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Controlled invariance for DAEs

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We study the concept of locally controlled invariant submanifolds for nonlinear descriptor systems. In contrast to classical approaches, we define controlled invariance as the property of solution trajectories to evolve in a given submanifold whenever they start in it. It is then shown that this concept is equivalent to the existence of a feedback which renders the closed-loop vector field invariant in the descriptor sense. This result is motivated by a preliminary consideration of the linear case.

Local controlled invariance leads to the concept of output zeroing submanifolds. We show that the outcome of the differential-algebraic version of the zero dynamics algorithm yields a locally maximal output zeroing submanifold.

1 Motivation - linear systems

We study controlled invariance for linear descriptor systems governed by differential-algebraic equations (DAEs),

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$. The set of these systems is denoted by $\Sigma_{l,n,m}$ and we write $[E, A, B] \in \Sigma_{l,n,m}$. Note that we do not assume regularity of the pencil $sE - A$. The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $x : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *state* of the system, resp. The *behavior* of (1.1) is the set

$$\mathfrak{B}_{(1.1)} := \{\, (x, u) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m) \mid Ex \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^l) \text{ and } (x, u) \text{ satisfies (1.1) for all } t \in \mathbb{R} \,\}.$$

Definition 1.1 Let $[E, A, B] \in \Sigma_{l,n,m}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace. Then $\mathcal{V}$ is called *controlled invariant*, if

$$\forall\, x^0 \in \mathcal{V}\ \exists\, (x, u) \in \mathfrak{B}_{(1.1)}\ \forall\, t \ge 0 : \quad x \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n)\ \wedge\ x(0) = x^0\ \wedge\ x(t) \in \mathcal{V}.$$

For ODEs, characterizations of controlled invariance can be found e.g. in [1]; the following is the DAE version.

Theorem 1.2 *For $[E, A, B] \in \Sigma_{l,n,m}$ and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ the following statements are equivalent:*

(i) *$\mathcal{V}$ is controlled invariant.*
(ii) $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$.
(iii) *There exists $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)\mathcal{V} \subseteq E\mathcal{V}$.*

For the proofs and more details on the results in the present paper see [2]. Note that a subspace $\mathcal{V}$ satisfying property (ii) in Theorem 1.2 is usually called a (A, E, B)-invariant subspace, see the survey [3] and the references therein.

2 Nonlinear systems

In this section we consider nonlinear descriptor systems governed by DAEs of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t)) + g(x(t))\,u(t),\ \ y(t) = h(x(t)), \tag{2.1}$$

where $X \subseteq \mathbb{R}^n$ is open, $0 \in X$, $f \in \mathcal{C}(X; \mathbb{R}^l)$, $h \in \mathcal{C}(X; \mathbb{R}^p)$, $E \in \mathcal{C}^1(X; \mathbb{R}^l)$ such that $f(0) = 0$, $h(0) = 0$, and $g \in \mathcal{C}(X; \mathbb{R}^{l\times m})$. The set of these systems is denoted by $\Sigma^X_{l,n,m,p}$; and we write $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$.

A trajectory $(x, u, y) \in \mathcal{C}(I; X \times \mathbb{R}^m \times \mathbb{R}^p)$ is called a *solution* of (2.1), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $E \circ x \in \mathcal{C}^1(I; \mathbb{R}^l)$ and (x, u, y) solves (2.1) for all $t \in I$. A solution (x, u, y) of (2.1) is called *maximal*, if any other solution $(\tilde{x}, \tilde{u}, \tilde{y})$ of (2.1) satisfies

$$J := \operatorname{dom} \tilde{x} \cap \operatorname{dom} x \neq \emptyset \wedge \tilde{x}|_J = x|_J \ \Rightarrow\ \operatorname{dom} \tilde{x} \subseteq \operatorname{dom} x.$$

The *behavior* of (2.1) is the set of maximal solutions

$$\mathfrak{B}_{(2.1)} := \{\, (x, u, y) \in \mathcal{C}(I; X \times \mathbb{R}^m \times \mathbb{R}^p) \mid I \subseteq \mathbb{R} \text{ open interval, } (x, u, y) \text{ is maximal solution of (2.1)} \,\}.$$

The concept of (locally) controlled invariant submanifolds has been introduced by Isidori and Moog [4], see also the textbooks [5, 6]. Loosely speaking, a connected submanifold M is locally controlled invariant, if it is invariant under the flow of the closed-loop vector field $f(x)+g(x)u(x)$ for some feedback $u(x)$. We show that this "classical" definition in terms of feedback is equivalent to the "natural" definition, that (locally) for any initial value in M there exists an input such that the corresponding state trajectory remains in the submanifold M for all times or reaches its boundary in finite time.

Definition 2.1 Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *locally controlled invariant*, if there exists an open neighborhood $U \subseteq X$ of the origin in $\mathbb{R}^n$ such that

$$\begin{aligned} &\forall\, x^0 \in M \cap U\ \exists\, (x, u, y) \in \mathfrak{B}_{(2.1)},\ x \in \mathcal{C}^1(\operatorname{dom} x; \mathbb{R}^n)\\ &\exists\, t_0 \in \operatorname{dom} x,\ x(t_0) = x^0 :\\ &\quad \big(\forall\, t \in \operatorname{dom} x,\ t \ge t_0 :\ x(t) \in M \cap U\big)\ \vee\ \big(\exists\, \hat{t} \in \operatorname{dom} x,\\ &\quad\ \hat{t} > t_0\ \forall\, t \in [t_0, \hat{t}) :\ x(t) \in M \cap U \wedge x(\hat{t}) \in \partial(M \cap U)\big). \end{aligned}$$

Theorem 2.2 *Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ be such that $E \in \mathcal{C}^2(X; \mathbb{R}^l)$, $f \in \mathcal{C}^1(X; \mathbb{R}^l)$ and $g \in \mathcal{C}^1(X; \mathbb{R}^{l\times m})$ and let M be a smooth connected submanifold of X such that $0 \in M$. Suppose that there exists an open neighborhood V of $0 \in X$ such that both $\dim E'(x)T_xM$ and $\dim \big(E'(x)T_xM + \operatorname{im} g(x)\big)$ are constant for $x \in M \cap V$. Then the following statements are equivalent:*

(i) *M is locally controlled invariant.*

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(ii) *There exists an open neighborhood* U *of* $0 \in X$ *such that* $f(x) \in E'(x)T_xM + \operatorname{im} g(x)$ *for all* $x \in M \cap U$.

(iii) *There exists an open neighborhood* U *of* $0 \in X$ *and* $u \in \mathcal{C}^1(M \cap U; \mathbb{R}^m)$ *such that* $f(x) + g(x)u(x) \in E'(x)T_xM$ *for all* $x \in M \cap U$.

In the remainder of this paper we consider the zero dynamics of (2.1), which is the set of trajectories $\mathcal{ZD}_{(2.1)} := \{ (x, u, y) \in \mathfrak{B}_{(2.1)} \mid y = 0 \}$. The concept of zero dynamics goes back to Byrnes and Isidori [7] and is studied extensively since then, see e.g. [5, 6]. For linear DAEs, the zero dynamics have been investigated in detail recently [8–11]. Zero dynamics are related to the concept of output zeroing submanifolds.

Definition 2.3 Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *output zeroing*, if M is locally controlled invariant and $h(x) = 0$ for all $x \in M$. An output zeroing submanifold M that is called *locally maximal*, if there exists an open neighborhood U of $0 \in X$ such that any output zeroing submanifold $\tilde{M}$ satisfies $\tilde{M} \cap U \subseteq M \cap U$.

We extend the zero dynamics algorithm developed in [4, 12] to nonlinear DAE systems (2.1).

Theorem 2.4 *Let* $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ *be such that* E, f, g *and* h *are smooth. Define* $M_0 := h^{-1}(0)$ *and for any* $k \in \mathbb{N}$ *the set* M_k *recursively as follows: Suppose that for some open neighborhood* U_{k-1} *of* $0 \in X$, $M_{k-1} \cap U_{k-1}$ *is a submanifold, define* $\tilde{M}_{k-1} := \bigcup \{ M_{k-1} \cap U \mid U \subseteq X \text{ open}, M_{k-1} \cap U \text{ is a submanifold} \}$, *let* M^c_{k-1} *be the connected component of* $\tilde{M}_{k-1}$ *which contains* $0 \in X$ *and define* $M_k := \left\{ x \in M^c_{k-1} \;\middle|\; f(x) \in E'(x)T_xM^c_{k-1} + \operatorname{im} g(x) \right\}$.
Then we have the following:

(i) *The sequence* (M_k) *is nested, terminates and satisfies*

$$\exists\, k^* \in \mathbb{N}_0\ \forall\, j \in \mathbb{N}:\ M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{k^*} \supseteq M^c_{k^*} = M_{k^*+j} = M^c_{k^*+j}.$$

(ii) *If* $Z^* := M^c_{k^*}$ *satisfies, for some open neighborhood* U *of* $0 \in \mathbb{R}$, *that* $\dim E'(x)T_xZ^*$ *and* $\dim \big(E'(x)T_xZ^* + \operatorname{im} g(x)\big)$ *are both constant for* $x \in Z^* \cap U$, *then* Z^* *is a locally maximal output zeroing submanifold.*

(iii) *There exists an open neighborhood* U *of* $0 \in X$ *such that for all open* $O \subseteq U$ *and all* $(x, u, y) \in \mathfrak{B}_{(2.1)}$ *with* $x \in \mathcal{C}^1(\operatorname{dom} x; X)$ *and* $x(t) \in O$ *for all* $t \in \operatorname{dom} x$

$$(x, u, y) \in \mathcal{ZD}_{(2.1)} \iff x(t) \in Z^* \cap O\ \ \forall\, t \in \operatorname{dom} x.$$

If the system (2.1) is linear, then the sequence (M_k) becomes an augmented Wong sequence, see [3, 13] and the references therein, which is based on the Wong sequences [14–16] and which have their origin in [17].

Output zeroing submanifolds can be exploited to study locally autonomous zero dynamics; the latter have been successively used for the analysis of linear time-varying ODEs in [18] and of linear time-invariant DAEs in [9]. Under the assumption of locally autonomous zero dynamics we aim to derive a local zero dynamics form for nonlinear DAE systems (2.1) which would provide the basis for the application of adaptive control techniques. In particular, we aim to use the results of [19] and show feasibility of funnel control for nonlinear descriptor systems which encompass nonlinear electrical circuits, extending the results for the linear case [20].

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