

EXPONENTIAL STABILITY AND ROBUST STABILITY FOR LINEAR TIME-VARYING SINGULAR SYSTEMS OF SECOND-ORDER DIFFERENCE EQUATIONS

VU HOANG LINH*, NGO THI THANH NGA[†], AND DO DUC THUAN[‡]

Abstract. In this paper, solvability, stability, and robust stability of linear time-varying singular systems of second order difference equations are studied. The leading coefficient is allowed to be singular, i.e., the system does not generate an explicit recursion. By transforming the system into an appropriate form, the existence and uniqueness of solutions are established under the so-called strangeness-free assumption. Consistent initial conditions are also explicitly constructed. Then, a criterion for exponential stability and a Bohl-Perron type theorem are presented. Finally, we investigate the robust stability when the system coefficients are subject to structured perturbations. Examples are also given for illustration.

Keywords. Singular system, second-order difference equation, exponential stability, robust stability structured perturbation, strangeness-free system, positive system.

Mathematics Subject Classifications: 15A99, 39A06, 39A30, 93D09

1. Introduction. In this paper, we consider the linear time-varying implicit difference equation of second order

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = f_n, \quad n = n_0, n_0 + 1, \dots, \quad (1.1)$$

with $n_0 \in \mathbb{N}$ and coefficients $A_n, B_n, C_n \in \mathbb{C}^{d,d}$, $n \geq n_0$. The leading coefficient A_n is supposed to be singular, but $\text{rank } A_n \equiv d_1 < d$ for all $n \geq n_0$. We also assume that the initial conditions

$$x(n_0) = x_0, \quad x(n_0 + 1) = x_1, \quad (1.2)$$

with $x_0, x_1 \in \mathbb{C}^d$, are given.

Singular difference equations (SDEs) are generalization of regular explicit difference equations, which have been well investigated in the literature, see [1, 16]. They arise as mathematical models in various fields such as population dynamics, economics, systems and control theory, and numerical analysis, e.g., see [21, 22, 23]. The singularity of the leading coefficient makes the analysis of the system (1.1) difficult since explicit computation of solutions is impossible at the first sight. Even the solvability of the initial value problem (IVP) (1.1)-(1.2) is doubtful. On the other hand, singular difference equations are discrete analogues of differential-algebraic equations (DAEs), see [19, 20]. For example, when a linear system of second order DAEs is discretized by appropriate difference schemes, difference systems of the form (1.1) arise. While the theory of differential-algebraic equations have become a mature topic, serious attentions on singular difference equations started just only in the last decade. Investigations of stability of singular difference equations are mostly limited to linear time-invariant systems, e.g., see [12, 25, 28]. Two approaches to mathematically rigorous analysis of linear time-varying systems have been discussed recently in [2, 9]. Solvability and stability problems for singular systems of linear time-varying

*Faculty of Mathematics, Mechanics and Informatics, Vietnam National University, 334 Nguyen Trai Str., Thanh Xuan, Hanoi, Vietnam (linhvh@vnu.edu.vn).

[†]Faculty of Mathematics and Informatics, Thang Long University, Nghiem Xuan Yem Str., Hoang Mai, Hanoi, Vietnam (nga.ngo@gmail.com).

[‡]School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, 1 Dai Co Viet Str., Hai Ba Trung, Hanoi, Vietnam (ducthuank7@gmail.com, thuan.doduc@hust.edu.vn).

and nonlinear difference equations of first order are investigated in [3, 4, 5, 15, 27]. For singular systems of high order difference equations, by using the strangeness-free form formulation, which is a well-known approach to the treatment of high order DAEs [24] and delay DAEs [14, 18], stability analysis for singular systems of linear time-invariant difference equations is given very recently in [25].

In this paper, we aim to study solvability, exponential stability, and robust stability for linear time-varying SDEs of the form (1.1). We propose an explicit construction of the so-called consistent initial conditions with which the IVP admits the unique solution. We also show the relation between the SDE (1.1) and a uniquely determined explicit difference equation. Combining this characterization and a comparison principle as used in [26], we obtain criteria for exponential stability and a Bohl-Perron type theorem for SDEs (1.1). Next, we make use of the recent result for linear time-invariant SDEs in [25] and obtain a bound for the robust stability of SDEs (1.1) when the coefficients are subject to structured perturbations. The approach in this paper can be extended to higher order SDEs and similar results are expected. However, for sake of simplicity, in this work we restrict the investigation to second order SDEs. The transformation into the strangeness-free form that are based on matrix decompositions plays the key role in our analysis. Up to our knowledge, this is the first work addressing the stability and robust stability of *linear time-varying singular* difference equations of *high order*. Though in this paper only second order systems are considered, the approach presented here can be extended to the analysis of higher order linear systems and linear delay systems.

The outline of the paper is as follows. In the next section, we deal with the solvability of the SDE (1.1). In Section 3, we introduce the notion of exponential stability for homogenous SDEs associated with (1.1) and establish criteria for the exponential stability. In Section 4, we consider the SDE (1.1) subject to structured perturbation and obtain a bound for the perturbations so that the exponential stability is preserved. We close the paper with a conclusion.

In the whole paper we will use the following notations. For $u, v \in \mathbb{R}^d$, we write $u \gg 0$ if each component of u is positive and we write $u \gg v$ if $u - v \gg 0$. By ℓ_p we denote the Banach space

$$\ell_p = \left\{ f = \{f_n\}_{n \geq n_0} : f_n \in \mathbb{C}^d \ \forall n \geq n_0 \text{ and } \left(\sum_{n=n_0}^{\infty} |f_n|^p \right)^{1/p} < \infty \right\},$$

endowed with the norm $\|f\|_p = \left(\sum_{n=n_0}^{\infty} |f_n|^p \right)^{1/p}$.

2. Initial value problems. First, we reformulate the SDE (1.1) in a form which is useful in the further analysis. For any fixed n , there exists a nonsingular matrix, denoted by W_n , such that

$$W_n^{-1} A_n = \begin{bmatrix} A_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, W_n^{-1} B_n = \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \\ 0 \end{bmatrix}, W_n^{-1} C_n = \begin{bmatrix} C_n^{(1)} \\ C_n^{(2)} \\ C_n^{(3)} \end{bmatrix}, \quad (2.1)$$

where $A_n^{(1)}, B_n^{(1)}, C_n^{(1)} \in \mathbb{C}^{d_1, d}$, $B_n^{(2)}, C_n^{(2)} \in \mathbb{C}^{d_2, d}$, $C_n^{(3)} \in \mathbb{C}^{d_3, d}$, $d_1 + d_2 + d_3 = d$. The matrices W_n can be constructed as follows. Suppose that a singular value decomposition (SVD) of A_n is $A_n = U_n \Sigma_n V_n^*$, where $U_n = \begin{bmatrix} U_{n1} & U_{n2} \end{bmatrix}$ with $U_{n1} \in \mathbb{C}^{d, d_1}$, $U_{n2} \in \mathbb{C}^{d, (d_2+d_3)}$ and $\Sigma_n = \text{diag}(\sigma_{n1}, \sigma_{n2}, \dots, \sigma_{nd_1}, 0, \dots, 0)$. The columns of

U_{n1} and U_{n2} are the unitary left singular vectors associated with nonzero and zero singular values of A_n , respectively. Assume that $\text{rank } U_{n2}^* B_n = d_2$. Let a singular value decomposition of $U_{n2}^* B_n$ be $\tilde{U}_n \tilde{\Sigma}_n \tilde{V}_n^*$, where $\tilde{U}_n = \begin{bmatrix} \tilde{U}_{n2} & \tilde{U}_{n3} \end{bmatrix}$ with $\tilde{U}_{n2} \in \mathbb{C}^{(d_2+d_3), d_2}$, $\tilde{U}_{n3} \in \mathbb{C}^{(d_2+d_3), d_3}$. We define $W_n = U_n \text{diag}(I_{d_1}, \tilde{U}_n)$. Consequently, we have $W_n^{-1} = \text{diag}(I_{d_1}, \tilde{U}_n^*) U_n^*$. Then, the form (2.1) obviously holds with

$$\begin{aligned} A_n^{(1)} &= U_{n1}^* A_n, \quad B_n^{(1)} = U_{n1}^* B_n, \quad C_n^{(1)} = U_{n1}^* C_n, \\ B_n^{(2)} &= \tilde{U}_{n2}^* U_{n2}^* B_n, \quad C_n^{(2)} = \tilde{U}_{n2}^* U_{n2}^* C_n, \quad C_n^{(3)} = \tilde{U}_{n3}^* U_{n2}^* C_n. \end{aligned}$$

From now on, we always assume that, in addition to the constant rank assumption of A_n , the condition $\text{rank } U_{n2}^* B_n \equiv d_2$ holds for all $n \geq n_0$, too.

DEFINITION 2.1. *The implicit difference equation (1.1) is called strangeness-free if there exists an sequence of invertible matrices $\{W_n\}$ such that the reformulation*

(2.1) holds and that $\begin{bmatrix} A_n^{(1)} \\ B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix}$ is invertible for all $n \geq n_0$.

It is easy to see that the above definition is independent of the choice of W_n . Due to the above construction, without loss of generality, we assume that $\{W_n\}$ are unitary matrices.

REMARK 2.2. Suppose that equation (1.1) is strangeness-free and W_n and \widehat{W}_n are two unitary matrices that both transform the coefficients of the equation to the form (2.1). Let $\widehat{A}_n^{(i)}, \widehat{B}_n^{(i)}, \widehat{C}_n^{(i)}$ be the transformed blocks corresponding to \widehat{W}_n . We introduce the block matrix $R_n = W_n^{-1} \widehat{W}_n$ and let $R_n = (R_{n,j}^{(i)})$ with $R_{n,j}^{(i)} \in \mathbb{C}^{d_i, d_j}$. Then, we have

$$R_n \begin{bmatrix} \widehat{A}_n^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} A_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, \quad R_n \begin{bmatrix} \widehat{B}_n^{(1)} \\ \widehat{B}_n^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \\ 0 \end{bmatrix}, \quad R_n \begin{bmatrix} \widehat{C}_n^{(1)} \\ \widehat{C}_n^{(2)} \\ \widehat{C}_n^{(3)} \end{bmatrix} = \begin{bmatrix} C_n^{(1)} \\ C_n^{(2)} \\ C_n^{(3)} \end{bmatrix}.$$

It is easy to verify (see [14, Remark 2.7]) that R_n is a block upper-triangular matrix, i.e., $R_{n,j}^{(i)}, 0 \leq j < i \leq 3$, are zero blocks. Since R_n is nonsingular, the diagonal blocks $R_{n,i}^{(i)}, i = 1, 2, 3$, are nonsingular. Thus, $\widehat{W}_n = W_n R_n$ with

$$R_n = \begin{bmatrix} R_{n,1}^{(1)} & R_{n,2}^{(1)} & R_{n,3}^{(1)} \\ 0 & R_{n,2}^{(2)} & R_{n,3}^{(2)} \\ 0 & 0 & R_{n,3}^{(3)} \end{bmatrix}.$$

Moreover, since both W_n, \widehat{W}_n are unitary, R_n is unitary, too. Therefore, $R_n R_n^* = R_n^* R_n = I$. This implies that $R_{n,2}^{(1)}, R_{n,3}^{(1)}$ and $R_{n,3}^{(2)}$ are zero blocks. Hence, $R_n = \text{diag}(R_{n,1}^{(1)}, R_{n,2}^{(2)}, R_{n,3}^{(3)})$ is actually a block diagonal matrix.

In the whole paper, we assume that equation (1.1) is strangeness free with unitary matrices W_n^{-1} constructed as above. Multiplying both sides of (1.1) with W_n^{-1} , we get

$$\begin{aligned} A_n^{(1)} x(n+2) + B_n^{(1)} x(n+1) + C_n^{(1)} x(n) &= f_n^{(1)}, \\ B_n^{(2)} x(n+1) + C_n^{(2)} x(n) &= f_n^{(2)}, \\ C_n^{(3)} x(n) &= f_n^{(3)}. \end{aligned}$$

for all $n \geq n_0$. Shifting the index in the second and the third equations, we obtain,
for all $n \geq n_0$,

$$\begin{aligned} A_n^{(1)}x(n+2) + B_n^{(1)}x(n+1) + C_n^{(1)}x(n) &= f_n^{(1)}, \\ B_{n+1}^{(2)}x(n+2) + C_{n+1}^{(2)}x(n+1) &= f_{n+1}^{(2)}, \\ C_{n+2}^{(3)}x(n+2) &= f_{n+2}^{(3)}. \end{aligned}$$

Putting

$$\hat{A}_n = \begin{bmatrix} A_n^{(1)} \\ B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix}, \quad \hat{B}_n = \begin{bmatrix} B_n^{(1)} \\ C_{n+1}^{(2)} \\ 0 \end{bmatrix}, \quad \hat{C}_n = \begin{bmatrix} C_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, \quad \hat{f}_n = \begin{bmatrix} f_n^{(1)} \\ f_{n+1}^{(2)} \\ f_{n+2}^{(3)} \end{bmatrix},$$

system (1.1) can be transformed into

$$\hat{A}_n x(n+2) + \hat{B}_n x(n+1) + \hat{C}_n x(n) = \hat{f}_n, \quad \forall n \geq n_0.$$

Due to Definition 2.1, the leading coefficient \hat{A}_n is invertible, we have

$$x(n+2) = -\hat{A}_n^{-1}\hat{B}_n x(n+1) - \hat{A}_n^{-1}\hat{C}_n x(n) + \hat{A}_n^{-1}\hat{f}_n, \quad \forall n \geq n_0 \quad (2.2)$$

PROPOSITION 2.3. *Assume that equation (1.1) is strangeness-free. Then, the explicit difference equation (2.2) is independent of the choice of unitary matrix sequence $\{W_n\}$ which transform the coefficients of (1.1) to the form (2.1).*

Proof. Assume that $\{W_n\}$ and $\{\widehat{W}_n\}$ are two arbitrary sequences of unitary matrices both of which transform the coefficients of (1.1) into the form (2.1). Then,

by Remark (2.2), we have $W_n^{-1} = R_n \widehat{W}_n^{-1}$, where $R_n = \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n,2}^{(2)} & 0 \\ 0 & 0 & R_{n,3}^{(3)} \end{bmatrix}$. Hence,

$$\begin{aligned} W_n^{-1}A_n &= \begin{bmatrix} A_n^{(1)} \\ 0 \\ 0 \end{bmatrix} = R_n \begin{bmatrix} \hat{A}_n^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)}\hat{A}_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, \\ W_n^{-1}B_n &= \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \\ 0 \end{bmatrix} = R_n \begin{bmatrix} \hat{B}_n^{(1)} \\ \hat{B}_n^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)}\hat{B}_n^{(1)} \\ R_{n,2}^{(2)}\hat{B}_n^{(2)} \\ 0 \end{bmatrix}, \\ W_n^{-1}C_n &= \begin{bmatrix} C_n^{(1)} \\ C_n^{(2)} \\ C_n^{(3)} \end{bmatrix} = R_n \begin{bmatrix} \hat{C}_n^{(1)} \\ \hat{C}_n^{(2)} \\ \hat{C}_n^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)}\hat{C}_n^{(1)} \\ R_{n,2}^{(2)}\hat{C}_n^{(2)} \\ R_{n,3}^{(3)}\hat{C}_n^{(3)} \end{bmatrix}, \\ W_n^{-1}f_n &= \begin{bmatrix} f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \end{bmatrix} = R_n \begin{bmatrix} \hat{f}_n^{(1)} \\ \hat{f}_n^{(2)} \\ \hat{f}_n^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)}\hat{f}_n^{(1)} \\ R_{n,2}^{(2)}\hat{f}_n^{(2)} \\ R_{n,3}^{(3)}\hat{f}_n^{(3)} \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned}
\hat{A}_n &= \begin{bmatrix} A_n^{(1)} \\ B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} \hat{A}_n^{(1)} \\ R_{n+1,2}^{(2)} \hat{B}_{n+1}^{(2)} \\ R_{n+2,3}^{(3)} \hat{C}_{n+2}^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n+1,2}^{(2)} & 0 \\ 0 & 0 & R_{n+2,3}^{(3)} \end{bmatrix} \hat{\hat{A}}_n, \\
\hat{B}_n &= \begin{bmatrix} B_n^{(1)} \\ C_{n+1}^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} \hat{B}_n^{(1)} \\ R_{n+1,2}^{(2)} \hat{C}_{n+1}^{(2)} \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n+1,2}^{(2)} & 0 \\ 0 & 0 & R_{n+2,3}^{(3)} \end{bmatrix} \hat{\hat{B}}_n, \\
\hat{C}_n &= \begin{bmatrix} C_n^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} \hat{C}_n^{(1)} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n+1,2}^{(2)} & 0 \\ 0 & 0 & R_{n+2,3}^{(3)} \end{bmatrix} \hat{\hat{C}}_n, \\
\hat{f}_n &= \begin{bmatrix} f_n^{(1)} \\ f_{n+1}^{(2)} \\ f_{n+2}^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} \hat{f}_n^{(1)} \\ R_{n+1,2}^{(2)} \hat{f}_{n+1}^{(2)} \\ R_{n+2,3}^{(3)} \hat{f}_{n+2}^{(3)} \end{bmatrix} = \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n+1,2}^{(2)} & 0 \\ 0 & 0 & R_{n+2,3}^{(3)} \end{bmatrix} \hat{\hat{f}}_n
\end{aligned}$$

Setting $\hat{R}_n := \begin{bmatrix} R_{n,1}^{(1)} & 0 & 0 \\ 0 & R_{n+1,2}^{(2)} & 0 \\ 0 & 0 & R_{n+2,3}^{(3)} \end{bmatrix}$, we have

$$\hat{A}_n = \hat{R}_n \hat{\hat{A}}_n, \quad \hat{B}_n = \hat{R}_n \hat{\hat{B}}_n, \quad \hat{C}_n = \hat{R}_n \hat{\hat{C}}_n, \quad \hat{f}_n = \hat{R}_n \hat{\hat{f}}_n.$$

Thus, the following equalities

$$\begin{aligned}
\hat{A}_n^{-1} \hat{B}_n &= \hat{\hat{A}}_n^{-1} \hat{R}_n^{-1} \hat{R}_n \hat{\hat{B}}_n = \hat{\hat{A}}_n^{-1} \hat{\hat{B}}_n, \\
\hat{A}_n^{-1} \hat{C}_n &= \hat{\hat{A}}_n^{-1} \hat{R}_n^{-1} \hat{R}_n \hat{\hat{C}}_n = \hat{\hat{A}}_n^{-1} \hat{\hat{C}}_n, \\
\hat{A}_n^{-1} \hat{f}_n &= \hat{\hat{A}}_n^{-1} \hat{R}_n^{-1} \hat{R}_n \hat{\hat{f}}_n = \hat{\hat{A}}_n^{-1} \hat{\hat{f}}_n,
\end{aligned}$$

hold. This implies that for two arbitrary sequences of unitary matrices W_n and \hat{W}_n , we obtain the same equation in form (2.2). The proof is complete. \square

PROPOSITION 2.4. *Consider the strangeness-free equation (1.1). The strangeness-free property is invariant with respect to change of variable $x(n) = Q_n y(n)$ with nonsingular Q_n , $n \geq n_0$.*

Proof. Consider the new system

$$A_n Q_{n+2} y(n+2) + B_n Q_{n+1} y(n+1) + C_n Q_n y(n) = f_n, \quad n \geq n_0. \quad (2.3)$$

We have

$$\begin{aligned} W_n^{-1} A_n Q_{n+2} &= \begin{bmatrix} A_n^{(1)} \\ 0 \\ 0 \end{bmatrix} Q_{n+2} = \begin{bmatrix} \bar{A}_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, \\ W_n^{-1} B_n Q_{n+1} &= \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \\ 0 \end{bmatrix} Q_{n+1} = \begin{bmatrix} \bar{B}_n^{(1)} \\ \bar{B}_n^{(2)} \\ 0 \end{bmatrix}, \\ W_n^{-1} C_n Q_{n+1} &= \begin{bmatrix} C_n^{(1)} \\ C_n^{(2)} \\ C_n^{(3)} \end{bmatrix} Q_n = \begin{bmatrix} \bar{C}_n^{(1)} \\ \bar{C}_n^{(2)} \\ \bar{C}_n^{(3)} \end{bmatrix}, \end{aligned}$$

where $\bar{A}_n^{(1)} = A_n^{(1)} Q_{n+2}$, $\bar{B}_n^{(1)} = B_n^{(1)} Q_{n+1}$, $\bar{B}_n^{(2)} = B_n^{(2)} Q_{n+1}$, $\bar{C}_n^{(1)} = C_n^{(1)} Q_n$, $\bar{C}_n^{(2)} = C_n^{(2)} Q_n$, $\bar{C}_n^{(3)} = C_n^{(3)} Q_n$. Therefore,

$$\begin{bmatrix} \bar{A}_n^{(1)} \\ \bar{B}_{n+1}^{(2)} \\ \bar{C}_{n+2}^{(3)} \end{bmatrix} = \begin{bmatrix} A_n^{(1)} \\ B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix} Q_{n+2} \text{ for all } n \geq n_0.$$

Since equation (1.1) is strangeness-free and Q_n is invertible for all $n \geq n_0$, it follows

that $\begin{bmatrix} \bar{A}_n^{(1)} \\ \bar{B}_{n+1}^{(2)} \\ \bar{C}_{n+2}^{(3)} \end{bmatrix}$ is invertible for all $n \geq n_0$. Thus, the new system (2.3) is strangeness-free, too. \square

DEFINITION 2.5. A sequence $\{x(n)\}_{n \geq n_0}$, $x(n) \in \mathbb{C}^d$ is called a solution of (1.1) if it satisfies (1.1) for all $n \geq n_0$. The SDE (1.1) is called solvable if it has at least a solution. The initial condition (1.2) is called consistent if the IVP (1.1)-(1.2) admits the unique solution.

Now, we analyze the solvability of the IVP (1.1)-(1.2). An arbitrary sequence $\{x(n)\}_{n \geq n_0}$, $x(n) \in \mathbb{C}^d$ is a solution of (1.1) if and only if it satisfies (2.2) for all $n \geq n_0$ as well as the following set of conditions

$$\begin{aligned} B_{n_0}^{(2)} x(n_0 + 1) + C_{n_0}^{(2)} x(n_0) &= f_{n_0}^{(2)}, \\ C_{n_0}^{(3)} x(n_0) &= f_{n_0}^{(3)}, \\ C_{n_0+1}^{(3)} x(n_0 + 1) &= f_{n_0+1}^{(3)}. \end{aligned} \tag{2.4}$$

Thus, a given initial condition (1.2) is consistent if and only if the initial functions $x^{(0)}$ and $x^{(1)}$ fulfil (2.4).

Next, we specify the consistent initialization. Since (1.1) is strangeness free, $\begin{bmatrix} B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix}$ and $C_{n+2}^{(3)}$ have full rank for all $n \geq n_0$. Assume that $\begin{bmatrix} B_{n_0}^{(2)} \\ C_{n_0+1}^{(3)} \end{bmatrix}$ and $C_{n_0}^{(3)}$ also have full rank. Then, there exists a sequence of invertible (even unitary) matrices $\{Q_n\}$, $Q_n \in \mathbb{C}^{d,d}$, such that

$$\begin{bmatrix} B_n^{(2)} \\ C_{n+1}^{(3)} \end{bmatrix} Q_{n+1} = \begin{bmatrix} 0 & \bar{B}_{n,2}^{(2)} & \bar{B}_{n,3}^{(2)} \\ 0 & 0 & \bar{C}_{n+1,3}^{(3)} \end{bmatrix} \text{ for all } n \geq n_0 \text{ and } C_{n_0}^{(3)} Q_{n_0} = \begin{bmatrix} 0 & 0 & \bar{C}_{n_0,3}^{(3)} \end{bmatrix},$$

where $\overline{B}_{n,2}^{(2)}$ and $\overline{C}_{n,3}^{(3)}$ are invertible matrices for all $n \geq n_0$. In practice, one can use the well-known QR factorization to calculate Q_n .

Put $x(n) = Q_n y(n)$ for all $n \geq n_0$, then (1.1) can be transform to

$$\begin{aligned} \begin{bmatrix} \overline{A}_{n,1}^{(1)} & \overline{A}_{n,2}^{(1)} & \overline{A}_{n,3}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y(n+2) + \begin{bmatrix} \overline{B}_{n,1}^{(1)} & \overline{B}_{n,2}^{(1)} & \overline{B}_{n,3}^{(1)} \\ 0 & \overline{B}_{n,2}^{(2)} & \overline{B}_{n,3}^{(2)} \\ 0 & 0 & 0 \end{bmatrix} y(n+1) \\ + \begin{bmatrix} \overline{C}_{n,1}^{(1)} & \overline{C}_{n,2}^{(1)} & \overline{C}_{n,3}^{(1)} \\ \overline{C}_{n,1}^{(2)} & \overline{C}_{n,2}^{(2)} & \overline{C}_{n,3}^{(2)} \\ 0 & 0 & \overline{C}_{n,3}^{(3)} \end{bmatrix} y(n) = \begin{bmatrix} f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \end{bmatrix}. \end{aligned} \quad (2.5)$$

According to Proposition 2.4, this system is strangeness-free, too. Similarly, by shifting the index of the coefficients in the second and third rows, we obtain,

$$y(n+2) = -\widehat{\overline{A}}_n^{-1} \widehat{\overline{B}}_n y(n+1) - \widehat{\overline{A}}_n^{-1} \widehat{\overline{C}}_n y(n) + \widehat{\overline{A}}_n^{-1} \widehat{f}_n, \quad \forall n \geq n_0, \quad (2.6)$$

where $\widehat{\overline{A}}_n = \begin{bmatrix} \overline{A}_{n,1}^{(1)} & \overline{A}_{n,2}^{(1)} & \overline{A}_{n,3}^{(1)} \\ 0 & \overline{B}_{n+1,2}^{(2)} & \overline{B}_{n+1,3}^{(2)} \\ 0 & 0 & \overline{C}_{n+2,3}^{(3)} \end{bmatrix}$, $\widehat{\overline{B}}_n = \begin{bmatrix} \overline{B}_{n,1}^{(1)} & \overline{B}_{n,2}^{(1)} & \overline{B}_{n,3}^{(1)} \\ \overline{C}_{n+1,1}^{(2)} & \overline{C}_{n+1,2}^{(2)} & \overline{C}_{n+1,3}^{(2)} \\ 0 & 0 & 0 \end{bmatrix}$, $\widehat{\overline{C}}_n = \begin{bmatrix} \overline{C}_{n,1}^{(1)} & \overline{C}_{n,2}^{(1)} & \overline{C}_{n,3}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Let $y(n) = \begin{bmatrix} y^{(1)}(n) \\ y^{(2)}(n) \\ y^{(3)}(n) \end{bmatrix}$. Then, the consistent initial conditions (2.4) are equivalent to

$$\begin{aligned} \overline{B}_{n_0,2}^{(2)} y^{(2)}(n_0+1) + \overline{B}_{n_0,3}^{(2)} y^{(3)}(n_0+1) + \overline{C}_{n_0,1}^{(2)} y^{(1)}(n_0) \\ + \overline{C}_{n_0,2}^{(2)} y^{(2)}(n_0) + \overline{C}_{n_0,3}^{(2)} y^{(3)}(n_0) = f_{n_0}^{(2)}, \\ \overline{C}_{n_0,3}^{(3)} y^{(3)}(n_0) = f_{n_0}^{(3)}, \\ \overline{C}_{n_0+1,3}^{(3)} y^{(3)}(n_0+1) = f_{n_0+1}^{(3)}. \end{aligned}$$

This implies that the constraints

$$\begin{aligned} y^{(3)}(n_0) = \left(\overline{C}_{n_0,3}^{(3)} \right)^{-1} f_{n_0}^{(3)}, \quad y^{(3)}(n_0+1) = \left(\overline{C}_{n_0+1,3}^{(3)} \right)^{-1} f_{n_0+1}^{(3)}, \\ y^{(2)}(n_0+1) = \left(\overline{B}_{n_0,2}^{(2)} \right)^{-1} \left[f_{n_0}^{(2)} - \overline{B}_{n_0,3}^{(2)} \left(\overline{C}_{n_0+1,3}^{(3)} \right)^{-1} f_{n_0+1}^{(3)} - \overline{C}_{n_0,3}^{(2)} \left(\overline{C}_{n_0,3}^{(3)} \right)^{-1} f_{n_0}^{(3)} \right. \\ \left. - \overline{C}_{n_0,1}^{(2)} y^{(1)}(n_0) - \overline{C}_{n_0,2}^{(2)} y^{(2)}(n_0) \right], \end{aligned} \quad (2.7)$$

hold, while the components $y^{(1)}(n_0) \in \mathbb{C}^{d_1}$, $y^{(2)}(n_0) \in \mathbb{C}^{d_2}$, $y^{(1)}(n_0+1) \in \mathbb{C}^{d_1}$ are arbitrarily given. Then, the consistent initial values $x(n_0), x(n_0+1)$ can be specified

by formulas

$$\begin{aligned} x(n_0) &= Q_{n_0} \begin{bmatrix} y^{(1)}(n_0) \\ y^{(2)}(n_0) \\ \left(\bar{C}_{n_0,3}^{(3)}\right)^{-1} f_{n_0}^{(3)} \end{bmatrix}, \\ x(n_0 + 1) &= Q_{n_0+1} \begin{bmatrix} y^{(1)}(n_0 + 1) \\ y^{(2)}(n_0 + 1) \\ \left(\bar{C}_{n_0+1,3}^{(3)}\right)^{-1} f_{n_0+1}^{(3)} \end{bmatrix}, \end{aligned} \quad (2.8)$$

where $y^{(2)}(n_0 + 1)$ is prescribed in (2.7). We conclude with the following theorem.

THEOREM 2.6. *Assume that the SDE (1.1) is strangeness-free and the initial functions in (1.2) satisfy (2.4). Then, the IVP (1.1)-(1.2) admits the unique solution which can be calculated by the explicit recursion (2.2). Moreover, if $\begin{bmatrix} B_{n_0}^{(2)} \\ C_{n_0+1}^{(3)} \end{bmatrix}$ and $C_{n_0}^{(3)}$ have full rank, then the consistent initial condition is specified in the explicit form (2.8).*

3. Exponential stability. Consider the homogeneous equations corresponding to (1.1)

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = 0, \quad \forall n \geq n_0, \quad (3.1)$$

where the initial conditions $x(n_0), x(n_0 + 1)$ are consistent.

3.1. Notion of exponential stability. We denote by $x(n, m, x^{(0)}, x^{(1)})$ a solution of (3.1) that satisfies $x(m) = x^{(0)}$ and $x(m+1) = x^{(1)}$ with consistent initial values $x^{(0)}, x^{(1)}$.

DEFINITION 3.1. *Equation (3.1) is called exponentially stable if there exist constants $K > 0$ and $0 < \omega < 1$ such that*

$$\|x(n, m, x(m), x(m+1))\| \leq K \omega^{n-m} \max\{\|x(m)\|, \|x(m+1)\|\}$$

for any solution $x(n, m, x(m), x(m+1))$ of (3.1), $n, m \in \mathbb{N}(n_0), n \geq m$.

Assume that equation (3.1) is strangeness-free. Then, there exists a sequence of unitary matrices $\{W_n\}$ transforming (3.1) to

$$x(n+2) = -\hat{A}_n^{-1} \hat{B}_n x(n+1) - \hat{A}_n^{-1} \hat{C}_n x(n), \quad \forall n \geq n_0, \quad (3.2)$$

with the consistent initial condition

$$\begin{aligned} B_{n_0}^{(2)} x(n_0 + 1) + C_{n_0}^{(2)} x(n_0) &= 0, \\ C_{n_0}^{(3)} x(n_0) &= 0, \\ C_{n_0+1}^{(3)} x(n_0 + 1) &= 0. \end{aligned} \quad (3.3)$$

ASSUMPTION 3.2. $\sup_{n \in \mathbb{N}(n_0)} \|\hat{A}_n^{-1} \hat{B}_n\| < \infty$ and $\sup_{n \in \mathbb{N}(n_0)} \|\hat{A}_n^{-1} \hat{C}_n\| < \infty$.

PROPOSITION 3.3. *Suppose that Assumption 3.2 holds, then equation (3.1) is exponentially stable if and only if equation (3.2) is exponentially stable.*

Proof. It is easy to see that if equation (3.2) is exponentially stable then equation (3.1) is exponentially stable. We will prove that if equation (3.1) is exponentially

stable then equation (3.2) is exponentially stable. Since (3.1) is exponentially stable, there exist constants $K_1 > 0$ and $0 < \omega < 1$ such that

$$\|x(n, m, x(m), x(m+1))\| \leq K_1 \omega^{n-m} \max\{\|x(m)\|, \|x(m+1)\|\}, \quad \forall n \geq m \in \mathbb{N}(n_0),$$

where $x(n, m, x(m), x(m+1))$ is an arbitrary solution of (3.1). Let $y(n, m, y(m), y(m+1))$ be an arbitrary solution of (3.2). Then $y(m+2), y(m+3)$ satisfy the consistent initial condition (3.3) with $n_0 = m+1$. It implies that $y(n, m+2, y(m+2), y(m+3))$ is a solution of (3.1) with $n_0 = m+1$. Thus, for all $n \geq m+2 \in \mathbb{N}(n_0)$,

$$\begin{aligned} \|y(n, m, y(m), y(m+1))\| &= \|y(n, m+2, y(m+2), y(m+3))\| \\ &\leq K_1 \omega^{n-m-2} \max\{\|y(m+2)\|, \|y(m+3)\|\}. \end{aligned} \quad (3.4)$$

On the other hand

$$\begin{aligned} \|y(m+2)\| &= \|\hat{A}_m^{-1} \hat{B}_m y(m+1) - \hat{A}_m^{-1} \hat{C}_m y(m)\| \\ &\leq \|\hat{A}_m^{-1} \hat{B}_m\| \|y(m+1)\| + \|\hat{A}_m^{-1} \hat{C}_m\| \|y(m)\| \\ &\leq 2K_2 \max\{\|y(m)\|, \|y(m+1)\|\}, \end{aligned}$$

where

$$K_2 = \max\left\{ \sup_{n \in \mathbb{N}(n_0)} \|\hat{A}_n^{-1} \hat{B}_n\|, \sup_{n \in \mathbb{N}(n_0)} \|\hat{A}_n^{-1} \hat{C}_n\| \right\}.$$

Moreover

$$\begin{aligned} \|y(m+3)\| &\leq \|\hat{A}_{m+1}^{-1} \hat{B}_{m+1} y(m+2) + \hat{A}_{m+1}^{-1} \hat{C}_{m+1} y(m+1)\| \\ &\leq 2K_2^2 \max\{\|y(m)\|, \|y(m+1)\|\} + K_2 \|y(m+1)\|, \\ &\leq (2K_2^2 + K_2) \max\{\|y(m)\|, \|y(m+1)\|\}. \end{aligned}$$

Thus,

$$\max\{\|y(m+2)\|, \|y(m+3)\|\} \leq \max\{2K_2, 2K_2^2 + K_2\} \max\{\|y(m)\|, \|y(m+1)\|\}.$$

Combining this estimate with inequality (3.4), we obtain

$$\begin{aligned} \|y(n, m, y(m), y(m+1))\| &\leq \frac{K_1 \max\{2K_2, 2K_2^2 + K_2\}}{\omega^2} \omega^{n-m} \max\{\|y(m)\|, \|y(m+1)\|\} \\ &= K \omega^{n-m} \max\{\|y(m)\|, \|y(m+1)\|\}, \end{aligned}$$

where $K = \frac{K_1 \max\{2K_2, 2K_2^2 + K_2\}}{\omega^2}$. The proof is complete. \square

3.2. A criterion for exponential stability. In this subsection, we derive a criterion for exponential stability of implicit difference equation (3.1) by the comparison principle.

PROPOSITION 3.4. *Assume that there exist constant matrices B, C such that $|\hat{A}_n^{-1} \hat{B}_n| \leq B, |\hat{A}_n^{-1} \hat{C}_n| \leq C$ for all $n \geq n_0$ and $\{s : \det(sB + C - s^2 I) = 0\} \subset B(0, 1)$. Then, equation (3.1) is exponentially stable.*

Proof. Let $x(n, m, x(m), x(m+1))$ be an arbitrary solution of equation (3.2) and $y(n)$ be a solution of equation

$$y(n+2) = By(n+1) + Cy(n), \quad (3.5)$$

with the initial condition $y(m) = |x(m)|$ and $y(m+1) = |x(m+1)|$. By induction, we will prove that

$$|x(n)| \leq y(n), \forall n \geq m. \quad (3.6)$$

Indeed, (3.6) is true for $n = m$ and $n = m+1$. Assume that (3.6) is true for all $n \leq k$. We have

$$\begin{aligned} |x(k+1)| &= |-\hat{A}_k^{-1}\hat{B}_k x(k) - \hat{A}_{k-1}^{-1}\hat{C}_{k-1}x(k-1)| \\ &\leq |-\hat{A}_k^{-1}\hat{B}_k||x(k)| + |\hat{A}_{k-1}^{-1}\hat{C}_{k-1}||x(k-1)| \\ &\leq By(k) + Cy(k-1) = y(k+1). \end{aligned}$$

Hence, (3.6) is proved. Since $\{s : \det(sB + C - s^2I) = 0\} \subset B(0, 1)$, equation (3.5) is exponentially stable, or equivalently, there exist $K > 0, 0 < \omega < 1$ such that

$$\|y(n)\| \leq K\omega^{n-m} \max\{\|y(m)\|, \|y(m+1)\|\}.$$

Since norms in finite-dimensional vector spaces are equivalent, without loss of generality, we assume that vector spaces are endowed with equivalent p -norms. Then $\|v\| = \|v\|$ and we get

$$\begin{aligned} \|x(n)\| &= \|x(n)\| \leq \|y(n)\| \\ &\leq K\omega^{n-m} \max\{\|y(m)\|, \|y(m+1)\|\} \\ &\leq K\omega^{n-m} \max\{\|x(m)\|, \|x(m+1)\|\} \\ &\leq K\omega^{n-m} \max\{\|x(m)\|, \|x(m+1)\|\}. \end{aligned}$$

It means that (3.2) is exponentially stable and by Proposition 3.3, so is equation (3.1). The proof is complete. \square

COROLLARY 3.5. *If $|\hat{A}_n^{-1}\hat{B}_n| \leq B, |\hat{A}_n^{-1}\hat{C}_n| \leq C$, for all $n \geq n_0$, where B, C are positive matrices such that $\rho(B + C) < 1$, then equation (3.1) is exponentially stable.*

Proof. We will prove that if $\rho(B + C) < 1$ then $\{s : \det(sB + C - s^2I) = 0\} \subset B(0, 1)$. Indeed, assume inverse that there exists $s_0 : |s_0| \geq 1$ such that $\det(s_0B + C - s_0^2I) = 0$. It follows that $\det(B + C/s_0 - s_0I) = 0$ and $\rho(B + C/s_0) \geq 1$. On the other hand, since $s_0 \geq 1$,

$$|B + C/s_0| \leq |B| + |C/s_0| \leq B + C.$$

This implies that $\rho(B + C) \geq \rho(B + C/s_0) \geq 1$ (contradictory). Thus, $\{s : \det(sB + C - s^2I) = 0\} \subset B(0, 1)$ and by Proposition 3.4, the proof is complete. \square

Moreover, we can prove

PROPOSITION 3.6. *Assume that there exists matrix D such that $|\hat{A}_n^{-1}\hat{B}_n| + |\hat{A}_n^{-1}\hat{C}_n| \leq D$ for all $n \geq n_0$ and $\rho(D) < 1$. Then, equation (3.1) is exponentially stable.*

Proof. Let $x(n, m, x(m), x(m+1))$ be an arbitrary solution of equation (3.1). Since $D \geq 0$ and $\rho(D) < 1$, by a direct corollary of Perron-Frobenius Theorem there exists a vector $v \gg 0$ such that $Dv \ll v$. By continuity, $Dv \ll \omega^2 v$ for some $\omega \in (0, 1)$. Since $v \gg 0, \omega > 0$, there exists $K > 0$ such that $K\omega\|v\| > 1$. It's easy to see that

$$\|x(m)\| = \|x(m)\| \leq K\|v\| \max\{\|x(m)\|, \|x(m+1)\|\}$$

and

$$\|x(m+1)\| = \|x(m+1)\| \leq K\omega\|v\| \max\{\|x(m)\|, \|x(m+1)\|\}.$$

From monotonic property of the norm, we imply that

$$|x(m)| \leq \bar{K}v, |x(m+1)| \leq \bar{K}\omega v, \text{ where } \bar{K} = K \max\{\|x(m)\|, \|x(m+1)\|\}.$$

By induction, we will prove that $|x(n)| \leq \bar{K}\omega^{n-m}v$ for all $n \geq m$. Indeed, this is true for $n = m, n = m+1$. Assume that this is true for all $m \leq n \leq k$. We have

$$\begin{aligned} |x(k+1)| &= |-\hat{A}_k^{-1}\hat{B}_k x(k) - \hat{A}_{k-1}^{-1}\hat{C}_{k-1}x(k-1)| \\ &\leq |\hat{A}_k^{-1}\hat{B}_k||x(k)| + |\hat{A}_{k-1}^{-1}\hat{C}_{k-1}||x(k-1)| \\ &\leq |\hat{A}_k^{-1}\hat{B}_k|\bar{K}\omega^k v + |\hat{A}_{k-1}^{-1}\hat{C}_{k-1}||x(k-1)|\bar{K}\omega^{k-1}v \\ &\leq \bar{K}\omega^{k-1}(|\hat{A}_k^{-1}\hat{B}_k| + |\hat{A}_{k-1}^{-1}\hat{C}_{k-1}|)v \\ &\leq \bar{K}\omega^{k-1}Dv = \bar{K}\omega^{k-1}\rho(D)v = \bar{K}\omega^{k+1}v. \end{aligned}$$

Thus, we get $\|x(n)\| = \|x(n)\| \leq \bar{K}\|v\|\omega^{n-m} = K\|v\|\omega^{n-m} \max\{\|x(m)\|, \|x(m+1)\|\}$. It means that (3.2) is exponentially stable and by Proposition 3.3, so is equation (3.1). \square

Now, we propose a criterion for exponential stability of the linear time-varying SDE (3.1) by comparing with a linear time-invariant SDE of the form

$$Ax(n+2) + Bx(n+1) + Cx(n) = 0, \forall n \geq n_0, \quad (3.7)$$

We assume that SDE (3.7) is already transformed into the positive difference equation

$$x(n+2) = -\hat{A}^{-1}\hat{B}x(n+1) - \hat{A}^{-1}\hat{C}x(n), \forall n \geq n_0, \quad (3.8)$$

by the nonsingular matrix $W \in \mathbb{C}^{d,d}$, where

$$\begin{aligned} W^{-1}A &= \begin{bmatrix} A^{(1)} \\ 0 \\ 0 \end{bmatrix}, W^{-1}B = \begin{bmatrix} B^{(1)} \\ B^{(2)} \\ 0 \end{bmatrix}, W^{-1}C = \begin{bmatrix} C^{(1)} \\ C^{(2)} \\ C^{(3)} \end{bmatrix}, \\ \hat{A} &= \begin{bmatrix} A^{(1)} \\ B^{(2)} \\ C^{(3)} \end{bmatrix}, \hat{B} = \begin{bmatrix} B^{(1)} \\ C^{(2)} \\ 0 \end{bmatrix}, \hat{C} = \begin{bmatrix} C^{(1)} \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

with $A^{(1)}, B^{(1)}, C^{(1)} \in \mathbb{C}^{d_1,d}$; $B^{(2)}, C^{(2)} \in \mathbb{C}^{d_2,d}$; $C^{(3)} \in \mathbb{C}^{d_3,d}$; $d_1 + d_2 + d_3 = d$.

THEOREM 3.7. *Assume that there exists a strangeness-free constant coefficient equation of the form (3.7) such that*

$$|-\hat{A}_n^{-1}\hat{B}_n| \leq -\hat{A}^{-1}\hat{B}, \quad |-\hat{A}_n^{-1}\hat{C}_n| \leq -\hat{A}^{-1}\hat{C}, \quad \forall n \geq n_0. \quad (3.9)$$

If SDE (3.7) is exponentially stable, then SDE (3.1) is exponentially stable, too.

Proof. By [25], equation (3.7) is exponentially stable if and only if $\{s : \det(s^2A + sB + C) = 0\} \subset B(0,1)$. On the other hand, we have

$$s^2A + sB + C = W \begin{bmatrix} s^2A^{(1)} + sB^{(1)} + C^{(1)} \\ sB^{(2)} + C^{(2)} \\ C^{(3)} \end{bmatrix}$$

and

$$s^2 I + \hat{A}^{-1} \hat{B} s + \hat{A}^{-1} \hat{C} = \hat{A}^{-1} \begin{bmatrix} s^2 A^{(1)} + s B^{(1)} + C^{(1)} \\ s^2 B^{(2)} + s C^{(2)} \\ s^2 C^{(3)} \end{bmatrix}.$$

Thus,

$$\{s : \det(s^2 I + s \hat{A}^{-1} \hat{B} + \hat{A}^{-1} \hat{C}) = 0\} = \{s : \det(s^2 A + s B + C) = 0\} \cup \{0\}.$$

This implies that $\{s : \det(s^2 I + s \hat{A}^{-1} \hat{B} + \hat{A}^{-1} \hat{C}) = 0\} \subset B(0, 1)$. By Proposition 3.4, equation (3.2) is exponentially stable. Thus, by Proposition 3.3, equation (3.1) is exponentially stable. \square

REMARK 3.8. By Proposition 2.3, assumption (3.9) is not depend on the choice of matrices W_n, W .

Now we give an example to illustrate Theorem 3.7.

EXAMPLE 3.9. Consider the system

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = 0, \forall n \geq 0, x(n) \in \mathbb{C}^3, \quad (3.10)$$

where

$$\begin{aligned} A_n &= \begin{bmatrix} -(3n+1)\cos n & 0 & -(n+1)\cos n \\ (3n+1)\cos n & 0 & (n+1)\cos n \\ 0 & 0 & 0 \end{bmatrix}, \\ B_n &= \begin{bmatrix} -n\cos n & n\cos n + (2n-1)\sin n & -n\cos n \\ n\sin n & -n\sin n + (2n-1)\cos n & n\sin n \\ 0 & 0 & 0 \end{bmatrix}, \\ C_n &= \begin{bmatrix} -(n-1)\cos n & -n\cos n + (n-2)\sin n & n\cos n + (n-1)\sin n \\ (n-1)\sin n & n\sin n + (n-2)\cos n & -n\sin n + (n-1)\cos n \\ 0 & 0 & n-1 \end{bmatrix}. \end{aligned}$$

Choose unitary matrix $W_n = \begin{bmatrix} -\cos n & \sin n & 0 \\ \sin n & \cos n & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then, we can caculate $\hat{A}_n, \hat{B}_n, \hat{C}_n, \hat{A}_n^{-1} \hat{B}_n$ and $\hat{A}_n^{-1} \hat{C}_n$ as follows

$$W_n^{-1} A_n = \begin{bmatrix} 3n+1 & 0 & n+1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, W_n^{-1} B_n = \begin{bmatrix} n & -n & n \\ 0 & 2n-1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$W_n^{-1} C_n = \begin{bmatrix} n-1 & n & -n \\ 0 & n-2 & n-1 \\ 0 & 0 & n-1 \end{bmatrix}.$$

Hence

$$\hat{A}_n = \begin{bmatrix} 3n+1 & 0 & n+1 \\ 0 & 2n+1 & 0 \\ 0 & 0 & n+1 \end{bmatrix}, \hat{B}_n = \begin{bmatrix} n & -n & n \\ 0 & n-1 & n \\ 0 & 0 & 0 \end{bmatrix}, \hat{C}_n = \begin{bmatrix} n-1 & n & -n \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widehat{A}_n^{-1}\widehat{B}_n = \begin{bmatrix} \frac{n}{3n+1} & \frac{-n}{3n+1} & \frac{n}{3n+1} \\ 0 & \frac{n-1}{2n+1} & \frac{n}{2n+1} \\ 0 & 0 & 0 \end{bmatrix}, \widehat{A}_n^{-1}\widehat{C}_n = \begin{bmatrix} \frac{n-1}{3n+1} & \frac{n}{3n+1} & \frac{-n}{3n+1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Consider the constant coefficients system

$$Ax(n+2) + Bx(n+1) + Cx(n) = 0, \quad (3.11)$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & -1 \\ -1 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & -1 & -2 \\ -1 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is easy to see that $\{s : \det(s^2A + sB + C) = 0\} = \left\{\frac{1}{6} \pm \frac{i\sqrt{11}}{6}; \frac{1}{2}\right\} \subset B(0, 1)$.

Therefore, this equation is exponentially stable. Choosing $W^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we

have

$$W^{-1}A = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, W^{-1}B = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}, W^{-1}C = \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$\widehat{A} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \widehat{B} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \widehat{C} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$-\widehat{A}^{-1}\widehat{B} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{0}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & 0 \end{bmatrix}, -\widehat{A}^{-1}\widehat{C} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{0}{3} & \frac{0}{3} & \frac{0}{3} \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $|\widehat{A}_n^{-1}\widehat{B}_n| \leq -\widehat{A}^{-1}\widehat{B}$ and $|\widehat{A}_n^{-1}\widehat{C}_n| \leq -\widehat{A}^{-1}\widehat{C}$. Thus, by Theorem 3.7, we imply that system (3.10) is exponentially stable.

3.3. Bohl-Perron theorem. In this subsection, we will present a Bohl-Perron type theorem for SDE (1.1). For explicit difference equations of first order, the following classical theorem is well known, e.g., see [6] and the references therein.

THEOREM 3.10. *Consider the system*

$$x(n+1) = \mathcal{A}_n x(n) + f_n, \quad \forall n \geq n_0, \quad (3.12)$$

and

$$x(n+1) = \mathcal{A}_n x(n), \quad \forall n \geq n_0, \quad (3.13)$$

where $x(n) \in \mathbb{C}^{d'}$, $\mathcal{A}_n \in \mathbb{C}^{d',d'}$. Suppose that $\sup_{n \geq n_0} \|\mathcal{A}_n\| < \infty$, then for every $\{f_n\} \in \ell_p$, ($1 \leq p \leq \infty$), the solution $x(n)$ of (3.12) associated with $\{f_n\}$ belong to ℓ_p if and only if (3.13) is exponentially stable.

A similar result also holds for explicit difference equations of second order. This is obtained as a special case of the main result in [7]. Here, we give a simple proof based on Theorem 3.10.

THEOREM 3.11. *Consider system*

$$y(n+2) = C_n y(n+1) + D_n y(n) + q_n, \quad \forall n \geq n_0, \quad (3.14)$$

and

$$y(n+2) = C_n y(n+1) + D_n y(n), \quad \forall n \geq n_0, \quad (3.15)$$

where $y(n), q_n \in \mathbb{C}^d$, $C_n, D_n \in \mathbb{C}^{d,d}$. Suppose that $\sup_{n \geq n_0} \|C_n\| < \infty$, $\sup_{n \geq n_0} \|D_n\| < \infty$. Then, for every $\{q_n\}_{n \geq n_0} \in \ell_p$ solution $y(n)$ of (3.14) associated with $\{q_n\}_{n \geq n_0}$ belongs to ℓ_p if and only if (3.15) is exponentially stable.

Proof. Firstly, we transform system (3.14) to first order system as follows. Put $z(n) = \begin{bmatrix} y(n+1) \\ y(n) \end{bmatrix}$ for all $n \geq n_0$, we have new system

$$z(n+1) = \begin{bmatrix} C_n & D_n \\ I & 0 \end{bmatrix} z(n) + \begin{bmatrix} q_n \\ 0 \end{bmatrix}, \quad \forall n \geq n_0. \quad (3.16)$$

It is not difficult to see that if $\begin{bmatrix} z^{(1)}(n) \\ z^{(2)}(n) \end{bmatrix}$ is a solution of

$$\begin{bmatrix} z^{(1)}(n+1) \\ z^{(2)}(n+1) \end{bmatrix} = \begin{bmatrix} C_n & D_n \\ I & 0 \end{bmatrix} \begin{bmatrix} z^{(1)}(n) \\ z^{(2)}(n) \end{bmatrix} + \begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \end{bmatrix}, \quad \forall n \geq n_0, \quad (3.17)$$

then $\begin{bmatrix} \bar{z}^{(1)}(n) \\ \bar{z}^{(2)}(n) \end{bmatrix} = \begin{bmatrix} z^{(1)}(n) + g_n^{(2)} \\ z^{(2)}(n) \end{bmatrix}$ is a solution of

$$\bar{z}(n+1) = \begin{bmatrix} C_n & D_n \\ I & 0 \end{bmatrix} \bar{z}(n) + \begin{bmatrix} \bar{q}_n \\ 0 \end{bmatrix}, \quad \forall n \geq n_0,$$

where $\bar{q}_n = g_{n+1}^{(2)} - C_n g_n^{(2)} - g_n^{(1)}$. Moreover, if $\begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \end{bmatrix} \in \ell_p$ then $\begin{bmatrix} \bar{q}_n \\ 0 \end{bmatrix} \in \ell_p$. Thus, following assertions are equivalent:

- (i) For every $\{q_n\}_{n \geq n_0} \in \ell_p$ solutions of (3.14) associated with $\{q_n\}_{n \geq n_0}$ belong to ℓ_p .
- (ii) For every $\left\{ \begin{bmatrix} q_n \\ 0 \end{bmatrix} \right\}_{n \geq n_0} \in \ell_p$ solutions of (3.16) associated with $\left\{ \begin{bmatrix} q_n \\ 0 \end{bmatrix} \right\}_{n \geq n_0}$ belong to ℓ_p .
- (iii) For every $\left\{ \begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \end{bmatrix} \right\}_{n \geq n_0} \in \ell_p$ solutions of (3.16) associated with $\left\{ \begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \end{bmatrix} \right\}_{n \geq n_0}$ belong to ℓ_p .

On the other hand, since $\sup_{n \geq n_0} \|C_n\| < \infty$, $\sup_{n \geq n_0} \|D_n\| < \infty$, it implies that $\sup_{n \geq n_0} \|\mathcal{A}_n\| < \infty$, where $\mathcal{A}_n = \begin{bmatrix} C_n & D_n \\ I & 0 \end{bmatrix}$. By Theorem 3.10, assertion (iii) is equivalent to that (3.15) is exponentially stable. \square

Now let us define

$$\widehat{\ell}_p = \{\{f_n\}_{n \geq n_0} : \{\widehat{A}_n^{-1} \widehat{f}_n\}_{n \geq n_0} \in \ell_p\}.$$

Note that

- the space $\widehat{\ell}_p$ is unique for the arbitrary sequence of unitary matrices $\{W_n\}$ which transforms the coefficients of equation (1.1) to the form (2.1);
- for every $\{g_n\}_{n \geq n_0} \in \ell_p$ there exists $\{f_n\}_{n \geq n_0} \in \widehat{L}_{n_0}^p$ such that $\{g_n\}_{n \geq n_0} = \{\widehat{A}_n^{-1} \widehat{f}_n\}_{n \geq n_0}$.

PROPOSITION 3.12. *Consider equation*

$$x(n+2) = -\widehat{A}_n^{-1} \widehat{B}_n x(n+1) - \widehat{A}_n^{-1} \widehat{C}_n x(n) + g_n, \forall n \geq n_0, \quad (3.18)$$

where $\{g_n\}_{n \geq n_0} \in \ell_p$. Suppose that $y(n)$ is a solution of (3.18) with $y(n_0), y(n_0+1)$ are an arbitrary initial values. Then, there exists $\{f_n\}_{n \geq n_0} \in \widehat{L}_{n_0}^p$ such that $y(n)$ is a solution of (1.1) associated with $\{f_n\}_{n \geq n_0}$.

Proof. We will construct $\{f_n\}_{n \geq n_0}$ such that $g_n = \widehat{A}_n^{-1} \widehat{f}_n, \forall n \geq n_0$ and

$$\begin{cases} B_{n_0}^{(2)} y(n_0+1) + C_{n_0}^{(2)} y(n_0) &= f_{n_0}^{(2)} \\ C_{n_0}^{(3)} y(n_0) &= f_{n_0}^{(3)} \\ C_{n_0+1}^{(3)} y(n_0+1) &= f_{n_0+1}^{(3)}. \end{cases}$$

Put $\widehat{A}_n g_n = \begin{bmatrix} g_n^{(1)} \\ g_n^{(2)} \\ g_n^{(3)} \end{bmatrix}$ for all $n \geq n_0$. We choose

$$\begin{aligned} f_n^{(1)} &= g_n^{(1)}, \forall n \geq n_0, \\ f_n^{(2)} &= g_{n-1}^{(2)}, \forall n \geq n_0+1, \\ f_n^{(3)} &= g_{n-2}^{(3)}, \forall n \geq n_0+2, \end{aligned}$$

and

$$\begin{aligned} f_{n_0}^{(2)} &= B_{n_0}^{(2)} y(n_0+1) + C_{n_0}^{(2)} y(n_0), \\ f_{n_0}^{(3)} &= C_{n_0}^{(3)} y(n_0), \\ f_{n_0+1}^{(3)} &= C_{n_0+1}^{(3)} y(n_0+1). \end{aligned}$$

With $f_n^{(i)}$ defined for all $n \geq n_0, i = 1, 2, 3$, we define f_n as follows

$$f_n = W_n \begin{bmatrix} f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \end{bmatrix}, \forall n \geq n_0.$$

It is easy to see that $\{\widehat{A}_n^{-1}\widehat{f}_n\}_{n \geq n_0} = \{g_n\}_{n \geq n_0} \in \ell_p$, hence $\{f_n\}_{n \geq n_0} \in \widehat{\ell}_p$. Moreover, $y(n_0), y(n_0 + 1)$ are consistent initial values corresponding to $\{f_n\}_{n \geq n_0}$. Thus, $y(n), n \geq n_0$ of (3.18) is also a solution of (1.1) associated with $\{f_n\}_{n \geq n_0}$. \square

THEOREM 3.13. (*Bohl - Perron type theorem*) *Let $1 \leq p \leq \infty$. Suppose that Assumption 3.2 holds. Then for every $\{f_n\}_{n \geq n_0} \in \widehat{\ell}_p$ solutions of (1.1) associated with $\{f_n\}_{n \geq n_0}$ belong to ℓ_p if and only if (3.1) is exponentially stable.*

Proof. Applying Proposition 3.12 we have the claim that for every $\{f_n\}_{n \geq n_0} \in \widehat{\ell}_p$, the solution of (1.1) associated with $\{f_n\}_{n \geq n_0}$ belong to ℓ_p if and only if for every $\{g_n\}_{n \geq n_0} \in \ell_p$, the solution of (3.18) associated with $\{g_n\}_{n \geq n_0}$ belong to ℓ_p . Moreover, according to Proposition 3.3 we have equation (3.1) is exponentially stable if and only if equation (3.2) is exponentially stable. Applying Theorem 3.11 we complete the proof of Theorem 3.13. \square

4. Robust stability. It is already known for perturbed differential-algebraic equations (DAEs) [10], see also [8, 11, 13, 14], that it is necessary to restrict the perturbation structure in order to get a meaningful problem of robust stability, since under infinitesimally small perturbations the solvability and/or the stability may fail when the index changes. Following the approach used for DAEs in [10, 14], we therefore introduce the notion of *allowable* perturbations.

Suppose that equation (3.1) is exponentially stable and consider a perturbed system

$$\widetilde{A}_n x(n+2) + \widetilde{B}_n x(n+1) + \widetilde{C}_n x(n) = 0, \forall n \geq n_0, \quad (4.1)$$

with

$$\begin{aligned} \widetilde{A}_n &= A_n + D_{n,1} \Delta_{n,1} E_n, \\ \widetilde{B}_n &= B_n + D_{n,2} \Delta_{n,2} E_n, \\ \widetilde{C}_n &= C_n + D_{n,3} \Delta_{n,3} E_n, \end{aligned}$$

where $\Delta_{n,i} \in \mathbb{C}^{l_i, q}$, $i = 1, 2, 3$ are perturbations and $D_{n,i} \in \mathbb{C}^{d, l_i}$, $E_n \in \mathbb{C}^{q, d}$, $i = 1, 2, 3$ are matrices that restrict the structure of the perturbations.

DEFINITION 4.1. *Consider a strangeness-free equation (3.1) and let $\{W_n\} \subset \mathbb{C}^{d, d}$ be such that (2.1) holds. A structured perturbation as in (4.1) is called allowable if (4.1) is still strangeness-free with the same triple (d_1, d_2, d_3) , i.e., there exists a sequence of unitary matrices $\{\widetilde{W}_n\} \subset \mathbb{C}^{d, d}$ such that*

$$\begin{aligned} \widetilde{W}_n^{-1}(A_n + D_{n,1} \Delta_{n,1} E_n) &= \begin{bmatrix} \widetilde{A}_n^{(1)} \\ 0 \\ 0 \end{bmatrix}, \quad \widetilde{W}_n^{-1}(B_n + D_{n,2} \Delta_{n,2} E_n) = \begin{bmatrix} \widetilde{B}_n^{(1)} \\ \widetilde{B}_n^{(2)} \\ 0 \end{bmatrix}, \\ \widetilde{W}_n^{-1}(C_n + D_{n,3} \Delta_{n,3} E_n) &= \begin{bmatrix} \widetilde{C}_n^{(1)} \\ \widetilde{C}_n^{(2)} \\ \widetilde{C}_n^{(3)} \end{bmatrix}, \end{aligned} \quad (4.2)$$

where $\widetilde{A}_n^{(1)}, \widetilde{B}_n^{(1)}, \widetilde{C}_n^{(1)} \in \mathbb{C}^{d_1, d}$, $\widetilde{B}_n^{(2)}, \widetilde{C}_n^{(2)} \in \mathbb{C}^{d_2, d}$, $\widetilde{C}_n^{(3)} \in \mathbb{C}^{d_3, d}$, are such that

$$\widehat{\widetilde{A}}_n = \begin{bmatrix} \widetilde{A}_n^{(1)} \\ \widetilde{B}_{n+1}^{(2)} \\ \widetilde{C}_{n+2}^{(3)} \end{bmatrix}$$

is invertible.

Suppose that the matrices $D_{n,i}$, $i = 1, 2, 3$, which restrict the perturbation structure, have the form

$$W_n^{-1}D_{n,1} = \begin{bmatrix} D_{n,1}^{(1)} \\ D_{n,1}^{(2)} \\ D_{n,1}^{(3)} \end{bmatrix}, W_n^{-1}D_{n,2} = \begin{bmatrix} D_{n,2}^{(1)} \\ D_{n,2}^{(2)} \\ D_{n,2}^{(3)} \end{bmatrix}, W_n^{-1}D_{n,3} = \begin{bmatrix} D_{n,3}^{(1)} \\ D_{n,3}^{(2)} \\ D_{n,3}^{(3)} \end{bmatrix}, \quad (4.3)$$

where $D_{n,i}^{(j)} \in \mathbb{C}^{d_j, l_i}$, $i, j = 1, 2, 3$. According to [10, Lemma 3.3], if the structured perturbation is allowable, then $D_{n,i}^{(j)}\Delta_{n,i}E_n = 0$ for all $n \geq n_0$ and $1 \leq i < j \leq 3$. This can be achieved by requiring that

$$D_i^{(j)} = 0, \quad 1 \leq i < j \leq 3. \quad (4.4)$$

Note that by Remark 2.2, condition (4.4) is invariant with respect to the choice of the transformation matrix W . Furthermore, it is easy to see that for structured perturbations satisfying (4.4), if the perturbation Δ is sufficiently small, then the strangeness-free property is preserved with the same sizes of the blocks.

Now, we have

$$\begin{aligned} W_n^{-1}\tilde{A}_n &= \begin{bmatrix} A_n^{(1)} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} D_{n,1}^{(1)} \\ 0 \\ 0 \end{bmatrix} \Delta_{n,1}E_n = \begin{bmatrix} A_n^{(1)} + D_{n,1}^{(1)}\Delta_{n,1}E_n \\ 0 \\ 0 \end{bmatrix} \\ W_n^{-1}\tilde{B}_n &= \begin{bmatrix} B_n^{(1)} \\ B_n^{(2)} \\ 0 \end{bmatrix} + \begin{bmatrix} D_{n,2}^{(1)} \\ D_{n,2}^{(2)} \\ 0 \end{bmatrix} \Delta_{n,2}E_n = \begin{bmatrix} B_n^{(1)} + D_{n,2}^{(1)}\Delta_{n,2}E_n \\ B_n^{(2)} + D_{n,2}^{(2)}\Delta_{n,2}E_n \\ 0 \end{bmatrix} \\ W_n^{-1}\tilde{C}_n &= \begin{bmatrix} C_n^{(1)} \\ C_n^{(2)} \\ C_n^{(3)} \end{bmatrix} + \begin{bmatrix} D_{n,3}^{(1)} \\ D_{n,3}^{(2)} \\ D_{n,3}^{(3)} \end{bmatrix} \Delta_{n,3}E_n = \begin{bmatrix} C_n^{(1)} + D_{n,3}^{(1)}\Delta_{n,3}E_n \\ C_n^{(2)} + D_{n,3}^{(2)}\Delta_{n,3}E_n \\ C_n^{(3)} + D_{n,3}^{(3)}\Delta_{n,3}E_n \end{bmatrix} \end{aligned}$$

Thus

$$\begin{aligned} \hat{\tilde{A}}_n &= \begin{bmatrix} A_n^{(1)} + D_{n,1}^{(1)}\Delta_{n,1}E_n \\ B_{n+1}^{(2)} + D_{n+1,2}^{(2)}\Delta_{n+1,2}E_{n+1} \\ C_{n+2}^{(3)} + D_{n+2,3}^{(3)}\Delta_{n+2,3}E_{n+2} \end{bmatrix} \\ &= \begin{bmatrix} A_n^{(1)} \\ B_{n+1}^{(2)} \\ C_{n+2}^{(3)} \end{bmatrix} + \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix} \\ \hat{\tilde{B}}_n &= \begin{bmatrix} B_n^{(1)} + D_{n,2}^{(1)}\Delta_{n,2}E_n \\ C_{n+1}^{(2)} + D_{n+1,3}^{(2)}\Delta_{n+1,3}E_{n+1} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} B_n^{(1)} \\ C_{n+1}^{(2)} \\ 0 \end{bmatrix} + \begin{bmatrix} D_{n,2}^{(1)} & 0 & 0 \\ 0 & D_{n+1,3}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,2}E_n \\ \Delta_{n+1,3}E_{n+1} \\ 0 \end{bmatrix} \\ \hat{\tilde{C}}_n &= \begin{bmatrix} C_n^{(1)} + D_{n,3}^{(1)}\Delta_{n,3}E_n \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} C_n^{(1)} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} D_{n,3}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,3}E_n \\ 0 \\ 0 \end{bmatrix} \end{aligned} \quad (4.5)$$

Motivated by the stability analysis in Section 3, for studying the robustness of exponential stability of (3.1), we assume that there exists a strangeness-free time-invariant implicit difference equation (3.7) satisfying assumption (3.9) in Theorem 3.7. In transformed equations (3.2), (3.8), let

$$\widehat{A}_n^{-1} = \begin{bmatrix} M_{n,1} & M_{n,2} & M_{n,3} \end{bmatrix}, \quad \widehat{A}^{-1} = \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix},$$

where $M_{n,1}, M_1 \in \mathbb{C}^{d,d_1}, M_{n,2}, M_2 \in \mathbb{C}^{d,d_2}, M_{n,3}, M_3 \in \mathbb{C}^{d,d_3}$. Suppose that for all $n \geq n_0$, there exist constant matrices $D_j^{(i)} \in \mathbb{C}^{d_i, l_j}, E \in \mathbb{R}_+^{q,d}, \Delta_i \in \mathbb{R}_+^{l_i, q}$ such that

$$|M_{n,i} D_{n+i-1,j}^{(i)}| \leq M_i D_j^{(i)}, \quad |E_n| \leq E, \quad |\Delta_{n,i}| \leq \Delta_i, \quad \forall i, j : 1 \leq i \leq j \leq 3. \quad (4.6)$$

Let

$$D_1 = W \begin{bmatrix} D_1^{(1)} \\ 0 \\ 0 \end{bmatrix}, \quad D_2 = W \begin{bmatrix} D_2^{(1)} \\ D_2^{(2)} \\ 0 \end{bmatrix}, \quad D_3 = W \begin{bmatrix} D_3^{(1)} \\ D_3^{(2)} \\ D_3^{(3)} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix}$$

and consider the perturbed equation of (3.7)

$$\tilde{A}x(n+2) + \tilde{B}x(n+1) + \tilde{C}x(n) = 0, \quad \forall n \geq n_0, \quad (4.7)$$

where

$$\tilde{A} = A + D_1(-\Delta_1)E, \quad \tilde{B} = B + D_2(-\Delta_2)E, \quad \tilde{C} = C + D_3(-\Delta_3)E.$$

We need the following lemma, which is easily checked by direct calculation.

LEMMA 4.2. *Let N, F be constant matrices, $N \in \mathbb{C}^{k,l}, F \in \mathbb{C}^{l,k}$. If $I + NF$ is invertible then $I + FN$ is also invertible and*

$$(I + FN)^{-1} = I - F(I + NF)^{-1}N.$$

For $s \in \mathbb{C} : \det(s^2A + sB + C) \neq 0$, we define

$$H(s) = E(s^2A + sB + C)^{-1} \begin{bmatrix} s^2D_1 & sD_2 & D_3 \end{bmatrix}. \quad (4.8)$$

By Proposition 4.4 in [25], it is known that

$$\|H(\infty)\|^{-1} = \left(\lim_{s \rightarrow \infty} \|H(s)\| \right)^{-1} = \left\| E \widehat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \right\|^{-1}.$$

PROPOSITION 4.3. *Let equation (3.1) be strangeness-free and let there exist a strangeness-free constant coefficient equation (3.7) satisfying assumption (3.9). If the equation is subjected to structured perturbations satisfying (4.4) and (4.6) with*

$$\|\Delta\| < \|H(\infty)\|^{-1} = \left(\lim_{s \rightarrow \infty} \|H(s)\| \right)^{-1},$$

then the structured perturbation is allowable, i.e., the perturbed equation (4.1) is strangeness-free with the same block-sizes d_1, d_2, d_3 .

Proof. By (4.5), we have

$$\widehat{\widehat{A}}_n = \widehat{A}_n \left(I + \widehat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix} \right).$$

By setting

$$K_n = \widehat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix}. \quad (4.9)$$

we get $\widehat{\widehat{A}}_n = \widehat{A}_n(I + K_n)$. Moreover, let

$$\begin{aligned} P_n &= \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix} \widehat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix}, \\ P &= \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \widehat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} = \Delta H(\infty). \end{aligned} \quad (4.10)$$

Then, it is easy to see that

$$\begin{aligned} |P_n| &\leq \left\| \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix} \right\| \left\| \widehat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} \right\|, \\ &\leq \begin{bmatrix} \Delta_1 E \\ \Delta_2 E \\ \Delta_3 E \end{bmatrix} \left\| \begin{bmatrix} M_{n,1}D_{n,1}^{(1)} & M_{n,2}D_{n+1,2}^{(2)} & M_{n,3}D_{n+2,3}^{(3)} \end{bmatrix} \right\|, \\ &\leq \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \begin{bmatrix} M_1 D_1^{(1)} & M_2 D_2^{(2)} & M_3 D_3^{(3)} \end{bmatrix}, \\ &= \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \\ &= \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \widehat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} = P = \Delta H(\infty). \end{aligned}$$

Therefore,

$$\|P_n\| \leq \|P_n\| \leq \|P\| = \|\Delta H(\infty)\| \leq \|\Delta\| \|H(\infty)\| < \frac{1}{\|H(\infty)\|} \|H(\infty)\| = 1.$$

This implies that $I + P_n$ is invertible for all $n \geq n_0$. Applying Lemma 4.2 with

$$N = \begin{bmatrix} \Delta_{n,1}E_n \\ \Delta_{n+1,2}E_{n+1} \\ \Delta_{n+2,3}E_{n+2} \end{bmatrix} \text{ and } F = \widehat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix}, \text{ we obtain } I + K_n$$

is also invertible for all $n \geq n_0$. Thus, $\widehat{\widehat{A}}_n = \widehat{A}_n(I + K_n)$ is invertible, which means that the structured perturbation is allowable. \square

THEOREM 4.4. *Let equation (3.1) be strangeness-free and let there exist a strangeness-free constant coefficient equation (3.7) which is exponentially stable and satisfies assumption (3.9). Moreover, we suppose that the structured perturbations satisfy (4.4) and (4.6) with*

$$\|\Delta\| < \frac{1}{\max\{\|H(1)\|, \|H(\infty)\|\}}. \quad (4.11)$$

Then, the perturbed equation (4.1) is exponentially stable.

Proof. By assumption (4.11) and using Proposition 4.4 in [25] we imply that the perturbed equation (4.7) is strangeness-free. Let

$$K = \widehat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E.$$

In equation (4.7), we have

$$\begin{aligned} \widehat{\widehat{A}} &= \widehat{A} + \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \begin{bmatrix} -\Delta_1 \\ -\Delta_2 \\ -\Delta_3 \end{bmatrix} E = \widehat{A}(I - K), \\ \widehat{\widehat{B}} &= \widehat{B} + \begin{bmatrix} D_2^{(1)} & 0 & 0 \\ 0 & D_3^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\Delta_2 \\ -\Delta_3 \\ 0 \end{bmatrix} E, \\ \widehat{\widehat{C}} &= \widehat{C} + \begin{bmatrix} D_3^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\Delta_3 \\ 0 \\ 0 \end{bmatrix} E. \end{aligned}$$

Therefore,

$$\widehat{\widehat{A}}^{-1} \widehat{\widehat{B}} = (I - K)^{-1} \widehat{A}^{-1} \widehat{B}, \quad \widehat{\widehat{A}}^{-1} \widehat{\widehat{C}} = (I - K)^{-1} \widehat{A}^{-1} \widehat{C}.$$

Moreover, in the proof of Proposition (4.3), it has been shown that $\widehat{\widehat{A}}_n^{-1} = (I + K_n)^{-1} \widehat{A}_n^{-1}$ and hence

$$\widehat{\widehat{A}}_n^{-1} \widehat{\widehat{B}}_n = (I + K_n)^{-1} \widehat{A}_n^{-1} \widehat{B}_n, \quad \widehat{\widehat{A}}_n^{-1} \widehat{\widehat{C}}_n = (I + K_n)^{-1} \widehat{A}_n^{-1} \widehat{C}_n,$$

where K_n is given by (4.9). Now, by using Lemma 4.2, we get

$$\begin{aligned}
(I + K_n)^{-1} &= \left(I + \hat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_{n,1} E_n \\ \Delta_{n+1,2} E_{n+1} \\ \Delta_{n+2,3} E_{n+2} \end{bmatrix} \right)^{-1} \\
&= I - \hat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} (I + P_n)^{-1} \begin{bmatrix} \Delta_{n,1} E_n \\ \Delta_{n+1,2} E_{n+1} \\ \Delta_{n+2,3} E_{n+2} \end{bmatrix}, \\
(I - K)^{-1} &= \left(I - \hat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \right)^{-1} \\
&= I + \hat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} (I - P)^{-1} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E,
\end{aligned}$$

where P_n, P are given in (4.10). Since (4.11) holds, it follows from the proof of Proposition (4.3) that $|P_n| \leq P, \|P_n\| \leq \|P\| < 1$. Therefore, we have

$$|(I + P_n)^{-1}| = \left| \sum_{i=0}^{\infty} (-P_n)^i \right| \leq \sum_{i=0}^{\infty} |P_n|^i \leq \sum_{i=0}^{\infty} P^i = (I - P)^{-1}.$$

By this inequality and assumption (4.6), we get

$$\begin{aligned}
|(I + K_n)^{-1}| &= \left| I - \hat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} (I + P_n)^{-1} \begin{bmatrix} \Delta_{n,1} E_n \\ \Delta_{n+1,2} E_{n+1} \\ \Delta_{n+2,3} E_{n+2} \end{bmatrix} \right| \\
&\leq I + \left| \hat{A}_n^{-1} \begin{bmatrix} D_{n,1}^{(1)} & 0 & 0 \\ 0 & D_{n+1,2}^{(2)} & 0 \\ 0 & 0 & D_{n+2,3}^{(3)} \end{bmatrix} (I + P_n)^{-1} \begin{bmatrix} \Delta_{n,1} E_n \\ \Delta_{n+1,2} E_{n+1} \\ \Delta_{n+2,3} E_{n+2} \end{bmatrix} \right| \\
&\leq I + \left[M_{n,1} D_{n,1}^{(1)} \quad M_{n,2} D_{n+1,2}^{(2)} \quad M_{n,3} D_{n+2,3}^{(3)} \right] \|(I + P_n)^{-1}\| \begin{bmatrix} \Delta_{n,1} E_n \\ \Delta_{n+1,2} E_{n+1} \\ \Delta_{n+2,3} E_{n+2} \end{bmatrix} \\
&\leq I + \begin{bmatrix} M_1 D_1^{(1)} & M_2 D_2^{(2)} & M_3 D_3^{(3)} \end{bmatrix} (I - P)^{-1} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E \\
&= I + \hat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} (I - P)^{-1} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} E = (I - K)^{-1}.
\end{aligned} \tag{4.12}$$

On the other hand, by assumption (3.9), (4.6), we imply

$$\begin{aligned}
| -\hat{A}_n^{-1} \hat{\hat{B}}_n | &= \left| \hat{A}_n^{-1} \left(\hat{B}_n + \begin{bmatrix} D_{n,2}^{(1)} & 0 & 0 \\ 0 & D_{n+1,3}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,2} E_n \\ \Delta_{n+1,3} E_{n+1} \\ 0 \end{bmatrix} \right) \right| \\
&= \left| \hat{A}_n^{-1} \hat{B}_n + \begin{bmatrix} M_{n,1} D_{n,2}^{(1)} & M_{n,2} D_{n+1,3}^{(2)} & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,2} E_n \\ \Delta_{n+1,3} E_{n+1} \\ 0 \end{bmatrix} \right| \\
&\leq | \hat{A}_n^{-1} \hat{B}_n | + \left| \begin{bmatrix} M_{n,1} D_{n,2}^{(1)} & M_{n,2} D_{n+1,3}^{(2)} & 0 \end{bmatrix} \right| \left| \begin{bmatrix} \Delta_{n,2} E_n \\ \Delta_{n+1,3} E_{n+1} \\ 0 \end{bmatrix} \right| \\
&\leq -\hat{A}^{-1} \hat{B} + \begin{bmatrix} M_1 D_2^{(1)} & M_2 D_3^{(2)} & 0 \end{bmatrix} \begin{bmatrix} \Delta_2 \\ \Delta_3 \\ 0 \end{bmatrix} E \\
&= -\hat{A}^{-1} \hat{B} + \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} D_2^{(1)} & 0 & 0 \\ 0 & D_3^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_2 \\ \Delta_3 \\ 0 \end{bmatrix} E \\
&= -\hat{A}^{-1} \left(\hat{B} + \begin{bmatrix} D_2^{(1)} & 0 & 0 \\ 0 & D_3^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\Delta_2 \\ -\Delta_3 \\ 0 \end{bmatrix} E \right) = -\hat{A}^{-1} \hat{\hat{B}},
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
| -\hat{A}_n^{-1} \hat{\hat{C}}_n | &= \left| \hat{A}_n^{-1} \left(\hat{C}_n + \begin{bmatrix} D_{n,3}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,3} E_n \\ 0 \\ 0 \end{bmatrix} \right) \right| \\
&= \left| \hat{A}_n^{-1} \hat{C}_n + \begin{bmatrix} M_{n,1} D_{n,3}^{(1)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n,3} E_n \\ 0 \\ 0 \end{bmatrix} \right| \\
&\leq | \hat{A}_n^{-1} \hat{C}_n | + \left| \begin{bmatrix} M_{n,1} D_{n,3}^{(1)} & 0 & 0 \end{bmatrix} \right| \left| \begin{bmatrix} \Delta_{n,3} E_n \\ 0 \\ 0 \end{bmatrix} \right| \\
&\leq -\hat{A}^{-1} \hat{C} + \begin{bmatrix} M_1 D_3^{(1)} & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_3 \\ 0 \\ 0 \end{bmatrix} E \\
&= -\hat{A}^{-1} \hat{C} + \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix} \begin{bmatrix} D_3^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta_3 \\ 0 \\ 0 \end{bmatrix} E \\
&= -\hat{A}^{-1} \left(\hat{C} + \begin{bmatrix} D_3^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\Delta_3 \\ 0 \\ 0 \end{bmatrix} E \right) = -\hat{A}^{-1} \hat{\hat{C}},
\end{aligned} \tag{4.14}$$

By (4.12), (4.13), (4.14) we obtain

$$\left| \hat{\hat{A}}_n^{-1} \hat{\hat{B}}_n \right| \leq -\hat{\hat{A}}^{-1} \hat{\hat{B}}, \quad \left| \hat{\hat{A}}_n^{-1} \hat{\hat{C}}_n \right| \leq -\hat{\hat{A}}^{-1} \hat{\hat{C}}.$$

By assumption (4.11) and using Theorem 4.6 in [25] we imply that the perturbed equation (4.7) is exponentially stable. Applying Theorem 3.7, we conclude that equation (4.1) is exponentially stable. The proof is complete. \square

Next, we discuss some consequences when the system is subjected to unstructured perturbations. We consider a perturbed system

$$\tilde{A}_n x(n+2) + \tilde{B}_n x(n+1) + \tilde{C}_n x(n) = 0, \forall n \geq n_0, \quad (4.15)$$

where

$$W_n^{-1} \tilde{A}_n = \begin{bmatrix} A_n^{(1)} + \Delta_{n,1}^{(1)} \\ 0 \\ 0 \end{bmatrix}, W_n^{-1} \tilde{B}_n = \begin{bmatrix} B_n^{(1)} + \Delta_{n,2}^{(1)} \\ B_n^{(2)} + \Delta_{n,2}^{(2)} \\ 0 \end{bmatrix}, W_n^{-1} \tilde{C}_n = \begin{bmatrix} C_n^{(1)} + \Delta_{n,3}^{(1)} \\ C_n^{(2)} + \Delta_{n,3}^{(2)} \\ C_n^{(3)} + \Delta_{n,3}^{(3)} \end{bmatrix}.$$

Assume that there exist matrices $\Delta_i^{(j)}$ such that

$$|\Delta_{n,i}^{(j)}| \leq \Delta_i^{(j)}, \quad \forall i, j : 1 \leq j \leq i \leq 3. \quad (4.16)$$

Let $\Delta = \begin{bmatrix} \Delta_1^{(1)T} & \Delta_2^{(1)T} & \Delta_2^{(2)T} & \Delta_3^{(1)T} & \Delta_3^{(2)T} & \Delta_3^{(3)T} \end{bmatrix}^T$, $(A + B + C)^{-1}W = \begin{bmatrix} G_1 & G_2 & G_3 \end{bmatrix}$, with $G_1 \in \mathbb{C}^{d,d_1}$, $G_2 \in \mathbb{C}^{d,d_2}$, $G_3 \in \mathbb{C}^{d,d_3}$. Let us define

$$G = \begin{bmatrix} G_1 & G_1 & G_2 & G_1 & G_2 & G_3 \end{bmatrix} \in \mathbb{C}^{d,(3d_1+2d_2+d_3)}.$$

Then, it is easy to check that

$$\|H(1)\| = \|G\|, \quad \|H(\infty)\| = \|\hat{A}^{-1}\|.$$

Thus, we obtain

COROLLARY 4.5. *Assume that equation (3.1) is strangeness-free and there exists a strangeness-free constant-coefficient equation (3.7) which is exponentially stable and satisfy $|\hat{A}_n^{-1}| \leq \hat{A}^{-1}$, $|\hat{B}_n| \leq \hat{B}$, $|\hat{C}_n| \leq \hat{C}$. Moreover, suppose that the perturbations $\Delta_{n,i}$ satisfy (4.16) with*

$$\|\Delta\| < \frac{1}{\max\{\|G\|, \|\hat{A}^{-1}\|\}}. \quad (4.17)$$

Then, the perturbed equation (4.15) is exponentially stable.

By using Theorem 4.4, we can study the exponential stability of a linear time-varying implicit difference equation, of which coefficients converge to constant coefficients of a linear time-invariant implicit difference equation.

COROLLARY 4.6. *Assume that equation (3.1) is strangeness-free and there exists a sequence of matrices $\{W_n\}$ transforming (3.1) to (3.2) such that*

$$\lim_{n \rightarrow \infty} A_n^{(1)} = A^{(1)}, \lim_{n \rightarrow \infty} B_n^{(1)} = B^{(1)}, \lim_{n \rightarrow \infty} B_n^{(2)} = B^{(2)}, \\ \lim_{n \rightarrow \infty} C_n^{(1)} = C^{(1)}, \lim_{n \rightarrow \infty} C_n^{(2)} = C^{(2)}, \lim_{n \rightarrow \infty} C_n^{(3)} = C^{(3)}.$$

Moreover, we suppose that the corresponding equation (3.7) is strangeness-free, positive and exponentially stable. Then, SDE (3.1) is exponentially stable, too.

Now, we give an example to illustrate Theorem 4.4.

EXAMPLE 4.7. Consider a perturbed equation of (3.10)

$$\tilde{A}_n x(n+2) + \tilde{B}_n x(n+1) + \tilde{C}_n x(n) = 0, \forall n \geq n_0, \quad (4.18)$$

with restricted perturbations of the form

$$\begin{aligned} \tilde{A}_n &= A_n + D_{n,1} \Delta_{n,1} E_n \\ \tilde{B}_n &= B_n + D_{n,2} \Delta_{n,2} E_n \\ \tilde{C}_n &= C_n + D_{n,3} \Delta_{n,3} E_n, \end{aligned}$$

where A_n, B_n, C_n are given as (3.10), and

$$\begin{aligned} D_{n,1} &= \begin{bmatrix} -n \cos n \\ n \sin n \\ 0 \end{bmatrix}, D_{n,2} = \begin{bmatrix} -2n \cos n + (n-1) \sin n \\ 2n \sin n + (n-1) \cos n \\ 0 \end{bmatrix}, \\ D_{n,3} &= \begin{bmatrix} -n \cos n + (n-1) \sin n \\ n \sin n + (n-1) \cos n \\ n-2 \end{bmatrix}, E_n = \begin{bmatrix} \sin n & 0 & 1 + \cos n \\ \cos n & 0 & 1 - \sin n \\ 0 & \sin n & \cos n \end{bmatrix}. \end{aligned}$$

Using the same matrices W_n as in example 3.9, we obtain

$$W_n^{-1} D_{n,1} = \begin{bmatrix} n \\ 0 \\ 0 \end{bmatrix}, W_n^{-1} D_{n,2} = \begin{bmatrix} 2n \\ n-1 \\ 0 \end{bmatrix}, W_n^{-1} D_{n,3} = \begin{bmatrix} n \\ n-1 \\ n-2 \end{bmatrix}.$$

Consider a perturbed equation of (3.11)

$$\tilde{A}x(n+2) + \tilde{B}x(n+1) + \tilde{C}x(n) = 0, \forall n \geq n_0, \quad (4.19)$$

where

$$\begin{aligned} \tilde{A} &= A + D_1(-\Delta_1)E, \\ \tilde{B} &= B + D_2(-\Delta_2)E, \\ \tilde{C} &= C + D_3(-\Delta_3)E, \end{aligned}$$

$$D_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Using the same matrices W as in Example 3.9, we have

$$W^{-1} D_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, W^{-1} D_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, W^{-1} D_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let vector spaces be endowed with $\|\cdot\|_\infty$. By some simple computations, we get

$$\begin{aligned}
\|H(1)\|_\infty &= \left\| E \left(\widehat{A} + \widehat{B} + \widehat{C} \right)^{-1} \begin{bmatrix} W^{-1}D_1 & W^{-1}D_2 & W^{-1}D_3 \end{bmatrix} \right\|_\infty \\
&= \left\| \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & \frac{20}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{20}{3} \\ \frac{1}{3} & \frac{4}{3} & \frac{20}{3} \\ 0 & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \right\|_\infty = \max \left\{ \frac{25}{3}, 10 \right\} = 10, \\
\|H(\infty)\|_\infty &= \left\| E \widehat{A}^{-1} \begin{bmatrix} D_1^{(1)} & 0 & 0 \\ 0 & D_2^{(2)} & 0 \\ 0 & 0 & D_3^{(3)} \end{bmatrix} \right\|_\infty \\
&= \left\| \begin{bmatrix} \frac{1}{3} & 0 & \frac{8}{3} \\ \frac{1}{3} & 0 & \frac{8}{3} \\ \frac{1}{3} & 0 & \frac{8}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\|_\infty = \max\{3, 2\} = 3.
\end{aligned}$$

Hence,

$$\max\{\|H(1)\|, \|H(\infty)\|\} = \max\{10, 3\} = 10.$$

We can verify that all conditions in Theorem 4.4 hold, so applying Theorem 4.4,

we conclude that if $\|\Delta\| = \left\| \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \right\| < \frac{1}{10}$ then the perturbed equation (4.18) is exponentially stable.

5. Conclusion. In this paper we have investigated the solvability, exponential stability, and robust stability of a class of linear time-varying singular systems of linear second order difference equations. The reformulation of the systems and the strangeness-free assumption enable us to establish the solvability of the IVPs as well as the consistency of initial conditions. The relation between the solution sets of the systems and those of the associated reduced regular systems is pointed out. By a comparison principle, criteria for exponential stability are obtained. The boundedness of solutions of non-homogeneous equations is characterized in term of a Bohl-Perron type theorem. Finally, the problem of robust stability is investigated. Making use of the comparison principle and the robust stability of positive systems, an explicit bound for perturbations under which the systems preserve their exponential stability is obtained. By the same approach, the results in this paper can be extended without difficulty to general high order linear systems and linear delay systems.

Acknowledgments. N.T.T. Nga and D.D. Thuan would like to thank Vietnam Institute for Advanced Study in Mathematics (VIASM) for providing a fruitful research environment and hospitality.

REFERENCES

- [1] R. P. Agarwal, *Difference Equations and Inequalities, Theory, Methods and Applications*, vol. 228 of Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, NY, USA, 2nd edition, 2000.

- [2] P. K. Anh, N. H. Du, and L. C. Loi, Singular difference equations: an overview, *Vietnam J. Math.*, **35** (2007), 339–372.
- [3] P. K. Anh and D. S. Hoang, Stability of a class of singular difference equations, *Inter. J. Difference Equ.*, **1** (2006), 181–193.
- [4] P.K. Anh and H.T.N. Yen, On the solvability of initial-value problems for nonlinear implicit difference equations, *Adv. Difference Equ.*, **3** (2004), 195–200.
- [5] P.K. Anh and H.T.N. Yen, Floquet theorem for linear nonautonomous difference systems, *J. Math. Anal. Appl.*, **321** (2006), no. 2, 921–929.
- [6] B. Aulbach and N.V. Minh, The concept of spectral dichotomy for linear difference equations II, *J. Differ. Equations Appl.*, **2**(1996), 251–262.
- [7] L. Berezansky, E. Braverman, On exponential dichotomy, Bohl-Perron type theorems and stability of difference equations, *J. Math. Anal. Appl.*, **304** (2005), 511–530.
- [8] T. Berger, Robustness of stability of time-varying index-1 DAEs, *Math. Control Signals Systems*, **26** (2014), no. 3, 403–433.
- [9] T. Brüll, Existence and uniqueness of solutions of linear variable coefficient discrete-time descriptor systems. *Linear Algebra Appl.*, **431** (2009), no. 1-2, 247–265.
- [10] R. Byers and N.K. Nichols, “On the stability radius of a generalized state-space systems,” *Linear Algebra Appl.*, vol. 188-189, pp. 113-114, 1993.
- [11] C. J. Chyan, N.H.Du, and V. H. Linh, On data-dependence of exponential stability and the stability radii for linear time-varying differential-algebraic systems, *J. Differential Equations*, **245** (2008), 2078–2102.
- [12] N.H. Du, D.T. Lien, V.H. Linh, On complex stability radii for implicit discrete-time systems, *Viet. J. Math.*, **31**(2003), 475–488.
- [13] N.H. Du, V. H. Linh, and V. Mehrmann, *Robust stability of differential-algebraic equations*, in: Surveys in Differential-Algebraic Equations I, DAE-F (2013), 63–95.
- [14] N.H. Du, V.H. Linh, V. Mehrmann and D.D.Thuan, Stability and robust stability of linear time-invariant delay differential-algebraic equations, *SIAM J. Matrix Anal. Appl.*, vol. 34, pp. 1631-1654, 2013.
- [15] N.H. Du, V.H. Linh, and N.T.T Nga, On stability and Bohl exponent of linear singular systems of difference equations with variable coefficients, *J. Differ. Equations Appl.*, 2016. DOI: 0.1080/10236198.2016.1198341
- [16] S. N. Elaydi, *An Introduction to Difference Equations*, Springer, London, UK, 3rd edition, 2005.
- [17] Y. Genin, R. Stefan, P. Van Dooren, Real and complex stability radii of polynomial matrices, *Linear Algebra Appl.*, **351-352** (2002), no. 1-2, 381–410.
- [18] P. Ha and V. Mehrmann, Analysis and numerical solution of linear delay differential-algebraic equations, *BIT Numer. Math.*, **56**(2016), 633–657.
- [19] P. Kunkel and V. Mehrmann, *Differential-Algebraic Equations. Analysis and Numerical Solution*, EMS Publishing House, Zürich, Switzerland, 2006.
- [20] R. Lamour, R. März, and C.Tischendorf, *Differential-Algebraic Equations: A Projector Based Analysis*, Springer, 2013.
- [21] D.G. Luenberger, Dynamic equations in descriptor form, *IEEE Trans. Automat. Control*, **22**(1977), 312–322.
- [22] D.G. Luenberger, Control of linear dynamic market systems, *J. Eco. Dyn. Control*, **10**(1986), 339–351.
- [23] D.G. Luenberger, A. Arbel, Singular dynamic Leontief systems, *Econometrica*, **45**(1977), 991–995.
- [24] V. Mehrmann and C. Shi, Transformation of high order linear differential-algebraic systems to first order, *Numer. Algorithms*, **42**(2006), 281–307.
- [25] V. Mehrmann, D.D. Thuan, Stability analysis of implicit difference equations under restricted perturbations. *SIAM J. Matrix Anal. Appl.*, **36** (2015), no.1, 178–202.
- [26] P. H. A. Ngoc and L. T. Hieu, New criteria for exponential stability of nonlinear difference systems with time-varying delay, *Int. J. Control*, **86**(2013), 1646–1651.
- [27] B. Rodjanadid, N. V. Sanh, N. T. Ha and N. H. Du, Stability radii for implicit difference equations, *Asian-European J. Math.*, **2** (2009), 95–115.
- [28] N.H. Sau, P. Niamsup, V.N. Phat, Positivity and stability analysis for linear implicit difference delay equations, *Linear Algebra Appl.*, **510** (2016), 25–41.