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System of fractional differential algebraic equations with applications

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ABSTRACT

One of the important classes of coupled systems of algebraic, differential and fractional differential equations (CSADFDEs) is fractional differential algebraic equations (FDAEs). The main difference of such systems with other class of CSADFDEs is that their singularity remains constant in an interval. However, complete classifying and analyzing of these systems relay mainly to the concept of the index which we introduce in this paper. For a system of linear differential algebraic equations (DAEs) with constant coefficients, we observe that the solvability depends on the regularity of the corresponding pencils. However, we show that in general, similar properties of DAEs do not hold for FDAEs. In this paper, we introduce some practical applications of systems of FDAEs in physics such as a simple pendulum in a Newtonian fluid and electrical circuit containing a new practical element namely fractors. We obtain the index of introduced systems and discuss the solvability of these systems. We numerically solve the FDAEs of a pendulum in a fluid with three different fractional derivatives (Liouville–Caputo's definition, Caputo–Fabrizio's definition and with a definition with Mittag–Leffler kernel) and compare the effect of different fractional derivatives in this modeling. Finally, we solved some existing examples in research and showed the effectiveness and efficiency of the proposed numerical method.

1. Introduction

Many interesting natural phenomena have been described by dynamical systems, involving algebraic and differential equations [1,2]. In the recent century, developments in fractional calculus have been introduced fractional derivative as a very important tool in the modeling of natural phenomena [3–6]. By the appearance of many application of fractional calculus in physics and almost all branches of science the appearing of complex dynamical systems included both fractional derivatives and algebraic equations is inevitable. In this paper, we study such complex systems which are known as fractional differential algebraic equations (FDAEs).

Almost all of the physical processes have a non–conservative feature since they involve irreversible dissipating effects such as friction. As a result of these dissipating effects, time–reversal symmetry fails for non–conservative systems. Hence, the experimental results do not comply with standard theoretical calculations. Their dynamic memory and hereditary effects. Therefore, the Caputo's fractional derivatives of the form

$$\mathfrak{D}^{\alpha} y = \int_0^t k_\alpha(t-\tau) y'(\tau) d\tau, \quad 0 < \alpha < 1 \tag{1}$$

is an appropriate tool for modeling the dynamic behavior of these systems. There are well–known kernels that have been studied in a broad range of researches. These kernels can somehow categorize into following classes:

- Singular (Liouville–Caputo's definition, see [7])

$$k_\alpha(t-\tau) = \frac{1}{\Gamma(1-\alpha)(t-\tau)^{\alpha}}. \tag{2}$$

- Nonsingular and local (Caputo–Fabrizio's definition, see [8])

$$k_\alpha(t-\tau) = \frac{1}{1-\alpha} \exp \frac{-\alpha}{1-\alpha}(t-\tau). \tag{3}$$

- Nonsingular and nonlocal (Mittag–Leffler kernel, see [9])

$$k_\alpha(t-\tau) = \frac{B(\alpha)}{1-\alpha} E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-\tau)^{\alpha}\right). \tag{4}$$

The recent application and properties of fractional derivatives with these kernels can be found in [10–30].

Constraints are the sources that impose algebraic parts in a dynamical system. For example, in a pendulum, the length of rod remain constant. Therefore, the modeling of such dynamical systems

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involves the coupled system of algebraic, differential and fractional differential equations. The coupled systems of algebraic and differential equations known as differential algebraic equations (DAEs) have been received much attention in the recent three decades. The coupled systems of algebraic and differential equations can be represented in the form

$$F(t, \mathbf{y}, \mathfrak{D}\mathbf{y}) = O, \tag{5}$$

where $\mathbf{y}$ is an unknown vector function and F is given multi–variable vector function. O is a zero vector function. The algebraic part appears when

$$\det\left(\frac{\partial}{\partial \mathfrak{D}\mathbf{y}}F\right) \equiv O,$$

on the interval [0, T], $T \in \mathbb{R}$. We note that if these singularities occur in discrete points, then the System (5) is not a system of DAEs. There are many books and papers that comprehensively introduce these systems [1,2,31].

Simultaneously, a good deal of progress and development in fractional calculus has been done in the last three decades [3,4,6,32,33]. However, there is an only ad–hoc study for the system of FDAEs [34–38]. These systems can be described by the following formula:

$$F(t, \mathbf{y}, \mathfrak{D}\mathbf{y}, \mathfrak{D}^{\alpha_1}, \dots, \mathfrak{D}^{\alpha_m}) = O \tag{6}$$

where $\mathbf{y}$ is an unknown vector function and F is given multi–variable vector function and $m \in \mathbb{N}$. The algebraic part appears when

$$\det\left(\frac{\partial}{\partial \mathfrak{D}\mathbf{y}}F\right) \equiv O,$$

and

$$\det\left(\frac{\partial}{\partial \mathfrak{D}^{\alpha_i}\mathbf{y}}F\right) \equiv O, \quad \text{for} \quad i = 1, \dots, m,$$

on the interval [0, T], $T \in \mathbb{R}$. An interesting class of the system of FDAEs is the following semi–implicit system of the FDAEs:

$$\begin{aligned} &\mathfrak{D}^{\alpha_i} y_i = F_i(t, \mathbf{y}, \mathfrak{D}\mathbf{y}), \quad i = 1, \dots, m, \\ &G(t, \mathbf{y}) = O, \end{aligned} \tag{7}$$

where $\mathbf{y} = [y_1, \dots, y_m]^T$, G is given multi–variable vector function and F_i for $i = 1, \dots, m$ are multi–variable functions.

The most studies on the topic of FDAEs are generally concentrated on the numerical solution of the System (7) and there are not any generalizations of the index concept for these systems. Also, there is not any solvability analysis similar to DAEs for this concept. Despite the investigation of many systems of FDAEs in researches, the applied model have not studied mostly.

The aim of this paper is to generalize the index concept for the systems of FDAEs and study the solvability of these systems. First, we investigate the linear case, and then we discuss the nonlinear case. We observe many difficulties for this generalization even for linear time–varying fractional differential algebraic equations due to the fact that Leibniz formula and chain rule do not have the similar patterns of ordinary derivatives. Nevertheless, we show that the linear fractional differential algebraic equations with constant coefficients have similar properties of solvability. Another aim of this paper is to provide important applicable examples of the systems of FDAEs. As far as we know, this is the first time that the dynamic system of a simple pendulum immersed in a Newtonian fluid is obtained by a fractional model using the principle of least action. Furthermore, besides the existing theoretical evidence of the fractor elements, we consider it as separate element in electrical circuits, for the first time. We proposed a simple and efficient numerical method for solving systems of FDAEs by means of all fractional derivative that we mentioned in this paper. We compared the result of applying different fractional derivatives in a model of simple pendulum immersed in a Newtonian fluid.

The structure of the paper is as follows: In Section 2, we give some preliminaries about coupled system of algebraic, differential and fractional differential equations and we recall the chain rule and Leibniz formula. In Section 3, the solvability of the introduced system are reviewed and investigated. In Section 4, we discuss about the solvability of fractional differential algebraic equations and the generalization if index concept for this systems. In Section 5, we introduce models of simple pendulum and RF circuit which lead to systems of fractional differential equations of index 1 to 3, and we apply the solvability conditions for these systems. In Section 6, we introduce a numerical method for solving some class of FDAEs with different definitions of fractional derivatives. By solving some existence examples of FDAEs in the researches, we show the introduced method is efficient.

2. Preliminaries

2.1. Some linear systems

Here, we suppose that $\mathbb{I} = [0, T]$ is the interval in which the solution will be sought and $T > 0$ is a real number. First, we can categorize the linear systems involving algebraic, ordinary or fractional derivatives into four parts.

- System of algebraic equations:

$$A\mathbf{y}(t) + \mathbf{f}(t) = 0, \quad t \in \mathbb{I}. \tag{8}$$

- System of ordinary differential equations:

$$\mathbf{y}'(t) + A\mathbf{y}(t) + \mathbf{f}(t) = 0, \quad t \in \mathbb{I}. \tag{9}$$

- System of fractional differential equations (FDEs):

$$\mathfrak{D}^{\alpha}\mathbf{y}(t) + A\mathbf{y}(t) + \mathbf{f}(t) = 0, \;\; 0 < \alpha < 1, \quad t \in \mathbb{I}. \tag{10}$$

- System of fractional differential algebraic equations (FDAEs):

$$A\mathfrak{D}^{\alpha}\mathbf{y}(t) + B\mathbf{y}(t) + \mathbf{f}(t) = 0, \;\; 0 < \alpha < 1, \quad t \in \mathbb{I}. \tag{11}$$

Here, A and B are matrices of dimension $\nu \times \nu$, and $\mathbf{f}$ is a column vector function of dimension ν. The solvability of these problems with initial condition

$$\mathbf{y}(0) = \mathbf{y}_0$$

are studied in the next section.

2.2. Chain rule and Leibniz formula

We might note that the chain rule and Leibniz formula for Liouville–Caputo's definition of fractional derivatives does not hold similar to non–fractional derivatives. Actually, it contains the sum of the infinite number of fractional terms. Here, we briefly recall these formulas:

Theorem 1 (Leibniz formula)**.** *[33] Let* $0 < \alpha < 1$, *and assume that* q *and* x *are analytic on* $(-h, h)$. *Then,*

$$\begin{aligned} \mathfrak{D}^{\alpha} f(t)g(t) = &\frac{t^{-\alpha}}{\Gamma(1-\alpha)}(f(t) - f(0))g(t) + f(t)\mathfrak{D}^{\alpha}g(t) \\ &+ \sum_{k=1}^{\infty}\binom{\alpha}{k}\mathfrak{D}^{k} f(t) I^{k-\alpha} g(t). \end{aligned} \tag{12}$$

A generalizations of these formulas by Mittag–Leffler kernel, has been done in [39].

Theorem 2 (Chain rule)**.** *[33,40] Let* $0 < \alpha < 1$, *and assume that* f *and* g *are analytic on* $(-h, h)$. *Then,*

$$\mathfrak{D}^\alpha fog(t) = \sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k!t^{k-\alpha}}{\Gamma(k-\alpha+1)} \sum_{m=1}^{k}(\mathfrak{D}^m f(t))(g(t))\sum\prod_{r=1}^{k}\frac{1}{a_r!}\left(\frac{\mathfrak{D}^r g(t)}{r!}\right)^{a_r} + \frac{t^{-\alpha}}{\Gamma(1-\alpha)}(fog(t)-fog(0)), \quad (13)$$

where the third sum Σ extends over all combinations of non–negative integer values of $a_1, \dots, a_k$ such that $\sum_{r=1}^{k} ra_r = k$ and $\sum_{r=1}^{k} a_r = m$.

2.3. Fractional integral operator

For each definition of Caputo's fractional derivatives (15)–(17), there exists a corresponding integral operator $\mathfrak{I}^\alpha$ such that

$$\mathfrak{I}^\alpha\mathfrak{D}^\alpha f(t) = f(t) - f(0). \quad (14)$$

- Integral operator for Liouville–Caputo's definition, see [3]

$$\mathfrak{I}^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau. \quad (15)$$

- Integral operator corresponding to Mittag–Leffler kernel, see [41]

$$\mathfrak{I}^\alpha f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{t_0}^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau. \quad (16)$$

- Integral operator for Caputo–Fabrizio's definition, see [11]

$$\mathfrak{I}^\alpha f(t) = \frac{2(1-\alpha)}{(2-\alpha)}f(t) + \frac{2\alpha}{(2-\alpha)}\int_{t_0}^t f(\tau)d\tau. \quad (17)$$

3. Solvability

In this section, we first define the solvability concept, and then we investigate the solvability of the Systems (8)–(11):

Definition 3. We say that a system of the form (8)–(11) is solvable in space X, if for every $\mathbf{f}\in X$, there exists a unique solution $\mathbf{y}\in X$ such that satisfies the related system and is well defined on the interval I.

Theorem 4. *The System (8) has a unique solution iff A is invertible matrix and in this case*

$$\mathbf{y}(t) = -A^{-1}\mathbf{f}(t). \quad (18)$$

Therefore, the System (8) is solvable on $(\mathcal{C}(\mathbb{I}))^\nu$.

Theorem 5. *The unique solution of the System (9) is*

$$\mathbf{y}(t) = e^{-At}\mathbf{y}_0 - \int_0^t e^{-A(t-\tau)}\mathbf{f}(\tau)d\tau \quad (19)$$

and this system is solvable on

$$X = \{f \in (\mathcal{C}(\mathbb{I}))^\nu : \int_0^T e^{A\tau}f_i(\tau)d\tau < \infty,\ i = 1, \dots, \nu\}.$$

Theorem 6. *The unique solution of the System (10) with Liouville–Caputo's definition of fractional derivative is*

$$\mathbf{y}(t) = E_{\alpha,1}(-At^\alpha)\mathbf{y}_0 - \int_0^t E_{\alpha,\alpha}(-A(t-\tau)^\alpha)\mathbf{f}(\tau)d\tau \quad (20)$$

and this system is solvable on

$$X = \{f \in (\mathcal{C}(\mathbb{I}))^\nu : \int_0^T E_{\alpha,\alpha}(A(\tau)^\alpha)f_i(\tau)d\tau < \infty,\ i = 1, \dots, \nu\}.$$

Diethelm and Ford [42] has investigated the solvability of the more general system of the form

$$\mathfrak{D}^\alpha y = F(t, y), \quad 0 < \alpha < 1. \quad (21)$$

They studied the uniqueness and existence of a local continuous solution $y(x)\in C[0, h]$, provided f be continuous and Lipschitzian. The book [3] contains almost comprehensive survey about the solvability of the Systems (21) and the related systems.

Theorem 7. *Suppose $0 < \alpha < 1$, be such that $1 - \frac{B(\alpha)}{\|A\|} < \alpha$, and $f \in \mathcal{C}^1$. Then, the System (10) with Mittag–Leffler kernel, has a unique solution of the form*

$$\begin{aligned}\mathbf{y}(t) =& E_\alpha\left(-\frac{\alpha}{B(\alpha)}E^{-1}At^\alpha\right)E^{-1}\mathbf{y}_0 \\ &- \frac{1-\alpha}{B(\alpha)}E_\alpha\left(-\frac{\alpha}{B(\alpha)}E^{-1}At^\alpha\right)E^{-1}\mathbf{f}(0) \\ &- \frac{1-\alpha}{B(\alpha)}\int_0^t E_\alpha\left(-\frac{\alpha}{B(\alpha)}E^{-1}A\tau^\alpha\right)E^{-1}\mathbf{f}'(t-\tau)d\tau \\ &- \frac{\alpha}{B(\alpha)}\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left(-\frac{\alpha}{B(\alpha)}E^{-1}A\tau^\alpha\right)E^{-1}\mathbf{f}(t-\tau)d\tau \end{aligned} \quad (22)$$

where $E := \mathbf{I} + \frac{1-\alpha}{B(\alpha)}A$. This system is solvable on

$$X = \left\{f \in (\mathcal{C}(\mathbb{I}))^\nu : \int_0^T E_{\alpha,\alpha}(A(\tau)^\alpha)f_i(\tau)d\tau < \infty \ \& \int_0^T E_\alpha(\frac{\alpha}{B(\alpha)}E^{-1}A\tau^\alpha)f_i(\tau)d\tau < \infty,\ i = 1, \dots, \nu\right\}.$$

Proof. By using Laplace transform the proof is straightforward. □

For the System (11), the pencil of matrices $A + \lambda B$ may be shown by (A,B).

Definition 8 [43]**.** A pencil of matrices $A + \lambda B$ is called regular if

- A and B are square matrices of the some dimension n.
- the determinant $A + \lambda B$ ($\det(A + \lambda B)$) does not vanish identically.

In all other case, the pencil is called nonsingular.

Definition 9 [43]**.** Two pencils $A + \lambda B$ and $A_1 + \lambda B_1$ of dimension $m \times n$ are said to be strictly equivalent when there exist constant invertible matrices P and Q of orders m and n respectively, such that

$$P(A + \lambda B)Q = A_1 + \lambda B_1.$$

We will denote this equivalence relation by $\sim$.

Theorem 10 [43]**.** *For singular pencils $A + \lambda B$ there exists invertible matrices P and Q such that*

$$P(A + \lambda B)Q = diag\{L_{\epsilon_1}, \dots, L_{\epsilon_p}, L^T_{\eta_1}, \dots, L^T_{\eta_q}, \lambda N + I, \lambda I + J\} \quad (23)$$

where

- *L_μ is a $\mu \times (\mu + 1)$ bidiagonal pencil*

$$\begin{pmatrix}\lambda & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \vdots\\ 0 & \dots & \lambda & 1\end{pmatrix},$$

- *L^T_μ is the transpose of the L_μ of dimension $(\mu + 1) \times \mu$,*
- *N and I are nilpotent Jordan matrix and identity matrix, respectively,*
- *J is in Jordan canonical form.*

Theorem 11 [43]**.** *For regular pencils $A + \lambda B$ there exists invertible matrices P and Q such that*

$$PAQ = \begin{pmatrix} I & 0\\ 0 & N\end{pmatrix}, \quad PBQ = \begin{pmatrix} J & 0\\ 0 & I\end{pmatrix}, \quad (24)$$

where N and I are nilpotent Jordan matrix (with nilpotency index k) and identity matrix, respectively. If $N = 0$, we define $k = 1$. If A

is nonsingular, then N does not exist in the canonical representation (24) and we set $k=0$.

Definition 12. We say to the nilpotency index of matrix N, the index of the system of FDAEs (ind).

The main theorem of this paper is about the solvability of FDAEs (11):

Theorem 13. *The linear constant coefficient FDAEs (11) is solvable if and only if* $A+\lambda B$ *is a regular pencil.*

Proof. Suppose that $A+\lambda B$ has a singular pencil. Setting $\mathbf{y}=Q\mathbf{x}$ and multiplying System (11) by P, we obtain

$$PAQ\mathfrak{D}^{\alpha}\mathbf{x}+PBQ\mathbf{x}+P\mathbf{f}(t)=0. \tag{25}$$

Using canonical decomposition introduced in Theorem 10, one of the matrices L_μ or L_μ^T for some $\mu\in\mathbb{N}$, should be present in first diagonal block. We suppose that the first diagonal block be L_μ. In this case, for the first μ components of the system, we have

$$\begin{aligned}\mathfrak{D}^{\alpha}x_2+x_1+h_1(t)&=0,\\ \mathfrak{D}^{\alpha}x_3+x_2+h_2(t)&=0,\\ &\vdots\\ \mathfrak{D}^{\alpha}x_{\mu+1}+x_\mu+h_\mu(t)&=0,\end{aligned}$$

where $h_i(t)$, $(i=1,\dots,\mu)$ are the components of $P\mathbf{f}(t)$. This system has infinitely many solution even with initial condition $\mathbf{x}(0)=\mathbf{x}_0=Q^{-1}\mathbf{y}_0$. Similarly, for the second case (i. e. L^T), the first μ components of this system are inconsistence and have no solution. For regular systems, we invoke the Theorem 11, and by a similar decomposition we have

$$\mathfrak{D}^{\alpha}X_1+JX_1=H_1(t), \tag{26}$$

$$N\mathfrak{D}^{\alpha}X_2+X_2=H_2(t), \tag{27}$$

where $[X_1,X_2]=(Q^{-1}\mathbf{y})^T$ and $[H_1(t),H_2(t)]=-(P\mathbf{f}(t))^T$. The System (26) with initial condition $[X_1(0),X_2(0)]=[X_{10},X_{20}]=(Q^{-1}\mathbf{y}_0)^T$ has a unique solution. Now, Let the nilpotency index of the matrix N be k. Then, The System (27) can be written in the form

$$(N\mathfrak{D}^{\alpha}+I)X_2=H_2(t).$$

Hence, the unique solution of this system can be obtained by

$$X_2(t)=(N\mathfrak{D}^{\alpha}+I)^{(-1)}H_2(t)=\sum_{i=0}^{k}N^i(\mathfrak{D}^{\alpha})^iH_2(t)$$

if the consistency condition

$$\sum_{i=0}^{k}N^i(\mathfrak{D}^{\alpha})^iH_2(0)=X_{20} \tag{28}$$

holds. □

Like the descriptor systems, we observe the following properties for a system of FDAEs

1. The order of fractional derivative in the solutions of a FDAE system (i. e., $\mathfrak{D}^{\alpha}$) increases with increasing the index of the System (11).
2. The consistency condition (28), implies that the degree of the freedom in the System (11) is less than $\nu-n+1$.
3. These systems can have hidden algebraic constraints.

For the last point consider for example the system of FDAEs

$$\begin{aligned}\mathfrak{D}^{\alpha}x_2(t)+x_1(t)&=f_1(t)\\ x_2(t)&=f_2(t)\end{aligned} \tag{29}$$

This system has zero degree of freedom. The index of this system is 1. The consistency condition restrict this system by 2 equation

$$\begin{aligned}x_1(0)&=f_1(0)-\mathfrak{D}^{\alpha}f_2(0),\\ x_2(0)&=f_2(0).\end{aligned} \tag{30}$$

4. Nonlinear and linear time varying FDAEs

Taking fractional derivative from both site of the System (29), we obtain

$$(\mathfrak{D}^{\alpha})^2x_2+\mathfrak{D}^{\alpha}x_1=\mathfrak{D}^{\alpha}f_1(t), \tag{31}$$

$$\mathfrak{D}^{\alpha}x_2=\mathfrak{D}^{\alpha}f_2(t). \tag{32}$$

Substituting (32) into (31), we obtain

$$\begin{aligned}\mathfrak{D}^{\alpha}x_1&=f_1(t)-(\mathfrak{D}^{\alpha})^2f_2(t),\\ \mathfrak{D}^{\alpha}x_2&=\mathfrak{D}^{\alpha}f_2(t),\end{aligned} \tag{33}$$

which is a system of FDAEs of the form (10). This system has two degrees of freedom with index 0. By applying the transform $\mathfrak{D}^{\alpha}$ the index of the system reduces by 1. This process known as index reduction process may imply that we can use it for defining the index for nonlinear systems of FDAEs. But, it is not possible because in fractional calculus the classical Leibniz formula and chain rule do not hold (see for example Theorems 1 and 2). Therefore, it is difficult (and in some case not possible) to find a left regular operator that convert a nonlinear fractional algebraic equations to a regular fractional equations of the type

$$\mathfrak{D}^{\alpha}\mathbf{y}=F(t,\mathbf{y}). \tag{34}$$

For example, consider a nonlinear system of FDAEs obtained by replacing the algebraic equations $x_1x_2=f_2$ in (29). Applying the Leibniz formula for $\mathfrak{D}^{\alpha}x_1x_2$, we obtain an expansion series including infinite numbers of the fractional integrals of order $k-\alpha$, for $k=1,2,\cdots$. Therefore, we can not find a left regular operator like

$$L(\mathbf{y})=\sum_{i=0}^{m}p_i(t)(\mathfrak{D}^{\alpha})^i\mathbf{y}$$

such that it converts the corresponding system to a regular system of the form (34).

Consider a linear time varying system of FDAEs of the form

$$A(t)\mathfrak{D}^{\alpha}\mathbf{y}(t)+B(t)\mathbf{y}(t)+\mathbf{f}(t)=0,\quad 0<\alpha<1, \tag{35}$$

where A and B are matrix functions and $\mathbf{f}$ is a vector function. Now, the important questions are how we could define the index of systems of FDAEs? What is the solvability conditions of the System (35)?

To answer this question, we first extend the regularity concept for a time–varying pencil of matrices. We say that a pencil of matrices $A(t)+\lambda B(t)$ is regular if it is regular for all $t\in I$. In the previous section, we observed that the solvability of (11) depends on the regularity of the pencil of matrices $A+\lambda B$. For the case $\alpha=1$, (when it is a DAE), this dependency no longer holds. Nonetheless, all the counterexamples given for DAEs depend on Leibniz rule which is not true for $0\le\alpha\le 1$. Therefore, we guess that for the fractional differential equations solvability depends on the regularity and we remain it as an open problem.

Following example show that in general, the solvability of linear vector order systems of FDAEs do not depend on regularity.

Example 14. Consider a systems of FDAEs of the form

$$\mathfrak{D}y_1+(t+1)\mathfrak{D}^{0.5}y_2=f_1, \tag{36}$$

$$y_1+(t+1)y_2=f_2, \tag{37}$$

which can be written in the standard matrix form

$$\begin{pmatrix}1 & t+1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathfrak{D}y_1(t)\\ \mathfrak{D}^{0.5}y_2(t)\end{pmatrix}+\begin{pmatrix}0 & 0\\ 1 & t\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}=\begin{pmatrix}f_1(t)\\ f_2(t)\end{pmatrix}.$$

It is evident that $det(A(t)+\lambda B(t))\equiv 0$ and this system is singular. Now, differentiating from both sides of (37), we obtain

$$\mathfrak{D}y_1=\mathfrak{D}f_2-(t+1)\mathfrak{D}y_2-y_2$$

and hence, we obtain

$$\begin{aligned}\mathfrak{D}y_1+(t+1)\mathfrak{D}y_2+y_2&=\mathfrak{D}f_2,\\ (t+1)\mathfrak{D}y_2-(t+1)\mathfrak{D}^{0.5}y_2+y_2&=\mathfrak{D}f_2-f_1.\end{aligned}\tag{38}$$

This system can be written in the matrix form

$$\begin{pmatrix}1 & t+1\\ 0 & t+1\end{pmatrix}\begin{pmatrix}\mathfrak{D}y_1(t)\\ \mathfrak{D}y_2(t)\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & -(t+1)\end{pmatrix}\begin{pmatrix}\mathfrak{D}y_1(t)\\ \mathfrak{D}^{0.5}y_2(t)\end{pmatrix}+\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_1(t)\\ y_2(t)\end{pmatrix}=\begin{pmatrix}\mathfrak{D}f_2\\ \mathfrak{D}f_2-f_1\end{pmatrix}\tag{39}$$

which is a solvable system, but the original system is not regular.

Taking into account that we could use ordinary (integer order) derivative for the algebraic part, we may use this fact for investigating FDAEs. For this purpose, we first define a vector order concept and we consider a well–posed problem with this vector order derivatives.

Definition 15. Suppose $\mathbf{y}=[y_1,\dots,y_\nu]^T$ be a vector function and $\overline{\alpha}=[\alpha_1,\dots,\alpha_\nu]$ be a vector of dimension $\nu\in\mathbb{N}$, with $\alpha_i\in\mathbb{R}$ for $i=1,\dots,\nu$. Then, the vector order fractional integral and derivative are defined by

$$\mathfrak{I}^{\overline{\alpha}}\mathbf{y}=[\mathfrak{I}^{\alpha_1}y_1,\dots,\mathfrak{I}^{\alpha_\nu}y_\nu]^T,$$

and

$$\mathfrak{D}^{\overline{\alpha}}\mathbf{y}=[\mathfrak{D}^{\alpha_1}y_1,\dots,\mathfrak{D}^{\alpha_\nu}y_\nu]^T.$$

Now, consider a multi–term vector order fractional differential system of the form

$$\mathfrak{D}^{\overline{\alpha}}\mathbf{y}=F(t,\mathbf{y},\mathfrak{D}^{\overline{\alpha}_i}\mathbf{y},\dots,\mathfrak{D}^{\overline{\alpha}_i}\mathbf{y})\tag{40}$$

where F is a given multi–variable vector function. Here and for the rest of the paper, we suppose $\overline{\alpha}\in(0,1]^\nu$ and $\overline{\alpha}_i\in(0,1]^\nu$ for $i=1,\dots,m$. Transforming the system (40) by using fractional integral of vector order $\overline{\alpha}$ we obtain

$$\mathbf{y}(t)=\mathbf{y}(0)+\mathfrak{I}^{\overline{\alpha}}(F(t,\mathbf{y},\mathfrak{D}^{\overline{\alpha}_i}\mathbf{y},\dots,\mathfrak{D}^{\overline{\alpha}_i}\mathbf{y}))(t),\quad t\in[0,T].\tag{41}$$

The solvability of the System (40), is guarantied then using contraction mapping theorem on metric space or fixed point theorem in Banach spaces.

Now, we can define the index of the System (35) to be the minimum number of fractional derivative of order $0<\alpha\le 1$ (including ordinary derivative) that is needed to transform the System (35) to a system of the form (40).

5. Application

According to the Heisenberg uncertainty principle, energy and time are closely related to each other. So, it is plausible to think that a similar relationship could be found between temperature and time. Therefore, increasing temperature of the resistance gives rise to fractionalization of time. This fact is confirmed somehow in some recent experiments in electric circuites [44].

For analyzing the physical processes by using the related differential equations, Planck unit have used to describe the mathematical expressions of physical laws in a non–dimensional form. Besides its dimensional effect, it is a constant unit and we consider the following equations without this constant.

5.1. Electrical RF circuit

The dissipative effects caused by electrical resistance or ohmic friction are also found in the electrical circuits [4,44]. Theoretically, many authors obtained elements with non–integer order impedance with [4,45–47]. They introduced these elements by an infinite self–similar circuit consisting of resistors and capacitors. Later experimental and further developments have been done in [11,44,48,49]. We note that a perfect resistor obeys following Ohm's law

$$I=\frac{1}{R}V,$$

and a perfect capacitor obeys

$$I=C\mathfrak{D}V.$$

Therefore, we expect the existence of elements showing both properties of resistance and capacitance we say to this elements fractors. Fig. 1 show these three elements. In fact, fractance is a property of an electrical circuit which behaves between resistance and capacitance [4]. An appropriate two parameter $(\mathfrak{F},\alpha)$ model for fractance is

$$I=\mathfrak{F}\mathfrak{D}^\alpha V,\quad \alpha\in(0,1).$$

Recently, Ertik et al. have experimentally showed that this model better describes the capacitors behaviors than previous models [44].

A typical electrical circuit consisting of resistors and fractors are plotted in Fig. 2. By using Kirchhoff's current and Voltage law we obtain a system of FDAEs of the form:

$$\begin{aligned}\mathfrak{D}^\alpha V_1-\frac{1}{R_2}V_2-\frac{1}{R_1}V_1&=0,\\ V_1+V_2+E&=0,\\ V_2+V_3&=0,\end{aligned}\tag{42}$$

where $\alpha\in(0,1)$. This system is equivalent to a System (11) with

$$A=\begin{pmatrix}\mathfrak{F} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad B=\begin{pmatrix}-\frac{1}{R_1} & -\frac{1}{R_2} & 0\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix},$$

$\mathbf{f}=[E,0,0]^T$ and $\mathbf{y}=[V_1,V_2,V3]^T$. The System (42) is a linear system of FDAEs with constant coefficient. The pencil of matrices $A+\lambda B$ is regular and hence by Theorem 13, this system is solvable. The index of this system is 1.

5.2. A pendulum in a Newtonian fluid

One of the interesting applied models in physics is a model of a simple pendulum. In this paper, we obtain the equations of a pendulum immersed in a Newtonian fluid which leads to a nonlinear system of FDAEs. We suppose a point mass m suspended from a massless rod of length l under the influence of gravity g on a Newtonian fluid (see Fig. 3).

It is known that the dissipating force in a Newtonian fluid is a proportion of the fractional derivatives [50]. By stress–strain relationship we have

$$F_x=c_x\mathfrak{D}^{1+\alpha}x$$

for some $0<\alpha\le 1$. Adding the algebraic constraints $(x^2+y^2-l^2=0)$ with the Lagrange multiplier λ to the total energy we obtain

$$\begin{aligned}L=&mgy+\frac{1}{2}m((\mathfrak{D}x)^2+(\mathfrak{D}y)^2+\lambda(x^2+y^2-l^2)\\ &+\int_0^t c_x\mathfrak{D}^\alpha\mathfrak{D}x\mathfrak{D}xdt+\int_0^t c_y\mathfrak{D}^\alpha\mathfrak{D}y\mathfrak{D}ydt.\end{aligned}\tag{43}$$

Using the principle of least action, we minimize the Lagrangian L.

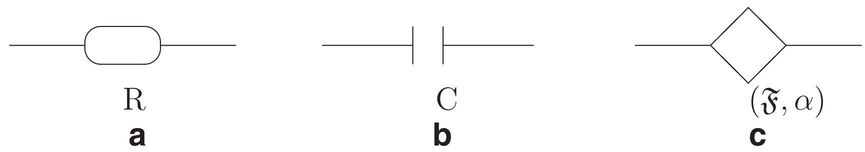

Fig. 1. (a) Resistor (b) Capacitor (c) Fractor.

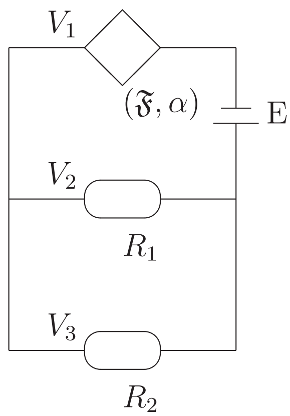

Fig. 2. An RF circuit.

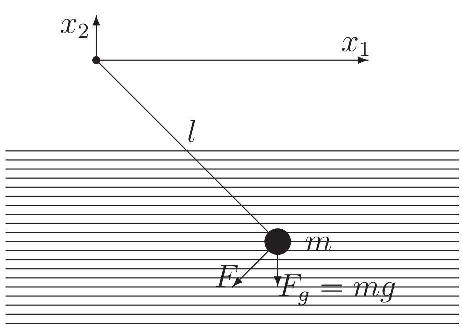

Fig. 3. Simple pendulum-scheme on Newtonian fluid.

Theorem 16 (Principle of Least Action)**.** *The actual path taken by the system is an extremum of S.*

The minimization problem is solved by well-known Euler–Lagrange equations:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial \mathfrak{D}x} = \frac{\partial L}{\partial x},\\
&\frac{d}{dt}\frac{\partial L}{\partial \mathfrak{D}y} = \frac{\partial L}{\partial y},\\
&\frac{d}{dt}\frac{\partial L}{\partial \mathfrak{D}\lambda} = \frac{\partial L}{\partial \lambda}.
\end{aligned} \tag{44}$$

Using Leibniz integral rule, we have

$$\frac{\partial \mathfrak{D}^\alpha \mathfrak{D}x}{\partial \mathfrak{D}x} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial}{\partial \mathfrak{D}x}\frac{\mathfrak{D}^2(\tau)}{(t-\tau)^\alpha}d\tau = 0$$

and hence we have

$$\frac{\partial L}{\partial \mathfrak{D}x} = m\mathfrak{D}x + \int_0^t c_x \mathfrak{D}^\alpha \mathfrak{D}x dt.$$

Taking into account that

$$\frac{\partial L}{\partial x} = 2\lambda x$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \mathfrak{D}x} = m\mathfrak{D}^2 x + c_x \mathfrak{D}^\alpha \mathfrak{D}x,$$

the first Euler–Lagrange equation leads to

$$m\mathfrak{D}^2 x + c_x \mathfrak{D}^\alpha \mathfrak{D}x = 2\lambda x.$$

Similarly, other equations are obtained as

$$m\mathfrak{D}^2 y + c_y \mathfrak{D}^\alpha \mathfrak{D}y = 2\lambda y + mg$$

and

$$x^2 + y^2 - l^2 = 0.$$

Now, by changing the variables $x_1 = x$, $x_2 = y$, $x_3 = \mathfrak{D}x$, and $x_4 = \mathfrak{D}y$ we obtain the following system of FDAEs

$$\begin{aligned}
\mathfrak{D}x_1 &= x_3,\\
\mathfrak{D}x_2 &= x_4,\\
m\mathfrak{D}x_3 &= -c_x\mathfrak{D}^\alpha x_3 + 2\lambda x_1,\\
m\mathfrak{D}x_4 &= -c_y\mathfrak{D}^\alpha x_4 + 2\lambda x_2 + mg,\\
x_1^2 + x_2^2 &= l^2.
\end{aligned} \tag{45}$$

Taking the first derivative we obtain

$$\begin{aligned}
\mathfrak{D}x_1 &= x_3,\\
\mathfrak{D}x_2 &= x_4,\\
m\mathfrak{D}x_3 &= -c_x\mathfrak{D}^\alpha x_3 + 2\lambda x_1,\\
m\mathfrak{D}x_4 &= -c_y\mathfrak{D}^\alpha x_4 + 2\lambda x_2 + mg,\\
x_1x_3 + x_2x_4 &= 0.
\end{aligned} \tag{46}$$

Taking the second derivative and adding two new variable x_6 and x_7, we obtain

$$\begin{aligned}
\mathfrak{D}x_1 &= x_3,\\
\mathfrak{D}x_2 &= x_4,\\
\mathfrak{D}^\alpha x_3 &= x_6,\\
\mathfrak{D}^\alpha x_4 &= x_7,\\
\mathfrak{D}^{1-\alpha}x_6 &= \frac{1}{m}(-c_x x_6 + 2\lambda x_1),\\
\mathfrak{D}^{1-\alpha}x_7 &= \frac{1}{m}(-c_y x_7 + 2\lambda x_2 + mg),\\
\lambda &= -\frac{m}{2l^2}(x_3^2 + x_4^2) + \frac{1}{2l^2}(c_x x_1 x_6 + c_y x_2 x_7) - \frac{mg}{2l^2}x_2.
\end{aligned} \tag{47}$$

Finally, taking the third derivative, we obtain

$$\begin{aligned}
\mathfrak{D}x_1 &= x_3,\\
\mathfrak{D}x_2 &= x_4,\\
\mathfrak{D}^\alpha x_3 &= x_6,\\
\mathfrak{D}^\alpha x_4 &= x_7,\\
\mathfrak{D}^{1-\alpha}x_6 &= \frac{1}{m}(-c_x x_6 + 2\lambda x_1),\\
\mathfrak{D}^{1-\alpha}x_7 &= \frac{1}{m}(-c_y x_7 + 2\lambda x_2 + mg),\\
D\lambda &= -\frac{1}{l^2}x_3(-c_x x_6 + 2\lambda x_1) - \frac{1}{l^2}x_4(-c_y x_7 + 2\lambda x_2 + mg)\\
&\quad + \frac{c_x}{2l^2}(x_1\mathfrak{D}x_6 + x_3x_6) + \frac{c_y}{2l^2}(x_2\mathfrak{D}x_7 + x_4x_7) - \frac{m}{2l^2 g}x_4
\end{aligned} \tag{48}$$

which is of the form (40) and hence is solvable. The index of the Systems (45)–(47) are 3, 2 and 1, respectively.

6. Numerical method and comparison

Let $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$ be a partition of the interval [0, T]. Let $h_n = t_{n+1} - t_n$ for $n = 0, 1, \ldots, N-1$. Suppose f_i be the value of $f(t)$ on $t_{i-1} + \theta h_{i-1}$ for $i = 1, \ldots, n$. Let we approximate f on $[t_n, t_{n+1}]$ by $f_n \sim f(t_n + sh_n) = f(t_n + \theta h_n)$. We first obtain an approximation of the fractional integral operator $\mathfrak{I}^\alpha$ on $t = t_n + \theta h$ for all fractional integrals that we introduced in the previous section, and then we introduce numerical methods for solving system of FDAEs corresponding to different definitions of fractional derivatives.

- Integral operator for Liouville–Caputo's definition, see [3]

$$\begin{aligned}\mathfrak{I}^\alpha f(t_n+\theta h) =& \frac{1}{\Gamma(\alpha)}\int_0^{t_n+\theta h_n}\frac{f(\tau)}{(t_n+\theta h_n-\tau)^{1-\alpha}}d\tau\\ =& \frac{1}{\Gamma(\alpha)}\left(\sum_{l=0}^{n-1} h_l\int_0^1 \frac{f(t_l+sh_l)}{(t_n+\theta h_n - t_l - sh_l)^{1-\alpha}}ds\right.\\ &\left.+h_n^\alpha\int_0^\theta \frac{f(t_n+sh_n)}{(\theta-s)^{1-\alpha}}ds\right)\\ \approx& \sum_{l=0}^{n-1} f_{l+1}\Delta_{n,l} + \frac{h_n^\alpha\theta^\alpha}{\Gamma(1+\alpha)}f_{n+1} \qquad (49)\end{aligned}$$

where

$$\begin{aligned}\Delta_{n,l} =& \frac{h_l}{\Gamma(\alpha)}\int_0^1 \frac{1}{(t_n+\theta h_n - t_l - sh_l)^{1-\alpha}}ds\\ =& \frac{1}{\Gamma(1+\alpha)}((t_n+\theta h_n - t_l)^\alpha - (t_n+\theta h_n - t_l - h_l)^\alpha).\end{aligned} \qquad (50)$$

- Integral operator corresponding to Mittag–Leffler kernel, see [41]

$$\mathfrak{I}^\alpha f(t_n+\theta h) \approx \frac{1-\alpha}{B(\alpha)}f_{n+1} + \frac{\alpha}{B(\alpha)}\left(\sum_{l=0}^{n-1} f_{l+1}\Delta_{n,l} + \frac{h_n^\alpha\theta^\alpha}{\Gamma(1+\alpha)}f_{n+1}\right). \qquad (51)$$

- Integral operator for Caputo–Fabrizio's definition, see [11]

$$\mathfrak{I}^\alpha f(t_n+\theta h) \approx \frac{2(1-\alpha)}{(2-\alpha)}f_{n+1} + \frac{2\alpha}{(2-\alpha)}\left(\sum_{l=0}^{n-1} h_l f_{l+1} + \theta h_n f_{n+1}\right). \qquad (52)$$

Therefore, all fractional integral operators can be approximated by a weighted sum of the form

$$\mathfrak{I}^\alpha f(t_n+\theta h) \approx \sum_{l=0}^{n} w_{n,l,\alpha} f_{l+1}$$

where $w_{n,\,l,\,\alpha}$ are suitable weighted constant that can be obtained using (49)–(52).

Now, suppose we could transform a system of FDAEs into the following system of fractional differential equations

$$\mathbf{y}(t) = \mathbf{y}(0) + \mathfrak{I}^{\overline{\alpha}}F(t, \mathbf{y}(t)), \quad t \in [0, T]. \qquad (53)$$

Then, by collocation on $t_{n,\theta} := t_n + \theta h_n$, we obtain a system of nonlinear equations of the form

$$\mathbf{y}_{n+1} = \mathbf{y}(0) + \sum_{l=0}^{n} w_{n,l,\overline{\alpha}} F(t_{n,\theta}, \mathbf{y}_{l+1}), \qquad (54)$$

where $w_{n,l,\overline{\alpha}}$ is diagonal matrix of the form

$$w_{n,l,\overline{\alpha}} = \begin{pmatrix} w_{n,l,\alpha_1} & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & w_{n,l,\alpha_\nu}\end{pmatrix}$$

and ν is the dimension of the FDAEs. Solving this system we can obtain the approximate solution in the next step $\mathbf{y}_{n+1}$.

Example 17. It is known that the graded mesh usually has a better convergence order for singular integral equations (see for example [10]). A graded mesh with exponent $\rho = 2$ can be determined by

$$t_j = T\left(\frac{j}{N}\right)^\rho, \quad j = 0, \ldots, N.$$

The system of FDAEs (47) has the form (54) and we can numerically solve this system with all three definitions of fractional derivative and compare them.

For all the numerical examples in this study, we set $N = 1500$, $\theta = 0.9$ and we solve them on the interval [0,20]. For physical parameters we set $g = 9.8$ (m/s^2), $m = 10$ (kg) and $l = 1$ (m).

Fig. 4(a)–(h) illustrate approximate solutions of the pendulum on a fluid modeled by the System (45) for various values of $c_x = x_y$ and α. The figures on the left hand side are the approximate solutions of the System (45) with $c_x = x_y = 50$ and the figures on the right hand side correspond to $c_x = x_y = 500$ with the same value of α.

All figures show that the System (45) by nonsingular kernels (Mittag–Leffler kernel and Caputo–Fabrizio's definition) has more decaying properties than oscillating and the System (45) with Liouville–Caputo's definition tend to oscillate even for $\alpha = 0$. We also observe that the order of fractional derivatives (α) plays an important role in dissipating energy and stabilizing the system than the parameters c_x and c_y. Furthermore, we observe that c_x and c_y effect on wavelength, and increasing these parameters increase the wavelength.

6.1. Numerical experiments for some chosen examples in researches

Finally, we survey some existing DAEs in research. We observe that almost all of the examples are system of FDAEs of index one. In the following examples, we denote by u_i and y_i $i = 1, \ldots, \nu$ the numerical approximation and the exact solution of a system of FDAEs of dimension ν, respectively on [0, T]. We define the maximum error by

$$e_i(N) = \max_{t\in[0,T]} |u_i(t) - y_i(t)|$$

for $i = 1, \ldots, \nu$ and $N \in \mathcal{N}$.

Example 18. [36] Consider a system of FDAEs of the form

$$\begin{aligned}\mathfrak{D}^\alpha y_1(t) - (1 - e^{y_2(t)})y_1(t) - \frac{\sqrt{\pi}}{2} + \sqrt{t}(1 - e^{t\sqrt{t}}) = 0,\\ y_2(t) - \sin(y_1(t)) - t\sqrt{t} + \sin(\sqrt{t}) = 0,\end{aligned} \qquad (55)$$

where $0 < \alpha \leq 1$ subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0.$$

Taking the derivative from both sides of the second equation we obtain

$$\mathfrak{D}y_2(t) = \cos(y_1(t))\mathfrak{D}y_1 - \frac{3}{2}\sqrt{t} - \frac{\cos\sqrt{t}}{2\sqrt{t}}$$

Hence, the index of this nonlinear system of FDAEs is 1 and it is solvable. Indeed, the exact solution of this system with $\alpha = 0.5$ by Liouville–Caputo's definition is $y_1(t) = \sqrt{t}$ and $y_2(t) = t\sqrt{t}$. Therefore, we can compare the numerical solution and exact solution. In Table 1, we reported the maximum error by the Numerical method (54) with the parameters $\theta = 1$ and $r = 1.5$ for various parameters of N. This table shows the efficiency and effectiveness of the proposed method.

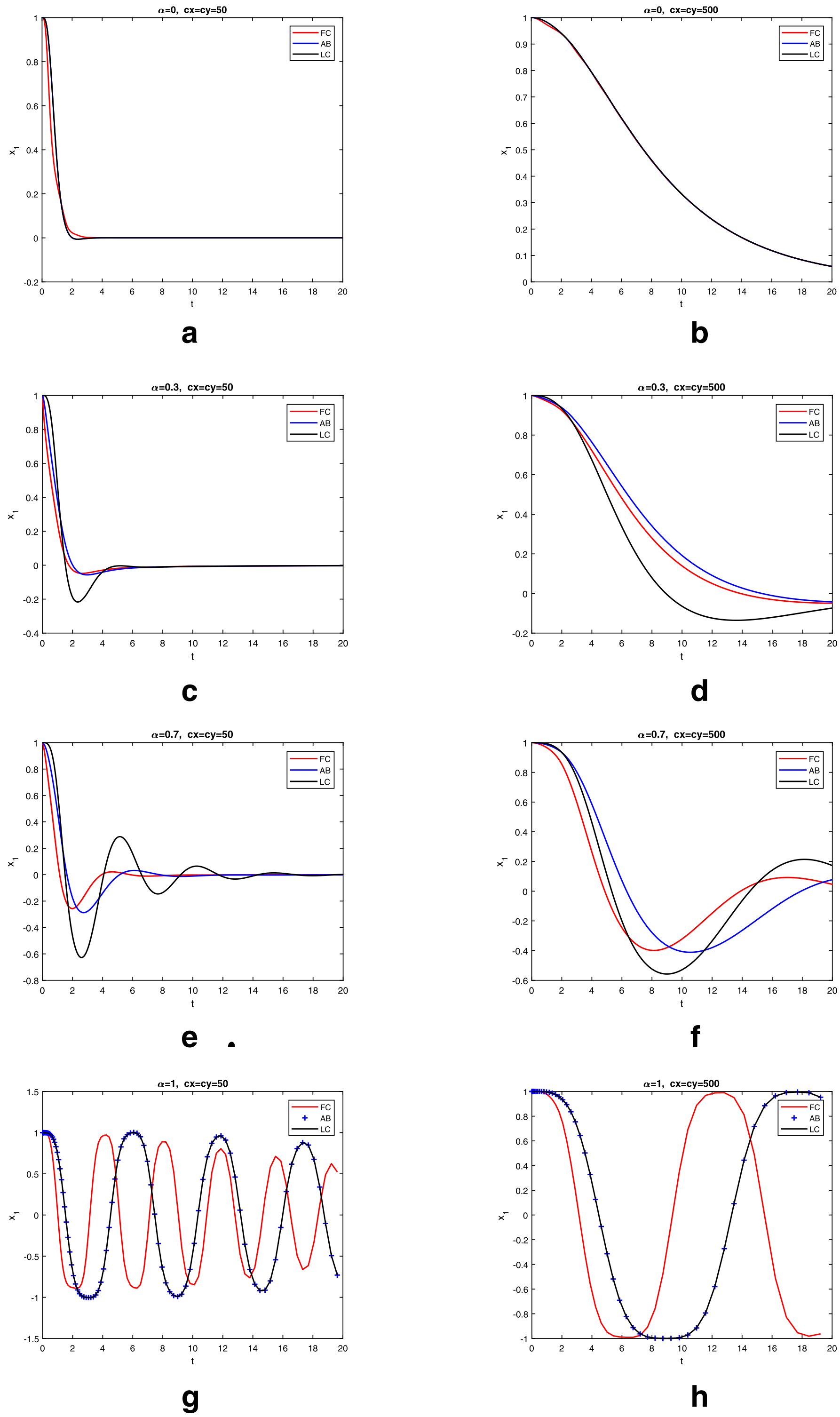

Fig. 4. Approximate solution of a pendulum on fluid modeled by the system of FDAEs (45) for $\alpha \in \{0, 0.3, 0.7, 1\}$ and $c_x = x_y \in \{50, 500\}$. Here CF stands for Caputo–Fabrizio's definition, AB stands for fractional derivative with Mittag–Leffler kernel and LC stand for Liouville–Caputo's definition.

Example 19. [38]

Consider a system of FDAEs of the form

$$\begin{aligned} \mathfrak{D}^{\alpha} y_1(t) + y_1(t) y_2(t) - y_3(t) - f_1(t) &= 0, \\ \mathfrak{D}^{\alpha} y_2(t) - \frac{\Gamma(5)}{\Gamma(9/2)} t^{1/2} y_1(t) + 2y_2(t) + y_1(t) y_3(t) - f_2(t) &= 0, \\ y_1^2(t) - t^2 y_2(t) + y_3(t) - f_3(t) &= 0, \end{aligned} \tag{56}$$

where

$$\begin{aligned} f_1(t) &= \frac{\Gamma(4)}{\Gamma(7/2)} t^{\frac{5}{2}} + 2t^4 + t^7 - e^t - t\sin(t), \\ f_2(t) &= \frac{2}{\Gamma(3/2)} t^{\frac{1}{2}} + 4t + 2t^4 + t^3 e^t + t^4 \sin(t), \\ f_3(t) &= e^t + t\sin(t) + 2t^3, \end{aligned} \tag{57}$$

Table 1
The maximum error of applying numerical method in example 18, with $\alpha = 0.5$ for various values of N.

N	16	32	64	128
$e_1(N)$	6.4419e-07	1.7342e-08	4.6269e-10	1.2279e-11
$e_2(N)$	6.3916e-07	1.7294e-08	4.6224e-10	1.2274e-11

Table 2
The maximum error of applying numerical method in example 19 with $\alpha = 0.5$ for various values of N.

N	16	32	64	128
$e_1(N)$	2.2964e-02	1.2979e-02	7.0446e-03	3.8244e-03
$e_2(N)$	1.0713e-01	5.0984e-02	2.5459e-02	1.3279e-02
$e_3(N)$	1.1090e-01	4.2116e-02	1.8460e-02	8.9850e-03

subject to the initial conditions

$$y_1(0) = 0, \quad y_2(0) = 0, \quad y_3(t) = 1.$$

This system is a time-varying linear system of FDAEs, and has index 1. The exact solution of this system with $\alpha = 0.5$ is $y_1(t) = t^3$, $y_2(t) = 2t + t^4$ and $y_3(t) = e^t + t\sin(t)$. In Table 2, we reported the maximum error by the Numerical method (54) with the parameters $\theta = 1$ and $r = 1.5$ for various parameters of N. This table confirms the efficiency and the effectiveness of the proposed method.

7. Conclusion

We obtained the solvability conditions of the system of linear FDAEs with the constant coefficient and we showed that it depends on regularity similar to DAEs. We showed that this similarity does not hold with general FDAEs including linear time–varying linear FDAEs and nonlinear DAEs. We observed that we could not define index concept similar to DAEs for nonlinear FDAEs because the chain rule for the fractional derivative is different and its expansion includes infinitely terms. We introduced the vector order fractional derivatives to overcome the mentioned problem and defined the index concept by this notation. Finally, by introducing some applied problem we showed the high importance of investigation of this new field. We expect this field will be received the same wide attention and application as the system of DAEs in the near future.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

Supplementary material

Supplementary material associated with this article can be found, in the online version, at doi:10.1016/j.chaos.2019.01.028

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