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Disturbance Decoupling with Stability for Linear Impulsive Systems

Anna Maria Perdon * **Elena Zattoni** ** **Giuseppe Conte** *

* *Department of Information Engineering, Polytechnic University of Marche, Via Brecce Bianche, 60131 Ancona, Italy.*
E-mail: perdon@univpm.it, gconte@univpm.it
** *Department of Electrical, Electronic, and Information Engineering "G. Marconi", Alma Mater Studiorum - University of Bologna, Viale Risorgimento 2, 40136 Bologna, Italy.*
E-mail: elena.zattoni@unibo.it

Abstract: The paper deals with the problem of decoupling the output of an impulsive linear system from a disturbance input by means of a stabilizing state feedback. Geometric methods are used to characterize the decoupling requirement from a structural point of view and in terms of feedback stabilizability. Assuming that the length of the time intervals between consecutive impulsive variations of the state, or jumps, is lower bounded by a positive constant, necessary and sufficient conditions for the existence of solutions are given and discussed.

Keywords: Hybrid systems, decoupling problems, feedback stabilization

1. INTRODUCTION

Hybrid dynamical systems whose state evolves according to a LTI dynamics, called the flow dynamics, except at isolated points on the time axis, in which it presents jump discontinuities, can arise in modeling phenomena like e.g. collisions in mechanical systems, the action of switches in electrical circuits or component failures in general situations (see Goebel et al. (2012), Liberzon (2003), Haddad et al. (2006)). Regulation problems and noninteracting control problems for systems of that kind, called linear impulsive systems (see e.g. Lakshmikantham et al. (1989), Yang (2001)), have been considered by many authors under various hypotheses, either assuming that jumps are equally spaced in time (i.e. they arise periodically) or not (see Marconi and Teel (2010), Carnevale et al. (2012a), Carnevale et al. (2012b), Carnevale et al. (2013), Carnevale et al. (2014b), Perdon et al. (2015)). In particular, Perdon et al. (2015) considered the problem of decoupling, by state feedback, the output of a linear impulsive system with periodic jumps from an unknown disturbance, while achieving asymptotic stability of the closed-loop dynamics. Here, we focus on the same problem for dynamical systems of a larger class: namely the class of linear impulsive systems with jumps that occur at unpredictable, possibly non-equally spaced times. More precisely, we look for solutions of the problem that are valid for any possible, a-priori unknown, time sequence of jumps, with the only limitation that the time between two consecutive jumps cannot be shorter than a known threshold.

The approach we follow in dealing with this problem employs geometric methods and tools that derive from the classical geometric approach to linear time-invariant systems Basile and Marro (1992); Wonham (1985). Lately, these methods have been extended to the framework of hybrid systems. In particular, they have been applied to switched linear systems in order to solve disturbance decoupling problems Otsuka (2010); Yurtseven et al. (2012); Zattoni et al. (2014a, 2016a), model matching problems Conte et al. (2014); Zattoni et al. (2014b); Perdon et al. (2016), and output regulation problems Zattoni et al. (2013a,b). Moreover, these techniques have been exploited to solve, more generally, noninteracting control problems for jumping hybrid systems or linear impulsive system in Medina and Lawrence (2006); Medina (2007); Carnevale et al. (2014b,a, 2015); Zattoni et al. (2015); Perdon et al. (2015).

Differently from Perdon et al. (2015), where only a sufficient condition for the solution of the problem was found, the more restrictive requirement that decoupling and stability hold for all possible time sequences of jumps gives us the possibility to state, in Theorem 1, necessary and sufficient conditions for the existence of solutions. Under a mild assumption, which is akin to, but weaker than left invertibility of the flow dynamics, different necessary and sufficient conditions for solvability, that are easier to check than those of Theorem 1, are given in Theorem 2. A stronger sufficient condition, that can be checked by a finite procedure is also given in Theorem 3.

The paper is organized as follows. In Section 2, we introduce the class of systems we consider and we state the control problem we study. In order to deal with the requirement of asymptotic stability of the closed-loop dynamics for any possible sequence of jumps, we give a result about feedback stabilization of the considered linear impulsive systems under the hypothesis that the flow dynamics is reachable. In Section 3, we describe the geometric tools we are going to use and their relevant properties. In Section 4, we state necessary and sufficient conditions for the solution of the problem and we discuss how to test them and to construct practically a solution. Section 5 contains an

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example. Conclusions and indications for future work are given in Section 6.

Notations. The symbols $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ are used for the sets of natural numbers (including 0), real numbers, positive real numbers, respectively. Matrices and linear maps between vector spaces are denoted by slanted capital letters like A. Sets, vector spaces and subspaces are denoted by calligraphic capital letters like $\mathcal{X}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V}\subseteq\mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The restriction of a linear map A to an A-invariant subspace $\mathcal{V}$ is denoted by $A|_{\mathcal{V}}$. The vector spaces image and kernel of a linear map A are denoted by $Im\ A$ and $Ker\ A$, respectively. The symbols A^{-1} is used to denote the inverse of the matrix A. For a vector $v \in \mathbb{R}^n$, $\|v\|_E$ denotes the Euclidean norm and, for a matrix A, $\|A\|$ denotes the matrix norm defined by $\|A\| = sup_{x\neq 0}\frac{\|Ax\|_E}{\|x\|_E}$.

2. THE DISTURBANCE DECOUPLING PROBLEM WITH STABILITY FOR IMPULSIVE SYSTEMS

Let us denote by $\mathcal{S}$ the set of all maps $\sigma : \mathbb{N} \to \mathbb{R}^+$ which, letting $\tau_\sigma = inf\{\sigma(0), \sigma(i+1) - \sigma(i); i \in \mathbb{N}, \sigma(i+1) \neq \sigma(i)\}$, satisfy the following conditions

$$\tau_\sigma > 0. \tag{1}$$

Condition (1) implies that $\sigma(i+1)$ is greater than or equal to $\sigma(i)$ for all $i \in \mathbb{N}$ and also that the set of points in the image of σ, i.e. the set $Im(\sigma) = \{t \in \mathbb{R}^+, t = \sigma(i)$ for some $i \in \mathbb{N}\}$, is a discrete, finite or countably infinite subset of $\mathbb{R}^+$, whose subsets (including $Im\,\sigma$ itself) have no accumulation points. We say that τ_σ, which describes the minimum interval between two points in $Im\ \sigma$, is the *dwell time* of σ. In the following, given $\tau \in \mathbb{R}^+$, we will denote by $\mathcal{S}_\tau$ the subset of $\mathcal{S}$ defined by $\mathcal{S}_\tau = \{\sigma \in \mathcal{S}, \tau_\sigma \geq \tau\}$.
The dynamical systems we consider are linear hybrid systems which present jumps in the state evolution, as described by the following equations:

$$\Sigma_\sigma \equiv \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \text{ for } t \neq \sigma(i),\ i \in \mathbb{N} \\ x(\sigma(i)) = Jx^-(\sigma(i)) \text{ for } i \in \mathbb{N} \\ y(t) = Cx(t) \end{cases} \tag{2}$$

where $t\in\mathbb{R}$ is the time variable; $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathcal{U} = \mathbb{R}^m$ and $y \in \mathcal{Y} = \mathbb{R}^p$ denote, respectively, the state, the input and the output variables; A, B, J, C are real matrices of suitable dimensions and $x^-(\sigma(i))$ denotes the limit of $x(t)$ for t which goes to $\sigma(i)$ from the left, that is $x^-(\sigma(i)) = lim_{t\to\sigma(i)^-}x(t)$; σ belongs to $\mathcal{S}$. In other words, the state $x(t)$ of Σ_σ, starting from an initial condition $x(0) = x_0$ at time $t = 0$, evolves continuously on the time interval $[0, \sigma(0))$ according to the dynamics given by the first block of equations in (2). Then, at time $t = \sigma(0)$, instead of taking the value $x^-(\sigma(0))$, the state jumps to $Jx^-(\sigma(0))$, as stated in the second block of equations in (2). The same behavior repeats on each one of the subsequent time intervals $[\sigma(i), \sigma(i+1))$, with initial condition $x(\sigma(i))$. We say that the equations in the first block in (2) represent the *flow dynamics* of Σ_σ, while the equations in the second block represent the *jump behavior* of Σ. Jumps occur at all points $\sigma(i) \in Im\ \sigma$ (it is understood that a single jump occurs at $\sigma(i)$, also in case $\sigma(i) = \sigma(j)$) and the dwell time τ_σ represents the minimum interval between consecutive, distinct jump times in Σ_σ. The jump behavior depends on J and on the choice of the map σ and, in particular, it is not necessarily periodic. Note that we do not assume that the interval between consecutive jumps has an upper bound. In case $Im\ \sigma$ is a finite set, no jumps occur for $t > max_{i\in\mathbb{N}}\sigma(i)$ and, on the interval $(max_{i\in\mathbb{N}}\sigma(i), +\infty)$, the dynamics of Σ_σ reduces to the flow dynamics.

Dynamical systems of the above kind are known as linear impulsive systems Lakshmikantham et al. (1989), Yang (2001). Recent results, ensuing from the geometric approach, on their structural and stability properties, under restrictions on the dwell time, are found in Medina and Lawrence (2009), Medina (2007). For impulsive control systems, that is systems of the above kind in which the input acts also on the jump behavior, regulation problems have been considered from a geometric point of view in Carnevale et al. (2014b).

Let us consider a linear impulsive system of the form (2) with an additional unknown disturbance input $d(t)$, that is the system $\Sigma_{\sigma d}$ given by the following equations

$$\Sigma_{\sigma d} \equiv \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \\ \qquad \text{for } t \neq \sigma(i),\ i \in \mathbb{N} \\ x(\sigma(i)) = Jx^-(\sigma(i)) \text{ for } i \in \mathbb{N} \\ y(t) = Cx(t) \end{cases} \tag{3}$$

where $d \in \mathcal{D} = \mathbb{R}^h$, and D is a real matrix of suitable dimensions. Then, the control problem we want to study is described as follows.

Problem 1. **Disturbance Decoupling with Global Asymptotic Stability.** Given a system $\Sigma_{\sigma d}$ of the form (3) and $\tau \in \mathbb{R}^+$, the Disturbance Decoupling Problem with Global Asymptotic Stability (DDPS) for $\Sigma_{\sigma d}$ and τ consists in finding a state feedback $F : \mathcal{X} \to \mathcal{U}$ such that the compensated impulsive system $\Sigma^F_{\sigma d}$ given by

$$\Sigma^F_{\sigma d} \equiv \begin{cases} \dot{x}(t) = (A + BF)x(t) + Dd(t) \\ \qquad \text{for } t \neq \sigma(i),\ i \in \mathbb{N} \\ x(\sigma(i)) = Jx^-(\sigma(i)) \text{ for } i \in \mathbb{N} \\ y(t) = Cx(t) \end{cases}$$

satisfies the following requirements for any $\sigma \in \mathcal{S}_\tau$:

$\mathcal{R}\,1$. the output $y(t)$ is decoupled from the disturbance input $d(t)$;
$\mathcal{R}\,2$. the impulsive system $\Sigma^F_{\sigma d}$ is globally asymptotically stable.

Requirement $\mathcal{R}\,1$ can be expressed by saying that $x(0) = 0$ implies $y(t) = 0$ for all $t \in \mathbb{R}^+$ and any disturbance $d(t)$. It has to be remarked that in solving the problem we will be able to characterize completely the set of states for which $y(t) = 0$ for all $t \in \mathbb{R}^+$ and any disturbance $d(t)$. Coming to the requirement of global asymptotic stability for any choice of $\sigma \in \mathcal{S}_\tau$, note that, since it must be satisfied, in particular, for $\bar{\sigma}$ defined by $\bar{\sigma}(i) = \tau$ for all $i \in \mathbb{N}$ (i.e in the case in which the system behavior presents a single jump in $t = \tau$), $\mathcal{R}\,2$ implies that all free motions of the compensated flow dynamics that are initialized in ImJ go asymptotically to 0. Hence, in order to satisfy $\mathcal{R}\,2$, (global) asymptotic stability of the compensated flow dynamics restricted to the smallest invariant subspace $\mathcal{W}$ such that of $Im\ J \subseteq \mathcal{W} \subseteq \mathcal{X}$ is necessary.

The disturbance decoupling problem has been considered in Perdon et al. (2015) for the class of linear impulsive systems of the form (2) in which jumps are equally spaced in time, that is $\sigma \in \mathcal{S}_P$, where $\mathcal{S}_P = \{\sigma : \mathbb{N} \to \mathbb{R}^+$, such that $\sigma(i+1) - \sigma(i) = \sigma(0)$ for all $i \in \mathbb{N}\}$.

Lemma 1. Let Σ_σ be a linear impulsive system of the form (2) whose flow dynamics is (globally) asymptotically stable and let $\mu, \lambda \in \mathbb{R}^+$ be such that $\|e^{At}\| \leq \mu e^{-\lambda t}$. Then, given $\tau \in \mathbb{R}^+$, Σ_σ is globally asymptotically stable for any choice of $\sigma \in \mathcal{S}_\tau$ if $\|J\|$ is such that $\|J\|\mu e^{-\lambda\tau} = \alpha < 1$.

Proposition 1. Let Σ_σ be a linear impulsive system of the form (2). If the pair (A, B) is reachable, for any $\tau \in \mathbb{R}^+$, there exists a feedback $F : \mathcal{X} \to \mathcal{U}$ such that the compensated impulsive system Σ_σ^F given by

$$\Sigma_\sigma^F \equiv \begin{cases} \dot{x}(t) = (A+BF)x(t) \ \text{ for } t \neq \sigma(i),\ i \in \mathbb{N} \\ x(\sigma(i)) = Jx^-(\sigma(i)) \ \text{ for } i \in \mathbb{N} \\ y(t) = Cx(t) \end{cases}$$

is globally asymptotically stable for any choice of $\sigma \in \mathcal{S}_\tau$.

Remark 1. Stability conditions, different from that of Lemma 1, are given, under suitable assumptions, in Lawrence (2013). Stabilizability of the system Σ_σ by means of a time varying feedback of the form $u(t) = F(t)x(t)$ has been recently studied in Medina and Lawrence (2009), Medina (2007), under the assumption that the dwell time is upper bounded. The results of those papers do not apply to our situation since the dwell time is not upper bounded and only time-invariant feedback matrices are considered here as solutions of the DDPS.

3. GEOMETRIC TOOLS

The methodology employed in this work to solve the DDPS relies on geometric notions that extend to the class of linear impulsive systems some basic concepts of the geometric approach Basile and Marro (1992); Wonham (1985). Such notions have already been considered in Medina and Lawrence (2006), Medina (2007), Carnevale et al. (2014b), Perdon et al. (2015), Zattoni et al. (2015) and their properties are further investigated here.

Given a linear impulsive system Σ_σ of the form (2), we denote by $\mathcal{B}$ the image of the input matrix B. A subspace $\mathcal{V} \subseteq \mathcal{X}$ is said to be

- an *h-invariant subspace* for Σ_σ if

$$A\mathcal{V} \subseteq \mathcal{V}, \tag{4}$$

$$J\mathcal{V} \subseteq \mathcal{V}. \tag{5}$$

- an *h-controlled invariant subspace* for Σ_σ if

$$A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B} \tag{6}$$

is satisfied together with (5).

If V is a basis matrix of $\mathcal{V} \subseteq \mathcal{X}$, then $\mathcal{V}$ is an h-invariant subspace if and only if there exist matrices L_A and L_J of suitable dimensions, such that

$$AV = VL_A, \tag{7}$$

$$JV = VL_J \tag{8}$$

hold and it is an h-controlled invariant subspace if and only if there exist matrices L_A, M, and L_J of suitable dimensions, such that

$$AV = VL_A + BM \tag{9}$$

and (8) hold.

Remark 2. If V is a basis matrix of the h-controlled invariant subspace $\mathcal{V} \subseteq \mathcal{X}$, since V is full column rank, the matrices L_A and L_J that satisfy (7), (8) are unique. The pairs of matrices that satisfy (9) are of the form $(L_A + N_1 H, M + N_2 H)$, where (L_A, M) is any pair of matrices that satisfy (9), $\left[\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right]$ is a matrix whose columns are a basis of $Ker[V \quad B]$ and H is an arbitrary matrix of suitable dimensions.

The notions of h-invariance and of h-controlled invariance are related by the fact that a subspace $\mathcal{V} \subseteq \mathcal{X}$ is an h-controlled invariant subspace for Σ_σ if and only if there exists a state feedback $F : \mathcal{X} \to \mathcal{U}$ such that the inclusion

$$(A+BF)\mathcal{V} \subseteq \mathcal{V} \tag{10}$$

is satisfied together with (5). Any F that verifies (10) is called a *friend* of $\mathcal{V}$.

Remark 3. If V is a basis matrix of the h-controlled invariant subspace $\mathcal{V} \subseteq \mathcal{X}$, any $F : \mathcal{X} \to \mathcal{U}$ such that $FV = -M$ is a friend of $\mathcal{V}$ and, conversely, for any friend F there is a unique matrix L_A such that the pair (L_A, FV) satisfies (9). Moreover, if $\left[\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right]$ is a matrix whose columns are a basis of $Ker[V \quad B]$, the columns of VN_1 are a basis of $\mathcal{B} \cap \mathcal{V}$. It is easy to see from this parametrization that there is a unique pair (L_A, M) that satisfies (9) if $\mathcal{B} \cap \mathcal{V} = \{0\}$.

Letting $\mathcal{W} \subseteq \mathcal{X}$ be a subspace, it is possible to show that there exists a maximum h-controlled invariant subspace for Σ contained in $\mathcal{W}$, that will be denoted by $\mathcal{V}_h^*(\mathcal{W})$. An algorithm for constructing $\mathcal{V}_h^*(\mathcal{W})$ is given e.g. in Zattoni et al. (2016b). Denoting by $\mathcal{K}$ the kernel of the output matrix C, the maximum h-controlled invariant subspace for Σ contained in $\mathcal{K}$ will be denoted simply by $\mathcal{V}_h^*$.

If $\mathcal{V}$ is an h-controlled invariant subspace for the impulsive system Σ_σ and F is one of its friends, by a change of basis $x = Tz$ in $\mathcal{X}$, where $T = [T_1 \ T_2]$ and the columns of T_1 span $\mathcal{V}$, the equations of the compensated system Σ_σ^F take the form

$$\Sigma_\sigma^F \equiv \begin{cases} \dot{z}(t) = \hat{A}z(t) \ \text{ for } t \neq \sigma(i),\ i \in \mathbb{N} \\ z(\sigma(i)) = \hat{J}z^-(\sigma(i)) \ \text{ for } i \in \mathbb{Z}^+ \\ y(t) = \hat{C}x(t) \end{cases}$$

with

$$\hat{A} = T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},\ \hat{J} = T^{-1}AT = \begin{bmatrix} J_{11} & J_{12} \\ 0 & J_{22} \end{bmatrix},$$

$$\hat{C} = CT = [C_1 \ C_2]$$

The above representation shows that the hybrid dynamics of Σ_σ^F defines a restricted dynamics on $\mathcal{V}$ and an induced dynamics on the quotient space $\mathcal{X}/\mathcal{V}$. The resulting hybrid systems will be denoted, respectively, by $\Sigma_{\sigma|\mathcal{V}}^F$ and by $\Sigma_{\sigma|\mathcal{X}/\mathcal{V}}^F$. In the basis defined by V, we have, by Remark 2, that the jump dynamics of $\Sigma_{\sigma|\mathcal{V}}^F$ is described by the unique matrix L_J that verifies (8). The flow dynamics of $\Sigma_{\sigma|\mathcal{V}}^F$ depends on the friend F and, by Remark 3, it can be described by the unique matrix L_A such that the pair (L_A, M) with $M = -FV$ verifies (9).

Given a linear impulsive system Σ_σ of the form (2), we call *unobservability subspace* of Σ_σ and we denote by $\mathcal{O}$

the subspace of states $x \in \mathcal{X}$ such that, for any $\sigma \in \mathcal{S}_\tau$, the trajectory $x_\sigma(t)$ starting from x is contained in $Ker\ C$ for all $t \in \mathbb{R}^+$ (i.e. such that $Cx_\sigma(t) = 0$ for all $t \in \mathbb{R}^+$ and for all $\sigma \in \mathcal{S}_\tau$).

Proposition 2. Given a linear impulsive system Σ_σ of the form (2), its unobservable subspace $\mathcal{O}$ is the maximal h-invariant subspace contained in the unobservability subspace $\mathcal{X}_O$ of the flow dynamics.

Hint of proof. It follows from the fact that the subspace $\mathcal{O}$ has to be a locus of trajectory for the flow dynamics, to be invariant with respect to J and to be contained in $\mathcal{X}_O$. Maximality is obvious. □

In analogy with Basile and Marro (1992), given $\tau \in \mathbb{R}^+$, we say that $\mathcal{V}$ is, respectively,

- *internally stabilizable* over $\mathcal{S}_\tau$ if there exists a friend F such that the hybrid system $\Sigma^F_{\sigma|\mathcal{V}}$, given by the restriction of the hybrid dynamics of Σ^F_σ to $\mathcal{V}$, is globally asymptotically stable for any $\sigma \in \mathcal{S}_\tau$;
- *externally stabilizable* over $\mathcal{S}_\tau$ if there exists a friend F such that the hybrid system $\Sigma^F_{|\mathcal{X}/\mathcal{V}}$, induced on the quotient space $\mathcal{X}/\mathcal{V}$ by the hybrid dynamic of Σ^F, is globally asymptotically stable for any $\sigma \in \mathcal{S}_\tau$.

Given an h-controlled invariant subspace $\mathcal{V}$ for Σ_σ, let us denote by $< A + BF|\mathcal{B} \cap \mathcal{V} >$ the subspace of $\mathcal{X}$ defined by

$$< A + BF|\mathcal{B} \cap \mathcal{V} >= (\mathcal{B} \cap \mathcal{V}) + (A + BF)(\mathcal{B} \cap \mathcal{V})+ (A + BF)^2(\mathcal{B} \cap \mathcal{V}) + ... + (A + BF)^{n-1}(\mathcal{B} \cap \mathcal{V})$$

where F is any friend of $\mathcal{V}$. Note that, by Remark 2 and Remark 3, $< A + BF|\mathcal{B} \cap \mathcal{V} >$ does not depend on the choice of F. Then, in addition to the above notions, we can consider the following one, which extends to the present framework the concept of controllability subspace for LTI systems (compare with Basile and Marro (1992), Wonham (1985)).

Definition 1. Given a linear impulsive system Σ_σ of the form (2), an h-controlled invariant subspace $\mathcal{V}$ is said to be an *h-controllability subspace* for Σ_σ if, for any friend F of $\mathcal{V}$, we have $< A + BF|\mathcal{B} \cap \mathcal{V} >= \mathcal{V}$.

The following proposition, whose proof follows the same lines as that of (Perdon et al., 2015, Proposition 4), provides a simple characterization of h-controllability subspaces.

Proposition 3. Any h-controlled invariance subspace $\mathcal{V}$ for Σ_σ is an h-controllability subspace if and only if, taken any matrix V whose columns are a basis of $\mathcal{V}$, any pair of matrices (L_A, M) that satisfies (9) and any matrix $\left[\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right]$ whose columns are a basis of $Ker[V \quad B]$, the pair (L_A, N_1) is reachable.

Remark 4. It is worth noting that, in the present framework, h-invariant, h-controlled invariant and flow-controllability subspaces are respectively invariant, controlled invariant and controllability subspaces for the flow dynamics. Moreover, if $Im\ J = \mathcal{X}$, internal and external stabilizability of $\mathcal{V}$ over $\mathcal{S}_\tau$ imply internal and external stabilizability with respect to the flow dynamics, since, as remarked in discussing requirement $\mathcal{R}\,2$ of the DDPS, global asymptotic stability for all $\sigma \in \mathcal{S}_\tau$ implies, in that case, (global) asymptotic stability of the compensated flow dynamics. The last statement does not hold in the framework of LTI invariant systems with periodic jumps considered in Perdon et al. (2015).

The following results have been stated in Perdon et al. (2015) in the framework of LTI systems with periodic jumps and the same proofs, thank to Proposition 1, apply to the present situation.

Proposition 4. (Perdon et al., 2015, Proposition 3) Given a linear impulsive system Σ_σ of the form (2) and $\tau \in \mathbb{R}^+$, any h-controllability subspace $\mathcal{V}$ for Σ is internally stabilizable over $\mathcal{S}_\tau$.

Proposition 5. (Perdon et al., 2015, Proposition 5) Given a linear impulsive system Σ_σ of the form (2) and $\tau \in \mathbb{R}^+$, any h-invariant subspace $\mathcal{V}$ for Σ_σ is internally stabilizable over $\mathcal{S}_\tau$ if, letting V be a matrix whose columns are a basis of $\mathcal{V}$ and letting (L_A, M) be a pair of matrices that satisfy (9), the pair (L_A, N_1), where $\left[\begin{smallmatrix} N_1 \\ N_2 \end{smallmatrix}\right]$ is a matrix whose columns are a basis of $Ker[V\ B]$, is reachable.

Proposition 6. (Perdon et al., 2015, Proposition 6) Given a linear impulsive system Σ_σ of the form (2) and $\tau \in \mathbb{R}^+$, any h-controlled invariant subspace $\mathcal{V}$ for Σ_σ is externally stabilizable over $\mathcal{S}_\tau$ if the pair (A, B) is reachable.

4. PROBLEM SOLUTION

The main result on the DDPS we state in this section is given by the theorem below.

Theorem 1. Given the impulsive system $\Sigma_{\sigma d}$ described by (3) and $\tau \in \mathbb{R}^+$, assume that the pair (A, B) is reachable. Then, the DDPS for $\Sigma_{\sigma d}$ and τ is solvable if and only if there exists an h-controlled subspace $\mathcal{V} \subseteq \mathcal{V}^*_h$, internally stabilizable over $\mathcal{S}_\tau$, such that $Im\ D \subseteq \mathcal{V}$.

Hint of proof. Sufficiency follows as in the classical case (see Basile and Marro (1992), Wonham (1985)). Necessity is proved by considering the inclusion of $Im\ D$ in the unobservability subspace $\mathcal{O}$ of $\Sigma^F_{\sigma d}$, where $F : \mathcal{X} \to \mathcal{U}$ is a solution of the DDPS. Stability results from representing $\Sigma^F_{\sigma d}$, in a suitable basis, as the interconnection of the restricted dynamics $\Sigma^F_{\sigma d|\mathcal{O}}$ and of the induced dynamics $\Sigma^F_{\sigma d|\mathcal{X}/\mathcal{O}}$. □

Clearly, the structural geometric condition $Im\ D \subseteq \mathcal{V}^*_h$ is necessary and sufficient for assuring the existence of a state feedback $F : \mathcal{X} \to \mathcal{U}$ that guarantees the decoupling requirement $\mathcal{R}\,1$. The above theorem shows also that the set of states for which $y(t) = 0$ for all $t \in \mathbb{R}^+$ and any disturbance $d(t)$ is given by $\mathcal{V}^*_h$.

Since the existence of an h-controlled invariant subspace that verifies the conditions of Theorem 1 may be difficult to ascertain, it is useful to state the necessary and sufficient solvability condition in a more explicit form, although in this way it applies to a slightly restricted situation. To this aim, let us assume that, given the system Σ_d described by (3), the following condition holds

$$\mathcal{B} \cap \mathcal{V}^*_h = \{0\}. \quad (11)$$

Note that condition (11) is generally weaker than $\mathcal{B} \cap \mathcal{V}^* = \{0\}$, where $\mathcal{V}^*$ is the maximum controlled invariant subspace for the flow dynamics of Σ contained in $KerC$, that in turn is equivalent to left invertibility of the flow dynamics. Hence, condition (11) is akin, but weaker, than left invertibility of the flow dynamics. If (11) holds and V is a matrix whose columns are a basis of an h-controlled invariant subspace $\mathcal{V}$, as already noted in Remark 3, there is a unique pair (L_A, M) that satisfies (9). Therefore, all friends F of $\mathcal{V}$, since $FV = -M$, coincide on $\mathcal{V}$. This fact has two important consequences. The first one is that all friends F of a given h-controlled invariant subspace $\mathcal{V}$ are also friends of all h-controlled invariant subspaces $\mathcal{V}'$ that are contained in $\mathcal{V}$. The second one is that the set of h-controlled invariant subspaces that are contained in a given subspace $\mathcal{K}$ and that contain a given subspace $\mathcal{W}$ is a complete lattice with respect to intersection and sum of subspaces and, therefore, it has a minimum element, denoted by $\mathcal{V}_{*h}(\mathcal{W})$ (compare with Conte and Perdon (1991)).

Theorem 2. Given the impulsive system $\Sigma_{\sigma d}$ described by (3) and $\tau \in \mathbb{R}^+$, assume that the pair (A, B) is reachable and that condition (11) holds. Then, the DDPS for $\Sigma_{\sigma d}$ and τ is solvable if and only if $Im D \subseteq \mathcal{V}_h^*$ and the minimum controlled h-invariant subspace $\mathcal{V}_{*h}$ that contains $Im\ D$ is internally stabilizable.

Hint of proof. Sufficiency follows from Theorem 1. Since any solution F of the DDPS is a friend of $\mathcal{V}_h^*$, the dynamics of $\Sigma_{\sigma d}^F$ can be represented, in a suitable basis, as the interconnection of the restricted dynamics $\Sigma^F_{\sigma d|\mathcal{V}_{*h}}$ and of the induced dynamics $\Sigma^F_{\sigma d|\mathcal{X}/\mathcal{V}_{*h}}$. Then, necessity follows. □

Assuming that $Im\ D$ is contained in $\mathcal{V}_h^*$ (otherwise the DDPS has no solution), it is possible to construct $\mathcal{V}_{*h}(Im\ D)$ and to check internal stabilizability by analyzing the restricted dynamics $\Sigma^F_{\sigma d|\mathcal{V}_{*h}}$, which is defined in terms of the first element of the unique pair (L_A, M) that satisfies (9). In order to construct $\mathcal{V}_{*h}(\mathcal{W})$, e.g. for $\mathcal{W} = Im\ D$, let us consider the sequence of subspaces $\mathcal{S}_i$, defined by $\mathcal{S}_0 = \mathcal{W} + Im\ B$ and $\mathcal{S}_i = \mathcal{S}_{i-1} + A(\mathcal{S}_{i-1} \cap Ker\ C) + J(\mathcal{S}_{i-1} \cap Ker\ C)$. The sequence converges, in the sense that it becomes stationary, in a finite number of steps since it is a nested sequence of subspaces of $\mathcal{X}$ of non decreasing dimension. Denoting by $\mathcal{S}_h^*(\mathcal{W})$ its limit, we have that $\mathcal{S}_h^*(\mathcal{W})$ obviously contains $\mathcal{W} + Im\ B$ and it can be proved by induction to be the smallest subspace of $\mathcal{X}$ having that property for which $A(\mathcal{S}_h^*(\mathcal{W}) \cap Ker\ C) \subseteq \mathcal{S}_h^*(\mathcal{W})$ and $J(\mathcal{S}_h^*(\mathcal{W}) \cap Ker\ C) \subseteq \mathcal{S}_h^*(\mathcal{W})$ hold. Using $\mathcal{S}_h^*(\mathcal{W})$ it is possible to construct the minimum h-controlled invariant subspace $\mathcal{V}_{*h}(\mathcal{W})$ for Σ_σ that contains $\mathcal{W}$ as stated in the next proposition.

Proposition 7. Given an impulsive system Σ_σ described by (2) and a subspace $\mathcal{W} \subseteq KerC$, assume that Condition (11) holds. Then, the minimum h-controlled invariant subspace $\mathcal{V}_{*h}(\mathcal{W})$ for Σ that contains $\mathcal{W}$ is given by $\mathcal{V}_{*h}(\mathcal{W}) = \mathcal{V}_h^* \cap \mathcal{S}_h^*(\mathcal{W})$.

Hint of proof. Computations show that $(\mathcal{V}_h^* \cap \mathcal{S}_h^*(\mathcal{W}))$ is an h-controlled invariant subspace for Σ_σ that is contained in $KerC$ and that contains $\mathcal{W}$. Minimality can be shown by taking an h-controlled invariant subspace $\mathcal{V}' \subseteq \mathcal{V}_h^*$ for Σ, such that $\mathcal{W} \subseteq \mathcal{V}'$, and showing that $\mathcal{S}_h^*(\mathcal{W}) \subseteq \mathcal{S}_h^*(\mathcal{V}')$. Then, $\mathcal{V}_h^* \cap \mathcal{S}_h^*(\mathcal{W}) \subseteq \mathcal{V}_h^* \cap \mathcal{S}_h^*(\mathcal{V}') = \mathcal{V}'$. □

In case condition (11) is not satisfied, a sufficient condition that guarantees solvability of the DDPS, which is stronger than that of Theorem 1 but is checkable by a finite procedure, is given by the following theorem, whose proof follows from Theorem 1 and Proposition 4.

Theorem 3. Given the impulsive system $\Sigma_{\sigma d}$ described by (3) and $\tau \in \mathbb{R}^+$, assume that the pair (A, B) is reachable. Then, the DDPS for $\Sigma_{\sigma d}$ and τ is solvable if there exists an h-controllability subspace $\mathcal{V} \subseteq \mathcal{V}_h^*$ such that $ImD \subseteq \mathcal{V}$.

Given the system Σ_d described by (3), in order to check the condition expressed by Theorem 3 and to find, if any exists, an h-controllability subspace $\mathcal{V} \subseteq \mathcal{V}_h^*$ such that $Im\ D \subseteq \mathcal{V}$, one can employ the recursive procedure already described in Perdon et al. (2015).

5. EXAMPLE

In this section we illustrate by a very simple academic example how the above approach to the DDPS applies. Simplicity is a consequence of low dimensionality and it has been purposefully chosen in order to let the reader follow easily all computations. Higher dimensional examples can be dealt with in the same manner, using the algorithms and procedures previously mentioned: they may require more involved computations but they do not present additional complexity.

Let us consider the DDPS problem for an impulsive linear system $\Sigma_{\sigma d}$ of the form (3), where

$$A = \begin{bmatrix} -4 & 4 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad J = \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $\tau = 0.3$. The pair (A, B) is easily seen to be reachable. The maximum h-controlled invariant subspace $\mathcal{V}_h^*$ for $\Sigma_{\sigma d}$ contained in $Ker\ C$ coincides with $Ker\ C$ and it is spanned by the columns of the matrix $V = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Note that $Im\ D = \mathcal{V}_h^*$ and, since $Im\ B \cap \mathcal{V}_h^* = \{0\}$, we can apply Theorem 2. We have that $L_A = [-5], M = [1]$ are the unique matrices that satisfy (9) and $L_J = [3]$ is the unique matrix that satisfies (8). Then, any friend F of $\mathcal{V}_h^*$ is of the form $F = [-1\ f_2]$, with $f_2 \in \mathbb{R}^+$. Moreover, the flow dynamics and the jump dynamics of the restriction to $\mathcal{V}_h^*$ of the compensated dynamics $\Sigma_{\sigma d}^F$ (namely: $\Sigma^F_{\sigma d|\mathcal{V}}$) are defined, respectively, by the matrices $L_A = [-5], L_J = [3]$. Since $\|L_J e^{L_a \tau}\| = \|3e^{-1.5}\| < 1$, the restricted dynamics $\Sigma^F_{\sigma|\mathcal{V}}$ is globally asymptotically stable for all $\sigma \in \mathcal{S}_{0.3}$ and $\mathcal{V}_h^*$ is internally stabilizable over $\mathcal{S}_{0.3}$. By Theorem 2, the DDPS is solvable. In fact, taking the feedback friend $F = \begin{bmatrix} -1 & -5 \end{bmatrix}$, we obtain $A + BF = \left[\begin{smallmatrix} -5 & 1 \\ 0 & -4 \end{smallmatrix}\right]$. Looking at the equations, it is clear that the disturbance excites only the component $x_1(t)$ of the state $x(t)$, while, starting from $x(0) = 0$, the second component $x_2(t)$, and hence the output $y(t) = Cx(t) = x_2(t)$, remains equal to 0. So, the decoupling requirement is satisfied. The behavior of $x_1(t)$ and of $y(t) = x_2(t)$ is shown in Figure 1 for $d(t) = 1$ for $t \in \mathbb{R}^+$. Then, since $\|J\| \leq 3.18$ and $\|e^{(A+BF)t}\| \leq 3e^{-8t}$, we have $\|Je^{(A+BF)\tau}\| = \|Je^{(A+BF)0.3}\| \leq 3.18 \times 3e^{-2.4} \leq 0.9 < 1$ and therefore, by Lemma 1, the compensated impulsive system $\Sigma^F_{\sigma d}$ is globally asymptotic stable for all $\sigma \in \mathcal{S}_{0.3}$.

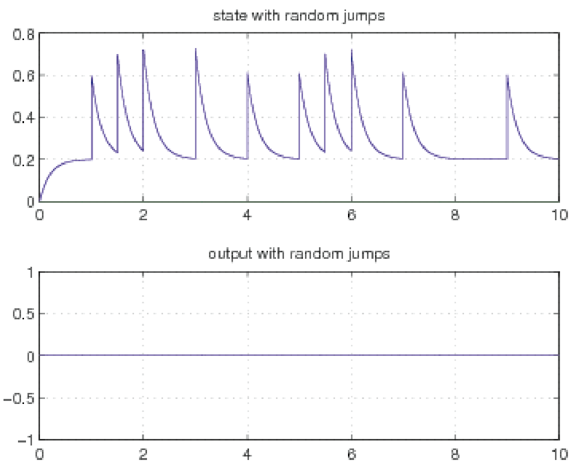

Fig. 1. Behavior of $x_1(t)$ and of $y(t) = x_2(t)$ in $\Sigma^F_{\sigma d}$, with $x(0) = 0$ and $d(t) = 1$, for a given σ with $\tau_\sigma \geq 0.3$

6. CONCLUSIONS

It has been shown that, for linear impulsive systems whose sequence of jumps satisfies a dwell time condition, geometric methods and tools provide necessary and sufficient solvability conditions for the problem of decoupling the output from an unknown disturbance by a stabilizing state feedback. The same approach can be employed to study solvability of other noninteracting control problems with stability, considering e.g. accessible disturbances, (dynamic) output feedback or more general classes of systems. These problems will be addressed in future works.

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