Observability and left invertibility of nonlinear discrete time systems with under-sampling

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Abstract

This paper deals with observability and left-invertibility of under-sampled systems. The concept of under-sampling of a discrete time system is introduced in the context of system theory for the first time in this article. Under-sampled systems are explained under a standard realization that allows to apply system theory. In particular, observability and left invertibility of these systems is investigated. It is shown that rewriting the under-sampled system under a standard form allows to characterize completely the observability and the left invertibility of the original system. One of the potential applications concerns the decrease of the number of output measurements without loss of observability of the original system.

Keywords: nonlinear systems, discrete time systems, under-sampling, observability, system theory

1. Introduction

The phenomenon of under-sampling appears in numerous engineering applications. Typically, under-sampling of the outputs is encountered in real-time controlled systems, where the sensors are connected by a network, subject to failures: the frames (i.e. the embedded output measurements) can be lost in a completely irregular way. Another example is the case of a sensor with temporal failures (e.g. supply problems), or a bloated network, where information packages can be arbitrarily lost. If the processing time of a system needs to be reduced, then an under-sampling may also be of interest since less data will be processed. An overview of networked control systems can be found in [1], where a stochastic approach has been applied. The lossy channel has been modelled by a stochastic process " theta " modelling the dropout, in order to do a state estimation over a lossy network. In [2] the stability of linear time-invariant sampled data system with uncertain and variable sampling intervals has been analyzed using Lyapunov functions.

Another application where the under-sampling can be useful are chaos-based secure communications. The chaotic carrier is chosen deliberately according to a chaotic sampling of the generator's output, i.e. not all output data are used for the transmission [3]. In all the above examples, the output information arrives at irregular, and unpredictable instants. However, the control (or transmission) system must still be reliable, i.e. able to generate the correct action, by reconstructing the missing samples.

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Such notions are popular in signal processing, for example in information compression, although no formal generally accepted concepts have been defined to date. For instance, an under-sampling [4] — or subsampling [5], downsampling [6] — of a discrete signal generates signal that looses some information according to a rule whereas an over-sampling [7] — or upsampling [8], interpolation [9] — adds some points between two instants according to a rule.

In the rest of the paper, we will focus on discrete time systems and the notion of under-sampling, which is sketched in Figure 1(b). More precisely, the figure shows the evolution of the time of a signal, indexed by time n and the one of an under-sampling signal, indexed by instant k. Figure 1(a) compares the normal evolution of the time n and the evolution of the time k of an over-sampled signal.

(a) Over-sampling

(b) Under-sampling

Figure 1: over and under-sampling time representation

Both concepts of over-sampling and under-sampling are widely used in signal processing. In opposition to the main stream of the current literature, a system theoretic approach is introduced in this paper; it is shown to be instrumental to analyze the above challenging problems and to derive appropriate solutions. The new approach is of paramount importance since it leads to a new formalism that highlightens both analysis and synthesis of such under-sampled systems. The possibility to reconstruct — or not — the missing output samples of the original system is naturally recasted in terms of observability matrices and left invertibility. As for the synthesis, if the under-sampled system is still observable, the number of output measurements can be reduced consequently without loss of observability. The possibility to rewrite into a so-called standard form is naturally stated as a realization problem. To our best knowledge, paralleling the open problem of reconstructed missing signal samples with such system theoretic notions as realization and observability has never been studied before, and therefore shall be introduced for the first time in this paper.

Sampling of continuous time systems is still studied including the case of non-uniform sampling [10, 11, 12] whereas under-sampling of discrete time systems has surprisingly never been considered to date. The first pioneering results on this topic are established in this paper. A realization study of under-sampled systems allows to apply systems theory and to announce new results. Amongst, special cases of constant, periodic or random, known or unknown samplings shall be considered. Full straight forward characterisations are derived from the observation in such special cases and surprisingly, they have not been reported in the current literature. Preliminary results of this work are available in [13].

This paper is organized as follows: the concept of under-sampling is formalized and the observability and left invertibility problems for the non-standard discrete time system under under-sampling is stated in section two. Section three recasts characterization of observability in terms of realization. Section four examines the observability of systems for different kinds of under-sampling.

2. Preliminaries

Consider a classical discrete time system:

$$\begin{cases} x(n+1) = f(x(n), u(n)) \\ y(n) = h(x(n)) \end{cases} \tag{1}$$

where $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ and $h : \mathbb{R}^N \to \mathbb{R}$ are analytic. x represents the states of the system, u its input and y its output. For simplicity, u and y are considered as being one-component vectors but this article can be generalised to multi-component vectors.

Under-sampled systems are considered in this paper so to define it, let define the under-sampling. This notion is usually used in signal processing theory. A signal s_e is said to be under-sampled of s if there exists an under-sampling step $p \in \mathbb{N}$ such that $s_e(n) = s(pn)$ [14]. This paper consider a more general case where p vary.

Definition 2.1 (Under-sampled signal)**.** Let $s : \mathbb{N} \to \mathbb{R}^N$ a signal.

$s_e : \mathbb{N} \to \mathbb{R}^N$ is said to be an under-sampled signal of s if the sequence $(s_e(n))_{n\in\mathbb{N}}$ is a subsequence of $(s(n))_{n\in\mathbb{N}}$.

s and s_e denote the same physical signal. Therefore, up to the rest of the paper, an under-sampled signal would be denoted by the same variable s. However, to avoid confusion, $s(n)$ would denote the original signal and $s[k]$ would denote the under-sampled signal.

Some definitions should be necessary to study under-sampled systems. Let first introduce the under-sampling.

Definition 2.2 (Under-sampling)**.** Let $s(.) : \mathbb{N} \to \mathbb{R}^N$ a signal and $s[.] : \mathbb{N} \to \mathbb{R}^N$ an under-sampled signal of $s(.)$.

$q : \mathbb{N} \to \mathbb{N}$ that verifies

$$s[k] = s(q[k]),\ \ k \in \mathbb{N}$$

is called under-sampling of signal $s(.)$.

Proposition 2.1 (Under-sampling)**.** *An under-sampling $k \to n = q[k]$, $q : \mathbb{N} \to \mathbb{N}$ is an injective and strictly increasing map.*

Proof. $(s[k])_{k\in\mathbb{N}}$ is a subsequence of $(s(n))_{n\in\mathbb{N}}$ and $s[k] = s(q[k]),\ k \in \mathbb{N}$ so q is an injective and strictly increasing map. □

These definitions allows to define under-sampled systems. The action of the under-sampling q on (1) results into:

$$\begin{cases} x(n+1) = f(x(n), u(n)) \\ y[k] = h(x(q[k])) \end{cases} \tag{2}$$

The particularity of this system is that it runs under two different time scales k and n: the state x evolves with the index n whereas the output y evolves with the index k. Indeed, the output of system (1) does not exist for all instants n but only at the instant $q[k]$. The limit case is attained when q is identity, there would not have no under-sampling.

Let us introduce the under-sampling step $p[k]$. This is instrumental for writing the realization of the under-sampled system under a standard form.

Definition 2.3 (Under-sampling step)**.** The step p of an under-sampling q is defined by

$$p[k] = q[k+1] - q[k], k \in \mathbb{N}$$

A visualisation of both under-samplings and under-sampling steps is in Figure 2.

Definition 2.4 (Left invertibility)**.** The discrete-time undersampled system (2) is said to be left invertible if the input $u(q[k])$ can be expressed as a function of the state vector, the output, and its forward iterates:

$$u[k] = G(x[k], y[k], y[k+1], \ldots) \tag{3}$$

Figure 2: under-sampling and under-sampling steps

Definition 2.5 (Relative degree)**.** The relative degree $\rho[k]$ of the discrete-time under-sampled system is defined as:

$$\rho := \min\{j \in \mathbb{N} \mid \frac{\partial y[k+j]}{\partial u[k]} \not\equiv 0\} \tag{4}$$

Note that $\rho[k]$ is necessarily $\leq p[k]$.

Example 2.1. Consider the following system with the under-sampling $p[k] = 8$.

$$\begin{cases} x_1(n+1) = x_2(n) \\ x_2(n+1) = x_3(n) \\ x_3(n+1) = x_4(n) \\ x_4(n+1) = u(n) \\ y[k] = x_1(8k) = x_1[k] \end{cases}$$

Here, $\rho = 1$ and $\rho < p$.

It should be noted that if the undersampling $p[k]$ is not constant, than the relative degree may not be constant either, and depends on the time instant $[k]$.

Problem statement 2.1 (Observability)**.** *Given an under-sampled system* (2)*, find if possible a function F and two integers ν_1 and ν_2 such that, locally,*

$$x[k] = F(y[k], y[k+1], \ldots, y[k+\nu_1], u(q[k]), u(q[k+1]), \ldots, u(q[k+\nu_2]))$$

This is an observability problem for the non standard system (2)*.*

Although there exists a tremendous literature on observation and observability of non-linear systems, no obvious solution of the above problem can be derived. Before defining observability, let firsty define distinguibility.

Definition 2.6 (Distinguishability of under-sampled systems)**.** Let $(x_0, \tilde{x}_0) \in \mathbb{R}^N$. Let y and $\tilde{y}$ denote respectively the output of the under-sampled system (2) (for the same under-sampling $q[k]$) with initial conditions x_0 and $\tilde{x}_0$. The states x_0 and $\tilde{x}_0$ are said to be distinguishable if there exists an integer k and an input $u[k]$ such that $\tilde{y}[k] \neq y[k]$.

Definition 2.7 (Observability of under-sampled systems)**.** An under-sampled system (2) is said to be locally observable in x_0 if there exists a neighborhood F of x_0 such that for all states $\tilde{x} \in F \backslash \{x_0\}$, $\tilde{x}$ and x_0 are distinguishable.

Therefore, if there exists an integer $k \in \mathbb{N}$ such that $\tilde{y}[k] \neq y[k]$ for $q[k] \geq n_0$ where y denotes the output of the system with initial condition x_0 and $\tilde{y}$ denotes the system with an initial condition $\tilde{x}$.

Computable criteria for observability of under-sampled systems are reviewed in Section 4. According to this definition, local observability of systems without under-sampling have been characterized. System (1) is locally observable if the observability matrix O_1 below has rank N.

$$O_1 = \begin{pmatrix} \frac{\partial y(n)}{\partial x_1(n)} & \frac{\partial y(n)}{\partial x_2(n)} & \cdots & \frac{\partial y(n)}{\partial x_N(n)} \\ \frac{\partial y(n+1)}{\partial x_1(n)} & \frac{\partial y(n+1)}{\partial x_2(n)} & \cdots & \frac{\partial y(n+1)}{\partial x_N(n)} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y(n+N-1)}{\partial x_1(n)} & \frac{\partial y(n+N-1)}{\partial x_2(n)} & \cdots & \frac{\partial y(n+N-1)}{\partial x_N(n)} \end{pmatrix} \tag{5}$$

$$rank(O_1) = N$$

The under-sampling can lose the observability. The following example show this:

Example 2.2. Consider the following system in the same form as (1).

$$\begin{cases} x_1(n+1) = x_2(n) \\ x_2(n+1) = x_3(n) \\ x_3(n+1) = x_1(n) \\ y[k] = x_1[k] \end{cases}$$

$y(3k) = x_1(0)$, $y(3k+1) = x_2(0)$ and $y(3k+2) = x_3(0)$. So this system is completely observable.

Now apply an under-sampling every three iterations: $q[k] = 3k$. $\forall k \in \mathbb{N}, y[k] = x_1[k]$; x_2 and x_3 are independent and never modify the output so both components x_2 and x_3 are not observable. So an under-sampling can make a system unobservable whereas the original system can be observable.

Now consider an other sampling $q(k) = 2k$. The output is defined by $y(3k) = x_1(0)$, $y(3k+1) = x_3(0)$ and $y(3k+2) = x_2(0)$. This system is completely observable so an under-sampling does not necessary make a system loose its observability.

Moreover, decreasing the number of output samples can preserve the observability of the original system, therefore lower number of measurements (i.e. less memory processing time) could be sufficient to observe the system states.

3. Realization of an under-sampled system under standard form

Control theory does not give any result to analyze non-uniformly sampled systems. The goal of this section is to find out the conditions under which an under-sampled system can be written under a standard form, more precisely, this is a realization problem using only one single time scale k. Two cases of realization shall be considered hereafter: a standard and a generalized one.

3.1. Standard realization

Problem statement 3.1 (Realization)**.** *Given* (2)*, find a realization* (6) *whenever it exists.*

$$\begin{cases} z[k+1] = g(z[k], \bar{u}[k]) \\ y[k] = h'(z[k]) \end{cases} \tag{6}$$

where $\bar{u}$ is a new input function depending on u, in the general case $\bar{u}[k] = [u(q[k]), u(q[k]+1), u(q[k]+2), ..., u(q[k+1]-1)$. z is a priori different from x and their dimensions are not necessary equal. A solution to problem 3.1 does not always exist if $\bar{u} = u$ as shown by the following example:

Example 3.1. Consider the following system with the under-sampling $q[k] = 2k$, $p[k] = 2$:

$$\begin{cases} x(n+1) = u(n) \\ y[k] = x(2k) = x(q[k]) = x[k] \end{cases}$$

The corresponding input-output equation is

$$y[k] = x(2k) = u(2k-1)$$

or equivalently

$$y[k+1] = x(4k) = u(4k-1)$$

$\mathcal{I}m(q) = 2\mathbb{N}$ so $\forall k \in \mathbb{N}, 2k-1 \notin \mathcal{I}m(q)$ so there does not exist any k' such that $u(2k-1) = u(q[k']) = u[k']$. Finally, problem 3.1 is not solvable for this system.

The following example show a system that can be written under a realization (6).

Example 3.2. Consider the following system with the under-sampling $q(k) = 2k$, $p[k] = 2$:

$$\begin{cases} x_1(n+1) = x_2(n) \\ x_2(n+1) = u(n) \\ y[k] = x_1(2k) \end{cases}$$

It can be reformulated under form (6) as

$$\begin{cases} z[k+1] = u[k] \\ y[k] = z[k] \end{cases}$$

Consider an application $h_k = h_O f^q[k]$ that links y to the initial state $x(0)$ and the inputs u:

$$y[k] = h_k(x(0); u(0); u(1); u(2); \ldots; u(q[k]-1))$$

Theorem 3.1. *System* (2) *can be written under a standard form* (6) *with* $\bar{u}[k] = u[k]$ *if and only if*

$$\forall n \in \mathbb{N} \setminus \mathcal{I}m(q),\ \forall k \in \mathbb{N}^* |\ \frac{\partial h_k[k]}{\partial u(n)} = 0$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

Proof. Consider f^i defined by induction: $f^1(x(n), u(n)) = f(x(n), u(n))$ and

$$f^2(x(n), u(n), u(n+1)) = f^1(x(n+1), u(n+1)) = f^1(f^1(x(n), u(n)), u(n+1))$$
$$f^3(x(n), u(n), u(n+1), u(n+2)) = f^1(f^2(x(n), u(n), u(n+1)), u(n+2))$$
$$f^{i+1}(x(n), u(n), \ldots, u(n+i)) = f^1(f^i(x(n), u(n), \ldots, u(n+i-1)), u(n+i))$$

Consider system (2) which satisfies condition of theorem 3.1. The following system describes the same output y.

$$\begin{cases} z(n+1) = f(z(n), \bar{u}(n)) \\ y[k] = h(z(q[k])) \end{cases}$$

with $\bar{u}(n) = \begin{cases} u(n) & \text{if } n \in \mathcal{I}m(q) \\ 0 & \text{else} \end{cases}$.

so the following realization is equivalent to the previous one

$$\begin{cases} z[k+1] = f^{p[k]}(z[k], u[k], 0, 0, \ldots, 0) \\ y[k] = h(z[k]) \end{cases}$$

This is a realization on the form of (6). □

From the theorem 3.1 it comes that

Corollary 3.1. *System* (2) *is left invertible in the sense of the definition 2.4 if and only if*

$$(i)\frac{\partial h_k[k]}{\partial u(n)} = 0 \quad \forall \quad n \notin \mathcal{I}m(q)$$

$$(ii)\exists k, \quad \frac{\partial h_k[k]}{\partial u(q[k])} \neq 0$$

Example 3.3. Consider the system defined in example 3.1 with $q(k) = 2k$.

$$\frac{\partial y(2k)}{\partial u(2k-1)} = 1, \;\; k \in \mathbb{N}^*$$

$\forall k \in \mathbb{N}^*, 2k-1 \notin \mathcal{I}m(q)$ so problem 3.1 is not solvable.

Example 3.4. Consider the system defined in example 3.2 with $q(k) = 2k$.

$$y[k] = u(2k-2), \;\; k \in \mathbb{N}^*$$

$\mathcal{I}m(q) = 2\mathbb{N}$ so $\forall k \in \mathbb{N}, 2k-2 \in \mathcal{I}m(q)$ so condition of Theorem 3.1 is satisfied.

Corollary 3.2. *Problem 3.1 is solvable if* (2) *is an autonomous system.*

Proof. If system (2) is an autonomous one,

$$\frac{\partial y[k]}{\partial u(n)} = 0, \;\; (k,n) \in \mathbb{N}^2$$

So conditions of Theorem 3.1 are verified.

An equivalent realization is then

$$\begin{cases} x[k+1] = f^{p[k]}(x[k]) \\ y[k] = h(x[k]) \end{cases}$$

□

Obviously, the condition is not necessary as shown by example (3.2).

Example 3.5. Consider the under-sampling $(q(0); q(1); q(2)) = (0; 3; 5)$, then $(p(0); p(1)) = (3; 2)$. The under-sampled system can be reformulated by

$$\begin{cases} x[1] = f \text{o} f \text{o} f(x[0]) \\ x[2] = f \text{o} f(x[1]) \\ \cdots \\ y[k] = h(x[k]) \end{cases}$$

3.2. Generalized realization

Not all systems can be written under realization (6). This section considers an other equivalent realization which is derived from some redefinition of the input.

Problem statement 3.2 (Realization)**.** *Find g and a prolongation $\bar{u}$ such that a realization* (7) *is equivalent to a given* (2) *whenever it exists.*

$$\begin{cases} z[k+1] = g(z[k], \bar{u}[k]) \\ y[k] = h'(z[k]) \end{cases} \tag{7}$$

Theorem 3.2. *System* (2) *has always a realisation* (7)*.*

Proof. Consider a $p[k]$-component vector $\bar{u}$ defined by

$$\bar{u}[k] = (u(q[k]); u(q[k]+1); u(q[k]+2); \ldots; u(q[k+1]-1))$$

Let f^i defined by induction: $f^1(x(n), u(n)) = f(x(n), u(n))$ and

$$f^2(x(n), u(n), u(n+1)) = f^1(x(n+1), u(n+1)) = f^1(f^1(x(n), u(n)), u(n+1))$$
$$f^3(x(n), u(n), u(n+1), u(n+2)) = f^1(f^2(x(n), u(n), u(n+1)), u(n+2))$$
$$f^{i+1}(x(n), u(n), \ldots, u(n+i)) = f^1(f^i(x(n), u(n), \ldots, u(n+i-1)), u(n+i))$$

Let g such that

$$g(x[k+1]) = f^{p[k]}(x[k], u(q[k]), u(q[k]+1), \ldots, u(q[k+1]-1))$$

Then, realizations (2) and (7) are equivalent. □

It can be noticed that if p only depends on the value of the state x, then the system is stationary.

Example 3.6. Consider the system defined in example 3.1 with $q(k) = 2k$. Let $\bar{u}[k]$ defined by $\forall k \in \mathbb{N}^*, \bar{u}[k] = (u(2k); u(2k+1))$.

Then, the system is equivalent to the following one:

$$\begin{cases} z[k+1] = \bar{u}_2[k] = u(2k+1) \\ y[k] = z[k] \end{cases}$$

The system (2) has been rewritten under form (6) by using the standard output function $y[k] = h(x[k])$ to allow to appear the same indexes in both state and output equations but not all systems can be written under this form. This has been solved by considering a more general equation (7) that characterises all under-sampled systems.

4. Observability and left-invertibility of under-sampled systems

After finding out an equivalent standard realization of any under-sampled systems, this section deals with observability and left-invertibility study whenever the conditions in Theorem (3.1) are fulfilled.

$$\begin{cases} x[k+1] = f^{p[k]}(x[k], u[k]) \\ y[k] = h(x[k]) \end{cases} \tag{8}$$

To our best knowledge, the observability and left-invertibility of such systems has never been investigated. This section considers two observability problems: the under-sampling is firstly considered as being known by the observer before considered as being unknown.

Without loss of generality, assume the system (1) to be left-invertible.

Theorem 4.1 (Existence of a left-inverse)**.** *Consider a non autonomous under-sampled system (8). The following three statements are equivalent:*

- *there exists a standard realization of the under-sampled system (8),*
- *the under-sampled system (8) is left-invertible,*
- *the under-sampling step p is a multiple of the relative degree ρ.*

Proof. If system (2) is left-invertible in the sense of definition 2.4, the input can be explicitely recovered and there exists a realization, and vice-versa. This proves the equivalence of the two first items in Theorem 4.1.

If the under-sampling step p is a multiple of the relative degree, then $q[k+1] - q[k] = m * \rho$ for some $m \in \mathbb{N}^*$. Thus

$$\frac{\partial h_{m*\rho}[k]}{\partial u(q[k])} \neq 0$$

and according to Corollary 3.1 the under-sampled system (8) is left invertible.

Conversely, it is obvious from Definition 2.5 that if the relative degree and the under sampling step are not multiple, then the under sampled system can not be left-invertible. □

Note that for the original system (1), a finite relative degree is necessary and sufficient for left-invertibility. This is absolutely not the case for the under-sampled system, where the relative degree varies as a function of the undersampling which can be unknown (aperiodic, chaotic etc).

Example 4.1. Consider the system (4.1), where the under-sampling is a multiple $m = 8$ of the relative degree: $q(k) = 8k$, $p[k] = 8$, $\rho = 1$.

Here $\rho \neq p$ since $\rho < p$. The input can be recovered after one iteration of the under-sampled system.

In the rest of this paper, it is assumed that the under-sampled system (8) is either autonomous or fulfils the conditions in Theorem 4.1.

Problem statement 4.1 (Observability of known under-sampling)**.** *Is a given under-sampled system* (9) *observable for an undistinguished known under-sampling?*

$$\begin{cases} x[k+1] = f^{p[k]}(x[k], u[k]) \\ y[k] = (h(x[k]); p[k]) \end{cases} \tag{9}$$

To do so, several under-sampling models are considered to make the under-sampling more and more complex. The first model is a classical one ($p = 1$ or $q = id$) as system (1). The second one is the periodic under-sampling (with constant p or linear q). Then, a regular under-sampling characterise a periodic step p. Finally, an undistinguished known under-sampling, a priori aperiodic, is considered.

The final aim of the work is to find out some observers for all of these under-sampling models. But before designing some observers, the conservation of the observability should be verified.

4.1. System without under-sampling ($p = 1$)

Consider the following system

$$\begin{cases} x(k+1) = g(x(k), u(k)) = f(x(k), u(k)) \\ y(k) = h(x(k)) \end{cases}$$

This kind of system is standard so the observability and the design of observers can be performed classically.

Observability criterion 4.1. *An under-sampled system* (9) *without under-sampling $q(k) = k$ or $p = 1$ is observable if and only if the rank of the observability matrix is N:*

$$rank \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_N} \\ \frac{\partial hof}{\partial x_1} & \frac{\partial hof}{\partial x_2} & \cdots & \frac{\partial hof}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial hof^{N-1}}{\partial x_1} & \frac{\partial hof^{N-1}}{\partial x_2} & \cdots & \frac{\partial hof^{N-1}}{\partial x_N} \end{pmatrix} = N$$

Remark 4.1. Consider a linear system (10) without under-sampling.

$$\begin{cases} x[k+1] = A^{p[k]}x[k] + Bu[k] \\ y[k] = (Cx[k]; p[k]) \end{cases} \tag{10}$$

The criterion becomes

$$\text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{pmatrix} = N$$

4.2. Periodic under-sampling

A periodic under-sampled system is written as:

$$\begin{cases} x[k+1] = g(x[k], u[k]) = f^p(x[k], u[k]) \\ y[k] = h(x[k]) \end{cases}$$

The step p is constant in this model. g is then f composed p times: $g = f^p$ so this problem is the same as for the system without any under-sampling but by changing f by f^p.

Observability criterion 4.2. *An under-sampled system* (9) *with a periodic under-sampling $q(k) = kp$ is observable if the rank of the observability matrix is N:*

$$rank \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_N} \\ \frac{\partial hof^p}{\partial x_1} & \frac{\partial hof^p}{\partial x_2} & \cdots & \frac{\partial hof^p}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial hof^{p(N-1)}}{\partial x_1} & \frac{\partial hof^{p(N-1)}}{\partial x_2} & \cdots & \frac{\partial hof^{p(N-1)}}{\partial x_N} \end{pmatrix} = N$$

Remark 4.2. For linear systems (10), the criterion becomes

$$\text{rank} \begin{pmatrix} C \\ CA^p \\ \vdots \\ CA^{p(N-1)} \end{pmatrix} = N$$

4.3. Alternate under-sampling

The alternate under-sampling is defined as being an under-sampled system with a step that alternates from two different steps p_1 and p_2. The system can be written under the form:

$$\begin{cases} x[k+1] = g(x[k], u[k]) \\ y[k] = h(x[k]) \end{cases}$$

with

$$\begin{cases} g(x[k]) = f^{p_1}(x[k], u[k]), \text{if } k \text{ is even} \\ g(x[k]) = f^{p_2}(x[k], u[k]), \text{if } k \text{ is odd} \end{cases}$$

The above equations are equivalent to (8) with $p(k) = p_1$ when k is even and $p(k) = p_2$ when k is odd.

Contrary to both afore under-sampled models, the state equation changes in the time so the study of observability must be different than in the usual case. Vidal, Chiuso et Soatto have studied observability of a particular trajectory of jump linear systems [15], which are linear systems whose state function change in the time.

The study of local observability of a system leads to consider a linearized system. By using the study of the observability of jump linear systems, the observability is determined by the rank of the observability matrix whose dimension is $(2(\sigma+1) \times N)$, where σ denotes the number of switches between the under-sampling steps p_1 and p_2.

Observability criterion 4.3. *An under-sampled system* (9) *without alternate under-sampling* p_1 *and* p_2*, as defined before, is locally observable if the rank of the observability matrix is* N*:*

$$rank \begin{pmatrix} \frac{\partial h}{\partial x_1} & \cdots & \frac{\partial h}{\partial x_N} \\ \frac{\partial hof^{p_1}}{\partial x_1} & \cdots & \frac{\partial hof^{p_1}}{\partial x_N} \\ \frac{\partial hof^{p_1+p_2}}{\partial x_1} & \cdots & \frac{\partial hof^{p_1+p_2}}{\partial x_N} \\ \frac{\partial hof^{2p_1+p_2}}{\partial x_1} & \cdots & \frac{\partial hof^{2p_1+p_2}}{\partial x_N} \\ \vdots & & \vdots \\ \frac{\partial hof^{\sigma p_1+\sigma p_2}}{\partial x_1} & \cdots & \frac{\partial hof^{\sigma p_1+\sigma p_2}}{\partial x_N} \\ \frac{\partial hof^{(\sigma+1)p_1+\sigma p_2}}{\partial x_1} & \cdots & \frac{\partial hof^{(\sigma+1)p_1+\sigma p_2}}{\partial x_N} \end{pmatrix} = N$$

Remark 4.3. For linear systems (10), the criterium becomes

$$\text{rank} \begin{pmatrix} C \\ CA^{p_1} \\ \vdots \\ CA^{(\sigma+1)p_1+\sigma p_2} \end{pmatrix} = N$$

4.4. General known under-sampling

In this section, the under-sampled step p evolves but is known for the observer.

$$\begin{cases} x[k+1] = g(x[k], u[k]) = f^{p[k]}(x[k], u[k]) \\ y[k] = (h(x[k]); p[k]) \end{cases} \tag{11}$$

Following the previous section, the criteria of observability can be generalized.

Proposition 4.1 (Observability)**.** *An under-sampled system* (11) *is locally observable if there exists an integer τ such that the rank of the observability matrix is N.*

$$rank \begin{pmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \cdots & \frac{\partial h}{\partial x_N} \\ \frac{\partial hof^{q[1]}}{\partial x_1} & \frac{\partial hof^{q[1]}}{\partial x_2} & \cdots & \frac{\partial hof^{q[1]}}{\partial x_N} \\ \frac{\partial hof^{q[2]}}{\partial x_1} & \frac{\partial hof^{q[2]}}{\partial x_2} & \cdots & \frac{\partial hof^{q[2]}}{\partial x_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial hof^{q[\tau]}}{\partial x_1} & \frac{\partial hof^{q[\tau]}}{\partial x_2} & \cdots & \frac{\partial hof^{q[\tau]}}{\partial x_N} \end{pmatrix} = N \tag{12}$$

Remark 4.4. For linear systems (10), the criterion becomes

$$\text{rank} \begin{pmatrix} C \\ CA^{q[1]} \\ \vdots \\ CA^{q[\tau]} \end{pmatrix} = N$$

The use of the integer τ is necessary for determining the observability because an under-sampled system defined by analytic functions f and h can be unobservable for an iteration $\tau - 1 > N$ and observable for an iteration τ. The following example illustrates this for a second order system defined by linear functions f and h.

Example 4.2. Consider the following system:

$$\begin{cases} x_1(n+1) = \frac{\sqrt{2}}{2}(x_1(n) + x_2(n)) \\ x_2(n+1) = \frac{\sqrt{2}}{2}(-x_1(n) + x_2(n)) \\ y[k] = x_1(q[k]) \end{cases}$$

Consider the under-sampling defined by: $(q[0]; q[1]; q[2]; q[3]) = (0; 4; 8; 9)$.

$$\begin{cases} x_1(n+4) = -x_1(n) \\ x_2(n+4) = -x_2(n) \end{cases}$$

So the observability matrix is:

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 1 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

Unlike the classical case, this second order system requires four iterations in order to be observable.

Proposition 4.2. *The ranks of the observability matrix O_1 of the system (1) and the observability matrix of O_2 of the under-sampled system (2) satisfy :*

$$rank(O_2) \leq rank(O_1)$$

Proof. Let O_1 be the observability matrix of system (1) without under-sampling and O_2 the observability matrix of the system with an under-sampling (2). The observability matrix of (1) is

$$O_1 = \begin{pmatrix} \frac{\partial y(n)}{\partial x_1(n)} & \frac{\partial y(n)}{\partial x_2(n)} & \cdots & \frac{\partial y(n)}{\partial x_N(n)} \\ \frac{\partial y(n+1)}{\partial x_1(n)} & \frac{\partial y(n+1)}{\partial x_2(n)} & \cdots & \frac{\partial y(n+1)}{\partial x_N(n)} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y(n+\sigma-1)}{\partial x_1(n)} & \frac{\partial y(n+\sigma-1)}{\partial x_2(n)} & \cdots & \frac{\partial y(n+\sigma-1)}{\partial x_N(n)} \end{pmatrix}$$

The under-sampled system (2) is defined as being (1) without some measurements. So the observability matrix O_2 is O_1 where some rows have been dropped out, thus $rank(O_2) \leq rank(O_1)$. □

4.5. Unknown under-sampling

Unlike the previous models, the under-sampling step p is now unknown for the observer. This is the most intricate case with respect to the characterization of observability. This problem is qualitatively different from the previous one where the under-sampling step was known.

Example 4.3. Let the second order system $f : x \mapsto Ax$ et $h : x \mapsto Cx$ where $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

$$A^n = \begin{pmatrix} \cos(\pi n\omega) & -\sin(\pi n\omega) \\ \sin(\pi n\omega) & \cos(\pi n\omega) \end{pmatrix}$$

The computation of the output iterates yields:

$$\Rightarrow \begin{cases} y[0] = x_1[0] \\ y[1] = \cos(\pi q[0]\omega)x_1[0] - \sin(\pi q[0]\omega)x_2[0] \\ y[2] = \cos(\pi q[1]\omega)x_1[0] - \sin(\pi q[1]\omega)x_2[0] \\ y[3] = \cos(\pi q[2]\omega)x_1[0] - \sin(\pi q[2]\omega)x_2[0] \\ \dots \end{cases}$$

The state $x_1[0]$ is easily obtained from the above. However, the state $x_2[0]$ can not be computed whenever $q[i]$ is unknown. The number of unknowns is always larger then the number of considered equations.

To circumvent this drawback, the state vector is extended with a new exogenous state variable: the under-sampling step. Namely the following problem is considered now.

Double state-step observability. Consider the following system.

$$\begin{cases} x[k+1] = f^{p[k]}(x[k], u[k]) \\ y[k] = h(x[k]) \end{cases} \tag{13}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$ and p is an unknown variable.

Problem statement 4.2 (Step and state observability of unknown under-sampling)**.** *Given the dynamical system* (13) *with an unknown under-sampling. Find if possible a function F and a σ such that*

$$(x[0]; p[1]; p[2]; \dots; p[\sigma - 1]) = F(y[0]; y[1]; \dots; y[\sigma])$$

So p, as well as the state x, must be observed. As a consequence, p becomes a component of the new state to observe. This will be explicitely stated in Proposition 4.4 below.

Definition 4.1 (local state and step observability)**.** A system is said to be locally state and step observable in x if $\exists \mathcal{V}(x), \forall \tilde{x} \in \mathcal{V}(x) \setminus \{x\}, \forall p : \mathbb{N} \to \mathbb{N}^*, \forall \tilde{p} : \mathbb{N} \to \mathbb{N}^*,$, (x, p) and $(\tilde{x}, \tilde{p})$ are distinguishable for an under-sampled system (13).

Example 4.4. Consider the following system:

$$\left\{\begin{array}{l} x_1(n+1) = x_2(n) \\ x_2(n+1) = \alpha x_2(n) \\ y[k] = h(x(q[k])) \end{array}\right.$$

where $0 < \alpha < 1$ and $x_2(0) > 0$.

The outputs are equal to:

$$\left\{\begin{array}{l} y[0] = x_1[0] \\ y[1] = \alpha^{q[0]-1} x_2[0] \\ y[2] = \alpha^{q[1]-1} x_2[0] \end{array}\right.$$

Global observability

This system is not globally state and step observable.

Let $\tilde{x}_2(0)$ being an other initial condition such that $x_2(0) = \alpha \tilde{x}_2(0)$, or equivalently, $x_2(0) = \tilde{x}_2(1)$.

Then let $\tilde{p}$ being the under-sampling defined such that $\tilde{q}[k] = q[k] + 1 \forall k \in \mathbb{N}$. It comes $\tilde{y}[1] = \alpha^{\tilde{q}[0]-1} \tilde{x}_2[0] = y[1]$. And $\tilde{y}[k] = y[k] \forall k \in \mathbb{N}$. So the system is globally state and step unobservable.

Local observability

More generally, a given initial condition $x(0)$ is undistinguishable with $\tilde{x}(0)$ if $x_1(0) = \tilde{x}_1(0)$ and $\tilde{x}_2(0) = \alpha^k x(0), k \in \mathbb{Z}$. So the system is locally observable.

Finally, the system (4.4) is not double state-step observable but is locally double state-step observable.

Linear systems

This section looks for conditions of unobservability of under-sampled linear system. Consider the under-sampled system (14) reformulated under the standard form (15).

$$\left\{\begin{array}{l} x(n) = A(x(n)) \\ y[k] = Cx(q[k]) \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l} x[k] = A^{q[k]}(x[k]) \\ y[k] = Cx[k] \end{array}\right. \tag{15}$$

Proposition 4.3. *If a linear system* (14) *is defined such that there exists an integer s such that A^s is similar to the right hand side of* (16) *and such that A and C verify* (4.5)*, then the under-sampled system* (14) *is unobservable for $q[k] = sk$.*

$$P^{-1} A^s P = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \tag{16}$$

$$CA^s = CP^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} P \tag{17}$$

Proof. Consider a linear system (14) which satisfies (16). It comes that for all $t \in \mathbb{N}^*$,

$$P^{-1}A^{ts}P = \begin{pmatrix} A_1^t & 0 \\ 0 & A_2^t \end{pmatrix}$$

Furthermore, if it satisfies (4.5), it comes

$$CA^{ts} = CP^{-1}\begin{pmatrix} A_1^t & 0 \\ 0 & 0 \end{pmatrix}P$$

Let D defined by

$$D = P^{-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

It comes that $CA^{ts}D = 0,\ t \in \mathbb{N}^*$. Then, there exists a periodic under-sampling $q[k] = sk$ such that the under-sampled system is unobservable. $\square$

The following example illustrates proposition .

Example 4.5. Consider a system defined by matrices A and C:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

A^n is calculable and it comes

$$A^n = \begin{pmatrix} \cos(\pi n\omega) & -\sin(\pi n\omega) \\ \sin(\pi n\omega) & \cos(\pi n\omega) \end{pmatrix} \mid \theta = \pi\omega$$

So, if $\omega \in \mathbb{Q}$, let $\omega = a/b, (a,b) \in \mathbb{N}^2$, there exists an integer n, $n = 2b$, such that

$$A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then the system is unobservable for sampling steps $p[k] \in 2b\mathbb{N}$.

This section we introduce an 'intelligent step sensor' to measure the unknown step by extending the state vector $x[k]$ with an additional state variable $z[k]$.

Proposition 4.4 (Intelligent Step Sensor)**.** *The dynamics of $z[k]$ is governed by the function $g(z[k])$, which plays the role of a step sensor: it is exogenous to the system and accounts explicitly for the step $p[k]$.*

$$\begin{cases} x[k+1] = f^{p[k]}(x[k]) \\ z[k+1] = \alpha^{p[k]} \\ y[k] = (h(x[k]), z[k]) \end{cases} \tag{18}$$

where $h : \mathbb{R}^n \times \mathbb{N}^* \to \mathbb{R} \times \mathbb{N}^*$, $\alpha > 1$, z is the extended state variable, and $p[k]$ is the under-sampling step. In this way the under-sampling step is now known. It is included in the output vector $y[k]$ and the problem falls in the case of undistinguished known under-sampling which has already been solved in section 4.4.

Conclusion

The open problems of observability and left invertibility of under-sampled systems have been defined and characterized. The notion of under-sampling has been introduced and formalized. This formalization has then permitted to consider the realization problem in order to equivalently rewrite an under-sampled system in a standard form from system theory point of view.

Under-sampled systems have been separated into two families: systems where the sampling is known and systems where the sampling is unknown. The study of observability in the context of known under-sampling has led to an extension of the classical observability criterion. Some examples have illustrated the case of unknown under-sampled systems. A potential application of this study is the decrease of required number of output measurements, without loss of observability of the original system.

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