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State derivative feedback in second-order linear systems: A comparative analysis of perturbed eigenvalues under coefficient variation

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ABSTRACT

This paper presents a comparative study of sensitivity to parameter variation in two feedback techniques applied in second-order linear systems: state feedback technique and the less conventional state derivative feedback technique. The former uses information on displacements and velocities whereas the latter uses velocities and accelerations. Several contributions on the problem of partial or full eigenvalue/eigenstructure assignment using the state feedback technique are presented in the literature. Recently, some interesting possibilities, such as solving the regularization problem in singular mass second-order systems, are approached using state derivative feedback. In this work, a general equivalence between state feedback and state derivative feedback is first established. Then, figures of merit on the resulting perturbed spectrum are proposed in order to assess the sensitivity of the closed-loop system to variations on the system matrices. Numerical examples are presented to support the obtained results.

1. Introduction

In the study of second-order linear models, the quadratic eigenvalue problem (QEP) arises in a natural manner. This problem has great relevance in the analysis and design of vibroacoustic systems, seismic analysis, and vibration control, among other topics [1–5]. In the problem studied in this work, active vibration control, the designer is interested in the assignment of part of or the entire spectrum (eigenvalues) or eigenstructure of the QEP. A series of relevant contributions on partial eigenvalue/eigenstructure assignment with no spillover can be found [6–8], including the additional minimum norm gains requisite. In all these works, state feedback is used to match the design. Very recently, the minimum-norm problem in the context of no spillover performance is addressed by Zhan [9] using state derivative feedback, and a small number of contributions using this technique can be found [10–12]. State derivative feedback has received attention in the last decade in the eigenvalue placement problem for first-order state-space systems [13–16]. The rapid development and diffusion of accelerometers contributes to the growing interest in this technique [17,18]. Some recent contributions show the advantages

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and applications of this technique to second-order systems [10–12,19] for example, in the regularization of singular mass matrix systems. The present work compares the sensitivity of those feedback techniques when uncertainty is present in the stiffness, damping, or mass matrices: a somewhat common occurrence in engineering applications. Figures of merit are introduced thus allowing a fair comparison between the state feedback and state derivative techniques. Since considerable knowledge is already available for the state feedback case, equivalences are constructed here to facilitate a rapid migration so that state derivative feedback gains may be applied once they are computed. A collection of examples is given to confirm the merit of the analyses.

2. Theory and calculations

The quadratic eigenvalue problem arises from second-order dynamic equations of the type

$$M\frac{d^2\mathbf{x}(t)}{dt^2}+C\frac{d\mathbf{x}(t)}{dt}+K\mathbf{x}(t)=B\mathbf{u}(t), \tag{1}$$

in which $M \in \mathbb{R}^{n\times n}$ is a positive definite matrix, which is diagonal in general, of mass; $C, K \in \mathbb{R}^{n\times n}$ are, respectively, positive semidefinite matrices of damping and stiffness; $B \in \mathbb{R}^{n\times p}$ is a full-rank column control matrix, $\mathbf{x} \in \mathbb{R}^n$ is the displacement vector, and $\mathbf{u} \in \mathbb{R}^p$ is a control vector.

For a solution as first order, an alternative representation that leads to a doubled dimension equivalent system can be constructed considering the augmented state vector $\tilde{\mathbf{x}}^T \triangleq \begin{bmatrix}\mathbf{x}^T & \dot{\mathbf{x}}^T\end{bmatrix}$:

$$\dot{\tilde{\mathbf{x}}}(t)=\begin{bmatrix}0 & I\\ -M^{-1}C & -M^{-1}K\end{bmatrix}\tilde{\mathbf{x}}(t)+\begin{bmatrix}0\\ B\end{bmatrix}\mathbf{u}(t) \tag{2}$$

The homogeneous solutions of the dynamic system given by Eq. (1) are given by $\mathbf{x}(t)=\mathbf{w}e^{\lambda t}$, in which $\mathbf{w}$ is a constant vector and the modes λ are the solutions of the eigenvalue-eigenvector equation

$$Q(\lambda_k)\mathbf{z}_k=0, \quad k=1,2\ldots,2n \tag{3}$$

with

$$Q(\lambda)=\lambda^2M+\lambda C+K \tag{4}$$

A better relative stability/damping characteristic or a desired transient response can be achieved with a feedback control law that assigns part of or the entire spectrum/eigenstructure. The traditional state feedback technique consists in applying the control law:

$$\mathbf{u}(t)=F_s\dot{\mathbf{x}}(t)+G_s\mathbf{x}(t) \tag{5}$$

in which $F_s, G_s \in \mathbb{R}^{p\times n}$ This leads to the following closed-loop pencil:

$$Q_s(\lambda)=\lambda^2M+\lambda(C-B_sF_s)+K-B_sG_s \tag{6}$$

Alternatively, the state derivative feedback control law is given by

$$\mathbf{u}(t)=F_d\ddot{\mathbf{x}}(t)+G_d\dot{\mathbf{x}}(t) \tag{7}$$

with $F_d, G_d \in \mathbb{R}^{p\times n}$ On then has the closed-loop quadratic pencil

$$Q_d(\lambda)=\lambda^2(M-B_dF_d)+\lambda(C-B_dG_d)+K \tag{8}$$

Notice that the matrix control B may be fixed, as it is in many situations; but sometimes it may be a design parameter since actuator technology permits making it non-sparse. This is discussed in [20].

A straightforward conclusion from this pencil is that null eigenvalues cannot be reassigned. Then, from now on, the open loop stiffness matrix K is assumed to be non-singular. Since null eigenvalues in closed-loop are undesirable, the closed-loop matrix $K-B_sG_s$ is also assumed to be non-singular.

3. Equivalence with state feedback and sensitivity analysis

Since the spectrum of pencils as Q_s and Q_d is invariant under multiplication by an invertible matrix, and assuming that M and $M-B_dG_d$ are nonsingular, one can define the following modified pencils:

$$\hat{Q}_s(\lambda)=\lambda^2I+\lambda M^{-1}(C-B_sF_s)+M^{-1}(K-B_sF_s) \tag{9}$$

$$\hat{Q}_d(\lambda)=\lambda^2I+\lambda(M-B_dG_d)^{-1}(C-B_dF_d)+(M-B_dG_d)^{-1}K \tag{10}$$

Expressions for fast migration between state feedback and state derivative feedback are proposed below. By imposing equality for the pencil coefficients, first we have

$$(M-B_dG_d)^{-1}K = M^{-1}(K-B_sF_s) \tag{11}$$

After multiplying by K^{-1} and then taking the inverse, it is possible to isolate the product B_dG_d as

$$B_dG_d = \left[I-(I-B_sG_sK^{-1})^{-1}\right]M \tag{12}$$

Taking the same steps, one can isolate the product B_dF_d as

$$B_dF_d = C-(I-B_sG_sK^{-1})^{-1}(C-B_sF_s) \tag{13}$$

Eqs. (12) and (13) can be used in a combined form to easily compute B_d, F_d and G_d using singular-value decomposition (SVD) as in the work of Datta [20]. Notice that, in the case of a fixed $B = B_s = B_d$, since every column in its matrix is of full rank, Eqs. (12) and (13) can be reduced to

$$G_d = B^+\left[I-(I-BG_sK^{-1})^{-1}\right]M \tag{14}$$

$$F_d = B^+\left[C-(I-BG_sK^{-1})^{-1}(C-BF_s)\right] \tag{15}$$

in which the superscript + denotes the Moore–Penrose inverse given by $B^+ = (B^TB)^{-1}B^T$.

We proceed now to analyzing the sensitivity of the two feedback techniques for a given design in which the spectrum is assigned in the same position in both cases. Recall that the following relations hold:

$$\prod \lambda_{cls} = \det\left[M^{-1}(K-B_sG_s)\right] = \frac{\det(K-B_sG_s)}{\det M} \tag{16}$$

$$\sum \lambda_{cls} = -\mathrm{tr}[M^{-1}(C-B_sF_s)] \tag{17}$$

$$\prod \lambda_{cld} = \det\left[(M-B_dG_d)^{-1}K\right] = \frac{\det K}{\det(M-B_dG_d)} \tag{18}$$

$$\sum \lambda_{cld} = -\mathrm{tr}[(M-B_dG_d)^{-1}(C-B_dF_d)] \tag{19}$$

in which the subscripts *cls* and *cld* stand, respectively, for state feedback and state derivative feedback. Consider the following sensitivity merit figures:

$$S_K = \frac{\partial \prod \lambda}{\partial K} \tag{20}$$

$$S_C = \frac{\partial \sum \lambda}{\partial C} \tag{21}$$

defined for variation in the stiffness and damping matrices. Since the M matrix may (but not in general) suffer uncertainty in its rated value, two more sensitivity figures are given as functions of these variations:

$$S_{Mp} = \frac{\partial \prod \lambda}{\partial M} \tag{22}$$

$$S_{Ms} = \frac{\partial \sum \lambda}{\partial M} \tag{23}$$

Returning to the identities for those derivatives [21], the following expressions are obtained in state feedback and state derivative feedback:

$$S_{Kcls} = \frac{\det(K-B_sG_s)}{\det M}(K-BG_s)^{-T} \tag{24}$$

$$S_{Ccls} = -M^{-T} \tag{25}$$

$$S_{Kcld} = \frac{\det K}{\det(M-B_dG_d)}K^{-T} \tag{26}$$

$$S_{Ccld} = -(M-B_dG_d)^{-T} \tag{27}$$

$$S_{Mpcls} = -\frac{\det(K-B_sG_s)}{\det M}M^{-T} \tag{28}$$

$$S_{Mscls} = -M^{-T}(C-B_sF_s)^TM^{-T} \tag{29}$$

$$S_{Mpcld} = -\frac{\det K}{\det(M - B_d G_d)}(M - BG_d)^{-T} \tag{30}$$

$$S_{Mscld} = -(M - B_d G_d)^{-T}(C - B_d G_d)^T (M - B_d G_d)^{-T} \tag{31}$$

One can now pursue consistent conditions under which state derivative feedback is less sensitive to changes in the matrices K, C, and sometimes M.

Let δK, δC, and δM be norm-bounded perturbations of the stiffness, damping, and mass matrices:

$$\|\delta K\| \le \mu_k,\ \|\delta C\| \le \mu_c,\ \|\delta M\| \le \mu_m. \tag{32}$$

We claim that comparing the directional derivatives computed with the sensitivities (Eqs. (24)–(31)) in the direction of the respective uncertainties is a fair and useful index for sensitivity comparison. Consider the directional derivative of a scalar function of a matrix X on another matrix Y:

$$D_Y f(X) = \mathrm{tr}\left(\frac{\partial f}{\partial X} Y\right) \tag{33}$$

Thus, by using each of the sensitivities (Eqs. (24)–(31)) as $\frac{\partial f}{\partial X}$, the absolute values of the computed directional derivatives provide a meaningful measure of perturbation intensity in the spectrum sum or product when some of the system matrices are perturbed.

Notice that the general structure of the uncertainties is the bound in their norm. Since the directional derivatives must be evaluated for all uncertainties that satisfy Eq. (32), the worst case for directional derivatives can be computed for the general case by solving the following optimization problem:

$$\begin{aligned} &\sup_{Y} \quad \mathrm{tr}\left(\frac{\partial f}{\partial X} Y\right) \\ &\quad \text{s.t.} \quad \|Y\| \le \mu_y \end{aligned} \tag{34}$$

This optimization problem is soft in the objective function, but may be hard in terms of constraints depending on the norm taken to structure the uncertainties. In the rest of the paper, we adopt the so-called Frobenius norm since we model a soft constraint in the problem (Eq. (34)). In fact, an analytical solution is possible when the constraint in Eq. (34) is taken as the squared Frobenius norm. Moreover, the problem is known to be concave in the objective function. Thus, by setting $\frac{\partial f}{\partial X} \equiv S$ and introducing a Lagrange multiplier η, we define the revised optimization problem for Eq. (34):

$$\max_{Y} \quad \Gamma(Y, \eta) \tag{35}$$

in which $\Gamma(Y, \lambda) = \mathrm{tr}(SY) + \eta\left[\mathrm{tr}(YY^T) - \mu_y^2\right]$. By applying the Karush–Kuhn–Tucker (KKT) first-order condition for the stationary points [22,23],

$$\begin{aligned} &\frac{\partial \Gamma}{\partial Y} = 0 \therefore \\ &Y^* = -\frac{1}{2\eta^*} S^T \end{aligned}$$

From the equality constraint, we have

$$\begin{aligned} &\mathrm{tr}\left(Y^* Y^{*T}\right) = \mu_y^2 \therefore \\ &\eta^* = -\frac{1}{2\mu_y} \|S\|_F \end{aligned}$$

where, for the maximum, we take the negative solution of η^* corresponding to the KKT second-order condition. The maximum computation gives

$$\begin{aligned} &\Gamma^* = \mathrm{tr}(SY^*) \therefore \\ &\Gamma^* = \mu_y \|S\|_F \end{aligned} \tag{36}$$

A simple procedure for guiding designer decisions in choosing between the two studied techniques is:

1. Apply a suitable technique for the eigenvalue assignment problem with state feedback.
2. Compute the state derivative feedback gains using Eqs. (12) and (13) or Eqs. (14) and (15).
3. Compute the relevant directional derivatives with Eqs. (34) or (36), choosing the adequate sensitivities Eqs. (24)–(31).
4. Compare the corresponding directional derivatives, e.g., computed with S_{Kcls} and S_{Kcld}.

An important feature in this analysis approach is that the resulting closed-loop eigenvectors matrices $Y_{cs} = [\mathbf{y}_{cs1}\ \mathbf{y}_{cs2} \dots \mathbf{y}_{cs2n}]$ and $Y_{cd} = [\mathbf{y}_{cd1}\ \mathbf{y}_{cd2} \dots \mathbf{y}_{cd2n}]$ are the same in the two designs. Thus, since that the condition number of the matrix:

$$\tilde{Y} = \begin{bmatrix} Y_c \\ Y_c \Lambda_c \end{bmatrix}$$

in which Λ_c stands for the diagonal matrix of closed-loop eigenvalues is a relevant indicator of robustness [24,11,12], the procedure is quite fair.

4. Results and discussion

In order to illustrate the previous discussion, three examples borrowed from the literature are explored. The first one is a classic example of a non-conservative vibrating rod finite-element model, and the second is the vibration phenomenon of a wing in an air stream.

4.1. Vibration control of a non-conservative vibrating rod

Consider the well-known example of an n-node finite-element model of a non-conservative vibrating rod. $S=[\delta_{i+1,j}], i,j=1\ldots n$ is the shift operator; $H=diag\{\gamma_i\},\ i=1\ldots n$ in which $\gamma_i=\sin\left(\frac{\pi i}{2n}\right)$; $J=I-S$, and we define

$$M=2\left(I+SS^T\right)+S+S^T,\quad C=\frac{1}{100}JHJ^T,\quad K=1000\,JJ^T$$

For this classical benchmark, two cases are analyzed below. In the first a full independent actuator is used [5] and in the second case a technique explored for single-input control with a node-distributed actuator is considered [25].

4.1.1. Full actuators at the nodes

As in the work of Henrion et al. [5], we suppose that there is one actuator at each of the nodes, resulting in $B=I$. The goal now is to compare spectrum sensitivity for the state feedback design [5] with the equivalent state derivative feedback by comparing the directional derivatives computed as Eqs. (34) or (36). Similarly, the desired spectrum is assigned in a robust design with specification that $-2<Re(\lambda_i)<0.5$. The gains computed for state feedback for $n=4$ nodes are:

$$F_s=\begin{bmatrix} -10.3100 & -1.4260 & -0.8742 & -0.0731 \\ -1.4270 & -11.1800 & -1.0810 & 0.1125 \\ -0.8741 & -1.0810 & -10.8400 & -2.5240 \\ -0.0729 & 0.1121 & -2.5230 & -4.8330 \end{bmatrix},$$

$$G_s=1000\begin{bmatrix} 1.989 & -0.9992 & -0.002612 & -0.0002172 \\ -0.9992 & 1.9870 & -0.9982 & 0.0003341 \\ -0.002612 & -0.9982 & 1.9880 & -1.0030 \\ -0.0002172 & 0.0003341 & -1.0030 & 0.9956 \end{bmatrix}$$

Then, applying Eqs. (14) and (15) the state derivative feedback gains can be obtained as:

$$F_d=1000\begin{bmatrix} -1.8076 & 0.6098 & 0.4842 & 0.0040 \\ 0.5393 & -1.4303 & 0.4310 & -0.0959 \\ 0.4963 & 0.3404 & -1.8413 & 1.3217 \\ -0.2080 & 0.3361 & 1.0640 & -1.2330 \end{bmatrix},$$

$$G_d=\begin{bmatrix} -683.0032 & 160.3117 & 281.6572 & 8.5203 \\ 126.6456 & -438.3136 & 47.6429 & -50.5881 \\ 298.0755 & -0.9273 & -630.7515 & 558.0064 \\ -125.5345 & 199.9087 & 386.6461 & -512.3799 \end{bmatrix},$$

In Table 1 a summary of the computed directional derivatives is presented considering structured uncertainty bounds in the stiffness and mass matrices of 0.1% and 50%, respectively:

We observe that a much less sensitive performance in the sense of directional derivatives, which is favorable to state derivative feedback, can be noted in Table 1.

The cloud for the eigenvalue locus when an additive, relative uncertainty of 0.1% is added to the stiffness matrix and when a linear, uniform uncertainty of $\pm 50\%$ in the mass matrix is taken into account, is seen in Fig. 1. This simulation was performed for 10,000 random δK and δM bounded as described.

A much more concentrated cloud is evident for state derivative feedback. The spectrum in the state feedback case touches the RHP dangerously in the worst case.

Table 1
Absolute value of the directional derivatives.

Technique	Sensitivity matrix		
State feedback	S_{Kcls} 88.0049	S_{Mcls} 7.4055	S_{Mpcls} 184.3639
State derivative feedback	S_{Kcld} 2.1727	S_{Mcld} 0.4041	S_{Mpcld} 4.5968

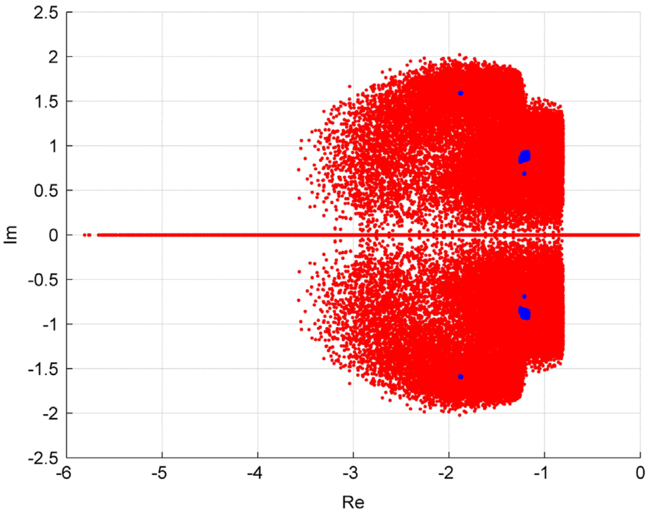

Fig. 1. Eigenvalue locus for the vibrating rod with perturbation of the stiffness matrix and uncertainty in the mass matrix in the full-actuator case: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

4.1.2. *Single-input case*

Now consider the example described by Datta [25], when a single-input control is applied by distributing it in the nodes, i.e., $B=[1\ \ 1\ \ 1\ \ 1]^T$. The gains computed for the state feedback case in order to allocate the two more unstable eigenvalues for the positions $-1\pm i$ are:

$$F_s=[-1.4850\ \ -2.7439\ \ -3.5852\ \ -1.9403],$$

$$G_s=[17.8477\ \ 32.9769\ \ 43.0857\ \ 23.3137]$$

We use then Eqs. (14) and (15) to compute:

$$F_d=[-19.3169\ \ -35.6999\ \ -46.6492\ \ -25.2468],$$

$$G_d=[-8.9185\ \ -16.4791\ \ -21.5308\ \ -11.6523]$$

A multiplicative, norm-bounded (Frobenius) perturbation $1-\epsilon$, with $\epsilon=0.0768$ in the stiffness matrix, is applied. The absolute values of the directional derivatives, normalized by the common factor $\frac{\det K}{\det(M-B_dG_d)}\equiv\frac{\det(K-B_sG_s)}{\det M}$, for state feedback and state derivative feedback, respectively, are evaluated as 35.8594 and 2.8008, indicating less sensitivity of state derivative feedback. The closed-loop system is able to quickly dampen some of present modes partially, as can be seen in Fig. 2a. In Fig. 2b, a very poor performance in the state feedback case is noted when the perturbation in the stiffness matrix is applied. The state derivative feedback, on the other hand, exhibits a better capacity to cope with this uncertainty, as seen in Fig. 2c.

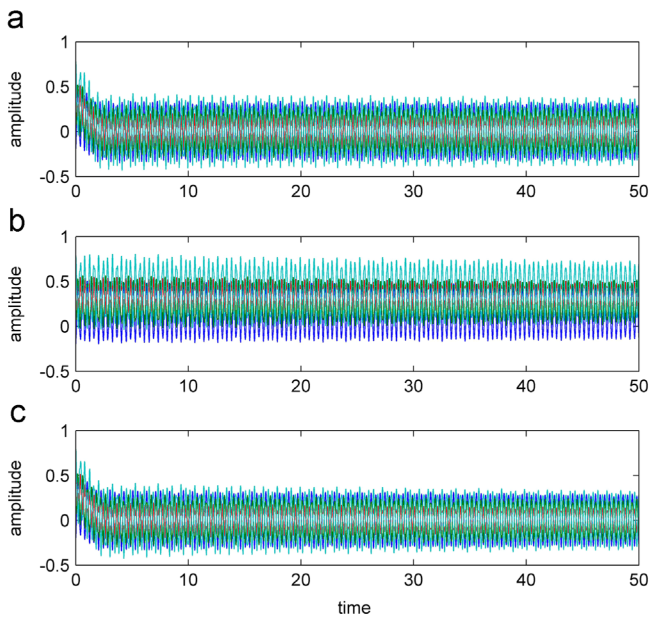

Fig. 2. Time-domain oscillation profile for the vibrating rod with a distributed actuator scheme: (a) nominal closed-loop performance, (b) perturbed stiffness matrix with state feedback, and (c) with state derivative feedback.

4.2. Wing vibration in an air stream

This example is also borrowed from the work of Henrion et al. [5] using the model given by Tisseur and Higham [26]. The system matrices for the QP that models wing vibration in an air stream are:

$$M=\begin{bmatrix}17.6 & 1.28 & 2.89\\ 1.28 & 0.824 & 0.413\\ 2.89 & 0.413 & 0.725\end{bmatrix},\quad C=\begin{bmatrix}7.66 & 2.45 & 2.1\\ 0.23 & 1.04 & 0.223\\ 0.60 & 0.756 & 0.658\end{bmatrix},\quad K=\begin{bmatrix}121 & 18.9 & 15.9\\ 0 & 2.7 & 0.145\\ 11.9 & 3.64 & 15.5\end{bmatrix},$$

In this example, three illustrative cases are examined next.

4.2.1. Full actuators scheme

In this study, $B=I_3$. The gain matrices for state feedback computes as in the original work [5] for a regional robust assignment of the closed-loop spectrum such that $-2<Re(\lambda_i)<0$ are:

$$F_s=\begin{bmatrix}-22.9900 & 0.2247 & -2.9400\\ -1.9950 & -0.4010 & -0.5176\\ -4.4400 & 0.0154 & -0.5585\end{bmatrix},\quad G_s=\begin{bmatrix}80.2500 & 15.9300 & 9.2000\\ -2.9670 & 0.8435 & -0.8139\\ 5.2000 & 2.6810 & 13.8800\end{bmatrix}$$

By applying Eqs. (14) and (15) for fast design of state derivative feedback:

$$F_d=\begin{bmatrix}-83.2639 & -11.8795 & -10.1168\\ 0.2450 & -1.0526 & 0.0696\\ -8.4230 & -2.3127 & -10.7795\end{bmatrix},\quad G_d=\begin{bmatrix}-34.7110 & -7.1411 & -3.4899\\ 1.2769 & -0.3907 & 0.3949\\ -2.1366 & -0.9579 & -6.8379\end{bmatrix}$$

The eigenvalue loci are given in Figs. 3–5. In Fig. 3, the damping matrix is randomly perturbed 10,000 times with a bounded uncertainty $\|\delta C\|_F=0.1918$. In Fig. 4, a similar experiment is carried out perturbing the stiffness matrix bounded as a damping matrix. Fig. 5 reflects the same procedure considering uncertainty in the mass matrix. A higher robustness can be noted for the three cases when one uses state derivative feedback. These results are in agreement with the summary presented in Table 2. Time domain performance is compared in Figs. 6 and 7. In Fig. 6, the almost independent performance of the state derivative technique is confirmed neglecting deviations from the nominal response under variation in the damping matrix that lead to the worst case using the state feedback method. A poorly damped response under the same conditions is depicted in Fig. 7 for the state feedback case.

4.2.2. Use of sensitivity for state derivative feedback designing

This example is taken from the work of Abdelaziz [11]. Even though the presented methodology is focused on comparing state feedback and state derivative feedback with the same eigenstructure design, the use of sensitivity figures

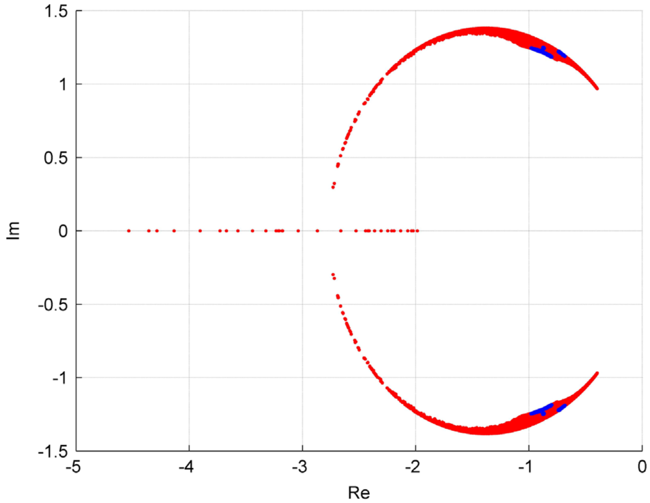

Fig. 3. Eigenvalue locus for the wing in an air stream with perturbation of the mass matrix: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

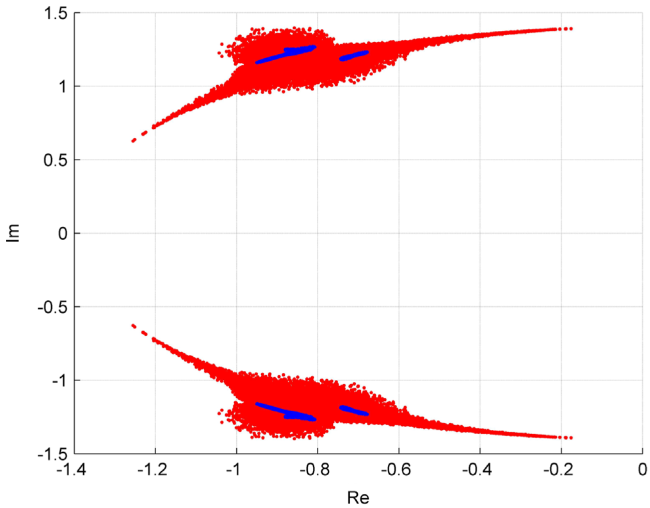

Fig. 4. Eigenvalue locus for the wing in an air stream with perturbation of the damping matrix: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

(Eqs. (24)–(31)) is likely to lead to improved design as shown in the following. The actuator matrix in this case is:

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In the compared work [11], the used cost function is a weighted combination of the condition number of $\tilde{Y}$, conditioning of the desired eigenvalues and the norms of the matrix gains. For a fair comparison, we consider here a cost function given by:

$$J(F_d, G_d) = \kappa(\tilde{Y}) + \|S_{Ccld}\|_F$$

The gains were then computed by minimizing this cost function with respect to them with the target eigenvalues $-1 \pm j$, $-2 \pm 2j$, $-4 \pm 3j$, resulting in:

$$F_d = \begin{bmatrix} -105.0357 & -32.6868 & -39.178 \\ -12.7366 & -4.3965 & -11.0766 \end{bmatrix}, \quad G_d = \begin{bmatrix} -24.1937 & -15.7037 & -10.1691 \\ -4.7082 & -2.8309 & -2.5416 \end{bmatrix}$$

One can verify that the condition number here is 19.0430 against 16.8897 in the previous work [11]. We remark again that a fair comparison can then be made based on the chosen sensitivity regarding the damping matrix. The Frobenius

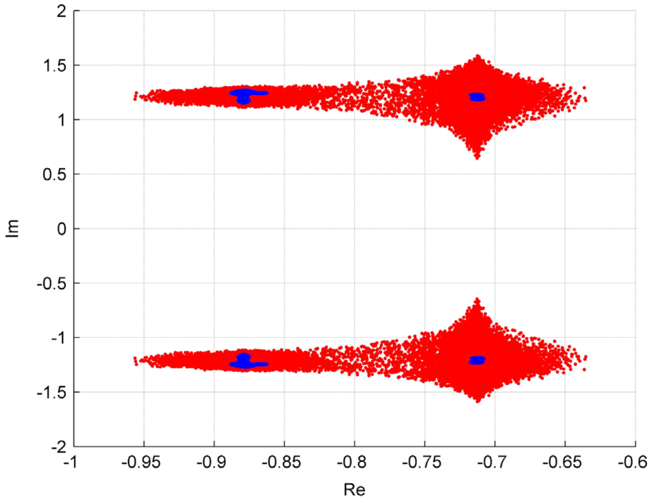

Fig. 5. Eigenvalue locus for the wing in an air stream with perturbation of the stiffness matrix in the full actuators case: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Table 2
Absolute value of the directional derivatives.

Technique	Sensitivity matrix			
State feedback	S_{Kcls} 6.0598	S_{Ccls} 1.1693	S_{Mscls} 1.6859	S_{Mpcls} 11.9920
State derivative feedback	S_{Kcld} 0.7851	S_{Ccld} 0.1682	S_{Mscld} 0.0893	S_{Mpcld} 1.7272

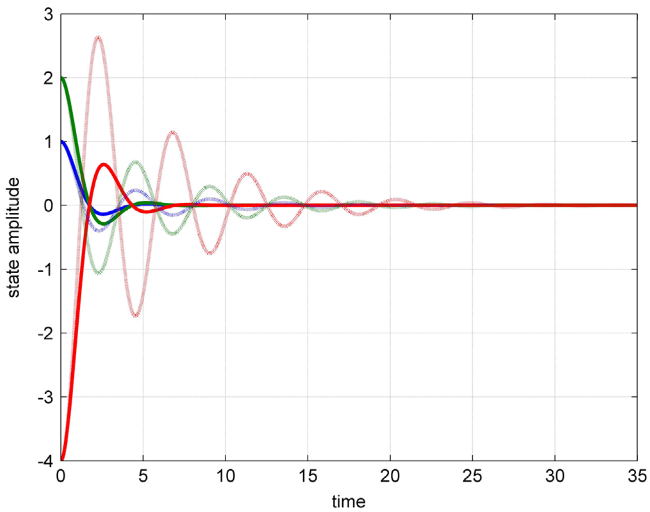

Fig. 6. Time-domain transient state response for state derivative feedback: nominal (continuous) and with damping matrix deviations (dashed).

norms of the sensitivity S_{Ccld} for the present analysis and the previous [11] are respectively 3.7580 and 7.8025. As in the full actuator case, a perturbation in the damping matrix as maximum $\|\delta C\|_F = 0.1918$ is simulated with 10,000 random points. The eigenvalue loci in Fig. 8 shows that our method gives slightly better results thus revealing that our method performs slightly better, in this example.

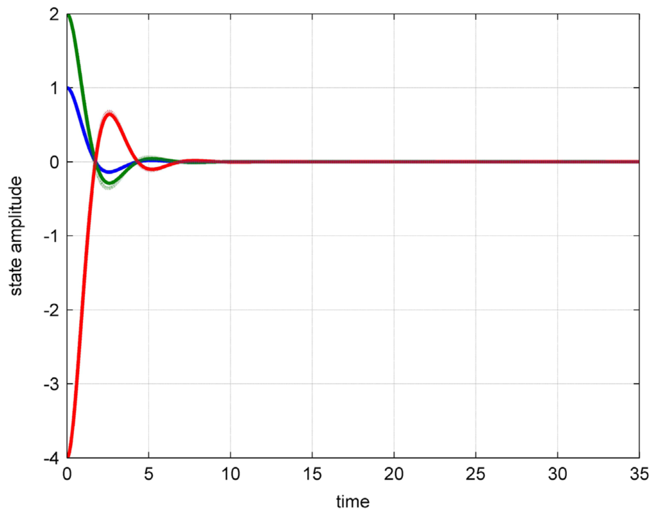

Fig. 7. Time-domain transient state response for state feedback: nominal (continuous) and with damping matrix deviations (dashed).

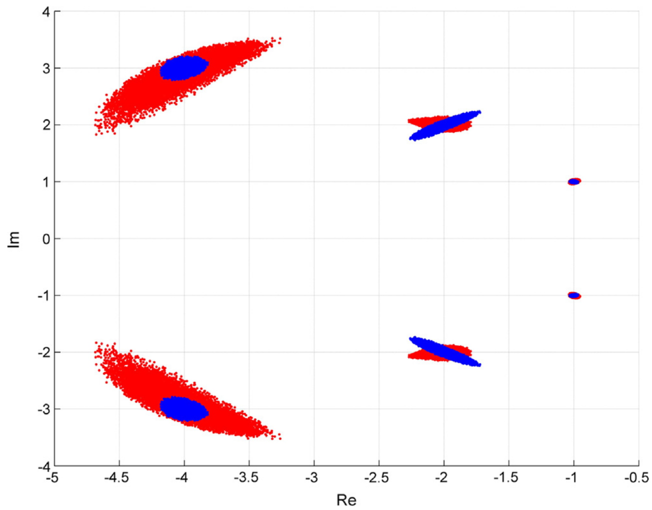

Fig. 8. Eigenvalue locus for the wing in an air stream with perturbation of the damping matrix in the design with sensitivity minimization (blue points) and the method of [11] (red points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

4.3. An ill-conditioned mass matrix example

This example is explored in different works with some variations in the matrices [27,28,5]. Chan et al. [27] propose the mass matrix in an ill-conditioned structure:

$$M = \begin{bmatrix} 5000 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1.00001 \end{bmatrix}, \quad C = \mathbf{0}, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix},$$

For the given desired spectrum $-1, -2, -3, -4, -5, -6$, this interesting approach can compute state feedback gains as [27]:

$$F_s = \begin{bmatrix} 1.3835e3 & 2.7532e0 & 3.6769e0 \\ 2.5939e-2 & -4.6269e-6 & 5.7691e-5 \end{bmatrix}, \quad Gs = \begin{bmatrix} 7.1424e2 & -6.1433e1 & 6.6875e1 \\ 1.9987e1 & 5.9999e1 & -5.9999e1 \end{bmatrix}$$

Returning to Eqs. (14) and (15), one can find the state derivative feedback gains as:

$$F_d = \begin{bmatrix} 1.0448e2 & -2.4283e1 & 4.6182e0 \\ -9.5266e1 & 2.082e1 & 3.9461e0 \end{bmatrix}, \quad Gd = \begin{bmatrix} -4.7392e3 & -2.0467e0 & 1.3048e-1 \\ 5.1299e3 & 1.9691e0 & 1.0164e-1 \end{bmatrix}$$

Now we consider uncertainties in the stiffness and mass matrices as follows: δK with the same structure (tridiagonal) of K and all uncertainties given by $\|\delta K\|_F \le 1e-3\|K\|_F$ and $|\delta M_{11}| \le 25$. Table 3 summarizes directional derivatives computed with these uncertainties. In this example, an equilibrium is evident between state and state derivative feedback methods. For stiffness matrix perturbation, state derivative feedback is much less sensitive. The eigenvalue locus generated in 10,000 random applications for the given uncertainty poses a serious threat to state feedback, while the state derivative feedback approach does not lose stability. This can be noted in Figs. 9 and 10. On the other hand, a more robust spectrum for mass matrix perturbation is seen when using state feedback, as deduced from Table 3 and corroborated in Fig. 11.

Table 3
Absolute value of the directional derivatives.

Technique	Sensitivity matrix		
State feedback	S_{Kcls} 5.7603e5	S_{Mscls} 1.3836e−3	S_{Mpcls} 3.6000
State derivative feedback	S_{Kcld} 13.2371	S_{Mscld} 3.1253	S_{Mpcld} 191.4157

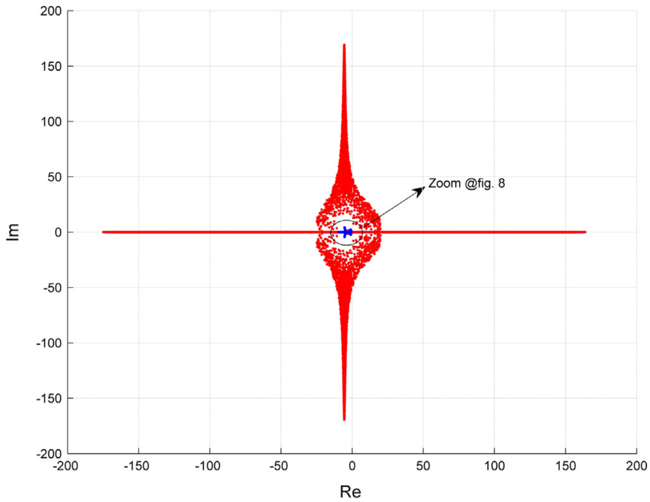

Fig. 9. Eigenvalue locus for the ill-conditioned mass matrix example with perturbation of the stiffness matrix: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

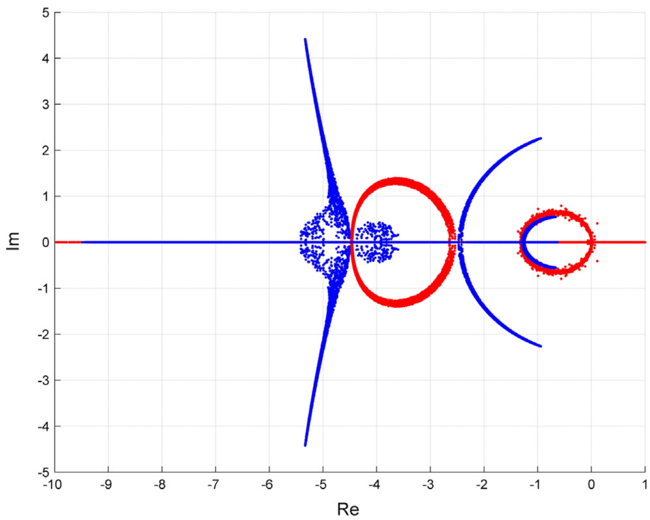

Fig. 10. Magnified view of the eigenvalue locus for the ill-conditioned mass matrix example with perturbation of the stiffness matrix: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

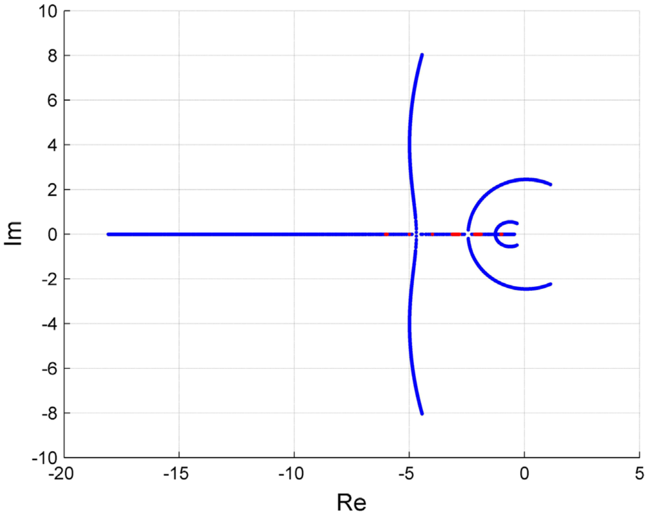

Fig. 11. Eigenvalue locus for the ill-conditioned mass matrix example with perturbation of the mass matrix: state feedback (red points) and state derivative feedback (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

4.4. A four degrees of freedom mass-spring-damp with singular mass matrix

This example is borrowed from the most work of Abdelaziz [19], in which the advantage of using state derivative feedback is evident to re-assign the infinite eigenstructure. We follow the same reasoning of Section 4.2.2, modifying the cost function in order to include the norm of the sensitivity S_{Ccld}, in substitution to the term concerning the sensitivity eigenvalues. The system matrices are:

$$M=\begin{bmatrix}3&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix},\quad C=\begin{bmatrix}15&-10&0&0\\-10&25&-15&0\\0&-15&35&-20\\0&0&-20&20\end{bmatrix},\quad K=\begin{bmatrix}20&-15&0&0\\-15&30&-15&0\\0&-15&35&-20\\0&0&-20&20\end{bmatrix},$$

$$B=\begin{bmatrix}1&0\\0&0\\0&0\\0&1\end{bmatrix}$$

Thus, by minimizing the following modified cost function with respect to F_d and G_d:

$$J(F_d,G_d)=\kappa(\tilde{Y})+\|S_{Ccld}\|_F+\|F_d\|_F+\|G_d\|_F$$

one obtains a solution for the gains of state derivative feedback with desired closed-loop eigenvalues $-1\pm0.5i$, $-2\pm i$, $-3\pm3i$, $-4\pm4i$:

$$F_d=\begin{bmatrix}-1.1456&20.1511&-23.2266&6.7678\\-1.1092&-4.4716&-8.5789&11.6063\end{bmatrix},\quad G_d=\begin{bmatrix}1.7218&6.1756&-6.6581&3.9700\\-0.8194&1.2816&-1.0348&0.1001\end{bmatrix}$$

Considering uncertainty in the stiffness matrix with a bound on Frobenius norm of 0.5% with respect to the nominal value, a simulation with 10,000 random points is performed. In Fig. 12, a more concentrated cloud in the closed loop eigenvalue locus is evident compared with the gains computed in the robust design [19].

5. Conclusion

A comparative analysis of sensitivity to parameter variation is proposed concerning second-order linear systems under state feedback and state derivative feedback. Some simple conditions for a fair comparison between the two feedback approaches are given with bases in the sensitivity of the spectrum location using the product and sum of the closed-loop eigenvalues. These products and sums can be related easily to the determinants and traces of the system matrices. Four representative examples of real systems taken from the specialized literature are used to validate our proposal and the results are fully consistent with the given conditions for the sensitivity comparison.

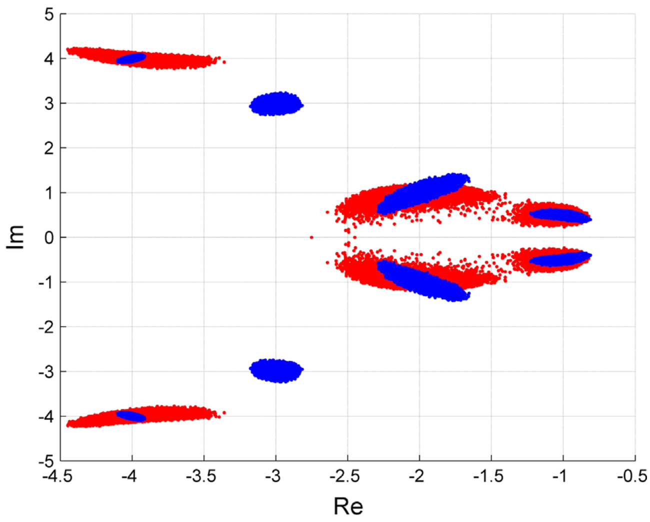

Fig. 12. Eigenvalue locus for the singular mass matrix example with perturbation of the stiffness matrix: Abedelaziz [19] technique (red points) and minimization of sensitivity (blue points). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

Along this line of inquiry, future works intend to investigate the combination of feedback involving position, velocity, and acceleration in order to aggregate the best features of the two techniques: the possibility of reducing sensitivity to parameters in state derivative feedback and the capacity of state feedback to re-allocate null eigenvalues (integrators) absent in the state derivative case.

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Appendix A. Supplementary data

Supplementary data associated with this paper can be found in the online version at http://dx.doi.org/10.1016/j.ymssp.2016.02.014.

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