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State bounding for positive coupled differential-difference equations with bounded disturbances

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Abstract: In this study, the problem of finding state bounds is considered, for the first time, for a class of positive time-delay coupled differential-difference equations (CDDEs) with bounded disturbances. First, the authors present a novel method, which is based on non-negative matrices and optimisation techniques, for computing a like-exponential componentwise upper bound of the state vector of the CDDEs without disturbances. The main idea is to establish bounds of the state vector on finite-time intervals and then, by using the solution comparison method and the linearity of the system, extend to infinite time horizon. Next, by using state transformations, they extend the obtained results to a class of CDDEs with bounded disturbances. As a result, componentwise upper bounds, ultimate bounds and invariant set of the perturbed system are obtained. The feasibility of the obtained results is illustrated through a numerical example.

1 Introduction

Coupled differential-difference equations (CDDEs) are dynamical systems, which include a differential equation coupled with a difference equation. Owing to the fact that there are many systems in engineering such as electrical systems, fluid systems, neutral systems and so on, which are described by (or reformulated into) CDDEs, the stability problem for classes of CDDEs has attracted much research attention during the past few decades [1–10]. The most widely used method to this problem is based on the Lyapunov–Krasovskii functionals combining with linear matrix inequality technique (see [3–7] and the references therein). Recently, by exploiting the properties of Metzler/Schur matrices combining with the comparison method, Shen and Zheng [8], for the first time, presented another method and reported a new result on the stability of a class of positive CDDEs with bounded time-varying delays. The topic on stability analysis of positive CDDEs by using the second method has gained a quickly increasing research attention in the recent years [9–14].

Disturbances are usually unavoidable in practical engineering systems due to many reasons such as external noises, measurement errors, modelling inaccuracies, linear approximation and so on. In the presence of disturbances, in general, it is hard to achieve asymptotic stability for perturbed dynamical systems. However, under the assumption that disturbances are bounded by a known bound, the boundedness/the convergence within a bounded set can be guaranteed. Hence, the topic on state bounding/reachable set bounding/robust convergence for classes dynamical systems perturbed by unknown-but-bounded disturbances has been an important issue in control theory and has attracted significant research attention during the past few decades [15–32]. There are two commonly used approaches to this problem. For classes of linear systems whose matrices are constant, the most widely used approach is based on the Lyapunov method combining with linear matrix inequality technique [17–26]. For classes of positive linear systems, the second widely used approach is based on the properties of Metzler/Schur matrices combining with the solution comparison method [27–32]. It is worthy to note that the second approach combining with the comparison method is also very useful for classes of non-linear/time-varying systems, which are bounded by positive linear systems [30–32].

By using the first approach, Feng and Lam [19] reported a result on reachable set bounding for a class of perturbed time-delay singular systems, which includes a class of CDDEs as a special case. Later, some of its extensions to more general classes of singular systems has also been given in [23, 24]. However, so far, there has not been any result, which is obtained by using the second approach, on state bounding for positive CDDEs/singular systems with bounded disturbances. Motivated by this, we study the problem of finding state bounds for a class of positive CDDEs with bounded disturbances by using the second approach. We present a novel method to derive componentwise state bounds for the positive CDDEs with bounded disturbances, including three main steps: (i) finite-time convergence for linear positive systems; (ii) like-exponential componentwise upper bound for CDDEs without disturbances; and (iii) state bounding for CDDEs with bounded disturbances.

The paper is organised as follows. After the introduction, the problem statement and preliminaries are introduced in Section 2. The main results are given in Section 3. A numerical example with simulation results are given in Section 4. Finally, a conclusion is drawn in Section 5.

2 Problem statement and preliminaries

Notations: $\mathbb{R}^n(\mathbb{R}^n_{0,+}, \mathbb{R}^n_{+})$ is the n-dimensional (non-negative, positive) vector space; $e_i = [0_{1\times(i-1)}\ 1\ 0_{1\times(n-i)}]^{\mathrm{T}} \in \mathbb{R}^n$ is the ith unit vector in $\mathbb{R}^n$; $l_j = [0_{1\times(j-1)}\ 1\ 0_{1\times(m-j)}]^{\mathrm{T}} \in \mathbb{R}^m$ is the jth unit vector in $\mathbb{R}^m$; $\overline{1,n} = \{1, 2, \cdots, n\}$; given three vectors

$$
\begin{aligned}
x &= [x_1\ \ x_2\ \ \cdots\ \ x_n]^{\mathrm{T}} \in \mathbb{R}^n,\\
y &= [y_1\ \ y_2\ \ \cdots\ \ y_n]^{\mathrm{T}} \in \mathbb{R}^n,\\
q &= [q_1\ \ q_2\ \ \cdots\ \ q_n]^{\mathrm{T}} \in \mathbb{R}^n_{0,+},
\end{aligned} \tag{1}
$$

two $n\times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, the following notations will be used in our development: $x \prec y(\preceq y)$ means that $x_i < y_i(\le y_i), \forall i \in \overline{1,n}$; $A \prec B(\preceq B)$ means that $a_{ij} < b_{ij}(\le b_{ij}), \forall i, j \in \overline{1,n}$; A is non-negative if $0 \preceq A$; A is a Metzler matrix if $a_{ij} \ge 0, \forall i, j \in \overline{1,n}, i \ne j$; $\mathbb{B}(0, q) = \{x \in \mathbb{R}^n_{0,+} : x \preceq q\}$ is a ball in $\mathbb{R}^n_{0,+}$; $s(A) = \max\{Re(\lambda) : \lambda \in \sigma(A)\}$ stands for the spectral abscissa of a matrix A; A is a Hurwitz matrix if $s(A) < 0$; $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$ stands for the spectral radius of a matrix A; A is a Schur matrix if $\rho(A) < 1$; the computations, such as minimum, maximum of a set of finite vectors, limitation of a vector-valued function etc., are understood in the component-wise sense.

Consider the linear CDDEs with bounded time-varying delays

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t - h_1(t)) + \omega(t), \quad t \ge t_0 \ge 0, \\ y(t) &= Cx(t) + Dy(t - h_2(t)) + d(t), \end{aligned} \tag{2}$$

where $x(.) \in \mathbb{R}^n_{0,+}$, $y(.) \in \mathbb{R}^m_{0,+}$ are the state vectors. Matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}_{0,+}$, $C \in \mathbb{R}^{m \times n}_{0,+}$ and $D \in \mathbb{R}^{m \times m}_{0,+}$ are known. D is assumed to be a Schur matrix. The disturbance vectors $\omega(.) \in \mathbb{R}^n_{0,+}$, $d(.) \in \mathbb{R}^m_{0,+}$ are unknown but assumed to be bounded by known bounds, i.e.

$$\begin{aligned} 0 \preceq \omega(t) \preceq \overline{\omega}, \quad \forall t \ge t_0, \\ 0 \preceq d(t) \preceq \overline{d}, \quad \forall t \ge t_0, \end{aligned} \tag{3}$$

where $\overline{\omega}, \overline{d}$ are two known vectors. The unknown time-varying delays, $h_1(.) \in \mathbb{R}_{0,+}$ and $h_2(.) \in \mathbb{R}_{0,+}$ are continuous, not necessary to be differential and are also assumed to be bounded, i.e.

$$\max_{t \ge t_0} \max \{h_1(t), h_2(t)\} \le h_M, \tag{4}$$

where h_M is a known constant. The initial condition of system (2) is given by $x(t_0) = \psi(t_0)$, $y(s) = \phi(s)$, $s \in [t_0 - h_M, t_0)$. The initial values $\psi(t_0)$ and $\phi(.)$ are unknown but assumed to be bounded by known bounds, i.e.

$$\begin{aligned} 0 \preceq \psi(t_0) \preceq \overline{\psi}, \\ 0 \preceq \phi(s) \preceq \overline{\phi}, \quad \forall s \in [t_0 - h_M, t_0), \end{aligned} \tag{5}$$

where $\overline{\psi}$ and $\overline{\phi}$ are known non-negative constant vectors. Let us denote by $x(t, t_0, \psi, \phi, \omega)$ and $y(t, t_0, \psi, \phi, d)$ the state vectors with the initial values (ψ, ϕ) and disturbances $(\omega(t), d(t))$ of system (2).

The objective of this study is to find as small as possible componentwise upper bounds of the state vectors of system (2). Concisely, we construct two decreasing vector-valued functions which are componentwise upper bounds of the two state vectors $x(t, t_0, \psi, \phi, \omega)$ and $y(t, t_0, \psi, \phi, d)$.

The definition of a positive system [33], the basic properties of non-negative matrices and Metzler matrices [34], the solution comparison properties of CDDEs (1) reported in [8, 10], the equivalent stability conditions of CDDEs (1) reported in [35] and the optimal value of a rational postive function reported in [31, 36] are needed for our development and they are re-stated respectively in the following definition and lemmas.

Definition 1: System (2) is said to be positive if for any non-negative initial values, $\psi(t_0) \succeq 0$, $\phi(s) \succeq 0$, $s \in [t_0 - h_M, t_0)$, the state trajectories of system (2) satisfy that $x(t, t_0, \psi, \phi, \omega) \succeq 0$ and $y(t, t_0, \psi, \phi, d) \succeq 0$, for all $t \ge t_0$.

Lemma 1:

(i) Let $M \in \mathbb{R}^{n \times n}$ be a non-negative matrix. Then, the following statements are equivalent: $(i_1) M$ is Schur stable; $(i_2)(M - I)q \prec 0$ for some $q \in \mathbb{R}^n_+$; $(i_3)(I - M)^{-1} \succeq 0$.

(ii) Let $M \in \mathbb{R}^{n \times n}$ be a Metzler matrix. Then, the following statements are equivalent: (ii_1) M is Hurwitz stable; (ii_2) $Mq \prec 0$ for some $q \in \mathbb{R}^n_+$; (ii_3) $M^{-1} \preceq 0$.

Lemma 2: Assume that A is a Metzler matrix, B, C, D are non-negative, D is a Schur matrix. Then,

(i) System (2) is positive.

(ii) For $\psi_1(t_0) \preceq \psi_2(t_0)$ and $\phi_1(s) \preceq \phi_2(s)$, $s \in [t_0 - h_M, t_0)$, we have

$$x(t, t_0, \psi_1, \phi_1, \omega) \preceq x(t, t_0, \psi_2, \phi_2, \omega), \quad \forall t \ge t_0, \tag{6}$$

$$y(t, t_0, \psi_1, \phi_1, d) \preceq y(t, t_0, \psi_2, \phi_2, d), \quad \forall t \ge t_0. \tag{7}$$

Lemma 3: Assume that A is a Metzler matrix, B, C, D are non-negative. Then, the following statements are equivalent:

(i) $\rho(D) < 1$ and $s(A + B(I - D)^{-1}C) < 0$;

(ii) there exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ such that

$$Ap + Bq \prec 0, \tag{8}$$

$$Cp + (D - I)q \prec 0; \tag{9}$$

(iii) $s(A) < 0$ and $\rho(C(-A)^{-1}B + D) < 1$.

Remark 1: Two matrix inequalities (8), (9) can be reformulated into the compact form

$$\begin{bmatrix} A & B \\ C & D - I \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \prec 0.$$

By (ii) of Lemma 1, matrix

$$-\begin{bmatrix} A & B \\ C & D - I \end{bmatrix}^{-1}$$

is non-negative and non-singular. This follows that all its row vectors are also non-negative and non-zero. Hence, the positive vector $[p^T \ q^T]^T$ can be computed by the following equation:

$$\begin{bmatrix} p \\ q \end{bmatrix} = -\begin{bmatrix} A & B \\ C & D - I \end{bmatrix}^{-1} \xi, \tag{10}$$

where $\xi \in \mathbb{R}^{n+m}_+$.

Remark 2: The authors of [8, 35] used inequalities (8) and (9) in order to derive asymptotic stability conditions for CDDEs (2). By using the completeness of the Euclidean space $\mathbb{R}^n$, Pathirana *et al.* [10] proposed tighter inequalities that

$$-A^{-1}Bq \preceq (1 - \mu)p \prec p, \tag{11}$$

$$(I - D)^{-1}Cp \preceq (1 - \mu)q \prec q, \tag{12}$$

$$Cp + Dq \preceq (1 - \mu)q \prec q, \tag{13}$$

for some positive scalar $\mu \in (0, 1)$, and used these inequalities to analyse the stability of CDDEs with unbounded time-delays. In the study, we also use these tighter inequalities to derive our main results (componentwise upper bounds of system (2)). Hence, in the following, we recall its proof. Assume that condition (i) of Lemma 3 holds. Then, by (iii) of Lemma 3, we have $s(A) < 0$, which implies that $A^{-1} \preceq 0$ due to (ii) of Lemma 1. Left multiplying A^{-1} on inequality (8), we obtain

$$-A^{-1}Bq \prec p. \tag{14}$$

Since D is a Schur matrix and non-negative, by (i) of Lemma 1, matrix $(I - D)^{-1}$ is non-negative. Left multiplying $(I - D)^{-1}$ on inequality (9), we obtain

$$(I - D)^{-1}Cp \prec q. \tag{15}$$

Matrix inequality (9) is also rewritten as

$$Cp + Dq \prec q. \tag{16}$$

Since (14)–(16) are strict inequalities, by using the completeness of the Euclidean space $\mathbb{R}^n$, there is a positive scalar $\mu \in (0, 1)$ such that inequalities (11)–(13) hold.

Lemma 4: For two given vectors $a, b \in \mathbb{R}^n_{0,+}$, $b \neq 0$ and a rational function $\Gamma: \mathbb{R}^n_+ \to \mathbb{R}_{0,+}$ given by

$$\Gamma(r) = \frac{a_1 r_1 + a_2 r_2 + \cdots + a_n r_n}{b_1 r_1 + b_2 r_2 + \cdots + b_n r_n}, \tag{17}$$

set $J = \{j \in \overline{1,n}: b_j > 0\}$, then

$$\inf_{r \in \mathbb{R}^n_+} \Gamma(r) = \min_{j \in J} \frac{a_j}{b_j}. \tag{18}$$

3 Main results

3.1 Finite-time convergence of linear positive systems

In this subsection, we present a method to find the smallest possible time, which guarantees the finite-time convergence (i.e. all the state vectors starting from a given bounded set converges within another given bounded set after a finite time) of linear positive systems. The result is needed in establishing a componentwise upper bound of CDDEs with/without bounded disturbances in Sections 3.2 and 3.3. Consider the following linear positive system:

$$\begin{aligned} \dot{u}(t) &= Au(t), \quad t \geq t_0 \geq 0, \\ u(t_0) &= \theta, \end{aligned} \tag{19}$$

where $u(t) \in \mathbb{R}^n_{0,+}$ is the state vector; $A \in \mathbb{R}^{n \times n}$ is a Metzler matrix; $\theta \in \mathbb{R}^n_{0,+}$ is the non-negative initial value. Let us denote by $u(t, t_0, \theta)$ the solution of system (19).

The object of this subsection is, for given two non-negative vectors $\overline{\theta} \in \mathbb{R}^n_{0,+}$ and $\delta = [\delta_1 \ldots \delta_n]^{\mathrm{T}} \in \mathbb{R}^n_{0,+}$, to find as small as possible time $T \geq 0$ such that, for all initial value $\theta \preceq \overline{\theta}$, the solution of system (19) satisfies $u(t, t_0, \theta) \preceq \delta, \quad \forall t \geq t_0 + T$.

3.2 Exponential componentwise estimate of linear positive system

Lemma 5: Assume that A is a Hurwitz stable. Then, there are a positive scalar $\alpha > 0$ and a vector-value function $\beta(\overline{\theta})$ such that the following exponential componentwise estimate holds:

$$u(t, t_0, \theta) \preceq \beta(\overline{\theta}) e^{-\alpha(t - t_0)}, \quad \forall t \geq t_0. \tag{20}$$

Proof: Since A is Hurwitz stable, there exists a positive scalar $\alpha > 0$ such that $A + \alpha I$ is Hurwitz stable. Note that $s(A^{\mathrm{T}} + \alpha I) = s(A + \alpha I)$. Therefore, matrix $A^{\mathrm{T}} + \alpha I$ is also Hurwitz stable. By (ii) of Lemma 1, there is a vector $v \succ 0$ such that

$$v^{\mathrm{T}}(A + \alpha I) \prec 0. \tag{21}$$

Let us consider the following Lyapunov functional:

$$V(t) = v^{\mathrm{T}} e^{\alpha t} u(t). \tag{22}$$

By a simple computation, the derivative of V along the solution $u(t, t_0, \theta)$ is given as below

$$\dot{V}(t) = v^{\mathrm{T}}(A + \alpha I) e^{\alpha t} u(t, t_0, \theta) \leq 0, \quad \forall t \geq t_0, \tag{23}$$

which follows that $V(t) \leq V(t_0), \quad \forall t \geq t_0$. Combining with $v \succ 0$, $u(t, t_0, \theta) \succeq 0, \quad \forall t \geq t_0$ and $u(t_0, t_0, \theta) = \theta \preceq \overline{\theta}$, we have, for each $i \in \overline{1,n}$, that

$$\begin{aligned} v_i u_i(t, t_0, \theta) e^{\alpha t} &\leq v^{\mathrm{T}} u(t, t_0, \theta) e^{\alpha t} \\ &\leq v^{\mathrm{T}} u(t_0, t_0, \theta) e^{\alpha t_0} \\ &\leq v^{\mathrm{T}} e^{\alpha t_0} \overline{\theta}, \quad \forall t \geq t_0, \end{aligned} \tag{24}$$

which follows that

$$u_i(t, t_0, \theta) \leq \frac{v^{\mathrm{T}} \overline{\theta}}{v_i} e^{-\alpha(t - t_0)}, \quad \forall t \geq t_0. \tag{25}$$

Set $\beta_i(v, \overline{\theta}) = v^{\mathrm{T}} \overline{\theta} / v_i$ and $\beta(v, \overline{\theta}) = [\beta_1(v, \overline{\theta}), \ldots, \beta_n(v, \overline{\theta})]^{\mathrm{T}}$. Then, from (25), we obtain a exponential componentwise estimate (20). The proof of Lemma 5 is completed. □

Remark 3: For a fixed decay rate α satisfying $s(A^{\mathrm{T}} + \alpha I) < 0$. Let us denote by Ω the set of all vectors $v \succ 0$ such that inequality (21) holds, i.e.

$$\Omega = \{v \in \mathbb{R}^n_+ : (A^{\mathrm{T}} + \alpha I) v \prec 0\}. \tag{26}$$

Then, by taking the minimum of the vector-value function $\beta(v, \overline{\theta})$ subject to $v \in \Omega$, i.e.

$$\min_{v \in \Omega} \beta(v, \overline{\theta}) = [\min_{v \in \Omega} \beta_1(v, \overline{\theta}), \ldots, \min_{v \in \Omega} \beta_n(v, \overline{\theta})]^{\mathrm{T}}, \tag{27}$$

we obtain the smallest exponential componentwise estimate with the fixed decay rate α of system (19) as below

$$u(t, t_0, \theta) \preceq \left(\min_{v \in \Omega} \beta(v, \overline{\theta})\right) e^{-\alpha(t - t_0)}, \quad \forall t \geq t_0. \tag{28}$$

Next, we present a method to find $\min_{v \in \Omega} \beta_i(v, \overline{\theta})$, $i \in \overline{1,n}$. For simplicity, we consider the case where $i = 1$.

3.3 Minimisation of partial factor $\beta_1(v, \overline{\theta})$

Let us denote

$$\Lambda = \{-(A^{\mathrm{T}} + \alpha I)^{-1} r : r \in \mathbb{R}^n_+\}. \tag{29}$$

Since matrix $A^{\mathrm{T}} + \alpha I$ is Metzler and Hurwitz, by part (ii) of Lemma 1, matrix $-(A^{\mathrm{T}} + \alpha I)^{-1}$ is non-negative and non-singular. Therefore, all its row vectors are also non-negative and non-zero. This implies that for every vector $r \in \mathbb{R}^n_+$, we have $-(A^{\mathrm{T}} + \alpha I)^{-1} r \in \mathbb{R}^n_+$ and $(A^{\mathrm{T}} + \alpha I)[-(A^{\mathrm{T}} + \alpha I)^{-1} r] = -r \prec 0$. This follows that $\Lambda \subseteq \Omega$. On the other hand, for every $v \in \Omega$, we have $(A^{\mathrm{T}} + \alpha I) v \prec 0$. Setting $r = -(A^{\mathrm{T}} + \alpha I) v$. Then, $r \in \mathbb{R}^n_+$ and $v = -(A^{\mathrm{T}} + \alpha I)^{-1} r$. This follows that $\Omega \subseteq \Lambda$. Hence, we have

$$\Omega = \Lambda, \tag{30}$$

which means that every vector $v \in \Omega$ has the following form:

$$v = -(A^{\mathrm{T}} + \alpha I)^{-1} r, \tag{31}$$

where $r \in \mathbb{R}^n_+$. By substituting (31) into formula $\beta_1(v, \overline{\theta}) = v^{\mathrm{T}} \overline{\theta} / v_1$ with some algebraic manipulations, $\beta_1(v, \overline{\theta}))$ is simplified into the following rational function of a vector variable r, which is denoted by $\Gamma_1(r)$

$$\beta_1(v, \overline{\theta}) = \frac{a_1 r_1 + a_2 r_2 + \cdots + a_n r_n}{b_1 r_1 + b_2 r_2 + \cdots + b_n r_n} \triangleq \Gamma_1(r). \tag{32}$$

This follows that $\min_{v \in \Omega} \beta_1(v, \overline{\theta}) = \min_{r \in \mathbb{R}^n_+} \Gamma_1(r)$. Hence, the problem of finding $\min_{v \in \Omega} \beta_1(v, \overline{\theta})$ is equivalent to the problem of finding $\min_{r \in \mathbb{R}^n_+} \Gamma_1(r)$. Since $\mathbb{R}^n_+$ is an open set in $\mathbb{R}^n$, the minimum of function $\Gamma_1(r)$ subject to $r \in \mathbb{R}^n_+$ may not exist. Therefore, instead of finding the minimum, we find the infimum of function $\Gamma_1(r)$ subject to $r \in \mathbb{R}^n_+$.

Note that by Lemma 1, matrix $-(A^{\mathrm{T}} + \alpha I)^{-1}$ is non-negative and non-singular. This follows that all its row vectors are non-negative and non-zero. Let us denote $a = [a_1\, a_2 \cdots a_n]^{\mathrm{T}}$ and $b = [b_1\, b_2 \cdots b_n]^{\mathrm{T}}$.

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By using formula $\beta_1(v, \overline{\theta}) = v^{\mathrm{T}}\overline{\theta}/v_1$, (31) and (32), we have $v^{\mathrm{T}}\overline{\theta} = \overline{\theta}^{\mathrm{T}}v = \overline{\theta}^{\mathrm{T}}[-(A^{\mathrm{T}} + \alpha I)^{-1}]r \equiv a^{\mathrm{T}}r$. This follows that $a^{\mathrm{T}} = \overline{\theta}^{\mathrm{T}}[-(A^{\mathrm{T}} + \alpha I)^{-1}]$. Since both vector $\overline{\theta}$ and matrix $-(A^{\mathrm{T}} + \alpha I)^{-1}$ are non-negative, vector $a = (\overline{\theta}^{\mathrm{T}}[-(A^{\mathrm{T}} + \alpha I)^{-1}])^{\mathrm{T}}$ is also non-negative. Similarly, we also have $v_1 = e_1^{\mathrm{T}}v = e_1^{\mathrm{T}}[-(A^{\mathrm{T}} + \alpha I)^{-1}]r \equiv b^{\mathrm{T}}r$. This follows that $b^{\mathrm{T}} = e_1^{\mathrm{T}}[-(A^{\mathrm{T}} + \alpha I)^{-1}]$, i.e. b^{T} is the first row of the non-negative and non-singular matrix $-(A^{\mathrm{T}} + \alpha I)^{-1}$. Hence, vector $b = [b_1\, b_2 \cdots b_n]^{\mathrm{T}}$ is non-negative and non-zero. Set $J = \{j \in \overline{1,n} : b_j > 0\}$, then by Lemma 4, we have

$$\inf_{r \in \mathbb{R}_+^n} \Gamma_1(r) = \min_{j \in J} \frac{a_j}{b_j} \triangleq \gamma_1 . \tag{33}$$

Thus, the smallest exponential estimate with a fixed decay rate α of the 1th partial state vector can be given as

$$u_1(t, t_0, \theta) \le \gamma_1 e^{-\alpha(t - t_0)}, \quad \forall t \ge t_0, \tag{34}$$

where γ_1 is computed by formula (33).

3.4 Finite-time convergence

For a given scalar $\delta_1 > 0$, set

$$t_\alpha^1 = \begin{cases} 0 & \text{if} \quad \gamma_1 \le \delta_1, \\ -\dfrac{1}{\alpha}\ln\dfrac{\delta_1}{\gamma_1} & \text{if} \quad \gamma_1 > \delta_1 . \end{cases} \tag{35}$$

From formula (34) with a simple computation, we can verify that

$$u_1(t, t_0, \theta) \le \delta_1, \quad \forall t \ge t_0 + t_\alpha^1 . \tag{36}$$

Note that $s(A^{\mathrm{T}} + \alpha I)$ is an increasing function with respect to variable α. By using the one-dimensional search method, we find the supremum $\alpha_{\max}$ of scalars $\alpha > 0$ such that $s(A^{\mathrm{T}} + \alpha I) < 0$. Hence, by increasing α gradually from 0 to $\alpha_{\max}$ with a chosen small step, e.g. 0.001, and comparing the times t_α^1 computed by (35), we find

$$T^1 = \min_{\alpha \in (0, \alpha_{\max}]} t_\alpha^1 . \tag{37}$$

Then, T^1 is the smallest time which guarantees that

$$u_1(t, t_0, \theta) \le \delta_1, \quad \forall t \ge t_0 + T^1 . \tag{38}$$

Similarly, for given scalars $\delta_i > 0, i = 2, \ldots, n$, we also compute the smallest times $T^i, i = 2, \ldots, n$ such that $u_i(t, t_0, \theta) \le \delta_i, \quad \forall t \ge t_0 + T^i$. Set

$$T = \max\{T^1, T^2, \ldots, T^n\} . \tag{39}$$

Then, T is the smallest time, which guarantees that the state vector $u(t, t_0, \theta)$ converges componentwisely within the ball $\mathbb{B}(0, \delta)$ after the finite-time T, i.e.

$$u(t, t_0, \theta) \le \delta, \quad \forall t \ge t_0 + T . \tag{40}$$

We have now summarised the above presented statements into the following theorem.

Theorem 1: Assume that A is a Metzler matrix and Hurwitz stable. Given two vectors $\overline{\theta} \in \mathbb{R}_{0,+}^n$, $\delta \in \mathbb{R}_{0,+}^n$. Then, all trajectories of system (19) converge componentwisely within the ball $\mathbb{B}(0, \delta)$ after the finite-time, T, computed by formula (39).

3.5 Componentwise bounds for positive CDDEs without disturbances

In this subsection, we present a new result on componentwise bound of system (2) for the case no disturbance, i.e. $\omega(t) \equiv d(t) \equiv 0$. For simplicity, we consider system (2) with $t_0 = 0$.

Theorem 2: Assume that A is a Metzler matrix, B, C, D are non-negative, D is a Schur matrix and $s(A + B(I - D)^{-1}C) < 0$. Then, there exist two positive vectors $p \in \mathbb{R}_+^n$, $q \in \mathbb{R}_+^m$, a scalar $\mu \in (0, 1)$, a time $T^* \ge h_M$, such that, for $k = 0, 1, 2, \ldots$, the following estimates hold:

$$\begin{aligned} x(t, 0, \psi, \phi, 0) &\le (1 - \mu)^k p, \quad \forall t \in [kT^*, (k+1)T^*), \\ y(t, 0, \psi, \phi, 0) &\le (1 - \mu)^{k+1} q, \quad \forall t \in [kT^*, (k+1)T^*) . \end{aligned} \tag{41}$$

Proof: Step 1: By Lemma 3 and Remark 1, there exist two positive vectors $\tilde{p} \in \mathbb{R}_+^n$ and $\tilde{q} \in \mathbb{R}_+^m$ such that three inequalities (14)–(16) hold. For given two non-negative vectors $\overline{\psi} \in \mathbb{R}_{+,0}^n$ and $\overline{\phi} \in \mathbb{R}_{+,0}^m$, set

$$\varrho = \begin{cases} \max\left\{\dfrac{\overline{\psi}_1}{\tilde{p}_1}, \ldots, \dfrac{\overline{\psi}_n}{\tilde{p}_n}, \dfrac{\overline{\phi}_1}{\tilde{q}_1}, \ldots, \dfrac{\overline{\phi}_m}{\tilde{q}_m}\right\} & \text{if} \quad \overline{\psi} \ne 0 \text{ or } \overline{\phi} \ne 0, \\ \epsilon & \text{if} \quad \overline{\psi} = 0 \text{ and } \overline{\phi} = 0, \end{cases}$$

where ϵ is an arbitrarily small positive scalar. Choose $p = \varrho\tilde{p}$ and $q = \varrho\tilde{q}$. Then, it is easy to see that $p > 0$, $q > 0$, $p \ge \overline{\psi}$, $q \ge \overline{\phi}$ and (14)–(16) hold. By using one-dimensional search, we find a scalar $\mu \in (0, 1)$ such that inequalities (11)–(13) hold. By (ii) of Lemma 2, we have

$$\begin{aligned} x(t, 0, \psi, \phi, 0) &\le x(t, 0, p, q, 0), \quad \forall t \ge 0, \\ y(t, 0, \psi, \phi, 0) &\le y(t, 0, p, q, 0), \quad \forall t \ge 0. \end{aligned} \tag{42}$$

Step 2: Next, we prove that there exist a time $T > 0$ such that

$$\begin{aligned} x(t, 0, p, q, 0) &\le (1 - \mu)p, \quad \forall t \ge T, \\ y(t, 0, p, q, 0) &\le (1 - \mu)^2 q, \quad \forall t \ge T, \end{aligned} \tag{43}$$

where *p, q μ* are computed in Step 1. Indeed, let us consider the linear positive system

$$\dot{u}(t) = Au(t), \quad \forall t \ge 0, \tag{44}$$

and by using Step 1 of the proof of Theorem 1 in [10] and (11)–(13), we obtain

$$x(t, 0, p, q, 0) \le p, \quad \forall t \ge 0, \tag{45}$$

$$y(t, 0, p, q, 0) \le (1 - \mu)q, \quad \forall t \ge 0, \tag{46}$$

and the following solution comparison:

$$x(t, 0, p, q, 0) \le -A^{-1}Bq + u(t, 0, p + A^{-1}Bq), \quad \forall t \ge 0. \tag{47}$$

By using Theorem 1 for system (44) with $\overline{\theta} = p + A^{-1}Bq$ and $\delta = (1 - \mu)p + A^{-1}Bq$, we find the smallest time T such that

$$u(t, 0, p + A^{-1}Bq) \le (1 - \mu)p + A^{-1}Bq, \quad \forall t \ge T, \tag{48}$$

which follows that

$$x(t, 0, p, q, 0) \le (1 - \mu)p, \quad \forall t \ge T . \tag{49}$$

Set $T^* = \max\{T, h_M\}$. Then, from (46) and (49), we have

$$\begin{aligned} x(t, 0, p, q, 0) &\le (1 - \mu)p, \quad \forall t \ge T^*, \\ y(t, 0, p, q, 0) &\le (1 - \mu)q, \quad \forall t \ge T^* - h_M . \end{aligned} \tag{50}$$

On the other hand, from (2), we have

$$y(t) = \begin{cases} Cx(t) + Dy(t - h_2(t)) & \text{if } h_2(t) > 0, \\ (I - D)^{-1}Cx(t) & \text{if } h_2(t) = 0. \end{cases} \tag{51}$$

Combining (12), (13), (50) and (51), we obtain

$$y(t, 0, p, q, 0) \preceq (1 - \mu)^2 q, \quad \forall t \ge T^*. \tag{52}$$

From (45) and (46), we obtain inequality (41) for the case where $k = 0$. From (50) and (52), we obtain inequality (41) for the case where $k = 1$.

Step 3: In this step, we prove that inequality (41) hold for the case where $k = 2$. Indeed, let us consider the comparison system with two new variables $x^1(t)$, $y^1(t)$ and the initial time $t_0 = T^*$ as below

$$\begin{aligned} \dot{x}^1(t) &= Ax^1(t) + By^1(t - h_1(t)), \quad t \ge T^* \ge 0, \\ y^1(t) &= Cx^1(t) + Dy^1(t - h_2(t)). \end{aligned} \tag{53}$$

Firstly, we find the state estimates of system (53). Similar to Step 2, we also prove that

$$\begin{aligned} x^1(t, T^*, p, q, 0) &\preceq (1 - \mu)p, \quad \forall t \in [2T^*, 3T^*), \\ y^1(t, T^*, p, q, 0) &\preceq (1 - \mu)^2 q, \quad \forall t \in [2T^*, 3T^*). \end{aligned} \tag{54}$$

By using the linearity of system (53), from (54), we also obtain, for any positive scalar λ, that

$$\begin{aligned} x^1(t, T^*, \lambda p, \lambda q, 0) &\preceq (1 - \mu)\lambda p, \quad \forall t \in [2T^*, 3T^*), \\ y^1(t, T^*, \lambda p, \lambda q, 0) &\preceq (1 - \mu)^2 \lambda q, \quad \forall t \in [2T^*, 3T^*). \end{aligned} \tag{55}$$

Choosing $\lambda = 1 - \mu$ and from (55), we have

$$\begin{aligned} x^1(t, T^*, (1 - \mu)p, (1 - \mu)q, 0) &\preceq (1 - \mu)^2 p, \quad \forall t \in [2T^*, 3T^*), \\ y^1(t, T^*, (1 - \mu)p, (1 - \mu)q, 0) &\preceq (1 - \mu)^3 q, \quad \forall t \in [2T^*, 3T^*). \end{aligned} \tag{56}$$

On the other hand, inequality (50) implies that we have

$$\begin{aligned} x(T^*, 0, p, q, 0) &\preceq (1 - \mu)p, \\ y(t, 0, p, q, 0) &\preceq (1 - \mu)q, \quad \forall t \in [T^* - h_M, T^*). \end{aligned} \tag{57}$$

Combining with part (ii) of Lemma 2, we obtain a comparison between a solution of system (2) and the one of system (53) as below

$$\begin{aligned} x(t, 0, p, q, 0) &\preceq x^1(t, T^*, (1 - \mu)p, (1 - \mu)q, 0), \quad \forall t \ge T^*, \\ y(t, 0, p, q, 0) &\preceq y^1(t, T^*, (1 - \mu)p, (1 - \mu)q, 0), \quad \forall t \ge T^*. \end{aligned} \tag{58}$$

From (56) and (58), we obtain

$$\begin{aligned} x(t, 0, p, q, 0) &\preceq (1 - \mu)^2 p, \quad \forall t \in [2T^*, 3T^*), \\ y(t, 0, p, q, 0) &\preceq (1 - \mu)^3 q, \quad \forall t \in [2T^*, 3T^*). \end{aligned} \tag{59}$$

This means that we have inequality (41) for the case where $k = 2$. By doing similarly as above, we also obtain inequality (41) for the cases where $k = 3, 4, \ldots$. The proof of Theorem 2 is completed. □

Remark 4: Under the same assumption as in Theorem 2, by using the solution comparison method, Pathirana *et al.* [10] showed that there exists two positive vectors p, q, a scalar $\mu \in (0, 1)$ and a time $\tilde{T} \ge h_M$ such that the following estimates hold:

$$\begin{aligned} x(t, 0, \psi, \phi, 0) &\preceq (1 - \mu)^k p, \quad \forall t \in [k\tilde{T}, (k + 1)\tilde{T}), \\ y(t, 0, \psi, \phi, 0) &\preceq (1 - \mu)^k q, \quad \forall t \in [k\tilde{T}, (k + 1)\tilde{T}). \end{aligned} \tag{60}$$

Here, vectors p, q and scalar $\mu \in (0, 1)$ are chosen such that conditions (11)–(13) hold and the time $\tilde{T}$ is chosen such that condition (49) hold. In the study, by adapting the method proposed in [10] and for the same vectors p, q and the same scalar $\mu \in (0, 1)$, we have derived an state estimate (41), where the time T^* is the smallest one which still guarantees that condition (49) hold, for system (2). Owing to $1 - \mu < 1$ and $T^* \le \tilde{T}$, it can see that the estimates on both $x(\,.\,)$ and $y(\,.\,)$ derived in Theorem 2 are always smaller than the ones derived in [10].

3.6 Componentwise bounds for positive CDDEs perturbed by bounded disturbances

This subsection is to extend the above obtained result to the class of CDDEs perturbed by bounded-non-zero-disturbances. For simplicity, we consider also system (2) with $t_0 = 0$.

Theorem 3: Assume that conditions given in Theorem 2 hold. Set

$$\begin{bmatrix} \eta \\ \varsigma \end{bmatrix} = -\begin{bmatrix} A & B \\ C & D - I \end{bmatrix}^{-1} \begin{bmatrix} \overline{\omega} \\ \overline{d} \end{bmatrix}. \tag{61}$$

(i) There exist two positive vectors $p \in \mathbb{R}^n_+$ and $q \in \mathbb{R}^m_+$, a scalar $\mu \in (0, 1)$, a time $T^* \ge h_M$, such that, for $k = 0, 1, 2, \ldots$, the following exponential componentwise estimates hold:

$$\begin{aligned} x(t, 0, \psi, \phi, \omega) &\preceq \eta + (1 - \mu)^k p, \quad \forall t \in [kT^*, (k + 1)T^*), \\ y(t, 0, \psi, \phi, d) &\preceq \varsigma + (1 - \mu)^{k+1} q, \quad \forall t \in [kT^*, (k + 1)T^*). \end{aligned} \tag{62}$$

(ii) The vector $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$ is the smallest componentwise ultimate bound of system (2).

(iii) The ball $\mathbb{B}\left(0, \begin{bmatrix} \eta \\ \varsigma \end{bmatrix}\right)$ is the smallest invariant set, which is different from $\{0\}$, of system (2).

Proof:

(i) Let us consider the following system:

$$\begin{aligned} \dot{\overline{x}}(t) &= A\overline{x}(t) + B\overline{y}(t - h_1(t)) + \overline{\omega}, \quad t \ge 0, \\ \overline{y}(t) &= C\overline{x}(t) + D\overline{y}(t - h_2(t)) + \overline{d}. \end{aligned} \tag{63}$$

Set

$$\begin{bmatrix} \hat{\psi} \\ \hat{\phi} \end{bmatrix} = \max\left\{ \begin{bmatrix} \overline{\psi} \\ \overline{\phi} \end{bmatrix}, \begin{bmatrix} \eta \\ \varsigma \end{bmatrix} \right\}.$$

By both (i) and (ii) of Lemma 2, we have

$$\begin{aligned} x(t, 0, \psi, \phi, \omega) &\preceq \overline{x}(t, 0, \hat{\psi}, \hat{\phi}, \overline{\omega}), \ t \ge 0, \\ y(t, 0, \psi, \phi, d) &\preceq \overline{y}(t, 0, \hat{\psi}, \hat{\phi}, \overline{d}), \ t \ge 0. \end{aligned} \tag{64}$$

Taking the following state transformation:

$$\begin{aligned} \check{x}(t) &= \overline{x}(t) - \eta, \\ \check{y}(t) &= \overline{y}(t) - \varsigma, \end{aligned} \tag{65}$$

then, we have

$$\begin{aligned} \dot{\check{x}}(t) &= A\check{x}(t) + B\check{y}(t - h_1(t)), \ t \ge 0, \\ \check{y}(t) &= C\check{x}(t) + D\check{y}(t - h_2(t)), \end{aligned} \tag{66}$$

and

$$\begin{aligned} \check{x}(t, 0, \hat{\psi} - \eta, \hat{\phi} - \varsigma, 0) &= \overline{x}(t, 0, \hat{\psi}, \hat{\phi}, \overline{\omega}) - \eta, \\ \check{y}(t, 0, \hat{\psi} - \eta, \hat{\phi} - \varsigma, 0) &= \overline{y}(t, 0, \hat{\psi}, \hat{\phi}, \overline{d}) - \varsigma. \end{aligned} \tag{67}$$

Now, we apply Theorem 2 for system (66) with the initial values $(\widehat{\psi} - \eta, \widehat{\phi} - \varsigma)$, there exist two positive vectors *p,q*, a scalar $\mu \in (0, 1)$ and the time $T^* \geq h_M$ such that, for $k = 0, 1, 2, \ldots$,

$$
\begin{aligned}
&\check{x}(t, 0, \widehat{\psi} - \eta, \widehat{\phi} - \varsigma, 0) \preceq (1 - \mu)^k p, \quad \forall t \in [kT^*, (k + 1)T^*), \\
&\check{y}(t, 0, \widehat{\psi} - \eta, \widehat{\phi} - \varsigma, 0) \preceq (1 - \mu)^{k+1} q, \quad \forall t \in [kT^*, (k + 1)T^*) .
\end{aligned}
\tag{68}
$$

From (64), (67) and (68), we obtain inequalities (62).

(ii) From (62), by letting *t* tend to infinity, we obtain

$$
\begin{aligned}
&\lim \sup_{t \to \infty} x(t, 0, \psi, \phi, \omega) \preceq \eta, \\
&\lim \sup_{t \to \infty} y(t, 0, \psi, \phi, d) \preceq \varsigma .
\end{aligned}
\tag{69}
$$

This follows that the vector $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$ is a componentwise ultimate bound of system (2). In order to prove that the vector $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$ is the smallest componentwise ultimate bound, we consider the case where $\omega(t) \equiv \overline{\omega}$ and $d(t) \equiv \overline{d}$ and take the following state transformation:

$$
\begin{aligned}
\widetilde{x}(t) &= \eta - x(t), \\
\widetilde{y}(t) &= \varsigma - y(t),
\end{aligned}
\tag{70}
$$

then, from (2), we obtain

$$
\begin{aligned}
&\dot{\widetilde{x}}(t) = A\widetilde{x}(t) + B\widetilde{y}(t - h_1(t)), \ t \geq 0, \\
&\widetilde{y}(t) = C\widetilde{x}(t) + D\widetilde{y}(t - h_2(t)),
\end{aligned}
\tag{71}
$$

and

$$
\begin{aligned}
&\widetilde{x}(t, 0, \eta, \varsigma, 0) = \eta - x(t, 0, 0, 0, \overline{\omega}), \\
&\widetilde{y}(t, 0, \eta, \varsigma, 0) = \varsigma - y(t, 0, 0, 0, \overline{d}) .
\end{aligned}
\tag{72}
$$

Now, we apply Theorem 2 for system (71) with the initial values (η, ς), there exist two positive vectors *p,q*, a scalar $\mu \in (0, 1)$ and the time $T^* \geq h_M$ such that, for $k = 0, 1, 2, \ldots$

$$
\begin{aligned}
&\widetilde{x}(t, 0, \eta, \varsigma, 0) \preceq (1 - \mu)^k p, \quad \forall t \in [kT^*, (k + 1)T^*), \\
&\widetilde{y}(t, 0, \eta, \varsigma, 0) \preceq (1 - \mu)^{k+1} q, \quad \forall t \in [kT^*, (k + 1)T^*) .
\end{aligned}
\tag{73}
$$

From (72) and (73), we have

$$
\begin{aligned}
\eta - (1 - \mu)^k p \preceq x(t, 0, 0, 0, \overline{\omega}), \quad \forall t \in [kT^*, (k + 1)T^*), \\
\varsigma - (1 - \mu)^{k+1} q \preceq y(t, 0, 0, 0, \overline{d}), \quad \forall t \in [kT^*, (k + 1)T^*) .
\end{aligned}
\tag{74}
$$

Letting *t* tend to infinity, we obtain

$$
\begin{aligned}
&\eta \preceq \lim \inf_{t \to \infty} x(t, 0, 0, 0, \overline{\omega}), \\
&\varsigma \preceq \lim \inf_{t \to \infty} y(t, 0, 0, 0, \overline{d}) .
\end{aligned}
\tag{75}
$$

This implies that the vector $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$ is the smallest componentwise ultimate bound.

(iii) For the case where $\begin{bmatrix} \overline{\psi} \\ \overline{\phi} \end{bmatrix} = \begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$, then $\begin{bmatrix} \widehat{\psi} \\ \widehat{\phi} \end{bmatrix} = \begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$. Substituting this equality into (67), we obtain

$$
\begin{aligned}
&\overline{x}(t, 0, \widehat{\psi}, \widehat{\phi}, \overline{\omega}) \equiv \eta, \\
&\overline{y}(t, 0, \widehat{\psi}, \widehat{\phi}, \overline{d}) \equiv \varsigma .
\end{aligned}
\tag{76}
$$

Combining with (64), we obtain

$$
\begin{aligned}
&x(t, 0, \psi, \phi, \omega) \preceq \eta, \ t \geq 0, \\
&y(t, 0, \psi, \phi, d) \preceq \varsigma, \ t \geq 0,
\end{aligned}
\tag{77}
$$

Step 1: input $A, B, C, D, n, m, \overline{\omega}, \overline{d}, h_M, \overline{\psi}, \overline{\phi}$,

Step 2: compute $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix} = -\begin{bmatrix} A & B \\ C & D - I \end{bmatrix}^{-1} \begin{bmatrix} \overline{\omega} \\ \overline{d} \end{bmatrix}$.

if $\begin{bmatrix} \overline{\psi} \\ \overline{\phi} \end{bmatrix} \preceq \begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$, the componentwise bound is $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$

else, proceed to Step 3

end

Step 3: (Finding vectors p, q and μ)

$\begin{bmatrix} \widehat{\psi} \\ \widehat{\phi} \end{bmatrix} = \max\left\{ \begin{bmatrix} \overline{\psi} \\ \overline{\phi} \end{bmatrix}, \begin{bmatrix} \eta \\ \varsigma \end{bmatrix} \right\}$, replace $\begin{bmatrix} \overline{\psi} \\ \overline{\phi} \end{bmatrix} = \begin{bmatrix} \widehat{\psi} - \eta \\ \widehat{\phi} - \varsigma \end{bmatrix}$,

set $\xi = [1 \cdots 1]^T$, compute $\begin{bmatrix} \widetilde{p} \\ \widetilde{q} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D - I \end{bmatrix}^{-1} \xi$,

$$
\varrho = \begin{cases} \max\{\frac{\overline{\psi}_1}{\widetilde{p}_1}, \cdots, \frac{\overline{\psi}_n}{\widetilde{p}_n}, \quad \frac{\overline{\phi}_1}{\widetilde{q}_1}, \cdots, \frac{\overline{\phi}_m}{\widetilde{q}_m}\} & \\ & \text{if } \overline{\psi} \neq 0 \text{ or } \overline{\phi} \neq 0, \\ \epsilon & \text{if } \overline{\psi} = 0 \text{ and } \overline{\phi} = 0, \end{cases}
$$

$p = \varrho\widetilde{p}, q = \varrho\widetilde{q}$.

Step 4: (Finding μ)

Compute $M_1 = -A^{-1}Bq$, $M_2 = (I - D)^1 Cp$

$M_3 = Cp + Dq$ and obtain

$\mu = 1 - \max\left\{ \frac{e_i^T M_1}{e_i^T p}, \frac{l_j^T M_2}{l_j^T q}, \frac{l_j^T M_3}{l_j^T q} \right\}_{i=\overline{1,n}, j=\overline{1,m}}$

Step 5: (Finding α_{max})

$step_1 = 0.001, \alpha = 0$

while $\mu(A + \alpha I) < 0$

$\alpha = \alpha + step_1$

end

$\alpha_{max} = \alpha - step_1$.

Step 6: (Finding $T^i, i = 1, \cdots, n$ and T^*)

Set $\overline{\theta} = p + ABq, \delta = (1 - \mu)p + ABq$,

for $i = 1 : 1 : n$

for $\alpha = 0 : step_1 : \alpha_{max}$

compute γ_i (by (32))

obtain t_α^i (by (34))

end

$T^i = \min_{\alpha \in [0, \alpha_{max}]} t_\alpha^i$

end

Obtain $T = \max\{T^1, \cdots, T^n\}$, $T^* = \max\{T, h_M\}$

and the componentwise state bound by (61).

Fig. 1 *Algorithm 1: Computing componentwise state bound*

From (75) and (77), we conclude that the ball $\mathbb{B}\left(0, \begin{bmatrix} \eta \\ \varsigma \end{bmatrix}\right)$ is the smallest invariant set, which is different from $\{0\}$, of system (2). The proof of Theorem 3 is completed.

□

Remark 5: It can be seen that Theorems 2 and 3 not only present the sufficient condition for the existence of both transient and ultimate bounds of the state vectors, bound of the invariant set but also derive the way to compute and optimise these bounds. From the above, an algorithm to compute the smallest possible componentwise state bound for system (2) is given as below:

Remark 6: In Step 5 of Algorithm 1 (see Fig. 1), for purpose of computation for a numerical example in next section, we choose $step_1 = 0.001$. In practice, we can choose a smaller size or a larger one. Note that the smaller chosen step is, the tighter derived bound is but the higher complexity is.

4 Numerical example

Example 1: Consider CDDEs (2), whose matrices are chosen as same as ones given in Shen and Zheng [8]

$$A = \begin{bmatrix} -2.5 & 0.3 & 0 \\ 0.5 & -2 & 0.1 \\ 0.4 & 0.6 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 0.2 & 0.1 \\ 0.5 & 0.3 \\ 0 & 0.4 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.2 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0.6 & 0.3 \\ 0.1 & 0.2 \end{bmatrix}.$$

Two disturbance vectors $\omega(t)$ and $d(t)$ are bounded by $\overline{\omega} = [0.5\ 0.3\ 0.1]^{\rm T}$ and $\overline{d} = [0.3\ 0.1]^{\rm T}$. The time-varying delays $h_1(t)$ and $h_2(t)$ are bounded by $h_M = 2$. The initial values $\psi(0)$ and $\phi(.)$ are bounded by $\overline{\psi} = [2 \quad 5 \quad 3]^{\rm T}$ and $\overline{\phi} = [15 \quad 5]^{\rm T}$.

By using (60), we compute $\eta = [0.7249\ 1.4756\ 0.5780]^{\rm T}$, $\varsigma = [3.7739\ 1.1469]^{\rm T}$. By choosing $\xi = [1 \dots 1]^{\rm T}$ and using Step 1 of the proof of Theorem 3, we find $p = [2.3951 \quad 5.5118 \quad 2.4220]^{\rm T}$, $q^{\rm T} = [14.1659 \quad 4.9990]^{\rm T}$ and $\mu = 0.0707$. By using Step 2 of the proof of Theorems 3 and 1, we find $T = 1.2056$. Hence, $T^* = \max\{T, h_M\} = 2$. As a result, we obtain componentwise state bounds of system (2) as below:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \preceq \begin{bmatrix} 0.7249 \\ 1.4756 \\ 0.5780 \end{bmatrix} + 0.9293^k \begin{bmatrix} 2.3951 \\ 5.5118 \\ 2.4220 \end{bmatrix}, \quad \forall t \in [k2, (k+1)2),\ k = 0, 1, \dots, \tag{78}$$

and

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \preceq \begin{bmatrix} 3.7739 \\ 1.1469 \end{bmatrix} + 0.9293^{k+1} \begin{bmatrix} 14.1659 \\ 4.9990 \end{bmatrix}, \quad \forall t \in [k2, (k+1)2),\ k = 0, 1, \dots . \tag{79}$$

For a visual simulation, we choose disturbances vectors as

$$\omega(t) = a\begin{bmatrix} 0.5|\sin(0.2t)| \\ 0.3|\sin(0.1t)| \\ 0.1|\sin(0.3t)| \end{bmatrix}, \quad d(t) = b\begin{bmatrix} 0.3|\cos(0.1t)| \\ 0.1|\cos(0.2t)| \end{bmatrix},$$

where $a \in \{0, 0.5, 1\}$; $b \in \{0, 1\}$, two time-varying delays as $h_1(t) = 1 + |\sin(t)|$, $h_2(t) = 1 + |\cos(t)|$, and initial values $\psi = \overline{\psi}$, $\phi = \overline{\phi}$. The following figures (Figs. 2–6) show that the trajectories of the partial state vectors of system (2) are bounded by upper bounds computed by Theorem 3. Furthermore, to illustrate the ultimate bound obtained in Theorem 3 is the smallest, we choose more a special case of the disturbances vectors where $\omega(t) = [0.5\ 0.3\ 0.1]^{\rm T}$ and $d(t) = [0.3\ 0.1]^{\rm T}$. Also, from these figures, it can see that the trajectories of the partial state vectors of system (2) with respect to this chosen special case converge to

$$\eta = \begin{bmatrix} 0.7249 \\ 1.4756 \\ 0.5780 \end{bmatrix} \quad \text{and} \quad \varsigma = \begin{bmatrix} 3.7739 \\ 1.1469 \end{bmatrix},$$

respectively. This shows that the vector $\begin{bmatrix} \eta \\ \varsigma \end{bmatrix}$ is the smallest ultimate componentwise bound of system (2).

5 Conclusion

This paper has studied the problem of finding state bounds for a class of positive CDDEs perturbed by unknown-but-bounded disturbances. A novel method to derive componentwise state bounds on infinite time horizon, the smallest ultimate bound and the smallest invariant set for the system has been presented. A numerical example is considered to illustrate the obtained result.

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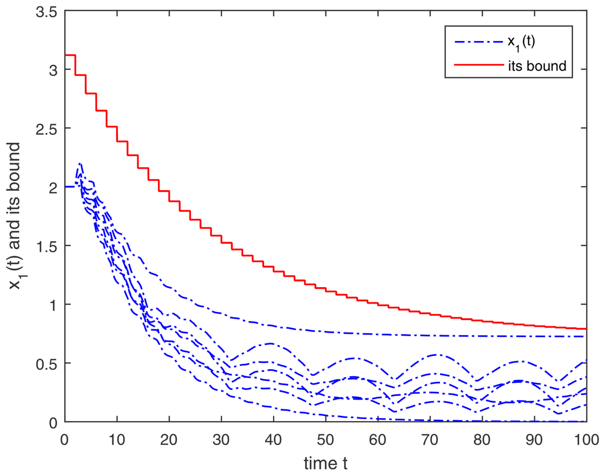

Fig. 2 *Trajectories of $x_1(t)$ and its bound*

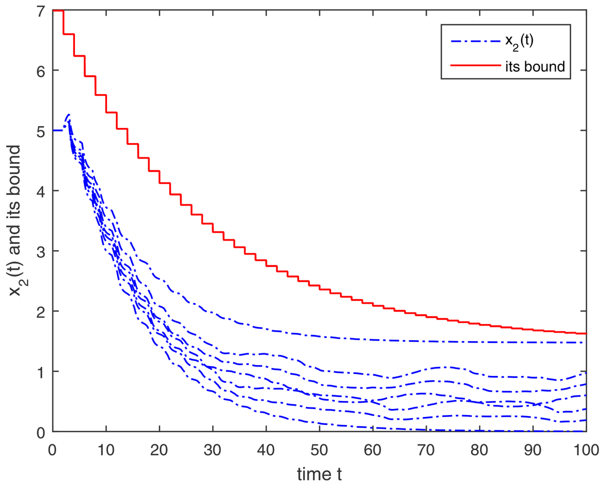

Fig. 3 *Trajectories of $x_2(t)$ and its bound*

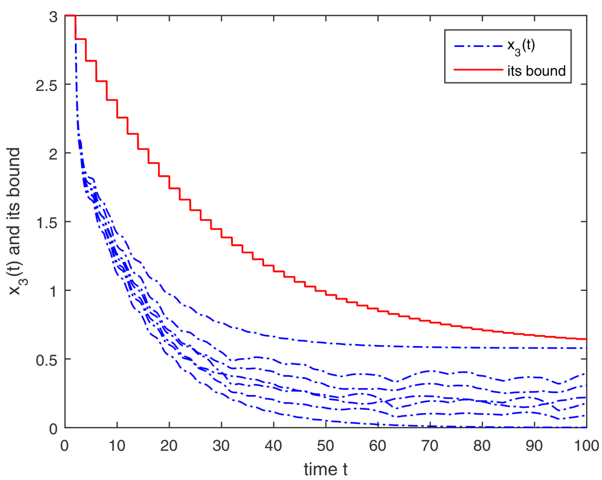

Fig. 4 *Trajectories of $x_3(t)$ and its bound*

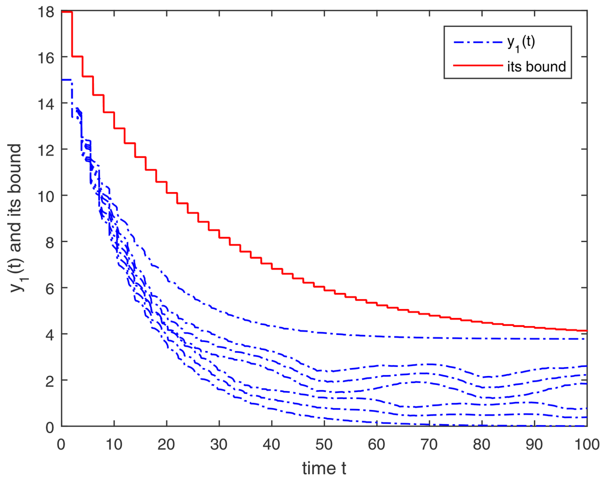

Fig. 5 *Trajectories of $y_1(t)$ and its bound*

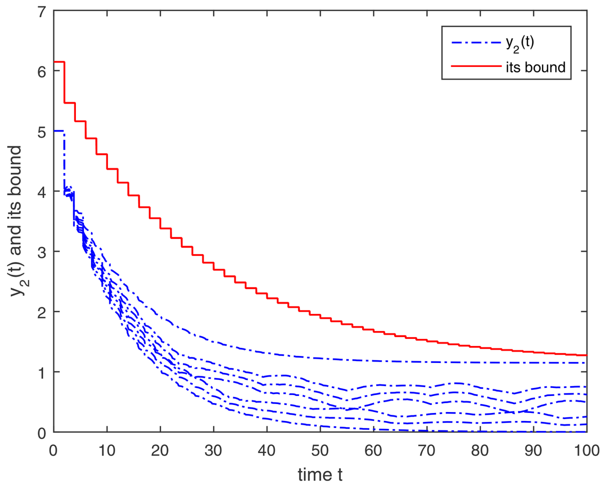

Fig. 6 *Trajectories of $y_2(t)$ and its bound*

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