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A-STABLE METHODS AND PADÉ APPROXIMATIONS TO THE EXPONENTIAL*

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Abstract. The set of Padé approximations to the exponential function is studied. It is shown that all entries on the first and second subdiagonal of the Padé table are analytic and bounded by 1 in the entire left half-plane. These results are then applied to the problem of producing A-stable numerical methods for solving initial value problems. It is shown that they easily permit one to generate several classes of methods of arbitrarily high order which are A-stable.

1. Introduction. There is currently considerable interest in numerical methods for solving systems of ordinary differential equations which exhibit the property of stiffness. A number of numerical methods have been proposed to solve such problems [8], [9], [12], [13]. Nearly all are designed to produce an approximation to the exponential function whose modulus is bounded by 1 when solving the initial value problem

$$y' = qy, \quad y(0) = 1, \tag{1}$$

with an arbitrary step size h, when q is any complex number with negative real part. Methods satisfying this condition are called A-stable [5]. A-stable methods generally permit the use of significantly larger step sizes than is possible with the classical fourth order Runge–Kutta or Adams' methods, for example, once the initial transient region is passed. This is because the A-stability condition guarantees that rapidly decaying terms will continue to decrease for any step size used. One difficulty in developing such methods has been a lack of suitable approximations to the exponential function which had moduli bounded by 1 in the entire left half-plane. Such approximations will be called A-acceptable in the remainder of this paper.

It has been shown by Varga [16] that the set of diagonal Padé approximations to the exponential are A-acceptable. Unfortunately, the moduli of all of these approximations approach 1 as $|z| \to \infty$, $\mathrm{Re}(z) < 0$ and this is not consistent with the behavior of e^z. A more satisfactory approximation to the exponential would be one that was not only A-acceptable but also satisfied the property that as $|z| \to \infty$, with $\mathrm{Re}(z) < 0$, its modulus approached zero. Such an approximation will be called L-acceptable. In a recent paper, Wright [19] has shown that the first eleven entries on the first subdiagonal of Padé approximations are L-acceptable.

In this paper it is shown, by an entirely different technique, that the set of all first and second subdiagonal Padé approximations to the exponential function are L-acceptable. Furthermore, evidence is given to suggest that these are the only L-acceptable Padé approximations to the exponential.

Finally, several classes of arbitrarily high order A-stable methods which produce L-acceptable approximations to the exponential are given.

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2. Preliminary theorems and definitions. A great deal is known about the Padé approximations to the exponential. In particular, if we denote by $P_{j,k}(z)$ the unique Padé approximation to the exponential with numerator $N_{j,k}(z)$ of degree k and denominator $D_{j,k}(z)$ of degree j, then it is known [11], [15] that

$$N_{j,k}(z) = \sum_{m=0}^{k} \frac{(j+k-m)!k!}{(j+k)!m!(k-m)!} z^m, \tag{2}$$

$$D_{j,k}(z) = \sum_{m=0}^{j} \frac{(j+k-m)!j!}{(j+k)!m!(j-m)!} (-z)^m.$$

In order to establish the L-acceptability of $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ we must establish three results:

(I) $P_{j,k}(z) - e^z = O(z^{m+1})$, $m \geqq 0$,

(II) $|P_{j,k}(z)| \to 0$ as $\operatorname{Re}(z) \to -\infty$,

(III) $|P_{j,k}(z)| \leqq 1$ for $\operatorname{Re}(z) \leqq 0$,

for $j = n+1$ or $n+2$, $k = n$, and $n = 0, 1, 2, \cdots$. Results (I) and (II) follow immediately from the fact [17, p. 394] that

$$P_{j,k}(z) - e^z = O(z^{j+k+1}),$$

and that $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ have denominators of higher degree than their numerators.

The proof that (III) is also satisfied is the subject of this paper. First it is established that $|P_{n+1,n}(z)|$ and $|P_{n+2,n}(z)|$ are bounded by 1 along the imaginary axis. Then it is shown that there are no zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ in the left half-plane. Consequently, $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ are analytic for $\operatorname{Re}(z) \leqq 0$, $n = 0, 1, 2, \cdots$, and the maximum modulus theorem may be applied to establish boundedness.

In order to effect the above proof, a number of relationships which hold between various $N_{j,k}(z)$ and $D_{j,k}(z)$ as given by (2) will be needed. In particular it is easily verified that:

(A) $D_{j,k}(z) = N_{k,j}(-z)$ for $j, k \geqq 0$ and all z.

(B) $\overline{N_{n,n}(iy)} = D_{n,n}(iy)$ and $\overline{D_{n,n}(iy)} = N_{n,n}(iy)$, y real.

(C) For all $j, k \geqq 1$ and all z,

(i) $N_{j,k}(z) = N_{j,k-1}(z) + AzN_{j-1,k-1}(z)$,

(ii) $D_{j,k}(z) = D_{j,k-1}(z) + AzD_{j-1,k-1}(z)$,

(iii) $N_{j,k}(z) = N_{j-1,k}(z) + BzN_{j-1,k-1}(z)$,

where

$$A = j/[(j+k)(j+k-1)] \quad \text{and} \quad B = -k/[(j+k)(j+k-1)].$$

(D) The polynomial $F_n(z) = N_{n,n}(z) \cdot D_{n,n}(z)$ has no odd terms.

(E) For all $n \geqq 2$,

$$N_{n,n}(z) = N_{n-1,n-1}(z) + Az^2 N_{n-2,n-2}(z)$$

and

$$D_{n,n}(z) = D_{n-1,n-1}(z) + Az^2 D_{n-2,n-2}(z),$$

where

$$A = 1/[4(2n-1)(2n-3)].$$

LEMMA 1. *For all $n \geqq 1$, the only term with an odd power of z in the product $D_{n,n}(z)N_{n-1,n-1}(z)$ is the term of highest power, namely,*

$$\frac{(-1)^n n!(n-1)!z^{2n-1}}{(2n)!(2n-2)!}.$$

Proof. The proof is by induction. For $n = 1$, we have

$$D_{1,1}(z)N_{0,0}(z) = (1 - z/2)(1).$$

Now by property (E),

$$D_{n,n}(z)N_{n-1,n-1}(z) = D_{n-1,n-1}N_{n-1,n-1} + \frac{z^2}{4(2n-1)(2n-3)}D_{n-2,n-2}N_{n-1,n-1}.$$

Assume the only odd term in $D_{n-1,n-1}(z)\,N_{n-2,n-2}(z)$ is of the form given by the theorem. By property (D), the first term on the right has no odd terms, and by property (A), the product has only the stated odd term. As an immediate corollary we have the following.

COROLLARY 1. *For all $n \geqq 1$ and all z,*

$$\frac{z}{(2n-1)}[D_{n,n}(z)N_{n-1,n-1}(z) - N_{n,n}(z)D_{n-1,n-1}(z)] = (-1)^n\left[\frac{(n-1)!z^n}{(2n-1)!}\right]^2.$$

3. Bounds on the imaginary axis. To establish that $|P_{n+1,n}(z)|$ and $|P_{n+2,n}(z)|$ are bounded by 1 on the imaginary axis, it is sufficient to show that $|D_{n+1,n}(iy)| \geqq |N_{n+1,n}(iy)|$ and that $|D_{n+2,n}(iy)| \geqq |N_{n+2,n}(iy)|$ for y real. To establish the first inequality we prove the following theorem.

THEOREM 1. *For all $n \geqq 1$, if $z = iy$, y real, then*

$$|D_{n,n-1}(z)|^2 - |N_{n,n-1}(z)|^2 = \left[\frac{(n-1)!}{(2n-1)!}\right]^2 y^{2n} \geqq 0,$$

and hence $|P_{n+1,n}(iy)| \leqq 1$ for y real and $n \geqq 0$.

Proof. Employing property (C) and observing that $|N_{n,n}(iy)| = |D_{n,n}(iy)|$ for y real, $n \geqq 0$, we have

$$\begin{aligned}|D_{n,n-1}(iy)|^2 &- |N_{n,n-1}(iy)|^2\\ &= \frac{iy}{2(2n-1)}[\overline{N_{n,n}(iy)}N_{n-1,n-1}(iy) - N_{n,n}(iy)\overline{N_{n-1,n-1}(iy)}\\ &\quad - \overline{D_{n,n}(iy)}D_{n-1,n-1}(iy) + D_{n,n}(iy)\,\overline{D_{n-1,n-1}(iy)}].\end{aligned}$$

We obtain the required result by applying property (B) to remove all conjugates and then applying Corollary 1.

In order to establish a similar result for $P_{n+2,n}(z)$, it is first necessary to observe the following.

LEMMA 2. *For all $n \geqq 2$ and all z,*

$$N_{n,n-2}(z) = \frac{1}{2(n-1)}[(4n-2)N_{n,n}(z) - (2n+z)N_{n-1,n-1}(z)]$$

and

$$D_{n,n-2}(z) = \frac{1}{2(n-1)}[(4n-2)D_{n,n}(z) - (2n+z)D_{n-1,n-1}(z)].$$

Proof. The two results follow from equations (2).

THEOREM 2. *For all $n \geqq 2$, if $z = iy$, y real, then*

$$|D_{n,n-2}(z)|^2 - |N_{n,n-2}(z)|^2 = \left[\frac{(n-2)!}{(2n-2)!}\right]^2 y^{2n} \geqq 0$$

and hence $|P_{n+2,n}(iy)| \leqq 1$ for y real and $n \geqq 0$.

Proof. Using Lemma 2 with $z = iy$, y real, we obtain

$$\begin{aligned}
&|D_{n,n-2}(iy)|^2 - |N_{n,n-2}(iy)|^2 \\
&\quad = \frac{2(2n-1)}{4(n-1)^2}[-(2n-iy)D_{n,n}\overline{D_{n-1,n-1}} - (2n+iy)\overline{D_{n,n}}D_{n-1,n-1} \\
&\qquad + (2n-iy)N_{n,n}\overline{N_{n-1,n-1}} + (2n+iy)\overline{N_{n,n}}N_{n-1,n-1}].
\end{aligned}$$

Applying property (B) and Corollary 1 completes the proof.

Before proceeding to the proof that $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ are analytic for $\mathrm{Re}(z) \leqq 0$, it seems appropriate to note that the following theorem can be established for $P_{n+3,n}(z)$.

THEOREM 3. *For all $n \geqq 3$, y real,*

$$|D_{n,n-3}(iy)|^2 - |N_{n,n-3}(iy)|^2 = (y^2 - n^2 + 2n)\left[\frac{(n-3)!y^{n-1}}{(2n-3)!}\right]^2$$

and hence $|P_{n,n-3}(iy)|$ is not bounded by one over the interval

$$-\sqrt{n^2-2n} < y < \sqrt{n^2-2n} \quad \text{for } n \geqq 3.$$

The proof of this theorem proceeds in a fashion similar to Theorems 1 and 2 after establishing that

$$N_{n,n-3}(z) = \frac{(n-z-2)}{(n-2)}N_{n-1,n-1}(z) + \frac{z(n+z)}{2(n-2)(2n-3)}N_{n-2,n-2}(z)$$

and that a similar result for $D_{n,n-3}(z)$ is true.

Since it is easily verified that $|P_{4,0}(z)|$ is not bounded by 1 on the imaginary axis, and since it is also known that $|P_{n,0}(z)|$ is not bounded by 1 in the left half-plane for $n \geqq 5$ [7, p. 25], there seems to be little likelihood that $P_{n+j,n}(z)$ can be L-acceptable for any $n \geqq 0$ with $j \geqq 3$. For this reason we direct our attention to establishing the L-acceptability of only $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ in the remainder of this paper.

4. The zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$. As was noted in § 2, the proof of the L-acceptability of $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ would be complete if we could establish that no zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ were in the left half-plane. By property (A), this is equivalent to showing that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ are in the left half-plane. We choose to work with the numerators because all the coefficients are positive while those of the denominator alternate in sign.

Wimp [18] has shown that all the zeros of the Bessel polynomials

$$P_n^{(\delta)}(z) = \sum_{k=0}^{n} \binom{n}{k} (n + \delta)_k z^{n-k}, \qquad \delta \geqq 0, \quad n \geqq 1,$$

where

$$(n + \delta)_k = (n + \delta)(n + \delta + 1) \cdots (n + \delta + k - 1),$$
$$(n + \delta)_0 = 1,$$

are in the left half-plane. It is easily verified that $N_{n,n+1}(z) = [n!/(2n + 1)!]P_{n+1}^{(0)}(z)$ and hence it follows that the set of first subdiagonal Padé approximations to the exponential are L-acceptable. Unfortunately, $N_{n,n+2}(z) = [n!/(2n + 2)!]P_{n+2}^{(-1)}(z)$, and Wimp's result does not apply to the second subdiagonal.

In trying to verify and extend Wimp's result, an attempt was made to prove that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ were in the left half-plane in a manner similar to Varga's proof [16] for $P_{n,n}(z)$. This was unsuccessful because the resulting continued fraction expansions were not easily related to one another. Instead, the proof is based on establishing the following two theorems.

THEOREM 4. *If for some j, $k \geqq 0$, $N_{j,k}(z)$ has all of its zeros in the open left half-plane, then for all $m \geqq j$, $N_{m,k}(z)$ has all of its zeros in the open left half-plane also.*

THEOREM 5. *For any $n \geqq 0$, if $N_{n+1,n+1}(z)$ has all of its zeros in the left half-plane, then $N_{n,n+2}(z)$ also has all of its zeros in the left half-plane.*

Assuming for the moment that the required proofs have been given, we note that the only zero of $N_{1,1}(z) = 1 + z/2$ is in the left half-plane. It then follows by repeated application of Theorems 5 and 4 as indicated by Fig. 1, that all the zeros of $N_{n,n+1}(z)$ and $N_{n,n+2}(z)$ are in the left half-plane. As noted above, it follows that none of the zeros of $D_{n+1,n}(z)$ and $D_{n+2,n}(z)$ are in the left half-plane. Thus we would have established the following result.

THEOREM 6. *For all $n \geqq 0$, $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$, the first and second subdiagonal Padé approximations to the exponential function, are analytic in the entire left half-plane. Furthermore, they are bounded in absolute value by* 1 *in the entire left half-plane and hence are L-acceptable approximations to the exponential function.*

Clearly we could have stated Theorem 5 in a more positive way since one of Varga's results in establishing the A-acceptability of the diagonal Padé approximations was that all the zeros of $N_{n+1,n+1}(z)$ were in the left half-plane. By stating the theorem without including this result, however, an alternative proof of the location of the zeros of $N_{n+1,n+1}(z)$ as well as $N_{n,n+1}(z)$ is provided.

5. Proof of Theorems 4 and 5. We begin by recalling a result given in Marden [14, p. 69].

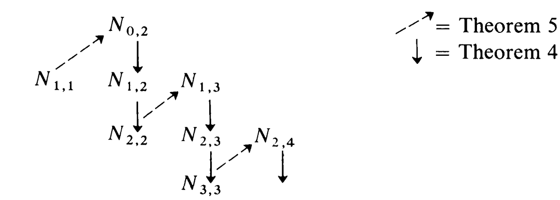

FIG. 1. *Basis of proof for Theorem* 6

THEOREM. *If* $f(z) = \sum_{k=0}^{n} a_k z^k$, $\beta_1 \neq \eta$ *and all the zeros of* $f(z)$ *lie in a circular region* C, *then every zero* Z *of the polynomial*

$$f_1(z) = \beta_1 f(z) - zf'(z)$$

may be written in the form $Z = \xi$ *or in the form*

$$Z = [\beta_1/(\beta_1 - \eta)]\xi,$$

where ξ *is a point of* C.

Observing that

$$N_{j,k}(z) = \frac{(j + k + 1)}{(k + 1)} N'_{j,k+1}(z) \quad \text{for all } j, k \geqq 0$$

follows immediately from (2), we can easily establish that

$$(j + k + 1)N_{j+1,k} = (j + k + 1)N_{j,k} - zN'_{j,k}$$

follows from property (C(iii)). With $\beta_1 = j + k + 1$ and $\eta = k$, it follows from the theorem [14] just given that all the zeros of $N_{j+1,k}$ are in the left half-plane provided all the zeros of $N_{j,k}$ are also. Thus Theorem 4 is established.

In order to prove Theorem 5, it is necessary to establish a relationship between $N_{n+1,n+1}(z)$ and $N_{n,n+2}(z)$. The following lemma will prove useful in this regard.

LEMMA 3. *For all* $j \geqq 0$, $k \geqq 0$,

$$N_{j,k+1}(z) = [1 + z/(j + 1)]N_{j+1,k}(z) - [z/(j + 1)]N'_{j+1,k}(z).$$

Proof. The proof follows directly from equation (2).

We now define $\theta_n(z)$ to be

$$\theta_n(z) = \frac{N_{n,n+2}(z)}{(-z/(n + 1))N_{n+1,n+1}(z)} = \left[\frac{-(n + 1)}{z} - 1\right] + \frac{N'_{n+1,n+1}(z)}{N_{n+1,n+1}(z)}.$$

The second equality is obtained using Lemma 3 with $j = n$ and $k = n + 1$.

Now, clearly all the zeros of $N_{n,n+2}(z)$ which are not zeros of $\theta_n(z)$ are zeros of $N_{n+1,n+1}(z)$. Conversely, all the zeros of $\theta_n(z)$ are zeros of $N_{n,n+2}(z)$. Thus, if $N_{n+1,n+1}(z)$ has zeros only in the left half-plane and we can show that all the zeros of $\theta_n(z)$ are in the left half-plane, then we will have shown that all the zeros of $N_{n,n+2}(z)$ are also in the left half-plane.

In order to study the zeros of $\theta_n(z)$ we consider the region S, bounded by the curve C, which is shown in Fig. 2. The boundary curve C is composed of the semicircle $|z| = R$, $\operatorname{Re}(z) \leqq 0$, R chosen so that all the zeros of $N_{n+1,n+1}(z)$ are inside $|z| = R$ together with the semicircle $|z| = r$, $r > 0$, $\operatorname{Re}(z) \geqq 0$, r chosen so that all the zeros of $N_{n,n+2}(z)$ are outside the circle $|z| = r$ together with the imaginary

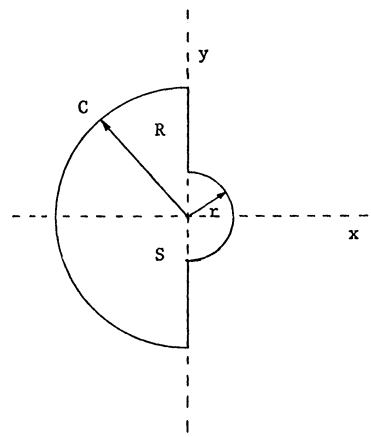

FIG. 2. *Region containing zeros of $\theta_n(z)$*

axis from $-R \leqq y \leqq -r$ and $r \leqq y \leqq R$. That values of $R < \infty$ and $r > 0$ can be found which satisfy these conditions follows at once from the known form of the polynomials $N_{n+1,n+1}(z)$ and $N_{n,n+2}(z)$ and several well-known results of Cauchy [14, p. 123–126].

The following well-known theorem [1, p. 123] from complex analysis will now be useful.

THEOREM. *Let $f(z)$ be meromorphic inside and on a simple closed curve C which does not pass through any of the zeros or poles of $f(z)$. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = N_C(f) - P_C(f),$$

where $N_C(f)$ and $P_C(f)$ are, respectively, the numbers of zeros and poles of $f(z)$ inside C.

Applying this theorem to $\theta_n(z)$, we obtain the following.

LEMMA 4. *If all the zeros of $N_{n+1,n+1}(z)$ are in the left half-plane, then*

$$N_C(N_{n,n+2}(z)) = (n+2) + \frac{1}{2\pi i}\int_C \frac{\theta_n'(z)}{\theta_n(z)}\,dz,$$

where C is the curve in Fig. 2.

Turning our attention to the evaluation of $\int_C (\theta_n'/\theta_n)\,dz$, we observe that this can be done by determining the index of $\theta_n(C)$ with respect to the origin, that is, the index of the curve into which C is mapped by $\theta_n(z)$ taken relative to the origin.[1] For convenience, we call this new curve C^*. We shall now show that as C^* is traversed, its real part is always negative and hence its index with respect to the origin is zero.

LEMMA 5. *For $|z| = R$, R sufficiently large,* $\mathrm{Re}\,(\theta_n(z)) < 0$.

Proof. For $|z|$ large enough, $\theta_n(z) = -1 + O(1/z)$; hence for sufficiently large R, the result follows.

[1] Dieudonné has used this technique in considering a problem which is similar in spirit to the one we are considering ([14, p. 87], [6]).

LEMMA 6. *For* $|z| = r$, $r > 0$, *r sufficiently small, and* $\operatorname{Re}(z) > 0$, $\operatorname{Re}(\theta_n(z)) < 0$.

Proof. Since as $|z| \to 0$, $(N'_{n+1,n+1}(z)/N_{n+1,n+1}(z)) \to \frac{1}{2}$, we have that for r sufficiently small, $\operatorname{Re}(N'_{n+1,n+1}(z)/N_{n+1,n+1}(z)) \leqq \frac{3}{4}$. Thus for r sufficiently small, we have

$$\operatorname{Re}(\theta_n(z)) \leqq \left[\frac{-(n+1)\operatorname{Re}(z)}{r^2} - 1\right] + \tfrac{3}{4} < 0.$$

LEMMA 7. *For y real and* $n \geqq 0$, *if* $N_{n+1,n+1}(z)$ *has all of its zeros in the left half-plane, then*

$$\operatorname{Re}\left(\frac{N'_{n+1,n+1}(iy)}{N_{n+1,n+1}(iy)}\right) < \tfrac{1}{2}$$

and hence $\operatorname{Re}(\theta_n(iy)) \leqq -\frac{1}{2}$ *for* $0 < r \leqq |y| \leqq R < \infty$.

Proof. With $j = k = n + 1$, property (C(i)) can be written

$$N_{n+1,n} = N_{n+1,n+1} - \frac{z}{2(2n+1)} N_{n,n} = 2N'_{n+1,n+1}.$$

It follows by Lemma 1 and property (B) that

$$\operatorname{Re}(N'_{n+1,n+1}(iy)\overline{N_{n+1,n+1}(iy)}) = \frac{1}{2}\left[N_{n+1,n+1}(iy)\overline{N_{n+1,n+1}(iy)} - \frac{[(n+1)!]^2 y^{2n+2}}{[(2n+2)!]^2}\right]$$

and thus

$$\operatorname{Re}(N'_{n+1,n+1}(iy)/N_{n+1,n+1}(iy)) = (1-t)/2,$$

where

$$t = \frac{[(n+1)!]^2 y^{2n+2}}{[(2n+2)!]^2 |N_{n+1,n+1}(iy)|^2} \geqq 0.$$

Thus the first inequality is established. The second inequality follows directly from the first using the definition of $\theta_n(z)$ with $z = iy$.

Lemmas 5, 6 and 7 establish the next lemma.

LEMMA 8. *If* $N_{n+1,n+1}(z)$ *has all of its zeros in the left half-plane, then the index* C^* *with respect to the origin is zero, and hence*

$$\frac{1}{2\pi i}\int_C \frac{\theta'_n(z)}{\theta_n(z)}\,dz = 0.$$

Lemmas 4 and 8 now establish that all the zeros of $N_{n,n+2}(z)$ are inside the region S of Fig. 2 provided all the zeros of $N_{n+1,n+1}(z)$ are in the left half-plane. But since all the zeros of $N_{n,n+2}(z)$ are outside the circle of radius r, we are also able to conclude that all the zeros of $N_{n,n+2}(z)$ are in the region S' given in Fig. 3. Noting that S' is entirely in the left half-plane, we have provided the necessary proof of Theorem 5.

6. Conclusions. Based on the results given above, it is possible to construct many new methods which are A-stable. We mention only two possibilities here, both being generalizations of ideas given in [8]. The first is to choose the coefficients

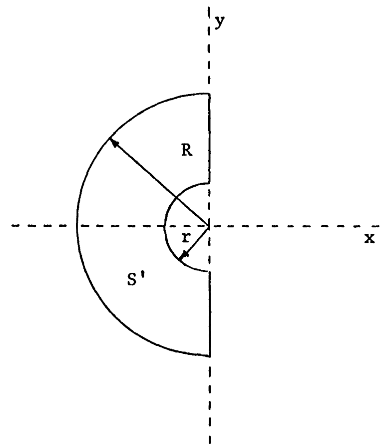

FIG. 3. *Region containing zeros of* $N_{n,n+2}(z)$

in the generalized one-step method

$$(3) \qquad y_{n+1} = y_n + \sum_{i=1}^{n} h^i(\alpha_i y_n^{(i)} + \beta_i y_{n+1}^{(i)}), \qquad n = 1, 2, 3, \cdots,$$

so that it reduces to a first or second subdiagonal Padé approximation to the exponential when solving the initial problem given by (1).

The appropriate choice for a first subdiagonal Padé approximation $P_{n,n-1}(z)$ is clearly

$$\alpha_i = (2n - 1 - i)!(n - 1)!/[(2n - 1)!i!(n - 1 - i)!], \quad i = 1, 2, \cdots, n - 1,$$

$$\alpha_n = 0,$$

and

$$\beta_i = (2n - 1 - i)!n!/[(2n - 1)!i!(n - i)!], \quad i = 1, 2, \cdots, n,$$

and for a second subdiagonal Padé approximation $P_{n,n-2}(z)$ the choice is

$$\alpha_i = (2n - 2 - i)!(n - 2)!/[(2n - 2)!i!(n - 2 - i)!], \quad i = 1, 2, \cdots, n - 2,$$

$$\alpha_n = \alpha_{n-1} = 0,$$

and

$$\beta_i = (2n - 2 - i)!n!/[(2n - 2)!i!(n - i)!].$$

Hermite [10], Hummel and Seebeck [11] and others have studied (3) and have observed that the coefficients given above are also those needed to make (3) a general method of order $2n - 1$ and $2n - 2$, respectively. Thus we have two sets of methods which are of arbitrarily high order and produce L-acceptable approximations to the exponential.

The second class of L-acceptable methods we consider is based on the implicit Runge–Kutta processes studied by Butcher [2], [3]. The obvious choice is to look at his methods based on Radau and Lobatto quadrature. Unfortunately, it is easily shown using the method given in [8] that none of the methods he proposes using these two quadratures are A-stable, since they reduce to above diagonal Padé approximations to the exponential when solving (1). Ehle [7], however, has shown that it is possible to construct implicit Runge–Kutta methods from these quadrature rules which reduce to subdiagonal Padé approximations $P_{n+1,n}(z)$ and $P_{n+2,n}(z)$ for small n. Chipman [4] has recently been able to show that the rules developed in [7] produce implicit Runge–Kutta processes which result in first and second subdiagonal Padé approximations for all n. This combined with Butcher's result about the order of such methods gives a second class of L-acceptable methods of arbitrarily high order.

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