



ELSEVIER

Linear Algebra and its Applications 299 (1999) 119–151

LINEAR ALGEBRA
AND ITS
APPLICATIONS

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Feedback design for regularizing descriptor systems

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Received 7 May 1998; accepted 2 August 1999

Submitted by P.A. Fuhrmann

Abstract

This paper surveys numerical techniques for the regularization of descriptor (generalized state-space) systems by proportional and derivative feedback. We review generalizations of controllability and observability to descriptor systems along with definitions of regularity and index in terms of the Weierstraß canonical form. Three condensed forms display the controllability and observability properties of a descriptor system. The condensed forms are obtained through orthogonal equivalence transformations and rank decisions, so they may be computed by numerically stable algorithms. In addition, the condensed forms display whether a descriptor system is regularizable, i.e., when the system pencil can be made to be regular

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¹ Partial support received from British Council/DAAD ARC (Project 597) under contract AZ 313-ARC-VIII-95/50.

² Partial support received from Deutsche Forschungsgemeinschaft under contract Me 790/7-2 Singuläre Steuerungsprobleme.

³ We thank Gene Golub for giving us the opportunity to work on this paper jointly by inviting us to Stanford.

⁴ Partially supported by National Science Foundation grants INT-8922444, DMS-9205538, CCR-9404425, DMS-9628626, CCR-9732671. Some support was also received from University of Kansas General Research Allocation 3514-20-0038 and from the NSF EPSCoR/K*STAR program through the Center for Advanced Scientific Computing.

by derivative and/or proportional output feedback, and, if so, what index can be achieved. Also included is a new characterization of descriptor systems that can be made to be regular with index 1 by proportional and derivative output feedback. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

Dynamic system representation gives rise to linear time-invariant *descriptor* (or *generalized state-space*) models of the form

$$E\dot{x} = Ax + Bu, \quad (1)$$

$$y = Cx, \quad (2)$$

where $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $\dot{x} = dx/dt$. For ease of notation, a descriptor system of the form (1), (2) is denoted here by (E, A, B, C) . Descriptor systems arise naturally in a variety of practical circumstances [15,29,39,47,52] and have recently been investigated in [12,14,15,18,30,31,36,43,44,46,48,49,53,54,56,58,59,61,62]. We consider only square systems, since they arise naturally from realizations [15] and also since the non-square case can be reduced to the square case [9]. With a little more technical effort most of our results could also be reformulated for the rectangular case. In contrast to *standard* systems, where $E = I$, the response of a descriptor system can have a complicated structure and can even have impulsive modes [21,58].

In this linear, time invariant context, we are interested in *proportional and derivative output feedback* control of the form $u = Fy + G\dot{y} + v = FCx + FC\dot{x} + v$ where $F, G \in \mathbb{R}^{m \times p}$ are chosen to give a closed loop system

$$(E + BGC)\dot{x} = (A + BFC)x + Bv \quad (3)$$

with desired properties, some of which are discussed below. *Proportional output feedback control* is the special case $G = 0$. *Derivative output feedback control* is the special case $F = 0$. *Direct state feedback controls* correspond to the special case $C = I$.

The response of a descriptor system can be described in terms of the eigenstructure of the matrix pencil $\alpha E - \beta A$. The pencil and the corresponding system (1) and (2) are said to be *regular* if $\det(\alpha E - \beta A) \neq 0$ for some $(\alpha, \beta) \in \mathbb{C}^2$. Regular systems are *solvability* in the sense that (1) admits a classical smooth solution $x(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ for all sufficiently smooth controls $u(t)$ and consistent initial conditions $x(t_0)$ [10,15,61].

For regular pencils, *generalized eigenvalues* are the pairs $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ for which $\det(\alpha E - \beta A) = 0$. If $\beta \neq 0$, then the pair represents the finite eigenvalue $\lambda = \alpha/\beta$. If $\beta = 0$, then (α, β) represents an “infinite” eigenvalue. A finite eigenvalue $\lambda = \alpha/\beta$ is a pole of the transfer function of the descriptor system (1) and (2),

so the generalized eigenvalues of $\alpha E - \beta A$ are sometimes called the *poles* of the system.

In the following we frequently need matrix representations of nullspaces of matrices. To simplify the notation, we denote a matrix with orthonormal columns spanning the right nullspace of the matrix M by $S_\infty(M)$ and a matrix with orthonormal columns spanning the left nullspace of M by $T_\infty(M)$. Note that these matrices are not uniquely determined although the corresponding spaces are. Nevertheless, for ease of discussion, we also speak of these matrices as the corresponding spaces.

For regular pencils the solution of the system equations can be characterized in terms of the Weierstraß canonical form (WCF). (The WCF is a special case of the Kronecker canonical form [19].)

Theorem 1 (Weierstraß Canonical Form [19]). *If $\alpha E - \beta A$ is a regular pencil, then there exist non-singular matrices $X = [X_r, X_\infty] \in \mathbb{R}^{n \times n}$ and $Y = [Y_r, Y_\infty] \in \mathbb{R}^{n \times n}$ for which*

$$Y^T E X = \begin{bmatrix} Y_r^T \\ Y_\infty^T \end{bmatrix} E \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \quad (4)$$

and

$$Y^T A X = \begin{bmatrix} Y_r^T \\ Y_\infty^T \end{bmatrix} A \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \quad (5)$$

where J is a matrix in Jordan canonical form whose diagonal elements are the finite eigenvalues of the pencil and N is a nilpotent matrix, also in Jordan form. J and N are unique up to permutation of Jordan blocks.

The *index* of the pencil is the index of nilpotency of the nilpotent matrix N in (4), i.e., the pencil is of index v if $N^{v-1} \neq 0$ and $N^v = 0$. By convention, if E is non-singular, the pencil is said to be of index zero. A descriptor system is regular and of index at most one if and only if it has exactly $q = \text{rank}(E)$ finite eigenvalues.

In the notation of (4) and (5), classical solutions of (1) take the form

$$x(t) = X_r z_1(t) + X_\infty z_2(t),$$

where

$$\begin{aligned} \dot{z}_1 &= J z_1 + Y_r^T B u, \\ N \dot{z}_2 &= z_2 + Y_\infty^T B u. \end{aligned} \quad (6)$$

This system admits the explicit solution

$$\begin{aligned} z_1(t) &= e^{tJ} z_1(0) + \int_0^t e^{(t-s)J} Y_r^T B u(s) ds, \\ z_2(t) &= - \sum_{i=0}^{v-1} \frac{d^i}{dt^i} \left(N^i Y_\infty^T B u(t) \right), \end{aligned} \quad (7)$$

where v is the index of the pencil.

Eq. (7) shows that for regular systems with an index larger than one, in order to have classical, continuous solutions, the input $u(t)$ has to be of a certain smoothness, that is, $u(t)$ must belong to some suitable function space \mathcal{U}_{ad} . Hence, the index is a fundamental characteristic of (1) involving existence and smoothness of solutions. (The concept of index generalizes in a variety of ways to linear differential algebraic equations with time varying coefficients [3,11,20,25,26,32,33,41,50] and to nonlinear differential algebraic equations [3,20,34]. For the linear differential algebraic equation (1), the index of the pencil $\alpha E - \beta A$ is identical to the common generalizations including the differentiation index [3,11,20] and (essentially) the strangeness index [33].)

If $u(t)$ is not sufficiently smooth, then impulses may arise in the response of the system [21,58]. To insure a smooth response for every continuous input $u(t)$, it is necessary for the system to be regular and have index less than or equal to one. If a descriptor system can be transformed into a closed loop system that is regular and of index at most one by feedback, then the system is said to be *regularizable*. (Note that we use the term “regularizable” in a stronger sense than it is used in [31,38,48]. There, “regularizable” describes a system that can be made to be regular by feedback but not necessarily of index at most one. In [38] the term “properizable” is used in the sense that we use “regularizable”.)

The following lemma gives a useful characterization of regular, index one pencils.

Lemma 2 [30]. *The following are equivalent:*

1. *The pencil $\alpha E - \beta A$ is regular and has index less than or equal to one.*
2. $\text{rank} \left(\begin{bmatrix} E \\ T_\infty^T(E)A \end{bmatrix} \right) = \text{rank} (E + T_\infty(E)T_\infty^T(E)A) = n.$
3. $\text{rank} ([E, AS_\infty(E)]) = \text{rank} (E + AS_\infty(E)S_\infty^T(E)) = n.$
4. $T_\infty(E)^T AS_\infty(E)$ is non-singular.
5. *If*

$$U^T EV = \begin{pmatrix} r & n-r \\ U_1 & U_2 \end{pmatrix}^T E \begin{pmatrix} r & n-r \\ V_1 & V_2 \end{pmatrix} = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

is the singular value decomposition of E (with orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$ and a non-singular, diagonal matrix $\Sigma_r \in \mathbb{R}^{r \times r}$), then the $(n-r)$ -by- $(n-r)$ matrix $A_{22} = U_2^T AV_2$ is non-singular.

Eq. (7) also shows that the possible values of the initial condition $x(0)$ are restricted. The initial state must be a member of the set of *consistent* initial conditions

$$\mathcal{A} \equiv \left\{ X_r z_1 + X_\infty z_2 \mid z_1 \in \mathbb{R}^r, z_2 = - \sum_{i=0}^{v-1} \left(\frac{d^i}{dt^i} (N^i Y_\infty^T Bu)(0) \right), u(t) \in \mathcal{U}_{ad} \right\}.$$

The set of *reachable* states of (1) from the set \mathcal{A} of consistent initial conditions is \mathcal{A} itself [61]. We refer to \mathcal{A} as the *solution space* of (1). The relationship between the

set of consistent initial conditions and the set of admissible input functions is studied in [21].

In this paper we discuss regularization of descriptor systems by state and output feedback. After some motivating examples in Section 2, we review generalizations of controllability and observability to descriptor systems in Section 3. Controllability and observability may be tested using condensed forms introduced in Section 4. A complete characterization of descriptor systems for which some derivative and/or proportional feedback yields a regular and index at most one closed loop system appears in Section 5. The characterization displays which ranks of $E + BGC$ are consistent with a regular, index at most one closed loop descriptor system.

The condensed forms discussed in the following sections are obtained from the original system matrices E , A , B , and C by multiplication by elementary orthogonal matrices and from rank decisions in which “small” singular values are set to zero. The well-known rounding error analysis of orthogonal matrix computations applies [24,60]. Consequently, numerical methods based on these forms are backward stable. In backward stable algorithms, the effect of finite precision arithmetic is equivalent to perturbing the original data matrices E , A , B , and C to nearby matrices $E + \delta E$, $A + \delta A$, $B + \delta B$, $C + \delta C$, where $\|\delta E\| < p(n)\epsilon\|E\|$, $\|\delta A\| < p(n)\epsilon\|A\|$, $\|\delta B\| < p(n)\epsilon\|B\|$, and $\|\delta C\| < p(n)\epsilon\|C\|$, ϵ is the unit round and $p(n)$ is a low degree polynomial that depends on the details of the underlying finite precision arithmetic and numerical methods.

We focus on condensed forms obtained through orthogonal equivalence transformations, because they lead to backward stable algorithms [24,60]. It is widely recognized that backward stable algorithms outperform polynomial and geometric methods with respect to accuracy in the presence of rounding errors and usually with respect to computation time too [35,57]. However, the mathematical structure of descriptor systems is sometimes more clearly displayed by condensed and canonical forms obtained through more general equivalence transformations. See, for example, the geometric approach in [48], the polynomial approach in [31], the state space canonical forms for descriptor systems in [22,28] and the controllability tests in [27,51]. A geometric analysis of canonical forms and state feedback orbits for descriptor systems appears in [23].

2. Two examples

Many practical examples of both discrete time and continuous time descriptor systems appear in the open literature. See, for example, [2,15,29,37,39,40,47,52]. To illustrate some of the problems that arise in the analysis of descriptor systems we use applications from the literature. One is a model of a multi-link constrained manipulator, representing a window cleaning robot [29]. The other is a model of a simple electrical circuit [15].

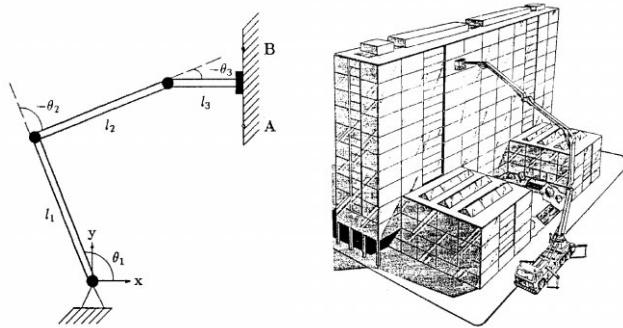


Fig. 1. A three link mobile manipulator [29].

Example 1. Consider a simplified, linearized model of a two-dimensional, three-link mobile manipulator [29] (see Fig. 1).

The Lagrangian equations of motion take the form

$$\begin{aligned} M(\Theta)\ddot{\Theta} + C(\Theta, \dot{\Theta}) + G(\Theta) &= u + F^T\mu, \\ \psi(\Theta) &= 0, \end{aligned}$$

where

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}$$

is the vector of joint displacements, $u \in \mathbb{R}^3$ the vector of control torques applied at the joints, $M \in \mathbb{R}^{3 \times 3}$ the mass matrix, $C \in \mathbb{R}^3$ the vector of centrifugal and Coriolis forces and $G \in \mathbb{R}^3$ is the gravity vector. The constraint function ψ is given by

$$\psi(\Theta) = \begin{bmatrix} l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3)l_3 - l \\ \theta_1 + \theta_2 + \theta_3 \end{bmatrix}.$$

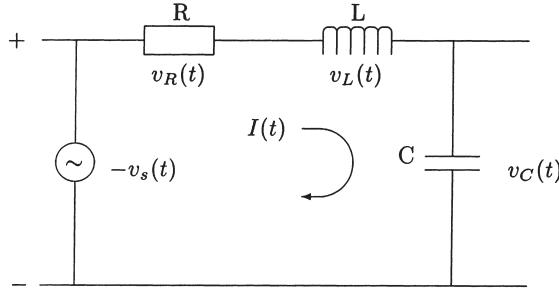
$F = (\partial\psi/\partial\Theta)$, $\mu \in \mathbb{R}^2$, represents the Lagrange multipliers and $F^T\mu$ is the generalized constraint force. Rewriting the system in Cartesian coordinates and linearizing yields a model of the form

$$\begin{aligned} M_0\delta\ddot{z} + D_0\delta\dot{z} + K_0\delta z &= S_0\delta u + F_0^T\delta\mu, \\ F_0\delta z &= 0. \end{aligned}$$

Letting

$$x = \begin{bmatrix} \delta z \\ \delta\dot{z} \\ \delta\mu \end{bmatrix}$$

and $u = \delta u$, one obtains a descriptor system of the form

Fig. 2. A simple RLC circuit.

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & I_3 & 0 \\ -K_0 & -D_0 & F_0^T \\ F_0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ S_0 \\ 0 \end{bmatrix} u.$$

Finally, one can add tracking output $y = Cx$ to the system as in [29]. Explicit data from [29] are given in Appendix A.

Example 2. Our second example is the simple RLC electrical circuit [15] in Fig. 2. The voltage source $v_s(t)$ is the control input, R , L and C are the resistance, inductance and capacitance, respectively. The corresponding voltage drops are denoted by $v_R(t)$, $v_L(t)$ and $v_C(t)$, respectively, and $I(t)$ denotes the current.

Applying Kirchoff's laws we obtain the following circuit equation:

$$\begin{aligned} & \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{I}(t) \\ \dot{v}_L(t) \\ \dot{v}_C(t) \\ \dot{v}_R(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/C & 0 & 0 & 0 \\ -R & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} I(t) \\ v_L(t) \\ v_C(t) \\ v_R(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} v_s(t). \end{aligned}$$

If we measure the voltage at the capacitor as output, we also have the output equation

$$y(t) = Cx(t) = [0 \quad 0 \quad 1 \quad 0] x(t).$$

3. Controllability and observability conditions

Given the descriptor system (1), (2), then one or more of the following conditions are essential for most classical design aims.

- C0:** $\text{rank}[\alpha E - \beta A, B] = n$ for all $(\alpha, \beta) \in \mathbb{C}^2$.
C1: $\text{rank}[\lambda E - A, B] = n$ for all $\lambda \in \mathbb{C}$.
C2: $\text{rank}[E, AS_\infty(E), B] = n$.

It is an immediate observation that **C0** implies **C1** and **C2**. Moreover, the condition **C1** together with the condition

$$\text{rank}[E, B] = n, \quad (9)$$

is equivalent to **C0**.

A regular system is *completely controllable or C-controllable* if **C0** holds [61] and is *strongly controllable or S-controllable* if **C1** and **C2** hold [5]. Complete controllability ensures that for any given initial and final states x_0, x_f there exists an admissible control that transfers the system from x_0 to x_f in finite time [61], while strong controllability ensures the same for any given initial and final states $x_0, x_f \in \mathcal{A}$ (the solution space).

Regular systems that satisfy Condition **C2** are called *controllable at infinity* or *impulse controllable* [14,30,58]. For these systems, impulsive modes can be excluded by a suitable linear feedback. Condition **C2** is closely related to the second condition in Lemma 2, which characterizes regular systems of index at most one. A regular descriptor system of index at most one is controllable at infinity.

Observability for descriptor systems is the dual of controllability. We define the following conditions:

- O0:** $\text{rank} \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} = n \quad \text{for all } (\alpha, \beta) \in \mathbb{C}^2$.
O1: $\text{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \text{for all } \lambda \in \mathbb{C}$.
O2: $\text{rank} \begin{bmatrix} E \\ T_\infty^T(E)A \\ C \end{bmatrix} = n$.

Again it is immediate that condition **O0** implies **O1** and **O2**. Moreover, **O1** and

$$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n, \quad (11)$$

together hold if and only if **O0** holds.

A regular descriptor system is called *completely observable or C-observable* if condition **O0** holds and is called *strongly observable or S-observable* if conditions **O1** and **O2** hold. A regular system that satisfies condition **O2** is called *observable at infinity or impulse-observable*.

Note that conditions (8)–(11) are preserved under non-singular equivalence transformations of the system and under state and output feedback, i.e., if the system satisfies **C0**, **C1**, or **C2**, then for any non-singular $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times m}$ and for any $F \in \mathbb{R}^{m \times p}$, the system $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C})$, where

$$\tilde{E} = UEV, \quad \tilde{A} = UAV, \quad \tilde{B} = UBW \quad (12)$$

or

$$\tilde{E} = E, \quad \tilde{A} = A + BF, \quad \tilde{B} = B$$

or

$$\tilde{E} = E, \quad \tilde{A} = A + BFC, \quad \tilde{B} = B$$

also satisfies these conditions. Analogous properties hold for **O0**, **O1** and **O2**.

Conditions **C0**, **C1** and **O0**, **O1** are also preserved under state and output derivative feedback transformations of the form

$$\tilde{E} = E + BG, \quad \tilde{A} = A, \quad \tilde{B} = B$$

or

$$\tilde{E} = E + BGC, \quad \tilde{A} = A, \quad \tilde{B} = B$$

Note, however, that the condition **C2** may not be preserved under derivative feedback as shown in the following example.

Example 3. We consider the system given by

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Here $[E, AS_\infty(E), B]$ has full rank and the system is regular. Therefore, it is controllable at infinity. With the feedback $G = I$, however, we obtain

$$E + BGC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S_\infty(E + BGC) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and $\text{rank}[E + BGC, AS_\infty(E + BGC), B] = 2$. Hence, condition **C2** is not invariant under derivative feedback.

A similar example demonstrates that condition **O2** is not invariant under derivative feedback either.

In this section we have introduced conditions that ensure the controllability and observability of descriptor systems. These conditions are essential in feedback design problems such as stabilization, pole assignment or linear quadratic control. Controllability and observability require *regularity* of the system in addition to conditions (8) and (10). In the literature, regularity of the open loop system is generally assumed [14,15,46,53,54,58,61,62], allowing the transformation to Weierstraß canonical form to be applied. Regularity is *not* needed, however, to obtain feedback designs that regularize the system, and conditions (8) and (10) alone are sufficient for most design problems. In the following we make no assumption about the regularity of system (1), (2).

In Section 4 we examine condensed forms that reveal the properties of the system and enable feedback design. Regularity of the system is not required. Unlike the Weierstraß canonical form, the condensed forms are computable by numerically reliable algorithms.

4. Condensed forms

Equivalence transformations such as (12) can be used to reduce the system (1), (2) to canonical or condensed forms that reveal the controllability and observability properties. This section presents condensed forms under orthogonal equivalence transformations. Section 5 shows how to use them to test whether a system can be made regular and to determine what is the minimum possible index. Regularizing feedbacks can also be constructed from the condensed forms.

The canonical form under arbitrary equivalence transformations derived in [38] displays more information than any of the condensed forms presented below. However, it is ill-suited to finite-precision computation, because arbitrarily ill-conditioned equivalence transformation matrices may be needed to reduce the original descriptor system to the canonical form. For finite precision computation, it is better to use well-conditioned equivalence transformations. Ideally, as in this paper, only real orthogonal (unitary in the complex case) transformations are used. We find condensed forms, like the Schur-form for matrices under unitary equivalence [24] or the staircase form [55]. Such condensed forms display most of the invariants of the problem. They can be computed using algorithms that are numerically stable in the sense that in the presence of rounding errors, the computed condensed form is what would have been obtained using exact arithmetic from a rounding-error-small perturbation of the original descriptor system.

We now present three condensed forms of this kind. In all these forms we assume for simplicity that the matrices B, C^T have full column rank. If this is not the case, then it can easily be achieved by introducing new input and output vectors. Also we adopt the notation that a matrix Σ_j is a non-singular j -by- j diagonal matrix, and 0 denotes the null-matrix of any size.

The proofs for the following condensed forms are given by construction, using a finite sequence of singular value decompositions [24] and rank decisions on transformed submatrices of E, A, B and C . The proofs translate directly into numerical algorithms and give numerically stable methods for computing the condensed forms. However, the algorithms must determine matrix dimensions from the ranks of submatrices. This is a deep and difficult problem, because arbitrarily small perturbations of a rank deficient matrix may change its rank. Any uncertainty in the data – even rounding errors – may obscure the rank. Ultimately, rank decisions in the presence of uncertainties are partially heuristic. One of the more reliable ways to decide the rank of a matrix $M \in \mathbb{R}^{m \times n}$, $m \geq n$, is as follows. Use a reliable numerical procedure to calculate the singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$, as in [1] or [42] and consider

a singular value σ_j to be zero if $\sigma_j \leq \mu\sigma_1$ where μ bounds the relative error in M . (If errors come only from rounding errors, then μ may be taken to be a modest multiple of the unit round-off.) The number of remaining non-zero singular values is then taken to be the (numerical) rank of the matrix. Precautions have to be taken, however, if the first neglected and last non-neglected σ_i are close together. (See [16].)

The first condensed form was introduced in [6].

Theorem 3. *Let E , $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{p \times n}$, where B and C are of full column and row rank, respectively. Then there exist orthogonal matrices U , $V \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times m}$, and $Y \in \mathbb{R}^{p \times p}$ such that*

$$U^T E V = \begin{matrix} t_1 & n - t_1 \\ \frac{t_1}{n - t_1} & \left[\begin{matrix} \Sigma_{t_1} & 0 \\ 0 & 0 \end{matrix} \right] \end{matrix}, \quad (13)$$

$$U^T B W = \begin{matrix} t_1 & k_1 & k_2 \\ \frac{t_2}{t_3} & \left[\begin{matrix} B_{11} & B_{12} \\ B_{21} & 0 \\ B_{31} & 0 \\ 0 & 0 \end{matrix} \right] \end{matrix}, \quad (14)$$

$$Y^T C V = \begin{matrix} t_1 & s_2 & t_5 & n - t_1 - s_2 - t_5 \\ \frac{\ell_1}{\ell_2} & \left[\begin{matrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & 0 & 0 & 0 \end{matrix} \right] \end{matrix}, \quad (15)$$

$$U^T A V = \begin{matrix} t_1 & s_2 & t_5 & t_4 & t_3 & s_6 \\ t_2 & \left[\begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} & \Sigma_{t_3} & 0 \\ A_{41} & A_{42} & A_{43} & \Sigma_{t_4} & 0 & 0 \\ A_{51} & 0 & \Sigma_{t_5} & 0 & 0 & 0 \\ A_{61} & 0 & 0 & 0 & 0 & 0 \end{matrix} \right] \end{matrix}. \quad (16)$$

The matrix B_{12} has full column rank, C_{21} has full row rank, and the matrices

$$\begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} \in \mathbb{C}^{k_1 \times k_1}, \quad [C_{12} \quad C_{13}] \in \mathbb{C}^{\ell_1 \times \ell_1}$$

are square and non-singular and are of dimension $k_1 = t_2 + t_3$ and $\ell_1 = s_2 + t_5$, respectively.

Here t_j , s_j , k_j and ℓ_j are non-negative integers displaying the number of rows or columns in the corresponding block row or column of the matrices. A zero value of one of these integers indicates that the corresponding block row or column does not appear.

Proof. A constructive proof is given in [6]. \square

An immediate implication of this result is the following corollary which characterizes the conditions introduced in Section 3.

Corollary 4. *Let the system (E, A, B, C) be in condensed form (13)–(16) of Theorem 3.*

1. *The pair (E, A) is regular and of index at most one if and only if $s_6 = t_6 = 0$ and $A_{22} - A_{24}\Sigma_{t_4}^{-1}A_{42}$ is non-singular.*
2. *Condition C2 holds if and only if $t_6 = 0$.*
3. *Condition O2 holds if and only if $s_6 = 0$.*
4. *$\text{rank}[E, B] = t_1 + t_2 + t_3$, and thus $\text{rank}[E, B] = n$ if and only if $t_4 = t_5 = t_6 = 0$.*
5. *$\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = t_1 + s_2 + t_5$, and thus $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$ if and only if $t_4 = t_3 = s_6 = 0$.*
6. *$\text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} = t_1 + t_2 + s_2 + t_3 + t_5 + \min(\ell_2, k_2)$.*

Proof. Clear from condensed form (13) and (16). \square

In the construction of the condensed form (13)–(16) we first determine the singular value decomposition of E and then modify the remaining matrices. This order of operations displays $S_\infty(E)$ and $AS_\infty(E)$ so that regular systems with index at most one are recognized early in the procedure. If regularization is necessary, a regularizing proportional feedback can be constructed immediately. However, it is often the case that we wish to use derivative feedback. For derivative feedback, it is more convenient to start with the singular value decompositions of B and C in order to split E into a set of components that are left invariant and a set of components that can be set to arbitrary values by derivative feedback. This leads to the following condensed form.

Theorem 5. *Let $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$, where B and C have full column and row rank, respectively. Then there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{p \times p}$ such that*

$$U^T E V = \tilde{E} = \begin{bmatrix} \tilde{s}_1 & \tilde{t}_4 & \tilde{t}_3 & \tilde{t}_2 & \tilde{t}_5 & \tilde{s}_6 \\ \tilde{t}_1 & \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & 0 & 0 & 0 \\ \tilde{t}_2 & \tilde{E}_{21} & \tilde{E}_{22} & \tilde{E}_{23} & \Sigma_{\tilde{t}_2} & 0 & 0 \\ \tilde{t}_3 & \tilde{E}_{31} & \tilde{E}_{32} & \Sigma_{\tilde{t}_3} & 0 & 0 & 0 \\ \tilde{t}_4 & 0 & \Sigma_{\tilde{t}_4} & 0 & 0 & 0 & 0 \\ \tilde{t}_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{t}_6 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (17)$$

$$U^T B W = \tilde{B} = \begin{matrix} \tilde{t}_1 & & m \\ \tilde{t}_2 & \left[\begin{matrix} \tilde{B}_1 \\ \tilde{B}_2 \\ 0 \end{matrix} \right] \\ n - \tilde{t}_1 - \tilde{t}_2 & \end{matrix}, \quad (18)$$

$$Y^T C V = \tilde{C} = p \begin{bmatrix} \tilde{s}_1 & \tilde{t}_4 & n - \tilde{s}_1 - \tilde{t}_4 \\ \tilde{C}_1 & \tilde{C}_2 & 0 \end{bmatrix}, \quad (19)$$

$$U^T A V = \tilde{A} = \begin{matrix} \tilde{s}_1 & \tilde{t}_4 & \tilde{t}_3 & \tilde{t}_2 & \tilde{t}_5 & \tilde{s}_6 \\ \tilde{t}_1 & \left[\begin{matrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} & \tilde{A}_{16} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} & \tilde{A}_{26} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} & \tilde{A}_{35} & \tilde{A}_{36} \\ \tilde{A}_{41} & \tilde{A}_{42} & \tilde{A}_{43} & \tilde{A}_{44} & \tilde{A}_{45} & \tilde{A}_{46} \\ \tilde{A}_{51} & \tilde{A}_{52} & \tilde{A}_{53} & \tilde{A}_{54} & \Sigma_{\tilde{t}_5} & 0 \\ \tilde{A}_{61} & \tilde{A}_{62} & \tilde{A}_{63} & \tilde{A}_{64} & 0 & 0 \end{matrix} \right] \\ \tilde{t}_2 & \\ \tilde{t}_3 & \\ \tilde{t}_4 & \\ \tilde{t}_5 & \\ \tilde{t}_6 & \end{matrix}, \quad (20)$$

where $\tilde{t}_1 + \tilde{t}_2 = m$, $\tilde{s}_1 + \tilde{t}_4 = p$, and the matrices $\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$ and $\begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}$ are square and non-singular.

Proof. The proof is a simple modification of the proof of the condensed form of Theorem 3 (see [6]). \square

We have an immediate corollary which characterizes the controllability and observability conditions of Section 3.

Corollary 6. *Let system (E, A, B, C) be in the condensed form of Theorem 5. Then we have the following properties:*

1. $\text{rank}[E, B] = \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$, and thus $\text{rank}[E, B] = n$ if and only if $\tilde{t}_6 = \tilde{t}_5 = 0$.
2. $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = \tilde{s}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$, and thus $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$ if and only if $\tilde{s}_6 = \tilde{t}_5 = 0$.
3. $\text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} = \tilde{t}_1 + \tilde{s}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$.
4. If $\tilde{t}_6 = 0$, then **C2** holds.
5. If $\tilde{s}_6 = 0$, then **O2** holds.

Proof. Clear from the condensed form (17)–(20). \square

The next condensed form that we present was first introduced in the context of pole-placement algorithms for descriptor systems [45] and independently discovered by several authors [7,13]. It is different from the previous condensed forms, in the sense that it does not display all the controllability properties. It can, however, be used to separate the parts of the system that have high index but are not controllable or observable at infinity.

Theorem 7. Let $E, A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{p \times n}$, where B and C have full column and row rank, respectively. Then there exist orthogonal matrices $U, V \in \mathbb{R}^{n \times n}$, $W \in \mathbb{R}^{m \times m}$ and $Y \in \mathbb{R}^{p \times p}$ such that

$$\begin{aligned} U^T E V &= \begin{matrix} \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ \hat{t}_1 & \left[\begin{matrix} E_{11} & 0 & 0 & E_{14} \\ 0 & 0 & 0 & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ 0 & 0 & 0 & E_{44} \end{matrix} \right] \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \end{matrix}, \\ U^T A V &= \begin{matrix} \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ \hat{t}_1 & \left[\begin{matrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{matrix} \right] \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \end{matrix}, \\ U^T B &= \begin{matrix} \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ \hat{t}_1 & \left[\begin{matrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{matrix} \right] \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \end{matrix}, \quad CV = p \begin{bmatrix} \hat{t}_1 & \hat{t}_2 & \hat{t}_3 & \hat{t}_4 \\ C_1 & C_2 & 0 & C_4 \end{bmatrix}, \end{aligned} \tag{21}$$

with the following properties:

1. $\text{rank}(E_{11}) = \hat{t}_1$,
2. $\text{rank}(C_2) = \hat{t}_2$,
3. A_{33} is block lower triangular,
4. E_{33} is block lower triangular with zero diagonal blocks, partitioned conformally with A_{33} ,
5. A_{44} is block upper triangular,
6. E_{44} is block upper triangular with zero diagonal blocks, partitioned conformally with A_{44} ,
7. the subsystem obtained by deleting the last block row and column in (21) satisfies **C2**, and
8. the subsystem obtained by deleting the last two block rows and columns in (21) satisfies **C2** and **O2**.

Proof. The proof is given in [7]. \square

We have seen that the three condensed forms of Theorems 3, 5 and 7 reveal different properties of the system.

The condensed form of Theorem 3 displays whether the system is regular and of index at most one and gives necessary and sufficient conditions for **C2** and **O2** to hold. The condensed form of Theorem 5 only gives sufficient conditions. The following example shows that these conditions are not necessary.

Example 4. Suppose

$$E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0].$$

This system is in condensed form (17)–(20) with $\tilde{s}_6 = \tilde{t}_6 = 1$. Nevertheless, **C2** and **O2** both hold.

Section 5 shows that the three condensed forms presented here can be used to test whether a system can be made regular or regular with index one by proportional or derivative feedback. Such a regularizing feedback can then be constructed from the condensed form. The condensed form (13)–(16) in Theorem 3 shows whether the system can be transformed into a regular system of index one by proportional feedback. The condensed form (17)–(20) from Theorem 5 shows whether a regular system with index one can be achieved by a derivative and proportional feedback of the form $u(t) = G\dot{y}(t) + Fy(t)$ and which ranks for the closed loop matrix $E + BGC$ are achievable. Finally, the condensed form (21) of Theorem 7 determines whether the system can be made regular (but not necessarily index one) and what is the minimal achievable index.

The treatment of systems that cannot be regularized or made index one can thus be analyzed from Theorem 7, but not from the other two condensed forms.

Each of the three condensed forms has its advantages and disadvantages. It is unsatisfying that not all the properties are displayed by a single condensed form. At this writing, it is an open problem whether there exists such a condensed form under orthogonal equivalence transformations. However, using non-orthogonal equivalence transformations, there do exist the highly refined condensed or canonical forms that display more information. (See, for example, [38].) The non-orthogonal equivalence transformations are sometimes ill-conditioned, so it is not always possible to compute such condensed or canonical forms in a numerically stable way.

For the three-link mobile manipulator with data given in Example 1 and for the electrical circuit in Example 2, we numerically computed the condensed forms (13)–(16) and (17)–(20) of Theorems 3 and 5 and list them in Appendix A.

In this section we have introduced several condensed forms that can be computed in a numerically stable way using real orthogonal equivalence transformations. From these condensed forms we can detect whether the controllability and observability conditions hold and furthermore also detect other properties of the system. In the next section we discuss the regularization via feedback.

5. Regularization

As mentioned in Section 1, it is desirable to have regular systems that are of index at most one. If we compute the condensed form (13)–(16) in Theorem 3, then we can check whether the system is regular and of index at most one. In case it is not

regular, we can often find a feedback control to make it regular and of index one and thus obtain more favorable behavior. In this section we discuss how the regularity of the system can be obtained via feedback.

5.1. Proportional feedback

The following characterization of systems for which a regularizing proportional feedback exists has been established for some time in the case where the open loop system (1), (2) is assumed to be regular. (See, for example, [15,46].) However, it is not necessary to assume regularity of the open loop system [6].

Theorem 8. *Consider the system (E, A, B, C) given by (1), (2). There exists a matrix $F \in \mathbb{R}^{m \times p}$ such that $\alpha E - \beta(A + BFC)$ is regular and has index at most one if and only if conditions **C2** and **O2** hold.*

Proof. A proof based on the condensed form (13)–(16) appears in [6]. \square

Theorem 8 gives necessary and sufficient conditions such that the system can be made regular and of index one by proportional output feedback, but often in practice these conditions do not hold. Example 1 is a typical case. (See Appendix A.) A natural question is whether the system can be made regular of any index by feedback. Here we can make use of Theorem 7.

Theorem 9. *Suppose (E, A, B, C) is in the condensed form (21) of Theorem 7. The system can be made regular by proportional output feedback, that is, there exists a feedback matrix F such that the closed loop pencil $\alpha E - \beta(A + BFC)$ is regular, if and only if A_{33} and A_{44} are non-singular.*

Proof. This follows from Theorem 7. (See also [7,13].) \square

Note that it is possible to decide directly from (21), whether the system can be made regular by proportional feedback. From (13)–(16) or (17)–(20) we can easily test whether the system can be made regular and of index at most one, but (13)–(16) or (17)–(20) do not openly display whether the system can be made regular and of higher index. Another advantage of the condensed form (21) is that it displays the minimum achievable index (see [7]).

Theorem 10. *Let the system (E, A, B, C) be in the condensed form (21) of Theorem 7. If A_{33} and A_{44} are non-singular, then there exists a proportional output feedback $F \in \mathbb{R}^{m \times p}$ such that $\alpha E - \beta(A + BFC)$ is regular and the index of $\alpha E - \beta(A + BFC)$ is equal to the maximum of the index of nilpotency of*

$$\Theta = \begin{pmatrix} \hat{t}_2 & \hat{t}_3 \\ \hat{t}_2 & 0 \\ \hat{t}_3 & E_{32} \\ E_{32} & E_{33} \end{pmatrix} \quad (22)$$

and the index of nilpotency of E_{44} .

Proof. Since E_{44} is strictly upper triangular and A_{44} is non-singular, it is clear that the index will be equal to the maximum of the index of nilpotency of E_{44} and the index of

$$\alpha \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ E_{31} & E_{32} & E_{33} \end{bmatrix} - \beta \left(\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} F \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix} \right).$$

The index of the latter has been shown in [7] to be equal to the index of Θ given by (22). \square

Another useful observation made in [7] is the following.

Theorem 11. Let (E, A, B, C) be in condensed form (21) of Theorem 7:

$$\begin{aligned} & \begin{bmatrix} E_{11} & 0 & 0 & E_{14} \\ 0 & 0 & 0 & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ 0 & 0 & 0 & E_{44} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix} u, \quad (23) \\ y &= [C_1 \ C_2 \ 0 \ C_4] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \end{aligned}$$

If this system can be made regular by a proportional output feedback and if the initial conditions are consistent, then x_4 is constrained to be zero and possible impulsive behavior in x_3 is not observed in the output.

Proof. For the first part see [7]. The second part follows trivially. \square

It follows that any regularizable system consists of a subsystem that is controllable and observable at infinity, solution components (modes) that are constrained to be zero in an appropriate coordinate system, i.e. modes that do not contribute to the system dynamics, plus modes that may display impulsive behavior but are not observed at output. The modes that are constrained to zero can be removed from the

system, and from a practical point of view systems with unobservable infinite modes should be avoided.

Note, however, that the results we have just discussed are very sensitive to perturbations from modeling errors as well as round-off errors in the numerical computation. In (23), x_4 is constrained to be zero, but a perturbation or unmodeled forcing function that excites x_4 may give rise to impulses.

5.2. Derivative and proportional feedback

Theorems 9 and 11 deal only with the case of proportional feedback but, as mentioned above, derivative feedback may also be used to alter the regularity of the system. Since derivative feedback of the form $u(t) = G\dot{y}(t) + v(t)$ yields a closed loop system with pencil $(E + BGC, A)$, we see that the left and right nullspaces S_∞ and T_∞ of E may be modified. In turn, this may change system properties like **C2** and **O2**. Also the rank of $E + BGC$ may be increased or decreased from the rank of E . If the closed loop system is regular with index 1, then $N = 0$ in (6) and the system breaks into $\text{rank}(E + BGC)$ differential equations and $n - \text{rank}(E + BGC)$ algebraic equations. In applications, it may be desirable to have more differential equations or it may be desirable to have fewer differential equations. In the special case of direct state feedback ($C = I$), the range of possible ranks of $F + BG$ are described in [5]. For the output feedback case, the complete range of possible ranks of $E + BGC$ is given by the following result.

Lemma 12. *Let a linear descriptor system (E, A, B, C) of the form (1), (2) be given. If r satisfies*

$$\begin{aligned} & \text{rank}[E, B] + \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} - \text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} \\ & \leq r \leq \min \left(\text{rank}[E, B], \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} \right), \end{aligned} \quad (24)$$

then there exists a feedback matrix $G \in \mathbb{R}^{m \times n}$ such that $\text{rank}(E + BGC) = r$.

Proof. A proof was given in [13] using Theorem 7. It can also be obtained more easily from Theorem 5. \square

The ranks in (24) are available from condensed form (13)–(16) as well as from (17)–(20). See Corollaries 4 and 6.

With the possible ranks of E being characterized, we can ask whether we can make the system regular and of index at most one for any $\text{rank}(E + BGC) = r$ the

range (24). It is not surprising that the answer is “no”. In some cases, the regular index 1 requirement may leave very little freedom in the choice of feedback indeed.

Example 5. Consider the system given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [1 \quad 0].$$

The pencil $(E + BGC, A + BFG)$ is regular with index 2 and $\text{rank}(E + BGC) = 1$ for all G and F except $G = [-1]$. For $G = [-1]$, $\text{rank}(E + BGC) = 0$, and for any F , the pencil $(E + BGC, A + BFG)$ is regular, index one (and purely algebraic).

To give a complete characterization of the achievable ranks for $E + BGC$ in closed loop systems that are regular and of index at most one we have to introduce some further notation. Let

$$\begin{aligned} T_b &= T_\infty \left(\begin{bmatrix} E, AS_\infty \begin{bmatrix} E \\ C \end{bmatrix}, B \end{bmatrix} \right), \\ S_b &= S_\infty \left(\begin{bmatrix} E \\ (T_\infty [E, B])^T A \\ C \end{bmatrix} \right), \end{aligned} \tag{25}$$

and let

$$\begin{aligned} T_a &= T_\infty (ES_\infty(C)), \\ S_a &= S_\infty ((T_\infty(B))^T E). \end{aligned} \tag{26}$$

Although these matrices look complicated they are easily described in terms of a slight modification of the condensed form (17)–(20). (See (29)–(32) below.)

Theorem 13. *Let (E, A, B, C) be a linear descriptor system in the form of (1), (2). The following are equivalent:*

- (i) *There exist feedback matrices $F, G \in \mathbb{R}^{m \times p}$ such that the closed loop pencil $(E + BGC, A + BFC)$ is regular and of index at most one.*
- (ii) *$T_a^T AS_b$ has full column rank and $T_b^T AS_a$ has full row rank.*

Moreover, if the closed loop pencil $(E + BGC, A + BFC)$ is regular and of index at most one, then

$$\begin{aligned} &\text{rank}[E, B] + \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} - \text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} \\ &\leq \text{rank}(E + BGC) \\ &\leq \text{rank}[E, B] - \text{rank}(T_a^T AS_b) = \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} - \text{rank}(T_b^T AS_a). \end{aligned} \tag{27}$$

Proof. Inequality (27) and statements (i) and (ii) are invariant under arbitrary equivalence transformations of the form (12), so without loss of generality, we may assume that the system is in condensed form (17)–(20). Using (non-unitary) block Gauß transformations we may use the third block row to eliminate \tilde{E}_{31} , and \tilde{E}_{32} and the third block column to eliminate \tilde{E}_{13} and \tilde{E}_{23} . (We use non-unitary Gauß transformations only as theoretical tools to simplify the exposition of this proof. For explicit numerical construction of feedback control, these transformations need not be carried out.) The matrix \check{E} now takes the form

$$\check{E} = \begin{bmatrix} \tilde{s}_1 & \tilde{t}_4 & \tilde{t}_3 & \tilde{t}_2 & \tilde{t}_5 & \tilde{s}_6 \\ \tilde{t}_1 & \check{E}_{11} & \check{E}_{12} & 0 & 0 & 0 \\ \tilde{t}_2 & \check{E}_{21} & \check{E}_{22} & 0 & \Sigma_{\tilde{t}_2} & 0 \\ \tilde{t}_3 & 0 & 0 & \Sigma_{\tilde{t}_3} & 0 & 0 \\ \tilde{t}_4 & 0 & \Sigma_{\tilde{t}_4} & 0 & 0 & 0 \\ \tilde{t}_5 & 0 & 0 & 0 & 0 & 0 \\ \tilde{t}_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (28)$$

The matrices \tilde{B} and \tilde{C} remain unchanged. The matrix \tilde{A} transforms to a new matrix \check{A} but the block structure (20) remains unchanged. The matrices in (25), (26), and (27) now take the forms

$$T_b = \begin{bmatrix} 0 \\ I_{\tilde{t}_6} \end{bmatrix}, \quad S_b = \begin{bmatrix} 0 \\ I_{\tilde{s}_6} \end{bmatrix}, \quad (29)$$

$$T_a = \begin{bmatrix} I_{\tilde{t}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{\tilde{t}_4} & 0 & 0 \\ 0 & 0 & I_{\tilde{t}_5} & 0 \\ 0 & 0 & 0 & I_{\tilde{t}_6} \end{bmatrix}, \quad S_a = \begin{bmatrix} I_{\tilde{s}_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I_{\tilde{t}_2} & 0 & 0 \\ 0 & 0 & I_{\tilde{t}_5} & 0 \\ 0 & 0 & 0 & I_{\tilde{s}_6} \end{bmatrix}, \quad (30)$$

$$T_a^T \check{A} S_b = \begin{bmatrix} \tilde{t}_1 & \check{A}_{16} \\ \tilde{t}_2 & 0 \\ \tilde{t}_3 & 0 \\ \tilde{t}_4 & \check{A}_{46} \\ \tilde{t}_5 & 0 \\ \tilde{t}_6 & 0 \end{bmatrix}, \quad T_b^T \check{A} S_a = \tilde{t}_6 \begin{bmatrix} \tilde{s}_1 & \tilde{t}_2 & \tilde{t}_5 & \tilde{s}_6 \\ \check{A}_{61} & \check{A}_{64} & 0 & 0 \end{bmatrix}, \quad (31)$$

and

$$\begin{bmatrix} \check{E} & \tilde{B} \\ \tilde{C} & 0 \end{bmatrix} = \begin{bmatrix} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 & \tilde{t}_5 & \tilde{t}_6 & \tilde{t}_1 + \tilde{t}_2 \\ \tilde{t}_2 & \tilde{t}_1 & \check{E}_{11} & \check{E}_{12} & 0 & 0 & \tilde{B}_1 \\ \tilde{t}_3 & \tilde{t}_4 & \check{E}_{21} & \check{E}_{22} & 0 & \Sigma_{\tilde{t}_2} & \tilde{B}_2 \\ \tilde{t}_4 & \tilde{t}_3 & 0 & 0 & \Sigma_{\tilde{t}_3} & 0 & 0 \\ \tilde{t}_5 & \tilde{t}_4 & 0 & \Sigma_{\tilde{t}_4} & 0 & 0 & 0 \\ \tilde{t}_6 & \tilde{t}_5 & 0 & 0 & 0 & 0 & 0 \\ \tilde{s}_1 + \tilde{t}_4 & \tilde{s}_1 & \tilde{C}_1 & \tilde{C}_2 & 0 & 0 & 0 \end{bmatrix}. \quad (32)$$

It follows from (32), $\tilde{t}_1 + \tilde{t}_6 = \tilde{s}_1 + \tilde{s}_6$ and the non-singularity of

$$\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \in \mathbb{R}^{(\tilde{t}_1 + \tilde{t}_2) \times (\tilde{t}_1 + \tilde{t}_2)}$$

and

$$\begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \in \mathbb{R}^{(\tilde{s}_1 + \tilde{t}_4) \times (\tilde{s}_1 + \tilde{t}_4)}$$

that the range of ranks in (27) is

$$\begin{aligned} \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4 &\leqslant \text{rank}(E + BG\tilde{C}) \\ &\leqslant \tilde{t}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4 - \tilde{s}_6 = \tilde{s}_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4 - \tilde{t}_6. \end{aligned} \quad (33)$$

Regardless of the choice of G , $S_\infty(\check{E} + \tilde{B}G\tilde{C})$ and $T_\infty(\check{E} + \tilde{B}G\tilde{C})$ take the forms

$$S_\infty(\check{E} + \tilde{B}G\tilde{C}) = \begin{bmatrix} \tilde{t}_5 & \tilde{t}_6 & \tilde{s}_1 - r_1 \\ \tilde{t}_1 & 0 & 0 & Y_1 \\ \tilde{t}_2 & 0 & 0 & 0 \\ \tilde{t}_3 & 0 & 0 & 0 \\ \tilde{t}_4 & 0 & 0 & Y_4 \\ \tilde{t}_5 & I_{\tilde{t}_5} & 0 & 0 \\ \tilde{t}_6 & 0 & I_{\tilde{s}_6} & 0 \end{bmatrix} \quad (34)$$

and

$$T_\infty(\check{E} + \tilde{B}G\tilde{C}) = \begin{bmatrix} \tilde{t}_5 & \tilde{t}_6 & \tilde{t}_1 - r_1 \\ \tilde{t}_1 & 0 & 0 & Z_1 \\ \tilde{t}_2 & 0 & 0 & 0 \\ \tilde{t}_3 & 0 & 0 & 0 \\ \tilde{t}_4 & 0 & 0 & Z_4 \\ \tilde{t}_5 & I_{\tilde{t}_5} & 0 & 0 \\ \tilde{t}_6 & 0 & I_{\tilde{s}_6} & 0 \end{bmatrix}, \quad (35)$$

where $r_1 = \text{rank}(\check{E}_{11} + \tilde{B}_1G\tilde{C}_1)$ is the rank of the $(1, 1)$ block of $\check{E} + \tilde{B}G\tilde{C}$.

To see that (i) implies (ii), observe that part 3 of Lemma 2 implies that $\check{A}_{61}Y_1 + \check{A}_{64}Y_4$ has full row rank \tilde{t}_6 . It follows that $[\check{A}_{61}, \check{A}_{64}]$ has full row rank \tilde{t}_6 . A similar argument involving $T_\infty(\check{E} + \tilde{B}G\tilde{C})$ shows that

$$Z_1^T \check{A}_{16} + Z_4^T \check{A}_{46} \quad \text{and} \quad \begin{bmatrix} \check{A}_{16} \\ \check{A}_{46} \end{bmatrix}$$

have full column rank \tilde{s}_6 . Statement (ii) now follows from (31).

To show that (ii) implies (i) and to establish (27), we will first use (ii) to construct feedback matrices F and G satisfying (i) with $\text{rank}(E + BGC)$ satisfying (33). Then we will show that no other value $\text{rank}(E + BGC)$ is possible. The construction starts with choosing $S_\infty(\check{E} + \tilde{B}G\tilde{C})$ and $T_\infty(\check{A} + \tilde{B}F\tilde{C})$ through the choice of Y_1 , Y_4 , Z_1 , and Z_4 . Then we make compatible choices of the (1, 1), (1, 2), (2, 1), and (2, 2) blocks of $\check{E} + \tilde{B}G\tilde{C}$. These choices then uniquely fix G . Finally, we make choices of the (1, 1), (1, 2), (2, 1), and (2, 2) blocks of $\check{A} + \tilde{B}F\tilde{C}$ and corresponding feedback F so that Lemma 2 implies the resulting pencil is regular with index at most 1.

Let r_1 be any integer in the interval

$$0 \leq r_1 \leq \tilde{s}_1 - \tilde{t}_6 = \tilde{t}_1 - \tilde{s}_6. \quad (36)$$

Assumption (ii) and (31) imply that $[\check{A}_{61} \quad \check{A}_{64}]$ has full row rank \tilde{t}_6 and $\begin{bmatrix} \check{A}_{16} \\ \check{A}_{46} \end{bmatrix}$ has full column rank \tilde{s}_6 . Hence, there exist matrices $Y_1 \in \mathbb{R}^{\tilde{s}_1 \times (\tilde{s}_1 - r_1)}$, $Y_4 \in \mathbb{R}^{\tilde{t}_2 \times (\tilde{s}_1 - r_1)}$, $Z_1 \in \mathbb{R}^{\tilde{t}_1 \times (\tilde{s}_1 - r_1)}$, and $Z_4 \in \mathbb{R}^{\tilde{t}_4 \times (\tilde{s}_1 - r_1)}$ such that Y_1 has full column rank $\tilde{s}_1 - r_1$, Z_1 has full column rank $\tilde{t}_1 - r_1$, $\check{A}_{61}Y_1 + \check{A}_{64}Y_4$ has full row rank \tilde{t}_6 , and $Z_1^T \check{A}_{16} + Z_4^T \check{A}_{46}$ has full column rank \tilde{s}_6 .

Select a rank r_1 matrix $\check{E}_{11} \in \mathbb{R}^{\tilde{t}_1 \times \tilde{s}_1}$ satisfying $\check{E}_{11}Y_1 = 0$ and $Z_1^T \check{E}_{11} = 0$. Set $\check{E}_{21} = -\Sigma_{\tilde{t}_2} Y_4(Y_1^T Y_1)^{-1} Y_1^T$, $\check{E}_{12} = -Z_1(Z_1^T Z_1)^{-1} Z_4^T \Sigma_{\tilde{t}_4}$, and $\check{E}_{22} = 0$. Note that if

$$G = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}^{-1} \begin{bmatrix} \check{E}_{11} & \check{E}_{12} \\ \check{E}_{21} & 0 \end{bmatrix} [\tilde{C}_1 \quad \tilde{C}_2]^{-1}, \quad (37)$$

then

$$\check{E} + \tilde{B}G\tilde{C} = \begin{bmatrix} \tilde{s}_1 & \tilde{t}_4 & \tilde{t}_3 & \tilde{t}_2 & \tilde{t}_5 & \tilde{s}_6 \\ \tilde{t}_1 & \begin{bmatrix} \check{E}_{11} & \check{E}_{12} & 0 & 0 & 0 & 0 \\ \check{E}_{21} & 0 & 0 & \Sigma_{\tilde{t}_2} & 0 & 0 \end{bmatrix} \\ \tilde{t}_2 & 0 & 0 & \Sigma_{\tilde{t}_3} & 0 & 0 \\ \tilde{t}_3 & 0 & 0 & 0 & 0 & 0 \\ \tilde{t}_4 & 0 & \Sigma_{\tilde{t}_4} & 0 & 0 & 0 \\ \tilde{t}_5 & 0 & 0 & 0 & 0 & 0 \\ \tilde{t}_6 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (38)$$

which implies (27), (34) and (35).

By construction, $\check{A}_{61}Y_1 + \check{A}_{64}Y_4$ has full row rank \tilde{t}_6 , and $Z_1^T \check{A}_{16} + Z_4^T \check{A}_{46}$ has full column rank \tilde{s}_6 , so there exists a matrix $\check{W} \in \mathbb{R}^{(\tilde{t}_1 - r_1) \times (\tilde{s}_1 - r_1)}$ so that

$$\begin{bmatrix} \tilde{s}_6 & \tilde{s}_1 - r_1 \\ \tilde{t}_6 & 0 \\ \tilde{t}_1 - r_1 & \begin{bmatrix} Z_1^T \check{A}_{16} + Z_4^T \check{A}_{46} & \check{A}_{61}Y_1 + \check{A}_{64}Y_4 \\ \check{W} \end{bmatrix} \end{bmatrix}$$

is non-singular. Set

$$F = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}^{-1} \check{W} \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}^{-1}. \quad (39)$$

With F and G as in (37) and (39), a tedious but elementary calculation shows that

$$\begin{aligned} T_\infty(\check{E} + \tilde{B}G\tilde{C})^T(\check{A} + \tilde{B}F\tilde{C})S_\infty(\check{E} + \tilde{B}G\tilde{C}) \\ = \tilde{t}_6 \begin{bmatrix} \tilde{t}_5 & \tilde{s}_6 & \tilde{s}_1 - r_1 \\ \Sigma_{\tilde{t}_5} & 0 & \check{A}_{51}Y_1 + \check{A}_{54}Y_4 \\ 0 & 0 & \check{A}_{61}Y_1 + \check{A}_{64}Y_4 \\ Z_1^T\check{A}_{15} + Z_4^T\check{A}_{45} & Z_1^T\check{A}_{16} + Z_4^T\check{A}_{46} & \check{W} \end{bmatrix} \end{aligned}$$

is non-singular. By Lemma 2, the closed loop pencil $(E + BGC, A + BFC)$ is regular and of index at most 1.

Finally, from (38), $\text{rank}(E + BGC) = \text{rank}(\check{E} + \tilde{B}G\tilde{C}) = r_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$. Inequality (36) now implies (33).

We have shown that $\text{rank}(E + BGC)$ may assume any value in (27) where the closed loop pencil is regular and of index at most one. Finally, we must show that no other value $\text{rank}(E + BGF)$ is possible. The lower bound in (27) is clear, because it is the lower bound in Lemma 12, i.e., it is the minimum possible rank of $E + BGC$ regardless of the regularity and index of the closed loop pencil. It remains to establish the upper bound.

As above, let $r_1 = \text{rank}(\check{E}_{11} + \tilde{B}_1G\tilde{C}_1)$ be the rank of the $(1, 1)$ block of $\check{E} + \tilde{B}G\tilde{C}$. Now, $\check{E} + \tilde{B}G\tilde{C}$ has the same block structure as (28), so $\text{rank}(E + BGC) = \text{rank}(\check{E} + \tilde{B}G\tilde{C}) = r_1 + \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4$. By Lemma 2, $\check{A}_{61}Y_1 + \check{A}_{64}Y_4$ is a \tilde{t}_6 -by- $(\tilde{s}_1 - r_1)$ matrix of full row rank. Hence, $\tilde{t}_6 \leq \tilde{s}_1 - r_1$ and $\text{rank}(E + BGC) \leq \tilde{t}_2 + \tilde{t}_3 + \tilde{t}_4 + \tilde{t}_6 - \tilde{s}_1$. The upper bound in (27) now follows from (29)–(32). \square

The following example illustrates Theorem 13.

Example 6. Consider the system given by

$$E = 0, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This system is already in the condensed form (17)–(20) with $\tilde{t}_1 = \tilde{s}_1 = 2$, $\tilde{t}_6 = \tilde{s}_6 = 1$, and $\tilde{t}_2 = \tilde{t}_3 = \tilde{t}_4 = \tilde{t}_5 = 0$. Lemma 12 states that regardless of regularity and index, the range of ranks of $E + BGC$ is $0 \leq \text{rank}(E + BGC) \leq 2$. This is obviously correct. The range of ranks (27) is $0 \leq \text{rank}(E + BGC) \leq 1$. So, according to Theorem 13, the closed loop system can be made to be regular with index 1 with either $\text{rank}(E + BGC) = 0$ or $\text{rank}(E + BGC) = 1$. It is easy to verify that this is correct. For example, the choices $G = 0$ and

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

give a regular, index 1 closed loop system with $\text{rank}(E + BGC) = 0$ and the choices

$$G = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $F = 0$ give a regular, index 1 closed loop system with $\text{rank}(E + BGC) = 1$.

Also, according to Theorem 13, it is impossible to make the closed loop pencil be regular, index 1 with $\text{rank}(E + BGC) = 2$ despite the fact that 2 is in the range (24). For this example, if $\text{rank}(E + BGC) = 2$, then

$$S_\infty(E + BGC) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and regardless of the choice of F , $[E + BGC, (A + BFC)S_\infty(E + BGC)]$ is not full rank, since it has a row of zeros.) By Lemma 2, the closed loop system can not be regular with index 1.

It is still difficult to explicitly construct feedback matrices that make the system regular and of index at most one. First of all, there is quite a lot of freedom in the choice of F and G , which has to be resolved.

Second, it is not enough to construct the feedback matrices in a numerically stable way. It is also important that the resulting closed loop system is robustly of index at most one. A complete analysis when this is the case is not known; see [4,5,8,17,30] for partial results.

6. Conclusions

Controllability, observability, and regularizability properties of the linear descriptor system (1), (2) are displayed by three different condensed forms. The condensed forms also lead to a new characterization of descriptor systems that can be made to be regular with index 1 by proportional and derivative output feedback along with the possible ranks of $E + BGC$ in the closed loop system (3). The condensed forms are obtained through orthogonal equivalence transformations. These lead to algorithms that are numerically stable in the sense that, in the presence of rounding errors, the computed condensed form is what would have been obtained using exact arithmetic from a rounding-error-small perturbation of the original descriptor system.

It is unsatisfactory that not all the properties are displayed by a single condensed form obtained through orthogonal equivalence transformations. At this writing, the existence of such a condensed form is an open question.

Appendix A

Condensed forms for Examples 1 and 2

Explicitly computed samples of the condensed forms of Theorems 3 and 5 are demonstrated in this appendix for the descriptor systems of Examples 1 and 2. Computations were performed MATLAB 5.1 [42] on an Pentium workstation with machine precision $\mu = 2.22 \times 10^{-16}$. For the singular value drop tolerance, we set singular values of submatrices of E , A , B , and C to zero if their computed value was less than $\mu \|A\|$, $\mu \|E\|$, $\mu \|B\|$, or $\mu \|C\|$ respectively. Here, $\|M\|$ denotes the spectral norm of the matrix M .

A.1. Condensed forms for Example 1

In natural variables, the three-link mobile manipulator of Example 1 is modeled [29] by a descriptor system of the form (1), (2), where

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 18.7532 & -7.9449 & 7.9449 & 0 & 0 \\ 0 & 0 & 0 & -7.9449 & 31.8182 & -26.8182 & 0 & 0 \\ 0 & 0 & 0 & 7.9449 & -26.8182 & 26.8182 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -67.4894 & -69.2393 & 69.2393 & 1.5214 & 1.5517 & -1.5517 & 1 & 0 \\ -69.8124 & -1.6862 & 1.6862 & -3.2206 & -3.2847 & 3.2847 & 0 & 0 \\ 69.8123 & 1.6862 & 68.2707 & 3.2206 & 3.2847 & -3.2847 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.2166 & -0.0338 & 0.5547 \\ 0.4585 & -0.8452 & 0.3866 \\ -0.4585 & 0.8452 & 0.6134 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

A.1.1. Example 1: Condensed form (13)–(16)

The condensed form of Theorem 3, (13)–(16), is partitioned as $t_1 = 6$, $s_2 = t_2 = t_3 = t_4 = t_5 = 0$, $t_6 = s_6 = 2$, $\ell_1 = 0$, $\ell_2 = 3$, $k_1 = 0$, and $k_2 = 3$. The transformed matrices are

$$U^T E V = \text{diag}(59.3556, 15.6697, 2.3643, 1, 1, 1, 0, 0),$$

$$U^T A V$$

$$= \begin{bmatrix} -5.3776 & -6.4410 & 0.5521 & 77.0846 & -16.1536 & 61.6916 & 0.2665 & 0.6509 \\ 0.2935 & 0.3515 & -0.0301 & 91.0970 & 67.3167 & -56.7237 & -0.9631 & 0.1514 \\ 0.2134 & 0.2556 & -0.0219 & -7.8515 & -2.6995 & -49.3394 & 0.0371 & -0.7439 \\ 0.2665 & -0.9631 & 0.0371 & 0 & 0 & 0 & 0 & 0 \\ -0.7108 & -0.2224 & -0.6673 & 0 & 0 & 0 & 0 & 0 \\ 0.6509 & 0.1514 & -0.7439 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$U^T B W = \begin{bmatrix} -1.3567 & -0.0429 & 0.0314 \\ -0.1524 & -0.6033 & -0.1172 \\ 0.2314 & -0.6491 & 0.1068 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$Y^T C V = \begin{bmatrix} 0.9638 & 0.2663 & -0.0103 & 0 & 0 & 0 & 0 & 0 \\ 0.0000 & 0.0007 & 0.0183 & 0 & 0.9998 & 0 & 0 & 0 \\ 0.0000 & -0.0385 & -0.9991 & 0 & 0.0183 & 0 & 0 & 0 \end{bmatrix}.$$

The orthogonal equivalence transformations are

$$U = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0.2665 & -0.9631 & 0.0371 & 0 & 0 & 0 & 0 & 0 \\ -0.7108 & -0.2224 & -0.6673 & 0 & 0 & 0 & 0 & 0 \\ 0.6509 & 0.1514 & -0.7439 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.2665 & -0.9631 & 0.0371 & 0 & 0 & 0 & 0 & 0 \\ -0.7108 & -0.2224 & -0.6673 & 0 & 0 & 0 & 0 & 0 \\ 0.6509 & 0.1514 & -0.7439 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$W = \begin{bmatrix} 0.4829 & -0.0137 & -0.8755 \\ -0.8437 & -0.2750 & -0.4611 \\ -0.2345 & 0.9614 & -0.1443 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & 0.9998 & 0.0183 \\ -0.7375 & -0.0124 & 0.6752 \\ 0.6754 & -0.0135 & 0.7374 \end{bmatrix}.$$

Since $s_6, t_6 \neq 0$, it follows that neither **C2** nor **O2** holds for this system.

A.1.2. Example 1: Condensed form (17)–(20)

The condensed form of Theorem 5, (17)–(20), is partitioned as $\tilde{s}_1 = \tilde{s}_6 = \tilde{t}_1 = \tilde{t}_3 = \tilde{t}_6 = 2$, $\tilde{t}_2 = 1$, $\tilde{t}_4 = 1$, $\tilde{t}_5 = 0$. The transformed matrices are

$$U^T E V = \begin{bmatrix} 5.5655 & -2.1964 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6.7080 & 4.6188 & 0 & 0 & 0 & 0 & 0 & 0 \\ 41.4580 & -38.4109 & 0 & 0 & 0 & 21.8615 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U^T A V$$

$$= \begin{bmatrix} -3.0583 & -0.0397 & -25.5347 & -33.0776 & 81.0042 & 2.1205 & 0.3487 & 0.1078 \\ -3.3107 & -0.0430 & -27.6423 & 37.0934 & 87.6899 & 2.2956 & 0.3775 & -0.9254 \\ -5.2583 & -0.0683 & 58.1691 & 83.5929 & -7.1513 & 3.6459 & -0.8578 & -0.3634 \\ 0.7162 & -0.6979 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0.6979 & 0.7162 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$U^T B W = \begin{bmatrix} 0.8754 & -0.3828 & -0.1040 \\ 0.5975 & 0.7860 & -0.0275 \\ 0.8911 & -0.1509 & 0.1206 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y^T C V = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6979 & 0.7162 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.7162 & 0.6979 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The orthogonal transformation matrices are:

$$U = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0.3487 & 0.3775 & -0.8578 & 0 & 0 & 0 & 0 & 0 \\ 0.9310 & -0.0342 & 0.3634 & 0 & 0 & 0 & 0 & 0 \\ 0.1078 & -0.9254 & -0.3634 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0.6979 & 0.7162 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.7162 & 0.6979 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$W = \begin{bmatrix} 0.4829 & -0.0137 & -0.8755 \\ -0.8437 & -0.2750 & -0.4611 \\ -0.2345 & 0.9614 & -0.1443 \end{bmatrix}$$

and $Y = I$.

We cannot conclude anything about the controllability and observability at infinity, since the conditions 4 and 5 in Corollary 6 are only sufficient conditions.

From both canonical forms we get via Corollaries 4 and 6, respectively, that

$$\text{rank}[E, B] = 6, \quad \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = 6, \quad \text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} = 8.$$

Hence, the range of possible ranks for $E + BGC$ in Lemma 12 is $4 \leq r \leq 6$.

The system cannot be made regular and of index at most one by feedback as we can see from Theorem 13, since in the condensed form of Theorem 5 $T_a^T A S_b$ and $T_b^T A S_a$ are 2×2 and both have zero rank. For this system, however, using the condensed form of Theorem 7, the reduction procedure of [7] can be applied to remove the parts of the system that are not controllable or observable at infinity.

A.2. Condensed forms for Example 2

In natural variables, the circuit of Example 1 is modeled [15] by a descriptor system of the form (1), (2) where

$$E = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 10^4 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

and

$$C = [0 \ 0 \ 1 \ 0].$$

A.2.1. Example 2: Condensed form (17)–(20)

The condensed form of Theorem 3, (13)–(16), is partitioned as $t_1 = 2, s_2 = t_2 = 0, t_3 = t_4 = 1, t_5 = s_6 = t_6 = 0, \ell_1 = 0, \ell_2 = 1, k_1 = 1, k_2 = 0$. The transformed matrices are:

$$U^T E V = \begin{bmatrix} 1.1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U^T A V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 10^4 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 \\ -2 & 0 & 1 & 0 \end{bmatrix},$$

$$U^T B W = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad Y^T C V = [0 \ 1 \ 0 \ 0].$$

The orthogonal transformation matrices are:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and $W = Y = [1]$. It follows that conditions **C2** and **O2** hold.

A.2.2. Example 2: Condensed form (17)–(20)

The condensed form of Theorem 5, (17)–(20), is partitioned as $\tilde{t}_1 = \tilde{t}_3 = \tilde{t}_4 = \tilde{t}_5 = \tilde{s}_6 = 1$ and $\tilde{t}_2 = \tilde{t}_6 = \tilde{s}_1 = 0$. The transformed matrices are:

$$U^T EV = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U^T AV = \begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -10^4 & 0 & 0 \\ 0 & -2 & 1 & 0 \end{bmatrix},$$

$$U^T BW = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad Y^T CV = [1 \ 0 \ 0 \ 0].$$

The orthogonal transformation matrices are

$$U = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and $W = Y = [1]$.

From both canonical forms we get via Corollaries 4 and 6, respectively, that

$$\text{rank}[E, B] = 3, \quad \text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = 2 \quad \text{and} \quad \text{rank} \begin{bmatrix} E & B \\ C & 0 \end{bmatrix} = 3.$$

Hence, the only possible rank for $E + BGC$ in Lemma 12 is 2.

The system can be made regular and of index at most one by feedback as we can see from Theorem 13, since in the condensed form of Theorem 5 $T_a^T AS_b$ and $T_b^T AS_a$ have full column and row rank, respectively.

References

- [1] E. Anderson, Z. Bai, C. Bischof, J.W. Demmel, J. Dongarra, J.D. Croz, A. Greenbaum, S. Hammarling, A. McKenney, S. Ostrouchov, D. Sorensen, LAPACK Users' Guide, Society for Industrial and Applied Mathematics, Philadelphia, 1992.
- [2] P. Bernhard, On singular implicit linear dynamical systems, SIAM J. Control Optim. 20 (1982) 612–633.
- [3] K. Brenan, S. Campbell, L. Petzold, Numerical Solution of Initial-Value Problems in Differential Algebraic Equations, Elsevier, Amsterdam, 1989.
- [4] A. Bunse-Gerstner, V. Mehrmann, N.K. Nichols, Numerical methods for the regularization of descriptor system by output feedback, Tech. Report 987, Institute for Mathematics and Its Applications, University of Minnesota, 1992.
- [5] A. Bunse-Gerstner, V. Mehrmann, N.K. Nichols, Regularization of descriptor systems by derivative and proportional state feedback, SIAM J. Matrix Anal. Appl. 13 (1992) 46–67.
- [6] A. Bunse-Gerstner, V. Mehrmann, N.K. Nichols, Regularization of descriptor systems by output feedback, IEEE Trans. Automat. Control, AC-39 (1994) 1742–1747.

- [7] R. Byers, T. Geerts, V. Mehrmann, Descriptor systems without controllability at infinity, SIAM J. Control Optim. 35 (1997) 462–479.
- [8] R. Byers, C. He, V. Mehrmann, Where is the nearest non-regular pencil? Linear Algebra Appl. 285 (1998) 81–105.
- [9] R. Byers, P. Kunkel, V. Mehrmann, Regularization of linear descriptor systems with variable coefficients, SIAM J. Control Optim. 35 (1997) 117–133.
- [10] S.L. Campbell, Singular Systems of Differential Equations, Pitman, San Francisco, 1980.
- [11] S.L. Campbell, A general form for solvable linear time varying singular systems of differential equations, SIAM J. Math. Anal. 18 (1987) 1101–1114.
- [12] M. Christodoulou, Decoupling in the design and synthesis of singular systems, Automatica 22 (1986) 245–249.
- [13] D.L. Chu, H.C. Chan, D.W.C. Ho, Regularization of singular systems by derivative and proportional output feedback, SIAM J. Matrix Anal. Appl. 19 (1998) 21–38.
- [14] D.J. Cobb, Feedback and pole placement in descriptor variable systems, Int. J. Control 33 (1981) 1135–1146.
- [15] L. Dai, Singular control systems, Lecture Notes in Control and Information Sciences, vol. 118, Springer, Berlin, 1989.
- [16] J.W. Demmel, B. Kågström, The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$, Parts I and II, ACM Trans. Math. Software 19 (1993) 160–174, 175–201.
- [17] L. Elsner, C. He, V. Mehrmann, Completion of a matrix so that the inverse has minimum norm. Application to the regularization of descriptor control problems, in: P.M. Van Dooren, B. Wyman (Eds.), Linear Algebra for Control Theory, Springer, New York, 1993, pp. 75–86.
- [18] L.R. Fletcher, J. Kautsky, N.K. Nichols, Eigenstructure assignment in descriptor systems, IEEE Trans. Automat. Control AC-31 (1986) 1138–1141.
- [19] F.R. Gantmacher, Theory of Matrices, vol. 2, Chelsea, New York, 1959.
- [20] C.W. Gear, Differential-algebraic index transformations, SIAM J. Sci. Statist. Comput. 9 (1988) 39–47.
- [21] T. Geerts, Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant linear systems: The general case, Linear Algebra Appl. 181 (1993) 111–130.
- [22] H. Glüsing-Lüerssen, D. Hinrichsen, A Jordan control canonical form for singular systems, Int. J. Control 48 (1988) 1769–1785.
- [23] H. Glüsing-Lüerssen, Gruppenaktionen in der Theorie singulärer Systeme, PhD thesis, Universität Bremen, 1991.
- [24] G.H. Golub, C.F.V. Loan, Matrix Computations, second ed., The Johns Hopkins University Press, Baltimore, Maryland, 1989.
- [25] E. Griepentrog, R. März, Differential-algebraic equations and their numerical treatment, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1986, with German, French and Russian summaries.
- [26] E. Griepentrog, R. März, Basic properties of some differential-algebraic equations, Z. Anal. Anwendungen 8 (1989) 25–41.
- [27] U. Helmke, J. Rosenthal, J.M. Schumacher, A controllability test for general first-order representations, Automatica J. IFAC 33 (1997) 193–201.
- [28] U. Helmke, M.A. Shayman, A canonical form for controllable singular systems, Systems Control Lett. 12 (1989) 111–122.
- [29] M. Hou, P.C. Müller, LQ and tracking control of descriptor systems with application to constrained manipulator, tech. report, Sicherheitstechnische Regelungs- und Meßtechnik, Universität Wuppertal, Gauß-Straße 20, D-5600 Wuppertal 1, Germany, 1994.
- [30] J. Kautsky, N.K. Nichols, E.K.-W. Chu, Robust pole assignment in singular control systems, Linear Algebra Appl. 121 (1989) 9–37.
- [31] V. Kucera, D. Zagálak, Fundamental theorem of state feedback for singular systems, Automatica 24 (1988) 653–658.

- [32] P. Kunkel, V. Mehrmann, Canonical forms for linear differential-algebraic equations with variable coefficients, *J. Comput. Appl. Math.* 56 (1994) 225–251.
- [33] P. Kunkel, V. Mehrmann, Local and global invariants of linear differential-algebraic equations and their relation, *Electron. Trans. Numer. Anal.* 4 (1996) 138–157 (electronic).
- [34] P. Kunkel, V. Mehrmann, Regular solutions of nonlinear differential-algebraic equations and their numerical determination, *Numer. Math.* 79 (1998) 581–600.
- [35] A.J. Laub, Numerical linear algebra aspects of control design computations, *IEEE Trans. Automat. Control* 30 (1985) 97–108.
- [36] F.L. Lewis, Descriptor systems: Fundamental matrix, reachability and observability matrices, subspaces, in: Proceedings of 23rd Conference on Decision and Control, Las Vegas, NV, December 1984, vol. I, 1984, pp. 293–298.
- [37] F.L. Lewis, A tutorial on the geometric analysis of linear time-invariant implicit systems, *Automatica J. IFAC* 28 (1992) 119–137.
- [38] J.J. Loiseau, K. Özçaldiran, M. Malabre, N. Karcianas, Feedback canonical forms of singular systems, *Kybernetika* 27 (1991) 289–305.
- [39] D.G. Luenberger, Dynamic equations in descriptor form, *IEEE Trans. Automat. Control* AC-22 (1977) 312–321.
- [40] D.G. Luenberger, A. Arbel, Singular dynamic Leontief systems, *Econometrica* 45 (1977) 991–995.
- [41] R. März, Some new results concerning index-3 differential-algebraic equations *J. Math. Anal. Appl.* 140 (1989) 177–199.
- [42] The MathWorks, PRO-MATLAB Users' Guide, 21 Eliot Street, South Natick, MA 01760, USA 1990.
- [43] V. Mehrmann, Existence, uniqueness and stability of solutions to singular linear quadratic control problems, *Linear Algebra Appl.* 121 (1989) 291–331.
- [44] V. Mehrmann, The Autonomous Linear Quadratic Control Problem: Theory and Numerical Algorithms, Lecture Notes in Control and Information Sciences, vol. 163, Springer, Heidelberg, 1991.
- [45] G. Miminis, Deflation in eigenvalue assignment of descriptor systems using state feedback, *IEEE Trans. Automat. Control* 38 (1993) 1322–1336.
- [46] R. Mukundan, W. Dayawansa, Feedback control of singular systems – proportional and derivative feedback of the state, *Int. J. Syst. Sci.* 14 (1983) 615–632.
- [47] R.W. Newcomb, B. Dziurla, Some circuits and systems applications of semistate theory, *Circuits Systems Signal Process.* 8 (1989) 235–260.
- [48] K. Özçaldiran, F.L. Lewis, On the regularizability of singular systems, *IEEE Trans. Automat. Control* 35 (1990) 1156–1160.
- [49] D.W. Pearson, M.J. Chapman, D.N. Shields, Partial singular value assignment in the design of robust observers for discrete time descriptor systems, *IMA. J. Math. Control and Inf.* 5 (1988) 203–213.
- [50] P.J. Rabier, W.C. Rheinboldt, Classical and generalized solutions of time-dependent linear differential-algebraic equations, *Linear Algebra Appl.* 245 (1996) 259–293.
- [51] M.S. Ravi, J. Rosenthal, A general realization theory for higher-order linear differential equations, *Systems Control Lett.* 25 (1995) 351–360.
- [52] H.H. Rosenbrock, Structural properties of linear dynamic systems, *Int. J. Control* 20 (1974) 191–202.
- [53] M.A. Shayman, Pole placement by dynamic compensation for descriptor systems, *Automatica* 24 (1988) 279–282.
- [54] M.A. Shayman, Z. Zhou, Feedback control and classification of generalized linear systems, *IEEE Trans. Automat. Control* AC-32 (1987) 483–494.
- [55] P.M. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, *Linear Algebra Appl.* 27 (1979) 103–141.
- [56] P.M. Van Dooren, The generalized eigenstructure problem in linear system theory, *IEEE Trans. Automat. Control* 6 (1981) 111–129.

- [57] P.M. Van Dooren, M. Verhaegen, On the use of unitary state-space transformations, *Contemp. Math.* 47 (1985) 447–463.
- [58] G.C. Verghese, B.C. Lévy, T. Kailath, A general state space for singular systems, *IEEE Trans. Automat. Control* AC-26 (1981) 811–831.
- [59] G.C. Verghese, P.M. Van Dooren, T. Kailath, Properties of the system matrix of a generalized state space system, *Int. J. Control* 30 (1979) 235–243.
- [60] J.H. Wilkinson, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford University Press, Walton Street, Oxford OX2 6DP, 1965.
- [61] E.L. Yip, R.F. Sincovec, Solvability, controllability and observability of continuous descriptor systems, *IEEE Trans. Automat. Control* AC-26 (1981) 702–707.
- [62] Z. Zhou, M.A. Shayman, T.-J. Tarn, Singular systems: A new approach in the time domain, *Int. J. Control* AC-32 (1987) 42–50.