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Robust Stability and Performance Analysis of Positive Systems Using Linear Programming

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***Abstract*— This work presents a new necessary and sufficient condition for robust stability and robust performance of positive linear time invariant (LTI) systems with structured uncertainties. It is known that robustness analysis of LTI systems with structured uncertainties requires evaluation of structured singular value (SSV), which is known to be NP-hard. This paper shows that for positive systems, the structured singular value can be estimated efficiently using linear programming. Thus, the robustness analysis of positive systems is simplified to easily verifiable conditions that scale linearly with the dimensions of the system. This property finds great utility in analysis and synthesis of large scale positive systems with distributed control strategies.**

I. Introduction

An externally positive system is defined as a dynamical system whose output trajectory remains non-negative for any non-negative initial condition and any non-negative input. The system is called internally positive if the non-negativeness applies to all the system states as well [1]. Positive systems are widespread in many applications including consensus problems over graphs [2], economic models [3], vehicular formation [4], chemical reactions [1] and biological systems [5].

In recent years, positive systems have attracted significant amount of interests for their practical relevance and more importantly, for their theoretic properties. It has been established that some modern NP-hard control problems are interestingly tractable for positive systems. That is basically on account of linear Lyapunov functions and storage functions that can be utilized for positive systems instead of quadratic ones [6]–[8].

For example, it is shown in [7] and [8] that by utilizing linear Lyapunov functions, analysis and synthesis of positive systems can be exploited for the L_1 and L_∞ gain settings in terms of linear programming. Similar results for robust analysis and synthesis of positive systems are studied in [9]–[11] for the L_1 and L_∞ gain settings and tractable necessary and sufficient conditions are proposed. All these studies enjoy the direct compatibility of the induced L_1 and L_∞ norms with linear copositive Lyapunov functions [8], [12] for the analysis of positive systems. Although, utilizing linear Lyapunov functions provides scalable frameworks for large-scale positive systems, their incorporation in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm settings is not straightforward.

$\mathcal{H}_\infty$ performance of positive systems is studied in [6], [13]–[15]. While the fixed-order $\mathcal{H}_\infty$ control design [16] and the structured control design [17], [18] are both NP-hard problems in general, decentralized $\mathcal{H}_\infty$ static output feedback controllers with positivity preserving property based on linear programming is proposed in [6]. However, the results of [6] are dedicated to $\mathcal{H}_\infty$ control of single-input single-output systems and are not applicable for multivariable systems wherein the concern of scalability is apparent. It is shown in [13] that the structured $\mathcal{H}_\infty$ state feedback design can be described by a semi-definite program (SDP). Robust state feedback control design of positive systems via SDP is studied in [14]. It is also demonstrated in [15] that whereas evaluation of the structured singular value μ [19] is NP-hard in general, for a positive system it can be computed using semi-definite programming. Nevertheless, the level of complexity in the SDP approach limits its utility to systems of medium size. This is on the grounds that even nominal stability verification of a linear system with n states generally requires a Lyapunov function involving n^2 quadratic terms even if the system matrices are sparse [8]. Linear programming fulfils greater scalability as it can be solved for larger systems and with higher accuracy.

This work offers a scalable approach to analyze robustness of positive systems. We first, use the copositive Lyapunov function to show that robust stability and robust performance of positive systems with arbitrary block diagonal uncertain structure can be equivalently described by the Hurwitz stability of a Metzler matrix. The results are further extended to analyze robust stability and robust performance of multivariable positive systems using linear programming.

Throughout the paper, $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ represent the space of $n\times m$ real and complex matrices respectively. $R_+^{n\times m}$ and $R_{++}^{n\times m}$ are the non-negative orthant and positive orthant of $\mathbb{R}^{n\times m}$, respectively. X_{ij} denotes the ij^{th} entry of matrix X. For $X, Y \in \mathbb{R}^{n\times m}$, $A \geq B$ $(A > B)$ indicates that $A_{ij} \geq B_{ij}$ $(A_{ij} > B_{ij})$ for all i, j. For matrix $X \in \mathbb{R}^{n\times n}$, $X \succ 0$ $(\succeq 0)$ indicates that X is positive-definite (semi-definite), and $X \prec 0 (\preceq 0)$ denotes that X is negative definite (semi-definite). Moreover, $\bar{\sigma}(X)$ represents the maximum singular value of X and $\rho(X)$ denotes the spectral radius of X given by $\rho(X) = \underset{i}{max}|\lambda_i(X)|$ where λ_i is the i^{th} eigenvalue of X.

II. Preliminaries

In this section, some necessary definitions and theorems are presented, which are utilized in the subsequent technical development of robust stability and robust performance criteria.

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Definition 1 ([20]): The matrix $S \in \mathbb{R}^{n\times n}$ is called a *Metzler* matrix if $A_{ij} \geq 0$ for all $i \neq j$.

Definition 2 ([20]): A linear time invariant system is called *externally positive* if its impulse response is nonnegative.

External positivity is an input output property and it is independent of the state space realization.

Definition 3 ([20]): A state space realization $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ of a LTI system described by,

$$\begin{aligned} \dot{x} &= \mathcal{A}x + \mathcal{B}u, \\ y &= \mathcal{C}x + \mathcal{D}u, \end{aligned} \tag{1}$$

is called *internally positive*, if and only if $\mathcal{A}$ is a Metzler matrix and $\mathcal{B}, \mathcal{C}, \mathcal{D} \geq 0$.

It is notable that internal positivity implies external positivity. However, the reverse is not necessarily true. That is, a system may be externally positive whilst some realizations of it does not show the internal positivity property [20]. For simplicity, throughout the rest of the paper, we call the systems with an internally positive realization a *positive system*.

Theorem 1 ([1]): Let A be a Metzler matrix. The following statements are equivalent.

1) The matrix A is Hurwitz.
2) There exists $\xi \in \mathbb{R}^n$ with $\xi > 0$ and $A\xi < 0$.
3) There exists $z \in \mathbb{R}^n$ with $z > 0$ and $z^T A < 0$.
4) There exists a diagonal positive-definite matrix P such that $A^T P + PA \prec 0$.
5) $-A^{-1} \geq 0$.

Theorem 2 ([13]): Let the single-input single-output externally positive system $g(s)$ have an internally positive realization given by (A, B, C, D) where $A \in \mathbb{R}^{n\times n}$ is Metzler and $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$, $D \in \mathbb{R}$ are all non-negative matrices. Then, the following statements are equivalent.

1) $\mathcal{A}$ is Hurwitz and $\|g\|_\infty < \gamma$.
2) The matrix $\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix}$ is Hurwitz.

Moreover, $\|g\|_\infty = D - CA^{-1}B$.

Theorem 3 ([11], [21]): Let M be the transfer function of a positive system. Then,

1) $\|M\|_\infty = \underset{\omega}{sup}\ \bar{\sigma}\left(M(j\omega)\right) = \bar{\sigma}\left(M(0)\right)$.
2) $M(0) \geq 0$.

III. The Structured Singular Value Framework

A. Robust Stability and robust performance of Structured Uncertain Systems

In this paper, we consider the structure of an uncertain matrix $\mathbf{\Delta}_s$ as the set of block diagonal complex matrices where the diagonal consists of f full uncertain blocks and k repeated scalar uncertain blocks as follows,

$$\begin{aligned} \mathbf{\Delta}_s = &\Big\{\Delta :\ \Delta = diag\left[\delta_1 I_{r_1}, \ldots, \delta_k I_{r_k},\ \Delta_1, \ldots, \Delta_f\right], \\ &\delta_i \in \mathbb{C},\ \triangle_j \in \mathbb{C}^{k_j \times k_j},\ \bar{\sigma}(\Delta)\gamma, \\ &i = 1 \ldots k,\ j = 1 \ldots f\Big\}. \end{aligned} \tag{2}$$

The structure of $\mathbf{\Delta}_s$ includes both dynamical and parametric uncertainties.

Let the uncertain system be described by the feedback interconnection of LTI system M and the unknown stable LTI system Δ given by,

$$\begin{aligned} q &= Mp \\ p &= \Delta q \end{aligned}$$

with $\Delta(j\omega) \in \mathbf{\Delta}_s \ \ \forall\omega \in \mathbb{R}$. The generic interconnection of $M - \Delta$ is shown in Figure. 1. The robust stability problem

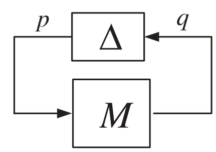

Fig. 1. Feedback interconnection of $M - \Delta$ structure.

with uncertainty $\Delta \in \mathbf{\Delta}_s$ is well established in the structured singular value (μ) framework [22]. It provides conditions to ensure the robust stability of a system with uncertain block $\Delta \in \mathbf{\Delta}_s$, and is defined by [22],

$$\mu(M, \Delta) = \frac{1}{min\{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0, \Delta \in \mathbf{\Delta}\}}, \tag{3}$$

The $M - \Delta$ Structure in robustly stable if and only if,

$$\mu(M, \Delta) < \gamma^{-1} \quad \forall\omega. \tag{4}$$

However, the exact calculation of μ in (4) is proved to be NP-hard for either pure real or mixed real-complex uncertain blocks [23]. The other difficulty of the robust stability verification in (4) is the necessity to calculate $\mu(M, \Delta)$ for the entire frequency range.

Despite the NP-hardness of μ calculation for general systems, the robust performance of LTI system M is also studied and well developed in terms of μ. The condition $\|T_{wz}\|_\infty < \beta^{-1}$ is satisfied for all $\Delta \in \mathbf{\Delta}_s$ if and only if,

$$\mu(M, \tilde{\Delta}) < min\left(\frac{1}{\gamma}, \frac{1}{\beta}\right) \quad \forall\omega, \tag{5}$$

where $\tilde{\Delta} = \begin{bmatrix} \Delta_F & 0 \\ 0 & \Delta \end{bmatrix}$, Δ represents the plant uncertainty block, and Δ_F is a fictitious perturbation defined to evaluate the robust performance.

The constraint (5) requires infinite evaluations of μ, each of which is a hard problem. In the next section it is shown that the SSV calculation can be simplified for positive systems.

B. Structured Singular Value for externally Positive systems

Suppose that the uncertainty structure $\Delta \in \mathbf{\Delta}_s$ is given. The following bounds are achieved for $\mu(M, \Delta)$ [24],

$$\rho(M) \leq \mu(M, \Delta) \leq \underset{\Theta \in \mathbf{\Theta}_s}{inf}\ \bar{\sigma}(\Theta^{\frac{1}{2}} M \Theta^{-\frac{1}{2}}). \tag{6}$$

In (6), $\mathbf{\Theta}_s$ represents the set of positive-definite matrices with the following property,

$$\mathbf{\Theta}_s : \ = \{\Theta \succ 0 \,|\Theta\Delta = \Delta\Theta, \ \forall \Delta \in \mathbf{\Delta}_s\}. \qquad (7)$$

Generally, the lower and upper bounds of $\mu(M, \Delta)$ in (6) are not tight. It is shown in [15] that if M is externally positive, the structure of $\mathbf{\Delta}_s$ is simplified to $\mathbf{\Delta}_p$ as follows,

$$\begin{aligned}\mathbf{\Delta}_p =\Big\{&\Delta : \ \Delta = diag\,[\delta_1, \ \delta_2, \ldots, \ \delta_{\bar{s}}, \ \Delta_1, \ldots, \Delta_f]\,,\\ &\delta_i \in \mathbb{R}_+, \ \Delta_j \in \mathbb{R}_+^{k_j \times k_j}, \bar{s} = \sum_{i=1}^{k} r_i, \ \bar{\sigma}(\Delta) < \gamma,\\ &i = 1 \ldots k, \ j = 1 \ldots f\Big\}. \end{aligned} \qquad (8)$$

In this case, the set $\mathbf{\Theta}_s$ satisfying the commutative property (7) simplifies to,

$$\begin{aligned}\mathbf{\Theta}_p = \big\{&\Theta : \Theta = diag\left[\theta_1, \ldots, \theta_{\bar{s}}, \ \theta_{\bar{s}+1} I_{k_1}, \ldots, \theta_{\bar{s}+f} I_{k_f}\right],\\ &\theta_i \in \mathbb{R}_{++}, \ i = 1, \ldots, \bar{s} + f. \end{aligned} \qquad (9)$$

Consequently, by considering the set $\mathbf{\Theta}_p$, the following Theorem presents a tractable method of finding $\mu(M, \Delta)$ for externally positive system M.

Theorem 4 ([15]): For an externally positive system M we have,

$$\sup_{\omega} \mu\left(M(j\omega), \Delta\right) = \inf_{\Theta \in \mathbf{\Theta}_p} \bar{\sigma}\left(\Theta^{\frac{1}{2}} M(0) \Theta^{-\frac{1}{2}}\right). \qquad (10)$$

Theorem 4 provides a tractable method for finding $\mu(M, \Delta)$ based on Linear Matrix Inequalities (LMIs). However, as mentioned earlier, some problems in positive systems can be expressed in terms of linear programming which outperforms the SDP approach in large scale systems regarding its scalability and reliability. This inspired us to propose a new approach to verify robust stability and robust performance of positive systems through linear programming.

IV. Robustness Analysis Using Linear Programming

In this section, a necessary and sufficient condition is proposed to establish robust stability and robust performance of the $M - \Delta$ structure in Figure 1 based on linear programming. To this end, consider the following Theorem which relates the structured singular value problem to Hurwitz stability of an augmented Metzler matrix $W(\Theta)$.

Theorem 5: Let the matrix $W(\Theta)$ be given by,

$$W(\Theta) = \begin{bmatrix} A & 0 & B & 0 \\ 0 & A^T & 0 & C^T \\ 0 & B^T & -\Theta\gamma^{-1} & D^T \\ C & 0 & D & -\Theta^{-1}\gamma^{-1} \end{bmatrix}. \qquad (11)$$

For a system M with internally positive realization (A, B, C, D) and an uncertainty structure $\mathbf{\Delta}_s$, the $M - \Delta$ structure in Figure 1 is robustly stable if and only if there exists a diagonal positive matrix $\Theta \in \mathbf{\Theta}_s$ which makes the matrix $W(\Theta)$ Hurwitz stable.

Proof: First, Suppose that the $M - \Delta$ interconnection is robustly stable for all $\Delta \in \mathbf{\Delta}_s$. Using Theorem 4, it is equivalent to the existence of $\Theta \in \mathbf{\Theta}_p$ such that,

$$\bar{\sigma}\left(\Theta^{\frac{1}{2}} M(0) \Theta^{-\frac{1}{2}}\right) < \gamma^{-1},$$

or equivalently,

$$M(0)^T \Theta M(0) \prec \gamma^{-2}\Theta. \qquad (12)$$

The constrain (12) can be represented by,

$$\begin{bmatrix} -\gamma^{-1}\Theta & M(0)^T \\ M(0) & -\gamma^{-1}\Theta^{-1} \end{bmatrix} \prec 0. \qquad (13)$$

According to (9), Θ always has a diagonal structure for all $\Delta \in \mathbf{\Delta}_s$. $M(0)$ is a positive matrix due to Theorem 3 and thus,

$$\begin{bmatrix} -\gamma^{-1}\Theta & M(0)^T \\ M(0) & -\gamma^{-1}\Theta^{-1} \end{bmatrix},$$

is a Metzler matrix with real and negative eigenvalues. Then, due to Theorem 1 there exist $\varphi_1 > 0$ and $\varphi_2 > 0$ such that,

$$\begin{bmatrix} -\gamma^{-1}\Theta & M(0)^T \\ M(0) & -\gamma^{-1}\Theta^{-1} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} < 0. \qquad (14)$$

Now, define $x_1 = \varphi_1 - A^{-1}B\varphi_1$. Since $-A^{-1} > 0$ and $B > 0$, we have $x_1 > 0$ and,

$$Ax_1 + B\varphi_1 = A\varphi_1 < 0. \qquad (15)$$

Similarly, define $x_2 = \varphi_2 - A^{-T}C^T\varphi_2 > 0$ to obtain,

$$A^T x_2 + C^T \varphi_2 = A^T \varphi_2 < 0. \qquad (16)$$

Moreover, from (14),

$$-\gamma^{-1}\Theta\varphi_1 + M(0)^T\varphi_2 < 0.$$

By substituting $M(0) = D - CA^{-1}B$,

$$\begin{aligned} &-\gamma^{-1}\Theta\varphi_1 + D^T\varphi_2 - B^T A^{-T} C^T \varphi_2 < 0\\ \Rightarrow &-\gamma^{-1}\Theta\varphi_1 + D^T\varphi_2 - B^T(\varphi_2 - x_2) < 0\\ \Rightarrow &\ B^T x_2 - \gamma^{-1}\Theta\varphi_1 + D^T\varphi_2 - B^T\varphi_2 < 0. \end{aligned}$$

From (14), φ_2 can be selected sufficiently small to have,

$$B^T x_2 - \gamma^{-1}\Theta\varphi_1 + D^T\varphi_2 < 0. \qquad (17)$$

Similarly, from (14),

$$-\gamma^{-1}\Theta^{-1}\varphi_2 + M(0)\varphi_1 < 0.$$

By substituting $M(0) = D - CA^{-1}B$,

$$\begin{aligned} &-\gamma^{-1}\Theta^{-1}\varphi_2 + D\varphi_1 - CA^{-1}B\varphi_1 < 0\\ \Rightarrow &-\gamma^{-1}\Theta^{-1}\varphi_2 + D\varphi_1 - C(\varphi_1 - x_1) < 0\\ \Rightarrow &\ Cx_1 - \gamma^{-1}\Theta^{-1}\varphi_2 + D\varphi_1 - C\varphi_1 < 0. \end{aligned}$$

From (14), φ_1 can be selected sufficiently small to have,

$$Cx_1 - \gamma^{-1}\Theta^{-1}\varphi_2 + D\varphi_1 < 0. \qquad (18)$$

The resultant inequalities (15), (16), (17), (18) can be represented by,

$$\begin{bmatrix} A & 0 & B & 0 \\ 0 & A^T & 0 & C^T \\ 0 & B^T & -\Theta\gamma^{-1} & D^T \\ C & 0 & D & -\Theta^{-1}\gamma^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \varphi_1 \\ \varphi_2 \end{bmatrix} < 0. \tag{19}$$

Thus the existence of the vector $\begin{bmatrix} x_1 & x_2 & \varphi_1 & \varphi_2 \end{bmatrix} > 0$ guarantees that the matrix $W(\Theta)$ is Hurwitz.

Conversely, suppose that there exists Θ such that $W(\Theta)$ is Hurwitz. It implies that there exists $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} > 0$ such that,

$$Ax_1 + Bx_3 < 0, \tag{20}$$

$$A^T x_2 + C^T x_4 < 0, \tag{21}$$

$$B^T x_2 + D^T x_4 - \Theta\gamma^{-1} x_3 < 0, \tag{22}$$

$$Cx_1 + Dx_3 - \gamma^{-1}\Theta^{-1} x_4 < 0. \tag{23}$$

Since A is Hurwitz and Metzler, $-A^{-1} > 0$ and thus from (20),

$$x_1 > -A^{-1}Bx_3. \tag{24}$$

Subsequently, from (24) and (23) we have,

$$\begin{aligned} Cx_1 + Dx_3 - \gamma^{-1}\Theta^{-1}x_4 < 0 \\ \Rightarrow -CA^{-1}Bx_3 + Dx_3 - \gamma^{-1}\Theta^{-1}x_4 < 0 \\ \Rightarrow M(0)x_3 - \gamma^{-1}\Theta^{-1}x_4 < 0 \end{aligned}$$

Similarly, we have$-A^{-T} > 0$ and then from (21),

$$x_2 > -A^{-T}C^T x_4. \tag{25}$$

Thus, from (25) and (22) we have,

$$\begin{aligned} B^T x_2 + D^T x_4 - \gamma^{-1}\Theta x_3 < 0 \\ \Rightarrow -B^T A^{-T} C^T x_4 + D^T x_4 - \gamma^{-1}\Theta x_3 < 0 \\ \Rightarrow M(0)^T x_4 - \gamma^{-1}\Theta x_3 < 0. \end{aligned}$$

Then it is straightforward to show that,

$$\left(M(0)^T \Theta M(0) - \gamma^{-2}\Theta\right) x_4 < 0.$$

Due to the positivity of $M(0)$ and diagonality of Θ, the matrix $M(0)^T\Theta M(0) - \gamma^{-2}\Theta$ is Metzler. Now we can use the stability property of Metzler matrices and from $x_4 > 0$ conclude that $M(0)^T\Theta M(0) - \gamma^{-2}\Theta$ is Hurwitz. Moreover, it is symmetric and thus it is negative definite,

$$M(0)^T\Theta M(0) - \gamma^{-2}\Theta \prec 0, \tag{26}$$

or equivalently,

$$\bar{\sigma}\left(\Theta^{\frac{1}{2}} M(0) \Theta^{-\frac{1}{2}}\right) < \gamma^{-1},$$

and the proof is completed. ∎

Theorem 5 provides a versatile tool for robust stability and robust performance analysis with linear programming proposed in Corollaries 1 and 2.

Corollary 1: For a system M with internally positive realization (A, B, C, D) and an uncertainty structure,

$$\begin{aligned} \mathbf{\Delta}_r = \Big\{\Delta :\ \Delta = diag\left[\delta_1 I_{r_1}, \ldots, \delta_k I_{r_k}\right], \\ \delta_i \in \mathbb{C},\ \bar{\sigma}(\Delta) < \gamma, i = 1, \ldots \bar{s}\Big\}. \end{aligned} \tag{27}$$

the $M - \Delta$ structure in Figure 1 is robustly stable if there exist $x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ z > 0$ such that the following linear program is solvable.

$$\begin{aligned} Ax_1 + Bx_3 < 0, \\ A^T x_2 + C^T z < 0, \\ B^T x_2 - \gamma^{-1} z + D^T z < 0, \\ Cx_1 + Dx_3 - \gamma^{-1} x_3 < 0. \end{aligned} \tag{28}$$

Proof: Suppose that the linear program (28) is solvable with the solution $(x_1^*, x_2^*, x_3^*, z^*)$. Due to the structure of $\mathbf{\Delta}_r$ in (27), the associated matrix Θ has the following structure,

$$\begin{aligned} \mathbf{\Theta}_r = \{\Theta : \Theta = diag\left[\theta_1, \ldots, \theta_{\bar{s}}\right], \\ \theta_i \in \mathbb{R}_{++},\ i = 1, \ldots, \bar{s}. \end{aligned} \tag{29}$$

Then we can set $\Theta^* = diag(\frac{z^*(1)}{x_3^*(1)}, \ldots, \frac{z^*(n)}{x_3^*(n)})$ to obtain,

$$\begin{aligned} Ax_1^* + Bx_3^* < 0, \\ A^T x_2^* + C^T z^* < 0, \\ B^T x_2^* - \gamma^{-1}\Theta^* x_3^* + D^T z^* < 0, \\ Cx_1^* + Dx_3^* - \gamma^{-1}\Theta^{*-1} z^* < 0. \end{aligned}$$

Thus, the following nonlinear program with variables $\Theta,\ x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ x_4 > 0$ is also solvable,

$$\begin{bmatrix} A & 0 & B & 0 \\ 0 & A^T & 0 & C^T \\ 0 & B^T & -\Theta\gamma & D^T \\ C & 0 & D & -\Theta^{-1}\gamma \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} < 0. \tag{30}$$

∎

Utilizing Theorem 5 it is evident that $W(\Theta^*)$ is Hurwitz and the $M - \Delta$ structure is robustly stable.

Corollary 2: Consider a system M with internally positive realization (A, B, C, D) and an uncertainty structure,

$$\begin{aligned} \mathbf{\Delta}_m = \Big\{\Delta :\ \Delta = diag\left[\Delta_1, \ldots, \Delta_f,\ \delta_1 I_{r_1}, \ldots, \delta_k I_{r_k}\right], \\ \delta_i \in \mathbb{C},\ \triangle_j \in \mathbb{C}^{k_j \times k_j}, \bar{\sigma}(\Delta) < \gamma \\ i = 1 \ldots k,\ j = 1 \ldots f\Big\}. \end{aligned} \tag{31}$$

The $M - \Delta$ structure in Figure 1 is robustly stable if there exist $x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ z > 0$ such that for $j =$

$1, \dots, f-1$ the following linear program is solvable.

$$\begin{aligned}
&Ax_1 + Bx_3 < 0,\\
&A^T x_2 + C^T z < 0,\\
&B^T x_2 - \gamma^{-1} z + D^T z < 0,\\
&Cx_1 + Dx_3 - \gamma^{-1} x_3 < 0,\\
&z(1) = \dots = z(k_1),\\
&x_3(1) = \dots = x_3(k_1),\\
&z(\sum_{l=1}^{j} k_l + 1) = \dots = z(\sum_{l=1}^{j+1} k_l),\\
&x_3(\sum_{l=1}^{j} k_l + 1) = \dots = x_3(\sum_{l=1}^{j+1} k_l).
\end{aligned} \tag{32}$$

Proof: The proof is similar to corollary 1 by considering the following structure for the associated matrix Θ,

$$\begin{aligned}
\mathbf{\Theta}_m =\Big\{ &\Theta : \Theta = diag\left[\theta_1 I_{k_1}, \dots, \theta_f I_{k_f},\ \theta_{f+1}, \dots, \theta_{f+\bar{s}}\right],\\
&\theta_i \in \mathbb{R}_{++},\ i = 1, \dots, \bar{s}+f \Big\}.
\end{aligned} \tag{33}$$

■

Robust performance of a positive system can be verified in a similar manner using linear programming. Consider the positive system $\tilde{M}$ described by,

$$\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 p,\\
z &= C_1 x + D_{11} w + D_{12} p,\\
q &= C_2 x + D_{21} w + D_{22} p,\\
p &= \Delta q,
\end{aligned}$$

where A is a Metzler matrix and B_1, B_2, C_1, C_2, D_{11}, D_{12}, D_{21}, D_{22} are all non-negative matrices. Let T_{wz} be the transfer function from w to z. The condition $\|T_{wz}\|_\infty < \beta^{-1}$ is satisfied for all $\Delta \in \mathbf{\Delta}_s$ if and only if

$$\mu(\tilde{M}(0), \tilde{\Delta}) < min\left(\frac{1}{\gamma}, \frac{1}{\beta}\right), \tag{34}$$

where $\tilde{\Delta} = \begin{bmatrix} \Delta_F & 0 \\ 0 & \Delta \end{bmatrix}$, Δ represents the plant uncertainty block, and Δ_F is a fictitious perturbation defined to evaluate the robust performance. In this case, the Δ_F block is a full block and the resultant uncertain set $\tilde{\mathbf{\Delta}}_s$ is given by,

$$\begin{aligned}
\tilde{\mathbf{\Delta}}_s =\Big\{ &\tilde{\Delta} :\ \tilde{\Delta} = diag\left[\Delta_F, \Delta_1, \dots, \Delta_f,\ \delta_1 I_{r_1}, \dots, \delta_k I_{r_k}\right],\\
&\delta_i \in \mathbb{C},\ \Delta_j \in \mathbb{C}^{k_j \times k_j} \Delta_F \in \mathbb{C}^{k_0 \times k_0}, \bar{\sigma}(\Delta) < \gamma \Big\}.
\end{aligned} \tag{35}$$

Accordingly, the associated $\tilde{\mathbf{\Theta}}_s$ is given by,

$$\begin{aligned}
\tilde{\mathbf{\Theta}}_s = \Big\{ &\Theta : \Theta = diag\left[\theta_0 I_{k_0}, \theta_1 I_{k_1}, \dots, \theta_f I_{k_f}, \phi_1, \dots, \phi_{\bar{s}}\right],\\
&\theta_i, \phi_j \in \mathbb{R}_{++}, i = 0, \dots, f,\ j = 1, \dots, \bar{s} \Big\}.
\end{aligned} \tag{36}$$

The system is robustly stable with $\|T_{wz}\|_\infty < \beta^{-1}$ if there exist $x_1 > 0,\ x_2 > 0,\ x_3 > 0,\ z > 0$ such that for $j = \{0 \dots,\ f-1\}$ the linear program (32) is solvable by replacing γ^{-1} by $\mu^{-1} = \min\left(\frac{1}{\gamma}, \frac{1}{\beta}\right)$.

V. Numerical Example

Consider the vehicle formation model given in [6], [8] described by,

$$\begin{aligned}
\dot{x}_1 &= -x_1 + l_{13}(x_3 - x_1) + w,\\
\dot{x}_2 &= l_{21}(x_1 - x_2) + l_{23}(x_3 - x_2) + w,\\
\dot{x}_3 &= l_{32}(x_2 - x_3) + l_{34}(x_4 - x_3) + w,\\
\dot{x}_4 &= -4x_4 + l_{43}(x_3 - x_4) + w.
\end{aligned}$$

This model represents a formation of four vehicles. The signal w is an external disturbance acting on the vehicles. The feedback gains l_{ij} adjust the position of four vehicles are selected to be $l_{13} = 0, l_{21} = 1, l_{23} = 1, l_{32} = 0, l_{34} = 1, l_{43} = 0$. The objective is to analyze stability of the formation and also the achievable $\mathcal{H}_\infty$ gain from disturbance w to $z = x_1 + x_2 + x_3 + x_4$ for the uncertain system with the uncertainty model as follows,

$$\begin{aligned}
\dot{x} &=Ax + B_1 w + B_2 p,\\
z &=C_1 x,\\
q &=C_2 x,\\
p &=\Delta q,
\end{aligned}$$

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T,$$
$$C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.3 \end{bmatrix},$$
$$D_{11} = 0,\ D_{12} = 0_{1\times 4},\ D21 = 0_{4\times 1},\ D_2 = 0_{4\times 4}.$$

The $\mathcal{H}_\infty$ performance of the system at the absence of any kinds of perturbations from w to z is equal to 4.1250. To apply the presented method, first consider the special case of unstructured 4×4 uncertainty block Δ_1. The uncertainty Δ_1 can represent the dynamical uncertainties in the system. To obtain the maximum tolerable $\|\Delta_1\|_\infty$ which maintains the robust stability of the system, note that the unstructured Δ_1 enforces the optimization parameter Θ to be in the form of θI_4. Then from Theorem 5, robust stability of the system is equivalent to the Hurwitz stability of the matrix,

$$\begin{bmatrix} A & 0 & B_2 \\ \theta C_2^T C_2 & A^T & \theta C_2^T D_2 \\ \theta D_2^T C_2 & B_2^T & \theta(-\gamma^{-1} I + D_2^T D_2) \end{bmatrix},$$

Which is subsequently verified by the existence of vectors $x_1 > 0,\ x_2 > 0,\ x_3 > 0$ and an scalar $\theta > 0$ such that,

$$\begin{bmatrix} A & 0 & B_2 \\ \theta C_2^T C_2 & A^T & \theta C_2^T D_2 \\ \theta D_2^T C_2 & B_2^T & \theta(-\gamma^{-1} I + D_2^T D_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} < 0.$$

It is easy to verify that the structure of $\Theta_1 = \theta I_4$ makes the value of γ^{-1} independent of the value of θ by considering $z_1 = \theta x_1,\ z_3 = \theta x_3$. Then the problem is equivalent to the linear program,

$$\begin{bmatrix} A & 0 & B_2 \\ C_2^T C_2 & A^T & C_2^T D_2 \\ D_2^T C_2 & B_2^T & (-\gamma^{-1} I + D_2^T D_2) \end{bmatrix} \begin{bmatrix} z_1 \\ x_2 \\ z_3 \end{bmatrix} < 0,$$

or equivalently the Hurwitz stability of,

$$\begin{bmatrix} A & 0 & B_2 & 0 \\ 0 & A^T & 0 & C_2^T \\ 0 & B_2^T & -\gamma^{-1} & D_2^T \\ C_2 & 0 & D_2 & -\gamma^{-1} \end{bmatrix},$$

equivalently represented by the problem of existence of $y_1 > 0,\ y_2 > 0,\ y_3 > 0,\ y_4 > 0$ such that,

$$\begin{bmatrix} A & 0 & B_2 & 0 \\ 0 & A^T & 0 & C_2^T \\ 0 & B_2^T & -\gamma^{-1} & D_2^T \\ C_2 & 0 & D_2 & -\gamma^{-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} < 0.$$

This result is in accordance with the small gain theorem [25] which introduces the inverse of the $\mathcal{H}_\infty$ norm of the transfer function from p to q, as the maximum tolerable perturbation norm of unstructured Δ_1. By this discussion, $\gamma = 2.7041$ and the system can tolerate any unstructured perturbations with $\|\Delta_1\|_\infty < 2.7041$. As the second scenario, let the perturbation block Δ_2 be given by,

$$\Delta_2 = \begin{bmatrix} \theta_1 & & & \\ & \theta_2 & & \\ & & \theta_3 & \\ & & & \theta_4 \end{bmatrix}.$$

The imposed structure on Δ_2 adds some degrees of freedom to the structure of Θ and increases the maximum tolerable perturbations to maintain robust stability of the system. In this case we obtain $\gamma = 3.3333$ and conclude that robust stability of the system with structured perturbation Δ_2 is guaranteed as long as $\|\Delta_2\|_\infty < 3.3333$. The third scenario is to obtain the maximum tolerable perturbations which restrict the $\mathcal{H}_\infty$ norm of the system. In this case we obtain $\min\left(\frac{1}{\gamma}, \frac{1}{\beta}\right) = 4.4024$ and thus, for $\|\Delta_2\|_\infty < 0.22714$, the system is guaranteed to be stable with the $\mathcal{H}_\infty$ norm less than or equal to 4.4024.

VI. Conclusion

In this paper necessary and sufficient conditions for robust stability and robust performance of an uncertain LTI positive system is proposed. The uncertain system is modeled as the feedback interconnection of a positive system and a block structured, norm bounded uncertainty. It is first shown that robust stability and robust performance of positive systems with arbitrary block diagonal uncertain structure can be equivalently described by the Hurwitz stability of a Metzler matrix. The results are further extended to analyze robust stability and robust performance of positive systems using linear programming. The results are easily extendable to positive discrete-time systems. We believe that the proposed framework finds great utility in the analysis and synthesis of distributed control structures in large scale systems.

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