Commun Nonlinear Sci Numer Simulat 16 (2011) 3855–3862

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Short communication

Root locus of fractional linear systems

J.A. Tenreiro Machado

Institute of Engineering of Porto, Dept. of Electrical Engineering, Rua Dr. António Bernardino de Almeida, 431, 4200-072 Porto, Portugal

ARTICLE INFO

Article history:
Received 20 September 2010
Received in revised form 24 January 2011
Accepted 25 January 2011
Available online 1 February 2011

Keywords:
Root locus
Fractional calculus

ABSTRACT

In this paper an algorithm for the calculation of the root locus of fractional linear systems is presented. The proposed algorithm takes advantage of present day computational resources and processes directly the characteristic equation, avoiding the limitations revealed by standard methods. The results demonstrate the good performance for different types of expressions.

1. Introduction

Fractional calculus (FC) generalizes integrodifferential operation to a non-integer order. During the last decades FC verified a strong development which is demonstrated by the large volume of research published. The mathematical formulation of the FC theory can be addressed in Refs. [1–5] and applications, such as viscoelasticity, biology, signal processing, diffusion, modeling and control, can be found in [6–10]. In spite of its popularity, FC requires some efforts towards the development and adoption of standard tools often adopted in system analysis and control. In fact, the modeling of fractional systems and the calculation of fractional derivatives leads to non-rational functions, often approximated through series or rational fractions [11–14]. This paper addresses the development of an algorithm for obtaining the Root Locus (RL) of fractional order expressions of any type, easy to implement using today's computational resources.

In this line of thought the manuscript is organized as follows. Section 2 introduces the algorithm for the calculation of the root locus of fractional order expressions and presents several examples. Section 3 outlines the main conclusions.

2. Root locus of fractional systems

The RL is a classical tool for the stability analysis of integer order linear systems [15–19], but its application in the fractional counterpart poses some difficulties. Therefore, researchers have mainly preferred to adopt frequency based methods, but we can mention several studies addressing the RL in the scope of fractional systems [20–23]. More recently the RL was considered for the stability analysis of fractional systems by tacking advantage of commensurable expressions that occur when truncating real valued integrodifferential orders up to a finite precision [24]. This strategy allows the use of built in routines available in several engineering packages, with a good level of integration of commands and editor, but, on the other

E-mail address: jtm@isep.ipp.pt

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2011.01.020

hand, limits the precision and the type of symbolic expressions. For example, several fractional systems are modeled by transcendental expressions and the adoption of polynomials of non-integer order is not the most appropriate way to construct the corresponding RL.

Bearing these ideas in mind, it was decided to implement a numerical tool for construction the RL without the previous limitations. The algorithm consists in searching the complex plane s for possible solutions of the closed-loop characteristic equation. In general, in the s-plane it is adopted a two stage calculation scheme, with a large spaced grid of points for a preliminary evaluation of candidate solutions and, for each solution, a more precise calculation by inserting a local small spaced grid of points. Due to the importance of solutions in the real and imaginary axis it was decided to search directly those cases. Moreover, after analyzing the RL of several experiments it was verified to be the best option to consider directly a small-spaced grid and a two stage grid for root evaluation in the real and the imaginary axis, respectively. Therefore, the structure of the algorithm is designed as follows:

1. Definition of the system characteristic equation.
2. Definition of parameters:
   2.1. The limits of the search in the complex plane.
   2.2. The number of points with the large spaced grid for testing possible solutions.
   2.3. The accuracy threshold for entering the high precision root evaluation.
   2.4. The number of points in the small spaced grid for calculating more precise solutions.
3. Calculation of solutions in the imaginary axis with two phases.
   3.1. Test solutions with large grid.
   3.2. If feasible solution found, then larger precision evaluation by searching within a local small spaced grid.
4. Calculation of solutions in the real axis, directly with small spaced grid.
5. Calculation of solutions in the complex plane with two phases.
   5.1. Test solutions with large spaced grid.
   5.2. If feasible solution found, then larger precision evaluation by searching within a local small spaced grid.

For the implementation of the algorithm were tested both compiler and interpreted computational packages. It was verified that interpreted codes were much slower and, therefore, in the sequel the experiments are based on code implemented with the Lazarus compiler [25]. However, given that many users use nowadays Matlab [26] in Appendix A is listed the main code and the definition of one fractional system adopted in the following examples.

For the purpose of testing the algorithm are considered several fractional characteristic equations $Q(s)$ proposed in the literature, namely:

– Example 1, by F. Merrikh-Bayat et al. [24] $Q(s) = s^2 - 3s^{1.5} - 2s + 2s^{0.5} + 12 + k(s^{0.5} - 1)$
– Example 2, by I. Podlubny et al. [4] $Q(s) = 0.7943s^{2.5708} + 5.2385s^{0.8372} + 1.5560 + k$
– Example 3, by I. Podlubny [27] $Q(s) = 14994s^{1.131} + 6009.5s^{0.97} + 1.69 + k$
– Example 4, by I. Jesus et al. [28] $Q(s) = 1 + ke^{-3.0\sqrt{\frac{s}{0.042}}}$
– Example 5, by F. Merrikh-Bayat et al. [29] $Q(s) = s + k(s^{0.5} + 1)e^{-\sqrt{s}}$
– Example 6, by M. Busłowicz [30] $Q(s) = s^{1.5} - 1.5s - 1.5s\ e^{-sk} + 4s^{0.5} + 8$

where s represents the Laplace transform variable.

Fig. 1 depicts the corresponding root locus for $k \geqslant 0$ obtained through the proposed computation scheme. The results are consistent with those reported based on the adaptation of the standard RL, when available, or with stability conditions based on other methods, for those cases where the RL was not obtained. It was also confirmed that the quality, precision and speed can be easily tuned through the parameters defining the grid points and the precision threshold. From the operational point of view we find that we loose the automatisms provided by common engineering packages, since we have to insert the code for each characteristic equation but, on the other hand, we gain computational speed and freedom of analyzing a larger set of fractional-order expressions.

In conclusion the proposed algorithm provides a simple platform for the stability analysis and control design of closed loop linear fractional systems.

3. Conclusions

The advances in FC demonstrate its superiority for an accurate system modeling and point towards important developments in applications, namely in control design. During the last years some algorithms for adapting classical RL packages were proposed. Nevertheless, the case of complex expressions was not yet tackled. In this paper a new algorithm for the calculation of the RL of fractional expressions was introduced. The results demonstrate the good performance for different types of systems.

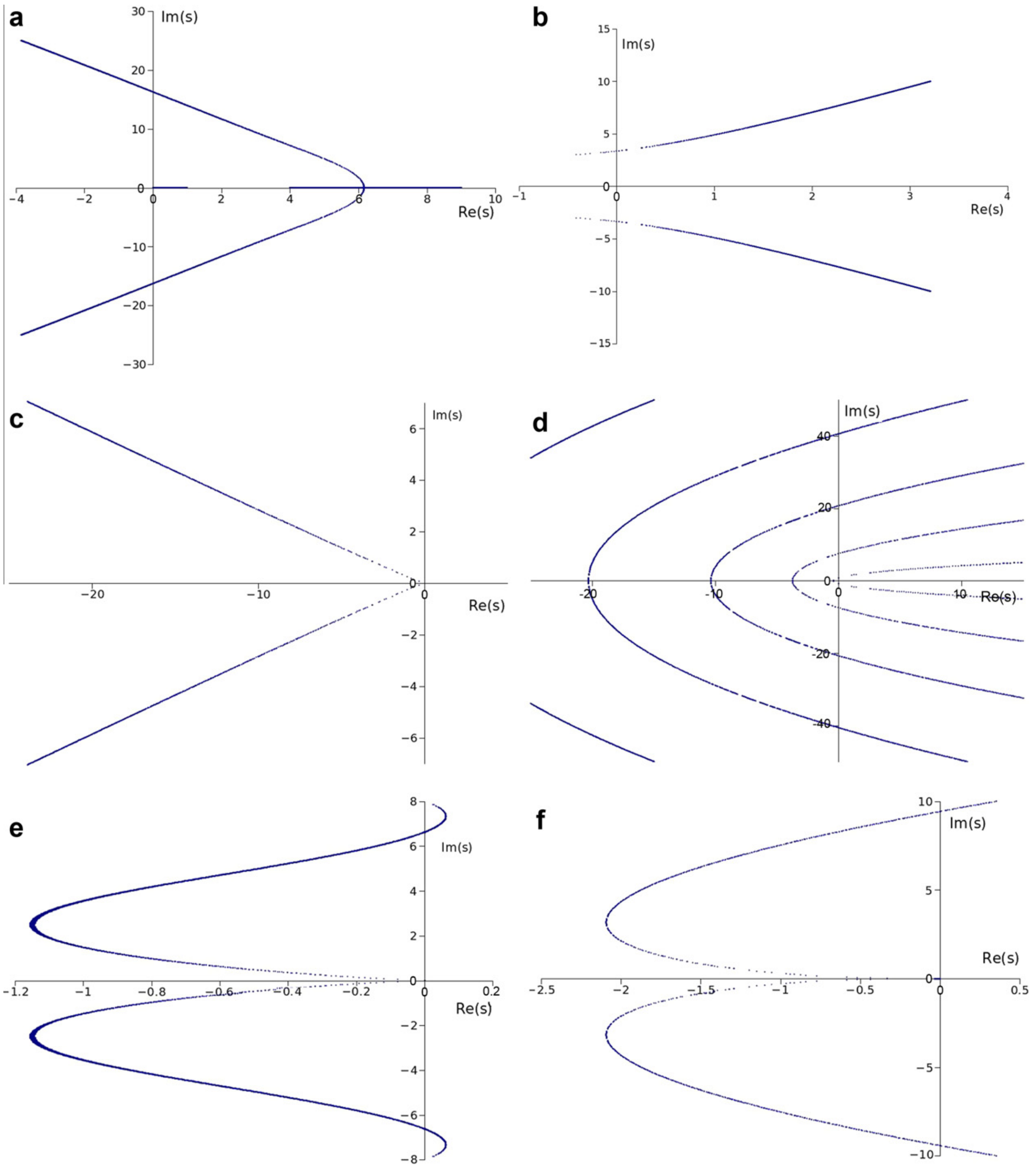

Fig. 1. Root locus for $k \geqslant 0$ of the fractional-order characteristic equations: (a) $Q(s) = s^2 - 3s^{1.5} - 2s + 2s^{0.5} + 12 + k(s^{0.5} - 1)$, (b) $Q(s) = 0.7943s^{2.5708} + 5.2385s^{0.8372} + 1.5560 + k$, (c) $Q(s) = 14994s^{1.131} + 6009.5s^{0.97} + 1.69 + k$, (d) $Q(s) = 1 + ke^{-3.0\sqrt{\frac{s}{0.042}}}$, (e) $Q(s) = s + k(s^{0.5} + 1)e^{-\sqrt{s}}$, (f) $Q(s) = s^{1.5} - 1.5s - 1.5s\,e^{-sk} + 4s^{0.5} + 8$.

Appendix A. Matlab code

A.1. The main program

```
% This program plots the root locus of a fractional order system
%
% The the width of the plot is defined by the variables: xmin, xmax, ymax
% The number of large cells for the numerical evaluation is variable: n
% The number of small cells within each large cell: ns
%
% The system is defined in syst.m
%
% 2011 J. Tenreiro Machado

close all
clear all

xmin=-10; % left limit
xmax=10; % right limit
ymax=20; % upper and lower limits
n=300; % large grid
ns=20; % small grid
ns1=ns+1;
ns2=ns+ns1;

dx=(xmax-xmin)/n;
dy=0.5*ymax/n;
dx2=dx/2;
dxs=dx/ns;
dy2=dy/2;
dys=dy/ns;
asarg_lim1=0.01;
asarg_lim2=asarg_lim1*0.1;

ip=1;
x=xmin;
while x<=xmax
  y=dy;
  while y<=ymax
    [mod_k,arg_k] = syst (x,y);
    asarg_k=abs (arg_k);

    if asarg_k<asarg_lim1
      asarg_min=asarg_k;
      arg_k_min=arg_k;
      asarg_k_min=asarg_k;

      % small grid
      xsmin=x;
      ysmin=y;
      for i1=1:ns2
        for j1=1:ns2
          k1(i1,j1)=0;
        end;
      end;
      i1min=ns1;
      j1min=ns1;
      k1(i1min,j1min)=1;
      flagk=0;
      while (xsmin<=x-dx2)|(xsmin>=x+dx2)|(ysmin<=y-dy2)|(ysmin>=y+dy2)|(flagk==0)
        for i1=1:3
```

```
        xs (i1)=xsmin+(i1-2)*dxs;
      end;
      for j1=1:3
        ys (j1)=ysmin+(j1-2)*dys;
      end;
      for i1=1:3
        for j1=1:3
          flagk=0;
          i10=i1min+i1-2;
          j10=j1min+j1-2;
          if (i10<=ns2)& (j10<=ns2)& (i10>0)& (j10>0)
            if k1(i10,j10)==0
              [mod_k,arg_k] =syst (xs (i1),ys (j1));
              i1min=i10;
              j1min=j10;
              k1(i10,j10)=1;
              flagk=1;
              asarg_k=abs (arg_k);
              if asarg_k<asarg_min
                asarg_min=asarg_k;
                arg_k_min=arg_k;
                k_min=mod_k;
                xsmin=xs (i1);
                ysmin=ys (j1);
              end;
            end;
          end;
        end;
      end;
    end;
    % small grid
    if abs (arg_k_min)<asarg_lim2
      ip=ip+1;
      px (ip)=xsmin;
      py (ip)=ysmin;
      ip=ip+1;
      px (ip)=xsmin;
      py (ip)=-ysmin;
    end;
  end;
  y=y+dy;
 end;
 x=x+dx;
end;
% Real axis
y=0.0;
x=xmin;
while x<=xmax
 [mod_k,arg_k] = syst (x,y);
 asarg_k=abs (arg_k);
 if asarg_k<asarg_lim2
  asarg_min=asarg_k;
  arg_k_min=arg_k;
  asarg_k_min=asarg_k;
```

```
      xsmin=x;
      ysmin=0;
      ip=ip+1;
      px (ip)=xsmin;
      py (ip)=ysmin;
    end;
    x=x+dx;
  end;
  if ip>1
    Fig. 1
    i=1;
    while i<=ip
      scatter (px (i),py (i),3,[1 0 0],'filled');
      i=i+1;
      hold on
    end;
    xlabel ('Re\{s\}')
    ylabel ('Im\{s\}')
    grid on
    hold off
  end
```

One fractional system definition:

```
  function [mod_k,arg_k] = syst (x,y)
   % System 1
   % numerator definition
   a (1)=-1; af (1)=0;
   a (2)=1; af (2)=0.5;
   imax=2;
   % denominator definition
   b (1)=12; bf (1)=0;
   b (2)=2; bf (2)=0.5;
   b (3)=-2; bf (3)=1;
   b (4)=-3; bf (4)=1.5;
   b (5)=1; bf (5)=2;
   jmax=5;
   % total
   logz=0.5*log (x*x+y*y);
   argz=atan2(y,x);
   % numerator calculation
   re (1)=0;
   im (1)=0;
  for i=1:imax
    if af (i)==0
      r=1;
      t=0;
    end;
    if af (i)==1
      r=exp (logz);
      t=argz;
    end;
    if (af (i)~=0)& (af (i)~=1)
      r=exp (af (i)*logz);
      t=af (i)*argz;
    end;
```

```
    ar=a (i)*r;
    re (1)=re (1)+ar*cos (t);
    im (1)=im (1)+ar*sin (t);
  end;

  m1=re (1)*re (1)+im (1)*im (1);
  % denominator calculation
   re (2)=0;
   im (2)=0;
  for j=1:jmax
    if bf (j)==0
      r=1;
      t=0;
    end;
    if bf (j)==1
      r=exp (logz);
      t=argz;
    end;
    if (bf (j)~=0)& (bf (j)~=1)
      r=exp (bf (j)*logz);
      t=bf (j)*argz;
    end;

    br=b (j)*r;
    re (2)=re (2)+br*cos (t);
    im (2)=im (2)+br*sin (t);
  end;

  m2=re (2)*re (2)+im (2)*im (2);

  %output
  mod_k=sqrt (m2/m1);
  arg_k=atan2(re (1)*im (2)-im (1)*re (2),re (1)*re (2)+im (1)*im (2))+pi;

  na=round (arg_k/6.283185307);
  arg_k=arg_k-6.283185307*na;
```

References

[1] Oldham KB, Spanier J. The fractional calculus: theory and application of differentiation and integration to arbitrary order. Academic Press; 1974.
[2] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: theory and applications. Gordon and Breach Science Publishers; 1993.
[3] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. John Wiley and Sons; 1993.
[4] Podlubny I. Fractional differential equations. In: An introduction to fractional derivatives. Fractional differential equations, to methods of their solution (mathematics in science and engineering), vol. 198. Academic Press; 1998.
[5] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. North-Holland Mathematics Studies, Elsevier; 2006. vol. 204.
[6] Bagley RL, Torvik PJ. Fractional calculus – a different approach to the analysis of viscoelastically damped structures. AIAA J 1983;21:741–8.
[7] Alain Oustaloup, La commande CRONE: commande robuste d'ordre non entier, Hermes, 1991.
[8] Podlubny Igor. Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. IEEE Trans Autom Control 1999;44(1):208–13.
[9] Hilfer Rudolf. Applications of fractional calculus in physics. World Scientific Publishing Company; 2000.
[10] Sabatier J, Agrawal OP, Tenreiro Machado JA, editors. Advances in fractional calculus: theoretical developments and applications in physics and engineering. Springer; 2007.
[11] Al-Alaoui Mohamad Adnan. Novel digital integrator and differentiator. Electron Lett 1993;29(4):376–8.
[12] Tseng Chien-Cheng. Design of fractional order digital fir differentiators. IEEE Signal Process Lett 2001;8(3):77–9.
[13] Chen Yang Quan, Moore KL. Discretization schemes for fractional-order differentiators and integrators. IEEE Trans Circuits Syst-I: Fundam Theory Appl 2002;49(3):363–7.
[14] Tenreiro Machado JA, Galhano AM, Oliveira AM, Tar JK. Approximating fractional derivatives through the generalized mean. Commun Nonlinear Sci Numer Simul 2009;14(11):3723–30.
[15] Evans WR. Graphical Analysis of Control Systems. Trans AIEE 1948;67:547–51.
[16] Evans WR. Contol systems synthesis by root locus method. Trans AIEE 1950;69:66–9.
[17] Krall AM. The root locus method: a survey. SIAM Rev 1970;12(1):64–72.
[18] Eydgahi AM, Ghavamzadeh M. Complementary root locus revisited. IEEE Trans Educ 2001;44(2):137–43.
[19] Eydgahi AM, Ghavamzadeh M. Complementary root locus revisited. IEEE Trans Educ 2001;44(2):137–43.
[20] Matignon Denis. Stability properties for generalized fractional differential systems. In: Systemes differentiels fractionnaires (Paris, 1998). ESAIM proceedings, vol. 5. Paris, France: SMAI; 1998. p. 145–58.
[21] Matignon Denis. Stability results on fractional differential equations with applications to control processing. In: Proceedings of the Computational Engineering in Systems Applications, CESA96 IMACSIEEE/SMC Multiconference; 1996, p. 963–968, Lille, France, July.
[22] Lorenzo CF, Hartley TT. Initialization, conceptualization, and application in the generalized (Fractional) calculus. NASA TP-1998-208415 1998.
[23] Petras Ivo. Stability of fractional-order systems with rational orders: a survey. Fract Calc Appl Anal 2009;12(3):269–98.

[24] Merrikh-Bayat Farshad, Afshar Mahdi. Extending the root-locus method to fractional-order systems. J Appl Math Hindawi 2008:13. doi:10.1155/2008/528934. Article ID 528934.
[25] <http://www.lazarus.freepascal.org/>.
[26] <http://www.mathworks.com/>.
[27] Podlubny I, Dorcak L, Kostial I. On fractional derivatives, fractional order dynamic system and $PI^{\lambda}D^{\mu}$-controllers. In: Proceedings of the 36th IEEE Conference on Decision and Control. San Diego, California, USA, 1997; vol. 5: p. 4985–4990.
[28] Jesus Isabel S, Tenreiro Machado JA. Fractional control of heat diffusion systems. Nonlinear Dyn 2008;54(3):263–82.
[29] Merrikh-Bayat Farshad, Karimi-Ghartemani Masoud. An efficient numerical algorithm for stability testing of fractional-delay systems. ISA Trans 2009;48(1):32–7.
[30] Busłowicz M. Stability of linear continuous-time fractional order systems with delays of the retarded type. Bull Pol Acad Sci Tech Sci 2008;56(4):319–24.