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The structured controllability radius of linear delay systems

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The structured controllability radius of linear delay systems

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In this article, we shall deal with the problem of calculation of the controllability radius of a delay dynamical systems of the form $x'(t)=A_0x(t)+A_1x(t-h_1)+\cdots+A_kx(t-h_k)+Bu(t)$. By using multi-valued linear operators, we are able to derive computable formulas for the controllability radius of a controllable delay system in the case where the system's coefficient matrices are subjected to structured perturbations. Some examples are provided to illustrate the obtained results.

Keywords: linear delay systems; multi-valued linear operators; structured perturbations; controllability radius

1. Introduction

In a lot of applications, there is a frequently arising question, namely, how robust is a characteristic qualitative property of a system (e.g. controllability) when the system is subject to uncertainty. This work concerns the robust controllability analysis which has attracted considerable attention of researchers recently. This article is concerned with linear delay systems of the form

$$x'(t)=A_0x(t)+A_1x(t-h_1)+\cdots+A_kx(t-h_k)+Bu(t),\quad t\geq 0, \tag{1.1}$$

where $A_i\in\mathbb{C}^{n\times n}$, $i\in\overline{0,k}$, $B\in\mathbb{C}^{n\times m}$, $x(t)\in\mathbb{C}^n$ and $u(t)\in\mathbb{C}^m$.

The so-called controllability radius is defined by the largest bound r such that the controllability is preserved for all perturbations Δ of norm strictly less than r. For the linear control system $\dot{x}=Ax+Bu$, one can define the controllability radius $r_{\mathbb{C}}(A,B)$ as

$$\begin{aligned} r_{\mathbb{C}}(A,B)=\inf\{\|[\Delta_1,\Delta_2]\| : [\Delta_1,\Delta_2]\in\mathbb{C}^{n\times(n+m)},\\ [A,B]+[\Delta_1,\Delta_2] \text{ is not controllable}\}. \end{aligned} \tag{1.2}$$

Here, $\|\cdot\|$ denotes a matrix norm. The problem of estimating (1.2) is of great importance in mathematical control theory, and there have been several works in this direction in recent years (Boley and Lu 1986; Gahinet and Laub 1992; Gu 2000; Burke, Lewis, and Overton 2004; Gu et al. 2006). One of the most well-known results was due to Eising (1984), who has proved the formula

$$r_{\mathbb{C}}(A,B)=\inf_{\lambda\in\mathbb{C}}\sigma_{\min}([A-\lambda I B]), \tag{1.3}$$

where σ_{min} denotes the smallest singular value of a matrix and the matrix norm in (1.2) is the spectral norm or Frobenius norm. The proof of (1.3) was based on the Hautus characterisation of controllability (Hautus 1969):

$$(A,B)\in\mathbb{K}^{n\times n}\times\mathbb{K}^{n\times m}\text{controllable}\Longleftrightarrow\operatorname{rank}[A-\lambda I,B]=n,\quad \forall\lambda\in\mathbb{C}. \tag{1.4}$$

Motivated by the recent development in the theory of stability radius (see, e.g. Hinrichsen and Pritchard 1986; Hinrichsen and Pritchard 1986; and the extensive literature therein), it is natural, and more general, to consider a problem of computing the structured controllability radius when the pair (A,B) is subjected to structured perturbations:

$$[A,B]\rightsquigarrow[\tilde{A},\tilde{B}]=[A,B]+D\Delta E, \tag{1.5}$$

where $D\in\mathbb{K}^{n\times l}$, $E\in\mathbb{K}^{q\times(n+m)}$ are given structure matrices. This problem has been solved in recent papers (Karow and Kressner 2009; Son and Thuan 2010) where some formulas of the structured controllability radius have been derived.

In this article, we shall study the measures of robust controllability of linear delay systems (1.1). By using the unified approach which we have developed in the

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previous work (Son and Thuan 2010), we are able to derive, as the main result of this article, some formulas for computing the structured controllability radius of linear delay systems under the assumption that the tuple of coefficient matrices $(A_0, A_1, \dots, A_k, B)$ is subjected to general structured perturbations of the form

$$\begin{aligned}&[A_0, A_1, \dots, A_k, B]\\ &\quad \rightsquigarrow [\tilde{A}_0, \tilde{A}_1, \dots, \tilde{A}_k, \tilde{B}]\\ &\quad = [A_0, A_1, \dots, A_k, B] + D\Delta E, \end{aligned} \tag{1.6}$$

where $D \in \mathbb{C}^{n\times l}$, $E \in \mathbb{C}^{q\times(n+kn+m)}$ are given matrices defining the structure of perturbations, $\Delta \in \mathbb{C}^{l\times q}$ is unknown disturbance matrix. Moreover, avoiding the restrictive assumption on the matrix norm used in the previous works, throughout this article the norm of matrices is assumed to be the operator norm induced by arbitrary vector norms on corresponding vector spaces. In some particular cases, the main result yield new computable formulas of structured controllability radius of linear delay systems.

The organisation of this aritcle is as follows. In the next section, we shall recall some notations and some known results from the theory of linear multi-valued operators (see, e.g. Cross 1998) which will be used in the sequence. Section 3 will be devoted to prove the main results of this article establishing formulas for the structured controllability radius under structured perturbations of the form (1.6) and deriving the computable formulas in some special cases. In conclusion, we summarise the obtained results and give some remarks of further investigation.

2. Preliminaries

Let n, m, k, l, q be positive integers. Throughout this article, $\mathbb{C}^{n\times m}$ will stand for the set of all $n\times m$–matrices, $A^* \in \mathbb{C}^{m\times n}$ denotes the adjoint matrix of $A \in \mathbb{C}^{n\times m}$, $\mathbb{C}^n (= \mathbb{C}^{n\times 1})$ is the n-dimensional vector space (of columns of n numbers from $\mathbb{C}$) equipped with the vector norm $\|\cdot\|$ and its dual space can be identified with $(\mathbb{C}^n)^* = (\mathbb{C}^{n\times 1})^* = \{u^*\colon\ u \in \mathbb{C}^n\}$, the vector space of rows of n numbers from $\mathbb{C}$, equipped with the dual norm. For $u^* \in (\mathbb{C}^n)^*$ we shall write $u^*(x) = u^*x$, $\forall x \in \mathbb{C}^n$. For a subset $M \subset \mathbb{C}^n$, we denote $M^\perp = \{u^* \in (\mathbb{C}^n)^*\colon\ u^*x = 0 \text{ for all } x \in M\}$. Let $\mathcal{F}\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ be a multi-valued operator. If the graph of $\mathcal{F}$, defined by

$$\operatorname{gr}\mathcal{F} = \big\{(x, y) \in \mathbb{C}^m \times \mathbb{C}^n : y \in \mathcal{F}(x)\big\}, \tag{2.1}$$

is a linear subspace of $\mathbb{C}^m \times \mathbb{C}^n$, then $\mathcal{F}$ is called a linear multi-valued operator. The domain and the nullspace of $\mathcal{F}$ are denoted, respectively, by $\operatorname{dom}\mathcal{F} = \big\{x \in \mathbb{C}^m : \mathcal{F}(x) \neq \emptyset\big\}$ and $\ker\mathcal{F} = \big\{x \in \operatorname{dom}\mathcal{F} : 0 \in \mathcal{F}(x)\big\}$. By definition, if $\mathcal{F}$ is a multi-valued linear operator then $\mathcal{F}(0)$ is a linear subspace, and for $x \in \operatorname{dom}\mathcal{F}$, we have the following equivalence

$$y \in \mathcal{F}(x) \Longleftrightarrow \mathcal{F}(x) = y + \mathcal{F}(0). \tag{2.2}$$

Let $\mathcal{F}\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$ be a multi-valued linear operator, then for given vector norms on $\mathbb{C}^n$ and $\mathbb{C}^m$, the norm of $\mathcal{F}$ is defined by

$$\|\mathcal{F}\| = \sup\Big\{\inf_{y\in\mathcal{F}(x)} \|y\| : x \in \operatorname{dom}\mathcal{F}, \|x\| = 1\Big\}. \tag{2.3}$$

It follows from the definition that

$$\inf_{y\in\mathcal{F}(x)} \|y\| \le \|\mathcal{F}\|\|x\| \quad \text{ for all } x \in \operatorname{dom}\mathcal{F},$$

and therefore, if $\mathcal{F}$ is single-valued,

$$\|\mathcal{F}(x)\| \le \|\mathcal{F}\|\|x\| \quad \text{for all } x \in \operatorname{dom}\mathcal{F}. \tag{2.4}$$

If the spaces under consideration are equipped with the Euclidean norms (i.e. $\|x\| = \sqrt{x^*x}$) then from (2.2) it follows obviously that the following implication holds

$$y \in \mathcal{F}(x),\ y^* \in \mathcal{F}(0)^\perp \Longrightarrow d(0, \mathcal{F}(x)) := \inf_{z\in\mathcal{F}(x)} \|z\| = \|y\|. \tag{2.5}$$

For a linear multi-valued operator $\mathcal{F}\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$, its adjoint operator $\mathcal{F}^*\colon (\mathbb{C}^n)^* \rightrightarrows (\mathbb{C}^m)^*$ and its inverse operator $\mathcal{F}^{-1}\colon \operatorname{Im}\mathcal{F} \rightrightarrows \mathbb{C}^m$ are defined, correspondingly, by

$$\mathcal{F}^*(v^*) = \big\{u^* \in (\mathbb{C}^n)^* : u^*x = v^*y \text{ for all } (x, y) \in gr\mathcal{F}\big\}, \tag{2.6}$$

$$\mathcal{F}^{-1}(y) = \big\{x \in \mathbb{C}^m : y \in \mathcal{F}(x)\big\}. \tag{2.7}$$

Clearly $\mathcal{F}^*$ and $\mathcal{F}^{-1}$ are also linear multi-valued operators and we have

$$(\mathcal{F}^*)^{-1} = (\mathcal{F}^{-1})^*, \quad \|\mathcal{F}\| = \|\mathcal{F}^*\|. \tag{2.8}$$

It can be proved that $\mathcal{F}$ is surjective (i.e. $\mathcal{F}(\mathbb{C}^m) = \mathbb{C}^n$) if and only if $\mathcal{F}^*$ is injective (i.e. $\mathcal{F}^{*-1}(0) = \{0\}$), or, equivalently, $\mathcal{F}^{*-1}$ is single-valued. Let $\mathcal{F}\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^n$, $\mathcal{G}\colon \mathbb{C}^n \rightrightarrows \mathbb{C}^l$ are the linear multi-valued operators, then the operator $\mathcal{G}\mathcal{F}\colon \mathbb{C}^m \rightrightarrows \mathbb{C}^l$, defined by $(\mathcal{G}\mathcal{F})(x) = \mathcal{G}(\mathcal{F}(x))$ for all $x \in \operatorname{dom}\mathcal{F}$, is a linear multi-valued operator and if $\operatorname{Im}\mathcal{F} \subset \operatorname{dom}\mathcal{G}$ or $\operatorname{Im}\mathcal{G}^* \subset \operatorname{dom}\mathcal{F}^*$ then

$$\begin{aligned}&(\mathcal{G}\mathcal{F})^* = \mathcal{F}^*\mathcal{G}^* \quad \text{and}\\ &\|(\mathcal{G}\mathcal{F})^*\| = \|\mathcal{F}^*\mathcal{G}^*\| \le \|\mathcal{F}^*\|\ \|\mathcal{G}^*\| = \|\mathcal{F}\|\ \|\mathcal{G}\|.\end{aligned} \tag{2.9}$$

If $\mathcal{F}$ is the linear single-valued operator defined by $\mathcal{F}(x) = \mathcal{F}_G(x) = Gx$, where $G \in \mathbb{C}^{n\times m}$ and $x \in \mathbb{C}^m$, then, clearly, the norm of $\mathcal{F}_G$ defined by (2.3) is just the operator norm of matrix G:

$$\|\mathcal{F}_G\| = \|G\|.$$

In the sequence, when dealing with this operator in the context of the theory of multi-valued linear operators, we shall use the notation $\mathcal{F}_G(x) = G(x)$. It is easily seen that the adjoint operator $(\mathcal{F}_G)^*$: $(\mathbb{C}^n)^* \to (\mathbb{C}^m)^*$ is also linear single-valued operator which is given by $(\mathcal{F}_G)^*(v^*) = v^*G$, $\forall v^* \in (\mathbb{C}^n)^*$. For the sake of simplicity, we shall identify $(\mathcal{F}_G)^*$ with G^*, that reads

$$(\mathcal{F}_G)^*(v^*) = G^*(v^*) = v^*G, \quad \forall v^* \in (\mathbb{C}^n)^*. \tag{2.10}$$

Remark that the notation G^*v is understood, as usual, the product of matrix $G^* \in \mathbb{C}^{m\times n}$ and column vector $v \in \mathbb{C}^n$ and we have $(G^*v)^* = G^*(v^*)$.

3. Main results

We consider the linear delay systems with constant delays $0 < h_1 < \cdots < h_k$,

$$\begin{cases} x'(t) = A_0x(t) + A_1x(t-h_1) \\ \qquad + \cdots + A_kx(t-h_k) + Bu(t), \\ x(0) = x_0, x(t) = g(t), \forall t \in [-h_k, 0), \end{cases} \tag{3.1}$$

where $A_i \in \mathbb{C}^{n\times n}$, $i = 0, 1, \ldots, k$, $B \in \mathbb{C}^{n\times m}$, and $g(t)$: $[-h_k, 0) \to \mathbb{R}^n$ is a continuous function. System (3.1) is called *controllable* if for any given initial conditions x_0, $g(t)$ and desired final state x_1, there exists a time t_1, $0 < t_1 < \infty$, and a measurable control function $u(t)$ for $t \in [0, t_1]$ such that $x\big(t_1; x_0, g(t), u(t)\big) = x_1$. Define

$$P(\lambda) = A_0 + e^{-h_1\lambda}A_1 + \cdots + e^{-h_k\lambda}A_k - \lambda I_n. \tag{3.2}$$

It is well known that, (Bhat and Koivo 1976; Rocha and Willems 1997),

$$\begin{aligned} &\text{system (3.1) is controllable} \\ &\quad\Longleftrightarrow \operatorname{rank}[P(\lambda), B] = n \quad \text{for all } \lambda \in \mathbb{C}. \end{aligned} \tag{3.3}$$

Assume that system (3.1) is subjected to structured perturbations of the form

$$x'(t) = \widetilde{A}_0x(t) + \widetilde{A}_1x(t-h_1) + \cdots + \widetilde{A}_kx(t-h_k) + \widetilde{B}u(t), \tag{3.4}$$

with

$$\begin{aligned} &[A_0, A_1 \ldots, A_k, B] \\ &\quad \rightsquigarrow [\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_k, \widetilde{B}] \\ &\quad = [A_0, A_1 \ldots, A_k, B] + D\Delta E. \end{aligned} \tag{3.5}$$

Here, $D \in \mathbb{C}^{n\times l}$, $E \in \mathbb{C}^{q\times(n(k+1)+m)}$ are the given matrices and $\Delta \in \mathbb{C}^{l\times q}$ is the perturbation matrix. The *structure matrices* D, E determine the structure of the perturbations $D\Delta E$. We use the notation $\underline{A} = [A_0, A_1, \ldots, A_k]$.

Definition 3.1: Let system (3.1) be controllable. Given a norm $\|\cdot\|$ on $\mathbb{C}^{l\times q}$, the controllability radius of system (3.1) with respect to structured perturbations of the form (3.5) is defined by

$$\begin{aligned} r_{\mathbb{C}}(\underline{A}, B; D, E) = \inf\big\{&\|\Delta\| : \Delta \in \mathbb{C}^{l\times q} \\ &\text{s.t. } [\underline{A}, B] + D\Delta E \text{ not controllable}\big\}. \end{aligned} \tag{3.6}$$

If $[\underline{A}, B] + D\Delta E$ is controllable for all $\Delta \in \mathbb{C}^{l\times q}$ then we set $r_{\mathbb{C}}(\underline{A}, B; D, E) = +\infty$.

We define

$$W(\lambda) = [P(\lambda), B], H(\lambda) = \begin{bmatrix} I_n & 0 \\ e^{-h_1\lambda}I_n & 0 \\ \vdots & \vdots \\ e^{-h_k\lambda}I_n & 0 \\ 0 & I_m \end{bmatrix}, \tag{3.7}$$

$$E(\lambda) = EH(\lambda),$$

and the multi-valued operators $E(\lambda)W(\lambda)^{-1}D$: $\mathbb{C}^l \rightrightarrows \mathbb{C}^q$ by setting

$$(E(\lambda)W(\lambda)^{-1}D)(u) = E(\lambda)(W(\lambda)^{-1}(Du)), \quad \forall u \in \mathbb{C}^l,$$

where $W(\lambda)^{-1}$: $\mathbb{C}^n \rightrightarrows \mathbb{C}^{n+m}$ is the (multi-valued) inverse operators of $W(\lambda)$.

Theorem 3.2: *Assume that system* (3.1) *is controllable and subjected to structured perturbations of the form* (3.5). *Then the controllability radius of* (3.1) *is given by the formula*

$$r_{\mathbb{C}}(\underline{A}, B; D, E) = \frac{1}{\sup_{\lambda\in\mathbb{C}} \|E(\lambda)W(\lambda)^{-1}D\|}. \tag{3.8}$$

Proof: Suppose that $[\widetilde{\underline{A}}, \widetilde{B}] = [\underline{A}, B] + D\Delta E$ is not controllable for $\Delta \in \mathbb{C}^{l\times q}$. It means, by (3.3), the operator $\widetilde{W}(\lambda) = [\widetilde{P}(\lambda), \widetilde{B}]$ is not surjective for some $\lambda_0 \in \mathbb{C}$, where $\widetilde{P}(\lambda) = \widetilde{A}_0 + e^{-h_1\lambda}\widetilde{A}_1 + \cdots + e^{-h_k\lambda}\widetilde{A}_k - \lambda I_n$. By definitions (3.2) and (3.7), we can deduce

$$\begin{aligned} \widetilde{W}(\lambda_0) &= [\widetilde{P}(\lambda_0), \widetilde{B}] = [\widetilde{A}_0, \widetilde{A}_1, \ldots, \widetilde{A}_k, \widetilde{B}]H(\lambda_0) - \lambda[I_n, 0] \\ &= ([A_0, A_1, \ldots, A_k, B] + D\Delta E)H(\lambda_0) - \lambda[I_n, 0] \\ &= [A_0, A_1, \ldots, A_k, B]H(\lambda_0) - \lambda[I_n, 0] + D\Delta EH(\lambda_0) \\ &= [P(\lambda_0), B] + D\Delta E(\lambda_0) = W(\lambda_0) + D\Delta E(\lambda_0). \end{aligned} \tag{3.9}$$

This implies that there exists $y_0^* \in (\mathbb{C}^n)^*, y_0^* \neq 0$ such that

$$\begin{aligned} &(W(\lambda_0) + D\Delta E(\lambda_0))^*(y_0^*) \\ &\quad = W(\lambda_0)^*(y_0^*) + (E(\lambda_0)^*\Delta^*D^*)(y_0^*) = 0. \end{aligned}$$

Since system (3.1) is controllable, by (3.3), $W(\lambda_0)$ is surjective, or equivalently $W(\lambda_0)^{*-1}$ is single-valued.

Therefore, we have

$$y_0^* = -(W(\lambda_0)^{*-1}E(\lambda_0)^*\Delta^*)(D^*(y_0^*)) \tag{3.10}$$

and, hence, $D^*(y_0^*) \neq 0$. By applying D^* to the left of the both sides of (3.10), we obtain

$$D^*(y_0^*) = -(D^*W(\lambda_0)^{*-1}E(\lambda_0)^*\Delta^*)(D^*(y_0^*)).$$

Therefore, by (2.4),

$$\begin{aligned}0 < \|D^*(y_0^*)\| &\leq \|D^*W(\lambda_0)^{*-1}E(\lambda_0)^*\|\|\Delta^*(D^*(y_0^*))\|\\ &\leq \|D^*W(\lambda_0)^{*-1}E(\lambda_0)^*\|\|\Delta^*\|\|D^*(y_0^*)\|.\end{aligned}$$

Since Im $W(\lambda_0)^{-1} \subset \operatorname{dom} E(\lambda_0) = \mathbb{C}^{n+m}$, we have, by using (2.9), $(E(\lambda_0)W(\lambda_0)^{-1})^* = W(\lambda_0)^{-1^*}E(\lambda_0)^* = W(\lambda_0)^{*-1}E(\lambda_0)^*$. Further, since $W(\lambda_0)$ is surjective, Im $D \subset \operatorname{dom}(E(\lambda_0)W(\lambda_0)^{-1}) = \mathbb{C}^n$ we have again by (2.9),

$$\begin{aligned}&(E(\lambda_0)W(\lambda_0)^{-1}D)^*\\ &\quad= D^*(E(\lambda_0)W(\lambda_0)^{-1})^*\\ &\quad= D^*W(\lambda_0)^{*-1}E(\lambda_0)^*.\end{aligned}$$

By (2.8), we get

$$\begin{aligned}\|\Delta^*\| &= \|\Delta\|\\ &\geq \frac{1}{\|D^*W(\lambda_0)^{*-1}E(\lambda_0)^*\|}\\ &= \frac{1}{\|E(\lambda_0)W(\lambda_0)^{-1}D\|}\\ &\geq \frac{1}{\sup_{\lambda\in\mathbb{C}}\|E(\lambda)W(\lambda)^{-1}D\|}.\end{aligned}$$

Since the above inequality holds for any disturbance matrix $\Delta \in \mathbb{C}^{l\times q}$ such that $D\Delta E$ destroys controllability of (3.1), we obtain by definition,

$$r_{\mathbb{C}}(\underline{A}, B; D, E) \geq \frac{1}{\sup_{\lambda\in\mathbb{C}}\|E(\lambda)W(\lambda)^{-1}D\|}. \tag{3.11}$$

To prove the converse inequality, for any small $\epsilon > 0$ such that $\sup_{\lambda\in\mathbb{C}} \|E(\lambda)W(\lambda)^{-1}D\| - \epsilon > 0$ there exists $\lambda_\epsilon \in \mathbb{C}$ such that $\|D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*\| = \|E(\lambda_\epsilon)W(\lambda_\epsilon)^{-1}D\| \geq \sup_{\lambda\in\mathbb{C}} \|E(\lambda)W(\lambda)^{-1}D\| - \epsilon$. We note further that $D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*$ is single-valued, therefore its norm is the operator norm and hence there exists $v_\epsilon^* \in (\mathbb{C}^q)^* : \|v_\epsilon^*\| = 1$, $v_\epsilon^* \in \operatorname{dom}(D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*)$ such that

$$\begin{aligned}\|E(\lambda_\epsilon)W(\lambda_\epsilon)^{-1}D\| &= \|D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*\|\\ &= \|(D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*)(v_\epsilon^*)\|.\end{aligned}$$

Denoting $u_\epsilon^* = -W(\lambda_\epsilon)^{*-1}(E(\lambda_\epsilon)^*(v_\epsilon^*)) \neq 0$, we have

$$\begin{aligned}W(\lambda_\epsilon)^*(u_\epsilon^*) &= -E(\lambda_\epsilon)^*(v_\epsilon^*) \text{ and } D^*(u_\epsilon^*)\\ &= -(D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*)(v_\epsilon^*) \neq 0.\end{aligned}$$

By Hahn–Banach Theorem, applying with $V = \{sD^*(u_\epsilon^*) : s \in \mathbb{C}\} \subset (\mathbb{C}^l)^*$, there exists $h_\epsilon \in \mathbb{C}^l$ such that $\|h_\epsilon\| = 1, (D^*(u_\epsilon^*))h_\epsilon = \|D^*(u_\epsilon^*)\|$. Thus, we can define a rank-one perturbation $\Delta_\epsilon \in \mathbb{C}^{l\times q}$ by setting

$$\Delta_\epsilon = \frac{1}{\|D^*(u_\epsilon^*)\|}h_\epsilon v_\epsilon^*.$$

Then, it is obvious that $\|\Delta_\epsilon\| \leq \|D^*(u_\epsilon^*)\|^{-1}$. Moreover, we have $D^*(u_\epsilon^*)\Delta_\epsilon = v_\epsilon^*$. This implies that $\|\Delta_\epsilon\| \geq \|D^*(u_\epsilon^*)\|^{-1}$. Thus, we obtain

$$\begin{aligned}\|\Delta_\epsilon\| = \|D^*(u_\epsilon^*)\|^{-1} &= \|(D^*W(\lambda_\epsilon)^{*-1}E(\lambda_\epsilon)^*)(v_\epsilon^*)\|^{-1}\\ &= \frac{1}{\|E(\lambda_\epsilon)W(\lambda_\epsilon)^{-1}D\|}.\end{aligned}$$

Using (2.10),

$$(\Delta_\epsilon^*D^*)(u_\epsilon^*) = \Delta_\epsilon^*(D^*(u_\epsilon^*)) = D^*(u_\epsilon^*)\Delta_\epsilon = v_\epsilon^*.$$

Hence, $(E(\lambda_\epsilon)^*\Delta_\epsilon^*D^*)(u_\epsilon^*) = E(\lambda_\epsilon)^*(v_\epsilon^*)$ and, therefore,

$$W(\lambda_\epsilon)^*(u_\epsilon^*) + (E(\lambda_\epsilon)^*\Delta_\epsilon^*D^*)(u_\epsilon^*) = 0,$$

with $u^* \neq 0$, which implies that the perturbed matrix $[\widetilde{P}(\lambda_\epsilon), \widetilde{B}] = \widetilde{W}(\lambda_\epsilon) = W(\lambda_\epsilon) + D\Delta_\epsilon E(\lambda_\epsilon)$ is non-surjective or, equivalently, by (3.3), system (3.1) is not controllable. Thus, by definition,

$$\begin{aligned}r_{\mathbb{C}}(\underline{A}, B; D, E) &\leq \|\Delta_\epsilon\|\\ &= \frac{1}{\|E(\lambda_\epsilon)W(\lambda_\epsilon)^{-1}D\|}\\ &\leq \frac{1}{\sup_{\lambda\in\mathbb{C}}\|E(\lambda)W(\lambda)^{-1}D\| - \epsilon}.\end{aligned}$$

Letting $\epsilon \to 0$, we get the required converse inequality. Thus, we obtain

$$r_{\mathbb{C}}(\underline{A}, B; D, E) = \frac{1}{\sup_{\lambda\in\mathbb{C}}\|E(\lambda)W(\lambda)^{-1}D\|}.$$

The proof is complete. □

The above theorem have been proved similarly with one result for higher order descriptor systems in Son and Thuan (2012), for the case when the norms of matrices under consideration are operator norms induced by arbitrary vector norms in corresponding vector spaces.

Formula (3.8) gives us a unified framework for computation of controllability radii, however, it is not easy to be used because this formula involves calculation of the norm of the multi-valued linear operator $E(\lambda)W(\lambda)^{-1}D$ which do not have an explicit representation. We now derive from this result more computable formulas for the particular case, where the norm of the matrices under consideration is the spectral norm (i.e. the operator norm induced by Euclidean vector norms of the form $\|x\| = \sqrt{x^*x}$). To this end, we need the following lemmas.

Lemma 3.3: *Assume that* $G \in \mathbb{C}^{n\times p}$ *has full row rank*: $\operatorname{rank} G = n$ *and* $\mathbb{C}^n$, $\mathbb{C}^p$ *are equipped with Euclidean norms. Then, for the linear operator* $\mathcal{F}_G(z) = Gz$, *we have*

$$d(0, \mathcal{F}_G^{-1}(y)) = \|G^\dagger y\|, \quad \|\mathcal{F}_G^{-1}\| = \|G^\dagger\|, \tag{3.12}$$

where $G^\dagger$ *denotes the Moore-Penrose pseudoinverse of* G.

Proof: See Lemma 3.3 in Son and Thuan (2010). □

Lemma 3.4: *Assume that* $\Sigma \in \mathbb{C}^{n\times(n+m)}$ *has full row rank,* $M \in \mathbb{C}^{q\times(n+m)}$ *has full column rank and the operator norms are induced by Euclidean vector norms. Then, we have*

$$\|M\Sigma^{-1}D\| = \|(\Sigma(M^*M)^{-1/2})^\dagger D\|, \tag{3.13}$$

where † *denotes the Moore-Penrose pseudoinverse,* $D \in \mathbb{C}^{n\times l}$ *and* Σ^{-1} *is the (multi-valued) inverse operator of* Σ.

Proof: See Corollary 3.7 in Son and Thuan (2010) or Lemma 4.2 in Son and Thuan (2012). □

We note that if system (3.1) is controllable, then $W(\lambda)$ have full row rank, and if E has full column rank, then $E(\lambda)$ have full column rank for all $\lambda \in \mathbb{C}$. By Lemma 3.4 and Theorem 3.2, we obtain

Theorem 3.5: *Assume that* E *has full column rank and the operator norms are induced by Euclidean vector norms. Then we have*

$$r_{\mathbb{C}}(\underline{A}, B; D, E) = \frac{1}{\sup_{\lambda\in\mathbb{C}} \left\| \left(W(\lambda)[E(\lambda)^*E(\lambda)]^{-1/2}\right)^\dagger D \right\|}. \tag{3.14}$$

The above theorem covers many existing results as particular cases. Indeed, for $k = 0$, we obtain the main result in Karow and Kressner (2009). Further, it is easy to see that if $k = 0$ and D, E are the identity matrices in $\mathbb{C}^{n\times n}$ and $\mathbb{C}^{(n(k+1)+m)\times(n(k+1)+m)}$, respectively, then Theorem 3.5 is reduced to the formula of Eising (1984) as a particular case.

It is worth to mention that if coefficient matrices A_i, B are subjected to separate structured perturbations, then it may not be possible to cover this case by the model (3.5) with the full block Δ. Next, we consider a particular case of separate structured perturbations, which can be covered by the model (3.5) and thus the above result are applicable. Assume that system (3.1) is subjected to separate perturbations of the form

$$\begin{aligned} &B \rightsquigarrow \widetilde{B} = B + D_B\Delta_B E_B, \\ &A_i \rightsquigarrow \widetilde{A}_i = A_i + D_{A_i}\Delta_{A_i}E_{A_i}, \quad \text{for all } i \in \overline{0,k}, \end{aligned} \tag{3.15}$$

where $D_{A_i} = D_B \in \mathbb{C}^{n\times l}, E_{A_i} \in \mathbb{C}^{q_{A_i}\times n}, E_B \in \mathbb{C}^{q_B\times m}$, for all $i \in \overline{0,k}$, are given matrices and $\Delta_B \in \mathbb{C}^{l\times q_B}$, $\Delta_{A_i} \in \mathbb{C}^{l\times q_{A_i}}$, for all $i \in \overline{0,k}$, are the perturbation matrices. It is easy to see that the perturbation model (3.15) can be rewritten in the form

$$\begin{aligned} &[A_0, A_1, \dots, A_k, B] \\ &\quad \rightsquigarrow [\widetilde{A}_0, \widetilde{A}_1, \dots, \widetilde{A}_k, \widetilde{B}] \\ &\quad = [A_0, A_1, \dots, A_k, B] + \widehat{D}\Delta\widehat{E}, \end{aligned}$$

where $\widehat{D} = D_B, \widehat{E} = \operatorname{diag}(E_{A_0}, E_{A_1}, \dots, E_{A_k}, E_B)$ and the perturbation

$$\Delta = [\Delta_{A_0}, \Delta_{A_1}, \dots, \Delta_{A_k}, \Delta_B].$$

In this situation, we define

$$\widehat{E}(\lambda) = \widehat{E}H(\lambda) = \begin{bmatrix} E_{A_0} & 0 \\ e^{-h_1\lambda}E_{A_1} & 0 \\ \vdots & \vdots \\ e^{-h_k\lambda}E_{A_k} & 0 \\ 0 & E_B \end{bmatrix}. \tag{3.16}$$

Theorem 3.6: *Assume that system* (3.1) *is subjected to separate structured perturbations of the form* (3.15). *Then, if system* (3.1) *is controllable then*

$$r_{\mathbb{C}}(\underline{A}, B; D_B, E_B, E_{A_i}, i \in \overline{0,k}) = \frac{1}{\sup_{\lambda\in\mathbb{C}} \|\widehat{E}(\lambda)W(\lambda)^{-1}\widehat{D}\|}. \tag{3.17}$$

Let us consider system (3.1) subjected to perturbations of the form

$$\begin{aligned} &B \rightsquigarrow \widetilde{B} = B + \Delta_B, \\ &A_i \rightsquigarrow \widetilde{A}_i = A_i + \alpha_i\Delta_{A_i}, \quad \text{for all } i \in \overline{0,k}, \end{aligned} \tag{3.18}$$

where $\alpha_i \in \mathbb{C}, i \in \overline{0,k}$ are given scalar parameters, not all zero, and $\Delta_{A_i} \in \mathbb{C}^{n\times n}, i \in \overline{0,k}, \Delta_B \in \mathbb{C}^{n\times m}$ are unknown matrices. Then, we can apply Theorem 3.6 to calculate the controllability radii of system (3.1) under structured perturbations (3.18). Define

$$\mu(\lambda) = |\alpha_0|^p + \sum_{i=1}^{k} |\alpha_i|^p |e^{-h_i\lambda p}|. \tag{3.19}$$

Now, we will derive the formula of the controllability radius for linear delay systems under affine perturbations (3.18), which is nearly similar with the one for higher order descriptor systems in Son and Thuan (2012).

Corollary 3.7: *Assume that the controllable system* (3.1) *is subjected to perturbations of the form* (3.18) *and* *the vector spaces are endowed with the p-norm with* $0 < p < \infty$. *Then,*

$$r_{\mathbb{C}}(\underline{A}, B; \alpha_i, i \in \overline{0,k}) = \frac{1}{\sup_{\lambda\in\mathbb{C}} \left\| \left[\frac{P(\lambda)}{\mu(\lambda)^{1/p}}, B \right]^{-1} \right\|}, \quad (3.20)$$

and if $p = 2$,

$$r_{\mathbb{C}}(\underline{A}, B; \alpha_i, i \in \overline{0,k}) = \frac{1}{\sup_{\lambda\in\mathbb{C}} \left\| \left[\frac{P(\lambda)}{\sqrt{\mu(\lambda)}}, B \right]^{\dagger} \right\|} = \inf_{\lambda\in\mathbb{C}} \sigma_{\min}\left[\frac{P(\lambda)}{\sqrt{\mu(\lambda)}}, B \right]. \quad (3.21)$$

Proof: We see in model (3.18) that

$$D_{A_i} = D_B = I_n, E_{A_i} = \alpha_i I_n, E_B = I_m, \text{ for all } i \in \overline{0,k}.$$

Therefore, we get $\widehat{D} = I_n, \widehat{E} = diag(\alpha_0 I_n, \alpha_1 I_n, \ldots, \alpha_k I_n, I_m)$, and by (3.16)

$$\widehat{E}(\lambda) = \begin{bmatrix} \alpha_0 I_n & 0 \\ \alpha_1 e^{-h_1\lambda} I_n & 0 \\ \vdots & \vdots \\ \alpha_k e^{-h_k\lambda} I_n & 0 \\ 0 & I_m \end{bmatrix}.$$

Therefore,

$$(\widehat{E}(\lambda)W(\lambda)^{-1}\widehat{D})(v) = \left\{ \begin{pmatrix} \alpha_0 y_1 \\ \alpha_1 e^{-h_1\lambda} y_1 \\ \vdots \\ \alpha_k e^{-h_k\lambda} y_1 \\ u_1 \end{pmatrix} : y_1 \in \mathbb{C}^n, u_1 \in \mathbb{C}^m \text{ s.t. } P(\lambda)y_1 + Bu_1 = v \right\}.$$

Let $w_1 := \mu(\lambda)^{1/p} y_1$ with $\mu(\lambda)$ defined by (3.19). We have

$$\left\| \begin{pmatrix} Z_1 \\ u_1 \end{pmatrix} \right\|^p = \left(|\alpha_0|^p + \sum_{i=1}^{k} |\alpha_i|^p |e^{-h_i\lambda p}| \right) \| y_1 \|^p + \| u_1 \|^p = \left\| \begin{pmatrix} w_1 \\ u_1 \end{pmatrix} \right\|^p,$$

for each $\binom{Z_1}{u_1} \in (\widehat{E}(\lambda)W(\lambda)^{-1}\widehat{D})(v)$. This implies that

$$d(0, \widehat{E}(\lambda)W(\lambda)^{-1}\widehat{D}(v)) = \inf\left\{ \left\| \begin{pmatrix} w_1 \\ u_1 \end{pmatrix} \right\| : \left[\frac{P(\lambda)}{\mu(\lambda)^{1/p}}, B \right] \begin{pmatrix} w_1 \\ u_1 \end{pmatrix} = v \right\} = d\left(0, \left[\frac{P(\lambda)}{\mu(\lambda)^{1/p}}, B \right]^{-1}(v)\right).$$

Therefore, by Theorem 3.6, we obtain formula (3.20). Note that for the matrix spectral norm and $G \in \mathbb{C}^{n\times m}$, $\frac{1}{\|G^\dagger\|} = \sigma_{\min}(G)$, the smallest singular value of G. Thus, by Lemma 3.3, we obtain formula (3.21). □

Example 3.8: Let us consider the second-order time-delay system

$$x'(t) = A_0 x(t) + A_1 x(t-1) + Bu(t), \quad (3.22)$$

where

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We see that

$$W(\lambda) = [P(\lambda), B] = \begin{bmatrix} 1-\lambda & 1 & 0 \\ e^{-\lambda} & e^{-\lambda}-\lambda & 1 \end{bmatrix}.$$

It follows that rank $W(\lambda) = 2$ for all $\lambda \in \mathbb{C}$. Therefore, by (3.3), the system is controllable. Assume that the control matrix $[A_0, A_1, B]$ is subjected to structured perturbation of the form

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1+\delta_1 & 1+\delta_1 & \delta_2 & \delta_2 & \delta_2 \\ \delta_1 & \delta_1 & 1+\delta_2 & 1+\delta_2 & 1+\delta_2 \end{bmatrix},$$

where $\delta_i \in \mathbb{C}, i \in \overline{1,2}$ are disturbance parameters. The above-perturbed model can be represented in the form

$$[A_0, A_1, B] \rightsquigarrow [A_0, A_1, B] + D\Delta E$$

with

$$D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and $\Delta = [\delta_1 \; \delta_2]$. It implies that

$$E(\lambda) = \begin{bmatrix} 1 & 1 & 0 \\ e^{-\lambda} & e^{-\lambda} & 1 \end{bmatrix}.$$

We have, for $v \in \mathbb{C}$,

$$E(\lambda)W(\lambda)^{-1}D(v) = E(\lambda)W(\lambda)^{-1}\binom{v}{v} = \left\{ E(\lambda)\begin{pmatrix} p \\ q \\ r \end{pmatrix} : (1-\lambda)p + q = e^{-\lambda}p + (e^{-\lambda}-\lambda)q + r = v \right\} = \left\{ \begin{pmatrix} v + \lambda p \\ (\lambda+1)v + \lambda(\lambda-1)p \end{pmatrix} : p \in \mathbb{C} \right\}.$$

Thus, for each $v \in \mathbb{C}$, the problem of computing $d(0, E(\lambda)W(\lambda)^{-1}D(v))$ is reduced to the calculation of the distance from the origin to the straight line in $\mathbb{C}^2$ whose equation can be rewritten in the form $x_2-(\lambda-1)x_1=2v$ with

$$x_1 = v + \lambda p, \quad x_2 = (\lambda+1)v + \lambda(\lambda-1)p.$$

Note that if $\lambda=0$ then this line is reduced to the point $\binom{v}{v}$. Assume that $\lambda\neq 0$ and let $\mathbb{C}^2$ be endowed with the vector norms $\|\cdot\|_\infty$, then we can deduce,

$$2|v| \le |\lambda-1||x_1| + |x_2| \le (|\lambda-1|+1)\max\{|x_1|,|x_2|\} = (|\lambda-1|+1)\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_\infty.$$

This implies

$$\left\|\begin{pmatrix}x_1\\x_2\end{pmatrix}\right\|_\infty \ge \frac{2|v|}{|\lambda-1|+1},$$

which yields the equality if $x_2=\frac{2v}{|\lambda-1|+1}$ and $x_1=e^{i\varphi}x_2$, where φ is chosen such that $(1-\lambda)e^{i\varphi}=|\lambda-1|$. Therefore,

$$\|E(\lambda)W(\lambda)^{-1}D\|_\infty = \sup_{|v|=1} \mathrm{d}\big(0, E(\lambda)W(\lambda)^{-1}D(v)\big) = \begin{cases} \dfrac{2}{|\lambda-1|+1} & \text{if } \lambda\neq 0,\\ 1 & \text{if } \lambda = 0.\end{cases}$$

Thus, by Theorem 3.2, we obtain $r_{\mathbb{C}}(A_0,A_1,B;D,E)=\frac{1}{2}$.

4. Conclusion

In this article, we developed a unifying approach to the problem of calculating the controllability radius of linear delay systems, which is based on the theory of linear multi-valued operators. We obtained some general formulas of complex controllability radii under the assumption that the system coefficient matrices are subjected to structured perturbations. These results unify and extend many existing results to more general cases. Moreover, it has been shown that from our general results, some easily computable formulas can be derived. Our approach can be developed further for calculating the distance from ill-posedness of conic systems of the form $Ax=b$, $x\in K\subset\mathbb{C}^m$, where K is a closed convex cone, as well as for controllability radius of convex processes $\dot{x}\in\mathcal{F}(x), t\ge 0$. These problems are the topics of our further study.

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