

¹ Solvability Analysis of Second Order, ² Discrete Time Descriptor Systems *

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⁶ **Abstract** This paper is devoted to the analysis of linear, second order *discrete*
⁷ *time descriptor systems* (or singular difference equations (SiDEs) with control).
⁸ Following the algebraic approach proposed in [10, 11], first we present a theo-
⁹ retical framework to analyze the corresponding initial value problem for SiDEs,
¹⁰ which is followed by the analysis of descriptor systems. We also describe numer-
¹¹ ical methods to determine the structural properties related to the solvability
¹² analysis of these systems. This work extends and completes the researches in
¹³ [2, 14, 17].

¹⁴ **Keywords:** Singular systems; Difference equation; Descriptor systems;
¹⁵ Strangeness-index; Regularization; Feedback.

¹⁶ **AMS Subject Classification:** 34A09, 34A12, 65L05, 65H10

¹⁷ 1 Introduction and Preliminaries

In this paper we study second order, discrete time descriptor systems of the form

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) = f(n), \quad \text{for all } n \geq n_0. \quad (1.1)$$

We will also discuss the initial value problem of the associated singular difference equation (SiDE)

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = f(n), \quad \text{for all } n \geq n_0, \quad (1.2)$$

together with some given initial conditions

$$x(n_0+1) = x_1, \quad x(n_0) = x_0. \quad (1.3)$$

¹⁸ Here the solution/state $x = \{x(n)\}_{n \geq n_0}$, the inhomogeneity $f = \{f(n)\}_{n \geq n_0}$,
¹⁹ the input function $u = \{u(n)\}_{n \geq n_0}$, where $x(n) \in \mathbb{R}^d$, $f(n) \in \mathbb{R}^m$ and $u(n) \in \mathbb{R}^p$

20 for each $n \geq n_0$. The coefficients contain three matrix sequences $\{A_n\}_{n \geq n_0}$,
 21 $\{B_n\}_{n \geq n_0}$, $\{C_n\}_{n \geq n_0}$ which always take values in $\mathbb{R}^{m,d}$, and $\{D_n\}_{n \geq n_0}$ which
 22 take values in $\mathbb{R}^{m,p}$. We notice, that all the results in this paper also carry
 23 over to the complex case, and they can also be easily extended to systems of
 24 higher than second order, but for ease of notation and because this is the most
 25 important case in practice, we restrict ourselves to the real second order case.

26 The SiDE (1.2), on one side, can be consider as the resulting equations,
 27 obtained by finite difference or discretization of some continuous-time DAEs or
 28 constrained PDEs. One the other side, there are also many models/applications
 29 in real-life, which lead to SiDEs, for example Leotief economic models, backward
 30 Leslie model in biology, etc.

31 While both DAEs and SiDEs of first order have been well-studied from both
 32 theoretical and numerical sides, the same maturity has not been reached for
 33 higher order systems. In classical literatures [1, 5, 9], usually new variables are
 34 introduced to present some chosen derivatives of the state variable x such that
 35 a high order system can be reformulated as a first order one. This method,
 36 however, is not only non-unique but also has presented some substantial disad-
 37 vantages. As have been fully discussed in [14, 17] for continuous time systems,
 38 these disadvantages include: (1st) increase the index of the system, and there-
 39 fore the complexity of the numerical method to solve it; (2nd) increase the
 40 computational effort, because of the bigger size of a new system; (3rd) affect
 41 the controllability/observability of the corresponding descriptor system, since
 42 there exist situations where a new system is uncontrollable while the original
 43 one is. Therefore, the *algebraic approach*, which treats the system directly with-
 44 out reformulating it, has been presented in [14, 17, 20, 21] in order to overcome
 45 the disadvantages mentioned above. Nevertheless, even for second order SiDEs,
 46 this method has not yet been considered.

47 Another motivation of this work comes from recent researches on the stability
 48 analysis of high order discrete time systems with time-dependent coefficients
 49 [13, 18]. There, considered systems are in either strangeness-free form or linear
 50 state-space form. Nevertheless, it is not always the case in applications, and
 51 hence, a reformulation procedure is necessary.

52 Therefore, the main aim of this article is to set up a comparable frame-
 53 work for second order SiDEs/descriptor systems. It is worth marking that the
 54 algebraic method proposed in [14, 17] is applicable theoretically but not numer-
 55 ically, due to two reasons: (1) The condensed form of the matrix coefficients
 56 are very big and complicated. (2) The system's transformations are not unitary.
 57 In this work, we will modify this method to make it more concise and also be
 58 computable in a stable way.

59 The outline of this paper is as follows. After recalling some preliminary
 60 concepts and some auxiliary lemmata, in Sections 2 and 3 we consecutively in-
 61 troduce *index reduction procedures* for SiDEs and for descriptor systems, based
 62 on condensed forms that allow us to determine structural properties such as
 63 existence and uniqueness of a solution, consistency and hidden constraints, etc.
 64 For the numerical solution of these systems, we consider in Section 4 the *differ-
 65 ence array approach* to bring the original system to its strangeness-free form.
 66 The presented algorithms are demonstrated by numerical experiments. Finally,
 67 we finish with some conclusion.

69 In the following example we demonstrate some difficulties that may arise in
70 the analysis of second order SiDEs.

Example 1 Consider the following second order SiDE, motivated from Example 2, [17].

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}, \quad n \geq n_0.$$

Clearly, from the second equation $\begin{bmatrix} 1 & 0 \end{bmatrix} x(n) = f_2(n)$, we can shift the time n to obtain

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x(n+1) = f_2(n+1) \quad \text{and} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x(n+2) = f_2(n+2).$$

Inserting these to the first equation of the original system, we find out the hidden constraint $f_2(n+2) + f_2(n+1) + \begin{bmatrix} 0 & 1 \end{bmatrix} x(n) = f_1(n)$. Consequently, we obtain the following system, which possess a unique solution

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) - f_2(n+2) - f_2(n+1) \\ f_2(n) \end{bmatrix}, \quad n \geq n_0.$$

71 Let $n = n_0$ in this new system, we obtain a constraint that $x(n_0)$ must obey.
72 This example showed us some important facts. Firstly, one can use some shift
73 operators and row-manipulation (Gaussian eliminations) to derive hidden con-
74 straints. Secondly, the solution only exists if the initial condition fulfills some
75 consistency conditions.

For matrices $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, the pair (Q, P) is said to *have no hidden redundancy* if

$$\text{rank} \left(\begin{bmatrix} Q \\ P \end{bmatrix} \right) = \text{rank}(Q) + \text{rank}(P).$$

76 Otherwise, (Q, P) is said to *have hidden redundancy*. The geometrical meaning
77 of this concept is that the intersection space $\text{span}(P^T) \cap \text{span}(Q^T)$ contains
78 only the zero-vector $\mathbf{0}$. Here by $\text{span}(P^T)$ (resp., $\text{span}(Q^T)$) we denote the real
79 vector space spanned by the rows of P (resp., rows of Q). We further notice
80 that, if $\begin{bmatrix} Q \\ P \end{bmatrix}$ is of full row rank then obviously, the pair (Q, P) has no hidden
81 redundancy. However, the converse is not true as is obvious for $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,
82 $P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

83 **Lemma 1** ([7]) Suppose that for $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, the pair (Q, P) has no
84 hidden redundancy. Then, for any matrix $U \in C^{q,q}$ and any $V \in C^{p,p}$, the pair
85 (UQ, VP) has no hidden redundancy.

86 **Lemma 2** ([7]) Consider $k+1$ full row rank matrices $R_0 \in \mathbb{R}^{r_0,d}, \dots, R_k \in$
87 $\mathbb{R}^{r_k,d}$, and assume that for $j = k, \dots, 1$ none of the matrix pairs $\left(R_j, \begin{bmatrix} R_{j-1} \\ \vdots \\ R_0 \end{bmatrix} \right)$
88 has a hidden redundancy. Then, $\begin{bmatrix} R_k \\ \vdots \\ R_0 \end{bmatrix}$ has full row rank.

⁸⁹ Lemma 3 below will be very useful later for our analysis, in order to remove
⁹⁰ hidden redundancy in the coefficients of (1.2).

⁹¹ **Lemma 3** *For $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, there exists $\begin{bmatrix} S & 0 \\ Z_1 & Z_2 \end{bmatrix} \in \mathbb{R}^{q,q+p}$ such that the*
⁹² *following conditions hold.*

⁹³ i) $\begin{bmatrix} S \\ Z_1 \end{bmatrix}$ is unitary, and $Z_1 P + Z_2 Q = 0$,

⁹⁴ ii) the matrix SP has full row rank, and the pair (SP, Q) has no hidden redundancy.
⁹⁵

Proof. First using SVD we factorize Q and then partition P conformably to get

$$U_1^H Q V_1 = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } P V_1 = [P_1 \ P_2], \quad (1.4)$$

where $U_1 = [U_{11} \ U_{12}] \in \mathbb{R}^{q,q}$, $V_1 = [V_{11} \ V_{12}] \in \mathbb{R}^{d,d}$ are unitary and $\Sigma \in \mathbb{R}^{r_Q, r_Q}$ is diagonal. Now we use a second SVD to factorize P_2 and to find a unitary matrix $U_2^H = \begin{bmatrix} S \\ Z_1 \end{bmatrix} \in \mathbb{R}^{p,p}$ such that $U_2^H P_2 = \begin{bmatrix} P_{12} \\ 0 \end{bmatrix}$, where P_{12} has full row rank. Thus, we obtain

$$\begin{bmatrix} S & 0 \\ Z_1 & 0 \\ 0 & U_{11}^H \\ 0 & U_{12}^H \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & 0 \\ \Sigma & 0 \\ 0 & 0 \end{bmatrix}.$$

Since P_{12} has full row rank, $SP = [P_{11} \ P_{12}] V_1^{-1}$ also has full row rank. Moreover, one sees that

$$\text{rank} \left(\begin{bmatrix} SP \\ Q \end{bmatrix} \right) = \text{rank} ([0 \ P_{12}]) + \text{rank} ([\Sigma \ 0]) = \text{rank}(SP) + \text{rank}(Q),$$

which follows that the pair (SP, Q) has no hidden redundancy.
Finally, setting $Z_2 := -P_{21} \Sigma^{-1} U_{11}^H$, we obtain

$$Z_1 P + Z_2 Q = ([P_{21} \ 0] - P_{21} \Sigma^{-1} [\Sigma \ 0]) V_1^{-1} = 0,$$

⁹⁶ which completes the proof. \square

⁹⁷ *Remark 1* It should be noted, that the matrices U_1 , U_2 , V_1 in the proof of
⁹⁸ Lemma 3 are orthogonal. Therefore, in case that the singular values of Q are
⁹⁹ neither too small nor too big, then Σ^{-1} is well-conditioned, and hence we can
¹⁰⁰ stably compute the matrix Z_2 . Both matrices Z_1 and Z_2 will play the key role
¹⁰¹ in our *index reduction procedure* presented in the next section.

¹⁰² For any given matrix M , by M^T we denote its transpose. By $T_0(M)$ we
¹⁰³ denote an orthogonal matrix whose columns span the left null space of M . By
¹⁰⁴ $T_\perp(M)$ we denote an orthogonal matrix whose columns span the vector space
¹⁰⁵ $\text{range}(M)$. From basic linear algebra, we have the following three lemmata.

Lemma 4 *The matrix $\begin{bmatrix} T_{\perp}^T(M) \\ T_0^T(M) \end{bmatrix}$ is nonsingular, the matrix $T_{\perp}^T(M)$ has full row rank, and the following identity holds*

$$\begin{bmatrix} T_{\perp}^T(M) \\ T_0^T(M) \end{bmatrix} M = \begin{bmatrix} T_{\perp}^T(M) & M \\ 0 & 0 \end{bmatrix}.$$

¹⁰⁶ *Proof.* A simple proof can be found in, for example, [6]. \square

Lemma 5 *Given four matrices $\check{A}, \check{B}, \check{C}$ in $\mathbb{R}^{m,d}$ and \check{D} in $\mathbb{R}^{m,p}$. Let us consider the following matrices whose columns span orthonormal bases of the associated vector spaces*

$$\begin{aligned} T_1 &\text{ basis of } \text{kernel}(\check{A}^T), & T_{1,\perp} &\text{ basis of } \text{range}(\check{A}), \\ W_1 &\text{ basis of } \text{kernel}(T_1^T \check{D})^T, & W_{1,\perp} &\text{ basis of } \text{range}(T_1^T \check{D}), \\ J_D &:= W_{1,\perp}^T T_1^T \check{D}, \\ J_{B1} &:= W_1^T T_1^T \check{B}, & J_{B2} &:= W_{1,\perp}^T T_{1,\perp}^T \check{B}, \\ T_2 &\text{ basis of } \text{kernel}(J_{B1})^T, & T_{2,\perp} &\text{ basis of } \text{range}(J_{B1}), \\ T_3 &\text{ basis of } \text{kernel}(J_{B2})^T, & T_{3,\perp} &\text{ basis of } \text{range}(J_{B2}), \\ J_{C1} &:= W_1^T T_1^T \check{C}, & J_{C2} &:= W_{1,\perp}^T T_1^T \check{C}, \\ T_4 &\text{ basis of } \text{kernel}(T_2^T J_{C1})^T, & T_{4,\perp} &\text{ basis of } \text{range}(T_2^T J_{C1}). \end{aligned}$$

¹⁰⁷ Then, the following assertions hold true.

¹⁰⁸ i) The matrices $\begin{bmatrix} T_{i,\perp} \\ T_i \end{bmatrix}$, $i = 1, \dots, 4$, $\begin{bmatrix} W_{1,\perp} \\ W_1 \end{bmatrix}$ are orthogonal.

¹⁰⁹ ii) The matrices $T_{1,\perp}^T \check{A}$, $T_{2,\perp}^T J_{B1}$, $T_{3,\perp}^T J_{B2}$, $T_{4,\perp}^T T_2^T J_{C1}$, and J_D have full row rank.

iii) Moreover, there exists a nonsingular matrix \check{U} such that

$$\check{U} \begin{bmatrix} \check{A} & \check{B} & \check{C} & | & \check{D} \end{bmatrix} = \left[\begin{array}{cc|c} T_{1,\perp}^T \check{A} & T_{1,\perp}^T \check{B} & T_{1,\perp}^T \check{C} & | & T_{1,\perp}^T \check{D} \\ 0 & T_{2,\perp}^T J_{B1} & T_{2,\perp}^T J_{C1} & | & 0 \\ 0 & 0 & T_{4,\perp}^T T_2^T J_{C1} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ \hline 0 & T_{3,\perp}^T J_{B2} & T_{3,\perp}^T J_{C2} & | & T_{3,\perp}^T J_D \\ 0 & 0 & T_3^T J_{C2} & | & T_3^T J_D \end{array} \right]. \quad (1.5)$$

Proof. The first two claims followed directly from Lemma 4, while the desired matrix \check{U} in the third part is

$$\check{U} := \left[\begin{array}{c|c} I & \\ \hline I & \\ \hline T_4^T & \\ \hline T_4^T & | I \end{array} \right] \cdot \left[\begin{array}{c|c} I & \\ \hline T_2^T & \\ \hline T_2^T & \\ \hline T_3^T & \end{array} \right] \cdot \left[\begin{array}{c|c} I & \\ \hline W_1^T & \\ \hline W_{1,\perp}^T & \end{array} \right] \cdot \left[\begin{array}{c} T_{1,\perp}^T \\ T_1^T \end{array} \right].$$

¹¹⁰ \square

¹¹¹ **Lemma 6** *Let $P \in \mathbb{R}^{p,d}$, $Q \in \mathbb{R}^{q,d}$ be two full row rank matrices and $p + q \leq d$.*
¹¹² *Then, the following assertions hold true.*

- ¹¹³ i) There exists a matrix $F \in \mathbb{R}^{d,d}$ such that $H := \begin{bmatrix} P \\ QF \end{bmatrix}$ has full row rank.
- ¹¹⁴ ii) For any $G \in \mathbb{R}^{q,d}$, there exists a matrix $F \in \mathbb{R}^{d,d}$ such that $\begin{bmatrix} P \\ G + QF \end{bmatrix}$ has full row rank.

Proof. i) First we consider the SVDs of P and G that reads

$$U_P PV_P = [\Sigma_P \ 0_{p,d-p}], \quad U_Q QV_Q = [\Sigma_Q \ 0_{q,d-q}],$$

where Σ_P, Σ_Q are nonsingular, diagonal matrices, and $0_{p,d-p}$ (resp. $0_{q,d-q}$) are the zero matrix of size p by $d-p$ (resp. q by $d-q$).

By choosing $F := V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^{-1}$ we see that

$$\begin{bmatrix} U_P & 0 \\ 0 & U_Q \end{bmatrix} \begin{bmatrix} P \\ QF \end{bmatrix} V = \begin{bmatrix} U_P PV_P \\ U_Q QFV \end{bmatrix} = \begin{bmatrix} \Sigma_P & 0_{p,d-p-q} & 0_{p,q} \\ 0_{q,p} & 0_{p,d-p-q} & \Sigma_Q \end{bmatrix},$$

and hence, the claim i) is proven.

ii) Clearly, in case that the matrix F is very big, then G is only a small perturbation, and hence for sufficiently large η , by choosing

$$F := \eta V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^{-1},$$

¹¹⁶ we obtain the full row rank property of $\begin{bmatrix} P \\ G + QF \end{bmatrix}$. □

¹¹⁷ *Remark 2* It should be noted that, the proof of Lemmata 5 and 6 are constructive, and all the matrices $T_{i,\perp}, T_i, i = 1, \dots, 4, W_{1,\perp}, W_1$ and F can be stably computed.

¹²⁰ 2 Strangeness-index of second-order SiDEs

¹²¹ In this section, we study the solvability analysis of the second-order SiDE (1.2)
¹²² and of its corresponding IVP (1.2)–(1.3). Many regularization procedures and
¹²³ their associated index concepts have been proposed for first order systems, see
¹²⁴ the survey [16] and the references therein. Nevertheless, for second order sys-
¹²⁵ tems, only the strangeness-index has been proposed for only continuous time
¹²⁶ systems in [17, 21]. Thus, it is our purpose to construct a comparable index
¹²⁷ concept for system (1.2). Furthermore, we will consider some modifications
¹²⁸ in order to make the *algebraic approach* more concise and better for not only
¹²⁹ theoretical analysis but also for numerical computation.

Let

$$M_n := [A_n \ B_n \ C_n], \quad X_n := \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix},$$

we call $\{M_n\}_{n \geq n_0}$ the *behavior matrix sequence* of system (1.2). Thus, (1.2) can be rewritten as

$$M_n X_n = f(n), \text{ for all } n \geq n_0. \tag{2.1}$$

Clearly, by scaling (1.2) with a nonsingular matrix $P_n \in \mathbb{R}^{d,d}$, we obtain a new system

$$[P_n A_n \ P_n B_n \ P_n C_n] X_n = P_n f(n), \text{ for all } n \geq n_0, \quad (2.2)$$

without changing the solution space. This motivates the following definition.

Definition 1 Two behavior matrix sequences $\{M_n = [A_n \ B_n \ C_n]\}_{n \geq n_0}$ and $\{\tilde{M}_n = [\tilde{A}_n \ \tilde{B}_n \ \tilde{C}_n]\}_{n \geq n_0}$ are called (*strongly*) left equivalent if there exists a nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that $\tilde{M}_n = P_n M_n$ for all $n \geq n_0$. We denote this equivalence by $\{M_n\}_{n \geq n_0} \xrightarrow{\ell} \{\tilde{M}_n\}_{n \geq n_0}$. If this is the case, we also say that two SiDEs (1.2), (2.2) are left equivalent.

Lemma 7 Consider the behavior matrix sequence $\{M_n\}_{n \geq n_0}$ of system (1.2). Then, for all $n \geq n_0$, we have that

$$\{M_n\}_{n \geq n_0} \xrightarrow{\ell} \left\{ \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \right\}_{n \geq n_0}, \quad \begin{array}{l} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ v \end{array} \quad (2.3)$$

where the matrices $A_{n,1}, B_{n,2}, C_{n,3}$ on the main diagonal have full row rank. Furthermore, the numbers $r_{2,n}, r_{1,n}, r_{0,n}, v$ are invariant under global left equivalent transformations. Thus, we can call them the local characteristic invariants of the SiDE (1.2).

Proof. The block diagonal form (2.3) is obtained directly by consecutively compressing the block columns A_n, B_n, C_n of M via Lemma 4. From (2.3), we obtain the following identities

$$\begin{aligned} r_{2,n} &= \text{rank}(A_n), \\ r_{1,n} &= \text{rank}([A_n \ B_n]) - \text{rank}(A_n), \\ r_{0,n} &= \text{rank}([A_n \ B_n \ C_n]) - \text{rank}([A_n \ B_n]), \end{aligned}$$

which proves the second claim. \square

Analogous to the continuous-time case, we will apply an *algebraic approach* (see [2, 17]), which aims to reformulate (1.2) into a so-called *strangeness-free* form, as stated in the following definition.

Definition 2 ([13]) System (1.2) is called *strangeness-free* if there exists a pointwise-nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that by scaling the SiDE (1.2) at each point n with P_n , we obtain a new system of the form

$$\begin{aligned} \hat{r}_2 &= \begin{bmatrix} \hat{A}_{n,1} \\ 0 \\ 0 \\ 0 \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{B}_{n,2} \\ 0 \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ \hat{C}_{n,2} \\ \hat{C}_{n,3} \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hat{f}_4(n) \end{bmatrix}, \text{ for all } n \geq n_0, \\ \hat{r}_1 &= \begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix} \end{aligned} \quad (2.4)$$

where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank for all $n \geq n_0$.

148 *Remark 3* We notice that, if the SiDE (1.2) is of the strangeness-free form (2.4),
149 then the existence and uniqueness of the solution $\{x(n)\}_{n \geq n_0}$ can be achieved
150 if and only if $\hat{r}_2 + \hat{r}_1 + \hat{r}_0 = d$. Furthermore, either the last block row equation
151 of (2.4) either do not appear , i.e. $\hat{v} = 0$, or $\hat{f}_4(n) = 0$ for all $n \geq n_0$.

152 In order to perform an algebraic approach, an additional assumption below
153 is usually needed.

154 **Assumption 1.** *Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$, $r_{0,n}$
155 become global, i.e., they are constant for all $n \geq n_0$.*

Due to Lemma 7, we see that Assumption 1 is satisfied if and only if $\text{rank}(A_n)$, $\text{rank}([A_n \ B_n])$, $\text{rank}([A_n \ B_n \ C_n])$ do not depend on n . Let us call the number

$$r_u := 3r_2 + 2r_1 + r_0$$

the *upper rank* of system (1.3). Clearly, r_u is invariant under left equivalence transformations. Rewriting (2.1) block row-wise, we obtain the following system for all $n \geq n_0$.

$$A_{n,1}x(n+2) + B_{n,1}x(n+1) + C_{n,1}x(n) = f_1(n), \quad r_2 \text{ equations}, \quad (2.5a)$$

$$B_{n,2}x(n+1) + C_{n,2}x(n) = f_2(n), \quad r_1 \text{ equations}, \quad (2.5b)$$

$$C_{n,3}x(n) = f_3(n), \quad r_0 \text{ equations}, \quad (2.5c)$$

$$0 = f_4(n), \quad v \text{ equations}. \quad (2.5d)$$

156 Since the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ have full row rank, the number of scalar
157 difference equations of order 2 (resp. 1, and 0) in (1.2) is exactly r_2 (resp. r_1
158 and r_0), while v is the number of redundant equations.

159 *Remark 4* In the context of continuous-time systems, the quantities r_2 , r_1 , and
160 r_0 are the dimensions of the second order derivative part, the first order derivative
161 part, and the algebraic part, respectively. Furthermore, $r_2 + r_1$ is exactly
162 the degree of freedoms of the considered system.

Now we are able to define the shift operator Δ , which acts on some or whole equations of system (2.5). This operator maps each equation of system (2.5) at the time instant n to the equation itself at the time $n + 1$, for example

$$\Delta : C_{n,3}x(n) = f_3(n) \mapsto C_{n+1,3}x(n+1) = f_3(n+1). \quad (2.6)$$

163 Clearly, only under Assumption 1, this shift operator can be applied to equations
164 of system (2.5).

In order to reveal all hidden constraints of (2.5) we propose the idea, that for each $j = 1, 2$, we use difference equations of order less than j to reduce the number of scalar difference equations of order j . This task will be performed in Lemmata 9 and 10 below. In details, if the matrix pair $(B_{n,2}, C_{n+1,3})$ has hidden redundancy then we will make use of the shifted equation (2.6). Analogously, if the pair $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix} \right)$ has hidden redundancy then we will make use of both the shifted equation

$$B_{n+1,2}x(n+2) + C_{n+1,2}x(n+1) = f_2(n+1),$$

and the double shifted equation

$$C_{n+2,3}x(n+2) = f_3(n+2).$$

165

Lemma 8 Consider the SiDE (1.2) and system (2.5). Then, (1.2) has an identical solution set as the extended system

$$\begin{array}{l} r_2 \\ r_1 \\ r_0 \\ v \\ \hline r_0 \\ r_1 \\ r_0 \end{array} \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ \hline 0 & C_{n+1,3} & 0 \\ B_{n+1,2} & C_{n+1,2} & 0 \\ C_{n+2,3} & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{bmatrix}, \quad (2.7)$$

166 for all $n \geq n_0$.

167 Proof. Since all equations in the lower part of (2.7) at any time point n is the
168 consequence of the upper part (which is exactly (2.5)) at the time instants $n+1$
169 and $n+2$, the proof is directly followed. \square

170 **Lemma 9** Consider the behavior matrix sequence $\{M_n\}_{n \geq n_0}$ as in equation
171 (2.3). Then, there exist matrix sequences $\{S_n^{(i)}\}_{n \geq n_0}$, $i = 1, 2$, and $\{Z_n^{(j)}\}_{n \geq n_0}$,
172 $j = 1, \dots, 5$, of appropriate sizes such that for all $n \geq n_0$, the following conditions
173 hold true.

174 i) For $i = 1, 2$, the matrices $\begin{bmatrix} S_n^{(i)} \\ Z_n^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$ are unitary.

ii) The following identities hold true.

$$Z_n^{(1)} B_{n,2} + Z_n^{(3)} C_{n+1,3} = 0, \quad (2.8a)$$

$$Z_n^{(2)} A_{n,1} + Z_n^{(4)} B_{n+1,2} + Z_n^{(5)} C_{n+2,3} = 0. \quad (2.8b)$$

175 iii) Both matrix pairs $\left(S_n^{(2)} A_n, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$, $\left(S_n^{(1)} B_{n,2}, C_{n+1,3}\right)$ have no hidden
176 redundancy.

177 Proof. The proof can be directly obtained by applying Lemma 3 to two matrix
178 pairs $(B_{n,2}, C_{n+1,3})$ and $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$. \square

For $i = 1, 2$, let us denote the row dimensions of $S_n^{(i)}$ and $Z_n^{(2)}$ by d_i and s_i , respectively. For notational convenience, we set

$$\begin{aligned} K_{n,1} &:= Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2}, & H_{n,1} &:= Z_n^{(2)} C_{n,1}, \\ \gamma(n+2) &:= Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2). \end{aligned} \quad (2.9)$$

Lemma 10 Consider the matrix sequences $\{Z_n^{(j)}\}_{n \geq n_0}$, $j = 1, \dots, 5$ as in Lemma 7. Then, there exist a unitary matrix $\begin{bmatrix} S_n^{(6)} \\ Z_n^{(6)} \end{bmatrix} \in \mathbb{R}^{s_2, s_2}$ and a matrix $Z_n^{(7)}$ of appropriate size, such that the pair $(S_n^{(6)}, C_{n+1,3})$ has no hidden redundancy and

$$Z_n^{(6)} K_{n,1} + Z_n^{(7)} C_{n+1,3} = 0. \quad (2.10)$$

179 Proof. The proof is directly followed by applying Lemma 3 to the matrix pair
180 $(K_{n,1}, C_{n+1,3})$. \square

Theorem 2 Consider the behavior matrix sequence $\{M_n\}_{n \geq n_0}$ in (2.3). Let the matrix sequences $\{S_n^{(i)}\}_{n \geq n_0}$, $i = 1, 2$ and $\{Z_n^{(j)}\}_{n \geq n_0}$, $j = 1, \dots, 5$, be defined as in Lemmata 9 and 10. Let the matrices $K_{n,1}$, $H_{n,1}$ and the function $\gamma(n+2)$ be defined as in (2.9). Then, the SiDE (1.3) has exactly the same solution set as the following system

$$\frac{d_2}{s_{21}} \frac{s_{20}}{\frac{d_1}{s_1}} \frac{r_0}{v} \begin{bmatrix} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ 0 & S_n^{(6)} K_{n,1} & S_n^{(6)} H_{n,1} \\ 0 & 0 & Z_n^{(6)} H_{n,1} \\ \hline 0 & S_n^{(1)} B_{n,2} & S_n^{(1)} C_{n,2} \\ 0 & 0 & Z_n^{(1)} C_{n,2} \\ \hline 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} S_n^{(2)} f_1(n) \\ S_n^{(6)} \gamma(n+2) \\ Z_n^{(6)} \gamma(n+2) + Z_n^{(7)} f_3(n+1) \\ \hline S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1) \\ \hline f_3(n) \\ f_4(n) \end{bmatrix}, \quad (2.11)$$

181 for all $n \geq 0$. Here we have $r_2 = d_2 + s_2$, $r_1 = d_1 + s_1$, $s_2 = s_{21} + s_{20}$. Furthermore,
182 both matrix pairs $\left(S_n^{(2)} A_n, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$ and, $\left(\begin{bmatrix} S_n^{(6)} \tilde{K}_{n,1} \\ S_n^{(1)} B_{n,2} \end{bmatrix}, C_{n+1,3}\right)$ have
183 no hidden redundancy.

184 Proof. The main idea of this proof is to apply elementary row transformations
185 to system (2.7) to obtain (2.11). Since the proof is quite long and technical, we
186 present it in Appendix ??.

187 *Remark 5* From (2.9), we notice that if the pair $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$ has hidden
188 redundancy, then $Z_n^{(2)}$ is not an empty matrix. Furthermore, if $Z_n^{(5)}$ is also not
189 an empty matrix, then we need two shifts to pass from (2.5) to (2.11).

Comparing system (2.11) with (2.5), we see that the number of second, first and zero-order difference equations are d_2 , $s_{21} + d_1$, and $s_{20} + s_1 + r_0$, respectively. The upper rank of this modified system is

$$r_u^{new} \leq 3d_2 + 2(s_{21} + d_1) + (s_{20} + s_1 + r_0) = r - (s_{21} + 2s_{20} + s_1) \leq r.$$

190 In conclusion, after performing this *index reduction step*, which passes from
191 (2.5) to (2.11), we have reduced the upper rank r_u at least by $s_{21} + 2s_{20} + s_1$.
192 Continuing in this way until $s_{21} = s_{20} = s_1 = 0$, we obtain the following
193 algorithm.

Algorithm 1 Index reduction steps for SiDEs at the time point n

- 1: **Input:** The SiDE (1.2) and its behavior form (2.1). Set $i = 0, \mu = 0$.
- 2: **Return:** The resulting system in a special form.
- 3: Transform the behavior matrix $[A_n \ B_n \ C_n]$ to the block upper triangular form

$$\tilde{M} := \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{matrix} r_2 \\ r_1 \\ r_0 \\ v \end{matrix}$$

where all the matrices $A_{n,1}, B_{n,2}, C_{n,3}$ on the main diagonal have full row rank.

- 4: **if** both matrix pairs $(A_{n,1}, [B_{n+1,2} \ C_{n+2,3}])$ and $(B_{n,2}, C_{n+1,3})$ have no hidden redundancy **then STOP**.
- 5: **else** set $i := i + 1$ and go to 6
- 6: Find the matrices $S_n^{(i)}, i = 1, 2$, and $Z_n^{(j)}, j = 1, \dots, 5$ as in Lemma 9.
- 7: **if** $Z_n^{(5)} \neq []$ **then** set $\mu := \mu + 2$.
- 8: **else** set $\mu := \mu + 1$
- 9: **end if**
- 10: Compute the matrices $K_{n,1}, H_{n,1}$ as in (2.9)
- 11: **if** the matrix pair $(K_{n,1}, C_{n+1,3})$ has no hidden redundancy **then** go to 14.
- 12: **else** find the matrices $S_n^{(6)}, Z_n^{(6)}, Z_n^{(7)}$ as in Lemma 10.
- 13: **end if**
- 14: Transform system (2.5) to system (2.11) as in Theorem 2.
- 15: **end if**
- 16: Go back to 3.

194 After each index reduction step the upper rank r_u^i has been decreased at
195 least by $s_2^i + s_1^i$, so Algorithm 1 terminates after a finite number μ of iterations,
196 which will be called the *strangeness-index* of the SiDE (1.2).

Theorem 3 Consider the SiDE (2.1) and assume that Assumption 1 is satisfied for any n and any considered i in the loop. Then, the SiDE (1.2) has the same solution set as the strangeness-free-SiDE

$$\begin{matrix} r_2^\mu \\ r_1^\mu \\ r_0^\mu \\ v^\mu \end{matrix} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} \hat{g}_1(n) \\ \hat{g}_2(n) \\ \hat{g}_3(n) \\ \hat{g}_4(n) \end{bmatrix}, \text{ for all } n \geq n_0, \quad (2.12)$$

197 where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank for all $n \geq n_0$. Here \hat{g}_2 and \hat{g}_3 ,
198 are functions of $f(n+1), \dots, f(n+\mu)$.

199 *Proof.* The proof is a direct consequence of Algorithm 1, where the matrix
200 $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank due to Lemma 2. \square

201 To illustrate Algorithm 1, we consider the following example.

Example 2 Consider the second order SiDE

$$\begin{bmatrix} 1 & n+1 & n+4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 0 & 0 & 2n+3 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & n+1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n+1 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \end{bmatrix}, \quad (2.13)$$

for all $n \geq n_0 = 0$. Fortunately, the behavior matrix

$$M = \left[\begin{array}{ccc|ccc|ccc} 1 & n+1 & n+4 & 0 & 0 & 2n+3 & 0 & n+1 & 0 \\ 0 & 0 & 0 & 1 & n & 1 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n+1 \end{array} \right] = \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \end{bmatrix}$$

is already in the block diagonal form, so we do not need to perform Step 3 in Algorithm 1. Furthermore, all constant rank conditions required in Assumption 1 are satisfied. We observe here that

$$\begin{aligned} B_{n+1,2} &= [1 \ n+1 \ 1], & C_{n+1,2} &= [0 \ 0 \ n+1], \\ C_{n+1,3} &= [0 \ 0 \ n+2], & C_{n+2,3} &= [0 \ 0 \ n+3]. \end{aligned}$$

By directly verifying, we see that the matrix pair $\left(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n,3} \end{bmatrix} \right)$ has hidden redundancy, while the pair $(B_{n,2}, C_{n+1,3})$ does not. Due to Lemma 9 we choose $S_n^{(2)} = []$, $Z_n^{(2)} = 1$, $Z_n^{(4)} = -1$, $Z_n^{(5)} = -1$. Notice, that $Z_n^{(5)}$ is non-empty, so this results in the appearance of $f_3(n+2)$ in the function γ of (2.9). In details, we have that

$$\begin{aligned} K_{n,1} &= [0 \ 0 \ n+2], & H_{n,1} &= [0 \ n+3 \ 0], \\ \gamma(n+2) &= f_1(n) - f_2(n+1) - f_3(n+2). \end{aligned}$$

Here, we observe that the pair $(K_{n,1}, C_{n+1,3})$ has hidden redundancy. Due to Lemma 10 we choose $S_n^{(6)} = []$, $Z_n^{(6)} = 1$, $Z_n^{(7)} = -1$. Therefore, system (2.11) reads

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 0 & 0 & 0 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & n+1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n+1 \end{bmatrix} x(n) \\ &= \begin{bmatrix} f_1(n) - f_2(n+1) - f_3(n+2) - f_3(n+1) \\ f_2(n) \\ f_3(n) \end{bmatrix}. \end{aligned}$$

Go back to Step 3, we obtain the following system

$$\begin{aligned} &\begin{bmatrix} 1 & n & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 0 & n \\ 0 & n+1 & 0 \\ 0 & 0 & n+1 \end{bmatrix} x(n) \\ &= \begin{bmatrix} f_2(n) \\ f_1(n) - f_2(n+1) - f_3(n+2) - f_3(n+1) \\ f_3(n) \end{bmatrix}. \end{aligned}$$

202 Algorithm 1 terminates here, and the strangeness-index is $\mu = 2$, which is
 203 exactly the number of time-shift applied to the inhomogeneity f in order to
 204 obtain the strangeness-free formulation.

205 A direct consequence of Theorem 3 is, that we can deduce the theoretical
 206 solvability for (1.2) as follows.

207 **Corollary 1** Consider the SiDE (1.2) and assume that Assumption 1 is satisfied
 208 for any n and any considered i in the loop, such that the strangeness-index
 209 μ exists. Then the following statements hold.

- 210 i) The corresponding IVP for the SiDE (1.2) is solvable if and only if $\hat{g}_4(n) = 0$
 211 for all $n \geq n_0$. Furthermore, it is uniquely solvable if, in addition, we have
 212 $r_2^\mu + r_1^\mu + r_0^\mu = d$.
- 213 ii) The initial condition $x_0 = x(n_0)$ is consistent if and only if the following
 214 equalities hold.

$$\begin{aligned}\hat{B}_{n_0,2}x_1 + \hat{C}_{n_0,2}x_0 &= \hat{g}_2(n_0), \\ \hat{C}_{n_0,3}x_0 &= \hat{g}_3(n_0).\end{aligned}$$

215 Another direct consequence of Theorem 2 is, that we can obtain an under-
 216 lying difference equation as follows.

217 **Corollary 2** Consider the SiDE (1.2) and assume that the corresponding IVP
 218 is uniquely solvable. Moreover, suppose that Assumption 1 is satisfied for any
 219 n and any considered i in the loop, such that the strangeness-index μ exists.
 220 Then the solution to this IVP is also the solution of the (implicit) underlying
 221 difference equation

$$\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{C}_{n+1,2} \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ 0 \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{g}_1(n) \\ \hat{g}_2(n+1) \\ \hat{g}_3(n+2) \end{bmatrix}, \quad (2.14)$$

222 where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ is invertible for all $n \geq n_0$.

- 223 i) Within one loop of Algorithm 1, for each n , we have used 4 SVDs
 224 to remove the hidden redundancies in two matrix pairs. The total cost depends
 225 on the problems itself, i. e., depending on sizes of the matrix pairs which applied
 226 SVDs. Nevertheless, the total cost would not exceed $\mathcal{O}(m^2d^2)$.
- 227 ii) Different from [17] (see Remark 17), due to Step 7 in Algorithm 1, μ is the
 228 exact number of shifts that have been used in order to achieve (2.12).
- 229 iii) Unfortunately, since Z_2 is not orthogonal, Algorithm 1 could not be stably
 230 implemented. For the numerical solution to the IVP (1.2)-(1.3), we will consider
 231 a suitable numerical scheme in Section 4.
- 232 iv) Unlike in [14, 17], we do not change the variable x . This trick permits us to
 233 simplify significantly the condensed form in [2, 17]. This trick is also useful for
 234 the control analysis of the descriptor system (1.1) as will be seen later.

235 3 Strangeness-index of second order descriptor systems

236 Based on the index reduction procedure for SiDEs in Section 2, in this section
 237 we construct the strangeness-index concept for the descriptor system (1.1). The

solvability analysis for first order descriptor systems with variable coefficients have been carefully discussed in [3, 12, 19]. Nevertheless, for second order descriptor systems, this problem has been rarely considered. We refer the interested readers to [14, 21] for continuous time systems.

In the index reduction procedure of continuous time systems, one should avoid differentiating equations that involve an input function, due to the fact that it may not be differentiable. Here, we will also keep this spirit, and hence, will not shift any equation that involve an input function, since it may destroy the causality of the considered system. In the following lemma, we give the condensed form for system (1.1).

Lemma 11 *Consider the descriptor system (1.1). Then, there exist two pointwise nonsingular matrix sequences $\{U_n\}_{n \geq n_0}$, $\{V_n\}_{n \geq n_0}$ such that the following identities hold.*

$$(U_n [A_n \ B_n \ C_n], \ U_n D_n V_n) = \left(\begin{array}{c|c} \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ \hline 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{bmatrix}, & \begin{bmatrix} D_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \\ 0 & 0 & 0 \end{bmatrix} \end{array} \right), \quad \begin{array}{ll} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ \varphi_{1,n} \\ \varphi_{0,n} \\ v_n \end{array} \quad \text{for all } n \geq n_0. \quad (3.1)$$

Here sizes of the block rows are $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n , the matrices $A_{n,1}$, $B_{n,2}$, $B_{n,4}$, $C_{n,3}$ are of full row rank and the matrices $\Sigma_{\varphi,1}$, $\Sigma_{\varphi,0}$ are nonsingular and diagonal.

Proof. First we apply Lemma 5 to four matrices A_n , B_n , C_n and D_n to obtain the matrix U_n that satisfies (1.5). Decompose the matrix $\begin{bmatrix} T_{3;\perp}^T \\ T_3^T \end{bmatrix} J_D$ via one SVD, we then obtain the block $\begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix}$. Finally, we use Gaussian elimination to cancel out all matrices on the two columns of \check{D} that contain $\Sigma_{\varphi,1}$ and $\Sigma_{\varphi,0}$, and hence, we obtain the desired form (3.1). \square

In order to build an index reduction procedure for (1.1), we also need the following assumption.

Assumption 4. *Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n , become global, i.e., they are constant for all $n \geq n_0$.*

Applying Lemma 11, we can transform the descriptor system (1.1) to the following system

$$\begin{array}{l} r_2 \\ r_1 \\ r_0 \\ \varphi_1 \\ \varphi_0 \\ v \end{array} \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ \hline 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} D_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} = \tilde{f}(n), \quad (3.2)$$

255 where $u(n) = V_n v(n)$, $v(n) := \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix}$, $\tilde{f}(n) := U_n f(n)$ for all $n \geq n_0$.
 256

257 In this decomposition, we notice that the third and fourth block rows, whose
 258 sizes are φ_1 and φ_0 , are related to the feedback regularization of (1.1), as shown
 259 in the following proposition.

Proposition 1 *i) Assume that for each $n \geq n_0$, the matrix $\begin{bmatrix} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}$ is of full row rank. Then, there exist two matrices $F_{n,1}$ and $F_{n,0}$ such that the following matrix has full row rank*

$$\begin{bmatrix} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \\ B_{n+1,4} + \begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \end{bmatrix} F_{n,1} \\ C_{n+2,5} + \begin{bmatrix} 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix} F_{n,0} \end{bmatrix}.$$

ii) Consequently, if the upper part of (3.2) is strangeness-free then there exists a first order feedback of the form

$$u(n) = F_{n,1}x(n+1) + F_{n,0}x(n), \text{ for all } n \geq n_0, \quad (3.3)$$

260 such that the closed loop system associated with (4.2) is strangeness-free.

261 *Proof.* Since the part ii) is a direct consequence of part i), we only need to
 262 prove i). The part i) is directly followed from Lemma 6 with $P = \begin{bmatrix} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}$,
 263 $Q = \begin{bmatrix} 0 & \Sigma_{\varphi,1} & 0 \\ 0 & 0 & \Sigma_{\varphi,0} \end{bmatrix}$ and $G = \begin{bmatrix} B_{n+1,4} \\ C_{n+2,5} \end{bmatrix}$. \square

264 From Proposition 1, we see that we only need to remove the hidden redundancies in the upper part of (3.2). This will be done as in the following lemma.
 265

Lemma 12 Consider the descriptor system (3.2). Then, for each input sequence $\{v(n)\}_{n \geq n_0}$, it has exactly the same solution set as the following system

$$\begin{array}{l} \tilde{r}_2 \left[\begin{array}{ccc} \tilde{A}_{n,1} & \tilde{B}_{n,1} & \tilde{C}_{n,1} \\ 0 & \tilde{B}_{n,2} & \tilde{C}_{n,2} \\ 0 & 0 & \tilde{C}_{n,3} \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \tilde{D}_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} = \tilde{f}(n), \quad (3.4) \\ \tilde{r}_1 \\ \tilde{r}_0 \\ \varphi_1 \\ \varphi_0 \\ v \end{array}$$

266 where $\tilde{r}_2 < r_2$, $\tilde{r}_0 > r_0$, $\sum_{i=0}^2 r_i = \sum_{i=0}^2 \tilde{r}_i$.

267 *Proof.* System (3.4) is directly obtained by applying Theorem 2 to the upper
 268 part of (3.2). To keep the brevity of this paper, we will omit the details here. \square

Similar to the observation made in Section 2, here we also see, that the so-called *index reduction step*, which passes system (3.2) to the new form (3.4) has reduced the upper rank r^u by at least $(\tilde{r}_0 - r_0) + (r_2 - \tilde{r}_2)$. Continuing in this way, finally we obtain the strangeness-free descriptor system in the next theorem.

Theorem 5 *Consider the descriptor system (1.1). Furthermore, assume that Assumption 4 is fulfilled whenever needed. Then, for each fixed input sequence $\{u(n)\}_{n \geq n_0}$, system (1.1) has the same solution set as the so-called strangeness-free descriptor system*

$$\begin{aligned} \hat{r}_2 & \left[\begin{array}{ccc} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \end{bmatrix} u(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix}, \text{ for all } n \geq n_0, \\ \hat{\varphi}_1 & \\ \hat{\varphi}_0 & \\ \hat{v} & \end{aligned} \quad (3.5)$$

where the matrices $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$, $\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix}$ have full row rank for all $n \geq n_0$.

Proof. By performing index reduction steps until the upper rank r^u stop decreasing, we obtain the system

$$\begin{aligned} \hat{r}_2 & \left[\begin{array}{ccc} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix}, \\ \hat{\varphi}_1 & \\ \hat{\varphi}_0 & \\ \hat{v} & \end{aligned}$$

for all $n \geq n_0$, where the matrix $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ has full row rank for all $n \geq n_0$. Here

the new input sequence $\{v(n)\}_{n \geq n_0}$ satisfies $u(n) = V_n v(n)$, V_n is nonsingular for all $n \geq n_0$. Transform back $v(n) = V_n^{-1} u(n)$, and set

$$\begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hat{D}_{n,4} \\ \hat{D}_{n,5} \\ 0 \end{bmatrix} := \begin{bmatrix} \hat{D}_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma_{\hat{\varphi}_1} & 0 \\ 0 & 0 & \Sigma_{\hat{\varphi}_0} \\ 0 & 0 & 0 \end{bmatrix} V^{-1},$$

we obtain exactly the strangeness-free descriptor system (3.5). \square

As a direct corollary of Theorem 5, we obtain the existence and uniqueness of a solution to the closed-loop system via feedback as follows.

- 278 **Corollary 3** Consider the descriptor system (1.1). Furthermore, assume that
 279 Assumption 4 is fulfilled whenever needed, so that the strangeness-free descrip-
 280 tor system (3.5) is well-defined. Then, the following statements hold true.
 281 i) There exists a first order feedback of the form (3.3) such that the closed-loop
 282 system is solvable if and only if $\hat{f}_6(n) = 0$ for all $n \geq n_0$.
 283 ii) Furthermore, the solution to the corresponding IVP (of the closed-loop sys-
 284 tem) is unique if and only if in addition, $d = \sum_{i=0}^2 \hat{r}_i + \sum_{i=0}^1 \hat{\varphi}_i$.

285 *Remark 7* It should be noted that, in analogous to SiDEs, each index reduction
 286 step of the descriptor system (1.1) makes use of Theorem 2, where the matri-
 287 ces $Z_n^{(i)}$, $i = 3, 4, 5$, may not be orthogonal. Furthermore, in Lemma 11, two
 288 matrices U_n, V_n are only nonsingular but not orthogonal. Therefore, in general,
 289 the strangeness-free formulation (3.5) could not be stably computed. For the
 290 numerical treatment of (continuous time) second order DAEs, in [21] a different
 291 approach was developed. We will modify it for handling SiDEs and descriptor
 292 systems in the next section.

293 4 Difference arrays of second-order SiDEs/descriptor systems

294 As have shown in two previous sections, to analyze the theoretical solvability
 295 of the SiDE (1.2) or of the descriptor system (1.1), first one needs to bring it
 296 to a strangeness-free formulation. Nevertheless, this task is not always doable,
 297 for example when Assumptions 1, 4 are violated at some index reduction steps.
 298 These difficulties have also been observed for continuous time systems of both
 299 first and higher orders, and they have been addressed in [12, 21]. The basic
 300 idea, thanks to Campbell [4], while considering DAEs, is to differentiate a given
 301 system a number of times and put every one of them, including the original one,
 302 into a so-called *an inflated system*. Then, the strangeness-free formulation will
 303 be determined by appropriate selection of equations inside this inflated system.
 304 In this section we will examine this approach to the descriptor system (1.1).
 305 The analysis for SiDEs of the form (1.2) can be obtained by simply setting an
 306 input u to be 0. We further assume the following condition.

307 **Assumption 6.** Consider the descriptor system (1.1). Assume that there ex-
 308 ists a first order feedback of the form (3.3) such that the closed-loop system is
 309 uniquely solvable.

310 It should be noted that, in case of the SiDE (1.2), Assumption 6 means that
 311 the corresponding IVP (1.2)-(1.3) is uniquely solvable. Now let us introduce the
 312 *difference-inflated system of level $\ell \in \mathbb{N}$* as follows.

$$\begin{aligned} A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) &= f(n), \\ A_{n+1} x(n+3) + B_{n+1} x(n+2) + C_{n+1} x(n+1) + D_{n+1} u(n+1) &= f(n+1), \\ &\dots \\ A_{n+\ell} x(n+\ell+2) + B_{n+\ell} x(n+\ell+1) + C_{n+\ell} x(n+\ell) + D_{n+\ell} u(n+\ell) &= f(n+\ell). \end{aligned}$$

We rewrite this system as

$$\underbrace{\begin{bmatrix} C_n & B_n & A_n \\ C_{n+1} & B_{n+1} & A_{n+1} \\ \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots \\ C_{n+\ell} & B_{n+\ell} & A_{n+\ell} \end{bmatrix}}_{=: \mathcal{M}} \underbrace{\begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \vdots \\ x(n+\ell) \end{bmatrix}}_{=: \mathcal{X}} \\
 + \underbrace{\begin{bmatrix} D_n & & \\ & D_{n+1} & \\ & \ddots & \\ & & D_{n+\ell} \end{bmatrix}}_{=: \mathcal{N}} \underbrace{\begin{bmatrix} u(n) \\ u(n+1) \\ \vdots \\ u(n+\ell) \end{bmatrix}}_{=: \mathcal{U}} = \underbrace{\begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(n+\ell) \end{bmatrix}}_{=: \mathcal{G}}, \quad \text{for all } n \geq n_0. \quad (4.1)$$

313

Definition 3 Suppose that the descriptor system (1.1) satisfies Assumption 6. The minimum number ℓ such that by using elementary matrix's row operations, a strangeness-free descriptor system of the form (3.5) can be extracted from (4.1) is called the *shift-index* of (1.1), and be denoted by ν .

318 Proposition 2 Suppose that the descriptor system (1.1) satisfies Assumption
319 6. If the strangeness-index μ is well-defined, then so is the shift-index ν . Fur-
320 thermore, we have that $\nu \leq \mu$.

323 Proof. The first claim is straight forward, since every reformulation step per-
324 formed in Theorem 2 is still a consequence of an inflated system (4.1) with
325 $\ell = \mu$. Clearly, by definition, $\nu \leq \mu$. \square

326 Remark 8 i) It should be noted that the difference array approach presented in
327 this section can be extended in such a way that Assumption 6 can be avoided. If
328 this is the case, then the index ν will be a function of the time n . Furthermore,
329 the strangeness-free form will be defined pointwise, and clearly, the characteristic
330 values $r_2^\mu, r_1^\mu, r_0^\mu$ will also vary in time.

331 ii) In all examples we have experienced, the equality $\mu = \nu$ holds. Nevertheless,
332 even in the continuous time case, we have not found any references on this
333 mater. This topic is currently under our investigation.

Next we construct an algorithm in order to select the strangeness-free de-
scriptor system (3.5) from the inflated system (4.1). For notational convenience,
we will follow the Matlab language, [15]. Consider the following spaces and
matrices

$$\mathcal{W} := [\mathcal{M}(:, 3n+1 : end) \ \mathcal{N}(:, n+1 : end)], \quad (4.2) \\
 U_1 \text{ basis of } \text{kernel}(\mathcal{W}^T), \text{ and } U_{1,\perp} \text{ basis of } \text{range}(\mathcal{W}),$$

we have that $U_1^T \mathcal{W} = 0$ and $U_{1,\perp}^T \mathcal{W}$ has full row rank. Furthermore, the matrix
 $\begin{bmatrix} U_1^T \\ U_{1,\perp}^T \end{bmatrix}$ is nonsingular, and hence system (4.1) is equivalent to the scaled-system

below.

$$U_1^T \mathcal{M}(:, 1 : 3n) \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \end{bmatrix} + U_1^T \mathcal{N}(:, 1 : n) u(n) = U_1^T \mathcal{G}, \quad (4.3)$$

$$U_{1,\perp}^T \mathcal{W} \begin{bmatrix} x(n+3) \\ \vdots \\ \frac{x(n+\nu)}{u(n+1)} \\ \vdots \\ u(n+\nu) \end{bmatrix} + U_{1,\perp}^T [\mathcal{M}(:, 1 : 3n) \ \mathcal{N}(:, 1 : n)] \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \vdots \\ u(n) \end{bmatrix} = U_{1,\perp}^T \mathcal{G}. \quad (4.4)$$

334 Notice that due to the full row rank property of $U_{1,\perp}^T \mathcal{W}$, (4.4) plays no role in
335 the determination of the strangeness-free descriptor system (3.5). Thus, (3.5) is
336 a consequence of (4.3). In the following proposition we show that system (4.3)
337 is not affected by left equivalence transformation.

338 **Proposition 3** Consider two left equivalent systems. Then, at the same level
339 ℓ , their difference-inflated systems of the form (4.1) are also left equivalent.
340 Consequently, system (4.3) is not affected by left equivalence transformation.

Proof. Let us assume that (1.1) is left equivalent to the SiDE

$$\tilde{A}_n x(n+2) + \tilde{B}_n x(n+1) + \tilde{C}_n x(n) + \tilde{D}_n u(n) = \tilde{f}(n), \text{ for all } n \geq n_0. \quad (4.5)$$

Thus, there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that

$$[\tilde{A}_n \ \tilde{B}_n \ \tilde{C}_n \ \tilde{D}_n] = P_n [A_n \ B_n \ C_n \ D_n] \text{ and } \tilde{f}(n) = P_n f(n) \text{ for all } n \geq n_0.$$

Therefore, the difference-inflated system of level ℓ to (4.5) takes the form

$$\tilde{\mathcal{M}}\mathcal{X} + \tilde{\mathcal{N}}\mathcal{U} = \tilde{\mathcal{G}}, \quad (4.6)$$

where the matrix coefficients are

$$\tilde{\mathcal{M}} = \text{diag}(P_n, \dots, P_{n+\ell}) \ \mathcal{M}, \ \tilde{\mathcal{N}} = \text{diag}(P_n, \dots, P_{n+\ell}) \ \mathcal{N}, \ \tilde{\mathcal{G}} = \text{diag}(P_n, \dots, P_{n+\ell}) \ \mathcal{G}.$$

341 This follows that two systems (4.1) and (4.6) are left equivalent, which finishes
342 the proof. \square

For notational convenience, let us rewrite system (4.3) as

$$[\check{A} \ \check{B} \ \check{C} \ | \ \check{D}] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{bmatrix} = \check{\mathcal{G}}. \quad (4.7)$$

Lemma 13 Consider the matrices $T_{i,\perp}$, T_i , $i = 1, \dots, 4$, $W_{1,\perp}$, W_1 , \check{U} as in Lemma 5. Then, system (4.7) has the same solution set as the following system

$$\left[\begin{array}{ccc|c} T_{1,\perp}^T \check{A} & T_{1,\perp}^T \check{B} & T_{1,\perp}^T \check{C} & T_{1,\perp}^T \check{D} \\ 0 & T_{2,\perp}^T J_{B1} & T_{2,\perp}^T J_{C1} & 0 \\ 0 & 0 & T_{4,\perp}^T T_2^T J_{C1} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & T_{3,\perp}^T J_{B2} & T_{3,\perp}^T J_{C2} & T_{3,\perp}^T J_D \\ 0 & 0 & T_3^T J_{C2} & T_3^T J_D \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{bmatrix} = \check{U} \check{\mathcal{G}}. \quad (4.8)$$

³⁴³ Here the matrices $T_{1,\perp}^T \check{A}$, $T_{2,\perp}^T J_{B1}$, $T_{3,\perp}^T J_{B2}$, $T_{2,\perp}^T J_{C1}$, and J_D have full row rank.

³⁴⁴ Proof. Scaling system (4.7) with the matrix \check{U} obtained from Lemma 5 iii), we
³⁴⁵ directly obtain (4.8). \square

³⁴⁶ In the following theorem we answer the question how to derive the strangeness-free formulation (3.5) from (4.8).

Theorem 7 Assume that the shift index ν to the descriptor system (1.1) is well-defined pointwise. Furthermore, suppose that (1.1) satisfies Assumption 6. Then, from system (4.3), we can extract the following system

$$\begin{array}{l} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix} u(n) = \begin{bmatrix} \hat{G}_1 \\ \hat{G}_2 \\ \hat{G}_3 \\ \hat{G}_4 \\ \hat{G}_5 \end{bmatrix}, \text{ for all } n \geq n_0, \quad (4.9)$$

³⁴⁸ where the matrices $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$, $\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix}$ have full row rank for all $n \geq n_0$. Consequently,
³⁴⁹ the descriptor system (1.1) has exactly the same solution set as the
³⁵⁰ strangeness-free descriptor system (4.9).

Proof. First we observe, that the third, fifth and sixth equations of (4.8) are the last three block row equations of (4.9). The remaining task is to extract the first two block row equations of system (4.9) from (4.8), by suitably removing the existence hidden redundancy. Applying Lemma 4 consecutively for two following matrix pairs

$$(T_{2,\perp}^T J_{B1}, T_{4,\perp}^T T_2^T J_{C1}), \left(T_{1,\perp}^T \check{A}, \begin{bmatrix} T_{2,\perp}^T J_{B1} \\ T_{4,\perp}^T T_2^T J_{C1} \end{bmatrix} \right),$$

we obtain two unitary matrices $\begin{bmatrix} S_n^{(i)} \\ Z_n^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$, $i = 1, 2$ such that both pairs

$$(S_n^{(1)} T_{2,\perp}^T J_{B1}, T_{4,\perp}^T T_2^T J_{C1}), \left(S_n^{(2)} T_{1,\perp}^T \check{A}, \begin{bmatrix} S_n^{(1)} T_{2,\perp}^T J_{B1} \\ T_{4,\perp}^T T_2^T J_{C1} \end{bmatrix} \right).$$

³⁵¹ have no hidden redundancy. Scale the first and second block row equations of
³⁵² (4.8) with $S_n^{(2)}$ and $S_n^{(1)}$ respectively, we obtain the first two block row equations
³⁵³ of (4.9). \square

³⁵⁴ We summarize our result in the following algorithm.

Algorithm 2 Strangeness-free formulation for SiDEs using difference arrays

1: **Input:** The SiDE (1.1).
 2: **Return:** The strangeness-free descriptor system (4.9).
 3: Set $\ell := 0$.
 4: Construct the difference-inflated system of level ℓ , and rewrite it in the form (4.1).
 5: Find U_1 as in (4.2) and construct system (4.3).
 6: Find T_i , $T_{i,\perp}$, $i = 1, \dots, 4$, W_1 , $W_{1,\perp}$ and construct (4.8) as in Lemma 5.
 7: Applying Lemma 3 to reduce the hidden redundancies in two matrix pairs
 $(T_{2,\perp}^T J_{B1}, T_{4,\perp}^T T_2^T J_{C1})$, $(T_{1,\perp}^T \check{A}, \begin{bmatrix} T_{2,\perp}^T J_{B1} \\ T_{4,\perp}^T T_2^T J_{C1} \end{bmatrix})$, and hence, to obtain system (4.9).
 8: **if** rank $\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix}$ + rank $\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix} = d$ **then** STOP.
 9: **else** set $\ell := \ell + 1$ and go to 4
 10: **end if**

355 In order to illustrate Algorithm 2, we consider a three link robot arm [8] in
 356 the following example.

Example 3 Our consider system is of the form

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(t).$$

Here M_0 represents the nonsingular mass matrix, G_0 the coefficient matrix associated with damping, centrifugal, gravity, and Coriolis forces, K_0 the stiffness matrix, and H_0 the constraint. A simple discretized version of this system takes the form

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - 2x(n+1) + x(n)}{h^2} + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - x(n+1)}{h} + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(n) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(n).$$

357 where h is the discretized stepsize.

358 As a simple example, let us take $M_0 = G_0 = K_0 = H_0 = B_0 = 1$, $h =$
 359 0.01. Then, Algorithm 2 terminates after two steps and hence, the shift index
 360 is $\nu = 2$ for all $n \geq n_0$. Furthermore, we notice that no matter forward or
 361 backward approximations has been chosen for the derivative $\dot{x}(t)$, the index
 362 remains unchanged $\nu = 2$. Nevertheless, the resulting strangeness-free descriptor
 363 systems are different.

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417 **A Proof of Lemma 10**

418 First we prove that any solution to (2.7) is also a solution to (2.9). Notice that, due
 419 to Lemma ??, two systems (2.5) and (2.7) have identical solution set. Thus, we only
 420 need to prove that (2.7) and (2.9) are equivalent.

421 **Necessity:** The main idea here is to apply elementary row transformations to system
 422 (2.7) to obtain (2.9). Notice that we use only two elementary block row operations:

- 423 i) scaling a block row equation with a nonsingular matrix,
 424 ii) add to one row a linear combinations of another rows.

Firstly, by scaling the first (resp., second) block row equation of (2.7) with a unitary
 matrix $\begin{bmatrix} S_n^{(2)} \\ Z_n^{(2)} \end{bmatrix}$ (resp., $\begin{bmatrix} S_n^{(1)} \\ Z_n^{(1)} \end{bmatrix}$), we obtain an equivalent system to (2.5), as follows

$$\left[\begin{array}{ccc} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ Z_n^{(2)} A_{n,1} & Z_n^{(2)} B_{n,1} & Z_n^{(2)} C_{n,1} \\ \hline 0 & S_n^{(1)} B_{n,2} & S_n^{(1)} C_{n,2} \\ 0 & Z_n^{(1)} B_{n,2} & Z_n^{(1)} C_{n,2} \\ \hline 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ \hline 0 & C_{n+1,3} & 0 \\ B_{n+1,2} & C_{n+1,2} & 0 \\ C_{n+2,3} & 0 & 0 \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} S_n^{(2)} f_1(n) \\ Z_n^{(2)} f_1(n) \\ \hline S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) \\ \hline f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{bmatrix}, \quad (A.1)$$

By adding the seventh row scaled with $Z_n^{(3)}$ to the fourth row of (A.1) and making
 use of (??) we obtain the first hidden constraint

$$Z_n^{(1)} C_{n,2} x(n) = Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1),$$

425 which is exactly the fourth row of (2.9).

We continue by adding the seventh row scaled with $Z_n^{(4)}$ and the eighth row scaled
 with $Z_n^{(5)}$ to the second row of (A.1) and making use of (??) to obtain

$$\begin{aligned} & (Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2}) x(n+1) + Z_n^{(2)} C_{n,1} x(n) \\ &= Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2). \end{aligned}$$

This is exactly the second row of (2.9). Therefore, any solution to (2.5) is also a solution
 to (2.9).

Sufficiency: Let x be an arbitrary solution to (2.9). Thus, x is also a solution to the

shifted system

$$\begin{aligned}
 & \frac{d_2}{s_2} \begin{bmatrix} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ 0 & Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2} & Z_n^{(2)} C_{n,1} \\ 0 & S_n^{(1)} B_{n,2} & S_n^{(1)} C_{n,2} \\ 0 & 0 & Z_n^{(1)} C_{n,2} \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} \\
 & \frac{s_1}{d_1} \begin{bmatrix} 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \\ 0 & C_{n+1,3} & 0 \end{bmatrix} \\
 & \frac{v}{r_0} \begin{bmatrix} C_{n+2,3} & 0 & 0 \end{bmatrix} \\
 & = \begin{bmatrix} S_n^{(2)} f_1(n) \\ Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2) \\ S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1) \\ f_3(n) \\ f_4(n) \\ f_3(n+1) \\ f_3(n+2) \end{bmatrix}, \quad \text{for all } n \geq n_0. \tag{A.2}
 \end{aligned}$$

426 Since elementary matrix row operations are reversible, we can reverse the transformations
427 performed in the necessity part. Consequently, we see that any solution to (A.2)
428 is also a solution to (A.1), and hence, this completes the proof.