

1 Index Reduction for Second Order Singular Systems
2 of Difference Equations

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6 **Abstract**

This paper is devoted to the analysis of linear second order *discrete-time descriptor systems* (or singular difference equations (SiDEs) with control). Following the algebraic approach proposed by Kunkel and Mehrmann for pencils of matrix valued functions, first we present a theoretical framework based on a procedure of reduction to analyze solvability of initial value problems for SiDEs, which is followed by the analysis of descriptor systems. We also describe methods to analyze structural properties related to the solvability analysis of these systems. Namely, two numerical algorithms for reduction to the so-called *strangeness-free forms* are presented. Two associated index notions are also introduced and discussed. This work extends and complements some recent results for high order continuous-time descriptor systems and first order discrete-time descriptor systems.

7 *Keywords:* Singular system, Second order difference equation, Descriptor
8 system, Strangeness-index, Index reduction, Regularization.

9 *2000 MSC:* 15A23, 39A05, 39A06, 93C05

10 **1. Introduction**

In this paper we study second order discrete-time descriptor systems of the form

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) = f(n) \quad \text{for all } n \geq n_0. \quad (1)$$

We will also discuss the initial value problem of the associated singular difference equation (SiDE)

$$A_n x(n+2) + B_n x(n+1) + C_n x(n) = f(n) \quad \text{for all } n \geq n_0, \quad (2)$$

together with some given initial conditions

$$x(n_0+1) = x_1, \quad x(n_0) = x_0. \quad (3)$$

11 Here the solution/state $x = \{x(n)\}_{n \geq n_0}$, the inhomogeneity $f = \{f(n)\}_{n \geq n_0}$,
12 the input $u = \{u(n)\}_{n \geq n_0}$, where $x(n) \in \mathbb{R}^d$, $f(n) \in \mathbb{R}^m$ and $u(n) \in \mathbb{R}^p$ for each

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13 $n \geq n_0$. Three matrix sequences $\{A_n\}_{n \geq n_0}$, $\{B_n\}_{n \geq n_0}$, $\{C_n\}_{n \geq n_0}$ take values
14 in $\mathbb{R}^{m,d}$, and $\{D_n\}_{n \geq n_0}$ takes values in $\mathbb{R}^{m,p}$. We notice that all the results in
15 this paper also can be carried over to the complex case and they can also be
16 easily extended to systems of higher order. However, for sake of simplicity and
17 because this is the most important case in practice, we restrict ourselves to the
18 case of real and second order systems.

19 The SiDE (2), on one hand, can be considered as the resulting equation
20 obtained by finite difference or discretization of some continuous-time DAEs or
21 constrained PDEs. On the other hand, there are also many models/applications
22 in real-life, which lead to SiDEs, for example Leontief economic models, biological
23 backward Leslie model, etc, see e.g. [1, 5, 10, 14].

24 While both DAEs and SiDEs of first order have been well-studied from
25 both theoretical and numerical points of view, the same maturity has not been
26 reached for higher order systems. In the classical literature for regular differ-
27 ence equations, e.g. [1, 5, 10], usually new variables are introduced to represent
28 some chosen derivatives of the state variable x such that a high order system
29 can be reformulated as a first order one. Unfortunately for singular systems,
30 this approach may induce some substantial disadvantages. As have been fully
31 discussed in [13, 17] for continuous-time systems, these disadvantages include:
32 (1st) increase the index of the singular system, and therefore the complexity of a
33 numerical method to solve it; (2nd) increase the computational effort due to the
34 bigger size of a new system; (3rd) affect the controllability/observability of the
35 corresponding descriptor system since there exist situations where a new system
36 is uncontrollable while the original one is. Therefore, the *algebraic approach*,
37 which treats the system directly without reformulating it, has been presented in
38 [13, 17, 22, 23] in order to overcome the disadvantages mentioned above. Never-
39 theless, even for second order SiDEs, this method has not yet been considered.

40 Another motivation of this work comes from recent research on the stability
41 analysis of high order discrete-time systems with time-dependent coefficients
42 [12, 18]. In these works, systems are supposed to be given in either strangeness-
43 free form or linear state-space form. This, however, is not always the case in
44 applications, and hence, a reformulation procedure would be required.

45 Therefore, the main aim of this article is to set up a comparable framework
46 for second order SiDEs and for discrete-time descriptor systems as well. It
47 is worth marking that the algebraic method proposed in [13, 17] is applicable
48 theoretically but not numerically due to two reasons: (1st) The condensed forms
49 of the matrix coefficients are really big and complicated; (2nd) The system's
50 transformations are not orthogonal, and hence, not numerically stable. In this
51 work, we will modify this method to make it more concise and also computable
52 in a stable way.

53 The outline of this paper is as follows. After giving some auxiliary results in
54 Section 2, in Sections 3 and 4 we consecutively introduce *index reduction pro-*
55 *cedures* for SiDEs and for descriptor systems. A desired *strangeness-free form*
56 and a constructive algorithm to get it will be presented in Theorem 3.8 and
57 Algorithm 1 (Section 3). A resulting system from this algorithm allows us to
58 fully analyze structural properties such as existence and uniqueness of a solu-

59 tion, consistency and hidden constraints, etc. For descriptor systems, where
 60 feedback also takes part in the regularization/solution procedure, besides the
 61 strangeness-free form presented in Theorem 4.7, regularization via first order
 62 feedback is discussed in Theorem 4.4. In order to get stable numerical solutions
 63 of these systems, in Section 5 we study the *difference array approach* in Algo-
 64 rithm 2 and Theorem 5.6 aiming at bringing out the strangeness-free form of a
 65 given system. Finally, we finish with some conclusions.
 66

67 2. Preliminaries

68 In the following example we demonstrate some difficulties that may arise in
 69 the analysis of second order SiDEs.

Example 2.1. Consider the following second order descriptor system, motivated from Example 2, [17].

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) - \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \end{bmatrix}, \quad n \geq n_0. \quad (4)$$

Clearly, from the second equation $\begin{bmatrix} 1 & 0 \end{bmatrix} x(n) = u(n) + f_2(n)$, we can shift the time n forward to obtain

$$\begin{bmatrix} 1 & 0 \end{bmatrix} x(n+1) = u(n+1) + f_2(n+1) \quad \text{and} \quad \begin{bmatrix} 1 & 0 \end{bmatrix} x(n+2) = u(n+2) + f_2(n+2).$$

Inserting these into the first equation of (4), we find out the hidden constraint

$$f_2(n+2) + u(n+2) + f_2(n+1) + u(n+1) + \begin{bmatrix} 0 & 1 \end{bmatrix} x(n) = f_1(n).$$

Consequently, we deduce the following system, which possess a unique solution

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) - f_2(n+2) - f_2(n+1) - u(n+2) - u(n+1) \\ u(n) + f_2(n) \end{bmatrix}, \quad n \geq n_0.$$

70 Let $n = n_0$ in this new system, we obtain a constraint that $x(n_0)$ must obey.
 71 This example showed us some important facts. Firstly, one can use some shift
 72 operators and row-manipulation (Gaussian eliminations) to derive hidden con-
 73 straints. Secondly, a solution only exists if initial conditions and an input fulfill
 74 certain consistency conditions. Finally, in this example the solution depends on
 75 the future input. This property is called *non-causality* and cannot happen in
 76 the case of regular difference equations.

For matrices $Q \in \mathbb{R}^{q,d}$, $P \in \mathbb{R}^{p,d}$, the pair (Q, P) is said to *have no hidden redundancy* if

$$\text{rank} \left(\begin{bmatrix} Q \\ P \end{bmatrix} \right) = \text{rank}(Q) + \text{rank}(P).$$

77 Otherwise, (Q, P) is said to *have hidden redundancy*. The geometrical meaning
 78 of this concept is that the intersection space $\text{span}(P^T) \cap \text{span}(Q^T)$ contains only
 79 the zero-vector 0. Here, for any given matrix M , by M^T we denote its transpose.
 80 We denote by $\text{span}(P^T)$ (resp., $\text{span}(Q^T)$) the real vector space spanned by the
 81 rows of P (resp., rows of Q).

82 **Lemma 2.2.** ([7]) Consider $k + 1$ full row rank matrices $R_0 \in \mathbb{R}^{r_0,d}$, $R_1 \in$
 83 $\mathbb{R}^{r_1,d} \dots, R_k \in \mathbb{R}^{r_k,d}$, and assume that for $j = k, \dots, 1$ none of the matrix pairs
 84 $(R_j, [R_{j-1}^T \dots R_0^T]^T)$ has a hidden redundancy. Then $[R_k^T \dots R_0^T]^T$ has full
 85 row rank.

86 Lemma 2.3 below will be very useful later for our analysis, in order to remove
 87 hidden redundancy in the coefficients of (2).

Lemma 2.3. Consider two matrix sequences $\{P_n\}_{n \geq n_0}$, $\{Q_n\}_{n \geq n_0}$ which take values in $\mathbb{R}^{p,d}$ and $\mathbb{R}^{q,d}$, respectively. Furthermore, assume that they satisfy the constant rank assumptions

$$\text{rank}(Q_n) = r_Q, \quad \text{and} \quad \text{rank}\left(\begin{bmatrix} P_n \\ Q_n \end{bmatrix}\right) = r_{[P;Q]} \quad \text{for all } n \geq n_0.$$

88 Then there exists a matrix sequence $\left\{\begin{bmatrix} S_n & 0 \\ Z_n^{(1)} & Z_n^{(2)} \end{bmatrix}\right\}_{n \geq n_0}$ in $\mathbb{R}^{p,p+q}$ such that the
 89 following conditions hold.

- 90 i) $S_n \in \mathbb{R}^{r_{[P;Q]}-r_Q, p}$, $Z_n^{(1)} \in \mathbb{R}^{p-r_{[P;Q]}+r_Q, p}$, $Z_n^{(2)} \in \mathbb{R}^{p-r_{[P;Q]}+r_Q, q}$,
- 91 ii) $\begin{bmatrix} S_n \\ Z_n^{(1)} \end{bmatrix} \in \mathbb{R}^{p,p}$ is orthogonal, and $Z_n^{(1)} P_n + Z_n^{(2)} Q_n = 0$,
- 92 iii) $S_n P_n$ has full row rank, and the pair $(S_n P_n, Q_n)$ has no hidden redundancy.

94 PROOF. Since the proof is essentially the same as in the continuous-time case,
 95 we refer the interested readers to the proof of Lemma 2.7, [8].

96 **Remark 2.4.** i) In the special case, where P_n has full row rank and the pair
 97 (P_n, Q_n) has no hidden redundancy, we will adapt the notation of an empty
 98 matrix and take $S_n = I_p$, $Z_n^{(1)} = [\]_{0,p}$, $Z_n^{(2)} = [\]_{0,q}$.
 99 ii) Furthermore, we notice, that whenever the smallest singular value of Q_n and
 100 the largest one do not differ very much in size, then we can stably compute the
 101 matrix $Z_n^{(2)}$. Both matrices $Z_n^{(1)}$ and $Z_n^{(2)}$ will play the key role in our *index*
 102 *reduction procedure* presented in the next section.

103 For any given matrix M , by $T_0(M)$ we denote an orthogonal matrix whose
 104 columns span the left null space of M . By $T_\perp(M)$ we denote an orthogonal ma-
 105 trix whose columns span the vector space $\text{range}(M)$. From basic linear algebra,
 106 we have the following lemma.

Lemma 2.5. The matrix $\begin{bmatrix} T_\perp^T(M) \\ T_0^T(M) \end{bmatrix}$ is nonsingular, the matrix $T_\perp^T(M)M$ has
 full row rank, and the following identity holds

$$\begin{bmatrix} T_\perp^T(M) \\ T_0^T(M) \end{bmatrix} M = \begin{bmatrix} T_\perp^T(M)M \\ 0 \end{bmatrix}.$$

107 PROOF. A simple proof can be found, for example, in [6].

¹⁰⁸ **3. Strangeness-index of second order SiDEs**

¹⁰⁹ In this section, we study the solvability analysis of the second order SiDE (2)
¹¹⁰ and that of its corresponding IVP (2)–(3). Many regularization procedures and
¹¹¹ their associated index notions have been proposed for first order systems, see
¹¹² the survey [16] and the references therein. Nevertheless, for high order systems,
¹¹³ only the strangeness-index has been proposed in the continuous-time case in
¹¹⁴ [17, 23]. Thus, it is our purpose to construct a comparable regularization and
¹¹⁵ index concept for discrete-time system (2).

Let

$$M_n := \begin{bmatrix} A_n & B_n & C_n \end{bmatrix}, \quad X(n) := \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix},$$

we call $\{M_n\}_{n \geq n_0}$ the *behavior matrix sequence* of system (2). Thus, (2) can be rewritten as

$$M_n X(n) = f(n) \quad \text{for all } n \geq n_0. \quad (5)$$

Clearly, by scaling (2) with a pointwise nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ in $\mathbb{R}^{m,m}$, we obtain a new system

$$\begin{bmatrix} P_n A_n & P_n B_n & P_n C_n \end{bmatrix} X(n) = P_n f(n) \quad \text{for all } n \geq n_0, \quad (6)$$

¹¹⁶ without changing the solution space. This motivates the following definition.

¹¹⁷ **Definition 3.1.** Two behavior matrix sequences $\{M_n = [A_n \ B_n \ C_n]\}_{n \geq n_0}$
¹¹⁸ and $\{\tilde{M}_n = [\tilde{A}_n \ \tilde{B}_n \ \tilde{C}_n]\}_{n \geq n_0}$ are called (*strongly*) *left equivalent* if there
¹¹⁹ exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that $\tilde{M}_n = P_n M_n$
¹²⁰ for all $n \geq n_0$. We denote this equivalence by $\{M_n\}_{n \geq n_0} \xrightarrow{\ell} \{\tilde{M}_n\}_{n \geq n_0}$. If this is
¹²¹ the case, we also say that two SiDEs (2), (6) are left equivalent.

Lemma 3.2. Consider the behavior matrix sequence $\{M_n\}_{n \geq n_0}$ of system (2). Then for all $n \geq n_0$, we have that

$$\{M_n\}_{n \geq n_0} \xrightarrow{\ell} \left\{ \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix} \right\}_{n \geq n_0}, \quad \begin{array}{l} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ v_n \end{array} \quad (7)$$

¹²² where the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ have full row rank. Here, the numbers
¹²³ $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, v_n are row-sizes of the block rows of M_n . Furthermore, these
¹²⁴ numbers are invariant under left equivalent transformations. Thus, we can call
¹²⁵ them the local characteristic invariants of the SiDE (2).

Proof. The block diagonal form (7) is obtained directly by consecutively compressing the block columns A_n , B_n , C_n of M_n via Lemma 2.5. In details, we

have that

- rows of $A_{n,1}$ form the basis of the space $\text{range}(A_n^T)$,
- rows of $B_{n,2}$ form the basis of the space $\text{range}(T_0^T(A_n)B_n)^T$,
- rows of $C_{n,3}$ form the basis of the space $\text{range}\left(T_0^T\left(\left[A_n^T \ B_n^T\right]^T\right)C_n\right)^T$.

¹²⁶ Moreover, from (7), we obtain the following identities

$$\begin{aligned} r_{2,n} &= \text{rank}(A_n), \\ r_{1,n} &= \text{rank}([A_n \ B_n]) - \text{rank}(A_n), \\ r_{0,n} &= \text{rank}([A_n \ B_n \ C_n]) - \text{rank}([A_n \ B_n]), \\ v_n &= m - r_{2,n} - r_{1,n} - r_{0,n}, \end{aligned}$$

¹²⁷ which proves the second claim. \square

¹²⁸ Analogous to the continuous-time case, we will apply an *algebraic approach*
¹²⁹ (see [2, 17]), which aims to reformulate (2) into a so-called *strangeness-free* form,
¹³⁰ as stated in the following definition.

Definition 3.3. ([12]) System (2) is called *strangeness-free* if there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that by scaling the SiDE (2) at each point n with P_n , then we obtain a new system of the form

$$\begin{bmatrix} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hat{v} \end{bmatrix} = \begin{bmatrix} \hat{A}_{n,1} \\ 0 \\ 0 \\ 0 \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{B}_{n,2} \\ 0 \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ \hat{C}_{n,2} \\ \hat{C}_{n,3} \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hat{f}_4(n) \end{bmatrix}, \quad (8)$$

¹³¹ for all $n \geq n_0$, where matrix $[\hat{A}_{n,1}^T \ \hat{B}_{n+1,2}^T \ \hat{C}_{n+2,3}^T]^T$ always has full row rank.

¹³² In order to perform an algebraic approach, an additional assumption below
¹³³ is usually needed.

¹³⁴ **Assumption 3.4.** Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$, $r_{0,n}$
¹³⁵ become global, i.e., they are constant for all $n \geq n_0$. Furthermore, assume that
¹³⁶ two matrix sequences $\{\left[A_{n,1}^T \ B_{n,2}^T \ C_{n,3}^T\right]^T\}_{n \geq n_0}$ and $\{\left[B_{n,2}^T \ C_{n,3}^T\right]^T\}_{n \geq n_0}$
¹³⁷ have constant rank for all $n \geq n_0$.

Remark 3.5. Following directly from the proof of Lemma 3.2, we see that Assumption 3.4 is satisfied if and only if five following constant rank conditions are satisfied

$$\begin{aligned} \text{rank}(A_n) &\equiv \text{const.}, \quad \text{rank}([A_n \ B_n]) \equiv \text{const.}, \quad \text{rank}([A_n \ B_n \ C_n]) \equiv \text{const.}, \\ \text{rank}(T_0^T(A_n)B_n) &\equiv \text{const.}, \quad \text{rank}\left(T_0^T\left(\left[A_n^T \ B_n^T\right]^T\right)C_n\right) \equiv \text{const.} \end{aligned} \quad (9)$$

¹³⁸ **Remark 3.6.** In (8), the quantities \hat{r}_2 , \hat{r}_1 , and \hat{r}_0 are dimensions of the second
¹³⁹ order dynamics part, the first order dynamics part, and the algebraic (zero
¹⁴⁰ order) part, respectively. Furthermore, $r_2 + r_1$ is exactly the degree of freedoms.

Let us call the number $r_u := 3r_2 + 2r_1 + r_0$ the *upper rank* of system (2). Clearly, r_u is invariant under left equivalence transformations. Rewrite (5) block row-wise, we obtain the following system for all $n \geq n_0$.

$$A_{n,1}x(n+2) + B_{n,1}x(n+1) + C_{n,1}x(n) = f_1(n), \quad r_2 \text{ equations,} \quad (10a)$$

$$B_{n,2}x(n+1) + C_{n,2}x(n) = f_2(n), \quad r_1 \text{ equations,} \quad (10b)$$

$$C_{n,3}x(n) = f_3(n), \quad r_0 \text{ equations,} \quad (10c)$$

$$0 = f_4(n), \quad v \text{ equations.} \quad (10d)$$

Since the matrices $A_{n,1}$, $B_{n,2}$, $C_{n,3}$ have full row rank, the number of scalar difference equations of order 2 (resp. 1, and 0) in (2) is exactly r_2 (resp. r_1 and r_0), while v is the number of redundant equations. Now we are able to define the shift-forward operator Δ , which acts on some or whole equations of system (10). This operator maps each equation of system (10) at the time instant n to the equation itself at the time $n + 1$, for example

$$\Delta : C_{n,3}x(n) = f_3(n) \mapsto C_{n+1,3}x(n+1) = f_3(n+1). \quad (11)$$

¹⁴¹ Clearly, under Assumption 3.4, this shift operator can be applied to equations of
¹⁴² system (10). In order to reveal all hidden constraints of (10) we propose the idea
¹⁴³ that for each $j = 1, 2$, we use equations of order less than j to reduce the number
¹⁴⁴ of scalar equations of order j . This task will be performed as follows. Firstly, by
¹⁴⁵ applying Lemma 2.3 to two matrix pairs $(B_{n,2}, C_{n+1,3})$ and $(A_{n,1}, [B_{n+1,2} \ C_{n+2,3}])$,
¹⁴⁶ we obtain matrix sequences $\{S_n^{(i)}\}_{n \geq n_0}$, $i = 1, 2$, and $\{Z_n^{(j)}\}_{n \geq n_0}$, $j = 1, \dots, 5$, of
¹⁴⁷ appropriate sizes such that for all $n \geq n_0$, the following conditions hold true.

- ¹⁴⁸ i) For $i = 1, 2$, the matrices $\begin{bmatrix} S_n^{(i)} \\ Z_n^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$ are orthogonal.
- ii) The following identities hold true.

$$Z_n^{(1)} B_{n,2} + Z_n^{(3)} C_{n+1,3} = 0, \quad (12a)$$

$$Z_n^{(2)} A_{n,1} + Z_n^{(4)} B_{n+1,2} + Z_n^{(5)} C_{n+2,3} = 0. \quad (12b)$$

- ¹⁴⁹ iii) Both matrix pairs $(S_n^{(2)} A_{n,1}, [S_n^{(1)} B_{n+1,2} \ C_{n+2,3}])$, $(S_n^{(1)} B_{n,2}, C_{n+1,3})$ have no
¹⁵⁰ hidden redundancy.

¹⁵¹ Now we will transform the SiDE (2) as in Lemma 3.7 below.

Lemma 3.7. *Assume that Assumption 3.4 is satisfied. Let the matrix sequences $\{S_n^{(i)}\}_{n \geq n_0}$, $i = 1, 2$, and $\{Z_n^{(j)}\}_{n \geq n_0}$, $j = 1, \dots, 5$ be defined as above. Then the*

SiDE (2) has exactly the same solution set as the transformed system

$$\begin{aligned}
& \frac{d_2}{s_2} \left[\begin{array}{ccc} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ 0 & Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2} & Z_n^{(2)} C_{n,1} \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \\
& = \left[\begin{array}{c} S_n^{(2)} f_1(n) \\ Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2) \\ \hline S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1) \\ \hline f_3(n) \\ f_4(n) \end{array} \right] \quad \text{for all } n \geq n_0. \quad (13)
\end{aligned}$$

152 Furthermore, both matrix pairs $\left(S_n^{(2)} A_{n,1}, \begin{bmatrix} S_n^{(1)} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix}\right)$, $\left(S_n^{(1)} B_{n,2}, C_{n+1,3}\right)$ have
153 no hidden redundancy.

154 Proof. The proof is not too difficult but rather lengthy and technical, so we
155 leave it to Appendix A. \square

156 Consider system (13), we see that the upper rank of the behavior matrix is

$$\begin{aligned}
r_u^{new} &\leq 3d_2 + 2(s_2 + d_1) + (s_1 + r_0) \\
&= 3(r_2 - s_2) + 2(s_2 + r_1 - s_1) + (s_1 + r_0) \\
&= r - (s_2 + s_1) \leq r.
\end{aligned}$$

157 In conclusion, after performing a so-called *index reduction step*, which passes
158 from (10) to (13), we have reduced the upper rank r_u at least by $s_2 + s_1$.
159 Continue in this fashion until $s_1 = s_2 = 0$, we obtain the following algorithm.

Algorithm 1 Index reduction steps for SiDEs at the time point n

Input: The SiDE (2) and its behavior form (5).

Output: A strangeness-free SiDE of the form (8) and the strangeness-index μ .

- 1: Set $i = 0$.
 - 2: Transform the behavior matrix $[A_n \ B_n \ C_n]$ to the block upper triangular form
$$\tilde{M}_n := \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{bmatrix},$$
where all the matrices $A_{n,1}, B_{n,2}, C_{n,3}$ on the main diagonal have full row rank. The system now takes the form (10).
 - 3: **if** both matrix pairs $(A_{n,1}, \begin{bmatrix} B_{n+1,2} \\ C_{n+2,3} \end{bmatrix})$ and $(B_{n,2}, C_{n+1,3})$ have no hidden redundancy **then** set $\mu = i$ and STOP.
 - 4: **else** set $i := i + 1$
 - 5: Find the matrices $S_n^{(j)}, j = 1, 2$, and $Z_n^{(j)}, j = 1, \dots, 5$.
 - 6: Transform the system to the new form (13) as in Lemma 3.7.
 - 7: **end if**
 - 8: Go back to Step 2 with the updated behavior matrix.
-

160 After each index reduction step the upper rank r_u^i has been decreased at
161 least by $s_2^i + s_1^i$, so Algorithm 1 terminates after a finite number μ of iterations,
162 which will be called the *strangeness-index* of the SiDE (2).

Theorem 3.8. Consider the SiDE (2) and assume that Assumption 3.4 is satisfied for any n and any i considered within the loop, such that the strangeness-index μ is well-defined by Algorithm 1. Then the SiDE (2) has the same solution set as the strangeness-free SiDE

$$\begin{array}{lcl} r_2^\mu & \left[\hat{A}_{n,1} \quad \hat{B}_{n,1} \quad \hat{C}_{n,1} \right] & \left[\hat{g}_1(n) \right] \\ r_1^\mu & \left[0 \quad \hat{B}_{n,2} \quad \hat{C}_{n,2} \right] & \left[\hat{g}_2(n) \right] \\ r_0^\mu & \left[0 \quad 0 \quad \hat{C}_{n,3} \right] & \left[\hat{g}_3(n) \right] \\ v^\mu & \left[0 \quad 0 \quad 0 \right] & \left[\hat{g}_4(n) \right] \end{array} \quad \text{for all } n \geq n_0, \quad (14)$$

163 where the matrix $[\hat{A}_{n,1}^T \ \hat{B}_{n+1,2}^T \ \hat{C}_{n+2,3}^T]^T$ has full row rank for all $n \geq n_0$, and
164 the functions \hat{g}_2 and \hat{g}_3 consist of the components of $f(n), f(n+1), \dots, f(n+2\mu)$
165 (at most).

166 *Proof.* The proof is a direct consequence of Algorithm 1, where the matrix
167 $[\hat{A}_{n,1}^T \ \hat{B}_{n+1,2}^T \ \hat{C}_{n+2,3}^T]^T$ has full row rank due to Lemma 2.2. \square

168 To illustrate Algorithm 1, we consider the following example.

Example 3.9. Given a parameter $\alpha \in \mathbb{R}$, we consider the second order SiDE

$$\begin{bmatrix} 1 & n+1 & n+4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+2) + \begin{bmatrix} 0 & \alpha & 2n+3 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & n+1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n+1 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \end{bmatrix}, \quad (15)$$

for all $n \geq 0$. Fortunately, the behavior matrix

$$M = \left[\begin{array}{ccc|ccc|ccc} 1 & n+1 & n+4 & 0 & \alpha & 2n+3 & 0 & n+1 & 0 \\ 0 & 0 & 0 & 1 & n & 1 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n+1 \end{array} \right] = \begin{bmatrix} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \end{bmatrix}$$

is already in the block diagonal form, so we do not need to perform Step 2 in Algorithm 1. Furthermore, all constant rank conditions required in Assumption 3.4 are satisfied. We observe that

$$\begin{aligned} B_{n+1,2} &= [1 \ n+1 \ 1], \quad C_{n+1,2} = [0 \ 0 \ n+1], \\ C_{n+1,3} &= [0 \ 0 \ n+2], \quad C_{n+2,3} = [0 \ 0 \ n+3]. \end{aligned}$$

By directly verifying, we see that the matrix pair $(A_{n,1}, [B_{n+1,2} \ C_{n+2,3}])$ has hidden redundancy, while the pair $(B_{n,2}, C_{n+1,3})$ does not. Now we choose $S_n^{(2)} = []$, $Z_n^{(2)} = 1$, $Z_n^{(4)} = -1$, $Z_n^{(5)} = -1$. Notice that the fact $Z_n^{(5)}$ is non-empty leads to the appearance of $f_3(n+2)$. Furthermore, the resulting system (13) reads

$$\begin{bmatrix} 0 & \alpha & n+2 \\ 1 & n & 1 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & n+1 & 0 \\ 0 & 0 & n \\ 0 & 0 & n+1 \end{bmatrix} x(n) = \begin{bmatrix} f_1(n) - f_2(n+1) - f_3(n+2) \\ f_2(n) \\ f_3(n) \end{bmatrix}. \quad (16)$$

Since the leading coefficient matrix associated with $x(n+2)$ becomes zero, so for notational convenience we do not write this term. Go back to Step 3, we see that the following two cases may happen.

- i) If $\alpha \neq 0$, then Algorithm 1 terminates here, and the strangeness-index is $\mu = 1$. The number of time-shift appear in the inhomogeneity f in the strangeness-free formulation (16) is 2.
- ii) If $\alpha = 0$, then the matrix pair $([0 \ \alpha \ n+2], [0 \ 0 \ n+2])$ have hidden redundancy. Now we choose $S_n^{(1)} = [1 \ 0]$, $Z_n^{(1)} = [0 \ 1]$, $Z_n^{(3)} = [-1 \ 0]$. The resulting system (13) now reads

$$\begin{aligned} &\begin{bmatrix} 1 & n & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(n+1) + \begin{bmatrix} 0 & 0 & n \\ 0 & n+1 & 0 \\ 0 & 0 & n+1 \end{bmatrix} x(n) \\ &= \begin{bmatrix} f_2(n) \\ f_1(n) - f_2(n+1) - f_3(n+2) - f_3(n+1) \\ f_3(n) \end{bmatrix}. \end{aligned} \quad (17)$$

¹⁶⁹ Algorithm 1 terminates here, and the strangeness-index is $\mu = 2$. However, the
¹⁷⁰ number of time-shifts appearing in the inhomogeneity f in the strangeness-free
¹⁷¹ formulation (17) remains 2.

172 As a direct consequence of Theorem 3.8, we obtain the solvability for (2) as
 173 follows.

174 **Corollary 3.10.** *Under the assumption of Theorem 3.8, the following state-
 175 ments hold true.*

- 176 i) *The corresponding IVP for the SiDE (2) is solvable if and only if either
 177 $v^\mu = 0$ or $\hat{g}_4(n) = 0$ for all $n \geq n_0$. Furthermore, it is uniquely solvable if,
 178 in addition, we have $d = m - v^\mu$.*
- 179 ii) *The initial condition (3) is consistent if and only if the following equalities
 180 hold.*

$$\begin{aligned}\hat{B}_{n_0,2}x_1 + \hat{C}_{n_0,2}x_0 &= \hat{g}_2(n_0), \\ \hat{C}_{n_0,3}x_0 &= \hat{g}_3(n_0).\end{aligned}$$

181 Another direct consequence of Theorem 5.3 is that we can obtain an inherent
 182 regular difference equation as follows.

Corollary 3.11. *Assume that the IVP (2)-(3) is uniquely solvable for any
 consistent initial condition. Under the assumption of Theorem 3.8, the solution
 to this IVP is also a solution to the (implicit) inherent regular difference
 equation*

$$\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3} \end{bmatrix} x(n+2) + \begin{bmatrix} \hat{B}_{n,1} \\ \hat{C}_{n+1,2} \\ 0 \end{bmatrix} x(n+1) + \begin{bmatrix} \hat{C}_{n,1} \\ 0 \\ 0 \end{bmatrix} x(n) = \begin{bmatrix} \hat{g}_1(n) \\ \hat{g}_2(n+1) \\ \hat{g}_3(n+2) \end{bmatrix}, \quad (18)$$

183 where the matrix $[\hat{A}_{n,1}^T \quad \hat{B}_{n+1,2}^T \quad \hat{C}_{n+2,3}^T]$ is invertible for all $n \geq n_0$.

184 **Remark 3.12.** Unlike the procedures in [2, 13, 17], we do not change the
 185 variable x . This approach permits us to simplify significantly the condensed
 186 forms in these references. We emphasize that as in (9), we only require five
 187 constant rank conditions within one step of index reduction, instead of seven as
 188 in [17]. Therefore, by this way the domain of application for SiDEs (and also
 189 for DAEs in the continuous-time case) will be enlarged. This approach is also
 190 useful for the control analysis of the descriptor system (1), as will be seen in the
 191 next section.

192 **Remark 3.13.** i) Within one loop of Algorithm 1, for each n , we have used four
 193 Singular Value Decompositions (SVDs) to remove the hidden redundancies in
 194 two matrix pairs. The total cost depends on the problems itself, i. e., depending
 195 on sizes of the matrix pairs which applied SVDs. Nevertheless, it does not exceed
 196 $\mathcal{O}(m^2d^2)$.

197 ii) Unfortunately, since $Z_n^{(3)}, Z_n^{(4)}, Z_n^{(5)}$ are not orthogonal, in general Algorithm
 198 1 could not be stably implemented. For the numerical solution to the IVP
 199 (2)-(3), we will consider a suitable numerical scheme in Section 5.

200 **4. Regularization of second order descriptor systems**

201 Based on the index reduction procedure for SiDEs in Section 3, in this sec-
 202 tion we construct the strangeness-index concept for the descriptor system (1).
 203 The solvability analysis for first order descriptor systems with variable coef-
 204 ficients have been carefully discussed in [3, 11, 19]. Nevertheless, for second
 205 order descriptor systems, this problem has been rarely considered. We refer the
 206 interested readers to [13, 23] for continuous-time systems.

207 It is well known that in regularization procedures of continuous-time systems,
 208 one should avoid differentiating equations that involve an input function, due
 209 to the fact that it may not be differentiable. We will also keep this spirit, and
 210 hence, will not shift any equation that involve an input function, since it may
 211 destroy the causality of the considered system, as in Example 2.1. Instead of it,
 212 we will also incorporate proportional state and first order feedback within each
 213 index reduction step of the regularization procedure, as will be seen later. Now
 214 let us present two auxiliary lemmas, which will be very useful later.

Lemma 4.1. *Given four matrices \check{A} , \check{B} , \check{C} in $\mathbb{R}^{m,d}$ and \check{D} in $\mathbb{R}^{m,p}$. Let us consider the following matrices whose columns span orthogonal bases of the associated vector spaces*

$$\begin{array}{lll} T_1 & \text{basis of } \text{kernel}(\check{A}^T), & \text{and} \\ W_1 & \text{basis of } \text{kernel}(T_1^T \check{D})^T, & \text{and} \\ J_{B_1} & := W_1^T T_1^T \check{B}, & \text{and} \\ J_{C_1} & := W_1^T T_1^T \check{C}, & \text{and} \\ T_2 & \text{basis of } \text{kernel}(J_{B_1}^T), & \text{and} \\ T_3 & \text{basis of } \text{kernel}(J_{B_2}^T), & \text{and} \\ T_4 & \text{basis of } \text{kernel}(T_2^T J_{C_1})^T, & \text{and} \end{array} \quad \begin{array}{lll} T_{1,\perp} & \text{basis of } \text{range}(\check{A}), & \\ W_{1,\perp} & \text{basis of } \text{range}(T_1^T \check{D}), & \\ J_D & := W_{1,\perp}^T T_1^T \check{D}, & \\ J_{B_2} & := W_{1,\perp}^T T_{1,\perp}^T \check{B}, & \\ J_{C_2} & := W_{1,\perp}^T T_1^T \check{C}, & \\ T_{2,\perp} & \text{basis of } \text{range}(J_{B_1}), & \\ T_{3,\perp} & \text{basis of } \text{range}(J_{B_2}), & \\ T_{4,\perp} & \text{basis of } \text{range}(T_2^T J_{C_1}). & \end{array}$$

215 Then the following assertions hold true.

216 i) The matrices $[T_{i,\perp}^T T_i^T]^T$, $i = 1, \dots, 4$, and $[W_{1,\perp}^T W_1^T]^T$ are orthogonal.
 217 ii) The matrices $T_{1,\perp}^T \check{A}$, $T_{2,\perp}^T J_{B_1}$, $T_{3,\perp}^T J_{B_2}$, $T_{4,\perp}^T T_2^T J_{C_1}$, and J_D have full row
 218 rank.

219 iii) Moreover, there exists an orthogonal matrix \check{U} such that

$$\check{U} \begin{bmatrix} \check{A} & \check{B} & \check{C} & | & \check{D} \end{bmatrix} = \left[\begin{array}{ccc|c} \check{A}_1 & \check{B}_1 & \check{C}_1 & \check{D}_1 \\ 0 & \check{B}_2 & \check{C}_2 & 0 \\ 0 & 0 & \check{C}_3 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & \check{B}_4 & \check{C}_4 & \check{D}_4 \\ 0 & 0 & \check{C}_5 & \check{D}_5 \end{array} \right], \quad (19)$$

219 where the matrices \check{A}_1 , \check{B}_2 , \check{B}_4 , \check{C}_3 , \check{D}_4^T , \check{D}_5^T have full row rank.

PROOF. The first two claims followed directly from Lemma 2.5. To prove the third claim, we construct the desired matrix \check{U} as follows

$$\check{U} := \begin{bmatrix} I & & & \\ & I & T_{4,\perp}^T & \\ & & T_4^T & \\ & & & I \end{bmatrix} \cdot \begin{bmatrix} I & & & \\ & T_{2,\perp}^T & & \\ & T_2^T & & \\ & & T_{3,\perp}^T & \\ & & & T_3^T \end{bmatrix} \cdot \begin{bmatrix} I & & \\ & W_1^T & \\ & W_{1,\perp}^T & \end{bmatrix} \cdot \begin{bmatrix} T_{1,\perp}^T \\ T_1^T \end{bmatrix} .$$

Thus, we have that

$$\check{U} \begin{bmatrix} \check{A} & \check{B} & \check{C} & | & \check{D} \end{bmatrix} = \left[\begin{array}{ccc|c} T_{1,\perp}^T \check{A} & T_{1,\perp}^T \check{B} & T_{1,\perp}^T \check{C} & T_{1,\perp}^T \check{D} \\ 0 & T_{2,\perp}^T J_{B1} & T_{2,\perp}^T J_{C1} & 0 \\ 0 & 0 & T_{4,\perp}^T T_2^T J_{C1} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & T_{3,\perp}^T J_{B2} & T_{3,\perp}^T J_{C2} & T_{3,\perp}^T J_D \\ 0 & 0 & T_3^T J_{C2} & T_3^T J_D \end{array} \right].$$

²²⁰ Due to the parts i) and ii), we see that this is exactly the desired form (19).

²²¹ **Lemma 4.2.** Let $P \in \mathbb{R}^{p,d}$, $Q \in \mathbb{R}^{q,d}$ be two full row rank matrices, where
²²² $p + q \leq d$. Then the following assertions hold true.

- ²²³ i) There exists a matrix $F \in \mathbb{R}^{d,d}$ such that $H := \begin{bmatrix} P \\ QF \end{bmatrix}$ has full row rank.
²²⁴ ii) For any $G \in \mathbb{R}^{q,d}$, there exists a matrix $F \in \mathbb{R}^{d,d}$ such that $\begin{bmatrix} P \\ G+QF \end{bmatrix}$ has
²²⁵ full row rank.

PROOF. i) First we consider the SVDs of P and Q that reads

$$U_P P V_P = [\Sigma_P \quad 0_{p,d-p}], \quad U_Q Q V_Q = [\Sigma_Q \quad 0_{q,d-q}],$$

where Σ_P , Σ_Q are nonsingular, diagonal matrices, and $0_{p,d-p}$ (resp. $0_{q,d-q}$) are the zero matrix of size p by $d-p$ (resp. q by $d-q$).

By choosing $F := V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^T$ we see that

$$\begin{bmatrix} U_P & 0 \\ 0 & U_Q \end{bmatrix} \begin{bmatrix} P \\ QF \end{bmatrix} V_P = \begin{bmatrix} U_P P V_P \\ U_Q Q F V_P \end{bmatrix} = \begin{bmatrix} \Sigma_P & 0_{p,d-p-q} & 0_{p,q} \\ 0_{q,p} & 0_{p,d-p-q} & \Sigma_Q \end{bmatrix},$$

and hence, the claim i) is proven.

ii) Clearly, in case that the matrix F is very big in norm, then G is only a small perturbation, and hence for sufficiently large η , by choosing

$$F := \eta V_Q \begin{bmatrix} 0 & I_q \\ I_{d-q} & 0 \end{bmatrix} V_P^T,$$

²²⁶ we obtain the full row rank property of $\begin{bmatrix} P \\ G+QF \end{bmatrix}$.

²²⁷ **Remark 4.3.** It should be noted that, the proof of Lemmas 4.1 and 4.2 are
²²⁸ constructive, and all the matrices $T_{i,\perp}$, T_i , $i = 1, \dots, 4$, $W_{1,\perp}$, W_1 and F can be
²²⁹ stably computed.

²³⁰ In the following theorem, we give the condensed form for system (1).

Theorem 4.4. *i) Consider the descriptor system (1). Then there exist two pointwise nonsingular matrix sequences $\{U_n\}_{n \geq n_0}$, $\{V_n\}_{n \geq n_0}$ such that by scaling (1) with U_n and changing $u(n) = V_n v(n)$, $f(n) := U_n f(n)$, we can transform (1) to the system*

$$\begin{array}{c} r_{2,n} \\ r_{1,n} \\ r_{0,n} \\ \varphi_{1,n} \\ \varphi_{0,n} \\ v_n \end{array} \left[\begin{array}{ccc} A_{n,1} & B_{n,1} & C_{n,1} \\ 0 & B_{n,2} & C_{n,2} \\ 0 & 0 & C_{n,3} \\ 0 & B_{n,4} & C_{n,4} \\ 0 & 0 & C_{n,5} \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \left[\begin{array}{ccc} D_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \Sigma_{n,1} & 0 \\ 0 & 0 & \Sigma_{n,0} \\ 0 & 0 & 0 \end{array} \right] \underbrace{\begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \\ v(n) \end{bmatrix}}_{\tilde{f}(n)} = \tilde{f}(n) \quad (20)$$

for all $n \geq n_0$. Here, sizes of the block rows are $r_{2,n}$, $r_{1,n}$, $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n , the matrices $A_{n,1}$, $B_{n,2}$, $B_{n,4}$, $C_{n,3}$ are of full row rank and the matrices $\Sigma_{n,1}$, $\Sigma_{n,0}$ are nonsingular and diagonal.

ii) Furthermore, if the matrix $[A_{n,1}^T \ B_{n+1,2}^T \ C_{n+2,3}^T]^T$ is of full row rank for all $n \geq n_0$ then there exists a first order feedback of the form

$$v(n) = F_{n,1}x(n+1) + F_{n,0}x(n) \quad (21)$$

such that the closed loop system

$$A_n x(n+2) + (B_n + D_n F_{n,1}) x(n+1) + (C_n + D_n F_{n,0}) x(n) = f(n)$$

²³¹ is strangeness-free.

Proof. i) First we apply Lemma 4.1 to four matrices A_n , B_n , C_n and D_n to obtain the matrix U_n that satisfies (19). Decompose the matrix $[\check{D}_4^T \ \check{D}_5^T]^T$ via one SVD, we then obtain the block $\begin{bmatrix} 0 & \Sigma_{n,1} & 0 \\ 0 & 0 & \Sigma_{n,0} \end{bmatrix}$. Finally, we use Gaussian elimination to nullify all non-zero matrices in the two columns of \check{D} that contain $\Sigma_{n,1}$ and $\Sigma_{n,0}$, and hence, we obtain

$$(U_n [A_n \ B_n \ C_n], \ U_n D_n V_n) = \left(\begin{array}{ccc|ccc} A_{n,1} & B_{n,1} & C_{n,1} & D_{n,1} & 0 & 0 \\ 0 & B_{n,2} & C_{n,2} & 0 & 0 & 0 \\ 0 & 0 & C_{n,3} & 0 & 0 & 0 \\ \hline 0 & B_{n,4} & C_{n,4} & 0 & \Sigma_{n,1} & 0 \\ 0 & 0 & C_{n,5} & 0 & 0 & \Sigma_{n,0} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \text{ for all } n \geq n_0. \quad (22)$$

This directly leads us the desired system (20).

ii) By applying Lemma 4.2 for $P = [A_{n,1}^T \ B_{n+1,2}^T \ C_{n+2,3}^T]^T$, $Q = \begin{bmatrix} 0 & \Sigma_{n,1} & 0 \\ 0 & 0 & \Sigma_{n,0} \end{bmatrix}$

and $G = \begin{bmatrix} B_{n+1,4} \\ C_{n+2,5} \end{bmatrix}$, we see that there exist two matrix sequences $\{F_{n,1}\}_{n \geq n_0}$, $\{F_{n,0}\}_{n \geq n_0}$ such that by choosing the feedback of the form (21) then the matrix

$$\left[\begin{array}{c} A_{n,1} \\ B_{n+1,2} \\ C_{n+2,3} \\ \hline B_{n+1,4} + \begin{bmatrix} 0 & \Sigma_{n,1} & 0 \end{bmatrix} F_{n+1,1} \\ C_{n+2,5} + \begin{bmatrix} 0 & 0 & \Sigma_{n,0} \end{bmatrix} F_{n+2,0} \end{array} \right]$$

²³² has full row rank for all $n \geq n_0$. This completes the proof. \square

²³³ In order to build an index reduction procedure for (1), we also need the
²³⁴ following assumption.

²³⁵ **Assumption 4.5.** Assume that the local characteristic invariants $r_{2,n}$, $r_{1,n}$,
²³⁶ $r_{0,n}$, $\varphi_{1,n}$, $\varphi_{0,n}$, v_n , become global, i.e., they are constant for all $n \geq n_0$.

²³⁷ From Theorem 4.4, we see that we only need to remove the hidden redundan-
²³⁸ cies in the upper part of (20) as follows. By performing one index reduction
²³⁹ step for the upper part of (20), as in section 3, we obtain the following lemma.

Lemma 4.6. *Assume that the upper part of the descriptor system (20) is not strangeness-free. Then for each input sequence $\{v(n)\}_{n \geq n_0}$, it has exactly the same solution set as the following system*

$$\begin{array}{l} \tilde{r}_2 \quad \begin{bmatrix} \tilde{A}_{n,1} & \tilde{B}_{n,1} & \tilde{C}_{n,1} \\ 0 & \tilde{B}_{n,2} & \tilde{C}_{n,2} \\ 0 & 0 & \tilde{C}_{n,3} \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \tilde{D}_{n,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \end{bmatrix} = \tilde{f}(n) \\ \tilde{r}_1 \\ \tilde{r}_0 \\ \varphi_1 \\ \varphi_0 \\ \tilde{v} \end{array} \quad (23)$$

²⁴⁰ for all $n \geq n_0$. Here, we have $\tilde{r}_2 = r_2 - s_2$, $\tilde{r}_1 = r_1 + s_2 - s_1$, $\tilde{r}_0 = r_0 + s_1$,
²⁴¹ $\tilde{v} \geq v$ for some $s_2 > 0$, $s_1 > 0$. Furthermore, both pairs $(\tilde{A}_{n,1}, [\tilde{B}_{n,2}^T \ \tilde{C}_{n,3}^T]^T)$
²⁴² and $(\tilde{B}_{n,2}, \tilde{C}_{n,3})$ have no hidden redundancy.

²⁴³ *Proof.* System (23) is directly obtained by applying Lemma 3.7 to the upper
²⁴⁴ part of (20). To keep the brevity of this paper, we will omit the details here. \square

²⁴⁵ Similar to the observation made in section 3, we also see that an *index
246 reduction step*, which passes system (20) to the new form (23) has reduced the
247 upper rank r^u by at least $s_2 + s_1$. Continue in this way until $s_2 = s_1 = 0$, finally
248 we obtain a strangeness-free descriptor system in the next theorem.

Theorem 4.7. Consider the descriptor system (1). Furthermore, assume that Assumption 4.5 is fulfilled whenever needed. Then for each fixed input sequence $\{u(n)\}_{n \geq n_0}$, system (1) has the same solution set as the so-called strangeness-free descriptor system

$$\begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \left[\begin{array}{ccc} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ \hline 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hline \hat{D}_{n,4} \\ \hat{D}_{n,5} \\ 0 \end{bmatrix} u(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hline \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix} \quad \text{for all } n \geq n_0, \quad (24)$$

249 where the matrices $[\hat{A}_{n,1}^T \quad \hat{B}_{n+1,2}^T \quad \hat{C}_{n+2,3}^T]^T$, $[\hat{D}_{n,4}^T \quad \hat{D}_{n,5}^T]^T$ have full row rank
250 for all $n \geq n_0$.

Proof. By repeating index reduction steps until the upper rank r^u stop decreasing, we obtain the system

$$\begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hline \hat{\varphi}_1 \\ \hat{\varphi}_0 \\ \hat{v} \end{array} \left[\begin{array}{ccc} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ \hline 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \\ 0 & 0 & 0 \end{array} \right] \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \hat{\Sigma}_{n,1} & 0 \\ 0 & 0 & \hat{\Sigma}_{n,0} \\ 0 & 0 & 0 \end{bmatrix} v(n) = \begin{bmatrix} \hat{f}_1(n) \\ \hat{f}_2(n) \\ \hat{f}_3(n) \\ \hline \hat{f}_4(n) \\ \hat{f}_5(n) \\ \hat{f}_6(n) \end{bmatrix}$$

for all $n \geq n_0$, where the matrix $[\hat{A}_{n,1}^T \quad \hat{B}_{n+1,2}^T \quad \hat{C}_{n+2,3}^T]^T$ has full row rank for all $n \geq n_0$. The new input sequence $\{v(n)\}_{n \geq n_0}$ satisfies $u(n) = V_n v(n)$, where V_n is nonsingular for all $n \geq n_0$. Transform back $v(n) = V_n^{-1} u(n)$ and set

$$\begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hline \hat{D}_{n,4} \\ \hat{D}_{n,5} \\ 0 \end{bmatrix} := \begin{bmatrix} \hat{D}_{n,11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & \hat{\Sigma}_{n,1} & 0 \\ 0 & 0 & \hat{\Sigma}_{n,0} \\ 0 & 0 & 0 \end{bmatrix} V_n^{-1},$$

251 we obtain exactly the strangeness-free descriptor system (24). \square

252 As a direct corollary of Theorem 4.7, we obtain the existence and uniqueness
253 of a solution to the closed-loop system via feedback as follows.

254 **Corollary 4.8.** Under the conditions of Theorem 4.7, the following statements
255 hold true.

- 256 i) There exists a first order feedback of the form (21) such that the closed-loop
257 system is solvable if and only if either $\hat{v} = 0$ or $\hat{f}_6(n) = 0$ for all $n \geq n_0$.
258 ii) Furthermore, the solution to the corresponding IVP (of the closed-loop sys-
259 tem) is unique if and only if in addition, $d = m - \hat{v}$.

260 **Remark 4.9.** It should be noted that, analogously to SiDEs, each index re-
 261 duction step of the descriptor system (1) also makes use of Lemma 3.7, where
 262 the matrices $Z_n^{(i)}$, $i = 3, 4, 5$, may not be orthogonal. Furthermore, in Theorem
 263 4.4, two matrices U_n , V_n are only nonsingular but not orthogonal. Therefore,
 264 in general, the strangeness-free formulation (24) could not be stably computed.
 265 For the numerical treatment of (continuous-time) second order DAEs, in [23]
 266 a different approach was developed. We will modify it for SiDEs/descriptor
 267 systems in the next section.

268 **Remark 4.10.** Another interesting method in the study of descriptor systems
 269 is the *behavior approach*, where we do not distinguish the state x and an input
 270 u but combine them in one *behavior vector*. Then (1) will become a SiDE of
 271 this behavior variable, and hence, we can apply the results in section 3 for this
 272 system. Nevertheless, due to the reinterpretation of variables, this approach
 273 may alter the strangeness-free form (24). To keep the brevity of this research,
 274 we will not present the details here. For the interested readers, we refer to
 275 [11, 19, 20] for the case of first order DAEs, and to [23] for the case of second
 276 order DAEs.

277 **5. Difference arrays associated with second order SiDEs/descriptor
 278 systems**

279 In two previous sections, to analyze the solvability of the SiDE (2) or of the
 280 descriptor system (1), first one needs to bring it into the strangeness-free form.
 281 Nevertheless, sometime this task is not feasible, for example when Assumptions
 282 3.4 or 4.5 is violated at some index reduction steps. These difficulties have also
 283 been observed for continuous-time systems of first or higher orders in [11, 23].
 284 A breakthrough, thanks to Campbell [4] while considering DAEs, is to differ-
 285 entiate a given system a number of times and put everyone of them, including
 286 the original one, into a so-called *inflated system*. Then the strangeness-free for-
 287 mulation will be determined by appropriate selection of equations inside this
 288 inflated system. In this section we will examine this approach to the descriptor
 289 system (1). The analysis for SiDEs of the form (2) can be obtained by simply
 290 setting D_n to be $0_{m,p}$ for all n . We further assume the following condition.

291 **Assumption 5.1.** Consider the descriptor system (1). Assume that there exists
 292 a first order feedback of the form (21) such that the corresponding IVP of the
 293 closed-loop system is uniquely solvable.

294 Notice that, in case of the SiDE (2), Assumption 5.1 means that the IVP
 295 (2)-(3) is uniquely solvable. Now let us introduce the *difference-inflated system
 296 of level $\ell \in \mathbb{N}$* as follows.

$$\begin{aligned}
 A_n x(n+2) + B_n x(n+1) + C_n x(n) + D_n u(n) &= f(n), \\
 A_{n+1} x(n+3) + B_{n+1} x(n+2) + C_{n+1} x(n+1) + D_{n+1} u(n+1) &= f(n+1), \\
 &\dots \\
 A_{n+\ell} x(n+\ell+2) + B_{n+\ell} x(n+\ell+1) + C_{n+\ell} x(n+\ell) + D_{n+\ell} u(n+\ell) &= f(n+\ell).
 \end{aligned}$$

We rewrite this system as

$$\begin{aligned}
& \underbrace{\begin{bmatrix} C_n & B_n & A_n \\ C_{n+1} & B_{n+1} & A_{n+1} \\ \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots \\ C_{n+\ell} & B_{n+\ell} & A_{n+\ell} \end{bmatrix}}_{=: \mathcal{M}} \underbrace{\begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \vdots \\ x(n+\ell) \end{bmatrix}}_{=: \mathcal{X}} + \\
& + \underbrace{\begin{bmatrix} D_n & & \\ & D_{n+1} & \\ & \ddots & \\ & & D_{n+\ell} \end{bmatrix}}_{=: \mathcal{N}} \underbrace{\begin{bmatrix} u(n) \\ u(n+1) \\ \vdots \\ u(n+\ell) \end{bmatrix}}_{=: \mathcal{U}} = \underbrace{\begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(n+\ell) \end{bmatrix}}_{=: \mathcal{G}}
\end{aligned} \quad \text{for all } n \geq n_0. \quad (25)$$

²⁹⁷ **Definition 5.2.** Suppose that the descriptor system (1) satisfies Assumption
²⁹⁸ 5.1. Let ℓ be the minimum number such that a strangeness-free descriptor
²⁹⁹ system of the form (24) can be extracted from (25) by using elementary matrix-
³⁰⁰ row operations. Then the so-called *shift-index* of (1), denoted by ν , is set by
³⁰¹ $\ell/2$ if ℓ is even and by $(\ell + 1)/2$ otherwise.

³⁰² We give the relation between this shift-index ν and the strangeness-index μ
³⁰³ in the following proposition.

³⁰⁴ **Proposition 5.3.** Suppose that the descriptor system (1) satisfies Assumption
³⁰⁵ 5.1. If the strangeness-index μ is well-defined, then so is the shift-index ν .
³⁰⁶ Furthermore, we have that $\nu \leq \mu$.

³⁰⁷ **Proof.** The claim is straight forward, since every reformulation step performed
³⁰⁸ in Algorithm 1 is a consequence of an inflated system (25) with $\ell = 2\mu$ or
³⁰⁹ $2\mu - 1$. \square

³¹⁰ **Remark 5.4.** As will be seen later in Example 5.7, for second order SiDEs, the
³¹¹ shift-index can be strictly smaller than the strangeness index.

³¹² **Remark 5.5.** Restricted to the case of first order SiDEs (i.e., $A_n = 0$ for
³¹³ all $n \geq n_0$), the strangeness-index μ defined in this paper is equal to the for-
³¹⁴ ward strangeness-index proposed by Brüll [2]. For second order system, our
³¹⁵ strangeness-index is analogous to the one for continuous-time systems proposed
³¹⁶ by Mehrmann and Shi [17], and by Wunderlich [23]. We, however, emphasize
³¹⁷ that the canonical forms constructed in this research is simpler and more
³¹⁸ convenient from the theoretical viewpoint. Besides that, similar to the case of
³¹⁹ continuous-time systems, the strangeness index μ only gives an upper bound for
³²⁰ the number of shift-forward operator that have been used, in order to achieve
³²¹ the strangeness-free form (14). For further details, see Remark 17, [17].

³²² In the following theorem we will answer the question how to derive the
³²³ strangeness-free formulation (24) from (25).

Theorem 5.6. *Assume that the shift index ν of the descriptor system (1) is well-defined. Furthermore, suppose that (1) satisfies Assumption 5.1. Then any solution to the descriptor system (1) is also a solution to the following system*

$$\begin{array}{c} \hat{r}_2 \\ \hat{r}_1 \\ \hat{r}_0 \\ \hat{\varphi}_1 \\ \hat{\varphi}_0 \end{array} \begin{bmatrix} \hat{A}_{n,1} & \hat{B}_{n,1} & \hat{C}_{n,1} \\ 0 & \hat{B}_{n,2} & \hat{C}_{n,2} \\ 0 & 0 & \hat{C}_{n,3} \\ 0 & \hat{B}_{n,5} & \hat{C}_{n,5} \\ 0 & 0 & \hat{C}_{n,6} \end{bmatrix} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} + \begin{bmatrix} \hat{D}_{n,1} \\ 0 \\ 0 \\ \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix} u(n) = \begin{bmatrix} \hat{G}_{n,1} \\ \hat{G}_{n,2} \\ \hat{G}_{n,3} \\ \hat{G}_{n,4} \\ \hat{G}_{n,5} \end{bmatrix} \quad \text{for all } n \geq n_0, \end{array} \quad (26)$$

where the matrices $[\hat{A}_{n,1}^T \quad \hat{B}_{n+1,2}^T \quad \hat{C}_{n+2,3}^T]^T$, $[\hat{D}_{n,4}^T \quad \hat{D}_{n,5}^T]^T$ have full row rank for all $n \geq n_0$. Furthermore, we have that $\sum_{i=0}^2 \hat{r}_i + \sum_{i=0}^1 \hat{\varphi}_i = d$, or equivalently,

$$\text{rank} \left(\begin{bmatrix} \hat{A}_{n,1} \\ \hat{B}_{n+1,2} \\ \hat{C}_{n+2,3}^T \end{bmatrix} \right) + \text{rank} \left(\begin{bmatrix} \hat{D}_{n,4} \\ \hat{D}_{n,5} \end{bmatrix} \right) = d. \quad (27)$$

Proof. Assume that ν is already known, we now construct an algorithm to select the strangeness-free descriptor system (24) from the inflated system (25). For notational convenience, we will follow the MATLAB language, [15]. Consider the following spaces and matrices

$$\begin{array}{ll} \mathcal{W} & := [\mathcal{M}(:, 3n+1 : end) \quad \mathcal{N}(:, n+1 : end)], \\ U_1 & \text{basis of kernel}(\mathcal{W}^T), \text{ and } U_{1,\perp} \text{ basis of range}(\mathcal{W}). \end{array} \quad (28)$$

Due to Lemma 2.5, we have that $U_1^T \mathcal{W} = 0$ and $U_{1,\perp}^T \mathcal{W}$ has full row rank. Furthermore, the matrix $[U_1 \quad U_{1,\perp}]^T$ is nonsingular, and hence, system (25) is equivalent to the coupled system below.

$$U_1^T \mathcal{M}(:, 1 : 3n) \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \end{bmatrix} + U_1^T \mathcal{N}(:, 1 : n) u(n) = U_1^T \mathcal{G}, \quad (29)$$

$$U_{1,\perp}^T \mathcal{W} \begin{bmatrix} x(n+3) \\ \vdots \\ x(n+\nu) \\ \hline u(n+1) \\ \vdots \\ u(n+\nu) \end{bmatrix} + U_{1,\perp}^T [\mathcal{M}(:, 1 : 3n) \quad \mathcal{N}(:, 1 : n)] \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ \hline u(n) \end{bmatrix} = U_{1,\perp}^T \mathcal{G}. \quad (30)$$

Due to the full row rank property of $U_{1,\perp}^T \mathcal{W}$, we see that (30) plays no role in the determination of the strangeness-free descriptor system (24). Thus, (24) is

a consequence of (29). For notational convenience, let us rewrite system (29) as

$$\left[\begin{array}{ccc|c} \check{A} & \check{B} & \check{C} & \check{D} \end{array} \right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{array} \right] = \check{G}.$$

Scaling this system with the matrix \check{U} obtained in Lemma 4.1, we have

$$\left[\begin{array}{ccc|c} \check{A}_1 & \check{B}_1 & \check{C}_1 & \check{D}_1 \\ 0 & \check{B}_2 & \check{C}_2 & 0 \\ 0 & 0 & \check{C}_3 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{array} \right] = \left[\begin{array}{c} \check{G}_1 \\ \check{G}_2 \\ \check{G}_3 \\ 0 \end{array} \right], \quad (31)$$

$$\left[\begin{array}{ccc|c} 0 & \check{B}_4 & \check{C}_4 & \check{D}_4 \\ 0 & 0 & \check{C}_5 & \check{D}_5 \end{array} \right] \left[\begin{array}{c} u(n) \\ \check{G}_4 \\ \check{G}_5 \end{array} \right]$$

where the matrices $\check{A}_1, \check{B}_2, \check{B}_4, \check{C}_3$, and $[\check{D}_4^T \check{D}_5^T]^T$ have full row rank. Notice that the presence of the 0 block on the right hand side vector is due to the existence of a solution (Assumption 5.1). Applying Lemma 2.5 consecutively for two following matrix pairs $(\check{B}_2, \check{C}_3)$, $(\check{A}_1, [\check{B}_2^T \check{C}_3^T]^T)$, we obtain two orthogonal matrices $\begin{bmatrix} S_n^{(i)} \\ Z_n^{(i)} \end{bmatrix} \in \mathbb{R}^{r_i, r_i}$, $i = 1, 2$ such that both pairs $(S_n^{(1)} \check{B}_2, \check{C}_3)$, $(S_n^{(2)} \check{A}_1, [\check{B}_2^T \check{C}_3^T]^T)$ have no hidden redundancy. Scaling the first and second block row equations of (31) with $S_n^{(2)}$ and $S_n^{(1)}$ respectively, we obtain

$$\left[\begin{array}{ccc|c} S_n^{(2)} \check{A}_1 & S_n^{(2)} \check{B}_1 & S_n^{(2)} \check{C}_1 & S_n^{(2)} \check{D}_1 \\ 0 & S_n^{(1)} \check{B}_2 & S_n^{(1)} \check{C}_2 & 0 \end{array} \right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{array} \right] = \left[\begin{array}{c} S_n^{(2)} \check{G}_1 \\ S_n^{(1)} \check{G}_2 \end{array} \right].$$

Combining these equations with the third, fifth and sixth block equations of (31), we obtain the system

$$\left[\begin{array}{ccc|c} S_n^{(2)} \check{A}_1 & S_n^{(2)} \check{B}_1 & S_n^{(2)} \check{C}_1 & S_n^{(2)} \check{D}_1 \\ 0 & S_n^{(1)} \check{B}_2 & S_n^{(1)} \check{C}_2 & 0 \\ 0 & 0 & \check{C}_3 & 0 \end{array} \right] \left[\begin{array}{c} x(n+2) \\ x(n+1) \\ x(n) \\ u(n) \end{array} \right] = \left[\begin{array}{c} S_n^{(2)} \check{G}_1 \\ S_n^{(1)} \check{G}_2 \\ \check{G}_3 \\ \check{G}_4 \\ \check{G}_5 \end{array} \right]. \quad (32)$$

³²⁴ which is exactly our desired system (26). Moreover, due to Lemma 2.2, the
³²⁵ matrix $[(S_n^{(2)} \check{A}_1)^T \ (S_n^{(1)} \check{B}_2)^T \ \check{C}_3^T]^T$ has full row rank, and the identity (27)
³²⁶ holds true due to Assumption 5.1.

³²⁷

Finally, we will prove that system (26) is not affected by left equivalence transformation. Let us assume that (1) is left equivalent to the SiDE

$$\tilde{A}_n x(n+2) + \tilde{B}_n x(n+1) + \tilde{C}_n x(n) + \tilde{D}_n u(n) = \tilde{f}(n) \quad \text{for all } n \geq n_0. \quad (33)$$

Thus, there exists a pointwise nonsingular matrix sequence $\{P_n\}_{n \geq n_0}$ such that

$$[\tilde{A}_n \ \tilde{B}_n \ \tilde{C}_n \ \tilde{D}_n] = P_n [A_n \ B_n \ C_n \ D_n] \text{ and } \tilde{f}(n) = P_n f(n) \text{ for all } n \geq n_0.$$

Therefore, the difference-inflated system of level ℓ for system (33) takes the form

$$\tilde{\mathcal{M}}\mathcal{X} + \tilde{\mathcal{N}}\mathcal{U} = \tilde{\mathcal{G}}, \quad (34)$$

where the matrix coefficients are

$$\tilde{\mathcal{M}} = \text{diag}(P_n, \dots, P_{n+\ell}) \mathcal{M}, \quad \tilde{\mathcal{N}} = \text{diag}(P_n, \dots, P_{n+\ell}) \mathcal{N}, \quad \tilde{\mathcal{G}} = \text{diag}(P_n, \dots, P_{n+\ell}) \mathcal{G}.$$

328 This follows that two systems (25) and (34) are left equivalent, which finishes
329 the proof. \square

330 We summarize our result in the following algorithm.

Algorithm 2 Strangeness-free formulation for SiDEs using difference arrays

Input: The SiDE (1).

Output: The strangeness-free descriptor system (26) and the minimal number of shifts ℓ .

- 1: Set $\ell := 0$.
 - 2: Construct the difference-inflated system of level ℓ , and rewrite it in the form (25).
 - 3: Find U_1 as in (28) and construct system (29).
 - 4: Find \check{U} as in Lemma 4.1 and construct system (31).
 - 5: Find the matrices $S_n^{(1)}, S_n^{(2)}$ in the process used to remove the hidden redundancies in two matrix pairs $(\check{B}_2, \check{C}_3), (\check{A}_1, [\check{B}_2^T \check{C}_3^T]^T)$, respectively.
 - 6: Construct the system (32).
 - 7: **if** $\text{rank} [\hat{A}_{n,1}^T \ \hat{B}_{n+1,2}^T \ \hat{C}_{n+2,3}^T]^T + \text{rank} [\hat{D}_{n,4}^T \ \hat{D}_{n,5}^T]^T = d$ **then STOP.**
 - 8: **else** set $\ell := \ell + 1$ and go to 2
 - 9: **end if**
-

331 In order to illustrate Algorithm 2, we consider the following two examples.

Example 5.7. Let us revisit system (15) for the case $\alpha = 0$. In this system, $D_n = 0$ for all $n \geq 0$. For $\ell = 2$, the inflated system (25) reads

$$\left[\begin{array}{ccc|cc} C_n & B_n & A_n & 0 & 0 \\ 0 & C_{n+1} & B_{n+1} & A_{n+1} & 0 \\ 0 & 0 & C_{n+2} & \underbrace{B_{n+2} & A_{n+2}}_{=: \mathcal{W}} \\ \end{array} \right] \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \\ x(n+3) \\ x(n+4) \end{bmatrix} = \begin{bmatrix} f(n) \\ f(n+1) \\ f(n+2) \\ f(n+3) \\ f(n+4) \end{bmatrix} \quad (35)$$

Let U_1 be the basis of $\text{kernel}(\mathcal{W}^T)$. We then determine system (29) by scaling (35) with U_1^T . The resulting system reads

$$U_1^T \begin{bmatrix} C_n & B_n & A_n \\ 0 & C_{n+1} & B_{n+1} \\ 0 & 0 & C_{n+2} \end{bmatrix} \begin{bmatrix} x(n) \\ x(n+1) \\ x(n+2) \end{bmatrix} = U_1^T \begin{bmatrix} f(n) \\ f(n+1) \\ f(n+2) \end{bmatrix}. \quad (36)$$

332 Finally, by performing Steps 6 to 10 we can extract the strangeness-free form
333 (17) from (36). Thus, we conclude that the shift index is $\nu = 1$, which is the
334 same as the shift index in the case $\alpha \neq 0$. We recall Example 3.9, in which it is
335 shown that the strangeness indices in the two cases are different.

Example 5.8. A singular system of second order differential equations, which describes a three link robot arm [9], is given by

$$\begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(t) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(t),$$

where M_0 represents the nonsingular mass matrix, G_0 the coefficient matrix associated with damping, centrifugal, gravity, and Coriolis forces, K_0 the stiffness matrix, and H_0 the constraint. A simple discretized version of this system with the stepsize h takes the form

$$\begin{aligned} & \begin{bmatrix} M_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - 2x(n+1) + x(n)}{h^2} + \begin{bmatrix} G_0 & 0 \\ 0 & 0 \end{bmatrix} \frac{x(n+2) - x(n)}{2h} \\ & + \begin{bmatrix} K_0 & H_0^T \\ H_0 & 0 \end{bmatrix} x(n+1) = \begin{bmatrix} B_0 \\ 0 \end{bmatrix} u(n+1). \end{aligned}$$

336 As a simple example, let us take $M_0 = G_0 = K_0 = H_0 = B_0 = 1$, $h = 0.01$.
337 Then Algorithm 2 terminates after two steps and hence, the shift index is $\nu = 1$
338 for all $n \geq n_0$. Furthermore, we notice that no matter central, forward or
339 backward difference is chosen for discretizing the derivative $\dot{x}(t)$, the shift index
340 remains unchanged $\nu = 1$. Of course, the resulting strangeness-free descriptor
341 systems are different.

342 6. Conclusion

343 By using the algebraic approach, we have analyzed the solvability of sec-
344 ond order SiDEs/descriptor systems, based on the derived condensed forms
345 constructed under certain constant rank assumptions. In comparison to the
346 previously known procedures [17, 22], we have reduced the number of constant
347 rank conditions in every index reduction step from seven to five. This would
348 enlarge the domain of application for SiDEs (and also for DAEs). However,
349 requiring constant rank assumptions in the discrete-time case seems less nature
350 than in the continuous-time case. To overcome this limitation, we also con-
351 sider the difference-array method, which is numerically stable, to obtain the
352 strangeness-free form. The index theory together with the two algorithms pre-
353 sented in this paper can be extended without difficulty to arbitrarily high order

354 SiDEs/descriptor systems. We also notice that the backward time case ($n \leq n_0$)
 355 can be directly extended from the forward time case, as it has been done in [2].
 356 The analysis of the two-way case, which happens while considering boundary
 357 value problems for SiDEs, is under our on-going research. Furthermore, the con-
 358 densed forms presented in this work also motivate further study on the staircase
 359 form for second order systems, which would be an interesting extension of the
 360 classical result for first order systems, e.g. [21].

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420 **Appendix A. Proof of Lemma 3.7**

PROOF. In order to prove this lemma, we will make use of the shifted equation (11) if the matrix pair $(B_{n,2}, C_{n+1,3})$ has hidden redundancy. Analogously, if the pair $\left(A_{n,1}, [B_{n+1,2}^T \ C_{n+2,3}^T]^T\right)$ has hidden redundancy then we will make use of the shifted equation

$$B_{n+1,2}x(n+2) + C_{n+1,2}x(n+1) = f_2(n+1), \quad (\text{A.1})$$

and may be also the double shifted equation

$$C_{n+2,3}x(n+2) = f_3(n+2). \quad (\text{A.2})$$

Now we observe that (2) has the same solution set as that of the following extended system

$$\begin{array}{c|ccc} r_2 & A_{n,1} & B_{n,1} & C_{n,1} \\ r_1 & 0 & B_{n,2} & C_{n,2} \\ r_0 & 0 & 0 & C_{n,3} \\ v & 0 & 0 & 0 \\ \hline r_0 & 0 & C_{n+1,3} & 0 \\ r_1 & B_{n+1,2} & C_{n+1,2} & 0 \\ r_0 & C_{n+2,3} & 0 & 0 \end{array} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{bmatrix}, \quad (\text{A.3})$$

for all $n \geq n_0$. Therefore, it suffices to prove that any solution to (A.3) is also a solution to (13) and vice versa.

Necessity: The main idea is to apply (only) two elementary row transformations below to system (A.3) to obtain (13).

- i) scaling a block row equation with a nonsingular matrix,
- ii) adding to one row a linear combination of some other rows.

By scaling the first (resp., second) block row equation of (A.3) with an orthogonal matrix $\begin{bmatrix} S_n^{(2)} \\ Z_n^{(2)} \end{bmatrix}$ (resp., $\begin{bmatrix} S_n^{(1)} \\ Z_n^{(1)} \end{bmatrix}$), we obtain an equivalent system to (10), as follows

$$\begin{array}{c|ccc} d_2 & S_n^{(2)}A_{n,1} & S_n^{(2)}B_{n,1} & S_n^{(2)}C_{n,1} \\ s_2 & Z_n^{(2)}A_{n,1} & Z_n^{(2)}B_{n,1} & Z_n^{(2)}C_{n,1} \\ \hline d_1 & 0 & S_n^{(1)}B_{n,2} & S_n^{(1)}C_{n,2} \\ s_1 & 0 & Z_n^{(1)}B_{n,2} & Z_n^{(1)}C_{n,2} \\ \hline r_0 & 0 & 0 & C_{n,3} \\ v & 0 & 0 & 0 \\ \hline r_0 & 0 & C_{n+1,3} & 0 \\ r_1 & B_{n+1,2} & C_{n+1,2} & 0 \\ r_0 & C_{n+2,3} & 0 & 0 \end{array} \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} S_n^{(2)}f_1(n) \\ Z_n^{(2)}f_1(n) \\ \hline S_n^{(1)}f_2(n) \\ Z_n^{(1)}f_2(n) \\ f_3(n) \\ f_4(n) \\ \hline f_3(n+1) \\ f_2(n+1) \\ f_3(n+2) \end{bmatrix}. \quad (\text{A.4})$$

By adding the seventh row scaled with $Z_n^{(3)}$ to the fourth row of (A.4) and making use of (12a) we obtain the first hidden constraint

$$Z_n^{(1)}C_{n,2}x(n) = Z_n^{(1)}f_2(n) + Z_n^{(3)}f_3(n+1),$$

which is exactly the fourth row of (13).

We continue by adding the seventh row scaled with $Z_n^{(4)}$ and the eighth row scaled with $Z_n^{(5)}$ to the second row of (A.4) and making use of (12b) to obtain

$$\begin{aligned} & \left(Z_n^{(2)}B_{n,1} + Z_n^{(4)}C_{n+1,2} \right) x(n+1) + Z_n^{(2)}C_{n,1}x(n) \\ &= Z_n^{(2)}f_1(n) + Z_n^{(4)}f_2(n+1) + Z_n^{(5)}f_3(n+2). \end{aligned}$$

This is exactly the second row of (13). Therefore, any solution to (10) is also a solution to (13).

Sufficiency: Let x be an arbitrary solution to (13). Thus, x is also a solution to the shifted system

$$\begin{array}{c}
 \frac{d_2}{s_2} \left[\begin{array}{ccc} S_n^{(2)} A_{n,1} & S_n^{(2)} B_{n,1} & S_n^{(2)} C_{n,1} \\ 0 & Z_n^{(2)} B_{n,1} + Z_n^{(4)} C_{n+1,2} & Z_n^{(2)} C_{n,1} \end{array} \right] \\
 \hline
 \frac{d_1}{s_1} \left[\begin{array}{ccc} 0 & S_n^{(1)} B_{n,2} & S_n^{(1)} C_{n,2} \\ 0 & 0 & Z_n^{(1)} C_{n,2} \end{array} \right] \\
 \hline
 \frac{r_0}{v} \left[\begin{array}{ccc} 0 & 0 & C_{n,3} \\ 0 & 0 & 0 \end{array} \right] \\
 \hline
 \frac{r_0}{r_0} \left[\begin{array}{ccc} 0 & C_{n+1,3} & 0 \\ C_{n+2,3} & 0 & 0 \end{array} \right]
 \end{array} = \begin{bmatrix} x(n+2) \\ x(n+1) \\ x(n) \end{bmatrix}$$

$$= \begin{bmatrix} S_n^{(2)} f_1(n) \\ Z_n^{(2)} f_1(n) + Z_n^{(4)} f_2(n+1) + Z_n^{(5)} f_3(n+2) \\ S_n^{(1)} f_2(n) \\ Z_n^{(1)} f_2(n) + Z_n^{(3)} f_3(n+1) \\ f_3(n) \\ f_4(n) \\ f_3(n+1) \\ f_3(n+2) \end{bmatrix} \quad \text{for all } n \geq n_0. \quad (\text{A.5})$$

⁴²⁸ Since elementary matrix-row operations are reversible, we can reverse the trans-
⁴²⁹formations performed in the necessity part. Consequently, we see that any so-
⁴³⁰lution to (A.5) is also a solution to (A.4), and hence, this completes the proof.