

Multi-Rate Sampled-Data Observer Design Based on a Continuous-Time Design

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Abstract—A multi-rate sampled-data observer for nonlinear systems under asynchronous sampling is developed in this work. The proposed multi-rate observer is based on a continuous-time design coupled with inter-sample output predictors for the sampled measurements. The sampled-data system with the multi-rate observer forms a hybrid system and it is shown that the error dynamics of the overall system is input-to-output stable with respect to measurement errors, by applying the Karafyllis-Jiang vector small-gain theorem. This sampled-data design also offers robustness with respect to perturbations in the sampling schedule. The proposed method is evaluated through linear systems and a nonlinear batch reactor example.

Index Terms—Multi-rate observers, nonlinear sampled-data systems, asynchronous sampling, input-to-output stability, vector small-gain theorem.

I. INTRODUCTION

THE objective of this work is to develop a state observer in a multi-rate sampled-data system under asynchronous sampling. The problem of nonlinear observer design has been intensively studied for systems under fast sampling [1]–[5], which can be potentially applied to estimation, process control, and fault detection and identification. Motivated by practical implementation needs, however, one of the biggest challenges is to design a state observer for general multi-rate systems (e.g., chemical processes, biological systems, networked control systems), where different sampling rates of sensors need to be accommodated in the observer design framework.

The observer design for linear multi-rate systems was studied in [6], [7], where an available continuous-time observer design was adopted but different methods (i.e., sample-and-hold strategy [6] and model-based prediction [7], respectively) were employed to approximate the inter-sample behavior between consecutive measurements. Both approaches provided robustness with respect to perturbations in the sampling schedule.

In nonlinear systems, the focus has been primarily on single-rate sampled-data observer design based on a mixed continuous and discrete strategy, which was inspired by the continuous-discrete Kalman filter in [8]. In [9], an observer for state affine systems with regularly persistent inputs was designed. The results in [9] were extended to observer design for state affine systems up to output injection in [10] and adaptive observer design in [11]. The continuous-discrete approach has been applied to high-gain observers in [12]–[17]. In general, the continuous-discrete observer consists of two steps: (i) open-loop prediction when no measurements are available, and

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(ii) impulsive correction once a new measurement is obtained. A similar idea was also used in [18], [19].

The problem of nonlinear multi-rate observer design has received little attention, given the challenge in stability analysis that arises from the asynchronous nature of different sensors and uncertainty in the sampling schedule. A nonlinear multi-rate observer design, based on an approximate discrete-time model, was developed in [20], with guaranteed semiglobal practical input-to-state stability (ISS) from exogenous disturbances to estimation errors. An artificial fast-rate sampler and a hold device were introduced to reconstruct the missing outputs as well as inputs between sampling times, which were then fed to a single-rate observer working at a base sampling period of the plant. The results were extended to one-sided Lipschitz systems in [21]. Recently, a hybrid observer was reported for a class of nonlinear systems with multi-rate sampled and delayed measurements [22], with global exponential stability of the error dynamics. However, it assumes a certain special structure of the nonlinear system for the method to be applicable.

In this work, the proposed multi-rate sampled-data observer design adopts the idea in [23], [24] of using a state predictor to approximate the inter-sample behavior, but in a more general context, where multiple inter-sample predictors are used for the multi-rate system. These predictors will be running asynchronously at the same time. Each predictor generates an estimate of the evolution of a sampled output between consecutive measurements, in the same spirit as [7] for linear systems. The existence of a continuous-time observer is a prerequisite for a multi-rate observer design. This is a common assumption in the continuous-discrete observer design for sampled-data systems [15]. Taking the measurement errors as inputs and the estimation errors as outputs, the notion of input-to-output stability (IOS), originated from [25] for systems described by ordinary differential equations, is adopted for stability analysis of the sampled-data system and the multi-rate observer. Since the overall system is a hybrid system in the sense that the classical semigroup property does not hold, the notion of weak semigroup property introduced in [26], [27] will be utilized, as it is more relaxed than the semigroup property and allows to study a very general class of systems (e.g., sampled-data systems, networked control systems, and hybrid systems). A direct product from this system theoretic framework is a small-gain theorem for two interconnected feedback systems [28], which played an important role in the stability analysis of the single-output sampled-data observer in [23]. This result was further generalized to a vector small-gain theorem in [29], which allows to study IOS and ISS properties for large-scale systems consisting of multiple interacting subsystems, such as

the proposed multi-rate sampled-data observer in section II.

The rest of the paper is organized as follows. The representation of a multi-rate sampled-data observer is formulated in Section II. The main results are stated in Section III and applied to linear and nonlinear systems in Section IV.

II. FORMULATION OF THE SAMPLED-DATA OBSERVER

A. Notations

- \mathcal{K}^+ denotes the class of positive, continuous functions defined on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. We adopt the notations of class \mathcal{K} , class \mathcal{K}_∞ and class \mathcal{KL} functions in [30]. The set of nonnegative integers is denoted by \mathbb{Z}_+ .
- $\mathbb{R}_+^n := \{[x_1, \dots, x_n]' \in \mathbb{R}^n : x_1 \geq 0, \dots, x_n \geq 0\}$. Let $x, y \in \mathbb{R}^n$. We say that $x \leq y$ if and only if $(y - x) \in \mathbb{R}_+^n$. We say that a function $\rho : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is of class \mathcal{N}_n , if ρ is continuous with $\rho(0) = 0$ and such that $\rho(x) \leq \rho(y)$ for all $x, y \in \mathbb{R}_+^n$ with $x \leq y$.
- For every positive integer l and an open, non-empty set $A \subseteq \mathbb{R}^n$, $C^l(A; \Omega)$ denotes the class of continuous functions on A with continuous derivatives of order l , which take values in $\Omega \subseteq \mathbb{R}^m$. $C^0(A; \Omega)$ denotes the class of continuous functions on A which take values in Ω .
- We denote by $\|\cdot\|_{\mathcal{X}}$ the norm of the normed linear space \mathcal{X} . By $|\cdot|$, we denote the ℓ_1 -norm of \mathbb{R}^n . Let $I \subseteq \mathbb{R}_+$ be an interval and $D \subseteq \mathbb{R}^l$ be a non-empty set. By $\mathcal{L}_{loc}^\infty(I; D)$, we denote the class of all Lebesgue measurable and locally bounded functions $u : I \rightarrow D$. For $u \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^n)$, we define the norm $\|u(t)\|_{\mathcal{U}} := \sum_{i=1}^n \sup_{\tau \in [0, t]} |u_i(\tau)|$. Notice that $\sup_{\tau \in [0, t]} |u_i(\tau)|$ denotes the actual supremum of $|u_i(t)|$ on $[0, t]$.

B. Problem Formulation

Consider a multi-output continuous-time autonomous system, where without loss of generality, the output is assumed to be a part of the state vector

$$\begin{aligned}\dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ y(t) &= x_M(t)\end{aligned}\tag{1}$$

with $x_R \in \mathbb{R}^{n-m}$ being the unmeasured state vector, $x_M \in \mathbb{R}^m$ being the remaining state vector that is directly measured, y denoting the output vector, and $f_R \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m; \mathbb{R}^{n-m})$, $f_M \in C^1(\mathbb{R}^{n-m} \times \mathbb{R}^m; \mathbb{R}^m)$ with $f_R(0, 0) = 0$, $f_M(0, 0) = 0$.

In the presence of multiple measurements, it makes more sense to use a reduced-order observer so that a significantly lower dimensionality can ease implementation of the observer. Therefore, a reduced-order observer formulation will be the focus of this work. Suppose that a continuous-time reduced-order observer design is available for system (1)

$$\begin{aligned}\dot{z}(t) &= F(z(t), y(t)) \\ \hat{x}_R(t) &= \Psi(z(t), y(t))\end{aligned}\tag{2}$$

with $z \in \mathbb{R}^k$ being the observer state, $\hat{x}_R \in \mathbb{R}^{n-m}$ being the state estimates, and $F \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^k)$, $\Psi \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^{n-m})$ with $F(0, 0) = 0$, $\Psi(0, 0) = 0$.

The output equation of system (1) should be modified under slow-sampled measurements, which yields the following multi-rate sampled-data system

$$\begin{aligned}\dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ y^i(t_j^i) &= x_M^i(t_j^i), \quad j \in \mathbb{Z}_+, i = 1, 2, \dots, m\end{aligned}\tag{3}$$

where t_j^i denotes the j -th sampling time for the i -th component in x_M , at some sequence of time instants $S = \{t_k\}_{k=0}^\infty$. The sampling times of the i -th sensor form an infinite subsequence that tends to infinity. These sampling times are not necessarily uniformly spaced, but satisfying $0 < t_{j+1}^i - t_j^i \leq r$ for all $j \in \mathbb{Z}_+$, where r is the maximum sampling period among all the sensors. The sampling times from all the subsequences will be considered as the sampling times of the multi-rate sampled-data system (3). Notice that S is the sequence of all sampling times in ascending order. There is a one-to-one mapping from $\{t_j^i : j \in \mathbb{Z}_+, i = 1, 2, \dots, m\}$ to $\{t_k\}_{k=0}^\infty$. Finally, we assume there is no measurement available at the initial time t_0 .

As mentioned in Section I, a continuous-time design (2) will be the basis of a multi-rate observer design in the presence of asynchronous sampled measurements, as long as the inter-sample behavior is taken care of. In this way, a continuous-time design from the literature can be reused in the context of a multi-rate observer so that we do not need to design it from scratch. System (3) can be used to predict the evolution of the output between consecutive measurements. As depicted in Fig. 1, we propose the multi-rate sampled-data observer design

$$\begin{aligned}\dot{z}(t) &= F(z(t), w(t)), & t \in [t_k, t_{k+1}) \\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)), & t \in [t_k, t_{k+1}) \\ w^i(t_{k+1}) &= y^i(t_{k+1}) \\ \hat{x}_R(t) &= \Psi(z(t), w(t)), & \hat{x}_R \in \mathbb{R}^{n-m}\end{aligned}\tag{4}$$

This observer has the same dynamics as the continuous-time observer (2). The predictors operate continuously at different time horizons, which generate additional signals $w(t)$ to approximate and replace the output $y(t)$ in the implementation of continuous-time observer (2). $w^i(t)$ will be reinitialized once a new measurement $y^i(t_{k+1})$ becomes available, while the rest of the predictor states do not change until their measurements are obtained. By integrating the predictor equations, a model-based correction is applied on the most-recent measurement. Notice that t_k , t_{k+1} are not necessarily the sampling times from the same sensor. It was seen in [7] that the model-based prediction offers a more meaningful approach to approximate the inter-sample behavior instead of a simple sample-and-hold strategy, especially under large sampling period.

It is important to point out that the sampled-data system (3) together with the multi-rate observer (4) is a hybrid system, which does not satisfy the classical semigroup property. However, the weak semigroup property still holds (see [26], [27]). The recent results in [26]–[29] for a wide class of systems will be used in the proof of the main theorem to be stated in the next section. From the main theorem, other important features of this multi-rate sampled-data observer include: (i) the missing inter-sample behavior can be reconstructed by using the inter-sample predictors, (ii) as long as the maximum

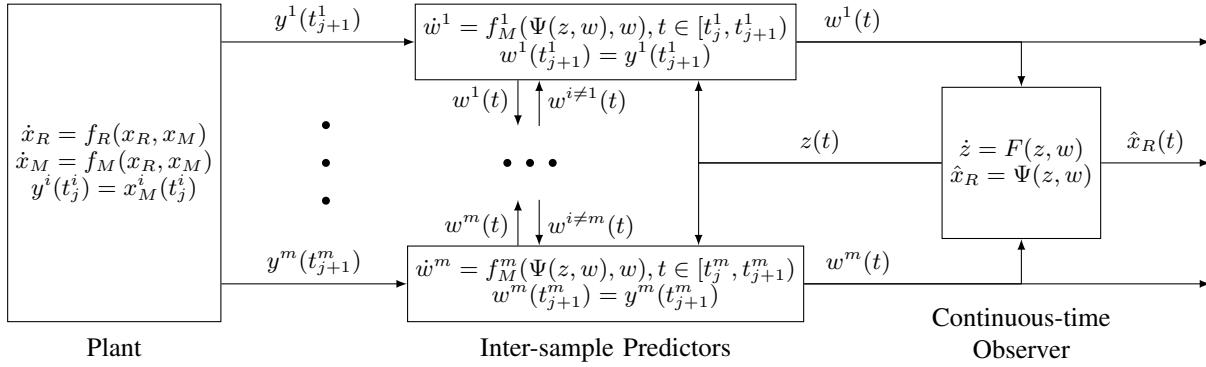


Fig. 1. A schematic of the multi-rate sampled-data observer with the plant (no measurement error).

sampling period is sufficiently small, if the continuous-time observer implementation guarantees stability of the error dynamics and robustness with respect to measurement errors, then the multi-rate observer will inherit these properties as well. These properties are unaffected by perturbations in the sampling schedule, which is a major advantage of the proposed hybrid implementation as opposed to an approximate discrete-time observer approach. Moreover, this continuous-discrete observer approach is able to use all possible measurements with different sampling rates, without making common assumptions when a discrete-time observer is used, such as that the sampling periods of the sensors are uniform and/or their ratios are rational numbers.

C. Basic Notions

We require that the following assumption holds.

Assumption 1: System (1) is forward complete.

Note that Assumption 1, according to the main results in [31], implies the existence of functions $\mu \in \mathcal{K}^+$ and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, the solution $(x_R(t), x_M(t))$ of (1) with initial condition $(x_R(0), x_M(0)) = (x_{R,0}, x_{M,0})$ exists for all $t \geq 0$ and satisfies

$$|(x_R(t), x_M(t))| \leq \mu(t)a(|(x_{R,0}, x_{M,0})|), \quad \forall t \geq 0 \quad (5)$$

In other words, a finite dimensional system described by ordinary differential equations is forward complete if and only if the corresponding solution exists for all $t \geq 0$ and for every initial condition [24].

Similarly to Definition 2.1 in [23] but in the context of a reduced-order observer, it is necessary to define the following notion of robust observer for system (1) with respect to measurement errors, which is important to develop the main results of this work.

Definition 1: The system

$$\begin{aligned} \dot{z}(t) &= F(z(t), y(t)), \quad z \in \mathbb{R}^k \\ \hat{x}_R(t) &= \Psi(z(t), y(t)), \quad \hat{x}_R \in \mathbb{R}^{n-m} \end{aligned} \quad (6)$$

where $F \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^k)$, $\Psi \in C^1(\mathbb{R}^k \times \mathbb{R}^m; \mathbb{R}^{n-m})$ with $F(0, 0) = 0$, $\Psi(0, 0) = 0$, is called a *robust observer* for system (1) with respect to measurement errors, if the following conditions are met:

(i) there exist functions $\sigma \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu \in \mathcal{K}^+$ and

$a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), x_M(t) + v(t)) \\ \hat{x}_R(t) &= \Psi(z(t), x_M(t) + v(t)) \end{aligned} \quad (7)$$

with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geq 0$ and satisfies the following estimates

$$\begin{aligned} |\hat{x}_R(t) - x_R(t)| &\leq \sigma(|(x_{R,0}, x_{M,0}, z_0)|, t) \\ &\quad + \gamma(\|v(t)\|_{\mathcal{U}}), \quad \forall t \geq 0 \end{aligned} \quad (8a)$$

$$\begin{aligned} |z(t)| &\leq \mu(t)[a(|(x_{R,0}, x_{M,0}, z_0)|) \\ &\quad + p(\|v(t)\|_{\mathcal{U}})], \quad \forall t \geq 0 \end{aligned} \quad (8b)$$

(ii) for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $z_0 \in \mathbb{R}^k$ such that the solution $(x_R(t), x_M(t), z(t))$ of system (7) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \equiv 0$, satisfies $x_R(t) = \Psi(z(t), x_M(t))$ for all $t \geq 0$.

Remark 1: If system (6) is a robust observer for system (1) with respect to measurement errors, then system (7) with the output $Y = \Psi(z, x_M + v) - x_R$ satisfies the uniform input-to-output stability (UIOS) property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ with gain $\gamma \in \mathcal{N}_1$ (see [28]).

Instead of using on-line, continuous-time outputs, a multi-rate sampled-data observer only uses outputs at discrete sampling times in $S = \{t_k\}_{k=0}^\infty$. The sampling partition is not necessarily uniform, but there exists a maximum sampling period r acting as the upper bound on each sampling interval. Next, we define the notion of a robust multi-rate sampled-data observer.

Definition 2: The system

$$\begin{aligned} \dot{\zeta}(t) &= g(\zeta(t), \zeta(t_k)), \quad t \in [t_k, t_{k+1}) \\ \zeta(t_{k+1}) &= G \left(\lim_{t \rightarrow t_{k+1}^-} \zeta(t), y^i(t_{k+1}) \right) \\ \hat{x}_R(t) &= \Psi(\zeta(t)) \end{aligned} \quad (9)$$

where $g \in C^1(\mathbb{R}^k \times \mathbb{R}^k; \mathbb{R}^k)$, $G \in C^0(\mathbb{R}^k \times \mathbb{R}; \mathbb{R}^k)$, $\Psi \in C^1(\mathbb{R}^k; \mathbb{R}^{n-m})$ with $g(0, 0) = 0$, $G(0, 0) = 0$, $\Psi(0) = 0$,

is called a *robust multi-rate sampled-data observer* for system (3) with respect to measurement errors, if the following conditions are met:

(i) there exist functions $\sigma \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu \in \mathcal{K}^+$ and $a \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, \zeta_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1]) \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), \zeta(t))$ of

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{\zeta}(t) &= g(\zeta(t), \zeta(t_k)), \quad t \in [t_k, t_{k+1}] \\ \zeta(t_{k+1}) &= G \left(\lim_{t \rightarrow t_{k+1}^-} \zeta(t), x_M^i(t_{k+1}) + v^i(t_{k+1}) \right) \quad (10) \\ t_{k+1} &= t_k + rd(t_k) \\ \hat{x}_R(t) &= \Psi(\zeta(t)) \end{aligned}$$

with initial condition $(x_R(0), x_M(0), \zeta(0)) = (x_{R,0}, x_{M,0}, \zeta_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ and $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geq 0$ and satisfies the following estimates

$$|\hat{x}_R(t) - x_R(t)| \leq \sigma(|(x_{R,0}, x_{M,0}, \zeta_0)|, t) + \gamma(\|v(t)\|_{\mathcal{U}}), \forall t \geq 0 \quad (11a)$$

$$|\zeta(t)| \leq \mu(t)[a(|(x_{R,0}, x_{M,0}, \zeta_0)|) + p(\|v(t)\|_{\mathcal{U}})], \forall t \geq 0 \quad (11b)$$

(ii) for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $\zeta_0 \in \mathbb{R}^k$ such that for all $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$, the solution $(x_R(t), x_M(t), \zeta(t))$ of (10) with initial condition $(x_R(0), x_M(0), \zeta(0)) = (x_{R,0}, x_{M,0}, \zeta_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ and $v \equiv 0$, satisfies $x_R(t) = \Psi(\zeta(t))$ for all $t \geq 0$.

Remark 2: If system (9) is a robust multi-rate sampled-data observer for system (3) with respect to measurement errors, then system (10) with the output $Y = \Psi(\zeta) - x_R$ satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, with gain $\gamma \in \mathcal{N}_1$ (see [28], where the notion of UIOS is defined for hybrid systems, e.g., system (10), which does not satisfy the classical semigroup property).

Remark 3: The sampling period in each subsequence is allowed to be time-varying. The equation $t_{k+1} = t_k + rd(t_k)$ generates the sampling instants in S sequentially with $0 \leq t_{k+1} - t_k \leq r$ for all $k \in \mathbb{Z}_+$. The value that $d(t_k)$ takes introduces uncertainty to the end-point of each sampling interval. Proving stability for any disturbance $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ will guarantee stability for all sampling schedules of system (10).

Remark 4: At a specific time t_k , there could be measurement of more than one output or the sampling of one sensor may coincide with another (i.e., $d(t_k) = 0$). Hence, some sampling instants may appear more than once in the sequence S , where the reinitialization step will occur repeatedly but on different elements in $w(t)$.

III. MAIN RESULTS

Recently, the nonlinear small-gain theorem was generalized from two interconnected systems to large-scale complex systems consisting of multiple, interacting input-to-output stable (or input-to-state stable) subsystems in [32]–[34]. In [29],

a generalization of several previously developed nonlinear small-gain theorems was obtained. Uniform and non-uniform IOS and ISS properties were studied for a wide class of nonlinear feedback systems that do not satisfy the semigroup property, such as hybrid and switched systems.

In this section, we assume there exists a robust observer for system (1) in the sense of Definition 1, and would like to give conditions so that stability of the error dynamics and robustness with respect to measurement errors still hold for the multi-rate design.

Theorem 1: Consider system (1) under Assumption 1 and suppose that system (6) is a robust observer for system (1) with respect to measurement errors. Moreover, suppose that there exist constants $C^i \geq 0$ and functions $\bar{\sigma}^i \in \mathcal{KL}$ for all $i = 1, 2, \dots, m$, such that for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t))$ of (7) with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ exists for all $t \geq 0$ and satisfies the following estimate

$$\begin{aligned} &|f_M^i(\Psi(z(t), x_M(t) + v(t)), x_M(t) + v(t)) \\ &- f_M^i(x_R(t), x_M(t))| \\ &\leq \bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + C^i \|v(t)\|_{\mathcal{U}}, \forall t \geq 0 \end{aligned} \quad (12)$$

Additionally, suppose that (i) $3rC^i m < 1$ for $i = 1, 2, \dots, m$; (ii) $3\gamma(ms) < s$ for all $s > 0$, where $\gamma \in \mathcal{N}_1$ is the gain function in the estimate (8a) of the robust observer. Then (4) is a robust multi-rate sampled-data observer for system (3) with respect to measurement errors.

Proof: We focus on the following hybrid system, consisting of a sampled-data system and a multi-rate observer

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), w(t)), \quad t \in [t_k, t_{k+1}) \\ \dot{w}(t) &= f_M(\Psi(z(t), w(t)), w(t)), \quad t \in [t_k, t_{k+1}) \\ w^i(t_{k+1}) &= x_M^i(t_{k+1}) + v^i(t_{k+1}) \\ t_{k+1} &= t_k + rd(t_k) \\ Y(t) &= \Psi(z(t), w(t)) - x_R(t) \end{aligned} \quad (13)$$

By virtue of Definition 2, it is necessary to show that system (13) with the output $Y = \Psi(z, w) - x_R$ satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$.

Notice that the hybrid system (13) has a distributed structure, where each inter-sample predictor is a subsystem operating asynchronously. Each subsystem receives the associated system output and reinitializes its own inter-sample predictor. These subsystems also communicate with each other as well as the continuous-time observer by transmitting the predicted outputs. We focus on the i -th subsystem and treat $w^j(t)$ ($j \neq i$) and $v^i(t)$ as inputs to this subsystem. First, the boundedness of $\|w(t) - x_M(t)\|_{\mathcal{U}}$ will be established. Next, we will focus on the overall hybrid system (13) and study the UIOS property from the actual input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. A vector small-gain theorem (see [29]) will be used to complete the proof.

Consider the i -th subsystem from the distributed structure, where $w^j(t)$ ($j \neq i$) and $v^i(t)$ is considered as input

$$\begin{aligned} \dot{x}_R(t) &= f_R(x_R(t), x_M(t)) \\ \dot{x}_M(t) &= f_M(x_R(t), x_M(t)) \\ \dot{z}(t) &= F(z(t), w(t)), \quad t \in [t_j^i, t_{j+1}^i] \\ \dot{w}^i(t) &= f_M^i(\Psi(z(t), w(t)), w(t)), \quad t \in [t_j^i, t_{j+1}^i] \quad (14) \\ w^i(t_{j+1}^i) &= x_M^i(t_{j+1}^i) + v^i(t_{j+1}^i) \\ t_{j+1}^i &= t_j^i + rd(t_j^i) \\ Y(t) &= \Psi(z(t), w(t)) - x_R(t) \end{aligned}$$

which satisfies the weak semigroup property. Since two different samplings can never occur at the same time in a subsystem (i.e., $t_{j_1}^i \neq t_{j_2}^i$ if $j_1 \neq j_2$), disturbance $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; (0, 1])$ is used to introduce perturbations in the sampling schedule of the i -th subsystem, which rules out the Zeno phenomenon. Because system (6) is a robust observer for (1) with respect to measurement errors, it follows from (8a), (8b), and (12) that for every $(x_{R,0}, x_{M,0}, z_0, w_0^i, d) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R} \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; (0, 1])$, the solution $(x_R(t), x_M(t), z(t), w^i(t))$ of (14) with initial condition $(x_R(0), x_M(0), z(0), w^i(0)) = (x_{R,0}, x_{M,0}, z_0, w_0^i)$ corresponding to the inputs $w^j(t)$ where $j \neq i$ (i.e., predicted outputs from other inter-sample predictors) satisfies the following estimates for all $t \in [0, t_{\max}]$

$$|Y(t)| \leq \hat{\sigma}(|(x_{R,0}, x_{M,0}, z_0)|, t) + \gamma(\|w(t) - x_M(t)\|_{\mathcal{U}}) \quad (15)$$

$$|z(t)| \leq \mu(t)[a(|(x_{R,0}, x_{M,0}, z_0)|) + p(\|w(t) - x_M(t)\|_{\mathcal{U}})] \quad (16)$$

$$\begin{aligned} &|f_M^i(\Psi(z(t), w(t)), w(t)) - f_M^i(x_R(t), x_M(t))| \\ &\leq \bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + C^i \|w(t) - x_M(t)\|_{\mathcal{U}} \quad (17) \end{aligned}$$

Since Assumption 1 holds, we obtain from (5) and (16)

$$\begin{aligned} &|(x_R(t), x_M(t), z(t))| \leq \bar{\mu}(t)[\bar{a}(|(x_{R,0}, x_{M,0}, z_0)|) \\ &\quad + p(\|w(t) - x_M(t)\|_{\mathcal{U}})], \forall t \in [0, t_{\max}] \quad (18) \end{aligned}$$

for appropriate functions $\hat{\sigma}, \bar{\sigma}^i \in \mathcal{KL}$, $\gamma, p \in \mathcal{N}_1$, $\mu, \bar{\mu} \in \mathcal{K}^+$ and $a, \bar{a} \in \mathcal{K}_\infty$, where $t_{\max} \in (0, +\infty]$ is the maximal existence time of the solution.

Consider those time intervals where the reinitialization step occurs at the beginning. For all $t \in [t_j^i, t_{j+1}^i] \cap [0, t_{\max})$ with $j \geq 1$, we have

$$\begin{aligned} &|w^i(t) - x_M^i(t)| \leq \int_{t_j^i}^t |f_M^i(\Psi(z(s), w(s)), w(s)) \\ &\quad - f_M^i(x_R(s), x_M(s))| ds + |v^i(t_j^i)| \\ &\leq r\bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, t_j^i) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \quad (19) \\ &\quad + |v^i(t_j^i)| \\ &\leq \sigma_1^i(|(x_{R,0}, x_{M,0}, z_0)|, t) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\quad + \sup_{0 \leq \tau \leq t} |v^i(\tau)| \end{aligned}$$

where $\sigma_1^i(s, t) = r\bar{\sigma}^i(s, t-r)$ for $t \geq r$ and $\sigma_1^i(s, t) = \exp(r-t)r\bar{\sigma}^i(s, 0)$ for $t < r$. Notice that $\sigma_1^i(s, t) \in \mathcal{KL}$. Since there is no measurement available at the initial time $t_0^i = 0$, we make

an initial guess $w^i(t_0^i) = w^i(0) = w_0^i$ for the i -th state of the predictor. For all $t \in [0, t_1^i] \cap [0, t_{\max})$, we get

$$\begin{aligned} &|w^i(t) - x_M^i(t)| \leq |w_0^i - x_{M,0}^i| \\ &\quad + r \sup_{0 \leq s \leq t} |f_M^i(\Psi(z(s), w(s)), w(s)) - f_M^i(x_R(s), x_M(s))| \\ &\leq |w_0^i| + |x_{M,0}^i| + r\bar{\sigma}^i(|(x_{R,0}, x_{M,0}, z_0)|, 0) \\ &\quad + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\leq \sigma_2^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, t) + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} \quad (20) \end{aligned}$$

where $\sigma_2^i(s, t) = [r\bar{\sigma}^i(s, 0) + s] \exp(r-t)$ and $\sigma_2^i(s, t) \in \mathcal{KL}$. Combining (19) and (20), we conclude that the following estimate holds for all $t \in [0, t_{\max})$ and for $i = 1, 2, \dots, m$

$$\begin{aligned} &|w^i(t) - x_M^i(t)| \leq \sigma^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, t) \\ &\quad + rC^i \|w(t) - x_M(t)\|_{\mathcal{U}} + \sup_{0 \leq \tau \leq t} |v^i(\tau)| \quad (21) \end{aligned}$$

From (21), the fact that $\sum_{i=1}^m rC^i < 1/3$, and the assumption that t_{\max} is finite, it suffices to show the boundedness of $\|w(t) - x_M(t)\|_{\mathcal{U}}$ for all $t \in [0, t_{\max})$. In fact,

$$\begin{aligned} &\|w(t) - x_M(t)\|_{\mathcal{U}} \\ &\leq \frac{\sum_{i=1}^m \sigma^i(|(x_{R,0}, x_{M,0}, z_0, w_0^i)|, 0) + \|v(t)\|_{\mathcal{U}}}{1 - \sum_{i=1}^m rC^i} \quad (22) \end{aligned}$$

From (18), (21), (22), and the Boundedness-Implies-Continuation (BIC) property (see [26]) for system (14), we conclude that $t_{\max} = +\infty$. Hence, all the above inequalities hold for all $t \geq 0$. Therefore, $t \in [0, t_{\max})$ can be replaced by $t \geq 0$ in the inequalities. In addition, the BIC property of each subsystem implies that the BIC property holds true for the overall hybrid system (13) with $t_{\max} = +\infty$.

Now, we are in a position to study the UIOS property of the overall hybrid, multi-rate system (13) from the actual input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. A vector small-gain theorem will be used to check stability of the large-scale hybrid systems composed of multiple interconnected subsystems.

Without loss of generality, we may assume that $\bar{\mu}(t) \geq 1$ in (18). From (18), (22), and the triangle inequality $|w(t)| \leq |w(t) - x_M(t)| + |x_M(t)|$, we obtain for all $t \geq 0$

$$\begin{aligned} &|(x_R(t), x_M(t), z(t), w(t))| \\ &\leq 2\bar{\mu}(t)[\hat{a}(|(x_{R,0}, x_{M,0}, z_0, w_0)|) + \hat{p}(\|v(t)\|_{\mathcal{U}})] \quad (23) \end{aligned}$$

for appropriate functions $\hat{a} \in \mathcal{K}_\infty$ and $\hat{p} \in \mathcal{N}_1$. Furthermore, (23) and the BIC property of the hybrid system (13) imply the following important properties:

- (i) system (13) is robustly forward complete from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$;
- (ii) $0 \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m$ is a robust equilibrium point from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, with any output function $H(t, x_R, x_M, z, w, v)$ with $H(t, 0, 0, 0, 0, 0) = 0$ for all $t \geq 0$.

Now, Hypotheses (H1)-(H4) of the vector small-gain theorem are satisfied by virtue of the inequalities (15), (21), (22), and (23). We conclude that the hybrid system (13) satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. In other words, there exist functions $\tilde{\sigma} \in \mathcal{KL}$, $\tilde{\gamma}, \tilde{p} \in \mathcal{N}_1$, $\tilde{\mu} \in \mathcal{K}^+$

and $\tilde{a} \in \mathcal{K}_\infty$ such that for every $(x_{R,0}, x_{M,0}, z_0, w_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1]) \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ and $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ satisfies the following estimates for all $t \geq 0$

$$|Y(t)| \leq \tilde{\sigma}(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) + \tilde{\gamma}(\|v(t)\|_{\mathcal{U}}) \quad (24a)$$

$$|(z(t), w(t))| \leq \tilde{\mu}(t)[\tilde{a}(|(x_{R,0}, x_{M,0}, z_0, w_0)|) + \tilde{\rho}(\|v(t)\|_{\mathcal{U}})] \quad (24b)$$

Moreover, for every $(x_{R,0}, x_{M,0}) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $(z_0, w_0) \in \mathbb{R}^k \times \mathbb{R}^m$ with $w_0 = x_{M,0}$ such that for all $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ and $v \equiv 0$, satisfies $x_R(t) = \Psi(z(t), w(t))$ for all $t \geq 0$. ■

Remark 5: If we consider the hybrid system (13) with the prediction error as the output map, i.e., $Y'(t) = w(t) - x_M(t)$, by applying the vector small-gain theorem again, we conclude that with this output, system (13) also satisfies the UIOS property from the input $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$. In other words, there exist functions $\tilde{\sigma} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{N}_1$ such that for every $(x_{R,0}, x_{M,0}, z_0, w_0, d, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^k \times \mathbb{R}^m \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1]) \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$, the solution $(x_R(t), x_M(t), z(t), w(t))$ of (13) with initial condition $(x_R(0), x_M(0), z(0), w(0)) = (x_{R,0}, x_{M,0}, z_0, w_0)$ corresponding to $d \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; [0, 1])$ and $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ satisfies

$$|w(t) - x_M(t)| \leq \tilde{\sigma}(|(x_{R,0}, x_{M,0}, z_0, w_0)|, t) + \tilde{\gamma}(\|v(t)\|_{\mathcal{U}}), \forall t \geq 0 \quad (25)$$

Without measurement errors, the error dynamics of the multi-rate sampled-data observer, including the inter-sample predictors, will converge to zero asymptotically.

Remark 6: The continuous-time observer design coupled with inter-sample output predictors can be applied to multi-rate full-order observer design, under appropriate modifications. The vector small-gain theorem is applicable to study the UIOS property of the overall system from measurement errors.

Remark 7: It would be possible to include exogenous inputs in the system definition (3) and (4), and subsequently extend the stability analysis mutatis mutandis. This is not included for brevity.

IV. APPLICATIONS

In this section, the performance of the proposed multi-rate sampled-data observer is illustrated through linear systems and a nonlinear reactor example. An explicit formula for estimating the maximum sampling period is derived for linear detectable systems with application to an oscillator example.

A. Linear Detectable Systems

Consider a linear detectable system, where without loss of generality, the output is assumed to be a part of the state vector

$$\begin{aligned} \dot{x}_R(t) &= A_{11}x_R(t) + A_{12}x_M(t), & x_R \in \mathbb{R}^{n-m} \\ \dot{x}_M(t) &= A_{21}x_R(t) + A_{22}x_M(t), & x_M \in \mathbb{R}^m \\ y(t) &= x_M(t) \end{aligned} \quad (26)$$

A reduced-order Luenberger observer design is available

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hy(t) \\ \dot{x}_R(t) &= T_R^{-1}(z(t) - T_My(t)) \end{aligned} \quad (27)$$

where F is a Hurwitz matrix with desired eigenvalues and forms a controllable pair with H . The transformation matrices T_R, T_M satisfy the following Sylvester equation

$$[T_R \quad T_M] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = F [T_R \quad T_M] + H [0 \quad I] \quad (28)$$

Consequently, there exists a positive definite matrix P such that $F'P + PF$ is negative definite and there exist constants $\alpha, \delta > 0$ such that

$$x'(F'P + PF)x + 2x'PHv \leq -2\alpha x'Px + \delta|v|^2 \quad (29)$$

for all $(x, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. Inequality (29) implies for every $(x_{R,0}, x_{M,0}, z_0, v) \in \mathbb{R}^{n-m} \times \mathbb{R}^m \times \mathbb{R}^{n-m} \times \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ the solution of (26) with

$$\hat{z}(t) = Fz(t) + H(y(t) + v(t)) \quad (30)$$

with initial condition $(x_R(0), x_M(0), z(0)) = (x_{R,0}, x_{M,0}, z_0)$ corresponding to $v \in \mathcal{L}_{loc}^\infty(\mathbb{R}_+; \mathbb{R}^m)$ satisfies the estimates for $i = 1, 2, \dots, m$

$$\begin{aligned} |\hat{x}_R - x_R| &\leq |T_R^{-1}| \exp(-\alpha t) \sqrt{\frac{K_2}{K_1}} |z_0 - T_R x_{R,0} - T_M x_{M,0}| \\ &+ \left(|T_R^{-1}| \sqrt{\frac{\delta}{2\alpha K_1}} + |T_R^{-1} T_M| \right) \sup_{0 \leq \tau \leq t} |v(\tau)|, \forall t \geq 0 \\ |A_{21}^i(\hat{x}_R - x_R) + A_{22}^i v| &\leq |A_{21}^i| |T_R^{-1}| \exp(-\alpha t) \sqrt{\frac{K_2}{K_1}} |z_0 - T_R x_{R,0} - T_M x_{M,0}| \\ &+ \left(|A_{21}^i| |T_R^{-1}| \sqrt{\frac{\delta}{2\alpha K_1}} + |A_{21}^i| |T_R^{-1} T_M| + |A_{22}^i| \right) \\ &\times \sup_{0 \leq \tau \leq t} |v(\tau)|, \forall t \geq 0 \end{aligned}$$

where $K_1, K_2 > 0$ are constants such that $K_1|x|^2 \leq x'Px \leq K_2|x|^2$ for all $x \in \mathbb{R}^{n-m}$, and A_{21}^i, A_{22}^i are the i -th row of the matrices A_{21}, A_{22} respectively. We conclude from Theorem 1 that the following system

$$\begin{aligned} \dot{z}(t) &= Fz(t) + Hw(t), & t \in [t_k, t_{k+1}] \\ \dot{w}(t) &= A_{21}\hat{x}_R(t) + A_{22}w(t), & t \in [t_k, t_{k+1}] \\ w^i(t_{k+1}) &= x_M^i(t_{k+1}) + v^i(t_{k+1}) \\ \hat{x}_R(t) &= T_R^{-1}(z(t) - T_Mw(t)) \end{aligned} \quad (31)$$

is a robust multi-rate sampled-data observer for system (26) with respect to measurement errors given that the maximum sampling period r satisfies the inequalities

$$3rm \left(|A_{21}^i| |T_R^{-1}| \sqrt{\frac{\delta}{2\alpha K_1}} + |A_{21}^i| |T_R^{-1} T_M| + |A_{22}^i| \right) < 1, \quad i = 1, 2, \dots, m \quad (32)$$

and

$$3m \left(|T_R^{-1}| \sqrt{\frac{\delta}{2\alpha K_1}} + |T_R^{-1} T_M| \right) < 1 \quad (33)$$

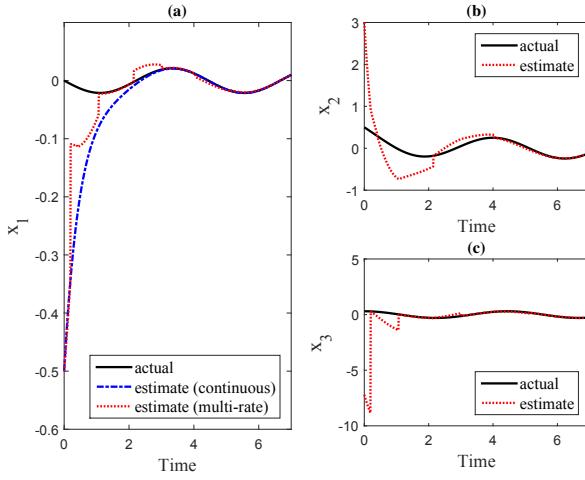


Fig. 2. (a) Comparison of the multi-rate sampled-data observer with a continuous-time observer using continuous-time measurements; (b),(c) Performance of the inter-sample output predictors for sampled measurements y^1 and y^2 respectively.

Consider a third-order linear oscillator

$$\begin{aligned} \dot{x}_1 &= -0.1x_3, & \dot{x}_2 &= 20x_1 - x_2, & \dot{x}_3 &= 20x_1 \\ y^1 &= x_2, & y^2 &= x_3 \end{aligned} \quad (34)$$

A reduced-order Luenberger observer is designed with $F = -2$, $H = [1 \ 2]$. Inequality (29) holds true with $P = K_1 = K_2 = 3$, $\alpha = 1$, $\delta = 7.5$. We conclude from (32), (33) that there exists a multi-rate sampled-data observer for system (34) provided that the maximum sampling period $r < 0.039$. Notice that conditions (32), (33) are very conservative. Indeed, simulations show the stability and robustness of the multi-rate observer will be preserved under much larger sampling period. The actual sampling subsequences of y^1 and y^2 are as follows

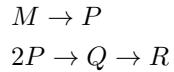
$$\begin{aligned} t_j^1 &= \{0.40, 2.15, 4.05, 5.88, 7.70\}, \\ t_j^2 &= \{0.20, 1.07, 2.17, 3.02, 4.07, 5.13, 6.20, 7.07\} \end{aligned}$$

where perturbations in the sampling schedule are considered.

Fig. 2 shows the performance of the multi-rate sampled-data observer with initial conditions $x(0) = [0 \ 0.5 \ 0.3]'$, $\hat{x}_1(0) = -0.5$, $w^1(0) = 3$, $w^2(0) = -7.2$. It is clear that the observer provides reliable estimation of the state.

B. Batch Chemical Reactor

Consider an isothermal batch reactor, where the following series reactions are taking place



The reaction rates of M , P , and Q are assumed to be

$$\begin{aligned} r_M &= -k_1 C_M, & r_P &= k_1 C_M - k_2 C_P^2, \\ r_Q &= k_2 C_P^2 - k_3 C_Q \end{aligned}$$

where $k_1 = 0.4 \text{ h}^{-1}$, $k_2 = 1 \text{ L}/(\text{mol}\cdot\text{h})$, and $k_3 = 0.5 \text{ h}^{-1}$. The concentrations of P , Q can be measured by on-line analytical instruments, with different sampling rates. Let x_1 , x_2 , x_3

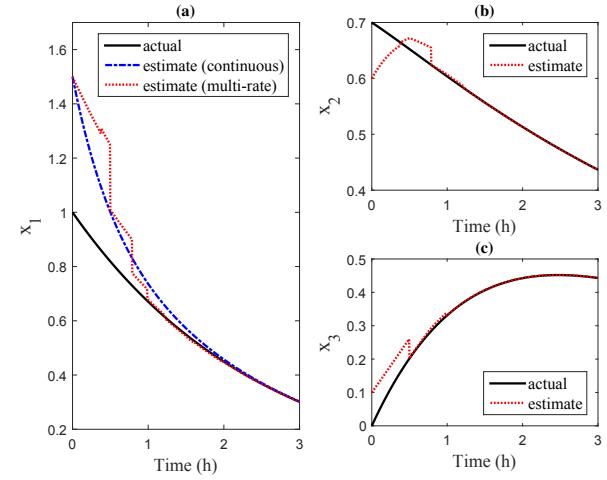


Fig. 3. (a) Comparison of the multi-rate sampled-data observer with a continuous-time observer using continuous-time measurements; (b),(c) Performance of the inter-sample output predictors for sampled measurements y^1 and y^2 respectively.

represent the concentrations of M , P , and Q respectively. The state-space model is given by

$$\begin{aligned} \dot{x}_1 &= -0.4x_1, & \dot{x}_2 &= 0.4x_1 - x_2^2, & \dot{x}_3 &= x_2^2 - 0.5x_3 \\ y^1(t_j^1) &= x_2(t_j^1), & y^2(t_j^2) &= x_3(t_j^2), & j &\in \mathbb{Z}_+ \end{aligned}$$

Sampling normally occurs every 0.4 h for y^1 and every 0.5 h for y^2 . However, perturbations in the sampling schedule are considered and the actual sampling subsequences of y^1 and y^2 for $j = 1, 2, \dots, 8$ are as follows

$$\begin{aligned} t_j^1 &= \{0.38, 0.79, 1.23, 1.60, 1.98, 2.41, 2.82, 3.20\}, \\ t_j^2 &= \{0.50, 0.99, 1.52, 2.01, 2.48, 3.01, 3.52, 4.01\} \end{aligned}$$

A continuous-time observer, which serves as the basis of the multi-rate sampled-data observer, will be designed by using the exact error linearization method (see [5] for the full-order observer formulation, and [35] for the reduced-order observer formulation) as follows

$$\dot{z} = Az + B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (35)$$

where $A = -2$ and $B = [2 \ 1.5]$. The immersion map $z = T(x)$ satisfies

$$\frac{\partial T}{\partial x_1} f_1(x) + \frac{\partial T}{\partial x_2} f_2(x) + \frac{\partial T}{\partial x_3} f_3(x) = AT + B \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \quad (36)$$

which admits a global solution

$$T(x) = -0.25x_1 + x_2 + x_3 \quad (37)$$

which is solvable with respect to the unmeasured state x_1 .

We design a multi-rate sampled-data observer based on (4) and the performance is shown in Fig. 3 with the initial conditions: $x(0) = [1 \ 0.7 \ 0]'$, $\hat{x}_1(0) = 1.5$, $w^1(0) = 0.6$, $w^2(0) = 0.1$. Fig. 3(a) shows that the speed of convergence of the multi-rate sampled-data observer and the continuous-time observer is comparable under the selected design parameters. From Fig. 3(b) and (c), the inter-sample predictors are able to

predict the inter-sample behavior with high accuracy after a few samplings by using model-based prediction.

V. CONCLUSIONS

This work developed a design method for nonlinear multi-rate sampled-data observer based on an available continuous-time design, coupled with inter-sample predictors. The main contributions are that: (i) the input-to-output stability property was established for the estimation and prediction errors with respect to measurement errors, (ii) the multi-rate design can handle non-uniform and asynchronous sampling without any assumption on the ratio of sampling periods to be an integer, as seen in the oscillator and reactor examples, (iii) as long as the maximum sampling period does not exceed a certain limit, the error dynamics of the proposed multi-rate observer is input-to-output stable, irrespective of perturbations in the sampling schedule. (ii) and (iii) are the major advantages of the proposed hybrid observer over an approximate discrete-time observer. Checkable sufficient conditions for stability and robustness were derived for linear detectable systems.

The theoretical framework of this study refers to a global observer design. For general nonlinear systems, a local observer formulation is of great interest. In addition, measurement delay and output feedback control based on a multi-rate observer will also be considered in our future work.

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