphi\omega + \mathbf{N}^{-1}\mathbf{n}^{-1}(\mathbf{A}\mathbf{a} + \mathbf{B}) = \mathbf{T}, \quad (9)$$

$$\dot{\mathbf{L}}\mathbf{T} + \mathbf{R}\mathbf{T} + \mathbf{C}_t\mathbf{C}_v\omega = \mathbf{C}_t\mathbf{u}, \quad (10)$$

$$\begin{aligned} \mathbf{J}\dot{\mathbf{a}} + 2\mathbf{J}\mathbf{a} + \dot{\mathbf{J}}\mathbf{v} - \dot{\mathbf{x}}_d &+ \mathbf{k}_2(\mathbf{J}\mathbf{a} + \dot{\mathbf{J}}\mathbf{v} - \ddot{\mathbf{x}}_d) \\ &+ \mathbf{k}_1(\mathbf{J}\mathbf{v} - \dot{\mathbf{x}}_d) + \mathbf{k}_0[\mathbf{f}(\mathbf{q}) - \mathbf{x}_d] = \mathbf{0}, \end{aligned} \quad (11)$$

where  $\omega = \dot{\varphi}$ ,  $\mathbf{v} = \dot{\mathbf{q}}$ ,  $\mathbf{a} = \dot{\mathbf{v}}$ ,  $\mathbf{k}_2$ ,  $\mathbf{k}_1$ ,  $\mathbf{k}_0$  are constant feedback gain matrices, and the other symbols have already been described. Eq. (11) represents the stabilized version of the equation obtained by taking the third order time derivative of the constraint Eq. (4).

Thus, after replacing  $\dot{\mathbf{a}}$  in Eq. (11) by the expression obtained from Eq. (8), the system (6)–(11) can be represented in the standard index three DAE form:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{g}_1(\mathbf{x}, \mathbf{y}), \\ \dot{\mathbf{y}} &= \mathbf{g}_2(\mathbf{x}, \mathbf{y}, \mathbf{u}), \\ \mathbf{g}_3(\mathbf{x}, \mathbf{t}) &= 0, \end{aligned} \quad (12)$$

with

$$\begin{aligned} \mathbf{x} &= [\mathbf{q}^T, \mathbf{v}^T, \mathbf{a}^T, \boldsymbol{\omega}^T]^T, \\ \mathbf{y} &= \mathbf{T}. \end{aligned} \quad (13)$$

The basic idea of using a numerical ordinary differential equation method for solving the system (6)–(11) consists of replacing the time derivatives  $\dot{\mathbf{q}}$ ,  $\dot{\mathbf{v}}$ ,  $\dot{\mathbf{a}}$ ,  $\dot{\boldsymbol{\omega}}$ , and  $\dot{\mathbf{T}}$  by difference approximations, and then solving the resulting system of algebraic nonlinear equations for approximation to  $\mathbf{q}$ ,  $\mathbf{v}$ ,  $\mathbf{a}$ ,  $\boldsymbol{\omega}$ ,  $\mathbf{T}$ , and  $\mathbf{u}$ . This will be illustrated here by using the backward Euler method which presents the simplest possible algorithm. Applying higher order techniques as backward differentiation formulae, implicit Runge-Kutta methods, or extrapolation methods is the generalization of this idea (Gear and Petzold, 1984). However, the use of  $k$ -step, constant stepsize BDF method is restricted to the following values of  $k$ :  $1 \leq k \leq 6$  (Gear and Petzold, 1984; Löstedt and Petzold, 1986). Further restrictions on applying the higher order BDF techniques to the nonlinear index three problems are discussed by Brenan (1983). She showed that the two-step BDF method is extremely sensitive to errors in the initial and starting values (the starting values are necessary for methods with  $k \geq 2$ ). As it will be described later, in control applications there will always be errors in the initial and starting values for each sampling period, thus in such cases one cannot recommend the use of higher order BDF methods. As an alternate way of solving index three systems the use of two singly-implicit Runge-Kutta methods was examined. The analysis revealed that these methods are good tools for the DAE system solution, but their use is associated with much more computational effort as compared to BDF methods. From these remarks and considerations presented in the next sections, it follows that using the backward Euler method in the proposed control scheme seems to be an optimal solution.

Discretizing the set (6)–(11) by the use of the backward Euler method gives:

$$\frac{\mathbf{q}_{n+1} - \mathbf{q}_n}{t_{n+1} - t_n} = \mathbf{v}_{n+1}, \quad (14)$$

$$\frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{t_{n+1} - t_n} = \mathbf{a}_{n+1}, \quad (15)$$

$$\begin{aligned} \mathbf{A}(\mathbf{q}_{n+1}) \frac{\mathbf{a}_{n+1} - \mathbf{a}_n}{t_{n+1} - t_n} + \dot{\mathbf{A}}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}) \mathbf{a}_{n+1} + \dot{\mathbf{B}}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}, \mathbf{a}_{n+1}) \\ + \mathbf{nK}(\mathbf{v}_{n+1} - \mathbf{N}^{-1} \boldsymbol{\omega}_{n+1}) = \mathbf{0}, \end{aligned} \quad (16)$$

$$\begin{aligned} \mathbf{I}_r \frac{\boldsymbol{\omega}_{n+1} - \boldsymbol{\omega}_n}{t_{n+1} - t_n} + \mathbf{B}_\varphi \boldsymbol{\omega}_{n+1} + \mathbf{N}^{-1} \mathbf{n}^{-1} [\mathbf{A}(\mathbf{q}_{n+1}) \mathbf{a}_{n+1} \\ + \mathbf{B}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1})] = \mathbf{T}_{n+1}, \end{aligned} \quad (17)$$

$$\mathbf{L} \frac{\mathbf{T}_{n+1} - \mathbf{T}_n}{t_{n+1} - t_n} + \mathbf{RT}_{n+1} + \mathbf{C}_t \mathbf{C}_v \boldsymbol{\omega}_{n+1} = \mathbf{C}_t \mathbf{u}_{n+1}, \quad (18)$$

$$\mathbf{J}(\mathbf{q}_{n+1}) \frac{\mathbf{a}_{n+1} - \mathbf{a}_n}{t_{n+1} - t_n} + \mathbf{g}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}, \mathbf{a}_{n+1}, t_{n+1}) = \mathbf{0}. \quad (19)$$

Given the values of  $\mathbf{q}_n$ ,  $\mathbf{v}_n$ ,  $\mathbf{a}_n$ ,  $\boldsymbol{\omega}_n$ , and  $\mathbf{T}_n$  at time  $t_n$ , the unknown values of  $\mathbf{q}_{n+1}$ ,  $\mathbf{v}_{n+1}$ ,  $\mathbf{a}_{n+1}$ ,  $\boldsymbol{\omega}_{n+1}$ ,  $\mathbf{T}_{n+1}$ , and  $\mathbf{u}_{n+1}$  at time  $t_{n+1}$  can be obtained from Eqs. (14)–(19). The steps leading to this goal are:

- $\mathbf{q}_{n+1}$  and  $\mathbf{v}_{n+1}$  are eliminated from Eqs. (16), (17), and (19) using the formulas (14) and (15);
- the set of algebraic equations (16) and (19) is solved for  $\mathbf{a}_{n+1}$  and  $\boldsymbol{\omega}_{n+1}$ ;
- $\mathbf{T}_{n+1}$  is calculated from Eq. (17);
- $\mathbf{u}_{n+1}$  is calculated from Eq. (18).

In this way, given the system configuration at time  $t_n$ , the controls necessary for the next instant of time  $t_{n+1}$  can be found.

In the above described simplified inverse dynamics approach, the controls  $\mathbf{u}$  are determined as the solution of the set of nonlinear algebraic equations which is much simpler than in the full analytical inverse dynamics formulation. The solution of this set should not cause difficulties since at each step of calculation one disposes of proper initial approximations to the unknown algebraic variables, which are simply their values from the previous step. The organization of the control algorithm will be described in the following section.

The implementation of the procedure described by Eqs. (14)–(19) in the control process requires the measurement of the following quantities:  $\mathbf{q}$ ,  $\mathbf{v}$ ,  $\mathbf{a}$ ,  $\boldsymbol{\omega}$ , and  $\mathbf{T}$ . This can be alternatively replaced by the measurement of:  $\mathbf{q}$ ,  $\mathbf{v}$ ,  $\varphi$ ,  $\boldsymbol{\omega}$ , and  $\mathbf{T}$ .

## 5 Control Process Organization

Suppose the state of the robot is given at a fixed time  $t_n$  as a result of real-time measurement or simulation. During the time step from  $t_n$  to  $t_{n+1}$ , the controls necessary to approximately follow a prescribed trajectory are calculated as the solutions of the inverse dynamics problem described by the DAE set. The controls are then held constant over the next time step (from  $t_{n+1}$  to  $t_{n+2}$ ). It means that the discontinuous control history is considered.

Several implementation difficulties have been overcome when developing the algorithms suitable for control purposes. They are outlined below.

At time  $t_{n+1}$ , the state variables obtained from the simulation process or real-time measurement are not consistent with the DAE set solutions. However, they should be used as the initial values for the DAE set, in order to produce the controls necessary to drive the current robot motion parameters to their desired values. The problem of using incorrect initial values for DAE systems was considered by Sincovec et al. (1981) for the case of linear systems, and the conclusions were expanded by Brenan (1983) for some class of nonlinear systems. It was shown that, in spite of incorrect initial conditions, after  $(m-1)k+1$  steps (where  $m$  is the index of the DAE system), the  $k$ -step backward differentiation method yields a numerical solution consistent with some admissible initial conditions. In the case considered in this paper the backward Euler method ( $k=1$ ) is used to the index three problem ( $m=3$ ), so after three steps the numerical method will produce values accurate to  $O(h)$  for all variables (where  $h$  is the stepsize). Thus, in order to calculate the proper controls, it becomes necessary to divide the time step  $H = t_{n+1} - t_n$  into at least three smaller time steps  $h$ .

Here, it becomes clear why using the Euler method in the control process seems to be the best option. First, the starting values which should be determined specially for each time step  $H$  for methods with  $k \geq 2$ , are not needed in this case. Second, the number of steps required to get correct solutions of the DAE set increases linearly with  $k$ .

During the numerical experiments it has been noticed that the prediction of state variables by the DAE set solving from  $t_n$  to  $t_{n+1}$ , and subsequent determination of the control variables, do not ensure small tracking errors. Taking more than three time steps does not improve this situation. It appears to be necessary to solve the DAE set for time from  $t_n$  to  $t_{n+1} + H/2$ , i.e. to predict the controls in the middle of the next time step. Analytically it means that in the considered case one should use instead of the truncated control variables, the rounded ones. Since the solution runs for a time period equal to  $\frac{3}{2}H$ , the minimum size of the time step for DAE set solving is  $h = H/2$ . The organization of this control process is presented in Fig. 1. As it is seen, solving an index three DAE system in a control problem naturally leads to a one-and-a-half-step-ahead predictive control scheme.

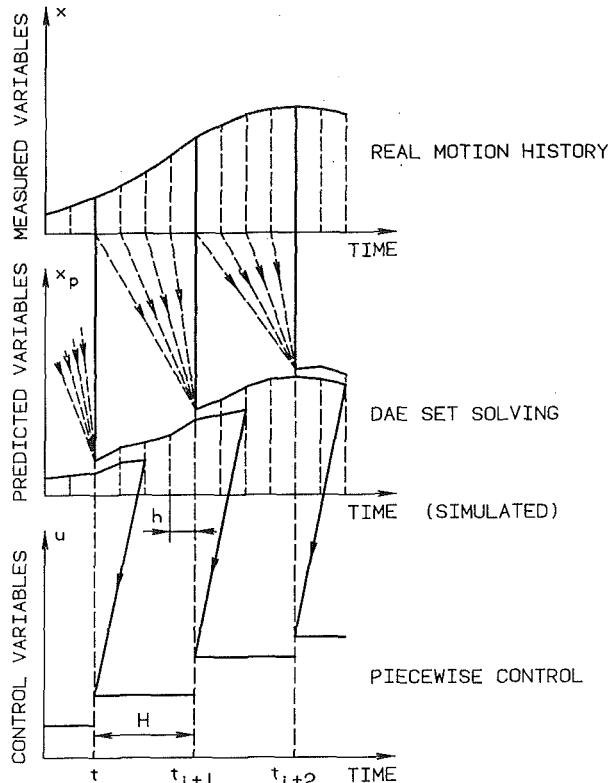


Fig. 1 Organization of the discrete-time control process

Interestingly, the feedback of measured values of torques  $T_n$  is not necessary in the control process. As it follows from the analysis of the control algorithm (14)–(19),  $T_n$  is needed only in the last step, when calculating the controls  $u_{n+1}$ . As the minimum number of steps equals three, the value of  $T_n$  results from the previous step calculation. Therefore, in the presented approach partial state feedback also works well.

## 6 Numerical Experiments

**6.1 Case Study.** Numerical methods for the solution of the control problem are tested on a particular example. The controlled motion of two-link planar robot is considered. However, the presented algorithms are quite easily applied to robots with more links.

For testing purposes, the robot is commanded to start from rest, and during the subsequent motion the end effector must move along a straight line trajectory to a new rest position. At the beginning ( $t = 0$ ) the end effector position is described by:  $X_{d0} = l_1$ ,  $Y_{d0} = -l_2$  (i.e.,  $q_1 = 0$ ,  $q_2 = -\pi/2$ ), and at the final time ( $t = t_f$ ):  $X_{df} = l_1$ ,  $Y_{df} = 0$ . Thus, the end effector trajectory is parallel to the  $O_1Y$  axis. The analytical expression for  $O_1X$  end effector position is:

$$X_d(t) = l_1, \quad (20)$$

and for  $O_1Y$  position, a Hermite polynomial of the ninth degree in  $t$  is used:

$$Y_d(t) = Y_{d0} + (Y_{df} - Y_{d0}) \left( \frac{70}{t_f^9} t^9 - \frac{315}{t_f^8} t^8 \right. \\ \left. + \frac{540}{t_f^7} t^7 - \frac{420}{t_f^6} t^6 + \frac{126}{t_f^5} t^5 \right), \quad (21)$$

to avoid unnecessary jumps in the control variables.

**6.2 Simulation Results.** Numerical experiments performed to examine the discrete-time control scheme have been

conceived to imitate faithfully a real process of digital robot control. Therefore, the piecewise-constant nature of the control inputs is considered and the sampling delays are taken into account.

Some results of the numerical simulation of the robot motion with piecewise control are presented below. The sampling frequency is chosen equal to 100 Hz ( $H = 0.01$  s), when the natural frequencies of the robotic system calculated along the desired trajectory are approximately equal to 11.1 Hz and 22.5 Hz. However, it should be noted that the satisfactory performance has also been reached for systems with other frequency characteristics and using different sampling frequencies, as reported by Jankowski and Van Brussel (1990, 1991). The coefficient  $\alpha$  in the feedback gain matrices equals 15.0. The maximum end effector position error which occurs during trajectory following equals 0.093 mm. This error can be decreased not only by the use of a higher sampling frequency, but also by using smaller time steps  $h$  when solving the DAE set. For example, if  $h = H/4$  this error equals 0.043 mm; and for  $h = H/6$ ,  $\epsilon_{pmax} = 0.028$  mm. However, decreasing the time step  $h$  means increasing the number of steps required for the prediction of control variables: for  $h = H/4$  six steps are used, and for  $h = H/6$ : nine. It causes an augmentation of the amount of computer time needed to determine the controls. Please note that, in each step, a set of nonlinear algebraic equations is solved by means of the Newton-Raphson iteration process. Fortunately, as the proper initial approximations are provided, the number of iterations necessary to get the satisfactory solutions does not exceed 3. In this context it seems that the reasonable partition of the sampling time is with  $h = H/4$ .

## 7 Conclusions

Due to problems with the continuous-time approach, the use of a discrete-time controller has been proposed to solve the inverse dynamics problem for flexible joint robots. To design the control algorithm, a numerical approach has been suggested, based on numerical methods conceived for solution of index three DAE systems. Among these methods, the backward Euler method has been chosen as its use gives several advantages, as discussed in the paper. Taking a model of a planar two-link robot as an example, extensive computer simulations have been carried out on the designed controller in order to find the best control algorithm. As yet, no corresponding theoretical analysis has been performed on such a nonlinear discrete-time control method, although the results of numerical experiments do indicate such work might be beneficial. Following these results, the proposed control scheme has been judged as assuring good tracking and stability properties. Among the advantages of the proposed discrete-time control method are:

- multiple analytical differentiation of system dynamic model is avoided, which results in the simplification of necessary on-line calculations and makes the method computationally efficient;
- it is not necessary to feedback higher order time derivatives of system variables, and it is possible to use a partial state feedback;
- using lower sampling frequencies than in the classical inverse dynamics approach is allowed.

The results of recently performed experiments on computer control of a two-link manipulator with one flexible joint (Jankowski and Van Brussel, 1990) confirm the effectiveness of the discrete-time inverse dynamics approach presented in this paper.

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