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On Roughness of Exponential Dichotomy

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We present new perturbation theorems on the roughness of exponential dichotomy, which improve previous results. The proofs given here are also much simpler compared with previous ones. The new results provide significant improvements of existing results in the case where the operator $A(t)$ is unbounded for $t \in J$. This is precisely the situation that is of interest from the point of view of the applications in lobe dynamics. The results are also valid in Banach spaces and useful for general purposes. © 2001 Academic Press

Key Words: linear non-autonomous differential equation; exponential dichotomy; perturbation; roughness.

1. INTRODUCTION

The theory of exponential dichotomies plays an important role in the analysis of non-autonomous differential equations. There have been extensive studies showing much significance in both theory and applications. For example, recently [7] a theoretical framework using the concept of exponential dichotomies has been set up for the study of lobe dynamics and Lagrangian transport of aperiodic flow in fluid dynamics. One of the most important and useful properties of exponential dichotomy in many theories and applications is its roughness under perturbations. It turns out that this is shown [6] to be true again for the study of lobe dynamics and Lagrangian transport of aperiodic flow in fluid dynamics. Especially, in [6], the persistence of the exponential dichotomy under conditions much less restrictive than which is given by the classical results, e.g., in [2], etc., is expected

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to obtain some very useful results for the central concept of *distinguished hyperbolic trajectories* in aperiodic flows. In fact, it appears that the roughness conditions provided by the traditional analysis will be *restrictive* for our study of distinguished hyperbolic trajectories, while the new conditions derived here will be *non-restrictive* as described in [6]. This is indeed the first motivation for the research done in this paper. In this note, we will derive these new perturbation results on exponential dichotomy which extend some of the previous classical results and provide much more simplified proofs. Obviously, the new results are also useful for general purposes. The new results provide significant improvements of existing results in the case where the operator $A(t)$ is unbounded for $t \in J$. For notations and details, see the later discussion on this issue. Notice that this is very important, as for many practical applications, the unboundedness of operator $A(t)$ for $t \in J$ is expected. This is precisely the situation that is of interest from the point of view of the applications for the study of lobe dynamics that we have mentioned above. See more details about the discussion of this application in [6].

Consider the following homogeneous linear system of ordinary differential equations,

$$\dot{x}(t) = A(t)x(t), \qquad t \in J, \tag{1.1}$$

and its perturbed system

$$\dot{y}(t) = [A(t) + B(t)]y(t), \qquad t \in J, \tag{1.2}$$

where J is a real interval, $A(t)$, $B(t)$ are matrix functions of $t \in J$, and $x(t)$, and $y(t)$ are vector functions of $t \in J$. $A(\cdot)$, $B(\cdot)$ are assumed to be regular enough so that $x(t)$ and $y(t)$ exist and are continuous for $t \in J$. For example, we can assume that $A(t)$, $B(t)$ are locally integrable. Obviously, this is not a strong assumption.

Now we recall the definition of *exponential dichotomy*.

DEFINITION 1.1. We say that (1.1) admits an *exponential dichotomy* if and only if there exists a projection matrix P and positive constants K_1, K_2, α_1, and α_2 such that for $t, s \in J$,

$$\begin{aligned} |X(t)PX^{-1}(s)| &\le K_1 \exp\{-\alpha_1(t-s)\}, \qquad t \ge s, \\ |X(t)(I-P)X^{-1}(s)| &\le K_2 \exp\{-\alpha_2(s-t)\}, \qquad s \ge t, \end{aligned} \tag{1.3}$$

where $X(t)$ is the fundamental solution matrix of system (1.1) and I is the identity matrix.

When $\alpha_1 = \alpha_2 = 0$, (1.1) is said to admit an *ordinary dichotomy*.

Notice that P is independent of t here. When $K_1 = K_2$ and $\alpha_1 = \alpha_2$, the above definition agrees with the classical definition [2]. The study of the case when $K_1 \neq K_2$ and $\alpha_1 \neq \alpha_2$ is important in that it can give us more accurate estimates which will usually be useful for various applications. For the case $K_1 \neq K_2, \alpha_1 \neq \alpha_2$, the definition above already appeared in Coppel's article [3]. See also [8].

It can be shown [2] that (1.3) is equivalent to the conditions

$$\begin{aligned} &|X(t)P\xi| \leq K_1' \exp\{-\alpha_1(t-s)\}|X(s)P\xi|, && t \geq s, \\ &|X(t)(I-P)\xi| \leq K_2' \exp\{-\alpha_2(s-t)\}|X(s)(I-P)\xi|, && s \geq t, \\ &|X(t)PX^{-1}(t)| \leq M, \qquad \forall\, t \in J, && \end{aligned} \tag{1.4}$$

where K_1', K_2', and M are positive constants and ξ is an arbitrary constant vector.

To study the *roughness* of the exponential dichotomy of (1.1) under the perturbation B, we are interested in finding out conditions on $B(t)$ under which (1.2) also admits an exponential dichotomy for some projection operator Q with the appropriate corresponding positive constants K's and α's. We hope to find out conditions as weak as possible.

In [2], it is shown that for $J = R_+, R_-$ or R with $K_1 = K_2 = K$ and $\alpha_1 = \alpha_2 = \alpha$ and if

$$\delta := |B|_\infty < \min\left\{\frac{\alpha}{4K^2}, \frac{\alpha}{2K}\right\}, \tag{1.5}$$

then there exists a projection operator Q similar to P in the usual sense such that

$$\begin{aligned} |Y(t)QY^{-1}(s)| &\leq \tfrac{5}{2}K^2 \exp\{-\alpha'(t-s)\}, && t \geq s, \\ |Y(t)(I-Q)Y^{-1}(s)| &\leq \tfrac{5}{2}K^2 \exp\{-\alpha'(s-t)\}, && s \geq t, \end{aligned} \tag{1.6}$$

where $Y(t)$ is the fundamental solution matrix of system (1.2) and $\alpha' = \alpha - 2K\delta (>0)$.

For $K_1 \neq K_2$ and $\alpha_1 \neq \alpha_2$, the best result is recently given in [9]. It is shown that for $J = R_+$ or R with

$$|B|_\infty \left\{\frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2}\right\} < \frac{1}{2}, \tag{1.7}$$

(1.2) still possesses an exponential dichotomy with the projection matrix Q similar to P.

The proofs of the above results given in [2, 9] utilized complicated functional analytic apparatus and a detailed study of some crucial inequalities.

In particular, they used the equivalence of (1.3) and (1.4). More specifically, [2] relies on the estimate of the positive real function $\phi(t)$ satisfying the inequality

$$\phi(t) \leq c\exp\{-\alpha t\} + \theta\alpha \int_0^\infty \exp\{-\alpha|t-\tau|\}\phi(\tau)d\tau, \tag{1.8}$$

while [9] relies on the estimate of the positive real function $\phi(t)$ satisfying the following more complicated inequalities:

$$\phi(t) \leq ce^{-\alpha_1(t-t_0)}\phi(t_0) + \epsilon c_1 \int_{t_0}^t e^{-\alpha_1(t-\tau)}\phi(\tau)d\tau$$
$$+ \epsilon c_2 \int_t^\infty e^{-\alpha_2(\tau-t)}\phi(\tau)d\tau, \qquad t \geq t_0 \tag{1.9}$$
$$\phi(t) \leq ce^{-\alpha_2(s-t)}\phi(s) + \epsilon c_1 \int_{t_0}^t e^{-\alpha_1(t-\tau)}\phi(\tau)d\tau$$
$$+ \epsilon c_2 \int_t^s e^{-\alpha_2(\tau-t)}\phi(\tau)d\tau, \qquad s \geq t \geq t_0. \tag{1.10}$$

See the original references for more details concerning these inequalities and the constants involved in their definitions.

In [9], it is also shown that for $J = R_+$ or R with

$$|B|_\infty \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\} < 1, \tag{1.11}$$

(1.2) still possesses an ordinary dichotomy.

Using a result of [2] (see also Proposition 2.1), [10] obtained a perturbation lemma showing that for $J = [t_0, \infty)$ and that

$$\lim_{t\to\infty} B(t) = 0, \tag{1.12}$$

(1.2) still possesses an exponential dichotomy.

In this note, we will improve all the above results in a systematic way. Moreover, our proofs are much simpler than the previous ones. Especially, we avoid using the equivalence of (1.3) and (1.4). Instead, we use Proposition 2.2, another equivalence of exponential dichotomy. In this way, we easily avoid solving the complicated inequalities of the type (1.8), (1.9), and (1.10) and thus simplify the proofs considerably with improved estimates of the roughness of perturbation. We note that our results provide significant improvement only when $A(t)$ is unbounded for t in the considered interval. See Remark 3.1 for more details.

Even though our discussion is in the setting of the finite dimensional case, it is easy to see that our results can be immediately extended to the case of Banach spaces. See Remark 3.2 at the end of Theorem 3.2.

The proofs of roughness using functional analytic characterizations of exponential dichotomy, as in this article as well, can be simpler than other proofs. On the other hand, they may provide less information about the perturbed systems.

The rest of the paper is organized as follows: In Section 2, we first recall some notations and preliminary results about exponential dichotomies. In Section 3, we state and prove our main theorems.

See also the interesting results of [11, 12].

2. PRELIMINARIES

We recall some useful results about exponential and ordinary dichotomies from [2].

The first one states that in order to prove the dichotomy on R_+, it is enough to prove it on a subinterval $[t_0, \infty)$ of R_+.

PROPOSITION 2.1. *If Eq.* (1.1) *has an exponential* (*resp. ordinary*) *dichotomy* (1.3) *on a subinterval* $[t_0, \infty)$ *of* R_+, *then it also has an exponential* (*resp. ordinary*) *dichotomy on the half line* R_+, *with the same projection* P *and the same exponents* α_1 *and* α_2, *and possibly different coefficients* K_1 *and* K_2.

Consider the following inhomogeneous equation with $J = R_+$,

$$\dot{x}(t) = A(t)x(t) + f(t), \qquad t \in J. \tag{2.1}$$

Define the function spaces $\mathbf{C}_J$, $\mathbf{L}_J$, and $\mathbf{M}_J$ as

$$\mathbf{C}_J := \left\{ f \,|\, \|f\|_{\mathbf{C}_J} := \sup_{s\in J} |f(s)|\, ds < \infty \right\}, \tag{2.2}$$

$$\mathbf{L}_J := \left\{ f \,|\, \|f\|_{\mathbf{L}_J} := \int_{s\in J} |f(s)|\, ds < \infty \right\}, \tag{2.3}$$

$$\mathbf{M}_{J,T} := \left\{ f \,|\, \|f\|_{\mathbf{M}_{J,T}} := \frac{1}{T} \sup_{t\in J} \int_t^{t+T} |f(s)|\, ds < \infty, f \in L^1_{loc} \right\}, \tag{2.4}$$

where $T > 0$ is a fixed constant. When J and T are well understood, we omit the subscripts of $\mathbf{C}_J$, $\mathbf{L}_J$, and $\mathbf{M}_{J,T}$.

The following result gives a necessary and sufficient condition for an exponential dichotomy to be valid on J, which will be very useful later.

PROPOSITION 2.2. *Suppose* $J = R_+$. *The homogeneous equation* (1.1) *has an exponential dichotomy if and only if* (2.1) *has at least one bounded solution* *for every function* $f \in \mathbf{M}_{J,T}$ *for some* $T > 0$. *Moreover,* (1.3) *holds with*

$$K_1 = K_2 = e \cdot r_{\mathbf{L}_J} > 0, \qquad \alpha_1 = \alpha_2 = r_{\mathbf{C}_J}^{-1} > 0,$$

where $r_{\mathbf{L}_J}$ *and* $r_{\mathbf{C}_J}$ *are the generic constants defined as the least positive numbers such that for every* $f \in \mathbf{B}$, *the unique bounded solution* $y(t)$ *of* (2.1) *with* $y(0) \in \mathbf{V}_2$ *satisfies*

$$\|y\|_{\mathbf{C}_J} \leq r_{\mathbf{B}} \|f\|_{\mathbf{B}},$$

where $\mathbf{B}$ *denotes the space* $\mathbf{C}_J, \mathbf{L}_J$, *or* $\mathbf{M}_{J,T}, \mathbf{V}_2$ *is any fixed subspace of* $\mathbf{V}$ *supplementary to* $\mathbf{V}_1, \mathbf{V}$ *is the underlying vector space* R^n (*or* C^n), *and* $\mathbf{V}_1$ *is the subspce of* $\mathbf{V}$ *consisting of the initial values of all bounded solutions of* (1.1).

Considering Proposition 2.1, Proposition 2.2 is still valid with J being an infinite subinterval of R_+, with the corresponding constants adjusted accordingly.

Proposition 2.2 was proved in [2] with $T = 1$. It is easy to see that it holds for $T > 0$ as any fixed number. It can also be shown that, with a slight modification of the definition of $\|\cdot\|_{\mathbf{M}_{J,T}}$ when necessary, Proposition 2.2 still holds for $J = R_+$, R_-, or R or any subinterval of them (provided that the bounded solution is unique for the case of $J = R$). It is interesting to note that K_i and α_i $(i = 1, 2)$ are independent of the choice of the interval J.

We remark that the existence of the generic constants $r_{\mathbf{L}_J}$ and $r_{\mathbf{C}_J}$ is well defined by the Proposition 4 of [2, p. 22].

Finally, we recall the following lemma, which will be useful later. The proof of the lemma for $T = 1$ can be found in [2]. The proof for $T \neq 1$ can be obtained similarly.

LEMMA 2.1. *Let* $\gamma(t)$ *be a non-negative and locally integrable function such that for* $T > 0$,

$$\frac{1}{T}\int_t^{t+T} \gamma(s)\, ds \leq C_0 \qquad \forall\ t \geq 0.$$

If $\alpha > 0$, *then for all* $t \geq 0$,

$$\int_0^t \exp\{-\alpha(t-s)\}\gamma(s)\, ds \leq \frac{C_0 T}{1 - e^{-\alpha T}}, \tag{2.5}$$

$$\int_t^\infty \exp\{-\alpha(s-t)\}\gamma(s)\, ds \leq \frac{C_0 T}{1 - e^{-\alpha T}}. \tag{2.6}$$

3. MAIN RESULTS

Now we state and prove our main results.

3.1. *Roughness on R_+*

We first deal with roughness on R_+. The case of R_- can be treated siminarly.

THEOREM 3.1. *Suppose* (1.1) *has an exponential dichotomy on* $J = [t_0, \infty)$ *with positive constants* K_1, K_2, α_1, *and* α_2. *Suppose*

$$\liminf_{t_0\to\infty} \inf_{T>0} \|B\|_{\mathbf{M}_{J,T}} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} < 1, \tag{3.1}$$

or

$$\inf_{t_0' \geq t_0} \inf_{T>0} \|B\|_{\mathbf{M}_{J',T}} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} < 1, \tag{3.2}$$

where $J' = [t_0', \infty)$.

Then (1.2) *also has an exponential dichotomy on* $J = R_+$ *with positive constants* $K_1' = K_2' = K_3 > 0$ *and* $\alpha_1' = \alpha_2' = \alpha_3 > 0$.

Moreover, the projection Q *is similar to* P *and one has that*

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \leq (K_1 + K_2)K_3 \qquad \forall\ t \in J. \tag{3.3}$$

Proof. By (3.1) or (3.2), there is a $t_0' \geq t_0$ and a $T > 0$ such that for $J' = [t_0', \infty)$,

$$\|B\|_{\mathbf{M}_{J',T}} < \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\}^{-1}. \tag{3.4}$$

Suppose $f \in \mathbf{M}_{J',T}$. Consider the operator $\mathcal{T}$ defined as

$$(\mathcal{T} y)(t) = \int_{J'} \Gamma(t, s)B(s)y(s)\, ds + \int_{J'} \Gamma(t, s)f(s)\, ds, \tag{3.5}$$

where $\Gamma(t, s)$ is defined as

$$\Gamma(t, s) = \begin{cases} X(t)PX^{-1}(s), & t \geq s, \\ \\ -X(t)(I - P)X^{-1}(s), & s \geq t. \end{cases} \tag{3.6}$$

Define $|y|_\infty := \|y\|_{c(J', \mathbf{R}^n)}$. Then, by Lemma 2.1,

$$\begin{aligned} |\mathcal{T}(y)(t)| &\leq \int_{J'} |\Gamma(t, s)|\, |B(s)|\, ds |y|_\infty + \int_{J'} |\Gamma(t, s)|\, |f(s)|\, ds \\ &\leq \left(\|B\|_{\mathbf{M}_{J',T}} |y|_\infty + \|f\|_{\mathbf{M}_{J',T}}\right) \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\}. \end{aligned} \tag{3.7}$$

Thus $\mathscr{T} : \mathscr{C}(J', \mathbf{R}^n) \mapsto \mathscr{C}(J', \mathbf{R}^n)$. Moreover, T is a contraction:

$$|(\mathscr{T}y)(t) - (\mathscr{T}z)(t)| \leq \int_{J'} |\Gamma(t,s)|\, ds |y - z|_\infty$$
$$\leq \|B\|_{\mathbf{M}_{J',T}} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} |y - z|_\infty. \quad (3.8)$$

Therefore, there exists a unique $y \in \mathscr{C}(J', \mathbf{R}^n)$ such that $y(t) = \mathscr{T}(y)(t)$, for $t \in J'$, which is a bounded solution of

$$\dot{y}(t) = [A(t) + B(t)]y(t) + f(t), \qquad t \in J. \quad (3.9)$$

By Proposition 2.2 and Proposition 2.1, the exponential dichotomy exists with some projection Q.

Next we estimate the difference between P and Q.

It is easy to verify that (see, e.g., [2, p. 33])

$$Y(t)QY^{-1}(t) - X(t)PX^{-1}(t) = X(t)(I - P)X^{-1}Y(t)QY^{-1}(t)$$
$$- X(t)PX^{-1}Y(t)(I - Q)Y^{-1}(t). \quad (3.10)$$

Thus, by (1.3) and Proposition 2.2, (3.3) is proved.

Now the only thing left to be shown is that Q is similar to P. This can again be done following the proof of Theorem 1 of [9]. ∎

Notice also that in [2], an estimate of $\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\|$ was given for the case when $K_1 = K_2$ and $\alpha_1 = \alpha_2$ as

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \leq \alpha^{-1} 4K^3 \sup_{t \in J} |B(t)| < K. \quad (3.11)$$

Now we show that Theorem 3.1 implies the following important corollary.

COROLLARY 3.1. *Suppose* (1.1) *has an exponential dichotomy on* $J = [t_0, \infty)$ *with positive constants* K_1, K_2, α_1, *and* α_2. *Suppose*

$$\limsup_{s \to \infty} |B(s)| < \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}^{-1}. \quad (3.12)$$

Then (1.2) *also has an exponential dichotomy on* $J = R_+$ *with positive constants* $K_1' = K_2' = K_3 > 0$ *and* $\alpha_1' = \alpha_2' = \alpha_3 > 0$.

Moreover, the projection Q *is similar to* P *and one has that*

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \leq (K_1 + K_2)K_3 \qquad \forall\, t \in J. \quad (3.13)$$

Proof. It is easy to see that for some J' and for all $T > 0$,

$$\|B\|_{\mathbf{M}_{J',T}} < \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}^{-1}. \tag{3.14}$$

Thus for J' and for all $T > 0$

$$\begin{aligned} &\|B\|_{\mathbf{M}_{J',T}} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} \\ &\quad < \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}^{-1} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\}. \end{aligned} \tag{3.15}$$

Define

$$g(t) = \frac{c_0 t}{1 - e^{-\alpha t}}.$$

It can be shown easily that

$$g'(t) = \frac{c_0 (e^{\alpha t} - 1 - \alpha t)}{e^{\alpha t}(1 - e^{-\alpha t})^2} > 0, \tag{3.16}$$

and that

$$\inf_{t>0} g(t) = \lim_{t \to 0^+} g(t) = \frac{c_0}{\alpha}. \tag{3.17}$$

By (3.15), (3.16), (3.17), and (3.12),

$$\begin{aligned} &\inf_{T>0} \|B\|_{\mathbf{M}_{J',T}} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} \\ &\quad < \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}^{-1} \inf_{T>0} \left\{ \frac{K_1 T}{1 - e^{-\alpha_1 T}} + \frac{K_2 T}{1 - e^{-\alpha_2 T}} \right\} \\ &\quad \le \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}^{-1} \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\} = 1. \end{aligned}$$

Thus (3.2) holds and therefore Corollary 3.1 is proved by applying Theorem 3.1. ∎

Notice that the results of the above corollary provide significant improvements over the results of (1.7), (1.11), and (1.12).

Remark 3.1. It is important to note that in Theorem 3.1 and Corollary 3.1, it is not assumed that $A(t)$ is uniformly bounded with respect to $t \in J$. Otherwise, Corollary 3.1 can be proved simply as follows.

By (3.12), there is a $J' = [t_0', \infty)$, such that

$$\sup_{t \in J'} |B(t)| \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\} < 1.$$

Let $f(t)$ be a bounded vector-valued continuous function. Then the mapping $\mathcal{T} : \mathbf{C}_{J'} \mapsto \mathbf{C}_{J'}$ defined by

$$(\mathcal{T}x)(t) = \int_{t'_0}^{t} X(t)PX^{-1}(s)[B(s)x(s) + f(s)]\,ds$$
$$- \int_{t}^{\infty} X(t)(I - P)X^{-1}(s)[B(s)x(s) + f(s)]\,ds$$

is a contraction with contraction constant

$$\sup_{t\in J'} |B(t)| \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\}.$$

The fixed point of $\mathcal{T}$ is a bounded solution of Eq. (3.9). Then it follows form Proposition 3 in [2, p. 22] that (1.2) also has an exponential dichotomy on J'. Notice, however, that Proposition 3 in [2, p. 22] requires that (1.1) has bounded growth, while bounded growth of (1.1) in J' is not required in Proposition 2.2, i.e., Proposition 2 in [2, p. 22].

3.2. *Roughness on R*

Now the roughness of exponential dichotomy on R can be obtained as follows.

THEOREM 3.2. *Suppose* (1.1) *has an exponential dichotomy on* $J = R$ *with positive constants* K_1, K_2, α_1, *and* α_2.

Suppose further

$$\sup_{t\in R} |B(s)| \left\{ \frac{K_1}{\alpha_1} + \frac{K_2}{\alpha_2} \right\} < 1. \tag{3.18}$$

Then (1.2) *also has an exponential dischotomy* (1.6) *on* $J = R$ *with positive constants* $K'_1 = K'_2 = K_3 > 0$ *and* $\alpha'_1 = \alpha'_2 = \alpha_3 > 0$.

Moreover, the projection Q *is similar to* P *and one has that*

$$\|Y(t)QY^{-1}(t) - X(t)PX^{-1}(t)\| \le (K_1 + K_2)K_3 \qquad \forall\ t \in J. \tag{3.19}$$

Proof. Indeed, it follows from Corollary 3.1 that (1.2) has an exponential dichotomy on R_+ and R_-. Now, to complete the argument, it suffices to show further that (1.2) has no nontrivial bounded solution. This follows from the proof of Lemma 7 of [9] and the condition (3.18). In [9], the stronger condition (1.7) was used to show the contraction of the mapping defined in the proof of Lemma 7 of [9], which proves that the zero solution is the only bounded solution to (1.2). ∎

Remark 3.2. Suppose that $A(t)$ and $B(t)$ are bounded linear operators in a Banach space for any fixed $t \in R$. Consider the mild solutions of the systems (1.1) and (1.2). (See, e.g., [1, 5].) Then, it is easy to see that all the results obtained above still hold.

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REFERENCES

1. C. Chicone and Y. Latushkin, "Evolution Semigroups in Dynamical Systems and Differential Equations," Amer. Math. Soc., Providence, 1999.
2. W. A. Coppel, "Dichotomy in Stability Theory," Lecture Notes in Mathematics, Vol. 629, Springer-Verlag, New York/Berlin, 1978.
3. W. A. Copple, Dichotomies and reducibility, *J. Differential Equations* **3** (1967), 500–521.
4. J. L. Daleckii and M. G. Krein, "Stability of Solutions of Differential Equations in Banach Space," Amer. Math. Soc., Providence, 1974.
5. K.-J. Engel and R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations," Springer-Verlag, New York, 2000.
6. N. Ju and S. Wiggins, Distinguished hyperbolic trajectories of aperiodic flows and their approximation, in preparation.
7. N. Malhotra and S. Wiggins, Geometric structures, lobe dynamics, and Lagrangian transport in flows with aperiodic time-dependence, with applications to Rossby wave flow, *J. Nonlinear Sci.* **8** (1998), 401–456.
8. J. L. Massera and J. J. Schäffer, "Linear Differential Equations and Function Spaces," Academic Press, New York, 1966.
9. R. Naulin and M. Pinto, Admissible perturbations of exponential dichotomy roughness, *Nonlinear Anal.* **31** (1998), 559–571.
10. K. J. Palmer, Exponential dichotomies and transversal homoclinic points, *J. Differential Equations* **55** (1984), 225–256.
11. K. J. Palmer, A perturbation theorem for exponential dichotomies, *Proc. Roy. Soc. Edinburgh Sect. A* **106** (1987), 25–37.
12. R. E. Vinograd, Exact bounds for exponential dichotomy roughness. I. Strong dichotomy, *J. Differential Equations* **71** (1988), 63–71; II, An example of attainability, *J. Differential Equations* **90** (1991), 203–210; III, Semistrong dichotomy, *J. Differential Equations* **91** (1991), 245–267.