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RESEARCH ARTICLE

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Interval observers for linear functions of state vectors of linear fractional-order systems with delayed input and delayed output

Dinh Cong Huong[1, 2]

[1]Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam

[2]Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam

Correspondence
Dinh Cong Huong, Department for Management of Science and Technology Development, Ton Duc Thang University, Ho Chi Minh City, Vietnam; or Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam.
Email: dinhconghuong@tdtu.edu.vn

Summary

This paper addresses the problem of interval observer design for linear functions of state vectors of linear fractional-order systems, which are subjected to time delays in the measured output as well as the control input. By using the information of both the delayed output and input, we design two linear functional state observers to compute two estimates, an upper one and a lower one, which bound the unmeasured linear functions of state vectors. As a particular case with output delay only, we design a linear functional state observer to estimate (asymptotically) the unmeasured linear functions of state vectors. Existence conditions of such observers are provided, and some of them are translated into a linear programming problem, in which the observers' matrices can be effectively computed. Constructive design algorithms are introduced. Numerical examples are provided to illustrate the design procedure, practicality, and effectiveness of the proposed design method.

KEYWORDS

fractional-order systems, linear functional state observers, linear programming, positive systems, time-delay systems

1 | INTRODUCTION

In recent decades, fractional calculus have evolved into an interesting and useful area of research in view of the extensive application of its modeling tools in applied and technical sciences. From the viewpoint of mathematics, fractional calculus generalizes integer-order calculus. Meanwhile, fractional-order derivatives can depict real situations more elaborately than integer-order derivatives, especially when the situations posses hereditary properties or have memory. Due to these facts, fractional-order systems have gained considerable popularity and importance due to their numerous applications in many fields of science and engineering including physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex medium, polymer rheology, and control of dynamical systems.[1-12]

In many practical applications, the states of the considered systems are not easily obtained because of the technical or economic reasons. In this case, the estimation of actual states and output feedback control law are very necessary. This problem for systems with integer order has been extensively investigated (see, for example, other works[13-27]). For fractional-order systems, some interesting results on the observer design for fractional-order systems were reported in related works.[28-36] As we know, in networked control system or in control processes where computation units are located far from the plant, measurement and control data have to be transmitted through a communication channel and time delay is certain to happen and may affect system stability (see, for example, other works[37-41]).

There are some state observer design methods and observer structures available in the literature for systems with delayed output and instantaneous input (see, for example, related works[19,20,22,23]) and for systems with delayed input and instantaneous output (see, for example, other works[11,25,26]). However, very little attention has been paid to the problem of state observer design for linear systems that are subjected to time delays in both the measurement output as well as the control input. So far, to the best of our knowledge, only the work of Ha et al[27] tackled the problem of estimating linear functions of the state vector in the case when the output information and control signal are both available not instantaneously but only after certain time delays. It is worth noting that the Lyapunov-Krasovskii functional method, which was used in the aforementioned works[27] for integer-order systems, cannot be extended to fractional-order ones. In fact, it is difficult to construct a Lyapunov function and calculate its fractional-order derivatives because the Leibniz rule does not hold for fractional-order derivatives. This is the main reason why the work of Ha et al[27] has not been extended to fractional-order systems with delayed input and delayed output. Meanwhile, as we have explained in the aforementioned paragraph, it is necessary and important to address the problem of estimating linear functions of the state vector of linear fractional-order systems with both input and output subject to time delays.

In this paper, we consider a new problem of designing interval observers for linear functions of the state vector of linear fractional-order systems with both input and output subject to time delays. The main contributions of this study are highlighted in the following. (1) We first introduce a pair of linear functional state observers, which constructs an interval observer for linear functions of the state vector of the considered system;. (2) We then derive new conditions for the existence of such linear functional state observers. (3) By using these conditions and some auxiliary lemmas, we then propose a computational approach based on linear programming (LP) for the determination of the unknown observer matrices. (4) The effectiveness of the proposed design method is supported by four examples and simulation results.

This paper is organized as follows. In Section 2, we provide the problem statement and preliminaries. The main results are given in Section 3. In Section 4, we provide four numerical examples to demonstrate the effectiveness of our obtained results. A conclusion is drawn in Section 5.

Notation. I_n denotes the $n \times n$ identity matrix and $0_{m,n}$ denotes the $m \times n$ zero matrix. $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. For a vector $x = (x_i) \in \mathbb{R}^n$, we denote $|x| = (|x_i|) \in \mathbb{R}^n_+$. For a real matrix M, M^T denotes the transpose and $M \geq 0$ is called a nonnegative matrix if all of its components are nonnegative (ie, $m_{ij} \geq 0$ for all i,j).

The Caputo derivative of function $f(t)$ with order $\alpha \in (0, 1]$ is defined by

$$ {}_0^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \dot{f}(\tau) d\tau, $$

where $\dot{f}$ is the first-order derivative of function f and the function $\Gamma(\cdot)$ is defined as

$$ \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \quad z \in \mathbb{R}. $$

2 PROBLEM STATEMENT AND PRELIMINARIES

This section presents some definitions and preliminary results that will be used throughout this paper.

Consider the following linear fractional-order system:

$$ {}_0^C D_t^\alpha x(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1} $$

$$ x(0) = x_0, \tag{2} $$

$$ y(t) = Cx(t), \tag{3} $$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, and $y(t) \in \mathbb{R}^p$ is the output vector. By ${}_0^C D_t^\alpha x(t)$, we mean that ${}_0^C D_t^\alpha x(t) = \begin{bmatrix} {}_0^C D_t^\alpha x_1(t) & {}_0^C D_t^\alpha x_2(t) & \dots & {}_0^C D_t^\alpha x_n(t) \end{bmatrix}^T$, $\alpha \in (0, 1]$. x_0 is the initial condition. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are constant matrices and C is assumed to be full row rank.

Definition 1. Let matrices $N \in \mathbb{R}^{r\times r}$, $N_d \in \mathbb{R}^{r\times r}$, and a function $g(t) \in \mathbb{R}^r$, $t \geq 0$. The system of the form

$$ {}_0^C D_t^\alpha w(t) = Nw(t) + N_d w(t-\tau) + g(t), \quad t \geq 0, \tag{4} $$

$$ w(\theta) = \psi(\theta), \theta \in [-\tau, 0], \tag{5} $$

is said to be positive if, for any nonnegative initial condition $\psi(\theta) \in \mathbb{R}_+^n, \theta \in [-\tau, 0]$ the corresponding trajectory $w(t) \in \mathbb{R}_+^r$ for all $t \geq 0$.

The following definition will be used in this paper.

Definition 2 (See the work of Luenberger[42]).
A square real matrix M is called a Metzler matrix if its off-diagonal elements are nonnegative, ie, $m_{ij} \geq 0, i \neq j$.

Let $\psi(\theta) \in \mathbb{R}_+^n$, $\forall\theta \in [-\tau, 0]$. Then, from the work of Shen and Lam,[43] we obtain the following condition, which ensures the positivity of system (4)-(5).

Lemma 1. *System (4)-(5) is positive if and only if N is a Metzler matrix, N_d is nonnegative matrix, and $g(t) \geq 0$ for all $t \geq 0$.*

3 | MAIN RESULTS

Let us consider the linear fractional-order system (1)-(3), where the input $u(t)$ and output $y(t)$ are both subjected to known and constant time delays τ_1 and τ_2, respectively. We denote $z(t) = Lx(t) \in \mathbb{R}^r$, $1 \leq r \leq n$, as a linear function of the state vector, where L is any given $r \times n$ matrix. Our objective in this paper is to design two linear functional observers to compute two estimates, an upper one $\hat{z}^+(t)$ and a lower one $\hat{z}^-(t)$, which bound the unmeasured linear function $z(t) = Lx(t)$, ie, $\hat{z}^-(t) \leq z(t) \leq \hat{z}^+(t)$ for all $t \geq 0$ and the estimated error $e(t) = \hat{z}^+(t) - \hat{z}^-(t)$ is bounded. To achieve the objective, we first introduce the following assumption.

(H) For all $t \geq 0$, there exists $\delta > 0$ such that

$$ f^-(t) \leq f(t) = u(t) - u(t-\tau) \leq f^+(t), \; 0 \leq \tau = \max\{\tau_1, \tau_2\} \leq \delta, \tag{6} $$

for some known $f^-(t) \in \mathbb{R}^m$ and $f^+(t) \in \mathbb{R}^m$.

We next consider the following rth-order linear functional observers:

$$ \hat{z}^+(t) = \omega^+(t) + Ey(t-\tau), \tag{7} $$

$$ \begin{aligned} {}_0^C D_t^\alpha \omega^+(t) =& N\omega^+(t) + N_d\omega^+(t-\tau) + J_1 y(t-\tau) \\ &+ J_2 y(t-2\tau) + Hu(t-\tau) + LBf^+(t), \quad t \geq 0, \end{aligned} \tag{8} $$

$$ \omega^+(\theta) = \phi^+(\theta) \in \mathbb{R}^r, \quad \forall\theta \in [-\tau, 0], \tag{9} $$

$$ \hat{z}^-(t) = \omega^-(t) + Ey(t-\tau), \tag{10} $$

$$ \begin{aligned} {}_0^C D_t^\alpha \omega^-(t) =& N\omega^-(t) + N_d\omega^-(t-\tau) + J_1 y(t-\tau) \\ &+ J_2 y(t-2\tau) + Hu(t-\tau) + LBf^-(t), \quad t \geq 0, \end{aligned} \tag{11} $$

$$ \omega^-(\theta) = \phi^-(\theta) \in \mathbb{R}^r, \quad \forall\theta \in [-\tau, 0], \tag{12} $$

where ${}_0^C D_t^\alpha \omega^+(t) = [{}_0^C D_t^\alpha \omega_1^+(t) \quad \cdots \quad {}_0^C D_t^\alpha \omega_r^+(t)]^T$, ${}_0^C D_t^\alpha \omega^-(t) = [{}_0^C D_t^\alpha \omega_1^-(t) \quad \cdots \quad {}_0^C D_t^\alpha \omega_r^-(t)]^T$, $\phi^+(\theta)$ and $\phi^-(\theta)$ are continuous initial functions. $N \in \mathbb{R}^{r\times r}$, $N_d \in \mathbb{R}^{r\times r}$, $J_1 \in \mathbb{R}^{r\times p}$, $J_2 \in \mathbb{R}^{r\times p}$, $E \in \mathbb{R}^{r\times p}$, and $H \in \mathbb{R}^{r\times m}$ are matrices to be determined.

Definition 3. Systems (7)-(9) and (10)-(12) are called an interval observer for linear function $z(t)$ if, for any initial conditions, $\phi^+(\theta), \phi^-(\theta) \in \mathbb{R}^r$, $\theta \in [-\tau, 0]$, $\hat{z}^+(t) \in \mathbb{R}^r$, $\hat{z}^-(t) \in \mathbb{R}^r$, and

$$ \hat{z}^-(t) \leq z(t) \leq \hat{z}^+(t), \tag{13} $$

for all $t \geq 0$.

Let us next define the upper error $e^+(t)$, the lower error $e^-(t)$, and the total error $e(t)$ as

$$\begin{aligned}
e^+(t) &= \hat{z}^+(t) - z(t), \quad t \geq 0, \\
e^+(\theta) &= 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0], \\
e^-(t) &= z(t) - \hat{z}^-(t), \quad t \geq 0, \\
e^-(\theta) &= 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0], \\
e(t) &= e^+(t) + e^-(t) = \hat{z}^+(t) - \hat{z}^-(t), \quad t \geq 0, \\
e(\theta) &= 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0].
\end{aligned}$$

The following theorem provides conditions that guarantee the existence of interval observers (13).

Theorem 1. *Assume that $LB \geq 0$ and there exist a vector $\lambda \in \mathbb{R}^r_+$, a Metzler matrix $N \in \mathbb{R}^{r\times r}$, nonnegative matrix $N_d \in \mathbb{R}^{r\times r}_+$, a fixed positive vector b satisfying the following:*

$$(N + N_d)\lambda < 0, \tag{14}$$

$$NL - LA = 0, \tag{15}$$

$$(NE - J_1)C - ECA - N_d L = 0, \tag{16}$$

$$(N_d E - J_2)C = 0, \tag{17}$$

$$LB - H - ECB = 0, \tag{18}$$

$$LB(f^+(t) - f^-(t)) \leq b. \tag{19}$$

Then,

$$\hat{z}^-(t) \leq z(t) \leq \hat{z}^+(t), \quad \forall t \geq 0, \tag{20}$$

and

$$e(t) \leq -(N + N_d)^{-1} b, \quad \forall t \geq 0. \tag{21}$$

Proof. Regarding (1), (7), and (10), the fractional-order derivatives of $e^+(t)$ and $e^-(t)$ are given by

$$\begin{aligned}
{}_0^C D_t^\alpha e^-(t) &= L {}_0^C D_t^\alpha x(t) - {}_0^C D_t^\alpha \hat{z}^-(t) \\
&= Ne^-(t) + N_d e^-(t-\tau) + (LA - NL)x(t) \\
&\quad + ((NE - J_1)C - ECA - N_d L)x(t-\tau) + (N_d E - J_2)Cx(t-2\tau) \\
&\quad + LB(f(t) - f^-(t)) + (LB - H - ECB)u(t-\tau), \quad t \geq 0,
\end{aligned} \tag{22}$$

$$e^-(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0], \tag{23}$$

and

$$\begin{aligned}
{}_0^C D_t^\alpha e^+(t) &= {}_0^C D_t^\alpha \hat{z}^+(t) - L {}_0^C D_t^\alpha x(t) \\
&= Ne^+(t) + N_d e^+(t-\tau) + (NL - LA)x(t) \\
&\quad + (N_d L + (J_1 - NE)C + ECA)x(t-\tau) + (J_2 - N_d E)Cx(t-2\tau) \\
&\quad + LB(f^+(t) - f(t)) + (H + ECB - LB)u(t-\tau), \quad t \geq 0,
\end{aligned} \tag{24}$$

$$e^+(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0]. \tag{25}$$

It follows from (22) and (24) that, if conditions (15)-(18) of Theorem 1 are satisfied, then we obtain the following:

$${}_0^C D_t^\alpha e^-(t) = Ne^-(t) + N_d e^-(t-\tau) + LB(f(t) - f^-(t)), \quad t \geq 0, \tag{26}$$

$$e^-(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0], \tag{27}$$

$${}_0^C D_t^\alpha e^+(t) = Ne^+(t) + N_d e^+(t-\tau) + LB(f^+(t) - f(t)), \quad t \geq 0, \tag{28}$$

$$e^+(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0], \tag{29}$$

$$ {}_0^C D_t^\alpha e(t) = Ne(t) + N_d e(t-\tau) + LB(f^+(t) - f^-(t)), \quad t \geq 0, \tag{30}$$

$$e(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0]. \tag{31}$$

Since N is Metzler, $N_d \geq 0, LB \geq 0, f^-(t) \leq f(t) \leq f^+(t)$ for all $t \geq 0$; and systems (26)-(27), (28)-(29), and (30)-(31) are positive. Hence, we obtain (20) and $e(t) \geq 0$ for all $t \geq 0$. Next, we prove (21). It is not hard to see that $e(t) \leq \bar{e}(t)$, for all $t \geq 0$, where $\bar{e}(t)$ is the solution of the following system:

$$ {}_0^C D_t^\alpha \bar{e}(t) = N\bar{e}(t) + N_d \bar{e}(t-\tau) + b, \quad t \geq 0, \tag{32}$$

$$\bar{e}(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0]. \tag{33}$$

Now, we consider the following positive system without delays:

$$ {}_0^C D_t^\alpha v(t) = (N + N_d)v(t) + b, \quad t \geq 0, \tag{34}$$

$$v(\theta) = 0 \in \mathbb{R}^r_+, \quad \forall \theta \in [-\tau, 0]. \tag{35}$$

We will prove that $\bar{e}(t) \leq v(t)$ for all $t \geq 0$. Let us first prove that $v(t) \geq v(s)$ for all $t > s \geq 0$. It suffices to show that, for any $r > 0$, $v(t+r) \geq v(t)$ for all $t \geq 0$. For any $r > 0$, by denoting $\epsilon(t) = v(t+r) - v(t)$, then $\epsilon(t)$ satisfies that

$$ {}_0^C D_t^\alpha \epsilon(t) = (N + N_d)\epsilon(t), \quad t \geq 0, \tag{36}$$

$$\epsilon(\theta) = v(\theta + r) - v(\theta), \quad \forall \theta \in [-\tau, 0]. \tag{37}$$

Since $v(\theta) = 0$ and $v(\theta + r) \geq 0$ for all $\theta \in [-\tau, 0]$, we have $\epsilon(\theta) \geq 0, \forall \theta \in [-\tau, 0]$. Therefore, system (36)-(37) is positive and we obtain $v(t) \geq v(s)$ for all $t > s \geq 0$.

Now, by denoting $\eta(t) = v(t) - \bar{e}(t)$, then $\eta(t)$ satisfies that

$$ {}_0^C D_t^\alpha \eta(t) = N\eta(t) + N_d \eta(t-\tau) + N_d(v(t) - v(t-\tau)), \quad t \geq 0, \tag{38}$$

$$\eta(\theta) = 0, \quad \forall \theta \in [-\tau, 0]. \tag{39}$$

Since $v(t) - v(t-\tau) \geq 0$ for all $t \geq 0$ and $\eta(\theta) = 0, \forall \theta \in [-\tau, 0]$, system (38)-(39) is positive and hence $v(t) \geq \bar{e}(t)$ for all $t \geq 0$.

Now, by denoting $\xi(t) = -(N + N_d)^{-1}b - v(t)$, we see that $\xi(t)$ satisfies the following system:

$$ {}_0^C D_t^\alpha \xi(t) = (N + N_d)\xi(t), \quad t \geq 0, \tag{40}$$

$$\xi(0) = -(N + N_d)^{-1}b \geq 0. \tag{41}$$

Since this system is positive, we obtain $v(t) \leq -(N + N_d)^{-1}b$. The proof is complete. □

Remark 1. From Theorem 1, the design of a pair of linear functional observers now rest with determining unknown observer parameters $N \in \mathbb{R}^{r\times r}$, $N_d \in \mathbb{R}^{r\times r}$, $J_1 \in \mathbb{R}^{r\times p}$, $J_2 \in \mathbb{R}^{r\times p}$, $E \in \mathbb{R}^{r\times p}$, and $H \in \mathbb{R}^{r\times m}$ such that conditions (14)-(18) of Theorem 1 hold.

For this, we first denote $U = J_1 - NE$ and represent the Equations (15)-(16) into the following form:

$$\chi X = Y, \tag{42}$$

where

$$\chi = \begin{bmatrix} N & N_d & U & E \end{bmatrix}, \tag{43}$$

$$X = \begin{bmatrix} L & 0 \\ 0 & L \\ 0 & C \\ 0 & CA \end{bmatrix} \in \mathbb{R}^{2(r+p)\times 2n}, \tag{44}$$

$$Y = \begin{bmatrix} LA & 0 \end{bmatrix} \in \mathbb{R}^{r\times 2n}. \tag{45}$$

Since X and Y are two known constant matrices, a solution for χ always exists if and only if

$$\operatorname{rank}\begin{bmatrix} X \\ Y \end{bmatrix} = \operatorname{rank}(X). \tag{46}$$

Under condition (46), a general solution for χ is given by

$$\chi = YX^{+} + Z\left(I_{2(r+p)} - XX^{+}\right), \tag{47}$$

where $X^{+} \in \mathbb{R}^{2n\times 2(p+r)}$ is the Moor-Penrose–inverse of X and $Z \in \mathbb{R}^{r\times 2(p+r)}$ is an arbitrary matrix to be determined. Moreover, matrices $N \in \mathbb{R}^{r\times r}$, $N_d \in \mathbb{R}^{r\times r}$, $U \in \mathbb{R}^{r\times p}$, and $E \in \mathbb{R}^{r\times p}$ can now be extracted from (47) and are expressed as

$$N = \Phi e_N + Z\Psi e_N, \tag{48}$$

$$N_d = \Phi e_{N_d} + Z\Psi e_{N_d}, \tag{49}$$

$$U = \Phi e_U + Z\Psi e_U, \tag{50}$$

$$E = \Phi e_E + Z\Psi e_E, \tag{51}$$

where

$$\Phi = YX^{+}, \quad \Psi = I_{2(p+r)} - XX^{+} \tag{52}$$

and $e_N, e_{N_d} \in \mathbb{R}^{2(p+r)\times r}$, $e_U \in \mathbb{R}^{2(p+r)\times p}$, $e_E \in \mathbb{R}^{2(p+r)\times p}$ are the following:

$$e_N = \begin{bmatrix} I_r \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_{N_d} = \begin{bmatrix} 0 \\ I_r \\ 0 \\ 0 \end{bmatrix}, e_U = \begin{bmatrix} 0 \\ 0 \\ I_p \\ 0 \end{bmatrix}, e_E = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_p \end{bmatrix}. \tag{53}$$

Next, we will formulate an LP-based problem for checking the design parameters. By using (48)-(49), conditions (14) can be represented as

$$\left(e_N^T + e_{N_d}^T\right)\Phi^T\lambda + \left(e_N^T + e_{N_d}^T\right)\Psi^T\Gamma\mathbf{1}_r < 0. \tag{54}$$

Based on the aforementioned discussion, we obtain the following theorem, which provides a computational approach that is based on LP for the determination of the parameters N, N_d, U, and E of linear functional observers.

Theorem 2. *If the following LP problem with the variables $\lambda \in \mathbb{R}^r$ and $\Gamma \in \mathbb{R}^{2(p+r)\times r}$ is feasible:*

$$\begin{cases} \lambda > 0, \\ \left(e_N^T + e_{N_d}^T\right)\Phi^T\lambda + \left(e_N^T + e_{N_d}^T\right)\Psi^T\Gamma\mathbf{1}_r < 0, \end{cases} \tag{55}$$

then the observer gains N, N_d, U, and E are obtained as in (48)-(51) where $Z = (diag(\lambda))^{-1}\Gamma^T$.

Remark 2. Provided that matrices N, N_d, U, and E are obtained, from condition (18), matrix H can be obtained as $H = LB - ECB$. On the other hand, since matrix C is full row rank, condition (17) holds if and only if $N_dE - J_2 = 0$, which implies that $J_2 = N_dE$. Finally, we obtain $J_1 = U + NE$.

Algorithm 1. *We now present an algorithm to design an interval observer for linear function $z(t)$.*

Step 1: Given matrices A, C, and L, check if condition (46) holds. If it does, proceed to Step 2.
Step 2: Compute the matrices Φ and Ψ from (52).
Step 3: Solve the LP (55) with respect to Γ and λ.
Step 4: Compute the matrix $Z = (diag(\lambda))^{-1}\Gamma^T$, where (λ, Γ) is a solution obtained in Step 3.
Step 5: Substitute Z into (48)-(51) to obtain observer gains N, N_d, U, and E.
Step 6: Compute the matrices H, J_1, and J_2 from Remark 2.

Remark 3. Let us now consider a particular case when the control input is available without delay, ie, $\tau_1 = 0$. In the following, we will show that the instantaneous information of the control input is useful for improving the estimation accuracy. That is, we will derive conditions for asymptotic convergence of the observer error.

For the case when $\tau_1 = 0$, we consider the following rth-order linear functional observer:

$$\hat{z}(t) = \omega(t) + Ey(t-\tau_2), \tag{56}$$

$${}_0^CD_t^\alpha \omega(t) = N\omega(t) + N_d\omega(t-\tau_2) + J_1y(t-\tau_2) + J_2y(t-2\tau_2) + Hu(t-\tau_2) + H_1u(t), \quad t \geq 0, \tag{57}$$

$$\hat{z}(\theta) = \phi(\theta) \in \mathbb{R}^r, \quad \forall \theta \in [-\tau_2, 0], \tag{58}$$

where $w(t) \in \mathbb{R}^r$ is the observer state with an initial condition $\phi(\theta)$, ${}_0^CD_t^\alpha \omega(t) = \begin{bmatrix} {}_0^CD_t^\alpha \omega_1(t) \\ \vdots \\ {}_0^CD_t^\alpha \omega_r(t) \end{bmatrix}$, $N \in \mathbb{R}^{r\times r}$, $N_d \in \mathbb{R}^{r\times r}$, $J_1 \in \mathbb{R}^{r\times p}$, $J_2 \in \mathbb{R}^{r\times p}$, $E \in \mathbb{R}^{r\times p}$, $H \in \mathbb{R}^{r\times m}$, and $H_1 \in \mathbb{R}^{r\times m}$ are matrices to be determined.

We now define the error $e(t)$ as

$$e(t) = \hat{z}(t) - z(t), \quad t \geq 0,$$
$$e(\theta) = 0 \in \mathbb{R}_+^r, \quad \forall \theta \in [-\tau_2, 0].$$

Theorem 3. *Assume that there exist a vector $\lambda \in \mathbb{R}_+^r$, a Metzler matrix $N \in \mathbb{R}^{r\times r}$, and nonnegative matrix $N_d \in \mathbb{R}_+^{r\times r}$, satisfying the following:*

$$(N + N_d)\lambda < 0, \tag{59}$$

$$NL - LA = 0, \tag{60}$$

$$(NE - J_1)C - ECA - N_dL = 0, \tag{61}$$

$$(N_dE - J_2)C = 0, \tag{62}$$

$$LB - H_1 = 0, \tag{63}$$

$$H + ECB = 0. \tag{64}$$

Then, the error $e(t)$ converges asymptotically to zero as $t \to \infty$ for any initial conditions $x(0)$, $\phi(\theta)$ and control input $u(t)$.

Proof. From (1) and (56), we obtain the following:

$$\begin{aligned} {}_0^CD_t^\alpha e(t) &= L{}_0^CD_t^\alpha x(t) - {}_0^CD_t^\alpha \hat{z}(t) \\ &= Ne(t) + N_de(t-\tau_2) + (NL - LA)x(t) + (N_dL + (J_1 - NE)C + ECA)x(t-\tau_2) \\ &\quad + (J_2 - N_dE)Cx(t-2\tau_2)(H_1 - LB)u(t) - (ECB + H)u(t-\tau_2), \quad t \geq 0, \end{aligned} \tag{65}$$

$$e(\theta) = 0 \in \mathbb{R}_+^r, \quad \forall \theta \in [-\tau_2, 0]. \tag{66}$$

It follows from (65) that, if conditions (60)-(64) of Theorem 3 are satisfied, then we obtain the following:

$${}_0^CD_t^\alpha e(t) = Ne(t) + N_de(t-\tau_2), \quad t \geq 0, \tag{67}$$

$$e(\theta) = 0 \in \mathbb{R}_+^r, \quad \forall \theta \in [-\tau_2, 0]. \tag{68}$$

Since N is Metzler, $N_d \geq 0$, this system is positive. Hence, from (59), the error $e(t)$ converges asymptotically to zero as $t \to \infty$. □

Algorithm 2. *We now present an algorithm to design a linear functional state observer for system (1)-(3) with delayed output and instantaneous input.*

Step 1: Given matrices A, C, and L. Obtain matrix $H_1 = LB$. Check if condition (46) holds. If it does, proceed to Step 2.
Step 2: Compute the matrices Φ and Ψ from (52).
Step 3: Solve LP (55) with respect to Γ and λ.
Step 4: Compute matrix $Z = (diag(\lambda))^{-1}\Gamma^T$, where (λ, Γ) is a solution obtained in Step 3.
Step 5: Substitute Z into (48)-(51) to obtain observer gains N, N_d, U, and E.
Step 6: Compute the matrices H, J_1, and J_2 from Remark 2.

4 | NUMERICAL EXAMPLES

Example 1. Let us first consider the following second-order system with one output and matrices A, B, C, and L:

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad C = [\,1 \quad 0\,], \quad L = [\,0 \quad 1\,],$$

where the input and the output are subjected to time delays $\tau_1 = 0.5$ s and $\tau_2 = 0.3$ s, respectively, and the difference $f(t) = u(t) - u(t - \tau_1)$ satisfies the condition

$$f^-(t) \le f(t) \le f^+(t), \quad t \ge 0, \tag{69}$$

with $f^-(t)$ and $f^+(t)$ are two known bounds.

According to Step 1 of Algorithm 1, we obtain matrices

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = 3 = \text{rank}[X].$$

Hence, condition (46) is satisfied. The LP problem (55) is feasible with

$$\lambda = 120.9631, \quad Z^T = \begin{bmatrix} 0.8433 \\ 0.8266 \\ 0.8106 \\ 0.8266 \end{bmatrix}. \tag{70}$$

Now, taking (70) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as $N = -1.3$, $N_d = 0.5457$, $U = 1.0915$, and $E = 0.5457$. According to Step 6 of Algorithm 1, we obtain $J_1 = 0.3820$, $J_2 = 0.2978$, and $H = 1.4543$. Hence, the following observers will provide estimates, an upper one and a lower one, which bound the unmeasured linear function $z(t)$:

$$\hat{z}^+(t) = \omega^+(t) + 0.5457y(t - 0.5), \tag{71}$$

$$\begin{aligned} {}_0^C D_t^\alpha \omega^+(t) = &- 1.3\omega^+(t) + 0.5457\omega^+(t - 0.5) + 0.3820y(t - 0.5) \\ &+ 0.2978y(t - 1) + 1.4543u(t - 0.5) + 2f^+(t), \quad t \ge 0, \end{aligned} \tag{72}$$

$$\omega^+(\theta) = 2, \quad \forall \theta \in [-0.5, 0), \tag{73}$$

$$\hat{z}^-(t) = \omega^-(t) + 0.5457y(t - 0.5), \tag{74}$$

$$\begin{aligned} {}_0^C D_t^\alpha \omega^-(t) = &- 1.3\omega^+(t) + 0.5457\omega^+(t - 0.5) + 0.3820y(t - 0.5) \\ &+ 0.2978y(t - 1) + 1.4543u(t - 0.5) + 2f^-(t), \quad t \ge 0, \end{aligned} \tag{75}$$

$$\omega^-(\theta) = -1, \quad \forall \theta \in [-0.5, 0). \tag{76}$$

For simulation, let us consider the control input $u(t) = 0.1 \sin t - 1,\ 0 \le t \le 20, f^-(t) = u(t) - u(t - 0.5) - 0.1$, $f^+(t) = u(t) - u(t - 0.5) + 0.1$, which satisfy assumption $f^-(t) \le f(t) \le f^+(t)$ and the initial conditions are $x_1(0) = 1$, $x_2(0) = 2$. Figure 1 shows the responses of $z(t) = x_2(t)$, $\hat{z}^-(t)$, and $\hat{z}^+(t)$ with fractional-order $\alpha = 0.75$.

Next, we demonstrate the case where only the output is subjected to time delay, that is, $\tau_2 = 0.5$ and $\tau_1 = 0$. By following the steps of Algorithm 2, we obtain $N = -1.3$, $N_d = 0.5457$, $U = 1.0915$, $E = 0.5457$, $J_1 = 0.382$, $J_2 = 0.2978$, $H = -0.0205$, and $H_1 = 2$. Hence, the following observers can estimate asymptotically $z(t)$:

$$\hat{z}(t) = \omega(t) + 0.5457y(t - 0.5), \tag{77}$$

$$\begin{aligned} {}_0^C D_t^\alpha \omega(t) = &- 1.3\omega(t) + 0.5457\omega(t - 0.5) + 0.382y(t - 0.5) \\ &+ 0.2978y(t - 1) - 0.0205u(t - 0.5) + 2u(t), \quad t \ge 0, \end{aligned} \tag{78}$$

$$\omega(\theta) = 2, \quad \forall \theta \in [-0.5, 0]. \tag{79}$$

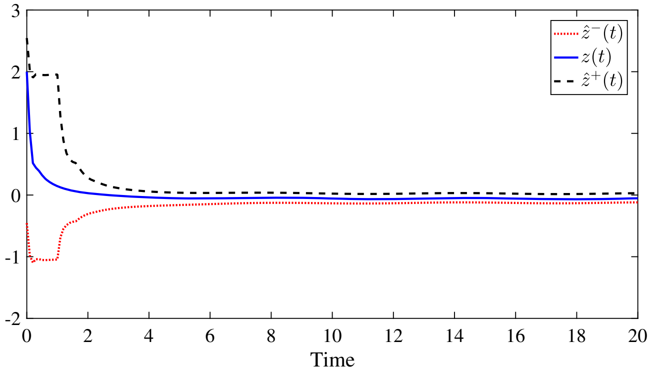

FIGURE 1 Responses of $z(t) = x_2(t)$, $\hat{z}^-(t)$, and $\hat{z}^+(t)$ with fractional-order $\alpha = 0.75$ [Colour figure can be viewed at wileyonlinelibrary.com]

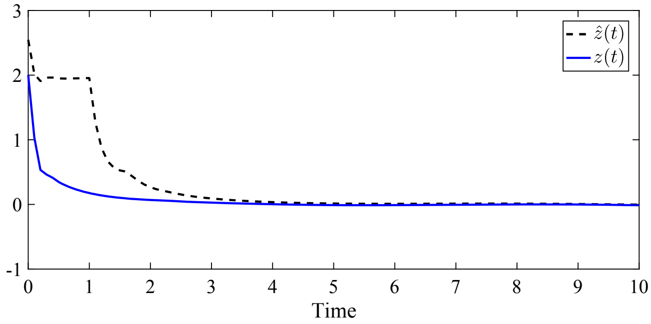

FIGURE 2 Responses of $z(t) = x_2(t)$ and $\hat{z}(t)$ with fractional-order $\alpha = 0.75$ [Colour figure can be viewed at wileyonlinelibrary.com]

For simulation, let us consider the control input $u(t) = 0.1 \sin t - 1,\ 0 \le t \le 10$, and the initial conditions are $x_1(0) = 1$, $x_2(0) = 2$. Figure 2 shows the responses of $z(t) = x_2(t)$, and its estimation $\hat{z}(t)$ with fractional-order $\alpha = 0.75$.

Example 2. Consider the following third-order system with two outputs and matrices A, B, C, and L are as given in the following:

$$A = \begin{bmatrix} 0.5 & -0.2 & 1 \\ 2 & -3 & 1 \\ 0 & 0 & -1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = [\,0 \quad 0 \quad 1\,],$$

where the input and the output are subjected to time delays $\tau_1 = 0.5$ s and $\tau_2 = 0.3$ s, respectively, and the difference $f(t) = u(t) - u(t - \tau_1)$ satisfies the condition

$$f^-(t) \le f(t) \le f^+(t), \quad t \ge 0,$$

with $f^-(t)$ and $f^+(t)$ are two known bounds.

According to Step 1 of Algorithm 1, we obtain

$$\operatorname{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = 4 = \operatorname{rank}[X],$$

which implies that condition (46) is satisfied. The LP problem (55) is feasible with

$$\lambda = 87.4424, \quad Z^T = \begin{bmatrix} 1.1523 \\ 1.2057 \\ 1.1765 \\ 1.1457 \\ 1.1017 \\ 1.1536 \end{bmatrix}. \tag{80}$$

Now, taking (80) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as $N = -1.5$, $N_d = 0.2197$, $U = [\,-0.0841 \quad 0.3181\,]$, $E = [\,-0.3491 \quad 0.1293\,]$, $J_1 = [\,0.4395 \quad 0.1242\,]$, $J_2 = [\,-0.0767 \quad 0.0284\,]$, and

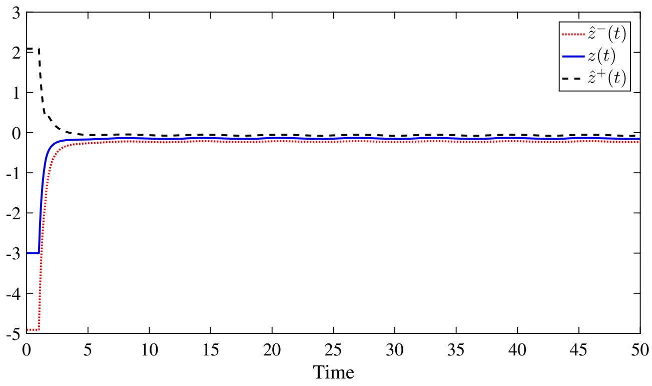

FIGURE 3 Responses of $z(t) = x_3(t), \hat{z}^-(t)$, and $\hat{z}^+(t)$ with fractional-order $\alpha = 0.91$ [Colour figure can be viewed at wileyonlinelibrary.com]

$H = 3.0904$. Hence, the following observers will provide estimates, an upper one and a lower one, which bound the unmeasured linear function $z(t)$:

$$\hat{z}^+(t) = \omega^+(t) + [\,-0.3491 \quad 0.1293\,]y(t-0.5), \tag{81}$$

$$\begin{aligned} {}_0^CD_t^\alpha \omega^+(t) = &- 1.5\omega^+(t) + 0.2197\omega^+(t-0.5) + [\,0.4395 \quad 0.1242\,]y(t-0.5) \\ &[\,-0.0767 \quad 0.0284\,]y(t-1) + 3.0904u(t-0.5) + 3f^+(t), \quad t \geq 0, \end{aligned} \tag{82}$$

$$\omega^+(\theta) = 2, \quad \forall\theta \in [-0.5, 0], \tag{83}$$

$$\hat{z}^-(t) = \omega^-(t) + [\,-0.3491 \quad 0.1293\,]y(t-0.5), \tag{84}$$

$$\begin{aligned} {}_0^CD_t^\alpha \omega^-(t) = &- 1.5\omega^+(t) + 0.2197\omega^+(t-0.5) + [\,0.4395 \quad 0.1242\,]y(t-0.5) \\ &+ [\,-0.0767 \quad 0.0284\,]y(t-1) + 3.0904u(t-0.5) + 3f^-(t), \quad t \geq 0, \end{aligned} \tag{85}$$

$$\omega^-(\theta) = -5, \quad \forall\theta \in [-0.5, 0]. \tag{86}$$

For simulation, let us consider the control input $u(t) = 0.1\sin t - 1, \quad 0 \leq t \leq 50, f^-(t) = u(t) - u(t-0.5) - 0.05$, $f^+(t) = u(t) - u(t-0.5) + 0.05$, which satisfy assumption $f^-(t) \leq f(t) \leq f^+(t)$, and the initial conditions are $x_1(0) = -1, x_2(0) = -2, x_3(0) = -3$. Figure 3 shows the responses of $z(t) = x_3(t), \hat{z}^-(t)$, and $\hat{z}^+(t)$ with fractional-order $\alpha = 0.91$.

Next, we illustrate the case where only the output is subjected to time delay, that is, $\tau_2 = 0.5$ and $\tau_1 = 0$. By following the steps of Algorithm 2, we obtain $N = -1.5, N_d = 0.2197, U = [\,-0.0841 \;\; 0.3181\,], E = [\,-0.3491 \;\; 0.1293\,]$, $J_1 = [\,0.4395 \;\; 0.1242\,], J_2 = [\,-0.0767 \;\; 0.0284\,], H = 0.0904$, and $H_1 = 3$. Hence, the following observers can estimate asymptotically $z(t)$:

$$\hat{z}(t) = \omega(t) + [\,-0.3491 \quad 0.1293\,]y(t-0.5), \tag{87}$$

$$\begin{aligned} {}_0^CD_t^\alpha \omega(t) = &- 1.5\omega(t) + 0.2197\omega(t-0.5) + [\,0.4395 \quad 0.1242\,]y(t-0.5) \\ &[\,-0.0767 \quad 0.0284\,]y(t-1) + 0.0904u(t-0.5) + 3u(t), \quad t \geq 0, \end{aligned} \tag{88}$$

$$\omega(\theta) = -5, \quad \forall\theta \in [-0.5, 0]. \tag{89}$$

For simulation, let us consider the control input $u(t) = 0.1\sin t - 1, \; 0 \leq t \leq 10$, and the initial conditions are $x_1(0) = -1, x_2(0) = -2, x_3(0) = -3$. Figure 4 shows the responses of $z(t) = x_3(t)$, and its estimation $\hat{z}(t)$ with fractional-order $\alpha = 0.91$.

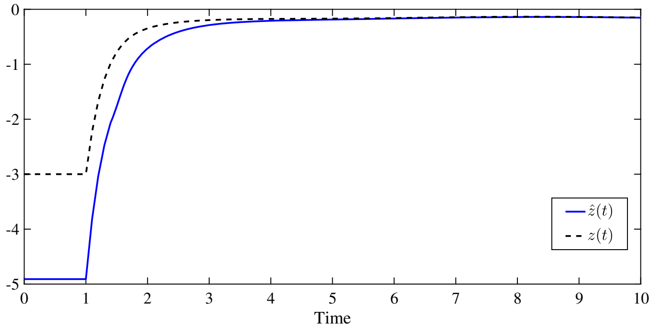

FIGURE 4 Responses of $z(t) = x_3(t)$ and $\hat{z}(t)$ with fractional-order $\alpha = 0.91$ [Colour figure can be viewed at wileyonlinelibrary.com]

Example 3. Let us consider the following positive fourth-order system with two outputs and matrices A, B, C, and L:

$$A = \begin{bmatrix} -5 & 3 & 1 & 2 \\ 4 & -7 & 1 & 3 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the input and the output are subjected to time delays $\tau_1 = 0.9$ s and $\tau_2 = 0.6$ s, respectively, and the difference $f(t) = u(t) - u(t - \tau_1)$ satisfies the condition

$$f^-(t) \le f(t) \le f^+(t), \quad t \ge 0,$$

with $f^-(t)$ and $f^+(t)$ are two known bounds.

According to Step 1 of Algorithm 1, we obtain

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = 6 = \text{rank}\,[X].$$

Hence, condition (46) is satisfied. The LP problem (55) is feasible with

$$\lambda = \begin{bmatrix} 72.7579 \\ 111.7689 \end{bmatrix}, \quad Z^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1.3959 & 0.9087 \\ 1.4116 & 0.9189 \\ -0.0309 & -0.0201 \\ -0.0222 & -0.0145 \\ -0.0133 & -0.0086 \\ -0.0089 & -0.0058 \end{bmatrix}. \tag{90}$$

Now, taking (90) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as

$$N = \begin{bmatrix} -4 & 4 \\ 1 & -5 \end{bmatrix}, \quad N_d = \begin{bmatrix} 0.4750 & 1.1484 \\ 0.3092 & 0.7476 \end{bmatrix}, \quad U = \begin{bmatrix} -0.5896 & -0.5588 \\ -0.3838 & -0.3637 \end{bmatrix}, \quad H = \begin{bmatrix} 3.6734 \\ 4.4383 \end{bmatrix},$$

$$E = \begin{bmatrix} -0.2766 & -0.1984 \\ -0.1801 & -0.1291 \end{bmatrix}, \quad J_1 = \begin{bmatrix} -0.2034 & -0.2818 \\ 0.2399 & 0.0836 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -0.3382 & -0.2425 \\ -0.2201 & -0.1579 \end{bmatrix}.$$

Hence, we obtain observers of the form (7)-(9) and (10)-(12), which provides estimates, an upper one and a lower one, which bound the unmeasured linear function $z(t)$. For simulation, let us consider the control input $u(t) = 0.1 \sin t - 1$, $0 \le t \le 50$, $f^-(t) = u(t) - u(t - 0.5) - 0.15$, $f^+(t) = u(t) - u(t - 0.5) + 0.15$, which satisfy assumption $f^-(t) \le f(t) \le f^+(t)$ and the initial conditions are $x_1(0) = -1$, $x_2(0) = -2$, $x_3(0) = -3$, $x_4(0) = -4$, $\omega_1^+(\theta) = 3$, $\omega_1^-(\theta) = -1$, $\omega_2^+(\theta) = 2$, $\omega_1^-(\theta) = -5$ for $\theta \in [-0.9, 0]$. Figure 5 shows the responses of $z_1(t) = x_3(t)$, $\hat{z}_1^-(t)$, and $\hat{z}_1^+(t)$, whereas Figure 6 shows the responses of $z_2(t) = x_2(t)$, $\hat{z}_2^-(t)$, and $\hat{z}_2^+(t)$ with fractional-order $\alpha = 0.84$.

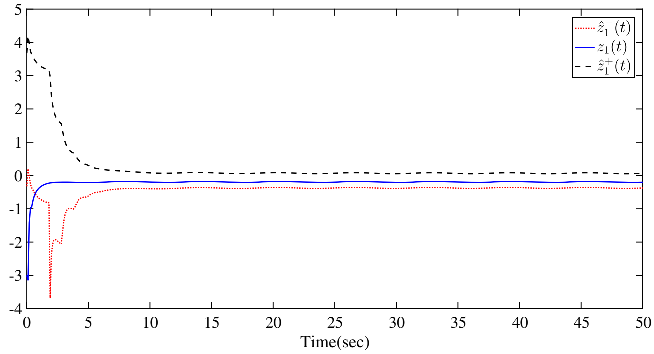

FIGURE 5 Responses of $z_1(t) = x_3(t)$, $\hat{z}_1^-(t)$, and $\hat{z}_1^+(t)$ with fractional-order $\alpha = 0.84$ [Colour figure can be viewed at wileyonlinelibrary.com]

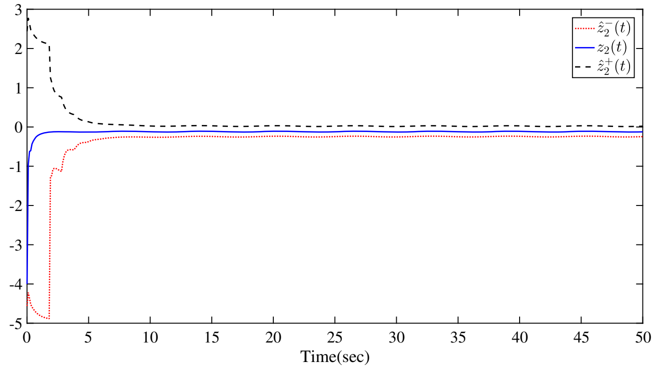

FIGURE 6 Responses of $z_2(t) = x_4(t)$, $\hat{z}_2^-(t)$, and $\hat{z}_2^+(t)$ with fractional-order $\alpha = 0.84$ [Colour figure can be viewed at wileyonlinelibrary.com]

Next, we illustrate the case where only the output is subjected to time delay, that is, $\tau_2 = 0.6$ and $\tau_1 = 0$. By following the steps of Algorithm 2, we obtain

$$N = \begin{bmatrix} -4 & 4 \\ 1 & -5 \end{bmatrix}, \quad N_d = \begin{bmatrix} 0.4750 & 1.1484 \\ 0.3092 & 0.7476 \end{bmatrix}, \quad U = \begin{bmatrix} -0.5896 & -0.5588 \\ -0.3838 & -0.3637 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.6734 \\ 0.4383 \end{bmatrix}, \quad E = \begin{bmatrix} -0.2766 & -0.1984 \\ -0.1801 & -0.1291 \end{bmatrix}, \quad J_1 = \begin{bmatrix} -0.2034 & -0.2818 \\ 0.2399 & 0.0836 \end{bmatrix},$$
$$J_2 = \begin{bmatrix} -0.3382 & -0.2425 \\ -0.2201 & -0.1579 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Hence, an observer of the form that can estimate asymptotically $z(t)$.

For simulation, let us consider the control input $u(t) = 0.1 \sin t - 1,\ 0 \le t \le 10$, and the initial conditions are $x_1 0) = -1, x_2(0) = -2, x_3(0) = -3, x_4(0) = -4, \hat{z}_1(\theta) = -2.5, \hat{z}_2(\theta) = -3.5$ for $\theta \in [-0.6, 0]$. Figures 7 and 8 show the responses of $z_1(t) = x_3(t)$, $z_2(t) = x_4(t)$ and their estimations $\hat{z}_1(t)$, $\hat{z}_2(t)$ with fractional-order $\alpha = 0.84$, respectively.

Example 4. We now consider the following fourth-order system with two outputs and matrices A, B, C, and L:

$$A = \begin{bmatrix} -4 & 3 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where the input and the output are subjected to time delays $\tau_1 = 0.9$ s and $\tau_2 = 0.6$ s, respectively, and the difference $f(t) = u(t) - u(t - \tau_1)$ satisfies the condition

$$f^-(t) \le f(t) \le f^+(t), \quad t \ge 0,$$

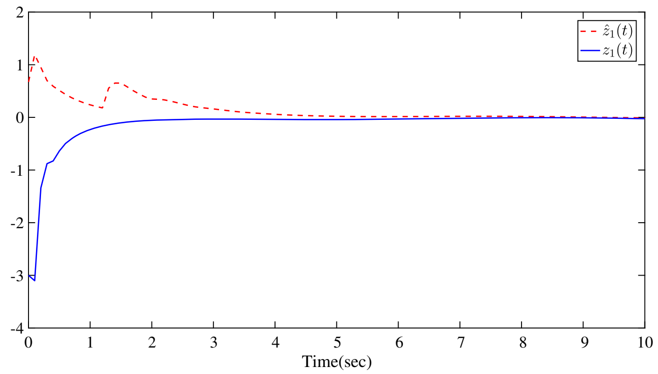

FIGURE 7 Responses of $z_1(t) = x_3(t)$ and $\hat{z}_1(t)$ with fractional-order $\alpha = 0.84$ [Colour figure can be viewed at wileyonlinelibrary.com]

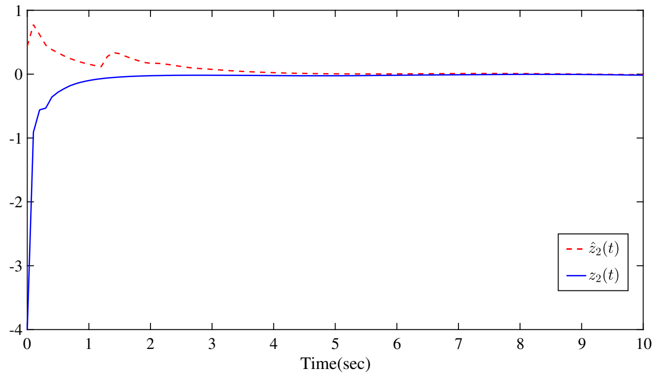

FIGURE 8 Responses of $z_2(t) = x_4(t)$ and $\hat{z}_2(t)$ with fractional-order $\alpha = 0.84$ [Colour figure can be viewed at wileyonlinelibrary.com]

with $f^-(t)$ and $f^+(t)$ are two known bounds.

Note that, for this example, the pair (A, C) is not observable. We will show that all conditions of Theorem 1 and Theorem 2 are satisfied. Therefore, it is not necessary to make assumption that the pair (A, C) is observable in this paper.

According to Step 1 of Algorithm 1, we obtain

$$\text{rank}\begin{bmatrix} X \\ Y \end{bmatrix} = 6 = \text{rank}[X].$$

Hence, condition (46) is satisfied. The LP problem (55) is feasible with

$$\lambda = \begin{bmatrix} 45.5485 \\ 27.3773 \end{bmatrix}, \quad Z^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2.2908 & 3.8112 \\ 0 & 0 \\ -0.1455 & -0.2421 \\ -0.2219 & -0.3693 \\ -0.0837 & -0.1392 \\ -0.0946 & -0.1574 \end{bmatrix}. \tag{91}$$

Now, taking (91) into account for Step 4 and Step 5 of Algorithm 1, the observer gains are obtained as

$$N = \begin{bmatrix} -4 & 0 \\ 1 & -5 \end{bmatrix}, \quad N_d = \begin{bmatrix} 0.3007 & 0 \\ 0.5004 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -0.4627 & -0.7056 \\ -0.7698 & -1.1739 \end{bmatrix}, \quad H = \begin{bmatrix} 3.8675 \\ 5.4433 \end{bmatrix},$$

$$E = \begin{bmatrix} -0.266 & -0.3007 \\ -0.4426 & -0.5004 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0.6015 & 0.4974 \\ 1.1773 & 1.0271 \end{bmatrix}, \quad J_2 = \begin{bmatrix} -0.08 & -0.0904 \\ -0.1331 & -0.1505 \end{bmatrix}.$$

Hence, we obtain observers of the form (7)-(9) and (10)-(12), which provides estimates, an upper one and a lower one, which bound the unmeasured linear function $z(t)$. For simulation, let us consider the control input $u(t) = 0.1$

$\sin t - 1,\ \ 0 \le t \le 50, f^{-}(t) = u(t) - u(t-0.5) - 0.15, f^{+}(t) = u(t) - u(t-0.5) + 0.15$, which satisfy assumption $f^{-}(t) \le f(t) \le f^{+}(t)$, and the initial conditions are $x_1(0) = -1, x_2(0) = -2, x_3(0) = -3, x_4(0) = -4, \omega_1^{+}(\theta) = 3, \omega_1^{-}(\theta) = -1, \omega_2^{+}(\theta) = 2, \omega_1^{-}(\theta) = -5$ for $\theta \in [-0.9, 0]$. Figure 9 shows the responses of $z_1(t) = x_3(t), \hat{z}_1^{-}(t)$, and $\hat{z}_1^{+}(t)$, whereas Figure 10 shows the responses of $z_2(t) = x_2(t), \hat{z}_2^{-}(t)$, and $\hat{z}_2^{+}(t)$ with fractional-order $\alpha = 0.79$.

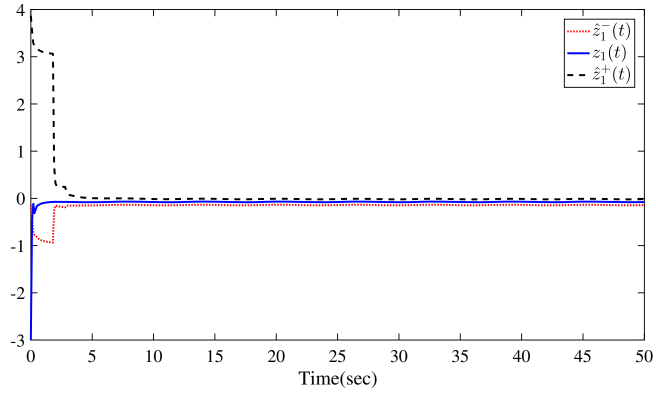

FIGURE 9 Responses of $z_1(t) = x_3(t), \hat{z}_1^{-}(t)$ and $\hat{z}_1^{+}(t)$ with fractional-order $\alpha = 0.79$ [Colour figure can be viewed at wileyonlinelibrary.com]

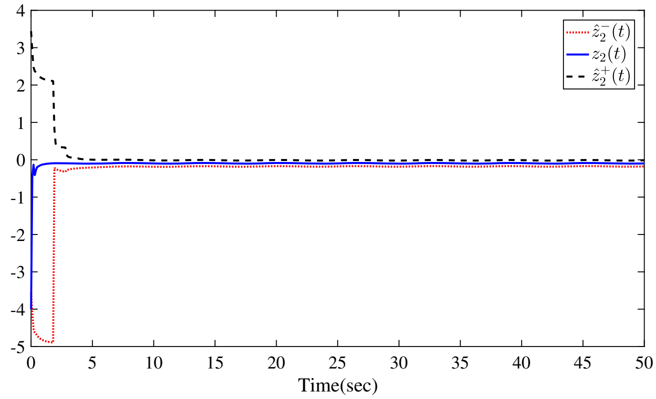

FIGURE 10 Responses of $z_2(t) = x_4(t), \hat{z}_2^{-}(t)$ and $\hat{z}_2^{+}(t)$ with fractional-order $\alpha = 0.79$ [Colour figure can be viewed at wileyonlinelibrary.com]

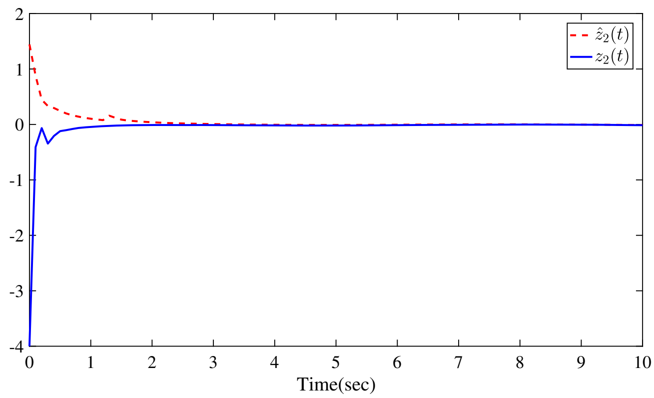

FIGURE 11 Responses of $z_1(t) = x_3(t)$ and $\hat{z}_1(t)$ with fractional-order $\alpha = 0.79$ [Colour figure can be viewed at wileyonlinelibrary.com]

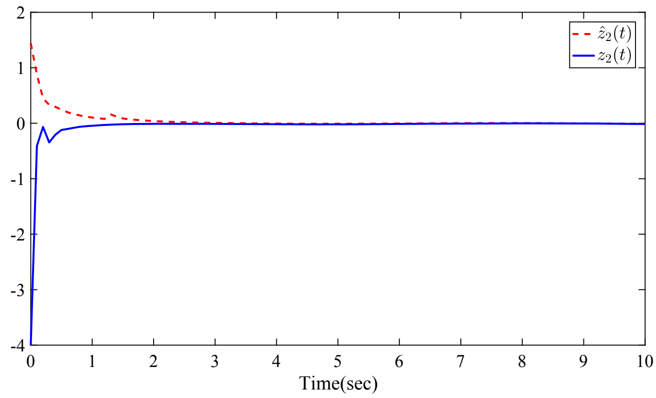

FIGURE 12 Responses of $z_2(t) = x_4(t)$ and $\hat{z}_2(t)$ with fractional-order $\alpha = 0.79$ [Colour figure can be viewed at wileyonlinelibrary.com]

Next, we illustrate the case where only the output is subjected to time delay, that is, $\tau_2 = 0.6$ and $\tau_1 = 0$. By following the steps of Algorithm 2, we obtain

$$N = \begin{bmatrix} -4 & 0 \\ 1 & -5 \end{bmatrix}, \quad N_d = \begin{bmatrix} 0.3007 & 0 \\ 0.5004 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} -0.4627 & -0.7056 \\ -0.7698 & -1.1739 \end{bmatrix},$$
$$H = \begin{bmatrix} 0.8675 \\ 1.4433 \end{bmatrix}, \quad E = \begin{bmatrix} -0.266 & -0.3007 \\ -0.4426 & -0.5004 \end{bmatrix}, \quad J_1 = \begin{bmatrix} 0.6015 & 0.4974 \\ 1.1773 & 1.0271 \end{bmatrix},$$
$$J_2 = \begin{bmatrix} -0.08 & -0.0904 \\ -0.1331 & -0.1505 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Hence, an observer of the form that can estimate asymptotically $z(t)$.

For simulation, let us consider the control input $u(t) = 0.1 \sin t - 1, \ 0 \leq t \leq 10$ and the initial conditions are $x_1(0) = -1, x_2(0) = -2, x_3(0) = -3, x_4(0) = -4, \hat{z}_1(\theta) = -2.5, \hat{z}_2(\theta) = -3.5$ for $\theta \in [-0.6, 0]$. Figures 11 and 12 show the responses of $z_1(t) = x_3(t)$, $z_2(t) = x_4(t)$ and their estimations $\hat{z}_1(t)$, $\hat{z}_2(t)$ with fractional-order $\alpha = 0.79$, respectively.

5 | CONCLUSION

In this paper, a new problem of designing interval observers for linear functions of the state vector of linear fractional-order systems with delayed input and delayed output has been investigated. We have designed two linear functional state observers to compute two estimates, ie, an upper one and a lower one, which bound the unmeasured linear functions of state vectors. In particular, when the control input is available without delay, a linear functional state observer to estimate (asymptotically) the unmeasured linear functions of state vectors has been presented. Existence conditions of such observers are provided, and some of them are translated into an LP problem in which the observers' matrices can be effectively computed. Four numerical examples are provided to illustrate the effectiveness of the proposed method.

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ORCID

Dinh Cong Huong https://orcid.org/0000-0003-2028-3784

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