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Relative controllability of fractional dynamical systems with multiple delays in control[☆]

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ABSTRACT

This paper is concerned with the global relative controllability of fractional dynamical systems with multiple delays in control for finite dimensional spaces. Sufficient conditions for controllability results are obtained using Schauder's fixed point theorem and the controllability Grammian matrix which is defined by the Mittag-Leffler matrix function. An example is provided to illustrate the theory.

1. Introduction

For the past three centuries, fractional calculus has been dealt with by the mathematicians, and only in the last few years was this pulled to several (applied) fields of engineering, science and economics. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. Perhaps the reason of interest is that the numerical value of the fraction parameter allows a closer characterization of eventual uncertainties present in the dynamic model. Fractional differentials and integrals provide more accurate models of systems under consideration. Many authors have demonstrated applications of fractional calculus in the frequency dependent damping behavior of many visco-elastic materials [1,2], dynamics of interfaces between nanoparticles and substrates [3], the nonlinear oscillation of earthquakes [4], bioengineering [5], continuum and statistical mechanics [6], signal processing [7], filter design, circuit theory [8] and robotics. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested in [9] and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena [10]. Some recent contributions to the theory of fractional differential equations can be seen in [11–13].

On the other hand, there is also an increasing interest in the recent issue related to dynamical fractional systems oriented towards the field of control theory concerning heat transfer, lossless transmission lines [14,8], the use of discretizing devices supported by fractional calculus. Controllability results for linear fractional differential equations have been considered by a few authors (see, for instance [15–18]). Balachandran and Dauer [19] discussed the controllability of nonlinear dynamical systems via fixed point approach. In the literature, controllability results for integer order nonlinear dynamical systems with several types of delays in control have been addressed by many monographs. Dauer and Gahl [20] studied the controllability for delay systems while in [21,22], the authors obtained the controllability of nonlinear systems with time varying multiple delays in control and implicit derivative by suitably adopting the technique of [23] and Darbo's fixed point theorem. Further, Balachandran and Somasundaram [24] derived the sufficient conditions for the global relative controllability of a nonlinear

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system consisting of a bilinear mode with time-varying multiple delays in control by using Schauder's fixed point theorem and Klamka [25,26] investigated the controllability of nonlinear systems with different types of delays in control variables.

It should be mentioned that the theory of controllability for nonlinear fractional dynamical systems is still in the development process and is a less satisfactory solution. Motivated by this fact, this paper deals with the global relative controllability of fractional dynamical systems with multiple delays in control variables. Sufficient conditions for the controllability results are obtained by using the Schauder fixed point theorem and fractional calculus. The paper is organised as follows: In Section 2, some well known fractional operators and special functions, along with a set of properties are defined and the solution representation of linear fractional differential equations are derived using Laplace transform for further discussion. In Section 3, the linear fractional dynamical system with multiple delays in control is proposed and the controllability condition is established using the controllability Grammian matrix which is defined by means of the Mittag-Leffler matrix function. In Section 4, the corresponding nonlinear fractional dynamical system is considered and the controllability results are examined with the natural assumption that the linear fractional system is relatively controllable. The results are obtained by using the Schauder fixed point theorem and the fractional calculus. Finally, Section 5 ends up with an example to illustrate the theory.

2. Preliminaries

Let $\alpha, \beta > 0$, with $n-1 < \alpha < n$, $n-1 < \beta < n$, and $n \in \mathbb{N}$, D is the usual differential operator. Let $\mathbb{R}^m$ be the m-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$, and suppose $f \in L_1(\mathbb{R}_+)$. The following definitions and properties are well known, for $\alpha, \ \beta > 0$ and f as a suitable function (see, for instance, [27,28]):

(a) Riemann–Liouville fractional operators:

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt,$$

$$(D_{0+}^{\alpha} f)(x) = D^n (I_{a+}^{n-\alpha} f)(x).$$

(b) Caputo fractional derivative:

$$({}^C D_{0+}^{\alpha} f)(x) = (I_{0+}^{n-\alpha} D^n f)(x),$$

in particular $I_{0+}^{\alpha}\ {}^C D_{0+}^{\alpha} f(t) = f(t) - f(0)$ $(0 < \alpha < 1)$.

The following is a well known relation, for finite interval $[a, b] \in \mathbb{R}_+$

$$(D_{a+}^{\alpha} f)(x) = ({}^C D_{a+}^{\alpha} f)(x) + \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(1+j-\alpha)} (x-a)^{j-\alpha}, \quad n = \Re(\alpha) + 1.$$

The Laplace transform of the Caputo fractional derivative is

$$\mathcal{L}\{{}^C D_{0+}^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}.$$

The Riemann–Liouville fractional derivatives have singularity at zero and the fractional differential equations in the Riemann–Liouville sense require initial conditions of special form lacking physical interpretation. To overcome this difficulty Caputo [29] introduced a new definition of fractional derivative but in general, both the Riemann–Liouville and the Caputo fractional operators possess neither semigroup nor commutative properties, which are inherent to the derivatives on integer order. Due to this fact, the concept of sequential fractional differential equations are discussed in [27,30].

(c) Linear Sequential Derivative:

For $n \in \mathbb{N}$, the sequential fractional derivative for suitable function $y(x)$ is defined by

$$y^{(k\alpha)} := (\mathbf{D}^{k\alpha} y)(x) = (\mathbf{D}^{\alpha} \mathbf{D}^{(k-1)\alpha} y)(x),$$

where $k = 1, \ldots, n$, $(\mathbf{D}^{\alpha} y)(x) = y(x)$, and $\mathbf{D}^{\alpha}$ is any fractional differential operator, here we mention it as ${}^C D_{0+}^{\alpha}$.

(d) Mittag-Leffler Function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \quad \text{for } \alpha, \beta > 0.$$

The general Mittag-Leffler function satisfies

$$\int_0^{\infty} e^{-t} t^{\beta-1} E_{\alpha,\beta}(t^{\alpha} z) dt = \frac{1}{1-z}, \quad \text{for } |z| < 1.$$

The Laplace transform of $E_{\alpha,\beta}(z)$ follows from the integral

$$\int_0^\infty e^{-st}\, t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha) dt = \frac{s^{\alpha-\beta}}{(s^\alpha \mp a)}.$$

That is

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\pm a t^\alpha)\} = \frac{s^{\alpha-\beta}}{(s^\alpha \mp a)},$$

for $\Re(s) > |a|^{\frac{1}{\alpha}}$ and $\Re(\beta) > 0$. In particular, for $\beta = 1$,

$$E_{\alpha,1}(\lambda z^\alpha) = E_\alpha(\lambda z^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k + 1)}, \quad \lambda, z \in \mathbb{C}$$

have the interesting property ${}^C D^\alpha E_\alpha(\lambda t^\alpha) = \lambda E_\alpha(\lambda t^\alpha)$ and

$$\mathcal{L}\{E_\alpha(\pm a t^\alpha)\} = \frac{s^{\alpha-1}}{(s^\alpha \mp a)}, \quad \text{for } \beta = 1.$$

For brevity of notation let us take I^q_{0+} as I^q and ${}^C D^q_{0+}$ as ${}^C D^q$ and the fractional derivative is taken as Caputo sense.

(d) Solution Representation:

Consider the linear fractional differential equation of the form

$$\begin{aligned} &{}^C D^q x(t) = Ax(t) + f(t), \quad t \in [0, T],\\ &x(0) = x_0, \end{aligned}$$

where $0 < q < 1, x \in \mathbb{R}^n$ and A is an $n \times n$ matrix. In order to find the solution, apply Laplace transform on both sides and use the Laplace transform of Caputo derivative, we get

$$s^q X(s) - s^{q-1}x(0) = AX(s) + F(s).$$

Apply inverse Laplace transform on both sides and by we have

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{s^{q-1}(s^q I - A)^{-1}\}x_0 + \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{(s^q I - A)^{-1}\}.$$

Finally, substituting Laplace transformation of the Mittag-Leffler function, we get the solution of the given system [31]

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s) ds$$

where $E_q(At^q)$ is the matrix extension of the mentioned Mittag-Leffler functions with the following representation:

$$E_q(At^q) = \sum_{k=0}^{\infty} \frac{A^k t^{kq}}{\Gamma(1+kq)}$$

with the property ${}^C D^q E_q(At^q) = AE_q(At^q)$.

3. Linear systems

Consider the linear fractional dynamical system with multiple delays in control represented by the fractional differential equation of the form

$$\begin{aligned} &{}^C D^q x(t) = Ax(t) + \sum_{i=0}^{M} B_i u(h_i(t)), \quad t \in [0, T] := J, \\ &x(0) = x_0, \end{aligned} \tag{3.1}$$

where $0 < q < 1, x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$ and A is an $n \times n$ matrix and B_i are $n \times p$ matrices, for $i = 0, 1, \ldots, M$.

Let us assume the following settings:

- Assume the function $h_i : [0, T] \to \mathbb{R}, i = 0, 1, 2, \ldots, M$ are twice continuously differentiable and strictly increasing in J. Moreover,

$$h_i(t) \le t \quad \text{for } t \in J,\ i = 0, 1, 2, \ldots, M. \tag{3.2}$$

- Introduce the time lead functions $r_i(t) : [h_i(0), h_i(T)] \to [0, T], i = 0, 1, 2, \ldots, M$ such that $r_i(h_i(t)) = t$ for $t \in J$. Further assume that $h_0(t) = t$ and for $t = T$, the following inequalities hold

$$h_M(T) \le h_{M_1}(T) \le \cdots h_{m+1}(T) \le 0 = h_m(T) < h_{m-1}(T) = \cdots h_1(T) = h_0(T) = T. \tag{3.3}$$

- Let $h > 0$ be given. For functions $u : [-h, T] \to \mathbb{R}^p$ and $t \in J$, we use the symbol u_t to denote the function on $[-h, 0]$, defined by $u_t(s) = u(t+s)$ for $s \in [-h, 0)$.

The following definitions of complete state of the system (3.1) at time t and relative controllability are assumed.

Definition 3.1. The set $y(t) = \{x(t), u_t\}$ is the complete state of the system (3.1) at time t.

Definition 3.2. System (3.1) is said to be globally relatively controllable on J if for every complete state $y(0)$ and every vector $x_1 \in \mathbb{R}^n$ there exists a control $u(t)$ defined on J such that the corresponding trajectory of the system (3.1) satisfies $x(T) = x_1$.

Then the solution of the system (3.1) can be expressed in the following form [31]

$$x(t) = E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sum_{i=0}^{M} B_i u(h_i(s))ds.$$

Using the time lead functions $r_i(t)$ the solution becomes,

$$x(t) = E_q(A(t)^q)x_0 + \sum_{i=0}^{M}\int_{h_i(0)}^{h_i(t)} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u(s)ds. \tag{3.4}$$

Now using the inequalities (3.3), the above equation can be expressed as

$$\begin{aligned} x(t) &= E_q(At^q)x_0 + \sum_{i=0}^{m}\int_{h_i(0)}^{0} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds \\ &\quad + \sum_{i=0}^{m}\int_{0}^{t} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u(s)ds \\ &\quad + \sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(t)} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds. \end{aligned} \tag{3.5}$$

For brevity, let us introduce the following notations:

$$\begin{aligned} G(t) &= \sum_{i=0}^{m}\int_{h_i(0)}^{0} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds \\ &\quad + \sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(t)} (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds. \end{aligned} \tag{3.6}$$

Now let us define the controllability Grammian matrix

$$W(0,T) = \sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)][E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]^* ds$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and the $*$ denotes the matrix transpose.

Theorem 3.3. *The linear control system (3.1) is relatively controllable on $[0, T]$ if and only if the controllability Grammian matrix*

$$W(0,T) = \sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)][E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]^* ds \tag{3.7}$$

is positive definite, for some $T > 0$.

Proof. Since W is positive definite, that is, it is non-singular and so its inverse is well-defined. Define the control function as,

$$u(t) = [B_i^* E_{q,q}(A^*(T-r_i(t))^q)\dot{r}_i(t)]W^{-1}[x_1 - E_q(At^q)x_0 - G(T)], \quad \text{for } i = 0, 1, \ldots, m \tag{3.8}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily. Inserting (3.8) in (3.5) and using (3.6), we have

$$\begin{aligned} x(T) &= E_q(AT^q)x_0 + G(T) + \sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)] \\ &\quad \times [B_i^* E_{q,q}(A^*(T-r_i(s))^q)\dot{r}_i(s)]W^{-1}[x_1 - E_q(AT^q)x_0 - G(T)]ds \\ &= x_1. \end{aligned}$$

Thus the control $u(t)$ transfers the initial state $y(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T. Hence the system (3.1) is controllable.

On the other hand, if it is not positive definite, there exists a nonzero y such that

$$\begin{aligned}
&y^*Wy = 0\\
&y^*\sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)][E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]^* y\,ds = 0\\
&y^*\sum_{i=0}^{m}(T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)] = 0,
\end{aligned}$$

on $[0, T]$. Let $x_0 = [E_q(AT^q)]^{-1}y$. By the assumption, there exists a control u such that it steers the complete initial state $y(0) = \{x_0, u_0(s)\}$ to the origin in the interval $[0, T]$. It follows that

$$\begin{aligned}
x(T) &= E_q(AT^q)x_0 + G(T) + \sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]\\
&\quad\times [B_i^*E_{q,q}(A^*(T-r_i(s))^q)\dot{r}_i(s)]W^{-1}[x_1 - E_q(AT^q)x_0 - G(T)]ds\\
&= y + G(T) + \sum_{i=0}^{m}\int_0^T (T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]\\
&\quad\times [B_i^*E_{q,q}(A^*(T-r_i(s))^q)\dot{r}_i(s)]W^{-1}[x_1 - E_q(AT^q)x_0 - G(T)]ds\\
&= 0.
\end{aligned}$$

Thus,

$$0 = y^*y + \sum_{i=0}^{m}\int_0^T y^*(T-r_i(s))^{q-1}[E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)]u(s)ds + y^*G(T).$$

But the second and third term are zero leading to the conclusion $y^*y = 0$. This is a contradiction to $y \neq 0$. Thus W is positive definite. Hence the desired result. □

4. Nonlinear systems

Consider the nonlinear fractional dynamical system with multiple delays in control represented by the fractional differential equation of the form

$$\begin{aligned}
&{}^CD^qx(t) = Ax(t) + \sum_{i=0}^{M}B_iu(h_i(t)) + f(t, x(t), u(t)), \quad t \in [0, T] := J,\\
&x(0) = x_0,
\end{aligned} \tag{4.1}$$

where $0 < q < 1, x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$ and A, B_i are defined as above and $f : J \times \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is a continuous function. Denote Q as the Banach space of continuous $\mathbb{R}^n \times \mathbb{R}^p$ valued functions defined on the interval J with the uniform norm $\|(z, v)\| = \|z\| + \|v\|$ where

$$\|z\| = \sup\{|z(t)| : t \in J\}.$$

That is, $Q = C_n(J) \times C_p(J)$, where $C_n(J)$ is the Banach space of continuous $\mathbb{R}^n$ valued functions defined on the interval J with the sup norm. For each $(z, v) \in Q$, consider the linear fractional dynamical system

$$\begin{aligned}
&{}^CD^qx(t) = Ax(t) + \sum_{i=0}^{M}B_iu(h_i(t)) + f(t, z(t), v(t)),\\
&x(0) = x_0.
\end{aligned} \tag{4.2}$$

Then the solution of the system (4.1) can be expressed in the following form [31]

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)\sum_{i=0}^{M}B_iu(h_i(s))ds\\
&\quad + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s, z(s), v(s))ds.
\end{aligned}$$

Using the time lead functions $r_i(t)$ the solution becomes,

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{i=0}^{M} \int_{h_i(0)}^{h_i(t)} (t-r_i(s))^{q-1} E_{q,q}(A(t-r_i(s))^q) B_i \dot{r}_i(s) u(s) ds \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s, z(s), v(s)) ds. \end{aligned} \tag{4.3}$$

Now using the inequalities (3.3), the above equation for $t = T$ can be expressed as

$$\begin{aligned} x(T) &= E_q(A(T)^q)x_0 + \sum_{i=0}^{m} \int_{h_i(0)}^{0} (T-r_i(s))^{q-1} E_{q,q}(A(T-r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \sum_{i=0}^{m} \int_0^T (T-r_i(s))^{q-1} E_{q,q}(A(T-r_i(s))^q) B_i \dot{r}_i(s) u(s) ds \\ &\quad + \sum_{i=m+1}^{M} \int_{h_i(0)}^{h_i(T)} (T-r_i(s))^{q-1} E_{q,q}(A(T-r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q) f(s, z(s), v(s)) ds. \end{aligned} \tag{4.4}$$

For brevity, let us introduce the following notation using (3.6):

$$\psi(y(0), x_1; z, v) = x_1 - E_q(AT^q)x_0 - G(T) - \int_0^T (T-s)^{q-1} E_{q,q}(A(T-s)^q) f(s, z(s), v(s)) ds. \tag{4.5}$$

Now let us define the controllability Grammian matrix and the control function

$$W(0,T) = \sum_{i=0}^{m} \int_0^T (T-r_i(s))^{q-1} [E_{q,q}(A(T-r_i(s))^q) B_i \dot{r}_i(s)][E_{q,q}(A(T-r_i(s))^q) B_i \dot{r}_i(s)]^* ds \tag{4.6}$$

$$u(t) = [B_i^* E_{q,q}(A^*(T-r_i(t))^q) \dot{r}_i(t)] W^{-1} \psi(y(0), x_1; z, v), \quad \text{for } i = 0, 1, \ldots, m \tag{4.7}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrarily and $*$ denotes the matrix transpose. Inserting (4.7) in (4.4) by using (4.5) and (4.6), it is easy to verify that the control $u(t)$ transfers the initial complete state $y(0)$ to the desired vector $x_1 \in \mathbb{R}^n$ at time T for each fixed $(z, v) \in Q$. Now observing (4.5) and substituting (4.7) in (4.3), we have

$$\begin{aligned} x(t) &= E_q(A(t)^q)x_0 + \sum_{i=0}^{m} \int_{h_i(0)}^{0} (t-r_i(s))^{q-1} E_{q,q}(A(t-r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \sum_{i=0}^{m} \int_0^t (t-r_i(s))^{q-1} E_{q,q}(A(t-r_i(s))^q) B_i \dot{r}_i(s) \\ &\quad \times B_i^* E_{q,q}(A^*(T-r_i(s))^q) \dot{r}_i(s) W^{-1} \psi(y(0), x_1; z, v) ds \\ &\quad + \sum_{i=m+1}^{M} \int_{h_i(0)}^{h_i(t)} (t-r_i(s))^{q-1} E_{q,q}(A(t-r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \int_0^t (t-s)^{q-1} E_{q,q}(A(t-s)^q) f(s, z(s), v(s)) ds. \end{aligned} \tag{4.8}$$

Theorem 4.1. *Let the continuous function f satisfies the following condition*

$$\lim_{|(x,u)| \to \infty} \frac{|f(t, x, u)|}{|(x, u)|} = 0,$$

uniformly in $t \in J$, and suppose that the linear fractional dynamical system (3.1) is globally relatively controllable. Then the nonlinear system (4.1) is globally relatively controllable on J.

Proof. Define the operator $\Phi : Q \to Q$ by

$$\Phi(z, v) = (x, u)$$

where

$$\begin{aligned}
u(t) &= B_i^* E_{q,q}(A^*(T-r_i(t))^q)\dot{r}_i(t)W^{-1}\psi(y(0),x_1;z,v)\\
&= B_i^* E_{q,q}(A^*(T-r_i(t))^q)\dot{r}_i(t)W^{-1}\\
&\quad\times\Bigg[x_1 - E_q(AT^q)x_0 - \sum_{i=0}^{m}\int_{h_i(0)}^{0}(T-r_i(s))^{q-1}E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds\\
&\quad - \sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(T)}(T-r_i(s))^{q-1}E_{q,q}(A(T-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds\\
&\quad - \int_0^T (T-s)^{q-1}E_{q,q}(A(T-s)^q)f(s,z(s),v(s))ds\Bigg]
\end{aligned}$$

and

$$\begin{aligned}
x(t) &= E_q(A(t)^q)x_0 + \sum_{i=0}^{m}\int_{h_i(0)}^{0}(t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds\\
&\quad + \sum_{i=0}^{m}\int_0^t (t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)\\
&\quad \times B_i^* E_{q,q}(A^*(T-r_i(s))^q)\dot{r}_i(s)W^{-1}\psi(y(0),x_1;z,v)ds\\
&\quad + \sum_{i=m+1}^{M}\int_{h_i(0)}^{h_i(t)}(t-r_i(s))^{q-1}E_{q,q}(A(t-r_i(s))^q)B_i\dot{r}_i(s)u_0(s)ds\\
&\quad + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)f(s,z(s),v(s))ds.
\end{aligned}$$

For our convenience, let us introduce the following constants,

$$\begin{aligned}
& a_i = \sup\|E_{q,q}(A(T-r_i(s))^q)\|; \qquad b_i = \sup\|\dot{r}_i(s)\|, \quad i=0,1,2,\ldots,M\\
& L_i = \int_0^T (T-r_i(s))^{q-1}ds, \qquad N_i = \int_{h_i(0)}^{0}(T-r_i(s))^{q-1}ds \quad i=0,1,2,\ldots,m\\
& M_i = \int_{h_i(0)}^{h_i(T)}(T-r_i(s))^{q-1}ds, \quad i=m+1,m+2,\ldots,M\\
& \alpha = \sup\|E_{q,q}(A(T-s)^q)\|; \qquad \beta = \sup\|E_q(At^q)x_0\|; \qquad \gamma = \sup\|u_0(s)\|;\\
& b = \sum_{i=0}^{m} a_i b_i L_i; \qquad a = \max\{bq^{-1}T^q\|B_i\|, 1\}; \qquad \mu = \sum_{i=0}^{m} a_i b_i\|B_i\|N_i + \sum_{i=m+1}^{M} a_i b_i\|B_i\|M_i;\\
& c_2 = 4\alpha q^{-1}T^q; \qquad d_2 = 4[\beta+\gamma\mu]; \qquad \sup|f| = \sup[|f(s,z(s),v(s))| : s\in J];\\
& \bar{c}_i = 4[a_i b_i\|B_i^*\|]\|W^{-1}\|\alpha q^{-1}T^q; \qquad c = \max\{\bar{c}_i, c_2\}; \quad i=0,1,2,\ldots,m;\\
& \bar{d}_i = 4[a_i b_i\|B_i^*\|]\|W^{-1}\|[|x_1|+\beta+\mu]; \qquad d = \max\{\bar{d}_i, d_2\}; \quad i=0,1,2,\ldots,m.
\end{aligned}$$

Then

$$\begin{aligned}
|u(t)| &\le a_i b_i\|B_i^*\|\,\|W^{-1}\|\Bigg[|x_1| + \beta + \gamma\sum_{i=0}^{m} a_i b_i\|B_i\|\int_{h_i(0)}^{0}(T-r_i(s))^{q-1}ds\\
&\quad + \gamma\sum_{i=m+1}^{M} a_i b_i\|B_i\|\int_{h_i(0)}^{h_i(T)}(T-r_i(s))^{q-1}ds\Bigg] + a_i b_i\|B_i^*\|\,\|W^{-1}\|\alpha q^{-1}T^q\sup|f|\\
&\le a_i b_i\|B_i^*\|\,\|W^{-1}\|[|x_1|+\beta+\gamma\mu] + a_i b_i\|B_i^*\|\,\|W^{-1}\|\alpha q^{-1}T^q\sup|f|\\
&\le \left[\frac{\bar{d}_i}{4a} + \frac{\bar{c}_i}{4a}\sup|f|\right]\\
&\le \frac{1}{4a}[d + c\sup|f|]
\end{aligned}$$

$$\begin{aligned}|x(t)| &\leq \beta + \gamma\mu + \left[\sum_{i=0}^{m} a_i b_i \|B_i\| L_i q^{-1} T^q\right] \frac{1}{4a}[d + c \sup|f|] + \alpha q^{-1} T^q \sup|f| \\ &\leq \frac{d}{4} + \frac{1}{4}[d + c \sup|f|] + \frac{c}{4} \sup|f| \\ &\leq \frac{d}{2} + \frac{c}{2} \sup|f|.\end{aligned}$$

By hypothesis the function f satisfies the following conditions [32]. For each pair of positive constants c and d, there exists a positive constant r such that, if $|\bar{p}| \leq r$, then

$$c|f(t, \bar{p})| + d \leq r, \quad \text{for all } t \in J. \tag{4.9}$$

Also for given c and d, if r is a constant such that the inequality (4.9) is satisfied, then any r_1 such that $r < r_1$ will also satisfy (4.9). Now, take c and d as given above, and let r be chosen so that (4.9) is satisfied. Therefore, if $\|z\| \leq \frac{r}{2}$ and $\|v\| \leq \frac{r}{2}$, then $|z(s)| + |v(s)| \leq r$, for all $s \in J$. It follows that $d + c \sup|f| \leq r$. Therefore, $|u(s)| \leq \frac{r}{4a}$, for all $s \in J$, and hence $\|u\| \leq \frac{r}{4a}$, which gives $\|x\| \leq \frac{r}{2}$. Thus, we have proved that, if $Q(r) = \{(z, v) \in Q : \|z\| \leq \frac{r}{2} \text{ and } \|v\| \leq \frac{r}{2}\}$, then Φ maps $Q(r)$ into itself. Since f is continuous, it implies that the operator is continuous, and hence is completely continuous by the application of Arzela–Ascoli's theorem. Since $Q(r)$ is closed, bounded and convex, the Schauder fixed point theorem guarantees that Φ has a fixed point $(z, v) \in Q(r)$ such that $\Phi(z, v) = (z, v) \equiv (x, u)$. Hence we have

$$\begin{aligned}x(t) &= E_q(A(t)^q)x_0 + \sum_{i=0}^{m} \int_{h_i(0)}^{0} (t - r_i(s))^{q-1} E_{q,q}(A(t - r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \sum_{i=0}^{m} \int_{0}^{t} (t - r_i(s))^{q-1} E_{q,q}(A(t - r_i(s))^q) B_i \dot{r}_i(s) u(s) ds \\ &\quad + \sum_{i=m+1}^{M} \int_{h_i(0)}^{h_i(t)} (t - r_i(s))^{q-1} E_{q,q}(A(t - r_i(s))^q) B_i \dot{r}_i(s) u_0(s) ds \\ &\quad + \int_{0}^{t} (t - s)^{q-1} E_{q,q}(A(t - s)^q) f(s, z(s), v(s)) ds.\end{aligned}$$

Thus, $x(t)$ is the solution of the system (4.1), and it is easy to verify that $x(T) = x_1$. Further the control function $u(t)$ steers the system (4.1) from initial complete state $y(0)$ to x_1 on J. Hence the system (4.1) is globally relatively controllable on J. □

5. Example

In this section, we apply the results obtained in the previous section for the following fractional dynamical systems with multiple delays in control which involves sequential Caputo derivative

$$\begin{aligned}&{}^C D^q x(t) = Ax(t) + B_1 u(t) + B_2 u(t - h) + f(t, x(t)), \quad 0 < q < 1,\ t \in [0, T] \\ &x(0) = x_0,\end{aligned} \tag{5.1}$$

where $A = \begin{pmatrix} -1 & 0 \\ 3 & -2 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$f(t, x(t)) = \begin{pmatrix} \dfrac{10x_1}{1 + x_1^2(t) + x_2^2(t)} \\ \dfrac{x_2}{1 + x_2^2(t)} \end{pmatrix}.$$

Here $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ with $x_1(t) = x(t);\ D^{\frac{q}{2}} x_1(t) = x_2(t)$. The Mittag-Leffler matrix of the given system is given by [31]

$$E_q(At^q) = \begin{pmatrix} E_q(-t^q) & 0 \\ 3E_q(-t^q) - 3E_q(-2t^q) & E_q(-2t^q) \end{pmatrix}.$$

Further

$$E_{q,q}(A(T - s)^q) = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix},$$

$$a = E_{q,q}(-(T - s)^q), \qquad b = E_{q,q}(-2(T - s)^q), \qquad c = 3a - 3b.$$

$$E_{q,q}(A(T - (s + h))^q) = \begin{pmatrix} \bar{a} & 0 \\ \bar{c} & \bar{b} \end{pmatrix},$$

$$\bar{a} = E_{q,q}(-(T - (s + h))^q), \qquad \bar{b} = E_{q,q}(-2(T - (s + h))^q), \qquad \bar{c} = 3\bar{a} - 3\bar{b}.$$

By simple matrix calculation one can see that the controllability matrix

$$W(0,T) = \sum_{i=0}^{m} \int_0^T (T - r_i(s))^{q-1}[E_{q,q}(A(T - r_i(s))^q)B_i\dot{r}_i(s)][E_{q,q}(A(T - r_i(s))^q)B_i\dot{r}_i(s)]^* ds$$
$$= \int_0^T \left[(T-s)^{q-1} \begin{pmatrix} a^2 & ac \\ ac & b^2 + c^2 \end{pmatrix} + (T - (s+h))^{q-1} \begin{pmatrix} \bar{a}^2 & \bar{a}\bar{c} \\ \bar{a}\bar{c} & \bar{b}^2 + \bar{c}^2 \end{pmatrix} \right] ds$$

is positive definite for any $T > h$. Further the nonlinear function $f(t, x(t))$ satisfies the hypothesis of the Theorem 4.1 and so the fractional system (5.1) is globally relatively controllable on $[0, T]$.

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