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Asymptotic expansions for second-order linear difference equations *

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In memory of Max Wyman

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Abstract

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Formal series solutions are obtained for the difference equation

$$y(n+2)+a(n)y(n+1)+b(n)y(n)=0,$$

where $a(n)$ and $b(n)$ have asymptotic expansions of the form

$$a(n)\sim\sum_{s=0}^{\infty}\frac{a_s}{n^s}\quad\text{and}\quad b(n)\sim\sum_{s=0}^{\infty}\frac{b_s}{n^s},$$

for large values of n, and $b_0\neq 0$. These solutions are characterized by the roots of the characteristic equation $\rho^2+a_0\rho+b_0=0$. Our discussion is divided into three cases, according to whether the roots are distinct, or equal and do not satisfy the auxiliary equation $a_1\rho+b_1=0$, or equal and do satisfy the auxiliary equation. The last case is further divided into three subcases, according to whether the roots of the indicial equation $\alpha(\alpha-1)\rho^2+(a_1\alpha+a_2)\rho+b_2=0$ do not differ by a nonnegative integer, or differ by a positive integer, or are equal. In all cases, the formal series solutions will be shown to be asymptotic. Our approach is based on the method of successive approximations.

Keywords: Asymptotic expansion; linear difference equation; method of successive approximations.

1. Introduction

Many special functions of mathematical physics and, in particular, orthogonal polynomials satisfy a three-term recurrence relation, which is a second-order linear difference equation.

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When a second-order linear differential equation is discretized, one obtains a corresponding difference equation. Many sequences that occur in enumerative combinatorics also satisfy a second-order linear difference equation. Thus, such equations arise frequently in many branches of mathematics. While the asymptotic theory for second-order linear differential equations has been satisfactorily developed, and can be found in many books (see, e.g., [7,10,12,13]), the corresponding theory for second-order linear difference equations is still not widely known. A probable explanation is that there does not seem to be an elementary method for verifying the asymptotic nature of formal series solutions to even the simplest and most familiar difference equation

$$y(n+2)+a(n)y(n+1)+b(n)y(n)=0, \tag{1.1}$$

where $a(n)$ and $b(n)$ have asymptotic expansions of the form

$$a(n)\sim\sum_{s=0}^{\infty}\frac{a_s}{n^s} \quad\text{and}\quad b(n)\sim\sum_{s=0}^{\infty}\frac{b_s}{n^s}, \tag{1.2}$$

for large values of n, and $b_0\neq 0$.

Asymptotic solutions to (1.1) are classified by the roots of the *characteristic equation*

$$\rho^2+a_0\rho+b_0=0. \tag{1.3}$$

Two possible values of ρ are

$$\rho_1,\rho_2=-\tfrac{1}{2}a_0\pm\left(\tfrac{1}{4}a_0^2-b_0\right)^{1/2}. \tag{1.4}$$

If $\rho_1\neq\rho_2$, i.e., $a_0^2\neq 4b_0$, then Birkhoff [4] showed that (1.1) has two linearly independent solutions, both of the form

$$y(n)\sim\rho^n n^{\alpha}\sum_{s=0}^{\infty}\frac{c_s}{n^s},\quad n\to\infty. \tag{1.5}$$

Motivated by the terminologies in differential equation theory [12, p.230], we shall call series of the form (1.5) *normal series* or *normal solutions*. If $\rho_1=\rho_2$ but their common value $\rho=-\frac{1}{2}a_0$ is not a root of the auxiliary equation

$$a_1\rho+b_1=0, \tag{1.6}$$

i.e., $2b_1\neq a_0a_1$, then Adams [1] showed that two linearly independent solutions of (1.1) are of the form

$$y(n)\sim\rho^n\,\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}\sum_{s=0}^{\infty}\frac{c_s}{n^{s/2}}, \tag{1.7}$$

which we shall call *subnormal solutions* (cf. [12, p.231]).

The papers of both Birkhoff and Adams are very long and complicated, and they are not easily understood even by most of the specialists in asymptotics. Adams also studied the case when the double root of (1.3) satisfies the auxiliary equation (1.6), i.e., when $2b_1=a_0a_1$, but his analysis was incomplete, as pointed out by Birkhoff [5, p.206]. A powerful asymptotic theory for difference equations was later given in [5,6], but these two papers are even lengthier and more difficult to read than the earlier ones [1,4]. For a short summary and some more recent comments on this massive work, see [14, Appendix] and [9], respectively. In recent years, desires to make the asymptotic theory of difference equations more accessible have been expressed in [2, p.511], [8,15,16], and the purpose of this paper is to make a first attempt in this direction. Here we shall give explicit formulas for the constants γ and α in (1.5) and (1.7), derive recurrence relations for the coefficients c_s in these expansions, and provide a new and elementary proof of these results, based on the method of successive approximations which is customarily used in the differential equation theory [12, pp. 229–235].

Furthermore, we shall discuss in detail the *exceptional case* in which the double root of (1.3) satisfies the auxiliary equation (1.6). Our discussion will be divided into three groups according as the zeros α_1, α_2 ($\operatorname{Re}\alpha_2 \geqslant \operatorname{Re}\alpha_1$) of the *indicial polynomial*

$$q(\alpha)=\alpha(\alpha-1)\rho^2+(a_1\alpha+a_2)\rho+b_2 \tag{1.8}$$

satisfy

$$\text{(i)}\quad \alpha_2-\alpha_1\neq 0,1,2,\ldots,\qquad \text{(ii)}\quad \alpha_2-\alpha_1=1,2,\ldots,\qquad \text{(iii)}\quad \alpha_2-\alpha_1=0. \tag{1.9}$$

In case (i), it will be shown that (1.1) has two linearly independent asymptotic solutions of the form

$$y(n)\sim\rho^n n^\alpha\sum_{s=0}^{\infty}\frac{c_s}{n^s}, \tag{1.10}$$

where ρ is the double root of the characteristic equation (1.3) satisfying (1.6), and $\alpha=\alpha_i$, $i=1,2$. The possibility that subnormal solutions may reduce to normal solutions in this case was also pointed out in [1, p.510]. In the other two cases, one of the two linearly independent solutions, say $y_1(n)$, is given by (1.10) with $\alpha=\alpha_1$; the other solution involves a logarithmic term. More precisely, we shall show that in case (ii), a second solution is given by

$$y_2(n)=z(n)+c(\log n)y_1(n), \tag{1.11}$$

where c is a constant which may happen to be zero, and

$$z(n)\sim\rho^n n^{\alpha_2}\sum_{s=0}^{\infty}\frac{d_s}{n^s}. \tag{1.12}$$

In the final case (iii), the second solution is again of the form

$$y_2(n)=z(n)+c(\log n)y_1(n), \tag{1.13}$$

but here c is a nonzero constant,

$$z(n)\sim\rho^n n^{\alpha_1-Q+2}\sum_{s=0}^{\infty}\frac{d_s}{n^s}, \tag{1.14}$$

and Q is an integer $\geqslant 3$ and is specified by (7.16) below. From (1.10), (1.11) and (1.13), it is evident that the exceptional case, in which $2b_1=a_0a_1$, corresponds to the case of regular singularity at infinity in the differential equation theory; cf. [12, p.231]. Explicit formulas will be given for the constant c in (1.11) and (1.13), and recurrence relations will be derived for the coefficients c_s and d_s in (1.10), (1.12) and (1.14). In all three cases listed in (1.9), the validity of the asymptotic solutions will be established again by the successive approximation method mentioned above.

It should be emphasized that it is the method, more than the result, that is important. We expect that this method will eventually lead us to the construction of error bounds associated with the various asymptotic solutions to (1.1). We also expect that this method will enable us to prove the validity of uniform asymptotic expansions for the solutions when the coefficient functions $a(n)$ and $b(n)$ in (1.1) depend on an additional parameter x, for instance, in the case of three-term recurrence relations satisfied by orthogonal polynomials. It should also be noted that it is precisely the idea behind our current method that was used by Olver and enabled him to develop a complete and satisfactory asymptotic theory for second-order linear differential equations; see [12, Chapters 7, 10–12].

In a subsequent paper [17], we shall consider the more general equation

$$y(n+2)+n^p a(n)y(n+1)+n^q b(n)y(n)=0, \tag{1.15}$$

where p and q are integers, and $a(n)$ and $b(n)$ are as given in (1.2) with the leading coefficients a_0 and b_0 both being nonzero. This will cover the case (1.1) when $p \leqslant 0$ and $q=0$. If $q=2p$, then (1.15) can be reduced to (1.1) by making the transformation

$$x(n)=[(n-2)!]^{\mu}y(n), \tag{1.16}$$

with $\mu=-p$. If $2p<q$ and $2p-q$ is even, then (1.15) can also be reduced to (1.1), by choosing $\mu=-\frac{1}{2}q$. However, not all cases can be reduced to (1.1), since the leading coefficient b_0 in (1.2) is required to be nonzero.

2. Normal and subnormal solutions: formal theory

To show that the infinite series in (1.5) is indeed a formal solution of (1.1), we substitute it into (1.1) and make use of the formula

$$(n+\mu)^{\alpha-s}=n^{\alpha}\sum_{k=0}^{\infty}\binom{\alpha-s}{k}\mu^k n^{-(s+k)}, \quad \mu=1, 2.$$

Comparing coefficients in the resulting expression, we obtain

$$\sum_{j=0}^{s}\left\{\rho^2 2^{s-j}\binom{\alpha-j}{s-j}+\rho\sum_{i=j}^{s}\binom{\alpha-j}{i-j}a_{s-i}+b_{s-j}\right\}c_j=0. \tag{2.1}$$

When $s=0$, (2.1) reduces to the characteristic equation (1.3). When $s=1$ in (2.1), it can be shown by using (1.3) that

$$\alpha=-\frac{a_1\rho+b_1}{2\rho^2+\rho a_0}=\frac{a_1\rho+b_1}{a_0\rho+2b_0}. \tag{2.2}$$

Since we are dealing with the case of distinct characteristic roots, the denominator in (2.2) is not zero. Now observe that the coefficient of c_s in (2.1) vanishes on account of (1.3). Therefore, (2.1) becomes

$$\sum_{j=0}^{s-1}\left\{\rho^2 2^{s-j}\binom{\alpha-j}{s-j}+\rho\sum_{i=j}^{s}\binom{\alpha-j}{i-j}a_{s-i}+b_{s-j}\right\}c_j=0, \tag{2.3}$$

for $s=2, 3,\ldots$. The coefficients c_s in (1.5) are now determined recursively by (2.3). Without loss of generality, one may assume that $c_0=1$.

In the case of subnormal solution (1.7), we make use of the formula

$$\mathrm{e}^{\gamma\sqrt{n+\mu}} = \mathrm{e}^{\gamma\sqrt{n}} \sum_{s=0}^{\infty} G_s^{(\mu)}(\gamma) n^{-s/2}, \quad \mu = 1, 2, \tag{2.4}$$

where

$$G_s^{(\mu)}(\gamma) = \sum_{\sigma(l_p)=s} \prod_{j=1}^{p} \frac{1}{l_j!} \left[\binom{\frac{1}{2}}{j} \mu^j \gamma \right]^{l_j}, \tag{2.5}$$

the summation being taken over all multi-indices $l_p = (l_1, \ldots, l_p)$ with $l_p \neq 0$ for $p > 1$ and such that

$$\sigma(l_p) \equiv l_1 + 3l_2 + \cdots + (2p-1)l_p = s. \tag{2.6}$$

Elementary computation gives

$$\begin{aligned}
&G_0^{(1)}(\gamma) = 1, && G_0^{(2)}(\gamma) = 1,\\
&G_1^{(1)}(\gamma) = \tfrac{1}{2}\gamma, && G_1^{(2)}(\gamma) = \gamma,\\
&G_2^{(1)}(\gamma) = \tfrac{1}{8}\gamma^2, && G_2^{(2)}(\gamma) = \tfrac{1}{2}\gamma^2,\\
&G_3^{(1)}(\gamma) = \tfrac{1}{48}\gamma^3 - \tfrac{1}{8}\gamma, && G_3^{(2)}(\gamma) = \tfrac{1}{6}\gamma^3 - \tfrac{1}{2}\gamma,\\
&G_4^{(1)}(\gamma) = \tfrac{1}{384}\gamma^4 - \tfrac{1}{16}\gamma^2, && G_4^{(2)}(\gamma) = \tfrac{1}{24}\gamma^4 - \tfrac{1}{2}\gamma^2.
\end{aligned} \tag{2.7}$$

For convenience, we also introduce the notation

$$F_s^{(\mu)}(\alpha) = \sum_{k=0}^{s} \tfrac{1}{2}\left(1 + (-1)^{s-k}\right) \mu^{(s-k)/2} \binom{\alpha - \frac{1}{2}k}{\frac{1}{2}(s-k)} c_k. \tag{2.8}$$

By the binomial expansion, we have

$$\sum_{s=0}^{\infty} c_s (n+\mu)^{\alpha - s/2} = n^{\alpha} \sum_{s=0}^{\infty} F_s^{(\mu)}(\alpha) n^{-s/2}, \quad \mu = 1, 2. \tag{2.9}$$

To show that the infinite series (1.7) is indeed a formal power series solution of (1.1), we substitute (1.2) and (1.7) in (1.1) and make use of (2.9). Equating coefficients of $n^{-s/2}$ to zero gives

$$\rho^2 \sum_{k=0}^{s} G_{s-k}^{(2)}(\gamma) F_k^{(2)}(\alpha) + \rho \sum_{k=0}^{s} \left[a_k^* \sum_{j=k}^{s} G_{s-j}^{(1)}(\gamma) F_{j-k}^{(1)}(\alpha) \right] + \sum_{k=0}^{s} b_{s-k}^* c_k = 0, \tag{2.10}$$

where $a_k^* = a_{k/2}$ and $b_k^* = b_{k/2}$ when k is a nonnegative even integer, and otherwise they are zero. Inserting (2.8) in (2.10), we have

$$\begin{aligned}
\sum_{l=0}^{s} \Bigg\{ & \rho^2 \sum_{k=0}^{s-l} \tfrac{1}{2}\left(1 + (-1)^k\right) 2^{k/2} \binom{\alpha - \frac{1}{2}l}{\frac{1}{2}k} G_{s-l-k}^{(2)}(\gamma) \\
& + \rho \sum_{k=0}^{s-l} a_k^* \sum_{j=0}^{s-l-k} \tfrac{1}{2}\left(1 + (-1)^j\right) \binom{\alpha - \frac{1}{2}l}{\frac{1}{2}j} G_{s-l-k-j}^{(1)}(\gamma) + b_{s-l}^* \Bigg\} c_l = 0.
\end{aligned} \tag{2.11}$$

Since $G_0^{(1)}(\gamma) = G_0^{(2)}(\gamma) = 1$, (2.11) reduces to (1.3) if $s = 0$. Similarly, since ρ is a double root of (1.3) and $\rho = -\frac{1}{2}a_0$ in this case, by using (2.7) it is easily seen that the coefficients of c_0 and c_1

in (2.11) are both zero if $s=1$. Therefore, we may restrict ourselves to $s \geqslant 2$ in (2.11). By the same reasoning, it can be shown that the coefficients of c_s and c_{s-1} in (2.11) are zero and, consequently, the equation becomes

$$\sum_{l=0}^{s-2}\left\{\sum_{k=0}^{s-l-1}\left[\tfrac{1}{2}(1+(-1)^k)\rho^2 2^{k/2}\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}k}G_{s-l-k}^{(2)}(\gamma)\right.\right.$$
$$\left.+\rho a_k^* \sum_{j=0}^{s-l-k}\tfrac{1}{2}(1+(-1)^j)\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}j}G_{s-l-k-j}^{(1)}(\gamma)\right]$$
$$\left.+\left[\tfrac{1}{2}(1+(-1)^{s-l})\rho^2 2^{(s-l)/2}\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}(s-l)}+\rho a_{s-l}^*+b_{s-l}^*\right]\right\}c_l=0. \tag{2.12}$$

Now, we set $s=2$ in (2.12). In view of (2.7) and the fact that $\rho=-\frac{1}{2}a_0$, we have

$$\tfrac{1}{4}\rho^2\gamma^2+\rho a_1+b_1=0. \tag{2.13}$$

Next, we set $s=3$ in (2.12). By the same argument, and using (2.13), we now have

$$\rho\left(\tfrac{1}{8}\gamma^2+\alpha-\tfrac{1}{4}\right)+\tfrac{1}{2}a_1=0. \tag{2.14}$$

The last two equations determine γ and α in (1.7). More explicitly, since

$$\rho=-\tfrac{1}{2}a_0 \quad\text{and}\quad \rho^2=b_0, \tag{2.15}$$

we have

$$\gamma=\pm 2\sqrt{\frac{a_0a_1-2b_1}{2b_0}} \tag{2.16}$$

and

$$\alpha=\frac{1}{4}+\frac{b_1}{2b_0}. \tag{2.17}$$

Finally, we note that the coefficients of c_{s-2} and c_{s-3} in (2.12) are, respectively,

$$\tfrac{1}{4}\gamma^2\rho^2+\rho a_1+b_1 \quad\text{and}\quad \rho\gamma\left[\rho\left(\tfrac{1}{8}\gamma^2-\tfrac{1}{4}+\alpha\right)+\tfrac{1}{2}a_1\right]-\tfrac{1}{2}\rho^2\gamma(s-3).$$

In view of (2.13) and (2.14), these are in turn equal to 0 and $-\frac{1}{2}\rho^2\gamma(s-3)$, respectively. Therefore, (2.12) can be written as

$$c_{s-3}=\frac{2}{(s-3)\rho^2\gamma}\sum_{l=0}^{s-4}\left\{\sum_{k=0}^{s-l-1}\left[\tfrac{1}{2}(1+(-1)^k)\rho^2 2^{k/2}\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}k}G_{s-l-k}^{(2)}(\gamma)\right.\right.$$
$$\left.+\rho a_k^* \sum_{j=0}^{s-l-k}\tfrac{1}{2}(1+(-1)^j)\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}j}G_{s-l-k-j}^{(1)}(\gamma)\right]$$
$$\left.+\left[\tfrac{1}{2}(1+(-1)^{s-l})\rho^2 2^{(s-l)/2}\binom{\alpha-\frac{1}{2}l}{\frac{1}{2}(s-l)}+\rho a_{s-l}^*+b_{s-l}^*\right]\right\}c_l. \tag{2.18}$$

The coefficients $c_1, c_2, \ldots$ in (1.7) are now determined recursively by (2.18) with $c_0 = 1$. In particular, we have

$$c_1 = \frac{1}{24b_0^2\gamma}\big(a_0^2a_1^2 - 24a_0a_1b_0 + 8a_0a_1b_1 - 24a_0a_2b_0 - 9b_0^2 - 32b_1^2 + 24b_0b_1 + 48b_0b_2\big). \tag{2.19}$$

As an illustration of the case of subnormal solutions (1.7), we consider the three-term recurrence relation

$$ny_n^{(\beta)}(x) + (x - 2n - \beta + 1)y_{n-1}^{(\beta)}(x) + (n + \beta - 1)y_{n-2}^{(\beta)}(x) = 0, \tag{2.20}$$

satisfied by the Laguerre polynomial $L_n^{(\beta)}(x)$. Writing (2.20) in the form (1.1), we have

$$y_{n+2}^{(\beta)}(x) + a(n)y_{n+1}^{(\beta)}(x) + b(n)y_n^{(\beta)}(x) = 0, \tag{2.21}$$

where

$$a(n) = \frac{x - 2n - \beta - 3}{n+2} = -2 + \frac{x - \beta + 1}{n} - \frac{2(x - \beta + 1)}{n^2} + \cdots$$

and

$$b(n) = \frac{n + \beta + 1}{n+2} = 1 + \frac{\beta - 1}{n} - \frac{2(\beta - 1)}{n^2} + \cdots.$$

The characteristic equation here is

$$\rho^2 - 2\rho + 1 = 0,$$

which has a double root $\rho_1 = \rho_2 = 1$ not satisfying the auxiliary equation (1.6) when $x \neq 0$. Thus equation (2.21) has two subnormal solutions of the form (1.7) with

$$\gamma = \pm 2\sqrt{x}\,\mathrm{i} \quad \text{and} \quad \alpha = \tfrac{1}{2}\beta - \tfrac{1}{4}. \tag{2.22}$$

Substituting (2.22) in (1.7) gives

$$y_{n,\pm}^{(\beta)}(x) \sim \mathrm{e}^{\pm 2\sqrt{nx}\,\mathrm{i}} n^{\beta/2 - 1/4} \sum_{s=0}^{\infty} \frac{c_s^{(\pm)}}{n^{s/2}},$$

where $c_0^{(\pm)} = 1$ and

$$c_1^{(\pm)} = \mp \frac{\mathrm{i}}{48\sqrt{x}}(4x^2 - 12\beta^2 - 24x\beta - 24x + 3).$$

From (2.18) it is readily seen that $y_{n,+}^{(\beta)}(x)$ and $y_{n,-}^{(\beta)}(x)$ are complex conjugates of each other. In view of the one-term expansion [11, p.245]

$$L_n^{(\beta)}(x) = \pi^{-1/2}\, \mathrm{e}^{x/2} x^{-\beta/2 - 1/4} n^{\beta/2 - 1/4} \cos\big(2\sqrt{nx} - \tfrac{1}{2}\beta\pi - \tfrac{1}{4}\pi\big) + \mathrm{O}(n^{\beta/2 - 3/4}),$$

we have

$$L_n^{(\beta)}(x) = \tfrac{1}{2}\pi^{-1/2}\, \mathrm{e}^{x/2} x^{-\beta/2 - 1/4}\big\{\mathrm{e}^{-(\beta\pi/2 + \pi/4)\mathrm{i}} y_{n,+}^{(\beta)}(x) + \mathrm{e}^{(\beta\pi/2 + \pi/4)\mathrm{i}} y_{n,-}^{(\beta)}(x)\big\}.$$

Upon simplification, we obtain

$$L_n^{(\beta)}(x) = \pi^{-1/2} \mathrm{e}^{x/2} x^{-\beta/2-1/4} n^{\beta/2-1/4}$$
$$\times \left\{ \cos\left(2\sqrt{nx} - \tfrac{1}{2}\beta\pi - \tfrac{1}{4}\pi\right) \sum_{s=0}^{\infty} \frac{A_s(x)}{n^{s/2}} + \sin\left(2\sqrt{nx} - \tfrac{1}{2}\beta\pi - \tfrac{1}{4}\pi\right) \sum_{s=0}^{\infty} \frac{B_s(x)}{n^{s/2}} \right\}, \tag{2.23}$$

where $A_0(x) = 1$, $A_1(x) = 0$, $B_0(x) = 0$ and

$$B_1(x) = \frac{1}{48\sqrt{x}}(4x^2 - 12\beta^2 - 24x\beta - 24x + 3).$$

3. Normal solutions: proof, I

If the characteristic equation (1.3) has two distinct nonzero roots ρ_1 and ρ_2, then, as we have shown in Section 2, equation (1.1) has two formal series solutions of the form

$$y_i(n) = \rho_i^n n^{\alpha_i} \sum_{s=0}^{\infty} \frac{c_{s,i}}{n^s}, \quad i = 1, 2, \tag{3.1}$$

where the exponent α_i is determined by (2.2) with ρ replaced by ρ_i and the coefficients $c_{s,i}$, $i = 1, 2$, are determined recursively by (2.3) with $c_{0,i} = 1$. In this and the following section, it will be proved that these formal solutions are indeed asymptotic.

Throughout this paper, we shall assume that $|\rho_2| \geqslant |\rho_1|$. First, we consider the case of $y_1(n)$. Following the argument in [12, p.233], we set

$$y_1(n) = L_N^{(1)}(n) + \epsilon_N^{(1)}(n), \tag{3.2}$$

with

$$L_N^{(1)}(n) = \rho_1^n n^{\alpha_1} \sum_{s=0}^{N-1} \frac{c_{s,1}}{n^s}. \tag{3.3}$$

Elementary calculation shows that the coefficients of $\rho_1^n n^{\alpha_1 - s}$, $s = 0, 1, \ldots, N$, in the formal series of

$$L_N^{(1)}(n+2) + a(n) L_N^{(1)}(n+1) + b(n) L_N^{(1)}(n)$$

are all zero, on account of (1.3) and (2.3). Consequently,

$$L_N^{(1)}(n+2) + a(n) L_N^{(1)}(n+1) + b(n) L_N^{(1)}(n) = \rho_1^n n^{\alpha_1} R_N^{(1)}(n), \tag{3.4}$$

where

$$R_N^{(1)}(n) = \mathrm{O}(n^{-N-1}), \quad \text{as } n \to \infty. \tag{3.5}$$

Substituting (3.2) in (1.1), we obtain from (3.4),

$$\epsilon_N^{(1)}(n+2) + a(n) \epsilon_N^{(1)}(n+1) + b(n) \epsilon_N^{(1)}(n) = -\rho_1^n n^{\alpha_1} R_N^{(1)}(n). \tag{3.6}$$

We shall show that (3.6) has a solution satisfying

$$\epsilon_N^{(1)}(n) = \mathrm{O}\big(\rho_1^n n^{\alpha_1 - N}\big), \quad \text{as } n \to \infty. \tag{3.7}$$

To do this, we rewrite (3.6) in the form

$$\begin{aligned}&\epsilon_N^{(1)}(n+2) + a_0\epsilon_N^{(1)}(n+1) + b_0\epsilon_N^{(1)}(n)\\ &\quad= -\rho_1^n n^{\alpha_1} R_N^{(1)}(n) - \{b(n) - b_0\}\epsilon_N^{(1)}(n) - \{a(n) - a_0\}\epsilon_N^{(1)}(n+1),\end{aligned} \tag{3.8}$$

i.e., we retain on the left-hand side of (3.6) only the leading terms in the expansion of $a(n)$ and $b(n)$, and transfer the rest to the right-hand side. Using the method of variation of parameters [3, p.49] (watch out for a minor error there), one can formally derive from (3.8) the equation

$$\epsilon_N^{(1)}(n) = \sum_{k=n}^{\infty} K(n, k)\big\{\rho_1^k k^{\alpha_1} R_N^{(1)}(k) + [b(k) - b_0]\epsilon_N^{(1)}(k) + [a(k) - a_0]\epsilon_N^{(1)}(k+1)\big\}, \tag{3.9}$$

where

$$K(n, k) = \frac{\rho_2^{n-k-1} - \rho_1^{n-k-1}}{\rho_2 - \rho_1}. \tag{3.10}$$

It can be verified by straightforward calculation that every solution of (3.9) is a solution of (3.8).

Equation (3.9) will be solved by the method of successive approximations. Define the sequence $\{h_s(n)\}$ by $h_0(n) = 0$ and

$$h_{s+1}(n) = \sum_{k=n}^{\infty} K(n, k)\big\{\rho_1^k k^{\alpha_1} R_N^{(1)}(k) + [b(k) - b_0]h_s(k) + [a(k) - a_0]h_s(k+1)\big\}. \tag{3.11}$$

We shall show that the series

$$\epsilon_N^{(1)}(n) = \sum_{s=0}^{\infty} \{h_{s+1}(n) - h_s(n)\} \tag{3.12}$$

is absolutely and uniformly convergent in n, and that its sum $\epsilon_N^{(1)}(n)$ is a solution to (3.9) satisfying (3.7).

Let n_0 be sufficiently large so that

$$|R_N(n)| \leqslant B_N n^{-N-1}, \tag{3.13}$$

for some constant B_N and for all $n \geqslant n_0$; see (3.5). Since $|\rho_2| \geqslant |\rho_1|$, a simple estimation gives

$$|K(n, k)| \leqslant \frac{2}{|\rho_2 - \rho_1|}|\rho_1|^{n-k-1}. \tag{3.14}$$

From (3.11) it follows that

$$|h_1(n)| \leqslant \frac{2B_N}{|\rho_2 - \rho_1|}|\rho_1|^{n-1} \sum_{k=n}^{\infty} k^{-(N+1-m_1)}, \tag{3.15}$$

for $n \geqslant n_0$, where $m_1 \equiv \operatorname{Re} \alpha_1$. If $p > 0$ and n is sufficiently large, then

$$\sum_{k=n}^{\infty} \frac{1}{k^{p+1}} \leqslant \int_{n-1}^{\infty} \frac{1}{x^{p+1}}\,\mathrm{d}x = \frac{1}{p}(n-1)^{-p} \leqslant \frac{2}{p}n^{-p}. \tag{3.16}$$

Without loss of generality, we may assume that (3.16) holds for $n \geqslant n_0$. Coupling (3.15) and (3.16), we obtain

$$|h_1(n)| \leqslant \frac{2^2 B_N}{|\rho_2-\rho_1|(N-m_1)} |\rho_1|^{n-1} n^{-(N-m_1)}, \tag{3.17}$$

if $N > m_1$. From (3.11) we also have

$$h_2(n)-h_1(n) = \sum_{k=n}^{\infty} K(n,k)\{[b(k)-b_0]h_1(k)+[a(k)-a_0]h_1(k+1)\}.$$

Applying (3.14) and (3.17) to the last equation gives

$$|h_2(n)-h_1(n)| \leqslant \frac{2^3 B_N}{|\rho_2-\rho_1|^2} \frac{\beta}{(N-m_1)} |\rho_1|^{n-2} \sum_{k=n}^{\infty} k^{-(N-m_1+1)},$$

where

$$\beta = \sup\{k[|b(k)-b_0|+|a(k)-a_0||\rho_1|]: k \geqslant 1\}. \tag{3.18}$$

From (3.16), it then follows that

$$|h_2(n)-h_1(n)| \leqslant \frac{2^4 B_N \beta}{|\rho_2-\rho_1|^2 (N-m_1)^2} |\rho_1|^{n-2} n^{-(N-m_1)}.$$

By induction, it can be proved that

$$|h_{s+1}(n)-h_s(n)| \leqslant \frac{2^{2(s+1)} B_N \beta^s}{|\rho_2-\rho_1|^{s+1}(N-m_1)^{s+1}} |\rho_1|^{n-s-1} n^{-(N-m_1)}, \tag{3.19}$$

for $s = 0, 1, 2, \ldots$. It is now evident that the series in (3.12) is uniformly convergent in n, if we choose N to satisfy $|\rho_1(\rho_2-\rho_1)|(N-m_1) > 4\beta$. Summation of (3.19) gives

$$\epsilon_N^{(1)}(n) = \mathrm{O}(\rho_1^n n^{m_1-N}). \tag{3.20}$$

Since (3.12) can also be written as

$$\epsilon_N^{(1)}(n) = \lim_{s\to\infty} h_s(n), \tag{3.21}$$

by taking $s \to \infty$ in (3.11) we have established that $\epsilon_N^{(1)}(n)$ is a solution to (3.9), and therefore to (3.6), satisfying (3.7). Consequently, for all sufficiently large N, equation (1.1) has a solution $y_{N,1}(n)$ with the property

$$y_{N,1} = \rho_1^n n^{\alpha_1} \left\{ \sum_{s=0}^{N-1} \frac{c_{s,1}}{n^s} + \mathrm{O}(n^{-N}) \right\}, \tag{3.22}$$

as $n \to \infty$. The fact that $y_{N,1}(n)$ is independent of N will be proved in the next section.

4. Normal solutions: proof, II

We next show that the formal series $y_2(n)$ in (3.1) is also asymptotic. The argument here is quite different from, and is in fact considerably more difficult than, that for second-order differential equations [12, pp. 233–235]. If $|\rho_2|=|\rho_1|$, then the analysis in Section 3 can be repeated with only the roles played by ρ_1 and ρ_2 being interchanged. However, if $|\rho_2|>|\rho_1|$, then this argument fails, and an alternative method must be sought. A natural attempt is to use the method of variation of parameters (also known as the method of reduction of order [3, p.43]), by setting $y_2(n)=v(n)y_1(n)$ and showing that the difference $w(n)\equiv v(n+1)-v(n)$ satisfies the first-order equation

$$y_1(n+2)w(n+1)-b(n)y_1(n)w(n)=0. \tag{4.1}$$

From (4.1) one readily obtains (see [3, pp. 38–39])

$$v(n)=v(0)+[v(1)-v(0)]\sum_{k=0}^{n-1}\prod_{l=0}^{k-1}\frac{b(l)y_1(l)}{y_1(l+2)}, \tag{4.2}$$

but the behaviour of $y_2(n)$ is difficult to derive from (4.2). Therefore, we make the following alterations. Set $v(n)=K_N(n)+\delta_N(n)$ so that

$$y_2(n)=[K_N(n)+\delta_N(n)]y_1(n), \tag{4.3}$$

where

$$K_N(n)=\left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1}\sum_{s=0}^{N-1}\frac{d_s}{n^s},$$

and the coefficients d_s are determined by

$$c_{s,2}=\sum_{t=0}^{s}d_t c_{s-t,1}.$$

Note that this can always be done. To demonstrate that (3.1) is asymptotic when $i=2$, it suffices to show

$$\delta_N(n)=\left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1}\mathrm{O}(n^{-N}). \tag{4.4}$$

As in (3.2) and (3.3), we set

$$y_2(n)=L_N^{(2)}(n)+\epsilon_N^{(2)}(n), \quad \text{with } L_N^{(2)}(n)=\rho_2^n n^{\alpha_2}\sum_{s=0}^{N-1}\frac{c_{s,2}}{n^s}.$$

It is easily shown that

$$K_N(n)y_1(n)=L_N^{(2)}(n)+\rho_2^n n^{\alpha_2}\mathrm{O}(n^{-N})$$

and

$$L_N^{(2)}(n+2)+a(n)L_N^{(2)}(n+1)+b(n)L_N^{(2)}(n)=\rho_2^n n^{\alpha_2}\mathrm{O}(n^{-N-1});$$

cf. (3.4) and (3.5). From these and the fact that ρ_2 satisfies the characteristic equation (1.3), it readily follows that $K_N(n)y_1(n)$ satisfies the equation

$$K_N(n+2)y_1(n+2)+a(n)K_N(n+1)y_1(n+1)+b(n)K_N(n)y_1(n)$$
$$=\rho_2^n n^{\alpha_2}\mathrm{O}(n^{-N-1}). \tag{4.5}$$

Since $y_2(n)$ is a solution of (1.1), by coupling (4.3) and (4.5), we have

$$y_1(n+2)\delta_N(n+2)+a(n)y_1(n+1)\delta_N(n+1)+b(n)y_1(n)\delta_N(n)$$
$$=\rho_2^n n^{\alpha_2}O(n^{-N-1}). \tag{4.6}$$

Furthermore, since $y_1(n)$ is also a solution of (1.1), it can be verified from (4.6) that the difference

$$\Delta_N(n)\equiv\delta_N(n+1)-\delta_N(n) \tag{4.7}$$

satisfies the first-order equation

$$y_1(n+2)\,\Delta_N(n+1)-b(n)y_1(n)\,\Delta_N(n)=\rho_2^n n^{\alpha_2}O(n^{-N-1}). \tag{4.8}$$

From (1.2) and (3.1) with $i=1$, we have

$$y_1(n+2)=\rho_1^{n+2}n^{\alpha_1}[1+\sigma_1(n)] \quad\text{and}\quad b(n)y_1(n)=\rho_1^n n^{\alpha_1}[b_0+\sigma_2(n)],$$

where

$$\sigma_1(n)=O(n^{-1}) \quad\text{and}\quad \sigma_2(n)=O(n^{-1}). \tag{4.9}$$

Now define $\xi_N(n)$ by

$$\Delta_N(n)=\left(\frac{\rho_2}{\rho_1}\right)^n\xi_N(n). \tag{4.10}$$

In terms of $\xi_N(n)$, equation (4.8) becomes

$$\rho_1\rho_2[1+\sigma_1(n)]\xi_N(n+1)-[b_0+\sigma_2(n)]\xi_N(n)=n^{\alpha_2-\alpha_1}O(n^{-N-1}).$$

Since $\rho_1\rho_2=b_0$, the last equation can be rewritten as

$$\xi_N(n+1)-\xi_N(n)=n^{\alpha_2-\alpha_1}E_N(n)-\sigma_1(n)\xi_N(n+1)+\sigma_2^*(n)\xi_N(n), \tag{4.11}$$

where $\sigma_2^*(n)=\sigma_2(n)/b_0$ and

$$|E_N(n)|\leqslant K_N n^{-N-1}, \tag{4.12}$$

K_N being some constant independent of n. Treating (4.11) as a first-order linear equation with the right-hand side being the nonhomogeneous term, one can formally derive the equation

$$\xi_N(n)=\sum_{k=n}^{\infty}[-k^{\alpha_2-\alpha_1}E_N(k)+\sigma_1(k)\xi_N(k+1)-\sigma_2^*(k)\xi_N(k)]. \tag{4.13}$$

Obviously, every solution of (4.13) is a solution of (4.11). We shall show that (4.13) has a solution satisfying

$$\xi_N(n)=n^{\alpha_2-\alpha_1}O(n^{-N}). \tag{4.14}$$

To establish (4.14), we again use the method of successive approximations. Define $g_0(n)\equiv 0$ and

$$g_{s+1}(n)=\sum_{k=n}^{\infty}[-k^{\alpha_2-\alpha_1}E_N(k)+\sigma_1(k)g_s(k+1)-\sigma_2^*(k)g_s(k)], \tag{4.15}$$

for $s=0, 1, 2, \ldots$. Put $m=\mathrm{Re}(\alpha_2-\alpha_1)$ and choose $N>m$. From (4.15), (4.12) and (3.16), it follows that

$$|g_1(n)| \leqslant K_N \sum_{k=n}^{\infty} k^{m-N-1} \leqslant \frac{K_N}{N-m}(n-1)^{m-N}. \tag{4.16}$$

Let $n_0(N)$ be sufficiently large so that $(n-1)^{m-N} \leqslant 2n^{m-N}$ for all $n \geqslant n_0(N)$. Consequently,

$$|g_1(n)| \leqslant \frac{2K_N}{N-m} n^{m-N}, \quad n \geqslant n_0(N). \tag{4.17}$$

From (4.16), we also have

$$|g_1(n+1)| \leqslant \frac{K_N}{N-m} n^{m-N} \leqslant \frac{2K_N}{N-m} n^{m-N}. \tag{4.18}$$

By the same argument, it can be established by induction that

$$|g_{s+1}(n)-g_s(n)|, \ |g_{s+1}(n+1)-g_s(n+1)| \leqslant \frac{K_N}{\beta}\left(\frac{2\beta}{N-m}\right)^{s+1} n^{m-N}, \tag{4.19}$$

for $n \geqslant n_0(N)$, $s=0, 1, 2, \ldots$, where

$$\beta = \sup\{k[\,|\sigma_1(k)|+|\sigma_2^*(k)|\,] : k \geqslant 1\}. \tag{4.20}$$

Let N be larger than $m+2\beta$ so that the series

$$\xi_N(n) = \sum_{s=0}^{\infty} [g_{s+1}(n)-g_s(n)] \tag{4.21}$$

is absolutely and uniformly convergent in n. (We first fix N and then choose the integer $n_0(N)$ in (4.17).) Since $\xi_N(n)$ can also be written in the form

$$\xi_N(n) = \lim_{s\to\infty} g_s(n), \tag{4.22}$$

by taking the limit as $s \to \infty$ in (4.15) we conclude that $\xi_N(n)$ is a solution to (4.13). From (4.21) and (4.19), it is evident that $\xi_N(n)$ satisfies (4.14).

By definition (4.7), we can rewrite (4.10) as

$$\delta_N(n+1)-\delta_N(n) = \left(\frac{\rho_2}{\rho_1}\right)^n \xi_N(n).$$

Solving this first-order equation, we obtain

$$\delta_N(n) = \delta_N(0) + \sum_{k=0}^{n-1} \left(\frac{\rho_2}{\rho_1}\right)^k \xi_N(k). \tag{4.23}$$

Since

$$\sum_{j=1}^{n} \rho^j \left(\frac{n}{n-j}\right)^p = \mathrm{O}(1),$$

for any positive integer p and any real number $\rho \in (0, 1)$, we have from (4.14),

$$\sum_{k=0}^{n-1} \left(\frac{\rho_2}{\rho_1}\right)^k \xi_N(k) = \left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1-N}\mathrm{O}(1). \tag{4.24}$$

Recalling the fact that $|\rho_2|>|\rho_1|$, the required result (4.4) now follows from (4.23) and (4.24). Therefore, for all sufficiently large N, equation (1.1) has a solution $y_{N,2}(n)$ satisfying

$$y_{N,2}(n)=\rho_2^n n^{\alpha_2}\left\{\sum_{s=0}^{N-1}\frac{c_{s,2}}{n^s}+\mathrm{O}(n^{-N})\right\}; \tag{4.25}$$

cf. (3.22).

It now remains to show that $y_{N,1}$ and $y_{N,2}$ are independent of N. Let both N_1 and N_2 be admissible values of N. Since $y_{N_1,1}$ and $y_{N_1,2}$ are linearly independent, there exist constants A and B such that

$$y_{N_2,1}(n)=Ay_{N_1,1}(n)+By_{N_1,2}(n), \tag{4.26}$$

which, in view of (3.22) and (4.25), can be written as

$$\rho_1^n n^{\alpha_1}\left\{\sum_{s=0}^{N_2-1}\frac{c_{s,1}}{n^s}+\mathrm{O}(n^{-N_2})\right\}$$
$$=A\rho_1^n n^{\alpha_1}\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,1}}{n^s}+\mathrm{O}(n^{-N_1})\right\}+B\rho_2^n n^{\alpha_2}\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,2}}{n^s}+\mathrm{O}(n^{-N_1})\right\}. \tag{4.27}$$

Divide both sides of this equation by $\rho_2^n n^{\alpha_2}$. If $|\rho_2|>|\rho_1|$, then, by letting $n\to\infty$, one readily sees that B must be zero and consequently A is equal to 1. Similarly, if we express

$$y_{N_2,2}(n)=Ay_{N_1,1}(n)+By_{N_1,2}(n), \tag{4.28}$$

then we have

$$\sum_{s=0}^{N_2-1}\frac{c_{s,2}}{n^s}+\mathrm{O}(n^{-N_2})$$
$$=A\left(\frac{\rho_1}{\rho_2}\right)^n n^{\alpha_1-\alpha_2}\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,1}}{n^s}+\mathrm{O}(n^{-N_1})\right\}+B\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,2}}{n^s}+\mathrm{O}(n^{-N_1})\right\}. \tag{4.29}$$

Letting $n\to\infty$, one immediately sees that $B=1$. Therefore, (4.28) becomes

$$y_{N_2,2}(n)-y_{N_1,2}(n)=Ay_{N_1,1}(n).$$

Without loss of generality, we may assume that $N_2>N_1$. Suppose that A is not zero. Then it follows from (4.29) that

$$\left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1}\mathrm{O}(n^{-N_1})=A\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,1}}{n^s}+\mathrm{O}(n^{-N_1})\right\}.$$

[The above derivation depends on the fact that A multiplied by $\mathrm{O}(n^{-N_1})$ is not zero.] As $n\to\infty$, the left-hand side becomes unbounded, while the right-hand side approaches A. This is impossible, and therefore $A=0$.

If $|\rho_2|=|\rho_1|$, then we may assume, without loss of generality, that $\mathrm{Re}\,\alpha_2\geqslant\mathrm{Re}\,\alpha_1$ and $N_1>\mathrm{Re}(\alpha_2-\alpha_1)$. Divide both sides of (4.26) and (4.28) by $\rho_2^n n^{\alpha_2}$, and let n tend to infinity. If $\mathrm{Re}\,\alpha_2>\mathrm{Re}\,\alpha_1$, then, by using similar arguments as in the case when $|\rho_2|>|\rho_1|$, one can show that $B=0$ and $A=1$ in (4.26), and that $B=1$ and $A=0$ in (4.28).

If $|\rho_2|=|\rho_1|$ and $\operatorname{Re}\alpha_2=\operatorname{Re}\alpha_1$, then we divide both sides of (4.27) by $\rho_1^n n^{\alpha_1}$ and obtain

$$\sum_{s=0}^{N_2-1}\frac{c_{s,1}}{n^s}\mathrm{O}(n^{-N_2})$$
$$=A\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,1}}{n^s}+\mathrm{O}(n^{-N_1})\right\}+B\left(\frac{\rho_2}{\rho_1}\right)^n n^{\alpha_2-\alpha_1}\left\{\sum_{s=0}^{N_1-1}\frac{c_{s,2}}{n^s}+\mathrm{O}(n^{-N_1})\right\}.$$

As $n\to\infty$, the left-hand side tends to $c_{0,1}$ and the first term on the right tends to $Ac_{0,1}$, but the second term does not have a limit if $B\neq 0$. Therefore, B must be zero and A is equal to 1. Similarly, we conclude from (4.29) that $A=0$ and $B=1$ in (4.28).

5. Subnormal solutions: proof, I

Here the roots of the characteristic equation (1.3) are equal, and their common value is given by (2.15). In Section 2, we have shown that two formal series solutions of (1.1) are both of the form

$$y(n)=\rho^n\,\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}\sum_{s=0}^{\infty}c_s n^{-s/2}, \tag{5.1}$$

where γ and α are determined by (2.16) and (2.17), and the coefficients c_s are determined recursively by (2.18). To show that these formal series solutions are asymptotic, we first observe that by writing $y(n)=\rho^n z(n)$, it is easily seen that we may assume without loss of generality that $\rho=1$ or, equivalently $a_0=-2$ and $b_0=1$; cf. (2.15).

As in (3.2), we now set

$$y(n)=L_N(n)+E_N(n), \tag{5.2}$$

with

$$L_N(n)=\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}\sum_{s=0}^{N-1}c_s n^{-s/2}. \tag{5.3}$$

Since the coefficients of c_s, c_{s-1} and c_{s-2} in (2.11) are all zero (see (2.18)), it is easily verified that

$$L_N(n+2)+a(n)L_N(n+1)+b(n)L_N(n)=\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}R_N(n), \tag{5.4}$$

where

$$R_N(n)=\mathrm{O}(n^{-(N+3)/2}). \tag{5.5}$$

Substituting (5.2) in (1.1), we obtain from (5.4),

$$E_N(n+2)+a(n)E_N(n+1)+b(n)E_N(n)=-\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}R_N(n). \tag{5.6}$$

Write

$$E_N(n)=\mathrm{e}^{\gamma\sqrt{n}}n^{\alpha}\epsilon_N(n). \tag{5.7}$$

Then $\epsilon_N(n)$ satisfies the equation

$$\epsilon_N(n+2)+a^*(n)\epsilon_N(n+1)+b^*(n)\epsilon_N(n)=R_N^*(n), \tag{5.8}$$

where

$$a^*(n) = \mathrm{e}^{\gamma\sqrt{n+1} - \gamma\sqrt{n+2}} \left(\frac{n+1}{n+2} \right)^{\alpha} a(n), \tag{5.9}$$

$$b^*(n) = \mathrm{e}^{\gamma\sqrt{n} - \gamma\sqrt{n+2}} \left(\frac{n}{n+2} \right)^{\alpha} b(n) \tag{5.10}$$

and

$$R_N^*(n) = -\mathrm{e}^{\gamma\sqrt{n} - \gamma\sqrt{n+2}} \left(\frac{n}{n+2} \right)^{\alpha} R_N(n). \tag{5.11}$$

Recall that we have assumed $\rho = 1$, or equivalently $a_0 = -2$ and $b_0 = 1$. Simple computation shows that

$$\begin{aligned} a^*(n) = {} & -2 + \gamma n^{-1/2} + \left(a_1 - \tfrac{1}{4}\gamma^2 + 2\alpha\right)n^{-1} + \left(-\tfrac{1}{2}a_1 - \tfrac{3}{4} + \tfrac{1}{24}\gamma^2 - \alpha\right)\gamma n^{-3/2} \\ & + R_a(n) \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} b^*(n) = {} & 1 - \gamma n^{-1/2} + \left(b_1 + \tfrac{1}{2}\gamma^2 - 2\alpha\right)n^{-1} + \left(\tfrac{1}{2} - \tfrac{1}{6}\gamma^2 + 2\alpha - b_1\right)\gamma n^{-3/2} \\ & + R_b(n), \end{aligned} \tag{5.13}$$

where

$$R_a(n),\ R_b(n) = \mathrm{O}(n^{-2}), \quad \text{as } n \to \infty. \tag{5.14}$$

Furthermore,

$$R_N^*(n) = \mathrm{O}(n^{-(N+3)/2}), \quad \text{as } n \to \infty. \tag{5.15}$$

Since $a_0 = -2$ and $b_0 = 1$, we have from (2.16) and (2.17),

$$\gamma^2 = -4(a_1 + b_1), \qquad \alpha = \tfrac{1}{4} + \tfrac{1}{2}b_1. \tag{5.16}$$

Insertion of (5.12) and (5.13) in (5.8) then yields

$$\begin{aligned} & \epsilon_N(n+2) + \left[-2 + \gamma n^{-1/2} + \left(\tfrac{1}{2} + 2a_1 + 2b_1\right)n^{-1} - \left(1 + \tfrac{2}{3}a_1 + \tfrac{2}{3}b_1\right)\gamma n^{-3/2}\right]\epsilon_N(n+1) \\ & \quad + \left[1 - \gamma n^{-1/2} - \left(\tfrac{1}{2} + 2a_1 + 2b_1\right)n^{-1} + \left(1 + \tfrac{2}{3}a_1 + \tfrac{2}{3}b_1\right)\gamma n^{-3/2}\right]\epsilon_N(n) \\ & \quad = R_N^*(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n), \end{aligned} \tag{5.17}$$

which can also be written as

$$\begin{aligned} & \Delta\epsilon_N(n+1) - \left[1 - \gamma n^{-1/2} - \left(\tfrac{1}{2} + 2a_1 + 2b_1\right)n^{-1} + \left(1 + \tfrac{2}{3}a_1 + \tfrac{1}{3}b_1\right)\gamma n^{-3/2}\right] \Delta\epsilon_N(n) \\ & \quad = R_N^*(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n), \end{aligned} \tag{5.18}$$

with $\Delta\epsilon_N(n) = \epsilon_N(n+1) - \epsilon_N(n)$. For convenience, we introduce the notations

$$\theta(n) = 1 - \gamma n^{-1/2} - \left(\tfrac{1}{2} + 2a_1 + 2b_1\right)n^{-1} + \left(1 + \tfrac{2}{3}a_1 + \tfrac{2}{3}b_1\right)\gamma n^{-3/2} \tag{5.19}$$

and

$$q(\epsilon_N(n+1), \epsilon_N(n), n) = R_N^*(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n). \tag{5.20}$$

Equation (5.18) then becomes

$$\Delta\epsilon_N(n+1) - \theta(n)\,\Delta\epsilon_N(n) = q(\epsilon_N(n+1), \epsilon_N(n), n). \tag{5.21}$$

To show that the formal series solution (5.1) is asymptotic, it suffices to prove that (5.17), or equivalently (5.21), has a solution $\epsilon_N(n)$ satisfying

$$\epsilon_N(n) = \mathrm{O}(n^{-N/2}), \quad \text{as } n \to \infty. \tag{5.22}$$

Considering (5.21) as a first-order linear nonhomogeneous equation, one can formally derive the equations

$$\epsilon_N(n) = \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} \theta^{-1}(i)\theta^{-1}(i+1) \cdots \theta^{-1}(j) q(\epsilon_N(j+1), \epsilon_N(j), j), \tag{5.23}$$

where $\theta^{-1}(j) \equiv 1/\theta(j)$, and

$$\epsilon_N(n) = -\sum_{i=n}^{\infty} \sum_{j=1}^{i-1} \theta(i-1)\theta(i-2) \cdots \theta(j+1) q(\epsilon_N(j+1), \epsilon_N(j), j), \tag{5.24}$$

where it is understood that $\theta(i-1)\theta(i-2) \cdots \theta(j+1) = 1$ when $j = i-1$. It is easily verified that every solution of (5.23) and every solution of (5.24) is a solution of (5.21). Now recall that the constant γ in (5.1) has two possible values which are given in (2.16). We shall show that when $\operatorname{Re}\gamma \leqslant 0$, then (5.23) has a solution satisfying (5.22), and that when $\operatorname{Re}\gamma > 0$, then (5.24) has such a solution. This will complete our investigation of subnormal solutions.

In this section, we are concerned only with the case $\operatorname{Re}\gamma \leqslant 0$. Before proceeding, we first record some preliminary results.

Lemma 1. *For positive integers $j \geqslant i \geqslant 1$, the function $\theta(n)$ in (5.19) satisfies*

$$\theta^{-1}(i)\theta^{-1}(i+1) \cdots \theta^{-1}(j) = \mathrm{e}^{2\gamma(\sqrt{j}-\sqrt{i})}\sqrt{\frac{j}{i}}\,[1 + \mathrm{O}(i^{-1/2})], \qquad i \to \infty. \tag{5.25}$$

where the O-term is uniform with respect to j.

Proof. First we recall the well-known asymptotic formulas [12, p.292]

$$\sum_{k=1}^{n-1} k^z - \zeta(-z) = \frac{n^{z+1}}{z+1} + \mathrm{O}(n^z), \quad n \to \infty, \ z \neq -1, \tag{5.26}$$

and

$$\sum_{k=1}^{n-1} \frac{1}{k} = \log n + C + \mathrm{O}(n^{-1}), \quad n \to \infty,$$

where $\zeta(z)$ is the Riemann Zeta function and C denotes the Euler constant. Since

$$\log(1-x) = -x - \tfrac{1}{2}x^2 + \mathrm{O}(x^3), \quad x \to 0,$$

we have from (5.19),

$$\log\theta(k) = -\gamma k^{-1/2} - \tfrac{1}{2}k^{-1} + \mathrm{O}(k^{-3/2}), \quad k \to \infty.$$

Here we have made use of the first equation in (5.16). Upon summation, we obtain

$$-\sum_{k=i}^{j}\log\theta(k)=2\gamma\left(\sqrt{j}-\sqrt{i}\right)+\tfrac{1}{2}\log\left(\frac{j}{i}\right)+\mathrm{O}(i^{-1/2}),\quad i\to\infty, \tag{5.27}$$

uniformly for $j\geqslant i\geqslant 1$. The result (5.25) now follows from (5.27) by exponentiation. □

Lemma 2. *For* $\operatorname{Re}\gamma\leqslant 0$, $\gamma\neq 0$ *and* $N\geqslant 1$, *we have*

$$\sum_{j=i}^{\infty}\mathrm{e}^{2\gamma(\sqrt{j}-\sqrt{i})}j^{-N/2-1}=-\frac{1}{\gamma}i^{-N/2-1/2}+\mathrm{O}(i^{-N/2-1}),\quad i\to\infty. \tag{5.28}$$

Proof. Recall the Euler–Maclaurin formula [12, p.281]

$$f(0)+\cdots+f(n)=\int_0^n f(x)\,\mathrm{d}x+\tfrac{1}{2}[f(0)+f(n)]+\int_0^n\tilde{\omega}_1(x)f'(x)\,\mathrm{d}x, \tag{5.29}$$

where $\tilde{\omega}_1(x)=x-\frac{1}{2}$ for $0\leqslant x<1$ and is periodic with period 1. Applying this formula to the function

$$f(x)=\mathrm{e}^{2\gamma(\sqrt{x+i}-\sqrt{i})}(x+i)^{-N/2-1}$$

and letting $n\to\infty$ yields

$$\sum_{j=i}^{\infty}\mathrm{e}^{2\gamma(\sqrt{j}-\sqrt{i})}j^{-N/2-1}=\int_i^{\infty}\mathrm{e}^{2\gamma(\sqrt{t}-\sqrt{i})}t^{-N/2-1}\,\mathrm{d}t+\tfrac{1}{2}i^{-N/2-1}+\mathrm{O}(i^{-N/2-1}).$$

Integration by parts twice gives

$$\int_i^{\infty}\mathrm{e}^{2\gamma(\sqrt{t}-\sqrt{i})}t^{-N/2-1}\,\mathrm{d}t=-\frac{1}{\gamma}i^{-N/2-1/2}+\mathrm{O}(i^{-N/2-1}).$$

Coupling the last two results, we obtain the approximation (5.28). □

We now return to (5.23), and define the successive approximants $h_0(n)\equiv 0$ and

$$h_{s+1}(n)=\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q(h_s(j+1),h_s(j),j), \tag{5.30}$$

for $s=0,1,\ldots$. In particular, we have

$$h_1(n)=\sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\theta^{-1}(i)\cdots\theta^{-1}(j)R_N^*(j), \tag{5.31}$$

where $R_N^*(j)$ is given in (5.11). From (5.4), it is readily seen that $R_N(n)$ has an asymptotic approximation of the form

$$R_N(n)=cn^{-N/2-3/2}+\mathrm{O}(n^{-N/2-2}),$$

where c is some constant whose exact value is immaterial for our purpose. Hence (5.11) gives

$$R_N^*(n)=-cn^{-N/2-3/2}+\mathrm{O}(n^{-N/2-2}). \tag{5.32}$$

Inserting (5.32) in (5.31), and applying Lemmas 1 and 2, we obtain

$$h_1(n) = O(n^{-N/2}),$$

where use has also been made of (3.16). Let M_0 be a positive number such that

$$|h_1(n)|,\ |h_1(n+1)| \leqslant M_0 n^{-N/2}.$$

In view of (5.14) and Lemma 1, it is possible to choose M so that

$$|R_a(n)| + |R_b(n)| \leqslant M n^{-2} \tag{5.33}$$

and

$$|\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)| \leqslant M\sqrt{\frac{j}{i}}, \quad j \geqslant i \geqslant 1.$$

A combination of (5.30), (5.20) and these estimates yields

$$|h_2(n) - h_1(n)| \leqslant M_0 M^2 \sum_{i=n}^{\infty} i^{-1/2} \sum_{j=i}^{\infty} j^{-N/2-3/2}.$$

By (3.16), we have

$$|h_2(n+1) - h_1(n+1)|,\ |h_2(n) - h_1(n)| \leqslant \frac{16 M_0 M^2}{N(N+1)} n^{-N/2},$$

if $n \geqslant n_0(N)$ for some positive integer $n_0(N)$. Using induction, it can be shown that

$$|h_{s+1}(n+1) - h_s(n+1)|,\ |h_{s+1}(n) - h_s(n)| \leqslant M_0 \left[\frac{16M^2}{N(N+1)}\right]^s n^{-N/2},$$

if $n \geqslant n_0(N)$, $s = 0, 1, 2, \ldots$. Now we choose N so that

$$\frac{16M^2}{N(N+1)} < 1.$$

The series

$$\epsilon_N(n) = \sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)] = \lim_{s\to\infty} h_{s+1}(n) \tag{5.34}$$

is then uniformly convergent in n, and its sum $\epsilon_N(n)$ satisfies (5.23) and (5.22).

6. Subnormal solutions: proof, II

When the constant γ in (5.1) has a positive real part, then the series in (5.28) is divergent, and hence (5.23) cannot be used to establish the existence of a solution to (5.21) satisfying (5.22). In this case, we shall use, instead, (5.24). Define the sequence $\{h_s(n)\}$ of successive approximations by $h_0(n) = 0$ and

$$h_{s+1}(n) = -\sum_{i=n}^{\infty} \sum_{j=1}^{i-1} \theta(i-1)\theta(i-2)\cdots\theta(j+1) q(h_s(j+1), h_s(j), j), \tag{6.1}$$

where it is understood that $\theta(i-1)\theta(i-2)\cdots\theta(j+1)=1$ when $j=i-1$. From (5.25), we have

$$\theta(i-1)\theta(i-2)\cdots\theta(j+1)=\mathrm{e}^{2\gamma(\sqrt{j+1}-\sqrt{i-1})}\sqrt{\frac{j+1}{i-1}}\left[1+\mathrm{O}(j^{-1/2})\right],\quad j\to\infty, \tag{6.2}$$

uniformly with respect to $i\geqslant j+2$. Consequently, it is possible to choose a constant $M>1$ such that

$$|\theta(i-1)\theta(i-2)\cdots\theta(j+1)|\leqslant M\,\mathrm{e}^{2\sigma(\sqrt{j}-\sqrt{i-1})}\sqrt{\frac{j}{i-1}},\quad i\geqslant j+1, \tag{6.3}$$

where $\sigma=\operatorname{Re}\gamma$,

$$|R_N^*(i)|\leqslant Mi^{-N/2-3/2}, \tag{6.4}$$

and (5.33) also holds; cf. (5.15). From (6.1) and (5.20), it then follows that

$$|h_1(n)|\leqslant M^2\sum_{i=n}^{\infty}\mathrm{e}^{-2\sigma\sqrt{i-1}}(i-1)^{-1/2}\sum_{j=1}^{i-1}\mathrm{e}^{2\sigma\sqrt{j}}j^{-N/2-1}. \tag{6.5}$$

Using the Euler–Maclaurin formula (5.29), it can be shown that

$$\sum_{j=1}^{i-1}\mathrm{e}^{2\sigma\sqrt{j}}j^{-N/2-1}\leqslant M_0\,\mathrm{e}^{2\sigma\sqrt{i-1}}(i-1)^{-N/2-1/2}, \tag{6.6}$$

for some constant $M_0>0$. Therefore

$$|h_1(n)|\leqslant M_0M^2\sum_{i=n}^{\infty}(i-1)^{-N/2-1}\leqslant\frac{4M_0M^2}{N}n^{-N/2}, \tag{6.7}$$

for sufficiently large n. By induction the same argument gives

$$|h_{s+1}(n)-h_s(n)|\leqslant\left(\frac{4}{N}M_0M^2\right)^{s+1}n^{-N/2},$$

for sufficiently large n. As before, we now choose $N>4M_0M^2$ so that the series

$$\epsilon_N(n)=\sum_{s=0}^{\infty}\left[h_{s+1}(n)-h_s(n)\right]=\lim_{s\to\infty}h_{s+1}(n)$$

converges uniformly in n, and its sum $\epsilon_N(n)$ satisfies (5.24) and (5.22).

The solutions established in this and the previous section depend on the positive integer N. Note that both ρ and α have the same values in the two subnormal solutions given in (1.7), whereas the values of γ are different (see (2.16)). Let γ_1 and γ_2 denote the two different values of γ. Then $\exp\{(\gamma_1-\gamma_2)\sqrt{n}\}\to 0$ as $n\to\infty$ if $\operatorname{Re}(\gamma_1-\gamma_2)<0$, and the limit of $\exp\{(\gamma_1-\gamma_2)\sqrt{n}\}$ does not exist as $n\to\infty$ if $\operatorname{Re}(\gamma_1-\gamma_2)=0$. Using these facts, one can prove that the solutions are independent of N, in a manner similar to that given at the end of Section 4.

7. The exceptional case: formal theory

In Section 2, we have shown that upon substituting (1.5) or (1.10) in (1.1) and equating coefficients of terms with like powers of n, we obtain

$$\sum_{j=0}^{s}\left\{\rho^2 2^{s-j}\binom{\alpha-j}{s-j}+\rho\sum_{i=j}^{s}\binom{\alpha-j}{i-j}a_{s-i}+b_{s-j}\right\}c_j=0. \tag{7.1}$$

Without loss of generality, we shall assume that $c_0=1$. When $s=0$, (7.1) reduces to the characteristic equation (1.3). Throughout this section, it will be assumed that ρ is the double root of (1.3), i.e., (2.16) holds, and that it satisfies (1.6). In view of these, (7.1) reduces to the trivial identity $0=0$ when $s=1$, and becomes the indicial equation (1.8) when $s=2$. The same argument shows that the coefficients of c_s and c_{s-1} in (7.1) are zero. Consequently, (7.1) becomes

$$c_{s-2}=-\frac{1}{q(\alpha-s+2)}\sum_{j=0}^{s-3}\left\{\rho^2 2^{s-j}\binom{\alpha-j}{s-j}+\rho\sum_{l=0}^{s-j}\binom{\alpha-j}{s-l-j}a_l+b_{s-j}\right\}c_j, \tag{7.2}$$

for $s=3,4,\ldots$, where $q(\alpha)$ is the indicial polynomial given by (1.8). In case (i), $q(\alpha)$ has two distinct zeros α_1 and α_2 with $\operatorname{Re}\alpha_2\geqslant\operatorname{Re}\alpha_1$ and $\alpha_2-\alpha_1\neq 1,2,\ldots$. Hence the denominator $q(\alpha-s+2)$ in (7.2) is not zero for all $s=3,4,\ldots$, and (1.10) yields two linearly independent formal series solutions to the difference equation (1.1).

We next consider case (ii), i.e., when $\alpha_2-\alpha_1$ is a positive integer, say p. The recursive formula (7.2) for the coefficients breaks down when α is replaced by α_2 and $s=p+2$. Hence the preceding argument by case (i) yields only one formal series solution, namely,

$$y_1(n)=\rho^n n^{\alpha_1}\sum_{s=0}^{\infty}\frac{c_s}{n^s}, \tag{7.3}$$

the coefficients c_s being determined by (7.2) with α replaced by α_1. In [5, p.213], Birkhoff presented the second solution in the form

$$y_2(n)=\rho^n\,\mathrm{e}^{\gamma\sqrt{n}}n^r[S_2(n)+(\log n)S_1(n)], \tag{7.4}$$

where γ, r are some unspecified constants, and

$$S_i(n)=\sum_{s=0}^{\infty}d_s^{(i)}n^{-s/2},\quad i=1,2; \tag{7.5}$$

see also [6, pp. 3,4]. Here we shall show that the second formal series solution in fact has the simple form given by (1.11) and (1.12), i.e.,

$$y_2(n)=z(n)+c(\log n)y_1(n), \tag{7.6}$$

and

$$z(n)=\rho^n n^{\alpha_2}\sum_{s=0}^{\infty}\frac{d_s}{n^s}, \tag{7.7}$$

c being a constant.

Let $\mathscr{L}$ denote the linear difference operator

$$\mathscr{L}\{y(n)\} = y(n+2) + a(n)y(n+1) + b(n)y(n). \tag{7.8}$$

To show that $y_2(n)$ formally satisfies (1.1), we shall make use of the expansion

$$\log(n+\mu) = \log n + \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{s}\left(\frac{\mu}{n}\right)^s, \quad \mu = 1, 2. \tag{7.9}$$

Substituting (7.6) and (7.7) in (1.1) gives

$$\begin{aligned}\mathscr{L}\{y_2(n)\} ={}& \mathscr{L}\{z(n)\} + c(\log n)\mathscr{L}\{y_1(n)\} \\ &+ c\rho^n \sum_{s=2}^{\infty}\left[\sum_{j=0}^{s-2}\left\{\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left[\rho^2 2^{s-j}\binom{\alpha_1-j}{l-j}\right.\right.\right. \\ &\left.\left.\left.+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]\right\}c_j\right]n^{\alpha_1-s},\end{aligned} \tag{7.10}$$

where the c_j are the coefficients of $y_1(n)$ in (7.3). In arriving at (7.10), use has also been made of the fact that $2\rho + a_0 = 0$, and hence that the coefficients of n^{α_1-1} and c_{s-1} are zero. By the argument used for case (i), we have

$$\begin{aligned}\mathscr{L}\{z(n)\} ={}& \rho^n \sum_{s=2}^{\infty}\left\{q(\alpha_2-s+2)d_{s-2} + \sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\binom{\alpha_2-j}{s-j}\right.\right. \\ &\left.\left.+\rho\sum_{l=0}^{s-j}\binom{\alpha_2-j}{s-l-j}a_l + b_{s-j}\right]d_j\right\}n^{\alpha_2-s},\end{aligned} \tag{7.11}$$

where, as usual, empty sums are understood to be zero; cf. (7.2). Note that $\mathscr{L}\{y_1(n)\} = 0$. Thus, inserting (7.11) in (7.10) and equating coefficients of n^{-s} to zero, we obtain the recursive formulas

$$d_{s-2} = \frac{-1}{q(\alpha_2-s+2)}\sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\binom{\alpha_2-j}{s-j} + \rho\sum_{l=0}^{s-j}\binom{\alpha_2-j}{s-l-j}a_l + b_{s-j}\right]d_j, \tag{7.12}$$

for $s = 3, 4, \ldots, p+1$, and

$$\begin{aligned}d_{s-2} ={}& \frac{-1}{q(\alpha_2-s+2)}\left\{\sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\binom{\alpha_2-j}{s-j} + \rho\sum_{l=0}^{s-j}\binom{\alpha_2-j}{s-l-j}a_l + b_{s-j}\right]d_j\right. \\ &+ c\sum_{j=0}^{s-p-2}\sum_{l=j}^{s-p-1}\frac{(-1)^{s-p+1-l}}{s-p-l}\left[\rho^2 2^{s-p-j}\binom{\alpha_1-j}{l-j}\right. \\ &\left.\left.+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\right\},\end{aligned} \tag{7.13}$$

for $s=p+3,\ p+4,\dots,$ where $p=\alpha_2-\alpha_1$ and

$$c=\frac{-1}{\rho(2\rho\alpha_1+a_1-\rho)}\sum_{j=0}^{p-1}\left[\rho^2 2^{p+2-j}\binom{\alpha_2-j}{p+2-j}+\rho\sum_{l=0}^{p+2-j}\binom{\alpha_2-j}{p+2-l-j}a_l\right.$$
$$\left.+b_{p+2-j}\right]d_j. \tag{7.14}$$

Note that since $q(\alpha)$ vanishes only at $\alpha=\alpha_1$ and $\alpha=\alpha_2$, the denominators in (7.12) and (7.13) are not zero. In arriving at (7.14), we have made use of the fact that $c_0=1$ and $q(\alpha_1)=0$. Also note that since the coefficient of d_p is $q(\alpha_1)=0$, d_p can be arbitrary constant. For convenience, one may choose $d_p=1$.

Finally we consider case (iii), in which $\alpha_1=\alpha_2$. Here, as in case (ii), equation (1.10) again gives only one formal series solution; cf. (7.3). To show that (1.13) and (1.14) provide a second formal series solution, we insert (7.11) in (7.10) with α_2 replaced by α_1-Q+2. Since $\mathscr{L}\{y_1(n)\}=0$, we obtain

$$\mathscr{L}\{y_2(n)\}=\rho^n\sum_{s=2}^{\infty}\left\{q(\alpha_1-Q-s+4)d_{s-2}\right.$$
$$+\sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\binom{\alpha_1-Q+2-j}{s-j}\right.$$
$$\left.\left.+\rho\sum_{l=0}^{s-j}\binom{\alpha_1-Q+2-j}{s-l-j}a_l+b_{s-j}\right]d_j\right\}n^{\alpha_1-Q+2-s}$$
$$+c\rho^n\sum_{s=2}^{\infty}\left\{\sum_{j=0}^{s-2}\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left[\rho^2 2^{s-j}\binom{\alpha_1-j}{l-j}\right.\right.$$
$$\left.\left.+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\right\}n^{\alpha_1-s}. \tag{7.15}$$

Note that α_1 is a repeated root of (1.8). Hence $2\rho\alpha_1+a_1-\rho=0$, and the coefficient of n^{α_1-2} in (7.15) is zero. Since the coefficient of n^{α_1-Q} in (7.15) is not zero, in order to have $\mathscr{L}\{y_2(n)\}=0$, the integer Q must be $\geqslant 3$. We shall choose Q to be the smallest integer $\geqslant 3$ such that

$$\sum_{j=0}^{Q-2}\sum_{l=j}^{Q-1}\frac{(-1)^{Q+1-l}}{Q-l}\left[\rho^2 2^{Q-j}\binom{\alpha_1-j}{l-j}+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\neq 0. \tag{7.16}$$

Equating coefficients of n^{α_1-Q} to zero now gives

$$c=-q(\alpha_1-Q+2)d_0$$
$$\times\left\{\sum_{j=0}^{Q-2}\sum_{l=j}^{Q-1}\frac{(-1)^{Q+1-l}}{Q-l}\left[\rho^2 2^{Q-j}\binom{\alpha_1-j}{l-j}+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\right\}^{-1} \tag{7.17}$$

and

$$d_{s-2}=\frac{-1}{q(\alpha_1-Q-s+4)}\left\{\sum_{j=0}^{s-3}\left[\rho^2 2^{s-j}\binom{\alpha_1-Q+2-j}{s-j}+\rho\sum_{l=0}^{s-j}\binom{\alpha_1-Q+2-j}{s-l-j}a_l\right.\right.$$
$$\left.+b_{s-j}\right]d_j$$
$$+c\sum_{j=0}^{s+Q-4}\sum_{l=j}^{s+Q-3}\frac{(-1)^{s+Q-1-l}}{s+Q-2-l}\left[\rho^2 2^{s+Q-2-j}\binom{\alpha_1-j}{l-j}\right.$$
$$\left.\left.+\rho\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\right\}, \tag{7.18}$$

for $s=3,4,\ldots$. Without loss of generality, one may take $d_0=1$.

8. The exceptional case: proof of expansion (1.10)

As in Section 5, by writing $y(n)=\rho^n z(n)$, it is easily seen that we may assume without loss of generality that $\rho=1$ or equivalently $a_0=-2$ and $b_0=1$; cf. (2.15). From (1.16), it follows that

$$a_1+b_1=0. \tag{8.1}$$

Following (3.2), we set

$$y(n)=L_N(n)+E_N(n), \tag{8.2}$$

with

$$L_N(n)=n^\alpha\sum_{s=0}^{N-1}c_s n^{-s}. \tag{8.3}$$

Using (7.2), it is easily verified that

$$L_N(n+2)+a(n)L_N(n+1)+b(n)L_N(n)=n^\alpha R_N(n), \tag{8.4}$$

where

$$R_N(n)=O(n^{-N-2}),\quad \text{as } n\to\infty; \tag{8.5}$$

cf. (3.5). Hence

$$E_N(n+2)+a(n)E_N(n+1)+b(n)E_N(n)=-n^\alpha R_N(n). \tag{8.6}$$

If we write

$$E_N(n)=n^\alpha\epsilon_N(n), \tag{8.7}$$

then $\epsilon_N(n)$ satisfies the equation

$$\epsilon_N(n+2)+a^*(n)\epsilon_N(n+1)+b^*(n)\epsilon_N(n)=R^*(n), \tag{8.8}$$

where

$$a^*(n) = \left(\frac{n+1}{n+2}\right)^{\alpha} a(n), \tag{8.9}$$

$$b^*(n) = \left(\frac{n}{n+2}\right)^{\alpha} b(n) \tag{8.10}$$

and

$$R_N^*(n) = -\left(\frac{n}{n+2}\right)^{\alpha} R_N(n). \tag{8.11}$$

Recall that we have assumed $\rho = 1$, or equivalently $a_0 = -2$ and $b_0 = 1$. Using (8.1), simple calculation shows that

$$a^*(n) = -2 + (2\alpha + a_1)n^{-1} + R_a(n), \tag{8.12}$$

$$b^*(n) = 1 - (2\alpha + a_1)n^{-1} + R_b(n), \tag{8.13}$$

where

$$R_a(n),\ R_b(n) = \mathrm{O}(n^{-2}), \quad \text{as } n \to \infty. \tag{8.14}$$

Furthermore,

$$R_N^*(n) = \mathrm{O}(n^{-N-2}), \quad \text{as } n \to \infty. \tag{8.15}$$

Inserting (8.12) and (8.13) in (8.8), we obtain

$$\begin{aligned} &\epsilon_N(n+2) + \left[-2 + (2\alpha + a_1)n^{-1}\right]\epsilon_N(n+1) + \left[1 - (2\alpha + a_1)n^{-1}\right]\epsilon_N(n) \\ &\quad = R_N^*(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n), \end{aligned} \tag{8.16}$$

which can be written as

$$\Delta\epsilon_N(n+1) - \theta(n)\ \Delta\epsilon_N(n) = q(\epsilon_N(n+1), \epsilon_N(n), n), \tag{8.17}$$

where

$$\Delta\epsilon_N(n) = \epsilon_N(n+1) - \epsilon_N(n), \tag{8.18}$$

$$\theta(n) = 1 - (2\alpha + a_1)n^{-1} \tag{8.19}$$

and

$$q(\epsilon_N(n+1), \epsilon_N(n), n) = R_N^*(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n). \tag{8.20}$$

To prove (1.10), it suffices to show that

$$\epsilon_N(n) = \mathrm{O}(n^{-N}), \quad \text{as } n \to \infty. \tag{8.21}$$

By considering (8.17) as a first-order linear nonhomogeneous equation one can formally derive the equation

$$\epsilon_N(n) = \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} \theta^{-1}(i)\theta^{-1}(i+1) \cdots \theta^{-1}(j) q(\epsilon_N(j+1), \epsilon_N(j), j). \tag{8.22}$$

It is easily seen that every solution of (8.22) is a solution of (8.17). To prove that (8.22) has a solution which satisfies (8.21), we first prove the following analogue of Lemma 1.

Lemma 3. *For positive integers $j > i \geqslant 1$, the function $\theta(n)$ in (8.19) satisfies*

$$\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j) = \left(\frac{j}{i}\right)^{2\alpha+a_1}\left[1+\mathrm{O}(i^{-1})\right], \quad \text{as } i\to\infty, \tag{8.23}$$

where the O-term is uniform with respect to j.

Proof. We first recall the well-known asymptotic approximations

$$\sum_{k=1}^{n-1} k^{-2} = \frac{\pi^2}{6} - \frac{1}{n} + \mathrm{O}(n^{-2}), \quad \text{as } n\to\infty,$$

and

$$\sum_{k=1}^{n-1} k^{-1} = \log n + C + \mathrm{O}(n^{-1}), \quad \text{as } n\to\infty,$$

where C is the Euler constant; cf. (5.26). Since $\log(1-x) = -x + \mathrm{O}(x^2)$ as $x\to 0$, we have from (8.19),

$$\log\theta(k) = -(2\alpha+a_1)k^{-1} + \mathrm{O}(k^{-2}), \quad \text{as } k\to\infty.$$

Upon summation, we obtain

$$-\sum_{k=i}^{j}\log\theta(k) = (2\alpha+a_1)\log\left(\frac{j}{i}\right) + \mathrm{O}(i^{-1}), \quad \text{as } i\to\infty,$$

uniformly for $j > i$. The result in (8.23) now follows by exponentiation. □

We now return to (8.22), and define the successive approximants $h_0(n) = 0$ and

$$h_{s+1}(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\theta^{-1}(i)\theta^{-1}(i+1)\cdots\theta^{-1}(j)q(h_s(j+1), h_s(j), j), \tag{8.24}$$

for $s = 0, 1, 2, \ldots$. When $s = 0$, we have

$$h_1(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}\theta^{-1}(i)\cdots\theta^{-1}(j)R_N^*(j), \tag{8.25}$$

where $R_N^*(j)$ is given in (8.11). From (8.15) and (8.23), it follows that there is a positive constant M_1 and a positive integer n_1 such that

$$|\theta^{-1}(i)\cdots\theta^{-1}(j)| \leqslant M_1\left(\frac{j}{i}\right)^{\mathrm{Re}(2\alpha+a_1)} \tag{8.26}$$

and

$$|R_N^*(j)| \leqslant M_1 j^{-N-2}, \tag{8.27}$$

for $j \geqslant i \geqslant n_1$. Consequently,

$$|h_1(n)| \leqslant M_1^2\sum_{i=n}^{\infty}\sum_{j=i}^{\infty} i^{-(2\alpha+a_1)} j^{-N-2+\mathrm{Re}(2\alpha+a_1)}, \tag{8.28}$$

for all $n \geqslant n_1$. Let $N+1 > \mathrm{Re}(2\alpha + a_1)$ and put $M_0 = 4M_1^2$. Choose $n_0 \geqslant n_1$ so that two applications of (3.16) to (8.28) give

$$|h_1(n+1)|, |h_1(n)| \leqslant \frac{M_0}{N[N+1-\mathrm{Re}(2\alpha+a_1)]} n^{-N},$$

for all $n \geqslant n_0$. In view of (8.14), the constant M_1 and the integer n_1 in (8.26) and (8.27) may be chosen so that

$$|R_a(j)| + |R_b(j)| \leqslant M_1 j^{-2}, \quad \text{for all } j \geqslant n_1.$$

Induction then shows that

$$|h_{s+1}(n+1) - h_s(n)| \leqslant \left\{ \frac{M_0}{N[N+1-\mathrm{Re}(2\alpha+a_1)]} \right\}^{s+1} n^{-N},$$

for $s = 1, 2, \ldots$, and for all $n \geqslant n_0$. As long as $N[N+1-\mathrm{Re}(2\alpha+a_1)] > M_0$, the series

$$\epsilon_N(n) = \sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)] = \lim_{s \to \infty} h_{s+1}(n)$$

is uniformly convergent in n; its sum $\epsilon_N(n)$ satisfies (8.22) and (8.21).

For each zero α of the indicial polynomial (1.8), the above construction provides a solution to (8.22) and hence a solution to (8.17). Since in the present case (1.8) has two distinct zeros α_1 and α_2, this establishes the existence of two asymptotic solutions to (1.1), both of the form (1.10). That the solutions are independent of N can be established in the same manner as that given at the end of Section 4.

9. The exceptional case: proof of (1.11), (1.12)

Let $y_1(n)$ be the solution of (1.1) which has the asymptotic expansion (7.3). As in the previous case, we may assume without loss of generality that $\rho = 1$, or equivalently $a_0 = -2$ and $b_0 = 1$. Set

$$y_2(n) = \sum_{s=0}^{N-1} d_s n^{\alpha_2 - s} + E_N(n) + c(\log n) y_1(n), \tag{9.1}$$

where the constant c and the coefficients d_s are given by (7.12)–(7.14); cf. (1.11) and (1.12). Applying the difference operator defined in (7.8) to both sides of (9.1), we obtain

$$\mathscr{L}\{y_2(n)\} = \mathscr{L}\left\{ \sum_{s=0}^{N-1} d_s n^{\alpha_2 - s} \right\} + \mathscr{L}\{E_N(n)\} + c\mathscr{L}\{(\log n) y_1(n)\}. \tag{9.2}$$

Straightforward but tedious computations show that

$$\mathscr{L}\left\{ \sum_{s=0}^{N-1} d_s n^{\alpha_2 - s} \right\} = \sum_{s=2}^{N+1} \left\{ \sum_{j=0}^{s-2} \left[2^{s-j} \binom{\alpha_2 - j}{s-j} + \sum_{l=0}^{s-j} \binom{\alpha_2 - j}{s-l-j} a_l + b_{s-j} \right] d_j \right\} n^{\alpha_2 - s}$$
$$+ O(n^{\alpha_2 - N - 2}); \tag{9.3}$$

see (7.11). It also gives

$$\mathscr{L}\{(\log n)y_1(n)\}$$

$$= \sum_{s=2}^{N+1-\alpha_2+\alpha_1}\left\{\sum_{j=0}^{s-2}\left[\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left(2^{s-l}\binom{\alpha_1-j}{l-j}+\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right)\right]c_j\right\}n^{\alpha_1-s}$$

$$+\mathrm{O}(n^{\alpha_2-N-2}), \tag{9.4}$$

for $N \geqslant \alpha_2-\alpha_1+1$; see (7.10). Upon substitution of (9.4) and (9.3) in (9.2), it follows that $y_2(n)$ is a solution of (1.1) if and only if $E_N(n)$ satisfies the equation

$$\mathscr{L}\{E_N(n)\}$$

$$= -\sum_{s=2}^{N+1}\left\{\sum_{j=0}^{s-2}\left[2^{s-j}\binom{\alpha_2-j}{s-j}+\sum_{l=0}^{s-j}\binom{\alpha_2-j}{s-l-j}a_l+b_{s-j}\right]d_j\right\}n^{\alpha_2-s}$$

$$-c\sum_{s=2}^{N+1-\alpha_2+\alpha_1}\left\{\sum_{j=0}^{s-2}\left[\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left(2^{s-j}\binom{\alpha_1-j}{l-j}+\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right)\right]c_j\right\}n^{\alpha_1-s}$$

$$+\mathrm{O}(n^{\alpha_2-N-2}). \tag{9.5}$$

In view of the recurrence equations for d_s given in (7.12)–(7.14) with $\rho=1$, the above two series over s cancel out, and (9.5) reduces to

$$E_N(n+2)+a(n)E_N(n+1)+b(n)E_N(n)=\mathrm{O}(n^{\alpha_2-N-2}), \tag{9.6}$$

which is exactly of the form (8.6) with $\alpha=\alpha_2$. Hence the existence of a solution to (9.6) satisfying

$$E_N(n)=\mathrm{O}(n^{\alpha_2-N}) \tag{9.7}$$

is guaranteed by the construction provided in Section 8. That the solution $y_2(n)$ to (1.1) is independent of N can be demonstrated in a manner similar to that given in Section 4.

10. The exceptional case: proof of (1.13), (1.14)

The analysis in this case proceeds in a manner similar to that given in Section 9. Instead of (9.1), we now set

$$y_2(n)=\sum_{s=0}^{N-1}d_sn^{\alpha_1-Q+2-s}+E_N(n)+c(\log n)y_1(n), \tag{10.1}$$

where the exponent Q is determined by (7.16), and where the constant c and the coefficients d_s are given by (7.17) and (7.18). Applying the difference operator $\mathscr{L}$ to both sides of (10.1) gives

$$\mathscr{L}\{y_2(n)\}=\mathscr{L}\left\{\sum_{s=0}^{N-1}d_sn^{\alpha_1-Q+2-s}\right\}+\mathscr{L}\{E_N(n)\}+c\mathscr{L}\{(\log n)y_1(n)\}. \tag{10.2}$$

By straightforward calculation, we obtain, as in (9.3) and (9.4),

$$\mathscr{L}\left\{\sum_{s=0}^{N-1} d_s n^{\alpha_1-Q+2-s}\right\}$$
$$= \sum_{s=2}^{N+1}\left\{\sum_{j=0}^{s-2}\left[2^{s-j}\binom{\alpha_1-Q+2-j}{s-j}+\sum_{l=0}^{s-j}\binom{\alpha_1-Q+2-j}{s-l-j}a_l+b_{s-j}\right]d_j\right\}n^{\alpha_1-Q+2-s}$$
$$+\,\mathrm{O}(n^{\alpha_1-Q-N}) \tag{10.3}$$

and

$$\mathscr{L}\{(\log n)y_1(n)\}$$
$$= \sum_{s=Q}^{N+Q-1}\left\{\sum_{j=0}^{s-1}\left[\sum_{l=j}^{s-1}\frac{(-1)^{s+1-l}}{s-l}\left(2^{s-j}\binom{\alpha_1-j}{l-j}+\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right)\right]c_j\right\}n^{\alpha_1-s}$$
$$+\,\mathrm{O}(n^{\alpha_1-Q-N}). \tag{10.4}$$

Since $a_0=-2$ (see the second sentence in Section 9), the coefficient of c_{s-1} in (10.4) is zero. Thus, upon re-indexing the summation over s, (10.4) becomes

$$\mathscr{L}\{(\log n)y_1(n)\}$$
$$= \sum_{s=2}^{N+1}\left\{\sum_{j=0}^{s+Q-4}\sum_{l=j}^{s+Q-3}\frac{(-1)^{s+Q-1-l}}{s+Q-2-l}\left[2^{s+Q-2-j}\binom{\alpha_1-j}{l-j}\right.\right.$$
$$\left.\left.+\sum_{k=j}^{l}\binom{\alpha_1-j}{k-j}a_{l-k}\right]c_j\right\}n^{\alpha_1-Q+2-s}$$
$$+\,\mathrm{O}(n^{\alpha_1-Q-N}). \tag{10.5}$$

Inserting (10.3) and (10.5) in (10.2), and making use of the recurrence equation (7.18) with $\rho=1$, we obtain

$$\mathscr{L}\{y_2(n)\}=\mathscr{L}\{E_N(n)\}+\mathrm{O}(n^{\alpha_1-Q-N}).$$

Consequently, for $y_2(n)$ to be a solution of (1.1), it is necessary and sufficient that $E_N(n)$ satisfies

$$E_N(n+2)+a(n)E_N(n+1)+b(n)E_N(n)=\mathrm{O}(n^{\alpha_1-Q-N}). \tag{10.6}$$

A solution to (10.6) satisfying

$$E_N(n)=\mathrm{O}(n^{\alpha_1-Q+2-N}) \tag{10.7}$$

can be constructed again by the method given in Section 8. Again, the independence of N of the solution $y_2(n)$ to (1.1) can be established by the method of Section 4.

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References

[1] C.R. Adams, On the irregular cases of linear ordinary difference equations, *Trans. Amer. Math. Soc.* **30** (1928) 507–541.

[2] E.A. Bender, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974) 485–515.

[3] C.M. Bender and S.A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978).

[4] G.D. Birkhoff, General theory of linear difference equations, *Trans. Amer. Math. Soc.* **12** (1911) 243–284.

[5] G.D. Birkhoff, Formal theory of irregular linear difference equations, *Acta Math.* **54** (1930) 205–246.

[6] G.D. Birkhoff and W.J. Trjitzinsky, Analytic theory of singular difference equations, *Acta Math.* **60** (1932) 1–89.

[7] A. Erdélyi, *Asymptotic Expansions* (Dover, New York, 1956).

[8] C. Hunter, Asymptotic solutions of certain linear difference equations, with applications to some eigenvalue problems, *J. Math. Anal. Appl.* **24** (1968) 279–289.

[9] G.K. Immink, *Asymptotics of Analytic Difference Equations*, Lecture Notes in Math. **1085** (Springer, Berlin, 1984).

[10] E.L. Ince, *Ordinary Differential Equations* (Longmans and Green, London, 1927); reprinted by (Dover, New York, 1956).

[11] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics* (Springer, Berlin, 1966).

[12] F.W.J. Olver, *Asymptotics and Special Functions* (Academic Press, New York, 1974).

[13] F.G. Tricomi, *Equazioni Differenziali* (Torino, 2nd ed., 1953).

[14] J. Wimp, *Computation with Recurrence Relations* (Pitman, London, 1983).

[15] J. Wimp, Review of R. Wong's book 'Asymptotic Approximations of Integrals', *Math. Comp.* **56** (1991) 388–394.

[16] J. Wimp and D. Zeilberger, Resurrecting the asymptotics of linear recurrences, *J. Math. Anal. Appl.* **111** (1985) 162–176.

[17] R. Wong and H. Li, Asymptotic expansions for second-order linear difference equations, II, *Stud. Appl. Math.*, to appear.