A Functional-Differential System of Neutral Type Arising in a Two-Body Problem of Classical Electrodynamics

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Introduction

Functional-differential equations of the form

$$y'(t) = f(t, y(t), y(g(t)), y'(g(t))) \qquad \text{for} \qquad t > t_0,$$

where f and g are given functions, $g(t) \leqslant t$, and $y'(g(t))$ stands for dy/dt evaluated at $g(t)$, are called equations of neutral type. The term apparently arose because in certain simple cases the equation could equally well be considered as having a retarded argument or an advanced argument. Such a case would occur, for example, if $g(t) = t - \tau$, where τ is a positive constant, and the equation could be solved for $y'(t - \tau)$.

The existence and uniqueness of solutions for equations or systems of equations of this type has been discussed by Èl'sgol'c [3], Kamenskiĭ [5], and others.

The present paper is concerned with a functional-differential system which arises in one approach to the two-body problem of classical electrodynamics. I refer to this as a system of neutral type because of its resemblence to the equation above. However, in this system, the retarded arguments will depend upon the dependent variables as well as the independent variable. We will consider the system

$$y_i'(t) = f_i(t, y(t), y(g_1(t, y(t))), \ldots, y(g_m(t, y(t))), y'(g_1(t, y(t))), \ldots, y'(g_m(t, y(t))))$$
$$\text{for} \quad t > t_0 \qquad (i = 1, \ldots, n),$$

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where $y(t) = (y_1(t), ..., y_n(t))$, the functions $f_1, ..., f_n$ and $g_1, ..., g_m$ are given, each $g_j(t, y) < t$—a strict inequality—for the values of t and y under consideration, and $y'(g_j(t, y(t)))$ stands for dy/dt evaluated at $g_j(t, y(t))$. More briefly we will write

$$y'(t) = f(t, y(t), y(g(t, y(t))), y'(g(t, y(t)))),$$

where $y(g(t, y(t)))$ stands for $(y(g_1(t, y(t))), ..., y(g_m(t, y(t))))$ and similarly for $y'(g(t, y(t)))$.

A *local* existence and uniqueness theorem for such a system is simple. Let $\alpha \leqslant g_j(t, y) < t$ for all values of t and y under consideration with $t \geqslant t_0$. One then sets $y(t) = \varphi(t)$, a given continuously differentiable function, on $[\alpha, t_0]$. Then $y(t)$ can be continuously extended for a short time to the right of t_0 so as to satisfy the functional-differential system, provided that the ordinary differential system

$$y'(t) = f(t, y(t), \varphi(g(t, y(t))), \varphi'(g(t, y(t))))$$

has a solution to the right of t_0 with $y(t_0) = \varphi(t_0)$. If the solution of the ordinary differential system is unique then so is the solution of the functional-differential system. Appropriate continuity and Lipschitz conditions on the given functions, f, g, φ, and φ', will assure the local existence and uniqueness.

Due to the arbitrariness of the initial function, $\varphi(t)$, there is no reason to expect that $y'(t_0 - 0) = y'(t_0 + 0)$. Thus the extension, $y(t)$, of $\varphi(t)$ will not have a continuous derivative at t_0. Thus one can expect some difficulty extending the solution past any instant, $t_1 > t_0$, at which $g_j(t_1, y(t_1)) = t_0$ for some j.

In the case of difference-differential equations of neutral type—i.e. if each $g_j(t, y) = t - \tau_j$, where each τ_j is a positive constant—this difficulty is easily overcome. More generally, the difficulty is easily overcome whenever each $g_j(t, y) = g_j(t)$, a given function of t alone, and, on some finite interval $[t_0, T]$, each $g_j(t) =$ any constant only a finite number of times. One can then define a solution to be a continuous extension of $\varphi(t)$ which satisfies the functional-differential system on (t_0, β) for some $\beta \in (t_0, T]$ except at the sequence of points, $t_0 < t_1 < t_2 < ... < \beta$, defined inductively by the condition $g_j(t_k) = t_0, t_1, ...,$ or t_{k-1} for some $j = 1, ...,$ or m $(k = 1, 2, ...)$. At these exceptional points, $y(t)$ will not, in general, be differentiable.

For systems in which $g_j(t, y)$ really depends upon t *and* y the appropriate hypothesis—in addition to continuity requirements—is not immediately clear. The following two examples illustrate the difficulty.

Consider the scalar equation

$$y'(t) = -y'(t - y^2(t)/4) \qquad \text{for} \qquad t > 0,$$

with

$$\varphi(t) = 1 - t \qquad \text{for} \qquad t \leqslant 0.$$

As long as $y > 0$, $g(t, y) = t - y^2/4 < t$. The functions f, g, and φ are analytic for all values of their arguments, so there is no question about the continuity or Lipschitz conditions. The obvious method of defining a solution gives $y(t) = 1 + t$ for $0 \leqslant t \leqslant 1$. However, for $1 < t < 3$ both $y(t) = 1 + t$ and $y(t) = 3 - t$ are solutions. Moreover, for each of these solutions we find that every equation $g(t, y(t)) =$ a constant has only a finite number of solutions.

Another innocent-looking example is the scalar equation

$$y'(t) = -y'(y(t) - 2) \qquad \text{for} \qquad t > 0,$$

with

$$\varphi(t) = 1 - t \qquad \text{for} \qquad t \leqslant 0.$$

To assure that $g(t, y) < t$ we consider this equation on a domain in which $y < t + 2$. One readily finds $y(t) = 1 + t$ for $0 \leqslant t \leqslant 1$. However, no solution exists for $t > 1$, since $y(t) > 2$ would imply $y'(t) < 0$ and $y(t) < 2$ would imply $y'(t) > 0$.

The additional hypothesis which we will impose to obtain unique solutions for the equations considered in this paper is that each $g_j(t, \eta(t))$ be a strictly increasing function of t whenever $\eta(t)$ belongs to a certain class of functions which includes all functions that could possibly be solutions. This hypothesis, as it will be stated, is satisfied in the electrodynamics problem to be discussed.

1. Extended Existence and Uniqueness

The equations of motion for the two-body problem of classical electrodynamics have certain singularities, e.g. in case of a collision or in case of a charge traveling with the speed of light. Thus it will be important to assume restricted domains of definition for the functions f and g. The work of Èl'sgol'c and Kamenskiĭ, cited in the introduction, essentially assumes that f is defined for all values of its arguments.

Since the solution will not have any more than a piecewise continuous derivative, nothing is gained by imposing more than piecewise continuity conditions on $\varphi'(t)$.

Let n and m be positive integers. Let

$$(t, Y) = (t, y_1, \dots, y_n, y_{11}, \dots, y_{1n}, \dots, y_{m1}, \dots, y_{mn}, z_{11}, \dots, z_{1n}, \dots, z_{m1}, \dots, z_{mn})$$

denote a typical vector in $E^{1+n+2nm}$, Euclidean space of $1 + n + 2nm$ dimensions. Let

$$(t, y) = (t, y_1, \dots, y_n)$$

be the projection of (t, Y) into E^{1+n}, the space of the first $1 + n$ coordinates of $E^{1+n+2nm}$.

Let $D = D^{1+n+nm}$ be a domain (a connected open set) in E^{1+n+nm}, the space of the first $1 + n + nm$ coordinates of $E^{1+n+2nm}$, and let

$$f(t, Y) = (f_1(t, Y), \dots, f_n(t, Y))$$

be a real, n-vector-valued function over $D \times E^{nm}$. Thus $f(t, Y)$ is defined over a domain in $E^{1+n+2nm}$ whose boundary puts no restrictions on the values of the z_{ji} $(i = 1, \dots, n,\ j = 1, \dots, m)$.

Let D^{1+n} be the domain obtained by projecting D into E^{1+n}, and let $g(t, y) = (g_1(t, y), \dots, g_m(t, y))$ be a real, m-vector-valued function over D^{1+n}.

In terms of the above notation we can now state

Problem 1 (A Functional-Differential System of Neutral Type). Let $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$ be a continuous, n-vector-valued function of t on an interval $[\alpha, t_0]$—interpreted as $(\alpha, t_0]$ in case $\alpha = -\infty$. Let $\varphi(t)$ be differentiable on (α, t_0) except at a finite number of points, $\alpha = t_{-p} < t_{-p+1} < \dots < t_{-1} < t_0$. We seek a continuous extension of $\varphi(t)$ to $y(t)$ on $[\alpha, \beta)$, where $\beta > t_0$, such that

(a) $(t, y(t)) \in D^{1+n}$ for $t_0 \leqslant t < \beta$,

(b) $\alpha \leqslant g_j(t, y(t)) < t$ for $t_0 \leqslant t < \beta, j = 1, \dots, m$,

(c) $(t, y(t), y(g(t, y(t)))) \in D$ for $t_0 \leqslant t < \beta$, and

(d) $y'(t) = f(t, y(t), y(g(t, y(t))), y'(g(t, y(t))))$ for $t_0 < t < \beta$ except at points $t_0 < t_1 < t_2 < \dots < \beta$, each t_k of which satisfies the condition $g_{j_k}(t_k, y(t_k)) = t_{-p}, t_{-p+1}, \dots, t_{-1}, t_0, t_1, \dots,$ or t_{k-1} for some $j_k = 1, 2, \dots,$ or m.

Definition. A solution, $y(t)$, of problem 1 will be called *unique* if every other solution coincides with $y(t)$ in their common region of definition.

Notation. Throughout this section, it is convenient to take the norm of a vector $z = (z_1, \ldots, z_p) \in E^p$ to be

$$\| z \| = \max_{i=1,\ldots,p} | z_i |.$$

For a function $h(t, z)$ defined in some region, R, the statement

$$h(t, z) \in \text{Lip in } R$$

will mean that $h(t, z)$ is Lipschitz continuous with respect to variations in all its arguments throughout R, i.e. there exists a (Lipschitz) constant, A, such that $\| h(t, z) - h(\hat{t}, \hat{z}) \| \leqslant A \max (| t - \hat{t} |, \| z - \hat{z} \|)$ whenever (t, z), $(\hat{t}, \hat{z}) \in R$.

Theorem 1. (Extended Existence and Uniqueness Theorem). Let $f(t, Y)$ and $g(t, y) \in$ Lip in each compact subset of $D \times E^{nm}$ and D^{1+n} respectively. Let $g_j(t, y) < t$ for all $(t, y) \in D^{1+n}$ $(j = 1, \ldots, m)$. Let each $g_j(t, \eta(t))$ be a strictly increasing function of t whenever, for some continuous functions $\bar{y}(t)$ from $(-\infty, \infty)$ into E^n and $z(t, y)$ from D^{1+n} into E^{nm}, $(t, \eta(t), \bar{y}(g(t, \eta(t)))) \in D$ and $\eta'(t) = f(t, \eta(t), \bar{y}(g(t, \eta(t))), z(t, \eta(t)))$. Let $\varphi(t)$ satisfy the conditions imposed on it in problem 1, and let conditions (a), (b), and (c) of that problem be satisfied at the instant t_0. Let $\varphi(t)$ be Lipschitz on $[\alpha, t_0]$ and let $\varphi'(t)$ be Lipschitz on each interval (t_k, t_{k+1}) $(k = -p, \ldots, -1)$.

Then problem 1 has a unique solution, $y(t)$, on $[\alpha, \beta)$, where $\beta > t_0$ and for any compact set $F \subset D$ there is a sequence of numbers, $t_0 < \xi_1 < \xi_2 < \ldots \to \beta$ such that

$$(\xi_i, y(\xi_i), y(g(\xi_i, y(\xi_i)))) \in D - F \quad \text{for } i = 1, 2, \ldots.$$

Remark. The hypotheses of this theorem could be weakened in various ways. For example, the region of definition of each $\varphi_i(t)$ could be reduced (cf. ref. [*2*] where the analogous problem for a delay-differential system is discussed). In particular, if some component of $y(t)$, say $y_i(t)$, and its derivative do not appear with retarded arguments on the right hand side of the functional-differential system, then, for that component, the initial value, $\varphi_i(t_0)$, is sufficient initial data. Also, in this case, f_i need not be Lipschitz continuous with respect to t.

Outline of Proof of Theorem 1. (1) Prove that there exists a unique solution on $[\alpha, t_1)$ where either t_1 satisfies the conditions on β asserted by the theorem, or else

$$\lim_{t \to t_1} g_j(t, y(t)) = t_{-p}, t_{-p+1}, \ldots, \text{ or } t_0 \qquad \text{for some} \qquad j = 1, 2, \ldots, \text{ or } m.$$

In this proof one must allow for the possibility that, for some $j = 1, \ldots,$ or m, it may be the case that

$$g_j(t_0, \varphi(t_0)) = t_{-p}, t_{-p+1}, \ldots, \text{or } t_{-1}.$$

(2) Prove that this basic existence and uniqueness step can be iterated to any finite number of points, $t_0 < t_1 < t_2 < \ldots < t_q$ as defined in (d) of problem 1, as long as none of these points satisfies the conditions on β asserted in the theorem.

(3) Suppose (for contradiction) that $y(t)$ is a solution on $[\alpha, \beta)$, where $t_0 < \beta < \infty$, β cannot be increased, and yet for some compact set $F \subset D$

$$(t, y(t), y(g(t, y(t)))) \in F \qquad \text{for all} \qquad t \in [t_0, \beta).$$

Then show that this implies that there are only a finite number of points in the sequence defined in (d) of problem 1 and that the solution can be extended beyond β.

(4) Prove that the solution is unique by a method quite analogous to the extended uniqueness argument for ordinary differential equations.

2. Application to a Two-Body Problem of Classical Electrodynamics

Modern physics references make extensive use of a 4-dimensional space-time notation in which time is the intrinsic (or proper) time of the particle under consideration. Here, however, we will use position vectors (indicated by bold-face type) in E^3, a Euclidean reference frame in some given inertial system, and the time, t, of an observer in that system.

Let $\mathbf{r}_i(t)$ be the position of the i'th point charge ($i = 1, 2$) at time t, and let $\mathbf{v}_i(t) = \mathbf{r}_i'(t)$ be its velocity. Let q_i be the (constant) magnitude and m_i the (constant positive) rest mass of the i'th charge. Let c be the speed of light and ϵ_0, the dielectric constant of free space (two universal positive constants). Then, using the M.K.S. system of units, the Lorentz-Abraham equation (or Dirac equation) for the motion of the i'th charge can be written in the form

$$\mathbf{v}_i(t) - \frac{q_i^2}{6\pi\epsilon_0 c^3 m_i}\left\{\frac{\mathbf{v}_i''(t)}{[1 - v_i^2(t)/c^2]^{1/2}} + \frac{3[\mathbf{v}_i(t)\cdot\mathbf{v}_i'(t)]\,\mathbf{v}_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}\,c^2}\right\}$$
$$= m_i^{-1}[1 - v_i^2(t)/c^2]^{1/2}\,[\mathbf{F}_i - (\mathbf{F}_i\cdot\mathbf{v}_i(t))\,\mathbf{v}_i(t)/c^2], \tag{2.1}$$

where $\mathbf{F}_i = q_i[\mathbf{E}_i + \mathbf{v}_i(t) \times \mathbf{B}_i]$, $\mathbf{E}_i$ being the electric field and $\mathbf{B}_i$ the magnetic field at $(t, \mathbf{r}_i(t))$ produced by all sources other than the i'th charge itself. In the two-body problem, these fields depend upon t, $\mathbf{r}_i(t)$, and the trajectory of the other charge, call it charge j. The above equation does not look the same as the equations found in the literature (cf. Schott [6], p. 246). This is because of the different notation, and because in (2.1) the relativistic expression for $\mathbf{F}_i$ has been essentially solved for the highest order derivative $\mathbf{v}_i''(t)$.

The second term on the left hand side of (2.1), { }, is the radiation reaction, representing the action of the i'th charge on itself. Its exact form is not universally accepted by physicists and it is sometimes omitted. Since the analysis of the problem turns out to be easier and the qualitative results appear to be more in agreement with our physical intuition when this term is not present, it will be omitted for the remainder of this paper. Then (2.1) expresses $\mathbf{v}_i'(t)$ in terms of $\mathbf{F}_i$, with all relativistic mass corrections incorporated in the right hand side.

Since electromagnetic effects propagate with speed c, the fields at $\mathbf{r}_i(t)$ at time t depend on the state of charge j at an earlier instant, $t - \tau_{ji}(t)$. The propagation time, $\tau_{ji}(t)$, must satisfy the functional equation

$$\tau_{ji}(t) = | \mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t)) |/c \qquad \text{for} \qquad (j, i) = (2,1), (1,2), \tag{2.2}$$

where $|\ |$ implies the Euclidean norm or length of a vector. This equation is not given explicitly in most physics references, but it, or an equivalent, is implied.

If (2.2) has a solution, and assuming that the electric and magnetic fields of a moving charge are those calculated from the Liénard-Wiechert potentials (cf. Schott [6], pp. 22, 23), $\mathbf{F}_i$ can be expressed as a function of $q_i q_j$, $\mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t))$, $\mathbf{v}_i(t)$, $\mathbf{v}_j(t - \tau_{ji}(t))$, and $\mathbf{v}_j'(t - \tau_{ji}(t))$. The acceleration of charge j at the retarded time, $t - \tau_{ji}(t)$, enters in an important way because it is one of the principle quantities determining the radiation field of a moving charge. If (2.2) has no solution then charge j does not influence charge i, thus $\mathbf{F}_i = 0$.

Without specifying $\mathbf{F}_i$ explicitly, (2.1) can now be written in the form

$$\mathbf{v}_i'(t) = \begin{cases} q_i q_j m_i^{-1} \mathbf{f}(\mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t)), \mathbf{v}_i(t), \mathbf{v}_j(t - \tau_{ji}(t)), \mathbf{v}_j'(t - \tau_{ji}(t))) \\ \qquad \text{whenever (2.2) has a solution,} \\ 0 \qquad \text{whenever (2.2) has no solution,} \end{cases} \tag{2.3}$$

where $(j, i) = (2, 1)$ or $(1, 2)$ and $\mathbf{f}$ is analytic as long as

$$\mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t)) \neq 0, \qquad | \mathbf{v}_i(t) | < c \qquad \text{and} \qquad | \mathbf{v}_j(t - \tau_{ji}(t)) | < c.$$

This system now resembles the system of neutral type considered in section 1 except for the fact that the retarded arguments depend upon the dependent variables according to an implicit functional condition, (2.2), rather than explicitly.

The problem we will pose for the system based upon (2.2) and (2.3) is

Problem 2 (Two-Body Problem Without Radiation Reaction). Let initial trajectories $\mathbf{r}_i(t)$ $(i = 1, 2)$, be given which are continuously differentiable for $\alpha \leqslant t \leqslant t_0$, where $\alpha < t_0$. Let each $\mathbf{r}_i(t)$ also be twice differentiable with a bounded second derivative on (α, t_0) except at a finite number of points, $\alpha = t_{-p} < t_{-p+1} < \ldots < t_{-1} < t_0$. We seek an extension, $\mathbf{r}_i(t)$ $(i = 1, 2)$, of the initial trajectories to $\alpha \leqslant t < \beta$, where $\beta > t_0$, such that each $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t) = \mathbf{r}_i'(t)$ is continuous for $t_0 \leqslant t < \beta$, and

(a) $\mathbf{r}_1(t) \neq \mathbf{r}_2(t)$ for $t_0 \leqslant t < \beta$,

(b) $|\mathbf{v}_i(t)| < c$ for $t_0 \leqslant t < \beta$, $i = 1, 2$, and

(c) $\mathbf{r}_i(t)$ and $\mathbf{v}_i(t)$ $(i = 1, 2)$ satisfy the equations of motion, (2.3) for $t_0 < t < \beta$ except at points $t_0 < t_1 < t_2 < \ldots < \beta$, each t_k of which satisfies the condition $t_k - \tau_{ji}(t_k) = t_{-p}, \ldots, t_{-1}, t_0, t_1, \ldots,$ or t_{k-1} for $(j, i) = (2, 1)$ or $(1, 2)$, where $\tau_{ji}(t_k)$ is a solution of (2.2) at $t = t_k$.

We will first state (without proof) conditions under which the functional equations (2.2) have a solution and can be transformed into delay-differential equations. This will lead to a differential system of the type considered in section 1.

Theorem 2.1. (Existence, Uniqueness, and Properties of Solutions of (2.2)). (a) Let $\mathbf{r}_j(t)$ be continuous on $[\alpha, \beta]$, where $\alpha < \beta < \infty$, and continuously differentiable with $|\mathbf{v}_j(t)| < c$ on (α, β). Then, for any $t \in [\alpha, \beta]$, a solution, $\tau_{ji}(t)$, of (2.2) exists if and only if, for some finite $\bar{\alpha} \geqslant \alpha$,

$$|\mathbf{r}_i(t) - \mathbf{r}_j(\bar{\alpha})| \leqslant c(t - \bar{\alpha}).$$

The solution of (2.2), if it exists, is unique and satisfies the inequalities

$$|\mathbf{r}_i(t) - \mathbf{r}_j(t)|/2c \leqslant \tau_{ji}(t) \leqslant t - \bar{\alpha}.$$

(b) Now, in addition to the continuity and differentiability of $\mathbf{r}_j(t)$, let $\mathbf{r}_i(t)$ be continuous on $[t_0, \beta]$ and continuously differentiable with $|\mathbf{v}_i(t)| < c$ on (t_0, β), for some $t_0 \in (\alpha, \beta)$. Let (2.2) have a solution, $\tau_{ji}(t_0)$, at t_0 and let $\mathbf{r}_i(t) \neq \mathbf{r}_j(t)$ on $[t_0, \beta]$. Then for all $t \in [t_0, \beta]$ there is a unique solution, $\tau_{ji}(t)$, of (2.2), $\tau_{ji}(t)$ is continuous on $[t_0, \beta]$, it is continuously differentiable on (t_0, β) and satisfies the delay-differential equation

$$\tau'_{ji}(t) = \frac{[\mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t))] \cdot [\mathbf{v}_i(t) - \mathbf{v}_j(t - \tau_{ji}(t))]}{| \mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t)) | c - [\mathbf{r}_i(t) - \mathbf{r}_j(t - \tau_{ji}(t))] \cdot \mathbf{v}_j(t - \tau_{ji}(t))} \quad (2.4)$$

and $\tau_{ji}(\beta) = 0$ if and only if $\mathbf{r}_i(\beta) = \mathbf{r}_j(\beta)$—a collision.

Theorem 2.2. (Extended Existence and Uniqueness Theorem for the Two-Body Problem Without Radiation Reaction). Let each $\mathbf{r}_i(t)$ $(i = 1, 2)$ satisfy the conditions imposed on it in problem 2 for $\alpha \leqslant t \leqslant t_0$ and let each $\mathbf{v}'_i(t)$ be Lipschitz continuous on each of the open intervals (t_k, t_{k+1}) $(k = -p, ..., -1)$. Also let

(1) $\mathbf{r}_1(t_0) \neq \mathbf{r}_2(t_0)$,

(2) $| \mathbf{v}_i(t) | < c$ for $\alpha \leqslant t \leqslant t_0$, $i = 1, 2$, and

(3) equations (2.2) have a solution $\tau_{ji}(t_0)$, at t_0 for $(j, i) = (2, 1)$ and $(1, 2)$.

Then problem 2 has a unique solution for $\alpha \leqslant t < \beta$, where either $\beta = +\infty$ or else

$$\lim_{t \to \beta - 0} \mathbf{r}_1(t) = \lim_{t \to \beta - 0} \mathbf{r}_2(t)$$

—a collision.

Method of Proof. One defines an appropriate problem 1, based on the system composed of (2.3) and (2.4), and shows that this problem 1 has a unique solution. One then shows an equivalence between the problem 1 and problem 2 under consideration, which implies the existence of a unique solution for problem 2. To show that the solution does indeed exist until a collision occurs, one must prove, among other things, that no delays, $\tau_{ji}(t)$, approach zero and that no speeds approach the speed of light in a finite time unless there is a collision. This latter calculation depends upon the specific form of (2.3) and hence requires a more detailed statement of the equations than is presented in this paper.

3. Further Remarks and Unsolved Problems

The N-body problem without radiation reaction can be treated quite similarly, in fact one can even allow the presence of a suitably restricted external field not due to the charges under consideration. This paper was restricted to the two-body problem partly for simplicity and partly to emphasize the fact that the existence theorems for even this most basic problem had not been given before.

When the radiation reaction term of (2.1) is included, a higher order derivative, $\mathbf{v}_i''(t)$, is introduced and the equation no longer appears to be of neutral type. However, due to the apparent necessity, for physical reasons, of allowing the accelerations to be discontinuous at t_0, the system still exhibits behavior something like that of a system of neutral type. It is possible to modify section 1 so that it applies to such system. However, the existence theorem for the two-body problem in this case is not yet in such a satisfactory state as Theorem 2.2.

Another unsettled question of existence is the following. If one omitted condition (3) of Theorem 2.2 would a unique solution exist? This would amount to assuming that the past trajectories had been such that one or both charges had not yet come under the influence of the other. See the example discussed by Havas [*4*].

For the simpler special case of the two-body problem without radiation reaction in which the initial trajectories of both charges are restricted to the x-axis the functional-differential system is no longer of neutral type, since the acceleration terms disappear from the right hand side of (2.3). In this case one can obtain rather specific qualitative information about the future trajectores [*1*]. One would expect that similar information could be obtained in the case of three-dimensional motion as considered here.

A priori, there is at least one other formulation of the two-body problem using the equations of motion discussed in this paper. Instead of specifying rather arbitrary initial trajectories as in problem 2, one might hope to find trajectories which satisfy the equations of motion for all $t < \beta$. Then a unique solution might be determined by specifying appropriate "initial" data at one point, t_0, or at a different point for each charge. Synge [*7*] discussed this approach heuristically, but apparently no existence theorem has been proved. If this could be treated successfully, one might also be able to deal with equations having both retarded and advanced arguments.

Returning to section 1, it is dissatisfying not to have some theorem on the dependence of the solution on the initial data and on the right hand sides of the equations. Continuous dependence can be asserted for delay-differential equations [*2*] and, with an appropriate norm, for equations of neutral type when the retarded arguments depend only on t.

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