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Adjoint Pairs of Differential-Algebraic Equations and Their Lyapunov Exponents

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Abstract This paper is devoted to the analysis of adjoint pairs of regular differential-algebraic equations with arbitrarily high tractability index. We consider both standard form DAEs and DAEs with properly involved derivative. We introduce the notion of factorization-adjoint pairs and show their common structure including index and characteristic values. We precisely describe the relations between the so-called inherent explicit regular ODE (IERODE) and the essential underlying ODEs (EUODEs) of a regular DAE. We prove that among the EUODEs of an adjoint pair of regular DAEs there are always those which are adjoint to each other. Moreover, we extend the Lyapunov exponent theory to DAEs with arbitrarily high index and establish the general class of DAEs being regular in Lyapunov's sense. The Perron identity which is well known in the ODE theory does not hold in general for adjoint pairs of Lyapunov regular DAEs. We establish criteria for the Perron identity to be valid. Examples are also given for illustrating the new results.

Keywords Differential-algebraic equation · Tractability index · Adjoint equation · Essential underlying ODE · Lyapunov exponent · Lyapunov regularity · Perron identity

Mathematics Subject Classification 34A09 · 34D08

Dedicated to the memory of Katalin Balla (1947–2005).

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List of symbols and abbreviations

$\mathbb{K}$	Set of real numbers $\mathbb{R}$ and set of complex numbers $\mathbb{C}$
$\mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)$	Set of $\mathbb{K}$-valued $n \times s$—matrices and linear operators from $\mathbb{K}^s$ to $\mathbb{K}^n$
$\mathcal{C}(\mathcal{I}, X)$	Space of continuous functions mapping $\mathcal{I}$ into X
$\mathcal{C}^1(\mathcal{I}, X)$	Space of continuously differentiable functions mapping $\mathcal{I}$ into X
$\mathcal{C}^1_M(\mathcal{I}, X)$	$\{x \in \mathcal{C}(\mathcal{I}, X) : Mx \in \mathcal{C}^1(\mathcal{I}, Y)$, with $M \in \mathcal{L}(X, Y)\}$
K^*	Adjoint matrix
K^-	Generalized inverse, $KK^-K = K$, $K^-KK^- = K^-$
K^+	Moore–Penrose inverse
K^{*-}	$[K^*]^-$
K^{-*}	$[K^-]^*$
K^{-*-}	$[[K^-]^*]^-$
$\ker K$	Nullspace (kernel) of K
$\operatorname{im} K$	Image (range) of K
$\langle\cdot, \cdot\rangle$	Scalar product in $\mathbb{K}^m$
$(\cdot, \cdot)$	Scalar product in function spaces
$\lvert \cdot \rvert$	Vector and matrix norms
$\lVert \cdot \rVert$	Norms on function spaces, operator norms
$\oplus$	Direct sum
$\chi^u(f)$	The upper Lyapunov characteristic exponent of f
$\chi^l(f)$	The lower Lyapunov characteristic exponent of f
DAE	Differential-algebraic equation
ODE	Ordinary differential equation
IVP	Initial value problem
IERODE	Inherent explicit regular ODE
EUODE	Essential underlying ODE

1 Introduction

In the classical theory of explicit ordinary differential equations (ODEs) the *adjoint equation* is introduced as equation satisfied by the adjoint inverses of the fundamental solution matrices, e.g. [2,13,16,17]. If

$$x'(t) + B(t)x(t) = 0, \quad t \in \mathcal{I}, \tag{1}$$

is the given ODE, with $X(t, t_0)$ being a fundamental solution matrix normalized at t_0, then

$$Y(t, t_0) := X(t, t_0)^{-1*}, \quad t \in \mathcal{I},$$

satisfies the adjoint ODE

$$-y'(t) + B(t)^* y(t) = 0, \quad t \in \mathcal{I}. \tag{2}$$

This property is closely related to the so-called *Lagrange identity*,

$$\langle x(t), y(t)\rangle = \text{constant}, \quad t \in \mathcal{I}, \tag{3}$$

which is valid for each arbitrary pair of solutions of Eqs. (1) and (2). Thereby the interval $\mathcal{I} \subseteq \mathbb{R}$ is arbitrary. We are most interested in an infinite one.[1] Particularly the Lagrange iden-

[1] In contrast, when looking for adjoint operators of the operator representing the given ODE, one supposes a compact interval and, additionally, boundary conditions.

tity accounts for the benefit of adjoint ODEs, for instance, when investigating asymptotics, boundary value problems, and also optimal control problems.

The nature of differential-algebraic equations (DAEs) is much more complicated. Except for the less interesting case of index-0 DAEs, all fundamental solution matrices of regular DAEs are everywhere singular matrix functions such that one is coerced into finding appropriate generalized inverses. It was Katalin Balla who initiated to clarify the relevant structure of regular index-1 and index-2 DAEs and who made profound contribution to this topic [4–10].

In the present paper, first we continue the investigations and take up the intentions of Katalin Balla concerning adjoint pairs of DAE, now for regular DAEs with arbitrarily high tractability index. We adress both standard form DAEs

$$E(t)x'(t) + F(t)x(t) = 0, \quad t \in \mathcal{I}, \tag{4}$$

and DAEs with properly involved derivative

$$A(t)(Dx)'(t) + B(t)x(t) = 0, \quad t \in \mathcal{I}. \tag{5}$$

Together with the DAEs (4) and (5) we consider the equations

$$-(E^*y)'(t) + F(t)^*y(t) = 0, \quad t \in \mathcal{I}, \tag{6}$$

and

$$-D^*(t)(A^*y)'(t) + B(t)^*y(t) = 0, \quad t \in \mathcal{I}, \tag{7}$$

later on justified as their *adjoint* counterparts.

The attempt [19] to treat DAEs as operator equations in appropriate function spaces provides the adjoint operators as a byproduct when looking for the biadjoint operators representing the closures, e.g.,[19, Theorem 1]. In this context, Eq. (6) is already justified as adjoint equation associated with (4); and (7) is justified as adjoint of (5), see also [26]. In contrast, here we do not make use of functional-analytic arguments, but we try to argue from appropriate aspects of the theory of differential equations.

For smooth coefficients E and F, the Eq. (6) is introduced as *dual* descriptor system in [11]; and the original DAE and its dual are shown to be solvable at the same time. In case of merely continuous coefficients, the DAE (6) and a generalized Lagrange identity are applied in [5,28] to describe solution manifolds of boundary problems for the DAE (4) with index 1. In [6] the notion *adjoint* DAE is used and justified for the index-1 case by rigorous solvability investigations. In particular, it is shown that, if $X(t)$ denotes the maximal fundamental solution matrix of the DAE (4) normalized at $t_0 \in \mathcal{I}$, then

$$Y(t, t_0) := E(t)^{*-}X(t, t_0)^{-*}E(t_0)^*, \quad t \in \mathcal{I},$$

is the maximal fundamental solution matrix of the adjoint equation, also normalized at t_0. The superscript "−" indicates special generalized inverses. At this point we add, that for the less interesting index-0 DAEs (the case of nonsingular E), one obtains this relations immediately by simple computations.

Correspondent results for index-1 and index-2 DAEs with properly stated leading term (5) and (7) are reached in [7,8]. In particular, it is shown that the adjoint pair shares in the index and the further characteristic values. For an important class of self-adjoint DAEs, the *inherent explicit regular ODE* (IERODE) is proved to be Hamiltonian in [10]. Supposing so-called completely decoupling projectors to define the generalized inverses D^- and A^{*-}, it is proved in [7] that

$$Y(t, t_0) := A(t)^{*-}D(t)^{*-}X(t, t_0)^{-*}D(t_0)^*A(t_0)^*$$

is the fundamental solution matrix of the adjoint DAE (7) normalized at the same point t_0. These investigations are continued for DAEs with index ≤ 2 in [4,9]. Conditions ensuring the inherent regular ODEs of the adjoint pair to be adjoint each to other are given. Moreover, also adjoint pairs of *essential underlying ODEs* (EUODE) are studied.
The standard form DAE (9) with smooth coefficients together with the standard form DAE

$$-E(t)^*y'(t) + (F(t)^* - E'(t)^*)y(t) = 0, \quad t \in \mathcal{I}, \tag{8}$$

resulting from (6) are revisited in [20], there called *formally adjoint* pair. It is shown that the DAEs (4) and (8) share in the differentiation index and the size of the differential part.

The second purpose of the paper is to characterize the stability of regular DAEs by using the Lyapunov exponent theory, which is well known for ODEs, see [1,17,25]. Recently, Lyapunov exponents and their properties have been extended to index-1 DAEs given in either the standard form or the strangeness-free form, see [14,15,22–24]. Now we aim to extend the Lyapunov exponent theory to regular DAEs of arbitrarily high index. In particular, we invesigate the relation between the sets of the Lyapunov exponents for adjoint pairs of regular DAEs which is known as the Perron identity.

The present paper is organized as follows: Sect. 2 describes the general assumptions and the Lagrange identity for DAEs. Section 3 collects required material concerning transformations and refactorizations. The basic structure of regular DAEs is exposed in Sect. 4. In particular, we discuss how the IERODEs and the EUODEs are related to each other. These preliminaries are followed by Sect. 5 which gives a definition of adjoint DAEs and provides results concerning the joint basic structure of adjoint pairs. It is also shown that an adjoint DAE pair possesses EUODEs adjoint each to other. Finally, Sect. 6 presents new insights concerning the stability analysis of regular DAEs by investigating their Lyapunov exponents. The list of symbols and abbreviations is given at the end. We drop the argument t if ever possible without causing confusion.

2 Basics and Lagrange Identity

We investigate standard form DAEs

$$E(t)x'(t) + F(t)x(t) = 0, \quad t \in \mathcal{I}, \tag{9}$$

and DAEs with properly involved derivative

$$A(t)(Dx)'(t) + B(t)x(t) = 0, \quad t \in \mathcal{I}. \tag{10}$$

The interval $\mathcal{I} \subseteq \mathbb{R}$ is arbitrary, possible infinite. The coefficients are supposed to be continuous, that is,

$$\begin{aligned} &E, F \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),\\ &\quad A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)),\ \ D \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^n)),\ \ B \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),\end{aligned}$$

with $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$ in the real and complex versions, respectively. Additionally, throughout the paper we assume the time-varying subspaces

$$\ker E(t), \quad \ker A(t), \quad \text{and} \quad \operatorname{im} D(t), \quad t \in \mathcal{I},$$

to be $\mathcal{C}^1$- subspaces. When dealing with a DAE of the form (10), we presume the transversality condition

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{K}^n, \quad t \in \mathcal{I}, \tag{11}$$

to be valid, which means that the *derivative is properly involved* and the DAE shows actually a *properly stated leading term*. The decomposition (11) determines the so-called *border projector function* $R \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^n))$ by

$$\ker R(t) = \ker A(t), \quad \operatorname{im} R(t) = \operatorname{im} D(t), \quad t \in \mathcal{I}. \tag{12}$$

Since both involved subspaces are $\mathcal{C}^1$-subspaces, the projector function R is actually continuously differentiable.

Since $\ker E$ is a $\mathcal{C}^1$-subspace, owing to [26, Theorem 3.1], we find a *proper factorization* $E =: AD$, $A \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m))$, $D \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^n))$ such that $\ker E(t) = \ker D(t)$, $t \in \mathcal{I}$ and the condition (11) is valid. For instance, one can use $A = E$, $D = P$, with a projector function P along $\ker E$ as applied already in [18]. Then we can rewrite the standard form DAE (9) as

$$A(t)(Dx)'(t) + (F(t) - A(t)D'(t))x(t) = 0, \quad t \in \mathcal{I}, \tag{13}$$

which is a DAE with properly stated leading term.
Conversely, each DAE (10) with a continuously differentiable coefficient D and $\mathcal{C}^1$-solutions can be written also as standard DAE

$$A(t)D(t)x'(t) + (B(t) + A(t)D'(t))x(t) = 0, \quad t \in \mathcal{I}. \tag{14}$$

We emphasize that the standard form DAE and the DAE with properly stated leading term share most their structural properties. However, though $\mathcal{C}^1$-solutions are supposed for standard form DAEs, the DAE (10) naturally admits continuous functions x showing a continuously differentiable part Dx.[2]

Together with the DAEs (9) and (10) we consider the equations

$$-(E^*y)'(t) + F(t)^*y(t) = 0, \quad t \in \mathcal{I}, \tag{15}$$

and

$$-D^*(t)(A^*y)'(t) + B(t)^*y(t) = 0, \quad t \in \mathcal{I}, \tag{16}$$

later on justified as their *adjoint* counterparts.
The DAE (16) has a properly stated leading term at the same time as (10), with the associated border projector function R^*.
The DAE (15) is obviously out of the scope of a standard form DAE, but, supposing additionally that E and y are continuously differentiable, one can turn to the standard form DAE

$$-E^*(t)y'(t) + (F(t)^* - E'(t)^*)y(t) = 0, \quad t \in \mathcal{I}. \tag{17}$$

On the other hand, applying the proper factorization $E^* = [AD]^* = D^*A^*$, Eq. (17) leads to

$$-D(t)^*(A^*y)'(t) + (F(t)^* - D'(t)^*A(t)^*)y(t) = 0, \quad t \in \mathcal{I}, \tag{18}$$

which is the precise counterpart of (13).

For any solution pair $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{K}^m)$ and $y \in \mathcal{C}^1_{A^*}(\mathcal{I}, \mathbb{K}^m)$ of the DAEs (10) and (16), respectively, we have

[2] In functional-analytic terms, the operator representing the DAE (13) with properly stated leading term is the closure of operator of the standard DAE (9).

$$\frac{\mathrm{d}}{\mathrm{dt}}\langle D(t)x(t),\ A(t)^*y(t)\rangle = \langle (Dx)'(t),\ A(t)^*y(t)\rangle + \langle D(t)x(t),\ (A^*y)'(t)\rangle$$
$$= \langle A(t)(Dx)'(t),\ y(t)\rangle + \langle x(t),\ D(t)^*(A^*y)'(t)\rangle$$
$$= \langle -B(t)x(t),\ y(t)\rangle + \langle x(t),\ B(t)^*y(t)\rangle = 0, \quad t \in \mathcal{I},$$

and this implies the *Lagrange identity* generalized for DAEs with properly stated leading terms (10) and (16),

$$\langle D(t)x(t),\ A(t)^*y(t)\rangle = \text{constant}, \quad t \in \mathcal{I}, \tag{19}$$

as well as the generalized Lagrange identity for the pair (9) and (15),

$$\langle x(t),\ E(t)^*y(t)\rangle = \langle D(t)x(t),\ A(t)^*y(t)\rangle = \text{constant}, \quad t \in \mathcal{I}. \tag{20}$$

The last identity (20) is valid for all solutions $x \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{K}^m)$ (including all $x \in \mathcal{C}^1(\mathcal{I}, \mathbb{K}^m)$) and $y \in \mathcal{C}^1_{A^*}(\mathcal{I}, \mathbb{K}^m)$ of the DAEs (13) and (15), respectively. If E is continuously differentiable, then (20) makes sense for all solutions $x \in \mathcal{C}^1(\mathcal{I}, \mathbb{K}^m)$ and $y \in \mathcal{C}^1(\mathcal{I}, \mathbb{K}^m)$ of (9) and (17) at least.

3 Transformations and Refactorizations

Let pointwise nonsingular matrix functions

$$L \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)), \quad K \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m))$$

be given. Multiplying the standard form DAE

$$Ex' + Fx = q \tag{21}$$

from left by L and transforming $x = K\tilde{x}$ yields the equivalent DAE

$$\underbrace{LEK}_{=:\tilde{E}}\tilde{x}' + \underbrace{(LFK + LEK')}_{=:\tilde{F}}\tilde{x} = Lq. \tag{22}$$

Multiplying the associated adjoint equation

$$-(E^*y)' + F^*y = p \tag{23}$$

from left by K^* and transforming $y = L^*\tilde{y}$ leads to

$$-((LEK)^*\tilde{y})' + ((LFK)^* + (LEK')^*\tilde{y} = K^*p,$$

that is,

$$-(\tilde{E}^*\tilde{y})' + \tilde{F}^*\tilde{y} = K^*p. \tag{24}$$

In summary the following relations are valid:

$$\begin{array}{ccc}
\boxed{Ex' + Fx = q} & \overset{adjoint}{\longleftrightarrow} & \boxed{-(E^*x)' + F^*y = p} \\
\Downarrow L, K \ \ \Uparrow L^{-1}, K^{-1} & & \Downarrow K^*, L^* \ \ \Uparrow K^{*-1}, L^{*-1} \\
\boxed{\tilde{E}\tilde{x}' + \tilde{F}\tilde{x} = Lq} & \overset{adjoint}{\longleftrightarrow} & \boxed{-(\tilde{E}^*\tilde{y})' + \tilde{F}^*\tilde{y} = K^*p}
\end{array}$$

Next we turn to DAEs with properly stated leading term

$$A(Dx)' + Bx = q \tag{25}$$

and consider multiplications from left and coordinate transformations $x = K\tilde{x}$ by pointwise nonsingular matrix functions

$$L \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)), \quad K \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)).$$

Additionally we allow *refactorizations of the leading term* $AD = (AH)(H^- D)$ by H with

$$\begin{aligned} &H \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)), \quad H^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^s)), \quad n, s \geq r := \operatorname{rank} D, \\ &HH^-H = H, \quad H^-HH^- = H^-, \quad RHH^-R = R. \end{aligned} \tag{26}$$

In particular, one can apply refactorizations with $n = s$ and nonsingular H.

The resulting DAE (cf. [21, Sect. 2.3])

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = Lq \tag{27}$$

has the coefficients

$$\tilde{A} := LAH, \quad \tilde{D} := H^- DK, \quad \tilde{B} := LBK - LAH(H^- R)' DK.$$

It inherits the properly stated leading term from (25), and its border projector function is $\tilde{R} = H^- RH$.

Observe that

$$H^{-*}H^*H^{-*} = H^{-*}, \quad H^*H^{-*}H^* = H^*, \quad R^*H^{-*}H^*R^* = R^*,$$

which means that $H^{-*-} := H^*$ is a generalized inverse of H^{-*} suitable for the refactorization $D^*A^* = (D^*H^{-*})(H^{-*-}A^*) = (D^*H^{-*})(H^*A^*)$.

Multiplying the adjoint equation

$$-D^*(A^*y)' + B^*y = p \tag{28}$$

by K^*, transforming $y = L^*\tilde{y}$ and refactoryzing by means of H^{-*} leads to the transformed DAE

$$-\tilde{D}^*(\tilde{A}^*\tilde{y})' + \tilde{B}^*\tilde{y} = K^*p. \tag{29}$$

In summary the following relations are valid:

$$\begin{array}{ccc} \boxed{A(Dx)' + Bx = q} & \overset{adjoint}{\longleftrightarrow} & \boxed{-D^*(A^*y)' + B^*y = p} \\ \Downarrow L, K, H \;\; \Uparrow L^{-1}, K^{-1}, H^- & & \Downarrow K^*, L^*, H^{-*} \;\; \Uparrow K^{*-1}, L^{*-1}, H^* \\ \boxed{\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = Lq} & \overset{adjoint}{\longleftrightarrow} & \boxed{-\tilde{D}^*(\tilde{A}^*\tilde{y})' + \tilde{B}^*\tilde{y} = K^*p} \end{array}$$

The following observation will play its role for Definition 1 below.

Let the matrix function $H \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^s, \mathbb{K}^n))$ describe a refactorization of the leading term in the DAE (25) and let H^{-*} induce a refactorization in (28). A refactorization does not change neither the DAE solutions nor the relevant function spaces housing the DAE solutions. In particular, we have

$$\mathcal{C}^1_D(\mathcal{I},\mathbb{K}^m)=\mathcal{C}^1_{H^-D}(\mathcal{I},\mathbb{K}^m), \text{ and } \mathcal{C}^1_{A^*}(\mathcal{I},\mathbb{K}^m)=\mathcal{C}^1_{H^*A^*}(\mathcal{I},\mathbb{K}^m).$$

For any solution pair $x\in\mathcal{C}^1_D(\mathcal{I},\mathbb{K}^m)$ and $y\in\mathcal{C}^1_{A^*}(\mathcal{I},\mathbb{K}^m)$ of the homogenous versions of the DAEs (25) and (28), respectively, it holds that

$$\begin{aligned}&\langle H(t)^-D(t)x(t),\ H(t)^*A(t)^*y(t)\rangle\\&\quad=\langle R(t)H(t)H(t)^-D(t)x(t),\ A(t)^*y(t)\rangle=\langle D(t)x(t),\ A(t)^*y(t)\rangle,\quad t\in\mathcal{I},\end{aligned}$$

and hence, next to (19), also

$$\langle H(t)^-D(t)x(t),\ H(t)^*A(t)^*y(t)\rangle=\ \text{constant},\quad t\in\mathcal{I}.\tag{30}$$

Thereby, the constant is the same as in (19).

4 The Basic Structure of a Regular DAE

In the context of the projector based analysis of DAEs, the basic structure of a regular DAE is determined by its tractability index μ and the characteristic values $r_0\le\cdots r_{\mu-1}<r_\mu=m$. We refer to [21] for general relations with other index notions.

The DAE with properly stated leading term

$$A(Dx)'+Bx=q\tag{31}$$

as described in Sect. 2 has continuous coefficients A, D, B. If necessary, the coefficients are supposed to be smooth enough for regularity and the existence of complete decouplings, e.g., [21, Sect. 2.4.3]. We apply the regularity notion given in [21, Definition 2.25], which is supported by several constant-rank requirements yielding the tractability index $\mu\in\mathbb{N}$ and the *characteristic values*

$$r_0\le\cdots\le r_{\mu-1}<r_\mu=m,$$

of a regular DAE. Regularity is formally determined by means of *admissible projector functions*

$$P_0,\ldots,P_{\mu-1}\in\mathcal{C}(\mathcal{I},\mathcal{L}(\mathbb{K}^m,\mathbb{K}^m))$$

associated with the construction of *admissible matrix functions sequences* starting from $G_0:=AD$ and ending up with a nonsingular G_μ, see [21, Definition 2.6].
The tractability index generalizes the Kronecker index of a regular matrix pencil, and, in case of such a matrix pencil, the characteristic values r_i provide a complete description of the formal structure of the corresponding Weierstraß–Kronecker form.

We use the further denotations

$$Q_0:=I-P_0,\ \Pi_0:=P_0,\ Q_i:=I-P_i,\ \Pi_i:=\Pi_{i-1}P_i,\ i=1,\ldots,\mu-1.$$

A regular DAE (31) accommodates also the projector functions

$$D\Pi_0D^-,\ldots,D\Pi_{\mu-1}D^-\in\mathcal{C}^1(\mathcal{I},\mathcal{L}(\mathbb{K}^n,\mathbb{K}^n)),$$

with the pointwise determined generalized inverse D^- such that

$$DD^-D=D,\ D^-DD^-=D^-,\ DD^-=R,\ D^-D=P_0.\tag{32}$$

The regularity notion applies to standard form DAEs

$$Ex'+Fx=q,\tag{33}$$

with sufficiently smooth coefficients E, F, as follows: the standard form DAE (33) is regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, if any (equivalently: each) proper factorization of the leading coefficient $E = AD$ yields a regular DAE of type (31),

$$A(Dx)' + (F - AD')x = q, \tag{34}$$

being regular with these characteristics, e.g., [21, Sect. 2.7]. Similarly, the equation

$$-(E^*y)' + F^*y = p, \tag{35}$$

with sufficiently smooth coefficients E, F, is called regular DAE with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, if any (equivalently: each) proper factorization $E = AD$ yields a regular DAE of type (31),

$$-D^*(A^*y)' + (F^* - D^{*\prime}A^*)y = q, \tag{36}$$

being regular with these characteristics.

The sequence of projector functions $P_0, \ldots, P_{\mu-1}$ serves as tool for the decoupling of the DAE itself and the decomposition of the solution x into their characteristic parts, see [21, Sect. 2.4]. In particular, the component $u = D\Pi_{\mu-1}x$ satisfies the so-called IERODE

$$u' - \left(D\Pi_{\mu-1}D^-\right)'u + D\Pi_{\mu-1}G_\mu^{-1}B\Pi_{\mu-1}D^-u = D\Pi_{\mu-1}G_\mu^{-1}q. \tag{37}$$

The components $v_0 = Q_0x, v_1 = \Pi_0Q_1x, \ldots, v_{\mu-1} = \Pi_{\mu-2}Q_{\mu-1}x$ satisfy the triangular subsystem involving several differentiations

$$\begin{bmatrix} 0 & \mathcal{N}_{01} & \cdots & \mathcal{N}_{0,\mu-1} \\ & 0 & \ddots & \vdots \\ & & \ddots & \mathcal{N}_{\mu-2,\mu-1} \\ & & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ (Dv_1)' \\ \vdots \\ (Dv_{\mu-1})' \end{bmatrix}$$
$$+ \begin{bmatrix} I & \mathcal{M}_{01} & \cdots & \mathcal{M}_{0,\mu-1} \\ & I & \ddots & \vdots \\ & & \ddots & \mathcal{M}_{\mu-2,\mu-1} \\ & & & I \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_{\mu-1} \end{bmatrix} + \begin{bmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \\ \vdots \\ \mathcal{H}_{\mu-1} \end{bmatrix} D^-u = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \\ \vdots \\ \mathcal{L}_{\mu-1} \end{bmatrix} q. \tag{38}$$

The subspace $\operatorname{im} D\Pi_{\mu-1}$ is an invariant subspace for the IERODE. The components $v_0, v_1, \ldots, v_{\mu-1}$ remain within their subspaces $\operatorname{im} Q_0$, $\operatorname{im} \Pi_{\mu-2}Q_1, \ldots, \operatorname{im} \Pi_0 Q_{\mu-1}$, respectively. The structural decoupling is associated with the decomposition

$$x = D^-u + v_0 + v_1 + \cdots + v_{\mu-1}.$$

The coefficients are continuous and explicitly given in terms of an admissible matrix function sequence as

$$\begin{aligned}
\mathcal{N}_{01} &:= -Q_0Q_1D^- \\
\mathcal{N}_{0j} &:= -Q_0P_1\cdots P_{j-1}Q_jD^-, && j = 2, \ldots, \mu-1, \\
\mathcal{N}_{i,i+1} &:= -\Pi_{i-1}Q_iQ_{i+1}D^-, \\
\mathcal{N}_{ij} &:= -\Pi_{i-1}Q_iP_{i+1}\cdots P_{j-1}Q_jD^-, && j = i+2, \ldots, \mu-1,\ i = 1, \ldots, \mu-2, \\
\mathcal{M}_{0j} &:= Q_0P_1\cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, && j = 1, \ldots, \mu-1,
\end{aligned}$$

$$
\begin{aligned}
\mathcal{M}_{ij} &:= \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{M}_jD\Pi_{j-1}Q_j, \quad && j=i+1,\dots,\mu-1,\ i=1,\dots,\mu-2,\\
\mathcal{L}_0 &:= Q_0P_1\cdots P_{\mu-1}G_\mu^{-1},\\
\mathcal{L}_i &:= \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}G_\mu^{-1}, && i=1,\dots,\mu-2,\\
\mathcal{L}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}G_\mu^{-1},\\
\mathcal{H}_0 &:= Q_0P_1\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1},\\
\mathcal{H}_i &:= \Pi_{i-1}Q_iP_{i+1}\cdots P_{\mu-1}\mathcal{K}\Pi_{\mu-1}, && i=1,\dots,\mu-2,\\
\mathcal{H}_{\mu-1} &:= \Pi_{\mu-2}Q_{\mu-1}\mathcal{K}\Pi_{\mu-1},
\end{aligned}
$$

with

$$
\mathcal{K} := (I-\Pi_{\mu-1})G_\mu^{-1}B_{\mu-1}\Pi_{\mu-1} + \sum_{l=1}^{\mu-1}(I-\Pi_{l-1})(P_l-Q_l)(D\Pi_lD^-)'D\Pi_{\mu-1},
$$

$$
\mathcal{M}_j := \sum_{k=0}^{j-1}(I-\Pi_k)\left\{P_kD^-(D\Pi_kD^-)' - Q_{k+1}D^-(D\Pi_{k+1}D^-)'\right\}D\Pi_{j-1}Q_lD^-,
$$
$$
l=1,\dots,\mu-1.
$$

The IERODE is always uncoupled of the second subsystem, but the latter is tied to the IERODE if among the coefficients $\mathcal{H}_0,\dots,\mathcal{H}_{\mu-1}$ is at least one who does not vanish. One speaks about a *fine decoupling*, if $\mathcal{H}_1=\cdots=\mathcal{H}_{\mu-1}=0$, and about a *complete decoupling*, if $\mathcal{H}_0=0$, additionally. A complete decoupling is given, exactly if the coefficient $\mathcal{K}$ vanishes identically.

If the DAE is regular and the original data are sufficiently smooth, then fine and complete decouplings exist and can be constructed, see [21, Sect. 2.4.3]. Below, we suppose at least a fine decoupling.

It should be added at this point, that the coefficients of the IERODE depend on the special choice of admissible projector functions. However, they are uniquely determined in the scope of fine decouplings.

The so-called *canonical projector function* Π_{can} of a regular DAE (see [21, Definition 2.37]) is actually a generalization of the spectral projector onto the finite eigenspace along the infinite eigenspace of a regular matrix pencil (cf. [21, Sect. 1.4]).

By means of fine decoupling projector functions $P_0,\dots,P_{\mu-1}$, the *canonical projector function of the DAE* ([21, Definition 2.37]) can be represented as

$$
\Pi_{can} = (I-\mathcal{H}_0)\Pi_{\mu-1}.
$$

It follows that

$$
D\Pi_{\mu-1} = D\Pi_{can}. \tag{39}
$$

We emphasize that Π_{can} itself is independent of the choice of projector functions. Therefore, also $D\Pi_{\mu-1}$ does not depend of the construction.

One can find fine decoupling projector functions $P_0,\dots,P_{\mu-1}$ with arbitrarily fixed start projector function P_0 along $\ker D$. This allows to prescribe the generalized inverse D^- in (32).

In contrast, complete decoupling projector functions $P_0,\dots,P_{\mu-1}$ yield the representation $\Pi_{can}=\Pi_{\mu-1}$, which is very useful in theory, but less comfortable in practice when dealing with D^- (cf. (32)).

If the DAE is regular, then the IVP

$$A(DX)' + BX = 0, \quad X(t_0) = \Pi_{can}(t_0), \tag{40}$$

possesses exactly one solution $X(\cdot, t_0)$ called *maximal fundamental solution matrix normalized at* t_0, cf. [6,7,21]. It holds that

$$\operatorname{im} X(t, t_0) = \operatorname{im} \Pi_{can}(t), \quad \ker X(t, t_0) = \ker \Pi_{can}(t_0), \tag{41}$$

$$X(t, t_0) = \Pi_{can}(t) D(t)^- U(t, t_0) D(t_0) \Pi_{can}(t_0), \quad t \in \mathcal{I}, \tag{42}$$

with $U(\cdot, t_0)$ being the classical (nonsingular) fundamental solution matrix of the IERODE (37) from a fine decoupling, $U(t_0, t_0) = I$. Furthermore,

$$X(t, t_0)^- := \Pi_{can}(t_0) D(t_0)^- U(t, t_0)^{-1} D(t) \Pi_{can}(t), \quad t \in \mathcal{I}, \tag{43}$$

turns out to be the appropriate generalized inverse concerning the flow, see [21, Sect. 2.6].

Example 1 The system comprising the $m = m_1 + m_2$ equations

$$\begin{aligned} &x_1' + B_{11}x_1 + B_{12}x_2 = q_1, \\ &B_{21}x_1 = q_2, \end{aligned}$$

is said to be a DAE in Hessenberg form of size 2, supposed the product $B_{21}B_{12}$ is nonsingular. Then this DAE is regular with tractability index 2, and with characteristic values $r_0 = r_1 = m_1$, $r_2 = m_1 + m_2$. Denoting

$$\Omega := B_{12}B_{12}^-, \quad B_{12}^- := (B_{21}B_{12})^{-1}B_{21}, \tag{44}$$

we obtain the projector function Ω onto $\operatorname{im} B_{12}$ along $\ker B_{21}$. Suppose Ω to be continuously differentiable. The canonical projector function of this DAE reads (e.g., [21, p. 107])

$$\Pi_{can} = \begin{bmatrix} I - \Omega & 0 \\ -B_{12}^-(B_{11} - \Omega')(I - \Omega) & 0 \end{bmatrix}.$$

It can be directly checked that all solutions of the homogenous DAE with $q = 0$ have the form

$$x = \begin{bmatrix} (I - \Omega)x_1 \\ -B_{12}^-(B_{11} - \Omega')(I - \Omega)x_1 \end{bmatrix} = \Pi_{can}x,$$

in which the component $u = (I - \Omega)x_1$ satisfies the ODE

$$u' - (I - \Omega)'u + (I - \Omega)B_{11}(I - \Omega)u = 0. \tag{45}$$

This ODE stated in $\mathbb{K}^{m_1}$ represents the associated IERODE, if one supposes the natural description

$$\underbrace{\begin{bmatrix} I \\ 0 \end{bmatrix}}_{=A} \Bigg(\underbrace{\begin{bmatrix} I & 0 \end{bmatrix}}_{=D} x \Bigg)' + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{bmatrix} x = q, \quad D^- = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Choosing a matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^{m_1}, \mathbb{K}^d))$, $d := m_1 - m_2$, whose columns form a basis of $\operatorname{im}(I - \Omega) = \ker B_{21}$, and determining the generalized inverse Γ_d^- so that

$$\Gamma_d^- \Gamma_d = I - \Omega, \ \Gamma_d \Gamma_d^- = I_d,$$

we may condense the IERODE to the so-called *essential underlying ODE* (cf. [3,9])

$$\eta' + (\Gamma_d B_{11}\Gamma_d^- + \Gamma_d \Gamma_d^{-\prime})\eta = 0,$$

Since there are different possibilities to choose a basis to form Γ_d, the essential underlying ODE depends on the specially chosen basis.

We will see below, that such kind of EUODEs can be associated with any regular DAE and how they are related to the IERODE.

As exposed in Sect. 3, the multiplication from left by L, the coordinate transformation $x = K\tilde{x}$, and the refactorization by H transforms the DAE (31) into the DAE

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = Lq \tag{46}$$

with coefficients

$$\tilde{A} := LAH, \quad \tilde{D} := H^- DK, \quad \tilde{B} := LBK - LAH(H^- R)' DK.$$

Theorem 1 *The tractability index μ and the characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu$ of any regular DAE with properly stated leading term persist under multiplications from left by continuous nonsingular matrix functions, continuous coordinate transformations and refactorizations with continuously differentiable matrix functions.*

Proof For $\mathbb{K} = \mathbb{R}$, the assertion is a direct consequence of [21, Theorems 2.18, 2.21]. The proof of the complex case can be carried out in the same way. □

In the context of standard form DAEs one supposes continuously differentiable solutions and, correspondingly, continuously differentiable coordinate transformations.

Corollary 1 *The tractability index μ and the characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu$ of any regular standard form DAE* (33) *persist under multiplications from left by continuous nonsingular matrix functions and continuously differentiable coordinate transformations.*

Proof The tractability index as well as the characteristic values of a standard form DAE are defined via the form (34). Thereby, the special choice of the factorization does not matter. Supposing K and x to be continuously differentiable in

$$LA(DK\tilde{x})' + (LFK - LAD'K)\tilde{x} = Lq$$

immediately yields

$$LADK\tilde{x}' + (LFK + LADK')\tilde{x} = Lq,$$

that is, (22). □

Proposition 1 *Let the DAE* (31) *be regular, and let $P_0, \ldots, P_{\mu-1}$ be associated admissible projector functions. Then the transformed DAE* (46) *is also regular and it has admissible projector functions $\tilde{P}_0, \ldots, \tilde{P}_{\mu-1}$ such that*

$$\tilde{\Pi}_i = K^{-1}\Pi_i K, \quad \tilde{D}\tilde{\Pi}_i\tilde{D}^- = H^- D\Pi_i D^- H, \quad i = 0, \ldots, \mu - 1,$$

with $\tilde{D}^- := K^{-1}D^-H$. Furthermore, the border projector function and the canonical projector function of the transformed DAE (46) *can be described by $\tilde{R} = H^-RH$ and $\tilde{\Pi}_{can} = K^{-1}\Pi_{can}K$.*

Proof The real case is verified in [21, Sect. 2.3]. The proof of the complex case can be carried out in the same way. □

The next theorem says that each regular DAE can be transformed into a form with decoupled fast and slow parts, similar to the Weierstraß–Kronecker form of a regular matrix pencil. This provides the main tool to be used in the next section for proving properties of adjoint DAE pairs.

Theorem 2 *Each regular DAE* (31) *with tractability index* μ *and characteristic values* $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$ *can be transformed by pointwise nonsingular matrix functions*

$$L \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)), \quad K \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m)),$$

and a refactorizations of the leading term by H *with*

$$H \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)), \quad H^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^s)), \quad n, s \geq r := \operatorname{rank} D,$$
$$HH^-H = H, \quad H^-HH^- = H^-, \quad RHH^-R = R,$$

into the structured form

$$\tilde{A}(\tilde{D}\tilde{x})' + \tilde{B}\tilde{x} = Lq, \tag{47}$$

with

$$\tilde{A} = LAH = \begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix}, \quad \tilde{D} = H^-DK = \begin{bmatrix} I_d & 0 \\ 0 & P_N \end{bmatrix},$$
$$\tilde{B} = LBK - LAH(H^-R)'DK = \begin{bmatrix} W & 0 \\ 0 & I_{m-d} \end{bmatrix},$$
$$d = m - \sum_{i=0}^{\mu-1}(m - r_i),$$
$$N = \begin{bmatrix} 0 & N_{0\,1} & \cdots & N_{0\,\mu-1} \\ & \ddots & \ddots & \\ & & 0 & N_{\mu-2\,\mu-1} \\ & & & 0 \end{bmatrix}, \; P_N = \begin{bmatrix} 0_{m-r_0} & & & \\ & I_{m-r_1} & & \\ & & \ddots & \\ & & & I_{m-r_{\mu-1}} \end{bmatrix},$$

whereby the entries $N_{i-1\,i}$ *have size* $(m - r_{i-1}) \times (m - r_i)$ *and full column-rank* $m - r_i$, $i = 1, \ldots, \mu - 1$.

Proof The real case is verified by [21, Theorem 2.65(2)]. In essence, completely decoupling projector functions are chosen and then the above large decoupled system is condensed to minimal size m. The proof of the complex case can be carried out in the same way. □

The transformed DAE (47) comprises the explicit ODE

$$\eta' + W\eta = \rho, \tag{48}$$

with size d. We call it *essential underlying ODE* (EUODE) of the originally given DAE (31) or (33) after [3,9,12].
Recall that the IERODE lives in $\mathbb{K}^n$, $n \geq d$, and it is unique in the scope of fine decouplings. Its coefficients are expressed in terms of the original DAE. In contrast, the EUODE has minimal size d, but it is accessible by suitable transformations only. An EUODE can be seen as condensed IERODE. However, depending on the transformations used, there is a variety of EUODEs. We describe the condensing in detail, which is actually part of the proof of Theorem 2 in [21].

The IERODE (37) resulting from a fine decoupling is independent of the construction. We emphasize this fact by rewriting the IERODE as

$$u' - (D\Pi_{can}D^-)'u + D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-u = D\Pi_{can}G_\mu^{-1}q. \tag{49}$$

Since the projector function $D\Pi_{can}D^-$ is continuously differentiable and has rank d, so is $(D\Pi_{can}D^-)^*$, and $\operatorname{im}(D\Pi_{can}D^-)^*$ is spanned by d continuously differentiable basis functions. This means that there is a matrix function

$$\Gamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^d)), \quad \operatorname{im}\Gamma_d^* = \operatorname{im}(D\Pi_{can}D^-)^*, \quad \ker\Gamma_d^* = \{0\}.$$

Then we determine the generalized $\Gamma_d^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^d, \mathbb{K}^n))$ by

$$\Gamma_d\Gamma_d^-\Gamma_d = \Gamma_d, \quad \Gamma_d^-\Gamma_d\Gamma_d^- = \Gamma_d^-, \quad \Gamma_d^-\Gamma_d = D\Pi_{can}D^-, \quad \Gamma_d\Gamma_d^- = I_d. \tag{50}$$

Letting $\eta = \Gamma_d u$ for the solutions $u = D\Pi_{can}D^-u = \Gamma_d^-\Gamma_d u$ of the IERODE leads to an EUODE (48) with (cf. [21, Sect. 2.8])

$$W = -\Gamma_d'\Gamma_d^- + \Gamma_d D\Pi_{can}G_\mu^{-1}B\Pi_{can}D^-\Gamma_d^- = -\Gamma_d'\Gamma_d^- + \Gamma_d D\Pi_{can}G_\mu^{-1}BD^-\Gamma_d^-.$$

Finally in this section, we emphasize again that, though the IERODE is unique, the EUODE is not, since it depends on the choice of the basis functions of $\operatorname{im}(D\Pi_{can}D^-)^*$ to construct Γ_d.

5 The Common Structure of Factorization-Adjoint Pairs

A standard form DAE and its adjoint are known to be regular with tractability index $\mu \le 2$ at the same time; the pair also shares the characteristic values $r_0 < r_1 = m$, resp. $r_0 \le r_1 < r_2 = m$, and it has the common dynamical degree of freedom $d = r_0$ for $\mu = 1$, $d = r_0 + r_1 - m$ for $\mu = 2$, see [6,27].
The same is shown for DAEs with properly stated leading term, see [7].
In the present section we verify and specify correspondent general properties of regular DAEs with arbitrary index. Being about to do this we take a second look to Hessenberg size 2 DAEs.

Example 2 (*Continuation of Example* 1) The Hessenberg form DAE from Example 1

$$\begin{aligned} x_1' + B_{11}x_1 + B_{12}x_2 &= q_1, \\ B_{21}x_1 &= q_2, \end{aligned}$$

and its adjoint

$$\begin{aligned} -y_1' + B_{11}^*y_1 + B_{21}^*y_2 &= p_1, \\ B_{12}^*y_1 &= p_2, \end{aligned}$$

have the canonical projector functions

$$\Pi_{can} = \begin{bmatrix} I-\Omega & 0 \\ -B_{12}^-(B_{11}-\Omega')(I-\Omega) & 0 \end{bmatrix}, \Pi_{*\,can} = \begin{bmatrix} I-\Omega^* & 0 \\ -B_{21}^{*-}(B_{11}^*+\Omega^{*\prime})(I-\Omega^*) & 0 \end{bmatrix}.$$

It holds that

$$D\Pi_{can}D^- = I-\Omega, \quad A^*\Pi_{*\,can}A^{*-} = I-\Omega^* = (D\Pi_{can}D^-)^*.$$

The associated IERODEs are

$$u' - (I - \Omega)'u + (I - \Omega)B_{11}(I - \Omega)u = 0 \tag{51}$$

and

$$-v' + (I - \Omega^*)'v + (I - \Omega^*)B_{11}^*(I - \Omega^*)v = 0. \tag{52}$$

Observe that the IERODEs (51) and (52) form a classical adjoint pair, if and only if the projector function Ω is constant, or, equivalently, if the associated subspaces im B_{12} and ker B_{21} do not vary with t.
One has $(D\Pi_{can}D^-)^* = I - \Omega^* = A^*\Pi_{*\,can}A^{*-}$, however, the canonical projector function $\Pi_{*\,can}$ does not equal Π_{can}^* unless very strong additional restrictions.
On the other hand, choosing a matrix function $\Gamma_d \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^{m_1}, \mathbb{K}^d))$, $d := m_1 - m_2$, whose columns form a basis of im $(I - \Omega) =$ ker B_{21}, and determining the generalized inverse Γ_d^- so that

$$\Gamma_d^-\Gamma_d = I - \Omega,\ \Gamma_d\Gamma_d^- = I_d,$$

we may simultaneously condense both IERODEs to EUODEs (cf. [9])

$$\eta' + (\Gamma_d B_{11}\Gamma_d^- + \Gamma_d\Gamma_d^{-\prime})\eta = 0$$

and

$$-\zeta' + (\Gamma_d^{-*}B_{11}^*\Gamma_d^* + \Gamma_d^{-*\prime}\Gamma_d^*)\zeta = 0,$$

which are actually adjoint each to other, without additional restrictions. Hereby, the second IERODE is condensed by means of

$$v = (I - \Omega^*)v = \Gamma_d^*\Gamma_d^{-*}v = \Gamma_d^*\Gamma_d\zeta, \quad \zeta = \Gamma_d^{-*}v.$$

We emphasize again that the EUODEs depend on the specially chosen Γ_d.

Correspondent properties as in Example 2 are verified for all regular DAEs with tractability index $\mu \le 2$ in [7,9]. Moreover, supposing completely decoupling projector functions to define also D^- and A^{*-}, it is proved in [7] that the relation

$$Y(t, t_0) = A(t)^{*-}D(t)^{-*}X(t, t_0)^{-*}D(t_0)^*A(t_0)^*$$

concerning the fundamental solution matrices of the DAE and the adjoint DAE normalized at the same point t_0 is valid, and in particular

$$\Pi_{*\,can} = A^{*-}D^{-*}\Pi_{can}^*D^*A^*.$$

We turn back to the general regular DAE with properly stated leading term

$$A(Dx)' + Bx = q. \tag{53}$$

The regular DAE (53) and the DAE

$$AH(H^-Dx)' + (B - AH(H^-R)'D)x = q \tag{54}$$

resulting from a refactorization of the leading term by H have exactly the same solutions. In particular, each fundamental solution matrix of (53) represents at the same time a fundamental solution of the DAE (54). Therefore, having in mind wanted appropriate relations of fundamental solution matrices, we regard this matter by extending the notion of adjoint pairs accordingly to a refactorization-tolerant version.[3] We add that, owing to Theorem 1 and Proposition 1, the DAEs (53) and (54) share their tractability index and all characteristic values, and also the canonical projector function. The difference refers solely to the shape of the IERODEs.

Definition 1 The DAE (53) with properly stated leading term is said to be factorization-adjoint to the DAE

$$-\mathcal{D}^*(\mathcal{A}^*y)' + \mathcal{B}^*y = p \tag{55}$$

if there is a refactorization of the leading term in (53)

$$\begin{aligned} &H \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^s, \mathbb{K}^n)), \quad H^- \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^n, \mathbb{K}^s)), \quad n, s \geq r := \operatorname{rank} D,\\ &HH^-H = H, \quad H^-HH^- = H^-, \quad RHH^-R = R, \end{aligned}$$

such that

$$\mathcal{A} = AH, \quad \mathcal{D} = H^-D, \quad \mathcal{B} = B - AH(H^-R)'D.$$

If the DAE (53) is factorization-adjoint to (55), then, conversely, the DAE (55) is factorization-adjoint to the DAE (53), since one can apply a refactorization in (55) with H^{*-}, $H^{*-*} := H^*$ (cf. Sect. 3). This justifies to say that the DAEs (53) and (55) form an *factorization-adjoint pair*.

As stated previously, e.g. [4,7,21], the DAEs (53) and

$$-D^*(A^*y)' + B^*y = p \tag{56}$$

are said to be adjoint each to other. Each adjoint pair (53) and (56) is at the same time factorization-adjoint for trivial reason. The converse does not necessarily hold. The notion of factorization-adjoint pairs is broader as the following scheme documents:

$$\begin{array}{ccc}
\boxed{A(Dx)' + Bx = q} & \overset{\textit{adjoint}}{\longleftrightarrow} & \boxed{-D^*(A^*y)' + B^*y = p} \\
 & \nwarrow & \\
\Downarrow H \quad \Uparrow H^- & \textit{factorization-adjoint} & \Downarrow H^{-*} \quad \Uparrow H^* \\
 & \searrow & \\
\boxed{\mathcal{A}(\mathcal{D}x)' + \mathcal{B}x = q} & \overset{\textit{adjoint}}{\longleftrightarrow} & \boxed{-\mathcal{D}^*(\mathcal{A}^*y)' + \mathcal{B}^*y = p}
\end{array}$$

If the DAEs (53) and (55) are an factorization-adjoint pair, then the *Lagrange identity*

$$\langle \mathcal{D}(t)x(t),\ \mathcal{A}(t)^*y(t)\rangle = \langle D(t)x(t),\ A(t)^*y(t)\rangle = \text{ constant}, \quad t \in \mathcal{I},$$

is valid for all solution pairs of the correspondent homogeneous equations (cf. (30)).

Next we recall the commonly accepted notion of the adjoint of a standard form DAE (e.g., [5,6,11,12,28]).

Definition 2 The equation

$$-(E^*x)' + F^*x = p, \tag{57}$$

[3] Correspondingly, the operator L representing a linear DAE on a compact interval $\mathcal{I}$ does not at all depend on the special proper factorization of the leading term, and the same is true for the adjoint operator L^*, [26].

is said to be the adjoint of the standard form DAE

$$Ex' + Fx = q, \tag{58}$$

and vice versa.

Replacing both DAEs (57) and (58) by proper versions (cf. (13) and (L.9))

$$-D^*(A^*x)' + (F^* - D'^* A^*)x = p,$$

and

$$\bar{A}(\bar{D}x)'(t) + (F - \bar{A}\bar{D}')x = q, \tag{59}$$

possibly with different proper factorizations $E = AD$ and $E = \bar{A}\bar{D}$, we know that these proper versions form a factorization-adjoint pair exactly if the DAEs (57) and (58) are adjoint to each other. Namely, the DAE

$$A(Dx)' + (F - AD')x = p$$

transforms by means of refactorization with $H = D\bar{D}^-$, $H^- = \bar{D}D^-$, into the DAE (59).

Theorem 3 *Let the DAE (53) have sufficiently smooth coefficients.*

(1) *If the DAE (53) is regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$, then so is each factorization-adjoint DAE (55). In particular, a factorization-adjoint pair shares in the dimension (dynamical degree of freedom) $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$.*
(2) *If the DAEs (53) and (56) are regular, then there exist EUODEs of size d of both being adjoint each to other in the classical sense.*

Proof Let the DAE (53) be regular with tractability index μ and characteristic values $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$. Owing to Theorem 2 it transforms by L, K, H into the structured form (47), that is,

$$\begin{bmatrix} I_d & 0 \\ 0 & N \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & P_N \end{bmatrix} \tilde{x} \right)' + \begin{bmatrix} W & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{x} = \tilde{q}, \tag{60}$$

and, by Theorem 1, the tractability index as well as the characteristic values persist. Then, owing to Lemma 1 below, the adjoint DAE

$$-\begin{bmatrix} I_d & 0 \\ 0 & P_N \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & N^* \end{bmatrix} \tilde{y} \right)' + \begin{bmatrix} W^* & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{y} = \tilde{p}, \tag{61}$$

also possesses the same characteristics. This DAE transforms by K^{*-1}, L^{*-1}, H^* to the DAE (56). Owing to Theorem 1 the DAE (56) has the same characteristics, too. Since the DAE (55) results from the DAE (56) by a further refactorization, both DAEs share in their tractability index and characteristic values. This proves Assertion (1).
The structured forms (60) and (61) show the EUODEs being adjoint each to other in the classical sense. This verifies Assertion (2). □

The following immediate corollary of Theorem 3 specifies and extends [20, Theorem 3.5].

Corollary 2 *Let the standard form DAE (58) have sufficiently smooth coefficients.*

(1) *If the DAE* (58) *is regular with tractability index* μ *and characteristic values* $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$*, then so is its adjoint* (57) *and vice versa. In particular, a adjoint pair share in the dimension (dynamical degree of freedom)* $d = m - \sum_{i=0}^{\mu-1}(m - r_i)$.
(2) *If the DAEs* (58) *and* (57) *are regular, then they possess EUODEs being adjoint each to other in the classical sense with size* d.

Lemma 1 *The DAE* (61)*, i.e.,*

$$-\begin{bmatrix} I_d & 0 \\ 0 & P_N \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & N^* \end{bmatrix} \tilde{y} \right)' + \begin{bmatrix} W^* & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{y} = \tilde{p},$$

with $W \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^d, \mathbb{K}^d))$*,* $P_N \in \mathcal{L}(\mathbb{K}^{m-d}, \mathbb{K}^{m-d})$*, and* $N \in \mathcal{C}^1(\mathcal{I}, \mathcal{L}(\mathbb{K}^{m-d}, \mathbb{K}^{m-d}))$ *having the structure described in Theorem* 2*, is regular with tractability index* μ *and characteristic values* $r_0 \leq \cdots \leq r_{\mu-1} < r_\mu = m$.
The canonical projector function associated with (61) *is simply*

$$\tilde{\Pi}_{can} = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof We introduce the permutation matrices

$$J_N := \begin{bmatrix} 0 & & I_{m-r_0} \\ & \cdot^{\cdot^{\cdot}} & \\ I_{m-r_{\mu-1}} & & 0 \end{bmatrix}, \quad J_N^{-1} = \begin{bmatrix} 0 & & I_{m-r_{\mu-1}} \\ & \cdot^{\cdot^{\cdot}} & \\ I_{m-r_0} & & 0 \end{bmatrix}$$

as well as

$$M := J_N^{-1} N^* J_N = \begin{bmatrix} 0 & N^*_{\mu-2\,\mu-1} & \cdots & N^*_{0\,\mu-1} \\ & \ddots & \ddots & \\ & & 0 & N^*_{0\,1} \\ & & & 0 \end{bmatrix},$$

$$R_M := J_N^{-1} P_N J_N = \begin{bmatrix} I_{m-r_{\mu-1}} & & & \\ & \ddots & & \\ & & I_{m-r_1} & \\ & & & 0_{m-r_0} \end{bmatrix}.$$

The matrix function M has strict upper block triangular structure, with entries $N^*_{i\,i+1}$ having full row-rank $m - r_{i+1}$, $i = 0, \ldots, \mu - 2$. Set

$$J := \begin{bmatrix} I_d & 0 \\ 0 & J_N \end{bmatrix}.$$

Multiplying (61) by J^{-1}, transforming the coordinate $\tilde{y} = J\tilde{\tilde{y}}$ and refactorizing the leading term by J yields

$$-\begin{bmatrix} I_d & 0 \\ 0 & R_M \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & M \end{bmatrix} \tilde{\tilde{y}} \right)' + \begin{bmatrix} W^* & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{\tilde{y}} = J^{-1} r. \tag{62}$$

Exploiting the special structure of M we construct a projector function V_M onto ker M having also upper block triangular structure,

$$V_M = \begin{bmatrix} I_{m-r_{\mu-1}} & & & & \\ & V_{\mu-1} & * & \cdots & * \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & V_1 \end{bmatrix}.$$

Thereby, the entries V_{i+1} are projector functions onto ker $N^*_{i\,i+1}$, $i = 0, \ldots, \mu - 2$. Denote further $U_M := I - V_M$ and determine the generalized inverse M^- by

$$MM^-M = M, \quad M^-MM^- = M^-, \quad M^-M = U_M, \quad MM^- = R_M.$$

Set

$$\tilde{H} = \begin{bmatrix} I_d & \\ & M \end{bmatrix}, \quad \tilde{H}^- = \begin{bmatrix} I_d & \\ & M^- \end{bmatrix},$$

and apply a further refactorization by $\tilde{H}$, which leads to

$$-\begin{bmatrix} I_d & 0 \\ 0 & M \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & U_M \end{bmatrix} \tilde{\tilde{y}} \right)' + \begin{bmatrix} W^* & 0 \\ 0 & I_{m-d} - M'U_M \end{bmatrix} \tilde{\tilde{y}} = J^{-1}r. \tag{63}$$

Owing to the structure of M and U_M, the matrix function $I_{m-d} - M'U_M$ is upper block triangular with identity diagonal blocks, and $(I_{m-d} - M'U_M)^{-1}M =: \mathfrak{M}$ has again strict upper block triangular structure. Multiplying the DAE (63) by

$$\tilde{L} = \begin{bmatrix} I_d & \\ & (I_{m-d} - M'U_M)^{-1} \end{bmatrix}$$

results in the DAE

$$-\begin{bmatrix} I_d & 0 \\ 0 & \mathfrak{M} \end{bmatrix} \left(\begin{bmatrix} I_d & 0 \\ 0 & U_M \end{bmatrix} \tilde{\tilde{y}} \right)' + \begin{bmatrix} W^* & 0 \\ 0 & I_{m-d} \end{bmatrix} \tilde{\tilde{y}} = \tilde{L}J^{-1}r. \tag{64}$$

Finally, the DAE (64) fits into the form of equation [21, (2.141), p.163]. which implies the first part of the assertion for the real case. The complex case can be treated in the same way. The canonical projector function associated with (64) is constant, namely,

$$\tilde{\Pi}_{can} = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, $\Pi_{can} = J\tilde{\Pi}_{can}J^{-1} = \tilde{\Pi}_{can}$ represents the constant canonical projector function associated with the DAE (61). □

6 Stability Issues

In this section, we study the qualitative behaviour of solutions of DAEs with properly involved derivative (10). We suppose the infinite interval $\mathcal{I} = [0, \infty)$ and extend some notions and results obtained for index-1 DAEs in [14,15,22–24] to regular DAEs with arbitrary index. First, we recall preliminary results from [21, Sect. 2.6.3].

Definition 3 Let the regular DAE (10) be given on the infinite interval $\mathcal{I} = [0, \infty)$. The DAE is said to be

(1) *stable*, if for every $\varepsilon > 0$, $t_0 \in \mathcal{I}$, there exists a value $\delta(\varepsilon, t_0)$ such that the conditions $x_0, \bar{x}_0 \in \operatorname{im} \Pi_{can}(t_0)$, $|x_0 - \bar{x}_0| < \delta(\varepsilon, t_0)$ imply the existence of solutions $x(., t_0, x_0),\ x(., t_0, \bar{x}_0) \in \mathcal{C}^1_D(\mathcal{I}, \mathbb{R}^m)$ as well as the inequality

$$|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| < \varepsilon, \quad t_0 \le t,$$

(2) *uniformly stable*, if $\delta(\varepsilon, t_0)$ in (1) is independent of t_0,
(3) *asymptotically stable*, if (1) holds true, and as $t \to \infty$

$$|x(t, t_0, x_0) - x(t, t_0, \bar{x}_0)| \longrightarrow 0 \text{ for all } x_0,\ \bar{x}_0 \in \operatorname{im} \Pi_{can}(t_0),\ t_0 \in \mathcal{I},$$

(4) *uniformly asymptotically stable*, if the limit in (3) is uniform with respect to t_0.

Similarly to the well-known results for ODEs, the stability properties of DAEs are characterized by the growth behaviour of the normalized maximal fundamental solution matrix defined by (40).

Theorem 4 *Let the regular DAE (10) be considered on the infinite interval $\mathcal{I} = [0, \infty)$. Then the following assertions hold true, with postive constants K_{t_0}, K and α:*

(1) *The DAE is stable, if and only if $|X(t, t_0)| \le K_{t_0},\ t \ge t_0$.*
(2) *The DAE is uniformly stable, if and only if $|X(t, t_0)X(s, t_0)^-| \le K,\ t_0 \le s \le t$.*
(3) *The DAE is asymptotically stable, if and only if $|X(t, t_0)| \longrightarrow 0$ as $t \to \infty$.*
(4) *The DAE is uniformly asymptotically stable, if and only if $|X(t, t_0)X(s, t_0)^-| \le Ke^{-\alpha(t-s)},\ t_0 \le s \le t$.*

Proof The proofs for the sufficiency statements are given in [21, p. 129]. For the reverse direction, we proceed similarly to the ODE case, see [1, Chap. IV.]. □

In turn, the representations (41) and (43) allow to trace back the stability question to the IERODE, cf. [21, Sect. 2.6].

Now, we consider possible changes in the stability properties under the transformations and the refactorizations discussed in Sect. 3. It is easy to see that a refactorization does not change the solutions of the DAE (10), therefore neither the stability properties of the DAE. However, a transformation may alter the stability properties of the DAE. Hence, we need the so-called kinematic equivalent transformation.

Definition 4 A pair of pointwise nonsingular matrix functions $L,\ K \in \mathcal{C}(\mathcal{I}, \mathcal{L}(\mathbb{K}^m, \mathbb{K}^m))$ is said to yield a *kinematic equivalent transformation* for the DAE (25) (i.e., (10)) if both K and K^{-1} are bounded on $\mathcal{I}$. If in addition, both L and L^{-1} are bounded, then it is a *strong kinematic equivalent transformation*.

It is easy to see that the stability property of a DAE does not alter under kinematic equivalent transformations. Then, in this case we say that the DAE (25) and the transformed one (27) are *kinematically equivalent*.

To characterize the growth rate of the solutions of the DAE (10), we use the notion of characteristic exponent introduced by Lyapunov [1,17,25]. For a non-vanishing function f : $[0, \infty) \longrightarrow \mathbb{R}^n$, the quantity $\chi^u(f) = \limsup_{t\to\infty} \frac{1}{t} \ln |f(t)|$, is called the *upper Lyapunov characteristic exponent of f*. Similarly, one can define the *lower Lyapunov characteristic exponent* $\chi^l(f)$ by taking lim inf instead of lim sup. Here, we focus on the upper Lyapunov characteristic exponent and refer to it as the Lyapunov exponent for brevity. In this context the Euclidean norm is used.

Theorem 5 *Let the DAE* (10) *be regular and the coefficients of its IERODE as well as* $\Pi_{can} D^-$ *be bounded. Then each nontrivial solution of the homogenous DAE has a finite Lyapunov exponent.*

Proof By [1, Theorem 2.3.1] each nontrivial solution u of the homogenous IERODE has a finite Lyapunov exponent. This transfers to the DAE solution by the representation $x = \Pi_{can} D^- u$. □

To get the complete information on the Lyapunov exponents of the solutions of (10), we use *minimal fundamental solution matrices* instead of maximal ones. We regard fundamental solution matrices of different sizes after the idea of Katalin Balla first introduced into [6]. Any matrix function $X \in C^1_D(\mathcal{I}, \mathcal{L}(\mathbb{K}^k, \mathbb{K}^m))$, with $d \le k \le m$ is called *a fundamental solution matrix* of the regular DAE (10) if each of its columns is a solution to (10) and rank $X(t) = d$, for all $t \ge 0$. A fundamental solution matrix is said to be *maximal* if $k = m$ and *minimal* if $k = d$.

One may construct a minimal fundamental solution matrix by solving initial value problems for (10) with d linearly independent, consistent initial vectors arbitrarily chosen from $\operatorname{im} \Pi_{can}(t_0)$.

It is now straightforward to generalize the classical notions of a *normal basis (normal fundamental solution matrix)* and the *Lyapunov spectrum* of the DAE. We refer to [23, Definition 4.2] for the case of strangeness-free DAEs.

Definition 5 For a given minimal fundamental solution matrix X of the regular DAE (10), and for $1 \le i \le d$, we introduce

$$\lambda_i = \limsup_{t\to\infty} \frac{1}{t} \ln |X(t)e_i|,$$

where e_i denotes the i-th unit vector. The columns of a minimal fundamental solution matrix form a *normal basis* if $\Sigma_{i=1}^d \lambda_i$ is minimal with respect to all possible minimal fundamental solution matrices. The λ_i, $i = 1, 2, ..., d$, belonging to a normal basis are called the *Lyapunov exponents* of (10). The set of the Lyapunov exponents is called the *Lyapunov spectrum* of the DAE (10) and denoted by Σ_L.

Simple consequences for the asymptotic stability of the DAE (10) are easily derived by looking at the largest Lyapunov exponent. Namely, if the largest Lyapunov exponent of the DAE (10) is negative, then the DAE is asymptotically stable. In contrary, if the largest Lyapunov exponent is positive, then the DAE is unstable. It is easy to see that the Lyapunov spectrum of a regular DAE is invariant under kinematic equivalent transformation.

Example 3 Given a regular time-invariant DAE (10), i. e., A, D, and B are constant matrices, it is not difficult to show that the Lyapunov spectrum of (10) is the set of the real parts of generalized eigenvalues of matrix pencil $\lambda AD + B$, i. e., $\Sigma_L = \{\operatorname{Re}\lambda, \det(\lambda AD + B) = 0\}$.

By the same argument as in the ODE case [1, Theorem 2.4.2], a normal basis can always be constructed from an arbitrary minimal fundamental solution matrix.

Proposition 2 *For any given minimal fundamental solution matrix X of the regular DAE* (10)*, for which the Lyapunov exponents of the columns are ordered decreasingly, there exists a constant, nonsingular, and upper triangular matrix* $C \in \mathbb{R}^{d\times d}$ *such that the columns of* XC *form a normal basis for* (10).

Next, we investigate the relation between the Lyapunov spectrum of the DAE (10) and that of the correspondent homogeneous EUODE (48).

Proposition 3 *Consider the regular DAE* (10)*. Let* Γ_d^* *be a basis of* $\mathrm{im}\,(D\Pi_{can}D^-)^*$ *and* Γ_d^- *be determined by* (50)*. If both* $\Gamma_d D\Pi_{can}$ *and* $\Pi_{can}D^-\Gamma_d^-$ *are bounded on* $\mathcal{I}$*, then the Lyapunov spectra of the DAE* (10) *and of the correspondent homogeneous EUODE* (48) *coincide.*

Proof Let x be an arbitrary nontrivial solution of (10) and η be the correspondent solution of the homogeneous version of the EUODE (48). By the construction, we have that $u = D\Pi_{can}x$ and $\eta = \Gamma_d u$. Hence, $\eta = \Gamma_d D\Pi_{can}x$, which implies $|\eta(t)| \le \|\Gamma_d D\Pi_{can}\||x(t)|,\ t \in \mathcal{I}$. By the definition, we have $\chi^u(\eta) \le \chi^u(x)$. Conversely, we have that $x = \Pi_{can}D^-u$ and $u = \Gamma_d^-\eta$. Thus, $x = \Pi_{can}D^-\Gamma_d^-\eta$. By a similar argument, the reverse estimate $\chi^u(x) \le \chi^u(\eta)$ holds. Consequently, we have $\chi^u(\eta) = \chi^u(x)$. This means that the Lyapunov exponent of an arbitrary nontrivial solution of (10) and that of the correspondent solution of EUODE (48) are equal. Hence, the columns of a minimal fundamental solution matrix X of (10) form a normal basis if and only those of the correspondent fundamental solution matrix of (48) do so and the sets of their Lyapunov exponents are equal. □

By construction, it holds that

$$\Gamma_d D\Pi_{can}\Pi_{can}D^-\Gamma_d^- = I_d, \quad \Pi_{can}D^-\Gamma_d^-\Gamma_d D\Pi_{can} = \Pi_{can}, \tag{65}$$

which makes clear that the factors $\Gamma_d D\Pi_{can}$ and $\Pi_{can}D^-\Gamma_d^-$ have constant rank d and constitute a factorization of Π_{can}. If both factors are bounded, then Π_{can} is necessarily bounded, too. If Π_{can} is unbounded, then one of these factors must be unbounded at least.

Here, we emphasize once again that the EUODE (48) depends on the choice of the basis Γ_d^*, thus, on the choice of Γ_d.

We say that an EUODE is *spectrum-preserving* if it inherits the Lyapunov spectrum of the DAE.

Clearly, if an EUODE is obtained by Γ_d satisfying the conditions of Proposition 3 then it is spectrum-preserving. Next, we show that, surprisingly, a spectrum-preserving EUODE can always be constructed by means of an appropriately chosen Γ_d. We first study a simple example.

Example 4 We consider the regular index-1 DAE

$$\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}x\right)'(t) + \begin{bmatrix}\alpha & 0\\ 1 & -\frac{1}{\beta(t)}\end{bmatrix}x(t) = 0, \quad t \in [0,\infty), \tag{66}$$

with $\mathbb{K} = \mathbb{R}$, $n = 1$, $m = 2$, $d = 1$, $\alpha \in \mathbb{R}$, and a continuous scalar function β with no zeros. We derive

$$Q_0 = \begin{bmatrix}0 & 0\\ -\beta & 1\end{bmatrix}, D^- = \begin{bmatrix}1\\ \beta\end{bmatrix}, G_1 = \begin{bmatrix}1 & 0\\ 1 & -\frac{1}{\beta}\end{bmatrix}, \Pi_{can} = \begin{bmatrix}1 & 0\\ \beta & 0\end{bmatrix}, \Pi_{can}D^- = \begin{bmatrix}1\\ \beta\end{bmatrix},$$

and further $\Pi_{can}G_1^{-1} = \Pi_{can}$, $D\Pi_{can}D^- = 1$, $D\Pi_{can}G_1^{-1}BD^- = \alpha$. The IERODE reads

$$u'(t) + \alpha u(t) = 0, \quad u = Dx = x_1.$$

The solutions of the DAE have the form

$$x(t) = \Pi_{can}(t)D(t)^{-}u(t) = \begin{bmatrix} 1 \\ \beta(t) \end{bmatrix} \mathrm{e}^{-\alpha t} c, \quad |x(t)| = \underbrace{(1+\beta^2)^{\frac{1}{2}}}_{=:f(t)} \mathrm{e}^{-\alpha t}|c|$$

with a constant c, so that the only Lyapunov exponent of the DAE is $\chi^u(f) - \alpha$.
Though the IERODE is uniquely determined, the EUODE is not. For an arbitrary nonvanishing function $\xi \in \mathcal{C}^1(\mathcal{I}, \mathbb{R})$, we obtain with $\Gamma_d^* = \xi$ a basis of $\operatorname{im}(D\Pi_{can}D^-)^* = \operatorname{im}(D\Pi_{can}D^-)$. This yields $\Gamma_d = \xi$ and $\Gamma_d^- = \frac{1}{\xi}$. Then the associated EUODE results as

$$\eta'(t) + \left(\alpha - \frac{\xi'(t)}{\xi(t)}\right)\eta(t) = 0.$$

The particular choice $\xi(t) \equiv 1$ leads to an EUODE which coincides with the IERODE. However, this EUODE is not necessarily spectrum-preserving, since its Lyapunov exponent is $-\alpha$. This EUODE preserves the spectrum of the DAE, exactly if $\chi^u(f) = 0$, which is the case for a polynomial or bounded function f.
Letting $\xi(t) = (1+\beta(t)^2)^{\frac{1}{2}} = f(t)$ – which seems to be strange for the first glance – we arrive at

$$\Pi_{can}(t)D^-\Gamma_d^-(t) = \begin{bmatrix} 1 \\ \beta(t) \end{bmatrix} \frac{1}{\xi(t)} =: U(t), \quad |U(t)| = 1,\ U(t)^*U(t) = 1,$$

such that

$$|x(t)| = |\Pi_{can}(t)D^-\Gamma_d^-(t)\eta(t)| = |U(t)\eta(t)| = |\eta(t)|.$$

This version of an EUODE is actually spectrum-preserving. Its solutions are

$$\eta(t) = \mathrm{e}^{-\alpha t}(1+\beta(t)^2)^{\frac{1}{2}}c.$$

If the function f is unbounded then so is Π_{can} and Proposition 3 does not apply.

Now we show that any regular DAE possesses a spectrum-preserving EUODE. We proceed as follows. Let U be a continuous matrix function such that its columns form an orthonormal basis of $\operatorname{im}\Pi_{can}D^- = \operatorname{im}\Pi_{can}$. This implies $\operatorname{im}DU = \operatorname{im}D\Pi_{can}D^-$, $U^*U = I_d$, and UU^* represents a projector function such that $\operatorname{im}UU^* = \operatorname{im}\Pi_{can}D^- = \operatorname{im}\Pi_{can}$. Next we put

$$\Gamma_d := U^*\Pi_{can}D^- = U^*\Pi_{can}^*\Pi_{can}D^-, \quad \Gamma_d^- := DU \tag{67}$$

and verify the required properties. First of all, $\Gamma_d^* = (\Pi_{can}D^-)^*U$ forms a basis of $\operatorname{im}(D\Pi_{can}D^-)^*$. Namely, $\Gamma_d^*z = 0$ implies $Uz \in \ker(\Pi_{can}D^-)^* = (\operatorname{im}\Pi_{can}D^-)^\perp$, thus $Uz = 0$, $z = 0$. Then Γ_d^* has full column-rank d. We have further

$$\operatorname{im}\Gamma_d^* \subseteq \operatorname{im}(\Pi_{can}D^-)^* = (\ker\Pi_{can}D^-)^\perp = (\ker D\Pi_{can}D^-)^\perp = \operatorname{im}(D\Pi_{can}D^-)^*.$$

For reasons of dimensions we have $\operatorname{im}\Gamma_d^* = \operatorname{im}(D\Pi_{can}D^-)^*$.
Next we show that Γ_d^- actually satisfies (50). Compute

$$\begin{aligned} \Gamma_d^-\Gamma_d &= D\underbrace{UU^*\Pi_{can}}_{=\Pi_{can}}D^- = D\Pi_{can}D^-,\ \Gamma_d\Gamma_d^- = U^*\Pi_{can}D^-DU \\ &= U^*\underbrace{\Pi_{can}U}_{=U} = U^*U = I_d. \end{aligned}$$

The remaining two relations in (50) are trivially fulfilled. It comes out that (67) determines a possible choice. Derive further

$$\Pi_{can} D^- \Gamma_d^- = \Pi_{can} D^- DU = \Pi_{can} U = U,$$

which proves that $|x(t)| = |U(t)\eta(t)| = |\eta(t)|$ so that the associated EUODE is spectrum-preserving.
Surely, U is not unique in this context. However, the solutions of the EUODEs $\eta' + W\eta = 0$ and $\tilde{\eta}' + \tilde{W}\tilde{\eta} = 0$ corresponding to choices U and $\tilde{U}$, respectively, are related via $\eta = U^*\tilde{U}\tilde{\eta}$, where $U^*\tilde{U}$ is a pointwise orthogonal matrix function. This proves their Lyapunov spectrum to be independent of the special choice of U in (67).

Theorem 6 *Consider the regular DAE* (10). *With a Γ_d chosen as in* (67), *the EUODE* (48) *preserves the Lyapunov spectrum of the DAE* (10). *Furthermore, the so-called Lyapunov's inequality*

$$\sum_{i=1}^{d} \lambda_i \geq \limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Re}\,\mathrm{Trace}\ (-W(s))\, ds \tag{68}$$

holds, where λ_i, $i = 1, 2, \ldots, d$, are the Lyapunov exponents of (10) *and W is the coefficient matrix of the associated EUODE* (48).

Proof Let again x be an arbitrary nontrivial solution of (10) and η be the correspondent solution of the homogeneous EUODE (48). The pointwise orthonormal property of $\Pi_{can} D^- \Gamma_d^-$ immediately implies that $|x(t)| = |\eta(t)|$, $t \in \mathcal{I}$. Hence, the spectra of (10) and of (48) coincide. The inequality (68) follows directly from the well-known Lyapunov's inequality for ODEs [1, Theorem 2.5.1]. □

We emphasize again that in the above construction U, and hence Γ_d, are not unique. However, the quantity on the right-hand side of (68) is independent of the choice of Γ_d chosen in this way.

Definition 6 Let W be the coefficient of a spectrum-preserving EUODE (48) constructed with such a Γ_d from (67) and assume that it is bounded on $\mathcal{I}$. The regular DAE (10) is said to be Lyapunov regular if its Lyapunov exponents satisfies the equality

$$\sum_{i=1}^{d} \lambda_i = \liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Re}\,\mathrm{Trace}\ (-W(s))\, ds.$$

This definition means exactly that the regular DAE (10) is Lyapunov regular if and only if the EUODE used in Theorem 6 is regular in Lyapunov's sense. It is true that this regularity property does not depend on the choice of the basis U in this scope.

Example 5 We continue to study Example 4. We find $W(t) = \alpha - \frac{f'(t)}{f(t)}$ for the DAE (66), further

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \mathrm{Re}\,\mathrm{Trace}\ (-W(s))\, ds = -\alpha + \liminf_{t\to\infty} \frac{1}{t} \ln|f(t)| =: -\alpha + \chi^l(f).$$

Therefore, the DAE (66) is regular in Lyapunov's sense if the upper and lower Lyapunov exponents of f coincide. Let us recall that $f(t) = (1+\beta(t))^{1/2}$ by definition. Thus, Lyapunov regularity is given, for example, if $\beta(t)$ equals e^{-t}, e^{t}, $\mathrm{e}^{\sin t}$, and e^{t^2}, yielding 0, 1, 0, and ∞, for $\chi^u(f) = \chi^l(f)$, respectively. In contrast, for $\beta(t) = \mathrm{e}^{t\sin t}$ it results that $\chi^u(f) = 1$, but $\chi^l(f) = 0$. Then the regular index-1 DAE (66) fails to be Lyapunov regular. □

Proposition 4 *Consider the regular DAE (10). If there exists a basis function $\hat{\Gamma}_d$ such that both $\hat{\Gamma}_d D\Pi_{can}$ and $\Pi_{can}D^-\hat{\Gamma}_d^-$ are bounded on $\mathcal{I}$, then the DAE (10) is Lyapunov regular if and only if the correspondent EUODE $\hat{\eta}' + \hat{W}\hat{\eta} = 0$ is regular in Lyapunov's sense.*

Proof Due to Proposition 3, the correspondent EUODE is spectrum-preserving. Now, we show that under the assumption, the equality

$$\liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \mathrm{Re\,Trace}\,\left(-\hat{W}(s)\right) ds = \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \mathrm{Re\,Trace}\,(-W(s))\, ds$$

holds, where W is the coefficient matrix of the EUODE $\eta' + W\eta = 0$ used in Definition 6. Indeed, since both Γ_d^- and $\hat{\Gamma}_d^-$ are bases in im $D\Pi_{can}D^-$, there exists a pointwise nonsingular function V such that $\hat{\Gamma}_d^- = \Gamma_d^- V$. Due to the pointwise orthonormal property of $\Pi_{can}D^-\Gamma_d^-$, we have

$$|\Pi_{can}D^-\hat{\Gamma}_d^-| = |\Pi_{can}D^-\Gamma_d^- V| = |V|.$$

Consequently V is bounded on $\mathcal{I}$. From the relation $u = \Gamma_d^-\eta = \hat{\Gamma}_d^-\hat{\eta}$, we have $\eta = V\hat{\eta}$.

Next, we prove that V^{-1} is bounded on $\mathcal{I}$, too. Indeed, let us take an arbitrary solution η of the EUODE $\eta' + W\eta = 0$. We consider also the correspondent x and $\hat{\eta}$. We have

$$|V^{-1}\eta| = |\hat{\eta}| = |\hat{\Gamma}_d D\Pi_{can}x| \le \|\hat{\Gamma}_d D\Pi_{can}\||x| = \|\hat{\Gamma}_d D\Pi_{can}\||\eta|.$$

Since η is arbitrarily chosen and $\hat{\Gamma}_d D\Pi_{can}$ is bounded, the boundedness of V^{-1} follows.

The classical Liouville formulas for the two EUODEs lead to

$$\exp\int_{t_0}^{t} \mathrm{Trace}\,(-W(s))\, ds = (\det V(t_0))^{-1}\det V(t)\exp\int_{t_0}^{t} \mathrm{Trace}\,\left(-\hat{W}(s)\right) ds.$$

By taking the logarithm of the modulus of both sides, then dividing by t, we have

$$\frac{1}{t}\int_{t_0}^{t} \mathrm{Re\,Trace}\,(-W(s))\, ds = \frac{1}{t}\ln\left|(\det V(t_0))^{-1}\det V(t)\right| + \frac{1}{t}\int_{t_0}^{t} \mathrm{Re\,Trace}\,\left(-\hat{W}(s)\right) ds.$$

The boundedness of V and V^{-1} implies the exact limit

$$\lim_{t\to\infty}\frac{1}{t}\ln\left|(\det V(t_0))^{-1}\det V(t)\right| = 0.$$

Finally, taking the limit inferior as $t \to \infty$, the required equality is obtained. Thus, the regularities of the EUODEs $\eta' + W\eta = 0$ and $\hat{\eta}' + \hat{W}\hat{\eta} = 0$ simultaneously hold. □

Example 6 Consider once more Example 4. If the expression $f(t) = (1+\beta(t)^2)^{\frac{1}{2}}$ remains bounded, and thus $|\Pi_{can}(t)| = f(t)$, we may turn to the basis $\xi \equiv 1$ yielding $\hat{\Gamma}_d(t) = 1$, $|\hat{\Gamma}_d(t)D\Pi_{can}(t)| = 1$ and $|\Pi_{can}(t)D(t)^-\hat{\Gamma}_d(t)^-| = f(t)$. In contrast, if $f(t)$ growths unboundedly, there is no such factorization with bounded factors. □

We also emphasize that the Lyapunov regularity of the DAE (10) is invariant with respect to kinematic equivalent transformations.

Analogously to the ODE case, see [1, Lemma 3.5.1], it is easy to see that the regular DAE (10) is Lyapunov regular if and only if

(1) there exists the exact limit

$$S = \lim_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} \mathrm{Re\,Trace}\,(-W(s))\, ds$$

and

(2) $\sum_{i=1}^{d} \lambda_i = S$.

As a consequence, we obtain a property of solutions of Lyapunov regular DAE (10) which is already well known in the ODE case.

Corollary 3 *Suppose that the regular DAE* (10) *is Lyapunov regular. Let x be an arbitrary nontrivial solution of* (10)*. Then, x has the sharp Lyapunov exponent, i.e. the exact limit*

$$\lim_{t\to\infty} \frac{1}{t} \ln |x(t)|$$

exists.

Proof The proof comes directly from the fact that $|x(t)| = |\eta(t)|,\ t \in \mathcal{I}$ and the property of solutions of Lyapunov regular ODEs, see [1, Theorem 3.9.1]. □

Finally, we investigate the relation between the Lyapunov regularity of DAE (10) and that of its adjoint DAE (16). To this end, we consider the EUODEs provided by Theorem 3 (2), which are adjoint to each other. We recall that the transformation matrices stated in Theorem 2 have the following explicit form (cf. [21, p. 146])

$$L = \begin{bmatrix} I_d & 0 \\ 0 & (I+\tilde{\mathcal{M}})^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_d D\Pi_{can} \\ \Gamma_0 Q_0 \\ \vdots \\ \Gamma_{\mu-1} D\Pi_{\mu-2} Q_{\mu-1} \end{bmatrix} G_\mu^{-1}$$

and

$$K = \begin{bmatrix} \Gamma_d D\Pi_{can} \\ \Gamma_0 Q_0 \\ \vdots \\ \Gamma_{\mu-1} D\Pi_{\mu-2} Q_{\mu-1} \end{bmatrix}^{-1},$$

where the complete decoupling projector functions are used and the functions $\Gamma_i,\ i = 0, 1, \ldots, \mu-1$, are defined as in [21, p. 143]. Though the matrix function G_μ and its inverse may depend on the special choice of the completely decoupling projector functions, the expression $\Pi_{can} G_\mu^{-1}$ and $G_\mu \Pi_{can} = AD\Pi_{can}$ are invariant.

The structured form (61) is transformed from (16) by $K^*,\ L^*$.

Lemma 2 *Let us denote the variable for the EUODE retrieved from* (61) *by ζ. Then, for the homogenous equations, we have the relations*

$$y = G_\mu^{-*}(\Gamma_d D\Pi_{can})^*\zeta = \left(\Pi_{can} G_\mu^{-1}\right)^* (\Gamma_d D\Pi_{can})^*\zeta, \tag{69}$$

$$\zeta = \left(\Pi_{can} D^- \Gamma_d^-\right)^* G_\mu^* y = \left(\Pi_{can} D^- \Gamma_d^-\right)^* (G_\mu \Pi_{can})^* y. \tag{70}$$

Proof Taking into account the relation $y = L^*\tilde{y}$ and the special structure of $\tilde{y}$ (all the components are zeros, except for the first component ζ), the first equality immediately follows. Now, we show that

$$K = \left[\Pi_{can} D^- \Gamma_d^- \ * \ldots *\right],$$

where $*$s denote certain unknown matrix functions. Indeed, due to Theorem 2, we have $\tilde{x} = K^{-1}x = K^{-1}\Pi_{can}D^{-}\Gamma_d^{-}\eta$, i.e.,

$$\begin{bmatrix} \eta \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \Gamma_d D\Pi_{can} \\ \Gamma_0 Q_0 \\ \vdots \\ \Gamma_{\mu-1}D\Pi_{\mu-2}Q_{\mu-1} \end{bmatrix} \Pi_{can}D^{-}\Gamma_d^{-}\eta.$$

Equivalently, we have

$$\begin{bmatrix} I_d \\ 0 \\ \vdots \\ 0 \end{bmatrix} \eta = K^{-1}\Pi_{can}D^{-}\Gamma_d^{-}\eta.$$

Since this is true for any $\eta \in \mathbb{R}^d$, the formula for K is proven. Therefore, from $\tilde{y} = L^{-*}y$, the second equality follows. □

The following lemma generalizes the relation of the canonical projector functions of the regular DAE (10) and its adjoint (16) in the spirit of Katalin Balla.

Lemma 3 *For the canonical projector functions Π_{can} and $\Pi_{*\,can}$ of the regular DAE (10) and its adjoint (16) it holds that*

$$A^*\Pi_{*\,can}A^{*-} = (D\Pi_{can}D^{-})^*,$$
$$\Pi_{*\,can} = A^{*-}(D\Pi_{can}D^{-})^*A^*.$$

Proof Supposing completely decoupling projector functions associated with the DAE (10) we apply Theorem 3. We find

$$K^{-1}\Pi_{can}K = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}, \quad L^{*\,-1}\Pi_{*\,can}L^* = \begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}.$$

This leads to

$$\Pi_{*\,can} = L^*K^{-1}\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}KL^{*-1} = G_\mu^{-*}\,(\Gamma_d D\Pi_{can})^*\left(\Pi_{can}D^{-}\Gamma_d^{-}\right)^* G_\mu^* = G_\mu^{-*}\Pi_{can}^*G_\mu^*.$$

Regarding $G_\mu\Pi_{can} = AD\Pi_{can}$ we find $\Pi_{*\,can} = G_\mu^{-*}\Pi_{can}^*D^*A^*$, further $A^*\Pi_{*\,can}A^{*-} = A^*G_\mu^{-*}\Pi_{can}^*D^*A^*A^{*-}$. Taking into account that $A^*A^{*-} = R^*$, $D^*A^*A^{*-} = D^*R^* = D^*$ we finally derive

$$\begin{aligned} A^*\Pi_{*\,can}A^{*-} &= A^*G_\mu^{-*}\Pi_{can}^*D^*, \\ A^*G_\mu^{-*}\Pi_{can}^* &= (ADD^{-})^*G_\mu^{-*}\Pi_{can}^* = \left(\Pi_{can}G_\mu^{-1}ADD^{-}\right)^* \\ &= \left(\Pi_{can}P_{\mu-1}\cdots P_0D^{-}\right)^* = \left(\Pi_{can}D^{-}\right)^*, \\ A^*\Pi_{*\,can}A^{*-} &= \left(\Pi_{can}D^{-}\right)^* D^* = \left(D\Pi_{can}D^{-}\right)^*. \end{aligned}$$

The second relation follows from $A^{*-}A^*\Pi_{*can}A^{*-}A^* = \Pi_{*can}$. □

Note that the expression $(AD)^{-} := D^{-}A^{*-*}$ represents a reflexive generalized inverse of AD. From Lemma 3 and its proof, we also have $\Pi_{*\,can}^* = AD\Pi_{can}(AD)^{-}$ or equivalently, $G_\mu\Pi_{can}G_\mu^{-1} = G_\mu\Pi_{can}(AD)^{-}$. Consequently, we obtain the relation $\Pi_{can}G_\mu^{-1} = \Pi_{can}(AD)^{-}$.

Theorem 7 *Let the DAE (10) be regular with tractability index μ and let the auxiliary matrix functions $AD\Pi_{can}$ and $\Pi_{can}(AD)^-$ be bounded on $\mathcal{I}$. Additionally, let such a basis Γ_d^* of* im $(D\Pi_{can}D^-)^*$ *exist, that both $\Gamma_d D\Pi_{can}$ and $\Pi_{can}D^-\Gamma_d^-$ are bounded on $\mathcal{I}$.*
Then the DAE (10) is Lyapunov regular if and only if its adjoint DAE (16) is so. Furthermore, in this case we have the Perron identity

$$\lambda_i + \beta_i = 0, \quad i = 1, 2, \ldots, d,$$

where λ_i are the Lyapunov exponents of (10) in decreasing order and β_i are the Lyapunov exponents of (16) in increasing order.

Proof By Proposition 4, the DAE (10) is Lyapunov regular if and only if the EUODE $\eta' + W\eta = 0$ is so. The solutions y of the adjoint DAE (16) are represented by (69), where ζ solves the adjoint ODE $-\zeta' + W^*\zeta = 0$, which serves as EUODE for (16). By Lemma 2, regarding the boundedness assumptions, the estimates $|y(t)| \leq \|(\Pi_{can}G_\mu^{-1})^*\|\|(\Gamma_d D\Pi_{can})^*\||\zeta(t)|$ and $|\zeta(t)| \leq \|\Pi_{can}D^-\Gamma_d^-)^*\|\|(G_\mu \Pi_{can})^*\||y(t)|$ hold for $t \in \mathcal{I}$. Thus, the EUODE $-\zeta' + W^*\zeta = 0$ is spectrum-preserving, too.
Next we show that Π_{*can} has a bounded factorization, so that Proposition 4 applies to the adjoint DAE (16). Then, the adjoint DAE (16) is Lyapunov regular if and only if the EUODE $-\zeta' + W^*\zeta = 0$ is so. From $\Pi_{*can} = A^{*-}(D\Pi_{can}D^-)^*A^* = A^{*-}\Gamma_d^*\Gamma_d^{-*}A^*$ we derive the factorization

$$\Pi_{*can} = \left(\Pi_{*can}A^{*-}\Gamma_d^*\right)\left(\Gamma_d^{-*}A^*\Pi_{*can}\right) =: F_1F_2,$$

which is associated with the choice of the basis $\Gamma_{*d}^* := \Gamma_d^{-*}$ of im $(D\Pi_{can}D^-)$, and $\Gamma_{*d}^- = \Gamma_d^*$. To show the factors to be bounded we derive

$$\begin{aligned} F_1^* &= \Gamma_d A^{*-*}\Pi_{*can}^* = \Gamma_d A^{*-*}AD\Pi_{can}D^-A^{*-*} = \Gamma_d D\Pi_{can}D^-A^{*-*} \\ &= (\Gamma_d D\Pi_{can})\left(\Pi_{can}D^-A^{*-*}\right), \end{aligned}$$

and

$$F_2 = \left(\Pi_{*can}^*A\Gamma_d^-\right)^* = \left((AD\Pi_{can})(\Pi_{can}D^-\Gamma_d^-)\right)^*.$$

The boundedness conditions agreed upon ensure the boundedness of F_1 and F_2, and Proposition 4 applies to the adjoint DAE.

Recalling the well-known fact in the ODE theory that an ODE is Lyapunov regular if and only if its adjoint is so, the first statement is proved. The second statement follows from the Perron identity, see [1, Theorem 3.6.1], for the Lyapunov exponents of an ODE and its adjoint. □

Remark 1 Under the assumptions of Theorem 7, by [1, Theorem 3.6.1], the Perron identity is not only a necessary condition, but also a sufficient one for the Lyapunov regularity of the DAE (10).

Remark 2 By multiplying both sides of the regular DAE (10) by G_μ^{-1}, we obtain a transformed DAE for which the equality $\tilde{G}_\mu \equiv I$ holds, supposing the same projector functions as before are chosen. Then, the boundedness assumptions stated in Theorem 7 are directed mainly to Π_{can}. However, if either G_μ or G_μ^{-1} are unbounded, this scaling is no longer a strong kinematic equivalent transformation and it may change the stability behavior of the adjoint DAE.
Furthermore, if we assume that the above pair of matrix functions L, K forms a strong kinematic equivalent transformation, then obviously the statements of Theorem 7 remain true.

Finally, we illustrate Theorem 7 by an example.

Example 7 We continue the analysis in Example 4 by checking the Lyapunov regularity and the Perron identity for the DAE

$$\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}x\right)'(t)+\begin{bmatrix}\alpha & 0\\ 1 & \frac{-1}{\beta(t)}\end{bmatrix}x(t)=0,\quad t\in[0,\infty) \tag{71}$$

and its adjoint equation

$$-\begin{bmatrix}1\\0\end{bmatrix}\left(\begin{bmatrix}1 & 0\end{bmatrix}y\right)'(t)+\begin{bmatrix}\alpha & 1\\ 0 & \frac{-1}{\beta(t)}\end{bmatrix}y(t)=0,\quad t\in[0,\infty).$$

The auxiliary matrix functions used in Theorem 7 are

$$AD\Pi_{can}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \Pi_{can}(AD)^{-}=\begin{bmatrix}1 & 0\\ \beta & 0\end{bmatrix},$$

so that the boundedness requirements result in the condition for $f=(1+\beta^2)^{\frac{1}{2}}$ to be bounded.

It is easy to calculate that the IERODE for the adjoint DAE is $v'-\alpha v=0$ and the solutions of the adjoint DAE are $y(t)=\begin{bmatrix}1 & 0\end{bmatrix}^T \mathrm{e}^{\alpha t}c$. Obviously a spectrum-preserving EUODE is $\zeta'-\alpha\zeta=0$. Thus, the adjoint DAE is Lyapunov regular with its only Lyapunov exponent α. Obviously, if f remains bounded, both DAEs are regular and the Perron identity is satisfied. Here, instances are $\beta(t)=\mathrm{e}^{-t}$, $\beta(t)=\mathrm{e}^{\sin t}$.
In contrast, if f is unbounded, the situation changes. For example, with $\beta(t)=\mathrm{e}^{t}$ both DAEs are Lyapunov regular, however, the Perron identity is not satisfied. For $\beta(t)=\mathrm{e}^{t\sin t}$, the adjoint DAE is Lyapunov regular but the original DAE is not.

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