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On data-dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems ☆

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ABSTRACT

This paper is addressed to some questions concerning the exponential stability and its robustness measure for linear time-varying differential-algebraic systems of index 1. First, the Bohl exponent theory that is well known for ordinary differential equations is extended to differential-algebraic equations. Then, it is investigated that how the Bohl exponent and the stability radii with respect to dynamic perturbations for a differential-algebraic system depend on the system data. The paper can be considered as a continued and complementary part to a recent paper on stability radii for time-varying differential-algebraic equations [N.H. Du, V.H. Linh, Stability radii for linear time-varying differential-algebraic equations with respect to dynamic perturbations, J. Differential Equations 230 (2006) 579–599].

1. Introduction

In this paper, we investigate the exponential stability and its robustness for time-varying systems of differential-algebraic equations (DAEs) of the form

$$E(t)x'(t) = A(t)x(t), \quad t \geqslant 0, \tag{1.1}$$

where $E(\cdot), A(\cdot) \in L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^{n\times n})$, $\mathbb{K} = \{\mathbb{C},\mathbb{R}\}$. The leading term $E(t)$ is supposed to be singular for almost all $t \geqslant 0$ and to have absolute continuous kernel. We suppose that (1.1) generates an exponen-

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tially stable evolution operator $\Phi = \{\Phi(t,s)\}_{t,s\geqslant 0}$, i.e., there exist positive constants M and α such that

$$\|\Phi(t,s)\|_{\mathbb{K}^{n\times n}} \leqslant Me^{-\omega(t-s)}, \quad t \geqslant s \geqslant 0. \tag{1.2}$$

Linear systems of the form (1.1) may occur when one linearizes a general nonlinear system of DAEs

$$F(t, y, y') = 0, \quad t \geqslant 0, \tag{1.3}$$

along a particular solution $y = y(t)$, where function F is assumed to be sufficiently smooth, see [22,26]. Differential-algebraic equations of the form (1.1) or (1.3) play an important role in mathematical modeling arising in multibody mechanics, electrical circuits, prescribed path control, chemical engineering, etc., see [4,5,22]. It is well known that, due to the fact that the dynamics of DAEs is constrained, extra difficulties appear in the analytical as well as numerical treatments of DAEs. These difficulties are typically characterized by one of many different index concepts, see [5,14,22].

Example 1.1. Consider the following nonlinear DAE system which mimics an example from [5]

$$\begin{aligned} y_1' &= -2y_1 + y_3, \\ 0 &= (2 - y_2)\big(y_2 - e^{-t}\big), \\ 0 &= y_1 y_2 + y_3(2 - y_2) + q(t), \end{aligned} \tag{1.4}$$

where $q(t) = -e^{-2t} + e^t - 4$. A particular solution of this system (but it is not the unique one) is $y(t) = (e^{-2t} + 1, e^{-t}, 2)^T$. If we want to know the asymptotic behaviour or the convergent (divergent) rate of nearby solutions, we investigate the corresponding homogeneous linearized DAE system, which reads

$$\begin{aligned} x_1' &= -2x_1 + x_3, \\ 0 &= \big(2 - e^{-t}\big)x_2, \\ 0 &= e^{-t}x_1 + \big(e^{-2t} + 1\big)x_2 + \big(2 - e^{-t}\big)x_3. \end{aligned} \tag{1.5}$$

This is a linear time-varying, but almost-constant coefficient, index-1 DAE system in semi-explicit form, see [5]. It is easy to check that the corresponding time-invariant system is exponentially stable, hence the asymptotic stability of (1.5) as well as that of the particular solution $y(t)$ of (1.4) are expected. However, linearization of general nonlinear DAE systems of the form (1.3) result, in general, fully implicit time-varying DAE systems of the form (1.1) which give rise to more difficulties in the stability analysis. Note that the index of the linearized DAE system may depend on the solution in consideration, as well. We refer to [1,7,8,13,15,23,24,27,28,31,32,34–36] for some recent stability results for DAEs and their numerical solutions.

In 1913, Bohl introduced a characteristic number for analyzing the uniform exponential growth of solutions of linear differential systems, see [9] and references therein. This characteristic number, later called Bohl exponent, has been proven to be a useful tool in the qualitative and the control theory of finite as well as infinite dimensional linear systems. Numerous interesting properties of Bohl exponent are discussed in [9]. Though less well-known than the famous characteristic number introduced by Lyapunov, the Bohl exponent has a more natural property. Namely, it is stable with respect to small perturbations occurring in the system coefficient. For this reason, the Bohl exponent was used for characterizing the stability robustness of linear systems in many papers, e.g., see [16,33]. We are interested in extending the Bohl exponent theory to linear DAEs of the general form (1.1) and expect that similar results hold for DAEs (under some extra assumptions, of course).

On the other hand, many problems arising from real life contain uncertainty, because there are parameters which can be determined only by experiments or the remainder part ignored during linearization process can also be considered uncertainty. That is why we are interested in investigating the uncertain system of the form

$$E(t)x'(t) = \big(A(t) + F(t)\big)x(t), \quad t \geqslant 0, \tag{1.6}$$

where $F \in L_\infty^{\mathrm{loc}}(0, \infty; \mathbb{K}^{n\times n})$ is assumed to be an uncertain perturbation. A natural question arises that under what condition the system (1.6) remains exponentially stable, i.e., how robust the stability of the nominal system (1.1) is.

More concretely, we consider the system (1.1) subjected to structured perturbation of the form

$$E(t)x'(t) = A(t)x(t) + B(t)\Delta\big(C(\cdot)x(\cdot)\big)(t), \quad t \geqslant 0, \tag{1.7}$$

where $B(\cdot) \in L_\infty(0, \infty; \mathbb{K}^{n\times m})$ and $C(\cdot) \in L_\infty(0, \infty; \mathbb{K}^{q\times n})$ are given matrices defining the structure of the perturbation and $\Delta : L_p(0, \infty; \mathbb{K}^m) \to L_p(0, \infty; \mathbb{K}^q)$ is an unknown disturbance operator which is supposed to be linear, dynamic, and causal.

The so-called stability radius is defined by the largest bound r such that the stability is preserved for all perturbations Δ of norm strictly less than r. This measure of the robust stability was introduced by Hinrichsen and Pritchard [17] for linear time-invariant systems of ordinary differential equations (ODEs) with respect to time- and output-invariant, i.e., static perturbations. See [17,19,29] for results on stability radii of time-invariant linear systems. Earlier results for the robust stability of time-varying systems can be found, e.g., in [16,20,21]. Therefore, it is natural to extend the notion of the stability radius to differential-algebraic equations. This problem has been solved for linear time-invariant DAEs, see [4,6,10,11,30]. Recently, in [12], Du and Linh have extended Jacob's result in [21] to systems of DAEs. It is worth mentioning that the index notion, which plays a key role in the qualitative theory and in the numerical analysis of DAEs, should be taken into consideration in the robust stability analysis, too. Namely, for the definition of the stability radii for DAEs, not only the stability, but also the index-1 property are required to be preserved. In this context, we follow the tractability index approach proposed by März et al., see [14,26]. See also [2] for a detailed analysis on fundamental solutions for DAEs.

The first aim of this paper is to extend the Bohl exponent theory to DAE system (1.1). An analogous extension for the Lyapunov exponent for DAEs was given in [7,8]. Then we intend to analyze how the exponential stability and the stability radii of system (1.1) depend on the second coefficient A and the perturbation structure $\{B, C\}$. We remark that the latter problem was solved for time-invariant and time-varying ODEs, see [16,18]. See also [20] for a closely related problem.

The paper is organized as follows. In the next section we summarize some preliminary results on the theory of linear DAEs. In Section 3, we give a short review on the robust stability result for (1.1) presented in [12] and recall a formula of the stability radii proven there. Section 4 deals with the Bohl exponent and its relevant properties for the DAE case. Generalization of a classical theorem on the relation between the exponential stability and the existence of a bounded solution to inhomogeneous DAEs is given. In Section 5, the stability of the Bohl exponent and the data-dependence of the stability radii are analyzed. As a practical consequence, the formula of the stability radii for linear DAE systems with asymptotically constant coefficients is reduced to a computable one. Some conclusions will close the paper.

2. Preliminaries

2.1. Notations

Throughout the paper we use the following standard notations as in [12,21]. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let X, Y be finite dimensional vector spaces and let $t_0 \geqslant 0$. For every p, $1 \leqslant p < \infty$, we denote by $L_p(s, t; X)$ the space of measurable function f with $\|f\|_p := (\int_s^t \|f(\rho)\|^p \, d\rho)^{1/p} < \infty$

and by $L_\infty(s,t;X)$ the space of measurable and essentially bounded functions f with $\|f\|_\infty := \operatorname{ess\,sup}_{\rho\in[s,t]}\|f(\rho)\|$, where $t_0 \leqslant s < t \leqslant \infty$. We also consider the spaces $L_p^{\mathrm{loc}}(t_0,\infty;X)$ and $L_\infty^{\mathrm{loc}}(t_0,\infty;X)$, which contain all functions f satisfying $f \in L_p(s,t;X)$ and $f \in L_\infty(s,t;X)$, respectively, for every $s,t,t_0 \leqslant s < t < \infty$. For example, all piecewise continuous functions defined on $[s,t]$ belong trivially to $L_p(s,t;X)$ $(1 \leqslant p \leqslant \infty)$.

We use the conventional notation $\mathcal{L}(L_p(t_0,\infty;X), L_p(t_0,\infty;Y))$ to denote the Banach space of linear bounded operators $\mathbb{P}$ from $L_p(t_0,\infty;X)$ to $L_p(t_0,\infty;Y)$ supplied with the norm

$$\|\mathbb{P}\| := \sup_{x\in L_p(t_0,\infty;X),\, \|x\|=1} \|\mathbb{P}x\|_{L_p(t_0,\infty;Y)}.$$

For $k \geqslant 0$ the operator of truncation π_k at k on $L_p(0,\infty;X)$ is defined by

$$\pi_k(u)(t) := \begin{cases} u(t), & t \in [0,k], \\ 0, & t > k. \end{cases}$$

An operator $\mathbb{P} \in \mathcal{L}(L_p(0,\infty;X), L_p(0,\infty;Y))$ is called to be causal, if $\pi_t\mathbb{P}\pi_t = \pi_t\mathbb{P}$ for every $t \geqslant 0$.

Finally, in the whole paper, let us omit for brevity the time variable t, where no confusion occurs. In Sections 4 and 5, for a bounded, piecewise continuous matrix function D defined on $[0,\infty)$, we will not indicate the subscript for the supremum norm of D, that is

$$\|D\| := \|D\|_\infty = \sup_{t\geqslant 0}\big\|D(t)\big\|.$$

2.2. Linear differential-algebraic equations

We consider the linear differential-algebraic system

$$E(t)x'(t) = A(t)x(t) + q(t), \quad t \geqslant 0, \tag{2.1}$$

where E, A are supposed as in Section 1, $q \in L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^n)$. Let $N(t)$ denote $\ker E$, then there exists an absolutely continuous projector $Q(t)$ onto $N(t)$, i.e., $Q \in C(0,\infty;\mathbb{K}^{n\times n})$, Q is differentiable almost everywhere, $Q^2 = Q$, and $\operatorname{Im} Q(t) = N(t)$ for all $t \geqslant 0$. We assume in addition that $Q' \in L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^{n\times n})$. Set $P = I - Q$, then $P(t)$ is a projector along $N(t)$. The system (2.1) is rewritten into the form

$$E(t)(Px)'(t) = \overline{A}(t)x(t) + q(t), \tag{2.2}$$

where $\overline{A} := A + EP' \in L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^{n\times n})$. We define $G := E - \overline{A}Q$.

Definition 2.1. (See also [14, Section 1.2].) The DAE (2.1) is said to be index-1 tractable if $G(t)$ is invertible for almost every $t \in [0,\infty)$ and $G^{-1} \in L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^{n\times n})$.

Let (2.1) be index-1. Note that the index-1 property does not depend on the choice of projectors $P(Q)$, see [14,26]. We now consider the homogeneous case $q = 0$ and construct the Cauchy operator generated by (2.1). Multiplying both sides of (2.2) by PG^{-1}, QG^{-1}, we obtain

$$\begin{cases} (Px)' = \big(P' + PG^{-1}\overline{A}\big)Px, \\ Qx = QG^{-1}\overline{A}Px. \end{cases}$$

Thus, the system is decomposed into two parts: a differential part and an algebraic one. Hence, it is clear that we need to address the initial value condition to the differential components, only. Denote $u = Px$, the differential part becomes

$$u' = \big(P' + PG^{-1}\overline{A}\big)u. \tag{2.3}$$

This equation is called the inherent ordinary differential equation (INHODE) of (2.1). The INHODE (2.3) has the invariant property that every solution starting in $\operatorname{im}(P(t_0))$ remains in $\operatorname{im}(P(t))$ for all t, see [14,26]. Let $\Phi_0(t,s)$ denote the Cauchy operator generated by the INHODE (2.3), i.e.,

$$\begin{cases} \dfrac{d}{dt}\Phi_0(t,s) = \big(P' + PG^{-1}\overline{A}\big)\Phi_0(t,s), & t > s \geqslant 0. \\ \Phi_0(s,s) = I, \end{cases}$$

Then, the Cauchy operator generated by system (2.1) is defined by

$$\begin{cases} E\dfrac{d}{dt}\Phi(t,s) = A\Phi(t,s), & t > s \geqslant 0, \\ P(s)\big(\Phi(s,s) - I\big) = 0, \end{cases}$$

and can be given as follows

$$\Phi(t,s) = \big(I + QG^{-1}\overline{A}(t)\big)\Phi_0(t,s)P(s), \quad t > s \geqslant 0.$$

By the arguments used in [14, Section 1.2], [26], the unique solution of the initial value problem (IVP) for (2.1) with the initial condition

$$P(t_0)\big(x(t_0) - x_0\big) = 0, \quad t_0 \geqslant 0, \tag{2.4}$$

can be given by the constant-variation formula

$$x(t) = \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t,\rho)PG^{-1}q(\rho)\,d\rho + QG^{-1}q(t). \tag{2.5}$$

Remark 2.2. In general, the equality $x(t_0) = x_0$ for a given $x_0 \in \mathbb{K}^n$ cannot be expected as in an initial value problem for ODEs. Finally, we remark that, due to very mild conditions on the data of (2.1), only the differential part $P(t)x(t)$ can be expected to be smooth.

3. Stability radii for differential-algebraic systems

From now on, let the following assumptions hold.

Assumption A1. System (1.1) is strongly index-1 in the sense that, supplied with a bounded projection Q, the matrix function G^{-1} and the so-called canonical projection $Q_s := -QG^{-1}\overline{A}$ are essentially bounded on $[0,\infty)$.

Assumption A2. There exist $\overline{M} > 0$, $\omega > 0$ such that

$$\big\|\Phi_0(t,s)P(s)\big\| \leqslant \overline{M}e^{-\omega(t-s)}, \quad t \geqslant s \geqslant 0.$$

Remark 3.1. We note that the above assumptions imply immediately the estimate

$$\big\|\Phi(t,s)\big\| = \big\|\big(I - Q_s(t)\big)\Phi_0(t,s)P(s)\big\| \leqslant \Big(1 + \operatorname*{ess\,sup}_{t\geqslant 0}\big\|Q_s(t)\big\|\Big)\overline{M}e^{-\omega(t-s)},$$

that is, (1.2) holds for almost all $t \geqslant s \geqslant 0$ with $M := (1 + \operatorname{ess\,sup}_{t\geqslant 0}\|Q_s(t)\|)\overline{M}$. Furthermore, due to the invariant property of the solutions of the INHODE (2.3), we have

$$P(t)\Phi(t,s) = P(t)\Phi_0(t,s)P(s) = \Phi_0(t,s)P(s).$$

It is also remarkable that the terms QG^{-1}, Q_s do not depend on the choice of projector Q (see [14,26]). Further, it is easy to see that the boundedness of G^{-1} does not depend on the choice of a bounded Q.

Remark 3.2. One may ask why we should restrict ourselves only to the investigation of index-1 DAEs. It is well know that higher-index DAEs are very sensitive to perturbations occurring in the coefficients and in the inhomogeneous part, because higher-index DAEs contain not only ordinary differential equations and algebraic constraints, but also hidden constraints which involve derivatives of several solution components and derivatives of the inhomogeneous part (or input) as well. An arbitrary small perturbation may destroy the stability as well as the existence and uniqueness of solutions, even in the case of the simplest class such as the class of linear constant-coefficient DAEs. That is why most stability results in the literature are obtained for DAEs of index 1, see [1,6–8,11,13,15,23,24,27, 30,32,34]. Stability results for higher-index DAEs exist only in the case if special structured problems are considered and(or) extra assumptions are necessary [28,32,35,36]. Another alternative way is to reformulate the DAE by applying some index reduction technique in order to obtain lower-index DAEs which possess the same solution set, e.g. see [22,23]. To our best knowledge, at this moment no perturbation result exists for general higher-index DAEs.

Furthermore, we choose the tractability index approach among many index definitions existing in the DAE theory, because this approach gives a nice decoupling of the DAE system and admits us to obtain the existence and uniqueness of generalized solution under very mild assumptions on coefficient functions. If the coefficient functions are sufficiently smooth, one may proceed in a very similar way with another index definition such as the differentiation index [5] or the strangeness index [22], of course after transforming the system into an appropriate form.

First, the index notion is extended to the perturbed system (1.3), where the disturbance operator $\Delta \in \mathcal{L}(L_p(0,\infty;\mathbb{K}^q), L_p(0,\infty;\mathbb{K}^m))$ is supposed to be causal.

Let the linear operator $\widetilde{G} \in \mathcal{L}(L_p^{\mathrm{loc}}(0,\infty;\mathbb{K}^n), L_p^{\mathrm{loc}}(0,\infty;\mathbb{K}^n))$ be defined as follows

$$(\widetilde{G}u)(t) = (E - \overline{A}Q)u(t) - B\Delta\big(CQ(\cdot)u(\cdot)\big)(t), \quad t \geqslant 0.$$

Writing formally, we have

$$\widetilde{G} = \big(I - B\Delta CQG^{-1}\big)G. \tag{3.1}$$

Definition 3.3. The functional differential-algebraic system (1.3) is said to be index-1 (in the generalized sense) if for every $T > 0$, the operator $\widetilde{G}$ restricted to $L_p(0,T;\mathbb{K}^n)$ is invertible and the inverse operator $\widetilde{G}^{-1}$ is bounded.

Definition 3.4. We say that the IVP for the perturbed system (1.3) with (2.4) admits a mild solution if there exists $x \in L_p^{\mathrm{loc}}(t_0,\infty;\mathbb{K}^n)$ satisfying

$$x(t) = \Phi(t,t_0)P(t_0)x_0 + \int_{t_0}^{t} \Phi(t,\rho)PG^{-1}B\Delta\big(\big[Cx(\cdot)\big]_{t_0}\big)(\rho)\,d\rho + QG^{-1}B\Delta\big(\big[Cx(\cdot)\big]_{t_0}\big)(t) \tag{3.2}$$

for $t \geqslant t_0$, where

$$\big[Cx(\cdot)\big]_{t_0} = \begin{cases} 0, & t \in [0,t_0), \\ C(t)x(t), & t \in [t_0,\infty). \end{cases}$$

Theorem 3.5. *(See [12].) Consider the IVP* (1.3), (2.4). *If* (1.3) *is index*-1, *then it admits a unique mild solution* $x \in L_p^{\mathrm{loc}}(t_0, \infty; \mathbb{K}^n)$ *with absolute continuous* Px *for all* $t_0 \geqslant 0$, $x_0 \in \mathbb{K}^n$. *Furthermore, for an arbitrary* $T > 0$, *there exists a constant* M_1 *such that*

$$\left\| P(t)x(t) \right\| \leqslant M_1 \left\| P(t_0)x_0 \right\| \quad \textit{for all } t \in [t_0, T].$$

We associate with (1.3) the following operators

$$\begin{aligned}
(\mathbb{L}_{t_0}u)(t) &= C(t)\int_{t_0}^{t} \Phi(t,\rho)PG^{-1}B(\rho)u(\rho)\,d\rho + CQG^{-1}B(t)u(t),\\
(\mathbb{M}_{t_0}u)(t) &= C(t)\int_{t_0}^{t} \Phi(t,\rho)PG^{-1}B(\rho)u(\rho)\,d\rho,\\
(\mathbb{N}_{t_0}u)(t) &= CQG^{-1}B(t)u(t)
\end{aligned} \tag{3.3}$$

for all $t \geqslant t_0 \geqslant 0$, $u \in L_p(0, \infty; \mathbb{K}^m)$. Due to Assumption A1–A2, it is not difficult to see that they are linear and bounded. The first operator is called the (artificial) input–output operator (or perturbation operator) associated with (1.3).

The following properties of the input–output operator $\mathbb{L}_t$ are established.

Proposition 3.6. *Suppose that Assumptions* A1–A2 *hold.*

(i) $\|\mathbb{L}_t\|$ *is monotone nonincreasing with respect to* t, *i.e.,*

$$\|\mathbb{L}_{t_0}\| \geqslant \|\mathbb{L}_{t_1}\| \quad \forall t_1 \geqslant t_0 \geqslant 0.$$

(ii) *If* E, A, B, C *are periodic of the same period, then*

$$\|\mathbb{L}_{t_0}\| = \|\mathbb{L}_{t_1}\| \quad \forall t_1, t_0 \geqslant 0.$$

In particular, if E, A, B, C *are time-invariant, then* $\|\mathbb{L}_t\| = \|\mathbb{L}_0\|$ *for all* $t \geqslant 0$.

(iii) $\|\mathbb{L}_t\| \leqslant \frac{M}{\omega}\|PG^{-1}\|_\infty \|B\|_\infty \|C\|_\infty + \|CQG^{-1}B\|_\infty,\ t \geqslant 0.$

Proof. The proof is straightforward and is quite similar to the ODE case in [16]. □

Definition 3.7. Let Assumptions A1–A2 hold. The trivial solution of (1.3) is said to be globally L_p-stable if there exist constants $M_2, M_3 > 0$ such that

$$\begin{aligned}
\left\| P(t)x(t; t_0, x_0) \right\|_{\mathbb{K}^n} &\leqslant M_2 \left\| P(t_0)x_0 \right\|_{\mathbb{K}^n},\\
\left\| x(\cdot; t_0, x_0) \right\|_{L_p(t_0,\infty;\mathbb{K}^n)} &\leqslant M_3 \left\| P(t_0)x_0 \right\|_{\mathbb{K}^n}
\end{aligned} \tag{3.4}$$

for all $t \geqslant t_0$, $x_0 \in \mathbb{K}^n$.

Note that due to [12, Proposition 1], the second inequality implies the first one. Further, this kind of stability notion is equivalent to the output stability. See [20] for some more details on different stability concepts in the ODE case.

Next, the notion of the stability radius introduced in [17,21] is extended to time-varying differential-algebraic system (1.1).

Definition 3.8. Let Assumptions A1–A2 hold. The complex (real) structured stability radius of (1.1) subjected to linear, dynamic and causal perturbation in (1.3) is defined by

$$r_{\mathbb{K}}(E,A;B,C) = \inf\left\{\begin{array}{l}\|\Delta\|,\ \text{the trivial solution of (1.3) is not globally } L_p\text{-stable}\\ \quad\text{or (1.3) is not index-1}\end{array}\right\},$$

where $\mathbb{K} = \mathbb{C}, \mathbb{R}$, respectively.

In [12], the following important results have been established.

Theorem 3.9. *Let Assumptions* A1–A2 *hold. Then*

$$\begin{aligned} r_{\mathbb{K}}(E,A;B,C) &= \min\Big\{\sup_{t_0\geqslant 0}\|\mathbb{L}_{t_0}\|^{-1}, \|\mathbb{N}_0\|^{-1}\Big\}\\ &= \min\Big\{\lim_{t_0\to\infty}\|\mathbb{L}_{t_0}\|^{-1}, \Big(\operatorname*{ess\,sup}_{t\geqslant 0}\big\|CQG^{-1}B(t)\big\|\Big)^{-1}\Big\}.\end{aligned}$$

Corollary 3.10. *Let the data* $(E,A;B,C)$ *be real and Assumptions* A1–A2 *hold. Then*

$$r_{\mathbb{C}}(E,A;B,C) = r_{\mathbb{R}}(E,A;B,C).$$

Furthermore, for time-invariant systems, we obtain a computable formula for the complex stability radius.

Theorem 3.11. *Let* E, A, B, C *be time-invariant, the system* (1.1) *be index*-1 *and exponentially stable. If* $p = 2$, *i.e., the space* L_2 *of square integrable functions is in consideration, then*

$$r_{\mathbb{C}}(E,A;B,C) = \|\mathbb{L}_0\|^{-1} = \Big(\sup_{w\in i\mathbb{R}}\big\|C(wE-A)^{-1}B\big\|\Big)^{-1}.$$

The function $C(wE-A)^{-1}B$ is called the artificial transfer functions associated with (1.1). We remark that the exponential stability of time-invariant system (1.1) means exactly that all finite generalized eigenvalues of matrix pencil (E,A) have negative real part. Thus, the transfer function is well defined on the imaginary axis $i\mathbb{R}$ of the complex plane. For time-invariant systems, the computation of the complex stability radius leads to a global optimization problem that can be solved numerically in principle.

4. Bohl exponent for DAEs

In this section, we aim to extend the Bohl exponent notion introduced by Bohl (see [9]) to the case of linear DAEs. For simplicity, we assume that in the remainder part of the paper, the coefficients E, A are piecewise continuous functions. We stress that all the results in this and in the next section can be extended to systems with coefficients E, A belonging to the space $L_\infty^{\mathrm{loc}}(0,\infty;\mathbb{K}^{n\times n})$ without difficulty.

Definition 4.1. The (upper) Bohl exponent for the index-1 system (1.1) is given by

$$k_B(E,A) = \inf\big\{-\omega\in\mathbb{R};\ \exists M_\omega>0\colon\ \forall t\geqslant t_0\geqslant 0\ \Rightarrow\ \big\|\Phi(t,t_0)\big\|\leqslant M_\omega e^{-\omega(t-t_0)}\big\}. \tag{4.1}$$

The Bohl exponent for the INHODE (2.3) as well as the Bohl exponent for (2.3) with respect to subspace $\operatorname{Im} P$ are defined in a similar manner, see [9, p. 118].

Remark 4.2. If (E, A) is a regular pair of constant matrices, then

$$k_B(E, A) = \max\{\Re\lambda;\ \lambda \in \sigma(E, A)\},$$

where $\sigma(E, A)$ denotes the spectrum of the pencil (E, A).

The following characterization follows immediately from the definition.

Lemma 4.3. *If the Bohl exponent of* (1.1) *is finite, then the canonical projection* $P_s := I - Q_s$ *is necessarily bounded.*

Proof. We simply set $t = t_0$ in (4.1), then obtain

$$\|\Phi(t, t)\| \leqslant M_\omega, \quad t \geqslant 0,$$

for some finite ω and constant M_ω. On the other hand

$$\|\Phi(t, t)\| = \|P_s(t)\Phi_0(t, t)P(t)\| = \|P_s(t)P(t)\| = \|P_s(t)\|,$$

hence the statement is verified. □

Analogously to the ODE case (see [9]), we have

Proposition 4.4. *The Bohl exponent of* (1.1) *is finite if and only if*

$$\sup_{0 \leqslant |t-s| \leqslant 1} \|\Phi(t, s)\| < \infty.$$

Furthermore, if the Bohl exponent of (1.1) *is finite, it can be determined by*

$$k_B(E, A) = \overline{\lim_{s,t-s\to\infty}} \frac{\ln\|\Phi(t, s)\|}{t - s}.$$

Proof. The first statement is easily verified by using the semi-group property of Φ

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0) \quad \forall t_2 \geqslant t_1 \geqslant t_0 \geqslant 0.$$

The second statement comes from the definition of Bohl exponents. □

Definition 4.5. The Bohl exponent of (1.1) is said to be strict if it is finite and

$$k_B(E, A) = \lim_{s,t-s\to\infty} \frac{\ln\|\Phi(t, s)\|}{t - s}.$$

Proposition 4.6. *Suppose that Assumption* A1 *holds. Then the Bohl exponent of* (1.1) *is exactly equal to the Bohl exponent of the INHODE* (2.3) *corresponding to the subspace* Im P*. Furthermore,*

$$k_B(E, A) = \overline{\lim_{s,t-s\to\infty}} \frac{\ln\|\Phi_0(t, s)P(s)\|}{t - s} \leqslant \overline{\lim_{s,t-s\to\infty}} \frac{\ln\|\Phi_0(t, s)\|}{t - s}.$$

Proof. Clearly, the Bohl exponent of the INHODE (2.3) corresponding to the subspace $\operatorname{Im} P$ is well defined and it has formula

$$k_P = \overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi_0(t,s)P(s)\|}{t-s}.$$

From the assumptions, P and P_s are bounded, as well. We have

$$\|\Phi(t,s)\| = \|P_s(t)\Phi_0(t,s)P(s)\| \leqslant \|P_s(t)\| \|\Phi_0(t,s)P(s)\|,$$

hence

$$k_B(E,A) = \overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi(t,s)\|}{t-s} \leqslant \overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi_0(t,s)P(s)\|}{t-s}.$$

Conversely,

$$\|\Phi_0(t,s)P(s)\| = \|P(t)P_s(t)\Phi_0(t,s)P(s)\| = \|P(t)\Phi(t,s)\| \leqslant \|P(t)\| \|\Phi(t,s)\|,$$

which yields

$$\overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi_0(t,s)P(s)\|}{t-s} \leqslant \overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi(t,s)\|}{t-s} = k_B(E,A).$$

The remainder inequality is trivial. Note that the Bohl exponent of the INHODE (2.3) given by

$$k_{INH} = \overline{\lim_{s,t-s\to\infty}} \frac{\ln \|\Phi_0(t,s)\|}{t-s}$$

provides us an estimate for the Bohl exponent of DAE system (1.1). $\square$

Corollary 4.7. *Suppose that Assumption* A1 *holds. Then, the Bohl exponent of* (1.1) *is strict if and only if so is the Bohl exponent of the INHODE* (2.3) *corresponding to the subspace* $\operatorname{Im} P$.

We obtain a sufficient condition for the finiteness of the Bohl exponent for (1.1).

Corollary 4.8. *Suppose that Assumption* A1 *holds. If the Bohl exponent of the INHODE* (2.3) *is finite then so is that of* (1.1)*. In particular, if $A_0 := P' + PG^{-1}\overline{A}$ is integrally bounded, i.e.,*

$$\sup_{t\geqslant 0} \int_t^{t+1} \|A_0(\tau)\| \, d\tau < \infty,$$

the Bohl exponent of (1.1) *is finite.*

Proof. The first statement comes directly from Proposition 4.6. Next, suppose that $A_0 = P' + PG^{-1}\overline{A}$ is integrally bounded. Invoking [9, Theorem 4.3], the INHODE (2.3) has finite Bohl exponent, hence so is the Bohl exponent of (1.1). $\square$

Remark 4.9. (i) It is easy to verify the shifting property

$$k_B(E, A + aE) = k_B(E, A + \alpha P) = k_B(E,A) + k_B(a),$$

provided that the scalar function $a(\cdot)$ has a strict Bohl exponent.

(ii) Under the boundedness assumption of Q, Q_s, dynamic behaviour of the DAE system (1.1) and that of the INHODE (2.3) with respect to subspace $\operatorname{Im} P$ have a lot of similar properties. See also [7,8] for a similar statement established for Lyapunov exponents. We remark in addition that the Bohl exponent of the system (1.1) does not depend on the choice of a (bounded) projector Q.

Definition 4.10. The DAE system (1.1) is said to be exponentially stable if there exist positive constants M, α such that

$$\|\Phi(t,t_0)\| \leqslant M e^{-\alpha(t-t_0)} \quad \forall t \geqslant t_0 \geqslant 0.$$

Lemma 4.11. *Let Assumption* A1 *hold. Then DAE system* (1.1) *is exponentially stable if and only if Assumption* A2 *holds.*

Proof. Because of the relation between two fundamental solutions

$$\Phi(t,t_0) = P_s(t)\Phi_0(t,t_0)P(t_0),$$

it is trivial to see the equivalence. Further, $\alpha = \omega$ and $M = \|P_s\|\overline{M}$, where $\overline{M}$ and ω are those constants in Assumption A2. □

The following theorem generalizes classical results that are well known for ODEs, see [9,16].

Theorem 4.12. *Let Assumption* A1 *hold and suppose that $A_0(\cdot)$ is integrally bounded. Then, the following statements are equivalent*:

(i) *The DAE system* (1.1) *is exponentially stable.*
(ii) *The Bohl exponent $k_B(E,A)$ is negative.*
(iii) *For any $q > 0$, there exists a positive constant C_q such that*

$$\int_{t_0}^{\infty} \|\Phi(t,t_0)\|^q \, dt \leqslant C_q \quad \forall t_0 \geqslant 0.$$

(iv) *For every bounded $f(\cdot)$, the solution of the IVP*

$$\begin{cases} E(t)x' = A(t)x + f(t), & t \geqslant 0, \\ P(0)x(0) = 0 \end{cases} \tag{4.2}$$

is bounded.

Proof. The main idea is to consider the corresponding statements for the INHODE (2.3). The equivalence of the first 3 statements is trivial, because of the equivalence of the corresponding statements for the INHODE (2.3), see [9,16]. The implication (i) ⇒ (iv) is easily verified by using the constant-variation formula (2.5). For the converse direction, we progress as follows. Using the decoupling technique as in Section 2.2 to (4.2), it is easy to see that (iv) is equivalent to

(iv*) *For every bounded $f(\cdot)$, the solution of the IVP*

$$\begin{cases} (Px)' = A_0 Px + PG^{-1} f, & t \geqslant 0, \\ P(0)x(0) = 0 \end{cases} \tag{4.3}$$

is bounded.

Note that the unique solution to this IVP remain in subspace $\operatorname{Im} P$, too. By repeating the arguments of [9, Theorems 5.1–5.2] (the only difference is that we consider initial value problems for an inhomogeneous INHODE with respect to subspace $\operatorname{Im} P$), one can prove without difficulty that (iv*) holds if and only if the Bohl exponent of INHODE (2.3) corresponding to subspace $\operatorname{Im} P$ is negative. By Proposition 4.6, the proof is complete. □

Remark 4.13. Under the weaker assumption $k_B(E, A) < \infty$, statements (i)–(iii) are equivalent. Unfortunately, in this case, the implication (iv) ⇒ (ii) does not hold, see a counter-example for ODEs in [9, p. 131]. That is, the integrally boundedness condition is essential and cannot be dropped.

By introducing the variable change $x(t) = T(t)z(t)$ and scaling Eq. (1.1) by W, where $W \in C(\mathbb{R}, \mathbb{K}^{n\times n})$, $T \in C^1(\mathbb{R}, \mathbb{K}^{n\times n})$ are nonsingular matrix functions, we arrive at a new system

$$\widehat{E}(t)z' = \widehat{A}(t)z, \tag{4.4}$$

where $\widehat{E} = WET$, $\widehat{A} = W(AT - ET')$.

Definition 4.14. The transformation with matrix functions $W \in C(\mathbb{R}, \mathbb{K}^{n\times n})$ and $T \in C^1(\mathbb{R}, \mathbb{K}^{n\times n})$ is said to be a Bohl transformation if

$$\inf\bigl\{\varepsilon \in \mathbb{R};\ \exists M_\varepsilon > 0:\ \bigl\|T^{-1}(t)\bigr\|\bigl\|T(s)\bigr\| \leqslant M_\varepsilon e^{\varepsilon|t-s|},\ \forall t, s \geqslant 0\bigr\} = 0.$$

It is easy to see that the fundamental matrix for (4.4) can be given by

$$\widehat{\Phi}(t, s) = T^{-1}(t)\Phi(t, s)T(s), \quad t \geqslant s \geqslant 0.$$

Remark 4.15. If the pair W, T gives a kinematically equivalent transformation, i.e., both T and T^{-1} are bounded (see [24]), then it is a Bohl transformation. In addition, under a Bohl transformation, all the assumptions on the system (1.1) remain true for (4.4) with a new projection $\widehat{Q}(t) = T^{-1}(t)Q(t)T(t)$.

The following statements are adopted from ODE case (see [16]) and easily verifiable.

Proposition 4.16. (i) *The set of Bohl transformations forms a group with respect to pointwise multiplication.*
(ii) *The Bohl exponent is invariant with respect to Bohl transformation.*

Proposition 4.17. *If $W \in C(\mathbb{R}, \mathbb{K}^{n\times n})$ and $T \in C^1(\mathbb{R}, \mathbb{K}^{n\times n})$ admit a Bohl transformation, then*

$$r_{\mathbb{K}}(\widehat{E}, \widehat{A}; WB, CT) = r_{\mathbb{K}}(E, A; B, C).$$

5. Data-dependence of the Bohl exponent and the stability radii

Given a perturbation matrix function $F(\cdot) \in L_\infty(0, \infty; \mathbb{K}^{n\times n})$, we consider the perturbed equation

$$E(t)(Px)'(t) = \bigl(\overline{A}(t) + F(t)\bigr)x(t). \tag{5.1}$$

Multiplying both sides of (5.1) by PG^{-1} and QG^{-1}, respectively, we obtain

$$(Px)' = PG^{-1}(\overline{A} + P')Px + PG^{-1}Fx, \tag{5.2}$$

$$Qx = QG^{-1}\overline{A}Px + QG^{-1}Fx. \tag{5.3}$$

For simplicity, we suppose that F is piecewise continuous and bounded. We note that this does not mean a restriction, since the result of this section can be extended to a general essentially bounded and measurable F without any difficulty. In addition to Assumptions A1–A2, let F be such that the perturbed system (5.1) satisfies a similar assumption like A1, that is,

Assumption A3. With the bounded projection Q chosen in Section 2, the matrix $\widetilde{G} = E - (\overline{A} + F)Q$ is invertible everywhere. Furthermore, let $\widetilde{G}^{-1}$ and $\widetilde{Q}_s = -Q\widetilde{G}^{-1}(\overline{A} + F)$ be bounded on $[0, \infty)$.

It is easy to give a sufficient condition for F such that this assumption holds true.

Assumption A3*. Let perturbation F be sufficiently small such that

$$\sup_{t\geqslant 0}\|F(t)\| < \Big(\sup_{t\geqslant 0}\|QG^{-1}\|\Big)^{-1}.$$

Lemma 5.1. *Let Assumption* A1 *hold. Then* $\widetilde{G}$ *is invertible if and only if* $(I - QG^{-1}F)$ *is invertible. Further, Assumption* A3* *implies Assumption* A3.

Proof. From the definition, we have

$$\widetilde{G}G^{-1} = \big(E - (\overline{A} + F)Q\big)G^{-1} = I - FQG^{-1}.$$

Hence, if $\widetilde{G}^{-1}$ is invertible, then $(I - FQG^{-1})$ is invertible. Further,

$$\big(I - FQG^{-1}\big)^{-1} = G\widetilde{G}^{-1}.$$

By direct calculations, it is easy to show that the inverse of $(I - QG^{-1}F)$ exists and

$$\big(I - QG^{-1}F\big)^{-1} = I + QG^{-1}\big(I - FQG^{-1}\big)^{-1}F.$$

The converse direction is proven similarly.

Under Assumption A3, by a well-known result in functional analysis, the inverse of $(I - QG^{-1}F)$ exists and

$$\big\|\big(I - QG^{-1}F\big)^{-1}\big\| \leqslant \frac{1}{1 - \|QG^{-1}\|\|F\|},$$

which implies immediately the boundedness of $\widetilde{G}^{-1}$. To see the boundedness of $\widetilde{Q}_s$, we manipulate as follows

$$\begin{aligned} Q\widetilde{G}^{-1}\overline{A} &= QG^{-1}\big(I - FQG^{-1}\big)^{-1}\overline{A} = QG^{-1}\Bigg(\sum_{i=0}^{\infty}\big(FQG^{-1}\big)^i\Bigg)\overline{A} \\ &= \Bigg(\sum_{i=0}^{\infty}\big(QG^{-1}F\big)^i\Bigg)QG^{-1}\overline{A} = \big(I - QG^{-1}F\big)^{-1}QG^{-1}\overline{A}. \end{aligned}$$

Hence $\widetilde{Q}_s = -Q\widetilde{G}^{-1}(\overline{A} + F) = -(Q\widetilde{G}^{-1}\overline{A} + Q\widetilde{G}^{-1}F)$ is bounded. $\square$

Theorem 5.2. *Let Assumptions* A1 *and* A3* *hold, for any* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon) > 0$ *such that*

$$\limsup_{s,t-s\to\infty} \frac{1}{t-s} \int_s^t \left\| PG^{-1}F(\tau) \right\| d\tau < \delta$$

implies

$$k_B(E, A+F) \leqslant k_B(E, A) + \varepsilon.$$

Proof. Denote by $\widetilde{\Phi}(t,s)$ the fundamental solution matrix of (5.1). By (5.2) and (5.3), for $t \geqslant s \geqslant 0$, we have

$$\frac{d}{dt}\big(P(t)\widetilde{\Phi}(t,s)\big) = PG^{-1}(\overline{A}+P')P(t)\widetilde{\Phi}(t,s) + PG^{-1}F(t)\widetilde{\Phi}(t,s), \tag{5.4}$$

$$Q(t)\widetilde{\Phi}(t,s) = QG^{-1}\overline{A}P(t)\widetilde{\Phi}(t,s) + QG^{-1}F(t)\widetilde{\Phi}(t,s). \tag{5.5}$$

Solving $Q\widetilde{\Phi} = (I - QG^{-1}F)^{-1}QG^{-1}(\overline{A}+F)P\widetilde{\Phi}$ from (5.5) and substituting it into (5.4), we obtain

$$\frac{d}{dt}(P\widetilde{\Phi}) = PG^{-1}(\overline{A}+P')P\widetilde{\Phi} + PG^{-1}F\big[I + \big(I - QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big]P\widetilde{\Phi}.$$

By using the constant-variation method we get

$$(P\widetilde{\Phi})(t,s) = \Phi_0(t,s)P(s) + \int_s^t \Phi_0(t,\tau)\big(PG^{-1}F\big[I + \big(I - QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big]P\big)(\tau)\widetilde{\Phi}(\tau,s)\,d\tau.$$

By virtue of Definition 4.1 and Proposition 4.6, there exists constant $\overline{M}$ such that

$$\left\|\Phi_0(t,s)P(s)\right\| \leqslant \overline{M}e^{-\alpha(t-s)}, \quad t \geqslant s \geqslant 0,$$

with $\alpha = -k_B(E,A) - \varepsilon/2$. It follows that

$$\left\|(P\widetilde{\Phi})(t,s)\right\| = \overline{M}e^{-\alpha(t-s)} + \overline{M}\int_s^t e^{-\alpha(t-\tau)}h(\tau)\left\|(P\widetilde{\Phi})(\tau,s)\right\| d\tau, \tag{5.6}$$

where h is a nonnegative scalar function defined by

$$h(t) := \left\|\big(PG^{-1}F\big[I + \big(I - QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big]\big)(t)\right\|. \tag{5.7}$$

Since $[I + (I - QG^{-1}F)^{-1}QG^{-1}(\overline{A}+F)]$ is bounded, it is clear that there exists constant $K > 0$ such that

$$h(t) \leqslant K\left\|PG^{-1}F(t)\right\|, \quad t \geqslant 0. \tag{5.8}$$

Multiplying both sides of (5.6) by $e^{\alpha t}$, we have

$$e^{\alpha t}\left\|(P\widetilde{\Phi})(t,s)\right\| \leqslant \overline{M}e^{\alpha s}\left\|P(s)\right\| + \overline{M}\int_s^t h(\tau)e^{\alpha\tau}\left\|(P\widetilde{\Phi})(\tau,s)\right\| d\tau.$$

An application of Gronwall–Bellman's inequality yields

$$\left\| P\widetilde{\Phi}(t,s)\right\| \leqslant \overline{M}e^{-\alpha(t-s)}e^{\overline{M}\int_s^t h(\tau)\,d\tau} \leqslant \overline{M}e^{-\alpha(t-s)}e^{\overline{M}K\int_s^t \|PG^{-1}F(\tau)\|\,d\tau} \tag{5.9}$$

for all $t \geqslant s \geqslant 0$. On the other hand, due to the assumption, there exist sufficiently large s_0 and T such that

$$\sup_{s\geqslant s_0} \frac{1}{T}\int\limits_s^{s+T} \left\| PG^{-1}F(\tau)\right\| d\tau \leqslant 2\delta.$$

Therefore, we have

$$\left\| P(t)\widetilde{\Phi}(t,s)\right\| \leqslant \overline{M}e^{2\overline{M}KT\delta}e^{-(\alpha-2\overline{M}K\delta)(t-s)}, \quad t \geqslant s \geqslant s_0.$$

The above estimate, together with Proposition 4.6 applied to (5.1), implies that

$$k_B(E, A+F) \leqslant -\alpha + 2\overline{M}K\delta \leqslant k_B(E+A) + \varepsilon/2 + 2\overline{M}K\delta.$$

Finally, it remains to choose

$$\delta = \frac{\varepsilon}{4\overline{M}K}.$$

The proof is complete. □

Corollary 5.3. *Suppose that Assumptions* A1 *and* A3* *hold.*

(i) *If*

$$\limsup_{s,t-s\to\infty} \frac{1}{t-s}\int\limits_s^t \left\| PG^{-1}F(\tau)\right\| d\tau = 0,$$

then $k_B(E, A+F) = k_B(E, A)$.

(ii) *In particular, if*

$$\lim_{t\to\infty}\left\| PG^{-1}F(t)\right\| = 0 \quad \textit{or} \quad \int\limits_0^\infty \left\| PG^{-1}F(\tau)\right\| d\tau < \infty,$$

then $k_B(E, A+F) = k_B(E, A)$.

Remark 5.4. Theorem 5.2 and Corollary 5.3 not only give stability criteria for time-varying DAEs, but also provide the mathematical basis for the numerical computation of Bohl exponents and exponential dichotomy spectral intervals, see [25].

Definition 5.5. Suppose that Assumptions A1 and A3* hold. The DAE system (1.1) and the perturbed system (5.1) are said to be asymptotically equivalent (or integrally comparable) if $\lim_{t\to\infty}\|PG^{-1}F(t)\| = 0$ (or $\int_0^\infty \|PG^{-1}F(\tau)\|\,d\tau < \infty$, respectively).

So, the two DAE systems (1.1) and (5.1) have the same Bohl exponent provided that they are asymptotically equivalent (or integrally comparable). It is worth remarking that the conditions on $PG^{-1}F$ do not depend on the choice of a bounded projector P.

Example 5.6. Consider the linear DAE of the form (1.1) with data

$$E=\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix},\qquad A=\begin{pmatrix}2\sin(\ln(t+1))+2\cos(\ln(t+1))+t & \sin(\ln(t+1))+\cos(\ln(t+1))+t\\ 2 & 2\end{pmatrix}. \tag{5.10}$$

Furthermore, take a time-varying perturbation defined by

$$F(t)=\begin{pmatrix}-\frac{1}{1+t^2} & \frac{1}{\sqrt{1+t}}\\ e^{-2t} & -e^{-t}\end{pmatrix}.$$

Choose

$$Q=\begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix},$$

then after some matrix manipulations, we obtain

$$PG^{-1}F(t)=\begin{pmatrix}-\frac{1}{1+t^2}-\frac{te^{-2t}}{2} & \frac{1}{\sqrt{1+t}}+\frac{te^{-t}}{2}\\ \frac{1}{1+t^2}+\frac{te^{-2t}}{2} & -\frac{1}{\sqrt{1+t}}-\frac{te^{-t}}{2}\end{pmatrix},$$

which fulfills the assumptions of Corollary 5.3. It is easy to check that the DAE system (5.10) has a fundamental solution

$$\Phi(t)=\begin{pmatrix}1\\ -1\end{pmatrix}e^{(t+1)\sin(\ln(t+1))}.$$

Hence, due to [9] and Corollary 5.3, both the unpertubed DAE system and the perturbed one have the same Bohl exponent which equals $\sqrt{2}$.

Corollary 5.7. *Let Assumptions* A1, A2, *and* A3* *hold. If* (1.1) *and* (5.1) *are asymptotically equivalent or integrally comparable, then the perturbed system* (5.1) *generates an exponentially stable Cauchy operator, too.*

We now deal with the continuity of the stability radius of (5.1) with respect to the coefficient matrix A. To this end, we first state a key theorem about asymptotic behaviour of the norm of input–output operators.

Theorem 5.8. *Let Assumptions* A1, A2, *and* A3* *hold. In addition, suppose that*

$$\lim_{t\to\infty}\|F(t)\|=0.$$

Then

$$\lim_{t\to\infty}\|\mathbb{L}_t\|=\lim_{t\to\infty}\|\widetilde{\mathbb{L}}_t\|,$$

where $\widetilde{\mathbb{L}}_t$ *denotes the input–output operator for the perturbed system* (5.1).

Proof. The proof is divided into two main steps.

Step 1. First, we give an estimate to the difference between two fundamental matrices $\widetilde{\Phi}$ and Φ. We will show that there exist positive constants K_1, K_2 such that

$$\big\|\widetilde{\Phi}(t,s)-\Phi(t,s)\big\| \leqslant K_1 e^{-\omega(t-s)}\int_s^t h(\tau)\,d\tau + K_2 e^{-\omega(t-s)}\big\|F(t)\big\|, \quad t\geqslant s\geqslant 0, \tag{5.11}$$

where h is defined by (5.7).

We have

$$\frac{d}{dt}\big(P(t)\Phi(t,s)\big)=\big(PG^{-1}\overline{A}+P'\big)P(t)\Phi(t,s), \tag{5.12}$$

$$\begin{aligned}\frac{d}{dt}\big(P(t)\widetilde{\Phi}(t,s)\big)&=\big(PG^{-1}\overline{A}+P'\big)P(t)\widetilde{\Phi}(t,s)\\ &\quad+PG^{-1}F\big(I+\big(I-QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big)P(t)\widetilde{\Phi}(t,s)\end{aligned} \tag{5.13}$$

with $P(s)(\Phi(s,s)-I)=P(s)(\widetilde{\Phi}(s,s)-I)=0$. Subtracting side by side (5.12) from (5.13), we obtain

$$\begin{aligned}\frac{d}{dt}P(t)\widetilde{\Phi}(t,s)-\frac{d}{dt}P(t)\Phi(t,s)&=\big(PG^{-1}\overline{A}+P'\big)P(t)\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)\\ &\quad+PG^{-1}F\big(I-\big(I+QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big)P(t)\widetilde{\Phi}(t,s).\end{aligned}$$

Putting $Z(t,s)=P(t)(\widetilde{\Phi}(t,s)-\Phi(t,s))$, it yields

$$\frac{d}{dt}Z(t,s)=\big(PG^{-1}\overline{A}+P'\big)Z(t,s)+PG^{-1}F\big(I+\big(I-QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big)P(t)\widetilde{\Phi}(t,s)$$

with $Z(s,s)=0$. By the constant-variation method, we get

$$Z(t,s)=\int_s^t P\Phi(t,\tau)PG^{-1}F\big(I+\big(I-QG^{-1}F\big)^{-1}QG^{-1}(\overline{A}+F)\big)P\widetilde{\Phi}(\tau,s)\,d\tau.$$

Due to Proposition 4.6 and Corollary 5.3, there exist positive constants M and bM such that

$$\begin{aligned}\big\|P\Phi(t,s)\big\| &\leqslant \overline{M}e^{-\omega(t-s)}, &\quad \big\|P\widetilde{\Phi}(t,s)\big\| &\leqslant \overline{M}e^{-\omega(t-s)},\\ \big\|\Phi(t,s)\big\| &\leqslant Me^{-\omega(t-s)}, &\quad \big\|\widetilde{\Phi}(t,s)\big\| &\leqslant Me^{-\omega(t-s)},\end{aligned}\qquad t\geqslant s\geqslant 0.$$

Therefore, the estimates

$$\begin{aligned}\big\|Z(t,s)\big\| &\leqslant \int_s^t \big\|P\Phi(t,\tau)\big\|h(\tau)\big\|P\widetilde{\Phi}(\tau,s)\big\|\,d\tau\\ &\leqslant \overline{M}^2\int_s^t e^{-\omega(t-\tau)}h(\tau)e^{-\omega(\tau-s)}\,d\tau=\overline{M}^2e^{-\omega(t-s)}\int_s^t h(\tau)\,d\tau\end{aligned} \tag{5.14}$$

hold. Using this estimate and (5.5), we have

$$\big\|Q(t)\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)\big\| \leqslant \big\|QG^{-1}\overline{A}(t)P(t)\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)\big\|+\big\|QG^{-1}F(t)\widetilde{\Phi}(t,s)\big\|. \tag{5.15}$$

It follows

$$\big\|\widetilde{\Phi}(t,s)-\Phi(t,s)\big\| \leqslant \big(1+\|Q_s\|\big)\overline{M}^2 e^{-\omega(t-s)}\int_s^t h(\tau)\,d\tau + M\big\|QG^{-1}\big\|e^{-\omega(t-s)}\big\|F(t)\big\|, \tag{5.16}$$

which implies the estimate (5.11) with

$$K_1=\big(1+\|Q_s\|\big)\overline{M}^2, \qquad K_2=M\big\|QG^{-1}\big\|.$$

Step 2. Next, denote by $\widetilde{\mathbb{L}}_{t_0}, \widetilde{\mathbb{M}}_{t_0}, \widetilde{\mathbb{N}}_{t_0}$ the corresponding operator triplet defined by (3.3) for the system data $\{E, A+F\}$. For any $t>t_0\geqslant 0$ and for any

$$u\in L_p\big(t_0,\infty;\mathbb{K}^n\big), \quad \|u\|_p=1,$$

we have

$$\begin{aligned}
(\widetilde{\mathbb{L}}_{t_0}u)(t)-(\mathbb{L}_{t_0}u)(t) &= C(t)\int_{t_0}^t\big(\widetilde{\Phi}(t,s)P\widetilde{G}^{-1}-\Phi(t,s)PG^{-1}\big)B(s)u(s)\,ds + C(t)Q\left(\widetilde{G}^{-1}-G^{-1}\right)B(t)u(t)\\
&= C(t)\int_{t_0}^t\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)P\widetilde{G}^{-1}B(s)u(s)\,ds\\
&\quad + C(t)\int_{t_0}^t \Phi(t,s)P\big(\widetilde{G}^{-1}-G^{-1}\big)B(s)u(s)\,ds + C(t)Q\left(\widetilde{G}^{-1}-G^{-1}\right)B(t)u(t).
\end{aligned}$$

Now, we are able to estimate term by term the difference in L_p norm between the unperturbed operator and the perturbed one. First, we have

$$\begin{aligned}
\Delta_1(t_0,u) &:= \left\|C(\cdot)\int_{t_0}^{\cdot}\big(\widetilde{\Phi}(\cdot,s)-\Phi(\cdot,s)\big)P\widetilde{G}^{-1}B(s)u(s)\,ds\right\|_p\\
&= \left[\int_{t_0}^{\infty}\left\|C(t)\int_{t_0}^t\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)P\widetilde{G}^{-1}B(s)u(s)\,ds\right\|^p dt\right]^{\frac{1}{p}}\\
&\leqslant \|C\|\big\|P\widetilde{G}^{-1}B\big\|\left[\int_{t_0}^{\infty}\left\|\int_{t_0}^t\big(\widetilde{\Phi}(t,s)-\Phi(t,s)\big)u(s)\,ds\right\|^p dt\right]^{\frac{1}{p}}\\
&\leqslant \|C\|\big\|P\widetilde{G}^{-1}B\big\|\left[\int_{t_0}^{\infty}\left[\int_{t_0}^t e^{-\omega(t-s)}\left(K_1\int_s^t h(\tau)\,d\tau+K_2\big\|F(t)\big\|\right)\big\|u(s)\big\|\,ds\right]^p dt\right]^{\frac{1}{p}}\\
&\leqslant \|C\|\big\|P\widetilde{G}^{-1}B\big\|\left[K_1\int_0^{\infty}\left[\int_0^t e^{-\omega(t-s)}\int_{s+t_0}^{t+t_0}h(\tau)\,d\tau\big\|u(s+t_0)\big\|\,ds\right]^p dt\right]^{\frac{1}{p}}\\
&\quad + \|C\|\big\|P\widetilde{G}^{-1}B\big\|\left[K_2\int_0^{\infty}\left[\int_0^t e^{-\omega(t-s)}\big\|F(t+t_0)\big\|\big\|u(s+t_0)\big\|\,ds\right]^p dt\right]^{\frac{1}{p}}
\end{aligned}$$

$$\leqslant K_1\|C\|\left\|P\widetilde{G}^{-1}B\right\|\sup_{t\geqslant t_0}h(t)\left[\int_0^\infty\left[\int_0^t e^{-\omega(t-s)}(t-s)\left\|u(s+t_0)\right\|ds\right]^p dt\right]^{\frac{1}{p}}$$
$$+K_2\|C\|\left\|P\widetilde{G}^{-1}B\right\|\sup_{t\geqslant t_0}\left\|F(t)\right\|\left[\int_0^\infty\left[\int_0^t e^{-\omega(t-s)}\left\|u(s+t_0)\right\|ds\right]^p dt\right]^{\frac{1}{p}}.$$

By applying Young's inequality to each convolution product, e.g., see [3], we get

$$\Delta_1(t_0,u)\leqslant K_1\|C\|\left\|P\widetilde{G}^{-1}B\right\|\sup_{t\geqslant t_0}h(t)\|u\|_p\int_0^\infty e^{-\omega t}t\,dt$$
$$+K_2\|C\|\left\|P\widetilde{G}^{-1}B\right\|\sup_{t\geqslant t_0}\left\|F(t)\right\|\|u\|_p\int_0^\infty e^{-\omega t}\,dt$$
$$=\|C\|\left\|P\widetilde{G}^{-1}B\right\|\left(\frac{K_1}{\omega^2}\sup_{t\geqslant t_0}h(t)+\frac{K_2}{\omega}\sup_{t\geqslant t_0}\left\|F(t)\right\|\right)\|u\|_p.$$

Using the estimate (5.8) for function h, we obtain

$$\Delta_1(t_0,u)\leqslant\left(\frac{K_1}{\omega^2}K\left\|PG^{-1}\right\|+\frac{K_2}{\omega}\right)\|C\|\left\|P\widetilde{G}^{-1}B\right\|\sup_{t\geqslant t_0}\left\|F(t)\right\|\|u\|_p. \tag{5.17}$$

Next, from the difference between $\widetilde{G}^{-1}$ and G^{-1}

$$\begin{aligned}\widetilde{G}^{-1}-G^{-1}&=G^{-1}\left(I-FQG^{-1}\right)^{-1}-G^{-1}\\&=G^{-1}\left(\left(I-FQG^{-1}\right)^{-1}-I\right)=G^{-1}FQG^{-1}\left(I-FQG^{-1}\right)^{-1},\end{aligned} \tag{5.18}$$

we have

$$\left\|P\widetilde{G}^{-1}(t)-PG^{-1}(t)\right\|\leqslant\frac{\|PG^{-1}\|\|QG^{-1}\|}{1-\|F\|\|QG^{-1}\|}\left\|F(t)\right\|$$

and

$$\left\|Q\widetilde{G}^{-1}(t)-QG^{-1}(t)\right\|\leqslant\frac{\|QG^{-1}\|^2}{1-\|F\|\|QG^{-1}\|}\left\|F(t)\right\|$$

for all $t\geqslant 0$. Then,

$$\Delta_2(t_0,u):=\left\|C(\cdot)\int_{t_0}^{\cdot}\Phi(\cdot,s)P\left(\widetilde{G}^{-1}-G^{-1}\right)B(s)u(s)\,ds\right\|_p$$
$$=\left[\int_{t_0}^\infty\left\|C(t)\int_{t_0}^{t}\Phi(t,s)P\left(\widetilde{G}^{-1}-G^{-1}\right)B(s)u(s)\,ds\right\|^p dt\right]^{\frac{1}{p}}$$

$$\leqslant \frac{M\|C\|\|B\|\|PG^{-1}\|\|QG^{-1}\|}{1-\|F\|\|QG^{-1}\|}\left[\int_{t_0}^{\infty}\left[\int_{t_0}^{t}\|F(s)\|e^{-\omega(t-s)}\|u(s)\|\,ds\right]^p dt\right]^{\frac{1}{p}}$$

$$\leqslant \frac{M\|C\|\|B\|\|PG^{-1}\|\|QG^{-1}\|}{1-\|F\|\|QG^{-1}\|}\sup_{t\geqslant t_0}\|F(t)\|\left[\int_{0}^{\infty}\left[\int_{0}^{t}e^{-\omega(t-s)}\|u(s+t_0)\|\,ds\right]^p dt\right]^{\frac{1}{p}}.$$

By applying Young's inequality once again, we get

$$\begin{aligned}\Delta_2(t_0,u) &\leqslant \frac{M\|C\|\|B\|\|PG^{-1}\|\|QG^{-1}\|}{1-\|F\|\|QG^{-1}\|}\sup_{t\geqslant t_0}\|F(t)\|\|u\|_p\int_0^{\infty}e^{-\omega t}\,dt\\ &\leqslant \frac{M\|C\|\|B\|\|PG^{-1}\|\|QG^{-1}\|}{\omega(1-\|F\|\|QG^{-1}\|)}\sup_{t\geqslant t_0}\|F(t)\|\|u\|_p.\end{aligned}\tag{5.19}$$

Finally, we have

$$\begin{aligned}\Delta_3(t_0,u) &:= \left\|C(\cdot)Q\left(\widetilde{G}^{-1}-G^{-1}\right)B(\cdot)u(\cdot)\right\|_p\\ &= \left[\int_{t_0}^{\infty}C(t)Q\left(\widetilde{G}^{-1}-G^{-1}\right)B(t)u(t)\,dt\right]^{\frac{1}{p}}\\ &\leqslant \frac{\|C\|\|B\|\|QG^{-1}\|^2}{1-\|F\|\|QG^{-1}\|}\sup_{t\geqslant t_0}\|F(t)\|\|u\|_p.\end{aligned}\tag{5.20}$$

Since $\sup_{t\geqslant t_0}\|F(t)\|$ tends to zero as t_0 tends to infinity, the estimates (5.17), (5.19), and (5.20) imply

$$\lim_{t_0\to\infty}\|\widetilde{\mathbb{L}}_{t_0}-\mathbb{L}_{t_0}\|=0.$$

The proof is complete. □

Corollary 5.9. *In addition to the assumptions of Theorem* 5.8, *if either*

$$\left(\sup_{t\geqslant 0}\left\|CQ\widetilde{G}^{-1}B(t)\right\|\right)^{-1}\geqslant\left(\sup_{t\geqslant 0}\left\|CQG^{-1}B(t)\right\|\right)^{-1}$$

or both of these quantities are not less than $\lim_{t_0\to\infty}\|\mathbb{L}_{t_0}\|^{-1}$, *then*

$$r_{\mathbb{K}}(E,A+F;B,C)=r_{\mathbb{K}}(E,A;B,C).$$

Proof. The statement comes as a direct consequence of Theorems 3.9 and 5.8. □

Theorem 5.10. *Let Assumptions* A1 *and* A2 *hold. Furthermore, let* $\{F_k(\cdot)\}_{k=1}^{\infty}$ *be a bounded sequence of measurable matrix functions such that the following assumptions hold*:

(i) $$\lim_{t\to\infty}\left\|F_k(t)\right\|=0\quad\forall k=1,2,\ldots,$$

(ii) $$\sup_{t\geqslant 0}\left\|F_k(t)\right\|<\left(\sup_{t\geqslant 0}\left\|QG^{-1}(t)\right\|\right)^{-1}\quad\forall k=1,2,\ldots,$$

(iii) $$\lim_{k\to\infty}\sup_{t\geqslant 0}\left\|QG^{-1}F_k(t)\right\|=0.\tag{5.21}$$

Then,

$$\lim_{k\to\infty} r_{\mathbb{K}}(E, A+F_k; B, C) = r_{\mathbb{K}}(E, A; B, C). \tag{5.22}$$

Proof. Denote by $\widetilde{\mathbb{L}}_{t_0}^{(k)}, \widetilde{\mathbb{M}}_{t_0}^{(k)}, \widetilde{\mathbb{N}}_{t_0}^{(k)}$, $k = 1, 2, \ldots$, the sequence of corresponding operator triplets defined by (3.3) to the system data $\{E, A + F_k\}$, respectively. By applying Theorem 5.8, we have

$$\lim_{t_0\to\infty} \big\|\widetilde{\mathbb{L}}_{t_0}^{(k)}\big\|^{-1} = \lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|^{-1}, \quad \forall k = 1, 2, \ldots.$$

It remains to show that

$$\lim_{k\to\infty} \sup_{t\geqslant 0} \big\|CQ\widetilde{G}_k^{-1}B(t)\big\| = \lim_{k\to\infty} \sup_{t\geqslant 0} \big\|CQG^{-1}B(t)\big\|.$$

Using the same estimate for the difference between $\widetilde{G}_k^{-1}$ and G^{-1} as in the proof of Theorem 5.8, we have

$$\begin{aligned}
\big\|CQG\big(\widetilde{G}_k^{-1} - G^{-1}\big)B(t)\big\| &\leqslant \big\|CQG^{-1}F_kQG^{-1}\big(I - F_kQG^{-1}\big)^{-1}B(t)\big\| \\
&\leqslant \big\|CQG^{-1}F_k\big(I - QG^{-1}F_k\big)^{-1}QG^{-1}B(t)\big\| \\
&\leqslant \|C\|\|B\|\big\|QG^{-1}\big\|\big\|\big(I - QG^{-1}F_k\big)^{-1}\big\|\big\|QG^{-1}F_k(t)\big\|.
\end{aligned}$$

Note that due to assumption (iii), it is not difficult to show that $\|(I - QG^{-1}F_k)^{-1}\|$ is uniformly bounded with respect to k. Hence, assumption (iii) implies

$$\lim_{k\to\infty} \operatorname*{ess\,sup}_{t\geqslant 0} \big\|CQG\big(\widetilde{G}_k^{-1} - G^{-1}\big)B(t)\big\| = 0,$$

or equivalently

$$\lim_{k\to\infty} \big\|\widetilde{\mathbb{N}}_0^{(k)}\big\| = \|\mathbb{N}_0\|.$$

Invoking Theorem 3.9, we get

$$\lim_{k\to\infty} r_{\mathbb{K}}(E, A+F_k; B, C) = r_{\mathbb{K}}(E, A; B, C).$$

The proof is complete. □

By simplifying the assumptions, we get an easier-to-check sufficient conditions such that the statement of the above theorem remains true.

Theorem 5.11. *Let Assumptions* A1 *and* A2 *hold and* $\{F_k(\cdot)\}_{k=1}^{\infty}$ *be a sequence of measurable matrix functions. In addition, we suppose*

(i) $$\sup_{t\geqslant 0} \big\|F_k(t)\big\| < \Big(\sup_{t\geqslant 0} \big\|QG^{-1}(t)\big\|\Big)^{-1} \quad \forall k = 1, 2, \ldots, \tag{5.23}$$

(ii) $$\lim_{k\to\infty} \sup_{t\geqslant 0} \big\|F_k(t)\big\| = 0. \tag{5.24}$$

Then

$$\lim_{k\to\infty} r_{\mathbb{K}}(E, A + F_k; B, C) = r_{\mathbb{K}}(E, A; B, C).$$

Proof. First, we recall that assumption (i) implies Assumption A3 for the DAE systems with data $\{E, A + F_k\}$. By using the same techniques as in the proof of Theorems 5.8 and 5.10, we obtain

$$\lim_{k\to\infty} \lim_{t_0\to\infty} \big\|\widetilde{\mathbb{L}}_{t_0}^{(k)}\big\| = \lim_{t_0\to\infty} \|\mathbb{L}_{t_0}\|,$$
$$\lim_{k\to\infty} \big\|\widetilde{\mathbb{N}}_0^{(k)}\big\| = \|\mathbb{N}_0\|.$$

By invoking Theorem 3.9, the proof is complete. □

The result of Theorem 5.11 means exactly that the stability radii for the system (1.1) depend continuously on the coefficient matrix function A. As a consequence of Theorem 3.11, we get a result for almost time-invariant systems.

Corollary 5.12. *Let E, A, B, C be constant matrices, the system* (1.1) *be index*-1 *and exponentially stable. Furthermore, the sequence of time-varying perturbation $\{F_k\}_{k=1}^{\infty}$ fulfills the conditions of either Theorem* 5.10 *or Theorem* 5.11. *Then, for $p = 2$, i.e. the Euclidean norm is used, we have*

$$\lim_{k\to\infty} r_{\mathbb{C}}(E, A + F_k; B, C) = \Big(\sup_{w\in i\mathbb{R}} \big\|C(wE - A)^{-1}B\big\|\Big)^{-1}.$$

Example 5.13. Consider the simple example of a linear constant coefficient DAE with data

$$E = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} -2 & -1 \\ 2 & 2 \end{pmatrix}, \qquad B = I, \qquad C = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5.25}$$

Let a sequence of time-varying perturbations be defined by

$$F_k(t) = \begin{pmatrix} -\frac{1}{3+t^2} & \frac{1}{4\sqrt{1+t}} \\ \frac{e^{-2t}}{k+1} & -\frac{e^{-t}}{2k} \end{pmatrix}, \quad k = 1, 2, \ldots. \tag{5.26}$$

Here, we choose

$$Q = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}, \qquad G = E - AQ = \begin{pmatrix} 2 & 1 \\ -2 & -2 \end{pmatrix}.$$

Then it is easy to check that $\lim_{t\to\infty} \|F_k(t)\| = 0$ and

$$\sup_{t\geqslant 0} \big\|F_k(t)\big\| < \left\|\begin{pmatrix} 1/3 & 1/4 \\ 1/2 & 1/2 \end{pmatrix}\right\| \approx 0.8192 < \big\|QG^{-1}\big\|^{-1} \approx 0.8945.$$

Furthermore, we have

$$QG^{-1}F_k(t) = \begin{pmatrix} \frac{e^{-2t}}{2(k+1)} & -\frac{e^{-t}}{4k} \\ -\frac{e^{-2t}}{k+1} & \frac{e^{-t}}{2k} \end{pmatrix},$$

thus $\lim_{k\to\infty}\sup_{t\geqslant 0}\|QG^{-1}F_k(t)\|=0$. That is, all the three assumptions of Theorem 5.10 hold. On the other hand, by elementary calculations, we obtain

$$\Big(\sup_{w\in i\mathbb{R}}\big\|C(wE-A)^{-1}B\big\|\Big)^{-1}=1.$$

Invoking Theorem 5.10 or Corollary 5.12, we have

$$\lim_{k\to\infty} r_{\mathbb{C}}(E, A+F_k; B, C)=1.$$

Remark 5.14. A practical consequence of Theorems 5.10, 5.11 and Corollary 5.12 which was also one of our motivations leading to this work is that the stability radii of a time-varying DAE system can be well approximated in principle by the stability radii of a time-invariant system, which can be calculated numerically. Note also that the computation of the stability radius of a general time-varying system using the norm of the input–output operator, see Theorem 3.9, seems to be very complicated in practice.

We turn to the case of regular explicit systems, i.e., E is the identity matrix. As a special case, the projector functions are chosen (uniquely) as $P=I$, $Q=0$. The index requirement becomes unnecessary. Then, one gets a result for ODEs which could also be available as a direct consequence of [16, Proposition 4.5] and [21, Theorem 4.1].

Theorem 5.15. *Let the Cauchy operator of explicit system* (1.1) *be exponentially stable. If* (1.1) *and* (5.1) *are asymptotically equivalent or integrally comparable, i.e., either* $F(\cdot)\in L_1(0,\infty;\mathbb{K}^{n\times n})$ *or* $\lim_{t\to\infty}\|F(t)\|=0$ *holds, then*

$$r_{\mathbb{K}}(I, A+F; B, C)=r_{\mathbb{K}}(I, A; B, C).$$

Finally, by similar arguments as in Theorems 5.10 and 5.11, we can analyze the dependence of the stability radii on the perturbation structure.

Theorem 5.16. *Suppose that Assumptions* A1–A2 *hold for general time-varying system* (1.1)*. Let* $B_k(t)$ *and* $C_k(t)$ *be two sequences of measurable and essentially bounded matrix functions satisfying*

$$\lim_{k\to\infty}\operatorname*{ess\,sup}_{t\geqslant 0}\big\|B_k(t)-B(t)\big\|=0,\qquad \lim_{k\to\infty}\operatorname*{ess\,sup}_{t\geqslant 0}\big\|C_k(t)-C(t)\big\|=0, \tag{5.27}$$

then

$$\lim_{k\to\infty} r_{\mathbb{K}}(E, A; B_k, C_k)=r_{\mathbb{K}}(E, A; B, C).$$

For regular explicit systems, the statement is still true under a less restrictive condition.

Corollary 5.17. *Let the Cauchy operator of explicit system* (1.1) *be exponentially stable. Let* $\overline{B}(t)$ *and* $\overline{C}(t)$ *be two measurable and essentially bounded matrix functions satisfying*

$$\lim_{s\to\infty}\operatorname*{ess\,sup}_{t\geqslant s}\big\|\overline{B}(t)-B(t)\big\|=0,\qquad \lim_{s\to\infty}\operatorname*{ess\,sup}_{t\geqslant s}\big\|\overline{C}(t)-C(t)\big\|=0. \tag{5.28}$$

Then,

$$r_{\mathbb{K}}(I, A; \overline{B}, \overline{C})=r_{\mathbb{K}}(I, A; B, C).$$

Remark 5.18. By comparing the results for DAEs with those for ODEs, one can see an essential difference between the robust stability of DAEs and that of ODEs. For DAEs, because the dynamics is constrained and the index-1 property should be kept to provide the existence and uniqueness of solution, only weaker results hold but under some extra assumptions.

6. Conclusions

In this paper we have analyzed the data-dependence of the exponential stability and of the stability radii for linear time-varying differential-algebraic systems of index 1. The Bohl exponent theory that is well known for ODEs has been generalized to DAEs. Relevant properties of the Bohl exponent as well as the relation between the exponential stability and the existence of a bounded solution to an inhomogeneous DAE have been investigated. As a main result, we have shown that the Bohl exponent and the stability radii depend continuously on the coefficient matrix A. As a practical consequence, the complex stability radius of DAE systems with asymptotically constant coefficients can be approximated by a computable formula.

One may ask a natural question what would happen with perturbations occurring in the leading coefficient matrix E and whether we could expect similar results as those in this work. Unfortunately, the exponential stability of DAE system is sensitive with respect to perturbations in the leading term E, even in the case with constant coefficients, e.g., see [6,11]. So a similar result can be expected only for the case of certain class of “admissible” structured perturbations. As a future work, an analysis of the exponential stability and the stability radii with respect to perturbations occurring in the first coefficient matrix E seems to be an interesting problem, for which more technical difficulties are expected. In particular, an investigation of the robust stability of singularly perturbed time-varying systems and(or) slowly time-varying DAE systems would be of interest, as well.

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