Asymptotic Expansions for Second-Order Linear Difference Equations, II*

By R. Wong and H. Li

Infinite asymptotic expansions are derived for the solutions to the second-order linear difference equation

$$y(n+2) + n^p a(n) y(n+1) + n^q b(n) y(n) = 0,$$

where p and q are integers, $a(n)$ and $b(n)$ have power series expansions of the form

$$a(n) = \sum_{s=0}^{\infty} \frac{a_s}{n^s} \quad \text{and} \quad b(n) = \sum_{s=0}^{\infty} \frac{b_s}{n^s}$$

for large values of n, and $a_0 \neq 0$, $b_0 \neq 0$. Recurrence relations are also given for the coefficients in the asymptotic solutions. Our proof is based on the method of successive approximations. This paper is a continuation of an earlier one, in which only the special case $p \leq 0$ and $q = 0$ is considered.

1. Introduction

This paper is a continuation of our earlier one [11], in which we study the second-order linear difference equation

$$y(n+2) + a(n) y(n+1) + b(n) y(n) = 0, \tag{1.1}$$

Address for correspondence: R. Wong, Department of Applied Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2.

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where $a(n)$ and $b(n)$ have power series expansions of the form

$$a(n) = \sum_{s=0}^{\infty} \frac{a_s}{n^s} \quad \text{and} \quad b(n) = \sum_{s=0}^{\infty} \frac{b_s}{n^s} \tag{1.2}$$

for large values of n, and $b_0 \neq 0$. In the present paper, we shall consider the more general equation

$$y(n+2) + n^p a(n) y(n+1) + n^q b(n) y(n) = 0, \tag{1.3}$$

where p and q are integers, and the leading coefficients a_0 and b_0 in (1.2) are both nonzero. This covers the earlier case when $p \leq 0$ and $q = 0$. Many orthogonal polynomials and sequences that occur in enumerative combinatorics satisfy difference equations of the form (1.3), but not of the form (1.1). For example, the recurrence relation for the Charlier polynomials is

$$C_{n+1}^{(a)}(x) + (n+a-x) C_n^{(a)}(x) + an C_{n-1}^{(a)}(x) = 0, \qquad a \neq 0, \tag{1.4}$$

and the recurrence relation for the Bessel polynomials $y_n(x)$ is

$$y_{n+1}(x) = (2n+1) x y_n(x) + y_{n-1}(x); \tag{1.5}$$

see [5, Chap. VI]. Also, if T_n denotes the number of idempotent elements in the symmetric group of order n, then T_n satisfies

$$T_n = T_{n-1} + (n-1) T_{n-2}, \qquad T_0 = T_1 = 1; \tag{1.6}$$

see [8].

If the exponents p and q in (1.3) are related in the manner $q = 2p$, then (1.3) can be reduced to (1.1) using the transformation

$$x(n) = [(n-2)!]^{\mu} y(n). \tag{1.7}$$

Indeed, substitution of (1.7) in (1.3) gives the equivalent equation

$$x(n+2) + n^{p+\mu} a(n) x(n+1) + n^{q+2\mu} b^*(n) x(n) = 0, \tag{1.8}$$

where

$$b^*(n) = b(n)\left(1 - \frac{1}{n}\right)^{\mu} = \sum_{s=0}^{\infty} \left\{ \sum_{l=0}^{s} (-1)^l \binom{\mu}{l} b_{s-l} \right\} n^{-s}. \tag{1.9}$$

(Note that the constant term in the expansion (1.9) is again not zero.) If $q = 2p$ then, by taking $\mu = -p$, (1.8) becomes an equation of the same form as (1.1). Thus, our discussion of Equation (1.3) consists of only two cases, namely, (i) $q < 2p$ and (ii) $q > 2p$. We shall set $k \equiv 2p - q$. Each of these two cases has two or three subcases depending on the values of k.

In case (i), i.e., when $k > 0$, we shall show that Equation (1.3) has a formal series solution of the form

$$y_1(n) = [(n-2)!]^p \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \tag{1.10}$$

where $\rho = -a_0$, $\alpha = a_1 / a_0$ if $k > 1$, and

$$\alpha = \frac{1}{a_0}\left(a_1 - \frac{b_0}{a_0}\right) \quad \text{if } k = 1. \tag{1.11}$$

A second formal series solution is of the form

$$y_2(n) = [(n-2)!]^{q-p} \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \tag{1.12}$$

where $\rho = -b_0 / a_0$,

$$\alpha = \frac{b_0}{a_0^2} - \frac{a_1}{a_0} + \frac{b_1}{b_0} - p + q \quad \text{if } k = 1, \tag{1.13}$$

and

$$\alpha = \frac{b_1}{b_0} - \frac{a_1}{a_0} - p + q \quad \text{if } k > 1. \tag{1.14}$$

In case (ii), i.e., when $k < 0$, we have three subcases to consider depending on whether $k = -1$, or $k \leq -3$ and is odd, or $k \leq -2$ and is even. If $k = -1$, then two formal series solutions of (1.3) are of the form

$$y(n) = [(n-2)!]^{q/2} \rho^n e^{\gamma\sqrt{n}} n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^{s/2}}, \tag{1.15}$$

where $\rho^2 = -b_0$, $\gamma = -a_0 / \rho$ and

$$\alpha = \frac{b_1}{2b_0} + \frac{q}{4}. \tag{1.16}$$

If $k \leq -3$ and k is odd, then the exponential factor in (1.15) is absent and Equation (1.15) becomes

$$y(n) = [(n-2)!]^{q/2} \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^{s/2}}, \tag{1.17}$$

where ρ and α are as given in (1.15)–(1.16). If $k \leq -2$ and k is even, then coefficients of odd powers of $n^{-1/2}$ all vanish and (1.15) simplifies to

$$y(n) = [(n-2)!]^{q/2} \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \tag{1.18}$$

with $\rho^2 = -b_0$ and α given by (1.16) if $k = -4, -6, \ldots,$ and

$$\alpha = -\frac{1}{2\rho^2}\left(b_1 + \tfrac{1}{2}qb_0 + \rho a_0\right) \quad \text{if } k = -2. \tag{1.19}$$

In all cases, we shall prove that the formal series solutions in (1.10), (1.12), (1.15), (1.17) and (1.18) are, in fact, asymptotic. As in our previous paper [11], our approach is based on the method of successive approximations and depends heavily on the availability of explicit recurrence relations for the coefficients c_s in the formal series.

2. Recursion Formulas for c_s; Case (i): $k > 0$

To derive the first formal series solution (1.10), we choose $\mu = -p$ in (1.7) as suggested by Adams [2, p. 511] and Birkhoff [3, p. 210]. Equation (1.8) then becomes

$$x(n+2) + a(n)x(n+1) + n^{-k}b^*(n)x(n) = 0. \tag{2.1}$$

The characteristic equation associated with (2.1) is $\rho^2 + a_0\rho = 0$, which has the nonvanishing simple root $\rho = -a_0$. Motivated by the results in [11], we try a series solution of the form

$$x(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} \frac{c_s}{n^s} \tag{2.2}$$

with $\rho = -a_0$; see also Adams [2, p. 513, Case 2a].

Let $\mathscr{L}$ denote the difference operator defined by

$$\mathscr{L}\{x(n)\} \equiv x(n+2) + n^{p+\mu}a(n)x(n+1) + n^{q+2\mu}b^*(n)x(n). \tag{2.3}$$

Substituting (1.2) and (2.2) in (2.3), and making use of the identity

$$(n+\lambda)^{\alpha-s} = n^{\alpha}\sum_{i=0}^{\infty}\binom{\alpha-s}{i}\lambda^{i}n^{-(s+i)}, \qquad \lambda = 1,2. \tag{2.4}$$

we have, when $\mu = -p$,

$$\begin{aligned}\mathscr{L}\{x(n)\} = {} & \sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[2^{s-j}\rho\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j\right\}\rho^{n+1}n^{\alpha-s} \\ & + \sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left[\sum_{l=j}^{s}(-1)^{l-j}\binom{-p}{l-j}b_{s-l}\right]c_j\right\}\rho^{n}n^{\alpha-k-s}. \end{aligned} \tag{2.5}$$

For (2.2) to be a formal solution, the coefficients of powers of n on the right-hand side of (2.5) must all vanish. The coefficient of the leading term n^{α} is $(\rho + a_0)c_0\rho^{n+1}$. Hence, we must choose $\rho = -a_0$. As a consequence, the coefficient of c_s in the first summation on j in (2.5) is zero and the coefficient of $n^{\alpha-1}$ in the first infinite series in (2.5) simplifies to $(-a_0\alpha + a_1)c_0\rho^{n+1}$.

If $k=1$ then (2.5) can be written as

$$\begin{aligned}\mathscr{L}\{x(n)\} = {} & \left[(a_0^2\alpha - a_1a_0)+b_0\right]c_0\rho^{n}n^{\alpha-1} \\ & + \sum_{s=2}^{\infty}\left\{\sum_{j=0}^{s-1}\left[2^{s-j}\rho^{2}\binom{\alpha-j}{s-j}+\sum_{l=j}^{s}\binom{\alpha-j}{l-j}\rho a_{s-l}\right.\right. \\ & \left.\left. + \sum_{l=j}^{s-1}(-1)^{l-j}\binom{-p}{l-j}b_{s-1-l}\right]c_j\right\}\rho^{n}n^{\alpha-s}. \end{aligned} \tag{2.6}$$

Here, we have again used the fact that $\rho = -a_0$. Equating the coefficient of $n^{\alpha-s}$ to zero, $s = 1,2,\ldots$, we obtain (1.11) and the recurrence relation

$$\begin{aligned} c_{s-1} = {} & \frac{1}{(s-1)a_0^2}\sum_{j=0}^{s-2}\left[2^{s-j}a_0^2\binom{\alpha-j}{s-j}-\sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_0a_{s-l}\right. \\ & \left. + \sum_{l=j}^{s-1}(-1)^{l-j}\binom{-p}{l-j}b_{s-1-l}\right]c_j, \end{aligned} \tag{2.7}$$

for $s = 2,3,\ldots$. In (2.7), we have also made use of (1.11).

If $k>1$ then (2.5) can be written as

$$
\begin{aligned}
\mathscr{L}\{x(n)\} = & \,(-a_0\alpha + a_1)c_0\,\rho^{n+1}n^{\alpha-1} \\
& + \sum_{s=2}^{k-1}\left\{\sum_{j=0}^{s-1}\left[2^{s-j}\rho\binom{\alpha-j}{s-j}\right.\right. \\
& \qquad\qquad \left.\left. + \sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j\right\}\rho^{n+1}n^{\alpha-s} \\
& + \sum_{s=k}^{\infty}\left\{\sum_{j=0}^{s-1}\left[2^{s-j}\rho^2\binom{\alpha-j}{s-j} + \sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\rho\right]c_j\right. \\
& \qquad\qquad \left. + \sum_{j=0}^{s-k}\left[\sum_{l=j}^{s-k}(-1)^{l-j}\binom{-p}{l-j}b_{s-k-l}\right]c_j\right\}\rho^n n^{\alpha-s}. \quad (2.8)
\end{aligned}
$$

Equating coefficients of $n^{\alpha-s}$ to zero for $s=1,2,\ldots$, we obtain $\alpha = a_1/a_0$ and

$$
c_{s-1} = \frac{1}{(s-1)a_0}\sum_{j=0}^{s-2}\left[2^{s-j}a_0\binom{\alpha-j}{s-j} - \sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_{s-l}\right]c_j, \quad (2.9)
$$

for $s=2,3,\ldots,k-1$, and

$$
\begin{aligned}
c_{s-1} = & \,\frac{1}{(s-1)a_0^2}\left\{\sum_{j=0}^{s-2}\left[2^{s-j}a_0^2\binom{\alpha-j}{s-j} - \sum_{l=j}^{s}\binom{\alpha-j}{l-j}a_0a_{s-l}\right]c_j\right. \\
& \qquad\qquad \left. + \sum_{j=0}^{s-k}\left[\sum_{l=j}^{s-k}(-1)^{l-j}\binom{-p}{l-j}b_{s-k-l}\right]c_j\right\}, \quad (2.10)
\end{aligned}
$$

for $s=k,k+1,\ldots$. This completes the derivation of the formal series solution (1.10).

To obtain the second formal series solution (1.12), we again follow the suggestion of Adams [2, p. 511] by taking $\mu = p-q$ so that (1.8) becomes

$$
x(n+2) + n^k a(n)x(n+1) + n^k b^*(n)x(n) = 0. \quad (2.11)
$$

(Recall: $k = 2p-q$.) The characteristic equation [2, p. 507] associated with (2.11) is $a_0\rho + b_0 = 0$, which has the nonzero root $\rho = -b_0/a_0$. Thus we try, as in (2.1), a formal series solution of the form

$$
x(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} c_s n^{-s}. \quad (2.12)
$$

Applying the difference operator $\mathscr{L}$ in (2.3) to (2.12) with $\mu = p - q$, we obtain by straight-forward computation

$$\begin{aligned}\mathscr{L}\{x(n)\} = &\sum_{s=0}^{\infty}\left[\rho^2 \sum_{j=0}^{s} 2^{s-j}\binom{\alpha-j}{s-j}c_j\right]\rho^n n^{\alpha-s}\\ &+\sum_{s=0}^{\infty}\left\{\sum_{j=0}^{s}\left(\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha-j}{l-j}\right.\right.\right.\\ &\left.\left.\left.+(-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right)c_j\right\}\rho^n n^{\alpha+k-s}, \quad (2.13)\end{aligned}$$

where use has been made of (1.9). For $\mathscr{L}\{x(n)\} = 0$, coefficients of powers of n on the right-hand side of (2.13) must all be zero. The coefficient of the leading term $n^{\alpha+k}$ is $(\rho a_0 + b_0)c_0\rho^n$. Hence, we choose $\rho = -b_0/a_0$. This forces the coefficient of c_s in the second series in (2.13) to vanish.

If $k = 1$ then the coefficient of n^α in (2.13) is

$$\left\{\left[\rho^2 + \rho a_1 + b_1 + \rho a_0\alpha - (p-q)b_0\right]c_0 + (\rho a_0 + b_0)c_1\right\}\rho^n.$$

Since $\rho = -b_0/a_0$, Equation (2.13) now becomes

$$\begin{aligned}\mathscr{L}\{x(n)\} = &\left[\rho^2 + \rho a_1 + b_1 + \rho a_0\alpha - (p-q)b_0\right]c_0\rho^n n^\alpha\\ &+\sum_{s=1}^{\infty}\left\{\sum_{j=0}^{s}\left[\rho^2 2^{s-j}\binom{\alpha-j}{s-j} + \sum_{l=j}^{s+1}\left(\rho a_{s+1-l}\binom{\alpha-j}{l-j}\right.\right.\right.\\ &\left.\left.\left.+(-1)^{l-j}\binom{p-q}{l-j}b_{s+1-l}\right)\right]c_j\right\}\rho^n n^{\alpha-s}. \quad (2.14)\end{aligned}$$

Equating the coefficients of $n^{\alpha-s}$ to zero for $s = 0, 1, \ldots$, we obtain (1.13) and

$$\begin{aligned}c_s = -\frac{1}{b_0 s}\sum_{j=0}^{s-1}&\left\{\rho^2 2^{s-j}\binom{\alpha-j}{s-j} + \sum_{l=j}^{s+1}\left[\rho a_{s+1-l}\binom{\alpha-j}{l-j}\right.\right.\\ &\left.\left.+(-1)^{l-j}\binom{p-q}{l-j}b_{s+1-l}\right]\right\}c_j, \quad (2.15)\end{aligned}$$

for $s = 1, 2, \ldots$. In arriving at (2.15), we have made use of (1.13) and the choice $\rho = -b_0/a_0$.

If $k>1$ then (2.13) can be written as

$$\mathscr{L}\{x(n)\} = \sum_{s=1}^{k-1}\left\{\sum_{j=0}^{s-1}\left(\left[\rho a_{s-l}\binom{\alpha-j}{l-j} + (-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right)c_j\right\}\rho^n n^{\alpha+k-s}$$
$$+\sum_{s=k}^{\infty}\left\{\sum_{j=0}^{s-1}\left(\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha-j}{l-j} + (-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right)c_j + \rho^2\sum_{j=0}^{s-k}2^{s-k-j}\binom{\alpha-j}{s-k-j}c_j\right\}\rho^n n^{\alpha+k-s}. \quad (2.16)$$

Since $\rho=-b_0/a_0$, the coefficient of $n^{\alpha+k-1}$ in (2.16) is

$$[\rho a_1+b_1+\rho a_0\alpha-(p-q)b_0]c_0.$$

Setting the coefficients of $n^{\alpha+k-s}$ to zero for $s=1,2,\ldots,$ we obtain (1.14) and the recurrence relations

$$c_{s-1}=\frac{1}{(1-s)b_0}\sum_{j=0}^{s-2}\left\{\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha-j}{l-j}+(-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right\}c_j, \quad (2.17)$$

for $s=2,3,\ldots,k-1$, and

$$c_{s-1}=\frac{1}{(1-s)b_0}\times\left\{\sum_{j=0}^{s-k}\left(\rho^2 2^{s-k-j}\binom{\alpha-j}{s-k-j}+\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha-j}{l-j} + (-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right)c_j\right.$$
$$\left.+\sum_{j=s-k+1}^{s-2}\left(\sum_{l=j}^{s}\left[\rho a_{s-l}\binom{\alpha-j}{l-j} + (-1)^{l-j}\binom{p-q}{l-j}b_{s-l}\right]\right)c_j\right\}, \quad (2.18)$$

for $s = k, k+1, \ldots$. In both (2.17) and (2.18), use has been made of (1.14) and $\rho = -b_0/a_0$. This completes the derivation of the second formal series solution (1.12).

3. Proof of (2.2)

We shall now show that the formal series solution (2.2) to Equation (2.1) is, indeed, asymptotic. By writing $x(n) = \rho^n z(n)$, it is easily verified that we may assume without loss of generality that $\rho = 1$, i.e., $a_0 = -1$. Write (2.2) as

$$x(n) = \sum_{s=0}^{\infty} c_s n^{\alpha-s} = L_N(n) + E_N(n) \tag{3.1}$$

with

$$L_N(n) = \sum_{s=0}^{N-1} c_s n^{\alpha-s}. \tag{3.2}$$

For $x(n)$ to be a solution of (2.1), we must have

$$\mathscr{L}\{L_N(n)\} + \mathscr{L}\{E_N(n)\} = 0, \tag{3.3}$$

where $\mathscr{L}$ is the difference operator defined in (2.3) with $\mu = -p$.

If $k = 1$ then, by putting $\rho = 1$ and $c_s = 0$ for $s \geq N$ in (2.6), we obtain

$$R_N(n) \equiv -\mathscr{L}\{L_N(n)\} = O(n^{\alpha-N-1}), \tag{3.4}$$

on account of (1.11) and the recurrence relation (2.7). If $k > 1$ then, by applying the same argument to (2.8), it follows that the order estimate in (3.4) again holds, in view of the choice $\alpha = a_1/a_0$ and the recurrence relations (2.9) and (2.10). Coupling (3.3) and (3.4), we have from (2.3) with $\mu = -p$,

$$E_N(n+2) + a(n)E_N(n+1) + n^{q-2p}b^*(n)E_N(n) = R_N(n). \tag{3.5}$$

We shall prove (2.2) by showing that (3.5) has a solution satisfying

$$E_N(n) = O(n^{\alpha-N}). \tag{3.6}$$

To this end, we rewrite (3.5) as

$$E_N(n+2) - E_N(n+1) = R_N(n) - [a(n)+1]E_N(n+1) - n^{-k}b^*(n)E_N(n). \tag{3.7}$$

Since every solution of the equation

$$E_N(n) = -\sum_{j=n-1}^{\infty} \{R_N(j) - [a(j)+1]E_N(j+1) - j^{-k}b^*(j)E_N(j)\} \tag{3.8}$$

is a solution of (3.7), for our purpose it suffices to show that (3.8) has a solution satisfying (3.6).

Define the sequence $\{h_s(n)\}$ successively by

$$h_{s+1}(n) = -\sum_{j=n-1}^{\infty} \{R_N(j) - [a(j)+1]h_s(j+1) - j^{-k}b^*(j)h_s(j)\} \tag{3.9}$$

with $h_0(n) \equiv 0$. It will be shown that the series

$$E_N(n) = \sum_{s=0}^{\infty} \{h_{s+1}(n) - h_s(n)\} \tag{3.10}$$

is absolutely and uniformly convergent in n, and that its sum $E_n(n)$ is a solution of (3.8) satisfying (3.6). Let n_0 be sufficiently large so that

$$|R_N(n)| \le Mn^{m-N-1}, \qquad m \equiv \operatorname{Re}\alpha \tag{3.11}$$

and

$$|a(n)+1| + |b^*(n)n^{-k}| \le Mn^{-1} \tag{3.12}$$

for some $M > 0$ and for all $n \ge n_0$; see (3.4), (1.2), and (1.9). (Recall that we have assumed without loss of generality that $a_0 = -1$.) From (3.9), it follows that

$$|h_1(n)| \le \sum_{j=n-1}^{\infty} |R_N(j)| \le M \sum_{j=n-1}^{\infty} j^{m-N-1} \tag{3.13}$$

for all $n \ge n_0$. Since

$$\sum_{k=n}^{\infty} \frac{1}{k^p} \le \int_{n-1}^{\infty} \frac{1}{x^p}\,dx = \frac{1}{p-1}(n-1)^{-p+1} \tag{3.14}$$

if $p > 1$, we let $N > m$ and assume without loss of generality that n_0 has already been chosen so that

$$\sum_{j=n-1}^{\infty} j^{m-N-1} \le \frac{1}{N-m}(n-2)^{m-N} \le \frac{2}{N-m}n^{m-N} \tag{3.15}$$

for all $n \geq n_0$. Coupling (3.13) and (3.15) gives

$$|h_1(n+1)|, \qquad |h_1(n)| \leq \frac{2M}{N-m} n^{m-N} \tag{3.16}$$

for all $n \geq n_0$. From (3.9), we also have

$$h_2(n) - h_1(n) = \sum_{j=n-1}^{\infty} \{[a(j)+1]h_1(j+1) + b^*(j) j^{-k} h_1(j)\}.$$

Applying (3.16), (3.12) and (3.15), we obtain

$$|h_2(n+1) - h_1(n+1)|, \qquad |h_2(n) - h_1(n)| \leq \left(\frac{2M}{N-m}\right)^2 n^{m-N}.$$

By induction, it can be proven that

$$|h_{s+1}(n+1) - h_s(n+1)|, \qquad |h_{s+1}(n) - h_s(n)| \leq \left(\frac{2M}{N-m}\right)^{s+1} n^{m-N}$$

for $s = 0, 1, \ldots$. We choose $N > m + 2M$. It is now evident that the series in (3.10) is uniformly convergent in n and its sum $E_n(n)$ satisfies (3.6). Furthermore, since

$$E_N(n) = \lim_{s \to \infty} \sum_{l=0}^{s} \{h_{l+1}(n) - h_l(n)\} = \lim_{s \to \infty} h_{s+1}(n), \tag{3.17}$$

it follows by taking the limit $s \to \infty$ in (3.9) that $E_n(n)$ is a solution to (3.8), and hence a solution to (3.5).

4. Proof of (2.12)

To verify that the formal series (2.12) is an asymptotic solution to (2.11), we again assume without loss of generality that $\rho = 1$. In the present case, this is equivalent to requiring that $a_0 = -b_0$; see the statement following (2.11). Now we proceed as in Section 3. Set

$$x(n) = \sum_{s=0}^{\infty} c_s n^{\alpha - s} = L_N(n) + E_N(n) \tag{4.1}$$

with

$$L_N(n) = \sum_{s=0}^{N-1} c_s n^{\alpha - s}, \tag{4.2}$$

and choose $N > k+1$. (Recall: $k = 2p - q$.) For $x(n)$ to be a solution of (2.11), we must have

$$\mathscr{L}\{L_N(n)\} + \mathscr{L}\{E_N(n)\} = 0, \tag{4.3}$$

where $\mathscr{L}$ is the difference operator defined in (2.3) with $\mu = p - q$.

If $k = 1$ then, by setting $c_j = 0$ for $j \geq N$ in (2.14), we obtain

$$\mathscr{L}\{L_N(n)\} = O(n^{\alpha - N}),$$

in view of (1.13) and the recurrence relation (2.15). If $k > 1$ then we set $c_j = 0$ for $j \geq N$ in (2.16). This gives

$$\mathscr{L}\{L_N(n)\} = O(n^{\alpha + k - N - 1})$$

by using (1.14) and the recurrence relations (2.17) and (2.18). Equation (4.3) can be written as

$$n^{-k} E_N(n+2) + a(n) E_N(n+1) + b^*(n) E_N(n) = R_N(n), \tag{4.4}$$

where

$$R_N(n) = -n^{-k} \mathscr{L}\{L_N(n)\}. \tag{4.5}$$

Thus, in both cases $k = 1$ and $k > 1$, we have

$$R_N(n) = O(n^{\alpha - N - 1}). \tag{4.6}$$

Recall that we have assumed $\rho = 1$ or, equivalently, $a_0 = -b_0$. Let $\xi = a_0 = -b_0$. Then Equation (4.4) can be written as

$$\begin{aligned} \xi[E_N(n+1) - E_N(n)] = R_N(n) &- n^{-k} E_N(n+2) \\ &- [a(n) - a_0] E_N(n+1) \\ &- [b^*(n) - b_0] E_N(n), \end{aligned} \tag{4.7}$$

from which one can formally derive the equation

$$\xi E_N(n) = -\sum_{j=n}^{\infty} \{R_N(j) - j^{-k} E_N(j+2) - [a(j) - a_0] E_N(j+1) - [b^*(j) - b_0] E_N(j)\}. \tag{4.8}$$

Note that every solution of (4.8) is a solution of (4.7). The argument now proceeds as in Section 3. We define the sequence $\{h_s(n)\}$ successively by

$$\xi h_{s+1}(n) = -\sum_{j=n}^{\infty} \{R_N(j) - j^{-k}h_s(j+2) - [a(j)-a_0]h_s(j+1) - [b^*(j)-b_0]h_s(j)\} \quad (4.9)$$

with $h_0(n)=0$, and we let n_0 be sufficiently large so that

$$|R_N(n)| \le Mn^{m-N-1}, \qquad m = \operatorname{Re}\alpha \quad (4.10)$$

and

$$n^{-k} + |a(n)-a_0| + |b^*(n)-b_0| \le Mn^{-1} \quad (4.11)$$

for all $n \ge n_0$ and for some $M > 0$; see (4.6), (1.2), and (1.9). Also, we choose $N > m$ and assume that the second inequality in (3.15) holds. From (4.8) it then follows that

$$|\xi h_1(n)| \le \sum_{j=n}^{\infty} R_N(j) \le M\sum_{j=n}^{\infty} j^{m-N-1}$$
$$\le \frac{2M}{N-m} n^{m-N}$$

for all $n \ge n_0$. By induction, it can be proven that

$$|\xi[h_{s+1}(n) - h_s(n)]| \le \left(\frac{2M}{N-m}\right)^{s+1} n^{m-N} \quad (4.12)$$

for all $n \ge n_0$. As long as $N > m + 2M$, the series

$$E_N(n) = \sum_{s=0}^{\infty} \{h_{s+1}(n) - h_s(n)\} = \lim_{s\to\infty} h_{s+1}(n) \quad (4.13)$$

is uniformly convergent in n, and its sum satisfies (4.4). Coupling (4.12) and (4.13), we also have

$$E_N(n) = O(n^{\alpha-N}). \quad (4.14)$$

This establishes the existence of a solution to (2.11) having an asymptotic expansion of the form (2.12).

5. Recursion Formulas for c_s; Case (ii): $k < 0$

Again suggested by the method indicated in Adams [2, pp. 511–512], we choose $\mu = -q/2$ in the present case. Equation (1.8) then becomes

$$x(n+2) + n^{k/2}a(n)x(n+1) + b^*(n)x(n) = 0 \tag{5.1}$$

(Recall: $k = 2p - q$.) The characteristic equation for (5.1) is

$$\rho^2 + b_0 = 0, \tag{5.2}$$

and we try a formal series solution of the form

$$x(n) \sim \rho^n e^{\gamma\sqrt{n}} n^\alpha \sum_{s=0}^{\infty} c_s n^{-s/2}, \tag{5.3}$$

where ρ is a root of equation (5.2); see Adams [2, Eq. (9)]. To derive explicit formulas for γ and α and recursion formulas for c_s, we make use of the formula

$$e^{\gamma\sqrt{n+\lambda}} = e^{\gamma\sqrt{n}} \sum_{s=0}^{\infty} G_s^{(\lambda)}(\gamma) n^{-s/2}, \qquad \lambda = 1, 2, \tag{5.4}$$

where

$$G_s^{(\lambda)}(\gamma) = \sum_{\sigma(\underset{\sim}{l}_p)=s} \prod_{j=1}^{p} \frac{1}{l_j!}\left[\binom{\frac{1}{2}}{j}\lambda^j\gamma\right]^{l_j}, \tag{5.5}$$

the summation being taken over all multi-indices $\underset{\sim}{l}_p = (l_1, \ldots, l_p)$ such that

$$\sigma(\underset{\sim}{l}_p) \equiv l_1 + 3l_2 + \ldots + (2p-1)l_p = s. \tag{5.6}$$

The first few values of $G_s^{(\lambda)}(\gamma)$ are given by

$$\begin{aligned} G_0^{(1)}(\gamma) &= 1, & G_0^{(2)}(\gamma) &= 1, \\ G_1^{(1)}(\gamma) &= \tfrac{1}{2}\gamma, & G_1^{(2)}(\gamma) &= \gamma, \\ G_2^{(1)}(\gamma) &= \tfrac{1}{8}\gamma^2, & G_2^{(2)}(\gamma) &= \tfrac{1}{2}\gamma^2; \end{aligned} \tag{5.7}$$

see [11]. For convenience, we also introduce the notation

$$F_s^{(\lambda)}(\alpha) = \sum_{j=0}^{s} \frac{1+(-1)^{s-j}}{2} \lambda^{(s-j)/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s-j)} c_j. \tag{5.8}$$

By the binomial expansion, we have

$$\sum_{s=0}^{\infty} c_s (n+\lambda)^{\alpha - s/2} = n^\alpha \sum_{s=0}^{\infty} F_s^{(\lambda)}(\alpha) n^{-s/2}, \qquad \lambda = 1, 2. \tag{5.9}$$

With $\mu = -q/2$ and $k = 2p - q$, the difference operator $\mathscr{L}$ in (2.3) becomes

$$\mathscr{L}\{x(n)\} \equiv x(n+2) + n^{k/2}a(n)x(n+1) + b^*(n)x(n). \quad (5.10)$$

Applying this operator to the formal series (5.3) gives

$$\begin{aligned}\mathscr{L}\{x(n)\} = e^{\gamma\sqrt{n}}\rho^n n^\alpha \Bigg\{ &\sum_{s=0}^{\infty} \rho^2 \left[\sum_{l=0}^{s} G_{s-l}^{(2)}(\gamma) F_l^{(2)}(\alpha)\right] n^{-s/2} \\ &+ \sum_{s=0}^{\infty} \rho \left[\sum_{i=0}^{s} \frac{1+(-1)^i}{2} a_{i/2} \sum_{j=i}^{s} G_{s-j}^{(1)}(\gamma) F_{j-i}^{(1)}(\alpha)\right] n^{(k-s)/2} \\ &+ \sum_{s=0}^{\infty} \left(\sum_{i=0}^{s} \frac{1+(-1)^{s-i}}{2} b^*_{(s-i)/2} c_i\right) n^{-s/2}\Bigg\}, \end{aligned} \quad (5.11)$$

where b_s^* is the coefficient of n^{-s} in (1.9). Expressing b_s^* in terms of b_s, and inserting (5.8) in (5.11), the quantity inside the curly brackets can be written as

$$\begin{aligned}&\sum_{s=0}^{\infty} \Bigg\{ \sum_{j=0}^{s} \Bigg[\rho^2 \sum_{i=0}^{s-j} \frac{1+(-1)^i}{2} 2^{i/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}i} G_{s-j-i}^{(2)}(\gamma) \\ &\qquad + \frac{1+(-1)^{s-j}}{2} \sum_{i=0}^{(s-j)/2} (-1)^i \binom{-\frac{1}{2}q}{i} b_{(s-j)/2-i}\Bigg] c_j \Bigg\} n^{-s/2} \\ &+ \sum_{s=0}^{\infty} \Bigg\{ \sum_{j=0}^{s} \Bigg[\rho \sum_{i=0}^{s-j} \frac{1+(-1)^i}{2} a_{i/2} \sum_{l=0}^{s-i-j} \frac{1+(-1)^l}{2} \\ &\qquad \times \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}l} G_{s-l-i-j}^{(1)}(\gamma)\Bigg] c_j \Bigg\} n^{(k-s)/2}. \end{aligned} \quad (5.12)$$

If $k = -1$, it follows that

$$\begin{aligned}\mathscr{L}\{x(n)\} = e^{\gamma\sqrt{n}}\rho^n n^\alpha \sum_{s=0}^{\infty} \Bigg\{ &\sum_{j=0}^{s} \Bigg[\rho^2 \sum_{i=0}^{s-j} \frac{1+(-1)^i}{2} 2^{i/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}i} G_{s-j-i}^{(2)}(\gamma) \\ &+ \frac{1+(-1)^{s-j}}{2} \sum_{i=0}^{(s-j)/2} (-1)^i \binom{-\frac{1}{2}q}{i} b_{(s-j)/2-i}\Bigg] c_j \\ &+ \sum_{j=0}^{s-1} \Bigg[\rho \sum_{i=0}^{s-1-j} \frac{1+(-1)^i}{2} a_{i/2} \sum_{l=i}^{s-1-j} \frac{1+(-1)^l}{2} \\ &\times \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(l-i)} G_{s-1-l-j}^{(1)}(\gamma)\Bigg] c_j \Bigg\} n^{-s/2}, \end{aligned} \quad (5.13)$$

where, as usual, empty sums are understood to be zero. As before, we now set coefficients of $n^{-s/2}$ to zero for all $s \geq 0$. The leading coefficient in (5.13) is $(\rho^2 + b_0)c_0$. Hence, ρ must be a root of the characteristic equation (5.2). This forces the coefficient of c_s to vanish. Consequently, in view of (5.7), the coefficient of $n^{-1/2}$ becomes $(\rho^2\gamma + \rho a_0)c_0$. Setting this to zero gives $\gamma = -a_0/\rho$, which in turn makes the coefficient of c_{s-1} zero and the coefficient of n^{-1} to be $(2\rho^2\alpha + b_1 + \frac{1}{2}qb_0)c_0$. The value of α is thus determined as $\alpha = \frac{1}{4}(q + 2b_1/b_0)$; compare (1.16). Equation (5.13) now reduces to

$$\begin{aligned}\mathscr{L}\{x(n)\} = e^{\gamma\sqrt{n}}\rho^n n^\alpha \sum_{s=3}^{\infty} \Bigg\{ \sum_{j=0}^{s-2} \Bigg[& \rho^2 \sum_{i=0}^{s-j} \frac{1+(-1)^i}{2} 2^{i/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}i} G^{(2)}_{s-j-i}(\gamma) \\ & + \frac{1+(-1)^{s-j}}{2} \sum_{i=0}^{(s-j)/2} (-1)^i \binom{-\frac{1}{2}q}{i} b_{(s-j)/2-i} \\ & + \rho \sum_{i=0}^{s-1-j} \frac{1+(-1)^i}{2} a_{i/2} \sum_{l=i}^{s-1-j} \frac{1+(-1)^l}{2} \\ & \times \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(l-i)} G^{(1)}_{s-1-l-j}(\gamma) \Bigg] c_j \Bigg\} n^{-s/2}.\end{aligned} \tag{5.14}$$

Using (5.7) and the values of α and γ, the coefficient of c_{s-2} can be simplified to $-(s-2)\rho^2$. The recursion formula for c_s is therefore given by

$$\begin{aligned}c_{s-2} = \frac{1}{(s-2)\rho^2} \sum_{j=0}^{s-3} \Bigg[& \rho^2 \sum_{i=0}^{s-j} \frac{1+(-1)^i}{2} 2^{i/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}i} G^{(2)}_{s-j-i}(\gamma) \\ & + \frac{1+(-1)^{s-j}}{2} \sum_{i=0}^{(s-j)/2} (-1)^i \binom{-\frac{1}{2}q}{i} b_{(s-j)/2-i} \\ & + \rho \sum_{i=0}^{(s-1-j)} \frac{1+(-1)^i}{2} a_{i/2} \sum_{l=i}^{s-1-j} \frac{1+(-1)^l}{2} \\ & \times \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(l-i)} G^{(1)}_{s-1-l-j}(\gamma) \Bigg] c_j,\end{aligned} \tag{5.15}$$

for $s = 3, 4, \ldots$.

If $k \le -3$ and k is odd, then the coefficient of the leading term is $(\rho^2 + b_0)c_0$. Hence, ρ must again be a root of the characteristic equation (5.2). This reduces the coefficient of $n^{-1/2}$ in (5.12) to $\rho^2\gamma c_0$. As a result, we must choose $\gamma = 0$. This makes $G_s^{(\mu)}(\gamma) = 0$ for all $s \ge 1$ and $\mu = 1,2$. The coefficient of n^{-1} in (5.12) now can be simplified to $(2\rho^2\alpha + b_1 + \frac{1}{2}qb_0)c_0$. Therefore, the value of α is again given by (1.16). From (5.11) and (5.12), it follows that

$$
\begin{aligned}
&\mathscr{L}\{x(n)\} \\
&= e^{\gamma\sqrt{n}}\rho^n n^\alpha\left[\sum_{s=3}^{-k-1}\left\{\sum_{j=0}^{s-2}\frac{1+(-1)^{s-j}}{2}\left[\rho^2 2^{(s-j)/2}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j)}\right.\right.\right. \\
&\qquad\left.\left. + \sum_{i=0}^{(s-j)/2}(-1)^i\binom{-q/2}{i}b_{(s-j)/2-i}\right]c_j\right\} n^{-s/2} \\
&\qquad + \sum_{s=-k}^{\infty}\left\{\sum_{j=0}^{s-2}\frac{1+(-1)^{s-j}}{2}\left[\rho^2 2^{(s-j)/2}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j)}\right.\right. \\
&\qquad\left. + \sum_{i=0}^{(s-j)/2}(-1)^i\binom{-\frac{1}{2}q}{i}b_{(s-j)/2-i}\right]c_j \\
&\qquad + \sum_{j=0}^{s+k}\frac{1+(-1)^{s+k-j}}{2} \\
&\qquad \times\left[\rho\sum_{i=0}^{s+k-j}\frac{1+(-1)^i}{2}a_{i/2}\right. \\
&\qquad\left.\left.\left. \times\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s+k-i-j)}\right]c_j\right\} n^{-s/2}\right]. \qquad (5.16)
\end{aligned}
$$

In view of (1.16), the coefficient of c_{s-2} is $(2-s)\rho^2$. Setting the coefficient of $n^{-s/2}$ in (5.16) to zero yields the recursion formulas

$$
\begin{aligned}
c_{s-2} = \frac{1}{\rho^2(s-2)}\sum_{j=0}^{s-3}\frac{1+(-1)^{s-j}}{2}&\left[\rho^2 2^{(s-j)/2}\binom{\alpha-\frac{1}{2}j}{\frac{1}{2}(s-j)}\right. \\
&\left. + \sum_{i=0}^{(s-j)/2}(-1)^i\binom{-\frac{1}{2}q}{i}b_{(s-j)/2-i}\right]c_j, \qquad (5.17)
\end{aligned}
$$

for $s = 3, \ldots, -k-1$, and

$$c_{s-2} = \frac{1}{\rho^2(s-2)} \left\{ \sum_{j=0}^{s-3} \frac{1+(-1)^{s-j}}{2} \left[\rho^2 2^{(s-j)/2} \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s-j)} + \sum_{i=0}^{(s-j)/2} (-1)^i \binom{-\frac{1}{2}q}{i} b_{(s-j)/2-i} \right] c_j + \sum_{j=0}^{s+k} \frac{1+(-1)^{s+k-j}}{2} \left[\rho \sum_{i=0}^{s+k-j} \frac{1+(-1)^i}{2} a_{i/2} \times \binom{\alpha - \frac{1}{2}j}{\frac{1}{2}(s+k-i-j)} \right] c_j \right\}, \quad (5.18)$$

for $s = -k, -k+1, \ldots$.

If $k \le -2$ and k is even, then one can show that the exponent γ in (5.3) is again zero, and that all odd-indexed coefficients $c_1, c_3, \ldots$ in (5.3) vanish. Thus, we try a formal series solution of the *simpler* form

$$x(n) \sim \rho^n n^\alpha \sum_{s=0}^{\infty} c_s n^{-s}. \quad (5.19)$$

Put $T = \frac{1}{2}k$ and apply the difference operator $\mathscr{L}$ in (5.10) to (5.19). The result is

$$\mathscr{L}\{x(n)\} = \rho^n n^\alpha \left\{ \sum_{s=0}^{-T-1} \left[\sum_{j=0}^{s} \left(2^{s-j} \binom{\alpha - j}{s-j} \rho^2 + \sum_{l=0}^{s-j} (-1)^l \binom{-\frac{1}{2}q}{l} b_{s-j-l} \right) c_j \right] n^{-s} + \sum_{s=-T}^{\infty} \left[\sum_{j=0}^{s} \left(2^{s-j} \binom{\alpha - j}{s-j} \rho^2 + \sum_{l=0}^{s-j} (-1)^l \binom{-\frac{1}{2}q}{l} b_{s-j-l} \right) c_j + \sum_{j=0}^{s+T} \left(\rho \sum_{i=j}^{s+T} \binom{\alpha - j}{i-j} a_{s+T-i} \right) c_j \right] n^{-s} \right\}. \quad (5.20)$$

The coefficient of the leading term is $(\rho^2 + b_0)c_0$. Hence, ρ satisfies (5.2), and the coefficient of c_s becomes zero. The coefficient of n^{-1} is $(2\alpha\rho^2 + b_1 + \frac{1}{2}qb_0)c_0$ if $k < -2$ (i.e., $T < -1$) and $(2\alpha\rho^2 + b_1 + \frac{1}{2}qb_0 + \rho a_0)c_0$ if $k = -2$. Accordingly, α is given by (1.16) if $k < -2$ and (1.19) if $k = -2$. This simplifies the coefficient of c_{s-1} to $-2\rho^2(s-1)$. Setting the right-hand side of (5.20) to zero, we obtain the recursion formulas

$$c_{s-1} = \frac{1}{2(s-1)\rho^2}\sum_{j=0}^{s-2}\left[2^{s-j}\binom{\alpha-j}{s-j}\rho^2 + \sum_{l=0}^{s-j}(-1)^l\binom{-\frac{1}{2}q}{l}b_{s-j-l}\right]c_j \tag{5.21}$$

for $s = 2,3,\ldots,-T-1$, and

$$c_{s-1} = \frac{1}{2(s-1)\rho^2}\left\{\sum_{j=0}^{s-2}\left[2^{s-j}\binom{\alpha-j}{s-j}\rho^2 + \sum_{l=0}^{s-j}(-1)^l\binom{-\frac{1}{2}q}{l}b_{s-j-l}\right]c_j + \sum_{j=0}^{s+T}\left[\rho\sum_{i=j}^{s+T}\binom{\alpha-j}{i-j}a_{s+T-i}\right]c_j\right\}, \tag{5.22}$$

for $s = -T, -T+1, \ldots$.

6. Proof of (5.3) When $k = -1$ and Re $\gamma < 0$

To prove (5.3), we shall again, as in Sections 3 and 4, assume without loss of generality that $\rho = 1$, i.e., $b_0^* = b_0 = -1$, and write

$$x(n) = L_N(n) + E_N(n) \tag{6.1}$$

with

$$L_N(n) = e^{\gamma\sqrt{n}}n^{\alpha}\sum_{s=0}^{N-1}c_s n^{-s/2}. \tag{6.2}$$

Putting $c_j = 0$ for $j \ge N$ in (5.14), the recurrence relation (5.15) gives

$$\mathscr{L}\{L_N(n)\} = e^{\gamma\sqrt{n}}n^{\alpha}O(n^{-N/2-1}). \tag{6.3}$$

Since $x(n)$ is a solution of (5.1) with $k = -1$, we have from (6.1)

$$E_N(n+2) + n^{-1/2}a(n)E_N(n+1) + b^*(n)E_N(n) = R_N(n), \tag{6.4}$$

where

$$R_N(n) \equiv -\mathscr{L}\{L_N(n)\} = e^{\gamma\sqrt{n}} n^\alpha O(n^{-N/2-1}). \tag{6.5}$$

Set

$$E_N(n) = e^{\gamma\sqrt{n}} n^\alpha \epsilon_N(n). \tag{6.6}$$

Then $\epsilon_N(n)$ satisfies the difference equation

$$\epsilon_N(n+2) + \tilde{a}(n)\epsilon_N(n+1) + \tilde{b}(n)\epsilon_N(n) = \tilde{R}_N(n), \tag{6.7}$$

where

$$\tilde{a}(n) = n^{-1/2} a(n) e^{\gamma\sqrt{n+1} - \gamma\sqrt{n+2}} \left(\frac{n+1}{n+2}\right)^\alpha, \tag{6.8}$$

$$\tilde{b}(n) = b^*(n) e^{\gamma\sqrt{n} - \gamma\sqrt{n+2}} \left(\frac{n}{n+2}\right)^\alpha, \tag{6.9}$$

and

$$\tilde{R}_N(n) = e^{-\gamma\sqrt{n+2}}(n+2)^{-\alpha} R_N(n). \tag{6.10}$$

Simple calculation shows that

$$\tilde{a}(n) = a_0 n^{-1/2} - \frac{\gamma}{2} a_0 n^{-1} + R_a(n) \tag{6.11}$$

and

$$\tilde{b}(n) = b_0^* - \gamma b_0^* n^{-1/2} + \left[b_0^*\left(\tfrac{1}{2}\gamma^2 - 2\alpha\right) + b_1^*\right] n^{-1} + R_b(n), \tag{6.12}$$

where $b_0^* = b_0$, $b_1^* = b_1 + \frac{1}{2} q b_0$ and

$$R_a(n), R_b(n) = O(n^{-3/2}). \tag{6.13}$$

Furthermore,

$$\tilde{R}_N(n) = O(n^{-N/2-1}). \tag{6.14}$$

Recall that we have assumed $\rho = 1$, or equivalently, $b_0 = -1$. Hence $\gamma = -a_0$

and $\alpha = -\frac{1}{2}b_1 + \frac{1}{4}q$. Consequently, Equation (6.7) can be written as

$$\epsilon_N(n+2) + \left[a_0 n^{-1/2} + \tfrac{1}{2}a_0^2 n^{-1} + R_a(n)\right]\epsilon_N(n+1) + \left[-1 - a_0 n^{-1/2} - \tfrac{1}{2}a_0^2 n^{-1} + R_b(n)\right]\epsilon_N(n) = \tilde{R}_N(n). \quad (6.15)$$

Let $\Delta\epsilon_N(n) = \epsilon_N(n+1) - \epsilon_N(n)$, and define

$$\theta(n) \equiv 1 + a_0 n^{-1/2} + \tfrac{1}{2}a_0^2 n^{-1} \quad (6.16)$$

and

$$q[\epsilon_N(n+1), \epsilon_N(n), n] = \tilde{R}_N(n) - R_a(n)\epsilon_N(n+1) - R_b(n)\epsilon_N(n). \quad (6.17)$$

Equation (6.15) can then be written in the form of a first order equation

$$\Delta\epsilon_N(n+1) + \theta(n)\Delta\epsilon_N(n) = q[\epsilon_N(n+1), \epsilon_N(n), n]. \quad (6.18)$$

To show that the formal series in (5.3) is indeed an asymptotic solution to (5.1) with $k = -1$, it suffices to show that Equation (6.15), or equivalently Equation (6.18), has a solution satisfying

$$\epsilon_N(n) = O(n^{-N/2}). \quad (6.19)$$

From (6.18), one can formally derive the equations

$$\epsilon_N(n) = \sum_{i=n}^{\infty}\sum_{j=i}^{\infty}(-1)^{j-i+1}\theta^{-1}(i)\ldots\theta^{-1}(j)q[\epsilon_N(j+1), \epsilon_N(j), j], \quad (6.20)$$

where $\theta^{-1}(j) \equiv 1/\theta(j)$, and

$$\epsilon_N(n) = -\sum_{i=n}^{\infty}\sum_{j=n_0}^{i-1}(-1)^{i-j+1}\theta(i-1)\ldots\theta(j+1)q[\epsilon_N(j+1), \epsilon_N(j), j], \quad (6.21)$$

where n_0 is any positive integer, and where it is understood that $\theta(i-1)\theta(i-2)\ldots\theta(j+1) = 1$ when $j = i-1$. It is easily verified that every solution of (6.20) and every solution of (6.21) is a solution of (6.18). It should also be noted that although Equations (6.20) and (6.21) appear similar in form to Equations (5.23) and (5.24) in our previous paper [11], the alternating sign $(-1)^{i-j+1}$ in the present case makes the following analysis considerably more difficult. Before proceeding, we first digress to discuss a summation formula due to G. Boole.

The Euler polynomials $E_s(x)$ are defined by the equation

$$\frac{2e^{tx}}{e^t+1} = \sum_{s=0}^{\infty} E_s(x)\frac{t^s}{s!}, \qquad |t| < \pi; \tag{6.22}$$

see [6, p. 40] or [1, p. 804]. It is well-known that each $E_s(x)$ is a polynomial of degree s, and that $E_s(1) = -E_s(0)$ for $s > 0$. We define the periodic Euler function $\overline{E}_s(x)$ by

$$\overline{E}_s(x+1) = -\overline{E}_s(x) \quad \text{for all } x, \tag{6.23}$$

and

$$\overline{E}_s(x) = E_s(x) \quad \text{for } 0 \le x < 1. \tag{6.24}$$

The following result is known as the *Boole summation formula* [9, p. 34].

LEMMA 1. *If $f(t)$ is an m-times continuously differentiable function on $x \le t \le x+w$ then for $0 \le h \le 1$,*

$$f(x+hw) = \frac{1}{2}\sum_{s=0}^{m-1}\frac{w^s}{s!}E_s(h)\left[f^{(s)}(x+w)+f^{(s)}(x)\right] + R_m(x), \tag{6.25}$$

where

$$R_m(x) = \tfrac{1}{2}w^m\int_0^1 \frac{\overline{E}_{m-1}(h-t)}{(m-1)!}f^{(m)}(x+wt)\,dt. \tag{6.26}$$

For our purpose, we need also the following consequences of Lemma 1. The first one that follows is stated in [4].

LEMMA 2. *Let $f(t)$ be an m-times continuously differentiable function on $x \le t < \infty$. If $f^{(k)}(t) \to 0$ as $t \to \infty$ for all $k = 0, 1, \ldots, m$, then for $0 \le h \le 1$,*

$$\sum_{l=0}^{\infty}(-1)^l f(x+h+l) = \frac{1}{2}\sum_{s=0}^{m-1}\frac{1}{s!}E_s(h)f^{(s)}(x) + \rho_m(x), \tag{6.27}$$

where

$$\rho_m(x) = \frac{1}{2}\int_0^{\infty}\frac{\overline{E}_{m-1}(h-t)}{(m-1)!}f^{(m)}(x+t)\,dt. \tag{6.28}$$

LEMMA 3. *Let n_0 and i be any two positive integers with $i > n_0$. If $f(t)$ is an m-times continuously differentiable function on $x+n_0 \le t \le x+i$, then for $0 \le h \le 1$,*

$$\sum_{l=n_0}^{i-1} (-1)^l f(x+h+l) = \frac{1}{2}\sum_{s=0}^{m-1} \frac{1}{s!} E_s(h)\Big[(-1)^{i-1} f^{(s)}(x+i-1) + (-1)^{n_0} f^{(s)}(x+n_0-1)\Big] + r_m(x), \quad (6.29)$$

where

$$r_m(x) = \frac{1}{2}\int_{n_0}^{i} \frac{\overline{E}_m(h-t)}{(m-1)!} f^{(m)}(x+t)\,dt. \quad (6.30)$$

For our latter use, we need the following special values:

$$E_0(x) \equiv 1, \qquad E_1(0) = -\tfrac{1}{2}, \qquad E_3(0) = \tfrac{1}{4}, \quad (6.31)$$

and $E_{2s}(0) = 0$ for $s = 1, 2, \ldots,$ and the estimates

$$|E_{2s}(x)| \le 4\frac{(2s)!}{\pi^{2s+1}}, \qquad 0 \le x \le 1, \quad (6.32)$$

and

$$|E_{2s-1}(x)| \le 4\frac{(2s-1)!}{\pi^2}\left(1+\frac{1}{2^{2s}-2}\right), \qquad 0 \le x \le 1, \quad (6.33)$$

for $s = 1, 2, \ldots.$

As an illustration of (6.27), we take

$$f(t) = e^{\gamma\sqrt{t}}\, t^{-\beta}, \quad (6.34)$$

where either $\operatorname{Re}\gamma < 0$ and β arbitrary real or $\operatorname{Re}\gamma = 0$ and $\beta > 0$. Clearly,

$$\sum_{j=i}^{\infty} (-1)^j e^{\gamma\sqrt{j}} j^{-\beta} = (-1)^i \sum_{l=0}^{\infty} (-1)^l f(i+l). \quad (6.35)$$

Thus, with $h = 0$, $x = i$ and $m = 1$, we obtain from (6.27)

$$\sum_{j=i}^{\infty} (-1)^j e^{\gamma\sqrt{j}} j^{-\beta} = (-1)^i\Big[\tfrac{1}{2} e^{\gamma\sqrt{i}} i^{-\beta} + \rho_1(i)\Big]. \quad (6.36)$$

Using integration-by-parts, it can be shown that

$$|\rho_1(i)| \le \begin{cases} -\dfrac{|\gamma|}{2\sigma} e^{\sigma\sqrt{i}} i^{-\beta} & \text{if } \sigma \equiv \operatorname{Re}\gamma < 0 \\ \tfrac{1}{2} i^{-\beta} & \text{if } \sigma \equiv \operatorname{Re}\gamma = 0 \quad \text{and} \quad \beta > 0. \end{cases} \tag{6.37}$$

Hence, when $\operatorname{Re}\gamma = 0$

$$\left| \sum_{j=i}^{\infty} (-1)^{j-i} e^{\gamma(\sqrt{j}-\sqrt{i})} j^{-\beta} \right| \le i^{-\beta}. \tag{6.38}$$

We shall make use of this result later. From (6.27), one also sees readily that the series in (6.36) has an asymptotic expansion of the form

$$\sum_{j=i}^{\infty} (-1)^j e^{\gamma\sqrt{j}} j^{-\beta} \sim \tfrac{1}{2}(-1)^i e^{\gamma\sqrt{i}} \sum_{s=0}^{\infty} c_s(\beta,\gamma) i^{-\beta-s/2} \tag{6.39}$$

for large values of i, where $c_0(\beta,\gamma) = 1$ and $c_1(\beta,\gamma) = -\frac{1}{4}\gamma$.

To establish the existence of a solution to (6.20), we need one further preliminary result that corresponds to Lemma 1 in [11].

LEMMA 4. *For positive integers $j \ge i \ge 1$, the function $\theta(n)$ in (6.16) satisfies*

$$\begin{aligned} &\theta^{-1}(i)\theta^{-1}(i+1)\ldots\theta^{-1}(j) \\ &\quad = e^{2\gamma(\sqrt{j}-\sqrt{i})}\left[1 + O(i^{-1/2})\right], \qquad i \to \infty, \end{aligned} \tag{6.40}$$

where the O-term is uniform with respect to j.

Proof: First, we recall the asymptotic formula [10, p. 292]

$$\sum_{k=1}^{n-1} k^z - \zeta(-z) = \frac{n^{z+1}}{z+1} + O(n^z), \qquad n \to \infty, \tag{6.41}$$

where $z \ne -1$ and $\zeta(z)$ is the Riemann Zeta function. Since

$$\log(1+x) = x - \tfrac{1}{2}x^2 + O(x^3)$$

as $x \to 0$, it follows from (6.16) that

$$\log\theta(n) = a_0 n^{-1/2} + O(n^{-3/2})$$

for $n \geq 1$. Since $\gamma = -a_0$ when we assume $\rho = 1$, upon summation we obtain

$$-\sum_{k=i}^{j} \log \theta(k) = 2\gamma\left(\sqrt{j} - \sqrt{i}\right) + O(i^{-1/2}), \qquad i \geq 1, \qquad (6.42)$$

uniformly for $j \geq i \geq 1$. The result (6.40) now follows by exponentiation. □

Returning to (6.20), we define the successive approximants $h_s(n)$ by

$$h_{s+1}(n) = \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} (-1)^{j-i+1} \theta^{-1}(i) \dots \theta^{-1}(j) q[h_s(j+1), h_s(j), j] \qquad (6.43)$$

with $h_0(n) \equiv 0$. In particular, we have

$$h_1(n) = \sum_{i=n}^{\infty} \sum_{j=i}^{\infty} (-1)^{j-i+1} \theta^{-1}(i) \dots \theta^{-1}(j) \tilde{R}_N(j). \qquad (6.44)$$

Retracing the argument leading to (6.5), it is easily seen from (6.10) that $\tilde{R}_N(n)$ has an asymptotic approximation of the form

$$\tilde{R}_N(n) = cn^{-N/2-1} + O(n^{-N/2-3/2}), \qquad (6.45)$$

where c is some constant whose exact value is immaterial for our purpose. From Lemma 2 in [11], we also have

$$\sum_{j=i}^{\infty} e^{2\gamma(\sqrt{j} - \sqrt{i})} j^{(-N/2)-1} = -\frac{1}{\gamma} i^{-N/2-1/2} + O(i^{-N/2-1}), \qquad i \to \infty, \qquad (6.46)$$

where $\operatorname{Re}\gamma \leq 0$ and N is any positive integer. Inserting (6.40) and (6.45) in (6.44), and using (6.39) and (6.46), we obtain

$$h_1(n) = O(n^{-N/2}), \quad \text{as } n \to \infty. \qquad (6.47)$$

In view of (6.47), (6.13), (6.40), and (6.46), there exists a positive constant M and a positive integer n_0 such that

$$|h_1(j)| \leq Mj^{-N/2} \qquad (6.48)$$

and

$$|R_a(j)| + |R_b(j)| \leq Mj^{-3/2} \qquad (6.49)$$

for all $j \geq n_0$,

$$\left|\theta^{-1}(i)\ldots\theta^{-1}(j)\right| \leq Me^{2\sigma(\sqrt{j}-\sqrt{i})} \tag{6.50}$$

for $j \geq i \geq n_0$, and

$$\sum_{j=i}^{\infty} e^{2\sigma(\sqrt{j}-\sqrt{i})} j^{-N/2-3/2} \leq Mi^{-N/2-1} \tag{6.51}$$

for $i \geq n_0$, where $\sigma = \operatorname{Re}\gamma$. (Note that M depends on γ and $\sigma < 0$.) We shall assume that n_0 is sufficiently large so that from (6.51) we also have

$$\sum_{i=n}^{\infty}\sum_{j=i}^{\infty} e^{2\sigma(\sqrt{j}-\sqrt{i})} j^{-N/2-3/2} \leq \frac{2^2 M}{N} n^{-N/2}, \qquad n \geq n_0; \tag{6.52}$$

compare (3.14) and (3.15). A combination of (6.43), (6.17), and the preceding estimates yields

$$|h_2(n) - h_1(n)| \leq \frac{2^2 M^3}{N} M n^{-N/2}, \qquad n \geq n_0. \tag{6.53}$$

By induction, it can be shown that

$$|h_{s+1}(n) - h_s(n)| \leq \left(\frac{2^2 M^3}{N}\right)^s M n^{-N/2}, \qquad n \geq n_0. \tag{6.54}$$

Thus, as long as $N > 4M^3$, the series

$$\epsilon_N(n) \equiv \sum_{s=0}^{\infty} \left[h_{s+1}(n) - h_s(n)\right] = \lim_{s\to\infty} h_{s+1}(n) \tag{6.55}$$

is uniformly convergent in n, and its sum $\epsilon_N(n)$ is a solution to (6.20) satisfying (6.19).

7. Proof of (5.3) When $k = -1$ and $\operatorname{Re}\gamma > 0$

In this case we shall work with Equation (6.21) instead of (6.20) and define the successive approximants $h_s(n)$ by

$$h_{s+1}(n) = -\sum_{i=n}^{\infty}\sum_{j=n_0}^{i-1} (-1)^{i-j+1}\theta(i-1) \times\theta(i-2)\ldots\theta(j+1)q\left[h_s(j+1), h_s(j), j\right], \tag{7.1}$$

again with $h_0(n) \equiv 0$. In (7.1), it is understood that $\theta(i-1)\theta(i-2)\dots\theta(j+1) = 1$ when $j = i-1$. Using the argument given in Lemma 4, it is readily seen that

$$\theta(i-1)\,\theta(i-2)\dots\theta(j+1) = e^{2\gamma(\sqrt{j}-\sqrt{i})}\left[1+O(j^{-1/2})\right], \qquad j \to \infty, \quad (7.2)$$

uniformly with respect to $i \geq j+1$. An analogue of (6.39) is

$$\sum_{j=n_0}^{i-1} (-1)^j e^{\gamma\sqrt{j}} j^{-\beta} \sim \tfrac{1}{2}(-1)^{i-1} e^{\gamma\sqrt{i}} \sum_{s=0}^{\infty} d_s(\beta,\gamma) i^{-\beta-s/2}, \qquad (7.3)$$

as $i \to \infty$, where $d_0(\beta,\gamma) = 1$ and $d_1(\beta,\gamma) = -\frac{3}{4}\gamma$. This can be proven by applying Lemma 3 to the function $f(t)$ in (6.34) with $x = h = 0$, and observing that, in the present case, we have $\operatorname{Re}\gamma > 0$. From (6.6) in [11], we also have

$$\sum_{i=n_0}^{i-1} e^{2\sigma\sqrt{j}} j^{-N/2-3/2} \leq M e^{2\sigma\sqrt{i}} i^{-N/2-1} \qquad (7.4)$$

for some positive constant M, where $\sigma = \operatorname{Re}\gamma > 0$. Since

$$h_1(n) = \sum_{i=n}^{\infty} \sum_{j=n_0}^{i-1} (-1)^{i-j} \theta(i-1)\,\theta(i-2)\dots\theta(j+1)\,\tilde{R}_N(j),$$

a combination of (6.45), (7.2), (7.3), and (7.4) gives

$$|h_1(n)| \leq M n^{-N/2} \qquad (7.5)$$

for some $M > 0$ and for all $n \geq n_0$. Without loss of generality, we shall assume that we have the same constant M and the same integer n_0 in (6.49), (7.4), and (7.5). In fact, we shall further assume that n_0 is sufficiently large so that (7.2) gives

$$|\theta(i-1)\,\theta(i-2)\dots\theta(j+1)| \leq M e^{2\sigma(\sqrt{j}-\sqrt{i})}, \qquad i-1 \geq j \geq n_0, \quad (7.6)$$

and (7.4) gives

$$\sum_{i=n}^{\infty} \sum_{j=n_0}^{i-1} e^{2\sigma(\sqrt{j}-\sqrt{i})} j^{-N/2-3/2} \leq \frac{2^2 M}{N} n^{-N/2}, \qquad n \geq n_0; \qquad (7.7)$$

compare (6.52). Induction then shows that

$$|h_{s+1}(n) - h_s(n)| \leq \left(\frac{2^2M^3}{N}\right)^s Mn^{-N/2}, \qquad n \geq n_0, \tag{7.8}$$

for $s = 0, 1, \dots$. The remaining argument now parallels that of the previous section.

8. Proof of (5.3) When $k = -1$ and Re $\gamma = 0$

In this section we shall show that Equation (6.20) has a solution satisfying (6.19) even when Re $\gamma = 0$. The problem with the argument given in Section 6 is that the exponential factor under the summation sign on the left-hand side of (6.51) is absent in this case, and consequently, the exponent of i in the estimate on the right-hand side increases by one half. To overcome this difficulty, we shall make use of the following version of the summation-by-parts formula (compare [10, p. 296]), whose proof is essentially that of the modified Abel theorem given in [7, p. 522].

LEMMA 5. *Let* Σu_j *be a convergent series, and let* $v_j(i)$ *be a sequence of numbers beginning with* $j \geq i$. *Put*

$$U_j = \sum_{s=j}^{\infty} u_s.$$

If for each fixed i, *we have* $v_j(i) \to 0$ *as* $j \to \infty$ *and*

$$\sum_{j=i+1}^{\infty} |v_j(i) - v_{j-1}(i)|$$

convergent, *then*

$$\sum_{j=i}^{\infty} u_j v_j(i) = \sum_{j=i+1}^{\infty} U_j\{v_j(i) - v_{j-1}(i)\} + U_i v_i(i). \tag{8.1}$$

To apply the preceding formula, we need a stronger asymptotic result for the product $\theta^{-1}(i)\dots\theta^{-1}(j)$ than that given in (6.40). First we recall the asymptotic expansion [10, p. 292]

$$\sum_{k=1}^{n-1} k^z - \zeta(-z) \sim \frac{n^{z+1}}{z+1} \sum_{s=0}^{\infty} \binom{z+1}{s} \frac{B_s}{n^s}, \tag{8.2}$$

where the coefficients B_s are the Bernoulli numbers with $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_2 = \frac{1}{6}$. Using (8.2) and the argument given in Lemma 4, it can be verified that

$$\theta^{-1}(i)\ldots\theta^{-1}(j) \sim e^{2\gamma(\sqrt{j}-\sqrt{i})}\left[1+\left(\frac{\gamma}{2}+\frac{\gamma^3}{3}\right)j^{-1/2} + \left(\frac{\gamma}{2}-\frac{\gamma^3}{3}\right)i^{-1/2}+\cdots\right] \tag{8.3}$$

for large values of i and j. Put

$$u_j = (-1)^j e^{2\gamma\sqrt{j}} j^{-N/2} \tag{8.4}$$

and

$$v_j(i) = e^{-2\gamma(\sqrt{j}-\sqrt{i})}\theta^{-1}(i)\ldots\theta^{-1}(j)\tilde{R}_N(j)j^{N/2}, \tag{8.5}$$

where $\tilde{R}_N(j)$ is as given in (6.10) and satisfies (6.45). It is readily seen that

$$\sum_{j=i}^{\infty}(-1)^j e^{2\gamma\sqrt{i}}\theta^{-1}(i)\ldots\theta^{-1}(j)\tilde{R}_N(j) = \sum_{j=i}^{\infty}u_j v_j(i). \tag{8.6}$$

From (6.38) with $\operatorname{Re}\gamma = 0$, we have

$$|U_j| = \left|\sum_{s=j}^{\infty}u_s\right| \le j^{-N/2}. \tag{8.7}$$

Inserting (6.45) and (8.3) in (8.5) yields $v_i(i) \sim ci^{-1}$ as $i\to\infty$ and $v_j(i) - v_{j-1}(i) \sim cj^{-2}$ as $j\to\infty$. Choose M and n_0 as in (6.49) and (6.50) with $\sigma = \operatorname{Re}\gamma = 0$, and assume without loss of generality that we also have

$$|v_i(i)| \le Mi^{-1} \quad \text{and} \quad |v_j(i) - v_{j-1}(i)| \le Mj^{-2} \tag{8.8}$$

for $j > i \ge n_0$.

Let $\{h_s(n)\}$ be the sequence of successive approximants defined by (6.43). Substituting (8.6) in (6.44), we obtain

$$h_1(n) = \sum_{i=n}^{\infty}(-1)^{1-i}e^{-2\gamma\sqrt{i}}\sum_{j=i}^{\infty}u_j v_j(i). \tag{8.9}$$

A combination of (8.1), (8.7), and (8.8) yields

$$\left|\sum_{j=i}^{\infty} u_j v_j(i)\right| \leq M \sum_{j=i+1}^{\infty} j^{-N/2-2}$$

$$+ Mi^{-N/2-1} \leq 2Mi^{-N/2-1}. \tag{8.10}$$

Consequently, for sufficiently large n_0, we have

$$|h_1(n)| \leq \frac{2^3 M}{N} n^{-N/2} \quad \text{for } n \geq n_0; \tag{8.11}$$

compare (3.15). From (8.9), it also follows that

$$h_1(n) - h_1(n-1) = (-1)^{n-1} e^{-2\gamma\sqrt{n-1}} \sum_{j=n-1}^{\infty} u_j v_j(n-1). \tag{8.12}$$

Since $\operatorname{Re}\gamma = 0$, coupling (8.10) and (8.12) gives

$$|h_1(n) - h_1(n-1)| \leq 2^2 M n^{-N/2-1} \quad \text{for } n \geq n_0. \tag{8.13}$$

For convenience, we introduce the notations

$$\Phi_a(i,j) \equiv \theta^{-1}(i) \ldots \theta^{-1}(j) e^{-2\gamma(\sqrt{j}-\sqrt{i})} R_a(j) j^{N/2} \tag{8.14}$$

and

$$\Phi_b(i,j) \equiv \theta^{-1}(i) \ldots \theta^{-1}(j) e^{-2\gamma(\sqrt{j}-\sqrt{i})} R_b(j) j^{N/2}. \tag{8.15}$$

By (6.49) and (6.50), we have

$$|\Phi_a(i,j)| + |\Phi_b(i,j)| \leq M^2 j^{(N-3)/2} \tag{8.16}$$

for $j \geq i \geq n_0$. From (6.8) and (6.11), it is also easily seen that $R_a(j)$ has an asymptotic expansion of the form $\alpha_0 j^{-3/2} + \alpha_1 j^{-2} + \cdots$ for some constants $\alpha_0, \alpha_1, \ldots$. Similarly, from (6.9) and (6.12), we have $R_b(j) \sim \beta_0 j^{-3/2} + \beta_1 j^{-2} + \cdots$ for some $\beta_0, \beta_1, \ldots$. These coupled with (8.3) imply that

$$\Phi_a(i,j) - \Phi_a(i,j-1) = O(j^{(N-5)/2})$$

and

$$\Phi_b(i,j) - \Phi_b(i,j-1) = O(j^{(N-5)/2})$$

uniformly with respect to $i \geq n_0$. Thus, without loss of generality, we may assume that

$$|\Phi_a(i,j)-\Phi_a(i,j-1)|+|\Phi_b(i,j)-\Phi_b(i,j-1)| \leq M^2 j^{(N-5)/2} \quad (8.17)$$

for all $j > i \geq n_0$ (compare (8.16)), where M and n_0 are, respectively, the same constant and integer chosen in (8.8). For $s = 1, 2, \ldots,$ we now define

$$v_j^{(s)}(i) = \Phi_a(i,j)\left[h_s(j+1)-h_{s-1}(j+1)\right]+\Phi_b(i,j)\left[h_s(j)-h_{s-1}(j)\right]. \quad (8.18)$$

In terms of u_j given in (8.4) and $v_j^{(s)}(i)$, we have from (6.43) and (6.17)

$$h_{s+1}(n)-h_s(n) = \sum_{i=n}^{\infty}(-1)^{-i}e^{-2\gamma\sqrt{i}}\sum_{j=i}^{\infty}u_j v_j^{(s)}(i) \quad (8.19)$$

and

$$\left[h_{s+1}(n)-h_s(n)\right]-\left[h_{s+1}(n-1)-h_s(n-1)\right]$$
$$= (-1)^{-n}e^{-2\gamma\sqrt{n-1}}\sum_{j=n-1}^{\infty}u_j v_j^{(s)}(n-1). \quad (8.20)$$

From (8.1) it follows that

$$\sum_{j=i}^{\infty}u_j v_j^{(s)}(i) = \sum_{j=i+1}^{\infty}U_j\left\{v_j^{(s)}(i)-v_{j-1}^{(s)}(i)\right\}+U_i v_i^{(s)}(i). \quad (8.21)$$

We first consider the case $s = 1$. Applying (8.11) and (8.16) to (8.18) gives

$$\left|v_i^{(1)}(i)\right| \leq \frac{2^2M^3}{(\frac{1}{2}N)}i^{-3/2}. \quad (8.22)$$

By adding and subtracting the terms $\Phi_a(i,j)h_1(j)$ and $\Phi_b(i,j)h_1(j-1)$ to the result for $v_j^{(1)}(i)-v_{j-1}^{(1)}(i)$ obtained from (8.18), and using (8.11), (8.13), and (8.17), we have

$$\left|v_j^{(1)}(i)-v_{j-1}^{(1)}(i)\right| \leq 3(2^2M^3)j^{-5/2} \quad (8.23)$$

as long as $N \geq 2$. From (8.19), (8.21), and (8.7), it follows that

$$|h_2(n) - h_1(n)| \leq 4\frac{(2M)^3}{(\frac{1}{2}N)^2} n^{-(N+1)/2} \tag{8.24}$$

and

$$|[h_2(n) - h_1(n)] - [h_2(n-1) - h_1(n-1)]| \leq 4\frac{(2M)^3}{(\frac{1}{2}N)} n^{-(N+3)/2}. \tag{8.25}$$

In the general case, i.e., when $s > 1$, we add and subtract the terms $\Phi_a(i,j)[h_s(j) - h_{s-1}(j)]$ and $\Phi_b(i,j)[h_s(j-1) - h_{s-1}(j-1)]$ to the result for $v_j^{(s)}(i) - v_{j-1}^{(s)}(i)$ obtained from (8.18), and show by induction that

$$|h_{s+1}(n) - h_s(n)| \leq A_{s+1} n^{-N/2-1/2} \tag{8.26}$$

and

$$|[h_{s+1}(n) - h_s(n)] - [h_{s+1}(n-1) - h_s(n-1)]| \leq A'_{s+1} n^{-N/2-3/2} \tag{8.27}$$

for $s = 2, 3, \ldots$, where the constants A_s and A'_s are defined successively by

$$A_{s+1} = \left[(A'_s + 2A_s)\frac{1}{\frac{1}{2}N} + A_s\right]\frac{2M^2}{\frac{1}{2}N} \tag{8.28}$$

and

$$A'_{s+1} = \frac{N}{2} A_{s+1}, \qquad s = 1, 2, \ldots, \tag{8.29}$$

with $A_2 = 4(2M)^3/(\frac{1}{2}N)^2$. Let $C \equiv 8M^2(N+2)/N^2 < 1$. Then, from (8.28) and (8.29), it follows immediately that $A_{s+1} = CA_s$, $A'_{s+1} = CA'_s$, and

$$A_{s+1} = \frac{2N^2}{M(N+2)^2} C^{s+1}. \tag{8.30}$$

Therefore, the series

$$\sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)]$$

is uniformly convergent in n, and its sum is a solution to (6.20) satisfying (6.19); compare (6.55).

9. Proof of (5.3) When $k \le -3$ and k is Odd

The analysis in this case is considerably simpler than that when $k=-1$. In Section 5, we have shown that when $k<-1$, the exponent γ in (5.3) is zero. Without loss of generality, we shall again assume, as in Section 6, that $\rho=1$, i.e., $b_0^*=b_0=-1$. Under these conditions, expansion (5.3) simplifies to

$$x(n) \sim n^{\alpha} \sum_{s=0}^{\infty} c_s n^{-s/2}. \tag{9.1}$$

As before, we write

$$x(n) = L_N(n) + E_N(n) \tag{9.2}$$

with

$$L_N(n) = n^{\alpha} \sum_{s=0}^{N-1} c_s n^{-s/2}, \tag{9.3}$$

and put $R_N(n) \equiv -\mathscr{L}\{L_N(n)\}$, where $\mathscr{L}$ is the difference operator defined in (5.10). Since $x(n)$ is a solution to (5.1), we have

$$E_N(n+2) + n^{k/2}a(n)E_N(n+1) + b^*(n)E_N(n) = R_N(n). \tag{9.4}$$

Choose $N>-k$. Applying the recurrence relations (5.17) and (5.18) to (5.16) gives

$$R_N(n) = O(n^{\alpha-N/2-1}); \tag{9.5}$$

compare (6.3). To show that Equation (9.4) has a solution satisfying

$$E_N(n) = O(n^{\alpha-N/2}), \tag{9.6}$$

we rewrite (9.4) in the form

$$E_N(n+2) - E_N(n) = R_N(n) - n^{-k/2}a(n)E_N(n+1) - [b^*(n)+1]E_N(n). \tag{9.7}$$

It is easily verified that any sequence $\{E_N(n)\}$ satisfying the equations

$$E_N(2n) = -\sum_{j=n}^{\infty}\Big\{R_N(2j)-(2j)^{k/2}a(2j)\,E_N(2j+1) \\ -\big[b^*(2j)+1\big]E_N(2j)\Big\} \qquad (9.8a)$$

and

$$E_N(2n+1) = -\sum_{j=n}^{\infty}\Big\{R_N(2j+1)-(2j+1)^{k/2}a(2j+1)\,E_N(2j+2) \\ -\big[b^*(2j+1)+1\big]E_N(2j+1)\Big\} \qquad (9.8b)$$

is automatically a solution to (9.7). Thus, it suffices to show that Equations (9.8a) and (9.8b) have a solution satisfying (9.6). To this end, we define the successive approximants $h_s(n)$ by

$$h_{s+1}(2n) = -\sum_{j=n}^{\infty}\Big\{R_N(2j)-(2j)^{k/2}a(2j)h_s(2j+1) \\ -\big[b^*(2j)+1\big]h_s(2j)\Big\} \qquad (9.9a)$$

and

$$h_{s+1}(2n+1) = -\sum_{j=n}^{\infty}\Big\{R_N(2j+1)-(2j+1)^{k/2}a(2j+1)h_s(2j+2) \\ -\big[b^*(2j+1)+1\big]h_s(2j+1)\Big\} \qquad (9.9b)$$

with $h_0(n)\equiv 0$. Since $k\le -3$ and $b_0^*=-1$, we have $n^{k/2}a(n)=O(n^{-1})$ and $b^*(n)+1=O(n^{-1})$. Hence, in view of (9.5), we may choose a positive constant M and a positive integer n_0 such that

$$|R_N(n)| \le Mn^{\mathrm{Re}\,\alpha-N/2-1} \qquad (9.10)$$

and

$$|n^{k/2}a(n)|+|b^*(n)+1| \le Mn^{-1} \qquad (9.11)$$

for all $n\ge n_0$. Following the analysis given in the previous sections, one can readily show that the series

$$\sum_{s=0}^{\infty} [h_{s+1}(n) - h_s(n)]$$

is uniformly convergent in n, and that its sum is a solution to (9.4) satisfying (9.6). The argument in the present case is, however, considerably simpler.

10. Proof of (5.3) When $k \le -2$ and k is Even

The formal series solution to (5.1) in this case is given in (5.19). Assuming $\rho = 1$ as before, this solution becomes

$$x(n) \sim n^{\alpha} \sum_{s=0}^{\infty} c_s n^{-s}. \tag{10.1}$$

We again set

$$x(n) = L_N(n) + E_N(n) \tag{10.2}$$

and

$$R_N(n) \equiv -\mathscr{L}\{L_N(n)\}, \tag{10.3}$$

where

$$L_N(n) = n^{\alpha} \sum_{s=0}^{N-1} c_s n^{-s}, \tag{10.4}$$

and $\mathscr{L}$ is the difference operator defined in (5.10). The remainder $E_N(n)$ then satisfies, as in Section 9, the equation

$$E_N(n+2) - E_N(n) = R_N(n) - n^{k/2} a(n) E_N(n+1) - [b^*(n)+1] E_N(n). \tag{10.5}$$

Choose $N > -T = -\frac{1}{2}k$. By (5.20) and the recurrence relations (5.21) and (5.22), we have

$$R_N(n) = O(n^{\alpha - N - 1}). \tag{10.6}$$

(Recall that the coefficient of c_s in (5.20) is zero.) The remaining argument proceeds as before, and the details will be omitted.

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