

Canonical forms for linear variable coefficient discrete-time descriptor systems*

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March 19, 2008

Abstract

We consider linear discrete-time descriptor systems, i.e., systems of linear equations of the form $E_{k+1}x^{k+1} = A_kx^k + f^k$ for $k \in \mathbb{Z}$, where all E_k and A_k are matrices, f_k are vectors and x_k are the vectors of the solution we are looking for. We study the existence and uniqueness of solutions. A strangeness index is defined for such systems. Compared to the continuous-time case, see [7], it turns out, that in the discrete-time case it makes a difference, if one has an initial condition and one wants a solution in the future or if one has an initial condition and one wants a solution into the past and future at the same time.

Key words. descriptor systems, strangeness index, linear time-varying discrete-time descriptor systems, two way canonical form

AMS subject classification. 39A05, 39A12, 15A06

1 Introduction

Let $\mathbb{I} \subset \mathbb{R}$ be an interval and let $\mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$ denote the space of all continuous functions mapping \mathbb{I} into the space of all complex valued m -by- n matrices. Also, let $\mathcal{C}(\mathbb{I}, \mathbb{C}^n)$ denote the space $\mathcal{C}(\mathbb{I}, \mathbb{C}^{n,1})$. Consider the *linear time-varying continuous-time descriptor system*

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \quad t \in \mathbb{I}, \quad (1)$$

where $E, A \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$, $x \in \mathcal{C}(\mathbb{I}, \mathbb{C}^n)$ is the state vector, $f \in \mathcal{C}(\mathbb{I}, \mathbb{C}^m)$ is the right hand side and $x_0 \in \mathbb{R}^n$ is an initial condition given at the point $t_0 \in \mathbb{I}$. Using some constant rank assumptions, a canonical form for systems of the form (1) is developed in [7, Chapter 3] and there the notion of the strangeness index is introduced. Note, that for continuous-time systems it has no meaning, where the initial condition is fixed, i.e., it does not matter if the initial condition is given at a point t_0 which belongs to the interior of \mathbb{I} or at a point t_0 which belongs to the boundary of \mathbb{I} .

The purpose of this paper is to obtain corresponding results for the discrete-time case. Therefore, let us first define two discrete intervals in the following way.

$$\begin{aligned} \mathbb{K} &:= \{k \in \mathbb{Z} : k_b \leq k \leq k_f\}, k_b \in \mathbb{Z} \cup \{-\infty\}, k_f \in \mathbb{Z} \cup \{\infty\}, \\ \mathbb{K}^+ &:= \begin{cases} \mathbb{K} & \text{if } k_f = \infty, \\ \mathbb{K} \cup \{k_f + 1\} & \text{if } k_f < \infty. \end{cases} \end{aligned}$$

*This research is supported by the DFG Research Center MATHEON in Berlin.

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With this definition we call

$$E_k x^{k+1} = A_k x^k + f^k, \quad x^{k_0} = x_0, \quad k \in \mathbb{K} \quad (2)$$

a *linear time-varying discrete-time descriptor system*, where $E_k, A_k \in \mathbb{C}^{m,n}$ for $k \in \mathbb{K}$, $x^k \in \mathbb{C}^n$ for $k \in \mathbb{K}^+$ are the state vectors, $f^k \in \mathbb{C}^m$ for $k \in \mathbb{K}$ are the right hand sides and $x_0 \in \mathbb{R}^n$ is an initial condition given at the point $k_0 \in \mathbb{K}^+$. Such equations arise naturally from equations of type (1) by approximating $\dot{x}(t)$ via an explicit finite difference. Other applications of equation (2) include Singular Leontief Systems [4, 11] and the Backward Leslie Model [3].

Systems of the form (2) have also received some theoretical attention, e.g., solvability of systems of the form (2) has already been studied in [9]. Periodic systems have been investigated in [14]. Some work regarding the associated control problem has been done, see [8, 12, 13, 15].

In [9] only finite sequences of solutions are considered, i.e., system (2) with \mathbb{K} being a finite subset of \mathbb{Z} . The problem of this approach is that one has to introduce two initial conditions (one at the beginning and one at the end) in order to fix a unique solution. We will almost only consider systems where one end is open, i.e., either $k_f = \infty$ or $k_b = -\infty$. As we will see, in contrast to the continuous-time case, it makes a difference if the initial condition is fixed in the interior and we are looking for a solution on all of \mathbb{Z} (also called the *two-way case*) or at the beginning of the interval k_b and we are only looking for a solution for all $k \geq k_b$ (also called the *forward case*). Curiously enough, the forward case is more closely related to the continuous-time case than the two-way case. For instance, we first define a strangeness index for the forward case, analogously to [7], where continuous-time systems are considered and then have some additional work to do in order to transfer the results to the two-way case.

Throughout the paper we will use the notation $\{a_k\}_{k \in \mathbb{K}}$ to denote a sequence and a_k to denote the k -th element of the sequence. The elements of the sequence can be vectors or matrices as well.

Considering a discrete-time descriptor system of the form (2) with variable coefficients and an initial condition we see that the original system

$$E_k x^{k+1} = A_k x^k + f^k, \quad x^0 = \hat{x},$$

is equivalent to the transformed system

$$P_k E_k Q_{k+1} Q_{k+1}^{-1} x^{k+1} = P_k A_k Q_k Q_k^{-1} x^k + P_k f^k, \quad Q_0^{-1} x^0 = Q_0^{-1} \hat{x},$$

as long as all P_k and Q_k are invertible. Defining $\tilde{E}_k := P_k E_k Q_{k+1}$, $\tilde{A}_k := P_k A_k Q_k$, $\tilde{x}^k := Q_k^{-1} x^k$, and $\tilde{f}_k := P_k f^k$ for all k , we can also write this system as

$$\tilde{E}_k \tilde{x}^{k+1} = \tilde{A}_k \tilde{x}^k + \tilde{f}_k, \quad \tilde{x}^0 = Q_0^{-1} \hat{x}.$$

This leads to the following equivalence relation.

Definition 1. Let $E_k, A_k, \tilde{E}_k, \tilde{A}_k \in \mathbb{C}^{m,n}$ for all $k \in \mathbb{K}$. Then two sequences of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{K}}$ and $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{K}}$ are called *globally equivalent* (on \mathbb{K}) if there exist two pointwise nonsingular matrix sequences

$$\begin{aligned} \{P_k\}_{k \in \mathbb{K}} & \quad \text{with } P_k \in \mathbb{C}^{m,m}, \\ \{Q_k\}_{k \in \mathbb{K}^+} & \quad \text{with } Q_k \in \mathbb{C}^{n,n}, \end{aligned}$$

such that

$$P_k E_k Q_{k+1} = \tilde{E}_k \quad \text{and} \quad P_k A_k Q_k = \tilde{A}_k,$$

for all $k \in \mathbb{K}$. We denote this equivalence by $\{(E_k, A_k)\}_{k \in \mathbb{K}} \sim \{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{K}}$.

Following the approach in [7] we first concentrate on one particular matrix pair in the sequence of matrix pairs to find out which transformations can be applied to this single matrix pair.

Definition 2. Two pairs of matrices $(E, A), (\tilde{E}, \tilde{A}) \in \mathbb{C}^{m,n}$ are called *locally equivalent* if there exist matrices $P \in \mathbb{C}^{m,m}$ and $Q, R \in \mathbb{C}^{n,n}$ that are all nonsingular, such that

$$\tilde{E} = PEQ \quad \text{and} \quad \tilde{A} = PAR.$$

Again, we denote this equivalence by $(E, A) \sim (\tilde{E}, \tilde{A})$.

Once we have seen that global equivalence is an equivalence relation it is easy to see that local equivalence is an equivalence relation, since we only have to consider the special case that $\mathbb{K} = \{1\}$.

Let us shortly review the notion of the echelon form that will be used in the next section. For any matrix $A \in \mathbb{C}^{m,n}$ there exist invertible matrices $P \in \mathbb{C}^{m,m}$ and $Q \in \mathbb{C}^{n,n}$ such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad (3)$$

is in *echelon form*. r is called the *rank* of A and is denoted by $\text{rank}(A) = r$.

For convenience, we say in the following that a matrix is a basis of a vector space if this is valid for its columns. For matrix pairs of block matrices we also use the convention that corresponding blocks (i.e., blocks in the same block row and block column) have the same number of rows and columns.

2 Local invariants

In this section we take a closer look at local equivalence and try to identify characteristic values, i.e., values that are invariant under local equivalence. We adjust the result of [6] to the discrete-time case.

Theorem 3. Let $E, A \in \mathbb{C}^{m,n}$ and introduce the following matrices:

$$\begin{aligned} Z \text{ basis of } \text{corange}(E) &= \text{kernel}(E^H), \\ Y \text{ basis of } \text{corange}(A) &= \text{kernel}(A^H). \end{aligned}$$

Then, the quantities

$$\begin{aligned} r_f &= \text{rank}(E), & (\text{corresponds to forward direction}) \\ r_b &= \text{rank}(A), & (\text{corresponds to backward direction}) \\ h_f &= \text{rank}(Z^H A), & (\text{rank of } Z^H A; \text{ forward}) \\ h_b &= \text{rank}(Y^H E) \\ &= r_f + h_f - r_b, & (\text{rank of } Y^H E; \text{ backward}) \\ c &= r_b - h_f, & (\text{common part}) \end{aligned}$$

$$\begin{aligned}
a &= \min(h_f, n - r_f), & (\text{algebraic part}) \\
s &= h_f - a, & (\text{strangeness}) \\
d &= r_f - c - s, & (\text{differential part}) \\
u &= n - r_f - a, & (\text{undetermined variables}) \\
v &= m - r_f - h_f, & (\text{vanishing equations})
\end{aligned}$$

are invariant under local equivalence, and (E, A) is locally equivalent to the canonical form

$$(\tilde{E}, \tilde{A}) = \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right). \quad (5)$$

We have that either $s = 0, u = 0$ or $s = u = 0$. The quantities defined above are called local characteristics or local invariants of the matrix pair (E, A) .

Proof. Let $(E_i, A_i), i = 1, 2$, be locally equivalent, i.e., let P, Q, R be invertible matrices of appropriate size such that

$$E_2 = PE_1Q \quad \text{and} \quad A_2 = PA_1R.$$

Since

$$\begin{aligned}
\text{rank}(E_2) &= \text{rank}(PE_1Q) = \text{rank}(E_1), \\
\text{rank}(A_2) &= \text{rank}(PA_1R) = \text{rank}(A_1),
\end{aligned}$$

it follows that r_f and r_b are invariant under local equivalence. First note that h_f is independent of the choice of the basis of Z . To see this let Z and \tilde{Z} be two bases of $\text{corange}(E)$. Then there exists a regular matrix M_Z with

$$\tilde{Z} = ZM_Z,$$

and from

$$\text{rank}(\tilde{Z}^H A) = \text{rank}(M_Z^H Z^H A) = \text{rank}(Z^H A)$$

the statement follows.

Let Z_2 be a basis of $\text{corange}(E_2)$, i.e $\text{kernel}(E_2^H) = \text{range}(Z_2)$. Then $Z_1 := P^H Z_2$ is a basis of $\text{corange}(E_1)$, since Z_1 has full column rank and

$$\text{kernel}(E_1^H) = \text{range}(Z_1).$$

To prove this, note that for any $x \in \text{kernel}(E_1^H)$ we have $0 = E_1^H x = Q^{-H} E_2^H P^{-H} x$ and thus $0 = E_2^H P^{-H} x$. This identity can be rewritten as $P^{-H} x \in \text{kernel}(E_2^H) = \text{range}(Z_2)$. This shows that there exists a z such that $P^{-H} x = Z_2 z$. Premultiplying this identity with P^H shows that $x = P^H Z_2 z = Z_1 z \in \text{range}(Z_1)$. Conversely, we have for any $x \in \text{range}(Z_1)$ that there exists a z such that $x = Z_1 z = P^H Z_2 z$. Premultiplying this identity with P^{-H} yields $P^{-H} x = Z_2 z \in \text{range}(Z_2) = \text{kernel}(E_2^H)$ which means $0 = E_2^H P^{-H} x = Q^H E_1^H P^H P^{-H} x = E_1^H x$. Thus we have shown that $x \in \text{kernel}(E_1^H)$. This implies

$$\text{rank}(Z_2^H A_2) = \text{rank}(Z_2^H P A_1 R) = \text{rank}(Z_1^H A_1 R) = \text{rank}(Z_1^H A_1),$$

which shows that also h_f is invariant under local equivalence. By exchanging the roles of A and E as well as the roles of Z and Y in the previous argument, we also see that h_b (defined as $\text{rank}(Y^H E)$) is invariant under local equivalence. Since all other quantities are functions of the first three invariant quantities all quantities are invariant under local equivalence. Let Z' be a matrix, such that the composed matrix $[Z' \ Z]$ is invertible. Also let U, V be matrices, such that $U((Z')^H E)V = \begin{bmatrix} I_{r_f} & 0 \end{bmatrix}$ is in echelon form (3). Note that $((Z')^H E)$ has full row rank, since Z' spans the vector space $\text{range}(E)$. Thus, the following equivalence transformations in the sense of Definition 2 can be applied to the matrix pair (E, A) (where all corresponding blocks of both matrices have the same size).

$$\begin{aligned}
(E, A) &\sim \left(\begin{bmatrix} (Z')^H E \\ 0 \end{bmatrix}, \begin{bmatrix} (Z')^H A \\ Z^H A \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_{r_f} & 0 \\ 0 \end{bmatrix}, \begin{bmatrix} U(Z')^H A \\ Z^H A \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_{d+c} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right). \tag{6}
\end{aligned}$$

Looking at matrix pair (6) one can (because of the Definition of local equivalence) permute the columns of the second matrix without permuting the columns of the first matrix. This shows that the size of the I_a block can be increased by decreasing the size of the I_s block (as long as the last block column does not vanish) and, conversely, that the size of the I_s block can be increased at the expense of the size of the I_a block (as long as the second block column does not vanish). To get a canonical form we choose the I_a block to be of maximum size. Hence, it is clear that either the last block column vanishes, i.e., $u = 0$, or that there is no I_s , i.e., $s = 0$ (or even $s = u = 0$), as stated in the assertion. These cases are considered separately in the following.

If $u = 0$, then (6) reads as

$$\begin{aligned}
(E, A) &\sim \left(\begin{bmatrix} I_s & 0 & 0 \\ 0 & I_{d+c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & I_a \\ I_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 \\ 0 & I_{d+c} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & 0 & I_a \\ I_s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 \\ 0 & 0 & 0 & I_a \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).
\end{aligned}$$

In this case we have $h_f = s + a \geq a = n - r_f$, which is consistent with the definition of a from the Theorem statement.

If $s = 0$, then (6) reads as

$$\begin{aligned} (E, A) &\sim \left(\begin{bmatrix} I_{d+c} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & * \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \sim \left(\begin{bmatrix} I_{d+c} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & 0 & * \\ 0 & I_a & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_d & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_c & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right). \end{aligned}$$

In this case we have $h_f = a = n - r_f - u \leq n - r_f$ which is also consistent with the definition of a from the Theorem statement.

Finally, the identity $h_b = r_f + h_f - r_b$ can be derived from the canonical form (5). Therefore, let \tilde{Z} and \tilde{Y} be bases of $\text{corange}(\tilde{E})$ and $\text{corange}(\tilde{A})$, where \tilde{E} and \tilde{A} are the matrices in the canonical form (5), respectively. Then we have

$$\tilde{Y} = \begin{bmatrix} I_s & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_v \end{bmatrix}.$$

From this we see that

$$\begin{aligned} \text{rank}(\tilde{Y}^H \tilde{E}) &= \text{rank} \left(\begin{bmatrix} I_s & 0 & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_v \end{bmatrix} \begin{bmatrix} I_s & 0 & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 & 0 \\ 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right) \\ &= s + d = s + r_f - c - s \\ &= r_f - c = r_f - r_b + h_f, \end{aligned}$$

and the proof is finished. \square

Note that the form (5) can also be obtained by first reducing (E, A) to Kronecker canonical form and then applying further local equivalence transformations.

Comparing this result to the analogous continuous-time result [7, Theorem 3.7] one notices the additional "common" part. This part cannot be eliminated, since local equivalence does not allow changes of the matrix A by means of the matrix E .

3 Forward global canonical form

Starting from the results in [1], where the time-invariant case has been studied, we first concentrate on the case where one starts at some time point (here this time point is always $k = 0$) and calculates into the future, i.e., one tries to get a solution for $k \geq 0$. In order to derive a global canonical form, some constant rank assumptions are introduced. Milder assumptions are necessary in this case, than in the case where one wants to get a solution for all $k \in \mathbb{Z}$. Despite the issue that we only want to get a solution for $k \geq 0$ we may also

consider linear descriptor systems with equations for all $k \in \mathbb{Z}$ (i.e., systems of the type (2) with $\mathbb{K} = \mathbb{Z}$), since this simplifies moving to the case where we want to get a solution for all $k \in \mathbb{Z}$. This is no restriction, since every linear descriptor system of the form $\{(E_k, A_k)\}_{k \in \mathbb{N}_0}$ can be extended to one of the form $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ by choosing $E_k = E_0$ and $A_k = A_0$ for all $k < 0$.

Note that we use here the term canonical form in a way that differs from the terminology of abstract algebra.

Lemma 4. *Consider system (2) and introduce the matrix sequence $\{Z_k\}_{k \in \mathbb{K}}$, where*

$$Z_k \text{ is a basis of } \text{corange}(E_k) = \text{kernel}(E_k^H) \text{ for all } k \in \mathbb{K}.$$

Let

$$\begin{aligned} r_f^k &= \text{rank}(E_k), \quad k \in \mathbb{K}, \\ r_b^k &= \text{rank}(A_k), \quad k \in \mathbb{K}, \\ h_f^k &= \text{rank}(Z_k^H A_k), \quad k \in \mathbb{K}, \end{aligned}$$

be the local characteristics of each matrix pencil (E_k, A_k) with $k \in \mathbb{K}$. Then, these characteristic sequences are invariant under global equivalence.

Assume further that the two local characteristic sequences

$$r_f \equiv r_f^k \text{ and } h_f \equiv h_f^k \tag{8}$$

are constant for all $k \in \mathbb{K}$. Then the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{K}}$ is globally equivalent to the sequence

$$\left\{ \left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{K}}, \tag{9}$$

where all matrices $[E_k^{(1)} \ E_k^{(2)}]$ have full row rank, i.e., they all are of rank r_f .

Proof. The invariance of the local characteristics follows directly from Theorem 3. Let

$$Z'_k \text{ be a basis of } \text{range}(E_k) \text{ for all } k \in \mathbb{K}.$$

Then $[Z'_k \ Z_k]$ is invertible for all $k \in \mathbb{K}$ and $Z_k'^H E_k$ has full row rank r_f . Transforming with the transpose of this sequence from the left yields

$$\begin{aligned} \{(E_k, A_k)\}_{k \in \mathbb{K}} &\sim \left\{ \left(\begin{bmatrix} Z_k'^H E_k \\ 0 \end{bmatrix}, \begin{bmatrix} Z_k'^H A_k \\ Z_k^H A_k \end{bmatrix} \right) \right\}_{k \in \mathbb{K}} \\ &\sim \left\{ \left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{K}} \\ &\sim \left\{ \left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{K}}. \quad \square \end{aligned}$$

Writing down the equations from (2) connected with the form (9) one obtains the equations

$$\begin{aligned} E_k^{(1)} x_1^{k+1} + E_k^{(2)} x_2^{k+1} &= A_k^{(1)} x_1^k + f_1^k, \\ 0 &= x_2^k + f_2^k, \\ 0 &= f_3^k, \end{aligned}$$

for $k \in \mathbb{K}$. Assuming that $\mathbb{K} = \mathbb{N}_0$, this system is equivalent to the system given by

$$\begin{aligned} E_k^{(1)} x_1^{k+1} &= A_k^{(1)} x_1^k + \tilde{f}_1^k, \\ 0 &= x_2^k + f_2^k, \\ 0 &= f_3^k, \end{aligned}$$

(where $\tilde{f}_1^k = f_1^k + E_k^{(2)} f_2^{k+1}$) which is connected with the sequence of matrix pairs

$$\left\{ \left(\begin{bmatrix} E_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{K}}. \quad (10)$$

Since this step is reversible the set of solution sequences is not altered. One may also notice that the new right hand side \tilde{f}^k can depend on the right hand side of the former next right hand side f^{k+1} . Looking at the results in [1] one can interpret this step as an index reduction.

Analogous to [7, Theorem 3.14], in the following Theorem it is shown, that the so obtained reduced sequences of matrix pairs are still globally equivalent, if the original sequences have been.

Theorem 5. *Assume that the sequences of matrix pairs*

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} = \left\{ \left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}$$

and

$$\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}} = \left\{ \left(\begin{bmatrix} \tilde{E}_k^{(1)} & \tilde{E}_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}$$

are globally equivalent on \mathbb{Z} and in the form (9). In particular, suppose that (8) holds and that all $[E_k^{(1)} E_k^{(2)}]$ and all $[\tilde{E}_k^{(1)} \tilde{E}_k^{(2)}]$ have full row rank r_f . Then the sequences of matrix pairs $\{(E_k^{(1)}, A_k^{(1)})\}$ and $\{(\tilde{E}_k^{(1)}, \tilde{A}_k^{(1)})\}$ are also globally equivalent on \mathbb{Z} .

Proof. By assumption, there exist two pointwise nonsingular matrix sequences

$$\{P_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}^{m,m} \text{ and } \{Q_k\}_{k \in \mathbb{Z}} \subset \mathbb{C}^{n,n},$$

such that

$$P_k E_k = \tilde{E}_k Q_{k+1} \text{ and } P_k A_k = \tilde{A}_k Q_k,$$

for all $k \in \mathbb{Z}$. By partitioning the transforming matrices appropriately we get

$$\begin{aligned}
\tilde{A}_k Q_k &= \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_k^{(1,1)} & Q_k^{(1,2)} \\ Q_k^{(2,1)} & Q_k^{(2,2)} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{A}_k^{(1)} Q_k^{(1,1)} & \tilde{A}_k^{(1)} Q_k^{(1,2)} \\ Q_k^{(2,1)} & Q_k^{(2,2)} \\ 0 & 0 \end{bmatrix} = \\
P_k A_k &= \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ P_k^{(2,1)} & P_k^{(2,2)} & P_k^{(2,3)} \\ P_k^{(3,1)} & P_k^{(3,2)} & P_k^{(3,3)} \end{bmatrix} \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_k^{(1,1)} A_k^{(1)} & P_k^{(1,2)} \\ P_k^{(2,1)} A_k^{(1)} & P_k^{(2,2)} \\ P_k^{(3,1)} A_k^{(1)} & P_k^{(3,2)} \end{bmatrix}, \tag{11}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{E}_k Q_{k+1} &= \begin{bmatrix} \tilde{E}_k^{(1)} & \tilde{E}_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{k+1}^{(1,1)} & Q_{k+1}^{(1,2)} \\ Q_{k+1}^{(2,1)} & Q_{k+1}^{(2,2)} \end{bmatrix} \\
&= \begin{bmatrix} \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)} + \tilde{E}_k^{(2)} Q_{k+1}^{(2,1)} & \tilde{E}_k^{(1)} Q_{k+1}^{(1,2)} + \tilde{E}_k^{(2)} Q_{k+1}^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \\
P_k E_k &= \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ P_k^{(2,1)} & P_k^{(2,2)} & P_k^{(2,3)} \\ P_k^{(3,1)} & P_k^{(3,2)} & P_k^{(3,3)} \end{bmatrix} \begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_k^{(1,1)} E_k^{(1)} & P_k^{(1,1)} E_k^{(2)} \\ P_k^{(2,1)} E_k^{(1)} & P_k^{(2,1)} E_k^{(2)} \\ P_k^{(3,1)} E_k^{(1)} & P_k^{(3,1)} E_k^{(2)} \end{bmatrix}. \tag{12}
\end{aligned}$$

From (11) we obtain that

$$P_k^{(3,2)} = 0 \text{ for all } k \in \mathbb{Z}.$$

Let $p \in \mathbb{C}^{1, r_f}$ be any row of one of the matrices $P_k^{(2,1)}$ or $P_k^{(3,1)}$. Then from (12) we get

$$p[E_k^{(1)} \ E_k^{(2)}] = 0.$$

But since all matrices $[E_k^{(1)} \ E_k^{(2)}]$ have full row rank as stated in Theorem 4, it follows that also $p = 0$. Thus, we get that

$$P_k^{(2,1)} = 0, \ P_k^{(3,1)} = 0 \text{ for all } k \in \mathbb{Z},$$

which means that the left transforming matrices take the form

$$P_k = \begin{bmatrix} P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)} \\ 0 & P_k^{(2,2)} & P_k^{(2,3)} \\ 0 & 0 & P_k^{(3,3)} \end{bmatrix}.$$

Hence, the diagonal matrices $P_k^{(1,1)}$, $P_k^{(2,2)}$, $P_k^{(3,3)}$ have to be nonsingular. Since from (11) we also get that

$$Q_k^{(2,1)} = P_k^{(2,1)} A_k^{(1)} = 0,$$

it follows that all matrices $Q_k^{(1,1)}$, $Q_k^{(2,2)}$ are invertible. With this, from (12) and (11) we finally get that

$$P_k^{(1,1)} E_k^{(1)} = \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)} + \tilde{E}_k^{(2)} Q_{k+1}^{(2,1)} = \tilde{E}_k^{(1)} Q_{k+1}^{(1,1)}$$

and

$$P_k^{(1,1)} A_k^{(1)} = \tilde{A}_k^{(1)} Q_k^{(1,1)},$$

which proves the claim by employing $\{P_k^{(1,1)}\}_{k \in \mathbb{Z}}$ and $\{Q_k^{(1,1)}\}_{k \in \mathbb{Z}}$ as transforming matrix sequences. \square

Corollary 6. *Under the assumptions of Theorem 5 the sequences of matrix pairs*

$$\left\{ \left(\begin{bmatrix} E_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}$$

and

$$\left\{ \left(\begin{bmatrix} \tilde{E}_k^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}$$

are globally equivalent on \mathbb{Z} .

Proof. Using the matrices from the proof of Theorem 5 one immediately sees that global equivalence is achieved by using

$$\left\{ \begin{bmatrix} P_k^{(1,1)} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right\} \text{ and } \left\{ \begin{bmatrix} Q_k^{(1,1)} & 0 \\ 0 & I \end{bmatrix} \right\}$$

as transforming matrices. \square

In this section we have obtained the forms (9) and (10) which both are some kind of canonical forms. A more advanced canonical form that requires the notion of the strangeness index will be introduced in the next chapter.

4 The strangeness index

The preceding results allow for an inductive procedure closely related to the corresponding procedure for continuous-time systems [7]. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}} =: \{(E_{k,0}, A_{k,0})\}_{k \in \mathbb{Z}}$ we define a sequence (of sequences of matrix pairs) $\{ \{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \}_{i \in \mathbb{N}_0}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}}$ by Lemma 4 to the form (9) assuming that its local invariants are constant and we set

$$r_f =: r_{f,i} \text{ and } h_f =: h_{f,i}. \quad (13)$$

Then we reduce the so obtained sequence of matrix pairs to the form (10) which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k \in \mathbb{Z}}$. This whole iterative process (although derived from [7]) is very similar to Luenberger's shuffle algorithm, which is described in [10] for discrete-time descriptor systems with constant coefficients.

Observe that we have to have the constant rank assumptions of the form (8) for every step of the procedure. Due to Corollary 6 the so obtained sequence of global invariants $\{(r_{f,i}, h_{f,i})\}_{i \in \mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs. Several properties of this sequence are summed up in the following Lemma.

Lemma 7. *Let the sequences $\{(r_{f,i}, h_{f,i})\}_{i \in \mathbb{N}_0}$ and $\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}}\}_{i \in \mathbb{N}_0}$ be defined as in (13). In particular, let the constant rank assumptions (8) hold for every $\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}}$. Defining the quantities*

$$h_{f,-1} := 0, \quad (14a)$$

$$a_i := h_{f,i} - h_{f,i-1}, \quad (14b)$$

$$d_i := r_{f,i} + h_{f,i}, \quad (14c)$$

$$s_i := r_{f,i} - r_{f,i+1}, \quad (14d)$$

$$w_0 := m - r_{f,0} - h_{f,0}, \quad (14e)$$

$$w_i := s_{i-1} - a_i, \quad (14f)$$

$$v_i := m - r_{f,i} - h_{f,i}, \quad (14g)$$

for all $i \in \mathbb{N}_0$ there exist $\xi, \mu \in \mathbb{N}_0$ so that:

$$r_{f,i} \geq r_{f,i+1}, \quad (15a)$$

$$d_i \geq d_{i+1}, \quad (15b)$$

$$s_{i+\mu} = 0, \quad (15c)$$

$$h_{f,i} \leq h_{f,i+1}, \quad (15d)$$

$$a_i \geq a_{i+1}, \quad (15e)$$

$$d_i \geq h_{f,i}, \quad (15f)$$

$$a_{i+\xi} = 0, \quad (15g)$$

$$v_i = v_0 + w_1 + \dots + w_i, \quad (15h)$$

$$s_i \leq a_i, \quad (15i)$$

$$a_{i+1} \leq s_i, \quad (15j)$$

$$s_i \geq s_{i+1}, \quad (15k)$$

$$a_{i+1} - w_{i+2} \leq a_i - w_{i+1}, \quad (15l)$$

$$a_i, d_i, s_i, w_i, v_i \geq 0, \quad (15m)$$

for all $i \in \mathbb{N}_0$. For any $\mu \in \mathbb{N}_0$ with the property (15c) we also have

$$a_{i+1} = 0, \quad r_{f,\mu} = r_{f,i}, \quad d_\mu = d_i \quad \text{for all } i \geq \mu. \quad (16)$$

Proof. See Appendix A.

Lemma 7 leads to the following Definition, according to [7, Definition 3.15].

Definition 8. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence of matrix pairs and let the corresponding sequence of characteristic values $\{(r_{f,i}, h_{f,i})\}_{i \in \mathbb{N}_0}$ as in (13) be well defined. In particular, let (8) hold for every entry $\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}}$ of the sequence (of sequences of matrix pairs). Then, with definition (14d) we call

$$\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\} \quad (17)$$

the *strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ and of the associated descriptor system (2). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ and (2) *strangeness-free*.

In the proof of Lemma 7 we see that (under some constant rank assumptions) after $\mu + 1$ reduction steps every sequence of matrix pairs $\{(E_{k,0}, A_{k,0})\}_{k \in \mathbb{Z}}$ is equivalent to a sequence

of the form

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 \\ 0 & I_{h_{f,\mu}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

with all $E_{k,\mu}^{(1)}$ having full row rank $r_{f,\mu}$. In a last step, one can further reduce all those matrices $E_{k,\mu}^{(1)}$ to echelon form (3) $[I_{r_{f,\mu}} \ 0]$ by global equivalence achieving (with adapted indexing)

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & A_{k,\mu}^{(2)} \\ 0 & I_{h_{f,\mu}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (18)$$

which can be regarded as a canonical form. One notices that in general not only μ but $\mu + 1$ reduction steps are necessary to get to the canonical form, although after μ reduction steps a strangeness-free sequence has already been reached. This situation can be avoided by introducing a further constant rank assumption in every step of the reduction process described at (13) (see [5]). Analogously to [7] one can derive a canonical form for sequences of matrix pairs with well defined strangeness index without performing the reduction from (9) to (10). For convenience, we denote unspecified blocks in a matrix by $*$.

Theorem 9. *Let the strangeness index μ of the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ as in (17) be well defined. Then, with the definitions from (14), $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to a sequence of the form*

$$\left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & F_k \\ 0 & 0 & G_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (19)$$

with

$$F_k = \begin{bmatrix} 0 & F_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & F_k^{(1)} \\ & & & 0 \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & G_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & G_k^{(1)} \\ & & & 0 \end{bmatrix},$$

where all $F_k^{(i)}$ and $G_k^{(i)}$ have sizes $w_i \times a_{i-1}$ and $a_i \times a_{i-1}$, respectively, and all $W_k = [* \cdots *]$ are partitioned accordingly. In particular, all $F_k^{(i)}$ and $G_k^{(i)}$ together have full row rank, i.e.,

$$\text{rank} \left(\begin{bmatrix} F_k^{(i)} \\ G_k^{(i)} \end{bmatrix} \right) = a_i + w_i = s_{i-1} \leq a_{i-1} \quad \text{for all } k \in \mathbb{Z}. \quad (20)$$

Proof. See Appendix B.

Further reduction of the F_k and G_k blocks in (19) are possible. This further reduced form will be needed in the proof of Lemma 14. Additionally to Theorem 9 we can show that every pencil which is equivalent to a pencil of the form (19) has a well defined strangeness index. This property will also be needed in the proof of Lemma 14.

Remark 10. For the discrete-time case studied here one can also obtain results corresponding to [7, Section 3.2], i.e., with the help of the canonical form (19) one can determine the characteristic values $\{(r_{f,i}, h_{f,i})\}_{i \in \mathbb{N}_0}$ from the local characteristic values (compare Theorem 3) of the inflated descriptor systems. The according statements and proofs can be found in [2].

4.1 Existence and uniqueness of solutions

Concerning existence and uniqueness of sequences of matrix pairs with well defined strangeness index we get similar results as in [7].

Theorem 11. *Let the strangeness index μ of the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ as in (17) be well defined. Then the discrete descriptor system (2) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete descriptor system of the form*

$$\begin{aligned} x_1^{k+1} &= A_k^{(1)} x_1^k + A_k^{(3)} x_3^k + f_1^k, & r_{f,\mu} \\ 0 &= x_2^k + f_2^k, & h_{f,\mu} \\ 0 &= f_3^k, & v_\mu \end{aligned}$$

where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_3^k \in \mathbb{C}^{u_\mu}$ and each inhomogeneity f_1^k, f_2^k, f_3^k is determined by the original right hand sides $f^k, \dots, f^{k+\mu+1}$ as in (2) for all $k \in \mathbb{Z}$. For the associated forward problem

$$E_k x^{k+1} = A_k x^k + f^k, \text{ for all } k \in \mathbb{N}_0, \quad (22)$$

we also have the following results:

1. System (22) is solvable if and only if the v_μ consistency conditions

$$f_3^k = 0$$

are fulfilled for all $k \in \mathbb{N}_0$.

2. An initial condition $x^0 = \hat{x}$ is consistent with system (22) if and only if in addition the $h_{f,\mu}$ conditions

$$x_2^0 = \hat{x}_2 = -f_2^0$$

are satisfied.

3. The corresponding initial value problem is uniquely solvable if and only if in addition

$$u_\mu = 0$$

holds.

Proof. Under the assumptions of the Theorem we see that the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ can be transformed to the form (18) by $\mu + 1$ reduction steps and proper global equivalence transformations. Both of these operations generate a one-to-one correspondence of solutions which proves the statement. \square

5 Backward global canonical form

In the previous section we have constructed a canonical form which allows for statements about the solvability of descriptor systems where one starts at some point k_0 and computes a solution into the future. Let us now have a short look at the case where one starts at a point in time k_0 , and calculates into the past, i.e., one calculates a solution $\{x^k\}_{k \leq k_0}$. This case is closely related to the first case. To see this, suppose that a descriptor system of the form

$$\begin{aligned} E_k x^{k+1} &= A_k x^k + f^k, & k \leq k_0 - 1, \\ x^{k_0} &= \hat{x}, \end{aligned}$$

is given and we are looking for a solution. Substituting k by $-k$ then yields

$$E_{-k} x^{-k+1} = A_{-k} x^{-k} + f^{-k}, \quad k \geq 1 - k_0.$$

Defining $y^k := x^{-k+1}$ and $g^k := -f^{-k}$, this system is equivalent to

$$A_{-k} y^{k+1} = E_{-k} y^k + g^k, \quad k \geq -k_0 + 1.$$

By calculating the solution of the very last system into the future with the initial condition $y^{-k_0+1} = \hat{x}$, i.e., by calculating $\{y^k\}_{k \geq -k_0+1}$, through resubstitution we see, that we have obtained a solution

$$\begin{aligned} \{y^k\}_{k \geq -k_0+1} &= \{x^{-k+1}\}_{k \geq -k_0+1} = \{x^{k+1}\}_{-k \geq -k_0+1} = \\ &= \{x^{k+1}\}_{k \leq k_0-1} = \{x^k\}_{k-1 \leq k_0-1} = \{x^k\}_{k \leq k_0}, \end{aligned}$$

i.e., a solution of the backward problem. Thus, we do not have to consider the backward case separately. We make the following definition.

Definition 12. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence of matrix pairs. Then

$$\{(A_{-k}, E_{-k})\}_{k \in \mathbb{Z}} \tag{23}$$

is called the *reversed sequence of matrix pairs*. Analogously, the descriptor system corresponding to (23) is called the *reversed descriptor system*. Also, the strangeness index of (23) is called *reversed strangeness index* and is denoted by μ_b (for backwards). In contrast to this, the strangeness index of the original sequence is also called *forward strangeness index* and denoted by μ_f .

The following Lemma is obvious.

Lemma 13. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ and $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ be two globally equivalent sequences of matrix pairs. Then the reversed sequences are also globally equivalent.

Proof. By assumption we know that there exist invertible matrices P_k, Q_k such that

$$\begin{aligned} E_k &= P_k \tilde{E}_k Q_{k+1}, \\ A_k &= P_k \tilde{A}_k Q_k, \end{aligned}$$

for all $k \in \mathbb{Z}$. Substituting k by $-k$ then yields

$$\begin{aligned} E_{-k} &= P_{-k} \tilde{E}_{-k} Q_{-k+1}, \\ A_{-k} &= P_{-k} \tilde{A}_{-k} Q_{-k}, \end{aligned}$$

for all $k \in \mathbb{Z}$. Setting $R_k := P_{-k}$ and $S_k := Q_{-k+1}$, this condition then is finally equivalent to

$$\begin{aligned} A_{-k} &= R_k \tilde{A}_{-k} S_{k+1}, \\ E_{-k} &= R_k \tilde{E}_{-k} S_k, \end{aligned}$$

for all $k \in \mathbb{Z}$, which proves the claim with the transformation matrix sequences $\{R_k\}$ and $\{S_k\}$. \square

6 A two-way global canonical form

Finally, we consider the case where we want to obtain a solution for all $k \in \mathbb{Z}$. This case is somehow different from the forward and backward case. To see this consider the form (18). The problem is that in this (strangeness-free) form (18) the $A_k^{(1)}$ are allowed to be arbitrary. Consider a descriptor system which only consists of the (1,1) block in (18). For such a system one can easily compute the unique value of x^{k_0+1} once the value of x^{k_0} is given. In contrast, if the value for x^{k_0} is given there may be many choices of appropriate x^{k_0-1} values (e.g., $x^{k_0} = 0x^{k_0-1}$, $x^{k_0-1} = x^{k_0-2}$, $x^{k_0-2} = x^{k_0-3}$, ...) or even no possible choice of an appropriate x^{k_0-1} value (e.g., $x^{k_0} = x^{k_0-1}$, $x^{k_0-1} = 0x^{k_0-2}$, given that $x^{k_0} \neq 0$), depending on the sequence of the $A_k^{(1)}$ matrices. Also, the solvability may vary from iterate to iterate. It seems that additional rank assumptions are appropriate to obtain a canonical form which allows for statements about solvability in the two-way case.

One approach that suggests itself is to not only demand the system itself to have well defined strangeness index but to also demand the reversed system to have well defined strangeness index. To study such systems, the following Lemma is very helpful.

Lemma 14. *For $k \in \mathbb{Z}$ let $E_k, A_k \in \mathbb{C}^{m,n}$ be such matrices, that the strangeness index μ_f and the reversed strangeness index μ_b of $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ are both well defined. Perform one step of index reduction from (9) to (10) on the reversed sequence $\{(A_{-k}, E_{-k})\}_{k \in \mathbb{Z}}$ and denote the so obtained sequence by $\{(\tilde{A}_{-k}, \tilde{E}_{-k})\}_{k \in \mathbb{Z}}$. Then, not only the reversed strangeness index $\tilde{\mu}_b$ (i.e., the strangeness index of $\{(\tilde{A}_{-k}, \tilde{E}_{-k})\}_{k \in \mathbb{Z}}$) but also the strangeness index $\tilde{\mu}_f$ of $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ is well defined. We have $\tilde{\mu}_f \leq \mu_f$ and $\tilde{\mu}_b \leq \mu_b$.*

Proof. See Appendix C.

The index reduction performed in Lemma 14 will be used frequently, which is why we introduce the following Definition.

Definition 15. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence of matrix pairs. Then performing one step of index reduction from form (9) to (10) on the reversed sequence $\{(A_{-k}, E_{-k})\}_{k \in \mathbb{Z}}$ and re-reversing the so obtained sequence is called one step of *reversed index reduction*. In contrast to this, the index reduction from (9) to (10) on the original sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is also called *forward index reduction*.

Lemma 14 shows that under the assumption that both the strangeness index and the reversed strangeness index are well-defined one can perform forward and reversed index reduction steps at will. One may conjecture, that one obtains globally equivalent sequences of matrix pairs, as long as one performs the same number of forward and reversed reduction steps, but as we will see in the next Example this is false. The next Example also shows that one step of reversed index reduction can alter the forward strangeness index.

Example 16. Consider the constant sequence of matrix pairs

$$\left\{ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}. \quad (24)$$

First performing one forward step of index reduction on (24) yields the sequence

$$\left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}. \quad (25)$$

Performing one step of reversed index reduction on this sequence does not alter the sequence any more. First performing one reversed step of index reduction on (24), however, yields the sequence

$$\left\{ \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (26)$$

which again is not altered anymore by one further step of forward index reduction. Comparing (25) with (26) clearly shows that these two sequences are not globally equivalent, since (corresponding to Lemma 4) those matrix pairs do not have the same characteristic values.

Let us first derive a canonical form under the assumption that both the strangeness index and the reversed strangeness index are well defined.

Theorem 17. For $k \in \mathbb{Z}$ let $E_k, A_k \in \mathbb{C}^{m,n}$ be matrices, such that the strangeness index and the reversed strangeness index of $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ are both well defined. Define the matrices

$$\begin{aligned} Z_k &\text{ as a basis of } \text{corange}(E_k) \text{ for } k \in \mathbb{Z}, \\ Y_k &\text{ as a basis of } \text{corange}(A_k) \text{ for } k \in \mathbb{Z}. \end{aligned}$$

Then, there exist $h_f, h_b, q \in \mathbb{N}_0$ such that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} h_f &= \text{rank}(Z_k^H A_k), & (\text{forward direction}) \\ h_b &= \text{rank}(Y_k^H E_k), & (\text{backward direction}) \\ q &= h_f + h_b - \text{rank} \left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right). \end{aligned} \quad (27)$$

These quantities in (27) are invariant under global equivalence and we have

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_k^{(1)} & 0 & 0 & E_k^{(2)} \\ 0 & I_{h_b-q} & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{h_f-q} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}, \quad (28)$$

where for all $k \in \mathbb{Z}$ the matrices $[E_k^{(1)} \ E_k^{(2)}]$ and $[A_k^{(1)} \ A_k^{(2)}]$ have full row rank.

Proof. See Appendix D.

From the form (28) one may conjecture that it is also possible to show Theorem 17 by defining

$$q = h_f + h_b - \text{rank} \left(\begin{bmatrix} Y_k^H E_k \\ Z_k^H A_k \end{bmatrix} \right), \quad (29)$$

instead of (27). This is not the case. If one would do so, q would not be invariant under global equivalence any more as shown by the following example.

Example 18. Define the (constant) sequence of matrix pairs

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} := \left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

which has $h_f = 1$, $h_b = 1$ and with both (29) or (27) $q = 1$. Transforming this sequence from the right by the sequence $\{Q_k\}_{k \in \mathbb{Z}}$ defined through

$$Q_{2k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } Q_{2k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for all } k \in \mathbb{Z},$$

will yield a sequence $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}} = \{(E_k Q_{k+1}, A_k Q_k)\}_{k \in \mathbb{Z}}$ which satisfies

$$\begin{aligned} (\tilde{E}_{2k}, \tilde{A}_{2k}) &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) \text{ and} \\ (\tilde{E}_{2k+1}, \tilde{A}_{2k+1}) &= \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) \text{ for all } k \in \mathbb{Z}. \end{aligned}$$

This sequence would have $q = 0$ if one would apply definition (29).

The same result as in Theorem 17 can be obtained under a weaker assumption.

Corollary 19. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence and define the matrices

Z_k as a basis of $\text{corange}(E_k)$ for $k \in \mathbb{Z}$,

Y_k as a basis of $\text{corange}(A_k)$ for $k \in \mathbb{Z}$.

Assume that the quantities

$$r_f = r_{f,k} \equiv \text{rank}(E_k), \quad (30a)$$

$$h_f = h_{f,k} \equiv \text{rank}(Z_k^H A_k), \quad (30b)$$

$$h_b = h_{b,k} \equiv \text{rank}(Y_k^H E_k), \quad (30c)$$

$$q = q_k \equiv h_{f,k} + h_{b,k} - \text{rank} \left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right), \quad (30d)$$

(which are invariant under global equivalence as shown in Theorem 17) are constant for all $k \in \mathbb{Z}$. Then we also have the relation (28), where for all $k \in \mathbb{Z}$ the matrices $[E_k^{(1)} \ E_k^{(2)}]$ and $[A_k^{(1)} \ A_k^{(2)}]$ have full row rank.

Proof. First we note that under the given assumptions the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to a sequence of the form

$$\left\{ \left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

with all $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ and all $A_k^{(1,1)}$ having full row rank. Since q is invariant under global equivalence, it is clear that

$$\text{rank} \left(\begin{bmatrix} E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & I_{h_f} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} E_k^{(2,1)} & 0 \\ 0 & I_{h_f} \end{bmatrix} \right)$$

has to be constant for all $k \in \mathbb{Z}$. Thus, also all $E_k^{(2,1)}$ have to have constant rank. The remainder of the proof can then be carried out analogous to the proof of Theorem 17. \square

Applying one step of forward and one step of reversed index reduction to the form (28) yields the form

$$\left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & 0 \\ 0 & I_{h_b-q} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{h_f-q} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}. \quad (31)$$

It is clear that first applying one step of reversed and then one step of forward index reduction will yield another form (i.e., the I_q block then stays in the left matrices and is therefore missing in the right matrices).

Remark 20. Using the preceding results we can adapt the process (13) to systems that fulfill even harder constant rank assumptions. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}} = \{(E_{k,0}, A_{k,0})\}_{k \in \mathbb{Z}}$ we define a sequence (of sequences of matrix pairs) $\{ \{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \}_{i \in \mathbb{Z}}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}}$ by Corollary 19 to the form (28) assuming that the local invariants $r_f =: r_{f,i}$, $h_f =: h_{f,i}$, $h_b =: h_{b,i}$ and $q =: q_i$ are constant for all matrix pairs on the whole interval \mathbb{Z} . Then we reduce the so obtained sequence of matrix pairs first by one step of forward and then by one step of reversed index reduction to the form (31), which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k \in \mathbb{Z}}$. Due to Corollary 6, Corollary 19 and Lemma 13 the so obtained sequence of quadruples $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i \in \mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs.

Remark 21. Under the assumption that the strangeness index and the reversed strangeness index of the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ are both well defined, all constant rank assumptions which are required in Remark 20 are satisfied, because of Lemma 14 and Theorem 17.

To define a strangeness index under the assumptions of Remark 20 we need a Lemma similar to Lemma 7.

Lemma 22. *Let the sequence $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i \in \mathbb{N}_0}$ and the sequence (of sequences of matrix pairs) $\{ \{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \}_{i \in \mathbb{N}_0}$ be defined as in Remark 20. In particular, let the constant rank assumptions (30) hold for every step of the reduction process in Remark 20. Defining the quantities*

$$\begin{aligned} r_{b,i} &:= r_{f,i} - h_{b,i} + h_{f,i}, \\ s_{E,i} &:= r_{f,i} - r_{f,i+1}, \\ s_{A,i} &:= r_{b,i} - r_{b,i+1}, \\ s_i &:= s_{E,i} + s_{A,i}, \end{aligned} \quad (32)$$

for all $i \in \mathbb{N}_0$, there exists a $\mu \in \mathbb{N}_0$ so that

$$r_{b,i} = \text{rank}(A_{k,i}), \quad (33a)$$

$$r_{f,i+1} \leq r_{f,i}, \quad (33b)$$

$$r_{b,i+1} \leq r_{b,i}, \quad (33c)$$

$$s_{E,\mu+i} = s_{A,\mu+i} = s_{\mu+i} = 0, \quad (33d)$$

for all $i \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.

Proof. (33a) follows directly from the identity $h_b = r_f + h_f - r_b$ in Theorem 3. Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Corollary 19 that we have the global equivalence relation

$$\{(E_{k,i}, A_{k,i})\} \sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & E_{k,i}^{(2)} \\ 0 & I_{h_{b,i}-q_i} & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & A_{k,i}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & I_{h_{f,i}-q_i} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\},$$

which implies

$$\{(E_{k,i+1}, A_{k,i+1})\} \sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & 0 \\ 0 & I_{h_{b,i}-q_i} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_i} & 0 \\ 0 & 0 & 0 & I_{h_{f,i}-q_i} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}.$$

This clearly shows that we have $\text{rank}(E_{k,i+1}) \leq \text{rank}(E_{k,i})$ and also that $\text{rank}(A_{k,i+1}) \leq \text{rank}(A_{k,i})$, which implies (33b) and (33c). Since we know that both of the sequences $\{r_{f,i}\}_{i \in \mathbb{N}_0}$ and $\{r_{b,i}\}_{i \in \mathbb{N}_0}$ are non-increasing and bounded by zero, they have to become stationary at some point μ , which finally shows (33d). \square

Lemma 22 leads to the following Definition.

Definition 23. Let $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ be a sequence of matrix pairs. Let the sequence $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i \in \mathbb{N}_0}$ (as described in Remark 20) be well defined. In particular, let (30) hold for every entry $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ of the sequence (of sequences of matrix pairs) in Remark 20. Then, with the definitions (32) we call

$$\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\} \quad (34)$$

the *two-way strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ and of the associated descriptor system (2). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ and (2) *two-way strangeness-free*.

Connections between the two-way strangeness index and the forward and backward strangeness index are investigated in the following. Since one step of the iterative procedure described in Remark 20 involves one step of forward and one step of reversed index reduction

it may happen that the two-way strangeness index is smaller than the forward strangeness index.

Example 24. Consider the sequence of (constant) matrix pairs

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} = \left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}.$$

With Definition 8 we get the sequences

$$\begin{aligned} (r_{f,0}, h_{f,0}, a_0, s_0) &= (2, 1, 1, 1), \\ (r_{f,1}, h_{f,1}, a_1, s_1) &= (1, 2, 1, 1), \\ (r_{f,2}, h_{f,2}, a_2, s_2) &= (0, 3, 1, 0), \\ (r_{f,3}, h_{f,3}, a_3, s_3) &= (0, 3, 0, 0), \\ (r_{f,4}, h_{f,4}, a_4, s_4) &= (0, 3, 0, 0), \\ &\vdots \end{aligned}$$

and thus an forward strangeness index of 2. With Definition 23, however, we face the reduction process

$$\begin{aligned} &\left\{ \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\ \text{reduction} \quad &\sim \left\{ \left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \end{aligned}$$

and thus the sequences

$$\begin{aligned} (r_{f,0}, h_{f,0}, h_{b,0}, q_0, r_{b,0}, s_{E,0}, s_{A,0}, s_0) &= (2, 1, 1, 0, 2, 1, 1, 2), \\ (r_{f,1}, h_{f,1}, h_{b,1}, q_1, r_{b,1}, s_{E,1}, s_{A,1}, s_1) &= (1, 1, 1, 0, 1, 0, 0, 0), \\ (r_{f,2}, h_{f,2}, h_{b,2}, q_2, r_{b,2}, s_{E,2}, s_{A,2}, s_2) &= (1, 1, 1, 0, 1, 0, 0, 0), \\ &\vdots \end{aligned}$$

which shows that the two-way strangeness index is 1.

From this Example we see that the forward strangeness index can be twice as big as the two-way strangeness index. From Example 16 we see that one step of forward index reduction can alter the backward strangeness index by one. Thus, if one first applies one step of forward index reduction this will definitely change the forward strangeness index (unless the system is forward strangeness-free). The subsequent step of backward index reduction can then again decrease the forward strangeness index by one. In other words, one step of two-way index reduction can decrease the forward strangeness index (or the backward strangeness index analogously) by two. This is why one could suppose that the forward and backward strangeness index of a system are always less than or equal to the two-way strangeness index multiplied by two. Also, we observe that one step of two-way index reduction involves one step of forward index reduction, which is why we can guess that the two-way strangeness

index is always less than or equal to the forward strangeness index.

Of course the forward strangeness index and the two-way strangeness index can also coincide as one can see by applying Definition 8 and Definition 23 to a (constant) sequence of matrix pairs of the form

$$\left\{ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}.$$

In fact, any constant sequence of regular pencils will do, as long as the pencil does not have the eigenvalue 0. Based on these observations we conjecture the following.

Conjecture 25. Consider the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$. Then the following statements hold.

1. If $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ has well defined two-way strangeness index it also has well defined forward and well defined reversed strangeness index.
2. Let the forward strangeness index μ_f , the reversed strangeness index μ_b and the two-way strangeness index μ of $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ all be well defined. Then we have $2\mu \geq \max(\mu_f, \mu_b) \geq \mu$.

In order to prove Conjecture 25 one could first prove a Theorem which adapts Theorem 9 to the two-way case. Then one could use this form to determine the forward strangeness index and thus show that it is well defined. The same technique was used in proof of Lemma 14. Conjecture 25 is easy to show if we require the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ to be two-way strangeness free, since in this case due to Theorem 17 we have

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_k^{(1)} & 0 & E_k^{(2)} \\ 0 & I_{h_b} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

with all $E_k^{(1)}$ and all $A_k^{(1)}$ having full row rank. Performing on step of forward index reduction on this sequence proves the claim. Anyway, it seems that at least the first part of Conjecture 25 is quite obvious, and it is not sure whether it is worthwhile to take the burden of the proof.

6.1 Existence and uniqueness of solutions

With the notation of Remark 20 and Corollary 19 we know that for the two-way strangeness index μ we have

$$\{(E_{k,\mu}, A_{k,\mu})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 & E_{k,\mu}^{(2)} \\ 0 & I_{h_{b,\mu}-q_\mu} & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & A_{k,\mu}^{(2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & I_{h_{f,\mu}-q_\mu} \end{bmatrix} \right) \right\},$$

and thus

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 & 0 \\ 0 & I_{h_{b,\mu}-q_\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q_\mu} & 0 \\ 0 & 0 & 0 & I_{h_{f,\mu}-q_\mu} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \right\}.$$

But we also know from the definitions (32) that $\text{rank}(E_{k,\mu}) = \text{rank}(E_{k,\mu+1})$ from which we see that $q_\mu = 0$ and that $E_{k,\mu}^{(1)}$ is a matrix with full row rank for all $k \in \mathbb{Z}$, since from Corollary 19 we know that all $[E_{k,\mu}^{(1)} \ E_{k,\mu}^{(2)}]$ had full row rank. From $\text{rank}(A_{k,\mu}) = \text{rank}(A_{k,\mu+1})$, we analogously see that all $A_{k,\mu}^{(1)}$ already have full row rank. Thus, every sequence with well defined two-way strangeness index can be transformed by $\mu + 1$ reduction steps and appropriate global equivalence transformations to a two-way strangeness-free sequence of the form

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 \\ 0 & I_{h_{b,\mu}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (35)$$

where all $A_{k,\mu}^{(1)}$ and all $E_{k,\mu}^{(1)}$ have full row rank. By transformations of the (1,1)-block in (35) one can also achieve that

$$E_{k,\mu}^{(1)} = [I \ 0] \text{ for all } k \in \mathbb{N}_0 \text{ and } A_{k,\mu}^{(1)} = [I \ 0] \text{ for all } k \leq -1. \quad (36)$$

From (35) with (36) one can derive a statement similar to Theorem 11 for the case where one wants to get a solution for all $k \in \mathbb{Z}$.

Theorem 26. *Assume that the two-way strangeness index μ of the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is well defined as in (34). Then the discrete descriptor system (2) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete descriptor system of the form*

$$\begin{aligned} x_1^{k+1} &= A_k^{(1)} x_1^k + A_k^{(2)} x_4^k + f_1^k, & k \geq 0, & \quad r_{f,\mu} - h_{b,\mu} \\ x_1^{k-1} &= E_{k-1}^{(1)} x_1^k + E_{k-1}^{(2)} x_4^k - f_1^{k-1}, & k \leq 0, & \quad r_{b,\mu} - h_{f,\mu} \\ x_2^{k+1} &= f_2^k, & & \quad h_{b,\mu} \\ 0 &= x_3^k + f_3^k, & & \quad h_{f,\mu} \\ 0 &= f_4^k, & & \quad m - r_{f,\mu} - h_{f,\mu} \end{aligned}$$

where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_4^k \in \mathbb{C}^{u_\mu}$ and each of the inhomogeneities $f_1^k, f_2^k, f_3^k, f_4^k$ is determined by the original right hand sides $f^{k-\mu-1}, \dots, f^k, \dots, f^{k+\mu+1}$ as in (2) for all $k \in \mathbb{Z}$. For the problem

$$E_k x^{k+1} = A_k x^k + f^k, \quad k \in \mathbb{Z}, \quad (38)$$

we also have the following results:

1. System (38) is solvable if and only if the $v_\mu := m - r_{f,\mu} - h_{f,\mu}$ consistency conditions

$$f_4^k = 0$$

are fulfilled for all $k \in \mathbb{Z}$.

2. An initial condition $x^0 = \hat{x}$ is consistent with (38) if and only if in addition the $h_{f,\mu} + h_{b,\mu}$ conditions

$$\begin{aligned} x_2^0 &= \hat{x}_2 = f_2^{-1}, \\ x_3^0 &= \hat{x}_3 = -f_3^0, \end{aligned}$$

are satisfied.

3. The corresponding initial value problem is uniquely solvable if and only if in addition

$$u_\mu = 0$$

holds.

Proof. The proof can be carried out as the proof of Theorem 11 by using (35) and (36). \square

7 Conclusion

Analogously to [7], in this text we have first derived a canonical form that allows for statements about the existence and uniqueness of solutions for forward discrete-time descriptor systems. To obtain this canonical form one has to make constant rank assumptions about the involved sequences of matrix pairs. An index was defined for descriptor systems that fulfill these constant rank assumptions. In every step of the reduction procedure only two characteristics (13) have to be constant for the sequence of matrix pairs, whereas in the continuous-time case there have to be three constant for the pair of matrix valued functions. However, it seems less natural to make constant rank assumptions in the discrete-time case than in the continuous-time case. In the discrete-time case one can never be sure if there is an interval in which the constant rank assumptions hold as one can be in the continuous-time case with smooth matrix valued functions (see [7, Theorem 3.25]).

After the analysis of the forward case we took a short look at the backward case and then continued to understand the two-way case. A different canonical form has been obtained for two-way descriptor systems and a two-way strangeness index has been defined. More constant rank assumptions than in the forward case were necessary to obtain the results and the proofs got a bit messy.

The processes that lead to the forward and the two-way canonical form both imply a way to numerically compute solutions of descriptor systems that satisfy the corresponding constant rank assumptions. A first attempt to do so has been done in [2]. The same algorithm (with a different implementation) could also be used to efficiently compute the solution of time-invariant descriptor systems, which always satisfy any constant rank assumption. Anyway, it is possible and desirable to generalize the time-variant algorithm to descriptor systems, that do not fulfill the constant rank assumptions, although this might be a quite complicated undertaking.

Acknowledgment. I wish to thank Volker Mehrmann for suggesting this topic to me, the many insightful discussions, and the repeated revisions of the manuscript.

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A Proof of Theorem 7

Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Lemma 4 that

$$\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}.$$

Thus from the Definition of the iterative process at (13) we have

$$\{(E_{k,i+1}, A_{k,i+1})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

which shows

$$r_{f,i} = \text{rank} \left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix} \right) \geq \text{rank} \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \end{bmatrix} \right) = r_{f,i+1},$$

and thus (15a) follows. Since $\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix}$ has full row rank $r_{f,i}$, we get

$$\dim \left(\text{range} \left(E_{k,i}^{(1)} \right) \right) + \dim \left(\text{corange} \left(E_{k,i}^{(1)} \right) \right) = r_{f,i}$$

and independent of this

$$\dim \left(\text{range} \left(E_{k,i}^{(1)} \right) \right) = r_{f,i+1}.$$

For $k \in \mathbb{Z}$ let Z_k be a basis of $\text{corange} \left(E_{k,i}^{(1)} \right)$. Then we know that

$$h_{f,i+1} = h_{f,i} + \text{rank} \left(Z_k^H A_{k,i}^{(1)} \right) \leq h_{f,i} + \dim \left(\text{corange} \left(E_{k,i}^{(1)} \right) \right). \quad (39)$$

Combining these equations yields

$$\begin{aligned} r_{f,i} + h_{f,i} &= \dim \left(\text{range} \left(E_{k,i}^{(1)} \right) \right) + \dim \left(\text{corange} \left(E_{k,i}^{(1)} \right) \right) + h_{f,i} \\ &\geq r_{f,i+1} + h_{f,i+1} \end{aligned}$$

which implies (15b). Since the sequence $\{r_{f,i}\}$ is non-increasing and bounded by zero, it becomes stationary at some point μ , which implies (15c). From (39) one can also see (15d). (15f) follows from (14c).

Now we show via induction that for all $i \in \mathbb{N}_0$ we have

$$\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 \\ 0 & I_{a_i} & 0 \\ 0 & 0 & I_{h_{f,i-1}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (40)$$

where all $\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix}$ have full row rank.

For $i = 0$ we have $h_{f,-1} = 0$ and because of Lemma 4 we get

$$\{(E_{k,0}, A_{k,0})\}_{k \in \mathbb{Z}} = \{(E_k, A_k)\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,0}^{(1)} & E_{k,0}^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & I_{a_0} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

since $a_0 = h_{f,0}$.

For the induction step we note that (40) is a special case of (9), and thus one can immediately perform the reduction from (9) to (10) by

$$\begin{aligned} \{(E_{k,i+1}, A_{k,i+1})\}_{k \in \mathbb{Z}} &\sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 \\ 0 & I_{a_i} & 0 \\ 0 & 0 & I_{h_{f,i-1}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\ &\sim \left\{ \left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{a_i+h_{f,i-1}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\ &\sim \left\{ \left(\begin{bmatrix} E_{k,i+1}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 \\ \hat{A}_{k,i+1}^{(2)} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \end{aligned}$$

where all $E_{k,i+1}^{(1)}$ have full row rank $r_{f,i+1}$. Adapting the indexing we can proceed with

$$\begin{aligned} &\{(E_{k,i+1}, A_{k,i+1})\}_{k \in \mathbb{Z}} \\ &\sim \left\{ \left(\begin{bmatrix} E_{k,i+1}^{(1)} & E_{k,i+1}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & A_{k,i+1}^{(2)} & 0 \\ 0 & I_{a_{i+1}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,i}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\ &\sim \left\{ \left(\begin{bmatrix} E_{k,i+1}^{(1)} & E_{k,i+1}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 & 0 \\ 0 & I_{a_{i+1}} & 0 \\ 0 & 0 & I_{h_{f,i}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \end{aligned}$$

which completes the induction step. From the form (40) we obtain $r_{f,i} - r_{f,i+1} \leq a_i$ for all $i \in \mathbb{N}_0$. From the induction step we can further see that $a_{i+1} = \text{rank}(\hat{A}_{k,i+1}^{(2)}) \leq r_{f,i} - r_{f,i+1}$ for all $i \in \mathbb{N}_0$, since $\hat{A}_{k,i+1}^{(2)}$ only has $r_{f,i} - r_{f,i+1}$ rows. This shows (15e). Let μ be as in (15c). Then (40) implies that

$$\{(E_{k,\mu}, A_{k,\mu})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & E_{k,\mu}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 \\ 0 & I_{a_\mu} & 0 \\ 0 & 0 & I_{h_{f,\mu-1}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}.$$

Further reduction steps show that for all $j \geq 1$ we have

$$\{(E_{k,\mu+j}, A_{k,\mu+j})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 \\ 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (41)$$

since $r_{f,\mu+j} = r_{f,\mu}$ which means that all $E_{k,\mu}^{(1)}$ have full row rank. Thus, applying a reduction step to (41) does not change anything. This shows that $h_{f,\mu+j} = h_{f,\mu}$, which is equivalent

to $a_{\mu+j} = 0$. From this (16) follows.

Note that there exists a positive integer ξ for which the sequence $h_{f,i}$ gets stationary, i.e., $h_{f,i} = h_f \in \mathbb{N}_0$ for all $i \geq \xi$, which follows from the boundedness of the sequence ($h_{f,i} \leq m$) and because of (15d). This implies (15g).

(15h) follows, since on the one hand we have

$$\begin{aligned} v_i - v_0 &= m - r_{f,i} - h_{f,i} - (m - r_{f,0} - h_{f,0}) \\ &= (r_{f,0} + h_{f,0}) - (r_{f,i} + h_{f,i}), \end{aligned}$$

and on the other hand we have by (14f), (14d), and (14b) that

$$\begin{aligned} w_1 + \dots + w_i &= s_0 + \dots + s_{i-1} - a_1 - \dots - a_i \\ &= r_{f,0} - r_{f,i} - (a_1 + \dots + a_i) \\ &= r_{f,0} - r_{f,i} - (h_{f,i} - h_{f,0}). \end{aligned}$$

(15i) can again be derived from (40). For this, note that we have by (14d) that

$$s_i = r_{f,i} - r_{f,i+1} = \text{rank} \begin{pmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{pmatrix} - \text{rank} \begin{pmatrix} E_{k,i}^{(1)} \end{pmatrix} \leq a_i,$$

where the last inequality holds, since the $E_{k,i}^{(2)}$ matrices only have a_i columns as one can see from the block structure of (40). Let $i \in \mathbb{N}$ be arbitrarily. Then by (14f) and (15i) we have that

$$w_i + a_i = s_{i-1} \leq a_{i-1} \leq a_{i-1} + w_{i+1},$$

which is equivalent to (15l). For this we have used that all w_i are non-negative, which is shown below.

To prove the non-negativity of all constants defined in (14) without using (15l), first observe that $a_i \geq 0$, since the sequence $\{h_{f,i}\}_{i \geq -1}$ is non-decreasing. All d_i are non-negative, since all $r_{f,i}$ and all $h_{f,i}$ are non-negative. All s_i are non-negative because the sequence $\{r_{f,i}\}$ is non-increasing.

We then see that $v_i = m - d_i$ and thus $\{v_i\}$ is a non-decreasing sequence. Since $v_0 = m - r_{f,0} - h_{f,0} = m - \text{rank}(E_k) - \text{rank}(Z_k^H A_k) \geq 0$ (where Z_k is a basis of corange(E_k)) all v_i have to be non-negative.

Also note, that for all $i \geq 1$ we have

$$\begin{aligned} v_i - v_{i-1} &= m - r_{f,i} - h_{f,i} - (m - r_{f,i-1} - h_{f,i-1}) \\ &= r_{f,i-1} - r_{f,i} - (h_{f,i} - h_{f,i-1}) \\ &= s_{i-1} - a_i = w_i, \end{aligned} \tag{42}$$

which shows the non-negativity of all w_i , since $\{v_i\}$ is non-decreasing.

To finally show (15j) and (15k) note that for all $i \in \mathbb{N}_0$ we have by (15i), (15m), and (14f) that

$$s_{i+1} \leq a_{i+1} \leq a_{i+1} + w_{i+1} = s_i. \quad \square$$

B Proof of Theorem 9

With an inductive argument we show that

$$\{(E_k, A_k)\}_{k \in \mathbb{Z}} \sim \{(\tilde{E}_{k,i}, \tilde{A}_{k,i})\}_{k \in \mathbb{Z}} := \left\{ \left(\begin{bmatrix} E_{k,i}^{(1,1)} & E_{k,i}^{(1,2)} & * & \cdots & * \\ 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix} \right) \right\}, \quad (43)$$

where:

1. All $[E_{k,i}^{(1,1)}, E_{k,i}^{(1,2)}]$ are of full row rank $r_{f,i}$ and $\text{rank}(E_{k,i}^{(1,1)}) = r_{f,i+1}$ for all $k \in \mathbb{Z}$.
2. With $Z_{k,i}$ being bases of corange $(E_{k,i}^{(1,1)})$, we have that $\text{rank}(Z_{k,i}^H A_{k,i}^{(1)}) = a_{i+1}$ for all $k \in \mathbb{Z}$.
3. $\text{rank}\left(\begin{bmatrix} F_{k,j} \\ G_{k,j} \end{bmatrix}\right)$ is full for all $j \in \{1, \dots, i\}$ and for all $k \in \mathbb{Z}$.

For $i = 0$, from Lemma 4 we see that

$$\begin{aligned} \{(E_k, A_k)\} &\sim \left\{ \left(\begin{bmatrix} E_{k,0}^{(1,1)} & E_{k,0}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & I_{h_{f,0}} \\ 0 & 0 \end{bmatrix} \right) \right\} \\ &\sim \left\{ \left(\begin{bmatrix} E_{k,0}^{(1,1)} & E_{k,0}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_0} \end{bmatrix} \right) \right\}, \end{aligned}$$

with $[E_{k,0}^{(1,1)}, E_{k,0}^{(1,2)}] = r_{f,0}$ from which the first part of 1. immediately follows. If we perform the step from (9) to (10), then we find that

$$r_{f,1} = \text{rank}(E_{k,0}^{(1,1)})$$

holds and thus 1. is shown. To see 2. let $Z_{k,0}$ be bases of corange $(E_{k,0}^{(1,1)})$ for all $k \in \mathbb{Z}$. Then

$$\begin{bmatrix} Z_{k,0} & 0 & 0 \\ 0 & I_{w_0} & 0 \\ 0 & 0 & I_{a_0} \end{bmatrix} \text{ is a basis of corange } \left(\begin{bmatrix} E_{k,0}^{(1,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

whereas the latter matrix is the one in (10). Thus,

$$h_{f,1} = \text{rank} \left(\begin{bmatrix} Z_{k,0}^H & 0 & 0 \\ 0 & I_{w_0} & 0 \\ 0 & 0 & I_{a_0} \end{bmatrix} \begin{bmatrix} A_{k,0}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_0} \end{bmatrix} \right) = \text{rank}(Z_{k,0}^H A_{k,0}^{(1)}) + a_0.$$

This gives

$$\text{rank} \left(Z_{k,0}^H A_{k,0}^{(1)} \right) = h_{f,1} - a_0 = h_{f,1} - h_{f,0} = a_1.$$

Suppose now that (43) holds for i . Applying Lemma 4 to the sequence of matrix pairs $\{(E_{k,i}^{(1,1)}, A_{k,i}^{(1)})\}_{k \in \mathbb{Z}}$ (in (43)) one obtains that

$$\{(E_{k,i}^{(1,1)}, A_{k,i}^{(1)})\}_{k \in \mathbb{Z}} \sim \left\{ \begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 \\ 0 & 0 \\ 0 & I_{a_{i+1}} \end{bmatrix} \right\}, \quad (44)$$

where $[E_{k,i+1}^{(1,1)}, E_{k,i+1}^{(1,2)}]$ is of full row rank $r_{f,i+1}$ due to part 1. and 2. of the inductive assumption. Applying the transformation corresponding to (44) to the original sequence of matrix pairs (43) yields that $\{(E_k, A_k)\}$ is globally equivalent to

$$\left\{ \left(\begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} & E_{k,i+1}^{(1,3)} & * & \cdots & * \\ 0 & 0 & E_{k,i+1}^{(2,3)} & * & & * \\ 0 & 0 & E_{k,i+1}^{(3,3)} & * & & * \\ 0 & 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & I_{a_{i+1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix} \right) \right\} \sim$$

$$\left\{ \left(\begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} & E_{k,i+1}^{(1,3)} & * & \cdots & * \\ 0 & 0 & F_{k,i+1} & * & & * \\ 0 & 0 & 0 & F_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,i+1} & * & & * \\ 0 & 0 & 0 & G_{k,i} & & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i+1}^{(1)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_{i+1}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & I_{a_i} & 0 & \cdots & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix} \right) \right\}, \quad (45)$$

by defining $F_{k,i+1} := E_{k,i+1}^{(2,3)}$ and $G_{k,i+1} := E_{k,i+1}^{(3,3)}$. Due to the nature of global equivalence it follows that

$$\begin{aligned} r_{f,i} &= \text{rank} \left(\begin{bmatrix} E_{k,i}^{(1,1)} & E_{k,i}^{(1,2)} \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} & E_{k,i+1}^{(1,3)} \\ 0 & 0 & E_{k,i+1}^{(2,3)} \\ 0 & 0 & E_{k,i+1}^{(3,3)} \end{bmatrix} \right). \end{aligned}$$

With regard to the fact that all $[E_{k,i+1}^{(1,1)}, E_{k,i+1}^{(1,2)}]$ also have full rank $r_{f,i+1}$, one observes that by (14d) it follows that

$$\text{rank} \left(\begin{bmatrix} E_{k,i+1}^{(2,3)} \\ E_{k,i+1}^{(3,3)} \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} F_{k,i+1} \\ G_{k,i+1} \end{bmatrix} \right) = r_{f,i} - r_{f,i+1} = s_i,$$

which means that part 3. is shown. Note that because of (15i) we always have $s_i \leq a_i$ which shows (20). Further, we notice that $G_{k,i+1}$ is in the same block row as $I_{a_{i+1}}$ and in the same block column as I_{a_i} which means that all $G_{k,i+1}$ are of size $a_{i+1} \times a_i$. Since the matrix

$$\begin{bmatrix} E_{k,i+1}^{(2,3)} \\ E_{k,i+1}^{(3,3)} \end{bmatrix}$$

has $(r_{f,i} - r_{f,i+1})$ rows, it follows by (14f) that all $F_{k,i+1}$ are of size $(r_{f,i} - r_{f,i+1} - a_{i+1}) \times a_i = w_{i+1} \times h_{f,i}$.

Performing $i + 1$ reductions from (9) to (10) for the sequence (45) gives us the identity

$$r_{f,i+1} = \text{rank} \left(\begin{bmatrix} E_{k,i+1}^{(1,1)} & E_{k,i+1}^{(1,2)} \end{bmatrix} \right).$$

Analogously performing $i + 2$ reductions from (9) to (10) on the sequence of matrix pairs (45) one finds that

$$r_{f,i+2} = \text{rank} \left(\begin{bmatrix} E_{k,i+1}^{(1,1)} \end{bmatrix} \right).$$

This shows part 1. To prove part 2., let $Z_{k,i+1}$ be bases of corange $(E_{k,i+1}^{(1,1)})$. Performing again $i + 2$ reductions from (9) to (10) on the sequence (45) and denoting the so obtained sequence by $\{(\hat{E}_k, \hat{A}_k)\}_{k \in \mathbb{Z}}$, we have that the matrices

$$\hat{Z}_k := \begin{bmatrix} Z_{k,i+1} & 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & I_{w_{i+1}} & \ddots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & I_{w_0} & \ddots & & \vdots \\ \vdots & & & \ddots & I_{a_{i+1}} & \ddots & \vdots \\ \vdots & & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 & I_{a_0} \end{bmatrix}$$

are bases of corange (\hat{E}_k) . Since all \hat{A}_k then only contain the $A_{k,i+1}^{(1)}$ and I_{a_j} block entries, it is clear that

$$h_{f,i+2} = \text{rank} \left(\hat{Z}_k^H \hat{A}_k \right) = \text{rank} \left(Z_{k,i+1} A_{k,i+1}^{(1)} \right) + a_0 + \dots + a_{i+1},$$

from which we see that

$$\text{rank} \left(Z_{k,i+1} A_{k,i+1}^{(1)} \right) = h_{f,i+2} - a_{i+1} - \dots - a_0.$$

Together with (14b) this gives

$$\text{rank} \left(Z_{k,i+1} A_{k,i+1}^{(1)} \right) = a_{i+2}.$$

This proves (43). Selecting $i = \mu$ in (43) and using (14b) yields

$$\begin{aligned} & \{(E_k, A_k)\} \\ & \sim \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1,1)} & E_{k,\mu}^{(1,2)} & * & \cdots & * \\ 0 & 0 & F_{k,\mu} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & F_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & G_{k,\mu} & & * \\ \vdots & \vdots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & G_{k,1} \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & I_{a_\mu} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & I_{a_0} \end{bmatrix} \right) \right\} \\ & = \left\{ \left(\begin{bmatrix} E_{k,\mu}^{(1,1)} & \tilde{W}_k \\ 0 & F_k \\ 0 & G_k \end{bmatrix}, \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}. \end{aligned}$$

By induction we obtain that all $\begin{bmatrix} E_{k,\mu}^{(1,1)} & E_{k,\mu}^{(1,2)} \end{bmatrix}$ have full row rank $r_{f,\mu}$ and that all $E_{k,\mu}^{(1,1)}$ have rank $r_{f,\mu+1}$. Further, from (16) we know that $r_{f,\mu} = r_{f,\mu+1}$. Thus, all $E_{k,\mu}^{(1,1)}$ have to have full row rank $r_{f,\mu}$ and one may reduce each of these matrices to echelon form (3) to obtain the form (19). \square

C Proof of Lemma 14

The proof will be carried out in three parts. The first part is a Corollary from Theorem 9, the second part is a Lemma which represents the converse of of Theorem 9, and the third part is the actual proof which employs the first two statements.

Corollary 27. *Let the strangeness index μ of the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ as in (17) be well defined. Then, with the definitions from (14), $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to a sequence of the form*

$$\left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (46)$$

with

$$C_k = \begin{bmatrix} 0 & 0 & I_{w_\mu} & 0 & & & \\ & & & I_{w_{\mu-1}} & 0 & & \\ & & & & \ddots & & \\ & & & & & I_{w_1} & 0 \\ & & & & & 0 & 0 \end{bmatrix},$$

$$D_k = \begin{bmatrix} 0 & 0 & 0 & D_k^{(\mu)} & 0 & * & \cdots & 0 & * \\ 0 & 0 & 0 & E_k^{(\mu)} & 0 & * & \cdots & 0 & * \\ & & 0 & 0 & 0 & D_k^{(\mu-1)} & \cdots & 0 & * \\ & & 0 & 0 & 0 & E_k^{(\mu-1)} & \cdots & 0 & * \\ & & & & & \ddots & \vdots & \vdots & \\ & & & & & & 0 & D_k^{(1)} & \\ & & & & & & 0 & E_k^{(1)} & \\ & & & & & & 0 & 0 & \\ & & & & & & 0 & 0 & \end{bmatrix},$$

where all $D_k^{(i)}$ and $E_k^{(i)}$ have sizes $w_{i+1} \times (a_{i-1} - w_i)$ and $(a_i - w_{i+1}) \times (a_{i-1} - w_i)$, respectively, and all $W_k = [* \cdots *]$ are partitioned accordingly. In particular, all $D_k^{(i)}$ and $E_k^{(i)}$ together have full row rank, i.e.,

$$\text{rank} \left(\begin{bmatrix} D_k^{(i)} \\ E_k^{(i)} \end{bmatrix} \right) = w_{i+1} + (a_i - w_{i+1}) = a_i \quad \text{for all } k \in \mathbb{Z}. \quad (47)$$

The diagonal blocks in D_k all are square matrices.

Proof. The only differences between the forms (19) and (46) are in the (2,3) and (3,3) blocks. We can transform these blocks without affecting the other structure (the W_k block is affected, but this does not matter, since its structure is not used). Thus, it is sufficient to only consider the sequence of matrix pairs which is built of the blocks mentioned before. From Theorem 9 we know that

$$\left\{ \left(\begin{bmatrix} F_k \\ G_k \end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_{f,\mu}} \end{bmatrix} \right) \right\} = \left\{ \left(\begin{bmatrix} 0 & F_k^{(\mu)} & * \\ & \ddots & \ddots \\ & & \ddots & F_k^{(1)} \\ 0 & G_k^{(\mu)} & * \\ & \ddots & \ddots \\ & & \ddots & G_k^{(1)} \\ & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \\ I_{a_\mu} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & I_{a_0} \end{bmatrix} \right) \right\},$$

where all $F_k^{(i)}$ have full row rank w_i , due to (20). Fixing any $i \in \{1, \dots, \mu\}$ we see that all $F_k^{(i)}$ can be reduced to echelon form by transforming with invertible matrices in block row $\mu + 1 - i$ and in block column $\mu + 2 - i$. This indeed affects the matrices $G_k^{(i)}$ and the identity matrices $I_{a_{i-1}}$ (by a multiplication with an invertible matrix from the right) but the $G_k^{(i)}$ matrices still have full row rank, together with the new $F_k^{(i)}$ matrices (which now are in echelon form), i.e., (20) still holds. Also the identity matrices $I_{a_{i-1}}$ can be restored by transforming with invertible matrices from the left. This alters the $G_k^{(i)}$ once more but,

again, condition (20) is not affected. From this we see that

$$\begin{aligned}
& \left\{ \left(\begin{bmatrix} F_k \\ G_k \end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_f, \mu} \end{bmatrix} \right) \right\} \\
& \sim \left\{ \left(\begin{bmatrix} 0 & I_{w_\mu} & 0 & & * & * & \cdots & \cdots & * \\ & & & I_{w_{\mu-1}} & 0 & & & & \vdots \\ & & & & & \ddots & * & & * \\ & 0 & G_k^{(\mu,1)} & G_k^{(\mu,2)} & * & * & \cdots & \cdots & 0 \\ & & & & G_k^{(\mu-1,1)} & G_k^{(\mu-1,2)} & & & 0 \\ & & & & & \ddots & * & & \vdots \\ & & & & & & G_k^{(1,1)} & G_k^{(1,2)} & * \\ & & & & & & & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_f, \mu} \end{bmatrix} \right) \right\} \\
& \sim \left\{ \left(\begin{bmatrix} 0 & I_{w_\mu} & 0 & 0 & * & & 0 & * \\ & & I_{w_{\mu-1}} & 0 & & & \vdots & \vdots \\ & & & \ddots & 0 & * & 0 & * \\ & 0 & 0 & G_k^{(\mu,2)} & 0 & * & 0 & 0 \\ & & 0 & & G_k^{(\mu-1,2)} & \vdots & \vdots & \vdots \\ & & & \ddots & 0 & 0 & * & * \\ & & & & 0 & G_k^{(1,2)} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_f, \mu} \end{bmatrix} \right) \right\} \\
& \sim \left\{ \left(\begin{bmatrix} 0 & I_{w_\mu} & 0 & 0 & 0 & \cdots & 0 & 0 \\ & & I_{w_{\mu-1}} & 0 & & & \vdots & \vdots \\ & & & \ddots & 0 & 0 & 0 & 0 \\ & 0 & 0 & G_k^{(\mu,2)} & 0 & * & 0 & * \\ & & 0 & & G_k^{(\mu-1,2)} & \vdots & \vdots & \vdots \\ & & & \ddots & 0 & 0 & * & * \\ & & & & 0 & G_k^{(1,2)} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_{h_f, \mu} + B_k \end{bmatrix} \right) \right\},
\end{aligned}$$

where all the matrices B_k are upper triangular and nilpotent. Thus, these B_k matrices can again be eliminated by adding multiples of a row k to a row l , where always $k > l$ (i.e., by transforming from the left). By splitting the block rows in the lower part (i.e., the part corresponding to the G_k block) and using

$$G^{(i,2)} = \begin{bmatrix} D_k^{(i)} \\ E_k^{(i)} \end{bmatrix},$$

we finally obtain the assertion, where (47) follows from (20). \square

Lemma 28. *Let $\mu \in \mathbb{N}_0$ and let $\{\hat{a}_i\}_{i \in \mathbb{N}_0}$ be a non-increasing sequence with $\hat{a}_{\mu+1} = 0$ and $\hat{a}_\mu \neq 0$. Further let $\{\hat{w}_i\}_{i \in \mathbb{N}_0}$ be a non-negative sequence with $\hat{w}_i + \hat{a}_i \leq \hat{a}_{i-1}$. Set $\hat{h}_f = \hat{a}_0 + \dots + \hat{a}_\mu$ and let $\hat{r}_f \in \mathbb{N}_0$ be an integer. Assume that $E_k, A_k \in \mathbb{C}^{m,n}$ are matrices such that the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to*

$$\left\{ \left(\begin{bmatrix} I_{\hat{r}_f} & 0 & W_k \\ 0 & 0 & F_k \\ 0 & 0 & G_k \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{\hat{h}_f} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (48)$$

with

$$F_k = \begin{bmatrix} 0 & F_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & F_k^{(1)} \\ & & & 0 \end{bmatrix}, \quad G_k = \begin{bmatrix} 0 & G_k^{(\mu)} & & * \\ & \ddots & \ddots & \\ & & \ddots & G_k^{(1)} \\ & & & 0 \end{bmatrix}, \quad (49)$$

where all $F_k^{(i)}$ and $G_k^{(i)}$ have sizes $\hat{w}_i \times \hat{a}_{i-1}$ and $\hat{a}_i \times \hat{a}_{i-1}$, respectively, and all $W_k = [* \dots *]$ are partitioned accordingly. Also assume that all $F_k^{(i)}$ and $G_k^{(i)}$ together have full row rank, i.e.,

$$\text{rank} \left(\begin{bmatrix} F_k^{(i)} \\ G_k^{(i)} \end{bmatrix} \right) = \hat{a}_i + \hat{w}_i \quad \text{for all } k \in \mathbb{Z}. \quad (50)$$

Then $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ has a well defined strangeness index μ as defined in (17). The sequence of characteristic values is thereby given as

$$r_{f,i} = \hat{r}_f + \sum_{j=i+1}^{\mu} \hat{w}_j + \sum_{j=i+1}^{\mu} \hat{a}_j, \quad (51)$$

$$h_{f,i} = \sum_{j=0}^i \hat{a}_j. \quad (52)$$

Proof. Performing i reductions from (9) to (10) on the sequence of pencils (49) shows that for the resulting sequence we have

$$\{(E_{k,i}, A_{k,i})\}_{k \in \mathbb{Z}} \sim \left\{ \left(\begin{bmatrix} I_{\hat{r}_f} & 0 & * & * & \dots & * & 0 \\ 0 & 0 & 0 & F_k^{(\mu)} & & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & F_k^{(i+1)} & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & G_k^{(\mu)} & & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & & 0 \\ \vdots & \vdots & \vdots & & \ddots & G_k^{(i+1)} & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 \end{bmatrix}, \begin{bmatrix} * & * & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & I_{\hat{a}_\mu} & & & & \\ \vdots & \vdots & & \ddots & & & \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & & & & I_{\hat{a}_i} & \\ 0 & 0 & & & & & I_{\hat{h}_{f,i-1}} \end{bmatrix} \right) \right\},$$

where $\hat{h}_{f,i-1} := \hat{a}_0 + \dots + \hat{a}_{i-1}$. Thus, one can see that the rank of all $E_{k,i}$ is constant in k and given by (51). Once Z_k are bases of corange($E_{k,i}$) one also notices, that the rank of $Z_k^H A_{k,i}$ is also constantly equal to $\hat{a}_i + \hat{h}_{f,i-1}$, which shows (52). \square

To start the actual proof we first recall that Corollary 27 shows that

$$\{(E_k, A_k)\} \sim \left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} \hat{A}_k^{(1)} & \hat{A}_k^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}. \quad (53)$$

We also know that all A_k have the same constant rank (since the strangeness index of $\{(A_{-k}, E_{-k})\}_{k \in \mathbb{Z}}$ is also well defined) and thus all $[\hat{A}_k^{(1)} \ \hat{A}_k^{(2)}]$ also have to have constant rank. This makes it possible to reduce all $[\hat{A}_k^{(1)} \ \hat{A}_k^{(2)}]$ to echelon form (3) $P_k [\hat{A}_k^{(1)} \ \hat{A}_k^{(2)}] Q_k$. Since all Q_k are invertible, for $k \in \mathbb{K}$ we have

$$P_k \begin{bmatrix} \hat{A}_k^{(1)} & \hat{A}_k^{(2)} \end{bmatrix} = \begin{bmatrix} \check{A}_k^{(1)} & \check{A}_k^{(2)} \\ 0 & 0 \end{bmatrix},$$

with $[\check{A}_k^{(1)} \ \check{A}_k^{(2)}]$ having full row rank, which will be called p in the following. It follows that there are p linear independent non-zeros rows in

$$P_k \begin{bmatrix} \hat{A}_k^{(1)} & \hat{A}_k^{(2)} \end{bmatrix} \begin{bmatrix} P_{k-1}^{-1} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \check{A}_k^{(1)} P_{k-1}^{-1} & \check{A}_k^{(2)} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \check{A}_k^{(1)} P_{k-1}^{-1} & \check{A}_k^{(2)} \\ 0 & 0 \end{bmatrix},$$

for $k \in \mathbb{K}$. This shows that

$$\begin{aligned} & \left\{ \left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} \hat{A}_k^{(1)} & \hat{A}_k^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\} \\ & \sim \left\{ \left(\begin{bmatrix} P_k & 0 & P_k W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} P_k \hat{A}_k^{(1)} & P_k \hat{A}_k^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\} \\ & \sim \left\{ \left(\begin{bmatrix} P_k P_k^{-1} & 0 & P_k W_k \\ 0 & 0 & C_k \\ 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} P_k \hat{A}_k^{(1)} P_k^{-1} & P_k \hat{A}_k^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\} \\ & = \left\{ \left(\begin{bmatrix} I_p & 0 & 0 & \hat{W}_k^{(1)} \\ 0 & I_{r_{f,\mu}-p} & 0 & \hat{W}_k^{(2)} \\ 0 & 0 & 0 & C_k \\ 0 & 0 & 0 & D_k \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{h_{f,\mu}} \end{bmatrix} \right) \right\}, \end{aligned}$$

where all matrices $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} \end{bmatrix}$ have full row rank p for $k \in \mathbb{Z}$.

With the notation of Corollary 27, the definitions from (14) and with $b_i := a_i - w_{i+1}$ for $i \in \mathbb{N}_0$ the sequence $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to a sequence of matrix pairs of

the form

$$\left(\left(\begin{bmatrix} I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & I_{r_f, \mu-p} & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\ 0 & \cdots & 0 & & & & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ 0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & & & & & \ddots & \vdots & \vdots \\ \vdots & & \vdots & & & & & & & 0 & D_k^{(1)} \\ \vdots & & \vdots & & & & & & & 0 & E_k^{(1)} \\ \vdots & & \vdots & & & & & & & & 0_{w_1} & 0 \\ \vdots & & \vdots & & & & & & & & 0 & 0_{b_0} \end{bmatrix} \right), \right. \\ \left. \begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\ \vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\ \vdots & & \vdots & & & I_{w_{\mu_f}} & & & & & & \\ \vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & & & & \\ \vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & \\ 0 & \cdots & 0 & & & & & & & & & I_{b_0} \end{bmatrix} \right), \right.$$

where all $[A_k^{(1)} A_k^{(2)} A_k^{(3)}]$ have full row rank. Then, with $I_{r_f, 0-p}$ all $*$ blocks in the same row can be eliminated. This may yield $*$ blocks in the first row of the left matrices (next to $[A_k^{(1)} A_k^{(2)} A_k^{(3)}]$), but these $*$ blocks can again be eliminated by the identity matrices below.

Thus, the system is also equivalent to the system given by

$$\left\{ \left(\begin{bmatrix} I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & I_{r_f, \mu-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\ 0 & \cdots & 0 & & & & & & & & & 0 \\ 0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & & & & & \ddots & \vdots & \vdots \\ \vdots & & \vdots & & & & & & & 0 & D_k^{(1)} \\ \vdots & & \vdots & & & & & & & 0 & E_k^{(1)} \\ \vdots & & \vdots & & & & & & & 0_{w_1} & 0 \\ 0 & \cdots & 0 & & & & & & & 0 & 0_{b_0} \end{bmatrix} \right\}, \quad (54)$$

$$\left\{ \left(\begin{bmatrix} A_k^{(1)} & A_k^{(2)} & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\ \vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\ \vdots & & \vdots & & & I_{w_{\mu_f}} & & & & & & \\ \vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & & & & \\ \vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & \\ 0 & \cdots & 0 & & & & & & & & & I_{b_0} \end{bmatrix} \right\}.$$

Next, we perform one reduction step on the reversed of system (54) and reverse the reduced system back. Thus, $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\left\{ \left(\begin{bmatrix} I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & I_{r_f, \mu-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\ 0 & \cdots & 0 & & & & & & & & & 0 \\ 0 & \cdots & 0 & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * \\ \vdots & & \vdots & & & & & & \ddots & \vdots & \vdots \\ \vdots & & \vdots & & & & & & & 0 & D_k^{(1)} \\ \vdots & & \vdots & & & & & & & 0 & E_k^{(1)} \\ \vdots & & \vdots & & & & & & & 0_{w_1} & 0 \\ 0 & \cdots & 0 & & & & & & & 0 & 0_{b_0} \end{bmatrix} \right), \left(\begin{bmatrix} A_k^{(1)} & 0 & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & I_{w_{\mu_f+1}} & & & & & & & & \\ \vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\ \vdots & & \vdots & & & 0_{w_{\mu_f}} & & & & & & \\ \vdots & & \vdots & & & & I_{b_{\mu_f-1}} & & & & & \\ \vdots & & \vdots & & & & & 0_{w_{\mu_f-1}} & & & & \\ \vdots & & \vdots & & & & & & I_{b_{\mu_f-2}} & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & 0_{w_1} & \\ 0 & \cdots & 0 & & & & & & & & & I_{b_0} \end{bmatrix} \right) \right\}.$$

Reordering of rows shows that $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\left\{ \left(\begin{bmatrix} I_p & 0 & 0 & * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & * \\ 0 & I_{r_f, \mu-p} & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & I_{w_{\mu_f}} & 0 & & & & & \\ \vdots & & \vdots & & & & & I_{w_{\mu_f-1}} & 0 & & & \\ \vdots & & \vdots & & & & & & & \ddots & & \\ \vdots & & \vdots & & & & & & & & I_{w_1} & 0 \\ 0 & \cdots & 0 & & & & & & & & & 0 \\ 0 & \cdots & 0 & & 0_{w_{\mu_f}} & 0 & 0 & D_k^{(\mu_f-1)} & \cdots & 0 & * & \\ \vdots & & \vdots & & & & & & \ddots & \vdots & \vdots & \\ \vdots & & \vdots & & & & & & & 0 & D_k^{(1)} & \\ \vdots & & \vdots & & & & & & & 0_{w_1} & 0 & \\ \vdots & & \vdots & 0_{w_{\mu_f+1}} & 0 & 0 & D_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & 0 & 0_{b_{\mu_f}} & 0 & E_k^{(\mu_f)} & 0 & * & \cdots & 0 & * \\ \vdots & & \vdots & & 0 & 0_{b_{\mu_f-1}} & 0 & E_k^{(\mu_f-1)} & \cdots & 0 & * & \\ \vdots & & \vdots & & & & & & \ddots & 0 & E_k^{(1)} & \\ 0 & \cdots & 0 & & & & & & & 0 & 0_{b_0} \end{bmatrix} \right) \right\},$$

$$\left[\begin{array}{cccccccccccccc} A_k^{(1)} & 0 & A_k^{(3)} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ \vdots & & \vdots & \vdots & & & & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & & & 0_{w_{\mu_f}} & & & & & & \\ \vdots & & \vdots & & & & & 0_{w_{\mu_f-1}} & & & & \\ \vdots & & \vdots & & & & & & \ddots & & & \\ \vdots & & \vdots & & & & & & & 0_{w_1} & & \\ \vdots & & \vdots & I_{w_{\mu_f+1}} & & & & & & & & \\ \vdots & & \vdots & & I_{b_{\mu_f}} & & & & & & & \\ \vdots & & \vdots & & & I_{b_{\mu_f-1}} & & & & & & \\ \vdots & & \vdots & & & & \ddots & & & & & \\ 0 & \cdots & 0 & & & & & & & I_{b_0} & & \end{array} \right] \Bigg\}.$$

Reordering of columns shows that $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\sim \left\{ \left(\left(\begin{bmatrix} I_p & 0 & * & \dots & 0 & * & \dots & \dots & \dots & \dots & * \\ 0 & I_{r_f, \mu-p} & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & I_{w_{\mu_f}} & & & & & & & & \\ & & & \ddots & & & & & & & \\ & & & & I_{w_1} & & & & & & \\ & & & & 0 & \dots & \dots & \dots & \dots & 0 & \\ & & & & & & & & & D_k^{(\mu_f-1)} & \\ & & & & & & & & & & \ddots \\ & & & & & & & & & & D_k^{(1)} \\ & & & & & & & & & & 0 \\ & & & & 0_{w_{\mu_f}+1} & 0 & D_k^{(\mu_f)} & * & \dots & * \\ & & & 0 & 0_{b_{\mu_f}} & E_k^{(\mu_f)} & * & \ddots & \vdots \\ & & & & 0_{b_{\mu_f}-1} & E_k^{(\mu_f-1)} & * & \ddots & * \\ & & & & & & & \ddots & E_k^{(1)} \\ & & & & & & & & 0_{b_0} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},$$

by setting $\hat{h}_f = b_{\mu_f} + \dots + b_0$, $\hat{r}_f = r_{f,0} + w_{\mu_f} + \dots + w_1$ and

$$\tilde{D}_k = \begin{bmatrix} 0 & D_k^{(\mu_f)} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & D_k^{(1)} \\ & & & 0 \end{bmatrix},$$

$$\tilde{E}_k = \begin{bmatrix} 0_{b_{\mu_f}} & E_k^{(\mu_f)} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & E_k^{(1)} \\ & & & 0_{b_0} \end{bmatrix},$$

since $w_{\mu_f+1} = 0$ (which can be seen from (42), (14g), (14c) and (16)).

By setting $\hat{a}_i := b_i$ and $\hat{w}_i := w_{i+1}$ for $i \in \mathbb{N}_0$ we finally see that the sequence $\{\hat{a}_i\}_{i \in \mathbb{N}_0}$ is

non-increasing due to (15l). Also it is clear that by (20)

$$\hat{w}_i + \hat{a}_i = w_{i+1} + b_i = w_{i+1} + a_i - w_{i+1} = a_i \leq a_{i-1} - w_i = b_i = \hat{a}_{i-1}.$$

It may happen that some $\hat{a}_{\mu_f} = \dots = \hat{a}_{\tilde{\mu}_f+1} = 0$ in which case the strangeness index $\tilde{\mu}_f$ of $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ has been decreased. Anyway, with (47), all assumptions of Lemma 28 are fulfilled, which shows that the strangeness index $\tilde{\mu}_f$ really is well defined.

That the reversed strangeness index $\tilde{\mu}_b$ is still well defined follows from the fact that we have performed the reduction step on the reversed system. To understand this, recall that Definition 8 demands the constant rank assumptions (8) to hold for every step of the reduction procedure. Actually performing one reduction step uses only the constant rank assumptions (8) of the first step and the so obtained system will still satisfy the constant rank assumptions (8) in every further reduction step. \square

D Proof of Theorem 17

That $\text{rank}(Z_k^H A_k)$ and $\text{rank}(Y_k^H E_k)$ are invariant under global equivalence follows from Lemma 3. That they are constant for all $k \in \mathbb{Z}$ follows from the fact that the strangeness index and the reversed strangeness index are both well defined.

To show that q is independent of the choice of the bases, let Y_k and \tilde{Y}_k be bases of $\text{corange}(A_k)$ and let Z_k and \tilde{Z}_k be bases of $\text{corange}(E_k)$ for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exists invertible matrices M_{Y_k} and M_{Z_k} such that $Y_k = \tilde{Y}_k M_{Y_k}$ and $Z_k = \tilde{Z}_k M_{Z_k}$. This shows that

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right) &= \text{rank} \left(\begin{bmatrix} M_{Y_k}^{-H} Y_k^H E_k \\ M_{Z_{k+1}}^{-H} Z_{k+1}^H A_{k+1} \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} \tilde{Y}_k^H E_k \\ \tilde{Z}_{k+1}^H A_{k+1} \end{bmatrix} \right), \end{aligned}$$

and thus, that q is independent of the choice of the bases. To show the invariance under global equivalence, let $\{(\tilde{E}_k, \tilde{A}_k)\}_{k \in \mathbb{Z}}$ be globally equivalent to $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$, i.e., let Q_k and P_k be invertible matrices, such that for all $k \in \mathbb{Z}$ we have

$$\begin{aligned} E_k &= P_k \tilde{E}_k Q_{k+1}, \\ A_k &= P_k \tilde{A}_k Q_k. \end{aligned}$$

Since

$$\begin{aligned} 0 &= Y_k^H A_k = Y_k^H P_k \tilde{A}_k Q_k = (P_k^H Y_k)^H \tilde{A}_k Q_k, \\ 0 &= Z_k^H E_k = Z_k^H P_k \tilde{E}_k Q_{k+1} = (P_k^H Z_k)^H \tilde{E}_k Q_{k+1}, \end{aligned}$$

it is clear that $\hat{Y}_k := P_k^H Y_k$ is a basis of $\text{corange}(\tilde{A}_k)$ and that $\hat{Z}_k := P_k^H Z_k$ is a basis of $\text{corange}(\tilde{E}_k)$. With this we see that

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} \hat{Y}_k^H \tilde{E}_k \\ \hat{Z}_{k+1}^H \tilde{A}_{k+1} \end{bmatrix} \right) &= \text{rank} \left(\begin{bmatrix} Y_k^H P_k \tilde{E}_k \\ Z_{k+1}^H P_{k+1} \tilde{A}_{k+1} \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} Y_k^H P_k \tilde{E}_k Q_{k+1} \\ Z_{k+1}^H P_{k+1} \tilde{A}_{k+1} Q_{k+1} \end{bmatrix} \right) \\ &= \text{rank} \left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix} \right), \end{aligned}$$

which means that q does only depend on the equivalence class.

Since the strangeness index of $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is well defined, it is clear that the sequence can be transformed to the form (9). Since the reversed strangeness index is also well defined, we also know that all A_k have constant rank. Thus, in (9) all $A_k^{(1)}$ matrices also have to have constant rank. Thus, by transforming the first block row of (9) from the left we have that $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is equivalent to

$$\left\{ \left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \quad (55)$$

with all $A_k^{(1,1)}$ having full (constant) row rank. Performing one (forward) reduction step from (9) to (10) on (55) yields the sequence

$$\left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 \\ E_k^{(2,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}. \quad (56)$$

Then it follows from Lemma 14 that (56) still has well-defined reversed strangeness index.

Let \tilde{Y}_k be bases of corange $\begin{pmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{pmatrix}$. Then clearly $\tilde{Y}_k = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}^T$, since all

$A_k^{(1,1)}$ have full row rank. Thus, since the reversed strangeness index of (56) is well defined, we know that for every $k \in \mathbb{Z}$ the matrix

$$\tilde{Y}_k^H \begin{bmatrix} E_k^{(1,1)} & 0 \\ E_k^{(2,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_k^{(2,1)} & 0 \\ 0 & 0 \end{bmatrix}$$

has to have constant rank, which means that all $E_k^{(2,1)}$ have to have constant rank. Let us say all $E_k^{(2,1)}$ matrices have constant rank \hat{g} . By reducing all $E_k^{(2,1)}$ in (55) to echelon form and adapting the indexing we then see that $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\begin{aligned} & \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} & E_k^{(1,3)} \\ 0 & I_{\hat{g}} & E_k^{(2,3)} \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\ & \sim \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & E_k^{(2,3)} \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \end{aligned}$$

$$\begin{aligned}
& \sim \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\
& \sim \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},
\end{aligned}$$

where all $\begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} \end{bmatrix}$ have full row rank. Also, all $\begin{bmatrix} E_k^{(1,1)} & 0 & \tilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \end{bmatrix}$ have full row rank, since those matrices are equivalent to $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ as in (55), which have full row rank. So all $E_k^{(3,3)}$ also have full row rank. Reducing all $E_k^{(3,3)}$ to echelon form then finally shows that $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ is globally equivalent to

$$\begin{aligned}
& \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & E_k^{(1,3)} & E_k^{(1,4)} \\ 0 & I_{\hat{g}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f - \hat{q}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}} \\
& \sim \left\{ \left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & E_k^{(1,4)} \\ 0 & I_{h_b - \hat{q}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f - \hat{q}} \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}},
\end{aligned}$$

where $h_b := \hat{g} + \hat{q}$ has been used. Finally, we have $q = \hat{q}$, since (as shown above) the quantity defined in (27) is invariant under global equivalence and q can directly be calculated from the last sequence of matrix pairs. \square