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# Stability Analysis for Continuous-Time Positive Systems With Time-Varying Delays

Xingwen Liu, Wensheng Yu, and Long Wang

***Abstract*—This note addresses the stability problem of continuous-time positive systems with time-varying delays. It is shown that such a system is asymptotically stable for any continuous and bounded delay if and only if the sum of all the system matrices is a Hurwitz matrix. The result is a time-varying version of the widely-known asymptotic stability criterion for constant-delay positive systems. A numerical example illustrates the correctness of our result.**

***Index Terms*—Asymptotic stability, linear copositive Lyapunov functional, positive system, time-varying delays.**

## NOMENCLATURE

| | |
|---|---|
| $A \succeq 0 (\preceq 0)$ | All entries of matrix $A$ are nonnegative (nonpositive). |
| $A \succ 0 (\prec 0)$ | All entries of matrix $A$ are positive (negative). |
| $A^T$ | Transpose of matrix $A$. |
| $\mathbb{R}_{0,+} (\mathbb{R}_+)$ | Set of all nonnegative (positive) real numbers. |
| $\mathbb{R}^n (\mathbb{R}^n_{0,+}, \mathbb{R}^n_+)$ | $n$-dimensional real (nonnegative, positive) vector space, where positive vector means all its elements are positive. |
| $\mathbb{R}^{n\times m} (\mathbb{R}^{n\times m}_{0,+})$ | Set of all real (nonnegative) matrices of $n \times m$-dimension. |
| $\mathbb{M}$ | Set of Metzler matrices. |

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## I. INTRODUCTION

This note considers the stability problem of positive systems [1], [2]. A system is said to be positive if its states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative. Positive systems model many real world physical systems that involve nonnegative variables, for example, population levels, absolute temperature, concentration of substances. Hence, in the past years researchers have paid much attention to analyze and synthesize positive systems [3]–[15].

As one of the most important properties of positive systems, stability has been actively studied. References [16]–[19] established some necessary and sufficient stability criteria for positive systems without delays, mainly using diagonal quadratic or linear co-positive Lyapunov functions. The linear copositive Lyapunov function method captures the very nature of positive systems, namely, that their states are always nonnegative, so it is a powerful tool for tackling positive systems [20]–[22]. [23]–[25] proposed some necessary and sufficient stability criteria for positive systems with constant delays by means of linear copositive Lyapunov functionals. It was shown that such a system is asymptotically stable if and only if the sum of the system matrices is a Hurwitz matrix (for continuous-time system) or a Schur matrix (for discrete-time system). [26] presented some sufficient stability conditions for delayed positive systems with uncertainties. Based on these results, the constrained control and observer designing problems are treated in [27]–[30].

Let us return to the above-mentioned sufficient and necessary condition. It considers constant delays, but it turns out that the system stability is independent of the delays. Hence it is reasonable to conjecture that the condition may still hold even if the delays are time-varying. It is nontrivial to prove or disprove the conjecture, because the constant-delay case heavily relies on linear copositive Lyapunov functionals, which would bring about excessive conservativeness when the delays are time-varying. Time-varying delays are universal and have intriguing impacts on system dynamics, so they have attracted much attention in studying the stability problem of general systems, see, for example, [31]–[34]. However little progress has been made in positive systems. Recently, [35] employed a novel idea and showed that the conjecture above stated is true for discrete-time positive systems with time-varying delays. On this ground, the present note tries to extend this result to continuous-time positive systems with time-varying delays.

Our main contribution is to show that the extension does hold. A novel lemma is proved which claims that under certain conditions, the derivative of the trajectory of a stable and constantly delayed positive system is negative. This lemma not only identifies an elegant property of such systems, but also enables us to reduce the stability problem of continuous positive systems with time-varying delays to that with constant delays.

The rest of this note is organized as follows. In Section II, necessary preliminaries are presented and some lemmas are provided. Section III proposes a necessary and sufficient stability criterion for continuous-time positive systems with time-varying delays. An example is given in Section IV, and Section V concludes this note.

## II. Preliminaries

The dimensions of matrices and vectors will not be explicitly mentioned if clear from context. For simplicity, let $\underline{\mathbf{p}} = \{1, 2, \cdots, p\}$ and $\underline{\mathbf{p}}_0 = \{0\} \cup \underline{\mathbf{p}}$, where $p$ is an arbitrary positive integer.

Consider the following system:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A_0\mathbf{x}(t) + \sum_{l=1}^{p} A_l \mathbf{x}\left(t - \tau_l(t)\right), \quad t \geq 0\\ \mathbf{x}(t) &= \boldsymbol{\varphi}(t) \succeq 0, \quad t \in [-\tau, 0]\end{aligned} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state variable, $A_l \in \mathbb{R}^{n\times n}, l \in \underline{\mathbf{p}}_0$, are system matrices. Delays $\tau_l(t) \geq 0$ are assumed to be bounded and continuous with respect to $t$, and satisfy

$$0 \leq \tau_l(t) \leq \tau_l, t \geq 0, \quad \tau = \max_{1\leq l\leq p}\{\tau_l\} \tag{2}$$

with constants $\tau_l \in \mathbb{R}_+$. $\boldsymbol{\varphi} : [-\tau, 0] \to \mathbb{R}^n_{0,+}$ is the vector-valued initial function. For simplicity, use $\dot{\mathbf{x}}(0)$ to denote the right-hand derivative of $\mathbf{x}(t)$ at zero.

*Definition 1:* System (1) is said to be positive if for any $\boldsymbol{\varphi} : [-\tau, 0] \to \mathbb{R}_{0,+}$, the corresponding trajectory satisfies $\mathbf{x}(t) \succeq 0$ for all $t \geq 0$.

*Lemma 1:* ([2, Theor. 1.18, Ch. 1]) Let $A \in \mathbb{R}^{n\times n}$. Then $e^{At} \succeq 0$, $\forall t \geq 0$, if and only if $A$ is a Metzler matrix.

*Lemma 2:* System (1) is positive if and only if $A_0 \in \mathbb{M}$ and $A_l \succeq 0$, $l \in \underline{\mathbf{p}}$.

*Proof:* This lemma can be seen as a natural extension of [23, Lemma 2.1]. ■

*Lemma 3:* Assume that system (1) is positive. Let $\boldsymbol{x}_a(t)$ and $\boldsymbol{x}_b(t)$, $t \geq 0$, be solution trajectories of (1) under the initial conditions $\boldsymbol{\varphi}_a(t)$ and $\boldsymbol{\varphi}_b(t)(t \in [-\tau, 0])$, respectively. Then $\boldsymbol{\varphi}_a(t) \preceq \boldsymbol{\varphi}_b(t)(t \in [-\tau, 0])$ implies that $\boldsymbol{x}_a(t) \preceq \boldsymbol{x}_b(t), t \geq 0$.

*Proof:* This lemma is a natural extension of [23, Lemma 2.2]. ■

## III. Main Results

This section will establish the asymptotic stability criterion for continuous-time positive systems with time-varying delays. From now on, we always assume that system (1) is positive, i.e., $A_0 \in \mathbb{M}$ and $A_l \succeq 0$, $l \in \underline{\mathbf{p}}$.

Introduce the next system, which is closely related to (1)

$$\begin{aligned}\dot{\mathbf{y}}(t) &= A_0\mathbf{y}(t) + \sum_{l=1}^{p} A_l\mathbf{y}(t - \tau_l), \quad t \geq 0\\ \mathbf{y}(t) &= \boldsymbol{\psi}(t) \succeq 0, \quad t \in [-\tau, 0].\end{aligned} \tag{3}$$

*Lemma 4:* ([23, Theor. 2.1]) System (3) is asymptotically stable if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ satisfying

$$\sum_{l=0}^{p} A_l\boldsymbol{\lambda} \prec 0. \tag{4}$$

*Remark 1:* ([18, Theor. 2.4]) Condition (4) holds if and only if $\sum_{l=0}^{p} A_l$ is a Hurwitz matrix.

The following lemma reveals an important property of system (3).

*Lemma 5:* Consider system (3). Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ satisfying (4) and that the initial function $\boldsymbol{\psi}(t) \equiv \boldsymbol{\lambda}$, $t \in [-\tau, 0]$. Then the solution $\mathbf{y}(t)$ to (3) satisfies $\dot{\mathbf{y}}(t) \prec 0, \forall t \in [0, +\infty)$.

*Proof:* Let $\delta = \sup\{t \geq 0 : \dot{\mathbf{y}}(s) \prec 0 \text{ for all } s \in [0, t)\}$. By (3), $\dot{\mathbf{y}}(0) = A_0\mathbf{y}(0) + \sum_{l=1}^{p} A_l\mathbf{y}(-\tau_l) = \sum_{l=0}^{p} A_l\boldsymbol{\lambda} \prec 0$, which implies that $\delta > 0$.

Suppose that $\delta < +\infty$. Then $\dot{\mathbf{y}}(t) \prec 0$ for all $t \in [0, \delta)$ and there exists an integer $1 \leq r \leq n$ such that there are exactly $r$ elements of $\dot{\mathbf{y}}(\delta)$ equal to zero. Without loss of generality, assume that $\mathbf{y}_1(\delta) = 0$ while $\mathbf{y}_2(\delta) \prec 0$, where $\mathbf{y}_1(\cdot) = [y_1(\cdot), \cdots, y_r(\cdot)]^T$ and $\mathbf{y}_2(\cdot) = [y_{r+1}(\cdot), \cdots, y_n(\cdot)]^T$. Partition the matrices $A_0$ and $A_l$, $1 \leq l \leq p$, into blocks

$$A_0 = \left[\begin{array}{c|c} A_{11}^{(0)} & A_{12}^{(0)} \\ \hline A_{21}^{(0)} & A_{22}^{(0)} \end{array}\right]$$

$$A_l = \left[\begin{array}{c} A_1^{(l)} \\ \hline A_2^{(l)} \end{array}\right], \; A_{11}^{(0)} \in \mathbb{R}^{r\times r}, A_1^{(l)} \in \mathbb{R}^{r\times n}.$$

Hereafter, use $a_{ij}^{(l)}$ to denote the element of $A_l$ located at the $i$th row and the $j$th column and let $L_1 = \{l : \tau_l < \delta\}$. Define

$$a = \max\left(\left\{a_{ij}^{(0)} : i \in \underline{\mathbf{r}}, r < j \leq n\right\} \cup \left\{a_{ij}^{(l)} : l \in L_1, i \in \underline{\mathbf{r}}, j \in \underline{\mathbf{n}}\right\}\right).$$

The only case when $a$ could not be defined would be when both $L_1$ is the empty set and $r = n$ (namely $\dot{\mathbf{y}}(0) = 0$). Since system (3) is positive, $a \geq 0$ if it exists. In fact, we have

*Claim 1:* $a$ exists and it is positive.

Suppose on contrary that either $a$ does not exist which means $L_1 = \emptyset$ and $r = n$, or $a = 0$. In either case, $\dot{\mathbf{y}}_1(t)$ satisfies

$$\dot{\mathbf{y}}_1(t) = A_{11}^{(0)}\mathbf{y}_1(t) + \mathbf{z}, t \in [0, \delta], \; \mathbf{y}_1(0) = \boldsymbol{\lambda}_1 = [\lambda_1, \cdots, \lambda_r]^T \tag{5}$$

with $\mathbf{z} = \sum_{l\in L_2} A_1^{(l)}\mathbf{y}(t - \tau_l)$ and $L_2 = \{l : \tau_l \geq \delta\}$. Note that if $r = n$, then $\mathbf{y}_1 = \mathbf{y}$ and $A_{11}^{(0)} = A_0$, and that if $L_2 = \emptyset$, then $\mathbf{z} = 0$. When $L_2 \neq \emptyset$, for any $l \in L_2$, since $t \in [0, \delta]$, we have that $t - \tau_l \leq 0$, implying that $\mathbf{y}(t - \tau_l) = \boldsymbol{\lambda}$. Anyway, $\mathbf{z}$ is constant and $\mathbf{y}_1(t) = e^{A_{11}^{(0)}t}\boldsymbol{\lambda}_1 + (\int_0^t e^{A_{11}^{(0)}(t-s)}ds)\mathbf{z}, t \in [0, \delta]$. Applying this into (5), we have

$$\begin{aligned}\dot{\mathbf{y}}_1(t) &= A_{11}^{(0)}\left(e^{A_{11}^{(0)}t}\boldsymbol{\lambda}_1 + \left(\int_0^t e^{A_{11}^{(0)}(t-s)}ds\right)\mathbf{z}\right) + \mathbf{z}\\ &= e^{A_{11}^{(0)}t}\left(A_{11}^{(0)}\boldsymbol{\lambda}_1 + \mathbf{z}\right) = e^{A_{11}^{(0)}t}\dot{\mathbf{y}}_1(0), \quad t \in [0, \delta].\end{aligned}$$

Since $A_{11}^{(0)} \in \mathbb{M}$, by Lemma 1, $e^{A_{11}^{(0)}t} \succeq 0$. Because $\dot{\mathbf{y}}_1(0) \prec 0$ and $\dot{\mathbf{y}}_1(\delta) = 0$, we have $e^{A_{11}^{(0)}\delta} = 0$, which contradicts the well-known fact that $e^{A_{11}^{(0)}\delta}$ is nonsingular. Hence, Claim 1 holds.

Then, choose an integer $k \in \underline{\mathbf{r}}$ such that $a = a_{kj}^{(0)}$ for some $r < j \leq n$, or $a = a_{kj}^{(l)}$ for some $l \in L_1$, $j \in \underline{\mathbf{n}}$. We have

*Claim 2:* $\dot{y}_k(\delta) < 0$.

Let $\hat{\tau} = \min_{l\in\underline{\mathbf{p}}}\{\tau_l\}$, $\theta_1 = \max(\{\sup_{0\leq t\leq\delta-\hat{\tau}} \dot{y}_j(t) : j \in \underline{\mathbf{n}}\} \cup \{\sup_{0\leq t\leq\delta} \dot{y}_j(t) : r < j \leq n\})$, and

$$\bar{\tau} = \begin{cases}\max\{\tau_l : l \in L_1\}, & \text{if } L_1 \neq \emptyset\\ 0, & \text{otherwise.}\end{cases}$$

Of course, $0 < \bar{\tau} < \delta$ if $L_1 \neq \emptyset$. According to Claim 1, $a$ exists, implying that $L_1 \neq \emptyset$ or $r < n$. As a result, $\delta - \hat{\tau} > 0$ or $r < n$, which guarantees that $\theta_1$ exists. Furthermore, by the definition of $\delta$ and $r$, $\theta_1 < 0$. Arbitrarily choose a positive number $b$ satisfying that $b > |a_{kj}^{(0)}|$ for all $j \in \underline{\mathbf{r}}$. Choose a sufficiently small, positive number $\xi < \min\{\delta - \bar{\tau}, \delta\}$ such that $\theta_2 = \max\{|\dot{y}_j(t)| : \delta - \xi \leq t \leq \delta, j \in \underline{\mathbf{r}}\} < (-a\theta_1/rb)$. Such a $\xi$ exists due to the continuity of $\dot{\mathbf{y}}_1(\cdot)$.

According to system (3)

$$\begin{aligned}\dot{y}_k(\delta) - \dot{y}_k(\delta-\xi) &= \sum_{j=1}^{r} a_{kj}^{(0)} \left(y_j(\delta) - y_j(\delta-\xi)\right) \\ &\quad + \sum_{j=r+1}^{n} a_{kj}^{(0)} \left(y_j(\delta) - y_j(\delta-\xi)\right) \\ &\quad + \sum_{l\in L_1}\sum_{j=1}^{n} a_{kj}^{(l)} \left(y_j(\delta-\tau_l) - y_j(\delta-\xi-\tau_l)\right) \\ &\quad + \sum_{l\in L_2}\sum_{j=1}^{n} a_{kj}^{(l)} \left(y_j(\delta-\tau_l) - y_j(\delta-\xi-\tau_l)\right).\end{aligned}$$

Because $\mathbf{y}(\cdot)$ is constant over $[-\tau, 0]$, when $l \in L_2$, $\delta - \tau_l \le 0$, $\delta - \xi - \tau_l \le 0$, thus $y_j(\delta-\tau_l) - y_j(\delta-\xi-\tau_l)$ is equal to zero for each $j \in \underline{\mathbf{n}}$. Hence

$$\begin{aligned}\dot{y}_k(\delta) - \dot{y}_k(\delta-\xi) &= \sum_{j=1}^{r} a_{kj}^{(0)} \left(y_j(\delta) - y_j(\delta-\xi)\right) \\ &\quad + \sum_{j=r+1}^{n} a_{kj}^{(0)} \left(y_j(\delta) - y_j(\delta-\xi)\right) \\ &\quad + \sum_{l\in L_1}\sum_{j=1}^{n} a_{kj}^{(l)} \left(y_j(\delta-\tau_l) - y_j(\delta-\xi-\tau_l)\right).\end{aligned}$$

Because $\delta > \delta - \xi > 0$ and for each $l \in L_1$, $\delta - \hat{\tau} \ge \delta - \tau_l > \delta - \xi - \tau_l \ge \delta - \xi - \bar{\tau} > 0$, we can apply the mean-value theorem to obtain

$$\dot{y}_k(\delta) - \dot{y}_k(\delta-\xi) \le \xi \left( \sum_{j=1}^{r} a_{kj}^{(0)} \dot{y}_j(\alpha_j) + \sum_{j=r+1}^{n} a_{kj}^{(0)} \dot{y}_j(\beta_j) + \sum_{l\in L_1}\sum_{j=1}^{n} a_{kj}^{(l)} \dot{y}_j(\gamma_{jl}) \right)$$

where each $\alpha_j \in [\delta-\xi, \delta]$, $\beta_j \in [\delta-\xi, \delta]$, $\gamma_{jl} \in [\delta-\xi-\tau_l, \delta-\tau_l]$. As a result

$$\begin{aligned}\dot{y}_k(\delta) - \dot{y}_k(\delta-\xi) &\le \xi \left( \sum_{j=1}^{r} \left|a_{kj}^{(0)}\right| \theta_2 + \left( \sum_{j=r+1}^{n} a_{kj}^{(0)} \theta_1 + \sum_{l\in L_1}\sum_{j=1}^{n} a_{kj}^{(l)} \theta_1 \right) \right) \\ &\le \xi (rb\theta_2 + a\theta_1) < 0.\end{aligned}$$

Since $\dot{y}_k(\delta - \xi) < 0$, we have $\dot{y}_k(\delta) < 0$. Claim 2 holds.

Now we reach a contradiction, so $\delta = +\infty$, and the lemma holds. ■

Lemma 5 shows that the solution to (3) is strictly monotonically decreasing, provided that the initial function satisfies (4). This lemma is crucial to the following one.

*Lemma 6:* Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfying (4), and that the initial conditions for systems (1) and (3) are $\boldsymbol{\varphi}(t) \equiv \boldsymbol{\lambda}$ and $\boldsymbol{\psi}(t) \equiv \boldsymbol{\lambda}$, $t \in [-\tau, 0]$, respectively. Then for all $t \ge 0$, it holds that $\mathbf{x}(t) \preceq \mathbf{y}(t)$, where $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are solutions to (1) and (3), respectively.

*Proof:* It holds that for $t \ge 0$

$$\mathbf{x}(t) - \mathbf{y}(t) = \int_0^t e^{A_0(t-s)} \left( \sum_{l=1}^{p} A_l \left(\mathbf{x}(s-\tau_l(s)) - \mathbf{y}(s-\tau_l)\right) \right) ds. \tag{6}$$

Because $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ satisfies (4), $\dot{\mathbf{x}}(0) \prec 0$. Hence, there is a sufficiently small number $0 < \varepsilon < \hat{\tau}$ such that $\mathbf{x}(t) \prec \mathbf{x}(0) = \boldsymbol{\lambda}$ for all $t \in (0, \varepsilon]$. So, for all $t \in (0, \varepsilon]$,

$$\begin{aligned}\mathbf{x}(t) - \mathbf{y}(t) &= \int_0^t e^{A_0(t-s)} \left( \sum_{l=1}^{p} A_l \left(\mathbf{x}(s-\tau_l(s))\right) \right) ds \\ &\quad - \int_0^t e^{A_0(t-s)} \left( \sum_{l=1}^{p} A_l \boldsymbol{\lambda} \right) ds \\ &\preceq \int_0^t e^{A_0(t-s)} \left( \sum_{l=1}^{p} A_l \boldsymbol{\lambda} \right) ds \\ &\quad - \int_0^t e^{A_0(t-s)} \left( \sum_{l=1}^{p} A_l \boldsymbol{\lambda} \right) ds \\ &= 0.\end{aligned}$$

Define $\delta = \sup\{\varepsilon \ge 0 : \mathbf{x}(t) - \mathbf{y}(t) \preceq 0 \text{ for all } t \in [0, \varepsilon]\}$. We have already shown that $\delta$ exists and that $\delta > 0$.

Assume that $\delta < +\infty$. Then $\mathbf{x}(\delta) \preceq \mathbf{y}(\delta)$ and at least one of the entries of $\mathbf{x}(\delta)$ coincides with the corresponding entry of $\mathbf{y}(\delta)$. By Lemma 5, $\mathbf{y}(\delta) \prec \mathbf{y}(\delta - \tau_l)$ for $l \in \underline{\mathbf{p}}$. Hence $\boldsymbol{\eta} = (\mathbf{y}(\delta - \hat{\tau}) - \mathbf{x}(\delta))/4 \succ 0$, and $\mathbf{x}(\delta) + \boldsymbol{\eta} \prec \mathbf{y}(\delta - \tau_l) - \boldsymbol{\eta}$ for $l \in \underline{\mathbf{p}}$. By continuity of $\mathbf{x}(\cdot)$ and $\mathbf{y}(\cdot)$, there exists a sufficiently small scalar $\varsigma > 0$ satisfying $\mathbf{x}(t) \preceq \mathbf{x}(\delta) + \boldsymbol{\eta}$, $\mathbf{y}(\delta - \tau_l) - \boldsymbol{\eta} \preceq \mathbf{y}(t - \tau_l)$ for $t \in [\delta, \delta + \varsigma]$ and $l \in \underline{\mathbf{p}}$.

Arbitrarily choose $s \in [\delta, \delta + \varsigma]$ and $l \in \underline{\mathbf{p}}$. If $s - \tau_l(s) \le \delta$, then $\mathbf{x}(s - \tau_l(s)) \preceq \mathbf{y}(s - \tau_l(s)) \preceq \mathbf{y}(s - \tau_l)$. If $s - \tau_l(s) > \delta$, then $\mathbf{x}(s - \tau_l(s)) \preceq \mathbf{x}(\delta) + \boldsymbol{\eta} \prec \mathbf{y}(\delta - \tau_l) - \boldsymbol{\eta} \preceq \mathbf{y}(s - \tau_l)$ holds. In either case, we have $\mathbf{x}(s - \tau_l(s)) \preceq \mathbf{y}(s - \tau_l)$. According to (6), $\mathbf{x}(t) \preceq \mathbf{y}(t)$ for $t \in [0, \delta + \varsigma]$, which contradicts the definition of $\delta$.

As a result, $\delta = +\infty$, which immediately means that for all $t \ge 0$, $\mathbf{x}(t) \preceq \mathbf{y}(t)$. ■

Lemma 6 shows that under specific conditions, the solution to system (3) *dominates* that to (1). This lemma plays a key role in the proof of Theorem 1.

*Theorem 1:* System (1) is asymptotically stable for any delays satisfying (2) if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (4) holds.

*Proof:* The proof is divided into two parts: sufficiency and necessity.

***Sufficiency***. Suppose that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (4) holds. By Lemma 4, system (3) is asymptotically stable. Below we show that the asymptotic stability of system (3) implies that of system (1).

Without loss of generality, in the rest of the proof, we take $\|\cdot\|_\infty$, denoted by $\|\cdot\|$ for simplicity, as the vector norm. Since system (3) is asymptotically stable, for any $\varepsilon > 0$, there exists a scalar $\delta > 0$ such that for any initial condition $\boldsymbol{\psi}(\cdot)$, if only $\sup_{t\in[-\tau,0]} \|\boldsymbol{\psi}(t)\| < \delta$, the corresponding solution $\mathbf{y}_{\boldsymbol{\psi}}(t)$ to (3) satisfies $\|\mathbf{y}_{\boldsymbol{\psi}}(t)\| < \varepsilon$ for all $t \ge 0$ and $\lim_{t\to+\infty} \|\mathbf{y}_{\boldsymbol{\psi}}(t)\| = 0$.

Choose constant $\alpha > 0$ such that $\|\alpha\boldsymbol{\lambda}\| < \delta$. Let $\delta' = \min\{\alpha\lambda_i | 1 \le i \le n\} > 0$. Let $\mathbf{x}_{\alpha\boldsymbol{\lambda}}(\cdot)$ and $\mathbf{y}_{\alpha\boldsymbol{\lambda}}(\cdot)$ be the solutions to systems (1) and (3) under the same initial condition $\alpha\boldsymbol{\lambda}$, respectively.

Now consider system (1) under an arbitrary initial condition $\boldsymbol{\varphi}(t) \succeq 0$, $t \in [-\tau, 0]$. Let $\mathbf{x}_{\boldsymbol{\varphi}}(\cdot)$ denote the corresponding solution to (1). If $\sup_{t\in[-\tau,0]} \|\boldsymbol{\varphi}(t)\| < \delta'$, then $\boldsymbol{\varphi}(t) \prec \alpha\boldsymbol{\lambda}$ for $t \in [-\tau, 0]$. By Lemma 3, $\mathbf{x}_{\boldsymbol{\varphi}}(t) \preceq \mathbf{x}_{\alpha\boldsymbol{\lambda}}(t)$ for $t \ge 0$. On the other hand, by Lemma 6, $\mathbf{x}_{\alpha\boldsymbol{\lambda}}(t) \preceq \mathbf{y}_{\alpha\boldsymbol{\lambda}}(t)$ for $t \ge 0$. Hence, $\mathbf{x}_{\boldsymbol{\varphi}}(t) \preceq \mathbf{y}_{\alpha\boldsymbol{\lambda}}(t)$ for $t \ge 0$. Because $\|\alpha\boldsymbol{\lambda}\| < \delta$, by the selection of $\delta$, we have $\|\mathbf{y}_{\alpha\boldsymbol{\lambda}}(t)\| < \varepsilon$ for all $t \geq 0$ and $\lim_{t\to+\infty} \|\mathbf{y}_{\alpha\boldsymbol{\lambda}}(t)\| = 0$. As a result, $\|\mathbf{x}_{\boldsymbol{\varphi}}(t)\| < \varepsilon$ for all $t \geq 0$ and $\lim_{t\to+\infty} \|\mathbf{x}_{\boldsymbol{\varphi}}(t)\| = 0$.

Altogether, we have shown that system (1) is asymptotically stable for any delays satisfying (2).

***Necessity***. Suppose that system (1) is asymptotically stable for any delays satisfying (2). Particularly, let $\tau_l(t) = \tau_l$. Then system (3) is necessarily asymptotically stable. By Lemma 4, (4) holds. ■

*Remark 2:* As shown in Theorem 1, the magnitude of delays has no any impact on the asymptotic stability of system (1).

*Remark 3:* For general dynamic systems with delays, it is easy to find such systems whose asymptotic stability is closely related to the magnitude of delays. Therefore, Theorem 1 reveals the important difference between positive systems and general dynamic systems in terms of asymptotic stability.

*Remark 4:* For the discrete-time positive systems with bounded time-varying delays, a conclusion similar to Theorem 1 also holds, where the magnitude of delays does not impact on the asymptotic stability—see [35] for details.

*Remark 5:* In system (1), $A_1$ is a Metzler matrix, and $A_l \succeq 0, l \in \underline{\mathbf{p}}$, so $\sum_{l=0}^{p} A_l$ is also a Metzler matrix. According to property of Metzler matrix, there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}_+^n$ such that (4) holds if and only if $\sum_{l=0}^{p} A_l$ is a Hurwitz matrix. Therefore, Theorem 1 shows that system (1) is asymptotically stable for any delays satisfying (2) if and only if $\sum_{l=0}^{p} A_l$ is a Hurwitz matrix.

## IV. Example

To illustrate the theoretical results, we study the following example. Consider

$$\begin{aligned} &\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}\left(t - \tau(t)\right), \quad t \geq 0 \\ &\boldsymbol{\varphi}(t) \succeq 0, \quad t \in [-\tau, 0] \end{aligned} \tag{7}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$, $\tau = \sup_{t\geq 0}\{\tau(t)\}$, and

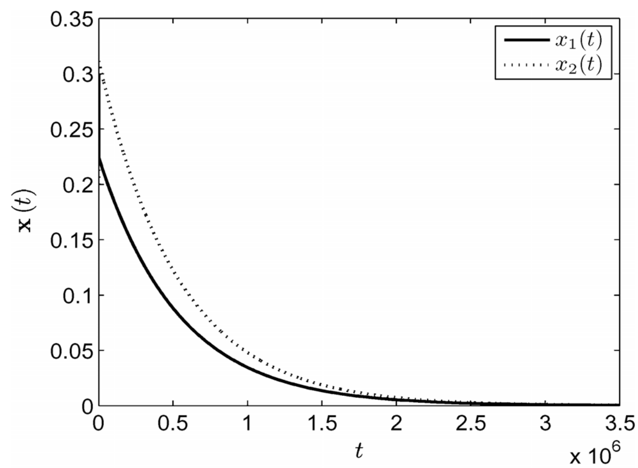
$$A_0 = \begin{bmatrix} -5 & 1.3 \\ 1.5 & -3 \end{bmatrix} \quad A_1 = \begin{bmatrix} a & 0.8 \\ 1 & 1.2 \end{bmatrix}$$

where $a$ is a nonnegative parameter. According to Lemma 2, system (7) is positive. One can check that two eigenvalues of $\sum_{l=0}^{1} A_l$ lie on the open left-half complex plane if $0 \leq a \leq 2.0833$ and one of them lies on the open right-half complex plane if $a \geq 2.0834$. By Theorem 1 and Remark 1, system (7) is asymptotically stable for any continuous and bounded delay $\tau(t)$ if $0 \leq a \leq 2.0833$ and unstable if $a \geq 2.0834$, as shown in Fig. 1. We take the initial conditions $\boldsymbol{\varphi}(t) = [0.3, 0.2]^T$ and delay $\tau(t) = 3 + 0.9\sin(t)$ in the simulation. From Fig. 1, one can conclude that system stability has nothing to do with the system delays, and is completely determined by the system matrices.

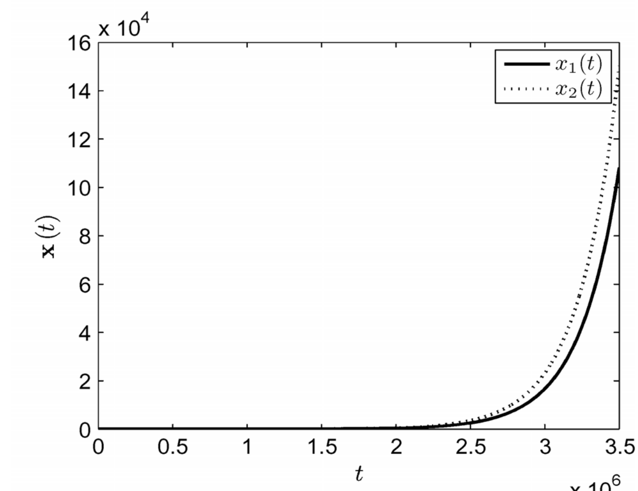
## V. Conclusion

Based on a novel approach, some necessary and sufficient stability conditions are proposed for continuous-time positive systems with continuous and bounded delays. An example shows that the obtained theoretical results are correct.

This note not only establishes necessary and sufficient stability criteria for positive systems with continuous and bounded delays, but also reveals some important properties of the solutions of positive systems with constant delays and the relationship between positive systems with constant delays and those with time-varying delays. The idea and main results in this note, as well as those interesting properties, are helpful for other branches of positive systems, such as 2-D positive systems and switched positive systems.



(a)



(b)

Fig. 1. Evolutions of system (7). (a) Evolution of system (7) with $a = 2.0833$. (b) Evolution of system (7) with $a = 2.0834$.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# Sliding Mode Control of Uncertain Multivariable Nonlinear Systems With Unknown Control Direction via Switching and Monitoring Function

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***Abstract*—A novel output-feedback tracking sliding mode control strategy is proposed for a class of uncertain multi-input-multi-output (MIMO) systems with strong nonlinearities and unknown high-frequency gain (HFG) matrix, that is, the control direction is assumed unknown. A switching scheme based on a monitoring function is designed to handle the control direction uncertainty. The proposed method provides global stability properties and exact output tracking. Simulation results about a robotics visual servoing system using a fixed but uncalibrated camera illustrate the robustness and practical viability of the proposed scheme.**

***Index Terms*—Global exact tracking, multivariable nonlinear systems, output-feedback, sliding mode control, unknown control direction.**

## I. Introduction

The design of output-feedback control of uncertain single-input-single-output (SISO) systems without the knowledge of the control direction (the sign of the scalar HFG) has been a challenging problem since the early 1980s [1]. In the adaptive control literature, the so-called Nussbaum gain has been used to design stable systems under this relaxed assumption, including the monovariable [2] as well as the multivariable case [3]. However, this approach is of arguable practical interest, due to the resulting poor transients, large control peaking and inherent lack of robustness [1], [4], [5].

More recently, tracking sliding mode control (SMC) designs for SISO uncertain *linear* and *nonlinear* plants with unknown control direction and arbitrary relative degree were introduced in [5] and [6], respectively. In lieu of the Nussbaum gain, the control sign was adjusted based on monitoring functions. The applicability of the proposed controllers in real-world conditions was corroborated by the DC motor control experiments presented in [7].

Other elaborate solutions can be found in the SMC literature, however they are restricted either to SISO first order or relative degree two systems [8], [9], or are based on full state measurement [10].

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