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Revisit of Linear-Quadratic Optimal Control

M. Pachter

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Abstract The classical finite-dimensional linear-quadratic optimal control problem is revisited. A new linear-quadratic control problem with linear state penalty terms but without quadratic state penalty terms, is introduced. An optimal control exists and the closed-form optimal solution is given. It is remarkable that feedback action plays no role and state information does not feature in the optimal control. The optimal cost function, rather than being quadratic, is linear in the initial state.

Keywords Linear quadratic control · Feedback control

1 Introduction

The classical finite-dimensional linear-quadratic (LQ) optimal control problem with free end-point is revisited. The finite-dimensional LQ optimal control problem in R^n has been studied extensively. It was elegantly solved by Kalman in 1960 [1] and a special issue of the IEEE Transactions on Automatic Control, dedicated to linear-quadratic control, was published in December 1971 [2]. Linear-quadratic optimal control features prominently in and is a staple of many textbooks on optimal control [see e.g. [3–8]]. Furthermore, monographs exclusively devoted to LQ optimal control have appeared [see e.g. [9–18]] and the solution of the matrix Riccati equation which features prominently in LQ control is addressed specifically in the monographs [19–22]. Therefore, it comes as somewhat of a surprise that one might come up with a new version of this old chestnut: in this paper, a more general cost

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M. Pachter (✉)
Department of Electrical and Computer Engineering, Air Force Institute of Technology, AFIT/ENG, 2950 Hobson Way, Bldg. 640, Wright-Patterson Air Force Base, OH 45433, USA
e-mail: Meir.Pachter@afit.edu

functional including linear-in-the state terms is considered. Thus, the cost functional is

$$J(u; x_0) = x^T(T)Hx(T) + 2\alpha h^T x(T) + \int_0^T (x^T Qx + 2\beta q^T x + u^T Ru)dt, \quad (1)$$

where h and q are unit vectors in R^n and the weights $\alpha \geq 0$, $\beta \geq 0$. The dynamics is

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \quad 0 \leq t \leq T. \quad (2)$$

However, our interest in generalization for the sake of generalization is not what motivates us to consider the LQ optimal control problem (1) and (2). It is rather the special case of the cost functional (1) where the quadratic state penalty terms are removed, viz., the real symmetric positive semidefinite matrices $H = 0$ and $Q = 0$, that we find particularly interesting.

In this paper, the LQ optimal control problem with the linear dynamics (2) is considered. We investigate the control energy—constrained optimal control problem where one is interested in steering the initial state x_0 into the interior of the halfspace

$$C_q = \{x \mid q^T x \leq 0\}$$

and/or

$$C_h = \{x \mid h^T x \leq 0\}.$$

Thus, the quadratic cost functional

$$J(u; x_0) = 2\alpha h^T x(T) + \int_0^T (2\beta q^T x + u^T Ru)dt \quad (3)$$

is employed. In particular, we can choose the parameters $\alpha = 0$, or $\beta = 0$, or we can set the vectors which determine the halfspace, $h = q$.

The novel LQ optimal control problem (2) and (3) is interesting. We show that a unique solution exists, yet feedback action plays no role here and the initial state information does not feature in the optimal control. Remarkably, the above holds true even in the presence of an unknown disturbance. Furthermore, rather than being quadratic, the optimal cost/value function

$$V(x_0) \equiv \min_u J(u; x_0),$$

is linear in the initial state.

The paper is organized as follows. The solution of the LQ optimal control problem (2) and (3) is given in Sect. 2. In Sect. 3, the optimal control problem of steering the initial state to a specified halfspace in R^n, say C_h, is considered. In Sect. 4, the linear-quadratic differential game (LQDG) of steering/countersteering the initial state x_0 into the half-space C_h is solved. Surprisingly, also in the game formulation, feedback action is not used. We conclude that the lack thereof is attributable exclusively to the absence in the cost functional of quadratic-in-the state penalty terms. For the sake of completeness, the solution of the general LQ optimal control problem (1) and (2) is given in Sect. 5. Concluding remarks are made in Sect. 6.

2 LQ Optimal Control

The LQ optimal control problem (2) and (3) is considered; we assume momentarily that the parameters $\alpha = \beta = 1$. The method of dynamic programming (DP) is applied. The DP value function satisfies the equation

$$V(t,x) = \min_u \left[(2q^T x + u^T Ru)dt + V(t,x) + \frac{\partial V}{\partial t} dt + \left(\frac{\partial V}{\partial x}\right)^T (Ax + Bu)dt \right],$$

so that

$$-\frac{\partial V}{\partial t} = 2q^T x + \left(\frac{\partial V}{\partial x}\right)^T Ax + \min_u \left[u^T Ru + \left(\frac{\partial V}{\partial x}\right)^T Bu \right]. \tag{4}$$

The value function is of the form

$$V(t,x) = x^T P(t)x + 2x^T p(t) + c(t), \quad 0 \le t \le T, \tag{5}$$

where P is a real symmetric $n \times n$ matrix, $p \in R^n$ and c is a scalar.

Inserting (5) into (4) yields

$$\begin{aligned} -x^T \dot{P} x - 2x^T \dot{p} - \dot{c} &= 2x^T q + 2x^T PAx + 2p^T Ax \\ &\quad + \min_u [u^T Ru + 2(x^T P + p^T)Bu], \end{aligned} \tag{6}$$

wherefrom we obtain the optimal control

$$u^* = -R^{-1} B^T (Px + p); \tag{7}$$

inserting (7) into (6) gives

$$\begin{aligned} -x^T \dot{P} x - 2x^T \dot{p} - \dot{c} &= 2x^T q + x^T PAx + x^T A^T Px + 2p^T Ax \\ &\quad - (x^T P + p^T) BR^{-1} B^T (Px + p) \\ &= x^T (A^T P + PA - PBR^{-1} B^T P)x + 2x^T q \\ &\quad + 2x^T A^T p - 2p^T BR^{-1} B^T Px \\ &\quad - p^T BR^{-1} B^T p. \end{aligned} \tag{8}$$

In view of (8), set

$$\begin{aligned} -\dot{P} &= A^T P + PA - PBR^{-1} B^T P, \qquad P(T) = 0, \\ -\dot{p} &= (A - BR^{-1} B^T P)^T p + q, \qquad p(T) = h, \\ \dot{c} &= p^T BR^{-1} B^T p, \qquad c(0) = 0, \quad 0 \le t \le T. \end{aligned}$$

Next, set

$$\begin{aligned} P(t) &:= P(T - t), \\ p(t) &:= p(T - t), \end{aligned}$$

and now

$$\dot{P} = A^T P + PA - PBR^{-1}B^T P, \qquad P(0) = 0, \tag{9}$$

$$\dot{p} = (A - BR^{-1}B^T P)^T p + q, \qquad p(0) = h, \quad 0 \le t \le T, \tag{10}$$

$$c(t) = -\int_t^T p^T(T-\sigma)BR^{-1}B^T p(T-\sigma)d\sigma. \tag{11}$$

We conclude immediately that

$$P(t) = 0, \quad \forall\, 0 \le t \le T. \tag{12}$$

Therefore the vector $p(t)$ is the solution of the $linear$ differential equation

$$\dot{p} = A^T p + q, \qquad p(0) = h, \quad 0 \le t \le T. \tag{13}$$

The solution of the linear differential equation (13) is

$$p(t) = \alpha e^{A^T t} h + \beta e^{A^T t} \cdot \left(\int_0^t e^{-A^T t} dt \right) q, \quad 0 \le t \le T. \tag{14}$$

If $h = q$, the solution is

$$p(t) = e^{A^T t} \left[I + \int_0^t e^{-A^T t} dt \right] q, \quad 0 \le t \le T;$$

if the dynamics matrix A is invertible, the explicit solution is

$$p(t) = (A^T)^{-1}[e^{A^T t}(A^T h + q) - q], \quad 0 \le t \le T.$$

We have obtained the following result.

Theorem 2.1 *Consider the LQ optimal control problem* (2) *and* (3). *The optimal control strategy is*

$$u^*(t,x) = -R^{-1}B^T p(T-t),$$

that is,

$$u^*(t) = -R^{-1}B^T p(T-t), \tag{15}$$

where $p(\cdot)$ *is given by* (14). *The value function is*

$$V(t,x) = 2p^T(T-t)x + c(t), \tag{16}$$

where $c(t)$ *is given by* (11).

The optimal control problem (2) and (3) is not trivial and yet, the optimal control (15) does *not* depend on x. There is *no* feedback action and the controller operates open-loop; moreover, the optimal control is not even dependent on the initial state x_0. All the controller needs is a clock. We see also that the value function (16) is *linear* in the state x.

2.1 Alternative Approach

That this is indeed the case is also corroborated by an analysis of the LQ optimal control problem (2) and (3) using Pontryagin's maximum principle. The absence in the cost functional (3) of the quadratic term $x^T Qx$ causes the costate differential equation to be *decoupled* from the state differential equation (2). Indeed, the costate $\lambda \in R^n$ satisfies the differential equation

$$\dot{\lambda} = -A^T \lambda + 2\beta q$$

and the absence in the cost functional (3) of the terminal quadratic term $x^T(T)Hx(T)$ renders the costate boundary value

$$\lambda(T) = 2\alpha h$$

independent of the terminal state $x(T)$. Hence, the solution of the costate's linear differential equation does not require the initial state x_0 information; the costate vector $\lambda(t)$, $0 \le t \le T$, is not dependent on the initial state information. Since in LQ control the optimal control is exclusively determined by the costate, we conclude that the initial state x_0 plays no role in determining the open-loop optimal control and $u^*(t; x_0) = u^*(t)$. This signals the conspicuous absence of feedback action—a "synthesis" of a feedback controller is not possible and is not needed.

3 Minimum Energy Control

The special case where there is only a terminal cost, namely $\beta = 0$, is of interest. It follows from Theorem 2.1 that

$$p(t) = e^{A^T t} h, \quad 0 \le t \le T,$$

so the optimal control

$$u^*(t) = -\alpha R^{-1} B^T e^{A^T(T-t)} h \tag{17}$$

and we calculate that

$$c(t) = -h^T \left(\int_0^{T-t} e^{A\sigma} B R^{-1} B^T e^{A^T \sigma} d\sigma \right) h.$$

Now, the controllability Grammian

$$W(t) \equiv \int_0^t e^{A\sigma} B R^{-1} B^T e^{A^T \sigma} d\sigma \tag{18}$$

is a real symmetric positive-definite matrix $\forall 0 < t \le T$, provided the pair (A, B) is controllable. $W(t)$ is the solution of the linear Lyapunov differential equation

$$\dot{W} = AW + WA^T + BR^{-1}B^T, \qquad W(0) = 0, \quad 0 \le t \le T.$$

Thus,

$$c(t) = -h^T W(T-t)h, \quad 0 \le t \le T,$$

and

$$c(0) = -h^T W(T)h,$$

so that the value function is

$$V(0, x_0) = 2\alpha h^T e^{AT} x_0 - \alpha^2 h^T W(T)h. \tag{19}$$

Hence, we have shown that the following holds.

Theorem 3.1 *Consider the LQ optimal control problem* (2), (3) *and assume the parameter* $\beta = 0$. *Then, the optimal control is given explicitly by* (17), *the expanded control energy is*

$$E^* = \alpha^2 h^T W(T)h,$$

and the value function is given by (19) *and* (18).

The following corollary concerns the value function dependence on the parameter α.

Corollary 3.1 *Consider the LQ optimal control problem* (2), (3) *and assume the parameter* $\beta = 0$. *The value function*

$$V(x_0; \alpha) \le 0, \quad \forall \alpha > \max\left(0, 2\frac{h^T e^{AT} x_0}{h^T W(T)h}\right)$$

and

$$V(x_0; \alpha) \le \frac{(h^T e^{AT} x_0)^2}{h^T W(T)h}, \quad \forall \alpha > 0.$$

Proof This follows from inspection of (19). □

Furthermore, we have the following corollary.

Corollary 3.2 *The goal of steering the initial state* x_0 *into the interior of the half-space* C_h *using an LQ optimal control problem formulation is achieved* iff *the weight*

$$\alpha > \frac{h^T e^{AT} x_0}{h^T W(T)h}.$$

Proof For our goal to be achieved, we need

$$V(0, x_0) - E^* < 0.$$

Using Theorem 3.1, we calculate that

$$V(0, x_0) - E^* = 2\alpha[h^T e^{AT} x_0 - \alpha h^T W(T)h],$$

and for the goal to be achieved, the weight α must be sufficiently high and satisfy the inequality above. □

The case where A is a stability matrix is interesting. When $T \to \infty$, a solution of the optimal control problem exists.

The optimal control (see e.g. (17)) is

$$u^*(t) = 0, \quad \forall 0 \le t,$$

and the value function (see e.g. (19)) is

$$V(0, x_0) = -h^T W h,$$

where W is the real symmetric positive-definite solution of the Lyapunov equation

$$AW + WA^T = -BR^{-1}B^T, \tag{20}$$

provided that the pair (A, B) is controllable. Thus, the following corollary holds.

Corollary 3.3 *Consider the LQ optimal control problem* (2) *and* (3) *with a terminal state penalty only*, *i.e.*, $\beta = 0$, *asymptotically stable dynamics*, *and an infinite planning horizon. Then*, *the optimal control* $u^*(t) \equiv 0$. *Let* W *be the solution of the Lyapunov equation* (20); *the value function* $V(0, x_0) = -\alpha^2 h^T W h$ *is negative*, *constant and the optimal cost does not depend on the initial state* x_0.

3.1 Alternative Formulation

The problem of steering the state into the interior of the half-space C_h can also be addressed by posing and solving the following LQ optimal control problem:

$$\min_u h^T x(T), \tag{21}$$

subject to the dynamics (2) and the control energy constraint

$$\int_0^T u^T R^{-1} u \, dt \le E, \tag{22}$$

where the specified control energy bound $E > 0$.

We shall require the following lemma [8].

Lemma 3.1 *The solution of the minimum energy optimal control problem*

$$J^*(x_0, x_f) = \min_u \left(\int_0^T u^T R^{-1} u \, dt \right) \tag{23}$$

s.t. the dynamics (2) *and s.t. the terminal state*

$$x(T) = x_f \tag{24}$$

is

$$u^*(t) = R^{-1} B^T e^{A^T(T-t)} W^{-1}(T)(x_f - e^{AT} x_0)$$

and the minimal cost is

$$J^*(x_0, x_f) = (x_f - e^{AT} x_0)^T W^{-1}(T)(x_f - e^{AT} x_0). \tag{25}$$

The application of Lemma 3.1 allows us to transform the LQ optimal control problem (2), (21), (22) into the following static LQ optimization problem in R^n:

$$\min_{x \in R^n} \quad h^T x, \tag{26}$$

$$\text{s.t.} \quad (x - e^{AT} x_0)^T W^{-1} (x - e^{AT} x_0) \le E. \tag{27}$$

We introduce the Lagrange multiplier $\lambda > 0$ and solve the unconstrained quadratic program

$$\min_{x \in R^n} [h^T x + \lambda (x - e^{AT} x_0)^T W^{-1} (x - e^{AT} x_0)],$$

which yields

$$x^* = e^{AT} x_0 - \frac{1}{2\lambda} W h; \tag{28}$$

thus, the optimal control is

$$u^*(t) = -\frac{1}{2\lambda} R^{-1} B^T e^{A^T(T-t)} h. \tag{29}$$

We use the expression (29) to calculate the expanded control energy, which we equate to E. Thus,

$$E = \frac{1}{4\lambda^2} h^T W h,$$

which allows us to calculate λ. Hence, the optimal control is

$$u^*(t) = -\frac{\sqrt{E}}{\sqrt{h^T W(T) h}} R^{-1} B^T e^{A^T(T-t)} h, \tag{30}$$

the terminal state is

$$x^* = e^{AT} x_0 - \frac{\sqrt{E}}{\sqrt{h^T W h}} W h,$$

and the value function is

$$V(x_0) = h^T e^{AT} x_0 - \sqrt{E}\sqrt{h^T W h}. \tag{31}$$

In summary, we have the following theorem.

Theorem 3.2 *Consider the LQ optimal control problem* (2), (21), (22), *which is parametrized by the control energy bound* $E > 0$. *The optimal control is given by* (30) *and the value function is given by* (31). *The value function is linear in the initial state and is monotonically decreasing in the parameter* E. *Evidently, for* E *sufficiently large,*

$$E > \frac{(h^T e^{AT} x_0)^2}{h^T W h},$$

the value function is negative, that is, the end state $x(T)$ *is in the interior of the halfspace* C_h.

Remark 3.1 Choosing the control energy bound

$$E > \frac{(h^T e^{AT} x_0 - b)^2}{h^T W h}$$

guarantees that the end state satisfies

$$h^T x(T) < b.$$

Theorems 3.1 and 3.2 address the same control problem using different methods and therefore it is interesting to compare the results. In Theorem 3.1, the parameter is the weight α and in Theorem 3.2 the parameter is the control energy bound E.

Comparing (17) and (30), we realize that, for the results of Theorems 3.1 and 3.2 to be equivalent, the following must hold:

$$\alpha = \frac{\sqrt{E}}{\sqrt{h^T W(T) h}}.$$

Now, according to Theorem 3.1, the expanded control energy during optimal play is

$$E^* = \alpha^2 h^T W(T) h \tag{32}$$

and so it transpires that setting the parameters

$$\alpha = 1 \tag{33}$$

and

$$E = h^T W(T) h \tag{34}$$

establishes the equivalence. Concerning the value functions, it is important to realize that the value function in Theorem 3.1, (19), is not the same value function as in

Theorem 3.2, (31), because the cost functionals are not the same: the value function in Theorem 3.1 is $2\alpha h^T x^* + E^*$, whereas the value function in Theorem 3.2 is $h^T x^*$. Taking this into account, the following must hold:

$$2(h^T e^{AT} x_0 - \sqrt{E}\sqrt{h^T W h}) + E = 2h^T e^{AT} x_0 - \alpha^2 h^T W h,$$

which is indeed the case, provided that (33) and (34) hold. Hence, we have established the following theorem.

Theorem 3.3 *The solution of the LQ optimal control problem* (2) *and* (3), *with the parameter $\beta = 0$ provided by Theorem* 3.1, *and the solution of the LQ optimal control problem* (2), (21), (22) *given by Theorem* 3.2 *are equivalent provided that the respective problem parameters are*

$$\alpha = 1$$

and

$$E = h^T W h.$$

4 LQ Differential Game

We have seen that, in LQ optimal control, when the state penalty in the cost functional is linear in the state, the optimal control law does not entail feedback action; moreover, the initial state information plays no role in determining the optimal control signal. This is somewhat bothersome and we would like to investigate the reason for this state of affairs.

At first blush, the lack of feedback action could be rationalized: in deterministic control, when no *unknown* disturbances are present, feedback action is not required. Even if a feedback control law could be synthesized, the use of feedback control would not yield an advantage and the performance functional would assume the same value as when open-loop optimal control is used. Indeed, it is somewhat serendipitous that feedback action is so easily obtained in classical LQ optimal control.

Hence, we are motivated to include an unknown disturbance input. The natural setting is the differential game paradigm. Thus, consider the zero-sum LQ differential game (LQDG) with linear dynamics

$$\dot{x} = Ax + Bu + Cv, \qquad x(0) = x_0, \quad 0 \le t \le T. \tag{35}$$

The minimizer control signal is u and the maximizer control signal is v. The quadratic cost functional is

$$J(u, v; x_0) = 2h^T x(T) + \int_0^T (2q^T x + u^T R_u u - v^T R_v v)dt \tag{36}$$

and, as in Sects. 2 and 3, it does not contain a quadratic state penalty: in the cost functional (36), the state penalty is linear in the state. In this setting, the value function is

$$V(x_0) = \min_u \max_v J(u, v; x_0).$$

Similar to the solution of the LQ optimal control problem in Sects. 2 and 3, the structure of the differential game value function is

$$V(t, x) = x^T P(T - t)x + 2x^T p(T - t) + c(t)$$

and the players' optimal strategies are

$$u^*(t, x) = -R_u^{-1} B^T [P(T - t)x + p(T - t)],$$
$$v^*(t, x) = R_v^{-1} C^T [P(T - t)x + p(T - t)],$$

where the matrix P and the vector p satisfy the differential equations

$$\dot{P} = A^T P + PA - P(BR_u^{-1}B^T - CR_v^{-1}C^T)P, \qquad P(0) = 0, \tag{37}$$
$$\dot{p} = [A - (BR_u^{-1}B^T - CR_v^{-1}C^T)P]^T p + q, \qquad p(0) = h, \quad 0 \le t \le T, \tag{38}$$

and the scalar

$$c(t) = -\int_t^T p^T(T - \sigma)(BR_u^{-1}B^T - CR_v^{-1}C^T)p(T - \sigma)d\sigma. \tag{39}$$

From (37), we conclude immediately that the matrix $P(t) \equiv 0$, which implies that state feedback is *not* used in the players controls. The value function of the differential game (35) and (36) is linear in the state and is

$$V(t, x) = 2x^T p(T - t) + c(t)$$

and the players strategies are the *open-loop* controls

$$u^*(t, x) = -R_u^{-1} B^T p(T - t),$$
$$v^*(t, x) = R_v^{-1} C^T p(T - t).$$

Similar to the optimal control case, also in the differential game paradigm, state feedback is not used. One would think that such a minimizer saddle point strategy could not possibly be optimal: if the maximizing player deviates from his optimal strategy and $v \neq v^*$, an optimal strategy of the minimizing player should allow him to take advantage immediately of the maximizer mistake. This however is not possible in the absence of feedback action and where the control is decided on in advance. Hence, the question is posed of how could the minimizer open-loop strategy be optimal. The answer is as follows. If the maximizing player deviates from optimal play sometime before time t, $0 < t < T$, then at time t the state $x(t)$ is different from the state reached under optimal play $u^*(t)$, $v^*(t)$. But in the differential game with $Q = 0$, state information is not needed, the players make do with a clock, and they do have the time. Hence, the minimizing player control at time t, $u^*(t)$, is no different than his control would be if the maximizer played optimally. The same considerations apply to the maximizing player strategy. Thus, in the LQDG with $Q=0$, the players' strategies are fixed open-loop controls and they form a saddle point.

The fact that, in a differential game setting, the state information plays no role whatsoever in both players strategies comes as somewhat of a surprise. Moreover, the vector function $p(t)$ is the same in the optimal control formulation as in the differential game formulation; therefore, the minimizing player optimal control $u^*(t)$ is exactly the same as in the one-sided optimal control problem, as if the maximizing player would be absent. This however does make sense, for otherwise the presence of the maximizing player would require the minimizing player to use feedback action. Note however that the presence of the maximizing player is felt in the value function because the function $c(t)$ in the LQDG is not the same as in the LQ optimal control problem.

In conclusion, we have shown that the absence of feedback action in optimal LQ control is caused exclusively by the absence of a quadratic state penalty in the cost functional. When $Q \neq 0$ and feedback action is used in the differential game, the gain of the minimizing player is reduced compared to the gain in the optimal control problem. Hence, the theory is consistent.

5 General Case

In this section, for the sake of completeness, more general LQ optimal control problems are addressed. First, the LQ optimal control problem with the linear dynamics (2) and the quadratic cost functional

$$J(u;x_0)=x^T(T)Hx(T)+2\alpha h^T x(T)+\int_0^T (2\beta q^T x+u^T Ru)\,dt, \tag{40}$$

where H is a real symmetric positive-definite matrix, is considered. The solution requires the integration of the differential equations (9) and (10) and the evaluation of the integral (11); however, in the differential equation (9), the initial condition

$$P(0)=H$$

is now used.

Since the matrix H is invertible, set

$$K \equiv P^{-1}$$

and replace the nonlinear Riccati differential equation by the *linear* Lyapunov differential equation in K,

$$\dot{K}=-AK-KA^T+BR^{-1}B^T, \qquad K(0)=H^{-1}, \quad 0\le t\le T. \tag{41}$$

The closed-form solution of the linear Lyapunov equation (41) is

$$K(t)=e^{-At}H^{-1}e^{-A^T t}+\int_0^t e^{-A(t-\sigma)}BR^{-1}B^T e^{-A^T(t-\sigma)}d\sigma. \tag{42}$$

Finally,

$$P(t) = K^{-1}(t), \quad 0 \le t \le T.$$

The point is not so much the availability of a closed-form solution, but rather the fact that, in the process of solving the LQ optimal control problem (2) and (40), and where $Q = 0$, one gets away with *linear* analysis and avoids the solution of a Riccati equation.

The more general LQ optimal control problem (1) and (2) is also discussed. This problem, with linear and quadratic state penalty terms, arises when the optimal tracking of a known signal is considered. The solution of this LQ optimal control problem entails the integration of the standard Riccati differential equation of conventional LQ optimal control

$$\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q, \quad P(0) = H, \tag{43}$$

and in addition the integration of the differential equation (10) and the calculation of the integral (11). The point here is that the Riccati equation is decoupled from (10) and (11). In addition, (10) is decoupled from (11). The offshoot of this triangular structure is that one first integrates the Riccati equation (43), as in conventional LQ optimal control. The solution $P(t)$ of the Riccati equation is needed in order to integrate the linear differential equation (10) and obtain its solution $p(t)$, $0 \le t \le T$, and finally the solution $p(t)$ of (10) features in the integrand in (11).

Now $P(t) \ne 0$ and the optimal control law entails feedback action, that is,

$$u^*(t, x) = -R^{-1}B^T[P(T - t)x + p(T - t)]$$

and the value function is quadratic in x, that is,

$$V(t, x) = x^T P(T - t)x + 2p(T - t)x + c(t).$$

The presence in the cost functional of linear-in-the state penalty terms causes the solution of the general LQ optimal control problem to require the integration of the additional differential equation (10) and the evaluation of the integral (11); this is in addition to the integration of the standard Riccati differential equation.

6 Conclusions

The LQ optimal control problem with a free endpoint is revisited. It is shown that, when quadratic state penalty terms are absent, an optimal solution exists; however, the optimal control is open loop and no state feedback action is employed. Furthermore, the value function is linear in the state. Moreover, the presence of an unknown disturbance, as in a differential game formulation, does not change the above conclusion: the value function is linear in the state and no sate feedback action is needed iff the state penalty term is linear in the state. The herein considered LQ problem formulation with linear-in-the state penalty terms is conducive to the solution of the minimum-energy optimal control problem where the end-state is to be steered into the interior of a specified half-space. An extension to the infinite-dimensional case is possible.

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