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# Stability Analysis of Positive Systems With Bounded Time-Varying Delays

Xingwen Liu, Wensheng Yu, and Long Wang

**Abstract**—This brief addresses stability of the discrete-time positive systems with bounded time-varying delays and establishes some necessary and sufficient conditions for asymptotic stability of such systems. It turns out that, for any bounded time-varying delays, the magnitude of the delays does not affect the stability of these systems. In other words, system stability is completely determined by the system matrices.

**Index Terms**—Linear copositive Lyapunov function (functional), positive linear system, stability, time-varying delays.

## I. INTRODUCTION

MANY physical systems in the real world involve variables that have nonnegative sign, e.g., population levels, absolute temperature, and concentration of substances. Such systems are referred to as positive systems [1]–[3], which means that their states and outputs are nonnegative whenever the initial conditions and inputs are nonnegative. The states of positive systems are confined within a “cone” located in the positive orthant rather than in the whole space  $\mathbb{R}^n$ . This feature makes analysis and synthesis of positive systems a challenging and interesting job [4]–[8].

Stability is one of the most important properties of dynamic systems, and a massive literature has been concentrated on this topic for positive systems. Reference [9] employed a characteristic equation to present some sufficient stability conditions for a class of delayed positive systems. Reference [10] established some necessary and sufficient stability criteria by using a quadratic diagonal Lyapunov function, and [11] proposed some necessary and sufficient stability conditions for

positive systems by means of a linear copositive Lyapunov function. The approach of the linear copositive Lyapunov function captures the nature of positivity; thus, it is widely used in research on positive systems [12]–[16]. References [12] and [15] established necessary and sufficient stability criteria for positive systems without delays. Reference [16] obtained a necessary and sufficient stability condition for continuous-time positive systems with delays. This result was then improved to design positive observers for delayed continuous-time positive systems in [17].

All the aforementioned references are concerned with positive systems without delays or with constant delays. In fact, many positive systems have time-varying delays. One example is the nonnegative and compartmental systems (a kind of positive systems playing a key role in biological and medical sciences) discussed in [16], where  $\tau$  denotes the time it takes for the mass to flow from one subsystem to another. In usual case,  $\tau$  is a function of time, i.e., it is time varying. Generally speaking, time-varying delays are more important and universal in real engineering processes and have more complex impacts on system dynamics than constant delays. For general dynamic systems with time-varying delays, the stability problem has extensively been studied (see, for example, [18]–[21]). To our knowledge, however, little attention has been paid to the same problem of positive systems with time-varying delays. Recently, it has been shown that the stability of continuous- or discrete-time positive systems with constant delays has nothing to do with the magnitude of the delays [16], [17], [22], [23]. Thus, a question immediately arises: Does a similar conclusion hold for positive systems with time-varying delays? As the first step to seek the answer, this brief investigates the stability of the discrete-time positive system with bounded time-varying delays and provides a positive answer.

The idea of this brief is original. We do not directly treat the original systems with time-varying delays, but the corresponding systems with constant delays. Hence, appropriate relations between the two types of systems must be found out, which should bridge the gap between their stability. As shown in Section III, such important relations do exist. This enables us to establish some necessary and sufficient conditions for positive systems with time-varying delays.

The remainder of this brief is organized as follows: In Section II, necessary preliminaries are presented, and some lemmas are provided. Section III proposes the necessary and sufficient stability criteria for discrete-time positive systems with time-varying delays. Section IV provides an example, and Section V concludes this brief.

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## II. PROBLEM STATEMENTS AND PRELIMINARIES

### Nomenclature

$A \succeq 0 (\preceq 0)$	All elements of matrix $A$ are nonnegative (nonpositive).
$A \succ 0 (\prec 0)$	All elements of matrix $A$ are positive (negative).
$A > 0 (< 0)$	Matrix $A$ is a symmetric positive (negative) definite matrix.
$A^T (A^{-1})$	Transpose (inverse) of matrix $A$ .
$\mathbb{R} (\mathbb{R}_{0,+}, \mathbb{R}_+)$	Set of all real (nonnegative, positive) numbers.
$\ \mathbf{x}\ $	Norm of vector $\mathbf{x}$ .
$\mathbb{R}^n (\mathbb{R}_{0,+}^n, \mathbb{R}_+^n)$	$n$ -dimensional real (nonnegative, positive) vector space.
$\mathbb{R}^{n \times m} (\mathbb{R}_{0,+}^{n \times m})$	Set of all real (nonnegative) matrices of $(n \times m)$ -dimension.
$\mathbb{N}$	$\{1, 2, 3, \dots\}$ .
$\mathbb{N}_0$	$\{0\} \cup \mathbb{N}$ .

The following notation of matrices will be used throughout this brief:  $A_i = [a_{jl}^{(i)}]$ . The dimensions of matrices and vectors will not explicitly be mentioned if clear from context. For simplicity, let  $\underline{\mathbf{p}} = \{1, 2, \dots, p\}$  and  $\underline{\mathbf{p}}_0 = \{0\} \cup \underline{\mathbf{p}}$ , where  $p$  is an arbitrary positive integer.

Consider the system

$$\mathbf{x}(t+1) = A\mathbf{x}(t), \quad t \in \mathbb{N}_0 \quad (1)$$

where  $\mathbf{x}(t) \in \mathbb{R}^n$  is the state vector, and  $A \in \mathbb{R}^{n \times n}$  is the system matrix.

**Definition 1 [10]:** System (1) is said to be positive if and only if for any initial condition  $\mathbf{x}(0) \succeq 0$ , the corresponding trajectory  $\mathbf{x}(t) \succeq 0$  holds for all  $t \in \mathbb{N}$ .

**Definition 2:** A square matrix  $A$  is called a Schur matrix if and only if system (1) is asymptotically stable. Equivalently, all the eigenvalues of  $A$  lie inside the unit circle on the complex plane.

**Lemma 1 [12], [15]:** A matrix  $A \succeq 0$  is a Schur matrix if and only if there exists  $\lambda \in \mathbb{R}_+^n$  such that  $(A - I)\lambda \prec 0$ .

**Lemma 2 [10]:** A matrix  $A \succeq 0$  is a Schur matrix if and only if there exists a diagonal matrix  $P > 0$  such that  $A^T P A - P < 0$ .

Consider the following delayed system:

$$\begin{aligned} \mathbf{x}(t+1) &= A_0 \mathbf{x}(t) + \sum_{i=1}^p A_i \mathbf{x}(t - \tau_i(t)), \quad t \in \mathbb{N}_0 \\ \mathbf{x}(t) &= \varphi(t) \succeq 0, \quad t = -\tau, \dots, 0 \end{aligned} \quad (2)$$

where  $A_i \in \mathbb{R}^{n \times n}$  for  $i \in \underline{\mathbf{p}}_0$ ,  $\tau = \max_{i \in \underline{\mathbf{p}}} \{\sup_{t \geq 0} \tau_i(t)\}$ ,  $\tau_i(t) \in \mathbb{N}_0$ , and  $\varphi : \{-\tau, \dots, 0\} \rightarrow \mathbb{R}_{0,+}^n$  is the vector-valued initial function.

**Definition 3:** System (2) is said to be positive if and only if for any initial condition  $\varphi(\cdot) \succeq 0$ , the corresponding trajectory  $\mathbf{x}(t) \succeq 0$  holds for all  $t \in \mathbb{N}$ .

**Lemma 3:** System (2) is positive if and only if  $A_i \succeq 0$ ,  $i \in \underline{\mathbf{p}}_0$ .

*Proof:* Represent (2) componentwise, i.e.,

$$x_l(t+1) = \sum_{j=1}^n a_{lj}^{(0)} x_j(t) + \sum_{i=1}^p \sum_{j=1}^n a_{lj}^{(i)} x_j(t - \tau_i(t)) \quad (3)$$

where  $x_l(\cdot)$  is the  $l$ th component of vector  $\mathbf{x}(\cdot)$ .

**Sufficiency:** Assume that  $A_i \succeq 0$  for  $i \in \underline{\mathbf{p}}_0$  and that the initial condition  $\varphi(\cdot) \succeq 0$ . Then, we have  $x_l(1) \geq 0$  by (3), and therefore,  $x_l(t) \geq 0$  for  $-\tau \leq t \leq 1$ .

Assume for induction that  $x_l(t) \geq 0$  for  $-\tau \leq t \leq k$  with  $k \in \mathbb{N}$ . Then, (3) implies that  $x_l(k+1) \geq 0$ ; thus,  $x_l(t) \geq 0$  for  $-\tau \leq t \leq k+1$ .

By induction,  $x_l(t) \geq 0$  for all  $t \in \mathbb{N}$ . Hence, system (2) is positive.

**Necessity:** Suppose that system (2) is positive. By Definition 3,  $\mathbf{x}(t) \succeq 0$  for all  $t \in \mathbb{N}$  if  $\varphi(\cdot) \succeq 0$ .

Assume for contradiction that there exists  $q \in \underline{\mathbf{p}}_0$  such that  $A_q \succeq 0$  does not hold. Thus, there exists an element  $a_{kg}^{(q)} < 0$ . Denote  $\varphi(-\tau_i(0)) = [\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{in}]^T$  and  $\mathbf{x}(0) = [\varphi_{01}, \varphi_{02}, \dots, \varphi_{0n}]^T$ . Choose  $\varphi_{ij}$  ( $i \in \underline{\mathbf{p}}_0$ ,  $j \in \underline{n}$ ) such that  $\varphi_{qg} > 0$  and  $\varphi_{ij} = 0$  for all  $i \neq q$  or  $j \neq g$ . It follows from (3) that

$$x_l(1) = \sum_{i=0}^p \sum_{j=1}^n a_{lj}^{(i)} \varphi_{ij} = a_{kg}^{(q)} \varphi_{qg} < 0$$

which contradicts the assumption that  $\mathbf{x}(t) \succeq 0$  for all  $t \in \mathbb{N}$ . Therefore,  $A_i \succeq 0$  for all  $i \in \underline{\mathbf{p}}_0$ . ■

**Lemma 4:** Assume that system (2) is positive. Let  $\mathbf{x}_a(t)$  and  $\mathbf{x}_b(t)$  be the trajectories solution of (2) under the initial conditions  $\varphi_a(\cdot)$  and  $\varphi_b(\cdot)$ , respectively. Then,  $\varphi_a(\cdot) \preceq \varphi_b(\cdot)$  implies that  $\mathbf{x}_a(t) \preceq \mathbf{x}_b(t)$  for all  $t \in \mathbb{N}$ .

*Proof:* It is not difficult to see that  $\mathbf{x}(t) = \mathbf{x}_b(t) - \mathbf{x}_a(t)$  is the solution to system (2) under the initial condition  $\varphi(\cdot) = \varphi_b(\cdot) - \varphi_a(\cdot) \succeq 0$ . Since (2) is positive,  $\mathbf{x}(t) \succeq 0$  for all  $t \in \mathbb{N}$ . ■

## III. MAIN RESULTS

The purpose of this section is to establish asymptotic stability criteria for discrete-time positive systems with bounded time-varying delays. Consider the following system:

$$\begin{aligned} \mathbf{x}(t+1) &= A_0 \mathbf{x}(t) + \sum_{i=1}^p A_i \mathbf{x}(t - \tau_i(t)), \quad t \in \mathbb{N}_0 \\ \mathbf{x}(t) &= \varphi(t) \succeq 0, \quad t = -\tau, \dots, 0 \end{aligned} \quad (4)$$

where  $A_i \in \mathbb{R}_{0,+}^{n \times n}$  for  $i \in \underline{\mathbf{p}}_0$ , and the delays  $\tau_i(t) \in \mathbb{N}_0$  satisfy

$$0 \leq \alpha_i \leq \tau_i(t) \leq \tau_i, \quad t \in \mathbb{N}_0 \quad (5)$$

with constants  $\alpha_i \in \mathbb{N}_0$  and  $\tau_i \in \mathbb{N}$ , and

$$\tau = \max\{\tau_i \mid i = 1, 2, \dots, p\} \quad (6)$$

$\varphi : \{-\tau, \dots, 0\} \rightarrow \mathbb{R}_{0,+}^n$  is the vector-valued initial function. By Lemma 3, system (4) is positive.

Now, we consider the following system closely related to (4):

$$\begin{aligned}\mathbf{y}(t+1) &= A_0\mathbf{y}(t) + \sum_{i=1}^p A_i\mathbf{y}(t-\tau_i), \quad t \in \mathbb{N}_0 \\ \mathbf{y}(t) &= \phi(t) \succeq 0, \quad t = -\tau, \dots, 0\end{aligned}\quad (7)$$

where all the system matrices are as in (4), and  $\tau_i$  in (7) is the supremum of  $\tau_i(t)$  in (4), as shown in (5).  $\tau$  is defined by (6), and  $\phi : \{-\tau, \dots, 0\} \rightarrow \mathbb{R}_{0,+}^n$  is an arbitrary initial condition. Note that the delays in (7) are constant.

*Lemma 5:* Consider system (7). Suppose that there exists a vector  $\lambda \in \mathbb{R}_+^n$  satisfying

$$\left( \sum_{i=0}^p A_i - I \right) \lambda \prec 0 \quad (8)$$

and that the initial condition is  $\phi(\cdot) \equiv \lambda$ . Then, for any  $t \in \{-\tau, \dots, 0\} \cup \mathbb{N}$ , the solution to system (7) satisfies

$$\mathbf{y}(t+1) \preceq \mathbf{y}(t). \quad (9)$$

*Proof:* Note that the initial condition  $\phi(\cdot) \equiv \lambda$ . By (7), it yields that

$$\mathbf{y}(1) = A_0\mathbf{y}(0) + \sum_{i=1}^p A_i\mathbf{y}(-\tau_i) = \sum_{i=0}^p A_i\lambda. \quad (10)$$

Since (8) holds and  $\lambda \in \mathbb{R}_+^n$ , (10) implies  $\mathbf{y}(1) \prec \lambda$ . Therefore

$$\mathbf{y}(1) \prec \mathbf{y}(0) = \mathbf{y}(-1) = \dots = \mathbf{y}(-\tau). \quad (11)$$

This means that (9) holds for any  $t \in \{-\tau, \dots, 0\}$ .

Now, suppose that (9) holds for any  $t \in \{-\tau, \dots, k\}$  with  $k \in \mathbb{N}_0$ , then

$$\begin{aligned}\mathbf{y}(k+2) &= A_0\mathbf{y}(k+1) + \sum_{i=1}^p A_i\mathbf{y}(k+1-\tau_i) \\ &\preceq A_0\mathbf{y}(k) + \sum_{i=1}^p A_i\mathbf{y}(k-\tau_i) \\ &= \mathbf{y}(k+1)\end{aligned}$$

which means that (9) holds for any  $t \in \{-\tau, \dots, k+1\}$ . By induction, (9) holds for any  $t \in \{-\tau, \dots, 0\} \cup \mathbb{N}$ . ■

The next lemma will show that, under certain conditions, the solution to system (4) is not greater than that to system (7), as stated in the following discussion.

*Lemma 6:* Suppose that there exists a vector  $\lambda \in \mathbb{R}_+^n$  satisfying (8) and that the initial conditions for systems (4) and (7) are the same, i.e.,  $\varphi(\cdot) \equiv \phi(\cdot) \equiv \lambda$ . Then

$$\mathbf{x}(t) \preceq \mathbf{y}(t) \quad (12)$$

holds for all  $t \in \mathbb{N}$ , where  $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  are solutions to (4) and (7), respectively.

*Proof:* By (4) and (7), it follows that

$$\begin{aligned}\mathbf{x}(1) &= A_0\mathbf{x}(0) + \sum_{i=1}^p A_i\mathbf{x}(-\tau_i(0)) = \sum_{i=0}^p A_i\lambda \\ \mathbf{y}(1) &= A_0\mathbf{y}(0) + \sum_{i=1}^p A_i\mathbf{y}(-\tau_i) = \sum_{i=0}^p A_i\lambda.\end{aligned}$$

Due to (8)

$$\mathbf{y}(1) = \mathbf{x}(1) \prec \lambda. \quad (13)$$

Moreover

$$\begin{aligned}\mathbf{x}(2) &= A_0\mathbf{x}(1) + \sum_{i=1}^p A_i\mathbf{x}(1-\tau_i(1)) \\ &\preceq A_0\mathbf{x}(1) + \sum_{i=1}^p A_i\mathbf{x}(1-\tau_i) \\ &= A_0\mathbf{y}(1) + \sum_{i=1}^p A_i\mathbf{y}(1-\tau_i) \\ &= \mathbf{y}(2)\end{aligned}$$

that is,  $\mathbf{x}(2) \preceq \mathbf{y}(2)$ . This, together with (13), shows that for  $t \in \{1, 2\}$ , (12) holds.

Suppose that (12) holds for any  $t \in \{1, 2, \dots, k\}$ , where  $k \in \mathbb{N}$  is not smaller than 2. Then, we have

$$\begin{aligned}\mathbf{x}(k+1) &= A_0\mathbf{x}(k) + \sum_{i=1}^p A_i\mathbf{x}(k-\tau_i(k)) \\ &\preceq A_0\mathbf{y}(k) + \sum_{i=1}^p A_i\mathbf{y}(k-\tau_i(k)).\end{aligned}$$

Since  $\tau_i(k) \leq \tau_i$ , Lemma 5 implies that  $\mathbf{y}(k-\tau_i) \succeq \mathbf{y}(k-\tau_i(k))$ . Therefore

$$\mathbf{x}(k+1) \preceq A_0\mathbf{y}(k) + \sum_{i=1}^p A_i\mathbf{y}(k-\tau_i) = \mathbf{y}(k+1)$$

i.e., if (12) holds for any  $t \in \{1, 2, \dots, k\}$ , then it holds for any  $t \in \{1, 2, \dots, k+1\}$ . By induction, (12) holds for all  $t \in \mathbb{N}$ . ■

*Lemma 7 [24]:* The positive system (7) is asymptotically stable if and only if there exists a vector  $\lambda \in \mathbb{R}_+^n$  such that

$$\left( \sum_{i=0}^p A_i - I \right) \lambda \prec 0.$$

We are now ready to provide the main results.

*Theorem 1:* The positive system (4) is asymptotically stable if and only if there exists a vector  $\lambda \in \mathbb{R}_+^n$  such that

$$\left( \sum_{i=0}^p A_i - I \right) \lambda \prec 0. \quad (14)$$

*Proof: Sufficiency.* Assume that there exists a vector  $\lambda \in \mathbb{R}_+^n$  such that (14) holds. By Lemma 7, system (7) is asymptotically stable.

Define  $\|\phi\| = \sup_{t \in \{-\tau, \dots, 0\}} \|\phi(t)\|$ . Since system (7) is asymptotically stable, for any given  $\varepsilon > 0$ , there exists a scalar  $\delta > 0$ , such that if  $\|\phi\| < \delta$ , then the corresponding solution  $\mathbf{y}_\phi(t)$  to (7) satisfies that  $\|\mathbf{y}_\phi(t)\| < \varepsilon$  for all  $t \in \mathbb{N}$ , and  $\lim_{t \rightarrow +\infty} \|\mathbf{y}_\phi(t)\| = 0$ .

In particular, choose the initial condition  $\phi(\cdot) = \alpha\lambda$ , where  $\alpha > 0$ , such that  $\|\alpha\lambda\| < \delta$ . Then, the corresponding solution  $\mathbf{y}_{\alpha\lambda}(t)$  must satisfy that  $\|\mathbf{y}_{\alpha\lambda}(t)\| < \varepsilon$  for all  $t \in \mathbb{N}$ , and  $\lim_{t \rightarrow +\infty} \|\mathbf{y}_{\alpha\lambda}(t)\| = 0$ .

For system (4), let the initial condition be  $\varphi(\cdot) = \alpha\lambda$ . By Lemma 6, the corresponding solution  $\mathbf{x}_{\alpha\lambda}(t)$  satisfies that  $\mathbf{x}_{\alpha\lambda}(t) \preceq \mathbf{y}_{\alpha\lambda}(t)$  for all  $t \in \mathbb{N}$ . In addition, by Lemma 4, for any initial condition  $\varphi(\cdot) \prec \alpha\lambda$ , the corresponding solution satisfies  $\mathbf{x}_\varphi(t) \preceq \mathbf{x}_{\alpha\lambda}(t)$  for all  $t \in \mathbb{N}$ .

Therefore, for system (4), arbitrarily given  $\varepsilon > 0$ , there exists  $\delta_1 = \|\alpha\lambda\| > 0$  such that  $\|\varphi\| < \delta_1$  implies  $\|\mathbf{x}_\varphi(t)\| \leq \|\mathbf{x}_{\alpha\lambda}(t)\| \leq \|\mathbf{y}_{\alpha\lambda}(t)\| < \varepsilon$  for all  $t \in \mathbb{N}$ .

Furthermore, since  $\|\mathbf{x}_\varphi(t)\| \leq \|\mathbf{y}_{\alpha\lambda}(t)\|$  for all  $t \in \mathbb{N}$ ,  $\lim_{t \rightarrow +\infty} \|\mathbf{y}_{\alpha\lambda}(t)\| = 0$  implies

$$\lim_{t \rightarrow +\infty} \|\mathbf{x}_\varphi(t)\| = 0$$

and therefore, system (4) is asymptotically stable.

*Necessity.* Suppose that system (4) is asymptotically stable for any delays satisfying (5). Particularly, let  $\tau_i(t) = \tau_i$ ; then system (7) is necessarily asymptotically stable. By Lemma 7, (14) must hold. ■

*Remark 1:* For the continuous-time positive systems with constant delays, [16] have established some necessary and sufficient stability criteria. Theorem 1 can be viewed as an extension to the discrete time-varying delayed case.

*Corollary 1:* For the positive system (4), the following statements are equivalent.

- (i) System (4) is asymptotically stable.
- (ii) Matrix  $\sum_{i=0}^p A_i$  is a Schur matrix.
- (iii) All the eigenvalues of  $\sum_{i=0}^p A_i$  lie inside the unit circle on the complex plane.
- (iv) There exists a diagonal matrix  $P > 0$  such that  $(\sum_{i=0}^p A_i)^T P (\sum_{i=0}^p A_i) - P < 0$ .
- (v) System  $\mathbf{x}(t+1) = \sum_{i=0}^p A_i \mathbf{x}(t)$ ,  $t \in \mathbb{N}_0$ ,  $\mathbf{x}(0) \succeq 0$ , is asymptotically stable.

*Proof:* (i)  $\Leftrightarrow$  (ii). By Theorem 1, system (4) is asymptotically stable if and only if there exists a vector  $\lambda \in \mathbb{R}_+^n$  such that (14) holds. Note that (14) is equivalent to  $\sum_{i=0}^p A_i$ , which is a Schur matrix due to Lemma 1. The equivalence holds.

(ii)  $\Leftrightarrow$  (iii). This obviously holds.

(ii)  $\Leftrightarrow$  (iv). Note that  $\sum_{i=0}^p A_i \succeq 0$ , and the equivalence between (ii) and (iv) holds due to Lemma 2.

(ii)  $\Leftrightarrow$  (v). This holds due to Definition 2. ■

*Remark 2:* As shown in Theorem 1 and Corollary 1, the magnitude of the delays does not affect the stability of system (4).

#### IV. EXAMPLE

To illustrate the theoretical results, we study the following example.

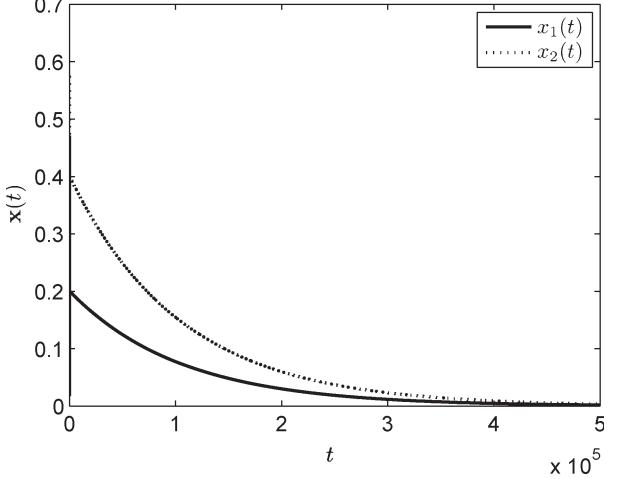


Fig. 1. Evolution of system (15),  $a = 0.4999$ ,  $\tau(t) \in \{0, 1, \dots, 10\}$ .

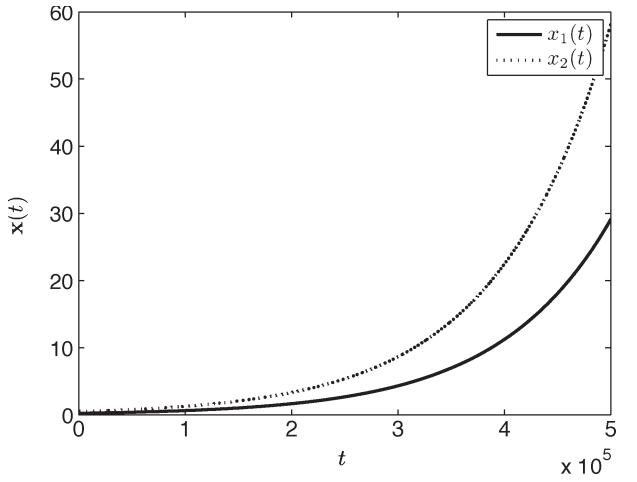


Fig. 2. Evolution of system (15),  $a = 0.5001$ .

*Example:* Consider

$$\begin{aligned} \mathbf{x}(t+1) &= A_0 \mathbf{x}(t) + A_1 \mathbf{x}(t - \tau(t)), & t \in \mathbb{N}_0 \\ \varphi(t) &\succeq 0, & t = -\tau, \dots, 0 \end{aligned} \quad (15)$$

where  $\mathbf{x}(t) = [x_1(t), x_2(t)]^T \in \mathbb{R}^2$ ,  $\tau = \sup_{t \geq 0} \{\tau(t)\}$ , and

$$A_0 = \begin{bmatrix} a & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}.$$

where  $a$  is a nonnegative parameter, and therefore, system (15) is positive. One can check that the two eigenvalues of  $A_0 + A_1$  are 0.4 and 1 if  $a = 0.5$ , and both of them lie inside the unit circle if  $0 \leq a < 0.5$ , and at least one of them lies outside if  $a > 0.5$ . By Corollary 1, system (15) is asymptotically stable for any bounded delay  $\tau(t)$  if  $0 \leq a < 0.5$ , unstable if  $a > 0.5$ , and critically stable if  $a = 0.5$ , as shown in Figs. 1–3, respectively.

In Figs. 1–3, all the initial conditions randomly take values on the interval  $[0,1]$ , and all the delays  $\tau(t)$  randomly take values in the set  $\{0, 1, \dots, 10\}$ , whereas in Fig. 4, the delay  $\tau(t)$  randomly takes values in the set  $\{0, 1, \dots, 15\}$ . From these figures, one can conclude that for bounded delays, the system's

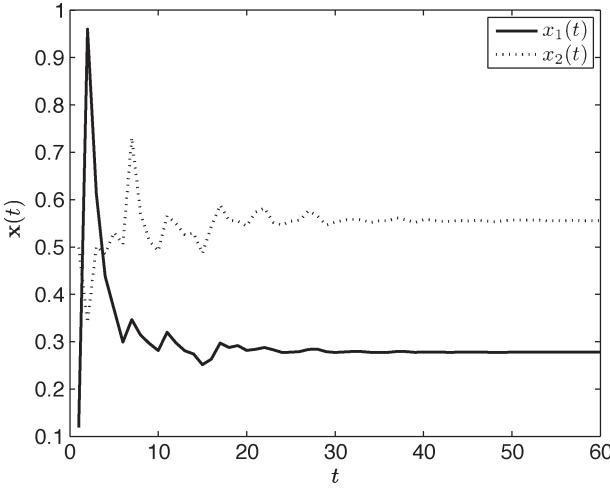


Fig. 3. Evolution of system (15),  $a = 0.5$ .

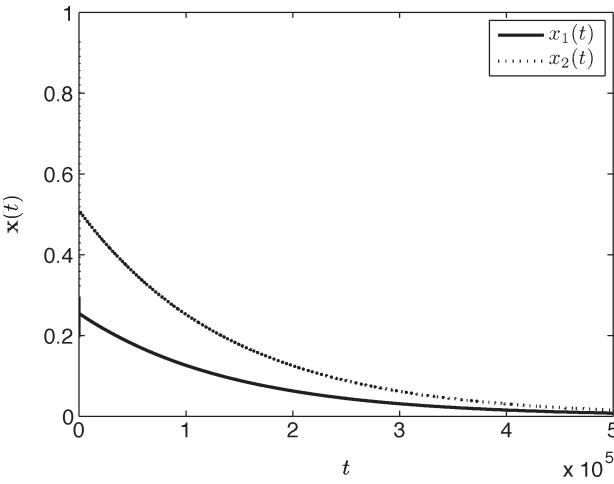


Fig. 4. Evolution of system (15),  $a = 0.4999$ ,  $\tau(t) \in \{0, 1, \dots, 15\}$ .

stability has nothing to do with the system delays, but it is completely determined by the system matrices.

## V. CONCLUSION

Based on a novel approach, some necessary and sufficient stability conditions have been established for discrete-time positive systems with bounded time-varying delays. An illustrative example shows the correctness of the obtained theoretical results. These results are computationally efficient. In the field of positive systems, to our knowledge, no results on stability of systems with time-varying delays have been reported. This brief reveals that stability of a discrete-time positive system with time-varying delays is completely determined by its system matrices and is not affected by the delays.

Based on the results obtained in this brief, many other important topics, such as bounded control, constrained control, and observer design, can further be investigated. Furthermore, the method used in this brief, as well as the obtained results, is helpful in addressing the stability of continuous-time positive systems with time-varying delays, which is another open problem in this area.

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