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PERIODIC SOLUTION FOR GENERALIZED HIGH-ORDER DELAY DIFFERENTIAL EQUATIONS*

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In this paper we consider a generalized n-th order delay differential equation, by applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions. Moreover, an example is given to illustrate the results.

Keywords: Periodic solution; high-order; delay differential equation.

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1. Introduction

Complicated behavior of models for technical applications is often described by nonlinear high-order differential equations (see [6]), for example, the Lorenz model of a simplified hydrodynamic flow, the dynamo model of erratic inversion of the earth's magnetic field, etc. Oftentimes high-order equations are a result of combinations of lower order equations. However, while there are plenty of results on the existence of periodic solutions for various types of second-order differential equations with

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(or without) delay (see [1]-[3] and references therein), studies on periodic solutions for high-order differential equation are rather infrequent, especially for high-order delay differential equation. In 1998, using the Schauder fixed-point theorem, Cong [4] studied the $(2k)$th-order differential equation

$$x^{(2k)} + \sum_{j=1}^{k-1} \alpha_j x^{(2j)} + (-1)^{k+1} f(t, x) = 0, \quad x \in \mathbb{R}^n, \tag{1.1}$$

and obtained sufficient conditions for the existence of periodic solutions. Afterwards, Cong, Huang and Shi [5] improved the results in [4] to $(2n+1)$th-order differential equations. Recently, in [8], Jazar and Mahinfalah investigated a general nonlinear third order differential equation

$$x''' + f(t, x, x', x'') = 0, \tag{1.2}$$

and showed that the equation has nontrivial periodic solutions by applying the Schauder fixed-point theorem. All the above results on high-order differential equations are confined to systems without delay, we study equations with delay which is known to be an important ingredient in more realistic models.

We consider a generalized n-th order delay differential equation of the following form

$$(\varphi_p(x^{(l)}(t)))^{(n-l)} = F(t, x(t), x(t - \tau(t)), x'(t), \cdots, x^{(l-1)}(t)), \tag{1.3}$$

where $1 \leq l \leq n$ and $n \in \mathbb{N}$, $\varphi_p : \mathbb{R} \to \mathbb{R}$ is given by $\varphi_p(s) = |s|^{p-2}s$ with a constant $p \geq 2$, in particular $\varphi_2 \equiv \mathrm{id}$, τ is a continuous periodic function defined on $\mathbb{R}$ with period T; F is a continuous function defined on $\mathbb{R}^{l+1}$ and is periodic in t, i.e. $F(t, \cdot, \cdots, \cdot) = F(t+T, \cdot, \cdots, \cdot)$, and $F(t, c, c, 0, \cdots, 0) \not\equiv 0$ for all $c \in \mathbb{R}$. Obviously, (1.1) and (1.2) are just special cases of (1.3) with $p = 2$. We will first transform (1.3) into a first-order differential equations and then by applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions for (1.3). Our results are new and our method is different from those in the above references. Moreover, an example is given to illustrate our results.

2. Preparation

First, we recall two lemmas. Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $Im\ L$ is closed in Y and $\dim Ker\ L = \dim(Y/Im\ L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y respectively, such that $X = Ker\ L \oplus X_1$, $Y = Im\ L \oplus Y_1$, and let $P : X \to Ker\ L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $Ker\ L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I - Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1. *(Gaines and Mawhin [7]) Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;
(2) $Nx \notin Im\ L, \forall x \in \partial\Omega \cap Ker\ L$;
(3) $deg\{JQN, \Omega \cap Ker\ L, 0\} \neq 0$, where $J : Im\ Q \to Ker\ L$ is an isomorphism.

Then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2. *(see [9]) If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then*

$$\int_0^T |\omega(t)|^p dt \leq \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'(t)|^p dt$$

where p is a fixed real number with $p > 1$, and $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

Now we consider (1.3). Define the conjugate index $q \in (1, 2]$ by $\frac{1}{p} + \frac{1}{q} = 1$. Introducing new variables

$$y_1(t) = x(t), \quad y_2(t) = x'(t), \quad y_3(t) = x''(t), \quad \ldots \quad, y_l(t) = x^{(l-1)}(t),$$

$$y_{l+1}(t) = \varphi_p(x^{(l)}(t)), \ \ y_{l+2}(t) = (\varphi_p(x^{(l)}(t)))', \ \ \ldots \ , \ \ y_n(t) = (\varphi_p(x^{(l)}(t)))^{(n-l-1)},$$

and using the fact that $\varphi_q \circ \varphi_p \equiv \text{id}$, (1.3) can be rewritten as

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y_3(t) \\ \quad\vdots \\ y_{l-1}'(t) = y_l(t) \\ y_l'(t) = \varphi_q(y_{l+1}(t)) \\ y_{l+1}'(t) = y_{l+2}(t) \\ \quad\vdots \\ y_{n-1}'(t) = y_n(t) \\ y_n'(t) = F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t)). \end{cases} \tag{2.1}$$

It is clear that if $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^\top$ is a T-periodic solution to (2.1), then $y_1(t)$ is a T-periodic solution to (1.3). Thus, the problem of finding a T-periodic solution for (1.3) reduces to finding one for (2.1). Define the linear spaces

$$X = Y = \{y = (y_1(\cdot), y_2(\cdot), \cdots, y_n(\cdot))^\top \in C^0(\mathbb{R}, \mathbb{R}^n) : y(t+T) \equiv y(t)\}$$

with norm $||y|| = \max\{||y_1||, ||y_2||, \cdots, ||y_n||\}$. Obviously, X and Y are Banach spaces. Define

$$L : D(L) = \{y \in C^1(\mathbb{R}, \mathbb{R}^n) : y(t+T) = y(t)\} \subset X \to Y$$

by

$$Ly = y' = \begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_l' \\ \vdots \\ y_n' \end{pmatrix}$$

and

$$N : X \to Y$$

by

$$Ny = \begin{pmatrix} y_2(t) \\ y_3(t) \\ \vdots \\ \varphi_q(y_{l+1}(t)) \\ y_{l+2}(t) \\ \vdots \\ F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t)) \end{pmatrix}. \tag{2.2}$$

Then Eq. (2.1) can be rewritten as the abstract equation $Ly = Ny$. From the definition of L, one can easily see that $Ker\ L = \{y \in C^1(\mathbb{R}, \mathbb{R}^n) : y \text{ is constant}\} \simeq \mathbb{R}^n$, $Im\ L = \{y : y \in X,\ \int_0^T y(s)ds = 0\}$. So L is a Fredholm operator with index zero. Let $P : X \to Ker\ L$ and $Q : Y \to Im\ Q$ be defined by

$$Py = \frac{1}{T}\int_0^T y(s)ds; \quad Qy = \frac{1}{T}\int_0^T y(s)ds.$$

It is easy to see that $Ker\ L = Im\ Q = \mathbb{R}^n$. Moreover, for all $y \in Y$, if we write $y^* = y - Q(y)$, we have $\int_0^T y^*(s)ds = 0$ and so $y^* \in Im\ L$. This is to say $Y = Im\ Q \oplus Im\ L$ and $\dim(Y/Im\ L) = \dim Im\ Q = \dim Ker\ L$. So, L is a Fredholm operator with index zero. Let K denote the inverse of $L|_{Ker\ p\ \cap D(L)}$, we have

$$[Ky](t) = \left(\int_0^T G_1(t,s)y_1(s)ds, \int_0^T G_2(t,s)y_2(s)ds, \ldots, \int_0^T G_n(t,s)y_n(s)ds\right)^\top$$

where

$$G_i(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s < t \le T \\ \frac{s-T}{T}, & 0 \le t \le s \le T \end{cases} \qquad i = 1,\ 2,\ \ldots,\ n\ . \tag{2.3}$$

From (2.2) and (2.3), it is clear that QN and $K(I-Q)N$ are continuous, and $QN(\overline{\Omega})$ is bounded and then $K(I-Q)N(\overline{\Omega})$ is compact for any open bounded $\Omega \subset X$ which means N is L-compact on $\bar{\Omega}$. For the functions in the domain of L we have

Lemma 2.3. *If $y(t) \in C^1(\mathbb{R}, \mathbb{R}^n)$ and $y(t+T) = y(t)$, then*

$$\int_0^T |y_1'(t)|^p dt \le \left(\frac{T}{\pi_p}\right)^{p(l-1)} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y_n'(t)|^q dt,$$

where $\frac{1}{p} + \frac{1}{q} = 1, \quad p \ge 2.$

Proof. From $y_1(0) = y_1(T)$, there is a point $t_1 \in [0,T]$ such that $y_1'(t_1) = 0$. Let $\omega_1(t) = y_1'(t+t_1)$, and then $\omega_1(0) = \omega_1(T) = 0$. From $y_2(0) = y_2(T)$, there is a point $t_2 \in [0,T]$ such that $y_2'(t_2) = 0$. Let $\omega_2(t) = y_2'(t+t_2)$, and then $\omega_2(0) = \omega_2(T) = 0$. Continuing this way we get from $y_{l-1}(0) = y_{l-1}(T)$ a point $t_{l-1} \in [0,T]$ such that $y_{l-1}'(t_{l-1}) = 0$. Let $\omega_{l-1}(t) = y_{l-1}'(t+t_{l-1})$, and then $\omega_{l-1}(0) = \omega_{l-1}(T) = 0$. From $y_l(0) = y_l(T)$, there is a point $t_l \in [0,T]$ such that $y_l'(t_l) = 0$, then we have $\varphi_p(y_l'(t_l)) = 0$. Let $\omega_l(t) = \varphi_p(y_l'(t+t_l)) = y_{l+1}(t+t_l)$, then we have $\omega_l(0) = \omega_l(T) = 0$. Continuing this way we get from $y_{n-1}(0) = y_{n-1}(T)$ a point $t_{n-1} \in [0,T]$ such that $y_{n-1}'(t_{n-1}) = 0$. Let $\omega_{n-1}(t) = y_{n-1}'(t+t_{n-1})$, and then $\omega_{n-1}(0) = \omega_{n-1}(T) = 0$. From Lemma 2.2, we have

$$\begin{aligned}
\int_0^T |y_1'(t)|^p dt &= \int_0^T |\omega_1(t)|^p dt \\
&\le \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega_1'(t)|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^p \int_0^T |y_2'(t)|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega_2(t)|^p dt \\
&\le \left(\frac{T}{\pi_p}\right)^{2p} \int_0^T |\omega_2'(t)|^p dt \\
&\cdots \\
&\le \left(\frac{T}{\pi_p}\right)^{p(l-1)} \int_0^T |\omega_{l-1}'(t)|^p dt \\
&= \left(\frac{T}{\pi_p}\right)^{p(l-1)} \int_0^T |y_l'(t)|^p dt
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\int_0^T |y_l'(t)|^p dt &= \int_0^T |\varphi_q(y_{l+1}(t))|^p dt \\
&= \int_0^T |y_{l+1}(t)|^{pq-p} dt \\
&= \int_0^T |y_{l+1}(t)|^q dt \\
&= \int_0^T |\omega_l(t)|^q dt \\
&\le \left(\frac{T}{\pi_q}\right)^q \int_0^T |\omega_l'(t)|^q dt \qquad (2.5)\\
&= \left(\frac{T}{\pi_q}\right)^q \int_0^T |y_{l+2}(t)|^q dt \\
&\cdots \\
&\le \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |\omega_{n-1}'(t)|^q dt \\
&= \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y_n'(t)|^q dt.
\end{aligned}$$

From (2.4) and (2.5), we can get

$$\int_0^T |y_1'(t)|^p dt \le \left(\frac{T}{\pi_p}\right)^{p(l-1)} \left(\frac{T}{\pi_q}\right)^{q(n-l)} \int_0^T |y_n'(t)|^q dt.$$

This completes the proof of Lemma 2.3. □

Remark 2.1. In particular, if we take $p = 2$, then $q = 2$ and $\pi_2 = \frac{2\pi(2-1)^{1/2}}{2\sin(\pi/2)} = \pi$.

3. Main Results

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) there is a constant $D > 0$ such that

$$z_1 F(t, z_1, z_2, \cdots, z_{l+1}) > 0, \quad \forall\ (t, z_1, z_2, \cdots, z_{l+1}) \in [0,T] \times \mathbb{R}^{l+1} \ \text{ with } \ |z_1| > D;$$

(H_2) there is a constant $M > 0$ such that

$$|F(t, z_1, z_2, \cdots, z_{l+1})| \le M, \quad \forall\ (t, z_1, z_2, \cdots, z_{l+1}) \in [0,T] \times \mathbb{R}^{l+1};$$

(H_3) there are non-negative constants $\alpha_1, \alpha_2, \cdots, \alpha_{n+1}, m$, such that

$$|F(t, z_1, z_2, \cdots, z_{l+1})| \le \alpha_1|z_1| + \alpha_2|z_2| + \cdots + \alpha_{l+1}|z_{l+1}| + m,$$

for all $(t, z_1, z_2, \cdots, z_{l+1}) \in [0,T] \times \mathbb{R}^{l+1}$.

Theorem 3.1. *If* (H_1) *and* (H_2) *hold, then* (3.1) *has at least one non-constant* T*-periodic solution.*

Proof. Consider the equation

$$Ly = \lambda N y, \quad \lambda \in (0,1).$$

Let $\Omega_1 = \{y : Ly = \lambda N y, \lambda \in (0,1)\}$. If $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^\top \in \Omega_1$, then

$$\begin{cases} y_1'(t) = \lambda y_2(t) \\ y_2'(t) = \lambda y_3(t) \\ \qquad \vdots \\ y_{l-1}'(t) = \lambda y_l(t) \\ y_l'(t) = \lambda \varphi_q(y_{l+1}(t)) \\ y_{l+1}'(t) = \lambda y_{l+2}(t) \\ \qquad \vdots \\ y_{n-1}'(t) = \lambda y_n(t) \\ y_n'(t) = \lambda F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t)). \end{cases} \tag{3.1}$$

We first claim that there is a constant $\xi \in \mathbb{R}$ such that

$$|y_1(\xi)| \le D. \tag{3.2}$$

Integrating the last equation of (3.1) over $[0,T]$, we have

$$\int_0^T F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t))dt = 0.$$

From the continuity of F, we know there exists a $\xi \in [0,T]$ such that

$$F(\xi, y_1(\xi), y_1(\xi - \tau(\xi)), \cdots, y_l(\xi)) = 0.$$

From assumption (H_1) we get (3.2). As a consequence, we have

$$|y_1(t)| = \left| y_1(\xi) + \int_\xi^t y_1'(s)ds \right| \le D + \int_0^T |y_1'(s)|ds. \tag{3.3}$$

On the other hand, multiplying both sides of the last equation of (3.1) by $y_n'(t)$ and integrating over $[0,T]$ we get, using assumption (H_2),

$$\begin{aligned} \int_0^T |y_n'(t)|^2 dt &= \lambda \int_0^T F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t)) y_n'(t)dt \\ &\le \int_0^T |F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t))||y_n'(t)|dt \\ &\le M \int_0^T |y_n'(t)|dt \\ &\le M T^{1/2} \left(\int_0^T |y_n'(t)|^2 dt \right)^{1/2}. \end{aligned}$$

It is easy to see that there is a constant $M_n' > 0$ (independent of λ) such that

$$\int_0^T |y_n'(t)|^2 dt \leq M_n'.$$

From $y_{n-1}(0) = y_{n-1}(T)$, there is a point $t_1 \in [0, T]$ such that $y_n(t_1) = 0$ and by applying Hölder's inequality, we have

$$|y_n(t)| \leq \int_0^T |y_n'(t)| dt \leq T^{1/2} \left(\int_0^T |y_n'(t)|^2 dt \right)^{1/2} \leq T^{1/2} (M_n')^{1/2} =: M_n.$$

From $y_{n-2}(0) = y_{n-2}(T)$, there is a point $t_2 \in [0, T]$ such that $y_{n-1}(t_2) = 0$, we have

$$|y_{n-1}(t)| \leq \int_0^T |y_{n-1}'(t)| dt = \int_0^T |\lambda y_n(t)| dt \leq \int_0^T |y_n(t)| dt \leq T M_n =: M_{(n-1)}.$$

Similarly,

$$|y_{n-2}(t)| \leq T M_{(n-1)} =: M_{(n-2)}.$$

And continuing this way for $y_{n-3}, \ldots, y_{l+1}$, we get

$$|y_{l+1}(t)| \leq T M_{(l+2)} =: M_{(l+1)}.$$

$$|y_l(t)| \leq \int_0^T |y_l'(t)| dt \leq \int_0^T |\lambda \varphi_q(y_{l+1}(t))| dt \leq \int_0^T |y_{l+1}(t)|^{q-1} dt \leq T (M_{(l+1)})^{q-1} =: M_l.$$

$$|y_{l-1}(t)| \leq T M_l =: M_{(l-1)}.$$

$$\vdots$$

$$|y_2(t)| \leq T M_3 =: M_2.$$

Moreover, from (3.3), we have

$$|y_1(t)| \leq D + \int_0^T |y_1'(t)| dt \leq D + T M_2 =: M_1.$$

Let $M_0 = \max\{M_1, M_2, \ldots, M_n\}$, obviously $||y_1|| \leq M_0$, $||y_2|| \leq M_0$, $\ldots$, $||y_n|| \leq M_0$.

Let $\Omega_2 = \{y \in Ker\ L : Ny \in Im\ L\}$. If $y \in \Omega_2$, then $y \in Ker\ L$ which means $y =$ constant and $QNy = 0$. We see that

$$\begin{cases} y_2 = 0, \\ y_3 = 0, \\ \quad \vdots \\ y_n = 0, \\ F(t, y_1, y_1, 0, \cdots, 0) = 0. \end{cases}$$

So

$$|y_1| \leq D \leq M_0, \qquad y_2 = y_3 = \cdots = y_n = 0 \leq M_0.$$

Now take $\Omega = \{y = (y_1, y_2, \cdots, y_n)^\top \in X : \|y_1\| < M_0 + 1, \ \|y_2\| < M_0 + 1, \ldots, \|y_n\| < M_0 + 1\}$. By the analysis of the above, it is easy to see that $\overline{\Omega}_1 \subset \Omega$, $\overline{\Omega}_2 \subset \Omega$ and conditions (1) and (2) of Lemma 2.1 are satisfied.

Next we show that condition (3) of Lemma 2.1 is also satisfied. Define the isomorphism $J : Im\ Q \to Ker\ L$ as follows:

$$J(y_1, y_2, \cdots, y_n)^\top = (y_n, y_1, \cdots, y_{n-1})^\top.$$

Let $H(\mu, y) = \mu y + (1-\mu)JQNy$, $(\mu, y) \in [0,1] \times \Omega$, then $\forall\ (\mu, y) \in (0,1) \times (\partial\Omega \cap Ker\ L)$,

$$H(\mu, y) = \begin{pmatrix} \mu y_1 + \frac{1-\mu}{T}\int_0^T F(t, y_1, y_1, 0, \cdots, 0)dt \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

From (H_1), it is obvious that $yH(\mu, y) > 0$, $\forall\ (\mu, y) \in (0,1) \times (\partial\Omega \cap Ker\ L)$. Therefore,

$$\begin{aligned} \deg\{JQN, \Omega \cap Ker\ L, 0\} &= \deg\{H(0,y), \Omega \cap Ker\ L, 0\} \\ &= \deg\{H(1,y), \Omega \cap Ker\ L, 0\} \\ &= \deg\{I, \Omega \cap Ker\ L, 0\} \neq 0, \end{aligned}$$

which means condition (3) of Lemma 2.1 is also satisfied. By applying Lemma 2.1, we conclude that equation $Ly = Ny$ has a solution $y(t)^* = (y_1^*(t), y_2^*(t), \cdots, y_n^*(t))^\top$ on $\bar{\Omega}$, i.e., (1.3) has a T-periodic solution $y_1^*(t)$ with $||y_1^*|| < M_0 + 1$.

Finally, observe that $y_1^*(t)$ is not constant. Otherwise, suppose $y_1^* \equiv c$ (constant), then from (1.3) we have $F(t, c, c, 0, \cdots, 0) \equiv 0$, which contradicts the assumption $F(t, c, c, 0, \cdots, 0) \not\equiv 0$, so the proof is complete. □

Theorem 3.2. *If* (H_1) *and* (H_3) *hold, then* (1.3) *has at least one non-constant* T*-periodic solution, if one of the following conditions hold:*

(1) $p > 2$*;*

or

(2) $p = 2$ *and* $\left[((\alpha_1 + \alpha_2)T + \alpha_3)(\frac{T}{\pi})^{l-1} + \alpha_4(\frac{T}{\pi})^{l-2} + \cdots + \alpha_{l+1}(\frac{T}{\pi})\right](\frac{T}{\pi})^{n-l} < 1.$

Proof. Let Ω_1 be defined as in Theorem 3.1. If $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^\top \in \Omega_1$, then from the proof of Theorem 3.1 we see that

$$y_n'(t) = \lambda F(t, y_1(t), y_1(t - \tau(t)), y_2(t), \cdots, y_l(t)), \tag{3.4}$$

and

$$|y_1|_0 \leq D + \int_0^T |y_1'(s)|ds.$$

We claim that $|y_n|_0$ is bounded.

Multiplying both sides of (3.4) by $\varphi_q(y_n'(t))$ and integrating it over $[0,T]$, by using assumption (H_3), we have

$$\begin{aligned}
&\int_0^T |y_n'(t)|^q dt \\
&\quad =\lambda \int_0^T F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t))\varphi_q(y_n'(t))dt \\
&\quad \leq \int_0^T |F(t, y_1(t), y_1(t-\tau(t)), y_2(t), \cdots, y_l(t))||\varphi_q(y_n'(t))|dt \\
&\quad \leq \alpha_1 |y_1|_0 \int_0^T |\varphi_q(y_n'(t))|dt + \alpha_2 |y_1|_0 \int_0^T |\varphi_q(y_n'(t))|dt \\
&\quad +\alpha_3 \int_0^T |y_2(t)||\varphi_q(y_n'(t))|dt + \cdots + \alpha_{l+1} \int_0^T |y_l(t)||\varphi_q(y_n'(t))|dt \\
&\quad +m \int_0^T |\varphi_q(y_n'(t))|dt \\
&\quad \leq (\alpha_1+\alpha_2)\left(D + \int_0^T |y_1'(t)|dt\right) \int_0^T |\varphi_q(y_n'(t))|dt \\
&\quad +\alpha_3 \int_0^T |y_2(t)||\varphi_q(y_n'(t))|dt + \cdots + \alpha_{l+1} \int_0^T |y_l(t)||\varphi_q(y_n'(t))|dt \\
&\quad +m \int_0^T |\varphi_q(y_n'(t))|dt\ .
\end{aligned}$$

Applying Hölder's inequality, we get

$$\begin{aligned}
&\int_0^T |y_n'(t)|^q dt \\
&\quad \leq (\alpha_1+\alpha_2)\left[D + T^{\frac{1}{p}} \left(\int_0^T |y_1'(t)|^q dt\right)^{\frac{1}{q}}\right] T^{\frac{1}{q}} \left(\int_0^T |\varphi_q(y_n'(t))|^p dt\right)^{\frac{1}{p}} \\
&\quad +\alpha_3 \left(\int_0^T |y_2(t)|^q dt\right)^{\frac{1}{q}} \cdot \left(\int_0^T |\varphi_q(y_n'(t))|^p dt\right)^{\frac{1}{p}} + \cdots \\
&\quad +\alpha_{l+1} \left(\int_0^T |y_l(t)|^q dt\right)^{\frac{1}{q}} \left(\int_0^T |\varphi_q(y_n'(t))|^p dt\right)^{\frac{1}{p}} + mT^{\frac{1}{q}} \left(\int_0^T |\varphi_q(y_n'(t))|^p dt\right)^{\frac{1}{p}} \\
&\quad \leq (\alpha_1+\alpha_2)T \cdot T^{\frac{p-2}{q(p-1)}} \left(\int_0^T |y_1'(t)|^p dt\right)^{\frac{1}{q(p-1)}} \left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}} \\
&\quad +\alpha_3 T^{\frac{p-2}{q(p-1)}} \left(\int_0^T |y_2(t)|^p dt\right)^{\frac{1}{q(p-1)}} \cdot \left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}} + \cdots \\
&\quad +\alpha_{l+1} T^{\frac{p-2}{q(p-1)}} \left(\int_0^T |y_l(t)|^p dt\right)^{\frac{1}{q(p-1)}} \left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}} \\
&\quad +((\alpha_1+\alpha_2)D + m)T^{\frac{1}{q}} \left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}\ .
\end{aligned} \tag{3.5}$$

Then by applying Lemma 2.3 and (3), we get

$$\begin{aligned}
&\textstyle\int_0^T |y_n'(t)|^q dt\\
&\le (\alpha_1+\alpha_2)T\cdot T^{\frac{p-2}{p}}\left(\int_0^T |y_1'(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}\\
&+\alpha_3 T^{\frac{p-2}{p}}\left(\int_0^T |y_2(t)|^p dt\right)^{\frac{1}{p}}\\
&\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}+\cdots+\alpha_{l+1}T^{\frac{p-2}{p}}\left(\int_0^T |y_l(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}\\
&+((\alpha_1+\alpha_2)D+m)T^{\frac{1}{q}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}\\
&\le (\alpha_1+\alpha_2)T\cdot T^{\frac{p-2}{p}}\left(\frac{T}{\pi_p}\right)^{l-1}\left(\frac{T}{\pi_q}\right)^{\frac{q(n-l)}{p}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{2}{p}}+\alpha_3 T^{\frac{p-2}{p}}\left(\frac{T}{\pi_p}\right)^{l-1}\\
&\cdot\left(\frac{T}{\pi_q}\right)^{\frac{q(n-l)}{p}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{2}{p}}+\cdots\\
&+\alpha_{l+1}T^{\frac{p-2}{p}}\left(\frac{T}{\pi_p}\right)\left(\frac{T}{\pi_q}\right)^{\frac{q(n-l)}{p}}\\
&\cdot\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{2}{p}}+((\alpha_1+\alpha_2)D+m)T^{\frac{1}{q}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}\\
&\le\left[((\alpha_1+\alpha_2)T+\alpha_3)\left(\frac{T}{\pi_p}\right)^{l-1}+\alpha_4\left(\frac{T}{\pi_p}\right)^{l-2}+\cdots+\alpha_{l+1}\left(\frac{T}{\pi_p}\right)\right]\\
&T^{\frac{p-2}{p}}\left(\frac{T}{\pi_q}\right)^{\frac{q(n-l)}{p}}\\
&\cdot\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{2}{p}}+((\alpha_1+\alpha_2)D+m)T^{\frac{1}{q}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Case (1): If $p = 2$ and $[((\alpha_1+\alpha_2)T+\alpha_3)(\frac{T}{\pi})^{l-1}+\alpha_4(\frac{T}{\pi})^{l-2}+\cdots+\alpha_{l+1}(\frac{T}{\pi})](\frac{T}{\pi})^{n-l}<1$, it is easy to see that there is a constant $M_n'>0$ (independent of λ) such that

$$\int_0^T |y_n'(t)|^q dt \le M_n'.$$

Case (2): If $p>2$, it is easy to see that there is a constant $M_n'>0$ (independent of λ) such that

$$\int_0^T |y_n'(t)|^q dt \le M_n'.$$

From $y_{n-1}(0)=y_{n-1}(T)$, there is a point $t_1\in[0,T]$ such that $y_n(t_1)=0$. By applying Hölder's inequality, we have

$$|y_n(t)|\le\int_0^T |y_n'(t)|dt\le T^{\frac{1}{p}}\left(\int_0^T |y_n'(t)|^q dt\right)^{\frac{1}{q}}\le T^{\frac{1}{p}}{M_n'}^{\frac{1}{q}}=:M_n.$$

This proves the claim and the rest of the proof of the theorem is identical to that of Theorem 3.1. $\square$

Remark 3.1. If the equation takes the form

$$(\varphi_p(x^{(l)}(t)))^{(n-l)} = F(t, x(t), x(t-\tau(t)), x'(t), \cdots, x^{(l-1)}(t)) + e(t),$$

where $e(t) \in C(\mathbb{R}, \mathbb{R})$, $e(t+T) = e(t)$ and $\int_0^T e(t)dt = 0$, the results of Theorem 3.1 and Theorem 3.2 still hold.

Finally, we present an example to illustrate our result.

Example 3.1. Consider the n-th order delay differential equation

$$(\varphi_p(x^{(4)}(t)))^{(n-4)} = \frac{1}{3\pi}x(t) + \frac{1}{6\pi}\cos 2x(t-\cos 2t) + \frac{1}{8}\sin x'(t) + \frac{1}{8}\cos x''(t)\sin 2t + \frac{1}{8}\sin x'''(t). \tag{3.6}$$

Here p is a constant with $p \geq 2$. It is clear that $T = \pi$, $F(t, z_1, z_2, z_3, z_4, z_5) = \frac{1}{3\pi}z_1 + \frac{1}{6\pi}\cos 2z_2 + \frac{1}{8}\sin z_3 + \frac{1}{8}\cos z_4 \sin 2t + \frac{1}{8}\sin z_5$, $\tau(t) = \cos 2t$ and $F(t, c, c, 0, 0, 0) = \frac{1}{3\pi}c + \frac{1}{6\pi}\cos 2c + \frac{1}{8}\sin 2t \not\equiv 0$. Then choose $D = 3\pi$ such that (H_1) holds, and it is obvious that (H_2) is not satisfied here. Now we consider the assumption (H_3). We have

$$|F(t, z_1, z_2, z_3, z_4, z_5)| \leq \frac{1}{3\pi}|z_1| + 1.$$

That is to say (H_3) holds with $\alpha_1 = \frac{1}{3\pi}$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = 0$, $m = 1$.
Case (1): If $p > 2$, by Theorem 3.2, we know (3.1) has at least one non-constant π-periodic solution.
Case (2): If $p = 2$,

$$\begin{aligned}
&\left[((\alpha_1+\alpha_2)T+\alpha_3)\left(\frac{T}{\pi}\right)^3 + \alpha_4\left(\frac{T}{\pi}\right)^2 + \alpha_5\left(\frac{T}{\pi}\right)\right]\left(\frac{T}{\pi}\right)^{n-4} \\
=&\left[\left(\left(\frac{1}{3\pi}+0\right)\pi+0\right)+0+0\right] \\
=&\frac{1}{3} < 1.
\end{aligned}$$

So by Theorem 3.2, we know (3.1) has at least one non-constant π-periodic solution.

References

1. W.-S. Cheung and J. Ren, On the existence of periodic solutions for p-Laplacian generalized Liénard equation, *Nonlinear Analysis TMA* **60** (2004) 65–75.
2. W.-S. Cheung and J. Ren, Periodic solutions for p-Laplacian differential equation with multiple deviating arguments, *Nonlinear Analysis TMA* **62** (2005) 727–742.
3. W.-S. Cheung and J. Ren, Periodic solutions for p-Laplacian Rayleigh equation, *Nonlinear Analysis TMA* **65** (2006) 2003–2012.
4. F. Z. Cong, Periodic solutions for $2k$th order ordinary differential equations with nonresonance, *Nonlinear Analysis TMA* **32** (1998) 787–793.

5. F. Z. Cong, Q. D. Huang, S. Y. Shi, Existence and uniqueness of periodic solution for $(2n+1)^{th}$-order differential equation, *J. Math. Anal. Appl.*, **241** (2000) 1–9.
6. E. A. Jackson, *Perspectives of nonlinear dynamics*, Cambridge University Press, 1991.
7. R.E. Gaines, J.L. Mawhin, *Coincidence Degree and Nonlinear Differential Equation*, Springer, Berlin, 1977.
8. G. N. Jazar, M. Mahinfalah, M. H. Alimi, A. Khazaei, Periodic behavior of a nonlinear third order vibrating system, *Communications in Nonlinear Science and Numerical Simulation* **10** (2005) 441–450.
9. M. R. Zhang, Nonuniform nonresonance at the first eigenvalue of the p-Laplacian, *Nonlinear Analysis TMA* **29** (1997) 41–51.