Received: 1 June 2016
DOI: 10.1002/mma.4507

RESEARCH ARTICLE

WILEY

Pseudo–almost-periodic solutions for delayed differential equations with integrable dichotomies and bi–almost-periodic Green functions

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Communicated by: C. Cuevas

Funding information
Fondecyt, Grant/Award Number: 1120709, 1170466

MOS Classification: 34G10; 47D06

Using the existence of integrable bi–almost-periodic Green functions of linear homogeneous differential equations and the contraction fixed point, we are able to prove the existence of almost and pseudo–almost-periodic mild solutions under quite general hypotheses for the differential equation with constant delay

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), x(t-\tau)), \quad t \in \mathbb{R},$$

in a Banach space X, where $\tau > 0$ is a fixed constant. The results extend the corresponding ones in the case of exponential dichotomy. Some examples illustrate the importance of the concepts.

KEYWORDS

almost-periodic solutions, functional differential equations, integrable dichotomy, pseudo–almost-periodic solutions, some examples illustrate the importance of the concepts

1 | INTRODUCTION

It is well known that periodicity and almost periodicity are natural and important phenomena in the real world. In fact, the existence of almost-periodic and pseudo–almost-periodic solutions is among the most attractive topics in qualitative theory of differential equations due to their applications, especially in biology, economics, and physics (see, for example, the literature[1-5]). The concept of almost periodicity, which is a generalization of periodicity, was first studied by Bohr[6] and later generalized by Stepanov,[7] Weyl,[8] and Besicovitch,[9], among others, while the concept of pseudo–almost periodicity was introduced by Zhang[5,10,11] in the early 1990s. Since then, such a notion became of great interest to the classical concept of almost periodicity in the sense of Bohr[6] and Bochner.[12] For more on the concepts of almost periodicity and pseudo–almost periodicity and related issues, we refer the reader to some studies.[1-3,5,10,11,13,14]

We consider the homogeneous linear system

$$\dot{x}(t) = A(t)x(t), \quad t \in \mathbb{R} \tag{1}$$

and the perturbed delayed system

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), x(t-\tau)), \quad t \in \mathbb{R} \tag{2}$$

in a Banach space X, where $\tau > 0$ is a fixed constant.

In this work, we investigate the existence of almost and pseudo–almost-periodic mild solutions of the nonlinear case (2) under the existence of an integrable dichotomy of linear system (1) (see Definition 4) and additional hypotheses; these results are summarized in Theorems 9 and 15, respectively. Consequences of these results are also analyzed.

First, we begin by studying the inhomogeneous Cauchy problem

$$\dot{x}(t) = A(t)x(t) + f(t), \quad t \in \mathbb{R} \tag{3}$$

in the Banach space X, which will be used to get our goal. We prove that if f is almost periodic (resp. pseudo–almost periodic), then under the existence of a bi–almost-periodic integrable dichotomy and appropriate assumptions, every bounded solution of (3) is also almost periodic (resp. pseudo–almost periodic). To get our results, first, we characterize the mild solutions in each case under the use of the Green function; second, we introduce an appropriate functional on an adequate subspace of X, and we prove that fixed points of this functional give us the solutions that we are looking for. During the proof, we observe that for (2), composition results are very important.

The concept of integrable dichotomy was introduced by Pinto in[15] for periodic integrodifferential equations, and here we adapt it to our linear system in order to prove the existence of a unique almost-periodic solution for the nonhomogeneous linear system. In Pinto,[15] the biperiodicity of the Green function is deduced; in our almost-periodic situation, the bi–almost periodicity of the Green function is again fundamental (see Definition 3).

Recently, we found several works in the literature where it is assumed the existence of some kind of dichotomy in order to prove the existence of some special or particular solutions of systems as in (2). For example, Lupa and Megan[16] consider 2 trichotomy concepts in the context of abstract evolution operators. The first one extends the notion of exponential trichotomy, and it is a natural extension of the generalized exponential dichotomy considered in the paper by Jiang.[17] The second concept is dual in a certain sense to the first one. They proved that these types of exponential trichotomy imply the existence of generalized exponential dichotomy on both half-lines. In Agarwal et al,[18] the existence and uniqueness of weighted pseudo–almost-periodic solutions to some partial neutral functional differential equations is obtained. Lupa and Megan[19] consider 3 dichotomy concepts: exponential dichotomy, uniform exponential dichotomy, and strong dichotomy. Using the notion of Green function, they gave necessary and sufficient conditions for strong exponential dichotomy. Xia et al[20] prove that if the nonlinear term is bounded, then the perturbed nonautonomous system with nonuniform exponential dichotomy has a unique solution. Xia[21] made use of the so-called Acquistapace-Terreni conditions associated with exponential dichotomy and derived some sufficient conditions to the existence and uniqueness of μ (or ergodic)-pseudo–antiperiodic (μ-pseudo-periodic, μ-pseudo–almost periodic, and μ-pseudo–almost automorphic) mild solution to nonautonomous integrodifferential equations with nondense domain. Here μ denotes a positive measure. Barreira et al[22] under the assumption of the existence of a nonuniform exponential dichotomy with a small nonuniformity, ie, a small deviation from the classical notion of (uniform) exponential dichotomy, showed that essentially any linear equation $\dot{v} = A(t)v$ admits a nonuniform exponential dichotomy, and thus, the above assumption only concerns the smallness of the nonuniformity of the dichotomy. The main result is that for any sufficiently small perturbation f, there exists a stable invariant manifold for the perturbed equation $\dot{v} = A(t)v + f(t, v)$, which corresponds to the set of negative Lyapunov exponents of the original linear equation. Barreira and Valls[23] give a unified presentation of a substantial body of work, which they have performed and which revolves around the concept of nonuniform exponential dichotomy.

The paper is organized as follows. In Section 2, we introduce the concept of integrable dichotomy with data (λ, P) for system (1) and the concept of λ-almost periodic (the "biperiodicity" of the Green function) of a function F on the set $BC_\lambda(\mathbb{R}^2, X)$. In Section 3, we prove the existence of a unique almost-periodic and pseudo–almost-periodic mild solution for the nonhomogeneous linear system in each situation under appropriate hypotheses. Also examples are presented in order to illustrate the existence of integrable dichotomy, which is not exponential dichotomy and the importance of the "bi–almost periodicity" of the Green function. Without this property, the almost-periodic solution cannot exist. In Section 4, we first establish a theorem that guarantees that nonlinear system (32) possesses a unique bounded solution, and then we give an existence theorem for almost-periodic and for pseudo–almost-periodic mild solution to the Equation 2.

The almost periodicity of inhomogeneous systems has been studied by several authors; here we call the attention to the paper of Maniar and Schnaubelt[24] (see references therein for this case). The authors investigate the almost periodicity of the solutions to the parabolic inhomogeneous evolution equations

$$\dot{x}(t) = A(t)u(t) + f(t)$$

in the situations $t \in \mathbb{R}$ and $t > 0$ in the Banach space X. They assumed that the linear operators $A(t)$ satisfy the Acquistapace-Terreni conditions, that is, the evolution family $U(t, s)$ solving the homogeneous problem has an exponential dichotomy, and the function $t \to R(\omega, A(t))$, for an $\omega \geq 0$, and $f(t)$ is almost periodic. These conditions are more general, but strong, one insuring, for the exponential dichotomy, the "bi–almost periodicity" of the Green function.

The existence of almost-periodic solutions has been studied in various works.[4,25-35] By using semigroup theory and the contraction mapping principle, Zaidman studied[25] the existence of almost-periodic solutions for the integral equation associated with the abstract partial differential equations

$$\dot{x}(t) = Ax(t) + f(t, x(t)),$$

where A is the infinitesimal generator of a C_0-semigroup of bounded linear operators on a Banach space. The case of pseudo–almost-periodic solutions has been considered in several works[10,11,36-42] under different points of views, such as ordinary differential equations, nonautonomous evolution equations with delay, nonautonomous neutral functional differential equations with unbounded delay, and neutral integral differential equations.

2 | PRELIMINARIES

Initially we will consider the homogeneous linear case given by the Equation 1. Throughout this paper, X stands for a Banach space with norm $\|\|$.

We will assume that $U(t,s)$ is an evolution family of system (1) and that there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t, such that

a. $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$.
b. The restriction $U_Q(t,s) : Q(s)X \to Q(t)X$ is invertible for all $t \geq s$ (and we set $U_Q(s,t) = U_Q(t,s)^{-1}$).

Here and below $Q(t) := I - P(t)$. By the work of Pazy,[43] we observe that

$$\frac{\partial U}{\partial t}(t,s) = A(t)U(t,s), \quad \frac{\partial U}{\partial s}(t,s) = -U(t,s)A(s), \ s \leq t.$$

Throughout this paper, we will assume that the initial value problem (1) with $x(t_0) = x_0$ has an associated evolution family $U(t,t_0)$. We also say that the evolution family $U(t,t_0)$ is generated by $A(t)$.

Let

$$G(t,s) = \begin{cases} U(t,s)P(s), & \text{if} \quad t \geq s, \\ -U_Q(t,s)Q(s), & \text{if} \quad t < s, \end{cases} \tag{4}$$

be the Green function corresponding to U and P of system (1).

We remember the concept of almost-periodic function (see details in the work of Bochner[12]).

Definition 1. A continuous function $f : \mathbb{R} \to X$ is called (Bochner) almost periodic if for every sequence of real numbers $\{\tau_n'\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}$ such that $\lim_{n\to\infty} f(t+\tau_n)$ converges uniformly on $\mathbb{R}$, ie, $\tilde{f}(t) = \lim_{n\to\infty} f(t+\tau_n)$ is well defined in $t \in \mathbb{R}$. We denote by $AP(\mathbb{R}, X)$ the set of all such functions.

Definition 2. A continuous function $f : \mathbb{R} \to X$ is called pseudo–almost periodic if it can be decomposed as

$$f(t) = g(t) + h(t), \ t \in \mathbb{R},$$

where $g \in AP(\mathbb{R}, X)$ and h is a bounded continuous function with vanishing mean value

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|h(s)\| ds = 0.$$

We denote by $PAP(\mathbb{R}, X)$ the set of all such functions. Also we denote

$$PAP_0(\mathbb{R}, X) = \left\{ h \in C(\mathbb{R}, X) \quad / \quad h \text{ is bounded and } \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|h(s)\| ds = 0 \right\}.$$

Let $\mathcal{F}(X,X)$ be the space of the functions of the Banach space X in X. We will denote $\mathcal{F}(\mathbb{R}^2, X, X) = \{F(t,s) \in \mathcal{F}(X,X) \quad / \quad (t,s) \in \mathbb{R}^2\}$.

Let $\lambda : \mathbb{R}^2 \to (0,\infty)$ be a function. $BC_\lambda(\mathbb{R}^2, X, X) \subset \mathcal{F}(\mathbb{R}^2, X, X)$ will denote the Banach space of bounded continuous function of $\mathbb{R}^2$ in X such that if $F : \mathbb{R}^2 \to X$, then $\sup_{(t,s,x)\in\mathbb{R}\times\mathbb{R}\times X} \frac{\|F(t,s)x\|}{\lambda(t,s)} < \infty$.

Definition 3. (Pinto[42])
A function $F \in BC_\lambda(\mathbb{R}^2, X, X)$ will be called λ-almost periodic ("bi–almost periodic with respect to λ") in $t, s \in \mathbb{R}$ uniformly in any bounded subset $K \subset X$ if for each $\{\tau_n'\}$ sequence of real numbers, we can extract a subsequence $\{\tau_n\}$ such that there is function $F_1 \in \mathcal{F}(\mathbb{R}^2, X, X))$ such that

$$\lim_{n\to\infty} \frac{F(t+\tau_n, s+\tau_n)x - F_1(t,s)x}{\lambda(t,s)} = 0, \text{ uniformly for each } \quad t, s \in \mathbb{R},\ x \in K. \tag{5}$$

The class of these functions F will be denoted by $AP_\lambda(\mathbb{R}^2, X, X)$.

Definition 4. We say that system (1) has an integrable dichotomy with data (λ, P) if there are projections $P(t)$, $t \in \mathbb{R}$, uniformly bounded and strongly continuous in t satisfying (a), (b) with $Q(t) = I - P(t)$ and there exists a function $\lambda : \mathbb{R}^2 \to (0,\infty)$ such that

$$\|G(t+\tau, s+\tau)\| \le \lambda(t,s), \quad \text{for each} \quad \tau \in \mathbb{R}, \tag{6}$$

and

$$L := \sup_{t\in\mathbb{R}} \int_{-\infty}^{\infty} \lambda(t,s)ds < \infty. \tag{7}$$

In particular if there is an integrable dichotomy, we have that

$$\mu := \sup_{\tau,t\in\mathbb{R}} \int_{-\infty}^{\infty} \|G(t+\tau, s+\tau)\|ds < \infty. \tag{8}$$

Remark 5. In the particular case $\|G(t,s)\| \le \lambda_0(t-s) := \lambda(t,s)$, where $\lambda_0 \in L^1(\mathbb{R})$, then the previous conditions (7) and (8) are reduced to $L = \mu = \int_{\mathbb{R}} \lambda_0$. Now, if we assume that $\|G(t,s)\| \le \lambda(t,s) = \lambda_0(t-s) + \lambda_1(s)$, and $\lambda_0, \lambda_1 \in L^1(\mathbb{R})$, conditions (7) and (8) are satisfied.

Remark 6. If (1) has an exponential dichotomy, ie, there are positive constant $M > 0$ and $\delta > 0$ such that

$$\|G(t,s)\| \le \lambda(t,s) := \begin{cases} Me^{-\delta(t-s)}, & \text{if} \quad t \ge s, \\ Me^{-\delta(s-t)}, & \text{if} \quad t < s. \end{cases} \tag{9}$$

Then it is an integrable dichotomy with data (λ, P), and in this situation $L = \mu = 2\frac{M}{\delta}$.

In the pseudo–almost-periodic situation, we will make the following additional assumption for the function $\lambda(t,s)$.

(A) For some constants $\Gamma_i, \tilde{\Gamma}_i > 0$ $(i = 1, 2)$, the function $\lambda : \mathbb{R}^2 \to (0,\infty)$ as in Definition 4, $\lambda_1 : (-\infty, -T) \to (0,\infty)$ and $\lambda_2 : (T,\infty) \to (0,\infty)$ $(T > 0)$ defined as $\lambda_1(s) = \int_{-T}^{T} \lambda(t,s)dt$, $\lambda_2(s) = \int_{-T}^{T} \lambda(t,s)dt$ satisfy

$$\begin{gathered} \int_s^T \lambda(t,s)dt \le \Gamma_1, \qquad \text{and} \qquad \int_{-T}^{s} \lambda(t,s)dt \le \Gamma_2, \quad \text{for} \quad |s| \le T \\ \int_{-\infty}^{-T} \lambda_1(s)ds \le \tilde{\Gamma}_1 \quad \text{and} \quad \int_T^{\infty} \lambda_2(s)ds \le \tilde{\Gamma}_2. \end{gathered} \tag{10}$$

Remark 7. If the Green function $G(t,s)$ has an exponential dichotomy as in (9), then we must have

1. $\Gamma_1 = \Gamma_2 = \frac{M}{\delta}$.
2. $\lambda_1(s) = \frac{M}{\delta} e^{\delta(T+s)}[1 - e^{-2\delta T}]$.
3. $\lambda_2(s) = \frac{M}{\delta} e^{\delta(T-s)}[1 - e^{-2\delta T}]$.
4. $\tilde{\Gamma}_1 = \tilde{\Gamma}_2 = \frac{M}{\delta^2}$.

3 | ALMOST-PERIODIC AND PSEUDO–ALMOST-PERIODIC SOLUTIONS IN THE NONHOMOGENEOUS LINEAR CASE

Initially we list some assumptions:

(H1) There exists a unique evolution family $\{U(t,s)\}_{-\infty<s\leq t<\infty}$ on X, which governs the linear version of (1).
(H2) The evolution family $U(t,s)$ generated by $A(t)$ has an integrable dichotomy satisfying (6) with function λ, dichotomy projections $P(t)$, $t \in \mathbb{R}$, and Green's function $G(t,s)$.
(H3) The Green function is λ-almost periodic in $t,s \in \mathbb{R}$ uniformly in any bounded subset $K \subset X$ as in Definition 3.
(H4) The function $\lambda(t,s)$ satisfies assumption (A).

Now, we will consider the nonhomogeneous linear case

$$\dot{x}(t) = A(t)x(t) + f(t), \tag{11}$$

in the Banach space X.

According to Xiao et al[44] and references therein, we can define a mild solution of Equation 11 as follows.

Definition 8. A continuous function $y : \mathbb{R} \to X$ is called a mild solution of Equation 11 if

$$y(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{12}$$

3.1 | Almost-periodic solutions of system (11)

Theorem 9. *Assume that (H1), (H2), and (H3) hold and $f \in AP(\mathbb{R},X)$. Then (11) has a unique almost-periodic (AP) mild solution given by*

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{13}$$

Proof. It is known by Pinto[15] that for f bounded (ie, $\|f\|_\infty = \sup_{t\in\mathbb{R}}\|f(t)\| < \infty$), Equation (11) has a unique bounded mild solution x given by (13).

We choose a bounded set $K \subset X$ such that $f(t) \in K$ for all $t \in \mathbb{R}$ and let $\{\tau_n'\}$ be a sequence of real numbers. By (H3) and since $f \in AP(\mathbb{R},X)$, we can extract a subsequence $\{\tau_n\}$ of $\{\tau_n'\}$ such that

(C1) $\lim\limits_{n\to\infty} \frac{G(t+\tau_n,s+\tau_n)x-G_1(t,s)x}{\lambda(t,s)}$, uniformly in $t,s \in \mathbb{R}$, $x \in K$.
(C2) $\lim\limits_{n\to\infty} f(t+\tau_n) = \tilde{f}(t)$, uniformly in $t \in \mathbb{R}$.

Note that (C2) implies that $\|\tilde{f}\|_\infty < \infty$. Now, we define

$$\tilde{x}(t) = \int_{-\infty}^{\infty} G_1(t,s)\tilde{f}(s)ds, \quad t \in \mathbb{R}, \tag{14}$$

which is well defined by (6) and condition (C1). Next we are going to prove that

$$x(t+\tau_n) \to \tilde{x}(t), \ t \in \mathbb{R}.$$

Note that

$$\begin{aligned} x(t+\tau_n) - \tilde{x}(t) &= \int_{-\infty}^{\infty} G(t+\tau_n,s)f(s)ds - \int_{-\infty}^{\infty} G_1(t,s)\tilde{f}(s)ds \\ &= \int_{-\infty}^{\infty} G(t+\tau_n,s+\tau_n)f(s+\tau_n)ds - \int_{-\infty}^{\infty} G_1(t,s)\tilde{f}(s)ds \\ &= \int_{-\infty}^{\infty} G(t+\tau_n,s+\tau_n)[f(s+\tau_n)-\tilde{f}(s)]ds \\ &\quad + \int_{-\infty}^{\infty} [G(t+\tau_n,s+\tau_n)-G_1(t,s)]\tilde{f}(s)ds. \end{aligned} \tag{15}$$

Let

$$I_1 = I_1(t,n) := \int_{-\infty}^{\infty} G(t+\tau_n, s+\tau_n)[f(s+\tau_n) - \tilde{f}(s)]ds \tag{16}$$

and

$$I_2 = I_2(t,n) := \int_{-\infty}^{\infty} [G(t+\tau_n, s+\tau_n) - G_1(t,s)]\tilde{f}(s)ds. \tag{17}$$

Since $f \in AP(\mathbb{R}, X)$ given $\varepsilon > 0$, there exists $N_1 > 0$ such that

$$\|f(s+\tau_n) - \tilde{f}(s)\| < \varepsilon, \ \forall\, n \geq N_1 \quad \text{uniformly in } s \in \mathbb{R},$$

and together to the fact that the dichotomy is integrable by (6), it follows that for $n \geq N_1$,

$$\begin{aligned} \|I_1\| &\leq \int_{-\infty}^{\infty} \|G(t+\tau_n, s+\tau_n)[f(s+\tau_n) - \tilde{f}(s)]\|ds \\ &< \varepsilon \sup_{t\in\mathbb{R},\quad n\geq N_1} \int_{-\infty}^{\infty} \|G(t+\tau_n, s+\tau_n)\|ds \\ &\leq \mu\varepsilon. \end{aligned} \tag{18}$$

Therefore,

$$I_1(t,n) \to 0, \quad \text{as} \quad n \to \infty \quad \text{uniformly in } t \in \mathbb{R}. \tag{19}$$

For I_2 since $G(t,s)$ is λ-almost periodic, given $\varepsilon > 0$, there is $N_2 > 0$ such that for $n \geq N_2$, we have

$$\frac{\|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - G_1(t,s)\tilde{f}(s)\|}{\lambda(t,s)} < \varepsilon, \qquad \text{uniformly in } t, s \in \mathbb{R},$$

so for $n \geq N_2$,

$$\begin{aligned} \|I_2\| &\leq \int_{-\infty}^{\infty} \|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - G_1(t,s)\tilde{f}(s)\|ds \\ &= \int_{-\infty}^{\infty} \frac{\|G(t+\tau_n, s+\tau_n)\tilde{f}(s) - G_1(t,s)\tilde{f}(s)\|}{\lambda(t,s)} \lambda(t,s)ds \\ &\leq \varepsilon \int_{-\infty}^{\infty} \lambda(t,s)ds \\ &\leq \varepsilon L, \end{aligned} \tag{20}$$

by (7) and hypotheses. Thus, we have that

$$\lim_{n\to\infty} I_2(t,n) = 0 \quad \text{uniformly in } t \in \mathbb{R}. \tag{21}$$

□

Therefore, we can conclude that $\lim_{n\to\infty} x(t+\tau_n) = \tilde{x}(t)$ uniformly in $t \in \mathbb{R}$, and as a conclusion, $x \in AP(\mathbb{R}, X)$.

3.2 | Pseudo–almost-periodic solutions of system (11)

Theorem 10. *Assume that (H1), (H2), (H3), and (H4) hold, $f \in PAP(\mathbb{R}, X)$. Then (11) has a unique pseudo–almost-periodic (PAP) mild solution given by*

$$x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds, \quad t \in \mathbb{R}. \tag{22}$$

Proof. It is known by Pinto[15] that equation 11 has a unique bounded mild solution x given by (22). Since $f \in PAP(\mathbb{R}, X)$, we suppose that

$$f(t) = g(t) + h(t),$$

where $g \in AP(\mathbb{R}, X)$ and $h \in PAP_0(\mathbb{R}, X)$. We define

$$u(t) = \int_{-\infty}^{\infty} G(t,s)g(s)ds, \quad t \in \mathbb{R}, \tag{23}$$

and

$$v(t) = \int_{-\infty}^{\infty} G(t,s)h(s)ds, \quad t \in \mathbb{R}, \tag{24}$$

then

$$x(t) = u(t) + v(t).$$

By Theorem 9, it follows that $u(t) \in AP(\mathbb{R}, X)$, and it remains to show that $v \in PAP_0(\mathbb{R}, X)$, ie,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|v(t)\| dt = 0 \Longleftrightarrow \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \| \int_{\mathbb{R}} G(t,s)h(s)ds\| dt = 0.$$

Now, we observe that by

$$\begin{aligned} \frac{1}{2T} \int_{-T}^{T} \| \int_{\mathbb{R}} G(t,s)h(s)ds\| dt \leq & \frac{1}{2T} \int_{-T}^{T} \| \int_{-\infty}^{t} G(t,s)h(s)ds\| dt + \\ & \frac{1}{2T} \int_{-T}^{T} \| \int_{t}^{\infty} G(t,s)h(s)ds\| dt. \end{aligned} \tag{25}$$

For any fixed $T > 0$, we have

$$\begin{aligned} I := & \frac{1}{2T} \int_{-T}^{T} \left\| \int_{-\infty}^{t} G(t,s)h(s)ds \right\| dt \\ \leq & \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{-T} \|G(t,s)h(s)\| dsdt + \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} \|G(t,s)h(s)\| dsdt \\ \leq & \frac{1}{2T} \int_{-T}^{T} \int_{-\infty}^{-T} \lambda(t,s)\|h(s)\| dsdt + \frac{1}{2T} \int_{-T}^{T} \int_{-T}^{t} \lambda(t,s)\|h(s)\| dsdt. \end{aligned} \tag{26}$$

To estimate I, we note that by (10) (or condition (H4))

$$\begin{aligned} I_1 := & \int_{-T}^{T} \left(\int_{-\infty}^{-T} \lambda(t,s)|h(s)| ds \right) dt = \int_{-\infty}^{-T} \left(\int_{-T}^{T} \lambda(t,s)dt \right) \|h(s)\| ds \\ \leq & \|h\|_\infty \int_{-\infty}^{-T} \lambda_1(s)ds \leq \|h\|_\infty \tilde{\Gamma}_1, \end{aligned} \tag{27}$$

and

$$\begin{aligned} I_2 := & \int_{-T}^{T} \left(\int_{-T}^{t} \lambda(t,s)\|h(s)\| ds \right) dt = \int_{-T}^{T} \left(\int_{s}^{T} \lambda(t,s)dt \right) \|h(s)\| ds \\ \leq & \Gamma_1 \int_{-T}^{T} \|h(s)\| ds, \end{aligned} \tag{28}$$

by changing the order of integration. Now, let

$$\begin{aligned} J := & \frac{1}{2T} \int_{-T}^{T} \| \int_{t}^{\infty} G(t,s)h(s)ds\| dt \\ \leq & \frac{1}{2T} \int_{-T}^{T} \int_{t}^{T} \|G(t,s)h(s)\| dsdt + \frac{1}{2T} \int_{-T}^{T} \int_{T}^{\infty} \|G(t,s)h(s)\| dsdt \\ \leq & \frac{1}{2T} \int_{-T}^{T} \int_{t}^{T} \lambda(t,s)\|h(s)\| dsdt + \frac{1}{2T} \int_{-T}^{T} \int_{T}^{\infty} \lambda(t,s)\|h(s)\| dsdt. \end{aligned} \tag{29}$$

Therefore by (10) (or condition (H4)), we have

$$\begin{aligned} J_1 := & \int_{-T}^{T} \left(\int_{t}^{T} \lambda(t,s)\|h(s)\| ds \right) dt = \int_{-T}^{T} \left(\int_{-T}^{s} \lambda(t,s)dt \right) |h(s)| ds \\ \leq & \Gamma_2 \int_{-T}^{T} \|h(s)\| ds \end{aligned} \tag{30}$$

and

$$\begin{aligned} J_2 := & \int_{-T}^{T} \left(\int_{T}^{\infty} \lambda(t,s)\|h(s)\| ds \right) dt = \int_{T}^{\infty} \left(\int_{-T}^{T} \lambda(t,s)dt \right) \|h(s)\| ds \\ \leq & \|h\|_\infty \int_{T}^{\infty} \lambda_2(s)ds \leq \|h\|_\infty \tilde{\Gamma}_2, \end{aligned} \tag{31}$$

by changing the order of integration. □

Taking limit as $T \to \infty$ in (26) and (29), we obtain the proof of the theorem.

4 | THE NONLINEAR CASE

In this section, we will consider the nonlinear differential equations considered in (2), where $f : \mathbb{R} \times X \times X \to X$ is a function under convenient conditions, and $\tau > 0$ is fixed, and also we will analyze the delayed case

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), x(t-\tau)), \quad t \in \mathbb{R}, \tag{32}$$

and the ordinary case

$$\dot{x}(t) = A(t)x(t) + g(t, x(t)), \quad t \in \mathbb{R}, \tag{33}$$

both in the Banach space X, where $g : \mathbb{R} \times X \to X$ is a function under convenient conditions.

4.1 | Definitions and mild solutions

Let X and Y be Banach spaces and $BC(\mathbb{R} \times X, Y)$ be the Banach space of all bounded continuous functions of $\mathbb{R} \times X$ in Y with the supremum norm $\| \ \|_\infty$, ie, $\|f\|_\infty = \sup_{(t,x)\in\mathbb{R}\times X} \|f(t,x)\|_Y$. Analogously, we define $BC(\mathbb{R} \times X \times X, Y)$ be the Banach space of bounded continuous function of $\mathbb{R} \times X \times X$ in Y with the supremum norm of $\| \ \|_\infty$, ie, $\|f\|_\infty = \sup_{(t,x,y)\in\mathbb{R}\times X\times X} \|f(t,x,y)\|_Y$.

Definition 11. A (bounded) continuous function $f : \mathbb{R} \times X \to Y$ is called almost periodic if for every sequence of real numbers $\{\tau_n'\}_{n=1}^\infty$, we can extract a subsequence $\{\tau_n\}$ such that $\lim_{n\to\infty} f(t+\tau_n, x)$ converges uniformly on $\mathbb{R}$ and uniformly for all x in any bounded subset of X, ie, $\tilde{f}(t,x) = \lim_{n\to\infty} f(t+\tau_n, x)$ is well defined in $t \in \mathbb{R}$ and $x \in K \subset X$ with K bounded subset of X. We denote by $AP(\mathbb{R} \times X, Y)$ the set of all such functions. Similarly, we define the set $AP(\mathbb{R} \times X \times X, Y)$.

Definition 12. A continuous function $u : \mathbb{R} \to X$ is called a mild solution of Equation 32 if

$$u(t) = \int_{\mathbb{R}} G(t,s) f(s, u(s), u(s-\tau)) ds, \quad t \in \mathbb{R}. \tag{34}$$

4.2 | Existence of almost-periodic solutions

Initially we will prove that under the existence of integrable dichotomy and additional hypotheses, system (32) and then system (33) possess a unique locally bounded solution.

Theorem 13. *Assume that the following hypotheses hold.*

(E_1) System (1) has an integrable dichotomy on $\mathbb{R}$ with data (μ, P).
(E_2) The function $f(t,\xi,\eta)$ is locally Lipschitz in $\xi, \eta \in X$, ie, for each positive number R, for all $\xi_1, \xi_2, \eta_1, \eta_2 \in X$, with $\|\xi_j\|, \|\eta_j\| \le R$, for $j = 1, 2$, it is valid

$$\|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)\| \le C(R)[\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\|],$$

where $C : [0,\infty) \to [0,\infty)$ is a function and there is a positive constant ρ, such that $2C(\rho) < \frac{1}{\mu}$ and $\sup_{t\in\mathbb{R}}\{\|f(t,0,0)\|\} \le \frac{\rho}{\mu}[1 - 2C(\rho) \quad \mu]$.

Then, there exists a unique bounded solution $y(t)$ with $\|y\|_\infty \le \rho$ of Equation (32) on $\mathbb{R}$.

Proof. Let $G(t,s)$ be the Green function associated with the Equation 1. For each $\rho > 0$, we denote by $X[\rho]$ the ball $\|\varphi\|_\infty \le \rho$ in X.

Now, we define the functional on X by

$$(\Gamma\varphi)(t) = \int_{-\infty}^{\infty} G(t,s) f(s, \varphi(s), \varphi(s-\tau)) ds, \ t \in \mathbb{R}. \tag{35}$$

It is clear that each fixed point of Γ gives us a mild solution of system (32). To obtain bounded solutions on $\mathbb{R}$, we will consider ρ constant satisfying (E_2) and we will prove that $\Gamma(\varphi) \in X[ho]$, for all $\varphi \in X[ho]$. In fact,

$$\begin{aligned}
|(\Gamma\varphi)(t)| &\leq \int_{-\infty}^{\infty} \|G(t,s)\| \quad \|f(s,\varphi(s),\varphi(s-\tau))\|ds \\
&\leq 2C(\rho)\int_{-\infty}^{\infty} \|G(t,s)\| \quad \|\varphi(s)\|ds + \int_{-\infty}^{\infty} \|G(t,s)\|\|f(s,0,0)\|ds \\
&\leq 2\mu \quad C(\rho)\|\varphi\|_\infty + \mu \sup_{t\in\mathbb{R}}\{\|f(t,0,0)\|\} \\
&\leq \rho.
\end{aligned}$$

This proves that $\Gamma\varphi \in X\,[\rho]$, for all $\varphi \in X\,[\rho]$.

Now, we will prove that Γ is a contraction. In fact,

$$\begin{aligned}
&\|(\Gamma\varphi)(t) - (\Gamma\psi)(t)\| \\
&\leq \int_{-\infty}^{\infty} \|G(t,s)\| \quad \|f(s,\varphi(s),\varphi(s-\tau) - f(s,\psi(s),\psi(s-\tau)\|ds \\
&\leq C(\rho)\int_{-\infty}^{\infty} \|G(t,s)\,[\|\varphi(s)-\psi(s)\| + \|\varphi(s-\tau)-\psi(s-\tau)\|]\,ds \\
&\leq 2\mu C(\rho)\ \|\varphi-\psi\|_\infty.
\end{aligned}$$

Now, using the contraction fixed point theorem, we deduce by (E_1)-(E_2) that Γ has a fixed point $\varphi \in X\,[\rho]$. Therefore, we conclude the proof. □

Remark 14. A similar result as in Theorem 13 holds for system (33).

Now, we are in position to prove the existence of an almost-periodic solution for system (32).

Theorem 15. *Assume that (H1), (H2), (H3), and (E_2) hold and $f \in AP(\mathbb{R}\times X\times X, X)$ as in Theorem 13. Then (32) has a unique $y(t)$ almost-periodic mild solution satisfying (34) and $\|y\|_\infty \leq \rho$.*

Proof. For each $\varphi \in AP(\mathbb{R}, X)$, using results in Amir and Maniar[36] and Ding et al[41] (a composition result in the almost-periodic case), it follows that $f_\varphi(\cdot) = f(\cdot,\varphi(\cdot),\varphi(\cdot-\tau)) \in AP(\mathbb{R}, X)$. Then by Theorem 9, the nonhomogeneous linear equation

$$\dot{x}(t) = A(t)x(t) + f_\varphi(t), \quad t \in \mathbb{R}$$

has a unique almost-periodic mild solution given by

$$\int_{-\infty}^{\infty} G(t,s)f(s,\varphi(s),\varphi(s-\tau))ds,\ t \in \mathbb{R}.$$

We define the functional Γ on X as in (35) for each $\varphi \in AP(\mathbb{R}, X)$, $t \in \mathbb{R}$, and we note that each fixed point of Γ gives us a mild solution of system (32). For each $\rho > 0$, we denote by $X[ho]$ the ball $\|\varphi\| \leq \rho$ in X. By Theorem 13, we have that Equation (33) has a unique bounded mild solution x given by (34). Our proof follows the following steps: (i) $\Gamma(AP(\mathbb{R}, X)) \subset AP(\mathbb{R}, X)$; (ii) $\Gamma(X[\rho] \subset X\,[\rho]$; and (iii) Γ restricted to $X\,[\rho] \cap AP(\mathbb{R}, X)$ is a contraction.

We choose bounded sets $K, K_1 \subset X$ such that $f(t,x,y) \in K$ for all $t \in \mathbb{R}$ and $x, y \in K_1$. Then, from the proof of Theorem 9, statement (i) follows. Now, analogously, to the proof of Theorem 13, we obtain (ii) and (iii). Therefore, Γ has a unique fixed point in $AP(\mathbb{R}\times X)$, which is an almost-periodic mild solution of (32). □

As a consequence of the previous result, we have the following.

Corollary 16. *Assume that (H1), (H2), and (H3) hold and $g \in AP(\mathbb{R}\times X, X)$. Furthermore assume that*

[(E_2^*)] The function $g(t,\xi)$ is locally Lipschitz in $\xi \in X$, ie, for each positive number R, for all $\xi_1, \xi_2 \in X$, with $\|\xi_j\| \leq R$, for $j = 1, 2$, it is valid

$$\|g(t,\xi_1) - g(t,\xi_2)\| \leq C(R)\|\xi_1 - \xi_2\|,$$

where $C : [0,\infty) \to [0,\infty)$ is a function and there is a positive constant ρ, such that $2C(\rho) < \frac{1}{\mu}$ and $\sup_{t\in\mathbb{R}}\{\|g(t,0)\|\} \leq \frac{\rho}{\mu}[1 - C(\lambda) \quad \mu]$.

Then (33) has a unique almost-periodic mild solution $y(t)$ satisfying (34) and $\|y\|_\infty \leq \rho$.

In the case of a global Lipschitz condition, we have the following result.

Corollary 17. *Assume that (H1), (H2), and (H3) hold and $f \in AP(\mathbb{R} \times X \times X, X)$. Furthermore assume that $[(E_2^{**})]$ The function $f(t, \xi, \eta)$ is Lipschitz in $\xi, \eta \in X$, ie, for all $\xi_1, \xi_2, \eta_1, \eta_2 \in X$, there is $C > 0$ such that $2\mu C < 1$ and*

$$\|f(t, \xi_1, \eta_1) - f(t, \xi_2, \eta_2)\| \leq C[\|\xi_1 - \xi_2\| + \|\eta_1 - \eta_2\|].$$

Then (32) has a unique almost-periodic mild solution $y(t)$ satisfying (34) and $\|y\|_\infty \leq \rho$.

If in Equation 32 we have that A and f in the first variable are T-periodic, then the Green function satisfies the biperiodicity conditions, ie, $G(t+T, s+T) = G(t, s)$. Thus following the same ideas of Theorem 13, the existence of periodic solutions follows.

Corollary 18. *Assume that all the hypotheses of Theorem 13 are satisfied, and that A, f in the first variable and τ are T-periodic. Then, there exists a unique T-periodic solution $y(t)$ of Equation (32) on $\mathbb{R}$.*

4.3 | Existence of pseudo–almost-periodic solutions

Theorem 19. *Assuming that the same assumptions given in Theorem 15 are satisfied, (H4) holds and $f \in PAP(\mathbb{R}\times X\times X, X)$. Then (32) has a unique pseudo–almost-periodic mild solution $y(t)$ of Equation 34 with $\|y\|_\infty \leq \rho$.*

Proof. The proof follows the same ideas of Theorem 15. In fact, by the composition result in the pseudo–almost-periodic case (see details in Amir and Maniar[36] and Ding et al[41]), we have that $f_\varphi(\cdot) = f(\cdot, \varphi(\cdot), \varphi(\cdot - h)) \in PAP(\mathbb{R}, X)$ for each $\varphi \in PAP(\mathbb{R}, X)$, and therefore by Theorem 10, the linear nonhomogeneous equation

$$\dot{x}(t) = A(t)x(t) + f_\varphi(t), \quad t \in \mathbb{R}$$

has a unique pseudo–almost-periodic mild solution given by

$$\int_{-\infty}^{\infty} G(t, s)f(s, \varphi(s), \varphi(s - \tau))ds, \ t \in \mathbb{R}.$$

Then, we define the nonlinear mapping Γ as in (35) on the space $PAP(\mathbb{R}, X)$, and we follow the same steps as in Theorem 15. □

Remark 20. Analogously to Section 4.2, we can prove in this case Corollaries 16, 17, and 18.

5 | EXAMPLES

At the present the characterizations concerning with the integrable dichotomy and the λ-bi–almost periodicity are open problems.[11] In the following examples, we show general diagonal systems having these properties.

Example 21. First, we show a large class of integrable dichotomy (see, for example, Coppel[46] [p. 73] and Pinto[15]). Let $\{a_k\}$xx-xx$_{k\in Z}$ be a positive sequence such that $\sum_{k\in\mathbb{Z}} a_k$ converges and $\inf_{k\in Z} a_k^{-1} = c > 0$. Define for $k \in Z$, $I_k = [k - a_k^2, k + a_k^2]$.

Let $\xi : \mathbb{R} \to (0, \infty)$ be a continuously differentiable function given by $\xi(t) \equiv c^{-1}$ except on I_k, where $\xi(k) = a_k$ and ξ on I_k lies between c^{-1} and a_k. We have

$$\sum_{k\in\mathbb{Z}} \int_{I_k} \xi^{-1}(s)ds \leq v < \infty.$$

Consider the scalar differential equation

$$\dot{x} = a(t)x, \quad a(t) = -\alpha + \dot{\xi}(t)\xi(t)^{-1}, \ \alpha > 0, \tag{36}$$

with solutions

$$x(t) = x_0 e^{-\alpha t}\xi(t) := x_0\phi(t).$$

It is clear that the evolution family of linear system (36) with projections $P(t) = I$, $t \in \mathbb{R}$ is given by $U(t, s) = \phi(t)\phi(s)^{-1}$.

We have

$$\phi(k + a_k^2)\phi(k)^{-1} \geq c^{-1} \quad a_k^{-1} e^{-\alpha a_k^2} \to \infty, \quad \text{as } k \to \infty,$$

and equation 36 is not exponentially stable. However, Equation 36 with Green's function $G(t,s) = U(t,s)$ has an integrable dichotomy, since

$$\int_{-\infty}^{t} \phi(t)\phi(s)^{-1}ds \le \int_{-\infty}^{t} e^{-\alpha(t-s)}ds + \sum_{k=-\infty}^{[t+1]} \int_{I_k} c^{-1}\xi(s)^{-1}ds \le \alpha^{-1} + c^{-1}v < \infty,$$

condition (8) is satisfied with $\mu = \alpha^{-1} + c^{-1}v$. Note that

$$|\phi(t)\phi(s)^{-1}| \le e^{-\alpha(t-s)} + \lambda_0(s),\ \ s \le t$$

with

$$\lambda_0(s) = \sum_{k\in\mathbb{Z}} c^{-1}\xi(s)^{-1}\chi_{I_k}(s),$$

where χ_{I_k} is the characteristic function on I_k. We verify that $\lambda_0 \in L^1(\mathbb{R})$. Then, equation 36 has an integrable dichotomy with $\lambda(t,s) = e^{-\alpha(t-s)} + \lambda_0(s)$, $t \ge s$ satisfying

$$L = \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} \lambda(t,s)ds \le \mu. \tag{37}$$

So, in this way, a big class of linear differential equations of type (36), satisfying (37), can be built. In a similar way, the above construction may be modified to obtain equation 36 satisfying

$$\int_{t}^{\infty} \phi(t)\phi(s)^{-1}ds \le \mu, \tag{38}$$

but not "exponentially" stable at $-\infty$.

Furthermore, if we construct the diagonal matrix

$$A(t) = \operatorname{diag}(a_1(t), a_2(t), \dots, a_n(t))$$

with a_i of different types satisfying (37) for $i = 1, \dots, k$ and satisfying (38) for $i = k+1, \dots, n$ $(k > 0)$, then the linear system

$$\dot{x} = A(t)x \tag{39}$$

has an integrable dichotomy (which is not exponential) with

$$\lambda(t,s) = e^{-\alpha|t-s|} + \lambda_0(s),\ \ t,s \in \mathbb{R},$$

λ_0 integrable in $\mathbb{R}$ and $P(t) = \operatorname{diag}(1, 1, \dots, 1, 0, 0, \dots, 0)$, where the number 1 appears k-times.

Clearly the diagonal and integrable character of the dichotomy of $A(t)$ in (39) can be extended to a diagonal infinite dimensional. See a concrete diagonal operator in the next example.

Example 22. Consider the following heat equation with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) =& \frac{\partial^2 u}{\partial^2 x}(t,x) + u(t,x) \quad h(t) + f(t,u(t,x)), \\ u(t,0) =& u(t,1) = 0,\ \ t \in \mathbb{R}, \end{aligned} \tag{40}$$

with $h \in AP(\mathbb{R})$.

We consider $X = L^2(0,1)$ and $D(B) = \{\xi \in C^1[0,1] \quad / \quad \xi'$ is absolutelycontinuous on $[0,1], \xi'' \in X, \xi(0) = \xi(1) = 0\}$. Let $B\xi(r) = \xi''(r),\ \ r \in (0,1),\ \ \xi \in D(B)$, and then B generates a C_0-semigroup $T(t)$ on X given by

$$(T(t)\xi) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle \xi, e_n\rangle_{L^2} e_n(t),$$

where $e_n(t) = \sqrt{2}\sin(n\pi t)$, $n = 1, 2, \dots$. Define the family of linear operator $A(t)$ by

$$\begin{aligned} &D(A(t)) = D(B),\ \ t \in \mathbb{R} \\ &A(t)\xi = (B + h(t)I)\,\xi, \quad \xi \in D(A(t)). \end{aligned}$$

Then $\{A(t),\ t \in \mathbb{R}\}$ generates an evolution family $\{U(t,s)\}$xx-xx$_{t\geq s}$ such that

$$U(t,s)\xi = T(t-s)\ e^{\int_s^t h(r)dr}\ \xi,$$

ie,

$$U(t,s)\xi = \sum_{n=1}^{\infty} e^{\int_s^t [h(r)-(n\pi)^2]dr} \langle \xi, e_n \rangle_{L^2} \quad e_n(t).$$

Since for $t > s$ exponential terms inside the sum tend to zero as $n \to +\infty$, we divide the series in the first N summands (the unstable part) and the next summands (the stable part).

Then, we consider the integrable dichotomy character of U. Let $N \in \mathbb{N}$ such that for $n > N$

$$\int_s^t [h(r) - (n\pi)^2]dr \leq -\int_s^t \mu_1(r)dr,$$

for $t \geq s$, t big enough, and $\mu_1 > 0$ is a locally integrable function with $\int^{\infty} \mu_1(r)dr = \infty$ and for $n \leq N$

$$\int_s^t [h(r) - (n\pi)^2]dr \leq \int_s^t \mu_2(r)dr,$$

for $t < s$, s big enough and $\mu_2 > 0$ a locally integrable function with $\int^{\infty} \mu_2(r)dr = \infty$.

We have built the following dichotomy

$$\begin{aligned} |U(t,s)P| &\leq Ce^{-\int_s^t \mu_1(r)dr}, \quad t \geq s \\ |U(t,s)(I-P)| &\leq Ce^{\int_s^t \mu_2(r)dr}, \quad t < s, \end{aligned} \tag{41}$$

where C is a positive constant, $Rank(I-P) = N$ and $Rank(P) = \infty$, since it can be seen as $I-P = diag(1, \dots, 1, 0, 0, \dots)$ with N numbers 1 at the diagonal. This dichotomy is an (h,k)-dichotomy.[15] Obviously, the dichotomy is not necessarily of exponential type. Particularly, this dichotomy is not exponential taking for the stable part $-\mu_1$ as a in Equation 36 and μ_2 for the unstable part for which $\phi(t) = e^{\int_0^t \mu_2(s)ds}$ satisfies (38). Then, $\dot{x} = A(t)x$ has an integrable dichotomy with data (λ, P), where λ is given by

$$\lambda = \lambda(t,s) = \begin{cases} Ce^{-\int_s^t \mu_1(r)dr}, & \text{if} \quad t \geq s, \\ Ce^{\int_s^t \mu_2(r)dr}, & \text{if} \quad t < s. \end{cases}$$

On the other hand, the Green function is λ-bi–almost periodic. In fact, for $t \geq s$

$$G(t,s)\xi = U(t,s)P\xi = T(t-s)P\ e^{\int_s^t h(r)dr} \quad \xi,$$

and for any sequence $\{\tau_n'\}$ in $\mathbb{R}$, we have

$$G(t+\tau_n', s+\tau_n')\xi = T(t-s)P\ e^{\int_{s+\tau_n'}^{t+\tau_n'} h(r)dr}\ \xi = T(t-s)P\ e^{\int_s^t h(r+\tau_n')dr} \quad \xi.$$

Since $h \in AP(\mathbb{R})$, over a subsequence $\{\tau_n\} \subset \{\tau_n'\}$, $G(t+\tau_n, s+\tau_n)\xi \to G_1(t,s)\xi = T(t-s)P\ e^{\int_s^t h(\tau)d\tau}\xi$ in the sense of Definition 3 with the previous λ.

Note that if h is not $AP(\mathbb{R})$, then the bounded solution of associated Equation 40 is not necessarily almost periodic. In fact, this is showed taking the equation $y' = -(c-h(t))y$ with $c > 1$ and $h(t) = 1/(1+t^2)$.

Theorem 23. *Under the previous notations and assumptions:*

1. *If $f \in AP(\mathbb{R}, \mathbb{R})$ and satisfies condition (E2), then system (40) has a unique AP mild solution.*
2. *If $f \in PAP(\mathbb{R}, \mathbb{R})$ and satisfies condition (E2), then system (40) has a unique PAP mild solution.*

Proof. The proof of item 1 follows, observing that under the previous remarks, all the conditions of Theorem 15 hold. Similarly, for item 2, we observe that all the conditions of Theorem 19 are valid. □

The function $f(t) = \cos t + \cos(\sqrt{2}t)$, is a typical example of AP function and $f(t) = \cos t + \cos(\sqrt{2}t) + e^{-|t|}$, is a typical example of PAP function.

In a similar way, we can study semilinear system (2) considering functions $f(t,x,y)$ satisfying the conditions of our main Theorems 13 and 19 to obtain unique almost and pseudo–almost-mild-periodic solutions.

Example 24. Consider the cellular neural networks with delays

$$\dot{u}_i = -r_i(t)u_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(u_j(t)) + \sum_{j=1}^{n} b_{ij}(t)g_j(u_j(t-\tau_{ij})) + I_i(t),\ i = 1, 2, \dots, n, \tag{42}$$

where n is the number of units in a neural networks; $u_i(t)$ denotes the state of the ith unit at the time t; $r_i(t) > 0$ and represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $a_{ij}(t)$ denotes the strength of the jth unit on the ith unit at the time t; $b_{ij}(t)$ denotes the strength of the jth unit on the ith unit at time $t-\tau_{ij}$; τ_{ij} corresponds to the transmission delay along the axon of the jth unit to the ith unit at the time t; f_j and g_j denote the measure of response or activation to its incoming potentials, and $I_i(t)$ represents varying external input signals (or stimulus) from outside the network to the ith unit at time t.

Here, the initial conditions associated with (42) read as

$$u_i(s) = \varphi_i(s),\ s \in [-\tau, 0],\ \tau = \max_{1\le i,j\le n} \tau_{ij},$$

where $\varphi_i \in C([-\tau,0],\mathbb{R})$, $i = 1, 2, \dots, n$.

The activation functions $f_i, g_i \in BC(\mathbb{R},\mathbb{R})$ are globally Lipschitz, ie, there exist constants $L_i > 0$ and $l_i > 0$ such that

$$|f_i(u) - f_i(v)| \le L_i|u-v|,\ |g_i(u) - g_i(v)| \le l_i|u-v|,\ u, v \in \mathbb{R}.$$

It is trivial to show that the usual sigmoidal activation functions and the piecewise linear function widely used in the literature (see for example, Xia and Fan[47]) satisfy the above assumptions.

To apply our results, we will assume that $u(t) = (u_1(t), \dots, u_n(t)) \in X$ with $X = \mathbb{R}^n$. Let $A(t) = \mathrm{diag}(-r_1(t), \dots, -r_n(t))$ represent a stable diagonal operator as in (39). Thus system (42) assumes the form

$$\dot{u}(t) = A(t)u(t) + f(t, u(t), u(t-\tau)),\ t \in \mathbb{R}, \tag{43}$$

in the Banach space X, where $f : \mathbb{R}\times X\times X \to X$ and is given by

$$f(t,u(t),u(t-\tau)) = \left(\sum_{j=1}^{n} a_{1j}(t)f_j(u_j(t)) + \sum_{j=1}^{n} b_{1j}(t)g_j(u_j(t-\tau_{1j})) + I_1(t), \dots, \right.$$
$$\left. \sum_{j=1}^{n} a_{nj}(t)f_j(u_j(t)) + \sum_{j=1}^{n} b_{nj}(t)g_j(u_j(t-\tau_{nj})) + I_n(t)\right)^T.$$

Theorem 25. *Let*

$$\nu = \max_{1\le i\le n}\left(\max_{t\in\mathbb{R}}\left(\sum_{j=1}^{n} L_j|a_{ij}(t)| + \sum_{j=1}^{n} l_j \quad |b_{ij}(t)|\right)\right),$$

and

$$\delta = \max_{1\le i\le n}\left(\sup_{t\in\mathbb{R}} |\sum_{j=1}^{n} a_{ij}(t)f_j(0) + \sum_{j=1}^{n} b_{ij}(t)g_j(0)) + I_i(t)|\right).$$

For any $\rho > 0$, suppose that $2\nu < \frac{1}{\mu}$ and $\delta \le \frac{\rho}{\mu}[1-2\nu\mu]$, and assume that r_i, $i = 1, \dots, n$ are $AP(\mathbb{R},\mathbb{R})$ satisfying for $s \le t$ and t big enough the inequality $\int_s^t r_i(\tau)d\tau \ge \int_s^t \mu_1(\tau)d\tau$ with $-\mu_1$ as a in (36) and f_i and g_i are Lipschitz as before.

1. *If $a_{ij}(t)$, $b_{ij}(t)$, $I_i(t) \in AP(\mathbb{R},\mathbb{R})$, $f_j, g_j \in AP(\mathbb{R},\mathbb{R})$, then system (42) has a unique AP mild solution.*
2. *If $a_{ij}(t)$, $b_{ij}(t)$, $I_i(t) \in PAP(\mathbb{R},\mathbb{R})$, $f_j, g_j \in PAP(\mathbb{R},\mathbb{R})$ and is valid condition (H4), then system (42) has a unique PAP mild solution.*

Proof. First we observe that

$$\|f(t,\xi_1,\eta_1) - f(t,\xi_2,\eta_2)\| \le \nu\,[\|\xi_1-\xi_2\| + \|\eta_1-\eta_2\|]\,.$$

Under the hypotheses of item 1, it follows that $f \in AP(\mathbb{R}\times X\times X, X)$. Thus, all the conditions of Theorem 15 are valid.

For item 2, it is clear that under the hypotheses $f \in PAP(\mathbb{R}\times X\times X, X)$. Thus, all the conditions of Theorem 19 are valid. □

ACKNOWLEDGMENTS

We thank the reviewers' comments and suggestions, which help us to improve the presentation of our paper.

The first author is partially supported by Fondecyt1120709 and 1170466 (Chile).

REFERENCES

1. Xiao TJ, Liang J. *The Cauchy problem for higher-order abstract differential equations*, Lecture Notes in Mathematics, vol. 1701. Springer: Berlin, 1998.
2. Corduneanu C. *Almost Periodic Functions* (second ed.) Chelsea: New York, 1989.
3. Fink A. *Almost periodic differential equations*, Lecture Notes in Mathematics, vol. 377. Springer-Verlag: New York, Berlin, 1974.
4. N'Guérékata G. *Almost Periodic Functions in Abstract Spaces*. Kluwer Academic, Plenum Publishers: New York, London, Moscow, 2001.
5. Zhang C. *Almost Periodic Type Functions and Ergodicity*. Science Press, Kluwer Academic Publishers: New York, 2003.
6. Bohr H. *Almost Periodic Functions*. Chelsea, 1947. reprint.
7. Stepanov V. Sur quelques généralisations des fonctions presque périodiques. *CR Acad Sci Paris*. 1925; 181:90-92.
8. Weyl H. Integralgleichungen und fastperiodische Funktionen. *Math Ann*. 1927; 97(1):338-356.
9. Besicovitch A.S. *Almost Periodic Functions*, Dover Publications, Inc., New York, 1955. xii+180 pp.
10. Zhang C. Pseudo almost periodic solutions of some differential equations. *J Math Anal Appl*. 1994; 151:62-76.
11. Zhang C. Pseudo almost periodic solutions of some differential equations II. *J Math Anal Appl*. 1995; 192:543-561.
12. Bochner S, A new approach to almost periodicity. *Proc Nat Acad Sci*. 1962;48:2039-2043.
13. N'Guérékata G, Pankov A. Integral operators in spaces of bounded, almost periodic, and almost automorphic functions. *Differ Integral Equ*. 2008; 21(11–12):1155-1176.
14. Pinto M, Robledo G. Diagonalizability of nonautonomous linear systems with bounded continuous coefficients. *J Math Anal Appl*. 2013;407(2):513-526.
15. Pinto M. Bounded and periodic solutions of nonlinear integro-differential equations with infinite delay. *Electron J Qual Theory Differ Equ*. 2009;46:1-20.
16. Lupa N, Megan M. Generalized exponential trichotomies for abstract evolution operators on the real line. *J Funct Spaces Appl*. 2013; Art. ID 409049:8 pp.
17. Jiang L. Generalized exponential dichotomy and global linearization,. *J Math Anal Appl*. 2006; 315(2):474-490.
18. Agarwal R, Diagana T, Hernández ME. Weighted pseudo almost periodic solutions to some partial neutral functional differential equations. *J Nonlinear Convex Anal*. 2007;8(3):397-415.
19. Lupa N, Megan M. Exponential dichotomies of evolution operators in Banach spaces. *Monatsh Math*. 2014;174(2):265-284.
20. Xia Y, Yuan X, Kou K, Wong P. Existence and uniqueness of solution for perturbed nonautonomous systems with nonuniform exponential dichotomy. *Abstr Appl Anal*. 2014;Art ID 725098:10 pp.
21. Xia Z. Pseudo asymptotic behavior of mild solution for nonautonomous integrodifferential equations with nondense domain. *J Appl Math*. 2014;Art. ID 419103:10, 45J05.
22. Barreira L, Valls C. Stable manifolds for nonautonomous equations without exponential dichotomy. *J Differ Equ*. 2006;221(1):58-90.
23. Barreira L, Valls C. *Stability of nonautonomous differential equations*, Lecture Notes in Mathematics, vol. 1926. Springer: Berlin, 2008. xiv+285.
24. Maniar L, Schnaubelt R. 2003. Almost periodicity of inhomogeneous parabolic evolution equations. In *Lecture Notes in Pure and Appl Math*, Evolution Equations, vol. 234 Dekker: New York; 299-318.
25. Zaidman S. A non linear abstract differential equation with almost-periodic solution. *Riv Mat Univ Parma*. 1984;4(10):331-336.
26. Zaidman S. *Topics in abstract differential equations*, Pitman Research Notes in Mathematics Series, vol. 304. Longman Scientific & Technical, Harlow, copublished in the United States with John Wiley & Sons Inc.: New York, 1994.
27. Zaidman S. Topics in abstract differential equations. *Nonlinear Anal*. 1994;23(7):849-870.
28. Zaidman S. *Topics in abstract differential equations II*, Pitman Research Notes in Mathematics Series, vol. 321. Longman Scientific & Technical, Harlow, copublished in the United States with John Wiley & Sons Inc.: New York, 1995.
29. Diagana T, Henriquez H, Hernández E. Asymptotically almost periodic solutions to some classes of second-order functional differential equations. *Differ Integral Equ*. 2008;21(5–6):575-600.
30. Bostan M. Almost periodic solutions for first-order differential equations. *Differ Integral Equ*. 2006;19(1):91-120.
31. Ding H, Liang J, Xiao T. Weighted pseudo almost periodic functions and applications to evolution equations with delay. *Appl Math Comput*. 2013;219:8949-8958.
32. Liang J, Xiao T, Zhang J. Decomposition of weighted pseudo-almost periodic functions. *Nonlinear Anal*. 2010;73:3456-3461.
33. Zhang J, Xiao T, Liang J. Weighted pseudo almost-periodic functions and applications to semilinear evolution equations. *Abstr Appl Anal*. 2012;2012:Article ID 179525, 15.
34. Hino Y. Almost periodic solutions of a linear Volterra system. *Differ Integral Equ*. 1990;3(3):495-501.
35. Figueroa P, Pinto M. Poincarés problem in the class of almost periodic type functions. *Bull Belgium Math Soc Simon Stevin*. 2015;22(2):177-198.
36. Amir A, Maniar L. Composition of pseudo-almost periodic functions and Cauchy problems with operators of non dense domain. *Ann Math Blaise Pascal*. 1999;6:1-11.

37. Cuevas C, Pinto M. Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with nondense domain. *Nonlinear Anal*. 2001;45:73-83.
38. Diagana T. Pseudo almost periodic solutions to some differential equations. *Nonlinear Anal*. 2005;60:1277-1286.
39. Diagana T, Hernández E, Rabello M. Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay. *Math Comput Model*. 2007;45:1241-1252.
40. Diagana T. *Pseudo Almost Periodic Functions in Banach Spaces*. Nova Science Publishers: New York, 2008.
41. Ding H, Liang J, N'Guérékata G, Xiao T. Pseudo almost periodicity of some nonautonomous evolution equations with delay. *Nonlinear Anal*. 2007;67:1412-1418.
42. Pinto M. Pseudo-almost periodic solutions of neutral integral and differential equations with applications. *Nonlinear Anal*. 2010;72:4377-4383.
43. Pazy A. *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, vol. 44. Springer-Verlag: New York, 1983.
44. Xiao T, Zhu X, Liang J. Pseudo-almost automorphic mild solutions to nonautonomous differential equations and applications. *Nonlinear Anal*. 2009;70:4079-4085.
45. Coronel A, Maulén C, Pinto M, Sepúlveda D. Almost automorphic delayed differential equations and Lasota-Wazewska model. *Discrete Contin Dyn Syst*. 2017;37(4):1959-1977.
46. Coppel W. *Stability and Asymptotic Behaviour of Differential Equations*. D C Heath: Boston, 1965.
47. Xia Z, Fan M. Weighted Stepanov-like pseudo almost automorphy and applications. *Nonlinear Anal*. 2012;75:2378-2397.
48. Liang J, Maniar L, N'Guérékata G, Xiao T. Existence and uniqueness of $c^{(n)}$-almost periodic solutions to some ordinary differential equations. *Nonlinear Anal*. 2007;66:1899-1910.
49. Liang J, Zhang J, Xiao T-J. Composition of pseudo almost automorphic and asymptotically almost automorphic functions. *J Math Anal Appl*. 2008;340:1493-1499.

How to cite this article: Pinto M, Vidal C. Pseudo–almost-periodic solutions for delayed differential equations with integrable dichotomies and bi–almost-periodic Green functions. *Math Meth Appl Sci*. 2017;1–15. https://doi.org/10.1002/mma.4507