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The Kalman–Yakubovich–Popov inequality for differential-algebraic systems: Existence of nonpositive solutions

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ABSTRACT

The Kalman–Yakubovich–Popov lemma is a central result in systems and control theory which relates the positive semidefiniteness of a Popov function on the imaginary axis to the solvability of a linear matrix inequality. In this paper we prove sufficient conditions for the existence of a nonpositive solution of this inequality for differential-algebraic systems. Our conditions are given in terms of positivity of a modified Popov function in the right complex half-plane. Our results also apply to non-controllable systems. Consequences of our results are bounded real and positive real lemmas for differential-algebraic systems.

1. Introduction

We consider linear time-invariant differential-algebraic control systems

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1}$$

where $sE - A \in \mathbb{K}[s]^{n\times n}$ is assumed to be *regular* (i. e., $\det(sE - A)$ is not the zero polynomial) and $B \in \mathbb{K}^{n\times m}$. The set of such systems is denoted by $\Sigma_{n,m}(\mathbb{K})$ and we write $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. The function $u : \mathbb{R} \to \mathbb{K}^m$ is a *control input*, whereas $x : \mathbb{R} \to \mathbb{K}^n$ denotes the state of the system. The set of all solution trajectories $(x, u) : \mathbb{R} \to \mathbb{K}^n \times \mathbb{K}^m$ induces the *behavior*

$$\mathfrak{B}_{[E,A,B]} := \left\{ (x, u) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^n) \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^m) : \frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu \right\},$$

where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the distributional derivative.

We consider the so-called *modified Popov function*

$$\Psi : \mathbb{C} \setminus \sigma(E, A) \to \mathbb{C}^{m\times m}, \tag{2a}$$

where $\sigma(E, A) = \{\lambda \in \mathbb{C} : \det(\lambda E - A) \neq 0\}$ (see the subsequent section for the notation) and

$$\Psi(\lambda) = \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \tag{2b}$$

with $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$. Note that on $\mathrm{i}\mathbb{R}$, $\Psi(\cdot)$ attains Hermitian values and coincides with the classical Popov function (where the λ in the first factor is replaced by $-\overline{\lambda}$). In contrast to the Popov function, $\Psi(\cdot)$ is neither rational nor meromorphic.

First we revisit a characterization for $\Psi(\cdot) \geq 0$ on the imaginary axis which is strongly related to the feasibility of infinite time horizon linear–quadratic optimal control problems with *zero final state* [1]. For standard state space systems (i. e., with $E = I_n$), the above property can be checked by the famous Kalman–Yakubovich–Popov (KYP) lemma, see [2–5]. The lemma states that if $[I_n, A, B]$ is controllable, then $\Psi(\mathrm{i}\omega) \geq 0$ holds true for all $\mathrm{i}\omega \notin \sigma(A)$ if and only if the so-called *KYP inequality*

$$\begin{bmatrix} A^*P + PA + Q & PB + S \\ B^*P + S^* & R \end{bmatrix} \geq 0 \tag{3}$$

has a Hermitian solution $P \in \mathbb{K}^{n\times n}$.

On the other hand, there are modifications of this lemma for special choices of Q, S, and R. For instance, for $Q = 0_{n\times n}$, $S = C^*$, and $R = D + D^*$, one can show that if $[I_n, A, B]$ is controllable, then with $G(s) = C(sI_n - A)^{-1}B + D$ it holds that

$$\Psi(\lambda) = G(\lambda) + G(\lambda)^* \geq 0 \quad \forall \lambda \in \mathbb{C}^+ \setminus \sigma(A)$$

if and only if the KYP inequality (3) with $Q = 0_{n\times n}$, $S = C^*$, and $R = D + D^*$ has a solution $P \leq 0$. This result is called *positive real lemma* and is of great importance in the context of passivity [6].

The KYP lemma states that positive semi-definiteness of $\Psi(\cdot)$ on $\mathrm{i}\mathbb{R} \setminus \sigma(A)$ is equivalent to the existence of *a* solution of the

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KYP inequality, whereas for the positive real lemma, positive semidefiniteness of $\Psi(\cdot)$ in $\mathbb{C}^+ \setminus \sigma(A)$ is equivalent to the existence of a *nonpositive* solution of the KYP inequality.

Thus, a natural question is whether $\Psi(\cdot) \geq 0$ in $\mathbb{C}^+ \setminus \sigma(A)$ is equivalent to the existence of a solution $P \leq 0$ of the KYP inequality. The great Jan C. Willems first casually claimed in his seminal article [1] that this statement holds true for controllable systems. However, in a successive erratum [7], this claim has been disproved by himself with the aid of a counter-example. Willems further stated in this erratum [7] that the equivalence holds, if an inertial decomposition

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \begin{bmatrix} C_1^*C_1 & C_1^*D_1 \\ D_1^*C_1 & D_1^*D_1 \end{bmatrix} - \begin{bmatrix} C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix}, \tag{4a}$$

exists, where $C_1 \in \mathbb{K}^{m\times n}$, $C_2 \in \mathbb{K}^{p_2\times n}$, $D_1 \in \mathbb{K}^{m\times m}$, $D_2 \in \mathbb{K}^{p_2\times m}$, and

$$G_1(s) := C_1(sI_n - A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)). \tag{4b}$$

However, no proof of this statement has been carried out. In [8], it has been proven that for controllable and stable systems with the additional property that the inverse of $G_1(s)$ is bounded in $\mathbb{C}_+$, all solutions of the KYP inequality are positive definite.

A further condition for the larger class of behavioral systems has been examined by Trentelman and Rapisarda in [9,10]. Under an additional assumption which translates to $\Psi(\cdot) > 0$ on $\mathrm{i}\mathbb{R}$, the existence of nonpositive solutions has been characterized by means of an associated *Pick matrix*. In this paper, we revisit Willems' condition (4) and prove it for differential-algebraic systems. Thereby we are also dealing with non-controllable systems. We further present conditions for all solutions of the KYP inequality being nonnegative. We will apply our results to formulate positive real and bounded real lemmas for differential-algebraic systems.

Notation

We use the standard notations i, $\overline{\lambda}$, A^*, A^{-*}, I_n, $0_{m\times n}$ for the imaginary unit, the complex conjugate of $\lambda \in \mathbb{C}$, the conjugate transpose of a complex matrix and its inverse, the identity matrix of size $n \times n$ and the zero matrix of size $m \times n$ (subscripts may be omitted, if clear from context). The symbol $\mathbb{K}$ stands for either the field $\mathbb{R}$ of real numbers, or the field $\mathbb{C}$ of complex numbers. The closure of $S \subset \mathbb{C}$ is denoted by $\overline{S}$.

By writing $A \geq (\leq) B$ we mean that for two Hermitian matrices $A, B \in \mathbb{K}^{n\times n}$, the matrix $A - B$ is positive semidefinite (negative semidefinite). The following concept, namely equality and semidefiniteness on some subspace will be frequently used in this article.

Definition 1.1 (*Equality and Semidefiniteness on a Subspace*)**.** Let $\mathcal{V} \subseteq \mathbb{K}^n$ be a subspace and $A, B \in \mathbb{K}^{n\times n}$ be Hermitian. Then we write

$$A =_{\mathcal{V}} (\geq_{\mathcal{V}}, \leq_{\mathcal{V}}) B,$$

if we have $v^*(A - B)v = (\geq, \leq)\, 0$ for all $v \in \mathcal{V}$.

The following sets are further used in this article:

$\mathbb{N}_0$	the set of natural numbers including zero
$\mathbb{C}^+, \mathbb{C}^-$	the open sets of complex numbers with positive and negative real parts, resp.
$\mathbb{K}[s], \mathbb{K}(s)$	the ring of polynomials and the field of rational functions with coefficients in $\mathbb{K}$, resp.
$\mathrm{Gl}_n(\mathcal{K})$	the group of invertible $n \times n$ matrices with entries in a field $\mathcal{K}$
$\sigma(A)$	spectrum of $A \in \mathbb{K}^{n\times n}$
$\sigma(E, A)$	$= \{\lambda \in \mathbb{C} : \det(\lambda E - A) = 0\}$, the set of generalized eigenvalues of the matrix pencil $sE - A \in \mathbb{K}[s]^{n\times n}$
$\mathcal{RH}_\infty^{p\times m}$	the space of rational $p \times m$ matrix-valued functions which are bounded in $\mathbb{C}^+$
$\mathcal{L}^2_{\mathrm{loc}}(\mathcal{I}, \mathbb{K}^n)$	the set of measurable and locally square integrable functions $f : \mathcal{I} \to \mathbb{K}^n$ on the set $\mathcal{I} \subseteq \mathbb{R}$.

2. Preliminaries

2.1. Differential-algebraic systems

We first introduce some systems theoretic concepts for differential-algebraic systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$. First we consider notions related to controllability and stabilizability, see also [11,12] and [13, Def. 5.2.2] for the definition and the respective algebraic conditions in terms of the system matrices.

Definition 2.1 (*Controllability and Stabilizability*)**.** A system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ is called

(a) *behaviorally (beh.) stabilizable* if for all $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$, there exists some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with

$$(x(t), u(t)) = (x_1(t), u_1(t)) \quad \text{if } t < 0 \quad \text{and}$$
$$\lim_{t\to\infty} (x(t), u(t)) = 0;$$

(b) *behaviorally (beh.) controllable* if for all $(x_1, u_1), (x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$, there exist some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ and some $T > 0$ with

$$(x(t), u(t)) = \begin{cases} (x_1(t), u_1(t)), & \text{if } t < 0, \\ (x_2(t), u_2(t)), & \text{if } t > T; \end{cases}$$

(c) *completely controllable* if for all $x_0, x_{\mathrm{f}} \in \mathbb{K}^n$, there exist some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ and some $T > 0$ with $x(0) = x_0$ and $x(T) = x_{\mathrm{f}}$.

Moreover, we also consider differential-algebraic systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ which have an additional *output equation*

$$y(t) = Cx(t) + Du(t),$$

where $C \in \mathbb{K}^{p\times n}$ and $D \in \mathbb{K}^{p\times m}$. We denote the set of all such systems by $\Sigma_{n,m,p}(\mathbb{K})$ and we write $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ (or $[E, A, B, C] \in \Sigma_{n,m,p}(\mathbb{K})$ if $D = 0$). The behavior is given by

$$\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B]} \times \mathcal{L}^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{K}^p) : y = Cx + Du \right\},$$

and the expression

$$G(s) = C(sE - A)^{-1}B + D \in \mathbb{K}(s)^{p\times m}$$

is called the *transfer function of* $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$.

Behavioral detectability means that the state can be asymptotically reconstructed from the knowledge of input and output, cf. [13, Def. 5.3.16]. See also [13, Thm. 5.3.17] for an equivalent algebraic criterion.

Definition 2.2 (*Behavioral Detectability*)**.** The system $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ is called *behaviorally (beh.) detectable* if

$$(x_1, u, y),\ (x_2, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \ \Rightarrow \lim_{t\to\infty} (x_1(t) - x_2(t)) = 0.$$

2.2. System equivalence form, system space, and space of consistent initial differential variables

In this paper we consider systems $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ under system equivalence which is defined as follows.

Definition 2.3 (*System Equivalence [11]*)**.** Two systems $[E_i, A_i, B_i] \in \Sigma_{n,m}(\mathbb{K})$, $i = 1, 2$, are called *system equivalent*, if there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ such that

$$\begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} = W \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}.$$

Remark 2.4. Beh. stabilizability, beh. controllability and complete controllability are invariant under system equivalence.

The following *system equivalence form* is basically consisting of a Kalman decomposition of the infinite eigenvalues of $sE - A$ and will be of great importance in our proofs.

Proposition 2.5 (*System Equivalence Form [14]*)**.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given. Then there exist $W, T \in \mathrm{Gl}_n(\mathbb{K})$ such that*

$$\begin{aligned} \begin{bmatrix} s\widetilde{E} - \widetilde{A} & \widetilde{B} \end{bmatrix} &:= W \begin{bmatrix} sE - A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \\ &= \begin{bmatrix} sI_{n_1} - A_{11} & 0 & sE_{13} - A_{13} & B_1 \\ 0 & sE_{22} - I_{n_2} & sE_{23} - A_{23} & B_2 \\ 0 & 0 & sE_{33} - I_{n_3} & 0 \end{bmatrix}, \end{aligned} \quad (5)$$

where $E_{22} \in \mathbb{K}^{n_2 \times n_2}$ and $E_{33} \in \mathbb{K}^{n_3 \times n_3}$ are nilpotent and

$$\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & I_{n_2} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right] \in \Sigma_{n_1+n_2,m}(\mathbb{K})$$

is completely controllable.

Remark 2.6. The system $[E, A, B]$ is beh. stabilizable (beh. controllable) if and only if the ODE system $[I_{n_1}, A_{11}, B_1]$ is stabilizable (controllable) [15].

Next we introduce two fundamental spaces of a system $[E, A, B]$, namely the system space and the space of consistent initial differential variables.

Definition 2.7 (*System Space, Space of Consistent Initial Differential Variables*)**.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given.

(a) The *system space* of $[E, A, B]$ is the smallest subspace $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ such that for all $(x, u) \in \mathfrak{B}_{[E,A,B]}$ it holds that

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \quad \text{for almost all } t \in \mathbb{R}.$$

(b) The *space of consistent initial differential variables* of $[E, A, B]$ is defined by

$$\mathcal{V}_{\mathrm{diff}} := \left\{ x_0 \in \mathbb{K}^n : \exists (x, u) \in \mathfrak{B}_{[E,A,B]} \text{ with } Ex(0) = Ex_0 \right\}.$$

In the next lemma we consider the behavior of the two above spaces under system equivalence.

Lemma 2.8 (*[15,16]*)**.** *Assume that $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ and the space of consistent initial differential variables $\mathcal{V}_{\mathrm{diff}} \subseteq \mathbb{K}^n$ is given. Moreover, let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ and let $\widetilde{\mathcal{V}}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ and $\widetilde{\mathcal{V}}_{\mathrm{diff}} \subseteq \mathbb{K}^n$ be the system space and the space of consistent initial differential variables of $[WET, WAT, WB] \in \Sigma_{n,m}(\mathbb{K})$, respectively. Then we have*

(a)

$$\mathcal{V}_{\mathrm{sys}} = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \cdot \widetilde{\mathcal{V}}_{\mathrm{sys}};$$

(b)

$$\mathcal{V}_{\mathrm{diff}} = T \cdot \widetilde{\mathcal{V}}_{\mathrm{diff}}.$$

Finally, we give an explicit form of the system space and the space of consistent initial differential variables for systems given in system equivalence form.

Lemma 2.9. *Let the system $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ be given in system equivalence form (5). Let $\nu \in \mathbb{N}_0$ be the index of nilpotency of E_{22}. Then we have*

(a)

$$\mathcal{V}_{\mathrm{sys}} = \operatorname{im} \mathcal{M}_{\mathcal{V}_{\mathrm{sys}}},$$

where

$$\mathcal{M}_{\mathcal{V}_{\mathrm{sys}}} = \begin{bmatrix} I_{n_1} & 0 & 0 & \cdots & 0 \\ 0 & -B_2 & -E_{22}B_2 & \cdots & -E_{22}^{\nu-1}B_2 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & I_m & 0 & \cdots & 0 \end{bmatrix}; \quad (6)$$

(b)

$$\mathcal{V}_{\mathrm{diff}} = \mathbb{K}^{n_1+n_2} \times \ker \begin{bmatrix} E_{13} \\ E_{23} \\ E_{33} \end{bmatrix}.$$

Proof. Statement (a) can be concluded from the fact that the equation $\frac{\mathrm{d}}{\mathrm{d}t} E_{33} x_3(t) = x_3(t)$ implies $x_3(t) = 0$, whereas from $\frac{\mathrm{d}}{\mathrm{d}t} E_{22} x_2(t) = x_2(t) + B_2 u(t)$ we obtain

$$x_2(t) = -\sum_{i=0}^{\nu-1} E_{22} B_2 u^{(i)}(t) \quad \text{for almost all } t \in \mathbb{R}.$$

Statement (b) is a consequence of [16, Prop. 2.9]. □

2.3. Differential-algebraic Kalman–Yakubovich–Popov lemma

We recall a version of the KYP lemma for differential-algebraic equations which has recently been shown in [15, Thm. 4.1].

Theorem 2.10 (*KYP Lemma for Differential-algebraic Systems*)**.** *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$, and $Q = Q^* \in \mathbb{K}^{n \times n}$, $S \in \mathbb{K}^{n \times m}$, $R = R^* \in \mathbb{K}^{m \times m}$ be given. Further, let the modified Popov function $\Psi(\cdot)$ be defined as in (2). Then the following statements hold:*

(a) *If the KYP inequality*

$$\begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix} \geq_{\mathcal{V}_{\mathrm{sys}}} 0, \quad P = P^* \quad (7)$$

has a solution $P \in \mathbb{K}^{n \times n}$, then

$$\Psi(\mathrm{i}\omega) \geq 0 \quad \forall \omega \in \mathbb{R} \text{ with } \mathrm{i}\omega \notin \sigma(E, A). \quad (8)$$

(b) *If (8) and at least one of the two properties*

(i) *$\operatorname{rank} \Psi(\mathrm{i}\omega) = m$ for some $\omega \in \mathbb{R}$ with $\mathrm{i}\omega \notin \sigma(E, A)$, and $\max\left\{\operatorname{rank}\begin{bmatrix} \lambda E - A & B \end{bmatrix}, \operatorname{rank}\begin{bmatrix} -\overline{\lambda} E - A & B \end{bmatrix}\right\} = n$ for all $\lambda \in \mathbb{C}$;*

(ii) *$[E, A, B]$ is beh. controllable*

is satisfied, then there exists some $P \in \mathbb{K}^{n \times n}$ that solves the KYP inequality (7).

3. Main results

In this section we investigate the existence of nonpositive solutions of the KYP inequality (7). First, we define what we mean by nonpositivity.

Definition 3.1 (*Nonpositive Solution*)**.** Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the space of consistent initial differential variables $\mathcal{V}_{\text{diff}} \subseteq \mathbb{K}^n$ be given. Then we call a solution P of the KYP inequality (7) *nonpositive*, if

$$E^*PE \leq_{\mathcal{V}_{\text{diff}}} 0. \tag{9}$$

Remark 3.2. We would like to present a brief systems theoretic motivation for our above definition: Let $(x, u) \in \mathfrak{B}_{[E,A,B]}$. By time-invariance and the definition of $\mathcal{V}_{\text{sys}}$ and $\mathcal{V}_{\text{diff}}$, we have $x(t) \in \mathcal{V}_{\text{diff}}$ and $\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \in \mathcal{V}_{\text{sys}}$ for almost all $t \in \mathbb{R}$. With the same argumentation as in [1], the KYP inequality (7) gives rise to the *dissipation inequality*

$$x(t_1)^*E^*PEx(t_1) - x(t_2)^*E^*PEx(t_2) \leq \int_{t_1}^{t_2} w(x(t), u(t))\mathrm{d}t$$
$$\forall (x, u) \in \mathfrak{B}_{[E,A,B]},\ t_1,\ t_2 \in \mathbb{R} \text{ such that } t_1 \leq t_2,$$

where $w(x, u) := x^*Qx + 2\,\mathrm{Re}\,x^*Su + u^*Ru$. In particular, if P is a nonpositive solution, then the above inequality can be simplified to

$$x(t_1)^*E^*PEx(t_1) \leq \int_{t_1}^{t_2} w(x(t), u(t))\mathrm{d}t.$$

First we show the simpler part: The positive semidefiniteness of $\Psi(\cdot)$ in $\mathbb{C}^+ \setminus \sigma(E, A)$ is necessary for the existence of a nonpositive solution. This requires the following technical lemma.

Lemma 3.3. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the systems space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ and the space of consistent initial variables $\mathcal{V}_{\text{diff}} \subseteq \mathbb{K}^n$ be given. Then for all $\lambda \in \mathbb{C} \setminus \sigma(E, A)$ the following statements hold true:*

(a)

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} = \lambda \begin{bmatrix} E & 0 \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix};$$

(b)

$$\mathrm{im} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \subseteq \mathcal{V}_{\text{sys}};$$

(c)

$$\mathrm{im}(\lambda E - A)^{-1}B \subseteq \mathcal{V}_{\text{diff}}.$$

Proof. Statements (a) and (b) have been shown in [15]. It remains to show (c): Assume that $\lambda \in \mathbb{C} \setminus \sigma(E, A)$ and let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ be transformation matrices leading to system equivalence form (5). Then we have

$$\begin{aligned} \mathrm{im}(\lambda E - A)^{-1}B &= \mathrm{im}\, T\big(\lambda\widetilde{E} - \widetilde{A}\big)^{-1}\widetilde{B} \\ &= \mathrm{im}\, T \begin{bmatrix} (\lambda I_{n_1} - A_{11})^{-1}B_1 \\ (\lambda E_{22} - I_{n_2})^{-1}B_2 \\ 0 \end{bmatrix} \\ &\subseteq T \cdot \left(\mathbb{K}^{n_1+n_2} \times \ker \begin{bmatrix} E_{13} \\ E_{23} \\ E_{33} \end{bmatrix} \right) = \mathcal{V}_{\text{diff}}. \quad \square \end{aligned}$$

Theorem 3.4. *Let $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ with the systems space $\mathcal{V}_{\text{sys}} \subseteq \mathbb{K}^{n+m}$ and the space of consistent initial variables $\mathcal{V}_{\text{diff}} \subseteq \mathbb{K}^n$ be given and assume that $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, and $R = R^* \in \mathbb{K}^{m\times m}$. Suppose that $P \in \mathbb{K}^{n\times n}$ is a nonpositive solution of the KYP inequality (7). Then the modified Popov function (2) fulfills*

$$\Psi(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{C}^+ \setminus \sigma(E, A). \tag{10}$$

Proof. By using Lemma 3.3(a), for all $\lambda \in \mathbb{C} \setminus \sigma(E, A)$ we have

$$\begin{aligned} &\begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} A^*PE + E^*PA & E^*PB \\ B^*PE & 0 \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \left(\begin{bmatrix} A^* \\ B^* \end{bmatrix} \begin{bmatrix} PE & 0 \end{bmatrix} + \begin{bmatrix} E^*P \\ 0 \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \right) \\ &\quad \cdot \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \left(\overline{\lambda} \begin{bmatrix} E^* \\ 0 \end{bmatrix} \begin{bmatrix} PE & 0 \end{bmatrix} + \lambda \begin{bmatrix} E^*P \\ 0 \end{bmatrix} \begin{bmatrix} E & 0 \end{bmatrix} \right) \\ &\quad \cdot \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix} \\ &= 2\,\mathrm{Re}(\lambda) \cdot B^*\big(\overline{\lambda}E^* - A^*\big)^{-1}E^*PE\big(\lambda E - A\big)^{-1}B. \end{aligned}$$

The KYP inequality (7) and Lemma 3.3(b) yield the inequality

$$\Psi(\lambda) \geq -2\,\mathrm{Re}(\lambda) \cdot B^*\big(\overline{\lambda}E^* - A^*\big)^{-1}E^*PE\big(\lambda E - A\big)^{-1}B$$
$$\forall \lambda \in \mathbb{C} \setminus \sigma(E, A).$$

Now plugging in some $\lambda \in \mathbb{C}^+ \setminus \sigma(E, A)$ and using (9) and Lemma 3.3(c), we obtain the desired result. $\square$

We now show that, under an additional assumption, the nonnegativity of the modified Popov function also implies the existence of a nonpositive solution. For this we need the following two lemmas.

Lemma 3.5. *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}(\mathbb{K})$ be beh. stabilizable and beh. detectable and assume that its transfer function fulfills $G(s) \in \mathcal{RH}_\infty^{p\times m}$. Then it holds that $\sigma(E, A) \subset \mathbb{C}^-$.*

Proof. Since generalized state-space transformations preserve beh. stabilizability and beh. detectability, we can w. l. o. g. assume that $sE - A$ is given in quasi-Weierstraßform [17], i. e.,

$$sE - A = \begin{bmatrix} sI_r - A_{11} & 0 \\ 0 & sE_{22} - I_{n-r} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where $E_{22} \in \mathbb{K}^{n-r\times n-r}$ is nilpotent, $B_1 \in \mathbb{K}^{r\times m}$, $B_2 \in \mathbb{K}^{n-r\times m}$, $C_1 \in \mathbb{K}^{p\times r}$, and $C_2 \in \mathbb{K}^{p\times n-r}$. This corresponds to a decomposition

$$G(s) = G_{\text{sp}}(s) + G_{\text{poly}}(s),$$

where $G_{\text{sp}}(s) = C_1(sI_r - A_{11})^{-1}B_1 \in \mathbb{K}(s)^{p\times m}$ is strictly proper and $G_{\text{poly}}(s) = C_2(sE_{22} - I_{n-r})^{-1}B_2 \in \mathbb{K}[s]^{p\times m}$. The assumption $G(s) \in \mathcal{RH}_\infty^{p\times m}$ then implies $G_{\text{sp}}(s) = C_1(sI_r - A_{11})^{-1}B_1 \in \mathcal{RH}_\infty^{p\times m}$. Beh. stabilizability and beh. detectability of $[E, A, B, C, D]$ imply stabilizability and detectability of $[I_r, A_{11}, B_1, C_1, D]$. Then [18, Lem. 8.3] yields $\sigma(A_{11}) \subset \mathbb{C}^-$, and hence $\sigma(E, A) \subset \mathbb{C}^-$. $\square$

Lemma 3.6. *Let two systems $[E, A, B, C_1, D_1] \in \Sigma_{n,m,m}(\mathbb{K})$, $[E, A, B, C_2, D_2] \in \Sigma_{n,m,p_2}(\mathbb{K})$ given with transfer functions*

$$G_1(s) = C_1(sE - A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)),$$
$$G_2(s) = C_2(sE - A)^{-1}B + D_2 \in \mathbb{K}(s)^{p_2\times m}.$$

Then the pencil

$$\begin{bmatrix} -sE + A & B \\ C_1 & D_1 \end{bmatrix}$$

is regular and the transfer function of the system

$$[E_{\mathrm{e}}, A_{\mathrm{e}}, B_{\mathrm{e}}, C_{\mathrm{e}}] := \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} 0 \\ -I_m \end{bmatrix}, \begin{bmatrix} C_2 & D_2 \end{bmatrix}\right] \in \Sigma_{n+m,m,p_2}(\mathbb{K}) \tag{11}$$

is $G_{\mathrm{e}}(s) = G_2(s)G_1^{-1}(s)$.

Proof. Regularity of $sE_{\mathrm{e}} - A_{\mathrm{e}}$ has been shown in [19, Lem. 8.9] for $E = I_n$. Our more general result follows by the same argumentation. The remaining result follows from

$$\begin{bmatrix} sE - A & -B \\ -C_1 & -D_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -I_m \end{bmatrix} = \begin{bmatrix} (sE-A)^{-1}BG_1^{-1}(s) \\ G_1^{-1}(s) \end{bmatrix},$$

and consequently

$$\begin{aligned} G_{\mathrm{e}}(s) &= C_{\mathrm{e}}(sE_{\mathrm{e}} - A_{\mathrm{e}})^{-1}B_{\mathrm{e}} \\ &= \begin{bmatrix} C_2 & D_2 \end{bmatrix} \begin{bmatrix} (sE-A)^{-1}BG_1^{-1}(s) \\ G_1^{-1}(s) \end{bmatrix} = G_2(s)G_1^{-1}(s). \quad \square \end{aligned}$$

Theorem 3.7. *Let* $[E, A, B] \in \Sigma_{n,m}(\mathbb{K})$ *with system space* $\mathcal{V}_{\mathrm{sys}} \subseteq \mathbb{K}^{n+m}$ *and space of consistent initial differential variables* $\mathcal{V}_{\mathrm{diff}} \subseteq \mathbb{K}^n$ *be given and assume that* $Q = Q^* \in \mathbb{K}^{n\times n}$, $S \in \mathbb{K}^{n\times m}$, *and* $R = R^* \in \mathbb{K}^{m\times m}$. *Suppose that the modified Popov function* (2) *fulfills* (10). *Further, assume that at least one of the following two assumptions holds:*

(i) $[E, A, B]$ *is beh. stabilizable and the modified Popov function* (2) *satisfies* $\operatorname{rank}\Psi(\mathrm{i}\omega) = m$ *for some* $\omega \in \mathbb{R}$ *with* $\mathrm{i}\omega \notin \sigma(E, A)$;
(ii) $[E, A, B]$ *is beh. controllable.*

Let $C_1 \in \mathbb{K}^{m\times n}$, $C_2 \in \mathbb{K}^{p_2\times n}$, $D_1 \in \mathbb{K}^{m\times m}$, *and* $D_2 \in \mathbb{K}^{p_2\times m}$ *be given such that*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} =_{\mathcal{V}_{\mathrm{sys}}} \begin{bmatrix} C_1^*C_1 & C_1^*D_1 \\ D_1^*C_1 & D_1^*D_1 \end{bmatrix} - \begin{bmatrix} C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix} \tag{12}$$

and

$$G_1(s) := C_1(sE-A)^{-1}B + D_1 \in \mathrm{Gl}_m(\mathbb{K}(s)).$$

Then the solution set of the KYP inequality (7) *is nonempty. Furthermore, the following holds true:*

(a) *There exists a nonpositive solution of the KYP inequality* (7).
(b) *If, furthermore,*

$$\operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} = n + m \quad \forall \lambda \in \mathbb{C}^+, \tag{13}$$

then all solutions of the KYP inequality (7) *are nonpositive.*

Proof. Define $G_2(s) := C_2(sE-A)^{-1}B + D_2 \in \mathbb{K}(s)^{p\times m}$. In the first part of the proof we show the theorem for a special case and then turn to the more general situation.

Step 1: *We show that Theorem 3.7 is satisfied under the additional assumption that* $C_1 = 0$ *and* $D_1 = I_m$:

Step 1.1: *We show that under assumptions of Theorem 3.7 and* $C_1 = 0$, $D_1 = I_m$ *we have* $G_2(s) \in \mathcal{RH}_\infty^{p_2\times m}$: From $C_1 = 0$ and $D_1 = I_m$ we obtain $G_1(s) = I_m$. Hence,

$$\Psi(\lambda) = I_m - G_2^*(\lambda)G_2(\lambda) \geq 0 \quad \forall\, \lambda \in \mathbb{C}^+ \setminus \sigma(E, A).$$

As a consequence, we obtain $G_2(s) \in \mathcal{RH}_\infty^{p_2\times m}$.

Step 1.2: *We show that under the additional assumption that* $C_1 = 0$, $D_1 = I_m$ *and* (13), *the solution set of the KYP inequality* (7) *is nonempty. Moreover, all its solutions are nonpositive:* First, by a continuity argument and the fact that the Popov functions coincide with the modified Popov function on $\mathrm{i}\mathbb{R}$, we obtain that the Popov function is pointwisely positive semi-definite on $\mathrm{i}\mathbb{R}$. Then, by Theorem 2.10, the KYP inequality (7) has a solution $P \in \mathbb{K}^{n\times n}$. We now show that this solution fulfills $E^*PE \leq_{\mathcal{V}_{\mathrm{diff}}} 0$: $C_1 = 0$ and $D_1 = I_m$ implies that

$$\begin{aligned} n + m &= \operatorname{rank}\begin{bmatrix} -\lambda E + A & B \\ 0 & I_m \\ C_2 & D_2 \end{bmatrix} \\ &= m + \operatorname{rank}\begin{bmatrix} \lambda E - A \\ C_2 \end{bmatrix} \quad \forall \lambda \in \mathbb{C}. \end{aligned} \tag{14}$$

Then, by [13, Thm. 5.3.17], $[E, A, B, C_2, D_2]$ is beh. detectable. We further know from Step 1.1 that $G_2(s) \in \mathcal{RH}_\infty^{p_2\times m}$. Since $[E, A, B]$ is beh. stabilizable by assumption, Lemma 3.5 gives $\sigma(E, A) \subset \mathbb{C}^-$. Let $W, T \in \mathrm{Gl}_n(\mathbb{K})$ be matrices leading to system equivalence form (5). Then $\sigma(A_{11}) = \sigma(E, A) \subset \mathbb{C}^-$. Since P and C_2 can be accordingly transformed, and, in particular, nonpositivity is preserved under this transformation, we see that it is no loss of generality to assume that $W = T = I_n$. Let

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix}, \qquad C_2 = \begin{bmatrix} C_{21} & C_{22} & C_{23} \end{bmatrix}$$

be partitioned according the block structure of (5). Assume that $\nu \in \mathbb{N}_0$ is the index of nilpotency of E_{22}. With $\mathcal{M}_{\mathcal{V}_{\mathrm{sys}}}$ as in (6), we obtain from Lemma 2.9(a) that $\mathcal{V}_{\mathrm{sys}} = \operatorname{im}\mathcal{M}_{\mathcal{V}_{\mathrm{sys}}}$. Thus, the KYP inequality (7) reduces to

$$\begin{aligned} 0 &\leq \mathcal{M}_{\mathcal{V}_{\mathrm{sys}}}^* \begin{bmatrix} A^*PE + E^*PA - C_2^*C_2 & E^*PB - C_2^*D_2 \\ B^*PE - D_2^*C_2 & I_m - D_2^*D_2 \end{bmatrix} \mathcal{M}_{\mathcal{V}_{\mathrm{sys}}} \\ &= \left[\begin{array}{ccc|ccc} K_{11} & K_{12} & 0 & N_{1,1} & \dots & N_{1,\nu-1} \\ K_{12}^* & K_{22} & 0 & N_{2,1} & \dots & N_{2,\nu-1} \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \hline N_{1,1}^* & N_{2,1}^* & 0 & M_{1,1} & \dots & M_{1,\nu-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N_{1,\nu-1}^* & N_{2,\nu-1}^* & 0 & M_{1,\nu-1}^* & \dots & M_{\nu-1,\nu-1} \end{array}\right] \end{aligned} \tag{15}$$

where

$$\begin{aligned} K_{11} &= A_{11}^*P_{11} + P_{11}A_{11} - C_{21}^*C_{21}, \\ K_{12} &= P_{11}B_1 - C_{21}^*D_2 + C_{21}^*C_{22}B_2 - A_{11}^*P_{12}E_{22}B_2, \\ K_{22} &= -B_2^*C_{22}^*C_{22}B_2 + B_2^*C_{22}^*D_2 + D_2^*C_{22}B_2 + I_m \\ &\quad - D_2^*D_2 - B_2^*E_{22}^*P_{12}^*B_1 - B_1^*P_{12}E_{22}B_2, \end{aligned}$$

and, for $1 \leq i \leq j \leq \nu - 1$,

$$\begin{aligned} N_{1,i} &= -P_{12}E_{22}^iB_2 + C_{21}^*C_{22}E_{22}^iB_2 - A_{11}^*P_{12}E_{22}^{i+1}B_2, \\ N_{2,i} &= -B_2^*C_{22}^*C_{22}E_{22}^iB_2 + D_2^*C_{22}E_{22}^iB_2 \\ &\quad + B_2^*E_{22}^*P_{12}^*P_{12}E_{22}^iB_1 - B_1^*P_{12}E_{22}^{i+1}B_2, \\ M_{i,j} &= B_2^*(E_{22}^*)^{i+1}P_{22}E_{22}^jB_2 + B_2^*(E_{22}^*)^iP_{22}E_{22}^{j+1}B_2 \\ &\quad - B_2^*(E_{22}^*)^iC_{22}^*C_{22}E_{22}^jB_2. \end{aligned}$$

We will now show by induction that

(1) $P_{12}E_{22}^iB_2 = 0$ for $i = 1, \dots, \nu - 1$, and
(2) $B_2^*(E_{22}^*)^iP_{22}E_{22}^jB_2 = 0$ for $i, j = 1, \dots, \nu - 1$.

First note that by properness of $G_2(s)$ we obtain

$$C_{22}E_{22}B_{22} = \dots = C_{22}E_{22}^{\nu-1}B_{22} = 0.$$

Together with the fact that $E_{22}^\nu = 0$ this implies $M_{\nu-1,\nu-1} = 0$. Since (15) is positive semidefinite, this yields

$$N_{1,\nu-1} = 0, \qquad N_{2,\nu-1} = 0, \qquad M_{1,\nu-1} = \dots = M_{\nu-2,\nu-1} = 0.$$

This however implies $P_{12}E_{22}^{\nu-1}B_2 = 0$ and $B_2^*(E_{22}^*)^i P_{22}E_{22}^{\nu-1}B_2 = 0$, $i = 1, \dots, \nu-1$. Repeating this process inductively leads to statements (1) and (2).

Now define the controllability matrix

$$\mathfrak{C}_2 := \begin{bmatrix} B_2 & E_{22}B_2 & \dots & E_{22}^{\nu-1}B_2 \end{bmatrix}.$$

By the system equivalence form (5) the system $[E_{22}, I_{n_2}, B_2] \in \Sigma_{n_2,m}(\mathbb{K})$ is completely controllable. Thus by [20, Thm. 2-2.1] we obtain

$$\operatorname{rank} \mathfrak{C}_2 = n_2. \tag{16}$$

By statements (1) and (2) it holds that $P_{12}E_{22}\mathfrak{C}_2 = 0$ and $\mathfrak{C}_2^* E_{22}^* P_{22} E_{22} \mathfrak{C}_2 = 0$, hence by (16) this already implies

$$P_{12}E_{22} = 0, \qquad E_{22}^* P_{22} E_{22} = 0. \tag{17}$$

Moreover, since

$$0 \le K_{11} = A_{11}^* P_{11} + P_{11} A_{11} - C_{21}^* C_{21},$$

the Lyapunov inequality $A_{11}^*(-P_{11}) + (-P_{11})A_{11} \le 0$ is satisfied. Using $\sigma(A_{11}) \subset \mathbb{C}^-$, we obtain from [21, Lem. 3.18] that $-P_{11} \ge 0$, i. e., $P_{11} \le 0$. This, together with (17) and the representation of $\mathcal{V}_{\mathrm{diff}}$ as in Lemma 2.9(b) implies

$$E^*PE =_{\mathcal{V}_{\mathrm{diff}}} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{22}^* & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$=_{\mathcal{V}_{\mathrm{diff}}} \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \le_{\mathcal{V}_{\mathrm{diff}}} 0.$$

Step 1.3: *We prove that statement* (a) *holds true for* $C_1 = 0$ *and* $D_1 = I_m$:

By the Kalman decomposition [20, Sect. 2.5], it is no loss of generality to assume that

$$sE - A = \begin{bmatrix} sE_{11} - A_{11} & 0 \\ sE_{21} - A_{21} & sE_{22} - A_{22} \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} C_{21} & 0 \end{bmatrix},$$

where $[E_{11}, A_{11}, B_1, C_{21}] \in \Sigma_{n_1,m,p_2}(\mathbb{K})$ with the system space $\mathcal{V}_{\mathrm{sys},1} \subseteq \mathbb{K}^{n_1+m}$ and the space of consistent initial differential variables $\mathcal{V}_{\mathrm{diff},1} \subseteq \mathbb{K}^{n_1+m}$ is beh. detectable. Then we obtain $G_2(s) = C_{21}(sE_{11} - A_{11})^{-1}B_1 + D_2$. By the assumptions and Theorem 2.10, there exists some solution $P_{11} \in \mathbb{K}^{n_1 \times n_1}$ of the KYP inequality

$$\begin{bmatrix} A_{11}^* P_{11} E_{11} + E_{11} P_{11} A_{11} - C_{21}^* C_{21} & E_{11}^* P_{11} B_1 - C_{21}^* D_2 \\ B_1^* P_{11} E_{11} - D_2^* C_{21} & I_m - D_2^* D_2. \end{bmatrix} \ge_{\mathcal{V}_{\mathrm{sys},1}} 0.$$

Since $[E, A, B, C_2, D_2]$ is beh. stabilizable and beh. detectable, we can make use of the results of Step 1.2 to see that $E_{11}^* P_{11} E_{11} \le_{\mathcal{V}_{\mathrm{diff},1}} 0$.

Using the block triangular structure of E, A, and B, we obtain

$$\mathcal{V}_{\mathrm{sys}} \subseteq \widetilde{\mathcal{V}}_{\mathrm{sys}} := \left\{ \begin{pmatrix} x_1 \\ x_2 \\ u \end{pmatrix} \in \mathbb{K}^{n+m} : \begin{pmatrix} x_1 \\ u \end{pmatrix} \in \mathcal{V}_{\mathrm{sys},1} \right\}, \tag{18}$$
$$\mathcal{V}_{\mathrm{diff}} \subseteq \widetilde{\mathcal{V}}_{\mathrm{diff}} := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{K}^{n} : x_1 \in \mathcal{V}_{\mathrm{diff},1} \right\}.$$

Now define

$$P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{K}^{n \times n}.$$

Then $E^*PE \le_{\widetilde{\mathcal{V}}_{\mathrm{diff}}} 0$ and

$$\begin{bmatrix} A^*PE + E^*PA - C_2^*C_2 & E^*PB - C_2^*D_2 \\ B^*PE - D_2^*C_2 & I_m - D_2^*D_2. \end{bmatrix}$$
$$= \begin{bmatrix} A_{11}^* P_{11} E_{11} + E_{11}^* P_{11} A_{11} - C_{21}^* C_{21} & 0 & E_{11}^* P_{11} B_1 - C_{21}^* D_2 \\ 0 & 0 & 0 \\ B_1^* P_{11} E_{11} - D_2^* C_{21} & 0 & I_m - D_2^* D_2. \end{bmatrix}$$
$$\ge_{\widetilde{\mathcal{V}}_{\mathrm{sys}}} 0.$$

Hence, by (18), we see that P solves the KYP inequality (7).

Step 2: *We infer the general result:*

Using Lemma 3.3(b), we obtain that the modified Popov function fulfills

$$\Psi(\lambda) = \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}$$
$$= \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C_1^*C_1 & C_1^*D_1 \\ D_1^*C_1 & D_1^*D_1 \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}$$
$$- \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}^* \begin{bmatrix} C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix} \begin{bmatrix} (\lambda E - A)^{-1}B \\ I_m \end{bmatrix}$$
$$= G_1^*(\lambda)G_1(\lambda) - G_2^*(\lambda)G_2(\lambda).$$

In particular, the function $G_e(s) = G_2(s)G_1^{-1}(s) \in \mathbb{K}(s)^{p_2 \times m}$ fulfills

$$I_m - G_e^*(\lambda)G_e(\lambda) \ge 0 \tag{19}$$

for all $\lambda \in \mathbb{C}^+ \setminus \sigma(E, A)$ which are no poles of $G_e(s)$. Consider the matrices E_e, A_e, B_e, C_e as in (11). By Lemma 3.6 we obtain that the pencil $sE_e - A_e \in \mathbb{K}[s]^{n+m \times n+m}$ is regular and

$$C_e(sE_e - A_e)^{-1}B_e = G_2(s)G_1^{-1}(s) = G_e(s).$$

The structure of E_e, A_e, and B_e yields

$$\operatorname{rank} \begin{bmatrix} \lambda E_e - A_e & B_e \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \lambda E - A & B \end{bmatrix} + m \quad \forall \lambda \in \mathbb{C}.$$

Hence, we can use [13, Thm. 5.2.30] and beh. stabilizability of $[E, A, B]$ to see that $[E_e, A_e, B_e]$ is as well beh. stabilizable.

Let $\mathcal{V}_{\mathrm{sys,e}}$ and $\mathcal{V}_{\mathrm{diff,e}}$ be the system space and the space of consistent initial differential variables of the system (11), respectively. Consider the KYP inequality

$$\begin{bmatrix} A_e^* P_e E_e + E_e^* P_e A_e + Q_e & E_e^* P_e B_e + S_e \\ B_e^{\mathsf{T}} P_e E_e + S_e^* & R_e \end{bmatrix} \ge_{\mathcal{V}_{\mathrm{sys,e}}} 0, \quad P_e = P_e^*, \tag{20}$$

where

$$Q_e = -\begin{bmatrix} C_2^*C_2 & C_2^*D_2 \\ D_2^*C_2 & D_2^*D_2 \end{bmatrix}, \qquad S_e = 0, \qquad R_e = I_m.$$

The modified Popov function associated to the KYP inequality (20) then reads $\Psi_e(\lambda) = I_m - G_e^*(\lambda)G_e(\lambda)$. By (19) and the results from Step 1, there exists a nonpositive solution $P_e \in \mathbb{K}^{n+m \times n+m}$ of (20). Partition

$$P_e = \begin{bmatrix} P & P_{e,12} \\ P_{e,21} & P_{e,22} \end{bmatrix}$$

with $P \in \mathbb{K}^{n \times n}$. Then the equation $E_e^* P_e E_e \le_{\mathcal{V}_{\mathrm{diff,e}}} 0$ implies $E^*PE \le_{\mathcal{V}_{\mathrm{diff}}} 0$. Hence, it suffices for the desired statement (a) to prove that P indeed solves the KYP inequality (7).

By definition of $\mathcal{V}_{\mathrm{sys,e}}$, E_e, A_e, and B_e, we have

$$\mathcal{V}_{\mathrm{sys,e}} = \left\{ \begin{pmatrix} x \\ u \\ z \end{pmatrix} \in \mathbb{K}^{n+2m} : \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{V}_{\mathrm{sys}} \text{ and } z = C_1x + D_1u \right\}.$$

Thus, the matrix

$$T_e := \begin{bmatrix} I_n & 0 \\ 0 & I_m \\ C_1 & D_1 \end{bmatrix}$$

fulfills $T_e \mathcal{V}_{sys} = \mathcal{V}_{sys,e}$. Therefore, we obtain

$$\begin{aligned} 0 &\le_{\mathcal{V}_{sys}} T_e^* \begin{bmatrix} A_e^* P_e E_e + E_e^* P_e A_e + Q_e & E_e^* P_e B_e + S_e \\ B_e^* P_e E_e + S_e^* & R_e \end{bmatrix} T_e \\ &=_{\mathcal{V}_{sys}} \begin{bmatrix} A^*PE + E^*PA + C_1^*C_1 - C_2^*C_2 & E^*PB + C_1^*D_1 - C_2^*D_2 \\ B^*PE + D_1^*C_1 - D_2^*C_2 & D_1^*D_1 - D_2^*D_2 \end{bmatrix} \\ &=_{\mathcal{V}_{sys}} \begin{bmatrix} A^*PE + E^*PA + Q & E^*PB + S \\ B^*PE + S^* & R \end{bmatrix}. \end{aligned}$$

Finally, for statement (b) it remains to be shown that all solutions of the KYP inequality are nonpositive, if (13) holds.

Now let $P \in \mathbb{K}^{n\times n}$ be a solution of the KYP inequality (7). Then

$$P_e = \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}$$

solves the KYP inequality (20). Further, by elementary column operations, we obtain

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} -\lambda E_e + A_e & B_e \\ C_e & 0 \\ 0 & I_m \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} + m \\ &= n + 2m \quad \forall \lambda \in \overline{\mathbb{C}^+}. \end{aligned}$$

Hence we are in the situation of Step 1.2, which leads to

$$\begin{bmatrix} E^*PE & 0 \\ 0 & 0 \end{bmatrix} = E_e^* P_e E_e \le_{\mathcal{V}_{diff,e}} 0,$$

and therefore, $E^*PE \le_{\mathcal{V}_{diff}} 0$. This completes the proof. □

4. Positive real and bounded real lemma

Now we apply our results to characterize positive realness and bounded realness of transfer functions.

Definition 4.1 (*Positive Realness, Bounded Realness*)**.** A rational function $G(s) \in \mathbb{K}(s)^{p\times m}$ is called

(a) *positive real* if $p = m$, $G(s)$ has no poles in $\mathbb{C}^+$ and

$$G(\lambda) + G^*(\lambda) \ge 0 \quad \forall\, \lambda \in \mathbb{C}^+;$$

(b) *bounded real* if $G(s) \in \mathcal{RH}_\infty^{p\times m}$ and

$$I_m - G^*(\lambda)G(\lambda) \ge 0 \quad \forall\, \lambda \in \mathbb{C}^+.$$

Remark 4.2. (a) Bounded realness is equivalent to the $\mathcal{H}_\infty$ norm [21, Sec 4.3] of $G(s)$ being less than or equal to one.

(b) The notions of positive realness and bounded realness are quite traditional in linear systems theory [6]. Note that, though the terminology suggests, we actually do not impose any realness assumption in the previous definition. Realness actually does not provide any noteworthy additional property, hence we use the traditional notion also for the complex case.

Theorem 4.3 (*Differential-algebraic Positive Real Lemma*)**.** *Let $[E, A, B, C, D] \in \Sigma_{n,m,m}(\mathbb{K})$ with the system space $\mathcal{V}_{sys} \subseteq \mathbb{K}^{n+m}$, the space of consistent initial differential variables $\mathcal{V}_{diff} \subseteq \mathbb{K}^n$, and the transfer function $G(s) \in \mathbb{K}(s)^{m\times m}$ be given.*

(a) *If there exists some Hermitian $P \in \mathbb{K}^{n\times n}$ with $E^*PE \ge_{\mathcal{V}_{diff}} 0$ and*

$$\begin{bmatrix} A^*PE + E^*PA & E^*PB - C^* \\ B^*PE - C & -D^* - D \end{bmatrix} \le_{\mathcal{V}_{sys}} 0, \tag{21}$$

then $G(s)$ is positive real.

(b) *Assume that $G(s)$ is positive real and at least one of the following properties is satisfied:*

(i) *$[E, A, B]$ is beh. stabilizable and $G(\mathrm{i}\omega)^* + G(\mathrm{i}\omega) > 0$ for some $\omega \in \mathbb{R}$ with $\mathrm{i}\omega \notin \sigma(E, A)$;*

(ii) *$[E, A, B]$ is beh. controllable.*

Then there exists some Hermitian $P \in \mathbb{K}^{n\times n}$ with $E^*PE \ge_{\mathcal{V}_{diff}} 0$ fulfilling (21).

(c) *If, in addition to the assumptions in (b), $[E, A, B, C, D]$ is beh. detectable, then all Hermitian $P \in \mathbb{K}^{n\times n}$ with (21) fulfill $E^*PE \ge_{\mathcal{V}_{diff}} 0$.*

Proof. Statement (a) follows from Theorem 3.4 with $Q = 0$, $S = C^*$, $R = D + D^*$ and a substitution $P \rightsquigarrow -P$.

To prove (b), we simply check whether we meet the assumptions of Theorem 3.7 after the above substitution: In the terminology of Theorem 3.7 we have $C_1 = C_2 = \frac{1}{\sqrt{2}}C$, $D_1 = \frac{1}{\sqrt{2}}(D + I_m)$ and $D_2 = \frac{1}{\sqrt{2}}(D - I_m)$, $G_1(s) = \frac{1}{\sqrt{2}}(G(s) + I_m)$, and $G_2(s) = \frac{1}{\sqrt{2}}(G(s) - I_m)$. Invertibility of $G_1(s)$ is a consequence of the fact that for all $\lambda \in \mathbb{C}^+$ and $u \in \mathbb{C}^m$ it holds that

$$u^*(I_m + G(\lambda))u = \|u\|^2 + \tfrac{1}{2}u^*(G^*(\lambda) + G(\lambda))u \ge u^*u.$$

To prove (c), it suffices to check that (13) is true. This however is a consequence of [13, Thm. 5.3.17] together with

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} -\lambda E + A & B \\ \frac{1}{\sqrt{2}}C & \frac{1}{\sqrt{2}}(D + I_m) \\ \frac{1}{\sqrt{2}}C & \frac{1}{\sqrt{2}}(D - I_m) \end{bmatrix} \\ &= m + \operatorname{rank} \begin{bmatrix} -\lambda E + A \\ C \end{bmatrix} \quad \forall \lambda \in \mathbb{C}. \quad \square \end{aligned}$$

Remark 4.4. We note that in [22], a similar version of the positive real lemma has been elaborated. However, for the existence of a solution $P \ge 0$ fulfilling (21), minimality (which implies complete controllability and the respectively dual property for observability) of $[E, A, B, C, D]$ has been presumed.

By using substitutions $P \rightsquigarrow -P$, $C_1 = 0_{m\times n}$, $D_1 = I_m$, $C_2 = C$, $D_2 = D$, $G_1(s) = I_m$ and $G_2(s) = G(s)$ in Theorem 3.4, and further making use of (14), we can analogously conclude the differential-algebraic bounded real lemma.

Theorem 4.5 (*Differential-algebraic Bounded Real Lemma*)**.** *Let $[E, A, B, C, D] \in \Sigma_{n,m,p}$ with the system space $\mathcal{V}_{sys} \subseteq \mathbb{K}^{n+m}$, the space of consistent initial differential variables $\mathcal{V}_{diff} \subseteq \mathbb{K}^n$, and the transfer function $G(s) \in \mathbb{K}(s)^{p\times m}$ be given.*

(a) *If there exists some Hermitian $P \in \mathbb{K}^{n\times n}$ with $E^*PE \ge_{\mathcal{V}_{diff}} 0$ and*

$$\begin{bmatrix} A^*PE + E^*PA + C^*C & E^*PB + C^*D \\ B^*PE + D^*C & D^*D - I_m \end{bmatrix} \le_{\mathcal{V}_{sys}} 0, \tag{22}$$

then $G(s)$ is bounded real.

(b) *Assume that $G(s)$ is bounded real and at least one of the following properties is satisfied:*

(i) *$[E, A, B]$ is beh. stabilizable and $I_m - G(\mathrm{i}\omega)^*G(\mathrm{i}\omega) > 0$ for some $\omega \in \mathbb{R}$ with $\mathrm{i}\omega \notin \sigma(E, A)$;*

(ii) *$[E, A, B]$ is beh. controllable.*

Then there exists some Hermitian $P \in \mathbb{K}^{n\times n}$ with $E^*PE \ge_{\mathcal{V}_{diff}} 0$ fulfilling (22).

(c) *If, in addition to the assumptions in* (b), $[E, A, B, C, D]$ *is beh. detectable, then all Hermitian* $P \in \mathbb{K}^{n\times n}$ *with* (22) *fulfill* $E^*PE \geq_{\mathcal{V}_{\rm diff}} 0$.

5. Conclusions

In this paper we have proven a new sufficient condition for the existence of nonpositive solutions of the Kalman–Yakubovich–Popov (KYP) inequality. This condition, going back to Willems, has been extended to differential-algebraic systems. The general results on the KYP inequality have further been used to formulate differential-algebraic versions of the bounded real and positive real lemma.

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