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Observability under sampling for bilinear system

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This work is a continuation of the study of the problem of the preservation of the observability under sampling. In this paper, we establish that, for a bilinear system, the property of observability is preserved after sampling provided that the controls take their values in a compact space $\mathscr{U}$ of $\mathbf{R}^m$ and do not vary too quickly.

Keywords: bilinear systems; discrete-time systems; sampling; observability

1. Introduction and problem formulation

Many physical processes or industrial devices can be modelled by a system of equations which write

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x). \end{cases} \quad (1)$$

In practice, the observation signal y is sent to a computer; however, given a time interval $[0, T]$, the computer is obviously unable to record the infinite number of values taken by the mapping $t \mapsto y(t)$, so this signal is converted in digital form and is used by the computer to calculate the necessary controls. An usual approach to compute these controls is based on the sample-and-hold technique: control decisions are restricted to be taken at times $0, \delta, 2\delta, \ldots$ (δ is called the sampling time). At every time $k\delta$, the computer gets the value $y(k\delta)$ of the observation function and updates the control law by a programme. The output of this programme, $u(k\delta)$, is then fed into the system as a control during the interval $[k\delta, (k + 1)\delta)$.

In summary, even if we can model a physical process by a continuous-time system, for practical reasons, we can use only the values of the observation signal at discrete times $0, \delta, 2\delta, \ldots$ and the control u remains constant on the intervals $[k\delta, (k + 1)\delta)$. It is thus of interest to characterise the various properties of system (1) when the control and the signal are so restricted. By considering the continuous-time system (1), one can design an observer or an output feedback law (static or dynamic). This observer and this feedback need the knowledge of the signal $y(t)$ at every time t but in the real world, the observation signal is known only

at times $t_1, t_2, \ldots$ and the control remains constant between the times t_k and t_{k+1}. So the problem of the conservation of the observability is not only of academic interest but can also have practical interest in some cases: e.g. as we shall see, the property of observability can be lost if the control varies too quickly.

Concerning the controllability property, the problem has already been studied: see Sontag (1998) for the linear systems and Sontag and Sussmann (1982) and Sontag (1983) for nonlinear systems. As far as we are concerned with bilinear systems, the issue of the preservation of the controllability has been studied in Sontag (1986). In this paper, E.D. Sontag proved that the sampled accessibility at a point ξ (which is a local notion) is equivalent to the strong accessibility condition. This condition bears on the dimension of some Lie algebra related to the vector fields defining the system.

For linear systems, it is well known that there is a duality between the notions of controllability and observability. For bilinear systems, this duality has been highlighted in Benner and Damm (2011). The interest of this paper is that the duality is stated in terms of the solution of a generalised Lyapunov equation. Nevertheless, notice that the paper is concerned with *local* notion of controllability and observability; moreover, the observed bilinear system is homogeneous and the drift of this system must be a stable matrix.

It is well known that the possibility to build an observer for a given system relies on the observability of this system and this is why this notion is important in control theory and has been intensively studied. In this paper, we answer to the question for bilinear systems, can the observability be preserved by the sampling process described above?

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We consider the general control system (1) where the state x evolves on $\mathbf{R}^n$ and whose controls u belong to a subset of the set of the measurable and essentially bounded mappings defined on $\mathbf{R}_+$ and which take their values on a subset $\mathscr{U}$ of $\mathbf{R}^m$. The set U of these controls will be called the set of *admissible controls.* Given a sampling time δ, we shall define what we call the δ-sampled system related to Equation (1). We denote by $\varphi(t, t_0, x_0, u(\,\cdot\,))$ the solution of system (1) with the control $t \mapsto u(t)$ and the initial conditions (t_0, x_0): that is to say φ is such that $\varphi(t_0, t_0, x_0, u(\,\cdot\,)) = x_0$ and $\mathrm{d}\varphi/\mathrm{d}t = f(\varphi, u)$. This solution is defined on a semi-open interval that we shall denote by $[0, e_{x_0, u(\cdot)})$ $(0 < e_{x_0, u(\cdot)} \leq +\infty)$. The δ-sampled system related to Equation (1) is the discrete-time system which writes

$$\begin{cases} x_{k+1} = \varphi\big((k+1)\delta, 0, x_0, u^\delta(\cdot)\big) \\ y_{k+1} = h(x_{k+1}), \end{cases} \quad (2)$$

where u^δ is a control which is constant on every interval $[k\delta, (k+1)\delta)$. For this system, the set of admissible controls is the set of functions from $\mathbf{R}_+$ to $\mathscr{U}$ which are constant on every interval $[k\delta, (k+1)\delta)$ and which take their values in $\mathscr{U}$.

In this paper, we deal with the question of the preservation of the observability, to be more precise the problem we want to address is the following: does the observability of system (1) imply the observability of its δ-sampled system (2)? There exist many notions of observability, (see, for example, Sontag, 1984) so hereafter we shall give some precise definitions. We recall first the notion of distinguishable points and universal controls.

Definition 1.1: We say that x_0 and $\bar{x}_0$ are distinguishable points for system (1) if there exist an admissible control u and a time t in the interval $[0, \min(e_{x_0, u(\cdot)}, e_{\bar{x}_0, u(\cdot)})$ such that $h(\varphi(t, 0, x_0, u(\cdot))) \neq h(\varphi(t, 0, \bar{x}_0, u(\cdot)))$.

An admissible control that permits to distinguish every pair of distinct points is called a universal control.

The definition for discrete-time systems (and thus for system (2)) is similar.

Definition 1.2: We say that x_0 and $\bar{x}_0$ are distinguishable points for the system (2) if there exist an admissible control u^δ, which is constant on every interval $[k\delta, (k+1)\delta)$, and an index k such that $k\delta$ belongs to the interval $[0, \min(e_{x_0, u(\cdot)}, e_{\bar{x}_0, u(\cdot)})$ and $h(\varphi(k\delta, 0, x_0, u^\delta(\cdot))) \neq h(\varphi(k\delta, 0, \bar{x}_0, u^\delta(\cdot)))$.

We shall work with this basic notion of observability for both continuous and sampled systems.

Definition 1.3: We say that the systems (1) or (2) are observable for all admissible inputs if every pair of distinct points is distinguishable by every admissible input.

The answer to our problem had been known for a long time for linear system (see, e.g., Sontag, 1998) and, as

pointed out by the author of this book, the result is closely related to Shannon's Sampling Theorem. As regards the linear systems, we would also like to mention some recent papers: in Kreisselmeier (1999), Ding, Qiu, and Chen (2009) or Wang, Li, Yin, Guo, and Xu (2011), the authors deal with the problem of the preservation of the observability under *irregular* sampling; in Hagiwara (1995), the author examines the problem when a linear system is sampled with a first-order hold. As regards the nonlinear system, we would like to mention (Sontag, 1984), in this paper, the author has introduced a particular notion of *local* observability for continuous and discrete-time systems. The first result is concerned with the local observability for continuous-time system whereas the second result states the preservation of the local observability under constant-rate sampling. To be more precise, the paper considers a continuous-time system and the discrete-time system obtained after sampling and shows that the continuous system is locally observable if and only if the same is true for the discrete-time system. This equivalence is realised under the hypothesis that the ideal (of the Lie algebra generated by all the $f(\,\cdot\,, u)$) generated by the vector fields $f(\,\cdot\,, u) - f(\,\cdot\,, v)$ is of full rank at the point of interest; these Lie algebras take place in the theory of controllability. As far as we are concerned with sampled observability, notice that the sampled system is locally observable for all sampling time $0 < \delta < \Delta$, the parameter Δ depending on the neighbourhood of the point of interest.

In Ammar and Vivalda (2004), we gave a positive answer to the problem of the preservation of the observability in the framework of compact manifolds, to be more precise, we proved the following result.

Theorem 1.4: *Assume that the state of the system* (1) *belongs to a compact manifold and that it is observable for every admissible input $u(\cdot)$ and uniformly infinitesimally observable, then for all $M > 0$, there exists a $\delta_0 > 0$ such that the δ-sampled system of Equation* (1) *is observable for all $\delta \leq \delta_0$ and all M D-bounded input u^δ.*

Roughly speaking, the infinitesimal observability means that if we linearise the system (1) along its trajectories, we obtain a (time-varying) linear system that is observable; the reader is referred to Gauthier and Kupka (2001) for a rigorous definition of this notion. The D-boundedness means that the control u does not vary too much, to be more precise saying that u^δ is M D-bounded means that M is an upper bound for the Newton's quotients of u^δ. We would like to insist on the fact that this result is no more true when we consider systems which evolve on non-compact manifolds. As a matter of fact, consider the following system defined on $\mathbf{R}^2$:

$$\begin{cases} \dot{x}_1(t) = -\|x(t)\|^2 x_2(t) \\ \dot{x}_2(t) = \|x(t)\|^2 x_1(t) \\ \quad y = x_1, \end{cases} \quad (3)$$

here $\|\cdot\|$ denotes the usual euclidean norm on $\mathbf{R}^2$. The trajectories of this system are circles centred at the origin; to be more precise, one can write explicitly the solutions of Equation (3) as:

$$\begin{cases} x_1(t) = x_1(0)\cos(\|x(0)\|^2 t) - x_2(0)\sin(\|x(0)\|^2 t) \\ x_2(t) = x_1(0)\sin(\|x(0)\|^2 t) + x_2(0)\cos(\|x(0)\|^2 t), \end{cases}$$

this shows clearly that the system (3) is observable. Take now $\delta > 0$, the δ-sampled system related to Equation (3) writes:

$$\begin{cases} x_1((k+1)\delta) = x_1(k\delta)\cos(\|x(k\delta)\|^2\delta) \\ \qquad\qquad\quad - x_2(k\delta)\sin(\|x(k\delta)\|^2\delta) \\ x_2((k+1)\delta) = x_1(k\delta)\sin(\|x(k\delta)\|^2\delta) \qquad (4) \\ \qquad\qquad\quad + x_2(k\delta)\cos(\|x(k\delta)\|^2\delta) \\ y(k\delta) = x_1(k\delta). \end{cases}$$

Let $r = \sqrt{\frac{2\pi}{\delta}}$ and consider the initial conditions $x^0 = (0, r)$ and $\bar{x}^0 = (0, -r)$. Clearly, we have $x(k\delta) = x^0$ and $\bar{x}(k\delta) = \bar{x}^0$ for every $k \geq 0$; as $h(x^0) = h(\bar{x}^0)$, we see that $h(x(k\delta)) = h(\bar{x}(k\delta))$ for every $k \in \mathbf{N}$, whereas $x_0 \neq \bar{x}_0$. This proves that it is not possible to find a sampling time δ such that the system (4) is observable.

If we relax the compactness assumptions and if we assume that the output function is analytic, we can get a result weaker than observability, the following result has been proved in Ammar (2006).

Theorem 1.5: *Assume that system* (1)*, defined on* $\mathbf{R}^n$*, is observable for all admissible inputs and that its output function is analytic, then there exists* δ_0 *such that for every* $\delta \leq \delta_0$*, the subset of* $\mathbf{R}^n \times \mathbf{R}^n$ *constituted by the pairs of distinguishable points is dense in* $\mathbf{R}^n \times \mathbf{R}^n$*.*

In this paper, we want to extend the above-mentioned result to a particular class of system defined in $\mathbf{R}^n$: the bilinear systems, that is to say systems that write:

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} u_i(B_i x + b_i) \\ y = Cx. \end{cases} \qquad (5)$$

Here the state x belongs to $\mathbf{R}^n$, A as well as the B_i's denote $n \times n$ squared matrices, the b_i's are constant vectors in $\mathbf{R}^n$ and C is a $p \times n$ matrix. The set of admissible controls is taken as the set of measurable and essentially bounded mappings $u = (u_1, \ldots, u_m)$ defined from $\mathbf{R}_+$ to $\mathscr{U}$ where $\mathscr{U}$ is a compact subset of $\mathbf{R}^m$; this set will be denoted by $L^\infty(\mathbf{R}_+, \mathscr{U})$. In the sequel, we shall assume that system (5) is observable for all admissible inputs. In Ammar et al. (2010), a result of observability for small controls is proved, to be more precise, we have the following theorem.

Theorem 1.6: *If the pair* (A, C) *is observable, then there exists* $\delta_0 > 0$ *such that for all* $\delta \leq \delta_0$*, there exist* $\eta_\delta > 0$ *and an integer N, such that every input* u^δ *that is constant on the intervals* $[k\delta, (k+1)\delta)$ *and that satisfies the inequality* $\|u^\delta\|_\infty \leq \eta_\delta$ *is N-universal for the* δ*-sampled system related to the system* (5)*.*

Here saying that the control u^δ is N-universal means that u^δ permits to distinguish two distinct points in less than N steps, to be more precise, given two initial conditions $x_0 \neq \bar{x}_0$, there exists an index $k \leq N$ such that $h(\varphi(k\delta, 0, x_0, u^\delta(\cdot))) \neq h(\varphi(k\delta, 0, \bar{x}_0, u^\delta(\cdot)))$.

Before stating our main result, we make the following remarks. Related to system (5), we consider the following system:

$$\begin{cases} \dot{x} = Ax + \sum_{i=1}^{m} u_i\, B_i x \\ y = Cx. \end{cases} \qquad (6)$$

It is well known that the observability of system (5) is equivalent to the one of the homogeneous-related bilinear system (6). Now, take u^δ a control which is constant on the intervals $[k\delta, (k+1)\delta)$ and let $M_k^{u^\delta} \triangleq A + \sum_{i=1}^{m} u_i^\delta(k\delta)B_i$; the δ-sampled system related to Equation (5) writes

$$\begin{cases} x_{k+1} = e^{\delta M_k^{u^\delta}} x_k + \int_{k\delta}^{(k+1)\delta} e^{((k+1)\delta - s)M_k^{u^\delta}} \sum_{i=1}^{m} u_i^\delta(k\delta)b_i\, \mathrm{d}s \\ y_{k+1} = C x_{k+1}. \end{cases}$$
$$(7)$$

So, clearly the difference of two solutions of Equation (7) is a solution of the δ-sampled system related to Equation (6), the homogeneous part of Equation (5), that is to say a solution of the discrete-time system

$$\begin{cases} x_{k+1} = e^{\delta M_k^{u^\delta}} x_k \\ y_{k+1} = C x_{k+1}. \end{cases} \qquad (8)$$

This fact implies easily that the observability of system (7) for every admissible input is equivalent to the observability of system (8); so in the sequel, without loss of generality, we limit ourselves to study the problem of the preservation of the observability of *homogeneous* bilinear systems.

2. Main result

In this section, we shall state our main theorem; recall that the set of admissible inputs for system (6) is the set $L^\infty(\mathbf{R}_+, \mathscr{U})$; for system (8) this set is the subset U^d of U defined as the union $U^d = \bigcup_{\delta > 0} U^\delta$ where U^δ is the set of the controls which are constant on the intervals $[k\delta, (k+1)\delta)$ and which take their values in $\mathscr{U}$. We recall the notion of D-boundedness introduced in Ammar and Vivalda (2004).

Definition 2.1: Let u be a control in U^δ, we shall say that u is D-bounded by M (or M D-bounded) if there exists a differentiable mapping v such that

- for every $k \in \mathbf{N}$ and every $t \in [k\delta, (k + 1)\delta)$, we have $u(t) = v(k\delta)$;
- the sup norm of the derivative of v, $\|v'\|_\infty = \sup_{t \geq 0} \|v'(t)\|$, is less than M.

Theorem 2.2: *Assume that $\mathscr{U}$ is a compact subset of $\mathbf{R}^m$ and take $M > 0$; if system (6) is observable for all admissible inputs, then there exists $\delta_0 > 0$ such that, for all $\delta \leq \delta_0$, the δ-sampled system (8) related to Equation (6) is observable for every admissible input which is D-bounded by M.*

In this theorem, the assumption of infinitesimal observability does not appear (contrary to the main result in Ammar and Vivalda (2004)). This is because a bilinear system always satisfies this property.

In order to prove this result, we need the three following lemmas. Hereafter, given an admissible control $u \in L^\infty(\mathbf{R}_+, \mathscr{U})$, we denote by $\phi_u(t, \tau)$ the fundamental matrix related to the linear time varying equation

$$\dot{x} = \left(A + \sum_{i=1}^m u_i(t)B_i\right)x,$$

that is to say ϕ_u is such that $\phi_u(t_0, t_0) = \mathrm{Id}$ and $d(\phi_u(t, t_0))/dt = (A + \sum_{i=1}^m u_i(t)B_i)\phi_u(t, t_0)$. Given an admissible control u and $T > 0$, we denote by $\|u\|_\infty^T$ the norm of u defined as

$$\|u\|_\infty^T = \mathrm{ess\,sup}_{t \in [0,T]} \sum_{i=1}^m |u_i(t)|,$$

here ess sup denotes the essential supremum norm. Moreover, notice that when the set $\mathscr{U}$ is compact, the matrix $A + \sum_{i=1}^m u_i B_i$ is bounded by a constant which depends only on A, the B_i's and $\mathscr{U}$. This implies that, on every interval $[0, T]$, the matrix $\phi_u(t, \tau)$ is bounded by a constant which depends only on A, the B_i's, $\mathscr{U}$ and T.

The following lemma is an easy consequence of the Gronwall's inequality.

Lemma 2.3: *Assume that $\mathscr{U}$ is compact, take two controls $u, v \in L^\infty(\mathbf{R}_+, \mathscr{U})$ and $T > 0$. Then, there exist two real constants K_1 and K_2 (which are independent from u and v) such that*

$$\|\phi_u(t, 0) - \phi_v(t, 0)\| \leq t\,K_1\|u - v\|_\infty^T\,e^{t\,K_2}$$

for every $t \in [0, T]$.

Proof: From the definition of ϕ_u and ϕ_v, we have

$$\phi_u(t, 0) = \int_0^t \left(A + \sum_{i=1}^m u_i(s)B_i\right)\phi_u(s, 0)\,\mathrm{d}s$$

$$\phi_v(t, 0) = \int_0^t \left(A + \sum_{i=1}^m v_i(s)B_i\right)\phi_v(s, 0)\,\mathrm{d}s.$$

As noticed above, the matrices $A + \sum_{i=1}^m v_i B_i$, ϕ_u and ϕ_v are bounded, therefore there exist two positive numbers K_1 and K_2 (which do not depend on u and v) such that $\|B\phi_u(s, 0)\| \leq K_1$ and $\|A + \sum_{i=1}^m v_i(s)B\| \leq K_2$ for every $s \in [0, T]$. This and the above equalities allow us to arrive to the following inequality:

$$\|\phi_u(t, 0) - \phi_v(t, 0)\| \leq K_2 \int_0^t \|\phi_u(s, 0) - \phi_u^\delta(s, 0)\|\,\mathrm{d}s + t\,K_1\|u - v\|_\infty^T \quad (9)$$

for every $t \in [0, T]$. It follows from the Gronwall's lemma that

$$\|\phi_u(t, 0) - \phi_v(t, 0)\| \leq t\,K_1\|u - v\|_\infty^T\,e^{t\,K_2}$$

for every $t \in [0, T]$.   □

Given a sampling time $\delta > 0$ and an admissible control $u \in L^\infty(\mathbf{R}_+, \mathscr{U})$, we denote by u^δ, the δ-sampled control related to u, that is to say the control which is constant and equal to $u(k\delta)$ on every interval $[k\delta, (k + 1)\delta)$; for the sake of readability, we denote then by $\phi_u^\delta(t, \tau)$ (instead of $\phi_{u^\delta}(t, \tau)$) the fundamental matrix related to the linear time varying system

$$\dot{x} = \left(A + \sum_{i=1}^m u_i^\delta(t)B_i\right)x.$$

For a control $u \in U$ defined on the interval $[0, T]$, we denote by $\mathscr{G}_u(T)$ the observability Gramian related to the system (6), namely $\mathscr{G}_u(T)$ is the symmetric matrix defined as:

$$\mathscr{G}_u(T) \triangleq \int_0^T \phi_u(t, 0)^\mathrm{T} C^\mathrm{T} C\phi_u(t, 0)\,\mathrm{d}t \quad (10)$$

(here $^\mathrm{T}$ denotes the transposition). If the control u belongs to U^δ, we denote by $\mathscr{G}_u^\delta(N)$ the observability Gramian related to system (8), $\mathscr{G}_u^\delta(N)$ is defined as

$$\mathscr{G}_u^\delta(N) \triangleq \sum_{k=0}^N \phi_u^\delta(k\delta, 0)^\mathrm{T} C^\mathrm{T} C\phi_u^\delta(k\delta, 0). \quad (11)$$

If u belongs to U^δ, for the sake of readability, we shall use the notation $\mathscr{G}_u^\delta$ instead of $\mathscr{G}_{u^\delta}^\delta$.

Lemma 2.4: *Assume that $\mathscr{U}$ is compact and take a continuous control $u : \mathbf{R}_+ \to \mathscr{U}$. If the control u is universal for system* (6) *then there exists $\delta_0 > 0$ such that, for every $\delta < \delta_0$, the control u^δ is universal for system* (8).

Proof: As the control u is universal for system (6), there exists $T > 0$ such that the observability Gramian $\mathscr{G}_u(T)$ is positive definite.

On the other hand, the control u^δ is universal for system (8) if and only if there exists an integer N such that the discrete observability Gramian $\mathscr{G}_u^\delta(N)$ is positive definite. As the set of positive definite matrices is an open subset of the set of symmetric matrices, there exists $\varepsilon_0 > 0$ such that any symmetric matrix S satisfying $\|\mathscr{G}_u - S\| < \varepsilon_0$ is positive definite. In order to prove that $\mathscr{G}_u^\delta$ is positive definite, it is then enough to prove the following inequality:

$$\|\mathscr{G}_u - \delta \, \mathscr{G}_u^\delta\| < \varepsilon_0,$$

we shall do this for $N = [T/\delta]$ with δ small enough. We let

$$\psi_u(t) = \phi_u(t, 0)^\mathrm{T} C^\mathrm{T} C \phi_u(t, 0),$$
$$\psi_u^\delta(t) = \phi_u^\delta(t, 0)^\mathrm{T} C^\mathrm{T} C \phi_u^\delta(t, 0).$$

We have

$$\|\psi_u(t) - \psi_u^\delta(t)\| = \|\phi_u(t, 0)^\mathrm{T} C^\mathrm{T} C \phi_u(t, 0)$$
$$- \phi_u^\delta(t, 0)^\mathrm{T} C^\mathrm{T} C \phi_u^\delta(t, 0)\|$$
$$\leq \|(\phi_u(t, 0)^\mathrm{T} - \phi_u^\delta(t, 0)^\mathrm{T}) C^\mathrm{T} C \phi_u(t, 0)\|$$
$$+ \|\phi_u^\delta(t, 0)^\mathrm{T} C^\mathrm{T} C (\phi_u(t, 0) - \phi_u^\delta(t, 0)^T)\|. \quad (12)$$

From lemma (2.3), there exist two positive constants K_1 and K_2 (which do not depend on u) such that

$$\|\phi_u(t, 0) - \phi_u^\delta(t, 0)\| < t \, \|u - u^\delta\|_\infty^T K_1 \, e^{t K_2} \quad (13)$$

for every $t \in [0, T]$. As the controls u and u^δ take their values in the compact subset $\mathscr{U}$, the matrices $A + \sum_{i=1}^m u_i B_i$ and $A + \sum_{i=1}^m u_i^\delta B_i$ are bounded, so from the Gronwall lemma, it follows that $\phi_u(t, 0)$ and $\phi_u^\delta(t, 0)$ are bounded on $[0, T]$ (the upper bound being independent from δ), so there exists a constant K_3 such that

$$\|C^\mathrm{T} C \phi_u(t, 0)\| \leq K_3, \quad \|\phi_u^\delta(t, 0)^\mathrm{T} C^\mathrm{T} C\| \leq K_3, \quad (14)$$

for every $t \in [0, T]$. Using inequalities (14) and (13) in Equation (12), we get

$$\|\psi_u(t) - \psi_u^\delta(t)\| \leq t \, \|u - u^\delta\|_\infty^T K_4 \, e^{t K_1} \quad (15)$$

for every $t \in [0, T]$; here K_4 is equal to $2 K_2 K_3$.

Now, the function ψ_u being continuous, its integral can be approximated by its Riemann sum, so there exists $\delta_1 > 0$ such that if $\delta < \delta_1$ we have

$$\left\| \int_0^T \psi_u(t) \, \mathrm{d}t - \sum_{k=0}^{[\frac{T}{\delta}]} \delta \, \psi_u(k\delta) \right\| < \frac{\varepsilon_0}{2}.$$

As the function $t \mapsto t \, e^{t K_1}$ is non-negative and increasing on the interval $[0, T]$, we also have

$$\int_0^T t \, e^{t K_1} \mathrm{d}t \geq \sum_{k=0}^{[\frac{T}{\delta}]} \int_{k\delta}^{(k+1)\delta} t \, e^{t K_1} \mathrm{d}t$$
$$\geq \sum_{k=0}^{[\frac{T}{\delta}]} k \, \delta^2 e^{k \delta K_1},$$

so

$$\sum_{k=0}^{[\frac{T}{\delta}]} k \, e^{k \delta K_1} \leq \frac{K_5}{\delta^2}, \quad (16)$$

where $K_5 = \int_0^T t \, e^{t K_1} \mathrm{d}t$.

The control u being continuous, it is uniformly continuous on the interval $[0, T]$; from this it follows that there exists $\delta_2 > 0$ such that

$$\|u - u^\delta\|_\infty^T < \frac{\varepsilon_0}{2 K_4 K_5}$$

as soon as $\delta < \delta_2$. We have then

$$\left\| \sum_{k=0}^{[\frac{T}{\delta}]} \delta \, \psi_u(k\,\delta) - \sum_{k=0}^{[\frac{T}{\delta}]} \delta \, \psi_u^\delta(k\,\delta) \right\|$$
$$\leq \sum_{k=0}^{[\frac{T}{\delta}]} \delta \, \|\psi_u(k\,\delta) - \psi_u^\delta(k\,\delta)\|$$
$$\leq \sum_{k=0}^{[\frac{T}{\delta}]} \delta(k\,\delta) \frac{\varepsilon_0}{2 K_4 K_5} K_4 e^{k \delta K_1} \qquad \text{from Equation (15)}$$
$$= \frac{\varepsilon_0}{2 K_5} \delta^2 \sum_{k=0}^{[\frac{T}{\delta}]} k \, e^{k \delta K_1}$$
$$\leq \frac{\varepsilon_0}{2} \qquad \text{from Equation (16).}$$

Take $\delta < \min(\delta_1, \delta_2)$ and $N = [T/\delta]$, we finally have

$$\|\mathscr{G}_u - \delta\,\mathscr{G}_u^\delta\| = \left\| \int_0^T \psi_u(t)\mathrm{d}t - \delta \sum_{k=0}^{\left[\frac{T}{\delta}\right]} \psi_u^\delta(k\,\delta) \right\|$$

$$\leq \left\| \int_0^T \psi_u(t)\mathrm{d}t - \sum_{k=0}^{\left[\frac{T}{\delta}\right]} \delta\,\psi_u(k\,\delta) \right\|$$

$$+ \left\| \sum_{k=0}^{\left[\frac{T}{\delta}\right]} \delta\,\psi_u(k\,\delta) - \sum_{k=0}^{\left[\frac{T}{\delta}\right]} \delta\,\psi_u^\delta(k\,\delta) \right\|$$

$$\leq \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2}$$

$$= \varepsilon_0.$$

This inequality completes the proof.   □

The third lemma gives an estimation of the difference between two discrete observability Gramians.

Lemma 2.5: *Let u and v two admissible controls in U^δ, then we have*

$$\|\mathscr{G}_u^\delta(N) - \mathscr{G}_v^\delta(N)\| \leq \frac{K_4 K_5}{\delta}\,\|u - v\|_\infty^T,$$

where $N = [\frac{T}{\delta}]$ and, K_4 and K_5 are the constants defined in the proof of Lemma 2.4.

Proof: We have

$$\|\mathscr{G}_u^\delta(N) - \mathscr{G}_v^\delta(N)\| = \left\| \sum_{k=0}^{N} \psi_u^\delta(k\delta) - \psi_v^\delta(k\delta) \right\|$$

$$\leq \sum_{k=0}^{N} (k\delta)\,\|u - v\|_\infty^T\, K_4\, e^{k\delta K_1} \quad \text{from Equation (15)}$$

$$= \delta\,K_4\|u - v\|_\infty^T \sum_{k=0}^{N} k\, e^{k\delta K_1}$$

$$\leq \delta\,K_4\|u - v\|_\infty^T \frac{K_5}{\delta^2} \quad \text{from Equation (16)}$$

$$= \frac{K_4 K_5}{\delta}\,\|u - v\|_\infty^T. \quad\quad □$$

2.1. Proof of Theorem 2.2

The space of the mappings from $\mathbf{R}_+$ to $\mathscr{U}$ endowed with the pointwise convergence topology is identified with the space $\prod_{t\in\mathbf{R}_+} \mathscr{U}$ endowed with the product topology; this space is compact and so every sequence $(f_p)_{p \geq 0}$ on this space admits (at least) one limit point but, generally, we cannot extract a subsequence which converges to this limit point; nevertheless, this is true if the f_p's are D-bounded by the same constant M. The proof of the following lemma can be found in Ammar and Vivalda (2004).

Lemma 2.6: *Let $(\delta_p)_{p\geq 0}$ be a sequence of positive numbers and $(u_p)_{p\geq 0}$ a sequence of M D-bounded admissible controls such that each u_p belongs to U^{δ_p}. Assume that $\lim_{p\to\infty}\delta_p=0$, then there exists a subsequence $(u_{p_k})_{k\geq 0}$ of $(u_p)_{p\geq 0}$ which converges to a continuous function u.*

This lemma together with Lemmas 2.4 and 2.5 will allow us to achieve the proof of Theorem 2.2. We shall argue by contradiction, let $M>0$ and assume there exists a sequence of positive numbers $(\delta_p)_{p\geq 0}$ and a sequence $(u_p)_{p\geq 0}$ of M D-bounded admissible controls such that $\lim_{p\to\infty}\delta_p = 0$ and, for every p, $u_p \in U^{\delta_p}$; we assume also that none of the u_p is universal for the discrete-time system (8).

Lemma 2.6 ensures the existence of a continuous control $u \in U$ such that a subsequence of $(u_p)_{p\geq 0}$ converges pointwise to u. Without loss of generality, we shall assume that the sequence $(u_p)_{p\geq 0}$ itself converges pointwise to u. According to the assumptions of Theorem 2.2, u is universal, so there exists a time $T > 0$ such that the observability Gramian $\mathscr{G}_u(T)$ is positive definite.

Now for every index p, there exists a differentiable admissible control v_p such that $u_p(t) = v_p(k\delta_p)$ for every $k \geq 0$ and $t \in [k\delta_p, (k+1)\delta_p)$; moreover, the derivatives of the v_p's are bounded by M. Consider the family of mappings $(v_p)_{p\geq 0}$ restricted to the interval $[0, T]$

- this family is equicontinuous because the derivatives of the v_p are all bounded by M;
- for any $t \geq 0$, the family $(v_p(t))_{p\geq 0}$ is bounded because the v_p's take their values in the compact set $\mathscr{U}$.

The Arzelà–Ascoli theorem implies the existence of a subsequence of $(v_p)_{p\geq 0}$ which converges uniformly to a continuous function v defined on $[0, T]$ and whose range is included in $\mathscr{U}$. Without loss of generality, we shall assume that $(v_p)_{p\geq 0}$ (restricted to $[0, T]$) converges uniformly to v.

We shall show that $v \equiv u$ on $[0, T]$. Take $\varepsilon > 0$ and $t \in [0, T]$ and endow $\mathbf{R}^m$ with the following norm:

$$\|x\|_1 \triangleq \sum_{i=1}^{m} |x_i|.$$

We have

$$\|v(t) - u(t)\|_1 \leq \|v(t) - v_p(t)\|_1 + \|v_p(t) - u_p(t)\|_1$$
$$+ \|u_p(t) - u(t)\|_1 \leq \|v - v_p\|_\infty^T + \|v_p(t) - u_p(t)\|_1$$
$$+ \|u_p(t) - u(t)\|_1. \quad\quad (17)$$

As the sequence $(v_p)_{p\geq 0}$ converges uniformly to v, there exists p_1 such that $\|v - v_p\|_\infty^T < \varepsilon/3$ as soon as $p > p_1$. For

every $p \geq 0$ there exists $k_p \in \mathbf{N}$ such that $k_p \delta_p \leq t < (k_p + 1)\delta_p$ so we have

$$\begin{aligned}
\|v_p(t) - u_p(t)\|_1 &= \|v_p(t) - v_p(k_p \delta_p)\|_1 \\
&\leq M(t - k_p \delta_p) \\
&\leq M \delta_p.
\end{aligned}$$

Since $\lim_{p \to \infty} \delta_p = 0$, we have $M\delta_p < \varepsilon/3$ as soon as p is greater than some integer p_2. Finally, as $(u_p)_{p \geq 0}$ converges pointwise to u, there exists an index p_3 such that $\|u_p(t) - u(t)\|_1 < \varepsilon/3$ as soon as $p > p_3$. If we take $p > \max(p_1, p_2, p_3)$, from Equation (17) we get $\|v(t) - u(t)\|_1 < \varepsilon$; as this inequality is true for any $\varepsilon > 0$ we have $u \equiv v$ on $[0, T]$.

The matrix $\mathscr{G}_u(T)$ is positive definite, so there exists $\varepsilon_0 > 0$ such that, in the set of $n \times n$ symmetric matrices, the ball of centre $\mathscr{G}_u(T)$ and of radius ε_0 contains only positive definite matrices. Let $N_p = [\frac{T}{\delta_p}]$, we have

$$\begin{aligned}
\|\mathscr{G}_u(T) - \delta_p \mathscr{G}_{u_p}^{\delta_p}(N_p)\| &\leq \|\mathscr{G}_u(T) - \delta_p \mathscr{G}_u^{\delta_p}(N_p)\| \\
&\quad + \delta_p \|\mathscr{G}_u^{\delta_p}(N_p) - \mathscr{G}_{u_p}^{\delta_p}(N_p)\|.
\end{aligned} \tag{18}$$

From Lemma 2.4, as the sequence $(\delta_p)_{p \geq 0}$ tends to 0, there exists an index p_1 such that the first term in the right-hand member of Equation (18) is bounded from above by $\varepsilon_0/2$ as soon as p is greater than some index p_1. The second term in the right-hand member of Equation (18) can be bounded from above as follows:

$$\begin{aligned}
&\|\mathscr{G}_u^{\delta_p}(N_p) - \mathscr{G}_{u_p}^{\delta_p}(N_p)\| \\
&\quad \leq \frac{K_4 K_5}{\delta_p} \|u^{\delta_p} - u_p\|_\infty^T \qquad \text{from Lemma 2.5} \\
&\quad = \frac{K_4 K_5}{\delta_p} \|v^{\delta_p} - u_p\|_\infty^T \qquad \text{since } u \equiv v \text{ on } [0, T] \\
&\quad \leq \frac{K_4 K_5}{\delta_p} \left(\|v^{\delta_p} - v\|_\infty^T + \|v - u_p\|_\infty^T \right).
\end{aligned} \tag{19}$$

As v is uniformly continuous on $[0, T]$ and $\lim_{p \to \infty} \|v - u_p\|_\infty^T = 0$, the right-hand member of Equation (19) can be made less than $\varepsilon_0/(2\delta_p)$ provided that p is greater than some index p_2. It follows that the right-hand member of Equation (18) can be made less than ε_0 provided that $p > \max(p_1, p_2)$; this implies that the matrix $\delta_p \mathscr{G}_{u_p}^{\delta_p}(N_p)$ is positive definite, which is a contradiction. This completes the proof of the theorem.

3. Counterexamples

We may wonder if it is possible to relax some assumptions in the statement of Theorem 2.2. The first counterexample shows that our main result is no more valid if $\mathscr{U}$ is not compact. As a matter of fact, consider the following bilinear system given in R^2:

$$\begin{cases}
\dot{x}_1 = -u\, x_2 \\
\dot{x}_2 = u\, x_1 \\
y = Cx = x_1,
\end{cases} \tag{20}$$

the space $\mathscr{U}$ being equal to $[1, +\infty[$ while U, the space of admissible controls, is $L^\infty(\mathbf{R}_+, [1, +\infty))$. First, we shall see that this system is observable for all inputs. Take $u \in L^\infty(\mathbf{R}_+, \mathscr{U})$ as well as $x^0 = (x_1^0, x_2^0)$ and $\bar{x}^0 = (\bar{x}_1^0, \bar{x}_2^0)$ two initial conditions. Denote by $x(t)$ and $\bar{x}(t)$ the solutions of the system (20) with initial conditions x^0 and $\bar{x}^0$ respectively, with the control u. Assume that $Cx(t) = C\bar{x}(t)$ for every $t \geq 0$, by deriving this equality it follows that

$$\begin{aligned}
\dot{x}_1(t) &= \dot{\bar{x}}_1(t) & \text{for almost every } t \geq 0 \\
-u(t)x_2(t) &= -u(t)\bar{x}_2(t) & \text{for every } t \geq 0 \\
x_2(t) &= \bar{x}_2(t) & \text{as } u(t) \neq 0.
\end{aligned}$$

So we have proved that $x^0 = \bar{x}^0$.

The solution of the system (20) can be easily computed:

$$\begin{aligned}
x_1(t) &= x_1(0)\cos(U(t)) - x_2(0)\sin(U(t)) \\
x_2(t) &= x_1(0)\sin(U(t)) + x_2(0)\cos(U(t)),
\end{aligned}$$

with $U(t) = \int_0^t u(s)\,ds$. From this expression of the general solution of the system (20) we infer that the related δ-sampled system writes

$$\begin{cases}
x_1((k+1)\delta) = x_1(k\delta)\cos(\delta\, u(k\delta)) - x_2(k\delta)\sin(\delta\, u(k\delta)) \\
x_2((k+1)\delta) = x_2(k\delta)\cos(\delta\, u(k\delta)) + x_1(k\delta)\sin(\delta\, u(k\delta)) \\
y(k\delta) = x_1(k\delta).
\end{cases} \tag{21}$$

For every $0 < \delta < \pi$, we consider the constant control u_δ equal to π/δ and the two initial conditions $x^0 = (0, 1)$ and $\bar{x}^0 = (0, -1)$. We denote by $(x^k)_{k \geq 0}$ and $(\bar{x}^k)_{k \geq 0}$ the solutions of the discrete-time system (21) with initial conditions x^0 and $\bar{x}^0$, respectively, with the control u_δ. Clearly, we have $Cx^k = C\bar{x}^k = 0$ for every index k, which proves that the system (21) is not observable for any input.

In the following counter-example, we show that the assumption of D-boundedness cannot be relaxed anymore. We consider the following system given in $\mathbf{R}^3$.

$$\begin{cases}
\dot{x} = u\, Bx \\
y = Cx,
\end{cases} \tag{22}$$

where B and C are the matrices

$$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

and $\mathscr{U} = \{-1, 1\}$. The solution of the system (22) with the initial condition x^0 and a control u which is constant and equal to 1 can be easily computed:

$$\begin{cases} x_1(t) = x_2^0 \sin t + x_3^0 \cos t + x_1^0 - x_3^0 \\ x_2(t) = x_2^0 \cos t - x_3^0 \sin t \\ x_3(t) = x_2^0 \sin t + x_3^0 \cos t. \end{cases}$$

Take a sampling time $\delta > 0$ and choose the initial condition $\delta > 0$ such that

$$x_1^0 = 0 \qquad x_2^0 = r \, \sin\frac{\delta}{2} \qquad x_3^0 = r \, \cos\frac{\delta}{2} \qquad (23)$$

here r is a positive number. At time $t = \delta$, the solution of Equation (22) is such that

$$\begin{aligned} x_1(\delta) &= r\left(\sin\frac{\delta}{2} \, \sin\delta + \cos\frac{\delta}{2} \, \cos\delta - \cos\frac{\delta}{2} \right) \\ &= r\left(\cos\left(\delta - \frac{\delta}{2}\right) - \cos\frac{\delta}{2} \right) \\ &= 0. \end{aligned}$$

If we take the control u to be equal to -1 on the interval $[\delta, 2\delta[$, we return to the initial point x^0 with $x_1^0(2\delta) = 0$. So, the control which takes the value 1 on the intervals $[2k\delta, (2k + 1)\delta[$ and -1 on the intervals $[(2k + 1)\delta, (2k + 2)\delta[$ is such that we have, for the solution with initial condition x^0, $x_1(k\delta) = 0$ for every integer $k \geq 0$. This proves that such a control does not permit to distinguish two distinct initial conditions chosen as in Equation (23) (take x^0 and $\bar{x}^0$ as in Equation (23) with $r \neq \bar{r}$).

4. Conclusion

In Ammar and Vivalda (2004), the problem of the preservation of the observability under sampling was investigated in the framework of compact manifolds. A natural generalisation of the cited result would be to extend it to the systems globally Lipschitzian defined on non-compact manifolds. A simple representative of this class of systems is consti-

tuted by the set of bilinear systems, for this particular class of systems the preservation of the property of observability has been shown in this paper.

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