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OBSERVABILITY OF LINEAR TIME-VARYING DESCRIPTOR SYSTEMS*

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Abstract. A characterization of observability for linear time-varying descriptor systems $E(t)x'(t) + F(t)x(t) = B(t)u(t)$, $y(t) = C(t)x(t)$, is given. E is not required to have constant rank. The characterization is designed to reduce symbolic computation and has potential advantages even when E is nonsingular. It is also shown that all observable analytic descriptor systems are smoothly observable even if they are not uniformly observable. Finally, the external behavior of time-varying descriptor systems is characterized.

Key words. descriptor, singular, observability, external behavior

AMS(MOS) subject classifications. 34A08, 93B07, 93B15, 93C15, 93C50

1. Introduction. In the last decade there has been increasing interest in the utilization of implicit differential equations

$$F(x', x, t) = 0, \tag{1}$$

known variously as descriptor, singular, or differential algebraic (DAE) systems [3]. There have been several motivations for this effort ranging from computational advantages to the fact that most physical systems are originally modeled in this form.

When confronted with an implicit system there are many questions that we can ask. In this paper we will be concerned with the observability of

$$E(t)x'(t) + F(t)x(t) = B(t)u(t), \tag{2a}$$

$$y(t) = C(t)x(t). \tag{2b}$$

Here, u is the control, y is the m-dimensional observation, and x is the n-dimensional state. For technical reasons we assume that E, F, B, C are infinitely differentiable although that will not usually be required in practice. We define *smooth*, then, to mean infinitely differentiable. The infinite differentiability is only used to make some of our conditions necessary as well as sufficient. Intervals are always assumed to be nontrivial.

Our goal is to develop characterizations and algorithms that can be reasonably rapid to apply. In one scenario for the use of our results, the researcher has formulated the equations (2) and wants to "quickly" know if the problem is observable. The coefficients E, F, B, C are assumed known functions. However, the researcher may want to try several different formulations of the underlying problem, perhaps by changing what the choices of control and observation are. Also, there may be design parameters in the problem description and observability will need to be tested for several values of these parameters. In this setting we want to reduce the amount of symbolic computation. Our goal is to develop procedures for which the only symbolic operation is simply differentiation. All other computation, including matrix multiplication, will then be done numerically.

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With the obvious notation, (2) can be written as a single system

$$\tilde{E}x' + \tilde{F}x = \tilde{B}\begin{bmatrix} u \\ y \end{bmatrix}. \tag{3}$$

However, we shall see that there are advantages in keeping the pair of equations (2).

Our presentation is self-contained. The proofs make frequent use of results from [7].

1.1. Observability. The system (2) is *observable* on the interval $\mathscr{I}$ if knowledge of the output y and the control u on any subinterval $\tilde{\mathscr{I}}$ of $\mathscr{I}$ uniquely determines *smooth* solutions x of (2a) on $\tilde{\mathscr{I}}$. If C in (2b) is not full column rank on a dense set, then the additional information to determine x is gotten (at least theoretically) by differentiating (2b). Observability has been frequently discussed when $E(t) = I$ since the early work in [15], [19], [23], and [27] and in the descriptor case when E, F, B, C are constant matrices. However, ours is the first discussion of observability of time-varying descriptor systems. While our initial formulation is similar in spirit to that in [15], our assumptions, methods, and goals are different.

For linear time-invariant descriptor systems there is some variance in the definitions of observability depending on how the authors wish to deal with the potential impulsive behavior (for example, see [1], [12], [13], [14], [16], [20], [21], [26]). We shall assume that the controls u are sufficiently smooth and the initial conditions for the descriptor system consistent so that no impulsive behavior is present.

It was noted early in the observability literature that there are different forms of observability [23]. We have just defined *total observability*. In some problems a stronger type of observability is needed.

DEFINITION 1. The system (2) is *smoothly observable* (of order (k, l)) on the interval $\mathscr{I}$, if there exists smooth $K_i(t)$, $L_i(t)$ on $\mathscr{I}$ such that

$$x = \sum_{i=0}^{k} K_i(t)y^{(i)}(t) + \sum_{i=0}^{l} L_i(t)(Bu)^{(i)}(t). \tag{4}$$

DEFINITION 2. The system (2) is *uniformly observable* if it is smoothly observable of order $(n-1, n-1)$.

Uniform observability [23] is usually defined differently. We shall relate the two definitions later when we discuss the E nonsingular case.

Example 1. The system

$$x' = x + u, \tag{5}$$

$$y = \phi x, \tag{6}$$

where $\phi(t)$ is an infinitely differentiable function such that $\phi^{(i)}(0) = 0$ for $0 \leqq i < \infty$ and $\phi(t) \neq 0$ if $t \neq 0$, is observable but not smoothly observable on every interval $\mathscr{I}$ containing zero.

Example 2. The system

$$x' = x + u, \tag{7a}$$

$$y = t^2 x \tag{7b}$$

is smoothly observable of order (2, 1) since

$$x = (2t^2 + 2t + 2)^{-1}[y'' - y' + 2y - 4tu - t^2u']. \tag{8}$$

However, (7) is not uniformly observable on any interval containing zero.

Using all of the information gotten in differentiating the output can be helpful, as indicated by the next example.

Example 3. Consider the system

$$x' = x + u, \tag{9}$$

$$y = tx. \tag{10}$$

Equation (10) implies that $x = t^{-1}y$, which is not smooth at $t = 0$. Differentiating (10) once and using (9) for x' gives

$$x = (1+t)^{-1}(-y' - tu), \tag{11}$$

which is not smooth at $t = -1$. However, if we use the differentiated equation *and* the original (10), and solve the overdetermined system that results, we get

$$x = y' - y + tu, \tag{12}$$

which is smooth on any interval.

There is a tradeoff here. By allowing extra differentiations of the inputs and outputs, we can obtain extra smoothness of the coefficients in the observation equation (4).

Our definition implies that any portions of the solution x which are completely determined by the control u are automatically observable.

Example 4. Let E be a constant matrix N which is nilpotent of index ν, and let $F = -I$. Then (2) is observable independent of B, C since

$$x = -\sum_{i=0}^{\nu-1} N^i (Bu)^{(i)}.$$

1.2. Terminology and background. The system of algebraic equations, $Ax = b$, written as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{13}$$

is 1-*full* with respect to x_1 if (13) uniquely determines x_1 for any consistent b. From basic linear algebra we have Lemma 1.

LEMMA 1. *The following are equivalent for the system of algebraic equations* (13):

(1) *The system* (13) *is* 1-*full with respect to* x_1.

(2) *The submatrices*

$$A_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \qquad A_2 = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix}$$

have disjoint ranges and A_1 *has full column rank.*

(3) *The row echelon form of* A *is*

$$\begin{bmatrix} I_{n\times n} & 0 \\ 0 & * \end{bmatrix} \tag{14}$$

where $*$ *is a possibly nonzero entry.*

(4) *The* x_1 *entry of any vector in the nullspace of* A *is zero.*

To obtain smooth observability we will need the next lemma.

LEMMA 2. *Suppose* A *in* (13) *is a smooth function of* t *defined on an interval* $\mathcal{I}$:

(1) *If* A *is* 1-*full with respect to* x_1 *for each* $t \in \mathcal{I}$ *and* A *has constant rank, then*

there is a smooth $\Theta(t)$ such that ΘA has the form (14) *and A is said to be smoothly* 1-*full* [7].

(2) *If A has constant rank on $\mathcal{I}$ and A is* 1-*full on a dense subset of $\mathcal{I}$, then A is* 1-*full for every $t \in \mathcal{I}$.*

Proof. The first statement is Lemma 3.1 of [7]. To prove the second statement, suppose that A has constant rank. Then the nullspace of A has a smooth basis

$$\left\{\begin{bmatrix} x_{1i}(t) \\ x_{2i}(t) \end{bmatrix} \text{for } i=0, \cdots, r\right\}.$$

By assumption, $x_{1i} = 0$ on a dense subset of $\mathcal{I}$. Hence $x_{1i} \equiv 0$ by continuity and A is 1-full for all $t \in \mathcal{I}$. $\square$

We need to restrict the class of descriptor systems that we consider.

DEFINITION 3. The descriptor system $E(t)x' + F(t)x = f(t)$ is *solvable* on the interval $\mathcal{I}$ if

(1) For every sufficiently smooth f on $\mathcal{I}$, there is a solution to the descriptor system.

(2) Solutions are defined on all of $\mathcal{I}$.

(3) Solutions are uniquely determined by their values at any t_0 in $\mathcal{I}$.

This definition of solvability does not require E to have constant rank nor for it to be possible to carry out the usual inversion algorithms involving coordinate changes and differentiations [7], [24]. For a given f, the initial values at time t_0 form a proper submanifold if E is singular. More detailed exposition on the basic properties of DAEs may be found in [3], [6], [9], and [17].

2. Observability characterization. In this section we will develop our characterization of observability for (2). For simplicity, let $b(t) = B(t)u(t)$ and assume that the descriptor system (2a) is solvable. Differentiating the equation (2a) j times and the equation (2b) k times gives the system of equations

$$[\mathcal{F}_j \quad \mathcal{E}_j]\begin{bmatrix} x \\ \mathbf{x}_j \end{bmatrix} = \mathbf{b}_j, \tag{15a}$$

$$\mathcal{C}_k\begin{bmatrix} x \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathbf{y}_k \tag{15b}$$

where

$$\mathcal{F}_j = \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(j)} \end{bmatrix}, \quad \mathbf{y}_k = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(k)} \end{bmatrix}, \quad \mathbf{b}_j = \begin{bmatrix} b \\ b' \\ \vdots \\ b^{(j)} \end{bmatrix}, \quad \mathbf{x}_j = \begin{bmatrix} x' \\ \vdots \\ x^{(j+1)} \end{bmatrix},$$

$$\mathcal{E}_j = \begin{bmatrix} E & 0 & \cdot & \cdot & 0 \\ E'+F & E & 0 & \cdot & \cdot \\ E''+2F' & 2E'+F & E & \ddots & \cdot \\ * & * & * & \ddots & 0 \\ E^{(j)}+jF^{(j-1)} & * & * & * & E \end{bmatrix},$$

$$\mathcal{C}_k = \left[\begin{array}{c|cccc} C & 0 & \cdot & \cdot & 0 \\ C' & C & 0 & \cdot & \cdot \\ C'' & 2C' & C & \ddots & \cdot \\ * & * & * & \ddots & 0 \\ C^{(k)} & * & * & * & C \end{array}\right] = [\tilde{\mathcal{C}}_k \mid \hat{\mathcal{C}}_k].$$

These equations suggest the following result.

PROPOSITION 1. *The descriptor system* (2) *is observable on the interval* $\mathcal{I}$ *if and only if there are* j, k, *with* $k \leqq j+1$ *such that the matrix*

$$\mathcal{O}_{j,k} = \left[\begin{array}{c|c} \mathcal{F}_j & \mathcal{E}_j \\ \tilde{\mathcal{C}}_k & [\,\hat{\mathcal{C}}_k \; 0_{(k+1)m \times (j+1-k)n}\,] \end{array}\right] \tag{16}$$

is 1-*full with respect to* x *on a dense subset of* $\mathcal{I}$.

Proof. It is clear that the 1-fullness of $\mathcal{O}_{j,k}$ on a dense subset of $\mathcal{I}$ implies observability. The necessity of $\mathcal{O}_{j,k}$ being 1-full will follow from the proof of Proposition 3 where bounds on j, k are derived. □

PROPOSITION 2. *If* $\mathcal{O}_{j,k}$ *is* 1-*full on a dense subset of* $\mathcal{I}$ *and has constant rank, then* (2) *is smoothly observable of order* (k, j).

Proof. Under the given assumption, using Lemma 2, there is a smooth Θ such that

$$\Theta\mathcal{O}_{j,k} = \begin{bmatrix} I_{n\times n} & 0 \\ 0 & * \end{bmatrix}.$$

Then (4) holds where $[L_0, \cdots, L_j, K_0, \cdots, K_k]$ are the first n rows of Θ. □

Propositions 1 and 2 are the types of results we are seeking in that the only symbolic operations that need to be performed are the differentiation of the given coefficients. However, since E can have variable rank, calculations need to be carried out in a certain order to avoid incorrect rank determinations in verifying the observability condition. Also, we need more precise information on the needed values of j, k. Finally, ways to reduce the amount of computation have to be considered.

All of these concerns can be carried out simultaneously as we investigate the structure of $\mathcal{O}_{j,k}$. The key turns out to be the following fundamental result from [7].

THEOREM 1. *Suppose that* (2a) *is solvable on the interval* $\mathcal{I}$ *and that* E, F *are* $2n$-*times differentiable. Then*

$$\mathcal{E}_i \text{ has constant rank on } \mathcal{I} \text{ for } i=n+1, \tag{17}$$

$$\mathcal{E}_i \text{ is 1-full with respect to } x' \text{ for } i=n+1, \tag{18}$$

$$[\mathcal{F}_i \; \mathcal{E}_i] \text{ has full row rank for } 1 \leqq i \leqq n+1. \tag{19}$$

If the coefficients E, F are infinitely differentiable, then Theorem 1 provides sufficient as well as necessary conditions for solvability. If (19) holds, then the smallest value of i that satisfies the conditions (17), (18) of Theorem 1 is called the *index* ν of the descriptor system (2a). For time-invariant descriptor systems, the index is the same as the index of the pencil $\lambda E + F$. However, for time-varying solvable descriptor systems, the pencil $\lambda E + F$ need not be regular, and if the pencil is regular, its index need not be that of the descriptor system.

Theorem 1 is important since it assures us that if the descriptor system (2a) is solvable, then $\mathcal{E}_j$ will have constant rank even if E does not. Thus a computation concerning $\mathcal{E}_j$ can be well conditioned.

We now need the following technical lemma.

LEMMA 3. *Suppose that* (2a) *is a solvable index* ν *descriptor system. Then for any* $\ell \geqq 0$, *the row echelon form of* $[\,\mathcal{E}_{\nu+\ell} \,|\, \mathcal{F}_{\nu+\ell} \,|\, \mathbf{b}_{\nu+\ell}]$ *is*

$$\left[\begin{array}{cc|c|c} I_{n(\ell+1)\times n(\ell+1)} & 0 & \tilde{Q}_1 & \tilde{b}_1 \\ 0 & R & \tilde{Q}_2 & \tilde{b}_2 \\ 0_{\rho\times n(\ell+1)} & 0 & \mathcal{M} & \tilde{b}_3 \end{array}\right], \tag{20}$$

where R and $\mathcal{M}$ have full row rank. Furthermore, the solutions of $\mathcal{M}x = \tilde{b}_3$ are independent of ℓ.

Proof. The independence of the solutions of $\mathcal{M}x = \tilde{b}_3$ follows from [7]. Index the $n \times n$ block entries of $\mathscr{E}_j$ by $0 \leqq r \leqq j$, $0 \leqq s \leqq j$. Thus, for example, $\mathscr{E}_{0,0} = E$. Let $r \geqq 1$, $s \geqq 1$. Then for any $j \geqq s$, the (r, s) $n \times n$ block entry of $\mathscr{E}_j$ is generated by the recursion

$$(\mathscr{E}_j)_{r,s} = (\mathscr{E}_j)_{r-1,s-1} + \frac{d}{dt}(\mathscr{E}_j)_{r-1,s}. \tag{21}$$

To see (21), note that if the $r-1$ block row of $\mathscr{E}_j$ is

$$+ \cdots + \mathscr{A}x^{(s-1)} + \mathscr{B}x^{(s)} + \cdots,$$

then, upon differentiation, we get that the coefficient of $x^{(s)}$ in the rth row of $\mathscr{E}_j$ is $\mathscr{A} + \mathscr{B}'$, which is (21). For $j \geqq 0$ partition $\mathscr{E}_{j+1}$ as

$$\mathscr{E}_{j+1} = \begin{bmatrix} E & 0_{n \times n(j+1)} \\ * & \hat{\mathscr{E}}_j \end{bmatrix}. \tag{22}$$

Suppose that ν is such that $\mathscr{E}_\nu$ is 1-full with respect to x' and of constant rank. We first show that $\mathscr{E}_{\nu+1}$ is 1-full with respect to x', x''. It will be of constant rank from [7]. The nullspace of $\mathscr{E}_{\nu+1}$ consists of the solutions of

$$\mathscr{E}_{\nu+1} \begin{bmatrix} z_0 \\ \vdots \\ z_{\nu+1} \end{bmatrix} = 0, \tag{23}$$

which implies that

$$\mathscr{E}_\nu \begin{bmatrix} z_0 \\ \vdots \\ z_\nu \end{bmatrix} = 0. \tag{24}$$

By the 1-fullness of $\mathscr{E}_\nu$, we have $z_0 = 0$. Hence (23) implies that

$$\hat{\mathscr{E}}_\nu \begin{bmatrix} z_1 \\ \vdots \\ z_{\nu+1} \end{bmatrix} = 0. \tag{25}$$

But by (21),

$$\hat{\mathscr{E}}_\nu = \mathscr{E}_\nu + \left[\frac{d}{dt}(\mathscr{E}_{\nu *}) \,|\, 0_{(\nu+1)n \times n} \right] \tag{26}$$

where $\mathscr{E}_{\nu *}$ denotes the last ν block columns of $\mathscr{E}_\nu$. Note that (24) implies that

$$\mathscr{E}_{\nu *} \begin{bmatrix} z_1 \\ \vdots \\ z_\nu \end{bmatrix} = 0. \tag{27}$$

Equations (25) and (26) then imply that

$$\mathscr{E}_\nu \begin{bmatrix} z_1 \\ \vdots \\ z_{\nu+1} \end{bmatrix} + \left[\frac{d}{dt}(\mathscr{E}_{\nu *}) \,|\, 0 \right] \begin{bmatrix} z_1 \\ \vdots \\ z_{\nu+1} \end{bmatrix} = 0. \tag{28}$$

Since $\mathscr{E}_\nu$ is 1-full with constant rank, there exists smooth $[U_0, \cdots, U_\nu]$ with U_i, which are $n \times n$ such that

$$[U_0, \cdots, U_\nu]\mathscr{E}_\nu = [I\, 0 \cdots 0] \tag{29}$$

and hence

$$[U_0, \cdots, U_\nu]\mathscr{E}_{\nu*} = [0 \cdots 0]. \tag{30}$$

Differentiating (30) yields

$$[U_0', \cdots, U_\nu']\mathscr{E}_{\nu*} + [U_0, \cdots, U_\nu]\mathscr{E}_{\nu*}' = 0. \tag{31}$$

Multiplying (28) by $[U_0, \cdots, U_\nu]$ and using (31), we have

$$z_1 - [U_0', \cdots, U_\nu'][\mathscr{E}_{\nu*}|0]\begin{bmatrix} z_1 \\ \vdots \\ z_{\nu+1} \end{bmatrix} = 0$$

or $z_1 = 0$ by (27), and $\mathscr{E}_{\nu+1}$ is 1-full with respect to x' and x''. The $\mathscr{E}_{\nu+\ell}$ case now follows by a simple induction argument. □

Suppose then that (2a) is solvable and index ν. As noted in the proof of Lemma 3, $[\mathscr{E}_i\ \mathscr{F}_i\ 0|\mathbf{b}_i]$ is a leading submatrix of $[\mathscr{E}_j\ \mathscr{F}_j|\mathbf{b}_j]$ for every $i \leqq j$. In particular, $[\mathscr{E}_\nu\ \mathscr{F}_\nu]$ has constant rank and is 1-full with respect to x'. Performing a QR (or singular value decomposition) or using row operations on $[\mathscr{E}_\nu\ \mathscr{F}_\nu]$, as discussed in [4] and [5] gives

$$\left[\begin{array}{cc|c|c} I_{n\times n} & 0 & \bar{Q}_1 & \bar{b}_1 \\ 0 & H & \bar{Q}_2 & \bar{b}_2 \\ 0_{\rho\times n} & 0 & \mathscr{M} & \bar{b}_3 \end{array}\right]. \tag{32}$$

The equation

$$\mathscr{M}x = \bar{b}_3 \tag{33}$$

determines the solution manifold of (2a) at time t. Furthermore, (33) shows that there is a ρ-dimensional projection of x that is observable since it is given by b and its derivatives. Thus we have only to observe the solutions of $x' = -\bar{Q}_1 x + \bar{b}_1$ on an $(n-\rho)$-dimensional invariant submanifold. From the classical theory for observability, we then have that it suffices to take $k = n - \rho - 1$. We shall give a rigorous justification of this argument after summarizing it in the next proposition.

PROPOSITION 3. *Suppose that* (2a) *is solvable with index* ν. *Let* $\rho = n(\nu+1) - \operatorname{rank}(\mathscr{E}_\nu)$. *Note that* $0 \leqq \rho \leqq n$ *with* $\rho = 0$ *if and only if* E *is nonsingular. Then* (2) *is observable if and only if* $\mathscr{O}_{j,k}$ *is* 1-*full with respect to* x *on a dense subset of* $\mathscr{I}$ *where* (j, k) *are any pair of nonnegative integers satisfying* $k \geqq (n-\rho-1)$, $j \geqq \nu + k - 1$. *In particular, since* $\nu \leqq n$, *we may take* $k = n-1$, $j = 2n-2$.

Proof. If we perform the time-varying coordinate changes, $x = S(t)\bar{x}$, and pre-multiplication by $T(t)$, the new derivative arrays are related to the old by

$$[\bar{\mathscr{F}}_j\ \bar{\mathscr{E}}_j] = \mathscr{T}_j[\mathscr{F}_j\ \mathscr{E}_j]\mathscr{S}_j, \tag{34}$$

$$[\bar{\mathscr{C}}_k\ 0] = [\mathscr{C}_k\mathscr{S}_k\ 0] = [\mathscr{C}_k\ 0]\mathscr{S}_j, \tag{35}$$

where

$$\mathscr{X}_i = \begin{bmatrix} X & 0 & \cdot & \cdot & 0 \\ X' & X & 0 & \cdot & \cdot \\ X'' & 2X' & X & \ddots & \cdot \\ * & * & * & \ddots & 0 \\ X^{(i)} & * & * & * & X \end{bmatrix} \quad \text{for } X = S, T.$$

Thus for a given k, j, the 1-fullness of $\mathcal{O}_{j,k}$ is unchanged by time-varying coordinate changes. Using the structure theorem for solvable linear DAEs developed in [7] we may assume that (2) has the form

$$x_1' + E_1(t)x_2' + G(t)x_1 = B_1(t)u, \tag{36a}$$

$$N(t)x_2' + x_2 = B_2(t)u, \tag{36b}$$

$$y = C_1x_1 + C_2x_2, \tag{36c}$$

where x_1 is n_1-dimensional and x_2 is n_2-dimensional. In general, N will have variable rank and nonsmooth nullspace and range [7]. However, the operator $N(d/dt) + I$ is an invertible operator of the space of infinitely differentiable functions onto itself. In particular, for each u there is only one solution of (36b). Let $\mathcal{M}_2$ be the $\mathcal{M}$ matrix for the derivative array $[\mathcal{E}_j \ \mathcal{F}_j]$ for (36b). But then rank $(\mathcal{M}_2) = n_2$ since $\mathcal{M}$ determines the solution manifold of a descriptor system and $\mathcal{M}_1 = 0$ for (36a). Thus $\rho = n_2$. Accordingly, we have that there exists smooth L_i such that

$$x_2 = \sum_{i=0}^{\rho-1} L_i(t)(B_2u)^{(i)} \tag{37}$$

and x_2 is already observable independent of C.

Thus observability of (36) reduces to considering (36a), (36c), which is a classical nonsingular observability problem in the form

$$x_1' = -Gx_1 + p_1, \tag{38a}$$

$$y = C_1x_1 + p_2. \tag{38b}$$

We know that the observability of (38) can be determined from the derivative array by no more than $n_1 - 1$ differentiations of (38b), which requires $n_1 - 2$ differentiations of G, p_1 in (38a) [23]. However, p_1 requires a differentiation of x_2 so that $n_1 - 1 = n - \rho - 1$ differentiations are needed. □

COROLLARY 1. *If the descriptor system satisfies the assumptions of Proposition* 3 *and* $\mathcal{O}_{j,k}$ *has constant rank, then* (2) *is smoothly observable.*

2.1. Nonsingular systems. Before continuing, we will briefly discuss what happens when E is nonsingular. In this case, we have that $\rho = 0$ and $\nu = 0$ so that

$$\mathcal{O}_{n-1,n-1} = \begin{bmatrix} [\mathcal{F}_{n-1} \quad \mathcal{E}_{n-1}] \\ [\mathcal{C}_{n-1}] \end{bmatrix}. \tag{39}$$

Since E is now assumed nonsingular, so is $\mathcal{E}_j$ for any j. Then $\mathcal{O}_{n-1,n-1}$ is 1-full with respect to x if and only if

$$\mathcal{W}_{n-1} = \tilde{C}_{n-1} - \hat{C}_{n-1}\mathcal{E}_{n-1}^{-1}\mathcal{F}_{n-1} \tag{40}$$

has full column rank. If we let $A = -E^{-1}F$ so that the differential equation is $x' = Ax + E^{-1}Bu$, then $\mathcal{W}_{n-1}$ is precisely the usual observability matrix [2]

$$\mathcal{W}_{n-1} = \begin{bmatrix} C \\ C' + CA \\ \vdots \\ C_{n-1} + C_{n-2}A \end{bmatrix} \tag{41}$$

where $C_i = C_{i-1}A + C_{i-1}'$ and $C_0 = C$. If $j > k - 1$, define

$$\mathcal{W}_k = \tilde{\mathcal{C}}_k - [\hat{\mathcal{C}}_k \ 0]\mathcal{E}_j^{-1}\mathcal{F}_j.$$

Uniform observability is defined in [23] to be that $\mathcal{W}_{n-1}$ has full rank for all $t \in \mathcal{I}$. Smooth observability will follow if $\mathcal{W}_j$ has full rank for all $t \in \mathcal{I}$ for some j that does not depend on t. Example 2 gives an example where $\mathcal{W}_j$ has full rank for all t but only for a $j > n$.

From a theoretical point of view, our approach is not saying anything new about the nonsingular case. However, there are other considerations that may make it preferable to utilize (15a), (15b) rather than (41). One such situation is when it is desirable to avoid the inversion of E. If E has some simple structure such as sparsity or bandedness, this structure can be lost in the inversion. Also, if E is time varying, then the inversion has to be done symbolically, and the resulting expressions differentiated repeatedly and multiplied symbolically. For even moderate-sized problems this can lead to a major expansion in the complexity of the expressions involved. In these types of problems it is much quicker to proceed numerically from the array $\mathcal{O}_{j,k}$, which has been computed with the minimal amount of expression expansion possible.

Many control problems have a structure that can be exploited in working with (15) [8].

Another advantage of working with $\mathcal{O}_{j,k}$ directly arises when we are dealing with problems where ranks, or perhaps even the index, change with parameter values. Many symbolic packages produce what are sometimes referred to as generic solutions. That is, given the equation $kx = 0$, the solution is given as $x = 0$ if k is a parameter. However, when dealing with descriptor systems, the case where $k = 0$ may be important. In the solution of the complicated nongeneric linear algebra problems that can occur in solving descriptor systems, this sort of behavior may be much more subtle.

2.2. Analytic systems. If the coefficients are real analytic, then we can make a stronger statement. The key is the following lemma.

LEMMA 4. *Suppose that $H(t)$ is an $m \times n$ real analytic matrix function defined on an open interval containing the closed bounded interval $\mathcal{I}$. Let*

$$H[j] = \begin{bmatrix} H \\ H' \\ \vdots \\ H^{(j)} \end{bmatrix}.$$

Suppose that $H[k]$ has full column rank at some $t_0 \in \mathcal{I}$. Then there is a j such that $H[j]$ has full column rank for all $t \in \mathcal{I}$.

Proof. Suppose that there is a k and a $t_0 \in \mathcal{I}$ such that $H[k]$ has full column rank at t_0. Then the real analyticity of H on an open set containing the closure of the interval $\mathcal{I}$ implies that $H[k]$ has full column rank at all but a finite number of points $t_1, \cdots, t_r$ in $\mathcal{I}$. Let $\mathcal{N}_{j,p}$ be the nullspace of $H[j]$ at time t_p. Clearly, $\mathcal{N}_{i,p} \subset \mathcal{N}_{j,p}$ if $i \geqq j$. Let $\mathcal{N}_p = \bigcap_{i \geqq 0} \mathcal{N}_{i,p}$. If $\mathcal{N}_p \neq 0$, let v_p be a nonzero vector in $\mathcal{N}_p$. Then $\psi(t) = H(t)v_p$ is a real analytic function, all of whose derivatives vanish at t_p. Thus $\psi(t) \equiv 0$. But this implies that $H[j]v_p = 0$ for all t, j, which is a contradiction. Suppose then that $\mathcal{N}_p = 0$. But then $\mathcal{N}_{\mu_p,p} = 0$ for some μ_p, since $\mathcal{N}_p$ is the intersection of a nonincreasing chain of subspaces of a finite-dimensional vector space. Thus $H[\mu_p]$ will have full column rank. Let $\mu = \max\{k, \mu_1, \cdots, \mu_r\}$. Then $H[\mu]$ will have full column rank for all $t \in \mathcal{I}$. $\square$

PROPOSITION 4. *The solvable system* (2) *with E, F, B, C real analytic is observable if and only if it is smoothly observable. Furthermore, it is smoothly observable if and only if $\mathcal{O}_{j,k}$ is* 1*-full with constant rank for some* (j, k).

Proof. Assume that (2) is solvable and that E, F, B, C are real analytic. It suffices to show that observable implies smoothly observable. The real analyticity implies [11] that we may take $E_1 = 0$ in (36) and G, B_i, N, C_i are real analytic, as are the L_i in (37). Another analytic coordinate change gives $G = 0$ in (36a). Thus we can consider the nonsingular case. The proposition now follows by applying Lemma 4 to (41) with $A = 0$ and $C_i = C^{(i)}$. □

If we modify Example 1 by letting $y = t^s x$ where s is a nonnegative integer, and assume $0 \in \mathcal{I}$, we see that it is impossible to get a general upper bound for the amount of differentiation needed for smooth observability. However, having to perform a large number of extra differentiations in order to get smooth observability seems unlikely.

3. External behavior. In the systems theory literature the problem of representing the external behavior of a system, and of determining the external behavior given a representation, is frequently discussed ([22], [25], [28]). In this section we shall show how our characterizations of observability can be used to derive characterizations of the external behavior for (2). Our results do not follow from those of [22] since that paper makes a constant rank assumption at intermediate steps of the derivation and we allow these submatrices to have variable rank.

DEFINITION 4. The *external behavior* of (2) is the set $\Sigma_e = \{(y, u) \mid y, u$ are functions satisfying (2) for some state function $x(t)\}$.

An external description is sometimes called an input–output representation ([25], [28]).

DEFINITION 5. An *external description* of the system (2) is a set of equations

$$R(t, y, y', \cdots, y^{(\ell)}, u, u', \cdots, u^{(r)}) = 0 \tag{42}$$

with R continuous, such that the external behavior Σ_e of (2) is precisely the set of (y, u) satisfying (42).

Grimm [18] defines external behavior in terms of the Laplace transforms of y, u. Alternatively, using the notation of (3), the external behavior can be defined as the set of (u, y) such that $\left[\begin{smallmatrix} Bu \\ y \end{smallmatrix}\right]$ is in the range of $\tilde{E}(d/dt) + \tilde{F}$.

Suppose that $\mathcal{O}_{j,k}$ for (2) is 1-full with respect to the x variable and constant rank. Then

$$\begin{aligned} x &= \Psi(t, y, \cdots, y^{(k)}, u, \cdots, u^{(j)}) \\ &= \pi_1 \mathcal{O}_{j,k}^{\dagger} \begin{bmatrix} \mathbf{b}_j \\ \mathbf{y}_k \end{bmatrix}, \end{aligned} \tag{43}$$

where π_1 is the projection onto the first n coordinates. Define the functions R_1, R_2 by

$$R_1(t, y, \cdots, y^{(k)}, u, \cdots, u^{(j)}) = E(t)\frac{d}{dt}(\Psi) - F(t)\Psi - B(t)u, \tag{44}$$

$$R_2(t, y, \cdots, y^{(k)}, u, \cdots, u^{(j)}) = [I - \mathcal{O}_{j,k}\mathcal{O}_{j,k}^{\dagger}] \begin{bmatrix} \mathbf{b}_j \\ \mathbf{y}_k \end{bmatrix}. \tag{45}$$

PROPOSITION 5. *Suppose that the system* (2) *has* $\mathcal{O}_{j,k}$ *1-full with respect to x and constant rank so that* (2) *is smoothly observable of order* (k, j). *Then the external behavior is characterized by*

$$R_1 = 0, \qquad R_2 = 0,$$

where R_1, R_2 *are given by* (44), (45).

Proof. That $(y, u) \in \Sigma_e$ implies (y, u) satisfies R_1 and R_2 is clear. Suppose then that (y, u) satisfies $R_1 = 0$, $R_2 = 0$. Since $R_2 = 0$, the equations (15) are algebraically consistent.

Since $\mathcal{O}_{j,k}$ is 1-full with respect to x and constant rank, there is a unique smooth $\bar{x}$ given by

$$\bar{x} = \pi_1 \mathcal{O}_{j,k}^{\dagger} \begin{bmatrix} \mathbf{b}_j \\ \mathbf{y}_k \end{bmatrix},$$

which satisfies $R_1 = 0$. There remains only to show that $y = C\bar{x}$. However, $R_2 = 0$ implies that (15a), (15b) are consistent and the first block equation in (15b) is $y = C\bar{x}$. □

3.1. Nonsingular case. To illustrate the previous result, consider (2) with E nonsingular and $A = -E^{-1}F$. From (39), (41) and (15a), (15b), we have

$$\mathbf{y}_k = \mathcal{W}_k x + \mathbf{b}_j. \tag{46}$$

By the observability assumption, $\mathcal{W}_k$ has full column rank for large enough k. The functions R_1, R_2 are defined by

(47a)

$$R_1(t, y, \cdots, y^{(k+1)}, u, \cdots, u^{(k)}) = \frac{d}{dt}[\mathcal{W}_k^{\dagger}(\mathbf{y}_k - \mathbf{b}_k)] - F(t)[\mathcal{W}_k^{\dagger}(\mathbf{y}_k - \mathbf{b}_k)] - B(t)u,$$

$$R_2(t, y, \cdots, y^{(k+1)}, u, \cdots, u^{(k)}) = [I - \mathcal{W}_k \mathcal{W}_k^{\dagger}](\mathbf{y}_k - \mathbf{b}_k). \tag{47b}$$

PROPOSITION 6. *If* (2) *with E nonsingular is smoothly observable on the interval* $\mathcal{I}$ *so that* $\mathcal{W}_k$ *has full column rank on* $\mathcal{I}$, *then the external behavior of* (2) *is characterized by the external description* $R_1 = 0$, $R_2 = 0$ *where the* R_i *are given by* (47).

Example 5. For the simple system

$$x' = bu, \qquad y = tx$$

we have

$$\mathcal{W}_k = \begin{bmatrix} t \\ 1 \end{bmatrix}, \qquad \mathcal{W}_k^{\dagger} = (1 + t^2)^{-1}[t \quad 1].$$

Then

$$R_1 = \frac{d}{dt}[(1 + t^2)^{-1}(ty + y' - tbu)] - bu$$

and

$$R_2 = \begin{bmatrix} 1 - (1 + t^2)^{-1}t^2 & -t(1 + t^2)^{-1} \\ -t(1 + t^2)^{-1} & 1 - (1 + t^2)^{-1} \end{bmatrix} \begin{bmatrix} y \\ y' - tbu \end{bmatrix}.$$

The equations R_1, R_2 simplify to

$$(t^2 + 1)y'' + (t^3 - t)y' + (1 - t^2)y - [(t^4 + t^2 + 2)b + (t^3 + t)b']u - (t^3 + t)bu' = 0$$

and

$$y - ty' + t^2 bu = 0,$$

respectively.

Other 1-inverses [10] can be used besides $\mathcal{W}_k^{\dagger}$. For example, (8) used

$$\begin{bmatrix} t^2 \\ 2t + t^2 \\ 2 + 4t + t^2 \end{bmatrix}^{-} = (2t^2 + 2t + 2)^{-1}[2 \quad -1 \quad 1].$$

4. Conclusion. This paper has examined the observability of time-varying descriptor systems. Characterizations of different types of observability have been given in terms of rank conditions on arrays made up of derivatives only of the original coefficients. All algebra can be carried out numerically.

The concept of smooth observability, which is weaker than uniform observability, has been introduced. It is shown that every real-analytic system that is (totally) observable is smoothly observable, even if it is not uniformly observable. This is a new result even in the nonsingular case when E is invertible. These ideas have been used to develop a characterization of the external behavior of smoothly observable descriptor systems.

Several problems remain. One is a discussion of how to actually carry out these procedures in an efficient manner. In the characterization of the external behavior, an alternative characterization that does not require differentiating the computed Ψ would be desirable. Finally, it would be interesting to establish what the natural dual concept is to smooth observability.

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