Energy-Based Modeling and Control of Physical Systems

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Outline

Today

- Port-based modeling and Dirac structures
- Definition of port-Hamiltonian systems
- Representations of port-Hamiltonian systems
- Port-Hamiltonian systems and passivity

Coverage of parts of Chapters 3, 4, 5, 6, 7 of

A.J. van der Schaft, D. Jeltsema,
'Port-Hamiltonian Systems Theory: An Introductory Overview,
NOW Publishers, 2014, pdf freely available from my homepage.

From port-based modeling of multi-physics systems to port-Hamiltonian systems: differential equation representations to be used for simulation, analysis and control.
Mathematical formalization of underlying (generalized) geometric (coordinate-free) structure. Dynamics is structured: Hamiltonian function, geometric structure, energy-dissipation, .. .

Outline

The basic picture

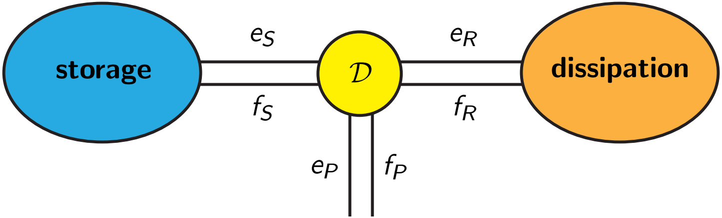


Figure : Port-Hamiltonian system.

The basic elements

Port-based modeling is based on viewing the physical system as the interconnection of ideal energy processing elements, all expressed in (vector) pairs of flow variables $f \in \mathcal{F}$, and effort variables $e \in \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{E}$ are linear spaces of equal dimension.
Furthermore, there is a pairing between $\mathcal{F}$ and $\mathcal{E}$ defining the power

$$< e \mid f >$$

Canonical choice: $\mathcal{E} = \mathcal{F}^*$ with $< e \mid f >= e^T f$.

- Energy-storing elements:

$$\begin{array}{rcl} \dot{x} & = & -f \\ e & = & \frac{\partial H}{\partial x}(x) \end{array}$$

- Energy-dissipating elements:

$$R(f, e) = 0, \quad e^T f \leq 0$$

The basic elements

- Energy-routing elements: generalized transformers, gyrators:

$$f_1 = Mf_2, \; e_2 = -M^T e_1, \quad f = Je, \; J = -J^T$$

They are power-conserving:

$$e^T f = 0$$

- Ideal interconnection constraints

0-junction :
$$e_1 = e_2 = \cdots = e_k, \quad f_1 + f_2 + \cdots + f_k = 0$$
1-junction :
$$f_1 = f_2 = \cdots = f_k, \quad e_1 + e_2 + \cdots + e_k = 0$$
Ideal flow or effort constraints :
$$f = 0, \qquad \text{or } e = 0$$

Also power-conserving:

$$e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0$$

From power-conserving elements to Dirac structures

All power-conserving elements/interconnection constraints have the following properties in common.
They are described by linear equations:

$$Ff + Ee = 0, \quad f, e \in \mathbb{R}^k$$

satisfying

$$e^T f = e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0,$$

while furthermore

$$\operatorname{rank} \begin{bmatrix} F & E \end{bmatrix} = k$$

All power-conserving elements/interconnection constraints will be grouped into one geometric object: the Dirac structure.

Definition of Dirac structures

Definition

A (constant) Dirac structure is a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

(i) $e^T f = 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

For any skew-symmetric map $J : \mathcal{E} \to \mathcal{F}$ its graph given as $\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je\}$ is a Dirac structure !

Alternative definition of Dirac structure

Symmetrized form of power

$$< e \mid f >= e^T f, \quad (f, e) \in \mathcal{F} \times \mathcal{E}.$$

Symmetrization leads to the indefinite bilinear form $\ll, \gg$ on $\mathcal{F} \times \mathcal{E}$:

$$\ll (f^a, e^a), (f^b, e^b) \gg \quad := \quad < e^a \mid f^b > + < e^b \mid f^a >,$$

$$(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{E}.$$

Alternative definition of Dirac structure

Definition

A (constant) Dirac structure is a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp},$$

where $\perp\!\!\!\perp$ denotes orthogonal companion with respect to the bilinear form $\ll, \gg$.

Outline

The three-port model

The port-Hamiltonian system is defined by closing the energy-storing and energy-dissipating ports of the Dirac structure $\mathcal{D}$ by their constitutive relations

$$-\dot{x} = f_S, \frac{\partial H}{\partial x}(x) = e_S$$

respectively

$$R(f_R, e_R) = 0$$

This leads to the DAEs

$$\begin{aligned} &(-\dot{x}(t), f_R(t), f_P(t), \tfrac{\partial H}{\partial x}(x(t)), e_R(t), e_P(t)) \in \mathcal{D} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in \mathbb{R} \\ &R(f_R(t), e_R(t)) = 0 \end{aligned}$$

Thus DAEs of a special form, determined by the Dirac structure and energy-conserving and energy-dissipating constitutive relations.

Energy-balance

Power-conservation

$$e_S^T f_S + e_R^T f_R + e_P^T f_P = 0$$

implies the energy-balance

$$\frac{dH}{dt}(x(t)) = \frac{\partial^T H}{\partial x}(x(t))\dot{x}(t) =$$

$$e_R^T(t) f_R(t) + e_P^T(t) f_P(t) \leq$$

$$e_P^T(t) f_P t)$$

This implies passivity, and that the Hamiltonian H is a candidate Lyapunov function for stability analysis.

Example (The ubiquitous mass-spring system)

Two storage elements:

- Spring Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

$$\begin{array}{rcccl} \dot{q} & = & -f_s & = & \text{velocity} \\ e_s & = & \frac{dH_s}{dq}(q) = kq & = & \text{force} \end{array}$$

- Mass Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

$$\begin{array}{rcccl} \dot{p} & = & -f_m & = & \text{force} \\ e_m & = & \frac{dH_m}{dp}(p) = \frac{p}{m} & = & \text{velocity} \end{array}$$

Example

Dirac structure linking f_s, e_s, f_m, e_m, u, y as

$$f_s = -e_m = -y, \quad f_m = e_s - u$$

Power-conserving since $f_s e_s + f_m e_m + uy = 0$. Yields the port-Hamiltonian system

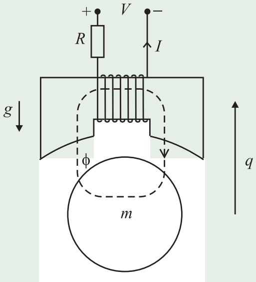
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with

$$H(q,p) = H_s(q) + H_m(p)$$

Example (Electro-mechanical systems)



$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p,\phi) \\ \frac{\partial H}{\partial p}(q,p,\phi) \\ \frac{\partial H}{\partial \varphi}(q,p,\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \quad I = \frac{\partial H}{\partial \varphi}(q,p,\phi)$$

Coupling electrical/mechanical domain via Hamiltonian $H(q,p,\phi)$

$$H(q,p,\varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2L(q)}$$

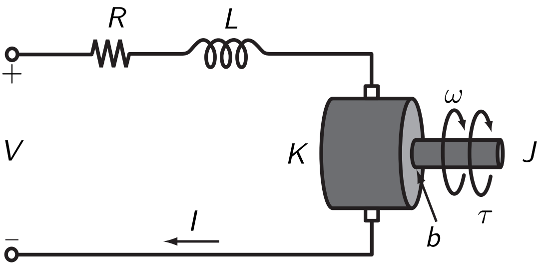


Figure : DC motor.

6 interconnected subsystems:

◦ 2 energy-storing elements: inductor L with state φ (flux), and rotational inertia J with state p (angular momentum);

◦ 2 energy-dissipating elements: resistor R and friction b;

◦ gyrator K;

◦ voltage source V.

The energy-storing elements (here assumed to be linear) are given by

$$\text{Inductor:} \begin{cases} \dot{\varphi} = -V_L \\ I = \dfrac{d}{d\varphi}\left(\dfrac{1}{2L}\varphi^2\right) = \dfrac{\varphi}{L}, \end{cases}$$

$$\text{Inertia:} \begin{cases} \dot{p} = -\tau_J \\ \omega = \dfrac{d}{dp}\left(\dfrac{1}{2J}p^2\right) = \dfrac{p}{J} \end{cases}$$

Hence, the corresponding total Hamiltonian reads $H(p, \phi) = \frac{1}{2L}\phi^2 + \frac{1}{2J}p^2$. The energy-dissipating relations (also assumed to be linear) are

$$V_R = -RI, \quad \tau_b = -b\omega,$$

with $R, b > 0$, where τ_b is a damping torque. The equations of the gyrator (converting magnetic power into mechanical, and conversely) are

$$V_K = -K\omega, \quad \tau_K = KI,$$

with K the gyrator constant.

The subsystems are interconnected by the equations

$$V_L + V_R + V_K + V = 0, \quad \tau_J + \tau_b + \tau_K + \tau = 0.$$

The Dirac structure is defined by the interconnection equation, together with the equations for the gyrator.

This results in the port-Hamiltonian model

$$\begin{bmatrix} \dot{\varphi} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -R & -K \\ K & -b \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V \\ \tau \end{bmatrix},$$

$$\begin{bmatrix} I \\ \omega \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\varphi}{L} \\ \frac{p}{J} \end{bmatrix}$$

Outline

Example (Bouncing pogo-stick)

Consider a vertically bouncing pogo-stick consisting of a mass m and a massless foot, interconnected by a linear spring (stiffness k and rest-length x_0) and a linear damper d.

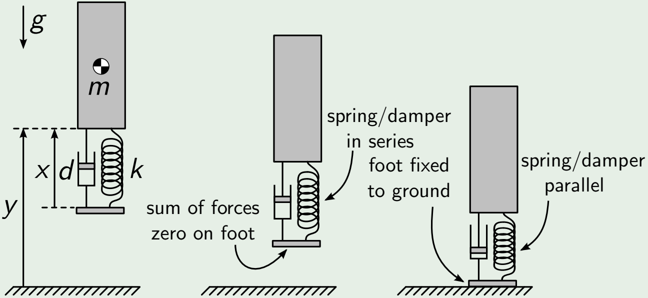


Figure : Model of a bouncing pogo-stick: definition of the variables (left), situation without ground contact (middle), and situation with ground contact (right).

Example

The mass moves vertically under the influence of gravity g until the foot touches the ground.

The state variables of the system are x (length of the spring), y (height of the bottom of the mass), and p (momentum of the mass, defined as $p := m\dot{y}$).

The Hamiltonian of the system equals

$$H(x, y, p) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}p^2$$

where y_0 is the distance from the bottom of the mass to its center of mass.

Furthermore, the switching is described by a binary variable s with values $s = 0$ (no contact) and $s = 1$ (contact).

Example

When the foot is not in contact with the ground total force on the foot is zero (since it is massless), which implies that the spring and damper force must be equal but opposite.

For $s = 0$ (no contact) the system is described by the port-Hamiltonian system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} y \\ p \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} mg \\ \frac{p}{m} \end{bmatrix} \\ -d\dot{x} &= k(x - x_0) \end{aligned}$$

while for $s = 1$ the port-Hamiltonian description is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -d \end{bmatrix}\begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}$$

Example

The two situations can be taken together into one port-Hamiltonian system with variable Dirac structure:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} \frac{s-1}{d} & 0 & s \\ 0 & 0 & 1 \\ -s & -1 & -sd \end{bmatrix} \begin{bmatrix} k(x-x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}$$

We obtain a switching port-Hamiltonian system, specified by a Dirac structure $\mathcal{D}_s$ depending on the switch position $s \in \{0,1\}^n$ (here n denotes the number of independent switches).

Every switching may be internally induced (like in the case of a diode in an electrical circuit or an impact in a mechanical system) or externally triggered (like an active switch in a circuit or mechanical system).

Switching dynamics

Definition

The dynamics of the switching port-Hamiltonian system is given as

$$(-\dot{x}(t), \frac{\partial H}{\partial x}(x(t)), -Re_R(t), e_R(t), f_P(t), e_P(t)) \in D_s$$

at all time instants t during which the system is in switch configuration s.

It follows from the power-conservation property of Dirac structures that during the time-interval in which the system is in a fixed switch configuration

$$\frac{d}{dt}H = -e_R^T Re_R + e_P^T f_P \leq e_P^T f_P,$$

showing passivity in every mode.

However, the conditions for a particular switch configuration s may entail algebraic constraints on the state variables x: need for reset rules, cf. Chapter 13.

Outline

Useful subclass: Input-state-output port-Hamiltonian systems

Consider a Dirac structure $\mathcal{D}$ given as the graph of the skew-symmetric map

$$\begin{bmatrix} f_S \\ f_P \end{bmatrix} = \begin{bmatrix} -J & -g \\ g^T & 0 \end{bmatrix} \begin{bmatrix} e_S \\ e_P \end{bmatrix},$$

leading ($f_S = -\dot{x}, e_S = \frac{\partial H}{\partial x}(x)$) to a port-Hamiltonian system as before

$$\begin{aligned} \dot{x} &= J\frac{\partial H}{\partial x}(x) + g e_P, \quad x \in \mathcal{X}, e_P \in \mathbb{R}^m \\ f_P &= g^T \frac{\partial H}{\partial x}(x), \qquad f_P \in \mathbb{R}^m \end{aligned}$$

with input $e_P = u$ and output $f_P = y$.

Energy-dissipation is included by terminating some of the ports by linear resistive elements

$$f_R = -\tilde{R} e_R, \quad \tilde{R} = \tilde{R}^T \geq 0$$

Writing out

$$\dot{x} = J \frac{\partial H}{\partial x}(x) + g_R f_R + g e_P, \quad e_R = g_R^T \frac{\partial H}{\partial x}(x)$$

this leads to an input-state-output port-Hamiltonian system given as

$$\begin{aligned} \dot{x} &= [J - R] \tfrac{\partial H}{\partial x}(x) + gu \\ y &= g^T \tfrac{\partial H}{\partial x}(x) \end{aligned}$$

where $u = e_P$, $y = f_P$ and

$$R(x) = g_R \tilde{R} g_R^T \geq 0$$

Extended definition of input-state-output port-Hamiltonian systems

$$
\begin{aligned}
\dot{x} &= [J - R]\tfrac{\partial H}{\partial x}(x) + [G - P]u \\
y &= [G^T + P^T]\tfrac{\partial H}{\partial x}(x) + [M + S]u
\end{aligned}
$$

with

$$
M = -M^T, \quad \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \geq 0
$$

implying

$$
\frac{d}{dt}H = u^T y - \begin{bmatrix} e_S^T & u^T \end{bmatrix} \begin{bmatrix} R & P \\ P^T & S \end{bmatrix} \begin{bmatrix} e_S \\ u \end{bmatrix} \leq u^T y
$$

Consider an LC-circuit consisting of two capacitors and one inductor

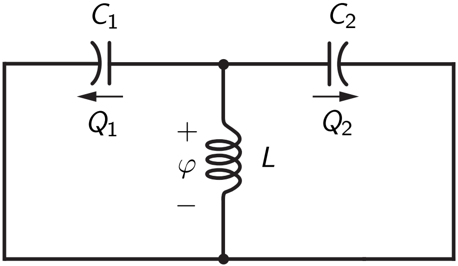


Figure : LC circuit.

The capacitors (first assumed to be linear) are described by the following dynamical equations

$$\dot{Q}_i = -I_i,$$
$$V_i = \frac{Q_i}{C_i},$$

for $i = 1, 2$. Here I_i and V_i are the currents through, respectively the voltages across, the two capacitors, and C_i are their capacitances. Furthermore, Q_i are the charges stored in the capacitors.
Similarly, the linear inductor is described by the dynamical equations

$$\dot{\varphi} = -V_L,$$
$$I_L = \frac{\varphi}{L},$$

where I_L is the current through the inductor, and V_L is the voltage across the inductor. Here, the (magnetic) flux-linkage φ is taken as the state variable of the inductor, and L denotes its inductance.

Parallel interconnection of these three subsystems by Kirchhoff's laws amounts to the interconnection equations

$$V_1 = V_2 = V_L, \quad I_1 + I_2 + I_L = 0,$$

where the equation $V_1 = V_2$ gives rise to the algebraic constraint

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2},$$

relating the two state variables Q_1, Q_2.

One way to represent the dynamics is to regard either I_1 or I_2 as a Lagrange multiplier for the constraint. By defining $\lambda = I_1$

$$\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \lambda,$$

$$0 = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix},$$

Eliminate the Lagrange multiplier λ by premultiplying by

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Together with the algebraic constraint this yields differential-algebraic system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix}.$$

The two equivalent equational representations result from two different representations of the Dirac structure of the system, namely

$$\mathcal{D} = \left\{ (f, e) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0 \right\},$$

and

$$\mathcal{D} = \left\{ (f, e) \in \mathbb{R}^3 \times \mathbb{R}^3 \;\middle|\; \exists \lambda \text{ such that} \right.$$

$$\left. -\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \lambda, \; 0 = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \right\}.$$

Furthermore, the energy-storing relations are given by

$$f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = - \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix}, \quad e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} Q_1/C_1 \\ Q_2/C_2 \\ \varphi/L \end{bmatrix},$$

where the last vector is the gradient vector of the total stored energy

$$H(Q_1, Q_2, \varphi) := \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{\varphi^2}{2L}.$$

Towards DAE analysis

In general, port-Hamiltonian systems will have index one.
Consider the nonlinear port-Hamiltonian system

$$\begin{aligned} \dot{x} &= J\tfrac{\partial H}{\partial x}(x) + g\lambda, \quad J = -J^T \\ 0 &= g^T\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

Then differentiation of the algebraic constraints yields

$$0 = \frac{d}{dt} g^T \frac{\partial H}{\partial x}(x) = g^T \frac{\partial^2 H}{\partial x^2}(x) g\lambda + \star,$$

which can be solved for λ as long as the rank of

$$g^T \frac{\partial^2 H}{\partial x^2}(x) g$$

is equal to the rank of g.
Thus if the Hessian $\frac{\partial^2 H}{\partial x^2}(x)$ is invertible the system has index one.

Nonlinear resistive structures

An input-state-output port-Hamiltonian system with nonlinear resistive structure is given as

$$\begin{array}{rcl}\dot{x} & = & Jz - \mathcal{R}(z) + gu, \quad z = \frac{\partial H}{\partial x}(x) \\ y & = & g^T z\end{array}$$

where $J = -J^T$, and $R(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$z^T R(z) \geq 0, \quad \text{for all } z \in \mathbb{R}^n$$

Like for linear resistive structures we obtain

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -\frac{\partial^T H}{\partial x}(x)R(\frac{\partial H}{\partial x}(x)) + y^T u \leq u^T y$$

Coulomb friction

Consider a mass-spring-damper system (mass m, spring constant k, momentum p, spring extension q, external force F) subject to ideal Coulomb friction

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} - \begin{bmatrix} 0 \\ c \operatorname{sign} \frac{p}{m} \end{bmatrix} + \begin{bmatrix} 0 \\ F \end{bmatrix},$$

where sign is the multi-valued function defined by

$$\operatorname{sign} v := \begin{cases} 1 & , \quad v > 0 \\ [-1, 1] & , \quad v = 0 \\ -1 & , \quad v < 0 \end{cases}$$

The multi-valued function $c \operatorname{sign}$ defines a (further generalized) nonlinear resistive structure, and the Hamiltonian $H(q,p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$ satisfies

$$\frac{d}{dt}H = -\frac{p}{m} \operatorname{sign} \frac{p}{m} + F\frac{p}{m} \leq F\frac{p}{m}$$

Extends directly to more complicated friction characteristics.

Outline

Network structure of chemical reaction networks

Motivation: systems biology deals with large-scale metabolic reaction networks.

Consider a chemical reaction network with m chemical species, among which r chemical reactions take place.
The dynamics of the vector of concentrations $x \in \mathbb{R}^m_+$ has the form

$$\dot{x} = Sv(x),$$

where the stoichiometric matrix S is an $m \times r$ matrix consisting of (positive and negative) integer elements, and $v(x) \in \mathbb{R}^r$ is the vector of reaction rates.

Network structure ?

The complexes are the left- and right-hand sides of the reactions, and define the vertices of a directed graph, with edges corresponding to the reactions, and $c \times r$ incidence matrix D. Basic relation

$$S = ZD$$

with Z the $m \times c$ composition matrix (i-th column specifies i-th complex).

Z defines a representation of the graph of complexes into $\mathbb{R}^m$.

Mass action kinetics

The most basic way to define the reaction rate $v(x)$ is mass action kinetics.

For example, for

$$X_1 + 2X_2 \to X_3$$

the mass action kinetics reaction rate is given as

$$v(x) = kx_1x_2^2,$$

with $k > 0$ a reaction constant.

This can be alternatively written as

$$v(x) = k \exp\left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} \ln x_1 \\ \ln x_2 \end{bmatrix} \right) = k \exp(Z^T \mathrm{Ln}\ x)$$

where $Z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathrm{Ln}\, x = \begin{bmatrix} \ln x_1 \\ \ln x_2 \end{bmatrix}$.

This leads to the description of the dynamics of the reaction network as

$$\dot{x} = ZDC\mathrm{Exp}\,(Z^T \mathrm{Ln}\; x)$$

where the $c \times c$ matrix $L := -DC$ is a flow-Laplacian matrix.

How to analyze these dynamics ?

The reaction network is called complex-balanced if there exists an equilibrium $x^* \in \mathbb{R}^m_+$, called a complex-balanced equilibrium, satisfying

$$Dv(x^*) = -L\mathrm{Exp}\,(Z^T \mathrm{Ln}\; x^*) = 0$$

Chemically this means that at the complex-balanced equilibrium x^* not only the chemical species but also the complexes remain constant; i.e., for each complex the total inflow (from the other complexes) equals the total outflow (to the other complexes).

Defining the diagonal matrix

$$\mathcal{K}(x^*) := \text{diag}\left(\exp(Z_i{}^T \mathrm{Ln}\, x^*)\right)_{i=1,\cdots,c},$$

this implies that the dynamics can be rewritten into the form

$$\dot{x} = -Z\mathcal{L}(x^*)\mathrm{Exp}\,(Z^T \mathrm{Ln}\,(\frac{x}{x^*})), \quad \mathcal{L}(x^*) := L\mathcal{K}(x^*),$$

with $\mathcal{L}(x^*)$ balanced.
By convexity of the exponential function not only $\mathcal{L}(x^*) + (\mathcal{L}(x^*))^T \geq 0$ but also

Proposition

For any $\gamma \in \mathbb{R}^r$ we have

$$\gamma^T \mathcal{L}(x^*)\mathrm{Exp}\,(\gamma) \geq 0$$

with equality if and only if $D^T\gamma = 0$.

Chemical interpretation

Up to a constant we may interpret

$$\mathrm{Ln}\left(\frac{x}{x^*}\right)$$

as the vector of chemical potentials, and the Gibbs' free energy as

$$G(x) = \sum_{i=1}^{m} x_i \ln \frac{x_i}{x_i^*} + x_i^* - x_i,$$

Then $\frac{\partial G}{\partial x}(x) = \mathrm{Ln}\left(\frac{x}{x^*}\right)$ and thus the dynamics is also given by

$$\dot{x} = -Z\mathcal{L}(x^*)\mathrm{Exp}\left(Z^T \frac{\partial G}{\partial x}(x)\right)$$

which is a port-Hamiltonian system with Hamiltonian G and nonlinear resistive structure $\mathcal{R}(z) = Z\mathcal{L}(x^*)\mathrm{Exp}\left(Z^T z\right)$.

Analysis

This enables to derive a number of key properties of the reaction network dynamics; similar to those for mass-damper systems.

- All positive equilibria are in fact complex-balanced equilibria, and given one positive complex-balanced equilibrium x^* the set of all positive equilibria is given by

$$\mathcal{E} := \{x^{**} \in \mathbb{R}^m_+ \mid S^T \mathrm{Ln}\,(x^{**}) = S^T \mathrm{Ln}\,(x^*)\}$$

- There exists for any initial condition $x_0 \in \mathbb{R}^m_+$ a unique $x^{**} \in \mathcal{E}$ such that $x^{**} - x_0 \in \operatorname{im} S$. By using

$$\frac{d}{dt} G \leq 0$$

  it follows that the vector of concentrations $x(t)$ starting from x_0 will converge for $t \to \infty$ to x^{**} if the reaction network is persistent.

Conclusion so far

Network modeling is prevailing in modeling and simulation of lumped-parameter physical systems (multi-body systems, electrical circuits, electro-mechanical systems, hydraulic systems, robotic systems, etc.), with many advantages:

- Modularity and flexibility. Re-usability ('libraries').
- Multi-physics approach.
- Suited to design/control.

Disadvantage of network modeling: it generally leads to a large set of DAEs, seemingly without any structure.

Main message:

Port-based modeling and port-Hamiltonian system theory identifies the underlying structure of network models of physical systems, to be used for analysis, simulation and control.

Outline

For many systems, especially those with 3-D mechanical components, the interconnection structure will be modulated by the energy or geometric variables.
This leads to the notion of (non-constant) Dirac structures on manifolds.

Definition

Consider a smooth manifold $\mathcal{X}$. A Dirac structure on $\mathcal{X}$ is a vector subbundle $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ such that for every $x \in \mathcal{X}$ the vector space

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$$

is a Dirac structure as before.

Example (Mechanical systems with kinematic constraints)

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$

This leads to *constrained* Hamiltonian equations

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)f \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \\
e &= B^T(q)\frac{\partial H}{\partial p}(q,p)
\end{aligned}$$

with $H(q,p)$ total energy, and $A(q)\lambda$ the constraint forces.
Dirac structure $\mathcal{D}$ is defined by the symplectic form on $\mathcal{X} = T^*Q$ together with constraints $A^T(q)\dot{q} = 0$ and force matrix $B(q)$.

Can be systematically extended to general multi-body systems.

Example (Rolling coin)

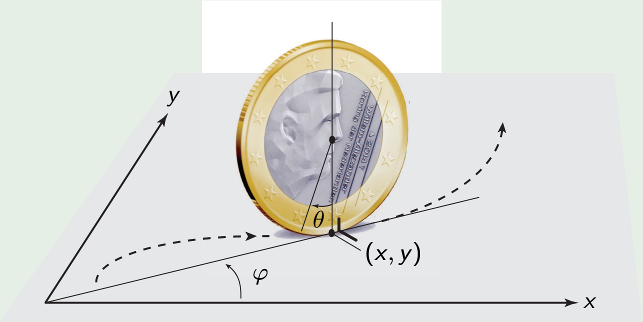


Figure : The geometry of the rolling euro

Example

Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, φ denotes the heading angle, and θ the angle of the coin. The rolling constraints (rolling without slipping) are (set all parameters equal to 1)

$$\dot{x} = \dot{\theta} \cos \varphi, \quad \dot{y} = \dot{\theta} \sin \varphi$$

The total energy is

$$H = \frac{1}{2} p_x^2 \; + \; \frac{1}{2} p_y^2 \; + \; \frac{1}{2} p_\theta^2 \; + \; \frac{1}{2} p_\varphi^2$$

and the constraints thus can be rewritten in the form $A^T(q)\frac{\partial H}{\partial p}(q,p) = 0$ as

$$p_x - p_\theta \cos \varphi = 0, \quad p_y - p_\theta \sin \varphi = 0.$$

Canonical coordinates

Any constant Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented as follows. There exist linear coordinates $x = (q, p, r, s)$ for $\mathcal{F}$ (with $\dim q = \dim p$), such that in the corresponding bases for (f_q, f_p, f_r, f_s) for $\mathcal{F}$ and (e_q, e_p, e_r, e_s) for $\mathcal{F}^*$, the Dirac structure is given as

$$\begin{cases} f_q & = & -e_p \\ f_p & = & e_q \\ f_r & = & 0 \\ e_s & = & 0 \end{cases}$$

In such canonical coordinates a port-Hamiltonian system on $\mathcal{X} = \mathcal{F}$ without energy-dissipating and external ports can be written as

$$\begin{array}{rcl} \dot{q} & = & \frac{\partial H}{\partial p}(q, p, r, s) \\ \dot{p} & = & -\frac{\partial H}{\partial q}(q, p, r, s) \\ \dot{r} & = & 0 \\ 0 & = & \frac{\partial H}{\partial s}(q, p, r, s) \end{array}$$

Excursion to integrability

A Dirac structure on a manifold $\mathcal{X}$ is integrable if it is possible to find local coordinates such that, in these coordinates, the Dirac structure becomes a constant Dirac structure, that is, it is not modulated anymore by the state variables.Thus then there also exist canonical coordinates.

First case

Let the modulated Dirac structure $\mathcal{D}$ be given for every $x \in \mathcal{X}$ as the *graph* of a skew-symmetric mapping $J(x)$ from the co-tangent space $T_x^*\mathcal{X}$ to the tangent space $T_x\mathcal{X}$. Integrability in this case means that $J(x)$ satisfies the conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \quad i, j, k = 1, \ldots, n$$

In this case we may find, by Darboux's theorem around any point x_0 where the rank of the matrix $J(x)$ is constant, local canonical coordinates $x = (q, p, r)$ in which the matrix $J(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & -I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $J(x)$ defines a Poisson bracket on $\mathcal{X}$, given for every $F, G : \mathcal{X} \to \mathbb{R}$ as

$$\{F, G\} = \frac{\partial^T F}{\partial x} J(x) \frac{\partial G}{\partial x}$$

Indeed, by the integrability condition the Jacobi-identity holds:

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$$

for all functions F, G, K.

Second case

A similar story holds for a Dirac structure given as the graph of a skew-symmetric mapping $\omega(x)$ from the tangent space $T_x\mathcal{X}$ to the co-tangent space $T^*_x\mathcal{X}$. In this case the integrability conditions take the form

$$\frac{\partial \omega_{ij}}{\partial x_k}(x) + \frac{\partial \omega_{ki}}{\partial x_j}(x) + \frac{\partial \omega_{jk}}{\partial x_i}(x) = 0, \quad i,j,k = 1,\ldots,n$$

The skew-symmetric matrix $\omega(x)$ can be regarded as the coordinate representation of a differential two-form ω on $\mathcal{X}$, that is $\omega = \sum_{i=1,j=1}^{n} dx_i \wedge dx_j$, and the integrability condition corresponds to the closedness of this two-form ($d\omega = 0$).

The differential two-form ω is called a pre-symplectic structure, and a symplectic structure if the rank of $\omega(x)$ is equal to the dimension of $\mathcal{X}$. By a version of Darboux's theorem we may find, around any point x_0 where the rank of the matrix $\omega(x)$ is constant, local coordinates $x = (q, p, s)$ in which the matrix $\omega(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For general Dirac structures, integrability is defined as

Definition

A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is integrable if for arbitrary pairs of smooth vector fields and differential one-forms $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$ there holds

$$< L_{X_1}\alpha_2 \mid X_3 > + < L_{X_2}\alpha_3 \mid X_1 > + < L_{X_3}\alpha_1 \mid X_2 >= 0$$

with L_{X_i} denoting the Lie-derivative.

The Dirac structure corresponding to mechanical systems with kinematic constraints

$$\mathcal{D} = \{(f_q, f_p, e_q, e_p) \mid f_q = -e_p, f_p = e_q + A(q)\lambda, A^T(q)e_p = 0\}$$

is integrable if and only if the kinematic constraints

$$A^T(q)\dot{q} = 0$$

are holonomic, which means that it is possible to find configuration coordinates $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0 \ ,$$

In this case one may eliminate the configuration variables $\overline{q}_{n-k+1}, \ldots, \overline{q}_n$, since the kinematic constraints are equivalent to the geometric constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \ldots, \overline{q}_n = c_n \ ,$$

for certain constants $c_{n-k+1}, \ldots, c_n$ determined by the initial conditions.

Input-state-output port-Hamiltonian systems on manifolds

Consider a modulated (non-constant) Dirac structure $\mathcal{D}$ given as the graph of the skew-symmetric map

$$\begin{bmatrix} f_S \\ f_P \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_S \\ e_P \end{bmatrix},$$

leading ($f_S = -\dot{x}, e_S = \frac{\partial H}{\partial x}(x)$) to a port-Hamiltonian system as before

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)e_P, \quad x \in \mathcal{X}, e_P \in \mathbb{R}^m \\ f_P &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad\quad f_P \in \mathbb{R}^m \end{aligned}$$

with input $e_P = u$ and output $f_P = y$.

Energy-dissipation is included by terminating some of the ports by linear resistive elements

$$f_R = -\tilde{R}(x)e_R, \quad \tilde{R}(x) = \tilde{R}^T(x) \geq 0$$

Writing out

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) + g_R(x)f_R + g(x)e_P, \quad e_R = g_R^T(x)\frac{\partial H}{\partial x}(x)$$

this leads to an input-state-output port-Hamiltonian system given as

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

where

$$R(x) = g_R(x)\tilde{R}(x)g_R^T(x) \geq 0$$

Common class of PH systems for control purposes.

Outline

Equational representations: start from the Dirac structure

Dirac structures, and therefore port-Hamiltonian systems, admit different equational representations, with different properties for analysis, simulation, and control.
Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with $\dim \mathcal{F} = n$, be a Dirac structure.

1. Kernel and Image representation

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\}$$

for $n \times n$ matrices F and E (possibly depending on x !) satisfying

$$(i) \quad EF^T + FE^T = 0,$$

$$(ii) \quad \operatorname{rank}[F \vdots E] = n.$$

It follows that $\mathcal{D}$ can be also written in image representation as
$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = E^T\lambda, e = F^T\lambda, \lambda \in \mathbb{R}^n\}$.

In case of the presence of an energy-conserving port S, an energy-dissipating port R and an external port the Dirac structure $\mathcal{D}$ can thus be given in kernel representation as

$$\begin{aligned}\mathcal{D} \; = \; & \Big\{ (f_S, e_S, f_R, e_R, f_P, e_P) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F}_R \times \mathcal{F}_R^* \times \mathcal{F} \times \mathcal{F}^* \mid \\ & F_S f_S + E_S e_S + F_R f_R + E_R e_R + F_P f_P + E_P e_P = 0 \Big\}\end{aligned}$$

with

$$\begin{array}{ll} (i) & E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E F^T + F E^T = 0 \\ (ii) & \operatorname{rank} \big[F_S \mid E_S \mid F_R \mid E_R \mid F \mid E \big] = \dim(\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}) \end{array}$$

Then the port-Hamiltonian system is given by the set of DAEs

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E_R e_R(t) + F_P f_P(t) + E_P e_P(t),$$

where the vectors f_R, e_R additionally satisfy the energy-dissipating constitutive relations; e.g., $f_R(t) = -R e_R(t), t \geq 0$.

2. Constrained input-output representation

Every Dirac structure $\mathcal{D}$ can be written as

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda, G^T e = 0\}$$

for a skew-symmetric matrix J and a matrix G such that

$$\text{im } G = \{f \mid (f,0) \in \mathcal{D}\}.$$

Equivalently

$$\ker\, G^T = \{e \mid \exists f \text{ s.t. } (f,e) \in \mathcal{D}\}.$$

Furthermore, $\ker J = \{e \mid (0,e) \in \mathcal{D}\}$.

In the absence of energy-dissipating and external ports it follows that any port-Hamiltonian system can be represented as

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + G(x)\lambda, \quad J(x) = -J^T(x) \\ 0 &= G^T(x)\frac{\partial H}{\partial x}(x) \end{aligned}$$

A typical example are mechanical systems with kinematic constraints

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \quad (= A^T(q)\dot{q}) \end{aligned}$$

where $A(q)\lambda$ are the constraint forces.

3. Hybrid input-output representation

Let $\mathcal{D}$ be given by square matrices E and F as in 1. Suppose rank $F = m (\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \vdots F_2]$, and correspondingly $E = [E_1 \vdots E_2], f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$.

Then the matrix $[F_1 \vdots E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \middle| \begin{bmatrix} f_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \right\}$$

with $J := -[F_1 \vdots E_2]^{-1}[F_2 \vdots E_1]$ skew-symmetric.

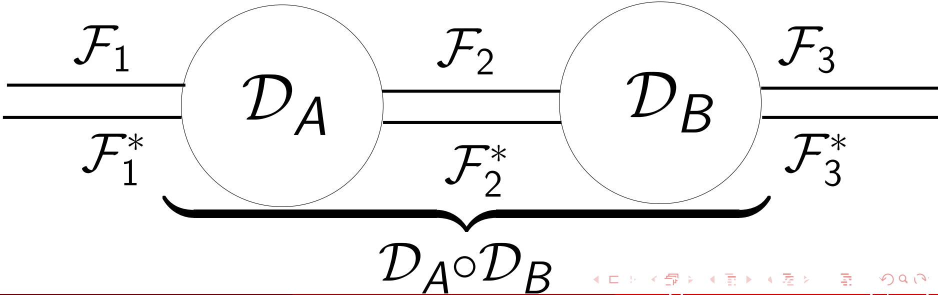
Outline

Interconnection port-Hamiltonian systems, and composition of Dirac structures

The composition of two Dirac structures with partially shared variables is again a Dirac structure:

$$\mathcal{D}_{12} \subset \mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_2 \times \mathcal{V}_2^*$$

$$\mathcal{D}_{23} \subset \mathcal{V}_2 \times \mathcal{V}_2^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$$



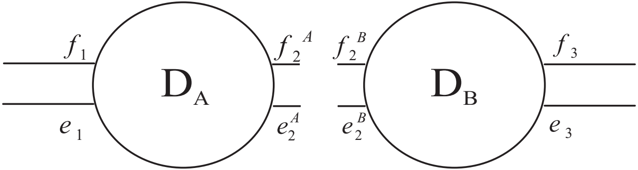


Figure : Standard interconnection

$$
\begin{array}{rcll}
f_2^A & = & -f_2^B & \in \mathcal{F}_2 \\
e_2^A & = & e_2^B & \in \mathcal{F}_2^*
\end{array}
$$

The feedback interconnection

$$
\begin{array}{rcl}
f_2^A & = & -e_2^B \\
e_2^B & = & f_2^B
\end{array}
$$

can be easily transformed to this case.

Thus

$$\mathcal{D}_A \circ \mathcal{D}_B := \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^*$$

$$\text{such that}$$

$$(f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\}$$

Theorem

Let $\mathcal{D}_A$, $\mathcal{D}_B$ be Dirac structures (defined with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, respectively $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$ and their bilinear forms). Then $\mathcal{D}_A \circ \mathcal{D}_B$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.

Explicit expressions for the composition

The composition $\mathcal{D}_A \circ \mathcal{D}_B$ results from considering

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0,$$

and then eliminating f_2, e_2. To this purpose, consider

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be matrices with

$$L = [L_A \,|\, L_B]\,, \qquad \ker L = \operatorname{im} M$$

Premultiplication of the equations by the matrix $L := [L_A \,|\, L_B]$ results in

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0$$

Consequence

The interconnection of a number of port-Hamiltonian systems $(\mathcal{X}_i, \mathcal{D}_i, H_i)$, $i = 1, \cdots, k$, through an interconnection Dirac structure $\mathcal{D}_I$ is a port-Hamiltonian system $(\mathcal{X}, \mathcal{D}, H)$, with

$$H = H_1 + \cdots + H_k,$$

$$\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$$

and $\mathcal{D}$ the composition of $\mathcal{D}_1, \cdots, \mathcal{D}_k, \mathcal{D}_I$.

Outline

Recall of passivity

A nonlinear system

$$\Sigma : \begin{array}{rcll} \dot{x} & = & f(x) + g(x)u, & u \in \mathbb{R}^m \\ y & = & h(x), & y \in \mathbb{R}^m \end{array}$$

is passive if there exists a storage function $S : \mathcal{X} \to \mathbb{R}$ with $S(x) \geq 0$ for every x, such that

$$S(x(t_2)) - S(x(t_1)) \leq \int_{t_1}^{t_2} y^T(t)u(t)dt$$

for all solutions $(u(\cdot), x(\cdot), y(\cdot))$ and times $t_1 \leq t_2$.

The system is lossless if $\leq$ is replaced by $=$.

If S is differentiable then being passive is equivalent to

$$\frac{d}{dt} S \leq y^T u$$

which reduces to

$$\begin{aligned} \frac{\partial^T S}{\partial x}(x) f(x) &\leq 0 \\ h(x) &= g^T(x) \frac{\partial S}{\partial x}(x) \end{aligned}$$

while in the lossless case $\leq$ is replaced by $=$.
In the linear case

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

is passive if there exists a quadratic storage function $S(x) = \frac{1}{2} x^T Q x$, with $Q = Q^T \geq 0$ satisfying the LMIs

$$A^T Q + QA \leq 0, \quad C = B^T Q$$

Theorem

Σ is passive if and only if the function

$$S_a(x) = \sup_{T,u(\cdot)} - \int_0^T u^T(t)y(t)dt$$

is finite for every initial condition $x \in \mathcal{X}$ at $t = 0$, where $y(t) = h(x(t))$ is the output resulting from initial condition x at time 0 resulting from applying the input function on $[0, T]$.
Furthermore, if S_a is well-defined then $S_a \geq 0$, and S_a is itself a storage function: the available storage function. In fact, it is the minimal storage function:

$$S_a(x) \leq S(x), \quad x \in \mathcal{X}$$

for every other storage function S.

Under additional conditions there also exists a maximal storage function. In the controllable linear case, there exists $0 \leq Q_a \leq Q_r$ such that any other storage function $S(x) = \frac{1}{2}x^T Qx$ satisfies

$$Q_a \leq Q \leq Q_r$$

Important conclusion: in general (except for the controllable lossless case) storage functions are not unique.

Any port-Hamiltonian system with non-negative Hamiltonian $H \geq 0$ is passive, since

$$\frac{d}{dt}H = -e_R^T f_R + e_P^T f_P \leq e_P^T f_P$$

and thus H is a storage function. Furthermore, if there are no power-dissipating elements R, then a port-Hamiltonian system with $H \geq 0$ is lossless.

In examples, the physical energy H will be somewhere 'in the middle' between S_a and S_r:

$$S_a \leq H \leq S_r$$

From passive to port-Hamiltonian; linear case

Fixing a particular storage matrix $Q > 0$ (with $Q_a \leq Q \leq Q_r$) and defining J to be the skew-symmetric part of the matrix AQ^{-1} and $-R$ its symmetric part, the passive system can be written as the port-Hamiltonian system

$$\begin{aligned} \dot{x} &= Ax + Bu &&= [J - R]\, Qx + Bu \\ y &= Cx &&= B^T Qx \end{aligned}$$

where

$$J = -J^T, \quad R = R^T \geq 0$$

Thus in general there are different port-Hamiltonian formulations of the same passive system. From a modeling point of view there is a unique port-Hamiltonian formulation.

Similarly, most nonlinear passive systems can be written as a port-Hamiltonian system.

Consider passive Σ_1 and Σ_2, with $U_1 = U_2 = Y_1 = Y_2$, interconnected by

$$\begin{aligned} u_1 &= e_1 - y_2, \\ u_2 &= e_2 + y_1 \end{aligned}$$

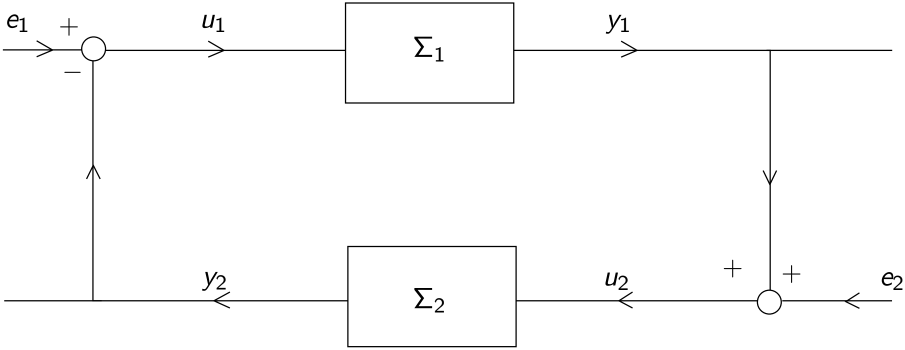


Figure : $\Sigma_1 \|_f \Sigma_2$

The Passivity Theorem continued

Addition of

$$
\begin{aligned}
S_1(x_1(t_1)) &\leq S_1(x_1(t_0)) + \int_{t_0}^{t_1} u_1^T(t) y_1(t) dt \\
S_2(x_2(t_1)) &\leq S_2(x_2(t_0)) + \int_{t_0}^{t_1} u_2^T(t) y_2(t) dt
\end{aligned}
$$

results in

$$
S_1(x_1(t_1)) + S_2(x_2(t_1)) \leq S_1(x_1(t_0)) + S_2(x_2(t_0)) +
$$

$$
+ \int_{t_0}^{t_1} u_1^T(t) y_1(t) + u_2^T(t) y_2(t) dt
$$

$$
= S_1(x_1(t_0)) + S_2(x_2(t_0)) +
$$

$$
+ \int_{t_0}^{t_1} e_1^T(t) y_1(t) + e_2^T(t) y_2(t) dt
$$

Hence the closed-loop system with inputs $e = (e_1, e_2)$ and outputs $y = (y_1, y_2)$ is again passive with storage function

$$S(x_1, x_2) = S_1(x_1) + S_2(x_2) \ , \qquad (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2.$$

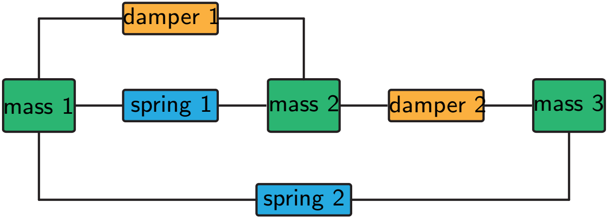
Corresponds to the theorem that the interconnection of port-Hamiltonian systems is again port-Hamiltonian.

Converse also holds: passivity of the interconnected system (with inputs $e = (e_1, e_2)$ and outputs $y = (y_1, y_2)$) implies passivity of the subsystems !

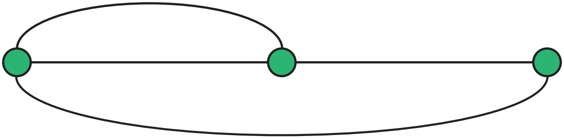
Outline

Mass-spring-damper systems as systems on graphs

Associate masses to the nodes, and springs and dampers to the edges.



(a)



(b)

Figure : (a) Mass-spring-damper system; (b) the corresponding graph.

Background on graphs

A (directed) graph $\mathcal{G}$ consists of a set of N nodes (vertices), and a set of M edges.
Every edge e corresponds to an ordered pair (v, w) of nodes (with $v \neq w$), representing the tail node v and the head node w of this edge.

The graph is specified by its $N \times M$ incidence matrix D, with (i, j)-th element equal to 1 if the j-th edge is an edge towards node i, equal to -1 if the j-th edge is an edge originating from node i, and 0 otherwise.

Basic property

$$\mathbb{1}^T D = 0\,,$$

where $\mathbb{1}$ is the vector of all ones.

The graph is connected if and only if $\ker D^T = \operatorname{span} \mathbb{1}$. This will be assumed throughout !

Mass-spring systems

For a mass-spring system with N masses and M springs in one-dimensional space $\mathbb{R}$

$$p \in \mathbb{R}^N \text{ node space}, \quad q \in \mathbb{R}^M \text{ edge space},$$

with dynamics

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D^T \\ -D & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with Hamiltonian (total energy)

$$H : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R},$$

with

$$H(q,p) = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{j=1}^{M} P_j(q_j)$$

Can be directly extended to motion in $\mathbb{R}^3$, or to multi-body systems.

Mass-spring-damper systems

Part of the edges correspond to springs; complementary part to dampers: thus we have

D_s spring incidence matrix, D_d damper incidence matrix

Dynamics of the mass-spring-damper system takes the Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

where R is a positive diagonal matrix (linear dampers).

Thus: contrary to multi-agent systems the dynamics is not only associated to the nodes but also to the edges !

Except for mass-damper systems, which reduce to

$$\dot{p} = -D_d R D_d^T \frac{\partial H}{\partial p}(p)$$

Dynamical analysis of mass-damper systems

$$\dot{p} = -D_d R D_d^T \frac{\partial H}{\partial p}(p)$$

The matrix

$$L_d := D_d R D_d^T$$

defines a weighted Laplacian matrix on the subgraph defined by the damper edges.

$$\frac{d}{dt} H(p) = -\frac{\partial^T H}{\partial p}(p) L_d \frac{\partial H}{\partial p}(p) = -v^T L_d v$$

Thus velocities $v := \frac{\partial H}{\partial p}(p)$ will converge to span $\mathbb{1}$, which is the set of equlibria.
While the total momentum $\mathbb{1}^T p$ is a Casimir.

For unit masses $v = p$, and the dynamics is equal to standard symmetric consensus dynamics.

Mass-spring-damper systems: equilibria

Proposition

The set of equilibria $\mathcal{E}$ of a mass-spring-damper system is given as

$$\mathcal{E} = \{(q,p) \mid \frac{\partial H}{\partial q}(q,p) \in \ker D_s,\ v = \frac{\partial H}{\partial p}(q,p) \in \operatorname{span} \mathbb{1}\}$$

Theorem

Let

$$H(q,p) = \frac{1}{2}q^T K q + \frac{1}{2}p^T G p, \quad K, \; G \textit{ diagonal positive matrices}$$

Define the spring Laplacian matrix $L_s := D_s K D_s^T$.

Then for every (q_0, p_0) *there exists a unique equilibrium point* $(q_\infty, p_\infty) \in \mathcal{E}$ *to which the system converges exponentially iff the largest* GL_s*-invariant subspace contained in* $\ker L_d$ *equals* $\operatorname{span} \mathbb{1}$.

Can be extended to the nonlinear case (nonlinear springs).

Mass-spring-damper systems with constant external forces

Consider the port-Hamiltonian system

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{f}_b \end{bmatrix},$$

with $\bar{f}_b$ the constant external forces acting on some of the (boundary) masses.

Steady-state analysis for constant $\bar{f}_b$:

Assume the existence of a $\bar{q}$ such that

$$D_s \frac{\partial H}{\partial q}(\bar{q}, 0) = \bar{f}_b$$

(*spring forces are able to counteract the external forces* ;
in particular $\mathbb{1}^T \bar{f}_b = 0$)

Constant external forces

Consider the shifted Hamiltonian (or availability function)

$$\bar{H}(q,p) := H(q,p) - \frac{\partial H}{\partial q}(\bar{q},0)(q-\bar{q}) - H(\bar{q},0)$$

The system can be rewritten as the system without external force $\bar{f}_b$

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & D_s^T \\ -D_s & -D_d R D_d^T \end{bmatrix} \begin{bmatrix} \frac{\partial \bar{H}}{\partial q}(q,p) \\ \frac{\partial \bar{H}}{\partial p}(q,p) \end{bmatrix}$$

If the potential energy $P(q)$ is convex then $\bar{H}$ has a minimum at $(\bar{q},0)$.

Constant external forces

We obtain the following extension of the case without external forces:

Proposition

The set of steady states is given by

$$\bar{\mathcal{E}} = \{(q,p) \mid D_s K q = \bar{f}_b, v = Gp \in \operatorname{span} \mathbb{1}\}$$

For every (q_0, p_0) there exists a unique equilibrium $(\bar{q}_\infty, p_\infty) \in \bar{\mathcal{E}}$ where $\bar{q}_\infty = \bar{q} + q_\infty$ to which the system converges exponentially iff the largest GL_s-invariant subspace contained in $\ker L_d$ is equal to $\operatorname{span} \mathbb{1}$.

The springs act as integral controllers, counteracting the unknown external force $\bar{f}_b$ so that the velocities $v = Gp$ still converge to a common value. By additional control external forces, this common value can be regulated to any value, e.g. zero.

Outline

Motivational context

- The structure and operation of the power network is changing: increasing share of renewable energy sources with large variability; consumers becoming 'prosumers'; towards distributed control instead of 'top-down' operation; ...
- Fundamental open stability questions; some classical, others stimulated by the changing operation of the power network near its capacity.
- Focus of power network theory on simple models; for different purposes; sometimes under conflicting assumptions.
- Current control schemes are structured in three layers; primary control dictated by local 'droop control' for the generators, secondary control for robust set-point regulation, tertiary control for optimal power dispatch.
- How can pricing mechanisms being used for supply/demand matching and optimization ?

Basic assumptions in the swing equation model

- All voltage and current in the network are pure sinusoids with same frequency $\widehat{\omega}$ (50 Hz). Then any voltage/current signal

$$V(t) = V \sin(\widehat{\omega} t + \delta), \quad t \in \mathbb{R},$$

can be represented by its phasor

$$Ve^{j\delta}$$

Crucial property: map from sinusoids (with same frequency) to their phasors is linear.

- Amplitudes $V_i, i = 1, \cdots, n$, of voltage potentials at all nodes are constant.
- All transmission lines (edges) are purely inductive.

Model the magnetic/electric part of the i-th generator/motor as a voltage source with corresponding voltage angle δ_i (and a reactance included in its transmission line).
Average power ('active power') flow from node i to node j is given by

$$\Gamma_{ij} \sin(\delta_i - \delta_j)$$

with $\Gamma_{ij} = S_{ij} V_i V_j, \quad S_{ij}$ susceptance of the line from i to j.
Define the phase differences across the lines

$$q_k := \delta_j - \delta_i, \quad k = 1, \cdots, m$$

Then

$$q = D^T \delta,$$

with D the $n \times m$ incidence matrix of the network: n number of nodes, m number of edges (lines).

It follows that the vector of power flows through the lines is given by

$$P_{\mathrm{network}} = -D\Gamma \,\mathrm{Sin}\, D^T \delta = -D\Gamma \,\mathrm{Sin}\, q$$

where $\mathrm{Sin} : \mathbb{R}^m \to \mathbb{R}^m$ is the element-wise sin function.

Network of generators modeled by swing equations

The swing equations model the balance between mechanical and electric power as

$$M\dot{\omega} = -A\omega + P_{\mathrm{network}} + u = -A\omega - D\Gamma \,\mathrm{Sin}\, q + u$$

where $u \in \mathbb{R}^n$ is the vector of produced/consumed power at all nodes, and $A\omega$ is the vector of damping torques, with A a positive diagonal matrix.

Let ω_i be the frequency deviation with respect to $\widehat{\omega}$ of node i, then the vector of phase differences $q = D^T\delta$ satisfies

$$\dot{q} = D^T\omega, \quad \omega = (\omega_1, \cdots, \omega_n)^T$$

Together, we obtain the system

$$\begin{aligned} \dot{q} &= D^T\omega \\ M\dot{\omega} &= -A\omega - D\Gamma \,\mathrm{Sin}\, q + u \end{aligned}$$

Favorite equations in the control literature on power networks.

This system is naturally written into port-Hamiltonian format:

$$
\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix}, \quad p = M\omega \\
y &= \frac{\partial H}{\partial p}(q,p) = \omega
\end{aligned}
$$

with u vector of generated/consumed power, and Hamiltonian

$$
H(q,p) = \frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \, \mathrm{Cos}\, q
$$

However:

- Note that u is power, and thus the conjugated output ω is dimensionless in order that $u^T y$ is power.
- Note furthermore that ω is frequency deviation, and $p = M\omega$ is momentum deviation.
- Furthermore, $\frac{1}{2} p^T M^{-1} p$ is shifted kinetic energy, and $A\omega$ is a restoring magnetic torque; not energy dissipation.

Stability analysis of the model for constant inputs is 'easy'

Let $\bar{u}$ be a constant input, yielding steady state values $(\bar{q}, \bar{p} = M\bar{\omega})$ determined by $D^T\bar{\omega} = 0$ and thus

$$\bar{\omega} = \mathbb{1}\omega_*$$

where

$$\mathbb{1}^T A \mathbb{1}\omega_* = \mathbb{1}^T \bar{u}$$

$$\text{(premultiply } 0 = -D\frac{\partial H}{\partial q}(\bar{q}, \bar{p}) - A\frac{\partial H}{\partial p}(\bar{q}, \bar{p}) + \bar{u} \text{ by } \mathbb{1}^T\text{)}$$

and furthermore

$$D\Gamma \, \mathrm{Sin}\, \bar{q} = -A\mathbb{1}\omega_* + \bar{u}$$

Note that $\omega_* = 0$ if and only if $\mathbb{1}^T\bar{u} = 0$.

Define the 'shifted Hamiltonian' (availability function)

$$\widetilde{H}(q,p) := \frac{1}{2}(p-\bar{p})^T M^{-1}(p-\bar{p}) - \mathbb{1}^T \Gamma \,\mathrm{Cos}\, q + \mathbb{1}^T \Gamma \,\mathrm{Sin}\, \bar{q}\,(q-\bar{q})$$

Has a strict minimum at $(\bar{q},\bar{p})$, whenever $\bar{q} \in (-\frac{\pi}{2}, \frac{\pi}{2})^n$.

In fact, the system can be rewritten as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & D^T \\ -D & -A \end{bmatrix} \begin{bmatrix} \frac{\partial \widetilde{H}}{\partial q}(q,p) \\ \frac{\partial \widetilde{H}}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ u-\bar{u} \end{bmatrix}, \quad p = M\omega \\ y-\bar{y} &= \frac{\partial \widetilde{H}}{\partial p}(q,p) \end{aligned}$$

and is shifted passive with respect to the shifted inputs $u-\bar{u}$ and outputs $y-\bar{y}$.

In particular, for $u=\bar{u}$ the steady state $(\bar{q},\bar{p})$ is asymptotically stable.

Outline

Control towards optimal supply-demand-matching

Write $u = u_g - u_d$, with u_g generated power and u_d consumed power.

Then maximize the social welfare function

$$U(u_d) - C(u_g),$$

with concave utility function $U(u_d)$ of the consumers u_d,
and convex generation cost $C(u_g)$ of the producers u_g,
under the constraint of zero frequency deviation.
Recall that 'if and only if' condition for zero frequency deviation is

$$\mathbb{1}^T u_d = \mathbb{1}^T u_g$$

Furthermore, (u_g, u_d) satisfies this equation if and only if there exists $v \in \mathbb{R}^{m_c}$ such that

$$D_c v - u_g + u_d = 0,$$

where $D_c \in \mathbb{R}^{n \times m_c}$ is the incidence matrix of an arbitrary connected communication graph with m_c edges and n nodes.

Leads to the convex minimization problem:

$$\min_{u_g, u_d, v} \quad C(u_g) - U(u_d)$$
$$\text{s.t.} \quad D_c v - u_g + u_d = 0$$

The corresponding Lagrangian is

$$L = C(u_g) - U(u_d) + \lambda^T (D_c v - u_g + u_d)$$

with Lagrange multipliers $\lambda \in \mathbb{R}^n$. The KKT optimality conditions ($\nabla L = 0$) are

$$\begin{aligned}
\nabla C(\bar{u}_g) - \bar{\lambda} &= 0 \\
-\nabla U(\bar{u}_d) + \bar{\lambda} &= 0 \\
D_c^T \bar{\lambda} &= 0 \\
D_c \bar{v} - \bar{u}_g + \bar{u}_d &= 0.
\end{aligned}$$

λ_i acts as a price in control area i, and v represents the information exchange of the differences of the prices λ along the edges the communication graph.

Primal-dual gradient controller

A continuous-time algorithm to converge to the saddle-point described by the KKT conditions is the steepest descent/ascent gradient dynamics

$$\begin{aligned}
\tau_g \dot{u}_g &= -\nabla C(u_g) + \lambda + w_g \\
\tau_d \dot{u}_d &= \nabla U(u_d) - \lambda + w_d \\
\tau_v \dot{v} &= -D_c^T \lambda \\
\tau_\lambda \dot{\lambda} &= D_c v - u_g + u_d
\end{aligned}$$

with additional inputs $w = (w_g, w_d)$.

Here $\tau_E = \mathrm{diag}(\tau_g, \tau_d, \tau_v, \tau_\lambda) > 0$ are the time-scales of the algorithm.

Result: dynamic pricing controller (to be connected to the physical network through u_g, u_d and w_g, w_d) !

Port-Hamiltonian formulation of primal-dual gradient controller

Define new coordinates ('from co-energy to energy variables')

$$x_E := (x_g, x_d, x_v, x_\lambda) = (\tau_g u_g, \tau_d u_d, \tau_v v, \tau_\lambda \lambda) =: \tau_E z_E$$

and the quadratic Hamiltonian

$$H_c(x_E) = \frac{1}{2} x_E^T \tau_E^{-1} x_E, \quad z_E = \frac{\partial H_c}{\partial x_E}(x_E)$$

Then the primal-dual gradient controller can be rewritten as the pH system

$$\dot{x}_E = \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & -D_c^T \\ -I & I & D_c & 0 \end{bmatrix} z_E - \frac{\partial R}{\partial z_E}(z_E) + \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} w, \quad z_E = \frac{\partial H_c}{\partial x_E}(x_E)$$

$$y_E = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} z_E$$

Since

$$R(u_g, u_d) := C(u_g) - U(u_d)$$

is convex it has the incremental passivity property

$$(z_1 - z_2)^T \left(\frac{\partial R}{\partial z}(z_1) - \frac{\partial R}{\partial z}(z_2)\right) \geq 0, \;\; \forall z_1, z_2$$

Note: can be extended to inequality constraints as well.

Interconnection

Interconnect the pH model of the physical power network with the pH representation of the primal-dual gradient controller by negative feedback

$$w = -y, u = y_E$$

Then the closed-loop system is again port-Hamiltonian, with state $x = (q, p, x_E)$, and Hamiltonian H the sum of the physical energy

$$\frac{1}{2} p^T M^{-1} p - \mathbb{1}^T \Gamma \, \mathrm{Cos}\, q$$

of the physical network,
and the 'cyber energy'

$$H_c(x_E) = \frac{1}{2} x_E^T \tau_E^{-1} x_E$$

of the primal-dual gradient controller.

We obtain the closed-loop port-Hamiltonian system

$$\dot{x} = \begin{bmatrix} 0 & D^T & 0 & 0 & 0 & 0 \\ -D & -A & I & -I & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & I \\ 0 & I & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & 0 & -D_c^T \\ 0 & 0 & -I & I & D_c & 0 \end{bmatrix} z - \frac{\partial R}{\partial z}(z), \quad z = \frac{\partial H}{\partial x}(x)$$

Using again the shifted Hamiltonian as Lyapunov function it is shown that the system converges to the optimal $\bar{u}_g, \bar{u}_d$ and frequency deviation $\omega = 0$.

Time-scales $\tau_E = \text{diag}(\tau_g, \tau_d, \tau_v, \tau_\lambda) > 0$ determine the 'stiffness' of the controller !

True 'cyber-physical system': physical network with control/ communication/economic network layer !

Reconsideration

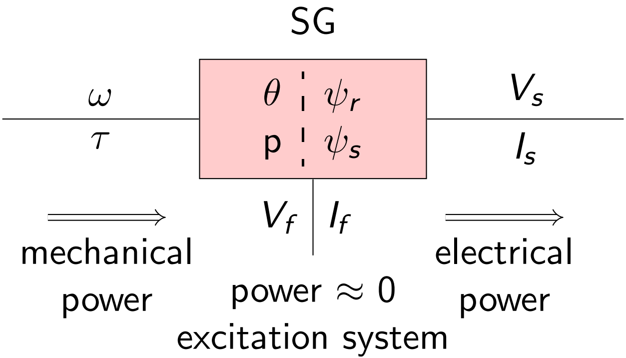
Recently, the same theory has been extended to the the third-order 'flux-decay' model of the physical network, incorporating voltage dynamics.

See:
T.W. Stegink, C. De Persis, A.J. van der Schaft,
'A Unifying Energy-Based Approach to Stability of Power Grids with Market Dynamics', to appear in IEEE-TAC.

Outline

The 'full' model of the working horse of the grid

Classical 8-dimensional model of the Synchronous Generator (SG) can be put into port-Hamiltonian form



'Same' model for synchronous motors.

(See Shaik Fiaz, D. Zonetti, R. Ortega, J. Scherpen, AJvdS, Eur. J. Control, 2013)

$$\begin{bmatrix} \dot{\psi}_s \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R_s & 0_3 & 0_{31} & 0_{31} \\ 0_3 & -R_r & 0_{31} & 0_{31} \\ 0_{13} & 0_{13} & -d & -1 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix} + \begin{bmatrix} I_3 & 0_{31} & 0_{31} \\ 0_3 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & 0_{31} \\ 0_{13} & 0 & 1 \\ 0_{13} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ V_f \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} I_s \\ I_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_3 & 0_3 & 0_{31} & 0_{31} \\ 0_{13} & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \psi_s} \\ \frac{\partial H}{\partial \psi_r} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \theta} \end{bmatrix}$$

$$\text{where } R_s = \begin{bmatrix} r_s & 0 & 0 \\ 0 & r_s & 0 \\ 0 & 0 & r_s \end{bmatrix}, \quad R_r = \begin{bmatrix} r_f & 0 & 0 \\ 0 & r_{kd} & 0 \\ 0 & 0 & r_{kq} \end{bmatrix}, \quad d$$

are the stator resistances, rotor resistances, mechanical friction constants.

- $\psi_s \in \mathbb{R}^3$ are stator fluxes
- $\psi_r \in \mathbb{R}^3$ are rotor fluxes: field winding and two damper windings
- p is angular momentum of rotor
- θ is the angle of the rotor
- $V_s \in \mathbb{R}^3, I_s \in \mathbb{R}^3$ are the three-phase stator terminal voltages and currents
- V_f, I_f are the rotor field winding voltage and current
- τ, ω are the mechanical torque and angular velocity

The Hamiltonian (total stored energy) is

$$\begin{aligned} H(\psi_s, \psi_r, p, \theta) &= \tfrac{1}{2} \begin{bmatrix} \psi_s^T & \psi_r^T \end{bmatrix} L^{-1}(\theta) \begin{bmatrix} \psi_s \\ \psi_r \end{bmatrix} + \tfrac{1}{2J} p^2 \\ &= \text{magnetic energy} + \text{kinetic energy} \end{aligned}$$

where $L(\theta)$ is the 6×6 inductance matrix.

In the round rotor case (no saliency)

$$L(\theta) = \begin{bmatrix} L_{ss} & L_{sr}(\theta) \\ L_{sr}^T(\theta) & L_{rr} \end{bmatrix}$$

where

$$L_{ss} = \begin{bmatrix} L_{aa} & -L_{ab} & -L_{ab} \\ -L_{ab} & L_{aa} & -L_{ab} \\ -L_{ab} & -L_{ab} & L_{aa} \end{bmatrix}, \; L_{rr} = \begin{bmatrix} L_{ffd} & L_{akd} & 0 \\ L_{akd} & L_{kkd} & 0 \\ 0 & 0 & L_{kkq} \end{bmatrix}$$

while

$$L_{sr}(\theta) = \begin{bmatrix} \cos\theta & \cos\theta & -\sin\theta \\ \cos(\theta - \frac{2\pi}{3}) & \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) \\ \cos(\theta + \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \begin{bmatrix} L_{afd} & 0 & 0 \\ 0 & L_{akd} & 0 \\ 0 & 0 & L_{akq} \end{bmatrix}$$

Can be extended to saliency case, and to saturation.

Note that

$$I_s = \frac{\partial H}{\partial \psi_s}, \quad I_r = \frac{\partial H}{\partial \psi_r}$$

are the vectors of stator and rotor currents, and

$$\omega = \frac{\partial H}{\partial p}$$

is the angular velocity of the rotor.

The torque balance (in the mechanical domain) is

$$\dot{p} = -d\omega - \tau_m + \tau$$

where the 'air-gap torque' τ_m, due to magnetism, is given by

$$\tau_m = \frac{\partial H}{\partial \theta} = \frac{\partial}{\partial \theta} \frac{1}{2} \psi^T L^{-1}(\theta) \psi$$

In principle, both (τ, ω) and (V_f, I_f) can be used as control ports.

Using co-energy variables, the port-Hamiltonian dynamics can be summarized as

$$\begin{bmatrix} \dot{\psi}_s \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R_s & 0 & 0 & 0 \\ 0 & -R_r & 0 & 0 \\ 0 & 0 & -d & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_s \\ I_r \\ \omega \\ \tau_m \end{bmatrix} + \begin{bmatrix} I_3 & 0 & 0 \\ 0 & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V_s \\ V_f \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} I_s \\ I_f \\ \omega \end{bmatrix} = \begin{bmatrix} I_3 & 0 & 0 & 0 \\ 0 & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_s \\ I_r \\ \omega \\ \tau_m \end{bmatrix}$$

Power-balance

The system, being port-Hamiltonian, satisfies the power-balance

$$
\begin{aligned}
\frac{d}{dt}H &= \omega\tau && \text{mechanical power} \\
&\quad + I_s^T V_s && \text{minus electrical power to the network} \\
&\quad + I_f V_f && \text{power corresponding to excitation system} \\
&\quad - I_s^T R_s I_s - I_r^T R_r I_r - d\omega^2 && \text{dissipated power (losses)}
\end{aligned}
$$

In typical (desired) operation

$$\omega\tau \approx I_s^T V_s$$

The synchronous generator can be coupled to the network via (I_s, V_s), leading to a complete time-domain model in port-Hamiltonian form.

Blondel-Park ($dq0$) transformation

Motivation: the 8-dimensional model is difficult to analyze; first of all since the magnetic energy is depending on the mechanical angle θ.

Define

$$
\begin{aligned}
\psi_{dq0} &= \begin{bmatrix} \psi_d \\ \psi_q \\ \psi_0 \end{bmatrix} := T_{dq0}(\theta)\psi_s = \\
&= \sqrt{\frac{2}{3}} \begin{bmatrix} \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ \sin\theta & \sin(\theta - \frac{2\pi}{3}) & \sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \psi_s
\end{aligned}
$$

Note that $T^{-1}_{dq0}(\theta) = T^{t}_{dq0}(\theta)$, and hence I_s and V_s transform in the same way:

$$I_{dq0} := T_{dq0}(\theta) I_s, \quad V_{dq0} := T_{dq0}(\theta) V_s$$

In the new coordinates the magnetic part of the transformed Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}_m = \frac{1}{2} \begin{bmatrix} \psi_{dq0}^T & \psi_r^T \end{bmatrix} \mathcal{L}^{-1} \begin{bmatrix} \psi_{dq0} \\ \psi_r \end{bmatrix}$$

where

$$\mathcal{L} = \begin{bmatrix} T_{dq0}(\theta) & 0 \\ 0 & I_3 \end{bmatrix} L(\theta) \begin{bmatrix} T_{dq0}^t(\theta) & 0 \\ 0 & I_3 \end{bmatrix} =$$

$$= \left[\begin{array}{ccc|ccc} L_d & 0 & 0 & \sqrt{\frac{3}{2}} L_{afd} & \sqrt{\frac{3}{2}} L_{akd} & 0 \\ 0 & L_q & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} L_{akq} \\ 0 & 0 & L_0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} L_{afd} & 0 & 0 & L_{ffd} & L_{akd} & 0 \\ \sqrt{\frac{3}{2}} L_{akd} & 0 & 0 & L_{akd} & L_{kkd} & 0 \\ 0 & -\sqrt{\frac{3}{2}} L_{akq} & 0 & 0 & 0 & L_{kkq} \end{array} \right]$$

and thus independent of θ !

Transformed dynamics

$$
\begin{bmatrix} \dot{\psi}_{dq0} \\ \dot{\psi}_r \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -R & 0_3 & \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} & 0_{31} \\ 0_3 & -R_r & 0_{31} & 0_{31} \\ \begin{bmatrix} \psi_q & -\psi_d & 0_3 \end{bmatrix} & 0_{13} & -d & -1 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_{dq0}} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \\ 0 \end{bmatrix} +
$$

$$
\begin{bmatrix} 0_{31} & 0_{31} \\ e_1 & 0_{31} \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_f \\ \tau \end{bmatrix}
$$

$$
\begin{bmatrix} I_f \\ \omega \end{bmatrix} = \begin{bmatrix} 0_{13} & e_1^T & 0 & 0 \\ 0_{13} & 0_{13} & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_s} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \\ 0 \end{bmatrix}
$$

Hence the air-gap torque due to magnetism is alternatively given as

$$\tau_m = \psi_d I_q - \psi_q I_d$$

Furthermore, we can split off the dynamics of θ, and consider steady states of the reduced port-Hamiltonian system

$$\begin{bmatrix} \dot{\psi}_{dq0} \\ \dot{\psi}_r \\ \dot{p} \end{bmatrix} = \begin{bmatrix} -R & 0 & \begin{bmatrix} -\psi_q \\ \psi_d \\ 0 \end{bmatrix} \\ 0 & -R_r & 0 \\ \begin{bmatrix} \psi_q & -\psi_d & 0 \end{bmatrix} & 0 & -d \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_{dq0}} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \end{bmatrix} + \begin{bmatrix} 0_{31} & 0_{31} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & 0_{31} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} V_f \\ \tau \end{bmatrix}$$

$$\begin{bmatrix} I_f \\ \omega \end{bmatrix} = \begin{bmatrix} 0_{13} & \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} & 0 \\ 0_{13} & 0_{13} & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \psi_s} \\ \frac{\partial \mathcal{H}}{\partial \psi_r} \\ \frac{\partial \mathcal{H}}{\partial p} \end{bmatrix}$$

Analysis and control of this model is still difficult!

Outline

Revisiting the swing equations

Write

$$H = H_m + H_k \quad := \quad \frac{1}{2} \begin{bmatrix} \psi_s^T & \psi_r^T \end{bmatrix} L^{-1}(\theta) \begin{bmatrix} \psi_s \\ \psi_r \end{bmatrix} + \frac{1}{2J} p^2$$

magnetic energy + kinetic energy

Note that in this decomposition

$$\frac{d}{dt} H_m = -I_s^T R_s I_s - I_r^T R_r I_r + I_s^T V_s + I_f V_f + \omega \tau_m$$

while

$$\frac{d}{dt} H_k = -d\omega^2 + \omega\tau - \omega \tau_m$$

Towards the swing equations: key assumption

Consider a fixed angular velocity value $\widehat{\omega}$ (i.e., 50 Hz). Key assumption:

$$\frac{d}{dt}H_m = -I_s^T R_s I_s - I_r^T R_r I_r + A(\omega - \widehat{\omega}) + I_f V_f, \quad A > 0,$$

or equivalently

$$\omega\tau_m = -I_s^T V_s + A(\omega - \widehat{\omega})$$

Interpretation: around steady-state $\bar{\omega}$ the mechanical power $\omega\tau_m$ delivered by the rotor is directly transformed into electrical power $-I_s^T V_s$ at the stator terminals transmitted to the network, plus the term $A(\omega - \widehat{\omega})$ due to the restoring magnetic torque of the damper windings.

$$\begin{aligned} \tfrac{d}{dt}H_k &= -d\omega^2 + \omega\tau - \omega\tau_m \\ &= -d\omega^2 - A(\omega - \widehat{\omega}) + I_s^T V_s + \omega\tau \\ &\approx -A(\omega - \widehat{\omega}) + I_s^T V_s + \omega\tau \\ &= -A(\omega - \widehat{\omega}) + P_{\text{electric}} + P_{\text{mech}} \end{aligned}$$

The swing equations continued

Furthermore, assuming ω remains close to $\bar{\omega}$

$$\frac{d}{dt} H_k = \frac{d}{dt} \frac{1}{2J} p^2 = \frac{d}{dt} \frac{1}{2} J \omega^2 = (J\omega)\, \dot{\omega} \approx M\dot{\omega},$$

with $M := J\widehat{\omega}$.

Combined this leads to the swing equations in the deviation $\tilde{\omega} := \omega - \widehat{\omega}$

$$M\dot{\tilde{\omega}} = P_{\text{electric}} + P_{\text{mech}} - A\,\tilde{\omega},$$

Formally equivalent to an actuated and damped rotating mass.
However, $\tilde{\omega}$ is deviation from nominal value, and $A\,\tilde{\omega}$ is a restoring force; not dissipation.

Furthermore, dimensions/units are peculiar: input is power instead of torque.

Some key references

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