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Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations

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In this paper, we propose a delayed perturbation of Mittag-Leffler type matrix function, which is an extension of the classical Mittag-Leffler type matrix function and delayed Mittag-Leffler type matrix function. With the help of the delayed perturbation of Mittag-Leffler type matrix function, we give an explicit formula of solutions to linear nonhomogeneous fractional delay differential equations.

KEYWORDS

delayed perturbation, fractional linear systems, Mittag-Leffler type matrix function

1 | INTRODUCTION

In recent decades, mathematical descriptions through fractional differential equations related to derivatives of nonintegral orders have proven to be a very useful tool in modeling various phenomena of viscoelasticity, anomalous diffusion, stability theory, control theory, and other areas. Delays are often associated with chemical processes, electrical and hydraulic networks, heredity in population dynamics, economics, and other industries. In general, a distinctive feature of the corresponding mathematical models is that the rate of evolution of these processes depends on past history. Differential equations that model these problems are called differential delay equations. The basic qualitative theory of these equations is well established, especially in the linear case. Unification of differential delay equations and fractional differential equation is provided by fractional time-delay differential equations, including both delays and noninteger derivatives. In technical applications, this approach is useful for creating highly realistic models of certain processes and systems with memory. It is involved, in particular, in the analysis of various systems with a time delay, stabilization, and control of which is carried out through state feedback. Therefore, in this respect, we study a fractional constant time-delay linear differential equation and present an explicit formula to these equations via a delayed perturbation of Mittag-Leffler type functions.

It is known that a solution of a linear system $y'(t) = Ay(t)$, $t \geq 0$ has the form $y(t) = e^{At}y(0)$, where exponential matrix e^{At} is also called fundamental matrix. However, it becomes more complex for seeking a fundamental matrix for linear delay system

$$\begin{aligned} &y'(t) = Ay(t) + By(t-h), \qquad t \geq 0, \quad h > 0, \\ &y(t) = \varphi(t), \quad -h \leq t \leq 0, \end{aligned} \tag{1}$$

where A, B are two constant square matrices. Under the assumptions that A and B are permutation matrices, Khusainov & Shuklin[1] give a representation of a solution of a linear homogeneous system with delay by introducing the concept of delayed matrix exponential e_h^{Bt} corresponding to delay h and matrix B:

$$e_h^{Bt} := \begin{cases} \Theta, & -\infty < t \leq -h, \\ I, & -h < t \leq 0, \\ I + Bt + B^2\frac{(t-h)^2}{2!} + \ldots + B^k\frac{(t-(k-1)h)^k}{k!}, & (k-1)h < t \leq kh. \end{cases}$$

They proved that fundamental matrix of linear delay system (1) (delayed perturbation of exponential matrix e^{At}) can be given by $e^{At}e_h^{B_1(t-h)}$, $B_1 = e^{-Ah}B$. Notice that the fractional analogue of the same problem was considered by Li and Wang[2] in the case $A = \Theta$. For more recent contributions on oscillating system with pure delay, relative controllability of system with pure delay, asymptotic stability of nonlinear multidelay differential equations, finite time stability of differential equations, one can refer to previous studies[1-14] and reference therein.

Motivated by Khusainov & Shuklin,[1] Li and Wang,[2] we extend to consider representation of solutions of a fractional delay differential equation of the form by introducing delayed perturbation of Mittag-Leffler function

$$\begin{aligned} \left({}^{C}D^{\alpha}_{-h^+}y\right)(t) &= Ay(t) + By(t-h) + f(t), \qquad t \in (0,T], h > 0, \\ y(t) &= \varphi(t), \quad -h \le t \le 0, \end{aligned} \tag{2}$$

where $\left({}^{C}D^{\alpha}_{-h^+}y\right)(\cdot)$ is the Caputo derivative of order, $\alpha \in (0,1)$, $A, B \in \mathbb{R}^{n\times n}$ denotes constant matrix, and $\varphi : \left[-h, 0\right] \to \mathbb{R}^n$ is an arbitrary Caputo differentiable vector function, $f \in C\left(\left[-h,T\right], \mathbb{R}^n\right)$, $T = lh$ for a fixed natural number l.

To end this section, we would like to state the main contribution as follows:

1. We propose delayed perturbation $X^{A,B}_{h,\alpha,\beta}(t)$ of Mittag-Leffler type functions, by means of the matrix equations (4). We show that for $B = \Theta$ the function $X^{A,B}_{h,\alpha,\beta}(t)$ coincide with Mittag-Leffler type function of two paramemters $t^{\alpha-1}E_{\alpha,\beta}(At^{\alpha})$. For $A = \Theta$ $X^{A,B}_{h,\alpha,\beta}(t)$ coincide with delayed Mittag-Leffler type matrix function of two parameters $E^{B}_{h,\alpha,\beta}(t-h)$.
2. We explicitly represent the solution of fractional delay linear system (2) via delayed perturbation of Mittag-Leffler type function.

Definition 1. Mittag-Leffler type matrix function of two parameters $\Phi_{\alpha,\beta}(A,z) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$\Phi_{\alpha,\beta}(A,z) := z^{\beta-1}E_{\alpha,\beta}(Az^{\alpha}) := z^{\beta-1}\sum_{k=0}^{\infty}\frac{A^k z^{\alpha k}}{\Gamma(k\alpha+\beta)}, \qquad \alpha, \beta > 0, z \in \mathbb{R}.$$

Definition 2. Delayed Mittag-Leffler type matrix function of two parameters $E^{B}_{h,\alpha,\beta}(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined by

$$E^{B}_{h,\alpha,\beta}(t) := \begin{cases} \Theta, & -\infty < t \le -h, \\ I\frac{(h+t)^{\beta-1}}{\Gamma(\beta)}, & -h < t \le 0, \\ I\frac{(h+t)^{\beta-1}}{\Gamma(\beta)} + B\frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + B^2\frac{(t-h)^{2\alpha+\beta-1}}{\Gamma(2\alpha+\beta)} + \ldots + B^p\frac{(t-(p-1)h)^{p\alpha+\beta-1}}{\Gamma(p\alpha+\beta)}, & (p-1)h < t \le ph. \end{cases} \tag{3}$$

In order to define delayed perturbation of Mittag-Leffler type matrix functions, we introduce the following matrix equation for $Q_k(s)$, $k = 1, 2, \ldots$

$$\begin{aligned} Q_{k+1}(s) &= AQ_k(s) + BQ_k(s-h), \\ Q_0(s) &= Q_k(-h) = \Theta, \; Q_1(0) = I, \\ k &= 0, 1, 2, \cdots, s = 0, h, 2h, \cdots \end{aligned} \tag{4}$$

Simple calculations show that

	$s = 0$	$s = h$	$s = 2h$	$s = 3h$	$\cdots$	$s = ph$
$Q_1(s)$	I	Θ	Θ	Θ	$\cdots$	Θ
$Q_2(s)$	A	B	Θ	Θ	$\cdots$	Θ
$Q_3(s)$	A^2	$AB + BA$	B^2	Θ	$\cdots$	Θ
$Q_4(s)$	A^3	$A(AB+BA)+BA^2$	$AB^2 + B(AB+BA)$	B^3	$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$	Θ
$Q_{p+1}(s)$	A^p	$\cdots$	$\cdots$	$\cdots$	$\cdots$	B^p

Definition 3. Delayed perturbation of two parameter Mittag-Leffler type matrix function $X^{A,B}_{h,\alpha,\beta}$ generated by A, B is defined by

$$X^{A,B}_{h,\alpha,\beta}(t) := \begin{cases} \Theta, & -h \le t < 0, \\ I, & t = 0, \\ \sum_{i=0}^{\infty}\sum_{j=0}^{p-1} Q_{i+1}(jh)\frac{(t-jh)^{i\alpha+\beta-1}}{\Gamma}(i\alpha+\beta), & (p-1)h < t \le ph. \end{cases} \tag{5}$$

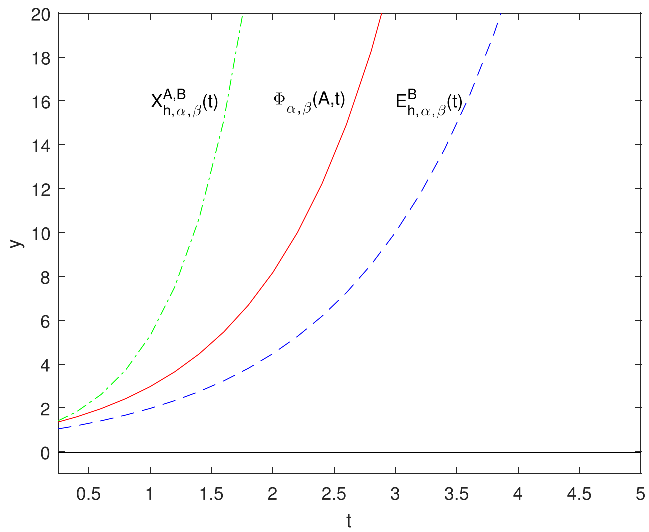

FIGURE 1 Comparison of functions $\Phi_{\alpha,\beta}(A,t)$, $E^B_{h,\alpha,\beta}(t)$, and $X^{A,B}_{h,\alpha,\beta}(t)$ for $\alpha = 0.9, \beta = 1, h = 1/4, A = 1, B = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

Remark 1. If A and B are permutable matrices, then $Q_{i+1}(jh)\begin{pmatrix} i \\ j \end{pmatrix} A^{i-j}B^j$. In this case, $X^{A,B}_{h,\alpha,\beta}(t)$ has a simple form:

$$X^{A,B}_{h,\alpha,\beta}(t) := \begin{cases} \Theta, & -h \le t < 0, \\ I, & t = 0, \\ \sum\limits_{i=0}^{\infty} A^i \frac{t^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} + \sum\limits_{i=1}^{\infty} \begin{pmatrix} i \\ 1 \end{pmatrix} A^{i-1}B\frac{(t-h)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} & \\ +\ldots + \sum\limits_{i=p-1}^{\infty} \begin{pmatrix} i \\ p-1 \end{pmatrix} A^{i-p+1}B^{p-1}\frac{(t-(p-1)h)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}, & (p-1)h < t \le ph. \end{cases}$$

Lemma 1. *Let $X^{A,B}_{h,\alpha,\beta}(t)$ be defined by (5). Then the following holds true:*

(i) if $A = \Theta$ then $X^{A,B}_{h,\alpha,\beta}(t) = E^B_{h,\alpha,\beta}(t-h)$, $(p-1)h \le t-h \le ph$,
(ii) if $B = \Theta$ then $X^{A,B}_{h,\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}(At^{\alpha}) = \Phi_{\alpha,\beta}(A,t)$,
(iii) if $\alpha = \beta = 1$ and $AB = BA$ then $X^{A,B}_{h,1,1}(t) = e^{At}e_h^{B_1(t-h)}$, $B_1 = e^{-Ah}B$, $(p-1)h < t \le ph$.

Proof.

(i) If $A = \Theta$, then

$$Q_{i+1}(jh) = \begin{cases} \Theta, & i \ne j, \\ B^i, & i = j, \end{cases}$$

and $X^{A,B}_{h,\alpha,\beta}(t)$ coincides with $E^B_{h,\alpha,\beta}(t-h)$:

$$X^{A,B}_{h,\alpha,\beta}(t) = \sum_{i=0}^{p} B^i \frac{(t-ih)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} = \frac{t^{\beta-1}}{\Gamma(\beta)} + B\frac{(t-h)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \ldots + B^p\frac{(t-ph)^{p\alpha+\beta-1}}{\Gamma(p\alpha+\beta)}$$
$$= E^B_{h,\alpha,\beta}(t-h), \; (p-1)h < t-h \le ph.$$

(ii) Trivially, from definition of $X^{A,B}_{h,\alpha,\beta}(t)$ we have: if $B = \Theta$, then

$$X^{A,B}_{h,\alpha,\beta}(t) = \sum_{i=0}^{\infty} A^i \frac{t^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)} = t^{\beta-1}E_{\alpha,\beta}(At^{\alpha}).$$

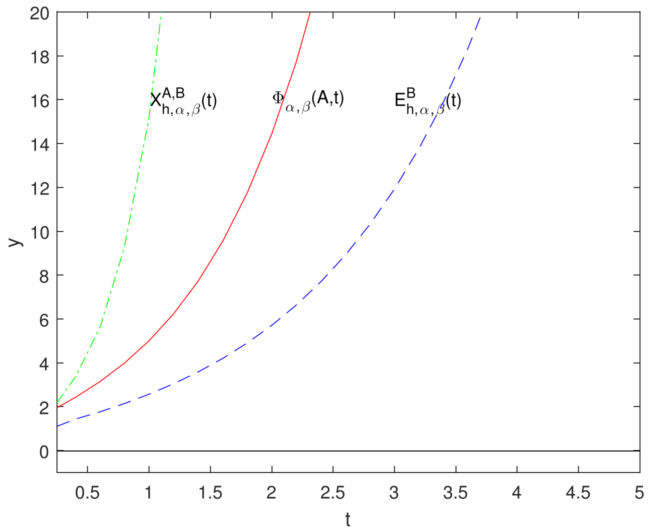

FIGURE 2 Comparison of functions $\Phi_{\alpha,\beta}(A,t)$, $E^{B}_{h,\alpha,\beta}(t)$, and $X^{A,B}_{h,\alpha,\beta}(t)$ for $\alpha = 0.5, \beta = 1, h = 1/4, A = 1, B = 1$ [Colour figure can be viewed at wileyonlinelibrary.com]

(iii) It can be easily shown that $Q_{i+1}(jh) = \binom{i}{j} A^{i-j}B^{j}$. So, for $(p-1)h < t \leq ph$ and $B_1 = e^{-Ah}B$ we have

$$\begin{aligned}
X^{A,B}_{h,1,1}(t) &= \sum_{i=0}^{\infty} Q_{i+1}(0)\frac{t^i}{i!} + \sum_{i=1}^{\infty} Q_{i+1}(h)\frac{(t-h)^i}{i!} + \ldots + \sum_{i=1}^{\infty} Q_{i+1}((p-1)h)\frac{(t-(p-1)h)^i}{i!} \\
&= \sum_{i=0}^{\infty} A^i\frac{t^i}{i!} + \sum_{i=1}^{\infty}\binom{i}{1}A^{i-1}B\frac{(t-h)^i}{i!} + \ldots + \sum_{i=p-1}^{\infty}\binom{i}{p-1}A^{i-p+1}B^{p-1}\frac{(t-(p-1)h)^i}{i!} \\
&= e^{At} + e^{A(t-h)}B(t-h) + \ldots + \sum_{i=0}^{\infty}\binom{i+p-1}{p-1}A^iB^{p-1}\frac{(t-(p-1)h)^{i+p-1}}{(i+p-1)!} \\
&= e^{At} + e^{A(t-h)}B(t-h) + \ldots + e^{A(t-(p-1)h)}B^{p-1}\frac{1}{(p-1)!}(t-(p-1)h)^{p-1} \\
&= e^{At}\left(I + e^{-Ah}B(t-h) + \ldots + e^{-A(p-1)h}B^{p-1}\frac{1}{(p-1)!}(t-(p-1)h)^{p-1}\right) = e^{At}e_h^{B_1(t-h)}.
\end{aligned}$$

□

Remark 2. Comparison of functions $\Phi_{\alpha,\beta}(A,t), E^{B}_{h,\alpha,\beta}(t)$ and $X^{A,B}_{h,\alpha,\beta}(t)$ is given in Figures 1 and 2.

Lemma 2. *Let $(p-1)h < t \leq ph, -h \leq s \leq t$. We have*

$$\int_s^t (t-r)^{-\alpha}X^{A,B}_{h,\alpha,\alpha}(r-s)\,dr = \sum_{i=0}^{\infty}\sum_{j=0}^{p-1} Q_{i+1}(jh)(t-s-jh)^{i\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(i\alpha+1)}.$$

Proof. Based on substitution, one has

$$\begin{aligned}
\int_s^t (t-r)^{-\alpha}X^{A,B}_{h,\alpha,\alpha}(r-s)\,dr &= \sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}\sum_{j=0}^{p-1}Q_{i+1}(jh)\int_s^t (t-r)^{-\alpha}(r-s-jh)^{(i+1)\alpha-1}dr \\
&= \sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}(0)\int_s^t (t-r)^{-\alpha}(r-s)^{(i+1)\alpha-1}dr \\
&\quad + \sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}(h)\int_{s+h}^t (t-r)^{-\alpha}(r-s-h)^{(i+1)\alpha-1}dr
\end{aligned}$$

$$+\dots+\sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}((p-1)h)\int_{s+(p-1)h}^{t}(t-r)^{-\alpha}(r-s-(p-1)h)^{(i+1)\alpha-1}dr$$

$$=\sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}(0)(t-s)^{i\alpha}B[1-\alpha,(i+1)\alpha]+\frac{1}{\Gamma((i+1)\alpha)}\sum_{i=0}^{\infty}Q_{i+1}(h)(t-s-h)^{i\alpha}B[1-\alpha,(i+1)\alpha]$$

$$+\dots+\sum_{i=0}^{\infty}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}((p-1)h)(t-s-(p-1)h)^{i\alpha}B[1-\alpha,(i+1)\alpha]$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{p-1}\frac{1}{\Gamma((i+1)\alpha)}Q_{i+1}(jh)(t-s-jh)^{i\alpha}B[1-\alpha,(i+1)\alpha]$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{p-1}Q_{i+1}(jh)(t-s-jh)^{i\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(i\alpha+1)}.$$

□

It turns out that $X_{h,\alpha,\beta}^{A,B}(t)$ is a delayed perturbation of the fundamental matrix of the Equation 2 with $f=0$.

Lemma 3. $X_{h,\alpha,\beta}^{A,B}: R\to R^n$ *is a solution of*

$$ {}^{C}D_{-h^+}^{\alpha}\quad X_{h,\alpha,\beta}^{A,B}(t)=AX_{h,\alpha,\beta}^{A,B}(t)+BX_{h,\alpha,\beta}^{A,B}(t-h). \tag{6}$$

Proof. We verify that $X_{h,\alpha,\alpha}^{A,B}(t)$ satisfies differential Equation 6 for $t\in\left(t_{p-1},t_p\right]$. We adopt mathematical induction to prove our result.

(i) For $p=1, 0<t\leq h$, we have

$$X_{h,\alpha,\beta}^{A,B}(t)=t^{\beta-1}E_{\alpha,\beta}(At^{\alpha}),\ X_{h,\alpha,\beta}^{A,B}(t-h)=\Theta,$$

$$ {}^{C}D_{-h^+}^{\alpha}X_{h,\alpha,\beta}^{A,B}(t)={}^{C}D_{-h^+}^{\alpha}\sum_{i=1}^{\infty}A^i\frac{t^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}=AX_{h,\alpha,\beta}^{A,B}(t)=AX_{h,\alpha,\beta}^{A,B}(t)+BX_{h,\alpha,\beta}^{A,B}(t-h).$$

(ii) Suppose $p=n$, $(n-1)h<t\leq nh$ the following relation holds:

$$X_{h,\alpha,\beta}^{A,B}(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}Q_{i+1}(jh)\frac{(t-jh)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}.$$

Next, for $p=n+1$, $nh<t\leq(n+1)h$, by elementary computation, one obtains

$$\begin{aligned}{}^{C}D_{-h^+}^{\alpha}\quad X_{h,\alpha,\beta}^{A,B}(t)&=\sum_{i=0}^{\infty}\sum_{j=0}^{n}Q_{i+1}(jh)\frac{\Gamma(i\alpha+\beta)}{\Gamma(i\alpha-\alpha+\beta)}\frac{(t-jh)^{i\alpha-\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{n}Q_{i+1}(jh)\frac{(t-jh)^{i\alpha-\alpha+\beta-1}}{\Gamma(i\alpha-\alpha+\beta)}=\sum_{i=0}^{\infty}\sum_{j=0}^{n}(AQ_i(jh)+BQ_i(jh-h))\frac{(t-jh)^{i\alpha-\alpha+\beta-1}}{\Gamma(i\alpha-\alpha+\beta)}\\&=\sum_{i=0}^{\infty}\sum_{j=0}^{n}AQ_{i+1}(jh)\frac{(t-jh)^{i\alpha+\beta-1}}{\Gamma(i\alpha+\beta)}+\sum_{i=0}^{\infty}\sum_{j=0}^{n-1}BQ_{i+1}(jh)\frac{(t-h-jh)^{i\alpha+\beta-}}{\Gamma(i\alpha+\beta)}\\&=AX_{h,\alpha,\beta}^{A,B}(t)+BX_{h,\alpha,\beta}{}^{A,B}(t-h).\end{aligned}$$

This ends the proof. □

Theorem 1. *The solution* $y(t)$ *of (2) satisfying zero initial condition* $y(t)=0,-h\leq t\leq 0$, *has a form*

$$y(t)=\int_{-h}^{t}X_{h,\alpha,\alpha}^{A,B}(t-s)f(s)ds,\ t\geq 0.$$

Proof. By using the method of variation of constants, any solution of nonhomogeneous system $y(t)$ should be satisfied in the form

$$y(t) = \int_{-h}^{t} X_{h,\alpha,\alpha}^{A,B}(t-s)\, c(s)\, ds,\ t \geq 0, \tag{7}$$

where $c(s),\ 0 \leq s \leq t$ is an unknown vector function and $y(0) = 0$. Having Caputo fractional differentiation on both sides of (7), we obtain the following cases:

(i) For $0 < t \leq h$ we have

$$\begin{aligned}
\left({}^{C}D_{-h^{+}}^{\alpha} y\right)(t) &= Ay(t) + By(t-h) + f(t) \\
&= A\int_{-h}^{t} X_{h,\alpha,\alpha}^{A,B}(t-s)\, c(s)\, ds + \int_{-h}^{t-h} X_{h,\alpha,\alpha}^{A,B}(t-h-s)\, c(s)\, ds + f(t) \\
&= A\int_{-h}^{t} X_{h,\alpha,\alpha}^{A,B}(t-s)\, c(s)\, ds + f(t).
\end{aligned}$$

According to Lemma 2, we have

$$\begin{aligned}
\left({}^{C}D_{-h^{+}}^{\alpha} y\right)(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-h}^{t}(t-r)^{-\alpha}\left(\int_{-h}^{r} X_{h,\alpha,\alpha}^{A,B}(r-s)\, c(s)\, ds\right) dr \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-h}^{t} c(s)\int_{s}^{t}(t-r)^{-\alpha}\sum_{i=0}^{\infty}A^{i}\frac{(r-s)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} drds \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-h}^{t} c(s)\int_{s}^{t}(t-r)^{-\alpha}\frac{(r-s)^{\alpha-1}}{\Gamma(\alpha)} drds \\
&\quad + \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{\infty}A^{i}\frac{d}{dt}\int_{-h}^{t} c(s)\int_{s}^{t}(t-r)^{-\alpha}\frac{(r-s)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)} drds \\
&= c(t) + \sum_{i=1}^{\infty}\frac{1}{\Gamma(1-\alpha)\Gamma((i+1)\alpha)}A^{i}\frac{d}{dt}\int_{-h}^{t} c(s)(t-s)^{i\alpha}B(\alpha(i+1), 1-\alpha)\, ds \\
&= c(t) + \sum_{i=1}^{\infty}\frac{\Gamma(1-\alpha)\Gamma((i+1)\alpha)}{\Gamma(1-\alpha)\Gamma((i+1)\alpha)\Gamma(i\alpha+1)}A^{i}\frac{d}{dt}\int_{-h}^{t} c(s)(t-s)^{i\alpha} ds \\
&= c(t) + \sum_{i=1}^{\infty}A^{i}\frac{1}{\Gamma(i\alpha+1)}\frac{d}{dt}\int_{-h}^{t} c(s)(t-s)^{i\alpha} ds \\
&= c(t) + \sum_{i=1}^{\infty}A^{i}\frac{\alpha i}{\Gamma(i\alpha+1)}\int_{-h}^{t} c(s)(t-s)^{i\alpha-1} ds = c(t) + \int_{-h}^{t}\sum_{i=1}^{\infty}A^{i}\frac{1}{\Gamma(i\alpha)}(t-s)^{i\alpha-1}c(s)\, ds \\
&= c(t) + A\int_{-h}^{t}\sum_{i=0}^{\infty}A^{i}\frac{(t-s)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\, ds = c(t) + A\int_{-h}^{t} X_{h,\alpha,\alpha}^{A,B}(t-s)\, c(s)\, ds.
\end{aligned}$$

Hence, we obtain $c(t) = f(t)$.

(ii) For $nh < t \leq (n+1)h$, according to (2), we have

$$
\begin{aligned}
\left({}^{C}D^{\alpha}_{-h^{+}}y\right)(t) &= Ay(t) + By(t-h) + f(t) \\
&= A\int_{-h}^{t} X^{A,B}_{h,\alpha,\alpha}(t-s)\,c(s)\,ds + B\int_{-h}^{t-h} X^{A,B}_{h,\alpha,\alpha}(t-h-s)\,c(s)\,ds + f(t) \\
&= A\int_{-h}^{t}\sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+1}(jh)\,\frac{(t-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds \\
&\quad + B\int_{-h}^{t-h}\sum_{i=0}^{\infty}\sum_{j=0}^{n-1} Q_{i+1}(jh)\,\frac{(t-s-h-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds + f(t) \\
&= A\sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+1}(jh)\int_{-h}^{t-jh}\frac{(t-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds \\
&\quad + B\sum_{i=0}^{\infty}\sum_{j=1}^{n} Q_{i+1}(jh-h)\int_{-h}^{t-jh}\frac{(t-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds + f(t) \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+2}(jh)\int_{-h}^{t-jh}\frac{(t-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds + f(t).
\end{aligned}
$$

According to Lemma 2, we have

$$
\begin{aligned}
&\left({}^{C}D^{\alpha}_{-h^{+}}y\right)(t) \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-h}^{t}(t-r)^{-\alpha}\left(\int_{-h}^{r} X^{A,B}_{h,\alpha,\alpha}(r-s)\,c(s)\,ds\right)dr \\
&= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{-h}^{t} c(s)\int_{s}^{t}(t-r)^{-\alpha}X_{h,\alpha,\alpha}{}^{A,B}(r-s)\,drds \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+1}(jh)\,\frac{d}{dt}\int_{-h}^{t} c(s)\int_{s}^{t}(t-r)^{-\alpha}\frac{(r-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}drds \\
&= \frac{1}{\Gamma(1-\alpha)}\sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+1}(jh)\,\frac{d}{dt}\int_{-h}^{t-jh} c(s)(t-s-jh)^{i\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(i\alpha+1)}ds \\
&= \sum_{j=0}^{n} Q_{1}(jh)\,\frac{d}{dt}\int_{-h}^{t-jh} c(s)\,ds + \sum_{i=1}^{\infty}\sum_{j=0}^{n} Q_{i+1}(jh)\int_{-h}^{t-jh} c(s)(t-s-jh)^{i\alpha-1}\frac{1}{\Gamma(i\alpha)}ds \\
&= c(t) + \sum_{i=0}^{\infty}\sum_{j=0}^{n} Q_{i+2}(jh)\int_{-h}^{t-jh}\frac{(t-s-jh)^{(i+1)\alpha-1}}{\Gamma((i+1)\alpha)}c(s)\,ds.
\end{aligned}
$$

Hence, we obtain $c(t) = f(t)$. The proof is completed. □

Theorem 2. *Let $p = 0, 1, \ldots, l$. A solution $y \in C\left(((p-1)h, ph], \mathbb{R}^{n}\right)$ of (2) with $f = 0$ has a form*

$$
y(t) = X^{A,B}_{h,\alpha,1}(t+h)\,\varphi(0) + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)\left(\left({}^{C}D^{\alpha}_{-h^{+}}\varphi\right)(s) - A\varphi(s)\right)ds.
$$

Proof. We are looking for a solution of the form

$$y(t) = X^{A,B}_{h,\alpha,1}(t+h)c + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)g(s)\,ds,$$

where c is an unknown constants, $g(t)$ is an unknown continuously differentiable function. Moreover, it satisfies initial condition $y(t) = \varphi(t), -h \le t \le 0$, ie,

$$y(t) = X^{A,B}_{h,\alpha,1}(t+h)c + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)g(s)\,ds := \varphi(t), \quad -h \le t \le 0.$$

Let $t = -h$, we have

$$X^{A,B}_{h,\alpha,1}(-h-s) = \begin{cases} \Theta, & -h < s \le 0, \\ I, & s = -h. \end{cases}$$

Thus, $c = \varphi(-h)$. Since $-h \le t \le 0$, one obtains

$$X^{A,B}_{h,\alpha,\alpha}(t-s) = \begin{cases} \Theta, & t < s \le 0, \\ (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha}), & -h \le s \le t, \quad 0 \le t-s \le t+h \le h. \end{cases}$$

Thus on interval $-h \le t \le 0$, one can derive that

$$\begin{aligned} \varphi(t) &= X^{A,B}_{h,\alpha,1}(t+h)\varphi(-h) + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)g(s)\,ds \\ &= X^{A,B}_{h,\alpha,1}(t+h)\varphi(-h) + \int_{-h}^{t} X^{A,B}_{h,\alpha,\alpha}(t-s)g(s)\,ds + \int_{t}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)g(s)\,ds \\ &= E_{\alpha,1}(A(t+h)^{\alpha})\varphi(-h) + \int_{-h}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})g(s)\,ds. \end{aligned} \tag{8}$$

Having differentiated (8), we obtain

$$\begin{aligned} \left({}^{C}D^{\alpha}_{-h^{+}}\varphi\right)(t) &= A\sum_{k=0}^{\infty}\frac{A^{k}(t+h)^{\alpha k}}{\Gamma(1+k\alpha)}\varphi(-h) + \int_{-h}^{t}\sum_{k=1}^{\infty}\frac{A^{k}(t-s)^{\alpha k-1}}{\Gamma(k\alpha)}g(s)\,ds + g(t) \\ &= AE_{\alpha,1}(A(t+h)^{\alpha})\varphi(-h) + A\int_{-h}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^{\alpha})g(s)\,ds + g(t) \\ &= A\varphi(t) + g(t). \end{aligned}$$

Therefore, $g(t) = \left({}^{C}D^{\alpha}_{-h^{+}}\varphi\right)(t) - A\varphi(t)$ and the desired result holds. □

Combining Theorems 1 and 2, we have the following result.

Corollary 1. *A solution* $y \in C\left(\left[-h,T\right] \cap ((p-1)h, ph\right], \mathbb{R}^{n})$ *of (2) has a form*

$$\begin{aligned} y(t) = X^{A,B}_{h,\alpha,1}(t+h)\varphi(-h) + \int_{-h}^{0} X^{A,B}_{h,\alpha,\alpha}(t-s)\left[\left({}^{C}D^{\alpha}_{-h^{+}}\varphi\right)(s) - A\varphi(s)\right]ds \\ + \int_{0}^{t} X^{A,B}_{h,\alpha,\alpha}(t-s)f(s)\,ds. \end{aligned}$$

2 | CONCLUSIONS AND FUTURE DIRECTIONS

We give an explicit formula for the solution to a Caputo type fractional order constant time-delay linear nonhomogeneous system using the concept of delayed perturbation of two parameter Mittag-Leffler type matrix function and the classical variation of constants method. For other kinds of fractional derivatives, such as the Riemann-Liouville fractional derivative and the Hadamard fractional derivative, the problem is much different and more stimulating. Say, if we replace the Caputo fractional derivative with the Riemann-Liouville fractional derivative, the initial condition needs to be changed to the fractional integral. In this case, the key point is the attempt to construct the corresponding fractional delayed perturbation of the matrix function of the Mittag-Leffler type. Another open problem is how to extend the results derived to the following fractional systems:

$$\begin{cases} {}^{C}D^{\alpha}_{-h^{+}}\left({}^{C}D^{\alpha}_{-h^{+}}\right)x(t)+A^{2}x(t)+\Omega^{2}x(t-h)=f(t), & t\geq 0, \quad h>0, \\ x(t)=\varphi(t),\ {}^{C}D^{\alpha}_{-h^{+}}x(t)= \ {}^{C}D^{\alpha}_{-h^{+}}\varphi(t), & -h\leq t\leq 0, \end{cases} \tag{9}$$

where $A,\Omega\in\mathbb{R}^{n\times n},\ f\in C([0,\infty),\mathbb{R}^{n}),\ \varphi\in C^{1}\left(\left[-h,0\right],\mathbb{R}^{n}\right)$. Our further work will be devoted to study finite time and exponential stability, Lyapunov type stability and controllability of the Caputo type fractional order time-delay linear nonhomogeneous systems.

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REFERENCES

1. Khusainov DY, Shuklin GV. Linear autonomous time-delay system with permutation matrices solving. *Stud Univ Žilina*. 2003;17:101-108.
2. Li M, Wang JR. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl Math Comput*. 2018;324:254-265.
3. Diblík J, Fečkan M, Pospíšil M. Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices. *Ukrainian Math J*. 2015;65:58-69.
4. Diblík J, Khusainov DY, Baštinec J, Sirenko AS. Exponential stability of linear discrete systems with constant coefficients and single delay. *Appl Math Lett*. 2016;51:68-73.
5. Khusainov DY, Shuklin GV. Relative controllability in systems with pure delay. *Int J Appl Math*. 2005;2:210-221.
6. Medveď M, Pospíšil M, Škripková L. Stability and the nonexistence of blowing-up solutions of nonlinear delay systems with linear parts defined by permutable matrices. *Nonlinear Anal*. 2011;74:3903-3911.
7. Medveď M, Pospíšil M. Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices. *Nonlinear Anal*. 2012;75:3348-3363.
8. Medveď M, Pospíšil M. Representation of solutions of systems linear differential equations with multiple delays and linear parts given by nonpermutable matrices. *J Math Sci*. 2018;228:276-229.
9. Li M, Wang JR. Finite time stability of fractional delay differential equations. *Appl Math Lett*. 2017;64:170-176.
10. Li M, Debbouche A, Wang JPR. Relative controllability in fractional differential equations with pure delay. *Math Methods Appl Sci*. 2017:1-9. https://doi.org/10.1002/mma.4651
11. Liang C, Wang JR, O'Regan D. Representation of solution of a fractional linear system with pure delay. *Appl Math Lett*. 2018;77:72-78.
12. Luo Z, Wang JR. Finite time stability analysis of systems based on delayed exponential matrix. *J Appl Math Comput*. 2017;55:335-351.
13. Pospíšil M. Representation and stability of solutions of systems of functional differential equations with multiple delays. *Electron J Qual Theory Differ Equ*. 2012:54.
14. Pospíšil M. Representation of solutions of delayed difference equations with linear parts given by pairwise permutable matrices via Z-transform. *Appl Math Comput*. 2017;294:180-194.

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