

Exponential dichotomy and (ℓ^p, ℓ^q) -admissibility on the half-line

Bogdan Sasu *, Adina Luminița Sasu

*Department of Mathematics, Faculty of Mathematics and Computer Science,
West University of Timișoara, Romania*

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Abstract

We give necessary and sufficient conditions for uniform exponential dichotomy of discrete evolution families in terms of the admissibility of the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$. We prove that the admissibility of the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ is a sufficient condition for uniform exponential dichotomy of a discrete evolution family. This condition becomes necessary for discrete evolution families with uniform exponential growth if and only if $p \geq q$. As consequences, we obtain necessary and sufficient conditions for uniform exponential dichotomy of evolution families.

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1. Introduction

Exponential dichotomy plays a very important role in the study of the asymptotic behaviour of time-varying differential equations. Starting with classical works in this field (see [9,10,12,18,19,30]) many research studies have been done to define, characterize and extend diverse concepts of exponential dichotomy for various evolution equations (see [3–8,13–17,21–28,31–33,35–37]). In the last decades important results on exponential di-

* Corresponding author.

E-mail addresses: sasu@math.uvt.ro, lbsasu@yahoo.com (B. Sasu).

chotomy in finite dimensional spaces were extended for the infinite dimensional case and valuable applications were provided (see [3–8,13–17,25,26,31,33]). Recently, for the case of evolution equations defined on real line (see [16,17]), Latushkin and Tomilov proved that the infinitesimal generator of the evolution semigroup associated to an evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s}$ on X is Fredholm on $C_0(\mathbb{R}, X)$ or on $L^p(\mathbb{R}, X)$, if and only if \mathcal{U} has exponential dichotomy on \mathbb{R}_+ and on \mathbb{R}_- and a pair of subspaces associated with the dichotomy projections is Fredholm. This result is an important generalization of a famous dichotomy theorem proved by Palmer (see [27,28]) and by Ben-Artzi and Gohberg (see [4]) for the finite dimensional case. Papers [16,17] bring into attention new perspectives concerning the applicability of the dichotomy properties, adding very nice contributions to the spectral theory of dynamical systems.

Among the tools used in the study of the asymptotic properties of evolution equations a significant method is represented by the input–output techniques (see [1,2,6,7,11,13,15,20–26,29,31,34–36]). In this context, one associates to an evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ the integral equation

$$f(t) = U(t, s)f(s) + \int_s^t U(t, \tau)v(\tau) d\tau, \quad t \geq s \geq 0, \quad (E_{\mathcal{U}})$$

and the exponential dichotomy of \mathcal{U} was expressed in terms of the solvability of $(E_{\mathcal{U}})$ on diverse function spaces. An important result was proved by Van Minh, Răbiger and Schnaubelt in [25] and it is given by

Theorem 1.1. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on a Banach space X . Then, \mathcal{U} is uniformly exponentially dichotomic if and only if for every $v \in C_0(\mathbb{R}_+, X)$ there is $f \in C_0(\mathbb{R}_+, X)$ such that the pair (f, v) satisfies Eq. $(E_{\mathcal{U}})$ and the space $V_1 = \{x \in X: U(\cdot, 0)x \in C_0(\mathbb{R}_+, X)\}$ is closed and complemented in X .*

The original proof of Theorem 1.1 used evolution semigroup techniques. Using a direct approach, this theorem was extended in [21] for the case of evolution families with nonuniform exponential growth. There we have shown that even in the nonuniform case, the solvability in $C_0(\mathbb{R}_+, X)$ of Eq. $(E_{\mathcal{U}})$ is a sufficient condition for the nonuniform exponential dichotomy of the evolution family \mathcal{U} . Using discrete-time methods, Theorem 1.1 was generalized in [22] where we gave discrete and integral characterizations for uniform exponential dichotomy of evolution families on the half-line in terms of the admissibility of the pairs $(c_0(\mathbb{N}, X), c_{00}(\mathbb{N}, X))$ and $(C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X))$, respectively. The results obtained in [22] were generalized in [23] for uniform exponential dichotomy of linear skew-product semiflows.

A version of Theorem 1.1 in terms of L^p -spaces was obtained by Van Minh and Huy in [26] and it is given by

Theorem 1.2. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X . Then, \mathcal{U} is uniformly exponentially dichotomic if and only if for every $v \in L^p(\mathbb{R}_+, X)$ there is $f \in L^p(\mathbb{R}_+, X) \cap C_b(\mathbb{R}_+, X)$ with the property that the pair (f, v) satisfies Eq. $(E_{\mathcal{U}})$ and the space $Y_1 = \{x \in X: U(\cdot, 0)x \in L^p(\mathbb{R}_+, X)\}$ is closed and complemented in X .*

The discrete methods in the study of the exponential dichotomy of evolution families were introduced by Henry in [13]. Significant generalizations for the results due to Henry were proved by Chow and Leiva in [7], where the authors introduced and characterized the concept of pointwise discrete dichotomy for a skew-product sequence $(\Phi_n(\theta), \sigma(\theta, n))_{n \in \mathbb{N}}$, over $X \times \Theta$, with X a Banach space and Θ a compact Hausdorff space. Important generalizations for the dichotomy and robustness theorems due to Henry, for linear skew-product semiflows have been presented by Pliss and Sell in [31]. Henry's ideas were the starting points for recent studies concerning the asymptotic behaviour of evolution equations using discrete-time techniques ([5,6,14,16,22,23,35]). In [16] Latushkin and Tomilov present a spectacular connection between discrete Fredholm operators and differential Fredholm operators, by means of discrete-time techniques.

The aim of this paper is to obtain necessary and sufficient conditions for uniform exponential dichotomy of evolution families on the half-line, presenting a unified approach for the study of this asymptotic property with l^p -spaces and L^p -spaces, respectively. First, we deduce characterizations for uniform exponential dichotomy of discrete evolution families in terms of the admissibility of the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$, with $p, q \in [1, \infty)$, where \mathbb{N} is the set of nonnegative integers. We prove that for a discrete evolution family $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ with the property that the space $X_1 = \{x \in X: \Phi(\cdot, 0)x \in \ell^p(\mathbb{N}, X)\}$ is closed and complemented, the admissibility of the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ implies the uniform exponential dichotomy of Φ . If $p \geq q$, the discrete admissibility becomes a necessary condition for uniform exponential dichotomy in the case of discrete evolution families with uniform exponential growth. After that, if $p \geq q$ we deduce that the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is admissible for an evolution family \mathcal{U} if and only if the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ is admissible for \mathcal{U} .

Next, we establish general characterizations for uniform exponential dichotomy of evolution families in terms of the solvability of the integral equation $(E_{\mathcal{U}})$, with the input space $W(\mathbb{R}_+, X) := L^{q_1}(\mathbb{R}_+, X) \cap \dots \cap L^{q_n}(\mathbb{R}_+, X) \cap C_{00}(\mathbb{R}, X)$ and the output space $L^p(\mathbb{R}_+, X)$, where $p, q_1, \dots, q_n \in [1, \infty)$. We apply our discrete-time results and we obtain that an evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is uniformly exponentially dichotomic if the pair $(L^p(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is admissible for it and the subspace $Y_1 := \{x \in X: U(\cdot, 0)x \in L^p(\mathbb{R}_+, X)\}$ is closed and complemented. We prove that these conditions become necessary for uniform exponential dichotomy if and only if $p \geq \min\{q_1, \dots, q_n\}$.

2. Exponential dichotomy for discrete evolution families

Let X be a real or a complex Banach space and let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on X . The norm on X and on $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. Let $\Delta = \{(m, n) \in \mathbb{N} \times \mathbb{N}: m \geq n\}$, where \mathbb{N} denotes the set of nonnegative integers. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. If $A \subset \mathbb{N}$ we denote by χ_A the characteristic function of the set A .

Definition 2.1. A family $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ of bounded linear operators on X is called discrete evolution family if $\Phi(n, n) = I$, for all $n \in \mathbb{N}$ and $\Phi(m, k)\Phi(k, n) = \Phi(m, n)$, for all $(m, k), (k, n) \in \Delta$. In addition, if there exist $M, \omega > 0$ such that $\|\Phi(m, n)\| \leq Me^{\omega(m-n)}$, for all $(m, n) \in \Delta$, then we say that Φ has uniform exponential growth.

Definition 2.2. A discrete evolution family $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ is said to be uniformly exponentially dichotomic if there exist a family of projections $\{P(n)\}_{n \in \mathbb{N}}$ and two constants $K, \nu > 0$ such that for every $(m, n) \in \Delta$, $\Phi(m, n)P(n) = P(m)\Phi(m, n)$, the restriction $\Phi(m, n)|_{\text{Ker } P(n)} : \text{Ker } P(n) \rightarrow \text{Ker } P(m)$ is an isomorphism and

$$\begin{aligned} \|\Phi(m, n)x\| &\leq K e^{-\nu(m-n)} \|x\|, \quad \forall x \in \text{Im } P(n), \quad \forall (m, n) \in \Delta, \\ \|\Phi(m, n)y\| &\geq \frac{1}{K} e^{\nu(m-n)} \|y\|, \quad \forall y \in \text{Ker } P(n), \quad \forall (m, n) \in \Delta. \end{aligned}$$

In what follows, for every $p \in [1, \infty)$ we denote $\ell_0^p(\mathbb{N}, X) = \{s \in \ell^p(\mathbb{N}, X) : s(0) = 0\}$. Let $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ be a discrete evolution family on X and let $p, q \in [1, \infty)$. We consider the discrete-time equation

$$\gamma(n+1) = \Phi(n+1, n)\gamma(n) + s(n+1), \quad n \in \mathbb{N}, \quad (E_d)$$

with $\gamma \in \ell^p(\mathbb{N}, X)$ and $s \in \ell_0^q(\mathbb{N}, X)$.

Definition 2.3. The pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is said to be admissible for the discrete evolution family $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$ if for every $s \in \ell_0^q(\mathbb{N}, X)$ there exists $\gamma \in \ell^p(\mathbb{N}, X)$ such that the pair (γ, s) satisfies Eq. (E_d) .

For every $n \in \mathbb{N}$ we consider the linear subspace

$$X_1(n) = \left\{ x \in X : \sum_{k=n}^{\infty} \|\Phi(k, n)x\|^p < \infty \right\}.$$

In what follows we suppose that $X_1 := X_1(0)$ is closed and complemented in X , i.e., there exists a closed linear subspace X_2 such that $X = X_1 \oplus X_2$. For every $n \in \mathbb{N}$, we denote $X_2(n) = \Phi(n, 0)X_2$. We denote $\Theta(\mathbb{N}, X) = \{\gamma \in \ell^p(\mathbb{N}, X) : \gamma(0) \in X_2\}$. Then $\Theta(\mathbb{N}, X)$ is a closed linear subspace of $\ell^p(\mathbb{N}, X)$.

Proposition 2.1. If the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ , then there is $\lambda > 0$ such that $\|\gamma\|_p \leq \lambda \|s\|_q$, for every $(\gamma, s) \in \Theta(\mathbb{N}, X) \times \ell_0^q(\mathbb{N}, X)$ with the property that (γ, s) satisfies Eq. (E_d) .

Proof. It is easy to see that for every $s \in \ell_0^q(\mathbb{N}, X)$ there is a unique $\gamma_s \in \Theta(\mathbb{N}, X)$ such that the pair (γ_s, s) satisfies Eq. (E_d) . Then it makes sense to consider the linear operator $Q : \ell_0^q(\mathbb{N}, X) \rightarrow \Theta(\mathbb{N}, X)$, $Q(s) = \gamma_s$. Then Q is closed, so for $\lambda = \|Q\|$ we obtain the conclusion. \square

Proposition 2.2. If the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ , then there are $\alpha, \beta > 0$ such that

- (i) $\|\Phi(m, n)x\| \leq \alpha \|x\|$, for all $x \in X_1(n)$ and all $(m, n) \in \Delta$;
- (ii) $\|\Phi(m, 0)x\| \geq \beta \|\Phi(n, 0)x\|$, for all $x \in X_2$ and all $(m, n) \in \Delta$.

Proof. Let $\lambda > 0$ be given by Proposition 2.1.

(i) Let $n \in \mathbb{N}^*$ and $x \in X_1(n)$. We consider $s, \gamma : \mathbb{N} \rightarrow X$ defined by

$$s(k) = \chi_{\{n\}}(k)x \quad \text{and} \quad \gamma(k) = (1 - \chi_{\{0, \dots, n-1\}}(k))\Phi(k, n)x, \quad \forall k \in \mathbb{N}.$$

Since $x \in X_1(n)$, we obtain that $\gamma \in \ell^p(\mathbb{N}, X)$. From $\gamma(0) = 0 \in X_2$ we have that $\gamma \in \Theta(\mathbb{N}, X)$. It is easy to see that the pair (γ, s) satisfies Eq. (E_d) , so we have that $\|\gamma\|_p \leq \lambda \|s\|_q$. This implies that

$$\|\Phi(m, n)x\| = \|\gamma(m)\| \leq \|\gamma\|_p \leq \lambda \|s\|_q = \lambda \|x\|, \quad \forall m \geq n.$$

Taking into account that λ does not depend on n or x and since $\Phi(1, 0)X_1 \subset X_1(1)$, we obtain the first inequality for $\alpha = \lambda(1 + \|\Phi(1, 0)\|)$.

(ii) Let $x \in X_2$ and let $m \in \mathbb{N}^*$. We consider the sequences $s, \gamma : \mathbb{N} \rightarrow X$,

$$s(k) = -\chi_{\{m\}}(k)\Phi(m, 0)x \quad \text{and} \quad \gamma(k) = \chi_{\{0, \dots, m-1\}}(k)\Phi(k, 0)x, \quad \forall k \in \mathbb{N}.$$

We have that $\gamma \in \Theta(\mathbb{N}, X)$ and the pair (γ, s) satisfies Eq. (E_d) . Then we deduce that $\|\gamma\|_p \leq \lambda \|s\|_q$. This implies that

$$\|\Phi(k, 0)x\| = \|\gamma(k)\| \leq \|\gamma\|_p \leq \lambda \|\Phi(m, 0)x\|, \quad \forall k \in \{0, \dots, m-1\}.$$

Setting $\beta = 1/\lambda$, we obtain the conclusion. \square

Proposition 2.3. *If the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ , then there are $K, v > 0$ such that*

$$\|\Phi(m, n)x\| \leq K e^{-v(m-n)} \|x\|, \quad \forall x \in X_1(n), \quad \forall (m, n) \in \Delta.$$

Proof. Let $\lambda > 0$ be given by Proposition 2.1 and let $\alpha > 0$ be given by Proposition 2.2(i). Let $l \in \mathbb{N}^*$ with $l \geq (e\lambda\alpha^2)^p$. Let $n \in \mathbb{N}$ and $x \in X_1(n)$. We analyze the following cases:

Case 1. $\Phi(n+l, n)x \neq 0$. Then $\Phi(k, n)x \neq 0$, for all $k \in \{n, \dots, n+l\}$. We consider the sequences

$$s : \mathbb{N} \rightarrow X, \quad s(k) = \chi_{\{n+1, \dots, n+l\}}(k) \frac{\Phi(k, n)x}{\|\Phi(k, n)x\|},$$

$$\gamma : \mathbb{N} \rightarrow X, \quad \gamma(k) = \sum_{i=0}^k \frac{\chi_{\{n+1, \dots, n+l\}}(i)}{\|\Phi(i, n)x\|} \Phi(k, n)x.$$

We have that $\gamma(0) = 0$ and $\gamma(k) = \xi \Phi(k, n)x$, for all $k \geq n+l$, where

$$\xi = \sum_{i=n+1}^{n+l} (1/\|\Phi(i, n)x\|).$$

Since $x \in X_1(n)$ we have that $\gamma \in \Theta(\mathbb{N}, X)$. It is easy to see that the pair (γ, s) satisfies Eq. (E_d) , so $\|\gamma\|_p \leq \lambda \|s\|_q$. Since $\xi \geq (l/\alpha \|x\|)$, we deduce that, for every $k \in \{n+l+1, \dots, n+2l\}$,

$$\|\Phi(n+2l, n)x\| \leq \alpha \|\Phi(k, n)x\| \leq \frac{\alpha^2 \|x\|}{l} \|\gamma(k)\|.$$

It follows that

$$\|\Phi(n+2l, n)x\|^{1/p} \leq \frac{\alpha^2 \|x\|}{l} \|\gamma\|_p \leq \frac{\alpha^2 \|x\|}{l} \lambda \|s\|_q \leq \lambda \alpha^2 \|x\|,$$

which implies that $\|\Phi(n+2l, n)x\| \leq (1/e)\|x\|$.

Case 2. $\Phi(n+2l, n)x = 0$. Then, obviously, $\|\Phi(n+2l, n)x\| \leq (1/e)\|x\|$. Taking into account that l does not depend on n or x it follows that $\|\Phi(n+2l, n)x\| \leq (1/e)\|x\|$, for all $x \in X_1(n)$ and all $n \in \mathbb{N}$. Then, for $v = 1/(2l)$ and $K = \alpha e$, we obtain the conclusion. \square

Proposition 2.4. *If the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ , then there are $K, v > 0$ such that*

$$\|\Phi(m, n)x\| \geq \frac{1}{K} e^{v(m-n)} \|x\|, \quad \forall x \in X_2(n), \quad \forall (m, n) \in \Delta.$$

Proof. Let $\lambda > 0$ be given by Proposition 2.1 and let $\beta > 0$ be given by Proposition 2.2(ii). Let $l \in \mathbb{N}^*$ with $l \geq (e\lambda/\beta^2)^p$. Let $x \in X_2 \setminus \{0\}$. Then $\Phi(k, 0)x \neq 0$, for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$. We consider the sequences

$$s: \mathbb{N} \rightarrow X, \quad s(k) = -\chi_{\{n+l+1, \dots, n+2l\}}(k) \frac{\Phi(k, 0)x}{\|\Phi(k, 0)x\|},$$

$$\gamma: \mathbb{N} \rightarrow X, \quad \gamma(k) = \sum_{j=k+1}^{\infty} \frac{\chi_{\{n+l+1, \dots, n+2l\}}(j)}{\|\Phi(j, 0)x\|} \Phi(k, 0)x.$$

We have that $\gamma \in \ell^p(\mathbb{N}, X)$. Setting $\xi = \sum_{j=n+l+1}^{n+2l} (1/\|\Phi(j, 0)x\|)$ we have that $\gamma(0) = \xi x \in X_2$, so $\gamma \in \Theta(\mathbb{N}, X)$. An easy computation shows that (γ, s) satisfies Eq. (E_d) . Then we have that $\|\gamma\|_p \leq \lambda \|s\|_q = \lambda l^{1/q} \leq \lambda l$. Moreover, we observe that $\gamma(k) = \xi \Phi(k, 0)x$, for all $k \in \{n+1, \dots, n+l\}$. Since $\xi \geq (\beta l / \|\Phi(n+2l, 0)x\|)$, we obtain that

$$\|\gamma(k)\| \geq \beta l \frac{\|\Phi(k, 0)x\|}{\|\Phi(n+2l, 0)x\|} \geq \beta^2 l \frac{\|\Phi(n, 0)x\|}{\|\Phi(n+2l, 0)x\|}, \quad \forall k \in \{n+1, \dots, n+l\}.$$

This implies that

$$l\beta^2 l^{1/p} \frac{\|\Phi(n, 0)x\|}{\|\Phi(n+2l, 0)x\|} \leq \|\gamma\|_p \leq \lambda l,$$

which shows that $\|\Phi(n+2l, 0)x\| \geq e\|\Phi(n, 0)x\|$.

Taking into account that l does not depend on n or x it follows that $\|\Phi(n+2l, 0)x\| \geq e\|\Phi(n, 0)x\|$, for all $(n, x) \in \mathbb{N} \times X_2$. Setting $v = 1/(2l)$ and $K = e/\beta$ we have that $\|\Phi(m, 0)x\| \geq (1/K) e^{v(m-n)} \|\Phi(n, 0)x\|$, for all $(m, n) \in \Delta$. Since $X_2(n) = \Phi(n, 0)X_2$, for all $n \in \mathbb{N}$, we obtain the conclusion. \square

The main result of this section is

Theorem 2.1. *Let $\Phi = \{\Phi(m, n)\}_{(m, n) \in \Delta}$ be a discrete evolution family, let $p, q \in [1, \infty)$ and let $X_1 = \{x \in X: \Phi(\cdot, 0)x \in \ell^p(\mathbb{N}, X)\}$. Then, the following assertions hold:*

- (i) If the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ and the subspace X_1 is closed and complemented in X , then Φ is uniformly exponentially dichotomic.
- (ii) If Φ has uniform exponential growth and $p \geq q$, then Φ is uniformly exponentially dichotomic if and only if the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ and the subspace X_1 is closed and complemented in X .

Proof. (i) From Propositions 2.3 and 2.4 we immediately obtain that $X_1(n)$ and $X_2(n)$ are closed linear subspaces and $X_1(n) \cap X_2(n) = \{0\}$, for all $n \in \mathbb{N}$.

Let now $n_0 \in \mathbb{N}^*$ and let $x \in X$. If $s: \mathbb{N} \rightarrow X$, $s(k) = -\chi_{\{n_0\}}(k)x$, let $\gamma \in \ell^p(\mathbb{N}, X)$ be such that the pair (γ, s) satisfies Eq. (E_d) . Since $\gamma(n) = \Phi(n, n_0)\gamma(n_0)$ for all $n \geq n_0$ and $\gamma \in \ell^p(\mathbb{N}, X)$, we deduce that $\gamma(n_0) \in X_1(n_0)$. If $\gamma(0) = x_1 + x_2$ with $x_k \in X_k$, $k \in \{1, 2\}$, then taking into account that $\gamma(n_0) = \Phi(n_0, 0)\gamma(0) - x$ it follows that $x = (\Phi(n_0, 0)x_1 - \gamma(n_0)) + \Phi(n_0, 0)x_2 \in X_1(n_0) + X_2(n_0)$. This shows that $X_1(n_0) \oplus X_2(n_0) = X$, for all $n_0 \in \mathbb{N}$.

For every $n \in \mathbb{N}$ let $P(n)$ denote the projection with $\text{Im } P(n) = X_1(n)$ and $\text{Ker } P(n) = X_2(n)$. Then $\Phi(m, n)P(n) = P(m)\Phi(m, n)$, for all $(m, n) \in \Delta$.

Let $(m, n) \in \Delta$. From the definition of the spaces $X_2(n)$ we immediately deduce that $\Phi(m, n)|_{\text{Ker } P(n)}: \text{Ker } P(n) \rightarrow \text{Ker } P(m)$ is surjective and from Proposition 2.4 we obtain that it is injective, so it is an isomorphism. Finally, using Propositions 2.3 and 2.4 the proof is complete.

(ii) *Necessity.* Let $\{P(n)\}_{n \in \mathbb{N}}$ be a family of projections and let $K, \nu > 0$ be given by Definition 2.2. From [22] we have that $L := \sup_{n \in \mathbb{N}} \|P(n)\| < \infty$.

Let $s \in \ell_0^q(\mathbb{N}, X)$. We consider the sequence $\gamma: \mathbb{N} \rightarrow X$ defined by

$$\gamma(n) = \sum_{k=0}^n \Phi(n, k)P(k)s(k) - \sum_{k=n+1}^{\infty} \Phi(k, n)|_{\text{Ker } P(n)}^{-1}(I - P(k))s(k),$$

where $\Phi(k, n)|_{\text{Ker } P(n)}^{-1}$ denotes the inverse of the operator $\Phi(k, n)|_{\text{Ker } P(n)}: \text{Ker } P(n) \rightarrow \text{Ker } P(k)$. It is easy to see that the pair (γ, s) satisfies Eq. (E_d) and using Hölder's inequality, we have that $\gamma \in \ell^p(\mathbb{N}, X)$. It follows that the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ .

Let $x \in X_1$. Then $h = \sup_{n \in \mathbb{N}} \|\Phi(n, 0)x\| < \infty$. From

$$\begin{aligned} \|x - P(0)x\| &\leq Ke^{-\nu n} \|\Phi(n, 0)(I - P(0))x\| \\ &\leq Ke^{-\nu n} [\|\Phi(n, 0)x\| + \|P(n)\Phi(n, 0)x\|] \\ &\leq K(L + 1)he^{-\nu n}, \quad \forall n \in \mathbb{N}, \end{aligned}$$

it follows that $x = P(0)x$, so $X_1 \subset \text{Im } P(0)$. Obviously we have that $\text{Im } P(0) \subset X_1$. Then we deduce that $X_1 = \text{Im } P(0)$, so X_1 is closed and complemented.

Sufficiency. This follows from (i). \square

Remark 2.1. Generally, if $p < q$ and the discrete evolution family Φ is uniformly exponentially dichotomic it does not follow that the pair $(\ell^p(\mathbb{N}, X), \ell_0^q(\mathbb{N}, X))$ is admissible for Φ , as shows the following example.

Example 2.1. Let $X = \mathbb{R}^2$ with respect to the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. We consider the discrete evolution family $\Phi = \{\Phi(m, n)\}_{(m,n) \in \Delta}$, where

$$\Phi(m, n)(x_1, x_2) = (e^{-(m-n)}x_1, e^{m-n}x_2), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \quad \forall (m, n) \in \Delta.$$

It is easy to see that Φ has uniform exponential growth and it is uniformly exponentially dichotomic with respect to the projections $P(n) = P$, for all $n \in \mathbb{N}$, where $P: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $P(x_1, x_2) = (x_1, 0)$.

If $p, q \in [1, \infty)$ with $p < q$ and $\delta \in (p, q)$ a simple computation shows that for the sequence $s: \mathbb{N} \rightarrow \mathbb{R}^2$, $s(n) = (0, s_2(n))$ where $s_2(0) = 0$ and $s_2(n) = (1/n^{1/\delta})$, for $n \in \mathbb{N}^*$, there is no $\gamma \in l^p(\mathbb{N}, \mathbb{R}^2)$ such that the pair (γ, s) satisfies Eq. (E_d) .

3. Uniform exponential dichotomy of evolution families

Let X be a real or a complex Banach space and let $\Delta = \{(m, n) \in \mathbb{N} \times \mathbb{N}: m \geq n\}$.

Definition 3.1. A family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ of bounded linear operators on X is called an evolution family if the following properties hold:

- (i) $U(t, t) = I$ and $U(t, s)U(s, t_0) = U(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$;
- (ii) there exist $M, \omega > 0$ such that $\|U(t, t_0)\| \leq Me^{\omega(t-t_0)}$, for all $t \geq t_0 \geq 0$;
- (iii) for every $x \in X$ and $t, t_0 \geq 0$, the mapping $s \mapsto U(s, t_0)x$ is continuous on $[t_0, \infty)$ and the mapping $s \mapsto U(t, s)x$ is continuous on $[0, t]$.

Remark 3.1. If $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is an evolution family, then one can associate to \mathcal{U} a discrete evolution family $\Phi_{\mathcal{U}} = \{U(m, n)\}_{(m,n) \in \Delta}$ called *the discrete evolution family associated with \mathcal{U}* .

Definition 3.2. An evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is said to be uniformly exponentially dichotomic if there exist a family of projections $\{P(t)\}_{t \geq 0}$ and two constants $K, \nu > 0$ such that for every $t \geq t_0 \geq 0$, $U(t, t_0)P(t_0) = P(t)U(t, t_0)$, the restriction $U(t, t_0)|_{\text{Ker } P(t_0)}: \text{Ker } P(t_0) \rightarrow \text{Ker } P(t)$ is an isomorphism and

$$\begin{aligned} \|U(t, t_0)x\| &\leq Ke^{-\nu(t-t_0)}\|x\|, \quad \forall x \in \text{Im } P(t_0), \quad \forall t \geq t_0 \geq 0, \\ \|U(t, t_0)y\| &\geq \frac{1}{K} e^{\nu(t-t_0)}\|y\|, \quad \forall y \in \text{Ker } P(t_0), \quad \forall t \geq t_0 \geq 0. \end{aligned}$$

Definition 3.3. Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let $p, q \in [1, \infty)$. The pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ is said to be admissible for the evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ if it is admissible for the discrete evolution family $\Phi_{\mathcal{U}}$ associated to \mathcal{U} .

Definition 3.4. Let $V(\mathbb{R}_+, X)$, $W(\mathbb{R}_+, X)$ be two Banach function spaces. The pair $(V(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is said to be admissible for the evolution family $\mathcal{U} =$

$\{U(t, s)\}_{t \geq s \geq 0}$ if for every $w \in W(\mathbb{R}_+, X)$ there exists a continuous function $f \in V(\mathbb{R}_+, X)$ such that (f, v) satisfies the equation

$$f(t) = U(t, s)f(s) + \int_s^t U(t, \tau)w(\tau) d\tau, \quad \forall t \geq s \geq 0. \quad (E_{\mathcal{U}})$$

Theorem 3.1. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let $p, q \in [1, \infty)$ with $p \geq q$. Then, the pair $(L^p(\mathbb{R}_+, X), L^q(\mathbb{R}_+, X))$ is admissible for \mathcal{U} if and only if the pair $(l^p(\mathbb{N}, X), l^q(\mathbb{N}, X))$ is admissible for \mathcal{U} .*

Proof. Let $M, \omega > 0$ be given by Definition 3.1. To prove the necessity, let $\alpha: [0, 1] \rightarrow [0, 2]$ be a continuous function with the support contained in $(0, 1)$ and $\int_0^1 \alpha(\tau) d\tau = 1$. Let $s \in l_0^q(\mathbb{N}, X)$ and let $v: \mathbb{R}_+ \rightarrow X$, $v(t) = U(t, [t])s([t])\alpha(t - [t])$. Then $v \in L^q(\mathbb{R}_+, X)$. Let $f \in L^p(\mathbb{R}_+, X)$ be such that the pair (f, v) satisfies Eq. $(E_{\mathcal{U}})$. Then, we have that $f(n+1) = U(n+1, n)(f(n) + s(n))$, for every $n \in \mathbb{N}$. Taking $\gamma(n) = f(n) + s(n)$, for all $n \in \mathbb{N}$, we obtain that the pair (γ, s) satisfies Eq. (E_d) .

Moreover, since $\|f([t] + 1)\| \leq Me^{\omega}(\|f(t)\| + 2\|s([t])\|)$, for all $t \geq 0$, it follows that $\gamma \in l^p(\mathbb{N}, X)$.

Sufficiency. Let $v \in L^q(\mathbb{R}_+, X)$. Then $\alpha: \mathbb{N} \rightarrow \mathbb{R}_+$, $\alpha(n) = (\int_n^{n+1} \|v(\tau)\|^q d\tau)^{1/q}$ belongs to $l^q(\mathbb{N}, \mathbb{R}_+)$. Let $s: \mathbb{N} \rightarrow X$ with $s(0) = 0$ and $s(n+1) = \int_n^{n+1} U(n+1, \tau)v(\tau) d\tau$, for all $n \in \mathbb{N}$. We have that $s \in l_0^q(\mathbb{N}, X)$ and let $\gamma \in l^p(\mathbb{N}, X)$ be such that the pair (γ, s) satisfies Eq. (E_d) . We consider the function

$$f: \mathbb{R}_+ \rightarrow X, \quad f(t) = U(t, [t])\gamma([t]) + \int_{[t]}^t U(t, \tau)v(\tau) d\tau.$$

It is easy to see that f is continuous and the pair (f, v) satisfies Eq. $(E_{\mathcal{U}})$. Observing that $\|f(t)\| \leq Me^{\omega}(\|\gamma([t])\| + \alpha([t]))$, for all $t \geq 0$, we deduce that $f \in L^p(\mathbb{R}_+, X)$ which completes the proof. \square

Lemma 3.1. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let $\Phi_{\mathcal{U}}$ be the discrete evolution family associated with \mathcal{U} . Then \mathcal{U} is uniformly exponentially dichotomic if and only if $\Phi_{\mathcal{U}}$ is uniformly exponentially dichotomic.*

Proof. Using similar arguments as in the proof of Theorem 3.1, we deduce that the pair $(C_0(\mathbb{R}_+, X), C_{00}(\mathbb{R}_+, X))$ is admissible for \mathcal{U} if and only if the pair $(c_0(\mathbb{N}, X), c_{00}(\mathbb{N}, X))$ is admissible for $\Phi_{\mathcal{U}}$. Using Theorems 2.3 and 3.2 from [22], we obtain the conclusion. \square

As consequences of the results presented in the previous section and using Lemma 3.1, we obtain

Theorem 3.2. *Let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X , let $p, q \in [1, \infty)$ and let $X_1 = \{x \in X: \Phi_{\mathcal{U}}(\cdot, 0)x \in l^p(\mathbb{N}, X)\}$. Then, the following assertions hold:*

- (i) If the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ is admissible for \mathcal{U} and the subspace X_1 is closed and complemented in X , then \mathcal{U} is uniformly exponentially dichotomic.
- (ii) If $p \geq q$, then \mathcal{U} is uniformly exponentially dichotomic if and only if the pair $(l^p(\mathbb{N}, X), l_0^q(\mathbb{N}, X))$ is admissible for \mathcal{U} and the subspace X_1 is closed and complemented in X .

Theorem 3.3. Let $p \in [1, \infty)$, let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution family on the Banach space X and let $X_1 = \{x \in X: U(\cdot, 0)x \in L^p(\mathbb{R}_+, X)\}$. Let $n \in \mathbb{N}^*$, let $q_1, \dots, q_n \in [1, \infty)$ and let

$$W(\mathbb{R}_+, X) = L^{q_1}(\mathbb{R}_+, X) \cap \dots \cap L^{q_n}(\mathbb{R}_+, X) \cap C_{00}(\mathbb{R}_+, X).$$

Then, the following assertions hold:

- (i) If the pair $(L^p(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is admissible for \mathcal{U} and the subspace X_1 is closed and complemented in X , then \mathcal{U} is uniformly exponentially dichotomic.
- (ii) If $\min\{q_1, \dots, q_n\} \leq p$ then \mathcal{U} is uniformly exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is admissible for \mathcal{U} and the subspace X_1 is closed and complemented in X .

Proof. (i) Let $\alpha: [0, 1] \rightarrow [0, 2]$ be a continuous function with the support contained in $(0, 1)$ and $\int_0^1 \alpha(\tau) d\tau = 1$. Let $s \in l_0^1(\mathbb{N}, X)$ and let $v: \mathbb{R}_+ \rightarrow X$, $v(t) = U(t, [t])s([t])\alpha(t - [t])$. Then $v \in W(\mathbb{R}_+, X)$. Using similar arguments as in the proof of Theorem 3.1, we obtain that the pair $(l^p(\mathbb{N}, X), l_0^1(\mathbb{N}, X))$ is admissible for \mathcal{U} . From Theorem 3.2 it follows that \mathcal{U} is uniformly exponentially dichotomic.

(ii) *Necessity.* Let $\{P(t)\}_{t \geq 0}$ be a family of projections given by Definition 3.2. Then $X_1 = \text{Im } P(0)$, so X_1 is closed and complemented in X . For $v \in W(\mathbb{R}_+, X)$, we define $f: \mathbb{R}_+ \rightarrow X$ by

$$f(t) = \int_0^t U(t, \tau)P(\tau)v(\tau) d\tau - \int_t^\infty U(\tau, t)|_t^{-1}(I - P(\tau))v(\tau) d\tau,$$

where $U(\tau, t)|_t^{-1}$ denotes the inverse of the operator $U(\tau, t)|_t: X_2(t) \rightarrow X_2(\tau)$, for every $\tau \geq t$. Since $q = \min\{q_1, \dots, q_n\} \leq p$, using Hölder's inequality it follows that $f \in L^p(\mathbb{R}_+, X)$. An easy computation shows that the pair (f, v) satisfies Eq. $(E_{\mathcal{U}})$, so the pair $(L^p(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is admissible for \mathcal{U} .

Sufficiency. This follows from (i). \square

Remark 3.2. The above theorem shows that in the study of the exponential dichotomy of evolution families by means of input–output techniques one can consider smaller and smaller input spaces.

Remark 3.3. Generally, if $p < \min\{q_1, \dots, q_n\}$ and the evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is uniformly exponentially dichotomic it does not follow that the pair $(L^p(\mathbb{R}_+, X), W(\mathbb{R}_+, X))$ is admissible for \mathcal{U} . This follows using similar arguments as in Example 2.1.

Remark 3.4. Theorem 3.3 extends a theorem proved by Van Minh and Huy in [26]. Their result states that an evolution family $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ is uniformly exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}_+, X) \cap C_b(\mathbb{R}_+, X), L^p(\mathbb{R}_+, X))$ is admissible for \mathcal{U} and the space $X_1 = \{x \in X: U(\cdot, 0)x \in L^p(\mathbb{R}_+, X)\}$ is closed and complemented in X .

Remark 3.5. For a different concept of exponential dichotomy for evolution families a theorem with L^p -spaces was recently proved by Preda, Pogan and Preda in [32].

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