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Stability radii for positive linear time-invariant systems on time scales

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ABSTRACT

We deal with dynamic equations on time scales, where we characterize the positivity of a system. Uniform exponential stability of a system is determined by the spectrum of its matrix. We investigate the corresponding stability radii with respect to structured perturbations and show that, for positive systems, the complex and the real stability radius coincide.

1. Introduction

We consider a d-dimensional time-invariant linear system of dynamic equations

$$x^{\Delta} = Ax \tag{1}$$

($A \in \mathbb{R}^{d\times d}$) on a time scale $\mathbb{T}$, where x^{Δ} denotes the derivative of x with respect to $\mathbb{T}$. Here a *time scale* is a non-empty closed subset of $\mathbb{R}$. For the basics of the dynamic equations on time scales we refer to [1]. System (1) is said to be positive if it leaves the cone $\mathbb{R}^d_+$ invariant, i.e. if every solution starting at a point $\xi \in \mathbb{R}^d_+$ at time $t_0 \in \mathbb{T}$ remains in $\mathbb{R}^d_+$ for all times $t \in \mathbb{T}$, $t \geq t_0$. Positive systems arise in the modeling of processes where the state variables only have a meaning if they are nonnegative. For the time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{N}$ the characterization of positive systems in terms of the system matrix is well-known. We provide a characterization for positivity of system (1) on time scales.

Since a dynamical model is never an exact portrait of the real process, it is important to investigate the robustness of a stable system (1) under perturbations. We deal with uniform exponential stability, which is determined by the spectrum of the system matrix. It is of interest to find the maximal $r > 0$ such that the family of systems

$$x^{\Delta} = (A + D)x, \quad \|D\| < r, \tag{2}$$

is uniformly exponentially stable, where the matrices D are complex, real or positive, respectively. This leads to the notions of complex, real and positive stability radius. We also study the case of structured perturbations

$$A \rightsquigarrow A + BDC$$

for given structure operators B and C.

For continuous- or discrete-time systems stability radii are well-investigated notions, see [2]. A discussion on the differences between the complex and the real stability radius can be found in [3]. The complex stability radius is more easily analyzed and computed than the real one. For positive systems the situation is simpler, since the complex and the positive stability radius coincide. The continuous and discrete time cases are established in [4], resp. [5]. In [6,7] both cases are considered for more general perturbation classes. The importance of monotonic norms in the context of positive systems is pointed out in [4,5]. In the setting of Banach lattices similar results are obtained in [8]. Stability radii of finite dimensional positive continuous- and discrete-time systems have first been studied in [9]. However, in this reference the condition that B and C have to be nonnegative is not explicitly stated. An example in [7] shows that this assumption is essential.

The stability of intervals of nonnegative matrices is studied in [10,11].

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In the present paper we deal with positive systems on arbitrary time scales. Combining the Perron–Frobenius theory for positive matrices and for Metzler matrices, respectively, we show that for such systems the complex and the real stability radius with respect to structured perturbations coincide.

2. Preliminaries

In the following $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. For $z \in \mathbb{C}$ and $r \in \mathbb{R}$ we define $B_r(z) = \{x \in \mathbb{C}: \|x - z\| \le r\}$. Consider on $\mathbb{C}^d$ the norm $\|x\| = (\sum_{i=1}^n |x_i|^2)^{\frac{1}{2}}$, such that one has $\|x + iy\|^2 = \|x\|^2 + \|y\|^2$ for $x, y \in \mathbb{R}^d$. As usual, $\mathbb{K}^{d\times d}$ denotes the space of square matrices with d rows, equipped with the according operator norm (spectral norm), and I_d is the identity mapping on $\mathbb{K}^d$. $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{d\times d}$. The *spectral radius*, respectively the *spectral abscissa* of A are given by

$$\rho(A) := \max\{|\lambda|: \lambda \in \sigma(A)\} \quad \text{and} \quad \mu(A) := \max\{\Re\lambda: \lambda \in \sigma(A)\}.$$

Let $\mathbb{R}^d$ be equipped with the standard entrywise ordering, i. e. $x \le y$ if and only if $x_i \le y_i$ for all $i \in \{1, \dots, d\}$, and denote by $\mathbb{R}^d_+ = \{x \in \mathbb{R}^d: 0 \le x\}$ the set of all nonnegative vectors. Analogously, the set of all nonnegative matrices in $\mathbb{R}^{n\times m}$ is denoted by $\mathbb{R}^{n\times m}_+$. For $A = (a_{ij})_{i,j} \in \mathbb{C}^{n\times m}$ we define $|A| := (|a_{ij}|)_{i,j}$, so that $|A|$ denotes the matrix obtained by taking the absolute value entrywise.

A *time scale* $\mathbb{T}$ is a non-empty, closed subset of the reals $\mathbb{R}$. For the purpose of this paper we assume from now on that $\mathbb{T}$ is unbounded from above, i.e. $\sup \mathbb{T} = \infty$. On $\mathbb{T}$ the *graininess* is defined by

$$\mu^*(t) := \inf\{s \in \mathbb{T}: t < s\} - t.$$

A point $t \in \mathbb{T}$ is called *right-dense* if $\mu^*(t) = 0$ and *right-scattered* if $\mu^*(t) > 0$. Similarly, $t \in \mathbb{T}$ is called *left-dense* if $t - \sup\{s \in \mathbb{T}: s < t\} = 0$ and *left-scattered* if $t - \sup\{s \in \mathbb{T}: s < t\} > 0$. For a function $f: \mathbb{T} \to \mathbb{K}$ and a point $t_0 \in \mathbb{T}$ we say that $f^\Delta(t_0) \in \mathbb{K}$ is the *derivative* of f in t_0 if for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta) \cap \mathbb{T}$ the inequality

$$\left|f(t_0 + \mu^*(t_0)) - f(t) - (t_0 + \mu^*(t_0) - t)f^\Delta(t_0)\right| \le \varepsilon \left|t_0 + \mu^*(t_0) - t\right|$$

is satisfied. If t_0 is right-scattered, one obtains

$$f^\Delta(t_0) = \frac{f(t_0 + \mu^*(t_0)) - f(t_0)}{\mu^*(t_0)}. \tag{3}$$

An introduction into dynamic equations on time scales can be found in [1].

For $A \in \mathbb{K}^{d\times d}$ we consider the d-dimensional linear system of dynamic equations on the time scale $\mathbb{T}$

$$x^\Delta = Ax. \tag{4}$$

We recall the classical examples for this setup.

Example 1. If $\mathbb{T} = \mathbb{R}$ we have a linear time-invariant system of the form $\dot{x}(t) = Ax(t)$. If $\mathbb{T} = h\mathbb{Z}$, then (4) reduces to $(x(t + h) - x(t))/h = Ax(t)$ or, equivalently, to $x(t + h) = [I_d + hA]x(t)$.

Let $e_A: \{(t, \tau) \in \mathbb{T} \times \mathbb{T}: t \ge \tau\} \to \mathbb{K}^{d\times d}$ denote the *transition matrix* corresponding to (4), that is, $x(t) = e_A(t, \tau)\xi$ solves the initial value problem (4) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^d$ and $t, \tau \in \mathbb{T}$ with $t \ge \tau$. Due to (3), for a right-scattered point $t_0 \in \mathbb{T}$ we have

$$e_A(t_0 + \mu^*(t_0), t_0) = I_d + \mu^*(t_0)A. \tag{5}$$

For a scalar system $x^\Delta = \lambda x$ ($\lambda \in \mathbb{C}$, $1 + \mu^*(t)\lambda \ne 0$ for all $t \in \mathbb{T}$) one obtains

$$|e_\lambda(t, \tau)| = \exp\left(\int_\tau^t \lim_{s\searrow\mu^*(u)} \frac{\log|1 + s\lambda|}{s} \,\Delta u\right), \tag{6}$$

cf. [1, Theorems 2.33 and 2.35].

The subsequent notions are recalled from [12].

Definition 2 (*Exponential Stability*). Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (4)

(i) *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \ge 0$ with

$$\|e_A(t, s)\| \le K(s) \exp(-\alpha(t - s)) \quad \text{for } t \ge s,$$

(ii) *uniformly exponentially stable* if K can be chosen independently of s in the definition of exponential stability.

Observe that $K(s) \ge 1$ follows from the definition for $t = s$.

In general, exponential stability does not imply uniform exponential stability [12]. The existence of a uniformly exponentially stable system can only be guaranteed if the time scale $\mathbb{T}$ has bounded graininess [13, Theorem 3.1].

In [13, Example 4.1] it is shown that exponential stability of (4) cannot be characterized by the spectrum of its matrix, whereas uniform exponential stability is determined by the spectrum. Note that although the following proposition is only proved for a real matrix in [13, Theorem 3.2] the statement remains true for an arbitrary complex matrix without any modification in the proof.

Proposition 3 (*[13, Theorem 3.2]*). *For $A \in \mathbb{K}^{d\times d}$ system (4) is uniformly exponentially stable if and only if system*

$$x^\Delta = \lambda x \tag{7}$$

is uniformly exponentially stable for every $\lambda \in \sigma(A)$.

Since we want to consider stability radii with respect to uniform exponential stability, we denote

$$\mathcal{US}_\mathbb{C}(\mathbb{T}) = \{\lambda \in \mathbb{C}: \text{system (7) is uniformly exponentially stable}\}.$$

So, for $A \in \mathbb{K}^{d\times d}$ system (4) is uniformly exponentially stable if and only if $\sigma(A) \subset \mathcal{US}_\mathbb{C}(\mathbb{T})$.

Remark 4. (i) Since uniform exponential stability is robust it follows that $\mathcal{US}_\mathbb{C}(\mathbb{T})$ is an open set [13, Proposition 3.1].

(ii) For any $h \ge \max\{\mu^*(t): t \in \mathbb{T}\}$ the system

$$x^\Delta = \frac{-1}{2h}x$$

is uniformly exponentially stable [13, Proof of Theorem 3.1]. On the other hand, for any $\alpha > 0$ the system

$$x^\Delta = \alpha x$$

is not exponentially stable. Therefore, 0 is contained in the boundary of $\mathcal{US}_\mathbb{C}(\mathbb{T})$.

(iii) Consider a scalar system (7). If there is $t_0 \in \mathbb{T}$ such that $1 + \mu^*(t_0)\lambda = 0$, then $x(t) = 0$ for all $t \in \mathbb{T}, t \ge t_0$, in particular (7) is uniformly exponentially stable (which follows directly from (5)). Such systems are called non-regressive, cf. [1, Definition 2.32].

In a particular case the notions of exponential stability and uniform exponential stability coincide. We call a time-scale *periodic* if there exists a constant $p > 0$ such that for every $t \in \mathbb{R}$ we have $t \in \mathbb{T}$ if and only if $t + p \in \mathbb{T}$. In this case p is called a *period* of the time-scale. Clearly, if a time scale is only given as a subset of $[a, \infty)$ and satisfies a periodicity condition there, it may be extended to a periodic time scale that is unbounded above and below.

The following proposition links the results in [13,12] and will be useful in the discussion of examples below.

Proposition 5. *If the time-scale $\mathbb{T}$ is periodic then (7) is exponentially stable if and only if it is uniformly exponentially stable.*

The proof is straightforward using periodicity and is thus omitted.

For a time scale with bounded graininess several essential features are captured by an associated characteristic ball. We define

$$C(\mathbb{T}) := \sup\{c \geq 0 \colon B_c(-c) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})\}.$$

It is clear that $C(\mathbb{T})$ is infinite if $\mathcal{US}_{\mathbb{C}}(\mathbb{T}) = \mathbb{C}_- := \{z \in \mathbb{C} \colon \Re z < 0\}$ and finite in every other case. The *ball of uniform exponential stability* $\mathcal{B}(\mathbb{T})$ is then defined as the maximal ball contained in $\mathcal{US}_{\mathbb{C}}(\mathbb{T})$ with real center and 0 on the boundary, i.e.

$$\mathcal{B}(\mathbb{T}) := B_{C(\mathbb{T})}(-C(\mathbb{T})).$$

In the case $C(\mathbb{T}) = \infty$ we put $\mathcal{B}(\mathbb{T}) = \mathbb{C}_-$. Our analysis of positive systems will yield a positive lower bound for $C(\mathbb{T})$, but a general characterization of this number is as yet elusive.

3. Positive systems

For the classical systems in Example 1 positivity can be characterized by a condition on the system matrix. Recall that a matrix $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ is said to be *Metzler* if $a_{ij} \geq 0$ for $i \neq j$, i.e. there exists $\lambda \in \mathbb{R}$ such that $A + \lambda I_d \geq 0$. For $\mathbb{T} = \mathbb{R}$ the system $\dot{x} = Ax$ is positive if and only if A is a Metzler matrix, whereas for $\mathbb{T} = h\mathbb{Z}$ the system $x(t+1) = [I_d + hA]x(t)$ is positive if and only if $A + \frac{1}{h} I_d \geq 0$. In this section we provide a similar characterization for positive systems on arbitrary time scales.

Definition 6 (*Positive System*). System (4) is said to be *positive* if for all $x \in \mathbb{R}^d_+$ and $s, t \in \mathbb{T}, s \leq t$, it follows that $e_A(t,s)x \in \mathbb{R}^d_+$.

To characterize the positivity of system (4) by a condition on the defining matrix, we distinguish two cases:

Case 1: $\mathbb{T}$ contains no right scattered points, i.e. there is $a \in \mathbb{R}$ such that $\mathbb{T} = [a, \infty)$. The classical result on continuous-time systems yields that system (4) is positive if and only if A is Metzler.

Case 2: $\mathbb{T}$ contains right scattered points. We define

$$\eta = \eta(\mathbb{T}) := \begin{cases} \dfrac{1}{\sup\{\mu^*(t) \colon t \in \mathbb{T}\}} & \text{if } \mathbb{T} \text{ has bounded graininess} \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Proposition 7 (*Characterization of Positive Systems*). *Let $\mathbb{T}$ contain right-scattered points. System (4) is positive if and only if $A + \eta I_d \geq 0$.*

Proof. *Necessity.* Due to (5), for every right-scattered point $t_0 \in \mathbb{T}$ we have

$$e_A(t_0 + \mu^*(t_0), t_0) = I_d + \mu^*(t_0)A.$$

The positivity of the system yields $A \geq -\frac{1}{\mu^*(t_0)} I_d$. This implies $A \geq -\eta I_d$.

Sufficiency. Assume $A \geq -\eta I_d$. In (i) we first show the positivity of system (4) in the case that all off-diagonal entries of A are strictly positive. In (ii) we use a continuity argument to establish the assertion for non-negative off-diagonal entries.

(i) We assume $A = (a_{ij})$ with $a_{ij} > 0$ for $i \neq j$. For fixed $s \in \mathbb{T}$ and $\xi \in \mathbb{R}^d_+$ and any $t \geq s$ we have to show

$$x(t) = e_A(t,s)\xi \in \mathbb{R}^d_+. \quad (9)$$

We use the Induction Principle [1, Theorem 1.7], where for $t \in [s, \infty) \cap \mathbb{T}$ the statement $S(t)$ corresponds to (9).

I. $S(s)$ is satisfied, since $e_A(s,s) = I$.

II. Let $t \in [s, \infty) \cap \mathbb{T}$ be right-scattered and let $S(t)$ be true, i.e. $e_A(t,s)\xi \in \mathbb{R}^d_+$. Then $e_A(t + \mu^*(t), s)\xi = (I + \mu^*(t)A)e_A(t,s)\xi \in \mathbb{R}^d_+$ due to the assumption, i.e. $S(t + \mu^*(t))$ is true.

III. Let $t \in [s, \infty) \cap \mathbb{T}$ be right-dense and assume that $S(t)$ is true. We have to show that there is a neighborhood U of t such that $S(\tau)$ is true for all $\tau \in U \cap (t, \infty) \cap \mathbb{T}$. If $x(t) = 0$ this is straightforward. In the case $x(t) \neq 0$ we show the assertion indirectly. Assume that there is a sequence (t_n) in $U \cap (t, \infty) \cap \mathbb{T}$ such that $t_n \downarrow t$ and $x(t_n) \notin \mathbb{R}^d_+$. Then there are $i \in \{1, \ldots, d\}$ and a subsequence (t_{n_k}) of (t_n) such that $x_i(t_{n_k}) < 0$ for all $k \in \mathbb{N}$. From $0 \leq x_i(t) = \lim_{k\to\infty} x_i(t_{n_k}) \leq 0$ follows $x_i(t) = 0$. The inequality

$$0 \geq \lim_{t_{n_k} \downarrow t} \frac{x_i(t_{n_k})}{t_{n_k} - t} = x_i^{\Delta}(t) = (Ax)_i(t) = \sum_{j=1}^{n} a_{ij}x_j(t) = \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j(t) > 0$$

yields a contradiction.

IV. Let $t \in [s, \infty) \cap \mathbb{T}$ be left-dense, i.e. there is a sequence (t_n) in $[s, \infty) \cap \mathbb{T}$ with $t_n \uparrow t$. Assume that $S(\tau)$ is true for all $\tau \in [s, t)$. In particular, we have $x(t_n) \in \mathbb{R}^d_+$, so $x(t) = \lim_{n\to\infty} x(t_n) \in \mathbb{R}^d_+$, i. e. $S(t)$ is true.

(ii) Now we deal with the case $a_{ij} \geq 0$ for all $i \neq j$. We define

$$M = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ & & \ddots & \\ 1 & 1 & \ddots & 1 \\ 1 & 1 & \dots & 0 \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

Using (i), we obtain that system

$$x^{\Delta} = (A + \varepsilon M)x \quad (10)$$

is positive for all $\varepsilon > 0$. Choose and fix $t_2 > t_1, t_2, t_1 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^d_+$. By variation of constants formula [1, pp. 195], we have

$$e_A(t_2, t_1)x_0 = e_{A+\varepsilon M}(t_2, t_1)x_0 + \varepsilon \int_{t_1}^{t_2} e_{A+\varepsilon M}(t_2, s + \mu^*(s))Me_A(s, t_1)x_0 \, \Delta s. \quad (11)$$

Since (10) is positive, it follows that $e_{A+\varepsilon M}(t_2, t_1)x_0 \in \mathbb{R}^d_+$ for all $\varepsilon > 0$. If ε tends to zero, we obtain

$$e_A(t_2, t_1)x_0 \in \mathbb{R}^d_+,$$

which completes the proof. □

Remark 8. Let $x^{\Delta} = \lambda x$ be a positive system, i.e. $\lambda \geq -\eta$. If there is $t_0 \in \mathbb{T}$ such that $1 + \mu^*(t_0)\lambda = 0$, then $\mu^*(t_0) = \sup\{\mu^*(t) \colon t \in \mathbb{T}\}$, which means $\lambda = -\eta$.

Fundamental spectral properties of positive matrices and Metzler matrices are provided by the classical Perron–Frobenius theory. For every positive matrix $B \in \mathbb{R}^{d\times d}_+$ the spectral radius is an eigenvalue of B, so

$$\rho(B) = \mu(B) \in \sigma(B). \quad (12)$$

For a Metzler matrix $A \in \mathbb{R}^{d\times d}$ we define

$$c(A) = \min\{\lambda \geq 0 \colon A + \lambda I_d \geq 0\}.$$

As a consequence of (12), we obtain for every Metzler matrix A

$$\mu(A) = \rho(A + \alpha I_d) - \alpha \quad \text{for all } \alpha \geq c(A).$$

We arrive at the following well-known properties of Metzler matrices [4, Proposition 1 and Lemma 2].

Lemma 9. *Let $A \in \mathbb{R}^{d\times d}$ be a Metzler matrix. Then*

(i) $\mu(A)$ *is an eigenvalue of* A *and there is a positive eigenvector* $x \in \mathbb{R}^n_+ \setminus \{0\}$ *such that* $Ax = \mu(A)x$.
(ii) $\mu(A) \le \mu(A+D)$ *for all* $D \in \mathbb{R}^{d\times d}_+$.
(iii) *Let* $\lambda \in \sigma(A)$. *Then*

$$|\lambda+\alpha| \le |\mu(A)+\alpha| \quad \text{for all } \alpha \ge c(A).$$

(iv) $(tI_d - A)^{-1}$ *exists and is positive if and only if* $t > \mu(A)$.

For positive matrices $A \in \mathbb{R}^{d\times d}_+, B \in \mathbb{R}^{d\times m}_+, C \in \mathbb{R}^{p\times d}_+$ and an arbitrary matrix $D \in \mathbb{R}^{m\times p}$ one has (see [5, Corollary 2.5])

$$\rho(A+BDC) \le \rho(A+B|D|C). \tag{13}$$

Finally, we recall the following property of rank-one matrices, which we quote from [4, Lemma 3(iii)].

Lemma 10. *If* $D \in \mathbb{R}^{m\times p}$ *has rank one, then* $\|D\| = \||D|\|$.

4. Uniform exponential stability

In this section we assume that $\mathbb{T}$ has bounded graininess, since we want to ensure the existence of a uniformly exponentially stable system.

We study the robustness of the matrix A of the nominal system with respect to structured perturbations. Assume $\sigma(A) \in \mathcal{US}_{\mathbb{C}}(\mathbb{T})$ and let structure matrices $B \in \mathbb{R}^{d\times m}_+$ and $C \in \mathbb{R}^{p\times d}_+$ be given and define the *structured complex*, *real* and *positive stability radius* by

$$r_{\mathbb{C}}(A,B,C) = \inf\{\|D\| : D \in \mathbb{C}^{m\times p}, \sigma(A+BDC) \not\subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})\},$$
$$r_{\mathbb{R}}(A,B,C) = \inf\{\|D\| : D \in \mathbb{R}^{m\times p}, \sigma(A+BDC) \not\subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})\},$$
$$r_{+}(A,B,C) = \inf\{\|D\| : D \in \mathbb{R}^{m\times p}_+, \sigma(A+BDC) \not\subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})\}.$$

Clearly, one has $r_{\mathbb{C}}(A,B,C) \le r_{\mathbb{R}}(A,B,C) \le r_+(A,B,C)$. It is of interest to investigate in which situations these stability radii coincide. We refer to [2] for a more detailed discussion of stability radii.

Unstructured stability radii are obtained by setting $B = C = I \in \mathbb{R}^{d\times d}$ and will be abbreviated by $r_{\mathbb{C}}(A), r_{\mathbb{R}}(A), r_+(A)$. Again, one has $r_{\mathbb{C}}(A) \le r_{\mathbb{R}}(A) \le r_+(A)$.

We intend to show that for a positive uniformly exponentially stable system on an arbitrary time scale the complex stability radius and the positive stability radius coincide. We start by showing the statement for scalar systems, which will be vital for the analysis of the d-dimensional case.

Note that it is immediate from the definition that if $\lambda \in \mathbb{R}$, $\lambda \in (-C(\mathbb{T}), 0)$, then the one-dimensional system $x^\Delta = \lambda x$ is uniformly exponentially stable and $r_{\mathbb{C}}(\lambda) = r_{\mathbb{R}}(\lambda) = r_+(\lambda) = |\lambda|$. We now show the same result for $\eta(\mathbb{T})$ (cf. (8)).

Proposition 11. *Let $\lambda \in \mathbb{R}$ be such that $\lambda \ge -\eta$. Suppose that the scalar positive system*

$$x^\Delta = \lambda x \tag{14}$$

is uniformly exponentially stable. Then $r_{\mathbb{C}}(\lambda) = r_{\mathbb{R}}(\lambda) = r_+(\lambda) = |\lambda|$.

Proof. We first establish $r_+(\lambda) \le r_{\mathbb{C}}(\lambda)$. Let $D_{\mathbb{C}} = a + bi \in \mathbb{C}$, $D_{\mathbb{C}} \ne 0$, be such that the system

$$x^\Delta = (\lambda + D_{\mathbb{C}})x$$

is not uniformly exponentially stable. We have to show that there exists $D \in \mathbb{R}_+$ such that $D \le |D_{\mathbb{C}}|$ and the corresponding system

$$x^\Delta = (\lambda + D)x$$

is not uniformly exponentially stable. We choose $D = \sqrt{a^2+b^2}$. According to the Remarks 4(iii) and 8, for $t_1, t_2 \in \mathbb{T}, t_1 \le t_2$, we obtain a representation of $|e_{\lambda+D_{\mathbb{C}}}(t_2,t_1)|$ and $|e_{\lambda+D}(t_2,t_1)|$ by means of (6), i.e.

$$|e_{\lambda+D_{\mathbb{C}}}(t_2,t_1)| = \exp\left(\int_{t_1}^{t_2} \lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+D_{\mathbb{C}})|}{s}\,\Delta u\right).$$

We verify

$$|e_{\lambda+D_{\mathbb{C}}}(t_2,t_1)| \le |e_{\lambda+D}(t_2,t_1)|, \quad \text{for all } t_1,t_2 \in \mathbb{T},\ t_1 \le t_2.$$

by showing the inequality

$$\lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+D_{\mathbb{C}})|}{s} \le \lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+D)|}{s}$$

for any $u \in \mathbb{T}$.

We consider two cases:

- $\mu^*(u) = 0$: A straightforward computation yields that

$$\begin{aligned}\lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+D_{\mathbb{C}})|}{s} &= \lim_{s\searrow 0} \frac{\log|1+s(\lambda+D_{\mathbb{C}})|}{s}\\ &= \lambda + a \le \lambda + \sqrt{a^2+b^2}\\ &= \lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+D)|}{s}.\end{aligned}$$

- $\mu^*(u) \ne 0$: Since (14) is positive it follows that $1+\mu^*(u)\lambda \ge 0$. We obtain

$$\begin{aligned}|1+\mu^*(u)\lambda+\mu^*(u)D_{\mathbb{C}}| &\le 1+\mu^*(u)\lambda+\mu^*(u)|D_{\mathbb{C}}|\\ &= |1+\mu^*(u)\lambda+\mu^*(u)D|.\end{aligned}$$

So far we have proved that $r_{\mathbb{C}}(\lambda) = r_{\mathbb{R}}(\lambda) = r_+(\lambda)$. To compute $r_+(\lambda)$ we first observe that the system $x^\Delta = 0x$ (i.e. system (14) with perturbation $D = |\lambda|$) is not uniformly exponentially stable. Therefore, $r_+(\lambda) \le |\lambda|$. As system (14) is uniformly exponentially stable we have $\lambda < 0$. Take any $\beta \in (0, |\lambda|)$ and consider the system

$$x^\Delta = (\lambda+\beta)x. \tag{15}$$

Note that for any $h \ge \max\{\mu^*(u) : u \in \mathbb{T}\}$ the system

$$x^\Delta = \frac{-1}{2h}x$$

is uniformly exponentially stable (see Remark 4), i.e. there exist $K, \alpha > 0$ such that the inequality

$$\begin{aligned}|e_{\frac{-1}{2h}}(t_2,t_1)| &= \exp\left(\int_{t_1}^{t_2} \lim_{s\searrow\mu^*(u)} \frac{\log|1+s\left(\frac{-1}{2h}\right)|}{s}\,\Delta u\right)\\ &\le K\exp(-\alpha(t_2-t_1))\end{aligned}$$

holds for all $t_2 \ge t_1$ (where the representation of the transition function exists due to Remark 8). Choose $h \ge \max\{\mu^*(u) : u \in \mathbb{T}\}$ such that $\lambda+\beta < \frac{-1}{2h}$. Due to

$$\lim_{s\searrow\mu^*(u)} \frac{\log|1+s(\lambda+\beta)|}{s} \le \lim_{s\searrow\mu^*(u)} \frac{\log|1+s\frac{-1}{2h}|}{s} \quad \text{for all } u \in \mathbb{T}$$

it follows that system (15) is uniformly exponentially stable. Consequently, the smallest positive perturbation that destabilizes (14) is $D = |\lambda|$, so $r_+(\lambda) = |\lambda|$, which completes the proof. □

We note a consequence for the ball of uniform exponential stability $\mathcal{B}(\mathbb{T})$.

Corollary 12. *Let $\mathbb{T}$ be a time scale with bounded graininess, then $\eta(\mathbb{T}) \le C(\mathbb{T})$, that is,*

$$B_\eta(-\eta) \subset \mathcal{B}(\mathbb{T}) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T}).$$

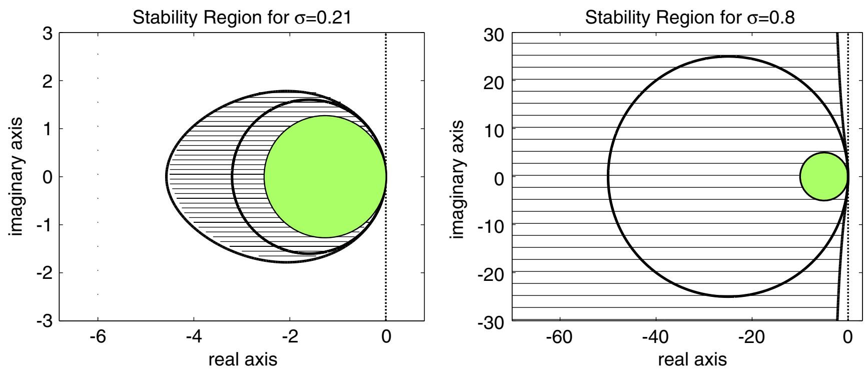

Fig. 1. Stability regions and positive balls as described in Example 14(iii), i.e. $\mathcal{US}_{\mathbb{C}}(T)$ (hatched), $B_\eta(-\eta)$ (full ball) and $\mathcal{B}(\mathbb{T})$ (only boundary shown).

Remark 13. On the other hand, if $\inf\{\mu^*(u): u \in \mathbb{T}\} > 0$, then with $\nu = \frac{1}{\inf\{\mu^*(u): u\in\mathbb{T}\}}$ the inclusion $\mathcal{US}_{\mathbb{C}}(\mathbb{T}) \subseteq B_\nu(-\nu)$ is satisfied.

Example 14. (i) In the classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ for a fixed $h > 0$ we obtain the standard results. In particular, if $\mathbb{T} = \mathbb{R}$, then A generates a positive system if and only if A is Metzler, $\eta(\mathbb{T}) = C(\mathbb{T}) = \infty$ and the ball of positivity and of uniform exponential stability coincide. Similar statements hold for $\mathbb{T} = h\mathbb{Z}$, namely A generates a positive system if and only if $A + hI \geq 0$, $\eta(\mathbb{T}) = C(\mathbb{T}) = h$ and again the ball of positivity and the ball of uniform exponential stability coincide; i.e. $B_\eta(-\eta) = \mathcal{B}(\mathbb{T})$.

(ii) Consider the time scale $\mathbb{T} = \{t_n\}_{n\in\mathbb{N}}$ of so-called *harmonic numbers* $t_0 := 0$, $t_n := \sum_{k=1}^{n} \frac{1}{k}$, $n \geq 1$. The graininess is given by $\mu^*(t_n) = \frac{1}{n+1}$. So $\eta = 1$, and a system is positive if and only if $A + I \geq 0$. On the other hand, the set of exponential stability is $\mathbb{C}_-$, [12]. We will use the techniques of Section 5 on eventually positive systems to establish $C(\mathbb{T}) = \infty$, thus the balls of positivity and uniform exponential stability are very different.

(iii) In the case of alternating continuous intervals and jumps of constant length we consider the time scale

$$\mathbb{T}_\sigma := \bigcup_{k\in\mathbb{Z}} [k, k+\sigma]$$

for a fixed $\sigma \in [0, 1]$. This time scale is periodic, so by Proposition 5 exponential and uniform exponential stability coincide. Note that Fig. 1 in [12] represents $\mathcal{US}_{\mathbb{C}}(\mathbb{T})$. We obtain $\eta = (1-\sigma)^{-1}$. It can be shown by simple calculations that $C(\mathbb{T}) = \eta^2$, so the inclusion of the positive ball in the ball of uniform exponential stability is strict, $B_\eta(-\eta) \subset \mathcal{B}(\mathbb{T})$. The cases $\sigma = 0.21$ and $\sigma = 0.8$ are depicted in the subsequent figure. The hatched area represents the set of exponential stability $\mathcal{US}_{\mathbb{C}}(\mathbb{T})$, the full ball is the ball of positivity $B_\eta(-\eta)$, and of the ball $\mathcal{B}(\mathbb{T})$ only the boundary is shown. Recall that for $\sigma = 0.21$ the set $\mathcal{US}_{\mathbb{C}}(\mathbb{T})$ is disconnected [12]. Here we only represent the connected component with 0 in its boundary (Fig. 1).

(iv) To give an example of a more exotic time scale, consider as in [12] the time scale obtained by gluing standard Cantor sets[1] together. In this case, the time-scale is periodic, so exponential stability coincides with uniform exponential stability and the stability set calculated in [12] coincides with the set of exponential stability. We have $\max \mu^*(t) = 1/3$, and by the previous considerations follows $B_3(-3) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$. This ball is by no means the largest one contained in the stability set (Fig. 2). We checked numerically that for $\gamma \approx 6.9969$ we have $B_\gamma(-\gamma) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$. It is tempting to conjecture that the real number in question is $C(\mathbb{T}) = 7$, but we have no proof of this.

Theorem 15. *Assume that the system (4) is positive. Then*

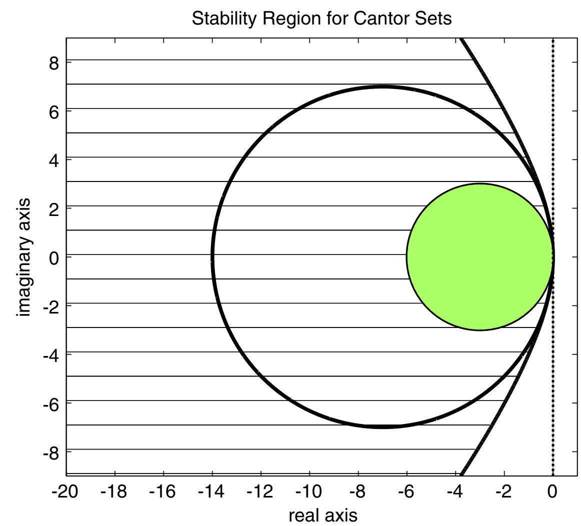

Fig. 2. Stability regions and positive balls for the Cantor set example as described in Example 14(iv), i.e. $\mathcal{US}_{\mathbb{C}}(T)$ (hatched), $B_\eta(-\eta)$ (full ball) and $\mathcal{B}(\mathbb{T})$ (only boundary shown).

(i) *the scalar system*

$$x^\Delta = \mu(A)x \tag{16}$$

is positive.

(ii) *system (4) is uniformly exponentially stable if and only if system (16) is uniformly exponentially stable.*

Proof. (i) By Lemma 9, there exists $x \in \mathbb{R}^d_+ \setminus \{0\}$ with $Ax = \mu(A)x$. Hence,

$$e_A(t_2, t_1)x = e_{\mu(A)I_d}(t_2, t_1)x, \quad \text{for all } t_2 \geq t_1,\ t_2, t_1 \in \mathbb{T}.$$

Since (4) is positive, we obtain positivity of (16) from

$$e_{\mu(A)I_d}(t_2, t_1) \geq 0, \quad \text{for all } t_2 \geq t_1,\ t_2, t_1 \in \mathbb{T}.$$

(ii) Due to Proposition 3 the uniform exponential stability of system (4) implies the uniform exponential stability of system (16). Conversely, assume that system (16) is uniformly exponentially stable and fix $\lambda \in \sigma(A)$. Clearly, $\operatorname{Re}\lambda \leq \mu(A)$, and by virtue of Lemma 9 we have

$$|\eta + \lambda| \leq |\eta + \mu(A)|. \tag{17}$$

Hence $\sigma(A) \subset B_{|\eta+\mu(A)|}(-\eta) \subset B_\eta(-\eta)$, because $\mu(A) \in (-\eta, 0)$ by Proposition 7 and Lemma 9. By Corollary 12 we have $\sigma(A) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$ and the claim follows. □

Theorem 16. *Let*

$$x^\Delta = Ax, \quad A \in \mathbb{R}^{d\times d}, x \in \mathbb{R}^d \tag{18}$$

be a positive uniformly exponentially stable system and let structure

[1] A Cantor set is constructed by removing middle thirds recursively starting from [0, 1].

matrices $B \in \mathbb{R}_+^{d\times m}$, $C \in \mathbb{R}_+^{p\times d}$ *be given. Then*

$$r_{\mathbb{C}}(A,B,C) = r_{\mathbb{R}}(A,B,C) = r_+(A,B,C) = \frac{1}{\|CA^{-1}B\|} \in \mathbb{R}\cup\{\infty\}.$$

Proof. Denote by Γ_{us} the boundary of the open set $\mathcal{US}_{\mathbb{C}}(\mathbb{T})$. According to Remark 4, we have $0 \in \Gamma_{us}$. Define the transfer matrix

$$G(s) = C(sI_d - A)^{-1}B$$

for all elements s of the resolvent set of A. First, a formula is recalled to compute the complex stability radius. Second, we estimate the positive stability radius.

Step 1: The complex stability radius is obtained by the formula

$$r_{\mathbb{C}}(A,B,C) = \frac{1}{\max\{\|G(s)\|: s \in \Gamma_{us}\}}. \tag{19}$$

This follows from a general result for the computation of the complex stability radius with respect to open subsets of the complex plane, see [14].

Step 2: We show that

$$r_+(A) \geq \frac{1}{\|BA^{-1}C\|}. \tag{20}$$

Indeed, let $D \in \mathbb{R}_+^{m\times p}$ be destabilizing, i.e. the system

$$x^{\Delta} = (A + BDC)x \tag{21}$$

is not uniformly exponentially stable. Obviously, system (21) is positive. Due to [14] the destabilizing $D \in \mathbb{C}^{m\times p}$ of minimal norm may be chosen to be of rank one and

$$x^{\Delta} = \mu x \tag{22}$$

with $\mu = \mu(A+BDC)$ is not uniformly exponentially stable. On the other hand, by virtue of Theorem 15 the system $x^{\Delta} = \mu(A)x$ is positive and uniformly exponentially stable, so $\mu(A) < 0$. By applying Proposition 11 we obtain

$$|\mu - \mu(A)| \geq -\mu(A).$$

Due to Lemma 9(ii) it follows that $\mu \geq \mu(A)$ and, hence, $\mu \geq 0$. Since $A + BDC$ is a Metzler matrix, Lemma 9(i) implies that $\mu \in \sigma(A + BDC)$ and that there exists $x \in \mathbb{R}_+^d \setminus \{0\}$ such that

$$(A + BDC)x = \mu x. \tag{23}$$

Since (18) is uniformly exponentially stable, Proposition 3 implies that the scalar systems corresponding to eigenvalues of A are uniformly exponentially stable. Because (22) is not uniformly exponentially stable, we obtain $\mu \notin \sigma(A)$, so $(\mu I_d - A)^{-1}$ exists. Hence, for $y := Cx \in \mathbb{R}_+^p \setminus \{0\}$ Eq. (23) reads as

$$G(\mu)Dy = y.$$

Thus, $\|D\| \geq 1/\|G(\mu)\|$ and to obtain (20), we have to show

$$\|D\| \geq \frac{1}{\|BA^{-1}C\|} = \frac{1}{\|G(0)\|}.$$

To this end it is sufficient to establish $\|G(\mu)\| \leq \|G(0)\|$. Indeed, by using Lemma 9(iv) we get that

$$G(0) = C(-A)^{-1}B \quad\text{and}\quad G(\mu) = C(\mu I_d - A)^{-1}B$$

are positive matrices. By the resolvent equation we obtain

$$C(-A)^{-1}B - C(\mu I_d - A)^{-1}B = C\left(\mu(-A)^{-1}(\mu I_d - A)^{-1}\right)B \geq 0,$$

which proves the desired inequality.

Step 3: It remains to show $r_{\mathbb{C}}(A,B,C) \geq r_+(A,B,C)$. Then we obtain

$$r_{\mathbb{C}}(A,B,C) \geq r_+(A,B,C) \geq \frac{1}{\|BA^{-1}C\|} \geq r_{\mathbb{C}}(A,B,C),$$

which proves the assertion of the theorem.

If $r_{\mathbb{C}}(A,B,C) = \infty$, there is nothing to show. So assume we have a destabilizing D, such that $\sigma(A + BDC) \not\subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$. Due to the remark in Step 1 it is no loss of generality to assume that D has rank one. As $B_\eta(-\eta) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$, we have by the choice of D and using (13) that

$$\eta \leq \rho(A + BDC + \eta I_d) \leq \rho(A + B|D|C + \eta I_d) =: \rho.$$

The Perron–Frobenius theorem yields $0 \leq \rho - \eta \in \sigma(A + B|D|C)$, so that $|D|$ is destabilizing. As D is of rank one, it follows from Lemma 10 that $\|D\| = \||D|\|$, which implies $r_{\mathbb{C}}(A,B,C) \geq r_+(A,B,C)$ as desired. □

As a corollary we treat the ball of uniform exponential stability.

Corollary 17. *Let system (4) be a uniformly exponentially stable system such that* $A + C(\mathbb{T})I_d \in \mathbb{R}_+^{d\times d}$. *Let structure matrices* $B \in \mathbb{R}_+^{d\times m}$, $C \in \mathbb{R}_+^{p\times d}$ *be given. Then*

$$r_{\mathbb{C}}(A,B,C) = r_{\mathbb{R}}(A,B,C) = r_+(A,B,C) = \frac{1}{\|CA^{-1}B\|} \in \mathbb{R}\cup\{\infty\}.$$

Proof. We simply need to retrace the steps of the proof of Theorem 16. Step 1 did not use the positivity of A, so is directly applicable under the assumption of the corollary. In Step 2 we can replace positivity of A by the assumption that A is Metzler and $\mathcal{B}(\mathbb{T}) \subset \mathcal{US}_{\mathbb{C}}(\mathbb{T})$. The same reasoning applies to Step 3. □

5. Eventually positive systems and robustness

For the sake of stability analysis of linear systems it turns out that a slightly more general system class has the same nice properties as positive systems.

Definition 18 (*Eventually Positive Systems*). System (4) is called *eventually positive* if there exists a $t_0 \in \mathbb{T}$ such that (4) restricted to $\mathbb{T} \cap [t_0, \infty)$ is positive.

From the previous sections we immediately obtain the following statements.

Proposition 19. (i) *A is eventually positive if and only if there exists a* $t_0 \in \mathbb{T}$ *such that for* $\eta_0 := 1/\sup\{\mu^*(t): t \geq t_0\}$ *we have* $A + \eta_0 I \geq 0$.

(ii) *If we set*

$$\mu_\infty := \inf\{\sup\{\mu^*(t): t \geq t_0\}: t_0 \geq 0\} = \lim_{t_0\to\infty} \sup\{\mu^*(t): t \geq t_0\}$$

and $\eta_\infty = 1/\mu_\infty$ *then A generates an eventually positive system if* $A + \eta I \geq 0$ *for some* $\eta > \eta_\infty$.

Proposition 20. *Let system (4) be an eventually positive, uniformly exponentially stable system and let structure matrices* $B \in \mathbb{R}_+^{d\times m}$, $C \in \mathbb{R}_+^{p\times d}$ *be given. Then*

$$r_{\mathbb{C}}(A,B,C) = r_{\mathbb{R}}(A,B,C) = r_+(A,B,C) = \frac{1}{\|CA^{-1}B\|} \in \mathbb{R}\cup\{\infty\}.$$

Proof. This follows immediately from Theorem 16, as a destabilizing solution in particular destabilizes the positive system on the time scale $\mathbb{T} \cap [t_0, \infty)$. □

Example 21. (i) For the harmonic time scale discussed in Example 14(ii) it is easy to see, that every Metzler matrix defines an eventually positive system, as $\mu^*(t_n) \to 0$. Thus it follows that $\eta_\infty = \infty$ and so $\mathcal{US}_{\mathbb{C}}(\mathbb{T}) = \mathbb{C}_-$. For A Hurwitz and Metzler, $B, C \geq 0$ we have that A defines a uniformly exponentially stable system and that $r_{\mathbb{C}}(A,B,C) = r_+(A,B,C) = \|CA^{-1}B\|^{-1}$.

(ii) For periodic time-scales a system is positive if and only if it is eventually positive. This shows, in particular, that $B_{\eta_\infty}(-\eta_\infty)$ can still be a strict subset of $\mathcal{B}(\mathbb{T})$ using the examples of Example 14(iii).

6. Conclusion

In this note we investigated positive linear systems on time scales. These systems are generated by a subset of the set of Metzler matrices, depending on the graininess of the time scale $\mathbb{T}$. Surprisingly, there is a difference between the ball in which the spectrum of a positive uniformly exponentially stable system may lie and the ball of uniform exponential stability. In the classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ this phenomenon does not occur. This difference and the ball of uniform exponential stability need to be analyzed further.

For positive systems we provided an easily computable formula for the stability radii. Although we have restricted ourselves to the case of the Euclidean norm, all results apply to monotone norms using the techniques provided in [4].

For systems of dynamic equations on time scales, stability radii with respect to block-diagonal perturbations or linear fractional perturbations will be the subject of further investigations (cf. also [6,7,15] and the references therein).

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