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Eigenvalue assignment in linear descriptor systems via output feedback

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Abstract: An approach for eigenvalue assignment in linear descriptor systems via output feedback is proposed. Sufficient conditions in order that a given set of eigenvalues is assignable are established. Parametric form of the desired output feedback gain matrix is also given. The approach assigns the full number of generalised eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous works.

1 Introduction

Consider the problem of output feedback eigenvalue assignment in descriptor systems of the form

$$E\dot{x} = Ax + Bu, \quad y = Cx \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$ are, respectively, the state vector, the input vector and the output vector; $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $C \in \mathbf{R}^{p\times n}$ are known matrices with $\text{rank}(E) = n_0 \le n$, $\text{rank}(B) = m$ and $\text{rank}(C) = p$. If an output feedback

$$u = Ky, \quad K \in \mathbf{R}^{m\times p} \tag{2}$$

is applied to system (1), the closed-loop system becomes

$$E\dot{x} = (A + BKC)x \tag{3}$$

Then the problem of eigenvalue assignment in the linear descriptor system (1) via output feedback (2) can be stated as follows: Given a self-conjugate set Λ of n_0 complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$, find an output feedback gain matrix $K \in \mathbf{R}^{m\times p}$ such that the generalised eigenspectrum of the closed-loop system (3), that is, the set of finite eigenvalues of the matrix pair $(E, A + BKC)$, is Λ. Since Λ is finite, so if such a matrix K exists, then the resulting closed-loop system (3) is regular, that is, $\det(\lambda E - A - BKC)$ is not identically zero for an arbitrary λ.

Eigenvalue assignment in descriptor systems is a very important problem in descriptor systems theory and has been studied during the past three decades by a lot of researchers [1–35]. In particular, eigenvalue assignment in descriptor systems by output feedback has been studied by several authors [28–35]. The pioneering work on this problem is duo to Fletcher [28], who gave a set of necessary and sufficient conditions for the existence of output feedback gain matrix, which assigns a given set of eigenvalues. In [29], an algorithm for the design of an output feedback control giving eigenvalue assignment under the condition $m + p > n$ was proposed. Conditions which ensure that the algorithm produces the required control were also given. In [30], Kimura's work [36] was generalised to descriptor systems. The basis of this work was a demonstration of the equivalence between eigenvalue assignment in descriptor systems and in state-space systems. Existing state-space results could then be used to draw conclusions about descriptor system problems. In [31], two workable sufficient rank conditions for eigenvalue assignability were proposed: one is the condition for complex feedback assignability that $m + p > 2n - n_0$; the other is the condition for real feedback assignability that $m + p > 2n - n_0 + 1$. In [32], an algorithm for the design of an output feedback control giving eigenvalue assignment under the condition $m + p > 2n - n_0$ was proposed. In [33] and [34], parametric approaches for eigenstructure assignment in descriptor systems via output feedback control were proposed. Unlike [28–32], the approaches assign both the whole sets of the left and right eigenvectors. Very recently, Zhang [35] proposed a new sufficient rank condition for real output feedback eigenvalue assignability that $mp > n_0$. This result improves considerably the ones obtained in [29] and [31] and is the best possible for general $m, p, \text{rank}(E)$. However, these reported studies on output feedback eigenvalue assignment are subject to the following defects.

(1) In [28], conditions for eigenvalue assignment involve constrained solutions of generalised Sylvester matrix equations. These conditions are usually difficult to check.
(2) In [33, 34], conditions for eigenvalue assignment contain a bilinear system of $n_0(n_0 - 1)$ algebraic equations (Constraint C2 in [33, 34]). It will be seen that if only eigenvalue assignment is concerned, then the number of algebraic equations in the system can be significantly reduced (see Remark 6).
(3) The work [30] is unable to give conditions to guarantee the assignability of the full number, $\text{rank}(E)$, of generalised eigenvalues.

(4) The works [29] and [32] impose the conditions $m+p>n$ and $m+p>2n-n_0$ on the system, respectively. It is known from [35] that both conditions are too conservative.
(5) The works [31] and [35] do not provide computational methods for determining the desired output feedback gain matrices.

This paper also studies the problem of eigenvalue assignment in the descriptor system (1) via output feedback (2). An approach for solving this problem is proposed. Sufficient conditions for the existence of an output feedback gain matrix, which assigns a self-conjugate set of rank(E) complex numbers to a generalised eigenspectrum of the closed-loop system while ensuring that the resulting system is regular are established. Parametric form of the desired output feedback gain matrix is also given. The approach assigns rank(E) finite closed-loop eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous works. Two illustrative examples show the effect of the proposed approach.

2 Eigenvalue assignment

We assume in this paper that the descriptor system (1) is strongly controllable (S-controllable) and strongly observable (S-observable), that is, the matrix quadruple (E, A, B, C) satisfies the following conditions (see [37, 38])

$$\operatorname{rank}\begin{bmatrix}\lambda E-A & B\end{bmatrix}=\operatorname{rank}\begin{bmatrix}\lambda E-A\\ C\end{bmatrix}=n,\quad \text{for all } \lambda\in\mathbf{C} \tag{4}$$

$$\operatorname{rank}\begin{bmatrix}E & AV_\infty & B\end{bmatrix}=\operatorname{rank}\begin{bmatrix}E\\ T_\infty^T A\\ C\end{bmatrix}=n \tag{5}$$

where V_∞ and T_∞ are $n\times(n-n_0)$ matrices defined by $EV_\infty=0$, $T_\infty^T E=0$, $\operatorname{rank}(V_\infty)=\operatorname{rank}(T_\infty)=n-n_0$.

Lemma 1 [28]: Given matrices E, V_∞ and T_∞ as described previously. Let M be an $n\times n$ matrix. Then $\deg\det(\lambda E-M)=\operatorname{rank}(E)$ if and only if $T_\infty^T M V_\infty$ is non-singular.

Denote the right and left eigenvectors associated with eigenvalue λ_i, respectively, by v_i and t_i. Then, we have by definition

$$(A+BKC-\lambda_i E)v_i=0,\quad i=1,2,\ldots,n_0 \tag{6}$$

$$(A+BKC-\lambda_i E)^T t_i=0,\quad i=1,2,\ldots,n_0 \tag{7}$$

Let

$$w_i=KCv_i, z_i=K^TB^Tt_i,\quad i=1,2,\ldots,n_0 \tag{8}$$

Then (6) and (7) become

$$(A-\lambda_i E)v_i+Bw_i=0,\quad i=1,2,\ldots,n_0 \tag{9}$$

$$(A-\lambda_i E)^T t_i+C^Tz_i=0,\quad i=1,2,\ldots,n_0 \tag{10}$$

Let the columns of $\begin{bmatrix}N(\lambda)\\ D(\lambda)\end{bmatrix}$ and $\begin{bmatrix}M(\lambda)\\ L(\lambda)\end{bmatrix}$ form bases for $\ker([A-\lambda E\ B])$ and $\ker([A^T-\lambda E^T C^T])$, respectively. Then the general parametric solutions for v_i, w_i satisfying (9) and t_i, z_i satisfying (10) are given by

$$v_i=N(\lambda_i)f_i, w_i=D(\lambda_i)f_i,\quad i=1,2,\ldots,n_0 \tag{11}$$

$$t_i=M(\lambda_i)g_i, z_i=L(\lambda_i)g_i,\quad i=1,2,\ldots,n_0 \tag{12}$$

where $f_i\in\mathbf{C}^m$, $g_i\in\mathbf{C}^p$, $i=1,2,\ldots,n_0$ are two groups of free parameter vectors.

Let the infinite eigenvalue of the matrix pair $(E, A+BKC)$ be denoted by λ_∞. Then $s_\infty=1/\lambda_\infty=0$ is the eigenvalue of the matrix pair $(A+BKC, E)$. Because of (6) and (7), s_∞ is a multiple eigenvalue with both geometric and algebraic multiplies being equal to $n-n_0$. Denote the right and left eigenvectors associated with s_∞, respectively, by v_j^∞, $j=1,2,\ldots,n-n_0$ and t_j^∞, $j=1,2,\ldots,n-n_0$. Then we have by definition

$$(E-s_\infty(A+BKC))v_j^\infty=0,\quad j=1,2,\ldots,n-n_0 \tag{13}$$

$$(E-s_\infty(A+BKC))^T t_j^\infty=0,\quad j=1,2,\ldots,n-n_0 \tag{14}$$

Let

$$w_j^\infty=KCv_j^\infty, z_j^\infty=K^TB^Tt_i^\infty,\quad j=1,2,\ldots,n-n_0 \tag{15}$$

Then (13) and (14) become

$$(E-s_\infty A)v_j^\infty+s_\infty Bw_j^\infty=0,\quad j=1,2,\ldots,n-n_0 \tag{16}$$

$$(E-s_\infty A)^T t_j^\infty+s_\infty C^T z_j^\infty=0,\quad j=1,2,\ldots,n-n_0 \tag{17}$$

Note that the columns of V_∞ and T_∞ form bases for $\ker(E)$ and $\ker(E^T)$, respectively. Then, from [18], the general parametric solutions for $v_j^\infty, w_j^\infty, j=1,2,\ldots,n-n_0$ satisfying (16) and $t_j^\infty, z_j^\infty, j=1,2,\ldots,n-n_0$ satisfying (17) are given by

$$v_j^\infty=V_\infty f_j^\infty, w_j^\infty \text{arbitrary},\ j=1,2,\ldots,n-n_0 \tag{18}$$

$$t_j^\infty=T_\infty g_j^\infty, z_j^\infty \text{arbitrary},\ j=1,2,\ldots,n-n_0 \tag{19}$$

where f_j^∞, $g_j^\infty\in\mathbf{R}^{n-n_0}$, $j=1,2,\ldots,n-n_0$ are two groups of free parameter vectors.

For $1\le k\le n_0$, we let

$$V_k=\begin{bmatrix}v_1 & v_2 & \ldots & v_k\end{bmatrix},\quad W_k=\begin{bmatrix}w_1 & w_2 & \cdots & w_k\end{bmatrix} \tag{20}$$

$$T_k=\begin{bmatrix}t_1 & t_2 & \cdots & t_k\end{bmatrix},\quad Z_k=\begin{bmatrix}z_1 & z_2 & \cdots & z_k\end{bmatrix} \tag{21}$$

where $v_i, w_i, t_i, z_i, i=1,2,\ldots,k$ are given by (11) and (12). For $1\le l\le n-n_0$, we let

$$V_l^\infty=\begin{bmatrix}v_1^\infty & v_2^\infty & \cdots & v_l^\infty\end{bmatrix},\quad W_l^\infty=\begin{bmatrix}w_1^\infty & w_2^\infty & \cdots & w_l^\infty\end{bmatrix} \tag{22}$$

$$T_l^\infty=\begin{bmatrix}t_1^\infty & t_2^\infty & \cdots & t_l^\infty\end{bmatrix},\quad Z_l^\infty=\begin{bmatrix}z_1^\infty & z_2^\infty & \cdots & z_l^\infty\end{bmatrix} \tag{23}$$

where v_j^∞, w_j^∞, t_j^∞, z_j^∞, $j=1,2,\ldots,l$ are given by (18) and (19).

In the following, we will develop an approach for eigenvalue assignment in the system (1) via output feedback (2). There are two different cases, that is, the case where $\max\{m,p\}\ge n_0$ and the case where $\max\{m,p\}<n_0$, to consider. First, we consider the case where $\max\{m,p\}\ge n_0$.

Theorem 1: Given the S-controllable S-observable system (1) with $\max\{m, p\} \geq n_0$. Let Λ be a self-conjugate set of n_0 distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$.

(1) If $p = \max\{m, p\}$ and there exist vectors $f_1, f_2, \ldots, f_{n_0}$ in $\mathbf{C}^m$, vectors $f_1^\infty, f_2^\infty, \ldots, f_{p-n_0}^\infty$ in $\mathbf{R}^{n-n_0}$ and vectors $w_1^\infty, w_2^\infty, \ldots, w_{p-n_0}^\infty$ in $\mathbf{R}^m$ satisfying the following conditions:

(a) $f_i = \bar{f}_j$ for $\lambda_i = \bar{\lambda}_j \in \Lambda$;
(b) $CN(\lambda_1)f_1, \quad CN(\lambda_2)f_2, \ldots, CN(\lambda_{n_0})f_{n_0}, CV_\infty f_1^\infty, CV_\infty f_2^\infty, \ldots, CV_\infty f_{p-n_0}^\infty$ are linearly independent in $\mathbf{C}^p$;
(c) $\det(T_\infty^T(A + B[W_{n_0} W_{p-n_0}^\infty](C[V_{n_0} V_{p-n_0}^\infty])^{-1}C)V_\infty) \neq 0$, where V_{n_0}, W_{n_0} and $V_{p-n_0}^\infty, W_{p-n_0}^\infty$ are given by (11), (20) and (18), (22), respectively;

then there exists an output feedback gain matrix $K \in \mathbf{R}^{m\times p}$ such that the generalised eigenspectrum of the closed-loop system (3) is Λ. This gain matrix is given by

$$K = \begin{bmatrix} W_{n_0} & W_{p-n_0}^\infty \end{bmatrix} \left(C \begin{bmatrix} V_{n_0} & V_{p-n_0}^\infty \end{bmatrix} \right)^{-1} \tag{24}$$

(2) If $m = \max\{m, p\}$ and there exist vectors $g_1, g_2, \ldots, g_{n_0}$ in $\mathbf{C}^p$, vectors $g_1^\infty, g_2^\infty, \ldots, g_{m-n_0}^\infty$ in $\mathbf{R}^{n-n_0}$ and $z_1^\infty, z_2^\infty, \ldots, z_{m-n_0}^\infty$ in $\mathbf{R}^p$ satisfying the following conditions:

(a) $g_i = \bar{g}_j$ for $\lambda_i = \bar{\lambda}_j \in \Lambda$;
(b) $B^TM(\lambda_1)g_1, \quad B^TM(\lambda_2)g_2, \ldots, B^TM(\lambda_{n_0})g_{n_0}, B^TT_\infty g_1^\infty, B^TT_\infty g_2^\infty, \ldots, B^TT_\infty g_{m-n_0}^\infty$ are linearly independent in $\mathbf{C}^m$;
(c) $\det(T_\infty^T(A + B([T_{n_0} T_{m-n_0}^\infty]^TB)^{-1}[Z_{n_0} Z_{m-n_0}^\infty]^TC)V_\infty) \neq 0$, where T_{n_0}, Z_{n_0} and $T_{m-n_0}^\infty$, $Z_{m-n_0}^\infty$ are given by (12), (21) and (19), (23), respectively;

then there exists an output feedback gain matrix $K \in \mathbf{R}^{m\times p}$ such that the generalised eigenspectrum of the closed-loop system (3) is Λ. This gain matrix is given by

$$K = \left(\begin{bmatrix} T_{n_0} & T_{m-n_0}^\infty \end{bmatrix}^T B \right)^{-1} \begin{bmatrix} Z_{n_0} & Z_{m-n_0}^\infty \end{bmatrix}^T \tag{25}$$

Proof: First, we prove part (1). Assume that $p = \max\{m, p\} \geq n_0$. Then, from (8), (15), (20) and (22), we have

$$\begin{bmatrix} W_{n_0} & W_{p-n_0}^\infty \end{bmatrix} = KC \begin{bmatrix} V_{n_0} & V_{p-n_0}^\infty \end{bmatrix} \tag{26}$$

Since $\det(C[V_{n_0} V_{p-n_0}^\infty]) \neq 0$ (condition 1(b)), the solution K to the matrix equation (26) exists and is given by (24). Since condition 1(a) is satisfied, the solution K is real. With this K, from (9), all the equations in (6) hold. Still with this K, from the condition 1(c) and Lemma 1, $\deg\det(\lambda E - A - BKC) = \mathrm{rank}(E)$. This result and equation (6) prove that with the K given by (24) the generalised eigenspectrum of the closed-loop system (3) is Λ. Thus, the proof of part (1) has been completed.

Part (2) can be proved by replacing (E, A, B, C) in the proof of part (1) by its dual (E^T, A^T, C^T, B^T). □

Remark 1: To solve the output eigenvalue assignment problem under the condition $\max\{m, p\} \geq n_0$, the approaches given in [28, 29, 33] and [34] may be used. However, these approaches need to solve a pair, not just one as in Theorem 1, of the generalised Sylvester matrix equations (or, equivalently, (9) and (10)). Thus, to solve the problem under the condition $\max\{m, p\} \geq n_0$, the approach of Theorem 1 is simpler and need less computational work than those reported in [28, 29, 33] and [34].

Next, we consider the case where $\max\{m, p\} < n_0$. In this case, we assume that the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ or the decomposition $\Lambda = \Lambda_1^{(m)} \bigcup \Lambda_2^{(m)}$ exists, where $\Lambda_1^{(p)} = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ and $\Lambda_2^{(p)} = \{\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{n_0}\}$, or $\Lambda_1^{(m)} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ and $\Lambda_2^{(m)} = \{\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_{n_0}\}$ are self-conjugate subsets. Clearly, the separation of the self-conjugate set Λ into two self-conjugate subsets $\Lambda_1^{(p)}, \Lambda_2^{(p)}$ or $\Lambda_1^{(m)}, \Lambda_2^{(m)}$ does not encompass all the possible cases. The case in which Λ cannot be decomposed into two self-conjugate subsets $\Lambda_1^{(p)}, \Lambda_2^{(p)}$ or $\Lambda_1^{(m)}, \Lambda_2^{(m)}$ occurs only Λ contains no real numbers (which implies that n_0 is an even number) and both p and m are odd numbers. Thus, the requirement that the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ or the decomposition $\Lambda = \Lambda_1^{(m)} \bigcup \Lambda_2^{(m)}$ exists is by no means a strong restriction from the practical viewpoint.

Theorem 2: Given the S-controllable S-observable system (1) with $\max\{m, p\} < n_0$. Let Λ be a self-conjugate set of n_0 distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_0}$.

(1) If the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ exists and there exist vectors $f_1, f_2, \ldots, f_p$ in $\mathbf{C}^m$ and vectors $g_{p+1}, g_{p+2}, \ldots, g_{n_0}$ in $\mathbf{R}^p$ satisfying the following conditions:

(a) $f_i = \bar{f}_j$ for $\lambda_i = \bar{\lambda}_j \in \Lambda_1^{(p)}$ and $g_k = \bar{g}_l$ for $\lambda_k = \bar{\lambda}_l \in \Lambda_2^{(p)}$;
(b) $CN(\lambda_1)f_1, CN(\lambda_2)f_2, \ldots, CN(\lambda_p)f_p$ are linearly independent in $\mathbf{C}^p$;
(c) $M(\lambda_{p+1})g_{p+1}, M(\lambda_{p+2})g_{p+2}, \ldots, M(\lambda_{n_0})g_{n_0}$ are linearly independent in $\mathbf{C}^n$;
(d) $g_j^TM^T(\lambda_j)EN(\lambda_i)f_i = 0, i = 1, 2, \ldots, p, j = p+1, p+2, \ldots, n_0$;
(e) $\det(T_\infty^T(A + BW_p(CV_p)^{-1}C)V_\infty) \neq 0$, where V_p and W_p are given by (11) and (20);

then there exists an output feedback gain matrix $K \in \mathbf{R}^{m\times p}$ such that the generalised eigenspectrum of the closed-loop system (3) is Λ. This gain matrix is given by

$$K = W_p(CV_p)^{-1} \tag{27}$$

(2) If the decomposition $\Lambda = \Lambda_1^{(m)} \bigcup \Lambda_2^{(m)}$ exists and there exist vectors $f_{m+1}, f_{m+2}, \ldots, f_{n_0}$ in $\mathbf{C}^m$ and vectors $g_1, g_2, \ldots, g_m$ in $\mathbf{R}^p$ satisfying the following conditions:

(a) $f_i = \bar{f}_j$ for $\lambda_i = \bar{\lambda}_j \in \Lambda_2^{(m)}$ and $g_k = \bar{g}_l$ for $\lambda_k = \bar{\lambda}_l \in \Lambda_1^{(m)}$;
(b) $B^TM(\lambda_1)g_1, B^TM(\lambda_2)g_2, \ldots, B^TM(\lambda_m)g_m$ are linearly independent in $\mathbf{C}^m$;
(c) $N(\lambda_{m+1})f_{m+1}, N(\lambda_{m+2})f_{m+2}, \ldots, N(\lambda_{n_0})f_{n_0}$ are linearly independent in $\mathbf{C}^n$;
(d) $g_j^TM^T(\lambda_j)EN(\lambda_i)f_i = 0, \ i = m+1, m+2, \ldots, n_0, \ j = 1, 2, \ldots, m$;
(e) $\det(T_\infty^T(A + B(T_m^TB)^{-1}Z_m^TC)V_\infty) \neq 0$, where T_m and Z_m are given by (12) and (21);

then there exists an output feedback gain matrix $K \in \mathbf{R}^{m\times p}$ such that the generalised eigenspectrum of the closed-loop system (3) is Λ. This gain matrix is given by

$$K = (T_m^TB)^{-1}Z_m^T \tag{28}$$

Proof: First, we prove part (1). Assume that the decomposition $\Lambda = \Lambda_1^{(m)} \bigcup \Lambda_2^{(m)}$ exists. Then, from (11) and (12), we have

$$(A - \lambda_iE)v_i + Bw_i = 0, \quad i = 1, 2, \ldots, p \tag{29}$$

$$(A - \lambda_iE)^Tt_j + C^Tz_j = 0, \quad j = p+1, \quad p+2, \ldots, n_0 \tag{30}$$

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Let

$$T_{n_0-p} = \begin{bmatrix} t_{p+1} & t_{p+2} \cdots & t_{n_0} \end{bmatrix},$$
$$Z_{n_0-p} = \begin{bmatrix} z_{p+1} & z_{p+2} \cdots & z_{n_0} \end{bmatrix}$$
$$J_p = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p),$$
$$J_{n_0-p} = \mathrm{diag}(\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_{n_0})$$

Then (29) and (30) and the condition 1(d) can be respectively written as

$$AV_p + BW_p = EV_pJ_p \tag{31}$$
$$T^T_{n_0-p}A + Z^T_{n_0-p}C = J_{n_0-p}T^T_{n_0-p}E \tag{32}$$
$$T^T_{n_0-p}EV_p = 0 \tag{33}$$

From (31)–(33), we can prove

$$T^T_{n_0-p}BW_p = Z^T_{n_0-p}CV_p \tag{34}$$

Since $\det(CV_p) \neq 0$ (the condition 1(b)) and the condition 1(a) is satisfied, the matrix K given by (27) is well defined and real. With this K, from (31), we have

$$(A + BKC)V_p = EV_pJ_p \tag{35}$$

and from (32) and (34), we have

$$\begin{aligned} T^T_{n_0-p}(A + BKC) &= T^T_{n_0-p}A + T^T_{n_0-p}BKC \\ &= T^T_{n_0-p}A + T^T_{n_0-p}BW_p(CV_p)^{-1}C \\ &= T^T_{n_0-p}A + Z^T_{n_0-p}CV_p(CV_p)^{-1}C \\ &= T^T_{n_0-p}A + Z^T_{n_0-p}C \\ &= J_{n_0-p}T^T_{n_0-p}E \end{aligned} \tag{36}$$

Still with this K, from the condition 1(e) and Lemma 1, $\deg\det(\lambda E - A - BKC) = \mathrm{rank}(E)$. This result and (35) and (36) prove that with the K given by (27) the generalised eigenspectrum of the closed-loop system (3) is Λ. Thus, the proof of part (1) has been completed.

Part (2) can be proved by replacing (E, A, B, C) in the proof of part (1) by its dual (E^T, A^T, C^T, B^T). □

Remark 2: Theorem 2 is a generalisation of Kimura's result (Theorem 1 in [36]) to descriptor systems. However, the proof of Theorem 2 is similar to the proof of Theorem 2 in [39].

Remark 3: Very recently, Zhang [35] proved that $mp > \mathrm{rank}(E)$ is a sufficient condition for generic eigenvalue assignability. It is known from [35] and Theorem 2 that if $mp > \mathrm{rank}(E)$ and $\max\{m, p\} < \mathrm{rank}(E)$, then for generic matrix triples (A, B, C) there exist vectors $f_1, f_2, \ldots, f_p$, $g_{p+1}, g_{p+2}, \ldots, g_{n_0}$ or vectors $f_{m+1}, f_{m+2}, \ldots, f_{n_0}, g_1, g_2, \ldots, g_m$ satisfying the conditions (a)–(e) in Theorem 2, provided that a slight modification of the eigenvalues to be assigned is allowed.

Remark 4: When complex eigenvalues are assigned to the closed-loop system, the associated eigenvectors are all complex. However, a well-known technique exists, which converts all these eigenvectors into real ones. For simplicity, we demonstrate this technique with the case where the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ exists and conjugate pairs of complex eigenvalues appear in $\Lambda_1^{(p)}$ and $\Lambda_2^{(p)}$, that is, $\lambda_k \in \Lambda_1^{(p)}$ and $\lambda_{k+1} = \bar{\lambda}_k \in \Lambda_1^{(p)}$, and $\lambda_l \in \Lambda_2^{(p)}$ and $\lambda_{l+1} = \bar{\lambda}_l \in \Lambda_2^{(p)}$. In this case, we can replace the right eigenvectors v_k and v_{k+1} by the real vectors $\mathrm{Re}(v_k)$ and $\mathrm{Im}(v_k)$, and replace the left eigenvectors t_l and t_{l+1} by the real vectors $\mathrm{Re}(t_l)$ and $\mathrm{Im}(t_l)$.

Remark 5: Unlike [28], our approach adopts parametric solutions to the generalised Sylvester matrix equations concerned. This makes the solution to the output feedback eigenvalue assignment problem much easier. Unlike [30], our approach assigns the full number, $\mathrm{rank}(E)$, of generalised eigenvalues. Unlike [29] and [32], our approach covers the general case where $mp \geq n_0$.

Remark 6: For the assignment of the desired set Λ, in [33] and [34], the condition defined by $g_j^T M^T(\lambda_j)E N(\lambda_i)f_i = 0$, $i \neq j$, $i, j = 1, 2, \ldots, n_0$ has to be solved. In this paper, the corresponding condition becomes the one defined by $g_j^T M^T(\lambda_j)EN(\lambda_i)f_i = 0$, $i = 1, 2, \ldots, p$, $j = p + 1, p + 2, \ldots, n_0$ or the one defined by $g_j^T M^T(\lambda_j)EN(\lambda_i)f_i = 0$, $i = m + 1, m + 2, \ldots, n_0$, $j = 1, 2, \ldots, m$ (when $\max\{m, p\} \geq n_0$ the condition vanishes). Thus our approach is simpler and need less computational work.

3 Algorithms for determining the output feedback gain matrix

In this section, we will give algorithms for determining the output feedback gain matrix K. We assume $mp \geq n_0$. It is shown in [35] that this condition is a necessary condition for output feedback eigenvalue assignability.

First, we consider the case where $\max\{m, p\} \geq n_0$. Without loss of generality, we assume $p = \max\{m, p\}$. For this case, a design algorithm based on Theorem 1 for finding a desired feedback gain matrix K is given in the following.

Algorithm 1: (1) Solve matrices $N(\lambda)$ and $D(\lambda)$ such that the columns of $\left[\begin{smallmatrix} N(\lambda) \\ D(\lambda) \end{smallmatrix}\right]$ form a basis for $\ker([A - \lambda E \;\, B])$ and solve matrix V_∞ satisfying $EV_\infty = 0$, $\mathrm{rank}(V_\infty) = n - n_0$.
(2) Construct the parametric forms of vectors $v_1, v_2, \ldots, v_{n_0}$, $w_1, w_2, \ldots, w_{n_0}$, $v_1^\infty, v_2^\infty, \ldots, v_{p-n_0}^\infty, w_1^\infty$, $w_2^\infty, \ldots, w_{p-n_0}^\infty$ using (11) and (18).
(3) Find parameters $f_1, f_2, \ldots, f_{n_0}, f_1^\infty, f_2^\infty, \ldots, f_{p-n_0}^\infty, w_1^\infty, w_2^\infty, \ldots, w_{p-n_0}^\infty$ such that the conditions 1(a)–1(c) in Theorem 1 are satisfied.
(4) Calculate matrices $V_{n_0}, W_{n_0}, V_{p-n_0}^\infty, W_{p-n_0}^\infty$ according to (20) and (22) based on the parameters obtained in step (3) and the vectors obtained in step (2).
(5) Calculate K according to (24) based on the found matrices $V_{n_0}, W_{n_0}, V_{p-n_0}^\infty, W_{p-n_0}^\infty$.

Next, we consider the case where $\max\{m, p\} < n_0$ and the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ or the decomposition $\Lambda = \Lambda_1^{(m)} \bigcup \Lambda_2^{(m)}$ exists. Without loss of generality, we assume that the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$ exists. For this case, a design algorithm based on Theorem 2 for finding a desired feedback gain matrix K is given in the following.

Algorithm 2: (1) Solve matrices $N(\lambda), D(\lambda)$ and $M(\lambda), L(\lambda)$ such that the columns of $\left[\begin{smallmatrix} N(\lambda) \\ D(\lambda) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M(\lambda) \\ L(\lambda) \end{smallmatrix}\right]$ form bases for $\ker([A - \lambda EB])$ and $\ker([A^T - \lambda E^T \;\, C^T])$ respectively and

solve matrices V_∞ and T_∞ satisfying $EV_\infty = 0$, $T_\infty^T E = 0$, $\mathrm{rank}(V_\infty) = \mathrm{rank}(T_\infty) = n - n_0$.
(2) Construct the parametric forms of vectors $v_1, v_2, \ldots, v_p$, $w_1, w_2, \ldots, w_p, t_{p+1}, t_{p+2}, \ldots, t_{n_0}, z_{p+1}, z_{p+2}, \ldots, z_{n_0}$ using (11) and (12).
(3) Find parameter vectors $f_1, f_2, \ldots, f_p$, $g_{p+1}, g_{p+2}, \ldots, g_{n_0}$ such that the conditions 1(a)–1(e) in Theorem 2 are satisfied.
(4) Calculate matrices V_p and W_p according to (20) based on the parameters obtained in step (3) and the vectors obtained in step (2).
(5) Calculate K according to (27) based on the found matrices V_p and W_p.

Remark 7: The main task in Algorithm 2 is to solve the system of algebraic equations in Theorem 2, 1(d). Clearly, the number of algebraic equations in the system is $p(n_0 - p)$. On the other hand, the vectors $v_1 = N(\lambda_1)f_1, v_2 = N(\lambda_2)f_2, \ldots, v_p = N(\lambda_p)f_p$ provide $p(m-1)$ free parameters and the vectors $t_{p+1} = M(\lambda_{p+1})g_{p+1}, t_{p+2} = M(\lambda_{p+2})g_{p+2}, \ \ldots, t_{n_0} = M(\lambda_{n_0})g_{n_0}$ provide $(n_0 - p)(p-1)$ free parameters. To solve the system it is clear that the number of equations must be equal or less than the sum of the free parameters. After simple algebraic manipulations, it is easily shown that this requirement is equivalent to the known condition $mp \geq n_0$ [35]. To solve the system under the condition $mp \geq n_0$ we consider two different cases, that is, the case where $m + p > n_0$ and the case where $m + p \leq n_0 \leq mp$. In the case where $m + p > n_0$, we can arbitrarily preselect the vectors $g_{p+1}, g_{p+2}, \ldots, g_{n_0}$ as constant vectors. Then the system reduces to a linear system of algebraic equations with the vectors $f_1, f_2, \ldots, f_p$ as unknown variables. Let

$$\Phi_i(g_{p+1}, g_{p+2}, \ldots, g_{n_0}) = \begin{bmatrix} g_{p+1}^T M^T(\lambda_{p+1}) EN(\lambda_i) \\ g_{p+2}^T M^T(\lambda_{p+2}) EN(\lambda_i) \\ \vdots \\ g_{n_0}^T M^T(\lambda_{n_0}) EN(\lambda_i) \end{bmatrix} \tag{37}$$

Then the linear system of algebraic equations with the vectors $f_1, f_2, \ldots, f_p$ as unknown variables can be written as

$$\Phi_i(g_{p+1}, g_{p+2}, \ldots, g_{n_0}) f_i = 0, \quad i = 1, 2, \ldots, p \tag{38}$$

Since all $\Phi_i(g_{p+1}, g_{p+2}, \ldots, g_{n_0})$ are $(n_0 - p) \times m$ matrices with $n_0 - p < m$, system (38) always has non-trivial solutions. In general, system (38) can be solved easily. In the case where $m + p \leq n_0 \leq mp$, the system of algebraic equations in Theorem 2, 1(d) is bilinear in nature. A suitable solution for this bilinear system of algebraic equations may not be easy to reach because of the nonlinearity of the system. In low-dimensional cases, the system can be solved by hand through directly operating on the system. However, in high-dimensional cases, some numerical methods or optimisation techniques may be called for help.

4 Illustrative examples

In this section, two examples are presented. The first example falls into the simple case $m + p > n_0$. The second example falls into the difficult case $m + p \leq n_0 \leq mp$.

Example 1: Consider a system in the form of (1) with the following coefficient matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For this system, we have $n = 4$, $n_0 = 3$, $m = p = 2$. It is easy to verify that the system is both S-controllable and S-observable. Clearly, the example system is not covered by the theory obtained by Goodwin and Fletcher [31] since $m + p = 4 < 6 = 2n - n_0 + 1$. However, arbitrary eigenvalue assignment for the example system is possible since $mp = 4 > 3 = n_0$ (see [35]). For simplicity we only consider the assignment of an arbitrary set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of distinct real numbers $\lambda_1, \lambda_2, \lambda_3$. Obviously, we have the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$, where $\Lambda_1^{(p)} = \{\lambda_1, \lambda_2\}$ and $\Lambda_2^{(p)} = \{\lambda_3\}$. Since $m + p = 4 > 3 = n_0$, the example falls into the simple case. In the following, we demonstrate step by step the proposed approach.

For the example system, we have

$$V_\infty = T_\infty = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$N(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\lambda & 0 \\ -\lambda^2 + \lambda & 0 \end{bmatrix}, \quad D(\lambda) = \begin{bmatrix} -1 & \lambda \\ \lambda & -1 \end{bmatrix}$$

$$M(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 0 \\ 0 & 1 \\ 1 & \lambda \end{bmatrix}, \quad L(\lambda) = \begin{bmatrix} \lambda^2 - 1 & -\lambda + 1 \\ 0 & -1 \end{bmatrix}$$

Denote

$$f_i = \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \end{bmatrix}, \ i = 1, 2, \quad g_3 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

Then, from (11), (12) and (20) we have

$$v_1 = \begin{bmatrix} \alpha_{11} \\ \alpha_{12} \\ -\lambda_1 \alpha_{11} \\ (-\lambda_1^2 + \lambda_1)\alpha_{11} \end{bmatrix}, \quad v_2 = \begin{bmatrix} \alpha_{21} \\ \alpha_{22} \\ -\lambda_2 \alpha_{21} \\ (-\lambda_2^2 + \lambda_2)\alpha_{21} \end{bmatrix},$$

$$t_3 = \begin{bmatrix} \beta_1 \\ \lambda_3 \beta_1 \\ \beta_2 \\ \beta_1 + \lambda_3 \beta_2 \end{bmatrix}$$

and

$$V_p = \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \\ -\lambda_1 \alpha_{11} & -\lambda_2 \alpha_{21} \\ (-\lambda_1^2 + \lambda_1)\alpha_{11} & (-\lambda_2^2 + \lambda_2)\alpha_{21} \end{bmatrix},$$

$$W_p = \begin{bmatrix} -\alpha_{11} + \lambda_1 \alpha_{12} & -\alpha_{21} + \lambda_2 \alpha_{22} \\ \lambda_1 \alpha_{11} - \alpha_{12} & \lambda_2 \alpha_{21} - \alpha_{22} \end{bmatrix}$$

Since all the closed-loop eigenvalues are real, the parameters $\alpha_{ij}, i,j = 1, 2$ and $\beta_i, i = 1, 2$ are restricted to be real. It is

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easy to verify that the conditions 1(b)–1(d) in Theorem 2 are, respectively

$$(\lambda_1^2 - \lambda_1)\alpha_{11}\alpha_{22} + (-\lambda_2^2 + \lambda_2)\alpha_{12}\alpha_{21} \neq 0 \tag{39}$$

$$\beta_1^2 + \beta_2^2 \neq 0 \tag{40}$$

and

$$\begin{cases} (\beta_1 - \lambda_1\beta_2)\alpha_{11} + \lambda_3\beta_1\alpha_{12} = 0 \\ (\beta_1 - \lambda_2\beta_2)\alpha_{21} + \lambda_3\beta_1\alpha_{22} = 0 \end{cases} \tag{41}$$

We can choose $\alpha_{11} = \alpha_{21} = \beta_1 = 1$ since there exists lack of uniqueness of eigenvectors of matrices. Then, (40) holds automatically, and (39) and (41) reduces to

$$(\lambda_1^2 - \lambda_1)\alpha_{22} + (-\lambda_2^2 + \lambda_2)\alpha_{12} \neq 0 \tag{42}$$

$$\begin{cases} 1 - \lambda_1\beta_2 + \lambda_3\alpha_{12} = 0 \\ 1 - \lambda_2\beta_2 + \lambda_3\alpha_{22} = 0 \end{cases} \tag{43}$$

In the following, there are two different cases, that is, the case where $0 \notin \Lambda$ and the case where $0 \in \Lambda$, to consider.

Consider the case where $0 \notin \Lambda$. From (43) we have

$$\alpha_{12} = \frac{\lambda_1\beta_2 - 1}{\lambda_3}, \quad \alpha_{22} = \frac{\lambda_2\beta_2 - 1}{\lambda_3} \tag{44}$$

and then

$$V_p = \begin{bmatrix} 1 & 1 \\ \dfrac{\lambda_1\beta_2 - 1}{\lambda_3} & \dfrac{\lambda_2\beta_2 - 1}{\lambda_3} \\ -\lambda_1 & -\lambda_2 \\ -\lambda_1^2 + \lambda_1 & -\lambda_2^2 + \lambda_2 \end{bmatrix},$$

$$W_p = \begin{bmatrix} \dfrac{\lambda_1^2\beta_2 - \lambda_1 - \lambda_2}{\lambda_3} & \dfrac{\lambda_2^2\beta_2 - \lambda_2 - \lambda_3}{\lambda_3} \\ \dfrac{1 + \lambda_1\lambda_3 - \lambda_1\beta_2}{\lambda_3} & \dfrac{1 + \lambda_2\lambda_3 - \lambda_2\beta_2}{\lambda_3} \end{bmatrix} \tag{45}$$

It is easily verified, using (44) and (45), that (42) is $\beta_2 \neq (\lambda_1 + \lambda_2 - 1)/\lambda_1\lambda_2$ and the condition 1(e) holds. Thus, if $\alpha_{11} = \alpha_{21} = \beta_1 = 1$ and $\beta_2 \neq (\lambda_1 + \lambda_2 - 1)/\lambda_1\lambda_2$, then the conditions 1(a)–1(e) in Theorem 2 are satisfied, and by (27) the output feedback gain matrix K is obtain as (see (46))

with d given by

$$d = \lambda_1 + \lambda_2 - 1 - \lambda_1\lambda_2\beta_2$$

Consider now the case where $0 \in \Lambda$. We can always choose $\lambda_1 = 0, \lambda_2 \neq 0, 1, \lambda_3 \neq 0$ since $\lambda_1, \lambda_2, \lambda_3$ are distinct.

From (43) we have

$$\alpha_{12} = -\frac{1}{\lambda_3}, \quad \alpha_{22} = \frac{\lambda_2\beta_2 - 1}{\lambda_3} \tag{47}$$

and then

$$V_p = \begin{bmatrix} 1 & 1 \\ -\dfrac{1}{\lambda_3} & \dfrac{\lambda_2\beta_2 - 1}{\lambda_3} \\ 0 & -\lambda_2 \\ 0 & -\lambda_2^2 + \lambda_2 \end{bmatrix},$$

$$W_p = \begin{bmatrix} -\dfrac{\lambda_2}{\lambda_3} & \dfrac{\lambda_2^2\beta_2 - \lambda_2 - \lambda_3}{\lambda_3} \\ \dfrac{1}{\lambda_3} & \dfrac{1 + \lambda_2\lambda_3 - \lambda_2\beta_2}{\lambda_3} \end{bmatrix} \tag{48}$$

It is easily verified, using (47) and (48), that (42) and condition 1(e) hold. Thus, if $\alpha_{11} = \alpha_{21} = \beta_1 = 1$ and β_2 arbitrary, then the conditions 1(a)–1(e) in Theorem 2 are satisfied, and by (27) the output feedback gain matrix K is obtain as

$$K = \begin{bmatrix} \dfrac{\lambda_2\lambda_3 - \lambda_3}{\lambda_2 - 1} & \dfrac{1 - (\lambda_2 - \lambda_3)\beta_2}{(\lambda_2 - 1)\lambda_3} \\ -1 & -\dfrac{1}{\lambda_2 - 1} \end{bmatrix} \tag{49}$$

From the above, we see that for the example system an arbitrary set $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$ of distinct real numbers $\lambda_1, \lambda_2, \lambda_3$ can always be assigned by output feedback. Specifically choosing $\lambda_1 = -1$, $\lambda_2 = -2$, $\lambda_3 = -3$ and $\beta_2 = -1$, we obtain, from (46),

$$K = \begin{bmatrix} -8 & 0.5 \\ -4 & 0.5 \end{bmatrix}$$

To check that this is as required note that

$$\det(\lambda E - A - BKC) = -0.5(\lambda + 1)(\lambda + 2)(\lambda + 3)$$

with roots $-1, -2, -3$.

Example 2: Consider a system in the form of (1) with the following coefficient matrices

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$K = \frac{1}{d}\begin{bmatrix} \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 - \lambda_3 - \lambda_1\lambda_2\beta_2 & \dfrac{1 - (\lambda_1 + \lambda_2 - \lambda_3)\beta_2 + \lambda_1\lambda_2\beta_2^2}{\lambda_3} \\ -(\lambda_1\lambda_2\lambda_3 + \lambda_1 + \lambda_2 - 1) + \lambda_1\lambda_2\beta_2 & -1 \end{bmatrix} \tag{46}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For this system we have $n = 6$, $n_0 = 5$, $m = 3$, $p = 2$. It is easy to verify that the system is both S-controllable and S-observable. Since $mp = 6 > 5 = n_0$, arbitrary eigenvalue assignment for this system is possible (see [35]). For simplicity, we only consider the assignment of the set $\Lambda = \{-1, -2, -3, -4, -5\}$. Obviously, we have the decomposition $\Lambda = \Lambda_1^{(p)} \bigcup \Lambda_2^{(p)}$, where $\Lambda_1^{(p)} = \{-1, -2\}$ and $\Lambda_2^{(p)} = \{-3, -4, -5\}$. Clearly, this example falls into the difficult case since $m + p = 5 = n_0$. For this difficult case, we focus on finding a numerical, but not analytical, solution.

For the example system, we have

$$V_\infty = T_\infty = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T$$

$$N(\lambda) = \begin{bmatrix} \lambda^2 & 0 & -1 \\ \lambda & 0 & 0 \\ 1 & 0 & 0 \\ -\lambda^3 & \lambda & \lambda \\ 0 & 1 & 0 \\ -\lambda^3 & 0 & \lambda \end{bmatrix}, \quad D(\lambda) = \begin{bmatrix} 0 & 0 & 1 \\ -\lambda^4 - 1 & \lambda^2 & \lambda^2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$M(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda & 0 \\ \lambda^2 & 0 \\ \lambda^3 & -1 \\ \lambda^4 & -\lambda \\ \lambda^5 & -\lambda^2 \end{bmatrix}, \quad L(\lambda) = \begin{bmatrix} 0 & 1 \\ \lambda^4 + 1 & -\lambda \end{bmatrix}$$

Then, from (11) and (12), we have

$$v_i = N(-i)f_i, i = 1, 2, \quad t_j = M(-j)g_j, \ j = 3, 4, 5$$

where $f_i \in \mathbf{R}^3$, $i = 1, 2$, $g_j \in \mathbf{R}^2$, $j = 3, 4, 5$. Let

$$f_i = \begin{bmatrix} 1 \\ \alpha_{i1} \\ \alpha_{i2} \end{bmatrix}, i = 1, 2, \quad g_j = \begin{bmatrix} 1 \\ \beta_j \end{bmatrix}, j = 3, 4, 5$$

Then condition 1(d) in Theorem 2 is

$$\begin{bmatrix} 1 & \beta_j \end{bmatrix} M^T(-j) E N(-i) \begin{bmatrix} 1 \\ \alpha_{i1} \\ \alpha_{i2} \end{bmatrix} = 0, \quad i = 1, 2, \ j = 3, 4, 5$$

By properly using the Matlab command *fsolve*, a solution to this bilinear system of algebraic equations is found to be

$$\alpha_{11} = 0.4278, \alpha_{12} = -1.9232, \ \alpha_{21} = 0.4867, \alpha_{22} = 2.0970$$
$$\beta_3 = -14.6851, \beta_4 = -34.7595, \ \beta_5 = -32.5477$$

With these parameters conditions 1(a)–1(e) in Theorem 2 are satisfied, and

$$V_p = \begin{bmatrix} 2.92316553 & 1.90304264 \\ -1 & -2 \\ 1 & 1 \\ 2.49532289 & 2.83267737 \\ 0.42784264 & 0.48670396 \\ 2.92316553 & 3.80608528 \end{bmatrix},$$

$$W_p = \begin{bmatrix} -1.92316553 & 2.09695736 \\ -3.49532289 & -6.66535473 \\ -0.42784264 & -0.48670396 \end{bmatrix}$$

Then by formula (27) the output feedback gain matrix K is obtain as

$$K = \begin{bmatrix} -15.23296494 & 4.55321440 \\ 7.00000000 & -3.59039636 \\ -0.23296494 & -0.06666667 \end{bmatrix}$$

It is easy to verify that with this K the closed-loop system is regular and has the finite eigenvalues -1.0000, -2.0000, -3.0000, -4.0000, -5.0000.

5 Conclusions

An approach for eigenvalue assignment in system (1) via output feedback (2) is proposed. Sufficient conditions for the existence of an output feedback gain matrix, which assigns a self-conjugate set of rank(E) complex numbers to a generalised eigenspectrum of the closed-loop system while ensuring that the resulting system is regular are established. Parametric form of the desired output feedback gain matrix is also given. The approach assigns rank(E) finite closed-loop eigenvalues, guarantees the closed-loop regularity and overcomes the defects of some previous works.

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7 References

1 Cobb, D.: 'Feedback and pole placement in descriptor variable systems', *Int. J. Control*, 1981, **33**, (6), pp. 1135–1146
2 Armentano, V.A.: 'Eigenvalue placement for generalized linear systems', *Syst. Control Lett.*, 1984, **4**, (4), pp. 199–202
3 Fletcher, L.R., Kautsky, J., Nichols, N.K.: 'Eigenstructure assignment in descriptor systems', *IEEE Trans. Autom. Control*, 1986, **31**, (12), pp. 1138–1141
4 Ozcaldiran, K., Lewis, F.: 'A geometric approach to eigenstructure assignment for singular systems', *IEEE Trans. Autom. Control*, 1987, **32**, (7), pp. 629–632
5 Wang, Y.Y., Shi, S.J., Zhang, Z.J.: 'Pole placement and compensator design of generalized systems', *Syst. Control Lett.*, 1987, **8**, (3), pp. 205–209
6 Chu, K.W.E.: 'A controllability condensed form and a state feedback pole assignment algorithm for descriptor systems', *IEEE Trans. Autom. Control*, 1988, **33**, (4), pp. 366–370
7 Ailon, A.: 'A approach for pole assignment in singular systems', *IEEE Trans. Autom. Control*, 1989, **34**, (8), pp. 889–893
8 Lewis, F.L., Ozcaldiran, K.S.: 'Geometric structure and feedback in singular systems', *IEEE Trans. Autom. Control*, 1989, **34**, (4), pp. 450–455

doi: 10.1049/iet-cta.2012.0842

9 Uetake, Y.: 'Pole assignment and observer design for continuous descriptor systems', *Int. J. Control*, 1989, **50**, (1), pp. 89–96
10 Fahmy, M.M., O'Reilly, J.: 'Parametric eigenstructure assignment for continuous-time descriptor systems', *Int. J. Control*, 1989, **49**, (1), pp. 129–143
11 Duan, G.R.: 'Solution to matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems', *Automatica*, 1992, **28**, (3), pp. 639–643
12 Koumboulis, F.N., Paraskevopoulous, P.N.: 'On the pole assignment of generalized state space systems via state feedback', Proc. Inst. Electr. Eng., D, 1992, **139**, pp. 106–108
13 Miminis, G.: 'Deflation in eigenvalue assignment of descriptor systems using state feedback', *IEEE Trans. Autom. Control*, 1993, **38**, (9), pp. 1322–1336
14 Zagalak, P., Kučera, V.: 'Eigenstructure assignment in linear descriptor systems', *IEEE Trans. Autom. Control*, 1995, **40**, (1), pp. 144–147
15 Chu, D.L., Chan, H.C., Ho, D.W.C.: 'A general framework for state feedback pole assignment of singular systems', *Int. J. Control*, 1997, **67**, (2), pp. 135–152
16 Duan, G.R.: 'Eigenstructure assignment and response analysis in descriptor linear systems with state feedback control', *Int. J. Control*, 1998, **69**, (5), pp. 663–694
17 Owens, T.J., Askarpour, S.: 'Integrated approach to eigenstructure assignment in descriptor systems by PD and P state feedback', *IEE Proc. Control Theory Appl.*, 2000, **147**, (4), pp. 407–415
18 Zhang, B.: 'Parametric eigenstructure assignment by state feedback in descriptor systems', *IET Control Theory Appl.*, 2008, **2**, (4), pp. 303–309
19 Zhang, B.: 'Infinite eigenvalue assignment in descriptor systems via state feedback', *Int. J. Syst. Sci.*, 2010, **41**, (9), pp. 1075–1084
20 Mukundan, R., Dayawansa, W.: 'Feedback control of singular systems—proportional and derivative feedback of the state', *Int. J. Syst. Sci.*, 1983, **14**, (6), pp. 615–632
21 Le, V.X.: 'Synthesis of proportional-plus-derivative feedbacks for descriptor systems', *IEEE Trans. Autom. Control*, 1992, **37**, (5), pp. 672–675
22 Chen, H.C., Chang, F.R.: 'Chained eigenstructure assignment for constraint-ratio proportional and derivative (CRPD) control law in controllable singular systems', *Syst. Control Lett.*, 1993, **21**, (5), pp. 405–411
23 Jing, H.Y.: 'Eigenstructure assignment by proportional-derivative state-feedback in singular systems', *Syst. Control Lett.*, 1994, **22**, (1), pp. 47–52
24 Duan, G.R., Patton, R.J.: 'Eigenstructure assignment in descriptor systems via proportional plus derivative state feedback', *Int. J. Control*, 1997, **68**, (5), pp. 1147–1162
25 Zhang, B.: 'Parametric eigenstructure assignment for descriptor systems via proportional plus derivative state feedback', *Int. J. Control, Autom. Syst.*, 2011, **9**, (1), pp. 15–22
26 Duan, G.R., Yu, H.H.: 'Robust pole assignment in high-order descriptor linear systems via proportional plus derivative state feedback', *IET Control Theory Appl.*, 2008, **2**, (4), pp. 277–287
27 Bai, J.J., Su, H.Y., Wang, J., Shi, B.X.: 'On pole placement in LMI region for descriptor linear systems', *Int. J. Innov. Comput., Inf. Control*, 2012, **8**, (4), pp. 2613–2624
28 Fletcher, L.R.: 'Eigenstructure assignment by output feedback in descriptor systems', *IEE Proc. Control Theory Appl.*, 1988, **135**, (4), pp. 302–308
29 Fletcher, L.R., Goodwin, M.S.: 'An intermediate algorithm for pole placement in descriptor systems',*Circuits Syst. Signal Proc.*, 1994, **13**, (2–3), pp. 329–345
30 Ailon, A.: 'An approach for pole assignment by gain output feedback in square singular systems', *Kybernetica*, 1991, **27**, pp. 317–332
31 Goodwin, M.S., Fletcher, L.R.: 'Exact pole assignment with regularity by output feedback in descriptor systems–Part 1', *Int. J. Control*, 1995, **62**, (2), pp. 379–411
32 Goodwin, M.S.: 'Exact pole assignment with regularity by output feedback in descriptor systems–Part 2', *Int. J. Control*, 1995, **62**, (2), pp. 413–441
33 Duan, G.R.: 'Parametric approach for eigenstructure assignment in descriptor systems via output feedback', *IEE Proc. Control Theory Appl.*, 1995, **142**, (6), pp. 36–41
34 Duan, G.R.: 'Eigenstructure assignment in descriptor systems via output feedback: a new complete parametric approach', *Int. J. Control*, 1999, **72**, (4), pp. 345–364
35 Zhang, B.: 'A sufficient condition for arbitrary eigenvalue assignment in linear descriptor systems by output feedback', *IEEE Trans. Autom. Control*, 2013, **58**, (8), pp. 2060–2064
36 Kimura, H.: 'Pole assignment by gain output feedback', *IEEE Trans. Autom. Control*, 1975, **20**, (4), pp. 509–516
37 Verghese, G.C., Levy, B.C., Kailath, T.: 'A generalized state space for singular systems', *IEEE Trans. Autom. Control*, 1981, **26**, (4), pp. 811–831
38 Dai, L.: 'Singular Control Systems' (*LNCS*, **118**) (Springer-Verlag, Berlin, 1989)
39 Kwon, B.H., Youn, M.J.: 'Eigenvalue-generalized eigenvector assignment by output feedback', *IEEE Trans. Autom. Control*, 1987, **32**, (5), pp. 417–421