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A note on the stability of fractional order systems

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Abstract

In this paper, a new approach is suggested to investigate stability in a family of fractional order linear time invariant systems with order between 1 and 2. The proposed method relies on finding a linear ordinary system that possesses the same stability property as the fractional order system. In this way, instead of performing the stability analysis on the fractional order systems, the analysis is converted into the domain of ordinary systems which is well established and well understood. As a useful consequence, we have extended two general tests for robust stability check of ordinary systems to fractional order systems.

Keywords: Fractional order LTI system; Stability; Robust stability; Interval systems

1. Introduction

Fractional calculus as an extension of ordinary calculus has a 300 years old history. It has been found that the behavior of many physical systems can be properly described by using the fractional order system theory. For example, heat conduction, dielectric polarization, electrode–electrolyte polarization, electromagnetic waves, visco-elastic systems, quantum evolution of complex systems, quantitative finance and diffusion wave are among the known dynamical systems that were modeled using fractional order equations. In fact, real world processes generally or most likely are fractional order systems. Furthermore, fractional order controllers such as CRONE controller [12], TID controller [7], fractional PID controller [15] and lead-lag compensator [18] have so far been implemented to improve the performance and robustness in the closed loop control systems.

Although the problem of stability is a very essential and crucial issue for control systems including fractional order systems, due to the complexity of the relations, it has been discussed and investigated only in some recent literature (for instance [6,8–10]). In the last 3 years, considerable attention has also been paid to obtain analytical robust stability conditions for fractional order linear time invariant (FO-LTI) systems with interval uncertainty [2,13,14].

In this paper, our aim is to analyze stability in an FO-LTI system by finding a linear time invariant (LTI) system with integer order that has equivalently the same stability property as of the FO-LTI system. The prominence of this object becomes clear when one notices that the analysis of stability in fractional systems is more complicated than in ordinary systems. Given results derived in this paper, instead of investigating stability of a fractional order system, one can analyze stability of its equivalent ordinary system. As a sample application for this subject, two general tests

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to check robust stability in the interval FO-LTI system are also proposed by extending their equivalent tests in LTI systems.

The paper is organized as follows. Section 2 briefs basic concepts in fractional calculus, fractional systems, and stability of these systems. The way to derive an equivalent (from the stability point of view) LTI system for a given FO-LTI system is discussed in Section 3. As a sample use from results obtained in Section 3, two robust stability tests for interval FO-LTI systems are proposed and discussed in Section 4. Conclusions in Section 5 close the paper.

2. Mathematical background

The differ-integral operator, denoted by ${}_aD_t^q$, is a combined differentiation and integration operator commonly used in fractional calculus. This operator is a notation for taking both the fractional derivative and the fractional integral in a single expression and is defined by:

$$ {}_aD_t^q = \begin{cases} \dfrac{d^q}{dt^q}, & q > 0 \\ 1, & q = 0 \\ \displaystyle\int_a^t (d\tau)^{-q}, & q < 0 \end{cases}. \tag{1} $$

There are different definitions for fractional derivatives [16]. The most commonly used definitions are the Grünwald–Letnikov, Riemann–Liouville and Caputo definitions. These definitions are briefly discussed in the following lines.

Grünwald–Letnikov definition:

$$ {}_aD_t^q f(t) = \frac{d^q f(t)}{d(t-a)^q} = \lim_{N\to\infty} \left[\frac{t-a}{N}\right]^{-q} \sum_{j=0}^{N-1} (-1)^j \binom{q}{j} f\left(t - j\left[\frac{t-a}{N}\right]\right). \tag{2} $$

Riemann–Liouville definition:

$$ {}_aD_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(-q)} \displaystyle\int_a^t (t-\tau)^{-q-1} f(\tau)\,d\tau, & q < 0 \\ f(t), & q = 0 \\ D^n[{}_aD_t^{q-n} f(t)], & q > 0 \end{cases}, \tag{3} $$

where n is the first integer which is not less than q, i.e., $n-1 \le q < n$ and $\Gamma(\cdot)$ is the Gamma function.

$$ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt. \tag{4} $$

For a wide class of functions, the Grünwald–Letnikov and the Riemann–Liouville definitions are equivalent [16]. The Laplace transform of the Riemann–Liouville fractional integral is given as follows:

$$ L\{{}_0D_t^q f(t)\} = s^q F(s), \quad q \le 0. \tag{5} $$

Also, the Laplace transform of the Riemann–Liouville fractional derivative is:

$$ L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum_{k=0}^{n-1} s^k\, {}_0D_t^{v-k-1} f(0), \quad n-1 < q \le n \in N. \tag{6} $$

Unfortunately, the Riemann–Liouville fractional derivative appears unsuitable to be treated by the Laplace transform technique in that it requires the knowledge of non-integer order derivatives of the function at $t=0$. This problem does not exist in the Caputo definition which is sometimes called smooth fractional derivative in literature.

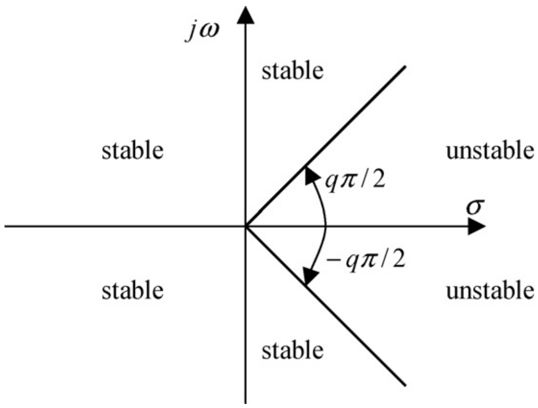

Fig. 1. Stability region of FO-LTI system with order $0<q\leq 1$.

Caputo definition:

$$ {}_0D_t^q f(t) = \begin{cases} \dfrac{1}{\Gamma(m-q)} \displaystyle\int_0^t \dfrac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}}\,\mathrm{d}\tau, & m-1<q<m \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m} f(t), & q=m \end{cases}, \tag{7} $$

where m is the first integer which is not less than q. It is found that the equations with Riemann–Liouville operators are equivalent to those with Caputo operators by homogeneous initial conditions assumption [16]. The Laplace transform of the Caputo fractional derivative is:

$$ L\{{}_0D_t^q f(t)\} = s^q F(s) - \sum\nolimits_{k=0}^{n-1} s^{q-1-k} f^{(k)}(0), \quad n-1<q\leq n \in N. \tag{8} $$

Contrary to the Laplace transform of the Riemann–Liouville fractional derivative, only integer order derivatives of function f appear in the Laplace transform of the Caputo fractional derivative. For zero initial conditions, (8) reduces to:

$$ L\{{}_0D_t^q f(t)\} = s^q F(s). \tag{9} $$

In the rest of the paper, D^q is used to denote the Caputo fractional derivative of order q. A fractional order linear time invariant (FO-LTI) system can be represented in the following state space like form:

$$ \begin{cases} D^q x = Ax + Bu \\ y = Cx \end{cases}, \tag{10} $$

where $x\in R^n$, $u\in R^r$, $y\in R^p$ are the state, input, and output vectors of the system and $A\in R^{n\times n}$, $B\in R^{n\times r}$, $C\in R^{p\times n}$. q represents the fractional commensurate order.

It has been shown that system $D^q x = Ax$ is asymptotically stable if the following condition is satisfied [8].

$$ |\arg(\mathrm{eig}(A))| > \frac{q\pi}{2}, \tag{11} $$

where $0<q<2$, and eig(A) are the eigenvalues of matrix A. The stable and unstable regions for $0<q\leq 1$ and $1<q<2$ are shown in Figs. 1 and 2, respectively.

3. An equivalent LTI system for an FO-LTI system in the sense of stability

The following theorem states a necessary and sufficient condition to place the eigenvalues of a real matrix in a specified sector. Using this theorem, we construct a link between stability of FO-LTI and ordinary LTI systems.

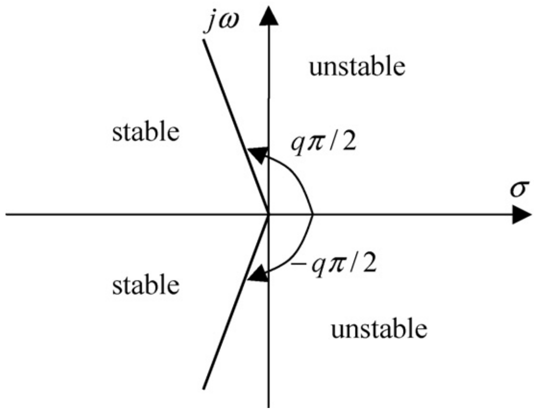

Fig. 2. Stability region of FO-LTI system with order $1 \leq q < 2$.

Theorem 1. [1,3]: *Eigenvalues of an* $n \times n$ *matrix* A *lie within the region* Ω (Fig. 3) $\Omega = \{\lambda | Re(\lambda)\cos\delta \pm Im(\lambda)\sin\delta \leq 0;\ 0 \leq \delta < \pi/2\}$ *if and only if the eigenvalues of the* $2n \times 2n$ *matrix*

$$A^* = \begin{bmatrix} A\ \cos\delta & -A\ \sin\delta \\ A\ \sin\delta & A\ \cos\delta \end{bmatrix}, \tag{12}$$

have negative real parts.

Theorem 1 has been proved using two different approaches in [1,3]. If the characteristic equation of matrix A is available, then one can state Theorem 1 with a more useful expression as in Theorem 2.

Theorem 2. [5]: *A necessary and sufficient condition that the zeros of an nth-order polynomial*

$$P(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \tag{13}$$

lie within the region Ω, *is that the 2nth-order polynomial*

$$F(s) = P(se^{j\delta})P(se^{-j\delta}), \tag{14}$$

has zeros in the left half of the complex plane.

If the coefficients of $P(s)$ (a_i, $0 \leq i < n$) *are real, then its roots are symmetric about the real axis of complex plane. Hence, the roots of* $F(s)$ *are also symmetric about the real axis and then polynomial* $F(s)$ *has real coefficients.*

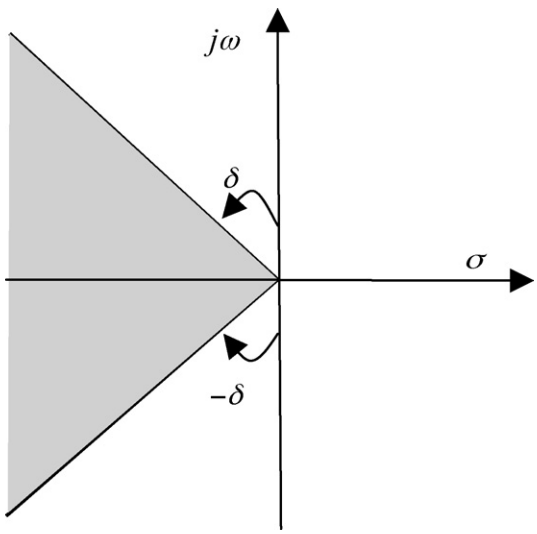

Fig. 3. Region Ω in Theorem 1 is shown in gray.

In fact, the region Ω can be interpreted as stable region of FO-LTI system $D^q x = Ax$ where $\delta = (q-1)\pi/2$. Now, we propose the following two theorems.

Theorem 3. *The FO-LTI system $D^q x = Ax (1 \le q < 2)$ is asymptotically stable if and only if the LTI system,*

$$\dot{\tilde{x}} = \begin{bmatrix} A \sin\left(\frac{q\pi}{2}\right) & A \cos\left(\frac{q\pi}{2}\right) \\ -A \cos\left(\frac{q\pi}{2}\right) & A \sin\left(\frac{q\pi}{2}\right) \end{bmatrix} \tilde{x}, \tag{15}$$

is asymptotically stable.

Proof. According to Theorem 1 and the stability condition (11), the proof is obvious. □

Theorem 4. *All eigenvalues of FO-LTI system $D^q x = Ax$ $(0 < q \le 1)$ settle in the unstable region* (Fig. 1) *if and only if the LTI system*

$$\dot{\tilde{x}} = -\begin{bmatrix} A \sin\left(\frac{q\pi}{2}\right) & -A \cos\left(\frac{q\pi}{2}\right) \\ A \cos\left(\frac{q\pi}{2}\right) & A \sin\left(\frac{q\pi}{2}\right) \end{bmatrix} \tilde{x}, \tag{16}$$

is asymptotically stable.

Proof. If all eigenvalues of the FO-LTI system $D^q x = Ax$ $(0 < q \le 1)$ lie on the unstable region $\Omega_1 = \{\lambda : |\arg(\lambda)| < q\pi/2\}$, the eigenvalues of matrix $-A$ settle in the region $\Omega_2 = \{\lambda : |\arg(\lambda)| > (2-q)\pi/2\}$. Therefore, according to Theorem 1, matrix

$$\begin{bmatrix} -A\cos\delta & A\sin\delta \\ -A\sin\delta & -A\cos\delta \end{bmatrix}$$

has eigenvalues in the left half of the complex plane, where $\delta = (1-q)\pi/2$. This means that system in (16) is asymptotically stable. Proof of the inverse is similar and thus not given. □

Theorems 3 and 4 relate stability of a fractional order system to stability of its equivalent ordinary system. Based on these two theorems, most of the stability related analysis in the ordinary systems is applicable to the fractional systems with commensurate order as well. As an illustration, in Example 1 the Routh–Hurwitz criterion is applied without any change to investigate stability of a fractional system.

Example 1. In this example, we want to find the parameters which make the following FO-LTI system stable.

$$D^q x = Ax$$

where $A = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$ and $1 < q < 2$ is given.

From (15), the equivalent LTI system in the sense of stability for the above system has the following characteristic polynomial.

$$P(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

where $a_0 = b^2$, $a_1 = 2ab\sin(q\pi/2)$, $a_2 = a^2 + 2b\cos(q\pi)$, $a_3 = -2a\sin(q\pi/2)$.

By the Routh–Hurwitz criterion, the real parts of the roots of $P(s)$ are negative if and only if $a < 0$, $b < 0$ and $a^2 + 4b\cos^2(q\pi/2) > 0$.Fig. 4 shows the stable and unstable regions in a–b plane for $q = 1.3$. If $a = b$, the above conditions reduce to $a < 0$, $b < 0$ and $a < -4\cos^2(q\pi/2)$. Time response for the initial condition $(x(0), \dot{x}(0), y(0), \dot{y}(0)) = (1, 0, 1, 0)$ and fractional order $q = 1.3$ is shown in Figs. 5 and 6. In Fig. 5, system parameters are $a = b = -0.9$, hence system is stable but in Fig. 6, system parameters are chosen as $a = b = -0.7$, and consequently the system is unstable. Numerical simulations in this example are performed based on Oustaloup frequency approximation for fractional order systems [11] (see Appendix A).

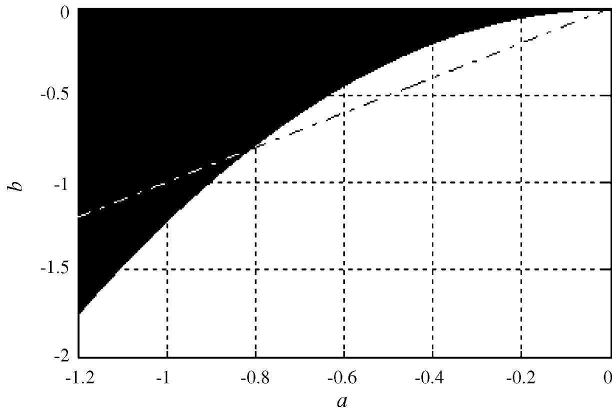

Fig. 4. Stable (white) and unstable (black) regions in a–b plane when $q = 1.3$ (Example 1).

4. Robust stability test for the interval FO-LTI systems

To illustrate the importance and effectiveness of the two given theorems, in this section we propose two robust stability tests for a class of FO-LTI systems represented by an interval matrix. In this regard, we first restate an existing theorem in robust stability of the LTI interval systems.

Theorem 5. [4]: *Consider the continuous LTI system described by* $\dot{x} = Ax + Bu$, *where* A *belongs to a given set defined by* $A \in [\underline{A}, \bar{A}]$ ($\underline{A}$ *and* $\bar{A} \in R^{n\times n}$ *are lower and upper boundaries of the uncertain matrix* A). *The system is Hurwitz stable if*

$$\alpha = \bar{\lambda}_C + n\,\bar{d} < 0, \tag{17}$$

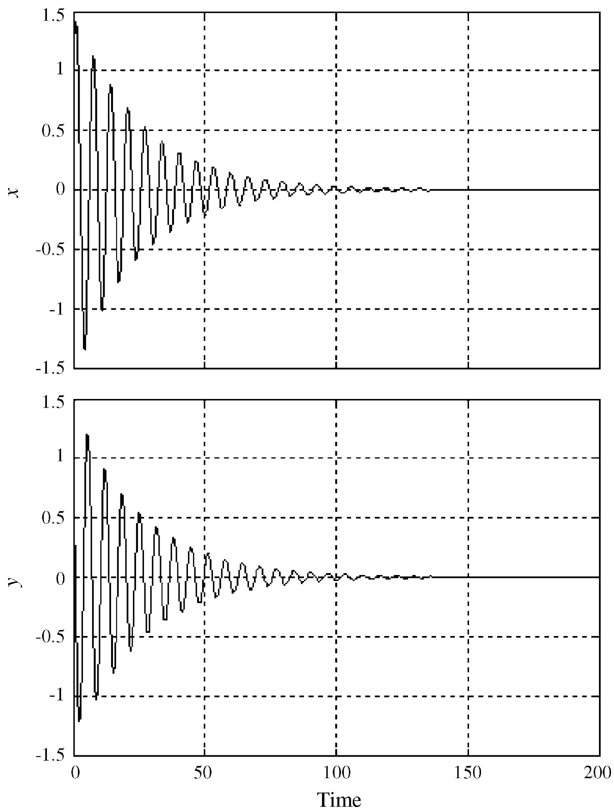

Fig. 5. Time response when $q = 1.3$ and $a = b = -0.9$ (Example 1).

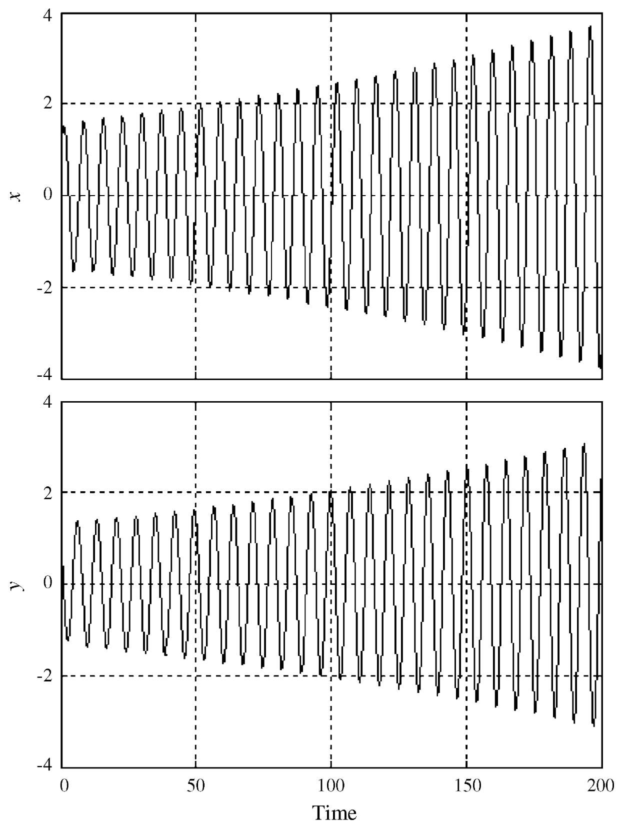

Fig. 6. Time response when $q = 1.3$ and $a = b = -0.7$ (Example 1).

where $\bar{\lambda}_C$ is the maximum eigenvalue of the Hermitian part of matrix $C = (\bar{A} + \underline{A})/2$ and $\bar{d}$ is the maximum element of matrix $D = (\bar{A} - \underline{A})/2$.

The proof of Theorem 5 has been given in [4]. Now consider the following interval FO-LTI system:

$$D^q x = Ax + Bu, \tag{18}$$

where the fractional commensurate order q is between 1 and 2, and the system matrix A is uncertain but limited to $A \in [\underline{A}, \bar{A}]$. Recently a robust stability test has been introduced for the interval FO-LTI system (18) in [2] which is based on Lyapunov inequality. Here we present another test for this system based on Theorems 3 and 5. A sufficient condition to guarantee the robust stability of the system (18) is given in Theorem 6.

Theorem 6. *Consider an FO-LTI system described by $D^q x = Ax + Bu$, where $1 < q < 2$ and A is confined to an interval, i.e., $A \in [\underline{A}, \bar{A}]$ where $\underline{A}$ and $\bar{A} \in R^{n\times n}$ are lower and upper boundaries of the uncertain matrix A. The system is internally stable if*

$$\alpha = \bar{\lambda}_C + 2n\,\bar{d} < 0, \tag{19}$$

where $\bar{\lambda}_C$ is the maximum eigenvalue of the Hermitian part of matrix

$$C = 0.5 \begin{bmatrix} (\bar{A} + \underline{A}) \sin\left(\dfrac{q\pi}{2}\right) & (\bar{A} + \underline{A}) \cos\left(\dfrac{q\pi}{2}\right) \\ -(\bar{A} + \underline{A}) \cos\left(\dfrac{q\pi}{2}\right) & (\bar{A} + \underline{A}) \sin\left(\dfrac{q\pi}{2}\right) \end{bmatrix},$$

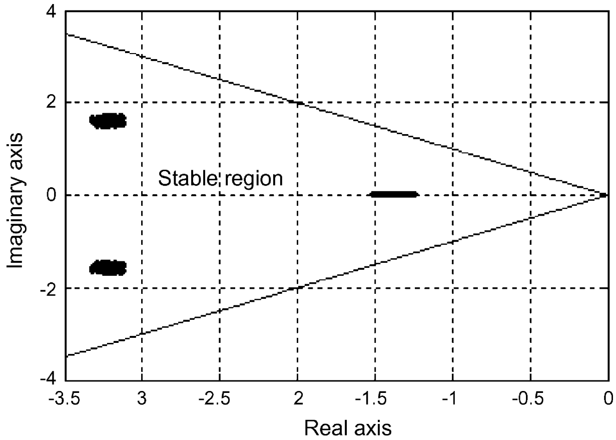

Fig. 7. Eigenvalues from random test for the uncertain FO-LTI system in Example 2.

and $\bar{d}$ is the maximum element of matrix

$$D = 0.5 \begin{bmatrix} (\bar{A} - \underline{A})\sin\left(\frac{q\pi}{2}\right) & -(\bar{A} - \underline{A})\cos\left(\frac{q\pi}{2}\right) \\ -(\bar{A} - \underline{A})\cos\left(\frac{q\pi}{2}\right) & (\bar{A} - \underline{A})\sin\left(\frac{q\pi}{2}\right) \end{bmatrix}.$$

Proof. Using results of Theorems 3 and 5, proof is straightforward. □

Example 2. In this example, we want to check the robust stability of the following interval FO-LTI system:

$$\frac{d^{1.5}x}{dt^{1.5}} = Ax,$$

where $A \in [\underline{A}, \bar{A}]$,

$$\underline{A} = \begin{bmatrix} -1.4 & 0.3 & 1 \\ -1.1 & -3.6 & 1 \\ -0.6 & -1.8 & -3 \end{bmatrix} \quad \text{and} \quad \bar{A} = \begin{bmatrix} -1.3 & 0.5 & 1.1 \\ -1 & -3.4 & 1.1 \\ -0.3 & -1.5 & -2.9 \end{bmatrix}.$$

From (19), the calculated α is -0.0103. Hence, according to Theorem 5, this system is robustly stable. Fig. 7 shows the eigenvalues loci for 600 randomly selected A matrixes in the given interval $[\underline{A}, \bar{A}]$. All the eigenvalues are in the stable region.

Another robust stability test to check the stability of uncertain LTI systems is given in the following theorem.

Theorem 7. [17]: *Consider a linear system defined by $\dot{x} = (A_0 + E)x$ in which $A_0 \in R^{n\times n}$ is stable and $E = [e_{ij}]$ represents the structured uncertainty such that the interval constraint conditions $-\Delta k_{ij} \le e_{ij} \le -\Delta k_{ij}$, where Δk_{ij}, i, $j = 1, 2, \ldots, n$ are known positive constants, are satisfied. Let P be the solution of the Lyapunov matrix equation*

$$A_0^T P + PA_0 + 2I = 0, \tag{20}$$

where I is the identity matrix. Also let P_{ij} be calculated from

$$P_{ij} = 0.5(A_{ij}^T P + PA_{ij}), \tag{21}$$

where A_{ij} is an $n \times n$ dimensional matrix with 1 in the ith and jth spot and 0 elsewhere. Then the linear system $\dot{x} = (A_0 + E)x$ is stable if

$$s = \sum_{i=1}^{n}\sum_{j=1}^{n} \Delta k_{ij}\sigma_{\max}(P_{ij}) < 1, \tag{22}$$

where $\sigma_{\max}(B)$ represents the largest singular value of the matrix B.

Proof of Theorem 7 has been given in [17]. Now, by use of Theorem 3, Theorem 7 can be extended to the domain of uncertain FO-LTI systems. Theorem 8 is an extension of Theorem 7 to check the stability of uncertain FO-LTI systems.

Theorem 8. *Consider an FO-LTI system defined by $D^q x = (A_0 + E)x$ with $1 < q < 2$ in which $A_0 \in R^{n\times n}$ is stable and $E = [e_{ij}]$ represents the structured uncertainty such that the interval constraint conditions $-\Delta k_{ij} \le e_{ij} \le -\Delta k_{ij}$, where Δk_{ij}, $i, j = 1, 2, \ldots, n$ are known positive constants, are satisfied. Define*

$$\tilde{A}_0 = \begin{bmatrix} A_0 \sin\left(\frac{q\pi}{2}\right) & A_0 \cos\left(\frac{q\pi}{2}\right) \\ -A_0 \cos\left(\frac{q\pi}{2}\right) & A_0 \sin\left(\frac{q\pi}{2}\right) \end{bmatrix}, \tag{23}$$

and

$$\Delta\tilde{K} = [\Delta\tilde{k}_{ij}] = \begin{bmatrix} \Delta K \sin\left(\frac{q\pi}{2}\right) & -\Delta K \cos\left(\frac{q\pi}{2}\right) \\ -\Delta K \cos\left(\frac{q\pi}{2}\right) & \Delta K \sin\left(\frac{q\pi}{2}\right) \end{bmatrix}. \tag{24}$$

Let P be the solution of the Lyapunov matrix equation

$$P\tilde{A}_0 + \tilde{A}_0^{\mathrm{T}} P + 2I = 0, \tag{25}$$

where I is the identity matrix. Also let P_{ij} be calculated from

$$P_{ij} = 0.5(A_{ij}^{\mathrm{T}} P + PA_{ij}), \tag{26}$$

where A_{ij} is an $2n \times 2n$ dimensional matrix with 1 in the ith and jth spot and 0 elsewhere. Then the FO-LTI system $D^q x = (A_0 + E)x$ is stable if the system $\dot{\tilde{x}} = \tilde{A}_0\tilde{x}$ is stable and

$$s = \sum_{i=1}^{2n}\sum_{j=1}^{2n} \Delta\tilde{k}_{ij}\sigma_{\max}(P_{ij}) < 1, \tag{27}$$

where $\sigma_{\max}(B)$ represents the largest singular value of the matrix B.

Proof. According to Theorem 3, stability of FO-LTI system $D^q x = (A_0 + E)x$ with $1 < q < 2$ is equivalent to stability of the following LTI system.

$$\dot{\tilde{x}} = \begin{bmatrix} (A_0 + E) \sin\left(\frac{q\pi}{2}\right) & (A_0 + E) \cos\left(\frac{q\pi}{2}\right) \\ -(A_0 + E) \cos\left(\frac{q\pi}{2}\right) & (A_0 + E) \sin\left(\frac{q\pi}{2}\right) \end{bmatrix} \tilde{x}.$$

The dynamic matrix of this LTI system can be divided into certain and uncertain parts as follows.

$$\begin{bmatrix} (A_0 + E) \sin\left(\frac{q\pi}{2}\right) & (A_0 + E) \cos\left(\frac{q\pi}{2}\right) \\ -(A_0 + E) \cos\left(\frac{q\pi}{2}\right) & (A_0 + E) \sin\left(\frac{q\pi}{2}\right) \end{bmatrix} = \tilde{A}_0 + \begin{bmatrix} E \sin\left(\frac{q\pi}{2}\right) & E \cos\left(\frac{q\pi}{2}\right) \\ -E \cos\left(\frac{q\pi}{2}\right) & E \sin\left(\frac{q\pi}{2}\right) \end{bmatrix}.$$

Since $\sin(q\pi/2) > 0$ and $\cos(q\pi/2) < 0$, the lower and the upper boundaries of the uncertain part of dynamic matrix are $-\Delta\tilde{K}$ and $\Delta\tilde{K}$. Rest of the proof is straightforward using results of Theorem 7. □

Example 3. Consider an uncertain system $D^q x = Ax$ with,

$$A = \begin{bmatrix} -1 + e_1 & 0.8 & 1.1 \\ -0.8 & -2 + e_2 & 0.9 \\ -0.3 & -1.2 & -1.6 + e_3 \end{bmatrix}$$

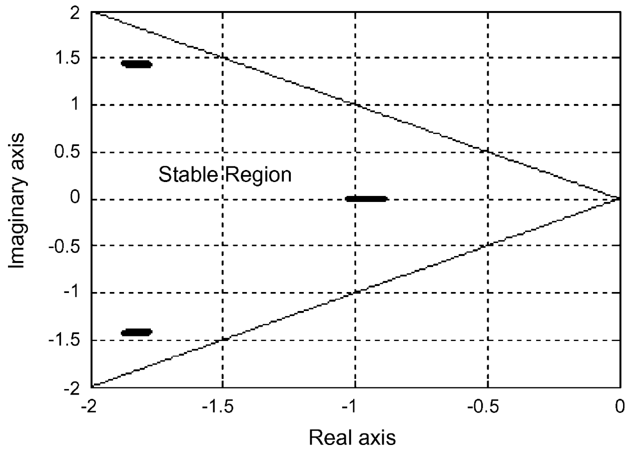

Fig. 8. Eigenvalues from random test of the uncertain FO-LTI system (Example 3).

where

$$|e_1| < 0.09, \qquad |e_2| < 0.05, \qquad |e_3| < 0.05.$$

For $q = 1.5$, this system is stable because s in (27) is equal to 0.9992. Fig. 8 shows eigenvalues loci for 600 randomly selected matrixes A. All the eigenvalues lie in the stable region.

5. Conclusions

In this paper, we showed that the stability of an FO-LTI system can be equivalent to the stability of a specific LTI system. This link between stability of FO-LTI and LTI systems is a powerful tool to extend stability results for ordinary systems to the domain of fractional order systems. As sample cases, we have extended two existing theorems from the robust stability of LTI systems to check the robust stability of FO-LTI systems in Section 4. By use of Theorem 3 mentioned in Section 3, other existing methods in robust stability analysis of ordinary LTI systems can be modified to support the studied fractional order systems. Extension of these methods is one of the future research topics in our group. Also, another related topic that can be considered as future work is finding an equivalent (from the stability point of view) LTI system for a given fractional order system with order between 0 and 1.

Appendix A

The Oustaloup method [11] to find integer order approximations of fractional transfer functions, used in numerical simulation of Example 1, is given by

$$s^v \approx k \prod_{n=1}^{N} \frac{1 + s/\omega_{z,n}}{1 + s/\omega_{p,n}}, \quad 0 < v < 1 \tag{28}$$

Gain k is adjusted so that both sides of (28) have unit gain at 1 rad/s. The number of poles and zeros of approximating transfer function (N) and the frequency range ($[\omega_l, \omega_h]$) must be selected beforehand. $\omega_{z,n}$ and $\omega_{p,n}$ are calculated by the following equations

$$\omega_{z,1} = \omega_l \sqrt{\eta}, \tag{29}$$

$$\omega_{p,n} = \omega_{z,n}\alpha, \quad n = 1, \ldots, N, \tag{30}$$

$$\omega_{z,n+1} = \omega_{p,n}\eta, \quad n = 1, \ldots, N-1, \tag{31}$$

where parameters α and η are given by

$$\alpha = \left(\frac{\omega_h}{\omega_l}\right)^{v/N}, \tag{32}$$

$$\eta = \left(\frac{\omega_h}{\omega_l}\right)^{(1-v)/N}. \tag{33}$$

For $-1 < \nu < 0$, the approximating transfer function is derived by inverting (28). To find the approximating transfer function of s^{ν} in case $|\nu| > 1$, first s^{ν} can be written in the form $s^{\nu} = s^{[\nu]} s^{\delta}$ and then the term s^{δ} is replaced by approximation transfer function (28).

In the numerical simulation of Example 1, the frequency range and the number of poles and zeros of the approximating transfer function have been chosen as $[\omega_l, \omega_h] = [10^{-2}, 10^{3}]$ and $N = 5$, respectively.

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