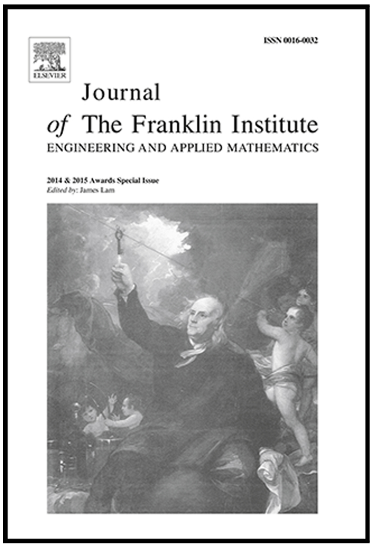

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New results on reachable sets bounding for delayed positive singular systems with bounded disturbances

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Abstract

This paper addresses the reachable sets bounding problem for time-delay positive singular systems subject to bounded disturbances. The time delays in the considered systems are assumed to be time-varying. Both invariant and time-varying singular systems are investigated in this paper. Existence conditions of componentwise ultimate bounds of the state vector of considered systems are derived and given in terms of the spectral abscissa of the system matrices, which are easy to be checked. The obtained results are demonstrated by two numerical examples.

Keywords: Componentwise ultimate bounds, singular systems, time-varying delays, positive systems.

1. Introduction

Due to the fact that time delay is often encountered in many practical control systems, in recent years, many significant research developments have been devoted to the problem of analyzing the stability of time-delay systems (see, for example, [1–3]). On the other hand, since external disturbances are usually unavoidable in practical engineering systems, the asymptotic stability for the systems cannot be achieved and therefore the problem of state bounding for dynamical systems has attracted considerable attention during the past decades (see, for example, [4–23]). The objective of the reachable sets bounding problem is to find an ultimate bound, which is a set such that the state vector converges within it when the time tends to infinity [24] or the time tends to prespecified time [9]. To solve this problem, two commonly methods are used. The first one is based on the Lyapunov method combining with linear matrix inequality [8, 10, 13–18] and the second one is based on positive systems combining with the solution comparison method [9, 19–23]. The first method is widely used for classes of linear systems whose matrices are constant, while, the second method is very useful for classes of positive

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linear systems and classes of nonlinear/time-varying systems which are bounded by positive linear systems.

Singular systems are formulated as a set of coupled differential and algebraic equations, which include information on the static as well as dynamic constraints of areal plant ([25–27]). Therefore, it can represent a much wider class of systems than its state-space counterpart. Singular systems can be found in many applications and real-world physical phenomena such as mechanical engineering systems, power systems, chemical processes, large-scale interconnected systems, circuit systems, robotics and economic systems ([25–30]). As a consequence, many important and interesting results on singular systems have been reported and various issues have been studied by many authors [25–28, 31, 32]. For examples, Ngoc [31] derived some explicit criteria for global exponential stability of singular nonlinear differential equations with constant time delays based on the spectral properties of Metzler combined with the comparison principle. Recently, the problems of admissibility analysis and stabilization were addressed in [32] for singular interval type-2 Takagi-Sugeno fuzzy systems with constant time delay by using Lyapunov functional methods and linear matrix inequalities. On the other hand, positive systems are dynamical systems whose state and output trajectories are always nonnegative for nonnegative inputs and initial states. Applications of positive systems can be found in ecological systems, biological systems and economic systems ([30, 33–35]). Positive singular systems are singular systems and positive systems. Since the states of singular positive systems are defined on cones rather than in the whole space, solutions of linear singular positive systems cannot be solved via the well-established methods for general systems [36, 37].This is the main reason that there are very few works devoted to singular positive systems [38–41]. In particular, Ngoc [38] studied the problem of exponential stability for coupled linear differential-difference equations with constant delays. It should be mentioned here that the author of the work [38] has to assume that the considered system satisfies regular condition and impulse-free condition to obtain the exponential stability results. Cui el al. considered the problem of stability analysis for linear positive singular systems with distributed delays [39] or the system with discrete-time delays [40]. It is notable that these works [39, 40] did not study reachable sets bounding problem for the considered systems. Recently, Nam et al. [41] considered the problem of reachable sets bounding for coupled differential-difference equations subject to bounded time-varying delays and bounded disturbances. Noting that the coupled differential-difference equations is just a particular case of positive singular systems. These results cannot be easily extend to delayed positive singular systems since the positive singular systems need more new techniques. To the best of our knowledge, there are no results on reachable sets bounding for positive singular systems with mixed time-varying delays in the literature. The main object of this paper is to fill this gap.

In this paper, we deal with the problem of reachable sets bounding for positive singular systems with mixed time-varying delays and bounded disturbances. Both discrete and distributed time-varying delays are considered in these systems. The main contributions of this work are highlighted in the following: (i) We derive several conditions for the existence of componentwise bounds and a condition for the exponential stability of positive singular systems subject to mixed time-varying delays without disturbances by using a novel approach. Our results improved and covered some existing results [39–41]; *(ii)* For the first time, we present some criteria conditions for the existence of componentwise ultimate bounds for the system with bounded disturbances via the spectral radius of the system matrices, which are easy to verify and allows us to compute directly the smallest componentwise ultimate bound; and *(iii)* We extend the obtained resuts to time-varying singular systems with time-varying discrete and distributed delays.

The remaining of this paper is organized as follows. In Section 2, we provide the problem statement and preliminaries. The main results are given in Section 3. In Section 4, we extend the obtained resuts in Section 3 to time-varying singular systems with time-varying discrete and distributed delays. Two numerical examples are provided in Section 5 to illustrate the effectiveness of the proposed method. A conclusion is presented in Section 6.

Notation: $\mathbb{R}^n_+$ ($\mathbb{R}^n_{0,+}$) denotes the set of all positive (nonnegative) vectors in $\mathbb{R}^n$; The set of real matrices of size $r\times h$ is denoted as $\mathbb{R}^{r\times h}$. The identity matrix of size $q\times q$ is denoted by I_q. $z\in\mathbb{R}^n$ is nonnegative (positive) if all coordinates of z are nonnegative (positive). For $M=(m_{ij})\in\mathbb{R}^{k\times k}$, M is the matrix Metzler if $m_{ij}\geq 0$ for all $i\neq j$; $i,j=1,2,\ldots,k$. The set of continuous functions on $[-h,0]$ that receives the value in $\mathbb{R}^k$ is denoted as $C([-h,0],\mathbb{R}^k)$. The infinity norm of function $\psi(\cdot)=(\psi_i(\cdot))\in C([-h,0],\mathbb{R}^k)$ is denoted by $\|\psi\|_\infty$ and is defined as follows: $\|\psi\|_\infty=\max\limits_{1\leq i\leq k}|\psi_i|$, where $|\psi_i|=\sup_{t\in[-h,0]}|\psi_i(t)|$. For $Q\in\mathbb{R}^{m\times n}$, the ∞-norm of Q is defined as follows $\|A\|_\infty=\max\limits_{1\leq i\leq m}\sum\limits_{j=1}^{n}|a_{ij}|$. For $H\in\mathbb{R}^{n\times n}$ the spectral abscissa of H is denoted by $s(H)=\max\{Re\lambda:\lambda\in\sigma(H)\}$ where $\sigma(H)$ is the spectrum of H. $H=(h_{ij})\in\mathbb{R}^{m\times n}$, $H\succeq 0(\succ 0)$ if $h_{ij}\geq 0(>0)$, $\forall i,j$. $H\succeq K(H\succ K)$ means that $H-K\succeq 0(H-K\succ 0)$. For $S\in\mathbb{R}^{l\times q}$ we define $|S|=(|s_{ij}|)$. Note that $|ST|\preceq|S||T|$, $\forall S\in\mathbb{R}^{k\times p}$, $\forall T\in\mathbb{R}^{p\times l}$. Let $G:=(a_{ij})\in\mathbb{R}^{n\times n}$, we associate the Metzler matrix $M(G):=(\bar{g}_{ij})\in\mathbb{R}^{n\times n}$ with $\bar{g}_{ij}=|g_{ij}|$, $i\neq j,i,j=1,\ldots,n$ and $\bar{g}_{ii}=g_{ii}$, $i=1,\ldots,n$.

2. Problem formulation and preliminaries

Consider the following singular system with time-varying discrete and distributed delays:

$$\begin{cases} E\dot{x}(t)=A_0x(t)+A_1x(t-\tau(t))+\int_{-d(t)}^{-\underline{d}}\mathscr{K}(s)x(t+s)ds+B\vartheta(t),\ t\geq 0,\\ x(s)=\psi(s),\ s\in[-M,\ 0], \end{cases} \tag{1}$$

where $x(t)=(x_1(t),x_2(t))\in\mathbb{R}^n$ is the state vector, in which $x_1(t)\in\mathbb{R}^r,x_2(t)\in\mathbb{R}^{n-r}$; the delay $\tau(t)\in[\underline{\tau},\ \overline{\tau}]$ and $d(t)\in[\underline{d},\ \overline{d}]$; $\underline{d},\ \overline{d},\ \underline{\tau},\ \overline{\tau}>0$, $M=\max\{\overline{d},\ \overline{\tau}\}$. The matrix $E\in\mathbb{R}^{n\times n}$ is singular and $\mathrm{rank}(E)=r<n$. $A_0,\ A_1,B$ are known constant matrices with appropriate dimensions; $\mathscr{K}(s)$, $s\in[-\overline{d},\ 0]$ is a known continuous matrix-valued function; the disturbance vector $\vartheta(t)\in\mathbb{R}^m_{0,+}$ satisfying

$$0\preceq\vartheta(t)\preceq\overline{\vartheta},\ \forall t\geq 0, \tag{2}$$

The initial condition $\psi(\cdot)$ of the system (1) is assumed to satisfy the following inequality:

$$0\preceq\psi(s)\preceq\overline{\psi},\forall s\in[-M,\ 0]. \tag{3}$$

Given an E matrix with $\text{rank}(E) = r < n$, , we always have two X, Y matrices so that there is the representation $XEY = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. And so, in this paper, we suppose that the matrices $E, A_0, A_1, \mathscr{K}(s), B$ have the following expression:

$$E := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, A_i := \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{pmatrix}, i = 0,1, \mathscr{K}(s) = \begin{pmatrix} K_1(s) & K_2(s) \\ K_3(s) & K_4(s) \end{pmatrix}, B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Putting $x(t, \psi, \vartheta)$ is the state trajectory with the initial function $\psi = (\psi_1, \psi_2)$ of system (1).

Definition 1. ([39])

(i) If the pair (E, A_0) is regular i.e., $\det(sE - A_0) \neq 0$, then the singular system (1) is regular.

(ii) If the pair (E, A_0) is impulse-free i.e., $deg(det(sE - A_0)) = rank(E)$, then the singular system (1) is impulse free.

Definition 2. ([27]) The system (1) ($\vartheta(t) = 0$) is said to be exponentially stable if there exist positive numbers $N,\ \theta > 0$ such that, for any initial conditions $\psi(t)$, the solution $x(t,\ \psi)$ satisfies $\| x(t, \psi) \| \leq Ne^{-\theta t} \| \psi \|, \quad \forall t \geq 0$.

Definition 3. ([33]) System (1) is positive if for all initial value $\psi \succeq 0$, and for any nonnegative input $\vartheta(\cdot) \succeq 0$ implies the corresponding trajectory $x(t, \psi, \vartheta) \succeq 0$ for all $t \geq 0$.

Lemma 1. ([37]) *Let D be a Metzler matrix. Then, the following statements are equivalent:*

1) $s(D) < 0$.
2) $\exists\, \lambda \in \mathbb{R}^n : \lambda \succ 0$ *and* $D\lambda \prec 0$.
3) $\exists \gamma \in \mathbb{R}^n : \gamma \succ 0$ *and* $\gamma^T D \prec 0$.
4) $det(D) \neq 0$ *and* $-D^{-1} \succeq 0$.

We denote the following

$$\begin{aligned}
&\overline{A}_{01} := A_{01} - A_{02}A_{04}^{-1}A_{03} = (a_{ij})_{r\times r}, && \overline{A}_{03} := -A_{04}^{-1}A_{03} = (d_{ij})_{(n-r)\times r},\\
&\overline{A}_{11} := A_{11} - A_{02}A_{04}^{-1}A_{13} = (b_{ij})_{r\times r}, && \overline{A}_{12} := A_{12} - A_{02}A_{04}^{-1}A_{14} = (c_{ij})_{r\times(n-r)},\\
&\overline{A}_{13} := -A_{04}^{-1}A_{13} = (e_{ij})_{(n-r)\times r}, && \overline{A}_{14} := -A_{04}^{-1}A_{14} = (f_{ij})_{(n-r)\times(n-r)},\\
&\overline{K}_1(s) := K_1(s) - A_{02}A_{04}^{-1}K_3(s) = (k_{ij}^{(1)}(s))_{r\times r}, && \overline{K}_2(s) := K_2(s) - A_{02}A_{04}^{-1}K_4(s) = (k_{ij}^{(2)}(s))_{r\times(n-r)},\\
&\overline{K}_3(s) := -A_{04}^{-1}K_3(s) = (k_{ij}^{(3)}(s))_{(n-r)\times r}, && \overline{K}_4(s) := -A_{04}^{-1}K_4(s) = (k_{ij}^{(4)}(s))_{(n-r)\times(n-r)},\\
&\overline{B}_1 := B_1 - A_{02}A_{04}^{-1}B_2 = (r_{ij}^{(1)})_{r\times m}, && \overline{B}_2 := -A_{04}^{-1}B_2 = (r_{ij}^{(2)})_{(n-r)\times m},\\
&\overline{\psi} := (\overline{\psi}_{11}, \overline{\psi}_{12}), \text{ where } \overline{\psi}_{11} = (\overline{\psi}_1, \ldots, \overline{\psi}_r), && \overline{\psi}_{12} = (\overline{\psi}_{r+1}, \ldots, \overline{\psi}_n), q_{\min} = \min_{1\leq i\leq n} q_i,\\
&\text{where } q = (q_1, \ldots, q_n) \in \mathbb{R}_+^n.
\end{aligned} \tag{4}$$

Lemma 2. *Suppose that A_{04} is Hurwitz, A_0 is Metzler, $A_1, B \succeq 0$ and $\mathscr{K}(s)$ is nonnegative for all $s \in [-\overline{d}, 0]$. Then, for all nonnegative inputs $\vartheta(t) \succeq 0, t \geq 0$ the following system is positive*

$$E\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s) x(t+s) ds + B\vartheta(t),\ t \geq 0. \tag{5}$$

Proof. We are applying Lemma 1 yields $-A_{04}^{-1} \succeq 0$. Hence $\overline{A}_{01}$ is Metzler, $\overline{B}_1, \overline{B}_2, \overline{A}_{1i}, \overline{K}_i(s), i = 1,2,3,4$ are nonnegative matrices. The system (5) is reduced to the system

$$\begin{aligned}
\dot{x}_1(t) &= \overline{A}_{01} x_1(t) + \overline{A}_{11} x_1(t-\tau(t)) + \overline{A}_{12} x_2(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \big(\overline{K}_1(s) x_1(t+s) + \overline{K}_2(s) x_2(t+s)\big) ds \\
&\quad + \overline{B}_1 \vartheta(t), \\
x_2(t) &= \overline{A}_{03} x_1(t) + \overline{A}_{13} x_1(t-\tau(t)) + \overline{A}_{14} x_2(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \big(\overline{K}_3(s) x_1(t+s) + \overline{K}_4(s) x_2(t+s)\big) ds \\
&\quad + \overline{B}_2 \vartheta(t).
\end{aligned} \tag{6}$$

We first prove that for any given $T > 0$, $x(t) = (x_1(t), x_2(t)) \succeq 0, t \in [0,T]$. Let us assume the opposite i.e. there exist $t_0 > 0 :\ x(t_0) \not\succeq 0$. Let $t_1 = \inf\{t > 0 : x(t) \not\succeq 0\} \in [0,T]$, by continuity, there exists $j_0 \in \{1,..,n\}$ such that for some $0 < \varepsilon < \min\{\underline{\tau}, \underline{d}\}$:

$$x(t) \succeq 0, t \in [-M, t_1), x_{j_0}(t_1) = 0,\ \ x_{j_0}(t) < 0, t \in (t_1, t_1+\varepsilon). \tag{7}$$

However, by the definition of t_1 and from (6) we obtain

$$\begin{aligned}
x_1(t_1+\varepsilon) &= e^{\overline{A}_{01}(t_1+\varepsilon)} x_1(0) + \int_0^{t_1+\varepsilon} e^{\overline{A}_{01}(t_1+\varepsilon-v)} \Big[\overline{A}_{11} x_1(v-\tau(v)) + \overline{A}_{12} x_2(v-\tau(v)) + \overline{B}_1 \vartheta(v)\Big] dv \\
&\quad + \int_0^{t_1+\varepsilon} e^{\overline{A}_{01}(t_1+\varepsilon-v)} \int_{-d(v)}^{-\underline{d}} \Big[\overline{K}_1(s) x_1(v+s) + \overline{K}_2(s) x_2(v+s)\Big] ds\, dv \succeq 0, \\
x_2(t_1+\varepsilon) &= \overline{A}_{03} x_1(t_1+\varepsilon) + \overline{A}_{13} x_1(t_1+\varepsilon-\tau(t_1+\varepsilon)) + \overline{A}_{14} x_2(t_1+\varepsilon-\tau(t_1+\varepsilon)) \\
&\quad + \int_{-d(t_1+\varepsilon)}^{-\underline{d}} \Big[\overline{K}_3(s) x_1(t_1+\varepsilon+s) + \overline{K}_4(s) x_2(t_1+\varepsilon+s)\Big] ds + \overline{B}_2 \vartheta(t_1+\varepsilon) \succeq 0.
\end{aligned} \tag{8}$$

Clearly, (8) contradicts (7). Thus $x(t) \succeq 0, \forall t \in [0,T]$. When $T \to \infty$, one can conclude that the system (5) is positive. $\square$

3. Main results

From now on, we assume that A_0 is Metzler, $A_1, B \succeq 0$ and $\mathscr{K}(s) \succeq 0$ for all $s \in [-\overline{d}, 0]$.

3.1. State bounding for positive singular systems

First, we consider the following system:

$$\begin{cases} E\dot{x}(t) = A_0x(t) + A_1x(t-\tau(t)) + \int_{-d(t)}^{-d} \mathscr{K}(s)x(t+s)ds, t \geq 0, \\ x(s) = \psi(s),\ s \in [-M,\ 0]. \end{cases} \tag{9}$$

The following theorem guarantees regularity, impulse-free, positivity and gives a componentwise bound for the state of the system (9).

Theorem 3. *Assume that A_0 is a Metzler matrix, $A_1, B \succeq 0$ and $\mathscr{K}(s) \succeq 0$ for all $s \in [-\bar{d}, 0]$ and*

$$s\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right) < 0. \tag{10}$$

Then, the system (9) *is regular, impulse-free, positive and there exist $\theta > 0$ and $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}_+^n$, such that*

$$x(t,\psi) \preceq \frac{e^{\theta\bar{\tau}}}{q_{\min}} \|\psi\|\, q\, e^{-\theta t}, t \geq 0, \tag{11}$$

where $q_{\min} = \min\limits_{1\leq j\leq n} q_j$.

Proof. Our proof is divided into two parts. In the first part we show the system (9) is regular, impulse-free and positive. In the second part, we present a new result on componentwise bound of system (9).

1. Positivity, regularity and impulse-free.

It follows from inequality (10) and Lemma 1, that $\exists\, \xi \succ 0$:

$$\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right)\xi \prec 0. \tag{12}$$

Let $\xi := (\xi_{11}, \xi_{12}) \in \mathbb{R}_+^n$, $\xi_{11} \in \mathbb{R}_+^r$, $\xi_{12} \in \mathbb{R}_+^{n-r}$, which satisfies the inequality (12). Using the conditions $A_1 \succeq 0$, $\int_{-\bar{d}}^{-d} \mathscr{K}(s)ds \succeq 0$, $\xi \succ 0$, together with the inequality (12) we obtain the following:

$$\begin{pmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{12} \end{pmatrix} \prec 0. \tag{13}$$

It follows from (13) that

$$A_{03}\xi_{11} + A_{04}\xi_{12} \prec 0. \tag{14}$$

From inequality (14), it is easy to deduce that $A_{04}\xi_{12} \prec 0$ because of $A_{03}\xi_{11} \succeq 0$. Combining this with Lemma 1, we get $det(A_{04}) \neq 0$ and A_{04} is Hurwitz matrix. Consequently, the system (9) is regular and impulse-free (see [27]) and from Lemma 2 we obtain the systems (9) is positive.

2. Componentwise estimate of linear positive system (9).

Now, the system (9) rewrites the following form

$$
\begin{aligned}
\dot{x}_1(t) &= \overline{A}_{01}x_1(t)+\overline{A}_{11}x_1(t-\tau(t))+\overline{A}_{12}x_2(t-\tau(t))+\int_{-d(t)}^{-\underline{d}}\left(\overline{K}_1(s)x_1(t+s)+\overline{K}_2(s)x_2(t+s)\right)ds,\\
x_2(t) &= \overline{A}_{03}x_1(t)+\overline{A}_{13}x_1(t-\tau(t))+\overline{A}_{14}x_2(t-\tau(t))+\int_{-d(t)}^{-\underline{d}}\left(\overline{K}_3(s)x_1(t+s)+\overline{K}_4(s)x_2(t+s)\right)ds.
\end{aligned}
\tag{15}
$$

From (12) there exist $\theta>0$ such that

$$
\left(\theta E+A_0+e^{\theta\overline{\tau}}A_1+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\mathscr{K}(s)ds\right)\xi\prec 0. \tag{16}
$$

Since matrix $\begin{pmatrix} I_r & -A_{02}A_{04}^{-1}\\ 0 & -A_{04}^{-1}\end{pmatrix}$ is nonnegative and nonsingular, using inequality (16) yields the following result:

$$
\begin{pmatrix} I_r & -A_{02}A_{04}^{-1}\\ 0 & -A_{04}^{-1}\end{pmatrix}\left(\theta E+A_0+e^{\theta\overline{\tau}}A_1+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\mathscr{K}(s)ds\right)\xi\prec 0. \tag{17}
$$

Using inequality (17), we get:

$$
\left(\theta I_r+\overline{A}_{01}+e^{\theta\overline{\tau}}\overline{A}_{11}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\overline{K}_1(s)ds\right)\xi_{11}+\left(e^{\theta h}\overline{A}_{12}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\overline{K}_2(s)ds\right)\xi_{12}\prec 0, \tag{18}
$$

$$
\left(\overline{A}_{03}+e^{\theta\overline{\tau}}\overline{A}_{13}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\overline{K}_3(s)ds\right)\xi_{11}+\left(e^{\theta\overline{\tau}}\overline{A}_{14}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}\overline{K}_4(s)ds\right)\xi_{12}\prec\xi_{12}, \tag{19}
$$

where $\xi=(\xi_{11},\ \xi_{12}),\xi_{11}=(\xi_1,\dots,\xi_r)\in\mathbb{R}^r_+,\ \xi_{12}=(\xi_{r+1},\dots,\xi_n)\in\mathbb{R}^{n-r}_+$.

Set $\delta=\max\left\{\frac{\overline{\psi}_1}{\xi_1},\ \cdots,\ \frac{\overline{\psi}_n}{\xi_n}\right\}$, $q:=\delta\xi$. Then obviously, we have $q\succeq\overline{\psi}$, and the inequalities (18), (19) holds. Let $q=(q_{11},q_{12})\in\mathbb{R}^n_+$, where $q_{11}=(q_1,\dots,q_r)\in\mathbb{R}^r_+$, $q_{12}=(q_{r+1},\dots,q_n)\in\mathbb{R}^{n-r}_+$. Then, from (18), (19) and (4) we obtain

$$
\begin{aligned}
&\sum_{j=1}^{r}\left(a_{ij}+e^{\theta\overline{\tau}}b_{ij}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}k_{ij}^{(1)}(s)ds\right)q_j+\sum_{j=r+1}^{n}\left(c_{ij}e^{\theta\overline{\tau}}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}k_{ij}^{(2)}(s)ds\right)q_j<-\theta q_i,\\
&i=1,2,..,r,\\
&\sum_{j=1}^{r}\left(d_{ij}+e^{\theta\overline{\tau}}e_{ij}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}k_{ij}^{(3)}(s)ds\right)q_j+\sum_{j=r+1}^{n}\left(e^{\theta\overline{\tau}}f_{ij}+\int_{-\overline{d}}^{-\underline{d}}e^{-\theta s}k_{ij}^{(4)}(s)ds\right)q_j<q_i,\\
&i=r+1,\dots,n.
\end{aligned}
\tag{20}
$$

Next, we prove (11). Consider the following functions

$$v_i(t) = \frac{e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|q_i e^{-\theta t}, t \in [-M, +\infty),\ i = 1,2,..,n. \tag{21}$$

From the function $v_i(t), i = 1,2,..,n$ in (20), for all $t \geq 0,\ j \in \{1,2,...,n\}$ we have the following result:

$$v_j(t-\tau(t)) = \frac{e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|q_j e^{-\theta(t-\tau(t))} \leq \frac{e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|q_j e^{-\theta t}e^{\theta\bar{\tau}} = e^{\theta\bar{\tau}}v_j(t), \tag{22}$$

and

$$\int_{-d(t)}^{-\underline{d}} v_j(t+s)ds = \int_{-d(t)}^{-\underline{d}} v_j(t)\, e^{-\theta s}ds \leq v_j(t) \int_{-\bar{d}}^{-\underline{d}} e^{-\theta s}ds. \tag{23}$$

We will prove the following inequality

$$x_i(t) \leq v_i(t),\ \forall t \geq 0,\ \forall i \in \{1,2,\ldots,n\}. \tag{24}$$

Let $g_i(t) = x_i(t) - v_i(t), t \in [-M,\infty)$. Then for $t \in [-M,0]$ we have

$$x_i(t) = \psi_i(t) < \frac{q_i e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|e^{-\theta t} = v_i(t),$$

which implies that $g_i(t) < 0, t \in [-M,0],\ i \in \{1,\ldots,n\}$. Next, we show that $g_i(t) \leq 0, i = 1,2,\ldots,n$, for $t > 0$. Let us assume the opposite i.e. $\exists\, t_0 > 0 :\ g(t_0) \npreceq 0$, where $g(t) = (x_1(t) - v_1(t),\ldots,x_n(t) - v_n(t)) \in \mathbb{R}^n$. Let $t_1 = \inf\{t > 0 : g(t) \npreceq 0\}$, and by continuity, there exists $i_1 \in \{1,\ldots,n\}$ such that for some $\varepsilon > 0$ sufficiently small

$$g(t) \preceq 0, t \in [-M, t_1), g_{i_1}(t_1) = 0, \quad g_{i_1}(t) \succ 0, t \in (t_1, t_1+\varepsilon). \tag{25}$$

Moreover, from the inequalities (20)-(23), we obtain

$$\begin{aligned}
&\sum_{j=1}^{r} a_{ij}v_j(t) + \sum_{j=1}^{r} b_{ij}v_j(t-\tau(t)) + \sum_{j=r+1}^{n} c_{ij}v_j(t-\tau(t)) + \sum_{j=1}^{r}\int_{-d(t)}^{-\underline{d}} k_{ij}^{(1)}(s)v_j(t+s)ds \\
&\quad + \sum_{j=r+1}^{n}\int_{-d(t)}^{-\underline{d}} k_{ij}^{(2)}(s)v_j(t+s)ds \leq \Bigg(\sum_{j=1}^{r}\big(a_{ij} + e^{\theta\bar{\tau}}b_{ij} + \int_{-\bar{d}}^{-\underline{d}} e^{-\theta s}k_{ij}^{(1)}(s)ds\big)q_j \\
&\qquad\qquad + \sum_{j=r+1}^{n}\big(c_{ij}e^{\theta\bar{\tau}} + \int_{-\bar{d}}^{-\underline{d}} e^{-\theta s}k_{ij}^{(2)}(s)ds\big)q_j\Bigg)\frac{e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|e^{-\theta t} \\
&\qquad\qquad < -\theta\,\frac{e^{\theta\bar{\tau}}}{q_{\min}}\|\psi\|e^{-\theta t}q_i = -\theta v_i(t) = \dot{v}_i(t),\ i = 1,2,...,r,
\end{aligned} \tag{26}$$

and

$$\sum_{j=1}^{r} d_{ij}v_j(t) + \sum_{j=1}^{r} e_{ij}v_j(t-\tau(t)) + \sum_{j=r+1}^{n} f_{ij}v_j(t-\tau(t)) + \sum_{j=1}^{r}\int_{-d(t)}^{-\underline{d}} k_{ij}^{(3)}(s)v_j(t+s)ds$$
$$+ \sum_{j=r+1}^{n}\int_{-d(t)}^{-\underline{d}} k_{ij}^{(4)}(s)v_j(t+s)ds \le \Big(\sum_{j=1}^{r} \big(d_{ij} + e^{\theta\overline{\tau}}e_{ij} + \int_{-\overline{d}}^{-\underline{d}} e^{-\theta s}k_{ij}^{(3)}(s)ds\big)q_j$$
$$+ \sum_{j=r+1}^{n} \big(e^{\theta\overline{\tau}}f_{ij} + \int_{-\overline{d}}^{-\underline{d}} e^{-\theta s}k_{ij}^{(4)}(s)ds\big)q_j \Big) \frac{e^{\theta\overline{\tau}}}{q_{\min}}\|\psi\|e^{-\theta t}$$
$$< \frac{e^{\theta\overline{\tau}}}{q_{\min}}\|\psi\|e^{-\theta t}\, q_i = v_i(t),\ i = r+1, r+2, ..., n. \tag{27}$$

We consider the following cases:
Case 1. If $i_1 \in \{1,\ldots,r\}$, then it follows from (15) and (26) that for $t \in [0,t_1)$, we have

$$\begin{aligned}
D^+g_{i_1}(t) &\le D^+\{|x_{i_1}(t)|\} - \dot{v}_{i_1}(t) < a_{i_1i_1}|x_{i_1}(t)| + \sum_{j=1, j\ne i_1}^{r} a_{i_1j}|x_j(t)| + \sum_{j=1}^{r} b_{i_1j}|x_j(t-\tau(t))| \\
&+ \sum_{j=r+1}^{n} c_{i_1j}|x_j(t-\tau(t))| + \sum_{j=1}^{r}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(1)}(s)|x_j(t+s)|ds + \sum_{j=r+1}^{n}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(2)}(s)|x_j(t+s)|ds \\
&- \Big[a_{i_1i_1}v_{i_1}(t) + \sum_{j=1,j\ne i_1}^{r} a_{i_1j}v_j(t) + \sum_{j=1}^{r} b_{i_1j}v_j(t-\tau(t)) + \sum_{j=r+1}^{n} c_{i_1j}v_j(t-\tau(t)) \\
&+ \sum_{j=1}^{r}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(1)}(s)v_j(t+s)ds + \sum_{j=r+1}^{n}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(2)}(s)v_j(t+s)ds\Big] \\
&= a_{i_1i_1}g_{i_1}(t) + \sum_{j=1,j\ne i_1}^{r} a_{i_1j}g_j(t) + \sum_{j=1}^{r} b_{i_1j}g_j(t-\tau(t)) + \sum_{j=r+1}^{n} c_{i_1j}g_j(t-\tau(t)) \\
&+ \sum_{j=1}^{r}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(1)}(s)g_j(t+s)ds + \sum_{j=r+1}^{n}\int_{-d(t)}^{-\underline{d}} k_{i_1j}^{(2)}(s)g_j(t+s)ds,
\end{aligned} \tag{28}$$

where D^+ denotes the Dini upper-right derivative. Therefore, from (25), (28) we obtain the following inequality

$$D^+g_{i_1}(t) < a_{i_1i_1}g_{i_1}(t).$$

Letting $t = t_1$, we obtain $D^+g_{i_1}(t_1) < 0$, which leads to a contradiction with (25) hence $|x_i(t)| \le v_i(t)$, holds for all $i \in \{1,\ldots r\}$. Since the systems (9) is positive, we have $|x_i(t)| = x_i(t), t \ge 0$ for all $i = 1,\ldots,n$. Then, we see that, for all $i \in \{1,\ldots r\}$:

$$x_i(t) \le v_i(t). \tag{29}$$

Case 2. If $i_1 \in \{r+1,\dots,n\}$, it follows from (15), (27) that

$$
\begin{aligned}
g_{i_1}(t_1) = x_{i_1}(t_1) - v_{i_1}(t_1) &< \sum_{j=1}^{r} d_{i_1 j} x_j(t_1) + \sum_{j=1}^{r} e_{i_1 j} x_j(t_1-\tau(t_1)) + \sum_{j=r+1}^{n} f_{i_1 j} x_j(t_1-\tau(t_1)) \\
&+ \sum_{j=1}^{r} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(3)}(s) x_j(t+s)ds + \sum_{j=r+1}^{n} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(4)}(s) x_j(t+s)ds - \Bigg[\sum_{j=1}^{r} d_{i_1 j} v_j(t_1) \\
&+ \sum_{j=1}^{r} e_{i_1 j} v_j(t_1-\tau(t_1)) + \sum_{j=r+1}^{n} f_{i_1 j} v_j(t_1-\tau(t_1)) + \sum_{j=1}^{r} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(3)}(s) v_j(t+s)ds \\
&+ \sum_{j=r+1}^{n} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(4)}(s) v_j(t+s)ds \Bigg] \\
= \sum_{j=1}^{r} d_{i_1 j} g_j(t_1) &+ \sum_{j=1}^{r} e_{i_1 j} g_j(t_1-\tau(t_1)) + \sum_{j=r+1}^{n} f_{i_1 j} g_j(t_1-\tau(t_1)) + \sum_{j=1}^{r} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(3)}(s) g_j(t+s)ds \\
&+ \sum_{j=r+1}^{n} \int_{-d(t_1)}^{-\underline{d}} k_{i_1 j}^{(4)}(s) g_j(t+s)ds.
\end{aligned} \tag{30}
$$

Combining (25), (29) and (30) we obtain $g_{i_1}(t_1) < 0$ which leads to a contradiction. Then, the following inequality holds:

$$
x_i(t) \le v_i(t), \ \forall i \in \{r+1, \dots n\}. \tag{31}
$$

Using the inequalities (29) and (31), we obtain (24), which leads to the inequality (11) hold. The proof is complete. □

Corollary 1. *Assume that* $s\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s)ds\right) < 0$. *Then, the system* (9) *is regular, impulse-free, positive and exponentially stable. Moreover, every solution* $x(t,\psi)$ *of* (9) *satisfies the following inequality* $\|x(t,\psi)\| \le N\|\psi\| \ e^{-\theta t}, t \ge 0$, *where* $N = \frac{e^{\theta \bar{\tau}}}{q_{\min}}\|q\|$ *as specified in Theorem 3.*

Proof. The proof of Corollary 1 is completed in the same way as in Theorem 3. □

Remark 1. In [38], the author addressed a class of singular systems with time-varying discrete delays. In order to obtain exponential stability criteria, the author of the work [38] have required the considered system satisfying the regular and impulse-free conditions. This condition, however, is no longer necessary in our approach. Different from the aforementioned works, in this paper, not only the regular and impulse-free conditions did not require but also the issue of time-varying discrete and distributed delays is considered. Therefore, Theorem 3 of this paper is new and generalized.

Remark 2. Recently, the authors in [39] have addressed the problem of positivity and asymptotic stability analysis of the singular system with distributed delay

$$
E\dot{x}(t) = A_0 x(t) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s)x(t+s)ds, \ t \ge 0, \tag{32}
$$

where the pair (E,A_0) is regular and impulse-free. For simply, we assume that $E := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $A_0 := \begin{pmatrix} A_{01} & 0 \\ 0 & I_{n-r} \end{pmatrix}$, $\mathscr{K}(s) = \begin{pmatrix} K_1(s) & K_2(s) \\ K_3(s) & K_4(s) \end{pmatrix}$. By Theorem 4.1 in Cui, Shen & Chen (2018) [39], system (32) is positive and asymptotically stable if $A_{d_i}(s) \succeq 0, i = 1,2, s \in [-\overline{d}, 0]$ and there exists a Metzler matrix H such that $\mathscr{A}_1 = HM$ and Π is Hurwitz, where

$$\Pi = \begin{bmatrix} H + \int_{-\overline{d}}^{-d} A_{d_1}(s)ds & \int_{-\overline{d}}^{-d} A_{d_1}(s)ds \\ \int_{-\overline{d}}^{-d} A_{d_2}(s)ds & \int_{-\overline{d}}^{-d} A_{d_2}(s)ds - I_n \end{bmatrix}, M = E,\ \mathscr{A}_1 = E^D A_0 = \begin{pmatrix} A_{01} & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_{d_1}(s) = E^D \mathscr{K}(s) = \begin{pmatrix} K_1(s) & K_2(s) \\ 0 & 0 \end{pmatrix}, A_{d_2}(s) = (M - I_n) A_0^D \mathscr{K}(s) = \begin{pmatrix} 0 & 0 \\ K_3(s) & K_4(s) \end{pmatrix},$$

with E^D is the Drazin inverse of E. By simple computation, we obtain the condition Π is Hurwitz is equivalent the condition $s(A_0 + \int_{-\overline{d}}^{-d} \mathscr{K}(s)ds) < 0$ of Corollary 1.

A necessary condition for the componentwise bounds of positive singular system without disturbances is established in the following theorem.

Theorem 4. *Assume that A_{04} is Hurwitz and there exists numbers G, $\alpha > 0$ and $q \in \mathbb{R}^n_+$, such that with any initial condition the following estimate hold: $x(t,\varphi,\omega) \preceq q\, Ge^{-\alpha t}, t \geq 0$. Then, there exists $\xi \in \mathbb{R}^n_{0,+}$ such that*

$$\left(A_0 + A_1 + \int_{-\overline{d}}^{-d} \mathscr{K}(s)ds\right) \xi \preceq 0. \tag{33}$$

Proof. Let $\tau(t) = \overline{\tau}$ and $d(t) = \overline{d}$, we consider the following singular system (9) with constant delay

$$\begin{cases} E\dot{x}(t) = A_0 x(t) + A_1 x(t - \overline{\tau}) + \int_{-\overline{d}}^{-d} \mathscr{K}(s) x(t+s) ds, \\ x(s) = \varphi(s),\ s \in [-M,\ 0]. \end{cases} \tag{34}$$

By the assumption, the system (9) is positive and exponentially stable for all delays $\tau(t) \in [\underline{\tau},\ \overline{\tau}]$ and $d(t) \in [\underline{d},\ \overline{d}]$, which implies that system (34) is also exponentially stable. Integrating the equation (34) in the interval $[0,T]$ we obtain

$$\begin{aligned} Ex(T) - Ex(0) =& A_0 \int_0^T x(t)dt + A_1 \int_0^T x(t - \overline{\tau})dt + \int_0^T \int_{-\overline{d}}^{-d} \mathscr{K}(s) x(t+s) ds dt. \\ =& A_0 \int_0^T x(t)dt + A_1 \int_{-\overline{\tau}}^{T-\overline{\tau}} x(t)dt + \int_{-\overline{d}}^{-d} \mathscr{K}(s) \int_s^{T+s} x(t) dt ds. \end{aligned} \tag{35}$$

On the other hand, we have

$$\begin{aligned} A_1 \int_{-\overline{\tau}}^{T-\overline{\tau}} x(t)dt &= A_1 \left(\int_{-\overline{\tau}}^{0} x(t)dt + \int_0^T x(t)dt \right) - A_1 \int_{T-\overline{\tau}}^{T} x(t)dt, \\ \int_{-\overline{d}}^{-d} \mathscr{K}(s) \int_s^{T+s} x(t)dtds &= \int_{-\overline{d}}^{-d} \mathscr{K}(s) \int_0^T x(t)dtds + \int_{-\overline{d}}^{-d} \mathscr{K}(s) \int_s^0 x(t)dtds \\ &\quad - \int_{-\overline{d}}^{-d} \mathscr{K}(s) \int_{T+s}^{T} x(t)dtds. \end{aligned} \tag{36}$$

Since system (34) is exponentially stable, $x(t) \to 0$ as $t \to \infty$. Therefore, there exists a large enough σ such that

$$\int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s) \int_{\sigma+s}^{\sigma} x(t)dtds = \mathscr{K}(v)x(\theta)S_D \to 0,$$

where $\theta \in [\sigma - \overline{d}, \sigma]$, $v \in [-\overline{d},\ -\underline{d}]$ and S_D is the finite domain on integral. With initial condition $x(s) = \varphi(s) \succ 0,\ s \in [-M, 0]$, we have

$$\begin{aligned} Ex(\sigma) - Ex(0) - A_1 \int_{-\overline{\tau}}^{0} x(t)dt + A_1 \int_{\sigma-\overline{\tau}}^{\sigma} x(t)dt - \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s) \int_{s}^{0} x(t)dtds \\ + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s) \int_{\sigma+s}^{\sigma} x(t)dtds \preceq 0. \end{aligned} \tag{37}$$

Combining this with (35), (36) we obtain

$$\begin{aligned} \left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right) \int_{0}^{\sigma} x(t)dt = Ex(\sigma) - Ex(0) - A_1 \int_{-\overline{\tau}}^{0} x(t)dt + A_1 \int_{\sigma-\overline{\tau}}^{\sigma} x(t)dt \\ - \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s) \int_{s}^{0} x(t)dtds + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s) \int_{\sigma+s}^{\sigma} x(t)dtds \preceq 0, \end{aligned} \tag{38}$$

which implies that (33) holds for $\xi = \int_0^{\sigma} x(t)dt$. This completes the proof. □

Under strictly positive initial condition, we obtain a necessary and sufficient condition for componentwise bounds of positive singular system (9).

Theorem 5. *Assume that there exists $v = (v_1, v_2) \in \mathbb{R}^n$ such that $v_2 \succeq 0$, and*

$$H := \begin{pmatrix} I_r & 0 \\ A_{13} + \int_{-d(0)}^{-\underline{d}} K_3(s)ds & A_{14} + \int_{-d(0)}^{-\underline{d}} K_4(s)ds \end{pmatrix} v \succ 0.$$

Then, the following statements are equivalent:

1) *A_{04} is Hurwitz matrix and there exists $\alpha \in \mathbb{R}_+$, $q \in \mathbb{R}_+^n$, such that with any initial condition $\varphi(\cdot) \succeq 0$ the following estimate hold: $x(t,\varphi) \preceq \frac{e^{\alpha\overline{\tau}}}{q_{\min}} \|\varphi\|\, q\, e^{-\alpha t}, t \geq 0$.*

2) *There exists $\eta \in \mathbb{R}_+^n$ such that $\left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\eta \prec 0$.*

Proof. From Theorem 3 we obtain 2) $\Rightarrow$ 1).
1) $\Rightarrow$ 2). From A_{04} is Metzler and Hurwitz and Lemma 1 we obtain $-A_{04}^{-1} \succeq 0$ and nonsingular. From this and $H \succ 0$ we get

$$\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix} H = \begin{pmatrix} I_r & 0 \\ \overline{A}_{13} + \int_{-d(0)}^{-\underline{d}} \overline{K}_3(s)ds & \overline{A}_{14} + \int_{-d(0)}^{-\underline{d}} \overline{K}_4(s)ds \end{pmatrix} v \succ 0.$$

From Theorem 4 we have the system (34) with initial condition $x_1(s) = v_1 \succ 0, x_2(s) = v_2 \succeq 0,\ \ s \in [-M, 0)$, and

$$\begin{aligned} x(0) &= \begin{pmatrix} I_r & 0 \\ \overline{A}_{03} + \overline{A}_{13} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_3(s)ds & \overline{A}_{14} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_4(s)ds \end{pmatrix} v \\ &\succeq \begin{pmatrix} I_r & 0 \\ \overline{A}_{13} + \int_{-d(0)}^{-\underline{d}} \overline{K}_3(s)ds & \overline{A}_{14} + \int_{-d(0)}^{-\underline{d}} \overline{K}_4(s)ds \end{pmatrix} v \succ 0. \end{aligned} \tag{39}$$

It follows from the continuity of $x(t)$ that $\xi = \int_0^\sigma x(t)dt \succ 0$. Let $\xi = (\xi_1, \xi_2),\ \xi_1 \in \mathbb{R}_+^r,\ \xi_2 \in \mathbb{R}_+^{n-r}$. From (37)-(38) we obtain

$$\left(A_{01} + A_{11} + \int_{-\overline{d}}^{-\underline{d}} K_1(s)ds\right)\xi_1 + \left(A_{02} + A_{12} + \int_{-\overline{d}}^{-\underline{d}} K_2(s)ds\right)\xi_2 \preceq -x_1(0) = -v_1 \prec 0, \tag{40}$$

and

$$\begin{aligned} &\left(A_{03} + A_{13} + \int_{-\overline{d}}^{-\underline{d}} K_3(s)ds\right)\xi_1 + \left(A_{04} + A_{14} + \int_{-\overline{d}}^{-\underline{d}} K_4(s)ds\right)\xi_2 \\ &\preceq -A_{13}\int_{-\overline{\tau}}^{0} x_1(t)dt - A_{14}\int_{-\overline{\tau}}^{0} x_2(t)dt - \int_{-\overline{d}}^{-\underline{d}} K_3(s)\int_s^0 x_1(t)dtds - \int_{-\overline{d}}^{-\underline{d}} K_4(s)\int_s^0 x_2(t)dtds. \end{aligned} \tag{41}$$

Note that, since the matrix A_{04} is Hurwitz and Metzler, by using Lemma 1 we obtain $-A_{04}^{-1} \succeq 0$. Pre-multiplying both sides of (41) with nonsingular matrix $-A_{04}^{-1} \succeq 0$ we have

$$\begin{aligned} &\left(\overline{A}_{03} + \overline{A}_{13} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_3(s)ds\right)\xi_1 + \left(\overline{A}_{14} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_4(s)ds - I_{n-r}\right)\xi_2 \\ &\preceq -\overline{A}_{13}\int_{-\overline{\tau}}^{0} x_1(t)dt - \overline{A}_{14}\int_{-\overline{\tau}}^{0} x_2(t)dt - \int_{-\overline{d}}^{-\underline{d}} \overline{K}_3(s)\int_s^0 x_1(t)dtds - \int_{-\overline{d}}^{-\underline{d}} \overline{K}_4(s)\int_s^0 x_2(t)dtds. \end{aligned} \tag{42}$$

From $\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix} H \succ 0$, we get

$$\begin{aligned} &\left(\overline{A}_{03} + \overline{A}_{13} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_3(s)ds\right)\xi_1 + \left(\overline{A}_{14} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_4(s)ds - I_{n-r}\right)\xi_2 \\ &\qquad \preceq -\left(\overline{A}_{13} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_3(s)\,ds\right) v_1 \min\{\underline{\tau}, \underline{d}\} - \left(\overline{A}_{14} + \int_{-\overline{d}}^{-\underline{d}} \overline{K}_4(s)ds\right) v_2 \min\{\underline{\tau}, \underline{d}\} \prec 0. \end{aligned} \tag{43}$$

From (40) and (43) we obtain

$$\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix} \left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\xi \prec 0. \tag{44}$$

It is easy to see that $\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix}\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right)$ is Metzler matrix. Hence, from (44) and Lemma 1 there exist $\gamma_1 = \begin{pmatrix} \gamma_{11}^T & \gamma_{12}^T \end{pmatrix}^T \in \mathbb{R}_+^n$ such that

$$\begin{pmatrix} \gamma_{11}^T & \gamma_{12}^T \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix}\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right) \prec 0. \tag{45}$$

Set $\gamma_2^T = \begin{pmatrix} \gamma_{11}^T & \gamma_{12}^T \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & -A_{04}^{-1} \end{pmatrix} = \begin{pmatrix} \gamma_{11}^T & \gamma_{12}^T(-A_{04}^{-1}) \end{pmatrix}$, then $\gamma_2^T \succ 0$ by $-A_{04}^{-1} \succeq 0$ and nonsingular. From (45) and Lemma 1 there exists $\eta \in \mathbb{R}_+^n$ such that $\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right)\eta \prec 0$. The proof is complete. $\square$

We also obtain the following corollary of Theorem 5.

Corollary 2. *Assume that there exists $v = (v_1, v_2) \in \mathbb{R}^n$ such that $v_2 \succeq 0$, and*

$$H := \begin{pmatrix} I_r & 0 \\ A_{13} + \int_{-d(0)}^{-d} K_3(s)ds & A_{14} + \int_{-d(0)}^{-d} K_4(s)ds \end{pmatrix} v \succ 0.$$

The following statements are equivalent:

(i) *A_{04} is Hurwitz matrix and system (9) is positive and exponentially stable with any initial condition $\varphi(\cdot) \succeq 0$.*

(ii) *There exists $\eta \in \mathbb{R}_+^n$ such that: $\left(A_0 + A_1 + \int_{-\bar{d}}^{-d} \mathscr{K}(s)ds\right)\eta \prec 0$.*

Now, we investigate the state bounding problem for the singular system (1). We first consider the following system:

$$\begin{cases} E\dot{z}(t) = A_0 z(t) + A_1 z(t-\tau(t)) + \int_{-d(t)}^{-d} \mathscr{K}(s) z(t+s)ds + B\overline{\vartheta},\ t \geq 0, \\ z(s) = \phi(s),\ s \in [-M,\ 0]. \end{cases} \tag{46}$$

The following lemma provides a relationship between the state trajectory of the system (1) and the state trajectory of the system (46).

Lemma 6. *Assume that A_{04} is Hurwitz, $x(t,\psi,\vartheta)$ and $z(t,\phi_i,\overline{\vartheta}), i = 1,2$ are the trajectories of systems (1) and (46) with the initial conditions $\psi(s)$ and $\phi_i(s), i = 1,2$, respectively. Then we have estimated as follows:*

i) If $\psi(s) \preceq \phi_1(s)\ \ \forall s \in [-\overline{\tau},\ 0]$ then $x(t,\psi,\vartheta) \preceq z(t,\phi_1,\overline{\vartheta}), \forall t \geq 0$.

ii) If $\phi_1(s) \preceq \phi_2(s)\ \ \forall s \in [-\overline{\tau},\ 0]$, then $z(t,\phi_1,\overline{\vartheta}) \preceq z(t,\phi_2,\overline{\vartheta}), \forall t \geq 0$.

Proof. (i) Define $q(t) = z(t) - x(t), e(t) = \overline{\vartheta} - \vartheta(t)$. Then we consider the system

$$E\dot{q}(t) = A_0 q(t) + A_1 q(t - \tau(t)) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s) q(t+s) ds + Be(t),\ t \geq 0,$$
$$q(s) = \phi_1(s) - \psi(s),\ s \in [-M,\ 0].$$

Apply Lemma 2 to the above system, we obtain $q(t) \succeq 0$, for all $t \geq 0$. Then we have $z(t,\ \phi_1,\ \overline{\vartheta}) - x(t, \psi, \vartheta) \succeq 0$, for all $t \geq 0$.
(ii) The proof of (ii) can be obtained similarly to the proof of (i). The proof is complete. $\square$

The following theorem provides a condition sufficient to ensure that system (1) is regular, impulse-free, and the existence of an upper bound for the system.

Theorem 7. *Suppose that* $s\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s) ds\right) < 0$. *Then, the systems* (1) *is regular, impulse-free, positive and*

(i) *There exist* $G, \theta > 0$, $q \in \mathbb{R}^n_+$, *and a nonnegative vector*

$$\zeta = -\left(A_0 + A_1 + \int_{-d(0)}^{-\underline{d}} \mathscr{K}(s) ds\right)^{-1} B\,\overline{\vartheta}, \tag{47}$$

such that

$$x(t, \psi, \vartheta) \preceq \zeta + q\, G e^{-\theta t}, t \geq 0. \tag{48}$$

(ii) $\zeta \in \mathbb{R}^n_{0,+}$ *is the smallest vector such that*

$$\limsup_{t \to \infty} x(t, \psi, \vartheta) \preceq \zeta. \tag{49}$$

Proof. The regularity, impulse-free, positivity of the singular system (1) is proved similar to that in Theorem 3, which we omitted here.
(i) Using condition $s\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s) ds\right) < 0$ and applying Lemma 1 implies that there exists a vector $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n_+$ such that:

$$\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s) ds\right) \xi \prec 0. \tag{50}$$

Similar in (12), from (50) we obtain A_{04} is Hurwitz which implies that systems (1) is regular and impulse-free.
Step 1. We show that there exists an initial function $\upsilon_\lambda(\cdot)$ of the system (46) such that the following estimate holds for all nonnegative initial functions $\psi(\cdot)$ of the system (1).

$$x(t, \psi, \vartheta) \preceq z(t, \upsilon_\lambda, \overline{\vartheta}),\ t \geq 0.$$

Indeed, let $\delta_1 = \max\left\{\frac{\overline{\psi}_1}{\xi_1}, \cdots, \frac{\overline{\psi}_n}{\xi n}\right\}, \ \lambda = \delta_1 \xi$. We get $\lambda \succeq \overline{\psi}$ and the following inequality holds:

$$\left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\lambda \prec 0. \tag{51}$$

Set $\upsilon_\lambda(s) = \lambda, \ s \in [-M,0]$. The estimate is easily obtained $\psi(s) \preceq \overline{\psi} \preceq \lambda, \ s \in [-M,0]$, for all $\psi(\cdot)$, the initial function satisfies the inequality (3). Furthermore, because $\psi(s) \preceq \lambda = \upsilon_\lambda(s), \ s \in [-M,0]$, we immediately estimate $\psi(s) \preceq \upsilon_\lambda(s), \ s \in [-M,0]$. The combination of this and Lemma 6 gives the following estimate:

$$x(t,\psi,\vartheta) \preceq z(t,\upsilon_\lambda,\overline{\vartheta}), \ t \geq 0. \tag{52}$$

Step 2. Because $\lambda \succ 0$ is strictly inequality, so there is always $\rho > 1$ so that we have inequality $\eta := \rho\lambda \succ \zeta$. Combining this with inequality (51) we have estimations as follow:

$$\left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\eta \prec 0. \tag{53}$$

Let $\upsilon_\eta(s) = \eta, \ s \in [-M,0]$. Because of $\lambda \prec \eta$, we immediately estimate

$$\upsilon_\lambda(s) \preceq \upsilon_\eta(s), \ s \in [-M,0]. \tag{54}$$

Then from Lemma 6 and (54), we get

$$z(t,\upsilon_\lambda,\overline{\vartheta}) \preceq z(t,\upsilon_\eta,\overline{\vartheta}), \ t \geq 0. \tag{55}$$

Note that, from (50) we have

$$\left(A_0 + A_1 + \int_{-d(0)}^{-\underline{d}} \mathscr{K}(s)ds\right)\xi \preceq \left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\xi \prec 0. \tag{56}$$

From Lemma 1 and (56), we see that ζ exists and nonnegative. Let $\upsilon_\zeta(s) = \zeta, \ s \in [-M,0]$, and by $\zeta \prec \eta$, it is easy to check

$$\upsilon_\zeta(s) \preceq \upsilon_\eta(s), \ s \in [-M,0]. \tag{57}$$

Set $\upsilon_{\eta-\zeta}(s) := \upsilon_\eta(s) - \upsilon_\zeta(s), \ s \in [-M,0]$. It follows from (57) that $\upsilon_{\eta-\zeta}(s) \succeq 0, s \in [-M,0]$. We now use the following variable transformation

$$z(t) = u(t) + \zeta, \tag{58}$$

then the system (46) is returned as follows

$$E\dot{u}(t) = A_0 u(t) + A_1 u(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s)u(t+s)ds, \tag{59}$$

and

$$z(t, \ \upsilon_\eta, \overline{\vartheta}) = \zeta + u(t,\upsilon_{\eta-\zeta}). \tag{60}$$

Let $\eta := (\eta_{11},\ldots,\eta_{1n}),\ \zeta := (\zeta_{11},\ldots,\zeta_{1n})$, and $\delta_2 = \max\left\{\frac{\eta_{11}-\zeta_{11}}{\eta_{11}},\ \cdots,\ \frac{\eta_{1n}-\zeta_{1n}}{\eta_{1n}}\right\}$. We immediately that $\eta = \rho\lambda \succ \zeta,\ \delta_2 > 0$, which indicates that there exists a positive number $\kappa > 0$, gives us an estimate $\vartheta := \delta_2\kappa > 1$. By setting $\nu := \vartheta\eta \succ \eta$, then easily deduce that $\nu \succ \eta - \zeta$, using this with inequality (53) we obtain inquality as follow

$$\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s)ds\right)\nu \prec 0.$$

Let $\upsilon_\nu(s) = \nu,\ s \in [-M,0]$, because of $\eta - \zeta \prec \nu$, then we get $\upsilon_{\eta-\zeta}(s) \preceq \upsilon_\nu(s),\ s \in [-M,0]$. Using Lemma 6 and (54) we obtain

$$u(t,\upsilon_{\eta-\zeta}) \preceq u(t,\upsilon_\nu),\ t \geq 0. \tag{61}$$

By applying Theorem 3 to system (59), we have

$$u(t,\upsilon_\nu) \preceq N\|\ \upsilon_\nu\|\ qe^{-\theta t}, \forall t \geq 0. \tag{62}$$

Combining (52), (55), (60), (61) and (62), we obtain (48).

(ii) In the inequality (48) we give $t \to \infty$ then we deduce the inequality (49), which indicates that ζ is a upper bound of the system (1). Below, we show that $\lim_{t\to\infty} z(t,\ \upsilon_0,\ \overline{\vartheta}) = \zeta$, where $\upsilon_0(s) = 0, s \in [-M,0]$. Indeed, using the following transform:

$$v(t) = \zeta - z(t), \tag{63}$$

then the system (46) becomes the following system:

$$E\dot{v}(t) = A_0 v(t) + A_1 v(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s)v(t+s)ds,\ t \geq 0, \tag{64}$$

and

$$v(t,\upsilon_{0\zeta}) = \zeta - z(t,\upsilon_0,\ \overline{\vartheta}), \forall t \geq 0, \tag{65}$$

where $v(t,\upsilon_{0\zeta})$ is the solution with initial function $\upsilon_{0\zeta}(s) = \zeta, s \in [-M,0]$. By Lemma 2, implies $v(t,\upsilon_{0\zeta}) \succeq 0, t \geq 0$. By applying Theorem 3 to system (64), implies that $\exists\, N,\ \theta > 0:$

$$0 \preceq v(t,\upsilon_{0\zeta}) \preceq Ne^{-\theta t}, \forall t \geq 0. \tag{66}$$

Combining (65) and (66) we get

$$\zeta - Ne^{-\theta t} \preceq z(t,\upsilon_0,\overline{\vartheta}) \preceq \zeta, \forall t \geq 0. \tag{67}$$

Letting $t \to \infty$, we obtain $\lim_{t\to\infty} z(t,\upsilon_0,\overline{\vartheta}) = \zeta$. Then we have ζ which is the smallest componentwise ultimate bound of (1). $\square$

Remark 3. Recently, the minimization of state bounding of the positive standard system has been investigated in [9]. The methods used in [9] were mainly based on the Lyapunov functional approach. Moreover, they did not consider the issues of singular systems. This work cannot be extended to positive singular systems easily. Recently, the problem of exponential stability of nonlinear singular equations was studied in [31]. However, Ngoc [31] had to assume that the considered singular systems to be regular and impulse-free. In this paper, we derived some new conditions for the state bounding problem of the positive singular system (1) without requiring the above limited conditions.

Remark 4. In [41], the authors considered the problem of state bounding for positive coupled differential-difference equations with bounded disturbances of the following form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t-\tau(t)) + \vartheta_1(t), \\ y(t) &= Cx(t) + Dy(t-\tau(t)) + \vartheta_2(t), \end{aligned} \tag{68}$$

the disturbance vector $\vartheta_i(t), i = 1,2$ is an unknown continuous vector-valued function satisfying $0 \preceq \vartheta_i(t) \preceq \overline{\vartheta}_i$, $\forall t \geq 0, i = 1,2$. It had been showed in Theorem 3 of [41] that the vector $\zeta = -\begin{bmatrix} A & B \\ C & D-I \end{bmatrix}^{-1} \begin{bmatrix} \overline{\vartheta}_1 \\ \overline{\vartheta}_2 \end{bmatrix}$ is the smallest upper bound of system (68) if the following conditions hold

i) A is a Metzler matrix, B, C, D are nonnegative, D is a Schur matrix,

ii) $s(A + B(I-D)^{-1}C) < 0$.

This work added considerable conservativeness into state bounding problem for positive coupled differential-difference equations (68). For example, the regular and impulse-free conditions of the considered system was required in this work. This condition, however, is no longer necessary in our approach. It is not hard to see that the criteria in [41] is equivalent to Theorem 7 of our results. Hence, the obtained results in this paper are novel.

4. An extension to time-varying singular systems

We extend the above results to the system of the following form:

$$\begin{cases} E\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-\tau(t)) + \int_{-d(t)}^{-d} \mathscr{K}(t,s)x(t+s)ds + B(t)\vartheta(t), \ t \geq 0, \\ x(s) = \psi(s), \ s \in [-M, \ 0], \end{cases} \tag{69}$$

where $E \in \mathbb{R}^{n\times n}, rank(E) = r < n$, $A_0(\cdot)$, $A_1(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $B(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $\mathscr{K}(\cdot,\cdot) : \mathbb{R} \times [-\overline{d}, 0] \to \mathbb{R}^{n\times n}$ are continuous functions. $\tau(t) \in [\underline{\tau}, \ \overline{\tau}]$ and $d(t) \in [\underline{d}, \ \overline{d}]$; $\underline{d}, \ \overline{d}, \ \underline{\tau}, \overline{\tau} > 0$, $M = \max\{\overline{d}, \ \overline{\tau}\}$. Functions $\vartheta(t) \in \mathbb{R}^m_{0,+}$ and $\psi(s) \in \mathbb{R}^n_{0,+}, s \in [-M, \ 0]$ satisfy the inequality (2) and (3), respectively. In this case, we assume that:
$E := \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, $A_i(t) := \begin{pmatrix} A_{i1}(t) & A_{i2}(t) \\ A_{i3}(t) & A_{i4}(t) \end{pmatrix}$, $i = 0,1$, $\mathscr{K}(t,s) = \begin{pmatrix} K_1(t,s) & K_2(t,s) \\ K_3(t,s) & K_4(t,s) \end{pmatrix}$, $B(t) :=$

$\begin{pmatrix} B_1(t) \\ B_2(t) \end{pmatrix}$, $t \in \mathbb{R}$, where $A_{01}(t),\ A_{11}(t),\ K_1(t,s) \in \mathbb{R}^{r\times r}, A_{02}(t) \in A_{12}(t),\ K_2(t,s) \in \mathbb{R}^{r\times(n-r)}$, $A_{03}(t), A_{13}(t),\ K_3(t,s) \in \mathbb{R}^{(n-r)\times r}, A_{04}(t),\ A_{14}(t),\ K_4(t,s) \in \mathbb{R}^{(n-r)\times(n-r)}, B_1(t) \in \mathbb{R}^{r\times m}, B_2(t) \in \mathbb{R}^{(n-r)\times m}$. $x(t) := (x_1(t),\ x_2(t))$ with $x_1(t) \in \mathbb{R}^r; x_2(t) \in \mathbb{R}^{n-r}$.

Definition 4. ([27]) (i) System (69) is said to be regular if $det(sE - A_0(t))$ is not identically zero for all $t \geq 0$. (ii) System (69) is said to be impulse-free if $deg(det(sE - A_0(t))) = rank(E) = r, \forall t \geq 0$.

Setting

$$\begin{aligned}
&\overline{A}_{01}(t) := A_{01}(t) - A_{02}(t)A_{04}^{-1}(t)A_{03}(t) = (a_{ij}(t))_{r\times r},\\
&\overline{A}_{11}(t) := A_{11}(t) - A_{02}(t)A_{04}^{-1}(t)A_{13}(t) = (b_{ij}(t))_{r\times r},\\
&\overline{A}_{03}(t) := -A_{04}^{-1}(t)A_{03}(t) = (d_{ij}(t))_{(n-r)\times r},\\
&\overline{A}_{12}(t) := A_{12}(t) - A_{02}(t)A_{04}^{-1}(t)A_{14}(t) = (c_{ij}(t))_{r\times(n-r)},\\
&\overline{A}_{13}(t) := -A_{04}^{-1}(t)A_{13}(t) = (e_{ij}(t))_{(n-r)\times r}, \quad \overline{A}_{14}(t) := -A_{04}^{-1}(t)A_{14}(t) = (f_{ij}(t))_{(n-r)\times(n-r)},\\
&\overline{K}_1(t,s) := K_1(t,s) - A_{02}(t)A_{04}^{-1}(t)K_3(t,s) = (k_{ij}^{(1)}(t,s))_{r\times r},\\
&\overline{K}_2(t,s) := K_2(t,s) - A_{02}(t)A_{04}^{-1}(t)K_4(t,s) = (k_{ij}^{(2)}(t,s))_{r\times(n-r)},\\
&\overline{K}_3(t,s) := -A_{04}^{-1}(t)K_3(t,s) = (k_{ij}^{(3)}(t,s))_{(n-r)\times r},\\
&\overline{K}_4(t,s) := -A_{04}^{-1}(t)K_4(t,s) = (k_{ij}^{(4)}(t,s))_{(n-r)\times(n-r)},\\
&\overline{B}_1(t) := B_1(t) - A_{02}(t)A_{04}^{-1}(t)B_2(t) = (r_{ij}^{(1)}(t))_{r\times m},\\
&\overline{B}_2(t) := -A_{04}^{-1}(t)B_2(t) = (r_{ij}^{(2)}(t))_{(n-r)\times m}, \quad \vartheta(t) = (\vartheta_1(t), \dots, \vartheta_m(t)),\ \overline{\vartheta} = (\overline{\vartheta}_1, \dots, \overline{\vartheta}_m).
\end{aligned}$$

Lemma 8. *([42]) For two Metzler matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, if $A_1 \succeq A_2$, then $s(A_1) \geq s(A_2)$. Moreover, if $s(A_1) < 0$, then $-A_1^{-1} \succeq -A_2^{-1}$.*

Lemma 9. *Assume that there exists $A_i := \begin{pmatrix} A_{i1} & A_{i2} \\ A_{i3} & A_{i4} \end{pmatrix}$, $i = 0,1$, $\mathscr{K}(s) = \begin{pmatrix} K_1(s) & K_2(s) \\ K_3(s) & K_4(s) \end{pmatrix}$, $B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ and $A_{04}(t)$ is Metzler $\forall t \in \mathbb{R}$ such that*

$$\begin{pmatrix} M(A_{01}(t)) & |A_{02}(t)| \\ |A_{03}(t)| & A_{04}(t) \end{pmatrix} \preceq \begin{pmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{pmatrix}; \quad \begin{pmatrix} |A_{11}(t)| & |A_{12}(t)| \\ |A_{13}(t)| & |A_{14}(t)| \end{pmatrix} \preceq \begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix}, \tag{70}$$

$$\begin{pmatrix} |K_1(t,s)| & |K_2(t,s)| \\ |K_3(t,s)| & |K_4(t,s)| \end{pmatrix} \preceq \begin{pmatrix} K_1(s) & K_2(s) \\ K_3(s) & K_4(s) \end{pmatrix}; \quad \begin{pmatrix} |B_1(t)| \\ |B_2(t)| \end{pmatrix} \preceq \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{71}$$

$t \in \mathbb{R},\ s \in [-\overline{d}, 0]$. If there exists $\lambda \succ 0$ such that $A_{04}\lambda \prec 0$, then we have

$$-A_{04}^{-1}(t) \succeq 0, \quad M(\overline{A}_{01}(t)) \preceq \overline{A}_{01}, \quad |\overline{A}_{03}(t)| \preceq \overline{A}_{03}, |\overline{A}_{1i}(t)| \preceq \overline{A}_{1i},$$

$$|\overline{K}_{1i}(t,s)| \preceq \overline{K}_{1i}(s), i = 1,2,3,4, \quad |\overline{B}_j(t)| \preceq \overline{B}_j,\ j = 1,2, t \in \mathbb{R}, s \in [-\overline{d}, 0].$$

Proof. From (70) we have $A_{04}(t) \preceq A_{04}$ hence A_{04} is Metzler by $A_{04}(t)$ Metzler. Using Lemma 1 and $A_{04}\lambda \prec 0$ implies A_{04} is Hurwitz, $-A_{04}^{-1} \succeq 0$. From $A_{04}(t) \preceq A_{04}$ and $\lambda \succ 0$ we have $A_{04}(t)\lambda \preceq A_{04}\lambda \prec 0$. Combining this with $A_{04}(t)$ is Metzler for all $t \geq 0$ and by using Lemma 1, we get that $-A_{04}^{-1}(t) \succeq 0,\ t \in \mathbb{R}$. By $A_{04}(t) \preceq A_{04}$ and Lemma 8, implies that $0 \preceq -A_{04}^{-1}(t) \preceq -A_{04}^{-1}$ for all $t \in \mathbb{R}$. Pre - multiplying both sides of (70), (71) with nonsingular matrix $\begin{pmatrix} I_r & -|A_{02}(t)|A_{04}^{-1}(t) \\ 0 & -A_{04}^{-1}(t) \end{pmatrix} \succeq 0$ we obtain

$$M(\overline{A}_{01}(t)) \preceq \overline{A}_{01}, \quad |\overline{A}_{03}(t)| \preceq \overline{A}_{03}, |\overline{A}_{1i}(t)| \preceq \overline{A}_{1i}, |\overline{K}_{1i}(t,s)| \preceq \overline{K}_{1i}(s), i = 1,2,3,4,$$

and $|\overline{B}_j(t)| \preceq \overline{B}_j,\ j = 1,2, t \in \mathbb{R}, s \in [-\overline{d}, 0]$. □

To obtain the componentwise ultimate bound of the time varying system (69), we will compare the solution of the system (69) with the solution of the following system:

$$\begin{cases} E\dot{\tilde{z}}(t) = A_0\tilde{z}(t) + A_1\tilde{z}(t-\tau(t)) + \int_{-d(t)}^{-\underline{d}} \mathscr{K}(s)\tilde{z}(t+s)ds + B\overline{\vartheta},\ t \geq 0, \\ \tilde{z}(s) = \psi(s),\ s \in [-M,\ 0], \end{cases} \tag{72}$$

Then, applying Theorem 7 to the system (72), we will get componentwise ultimate bound for the system (72), from which we can find componentwise ultimate bound for the system (69). The following theorem will give us sufficient conditions for the existence of the bound of the solution of the system (69).

Theorem 10. *Suppose that the conditions* (70) *and* (71) *in Lemma 9 are satisfied. If*

$$s\left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right) < 0,$$

then we have $\zeta = -\left(A_0 + A_1 + \int_{-d(0)}^{-\underline{d}} \mathscr{K}(s)ds\right)^{-1} B\,\overline{\vartheta}$ *is the componentwise ultimate bound of* (69).

Proof. To prove Theorem 10, it suffices to show the solution of the system (69) satisfying the following evaluation:

$$|x(t,\psi,\vartheta)| \preceq \tilde{z}(t,\psi,\overline{\vartheta}),\ t \geq 0. \tag{73}$$

Indeed, condition $s\left(A_0 + A_1 + \int_{-\overline{d}}^{-\underline{d}} \mathscr{K}(s)ds\right) < 0$ implies that A_{04} is Metzler and Hurwitz matrix, and $-A_{04}^{-1} \succeq 0$. Lemma 9 implies that $A_{04}(t)$ is Hurwitz and $-A_{04}(t)^{-1} \succeq 0$ for all $t \in \mathbb{R}$. The system (69) rewrites as:

$$\begin{aligned} \dot{x}_1(t) = {} & \overline{A}_{01}(t)x_1(t) + \overline{A}_{11}(t)x_1(t-\tau(t)) + \overline{A}_{12}(t)x_2(t-\tau(t)) \\ & + \int_{-d(t)}^{-\underline{d}} \left(\overline{K}_1(t,s)x_1(t+s) + \overline{K}_2(t,s)x_2(t+s)\right)ds + \overline{B}_1(t)\vartheta(t), \\ x_2(t) = {} & \overline{A}_{03}(t)x_1(t) + \overline{A}_{13}(t)x_1(t-\tau(t)) + \overline{A}_{14}(t)x_2(t-\tau(t)) \\ & + \int_{-d(t)}^{-\underline{d}} \left(\overline{K}_3(t,s)x_1(t+s) + \overline{K}_4(t,s)x_2(t+s)\right)ds + \overline{B}_2(t)\vartheta(t). \end{aligned} \tag{74}$$

Assume on the contrary that there exists $t_0 > 0$ such that $|x(t)| \npreceq \tilde{z}(t)$. Let $t_1 = \inf\{t > 0 : |x(t)| \npreceq \tilde{z}(t)\} \in (0,\infty)$, by continuity, there exists $j_0 \in \{1,..,n\}$ such that for some $0 < \varepsilon < \min\{\underline{\tau}, \underline{d}\}$:

$$|x(t)| \preceq \tilde{z}(t), \forall t \le t_1; \ \ |x_{j_0}(t_1)| = \tilde{z}(t_1), \quad |x_{j_0}(t)| > \tilde{z}_{j_0}(t), t \in (t_1, t_1+\varepsilon). \tag{75}$$

We consider the following cases:
Case 1. If $j_0 \in \{1,\dots,r\}$, by the definition of t_1, together with (74) we get

$$\begin{aligned}
D^+|x_{j_0}(t_1)| =& sgn(x_{j_0}(t_1))\dot{x}_{j_0}(t) \\
\le& a_{j_0j_0}(t_1)|x_{j_0}(t_1)| + \sum_{j=1, j\neq j_0}^{r} |a_{j_0j}(t_1)||x_j(t_1)| + \sum_{j=1}^{r} |b_{j_0j}(t_1)||x_j(t_1-\tau(t_1))| \\
&+ \sum_{j=r+1}^{n} |c_{j_0j}(t_1)||x_j(t_1-\tau(t_1))| + \sum_{j=1}^{r}\int_{-d(t_1)}^{-\underline{d}} |k^{(1)}_{j_0j}(t_1,s)||x_j(t_1+s)|ds \\
&+ \sum_{j=r+1}^{n}\int_{-d(t_1)}^{-\underline{d}} |k^{(2)}_{j_0j}(t_1,s)||x_j(t_1+s)|ds + \sum_{j=1}^{m} |r^{(1)}_{j_0j}(t_1)||\vartheta_j(t_1)|.
\end{aligned} \tag{76}$$

Using Lemma 9, we obtain

$$\begin{aligned}
D^+|x_{j_0}(t_1)| \le& a_{j_0j_0}|x_{j_0}(t_1)| + \sum_{j=1, j\neq j_0}^{r} a_{j_0j}|x_j(t_1)| + \sum_{j=1}^{r} b_{j_0j}|x_j(t_1-\tau(t_1))| \\
&+ \sum_{j=r+1}^{n} c_{j_0j}|x_j(t_1-\tau(t_1))| + \sum_{j=1}^{r}\int_{-d(t_1)}^{-\underline{d}} k^{(1)}_{j_0j}(s)|x_j(t_1+s)|ds \\
&+ \sum_{j=r+1}^{n}\int_{-d(t_1)}^{-\underline{d}} k^{(2)}_{j_0j}(s)|x_j(t_1+s)|ds + \sum_{j=1}^{m} r^{(1)}_{j_0j}|\vartheta_j(t_1)|.
\end{aligned} \tag{77}$$

Combining (75), (77) we obtain

$$\begin{aligned}
D^+|x_{j_0}(t_1)| \le& \ a_{j_0j_0}\tilde{z}_{j_0}(t_1) + \sum_{j=1, j\neq j_0}^{r} a_{j_0j}\tilde{z}_j(t_1) + \sum_{j=1}^{r} b_{j_0j}\tilde{z}_j(t_1-\tau(t_1)) + \sum_{j=r+1}^{n} c_{j_0j}\tilde{z}_j(t_1-\tau(t_1)) \\
&+ \sum_{j=1}^{r}\int_{-d(t_1)}^{-\underline{d}} k^{(1)}_{j_0j}(s)\tilde{z}_j(t_1+s)ds + \sum_{j=1}^{m} r^{(1)}_{j_0j}\overline{\vartheta}_j + \sum_{j=r+1}^{n}\int_{-d(t_1)}^{-\underline{d}} k^{(2)}_{j_0j}(s)\tilde{z}_j(t_1+s)ds = D^+\tilde{z}_{j_0}(t_1).
\end{aligned} \tag{78}$$

Clearly, (78) contradicts with (75), which implies that

$$|x_i(t,\psi,\vartheta)| \preceq \tilde{z}_i(t,\psi,\overline{\vartheta}), i \in \{1,\dots,r\}, \ t \ge 0. \tag{79}$$

Case 2. If $j_0 \in \{r+1,\dots,n\}$, for any $0<\varepsilon_1<\varepsilon$, together with (74) we get

$$\begin{aligned}|x_{j_0}(t_1+\varepsilon_1)| \le \sum_{j=1}^{r}|d_{j_0j}(t)||x_j(t_1+\varepsilon_1)| + \sum_{j=1}^{r}|e_{j_0j}(t)||x_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1))| \\ + \sum_{j=r+1}^{n}|f_{j_0j}(t)||x_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1))| + \sum_{j=1}^{r}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}|k_{j_0j}^{(3)}(t_1+\varepsilon_1,s)||x_j(t_1+\varepsilon_1+s)|ds \\ + \sum_{j=r+1}^{n}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}|k_{j_0j}^{(4)}(t_1+\varepsilon_1,s)||x_j(t_1+\varepsilon_1+s)|ds + \sum_{j=1}^{m}|r_{j_0j}^{(2)}(t_1+\varepsilon_1)||\vartheta_j(t_1+\varepsilon_1)|.\end{aligned} \tag{80}$$

Combining the conditions (75), (79), (80) with Lemma 9, we get the following estimate:

$$\begin{aligned}|x_{j_0}(t_1+\varepsilon_1)| &\le \sum_{j=1}^{r}d_{j_0j}|x_j(t_1+\varepsilon_1)| + \sum_{j=1}^{r}e_{j_0j}|x_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1))| \\ &\quad + \sum_{j=r+1}^{n}f_{j_0j}|x_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1))| + \sum_{j=1}^{r}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}k_{j_0j}^{(3)}(s)|x_j(t_1+\varepsilon_1+s)|ds \\ &\quad + \sum_{j=r+1}^{n}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}k_{j_0j}^{(4)}(s)|x_j(t_1+\varepsilon_1+s)|ds + \sum_{j=1}^{m}r_{j_0j}^{(2)}|\vartheta_j(t_1+\varepsilon_1)| \\ &\le \sum_{j=1}^{r}d_{j_0j}\tilde{z}_j(t_1+\varepsilon_1) + \sum_{j=1}^{r}e_{j_0j}\tilde{z}_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1)) \\ &\quad + \sum_{j=r+1}^{n}f_{j_0j}\tilde{z}_j(t_1+\varepsilon_1-\tau(t_1+\varepsilon_1)) + \sum_{j=1}^{r}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}k_{j_0j}^{(3)}(s)\tilde{z}_j(t_1+\varepsilon_1+s)ds \\ &\quad + \sum_{j=r+1}^{n}\int_{-d(t_1+\varepsilon_1)}^{-\underline{d}}k_{j_0j}^{(4)}(s)\tilde{z}_j(t_1+\varepsilon_1+s)ds + \sum_{j=1}^{m}r_{j_0j}^{(2)}\overline{\vartheta}_j = \tilde{z}_{j_0}(t_1+\varepsilon_1).\end{aligned}$$

This is contradicts (75). Then we have

$$|x_i(t,\psi,\vartheta)| \preceq \tilde{z}_i(t,\psi,\overline{\vartheta}), i\in\{1,\dots,n\},\ t\ge 0. \tag{81}$$

The inequality (81) implies that estimate (73) holds. Using estimate (73) and Theorem 7, we can find the componentwise bound of the absolute solution of the time-varying singular system (69). □

The following corollary of Theorem 10 gives a sufficient condition for exponential stability of (69) with $\vartheta(t)=0$.

Corollary 3. *Suppose that the conditions* (70) *and* (71) *in Lemma 9 are satisfied. The system* (69) *with* $\vartheta(t)=0$ *is exponentially stable if* $s\left(A_0+A_1+\int_{-\overline{d}}^{-\underline{d}}\mathscr{K}(s)ds\right)<0$.

Remark 5. The coupled linear delay differential-difference equation of the form

$$\begin{aligned}\dot{x}_1(t) &= A_{01}(t)x_1(t)+A_{11}(t)x_1(t-h)+A_{12}(t)x_2(t-h),\\ x_2(t) &= A_{03}(t)x_1(t)+A_{13}(t)x_1(t-h)+A_{14}(t)x_2(t-h),\end{aligned} \tag{82}$$

was studied in [38]. In particular, the exponential stability of (82) have been analysed in [38]. It has been showed in [38], Theorem III.4 that (82) is exponentially stable provided the following conditions hold:

1. $M(A_{01}(t)) \preceq A_{01}$, $|A_{1i}(t)| \preceq A_{1i}, i = 1,2,3,4$; $|A_{03}(t)| \preceq A_{03}, \forall t \in \mathbb{R}$.
2. There exist $p \succ 0$ and $q \succ 0$ such that $(A_{01}+A_{11})p+A_{12}q \prec 0,\ (A_{03}+A_{13})p+A_{14}q \prec q$.

Clearly, Theorem III.4 of [38] is just a particular case of Corollary 3.

5. Numerical examples

In this section, we present two examples to demonstrate the usefulness of the obtained results. Example 1 is given to show the effectiveness of Theorem 7. For time-varying singular systems, the efficiency of the proposed results is illustrated by Example 2.

Example 1. We consider the system (1) with the following parameters:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ A_0 = \begin{pmatrix} -2 & 1 & 0 \\ 0.5 & -2.8 & 0 \\ 0.2 & 0.4 & -1 \end{pmatrix},\ A_1 = \begin{pmatrix} 0.1 & 0.4 & 0 \\ 0.6 & 0.2 & 0 \\ 0.2 & 0.1 & 0 \end{pmatrix},$$

$\mathscr{K}(s) = \begin{pmatrix} -0.1s & -s^3 & 0 \\ -0.1s & -0.1s & 0 \\ -0.1s & -0.5s & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0.6 \\ 0.4 \\ 1 \end{pmatrix}$, $\tau(t) = 1.1+\cos(t)$, $d(t) = 0.6+0.5\sin(t)$, $0 \leq \vartheta(t) \leq 1$, $t \geq 0$. We can verify that

$$s\left(A_0 + A_1 + \int_{-\bar{d}}^{-\underline{d}} \mathscr{K}(s)ds\right) = -0.7165 < 0.$$

Therefore, using Theorem 7, we show that the smallest upper bound for the system as follows:

$$\zeta = -\left(A_0 + A_1 + \int_{-d(0)}^{-\underline{d}} \mathscr{K}(s)ds\right)^{-1} B\,\overline{\vartheta} = \begin{pmatrix} 0.6509 \\ 0.4365 \\ 0.6282 \end{pmatrix}.$$

Figures 1-3 show the trajectories of $x_1(t), x_2(t)\ x_3(t)$, and its bounds, respectively. The following figures shows that the trajectories of the partial state vectors of the system are bounded by the smallest upper bounds computed by Theorem 7.

Example 2. Consider the following linear positive singular time-varying system

$$E\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-\tau(t)) + B(t)\vartheta(t),\ t \geq 0, \tag{83}$$

where $x(t) \in \mathbb{R}^3$, $\tau(t)$ is a bounded time-varying delay, $\vartheta(t) \in [0,1]$ and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_0(t) = \begin{bmatrix} -3.5 - |\cos(t)| & 1 + \sin^2(t) & 0 \\ sin(t) & -4 - |t| & 0 \\ cos(t) & sin(t) & -3 - t^2 \end{bmatrix},$$

$$A_1(t) = \begin{bmatrix} 1 & e^{-2t} & 0 \\ 1.2 & \sin(t) & 0 \\ 0.3e^{-t} & 0.2 & 0 \end{bmatrix},\ B(t) = \begin{bmatrix} 0.2 \\ 0.3 + 0.1\sin^2(t) \\ 0.1 \end{bmatrix}.$$

We have $A_{01}(t) = \begin{bmatrix} -3.5 - |\cos(t)| & 1 + \sin^2(t) \\ sin(t) & -4 - |t| \end{bmatrix}$, $A_{02}(t) = [0 \ \ 0]^T$, $A_{03}(t) = [cos(t) \ \ sin(t)]$, $A_{04}(t) = -3 - t^2$. By using some simple computations, we can verify that all conditions of Lemma 9 are satisfied and

$$\begin{pmatrix} M(A_{01}(t)) & |A_{02}(t)| \\ |A_{03}(t)| & A_{04}(t) \end{pmatrix} \preceq A_0 := \begin{bmatrix} -3.5 & 2 & 0 \\ 1 & -4 & 0 \\ 1 & 1 & -3 \end{bmatrix},$$

$$|A_1(t)| \preceq A_1 := \begin{bmatrix} 1 & 1 & 0 \\ 1.2 & 1 & 0 \\ 0.3 & 0.2 & 0 \end{bmatrix}, |B(t)| \preceq B := \begin{bmatrix} 0.2 \\ 0.4 \\ 0.1 \end{bmatrix}.$$

We see that A_0 is a Metzler matrix, $A_1, B \succeq 0$, and

$$s(A_0 + A_1) = -0.1688 < 0.$$

Using Theorem 10, we can compute the componentwise ultimate bound of (83)

$$\zeta = -(A_0 + A_1)^{-1} B\,\overline{\vartheta} = \begin{pmatrix} 2 \\ 1.6 \\ 1.54 \end{pmatrix}.$$

Figures 4-6 show the trajectories of $x_1(t), x_2(t)\ x_3(t)$, and its bounds, respectively. The following figures shows that the trajectories of the partial state vectors of system (83) are bounded by the smallest upper bounds computed by Theorem Theorem 10.

6. Conclusions

This paper has addressed the componentwise ultimate bounds problem for a singular class system with bounded disturbances. By using a novel approach which mainly based on new mathematical techniques, we derived some criteria for the existence of componentwise ultimate bounds of the singular systems with bounded disturbances. We also extend the obtained results to time-varying singular systems with mixed time-varying delays. In order to illustrate

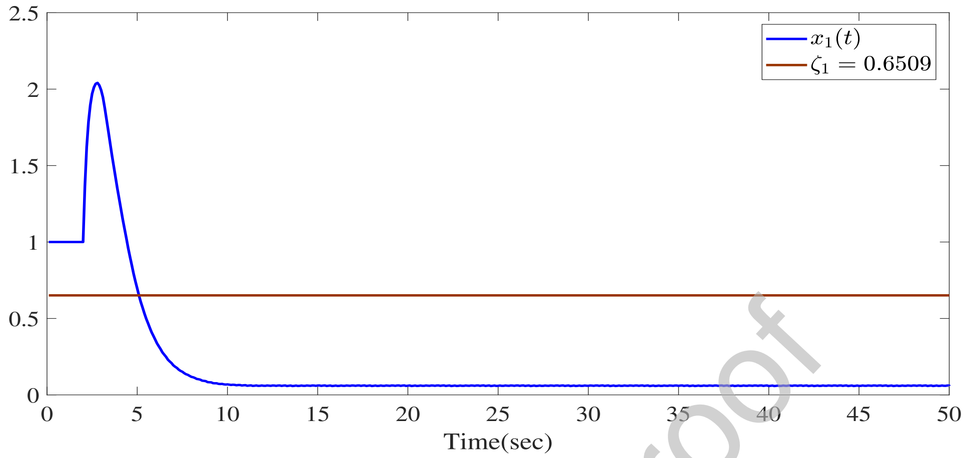

Figure 1: Responses of $x_1(t)$ and its bound $\zeta_1 = 0.6509$.

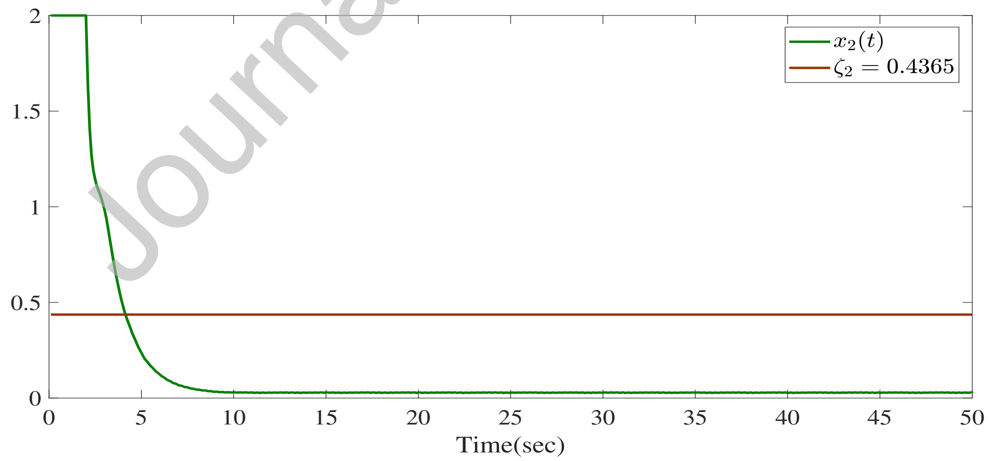

Figure 2: Responses of $x_2(t)$ and its bound $\zeta_2 = 0.4365$.

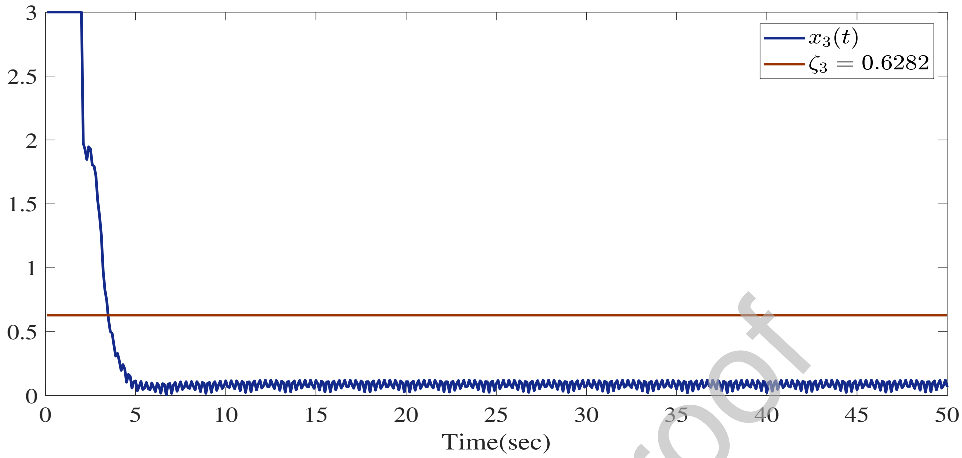

Figure 3: Responses of $x_3(t)$ and its bound $\zeta_3 = 0.6282$.

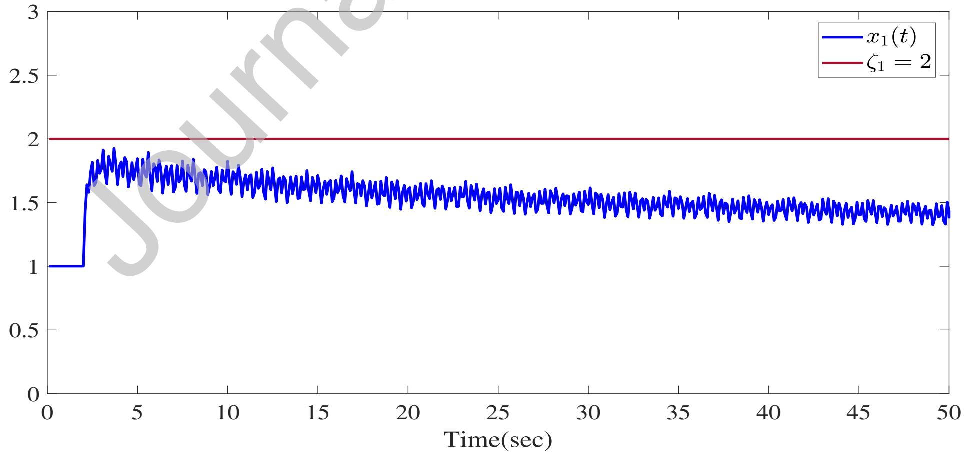

Figure 4: Responses of $x_1(t)$ and its bound $\zeta_1 = 2$.

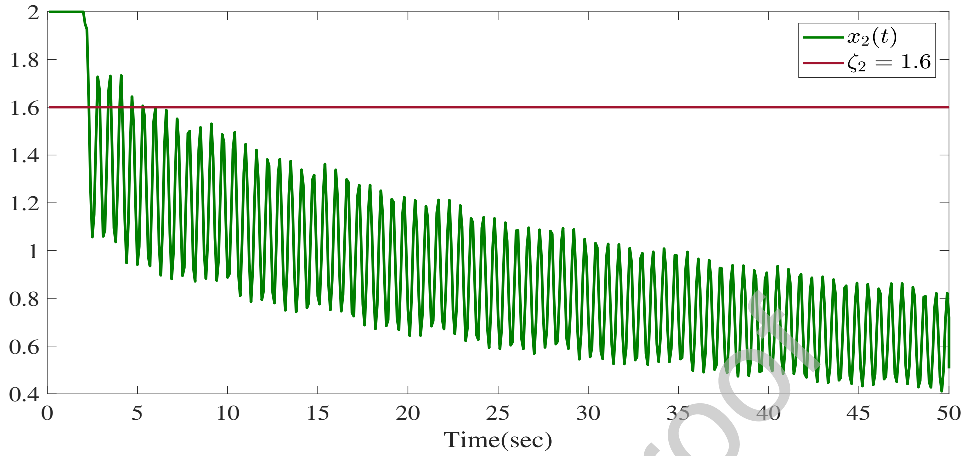

Figure 5: Responses of $x_2(t)$ and its bound $\zeta_2 = 1.6$.

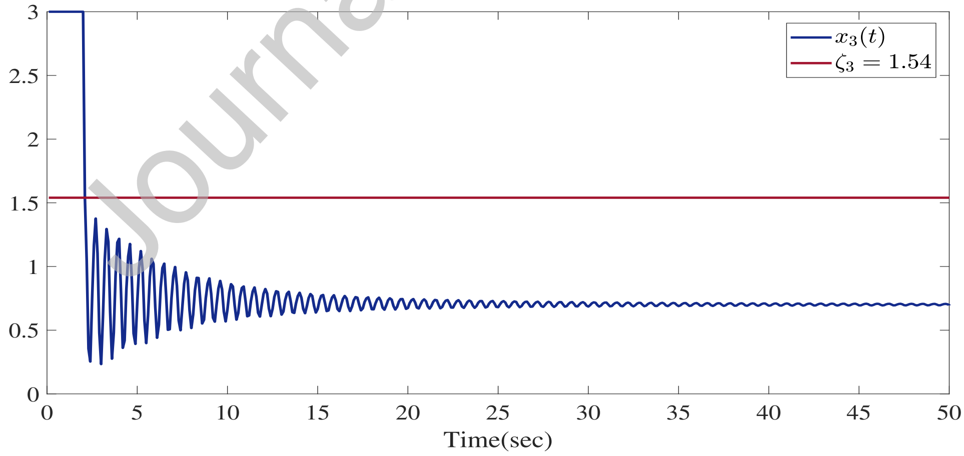

Figure 6: Responses of $x_3(t)$ and its bound $\zeta_3 = 1.54$.

the effectiveness of the proposed results, two numerical examples with simulation results are presented. It is notable that the time-varying functions in the considered system are bounded, then our future work is to extent the proposed results to the positive singular systems with unbounded time-varying delays. The second proposal for future work is the study of the control problem for positive singular systems with unbounded time-varying delays.

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Conflict of Interest Form

The authors (Nguyen Huu Sau, Dinh Cong Huong and Mai Viet Thuan) certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honoraria; educational grants; participation in speakers bureaus; membership, employment, consultancies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

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