CHARACTERIZATIONS OF EXPONENTIAL DICHOTOMY OF LINEAR SKEW-PRODUCT SEMIFLOWS OVER SEMIFLOWS

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Abstract. Using the method of discretization, we investigate the necessary and sufficient conditions for the existence of exponential dichotomy of linear skew-product semiflows over semiflows through the existence of discrete exponential dichotomy of the discretized linear-skew product semiflows. We then apply the obtained results to consider the roughness of exponential dichotomy of linear-skew product semiflows.

1. INTRODUCTION AND PRELIMINARIES

The notion of linear skew-product semiflows (LSPS) arises naturally when one considers the linearization along an invariant manifold of a dynamical system (generated by an autonomous differential equation), especially in the infinite dimensional case [18, Chapt. 4]. On the other hand, one can associate to a linear nonautonomous differential equation a linear skew-product semiflow whose asymptotic behavior such as exponential stability or dichotomy is closely related to that of the equation under consideration (see Sacker and Sell [10, 11, 12, 13, 15, 16] and Hale [4]). Recently (see, e.g., Chow and Leiva [2], Latushkin and Schnaubelt [3], Sell and You [18]), there has been an increasing interest in the asymptotic behavior of linear skew-product semiflows over flows. To the best of our knowledge, the main results in this direction are focused on the characterization of exponential dichotomy of linear skew-product semiflows over flows in terms of Sacker-Sell spectral properties [10] or the hyperbolicity of the associated evolution semigroups and their generators [3], [2]. In particular, a characterization of exponential dichotomy for linear skew-product semiflows over flows was given in [10] assuming the dimension of the unstable manifold to be finite. Meanwhile, in [3] a characterization is given through the hyperbolicity of the associated evolution semigroup and its generator. Another characterization in [2] uses discrete linear skew-product semiflows over discretized flows.

Once the exponential dichotomy is characterized, it is natural to study its robustness (linear perturbation) and the existence of invariant manifolds (nonlinear perturbation). For more information in this direction we refer the reader to [1], [2], [10] and the references therein.

In this paper we will make an attempt to characterize exponential dichotomy in a more general setting and consider *linear skew-product semiflows over semiflows*, i.e., there is only a semiflow on the base space. This setting is particularly appropriate in the infinite dimensional case since in this case the dynamical systems restricted to invariant manifolds are only semiflows in general.

Key words and phrases. Linear Skew-product Semiflows, Discrete skew-product, uniform discrete dichotomy, pointwise discrete dichotomy, exponential dichotomy, perturbations.

Our method is to discretize the linear skew-product semiflow to obtain the corresponding discrete linear skew-product semiflow. This is a natural extension of the discretizing technique which can be traced back to Henry [5]. As a result, we obtain a characterization of the exponential dichotomy of a linear skew-product semiflow over a semiflow. We then use this characterization to prove the robustness of exponential dichtomy. Our results are contained in theorems 3.5, 3.7, 3.10, 4.2 and 4.3. We now start by some preliminaries.

Consider the (trivial) Banach bundles $\mathcal{E} := X \times \Theta$ where X is a fixed Banach space(the state space) and Θ is Hausdorff metric space(the base space). Throughout this paper we shall consider the following sequence spaces endowed with the sup-norm.

$$\begin{aligned} l_\infty(\mathbb{N}, X) &:= \{v = \{v_n\}_{n\in\mathbb{N}} : v_n \in X : \sup_{n\in\mathbb{N}} \|v_n\| < \infty\} := l_\infty \\ l^0_\infty(\mathbb{N}, X) &:= \{v = \{v_n\} \in l_\infty;\ v_0 = 0\} := l^0_\infty \\ l_\infty([n_0, \infty), X) &:= \{v = \{v_n\} \in l_\infty;\ 0 < n_0 \leqslant n \in \mathbb{N}\}. \end{aligned}$$

On the base space Θ, the semiflow is now defined as follows.

Definition 1.1. A family $(\varphi^t)_{t\geq 0}$ of maps $\varphi^t : \Theta \to \Theta$ is called a continuous semiflow on Θ if the following properties hold:

(i) The map $(\theta, t) \to \varphi^t\theta$ is continuous,
(ii) $\varphi^0\theta = \theta; \varphi^{t+s} = \varphi^t\varphi^s$ for all $t, s \in \mathbb{R}_+$ and $\theta \in \Theta$.

Given a semiflow, the linear skew-product semiflow (LSPS) can then be defined as follows.

Definition 1.2. A linear skew-product semiflow $\Pi = (\Phi; \varphi)$ on $\mathcal{E} = X \times \Theta$ is a mapping:

$$\begin{aligned} \Pi : \mathcal{E} \times \mathbb{R}_+ &\to \mathcal{E} \\ \Pi(x, \theta, t) &= (\Phi(\theta; t)x; \varphi^t\theta) \end{aligned}$$

where $(\varphi^t)_{t\geq 0}$ is the semiflow on Θ and $(\Phi(\theta; t))_{\theta\in\Theta;t\in\mathbb{R}_+}$ is the so-called (strongly continuous, exponential bounded) cocycle satisfying the following properties.

(1) $\Phi(\theta; t) \in \mathcal{L}(X)$; $\Phi(\theta; 0) = Id$ (the indentity operator on X) for all $\theta \in \Theta; t \in \mathbb{R}_+$.
(2) The map $(\theta; t) \to \Phi(\theta; t)x$ are continuous for each $x \in X$.
(3) The cocycle identity satisfied ,i.e.,

$$\Phi(\theta; t + s) = \Phi(\varphi^t\theta; s)\Phi(\theta; t) \text{ for all } t, s \geq 0 \text{ and } \theta \in \Theta.$$

(4) There exist constants N, α such that

$$\|\Phi(\theta; t)\| \leqslant Ne^{\alpha t} \qquad \text{for} \quad t \geq 0 \quad \text{and} \quad \theta \in \Theta.$$

To define and study the exponential dichotomy of LSPS we need the following concept of projector on $\mathcal{E}$.

Definition 1.3. (i) A mapping $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is said to be a projector if $\mathbf{P}$ has the form

$\mathbf{P}(x; \theta) = (P(\theta)x; \theta)$ where, $P(\theta) \in \mathcal{L}(X)$ is a projection on X for each $\theta \in \Theta$.

(ii) The projector $\mathbf{P}$ is said to be invariant with respect to the cocycle $(\Phi(\theta; t))_{\theta\in\Theta;t\geq 0}$ if it satisfies

$$P(\varphi^t\theta)\Phi(\theta; t) = \Phi(\theta; t)P(\theta); \qquad t \geq 0; \theta \in \Theta.$$

We now define the exponential dichotomy of a LSPS over semiflows.

Definition 1.4. A linear skew-product semiflow (LSPS) Π on a Banach bundles $X \times \Theta$ is said to have an exponential dichotomy if there exists an invariant projector $\mathbf{P}$ such that :

(a) $\Phi(\theta;t)\mid_{\ker P(\theta)}$: $\ker P(\theta) \to \ker P(\varphi^t\theta)$ is an isormorphirsm whose inverse is denoted by
$$\Phi(\varphi^t\theta;-t) : \ker P(\varphi^t\theta) \to \ker P(\theta),$$
(b) There exist constants $N;\nu > 0$ such that
$\|\Phi(\theta;t)x\| \leqslant Ne^{-\nu t}\|x\|$ for $x \in P(\theta)X; t \geq 0$,
$\|\Phi(\theta;t)x\| \leqslant Ne^{\nu t}\|x\|$ for $x \in \ker P(\theta); t < 0$.

2. DISCRETE LINEAR SKEW-PRODUCT SEMIFLOWS OVER DISCRETIZED SEMIFLOWS

As said above, to investigate the exponential dichotomy of a LSPS we use the method of discretization. That is, we discretize the (continuous) LSPS to obtain some kind of discrete skew-product. We next use the so-called input-output technique (see [5, 2]) to obtain characterizations of the exponential dichotomy of discrete skew-product. We then convert the results to the (continuous) LSPS by proving the equivalence between the existence of exponential dichotomy of LSPS and that of its dicretized skew-product. We now start with the basic definition and properties of discrete LSPS.

Definition 2.1. Let $(\varphi^t\theta)_{t\geq 0}$ is a semiflow on Θ. A discrete skew-product is a mapping $\Pi^* = (\Phi;\varphi) : \mathcal{E} \times \mathbb{N} \to \mathcal{E}$ which is given by
$$\Pi^*(x,\theta,n) = (\Phi_n(\theta)x;\varphi^n\theta)$$
where, $\Phi_n(\theta) \in \mathcal{L}(X)$ for each $(n,\theta) \in \mathbb{N}\times\Theta$, satisfying the following properties:
(i) there exists $\rho > 0$ such that $\|\Phi_n(\theta)\| \leqslant \rho$ for all $n \in \mathbb{N}$ and $\theta \in \Theta$,
(ii) the mapping $(x;\theta) \to \Phi_n(\theta)x$ is continuous for each $n \in \mathbb{N}$.

Remark 2.2. For a (continuous) LSPS $\Pi = (\Phi,\varphi)$ we will discretize it by simply setting $\Phi_n(\theta) := \Phi(\varphi^n\theta, 1)$. Then we obtain the corresponding discrete skew-product $\Pi^* = (\Phi;\varphi)$.

We now define the exponential dichotomy of a discrete skew-product.

Definition 2.3. We say that a discrete skew product Π^* has an pointwise discrete dichotomy over Θ if for each $\theta \in \Theta$ there exist the positive constants M_θ, $\alpha_\theta < 1$; and the family of projections $\{P_n(\theta)\}_{n\in\mathbb{N}}$ in X such that:
(1) $\Phi_n(\theta)P_n(\theta) = P_{n+1}(\theta)\Phi_n(\theta)$
(2) $\Phi_n(\theta)\mid_{KerP_n(\theta)}$: $KerP_n(\theta) \to KerP_{n+1}(\theta)$ is an isomorphirsm, and we denote its inverse by $\Phi_n^{-1} : KerP_{n+1}(\theta) \to KerP_n(\theta) \qquad n \geq 0$
(3)
$$\|\Phi_{n,m}(\theta)x\| \leqslant M_\theta\alpha_\theta^{n-m}\|x\| \text{ for } x \in P_m(\theta)X \text{ and } n \geq m \geq 0$$
$$\text{where} \quad \Phi_{n,m}(\theta) = \Phi_{n-1}(\theta)\Phi_{n-2}(\theta)...\Phi_m(\theta)(n > m) \quad \text{and} \quad \Phi_{m,m} = I$$
(4)
$$\|\Phi_{n,m}(\theta)x\| \leqslant M_\theta\alpha_\theta^{n-m}\|x\| \text{ for } x \in KerP_m(\theta) \text{ and } n < m$$
$$\text{where} \quad \Phi_{n,m}(\theta) = \Phi_{m+1}^{-1}(\theta)\Phi_{m+2}^{-1}(\theta)...\Phi_n^{-1}(\theta)(n < m)$$

The projections $P_n(\theta)$ and the constants M_θ, α_θ are called dichotomy projections and dichotomy constants, respectively.

Remark 2.4. If Π_* has a pointwise discrete dichotomy over Θ then by the same argument as in [3] we can see that $\sup_{n\geq 0}\|P_n(\theta)\| < \infty$. Therefore in definition 2.2, the condition (3) is equivalent to the condition $\|\Phi_{n,m}(\theta)P_m(\theta)\| \leqslant M_\theta\alpha_\theta^{n-m}(n \geq m \geq 0)$ and,the condition(4) is equivalent to the condition $\|\Phi_{n,m}(\theta)(Id - P_m(\theta))\| \leqslant M_\theta\alpha_\theta^{n-m}(n < m)$.

Definition 2.5. We say that a discrete skew product Π^* has a uniform discrete dichotomy over Θ if in definition 2.3 the dichotomy contants M_θ, α_θ are independent of θ (i.e, $M_\theta = M; \alpha_\theta = \alpha \quad$ for all $\quad \theta \in \Theta$).

From definition 2.3 the following properties are obivious.

Remark 2.6. For the discrete skew-product having an exponential dichotomy, the following assertions hold.

i) $\Phi_{n,m}(\theta)\Phi_{m,k}(\theta) = \Phi_{n,k}(\theta)$,
ii) $\Phi_n(\theta)\Phi_{n,m}(\theta) = \Phi_{n+1,m}(\theta) \quad$ for all $n > m \geq k$,
iii) $\Phi_{n,m}(\theta)P_m(\theta) = P_n(\theta)\Phi_{n,m}(\theta) \quad$ for all $n \geq m$,
iv) $\Phi_{n,n-1}(\theta) = \Phi_{n-1}(\theta); \qquad \Phi_{n,r}(\theta)\Phi_{r-1}(\theta) = \Phi_{n,r-1}(\theta)$ for $1 \leqslant r < n$.

For later use, we state the following lemma whose proof can be done easily by induction.

Lemma 2.7. *Let $\Pi^* : \mathcal{E} \times \mathbb{N} \to \mathcal{E}$ be a discrete skew product.*
If for a sequence $y = \{y_n\}$ in X and $\theta \in \Theta$ we have $\quad x_{n+1} = \Phi_n(\theta)x_n + y_n \quad \forall n \in \mathbb{N} \quad$ then

$$x_n = \Phi_{n,m}(\theta)x_m + \sum_{k=m}^{n-1}\Phi_{n,k+1}(\theta)y_k; \quad n > m \geq 0.$$

We also need the following notion of the discrete Green's function.

Definition 2.8. Assume that a discrete skew product Π^* has pointwise discrete dichotomy with corresponding dichotomy constants $M_\theta; \alpha_\theta$ and projections $\{P_n(\theta)\}_{n\in\mathbb{N}}$. We define the discrete Green's function as follows:

$$G_{n,m}(\theta) = \begin{cases} \Phi_{n,m}(\theta)P_m(\theta) & for \quad n \geq m \geq 0, \\ -\Phi_{n,m}(\theta)(Id - P_m(\theta)) & for \quad 0 \leqslant n < m. \end{cases}$$

Then, it follows directly from the definition 2.3 that $\|G_{n,m}(\theta)\| \leqslant M_\theta\alpha_\theta^{|n-m|} \quad \forall n; m \in \mathbb{N}$.
The following lemma plays an important role in our strategy.

Lemma 2.9. *Let $\quad \Pi^* : \mathcal{E} \times \mathbb{N} \to \mathcal{E}$ be a discrete skew product which has pointwise discrete dichotomy over Θ. Consider $y = \{y_n\} \in l_\infty(\mathbb{N}; X)$ and let $x = \{x_n\}$ be a sequence with each element x_n belonging to X. Then, for each $\theta \in \Theta$ the following assertions are equivalent.*

i) *x and y satisfy $x_{n+1} = \Phi_n(\theta)x_n + y_n \quad \forall n \in \mathbb{N}$, and $x \in l_\infty(\mathbb{N}; X)$.*
ii) *x and y satisfy $x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum\limits_{k=0}^{\infty} G_{n,k+1}(\theta)y_k$.*

Proof. $(i) \Rightarrow (ii)$: By lemma 2.7 we have:

$$x_r = \Phi_{r,n}(\theta)x_n + \sum_{k=n}^{r-1}\Phi_{r,k+1}(\theta)y_k \quad r > n \geq 0$$

Then $\Phi_{n,r}(\theta)[I - P_r(\theta)]x_r = \Phi_{n,r}(\theta)x_r - \Phi_{n,r}(\theta)P_r(\theta)x_r$
$= \Phi_{n,r}(\theta)[\Phi_{r,n}(\theta)x_n + \sum_{k=n}^{r-1}\Phi_{r,k+1}(\theta)y_k] - \Phi_{n,r}(\theta)P_r(\theta)[\Phi_{r,n}(\theta)x_n + \sum_{k=n}^{r-1}\Phi_{r,k+1}(\theta)y_k]$

$= x_n + \sum_{k=n}^{r-1} \Phi_{n,k+1}(\theta)y_k - \Phi_{n,r}(\theta)P_r(\theta)\Phi_{r,n}(\theta)x_n - \sum_{k=n}^{r-1} \Phi_{n,r}(\theta)P_r(\theta)\Phi_{r,k+1}(\theta)y_k.$
By (iii) in remark 2.6 we have:

$$\begin{aligned}\Phi_{n,r}(\theta)[I - P_r(\theta)]x_r =& x_n + \sum_{k=n}^{r-1} \Phi_{n,k+1}(\theta)y_k - \Phi_{n,r}(\theta)\Phi_{r,n}(\theta)P_n(\theta)x_n \\ &- \sum_{k=n}^{r-1} \Phi_{n,r}(\theta)\Phi_{r,k+1}(\theta)P_{k+1}(\theta)y_k \\ =& x_n + \sum_{k=n}^{r-1} \Phi_{n,k+1}(\theta)y_k - P_n(\theta)x_n - \sum_{k=n}^{r-1} \Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k.\end{aligned}$$

Hence $-G_{n,r}(\theta)x_r = [I - P_n(\theta)]x_n - \sum_{k=n}^{r-1} G_{n,k+1}(\theta)y_k$.
Since $\|G_{n,r}(\theta)x_r\| \leqslant M_\theta \alpha_\theta^{|r-n|}\|x_r\|$ and $x = \{x_n\} \in l_\infty(\mathbb{N}; X)$, we obtain that $\lim\limits_{r\to\infty} \|G_{n,r}(\theta)x_r\| \to \infty$ and the series $\sum_{k=n}^{r-1} G_{n,k+1}(\theta)y_k$ converges. Therefore,

$$[I - P_n(\theta)]x_n = \sum_{k=n}^{r-1} G_{n,k+1}(\theta)y_k. \tag{2.1}$$

On other hand $x_n = \Phi_{n,0}(\theta)x_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1}(\theta)y_k$.
Then $P_n(\theta)x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum_{k=0}^{n-1} \Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k$.
Thus

$$P_n(\theta)x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum_{k=0}^{n-1} G_{n,k+1}(\theta)y_k. \tag{2.2}$$

From (2.1) and (2.2) we get $x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum_{k=0}^{\infty} G_{n,k+1}(\theta)y_k$.

$(ii) \Rightarrow (i)$: Using remark 2.6, one can easily see that if the sequences $x = \{x_n\}$ and $y = \{y_n\}$ satisfy $x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum_{k=0}^{\infty} G_{n,k+1}(\theta)y_k$ then $x_{n+1} = \Phi_n(\theta)x_n + y_n \quad \forall n \in \mathbb{N}$.

To complete this assertion we shall prove $x \in l_\infty(\mathbb{N}; X)$. By definition 2.8 we have that $\|x_n\| \leqslant M_\theta \alpha_\theta^n \|x_0\| + \sum_{k=0}^{\infty} M_\theta \alpha_\theta^{|n-k-1|}\|y\|$. Hence, $\|x_n\| \leqslant M_\theta \alpha_\theta^n \|x_0\| + M_\theta \|y\| \frac{1+\alpha_\theta}{1-\alpha_\theta}$ finishing the proof.

Remark 2.10. Under the hypothesis of lemma 2.9, let $x, y \in l_\infty(\mathbb{N}; X)$ satisfy one of the equivalent assertions of this lemma, and let $x_0 \in \ker P_0(\theta)$. Then, by the above proof one can easily see that

$$\|x\| \leqslant \frac{M_\theta(1 + \alpha_\theta)}{1 - \alpha_\theta}\|y\| \tag{2.3}$$

for M_θ and α_θ being the dichotomy constants of discrete LSPS Π^*.

□

3. DISCRETE DICHOTOMY OF SKEW-PRODUCT ON THE HALF LINE

In this section we shall give the necessary and sufficient conditions for discrete skew product to have a pointwise discrete dichotomy and uniform discrete dichotomy. We begin with the definitions of some difference operators which are key tools in our strategy.

For each $\theta \in \Theta$ we define the operator $T_\theta : l_\infty(\mathbb{N}; X) \to l_\infty(\mathbb{N}; X)$ by

$$T_\theta(x)_n = x_{n+1} - \Phi_n(\theta)x_n \text{ for } x = \{x_n\} \in l_\infty(\mathbb{N}; X).$$

We can see that $\|x_{n+1} - \Phi_n(\theta)x_n\| \leqslant (1+\rho)\|x\|_{l_\infty}$. Therefore, we obtain that T_θ is a bounded linear operator from l_∞ to l_∞.

We denote the restriction of T_θ on l^0_∞ by $T_{0\theta}$, i.e., $T_{0\theta} : l^0_\infty \to l_\infty(\mathbb{N}; X)$ and $T_{0\theta}x = T_\theta x$ for $x \in l^0_\infty$. We also have that $T_{0\theta}$ is a bounded linear operator from l^0_∞ to l_∞.
Moreover, since $\ker T_\theta = \{\, u = \{u_n\} \in l_\infty(\mathbb{N}; X) : u_n = \Phi_{n,0}(\theta)u_0\}$ it follows that $T_{0\theta}$ is injective.

To estimate the growth and decay of the solutions we need the following lemmas which are taken from [7].

Lemma 3.1. *Let $\{\chi_n\}_{n_1 > n \geq n_0}$ be positive real numbers and let $c > 1$ and $K, \alpha > 0$ be constants such that $\chi_n \leqslant Ke^{\alpha(n-n_0)}$ and $\sum_{k=n_0}^{n} \chi_n \chi_k^{-1} \leqslant c$ with $n_0 \leqslant n < n_1$.
Then exist N, ν dependent only on K, c, α such that $\chi_n \leqslant Ne^{-\nu(n-n_0)}$ for all $n_0 \leqslant n < n_1$.*

Lemma 3.2. *Let $\{\chi_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Assume that there is a constant $c > 1$ such that $\sum_{k=m}^{n} \chi_m \chi_k^{-1} \leqslant c \quad \forall n \geq m \geq 0$.
Then exist N, ν dependent only on c such that $\chi_n \geq Ne^{\nu(n-m)}\chi_m \quad \forall n \geq m \geq 0$.*

Recall that for an operator B on a Banach space Y the approximation point spectrum $A\sigma(B)$ of B is the set of all complex number λ such that for every $\epsilon > 0$ there exists $y \in D(B)$ with $\|y\| = 1$ and $\|(\lambda - B)y\| \leqslant \epsilon$.

To characterize the stability and dichotomy of a discrete skew-product $\Pi^* = (\Phi_n, \varphi^n)$ we also need the following notion of stable subspaces $X_0(n_0, \theta)$ of X which are defined by

$$X_0(n_0, \theta) := \{\, x \in X : \sup \|\Phi_{n,n_0}(\theta)x\| < \infty\} \text{ for each } (n_0, \theta) \in \mathbb{N} \times \Theta.$$

The following theorem connects the spectral properties of $T_{0\theta}$ to the exponential stability of discrete bounded orbits.

Theorem 3.3. *Let the operator $T_{0\theta}$ define as above and $0 \notin A\sigma(T_{0\theta})$ then every discrete bounded orbits of the family $\Phi(\theta) = \{\Phi_{n,m}(\theta)\}_{n\geq m\geq 0}$ is exponential stable.*

Precisely, if $\sup_{n_0 \leq n \in \mathbb{N}}\{\|\Phi_{n,n_0}(\theta)x\|\} < \infty$ with $x \in X$ and $n_0 \geq 0$, then there exist positive constants $N_\theta; \nu_\theta < 1$ independent of n_0 and x such that:

$$\|\Phi_{n,n_0}(\theta)x\| \leqslant N_\theta \nu_\theta^{n-s} \|\Phi_{s,n_0}(\theta)x\|; \quad n \geq s \geq n_0$$

Proof. Firstly, we shall prove for the case $s = n_0$.
Since $0 \notin A\sigma(T_{0\theta})$ there exists $\delta_\theta > 0$ such that $\|T_{0\theta}v\| \geq \delta_\theta \|v\| \quad \forall v \in l_\infty(\mathbb{N}; X)$.
Replacing δ_θ by a smaller one, if necessary, we can suppose that $\delta_\theta < 1$. Consider $0 \neq x \in X$. Without loss of generality we can let $\|x\| = 1$. Put $u_n = \Phi_{n,n_0}x$; $n \geq n_0$; $n_1 = \sup\{n \geq n_0 : \Phi_{n,n_0}(\theta)x \neq 0\}$.
For any natural number n_2 satisfying $n_0 \leqslant n_2 \leqslant n_1$ we take

$$v = \{v_n\} \quad \text{with} \quad v_n = \begin{cases} 0 & 0 \leqslant n < n_0, \\ u_n \sum_{k=n_0}^{n} \frac{1}{\|u_k\|} & n_0 \leqslant n \leqslant n_2, \\ u_n \sum_{k=n_0}^{n_2} \frac{1}{\|u_k\|} & n \geq n_2; \end{cases}$$

$$f = \{f_n\} \quad \text{with} \quad f_n = \begin{cases} 0 & 0 \leqslant n < n_0 - 1, \\ \frac{u_{n+1}}{\|u_{n+1}\|} & n_0 - 1 \leqslant n < n_2, \\ 0 & n \geq n_2. \end{cases}$$

Then we have $v_{n+1} = \Phi_n(\theta)v_n + f_n \quad \forall n \geq 0; v \in l^0_\infty$ and $f \in l_\infty(\mathbb{N}; X)$.
It follows that $T_{0\theta}v = f$ and $\|f\| \geq \delta_\theta \|v\|$. That means that
$1 = \|f\| \geq \delta_\theta \sup_{n \geq n_0} \|u_n\| \sum_{k=n_0}^{n} \frac{1}{\|u_k\|}$, and hence, $\sum_{k=n_0}^{n} \frac{1}{\|u_k\|} \leqslant \frac{1}{\delta_\theta}$ for all $n \geq n_0$.
Moreover, from definition 2.1 (i) we get

$$\|u_n\| = \|\Phi_{n,n_0}(\theta)x\| \leqslant e^{(n-n_0)ln(\rho)}\|x\| = e^{(n-n_0)ln\rho}.$$

Lemma 3.1 yields the existences of the constants $N_\theta = \frac{1}{\delta_\theta}; \alpha_\theta = ln\left(\frac{1}{1-\delta_\theta}\right)$ depending only on θ such that $\|u_n\| \leqslant N_\theta e^{-\alpha_\theta(n-n_0)}$.
Now, we fix $s \geq n_0$, set $y := \Phi_{s,n_0}x$ then $\sup_{n \geq s} \|\Phi_{n,s}(\theta)y\| < \infty$ and

$$\|\Phi_{n,n_0}x\| = \|\Phi_{n,s}y\| \leqslant N_\theta e^{-\alpha_\theta(n-s)} \|\Phi_{s,n_0}x\| \quad \forall n \geq s \geq 0.$$

□

From this theorem we obtain the following corollary properties of stable subspace $X_0(n_0, \theta)$.

Corollary 3.4. Under the conditions of Theorem 3.3 we have

$$X_0(n_0, \theta) = \{\ x \in X : \|\Phi_{n,n_0}(\theta)x\| \leqslant N_\theta e^{-\alpha_\theta(n-n_0)}\|x\| : n \geq n_0\}$$

for certain positive constants $N_\theta; \alpha_\theta$. Therefore, $X_0(n_0, \theta)$ is a closed subspace of X.

We now come to our first main result. It characterizes the pointwise dichotomy of a discrete skew-product via properties of the operators T_θ and the subspaces $X_0(0, \theta)$ of X.

Theorem 3.5. *For the discrete skew product Π^* the following assertions are equivalent.*

(i) Π^* *has a pointwise discrete dichotomy over* Θ.
(ii) *For each $\theta \in \Theta$ the linear operator T_θ is surjective and the stable subspace $X_0(0, \theta)$ is complemented in X.*

Proof. (**i**) ⇒ (**ii**) Let Π^* has a pointwise discrete dichotomy over Θ with the corresponding family of projections $(P_n(\theta))_{(n,\theta)\in\mathbb{N}\times\Theta}$.

For $f \in l_\infty(\mathbb{N}; X)$ we define $v = (v_n) \in l_\infty(\mathbb{N}; X)$as follow:

$$v_n = \begin{cases} \sum_{k=1}^{n} \Phi_{n,k}(\theta)P_k(\theta)f_{k-1} - \sum_{k=n+1}^{\infty} \Phi^{-1}_{k,n}(\theta)(I - P_k(\theta))f_{k-1} & n \geq 1, \\ -\sum_{k=1}^{\infty} \Phi_{0,k}(\theta)(I - P_k(\theta))f_{k-1} & n = 0. \end{cases}$$

Then we have $v_{n+1} = \Phi_n(\theta)v_n + f_n$ and $v \in l_\infty(\mathbb{N}; X)$.
So $T_\theta v = f$, hence T_θ is surjective.

We now prove that$X_0(0, \theta) = P_0(\theta)X$. It is clear that $P_0(\theta)X \subset X_0(0, \theta)$.
Conversely, take $x \notin P_0(\theta)X$ then $(I - P_0(\theta))x \neq 0$ and we get the inequality

$$\|\Phi_{n,0}(\theta)x\| \geq \|\Phi_{n,0}(\theta)[I - P_0(\theta)]x\| - \|\Phi_{n,0}(\theta)P_0(\theta)x\| \geq [M^{-1}_\theta \alpha^{-n}_\theta - M_\theta \alpha^n_\theta]\|x\|.$$

Since $0 < \alpha_\theta < 1$ then $\|\Phi_{n,0}(\theta)x\| \to +\infty$ as $n \to +\infty$.
It follows that $x \notin X_0(0, \theta)$. So, we have $X_0(0, \theta) \subset P_0(\theta)X$. Therefore, $X_0(0, \theta) = P_0(\theta)X$.
This yields that $X_0(0, \theta)$ is complemented in X.

(**ii**) ⇒ (**i**) : We prove this in several steps.

A) Let $Z(\theta) \subset X$ be a complement of $X_0(0,\theta)$,i.e., $Z(\theta) \oplus X_0(0,\theta) = X$
For $n \geq 0$ set $X_1(n,\theta) = \Phi_{n,0}(\theta)Z(\theta)$. Then

$$\Phi_{n,s}(\theta)X_0(s,\theta) \subseteq X_0(n,\theta); \qquad \Phi_{n,s}(\theta)X_1(s,\theta) = X_1(n,\theta) \quad (n \geq s \geq 0)$$

B) We shall prove that there exist constants $N_\theta; \alpha_\theta > 0$ such that

$$\|\Phi_{n,m}(\theta)y\| \geq N_\theta e^{v_\theta(n-m)}\|y\| \quad \text{for} \quad n \geq s \geq 0; y \in X_1(m,\theta).$$

In fact ,let $Y(\theta) := \{\ v = \{v_n\}_{n\in\mathbb{N}} \in l_\infty(\mathbb{N};X) : v_0 \in Z(\theta)\}$ endowed with the l_∞-norm. Then $Y(\theta)$ is a closed subspace of the Banach space l_∞ and hence $Y(\theta)$ is complete.
Since $X = X_1(0,\theta) \oplus X_0(0,\theta)$ and T_θ is surjective, we obtain $T_\theta\ |_{Y(\theta)}: Y(\theta) \to l_\infty(\mathbb{N};X)$ is bijective so it is an isomorphism. Thus there is a constant $\delta_\theta > 0$ such that

$$\|T_\theta v\|_{l_\infty(\mathbb{N};X)} \geq \delta_\theta \|v\|_{l_\infty(\mathbb{N};X)} \quad \text{for all} \quad v \in Y(\theta) \tag{3.1}$$

For $x \neq 0; x \in X_1(0,\theta)$ set $u_n = \Phi_{n,0}(\theta)x; n \geq 0$. Then , we have $u_n \neq 0$ for every $n \geq 0$ (because if there exists $u_k = 0 (k \geq 0)$ then $x \in X_0(0,\theta)$ which is a contradiction since $x \neq 0; x \in X_1(0,\theta)$)
For a natural large number ξ we take $v = \{v_n\}$ and $f = \{f_n\}$ where:

$$v_n = \begin{cases} u_n \sum_{k=n+1}^{\xi} \frac{1}{\|u_k\|} & 0 \leqslant n < \xi, \\ 0 & n \geq \xi; \end{cases}$$

$$f_n = \begin{cases} \frac{-u_{n+1}}{\|u_{n+1}\|} & 0 \leqslant n < \xi, \\ 0 & n \geq \xi. \end{cases}$$

Then $v \in Y(\theta); f \in l_\infty(\mathbb{N};X)$ which satisfy the equation $v_{n+1} = \Phi_n v_n + f_n \quad \forall n \in \mathbb{N}$.
It follows that $T_\theta v = f \Rightarrow \|f\| \geq \delta_\theta \|v\|$. That means

$$1 \geq \delta_\theta \|u_n\| \sum_{k=n+1}^{\xi} \frac{1}{\|u_k\|} \text{ hence, } \|u_n\| \sum_{k=n+1}^{\xi} \frac{1}{\|u_k\|} \geq \frac{1}{\delta_\theta} + 1 \quad .$$

From lemma 3.2 we have that $\|u_n\| \geq N_\theta e^{v_\theta(n-m)}\|u_m\|; \quad n \geq m \geq 0$, where

$$N_\theta = \frac{\delta_\theta}{1+\delta_\theta}; v_\theta = ln(1+\delta_\theta).$$

Moreover, since $\Phi_{m,0}(\theta)X_1(0,\theta) = X_1(m,\theta)$, we have

$$\|\Phi_{n,m}(\theta)y\| \geq N_\theta e^{\nu_\theta(n-m)}\|y\| \qquad \forall y \in X_1(m,\theta), n \geq m \geq 0.$$

C) We prove $X = X_0(n,\theta) \oplus X_1(n,\theta) : n \in \mathbb{N}$.
Let $Y(\theta) \subset l_\infty(\mathbb{N};X)$as in **B**). Then $l_\infty^0 \subset Y(\theta)$ so we have $\|T_{0\theta}v\| \geq \delta_\theta\|v\|$ for $v \in l_\infty^0$.
Thus $0 \notin A\sigma(T_{0\theta})$ and corollary 3.4 implies that $X_0(n,\theta)$ is a closed subspace of X.
Moreover $X_0(n,\theta) \cap X_1(n,\theta) = \{\ 0\} \quad \forall n \in \mathbb{N}$ by parts **A**) and **B**) .
Finally, fix $n_0 > 0$ and $x \in X$; for large natural number n_1 set

$$v = \{v_n\} \quad \text{with} \quad v_n = \begin{cases} (n - n_0 + 1)\Phi_{n,n_0}(\theta)x & n_0 \leqslant n \leqslant n_1, \\ 0 & n > n_1. \end{cases}$$

$$f = \{f_n\} \quad \text{with} \quad f_n = \begin{cases} \Phi_{n+1,n_0}(\theta)x & n_0 \leqslant n < n_1, \\ -(n_1 + 1 - n_0)\Phi_{n+1,n_0}(\theta)x & n = n_1, \\ 0 & n > n_1. \end{cases}$$

Thus $v_{n+1} = \Phi_n(\theta)v_n + f_n : n \geq n_0 > 0$ and $v \in l_\infty([n_0;\infty);X)$. Set $f_n = 0$ for $0 \leqslant n_0 < n$. Then $f \in l_\infty(\mathbb{N};X)$. By assumption there exists $\omega \in l_\infty(\mathbb{N};X)$ such that $T_{0\theta}\omega = f$. By the definition of T_θ then $v_n - \omega_n = \Phi_{n,n_0}(\theta)(v_{n_0} - \omega_{n_0}) = \Phi_{n,n_0}(\theta)(x - \omega_{n_0})$ for $n \geq n_0$. It follows that $x - \omega_{n_0} \in X_0(\theta)(n_0)$. Since $f_n = 0$ for $0 \leqslant n < n_0$ then $\omega_{n_0} = \Phi_{n_0;0}(\theta)\omega_0$. Writing ω_0 in the form $\omega_0 + \omega_1$; for $\omega_k \in X_k(0)$ $(k = 0;1)$ we have

$$\omega_{n_0} = \Phi_{n_0;0}(\theta)\omega_0^0 + \Phi_{n_0;0}(\theta)\omega_0^1 \in X_0(\theta)(n_0) + X_1(\theta)(\theta)(n_0).$$

Therefore $x = [x - \omega_{n_0} + \Phi_{n_0;0}(\theta)\omega_0^0] + \Phi_{n_0;0}(\theta)\omega_0^1 \in X_0(\theta)(n_0) + X_1(\theta)(n_0)$.
Since $X_0(n_0,\theta) \cap X_1(n_0,\theta) = \{0\}$ we obtain that $X = X_0(n_0,\theta) \oplus X_1(n_0,\theta)$.
D) Let $P_n(\theta)$ be the projection from X on to $X_0(n,\theta)$ with kernel $X_1(n,\theta)$ then from part (**A**) we have

$$P_{n+1}(\theta)\Phi_n(\theta) = \Phi_n(\theta)P_n(\theta).$$

From part (**B**) and corollary (3.4) we have:

$$\begin{aligned} \|\Phi_{n,m}(\theta)x\| &\leqslant \frac{1+\delta_\theta}{\delta_\theta}e^{[ln(1+\delta_\theta)](n-m)}\|x\| \qquad \text{for} \quad x \in KerP_n(\theta); n < m, \\ \|\Phi_{n,m}(\theta)x\| &\leqslant \frac{1}{\delta_\theta}e^{(n-m)ln(1-\delta_\theta)}\|x\| \qquad \text{for} \quad x \in P_n(\theta)X; n \geq m. \end{aligned}$$

Thus Π^* has an pointwise discrete dichotomy with the dichotomy constants $\alpha_\theta = 1-\delta_\theta < 1$ and $M_\theta = \frac{1+\delta_\theta}{\delta_\theta}$. □

To characterize a discrete skew-product having uniform discrete dichotomy we need the following notion of uniform correctness of a family of operators.

Definition 3.6. The family $\{A_\theta\}_{\theta\in\Theta}$ is called uniformly correct if there exists a positive constant ν independent on θ such that $\|A_\theta x\| \geq \nu\|x\|$ for all $\theta \in \Theta$ and all $x \in D(A)$.

Theorem 3.7. *For a discrete skew product Π^* the following assertions are equivalent:*

(i) *Π^* has an uniform discrete dichotomy*
(ii) *For each $\theta \in \Theta$ we have that T_θ is surjective and $X_0(0,\theta)$ has a complement $Z(\theta)$ in X. Moreover, setting $Y(\theta) := \{\{v_n\}_{n\in\mathbb{N}} \in l_\infty(\mathbb{N};X) : v_0 \in Z(\theta)\}$ and $\{T_{Y\theta}\}$ being the restriction of $\{T_\theta\}$ on $Y(\theta)$, then the family $\{\ T_{Y\theta}\}_{\theta\in\Theta}$ is uniformly correct.*

Proof. (**ii**) ⇒ (**i**) : From the part **D**) in theorem 3.5 we have that $N_\theta; v_\theta$ depend only on θ. By the same technique which is given in [7, Lemma 3.1] we obtain that $M_\theta = \sup\{\ \|P_n(\theta)\| : n \geq 0\}$ is finite and depends only on θ. Since the family $\{\ T_{0\theta}\}_{\theta\in\Theta}$ is uniformly correct, there exists a positive constant δ independent of θ such that $\|T_{0\theta}\| \geq \delta$ for all $\theta \in \Theta$. Therefore, the constant δ_θ in inequality (3.1) can be replaced by δ for all $\theta \in \Theta$.
That means there exist $N;\nu > 0$ independent of θ such that:

$$\begin{aligned} \|\Phi_{n,m}(\theta)P_m(\theta)\| &\leqslant Ne^{-\nu(n-m)} \qquad n \geq m \geq 0, \\ \|\Phi_{n,m}(\theta)(I - P_m(\theta))\| &\leqslant Ne^{\nu(n-m)} \qquad 0 \leqslant n < m. \end{aligned}$$

Then it follows that Π^* has uniform discrete-dichotomy.
(**i**) ⇒ (**ii**): Let Π^* has uniform discrete dichotomy with the associated constants $N; \alpha = e^{-\nu} < 1$.
Due to theorem 3.5 we need only to prove that the family $\{\ T_{Y\theta}\}_{\theta\in\Theta}$ is uniformly correct. By the proof of theorem 3.5 (implication (ii)⇒(i)) we obtain that $T_{Y\theta}$ is an isomorphism. For $y \in l_\infty(\mathbb{N};X)$ take $x = T_{Y\theta}^{-1}y$ then, by the definition of $T_{Y\theta}$ and lemma 2.9 we have that $x_n = \Phi_{n,0}(\theta)P_0(\theta)x_0 + \sum_{k=0}^{\infty} G_{n,k+1}(\theta)y_k$.

Since $x_0 \in X_1(0,\theta)$ we obtain that $P_0(\theta)x_0 = 0$, and hence,
$\|T_{Y\theta}^{-1}y\| \leqslant \sum_{k=0}^{n-1} Ne^{-\nu(n-k-1)}\|y\| + \sum_{k=n}^{\infty} Ne^{\nu(n-k-1)}\|y\|$
$= \|T_{Y\theta}^{-1}y\| \leqslant N\|y\|e^{-\nu n}\frac{e^{\nu n}-e^{\nu}}{e^{\nu}-1} + N\|y\|\frac{e^{-\nu}}{1-e^{-\nu}} \leqslant 2N\frac{\|y\|}{\nu}$. Therefore, $\|T_{Y\theta}\| \geq \frac{\nu}{2N} > 0$.

□

From this theorem we obtain the following corollary.

Corollary 3.8. If the discrete dichotomy Π^* satisfy $\|\Phi_n(\theta)\| \geq \delta > 1$ (or $\|\Phi_n(\theta)\| \leqslant \delta < 1$) for all $(n,\theta) \in (\mathbb{N} \times \Theta)$ then the following assertions are equivalent.

(i) Π^* has an uniform discrete dichotomy.

(ii) For each $\theta \in \Theta$ the operator T_θ is surjective and $X_{0\theta}$ is complemented in X.

Definition 3.9. Let $(Z(\theta))_{\theta\in\Theta}$ be a family of closed subspaces of X. We define $l_\infty^{Z(\theta)} := \Big\{ x \in l_\infty(\mathbb{N};X) : x_0 \in Z(\theta) \Big\}$ and $T_{Z(\theta)} := T_\theta \mid_{l_\infty^{Z(\theta)}}$.

Theorem 3.10. *Let Π^* be a discrete skew product and $(Z(\theta))_{\theta\in\Theta}$ be the family of closed subspaces of X. Then, the following assertions are equivalent.*

(i) Π^* *has an uniform discrete dichotomy with the corresponding projections $\{P_n(\theta)\}$ satisfying* $\ker P_0(\theta) = Z(\theta)$.

(ii) *For each $\theta \in \Theta$ the operator $T_{Z(\theta)}$ is invertible and the family $\{ T_{Z(\theta)}\}_{\theta\in\Theta}$ is uniformly correct.*

Proof. $(i) \Rightarrow (ii)$: We first prove that $T_{Z(\theta)}$ is invertible. Indeed, for $f \in l_\infty(\mathbb{N};X)$, since T_θ is surjective (by theorem 3.5), there exists $x \in l_\infty(\mathbb{N};X)$ such that $T_\theta x = f$. Let $u \in l_\infty(\mathbb{N};X)$ be defined as follow $u_n = \Phi_{n,0}(\theta)P_0(\theta)x_0$; $n \in \mathbb{N}$. Then $T_\theta(u) = 0$ because $\ker T_\theta = \Big\{ u \in l_\infty(\mathbb{N};X) : u_n = \Phi_{n,0}u_0 \Big\}$. Note that $x - u \in l_\infty^{Z(\theta)}$ and $T_{Z(\theta)}(x-u) = T_\theta(x-u) = T_\theta x = f$ then $T_{Z(\theta)}$ is surjective. Moreover, $T_{Z(\theta)}$ is injective since
$\ker T_{Z(\theta)} = \Big\{ u \in l_\infty(\mathbb{N};X) : u_n = \Phi_{n,0}u_0 \text{ for } u_0 \in P_0(\theta)X \Big\} \cap \Big\{ u \in l_\infty(\mathbb{N};X) : u_0 \in Z(\theta) = \ker P_0(\theta) \Big\} = \{0\}$. Thus, $T_{Z(\theta)}$ is invertible.

To complete this part we shall prove that the family $\{ T_{Z(\theta)}\}_{\theta\in\Theta}$ is uniformly correct. In fact, since Π^* has an uniform discrete dichotomy, we have that the corresponding dichotomy constants M, α do not depend on θ. By remark 2.10 we obtain that $\|T_{Z(\theta)}\| \geq \frac{1-\alpha}{M(1+\alpha)}$ $\forall \theta \in \Theta$. Therefore, the family $\{ T_{Z(\theta)}\}_{\theta\in\Theta}$ is uniformly correct.

$(ii) \Rightarrow (i)$: Since $T_{Z(\theta)}$ is invertible and the family $\{ T_{Z(\theta)}\}_{\theta\in\Theta}$ is uniformly correct, it follows that T_θ is surjective, and there exists $\delta > 0$ such that $\|T_{Z(\theta)}(x)\| \geq \delta\|x\|$ $\forall x \in l_\infty^{Z(\theta)}$. We note that $T_{0\theta}$ is the restriction of $T_{Z(\theta)}$ on l_∞^0. Therefore, $0 \notin A\sigma(T_{0\theta})$. By corollary 3.4 $X_0(0,\theta)$ is a closed subspace of X.
We now prove that $X = X_0(0,\theta) \oplus Z(\theta)$.
For any $x \in X$ let $u = \{u_n\}$ and $f = \{f_n\}$ be defined by $u_0 = x$; $u_n = 0$ for $n > 0$ and $f_0 = -\Phi_0(\theta)x$; $f_n = 0$ for $n > 0$, respectively. Then, $u, f \in l_\infty(\mathbb{N};X)$ and $T_\theta u = f$. Since $T_{Z(\theta)}$ is invertible then there exists $v = \{v_n\} \in l_\infty(\mathbb{N};X)$ such that $T_\theta(v) = T_{Z(\theta)}(v) = f$. Hence, $u - v \in \ker T_\theta$. Therefore, $u_0 - v_0 = x - v_0 \in X_0(0,\theta)$ and $v_0 \in Z(\theta)$ then $x \in X_0(0,\theta) + Z(\theta)$. If now $y \in X_0(0,\theta) \cap Z(\theta)$ then the sequence $\omega = \{\omega_n\}$ defined by $\omega_n = \Phi_{n,0}(\theta)y$; $n \in \mathbb{N}$ belongs to $l_\infty^{Z(\theta)} \cap \ker T(\theta)$. Hence, $T_{Z(\theta)}\omega = 0$. This implies that $\omega = 0$. That yields $X = X_0(0,\theta) \oplus Z(\theta)$; so $X_0(0,\theta)$ is complemented in X. The assertion is now followed from theorem 3.7. □

4. APPLICATIONS TO LINEAR SKEW PRODUCT SEMIFLOWS

In this part we give the necessary and sufficient conditions for linear skew-product semiflows to have exponential dichotomy. The equivalence between uniform discrete dichotomy and exponential dichotomy will be given in the following theorem whose proof can be done in the same way as in [2, Theorem 4.1].

Theorem 4.1. *Assume that* $\Pi = (\mathit{\Phi}, \varphi)$ *is a linear skew-product semiflows on* $\mathcal{E} = X \times \Theta$. *Then, the following assertions are equivalent.*

(i) *The discretized skew-product* Π^* *given as follow* $\Pi^*(x;\theta;n) = (\mathit{\Phi}(\varphi^n\theta;1)x;\varphi^n(\theta))$ *has a uniform discrete dichotomy.*

(ii) Π *has an exponential dichotomy.*

By theorem 3.10 and 4.1 we obtain the following characterization of a linear skew product semiflow having an exponential dichotomy.

Theorem 4.2. *Let* $\Pi = (\Phi, \varphi)$ *be a linear skew product semiflow on* $\mathcal{E}$ *and* $(Z(\theta))_{\theta\in\Theta}$ *be a family of closed subspaces of* X. *Then, the following assertions are equivalent.*

a) Π *has an exponential dichotomy with the corresponding projector* $\mathbf{P}$ *which, by definition, has the form* $\mathbf{P}(x,\theta) = (P(\theta)x,\theta)$ *for* $(x,\theta) \in X \times \Theta$ *and* $P(\theta)$ *being projections on* X *satisfying* $\ker P(\theta) = Z(\theta)$.

b) *For each* $\theta \in \Theta$ *the operator* $T_{Z(\theta)} : l_\infty^{Z(\theta)} \to l_\infty(\mathbb{N};X)$ *defined by*

$$(T_{Z(\theta)}x)_n = x_{n+1} - \Phi(\varphi^n\theta, 1)x_n \text{ for } x \in l_\infty^{Z(\theta)}$$

is invertible. Moreover, the family $\{T_{Z(\theta)}\}_{\theta\in\Theta}$ *is uniformly correct.*

To finish this article we shall prove exponential dichotomy is not affected by small pertubations.

Theorem 4.3. *Assume that* $\Pi = (\mathit{\Phi}, \varphi)$ *is a linear skew product semiflows on* $\mathcal{E}$ *having an exponential dichotomy with dichotomy* M *and* α. *Then, there exists* $\delta = \delta(N;\alpha) > 0$ *small enough such that any linear skew product semiflows* $\Gamma = (\Psi, \varphi)$ *on* $\mathcal{E}$ *satisfying* $\sup\{\,\|\mathit{\Phi}(\theta;t) - \Psi(\theta;t)\| : 0 \leqslant t \leqslant 1; \theta \in \Theta\} \leqslant \delta$ *has an exponential dichotomy.*

Proof. By assumption, $\Pi = (\mathit{\Phi}, \varphi)$ has exponential dichotomy. Let $\mathbf{P}$ be the projector corresponding the exponential dichotomy of $\Pi = (\mathit{\Phi}, \varphi)$. By definition, $\mathbf{P}$ has the form $\mathbf{P}(x,\theta) = (P(\theta)x,\theta)$ for $(x,\theta) \in X \times \Theta$ and $P(\theta)$ being projections on X. Putting $\ker P(\theta) = Z(\theta);\ \theta \in \Theta$, by theorem 4.2 we have that the operator $T_{Z(\theta)} : l_\infty^{Z(\theta)} \to l_\infty(\mathbb{N};X)$ defined by

$$(T_{Z(\theta)}x)_n = x_{n+1} - \Phi(\varphi^n\theta, 1)x_n \text{ for } x \in l_\infty^{Z(\theta)}$$

is invertible, also the family $(T_{Z(\theta)})_{\theta\in\Theta}$ is uniformly correct. Let now $T^\Psi_{Z(\theta)}$ be the operators corresponding to the discretized skew-product Γ^* of LSPS Γ. That means that $T^\Psi_{Z(\theta)} : l_\infty^{Z(\theta)} \to l_\infty(\mathbb{N};X)$ is defined by

$$(T^\Psi_{Z(\theta)}x)_n = x_{n+1} - \Psi(\varphi^n\theta, 1)x_n \text{ for } x \in l_\infty^{Z(\theta)}.$$

We now see that

$$T^\Psi_{Z(\theta)} = T_{Z(\theta)} + T^\Psi_{Z(\theta)} - T_{Z(\theta)} = T_{Z(\theta)}\left(I + T^{-1}_{Z(\theta)}(T^\Psi_{Z(\theta)} - T_{Z(\theta)})\right).$$

These identities yield that $T^\Psi_{Z(\theta)}$ is invertible provided that $\|T^\Psi_{Z(\theta)} - T_{Z(\theta)}\| < \frac{1}{\|T^{-1}_{Z(\theta)}\|}$.

We next estimate the norm $\|T^{\Psi}_{Z(\theta)} - T_{Z(\theta)}\|$. In fact, for each $x \in l_\infty^{Z(\theta)}$ we have that

$$\begin{aligned} \|T^{\Psi}_{Z(\theta)}x - T_{Z(\theta)}x\| &= \sup_{n\in\mathbb{N}} \|\Psi(\varphi^n\theta;1)x_n - \Phi(\varphi^n\theta;1)x_n\| \\ &\leqslant \sup_{n\in\mathbb{N}} \|\Psi(\varphi^n\theta;1) - \Phi(\varphi^n\theta;1)\|\|x\| \\ &\leqslant \delta\|x\|. \end{aligned}$$

Therefore, $\|T^{\Psi}_{Z(\theta)} - T_{Z(\theta)}\| \le \delta$. Since the family $(T_{Z(\theta)})_{\theta\in\Theta}$ is uniformly correct we have that the exists ν independent of θ such that $\|T_{Z(\theta)}\| \ge \nu$. Hence, if $\delta < \frac{1}{\nu}$ then the operator $T^{\Psi}_{Z(\theta)}$ is invertible. Moreover, $\|T^{\Psi}_{Z(\theta)}\| \ge \|T_{Z(\theta)}\| - \|T^{\Psi}_{Z(\theta)} - T_{Z(\theta)}\| \ge \|T_{Z(\theta)}\| - \delta \ge \nu - \delta$. It follows that, if $\delta < \nu$ then the family $(T^{\Psi}_{Z(\theta)})_{\theta\in\Theta}$ is also uniformly correct.

Thus, if $\delta < \min\{\nu, \frac{1}{\nu}\}$ then, by theorem 4.2, the LSPS Γ also has an exponential dichotomy. □

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