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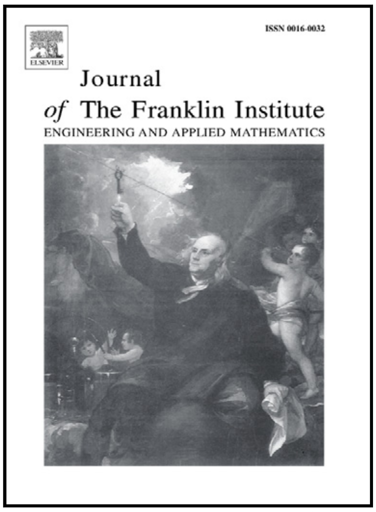

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Discrete Wirtinger-based inequality and its application

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Abstract

In this paper, we derive a new inequality, which encompasses the discrete Jensen inequality. The new inequality is applied to analyze stability of linear discrete systems with an interval time-varying delay and a less conservative stability condition is obtained. Two numerical examples are given to show the effectiveness of the obtained stability condition.

1. Introduction

Stability analysis of time-delay systems has received extensive attention in the literature [1-42]. The second Lyapunov method combining with Krasovskii and Razumikhin techniques is one of the most common approaches used in the stability analysis of linear time-delay systems [1, 2, 3, 4] and many significant developments of this approach have been reported in recent years. In order to obtain less conservative stability conditions, many researchers have pursued along two main research directions: (i) enlarging on the classes of Lyapunov functionals; and (ii) providing tighter estimations of the derivative of the Lyapunov functionals. With regard to the first research direction, many significant results such as the descriptor model transformations [5], delay-decomposition technique [6, 7, 8], the neutral model transformations [9, 10], triple (multiple) integral terms [11, 12], Razumikhin technique [13], and delay-dependent matrix technique [14] have been reported. With regard to the second research direction, significant developments such as the free-weighting matrix technique [15], and widely used inequalities such as the Jensen inequality [1], improved Jensen inequalities [16, 17], reciprocally convex combination inequality [18], Wirtinger's inequality [19] and Wirtinger-based integral inequality [20, 21] have been reported. In parallel, the above mentioned developments have also been applied to discrete time-delay systems [7, 8, 23, 24, 25, 26, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41]. To our knowledge, so far, the results reported in [40, 41] are, respectively, the best two results in the first and second research direction for discrete-time systems with time delays. Note that, up to now, the inequalities in [41] are the most improved of the discrete Jensen inequality [26].

Recently, the authors [20, 21] introduced a novel Wirtinger-based integral inequality in combination with the reciprocally convex technique to derive a less conservative stability condition. To our knowledge, up to now, this inequality is the most improved of the Jensen inequality and there has not been any reported discrete version of this inequality. In this paper, motivated

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by the work in [20, 21], we derive a discrete version of the Wirtinger-based integral inequality. The newly derived inequality is applied to obtain a new stability condition for linear discrete systems with an interval time-varying delay. Two numerical examples with extensive comparison to existing results are given to show the effectiveness of the obtained stability condition.

2. A new inequality

In this section, we derive a new inequality, which is a discrete version of the Wirtinger-based integral inequality. The following lemmas will be used in the derivation of our new inequality.

Lemma 1. *For a given positive integer n, we have*

$$i)\quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}, \tag{1}$$

$$ii)\quad 2^2 + 4^2 + \cdots + (2n)^2 = \frac{2n(n+1)(2n+1)}{3}, \tag{2}$$

$$iii)\quad 1^2 + 3^2 + \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}. \tag{3}$$

Proof. The proof of Lemma 1 is omitted since these equalities are fundamental results.

Lemma 2. *For two positive integers b and k, the following inequalities hold*

$$i)\quad \sum_{s=k-b}^{k-1} (2k - b - 1 - 2s) = 0, \tag{4}$$

$$ii)\quad \sum_{s=k-b}^{k-1} \frac{(2k - b - 1 - 2s)^2}{b^4} \le \frac{1}{3b}. \tag{5}$$

Proof. i) It is obvious.

ii) By computation, we have

$$\begin{aligned} \sum_{s=k-b}^{k-1} \frac{(2k-b-1-2s)^2}{b^4} &= \frac{1}{b^4}\Big((b-1)^2 + (b-3)^2 + \cdots + (-b+3)^2 + (-b+1)^2\Big) \\ &= \frac{2}{b^4} \times \begin{cases} 1^2 + 3^2 + \cdots + (b-1)^2, & b \text{ is even} \\ 2^2 + 4^2 + \cdots + (b-1)^2, & b \text{ is odd.} \end{cases} \end{aligned} \tag{6}$$

Case 1: When b is even, let $b = 2n+2$ and use iii) of Lemma 1, we obtain

$$\begin{aligned} \sum_{s=k-b}^{k-1} \frac{(2k-b-1-2s)^2}{b^4} &= \frac{2}{b^4}\Big(1^2 + 3^2 + \cdots + (2n+1)^2\Big) \\ &= \frac{2}{b^4}\frac{(n+1)(2n+1)(2n+3)}{3} \\ &= \frac{1}{b^4}\frac{b(b-1)(b+1)}{3} \\ &= \frac{1}{3b}\frac{b^2-1}{b^2} < \frac{1}{3b}. \end{aligned} \tag{7}$$

Case 2: When b is odd, let $b = 2n + 1$ and use ii) of Lemma 1, we obtain

$$\begin{aligned}\sum_{s=k-b}^{k-1} \frac{(2k-b-1-2s)^2}{b^4} &= \frac{2}{b^4}\Big(2^2 + 4^2 + \cdots + (2n)^2\Big) \\ &= \frac{2}{b^4}\frac{2n(n+1)(2n+1)}{3} \\ &= \frac{1}{b^4}\frac{(b-1)(b+1)b}{3} \\ &= \frac{1}{3b}\frac{b^2-1}{b^2} \le \frac{1}{3b}. \end{aligned} \tag{8}$$

From (6), (7) and (8), we obtain (5). The proof of Lemma 2 is completed. □.

Denote $y(k) = x(k+1) - x(k)$ and we now derive a new inequality in the form of the following lemma.

Lemma 3. *For a given positive-definite matrix R and three given non-negative integers a, b, k satisfying $a \le b \le k$, let us denote*

$$\chi(k,a,b) = \begin{cases} \frac{1}{b-a}\left[\left(2\sum_{s=k-b}^{k-a-1} x(s)\right) + x(k-a) - x(k-b)\right], & a < b, \\ 2x(k-a), & a = b. \end{cases}$$

Then, we have

$$-(b-a)\sum_{s=k-b}^{k-a-1} y^T(s)Ry(s) \le -\begin{bmatrix}\Omega_0 \\ \Omega_1\end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & 3R\end{bmatrix} \begin{bmatrix}\Omega_0 \\ \Omega_1\end{bmatrix}, \tag{9}$$

where

$$\begin{aligned}\Omega_0 &= x(k-a) - x(k-b) \\ \Omega_1 &= x(k-a) + x(k-b) - \chi(k,a,b).\end{aligned}$$

Proof. It is easy to verify that inequality (9) holds when $a = b$. Now, we will prove that inequality (9) also holds for the case where $a < b$. First, we consider the case where $a = 0$ and $b > 0$. If $a = 0$ and $b = 1$, then we have $-\sum_{s=k-1}^{k-1} y^T(s)Ry(s) = -(x(k) - x(k-1))^T R(x(k) - x(k-1))$, $\Omega_0 = x(k) - x(k-1)$ and $\Omega_1 = 0$, which follow that inequality (9) holds. Now for $a = 0$, $b \ge 2$, let us denote

$$z(s) = y(s) - \frac{1}{b}\Omega_0 + \frac{3}{b^2}(2k + b - 1 - 2s)\Omega_1, \tag{10}$$

then we have

$$0 \le z^T(s)Rz(s), \ \forall s \in \mathbb{Z}. \tag{11}$$

Taking the sum in s of (11) from $k - b$ to $k - 1$, we obtain

$$
\begin{aligned}
0 \leq \sum_{s=k-b}^{k-1} z^T(s)Rz(s) &= \sum_{s=k-b}^{k-1} y^T(s)Ry(s) - \frac{2}{b}\sum_{s=k-b}^{k-1} y^T(s)R\Omega_0 \\
&+\frac{6}{b^2}\sum_{s=k-b}^{k-1}(2k-b-1-2s)y^T(s)R\Omega_1 + \frac{1}{b^2}\Omega_0^T R\Omega_0 \\
&-\frac{6}{b^3}\sum_{s=k-b}^{k-1}(2k-b-1-2s)\Omega_0^T R\Omega_1 \\
&+\frac{9}{b^4}\sum_{s=k-b}^{k-1}(2k-b-1-2s)^2\Omega_1^T R\Omega_1. \qquad (12)
\end{aligned}
$$

By some computations, we have

$$
\begin{aligned}
\sum_{s=k-b}^{k-1}(2k-b-1-2s)y^T(s) &= (b-1)\Big(x^T(k-b+1) - x^T(k-b)\Big) \\
&+(b-3)\Big(x^T(k-b+2) - x^T(k-b+1)\Big) \\
&+\cdots \\
&+(-b+3)\Big(x^T(k-1) - x^T(k-2)\Big) \\
&+(-b+1)\Big(x^T(k) - x^T(k-1)\Big) \\
&= -b\Big\{x^T(k) + x^T(k-b) - \frac{1}{b}\Big(2\sum_{s=k-b+1}^{k-1} x^T(s) + x^T(k) \\
&\quad -x^T(k-b)\Big)\Big\} \\
&= -b\Omega_1^T \qquad (13)
\end{aligned}
$$

and

$$
\sum_{s=k-b}^{k-1} y^T(s) = x^T(k) - x^T(k-b) = \Omega_0^T. \qquad (14)
$$

By Lemma 2, (12), (13), (14), we obtain

$$
-\sum_{s=k-b}^{k-1} y^T(s)Ry(s) \leq -\frac{1}{b}(\Omega_0^T R\Omega_0 + 3\Omega_1^T R\Omega_1). \qquad (15)
$$

Note that inequality (15) holds for all $k, b \in \mathbb{Z}$, $0 < b \leq k$. Therefore, for the case where $0 \leq a < b$, by replacing b by $b-a$ and k by $k-a$ into inequality (15), we easily obtain inequality (9). The proof of Lemma 3 is completed. □

Remark 1. By setting $\Omega_1 = 0$ in the definition of the function $z(s)$ as defined in (10), we obtain an alternative proof of the discrete Jensen inequality,

$$
-(b-a)\sum_{s=k-b}^{k-a-1} y^T(s)Ry(s) \leq -\Omega_0^T R\Omega_0. \qquad (16)
$$

3. A stability condition

In this section, we use the Lyapunov-Krasovskii method in combining with the inequality (9) to derive a new stability condition for the following linear discrete time-delay system

$$\begin{aligned} x(k+1) &= Ax(k) + A_1 x(k-\tau(k)), \quad k \geq 0, \\ x(k) &= \phi(k), \ k = -\tau_M, \cdots, 0, \end{aligned} \tag{17}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, A, A_1 are known constant matrices with appropriate dimensions and $\phi(k) \in \mathbb{R}^n, k = -\tau_M, \cdots, 0$ are initial values, the time-varying delay $\tau(k)$ is assumed to satisfy $0 \leq \tau_m \leq \tau(k) \leq \tau_M$, where τ_m and τ_M are known integers.

The reciprocally convex combination inequality [18], which was reformulated in [21] and Finsler's Lemma [42] are used in our development:

Lemma 4. *[18, 21] For given positive integers n, m, a scalar $\alpha \in (0,1)$, a $n \times n$-matrix $R > 0$, two $n \times m$-matrices W_1, W_2. Define, for all vector $\xi \in \mathbb{R}^m$, the function $\Theta(\alpha, R)$ given by:*

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi.$$

If there is a matrix $X \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} R & X \\ \star & R \end{bmatrix} > 0$, *then the following inequality holds*

$$\min_{\alpha \in (0,1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ \star & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

Lemma 5. *[42] Let $\zeta \in \mathbb{R}^n$, $\Phi = \Phi^T \in \mathbb{R}^{n\times n}$ and $\Upsilon \in \mathbb{R}^{m\times n}$ such that rank$(\Upsilon) < n$. The following statements are equivalent:*

i) $\zeta^T \Phi \zeta < 0, \ \forall \Upsilon\zeta = 0, \zeta \neq 0,$

ii) ${\Upsilon^{\perp}}^T \Phi \Upsilon^{\perp} < 0,$

iii) $\exists X \in \mathbb{R}^{n\times m}: \ \Phi + X\Upsilon + \Upsilon^T X^T < 0.$

We also use the delay-decomposition technique to reduce the conservatism of derived stability criterion. For simplicity we divide interval $[\tau_m, \tau_M]$ into only two subintervals $[\tau_m, \tau_a]$, $[\tau_a, \tau_M]$. The following notations are used in our development:

$$\begin{aligned} \mu_1(k) &= \chi(k, 0, \tau_m), \\ \mu_2(k) &= \chi(k, \tau_m, \tau_a), \\ \mu_3(k) &= \chi(k, \tau_a, \tau_M), \\ \mu_4(k) &= \chi(k, \tau(k), \tau_a), \\ \mu_5(k) &= \chi(k, \tau_m, \tau(k)), \\ \mu_6(k) &= \chi(k, \tau(k), \tau_M), \\ \mu_7(k) &= \chi(k, \tau_a, \tau(k)), \end{aligned}$$

$$
\begin{aligned}
\zeta^T(k) &= \Big[x^T(k) \quad \sum_{s=k-\tau_m}^{k-1} x^T(s) \quad \sum_{s=k-\tau_a}^{k-\tau_m-1} x^T(s) \quad \sum_{s=k-\tau_M}^{k-\tau_a-1} x^T(s)\Big],\\
\xi^T(k) &= \Big[x^T(k) \;\; x^T(k-\tau(k)) \;\; x^T(k-\tau_m) \;\; x^T(k-\tau_a) \;\; x^T(k-\tau_M) \quad y^T(k)\\
&\qquad \mu_1^T(k) \quad \mu_2^T(k) \;\; \mu_3^T(k) \quad \mu_4^T(k) \quad \mu_5^T(k) \quad \mu_6^T(k) \quad \mu_7^T(k)\Big],\\
e_i &= [0_{n\times(i-1)n} \;\; I_n \;\; 0_{n\times(13-i)n}]^T, \quad i = 1, 2, \cdots, 13,\\
\rho_1 &= [e_6 \quad e_1 - e_3 \quad e_3 - e_4 \quad e_4 - e_5],\\
\rho_2 &= \Big[\frac{1}{2}((\tau_a - \tau(k))e_{10} + (\tau(k) - \tau_m)e_{11} - e_3 + e_4)\Big],\\
\rho_3 &= \Big[e_1 \;\; \frac{1}{2}(e_3 - e_1 + \tau_m e_7) \quad \rho_2 \quad \frac{1}{2}(e_5 - e_4 + (\tau_M - \tau_a)e_9)\Big],\\
\rho_4 &= [e_1 - e_3 \quad \sqrt{3}(e_1 + e_3 - e_7)],\\
\rho_5 &= [e_2 - e_4 \quad \sqrt{3}(e_2 + e_4 - e_{10})],\\
\rho_6 &= [e_3 - e_2 \quad \sqrt{3}(e_3 + e_2 - e_{11})],\\
\rho_7 &= [\rho_5 \quad \rho_6],\\
\rho_8 &= [e_4 - e_5 \quad \sqrt{3}(e_4 + e_5 - e_9)],\\
\rho_9 &= \Big[\frac{1}{2}((\tau_M - \tau(k))e_{12} + (\tau(k) - \tau_a)e_{13} - e_4 + e_5)\Big],\\
\rho_{10} &= \Big[e_1 \;\; \frac{1}{2}(e_3 - e_1 + \tau_m e_7) \quad \frac{1}{2}(e_4 - e_3 + (\tau_a - \tau_m)e_8) \quad \rho_9\Big],\\
\rho_{11} &= [e_3 - e_4 \quad \sqrt{3}(e_3 + e_4 - e_8)],\\
\rho_{12} &= [e_4 - e_2 \quad \sqrt{3}(e_4 + e_2 - e_{13})],\\
\rho_{13} &= [e_2 - e_5 \quad \sqrt{3}(e_2 + e_5 - e_{12})],\\
\rho_{14} &= [\rho_{12} \quad \rho_{13}],\\
\Upsilon &= [A - I_n \;\; A_1 \;\; 0_{n\times 3n} \quad -I_n \;\; 0_{n\times 7n}].
\end{aligned}
$$

We now derive a stability criterion for system (17) in the following theorem.

Theorem 1. *System* (17) *is stable if there exist a $4n \times 4n$ positive-definite matrix P, six $n \times n$ positive-definite matrices Q_1, Q_2, Q_3, S_1, S_2, S_3 and two $2n \times 2n$ matrices Y_2, Y_3, such that the following linear matrix inequalities hold*

$$
\Theta_i = \begin{bmatrix} \widetilde{S}_i & Y_i \\ \star & \widetilde{S}_i \end{bmatrix} > 0, \; i = 2, 3, \tag{18}
$$

$$
[\Upsilon^\perp]^T \Sigma_1(\tau(k))[\Upsilon^\perp] \le 0, \quad \forall \tau(k) \in \{\tau_m, \tau_a\}, \tag{19}
$$

$$
[\Upsilon^\perp]^T \Sigma_2(\tau(k))[\Upsilon^\perp] \le 0, \quad \forall \tau(k) \in \{\tau_a, \tau_M\}, \tag{20}
$$

where $\Upsilon^{\perp}$ *is the right orthogonal complement of* Υ *and*

$$\begin{aligned}
\widetilde{S}_i &= \begin{bmatrix} S_i & 0 \\ 0 & S_i \end{bmatrix}, \ i=1,2,3, \\
\Sigma_1(\tau(k)) &= \rho_1 P \rho_1^T + 2\rho_1 P \rho_3^T + e_1 Q_1 e_1^T + e_3(Q_2 - Q_1)e_3^T - e_4(Q_2 - Q_3)e_4^T - e_5 Q_3 e_5^T \\
&\quad + e_6(\tau_m^2 S_1 + (\tau_a - \tau_m)^2 S_2 + (\tau_M - \tau_a)^2 S_3)e_6^T - \rho_4 \widetilde{S}_1 \rho_4^T - \rho_7 \Theta_2 \rho_7^T - \rho_8 \widetilde{S}_3 \rho_8^T, \\
\Sigma_2(\tau(k)) &= \rho_1 P \rho_1^T + 2\rho_1 P \rho_{10}^T + e_1 Q_1 e_1^T + e_3(Q_2 - Q_1)e_3^T - e_4(Q_2 - Q_3)e_4^T - e_5 Q_3 e_5^T \\
&\quad + e_6(\tau_m^2 S_1 + (\tau_a - \tau_m)^2 S_2 + (\tau_M - \tau_a)^2 S_3)e_6^T - \rho_4 \widetilde{S}_1 \rho_4^T - \rho_{11} \widetilde{S}_2 \rho_{11}^T - \rho_{14} \Theta_3 \rho_{14}^T.
\end{aligned}$$

Proof. Consider the following Lyapunov-Krasovskii functional

$$V = V_1 + V_2 + V_3, \tag{21}$$

where

$$\begin{aligned}
V_1 &= \zeta(k)^T P \zeta(k), \\
V_2 &= \sum_{s=k-\tau_m}^{k-1} x^T(s) Q_1 x(s) + \sum_{s=k-\tau_a}^{k-\tau_m-1} x^T(s) Q_2 x(s) + \sum_{s=k-\tau_M}^{k-\tau_a-1} x^T(s) Q_3 x(s), \\
V_3 &= \tau_m \sum_{s=-\tau_m}^{-1} \sum_{v=k+s}^{k-1} y^T(v) S_1 y(v) + (\tau_a - \tau_m) \sum_{s=-\tau_a}^{\tau_m-1} \sum_{v=k+s}^{k-1} y^T(v) S_2 y(v) \\
&\quad + (\tau_M - \tau_a) \sum_{s=-\tau_M}^{\tau_a-1} \sum_{v=k+s}^{k-1} y^T(v) S_3 y(v).
\end{aligned}$$

Since $||\zeta(k)|| \geq ||x(k)||$, we have

$$V \geq \lambda_{min}(P)||x(k)||^2. \tag{22}$$

Taking the the forward difference of V, we have

$$\begin{aligned}
\Delta V_1 &= \Delta\zeta^T(k) P \Delta\zeta(k) + 2\Delta\zeta^T(k) P \zeta(k) = \xi^T(k)\Big\{\rho_1 P \rho_1^T + 2\rho_1 P \rho_3^T\Big\}\xi(k), && (23) \\
\Delta V_2 &= x^T(k) Q_1 x(k) + x^T(k-\tau_m)(Q_2 - Q_1)x(k-\tau_m) - x^T(k-\tau_a)(Q_3 - Q_2)x(k-\tau_a) \\
&\quad - x^T(k-\tau_M) Q_3 x(k-\tau_M) \\
&= \xi^T(k)\Big\{e_1 Q_1 e_1^T + e_3(Q_2 - Q_1)e_3^T + e_4(Q_3 - Q_2)e_4^T - e_5 Q_3 e_5^T\Big\}\xi(k), && (24) \\
\Delta V_3 &= y^T(k)[\tau_m^2 S_1 + (\tau_a - \tau_m)^2 S_2 + (\tau_M - \tau_a)^2 S_3]y(k) - \tau_m \sum_{s=k-\tau_m}^{k-1} y^T(s) S_1 y(s) \\
&\quad - (\tau_a - \tau_m) \sum_{s=k-\tau_a}^{k-\tau_m-1} y^T(s) S_2 y(s) - (\tau_M - \tau_a) \sum_{s=k-\tau_M}^{k-\tau_a-1} y^T(s) S_3 y(s). && (25)
\end{aligned}$$

Now, we consider system (17) for two cases: (i) $\tau_m \leq \tau(k) \leq \tau_a$; and (ii) $\tau_a \leq \tau(k) \leq \tau_M$. For Case (i), using (9), (18), Lemma 3 and Lemma 4, we have

$$-\tau_m \sum_{s=k-\tau_m}^{k-1} y^T(s) S_1 y(s) \leq -\xi^T(k) \rho_4 \widetilde{S}_1 \rho_4^T \xi(k), \tag{26}$$

$$\begin{aligned}
-(\tau_a-\tau_m)\sum_{s=k-\tau_a}^{k-\tau_m-1}y^T(s)S_2y(s) &= -(\tau_a-\tau_m)\Big(\sum_{s=k-\tau_a}^{k-\tau(k)-1}y^T(s)S_2y(s)+\sum_{s=k-\tau(k)}^{k-\tau_m-1}y^T(s)S_2y(s)\Big)\\
&\le -\xi^T(k)\Big\{\frac{\tau_a-\tau_m}{\tau_a-\tau(k)}\rho_5\widetilde{S}_2\rho_5^T+\frac{\tau_a-\tau_m}{\tau(k)-\tau_m}\rho_6\widetilde{S}_2\rho_6^T\Big\}\xi(k)\\
&\le \xi^T(k)\rho_7\Theta_2\rho_7^T\xi(k), \qquad (27)
\end{aligned}$$

and

$$-(\tau_M-\tau_a)\sum_{s=k-\tau_M}^{k-\tau_a-1}y^T(s)S_3y(s)\le -\xi^T(k)\rho_8\widetilde{S}_3\rho_8^T\xi(k). \qquad (28)$$

From (25), (26), (27) and (28), we obtain

$$\begin{aligned}
\Delta V_3 \le\ & \xi^T(k)\Big\{e_6(\tau_m^2S_1+(\tau_a-\tau_m)^2S_2+(\tau_M-\tau_a)^2S_3)e_6^T\\
&-\rho_4\widetilde{S}_1\rho_4^T-\rho_7\Theta_2\rho_7^T-\rho_8\widetilde{S}_3\rho_8^T\Big\}\xi(k). \qquad (29)
\end{aligned}$$

Adding (23), (24), (29), we obtain

$$\Delta V\le\xi^T(k)\Sigma_1(\tau(k))\xi(k). \qquad (30)$$

On the other hand, we can verify that $[\Upsilon^\perp]^T\Sigma_1(\tau(k))[\Upsilon^\perp]$ is linear with respect to $\tau(k)$. Consequently, $[\Upsilon^\perp]^T\Sigma_1(\tau(k))[\Upsilon^\perp]$ is convex with respect to $\tau(k)$. Hence, if condition (19) holds then we have

$$[\Upsilon^\perp]^T\Sigma_1(\tau(k))[\Upsilon^\perp]\le 0,\quad \forall\tau(k)\in\{\tau_m,\tau_m+1,\cdots,\tau_a\}. \qquad (31)$$

Combining (30), (31) and Lemma 5, we have

$$\Delta V\le\xi^T(k)\Sigma_1(\tau(k))\xi(k)\le 0,\quad \forall\tau(k)\in\{\tau_m,\tau_m+1,\cdots,\tau_a\}. \qquad (32)$$

For Case (ii), similarly, we also obtain

$$-(\tau_a-\tau_m)\sum_{s=k-\tau_a}^{k-\tau_m-1}y^T(s)S_2y(s)\ \le\ \xi^T(k)\rho_{11}\widetilde{S}_2\rho_{11}^T\xi(k), \qquad (33)$$

$$\begin{aligned}
-(\tau_M-\tau_a)\sum_{s=k-\tau_M}^{k-\tau_a-1}y^T(s)S_3y(s) &= -(\tau_M-\tau_a)\Big(\sum_{s=k-\tau_M}^{k-\tau(k)-1}y^T(s)S_3y(s)+\sum_{s=k-\tau(k)}^{k-\tau_a-1}y^T(s)S_3y(s)\Big)\\
&\le -\xi^T(k)\rho_{14}\Theta_3\rho_{14}^T\xi(k). \qquad (34)
\end{aligned}$$

$$\begin{aligned}
\Delta V_3 \le\ & \xi^T(k)\Big\{e_6(\tau_m^2S_1+(\tau_a-\tau_m)^2S_2+(\tau_M-\tau_a)^2S_3)e_6^T\\
&-\rho_4\widetilde{S}_1\rho_4^T-\rho_{11}\widetilde{S}_2\rho_{11}^T-\rho_{14}\Theta_3\rho_{14}^T\Big\}\xi(k). \qquad (35)
\end{aligned}$$

and

$$\Delta V\le\xi^T(k)\Sigma_2(\tau(k))\xi(k)\le 0,\quad \forall\tau(k)\in\{\tau_a,\tau_a+1,\cdots,\tau_M\}. \qquad (36)$$

Combining (32), (36) and (22), it follows that system (17) is stable. The proof of Theorem 1 is completed. □

Remark 2. Different from existing results where the discrete Jensen inequality (16) is used to estimate the derivative of Lyapunov-Krasovskii functionals, in this paper, a new inequality (9), which is tighter than the inequality (16), is used to estimate the derivative of the Lyapunov-Krasovskii functional (21). Consequently, our derived stability criterion using the new inequality (9) is less conservative than the one derived using the inequality (16) [8, Theorem 3]. In the numerical examples, we demonstrate this improvement by comparing our derived stability criterion to a stability criterion reported in [8, Theorem 3] which is based on the discrete Jensen inequality.

Remark 3. The delay-decomposition technique is used in this paper for the purpose of reducing the conservatism of derived stability criterion. The value of τ_a has an important impact on the conservativeness of the derived stability criterion. Since τ_a is an integer and belongs to an interval $[\tau_m, \tau_M]$, its value is in a finite set. Therefore, we can use one-dimensional search method to find the optimal value τ_a such that the derived stability criterion is the least conservative one. Note that the optimal genetic algorithm proposed in [8] can also give the optimal value τ_a. Furthermore, for a given discrete-time linear system with time-varying delay $\tau(k) \in [\tau_m, \tau_M]$ where τ_m is given, the following optimization problem produces the maximum allowable τ_M:

$$\begin{aligned} (OP): \quad & \max \ \tau_M \\ & \text{subject to} \quad a)\ \tau_a \in [\tau_m, \tau_M] \\ & \qquad\qquad\qquad b)\ (18), (19), (20). \end{aligned}$$

Note that if τ_m, τ_a, τ_M are fixed then (18),(19),(20) are linear matrix inequalities. Since τ_a and τ_M are finite, we can incorporate a two-dimensional search method into the LMI toolbox in Matlab to solve the above optimization problem (OP). This is given in the following algorithm.

Algorithm
Step 1: Set $\tau_M = \tau_m$. Check conditions (18),(19),(20) for $\tau_a = \tau_m$. If they hold, go to Step 2.
Step 2: Set $\tau_M = \tau_M + 1$. Check conditions (18),(19),(20) for integer τ_a varying from τ_m to τ_M.

- If there exists an integer $\tau_a \in [\tau_m, \tau_M]$ such that (18),(19),(20) hold, then repeat Step 2.
- If not, stop the algorithm and obtain the maximum allowable value of delay as $\tau_M - 1$.

Remark 4. The disadvantage of the delay-decomposition technique is that the number of variables can be very large. By setting $\tau_a = \tau_M$, $Q_3 = S_3 = 0$, $Y_3 = 0$, $P = \begin{bmatrix} P_{3n\times 3n} & 0_{3n\times n} \\ 0_{n\times 3n} & 0_{n\times n} \end{bmatrix}$, i.e., the delay-decomposition technique is not used, then stability criterion obtained in Theorem 1 is reduced to a stability criterion with a smaller number of variables. Although this stability criterion is more conservative than the one given in Theorem 1, it is still less conservative than most existing stability criteria. This is shown through numerical examples in the next section.

Remark 5. The number of variables in Theorem 1, Theorem 3 in [8], and Remark 4 are $19n^2+5n$, $\frac{11n^2+7n}{2}$ and $\frac{21n^2+7n}{2}$, respectively.

4. Numerical examples

In this section, we consider two numerical examples to show the effectiveness of our newly derived stability condition.

Table 1
The allowable upper bounds τ_M with different τ_m for Example 1.

Methods\ τ_m	2	4	6	10	15	20	25	30	Number of variables
[24]	13	13	14	15	18	22	26	30	143
[25]	13	13	14	17	20	24	29	33	42
[7](Thm. 3)	20	20	20	21	24	27	31	35	76
[31]	14	15	16	18	21	25	30	34	18
[33](Prop.2)	14	15	16	18	21	25	30	34	18
[33](Prop.1)	17	17	18	20	23	27	31	35	38
[34]	17	17	18	20	23	27	31	35	22
[35]	18	18	19	20	23	26	30	35	21
[37]	17	17	18	20	23	27	31	35	42
[38](Coro.1)	17	17	18	20	23	27	31	35	19
[38](Thm. 1)	19	19	20	21	24	27	31	35	31
[39]	20	21	21	22	24	27	29	34	49
[8](Thm. 3, l=2)	19	19	20	21	24	27	31	35	29
[8](Thm. 4, l=4)	21	21	21	22	24	27	31	35	49
[40](Thm.1)	17	17	18	20	24	27	31	36	117
[40](Thm. 2)	22	22	22	23	25	28	32	36	126
Remark 4	20	21	21	23	25	29	32	36	49
Theorem 1	22	22	22	23	26	29	32	36	86

Table 2
The allowable upper bounds τ_M with $\tau_m = 4, 12, 16$ for comparison with [30, 36].

	τ_M				Number of variables			
τ_m	[30]	[36]	Remark 4	Theorem 1	[30]	[36]	Remark 4	Theorem 1
4	18	19	21	22	$345(m = 4, \tau = 1)$	82	49	86
12	22	22	24	24	$185(m = 2, \tau = 6)$	82	49	86
16	25	25	26	26	$345(m = 4, \tau = 4)$	82	49	86

Example 1. Consider system (18) with

$$A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \ A_1 = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}.$$

In this example, we compare our result with most of the recent existing results by computing the allowable upper bounds τ_M with different τ_m and they are listed in Table 1. Since the number of variables of the approach in [30] is not consistent, we also provide Table 2, which is adopted from the table in [36], in order to compare our result with [30, 36]. From Table 1 and Table 2, it can be seen that Theorem 1 and Remark 4 of this paper mostly provide larger upper bounds and also with a moderate number of variables than those reported in the existing literature.

Example 2. Consider system (17), which was studied in [38], where

$$A = \begin{bmatrix} 0.7 & 0.1 \\ 0.05 & 0.7 \end{bmatrix}, \ A_1 = \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & -0.2 \end{bmatrix}.$$

Table 3
The allowable upper bounds τ_M with different τ_m for Example 2.

Methods\ τ_m	2	5	6	7	10	20	Number of variables
[25]	7	9	10	11	14	24	42
[31]	8	10	11	12	15	25	18
[34]	9	11	12	13	16	26	22
[37]	9	11	12	13	16	26	42
[38](Coro.1)	9	11	12	13	16	26	19
[38](Thm. 1)	12	14	15	16	19	29	31
[7](Thm. 3)	12	14	15	16	19	29	76
[35]	10	12	13	14	17	27	21
[39]	13	14	15	16	19	29	49
[8](Thm. 3, l=2)	11	13	14	15	18	28	29
[8](Thm. 4, l=4)	13	14	15	16	19	29	49
[40](Thm.1)	9	12	13	14	17	27	117
[40](Thm. 2)	16	19	20	21	23	33	126
Remark 4	13	16	17	18	20	30	49
Theorem 1	14	17	18	18	21	31	86

In this example, we create Table 3 to list the allowable upper bounds τ_M with different τ_m. Generally, from Table 3 it can be seen that Theorem 1 and Remark 4 of this paper provide larger upper bounds τ_M. Note that, in this example, Theorem 2 in [40] with the largest number of variables gives the largest upper bounds τ_M.

5. Conclusion

This paper has presented a new inequality, which is a discrete version of the Wirtinger-based integral inequality, and a new stability condition for linear discrete systems with an interval time-varying delay. It is envisaged that inequality (9) will find many applications in stability analysis and stabilization of discrete-time systems with time-varying delays. Two numerical examples with extensive comparison to recent existing results have been studied to show the effectiveness of the derived result.

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