Asian Journal of Control, Vol. 17, No. 1, pp. 1–9, January 2015
Published online in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.861

–Brief Paper–

IMPROVED STABILITY CRITERIA FOR LINEAR NEUTRAL TIME-DELAY SYSTEMS

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ABSTRACT

This paper is concerned with the problem of stability analysis of linear neutral time delay systems. First, a new simple Lyapunov-Krasovskii functional is constructed based on delay partitioning approach, then, a novel improved delay-dependent stability criterion is established to guarantee the asymptotic stability of the neutral system. The obtained stability criterion is expressed in terms of linear matrix inequalities, which can be verified easily by means of standard software. Numerical examples are given to illustrate the effectiveness of the proposed method and the advantage over some existing results.

Key Words: Neutral systems, stability criterion, delay partitioning approach, LMI.

I. INTRODUCTION

Since time delays can degrade system performance, and even result in instability, stability analysis and stabilization synthesis have been at the forefront of research on time delay systems in the recent decades [1–4]. In practice, time delays always appear in various engineering, biological, and economical systems [3–6]. Stability analysis and stabilization synthesis of neutral time-delay systems are more complex because the delays appear both in its states and in its derivatives of states. Many remarkable research results have been achieved in the literature [7–10].

It should be noted that all these stability criteria can be generally classified as delay-independent and delay-dependent types [11–13]. It is well known that the delay-dependent stability criteria, which utilize information of the upper and lower bounds of delays, are less conservative than delay-independent ones [10–15]. Obviously, there is an expectation to establish delay-dependent stability criteria with less conservatism. The Lyapunov functional approach is the main method employed to derive delay-dependent criteria [14–17]. It should be pointed out that the choices of specific Lyapunov functionals and bounding techniques are the main origin of reducing conservatism. Therefore, how to construct Lyapunov functionals, augmented vectors, new bounding techniques and derivation of the stability condition from the time-derivative of such a functional play key roles in the field of analysis of delay systems [16–20]. Towards this objective, the new cross-term bounding technique [21], parameterized neutral model transformation method [22,23], free weighting matrices technique [24] had contributed greatly to reduce the conservatism of stability criteria for time-delay systems, as discussed by Kwon *et al*. in [20]. To mention a few, He *et al*. employed free-weighting matrices to express the relationships between the terms in the Leibinz-Newton formula [25–26]. Kwon *et al*. established some delay-dependent stability criteria by using new augmented Lyapunov functional and delay-partitioning approaches [20]. More results can be found in [27–29] and references therein. Recently, Luis *et al*. [30] and Fridman *et al*. [23] have proposed less conservative criteria for time-delay systems by the introduction of new Lyapunov-Krasovskii functionals. A similar idea can be extended to deal with stability analysis and synthesis of neutral time-delay systems.

Inspired by the works of [30], in this paper we study the delay-dependent stability for a class of neutral time-delay systems based on the delay partitioning technique. The remainder of the paper is organized as follows. Section II gives problem formulation and some preliminary lemmas. In Section III, by dividing the delay interval into multiple segments, a Lyapunov functional is first constructed for the considered system, and then a new delay-dependent stability criterion is proposed by using some matrix inequality techniques. Numerical examples are provided in Section IV, followed by the concluding remarks, which are presented in Section V.

Notation. Throughout this paper, "T" stands for matrix transposition. "I" denotes the identity matrix of appropriate dimensions. "$P > 0$" means that P is positive definite. "*" represents the elements below the main diagonal of a symmetric matrix.

II. PROBLEM FORMULATION

Consider the following linear time-varying discrete neutral system described by

Received March 18, 2013; Revised October 8, 2013; Accepted January 1, 2014.
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The work is supported by National Natural Science Foundation of China (No.61304108, No.61273094, No.51307035) and the Project for Distinguished Young Scholars of the Basic Research Plan in Shenzhen City under Contract No. JCJ201110001. The authors are very thankful to the reviewers for their valuable suggestions and comments.

$$\begin{cases} \dot{x}(t) - C_d\dot{x}(t-\tau) = Ax(t) + A_d x(t-d(t)), & t > 0 \\ x(t) = \varphi(t), & t \in [-r, 0] \end{cases} \tag{1}$$

where $x(t)$ is the state vector; the matrices A, A_d, C_d are known constant matrices of appropriate dimensions, and the eigenvalue of the matrix C_d, $\rho(C_d)$ satisfies $\rho(C_d) < 1$. $\phi(t)(t \in [-r, 0])$ is the initial function which is continuous differentiable on $[-r, 0]$. The scalar τ is a positive constant time-delay. Time delay $d(t)$ is a continuously differentiable function satisfying the following conditions

$$0 \le h_{\min} \le d(t) \le h_{\max} \tag{2}$$

$$\mu_{min} \le \dot{d}(t) \le \mu_{max} \tag{3}$$

where $h_{\min}$, $h_{\max}$, $\mu_{\min}$, $\mu_{\max}$ are known positive real constants. In addition, we denote $r = \max\{h_{\max}, \tau\}$.

To facilitate further developments, we introduce the following lemma that will be frequently used in deriving the main results [30].

Lemma 1. Given scalars r_1, r_2 such that $(r_2 - r_1) \ge 0$, matrix $M = M^T > 0$, and any vectorial function $x : [r_1, r_2] \to \mathbb{R}^m$, we have:

$$(r_2 - r_1)\int_{r_1}^{r_2} x^{\mathrm{T}}(\beta) M x(\beta) d\beta \ge \left(\int_{r_1}^{r_2} x(\beta) d\beta\right)^{\mathrm{T}} M \left(\int_{r_1}^{r_2} x(\beta) d\beta\right)$$

III. MAIN RESULTS

This section discusses the problem of stable analysis of the system (1). Similarly to [23, 30], the time-varying delay interval $[h_{min}, h_{max}]$ is divided into two segments: $[h_1, h_2]$ and $[h_2, h_3]$, where $h_1 = h_{min}$, $h_3 = h_{max}$, $h_2 = \dfrac{h_1 + h_3}{2}$. With such an idea, a novel delay-dependent stability criterion for neutral systems is derived.

Theorem 1. Given positive scalars $0 \le h_m \le h_M$, the neutral system (1) with delay restrictions (2) and (3) is asymptotically stable if there exist matrices $P_i > 0 (i = 1, 2, 3)$, $R_1 > 0$, $R_3 > 0$, $(R_2 + R_3) > 0$, $Q_j > 0 (i = 1, 2, 3)$, $Z_k > 0 (k = 1, \ldots, 4)$, $S_1 > 0$, $S_2 > 0$, $\begin{bmatrix} T_{11} & T_{12} \\ * & T_{22} \end{bmatrix} \ge 0$, $\begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} \ge 0$, and

$$\begin{aligned} &R_1 + (1 - \mu_{min})R_2 + R_3 > 0 \\ &R_1 + (1 - \mu_{max})R_2 + R_3 > 0 \end{aligned} \tag{4}$$

and free-weighting matrices N_{1a}, N_{2a}, M_{1a}, M_{2a}, $a = (1, \ldots, 9)$, F_{11}, F_{12}, F_{21}, F_{22} with appropriate dimensions, such that the following LMIs (5) hold:

$$\begin{aligned} &\Omega_1\big|_{\dot{d}(t)\to\mu_{min}} < 0; \quad \Omega_1\big|_{\dot{d}(t)\to\mu_{max}} < 0; \\ &\Omega_2\big|_{\dot{d}(t)\to\mu_{min}} < 0; \quad \Omega_2\big|_{\dot{d}(t)\to\mu_{max}} < 0; \\ &\Omega_3\big|_{\dot{d}(t)\to\mu_{min}} < 0; \quad \Omega_3\big|_{\dot{d}(t)\to\mu_{max}} < 0; \\ &\Omega_4\big|_{\dot{d}(t)\to\mu_{min}} < 0; \quad \Omega_4\big|_{\dot{d}(t)\to\mu_{max}} < 0; \end{aligned} \tag{5}$$

where

$$\Omega_1 = \Gamma_{11} + \Phi_{11} + \Phi_{11}^{\mathrm{T}}; \Omega_2 = \Gamma_{12} + \Phi_{12} + \Phi_{12}^{\mathrm{T}}$$

$$\Omega_3 = \Gamma_{21} + \Phi_{21} + \Phi_{21}^{\mathrm{T}}; \Omega_4 = \Gamma_{22} + \Phi_{22} + \Phi_{22}^{\mathrm{T}}$$

$$\Phi_{11}^{\mathrm{T}} = \begin{bmatrix} 0 & \Theta_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -N_1^{\mathrm{T}} & M_1^{\mathrm{T}} & 0 & \Theta_{12}^{\mathrm{T}} \end{bmatrix}$$

$$\Phi_{12}^{\mathrm{T}} = \begin{bmatrix} 0 & \Theta_{11}^{\mathrm{T}} & 0 & 0 & 0 & 0 & -N_1^{\mathrm{T}} & M_1^{\mathrm{T}} & 0 & \Theta_{13}^{\mathrm{T}} \end{bmatrix}$$

$$\Phi_{21}^{\mathrm{T}} = \begin{bmatrix} 0 & \Theta_{21}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & -N_2^{\mathrm{T}} & M_2^{\mathrm{T}} & \Theta_{22}^{\mathrm{T}} \end{bmatrix}$$

$$\Phi_{22}^{\mathrm{T}} = \begin{bmatrix} 0 & \Theta_{21}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & -N_2^{\mathrm{T}} & M_2^{\mathrm{T}} & \Theta_{23}^{\mathrm{T}} \end{bmatrix}$$

$$\Gamma_{11} = \begin{bmatrix} \Omega_{11}^1 & F_{11}^{\mathrm{T}}A_d & \Pi_{11}^1 & \frac{1}{\tau}S_2 & F_{11}^{\mathrm{T}}C_d & \Omega_{16} & U_1 & 0 & 0 & 0 \\ * & \Omega_{22} & A_d^{\mathrm{T}}F_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33}^1 & 0 & F_{12}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77}^1 & Q_{12} & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88}^1 & \Omega_{89}^1 & 0 \\ * & * & * & * & * & * & * & * & \Omega_{99}^1 & 0 \\ * & * & * & * & * & * & * & * & * & \Pi_{b1}^1 \end{bmatrix}$$

$$\Gamma_{12} = \begin{bmatrix} \Omega_{11}^1 & F_{11}^{\mathrm{T}}A_d & \Pi_{12}^1 & \frac{1}{\tau}S_2 & F_{11}^{\mathrm{T}}C_d & \Omega_{16} & U_1 & 0 & 0 & 0 \\ * & \Omega_{22} & A_d^{\mathrm{T}}F_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33}^1 & 0 & F_{12}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77}^1 & Q_{12} & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88}^1 & \Omega_{89}^1 & 0 \\ * & * & * & * & * & * & * & * & \Omega_{99}^1 & 0 \\ * & * & * & * & * & * & * & * & * & \Pi_{a1}^1 \end{bmatrix}$$

$$\Gamma_{21} = \begin{bmatrix} \Omega_{11}^2 & F_{21}^{\mathrm{T}}A_d & \Pi_{21}^2 & \frac{1}{\tau}S_2 & F_{21}^{\mathrm{T}}C_d & \Omega_{16} & U_2 & 0 & 0 & 0 \\ * & \Omega_{22} & A_d^{\mathrm{T}}F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33}^2 & 0 & F_{22}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77}^2 & \Omega_{78}^2 & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88}^2 & -Q_{12} & 0 \\ * & * & * & * & * & * & * & * & -Q_{22} & 0 \\ * & * & * & * & * & * & * & * & * & \Pi_{b1}^2 \end{bmatrix}$$

$$\Gamma_{22}=\begin{bmatrix}\Omega_{11}^2 & F_{21}^{\mathrm{T}}A_d & \Pi_{22}^2 & \frac{1}{\tau}S_2 & F_{21}^{\mathrm{T}}C_d & \Omega_{16} & U_2 & 0 & 0 & 0\\ * & \Omega_{22} & A_d^{\mathrm{T}}F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & \Omega_{33}^2 & 0 & F_{22}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0\\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0\\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0\\ * & * & * & * & * & * & \Omega_{77}^2 & \Omega_{78}^2 & 0 & 0\\ * & * & * & * & * & * & * & \Omega_{88}^2 & -Q_{12} & 0\\ * & * & * & * & * & * & * & * & -Q_{22} & 0\\ * & * & * & * & * & * & * & * & * & \Pi_{a1}^2\end{bmatrix}$$

and, for $\alpha \in \{1, 2\}$,

$$N_\alpha=\begin{bmatrix}N_{\alpha 1} & N_{\alpha 2} & N_{\alpha 3} & N_{\alpha 4} & N_{\alpha 5} & N_{\alpha 6} & N_{\alpha 7} & N_{\alpha 8} & N_{\alpha 9} & 0\end{bmatrix}$$

$$M_\alpha=\begin{bmatrix}M_{\alpha 1} & M_{\alpha 2} & M_{\alpha 3} & M_{\alpha 4} & M_{\alpha 5} & M_{\alpha 6} & M_{\alpha 7} & M_{\alpha 8} & M_{\alpha 9} & 0\end{bmatrix}$$

$$\Theta_{\alpha 1}=N_\alpha-M_\alpha;\quad \Theta_{\alpha 2}=(h_{\alpha+1}-h_\alpha)M_\alpha;\quad \Theta_{\alpha 3}=(h_{\alpha+1}-h_\alpha)N_\alpha;$$

$$U_1=\frac{1}{h_1}\left[R_1+\left(1-\dot d(t)\right)R_2+R_3\right];U_2=\frac{1}{h_1}\left[\left(1-\dot d(t)\right)R_1+R_2+R_3\right];$$

$$\begin{aligned}\Omega_{11}^1=&\frac{\dot d(t)}{h_2-h_1}(P_1-P_2)+Q_1+Q_3+T_{11}\\ &-Z_1-U_1-\frac{1}{\tau}S_2+F_{11}^{\mathrm{T}}A+A^{\mathrm{T}}F_{11}\end{aligned}$$

$$\Pi_{11}^1=P_2+A^{\mathrm{T}}F_{12}-F_{11}^{\mathrm{T}};\Pi_{12}^1=P_1+A^{\mathrm{T}}F_{12}-F_{11}^{\mathrm{T}}$$

$$\Omega_{16}=Z_1+T_{12};\Omega_{22}=-\left(1-\dot d(t)\right)Q_2;\Omega_{44}=-Q_3-\frac{1}{\tau}S_2$$

$$\begin{aligned}\Omega_{33}^1=&\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)+(h_2-h_1)Z_3+(h_3-h_2)Z_4\\ &+h_2(R_1+R_2+R_3)+S_1+\tau S_2-F_{12}-F_{12}^{\mathrm{T}}\end{aligned}$$

$$\Omega_{66}=T_{22}-T_{11}-Z_1-Z_2;\Omega_{67}=Z_2-T_{12};$$
$$\Omega_{77}^1=-Q_1+Q_2-T_{22}+Q_{11}-Z_2-U_1$$

$$\Omega_{88}^1=Q_{22}-Q_{11}-\frac{1}{h_3-h_2}Z_4;\Omega_{89}^1=-Q_{12}+\frac{1}{h_3-h_2}Z_4;$$
$$\Omega_{99}^1=-Q_{22}-\frac{1}{h_3-h_2}Z_4$$

$$\Pi_{a1}^1=-(h_2-h_1)\left(Z_3+R_1+\left(1-\dot d(t)\right)R_2+R_3\right);$$
$$\Pi_{b1}^1=-(h_2-h_1)(Z_3+R_1+R_3)$$

$$\begin{aligned}\Omega_{11}^2=&\frac{\dot d(t)}{h_3-h_2}(P_3-P_1)+Q_1+Q_3\\ &+T_{11}-Z_1-U_2-\frac{1}{\tau}S_2+F_{21}^{\mathrm{T}}A+A^{\mathrm{T}}F_{21}\end{aligned}$$

$$\Pi_{21}^2=P_1+A^{\mathrm{T}}F_{22}-F_{21}^{\mathrm{T}};\Pi_{22}^2=P_3+A^{\mathrm{T}}F_{22}-F_{21}^{\mathrm{T}}$$

$$\begin{aligned}\Omega_{33}^2=&\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)+(h_2-h_1)Z_3+(h_3-h_2)Z_4+h_2(R_2+R_3)\\ &+h_3R_1+S_1+\tau S_2-F_{22}-F_{22}^{\mathrm{T}}\end{aligned}$$

$$\begin{aligned}\Omega_{77}^2=&-Q_1+Q_2-T_{22}+Q_{11}-Z_2-U_2\\ &-\frac{1}{h_2-h_1}\left(Z_3+\left(1-\dot d(t)\right)R_1+R_2+R_3\right)\end{aligned}$$

$$\Omega_{78}^2=Q_{12}+\frac{1}{h_2-h_1}\left[Z_3+\left(1-\dot d(t)\right)R_1+R_2+R_3\right]$$

$$\Omega_{88}^2=Q_{22}-Q_{11}-\frac{1}{h_2-h_1}\left[Z_3+\left(1-\dot d(t)\right)R_1+R_2+R_3\right]$$

$$\Pi_{a1}^2=-(h_3-h_2)\left(Z_4+\left(1-\dot d(t)\right)R_1\right);\Pi_{b1}^2=-(h_3-h_2)Z_4.$$

Proof. Let us choose a Lyapunov-Krasovskii functional candidate as follows:

$$V(t)=\sum_{i=1}^{7}V_i(t) \tag{6}$$

where

$$V_1(t)\big|_{d(t)<h_2}=x^{\mathrm{T}}(t)\left(\frac{d(t)-h_1}{h_2-h_1}P_1+\frac{h_2-d(t)}{h_2-h_1}P_2\right)x(t)$$

$$V_1(t)\big|_{d(t)>h_2}=x^{\mathrm{T}}(t)\left(\frac{d(t)-h_2}{h_3-h_2}P_3+\frac{h_3-d(t)}{h_3-h_2}P_1\right)x(t)$$

$$\begin{aligned}V_2(t)=&\int_{t-h_1}^{t}x^{\mathrm{T}}(s)Q_1x(s)ds+\int_{t-d(t)}^{t-h_1}x^{\mathrm{T}}(s)Q_2x(s)ds\\ &+\int_{t-\tau}^{t}x^{\mathrm{T}}(s)Q_3x(s)ds\end{aligned}$$

$$V_3(t)=\int_{t-\frac{h_1}{2}}^{t}\begin{bmatrix}x(s)\\ x\left(s-\frac{h_1}{2}\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}T_{11} & T_{12}\\ * & T_{22}\end{bmatrix}\begin{bmatrix}x(s)\\ x\left(s-\frac{h_1}{2}\right)\end{bmatrix}ds$$

$$V_4(t)=\int_{t-h_3}^{t-h_2}\begin{bmatrix}x(s)\\ x(s-h_2+h_1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q_{11} & Q_{12}\\ * & Q_{22}\end{bmatrix}\begin{bmatrix}x(s)\\ x(s-h_2+h_1)\end{bmatrix}ds$$

$$V_5(t)=\frac{h_1}{2}\int_{-\frac{h_1}{2}}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)dsd\theta+\frac{h_1}{2}\int_{-h_1}^{-\frac{h_1}{2}}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)dsd\theta$$
$$+\int_{-h_2}^{-h_1}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_3\dot{x}(s)dsd\theta+\int_{-h_3}^{-h_2}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)Z_4\dot{x}(s)dsd\theta$$

$$V_6(t)\big|_{d(t)<h_2}=\int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)dsd\theta+\int_{-h_2}^{-d(t)}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_3\dot{x}(s)dsd\theta+\int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)(R_2+R_3)\dot{x}(s)dsd\theta$$

$$V_6(t)\big|_{d(t)>h_2}=\int_{-d(t)}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)dsd\theta+\int_{-h_2}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)(R_2+R_3)\dot{x}(s)dsd\theta$$

$$V_7(t)=\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)S_1\dot{x}(s)ds+\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)dsd\theta$$

V_1–V_6 in (6) are chosen as (5) in [30]. V_7 in (6) is specially constructed for neutral time-delay systems.

First, consider $d(t)\neq h_2$. Taking the time derivative of the LKF (6) with $h_1\le d(t)<h_2$, we have

$$\dot{V}_1(t)\big|_{d(t)<h_2}=2x^{\mathrm{T}}(t)\left(\frac{d(t)-h_1}{h_2-h_1}P_1+\frac{h_2-d(t)}{h_2-h_1}P_2\right)\dot{x}(t)+x^{\mathrm{T}}(t)\frac{\dot{d}(t)}{h_2-h_1}(P_1-P_2)x(t)$$

$$\dot{V}_2(t)=x^{\mathrm{T}}(t)(Q_1+Q_3)x(t)-x^{\mathrm{T}}(t-h_1)Q_1x(t-h_1)+x^{\mathrm{T}}(t-h_1)Q_2x(t-h_1)-\left(1-\dot{d}(t)\right)x^{\mathrm{T}}(t-d(t))Q_2x(t-d(t))-x^{\mathrm{T}}(t-\tau)Q_3x(t-\tau)$$

$$\dot{V}_3(t)=\begin{bmatrix}x(t)\\ x\left(t-\frac{h_1}{2}\right)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}T_{11}&T_{12}\\ *&T_{22}\end{bmatrix}\begin{bmatrix}x(t)\\ x\left(t-\frac{h_1}{2}\right)\end{bmatrix}-\begin{bmatrix}x\left(t-\frac{h_1}{2}\right)\\ x(t-h_1)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}T_{11}&T_{12}\\ *&T_{22}\end{bmatrix}\begin{bmatrix}x\left(t-\frac{h_1}{2}\right)\\ x(t-h_1)\end{bmatrix}$$

$$\dot{V}_4(t)=\begin{bmatrix}x(t-h_1)\\ x(t-h_2)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q_{11}&Q_{12}\\ *&Q_{22}\end{bmatrix}\begin{bmatrix}x(t-h_1)\\ x(t-h_2)\end{bmatrix}-\begin{bmatrix}x(t-h_2)\\ x(t-h_3)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}Q_{11}&Q_{12}\\ *&Q_{22}\end{bmatrix}\begin{bmatrix}x(t-h_2)\\ x(t-h_3)\end{bmatrix}$$

$$\dot{V}_5(t)=\dot{x}^{\mathrm{T}}(t)\left[\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)+(h_2-h_1)Z_3+(h_3-h_2)Z_4\right]\dot{x}(t)$$
$$-\frac{h_1}{2}\int_{t-\frac{h_1}{2}}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds-\frac{h_1}{2}\int_{t-h_1}^{t-\frac{h_1}{2}}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)ds-\int_{t-h_2}^{t-h_1}\dot{x}^{\mathrm{T}}(s)Z_3\dot{x}(s)ds-\int_{t-h_3}^{t-h_2}\dot{x}^{\mathrm{T}}(s)Z_4\dot{x}(s)ds$$

$$\dot{V}_6(t)\big|_{d(t)<h_2}=\dot{x}^{\mathrm{T}}(t)\left[h_2R_1+d(t)(R_2+R_3)+(h_2-d(t))R_3\right]\dot{x}(t)-\int_{t-h_2}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)ds-\left(1-\dot{d}(t)\right)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)(R_2+R_3)\dot{x}(s)ds$$
$$-\dot{d}(t)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)R_3\dot{x}(s)ds-\int_{t-h_2}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)R_3\dot{x}(s)ds$$

$$\dot{V}_7(t)=\dot{x}^{\mathrm{T}}(t)(S_1+\tau S_2)\dot{x}(t)-\dot{x}^{\mathrm{T}}(t-\tau)S_1\dot{x}(t-\tau)-\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)S_2\dot{x}(s)ds$$

For $h_1\le d(t)<h_2$, one has

$$\int_{t-h_3}^{t}f(s)ds=\int_{t-h_1}^{t}f(s)ds+\int_{t-d(t)}^{t-h_1}f(s)ds+\int_{t-h_2}^{t-d(t)}f(s)ds+\int_{t-h_3}^{t-h_2}f(s)ds \tag{7}$$

According to (7), we lead to

$$\begin{aligned}&\dot{V}_5(t)+\dot{V}_6(t)\big|_{d(t)<h_2}\\ &\le-\frac{h_1}{2}\int_{t-\frac{h_1}{2}}^{t}\dot{x}^{\mathrm{T}}(s)Z_1\dot{x}(s)ds-\frac{h_1}{2}\int_{t-h_1}^{t-\frac{h_1}{2}}\dot{x}^{\mathrm{T}}(s)Z_2\dot{x}(s)ds\\ &-\int_{t-h_1}^{t}\dot{x}^{\mathrm{T}}(s)\left(R_1+\left(1-\dot{d}(t)\right)R_2+R_3\right)\dot{x}(s)ds\\ &-\int_{t-d(t)}^{t-h_1}\dot{x}^{\mathrm{T}}(s)\left(Z_3+R_1+\left(1-\dot{d}(t)\right)R_2+R_3\right)\dot{x}(s)ds\\ &-\int_{t-h_2}^{t-d(t)}\dot{x}^{\mathrm{T}}(s)(Z_3+R_1+R_3)\dot{x}(s)ds\\ &-\int_{t-h_3}^{t-h_2}\dot{x}^{\mathrm{T}}(s)Z_4\dot{x}(s)ds+\dot{x}^{\mathrm{T}}(t)\left[\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)\right.\\ &\left.+(h_2-h_1)Z_3+(h_3-h_2)Z_4+h_2(R_1+R_2+R_3)\right]\dot{x}(t)\end{aligned} \tag{8}$$

By Jensen's inequality (Lemma 1), it can be seen that:

$$\begin{aligned}&\dot{V}_5(t)+\dot{V}_6(t)\big|_{d(t)<h_2}\\ &\le-\left[x(t)-x\left(t-\frac{h_1}{2}\right)\right]^{\mathrm{T}}Z_1\left[x(t)-x\left(t-\frac{h_1}{2}\right)\right]\\ &-\left[x\left(t-\frac{h_1}{2}\right)-x(t-h_1)\right]^{\mathrm{T}}Z_2\left[x\left(t-\frac{h_1}{2}\right)-x(t-h_1)\right]\\ &-\frac{1}{h_1}[x(t)-x(t-h_1)]^{\mathrm{T}}\left(R_1+\left(1-\dot{d}(t)\right)R_2+R_3\right)[x(t)-x(t-h_1)]\\ &-\frac{1}{h_3-h_2}[x(t-h_2)-x(t-h_3)]^{\mathrm{T}}Z_4[x(t-h_2)-x(t-h_3)]\\ &-\gamma_{1d}^{\mathrm{T}}(d(t)-h_1)\left(Z_3+R_1+\left(1-\dot{d}(t)\right)R_2+R_3\right)\gamma_{1d}\\ &-\gamma_{d2}^{\mathrm{T}}(h_2-d(t))(Z_3+R_1+R_3)\gamma_{d2}+\dot{x}^{\mathrm{T}}(t)\left[\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)\right.\\ &\left.+(h_2-h_1)Z_3+(h_3-h_2)Z_4+h_2(R_1+R_2+R_3)\right]\dot{x}(t)\end{aligned} \tag{9}$$

where $\gamma_{1d}=\frac{1}{d(t)-h_1}\int_{t-d(t)}^{t-h_1}\dot{x}(s)ds$, $\gamma_{d2}=\frac{1}{h_2-d(t)}\int_{t-h_2}^{t-d(t)}\dot{x}(s)ds$.
Note that $\lim_{d(t)\to h_1}\gamma_{1d}=\dot{x}(t-h_1)$, $\lim_{d(t)\to h_2}\gamma_{d2}=\dot{x}(t-h_2)$.

By using the Leibniz-Newton formula, and denoting

$$\begin{aligned}\eta_1^{\mathrm{T}}(t) = [&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\ &x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{1d}^{\mathrm{T}} \quad \gamma_{d2}^{\mathrm{T}}]\end{aligned} \tag{10}$$

the following equations hold for any free-weighting matrices N_1 and M_1

$$\begin{aligned}&2\eta_1^{\mathrm{T}}(t)N_1^{\mathrm{T}}[-x(t-h_1)+x(t-d(t))+(d(t)-h_1)\gamma_{1d}]=0 \\ &2\eta_1^{\mathrm{T}}(t)M_1^{\mathrm{T}}[-x(t-d(t))+x(t-h_2)+(h_2-d(t))\gamma_{d2}]=0\end{aligned} \tag{11}$$

where $N_{1i} \in \mathbb{R}^{n\times n}$, $M_{1i} \in \mathbb{R}^{n\times n}$, $i \in \{1, \ldots, 9\}$.

Corresponding to (1), for any matrices $F_{1j} \in$ $(j = 1, 2)$ with appropriate dimensions, one has

$$\begin{aligned}&2[x^{\mathrm{T}}(t)F_{11}^{\mathrm{T}}+\dot{x}(t)F_{12}^{\mathrm{T}}] \\ &\quad\times[Ax(t)+A_d x(t-d(t))-\dot{x}(t)+C_d\dot{x}(t-\tau)]=0\end{aligned} \tag{12}$$

From $\dot{V}_k(t)(k=1,\cdots,7)$, the combination of (10) and (11), one can obtain

$$\dot{V}\big|_{d(t)<h_2} \le \eta_1^{\mathrm{T}}(t)\left(\Omega\big|_{d(t)<h_2}\right)\eta_1(t) \tag{13}$$

where

$$\Omega\big|_{d(t)<h_2} = \Gamma_1+\Phi_1+\Phi_1^{\mathrm{T}} \tag{14}$$

$$\begin{aligned}\Phi_1^{\mathrm{T}} = [&0 \quad N_1^{\mathrm{T}}-M_1^{\mathrm{T}} \quad 0 \quad 0 \quad 0 \quad 0 \quad -N_1^{\mathrm{T}} \\ &M_1^{\mathrm{T}} \quad 0 \quad (d(t)-h_1)N_1^{\mathrm{T}} \quad (h_2-d(t))M_1^{\mathrm{T}}]\end{aligned}$$

$$\Pi_1^1 = \frac{d(t)-h_1}{h_2-h_1}P_1+\frac{h_2-d(t)}{h_2-h_1}P_2+A^{\mathrm{T}}F_{12}-F_{11}^{\mathrm{T}}$$

$$\Pi_a^1 = -(d(t)-h_1)\left(Z_3+R_1+(1-\dot{d}(t))R_2+R_3\right)$$

$$\Pi_b^1 = -(h_2-d(t))(Z_3+R_1+R_3)$$

$$\Gamma_1 = \begin{bmatrix} \Omega_{11}^1 & F_{11}^{\mathrm{T}}A_d & \Pi_1^1 & \frac{1}{\tau}S_2 & F_{11}^{\mathrm{T}}C_d & \Omega_{16} & U_1 & 0 & 0 & 0 & 0 \\ * & \Omega_{22} & A_d^{\mathrm{T}}F_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \Omega_{33}^1 & 0 & F_{12}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Omega_{77}^1 & Q_{12} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \Omega_{88}^1 & \Omega_{89}^1 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Omega_{99}^1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & \Pi_a^1 & 0 \\ * & * & * & * & * & * & * & * & * & * & \Pi_b^1 \end{bmatrix}$$

By combining those results above, now it is interesting to note that

$$\begin{aligned}\eta_1^{\mathrm{T}}(t)\left(\Omega\big|_{d(t)<h_2}\right)\eta_1(t) = &\frac{d(t)-h_1}{h_2-h_1}\eta_{11}^{\mathrm{T}}(t)\Omega_1\eta_{11}(t) \\ &+\frac{h_2-d(t)}{h_2-h_1}\eta_{12}^{\mathrm{T}}(t)\Omega_2\eta_{12}(t)\end{aligned}$$

where Ω_1 and Ω_2 results from $\Omega\big|_{d(t)<h_2}$ for $d(t) \to h_1$ and $d(t) \to h_2$, respectively, and

$$\begin{aligned}\eta_{11}^{\mathrm{T}}(t) = [&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\ &x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{d2}^{\mathrm{T}}]\end{aligned}$$

$$\begin{aligned}\eta_{12}^{\mathrm{T}}(t) = [&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\ &x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{1d}^{\mathrm{T}}]\end{aligned}$$

What's more, considering (3), one has

$$\Omega_1 = \frac{\mu_{max}-\dot{d}(t)}{\mu_{max}-\mu_{min}}\Omega_1\big|_{\dot{d}(t)\to u_{min}}+\frac{\dot{d}(t)-\mu_{min}}{\mu_{max}-\mu_{min}}\Omega_1\big|_{\dot{d}(t)\to u_{max}}$$

$$\Omega_2 = \frac{\mu_{max}-\dot{d}(t)}{\mu_{max}-\mu_{min}}\Omega_2\big|_{\dot{d}(t)\to u_{min}}+\frac{\dot{d}(t)-\mu_{min}}{\mu_{max}-\mu_{min}}\Omega_2\big|_{\dot{d}(t)\to u_{max}}$$

Hence, for $h_2 < d(t) < h_3$, the following expressions hold

$$\begin{aligned}\dot{V}_1(t)\big|_{d(t)>h_2} = &x^{\mathrm{T}}(t)\left(\frac{\dot{d}(t)}{h_3-h_2}(P_3-P_1)\right)x(t) \\ &+2x^{\mathrm{T}}(t)\left(\frac{d(t)-h_2}{h_3-h_2}P_3-\frac{h_3-d(t)}{h_3-h_2}P_1\right)\dot{x}(t)\end{aligned}$$

$$\begin{aligned}&\dot{V}_6(t)\big|_{d(t)>h_2} = \dot{x}^{\mathrm{T}}(t)(d(t)R_1+h_2(R_2+R_3))\dot{x}(t) \\ &-\left(1-\dot{d}(t)\right)\int_{t-d(t)}^{t}\dot{x}^{\mathrm{T}}(s)R_1\dot{x}(s)ds-\int_{t-d_2}^{t}\dot{x}^{\mathrm{T}}(s)(R_2+R_3)\dot{x}(s)ds\end{aligned}$$

For $h_2 < d(t) < h_3$, one can obtain

$$\begin{aligned}\int_{t-h_3}^{t}f(s)ds = &\int_{t-h_1}^{t}f(s)ds+\int_{t-h_2}^{t-h_1}f(s)ds \\ &+\int_{t-d(t)}^{t-h_2}f(s)ds+\int_{t-h_3}^{t-d(t)}f(s)ds\end{aligned} \tag{15}$$

By Jensen's inequality (Lemma 1) and combining (15), it can be seen that:

$$
\begin{aligned}
\dot{V}_5(t)+\dot{V}_6(t)\big|_{d(t)>h_2} \le & -\left[x(t)-x\left(t-\frac{h_1}{2}\right)\right]^{\mathrm{T}} Z_1\left[x(t)-x\left(t-\frac{h_1}{2}\right)\right] \\
& -\left[x\left(t-\frac{h_1}{2}\right)-x(t-h_1)\right]^{\mathrm{T}} Z_2\left[x\left(t-\frac{h_1}{2}\right)-x(t-h_1)\right] \\
& -\frac{1}{h_1}[x(t)-x(t-h_1)]^{\mathrm{T}}\left(\left(1-\dot{d}(t)\right)R_1+R_2+R_3\right)[x(t)-x(t-h_1)] \\
& -\frac{1}{h_2-h_1}\Gamma_a^T\left(Z_3+\left(\left(1-\dot{d}(t)\right)R_1+R_2+R_3\right)\right)\Gamma_a \\
& -\gamma_{2d}^{\mathrm{T}}(d(t)-h_2)\left(Z_4+\left(1-\dot{d}(t)\right)R_1\right)\gamma_{2d}-\gamma_{d3}^{\mathrm{T}}(h_2-d(t))Z_4\gamma_{d3} \\
& +\dot{x}^{\mathrm{T}}(t)\left[\left(\frac{h_1}{2}\right)^2(Z_1+Z_2)\right. \\
& \left.+(h_2-h_1)Z_3+(h_3-h_2)Z_4+h_2(R_2+R_3)+h_3R_1\right]\dot{x}(t)
\end{aligned} \tag{16}
$$

where $\Gamma_a = x(t) - x(t - h_1)$, $\gamma_{2d}=\frac{1}{d(t)-h_2}\int_{t-d(t)}^{t-h_2}\dot{x}(s)ds$, $\gamma_{d3}=\frac{1}{h_3-d(t)}\int_{t-h_3}^{t-d(t)}\dot{x}(s)ds.$

Note that $\lim_{d(t)\to h_2}\gamma_{2d}=\dot{x}(t-h_2)$, and $\lim_{d(t)\to h_3}\gamma_{d3}=\dot{x}(t-h_3)$.

By using the Leibniz-Newton formula, and denoting

$$
\begin{aligned}
\eta_2^{\mathrm{T}}(t)=\Big[&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\
&x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{2d}^{\mathrm{T}} \quad \gamma_{d3}^{\mathrm{T}}\Big]
\end{aligned} \tag{17}
$$

the following equations hold for any free-weighting matrices N_2 and M_2:

$$
\begin{aligned}
&2\eta_2^{\mathrm{T}}(t)N_2^{\mathrm{T}}[-x(t-h_2)+x(t-d(t))+(d(t)-h_2)\gamma_{2d}]=0 \\
&2\eta_2^{\mathrm{T}}(t)M_2^{\mathrm{T}}[-x(t-d(t))+x(t-h_3)+(h_3-d(t))\gamma_{d3}]=0
\end{aligned} \tag{18}
$$

where $N_{2i} \in \mathbb{R}^{n\times n}$, $M_{2i} \in \mathbb{R}^{n\times n}$, $i \in \{1, \ldots, 9\}$.

Corresponding to (1), for any matrices F_{2j} $(j = 1, 2)$ with appropriate dimensions, one has

$$
\begin{aligned}
&2\left[x^{\mathrm{T}}(t)F_{21}^{\mathrm{T}}+\dot{x}(t)F_{22}^{\mathrm{T}}\right] \\
&\quad\times[Ax(t)+A_dx(t-d(t))-\dot{x}(t)+C_d\dot{x}(t-\tau)]=0
\end{aligned} \tag{19}
$$

Combining (17)–(19) and $\dot{V}_k(t)(k=1,\cdots,7)$, one can obtain

$$
\dot{V}\big|_{d(t)>h_2} \le \eta_2^{\mathrm{T}}(t)\left(\Omega\big|_{d(t)>h_2}\right)\eta_2(t) \tag{20}
$$

where

$$
\Omega\big|_{d(t)>h_2}=\Gamma_2+\Phi_2+\Phi_2^{\mathrm{T}} \tag{21}
$$

$$
\Gamma_2=\begin{bmatrix}
\Omega_{11}^2 & F_{21}^{\mathrm{T}}A_d & \Pi_2^2 & \frac{1}{\tau}S_2 & F_{21}^{\mathrm{T}}C_d & \Omega_{16} & U_2 & 0 & 0 & 0 & 0 \\
* & \Omega_{22} & A_d^{\mathrm{T}}F_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & \Omega_{33}^2 & 0 & F_{22}^{\mathrm{T}}C_d & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Omega_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -S_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & \Omega_{66} & \Omega_{67} & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Omega_{77}^2 & \Omega_{78}^2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & \Omega_{88}^2 & -Q_{12} & 0 & 0 \\
* & * & * & * & * & * & * & * & -Q_{22} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & \Pi_a^2 & 0 \\
* & * & * & * & * & * & * & * & * & * & \Pi_b^2
\end{bmatrix}
$$

$$
\begin{aligned}
\Phi_2^{\mathrm{T}}=\Big[&0 \quad N_2^{\mathrm{T}}-M_2^{\mathrm{T}} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -N_2^{\mathrm{T}} \\
&M_2^{\mathrm{T}} \quad (d(t)-h_2)N_2^{\mathrm{T}} \quad (h_3-d(t))M_2^{\mathrm{T}}\Big]
\end{aligned}
$$

$$
\Pi_2^2=\frac{d(t)-h_2}{h_3-h_2}P_3+\frac{h_3-d(t)}{h_3-h_2}P_1+A^{\mathrm{T}}F_{22}-F_{21}^{\mathrm{T}}
$$

$$
\Pi_a^2=-(d(t)-h_2)\left(Z_4+\left(1-\dot{d}(t)\right)R_1\right),\ \Pi_b^2=-(h_3-d(t))Z_4
$$

By the results above-mentioned, we can conclude that

$$
\begin{aligned}
\eta_2^{\mathrm{T}}(t)\left(\Omega\big|_{d(t)>h_2}\right)\eta_2(t)=&\frac{d(t)-h_2}{h_3-h_2}\eta_{21}^{\mathrm{T}}(t)\Omega_3\eta_{21}(t) \\
&+\frac{h_3-d(t)}{h_3-h_2}\eta_{22}^{\mathrm{T}}(t)\Omega_4\eta_{22}(t)
\end{aligned} \tag{22}
$$

where Ω_3 and Ω_4 results from $\Omega\big|_{d(t)>h_2}$ for $d(t) \to h_2$ and $d(t) \to h_3$, respectively, and

$$
\begin{aligned}
\eta_{21}^{\mathrm{T}}(t)=\Big[&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\
&x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{d3}^{\mathrm{T}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
\eta_{22}^{\mathrm{T}}(t)=\Big[&x^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-d(t)) \quad \dot{x}^{\mathrm{T}}(t) \quad x^{\mathrm{T}}(t-\tau) \quad \dot{x}^{\mathrm{T}}(t-\tau) \\
&x^{\mathrm{T}}\left(t-\frac{h_1}{2}\right) \quad x^{\mathrm{T}}(t-h_1) \quad x^{\mathrm{T}}(t-h_2) \quad x^{\mathrm{T}}(t-h_3) \quad \gamma_{2d}^{\mathrm{T}}\Big]
\end{aligned}
$$

Moreover, using (3), one can obtain

$$
\Omega_3=\frac{\mu_{max}-\dot{d}(t)}{\mu_{max}-\mu_{min}}\Omega_3\big|_{\dot{d}(t)\to u_{min}}+\frac{\dot{d}(t)-\mu_{min}}{\mu_{max}-\mu_{min}}\Omega_3\big|_{\dot{d}(t)\to u_{max}}
$$

$$
\Omega_4=\frac{\mu_{max}-\dot{d}(t)}{\mu_{max}-\mu_{min}}\Omega_4\big|_{\dot{d}(t)\to u_{min}}+\frac{\dot{d}(t)-\mu_{min}}{\mu_{max}-\mu_{min}}\Omega_4\big|_{\dot{d}(t)\to u_{max}}
$$

According to [15] and [21], for $d(t) = h_2$, the following expression holds:

$$
\dot{V}(t)\big|_{d(t)=h_2}=\max\left\{\eta_1^{\mathrm{T}}\left(\Omega\big|_{d(t)<h_2}\right)\eta_1,\ \eta_2^{\mathrm{T}}\left(\Omega\big|_{d(t)>h_2}\right)\eta_2\right\} \tag{23}
$$

The proof is thus completed.

Table I. Allowable upper bound of $h_{\max}$ for various μ_{max}.

μ_{max}	0	0.5	0.9	1
Fridman *et al*. [31]	1.59	1.26	0.97	—
He *et al*. [32]	1.96	1.51	1.07	—
Xiao *et al*. [33]	2.05	1.82	1.74	1.72
Bai *et al*. [34]	2.05	1.89	1.85	1.85
Theorem 1	3.37	3.27	3.16	3.15

Remark 1. It should be pointed out that Theorem 1 presents an improved delay-dependent stability criterion for linear neutral systems with time-varying delay by constructing a novel LKF, which is much less conservative than most results in the literature, such as the results in [31–36]. The reduced conservatism of Theorem 1 benefits from the construction of the new LKF in (6) based on the idea of delay partitioning.

Remark 2. It is interesting to consider the case when μ_{min} is unknown. In this case, form Theorem 1, by fulfilling the restrictions:

$$P_1 > P_2,\ P_3 > P_1,\ R_2 > 0 \tag{24}$$

the neutral system (1) with delay restrictions (2) and (3) is asymptotically stable.

Remark 3. Another important case to consider is the case when no restrictions are cast upon the time-varying delay derivative's characteristics. In this case, as we have no information about $\dot{d}(t)$, from Theorem 1, we must consider the following restrictions (25) such that the neutral system (1) with delay restrictions (2) and (3) is asymptotically stable.

$$P_1 = P_2 = P_3,\ Q_1 = 0,\ R_1 = 0,\ R_2 = 0 \tag{25}$$

Remark 4. Because of the terms U_1 and U_2, Theorem 1 is valid only for minimum delay strictly greater than zero. However, it is straightforward to extend these results to the case where $h_{\min} = 0$ by considering $U_1 = 0$ and $U_2 = 0$.

IV. NUMERICAL EXAMPLES

This section presents some examples to illustrate the effectiveness of the methods derived above.

Example 1. Consider the neutral system (1) with [33]

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -0.9 \end{bmatrix},\quad A_d = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},\quad C_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

For $h_{\min} = 0$, $S_2 = 0$, and unknown μ_{min}, and various values of μ_{max}, the admissible upper bound value of $h_{\max}$ to make the system (1) asymptotically stable are shown in Table I. It can be seen that Theorem 1 can give less conservative results than those from [31–34]. This fact implies that the proposed method in this paper is effective and feasible.

Table II. Allowable upper bound of $h_{\max}$ for various τ.

τ	0.1	0.2	0.3	0.4	0.5	0.6
Liu *et al*. [35]	1.772	1.764	1.755	1.746	1.737	1.729
Qian *et al*. [36]	1.820	1.807	1.793	1.779	1.766	1.752
Theorem 1	2.706	2.706	2.706	2.706	2.706	2.706
τ	0.7	0.8	0.9	1.0	1.1	10000
Liu *et al*. [35]	1.722	1.715	1.711	1.708	1.707	1.707
Qian *et al*. [36]	1.739	1.727	1.718	1.712	1.712	1.710
Theorem 1	2.706	2.706	2.706	2.706	2.706	2.703

Table III. Allowable upper bound of $h_{\max}$ for various $h_{\min}$

$h_{\min}$	0	1	2	3	4	5
$h_{\max}$	2.732	2.041	2.108	3.094	4.094	5.094

Example 2. Consider the neutral system (1) with [36]

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix},\quad A_d = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix},\quad C_d = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}.$$

For $h_{\min} = 0$, $\mu_{max} = 0.01$, and unknown μ_{min}, and various τ, the results obtained by various methods in the literature for the admissible upper bound value of $h_{\max}$ for which the system (1) remains stable, are shown in Table II, which implies that the proposed method is much less conservative than the existing results.

Example 3. Consider the neutral system (1) with the same system matrices as Example 2. For $\mu_{min} = -0.1$, $\mu_{max} = 0.1$, and various $h_{\min}$, the results obtained by the proposed method for the admissible upper bound value of $h_{\max}$ for which the system (1) remains stable are shown in Table III.

V. CONCLUSION

In this paper, the stability analysis for linear neutral systems with time-varying delay are discussed. The main contribution of this paper is the construction of a new LKF based on the idea of delay partitioning. Introducing appropriate free-weighting matrices, those delay partitioning LKFs lead to a novel stability condition that dependent on both the upper and lower bounds on the delay derivative, which is established in terms of LMIs and can be solved readily by using existing LMI optimization techniques. Numerical examples demonstrate that the stability criteria are less conservative than existing results in the literature. In addition, how to achieve a more appropriate tradeoff between constructing the Lyapunov functional and decreasing the unnecessary decision variables in the results is the aim of our future work.

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