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Impulse Elimination by Derivative Feedback for Singular Systems with Delay

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Abstract— **The impulse elimination problem by derivative feedback is investigated for linear singular systems with a finite number of commensurable point delays. Using as models linear singular systems with coefficients in a ring, an equivalent impulse elimination problem by derivative feedback is formulated for this class of systems and solvability conditions are provided. Several examples are worked out in details.**

I. Introduction

Singular system models (also referred to as generalized state, descriptor, degenerate, differential-algebraic or semi-state systems), introduced by Rosenbrock and Lueberger in [1] and [2], provide, in many cases, a description of physical problems more useful and natural than state-space systems, for instance dealing with constrained robots, networks, chemical processes, social-economic systems or biological systems.

Singular systems are described by time-invariant differential and algebraic equations of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

$t \geq 0$; $E, A \in \mathcal{K}^{n\times n}$, $B \in \mathcal{K}^{q\times m}$ where $\mathcal{K}$ is either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. If E is the identity matrix I_n, it reduces to the classical case of state space system.

It is well known that a singular system has complicated structures and contains not only finite poles but also infinite poles which may generate undesired impulse behaviors ([3]). Even if a singular system has no impulsive modes, it can still have initial finite discontinuities due to inconsistent initial conditions. Therefore many efforts have been devoted to the problem of designing a state feedback law F such that the closed loop system has no impulsive behavior.

State feedback impulse elimination problem has been solved in [4] and its extension to the time varying setting, where the elements of $E,\ A$ and B are real analytical functions, in [5] and [6]. Furthermore, it has been proved that, contrarily to what happens for state space systems, derivative plus proportional feedbacks are more powerful than purely proportional feedbacks in the singular case, for instance shifting the infinite frequencies to finite frequencies (see [7] and [8] for a survey of recent results on derivative feedback control for singular systems over a field).

We will investigate the problem of impulse elimination by derivative feedback for singular systems with a finite number of commensurable point delays. In general, the introduction of time delay factors makes the analysis much more complicated, but in modeling a dynamical system, the presence of inherent delays in the transmission of signals and/or in sensors' and actuators' behavior not always can be neglected. Even restricting the attention to linear systems, this fact complicates considerably the study, since the state space of the model at issue becomes infinite dimensional. A great research effort has been devoted also to extending techniques and tools for dealing with delay-differential systems (see, for an overview of the state-of-art in this field and updated references the Proceedings of the IFAC Workshops on Time Delay Systems in the years 2005, 2006 and 2007).

We will tackle the impulse elimination problem for linear singular systems with delays using as models systems with coefficients in a ring. The main advantage of this approach is the possibility of working with finite dimensional state spaces, although these are no longer vector spaces over a field, but modules over a ring.

D. Cobb [9] investigated the problem of impulse elimination for singular systems over Hermite domains via state feedback, our intent is to investigate the impulse elimination problem for singular by derivative feedback.

Computational issues in the theory of systems with coefficients in a ring can be dealt with using symbolic computation software such as MAPLE, Mathematica or CoCoA (see for instance [10]).

II. Singular systems over rings and singular time delay systems

Let $\mathcal{R}$ be a commutative ring with identity. A *singular linear system* Σ over $\mathcal{R}$ is given by a finitely generated free $\mathcal{R}$-module $\mathcal{X}$ of dimension n, and $\mathcal{R}$-linear maps $E, A : \mathcal{X} \to \mathcal{X}$, $\quad B : \mathcal{R}^m \to \mathcal{X}$, E may be singular.

The system $\Sigma = (E,\ A,\ B)$ can be thought as the abstract discrete time system:

$$Ex(t+1) = Ax(t) + Bu(t) \tag{2}$$

where $E,\ A$ and B are matrices of suitable dimension with entries in $\mathcal{R}$.

Assume that Σ_d is the linear, time invariant system with a finite number of commensurable point delays, described by the equations

$$\Sigma_d = \begin{cases} \sum_{i=0}^{e}\ E_i\dot{x}(t-ih) = \sum_{i=0}^{a}\ A_i x(t-ih) + \\ \qquad + \sum_{i=0}^{b} B_i u(t-ih) \\ x(t) = \varphi(t),\ t \in [-\alpha\ h, 0] \quad \alpha > 0 \end{cases} \tag{3}$$

where, denoting by $\mathbb{R}$ the field of real numbers, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $h \in \mathbb{R}^+$ is the delay, $\alpha = max(e,\ a,\ b)$, $\varphi(t)$

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is a consistent initial condition, E_i, A_i and B_i are matrices of suitable dimensions with entries in $\mathbb{R}$.

Remark that the class of singular systems we consider is more general than that usually considered in the literature and also contains delay systems of neutral type.

By introducing the delay operator δ defined, for any time function $f(t)$, by $\delta f(t) := f(t-h)$, we can write

$$\sum_{i=0}^{e} E_i\delta^i \dot{x}(t) = \sum_{i=0}^{a} A_i\delta^i x(t) + \sum_{i=0}^{b} B_i\delta^i u(t)$$

By formally replacing the delay operator δ with the algebraic indeterminate Δ and defining

$$E = \sum_{i=0}^{e} E_i\Delta^i, \quad A = \sum_{i=0}^{a} A_i\Delta^i, \quad B = \sum_{i=0}^{b} B_i\Delta^i$$

we can associate to (3) the system $\Sigma = (E,\ A,\ B)$ of the form (2) over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ of real polynomials in one indeterminate.

Actually, the two systems Σ_d and Σ are different objects from a dynamical point of view, but they share the "structural" properties that depend only on the defining matrices $(E,\ A,\ B)$. Therefore many control problems concerning systems with commensurable delays can be modeled by means of abstract objects of the form (2) with whom they share the structural properties that depend only on the defining matrices (see, for instance, [11], [12], [13], [14]).

In the literature on singular systems over the $\mathbb{R}$ usually the pencil $\lambda E - A$ is assumed to be *regular*, i.e. $det(sE-A) \neq 0$. A regular pencil with entries in $\mathbb{R}$, has a *standard canonical form* also called *Weierstrass decomposition*, (see [15]),

$$P(\lambda E - A)Q = \lambda \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix}$$

where P, Q are nonsingular real $n \times n$ matrices, J is strictly upper triangular, i.e. nilpotent, with E, A identically partitioned. The *index* of a pencil $\lambda E - A$ is the index of nilpotency of the nilpotent matrix J in (5), namely the minimum non negative integer ν such that $J^{\nu-1} \neq 0$ and $J^{\nu} = 0$. By convention, if E is non singular, the pencil (E, A) is said to be of index $\nu = 0$.

Then any regular singular system can be decomposed into a dynamic part (differential equations) and a possibly anticipative part (see, for instance, [3], [16]).

If the system (1) has index larger than one, then it has classical continuous solutions only if the input $u(t)$ has a certain smoothness, otherwise impulses may arise in the response of the system. The system is *impulse-free*, namely a smooth response is assured for every continuous $u(t)$, only if $J = 0$, i.e. the system is regular and of index at most one.

The regularity and the absence of impulses of the pair (E, A) ensure the existence and uniqueness of an impulse free solution to the system, therefore it is important to investigate wether such properties can be achieved or maintained applying to the system the usual transformation change of basis and/or proportional/derivative state/output feedbacks. Many authors considered this problem and several results are available (see for instance, [17],[18] [19]. A recent survey on singular systems over a field can be found in [8])

The regularity of a singular system ensures that, given any fixed initial condition $x(0^-)$, the solution $x(t)$ of $E\dot{x}(t) = Ax(t)$ is unique, and, given any fixed consistent initial condition $x(0^-)$ and sufficiently smooth input function $u(t)$, the solution $x(t)$ of $E\dot{x}(t) = Ax(t) + Bu(t)$ exists and it is unique.

In the literature, a singular systems with delays only in the state is said *regular* if the matrix pencil $\lambda E_0 - A_0$ is regular, namely if and only if $\det(\lambda E_0 - A_0)$ is not identically equal to zero and *impulse free* if $\deg(\det(\lambda E_0 - A_0)) = \mathrm{rank} E$ (see, for instance, [20][21]). Such system may have impulsive solutions, but the regularity and the absence of impulses of the pair (E_0, A_0) ensure the existence and uniqueness of an impulse free solution to this system (see [20, Lemma 1]).

III. Generalized state systems over rings

Impulse elimination by state feedback for generalized state systems over Hermitian rings have been investigated in [9]. We shall recall a few definitions and results that will be used in the following.

Let $\mathcal{R}$ be a commutative ring with identity. The set G of all pair $(P,\ Q)$ where $P,\ Q \in \mathcal{R}^{n\times n}$ are unimodular, with the binary operation

$$(P_1,\ Q_1) * (P_2,\ Q_2) = (P_2P_1, Q_1Q_2)$$

is a group acting on the set of all pairs $(E,\ A)$, $E,\ A \in \mathcal{R}^{n\times n}$, as follows

$$(E,\ A) * (P, Q) = (PEQ,\ PAQ) \tag{4}$$

The orbit of a pair $(E,\ A)$ with respect to the action (4) is the set of all pair $(\tilde{E},\ \tilde{A}) = (PEQ,\ PAQ)$ for some P, Q. It can be easily shown that the set of all orbits form a partition of $\mathcal{R}^{n\times n} \times \mathcal{R}^{n\times n}$.

Remark that left multiplication and right multiplication by unimodular matrices P and Q respectively correspond to row operations on the system of equations (2) and to a change of coordinates in the state module respectively.

Definition *1:* [9] The pair (E, A) is *algebraically solvable* if its orbit under (4) contains an element in standard canonical form

$$(\tilde{E},\ \tilde{A}) = (PEQ,\ PAQ) = \left(\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix} \right) \tag{5}$$

where J is strictly upper triangular with PEQ, PAQ identically partitioned for some $(P,\ Q) \in G$. The pair (E, A) has *unit index* if its orbit contains an element in standard canonical form with $J = 0$.

A system Σ over an Hermite ring has *unit index* if it is *algebraically solvable* and in its standard canonical form $J = 0$ and is *impulse free* if $\deg(\det(sE - A)) = \mathrm{rank}(E)$.

The group action (4) can easily be extended to triples $(E,\ A,\ B)$, where $B \in \mathcal{R}^{n\times m}$ by

$$(E,\ A,\ B) * (P, Q) = (PEQ,\ PAQ,\ PB) \tag{6}$$

Algebraic solvability is invariant under the group action (4) and it is not easy to establish but a necessary condition, easier to check, is available.

Proposition 3.1: [9] A necessary condition for a pair (E, A) to be algebraic solvability is pre-solvability.

DEFINITION 2: A pair (E, A) is *pre-solvable* if one of this properties holds.

$PS1)$ $\mathrm{Im}E + A\ker E = \mathcal{R}^n$
$PS2)$ $\mathrm{Im}E \cap A\ker E \neq \{0\}$
$PS3)$ $\ker E \cap \ker A \neq \{0\}$.

A crucial aspect to understand the peculiarity of the matrices over a ring is captured by the following notions.

$$\begin{aligned} \mathrm{rank}\,(E) &= \max\{k|E \text{ has a nonzero } k\text{th-order minor}\} \\ \rho(E) &= \max\{k| \text{ the } k\text{th-order minors of } E \\ &\qquad\qquad \text{satisfy a Bézout identity } \} \end{aligned}$$

where, if $x_1, \ldots, x_k \in \mathcal{R}$, a Bézout identity is an equation of the form $\sum_{i=1}^{k} a_i x_i = 1$ ($a_i \in \mathcal{R}$). An integral domain in which Bézout's identity holds is called a Bézout domain. Principal ideal domains (PID), such as the polynomial ring $\mathbb{R}[\Delta]$, are Bézout domains and Hermite rings.

Lemma 3.2: [9] Let $M \in \mathcal{R}^{p\times q}$, $p < q$. The following are equivalent. $i)$ $\rho(M) = p$, $ii)$ Im $M = \mathcal{R}^p$, $iii)$ M has a right inverse.

DEFINITION 3: The triple $(E,\ A,\ B)$ is *impulse controllable (by proportional feedback)* if there exists an $F \in \mathcal{R}^{m\times n}$ such that $(E,\ A+BF)$ has unit index. This property is invariant with respect to the action (6).

Proposition 3.3: [9] Let $\mathcal{R}$ be an Hermite ring. The pair (E, A) is *impulse controllable* (by proportional feedback) if and only if

$i)$ rank $(E) = \rho(E)$
$ii)$ $\mathrm{Im}E + A\ker E + \mathrm{Im}B = \mathcal{R}^n$
$iii)$ (E, A) is presolvable.

The impulse elimination by state feedback problem is therefore solved for all singular delay systems whose associated system over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ satisfies the hypotheses of the above Proposition. However, the following example shows that there are systems that do not satisfy the hypotheses of Proposition 3.3.

Example 3.4: Consider the generalized state system with delays

$$\Sigma_d = \begin{cases} \dot{x}_1(t) = -x_1(t) + x_2(t) \\ \dot{x}_3(t) = x_1(t) - \dot{x}_3(t-h) + u_1(t) \\ \dot{x}_4(t) = x_1(t) - x_3(t-h) \\ 0 = -x_1(t) + x_2(t) + x_4(t) + u_2(t) \end{cases} \tag{7}$$

The associated system $\Sigma = (E,\ A,\ B)$ over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ is given by equations (2) where

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1+\Delta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & \Delta & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^t$$

Σ is not algebraically solvable, namely it does not have a canonical form, since it is not pre-solvable. In fact, ker $E = \{e_2\}$, Im $E + A\ker\ E \neq \mathcal{R}^4$, Im $E \cap A\ker\ E = \{0\}$, ker $E \cap$ker $A = \{0\}$. Moreover, we have Im $E + A\ker\ E +$ Im $B = \mathcal{R}^4$, but rank $(E) = 3 > \rho(E) = 2$, then Proposition 3.3 does not apply and the impulsive behavior cannot be eliminated by state feedback.

IV. OUR CONTRIBUTE

We will now investigate what can be achieved to eliminate the impulsive behavior of a generalized state system with delays defined by equation of the form (3), using *proportional-plus-derivative feedback* of the form

$$u(t) = -K\dot{x}(t) + Fx(t) + Ww(t) \tag{8}$$

by studying the generalized state system Σ over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ associated to it as described in Section II. In analogy with what has been done in the algebraic setting by [9] we will use a feedback characterization.

Proposition 4.1: Denote by $\mathcal{D}$ the set of all pentuple $(P,\ Q,\ W,\ F,\ K)$, where $P,\ Q \in \mathcal{R}^{n\times n}$ and $W \in \mathcal{R}^{m\times m}$ are unimodular, $F,\ K \in \mathcal{R}^{m\times n}$. Then, the set $\mathcal{D}$ together with the binary operation

$$\begin{aligned} &(P_1, Q_1, W_1, F_1, K_1) \diamond (P_2, Q_2, W_2, F_2, K_2) = \\ &= (P_2P_1, Q_1Q_2, W_1W_2, F_1Q_2 + W_1F_2, K_1Q_2 + W_1K_2) \end{aligned} \tag{9}$$

has the structure of a non-commutative group, with neutral element $(I_n, I_n, I_m, 0, 0)$.

Proposition 4.2: The group $\mathcal{D}$ acts on the right on the set of triples (E, A, B), as follows:

$$\begin{aligned} &(E, A, B) \star (P, Q, W, F, K) = \\ &= (PEQ + PBK, PAQ + PBF, PBW) \end{aligned} \tag{10}$$

Sketch of the proof The orbit of any triple (E, A, B) with respect to the group action (10) is the set of all triples $(\tilde{E}, \tilde{A}, \tilde{B})$ such that $(\tilde{E}, \tilde{A}, \tilde{B}) = (E, A, B) \star (P, Q, W, F, K)$ for some P, Q, W, F, K.

If $(E_1, A_1, B_1) = (E, A, B) \star (P_1, Q_1, W_1, F_1, K_1)$ and $(E_2, A_2, B_2) = (E, A, B) \star (P_2, Q_2, W_2, F_2, K_2)$ are in the orbit of the same triple (E, A, B), then by taking $\tilde{P} = P_2P_1^{-1}, \tilde{Q} = Q_1^{-1}Q_2, \tilde{W} = W_1^{-1}W_2, \tilde{F} = W_1^{-1}(F_2 - F_1Q_1^{-1}Q_2), K = W_1^{-1}(K_2 - K_1Q_1^{-1}Q_2)$ we have $(E_2, A_2, B_2) = (E_1, A_1, B_1) \star (\tilde{P}, \tilde{Q}, \tilde{W}, \tilde{F}, K)$, therefore the set of all orbits forms a partition of $\mathcal{R}^{n\times n} \times \mathcal{R}^{n\times n} \times \mathcal{R}^{n\times m}$.

The group action (10) describes the combined application of the five transformations: row operations on the system equations, change of basis in the state/input modules, proportional and/or derivative feedback.

DEFINITION 4: Two generalized state systems Σ_1 and Σ_2 described by equations of the form (2) over the ring $\mathcal{R}$ are *equivalent* if and only if the two system triples $(E_i,\ A_i,\ B_i)$ for $i = 1, 2$ lie in the same $\mathcal{D}$-orbit.

The regularity of generalized state systems, also without delays, is not invariant with respect to proportional feedback or derivative or proportional-plus-derivative feedback. In [22] a generalized space system Σ (without delays) is called *regularizable* if it is equivalent to a regular one.

DEFINITION 5: A generalized state systems $\Sigma = (E,\ A,\ B)$ over an Hermite ring $\mathcal{R}$ is *proportional-derivative algebraically solvable* if there exist suitable matrices $K \in \mathcal{R}^{n\times m}$ and $F \in \mathcal{R}^{n\times m}$ such that $(E+BK,\ A+BF,\ B,\ C)$ is equivalent to an algebraically solvable system.

Remark that, the fact that K and F are matrices with elements in the ring, means that, going back to the original delay system we will have feedbacks possibly with delays, as will be shown in the examples below.

DEFINITION 6: The triple $(E,\ A,\ B)$ is *derivative feedback (d.f.) impulse controllable* if there exists a matrix $K \in \mathcal{R}^{m\times n}$ such that $(E+BK,\ A)$ is algebraically solvable with $J = 0$.

We will start by investigating the possibility to obtain a state space system by a suitable derivative feedback.

Proposition 4.3: A generalized state systems Σ described by equations of the form (2) over the ring $\mathcal{R}$ is equivalent to a state space system over the ring $\mathcal{R}$ if and only

$$\mathrm{Im}E + \mathrm{Im}B = \mathcal{R}^n \tag{11}$$

Necessity Assume that Σ is equivalent to a state space system, then there exists a pentuple $(P, Q, W, F, K) \in \mathcal{D}$ such that $(PEQ + PBK, PAQ + PBF, PBW)$ is a state space system, in particular such that $P(EQ + BK) = I_n$, therefore $\tilde{Q} = EQ + BK$, is unimodular. Then, $EQ\tilde{Q}^{-1} + BK\tilde{Q}^{-1} = I_n$, namely equation

$$[E\ B]\begin{bmatrix} Q\tilde{Q}^{-1} \\ K\tilde{Q}^{-1} \end{bmatrix} = I_n$$

is solvable, and the matrix $[E\ B]$ has a right inverse. Then (11) follows by Lemma 3.2.

Sufficiency Assume that (11) holds, then by Lemma 3.2 there exists a matrix $Q \in \mathcal{R}^{n\times n}$ and $K \in \mathcal{R}^{m\times n}$ such that $EQ + BK = I$, where I is the identity matrix of order n. Then Σ is equivalent to the state space system $(I,\ AQ,\ B)$.

Example 4.4: Consider the neutral system with delay defined by:

$$\begin{cases} \dot{x}_1(t) = x_1(t-h) + 2x_2(t) + u_1(t-h) - \dot{x}_1(t-h) \\ \dot{x}_2(t) = x_1(t) + x_1(t-2h) + x_2(t) + u_2(t) - \dot{x}_2(t-h) \end{cases} \tag{12}$$

The associated system over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ is given by equations (2) where

$$E = \begin{pmatrix} 1+\Delta & 0 \\ 0 & 1+\Delta \end{pmatrix},\ A = \begin{pmatrix} \Delta & 2 \\ 1+\Delta^2 & 1 \end{pmatrix},\ B = \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix}$$

Since $\rho(E) = 0 < \mathrm{rank}\,(E) = 2$ the system is not impulse controllable by state feedback, but $\mathrm{Im}E + \mathrm{Im}B = \mathcal{R}^2$. Then, by Lemma 3.2, the matrix $E\ B$ is right invertible, namely equation

$$[E\ B]\begin{bmatrix} Q \\ R \end{bmatrix} = I$$

is solvable. A solution is, for instance,

$$Q = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\quad K = \begin{pmatrix} -1 & 0 \\ 1+\Delta & -\Delta \end{pmatrix}$$

By applying the derivative feedback K and the change of basis Q in the state space we obtain the system defined by equations $(EQ + BK = I,\ AQ\ B)$. Going back to the original delay system, by applying the derivative feedback

$$\begin{cases} u_1(t) = -x_1(t) \\ u_2(t) = x_1(t) + x_1(t-h) - x_2(t-h) \end{cases}$$

and the change of basis $x(t) = Qz(t)$ we obtain the state space system with delay

$$\begin{cases} \dot{z}_1(t) = -2z_1(t) + z_1(t-h) + 2z_2(t) \\ \dot{z}_2(t) = z_1(t-2h) + z_2(t) \end{cases}$$

Example 4.5: Consider the system Σ_d defined in Example 3.4 and the associated system Σ over the ring $\mathcal{R} = \mathbb{R}[\Delta]$. As we have seen, the system is not impulse controllable by state feedback. However, $\mathrm{Im}E + \mathrm{Im}\ B = \mathcal{R}^4$. Then, by Lemma 3.2, the matrix $[E\ B]$ is right invertible over the ring $\mathbb{R}[\Delta]$ and we can find matrices

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ q1 & q2 & q3 & q4 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 1+\Delta & -\Delta & 1+\Delta & 1+\Delta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

such that $EQ+BK=I$. We chose, for instance, $q1=q2=1$ and $q3 = q4 = 0$. Applying to the original delay system Σ_d the derivative feedback

$$\begin{aligned} u_1(t) &= -\dot{x}_1(t) - \dot{x}_1(t-h) + \dot{x}_2(t-h) - \dot{x}_3(t) + \\ &\qquad -\dot{x}_3(t-h) - \dot{x}_4(t) - \dot{x}_4(t-h) \\ u_2(t) &= -\dot{x}_4(t) \end{aligned}$$

and the change of basis in the state space $x(t) = Qz(t)$ we obtain the state space system with delays defined by equations

$$\begin{cases} \dot{z}_1(t) = z_2(t) \\ \dot{z}_2(t) = z_1(t) \\ \dot{z}_3(t) = z_1(t) - z_1(t-h) + z_2(t-h) + \\ \qquad\qquad -z_3(t-h) - z_4(t-h) \\ \dot{z}_4(t) = z_2(t) + z_3(t) \end{cases}$$

Proposition 4.6: Assume that the generalized state Σ described by equations of the form (2) over the ring $\mathcal{R}$ satisfies the following condition

$$\mathrm{rank}\,([E \mid B]) = n, \quad \rho([E \mid B]) < n \tag{13}$$

i.e. the columns of E and B span an n-dimensional submodule of $\mathcal{R}^n$ strictly contained in $\mathcal{R}^n$. Then, Σ is equivalent to a neutral system.

Proof If (13) holds, the matrix $[E\ B]$ is full row rank but not invertible over the ring. An inverse exists on the field of rational function $\mathbb{R}(\Delta)$, the quotient field of $\mathbb{R}[\Delta]$, i.e. we can find matrices $\tilde{Q}$ and $\tilde{K}$ with rational element such that $E\tilde{Q} + B\tilde{K} = I$. Denote by $\phi_i(\Delta)$ the g.c.m. of denominators of i-th columns of $\tilde{Q}$ and $\tilde{K}$. Define

$P = diag\{\phi_1(\Delta), \dots, \phi_n(\Delta)\}$, $Q = \tilde{Q}P$ and $K = \tilde{K}P$ Then $EQ + BK$ is a full rank, diagonal matrix. If Σ is a system over $\mathcal{R}$ associated to a delay singular system, P is usually of the form $I + \Delta\tilde{E}$, i. e. the closed loop system is a neutral system.

In the following examples, for lack of space, we will start directly by the generalized state system over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ associated to the generalized state system with delay.

Example 4.7: Consider the linear singular delay system whose associated system over the ring $\mathbb{R}[\Delta]$ is described by the matrices E and A of example (3.4), and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

In this case $\text{Im}E + \text{Im}B = span\{e_1, (1+\Delta)e_2, e_3, e_4\}$, therefore (13) is satisfied but (11) does not hold. An inverse for the matrix $[E\ B]$ exists on the field of rational function $\mathbb{R}(\Delta)$. A solution is, for instance,

$$\tilde{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/(1+\Delta) & 0 & 0 \\ -\Delta & 0 & -1 & -1 \end{pmatrix}$$

$$\tilde{K} = \begin{pmatrix} \Delta & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Write $P = diag\{1, 1+\Delta, 1, 1\}$, $Q = \tilde{Q}P$ and $K = \tilde{K}P$. Then, we have $EQ + BK = P$ and

$$AQ = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & \Delta & 1 & 0 \\ -1-\Delta & 0 & 0 & -1 \end{pmatrix}$$

Applying to the original delay system Σ_d the derivative feedback

$$\begin{cases} u_1(t) = -\dot{x}_1(t-h) - 2\dot{x}_3(t) - \dot{x}_4(t) \\ u_2(t) = -\dot{x}_4(t) \end{cases}$$

and the change of basis in the state space $x(t) = Qz(t)$ we obtain the neutral system defined by equations

$$\begin{cases} \dot{z}_1(t) = -z_1(t) + z_3(t) \\ \dot{z}_2(t) = z_1(t) - \dot{z}_2(t-h) \\ \dot{z}_3(t) = z_1(t) + z_2(t-h) + z_3(t) \\ \dot{z}_4(t) = -z_1(t) - z_1(-h) - z_4(t) \end{cases}$$

Proposition 4.8: Assume that for the generalized state systems Σ described by equations of the form (2) over the ring $\mathcal{R}$

$$\text{Im } E + \text{Im } B = \mathcal{R}^t, \qquad t < n \tag{14}$$

i.e. the columns of E and B span a submodule isomorphic to $\mathcal{R}^t$. Then the system is d.f. algebraically solvable with $J = 0$, i.e. a state space system plus $n - t$ algebraic equations.

Proof If (14) holds, then the matrix $\rho([E\ \ B]) = \text{rank}([E\ \ B]) = t$ and there exist unimodular $\tilde{P}$ and $\tilde{Q}$ such that $\tilde{P}[E\ \ B]\tilde{Q} = \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix}$ (up to a row permutation, the Smith form of the matrix). In particular, writing $\tilde{Q} = \begin{pmatrix} Q & Q_1 \\ K & K_1 \end{pmatrix}$, with $Q \in \mathcal{R}^{n\times n}$ and $K \in \mathcal{R}^{m\times n}$, we have that $PEQ + PBK = \begin{pmatrix} I_t & 0_{n-t} \\ 0_{t\times(n-t)} & 0_{(n-t)\times(n-t)} \end{pmatrix}$. Then, the closed loop system $(P(EQ + BK), PAQ)$ has no impulsive behavior.

Example 4.9: Consider the linear singular delay system defined by

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1+\Delta & 0 \\ \Delta & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^t$$

In this case Proposition 3.3 does not apply and $\text{Im}E + \text{Im}B = span\{e_1, e_2, e_3\}$. Solving the equation

$$[E\ B]\begin{bmatrix} Q \\ R \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we have, for instance,

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ -\Delta & -1 & 1 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} 1+\Delta & -\Delta & 1+\Delta & 1+\Delta \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Applying to the delay system Σ_d the derivative feedback

$$\begin{aligned} & u_1(t) = -\dot{x}_1(t) - \dot{x}_1(t-h) + \dot{x}_2(t) + \\ & \qquad -\dot{x}_3(t) - \dot{x}_3(t-h) - \dot{x}_4(t) - \dot{x}_4(t-h) \\ & u_2(t) = -\dot{x}_2(t) \end{aligned}$$

and the change of basis in the state space $x(t) = Qz(t)$ we obtain the system defined by equations

$$\begin{cases} \dot{z}_1(t) = -z_1(t) + z_2(t)) \\ \dot{z}_2(t) = z_1(t-h) \\ \dot{z}_3(t) = z_3(t) \\ 0 = -z_1(t) - z_1(t-h) - z_2(t) + z_3(t) \end{cases}$$

Proposition 4.10: Assume that for the generalized state systems Σ described by equations of the form (2) over the ring $\mathcal{R}$

$$\rho([E\ B]) < \text{rank}([E\ B]) = t, \qquad t < n \tag{15}$$

i.e. the columns of E and B span a submodule $\mathcal{R}^n$ of dimension t strictly included in $\mathcal{R}^t$. Then the system is equivalent to a system in *generalized canonical form*, namely a neutral system plus $n-t$ algebraic equations, i.e. the system has no impulsive behavior

Proof If (15) holds, then there exist unimodular $\tilde{P}$ and $\tilde{Q}$ such that
$\tilde{P}[E\ B]\tilde{Q} = \begin{pmatrix} L & 0 \\ 0 & 0 \end{pmatrix}$, with $\rho(L) < rank(L) = t$ (up to a row permutation, the Smith form of the matrix). As in the proof of Proposition 4.8, we can find matrices P and Q such that $PEQ + PBK = \begin{pmatrix} L & 0_{n-t} \\ 0_{t\times(n-t)} & 0_{(n-t)\times(n-t)} \end{pmatrix}$. Then, the closed loop system has no impulsive behavior.

Example 4.11: Consider the generalized state system with delays

$$\Sigma_d = \begin{cases} \dot{x}_1(t) = x_1(t) - x_2(t) + u_1(t) \\ \dot{x}_3(t) = x_1(t-h) - \dot{x}_3(t-h) \\ \dot{x}_4(t) = x_2(t) - \dot{x}_3(t-h) + u_2(t) \\ 0 = -x_1(t) + x_4(t) \end{cases} \tag{16}$$

The associated system $\Sigma = (E,\ A,\ B)$ over the ring $\mathcal{R} = \mathbb{R}[\Delta]$ is given by

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1+\Delta & 0 \\ 0 & 0 & \Delta & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 0 \\ \Delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^t$$

Proposition 3.3 does not apply, then the system is not impulse controllable by proportional state feedback. We have in fact $rank\ E = 3 \neq \rho(E) = 1$ and $[\mathrm{Im}E + A\ker E + \mathrm{Im}B] = span\{e_1, (1+\Delta)e_2, e_3\}$ But we can apply Proposition 4.10. We can chose, for instance,

$$Q = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\Delta & 1-\Delta & -1 \end{pmatrix}, K = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \Delta & 1 \end{pmatrix}$$

obtaining the a system in generalized canonical form:

$$EQ + BK = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\Delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$AQ = \begin{pmatrix} -1 & 1 & 1 & 0 \\ \Delta & -\Delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1-\Delta & 1-\Delta & -1 \end{pmatrix}$$

Applying to the original delay system Σ_d the derivative feedback

$$\begin{aligned} u_1(t) &= -\dot{x}_2(t) \\ u_2(t) &= -\dot{x}_3(t-h) - \dot{x}_4(t) \end{aligned}$$

and the change of basis in the state space $x(t) = Qz(t)$ we obtain the state space system with delays defined by equations

$$\begin{cases} \dot{z}_1(t) = -z_1(t) + z_2(t) + z_3(t) \\ \dot{z}_2(t) = z_1(t-h) - z_2(t-h) - \dot{z}_2(t-h) \\ \dot{z}_3(t) = z_3(t) \\ 0 = -z_1(t) + z_2(t) - z_2(t-h) + z_3(t) + \\ \qquad -z_3(t-h) - z_4(t) \end{cases}$$

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