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This paper aims to give an overview and summary of numerical methods for the solution of stochastic differential equations. It covers discrete time strong and weak approximation methods that are suitable for different applications. A range of approaches and results is discussed within a unified framework. On the one hand, these methods can be interpreted as generalizing the well-developed theory on numerical analysis for deterministic ordinary differential equations. On the other hand they highlight the specific stochastic nature of the equations. In some cases these methods lead to completely new and challenging problems.

CONTENTS

1. Introduction

About three hundred years ago, Newton and Leibniz developed the differential calculus, allowing us to model continuous time dynamical systems in mechanics, astronomy and many other areas of science. This calculus has formed the basis of the revolutionary developments in science, technology and manufacturing that the world has experienced over the last two centuries.

As we try to build more realistic models, stochastic effects need to be taken into account. In areas such as finance, the randomness in the system dynamics is in fact the essential phenomenon to be modelled. Continuous time stochastic dynamics need to be modelled in many areas of application, including microelectronics, signal processing and filtering, several fields of biology and physics, population dynamics, epidemiology, psychology, economics, finance, insurance, fluid dynamics, radio astronomy, hydrology, structural mechanics, chemistry and medicine. Practical problems arising in some of these areas in the mid-1900s led to the development of a corresponding stochastic calculus.

Almost a hundred years ago, Bachelier (1900) used what we now call Brownian motion or the Wiener process to model stock prices in the Paris Bourse. Later Einstein (1906), in his work on Brownian motion, used an equivalent mathematical construct. Wiener (1923) then developed more fully the mathematical theory of Brownian motion. A further advance was made by Itô (1944), who laid the foundation of a stochastic calculus known today as the Itô calculus. This represents the stochastic generalization of the classical differential calculus, allowing us to model in continuous time such phenomena as the dynamics of stock prices or the motions of a microscopic particle subject to random fluctuations. The corresponding stochastic differential equations (SDEs) generalize the ordinary deterministic differential equations (ODEs).

A most striking example, where Itô SDEs provide the essential modelling device, is given by modern financial theory. The Nobel prize-winning work of Merton (1973) and Black and Scholes (1973) initiated the entire derivatives and risk management industry that we see today. For the development of corresponding financial markets, it is vital to improve our understanding of its underlying stochastic dynamics, and to calculate efficiently relevant financial quantities such as derivative prices and risk measures.

After the earlier technological revolution in manufacturing, it is the author's view that we are likely to experience now and into the next century a revolution in commercial technologies. The finance area is the most notable example where the new changes have occurred. In the insurance area a similar development has already started. Marketing can be expected to base its future models on SDEs. We are at the beginning of a development where commercial and economic activities will become subject to detailed stochastic modelling and quantitative analysis. This global phenomenon will be a major driving force in the development of appropriate numerical methods for the solution of SDEs.

This paper provides a very basic introduction, as well as a brief overview of the area of numerical methods for SDEs. The rapidly increasing literature on the topic makes it impossible to give a comprehensive survey. However, an attempt has been made to highlight key approaches, and note results

that have been instrumental in the development of the field, or may be of major significance in future research.

Books on numerical solutions of SDEs that provide systematic information on the subject include Gard (1988), Milstein (1988*a*, 1995*a*), Kloeden and Platen (1992/1995*b*), Bouleau and Lépingle (1993), Janicki and Weron (1994) and Kloeden, Platen and Schurz (1994/1997).

Given the diversity of numerical problems that arise in SDEs, there is a strong need to extend the wealth of expertise accumulated in the numerical analysis of ODEs to the field of SDEs. Important monographs on the numerical analysis of ODEs that had an impact on the numerical analysis of SDEs include those by Gear (1971), Björck and Dahlquist (1974), Butcher (1987), Hairer, Nørsett and Wanner (1987), Hairer and Wanner (1991) and Stoer and Bulirsch (1993).

As this paper will show, a multi-faceted variety of research topics on numerical methods for SDEs has emerged over the last twenty years. These topics can be linked to complexity theory: see, for instance, Traub, Wasilkowski and Wozniakowski (1988), Wozniakowski (1991) and Sloan and Wozniakowski (1998), where it was shown that simulation approaches, including those of stochastic numerical analysis, are optimal with respect to average case complexity.

2. Stochastic differential equations

Let us consider an *Itô SDE* of the form

$$\mathrm{d}X_t = a(X_t)\,\mathrm{d}t + b(X_t)\,\mathrm{d}W_t \tag{2.1}$$

for $t \in [0,T]$, with initial value $X_0 \in \mathcal{R}$. The stochastic process $X = \{X_t,\, 0 \le t \le T\}$ is assumed to be a unique solution of the SDE (2.1) which consists of a slowly varying component governed by the drift coefficient $a(\cdot)$ and a rapidly fluctuating random component characterized by the diffusion coefficient $b(\cdot)$. The second differential in (2.1) is an Itô stochastic differential with respect to the *Wiener process* $W = \{W_t,\, 0 \le t \le T\}$. It is defined via the corresponding stochastic integral by using a limit of Riemann sums with values of the integrands taken on the left-hand side of the discretization intervals. Another stochastic differential, the Stratonovich stochastic differential, would result if the values of the integrands were taken at the centre of the interval.

As introductory textbooks on SDEs, one can refer to Arnold (1974), Gard (1988), Oksendahl (1985) and Kloeden *et al.* (1994/1997). More advanced material on SDEs is presented, for instance, in Elliott (1982), Karatzas and Shreve (1988), Ikeda and Watanabe (1989), Protter (1990) and Kloeden and Platen (1992/1995*b*).

To keep our formulae simple in this introductory exposition, we discuss mainly the simple case of the one-dimensional SDE driven by a one-dimensional standard Wiener process. In principle, most of the numerical methods we mention can be generalized to multi-dimensional X and W.

Since the path of a Wiener process is not differentiable, the Itô calculus differs in its properties from classical calculus. This is most obvious in the stochastic chain rule, the *Itô formula*, which for a twice continuously differentiable function f has the form

$$\mathrm{d}f(X_t) = \left(f'(X_t)\,a(X_t) + \tfrac{1}{2}\,f''(X_t)\,b^2(X_t)\right)\,\mathrm{d}t + f'(X_t)\,b(X_t)\,\mathrm{d}W_t \quad (2.2)$$

for $0 \le t \le T$. We remark that the extra term $\frac{1}{2}\,f''b^2$ in the drift function of the resulting SDE (2.2) is characteristic of the Itô calculus, and this consequently also has a substantial impact on numerical methods for SDEs, as will be seen later.

The Stratonovich calculus follows the rules of classical calculus more closely. However, it does not conveniently relate to martingale theory, a fundamental part of stochastic analysis. Both the Itô and the Stratonovich stochastic calculus can be related to each other, and one can switch from one to the other if necessary. The above stochastic process X in (2.1) can be written as the solution of the *Stratonovich SDE* in the form

$$\circ\,\mathrm{d}X_t = \underline{a}(X_t)\,\mathrm{d}t + b(X_t) \circ \mathrm{d}W_t, \quad (2.3)$$

where, assuming b' exists, we have the Stratonovich drift function

$$\underline{a}(x) = a(x) - \tfrac{1}{2}\,b(x)\,b'(x), \quad (2.4)$$

with the notation '$\circ$' in (2.3) referring to the Stratonovich stochastic differential. This differential also arises as the limit of classical differentials when the path of the Wiener process is smoothed, as in the Wong–Zakai approximation. Such approximations of SDEs are studied in Wong and Zakai (1965), Kurtz and Protter (1991*b*) and Saito and Mitsui (1995), for instance.

The strong similarity of the Stratonovich calculus with the classical calculus is made clear by the *Stratonovich chain rule*, which for a differentiable function f has the form

$$\begin{aligned} \circ\,\mathrm{d}f(X_t) &= f'(X_t)\,(\underline{a}(X_t)\,\mathrm{d}t + b(X_t) \circ \mathrm{d}W_t) \\ &= f'(X_t) \circ \mathrm{d}X_t, \end{aligned} \quad (2.5)$$

with '$\circ$' again denoting the Stratonovich differential. We note that in (2.5) only first-order derivatives of f appear, as in deterministic calculus.

It turns out that for some numerical tasks the Itô, and for others the Stratonovich formulation, of an SDE is more convenient, as we shall see later. Usually only the Itô calculus allows us to exploit powerful martingale results for numerical analysis.

3. Euler approximation

Since analytical solutions of SDEs are rare, numerical approximations have been developed. These are typically based on a *time discretization* with points

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \cdots < \tau_N = T$$

in the time interval $[0, T]$, using a step-size $\Delta = T/N$. More general time discretizations can be used, that could, for instance, be random.

Simulation experiments and theoretical studies have shown that not all classical or heuristic time discrete approximations of SDEs converge in a useful sense to the corresponding solution process as the step-size Δ tends to zero: see, for instance, Clements and Anderson (1973), Wright (1974), Fahrmeier (1976), Clark and Cameron (1980) and Rümelin (1982). Consequently a systematic approach is needed in order to select an efficient and reliable numerical method for the problem at hand.

Several different approaches have been proposed in the literature to handle SDEs numerically. Without relying much on the specific structure of SDEs, Kohler and Boyce (1974), Boyle (1977) and Boyce (1978) have suggested general Monte Carlo simulation of the given random system. Kushner (1974) and Kushner and Dupuis (1992) proposed, as the approximating process for solutions of SDEs, discrete, finite state Markov chains. Platen (1992) developed higher-order Markov chain approximations. When digital computers were still in their infancy, Dashevski and Liptser (1966) and Fahrmeier (1976) also used analogue computers to handle SDEs numerically.

Both in the literature and in practice, most attention has been directed to discrete time approximations of SDEs. The *Euler approximation* that was first studied in Maruyama (1955) is the simplest example of such a method, and is ideally suited for implementation on a digital computer. For the SDE (2.1) the Euler approximation Y is given by the recursive equation

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n \tag{3.1}$$

for $n = 0, 1, \ldots, N-1$ with $Y_0 = X_0$. Here $\Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}$ denotes the increment of the Wiener process in the time interval $[\tau_n, \tau_{n+1}]$ and are represented by independent $N(0, \Delta)$ Gaussian random variables with mean zero and variance Δ.

It has been shown in the literature that the Euler approximation converges for vanishing $\Delta \to 0$, under rather different types of convergence, to the solution X of the Itô SDE (2.1). Some of the papers in which the Euler method has been studied are Allain (1974), Yamada (1976), Gikhman and Skorokhod (1979), Clark and Cameron (1980), Ikeda and Watanabe (1989), Janssen (1984*a*, 1984*b*), Atalla (1986), Jacod and Shiryaev (1987), Kaneko and Nakao (1988), Kanagawa (1988, 1989, 1995, 1996, 1997), Golec and Ladde (1989), Mikulevicius and Platen (1991), Mackevicius (1994), Camba-

nis and Hu (1996), Gelbrich (1995), Bally and Talay (1995, 1996*a*, 1996*b*), Jacod and Protter (1998), Kohatsu-Higa and Ogawa (1997) and Chan and Stramer (1998). This is certainly not a complete list of references on the Euler method. It is always an interesting task to study a new technique based on this simple discrete time approximation of an SDE. For instance, Gorostiza (1980) and Newton (1990) suggested Euler type approximations with random step-size, where the approximate path jumps from threshold to threshold.

To simulate a realization of the Euler approximation, one needs to generate the independent random variables involved. In practice, linear or nonlinear congruential pseudo-random number generators are often used. An introduction to this area is given by Ripley (1983*a*). Books that include chapters on random number generation include Ermakov (1975), Yakowitz (1977), Rubinstein (1981), Ripley (1983*b*), Morgan (1984), Ross (1991), Mikhailov (1992), Fishman (1992) and Gentle (1998). We mention also the papers by Box and Muller (1958), Marsaglia and Bray (1964), Brent (1974), Eichenauer and Lehn (1986), Niederreiter (1988), Sugita (1995) and Antipov (1995, 1996). Random number generation on supercomputers was considered by Petersen (1988), Anderson (1990), Petersen (1994*a*) and Entacher, Uhl and Wegenkittl (1998).

As in the deterministic case, it turns out that the Euler method is rather simple and crude, somewhat inefficient and often exhibits poor stability properties. Much better stochastic numerical methods can be constructed systematically.

4. Strong and weak convergence

It is convenient to have some measure of the efficiency of a numerical scheme by identifying its order of convergence.

Unlike in the typical deterministic modelling situation, there exist in the stochastic environment many different types of convergence that make theoretical or practical sense. Therefore in stochastic numerical analysis, one has to specify the class of problems that one wishes to investigate, before starting to construct a numerical method and seeking to optimize its efficiency with respect to one or another convergence criterion.

In stochastic numerical analysis, the order of convergence plays a crucial role in the design of numerical algorithms. However, as already explained, the choice of the convergence criterion depends on the type of the problem. Roughly speaking, there are two major types of convergence to be distinguished. These can be identified by whether one requires

(a) approximations to the sample paths, or
(b) approximations to the corresponding distributions.

For convenience we choose a rather simple characterization of each of these

two types of convergence for the classification of numerical algorithms, and call them the *strong* and the *weak* convergence criterion, respectively.

Tasks involving direct simulations of paths, such as the generation of a stock market price scenario, the computation of a filter estimate for some hidden unobserved variable, or the testing of a statistical estimator for parameters in some SDEs, require that the simulated sample paths be close to those of the solution of the original SDE. This implies that in these cases, among others, some strong convergence criterion should be used. The following simple criterion allows us to classify numerical methods according to their strong order γ of convergence, using the *absolute error* $E|X_T - Y_N|$ at the terminal time T.

We shall say that a discrete time approximation Y of the exact solution X of an SDE *converges in the strong sense with order* $\gamma \in (0, \infty]$ if there exists a constant $K < \infty$ such that

$$E|X_t - Y_N| \leq K \, \Delta^{\gamma} \tag{4.1}$$

for all step-sizes $\Delta \in (0, 1)$. In the deterministic case with vanishing diffusion coefficient $b \equiv 0$ this criterion reduces to the usual deterministic convergence order criterion, as used, for instance, in Gear (1971) or Butcher (1987).

Fortunately, in a large variety of practical problems a pathwise approximation of the solution X of an SDE is not required. Much computational effort has been wasted on simulations by missing this point. If one aims to compute, for instance, a moment of X, a probability related to X, an option price on a stock price X or a general functional of the form $E(g(X_T))$, then no strong approximation is required. The simulation of such functionals does not force us to approximate the path of X. Rather, it is sufficient to approximate adequately the probability distribution that corresponds to X. Consequently we need only a much weaker type of convergence than that expressed by the strong convergence criterion (4.1).

We shall say that a discrete time approximation Y of a solution X of an SDE *converges in the weak sense with order* $\beta \in (0, \infty]$ if, for any polynomial g, there exists a constant $K_g < \infty$ such that

$$|E(g(X_T)) - E(g(Y_N))| \leq K_g \, \Delta^{\beta}, \tag{4.2}$$

for all step-sizes $\Delta \in (0, 1)$, provided that these functionals exist. Clearly this criterion covers the convergence of pth moments because we can set $g(x) = x^p$. It reduces to the deterministic order criterion in the case $b \equiv 0$ and $g(x) = x$.

As we shall see later, the numerical methods that can be constructed with respect to this weak convergence criterion are much easier to implement than those required by the strong convergence criterion. In any practical simulation, one should try, if possible, to identify directly the task at hand as being one that requires a weak approximation method.

5. Stochastic Taylor expansions

The key to the construction of most higher-order numerical approximations is usually obtained from the truncated expansion of the variables of interest over small increments. The well-known Taylor formula provides the basis for the derivation of most deterministic numerical algorithms. In the stochastic case, a stochastic Taylor expansion for Itô SDEs was first described in Wagner and Platen (1978). This result was then extended and generalized in Platen (1981, 1982*b*), Platen and Wagner (1982), Azencott (1982), Sussmann (1988), Yen (1988), BenArous (1989), Kloeden and Platen (1991*a*, 1991*b*), Hu (1992, 1996), Hu and Watanabe (1996), Kohatsu-Higa (1997), Liu and Li (1997), and Kuznetsov (1998).

The *Wagner–Platen formula* is obtained by iterated applications of the Itô formula to the integrands in the integral version of the SDE (2.1). For example, in a simple case, we obtain the expansion

$$\begin{aligned} X_t &= X_{t_0} + a(X_{t_0}) \int_{t_0}^{t} \mathrm{d}s + b(X_{t_0}) \int_{t_0}^{t} \mathrm{d}W_s \\ &\quad + b(X_{t_0})\, b'(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s_2} \mathrm{d}W_{s_1}\, \mathrm{d}W_{s_2} + R_{t_0,t}, \end{aligned} \tag{5.1}$$

where $R_{t_0,t}$ represents some remainder term consisting of higher-order multiple stochastic integrals. *Multiple Itô integrals* of the type

$$I_{(1)} = \int_{\tau_n}^{\tau_{n+1}} \mathrm{d}W_s, \qquad I_{(1,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} \mathrm{d}W_{s_1}\, \mathrm{d}W_{s_2} = \frac{1}{2} \left((I_{(1)})^2 - \Delta \right),$$

$$I_{(0,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} \mathrm{d}s_1\, \mathrm{d}W_{s_2}, \qquad I_{(1,0)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} \mathrm{d}W_{s_1}\, \mathrm{d}s_2,$$

$$I_{(1,1,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \mathrm{d}W_{s_1}\, \mathrm{d}W_{s_2}\, \mathrm{d}W_{s_3} \tag{5.2}$$

on the interval $[\tau_n, \tau_{n+1}]$ form the random building blocks in the Wagner–Platen expansions.

For comparison, an application of the *Stratonovich–Taylor formula*, developed in Kloeden and Platen (1991*a*) and (1991*b*), to the integral version of the Stratonovich SDE (2.3) yields the expansion

$$\begin{aligned} X_t &= X_{t_0} + \underline{a}(X_{t_0}) \int_{t_0}^{t} \mathrm{d}s + b(X_{t_0}) \int_{t_0}^{t} \circ\, \mathrm{d}W_{s_1} \\ &\quad + b(X_{t_0})\, b'(X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s_2} \circ\, \mathrm{d}W_{s_1} \circ\, \mathrm{d}W_{s_2} + \underline{R}_{t_0,t}, \end{aligned} \tag{5.3}$$

where $\underline{R}_{t_0,t}$ is some remainder term with higher-order multiple Stratonovich

integrals. In this case *multiple Stratonovich integrals* of the form

$$J_{(1)} = I_{(1)}, \qquad J_{(1,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{s_1} \circ \mathrm{d}W_{s_2} = \frac{1}{2!}\,(I_{(1)})^2,$$

$$J_{(0,1)} = I_{(0,1)}, \qquad J_{(1,0)} = I_{(1,0)},$$

$$J_{(1,1,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{s_1} \circ \mathrm{d}W_{s_2} \circ \mathrm{d}W_{s_3} = \frac{1}{3!}\,(J_{(1)})^3 \tag{5.4}$$

on the interval $[\tau_n, \tau_{n+1}]$ represent the basic random elements of the expansion.

Close relationships exist between multiple Itô and Stratonovich integrals which form some kind of algebra. This algebra and certain approximations of multiple stochastic integrals have been described in Platen and Wagner (1982), Liske (1982), Platen (1984), Milstein (1988*a*, 1995*a*), Kloeden and Platen (1991*a*, 1991*b*, 1992/1995*b*), Kloeden, Platen and Wright (1992*c*), Hu and Meyer (1993), Hofmann (1994), Gaines and Lyons (1994), Gaines (1994, 1995*a*), Castell and Gaines (1995), Li and Liu (1997), Burrage (1998) and Kuznetsov (1998).

6. Strong approximation methods

In this section, we focus on strong discrete time approximations of SDEs. These are suitable for scenario simulations. They are usually more expensive to implement, both in development and computing time, than their weak counterparts.

6.1. Strong Taylor approximation

If, from the Wagner–Platen formula (5.1) we select only the first two integral terms, then we obtain the Euler approximation (3.1). It has been shown (see, for instance, Milstein (1974) or Platen (1981)) that in general the Euler approximation has strong order of only $\gamma = 0.5$, as a consequence of the Hölder continuity of order 0.5 of the paths of X.

Taking one more term in the expansion (5.1) we obtain the well-known *Milstein scheme*

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n + b(Y_n)\,b'(Y_n)\,I_{(1,1)}, \tag{6.1}$$

proposed by Milstein (1974), where the double Itô integral $I_{(1,1)}$ is given in (5.2) as $((\Delta W_n)^2 - \Delta)/2$. In general this scheme has strong order $\gamma = 1.0$. Thus adding one more term from the Wagner–Platen formula to the Euler scheme already provides an improvement in efficiency. The Milstein scheme can be obtained alternatively from the Stratonovich–Taylor formula (5.3) by selecting the first three integral terms of that expansion.

For multi-dimensional driving Wiener processes, the double stochastic integrals appearing in the Milstein scheme have to be approximated, unless the drift and diffusion coefficients fulfil a certain commutativity condition. Characterizations and approximations of such double Itô integrals are given, for instance, in Milstein (1988*a*, 1995*a*), Kloeden and Platen (1992/1995*b*), Gaines and Lyons (1994) and Kuznetsov (1998).

Wagner and Platen (1978), Platen (1981) and Platen and Wagner (1982) have described which terms of the Wagner–Platen formula have to be chosen to obtain a desired higher strong order of convergence. Thus, for instance, the *strong Taylor approximation of order* $\gamma = 1.5$ has the form

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta W_n + b\,b'\,I_{(1,1)} + b\,a'\,I_{(1,0)} + \left(a\,a' + \tfrac{1}{2}\,b^2\,a''\right)\tfrac{\Delta^2}{2} \\ &\quad + \left(a\,b' + \tfrac{1}{2}\,b^2\,b''\right)\,I_{(0,1)} + b\,\left(b\,b'' + (b')^2\right)\,I_{(1,1,1)}, \end{aligned} \tag{6.2}$$

where we suppress the dependence of the coefficients on Y_n and use the multiple Itô integrals mentioned in (5.2).

The integer strong order Taylor schemes given in Kloeden and Platen (1992/1995*b*) can be conveniently derived from a Stratonovich–Taylor formula of the type (5.3). For instance, the *strong order* 2.0 *Taylor scheme* is

$$\begin{aligned} Y_{n+1} &= Y_n + \underline{a}\,\Delta + b\,\Delta W_n + b\,b'\,J_{(1,1)} + b\,\underline{a}'\,J_{(1,0)} + \underline{a}\,b'\,J_{(0,1)} + \underline{a}\,\underline{a}'\,\tfrac{\Delta^2}{2} \\ &\quad + b\,(b\,b')'\,J_{(1,1,1)} + \underline{a}\,(b\,b')'\,J_{(0,1,1)} + b\,(\underline{a}\,b')'\,J_{(1,0,1)} + b\,(b\,\underline{a}')'\,J_{(1,1,0)} \\ &\quad + b\,(b\,(b\,b')')'\,J_{(1,1,1,1)}; \end{aligned} \tag{6.3}$$

here, in addition to those multiple Stratonovich integrals already mentioned in (5.4), we have also used

$$\begin{aligned} J_{(0,1,1)} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \mathrm{d}s_1 \circ \mathrm{d}W_{s_2} \circ \mathrm{d}W_{s_3}, \\ J_{(1,0,1)} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{s_1}\,\mathrm{d}s_2 \circ \mathrm{d}W_{s_3}, \\ J_{(1,1,0)} &= \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{s_1} \circ \mathrm{d}W_{s_2}\,\mathrm{d}s_3, \end{aligned}$$

and

$$J_{(1,1,1,1)} = \int_{\tau_n}^{\tau_{n+1}} \int_{\tau_n}^{s_4} \int_{\tau_n}^{s_3} \int_{\tau_n}^{s_2} \circ \mathrm{d}W_{s_1} \circ \mathrm{d}W_{s_2} \circ \mathrm{d}W_{s_3} \circ \mathrm{d}W_{s_4} = \frac{1}{4!}\,(J_1)^4. \tag{6.4}$$

Milstein (1988*a*, 1995*a*), Kloeden and Platen (1991*a*), Hofmann (1994), Gaines (1994), Liu and Li (1997), Burrage (1998) and Kuznetsov (1998) point out that certain multiple Stratonovich integrals can be expressed using some minimal set of random variables. This is important for efficient practical implementations.

In general, one can say that for higher-order numerical schemes one requires adequate smoothness of the drift and diffusion coefficients, but also adequate information about the driving Wiener processes. This information is contained in the multiple stochastic integrals appearing in the Wagner–Platen and Stratonovich–Taylor formulae.

For specific types of drift and diffusion coefficients, for instance, when they fulfil a certain commutativity condition, higher-order strong Taylor schemes can be considerably simplified: see Kloeden and Platen (1992/1995*b*). Only a reduced set of multiple stochastic integrals is then needed to achieve the corresponding strong order. In any given problem with several driving Wiener processes, it is worthwhile checking whether this might apply.

Another situation where considerable extra efficiency can be gained occurs when the SDE has only a *small noise* term: that is, the diffusion coefficient is small and the noise can be interpreted as a perturbation. This situation was studied by Milstein and Tretjakov (1994). Such an approximation has to focus on the drift part of the dynamics. One then usually achieves only a low theoretical strong order, but owing to the smallness of the noise a reasonable overall performance of the algorithm is achieved.

A relatively simple approach to constructing discrete time approximations is the *splitting method* applied by Bensoussan, Glowinski and Rascanu (1990, 1992), LeGland (1992), Sun and Glowinski (1994) and Petersen (1998), who treat the drift term and the diffusion term separately in their algorithms. This method in general achieves only the strong order of the Euler approximation, but is convenient in its implementation.

As an illustration, in Figure 6.1, we approximate a simulated path for the geometric Brownian motion that follows the SDE

$$\mathrm{d}X_t = r\, X_t\, \mathrm{d}t + \sigma\, X_t\, \mathrm{d}W_t \tag{6.5}$$

for $t \in [0, 1]$ with $X_0 = 1$. This process represents the standard model for asset prices in mathematical finance. Fortunately, in this special case we have an explicit solution of the form

$$X_t = X_0 \exp\left\{\left(r - \tfrac{1}{2}\sigma^2\right) t + \sigma\, W_t\right\}.$$

This was used in Figure 6.1 to plot a sample path of X for an interest rate $r = 0.05$ and volatility $\sigma = 0.2$. For the time step-size $\Delta = 0.1$, we also show in Figure 6.1 the linearly interpolated path of the Milstein approximation.

A disadvantage of higher-order strong Taylor approximations is the fact that derivatives of the drift and diffusion coefficients have to be calculated at each step. This can be avoided by considering derivative-free approximations, such as the Runge–Kutta-type methods to be discussed in the following section.

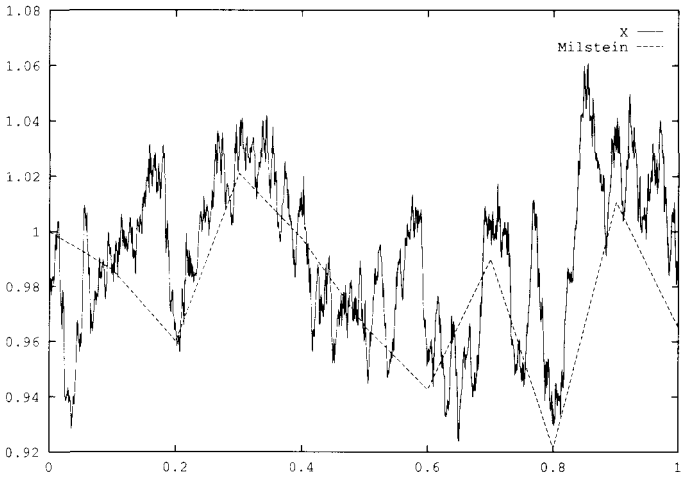

Fig. 6.1. Paths of exact solution X and the Milstein approximation

6.2. Strong Runge–Kutta approximations

As has been previously mentioned at the beginning of Section 3, one cannot simply take well-known deterministic Runge–Kutta schemes and adapt them to an SDE. These only converge with a given strong order towards the correct solution if they also approximate the corresponding strong Taylor scheme. Any viable scheme that aims to achieve a certain strong order must in general involve the appropriate multiple stochastic integrals appearing in the corresponding Taylor scheme.

As a first attempt at avoiding derivatives in the scheme, one can use the following simple method suggested by Platen (1984), which approximates the Milstein scheme. It has the Itô form

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta W_n + \left(b(\hat{Y}_n) - b'(Y_n)\right) \frac{1}{2\sqrt{\Delta}} \left((\Delta W_n)^2 - \Delta\right), \tag{6.6}$$

or the Stratonovich form

$$Y_{n+1} = Y_n + \underline{a}(Y_n)\,\Delta + b(Y_n)\,\Delta W_n + \left(b(\hat{Y}_n) - b'(Y_n)\right) \frac{1}{2\sqrt{\Delta}} (\Delta W_n)^2, \tag{6.7}$$

with

$$\hat{Y}_n = Y_n + b(Y_n)\sqrt{\Delta}.$$

In Rümelin (1982), Gard (1988), Kloeden and Platen (1992/1995*b*, 1992), and Artemiev (1993*a*, 1993*b*) further Runge–Kutta-type schemes can be found.

It is natural to ask whether the tree approach developed in Butcher (1987) can be translated to the stochastic setting. Some results along these lines

were given by Saito and Mitsui (1993*b*), Burrage and Platen (1994), Komori, Saito and Mitsui (1994), Komori and Mitsui (1995), Saito and Mitsui (1996), Burrage and Burrage (1996, 1998), Burrage, Burrage and Belward (1997), Komori, Mitsui and Sugiura (1997) and Burrage (1998). For instance, in the case of a single driving Wiener process, a rooted tree methodology has been described for Stratonovich SDEs by Burrage (1998). Following this approach, a $\gamma = 1.0$ *strong order two-stage Runge–Kutta method* has the form

$$Y_{n+1} = Y_n + (\underline{a}(Y_n) + 3\underline{a}(\bar{Y}_n)) \, \tfrac{\Delta}{4} + (b(Y_n) + 3b(\bar{Y}_n)) \, \tfrac{\Delta W_n}{4} \tag{6.8}$$

with

$$\bar{Y}_n = Y_n + \tfrac{2}{3} \, (\underline{a}(Y_n) \, \Delta + b(Y_n) \, \Delta W_n) \, .$$

The advantage of the method (6.8) compared, for instance, with the Platen method (6.6) is that the principal error constant has been minimized within a class of one-stage first-order Runge–Kutta methods. Four-stage Runge–Kutta methods of strong order $\gamma = 1.5$ can also be found in Burrage (1998). Similarly, in the context of filtering problems Newton (1986*a*, 1986*b*, 1991) and Castell and Gaines (1996) have proposed approximations that are, in some sense, asymptotically efficient with respect to the leading error coefficient within a class of Runge–Kutta-type methods. One *strong order* $\gamma = 1.0$ *method* proposed by Newton has the form

$$Y_{n+1} = Y_n + (\underline{a}(Y_n) + a_1) \, \tfrac{\Delta}{2} + (b(Y_n) + 2b_1 + 2b_2 + b_3) \, \tfrac{\Delta W_n}{6}, \tag{6.9}$$

where

$$a_{\pm} = \underline{a} \left(3\Delta \pm (\Delta W_n)^2\right), \qquad b_1 = b \left(Y_n + \tfrac{1}{2} \, b(Y_n) \, \Delta W_n\right),$$

$$a_1 = \underline{a} \left(Y_n + \tfrac{1}{2} \, \underline{a} + b_2 \, \Delta W_n\right), \qquad b_2 = b \left(Y_n + \tfrac{1}{4} \, a_+ + b_1 \, \tfrac{\Delta W_n}{2}\right),$$

$$b_3 = b \, (Y_n + a_- + b_2 \, \Delta W_n) \, .$$

Lépingle and Ribémont (1991) suggested a two-step strong scheme of first order. In Kloeden and Platen (1992/1995*b*) further two-step strong schemes have been proposed.

We now present a long list of publications that deal with higher-order discrete time approximations of Itô or Stratonovich SDEs; these contain many ideas and diverse approaches that may prove of interest in future research. They include Franklin (1965), Shinozuka (1971), Kohler and Boyce (1974), Rao, Borwankar and Ramkrishna (1974), Dsagnidse and Tschitashvili (1975), Harris (1976), Glorennec (1977), Kloeden and Pearson (1977), Clark (1978), Nikitin and Razevig (1978), Helfand (1979), Platen (1980*a*), Razevig (1980), Greenside and Helfand (1981), Casasus (1982), Clark (1982*a*), Guo (1982), Talay (1982*a*, 1982*b*, 1983*a*, 1983*b*), Drummond,

Duane and Horgan (1983), Casasus (1984), Guo (1984), Janssen (1984*a*, 1984*b*), Shimizu and Kawachi (1984), Tetzlaff and Zschiesche (1984), Unny (1984), Clark (1982*b*), Averina and Artemiev (1986), Drummond, Hoch and Horgan (1986), Kozlov and Petryakov (1986), Greiner, Strittmatter and Honerkamp (1987), Liske and Platen (1987), Platen (1987), Milstein (1987), Shkurko (1987), Römisch and Wakolbinger (1987), Averina and Artemiev (1988), Milstein (1988*b*), Golec and Ladde (1989), Feng (1990), Nakazawa (1990), Bensoussan, Glowinski and Rascanu (1992), Feng, Lei and Qian (1992), Artemiev (1993*b*), Kloeden, Platen and Schurz (1993), Saito and Mitsui (1993*a*), Petersen (1994*b*), Török (1994), Ogawa (1995), Gelbrich and Rachev (1996), Grecksch and Wadewitz (1996), Newton (1996), Saito and Mitsui (1996), Schurz (1996*b*), Yannios and Kloeden (1996), Artemiev and Averina (1997), Denk and Schäffer (1997), Abukhaled and Allen (1998) and Schein and Denk (1998).

To illustrate the strong order of convergence for the strong Taylor schemes mentioned earlier, let us perform a simulation study that uses geometric Brownian motion (6.5) introduced in the previous section. We estimate the absolute error (4.1) for different step-sizes Δ and different schemes, including the Euler scheme (3.1), the Milstein scheme (6.1), the order 1.5 strong Taylor approximation (6.2) and the order 2.0 strong Taylor approximation (6.3). In Figure 6.2 the logarithm of the respective estimated absolute errors from 2000 simulated paths are plotted against the logarithm of the step-size. We note that, for the different schemes, the slopes of the linearly interpolated absolute errors correspond to the theoretical strong orders of the schemes. Results for corresponding Runge–Kutta methods are almost identical.

Simulation studies involving higher-order schemes can be found, for instance, in Klauder and Petersen (1985), Pardoux and Talay (1985), Liske and Platen (1987), Newton (1991) and Kloeden *et al.* (1994/1997).

6.3. A-stability and implicit strong methods

What really matters in a numerical scheme is that it should be numerically stable, can be conveniently implemented, and generates fast highly accurate results.

Since SDEs generalize ODEs, their numerical analysis must encounter at least all the problems known for the deterministic case. Before any properties of higher order of convergence can be studied the question of numerical stability of a scheme has to be satisfactorily answered. Many practical problems turn out to be multi-dimensional: see Hofmann, Platen and Schweizer (1992) or Heath and Platen (1996) for examples from finance, or Schein and Denk (1998) for an example from microelectronics. We know from the numerical analysis of ODEs that stiff systems can easily occur which cause numerical instabilities for most explicit methods.

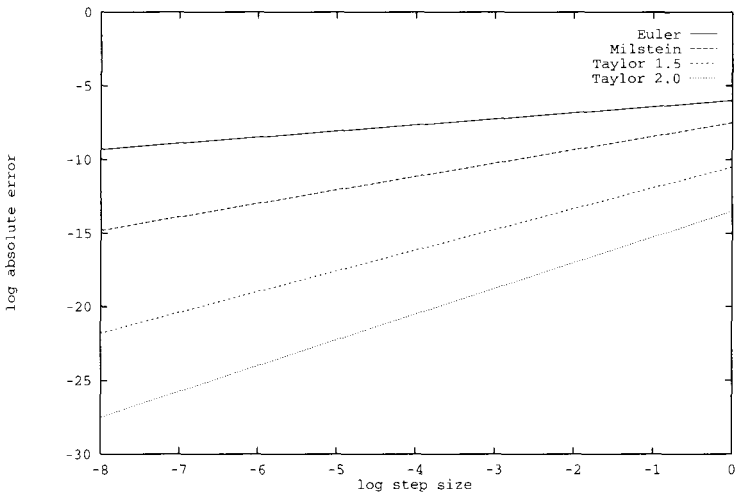

Fig. 6.2. Log absolute error versus log step-size

The propagation of errors in the stochastic case also depends on the specific nature of the stochastic part of the dynamics. It is quite a delicate matter to provide reasonable answers with respect to the stability of numerical schemes for general SDEs. Therefore it is useful to study important classes of test equations that provide insight into typical instability patterns.

The well-known concept of *A-stability* (see Björck and Dahlquist (1974)) can be directly generalized to the case of SDEs with additive noise, that is, $b(x) = \text{const}$ in equation (2.1): see Milstein (1988*a*), Hernandez and Spigler (1992), Kloeden and Platen (1992) or Milstein (1995*a*). One introduces a complex-valued test equation of the form

$$\mathrm{d}X_t = \lambda X_t \,\mathrm{d}t + \mathrm{d}W_t, \tag{6.10}$$

with additive noise, where λ is a complex number with real part $\mathrm{Re}(\lambda) < 0$ and W is a real-valued Wiener process. A one-step numerical approximation Y applied to (6.10) then usually yields a recursive relation of the form

$$Y_{n+1} = G(\lambda \Delta)\, Y_n + Z_n, \tag{6.11}$$

where the Z_n represent random terms that do not depend on λ or Y_0, Y_1, $\ldots$, Y_n. The *region of A-stability* of a scheme is defined as the subset of the complex plane consisting of those complex numbers $\lambda\Delta$ with $\mathrm{Re}(\lambda) < 0$ and $\Delta > 0$ which are mapped by the function G from (6.11) into the unit circle, that is, those $\lambda\Delta$ for which

$$|G(\lambda \Delta)| < 1. \tag{6.12}$$

If the A-stability region covers the left half of the complex plane, then we say that the scheme is A-stable.

Owing to the additive noise in the test equation, the concept of A-stability does not say much about instabilities that may arise, for example, from multiplicative noise, that is, $b(x) = \sigma x$, or from other non-constant diffusion coefficients. We shall discuss the problem of stability under multiplicative noise in Section 7.4. A-stability is a rough indicator for basic stability properties of any discrete time approximation.

On the basis of implicit stochastic Taylor expansions (see Kloeden and Platen (1992/1995*b*)), it is possible to construct implicit discrete time approximations. As an example, we mention the Stratonovich version of a family of implicit Milstein schemes described in Milstein (1988*a*, 1995*a*) and Kloeden and Platen (1992/1995*b*). It has the form

$$\begin{aligned} Y_{n+1} &= Y_n + \{\alpha\, \underline{a}(Y_{n+1}) + (1-\alpha)\, \underline{a}(Y_n)\}\, \Delta + b(Y_n)\, \Delta W_n \\ &\quad + b(Y_n)\, b'(Y_n)\, \frac{(\Delta W_n)^2}{2}, \end{aligned} \tag{6.13}$$

where $\alpha \in [0,1]$ represents the degree of implicitness. This family of schemes is of strong order $\gamma = 1.0$ and A-stable for $\alpha \geq \frac{1}{2}$.

A major difficulty arises from the fact that in a strong scheme it is almost impossible to construct implicit expressions for the noise terms, because in the actual discrete time approximation these would usually lead to terms with inverted Gaussian random variables. Such terms miss crucial moment properties. A possible research direction that seems to overcome part of the problem has been suggested by Milstein, Platen and Schurz (1998), who have proposed a family of *balanced implicit methods*. A balanced implicit method can be written in the form

$$Y_{n+1} = Y_n + a(Y_n)\, \Delta + b(Y_n)\, \Delta W_n + (Y_n - Y_{n+1})\, C_n, \tag{6.14}$$

where

$$C_n = c^0(Y_n)\, \Delta + c^1(Y_n)\, |\Delta W_n|$$

and c^0, c^1 represent positive real-valued uniformly bounded functions. One can also choose more general functions c^0 and c^1 that must fulfil conditions described in Milstein *et al.* (1998). The freedom in choosing c^0 and c^1 can be exploited to construct a method with stability properties tailored to the dynamics of the underlying SDE. However, one must pay a price: this method is only of strong order $\gamma = 0.5$. In a number of applications, especially those associated with multiplicative noise, the balanced implicit method showed much better stability behaviour than other methods: see, for instance, Schurz (1996*a*) and Fischer and Platen (1998).

Implicit schemes or different concepts of numerical stability have been suggested and studied in a variety of papers, and we again mention a long list, including Talay (1982*b*, 1984), Klauder and Petersen (1985), Pardoux

and Talay (1985), Milstein (1988*a*, 1995*a*), Artemiev and Shkurko (1991), Drummond and Mortimer (1991), Kloeden and Platen (1992), Hernandez and Spigler (1992, 1993), Artemiev (1993*a*, 1993*b*, 1994), Saito and Mitsui (1993*b*), Hofmann and Platen (1994), Milstein and Platen (1994), Komori and Mitsui (1995), Hofmann and Platen (1996), Saito and Mitsui (1996), Schurz (1996*a*), Schurz (1996*c*), Ryashko and Schurz (1997), Burrage (1998), Higham (1998) and Petersen (1998). Despite all this work, stochastic numerical stability remains an open and challenging area of research.

7. Weak approximation methods

As previously mentioned, in many applications it is not necessary to generate an almost exact replica of the sample path of the solution of the underlying SDE. The Monte Carlo simulation of option prices is a typical example, where simple random walks can be used to approximate option pricing functionals. Within this section we discuss numerical methods that focus on approximating the probability distributions of solutions of SDEs, allowing us to handle wide classes of functionals. We then need to study the weak order of convergence of several stochastic numerical methods.

7.1. Weak Taylor approximation

The weak convergence criterion (4.2) allows us more degrees of freedom in constructing a discrete time approximation than the strong convergence criterion (4.1). For instance, under weak convergence, the random increments ΔW_n of the Wiener process can be replaced by simpler random variables $\Delta \bar{W}_n$ which are similar to these in distribution. By substituting the $N(0, \Delta)$ Gaussian distributed random variable ΔW_n in the Euler approximation (3.1) by an independent two-point distributed random variable $\Delta \bar{W}_n$ with

$$P(\Delta \bar{W}_n = \pm\sqrt{\Delta}) = 0.5, \tag{7.1}$$

we obtain the *simplified Euler method*

$$Y_{n+1} = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta \bar{W}_n. \tag{7.2}$$

The key point for this choice of the two-point random variable $\Delta \bar{W}_n$ is that its first two moments match the corresponding ones for ΔW_n. It can be shown that this method (7.2) converges with weak order $\beta = 1.0$ if sufficient regularity conditions are imposed. This weak order is higher than the strong order $\gamma = 0.5$ achieved by the Euler approximation (3.1). In Mikulevicius and Platen (1991), a lower order, $\beta \leq 1.0$, of weak convergence has been proved if there are only Hölder continuous drift and diffusion coefficients.

The Euler approximation (3.1) can be interpreted as the *order* 1.0 *weak Taylor scheme.* One can select additional terms from the Wagner–Platen formula to obtain weak Taylor schemes of higher order. Platen (1984, 1992)

and Kloeden and Platen (1992/1995*b*) have described how to construct the weak Taylor scheme corresponding to a given weak order $\beta \in \{1, 2, 3, \ldots\}$. It turns out that one has to include all terms from the Wagner–Platen formula with multiple Itô integrals of multiplicity equal to or less than the desired weak order β. Thus the Euler method, that is, the order 1.0 weak Taylor scheme, is constructed using the multiple integrals of multiplicity one. The *order* 2.0 *weak Taylor scheme* must then include all terms with single and double integrals, and therefore has the form

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta W_n + b\,b'\,I_{(1,1)} + b\,a'\,I_{(1,0)} \\ &\quad + (a\,b' + \tfrac{1}{2}\,b^2\,b'')\,I_{(0,1)} + (a\,a' + \tfrac{1}{2}\,b^2\,a')\,\tfrac{\Delta^2}{2}. \end{aligned} \tag{7.3}$$

This scheme was first proposed by Milstein (1978) and later studied by Platen (1984) and Talay (1984). We still obtain a scheme of weak order $\beta = 2.0$ if we replace the random variable ΔW_n in (7.3) by $\Delta \hat{W}_n$, the double integrals $I_{(1,0)}$ and $I_{(0,1)}$ by $\frac{\Delta}{2}\Delta \hat{W}_n$ and the double Wiener integral $I_{(1,1)}$ by $\frac{1}{2}\,((\Delta \hat{W}_n)^2 - \Delta)$. Here $\Delta \hat{W}_n$ might be a three-point distributed random variable with

$$P\left(\Delta \hat{W}_n = \pm\sqrt{3\,\Delta}\right) = \frac{1}{6} \qquad \text{and} \qquad P\left(\Delta \hat{W}_n = 0\right) = \frac{2}{3}. \tag{7.4}$$

Note that the first four moments of $\Delta \hat{W}_n$ match the corresponding ones of ΔW_n. By approximating all triple Itô integrals in the order 3.0 weak Taylor scheme, a *simplified order* 3.0 *weak Taylor scheme* was derived by Platen (1984), with the form

$$\begin{aligned} Y_{n+1} &= Y_n + a\,\Delta + b\,\Delta \tilde{W} + b\,b'((\Delta \tilde{W})^2 - \Delta)/2 + b\,a'\,\Delta \tilde{Z} \\ &\quad + \tfrac{\Delta^2}{2}\left(a\,a' + \tfrac{1}{2}\,b^2\,a''\right) + \left(a\,b' + \tfrac{1}{2}\,b^2\,b''\right)(\Delta \tilde{W}\Delta - \Delta \tilde{Z}) \\ &\quad + \left\{a\left(a\,b' + b\,a' + \tfrac{1}{2}\,b^2\,b''\right)' + \tfrac{1}{2}\,b^2\left(a\,b' + b\,a' + \tfrac{1}{2}\,b^2\,b''\right)'' \right. \\ &\qquad \left. + b\left(a\,a' + \tfrac{1}{2}\,b^2\,a''\right)'\right\}\frac{\Delta \tilde{W}\Delta^2}{6} \\ &\quad + \left\{b\left(b\,a' + a\,b' + \tfrac{1}{2}\,b^2\,b''\right)' + a(b\,b')' + \tfrac{1}{2}\,b^2(b\,b')''\right\}\left((\Delta \tilde{W})^2 - \Delta\right)\frac{\Delta}{6} \\ &\quad + \left\{a\left(a\,a' + \tfrac{1}{2}\,b^2\,a''\right)' + \tfrac{1}{2}\,b^2\left(a\,a' + \tfrac{1}{2}\,b^2\,a''\right)''\right\}\frac{\Delta^3}{6} \\ &\quad + b\,(b\,b')'\left((\Delta \tilde{W})^2 - 3\,\Delta\right)\frac{\Delta \tilde{W}}{6}. \end{aligned} \tag{7.5}$$

Here $\Delta\tilde{W}$ and $\Delta\tilde{Z}$ can be chosen, for instance, as correlated zero mean Gaussian random variables with

$$E(\Delta\tilde{W})^2 = \Delta, \qquad E((\Delta\tilde{Z})^2) = \Delta^3/3, \qquad E(\Delta\tilde{Z}\Delta\tilde{W}) = \Delta^2/2.$$

As in the case of strong approximations, weak higher-order schemes can be constructed only if there is adequate smoothness of the drift and diffusion coefficients, and is a sufficiently rich set of random variables approximating the multiple stochastic integrals of the corresponding weak Taylor schemes, generated at each time step. Under the weak convergence criterion, we not only have considerable freedom to approximate multiple stochastic integrals, but also need fewer such integrals to achieve a certain order of weak convergence than for the same order of strong convergence: see Kloeden and Platen (1992/1995*b*) and Hofmann (1994).

We note that the weak higher-order Taylor schemes involve higher-order derivatives of a and b. Obviously, it would be desirable to have derivative-free or Runge–Kutta-type weak schemes. We discuss these in the following section.

7.2. Weak Runge–Kutta approximations

A *weak second-order Runge–Kutta approximation* that avoids derivatives in a and b is given by the algorithm

$$\begin{aligned} Y_{n+1} &= Y_n + (a(\hat{Y}_n) + a(Y_n))\,\frac{\Delta}{2} + (b(Y_n^+) + b(Y_n^-))\,\frac{\Delta\hat{W}_n}{4} \\ &\quad + (b(Y_n^+) - b(Y_n^-))\,((\Delta\hat{W}_n)^2 - \Delta)\,\frac{1}{4\sqrt{\Delta}} \end{aligned} \tag{7.6}$$

with

$$\hat{Y}_n = Y_n + a(Y_n)\,\Delta + b(Y_n)\,\Delta\hat{W}_n$$

and

$$Y_n^{\pm} = Y_n + a(Y_n)\,\Delta \pm b(Y_n)\,\sqrt{\Delta},$$

where $\Delta\hat{W}_n$ can be chosen as in (7.4): see Platen (1984).

Talay (1984) suggested a weak second-order scheme which is not completely derivative-free, and also requires two random variables at each step. Another weak second-order scheme that involves the derivative b' has been proposed by Milstein (1985), with the form

$$\begin{aligned} Y_{n+1} &= (a(\hat{Y}_n) + a(Y_n))\,\frac{\Delta}{2} + b(Y_n)\,b'(Y_n)\,\frac{((\Delta\hat{W}_n)^2 - \Delta)}{2} \\ &\quad + \left(\tfrac{1}{4}\,b(\hat{Y}_n^+) + \tfrac{1}{2}\,b(Y_n) + \tfrac{1}{4}\,b(\hat{Y}_n^-)\right)\Delta\hat{W}_n, \end{aligned} \tag{7.7}$$

where $\hat{Y}_n$ and $\Delta\hat{W}_n$ are as in (7.6), and

$$\hat{Y}_n^{\pm} = Y_n + a(Y_n) \pm b(Y_n) \frac{\Delta\hat{W}_n}{3}.$$

Weak second- and third-order Runge–Kutta-type schemes have been proposed, for instance, by Kloeden and Platen (1992/1995*b*), Mackevicius (1994) and Komori and Mitsui (1995). There appears to be some scope for future research in weak higher-order Runge–Kutta schemes, possibly generalizing Butcher's rooted tree methods as described, for instance, by Komori *et al.* (1997) and Burrage (1998).

7.3. Extrapolation methods

In deterministic numerical analysis, extrapolation methods represent an elegant way of achieving higher-order convergence by using lower-order methods, provided the numerical stability of these for a range of step-sizes can be guaranteed. For the weak second-order approximation of the functional $E(g(X_T))$, Talay and Tubaro (1990) proposed a *Richardson extrapolation* of the form

$$V_{g,2}^{\Delta}(T) = 2\,E\left(g\left(Y^{\Delta}(T)\right)\right) - E\left(g\left(Y^{2\Delta}(T)\right)\right), \tag{7.8}$$

where $Y^{\delta}(T)$ denotes the value at time T of an Euler approximation with step-size δ. Using Euler approximations with step-sizes $\delta = \Delta$ and $\delta = 2\Delta$, and then taking the difference (7.8) of their respective functionals, the leading error coefficient cancels out, and $V_{g,2}^{\Delta}(T)$ ends up being a weak second-order approximation.

Further weak higher-order extrapolations have been developed by Kloeden and Platen (1989). For instance, one obtains a *weak fourth-order extrapolation method* using

$$V_{g,4}^{\Delta} = \frac{1}{21}\left[32\,E\left(g\left(Y^{\Delta}(T)\right)\right) - 12\,E\left(g\left(Y^{2\Delta}(T)\right)\right) + E\left(g\left(Y^{4\Delta}(T)\right)\right)\right], \tag{7.9}$$

where $Y^{\delta}(T)$ is the value at time T of the weak second-order Runge–Kutta scheme (7.6) with step-size δ. Such weak high-order extrapolations require the existence of a leading error expansion for functionals of the underlying discrete time weak approximation Y^{δ}. Further results on extrapolation methods can be found in Hofmann (1994), Goodlett and Allen (1994), Kloeden, Platen and Hofmann (1995) and Mackevicius (1996).

Artemiev (1985), Müller-Gronbach (1996), Gaines and Lyons (1997), Burrage (1998) and Mauthner (1998) have derived results on step size control. Furthermore, Hofmann (1994), Hofmann, Müller-Gronbach and Ritter (1998) have considered extrapolation methods with both step-size and order control. This is another challenging area of practical importance.

7.4. M-stability and implicit weak methods

The comments in Section 6.3 on numerical stability of discrete time approximations in the context of strong convergence apply equally to weak schemes. Numerical instability is a problem that arises in both strong and weak schemes, and similar methods can be used to study it. However, as we shall see below, it is again much easier to construct a weak method with satisfactory stability properties than a corresponding strong one.

The crucial advantage in the construction of implicit schemes under the weak convergence criterion (4.2) lies in the freedom to choose the necessary random variables to be bounded. This allows us to construct weak schemes that have fully implicit terms for the noise part of the SDE.

To highlight the importance of this fact, Hofmann and Platen (1994), Hofmann (1995) and Hofmann and Platen (1996) have considered a complex-valued test equation with multiplicative noise of Stratonovich type

$$\mathrm{d}X_t = (1-\alpha)\,\lambda\, X_t\,\mathrm{d}t + \sqrt{\alpha}\,\gamma\, X_t \circ \mathrm{d}W_t, \tag{7.10}$$

where $\lambda = \lambda_1 + \lambda_2\, i$ and $\gamma = \gamma_1 + \gamma_2\, i$ are complex numbers such that $\gamma^2 = \lambda$. Here W again denotes a real-valued standard Wiener process. The real-valued parameter $\alpha \in [0,2]$ describes the degree of stochasticity in the test equation (7.10). For $\alpha = 0$ we have a purely deterministic equation. For $\alpha = 1$, (7.10) represents a Stratonovich SDE without drift, while for $\alpha = 2$ it can be written as an Itô SDE with no drift.

Suppose that we can express a given stochastic numerical scheme, to be applied to the test equation (7.10) with equidistant step-size Δ, in the recursive form

$$Y_{n+1} = G(\lambda\,\Delta, \alpha)\, Y_n, \tag{7.11}$$

where G is a complex-valued random function that does not depend on Y_0, $Y_1, \ldots, Y_{n-1}, Y_{n+1}$. Then we can introduce the *M-stability set*

$$\Gamma = \{\Gamma_\alpha : 0 \le \alpha \le 2\},$$

with stability region

$$\Gamma_\alpha = \{\lambda\,\Delta \in \mathcal{C} : \mathrm{Re}(\lambda) < 0, \mathrm{ess}_\omega \sup |G(\lambda\,\Delta, \alpha)| < 1\}, \tag{7.12}$$

for $\alpha \in [0,2]$. Whereas the A-stability discussed in Section 6.3 can be linked to test equations with 'additive noise', the term M-stability is used with test equations that have 'multiplicative noise'. The $\mathrm{ess}_\omega \sup$ in (7.12) denotes the essential supremum with respect to all $\omega \in \Omega$, and in practice refers to the worst case scenario. To check for the worst possible paths is important because we have to protect a simulation against unstable scenarios. A single overflow in a large number of simulations can make the whole simulation study questionable. On the other hand, excluding some extreme simulated

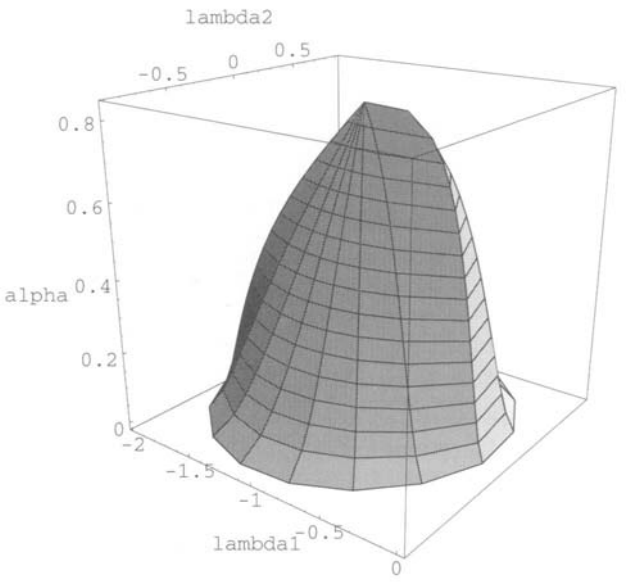

Fig. 7.1. M-stability set for the simplified Euler scheme

scenarios would certainly bias the result. Therefore it seems natural to judge stability on a worst case basis.

For the simplified Euler scheme (7.2) the M-stability region is shown in Figure 7.1. We note that in the deterministic case, $\alpha = 0$, the region of stability Γ_0 is a circle and coincides with the A-stability region discussed in Section 6.3. For $\alpha > 0$ we note that the stability region of the simplified Euler scheme shrinks, and no longer includes the α-axis. This is a crucial observation telling us that reduction of the step-size Δ might lead us to exit the stability region. In the deterministic case, this is not the typical behaviour of a scheme. Such behaviour is usually observed only when the step-sizes are close to machine precision. In the stochastic case, the noise modelled by the dynamics of the SDE can already generate this type of instability for large step sizes, and has to be taken rather seriously. We see from Figure 7.1 that, for $\alpha = 2$, the simplified Euler scheme has no M-stability at all. This is the martingale case for X, which is typical in asset price modelling in finance, where multiplicative noise arises naturally: see Hofmann *et al.* (1992). We also note that random walks and binomial trees which are typically implemented in many applications, particularly in finance, have the structure of the simplified Euler scheme and can therefore suffer serious instabilities.

7.5. *Implicit and predictor–corrector methods*

Implicit and predictor–corrector methods have much larger stability regions than most explicit schemes, and turn out to be better suited to simulation task with potential stability problems. Some results on implicit weak schemes in a weak context can be found in Milstein (1985, 1988*a*, 1995*a*), Drummond and Mortimer (1991) and Platen (1995). In Kloeden and Platen (1992/1995*b*) the following family of *weak implicit Euler schemes*, converging with weak order $\beta = 1.0$, has been discussed:

$$\begin{aligned} Y_{n+1} &= Y_n + \{\xi\, \bar{a}_\eta(Y_{n+1}) + (1-\xi)\, \bar{a}_\eta(Y_n)\}\, \Delta \\ &\quad + \{\eta\, b(Y_{n+1}) + (1-\eta)\, b(Y_n)\}\, \Delta \bar{W}_n. \end{aligned} \tag{7.13}$$

Here $\Delta \bar{W}_n$ can be chosen as in (7.1), and we have to set

$$\bar{a}_\eta(y) = a(y) - \eta\, b(y)\, b'(y) \tag{7.14}$$

for $\xi \in [0,1]$ and $\eta \in [0,1]$. For $\xi = 1$ and $\eta = 0$, (7.13) leads to the *drift implicit Euler scheme*. The choice $\xi = \eta = 0$ gives us the simplified Euler scheme, whereas for $\xi = \eta = 1$ we have the *fully implicit Euler scheme*. It can be shown, for instance, that the fully implicit Euler scheme is A-stable in the sense of Section 6.3. The exterior of the M-stability set of the drift implicit Euler scheme with respect to test equation (7.10) is shown in Figure 7.2. One notes that the M-stability set is much larger in Figure 7.2 than in Figure 7.1. However, the α-axis is not included in this set and one has to choose a step-size with a value above a critical minimal size to guarantee stability.

A family of *implicit weak order* 2.0 *schemes* has been proposed by Milstein (1995*a*) with

$$\begin{aligned} Y_{n+1} &= Y_n + \{\xi\, a(Y_{n+1}) + (1-\xi)\, a(Y_n)\}\, \Delta \\ &\quad + \tfrac{1}{2}\, b(Y_n)\, b'(Y_n)\, ((\Delta \hat{W}_n)^2 - \Delta)/2 \\ &\quad + \left\{ b(Y_n) + \tfrac{1}{2}\, (b'(Y_n) + (1-2\xi)\, a'(Y_n))\, \Delta \right\} \Delta \hat{W}_n \\ &\quad + (1-2\xi)\{\beta\, a'(Y_n) + (1-\beta)\, a'(Y_{n+1})\}\, \frac{\Delta^2}{2}, \end{aligned} \tag{7.15}$$

where $\Delta \hat{W}_n$ is chosen as in (7.4).

Kloeden and Platen (1992/1995*b*) suggested the following *Runge–Kutta-type implicit weak order* 2.0 *scheme*:

$$\begin{aligned} Y_{n+1} &= Y_n + \tfrac{1}{2}\, (a(Y_n) + a(Y_{n+1}))\, \Delta \\ &\quad + (b(\bar{R}_+) + b(\bar{R}_-) + 2\, b(Y_n))\, \frac{\Delta \hat{W}_n}{4} \\ &\quad + (b(\bar{R}_+) - b(\bar{R}_-))\, ((\Delta \hat{W}_n)^2 - \Delta)/4, \end{aligned} \tag{7.16}$$

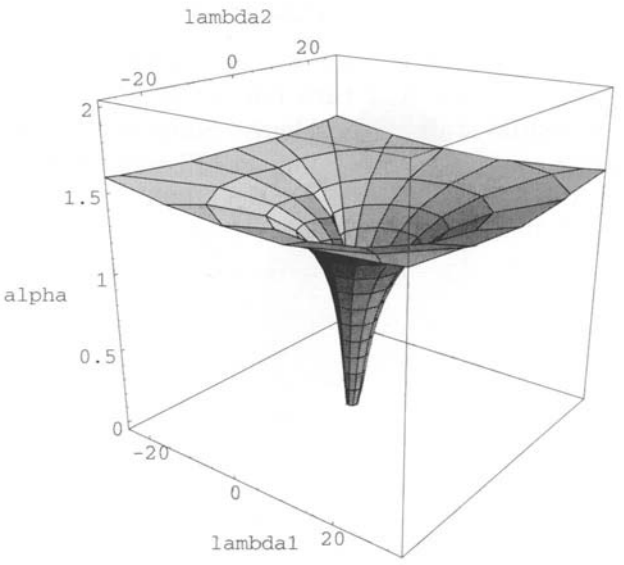

Fig. 7.2. Exterior of M-stability set for the drift implicit Euler scheme

$$\bar{R}_{\pm} = Y_n + a(Y_n)\,\Delta \pm b(Y_n)\,\sqrt{\Delta}$$

and

$$\bar{U}_{\pm} = Y_n \pm b(Y_n)\,\sqrt{\Delta},$$

where $\Delta\hat{W}_n$ is chosen as in (7.4).

In deterministic numerical analysis, predictor–corrector methods are often used because of their numerical stability, inherited from the implicit counterpart of their corrector scheme. With a predictor–corrector method one is not forced to solve an algebraic equation at each time step as with an implicit method. For instance, a *weak second-order predictor–corrector* method (see Platen (1995)) is given by the corrector

$$Y_{n+1} = Y_n + \tfrac{\Delta}{2}\,(a(\bar{Y}_{n+1}) + a(Y_n)) + \psi_n \tag{7.17}$$

with

$$\begin{aligned}\psi_n \;=\; & b(Y_n) + \tfrac{1}{2}\left(a(Y_n)\,b'(Y_n) + \tfrac{1}{2}\,b^2(Y_n)\,b''(Y_n)\right)\Delta\hat{W}_n \\ & + b(Y_n)\,b'(Y_n)\left((\Delta\hat{W}_n)^2 - \Delta\right)/2\end{aligned}$$

and the predictor

$$\bar{Y}_{n+1} = Y_n + a(Y_n)\,\Delta + \psi_n + \left(a(Y_n)\,a'(Y_n) + \tfrac{1}{2}\,b^2(Y_n)\,a''(Y_n)\right)\tfrac{\Delta^2}{2} + b(Y_n)\,a'(Y_n)\,\Delta\hat{W}_n\Delta/2,$$

where $\Delta\hat{W}_n$ is as in (7.4).

A list of references on schemes with implicit features and stochastic numerical stability has already been given in Section 6.3. Further publications dealing with aspects of weak approximations include Fahrmeier (1974), Milstein (1978), Platen (1980*b*), Gladyshev and Milstein (1984), Platen (1984), Talay (1984), Milstein (1985), Ventzel, Gladyshev and Milstein (1985), Haworth and Pope (1986), Talay (1986), Milstein (1988*a*), Talay (1990), Talay and Tubaro (1990), Kloeden and Platen (1991*b*), Mikulevicius and Platen (1991), Kloeden, Platen and Hofmann (1992*a*), Kannan and Wu (1993), Hofmann (1994), Hofmann and Platen (1994), Mackevicius (1994), Komori and Mitsui (1995), Bally and Talay (1996*a*, 1996*b*), Hofmann and Platen (1996), Kohatsu-Higa and Ogawa (1997) and Milstein and Tretjakov (1997).

Let us emphasize again that there is no point in trying to improve the efficiency of a simulation if its stability is not satisfactorily established.

7.6. Monte Carlo simulations of SDEs

There exists a well-developed literature on general Monte Carlo methods. We might mention, among others, Hammersley and Handscomb (1964), Ermakov (1975), Sabelfeld (1979), Rubinstein (1981), Ermakov and Mikhailov (1982), Kalos and Whitlock (1986), Bratley, Fox and Schrage (1987), Bouleau (1990), Law and Kelton (1991), Ross (1991), Mikhailov (1992) and Fishman (1992). In focusing on weak numerical discrete time approximations of SDEs, one obtains greater insight into the stochastic analytic structure of the problem than one usually does in general Monte Carlo problems. One can exploit martingale representations as in Newton (1994), or measure transformations as discussed in Milstein (1988*a*) or Kloeden and Platen (1992/1995*b*). These structures allow one to develop highly sophisticated Monte Carlo methods. In many cases these perform extremely well in circumstances where other methods fail, are difficult to implement or exceed available computing time.

Functionals of the form

$$u = E(g(X_T))$$

can be approximated by weak approximations, as discussed in the context of the weak convergence criterion (4.2). One can form a straightforward

Monte Carlo estimate using the sample average

$$u_{N,\Delta} = \frac{1}{N} \sum_{k=1}^{N} g(Y_T(\omega_k)), \tag{7.18}$$

with N independent simulated realizations $Y_T(\omega_1), Y_T(\omega_2), \ldots, Y_T(\omega_N)$ of a discrete time weak approximation Y at time T. The mean error $\hat{\mu}$ then has the form

$$\hat{\mu} = u_{N,\Delta} - E(g(X_T))$$

which we can decompose (see Kloeden and Platen (1992/1995*b*)) into a *systematic error* μ_{sys} and a *statistical error* μ_{stat}, such that

$$\hat{\mu} = \mu_{\text{sys}} + \mu_{\text{stat}}, \tag{7.19}$$

where

$$\begin{aligned} \mu_{\text{sys}} &= E(\hat{\mu}) \\ &= E\left(\frac{1}{N} \sum_{k=1}^{N} g(Y_T(\omega_k))\right) - E(g(X_T)) \\ &= E(g(Y_T)) - E(g(X_T)). \end{aligned} \tag{7.20}$$

Obviously, the absolute systematic error $|\hat{\mu}|$ represents the critical variable under the weak-order convergence criterion (4.2).

For a large number N of simulated independent sample paths of Y, we can conclude from the Central Limit Theorem that the statistical error μ_{stat} becomes asymptotically Gaussian with mean zero and variance of the form

$$\text{Var}(\mu_{\text{stat}}) = \text{Var}(\hat{\mu}) = \tfrac{1}{N} \text{Var}(g(Y_T)). \tag{7.21}$$

This reveals a significant disadvantage of Monte Carlo methods, because its deviation

$$\text{Dev}(\mu_{\text{stat}}) = \sqrt{\text{Var}(\mu_{\text{stat}})} = \frac{1}{\sqrt{N}} \sqrt{\text{Var}(g(Y_T))}, \tag{7.22}$$

decreases at only the slow rate $N^{-1/2}$ as $N \to \infty$. Thus the length of a corresponding confidence interval for the error is, for instance, only halved by a fourfold increase in the number N of simulated realizations.

The Monte Carlo approach is very general and works in almost all circumstances. For high-dimensional functionals it is sometimes the only method of obtaining a result. One pays for this generality by the large sample sizes required to achieve reasonably accurate estimates.

We note from (7.22) that the length of a confidence interval is also proportional to the square root of the variance of the simulated functional. As we shall see in the next section, this provides us with an opportunity to

construct unbiased estimates for $u = E(g(Y_T))$ with much smaller variances than the raw Monte Carlo functional (7.18).

7.7. Variance reduction techniques

One can increase the efficiency in Monte Carlo simulation for SDEs considerably by using various variance reduction techniques. These reduce primarily the variance of the random variable actually simulated. There exist many ways of achieving substantial variance reduction. Only some of them can be mentioned here. Experience has shown that, to be effective, variance reduction techniques need to be adapted and engineered to the given specific problem.

Some general variance reduction techniques from classical Monte Carlo theory usually result in only moderate improvements. Techniques that exploit to a high degree the stochastic analytic structure of the given functional of an SDE can easily yield savings in computer time corresponding to factors of several thousands. Useful references on variance reduction techniques in a more classical setting include Hammersley and Handscomb (1964), Ermakov (1975), Boyle (1977), Maltz and Hitzl (1979), Rubinstein (1981), Ermakov and Mikhailov (1982), Ripley (1983*b*), Kalos and Whitlock (1986), Bratley *et al.* (1987), Chang (1987), Wagner (1987, 1988*a*, 1988*b*, 1989*a*, 1989*b*), Law and Kelton (1991) and Ross (1991).

In what follows, we first mention some more classical Monte Carlo variance reduction techniques and then point to stochastic numerical variance reduction methods that use martingale representations or measure transformations for functionals of SDEs.

The method of *antithetic variates* (see, for instance, Law and Kelton (1991) or Ross (1991)) is a very general method. It uses repeatedly the random variables originally generated in some symmetric pattern to construct sample paths, say for the driving Wiener process, in such a way that these offset each others' noise to some extent in the estimate. The simplest version of antithetic variates is obtained by using together a Wiener path realization $W(\omega_+)$, and its negative counterpart $W(\omega_-) = -W(\omega_+)$ in the sample. This reduces the time for the computation of the sample and, using certain symmetries, one can reduce the variance of the simulated estimators substantially.

Another general method is through *variance reduction by conditioning*: see Law and Kelton (1991). For some σ-algebra, or information set, $\mathcal{F}$, we can interpret the conditional expectation $E(g(X_T) \mid \mathcal{F})$ as a variance-reduced unbiased estimator for the functional $E(g(X_T))$. The variance is then reduced according to the inequality

$$\mathrm{Var}(E(g(X_T) \mid \mathcal{F})) \leq \mathrm{Var}(g(X_T)).$$

Here $\mathcal{F}$ represents some information about the path of X, for instance that this path remains in a certain region.

Stratified sampling is another technique that has been widely used in Monte Carlo simulation: see, for instance, Glynn and Iglehart (1989), Ross (1991) and Fournie, Lebuchoux and Touzi (1997). A simple version of it can be described by dividing the whole sample space into M sets of disjoint events $A_1, \ldots, A_M$ with $P(A_i) = \frac{1}{M}$ for all $i \in \{1, \ldots, M\}$. For example, assume that the first step of the discrete time weak approximation ends up in one of the M equally probable states, where each of these events is indicated by writing Y_{T,A_i}, $i \in \{1, \ldots, M\}$ for the final value of Y at time T. Then we can use the unbiased estimator

$$Z_1 = \frac{1}{M} \sum_{i=1}^{M} g(Y_{t,A_i}),$$

where A_i is the above-mentioned random event. This estimator has variance

$$\begin{aligned} \mathrm{Var}(Z_1) &= \sum_{i=1}^{M} \frac{\mathrm{Var}(g(Y_{T,A_i}))}{M^2} \\ &= \frac{1}{M} \mathrm{Var}(g(Y_T)). \end{aligned}$$

We note that for large M the variance of the estimate Z_1 will be considerably smaller than that of the random variable $g(Y_T)$.

The rather standard *control variate technique* is based on the selection of a random variable ξ with known mean $E(\xi)$ that allows one to construct the unbiased estimate

$$Z_2 = g(Y_T) - \alpha(\xi - E(\xi)),$$

where the parameter

$$\alpha = \frac{\mathrm{Cov}(g(Y_T), \xi)}{\mathrm{Var}(\xi)}$$

is chosen to minimize the variance

$$\mathrm{Var}(Z_2) = \mathrm{Var}(g(Y_T)) + \alpha^2 \mathrm{Var}(\xi) - 2\alpha \mathrm{Cov}(g(Y_T), \xi).$$

Such a control variate can strongly exploit the stochastic analytic structure of the given functional. It turns out to be very powerful, as is shown, for example, in Hull and White (1988), Goodlett and Allen (1994), Newton (1994, 1997) and Heath and Platen (1996).

Another variance reduction technique, the *measure transformation method*, was proposed and studied by Milstein (1988*a*, 1995*a*), Kloeden and Platen (1992/1995*b*) and Hofmann *et al.* (1992). This introduces a new probability measure $\tilde{P}$ via a Girsanov transformation. The underlying Wiener process W is then no longer a Wiener process under the measure $\tilde{P}$. This

method formally computes the same functional as before, but uses the new measure $\tilde{P}$ and thus some corresponding Wiener process $\tilde{W}$. The measure-transformed estimate can now be expressed under the original measure P, which then provides an unbiased estimate

$$Z_3 = g(\tilde{Y}_T)\,\frac{d\tilde{P}}{dP},$$

where

$$E(g(Y_T)) = E\left(g(\tilde{Y}_T)\,\frac{d\tilde{P}}{dP}\right)$$

and $\frac{d\tilde{P}}{dP}$ represents the Radon–Nikodym derivative of $\tilde{P}$ with respect to the original measure P. Since the last relation can be fulfilled by a whole class of measure transformations, we have gained some degree of freedom and can seek a 'best' choice for $\tilde{P}$ that reduces the variance significantly. With reasonable knowledge about the qualitative properties of the functional $E(g(Y_T))$, this method can achieve considerable variance reductions.

If we summarize the variance reduction techniques discussed above, then it is apparent that all of them are fairly general and most of them can be combined with each other. This turns out to be an important property, because great flexibility is needed to tailor efficient Monte Carlo estimates for specific functionals. It should be mentioned, however, that there seems to be no specific generally suitable method that is also highly efficient. Experience is required to find an appropriate variance-reduced estimator for a given functional.

7.8. Quasi-Monte Carlo approach

Within this section we add some comments on the *quasi-Monte Carlo approach*, which is another technique for enhancing weak approximation methods. There is a rich literature on this subject with some reviews, for instance in Ripley (1983*b*), Niederreiter (1992) and Niederreiter and Shine (1995). Applications can be found in Barraquand (1995), Paskov and Traub (1995) or Joy, Boyle and Tan (1996), among others.

The approach can be illustrated by considering the probability density function p_X of the random variable X_T in such a way that the functional to be computed is expressed in the form

$$u = E(g(X_T)) = \int_{-\infty}^{\infty} g(x)\,p_X(x)\,\mathrm{d}x.$$

Consequently the estimation of the functional u appears as a numerical integration problem over $(-\infty, \infty)$. If we denote by F_{X_T} the distribution

function of X_T, then u can be expressed as

$$u = \int_0^1 g\left(F_{X_T}^{-1}(z)\right) \, \mathrm{d}z.$$

A standard Monte Carlo simulation could now evaluate the sum

$$S_N = \frac{1}{N} \sum_{i=1}^{N} g\left(F_{X_T}^{-1}(R_i)\right),$$

where the R_i, $i \in \{1, \ldots, N\}$, are independent uniformly distributed random variables. In a quasi-Monte Carlo method these random variables are replaced by elements from some low-discrepancy sequence or point set: see, for instance, the book by Niederreiter (1992). Low-discrepancy point sets such as Sobol, Halton or Faure sequences, discussed for instance in Halton (1960), Sobol (1967), Tezuka (1993), Tezuka and Tokuyama (1994), Radovic, Sobol and Tichy (1996), Tuffin (1996, 1997) and Mori (1998), exhibit fewer deviations from uniformity compared to uniformly distributed random point sets. This can generally lead to faster rates of convergence compared to random sequences as discussed in Hofmann and Mathé (1997) and Sloan and Wozniakowski (1998). However, the gain in efficiency is not always balanced with the bias that may result from the use of these methods. Caution has to be exerted in dealing with simplistic quasi-Monte Carlo estimates that could lead to undesirable biases.

8. Further developments and conclusions

In this final section, we comment on promising directions for further research, and briefly mention relevant literature that we have not included so far. A number of new research areas have been opened up in recent years, closely related to the area of numerical methods for SDEs.

Discrete time approximations for the numerical analysis of functionals of ergodic diffusion processes that depend on the corresponding invariant law were studied by Grorud and Talay (Talay 1987, 1990, 1991, 1995, and Grorud and Talay 1990, 1996) and Arnold and Kloeden (1996). Here the time horizon becomes *de facto* infinite, and one aims to tackle questions related to the computation of Lyapunov exponents, rotation numbers and other characteristics of stochastic dynamical systems. The numerical solution of nonlinear stochastic dynamical systems has been studied by Kloeden, Platen and Schurz (1991, 1992*b*) and Kloeden and Platen (1995*a*).

SDEs with coloured noise were approximated by Manella and Palleschi (1989), Fox (1991) and Milstein and Tretjakov (1994).

Weak approximations on a bounded domain, which relate to the solution of a corresponding parabolic partial differential equation, are constructed in Platen (1983), Milstein (1995*b*, 1995*c*, 1996, 1997) and Hausenblas (1999*a*).

This appears to be a very promising direction of future research, where stochastic numerical techniques provide access to efficient numerical solutions of partial differential equations with difficult boundary conditions. These methods seem to be also applicable in higher dimensions.

Approximations to first exit times of diffusion processes from a region were considered, for instance, by Platen (1983, 1985) and Abukhaled and Allen (1998). Related to this are numerical methods for SDEs with reflection or boundary conditions. These were studied, for instance, by Gerardi, Marchetti and Rosa (1984), Lépingle (1993), Slominski (1994), Asmussen, Glynn and Pitman (1995), Petterson (1995), Lépingle (1995) and Hausenblas (1999*b*). This is a technically demanding and growing area of research, where quantities such as local times have to be approximated.

Discrete time approximations for Itô processes with jump component have already been studied, for instance by Wright (1980), Platen (1982*a*, 1984), Maghsoodi and Harris (1987), Mikulevicius and Platen (1988) and Maghsoodi (1994). Driven by practical applications in finance and insurance, this area can be expected to develop further in the long-term future.

More generally, the discrete time strong and weak approximation of solutions of SDEs that represent semimartingales was studied by Marcus (1981), Platen and Rebolledo (1985), Protter (1985), Jacod and Shiryaev (1987), Mackevicius (1987), Bally (1989*a*, 1989*b*, 1990), Gyöngy (1991) and Kurtz and Protter (1991*a*, 1991*b*). Special emphasis on semimartingale SDEs driven by Lévy processes, including α-stable processes, was given in the book by Janicki and Weron (1994), and in papers by Kohatsu-Higa and Protter (1994), Janicki (1996), Janicki, Michna and Weron (1996), Protter and Talay (1997) and Tudor and Tudor (1997). Tudor and Tudor (1987) and Tudor (1989) have also approximated stochastic delay equations.

Approximation schemes for two-parameter SDEs were suggested by Tudor and Tudor (1983), Yen (1988) and Tudor (1992). Numerical experiments and numerical schemes for stochastic partial differential equations are discussed by Liske (1985), Elliott and Glowinski (1989), Bensoussan, Glowinski and Rascanu (1990), LeGland (1992), Gaines (1995*b*), Grecksch and Kloeden (1996), Ogorodnikov and Prigarin (1996), Gyöngy and Nuarlart (1997), Werner and Drummond (1997) and Allen, Novosel and Zhang (1998). In Ma, Protter and Yong (1994), Douglas, Ma and Protter (1996) and Chevance (1997), numerical methods for forward–backward SDEs have been studied. This represents yet another new direction of research.

Nonlinear diffusion processes that depend on related temporal and spatial partial differential equations were approximated by Ogawa (1992, 1994, 1995). Approximation schemes for Itô–Volterra SDEs have been suggested by Makroglou (1991) and Tudor and Tudor (1995). Averaging principles were applied to systems of singularly perturbed SDEs by Golec and Ladde (1990) and Golec (1995, 1997).

Almost in every area of stochastic modelling with finite-dimensional or infinite-dimensional dynamics, numerical methods have been or will soon be developed to provide quantitative results. The difficulties are often very similar in the different fields, and concern numerical stability, higher-order efficiency and variance reduction. For well-researched problems the development of standard software tools is becoming part of the general scientific work in the area. The construction of stochastic numerical schemes through symbolic manipulation and related questions were considered, for instance, by Valkeila (1991), Kloeden *et al.* (1992*c*), Kloeden and Scott (1993), Kendall (1993), Steele and Stine (1993), Xu (1995) and Cyganowski (1995, 1996).

It should be emphasized that Monte Carlo simulation in general, and particularly when it uses discrete time weak approximations of SDEs, represents by its very nature a *parallel algorithm.* The numerical analysis for ODEs is well developed, with an established literature on parallel computation and supercomputing: see, for example, Burrage (1995). Software packages and tools for it are already available. Stochastic numerical methods applied in parallel computation, as discussed in Petersen (1987, 1988), Anderson (1990) and Hausenblas (1999*b*), represent another promising area of research.

It is expected that the numerical analysis of SDEs will experience a diverse and rapid development during the next few years. One aim of this paper is to encourage research in this rewarding but demanding interdisciplinary field. It involves stochastic calculus, numerical analysis, scientific computing, statistics and is linked to many applied areas, including finance, physics and microelectronics. The progress of stochastic modelling in important fields of application will depend to some extent on our ability to master the resulting quantitative challenges.

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REFERENCES

M. I. Abukhaled and E. J. Allen (1998), 'A recursive integration method for approximate solution of stochastic differential equations', *Intern. J. Comput. Math.* **66**, 53–66.

M. F. Allain (1974), Sur quelques types d'approximation des solutions d'équations différentielles stochastiques, PhD thesis, Univ. Rennes.

E. J. Allen, S. J. Novosel and Z. Zhang (1998), 'Finite element and difference approximation of some linear stochastic partial differential equations', *Stochastics and Stochastics Reports* **64**, 117–142.

S. L. Anderson (1990), 'Random number generators on vector supercomputers and other advanced structures', *SIAM Review* **32**, 221–251.

M. V. Antipov (1995), 'Congruence operator of the pseudo-random numbers generator and a modification of Euclidean decomposition', *Monte Carlo Methods Appl.* **1**, 203–219.

M. V. Antipov (1996), 'Sequences of numbers for Monte Carlo methods', *Monte Carlo Methods Appl.* **2**, 219–235.

L. Arnold (1974), *Stochastic Differential Equations*, Wiley, New York.

L. Arnold and P. E. Kloeden (1996), 'Discretization of a random dynamical system near a hyperbolic point', *Mathematische Nachrichten* **181**, 43–72.

S. S. Artemiev (1985), 'A variable step algorithm for numerical solution of stochastic differential equations', *Chisl. Metody Mekh. Sploshn. Sredy* **16**, 11–23. In Russian.

S. S. Artemiev (1993*a*), Certain aspects of application of numerical methods of solving SDE systems, in *Numer. Anal.*, Vol. 1 of *Bulletin of the Novosibirsk Computing Center*, NCC Publisher, pp. 1–16.

S. S. Artemiev (1993*b*), The stability of numerical methods for solving stochastic differential equations, in *Numer. Anal.*, Vol. 2 of *Bulletin of the Novosibirsk Computing Center*, NCC Publisher, pp. 1–10.

S. S. Artemiev (1994), 'The mean square stability of numerical methods for solving stochastic differential equations', *Russ. J. Numer. Anal. Math. Model.* **9**, 405–416.

S. S. Artemiev and A. T. Averina (1997), *Numerical Analysis of Systems of Ordinary and Stochastic Differential Equations*, VSP, Utrecht.

S. S. Artemiev and I. O. Shkurko (1991), 'Numerical analysis of dynamics of oscillatory stochastic systems', *Soviet J. Numer. Anal. Math. Model.* **6**, 277–298.

S. Asmussen, P. Glynn and J. Pitman (1995), 'Discretization error in simulation of one-dimensional reflecting Brownian motion', *Ann. Appl. Probab.* **5**, 875–896.

M. A. Atalla (1986), Finite-difference approximations for stochastic differential equations, in *Probabilistic Methods for the Investigation of Systems with an Infinite Number of Degrees of Freedom*, Collection of Scientific Works, Kiev, pp. 11–16. In Russian.

A. T. Averina and S. S. Artemiev (1986), 'A new family of numerical methods for solving stochastic differential equations', *Soviet. Math. Dokl.* **33**, 736–738.

A. T. Averina and S. S. Artemiev (1988), 'Numerical solutions of systems of stochastic differential equations', *Soviet J. Numer. Anal. Math. Model.* **3**, 267–285.

R. Azencott (1982), Stochastic Taylor formula and asymptotic expansion of Feynman integrals, in *Séminaire de probabilités XVI, Supplement*, Vol. 921 of *Lecture Notes in Math.*, Springer, pp. 237–285.

L. Bachelier (1900), 'Théorie de la spéculation', *Annales de l'Ecole Normale Supérieure, Series 3* **17**, 21–86.

V. Bally (1989*a*), 'Approximation for the solution of stochastic differential equations. I: L^p-convergence', *Stochastics and Stochastics Reports* **28**, 209–246.

V. Bally (1989*b*), 'Approximation for the solution of stochastic differential equations. II: Strong-convergence', *Stochastics and Stochastics Reports* **28**, 357–385.

V. Bally (1990), 'Approximation for the solutions of stochastic differential equations. III: Jointly weak convergence', *Stochastics and Stochastics Reports* **30**, 171–191.

V. Bally and D. Talay (1995), 'The Euler scheme for stochastic differential equations: Error analysis with Malliavin calculus', *Math. Comput. Simul.* **38**, 35–41.

V. Bally and D. Talay (1996*a*), 'The law of the Euler scheme for stochastic differential equations I: Convergence rate of the distribution function', *Probability Theory Related Fields* **104**, 43–60.

V. Bally and D. Talay (1996*b*), 'The law of the Euler scheme for stochastic differential equations II: Convergence rate of the density function', *Monte Carlo Methods Appl.* **2**, 93–128.

J. Barraquand (1995), 'Monte Carlo integration, quadratic resampling, and asset pricing', *Math. Comput. Simul.* **38**, 173–182.

G. BenArous (1989), 'Flots et series de Taylor stochastiques', *Probability Theory Related Fields* **81**, 29–77.

A. Bensoussan, R. Glowinski and A. Rascanu (1990), 'Approximation of the Zakai equation by the splitting up method', *SIAM J. Control Optimiz.* **28**, 1420–1431.

A. Bensoussan, R. Glowinski and A. Rascanu (1992), 'Approximation of some stochastic differential equations by the splitting up method', *Appl. Math. Optim.* **25**, 81–106.

A. Björck and G. Dahlquist (1974), *Numerical Methods. Series in Automatic Computation*, Prentice-Hall, New York.

F. Black and M. Scholes (1973), 'The pricing of options and corporate liabilities', *J. Political Economy* **81**, 637–659.

N. Bouleau (1990), 'On effective computation of expectations in large or infinite dimension: Random numbers and simulation', *J. Comput. Appl. Math.* **31**, 23–34.

N. Bouleau and D. Lépingle (1993), *Numerical Methods for Stochastic Processes*, Wiley, New York.

G. Box and M. Muller (1958), 'A note on the generation of random normal variables', *Ann. Math. Statist.* **29**, 610–611.

W. E. Boyce (1978), 'Approximate solution of random ordinary differential equations', *Adv. Appl. Probab.* **10**, 172–184.

P. P. Boyle (1977), 'A Monte Carlo approach', *J. Financial Economics* **4**, 323–338.

P. Bratley, B. L. Fox and L. Schrage (1987), *A Guide to Simulation*, 2nd edn, Springer, New York.

R. P. Brent (1974), 'A Gaussian pseudo number generator', *Commun. Assoc. Comput. Mach.* **17**, 704–706.

K. Burrage (1995), *Parallel and Sequential Methods for Ordinary Differential Equations*, Clarendon Press, Oxford University Press.

K. Burrage and P. M. Burrage (1996), 'High strong order explicit Runge–Kutta methods for stochastic ordinary differential equations', *Appl. Numer. Math.* **22**, 81–101.

K. Burrage and P. M. Burrage (1998), 'General order conditions for stochastic Runge–Kutta methods for both commuting and non-commuting stochastic ordinary differential equation systems', *Appl. Numer. Math.* **28**, 161–177.

K. Burrage and E. Platen (1994), 'Runge–Kutta methods for stochastic differential equations', *Ann. Numer. Math.* **1**, 63–78.

K. Burrage, P. M. Burrage and J. A. Belward (1997), 'A bound on the maximum strong order of stochastic Runge–Kutta methods for stochastic ordinary differential equations', *BIT* **37**, 771–780.

P. M. Burrage (1998), Runge–Kutta methods for stochastic differential equations, PhD thesis, University of Queensland, Brisbane, Australia.

J. C. Butcher (1987), *The Numerical Analysis of Ordinary Differential Equations: Runge–Kutta and General Linear Methods*, Wiley, Chichester.

S. Cambanis and Y. Z. Hu (1996), 'Exact convergence rate of the Euler–Maruyama scheme and application to sample design', *Stochastics and Stochastics Reports* **59**, 211–240.

L. L. Casasus (1982), On the numerical solution of stochastic differential equations and applications, in *Proceedings of the Ninth Spanish–Portuguese Conference on Mathematics*, Vol. 46 of *Acta Salmanticensia Ciencias*, Univ. Salamanca, pp. 811–814. In Spanish.

L. L. Casasus (1984), On the convergence of numerical methods for stochastic differential equations, in *Proceedings of the Fifth Congress on Differential Equations and Applications*, Univ. La Laguna, pp. 493–501. Puerto de la Cruz (1982), in Spanish, Informes 14.

F. Castell and J. Gaines (1995), 'An efficient approximation method for stochastic differential equations by means of the exponential Lie series', *Math. Comput. Simul.* **38**, 13–19.

F. Castell and J. Gaines (1996), 'The ordinary differential equation approach to asymptotically efficient schemes for solution of stochastic differential equations', *Ann. Inst. H. Poincaré Probab. Statist.* **32**, 231–250.

K. S. Chan and O. Stramer (1998), 'Weak consistency of the Euler method for numerically solving stochastic differential equations with discontinuous coefficients', *Stochastic Process. Appl.* **76**, 33–44.

C. C. Chang (1987), 'Numerical solution of stochastic differential equations with constant diffusion coefficients', *Math. Comput.* **49**, 523–542.

D. Chevance (1997), Numerical methods for backward stochastic differential equations, in *Numerical Methods in Finance* (L. C. G. Rogers and D. Talay, eds), Cambridge University Press, pp. 232–244.

J. M. C. Clark (1978), The design of robust approximations to the stochastic differential equations of nonlinear filtering, in *Communication Systems and Random Processes Theory* (J. K. Skwirzynski, ed.), Vol. 25 of *NATO ASI Series E: Applied Sciences*, Sijthoff and Noordhoff, Alphen aan den Rijn, pp. 721–734.

J. M. C. Clark (1982*a*), An efficient approximation scheme for a class of stochastic differential equations, in *Advances in Filtering and Optimal Stochastic Control*, Vol. 42 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 69–78.

J. M. C. Clark (1982*b*), A nice discretization for stochastic line integrals, in *Stochastic Differential Systems*, Vol. 69 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 131–142.

J. M. C. Clark and R. J. Cameron (1980), The maximum rate of convergence of discrete approximations for stochastic differential equations, in *Stochastic Differential Systems* (B. Grigelionis, ed.), Vol. 25 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 162–171.

D. J. Clements and B. D. O. Anderson (1973), 'Well behaved Itô equations with simulations that always misbehave', *IEEE Trans. Automat. Control* **18**, 676–677.

S. O. Cyganowski (1995), A Maple package for stochastic differential equations, in *Computational Techniques and Applications: CTAC95* (A. K. Easton and R. L. May, eds), World Scientific.

S. O. Cyganowski (1996), 'Solving stochastic differential equations with Maple', *Maple Tech.* **3**, 38.

M. I. Dashevski and R. S. Liptser (1966), 'Simulation of stochastic differential equations connected with the disorder problem by means of analog computer', *Autom. Remote Control* **27**, 665–673. In Russian.

G. Denk and S. Schäffer (1997), 'Adam's methods for the efficient solution of stochastic differential equations with additive noise', *Computing* **59**, 153–161.

J. Douglas, J. Ma and P. Protter (1996), 'Numerical methods for forward-backward stochastic differential equations', *Ann. Appl. Probab.* **6**, 940–968.

I. T. Drummond, S. Duane and R. R. Horgan (1983), 'The stochastic method for numerical simulations: Higher order corrections', *Nuc. Phys.* **B220 FS8**, 119–136.

I. T. Drummond, A. Hoch and R. R. Horgan (1986), 'Numerical integration of stochastic differential equations with variable diffusivity', *J. Phys. A: Math. Gen.* **19**, 3871–3881.

P. D. Drummond and I. K. Mortimer (1991), 'Computer simulation of multiplicative stochastic differential equations', *J. Comput. Phys.* **93**, 144–170.

A. A. Dsagnidse and R. J. Tschitashvili (1975), Approximate integration of stochastic differential equations, Tbilisi State, University, Inst. Appl. Math. 'Trudy IV', Tbilisi, pp. 267–279. In Russian.

J. Eichenauer and J. Lehn (1986), 'A non-linear congruential pseudo random number generator', *Statist. Paper* **27**, 315–326.

A. Einstein (1906), 'Zur Theorie der Brownschen Bewegung', *Ann. Phys. IV* **19**, 371.

R. J. Elliott (1982), *Stochastic Calculus and Applications*, Springer.

R. J. Elliott and R. Glowinski (1989), 'Approximations to solutions of the Zakai filtering equation', *Stoch. Anal. Appl.* **7**, 145–168.

K. Entacher, A. Uhl and S. Wegenkittl (1998), 'Linear congruential generators for parallel Monte Carlo: the leap-frog case', *Monte Carlo Methods Appl.* **4**, 1–16.

S. M. Ermakov (1975), *Die Monte-Carlo-Methode und verwandte Fragen*, Hochschulbücher für Mathematik, Band 72, VEB Deutscher Verlag der Wissenschaften, Berlin. In German: translation from Russian by E. Schincke and M. Schleiff.

S. M. Ermakov and Mikhailov (1982), *Statistical Modeling*, 2nd edn, Nauka, Moscow.

L. Fahrmeier (1974), 'Schwache Konvergenz gegen Diffusionsprozesse', *Z. Angew. Math. Mech.* **54**, 245.

L. Fahrmeier (1976), 'Approximation von stochastischen Differenzialgleichungen auf Digital- und Hybridrechnern', *Computing* **16**, 359–371.

J. F. Feng (1990), 'Numerical solution of stochastic differential equations', *Chinese J. Numer. Appl.* **12**, 28–41.

J. F. Feng, G. Y. Lei and M. P. Qian (1992), 'Second order methods for solving stochastic differential equations', *J. Comput. Math.* **10**, 376–387.

P. Fischer and E. Platen (1998), Applications of the balanced method to stochastic differential equations in filtering, Technical report FMRR 005-98, Australian National University, Canberra, Financial Mathematics Research Reports.

G. S. Fishman (1992), *Monte Carlo: Concepts, Algorithms and Applications. Series in Operations Research*, Springer.

E. Fournie, J. Lebuchoux and N. Touzi (1997), 'Small noise expansion and importance sampling', *Asympt. Anal.* **14**, 331–376.

R. F. Fox (1991), 'Second-order algorithm for the numerical integration of colored-noise problems', *Phys. Rev. A* **43**, 2649–2654.

J. N. Franklin (1965), 'Difference methods for stochastic ordinary differential equations', *Math. Comput.* **19**, 552–561.

J. G. Gaines (1994), 'The algebra of iterated stochastic integrals', *Stochastics and Stochastics Reports* **49**, 169–179.

J. G. Gaines (1995*a*), 'A basis for iterated stochastic integrals', *Math. Comput. Simul.* **38**, 7–11.

J. G. Gaines (1995*b*), Numerical experiments with S(P)DE's, in *Proceedings of the ICMS Conference March 1994*, Cambridge University Press.

J. G. Gaines and T. J. Lyons (1994), 'Random generation of stochastic area integrals', *SIAM J. Appl. Math.* **54**, 1132–1146.

J. G. Gaines and T. J. Lyons (1997), 'Variable step size control in the numerical solution of stochastic differential equations', *SIAM J. Appl. Math.* **57**, 1455–1484.

T. C. Gard (1988), *Introduction to Stochastic Differential Equations*, Marcel Dekker, New York.

C. W. Gear (1971), *Numerical Initial Value Problems in Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, NJ.

M. Gelbrich (1995), 'Simultaneous time and chance discretization for stochastic differential equations', *J. Comput. Appl. Math.* **58**, 255–289.

M. Gelbrich and S. T. Rachev (1996), Discretization for stochastic differential equations, L^p Wasserstein metrics, and econometrical models, in *Distributions with fixed marginals and related topics*, Vol. 28 of *IMS Lecture Notes Monogr. Ser.*, Inst. Math. Statist., Hayward, CA, pp. 97–119.

J. E. Gentle (1998), *Random Number Generation and Monte Carlo Methods. Series in Statistics and Computing*, Springer.

A. Gerardi, F. Marchetti and A. M. Rosa (1984), 'Simulation of diffusions with boundary conditions', *Systems Control Lett.* **4**, 253.

I. I. Gikhman and A. V. Skorokhod (1979), *The Theory of Stochastic Processes*, Vol. I–III, Springer.

S. A. Gladyshev and G. N. Milstein (1984), 'The Runge–Kutta method for calculation of Wiener integrals of functionals of exponential type', *Zh. Vychisl. Mat. Mat. Fiz* **24**, 1136–1149. In Russian.

P. Y. Glorennec (1977), 'Estimation a priori des erreurs dans la résolution numérique d'équations différentielles stochastiques', *Séminaire de Probabilités, Univ. Rennes* **1**, 57–93.

P. W. Glynn and O. L. Iglehart (1989), 'Importance sampling for stochastic simulations', *Management Science* **35**, 1367–1392.

J. Golec (1995), 'Stochastic averaging principle for systems with pathwise uniqueness', *Stoch. Anal. Appl.* **13**, 307–322.

J. Golec (1997), 'Averaging Euler-type difference schemes', *Stoch. Anal. Appl.* **15**, 751–758.

J. Golec and G. S. Ladde (1989), 'Euler-type approximation for systems of stochastic differential equations', *J. Appl. Math. Simul.* **2**, 239–249.

J. Golec and G. S. Ladde (1990), 'Averaging principle and systems of singularly perturbed stochastic differential equations', *J. Math. Phys.* **31**, 1116–1123.

S. T. Goodlett and E. J. Allen (1994), 'A variance reduction technique for use with the extrapolated Euler method for numerical solution of stochastic differential equations', *Stoch. Anal. Appl.* **12**, 131–140.

L. G. Gorostiza (1980), 'Rate of convergence of an approximate solution of stochastic differential equations', *Stochastics* **3**, 267–276. Erratum in *Stochastics* **4** (1981), 85.

W. Grecksch and P. E. Kloeden (1996), 'Time-discretised Galerkin approximations of parabolic stochastic PDEs', *Bull. Austral. Math. Soc.* **54**, 79–85.

W. Grecksch and A. Wadewitz (1996), 'Approximation of solutions of stochastic differential equations by discontinuous Galerkin methods', *J. Anal. Appl.* **15**, 901–916.

H. S. Greenside and E. Helfand (1981), 'Numerical integration of stochastic differential equations', *Bell Syst. Techn. J.* **60**, 1927–1940.

A. Greiner, W. Strittmatter and J. Honerkamp (1987), 'Numerical integration of stochastic differential equations', *J. Statist. Phys.* **51**, 95–108.

A. Grorud and D. Talay (1990), Approximation of Lyapunov exponents of stochastic differential systems on compact manifolds, in *Analysis and Optimization of Systems*, Vol. 144 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 704–713.

A. Grorud and D. Talay (1996), 'Approximation of Lyapunov exponents of nonlinear stochastic differential equations', *SIAM J. Appl. Math.* **56**, 627–650.

S. J. Guo (1982), 'On the mollifier approximation for solutions of stochastic differential equations', *J. Math. Kyoto Univ.* **22**, 243–254.

S. J. Guo (1984), 'Approximation theorems based on random partitions for stochastic differential equations and applications', *Chinese Ann. Math.* **5**, 169–183.

I. Gyöngy (1991), 'On approximation of Itô stochastic equations', *Math. SSR Sbornik* **70**, 165–173.

I. Gyöngy and D. Nuarlart (1997), 'Implicit scheme for stochastic partial differential equations driven by space-time white noise', *Potential Analysis* **7**, 725–757.

E. Hairer and G. Wanner (1991), *Solving ordinary differential equations II: Stiff and differential algebraic systems*, Springer.

E. Hairer, S. P. Nørsett and G. Wanner (1987), *Solving ordinary differential equations I: Nonstiff problems*, Springer.

J. H. Halton (1960), 'On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals', *Numer. Math.* **2**, 84–90.

J. M. Hammersley and D. C. Handscomb (1964), *Monte Carlo Methods*, Methuen, London.

C. J. Harris (1976), Simulation of nonlinear stochastic equations with applications in modelling water pollution, in *Mathematical Models for Environmental Problems* (C. A. Brebbi, ed.), Pentech Press, London, pp. 269–282.

E. Hausenblas (1999*a*), A Monte-Carlo method with inherited parallelism for solving partial differential equations with boundary conditions numerically, Dept. Math., University of Salzburg, Austria. Paper in progress.

E. Hausenblas (1999*b*), A numerical scheme using excursion theory for simulating stochastic differential equations with reflection and local time at a boundary, Dept. Math., University of Salzburg, Austria. Paper in progress.

D. C. Haworth and S. B. Pope (1986), 'A second-order Monte-Carlo method for the solution of the Itô stochastic differential equation', *Stoch. Anal. Appl.* **4**, 151–186.

D. Heath and E. Platen (1996), 'Valuation of FX barrier options under stochastic volatility', *Financial Engineering and the Japanese Markets* **3**, 195–215.

E. Helfand (1979), 'Numerical integration of stochastic differential equations', *Bell Syst. Techn. J.* **58**, 2289–2299.

D. B. Hernandez and R. Spigler (1992), 'A-stability of implicit Runge–Kutta methods for systems with additive noise', *BIT* **32**, 620–633.

D. B. Hernandez and R. Spigler (1993), 'Convergence and stability of implicit Runge–Kutta methods for systems with multiplicative noise', *BIT* **33**, 654–669.

D. J. Higham (1998), Mean-square and asymptotic stability of numerical methods for stochastic ordinary differential equations, Strathclyde Mathematics Research Report 39, University of Strathclyde, Glasgow, UK.

N. Hofmann (1994), Beiträge zur schwachen Approximation stochastischer Differentialgleichungen, PhD thesis, Dissertation Humboldt Universität Berlin.

N. Hofmann (1995), 'Stability of weak numerical schemes for stochastic differential equations', *Math. Comput. Simul.* **38**, 63–68.

N. Hofmann and P. Mathé (1997), 'On quasi-Monte Carlo simulation of stochastic differential equations', *Math. Comput.* **66**, 573–589.

N. Hofmann and E. Platen (1994), 'Stability of weak numerical schemes for stochastic differential equations', *Comput. Math. Appl.* **28**, 45–57.

N. Hofmann and E. Platen (1996), 'Stability of superimplicit numerical methods for stochastic differential equations', *Fields Institute Communications* **9**, 93–104.

N. Hofmann, T. Müller-Gronbach and K. Ritter (1998), Optimal approximation of stochastic differential equations by adaptive step-size control, Preprint Nr. A-9-98, Fachbereich Mathematik, Freie Universität Berlin.

N. Hofmann, E. Platen and M. Schweizer (1992), 'Option pricing under incompleteness and stochastic volatility', *Mathematical Finance* **2**, 153–187.

Y. Z. Hu (1992), Series de Taylor stochastique et formule de Campbell-Hausdorff d'après Ben Arous, in *Séminaire de Probabilités XXVI*, Vol. 1526 of *Lecture Notes in Math.*, Springer, pp. 587–594.

Y. Z. Hu (1996), Strong and weak order of time discretization schemes of stochastic differential equations, in *Séminaire de Probabilités XXX*, Vol. 1626 of *Lecture Notes in Math.*, Springer, pp. 218–227.

Y. Z. Hu and P. A. Meyer (1993), 'On the approximation of multiple Stratonovich integrals', in *Stochastic Processes*, Springer, pp. 141–147.

Y. Z. Hu and S. Watanabe (1996), 'Donsker's delta functions and approximation of heat kernels by the time discretization methods', *J. Math. Kyoto Univ.* **36**, 499–518.

J. Hull and A. White (1988), 'The use of control variate techniques in option pricing', *J. Financial and Quantitative Analysis* **23**, 237–251.

N. Ikeda and S. Watanabe (1989), *Stochastic Differential Equations and Diffusion Processes*, 2nd edn, North-Holland, Amsterdam. (1st edn (1981).)

K. Itô (1944), 'Stochastic integral', *Proc. Imp. Acad. Tokyo* **20**, 519–524.

J. Jacod and P. Protter (1998), 'Asymptotic error distribution for the Euler method for stochastic differential equations', *Ann. Probab.* **26**, 267–307.

J. Jacod and A. N. Shiryaev (1987), *Limit Theorems for Stochastic Processes*, Springer.

A. Janicki (1996), *Numerical and Statistical Approximation of Stochastic Differential Equations with Non-Gaussian Measures*, H. Steinhaus Center for Stochastic Methods in Science and Technology, Wroclaw, Poland.

A. Janicki and A. Weron (1994), *Simulation of Chaotic Behavior of α-stable Stochastic Processes*, Vol. 178 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York.

A. Janicki, Z. Michna and A. Weron (1996), 'Approximation of stochastic differential equations driven by α-stable Lévy motion', *Applicationes Mathematicae* **24**, 149–168.

R. Janssen (1984*a*), 'Difference-methods for stochastic differential equations with discontinuous coefficients', *Stochastics* **13**, 199–212.

R. Janssen (1984*b*), 'Discretization of the Wiener process in difference methods for stochastic differential equations', *Stochastic Process. Appl.* **18**, 361–369.

C. Joy, P. P. Boyle and K. S. Tan (1996), 'Quasi Monte Carlo methods in numerical finance', *Management Science* **42**, 926–938.

M. H. Kalos and P. A. Whitlock (1986), *Monte Carlo Methods*, Wiley-Interscience, New York.

S. Kanagawa (1988), 'The rate of convergence for Maruyama's approximate solutions of stochastic differential equations', *Yokohoma Math. J.* **36**, 79–85.

S. Kanagawa (1989), 'The rate of convergence for approximate solutions of stochastic differential equations', *Tokyo J. Math.* **12**, 33–48.

S. Kanagawa (1995), 'Error estimation for the Euler–Maruyama approximate solutions of stochastic differential equations', *Monte Carlo Methods Appl.* **1**, 165–171.

S. Kanagawa (1996), Convergence rates for the Euler–Maruyama type approximate solutions of stochastic differential equations, in *Probability Theory and Mathematical Statistics, Proceedings of the Seventh Japan–Russia Symposium*, World Scientific, Singapore, pp. 183–192.

S. Kanagawa (1997), 'Confidence intervals of discretized Euler–Maruyama approximate solutions of SDE's', *Nonlinear Analysis, Theory, Methods and Applications*, Vol. 30, pp. 4101–4103.

T. Kaneko and S. Nakao (1988), A note on approximations for stochastic differential equations, in *Séminaire de Probabilités XXII*, Vol. 1321 of *Lecture Notes in Math.*, Springer, pp. 155–162.

D. Kannan and D. T. Wu (1993), 'A numerical study of the additive functionals of solutions of stochastic differential equations', *Dyn. Sys. Appl.* **2**, 291–310.

I. Karatzas and S. E. Shreve (1988), *Brownian Motion and Stochastic Calculus*, Springer.

W. S. Kendall (1993), Doing stochastic calculus with Mathematica, in *Economic and financial modeling with Mathematica*, TELOS, Santa Clara, CA, pp. 214–238.

J. R. Klauder and W. P. Petersen (1985), 'Numerical integration of multiplicative-noise stochastic differential equations', *SIAM J. Numer. Anal.* **6**, 1153–1166.

P. E. Kloeden and R. A. Pearson (1977), 'The numerical solution of stochastic differential equations', *J. Austral. Math. Soc. Ser. B* **20**, 8–12.

P. E. Kloeden and E. Platen (1989), 'A survey of numerical methods for stochastic differential equations', *J. Stochastic Hydrology and Hydraulics* **3**, 155–178.

P. E. Kloeden and E. Platen (1991*a*), 'Relations between multiple Itô and Stratonovich integrals', *Stoch. Anal. Appl.* **IX**, 86–96.

P. E. Kloeden and E. Platen (1991*b*), 'Stratonovich and Itô stochastic Taylor expansions', *Mathematische Nachrichten* **151**, 33–50.

P. E. Kloeden and E. Platen (1992), 'Higher order implicit strong numerical schemes for stochastic differential equations', *J. Statist. Phys.* **66**, 283–314.

P. E. Kloeden and E. Platen (1992/1995*b*), *Numerical Solution of Stochastic Differential Equations*, Vol. 23 of *Appl. Math.*, Springer.

P. E. Kloeden and E. Platen (1995*a*), Numerical methods for stochastic differential equations, in *Nonlinear Dynamics and Stochastic Mechanics*, *CRC Math. Model. Series*, CRC Press, Boca Raton, pp. 437–461.

P. E. Kloeden and W. D. Scott (1993), 'Construction of stochastic numerical schemes through Maple', *Maple Technical Newspaper* **10**, 60–65.

P. E. Kloeden, E. Platen and N. Hofmann (1992*a*), Stochastic differential equations: Applications and numerical methods, in *Proceedings of the 6th IAHR International Symposium on Stochastic Hydraulics, National Taiwan University, Taipeh*, pp. 75–81.

P. E. Kloeden, E. Platen and N. Hofmann (1995), 'Extrapolation methods for the weak approximation of Itô diffusions', *SIAM J. Numer. Anal.* **32**, 1519–1534.

P. E. Kloeden, E. Platen and H. Schurz (1991), 'The numerical solution of nonlinear stochastic dynamical systems: A brief introduction', *J. Bifur. Chaos* **1**, 277–286.

P. E. Kloeden, E. Platen and H. Schurz (1992*b*), 'Effective simulation of optimal trajectories in stochastic control', *Optimization* **1**, 633–644.

P. E. Kloeden, E. Platen and H. Schurz (1993), Higher order approximate Markov chain filters, in *Stochastic Processes: A Festschrift in Honour of Gopinath Kallianpur* (S. Cambanis *et al.*, eds), Springer, pp. 181–190.

P. E. Kloeden, E. Platen and H. Schurz (1994/1997), *Numerical Solution of SDEs Through Computer Experiments*, Universitext, Springer.

P. E. Kloeden, E. Platen and I. Wright (1992*c*), 'The approximation of multiple stochastic integrals', *Stoch. Anal. Appl.* **10**, 431–441.

A. Kohatsu-Higa (1997), 'High order Itô–Taylor approximations to heat kernels', *J. Math. Kyoto Univ.* **37**, 129–150.

A. Kohatsu-Higa and S. Ogawa (1997), 'Weak rate of convergence for an Euler scheme of nonlinear SDE's', *Monte Carlo Methods Appl.* **3**, 327–345.

A. Kohatsu-Higa and P. Protter (1994), The Euler scheme for SDEs driven by semimartingales, in *Stochastic Anal. on Infinite Dimensional Spaces* (H. Kunita and H. H. Kuo, eds), Pitman, pp. 141–151.

W. E. Kohler and W. E. Boyce (1974), 'A numerical analysis of some first order stochastic initial value problems', *SIAM J. Appl. Math.* **27**, 167–179.

Y. Komori and T. Mitsui (1995), 'Stable ROW-type weak scheme for stochastic differential equations', *Monte Carlo Methods Appl.* **1**, 279–300.

Y. Komori, T. Mitsui and H. Sugiura (1997), 'Rooted tree analysis of the order conditions of ROW-type scheme for stochastic differential equations', *BIT* **37**, 43–66.

Y. Komori, Y. Saito and T. Mitsui (1994), 'Some issues in discrete approximate solution for stochastic differential equations', *Comput. Math. Appl.* **28**, 269–278.

R. I. Kozlov and M. G. Petryakov (1986), 'The construction of comparison systems for stochastic differential equations and numerical methods', *Nauka Sibirsk Otdel. Novosibirsk* pp. 45–52. In Russian.

T. G. Kurtz and P. Protter (1991*a*), 'Weak limit theorems for stochastic integrals and stochastic differential equations', *Ann. Probab.* **19**, 1035–1070.

T. G. Kurtz and P. Protter (1991*b*), Wong–Zakai corrections, random evolutions and simulation schemes for SDE's, in *Stochastic Analysis* (E. M. E. Meyer-Wolf and A. Schwartz, eds), Academic Press, pp. 331–346.

H. J. Kushner (1974), 'On the weak convergence of interpolated Markov chains to a diffusion', *Ann. Probab.* **2**, 40–50.

H. J. Kushner and P. G. Dupuis (1992), *Numerical Methods for Stochastic Control Problems in Continuous Time*, Vol. 24 of *Applications of Mathematics*, Springer, New York.

D. F. Kuznetsov (1998), *Some Questions in the Theory of Numerical Solution of Itô Stochastic Differential Equations*, Saint Petersburg, State Technical University Publisher. In Russian.

A. M. Law and W. D. Kelton (1991), *Simulation Modeling and Analysis*, 2nd edn, McGraw-Hill, New York.

F. LeGland (1992), Splitting-up approximation for SPDEs and SDEs with application to nonlinear filtering, in *Stochastic Partial Differential Equations and their Applications*, Vol. 176 of *Lecture Notes in Control and Inform. Sci.*, Springer, Berlin, pp. 177–187.

D. Lépingle (1993), 'An Euler scheme for stochastic differential equations with reflecting boundary conditions', *Computes Rendus Acad. Sci. Paris, Séries I Math.* **316**, 601–605.

D. Lépingle (1995), 'Euler scheme for reflected stochastic differential equations', *Math. Comput. Simul.* **38**, 119–126.

D. Lépingle and B. Ribémont (1991), 'A multistep approximation scheme for the Langevin equation', *Stochastic Process. Appl.* **37**, 61–69.

C. W. Li and X. Q. Liu (1997), 'Algebraic structure of multiple stochastic integrals with respect to Brownian motions and Poisson processes', *Stochastics and Stochastics Reports* **61**, 107–120.

H. Liske (1982), 'Distribution of a functional of a Wiener process', *Theory of Random Processes* **10**, 50–54. In Russian.

H. Liske (1985), 'Solution of an initial-boundary value problem for a stochastic equation of parabolic type by the semi-discretization method', *Theory of Random Processes* **113**, 51–56. In Russian.

H. Liske and E. Platen (1987), 'Simulation studies on time discrete diffusion approximations', *Math. Comput. Simul.* **29**, 253–260.

X. Q. Liu and C. W. Li (1997), 'Discretization of stochastic differential equations by the product expansion for the Chen series', *Stochastics and Stochastics Reports* **60**, 23–40.

J. Ma, P. Protter and J. M. Yong (1994), 'Solving forward–backward stochastic differential equations explicitly: a four step scheme', *Probability Theory Related Fields* **98**, 339–359.

V. Mackevicius (1987), 'S^P-stability of solutions of symmetric stochastic differential equations with discontinuous driving semimartingales', *Ann. Inst. H. Poincaré Probab. Statist.* **23**, 575–592.

V. Mackevicius (1994), 'Second order weak approximations for Stratonovich stochastic differential equations', *Lietuvos Matem. Rink.* **34**, 226–247. Translation in *Lithuanian Math. Journal*, **34**, 183-200.

V. Mackevicius (1996), Extrapolation of approximations of solutions of stochastic differential equations, in *Probability Theory and Mathematical Statistics*, World Scientific, River Edge, NJ, pp. 276–297.

Y. Maghsoodi (1994), Mean-square efficient numerical solution of jump-diffusion stochastic differential equations, Preprint OR72, University of Southampton, UK.

Y. Maghsoodi and C. J. Harris (1987), 'In-probability approximation and simulation of nonlinear jump-diffusion stochastic differential equations', *IMA J. Math. Control Inform.* **4**, 65–92.

A. Makroglou (1991), 'Numerical treatment of stochastic Volterra integro-differential equations', *J. Comput. Appl. Math.* **II**, 307–313.

F. H. Maltz and D. L. Hitzl (1979), 'Variance reduction in Monte-Carlo computations using multi-dimensional Hermite polynomials', *J. Comput. Phys.* **32**, 345–376.

R. Manella and V. Palleschi (1989), 'Fast and precise algorithm for computer simulation of stochastic differential equations', *Phys. Rev. A* **40**, 3381–3386.

S. I. Marcus (1981), 'Modeling and approximation of stochastic differential equations driven by semimartingales', *Stochastics* **4**, 223–245.

G. Marsaglia and T. A. Bray (1964), 'A convenient method for generating normal variables', *SIAM Review* **6**, 260–264.

G. Maruyama (1955), 'Continuous Markov processes and stochastic equations', *Rend. Circolo Math. Palermo* **4**, 48–90.

S. Mauthner (1998), 'Step size control in the numerical solution of stochastic differential equations', *J. Comput. Appl. Math.* **100**, 93–109.

R. Merton (1973), 'The theory of rational option pricing', *Bell Journal of Economics and Management Science* **4**, 141–183.

G. A. Mikhailov (1992), *Optimization of Weighted Monte Carlo Methods. Series in Computational Physics*, Springer.

R. Mikulevicius and E. Platen (1988), 'Time discrete Taylor approximations for Itô processes with jump component', *Mathematische Nachrichten* **138**, 93–104.

R. Mikulevicius and E. Platen (1991), 'Rate of convergence of the Euler approximation for diffusion processes', *Mathematische Nachrichten* **151**, 233–239.

G. N. Milstein (1974), 'Approximate integration of stochastic differential equations', *Theory Probab. Appl.* **19**, 557–562.

G. N. Milstein (1978), 'A method of second order accuracy integration of stochastic differential equations', *Theory Probab. Appl.* **23**, 396–401.

G. N. Milstein (1985), 'Weak approximation of solutions of systems of stochastic differential equations', *Theory Probab. Appl.* **30**, 750–766.

G. N. Milstein (1987), 'A theorem on the order of convergence of mean-square approximations of solutions of systems of stochastic differential equations', *Teor. Veroyatnost. i Primenen* **32**, 809–811. In Russian.

G. N. Milstein (1988*a*), *Numerical Integration of Stochastic Differential Equations*, Urals Univ. Press, Sverdlovsk. In Russian.

G. N. Milstein (1988*b*), 'A theorem of the order of convergence of mean square approximations of systems of stochastic differential equations', *Theory Probab. Appl.* **32**, 738–741.

G. N. Milstein (1995*a*), *Numerical Integration of Stochastic Differential Equations*, Mathematics and its Applications, Kluwer, Dordrecht/Boston/London.

G. N. Milstein (1995*b*), 'The solving of boundary value problems by numerical integration of stochastic equations', *Math. Comput. Simul.* **38**, 77–85.

G. N. Milstein (1995*c*), 'Solving the first boundary value problem of parabolic type by numerical integration of stochastic differential equations', *Theory Probab. Appl.* **40**, 657–665.

G. N. Milstein (1996), 'Application of numerical integration of stochastic equations for solving boundary value problems with Neumann boundary conditions', *Theory Probab. Appl.* **41**, 210–218.

G. N. Milstein (1997), 'Weak approximation of a diffusion process in a bounded domain', *Stochastics and Stochastics Reports* **62**, 147–200.

G. N. Milstein and E. Platen (1994), The integration of stiff stochastic differential equations with stable second moments, Technical report SRR 014-94, Australian National University Statistics Report Series.

G. N. Milstein and M. V. Tretjakov (1994), 'Numerical solution of differential equations with colored noise', *J. Statist. Phys.* **77**, 691–715.

G. N. Milstein and M. V. Tretjakov (1997), 'Numerical methods in the weak sense for stochastic differential equations with small noise', *SIAM J. Numer. Anal.* **34**, 2142–2167.

G. N. Milstein, E. Platen and H. Schurz (1998), 'Balanced implicit methods for stiff stochastic systems', *SIAM J. Numer. Anal.* **35**, 1010–1019.

B. J. Morgan (1984), *Elements of Simulation*, Chapman & Hall, London.

M. Mori (1998), 'Low discrepancy sequences generated by piecewise linear maps', *Monte Carlo Methods Appl.* **4**, 141–162.

T. Müller-Gronbach (1996), 'Optimal design for approximating the path of a stochastic process', *J. Statist. Planning Inf.* **49**, 371–385.

H. Nakazawa (1990), 'Numerical procedures for sample structures on stochastic differential equations', *J. Math. Phys.* **31**, 1978–1990.

N. J. Newton (1986*a*), 'An asymptotic efficient difference formula for solving stochastic differential equations', *Stochastics* **19**, 175–206.

N. J. Newton (1986*b*), Asymptotically optimal discrete approximations for stochastic differential equations, in *Theory and Applications of Nonlinear Control Systems*, North-Holland, pp. 555–567.

N. J. Newton (1990), 'An efficient approximation for stochastic differential equations on the partition of symmetrical first passage times', *Stochastics* **29**, 227–258.

N. J. Newton (1991), 'Asymptotically efficient Runge–Kutta methods for a class of Itô and Stratonovich equations', *SIAM J. Appl. Math.* **51**, 542–567.

N. J. Newton (1994), 'Variance reduction for simulated diffusions', *SIAM J. Appl. Math.* **54**, 1780–1805.

N. J. Newton (1996), 'Numerical methods for stochastic differential equations', *Z. Angew. Math. Mech.* **76**, 211–214. Suppl. 3, I–XVI.

N. J. Newton (1997), Continuous-time Monte Carlo methods and variance reduction, in *Numerical Methods in Finance*, Newton Institute, Cambridge University Press, Cambridge, pp. 22–42.

H. Niederreiter (1988), 'Remarks on nonlinear pseudo random numbers', *Metrika* **35**, 321–328.

H. Niederreiter (1992), *Random Number Generation and Quasi-Monte-Carlo Methods*, SIAM, Philadelphia, PA.

H. Niederreiter and P. J. Shine (1995), *Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing*, Vol. 106 of *Lecture Notes in Statistics*, Springer.

N. N. Nikitin and V. D. Razevig (1978), 'Methods of numerical modelling of stochastic differential equations and estimates of their error', *Zh. Vychisl. Mat. Mat. Fiz* **18**, 106–117. In Russian.

S. Ogawa (1992), 'Monte Carlo simulation of nonlinear diffusion processes', *Japan J. Industrial and Appl. Math.* **9**, 25–33.

S. Ogawa (1994), 'Monte Carlo simulation of nonlinear diffusion processes II', *Japan J. Industrial and Appl. Math.* **2**, 31–45.

S. Ogawa (1995), 'Some problems in the simulation of nonlinear diffusion processes', *Math. Comput. Simul.* **38**, 217–223.

V. A. Ogorodnikov and S. M. Prigarin (1996), *Numerical Modelling of Random Processes and Fields: Algorithms and Applications*, VSP, Utrecht.

B. Oksendahl (1985), *Stochastic Differential Equations*, Springer.

E. Pardoux and D. Talay (1985), 'Discretization and simulation of stochastic differential equations', *Acta Appl. Math.* **3**, 23–47.

S. Paskov and J. Traub (1995), 'Faster valuation of financial derivatives', *J. Portfolio Manag.*, pp. 113–120.

W. P. Petersen (1987), Numerical simulation of Itô stochastic differential equations on supercomputers, in *Random Media*, Vol. 7 of *IMA Vol. Math. Appl.*, Springer, pp. 215–228.

W. P. Petersen (1988), 'Some vectorized random number generators for uniform, normal and Poisson distributions for CRAY X-MP', *J. Supercomputing* **1**, 318–335.

W. P. Petersen (1994*a*), 'Lagged Fibonacci series random number generators for the NEC SX-3', *Intern. J. High Speed Computing* **6**, 387–398.

W. P. Petersen (1994*b*), 'Some experiments on numerical simulations of stochastic differential equations and a new algorithm', *J. Comput. Phys.* **113**, 75–81.

W. P. Petersen (1998), 'A general implicit splitting for stabilizing numerical simulations of Itô stochastic differential equations', *SIAM J. Numer. Anal.* **35**, 1439–1451.

R. Petterson (1995), 'Approximations for stochastic differential equations with reflecting convex boundaries', *Stochastic Process. Appl.* **59**, 295–308.

E. Platen (1980*a*), Approximation of Itô integral equations, in *Stochastic differential systems*, Vol. 25 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 172–176.

E. Platen (1980*b*), 'Weak convergence of approximations of Itô integral equations', *Z. Angew. Math. Mech.* **60**, 609–614.

E. Platen (1981), 'An approximation method for a class of Itô processes', *Lietuvos Matem. Rink.* **21**, 121–133.

E. Platen (1982*a*), 'An approximation method for a class of Itô processes with jump component', *Lietuvos Matem. Rink.* **22**, 124–136.

E. Platen (1982*b*), 'A generalized Taylor formula for solutions of stochastic differential equations', *SANKHYA A* **44**, 163–172.

E. Platen (1983), 'Approximation of first exit times of diffusions and approximate solution of parabolic equations', *Mathematische Nachrichten* **111**, 127–146.

E. Platen (1984), Zur zeitdiskreten Approximation von Itoprozessen, Diss. B., IMath, Akad. der Wiss. der DDR, Berlin.

E. Platen (1985), On first exit times of diffusions, in *Stochastic differential systems*, Vol. 69 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 192–195.

E. Platen (1987), Derivative free numerical methods for stochastic differential equations, Vol. 96 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 187–193.

E. Platen (1992), 'Higher-order weak approximation of Itô diffusions by Markov chains', *Probability in the Engineering and Information Sciences* **6**, 391–408.

E. Platen (1995), 'On weak implicit and predictor–corrector methods', *Math. Comput. Simul.* **38**, 69–76.

E. Platen and R. Rebolledo (1985), 'Weak convergence of semimartingales and discretization methods', *Stochastic Process. Appl.* **20**, 41–58.

E. Platen and W. Wagner (1982), 'On a Taylor formula for a class of Itô processes', *Probability and Math. Statistics* **3**, 37–51.

P. Protter (1985), 'Approximations of solutions of stochastic differential equations driven by semimartingales', *Ann. Probab.* **13**, 716–743.

P. Protter (1990), *Stochastic Integration and Differential Equations*, Springer.

P. Protter and D. Talay (1997), 'The Euler scheme for Levy driven stochastic differential equations', *Ann. Probab.* **25**, 393–423.

I. Radovic, I. M. Sobol and R. F. Tichy (1996), 'Quasi-Monte Carlo methods for numerical integration: Comparison of different low discrepancy sequences', *Monte Carlo Methods Appl.* **2**, 1–14.

N. J. Rao, J. D. Borwankar and D. Ramkrishna (1974), 'Numerical solution of Itô integral equations', *SIAM J. Control Optimiz.* **12**, 124–139.

V. D. Razevig (1980), 'Digital modelling of multi-dimensional dynamics under random perturbations', *Autom. Remote Control* **4**, 177–186. In Russian.

B. D. Ripley (1983*a*), 'Computer generation of random variables: A tutorial letter', *Inter. Statist. Rev.* **45**, 301–319.

B. D. Ripley (1983*b*), *Stochastic Simulation*, Wiley, New York.

W. Römisch and A. Wakolbinger (1987), On the convergence rates of approximate solutions of stochastic equations, in Vol. 96 of *Lecture Notes in Control and Inform. Sci*, Springer, pp. 204–212.

S. M. Ross (1991), *A Course in Simulation*, MacMillan, New York.

R. Y. Rubinstein (1981), *Simulation and the Monte Carlo Method*, Wiley.

W. Rümelin (1982), 'Numerical treatment of stochastic differential equations', *SIAM J. Numer. Anal.* **19**, 604–613.

L. B. Ryashko and H. Schurz (1997), 'Mean square stability analysis of some linear stochastic systems', *Dyn. Sys. Appl.* **6**, 165–189.

K. K. Sabelfeld (1979), 'On the approximate computation of Wiener integrals by Monte-Carlo method', *Zh. Vychisl. Mat. Mat. Fiz* **19**, 29–43. In Russian.

Y. Saito and T. Mitsui (1993*a*), 'Simulation of stochastic differential equations', *Ann. Inst. Statist. Math.* **45**, 419–432.

Y. Saito and T. Mitsui (1993*b*), 'T-stability of numerical schemes for stochastic differential equations', *World Sci. Ser. Appl. Anal.* **2**, 333–344.

Y. Saito and T. Mitsui (1995), 'S-series in the Wong–Zakai approximation for stochastic differential equations', *Vietnam J. Math.* **23**, 303–317.

Y. Saito and T. Mitsui (1996), 'Stability analysis of numerical schemes for stochastic differential equations', *SIAM J. Numer. Anal.* **33**, 2254–2267.

O. Schein and G. Denk (1998), 'Numerical solution of stochastic differential-algebraic equations with applications to transient noise simulation of microelectronic circuits', *J. Comput. Appl. Math.* **100**, 77–92.

H. Schurz (1996*a*), 'Asymptotical mean square stability of an equilibrium point of some linear numerical solutions with multiplicative noise', *Stoch. Anal. Appl.* **14**, 313–354.

H. Schurz (1996*b*), 'Numerical regularization for SDEs: Construction of nonnegative solutions', *Dyn. Sys. Appl.* **5**, 323–351.

H. Schurz (1996*c*), Stability, stationarity and boundedness of some implicit numerical methods for stochastic differential equations, PhD thesis, Humboldt University, Berlin.

A. Shimizu and T. Kawachi (1984), 'Approximate solutions of stochastic differential equations', *Bull. Nagoya Inst. Tech.* **36**, 105–108.

M. Shinozuka (1971), 'Simulation of multivariate and multidimensional random differential processes', *J. Acoust. Soc. Amer.* **49**, 357–367.

I. O. Shkurko (1987), Numerical solution of linear systems of stochastic differential equations, in *Numerical Methods for Statistics and Modeling*, Novosibirsk, pp. 101–109. Collected Scientific Works, in Russian.

I. H. Sloan and H. Wozniakowski (1998), 'When are quasi-Monte-Carlo algorithms efficient for high dimensional integrals?', *J. Complexity* **14**, 1–33.

L. Slominski (1994), 'On approximation of solutions of multidimensional SDEs with reflecting boundary conditions', *Stochastic Process. Appl.* **50**, 197–219.

I. M. Sobol (1967), 'The distribution of points in a cube and the approximate evaluation of integrals', *USSR Comput. Math. Math. Phys.* **19**, 86–112.

J. M. Steele and R. A. Stine (1993), Mathematica and diffusions, in *Economic and Financial Modeling with Mathematica*, TELOS, Santa Clara, CA, pp. 192–213.

J. Stoer and R. Bulirsch (1993), *Introduction to Numerical Analysis*, 2nd edn, Springer. (1st edn (1980).)

H. Sugita (1995), 'Pseudo-random number generator by means of irrational rotation', *Monte Carlo Methods Appl.* **1**, 35–57.

M. Sun and R. Glowinski (1994), 'Pathwise approximation and simulation for the Zakai filtering equation through operator splitting', *Calcolo* **30**, 219–239.

H. J. Sussmann (1988), Product expansions of exponential Lie series and the discretization of stochastic differential equations, in *Stochastic Differential Systems, Stochastic Control Theory and Applications* (W. Fleming and P. I. Lions, eds), Vol. 10 of *IMA Vol. Math. Appl.*, Springer, pp. 563–582.

D. Talay (1982*a*), Analyse Numérique des Equations Différentielles Stochastiques, PhD thesis, Université de Provence, Centre Saint Charles. Thèse 3ème Cycle.

D. Talay (1982*b*), 'Convergence pour chaque trajectoire d'un schéme d'approximation des EDS', *Computes Rendus Acad. Sci. Paris, Séries I Math* **295**, 249–252.

D. Talay (1983*a*), How to discretize stochastic differential equations, in *Nonlinear filtering and stochastic control*, Vol. 972 of *Lecture Notes in Math.*, Springer, pp. 276–292.

D. Talay (1983*b*), 'Résolution trajectorielle et analyse numérique des équations différentielles stochastiques', *Stochastics* **9**, 275–306.

D. Talay (1984), Efficient numerical schemes for the approximation of expectations of functionals of the solution of an SDE and applications, in *Filtering and Control of Random Processes*, Vol. 61 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 294–313.

D. Talay (1986), 'Discrétisation d'une équation différentielle stochastique et calcul approché d'espérances de fonctionelles de la solution', *Modél Math. et Anal. Numér.* **20**, 141–179.

D. Talay (1987), Classification of discretization of diffusions according to an ergodic criterion, in *Stochastic Modelling and Filtering*, Vol. 91 of *Lecture Notes in Control and Inform. Sci.*, Springer, pp. 207–218.

D. Talay (1990), 'Second order discretization schemes of stochastic differential systems for the computation of the invariant law', *Stochastics and Stochastics Reports* **29**, 13–36.

D. Talay (1991), 'Approximation of upper Lyapunov exponents of bilinear stochastic differential equations', *SIAM J. Numer. Anal.* **28**, 1141–1164.

D. Talay (1995), Simulation of stochastic differential systems, in *Probabilistic Methods in Applied Physics* (P. Krée and W. Wedig, eds), Vol. 451 of *Lecture Notes in Physics.*, Springer, Chapter 3, pp. 54–96.

D. Talay and L. Tubaro (1990), 'Expansion of the global error for numerical schemes solving stochastic differential equations', *Stoch. Anal. Appl.* **8**, 483–509.

U. Tetzlaff and H.-U. Zschiesche (1984), 'Näherungslösungen für Itô-Differentialgleichungen mittels Taylorentwicklungen für Halbgruppen von Operatoren', *Wiss. Z. Techn. Hochschule Leuna-Merseburg* **2**, 332–339.

S. Tezuka (1993), 'Polynomial arithmetic analogue of Halton sequences', *ACM Trans. Model. Comput. Simul.* **3**, 99–107.

S. Tezuka and T. Tokuyama (1994), 'A note on polynomial arithmetic analogue of Halton sequences', *ACM Trans. Model. Computer Simul.* **4**, 279–284.

C. Török (1994), 'Numerical solution of linear stochastic differential equations', *Comput. Math. Appl.* **27**, 1–10.

J. F. Traub, G. W. Wasilkowski and H. Wozniakowski (1988), *Information-Based Complexity*, Academic Press, New York.

C. Tudor (1989), 'Approximation of delay stochastic equations with constant retardation by usual Itô equations', *Rev. Roumaine Math. Pures Appl.* **34**, 55–64.

C. Tudor and M. Tudor (1983), 'On approximation in quadratic mean for the solutions of two parameter stochastic differential equations in Hilbert spaces', *An. Univ. Bucuresti Mat.* **32**, 73–88.

C. Tudor and M. Tudor (1987), 'On approximation of solutions for stochastic delay equations', *Stud. Cerc. Mat.* **39**, 265–274.

C. Tudor and M. Tudor (1995), 'Approximation schemes for Itô–Volterra stochastic equations', *Bol. Soc. Mat. Mexicana* (3) **1**, 73–85.

C. Tudor and M. Tudor (1997), 'Approximate solutions for multiple stochastic equations with respect to semimartingales', *Z. Anal. Anwendungen* **16**, 761–768.

M. Tudor (1992), 'Approximation schemes for two-parameter stochastic equations', *Probability and Math. Statistics* **13**, 177–189.

B. Tuffin (1996), 'On the use of low discrepancy sequences in Monte Carlo methods', *Monte Carlo Methods Appl.* **2**, 295–320.

B. Tuffin (1997), 'Comments on "On the use of low discrepancy sequences in Monte Carlo methods"', *Monte Carlo Methods Appl.* **4**, 87–90.

T. E. Unny (1984), 'Numerical integration of stochastic differential equations in catchment modelling', *Water Res.* **20**, 360–368.

E. Valkeila (1991), 'Computer algebra and stochastic analysis', *CWI* **4**, 229–238.

A. D. Ventzel, S. A. Gladyshev and G. N. Milstein (1985), 'Piecewise constant approximation for the Monte-Carlo calculation of Wiener integrals', *Theory Probab. Appl.* **24**, 745–752.

W. Wagner (1987), 'Unbiased Monte-Carlo evaluation of certain functional integrals', *J. Comput. Phys.* **71**, 21–33.

W. Wagner (1988*a*), 'Monte-Carlo evaluation of functionals of solutions of stochastic differential equations. Variance reduction and numerical examples', *Stoch. Anal. Appl.* **6**, 447–468.

W. Wagner (1988*b*), 'Unbiased multi-step estimators for the Monte-Carlo evaluation of certain functionals', *J. Comput. Phys.* **79**, 336–352.

W. Wagner (1989*a*), Stochastische numerische Verfahren zur Berechnung von Funktionalintegralen, Habilitation, Report **02/89**, IMATH, Berlin.

W. Wagner (1989*b*), 'Unbiased Monte-Carlo estimators for functionals of weak solutions of stochastic differential equations', *Stochastics and Stochastics Reports* **28**, 1–20.

W. Wagner and E. Platen (1978), Approximation of Itô integral equations, Preprint ZIMM, Akad. Wissenschaften, DDR, Berlin.

M. J. Werner and P. D. Drummond (1997), 'Robust algorithms for solving stochastic partial differential equations', *J. Comput. Phys.* **132**, 312–326.

N. Wiener (1923), 'Differential space', *J. Math. Phys.* **2**, 131–174.

E. Wong and M. Zakai (1965), 'On the convergence of ordinary integrals to stochastic integrals', *Ann. Math. Statist.* **36**, 1560–1564.

H. Wozniakowski (1991), 'Average case complexity of multivariate integration', *Bull. Amer. Math. Soc.* **24**, 185–194.

D. J. Wright (1974), 'The digital simulation of stochastic differential equations', *IEEE Trans. Automat. Control* **19**, 75–76.

D. J. Wright (1980), 'Digital simulation of Poisson stochastic differential equations', *Intern. J. Systems. Sci.* **11**, 781–785.

K. Xu (1995), 'Stochastic pitchforkbifurcation: numerical simulations and symbolic calculations using Maple', *Math. Comput. Simul.* **38**, 199.

S. J. Yakowitz (1977), *Computational Probability and Simulation*, Addison Wesley, Reading, MA.

T. Yamada (1976), 'Sur l'approximation des solutions d'équations différentielles stochastiques', *Z. Wahrsch. Verw. Gebiete* **36**, 153–164.

N. Yannios and P. E. Kloeden (1996), Time-discretization solution of stochastic differential equations, in *Proc. CTAC 95* (R. L. May and A. K. Easton, eds), Computational Techniques and Applications: CTAC95, World Scientific, pp. 823–830.

Y. Y. Yen (1988), 'A stochastic Taylor formula for functionals of two-parameter semimartingales', *Acta Vietnamica* **13**, 45–54.