Proceedings of the 10-th IFAC Workshop on Time Delay Systems
The International Federation of Automatic Control
Northeastern University, Boston, USA. June 22-24, 2012

Spectral Analysis for Delay Differential-Algebraic Systems

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Abstract: The aim of this paper is the study of the stability properties of a class of systems expressed by functional differential equations coupled with difference equations. Starting by the linear case, we extend the spectral projection methodology for Lossless systems by introducing an appropriate bilinear form. A sufficient criteria for convergence of series expansion for the considered system is established. For the nonlinear case, the center Manifold theorem is extended to functional differential equations coupled with difference equations leading to reducing the dimension of the initial system. Finally, we illustrate the obtained results by computational example.

1. INTRODUCTION

In this paper we are concerned with systems of functional differential equations coupled with difference equations where the elements of interconnection are represented by the delay terms. This class of systems is frequently encountered in control problems where the delay terms come naturally from the feedback. In general by Lossless propagation it is understood the phenomenon associated with long transmission lines for physical signals. In engineering, this problem is strongly related to electric and electronic applications, e.g. circuit structures consisting of multipoles connected through LC transmission lines, this can also be seen in steam and water pipes Niculescu [2001], Fu & Niculescu & Chen [2006]. The linear and simpler form of such systems can be written

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,y(t-\tau) \\ y(t) &= C\,x(t) + D\,y(t-\tau) \end{aligned}\right\} \tag{1}$$

where the variables x and y are not necessarily of the same dimension, and the matrix D not necessarily invertible. In Fu & Niculescu & Chen [2006], the authors establish an efficient matrix pencil method with high precision in the study of stability.
From another point of view, this type of systems implies but not equivalent to Functional Differential Equations of Neutral type (NFDE)

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,y(t-\tau) \\ \dot{y}(t) - D\,\dot{y}(t-\tau) &= C\,A\,x(t) + C\,B\,y(t-\tau) \end{aligned}\right\}. \tag{2}$$

To the best of our knowledge this treatment by letting $y(t) = \dot{z}(t)$ was suggested first in Abolinia & Myshkis [1960] also applied in Răsvan [1975] and in Hale & Martinez-Amores [1977]. This transformation will be called in the sequel *Standard Transformation.*

This type of systems is subject to prolific interest due to its frequent use in the modeling for physical and biological phenomenas Răsvan & Niculescu [2002], Michiels & Niculescu [2007], Faria & Magales [2001], Gopalsamy [1992]. In Hale & Verduyn Lunel [1993] a bilinear form associated to this systems class is defined, which allows Frasson [2005] to establish an efficient spectral projection procedure. Indeed, Frasson [2005] is concerned with large time behavior of solutions of linear autonomous NFDE.

We underline the fact that defining such spectral projections represents an important tool to the study of nonlinear systems. Indeed, it permits to restrict the flow to a finite dimensional subspace which is invariant under the solution semigroup. For instance the study of local bifurcations often needs the computation of the center manifold which is characterized by spectral values with zero real part. Unfortunately, spectral projection defined for (2) can not be directly invested in the study of the center manifold associated to system (1) since the algebraic multiplicity of the zero spectral value increase when passing from (1) to (2).

Motivated by this fact, we focus this work on establishing first a bilinear form associated to (1) and then to describe a procedure scheme for computing associated spectral projection. Then we shall address the question whether the solution of (1) can be represented by a series of elementary solutions. This question was tackled for the retarded case Banks & Manitius [1975] and neutral one Verduyn Lunel [1995] but never asked for Lossless systems. We first start by recalling a theorem giving necessary and sufficient conditions for the solution of NFDE to be represented by a series of elementary solutions.

Theorem 1. (Verduyn Lunel [1995]). Let C be the banach space $C = C([-r, 0], \mathbb{R}^n)$ and let $\mathcal{T}(t)\ : C \to C$ denote the semigroup of solution operators associated with

$$\frac{d}{dt}[x(t)-D_0x(t-r)-\sum_{k=1}^{\infty}D_k\,x(t-\tau_n)]=\int_0^h d\zeta(\theta)x(t-\theta)$$

where ζ is an $n\times n$ matrix function of bounded variation, for $j=1,2,\ldots,\ \tau_j<r$ and D_0 is a nonsingular matrix, and $\mid D_j\mid\to 0$ as $j\to\infty$. such that the matrices $A\in\mathcal{M}_n(\mathbb{R})$ and $D\in\mathcal{M}_m(\mathbb{R})$ are invertible, λ_j, $j=0,1,\ldots$ the associated set of eigenvalues and P_{λ_j} the associated spectral projection given by

$$P_\lambda=\frac{1}{2\Pi i}\int_{\Gamma_\lambda}(zI-\mathcal{A})^{-1}\,dz. \tag{3}$$

Then the semigroup $\mathcal{T}(t)$ extends to a group of bounded linear operators and for every $\varphi\in C$, we have

$$\lim_{N\to\infty}\|\,\mathcal{T}(t)\varphi-\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{k}\mathcal{T}(t)P_{\lambda_j}\varphi\,\|=0,\qquad t\in\mathbb{R}^+ \tag{4}$$

uniformly on compact t-sets.

The invertibility of D_0 ensure that the generator semigroup $\mathcal{T}(t)$ has a complete system of eigenvectors and generalized eigenvectors, see Verduyn Lunel [1995]. Accordingly, the invertibility of D_0 guaranties the exclusion of small solutions, thus, the backward integration is possible via

$$\dot{x}(t)=D_0^{-1}[\dot{x}(t+r)-\sum_{k=1}^{n}A_k\,\dot{x}(t-\tau_n+r)+\int_0^h d\zeta(\theta)x(t-\theta)].$$

For more details about completeness of system of eigenvectors and generalized eigenvectors see Verduyn Lunel 2 [1995].

Unfortunately, this theorem can not be directly applied to Lossless systems (1) by the only use of the Standard transformation. Indeed, the difference matrix $D_0=\begin{pmatrix}0&0\\0&D\end{pmatrix}$ of the transformed system (2) is not invertible.

The Second section is devoted to the main results: we establish a bilinear form associated with Lossless systems as well as some sufficient conditions for series convergence although in some special cases when the matrix D is singular. Thus, we extend the center manifold theorem to nonlinear functional differential equations coupled with difference equations:

$$\left.\begin{array}{l}\dot{x}(t)=A\,x(t)+B\,y(t-\tau)+f_1(x(t),y(t),y(t-\tau))\\ y(t)=C\,x(t)+D\,y(t-\tau)+f_2(x(t),y(t),y(t-\tau))\end{array}\right\}. \tag{5}$$

2. MAIN RESULT: SPECTRAL PROJECTION FOR LOSSLESS PROPAGATION MODEL

Consider the system (1) where $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$ and A, B, C, D are real valued matrices. Thus $A\in\mathcal{M}_{m,m}(\mathbb{R})$, $B\in\mathcal{M}_{m,n}(\mathbb{R})$, $C\in\mathcal{M}_{n,m}(\mathbb{R})$, $D\in\mathcal{M}_{n,n}(\mathbb{R})$. Analogously, to the well prescribed general theory for functional differential equations Hale [1977], Hale & Verduyn Lunel [1993], Michiels & Niculescu [2007], Gopalsamy [1992], the infinitesimal generator associated with the semigroup of the solution operator $\mathcal{T}(t)$ is given by

$$\begin{aligned}&\mathcal{D}om(\mathcal{A})=\\&\quad\{\varphi=(\varphi_1,\varphi_2)^T\in\mathcal{C},\ \text{s.t.}\ \frac{d\varphi}{d\theta}\in\mathcal{C},\ \mathcal{D}\frac{d\varphi}{d\theta}=\mathcal{L}\varphi\},\\&\mathcal{A}\varphi=\frac{d\varphi}{d\theta}\ \text{and}\\&\mathcal{L}\,\varphi=(A\varphi_1(0)+B\varphi_2(-\tau),C\varphi_1(0)-\varphi_2(0)+D\varphi_2(-\tau))^T,\\&\mathcal{D}\,\varphi=(\varphi_1(0),0)^T.\end{aligned} \tag{6}$$

The associated characteristic matrix is

$$\Delta(z)=\begin{pmatrix}z\,I_n-A&-e^{-z\,\tau}\,B\\-C&I_m-e^{-z\,\tau}\,D\end{pmatrix}, \tag{7}$$

thus λ is said to be a spectral value if $\mathbb{H}(\lambda)=det(\Delta(\lambda))=0$.

2.1 Bilinear Form for Lossless propagation model

Let us consider φ an $m+n$ column vector and ψ an $m+n$ row vector,

$$\varphi=\begin{pmatrix}v_1\\\vdots\\v_n\\v_{n+1}\\\vdots\\v_{n+m}\end{pmatrix}\ \text{and}\ \psi=(u_1,\ldots,u_n,u_{n+1},\ldots,u_{n+m}).$$

Moreover, let us set $\psi=\psi_1+\psi_2=(u_1,\ldots,u_n,0,\ldots,0)+(0,\ldots,0,u_{n+1},\ldots,u_{n+m})$.

Analogously, let us set for a block matrix

$$J=\begin{pmatrix}L&M\\N&Q\end{pmatrix}$$

then $J=\tilde{J}_1+\tilde{J}_2=\begin{pmatrix}L&M\\0&0\end{pmatrix}+\begin{pmatrix}0&0\\N&Q\end{pmatrix}$.

Let denote by $\Gamma(\theta)$ the $m+n$ square matrix where

$$\Gamma(-\tau)=\begin{pmatrix}0\ldots0&0\ldots0\\\vdots\ \ldots\ \vdots&\vdots\ \ldots\ \vdots\\0\ldots0&0\ldots0\\0\ldots0&\\\vdots\ \ldots\ \vdots&D\\0\ldots0&\end{pmatrix},$$

and

$$\Gamma(0)=I_d-\tilde{I}_{d1},$$

and for all $s\in[-\tau,0]$ $\Gamma(s)=\Gamma(0)+\frac{s}{\tau}(\Gamma(0)-\Gamma(-\tau))$ and thus we define the bilinear form $<.,.>:\mathcal{C}^*\times\mathcal{C}\to\mathbb{C}$ by

$$\begin{aligned}<\psi,\varphi>=&-\psi_1(0)\varphi(0)-\int_{-\tau}^{0}\frac{d}{d\theta}[\frac{d\psi(\theta)}{d\theta}\,\Gamma(\theta)\,\varphi(\theta)]d\theta\\&+\int_{-\tau}^{0}\psi_2(\tau+\theta)\varphi(\theta)d\theta.\end{aligned} \tag{8}$$

2.2 Computing spectral projections using duality: Simple non zero spectral values

By analogy with the theory of NFDE, when λ is a simple nonzero eigenvalue of an operator $\mathcal{A}$, then the spectral projection onto the eigenspace $\mathcal{M}_\lambda$ is given by

$$P_\lambda(\varphi)=<\psi_\lambda,\varphi>\ \varphi_\lambda,$$

where

$$\psi_\lambda(\xi) = e^{-\lambda\,\xi} d_\lambda, \qquad 0 \le \xi \le r, \qquad d_\lambda\,\Delta(\lambda) = 0 \tag{9}$$

$$\varphi_\lambda(\theta) = e^{\lambda\,\theta} c_\lambda, \qquad -r \le \theta \le 0, \qquad \Delta(\lambda)\,c_\lambda = 0. \tag{10}$$

2.3 Series Expansion for Lossless Systems

A natural question arise: whether the solution of a given Lossless system (1) can be represented by a series of elementary solutions and under which conditions the convergence is insured. The main idea is to expand the state into a linear combination of eigenvectors ($\varphi_{\lambda_k}(\theta) = p_k(\theta)e^{\lambda_k\theta}$) and generalized eigenvectors:

$$x_t(\theta) = \sum_{k=0}^{\infty} p_k(t+\theta)e^{\lambda_k(t+\theta)}, \quad -h \le \theta \le 0,\ t \ge 0 \tag{11}$$

in other words, in operator language,

$$\mathcal{T}(t)\varphi = \sum_{k=0}^{\infty} \mathcal{T}(t)\,P_{\lambda_k}\varphi,\ t \ge 0. \tag{12}$$

The following two theorems give sufficient conditions for convergence of the power series associated with solutions for (1).

Theorem 2. Let $\mathcal{C} = C([-r,0],\mathbb{R}^{n+m})$ and let $\mathcal{T}(t)\ :\mathcal{C} \to \mathcal{C}$ denote the semigroup of solution operators associated with Lossless system (1) such that the matrix $D \in \mathcal{M}_m(\mathbb{R})$ is invertible, $\lambda_j,\ j = 0, 1, \ldots$ the associated set of eigenvalues and P_{λ_j} the associated spectral projection given by (3) . Then the semigroup $\mathcal{T}(t)$ extends to a group of bounded linear operators and for every $\varphi \in \mathcal{C}$, we have

$$\lim_{N\to\infty} \|\ \mathcal{T}(t)\varphi - \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{k}\mathcal{T}(t)P_{\lambda_j}\varphi\ \| = 0, \qquad t \in \mathbb{R}^+ \tag{13}$$

uniformly on compact t-sets.

The trick here is to identify for (1) its system of eigenvalues, and to prove that it is a complete system. In other words, we design an appropriate Neutral system having the same set of eigenvectors as (1) (apart from those associated with the zero spectral value), for which we can prove the convergence of its series expansion by using Verduyn-Lunel theorem 1, see Verduyn Lunel [1995].

Proof 1. Let us consider system (1) and apply the Standard transformation which leads (2). Here we can not directly apply the result of the theorem 1 since the associated matrix $D_0 = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$, (the matrix of the difference equation associated with the greatest delay) is singular.

However, we write down $y(t-\tau) = D^{-1}y(t) - D^{-1}Cx(t)$ from the second equation of (1). We substitute this fact in the first equation, we have

$$\left.\begin{aligned} \dot{x}(t) &= (A + BD^{-1}C)\,x(t) + BD^{-1}\,y(t) \\ y(t) &= C\,x(t) + D\,y(t-\tau). \end{aligned}\right\}$$

A rescaling of time by τ of the first equation gives

$$\dot{x}(t-\tau) = (A + BD^{-1}C)\,x(t-\tau) + BD^{-1}\,y(t-\tau).$$

Subtracting the last equality from the first equation of (1) and derivating its second equation lead to

$$\left.\begin{aligned} \dot{x}(t) - \dot{x}(t-\tau) &= (A + BD^{-1}C)\,(x(t) - x(t-\tau)) \\ &\quad + BD^{-1}\,(y(t) - y(t-\tau)) \\ \dot{y}(t) - D\,\dot{y}(t-\tau) &= CA\,x(t) + CB\,y(t-\tau). \end{aligned}\right\} \tag{14}$$

Then we obtain a neutral system with non-singular matrix $D_0 = \begin{pmatrix} I_d & 0 \\ 0 & D \end{pmatrix}$, thus the theorem 1 can be applied.

□

Note that the set of the spectral values of the characteristic equation associated with (1) belongs to the set of spectral values of the characteristic equation associated with (14). Moreover, the only additional spectral value is zero with multiplicity m. Thus the set of eigenvalues associated with (14) represents a complete set. But for $\tau \ne 0$, zero is not a spectral value for (1), then $P_0\varphi = 0,\ \forall\varphi \in C$.

Let us consider the system (1) with a two-block singular matrix $D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$ such that D_{11} is a non-singular matrix and $B = (B_1\ B_2)$ and $C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$. Under some restrictions we are able to prove the convergence of the series expansion of the solution. The same trick as the one used to prove the last theorem is applied here; that is the design of an appropriate Neutral system having the same set of eigenvectors as (1) (apart from those associated with the zero spectral value), for which we are able to prove the convergence of its series expansion (invertible matrix D_0).

Theorem 3. Let $\mathcal{C} = C([-r,0],\mathbb{R}^{n+m})$ and let $\mathcal{T}(t)\ :\mathcal{C} \to \mathcal{C}$ denote the semigroup of solution operators associated with Lossless system (1) such that $\lambda_j,\ j = 0, 1, \ldots$ the associated set of eigenvalues and P_{λ_j} the associated spectral projection given by (3) and one of the following sets of conditions hold

(1)

$$\left.\begin{aligned} B_2 &= B_1\,D_{11}^{-1}\,D_{12} \\ D_{22} &= D_{21}\,D_{11}^{-1}\,D_{12} \\ C_2 &= D_{21}\,D_{11}^{-1}\,C_1 \\ D_{11}^2 &= -D_{12}\,D_{21} \end{aligned}\right\}, \tag{15}$$

(2)

$$\left.\begin{aligned} B_2 &= B_1\,D_{11}^{-1}\,D_{12} \\ D_{22} &= 0 \\ D_{21} &= 0 \end{aligned}\right\}. \tag{16}$$

Then the semigroup $\mathcal{T}(t)$ extends to a group of bounded linear operators and for every $\varphi \in \mathcal{C}$, we have

$$\lim_{N\to\infty} \|\ \mathcal{T}(t)\varphi - \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{k}\mathcal{T}(t)P_{\lambda_j}\varphi\ \| = 0, \qquad t \in \mathbb{R}^+ \tag{17}$$

uniformly on compact t-sets.

The proof of the above theorem is in the same spirit of the proof of Theorem 2.

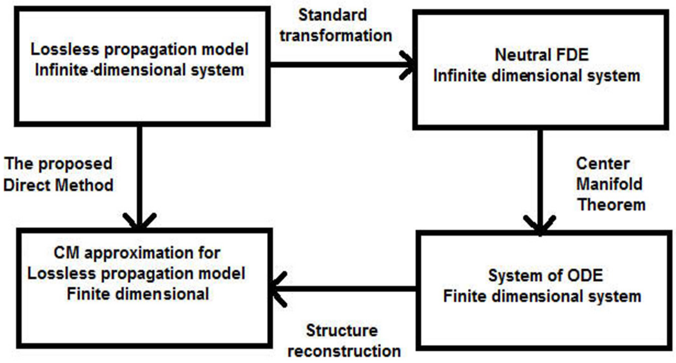

Fig. 1. Model reduction: Center manifold theorem for Lossless systems

2.4 Finite dimensional approximation: Center Manifold theorem for Lossless propagation model

As recalled in the first section, using the transformation (by letting $y(t) = \dot{z}(t)$) one can transform a Lossless system to a NFDE. Certainly this approach increases the number of spectral values to the system and thus the computation of the center manifold for the NFDE will not be valid for Lossless system but detailed inspection shows that there are only a zero additional spectral value with multiplicity $m = dim(y)$. When the left eigenvector associated with a given spectral value (other than zero) is the same for both the neutral structure as for the Lossless structure, then one can carefully distinguish the concrete spectral projections and reject the generalized eigenspace associated to the zero value and then compute a center manifold expansion. Finally, one has to translate the obtained result in terms of Lossless system structure. But in the the remaining case (different left eigenvectors) this approach becomes very difficult. In conclusion, this procedure is possible but still a complicated task. Figure 1 represents the described procedure.

In the light of the obtained results in the previous subsections for Lossless propagation model: the design of an appropriate bilinear form, the definition of the spectral projection and the study of the series expansion for a given solution, we show that the extension of the center manifold theorem to the system (5) is possible. Thus, we are able to present a direct center manifold computations for Lossless systems (5). In what follows, we give the computations schemes that will be illustrated by an example for Hopf-bifurcation Like case.

In this section our aim is to reduce the dimension of the system (5), where $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ and A, B, C, D are real valued matrices. Thus $A \in \mathcal{M}_{m,m}(\mathbb{R})$, $B \in \mathcal{M}_{m,n}(\mathbb{R})$, $C \in \mathcal{M}_{n,m}(\mathbb{R})$, $D \in \mathcal{M}_{n,n}(\mathbb{R})$.

Let us set $z(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $\frac{\hat{d}}{dt} w = \frac{\hat{d}}{dt} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{d}{dt} w_1 \\ 0 \end{bmatrix}$ and consider the linear part of (5), namely, (1) which can be represented as

$$\frac{\hat{d}}{dt} \mathcal{D} z_t = \mathcal{L} z_t \tag{18}$$

where the infinitesimal generator $\mathcal{A}$ and the operators $\mathcal{D}$ and $\mathcal{L}$ are such that

$\mathcal{D}om(\mathcal{A}) =$

$$\{\varphi = (\varphi_1 + \varphi_2) \in \mathcal{C}, \text{ s.t. } \frac{d\varphi}{d\theta} \in \mathcal{C},\ \mathcal{D}\frac{d\varphi}{d\theta} = \mathcal{L}\varphi\}$$

$\mathcal{A}\varphi = \dfrac{d\varphi}{d\theta}$ and

$\mathcal{L}\varphi = (A\varphi_1(0) + B\varphi_2(-\tau), C\varphi_1(0) - \varphi_2(0) + D\varphi_2(-\tau))^T$,

$\mathcal{D}\varphi = (\varphi_1(0), 0)^T$.

The associated characteristic matrix is

$$\Delta(\lambda) = \begin{pmatrix} \lambda I_n - A & -e^{-\lambda\tau} B \\ -C & I_m - e^{-\lambda\tau} D \end{pmatrix} \tag{19}$$

and the solution operator $\mathcal{T}(t)$ defined by

$$\mathcal{T}(t)(\phi) = z_t(.\,,\phi) \tag{20}$$

such that $z_t(.\,,\phi)(\theta) = z(t+\theta,\phi)$ for $\theta \in [-r, 0]$ is a strongly continuous semigroup. Obviously, $\sigma(\mathcal{A}) = \sigma_p(\mathcal{A})$ (since the system can be written as a neutral type) and the spectrum of $\mathcal{A}$ consists of complex values $\lambda \in \mathbb{C}$ which satisfy the characteristic equation $p(\lambda) = \det\Delta(\lambda) = 0$. Let us denote by $\mathcal{M}_\lambda$ the eigenspace associated with $\lambda \in \sigma(\mathcal{A})$. We define $\mathcal{C}^* = C([-r, 0], \mathbb{R}^{n+m^*})$ where $\mathbb{R}^{n+m^*}$ is the space of $(n+m)$-dimensional row vectors and consider the defined bilinear form. Let $\mathcal{A}^T$ be the transposed operator of $\mathcal{A}$, i.e., $< \psi, \mathcal{A}\varphi > = < \mathcal{A}^T\psi, \varphi >$. We underline also the fact that the Banach space decomposition Theorem Hale & Verduyn Lunel [1993] is extended to lossless propagation model under the prescribed sttings. Thus we obtain a decomposition of the space $\mathcal{C} = P \bigoplus Q$ where P ($P = span\{\mathcal{M}_\lambda(\mathcal{A}),\ \lambda \in \Lambda\}$) and Q are invariant under $\mathcal{T}(t), t \geq 0$. Furthermore, if $\Phi = (\phi_1, \ldots, \phi_m)$ forms a basis of P, $\Psi = col(\psi_1, \ldots, \psi_m)$ is a basis of P^T in $\mathcal{C}^*$ such that $(\Phi, \Psi) = Id$, then

$$\begin{aligned} Q &= \{\phi \in \mathcal{C} \setminus (\Psi, \phi) = 0\} \text{ and} \\ P &= \{\phi \in \mathcal{C} \setminus \exists b \in \mathbb{R}^d : \phi = \Phi b\}. \end{aligned} \tag{21}$$

Note also, $\mathcal{T}(t)\Phi = \Phi\, e^{Lt}$, where L is a $d \times d$ matrix such that $\sigma(L) = \Lambda$.

Analogously to the theory for NFDE, we consider the extension of the space $\mathcal{C}$ that contains continuous functions on $[-r, 0)$ with possible jump discontinuity at 0, we denote this space $\mathcal{BC}$. A given function $\xi \in \mathcal{BC}$ can be written $\xi = \varphi + X_0\,\alpha$, where $\varphi \in \mathcal{C}$, $\alpha \in \mathbb{R}^d$ and X_0 is defined by $X_0(\theta) = 0$ for $-r \leq \theta < 0$ and $X_0(0) = Id_{d\times d}$. Then the bilinear form from Sec. 2 can be extended to the space $\mathcal{C}^* \times \mathcal{BC}$ by $< \psi, X_0 > = \psi(0)$ and the infinitesimal generator $\mathcal{A}$ extends to an operator $\tilde{\mathcal{A}}$ (defined in C^1) onto the space $\mathcal{BC}$ as follow

$$\tilde{\mathcal{A}}\phi = \mathcal{A}\phi + X_0[\mathcal{L}\phi - \mathcal{D}\phi'] \tag{22}$$

Under the above consideration one can write equation (5) as an abstract ODE

$$\frac{\hat{d}}{dt} z_t = \tilde{\mathcal{A}} z_t + X_0 \mathcal{F}(z_t). \tag{23}$$

where $\mathcal{F}$ represents the nonlinear part of the system. Thanks to the projection $\Pi : \mathcal{BC} \to P$ such that $\Pi(\varphi + X_0\alpha) = \Phi[(\Psi, \varphi) + \Psi(0)\alpha]$ we obtain $z_t = \Phi v(t) + w_t$ where $v(t) \in \mathbb{R}^d$ and then equation (5) can be split to

$$\begin{aligned} \frac{\hat{d}}{dt} v &= Lv + \Psi(0) F(\Phi v + w) \\ \frac{\hat{d}}{dt} w &= \tilde{\mathcal{A}}_Q + (I - \Pi) X_0 \mathcal{F}(\Phi v + w), \end{aligned} \tag{24}$$

The above decomposition theorem can be easily exploited in the aim of computing the the evolution of solutions on the center manifold (the dimensional reduced system) as well as the explicit expression of the center manifold.

Definition 4. Given a C^1 map h from $\mathbb{R}^d$ into Q. The graph of h is said to be a local manifold if $h(0) = Dh(0) = 0$ and there exists a neighborhood V of $0 \in \mathbb{R}^d$ such that for each $\xi \in V$, there exists $\delta = \delta(\xi) > 0$ and the solution z of (18) with initial data $\Phi\xi + h(\xi)$ exists on the interval $]-\delta - r, \delta[$ and it is given by $z_t = \Phi u(t) + h(u(t))$ for $t \in [0, \delta[$ where $u(t)$ is the unique solution of the ODE

$$\frac{\hat{d}}{dt}u = Lu + \Psi(0)F(\Phi u + h(u)) \quad \text{where} \quad u(0) = \xi. \quad (25)$$

3. SYMBOLIC COMPUTATIONS FOR CENTER MANIFOLDS: HOPF-BIFURCATION LIKE CASE

The aim of this example is not the computations of the center manifold itself but is to provide a finite dimensional approximation of (5) in the sense of the convergence criteria based on Cesaro Sum given by (13). Thus, for the sake of simplicity, we ask for a finite dimensional approximation up to order three and we select the case where the nonlinearity of (5) does not contain quadratic terms, see Campbell [2009] for more hints about such a choice.

Consider system (5) with the delay $\tau = \pi$ and matrices

$$A = 1,\ B = \begin{bmatrix} 1 & -\frac{5}{2} \end{bmatrix},\ C = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ D = \begin{bmatrix} \frac{1}{5} & 0 \\ 5 & \\ 0 & -2 \end{bmatrix},$$

and the nonlinear part is defined by $f_1(x(t), y(t), y(t-\tau)) = x^3(t)$ and $f_2(x(t), y(t), y(t-\tau)) = (0, y_2^3(t-\tau))^T$.

By straightforward computations made by *QPMR*-algorithm (Quasi-Polynomial Mapping based Root-finder, developed for computing the spectrum of both Retarded and Neutral Time-Delay Systems) presented in Vyhlídal & Zítek [2003] show that $\lambda_0 = \pm\frac{i}{2}$ are the only spectral values with zero real part and are of algebraic and geometric multiplicity 1. Moreover, all remaining spectral values are of negative real part. Thus the phase space can be split into an invariant center variety of dimension 2 and a stable variety. Here we compute some important elements for describing the solutions dynamics on the center manifold. For $\lambda_0 = \pm\frac{i}{2}$, the associated right-eigenvector is

$$\varphi_{\frac{i}{2}}(\theta) = \begin{bmatrix} e^{\frac{i\theta}{2}} \\ 0 \\ \frac{1}{5}(1+2i)\, e^{\frac{i\theta}{2}} \end{bmatrix}$$

Thus we consider the real part and the imaginary part separately:

$$\varphi^1_{\frac{i}{2}}(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ 0 \\ \frac{1}{5}\cos(\frac{\theta}{2}) - \frac{2}{5}\sin(\frac{\theta}{2}) \end{bmatrix}$$

and

$$\varphi^2_{\frac{i}{2}}(\theta) = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ 0 \\ \frac{2}{5}\cos(\frac{\theta}{2}) + \frac{1}{5}\sin(\frac{\theta}{2}) \end{bmatrix}.$$

then the centre eigenspace of this steady state $(0, 0, 0)$ can be written in the form
$X = a_1\,\varphi^1_{\frac{i}{2}}(\theta) + a_2\,\varphi^2_{\frac{i}{2}}(\theta)$. Working through it in details allows to

$$X = \begin{bmatrix} x \\ y_1 \\ y_2 \end{bmatrix} =$$

$$= \begin{bmatrix} a_1\cos(\frac{\theta}{2}) + a_2\sin(\frac{\theta}{2}) \\ 0 \\ \frac{a_1}{5}\left(\cos(\frac{\theta}{2}) - 2\sin(\frac{\theta}{2})\right) + \frac{a_2}{5}\left(2\cos(\frac{\theta}{2}) + \sin(\frac{\theta}{2})\right) \end{bmatrix},$$

which proves that $y_1 = 0$ in the center variety.

Thus, approximating (5) on the two-dimensional center variety turns to approximate $(5)|_{y_1=0}$ to obtain after renaming y_2 by y

$$\left.\begin{aligned} \dot{x}(t) &= x(t) - \frac{5}{2}y(t-\pi) + x^3(t) \\ y(t) &= x(t) - 2\,y(t-\pi) + y^3(t-\pi) \end{aligned}\right\}, \quad (26)$$

and $\pm\frac{i}{2}$ are two simple spectral values characterizing the center variety, with left-eigenvectors

$$\varphi^1_{\frac{i}{2}}(\theta) = \begin{bmatrix} \cos(\frac{\theta}{2}) \\ \frac{1}{5}\cos(\frac{\theta}{2}) - \frac{2}{5}\sin(\frac{\theta}{2}) \end{bmatrix}$$

$$\varphi^2_{\frac{i}{2}}(\theta) = \begin{bmatrix} \sin(\frac{\theta}{2}) \\ \frac{2}{5}\cos(\frac{\theta}{2}) + \frac{1}{5}\sin(\frac{\theta}{2}) \end{bmatrix}.$$

then the centre eigenspace of this steady state $(0, 0)$ can be written in the form
$X = a_1\,\varphi^1_{\frac{i}{2}}(\theta) + a_2\,\varphi^2_{\frac{i}{2}}(\theta)$.

The basis of the generalized eigenspace P associated with $\pm\frac{i}{2}$ is

$$\Phi = \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ \frac{1}{5}\cos(\frac{\theta}{2}) - \frac{2}{5}\sin(\frac{\theta}{2}) & \frac{2}{5}\cos(\frac{\theta}{2}) + \frac{1}{5}\sin(\frac{\theta}{2}) \end{bmatrix}.$$

The computation of the left-eigenvectors allows us to

$$\psi_{\frac{i}{2}}(\xi) = \begin{bmatrix} -(4+2i)\,e^{-\frac{i\xi}{2}} & 5\,e^{-\frac{i\xi}{2}} \end{bmatrix}.$$

Analogously to the prescribed FDE theory described in the first section, this leads to a basis for the complementary space Q, which is given by

$$\tilde{\Psi} = \begin{bmatrix} -4\cos\left(\frac{\xi}{2}\right) - 2\sin\left(\frac{\xi}{2}\right) & 5\cos\left(\frac{\xi}{2}\right) \\ 4\sin\left(\frac{\xi}{2}\right) - 2\cos\left(\frac{\xi}{2}\right) & -5\sin\left(\frac{\xi}{2}\right) \end{bmatrix}.$$

The normalized basis of Q denoted Ψ is obtained by using the defined bilinear form (8) such that $<\Psi,\Phi>=I_d$, thus we have

$$\Psi=\begin{bmatrix}\psi_{1,1} & \psi_{1,2}\\ \psi_{2,1} & \psi_{2,2}\end{bmatrix}.$$

such that:

$$\psi_{1,1}=\frac{-256\cos\left(\frac{\xi}{2}\right)+272\sin\left(\frac{\xi}{2}\right)+104\,\pi\cos\left(\frac{\xi}{2}\right)-128\,\pi\sin\left(\frac{\xi}{2}\right)}{-876+184\,\pi+85\,\pi^2}$$

$$\psi_{1,2}=\frac{120\cos\left(\frac{\xi}{2}\right)-40\,\pi\cos\left(\frac{\xi}{2}\right)-400\sin\left(\frac{\xi}{2}\right)+180\,\pi\sin\left(\frac{\xi}{2}\right)}{-876+184\,\pi+85\,\pi^2}$$

$$\psi_{2,1}=\frac{368\cos\left(\frac{\xi}{2}\right)+704\sin\left(\frac{\xi}{2}\right)+128\,\pi\cos(1/2\,\xi)+104\,\pi\sin\left(\frac{\xi}{2}\right)}{-876+184\,\pi+85\,\pi^2}$$

$$\psi_{2,2}=\frac{-720\cos\left(\frac{\xi}{2}\right)-180\,\pi\cos\left(\frac{\xi}{2}\right)-520\sin\left(\frac{\xi}{2}\right)-40\,\pi\sin\left(\frac{\xi}{2}\right)}{-876+184\,\pi+85\,\pi^2}.$$

Hence, the dynamics of (5) in the center manifold is given by the top equation of (25) which can be written

$$\begin{bmatrix}\dfrac{du_1}{dt}\\ 0\end{bmatrix}=L\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\Psi(0)F(\Phi u+h(u)). \qquad (27)$$

where the matrix L characterizing the linear part of the approximation is given by

$$L=\begin{bmatrix}0 & \dfrac{1}{2}\\ -\dfrac{1}{2} & 0\end{bmatrix}.$$

Thus using (27), straightforward computations allows us to the following finite dimensional approximation

$$\begin{aligned}\frac{d\,u_1}{dt}&=\frac{u_2}{2}+\frac{8}{25}\,\frac{(-776+317\,\pi)\,u_1{}^3}{-876+184\,\pi+85\,\pi^2}\\
&-\frac{48}{25}\,\frac{(-3+\pi)\,u_2{}^2u_1}{-876+184\,\pi+85\,\pi^2}+\frac{8}{25}\,\frac{(-3+\pi)\,u_2{}^3}{-876+184\,\pi+85\,\pi^2}\\
&+\frac{96}{25}\,\frac{(-3+\pi)\,u_2\,u_1{}^2}{-876+184\,\pi+85\,\pi^2}\\
u_1&=\frac{32}{25}\,\frac{(503+182\,\pi)\,u_1{}^3}{-876+184\,\pi+85\,\pi^2}+\frac{864}{25}\,\frac{(4+\pi)\,u_2\,u_1{}^2}{-876+184\,\pi+85\,\pi^2}\\
&-\frac{432}{25}\,\frac{(4+\pi)\,u_2{}^2u_1}{-876+184\,\pi+85\,\pi^2}+\frac{72}{25}\,\frac{(4+\pi)\,u_2{}^3}{-876+184\,\pi+85\,\pi^2}\end{aligned} \qquad (28)$$

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers for valuable comments.

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