choice of state variables to be approximated in the reduced model has an important bearing on this problem.

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Controllability and Observability Criteria for Multivariable Linear Second-Order Models

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Abstract—**Criteria are discussed for the determination of controllability, stabilizability, observability, or detectability of linear second-order multivariable models of, for example, large space structures. An initial modal transformation is not required and the criteria are thus applicable to models with arbitrary damping coefficients. Moreover, the criteria are modal in the sense that some or all of the modes may be tested for controllability or observability. This aspect has advantages if not all the modes are known or easily computable. The criteria are further illustrated for a number of important special cases in a series of corollaries.**

I. INTRODUCTION

The behavior of many physical systems in engineering can be modeled by the following system of equations:

$$M\ddot{x}+(D_s+G)\dot{x}+Kx=f \tag{1}$$

where x and f are $n\times 1$ and M, D_s, G, and K are $n\times n$. Moreover, in describing physical systems one can make the following assumptions without loss of generality:

$$\begin{array}{ll} M=M^T>0 & \text{generalized capacitive storage} \\ D_s=D_s^T\geqslant 0 & \text{generalized energy dissipators} \\ G=-G^T & \text{generalized conservative elements} \\ K=K^T\geqslant 0 & \text{generalized inductive storage.} \end{array} \tag{2}$$

The model (1) can describe electrical, mechanical, thermal, and other systems by appropriate choice of "through" variables, f(current in electrical systems, force in mechanical systems, etc.) and "across" variables, x(voltage, displacement, etc.) [1]. Analogies exist among the various types of systems.

The model (1) can result directly from lumped parameter models, or finite approximations to distributed parameter systems described by partial differential equations. One large class of systems of current importance are large space structures (LSS), which are large distributed parameter systems that are most often discretized by the finite element method (FEM) into the form (1) [2]. The problem of controlling LSS motivated the studies in this note, although the results are applicable to any system described by (1).

In the LSS framework x is a displacement vector, f is a force vector, M is the mass matrix, K is the stiffness matrix, D_s is the damping matrix, and the G matrix gives rise to gyroscopic forces. In general, n is initially of very high order and the aforementioned matrices may be sparse.

In this note interest centers on the controllability of (1) when f is of the form

$$f=Bu \tag{3}$$

where u is an $m\times 1$ control vector, and on the observability of x and $\dot{x}$ from

$$y=Px+Q\dot{x} \tag{4}$$

where y is a $p\times 1$ output vector. Controllability and observability of (1), (3), and (4) has been shown to provide important insights into modal behavior of the system and to furnish information on the number and positioning of sensors and actuators [3], [4]. Also, controllability and observability information can be used in determining which modes to retain when performing model reduction [5].

The traditional method for determining controllability and observability is to transform to the equivalent standard matrix first-order (state variable) form with state $\begin{pmatrix} x \\ \dot{x}\end{pmatrix}$ of dimension $2n$. Unfortunately, the resulting standard controllability and observability tests (cast in terms of a $2n$th order "A" matrix) do not take advantage of the structure of the matrices M, D_s, G, K (i.e., symmetry, definiteness, and sparsity) and may present difficult computational problems if M is near singular or n is very large.

Other conditions have been derived by Hughes and Skelton [3] which exploit the specialized structure of (1) and (2) for the cases $D=0$ and $D_s=0$. However, these conditions could suffer from computational difficulties since they require knowledge of the full modal transformation matrix, whose columns are the eigenvectors corresponding to the eigenvalues λ of the generalized eigenproblem

$$[\lambda M+K]x=0 \tag{5}$$

where $\pm\lambda^{1/2}$ are the modal frequencies. An equivalent form of (5) is the simple eigenproblem

$$-M^{-1}Kx=\lambda x \tag{6}$$

since M is nonsingular. But this form is computationally undesirable since $M^{-1}K$ may, in general, be dense even though M and K are sparse. Moreover, $M^{-1}K$ will be computationally ill-determined for the eigenproblem if M is nearly singular (i.e., ill-conditioned with respect to inversion). Moreover, computation of the eigenvectors of (5) or (6) is ill-conditioned whenever the λ are repeated or nearly equal [6], which is often the case in LSS.

This note focuses on conditions for controllability (or, more generally, stabilizability) and observability (or, more generally, detectability) which take advantage of the structure of (1) and (2) but extend the results of [3] and are computationally more tractable. Most of the computational attractiveness of our criteria accrues from the fact that an initial modal transformation is not necessary. Thus, if just a few "important" modes are known (and there exist techniques to determine just selected modes, e.g., [7], [8]) these modes can be tested for, say controllability, by a test involving just the model matrices M, D, K, and B. ($D:=D_s+G$.)

II. MODAL CONTROLLABILITY AND OBSERVABILITY CRITERIA

Consider the following generalized first-order realization of (1), (3), and (4):

$$\begin{pmatrix} I & 0 \\ 0 & M\end{pmatrix}\begin{pmatrix} \dot{x} \\ \ddot{x}\end{pmatrix}=\begin{pmatrix} 0 & I \\ -K & -D\end{pmatrix}\begin{pmatrix} x \\ \dot{x}\end{pmatrix}+\begin{pmatrix} 0 \\ B\end{pmatrix}u \tag{7}$$

$$y=(P,Q)\begin{pmatrix} x \\ \dot{x}\end{pmatrix}. \tag{8}$$

Premultiplying (7) by $\begin{pmatrix} I & 0 \\ 0 & M\end{pmatrix}^{-1}$ yields the "standard" first-order model.

Manuscript received August 16, 1982; revised January 1, 1983. This work was supported in part by the U.S. Army Research Office under Contract DAAG29-81-K-0131.

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System (7), (8) is of the generalized first-order form

$$E\dot{z} = Az + \hat{B}u \tag{9}$$

$$y = Cz \tag{10}$$

by appropriate definition of z, A, $\hat{B}$, C, and E.

Other first-order realizations of this form which are of potential interest include:

$$\begin{pmatrix} D & M \\ M & 0 \end{pmatrix}\dot{z} = \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix}z + \begin{pmatrix} B \\ 0 \end{pmatrix}u; \tag{11}$$

$$\begin{pmatrix} D & M \\ -M & 0 \end{pmatrix}\dot{z} = \begin{pmatrix} -K & 0 \\ 0 & -M \end{pmatrix}z + \begin{pmatrix} B \\ 0 \end{pmatrix}u; \tag{12}$$

and

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix}\dot{z} = \begin{pmatrix} 0 & -K \\ -K & -D \end{pmatrix}z + \begin{pmatrix} 0 \\ B \end{pmatrix}u. \tag{13}$$

Note that (11) and (13) have symmetric E and A matrices when $G = 0$ while (12) has skew-symmetric E and symmetric A when $D_s = 0$. These properties are computationally advantageous in the eigenproblem that follows.

Definition 1:

$$W := \left\{ \lambda_i : \lambda_i \text{ is a generalized eigenvalue of the problem } \begin{pmatrix} 0 & I \\ -K & -D \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} \right\}$$

$$:= \{\text{modes of the system (7), (8)}\}.$$

Also,

$$W^+ := \{\lambda \in W : \operatorname{Re}\lambda \geqslant 0\} \tag{14}$$

$$W^- := \{\lambda \in W : \operatorname{Re}\lambda < 0\}. \tag{15}$$

For M nonsingular, there are $2n$ modes and W^+ and W^- are the sets of unstable and stable modes, respectively. Alternatively, W can be determined in terms of the generalized eigenvalue problems

$$\begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} - \lambda \begin{pmatrix} D & M \\ M & 0 \end{pmatrix}, \tag{16}$$

$$\begin{pmatrix} -K & 0 \\ 0 & -M \end{pmatrix} - \lambda \begin{pmatrix} D & M \\ -M & 0 \end{pmatrix}, \tag{17}$$

or

$$\begin{pmatrix} 0 & -K \\ -K & -D \end{pmatrix} - \lambda \begin{pmatrix} -K & 0 \\ 0 & M \end{pmatrix} \tag{18}$$

whichever yields the greatest computational advantage. Computation of eigenvalues for problems of the form (18) is discussed in [7] for the case $G = 0$. The modes W can also be determined as the regular eigenvalues of

$$\begin{pmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \tag{19}$$

but this can encounter numerical difficulty as stated previously.

Theorem 1 [9]: The system

$$\dot{x} = Ax + Bu; \qquad x(t) \in \mathbb{R}^n \tag{20}$$

is

a) controllable (stabilizable) if and only if

$$\operatorname{rank}[A - \lambda I, B] = n; \qquad \text{for all } \lambda \in \Lambda(A)(\Lambda^+(A)) \tag{21}$$

b) observable (detectable) if and only if

$$\operatorname{rank}\begin{pmatrix} C \\ A - \lambda I \end{pmatrix} = n; \qquad \text{for all } \lambda \in \Lambda(A)(\Lambda^+(A)), \tag{22}$$

where

$$\Lambda(A) := \{\lambda : (A - \lambda I)x = 0, x \neq 0\} := \text{Spectrum of } A. \tag{23}$$

Proof: See [9].

Clearly then, the system (9), (10) with E nonsingular is controllable (stabilizable) if and only if

$$\operatorname{rank}[A - \lambda E, \hat{B}] = 2n \qquad \text{for all } \lambda \in \Lambda(E^{-1}A)(\Lambda^+(E^{-1}A)) \tag{24}$$

and observable (detectable) if and only if

$$\operatorname{rank}\begin{pmatrix} C \\ A - \lambda E \end{pmatrix} = 2n \qquad \text{for all } \lambda \in \Lambda(E^{-1}A)(\Lambda^+(E^{-1}A)). \tag{25}$$

We now exploit the structure of A, $\hat{B}$, and C to derive controllability, etc., criteria directly in terms of M, D, K, B, P, and Q.

Theorem 2: The system (1), (3) is controllable (stabilizable) if and only if

$$\operatorname{rank}[\lambda^2 M + \lambda D + K, B] = n; \qquad \text{for all } \lambda \in W(W^+). \tag{26}$$

Proof: By Theorem 1 the system (1), (3) is controllable (stabilizable) if and only if

$$\begin{aligned} 2n &= \operatorname{rank}[A - \lambda E, \hat{B}]; \qquad \text{for all } \lambda \in W(W^+) \\ &= \operatorname{rank}\begin{pmatrix} -\lambda I & I & 0 \\ -K & -D - \lambda M & B \end{pmatrix} \qquad \text{from (7)–(10)} \\ &= \operatorname{rank}\begin{pmatrix} \lambda M + D & I \\ I & 0 \end{pmatrix}\begin{pmatrix} -\lambda I & I & 0 \\ -K & -D - \lambda M & B \end{pmatrix}\begin{pmatrix} -I & 0 & 0 \\ -\lambda I & I & 0 \\ 0 & 0 & I \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} \lambda^2 M + \lambda D + K & 0 & B \\ 0 & I & 0 \end{pmatrix}. \end{aligned}$$

Clearly this obtains if and only if

$$\operatorname{rank}[\lambda^2 M + \lambda D + K, B] = n; \qquad \text{for all } \lambda \in W(W^+). \qquad \blacksquare$$

Note that λ is a scalar, so that sparsity in the problem is preserved. Also, no inverses and no initial transformations are necessary. Finally, note that each mode of the system can be checked individually without transforming the system to modal coordinates.

Theorem 3: The system (1), (4) is observable (detectable) if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda Q + P \\ \lambda^2 M + \lambda D + K \end{pmatrix} = n \qquad \text{for all } \lambda \in W(W^+). \tag{27}$$

Proof: By Theorem 1 the system (1), (4) is observable (detectable) if and only if

$$\begin{aligned} 2n &= \operatorname{rank}\begin{pmatrix} C \\ A - \lambda E \end{pmatrix}; \qquad \text{for all } \lambda \in W(W^+) \\ &= \operatorname{rank}\begin{pmatrix} P & Q \\ -\lambda I & I \\ -K & -D - \lambda M \end{pmatrix} \qquad \text{from (7)–(10)} \\ &= \operatorname{rank}\begin{pmatrix} I & -Q & 0 \\ 0 & -\lambda M - D & -I \\ 0 & I & 0 \end{pmatrix}\begin{pmatrix} P & Q \\ -\lambda I & I \\ -K & -D - \lambda M \end{pmatrix}\begin{pmatrix} I & 0 \\ \lambda I & I \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} \lambda Q + P & 0 \\ \lambda^2 M + \lambda D + K & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

Clearly this obtains if and only if

$$\operatorname{rank}\begin{pmatrix} \lambda Q + P \\ \lambda^2 M + \lambda D + K \end{pmatrix} = n; \qquad \text{for all } \lambda \in W(W^+). \qquad \blacksquare$$

Note that an alternative proof of Theorems 2 and 3 is to observe that the results are essentially restatements of the spectral criteria for left or right coprimeness of the appropriate polynomial matrices (quadratic, in this case). However, we have exploited here the numerically useful characterization of W in Definition 1 and our proofs are direct and require no polynomial matrix theory. Several special cases of Theorems 2 and 3 are of interest in many systems and are now stated as corollaries.

Corollary 2.1: When $D = 0$ (i.e., no damping or gyroscopic forces) the system (1), (3) is controllable if and only if

$$\operatorname{rank}\left[-\omega_i^2 M + K, B\right] = n; \qquad i = 1, \cdots, n \tag{28}$$

where

$$\omega_i = \sqrt{\lambda_i}; \quad \lambda_i \in \Lambda(M^{-1}K) \quad (\text{note: } \lambda_i \geqslant 0).$$

Proof: When $D = 0$

$$W = \{\pm j\omega_i; i = 1, \cdots, n\}$$

and (28) follows directly from Theorem 2. ■

Corollary 2.2: When $D = 0$ and the system (1), (3) is in modal form, then (1), (3) is controllable if and only if

$$\operatorname{rank} B_r = n_r \qquad (r = 1, \cdots, R) \tag{29}$$

where B_r are partitioned rows of the modally transformed B matrix corresponding to the multiplicities n_i of the ω_i; $n_1 + \cdots + n_R = n$, and R is the number of distinct ω_i. This is Theorem 1 of [3].

Proof: In modal form

$$M = I, \quad K = \operatorname{diag}\left[\omega_1^2, \cdots, \omega_n^2\right],$$
$$B = \text{modally transformed } B \text{ matrix}.$$

Then (29) follows directly from Corollary 2.1. ■

Corollary 2.3: When $K = 0$ the system (1), (3) is controllable if and only if rank $B = n$.

Proof: When $K = 0$, $\lambda = 0 \in W$ and the corollary follows directly from Theorem 2. ■

Similarly, we can state

Corollary 3.1: When $D = 0$ the system (1), (4) is observable if and only if

$$\operatorname{rank}\begin{pmatrix} j\omega_i Q + P \\ -\omega_i^2 M + K \end{pmatrix} = n; \qquad i = 1, \cdots, n. \tag{30}$$

Proof: This result follows directly from the proof of Corollary 2.1 and Theorem 3. ■

Corollary 3.1a: When $D = 0$ and $Q = 0$ (i.e., no rate feedback) the system (1), (4) is observable if and only if

$$\operatorname{rank}\begin{pmatrix} P \\ -\omega_i^2 M + K \end{pmatrix} = n, \qquad i = 1, \cdots, n. \tag{31}$$

Corollary 3.2: When $D = 0$ and the system (1), (4) is in modal form, then (1), (4) is observable if and only if

$$\operatorname{rank}[j\omega_i Q_r + P_r] = n_r \qquad (r = 1, \cdots, R) \tag{32}$$

where $[j\omega_i Q_r + P_r]$ are the suitably partitioned columns of the modally transformed $[j\omega_i Q + P]$ matrix.

Proof: In modal form $M = I$, $K = \operatorname{diag}[\omega_1^2, \cdots, \omega_N^2]$, $[j\omega_i Q + P] =$ modally transformed $[j\omega_i Q + P]$ matrix and (32) follows directly from (30). ■

Corollary 3.2a: When $D = 0$, the system (1), (4) is in modal form, and $Q = 0$, then (1), (4) is observable if and only if

$$\operatorname{rank} P_r = n_r \qquad (r = 1, \cdots, R). \tag{33}$$

Corollary 3.3: When $K = 0$ the system (1), (4) is observable if and only if rank $P = n$.

Proof: Set $\omega_i = 0$ and $K = 0$ in (30). ■

Note that the above corollaries have obvious analogs in terms of stabilizability and detectability as appropriate.

III. Conclusions

Criteria have been provided for the determination of controllability, stabilizability, observability, or detectability of linear, time-invariant, second-order multivariable systems including systems with arbitrary damping terms. The criteria are modal in the sense that some or all the modes of such systems can be tested for controllability or observability but a full modal transformation need not be performed *a priori*. This has important numerical consequences should certain intermediate problems be ill-conditioned or if the matrices are sparse and it is desired to maintain the sparsity. Moreover, the criteria apply even if not all the modes are known —which would obviously rule out an initial modal transformation. For example, sometimes only some of the modes are known accurately, typically the lower frequency modes, or only some of the modes are known at all in the large, sparse case.

The criteria can also be used to provide a test for nearness to uncontrollability, etc., in a parametric sense, by examining the smallest singular value of $[\lambda^2 M + \lambda D + K, B]$, etc. This subject is under continuing investigation.

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A Note on Constructing Minimal Linear-Analytic Realizations for Polynomial Systems

ROLAND LEJEUNE AND WILSON J. RUGH

Abstract—**A procedure is given for constructing a simplified version of Crouch's form for minimal linear-analytic realizations of degree-3 polynomial systems.**

I. Introduction

In [1] a procedure is given for computing minimal-dimension linear-analytic realizations for polynomial nonlinear systems. The procedure is based on Crouch's form [2] and, in the degree-3 case explicitly treated, yields a realization of the form:

$$\begin{aligned}
\dot{x}_1(t) &= A_1 x_1(t) + b_1 u(t) \\
\dot{x}_2(t) &= A_2 x_2(t) + A_{21} x_1(t) + A_{211} x_1(t) \otimes x_1(t) \\
&\quad + D_{21} x_1(t) u(t) + b_2 u(t) \\
\dot{x}_3(t) &= A_3 x_3(t) + A_{31} x_1(t) + A_{311} x_1(t) \otimes x_1(t) + A_{32} x_2(t) \\
&\quad + A_{3111} x_1(t) \otimes x_1(t) \otimes x_1(t) + A_{312} x_1(t) \otimes x_2(t) \\
&\quad + D_{31} x_1(t) u(t) \\
&\quad + D_{311} x_1(t) \otimes x_1(t) u(t) + D_{32} x_2(t) u(t) + b_3 u(t)
\end{aligned}$$

Manuscript received January 21, 1983.
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