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Liouville–Green (WKB) asymptotics for almost-diagonal linear second-order matrix difference equations

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ABSTRACT

A Liouville–Green (or WKB) asymptotic approximation theory is developed for a class of almost-diagonal ('asymptotically diagonal') linear second-order *matrix difference* equations. Rigorous and explicitly computable bounds for the error terms are obtained, the asymptotics being made with respect to both, the index and some parameter affecting the equation. The case of the associated inhomogeneous equations is also considered in detail. Some examples and a number of applications are presented for the purpose of illustration.

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1. Introduction

Applications of difference equations and, in particular of linear recurrences, and systems of them, are important in a number of fields such as Biology, Computer Science, Economics, digital signal processing, and more, within the theory of orthogonal polynomials, continued fractions, and combinatorics. A number of issues involving matrix difference equations, pertaining to linear systems theory and controls (but not only) can be found in [3]. See [7] for some results of oscillation theory for second order linear matrix difference equations. Last but not least, these equations and the asymptotic behavior of their solutions play a role in the scattering theory for discrete Sturm–Liouville operators [2]. Despite their apparent simplicity, there are several issues pertaining to them, one of these being the asymptotic behavior of their solutions, which would avoid their automatic recursive computing, providing instead some close-form approximations. The associated *inhomogeneous* equations or systems also occur in a number of applications, for instance in the framework of the theory of *discrete-time control* systems.

In this paper, a Liouville–Green (also called WKB, or WKBJ) asymptotic approximation theory is developed for the linear second-order *matrix difference* equation

$$\Delta^2 Y_n + (D_n + G_n) Y_n = 0, \qquad n \in \mathbf{Z}_\nu \tag{1}$$

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besides the inhomogeneous case, with a term F_n on the right-hand side, where $\mathbf{Z}_\nu := \{n \in \mathbf{Z} : n \geq \nu\}$, for a fixed $\nu \in \mathbf{Z}$, $Y_n, D_n, G_n, F_n \in \mathbb{M}_d(\mathbf{C})$, $\mathbb{M}_d(\mathbf{C})$ denoting the set of all $d \times d$ matrices on the complex field, and $\Delta^2 Y_n = \Delta\left(\Delta Y_n\right)$, $\Delta Y_n := Y_{n+1} - Y_n$; D_n is, for each $n \in \mathbf{Z}_\nu$, a (complex- or real-valued) *diagonal* matrix such that $D_n \to D$ as $n \to \infty$, D being a (real-valued) either positive or negative *diagonal* matrix. Sometimes, in the literature, this class of equations is called 'almost-diagonal', since the coefficient $D_n + G_n$ becomes diagonal, hence the system decoupled, whenever $G_n \to 0$ as $n \to \infty$.

All this follows closely the theory developed in [9,10] for a class of equations like that in (1) but with the diagonal matrix D_n replaced by a constant matrix. These problems parallel the analogous cases concerning linear second-order matrix *differential* equations [18,19], and these works, in turn, generalize some previously existing theories for the corresponding scalar cases (see, e.g. [6,14] for scalar differential equations, and [11,14–17] for scalar difference equations). Other works concerning asymptotic approximations of solutions to linear second-order difference equations [4,20] and applications to the so-called matrix orthogonal polynomials [8,10] should be mentioned, without any claim to be exhaustive.

Clearly, the choice of *matrix* equations is in both cases, of difference or differential equations, equivalent to that of the corresponding *vector* equations, that is of *systems* (of arbitrary order) of linear scalar second-order differential or difference equations, respectively. In fact, if $y_n^{(k)}$, with $k = 1, 2, \ldots, d$, denotes the k-th column of Y_n, we obtain from (1)

$$\Delta^2 y_n^{(k)} + (D_n + G_n)\, y_n^{(k)} = 0, \quad k = 1, 2, \ldots, d, \quad n \in \mathbf{Z}_\nu \tag{2}$$

but the matrix formulation presents some technical advantages.

Throughout the paper, we adopt for all matrices $M \in \mathbb{M}_d(\mathbf{C})$ the *spectral norm*, defined as $\|M\|_2 := \sup_{x \neq 0} \frac{\|Mx\|_2}{\|x\|_2}$. This norm will be denoted hereafter simply by $\|\cdot\|$. As is well known, the spectral norm has the important properties that $\|I\| = 1$, where I is the identity matrix, and $\|D\| = \max_{1 \leq k \leq d} |\lambda_k|$ for every diagonal matrix $D := \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$.

The plan of the paper is the following. In Section 2, a preliminary result is established. Section 3 is devoted to the two different cases $D > 0$ (Subsection 3.1) and $D < 0$ (Subsection 3.2), while in Section 4 the corresponding inhomogeneous problems are considered (Subsections 4.1 and 4.2). Some examples and a number of applications are given in Section 5 to illustrate the theory, and finally, in the short Section 6 the high points of the paper are summarized.

2. A preliminary result

In this section, we establish a preliminary result, namely, we show that Equation (1), with $Q_n := D_n + G_n$, represents a kind of *canonical form* for all linear second-order matrix difference equations. This generalizes the same property that was shown to hold in the scalar case in [11] and used in [15,17]. We state this as a lemma:

Lemma 2.1: *Every linear homogeneous second-order matrix difference equation,*

$$U_{n+2} + A_n U_{n+1} + B_n U_n = 0, \qquad n \in \mathbf{Z}_\nu, \tag{3}$$

where A_n, B_n, $U_n \in \mathbb{M}_d(\mathbf{C})$, and such that the matrices A_n are invertible for all $n \in \mathbf{Z}_\nu$, can be taken into the form (1) setting

$$U_n := \Theta_n Y_n, \qquad n \geq \nu, \tag{4}$$

where

$$\Theta_n := \left(-\frac{1}{2}\right)^{n-\nu-1} \left(\prod_{k=2}^{n-\nu} A_{n-k}\right)\Theta_{\nu+1}, \qquad n \geq \nu + 2, \tag{5}$$

and $\Theta_{\nu+1}, \Theta_\nu$ are arbitrary invertible constant matrices in $\mathbb{M}_d(\mathbf{C})$.

The coefficient Q_n in (1) turns out to be

$$Q_n = -I + 4\Theta_{\nu+1}^{-1}\left(\prod_{k=\nu}^{n} A_k^{-1}\right)B_n\left(\prod_{k=2}^{n-\nu} A_{n-k}\right)\Theta_{\nu+1}, \qquad n \geq \nu + 2. \tag{6}$$

Proof: Setting $U_n := \Theta_n Y_n$ in (3), with Θ_n invertible at least for $n \geq \nu - 2$,

$$Y_{n+2} + \Theta_{n+2}^{-1}A_n\Theta_{n+1}Y_{n+1} + \Theta_{n+2}^{-1}B_n\Theta_n Y_n = 0, \qquad n \geq \nu + 2.$$

that is equivalent to

$$\Delta^2 Y_n + (2I + \Theta_{n+2}^{-1}A_n\Theta_{n+1})Y_{n+1} + (-I + \Theta_{n+2}^{-1}B_n\Theta_n)Y_n = 0, \tag{7}$$

and then look for a matrix Θ_n such that

$$\Theta_{n+2} = -\frac{1}{2}A_n\Theta_{n+1}, \tag{8}$$

for all $n \geq \nu - 1$. Therefore, $\Theta_{n+1} = -\frac{1}{2}A_{n-1}\Theta_n$ for $n \geq \nu - 1$, and hence Θ_n will have the form given in Equation (5). On the other hand, with such a choice of Θ_n Equation 7 takes the form of (1) with

$$Q_n := -I + \Theta_{n+2}^{-1}B_n\Theta_n, \tag{9}$$

i.e. that in (6). □

Remark 2.2: It is noteworthy to observe that, when A_n and B_n are diagonal matrices for every n, also Θ_n and Θ_n^{-1} are diagonal, and hence so is Q_n.

3. The main result for the homogeneous equation

It is convenient to treat separately Equation (1) in the two cases, of $D > 0$ and $D < 0$, since they present some peculiarities.

3.1. The case $D > 0$

We consider first the case $D > 0$, that is that of an asymptotically diagonal positive coefficient. We can establish the following

Theorem 3.1: *Let Equation (1) be given with D_n a (complex- or real-valued)* diagonal *matrix, with $D_n \to D$ as $n \to \infty$, D being a (real-valued) positive* diagonal *matrix, $D := diag(\lambda_1, \lambda_2, \dots, \lambda_d)$. If*

$$H_{\pm} = I \pm iD^{1/2} \tag{10}$$

are the roots of the 'characteristic polynomial' associated to (1) with $G_n \equiv 0$, assume that

$$\sum_{n=\nu}^{\infty} \|D - D_n - H_+^{-n} G_n H_+^n\| < \infty. \tag{11}$$

Then, there exists $n_+ \in \mathbf{Z}_\nu$ such that

$$Y_n^+ = H_+^n (I + E_n^+), \quad n \geq n_+, \tag{12}$$

is a solution of Equation (1), with the 'error term' estimated as

$$\|E_n^+\| \leq \frac{V_n^+}{1 - V_n^+}, \quad n \geq n_+, \tag{13}$$

where the 'error control function' is defined as

$$V_n^+ := \sigma \sum_{k=n}^{\infty} \|D - D_k - H_+^{-k} G_k H_+^k\|, \tag{14}$$

with

$$\sigma := \left[\lambda_m \left(1 + \lambda_m\right)\right]^{-1/2}, \tag{15}$$

where $\lambda_m := \min_j \lambda_j$ is the smallest eigenvalue of D.

A completely similar result holds if we replace the sign $+$ with $-$ everywhere, throughout in (11)–(15).

Moreover, if both conditions, (11) and the same with $-$ in place of $+$ hold, the pair (Y_n^+, Y_n^-) generates the set of solutions to Equation (1). The set of solutions to Equation (1) is a right-module on any given Banach algebra (in particular, in a matrix algebra), and it is free, and of rank 2 (that is, 2 is the minimum number of independent generators). A pair of linearly independent generators for such equation is called 'a basis' of the module of solutions. In this paper, we will confine our results to matrix algebras.

Above, the square root of the matrix D, that is positive definite, is assumed to be the unique (diagonal) positive definite matrix whose entries are the arithmetic square roots of the entries of D.

Remark 3.2: Below, proving Theorem 3.1, we shall see that

$$n_+ = \min\left\{n \in \mathbf{Z}_\nu : V_n^+ < 1\right\},$$

and

$$n_- = \min\left\{n \in \mathbf{Z}_\nu : V_n^- < 1\right\}.$$

Proof of Theorem 3.1: Equation (1) can be written as

$$Y_{n+2} - 2Y_{n+1} + (I + D_n + G_n)Y_n = 0. \tag{16}$$

Since $D_n \to D$ as $n \to \infty$, the associated characteristic equation with $G_n \equiv 0$ is

$$H^2 - 2H + I + D = 0, \tag{17}$$

whose roots are given by (10). Here after we will consider the case where (11) is satisfied. Inserting (12) in (16), we obtain:

$$H_+^{n+2}(I + E_{n+2}^+) - 2H_+^{n+1}(I + E_{n+1}^+) + (I + D_n + G_n)H_+^n(I + E_n^+) = 0,$$

i.e.

$$H_+^2 E_{n+2}^+ - 2H_+ E_{n+1}^+ + (I + D_n)E_n^+ + D_n - D = -H_+^{-n} G_n H_+^n (I + E_n^+).$$

Adding to both sides the quantity DE_n^+, we obtain the *error equation*

$$E_{n+2}^+ - 2H_+^{-1} E_{n+1}^+ + H_+^{-2}(I + D)E_n^+ = H_+^{-2}(D - D_n - \widehat{G}_n^+)(I + E_n^+), \tag{18}$$

where we set

$$\widehat{G}_n^+ := H_+^{-n} G_n H_+^n. \tag{19}$$

In order to solve Equation (18), we consider first the 'unperturbed problem' associated to it, obtained setting equal to zero its full right-hand side,

$$E_{n+2}^+ - 2H_+^{-1} E_{n+1}^+ + H_+^{-2}(I + D)E_n^+ = 0. \tag{20}$$

It can be checked immediately that the general solution to such equation is generated by $\{I, \Theta_+^n\}$, where

$$\Theta_+ := H_+^{-1} H_-, \tag{21}$$

which means that such a general solution can be represented as $R + \Theta_+^n S$, R and S being arbitrary matrices. We can then obtain a solution to the original ('perturbed') equation by the method of variation of parameters [1,5]. Thus, we look for a solution to (18) of the form

$$E_n^+ = R_n + \Theta_+^n S_n, \tag{22}$$

where R_n and S_n are two matrices, depending on n, to be determined. According to the method of variation of parameters, we impose on R_n and S_n the condition

$$\Delta R_n + \Theta_+^{n+1} \Delta S_n = 0 \tag{23}$$

We then evaluate

$$\Delta E_n^+ = \Theta_+^n (\Theta_+ - I) S_n, \tag{24}$$

and then

$$\Delta^2 E_n^+ = (\Theta_+ - I)\Theta_+^n \left(\Theta_+ S_{n+1} - S_n\right). \tag{25}$$

Defining

$$X_n^+ := H_+^{-2}(D - D_n - \widehat{G}_n^+)(I + E_n^+), \tag{26}$$

Equation (18) can be rewritten as

$$\Delta^2 E_n^+ + 2(I - H_+^{-1})\Delta E_n^+ = X_n^+,$$

that is, by (24) and (25), as

$$\Delta S_n = \Theta_+^{-n-1}(\Theta_+ - I)^{-1} X_n^+. \tag{27}$$

We now look for a solution R_n, S_n as follows. Summing both sides of (27) from n to ∞ we obtain

$$\sum_{k=n}^{\infty} \Delta S_k = -S_n = (\Theta_+ - I)^{-1} \sum_{k=n}^{\infty} \Theta_+^{-k-1} X_k^+,$$

hence

$$S_n = -(\Theta_+ - I)^{-1} \sum_{k=n}^{\infty} \Theta_+^{-k-1} X_k^+, \tag{28}$$

assuming that $S_n \to 0$ as $n \to \infty$ (which corresponds to choose a 'constant of integration'), and then, from (23), proceeding similarly,

$$R_n = (\Theta_+ - I)^{-1} \sum_{k=n}^{\infty} X_k^+,$$

where again we assumed that $R_n \to 0$ as $n \to \infty$, so that the solution to Equation (18) of the form in (22) is given by

$$\begin{aligned} E_n^+ &= (\Theta_+ - I)^{-1} \sum_{k=n}^{\infty} (I - \Theta_+^{n-k-1}) X_k^+ \\ &= (\Theta_+ - I)^{-1} H_+^{-2} \sum_{k=n}^{\infty} (I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}_k^+)(I + E_k^+). \end{aligned} \tag{29}$$

Equation (29) can be solved by successive approximations (as was done in [9,10]). Setting, for clarity, $E^+(n) := E_n^+$, we define the sequence $\{E_s^+(n)\}_{s=0}^{\infty}$ as

$$\begin{aligned} &E_0^+(n) \equiv 0, \\ &E_{s+1}^+(n) := (\Theta_+ - I)^{-1} H_+^{-2} \sum_{k=n}^{\infty} (I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}_k^+)(I + E_s^+(k)) \end{aligned}$$

Our purpose is to estimate $\|E^+(n)\|$ defined by (29). We have

$$E^+_{s+1}(n) - E^+_s(n) = (\Theta_+ - I)^{-1}H_+^{-2}\sum_{k=n}^{\infty}(I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}^+_k)(E^+_s(k) - E^+_{s-1}(k)), \quad s \geq 1.$$

Then, we prove, by induction on s, that

$$\|E^+_{s+1}(n) - E^+_s(n)\| \leq (V^+_n)^{s+1}, \quad s = 0, 1, 2, \ldots, \quad n \in \mathbf{Z}_\nu. \tag{30}$$

The basic case ($s = 0$) can be checked immediately,

$$\begin{aligned}\|E^+_1(n) - E^+_0(n)\| &= \|E^+_1(n)\|\\ &\leq \|(\Theta_+ - I)^{-1}H_+^{-2}\|\sum_{k=n}^{\infty}\|I - \Theta_+^{n-k-1}\|\|D - D_k - \widehat{G}^+_k\|.\end{aligned}$$

Since

$$\|\Theta_+\| = \max_{1\leq j\leq m}\left|\frac{1 - i\lambda_j^{1/2}}{1 + i\lambda_j^{1/2}}\right| = 1,$$

we obtain

$$\|E^+_1(n) - E^+_0(n)\| \leq 2\|(\Theta_+ - I)^{-1}H_+^{-2}\|\sum_{k=n}^{\infty}\|D - D_k - \widehat{G}^+_k\| =: V^+_n \tag{31}$$

We now use the inductive assumption for some $s \geq 1$ and prove the validity of the same assumption for $s + 1$: assuming that

$$\|E^+_s - E^+_{s-1}\| \leq (V^+_n)^s$$

for some $s \geq 1$, we obtain

$$\begin{aligned}&\|E^+_{s+1}(n) - E^+_s(n)\|\\ &\quad\leq 2\|(\Theta_+ - I)^{-1}H_+^{-2}\|\sum_{k=n}^{\infty}\|D - D_k - \widehat{G}^+_k\|\|E^+_s(k) - E^+_{s-1}(k)\|\\ &\quad\leq 2\|(\Theta_+ - I)^{-1}H_+^{-2}\|\sum_{k=n}^{\infty}\|D - D_k - \widehat{G}^+_k\|(V^+_k)^s \leq (V^+_n)^{s+1}.\end{aligned} \tag{32}$$

Now, the sequence

$$E^+_n := \sum_{s=0}^{\infty}(E^+_{s+1}(n) - E^+_s(n)), \tag{33}$$

is well-defined, in view of (30), whenever $V^+_n < 1$. This condition is satisfied for all $n \geq n_+$, n_+ being defined in Remark 3.2, since V^+_n decreases.

Moreover, we get from (30) and (33) that

$$\|E_n^+\| \leq \sum_{s=0}^{\infty} \left\|E_{s+1}^+(n) - E_s^+(n)\right\| \leq \sum_{s=1}^{\infty} \left(V_n^+\right)^s = \frac{V_n^+}{1-V_n^+}, \quad n \geq n_+.$$

Note that the series in (33) converges *uniformly* with respect to n, for $n \geq n_+$, and solves the *error equation* (18). In fact, we can write

$$\begin{aligned}
E_n^+ &= E_1^+(n) + \sum_{s=1}^{\infty} \left(E_{s+1}^+(n) - E_s^+(n)\right) \\
&= (\Theta_+ - I)^{-1} H_+^{-2} \sum_{k=n}^{\infty} (I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}_k^+) \\
&\quad + (\Theta_+ - I)^{-1} H_+^{-2} \sum_{s=1}^{\infty} \sum_{k=n}^{\infty} (I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}_k^+)(E_s^+(k) - E_{s-1}^+(k)).
\end{aligned}$$

We can now interchange the order of summation in view of the Lebesgue dominated convergence theorem, obtaining

$$\begin{aligned}
&\|(\Theta_+ - I)^{-1} H_+^{-2} \sum_{s=1}^{S} (I - \Theta_+^{n-k-1})(D - D_k - \widehat{G}_k^+)(E_s^+(k) - E_{s-1}^+(k))\| \\
&\quad \leq 2\|(\Theta_+ - I)^{-1} H_+^{-2}\| \|D - D_k - \widehat{G}_k^+\| \frac{V_{n_+}}{1 - V_{n_+}}
\end{aligned}$$

for any fixed $S > 1$ and $k > n_+$, and using (11).

Therefore, we have proved that the sequence $E^+(n)$ in (33) solves Equation (29).

A similar result can be obtained assuming the validity of (11) with H_- replacing H_+. We obtain the same conclusions just replacing throughout the sign $+$ with $-$, as one can easily check.

To conclude the proof of Theorem 3.1, we should show that the pair (Y_n^+, Y_n^-) generates the full set (the right-module) of solutions of Equation (1). It is easy to check that they are *linearly independent*, in view of their asymptotic behavior. Therefore, the *Casorati* matrix,

$$C(n) := \begin{pmatrix} Y_n^+ & Y_n^- \\ Y_{n+1}^+ & Y_{n+1}^- \end{pmatrix}$$

considered as a linear operator, is injective for every fixed n. In $\mathbb{M}_d(\mathbf{C})$, this suffices to guarantee the invertibility of $C(n)$, cf. [10, Equation (22)]. That the solutions Y_n^+ and Y_n^- are linearly independent can also be proven directly. In fact, the condition $Y_n^+ C_1 + Y_n^- C_2 = 0$ for every $n \geq \nu$ implies that $C_1 = C_2 = 0$, and it is also true that $Y_{n+1}^+ C_1 + Y_{n+1}^- C_2 = 0$ for every $n \geq \nu$. The linear system

$$\begin{cases} Y_n^+ C_1 + Y_n^- C_2 = 0 \\ Y_{n+1}^+ C_1 + Y_{n+1}^- C_2 = 0 \end{cases}$$

can be solved directly, obtaining first $C_2 = -(Y_n^-)^{-1}Y_n^+C_1$ from the first equation and then $[Y_{n+1}^+ - Y_{n+1}^-(Y_n^-)^{-1}Y_n^+]C_1 = 0$ from the second one. The latter will be satisfied *for every* matrix C_1 if and only if $Y_{n+1}^+ - Y_{n+1}^-(Y_n^-)^{-1}Y_n^+ \neq 0$, at least for n sufficiently large. But in view of Theorem 3.1, we have

$$\begin{aligned} Y_{n+1}^+ - Y_{n+1}^-(Y_n^-)^{-1}Y_n^+ &\sim H_+^{n+1} - H_-^{n+1}H_n^{-n}H_+^n \\ &= (H_+ - H_-)H_+^n = 2iD^{1/2}(I + iD^{1/2})^n, \end{aligned}$$

which goes to infinity (in norm, it diverges to $+\infty$) as $n \to \infty$. The same result can be established in a similar way in case $D < 0$ considered below. □

3.2. The case $D < 0$

Consider now the case of Equation (1) with $D < 0$. The following steps are essentially the same of Theorem 3.1.

Similarly to the scalar case, we call *recessive* a solution Y_n^- such that another, independent solution, Y_n^+, called *dominant*, exists, is invertible in a neighborhood of ∞, and such that $\left(Y_n^+\right)^{-1} Y_n^- \to 0$, as $n \to \infty$.

Theorem 3.2: *Let Equation (1) be given with $D_n < 0$, D_n diagonal and such that $D_n \to D := diag(\lambda_1, \lambda_2, \ldots, \lambda_d) < 0$ (hence, $\lambda_j < 0$ for every j) as $n \to \infty$. Set $P_n := -D_n$ and $P := -D$, for convenience. Let*

$$H_- = I - P^{1/2} \tag{34}$$

be one of the roots of the characteristic polynomial associated to (1) with $G_n \equiv 0$, and assume that

$$\sum_{n=\nu}^{\infty} \left\| P_n - P - H_-^{-n}G_nH_-^n \right\| < \infty. \tag{35}$$

We also require that $-\lambda_j \neq 1$ for ever j, in order to make H_- nonsingular.

Then, there exists $n_- \in \mathbf{Z}_\nu$ such that

$$Y_n^- = H_-^n \left(I + E_n^-\right), \quad n \geq n_-, \tag{36}$$

is a solution of Equation (1) with the 'error term' estimated as

$$\|E_n^-\| \leq \frac{V_n^-}{1 - V_n^-}, \quad n \geq n_-, \tag{37}$$

where

$$V_n^- := \sigma_- \sum_{k=n}^{\infty} \|P_k - P - H_-^{-k}G_kH_-^k\|, \tag{38}$$

where

$$\begin{aligned} \sigma_- &:= 2\left\| \left(\Theta_- - I\right)^{-1} H_-^{-2}\right\| = \|P^{-1/2}H_-^{-1}\| \\ &= \max_{1\leq j\leq d} \left| \frac{1}{(-\lambda_j)^{1/2}\left(1 - (-\lambda_j)^{1/2}\right)} \right| = \frac{1}{\sqrt{p}(1 - \sqrt{p})}, \end{aligned} \tag{39}$$

being p the absolute value of the eigenvalue which maximizes the function $(x^{1/2}|1-x^{1/2}|)^{-1}$ *for x in the spectrum of P. Note that* $n_- := \min\{n \in \mathbf{Z}_\nu : V_n^- < 1\}$.

Proof: Following the lines of the proof of Theorem 3.1, we obtain the error equation

$$E_{n+2}^- - 2H_-^{-1}E_{n+1}^- + H_-^{-2}(I-P)E_n^- = H_-^{-2}\left(P_n - P - \widehat{G}_n^-\right)\left(I + E_n^-\right) \tag{40}$$

where we set

$$\widehat{G}_n^- := H_-^{-n} G_n H_-^n. \tag{41}$$

Its solution satisfies, similarly to the previous case, the equation

$$E_n^- = (\Theta_- - I)^{-1} H_-^{-2} \sum_{k=n}^{\infty} (I - \Theta_-^{n-k-1})(P_k - P - \widehat{G}_k^-)\left(I + E_k^-\right) \tag{42}$$

where

$$\Theta_\pm := H_\pm^{-1} H_\mp. \tag{43}$$

Note that $\Theta_- = (\Theta_+)^{-1}$. Since now $\|\Theta_-\| > 1$ and $n-k-1 < 0$ for any $k \geq n$, applying the method of successive approximation we obtain the estimate given in (37). This completes the proof. □

When $D < 0$, the issue of existence of a second solution in some neighborhood of ∞, possibly spanning, along with the first solution, the whole set of solutions to Equation (1), with an error term which can be estimated rigorously as in Theorem 3.1, is more subtle, as it was even in the corresponding case of scalar differential equations [6]; see also [10]. An answer is provided by the following

Theorem 3.3: *Consider Equation* (1) *with* $D_n < 0$, *set* $P_n := -D_n > 0$, *and assume that*

$$\sum_{n=\nu}^{\infty} \|P_n - P - H_+^{-n} G_n H_+^n\| < \infty, \qquad H_+ := I + P^{1/2}. \tag{44}$$

Then, a second solution of Equation (1) *of the form*

$$Y_n^+ := H_+^n \left(I + E_n^+\right) \tag{45}$$

can be found to hold on a halfline $n \geq \nu^* \geq \nu$, *with the error term estimated as*

$$\|E_n^+\| \leq \frac{1}{1 - V_\infty^+(\nu^*)} \left[V_\infty^+(\nu^*) - V_{[\frac{n}{2}]}^+(\nu^*) + \|\Theta_+\|^{\,n-[\frac{n}{2}]} V_{[\frac{n}{2}]}^+(\nu^*)\right], \tag{46}$$

where

$$V_n^+ := \sigma_+ \sum_{k=\nu}^{n-1} \|P_k - P - H_+^{-k} G_k H_+^k\|, \tag{47}$$

for $n \geq \nu^* \geq \nu + 1$, *being*

$$\nu^* := \min\{n \in \mathbf{Z}_\nu : V_\infty^+(n) < 1/2\} \tag{48}$$

and

$$\sigma_+ := 2\,\|(\Theta_+ - I)^{-1}H_+^{-2}\| = \|P^{-1/2}H_+^{-1}\| = \frac{1}{\sqrt{|\lambda_M|}\left(1+\sqrt{|\lambda_M|}\right)}. \tag{49}$$

The pair of solutions (Y_n^+, Y_n^-) *generates the set of all solutions to Equation* (1), *at least for* n *sufficiently large.*

Proof of Theorem 3.3: As in Theorem 3.1, we derive the error equation

$$E_{n+2}^+ - 2H_+^{-1}E_{n+1}^+ + H_+^{-2}(I-P)E_n^+ = H_+^{-2}\left(P_n - P - \widehat{G}_n^+\right)\left(I + E_n^+\right) \tag{50}$$

where

$$\widehat{G}_n^+ := H_+^{-n}G_nH_+^n. \tag{51}$$

We follow the same procedure adopted in the proof of Theorem 3.1. Solving Equation (50) by the method of variation of parameters [1,5]), and imposing on R_n and S_n (similarly to what was done in Theorem 3.1) the conditions

$$\Delta R_n + \Theta_+^{n+1}\Delta S_n = 0$$

and

$$R_\nu = S_\nu = 0,$$

we obtain

$$E_n^+ = \left(\Theta_+ - I\right)^{-1}H_+^{-2}\sum_{k=\nu}^{n-1}\left(\Theta_+^{n-k-1} - I\right)\left(P_k - P - \widehat{G}_k^+\right)\left(I + E_k^+\right) \tag{52}$$

with $n \geq \nu + 1$ and

$$\Theta_+ := H_+^{-1}H_-.$$

Writing, for clarity, $E^+(n) := E_n^+$, and defining the sequence $\left\{E_s^+(n)\right\}_{s=0}^{\infty}$ as

$$E_0^+(n) \equiv 0,$$
$$E_{s+1}^+(n) := (\Theta_+ - I)^{-1}H_+^{-2}\sum_{k=\nu}^{n-1}(\Theta_+^{n-k-1} - I)(P_k - P - \widehat{G}_k^+)(I + E_s^+(k)),$$

by successive approximations, is easy to check that

$$\|E_{s+1}^+(n) - E_s^+(n)\| \leq \left(V_n^+\right)^{s+1} \tag{53}$$

with V_n^+ as in (47), since $\|\Theta_+\| < 1$ and $k \leq n-1$.

Unlike the previous cases, we should now face the problem to establish that E_n^+ has a finite limit as $n \to \infty$. To show this, we follow closely the method applied by Olver in the case of *differential* equations [6,10]).

We start showing that $\Delta E_n^+ \to 0$ as $n \to \infty$. We obtain by a little algebra

$$\Delta E_s^+(n) = H_+^{-2} \sum_{k=\nu}^{n-1} \Theta_+^{n-k-1} \left(P_k - P - \widehat{G}_k^+\right) \left(I + E_{s-1}^+(k)\right), \quad s \geq 1.$$

Defining

$$\Delta E_n^+ := \sum_{s=0}^{\infty} \left(\Delta E_{s+1}^+(n) - \Delta E_s^+(n)\right),$$

we can estimate ΔE_n^+ as

$$\begin{aligned}
\|\Delta E_n^+\| &\leq \sum_{s=0}^{\infty} \|\Delta E_{s+1}^+(n) - \Delta E_s^+(n)\| \\
&\leq \|H_+^{-2}\| \sum_{k=\nu}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| \sum_{s=0}^{\infty} \|E_s^+(k) - E_{s-1}^+(k)\| \\
&\leq \|H_+^{-2}\| \sum_{k=\nu}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| \sum_{s=0}^{\infty} \left(V_k^+\right)^s \\
&\leq \frac{1}{1 - V_\infty^+(\nu^*)} \|H_+^{-2}\| \sum_{k=\nu^*}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\|, \qquad (54)
\end{aligned}$$

since V_k^+ grows monotonically to V_∞^+, and $\nu^* \geq \nu$ can be chosen such that $V_\infty^+(\nu^*) := \lim_{n\to\infty} V_n^+ < 1$.

Inequality (54) can be written splitting the sum on the right-hand side in two parts, namely

$$\begin{aligned}
&\frac{1}{1 - V_\infty^+(\nu^*)} \|H_+^{-2}\| \sum_{k=\nu^*}^{N-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| \\
&\quad + \frac{1}{1 - V_\infty^+(\nu^*)} \|H_+^{-2}\| \sum_{k=N}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\|,
\end{aligned}$$

where N is chosen with $\nu^* \leq N - 1 < N \leq n - 1$, such that, for every $\varepsilon > 0$, the second summand is less that ε. This is possible in view of the assumption (44) and the fact that $\|\Theta_+\|^{\,n-k-1} < 1$. The first sum, instead, can be made arbitrarily small taking n sufficiently large. It follows that $\Delta E_n^+ \to 0$ as $n \to \infty$.

We can then write

$$E_n^+ - \left(\Theta_+ - I\right)^{-1} \Delta E_n^+ = -\left(\Theta_+ - I\right)^{-1} H_+^{-2} \sum_{k=\nu^*}^{n-1} \left(P_k - P - \widehat{G}_k^+\right) \left(I + E_k^+\right) \qquad (55)$$

obtaining

$$
\begin{aligned}
E_n^+ &= -\left(\Theta_+ - I\right)^{-1} H_+^{-2} \sum_{k=\nu^*}^{n-1} \left(P_k - P - \widehat{G}_k^+\right)\left(I + E_k^+\right) + \left(\Theta_+ - I\right)^{-1} \Delta E_n^+ \\
&= \left(\Theta_+ - I\right)^{-1} \Bigg\{ - H_+^{-2} \sum_{k=\nu^*}^{n-1} \left(P_k - P - \widehat{G}_k^+\right) [I + \sum_{r=0}^{\infty} \left(E_{r+1}^+(k) - E_r^+(k)\right)] \\
&\quad + \Delta E_n^+ \Bigg\} = \left(\Theta_+ - I\right)^{-1} \left(- \sum_{r=0}^{\infty} L_r(n) + \Delta E_n^+ \right),
\end{aligned}
$$

having set

$$
\begin{aligned}
L_0(n) &:= H_+^{-2} \sum_{k=\nu^*}^{n-1} \left(P_k - P - \widehat{G}_k^+\right), \\
L_r(n) &:= H_+^{-2} \sum_{k=\nu^*}^{n-1} \left(P_k - P - \widehat{G}_k^+\right)\left(E_r^+(k) - E_{r-1}^+(k)\right), \quad r \geq 1.
\end{aligned}
$$

We are now able to prove that $E_\infty^+ := \lim_{n\to\infty} E_n^+ < \infty$. Assuming, without loss of generality, that $n > m$, we find

$$
\|L_0(n) - L_0(m)\| \leq \frac{1}{2} \|\Theta_+ - I\| \left(V_n^+ - V_m^+\right), \tag{56}
$$

$$
\sum_{r=1}^{\infty} \|L_r(n) - L_r(m)\| \leq \frac{1}{1 - V_\infty^+(\nu^*)} \frac{1}{2} \|\Theta_+ - I\| \left(V_n^+ - V_m^+\right). \tag{57}
$$

Therefore, using the inequalities in (56), (57), we obtain

$$
\begin{aligned}
\|E_n^+ - E_m^+\| &\leq \frac{1}{2}\left(1 + \frac{1}{1 - V_\infty^+(\nu^*)}\right)\left(V_n^+ - V_m^+\right) \\
&\quad + \|(\Theta_+ - I)^{-1}\| \|\Delta E_n^+ - \Delta E_m^+\|.
\end{aligned} \tag{58}
$$

Having proved that $\Delta E_n^+ \to 0$ as $n \to \infty$ and that the scalar sequence $\{V_n^+\}$ has a finite limit when $n, m \to \infty$ independently (with $n > m$), it is proved that E_n^+ has a finite limit, say E_∞^+, as $n \to \infty$.

At this point, we showed that a second solution to Equation (1) with $D_n < 0$, of the form $Y_n^+ = H_+^n \left(I + E_n^+\right)$, does exist, with $E_n^+ \to E_\infty^+$ as $n \to \infty$. This means that

$$
\left(I + E_n^+\right)\left(I + E_\infty^+\right)^{-1} \sim I, \qquad \text{as } n \to \infty,
$$

i.e.

$$
\left(I + E_n^+\right)\left(I + E_\infty^+\right)^{-1} =: I + \widehat{E}_n^+, \quad \text{as } n \to \infty, \tag{59}
$$

provided that $I + E_n^+$ does not vanish in a neighborhood of ∞.

By Equation (59), we are able to estimate $\widehat{E}_n^+$,

$$\widehat{E}_n^+ = \left(E_n^+ - E_\infty^+\right)\left(I + E_\infty^+\right)^{-1} = \left(E_n^+ - E_\infty^+\right)\sum_{k=0}^{\infty}\left(-E_\infty^+\right)^k,$$

provided that $\left\|E_\infty^+\right\| < 1$, which can be realized taking a suitably large value of $\nu^* \geq \nu$. Therefore,

$$\|\widehat{E}_n^+\| \leq \frac{\|E_n^+ - E_\infty^+\|}{1 - \|E_\infty^+\|}.$$

Recall that, by Equations (52) and (55) and knowing that $\Delta E_n^+ \to 0$ as $n \to \infty$, we have

$$E_n^+ = \left(\Theta_+ - I\right)^{-1} H_+^{-2} \sum_{k=\nu^*}^{n-1}\left(\Theta_+^{n-k-1} - I\right)\left(P_k - P - \widehat{G}_k^+\right)\left(I + E_k^+\right),$$

and

$$E_\infty^+ = \lim_{n\to\infty} E_n^+ = -\left(\Theta_+ - I\right)^{-1} H_+^{-2} \sum_{k=\nu^*}^{\infty}\left(P_k - P - \widehat{G}_k^+\right)\left(I + E_k^+\right).$$

Then,

$$\begin{aligned}\|E_n^+ - E_\infty^+\| \leq \|\left(\Theta_+ - I\right)^{-1} H_+^{-2}\| &\left[\sum_{k=n}^{\infty}\|P_k - P - \widehat{G}_k^+\|\|I + E_k^+\|\right.\\ &\left.+ \sum_{k=\nu^*}^{n-1}\|\Theta_+\|^{n-k-1}\|P_k - P - \widehat{G}_k^+\|\|I + E_k^+\|\right].\end{aligned}$$

Since

$$\|I + E_k^+\| \leq 1 + \|E_k^+\| \leq 1 + \sum_{s=0}^{\infty}\|E_{s+1}^+(k) - E_s^+(k)\| \leq \frac{1}{1 - V_\infty^+(\nu^*)},$$

we have

$$\begin{aligned}\|E_n^+ - E_\infty^+\| \leq &\frac{1}{1 - V_\infty^+(\nu^*)}\|\left(\Theta_+ - I\right)^{-1} H_+^{-2}\|\\ &\times\left[\sum_{k=n}^{\infty}\|P_k - P - \widehat{G}_k^+\| + \sum_{k=\nu^*}^{n-1}\|\Theta_+\|^{n-k-1}\|P_k - P - \widehat{G}_k^+\|\right].\end{aligned}$$

Note that

$$
\begin{aligned}
&\sum_{k=\nu^*}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| \\
&\quad = \sum_{k=\nu^*}^{N-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| + \sum_{k=N}^{n-1} \|\Theta_+\|^{n-k-1} \|P_k - P - \widehat{G}_k^+\| \\
&\quad \le \|\Theta_+\|^{n-N} \sum_{k=\nu^*}^{N-1} \|P_k - P - \widehat{G}_k^+\| + \sum_{k=N}^{n-1} \|P_k - P - \widehat{G}_k^+\|.
\end{aligned}
$$

In order to better estimate the summand above, we choose N such that $\nu^* \le N-1 < N \le n-1$ and so large that, for every $\varepsilon > 0$, the second sum be less than ε. Having fixed N, the first sum will be made smaller than ε taking n sufficiently large.

We have finally found that

$$
\begin{aligned}
\|E_n^+ - E_\infty^+\| &\le \frac{1}{1 - V_\infty^+(\nu^*)} \| \left(\Theta_+ - I\right)^{-1} H^{-2} \| \\
&\quad \times \left[\sum_{k=N}^{\infty} \|P_k - P - \widehat{G}_k^+\| + \|\Theta_+\|^{n-N} \sum_{k=\nu^*}^{N-1} \|P_k - P - \widehat{G}_k^+\| \right] \\
&= \frac{1}{2} \frac{1}{1 - V_\infty^+(\nu^*)} \left[V_\infty^+(\nu^*) - V_N^+(\nu^*) + \|\Theta_+\|^{n-N} V_N^+(\nu^*) \right]
\end{aligned} \tag{60}
$$

To conclude the proof we should estimate

$$
\begin{aligned}
\|E_\infty^+\| &\le \| \left(\Theta_+ - I\right)^{-1} H_+^{-2} \| \sum_{k=\nu^*}^{\infty} \|P_k - P - \widehat{G}_k^+\| \frac{1}{1 - V_\infty^+(\nu^*)} \\
&= \frac{1}{2} \frac{1}{1 - V_\infty^+(\nu^*)} V_\infty^+(\nu^*).
\end{aligned} \tag{61}
$$

If we choose, for instance, $N := \left[\frac{n}{2}\right]$, we obtain the inequality in (46), using (60) and (61),

$$
\|\widehat{E}_n^+\| \le \frac{1}{1 - V_\infty^+(\nu^*)} \left[V_\infty^+(\nu^*) - V_{\left[\frac{n}{2}\right]}^+(\nu^*) + \|\Theta_+\|^{n-\left[\frac{n}{2}\right]} V_{\left[\frac{n}{2}\right]}^+(\nu^*) \right],
$$

ν^* is the smallest $n \in \mathbf{Z}_\nu$ such that $V_\infty^+(n) < 1/2$, so that

$$
1 - \|E_\infty^+\| \ge 1 - \frac{1}{2} \frac{V_\infty^+(\nu^*)}{1 - V_\infty^+(\nu^*)} > \frac{1}{2}.
$$

Proceeding as in the proof of Theorem 3.1, it is easy to check that the pair (Y_n^+, Y_n^-) is a basis of all solutions to equation (1) with $D < 0$.

4. The inhomogeneous equation

Here we consider the inhomogeneous equation associated to (1), i.e.

$$\Delta^2 Y_n + (D_n + G_n)Y_n = F_n, \qquad n \in \mathbf{Z}_\nu, \tag{62}$$

F_n being some given $d \times d$ matrix, playing the role of a forcing term. We assume first that $D > 0$.

4.1. The inhomogeneous equation with D > 0

In this subsection, we construct *a particular* solution to (62), say $W(n)$, and hence, by general consideration, we can infer that, similarly to the analogous case of differential equations,

$$Y_n = Y_1(n)C_1 + Y_2(n)C_2 + W(n) \tag{63}$$

will be the *general* solution to (62). Here, $Y_1(n)$ and $Y_2(n)$ denote two linearly independent solutions to the associated homogeneous equation, and we will write the dependence on n in parentheses instead of as an index, whenever convenient.

We will construct the solution $W(n)$ by a method of variation of parameters [1,5], proceeding as in the case of differential equations, that is, we look for a solution to (62) of the form

$$W(n) = Y_1(n)C_1(n) + Y_2(n)C_2(n), \tag{64}$$

where $C_1(n)$ and $C_2(n)$ are $d \times d$ matrices to be determined.

We first evaluate

$$\begin{aligned} \Delta W(n) &= \Delta[Y_1(n)C_1(n) + Y_2(n)C_2(n)] \\ &= Y_1(n+1)\Delta C_1(n) + (\Delta Y_1(n))C_1(n) + Y_2(n+1)\Delta C_2(n) + (\Delta Y_2(n))C_2(n), \end{aligned} \tag{65}$$

and impose the condition

$$Y_1(n+1)\Delta C_1(n) + Y_2(n+1)\Delta C_2(n) \equiv 0. \tag{66}$$

Therefore, we are left with

$$\Delta W(n) = (\Delta Y_1(n))C_1(n) + (\Delta Y_2(n))C_2(n), \tag{67}$$

and then we evaluate

$$\begin{aligned} \Delta^2 W(n) &= \Delta[(\Delta Y_1(n))C_1(n) + (\Delta Y_2(n))C_2(n)] \\ &= (\Delta Y_1(n+1))\Delta C_1(n) + (\Delta^2 Y_1(n))C_1(n) \\ &\quad + (\Delta Y_2(n+1))\Delta C_2(n) + (\Delta^2 Y_2(n))C_2(n). \end{aligned} \tag{68}$$

Finally, using the fact that $\Delta^2 Y_{1,2}(n) = -Q_n Y_{1,2}(n)$, we obtain

$$\Delta^2 W(n) + Q_n W(n) = (\Delta Y_1(n+1)))\Delta C_1(n) + (\Delta Y_2(n+1)))\Delta C_2(n) = F_n. \tag{69}$$

The two Equations (66), (69) allow to determine $C_1(n)$ and $C_2(n)$ as follows. We first obtain $\Delta C_2(n)$ from (66),

$$\Delta C_2(n) = -Y_2^{-1}(n+1)Y_1(n+1)\Delta C_1(n), \tag{70}$$

observing that $Y_2(n)$ is invertible for n sufficiently large, in view of Theorem 3.1.

Inserting this in (69), we obtain

$$K(n)Y_1(n+1)\Delta C_1(n) = F_n, \tag{71}$$

having set

$$\begin{aligned} K(n) &:= \big(\Delta Y_1(n+1)\big)\ Y_1^{-1}(n+1) - \big(\Delta Y_2(n+1)\big)\ Y_2^{-1}(n+1) \\ &= Y_1(n+2)Y_1^{-1}(n+1) - Y_2(n+2)Y_2^{-1}(n+1), \end{aligned} \tag{72}$$

and hence

$$\Delta C_1(n) = Y_1^{-1}(n+1)K^{-1}(n)F_n. \tag{73}$$

We can solve this to obtain $C_1(n)$, summing both sides from n to ∞, assuming that $C_1(n) \to 0$ as $n \to \infty$. This correspond to choose a 'constant of integration':

$$\sum_{k=n}^{\infty} \Delta C_1(k) = -C_1(n) = \sum_{k=n}^{\infty} Y_1^{-1}(k+1)\, K^{-1}(k)\, F_k, \tag{74}$$

i.e.

$$C_1(n) = -\sum_{k=n}^{\infty} Y_1^{-1}(k+1)\, K^{-1}(k)\, F_k. \tag{75}$$

Thus, from (70) and (73),

$$\Delta C_2(n) = -Y_2^{-1}(n+1)K^{-1}(n)F_n, \tag{76}$$

and proceeding similarly as before, we obtain

$$\sum_{k=n}^{\infty} \Delta C_2(k) = -C_2(n) = -\sum_{k=n}^{\infty} Y_2^{-1}(k+1)\, K^{-1}(k)\, F_k, \tag{77}$$

i.e.

$$C_2(n) = \sum_{k=n}^{\infty} Y_2^{-1}(k+1)\, K^{-1}(k)\, F_k, \tag{78}$$

and hence the *particular* solution to Equation (62) is

$$\begin{aligned} W(n) &:= Y_1(n)C_1(n) + Y_2(n)C_2(n) \\ &= \sum_{k=n}^{\infty} [Y_2(n)Y_2^{-1}(k+1) - Y_1(n)Y_1^{-1}(k+1)]\, K^{-1}(k)\, F_k \end{aligned}$$

$$= \sum_{k=n}^{\infty} [Y_2(n)Y_2^{-1}(k+1) - Y_1(n)Y_1^{-1}(k+1)] \times [Y_1(k+2)Y_1^{-1}(k+1) - Y_2(k+2)Y_2^{-1}(k+1)]^{-1}F_k. \tag{79}$$

By general results concerning linear equations (and vector spaces or modules of solutions), we conclude that the *general* solution to the *inhomogeneous* Equation (62) is given by Equations (63), (79), where C_1 and C_2 are two constant matrices.

We wish now to provide a representation of the particular solution, $W(n)$, valid for large n, accompanied by estimates for the error terms involved. This goal can be attained by a rather lengthy though simple process, exploiting the results established in Theorems 3.1 and 3.3, concerning two linearly independent solution to the homogeneous Equation (1).

We start estimating $K^{-1}(k)$. We have first

$$\begin{aligned}
K(k) &= Y_1(k+2)Y_1^{-1}(k+1) - Y_2(k+2)Y_2^{-1}(k+1) \\
&= H_+^{k+2}(I + E_+(k+2)) \left[I + E_+(k+1)\right]^{-1} H_+^{-k-1} \\
&\quad - H_-^{k+2}(I + E_-(k+2)) \left[I + E_-(k+1)\right]^{-1} H_-^{-k-1} \\
&= H_+^{k+2}[I + E_+(k+2) + B_+(k+1) + E_+(k+2)B_+(k+1)]H_+^{-k-1} \\
&\quad - H_-^{k+2}[I + E_-(k+2) + B_-(k+1) + E_-(k+2)B_-(k+1)]H_-^{-k-1} \\
&= H_+^{k+2}[I + \mathcal{E}_+(k)]H_+^{-k-1} - H_-^{k+2}[I + \mathcal{E}_-(k)]H_-^{-k-1} \\
&= H_+ - H_- + H_+^{k+2}\mathcal{E}_+(k)H_+^{-k-1} - H_-^{k+2}\mathcal{E}_-(k)H_-^{-k-1} \\
&= R + H_+^{k+2}\mathcal{E}_+(k)H_+^{-k-1} - H_-^{k+2}\mathcal{E}_-(k)H_-^{-k-1},
\end{aligned}$$

where we set

$$R := 2iD^{1/2}, \tag{80}$$

$$[I + E_\pm(k+1)]^{-1} = I + B_\pm(k+1), \tag{81}$$

and

$$\mathcal{E}_\pm(k) := E_\pm(k+2) + B_\pm(k+1) + E_\pm(k+2)B_\pm(k+1). \tag{82}$$

Above, we observed that

$$(I + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k = I + B, \tag{83}$$

provided that $\|A\| < 1$, and

$$\|B\| \leq \frac{\|A\|}{1 - \|A\|}, \tag{84}$$

with

$$\|B\| \leq 2\,\|A\|, \quad \text{if } \|A\| < 1/2. \tag{85}$$

We also noted that

$$\begin{aligned} H_+H_- &= (I+iD^{1/2})(I-iD^{1/2}) = I+D,\\ H_-^{-1}-H_+^{-1} &= H_-^{-1}H_+^{-1}(H_+-H_-) = 2iD^{1/2}(I+D)^{-1}. \end{aligned} \tag{86}$$

Then setting for short

$$S(k) := H_+^{k+2}\mathcal{E}_+(k)H_+^{-k-1} - H_-^{k+2}\mathcal{E}_-(k)H_-^{-k-1}, \tag{87}$$

we can write

$$K(k) = R\left[I + R^{-1}S(k)\right]. \tag{88}$$

Now, we can estimate $S(k)$ as

$$\begin{aligned} \|S(k)|\| &\le \|H_+\|^{k+2}\cdot\|\mathcal{E}_+(k)\|\cdot\|H_+^{-1}\|^{k+1} + \|H_-\|^{k+2}\cdot\|\mathcal{E}_-(k)\|\cdot\|H_-^{-1}\|^{k+1}\\ &\le (1+\lambda_M)^{\frac{k+2}{2}}\left(\|\mathcal{E}_+(k)\|+\|\mathcal{E}_-(k)\|\right)(1+\lambda_m)^{-\frac{k+1}{2}}\\ &\le 16\,(1+\lambda_M)^{\frac{k+2}{2}}(1+\lambda_m)^{-\frac{k+1}{2}}V(k), \end{aligned} \tag{89}$$

since

$$\begin{aligned} \|H_\pm\| &= \max_j\{(1+\lambda_j)^{1/2}\} = (1+\lambda_M)^{1/2},\\ \|H_\pm^{-1}\| &= \max_j\{(1+\lambda_j)^{-1/2}\} = (1+\lambda_m)^{-1/2}, \end{aligned}$$

being $\lambda_M := \max_j\{\lambda_j\}$, $\lambda_m := \min_j\{\lambda_j\}$, $V(k) := \max\{V_+(k), V_-(k)\}$, and

$$\begin{aligned} \|\mathcal{E}_\pm(k)\| &\le \|E_\pm(k+2)\| + \|B_\pm(k+1)\| + \|E_\pm(k+2)\|\cdot\|B_\pm(k+1)\|\\ &\le 2\left[V_\pm(k) + 2\,V_\pm(k+1)\right] + 8\,V_\pm(k)V_\pm(k+1) \le 8\,V_\pm(k). \end{aligned} \tag{90}$$

In fact,

$$\|E_\pm(k)\| \le \frac{V_\pm(k)}{1-V_\pm(k)} \le 2\,V_\pm(k) \tag{91}$$

for $k \ge n' \ge n$, being n' the smallest index such that $V_\pm(n') < 1/2$, and

$$\|B_\pm(k+1)\| \le 2\,\|E_\pm(k+1)\| \le 4\,V_\pm(k+1) \le 4\,V_\pm(k), \tag{92}$$

for $k \ge n'' \ge n \quad (n'' > n')$, being n'' the smallest index such that $V_\pm(n'') < 1/4$. Here, recall that

$$V_\pm(k) = \sigma\sum_{h=k}^{\infty}\|D - D_h - H_\pm^{-h}G_hH_\pm^h\|,$$

where

$$\sigma := [\lambda_m(1+\lambda_m)]^{-1/2}.$$

We also need to evaluate

$$K^{-1}(k) = \left[I + R^{-1}S(k)\right]^{-1} R^{-1}, \tag{93}$$

hence,

$$\|K^{-1}(k)\| \le \|R^{-1}\| \, \left\| \left[I + R^{-1}S(k)\right]^{-1} \right\|, \tag{94}$$

provided that $\|R^{-1}S(k)\| < 1$, and more, we can write

$$\left[I + R^{-1}S(k)\right]^{-1} = I + C \tag{95}$$

with

$$\|C\| \le \frac{\|R^{-1}S(k)\|}{1 - \|R^{-1}S(k)\|} \le 2\,\|R^{-1}S(k)\| \le 1, \tag{96}$$

provided that $\|R^{-1}S(k)\| < 1/2$. This can be realized for k sufficiently large (see (89)). Thus,

$$\| \left[I + R^{-1}S(k)\right]^{-1} \| \le 2,, \tag{97}$$

and

$$\|K^{-1}(k)\| \le 2\,\|R^{-1}\| = \frac{1}{\lambda_m^{1/2}} \tag{98}$$

We also need to estimate

$$\begin{aligned} K(n,k) &:= Y_2(n)Y_2^{-1}(k+1) - Y_1(n)Y_1^{-1}(k+1) \\ &= H_-^n[I + E_-(n)][I + E_-(k+1)]^{-1}H_-^{-k-1} - H_+^n[I + E_+(n)] \\ &\quad \times [I + E_+(k+1)]^{-1}H_+^{-k-1} \\ &= H_-^n[I + E_-(n) + B_-(k+1) + E_-(n)B_-(k+1)]H_-^{-k-1} \\ &\quad - H_+^n[I + E_+(n) + B_+(k+1) + E_+(n)B_+(k+1)]H_+^{-k-1} \\ &= H_-^{n-k-1} - H_+^{n-k-1} + H_-^n\mathcal{E}_-(n,k)H_-^{-k-1} - H_+^n\mathcal{E}_+(n,k)H_+^{-k-1}, \end{aligned} \tag{99}$$

where we defined $B_\pm(k+1)$ as in (81) and

$$\mathcal{E}_\pm(n,k) := E_\pm(n) + B_\pm(k+1) + E_\pm(n)B_\pm(k+1). \tag{100}$$

Therefore,

$$K(n,k) = 2iD^{1/2}(I + D)^{n-k-1}R(n,k) + S(n,k), \tag{101}$$

where we set, for short,

$$S(n,k) := H_-^n\mathcal{E}_-(n,k)H_-^{-k-1} - H_+^n\mathcal{E}_+(n,k)H_+^{-k-1}, \tag{102}$$

and

$$R(n,k) := \sum_{j=0}^{k-n} H_+^j H_-^{k-n-j}. \tag{103}$$

The latter comes from the following considerations (similar to those made above). In (99), we have

$$
\begin{aligned}
H_-^{n-k-1} - H_+^{n-k-1} &= H_-^{n-k-1} H_+^{n-k-1} \left(H_+^{-n+k+1} - H_-^{-n+k+1} \right) \\
&= \left(H_- H_+ \right)^{n-k-1} (H_+ - H_-) \sum_{j=0}^{k-n} H_+^j H_-^{k-n-j} \\
&= (I + D)^{n-k-1} 2iD^{1/2} \sum_{j=0}^{k-n} H_+^j H_-^{k-n-j}, \qquad (104)
\end{aligned}
$$

being $k \geq n$. We can then evaluate

$$
\begin{aligned}
\|R(n,k)\| &\leq \sum_{j=0}^{k-n} \|H_+\|^j \|H_-\|^{k-n-j} = \sum_{j=0}^{k-n} \|H_+\|^{k-n} \\
&= (k - n + 1)(1 + \lambda_M)^{(k-n)/2}. \qquad (105)
\end{aligned}
$$

We have also

$$
\begin{aligned}
\|S(n,k)\| &\leq \|H_-^n\| \, \|H_-^{-k-1}\| \, \|\mathcal{E}_-(n,k)\| + \|H_+^n\| \, \|H_+^{-k-1}\| \, \|\mathcal{E}_+(n,k)\| \\
&\leq (1 + \lambda_M)^{n/2} (1 + \lambda_m)^{-(k+1)/2} \left(\|\mathcal{E}_-(n,k)\| + \|\mathcal{E}_+(n,k)\| \right) \\
&\leq 16 \, (1 + \lambda_M)^{n/2} (1 + \lambda_m)^{-(k+1)/2} V(n), \qquad (106)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|K(n,k)\| &\leq \|(I + D)^{-1}\|^{k-n+1} \|2iD^{1/2}\| \, \|R(n,k)\| + \|S(n,k)\| \\
&\leq (1 + \lambda_m)^{n-k-1} \, 2 \, \lambda_M^{1/2} (k - n + 1)(1 + \lambda_M)^{(k-n)/2} \\
&\quad + 16 (1 + \lambda_M)^{n/2} (1 + \lambda_m)^{-(k+1)/2} V(n). \qquad (107)
\end{aligned}
$$

Finally, we have for the particular solution $W(n)$ the estimate

$$
\begin{aligned}
\|W(n)\| &\leq \sum_{k=n}^{\infty} \|K(n,k)\| \, \|K^{-1}(k)\| \, \|F_k\| \\
&\leq \frac{1}{\sqrt{\lambda_m}} \sum_{k=n}^{\infty} \Big\{ (1 + \lambda_m)^{n-k-1} \, 2 \, \lambda_M^{1/2} (k - n + 1) \times \\
&\quad \times (1 + \lambda_M)^{(k-n)/2} + 16 (1 + \lambda_M)^{n/2} (1 + \lambda_m)^{-(k+1)/2} V(n) \Big\} \, \|F_k\|. \qquad (108)
\end{aligned}
$$

Writing this as

$$\|W(n)\| \leq \frac{1}{\sqrt{\lambda_m}} \left[2 \frac{\lambda_M^{1/2}}{1+\lambda_m} \sum_{k=n}^{\infty} (k-n+1) \left(\frac{\sqrt{1+\lambda_M}}{1+\lambda_m} \right)^{k-n} \|F_k\| \right.$$
$$\left. +16 \frac{(1+\lambda_M)^{n/2}}{\sqrt{1+\lambda_m}} V(n) \sum_{k=n}^{\infty} (1+\lambda_m)^{-k/2} \|F_k\| \right].$$

In order to control this particular solution, i.e. to impose that $W(n)$ is bounded as $n \to \infty$, it suffices to choose F_n such that

$$\sum_{k=n}^{\infty} (k-n+1)\, r^{k-n} \|F_k\| < \infty \tag{109}$$

with $r := \frac{\sqrt{1+\lambda_M}}{1+\lambda_m}$. Clearly, for any $r > 0$, the condition $k\, r^k\, \|F_k\| \in \ell^1$ suffices.

4.2. The inhomogeneous equation with $D < 0$

Following the steps of the previous subsection, we consider equation (62) with $D < 0$. Again, we find that a *particular* solution can be written as

$$W(n) := \sum_{k=n}^{\infty} K(n,k)\, K^{-1}(k)\, F_k, \tag{110}$$

where

$$K(k) := Y_1(k+2)\, Y_1^{-1}(k+1) - Y_2(k+2)\, Y_2^{-1}(k+1),$$
$$K(n,k) := Y_2(n)\, Y_2^{-1}(k+1) - Y_1(n)\, Y_1^{-1}(k+1),$$

being $Y_1(n)$ and $Y_2(n)$ the two linearly independent solutions of the associated homogeneous Equation (1).

We write

$$\begin{aligned} K(k) &:= Y_1(k+2)Y_1^{-1}(k+1) - Y_2(k+2)Y_2^{-1}(k+1) \\ &= H_+ - H_- + H_+^{k+2}\mathcal{E}_+(k)H_+^{-k-1} - H_-^{k+2}\mathcal{E}_-(k)H_-^{-k-1} \\ &= R + S(k) = R\left[I + R^{-1}S(k)\right], \end{aligned}$$

upon defining

$$R := H_+ - H_- = 2P^{1/2},$$

for short, and

$$S(k) := H_+^{k+2}\mathcal{E}_+(k)H_+^{-k-1} - H_-^{k+2}\mathcal{E}_-(k)H_-^{-k-1},$$

with

$$\mathcal{E}_{\pm}(k) := E_{\pm}(k+2) + B_{\pm}(k+1) + E_{\pm}(k+1)B_{\pm}(k+1),$$

and $B_{\pm}(k+1)$ such that $\|B_{\pm}(k+1)\| \leq 2\|E_{\pm}(k+1)\|$.

By the definition of $H_{\pm}$, it is easy to check that

$$\begin{aligned} &\|R^{-1}\| = \frac{1}{2\sqrt{-\lambda_M}}, \qquad \|H_+\| = 1+\sqrt{-\lambda_m}, \qquad \|H_+^{-1}\| = \frac{1}{1+\sqrt{-\lambda_M}}, \\ &\|H_-\| = \max_j \left\{ \left| 1 - \sqrt{-\lambda_j} \right| \right\} \leq \|H_+\|, \end{aligned} \tag{111}$$

and if p is the eigenvalue of P for which the function $|1-\sqrt{x}|$ attains its minimum value for x in the spectrum of P, we have

$$\|H_-^{-1}\| := \max_j \left\{ \left| \frac{1}{1-\sqrt{-\lambda_j}} \right| \right\} := \frac{1}{1-\sqrt{p}} \geq \|H_+^{-1}\|, \tag{112}$$

and

$$\begin{aligned} \|S(k)\| &\leq \|H_+\|^{k+2}\|\mathcal{E}_+(k)\|\|H_+^{-1}\|^{k+1} + \|H_-\|^{k+2}\|\mathcal{E}_-(k)\|\|H_-^{-1}\|^{k+1} \\ &\leq \|H_+\|^{k+2}\left[\|\mathcal{E}_+(k)\| + \|\mathcal{E}_-(k)\|\right]\|H_-^{-1}\|^{k+1} \\ &\leq \left(1+\sqrt{-\lambda_m}\right)^{k+2}\left\{\|E_+(k)\|\left[3+2\|E_+(k)\|\right]\right. \\ &\quad \left. +\|E_-(k)\|\left[3+2\|E_-(k)\|\right]\right\}\left(\frac{1}{1-\sqrt{p}}\right)^{k+1}. \end{aligned} \tag{113}$$

Applying Theorems 3.2 and 3.3, concerning the case $D<0$, we obtain

$$\begin{aligned} &\|E_+(k)\|\left[3+2\,\|E_+(k)\|\right] \\ &\quad \leq \frac{1}{1-V_\infty^+(\nu^*)}\left[V_\infty^+(\nu^*) - V_{[k/2]}^+(\nu^*) + \|\Theta_+\|^{k-[k/2]}V_{[k/2]}^+(\nu^*)\right] \\ &\quad \times \left[3 + \frac{2}{1-V_\infty^+(\nu^*)}\left[V_\infty^+(\nu^*) - V_{[k/2]}^+(\nu^*) + \|\Theta_+\|^{k-[k/2]}V_{[k/2]}^+(\nu^*)\right]\right] := \Lambda(k), \end{aligned}$$

and

$$\|E_-(k)\|\left[3+2\,\|E_-(k)\|\right] \leq 8V_-(k).$$

Using the condition (113), we obtain

$$\|S(k)\| \leq \left(1+\sqrt{-\lambda_m}\right)^{k+2}\left[\Lambda(k) + 8V_-(k)\right]\left(\frac{1}{1-\sqrt{p}}\right)^{k+1}.$$

As in the case with D positive, we have again that $\|\left[I+R^{-1}S(k)\right]^{-1}\| \leq 2$ letting $\|R^{-1}S(k)\| \leq \frac{1}{2}$, and hence $\|K^{-1}(k)\| \leq 2\,\|R^{-1}\|$.

Proceeding in a similar way for $K(n,k)$, we have

$$\begin{aligned} K(n,k) &:= Y_2(n)Y_2^{-1}(k+1) - Y_1(n)Y_1^{-1}(k+1) \\ &= H_-^{n-k-1} - H_+^{n-k-1} + H_-^n\mathcal{E}_-(n,k)H_-^{-k-1} - H_+^n\mathcal{E}_+(n,k)H_+^{-k-1} \\ &:= (I-P)^{n-k-1}\, 2P^{1/2}\, R(n,k) + S(n,k), \end{aligned}$$

being

$$R(n,k) := \sum_{j=0}^{k-n} H_+^j H_-^{k-n-j},$$

and

$$S(n,k) := H_-^n\mathcal{E}_-(n,k)H_-^{-k-1} - H_+^n\mathcal{E}_+(n,k)H_+^{-k-1},$$

where we set

$$\mathcal{E}_\pm(n,k) := E_\pm(n) + B_\pm(k+1) + E_\pm(n)B_\pm(k+1),$$

with

$$\|B_\pm(k+1)\| \le 2\,\|E_\pm(k+1)\| \le 2\|E_\pm(n)\|.$$

Since

$$\|(I-P)^{-1}\| := \max_j \left|\frac{1}{1+\lambda_j}\right| := \frac{1}{1-q},$$

where q is defined as the value maximizing the function $(1-x)^{-1}$ for x in the spectrum of P, we obtain

$$\begin{aligned} \|P^{1/2}\| &= \max_j \left\{\sqrt{-\lambda_j}\right\} = \sqrt{-\lambda_m}, \\ \|R(n,k)\| &\le \sum_{j=0}^{k-n} \|H_+\|^j \|H_-\|^{k-n-j} \le (k-n+1)(1+\sqrt{-\lambda_m})^{k-n}, \\ \|\mathcal{E}_\pm(n,k)\| &\le \|E_\pm(n)\|\left[3 + 2\,\|E_\pm(n)\|\right], \\ \|S(n,k)\| &\le \|H_+\|^n[\|\mathcal{E}_-(n,k)\| + \|\mathcal{E}_+(n,k)\|]\|H_-^{-1}\|^{k+1} \\ &\le \left(1+\sqrt{-\lambda_m}\right)^n \left(\|E_-(n)\|\,\left[3+2\|E_-(n)\|\right] + \|E_+(n)\|\,\left[3+2\|E_+(n)\|\right]\right) \\ &\quad \times \left(\frac{1}{1-\sqrt{p}}\right)^{k+1} \le \left(1+\sqrt{-\lambda_m}\right)^n \left[8V_-(n) + \Lambda(n)\right]\left(\frac{1}{1-\sqrt{p}}\right)^{k+1}, \end{aligned}$$

then

$$\|K(n,k)\| \leq 2\sqrt{-\lambda_m}\left(\frac{1}{1-q}\right)^{k-n+1}(k-n+1)(1+\sqrt{-\lambda_m})^{k-n} + \left(1+\sqrt{-\lambda_m}\right)^n\left[8\,V_-(n)+\Lambda(n)\right]\left(\frac{1}{1-\sqrt{p}}\right)^{k+1}. \tag{114}$$

Using the second theorem concerning the case $D<0$, we finally obtain an estimate for the particular solution (110) of the inhomogeneous Equation (63),

$$\begin{aligned}\|W(n)\| &\leq \sum_{k=n}^{\infty}\|K(n,k)\|\,\|K^{-1}(k)\|\,\|F_k\| \\ &\leq \frac{1}{\sqrt{-\lambda_M}}\left[\frac{2\sqrt{-\lambda_m}}{1-q}\sum_{k=n}^{\infty}(k-n+1)\left(\frac{(1+\sqrt{-\lambda_m})}{1-q}\right)^{k-n}\|F_k\|\right. \\ &\quad\left.+\left(1+\sqrt{-\lambda_m}\right)^n\left[8V_n(n)+\Lambda(n)\right]\frac{1}{1-\sqrt{p}}\sum_{k=n}^{\infty}\frac{1}{(1-\sqrt{p})^k}\|F_k\|\right].\end{aligned} \tag{115}$$

Again, we can derive from this some criterion by which the solution $W(n)$ will be bounded as $n\to\infty$.

5. Examples and applications

In this section, we present some examples, for the purpose of illustration. We will also show some applications relevant to the subject.

Example 5.1: Let us consider Equation (1) with

$$G_n := \begin{pmatrix} 0 & g_{12}(n) \\ g_{21}(n) & 0 \end{pmatrix},$$

and

$$D_n := \begin{pmatrix} \frac{1}{n^\alpha}+2 & 0 \\ 0 & \frac{1}{n^\alpha}+3 \end{pmatrix},$$

$\alpha>0$. Then, according to Theorem 3.1, a sufficient condition which ensures existence of both solutions, (12) and the similar one obtained from (12) by replacing the sign $+$ with $-$, is

$$\sum_{n=\nu}^{\infty}\left\|D-D_n-H_\pm^{-n}G_nH_\pm^n\right\| < \infty. \tag{116}$$

We are now interested in finding conditions on α and G_n such that Theorem 3.1 can be applied. We obtain, after a little algebra,

$$D-D_n-H_\pm^{-n}G_nH_\pm^n = \begin{pmatrix} -\frac{1}{n^\alpha} & \left(\frac{1\pm i\sqrt{3}}{1\pm i\sqrt{2}}\right)^n g_{12}(n) \\ \left(\frac{1\pm i\sqrt{2}}{1\pm i\sqrt{3}}\right)^n g_{21}(n) & -\frac{1}{n^\alpha} \end{pmatrix}, \tag{117}$$

and the series in (116) converges if and only if the series in all entries of such matrix converge absolutely. This happens if, e.g. $\alpha > 1$, $g_{12}(n) = \mathcal{O}\left(2^{-n}\right)$, and $g_{21}(n) \in \ell^1$.

Example 5.2: Let us assume that the key condition in (11) is satisfied, while

$$\sum_{n=\nu}^{\infty} \|D - D_n\| = \sum_{n=\nu}^{\infty} \|H_+^{-n} G_n H_+^n\| = +\infty \tag{118}$$

This is the case, e.g. if

$$D_n := \begin{pmatrix} \frac{1}{n^2} - \frac{1}{n} + a & 0 \\ 0 & \frac{1}{n^2} - \frac{1}{n} + b \end{pmatrix}$$

and

$$G_n := \begin{pmatrix} \frac{1}{n} & \frac{1}{n} \\ \mathcal{O}((1+\sqrt{a})^{-n}) & \frac{1}{n} \end{pmatrix},$$

with $a > b > 0$. Now, (118) holds since

$$D - D_n = \begin{pmatrix} \frac{1}{n} - \frac{1}{n^2} & 0 \\ 0 & \frac{1}{n} - \frac{1}{n^2} \end{pmatrix}$$

and

$$H_+^{-n} G_n H_+^n := \widehat{G}_n^+ = \begin{pmatrix} \frac{1}{n} & \frac{1}{n}\left(\frac{1+i\sqrt{b}}{1+i\sqrt{a}}\right)^n \\ \left(\frac{1+i\sqrt{a}}{1+i\sqrt{b}}\right)^n \mathcal{O}((\sqrt{1+a})^{-n}) & \frac{1}{n} \end{pmatrix}.$$

However, Theorem 3.1 holds true since

$$D - D_n - \widehat{G}_n^+ = \begin{pmatrix} -\frac{1}{n^2} & -\frac{1}{n}\left(\frac{1+i\sqrt{b}}{1+i\sqrt{a}}\right)^n \\ -\left(\frac{1+i\sqrt{a}}{1+i\sqrt{b}}\right)^n \mathcal{O}((\sqrt{1+a})^{-n}) & -\frac{1}{n^2} \end{pmatrix}$$

satisfies (11).

Example 5.3: To exhibit a case in which Theorem 3.2 can be applied despite Theorem 3.3 does not hold, it suffices to consider two matrices, D_n and G_n, satisfying the assumption (35) while (44) is not met.

Consider again the matrices in $\mathbb{M}_2(\mathbf{C})$,

$$P_n := \begin{pmatrix} p_{11}(n) + p_1 & 0 \\ 0 & p_{22}(n) + p_2 \end{pmatrix}, \quad G_n := \begin{pmatrix} g_{11}(n) & g_{12}(n) \\ g_{21}(n) & g_{22}(n) \end{pmatrix},$$

where $0 < p_2 < p_1 < 1$ are arbitrary constants, and for $i = 1, 2$, $p_{ii}(n) \in \ell^1$, and $p_{ii}(n) \to 0$ as $n \to \infty$. Under these assumptions, we find by a little algebra that $P := \lim_{n\to\infty} P_n =$

$\mathrm{diag}(p_1, p_2)$, and then

$$P_n - P - \widehat{G}_n^{\pm} = \begin{pmatrix} p_{11}(n) - g_{11}(n) & -\left(\frac{1\pm\sqrt{p_2}}{1\pm\sqrt{p_1}}\right)^n g_{12}(n) \\ \left(\frac{1\pm\sqrt{p_1}}{1\pm\sqrt{p_2}}\right)^n g_{21}(n) & p_{22}(n) - g_{22}(n) \end{pmatrix}.$$

Therefore, if $g_{11}(n), g_{22}(n) \in \ell^1$, $g_{21}(n) = \frac{1}{n}$, and $g_{12}(n) = \mathcal{O}((1 - \sqrt{p_2})^{-n})$, it is easy to check that $\sum_{n=\nu}^{\infty} \|P_n - P - \widehat{G}_n^{+}\|$ does not converge, being $\sum_{n=\nu}^{\infty} \left(\frac{1+\sqrt{p_1}}{1+\sqrt{p_2}}\right)^n \frac{1}{n} = +\infty$, while $\sum_{n=\nu}^{\infty} \|P_n - P - \widehat{G}_n^{-}\| < \infty$.

Example 5.4: A peculiarity of the Liouville–Green approximation is its 'double asymptotic nature' [6], since it provides error estimates in both cases, as the independent variable goes to some limit (e.g. to infinity), and as some parameter does the same. Consider for instance the equation

$$\Delta^2 Y_n + (u\, D_n + G_n) Y_n = F_n, \quad n \in \mathbf{Z}_\nu, \;\; u > 0 \tag{119}$$

under the same assumptions of Theorem 3.1, and to be interested in the limiting behavior of its solutions as $u \to +\infty$. Since D becomes $u\,D$, λ_j becomes $u\,\lambda_j$, and $H_\pm$ becomes $I \pm iu^{1/2}D^{1/2}$, it is straightforward to see that we will have two linearly independent solutions

$$Y_n^{\pm}(u) = \left(I \pm iu^{1/2}D^{1/2}\right)^n \left(I + E_n^{\pm}\right), \tag{120}$$

with the error estimates

$$\|E_n^{\pm}(u)\| \le \frac{V_n^{\pm}(u)}{1 - V_n^{\pm}(u)}, \tag{121}$$

being

$$V_n^{\pm}(u) = \sigma(u) \sum_{k=n}^{\infty} \|u\,(D_k - D) + H_{\pm}^{-k} G_k H_{\pm}^{k}\|, \tag{122}$$

with

$$\sigma(u) := u^{-1/2}[\lambda_m\,(1 + u\,\lambda_m)]^{-1/2}. \tag{123}$$

It follows that

$$\|E_n^{\pm}(u)\| = \mathcal{O}(u^{-2}) \;\text{ as } u \to +\infty. \tag{124}$$

Similar results can be obtained when $D < 0$.

Example 5.5 (Perturbation of nonlinear equations):
Consider the *nonlinear* problem

$$\Delta^2 Y_n^{\varepsilon} + Q_n Y_n^{\varepsilon} + \varepsilon F(Y_n^{\varepsilon}) = 0 \quad n \in \mathbf{Z}_\nu, \tag{125}$$

subject to the 'initial data', $Y_\nu^{\varepsilon} = A$, $Y_{\nu+1}^{\varepsilon} = B$. Assume that $Q_n = D_n + G_n$, where D_n and G_n obey the properties required by Theorems 3.1–3.3. A formal expansion in powers of ε yields $Y_n^{\varepsilon} = Y_n^0 + \varepsilon Y_n^1 + \varepsilon^2 Y_n^2 + \cdots$, and then, to the lowest order,

$$\Delta^2 Y_n^0 + Q_n Y_n^0 = 0, \quad Y_\nu^0 = A, \; Y_{\nu+1}^0 = B, \tag{126}$$

$$\Delta^2 Y_n^1 + Q_n Y_n^1 = -F(Y_n^0), \quad Y_\nu^1 = Y_{\nu+1}^1 = 0. \tag{127}$$

Equation (126) coincides with the homogeneous equation studied previously in Section 3, hence its solution can be given along with estimates for the error terms. Equation (127), which provides the first correction to Y_n^ε, has the form of the *inhomogeneous* equation considered in Section 4 and therefore we are able to treat it too as made above.

Example 5.6: Other problems where *inhomogeneous* difference equations or systems may appear are certain linear *partial differential-difference equations*. Consider the equation

$$L_n \frac{\partial^2 U_n}{\partial t^2} + M_n \frac{\partial U_n}{\partial t} = A_n U_{n+2} + B_n U_{n+1} + C_n U_n - F_n(t), \tag{128}$$

for the unknown matrix-valued function $U_n(t)$, all coefficients being independent of t.

Under suitable assumptions, Laplace transforming such equation with respect to t, denoting with $V_n(s)$ the Laplace transform of $U_n(t)$, with $\Phi_n(s)$ that of $F_n(t)$, and writing simply $U_n' = \frac{\partial U_n}{\partial t}$, we obtain the equation

$$\begin{aligned} L_n \left(s^2 V_n(s) - sU_n(0) - U_n'(0)\right) + M_n \left(sV_n(s) - U_n(0)\right) \\ = A_n V_{n+2}(s) + B_n V_{n+1}(s) + C_n V_n(s) - \Phi_n(s), \end{aligned}$$

that is the inhomogeneous equation for V_n (parametrized by the Laplace variable s)

$$A_n V_{n+2}(s) + B_n V_{n+1}(s) + \widetilde{C}_n V_n(s) = \Psi_n(s), \tag{129}$$

where we set

$$\widetilde{C}_n := C_n - s^2 L_n - sM_n, \tag{130}$$

and

$$\Psi_n(s) := \Phi(s) - sL_n U_n(0) - L_n U_n'(0) - M_n U_n(0). \tag{131}$$

6. Summary

A Liouville–Green (or WKB) asymptotic theory has been developed for linear second-order *matrix* difference equations that is asymptotically diagonal. This means that such systems can be considered as a perturbation of decoupled systems. Precise and computable bounds have been derived for the error terms, appearing in the asymptotic representations. These may be useful for the numerical evaluation of solutions that have been obtained, as in the analogous case of second-order matrix differential equations. These results extend the corresponding ones established for second-order *scalar* difference equations. Needless to say, the theory can be applied to systems of linear scalar difference equations, each of the second order, i.e. to *vector* equations. In addition, we worked out the corresponding *inhomogeneous* equations, establishing, in particular, conditions under which a forcing term can control the system. The subject of second order linear *matrix* difference equations is also related to the spectral theory, more precisely to direct and inverse scattering theory, for discrete Sturm–Liouville equations; see, e.g. [2]. Here, the Marchenko condition in [2, (1.10)] recalls conditions like those in (11) or (35) of this paper. However, the relation among the coefficients in the two papers is unclear, and, in any case, Marchenko condition looks more similar to the 'finite moments' assumptions made in [12] for the scalar case and in [13] in an abstract framework.

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