

1                   **EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR**  
 2                   **DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE**

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ABSTRACT. We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line  $\mathbb{R}_+$ . Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform  $\varphi$ -Lipschitz condition, in which the Lipschitz function  $\varphi$  belongs to wide classes of admissible function spaces such as  $L_p$ ,  $1 \leq p \leq \infty$ ,  $L_{p,q}$ , etc.

4                   1. INTRODUCTION AND PRELIMINARIES

5         The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear  
 6         non-autonomous differential-algebraic equations (DAEs) of the form

$$d \text{ rows} \quad \underbrace{\begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}}_{E(t)} \dot{x}(t) = \underbrace{\begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}}_{A(t)} x(t) + \underbrace{\begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}}_{f(t, x(t))}, \quad t \in \mathbb{R}_+ := [0, +\infty). \quad (1.1)$$

7         To do that, we start by investigating the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \quad (1.2)$$

8         Here  $E = \begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}$  are matrix-valued functions acting on  $\mathbb{R}_+$  to  $\mathbb{R}^{n,n}$ ,  $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ , and  
 9          $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Furthermore, we assume that for all  $t$ , the matrices  $\mathbf{E}_1(t)$ ,  $\mathbf{A}_2(t)$  have full row rank.

10         DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical  
 11         circuits, chemical engineering, and many other applications. Due to the rank-deficiency of  $E(t)$ , the qualitative  
 12         behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the  
 13         reader to recent monographs [3, 15–17] and the references therein. In particular, even though the stability  
 14         analysis for DAEs have been intensively discussed (see the survey [15, Chapter 2]), the spectral theory of  
 15         DAEs, (in particular, the exponential dichotomy for DAEs), has attracted much attention only recently and  
 16         is still an active research area. We refer to [18] for the concept of exponential dichotomy and its relation to  
 17         the well conditioning of the associated boundary value problem, to [20–22] for Lyapunov and other spectra  
 18         for linear DAEs, to [2, 5, 10] for the robustness of exponential stability and Bohl exponents, to [9, 29, 30]  
 19         for the study of singular difference systems, and to [32] for the exponential dichotomy of infinite dimensional  
 20         systems.

21         On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is  
 22         characterized, the next important question in the qualitative theory of DAEs is to study the existence of  
 23         integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1)

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*Date:* May 19, 2021.

*Key words and phrases.* Exponential dichotomy, semilinear, differential-algebraic equation, admissibility of function spaces, stable manifold.

[4, 7]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps, this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1). Our method is based on the classical "Lyapunov-Perron method" ([7, 33]) and the admissibility of function spaces ([13, 14]).

The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for later use, including the notion of the exponential dichotomy and its properties, as well as some important features of admissible function spaces. In Section 2 we give a characterization for the existence of exponential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations, and we conclude this research by a summary and some open problems.

35

**1.1. Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By  $(\mathbb{R}^n, \|\cdot\|)$  we denote the n-dimensional real vector space equipped with the Euclidean norm. For any matrix  $V$ , by  $V^T$  we denote its transpose. For any  $p \in \mathbb{N}$ , by  $C^p([0, \infty), \mathbb{R}^n)$  we denote the space of  $p$ -times continuously differentiable functions acting on  $[0, \infty)$  with values in  $\mathbb{R}^n$ . By  $C_b([0, \infty), \mathbb{R}^n)$  we denote the space of continuous and bounded functions mapping from  $[0, \infty)$  into  $\mathbb{R}^n$ . This space is a Banach space with the *ess sup*-norm  $\|f\|_\infty := \sup\{\|f(t)\|, t \geq 0\}$ .

It is well-known (e.g. [4]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \geq s \geq 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

is well-posed, then there exists a pointwise nonsingular matrix-valued function  $(t, s) \mapsto X(t, s) \in \mathbb{R}^{n,n}$  such that the solution of (1.3) is given by  $x(t) = X(t, s)x_s$ . This fact motivates the existence of an evolution family  $(X(t, s))_{t \geq s \geq 0}$  associated with the matrix function  $A(t)$ . This family satisfies the condition  $X(t, t) = Id$  and the so-called *semi-group property*

$$X(t, r)X(r, s) = X(t, s), \quad \text{for all } t \geq r \geq s \geq 0. \tag{1.4}$$

Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), \quad \text{for all } t \geq s \geq 0,$$

also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \tau)f(\tau, x(\tau))d\tau, \quad \text{for all } t \geq s \geq 0. \tag{1.5}$$

For more details on the notion and discussion on properties and applications of evolution families we refer the readers to Pazy [26].

**Definition 1.1.** A given evolution family  $\{X(t, s)\}_{t \geq s \geq 0}$  of the ODE (1.3) is said to have an *exponential dichotomy* on the half-line if there exist a family of projection matrices  $\{P(t)\}_{t \geq 0}$  and two positive constants  $N, \nu$  such that the following conditions are satisfied.

- i)  $P(t)X(t, s) = X(t, s)P(s)$  for all  $t \geq s \geq 0$ ,
- ii) for all  $t \geq s \geq 0$ , the restriction  $X(t, s)| : \ker P(s) \rightarrow \ker P(t)$  is an isomorphism, and we denote its inverse by  $X(s, t)|$ ,
- iii)  $\|X(t, s)P(s)x\| \leq N e^{-\nu(t-s)} \|P(s)x\|$ , for all  $t \geq s \geq 0$ ,  $x \in \mathbb{R}^n$ ,
- iv)  $\|X(t, s)| (I - P(s))x\| \leq N e^{\nu(t-s)} \|(I - P(s))x\|$ , for all  $s \geq t \geq 0$ ,  $x \in \mathbb{R}^n$ .

58 Here  $\{P(t)\}_{t \geq 0}$  (reps.  $N, \nu$ ) are called *dichotomy projections* (resp. *dichotomy constants*).

59 The concept exponential dichotomy means that the state space  $\mathbb{R}^n$  has been splitted into the (exponentially) stable subspace ( $\text{Im}(P(t))$ ) and the (exponentially) unstable subspace ( $\ker(P(t))$ ).

61 **1.2. A short review of DAE solvability theory.** Linear DAEs of the form 1.2 have been extensively  
 62 studied in the last thirty years, see [16] and the references therein. In order to understand the solution  
 63 behavior and to obtain numerical solutions, the necessary information about derivatives of equations has to  
 64 be utilized. This has led to the concept of the strangeness index, which allows to use the DAE and (some of)  
 65 its derivatives to be reformulated as a system with the same solution that is *strangeness-free*, i.e., for which  
 66 the algebraic and differential part of the system are easily separated. In this paper we restrict ourselves to  
 67 regular DAEs with sufficiently smooth coefficients, i.e., we require that (1.2) (or the nonlinear DAE (1.1)  
 68 locally) has a unique solution for appropriately chosen (consistent) initial conditions, see [16] for a discussion  
 69 of more general nonregular DAEs. With the theory and appropriate numerical methods available, then  
 70 throughout this paper, for regular DAEs we may assume that the homogeneous DAE 1.2 in consideration  
 71 fulfills the following presumption.

72 **Assumption 1.2.** Assume that the function pair  $(E, A)$  in the DAEs (1.1), (1.2) is *strangeness-free*, i.e.,

$$\text{rank} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} = n,$$

73 for all  $t \geq 0$ . Furthermore, we assume that  $E \in C^1([0, \infty), \mathbb{R}^{n,n})$  and  $A \in C^0([0, \infty), \mathbb{R}^{n,n})$ .

74 **Definition 1.3.** The DAE

$$\tilde{E}(t)\dot{y}(t) = \tilde{A}(t)x(t) + \tilde{f}(t, y(t)) \quad (1.6)$$

is called *orthogonally equivalent* to the DAE (1.1) if there exist pointwise-orthogonal matrix-valued functions  $U \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that after changing variable  $x(t) = V(t)y(t)$ , and scaling (1.2) with  $U(t)$ , we obtain exactly (1.6). In details, this means that the following identities hold true.

$$\tilde{E} = UEV, \quad \tilde{A} = UAV - UE\dot{V}, \quad \tilde{f}(t, y(t)) = U(t)f(t, Vy(t)), \quad \text{for all } t \geq 0.$$

75 We denote this orthogonal equivalence by  $(E, A, f) \overset{\circ}{\sim} (\tilde{E}, \tilde{A}, \tilde{f})$  and omit the terms  $f, \tilde{f}$  if the homogeneous  
 76 system (1.2) is considered.

77 Indeed, one can directly verify that this orthogonal equivalence concept is an equivalent relation, i.e., it  
 78 fulfills three properties: reflexivity, symmetry and transitivity. We omit the detailed proof here in order to  
 79 keep the brevity of this work. By making use of some smooth factorizations, for example QR or SVD ([8] or  
 80 [16, Thm 3.9]), we can decouple the differential and algebraic parts of the DAE (1.2) in the following lemma.

81 **Lemma 1.4.** Consider the DAE (1.2) and assume that it satisfies Assumption 1.2. Then, there exist  
 82 pointwise-orthogonal matrix-valued functions  $U \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that by  
 83 changing variable  $x(t) = V(t)y(t)$ , and scaling (1.2) with  $U(t)$ , we obtain the so-called semi-explicit system

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad (1.7)$$

84 with pointwise nonsingular matrix-valued functions  $\Sigma(t) \in \mathbb{R}^{d,d}$  and  $A_4(t) \in \mathbb{R}^{a,a}$ .

85 *Proof.* Applying an SVD factorization for  $\mathbf{E}_1(t)$  we can find pointwise-orthogonal matrix functions  $U_1(t) \in$   
 86  $C^1([0, \infty), \mathbb{R}^{d,d})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$  such that  $U_1(t)\mathbf{E}_1(t)V(t) = [\Sigma(t) \ 0]$ , where  $\Sigma(t)$  is a continuous,

87 pointwise nonsingular function with values in  $\mathbb{R}^{d,d}$ . Changing the variable  $x(t) = V(t)y(t)$  and scaling (1.2)  
88 with  $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$ , we obtain a new system exactly of the form (1.7). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ A_3(t) & A_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} V(t),$$

89 then Assumption 1.2 yields that both  $\Sigma$  and  $A_4$  are nonsingular. This completes the proof.  $\square$

Let  $\hat{A}_3 := -A_4^{-1}A_3$ ,  $\hat{A}_1 := \Sigma^{-1}(A_1 - A_2A_4^{-1}A_3)$ , we rewrite system (1.7) as

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t), \quad (1.8a)$$

$$y_2(t) = \hat{A}_3(t)y_1(t). \quad (1.8b)$$

90 Since  $V(t)$  is orthogonal for all  $t \geq 0$ , we see that all important qualitative properties of  $x(t)$ , such as  
91 boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function  $y(t)$   
92 without any difficulty. Clearly, we see that (1.8b) gives an *algebraic constraint* that the solution to (1.7)  
93 must obey, while (1.8a) gives the dynamic of (1.7). For this reason, we call it *an underlying ODE* to (1.7).

94 **Definition 1.5.** (i) Consider the DAE (1.2). A matrix function  $X \in C([0, \infty), \mathbb{R}^{n,k})$ ,  $d \leq k \leq n$ , is called a  
95 *fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and  $\text{rank } X(t) = d$ , for all  
96  $t \geq 0$ .

97 (ii) A fundamental solution matrix is said to be *maximal* if  $k = n$  and *minimal* if  $k = d$ , respectively. A  
98 maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)(X(0) - Id) = 0. \quad (1.9)$$

99 We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of  
100 full rank. Furthermore, each fundamental solution matrix has exactly  $d$ -linear independent columns, and  
101 a minimal fundamental solution matrix can be made maximal by adding  $n - d$  zero columns. This is the  
102 major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a  
103 DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been  
104 proposed and carefully discussed in [20]. We recall it below, and notice that this concept is equivalent to the  
105 one proposed by Lentini and März in [18] within the context of the matrix chains approach and tractability  
106 index.

107 Let  $\{\hat{Y}_1(t, s)\}_{t \geq s \geq 0}$  be the evolution family associated with (1.8a), then we can define the corresponding  
108 evolution families for two DAEs (1.7), (1.2) consecutively as follows.

$$\hat{Y}(t, s) := \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, s) & 0 \end{bmatrix}, \quad \hat{X}(t, s) := V(t)\hat{Y}(t, s)V^T(s), \quad \text{for all } t \geq s \geq 0. \quad (1.10)$$

109 Nevertheless, since  $X(t, s)$  is not invertible, we will define the *reflexive generalized inverse matrix function*  
110 as in [20] by

$$\hat{Y}^-(t, s) := \begin{bmatrix} \hat{Y}_1^{-1}(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t, s) & 0 \end{bmatrix}, \quad \hat{X}^-(t, s) := V(s)\hat{Y}^-(t, s)V^T(t), \quad \text{for all } t \geq s \geq 0. \quad (1.11)$$

Then, we can directly verify the semigroup properties, i.e.

$$\hat{X}(t, r) = \hat{X}(t, s)\hat{X}(s, r), \quad \text{for all } t \geq s \geq r \geq 0,$$

$$\hat{X}(t, s) = \hat{X}(t, 0)\hat{X}^-(s, 0), \quad \text{for all } t \geq s \geq 0.$$

111 Furthermore, Lemmas 1.6 and 1.7 below show that the family  $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$  does not depend on the  
112 choice of orthogonal transformations, and it plays the same role as the evolution family  $\{X(t, s)\}_{t \geq s \geq 0}$ , in  
113 comparison to (1.5).

<sup>114</sup> **Lemma 1.6.** *The families  $\{X(t, s)\}_{t \geq s \geq 0}$ ,  $\{X^-(t, s)\}_{t \geq s \geq 0}$  defined by (1.10), (1.11) do not depend on the  
choice of orthogonal transformations.*

*Proof.* We will prove this claim only for the first family  $\{X(t, s)\}_{t \geq s \geq 0}$ , since for the second family the proof is essentially the same. Let us assume that we have two semi-explicit forms of system (1.2) obtained under orthogonal transformations, i.e.,

$$(E, A) \sim \left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \quad (1.12a)$$

$$(E, A) \sim \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right). \quad (1.12b)$$

Now we will prove that the two corresponding systems have the same evolution family  $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$ . Without loss of generality, we assume that  $(E, A)$  is already in the form of the right hand side of (1.12a), so  $U$  and  $V$  in Lemma 1.4 are identity matrices and  $\hat{X}(t, s) = \hat{Y}(t, s)$  for all  $t \geq s \geq 0$ . The corresponding system to the right hand side of (1.12b) reads

$$\dot{\tilde{y}}_1(t) = \hat{A}_1(t)\tilde{y}_1(t), \quad (1.13a)$$

$$\dot{\tilde{y}}_2(t) = \hat{A}_3(t)\tilde{y}_1(t). \quad (1.13b)$$

where  $\hat{A}_3 := -\tilde{A}_4^{-1}\tilde{A}_3$ ,  $\hat{A}_1 := \tilde{\Sigma}^{-1}(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3)$ .

The transitivity applied to (1.12) implies that there exist pointwise-orthogonal matrix-valued functions  $S = [S_1 \ S_2 \ S_3 \ S_4] \in C^0([0, \infty), \mathbb{R}^{n,n})$  and  $T = [T_1 \ T_2 \ T_3 \ T_4] \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that  $y(t) = T(t)\tilde{y}(t)$  and

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad (1.14a)$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \left( \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T}_1 & \dot{T}_2 \\ \dot{T}_3 & \dot{T}_4 \end{bmatrix} \right). \quad (1.14b)$$

<sup>116</sup> Let  $\{\hat{\mathcal{Y}}_1(t, s)\}_{t \geq s \geq 0}$  be the evolution family associated with (1.13a), then the evolution family associated with  
<sup>117</sup> system (1.13) is

$$\hat{\mathcal{Y}}(t, s) = \begin{bmatrix} \hat{\mathcal{Y}}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{\mathcal{Y}}_1(t, s) & 0 \end{bmatrix}, \quad \hat{\mathcal{X}}(t, s) := T(t)\hat{\mathcal{Y}}(t, s)T^T(s) \text{ for all } t \geq s \geq 0. \quad (1.15)$$

<sup>118</sup> Thus, we need to prove that  $\hat{\mathcal{X}}(t, s) = \hat{X}(t, s)$ .

<sup>119</sup> From (1.14a), it implies that  $S_3\Sigma[T_1 \ T_2] = 0$ . Thus, we have

$$\begin{bmatrix} S_3 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = [0 \ 0],$$

<sup>120</sup> and hence, this follows that  $S_3 = 0$ . Thus,  $S = [S_1 \ S_2 \ 0 \ S_4]$ , and then, due to the orthogonality of  $S$ ,  $S_1$  is  
<sup>121</sup> nonsingular and  $S_4$  is orthogonal. Also from (1.14a), we see that  $S_1\Sigma T_2 = 0$ , which yields that  $T_2 = 0$ .  
<sup>122</sup> Moreover, due to the orthogonality of  $S$  and  $T$ , from (1.13a) we have

$$\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^T & S_3^T \\ S_2^T & S_4^T \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^T & T_3^T \\ T_2^T & T_4^T \end{bmatrix}.$$

Therefore, using similar arguments as above, we can prove that  $S_2 = 0$  and  $T_3 = 0$ . Consequently, by inserting  $S_3 = T_3 = 0$  and  $S_2 = T_2 = 0$  into (1.14a) and (1.14b) we obtain

$$\tilde{\Sigma} = S_1 \Sigma T_1, \quad (1.16a)$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 (A_1 T_1 - \Sigma \dot{T}_1) & S_1 A_2 T_4 \\ S_4 A_3 T_1 & S_4 A_4 T_4 \end{bmatrix}, \quad (1.16b)$$

where the matrix-valued function  $S_i, T_i$  ( $i = 1, 4$ ) are pointwise orthogonal. Thus, we have

$$\hat{A}_3 = -\tilde{A}_4^{-1} \tilde{A}_3 = -(S_4 A_4 T_4)^{-1} S_4 A_3 T_1 = T_4^{-1} \hat{A}_3 T_1, \quad (1.17a)$$

$$\hat{A}_1 = \tilde{\Sigma}^{-1} (\tilde{A}_1 - \tilde{A}_2 \tilde{A}_4^{-1} \tilde{A}_3) = T_1^{-1} (\hat{A}_1 T_1 - \dot{T}_1). \quad (1.17b)$$

Furthermore, since  $y = T\tilde{y}$ , the structure of  $T$  implies that  $y_1 = T_1 \tilde{y}_1$  and  $y_4 = T_4 \tilde{y}_4$ . Therefore, the underlying ODE (1.13a) is directly obtained from (1.8a) by applying the variable transformation  $\tilde{y}_1(t) = T_1(t)y_1(t)$  and scaling the system with  $T_1^{-1}$ . So, we have that  $\hat{Y}_1(t, s) = T_1^{-1} \hat{Y}_1(t, s) T_1(s)$ . Making use of (1.17), we can deduce the evolution family  $\{\hat{\mathcal{X}}(t, s)\}_{t \geq s \geq 0}$  as follows

$$\hat{\mathcal{X}}(t, s) = \begin{bmatrix} T_1(t) & 0 \\ 0 & T_4(t) \end{bmatrix} \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3 \hat{Y}_1(t, s) & 0 \end{bmatrix} \begin{bmatrix} T_1^T(s) & 0 \\ 0 & T_4^T(s) \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3 \hat{Y}_1(t, s) & 0 \end{bmatrix},$$

and hence, this completes the proof.  $\square$

**Lemma 1.7.** Consider the DAE (1.1) and the evolution family  $(X(t, s))_{t \geq s \geq 0}$  defined by (1.10). Furthermore, we also consider the pointwise-orthogonal matrix-valued functions  $U, V$  defined in Lemma 1.7. Then, the solution to (1.1), if exists, also satisfies the so-called mild equation

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \hat{X}(t, s) \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t, \tau) \begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t) A_2(t) A_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau) f(\tau, x(\tau)) d\tau \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t) f(t, x(t)), \end{aligned}$$

for all  $t \geq s \geq 0$ .

*Proof.* The proof can be obtained directly by using Lemma 1.4. Thus, in order to keep the brevity we will omit the details here.  $\square$

In the following, for ease of notation, we will use the abbreviation  $\hat{X}(t) := \hat{X}(t, 0)$ ,  $\hat{X}^-(t) := \hat{X}^-(t, 0)$ ,  $\hat{Y}(t) := \hat{Y}(t, 0)$  and  $\hat{Y}^-(t) := \hat{Y}^-(t, 0)$ . The concept of exponential dichotomy for the DAE (1.7) is given as below.

**Definition 1.8.** ([20]) The DAE (1.7) is said to have an *exponential dichotomy* if there exist a family of projection matrices  $\{P_y(t)\}_{t \geq 0}$  in  $\mathbb{R}^{d,d}$  and positive constants  $N, \nu$  such that

$$\begin{aligned} \left\| \hat{Y}(t) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0, \\ \left\| \hat{Y}(t) \begin{bmatrix} I_d - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0, \end{aligned} \quad (1.18)$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.10) and (1.18) we have

$$\left\| \hat{X}(t) V^T(0) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0) \hat{X}^-(s) \right\| \leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0,$$

and

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0.$$

In addition, since  $V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$  is also a projection matrix for any  $t \geq 0$ , we can interpret the exponential dichotomy of (1.2) as the one of (1.7).

**1.3. Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces that play a fundamental role in the study of differential equations and refer to Nguyen [13], Massera and Schäffer [23, Chap. 2] and Räbiger and Schnaubelt [27, §1] for various applications.

Let  $E$  (endowed with the norm  $\|\cdot\|_E$ ) be Banach function space of real-valued functions defined as in [13]. We then recall the Banach space corresponding to the space  $E$  as follows.

**Definition 1.9** ([13]). Consider the Banach space  $(\mathbb{R}^n, \|\cdot\|)$ . For a Banach function space  $E$  we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

endowed with the norm  $\|f\|_{\mathcal{E}} := \|f(\cdot)\|_E$ . Thus, one can directly see that  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  is a Banach space. We call it *the Banach space corresponding to  $E$* .

We now introduce the notion of admissibility in the following definition.

**Definition 1.10** ([13]). The Banach function space  $E$  is called *admissible* if for any  $\varphi \in E$  the following conditions hold.

(i) There exists a constant  $M \geq 1$  such that for every compact interval  $[a, b] \subset \mathbb{R}_+$  we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \quad (1.19)$$

where  $\chi_{[a,b]}$  is the indicator function of  $[a, b]$ .

(ii) The function  $\Lambda_1 \varphi$  defined by  $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$  belongs to  $E$ .

(iii) For any  $\tau \geq 0$ , the space  $E$  is  $T_\tau^+$ -invariant and  $T_\tau^-$ -invariant, where  $T_\tau^+$  and  $T_\tau^-$  are defined as

$$T_\tau^+ \varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \quad (1.20)$$

$$T_\tau^- \varphi(t) := \varphi(t+\tau) \text{ for } t \geq 0.$$

Furthermore, there exist constants  $N_1, N_2$  such that  $\|T_\tau^+\|_E \leq N_1, \|T_\tau^-\|_E \leq N_2$  for all  $\tau \in \mathbb{R}_+$ .

**Example 1.11.** Besides the spaces  $L_p(\mathbb{R}_+)$ ,  $1 \leq p \leq \infty$ , and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau < \infty\},$$

(for any fixed  $\alpha > 0$ ), endowed with the norm  $\|h\|_{\mathbf{M}_\alpha} := \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau$ , many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces  $L_{p,q}$ ,  $1 < p < \infty, 1 \leq q < \infty$  (see [4], [31]) and, more general, the class of rearrangement invariant function spaces (see [19]) are admissible.

**Remark 1.12.** Following directly from Definition 1.10 we have that

$$\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau \leq \frac{M}{\inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

and hence,  $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$ . Furthermore,  $C_b(\mathbb{R}^+)$ , the Banach space of bounded, continuous function from  $\mathbb{R}_+$  to  $\mathbb{R}^n$ , is dense in  $\mathbf{M}_1$ .

<sup>161</sup> We present here some important features of admissible spaces in the following proposition (see [13, Propo-  
<sup>162</sup> sition 2.6] and originally in [23, 23.V.(1)]).

**Proposition 1.13** ([13]). Let  $E$  be an admissible Banach function space. Then the following assertions hold.

a) Let  $\varphi \in L_{1,loc}(\mathbb{R}_+)$  such that  $\varphi \geq 0$  and  $\Lambda_1\varphi \in E$ , where,  $\Lambda_1$  is defined as in definition 1.10 (ii). For  $\sigma > 0$  we define functions  $\Lambda'_\sigma\varphi$  and  $\Lambda''_\sigma\varphi$  by

$$\begin{aligned}\Lambda'_\sigma\varphi(t) &:= \int_0^t e^{-\sigma(t-s)}\varphi(s)ds, \\ \Lambda''_\sigma\varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)}\varphi(s)ds.\end{aligned}$$

<sup>163</sup> Then,  $\Lambda'_\sigma\varphi$  and  $\Lambda''_\sigma\varphi$  belong to  $E$ . In particular, if  $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau)d\tau < \infty$  (this will be satisfied if  $\varphi \in E$  (see  
<sup>164</sup> remark 1.12)) then  $\Lambda'_\sigma\varphi$  and  $\Lambda''_\sigma\varphi$  are bounded. Moreover, denoted by  $\|\cdot\|_\infty$  for *ess sup*-norm, we have

$$\|\Lambda'_\sigma\varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma\varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty \quad (1.21)$$

<sup>165</sup> for operator  $T_1^+$  and constants  $N_1, N_2$  defined as in Definition 1.10.

<sup>166</sup> b)  $E$  contains exponentially decaying functions  $\psi(t) = e^{-\alpha t}$  for any constant  $\alpha > 0$ .

<sup>167</sup> c)  $E$  does not contain exponentially growing functions  $f(t) := e^{bt}$  for any constant  $b > 0$ .

## 2. EXPONENTIAL DICHOTOMY FOR LINEAR DAEs

<sup>169</sup> In the qualitative analysis of ODEs, one of the central topics is to find necessary and sufficient conditions  
<sup>170</sup> such that the considered system admits an exponential dichotomy. Many researches have been devoted to  
<sup>171</sup> this topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces  
<sup>172</sup> (e.g. [7, Chap. 4], [33]). For DAEs, the only result that we are aware of is recalled below.

<sup>173</sup> **Proposition 2.1.** ([20]) The DAE (1.7) has an exponential dichotomy if and only if the corresponding  
<sup>174</sup> underlying ODE (1.8a) also has exponential dichotomy and the matrix function  $\hat{A}_3(t)$  is bounded. Moreover,  
<sup>175</sup> the existence of an exponential dichotomy implies that  $\sup_{t \geq 0} \|P_y(t)\| < \infty$ .

<sup>176</sup> Notice that, Proposition 2.1 is only valid for finite-dimensional but it is very hard to generalize for infinite  
<sup>177</sup> dimensional DAE systems. For this reason, we aim at another approach, motivated from one classical result  
<sup>178</sup> stated below.

<sup>179</sup> **Proposition 2.2.** ([6, Chap.3]) The ODE (1.3) has an exponential dichotomy if and only if one of the  
<sup>180</sup> following conditions is satisfied.

<sup>181</sup> i) For any function  $g \in \mathbf{M}_1(\mathbb{R}_+)$  there exists a continuous, bounded solution  $x(t)$  to the inhomogeneous  
<sup>182</sup> system

$$\dot{x}(t) = A(t)x(t) + g(t). \quad (2.1)$$

<sup>183</sup> ii) For any function  $g \in C_\infty(\mathbb{R}_+)$ , there exists a continuous, bounded solution  $x(t)$  to the inhomogeneous  
<sup>184</sup> system (2.1), provided that the ODE (1.3) has bounded growth.

<sup>185</sup> Comparable results to Proposition 2.2 have not been achieved for DAEs, and hence, this will be our main  
<sup>186</sup> aim in this section. Together with (1.2), let us consider the following system

$$E(t)\dot{x}(t) = A(t)x(t) + g(t). \quad (2.2)$$

<sup>187</sup> Since the existence of an exponential dichotomy is a characteristic property of the evolution family  $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$ ,  
<sup>188</sup> which is invariant with respect to orthogonal transformations, we may assume that system (2.2) is already in

189 the semi-explicit form (1.7). The following example shows that Proposition 2.2 could not be directly applied  
 190 to the DAE (2.2).

191 **Example 2.3.** Consider the system (2.2) with  $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} -1 & 0 \\ 0 & e^{-t} \end{bmatrix}$ ,  $f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Clearly,  $f$  is bounded. On  
 192 the other hand, the homogeneous system clearly has an exponential dichotomy. Nevertheless, the explicit  
 193 solution  $x(t) = \begin{bmatrix} e^{-t}x_1(0) \\ e^t \end{bmatrix}$  is unbounded no matter how an initial condition  $x(0)$  is chosen.

194 We define the linear space  $C_b^{sys}(\mathbb{R}_+)$  associated with the system (1.2) as follows.

$$C_b^{sys}(\mathbb{R}_+) := \left\{ g \in C(\mathbb{R}_+) \mid \sup_{t \geq 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} g(t) \right\| \right\} < +\infty \right\}, \quad (2.3)$$

195 *Remark 2.4.* In Theorem 2.6 below we will see that  $C_b^{sys}(\mathbb{R}_+)$  plays the role of  $C_b(\mathbb{R}_+)$  in Proposition 2.1.  
 196 In fact, for the ODE case,  $A_4$  is an empty matrix and  $\Sigma = I_n$ , so  $C_b^{sys}(\mathbb{R}_+)$  coincides with  $C_b(\mathbb{R}_+)$ .

197 **Lemma 2.5.** *The space  $C_b^{sys}(\mathbb{R}_+)$  is invariant with respect to system orthogonal transformations.*

*Proof.* Let us consider two orthogonally equivalent systems

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + g(t), \\ \tilde{E}(t)\dot{\tilde{x}}(t) &= \tilde{A}(t)\tilde{x}(t) + \tilde{g}(t), \end{aligned}$$

198 where  $(E, A, g) \sim (\tilde{E}, \tilde{A}, \tilde{g})$ , and

$$(E, A) = \left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \text{ and } (\tilde{E}, \tilde{A}) = \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.16) hold true for some pointwise orthogonal matrix-valued functions  $S_i, T_i$  ( $i = 1, 4$ ). Thus, we see that

$$\begin{aligned} \begin{bmatrix} \tilde{\Sigma}^{-1} & -\tilde{\Sigma}^{-1}\tilde{A}_2\tilde{A}_4^{-1} \\ 0 & \tilde{A}_4^{-1} \end{bmatrix} \tilde{g} &= \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} \begin{bmatrix} S_1^T & 0 \\ 0 & S_4^T \end{bmatrix} Sg \\ &= \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} g. \end{aligned}$$

199 Since Euclidean norm is preserved under orthogonal transformation, this identity completes the proof.  $\square$

200 The main result of this section is to prove a characterization of the exponential dichotomy for DAEs.  
 201 Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping  $\mathcal{L} := E \frac{d}{dt} - A$   
 202 is surjective on the space  $C_b^{sys}(\mathbb{R}_+)$ . We formulate our main result in this section as follows.

203 **Theorem 2.6.** *Consider the linear, strangeness-free DAE (1.2) and the associated inhomogeneous DAE  
 204 (2.2). Then the following assertions hold.*

- 205 (i) *If the DAE (1.2) admits an exponential dichotomy then for any function  $g \in C_b^{sys}(\mathbb{R}_+)$ , there exists a  
 206 continuous, bounded solution  $x(t)$  to the DAE (2.2).*  
 207 (ii) *If the matrix function  $\hat{A}_3(t)$  is bounded, then the converse of assertion (i) holds true.*

*Proof.* Firstly, we notice that, since  $\hat{g}(t) = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} g(t)$ , so the fact  $g \in C_b^{sys}(\mathbb{R}_+)$  implies the boundedness of  $\hat{g}$ . Recall that the semi-explicit system (1.7) reads

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{g}_1(t), \quad (2.4a)$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{g}_2(t). \quad (2.4b)$$

(i) Assuming that the DAE (1.2) admits an exponential dichotomy, then (1.7) also has an exponential dichotomy. Proposition 2.1 implies that equation (2.4a) has an exponential dichotomy, and the function  $\hat{A}_3$  is bounded. Therefore, Proposition 2.2 implies that  $y_1$  is bounded, and consequently,  $y_2$  is also bounded.

(ii) Notice that the mapping

$$g \mapsto \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -A_4^{-1} \end{bmatrix} g$$

is a surjection from  $C_b^{sys}(\mathbb{R}_+)$  to  $C_b(\mathbb{R}_+, \mathbb{R}^n)$ , so  $g(t)$  can be freely chosen in the space  $C_b(\mathbb{R}_+, \mathbb{R}^n)$ . Proposition 2.2 applied to system (2.4a) follows that (2.4a) has exponential dichotomy. If, in addition, the boundedness of  $\hat{A}_3$  is presumed then Proposition 2.1 implies that system (1.2) admits exponential dichotomy.  $\square$

### 3. LOCAL STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout this section we assume the following.

**Assumption 3.1.** Assume that the evolution family  $(X(t, s))_{t \geq s \geq 0}$  associated with the linear, homogeneous DAE (1.2) admits an exponential dichotomy on  $\mathbb{R}_+$  with the dichotomy constants  $N, \nu > 0$  as in Definition 1.8.

Due to Proposition 2.1, let us denote by  $H_1 := \sup_{t \geq 0} \|\hat{A}_3(t)\|$  and  $H_2 := \sup_{t \geq 0} \|P_y(t)\|$ . In the following example, we illustrate the sensitivity of an exponential dichotomy and a *constraint manifold*

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \mathbf{A}_2(t)x(t) = 0\}. \quad (3.1)$$

with respect to small disturbance  $f(t, x)$ .

**Example 3.2.** Consider the disturbed DAE with the parameter function  $\alpha(t)$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & \alpha(t) \\ 0 & 0 & 1 \end{bmatrix} x(t) + f(t, x) \quad \text{for all } t \geq 0. \quad (3.2)$$

Clearly, we see that the homogeneous system has an exponential dichotomy with the projection  $P_x(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ . Moreover, every solution to the homogeneous system must remain on the constraint manifold

$$\mathbb{L}(t, x) = \{x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \mid x_3 = 0\}.$$

For an arbitrarily large  $t_1 > 0$ , we consider the disturbance  $f(t, x) = [0 \ 0 \ 0]^T$  for  $t < t_1$  and  $f(t, x) = [0 \ 0 \ \frac{x_2}{t}]^T$  for  $t \geq t_1$ . Thus, for  $t \geq t_1$  system (3.2) becomes

$$\dot{x}_1 = -2x_1, \quad (3.3a)$$

$$\dot{x}_2 + t\dot{x}_3 = 2x_2 + \alpha(t)x_3, \quad (3.3b)$$

$$0 = \frac{1}{t}x_2 + x_3. \quad (3.3c)$$

Scaling equation (3.3c) with  $t$ , and differentiating it, we obtain  $\dot{x}_2 + t\dot{x}_3 + x_3 = 0$ . Then, substituting  $\dot{x}_2 + t\dot{x}_3 = -x_3$  into (3.3b) we achieve the *hidden constraint*  $0 = 2x_2 + (1 + \alpha(t))x_3$ . Consequently, the disturbance  $f(t, x)$  above has drastically changed both the dynamic and the constraint of the homogeneous system. Now the constraint manifold is

$$\mathbb{L}(t, x) = \{x = [x_1 \ x_2 \ x_3]^T \in \mathbb{R}^3 \mid [\begin{smallmatrix} 2 & 1+\alpha(t) \\ 1 & t \end{smallmatrix}] [\begin{smallmatrix} x_2 \\ x_3 \end{smallmatrix}] = 0\} \quad \text{for all } t \geq t_1,$$

and the dynamic part involves only  $x_1$  which is exponential stable. In the worst case, if  $\alpha(t) = 2t - 1$  then system (3.2) is not even uniquely solvable.

233 *Remark 3.3.* Example 3.2 demonstrates one fact different to the ODE theory, that even though the dis-  
 234 turbance  $f(t, x)$  satisfies the Lipschitz property with an arbitrarily small Lipschitz constant (since  $t_1$  can  
 235 be chosen to be arbitrarily large), the dynamic and constraints of the system may be drastically altered.  
 236 Therefore, to study the existence of invariant manifolds for (1.1), it is necessary to presume some additional  
 237 assumptions, in order to guarantee, that both the dynamic and the constraint manifold do not change too  
 238 much with respect to disturbance.

As in Lemma 1.4, by using orthogonal transformation  $x(t) = V(t)y(t)$  and scaling system (1.1) with  
 $U(t) = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$ , where  $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$ , we derive the coupled system

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), \quad (3.4a)$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), \quad (3.4b)$$

where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -A_4^{-1} \end{bmatrix} \begin{bmatrix} U_1(t)f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}.$$

239 Notice that equation (3.4b) only gives an implicit algebraic constraint in terms of  $y_1$  and  $y_2$ .

240 **Assumption 3.4.** Assume that for some  $\rho > 0$ , the function  $A_4^{-1}(t)f_2(t, x)$  is a contraction mapping in the  
 241 ball  $B_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$  (uniformly in time), i.e.,

$$\|A_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

242 for a.e.  $t \in \mathbb{R}_+$ , and for all  $x, \tilde{x} \in B_\rho$  where the Lipschitz constant  $L$  satisfies that  $L < 1$ .

243 **Lemma 3.5.** Let  $\rho_0 := \sqrt[2]{1 + \left(\frac{H_1+L}{1-L}\right)^2} \rho$  and  $\bar{\rho}_0 := \frac{H_1+L}{1-L} \rho_0$ . Under Assumption 3.4 and any given  
 244  $y_1 \in B_{\rho_0}$ , there exists a unique function  $y_2 \in B_{\bar{\rho}_0}$  satisfying (3.4b). Consequently,  $y \in B_\rho$ .

245 *Proof.* Firstly, notice that Assumption 3.4 implies that

$$\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| = \| - A_4^{-1}(f_2(t, V y) - f_2(t, V \tilde{y})) \| \leq L\|V y - V \tilde{y}\| = L\|y - \tilde{y}\|.$$

246 Thus,  $\hat{f}_2(t, y)$  is also Lipschitz in  $y$  with the same constant  $L$ . Then, the desired claim is obtained directly  
 247 by making use of [24, Lem. 2.7].  $\square$

248 We also need the following property of the nonlinear part  $f_1$  defined as follows.

249 **Definition 3.6.** Let  $\varphi$  be a positive function belonging to an admissible Banach function space  $E$ . A  
 250 function  $h : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to belong to the class  $(M, \varphi, \rho)$  for some positive constant  $M$ ,  $\rho$  if  $h$   
 251 satisfies

- 252 (i)  $\|h(t, x)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$  and for all  $x \in B_\rho$ ,
- 253 (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$ , for all  $x, \tilde{x} \in B_\rho$ .

254 **Assumption 3.7.** Assume that the function  $t \mapsto \Sigma^{-1}(t)[I_d \ -A_2(t)A_4^{-1}(t)]f(t, x(t))$  belongs to class  
 255  $(M, \varphi, \rho)$  for some positive constants  $M$ ,  $\rho$  and a positive function  $\varphi \in E$ .

256 The following lemma shows that Assumptions 3.4, 3.7 are invariant with respect to system orthogonal  
 257 transformations.

258 **Lemma 3.8.** Assumptions 3.4, 3.7 are also invariant with respect to system orthogonal transformations.

259 *Proof.* Let us consider two orthogonally equivalent systems

$$\left( \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \sim \left( \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.16) hold true for some pointwise orthogonal matrix-valued function  $S_i, T_i$  ( $i = 1, 4$ ). Therefore, we have that

$$\|\tilde{A}_4^{-1}(t)(\tilde{f}_2(t, x) - \tilde{f}_2(t, \tilde{x}))\| = \|T_4^{-1}(t)A_4^{-1}(t)S_4^{-1}(t)S_4(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

260 On the other hand, we have that

$$\tilde{\Sigma}^{-1} [I_d \quad -\tilde{A}_2 \tilde{A}_4^{-1}] = T_1 \Sigma^{-1} [I_d \quad -A_2 A_4^{-1}] \begin{bmatrix} S_1 & 0 \\ 0 & S_4 \end{bmatrix}^T,$$

261 and hence

$$\tilde{\Sigma}^{-1} [I_d \quad -\tilde{A}_2 \tilde{A}_4^{-1}] (\tilde{f}_2(t, x) - \tilde{f}_2(t, \tilde{x})) = T_1 \Sigma^{-1} [I_d \quad -A_2 A_4^{-1}] (f(t, x) - f(t, \tilde{x})).$$

262 Then, due to the orthogonality of  $T_1, T_4, S_1, S_4$ , the desired claims are directly followed.  $\square$

263 The following lemma gives one sufficient condition for Assumptions 3.4, 3.7.

264 **Proposition 3.9.** Consider the semi-linear DAE (1.1). Then, the following claims hold true.

- 265 i) Assume that the function  $\|\Sigma^{-1} [I_d \quad -A_2 A_4^{-1}]\|$  is bounded by  $\mathbf{M}$ . If  $f$  belongs to the class  $(M, \frac{\varphi}{\mathbf{M}}, \rho)$   
266 then  $\hat{f}_1$  belongs to the class  $(M, \varphi, \rho)$ .  
267 ii) Assumption (3.4) holds true of  $f(t, x)$  satisfies that inequality

$$\|f_2(t, x) - f_2(t, \tilde{x})\| \leq \frac{L}{\|A_4^{-1}(t)\|} \|x - \tilde{x}\|. \quad (3.5)$$

268 For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.4a)-  
269 (3.4b). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (3.6)$$

270 for all  $t \geq s \geq 0$ .

271 **Lemma 3.10.** Let Assumptions 3.4 and 3.7 hold true. Then, for all  $y, \tilde{y} \in B_\rho$  the following assertions hold.

- 272 (i)  $\|\hat{f}_1(t, y)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$ ,  
273 (ii)  $\|\hat{f}_1(t, y) - \hat{f}_1(t, \tilde{y})\| \leq \varphi(t)\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ ,  
274 (iii)  $\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| \leq L\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ .

275 *Proof.* The proof is trivially followed from Assumptions 3.4 and 3.7 due to the fact that  $\|y\| = \|Qy\|$  for any  
276 orthogonal matrix  $V$ .  $\square$

277 Under Assumption 3.1, the family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  also has an exponential dichotomy with the corresponding  
278 projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the same dichotomy constants  $N, \nu > 0$ . Thus, we can define the Green  
279 function on the half-line as follows

$$G(t, \tau) := \begin{cases} \hat{Y}(t, \tau) \begin{bmatrix} P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)P_y(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, \tau)P_y(\tau) & 0 \end{bmatrix}, & \text{for all } t \geq \tau \geq 0, \\ -\hat{Y}(t, \tau) \begin{bmatrix} I_d - P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \end{bmatrix}, & \text{for all } 0 \leq t < \tau. \end{cases} \quad (3.7)$$

280 Then, we have

$$\|G(t, \tau)\| \leq (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geq 0. \quad (3.8)$$

281 In the following lemma, we give an explicit form for bounded solutions to system (3.6).

282 **Lemma 3.11.** *Assume that Assumptions 3.1, 3.4, 3.7 hold true. Let  $y(t)$  be any solution to (3.6) such that  
283  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$  for fixed  $t_0 \geq 0$  and some  $\rho > 0$ . Then, for  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form  
284*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (3.9)$$

285 for some  $v_0 \in \text{Im}P_y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.7).

*Proof.* Put

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}.$$

286 By direct computation, we can verify that  $z$  satisfies the integral equation

$$z(t) = \hat{Y}(t, t_0) \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix},$$

287 for all  $t \geq t_0$ . Now let us estimate  $\|z(t)\|$ . Making use of Lemma 3.10 and (3.8), we see that

$$\|z(t)\| \leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} M\varphi(\tau) d\tau + L\rho,$$

288 and then, from (1.21) it follows that

$$\|z(t)\| \leq M (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho,$$

289 for all  $t \geq t_0$ . Thus,  $z(t) - y(t)$  is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t, t_0) (z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t) \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

290 we see that  $v_0 := z_1(t_0) - y_1(t_0) \in \text{Im}P_y(t_0)$ . Finally, since  $z(t) = y(t) + \hat{Y}(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$  for all  $t \geq t_0$ , equality  
291 (3.9) follows.  $\square$

292 *Remark 3.12.* By computing directly, we can see that the converse of Lemma 3.11 is also true. It means,  
293 that all solutions to (3.9) also satisfy equation (3.6) for all  $t \geq t_0$ .

294 Let us denote by

$$H_3 := (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)} \rho. \quad (3.10)$$

295 **Lemma 3.13.** *Under the assumptions of Lemma 3.11, let  $y(t)$ ,  $\tilde{y}(t)$  be any two functions lying in the ball  
296  $B_\rho$  and satisfy (3.9) for  $v_0, \tilde{v}_0 \in \text{Im}P_y(t_0)$ . If  $H_3$  defined as in (3.10) satisfies  $H_3 + L < 1$  then the following  
297 estimate holds true:*

$$\|y - \tilde{y}\|_\infty \leq \frac{N}{1 - H_3 - L} \|v_0 - \tilde{v}_0\|. \quad (3.11)$$

*Proof.* Using the same arguments as in the proof of Lemma 3.10, we see that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq N\|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (H_3 + L) \|y - \tilde{y}\|_\infty, \end{aligned}$$

which directly implies (3.11).  $\square$

In the following theorem, we exploit the local structure of bounded solutions to (3.6).

**Theorem 3.14.** *Consider the semi-explicit (1.1) and the transformed system (3.4). Furthermore, assume that Assumptions 3.1, 3.4, 3.7 hold true, and constant  $H_3$  defined as in (3.10). Then, the following assertions hold true.*

(i) If

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)\rho}{2M} \right\}, \quad (3.12)$$

then there corresponds to each  $v_0 \in B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$  one and only one solution  $y(t)$  to (3.6) on  $[t_0, \infty)$  satisfying  $P_y(t_0)y_1(t_0) = v_0$  and  $\text{esssup}_{t \geq t_0} \|y(t)\| \leq \rho$ .

(ii) Moreover, any two solutions  $y(t), \tilde{y}(t)$  corresponding to different  $v_0, \tilde{v}_0$  in  $B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$  attract each other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0, \quad (3.13)$$

for some positive constants  $H_4, \mu$ .

*Proof.* (i) Consider in the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  the ball  $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) : \|y(\cdot)\|_\infty := \text{esssup}_{t \geq 0} \|y(t)\| \leq \rho\}$ .

For each fixed  $v_0 \in B_{\tilde{\rho}}$  we will prove the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix} & \text{for all } t \geq t_0, \\ 0 & \text{for all } t < t_0, \end{cases} \quad (3.14)$$

is a contraction mapping from  $\mathcal{B}_\rho$  to itself. Using the same argument as in the proof of Lemma 3.10, we see that

$$\begin{aligned} \|(Ty)(t)\| &\leq (1 + H_1)Ne^{-\nu(t-t_0)} \|v_0\| + M(1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho, \\ &\leq (1 + H_1)N\|v_0\| + MH_3 + L\rho \quad \text{for all } t \geq 0, \end{aligned}$$

and by (3.12) we see that

$$\|(Ty)(t)\| \leq (1 + H_1)N\tilde{\rho} + \frac{(1 - L)\rho}{2} + L\rho = \rho \quad \text{for all } t \geq 0.$$

Therefore,  $T$  is a mapping from  $\mathcal{B}_\rho$  to itself. Now we prove its contraction property. Indeed, making use of (3.8), we obtain the following estimate:

$$\begin{aligned} \|Ty(t) - T\tilde{y}(t)\| &\leq \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L\|y(t) - \tilde{y}(t)\|, \\ &\leq (H_3 + L) \|y(\cdot) - \tilde{y}(\cdot)\|_\infty \quad \text{for all } t \geq 0. \end{aligned}$$

314 Consequently, due to (3.12), we see that  $T$  is a contraction mapping with the contraction constant  $H_3 + L$ .  
 315 Thus, there exist a unique function  $y \in \mathcal{B}_\rho$  such that  $y = Ty$ , and hence, due to the definition of  $T$ ,  $y$  is the  
 316 solution to the mild/integral-algebraic system (3.6).

(ii) The proof of the estimate (3.13) can be done in a similar way as in [14, Thm 3.7]. We present here  
 for seek of completeness. Let  $y(t)$  and  $\tilde{y}(t)$  be two essentially bounded solutions of (3.6) corresponding to  
 different values  $v_0, \tilde{v}_0 \in B_{\bar{\rho}} \cap \text{Im}P_y(t_0)$ . Then, we have that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq Y(t, t_0) \|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq (1 + H_1) Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \end{aligned}$$

317 and hence,

$$\|y(t) - \tilde{y}(t)\| \leq \frac{1 + H_1}{1 - L} Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty} \frac{(1 + H_1)(1 + H_2)}{1 - L} Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau.$$

Then, due to the Cone Inequality, [7, Theorem 1.9.3], in analogue to [25, Theorem 3.7], we obtain the estimation (3.13) with  $H_4, \mu$  are given by

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu-\nu}}}.$$

318 Furthermore, notice that from (3.12) it follows that  $\mu < \nu$  implying the positivity of  $H_4$ . This completes the  
 319 proof.  $\square$

320 Let

$$\mathbb{L}(t, y) := \{(t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t, y_1, y_2)\}, \quad (3.15)$$

321 be the constrained manifold, which all solutions to (3.4a)-(3.4b) must belong to. We further notice that this  
 322 manifold is of dimension  $d$ , which is the degree of freedom to the DAE (3.6). Now, we are able to introduce  
 323 the concept of a local stable manifold for the solutions of the integral-algebraic system (3.6).

**Definition 3.15.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *local stable manifold* for  
 solutions to (3.6) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such  
 that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf \{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exist positive constants  $\rho, \rho_1, \rho_2$  and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t), \quad t \in \mathbb{R}_+,$$

324 with a common Lipschitz constant independent of  $t$  such that

- 325 (i)  $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$ , and we denote by  
 $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\},$
- 326 (ii)  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$  for all  $t \geq 0$ ,
- 327 (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.6) satisfying  $y_1(t_0) = \tilde{w}$  and  
 $ess\sup_{t \geq t_0} \|y(t)\| \leq \rho.$

330 We now state and prove our main result on the existence of a local stable manifold for DAEs.

**Theorem 3.16.** Consider the semi-linear DAE (1.1) and the semi-explicit system (3.4). Furthermore, assume that Assumptions 3.1, 3.4, 3.7 hold true. If

$$H_3 < \min \left\{ 1 - L, \frac{(1-L)(1+H_1)\rho}{2M}, \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)} \right\},$$

then there exists a local stable manifold for the solutions of (3.6). Moreover, every two solutions  $y(t)$ ,  $\tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\|, \quad \text{for all } t \geq t_0. \quad (3.16)$$

*Proof.* First we notice that the phase subspace  $\mathbb{R}^d$  splits into the direct sum  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel}P_y(t)$  for all  $t \geq 0$ . We set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{kernel}P_y(t)$ , then due to Proposition 2.1, we see that  $\sup_{t \geq 0} \|P_y(t)\| < \infty$ , and hence,  $\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$ .

For any  $\rho > 0$  defined as in Assumptions 3.4, 3.7, let  $\rho_1 := \frac{1-L}{2N(1+H_1)}\rho$  and  $\rho_2 := \frac{(1-L)\rho}{2}$ . For each  $t \geq 0$  we define the mapping  $g_t$  acting on  $B_{\rho_1} \cap W_1(t)$  as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function  $y(t)$  is uniquely defined via Theorem 3.14 i). Clearly,  $g_t(w_1) \in \text{ker}P_y(t) = W_2(t)$ .

Now, we prove that  $\|g_t(w_1)\| \leq \rho_2$ . Due to Theorem 3.14 (i) and Lemma 3.10 (i), we have that  $\|y(t)\| \leq \rho$  and  $\|f_1(\tau, y(\tau))\| \leq M\varphi(\tau)$  for a.e.  $t \geq 0$ . Therefore,

$$\begin{aligned} \|g_t(w_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} M\varphi(\tau) d\tau, \\ &\leq M (1+H_2) \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) = \frac{MH_3}{1+H_1} \leq \frac{(1-L)\rho}{2}, \end{aligned}$$

and hence,  $g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t)$ .

Notice that both part (iii) in Definition 3.15 and estimation (3.16) are followed directly from Theorem 3.14. We now only need to prove that  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$ . We first prove that  $g_t$  is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

and hence, (3.11) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

Finally,  $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$  yields that  $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$ , and hence,  $g_t$  is a contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous mappings ([24, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow B_{\rho_1} \cap W_1(t)$  is a homeomorphism. This implies the condition (ii) of Definition 3.15 finishing the proof.  $\square$

## 350 4. GLOBAL INVARIANT STABLE MANIFOLDS FOR SEMI-LINEAR DAES

351 In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1).  
 352 We begin with the concept of  $\varphi$ -Lipschitz functions.

353 **Definition 4.1.** Let  $E$  be an admissible Banach function space and  $\varphi \in E$  be a positive function. A function  
 354  $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^d$  is said to be  $\varphi$ -Lipschitz if the following conditions hold true.

- 355 (i)  $\|h(t, 0)\| = 0$  for a.e.  $t \in \mathbb{R}_+$ ,
- 356 (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$  and all  $x, \tilde{x} \in \mathbb{R}^n$ .

357 In comparability to Assumptions 3.4, 3.7, we also need some global properties of the nonlinear term  $f$ .

358 **Assumption 4.2.** Assume that the following hypotheses hold true.

- 359 (i) The function  $t \mapsto \Sigma^{-1}(t)[I_d - A_2(t)A_4^{-1}(t)]f(t, x(t))$  is  $\varphi$ -Lipschitz.
- (ii) The function  $A_4^{-1}(t)f_2(t, x(t))$  is a contraction mapping with the Lipschitz constant  $L < 1$  for all  
 $(t, x(t))$  lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid \mathbf{A}_2(t)x + f_2(t, x) = 0\}.$$

360 In analogous to Lemma 3.8, we can directly verify that Assumption 4.2 is invariant with respect to  
 361 system orthogonal transformations. Hence, function  $\hat{f}_1$  in (3.4a) is  $\varphi$ -Lipschitz, and function  $\hat{f}_2$  in (3.4b) is  
 362 a contraction mapping with the Lipschitz constant  $L < 1$ . For notational simplicity, now we will study the  
 363 semi-explicit system (1.7) and the integral-algebraic system (3.6).

**Definition 4.3.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *global, invariant stable manifold* for solutions to (3.6) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \rightarrow W_2(t), \quad t \in \mathbb{R}_+,$$

364 with the Lipschitz constants independent of  $t$  such that

- 365 (i)  $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$ , and we denote by  
 $\mathbb{M}_t := \{(y_1, y_2) \mid (t, y_1, y_2) \in \mathbb{M}\},$
- 366 (ii)  $\mathbb{M}_t$  is homeomorphic to  $W_1(t)$  for all  $t \geq 0$ ,
- 367 (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.6) satisfying  $y_1(t_0) = \tilde{w}$  and  
 $ess\sup_{t \geq t_0} \|y(t)\| < \infty$ ,
- 368 (iv)  $\mathbb{M}$  is invariant under system (3.6), i.e., if  $y$  is a solution to (3.6), and  $ess\sup_{t \geq t_0} \|y(t)\| < \infty$ , then  
 $y(s) \in \mathbb{M}_s$  for all  $s \geq t_0$ .

372 Analogously to Lemma 3.11, we give the explicit form of bounded solutions to system (3.6) as below.

373 **Lemma 4.4.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the  
 374 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
 375 that Assumption 4.2 holds true. Let  $y(t)$  be any solution to (3.6) such that  $ess\sup_{t \geq t_0} \|y(t)\| < \infty$  for a fixed  
 376  $t_0 \geq 0$ . Then, for all  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (4.1)$$

377 for some  $v_0 \in \text{Im } P_y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.7).

378 *Proof.* The proof can be done by using similar arguments as in the proof of Lemma 3.5.  $\square$

379 In the following two theorems, we present the global versions of Theorems 3.14 and 3.16, where we  
380 construct the structure of bounded solutions to (3.6) and prove the existence of a global, stable manifold,  
381 respectively.

382 **Theorem 4.5.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the  
383 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
384 that Assumption 4.2 holds true.*

- 385 (i) *For any fixed  $t_0 \geq 0$ , if  $H_3 < 1 - L$  then there corresponds to each  $v_0 \in \text{Im}P_y(t_0)$  one and only one  
386 solution  $y(t)$  to (3.6) on  $[t_0, \infty)$  satisfying  $P_y(t_0)y_1(t_0) = v_0$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ .*  
(ii) *Any two solutions  $y(t), \tilde{y}(t)$  corresponding to different initial conditions  $v_0, \tilde{v}_0$  in  $\text{Im}P_y(t_0)$ , are exponentially attracted to each other, i.e.,*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0,$$

with some positive constants  $H_4, \mu$  satisfying

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu-\nu}}}.$$

*Proof.* The proof of this theorem is essentially the same as the proof of Theorem 3.14. The only change is, that instead of considering the ball  $B_\rho$  we will work with the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  itself. Then, we can prove (without any difficulty) that for each fixed  $v_0 \in \text{Im}P_y(t_0)$ , the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geq t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

387 is a contraction mapping, and therefore, all the assertions of the theorem follows.  $\square$

388 **Theorem 4.6.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the  
389 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
390 that Assumption 4.2 holds true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\}, \quad (4.2)$$

*then there exists a global invariant stable manifold for the solutions of (3.6). Moreover, every two solutions  $y(t), \tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\| \quad \text{for all } t \geq t_0.$$

391 *Proof.* Analogous to the proof of Theorem 3.16, we consider the decomposition  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel } P_y(t)$   
392 and set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{kernel } P_y(t)$ . Thus, we see that  $\inf_{t \in \mathbb{R}_+} \text{Sn}(W_1(t), W_2(t)) > 0$ .

393 Now we define the family of mappings  $(g_t)_{t \geq 0}$  acting on  $W_1$  as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function  $y(t)$  is bounded and be uniquely defined via Theorem 4.5 i). Clearly,  $g_t(w_1) \in \text{ker } P_y(t) = W_2(t)$ . To verify the Lipschitz property of  $g_t$ , let us consider two arbitrary elements  $w_1$  and  $\tilde{w}_1$  in  $W_1$  and

let  $y$  and  $\tilde{y}$  be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau , \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty , \end{aligned}$$

394 and hence, (3.11) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

395 Finally,  $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$  yields that  $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$ , and hence,  $g_t$  is a  
396 contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous  
397 mapping ([24, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow W_1(t)$  is a homeomorphism. This  
398 implies the condition ii) of Definition 3.15, and hence, the proof is finished.  $\square$

399 Now let us illustrate our results by the following examples.

400 **Example 4.7.** The dynamical behavior of a system in fluid mechanics and turbulence modeling is often  
401 described by the incompressible Navier-Stokes equation on an open, bounded domain  $\Omega \subset \mathbb{R}^k$ ,  $k = 2$  or  $3$ ,  
402 of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p - (u \cdot \nabla) u + f(t, u, p), \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned}$$

where  $\nu > 0$  is the viscosity,  $u = u(t, \xi)$  is the velocity field which is a function of the time  $t$  and the position  $\xi$ ,  $p$  is the pressure,  $f$  is the external force or disturbance. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [12], one obtains a differential-algebraic system of the following form.

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} K & -C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} + \begin{bmatrix} F(t, U, P) \\ 0 \end{bmatrix},$$

403 where  $U(t)$ ,  $P(t)$  approximate the velocity  $u(t, \xi)$  and the pressure  $p(t, \xi)$ , respectively. Here the leading  
404 matrix  $M$  is either an identity matrix or a symmetric positive definite matrix depending on the spatial  
405 discretization scheme,  $K$  and  $C$  are constant matrices of appropriate dimensions. Furthermore, in many  
406 applications, the matrix  $C^T M^{-1} C$  is nonsingular. We notice, that this system also occurs in many fluid  
407 dynamic models, see e.g. [1, 11]. It is well-known, that the differentiation index of this system is two, and  
408 hence, it is not strangeness-free, so Assumption 1.2 is violated. Nevertheless, we can transform it first in  
409 order to obtain a strangeness-free DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} K & -C \\ C^T M^{-1} K & -C^T M^{-1} C \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} + \begin{bmatrix} F(t, U, P) \\ C^T M^{-1} F(t, U, P) \end{bmatrix}, \quad (4.3)$$

410 Fortunately, we do not need any system transformation, since (4.3) is already in the semi-explicit form. Since  
411  $M$ ,  $K$ ,  $C$  are constant matrices, due to Proposition 2.1, the homogeneous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} K & -C \\ C^T M^{-1} K & -C^T M^{-1} C \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

412 admits an exponential dichotomy if and only if the matrix  $\hat{A}_1 := M^{-1} \left( I - C (C^T M^{-1} C)^{-1} C^T M^{-1} K \right)$  is  
 413 hyperbolic. If, in addition,  $F(t, U, P)$  satisfies the  $\frac{\varphi}{M}$ -Lipschitz condition, where

$$\mathbf{M} := \| M^{-1} \left( I - C (C^T M^{-1} C)^{-1} C^T M^{-1} \right) \|$$

414 and  $F(t, U, P)$  is a contraction mapping with the Lipschitz constant  $L < \frac{1}{\| (C^T M^{-1} C)^{-1} C^T M^{-1} \|}$ , then  
 415 Theorem 4.6 and Proposition 3.9 imply that there exists a stable manifold for the solution to (4.3), whenever  
 416 the function  $\varphi$  satisfies the condition (4.2).

417 **Example 4.8.** Motivated from [28], we consider the nonlinear electrical circuit with Josephson junction in  
 418 Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors  
 419 separated by an oxide barrier, is characterized by the sinusoidal relation  $i_2 = I_0 \sin(k\phi_2)$ , where  $I_0$  and  $k$  are  
 420 positive constants depend on the device itself. Moreover, the resistance  $R$ , inductance  $L$  and conductance  
 421  $G$  are positive. Furthermore,  $i_1$  is the current going through the inductance,  $v_1$  and  $v_2$  are voltage drops  
 across the inductance and the Josephson junction, respectively. As shown in [28], for the linear devices (R,

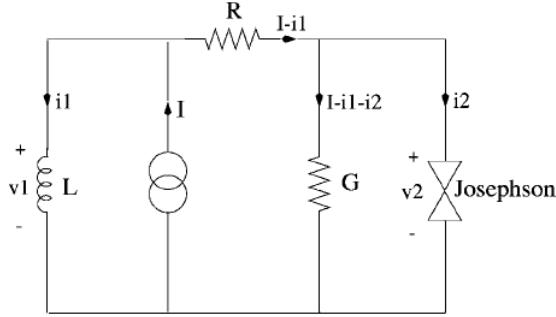


FIGURE 1. Electric circuit with Josephson junction, [28]

422  
 423  $L, C$ ), we obtain the following system, which completely describes the behavior of this circuit.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{L} & I_0 k & G \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ v_2 \end{bmatrix} \quad (4.4)$$

424 This system posses an exponential dichotomy for any  $k < 0$ . In case of nonlinear devices, we obtain a  
 425 nonlinear system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{R}{L} & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{L} & I_0 k & G \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} f_1(t, \phi_1) \\ 0 \\ f_2(t, \phi_1, v_2) \end{bmatrix}. \quad (4.5)$$

426 Fortunately, we do not need any system transformation, since (4.3) is already in the semi-explicit form. In  
 427 this case, we have that  $A_4 = 1$ ,  $[I - A_2 A_4^{-1}] = \begin{bmatrix} 1 & 0 & -1/G \\ 0 & 1 & -1/G \end{bmatrix}$ , and hence for any  $\varphi$ -Lipschitz function  
 428  $f_1$  and any contraction mapping  $f_2$ , we obtain a stable manifold for (4.5) whenever the condition (4.2) is  
 429 satisfied.

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## REFERENCES

- [1] R. Altmann and J. Heiland. *Continuous, Semi-discrete, and Fully Discretised Navier-Stokes Equations*, pages 277–312. Springer International Publishing, Cham, 2019.
- [2] Thomas Berger. Bohl exponent for time-varying linear differential-algebraic equations. *International Journal of Control*, 85(10):1433–1451, 2012.
- [3] Lorenz T. Biegler, Stephen L. Campbell, and Volker Mehrmann, editors. *Control and Optimization with Differential-Algebraic Constraints*. SIAM Publications, Philadelphia, PA, 2012.
- [4] C. Chicone. *Ordinary Differential Equations with Applications*. Texts in Applied Mathematics. Springer New York, 2013.
- [5] Chuan-Jen Chyan, Nguyen Huu Du, and Vu Hoang Linh. On data-dependence of exponential stability and stability radii for linear time-varying differential-algebraic systems. *Journal of Differential Equations*, 245(8):2078 – 2102, 2008.
- [6] W. A. Coppel. *Dichotomies in Stability Theory*. Springer-Verlag, New York, NY, 1978.
- [7] J.L. Daleckii and M.G. Krein. *Stability of Solutions of Differential Equations in Banach Space*. Translations of mathematical monographs. American Mathematical Society, 2002.
- [8] L. Dieci and T. Eirola. On smooth decompositions of matrices. *SIAM J. Matr. Anal. Appl.*, 20:800–819, 1999.
- [9] N. H. Du, V. H. Linh, and N. T. T. Nga. On stability and bohl exponent of linear singular systems of difference equations with variable coefficients. *Journal of Difference Equations and Applications*, 22(9):1350–1377, 2016.
- [10] Nguyen Huu Du and Vu Hoang Linh. Stability radii for linear time-varying differential-algebraic equations with respect to dynamic perturbations. *Journal of Differential Equations*, 230(2):579 – 599, 2006.
- [11] Etienne Emmrich and Volker Mehrmann. Operator differential-algebraic equations arising in fluid dynamics. *Computational Methods in Applied Mathematics*, 13(4):443–470, 2013.
- [12] C. Grossmann, H.G. Roos, and M. Stynes. *Numerical Treatment of Partial Differential Equations*. Springer-Verlag Berlin Heidelberg, 2007.
- [13] Nguyen Thieu Huy. Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line. *Journal of Functional Analysis*, 235(1):330 – 354, 2006.
- [14] Nguyen Thieu Huy. Invariant manifolds of admissible classes for semi-linear evolution equations. *Journal of Differential Equations*, 246(5):1820 – 1844, 2009.
- [15] A. Ilchmann and T. Reis. *Surveys in Differential-Algebraic Equations I*. Differential-Algebraic Equations Forum. Springer, 2013.
- [16] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations – Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.
- [17] R. Lamour, R. März, and C. Tischendorf. *Differential-algebraic equations: A projector based analysis*. Differential-Algebraic Equations Forum 1. Berlin: Springer, 2013.
- [18] M. Lentini and R. Marz. Conditioning and dichotomy in differential algebraic equations. *SIAM Journal on Numerical Analysis*, 27(6):1519–1526, 1990.
- [19] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces II: Function Spaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Berlin Heidelberg, 2013.
- [20] Vu Hoang Linh and Volker Mehrmann. Lyapunov, Bohl and Sacker-Sell spectral intervals for differential-algebraic equations. *Journal of Dynamics and Differential Equations*, 21(1):153–194, Mar 2009.
- [21] Pierangelo Masarati. Estimation of lyapunov exponents from multibody dynamics in differential-algebraic form. *Proceedings of the Institution of Mechanical Engineers, Part K: Journal of Multi-body Dynamics*, 227(1):23–33, 2013.
- [22] Pierangelo Masarati and Aykut Tamer. Sensitivity of trajectory stability estimated by lyapunov characteristic exponents. *Aerospace Science and Technology*, 47:501–510, 2015.

476

- [23] J.L. Massera and J.J. Schäffer. *Linear differential equations and function spaces*. Pure and applied mathematics. Academic Press, 1966.
- [24] Nguyen Van Minh and Jianhong Wu. Invariant manifolds of partial functional differential equations. *Journal of Differential Equations*, 198(2):381 – 421, 2004.
- [25] Thieu Huy Nguyen. Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line. *Journal of Mathematical Analysis and Applications*, 354(1):372 – 386, 2009.
- [26] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Applied Mathematical Sciences. Springer New York, 2012.
- [27] Frank Räbiger and Roland Schnaubelt. The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions. *Semigroup Forum*, 52(1):225–239, Dec 1996.
- [28] R. Riaza. A matrix pencil approach to the local stability analysis of non-linear circuits. *Internat. J. Circ. Theor. Appl.*, 32:23–46, 2004.
- [29] Diagana T. *Semilinear Evolution Equations and Their Applications*. Springer, Cham, 2018.
- [30] Do Duc Thuan, Khong Chi Nguyen, Nguyen Thu Ha, and Pham Van Quoc. On stability, bohl exponent and bohl–perron theorem for implicit dynamic equations. *International Journal of Control*, 0(0):1–13, 2020.
- [31] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. Carnegie-Rochester Conference Series on Public Policy. North-Holland Publishing Company, 1978.
- [32] Sascha Trostorff and Marcus Waurick. On differential–algebraic equations in infinite dimensions. *Journal of Differential Equations*, 266(1):526 – 561, 2019.
- [33] V.V. Zhikov. On the theory of admissibility of pairs of function spaces. *Soviet Math. Dokl.*, 13(4):1108–1111, 1972.

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