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Stability condition of discrete-time linear Hamiltonian systems with time-varying delay feedback interconnection ⋆

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Abstract: This paper is concerned with stability analysis of delay feedback structures within discrete port-Hamiltonian framework. We introduce a discrete dynamics that approximates linear port-Hamiltonian systems and is passive relatively to the same storage and dissipation functions. Stability of interconnected discrete systems is then addressed when considering time-varying delay feedback interconnection structure. A delay bounds-dependent stability condition is derived for variable and bounded delayed interconnection, reducing to a delay-independent condition for constant delay. A sufficient condition is formulated in terms of a feasibility problem under Linear Matrix Inequality (LMI) constraints. It is noticeable that the LMI parameters linearly depend on the network characteristics (damping and input matrices). Moreover, computing and storing past history of the discrete flow is no longer required. A numerical example illustrates the feasibility of the approach.

1. INTRODUCTION

Port-Hamiltonian systems (PHSs) [Maschke and van der Schaft (1992)] form a class of passive systems known to be composable: a network of PHSs belongs to the class. As stability follows from passivity, composability turns out to be an essential property since the network inherits passivity from its components. To set out stability issues including delays in this context, simply note that composability is derived from the network's topology which is characterized by a power-conserving interconnection structure, and that delay interconnection structures are no longer power-conserving.

Basically, the port-Hamiltonian framework draws a complex system as an oriented graph of energy exchanges (supported by edges) between subcomponents (associated with nodes). This energy-based description of physical systems is met in many engineering fields (as in mechanics, electronics, robotics and haptics). The topology of the graph (the set of energy links) defines the interconnection structure. Communication delays in this framework (encountered for instance in telemanipulation) are thus encoded by a delayed interconnection structure. We shall consider time-delay systems (TDSs) where the delays precisely and only occur in this structure.

In closed-loop schemes, it is well-known that delays may induce oscillations and instabilities [Niculescu (2001); Normey-Rico and Camacho (2007)]. Stability analysis is classically tackled by using Lyapounov-Krasovskii (LK) functionals to derive sufficient conditions in terms of a feasibility problem under linear matrix inequalities (LMI) constraints. Generally speaking, stability of TDSs is addressed from a state-space representation gathering the present and the past history of the trajectory. Considering linear discrete-time systems with time-varying (upper-bounded) delay, numerically tractable conditions are carried out based on the generator of the discrete flow using delay dependent LK functionals [Hetel et al. (2008)]. Note that the LMI grows as the upper bound of the delays does. We shall see that, with the class of linear discrete-time systems considered here, the size of the LMI does not depend anymore on the upper bound: the LMI criterion is directly derived from the network characteristics.

Regarding continuous-time delay interconnected nonlinear PHS, stability analysis has been investigated by LK functionals and Jensen's inequality in [Kao and Pasumarthy (2012)], and a less conservative criterion can be derived using improved Wirtinger-based inequality [Aoues et al. (2014)]. Both results propose a construction of LK functionals based on the subsystems energies and the network characteristics, leading to a LMI criterion.

In this paper, we consider the class of linear discrete-time port-controlled Hamiltonian systems with dissipation (LPCHD), that is a PHS with a quadratic energy and a damping matrix. Stability analysis of the feedback interconnection is achieved in terms of LMI condition. Delay bound-dependent (resp. -independent) condition is derived for variable and bounded (resp. constant) time-delay located in the interconnection structure. Opposite to the general approach (derived with delay-dependent LK), the criterion is derived from the network damping and input matrices which is noticeable from a dimensionality point of view: the computational cost becomes independent from the

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size of the delay variation. This result extends the ideal case (meaning without delay) processed in [Aoues et al. (2013)].

Content is as follows. Section 2 introduces the linear discrete-time LPCHD systems and states the stability issue. The main result is presented in Section 3: LK functional is proposed and LMI stability conditions are derived. Section 4 compares the result with the classical approach from dimensionality viewpoint. A numerical example is given in Section 5. Conclusion and proofs in Appendix end the paper.

2. LINEAR PORT-CONTROLLED HAMILTONIAN SYSTEMS WITH DISSIPATION (LPCHD)

2.1 Continuous-time settings

Throughout the paper, we shall consider the class of *linear port-controlled Hamiltonian systems with dissipation* (LPCHD), which dynamics is described by the following equations [van der Schaft (1999)]

$$\Sigma(x) : \begin{cases} \dot{x}(t) = [J-R]\,Qx(t) + gu(t) \\ y(t) = g^T Qx(t) \end{cases} . \tag{1}$$

$x \in \mathbb{R}^n$ is the state vector and $H(x) = \frac{1}{2}x^T Qx$ is the total energy with $Q = Q^T > 0$. The structure matrix J is a symplectic matrix (full rank and $J + J^T = 0$), and the damping matrix R satisfies $R = R^T \geqslant 0$. $g \in \mathbb{R}^{n\times m}$ is the input matrix. $u \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^m$ its conjugate port-output.

By integrating $\frac{\mathrm{d}}{\mathrm{d}t}H$ from t_0 to $t > t_0$, one gets the energy balance equation

$$H(t) - H(t_0) = \int_{t_0}^{t} y^T(s)u(s)\mathrm{d}s - \int_{t_0}^{t} x^T(s)Q^T RQx(s)\mathrm{d}s. \tag{2}$$

Equation (2) reflects that H is a storage function. Σ given by (1) is thus a *passive* system whenever H is bounded from below. Moreover when $R = 0$, Σ is said to be *lossless*.

Remark 2.1. Observe that equation (2) states $\frac{\mathrm{d}}{\mathrm{d}t}H = -x^T Q^T RQx$ in the unforced case. Then first when R is only symmetric and positive, the passivity equation translates Lyapunov stability. Second, when R is now positive definite, asymptotic stability follows.

Consider $\Sigma_1(x)$ and $\Sigma_2(z)$ given by (1) with constitutive elements $(J_i, R_i, Q_i, g_i)_{i=1,2}$. The feedback interconnection is characterized by the constraints on the port variables $u = -w$ and $v = y$ as drawn in Fig. 1.

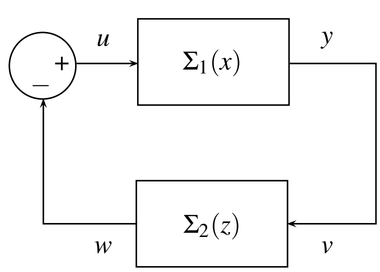

Fig. 1. Feedback interconnection structure

The resulting system, denoted by $\Sigma_{12}(X)$, is governed by the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} J_1 - R_1 & -g_1 g_2^T \\ g_2 g_1^T & J_2 - R_2 \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \tag{3}$$

or equivalently

$$\dot{X} = [J_{12} - R_{12}]Q_{12}X,$$

where $X = \begin{bmatrix} x^T & z^T \end{bmatrix}^T$, $J_{12} = \begin{bmatrix} J_1 & -g_1 g_2^T \\ g_2 g_1^T & J_2 \end{bmatrix}$, $R_{12} = \mathrm{diag}(R_1, R_2)$ and $Q_{12} = \mathrm{diag}(Q_1, Q_2)$.

Noting that the total energy H_{12} is given as the sum of subsystems energies $H_{12}(X) = H_1(x) + H_2(z) = \frac{1}{2}X^T Q_{12} X$, the structure matrix J_{12} gathers the skew-symmetric part and R_{12} the damping part, one concludes that Σ_{12} belongs to the class of LPCHD. The class invariance under interconnection is referred to as *composability*.

Therefore, thanks to composability, stability analysis of the interconnected system is conducted as previously: integrating $\frac{\mathrm{d}}{\mathrm{d}t}H_{12}$ leads to

$$H_{12}(X(t)) - H_{12}(X(t_0)) = -\int_{t_0}^{t} X(s)^T Q_{12}^T R_{12} Q_{12} X(s)\mathrm{d}s, \tag{4}$$

and one concludes as in Remark 2.1.

2.2 Discrete-time settings

Discrete-time approximation of Hamiltonian systems can not be performed using arbitrary numerical schemes. Indeed, the *unforced* case deserves dedicated schemes as energetic and geometric integrators [Feng and Qin (2002)], meaning that either the Hamiltonian or the volume is preserved. The *input-output* case remains an open issue, except for the linear case treated in [Greenhalgh et al. (2013)] from a dissipative viewpoint by a $\theta - \lambda$ method and in [Aoues et al. (2013)] from a lossless viewpoint by a midpoint scheme.

We introduce discrete-time LPCHD as follows.

Definition 2.1. A discrete-time LPCHD with state x_k is given by the set of equations

$$\Sigma(x_k) : \begin{cases} \dfrac{x_{k+1} - x_k}{\Delta t} = [J-R]\,Q\dfrac{x_{k+1} + x_k}{2} + gu_k \\ y_k = g^T Q\dfrac{x_{k+1} + x_k}{2} \end{cases}, \tag{5}$$

where $Q = Q^T > 0$, $J = -J^T$ is full rank, $R = R^T \geqslant 0$ and $\Delta t > 0$. (u_k, y_k) are called discrete conjugate ports.

Proposition 1. $\Sigma(x_k)$ given by (5) is a discrete-time approximation of $\Sigma(x)$ given by (1).

Proposition 2. $\Sigma(x_k)$ is a passive system relatively to the storage function $H(x_k) = \frac{1}{2}x_k^T Qx_k$. Moreover, (5) encodes losslessness when $R = 0$.

Proof. The discrete energy balance $\Delta H_k := H_{k+1} - H_k$ along the trajectories of (5) writes

$$\begin{aligned} \Delta H_k &= \left\langle Q\frac{x_{k+1} + x_k}{2}, (x_{k+1} - x_k) \right\rangle \\ &= \Delta t \left\langle Q\frac{x_{k+1} + x_k}{2}, [J-R]\,Q\frac{x_{k+1} + x_k}{2} + gu_k \right\rangle \\ &= \Delta t\, y_k^T u_k - \Delta t \left[Q\frac{x_{k+1} + x_k}{2}\right]^T R \left[Q\frac{x_{k+1} + x_k}{2}\right]. \end{aligned} \tag{6}$$

Equation (6) shows that $\Sigma(x_k)$ is a passive system relatively to the storage function H. It is the discrete counterpart of equation (2). The system is clearly lossless when $R = 0$. ■

Proposition 3. The unforced system (5) is stable when $R \geq 0$ and asymptotically stable when $R > 0$.

Proof. From (6) with $u_k \equiv 0$ one has $\Delta H_k \leq 0$, hence stability. As J is symplectic, if $R > 0$ then $[J - R]$ is invertible. Therefore, $x_{k+1} + x_k = 0 \iff x_{k+1} = x_k = 0$ by (5), meaning ΔH_k is strictly negative, hence asymptotic stability. ■

Consider now two discrete LPCHD (5) denoted by $\Sigma_1(x_k)$ and $\Sigma_2(z_k)$ with state respectively x_k and z_k. The discrete feedback interconnection structure is given by

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} y_k \\ w_k \end{bmatrix}, \tag{7}$$

with I the identity matrix of appropriate dimension. A straightforward computation leads to the interconnected dynamics

$$\begin{bmatrix} \dfrac{x_{k+1} - x_k}{\Delta t} \\ \dfrac{z_{k+1} - z_k}{\Delta t} \end{bmatrix} = \begin{bmatrix} J_1 - R_1 & -g_1 g_2^T \\ g_2 g_1^T & J_2 - R_2 \end{bmatrix} \begin{bmatrix} Q_1 \dfrac{x_{k+1} + x_k}{2} \\ Q_2 \dfrac{z_{k+1} + z_k}{2} \end{bmatrix} \tag{8}$$

or equivalently

$$\frac{X_{k+1} - X_k}{\Delta t} = [J_{12} - R_{12}]\, Q_{12} \frac{X_{k+1} + X_k}{2},$$

where $X_k = \begin{bmatrix} x_k^T & z_k^T \end{bmatrix}^T$ and J_{12}, R_{12}, Q_{12} defined as previously.

Noting that (8) is the discrete equivalent of (3), one deduces that the class of LCHPD given by (5) has the composability property. Stability analysis is thus addressed with the same arguments as in the continuous-time case, using propositions 2 and 3. Next, stability analysis is addressed for time-delay feedback structure.

3. TIME-VARYING DELAY FEEDBACK INTERCONNECTION OF DISCRETE-TIME LPCHD

For physical reasons (communication channel in teleoperation, network queuing etc.) delays may have to be taken into account in the modeling assumptions in order to better calk reality. We assume here that delays occur in the interconnection structure so that the constraints are given by

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} y_{k-d(k)} \\ w_{k-d(k)} \end{bmatrix}, \tag{9}$$

(compare with (7)) where the delay $d(k) \in \mathbb{N}$ is assumed to be time-varying and to satisfy $0 < d_m \leq d(k) \leq d_M$ for all $n \in \mathbb{N}$. The integers d_m and d_M are associated with the minimum and the maximum delays respectively. This delay network is depicted in Fig. 2.

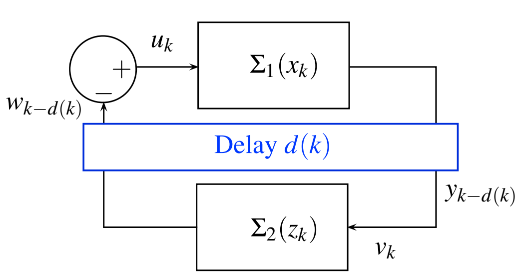

Fig. 2. Time-varying delay feedback interconnection

Assume that initial conditions over $[-d_M, 0]$ are given, and compute the interconnected dynamics. It follows

$$\begin{aligned} &\left[\frac{X_{i+1} - X_i}{\Delta t}\right]_{i=k} = \\ &[J - R]\, Q \left[\frac{X_{i+1} + X_i}{2}\right]_{i=k} + MQ \left[\frac{X_{i+1} + X_i}{2}\right]_{i=k-d(k)}, \end{aligned} \tag{10}$$

where $J = \mathrm{diag}(J_1, J_2)$, $R_{12} = \mathrm{diag}(R_1, R_2)$, $Q_{12} = \mathrm{diag}(Q_1, Q_2)$ and $M = \begin{bmatrix} 0 & -g_1 g_2^T \\ g_2 g_1^T & 0 \end{bmatrix}$ describes the topology of the network.

Let us address stability analysis as previously by studying the discrete energy balance equation.

$$\begin{aligned} \Delta H(X_i)_{|i=k} &:= H(X_{k+1}) - H(X_k) \\ &= -\Delta t \left[Q \frac{X_{i+1} + X_i}{2}\right]_{i=k}^T R \left[Q \frac{X_{i+1} + X_i}{2}\right]_{i=k} \\ &\quad + \Delta t \left[Q \frac{X_{i+1} + X_i}{2}\right]_{i=k}^T M \left[Q \frac{X_{i+1} + X_i}{2}\right]_{i=k-d(k)}. \end{aligned} \tag{11}$$

It is obvious that one is not able to conclude due to cross delayed states arising with M. An alternative is to choose candidate in the Lyapunov-Krasovskii sense. We claim the following.

Proposition 4. The time-varying delay interconnected system given by (10) is asymptotically stable if there exists a real symmetric positive definite matrix $\tilde{S} = \tilde{S}^T > 0$ such that

$$\begin{bmatrix} -R + (d_M - d_m + 1)\tilde{S} & M/2 \\ M^T/2 & -\tilde{S} \end{bmatrix} < 0. \tag{12}$$

The proof of Proposition 4 makes use of the two following lemmas (proofs are given in appendix).

Lemma 5. For any constant positive semi-definite symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P^T = P \geqslant 0$ and any vectors w_i in $\mathbb{R}^n$, the following holds

$$\sum_{i=k+1-d(k+1)}^{k-1} w_i^T P w_i - \sum_{i=k+1-d(k)}^{k-1} w_i^T P w_i \leqslant \sum_{i=k+1-d_M}^{k-d_m} w_i^T P w_i.$$

Lemma 6. For any constant positive semi-definite symmetric matrix $P \in \mathbb{R}^{n \times n}$, $P^T = P \geqslant 0$ any vectors w_i in $\mathbb{R}^n$, the following holds

$$\begin{aligned} &\sum_{j=k+2-d_M}^{k+1-d_m} \sum_{i=j}^{k} w_i^T P w_i - \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j}^{k-1} w_i^T P w_i \\ &\qquad = (d_M - d_m)\, w_k^T P w_k - \sum_{i=k+1-d_M}^{k-d_m} w_i^T P w_i. \end{aligned}$$

Proof. Consider a Lyapunov-Krasovskii-like candidate $V(k) = V_1(k) + V_2(k) + V_3(k)$ where

$$\begin{aligned} V_1(k) &= \frac{1}{2} X_k^T Q X_k, \\ V_2(k) &= \sum_{i=k-d(k)}^{k-1} \left[Q \frac{X_{i+1} + X_i}{2}\right]^T S \left[Q \frac{X_{i+1} + X_i}{2}\right] \\ V_3(k) &= \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j}^{k-1} \left[Q \frac{X_{i+1} + X_i}{2}\right]^T S \left[Q \frac{X_{i+1} + X_i}{2}\right], \end{aligned} \tag{13}$$

for a symmetric positive definite matrix $S = S^T > 0$, and compute its difference along the trajectories of (10). Let us compute each component separately.

The difference of V_1 along trajectories is

$$\begin{aligned}\Delta V_1(k) &:= V_1(k+1) - V_1(k)\\ &= -\Delta t \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} R \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k}\\ &\quad + \Delta t \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} M \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k-d(k)}.\end{aligned}$$

The difference of V_2 along trajectories is

$$\begin{aligned}\Delta V_2(k) &:= V_2(k+1) - V_2(k)\\ &= \sum_{i=k+1-d(k+1)}^{k} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right]\\ &\quad - \sum_{i=k-d(k)}^{k-1} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right]\\ &= \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} S \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k}\\ &\quad - \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k-d(k)} S \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k-d(k)}\\ &\quad + \sum_{i=k+1-d(k+1)}^{k-1} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right]\\ &\quad - \sum_{i=k+1-d(k)}^{k-1} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right],\end{aligned}$$

which can be bounded by Lemma 5 as

$$\begin{aligned}\Delta V_2(k) &\leqslant \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} S \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k}\\ &\quad - \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k-d(k)} S \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k-d(k)}\\ &\quad + \sum_{i=k+1-d_M}^{k-d_m} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right].\end{aligned}$$

And finally, difference of V_3 along trajectories writes (thanks to Lemma 6)

$$\begin{aligned}\Delta V_3(k) &:= V_3(k+1) - V_3(k)\\ &= (d_M - d_m) \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} S \left[Q\frac{X_{i+1}+X_i}{2}\right]_{i=k}\\ &\quad - \sum_{i=k+1-d_M}^{k-d_m} \left[Q\frac{X_{i+1}+X_i}{2}\right]^T S \left[Q\frac{X_{i+1}+X_i}{2}\right].\end{aligned}$$

Eventually, gathering and simplifying the increments of the V_i's, the difference of V along trajectories satisfies

$$\Delta V(k) \leq \xi^T(k) \begin{bmatrix} -\Delta t R + (d_M - d_m + 1)S & \dfrac{\Delta t}{2}M \\ \dfrac{\Delta t}{2}M^T & -S \end{bmatrix} \xi(k) \quad (14)$$

where

$$\xi^T(k) = \left[\left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k} \quad \left[Q\frac{X_{i+1}+X_i}{2}\right]^T_{i=k-d(k)}\right]. \quad (15)$$

To conclude, if $\tilde{S}$ solves (12) then define V with $S = \Delta t\, \tilde{S}$ so that (14) implies $\Delta V(k) < 0$, hence asymptotic stability. ■

It is worth noting that the stability condition is **time step-independent**. That is to say, by solving (12), $V(k)$ is adjusted w.r.t. the choice of Δt and the stability analysis does not dependent on the discretization step size. Moreover the LMI condition linearly depends on the system network representation, namely its damping R and interconnection M matrices. This sounds tractable compared with classical discrete-time delay stability analysis where the discrete flow expression is required (see discussion section 4).

Remark 3.1. To simplify the notation here, only one delay term has been considered. However, within simple manipulations, the interconnection (9) can be replaced by

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} y_{k-d_1(k)} \\ w_{k-d_2(k)} \end{bmatrix} \quad (16)$$

with $d_1(k)$ and $d_2(k)$ being two different time-varying delays.

As a by-product of Proposition 4, the constant delay case is derived. Consider an integer $d > 0$ such that $d(k) = d$ for any k. It follows

Corollary 7. The constant time delay interconnected system (10) is asymptotically stable if there exists a real matrix $S = S^T > 0$ in $\mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} -R+S & M/2 \\ M^T/2 & -S \end{bmatrix} < 0. \quad (17)$$

Notice that the stability condition is delay-independent. The proof follows from the previous theorem with $d_m = d_M = d$. This result can be easily checked by taking the function $V(k) = V_1(k) + V_2(k)$ (given in (13)) as a Lyapunov-Krasovskii candidate.

Remark 3.2. Note that, due to the conservatism [1] of this method, the ideal case $d(k) \equiv 0$ cannot be directly recovered from (12). However, it can be seen that $V_2 \equiv 0$, $V_3 \equiv 0$ and the interconnected dynamics (10) reduces to the undelayed case (8) which is stable using the energy $V(k) = V_1(k)$ as Lyapunov.

Eventually, let us discuss the lossless case $R \equiv 0$. Proposition 4 then reduces to

$$\text{find} \quad S = S^T > 0 \quad \text{s.t.} \quad \begin{bmatrix} \alpha S & M/2 \\ M^T/2 & -S \end{bmatrix} < 0, \alpha > 0$$

which has no solution since by Schur complement generalization it would also imply $S \leq 0$. This illustrates the conservatism and the limitation of this approach.

4. DISCUSSION: DIMENSIONALITY COMPARISON WITH THE CLASSICAL APPROACH

Let us recall the general result presented in [Hetel et al. (2008)] and dedicated to the stability analysis of a discrete-time system with time-delay given as

$$X(k+1) = AX(k) + A_d X(k-d(k)), k \in \mathbb{N} \quad (18)$$

with initial conditions given over $[-d_M, 0]$ and $0 < d_m \leq d(k) \leq d_M$. A and A_d are known constant $\mathbb{R}^{n\times n}$ matrices.

Stability analysis is addressed with delay-dependent Lyapunov-Krasovskii functionals of the form

$$V(k) = \sum_{i=0}^{d_M} \sum_{j=0}^{d_M} X^T(k-i) P^{i,j}_{d(k)} X(k-j), \quad (19)$$

with $P^{i,j}_d = P^{j,i\,T}_d$.

[1] The conservatism in a LMI problem reflects the tradeoff between the numerical complexity and the set of admissible solutions.

For all $d \in \{1, \cdots, d_M\}$, introduce the notations

$$\Phi(d) = \begin{bmatrix} P_d^{0,0} & P_d^{0,1} & \cdots & P_d^{0,d_M} \\ P_d^{0,1} & P_d^{1,1} & & \vdots \\ \vdots & & \ddots & \\ P_d^{0,d_M} & \cdots & & P_d^{d_M,d_M} \end{bmatrix} \quad (20)$$

and

$$\Lambda(d) = \begin{bmatrix} A & \Xi_1(d) & \Xi_2(d) & \cdots & \Xi_{d_M}(d) \\ \mathrm{I} & 0 & \cdots & \cdots & 0 \\ 0 & \mathrm{I} & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \mathrm{I} & 0 \end{bmatrix} \quad (21)$$

where $\Xi_i(d) = A_d$ if $d = i$ and 0 else.

Theorem 8. (Hetel et al. (2008)). The following statements are equivalent

- There exist $d_M(d_M+1)^2$ matrices $P_d^{i,j}$ such that the block matrices $\Phi(d), d = \{1, \cdots, d_M\}$ defined by (20) satisfy the LMI
$$\begin{bmatrix} \Phi(d_1) & \Lambda^T(d_1)\Phi(d_2) \\ \Phi(d_2)\Lambda(d_1) & \Phi(d_2) \end{bmatrix} > 0 \quad (22)$$
for all $d_1, d_2 \in \{1, \cdots, d_M\}$.
- There exists a delay dependent Lyapunov-Krasovskii functional (19) whose difference along system (18) solution is stricly negative definite.

Let us now compare the criteria (12) and (22) from a dimensionality point of view. Dimensionality will give an estimate of complexity of each approach multiplying the number of unknowns by the number of lines of the LMI.

Solving (12) involves $\frac{n(n+1)}{2}$ variables (constituting S) in $2n$ lines, that is a complexity of

$$n^3 + n^2 . \quad (23)$$

In (22), each $P_d^{i,j}$ requires $\frac{n(n+1)}{2}$ variables, and there exist $d_M(d_M+1)^2$ such matrices. For each d, there are $2n(d_M+1)$ lines in the LMI. As (22) has to be satisfied for all d_1, d_2, complexity eventually is

$$d_M^3(d_M+1)^3(n^3+n^2) . \quad (24)$$

Notice that the complexity of the stability condition derived in this paper is independent of the maximal size of the delay d_M, whereas the complexity of the classical approach does depend on it. The computational cost is thus independent from the size of the delay variation. In return, condition (12) introduces conservatism compared with (22). For instance, one may modify the candidate (13) by a delay-dependent one in the spirit of (19). This aspect deserves to be clarified and is beyond the scope of the paper. Moreover it does not seem to be straightforward since the discrete-time system (10) does not fit the classical representation (18).

5. NUMERICAL EXAMPLE

Consider Σ_{12} a time-varying delay feedback interconnection of two LPCHD as in Fig. 2. This network is characterized by the damping R and interconnection M matrices (arbitrarily) given by

$$R = \begin{bmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.1 & 0.3 & 0 & 0 \\ 0 & 0 & 0.25 & .05 \\ 0 & 0 & 0.05 & 0.4 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.15 \\ 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0 & 0 \end{bmatrix}, \quad (25)$$

and $d(k) \in \{1, 2, 3, 4\}$ is randomly generated.

Condition (12) is solved using a LMI routine yielding

$$S = \begin{bmatrix} 0.0668 & 0.0135 & 0 & 0 \\ 0.0135 & 0.039 & 0 & 0 \\ 0 & 0 & 0.0329 & 0.006 \\ 0 & 0 & 0.006 & 0.0521 \end{bmatrix}, \quad (26)$$

with eigenvalues $\{0.0312, 0.0335, 0.0538, 0.0723\}$. One concludes that V given by (13) is strictly decreasing along system solution and the system is asymptotically stable. The trajectories of $\Sigma_1(q_x, p_x)$ and $\Sigma_2(q_z, p_z)$ are depicted in Fig. 3

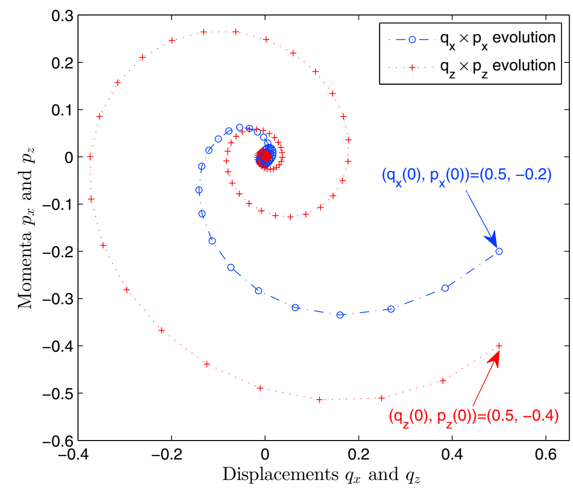

Fig. 3. Phase diagram of Σ_{12}

As claimed before, the stability criterion does not depend on the time step Δt. Indeed, the matrix $\frac{S}{\Delta t}$ remains symmetric positive definite for any (positive) time step. As an illustration, set $\Delta t_1 = 0.05, \Delta t_2 = 0.1$, and $V_{\Delta t_1}, V_{\Delta t_2}$ the associated candidates whose behaviors are drawn in Fig. 4

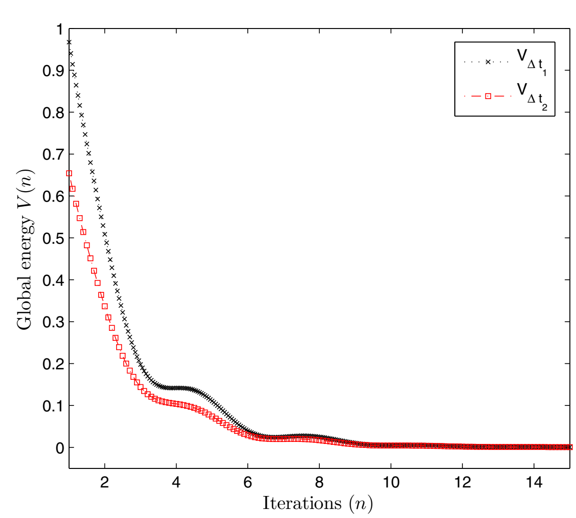

Fig. 4. Lyapounov-Krasovskii functions behaviours

6. CONCLUSION

In this paper, delay-dependant stability condition has been derived for time-varying delay interconnection of two discrete-time linear port-controlled Hamiltonian systems with dissipation. Using the Lyapunov-Krasovskii theory, sufficient stability condition is formulated in terms of LMI. The originality of this work relies on the LMI formulation which linearly depends on system network characteristics (damping and interconnection matrices) regardless explicit discrete trajectories computation. This approach sounds tractable from a dimensionality viewpoint when compared with classical stability analysis where the criterion requires past history of the trajectory increasing complexity. Simulation results are processed on simple example of time-varying delay feedback structure.

However, when considering the ideal case, the criterion does not conclude. This illustrates the conservatism, and thereby limitation, of such the approach. Accordingly, the later remark may suggest that stability analysis within delay port-Hamiltonian framework should be closely related to its intrinsic properties, aiming at unifying all cases.

It would also be challenging to formulate the general case where a collection of subcomponents is considered.

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8. APPENDIX

Proofs of lemmas 5 and 6 are as follows. Assume that a time delay $d(k)$ satisfies $0 < d_m \leq d(k) \leq d_M$ for any k.

Lemma 5: For any matrix $P \in \mathbb{R}^{n\times n}$, $P^T = P \geqslant 0$ and any vectors $w_i \in \mathbb{R}^n$, the following holds

$$\sum_{i=k+1-d(k+1)}^{k-1} w_i^T P w_i - \sum_{i=k+1-d(k)}^{k-1} w_i^T P w_i \leqslant \sum_{i=k+1-d_M}^{k-d_m} w_i^T P w_i \,.$$

Proof. Set $\alpha_i = w_i^T P w_i$ and note that $\alpha_i \geq 0$. Then

$$\sum_{k+1-d(k+1)}^{k-1} \alpha_i - \sum_{k+1-d(k)}^{k-1} \alpha_i = \sum_{k+1-d(k+1)}^{k-d(k)} \alpha_i \leq \sum_{k+1-d_M}^{k-d_m} \alpha_i \,,$$

where the last inequality follows from $\alpha_i \geq 0, \forall i$. ■

Lemma 6: For any matrix $P \in \mathbb{R}^{n\times n}$, $P^T = P \geqslant 0$ and any vectors $w_i \in \mathbb{R}^n$, the following holds

$$\sum_{j=k+2-d_M}^{k+1-d_m} \sum_{i=j}^{k} w_i^T P w_i - \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j}^{k-1} w_i^T P w_i$$
$$= (d_M - d_m)\, w_k^T P w_k - \sum_{i=k+1-d_M}^{k-d_m} w_i^T P w_i \,.$$

Proof. Set $\alpha_i = w_i^T P w_i$ and note that $\alpha_i \geq 0$. Then

$$\sum_{j=k+2-d_M}^{k+1-d_m} \sum_{i=j}^{k} \alpha_i - \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j}^{k-1} \alpha_i$$
$$= \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j+1}^{k} \alpha_i - \sum_{j=k+1-d_M}^{k-d_m} \sum_{i=j}^{k-1} \alpha_i$$
$$= \sum_{j=k+1-d_M}^{k-d_m} (\alpha_k - \alpha_j) \,,$$

hence the result. ■

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