Bull. Sci. math. 139 (2015) 538–557

Contents lists available at ScienceDirect

Bulletin des Sciences Mathématiques

www.elsevier.com/locate/bulsci

Nonuniform dichotomy spectrum and reducibility for nonautonomous equations ☆

Jifeng Chu [a,b,*], Fang-Fang Liao [a], Stefan Siegmund [b], Yonghui Xia [c], Weinian Zhang [d]

[a] *Department of Mathematics, College of Science, Hohai University, Nanjing 210098, China*
[b] *Center for Dynamics & Institute for Analysis, Department of Mathematics, TU Dresden, Germany*
[c] *College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua, China*
[d] *College of Mathematics, Sichuan University, Chengdu, China*

ARTICLE INFO

Article history:
Received 5 February 2014
Available online 7 November 2014

MSC:
37D25
37B55

Keywords:
Nonuniform dichotomy spectrum
Nonautonomous differential equations
Nonuniform exponential dichotomy
Reducibility

ABSTRACT

For nonautonomous linear differential equations with nonuniform hyperbolicity, we introduce a definition for nonuniform dichotomy spectrum, which can be seen as a generalization of Sacker–Sell spectrum. We prove a spectral theorem and use the spectral theorem to prove a reducibility result.

☆ Jifeng Chu was supported by the National Natural Science Foundation of China (Grant Nos. 11171090, 11271078) and the Alexander von Humboldt-Stiftung of Germany. Yonghui Xia was supported by the Natural Science Foundation of China (Grant No. 11271333) and China Postdoctoral Science Foundation (Grant No. 2014M562320).
* Corresponding author.
E-mail addresses: jifengchu@126.com (J. Chu), liaofangfang8178@sina.com (F.-F. Liao), stefan.siegmund@tu-dresden.de (S. Siegmund), yhxia@zjnu.cn, xiadoc@163.com (Y. Xia), wnzhang@scu.edu.cn (W. Zhang).

http://dx.doi.org/10.1016/j.bulsci.2014.11.002
0007-4497/

1. Introduction

Let $\mathfrak{L}^1_{loc} = \mathfrak{L}^1_{loc}(\mathbb{R}, \mathbb{R}^{N\times N})$, $N \in \mathbb{N}$, be the space of locally integrable matrix functions. Given $A \in \mathfrak{L}^1_{loc}$, we consider the following nonautonomous linear differential equation

$$x' = A(t)x. \tag{1.1}$$

Let $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{N\times N}$, $(t, s) \mapsto \Phi(t, s)$ denote the associated evolution operator of (1.1), i.e., $\Phi(t, s)x(s) = x(t)$ for every $t, s \in \mathbb{R}$, where x is any solution of (1.1). Clearly, $\Phi(t, \tau)\Phi(\tau, s) = \Phi(t, s)$, $t, \tau, s \in \mathbb{R}$.

The classical notion of exponential dichotomy introduced by Perron in [29] plays an important role in the study of dynamical behaviors of (1.1), particularly in what concerns the study of stable and unstable invariant manifolds, and therefore has attracted much attention (see, for example, [16,17,23,28,30,32–34,38]) during the last few decades. We also refer to the books [15,20,24] for details and further references related to exponential dichotomies. On the other hand, as Barreira and Valls mentioned in [12], the classical notion of exponential dichotomy substantially restricts some dynamics and it is important to look for more general types of hyperbolic behaviors. During the last several years, inspired by both the classical notion of exponential dichotomy and the notion of nonuniformly hyperbolic trajectory introduced by Pesin in [7,8], Barreira and Valls introduced the concept of nonuniform exponential dichotomy and investigated some related problems [9–11,13]. In particular, they discussed the existence and the smoothness of invariant manifolds for nonautonomous differential equations, a version of the Grobman–Hartman theorem, the existence of center manifolds and the theory of Lyapunov regularity. A more general nonuniform exponential dichotomy, considered in [5,6,18], admits different growth rates in the uniform and nonuniform parts. Barreira and Valls explained in [7,12] that, from the point of view of ergodic theory, almost all linear variational equations have a nonuniform exponential behavior.

Based on the study of classical exponential dichotomy, the dichotomy spectral theory was introduced by Sacker and Sell in [33]. The dichotomy spectrum is an important object in the theory of dynamical systems because the spectral intervals, together with the spectral manifolds, completely describe the dynamical skeleton of a linear system. A spectral theory based on finite-time hyperbolicity has been studied in [14,21,22]. Some other related results can be seen from [1–4,16,25,30,31,35]. The dichotomy spectral theory was applied in [36,37] to give block diagonalization and normal forms for nonautonomous differential equations.

In this paper we investigate the dichotomy spectrum in the setting of nonuniform exponential dichotomies, called *nonuniform dichotomy spectrum*. We show the topological structure of the nonuniform dichotomy spectrum and give the corresponding decomposition in spectral manifolds. At last, we use the above results on spectrum to prove the reducibility for (1.1), i.e., a kinematical similarity to a diagonal system in proper blocks.

2. Nonuniform dichotomy spectrum

Let $\Phi(t,s)$ be the evolution operator of (1.1). An *invariant projector* of (1.1) is defined to be a function $P:\mathbb{R}\to\mathbb{R}^{N\times N}$ of projections $P(t)$, $t\in\mathbb{R}$, such that

$$P(t)\Phi(t,s)=\Phi(t,s)P(s),\quad \text{for } t,s\in\mathbb{R}.$$

Clearly, P is continuous due to the identity $P\equiv\Phi(\cdot,s)P(s)\Phi(s,\cdot)$.

Definition 2.1. We say that Eq. (1.1) admits a nonuniform exponential dichotomy on $\mathbb{R}$, if there exist constants $\alpha>0$, $K>0$, $\varepsilon\geq 0$ with $\varepsilon<\alpha$ and an invariant projector P such that

$$\left\|\Phi(t,s)P(s)\right\|\leq Ke^{-\alpha(t-s)}e^{\varepsilon|s|},\quad t\geq s, \tag{2.1}$$

and

$$\left\|\Phi(t,s)Q(s)\right\|\leq Ke^{\alpha(t-s)}e^{\varepsilon|s|},\quad t\leq s, \tag{2.2}$$

where $Q(t)=\mathrm{Id}-P(t)$ is the complementary projection.

When one can take $\varepsilon=0$ in (2.1)–(2.2), we say that Eq. (1.1) admits a (uniform) exponential dichotomy, and thus a classical exponential dichotomy is a particular case of a nonuniform one. As illustrated in [12], in most cases, the nonuniform part $e^{\varepsilon|s|}$ in (2.1)–(2.2) cannot be removed. In particular, Barreira and Valls have proved that in finite-dimensional spaces essentially any linear equation with nonzero Lyapunov exponents admits a nonuniform exponential dichotomy, and as a consequence of Oseledets multiplicative ergodic theorem [26], the nonuniformity of most equations is very small. See [12] for the details. We remark that [7, Theorem 1.4.2] indicates that the condition $\varepsilon<\alpha$ is reasonable, which means that the nonuniform parts are small.

For example, if $\lambda>3a>0$, then the linear equation in $\mathbb{R}^2$ given by

$$u'=(-\lambda-at\sin t)u,\qquad v'=(\lambda+at\sin t)v,$$

admits a nonuniform exponential dichotomy, but it does not admit a (uniform) exponential dichotomy.

Lemma 2.2. *The projector of Eq.* (1.1) *can be chose as*

$$\tilde{P}=\begin{pmatrix} I_{N_1} & 0\\ 0 & 0_{N_2}\end{pmatrix},$$

with $N_1=\dim\operatorname{im}\tilde{P}$ *and* $N_2=\dim\ker\tilde{P}$*, and the fundamental matrix* $X(t)$ *can be chosen appropriately such that the estimates* (2.1)–(2.2) *can be rewritten as*

$$\left\|X(t)\tilde{P}X^{-1}(s)\right\|\leq Ke^{-\alpha(t-s)}e^{\varepsilon|s|},\quad t\geq s, \tag{2.3}$$

and

$$\|X(t)\tilde{Q}X^{-1}(s)\| \le Ke^{\alpha(t-s)}e^{\varepsilon|s|}, \quad t \le s, \tag{2.4}$$

where $\tilde{Q} = \mathrm{Id} - \tilde{P}$.

Proof. Let $\tau \in \mathbb{R}$ be arbitrarily chosen but fixed. Then there exists a non-singular matrix $T \in \mathbb{R}^{N\times N}$ such that

$$TP(\tau)T^{-1} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0_{N_2} \end{pmatrix}.$$

For the evolution operator $\Phi(t,\tau)$ of (1.1), we define

$$X(t) := \Phi(t,\tau)T^{-1}, \quad t \in \mathbb{R},$$

and

$$\tilde{P} := \begin{pmatrix} I_{N_2} & 0 \\ 0 & 0_{N_2} \end{pmatrix} = TP(\tau)T^{-1}.$$

Then

$$\begin{aligned} \|X(t)\tilde{P}X^{-1}(s)\| &= \|\Phi(t,\tau)T^{-1}\tilde{P}T\Phi^{-1}(s,\tau)\| \\ &= \|\Phi(t,\tau)P(\tau)\Phi^{-1}(s,\tau)\|. \end{aligned} \tag{2.5}$$

On the other hand, one has

$$\begin{aligned} \|\Phi(t,s)P(s)\| &= \|\Phi(t,\tau)\Phi(\tau,s)P(s)\| \\ &= \|\Phi(t,\tau)P(\tau)\Phi(\tau,s)\| \\ &= \|\Phi(t,\tau)P(\tau)\Phi^{-1}(s,\tau)\|. \end{aligned} \tag{2.6}$$

It follows from (2.5)–(2.6) that (2.1)–(2.2) can be rewritten in the equivalent form (2.3)–(2.4). $\square$

For fixed $\gamma \in \mathbb{R}$, consider the shifted system

$$\dot{x} = [A(t) - \gamma I]x, \tag*{$(2.7)_\gamma$}$$

which has the evolution operator

$$\Phi_\gamma(t,s) := e^{-\gamma(t-s)}\Phi(t,s).$$

If $(2.7)_\gamma$ admits a nonuniform exponential dichotomy, then its invariant projector $P(t)$ is also invariant for (1.1). The dichotomy estimates are equivalent to

$$\left\|\Phi(t,s)P(s)\right\| \le Ke^{(\gamma-\alpha)(t-s)}e^{\varepsilon|s|}, \quad t \ge s \tag{2.8}$$

and

$$\left\|\Phi(t,s)Q(s)\right\| \le Ke^{(\gamma+\alpha)(t-s)}e^{\varepsilon|s|}, \quad t \le s. \tag{2.9}$$

By Lemma 2.2, equivalently,

$$X_\gamma(t) := e^{-\gamma t}X(t) = e^{-\gamma t}\Phi(t,\tau)T^{-1}$$

is the fundamental matrix of the shifted system $(2.7)_\gamma$, and its invariant projection is

$$\tilde{P} = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The corresponding estimates are equivalent to

$$\left\|X_\gamma(t)\tilde{P}X_\gamma^{-1}(s)\right\| \le Ke^{-\alpha(t-s)}e^{\varepsilon|s|}, \quad t \ge s, \tag{2.10}$$

and

$$\left\|X_\gamma(t)\tilde{Q}X_\gamma^{-1}(s)\right\| \le Ke^{\alpha(t-s)}e^{\varepsilon|s|}, \quad t \le s. \tag{2.11}$$

We will use the estimates (2.8)–(2.9) as well as the equivalent formulation (2.10)–(2.11).

Definition 2.3. The nonuniform dichotomy spectrum of (1.1) is the set

$$\Sigma_{NED}(A) = \left\{\gamma \in \mathbb{R} : (2.7)_\gamma \text{ admits no nonuniform exponential dichotomy}\right\},$$

and the *resolvent set* $\rho_{NED}(A) = \mathbb{R} \setminus \Sigma_{NED}(A)$ is its complement.

Let $\Sigma_{ED}(A)$ denote the classical dichotomy spectrum of (1.1). Obviously, $\Sigma_{NED}(A) \subset \Sigma_{ED}(A)$. For $\gamma \in \rho_{NED}(A)$, define

$$\mathcal{S}_\gamma := \left\{(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^N \,:\, \sup_{t\ge 0}\left\{\left\|\Phi(t,\tau)\xi\right\|e^{-\gamma t}\right\}e^{-\varepsilon\tau} < \infty\right\},$$

and

$$\mathcal{U}_\gamma := \left\{(\tau,\xi) \in \mathbb{R} \times \mathbb{R}^N \,:\, \sup_{t\le 0}\left\{\left\|\Phi(t,\tau)\xi\right\|e^{-\gamma t}\right\}e^{-\varepsilon\tau} < \infty\right\}.$$

One may readily verify that $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ are linear integral manifold of (1.1). As defined in [35], a nonempty set $\mathcal{W} \subset \mathbb{R} \times \mathbb{R}^N$ is a *linear integral manifold of* (1.1) if: (a) it is *invariant*, i.e., $(\tau, \xi) \in \mathcal{W} \Rightarrow (t, \Phi(t,\tau)\xi) \in \mathcal{W}$ for all $t \in \mathbb{R}$, (b) for every $\tau \in \mathbb{R}$, the *fiber* $\mathcal{W}(\tau) = \{\xi \in \mathbb{R}^N : (\tau, \xi) \in \mathcal{W}\}$ is a linear subspace of $\mathbb{R}^N$.

At first glance, $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ are not well defined because they seem to depend on the constant ε, which may not be unique in (2.1)–(2.2). However, the following result ensures that $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ are well defined in the setting of a nonuniform exponential dichotomy and they do not depend on the choice of the constant ε. First we recall that the invariant projector P is unique for (1.1) following the arguments in [20, Chapter 2]. Although the arguments in [20] are done in the setting of exponential dichotomies, it is not difficult to verify that they are also applicable to the case of nonuniform exponential dichotomies.

Lemma 2.4. *Assume that* $(2.7)_\gamma$ *admits a nonuniform exponential dichotomy with invariant projector* P *for* $\gamma \in \rho_{NED}(A)$*. Then*

$$\mathcal{S}_\gamma = \operatorname{im} P, \qquad \mathcal{U}_\gamma = \ker P \quad \textit{and} \quad \mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{R} \times \mathbb{R}^N.$$

Proof. We show only $\mathcal{S}_\gamma = \operatorname{im} P$. The fact $\mathcal{U}_\gamma = \ker P$ is analog and the fact $\mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{R} \times \mathbb{R}^N$ is clear.

First we show $\mathcal{S}_\gamma \subset \operatorname{im} P$. Let $\tau \in \mathbb{R}$ and $\xi \in \mathcal{S}_\gamma(\tau)$. Then there exists a positive constant C such that

$$\|\Phi(t,\tau)\xi\| \le Ce^{\gamma t}e^{\varepsilon\tau}, \quad t \ge \tau.$$

We write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \operatorname{im} P(\tau)$ and $\xi_2 \in \ker P(\tau)$. We show that $\xi_2 = 0$. The invariance of P implies for $t \in \mathbb{R}$ that we have the equivalence

$$\xi_2 = \Phi_\gamma(\tau,t)\Phi_\gamma(t,\tau)Q(\tau)\xi = \Phi_\gamma(\tau,t)Q(t)\Phi_\gamma(t,\tau)\xi.$$

Since $(2.7)_\gamma$ admits a nonuniform exponential dichotomy, the following inequality holds

$$\|\Phi_\gamma(\tau,t)Q(t)\| \le Ke^{\alpha(\tau-t)}e^{\varepsilon|t|}, \quad t \ge \tau.$$

Thus

$$\begin{aligned}
\|\xi_2\| &\le Ke^{\alpha(\tau-t)}e^{\varepsilon|t|}\|\Phi_\gamma(t,\tau)\xi\| \\
&\le KCe^{\alpha(\tau-t)}e^{\varepsilon|t|}e^{\gamma t}e^{\varepsilon\tau}e^{-\gamma(t-\tau)} \\
&= KCe^{\alpha(\tau-t)}e^{\varepsilon|t|}e^{\varepsilon\tau}e^{\gamma\tau} \\
&\le KCe^{(\alpha-\varepsilon)(\tau-t)}e^{\varepsilon|\tau|}e^{\varepsilon\tau}e^{\gamma\tau}
\end{aligned}$$

which implies that $\xi_2 = 0$ by letting $t \to \infty$, since $\varepsilon < \alpha$.

Next we show $\operatorname{im} P \subset \mathcal{S}_\gamma$. Let $\tau \in \mathbb{R}$ and $\xi \in \operatorname{im} P(\tau)$, i.e., $P(\tau)\xi = \xi$. The nonuniform exponential dichotomy implies that

$$\|\Phi_\gamma(t,\tau)\xi\| \le K e^{-\alpha(t-\tau)} e^{\varepsilon|\tau|}\|\xi\| \le K e^{\varepsilon|\tau|}\|\xi\|, \quad t \ge \tau,$$

since $\alpha > 0$, which implies that

$$\|\Phi(t,\tau)\xi\| \le K e^{-\gamma(t-\tau)} e^{\varepsilon|\tau|}\|\xi\|,$$

and hence $\xi \in \mathcal{S}_\gamma(\tau)$. □

Lemma 2.5. *The resolvent set is open, i.e., for every $\gamma \in \rho_{NED}(A)$, there exists a constant $\beta = \beta(\gamma) > 0$ such that $(\gamma - \beta, \gamma + \beta) \subset \rho_{NED}(A)$. Furthermore,*

$$\mathcal{S}_\zeta = \mathcal{S}_\gamma, \qquad \mathcal{U}_\zeta = \mathcal{U}_\gamma \quad \text{for } \zeta \in (\gamma - \beta, \gamma + \beta).$$

Proof. Let $\gamma \in \rho_{NED}(A)$. Then $(2.7)_\gamma$ admits a nonuniform exponential dichotomy, i.e., the estimates (2.10)–(2.11) hold with an invariant projector $\tilde{P}$ and constants $K \ge 0$, $\alpha > 0$ and $\varepsilon \ge 0$. For $\beta := \alpha/2 > 0$ and $\zeta \in (\gamma - \beta, \gamma + \beta)$ we have

$$X_\zeta(t) = e^{(\gamma-\zeta)t} X_\gamma(t).$$

Now $\tilde{P}$ is also an invariant projector for

$$\dot{x} = [A(t) - \zeta I]x$$

and we have the estimates

$$\|X_\zeta(t)\tilde{P}X_\zeta^{-1}(s)\| \le K e^{(\gamma-\zeta-\alpha)(t-s)} e^{\varepsilon|s|} \le K e^{-\beta(t-s)} e^{\varepsilon|s|}, \quad t \ge s,$$

and

$$\|X_\zeta(t)\tilde{P}X_\zeta^{-1}(s)\| \le K e^{(\gamma-\zeta+\alpha)(t-s)} e^{\varepsilon|s|} \le K e^{\beta(t-s)} e^{\varepsilon|s|}, \quad t \le s.$$

Hence $\zeta \in \rho_{NED}(A)$ and therefore $\rho_{NED}(A)$ is an open set. □

Corollary 2.6. *$\Sigma_{NED}(A)$ is a closed set.*

Using the facts proved above, we can obtain the following result, whose proof is similar to [35, Lemma 3.2], and therefore we omit the proof here.

Lemma 2.7. *Let $\gamma_1, \gamma_2 \in \rho_{NED}(A)$ with $\gamma_1 < \gamma_2$. Then $\mathcal{F} = \mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}$ is a linear integral manifold which satisfies exactly one of the following two alternatives and the statements given in each alternative are equivalent:*

Alternative I	*Alternative II*
(A) $\mathcal{F} = \mathbb{Z} \times \{0\}$.	(A') $\mathcal{F} \neq \mathbb{Z} \times \{0\}$.
(B) $[\gamma_1, \gamma_2] \subset \rho_{NED}(A)$.	(B') *There is a* $\zeta \in (\gamma_1, \gamma_2) \cap \Sigma_{NED}(A)$.
(C) $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *and* $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$.	(C') $\dim \mathcal{S}_{\gamma_1} < \dim \mathcal{S}_{\gamma_2}$.
(D) $\mathcal{S}_{\gamma} = \mathcal{S}_{\gamma_2}$ *and* $\mathcal{U}_{\gamma} = \mathcal{U}_{\gamma_2}$ *for* $\gamma \in [\gamma_1, \gamma_2]$.	(D') $\dim \mathcal{U}_{\gamma_1} > \dim \mathcal{U}_{\gamma_2}$.

Now we are in a position to state and prove our main theorem on the nonuniform dichotomy spectrum.

Theorem 2.8. *The nonuniform dichotomy spectrum* $\Sigma_{NED}(A)$ *of* (1.1) *is a disjoint union of* n *closed intervals* (*called spectral intervals*) *where* $0 \leq n \leq N$, *i.e., either* $\Sigma_{NED}(A) = \emptyset$, *or* $\Sigma_{NED}(A) = \mathbb{R}$, *or* $\Sigma_{NED}(A)$ *is in one of the four cases*

$$\Sigma_{NED}(A) = \begin{Bmatrix} [a_1, b_1] \\ \text{or} \\ (-\infty, b_1] \end{Bmatrix} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \begin{Bmatrix} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{Bmatrix},$$

where $0 < a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$. *Furthermore, choose a*

$$\gamma_0 \in \rho_{NED}(A) \quad \text{with } (-\infty, \gamma_0) \subset \rho_{NED}(A); \tag{2.12}$$

otherwise define $\mathcal{U}_{\gamma_0} := \mathbb{R} \times \mathbb{R}^N$, $\mathcal{S}_{\gamma_0} := \mathbb{R} \times \{0\}$, *and choose a*

$$\gamma_n \in \rho_{NED}(A) \quad \text{with } (\gamma_n, +\infty) \subset \rho_{NED}(A); \tag{2.13}$$

otherwise define $\mathcal{U}_{\gamma_n} := \mathbb{R} \times \{0\}$, $\mathcal{S}_{\gamma_0} := \mathbb{R} \times \mathbb{R}^N$. *Then the sets*

$$\mathcal{W}_0 := \mathcal{S}_{\gamma_0} \quad \text{and} \quad \mathcal{W}_{n+1} := \mathcal{S}_{\gamma_n}$$

are both linear integral manifolds of (1.1). *For* $n \geq 2$, *choose* $\gamma_i \in \rho_{NED}(A)$ *with*

$$b_i < \gamma_i < a_{i+1} \quad \text{for } i = 1, \ldots, n-1. \tag{2.14}$$

Then for every $i = 1, \ldots, n-1$ *the intersection*

$$\mathcal{W}_i := \mathcal{U}_{\gamma_{i-1}} \cap \mathcal{S}_{\gamma_i}$$

is a linear integral manifold of (1.1) *with* $\dim \mathcal{W}_i \geq 1$. *Moreover, those linear integral manifolds* $\mathcal{W}_i$, $i = 0, \ldots, n+1$, *called spectral manifolds, are independent of the choice of* $\gamma_0, \ldots, \gamma_n$ *in* (2.12), (2.13) *and* (2.14) *and satisfy*

$$\mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{n+1} = \mathbb{R} \times \mathbb{R}^N$$

in the sense of Whitney sum, i.e., $\mathcal{W}_0 + \cdots + \mathcal{W}_{n+1} = \mathbb{R} \times \mathbb{R}^N$ *but* $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{R} \times \{0\}$ *for* $i \neq j$.

Proof. Recall that the resolvent set $\rho_{NED}(A)$ is open and therefore $\Sigma_{NED}(A)$ is the disjoint union of closed intervals. Next we will show that $\Sigma_{NED}(A)$ consists of at most N intervals. Indeed, if $\Sigma_{NED}(A)$ contains $N+1$ components, then one can choose a collection of points $\zeta_1, \dots, \zeta_N$ in $\rho_{NED}(A)$ such that $\zeta_1 < \cdots < \zeta_N$ and each of the intervals $(-\infty, \zeta_1), (\zeta_1, \zeta_2), \dots, (\zeta_{N-1}, \zeta_N), (\zeta_N, \infty)$ has nonempty intersection with the spectrum $\Sigma_{NED}(A)$. Now Alternative II of Lemma 2.7 implies

$$0 \le \dim \mathcal{S}_{\zeta_1} < \cdots < \dim \mathcal{S}_{\zeta_N} \le N$$

and therefore either $\dim \mathcal{S}_{\zeta_1} = 0$ or $\dim \mathcal{S}_{\zeta_N} = N$ or both. Without loss of generality, $\dim \mathcal{S}_{\zeta_N} = N$, i.e., $\mathcal{S}_{\zeta_N} = \mathbb{R} \times \mathbb{R}^N$. Assume that $\dot{x} = [A(t) - \zeta_N I]x$ admits a strong nonuniform exponential dichotomy with invariant projector $P \equiv \mathrm{Id}$, then

$$\dot{x} = \big[A(t) - \zeta I\big]x$$

also admits a nonuniform exponential dichotomy with the same projector for every $\zeta > \zeta_N$. Now we have the conclusion $(\zeta_N, \infty) \subset \rho_{NED}(A)$, which is a contradiction. This proves the alternatives for $\Sigma_{NED}(A)$.

Due to Lemma 2.7, the sets $\mathcal{W}_0, \dots, \mathcal{W}_{n+1}$ are linear integral manifolds. To prove that $\dim \mathcal{W}_1 \ge 1, \dots, \dim \mathcal{W}_n \ge 1$ for $n \ge 1$, we assume that $\dim \mathcal{W}_1 = 0$, i.e., $\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1} = \mathbb{R} \times \{0\}$. If $(0, b_1]$ is a spectral interval this implies that $\mathcal{S}_{\gamma_1} = \mathbb{R} \times \{0\}$. The projector of the nonuniform exponential dichotomy of

$$\dot{x} = \big[A(t) - \gamma_1 I\big]x$$

is 0 and then we get the contradiction $(-\infty, \gamma_1) \subset \rho_{NED}(A)$. If $[a_1, b_1]$ is a spectral interval then $[\gamma_0, \gamma_1] \cap \Sigma_{NED}(A) \neq \emptyset$ and Alternative II of Lemma 2.7 yields a contradiction. Therefore $\dim \mathcal{W}_1 \ge 1$ and similarly $\dim \mathcal{W}_n \ge 1$. Furthermore for $n \ge 3$ and $i = 2, \dots, n-1$ one has $[\gamma_{i-1}, \gamma_i] \cap \Sigma_{NED}(A) \neq \emptyset$ and again Alternative II of Lemma 2.7 yields $\dim \mathcal{W}_i \ge 1$.

For $i < j$ we have $\mathcal{W}_i \subset \mathcal{S}_{\gamma_i}$ and $\mathcal{W}_j \subset \mathcal{U}_{\gamma_{j-1}} \subset \mathcal{U}_{\gamma_i}$. Using Lemma 2.4, we have $\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{S}_{\gamma_i} \cap \mathcal{U}_{\gamma_i} = \mathbb{R} \times \{0\}$ and therefore $\mathcal{W}_i \cap \mathcal{W}_j = \mathbb{R} \times \{0\}$ for $i \neq j$.

To show that $\mathcal{W}_0 \oplus \cdots \oplus \mathcal{W}_{n+1} = \mathbb{R} \times \mathbb{R}^N$, recall the monotonicity relations $\mathcal{S}_{\gamma_0} \subset \cdots \subset \mathcal{S}_{\gamma_n}$, $\mathcal{U}_{\gamma_0} \supset \cdots \supset \mathcal{U}_{\gamma_n}$, and the identity $\mathcal{S}_\gamma \oplus \mathcal{U}_\gamma = \mathbb{R} \times \mathbb{R}^N$ for $\gamma \in \mathbb{R}$. Therefore $\mathbb{R} \times \mathbb{R}^N = \mathcal{W}_0 \times \mathcal{U}_{\gamma_0}$. Now we have

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^N &= \mathcal{W}_0 + \mathcal{U}_{\gamma_0} \cap [\mathcal{S}_{\gamma_1} + \mathcal{U}_{\gamma_1}] \\ &= \mathcal{W}_0 + [\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}] + \mathcal{U}_{\gamma_1} \\ &= \mathcal{W}_0 + \mathcal{W}_1 + \mathcal{U}_{\gamma_1}. \end{aligned}$$

Doing the same for $\mathcal{U}_{\gamma_1}$, we get

$$\begin{aligned}\mathbb{R}\times\mathbb{R}^N &= \mathcal{W}_0+\mathcal{W}_1+\mathcal{U}_{\gamma_1}\cap[\mathcal{S}_{\gamma_2}+\mathcal{U}_{\gamma_2}]\\ &= \mathcal{W}_0+\mathcal{W}_1+[\mathcal{U}_{\gamma_1}\cap\mathcal{S}_{\gamma_2}]+\mathcal{U}_{\gamma_2}\\ &= \mathcal{W}_0+\mathcal{W}_1+\mathcal{W}_2+\mathcal{U}_{\gamma_2},\end{aligned}$$

and mathematical induction yields $\mathbb{R}\times\mathbb{R}^N=\mathcal{W}_0+\cdots+\mathcal{W}_{n+1}$. To finish the proof, let $\widetilde{\gamma}_0,\dots,\widetilde{\gamma}_n\in\rho_{NED}(A)$ be given with the properties (2.12), (2.13) and (2.14). Then Alternative I of Lemma 2.7 implies

$$\mathcal{S}_{\gamma_i}=\mathcal{S}_{\widetilde{\gamma}_i}\quad\text{and}\quad\mathcal{U}_{\gamma_i}=\mathcal{U}_{\widetilde{\gamma}_i}\quad\text{for } i=0,\dots,n$$

and therefore the linear integral manifolds $\mathcal{W}_0,\dots,\mathcal{W}_{n+1}$ are independent of the choice of $\gamma_0,\dots,\gamma_n$ in (2.12), (2.13) and (2.14). □

Definition 2.9. We say that (1.1) is nonuniformly exponentially bounded if there exist constants $K>0$, $\varepsilon\ge 0$ and $a\ge 0$ such that

$$\big\|\Phi(t,s)\big\|\le Ke^{a|t-s|}e^{\varepsilon|s|},\quad\text{for } t,s\in\mathbb{R}. \tag{2.15}$$

Lemma 2.10. *Assume that (1.1) is nonuniformly exponentially bounded. Then $\Sigma_{NED}(A)$ is a bounded closed set and $\Sigma_{NED}(A)\subset[-a,a]$.*

Proof. Assume that (2.15) holds. Let $\gamma>a$ and $\alpha:=\gamma-a>0$, estimate (2.15) implies

$$\big\|\Phi_\gamma(t,s)\big\|\le Ke^{-\alpha(t-s)}e^{\varepsilon|s|},\quad\text{for } t\ge s$$

and therefore $(2.7)_\gamma$ admits a nonuniform exponential dichotomy with invariant projector $P=\mathrm{Id}$. We have $\gamma\in\rho_{NED}(A)$ and similarly for $\gamma<-a$, therefore $\Sigma_{NED}(A)\subset[-a,a]$. □

Corollary 2.11. *Assume that (1.1) is nonuniformly exponentially bounded. Then the nonuniform dichotomy spectrum $\Sigma_{NED}(A)$ of (1.1) is the disjoint union of n closed intervals where $0\le n\le N$, i.e.,*

$$\Sigma_{NED}(A)=[a_1,b_1]\cup[a_2,b_2]\cup\cdots\cup[a_{n-1},b_{n-1}]\cup[a_n,b_n],$$

where $a_1\le b_1<a_2\le b_2<\cdots<a_n\le b_n$.

Finally we present an example to illustrate that $\Sigma_{NED}(A)\ne\Sigma_{ED}(A)$ can occur.

Example 2.1. Consider the scalar equation $\dot{x}=A(t)x$ with $A(t)=\lambda_0+at\sin t$, where $\lambda_0<a<0$ ($|a|\ll 1$ is sufficiently small). Then $\Sigma_{NED}(A)=[\lambda_0+a,\lambda_0-a]$ and $\Sigma_{ED}(A)=\mathbb{R}$.

In fact, the evolution operator of $\dot{x} = A(t)x$ is given by

$$\Phi(t,s) = e^{\lambda_0(t-s)-a\cos t(t-s)-as(\cos t-\cos s)+a(\sin t-\sin s)}.$$

For any $\gamma \in \mathbb{R}$, the evolution operator of the shifted system $\dot{x} = [A(t) - \gamma]x$ is given by

$$\Phi_\gamma(t,s) = e^{(-\gamma+\lambda_0)(t-s)-a\cos t(t-s)-as(\cos t-\cos s)+a(\sin t-\sin s)}. \tag{2.16}$$

For any $\gamma \in (\lambda_0 - a, +\infty)$, it follows from (2.16) that

$$\left|\Phi_\gamma(t,s)\right| \le e^{2|a|}e^{-(\gamma-\lambda_0+a)(t-s)}e^{2|a|\cdot|s|}, \quad t \ge s,$$

which implies that the shifted system $\dot{x} = [A(t) - \gamma]x$ admits a nonuniform exponential dichotomy with $P = 1$, by taking

$$K = e^{2|a|}, \qquad \alpha = \gamma - \lambda_0 + a > 0, \qquad \varepsilon = 2|a| > 0.$$

Thus,

$$(\lambda_0 - a, +\infty) \subset \rho_{NED}(A). \tag{2.17}$$

For any $\widetilde{\gamma} \in (-\infty, \lambda_0 + a)$, it follows from (2.16) that

$$\left|\Phi_{\widetilde{\gamma}}(t,s)\right| \le e^{2|a|}e^{(-\widetilde{\gamma}+\lambda_0+a)(t-s)}e^{2|a|\cdot|s|}, \quad \text{for } t \le s,$$

which implies that the shifted system $\dot{x} = [A(t) - \widetilde{\gamma}]x$ admits a nonuniform exponential dichotomy with $P = 0$, by taking

$$K = e^{2|a|}, \qquad \widetilde{\alpha} = -\widetilde{\gamma} + \lambda_0 + a > 0, \qquad \varepsilon = 2|a| > 0.$$

Thus,

$$(-\infty, \lambda_0 + a) \subset \rho_{NED}(A). \tag{2.18}$$

It follows from (2.17)–(2.18) that

$$(-\infty, \lambda_0 + a) \cup (\lambda_0 - a, +\infty) \subset \rho_{NED}(A),$$

which implies that

$$\Sigma_{NED}(A) \subset [\lambda_0 + a, \lambda_0 - a].$$

Now we show that

$$[\lambda_0 + a, \lambda_0 - a] \subset \Sigma_{NED}(A).$$

To show this, we first prove that $\lambda_0 - a \in \Sigma_{NED}(A)$. On the contrary, assume that $\gamma_2 = \lambda_0 - a$ such that $\dot{x} = [A(t) - \gamma_2]x$ admits a nonuniform exponential dichotomy. We know that either the projector $P = 0$ or $P = 1$. If $P = 1$, then there exist constants $K, \alpha > 0$ and $\varepsilon > 0$ such that the following estimate holds

$$\begin{aligned}\left|\Phi_{\gamma_2}(t,s)\right| &= e^{[-\gamma_2+\lambda_0](t-s)-a\cos t(t-s)-as(\cos t-\cos s)+a(\sin t-\sin s)}\\ &\le Ke^{-\alpha(t-s)}e^{\varepsilon|s|}, \quad t \ge s.\end{aligned}$$

Substituting $\gamma_2 = \lambda_0 - a$, we have

$$e^{a(1-\cos t)(t-s)-as(\cos t-\cos s)+a(\sin t-\sin s)} \le Ke^{-\alpha(t-s)}e^{\varepsilon|s|}, \quad t \ge s,$$

which yields a contradiction for $s = 0$ and $t \to +\infty$. If $P = 0$, the dichotomy estimate is

$$e^{a(1-\cos t)(t-s)-as(\cos t-\cos s)+a(\sin t-\sin s)} \le Ke^{\alpha(t-s)}e^{\varepsilon|s|}, \quad t \le s,$$

which also yields a contradiction for $s = -(2k-1)\pi$ and $t = -2k\pi$ and $k \to +\infty$. Therefore, $\lambda_0 - a \in \Sigma_{NED}(A)$. Analogously, we can prove that $\lambda_0 + a \in \Sigma_{NED}(A)$. By Theorem 2.8, we know that $\Sigma_{NED}(A)$ is an interval. Thus, for any $\gamma \in [\lambda_0 + a, \lambda_0 - a]$, it follows from the connectedness that $\gamma \in \Sigma_{NED}(A)$. Consequently,

$$[\lambda_0 + a, \lambda_0 - a] \subset \Sigma_{NED}(A).$$

Therefore, $\Sigma_{NED}(A) = [\lambda_0 + a, \lambda_0 - a]$.

On the other hand, we can show that, for any $\gamma \in (\lambda_0 - a, +\infty) \cup (-\infty, \lambda_0 + a)$, the shifted system $\dot{x} = [A(t) - \gamma]x$ admits no exponential dichotomy. From the above proof, $\Sigma_{NED}(A) = [\lambda_0 + a, \lambda_0 - a]$, which implies that the shifted system $\dot{x} = [A(t) - \gamma]x$ admits no nonuniform exponential dichotomy. Consequently, for $\gamma \in [\lambda_0 + a, \lambda_0 - a]$, the shifted system $\dot{x} = [A(t) - \gamma]x$ admits no exponential dichotomy. Therefore, $\Sigma_{ED}(A) = \mathbb{R}$.

3. Reducibility

In this section we employ Theorem 2.8 to prove a reducibility result. We refer to [19,27,36] and the references therein for some reducibility results in the setting of classic exponential dichotomies. First we recall the definition of *kinematic similarity* and several results in [36].

Definition 3.1. Given $A, B \in \mathfrak{L}^1_{loc}$. Eq. (1.1) is said to be kinematically similar to another system

$$y' = B(t)y, \tag{3.1}$$

if there exists an absolutely continuous function $S : \mathbb{R} \to GL_N(\mathbb{R})$ with

$$\sup_{t\in\mathbb{R}} \| S(t) \| < \infty \quad \text{and} \quad \sup_{t\in\mathbb{R}} \| S^{-1}(t) \| < \infty$$

which satisfies the differential equation

$$S'(t) = A(t)S - SB(t). \tag{3.2}$$

The transformation $x = S(t)y$ which transforms (1.1) into (3.1) is called the Lyapunov transformation.

Lemma 3.2. *(See [36, Lemma A.5].) Let $P \in \mathbb{R}^{N\times N}$ be a symmetric projection and $X : \mathbb{R} \to GL_N(\mathbb{R})$ be an absolutely continuous matrix. Then:*

(A) *The mapping*

$$\widetilde{R} : \mathbb{R} \to \mathbb{R}^{N\times N}, \quad t \mapsto PX(t)^T X(t)P + QX(t)^T X(t)Q$$

is absolutely continuous and $\widetilde{R}(t)$ is a positive definite, symmetric matrix for every $t \in \mathbb{R}$. Moreover there is a unique absolutely continuous function $R : \mathbb{R} \to \mathbb{R}^{N\times N}$ of positive definite symmetric matrices $R(t)$, $t \in \mathbb{R}$, with

$$R(t)^2 = \widetilde{R}(t), \qquad PR(t) = R(t)P.$$

(B) *The mapping*

$$S : \mathbb{R} \to \mathbb{R}^{N\times N}, \quad t \mapsto X(t)R(t)^{-1}$$

is absolutely continuous and $S(t)$ is invertible, satisfying

$$\begin{gathered} S(t)PS^{-1}(t) = X(t)PX^{-1}(t), \\ S(t)QS^{-1}(t) = X(t)QX^{-1}(t), \\ \| S(t) \| \le \sqrt{2}, \\ \| S^{-1}(t) \| \le \left[\| X(t)PX^{-1}(t) \|^2 + \| X(t)QX^{-1}(t) \|^2 \right]^{\frac{1}{2}}, \quad t \in \mathbb{R}. \end{gathered}$$

In the setting of classical exponential dichotomies, $S^{-1}(t)$ is bounded, which follows from the properties $\|X(t)PX^{-1}(t)\| < \infty$ and $\|X(t)QX^{-1}(t)\| < \infty$. However, in the setting of nonuniform exponential dichotomies, $S^{-1}(t)$ can be unbounded, because

$$\| X(t)PX^{-1}(t) \| \le Ke^{\varepsilon t}, \quad t \ge 0.$$

Such a fact will make difficulties to the analysis. To overcome it, we introduce the new notions of *nonuniform Lyapunov transformation* and *nonuniform kinematical similarity*.

Definition 3.3. Suppose that $S : \mathbb{R} \to GL_N(\mathbb{R})$ is an absolutely continuous matrix. $S(t)$ is said to be a nonuniform Lyapunov matrix if there exists a constant $M = M_\varepsilon > 0$ such that

$$\|S(t)\| \le Me^{\varepsilon|t|} \quad \text{and} \quad \|S^{-1}(t)\| \le Me^{\varepsilon|t|}, \quad \text{for all } t \in \mathbb{R}.$$

Definition 3.4. Eq. (1.1) is said to be nonuniformly kinematically similar to Eq. (3.1) if there exists a nonuniform Lyapunov matrix $S(t)$ satisfying the differential equation (3.2). For short, we write (1.1) $\sim$ (3.1) or $A(t) \sim B(t)$.

For the sake of comparison, we denote kinematical similarity by (1.1) $\approx$ (3.1) or $A(t) \approx B(t)$.

Definition 3.5. We say that system (1.1) is reducible, if it is nonuniformly kinematically similar to system (3.1) whose coefficient matrix $B(t)$ has the block form

$$\begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix}, \tag{3.3}$$

where $B_1(t)$ and $B_2(t)$ are both matrices of smaller size than $B(t)$.

In [20], Coppel proved that if system (1.1) admits an exponential dichotomy, then there exists a Lyapunov transformation such that $A(t) \approx B(t)$ and $B(t)$ has the block form (3.3), i.e., system (1.1) is *reducible*. The following theorem shows that if system (1.1) admits a nonuniform exponential dichotomy, then there exists a nonuniform Lyapunov transformation such that $A(t) \sim B(t)$ and $B(t)$ has the block form (3.3), i.e., system (1.1) is reducible.

Lemma 3.6. *Suppose that system (1.1) admits a strong nonuniform exponential dichotomy with the form of estimates (2.1)–(2.2) and rank($\tilde{P}$) = k ($0 \le k \le N$). If there exists a nonuniform Lyapunov transformation $S(t)$ such that $A(t) \sim B(t)$, then system (3.1) also admits a nonuniform exponential dichotomy and the projector has the same rank.*

Proof. Suppose that $S(t)$ is the nonuniform Lyapunov matrix with $\|S(t)\| \le Me^{\varepsilon|t|}$, $\|S^{-1}(t)\| \le Me^{\varepsilon|t|}$ and such that $A(t) \sim B(t)$. Let $Y(t) = S(t)X(t)$. Then it is easy to see that $Y(t)$ is the fundamental matrix of system (3.1). To prove that system (3.1) admits a nonuniform exponential dichotomy, we first consider the case $t \ge 0$. For $t \ge 0$,

$$\begin{aligned} \|Y(t)\tilde{P}Y^{-1}(s)\| &= \|S(t)X(t)\tilde{P}X^{-1}(s)S^{-1}(s)\| \\ &\le \|S(t)\| \cdot \|X(t)\tilde{P}X^{-1}(s)\| \cdot \|S^{-1}(s)\| \\ &\le KM^2e^{\varepsilon|t|}e^{-\alpha(t-s)}e^{\varepsilon|s|}e^{\varepsilon|s|} \\ &\le KM^2e^{\varepsilon(t-s)}e^{-\alpha(t-s)}e^{3\varepsilon|s|} \\ &= KM^2e^{-(\alpha-\varepsilon)(t-s)}e^{3\varepsilon|s|}, \quad t \ge s. \end{aligned} \tag{3.4}$$

A similar argument shows that

$$\|Y(t)\tilde{Q}Y^{-1}(s)\| \leq KM^2 e^{(\alpha+\varepsilon)(t-s)} e^{3\varepsilon|s|}, \quad t \leq s. \tag{3.5}$$

It follows from (3.4)–(3.5) that (3.1) admits a nonuniform exponential dichotomy for $t \geq 0$ due to $\varepsilon < \alpha$. Similarly, we see that system (3.1) admits a nonuniform exponential dichotomy for $t \leq 0$. Thus (3.1) admits a nonuniform exponential dichotomy and the rank of the projector is k. □

Corollary 3.7. *Assume that there exists a nonuniform Lyapunov transformation $S(t)$ such that $A(t) \sim B(t)$. Then $\Sigma_{NED}(A) = \Sigma_{NED}(B)$.*

Theorem 3.8. *Assume that Eq. (1.1) admits a nonuniform exponential dichotomy of the form (2.1)–(2.2) with invariant projector $P(t) \neq 0, I$. Then (1.1) is nonuniformly kinematically similar to a decoupled system*

$$\dot{x} = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} x$$

for some locally integrable matrix functions

$$B_1 : \mathbb{R} \to \mathbb{R}^{N_1 \times N_1} \quad \text{and} \quad B_2 : \mathbb{R} \to \mathbb{R}^{N_2 \times N_2}$$

where $N_1 := \dim \operatorname{im} P$ and $N_2 := \dim \ker P$.

Proof. Since Eq. (1.1) admits a nonuniform exponential dichotomy of the form (2.1)–(2.2) with invariant projector $P(t) \neq 0, I$, by Lemma 2.2, we can choose a fundamental matrix $X(t)$ and the projector $P_0 = \begin{pmatrix} I_{N_1} & 0 \\ 0 & 0 \end{pmatrix}$ $(0 < k < N)$ such that the estimates (2.10)–(2.11) hold. For the given nonsingular matrix $X(t)$, by Lemma 3.2, there exists an absolutely continuous and invertible matrix $S(t)$ satisfying

$$\begin{aligned} \|S(t)\| &\leq \sqrt{2}, \\ \|S^{-1}(t)\| &\leq \left[\|X(t)\tilde{P}X^{-1}(t)\|^2 + \|X(t)\tilde{Q}X^{-1}(t)\|^2\right]^{\frac{1}{2}}, \end{aligned}$$

which combined with the estimates (2.3)–(2.4) gives

$$\begin{aligned} \|S(t)\| &\leq \sqrt{2} \leq Me^{\varepsilon|t|}, \\ \|S^{-1}(t)\| &\leq \left[\|X(t)\tilde{P}X^{-1}(t)\|^2 + \|X(t)\tilde{Q}X^{-1}(t)\|^2\right]^{\frac{1}{2}} \leq \sqrt{2}Ke^{\varepsilon|t|}. \end{aligned}$$

Thus we can take $M = M_\varepsilon \geq \max\{\sqrt{2}, \sqrt{2}K\}$ such that

$$\|S(t)\| \leq Me^{\varepsilon|t|}, \qquad \|S^{-1}(t)\| \leq Me^{\varepsilon|t|},$$

which implies that $S(t)$ is a nonuniform Lyapunov matrix. Setting

$$B(t) = \dot{R}(t)R^{-1}(t), \tag{3.6}$$

where $R(t) = S(t)X(t)$ and define $B(t) = 0$ for $t \in \mathbb{R}$ for which $\dot{S}(t)$ does not exist. Obviously, $R(t)$ is the fundamental matrix of the linear equation

$$\dot{y} = B(t)y.$$

Now we show that $A(t) \sim B(t)$ and $B(t)$ has the block diagonal form

$$B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

In fact,

$$\begin{aligned} S'(t) &= \big(X(t)R^{-1}(t)\big)' \\ &= X'(t)R^{-1}(t) + X(t)\big(R^{-1}(t)\big)' \\ &= A(t)X(t)R^{-1}(t) - X(t)R^{-1}(t)R'(t)R^{-1}(t), \end{aligned}$$

which, combining with (3.6) gives

$$S'(t) = A(t)S(t) - S(t)B(t).$$

Therefore, $A(t) \sim B(t)$. Now we show that $B(t)$ has the block diagonal form of (3.3). By Lemma 3.2, $R(t)$ and $R(t)^{-1}$ commute with the matrix $\tilde{P}$ for every $t \in \mathbb{R}$. The derivatives $\dot{R}(t)$ also commute with $\tilde{P}$, and then

$$\tilde{P}B(t) = B(t)\tilde{P} \tag{3.7}$$

for almost all $t \in \mathbb{R}$. Now we decompose $B : \mathbb{R} \to \mathbb{R}^{N\times N}$ into four functions

$$\begin{aligned} &B_1 : \mathbb{R} \to \mathbb{R}^{N_1\times N_1}, \qquad && B_2 : \mathbb{R} \to \mathbb{R}^{N_2\times N_2}, \\ &B_3 : \mathbb{R} \to \mathbb{R}^{N_1\times N_2}, \qquad && B_4 : \mathbb{R} \to \mathbb{R}^{N_2\times N_1}, \end{aligned}$$

with

$$B(t) = \begin{pmatrix} B_1(t) & B_3(t) \\ B_4(t) & B_2(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

Identity (3.7) implies that

$$\begin{pmatrix} B_1(t) & B_3(t) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1(t) & 0 \\ B_4(t) & 0 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Therefore $B_3(t) \equiv 0$ and $B_4(t) \equiv 0$. Thus B has the block diagonal form

$$B_k = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix}, \quad t \in \mathbb{R}$$

and the proof is finished. □

Now we are in a position to prove the reducibility result.

Theorem 3.9. *Assume that (1.1) admits a nonuniform exponential dichotomy. Due to Theorem 2.8, the dichotomy spectrum is either empty or the disjoint union of n closed spectral intervals $\mathcal{I}_1, \ldots, \mathcal{I}_n$ with $1 \le n \le N$, i.e.,*

$$\Sigma_{NED}(A) = \emptyset \quad (n = 0) \quad or \quad \Sigma_{NED}(A) = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_n.$$

Then there exists a weakly kinematic similarity action $S : \mathbb{R} \to \mathbb{R}^{N \times N}$ between (1.1) and a block diagonal system

$$\dot{x} = \begin{pmatrix} B_0(t) & & \\ & \ddots & \\ & & B_{n+1}(t) \end{pmatrix} x$$

with locally integrable functions $B_i : \mathbb{R} \to \mathbb{R}^{N_i \times N_i}$, $N_i = \dim \mathcal{W}_i$, and

$$\Sigma_{NED}(B_0) = \emptyset, \qquad \Sigma_{NED}(B_1) = \mathcal{I}_1, \quad \cdots, \quad \Sigma_{NED}(B_n) = \mathcal{I}_n, \qquad \Sigma_{NED}(B_{n+1}) = \emptyset.$$

Proof. If for any $\gamma \in \mathbb{R}$, system $(2.7)_\gamma$ admits a nonuniform exponential dichotomy, then $\Sigma_{NED}(A) = \emptyset$. Conversely, for any $\gamma \in \mathbb{R}$, the weighted system $(2.7)_\gamma$ does not admit a nonuniform exponential dichotomy, then $\Sigma_{NED}(A) = \mathbb{R}$. Now, we prove the theorem for the nontrivial case ($\Sigma_{NED}(A) \ne \emptyset$ and $\Sigma_{NED}(A) \ne \mathbb{R}$). First, recall that the resolvent set $\rho_{NED}(A)$ is open and therefore the dichotomy spectrum $\Sigma_{NED}(A)$ is the disjoint union of closed intervals. Using Theorem 2.8, we can assume

$$\mathcal{I}_1 = \left\{ \begin{matrix} [a_1, b_1] \\ \text{or} \\ (-\infty, b_1] \end{matrix} \right\}, \qquad \mathcal{I}_2 = [a_2, b_2], \quad \cdots, \quad \mathcal{I}_{n-1} = [a_{n-1}, b_{n-1}], \qquad \mathcal{I}_n = \left\{ \begin{matrix} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{matrix} \right\}$$

with $a_1 \le b_1 < a_2 \le b_2 < \cdots < a_n \le b_n$ due to $\varepsilon < \alpha/2$.

If $\mathcal{I}_1 = [a_1, b_1]$ is a spectral interval, then we have $(-\infty, \gamma_0) \subset \rho_{NED}(A)$ and $\mathcal{W}_0 = \mathcal{S}_{\gamma_0}$ for some $\gamma_0 < a_1$ due to Theorem 2.8, which implies that

$$\dot{x} = \big[A(t) - \gamma_0 I\big] x$$

admits a nonuniform exponential dichotomy with an invariant projector $\tilde{P}_0$. By Corollary 3.7 and Theorem 3.8, there exists a nonuniform Lyapunov transformation $x = S_0(t)x_0$ with $\|S_0(t)\| \leq M_0 e^{\epsilon_0|t|}$ and $\|S_0(t)^{-1}\| \leq M_0 e^{\epsilon_0|t|}$ such that $A(t) \sim A_0(t)$ and $A_0(t)$ has two blocks of the form $A_0(t) = \begin{pmatrix} B_0(t) & 0 \\ 0 & B_{0,*}(t) \end{pmatrix}$ with $\dim B_0(t) = \dim \operatorname{im} \tilde{P}_0 = \dim \mathcal{S}_{\gamma_0} = \dim \mathcal{W}_0 := N_0$ due to Theorem 3.8, Lemma 2.4 and Theorem 2.8. If $\mathcal{I}_1 = (-\infty, b_1]$ is a spectral interval, a block $B_0(t)$ is omitted.

Now we consider the following system

$$\dot{x}_0 = A_0(t)x_0 = \begin{pmatrix} B_0(t) & 0 \\ 0 & B_{0,*}(t) \end{pmatrix} x_0.$$

By using Lemma 2.7, we take $\gamma_1 \in (b_1, a_2)$. In view of $(b_1, a_2) \subset \rho_{NED}(B_{0,*}(t))$, $\gamma_1 \in \rho_{NED}(B_{0,*}(t))$, which implies that

$$\dot{x}_0 = \left[\begin{pmatrix} B_0(t) & 0 \\ 0 & B_{0,*}(t) \end{pmatrix} - \gamma_0 I \right] x_0$$

admits a nonuniform exponential dichotomy with an invariant projector $\tilde{P}_1$. From the claim above, we know that $\tilde{P}_1 \neq 0, I$. Similarly by Corollary 3.7 and Theorem 3.8, there exists a nonuniform Lyapunov transformation

$$x_0 = S_1(t)x_1 = \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_1(t) \end{pmatrix} x_1$$

with $\|\tilde{S}_1(t)\| \leq M_1 e^{\epsilon_1|t|}$ and $\|(\tilde{S}_1(t))^{-1}\| \leq M_1 e^{\epsilon_1|t|}$ such that $B_{0,*}(t) \sim \tilde{B}_{0,*}(t)$ and $\tilde{B}_{0,*}(t)$ has two blocks of the form $\tilde{B}_{0,*}(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_{1,*}(t) \end{pmatrix}$ with $B_1(t) = \dim \operatorname{im} \tilde{P}_1 = \dim \mathcal{S}_{\gamma_1} \geq \dim(\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1}) = \dim \mathcal{W}_1 := N_1$ due to Theorem 3.8, Lemma 2.4 and Theorem 2.8. In addition, using Corollary 3.7 and Theorem 3.8, we have

$$\Sigma_{NED}\big(B_1(t)\big) = \left\{ \begin{matrix} [a_1, b_1] \\ \text{or} \\ (-\infty, b_1] \end{matrix} \right\},$$

$$\Sigma_{NED}\big(B_{1,*}(t)\big) = [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \left\{ \begin{matrix} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{matrix} \right\}.$$

Now we can construct a nonuniform Lyapunov transformation $x = \tilde{S}(t)x_1$ with $\tilde{S}(t) = S_0(t)S_1(t) = S_0(t) \begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_1(t) \end{pmatrix}$, where $\|\tilde{S}(t)\| \leq M_0 M_1 e^{(\epsilon_0+\epsilon_1)|t|}$ and $\|\tilde{S}(t)^{-1}\| \leq M_0 M_1 e^{(\epsilon_0+\epsilon_1)|t|}$. Then $A(t) \sim A_1(t)$ and $A_1(t)$ has three blocks of the form

$$A_1(t) = \begin{pmatrix} B_0(t) & & \\ & B_1(t) & \\ & & B_{1,*}(t) \end{pmatrix}.$$

Applying similar procedures to $\gamma_2 \in (b_2, a_3)$, $\gamma_3 \in (b_3, a_4)$, $\dots$, we can construct a weakly non-degenerate transformation $x = S(t)x_n$ with

$$S(t) = S_0(t)\begin{pmatrix} I_{N_0} & 0 \\ 0 & \tilde{S}_1(t) \end{pmatrix}\begin{pmatrix} I_{N_0+N_1} & 0 \\ 0 & \tilde{S}_2(t) \end{pmatrix}\cdots\begin{pmatrix} I_{N_0+\ldots+N_{n-1}} & 0 \\ 0 & \tilde{S}_n(t) \end{pmatrix}$$

such that $\|S(t)\| \le Me^{\epsilon|t|}$ and $\|S(t)^{-1}\| \le Me^{\epsilon|t|}$ with $M = M_0 \times \cdots \times M_n$ and $\epsilon = \epsilon_0 + \cdots + \epsilon_n$. Now we can prove

$$A(t) \sim A_n(t) := B(t) = \begin{pmatrix} B_0(t) & & \\ & \ddots & \\ & & B_{n+1}(t) \end{pmatrix}$$

with locally integrable functions $B_i : \mathbb{R} \to \mathbb{R}^{N_i \times N_i}$ and

$$\Sigma_{NED}(B_0) = \emptyset, \qquad \Sigma_{NED}(B_1) = \mathcal{I}_1, \quad \dots, \quad \Sigma_{NED}(B_n) = \mathcal{I}_n, \qquad \Sigma_{NED}(B_{n+1}) = \emptyset.$$

Finally, we show that $N_i = \dim \mathcal{W}_i$. From the claim above, we note that $\dim B_0(t) = \dim \mathcal{W}_0$, $\dim B_1(t) \ge \dim \mathcal{W}_1$, $\dots$, $\dim B_n(t) \ge \dim \mathcal{W}_n$, $\dim B_{n+1}(t) = \dim \mathcal{W}_{n+1}$ and with Theorem 2.8 this gives $\dim \mathcal{W}_0 + \cdots + \dim \mathcal{W}_{n+1} = N$, so $\dim B_k^i = \dim \mathcal{W}_i$ for $i = 0, \dots, n+1$. Now the proof is finished. $\square$

Conflict of interest statement

The article has no conflict of interest with any other article.

References

[1] B. Aulbach, N.V. Minh, The concept of spectral dichotomy for linear difference equations II, J. Differ. Equ. Appl. 2 (1996) 251–262.
[2] B. Aulbach, N.V. Minh, P.P. Zabreiko, The concept of spectral dichotomy for linear difference equations, J. Math. Anal. Appl. 185 (1994) 275–287.
[3] B. Aulbach, S. Siegmund, The dichotomy spectrum for noninvertible systems of linear difference equations, J. Differ. Equ. Appl. 7 (2001) 895–913.
[4] B. Aulbach, S. Siegmund, A spectral theory for nonautonomous difference equations, in: New Trends in Difference Equations, Temuco, 2000, Taylor & Francis, London, 2002, pp. 45–55.
[5] L. Barreira, J. Chu, C. Valls, Robustness of nonuniform dichotomies with different growth rates, São Paulo J. Math. Sci. 5 (2011) 203–231.
[6] L. Barreira, J. Chu, C. Valls, Lyapunov functions for general nonuniform dichotomies, Milan J. Math. 81 (2013) 153–169.
[7] L. Barreira, Ya. Pesin, Lyapunov Exponents and Smooth Ergodic Theory, Univ. Lect. Ser., vol. 23, Amer. Math. Soc., 2002.
[8] L. Barreira, Ya. Pesin, Nonuniform Hyperbolicity, Encycl. Math. Appl., vol. 115, Cambridge University Press, 2007.
[9] L. Barreira, C. Valls, Stability of nonautonomous differential equations in Hilbert spaces, J. Differ. Equ. 217 (2005) 204–248.
[10] L. Barreira, C. Valls, Stable manifolds for nonautonomous equations without exponential dichotomy, J. Differ. Equ. 221 (2006) 58–90.

[11] L. Barreira, C. Valls, Stability of nonautonomous dynamics: a survey of recent results, São Paulo J. Math. Sci. 1 (2007) 133–174.
[12] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations, Lect. Notes Math., vol. 1926, Springer, 2008.
[13] L. Barreira, C. Valls, Smooth robustness of parameterized perturbations of exponential dichotomies, J. Differ. Equ. 249 (2010) 2021–2043.
[14] A. Berger, T.S. Doan, S. Siegmund, A definition of spectrum for differential equations on finite time, J. Differ. Equ. 246 (2009) 1098–1118.
[15] C. Chicone, Yu. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, Math. Surv. Monogr., vol. 70, Amer. Math. Soc., 1999.
[16] S.-N. Chow, H. Leiva, Dynamical spectrum for time dependent linear systems in Banach spaces, Jpn. J. Ind. Appl. Math. 11 (1994) 379–415.
[17] S.-N. Chow, H. Leiva, Unbounded perturbation of the exponential dichotomy for evolution equations, J. Differ. Equ. 129 (1995) 509–531.
[18] J. Chu, Robustness of nonuniform behavior for discrete dynamics, Bull. Sci. Math. 137 (2013) 1031–1047.
[19] W.A. Coppel, Dichotomies and reducibility, J. Differ. Equ. 3 (1967) 500–521.
[20] W.A. Coppel, Dichotomy in Stability Theory, Lect. Notes Math., vol. 629, Springer-Verlag, New York/Berlin, 1978.
[21] T.S. Doan, D. Karrasch, T.Y. Nguyen, S. Siegmund, A unified approach to finite-time hyperbolicity which extends finite-time Lyapunov exponents, J. Differ. Equ. 252 (2012) 5535–5554.
[22] T.S. Doan, K. Palmer, S. Siegmund, Transient spectral theory, stable and unstable cones and Gershgorin's theorem for finite-time differential equations, J. Differ. Equ. 250 (2011) 4177–4199.
[23] Y. Latushkin, S. Montgomery-Smith, T. Randolph, Evolutionary semigroups and dichotomy of linear skew-product flows on locally compact spaces with Banach fibers, J. Differ. Equ. 125 (1996) 73–116.
[24] J. Massera, J. Schäffer, Linear Differential Equations and Function Spaces, Pure Appl. Math., vol. 21, Academic Press, 1966.
[25] D.C. Nguyen, S. Siegmund, Dichotomy spectrum of nonautonomous linear stochastic differential equations, Stoch. Dyn. 2 (2002) 175–201.
[26] V. Oseledets, A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems, Trans. Mosc. Math. Soc. 19 (1968) 197–221.
[27] K.J. Palmer, On the reducibility of almost periodic systems of linear differential equations, J. Differ. Equ. 36 (1980) 374–390.
[28] K.J. Palmer, Exponential dichotomies and transversal homoclinic points, J. Differ. Equ. 55 (1984) 225–256.
[29] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703–728.
[30] C. Pötzsche, A note on the dichotomy spectrum, J. Differ. Equ. Appl. 15 (2009) 1021–1025.
[31] C. Pötzsche, Fine structure of the dichotomy spectrum, Integral Equ. Oper. Theory 73 (2012) 107–151.
[32] H.M. Rodrigues, M. Silveira, On the relationship between exponential dichotomies and the Fredholm alternative, J. Differ. Equ. 73 (1988) 78–81.
[33] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differ. Equ. 27 (1978) 320–358.
[34] R. Sacker, G.R. Sell, Dichotomies for linear evolutionary equations in Banach spaces, J. Differ. Equ. 113 (1994) 17–67.
[35] S. Siegmund, Dichotomy spectrum for nonautonomous differential equations, J. Dyn. Differ. Equ. 14 (2002) 243–258.
[36] S. Siegmund, Reducibility of nonautonomous linear differential equations, J. Lond. Math. Soc. 65 (2002) 397–410.
[37] S. Siegmund, Normal forms for nonautonomous differential equations, J. Differ. Equ. 178 (2002) 541–573.
[38] W. Zhang, The Fredholm alternative and exponential dichotomies for parabolic equations, J. Math. Anal. Appl. 191 (1995) 180–201.