Robust Pole Assignment in Singular Control Systems

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ABSTRACT

Necessary conditions are given for the problem of pole assignment by state feedback in singular linear systems (descriptor systems) to have a solution which is *regular* and nondefective. For a *robust* solution, such that the assigned closed-loop poles are insensitive to perturbations in the system data, the same conditions must hold. It can be shown that these conditions are also sufficient for the existence of a feedback which assigns the maximum possible number of finite poles with regularity. These results provide the basis of a procedure for constructing closed-loop semistate systems with given poles, guaranteed regularity, and maximum robustness.

1. INTRODUCTION

In singular, or degenerate, time-invariant multiinput linear control systems (descriptor systems), pole assignment by feedback requires not only that the closed-loop system have prescribed poles, but also that it be *regular*, and that it be *robust*, in the sense that its assigned poles are as insensitive as

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possible to perturbations in the system data. In this paper we give a detailed derivation of results which we have previously reported [8] on conditions for the pole assignment problem to have a *regular*, *nondefective*, solution. These results form the basis of numerical procedures for generating robust feedback systems with prescribed poles. The procedures are extensions of earlier techniques which we have developed for robust pole assignment in *non*degenerate systems [6, 9].

We begin by examining open-loop singular systems in Section 2, and in Section 3 we apply the results to closed-loop systems, in order to obtain necessary conditions for arbitrary pole assignment with regularity. These conditions are equivalent to the "finite" and "infinite" pole controllability conditions derived in [1, 2, 10, 14, 16], but the proof given here is very simple and does not require transformation of the system into decomposed "slow" and "fast" subsystems. These conditions are also sufficient for arbitrary pole assignability with guaranteed regularity [3, 5].

In Section 4 we give conditions under which a specified *nondefective* set of eigenvectors can be assigned to correspond with the required closed-loop poles, and an explicit form for the feedback matrix is derived. These results demonstrate that the "infinite" pole controllability condition can be used also to guarantee *regularity* of the closed-loop system pencil, and an algorithm based on these results for generating the feedback is described. In [1] and [2] algorithms are also suggested for the solution of the pole assignment problem. The method of [2], however, is based on the canonical decomposition of the system, which should be avoided for reasons of numerical stability (see, for example, [7]); and the method of [1] does not guarantee regularity of the closed-loop system. The new algorithm presented here does not require any transformations of the system, and it guarantees regularity of the closed-loop pencil. Moreover, the feedback is obtained by selecting independent eigenvectors corresponding to the assigned poles, and since it is known [12, 15] that the sensitivities of the closed-loop poles depend on the conditioning of the eigenvectors, the extra degrees of freedom in the feedback can be selected to give a *robust* solution to the pole assignment problem.

Measures of robustness are defined in Section 5 and properties of the robust-pole-assignment problem are discussed in Section 6. It is shown that optimizing robustness also minimizes bounds on the magnitude of the feedback matrix and on the transient response of the closed-loop system. In Section 7 a detailed procedure is described for selecting the eigenvectors to give a *robust*, *regular* solution to the pole assignment problem for singular systems, based on techniques which we have previously developed for nondegenerate systems [6, 9]. In Section 8 we present some applications and numerical results, and in Section 9 concluding remarks are given.

2. OPEN-LOOP REGULARITY

We first consider systems described by the dynamic equations

$$E\mathscr{D}\mathbf{x} = A\mathbf{x} \tag{2.1}$$

where $E, A \in \mathbb{R}^{n\times n}$ and $\operatorname{rank}[E] = q \leqslant n$. Here $\mathscr{D}$ denotes the differential operator d/dt for continuous systems, or the delay operator for discrete systems. We are specifically interested in the singular, or degenerate, case where $q < n$. The behavior of the system (2.1) is governed by the poles, or generalized eigenvalues, of the matrix pencil $A - \lambda E$, denoted by $[A, E]$. Solutions to the equations (2.1) which satisfy given initial conditions are unique provided the pencil $[A, E]$ is *regular*, that is,

$$\det[A - \lambda E] \neq 0 \tag{2.2}$$

(regarded as a polynomial in λ). It is well known [15] that a regular pencil has *at most* q finite eigenvalues and that the number of finite eigenvalues is given precisely by $r = \deg\det[A - \lambda E]$. Furthermore, the pencil $[E, A]$ then has precisely $n - r$ zero eigenvalues, as shown in the following lemma [13].

LEMMA 1. *Assume* $[A, E]$ *regular. Then* $[E, A]$ *has precisely* $n - r$ *zero eigenvalues, where* $r = \deg\det[A - \lambda E]$.

Proof. We let $p(\lambda) = \det[A - \lambda E]$ and $\hat{p}(\lambda) = \det[E - \lambda A]$. Then, since

$$\det[A - \lambda E] = \det[-\lambda(E - \lambda^{-1}A)] = (-\lambda)^n \det[E - \lambda^{-1}A],$$

we have $p(\lambda) = (-\lambda)^n \hat{p}(\lambda^{-1})$. Moreover, $[E, A]$ has precisely $n - r$ zero eigenvalues if and only if $\hat{p}(\lambda) = \lambda^{n-r} t(\lambda)$ where $t(0) \neq 0$. It follows that $p(\lambda) = (-1)^n \lambda^r t(\lambda^{-1})$ and $p(\lambda)$ is of exact degree r. ∎

The eigenvectors of the pencil $[E, A]$ associated with the zero eigenvalues must belong to the null space $\mathscr{N}\{E\}$, which has dimension $n - q$. Thus it follows from Lemma 1 that the regular pencil $[A, E]$ has q finite eigenvalues if and only if the zero eigenvalues of $[E, A]$ are *nondefective*. We have thus shown

LEMMA 2. *If the pencil* $[A, E]$ *is regular, then it has* $q \equiv \operatorname{rank}[E]$ *finite eigenvalues if and only if*

$$\mathrm{v}^T E = 0 \ \textit{and} \ \mathrm{v}^T A = \mathrm{z}^T E \ \textit{for any} \ \mathrm{z} \in \mathbb{C}^n \quad \Rightarrow \quad \mathrm{v} = 0, \tag{2.3}$$

or, equivalently,

$$E\mathrm{v} = 0 \ \textit{and} \ A\mathrm{v} = E\mathrm{z} \ \textit{for any} \ \mathrm{z} \in \mathbb{C}^n \quad \Rightarrow \quad \mathrm{v} = 0. \tag{2.4}$$

We next show that the condition (2.3) is *necessary* for regularity of the open-loop system. We write

$$E = [R_E, 0][S_E, S_\infty]^T = R_E S_E^T, \tag{2.5}$$

where $R_E \in \mathbb{R}^{n \times q}$, R_E is of full rank, and the matrix $[S_E, S_\infty]$ is orthogonal. Then the columns of S_∞ and S_E give orthonormal bases for $\mathcal{N}\{E\}$ and $\mathcal{R}\{E^T\}$, respectively, where $\mathcal{N}\{\cdot\}$ denotes null space and $\mathcal{R}\{\cdot\}$ denotes range. We use the following lemma.

LEMMA 3. *The condition* (2.3) *is equivalent to each of the following conditions*:

$$\operatorname{rank}[E, AS_\infty] = n; \tag{2.6}$$

$$\operatorname{rank}\left[E + AS_\infty S_\infty^T\right] = n. \tag{2.7}$$

Proof. The equivalence of (2.3) and (2.6) is demonstrated by contradiction. If (2.6) does not hold, then there exists $\mathrm{v} \neq 0$ such that $\mathrm{v}^T[E, AS_\infty] = 0$. Hence, $\mathrm{v}^T E = 0$ and $\mathrm{v}^T A = \mathrm{z}^T E$, where either $\mathrm{z} = 0$ or z satisfies $\mathrm{z}^T R_E = \mathrm{v}^T A S_E \neq 0$, and the condition (2.3) is violated. Conversely, if (2.3) does not hold, then there exists $\mathrm{v} \neq 0$ such that $\mathrm{v}^T E = 0$ and $\mathrm{v}^T A S_\infty = \mathrm{z}^T E S_\infty = 0$, and hence (2.6) is not satisfied.

To show the second part, we observe that if (2.6) is violated, then there exists $v \neq 0$ such that $v^T E = 0$ and $v^T A S_\infty = 0$, and (2.7) is clearly not satisfied. Finally, if (2.7) fails to hold, then there exists $v \neq 0$ with $v^T E = -v^T A S_\infty S_\infty^T$. It follows that $-v^T A S_\infty = v^T E S_\infty = 0$ and (2.6) is violated. The conditions (2.3), (2.6) and (2.7) are therefore all equivalent. ■

From the equivalence property of Lemma 3 we can now easily prove

LEMMA 4. *If the condition* (2.3) *holds, then the pencil* $[A, E]$ *is regular.*

Proof. The condition (2.3) implies (2.6), from which it follows that $\mathrm{rank}[ES_E, AS_\infty] = n$ and therefore there exist unique matrices Z_1, Z_2 satisfying

$$[ES_E, AS_\infty]\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = AS_E.$$

We thus have

$$[A - \lambda E][S_E, S_\infty] = [ES_E, AS_\infty]\begin{bmatrix} Z_1 - \lambda I & 0 \\ Z_2 & I \end{bmatrix},$$

and the pencil $[A, E]$ is clearly regular (with q finite eigenvalues). ■

From Lemmas 2 and 4 we have immediately

THEOREM 1. *The pencil* $[A, E]$ *is regular and has* $q \equiv \mathrm{rank}[E]$ *finite eigenvalues* if and only if *the condition* (2.3) (*or* (2.6) *or* (2.7)) *holds.*

Theorem 1 gives a necessary and sufficient condition for the pencil $[A, E]$ to be regular and have a full complement of finite eigenvalues (multiple or simple). A necessary condition can also be given for the pencil to be *nondefective*, that is, for $[A, E]$ to have a full independent set of corresponding eigenvectors. We have

LEMMA 5. *If the pencil* $[A, E]$ *is regular and there exists* $X_q \in \mathbb{C}^{n \times q}$ *with* $\mathrm{rank}[X_q] = q \equiv \mathrm{rank}[E]$ *such that*

$$AX_q = EX_q\Lambda_q, \qquad \Lambda_q = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_q\}, \tag{2.8}$$

where $\lambda_j \in \mathbb{C}\ \forall j$, *then* $\mathrm{rank}\{[X_q, S_\infty]\} = n$.

Proof. Since the matrix $[S_E, S_\infty]$ is orthogonal, we may write

$$[X_q, S_\infty] = [S_E, S_\infty]\begin{bmatrix} S_E^T X_q & 0 \\ S_\infty^T X_q & I \end{bmatrix},$$

and it follows that $[X_q, S_\infty]$ is nonsingular $\Leftrightarrow$ $S_E^T X_q$ is nonsingular $\Leftrightarrow$ $R_E S_E^T X_q \equiv EX_q$ and X_q have full rank. The result then follows by contradiction. If $\operatorname{rank}[X_q] = q$ and $\operatorname{rank}[EX_q] < q$, there exists $\mathbf{w} \neq 0$ such that $\mathbf{v} = X_q\mathbf{w} \neq 0$ and $E\mathbf{v} = 0$. Then for $\mathbf{z} = X_q\Lambda_q\mathbf{w}$ we have

$$A\mathbf{v} = AX_q\mathbf{w} = EX_q\Lambda_q\mathbf{w} = E\mathbf{z},$$

and the condition of Lemma 2 is violated. ∎

This lemma implies that if the regular pencil $[A, E]$ has q independent eigenvectors corresponding to finite eigenvalues, then these eigenvectors remain independent under the application of E, or equivalently, no linear combination of them lies in the null space of E. This lemma also gives, therefore, a necessary condition for a regular pencil to have $q \equiv \operatorname{rank}[E]$ *nondefective* finite eigenvalues.

In the next section we apply Theorem 1 to obtain conditions for the existence of regular solutions to the problem of pole assignment in singular systems. In Section 4 we examine eigenvector assignment by state feedback.

3. POLE ASSIGNMENT IN SINGULAR SYSTEMS

We now consider singular control systems governed by the open-loop equations

$$E\mathscr{D}\mathbf{x} = A\mathbf{x} + B\mathbf{u}, \tag{3.1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\operatorname{rank}[E] = q < n$, and $\operatorname{rank}[B] = m$. (Here $\mathscr{D}$ again denotes either the continuous differential or the discrete delay operator.) The poles, or generalized eigenvalues, of the pencil $[A, E]$ govern the behavior of the system and may be modified by state feedback. The pole assignment problem is specified as follows.

PROBLEM 1. Given real matrices E, A, B where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\operatorname{rank}[E] = q < n$, and $\operatorname{rank}[B] = m$, and a set of q self-conjugate complex numbers $\mathscr{L} = \{\lambda_1, \lambda_2, \dots, \lambda_q\}$, find $F \in \mathbb{R}^{m\times n}$ such that

$$\det[A + BF - \lambda E] = 0 \qquad \forall\lambda \in \mathscr{L} \tag{3.2}$$

and such that

$$\det[A + BF - \lambda E] \neq 0 \qquad \forall\lambda \notin \mathscr{L}. \tag{3.3}$$

The Equation (3.2) implies $\lambda_j \in \mathscr{L}$ is a generalized eigenvalue of the pencil $[M, E]$, where $M = A + BF$, and Equation (3.3) guarantees that the pencil is regular.

The following two conditions are easily shown to be necessary for the pole assignment problem, Problem 1, to have a solution for any arbitrary self-conjugate set $\mathscr{L}$ of q eigenvalues.

CONDITION C1. If $\mathbf{v}^TA = \mu\mathbf{v}^TE$ and $\mathbf{v}^TB = 0$, then $\mathbf{v} = 0$.

CONDITION C2. If $\mathbf{v}^TE = 0$, $\mathbf{v}^TB = 0$, and $\mathbf{v}^TA = \mathbf{z}^TE$, then $\mathbf{v} = 0$.

If Condition C1 does not hold, then there exists a vector $\mathbf{v}$ such that $\mathbf{v}^T(A + BF) = \mu\mathbf{v}^TE$ for any choice of matrix F, and hence (3.2) and (3.3) cannot both be satisfied unless $\mu \in \mathscr{L}$, and the problem cannot be solved for arbitrary $\mathscr{L}$. Similarly, if C2 is not satisfied, then there exists $\mathbf{v} \neq 0$ and a vector $\mathbf{z}$ such that $\mathbf{v}^TE = 0$ and $\mathbf{v}^T(A + BF) = \mathbf{z}^TE$ for any choice of F, and, by Theorem 1, a regular solution to the feedback problem cannot exist.

The Conditions C1 and C2 are thus necessary for the existence of a solution to the pole assignment problem, Problem 1 (see also [1, 2, 5, 10, 14, 16]). As shown in [5], these two conditions are also *sufficient* for the existence of a feedback which assigns precisely $q \equiv \operatorname{rank}[E]$ given finite eigenvalues with regularity, and we have the following theorem.

THEOREM 2. *The pole assignment problem, Problem 1, has a solution for an arbitrary self-conjugate set of poles $\mathscr{L}$ if and only if Conditions C1 and C2 hold.*

We remark that Conditions C1 and C2 have various equivalent formulations. Condition C1 is clearly equivalent to

CONDITION C1′. $\operatorname{rank}([B, A - \lambda E]) = n \quad \forall \lambda \in \mathbb{C}$.

From Lemma 3 of Section 2 it can also be seen that Condition C2 is equivalent to

CONDITION C3. If $\mathbf{v}^T[E + AS_\infty S_\infty^T] = 0$ and $\mathbf{v}^T B = 0$, then $\mathbf{v} = 0$;

and that C2 and C3 are both equivalent to the conditions

CONDITION C2′. $\operatorname{rank}[B, E, AS_\infty] = n$;

CONDITION C3′. $\operatorname{rank}[B, E + AS_\infty S_\infty^T] = n$.

Condition C1 (or C1′) corresponds to the "finite-pole controllability" condition as given in [2, 16], and implies that all the finite modes of the open-loop system (3.1) are controllable. Condition C2 (or C3, C2′, or C3′) corresponds to the "infinite-pole controllability" condition of [1, 2, 10, 14], and it guarantees that poles at infinity can be shifted into arbitrary finite positions and implies that impulses in the solutions may be eliminated. The formulation of Condition C2 given here does not, however, require the transformation of the system into canonical form in order to obtain a decomposition into "fast" and "slow" subsystems. For computational purposes it is important to avoid such transformations, as they are, in general, unreliable numerically (see e.g. [7]).

We remark, further, that Condition C2 guarantees both *regularity* of the closed-loop system and complete *controllability* of the open-loop "infinite" poles. Fletcher [3] points out that when C2 does not hold, then it is still possible to assign *fewer* than $q \equiv \operatorname{rank}[E]$ eigenvalues with regularity. Condition C1 simply guarantees controllability of the open-loop "finite" eigenvalues and is not needed to ensure regularity. Indeed, if C2 holds and all the uncontrollable modes which violate C1 are included in the set $\mathscr{L}$, with their appropriate multiplicities, then a *regular* solution to the pole assignment problem (Problem 1) can still be found. Moreover, although the uncontrollable open-loop poles may not be reassigned, their corresponding eigenvectors can be. This is significant because the sensitivities of the poles to perturbations in the system data are dependent on the conditioning of the corresponding eigenvectors [12, 13, 15]. In practice, therefore, we are interested in constructing a feedback which assigns both eigenvalues *and eigenvectors* so as to ensure robustness of the closed-loop matrix pencil. In the next section we examine conditions for complete eigenstructure assignment.

4. EIGENSTRUCTURE ASSIGNMENT IN SINGULAR SYSTEMS

In nonsingular systems, pole assignment by state feedback can be achieved by assigning the eigenvectors associated with the assigned eigenvalues of the closed-loop system. The selected eigenvectors then uniquely determine the required feedback matrix [9, 11]. In singular systems generalized eigenvalue-eigenvector assignment alone is not sufficient to determine the feedback. Furthermore, to obtain regularity of the closed-loop pencil, certain restrictions on the eigenstructure must be satisfied. In this section we derive conditions for determining a feedback such that the closed-loop system has a specified nondefective eigenstructure and is regular.

We first give a *necessary* condition for nondefective eigenstructure assignment with regularity. From Lemma 5 of Section, 2, we have immediately

LEMMA 6. *If there exists* $F \in \mathbb{R}^{m\times n}$ *such that the pencil* $[A + BF, E]$ *is regular, and* $X_q \in \mathbb{C}^{n\times q}$ *such that* $\operatorname{rank}[X_q] = q$ *and*

$$(A + BF)X_q = EX_q\Lambda_q, \qquad \Lambda_q = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_q\}, \tag{4.1}$$

where $\lambda_j \in \mathbb{C}$ $\forall j$, *then the matrix* $[X_q, S_\infty]$ *(equivalently,* EX_q*) is of full rank.*

The next theorem provides necessary and sufficient conditions under which a given set of nondefective eigenvalues and corresponding eigenvectors can be assigned.

THEOREM 3. *Given* $\Lambda_q = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_q\}$, $\lambda_j \in \mathscr{L}$, *and a matrix* X_q *such that* $[X_q, S_\infty]$ *is nonsingular, then there exists* F *satisfying* (4.1) *and such that the pencil* $(A + BF, E)$ *is regular* if and only if

$$U_1^T(AX_q - EX_q\Lambda_q) = 0 \tag{4.2}$$

and

$$U_1^T(E + AS_\infty S_\infty^T) \text{ has full rank}, \tag{4.3}$$

where

$$B = [U_0, U_1]\begin{bmatrix} Z \\ 0 \end{bmatrix}, \tag{4.4}$$

with $U = [U_0, U_1]$ orthogonal and Z nonsingular. Then F is given explicitly by

$$F = Z^{-1}\left[U_0^T(EX_q\Lambda_q - AX_q), W\right]\left[X_q, S_\infty\right]^{-1}, \tag{4.5}$$

where W is any matrix such that

$$\operatorname{rank}\left[E + AS_\infty S_\infty^T + U_0 W S_\infty^T\right] = n. \tag{4.6}$$

Proof. The assumption that B is of full rank implies the existence of decomposition (4.4). From (4.1) F must satisfy

$$BFX_q = EX_q\Lambda_q - AX_q, \tag{4.7}$$

and premultiplication by U^T gives

$$ZFX_q = U_0^T(EX_q\Lambda_q - AX_q) \tag{4.8}$$

and

$$0 = U_1^T(EX_q\Lambda_q - AX_q) \tag{4.9}$$

from which (4.2) follows.

From Theorem 1, the pencil $[A + BF, E]$ is regular, under the given conditions, if and only if the matrix $E + (A + BF)S_\infty S_\infty^T$ has full rank, or equivalently $E + AS_\infty S_\infty^T + U_0 W S_\infty^T$ has full rank, where

$$ZFS_\infty = W. \tag{4.10}$$

This condition holds if and only if W can be chosen such that the matrix

$$\begin{bmatrix} U_0^T(E + AS_\infty S_\infty^T + WS_\infty^T) \\ U_1^T(E + AS_\infty S_\infty^T) \end{bmatrix} \tag{4.11}$$

has full rank. Clearly the condition (4.3) is necessary and sufficient for this to be possible. The expression (4.5) for the feedback matrix F then follows directly from (4.8) and (4.10), and if W is chosen to satisfy (4.6), the pencil $[A + BF, E]$ has the given finite eigenvalues and is regular. ∎

The significance of this theorem for the construction of a feedback which achieves pole assignment with regularity is considerable. The condition (4.3) of the theorem holds if and only if Condition C3′, or equivalently C2, C2′, or C3, holds. (This follows because we have C3′ if and only if the matrix

$$U^T\left[B, E + AS_\infty S_\infty^T\right] = \begin{bmatrix} Z & U_0^T\left(E + AS_\infty S_\infty^T\right) \\ 0 & U_1^T\left(E + AS_\infty S_\infty^T\right) \end{bmatrix}$$

has full rank, which holds if and only if (4.3) holds.) The condition (4.3) can be tested independently of any choice of F, and if it is not satisfied, then a feedback assigning q finite eigenvalues and giving a *regular* closed-loop pencil cannot be found. Conversely, if a set of q independent eigenvectors corresponding to the required closed-loop poles can be selected such that $[X_q, S_\infty]$ is nonsingular, then the condition (4.3) *guarantees* that a feedback F can be found such that the pencil $[A + BF, E]$ is *regular*. Previously it has been recognized that this condition is necessary for "infinite-pole shifting" [1, 2, 10, 14], but its importance in guaranteeing regularity has not, hitherto, been appreciated or exploited.

From the condition (4.2) of Theorem 3 the eigenvectors corresponding to a distinct closed-loop eigenvalue λ_j must belong to the space

$$\mathscr{S}_j = \mathscr{N}\left\{U_1^T(A - \lambda_j E)\right\}. \tag{4.12}$$

(This, together with the requirement that a closed-loop finite pole must be nondefective, implies a minor restriction on the multiplicity of λ_j.) A feedback matrix F which solves the pole assignment problem, Problem 1, can therefore be constructed as follows: Given the set $\mathscr{L} = \{\lambda_j,\ j = 1,2,\dots,q\}$, select q independent vectors $\mathbf{x}_j \in \mathscr{S}_j$, $j = 1,2,\dots,q$, such that $[X_q, S_\infty]$ is nonsingular, where $X_q = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_q]$, and select W such that (4.6) holds. Then the matrix F given by (4.5) is the required solution.

By this algorithm, *regularity* of the closed-loop pencil is *guaranteed*. We note that *no* restriction on the controllability of the open-loop finite eigenvalues (Condition C1) is made. Provided any uncontrollable modes are included in $\mathscr{L}$ (with correct multiplicity), the algorithm can be applied (although the existence of a nondefective solution cannot, of course, be ensured).

The degrees of freedom in the choice of F correspond to the degrees of freedom associated with the selection of the eigenvectors $\{\mathbf{x}_j\}$ and the matrix W. Since the robustness of the closed-loop system depends on the selected eigenvectors, we may select the set $\{\mathbf{x}_j\}$ such as to *optimize* robustness. In the next sections we describe a measure of robustness and give an explicit algorithm for selecting the set $\{\mathbf{x}_j\}$ and the matrix W such as to obtain a *robust* feedback solution to the pole assignment problem.

We remark that Theorem 3 gives conditions for assigning a maximum number of finite poles, $q \equiv \operatorname{rank}[E]$, with regularity. In the case where fewer finite poles can be assigned with regularity, similar results hold (see [5]).

5. MEASURES OF ROBUSTNESS FOR SINGULAR SYSTEMS

The matrix pencil $[M, E]$ of a closed-loop system, where $M = A + BF$, is defined to be *robust* if its eigenvalues, or poles, are as insensitive to perturbations in M and E as possible. Both "finite" and "infinite" poles must be considered, and, in order to avoid special distinctions, we define a generalized pole, or eigenvalue, of the pencil to be a pair $(\lambda, \delta) \in \mathbb{C} \times \mathbb{R}$ where the pole takes the finite "value" λ/δ for $\delta \neq 0$, and becomes infinite for $\delta = 0$. We denote the right and left eigenvectors associated with the eigenvalue (λ, δ) by $\mathbf{x}, \mathbf{y}$; that is, $\mathbf{x}, \mathbf{y}$ satisfy

$$\delta M\mathbf{x} = \lambda E\mathbf{x}, \qquad \delta \mathbf{y}^T M = \lambda \mathbf{y}^T E. \tag{5.1}$$

If the pencil $[M, E]$ is nondefective, that is, it has a full set of n linearly independent eigenvectors, then it can be shown [12] that the sensitivity of a *simple* eigenvalue (λ, δ) to perturbations in the components of M and E depends upon the *condition number*

$$c(\lambda, \delta) = \frac{\|\mathbf{y}\|_2 \|\mathbf{x}\|_2}{\left(|\lambda|^2 + \delta^2\right)^{1/2}}, \tag{5.2}$$

where $\|\cdot\|_2$ denotes the L_2 vector norm, and the eigenvectors $\mathbf{x}, \mathbf{y}$ are normalized so that

$$\mathbf{y}^T E\mathbf{x} = \delta, \qquad \mathbf{y}^T M\mathbf{x} = \lambda. \tag{5.3}$$

More precisely, if a perturbation $O(\epsilon)$ is made in the coefficients of M or E, then the corresponding first-order perturbation in (λ, δ) is of order $\epsilon c(\lambda, \delta)$.

Here the distance between (λ, δ) and the perturbed eigenvalue $(\tilde{\lambda}, \tilde{\delta})$ is measured by

$$|(\lambda, \delta) - (\tilde{\lambda}, \tilde{\delta})| = \frac{|\lambda\tilde{\delta} - \tilde{\lambda}\delta|}{\{(|\lambda|^2 + \delta^2)(|\tilde{\lambda}|^2 + \tilde{\delta}^2)\}^{1/2}}.$$

If $[M, E]$ is defective, then the corresponding perturbation in *some* eigenvalue is at least an order of magnitude worse in ϵ, and therefore system matrices which are defective are necessarily less robust than those which are nondefective.

In the case of a multiple eigenvalue, if $[M, E]$ is nondefective, then the sensitivity, or condition number, of the distinct eigenvalue (λ, δ), of multiplicity p, depends on certain canonical angles associated with its right and left invariant subspaces, denoted $\mathscr{X}$ and $\mathscr{Y}$. If $X = \{\mathbf{x}_i\}_1^p$ and $Y = \{\mathbf{y}_i\}_1^p$ are bases for $\mathscr{X}$ and $\mathscr{Y}$ such that

$$Y^T E X = \delta I_p, \qquad Y^T M X = \lambda I_p, \tag{5.4}$$

then, from [12], first-order perturbations in (λ, δ) due to $O(\epsilon)$ perturbations in the pencil are of order $\epsilon p c(\lambda, \delta)$ where

$$c(\lambda, \delta) = \max_i \left\{ \frac{\|\mathbf{y}_i\|_2 \|\mathbf{x}_i\|_2}{(|\lambda|^2 + \delta^2)^{1/2}} \right\}. \tag{5.5}$$

It is easily seen that in the case where (λ, δ) is *simple* $(p = 1)$, then (5.5) is equivalent to (5.2).

We remark that $c(\lambda, \delta)$, as defined in (5.5), is not invariant under changes of bases for $\mathscr{X}$ and $\mathscr{Y}$. To define $c(\lambda, \delta)$ uniquely we require X and Y to be such that $X = \hat{X}\hat{\Gamma}^{-1}$ and $Y^T = \hat{\Gamma}\hat{\Sigma}^{-1}\hat{Y}^T$, where $\hat{\Gamma} = \operatorname{diag}\{\hat{\gamma}_i\}$ with $\hat{\gamma}_i = \|\hat{X}\mathbf{e}_i\|_2$, $\hat{\Sigma} = \operatorname{diag}\{\hat{\sigma}_i\}$ with $\hat{\sigma}_i > 1$, $i = 1, 2, \ldots, p$, and $\hat{X}, \hat{Y}$ are the bases for $\mathscr{X}$ and $\mathscr{Y}$ which satisfy

$$\hat{X}^* E^T E \hat{X} = \delta^2 I, \qquad \hat{X}^* M^T M \hat{X} = |\lambda|^2 I, \qquad \hat{Y}^* \hat{Y} = I, \tag{5.6}$$

and

$$\hat{Y}^T E \hat{X} = \delta \hat{\Sigma}, \qquad \hat{Y}^T M \hat{X} = \lambda \hat{\Sigma}. \tag{5.7}$$

Then from (5.5) the condition number is given uniquely as

$$c(\lambda,\delta) = \max_i \left\{ \frac{\hat{\gamma}_i \hat{\sigma}_i^{-1}}{\left(|\lambda|^2+\delta^2\right)^{1/2}} \right\}, \tag{5.8}$$

where, by definition, $\hat{\sigma}_i = \cos\theta_i$, $i = 1,2,\dots,p$, are the cosines of the canonical angles between the subspaces $\mathcal{Y}$ and $E\mathcal{X}$ if $\delta \neq 0$, or between $\mathcal{Y}$ and $M\mathcal{X}$ if $\delta = 0$. Furthermore, since $|\lambda|\,\|E\mathbf{x}_i\|_2 = |\delta|\,\|M\mathbf{x}_i\|^2$ and $\hat{\gamma}_i = |\delta|\,\|E\mathbf{x}_i\|_2^{-1}$ ($\delta \neq 0$), or $\hat{\gamma}_i = |\lambda|\,\|M\mathbf{x}_i\|_2^{-1}$ ($\delta = 0$), with $\mathbf{x}_i = \hat{X}\hat{\Gamma}^{-1}\mathbf{e}_i \equiv \hat{\mathbf{x}}_i/\|\hat{\mathbf{x}}_i\|_2$, it follows that

$$c(\lambda,\delta) = \max_i \left\{ \frac{\sec\theta_i}{\rho_i} \right\} \geqslant \max_i \left\{ \rho_i^{-1} \right\}, \tag{5.9}$$

where

$$\rho_i = \left(\|E\mathbf{x}_i\|_2^2 + \|M\mathbf{x}_i\|_2^2 \right)^{1/2}.$$

Equality holds in (5.9) if and only if the subspaces $\mathcal{X}$ and $\mathcal{Y}$ are biorthogonal *with respect to E ($\delta \neq 0$) or M ($\delta = 0$). As indicated in* [12], the quantity ρ_i measures how nearly the vector $\mathbf{x}_i$ is an approximate null vector of both E and M, and hence how close the pencil is to being irregular. The condition number (5.8) of a generalized eigenvalue (λ,δ) is thus inversely proportional to the cosine of the smallest canonical angle between its E- (or M-) invariant subspaces and to a measure of the distance of the pencil from irregularity.

We can also derive a relation between the Frobenius norm of certain bases for the invariant subspaces and the condition numbers as defined by (5.5). If $X = \{\mathbf{x}_i\}_1^p$ and $Y = \{\mathbf{y}_i\}_i^p$ are any bases for $\mathcal{X}$ and $\mathcal{Y}$ satisfying (5.4) and such that $\|\mathbf{x}_j\| = 1$, then

$$\frac{\|Y^T\|_F^2}{|\lambda|^2+\delta^2} = \sum_j \frac{\|\mathbf{y}_j\|_2\,\|\mathbf{x}_j\|_2}{|\lambda|^2+\delta^2}.$$

It follows that

$$c(\lambda,\delta) \leqslant \frac{\|Y^T\|_F}{\left(|\lambda|^2+\delta^2\right)^{1/2}} \leqslant p^{1/2} c(\lambda,\delta), \tag{5.10}$$

and $\|Y^T\|_F$ gives a measure of the sensitivity of the eigenvalue equivalent mathematically to its condition number. If in addition we assume that X gives an *orthonormal* basis for $\mathscr{X}$, then we can show that

$$\frac{\|Y^T\|_F^2}{|\lambda|^2+\delta^2}=\sum_j \frac{\sec^2\theta_j}{\rho_j^2}. \tag{5.11}$$

Since X, Y satisfy (5.4), we may write

$$X^*E^TEX=\delta^2U\Gamma^2U^*, \qquad X^*M^TMX=|\lambda|^2U\Gamma^2U^*,$$

where U is unitary and Γ is diagonal. Then from (5.6) it follows that we can express $\hat{X}$ in the form $\hat{X}=XU\Gamma^{-1}Z$, where Z is also a unitary matrix. Furthermore, $\hat{\gamma}_i=\|\hat{X}\mathbf{e}_i\|=\|XU\Gamma^{-1}Z\mathbf{e}_i\|_2=\|\Gamma^{-1}Z\mathbf{e}_i\|_2$. From (5.4) and (5.7) we then have that $Y^T=U\Gamma^{-1}Z\hat{\Sigma}^{-1}\hat{Y}^T$, and therefore,

$$\begin{aligned}\|Y^T\|_F^2&=\|\Gamma^{-1}Z\hat{\Sigma}^{-1}\|_F^2=\sum_j\|\Gamma^{-1}Z\hat{\Sigma}^{-1}\mathbf{e}_j\|_2^2\\&=\sum_j\hat{\sigma}_j^{-2}\|\Gamma^{-1}Z\mathbf{e}_j\|_2^2=\sum_j\hat{\sigma}_j^{-2}\hat{\gamma}_j^2.\end{aligned}$$

The result (5.11) follows immediately from the definitions of $\hat{\sigma}_i, \hat{\gamma}_i$, and we conclude that $\|Y^T\|_F^2$ is precisely equal to a weighted sum of the inverse squares of the cosines of the canonical angles between the invariant subspaces associated with (λ,δ). Furthermore, $\|Y^T\|_F$ satisfies (5.10), where, in this case, $c(\lambda,\delta)$ is uniquely defined by (5.8).

We now consider measures of the *robustness* of the nondefective closed-loop pencil $[M,E]$. Without loss of generality we let the eigenvalues of $[M,E]$, denoted by (λ_j,δ_j), be scaled and ordered so that $\delta_j=1$ for $j=1,2,\dots,q$, and $\lambda_j=1$, $\delta_j=0$ for $j=q+1,\dots,n$. We also let $X=\{\mathbf{x}_j\}_1^n$, $Y=\{\mathbf{y}_j\}_1^n$ denote the modal matrices of right and left eigenvectors $\mathbf{x}_j,\mathbf{y}_j$ corresponding to (λ_j,δ_j), where $\mathbf{x}_j$ is normalised to unit length ($\|\mathbf{x}_j\|_2=1$) and X, Y satisfy

$$Y^TEX=\begin{bmatrix}I_q&0\\0&0\end{bmatrix}, \qquad Y^TMX=\begin{bmatrix}\Lambda_q&0\\0&I_{n-q}\end{bmatrix}, \tag{5.12}$$

with $\Lambda_q=\operatorname{diag}\{\lambda_1,\lambda_2,\dots,\lambda_q\}$. [We note that the eigenvectors corresponding to a multiple eigenvalue then form bases satisfying (5.4).]

We observe that we may write $X = [X_q, S_\infty]$ where the columns of X_q satisfy (4.1) and are the right eigenvectors of unit length corresponding to finite eigenvalues $(\lambda_j, 1)$, $j = 1,2,\dots,q$, and the columns of S_∞ form an orthonormal basis for the null space $\mathcal{N}\{E\}$, as defined by (2.5), and are the right eigenvectors of unit length corresponding to the $n-q$ infinite eigenvalues $(1,0)$. Furthermore, from (5.12) it then follows that

$$Y^T E X_q = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \qquad Y^T M S_\infty = \begin{bmatrix} 0 \\ I_{n-q} \end{bmatrix}, \tag{5.13}$$

and hence

$$Y^T = \left[E X_q, M S_\infty \right]^{-1}. \tag{5.14}$$

As a global measure of the robustness we now take

$$\nu(\omega) = \|D_\omega Y^T\|_F = \left(\sum_{j=1}^{n} d_j^2 \|\mathbf{y}_j\|_2^2 \right)^{1/2}, \tag{5.15}$$

where

$$D_\omega = \operatorname{diag}\{d_j\}, \qquad d_j = \frac{\omega_j}{\left(|\lambda_j|^2 + \delta_j^2\right)^{1/2}},$$

and the weights $\omega_j > 0$ satisfy $\omega_j = \omega_k$ if $|(\lambda_j, \delta_j) - (\lambda_k, \delta_k)| = 0$, and $\sum_{j=1}^{n} \omega_j^2 = 1$. By the assumption $\|\mathbf{x}_j\|_2 = 1$, we then have

$$\nu(\omega)^2 = \sum_{j=1}^{n} \frac{\omega_j^2 \|\mathbf{y}_j\|_2^2 \|\mathbf{x}_j\|_2^2}{|\lambda_j|^2 + \delta^2}, \tag{5.16}$$

and, using the definition (5.5) for the condition number, we obtain

$$\sum_{(\lambda,\delta)} \omega_j^2 c^2(\lambda_j, \delta_j) \leqslant \nu(\omega)^2 \leqslant \sum_{(\lambda,\delta)} \omega_j^2 p_j c^2(\lambda_j, \delta_j), \tag{5.17}$$

where $\sum_{(\lambda,\delta)}$ denotes the sum over all *distinct* eigenvalues (λ_j, δ_j) of multiplicity p_j. It follows that $\nu(\omega)^2$ is precisely equal to a weighted sum of the squares of the condition numbers $c(\lambda_j, \delta_j)$ of the eigenvalues, where the corresponding weights lie in the ranges $[\omega_j, \omega_j p_j^{1/2}]$.

In the case where the right eigenvectors which correspond to multiple eigenvalues form *orthonormal* bases for the invariant subspaces, the measure becomes

$$\nu(\omega)^2 = \sum_{j=1}^{n} \frac{\omega_j^2 \hat{\gamma}_j^2 \hat{\sigma}_j^{-2}}{|\lambda_j|^2 + \delta_j^2} \equiv \sum_{j=1}^{n} \frac{\omega_j^2 \sec^2 \theta_j}{\rho_j^2}, \tag{5.18}$$

and $\nu(\omega)^2$ is equal to the weighted sum of the inverse squares of the cosines of all the canonical angles between the left and right E- (or M-) invariant subspaces associated with the distinct eigenvalues. In this case $\nu(\omega)$ satisfies (5.17) with $c(\lambda_j, \delta_j)$ uniquely defined by (5.8).

We may also define as a measure of robustness

$$\nu_\infty = \max_{(\lambda, \delta)} c(\lambda, \delta). \tag{5.19}$$

Then from (5.12) we have

$$\hat{\omega}\nu_\infty^2 \leqslant \sum_{(\lambda, \delta)} \omega_j^2 c^2(\lambda_j, \delta_j) \leqslant \nu(\omega)^2 \leqslant \nu_\infty^2, \tag{5.20}$$

where $\hat{\omega} = \min_j\{\omega_j\}$, and the measure $\nu(\omega)$ and ν_∞ are thus mathematically equivalent. Furthermore, minimizing either of the measures $\nu(\omega)^2$ or ν_∞^2 minimizes a bound on the weighted sum of the squares of the condition numbers of the pencil $[M, E]$, with corresponding weights ω_j, and either the measure $\nu(\omega)$ or ν_∞ gives an overall measure of the sensitivity of the poles of the closed-loop pencil $[M, E]$.

In the next section we examine properties of *robust* closed-loop singular systems, and in Section 7 we describe procedures for constructing feedback matrices which minimize the robustness measures.

6. ROBUST POLE ASSIGNMENT IN SINGULAR SYSTEMS

For the singular time-invariant linear multivariable control system (3.1), described by the matrix triple $[E, A, B]$, the problem of *robust* pole assignment is now defined as follows.

PROBLEM 2. Given real matrices E, A, B where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $\operatorname{rank}[E] = q < n$, and $\operatorname{rank}[B] = m$, and a set $\mathcal{L} = \{\lambda_j \in \mathbb{C},\ j = 1,2,\dots,q\}$ where $\lambda_j \in \mathcal{L} \Leftrightarrow \bar{\lambda}_j \in \mathcal{L}$, find a matrix $F \in \mathbb{R}^{m\times n}$ and a matrix $X_q \in \mathbb{C}^{n\times q}$ of full rank such that

$$(A + BF)X_q = EX_q\Lambda_q, \qquad \Lambda_q = \operatorname{diag}\{\lambda_j\}, \tag{6.1}$$

$$\operatorname{rank}\left[X_q, S_\infty\right] = n, \tag{6.2}$$

$$\operatorname{rank}\left[E + (A + BF)S_\infty S_\infty^T\right] = n, \tag{6.3}$$

and such that some *robustness* measure ν of the sensitivity of the generalized eigenproblem is optimized.

Here S_∞ is defined as in (2.5) to give an orthonormal basis for $\mathcal{N}\{E\}$, and the condition (6.2) is equivalent to $\operatorname{rank}[EX_q] = q$. The condition (6.3) guarantees that the pencil $[A + BF, E]$ is regular. The measure ν could be taken to be either of the measures described in Section 5, but here we are mainly interested in $\nu(\omega)$.

We remark that for the pole assignment to be robust it is necessary not only that the poles be insensitive to perturbations, but also that the rank conditions (6.2) and (6.3) be insensitive—that is, we require the matrices $[X_q, S_\infty]$ and $E + MS_\infty S_\infty^T$, where $M = A + BF$, to be *far* from singular. This is the case if the condition numbers κ_1, κ_2 respectively, of these matrices are small, where the condition number κ of a matrix H is defined by $\kappa(H) = \|H\|\,\|H^{-1}\|$ for some norm $\|\cdot\|$ [15]. We show now that the measure $\nu(\omega)$ of the conditioning of the poles is directly related to κ_1, κ_2, defined with respect to the Frobenius and L_2 norms respectively, and hence that the sensitivities of the rank requirements and the poles are minimized simultaneously.

Assuming the conditions of Section 5 (specifically $\|X_q e_j\|_2 = 1$, $j = 1,2,\dots,q$), then by definition

$$\nu(\omega) = \|D_\omega Y^T\|_F \equiv \left\|D_\omega\left[EX_q, MS_\infty\right]^{-1}\right\|_F, \tag{6.4}$$

where $D_\omega Y^T$ may be regarded as a scaling of the left generalized eigenvectors of the pencil $[M, E]$. We observe that

$$Y^{-T} \equiv \left[EX_q, MS_\infty\right] = \left(E + MS_\infty S_\infty^T\right)\left[X_q, S_\infty\right]\begin{bmatrix} I_q & 0 \\ -S_\infty^T X_q & I_{n-q} \end{bmatrix}, \tag{6.5}$$

and therefore, if $\nu(\omega)$ takes a finite value for some choice of X_q and F, then the rank conditions (6.2) and (6.3) are necessarily satisfied. Moreover, from the choice of scaling we have $\|[X_q, S_\infty]\|_F = n^{1/2}$ and $\|S_\infty^T X_q\|_F^2 \leqslant \|X_q\|_F^2 = q$, and by rearranging the equality (6.5), taking norms, and applying the inequality $\|GHK\|_F \leqslant \|G\|_2 \|H\|_F \|K\|_2$, we find that κ_1 and κ_2 both satisfy

$$\kappa_1, \kappa_2 \leqslant \|D_\omega^{-1}\|_2 n^{1/2}(n+q)^{1/2}\nu(\omega)\|E + MS_\infty S_\infty^T\|_2. \tag{6.6a}$$

Hence the condition numbers κ_1, κ_2 are bounded in terms of $\nu(\omega)$ and the magnitude of the matrix $E + MS_\infty S_\infty^T$. Conversely, we can bound $\nu(\omega)$ in terms of κ_1 and κ_2. Using (6.5) in (6.4) and taking norms, we obtain

$$\begin{aligned}\nu(\omega) &\leqslant \|D_\omega\|_2(n+q)^{1/2}\left\|[X_q, S_\infty]^{-1}\right\|_F\left\|[E + MS_\infty S_\infty^T]^{-1}\right\|_2 \\ &\leqslant \frac{\|D_\omega\|_2(1+q/n)^{1/2}\kappa_1\kappa_2}{\|E + MS_\infty S_\infty^T\|_2}. \end{aligned} \tag{6.6b}$$

The ratio $\kappa_2/\|E + MS_\infty S_\infty^T\|_2$ measures a balance between the magnitude of the norm of the matrix and its distance from singularity and may be interpreted as a measure of the regularity of the pencil. This ratio and $\kappa_1 \equiv n^{1/2}\|[X_q, S_\infty]^{-1}\|_F$ together give an upper bound on $\nu(\omega)$ and, therefore, on a measure of the sensitivity of the closed-loop poles. Conversely the sensitivity measure $\nu(\omega)$ bounds the product of these two measures. A robust solution to the pole placement problem is thus achieved either by minimizing $\nu(\omega)$ directly or by minimizing κ_1, κ_2 separately, subject to $\|E + MS_\infty S_\infty^T\|_2$ remaining bounded. We now show that optimizing these quantities leads to other desirable properties of the closed-loop system.

First we derive bounds on the feedback matrix F. We have

THEOREM 4. *The gain matrix F satisfies the inequality*

$$\|F\|_2 \leqslant \frac{\|A\|_2 + \max\{|\lambda_j|, 1\}\,\|Y^{-T}\|_F\left\|[X_q, S_\infty]^{-1}\right\|_F}{\sigma_{\min}\{B\}}, \tag{6.7}$$

where $\sigma_{\min}\{B\}$ *is the smallest singular value of* B, *and* $Y^{-T} = [EX_q, MS_\infty]$.

Proof. From the definition of Y we find

$$Y^T M[X_q, S_\infty] = \begin{bmatrix} \Lambda_q & 0 \\ 0 & I \end{bmatrix},$$

and therefore, since $M = A + BF$,

$$BF = \left(Y^{-T} \begin{bmatrix} \Lambda_q & 0 \\ 0 & I \end{bmatrix} [X_q, S_\infty]^{-1} - A \right). \tag{6.8}$$

We note that, from the singular-value decomposition of B, $\|\mathrm{BF}\|_2 \geqslant \sigma_{\min}\{\mathrm{B}\} \|\mathrm{F}\|_2$ and that $\|\cdot\|_2 \leqslant \|\cdot\|_F$ [15]; the result (6.7) then follows immediately by taking norms in (6.8). ■

Using now the expression (6.5) in the bound (6.7), we obtain

$$\|F\|_2 \leqslant \sigma_{\min}^{-1}\{B\}\left(\|A\|_2 + \max\{|\lambda_j|, 1\}(n+q)^{1/2}\kappa_1 \|E + MS_\infty S_\infty^T\|_2\right). \tag{6.9}$$

An upper bound on the magnitude of F is thus minimized if κ_1 and $\|E + MS_\infty S_\infty^T\|_2$ are minimized. However, to maintain regularity of the solution, the matrix $E + MS_\infty S_\infty^T$ must remain nonsingular, that is, $\kappa_2 / \|E + MS_\infty S_\infty^T\|_2$ must remain bounded. In effect then, there is a tradeoff between the conditioning $\nu(\omega)$ of the poles that can be achieved, and the magnitude of the gains. In practice, to obtain a robust solution to the pole assignment problem we select the matrix of eigenvectors X_q to minimize the conditioning κ_1 of the modal matrix $[X_q, S_\infty]$ and choose the remaining degrees of freedom to minimize the ratio $\kappa_2 / \|E + MS_\infty S_\infty^T\|_2$, subject to the condition $\|E + MS_\infty S_\infty^T\|_2 \leqslant c$, where c is some positive tolerance. Essentially we then optimize the sensitivity of the poles and the regularity of the pencil, subject to the magnitude of the gains being bounded.

Bounds on the transient response of the closed-loop system (3.1) can also be derived in terms of the conditioning measures. We have

THEOREM 5. *The transient response* $\mathbf{x}(t)$, *or* $\mathbf{x}(k)$, *of the closed-loop continuous, or discrete, time system*

$$E\mathscr{D}\mathbf{x} = (A + BF)\mathbf{x}, \tag{6.10}$$

$$\mathbf{x}(0) = \mathbf{x}_0 \in \mathscr{R}\{X_q\}, \tag{6.11}$$

is bounded by

$$\|\mathbf{x}(t)\|_2 \leqslant \max_j \{|e^{\lambda_j t}|\} \|X_q\|_F \|Y_1^T E\|_F \|\mathbf{x}_0\|_2 \tag{6.12}$$

or

$$\|\mathbf{x}(k)\|_2 \leqslant \max_j \{|\lambda_j|^k\} \|X_q\|_F \|Y_1^T E\|_F \|\mathbf{x}_0\|_2, \tag{6.13}$$

where $Y_1^T = [I_q, 0]Y^T$.

Proof. By definition, the columns of X_q form a normalized basis for the unique maximal invariant subspace of the pencil $[M, E]$, where $M = A + BF$, and by [4] the equation (6.10) has a unique solution if and only if the initial state $\mathbf{x}_0 \in \mathscr{R}\{X_q\}$. Then, also from [4], the solution takes the form

$$\mathbf{x}(t) = X_q e^{\Lambda_q t} X_q^+ \mathbf{x}_0, \qquad \text{or} \quad \mathbf{x}(k) = X_q \Lambda_q^k X_q^+ \mathbf{x}_0, \tag{6.14}$$

where X_q^+ is such that $\mathbf{x}(t)$, or $\mathbf{x}(k)$, $\in \mathscr{R}\{X_q\}$ $\forall t$, or $\forall k$. It is easy to see that the solutions (6.14) satisfy the system equation (6.10), and that with $\mathbf{x}_0 = X_q \mathbf{w}_0 \in \mathscr{R}\{X_q\}$, the matrix X_q^+ must be such that $X_q X_q^+ X_q = X_q$. Now from (5.13) it follows that $Y_1^T E X_q = I_q$ and hence we may take $X_q^+ = Y_1^T E$. The inequalities (6.12) and (6.13) then follow directly by taking norms in (6.14). ■

Using (5.12) we now have $Y_1^T E[X_q, S_\infty] = [I_q, 0]$ and, hence, we obtain from (6.12)

$$\begin{aligned}\|\mathbf{x}(t)\|_2 &\leqslant \max_j \{|e^{\lambda_j t}|\} \|X_q\|_F \left\| [X_q, S_\infty]^{-1} \right\|_F \|\mathbf{x}_0\|_2 \\ &\leqslant \max_j \{|e^{\lambda_j t}|\} q^{1/2} n^{-1/2} \kappa_1 \|\mathbf{x}_0\|_2,\end{aligned} \tag{6.15}$$

or, similarly, from (6.13),

$$\|\mathbf{x}(k)\| \leqslant \max_j \{|\lambda_j|^k\} q^{1/2} n^{-1/2} \kappa_1 \|\mathbf{x}_0\|. \tag{6.16}$$

It follows that a bound on the transient responses of the closed-loop system, denoted by the triple $[E, A+BF, B]$ is minimized if the conditioning κ_1 of the modal matrix of eigenvectors, $[X_q, S_\infty]$, is minimized.

We conclude that a *robust* solution to the pole assignment problem (Problem 2) is obtained by minimizing the conditioning measures

$$\kappa_1 = \|[X_q, S_\infty]\|_F \|[X_q, S_\infty]^{-1}\|_F \equiv n^{1/2} \|[X_q, S_\infty]^{-1}\|_F \tag{6.17}$$

and

$$\frac{\kappa_2}{\|E + MS_\infty S_\infty^T\|_2} \equiv \|(E + MS_\infty S_\infty^T)^{-1}\|_2, \tag{6.18}$$

subject to

$$\|E + MS_\infty S_\infty^T\|_2 \leqslant c, \qquad c > 0. \tag{6.19}$$

Then the robustness measure $\nu(\omega)$ of the sensitivities of the closed-loop poles is effectively minimized, and the regularity of the pencil is guaranteed within a certain tolerance. Moreover, a bound on the magnitude of the gains is minimized, subject to the regularity of the pencil being maintained, and a bound on the transient responses of the closed-loop system equation is also minimized.

We remark that in place of the measure (6.17) we could choose to minimize the norm of

$$\left([X_q, S_\infty] \begin{bmatrix} I_q & 0 \\ -S_\infty^T X_q & I_{n-q} \end{bmatrix} \right)^{-1} \equiv [S_E S_E^T X_q, S_\infty]^{-1},$$

or even $\nu(\omega)$ itself, in order to minimize the pole sensitivities more precisely. In this case we minimize simultaneously an upper bound on (6.17), which measures the sensitivity of the rank condition (6.2). The procedures for selecting the matrix of eigenvectors X_q remain, in principle, the same.

We observe that the measures (6.18) and (6.19), which guarantee regularity, are implicitly dependent upon the choice of F. This condition essentially fixes the extra degrees of freedom in the solution after eigenvector assignment, and can be treated *explicitly* using the results of Theorem 3. In the next section we describe procedures for determining F and X_q to solve the pole assignment problem and optimize the *robustness* of the closed-loop system.

7. NUMERICAL ALGORITHMS

In essence, now, the objective of the problem of robust pole placement is to select a nondefective system of eigenvectors (each of unit length) to minimize $\|[X_q, S_\infty]^{-1}\|_F$, and to choose the remaining degrees of freedom such that the pencil is as "regular" as possible. From Theorem 3, if an independent set of eigenvectors, given by $X_q = \{x_i\}_1^q$, can be selected such that $x_j \in \mathscr{S}_j$, $j = 1, 2, \dots, q$ [where $\mathscr{S}_j$ is defined by (4.12)], and $\operatorname{rank}[X_q, S_\infty] = n$, then, provided Condition C2 [or equivalently (4.3)] holds, the closed-loop pencil can be made regular by an appropriate choice of a matrix W which satisfies (4.6); the feedback F then is given by (4.5). By the definition of W we have

$$E + MS_\infty S_\infty^T \equiv E + AS_\infty S_\infty^T + U_0 W S_\infty^T,$$

and to optimize regularity, subject to the gains being bounded, we now select W to maximize $\|[E + AS_\infty S_\infty^T + U_0 W S_\infty^T]^{-1}\|_2$, subject to $\|E + AS_\infty S_\infty^T + U_0 W S_\infty^T\|_2 \leqslant c_w$, $c_w > 0$. We observe that the matrices W and X_q can be chosen independently and the conditioning measures (6.17) and (6.18) can be optimized in separate stages.

We now consider practical implementation of these results. The basic numerical algorithm consists of four steps:

Step A: Compute the decompositions of matrices E and B, given by (2.5) and (4.4), respectively, to find S_∞, U_0, U_1, and Z; construct orthonormal bases, comprising the columns of matrices S_j and $\hat{S}_j$ for the space $\mathscr{S}_j \equiv \mathscr{N}\{U_1^T[A - \lambda_j E]\}$ and its complement $\hat{\mathscr{S}}_j$ for $\lambda_j \in \mathscr{L}$, $j = 1, 2, \dots, q$.

Step W: Select the matrix W to minimize $\|[E + AS_\infty S_\infty^T + U_0 W S_\infty^T]^{-1}\|_2$, subject to $\|E + AS_\infty S_\infty^T + U_0 W S_\infty^T\|_2 \leqslant c_w$.

Step X: Select vectors $x_j = S_j v_j \in \mathscr{S}_j$ with $\|x_j\|_2 = 1$, $j = 1, 2, \dots, q$, to minimize $\|[X_q, S_\infty]^{-1}\|_F \equiv \kappa_1 n^{-1/2}$.

Step F: Determine the matrix F by solving the equation

$$ZF\left[X_q, S_\infty\right] = \left[U_0^T(EX_q\Lambda_q - AX_q), W\right].$$

Standard library software with reliable procedures for problems in numerical linear algebra is used to accomplish these steps. We discuss first the initial and final steps, step A and step F, and then describe techniques for the two key steps, step W and step X.

7.1. Step A

The required decompositions of B and E are found by either the QR (Householder) or SVD (singular-value) decomposition method. Construction of the bases for S_j and $\hat{S}_j$ is achieved similarly. With obvious modifications for the descriptor systems, the details of the techniques and operation counts are given in [9].

7.2. Step F

The feedback F is most efficiently and accurately found in two steps. First H is determined by solving the equations

$$ZH = \left[U_0^T(EX_q\Lambda_q - AX_q), W\right].$$

In the case where Z is obtained by the QR process, the coefficient matrix is upper triangular and H is found by back substitution. In the case where Z is given by the SVD method, H is found by straightforward matrix multiplication using Z^{-1}. Then F is computed by solving the equations $[X_q, S_\infty]^T F^T = H^T$ using a direct LU decomposition (or Gaussian elimination) method. This process is numerically stable for a well-conditioned matrix $[X_q, S_\infty]$ (that is, for κ_1 small). Operation counts are equivalent to those given in [9] for nonsingular systems.

7.3. Step W

The objective of this step is to select W to minimize $\|G^{-1}\|_2$ subject to $\|G\|_2 \leqslant c_w$, where $G = E + AS_\infty S_\infty^T + U_0 W S_\infty^T$. In practice the result is achieved only approximately. We observe that it is not necessary to determine W with great accuracy, as we are primarily concerned to ensure simply that G is nonsingular, where $\|G\|_2$ is reasonably bounded. We may write $\|G\|_2 \leqslant \|G_0\|_2 + \|\hat{W}\|_2$, where $G_0 \equiv E + AS_\infty S_\infty^T$ and $\hat{W} = U_0 W S_\infty^T$, and aim to select $\hat{W}$ such that $\|\hat{W}\| \leqslant \beta \|G_0\|_2$, where $\beta > 0$ and $1 + \beta \leqslant c_w / \|G_0\|_2$. The minimum value of $\|G\|_2$, attained with $\hat{W} \equiv 0$, is given by $\|G_0\|_2$, and this condition ensures that the choice of $\hat{W}$ gives only a proportionate increase in the norm of G over its minimum.

A simple algorithm for constructing $\hat{W}$ uses the SVD

$$G_0 \equiv E + AS_\infty S_\infty^T = \tilde{U}\tilde{\Sigma}\tilde{V}^T,$$

where $\tilde{\Sigma} = \operatorname{diag}\{\tilde{\sigma}_i\}$. We then set $\hat{W} = \tilde{U}\Sigma\tilde{V}^T$, where $\Sigma = \operatorname{diag}\{\sigma_i\}$ is chosen to minimize

$$\left\|(\tilde{\Sigma}+\Sigma)^{-1}\right\|_2 \equiv \max_i \left\{(\tilde{\sigma}_i+\sigma_i)^{-1}\right\}$$

subject to

$$\|\Sigma\|_2 \equiv \max_i \{\sigma_i\} \leqslant \beta \max_i \{\tilde{\sigma}_i\}.$$

Then, since $\|G_0\|_2 = \|\tilde{\Sigma}\|_2$ and $\|\hat{W}\|_2 = \|\Sigma\|_2$, it follows that

$$\|G\|_2 \leqslant (1+\beta)\max_i \{\tilde{\sigma}_i\} = (1+\beta)\|G_0\|_2 \tag{7.1}$$

and $\|G^{-1}\|_2 = \|(\tilde{\Sigma}+\Sigma)^{-1}\|_2$ is minimized. A simple choice of Σ is given by setting $\sigma_j = \beta \max_i\{\tilde{\sigma}_i\} - \tilde{\sigma}_j$ if this quantity is positive, or $\sigma_j = 0$ otherwise. Then

$$\max_i \{\tilde{\sigma}_i + \sigma_j\} = \max\{1,\beta\} \max_i \{\tilde{\sigma}_i\},$$

$$\min_i \{\tilde{\sigma}_i + \sigma_i\} \equiv \beta \max_i \{\sigma_i\},$$

and it follows that

$$\|G\|_2 \leqslant \max\{1,\beta\}\, \|G_0\|_2, \tag{7.2}$$

$$\|G^{-1}\|_2 \leqslant \left(\beta \max_i \{\tilde{\sigma}_i\}\right)^{-1} = \beta^{-1}\|G_0\|_2^{-1}, \tag{7.3}$$

and

$$\kappa_2 \equiv \|G^{-1}\|_2 \|G\|_2 \leqslant \begin{cases} \beta^{-1} & \text{if} \quad \beta \leqslant 1, \\ \beta & \text{if} \quad \beta > 1. \end{cases}$$

We see that if $\beta \leqslant 1$, then this choice of $\hat{W}$ does not increase the norm of G over its minimum, whilst achieving an explicit bound on κ_2.

Finally, in order to construct the matrix W from $\hat{W}$, we simply set

$$W = U_0^T \hat{W} S_\infty \equiv U_o^T \tilde{U} \Sigma \tilde{V}^T S_\infty.$$

We observe that for this choice of W, $\|U_0 W S_\infty^T\| \leqslant \|\Sigma\|_2$ and the constructed matrix $G = G_0 + U_0 W S_\infty^T$ satisfies the inequality (7.1). The inequality (7.3) for G^{-1} is, however, only satisfied approximately. Denoting the residual matrix $\Delta = \Sigma - P_1 \Sigma P_2$, where P_1, P_2 are the projection matrices $P_1 = \tilde{U} + U_0 U_0^T \tilde{U}$ and $P_2 = \tilde{V}^T S_\infty S_\infty^T \tilde{V}$, we find that

$$\|G^{-1}\|_2 \leqslant \alpha \beta^{-1} \|G_0\|^{-1}, \qquad \text{where} \quad \alpha \leqslant 1 - \beta^{-1} \|G_0\|^{-1} \|\Delta\|_2,$$

and α is close to unity if $\|\Delta\|_2$ is sufficiently small. The condition number κ_2 remains bounded, in any case. The construction of W is thus accomplished by one SVD, followed by a simple projection. These operations are all numerically stable.

7.4 Step X

To accomplish this step we use one of the iterative methods described in [9] for selecting a set of vectors $\mathbf{x}_j$ from given subspaces $\mathscr{S}_j$ such that the matrix $X = \{\mathbf{x}_j\}$ is *well conditioned*. These procedures all apply update techniques to modify the columns of X in turn, so as to minimize a specific measure of the conditioning.

The most appropriate of the procedures here is Method 1 of [9]. An initial set of independent vectors $\mathbf{x}_j \in \mathscr{S}_j$, $j = 1, 2, \ldots, q$, is chosen to form $X_q = \{\mathbf{x}_j\}_1^q$, and then a rank-one update is made to each column of X_q in turn so as to minimize the measure $\|[X_q, S_\infty]^{-1}\|_F$. For multiple eigenvalues of multiplicity p, an initial set of p orthonormal vectors for the corresponding invariant subspace is selected from $\mathscr{S}_j$, and then rank-p updates are made, such that the basis remains orthonormal, using the Modified Method 1 described in [6]. The only alteration to the process required for the descriptor case is that updates to the columns of S_∞ are not made, and the operation counts are correspondingly reduced.

Method 2/3 of [9] can also be used to determine X_q. This process is generally more efficient than Method 1, but in this case it does not minimize the precise measure we require. With this method an initial set of fully orthonormal vectors, constituting the matrix $[\tilde{X}_q, S_\infty]$, is chosen, and pairs of vectors are updated by applying rotations to minimize the sum of the squares of the distances of these vectors from the required subspaces $\mathscr{S}_j$. In [9] it is shown that if this measure can be made reasonably small, then it provides a good upper bound on $\kappa_1 n^{-1/2} = \|[X_q, S_\infty]^{-1}\|_F$, where $X_q \mathbf{e}_j$ is the projection of $\tilde{X}_q \mathbf{e}_j$ into the subspace $\mathscr{S}_j$.

Overall, we regard Method 1 as the more reliable of these methods, and as only a few iterations are usually required to obtain good solutions, we generally apply this method in practice.

7.5. *Implementation*

The four steps, step A, step W, step X, and step F, of the algorithm have all been implemented using a high-level matrix manipulation system based on stable numerical procedures from standard library software. A small executive package has been developed, and the algorithm has been applied to a number of examples. In the next section results of a test case are given.

8. RESULTS

To illustrate the form of the robust solutions determined by the algorithm described in Section 7, we now give results obtained for a test problem.

TEST EXAMPLE. $n = 5,\ m = 3,\ q = 3,$

$$E = \begin{bmatrix} 0 & 0 & 0 & 1.72 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.82 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1.1 & 0 & 0 & 0 \\ 0 & 0 & 1.56 & 0 & 0 \\ 1.23 & 0 & 0 & 1.98 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.01 & 0 & 0 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 0 & 1.55 & 0 & 0 & 0 \\ 0 & 0 & 1.07 & 0 & -2.5 \\ 0 & 0 & 0 & -1.11 & 0 \end{bmatrix}.$$

We assign the stable eigenvalue set $\mathscr{L} = \{-0.5, -1, -2\}$. We set the tolerance $\beta = 0.2$. Then the computed matrix G actually has condition $\kappa_2^{-1} = 0.141$. Using Method 2/3 to accomplish step X, we find the conditioning of the computed matrix $[X_q, S_\infty]$, after two sweeps of the process, is $\kappa_1 = 4.1683$. The computed feedback matrix F has magnitude $\|F\|_2 = 0.7327$ and is given to five figures by

$$F = \begin{bmatrix} 0.028710 & 0.0 & 0 & 0.35925 & 0.047441 \\ 0.075580 & 0.0 & 0 & -0.24315 & 0.30906 \\ 0.075633 & 0.30991 & 0 & 0.52389 & -0.0221967 \end{bmatrix}.$$

To demonstrate the effects of perturbations, random errors of maximum order $\pm 10^{-3}$ are introduced into the closed-loop system matrix, and the eigenvalues of the resulting matrix pencil are computed. For a robust feedback solution such perturbations should only cause errors of the same order of magnitude in the poles of the closed-loop system. For this test example the absolute errors in the assigned eigenvalues due to these perturbations are $\{0.4_{10^{-4}}, 0.4_{10^{-4}}, 0.2_{10^{-3}}\}$. A maximum relative error of 0.01% is thus obtained in the assigned poles, indicating that the solution is very *robust*.

With Method 1, the results are similar after two sweeps of the procedure. The condition of $[X_q, S_\infty]$ is now $\kappa_1 = 4.6711$, and F has magnitude $\|F\|_2 = 1.7806$ and is given by

$$F = \begin{bmatrix} 0.27000 & 0.0 & 0 & 0.79935 & 1.4705 \\ -0.18432 & 0.0 & 0 & -0.71963 & -0.13059 \\ 0.072572 & 0.30991 & 0 & 0.52885 & -0.15686 \end{bmatrix}.$$

The introduction of perturbations of order 10^{-3} (due to rounding matrix F to three figures) causes perturbations $\{0.1_{10^{-2}}, 0.4_{10^{-3}}, 0.3_{10^{-3}}\}$ in the closed-loop poles, with a maximum relative error of 0.2%, and it is seen that this solution is also highly robust. Additional iterations could be expected to improve the conditioning still further.

5. CONCLUSIONS

Novel necessary conditions for the solution of the pole assignment problem by state feedback in singular systems are given in this paper. These conditions must be satisfied in order to assign the maximum possible number of finite poles by feedback and also obtain a closed-loop system pencil which is regular and nondefective. It can be shown that these conditions are also sufficient for the existence of a feedback which assigns q finite poles with regularity. The prime significance of these results is that they provide conditions for the construction of a feedback which assigns given poles with *guaranteed regularity*, and such that the closed-loop system is *robust*, in the sense that its poles are insensitive to perturbations in the system data.

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