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Observer Design for Linear Multi-rate Sampled-data Systems

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Abstract— **This paper addresses observer design for linear systems with multi-rate sampled output measurements. The sensors are assumed to be asynchronous and to have uncertain nonuniform sampling intervals. The contributions of this paper are twofold. Given the maximum allowable sampling period (MASP) for each sensor, the main contribution of the paper is to propose sufficient Krasovskii-based conditions for design of linear observers. The designed observers guarantee exponential convergence of the estimation error to the origin. Most importantly, the sufficient conditions are cast as a set of linear matrix inequalities (LMIs) that can be solved efficiently. As a second contribution, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error is also formulated as a convex optimization program in terms of LMIs. The theorems are applied to a unicycle path following example.**

I. Introduction

In multi-rate sampled-data systems, data is gathered through several sensors that work at different sampling rates. One reason is that different phenomena (e.g. temperature, pressure, or voltage) are measured with different sensors that work at different sampling rates. As a second reason, different methods of sensing the same phenomenon can lead to different sampling frequencies (e.g. measuring an angle with a potentiometer, an encoder, or a camera through image processing). Finally, even if the sensors are synchronized, the inevitable delays and packet losses in non-ideal communication links result in the data arriving at the controller at different rates. The reader is referred to [1]–[4] and the references therein for stability analysis and controller synthesis of multi-rate sampled-data systems.

Design of single-rate sampled-data observers and single-rate sampled-data output feedback controllers have been the subject of numerous research (see [5]–[7] and the references therein). Observer-based robust fault detection of linear multi-rate sampled-data systems is addressed in [8]–[11]. A common drawback of [8]–[12] is the assumption that the sampling rate of the sensors are uniform and their ratios are rational numbers (commensurate samplings). However, the uniform and commensurate sampling assumptions do not hold in practice. For instance, in the servo control of brushless DC motors via Hall-effect sensors, the sampling intervals depend on the motor speed and are not uniform [2]. Reference [13] addresses observer design for a class of nonlinear single-rate sampled-data systems with nonuniform samplings using a Krasovskii-based theorem and linear matrix inequalities (LMIs). In contrast to [5]–[13], we address observer design for linear systems with multi-rate sampled output measurements, where the sensors are assumed to be asynchronous and to have uncertain nonuniform sampling intervals. To the best of the authors' knowledge, the continuous-time state estimation problem using asynchronous multi-rate discrete-time output measurements was not studied before.

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The contributions of this paper are twofold. Given the maximum allowable sampling period (MASP) for each sensor, the main contribution of the paper is to propose sufficient Krasovskii-based conditions for design of linear multi-rate sampled-data observers. The designed observers guarantee exponential convergence of the estimation error to the origin. The sufficient conditions are cast as a set of LMIs that can be solved efficiently using available optimization software [14], [15]. As a second contribution, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error is formulated as a convex optimization program in terms of LMIs. The importance of choosing the right sensors with adequate sampling frequencies is shown through a path following example.

Notation. The zero matrix and the identity matrix of the appropriate size are represented by 0 and I, respectively. The column vector with all elements equal to 1 is denoted by $\mathbf{1}$. The notation $Z_1 > Z_2$ (or $Z_1 < Z_2$), where Z_1 and Z_2 are symmetric matrices, denotes that $Z_1 - Z_2$ is positive (or negative) definite. A block diagonal matrix with diagonal entries $d_1, \ldots, d_m$ is denoted by $\operatorname{diag}\big(d_1, \ldots, d_m\big)$. Where there is no confusion, a vector $x(t)$ is written as x. The notation $|.|$ denotes the norm of a vector. The Kronecker product is represented by $\otimes$.

II. Problem Statement

Let an observable continuous-time linear system be defined as

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$

$$y(t) = Cx(t) + Du(t), \tag{1b}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state vector, $y \in \mathbb{R}^{n_y}$ represents the output vector, $u \in \mathbb{R}^{n_u}$ is the control input, and A, B, C, and D are matrices of the appropriate dimension. Consider a continuous-time Luenberger observer with gain L. Let $\hat{x}(t)$ denote the observer state vector and $e(t) = x(t) - \hat{x}(t)$ be the estimation error. Therefore, the rate of change of $\hat{x}$ and e can be written as

$$\begin{aligned}\dot{\hat{x}}(t) &= A\hat{x}(t) - L\Big(y(t) - [C\hat{x}(t) + Du(t)]\Big) + Bu(t)\\ &= A\hat{x}(t) - LCe(t) + Bu(t),\end{aligned}$$

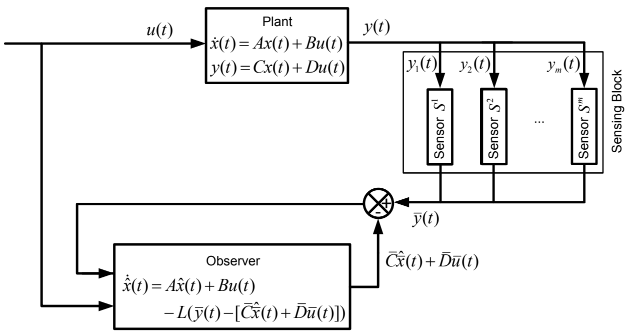

Fig. 1. The schematic diagram of a linear multi-rate sampled-data observer.

$$\dot{e}(t) = (A + LC)e(t),$$

In practice, the output vector is measured via different sensors (see Fig. 1). In this paper, the sensors are modeled as asynchronous sample and hold devices. Assume that the output vector comprises m components, i.e. $y^T = \begin{bmatrix} y_1^T & y_2^T & \dots & y_m^T \end{bmatrix}^T$. Each of these components is measured by a dedicated sensor S^i, $i \in \{1, \dots, m\}$, with an uncertain nonuniform sampling interval.

Assumption 1: The sensor S^i, $i \in \{1, \dots, m\}$, samples the i^{th} component of the output vector (i.e. y_i) at sampling instants s_k^i, where $0 < \epsilon < s_{k+1}^i - s_k^i < \tau^i$, $\forall k \in \mathbb{N}$.

The positive constant ϵ models the fact that a sensor cannot measure an output twice at the same instant. It is used later in the proof of the main result to rule out the occurrence of the Zeno phenomenon. The number τ^i denotes the longest interval between two consecutive samplings by sensor S^i, $i \in \{1, \dots, m\}$. For each sensor, the time elapsed since the sensor's last sampling instant is denoted by a sawtooth function $\rho^i(t)$ (see Fig. 2) defined as

$$\rho^i(t) = t - s_k^i, \text{ for } t \in [s_k^i, s_{k+1}^i). \tag{2}$$

Based on Assumption 1, equation (2) yields

$$0 \leq \rho^i < \tau^i. \tag{3}$$

The sensors are assumed to be asynchronous. Hence, in the multi-rate sampled-data structure, the output of the sample and hold devices (the sensors) is written as

$$\overline{y}(t) = \begin{bmatrix} y_1^T(t - \rho^1(t)) & \dots & y_m^T(t - \rho^m(t)) \end{bmatrix}^T.$$

Let C_i represent the row of the matrix C corresponding to the output y_i, i.e. $C = \begin{bmatrix} C_1^T & \dots & C_m^T \end{bmatrix}^T$. Similarly, let D_i, represent the rows of the matrix D corresponding to the output y_i. Considering (1b), the vector $\overline{y}(t)$ can be rewritten as

$$\overline{y}(t) = \overline{C}\overline{x}(t) + \overline{D}\overline{u}(t),$$

where

$$\begin{aligned} &\overline{C} = \operatorname{diag}(C_1, C_2, \dots, C_m), \ \overline{D} = \operatorname{diag}(D_1, D_2, \dots, D_m), \\ &\overline{x}(t) = \begin{bmatrix} x^T(t - \rho^1(t)) & \dots & x^T(t - \rho^m(t)) \end{bmatrix}^T, \\ &\overline{u}(t) = \begin{bmatrix} u^T(t - \rho^1(t)) & \dots & u^T(t - \rho^m(t)) \end{bmatrix}^T. \end{aligned}$$

In the multi-rate sampled-data structure, the rate of change of $\hat{x}$ and e can be written as

$$\begin{aligned} \dot{\hat{x}}(t) =& A\hat{x}(t) - L\Big(\overline{y}(t) - [\overline{C}\hat{\overline{x}}(t) + \overline{D}\overline{u}(t)]\Big) + Bu(t) \\ =& A\hat{x}(t) - L\overline{C}\overline{e}(t) + Bu(t), \\ \dot{e}(t) =& Ae(t) + L\overline{C}\overline{e}(t), \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\hat{\overline{x}}(t) = \begin{bmatrix} \hat{x}^T(t - \rho^1(t)) & \dots & \hat{x}^T(t - \rho^m(t)) \end{bmatrix}^T, \\ &\overline{e}(t) = \overline{x} - \hat{\overline{x}} = \begin{bmatrix} e^T(t - \rho^1(t)) & \dots & e^T(t - \rho^m(t)) \end{bmatrix}^T. \end{aligned} \tag{5}$$

In order to compute the observer output at time t, one only needs to store $\hat{x}$ and u at instants $t - \rho^i(t)$, $i \in \{1, \dots, m\}$, and not in the whole interval $[t, t - T]$, where $T = \max_i \tau^i$, $i \in \{1, \dots, m\}$. Also note that no stability property is assumed for system (1).

The instants at which (at least) one of the m sensors performs a sampling action constitute an increasing sequence in time, represented by $\{t_n\}$, $n \in \mathbb{N}$ (see Fig. 2 for a system with two sensors). Therefore, each time instant t_n, $n \in \mathbb{N}$, is associated with (i.e. is equal to) at least one and at most m instants s_k^i, $k \in \mathbb{N}$, with different $i \in \{1, \dots, m\}$. Based on Assumption 1, there exists a lower bound on the length of the interval (s_k^i, s_{k+1}^i), $\forall k \in \mathbb{N}$, $i \in \{1, \dots, m\}$. The length of the interval (t_n, t_{n+1}), $\forall n \in \mathbb{N}$, however, can approach zero because two sensors might possibly sample right after each other. Nonetheless, the following lemma is valid based on Assumption 1.

Lemma 1: For any time interval with length ϵ, there exists at least one interval (t_n, t_{n+1}), $n \in \mathbb{N}$, with a length greater than $\epsilon/(m + 1)$.

Proof: Let T_ϵ be a time interval with length ϵ. Based on Assumption 1, the interval T_ϵ contains at most m sampling instants $t_{n'}$, $n' \in \mathbb{N}$, that divide T_ϵ into sub-intervals. At least one of these sub-intervals has a length greater than $\epsilon/(m+1)$, and is a (potentially strict) subset of an interval (t_n, t_{n+1}), $n \in \mathbb{N}$, with a length greater than $\epsilon/(m + 1)$. ■

The time elapsed since the last sampling instant by any of the m sensors is denoted by $\rho(t)$ defined as

$$\begin{aligned} \rho(t) &= t - t_n, \ t \in [t_n, t_{n+1}) \\ &= \min_i \rho^i(t), \ i \in \{1, \dots, m\}. \end{aligned} \tag{6}$$

Therefore, based on (3),

$$0 \leq \rho(t) < \tau, \tag{7}$$

where

$$\tau = \min_i \tau^i, \ i \in \{1, \dots, m\}. \tag{8}$$

Figure 2 illustrates $\rho(t)$, for a system with two sensors. We conclude this section by presenting the definition of exponential stability and a lemma. Let $\mathcal{W}([-T, 0], \mathbb{R}^{n_x})$ be the space of absolutely continuous functions, with square integrable first-order derivatives, mapping the interval $[-T, 0]$ to $\mathbb{R}^{n_x}$. We define the function $e_t \in \mathcal{W}$ as

$$e_t(r) = e(t + r), \quad -T \leq r \leq 0,$$

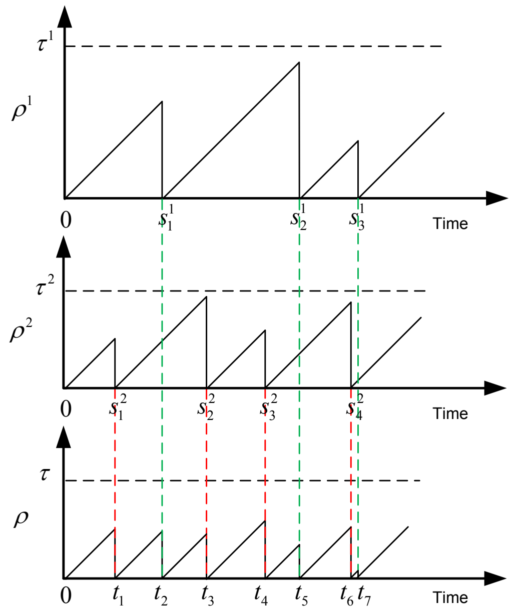

Fig. 2. The sawtooth functions $\rho^1(t)$, $\rho^2(t)$, and $\rho(t)$ in a multi-rate sampled-data structure with two sensors.

and denote its norm by $||e_t||_{\mathcal{W}} = \max_{r\in[-T,0]} |e_t(r)| + \left[\int_{-T}^{0} |\dot{e}_t(r)|^2 \,\mathrm{d}r\right]^{\frac{1}{2}}$.

Definition 1: The solution $e(t)$ of system (4) is said to be *globally uniformly exponentially stable* with *decay rate* λ if there exist $\delta > 0$ and $\lambda > 0$ such that for any initial condition $e_0 \in \mathcal{W}$ the solution is globally defined and satisfies $|e(t)| \leq \delta \exp(-\lambda t)||e_0||_{\mathcal{W}}$, for all $t \geq 0$.

The following lemma presents a useful property of Kronecker products and will be used in Section III (see [16] for the proof of the Lemma 2).

Lemma 2: Let H and ξ be a matrix and a vector, respectively, of the appropriate dimensions. Then $\mathbf{1} \otimes (H\xi) = (\mathbf{1} \otimes H)\xi$.

III. Main Results

The following theorem provides a set of sufficient conditions for which the estimation error of a linear multi-rate sampled-data observer exponentially converge to the origin.

Theorem 1: Consider the linear system defined in (1) and a multi-rate sampled-data observer with estimation error (4). Under Assumption 1, the estimation error is globally uniformly exponentially stable with a decay rate greater than $\alpha/2$ if there exist symmetric positive definite matrices P, R_0, R_i, $i \in \{1,\dots,m\}$, and X, and matrices N and $\overline{N}$, with appropriate dimensions, satisfying

$$\begin{bmatrix} \Psi + \tau M_1 & \star & \star & \star \\ \tau R_0 \begin{bmatrix} A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} & -\tau R_0 & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & 0 & -\overline{E}\overline{\tau}\overline{R} & \star \\ \overline{R}\overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}) & 0 & 0 & -\overline{R}\overline{\tau} \end{bmatrix} < 0 \tag{9}$$

$$\begin{bmatrix} \Psi + \tau M_2 & \star & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & \star & \star \\ \overline{R}\overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}) & 0 & -\overline{R}\overline{\tau} & \star \\ \tau N^T & 0 & 0 & \frac{-\tau R_0}{\exp(\alpha\tau)} \end{bmatrix} < 0 \tag{10}$$

where τ is defined in (8) and

$$\overline{\tau} = \mathrm{diag}(\tau^1 I, \tau^2 I, \dots, \tau^m I), \tag{11}$$
$$\overline{E} = \mathrm{diag}(\exp(\alpha\tau^1)I, \dots, \exp(\alpha\tau^m)I), \tag{12}$$
$$\overline{R} = \mathrm{diag}(R_1, R_2, \dots, R_m), \tag{13}$$

$$\begin{aligned} \Psi = &\begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \end{bmatrix} + \begin{bmatrix} P & 0 & 0 \end{bmatrix}^T \times \\ &\begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &- \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix}^T \overline{N}^T - \overline{N} \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix} \\ &- \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\ M_1 = &\begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix} + \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \times \\ &\begin{bmatrix} A & L\overline{C} & 0 \end{bmatrix} + \alpha \begin{bmatrix} I & 0 & -I \end{bmatrix}^T X \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\ M_2 = &- \mathrm{diag}(0, I, I)N^T - N\,\mathrm{diag}(0, I, I). \end{aligned}$$

Proof: Consider the candidate Lyapunov-Krasovskii functional

$$V(t, e_t) = V_1 + V_2 + V_3 + V_4,\ t \in [t_n, t_{n+1}), \tag{14}$$

where

$$\begin{aligned} V_1 = &e^T(t)Pe(t), \\ V_2 = &(\tau - \rho) \int_{t-\rho}^{t} \exp(\alpha(s-t)) \begin{bmatrix} \dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n) \end{bmatrix} \\ &\times R_0 \begin{bmatrix} \dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n) \end{bmatrix}^T \mathrm{d}s, \\ V_3 = &\sum_{i=1}^{m} (\tau^i - \rho^i) \int_{t-\rho^i}^{t} \exp(\alpha(s-t)) \dot{e}^T(s) R_i \dot{e}(s)\,\mathrm{d}s, \\ V_4 = &(\tau - \rho)(e(t) - e(t_n))^T X (e(t) - e(t_n)), \end{aligned}$$

where P, R_0, R_i, $i \in \{1,\dots,m\}$, and X are symmetric positive definite matrices, α is a positive scalar representing the decay rate, and functions ρ^i and ρ are defined in (2) and (6), respectively. It can be shown (see [16] for the details) that the Lyapunov-Krasovskii functional (14) satisfies

$$c_1|e_t(0)|^2 \leq V(t, e_t) \leq c_2||e_t||_{\mathcal{W}}^2, \tag{15}$$
$$V(t_n, e_{t_n}) \leq V(t_n^-, e_{t_n^-}),\ \forall n \in \mathbb{N}, \tag{16}$$

where c_1 and c_2 are positive scalars, and $V(t_n^-, e_{t_n^-}) = \lim_{t \nearrow t_n} V(t, e_t)$. In order to prove exponential stability, we next prove that the Lyapunov-Krasovskii functional (14) satisfies

$$\dot{V}(t, e_t) + \alpha V(t, e_t) < 0,\ \forall\, t \neq t_n,\ n \in \mathbb{N}. \tag{17}$$

The time derivative of V in the interval between two consecutive sampling instants $t \in (t_n, t_{n+1})$ is composed of four terms computed as follows. The time derivative of V_1 is

$$\dot{V}_1 = \dot{e}^T Pe + e^T P\dot{e}. \tag{18}$$

From (6) we have $\dot{\rho} = 1$. Hence, applying the Leibniz integral rule to V_2 yields

$$\begin{aligned}\dot{V}_2 =& -\int_{t-\rho}^{t} \exp(\alpha(s-t)) \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} \\ & \times R_0 \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T \mathrm{d}s \\ & + (\tau-\rho) \begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix} R_0 \begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix}^T \\ & - \alpha V_2. \end{aligned} \tag{19}$$

Since R_0 is positive definite, $\alpha > 0$, and $\rho < \tau$ (see (7)), for any $s \in [t-\rho, t]$ and any arbitrary time varying vector $h(t) \in \mathbb{R}^{(2+m)n_x}$ we can write

$$\begin{aligned}&\begin{bmatrix}\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} & h^T\end{bmatrix} \times \\ &\begin{bmatrix}\exp(\alpha(s-t))R_0 & -I \\ -I & \exp(\alpha\tau)R_0^{-1}\end{bmatrix} \times \\ &\begin{bmatrix}\begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} & h^T\end{bmatrix}^T \geq 0. \end{aligned} \tag{20}$$

This inequality can be verified using Schur complement. Hence, for all $s \in [t-\rho, t]$,

$$\begin{aligned}& -\exp(\alpha(s-t)) \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} \\ & \quad \times R_0 \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T \\ & \leq h^T \exp(\alpha\tau) R_0^{-1} h - \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} h \\ & \quad - h^T \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T. \end{aligned}$$

For s varying between $t-\rho$ and t, the vectors $\overline{e}(s)$ and $e(t_n)$ are constant, and $e(s) = e_s(0) \in \mathcal{W}$ is absolutely continuous. Therefore, integrating both sides with respect to s, yields

$$\begin{aligned}& -\int_{t-\rho}^{t} \exp(\alpha(s-t)) \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix} \\ & \quad \times R_0 \begin{bmatrix}\dot{e}^T(s) & \overline{e}^T(s) & e^T(t_n)\end{bmatrix}^T \mathrm{d}s \\ & \leq \rho h^T \exp(\alpha\tau) R_0^{-1} h \\ & \quad - \begin{bmatrix}e^T - e^T(t-\rho) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix} h \\ & \quad - h^T \begin{bmatrix}e^T - e^T(t-\rho) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix}^T. \end{aligned} \tag{21}$$

Based on (6), $t-\rho = t_n$. Replacing (21) in (19), yields

$$\begin{aligned}\dot{V}_2 \leq & \rho h^T \exp(\alpha\tau) R_0^{-1} h - \begin{bmatrix}e^T - e^T(t_n) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix} h \\ & - h^T \begin{bmatrix}e^T - e^T(t_n) & \rho\overline{e}^T & \rho e^T(t_n)\end{bmatrix}^T \\ & + (\tau-\rho) \begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix} R_0 \begin{bmatrix}\dot{e}^T & \overline{e}^T & e^T(t_n)\end{bmatrix}^T \\ & - \alpha V_2. \end{aligned} \tag{22}$$

Similarly, we can write the following inequality

$$\begin{aligned}\dot{V}_3 \leq & \sum_{i=1}^{m} \Big(\rho^i h_i^T \exp(\alpha\tau^i) R_i^{-1} h_i - \begin{bmatrix}e - e(t-\rho^i)\end{bmatrix}^T h_i \\ & - h_i^T \begin{bmatrix}e - e(t-\rho^i)\end{bmatrix} + (\tau^i - \rho^i) \dot{e}^T R_i \dot{e} \Big) \\ & - \alpha V_3, \end{aligned} \tag{23}$$

where $h_i(t) \in \mathbb{R}^{n_x}$, $i \in \{1, \ldots, m\}$, are arbitrary time-varying vectors. Based on (3), $\tau^i h_i^T \exp(\alpha\tau^i) R_i^{-1} h_i$ and $\tau^i \dot{e}^T R_i \dot{e}$ are upper bounds for $\rho^i h_i^T \exp(\alpha\tau^i) R_i^{-1} h_i$ and $(\tau^i - \rho^i) \dot{e}^T R_i \dot{e}$, respectively. Hence, inequality (23) can be rewritten in a more compact form as

$$\begin{aligned}\dot{V}_3 \leq & \overline{h}^T \overline{\tau} \overline{ER}^{-1} \overline{h} - [\mathbf{1} \otimes e - \overline{e}]^T \overline{h} - \overline{h}^T [\mathbf{1} \otimes e - \overline{e}] \\ & + (\mathbf{1} \otimes \dot{e})^T \overline{\tau} \overline{R} (\mathbf{1} \otimes \dot{e}) - \alpha V_3, \end{aligned} \tag{24}$$

where $\overline{e}$, $\overline{\tau}$, $\overline{E}$, and $\overline{R}$ are defined in (5), (11)-(13), and $\overline{h} = \begin{bmatrix}h_1^T & h_2^T & \ldots & h_m^T\end{bmatrix}^T$. The time derivative of V_4 is computed as

$$\begin{aligned}\dot{V}_4 =& -(e - e(t_n))^T X (e - e(t_n)) + (\tau-\rho) \dot{e}^T X (e - e(t_n)) \\ & + (\tau-\rho)(e - e(t_n))^T X \dot{e}. \end{aligned} \tag{25}$$

We now define an augmented vector $\zeta(t)$ as

$$\zeta(t) = \begin{bmatrix}e^T(t) & \overline{e}^T(t) & e^T(t_n)\end{bmatrix}^T, \; t \in [t_n, t_{n+1}). \tag{26}$$

Therefore, recalling (4),

$$\dot{e}(t) = \begin{bmatrix}A & L\overline{C} & 0\end{bmatrix} \zeta(t). \tag{27}$$

Replacing (27) in (18), (22), (24), and (25), setting $h(t) = N^T \zeta(t)$ and $\overline{h}(t) = \overline{N}^T \zeta(t)$, where $N \in \mathbb{R}^{(2+m)n_x \times (2+m)n_x}$ and $\overline{N} \in \mathbb{R}^{(2+m)n_x \times mn_x}$, and using Lemma 2, yields

$$\begin{aligned}\dot{V} + \alpha V =& \; \textstyle\sum_{j=1}^{4} (\dot{V}_j + \alpha V_j) \\ \leq & \; \zeta^T \Big(\Psi + (\tau-\rho) M_1 + \rho M_2 + \rho N \exp(\alpha\tau) R_0^{-1} N^T \\ & + (\tau-\rho) \begin{bmatrix}A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I\end{bmatrix}^T R_0 \begin{bmatrix}A & L\overline{C} & 0 \\ 0 & I & 0 \\ 0 & 0 & I\end{bmatrix} \\ & + (\mathbf{1} \otimes \begin{bmatrix}A & L\overline{C} & 0\end{bmatrix})^T \overline{\tau} \overline{R} (\mathbf{1} \otimes \begin{bmatrix}A & L\overline{C} & 0\end{bmatrix}) \\ & + \overline{N} \overline{\tau} \overline{ER}^{-1} \overline{N}^T \Big) \zeta, \end{aligned} \tag{28}$$

where Ψ, M_1, and M_2 are defined in Theorem 1. For $\rho = 0$, using Schur complement, inequality (9) implies $\dot{V} + \alpha V < 0$. Similarly, inequality (10) implies $\dot{V} + \alpha V < 0$ for $\rho = \tau$. Since (28) is affine in ρ, inequalities (9) and (10) are sufficient conditions for $\dot{V} + \alpha V < 0$ to hold for any $\rho \in (0, \tau)$. Therefore, based on (6), inequalities (9) and (10) are sufficient conditions for inequality (17) to be satisfied. Solving (17) for $t \in (t_n, t_{n+1})$ and using (16) yields

$$\begin{aligned}& V(t, e_t) \leq \exp(-\alpha(t - t_n)) V(t_n, e_{t_n}) \\ & \leq \exp(-\alpha(t - t_n)) V(t_n^-, e_{t_n^-}) \leq \ldots \leq \exp(-\alpha t) V(0, e_0). \end{aligned}$$

The Krasovskii functional V strictly decreases in intervals (t_n, t_{n+1}) that have a nonzero length (note that, according to Lemma 1, for any time interval with length ϵ, there exists at least one interval (t_n, t_{n+1}), $n \in \mathbb{N}$, with a length greater than $\epsilon/(m+1)$). Now, inequality (15) yields

$$\begin{aligned}|e(t)| & \leq (V(t, e_t)/c_1)^{\frac{1}{2}} \leq (\exp(-\alpha t) V(0, e_0)/c_1)^{\frac{1}{2}} \\ & \leq (c_2/c_1)^{\frac{1}{2}} \exp(-\alpha t/2) ||e_0||_{\mathcal{W}}. \end{aligned}$$

Hence, the estimation error is exponentially stable with a decay rate greater than $\alpha/2$. Note that the Zeno phenomenon does not occur since, by Assumption 1, for any time interval with length ϵ, there exists a finite number of (at most m) sampling instants t_n, $n \in \mathbb{N}$. ■

Next, the sufficient conditions in Theorem 1 are used to address two problems in sampled-data observers. In the first problem, it is assumed that an observer gain L is available which exponentially stabilizes the estimation error in continuous-time. The objective is to find the MASP for the sensor S^j, $j \in \{1, ..., m\}$, such that exponential stability of the estimation error is preserved. Given L, α, and τ^i, $i \in \{1, ..., m\}$, the sufficient conditions in Theorem 1 become LMIs. These LMIs can be solved efficiently using available optimization software [14], [15]. Following the line search strategy, the problem of finding a lower bound[1] on the MASP for the sensor S^j, $j \in \{1, ..., m\}$, that guarantees exponential stability of the estimation error is formulated as

Problem 1:

$$\begin{aligned}
&\text{maximize} \quad \tau^j \\
&\text{subject to} \quad P > 0,\ R_0 > 0,\ R_i > 0,\ i \in \{1, ..., m\}, \\
&\qquad\qquad\quad X > 0,\ (9) \text{ and } (10).
\end{aligned}$$

We denote the computed lower bound on the MASP that guarantees exponential stability of the estimation error by $\tau^j_{\max}$.

In the second problem, it is assumed that the upper bound on the sampling intervals for each sensor (i.e. τ^i, $i \in \{1, ..., m\}$) is known and the decay rate α is given. The objective is to design an observer gain L such that exponential stability of the estimation error is guaranteed. With L as an optimization variable, the conditions in Theorem 1 are bilinear matrix inequalities. Using a change of variables, however, these conditions can be written in the form of LMIs as shown in the following corollary.

Corollary 1: Given τ^i, $i \in \{1, ..., m\}$, and $\alpha > 0$, suppose there exist positive definite matrices P and $\mathcal{R}_0$, matrices Y, N, and $\overline{N}$, with appropriate dimensions, and a positive scalar ϵ_X, satisfying

$$\begin{bmatrix} \Psi_s + \tau M_{1s} & \star & \star & \star \\ \begin{bmatrix} PA & \begin{bmatrix} Y\overline{C} & 0 \end{bmatrix} \\ 0 & \mathcal{R}_0 \end{bmatrix} & -\tau R_{0s} & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & 0 & -\overline{E}\overline{\tau}\overline{R} & \star \\ \overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} PA & Y\overline{C} & 0 \end{bmatrix}) & 0 & 0 & -\overline{R}\overline{\tau} \end{bmatrix} < 0 \tag{29}$$

$$\begin{bmatrix} \Psi_s + \tau M_{2s} & \star & \star & \star \\ \overline{E}\overline{\tau}\overline{N}^T & -\overline{E}\overline{\tau}\overline{R} & \star & \star \\ \overline{\tau}(\mathbf{1} \otimes \begin{bmatrix} PA & Y\overline{C} & 0 \end{bmatrix}) & 0 & -\overline{R}\overline{\tau} & \star \\ \tau N^T & 0 & 0 & \frac{-\tau R_{0s}}{\exp(\alpha\tau)} \end{bmatrix} < 0 \tag{30}$$

where τ, $\overline{\tau}$, $\overline{E}$, and $\overline{R}$ are defined in (8), (11)-(13), and

$$\begin{aligned}
R_{0s} =& \operatorname{diag}(P, \mathcal{R}_0), \qquad (31)\\
\Psi_s =& \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T + \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&+ \alpha \begin{bmatrix} I & 0 & 0 \end{bmatrix}^T P \begin{bmatrix} I & 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T N^T - N \begin{bmatrix} I & 0 & -I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&- \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix}^T \overline{N}^T - \overline{N} \begin{bmatrix} \mathbf{1} \otimes I & -I & 0 \end{bmatrix} \\
&- \epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T P \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\
M_{1s} =& \epsilon_X \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ -PA & -Y\overline{C} & 0 \end{bmatrix}^T + \epsilon_X \begin{bmatrix} PA & Y\overline{C} & 0 \\ 0 & 0 & 0 \\ -PA & -Y\overline{C} & 0 \end{bmatrix} \\
&+ \alpha\epsilon_X \begin{bmatrix} I & 0 & -I \end{bmatrix}^T P \begin{bmatrix} I & 0 & -I \end{bmatrix}, \\
M_{2s} =& -\operatorname{diag}(0, I, I) N^T - N \operatorname{diag}(0, I, I).
\end{aligned}$$

Then there exists an observer gain $L = P^{-1}Y$ and a set of matrix variables R_0, R_i, $i \in \{1, ..., m\}$, and X that satisfy the conditions in Theorem 1, for the same values of τ^i, $i \in \{1, ..., m\}$, α, P, N, and $\overline{N}$.

Proof: The proof is straightforward and consists of verifying that inequalities (29) and (30) are equivalent to (9) and (10) with the change of variables $L = P^{-1}Y$, $R_0 = R_{0s}$ (see (31)), $R_i = P$, $i = \{1, \ldots, m\}$, and $X = \epsilon_X P$. ■

Therefore, given τ^i, $i \in \{1, \ldots, m\}$, and $\alpha > 0$ the problem of designing an observer gain L that guarantees exponential stability of the estimation error is formulated as

Problem 2:

$$\begin{aligned}
&\text{find} \qquad Y \\
&\text{subject to } P > 0,\ \mathcal{R}_0 > 0,\ \epsilon_X > 0,\ (29),\ \text{and } (30).
\end{aligned}$$

The observer gain is then computed as $L = P^{-1}Y$. The conditions in Corollary 1 are sufficient conditions for the inequalities in Theorem 1 and therefore are more conservative. However, they can be used to design linear observers by solving a convex optimization program that can be solved efficiently using available software packages.

IV. Numerical Example

Consider the path following problem where the objective is to control a unicycle to follow the line $y = 0$ in the $x - y$ plane (see Fig. 3(a)). The dynamics of the system are represented by

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -k/I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ v\sin(\psi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/I \\ 0 \end{bmatrix} u, \tag{32}$$

where y represents the distance from the line $y = 0$, ψ and r are the heading angle and its time derivative, respectively, $v = 1$ (m/s) is the unicycle's velocity, u denotes the torque input about the z axis, $I = 1$ (kgm^2) is the unicycle's moment of inertia with respect to its center of mass, and $k = 0.01$ (Nms) is the damping coefficient. Linearizing the system about the origin leads to the linear system (1), with

$$x = \begin{bmatrix} y \\ \psi \\ r \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -0.01 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

[1]The computed value is *a lower bound* on the MASP because the LMIs in Theorem 1 are sufficient conditions.

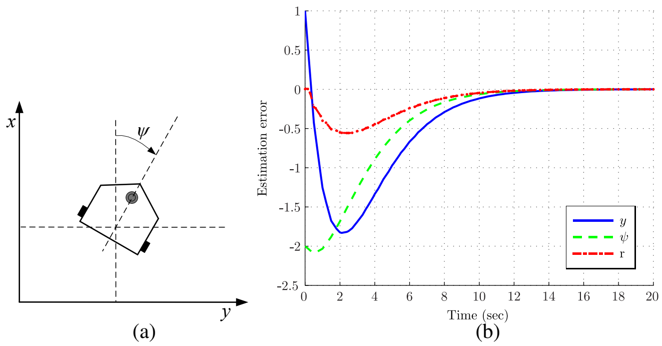

Fig. 3. (a) Unicycle path following problem, (b) state estimation error for $\tau^1 = 0.5$ (s) and $\tau^2 = 0.3$ (s) and observer gain L' defined in (34).

TABLE I

THE COMPUTED MASP (τ^2_{MAX}) FOR SENSOR S^2 THAT GUARANTEES EXPONENTIAL STABILITY OF THE ESTIMATION ERROR WITH $\alpha = 0.1$

	$\tau^2_{\max}$
For $\tau^1 = 0.1$ (s)	0.24 (s)
For $\tau^1 = 0.15$ (s)	0.18 (s)
For $\tau^1 = 0.17$ (s)	0.05 (s)

Assume that the states y and r are measured by asynchronous dedicated sensors S^1 and S^2, respectively. Let L be an observer gain that places the poles of the continuous-time estimation error vector field at $-0.25 \pm 1.5j$ and -3.5. Therefore,

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; L = -\begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix}^T. \quad (33)$$

Assume that the sampling intervals of sensor S^1 have a known upper bound, i.e. τ^1 is fixed. Solving Problem 1 for different values of τ^1, the lower bound on the MASP (that guarantees exponential stability of the estimation error) for sensor S^2 is presented in Table I. As expected, when the allowable length of sampling intervals for sensor S^1 increases, the computed value for $\tau^2_{\max}$ decreases. In other words, as the sampling frequency of the first sensor decreases, the second sensor must sample faster to guarantee convergence of the estimation error to the origin.

Now suppose that sensors S^1 and S^2 perform measurements at unknown nonuniform sampling intervals smaller than $\tau^1 = 0.5$ (s) and $\tau^2 = 0.3$ (s), respectively. In this case and with the choice of observer gain (33), the LMIs of Theorem 1 are infeasible and simulation results show that the estimation error does not converge to the origin. Solving Problem 2 with $\alpha = 0.1$ and $\epsilon_X = 1$, we find a new observer gain

$$L' = \begin{bmatrix} -0.8079 & -0.2071 & -0.7609 \\ -0.2555 & -0.0550 & -0.7714 \end{bmatrix}^T. \quad (34)$$

The new observer gain L' guarantees exponential stability of the estimation error as shown in Fig. 3(b).

V. Conclusion

The observer design problem for linear systems with multi-rate sampled output measurements was addressed. Given the MASP for each sensor, sufficient Krasovskii-based conditions for design of linear observers were proposed in terms of LMIs. Furthermore, given an observer gain, the problem of finding MASPs that guarantee exponential stability of the estimation error was formulated as a convex optimization program subject to LMIs.

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