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ORIGINAL RESEARCH

Finite time stability analysis of systems based on delayed exponential matrix

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Abstract In this paper, we analyze finite time stability for a class of differential equations with finite delay. Some sufficient conditions for the finite time stability results are derived based on delayed matrix exponential approach and Jensen's and Coppel's inequalities. Finally, we demonstrate the validity of designed method and make some discussions by using a numerical example.

Keywords Differential equations with delay · Finite time stability · Delayed exponential matrix

Mathematics Subject Classification 34A30 · 34D20

1 Introduction

In order to maintain a stable state of system, feedback control idea is used to deal with modeling automatic engines and physiological systems. In general, feedback control require a finite time to sense information and react to it. Thus, delay differential equation involving the evolution of a dependent variable at time t depends on its value at time $t-\tau$ is formulated to describe this process. However, deriving exact formula of solution to delay differential equation $\dot{x}(t) = Ax(t) + Bx(t-\tau)$ is a mathematically difficult task.

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In recent years, stability analysis of delay differential equations has been gradually attracted in the fields of mathematics and automatic control. Various mathematical methods as well as new stability concepts are explored to deal with such problem. For example, a powerful technique, namely Razumikhin technique, is successfully applied to study exponential stability problem [1–4]. Meanwhile, Gronwall's integral inequality is always regarded as a vital method to discuss finite time stability [5–7]. In addition, some authors propose sufficient conditions to guarantee finite time stability in the forms of linear matrix inequality (LMI), Lyapunov differential matrix equation, or algebraic inequality [8–14]. For more recent results on complex network, one can refer to [15–18].

After reviewing the previous literatures dealing with finite time stability problems for delay systems, we observe the following facts:

- Delay system $\dot{x}(t) = Ax(t) + Bx(t-\tau), t > 0$ is considered mostly as integral system, where A, B are suitable matrices.
- A uniform transition matrix associated with A, B is not computed directly and the structure of solution $x(t)$ is not well characterized on every subintervals $[j\tau, (j+1)\tau]$, $j \in N$.
- Gronwall inequality, LMI, Lyapunov function methods are used to obtain finite time stability results.

Very recently, Khusainov and Shuklin [19] initially introduce delayed exponential matrix $e_\tau^{B_1 t}$, $B_1 = e^{-A\tau}B$ to give a representation of a solution of a linear system with $AB = BA$ and one delay term. This extend the associated results in text books on ordinary differential equations that in the linear case without the delay term. Thereafter, oscillating system with pure delay, relative controllability of system with pure delay and multidelay differential equations and linear discrete delay systems are studied [20–24].

Now, we are concerned with the issue: Can we apply delayed exponential matrix in [19] to establish directly method to find the sufficient conditions to guarantee the finite time stability? If yes, we will provide another directly method to find some conditions to make the desired delay system to be finite time stability, which is a supplement of the current methods in some sense.

In this paper, we study finite time stability of the following linear system with delay:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau), & t \in J := [0, T], \\ x(t) = \varphi(t), & -\tau \le t \le 0,\ \tau > 0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, T denotes pre-fixed iteration domain length and without loss of generality, we set $T = N\tau$ and $N \in \mathbb{N}$, $\varphi \in C_\tau^1 := C^1([-\tau, 0], \mathbb{R}^n)$, and the $n \times n$ matrices A and B satisfy $AB = BA$. According to [19, Remark 1.3], any solution of system (1) has the form

$$x(t) = X_0(t)e^{A\tau}\varphi(-\tau) + \int_{-\tau}^{0} X_0(t-\tau-s)e^{A\tau}[\varphi'(s) - A\varphi(s)]ds, \tag{2}$$

where

$$X_0(t) = e^{At} e_\tau^{B_1 t},\ B_1 = e^{-A\tau} B, \tag{3}$$

is called fundamental matrix of (1) and delayed exponential matrix $e_\tau^{B_1 t}$ is defined by

$$e_\tau^{B_1 t} = \begin{cases} \Theta,\ t < -\tau, \\ E,\ \ -\tau \le t < 0, \\ E + B_1 t + B_1^2 \dfrac{(t-\tau)^2}{2} + \cdots + B_1^k \dfrac{(t-(k-1)\tau)^k}{k!}, \\ \quad (k-1)\tau \le t < k\tau,\ k = 1, 2, \cdots, \end{cases} \tag{4}$$

where Θ and E are the zero and identity matrices, respectively. Here we remark that the condition $AB = BA$ and $T = N\tau$ provide the convenient to derive the formula of (2). If not, the formula of (2) will be very complex due to (4). Of course, $T = N\tau$ is not essential, one can extend to $T > 0$ and one can also use (4) to describe delayed exponential matrix.

The main objective of this paper is to utilize delayed exponential matrix methods with Jensen's and Coppel's inequalities to discuss the finite time stability of the equations with single delay.

2 Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional spaces with the scalar product $x^\top y$ and the vector norm $\|\cdot\|$, $R^{n\times n}$ denotes an $n\times n$ matrix with real value elements, $X^\top$ denotes the transpose of the matrix X, and $\lambda_{\max}(X)$ denotes the maximum eigenvalue of the matrix X. Denote by $C(J, \mathbb{R}^n)$ the Banach space of vector-value continuous functions from $J \to \mathbb{R}^n$ endowed with the norm $\|x\|_C = \max_{t\in J} \|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(J, \mathbb{R}^n) = \{x \in C(J, \mathbb{R}^n) : \dot{x} \in C(J, \mathbb{R}^n)\}$. For $A:\mathbb{R}^n \to \mathbb{R}^n$, we consider its matrix norm $\|A\| = \max_{\|x\|=1} \|Ax\|$ generated by $\|\cdot\|$. In addition, we note $\|\varphi\|_C = \max_{\theta\in[-\tau,0]} \|\varphi(\theta)\|$.

Definition 2.1 (see [5, Definition 1]) The system (1) is finite time stable with respect to $\{0, J, \alpha, \beta, \tau\}$, if and only if $\varphi^\top(t)\varphi(t) < \alpha,\ \forall\, t \in [-\tau, 0]$, implies

$$x^\top(t)x(t) < \beta,\ \forall\, t \in J,$$

where α is a real positive number and $\beta \in \mathbb{R}$ with $\beta > \alpha$.

Definition 2.2 (see [5, Definition 2]) The system (1) is finite time stable with respect to $\{0, J, \alpha, \beta, \tau\}$, if and only if $\|\varphi\|_C^2 < \alpha$ implies

$$\|x(t)\|^2 < \beta,\ \forall\, t \in J.$$

Remark 2.3 Definition 2.1 coincides with Definition 2.2 when $\|\cdot\|$ be an Euclidean norm.

Definition 2.4 (see [25] or [8]) The matrix measure (or matrix logarithmic norm) of X is defined as

$$\mu(X) = \lim_{h \to 0^+} \frac{\|E + hX\| - 1}{h}.$$

In particular, if $\|\cdot\|$ denotes the 2-norm of matrix, then

$$\mu(X) = \frac{1}{2}\lambda_{\max}(X + X^\top),$$

where $\lambda_{\max}(Y)$ denotes the maximum eigenvalue of the matrix Y.

This norm is called the Lozinski matrix norm and this norm was then used to characterize stability and error bounds in the numerical integration of ordinary differential equations [26].

Lemma 2.5 (see [21, Lemma 3]) *For all* $t \in \mathbb{R}$*,*

$$\|e_\tau^{Bt}\| \le e^{\|B\|(t+\tau)}.$$

Lemma 2.6 (see [19, Lemma 4]) *For all* $t \in \mathbb{R}$*,*

$$\frac{d}{dt} e_\tau^{Bt} = B e_\tau^{B(t-\tau)}.$$

Lemma 2.7 (see [10, Lemma 1, Jensen's integral inequality]) *For any positive symmetric constant matrix* $W \in \mathbb{R}^{n\times n}$*, scalars* a*,* b *satisfying* $a < b$*, a vector function* $f : [a, b] \to \mathbb{R}^n$ *such that the integrations are well defined, then*

$$\left(\int_a^b f(s)ds\right)^\top W\left(\int_a^b f(s)ds\right) \le (b-a)\int_a^b f^\top(s)Wf(s)ds.$$

Lemma 2.8 (see [27] or [9, Lemma 2, Coppel's inequality]) *For any real square matrix* $W \in \mathbb{R}^{n\times n}$ *and scalar variable* t*, the expression*

$$\lambda_{\max}(e^{Wt}e^{W^\top t}) \le e^{2\mu(W)t}$$

holds, where $\mu(W)$ *is matrix measure of the matrix* W *defined in Definition 2.4.*

Lemma 2.9 (see [9, Lemma 3]) *For any vector* $u, v \in \mathbb{R}^n$*, symmetric positive definite matrix* $\mathbb{R}^{n\times n} \ni \Gamma = \Gamma^{-1} > 0$*, the inequalities*

$$2u(t)v(t) \le u^\top(t)\Gamma u(t) + v^\top(t)\Gamma^{-1}v(t),$$

and

$$-2u(t)v(t) \le u^\top(t)\Gamma u(t) + v^\top(t)\Gamma^{-1}v(t)$$

hold.

3 Finite time stability results

In this section, we are ready to present the finite time stability results by providing a different approach which is differ from previous methods.

Case I. Finite time stability results based on Definition 2.1.

Theorem 3.1 *The system* (1) *is finite time stable with respect to* $\{0, J, \alpha, \beta, \tau\}$, *if*

$$\|X_0(t)\| < \frac{\sqrt{\beta}}{\|e^{A\tau}\|\sqrt{N(\alpha)}}, \ \forall\, t \in J, \tag{5}$$

where $X_0(t)$ *is given in* (3), α, β *are defined in Definition* 2.1 *and*

$$N(\alpha) := \alpha + 2\sqrt{\alpha}\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|ds + \left(\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|ds\right)^2. \tag{6}$$

Proof Let x be the solution of system (1) satisfying (2). Denote $a(t) := X_0(t)e^{A\tau}\varphi(-\tau) \in \mathbb{R}^n$, and $d(t) := \int_{-\tau}^{0} X_0(t-\tau-s)e^{A\tau}[\varphi'(s) - A\varphi(s)]ds \in \mathbb{R}^n$. Clearly, $a^\top(t)d(t) = d^\top(t)a(t)$. For any $t \in J$, one can get

$$\begin{aligned}
x(t)^\top x(t) &= \left(X_0(t)e^{A\tau}\varphi(-\tau) + \int_{-\tau}^{0} X_0(t-\tau-s)e^{A\tau}[\varphi'(s) - A\varphi(s)]ds\right)^\top \\
&\quad \times\left(X_0(t)e^{A\tau}\varphi(-\tau) + \int_{-\tau}^{0} X_0(t-\tau-s)e^{A\tau}[\varphi'(s) - A\varphi(s)]ds\right) \\
&= \left(e^{A\tau}\varphi(-\tau)\right)^\top X_0^\top(t)X_0(t)e^{A\tau}\varphi(-\tau) + a^\top(t)d(t) \\
&\quad + d^\top(t)a(t) + d^\top(t)d(t) \\
&\le \lambda_m(t)(e^{A\tau}\varphi(-\tau))^\top e^{A\tau}\varphi(-\tau) + 2a^\top(t)d(t) + \|d(t)\|^2,
\end{aligned} \tag{7}$$

where $\lambda_m(t) = \max_{t\in J}\rho\{X_0^\top(t)X_0(t)\}$ and $\rho\{Y\}$ denotes the spectrum of the matrix Y.

For the term $a^\top(t)d(t)$, we have

$$\begin{aligned}
a^\top(t)d(t) &= (e^{A\tau}\varphi(-\tau))^\top X_0^\top(t)\int_{-\tau}^{0} X_0(t-\tau-s)e^{A\tau}[\varphi'(s) - A\varphi(s)]ds \\
&\le \|\varphi(-\tau)^\top\|\|e^{A\tau}\|\|X_0^\top(t)\|\int_{-\tau}^{0} \|X_0(t-\tau-s)\|\|e^{A\tau}\|\|\varphi'(s) - A\varphi(s)\|ds.
\end{aligned} \tag{8}$$

Let $\eta = t - \tau - s$, $s \in [-\tau, 0]$. Then, $\eta \in [t-\tau, t] \subseteq [t-\tau, T] \subseteq J$. Thus,

$$\|X_0(\eta)\| \mid_{\eta \in [t-\tau,t]} \leq \|X_0(t)\| \mid_{t \in [t-\tau,T]} \leq \|X_0(t)\| \mid_{t \in J} . \tag{9}$$

Liking (7) and (8) via (9) we can obtain

$$\begin{aligned} x(t)^\top x(t) \leq \lambda_m(t)(e^{A\tau}\varphi(-\tau))^\top e^{A\tau}\varphi(-\tau) \\ +2\|\varphi(-\tau)^\top\|\|e^{A\tau}\|\|X_0^\top(t)\|\|X_0(t)\|\|e^{A\tau}\| \int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\| ds \\ +\|X_0(t)\|^2\|e^{A\tau}\|^2 \left(\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\| ds\right)^2 . \end{aligned} \tag{10}$$

By using the fact

$$\lambda_m(t) \leq \|X_0^\top(t) X_0(t)\| \leq \|X_0^\top(t)\|\|X_0(t)\| = \|X_0(t)\|^2,$$

the formula (10) becomes to

$$\begin{aligned} x(t)^\top x(t) \leq \|X_0(t)\|^2\|e^{A\tau}\|^2\varphi^\top(-\tau)\varphi(-\tau) \\ +2\|\varphi(-\tau)^\top\|\|X_0(t)\|^2\|e^{A\tau}\|^2 \int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\| ds \\ +\|X_0(t)\|^2\|e^{A\tau}\|^2 \left(\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\| ds\right)^2 . \end{aligned} \tag{11}$$

Next, we choose $\varphi^\top(t)\varphi(t) < \alpha, \ \forall\, t \in [-\tau, 0]$ which implies $\|\varphi^\top(-\tau)\| < \sqrt{\alpha}$. Finally, applying the condition (5) to the preceding the above inequality, then (11) reduces

$$x(t)^\top x(t) \leq \|X_0(t)\|^2\|e^{A\tau}\|^2 N(\alpha) < \beta,$$

where $N(\alpha)$ is given in (6). According to Definition 2.1, the system (1) is finite time stable.

Theorem 3.2 *The system* (1) *is finite time stable with respect to* $\{0, J, \alpha, \beta, \tau\}$, *if*

$$e^{M(t+\tau)} < \frac{\beta}{2\left(\alpha + \tau \int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|^2 ds\right)}, \ t \in J, \tag{12}$$

where $M = 2\mu(A^\top) + 2\|B_1\|$, α, β *are defined in Definition* 2.1.

Proof Let x be the solution of system (1) satisfying (2). By using Lemma 2.9, we have

$$
\begin{aligned}
&x(t)^\top x(t)\\
&\quad\le \varphi^\top(-\tau)(e_\tau^{B_1t})^\top (e^{A(t+\tau)})^\top e^{A(t+\tau)} e_\tau^{B_1t}\varphi(-\tau)\\
&\qquad + \left[e^{A(t+\tau)} e_\tau^{B_1t}\varphi(-\tau)\right] E \left[e^{A(t+\tau)} e_\tau^{B_1t}\varphi(-\tau)\right]^\top + \Upsilon\\
&\qquad + \int_{-\tau}^0 (e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)])^\top ds\\
&\qquad \times \int_{-\tau}^0 e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)] ds.
\end{aligned}
$$

where

$$
\begin{aligned}
\Upsilon = &\int_{-\tau}^0 (e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)])^\top ds E^{-1}\\
&\times \int_{-\tau}^0 e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)] ds.
\end{aligned}
$$

Further, by Lemmas 2.7 and 2.8,

$$
\begin{aligned}
&x(t)^\top x(t)\\
&\quad\le 2\lambda_{\max}\left(\left(e^{A(t+\tau)}\right)^\top e^{A(t+\tau)}\right)\varphi^\top(-\tau)(e_\tau^{B_1t})^\top e_\tau^{B_1t}\varphi(-\tau)\\
&\qquad +2\tau \int_{-\tau}^0 (e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)])^\top e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)] ds\\
&\quad\le 2e^{2\mu(A^\top)(t+\tau)}\varphi^\top(-\tau)(e_\tau^{B_1t})^\top e_\tau^{B_1t}\varphi(-\tau)\\
&\qquad +2\tau \int_{-\tau}^0 (e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)])^\top e^{A(t-s)} e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)] ds\\
&\quad\le 2e^{2\mu(A^\top)(t+\tau)}\varphi^\top(-\tau)(e_\tau^{B_1t})^\top e_\tau^{B_1t}\varphi(-\tau)\\
&\qquad +2\tau \int_{-\tau}^0 e^{2\mu(A^\top)(t-s)}([\varphi'(s) - A\varphi(s)])^\top (e_\tau^{B_1(t-\tau-s)})^\top e_\tau^{B_1(t-\tau-s)}[\varphi'(s) - A\varphi(s)] ds.
\end{aligned} \tag{13}
$$

Notice that $\|Y^\top\| = \|Y\|$, $Y \in \mathbb{R}^{n\times n}$, $\|\varphi(-\tau)\|^2 = \varphi^\top(-\tau)\varphi(-\tau)$. Applying Lemma 2.5 to (13), one has

$$
\begin{aligned}
x(t)^\top x(t) &\le 2e^{M(t+\tau)}\varphi^\top(-\tau)\varphi(-\tau) + 2\tau \int_{-\tau}^0 e^{M(t-s)}\|\varphi'(s) - A\varphi(s)\|^2 ds\\
&\le 2e^{M(t+\tau)}\alpha + 2\tau e^{M(t+\tau)} \int_{-\tau}^0 \|\varphi'(s) - A\varphi(s)\|^2 ds\\
&\le 2e^{M(t+\tau)}\left[\alpha + \tau \int_{-\tau}^0 \|\varphi'(s) - A\varphi(s)\|^2 ds\right].
\end{aligned} \tag{14}
$$

Linking (14) and (12), we obtain $x^\top(t)x(t) < \beta$, $\forall\, t \in J$. According to Definition 2.1, the system (1) is finite time stable. $\square$

Theorem 3.3 *The system* (1) *is finite time stable with respect to* $\{0, J, \alpha, \beta, \tau\}$, *if*

$$4e^{Mt}\left[1+\frac{2\tau\lambda_{\max}(A^{\top}A)}{M}(e^{M\tau}-1)+\frac{\tau\lambda_{\max}(B^{\top}B)}{M}(1-e^{-M\tau})\right]<\frac{\beta}{\alpha},\ t\in J, \tag{15}$$

where $M\neq 0$, α, β *are defined in Definition 2.1.*

Proof According to (2) and integration by parts via Lemma 2.6, the solution of system (1) can be expressed

$$\begin{aligned}x(t)&=e^{At}e_{\tau}^{B_1(t-\tau)}\varphi(0)-2\int_{-\tau}^{0}Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s)ds\\&\quad-\int_{-\tau}^{0}Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s)ds.\end{aligned}$$

Analogy to the proof process of the Theorem 3.2, one can get

$$\begin{aligned}x^{\top}(t)x(t)\le{}&(e^{At}e_{\tau}^{B_1(t-\tau)}\varphi(0))^{\top}e^{At}e_{\tau}^{B_1(t-\tau)}\varphi(0)\\&-4(e^{At}e_{\tau}^{B_1(t-\tau)}\varphi(0))^{\top}\int_{-\tau}^{0}Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s)ds\\&-2(e^{At}e_{\tau}^{B_1(t-\tau)}\varphi(0))^{\top}\int_{-\tau}^{0}Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s)ds\\&+4\int_{-\tau}^{0}(Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s))^{\top}ds\\&\times\int_{-\tau}^{0}Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s)ds\\&+4\tau\int_{-\tau}^{0}(Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s))^{\top}Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s)ds\\&+\tau\int_{-\tau}^{0}(Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s))^{\top}Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s)ds.\end{aligned}$$

By Lemmas 2.7 and 2.9, we have

$$x^{\top}(t)x(t)\le 4e^{2\mu(A^{\top})t}\varphi^{\top}(0)(e_{\tau}^{B_1(t-\tau)})^{\top}e_{\tau}^{B_1(t-\tau)}\varphi(0)+8\tau P_1+4\tau P_2, \tag{16}$$

where

$$\begin{aligned}P_1&=\int_{-\tau}^{0}(Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s))^{\top}Ae^{A(t-s)}e_{\tau}^{B_1(t-\tau-s)}\varphi(s)ds,\\P_2&=\int_{-\tau}^{0}(Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s))^{\top}Be^{A(t-\tau-s)}e_{\tau}^{B_1(t-2\tau-s)}\varphi(s)ds.\end{aligned}$$

By Lemma 2.8,

$$
\begin{aligned}
P_1 &\le \lambda_{\max}(A^\top A)\int_{-\tau}^{0}(e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}\varphi(s))^\top e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}\varphi(s)ds\\
&\le \lambda_{\max}(A^\top A)\int_{-\tau}^{0}\lambda_{\max}[(e^{A(t-s)})^\top(e^{A(t-s)})](e_\tau^{B_1(t-\tau-s)}\varphi(s))^\top e_\tau^{B_1(t-\tau-s)}\varphi(s)ds\\
&\le \lambda_{\max}(A^\top A)\int_{-\tau}^{0}e^{2\mu(A^\top)(t-s)}(e_\tau^{B_1(t-\tau-s)}\varphi(s))^\top e_\tau^{B_1(t-\tau-s)}\varphi(s)ds\\
&\le \lambda_{\max}(A^\top A)\int_{-\tau}^{0}e^{2\mu(A^\top)(t-s)}e^{\|B_1\|(t-s)}\varphi(s)^\top\varphi(s)ds\\
&\le \lambda_{\max}(A^\top A)\alpha\int_{-\tau}^{0}e^{M(t-s)}ds\\
&\le \frac{\lambda_{\max}(A^\top A)\alpha}{M}e^{Mt}(e^{M\tau}-1), \qquad (17)
\end{aligned}
$$

where $M=2\mu(A^\top)+2\|B_1\|$. Similarly, we can get

$$
P_2 \le \frac{\lambda_{\max}(B^\top B)\alpha}{M}e^{Mt}(1-e^{-M\tau}). \qquad (18)
$$

Submitting (17) and (18) to (16), one can get

$$
\begin{aligned}
&x^\top(t)x(t)\\
&\le 4e^{2\mu(A^\top)t}e^{2\|B_1\|t}\alpha+\frac{8\tau\lambda_{\max}(A^\top A)\alpha}{M}e^{Mt}(e^{M\tau}-1)\\
&\quad+\frac{4\tau\lambda_{\max}(B^\top B)\alpha}{M}e^{Mt}(1-e^{-M\tau})\\
&\le 4e^{Mt}\alpha\left[1+\frac{2\tau\lambda_{\max}(A^\top A)\alpha}{M}(e^{M\tau}-1)+\frac{\tau\lambda_{\max}(B^\top B)\alpha}{M}(1-e^{-M\tau})\right]<\beta,
\end{aligned}
$$

due to the condition (15). According to Definition 2.1, the system (1) is finite time stable.

Case II. Finite time stability results based on Definition 2.2.

Theorem 3.4 *The system* (1) *is finite time stable with respect to* $\{0, J, \alpha, \beta, \tau\}$, *if*

$$
e^{(\|A\|+\|B_1\|)(t+\tau)} < \frac{\sqrt{\beta}}{\sqrt{\alpha}+\int_{-\tau}^{0}\|\varphi'(s)-A\varphi(s)\|ds},\ t\in J, \qquad (19)
$$

where α, β *are defined in Definition* 2.2.

Proof Link to (2) and Lemma 2.5, it is easy to get

$$\begin{aligned}
\|x(t)\| &\le \|e^{A(t+\tau)}e_\tau^{B_1 t}\varphi(-\tau)\| + \int_{-\tau}^{0} \|e^{A(t-s)}\|\|e_\tau^{B_1(t-\tau-s)}\|\|\varphi'(s) - A\varphi(s)\|ds \\
&\le e^{(\|A\|+\|B_1\|)(t+\tau)}\|\varphi(-\tau)\| + e^{(\|A\|+\|B_1\|)t}\int_{-\tau}^{0} e^{-(\|A\|+\|B_1\|)s}\|\varphi'(s) - A\varphi(s)\|ds \\
&\le e^{(\|A\|+\|B_1\|)(t+\tau)}\|\varphi(-\tau)\| + e^{(\|A\|+\|B_1\|)(t+\tau)}\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|ds \\
&\le e^{(\|A\|+\|B_1\|)(t+\tau)}\sqrt{\alpha} + e^{(\|A\|+\|B_1\|)(t+\tau)}\int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|ds \\
&\le e^{(\|A\|+\|B_1\|)(t+\tau)}\left[\sqrt{\alpha} + \int_{-\tau}^{0} \|\varphi'(s) - A\varphi(s)\|ds\right].
\end{aligned} \tag{20}$$

Using the condition (19), one can get $\|x(t)\| < \sqrt{\beta}$, $\forall\, t \in J$. According to Definition 2.2,

$$\|x(t)\|^2 < \beta, \ \forall\, t \in J.$$

Thus, the system (1) is finite time stable.

Theorem 3.5 *The system* (1) *is finite time stable with respect to* $\{0, J, \alpha, \beta, \tau\}$, *if*

$$e^{2Nt}\left[1 + (2\|A\| + \|B\|)\frac{1}{N}(1 - e^{-N\tau})\right]^2 < \frac{\beta}{\alpha}, \ \forall\, t \in J, \tag{21}$$

where $N = \|A\| + \|B_1\| \ne 0$ *and* α, β *are given in Definition* 2.2.

Proof According to (2), the solution of system (1) can be expressed

$$\begin{aligned}
x(t) = e^{A(t+\tau)}e_\tau^{B_1 t}\varphi(-\tau) &+ \int_{-\tau}^{0} e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}\varphi'(s)ds \\
&- \int_{-\tau}^{0} e^{A(t-s)}e_\tau^{B_1(t-\tau-s)}A\varphi(s)ds.
\end{aligned} \tag{22}$$

Integration by parts via Lemma 2.6, one can change the expression (22) to the form

$$\begin{aligned}
x(t) = e^{At}e_\tau^{B_1(t-\tau)}\varphi(0) &- 2\int_{-\tau}^{0} Ae^{A(t-s)}e_\tau^{B_1(t-\tau-s)}\varphi(s)ds \\
&- \int_{-\tau}^{0} Be^{A(t-\tau-s)}e_\tau^{B_1(t-2\tau-s)}\varphi(s)ds.
\end{aligned} \tag{23}$$

Taking norm on the both side of (23) and using Lemma 2.5, we have

$$\begin{aligned}\|x(t)\| &\leq e^{Nt}\|\varphi\|_C + 2\|\varphi\|_C \int_{-\tau}^{0} \|A\| e^{N(t-s)} ds + \|\varphi\|_C \int_{-\tau}^{0} \|B\| e^{N(t-\tau-s)} ds \\ &\leq \|\varphi\|_C e^{Nt}\left[1 + (2\|A\| + \|B\|)\frac{1}{N}(1 - e^{-N\tau})\right]. \end{aligned} \tag{24}$$

Since $\|\varphi\|_C \leq \alpha$, (24) implies

$$\|x(t)\|^2 \leq \alpha e^{2Nt}\left[1 + (2\|A\| + \|B\|)\frac{1}{N}(1 - e^{-N\tau})\right]^2 < \beta,$$

due to the condition (21). Thus, the system (1) is finite time stable. □

4 Numerical examples and discussions

In this section, we give numerical example to demonstrate the validity of our method and make some discussions with the inspired literature [5].

Let us consider the following linear delay system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - 0.2), \quad x(t) \in \mathbb{R}^2, \ t \in [0, 1.4], \\ \varphi(t) = (-0.3, -0.4)^\top, \quad -0.2 \leq t \leq 0, \end{cases} \tag{25}$$

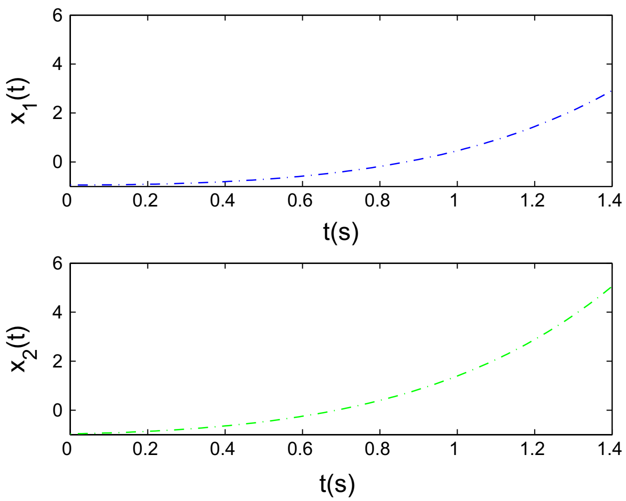

Fig. 1 The state response $x(t)$ of (25) when $T = 1.4$

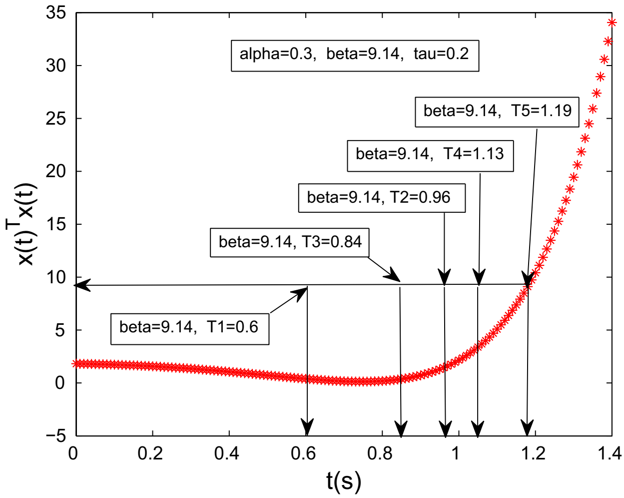

Fig. 2 The square norm of the state vector of (25) when $T = 1.4$

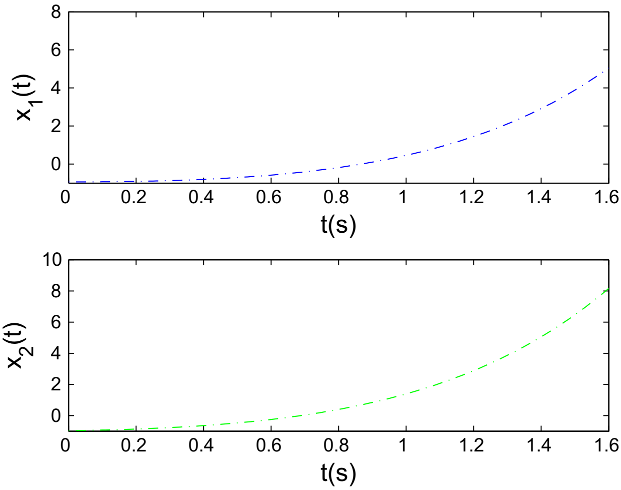

Fig. 3 The state response $x(t)$ of (25) when $T = 1.6$

where we set $T = 1.4$, $\tau = 0.2$, and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad AB = BA.$$

It is obvious that: $\varphi^\top(t)\varphi(t) = 0.25 < \alpha := 0.3,\ t \in [-0.2, 0]$. By elementary calculation, one has

$$B_1 = \begin{pmatrix} 0.1637 & 0 \\ 0 & 0.1637 \end{pmatrix}, \ e^{A\tau} = \begin{pmatrix} 1.2214 & 0 \\ 0 & 1.2214 \end{pmatrix}, \ \|e^{A\tau}\| = 1.2214.$$

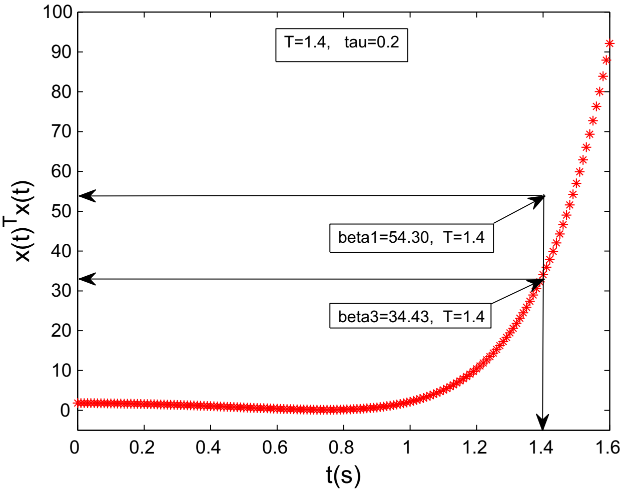

Fig. 4 The square norm of the state vector of (25) when $T = 1.4$

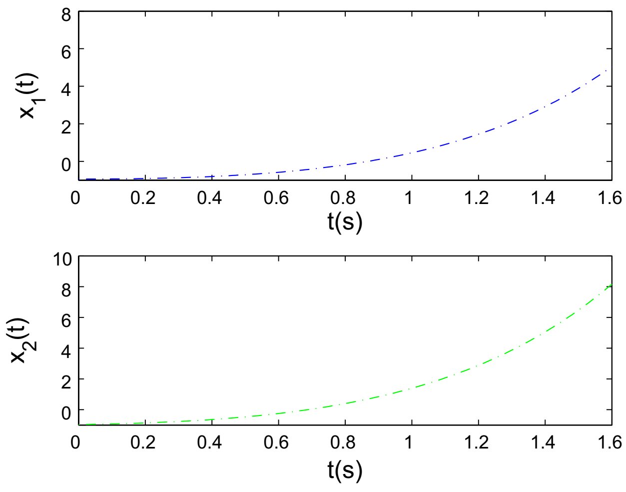

Fig. 5 The state response $x(t)$ of (25) when $T = 1.6$

Next, $\mu(A^\top) = 1, \ \|B_1\| = 0.1637,$

$$\int_{-0.2}^{0} \|\varphi'(s) - A\varphi(s)\|^2 ds = 0.05, \quad e^{2(\mu(A^\top)+\|B_1\|)(T+\tau)} = 6.4361.$$

Moreover, $M = 2.3274$, $\lambda_{\max}(A^\top A) = 1$, $\lambda_{\max}(B^\top B) = 0.04$, $4e^{MT} = 16.1625$, $e^{M\tau} = 1.5928$ and $e^{-M\tau} = 0.6278$.

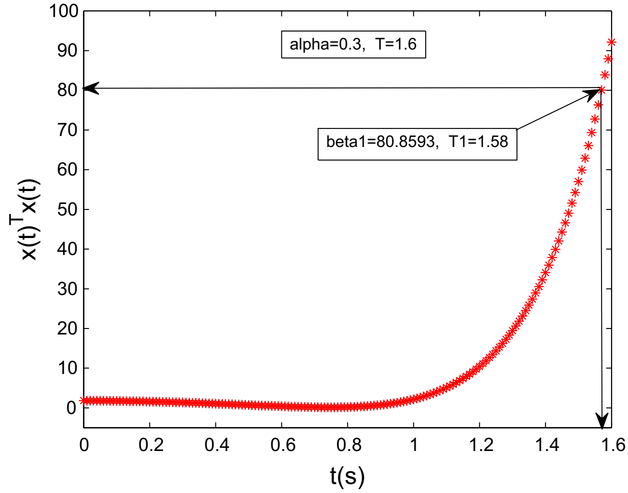

Fig. 6 The square norm of the state vector of (25) when $T = 1.6$

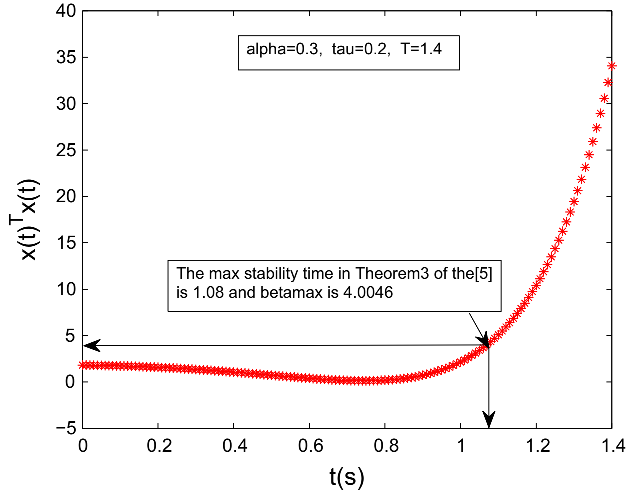

Fig. 7 The square norm of the state vector of (25) when $T = 1.4$

4.1 Discussions

(i) From Fig. 2, one can find the corresponding finite stability time when choosing $\beta = 9.14$. According to Theorem 3.5, we can conclude t firstly achieves the maximum stability time at 1.19s. In addition, Theorems 3.2, 3.4 and 3.5 are invalid by elementary computation via Fig. 4. Meanwhile, Theorems 3.1 and 3.3 are still available. However, when $T > 1.4s$, Theorem 3.3 is invalid too. Next, one can find maximum finite stability time at 1.81s in the sense of Theorem 3.1 from Fig. 6. Figure 7 shows the maximum finite stability time $T_w = 1.08s$ according to Theorem 3 in [5] (Figs. 1, 3, 5).

Table 1 $\alpha = 0.3, \tau = 0.2$

Theorem	$T_{\max}$	$\beta_{\max}$
3.1	1.58	80.86
3.2	1.31	21.45
3.3	1.40	34.43
3.4	1.20	10.75
3.5	1.19	9.14
Theorem 2 in [5]	1.81	225.24
Theorem 3 in [5]	1.08	4.0046

Table 2 $\alpha = 0.3,$ $\tau = 0.2,\ \beta = 4.0046$

Theorem	T_w (stability time)
3.1	0.22
3.2	0.61
3.3	0.48
3.4	0.77
3.5	0.83
Theorem 2 in [5]	0.62
Theorem 3 in [5]	1.08

Table 3 $\alpha = 0.3,$ $s\tau = 0.2,\ \beta = 9.14$

Theorem	T_w
3.1	0.6
3.2	0.96
3.3	0.84
3.4	1.13
3.5	1.19
Theorem 2 in [5]	0.85
Theorem 3 in [5]	Invalid

Table 4 $\alpha = 0.3,$ $\tau = 0.2,\ T_w = 0.6$

Theorem	β
3.1	9.2
3.2	3.9
3.3	5.23
3.4	2.05
3.5	2.30
Theorem 2 in [5]	3.74
Theorem 3 in [5]	1.26

(ii) Table 1 shows that we can find the maximum stability time and the corresponding maximum β; Tables 2 and 3 tell us the corresponding stability time T_w when α and β are given; Table 4 shows that we can find the corresponding β when α, τ and T_w are given.

5 Conclusions

By virtue of delayed exponential matrix corresponding to linear delay continuous system, like transition matrix for classical linear continuous system, we provide another method different from Gronwall inequality, Lyapunov function and LMI approach to study finite time stability of linear delay continuous system. New criterias to guarantee finite time stability are derived by using a direct formula solution.

- Theorem 3.1 shows that one can make (1) is finite time stable by putting restriction on the fundamental delay matrix $X_0(t)$ directly.
- Theorem 3.2 shows that one can make (1) is finite time stable by putting conditions associated with matrix measure for $A^\top$ and $\|B_1\|$.
- Theorem 3.3 shows that one can make (1) is finite time stable by putting conditions associated with matrix measure for $A^\top$ and $\|B_1\|$ and maximum eigenvalue for $A^\top A$. Thus, the conditions in Theorem 3.3 seem a little strong than Theorem 3.2.
- Theorems 3.4, 3.5 show that one can make (1) is finite time stable by putting conditions associated with matrix norm $\|A\|$ and $\|B_1\|$. Theorems 3.1, 3.4, 3.5 are related, however, presentation are different.

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