This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2749524, IEEE Transactions on Automatic Control

Stability Analysis for Positive Singular Systems with Time-varying Delays

Yukang Cui, Jun Shen, Zhiguang Feng, and Chen Yong

***Abstract*—This paper is concerned with the stability analysis for continuous-time positive singular systems with time-varying delays. First, an auxiliary system is introduced to establish the positivity condition for singular time-delay systems. Based on the positivity condition, the stability criterion is obtained for the positive singular systems with constant delays. By analyzing the monotonic property of the system trajectory, we extend the stability condition to the cases with time-varying delays. A numerical simulation is employed to illustrate the effectiveness of our results.**

***Index Terms*—positive singular systems, delay systems, time-varying systems, stability of linear systems.**

I. Introduction

Singular systems, also referred to as implicit systems, descriptor systems, or generalized state-space systems, have drawn considerable attention. As a generalization of standard state-space description, singular systems are widely applied to engineering systems, biological systems, and economic systems. They provide more precise descriptions of dynamic systems, as they may contain impulsive elements and non-dynamic constraints [1]. Singular systems whose states represent quantities that are intrinsically nonnegative, for example, volumes of liquids, number of molecules, and species population, are called positive singular systems. Such system has a nonnegative state and output as long as the initial condition and the input are nonnegative. With standard positive systems [2], [3], [4] being an active field of research, increasing attention is paid to the study of positive singular systems. Some results have been published on the fundamental properties of positive singular systems, such as solvability, positivity and controllability [5], [6]. In [7], the positivity of singular systems in the continuous-time framework was discussed and the stability issue was investigated by virtue of the Lyapunov functional method. However, those positivity conditions contained an unnecessary assumption that was later removed in [8].

This work was partially supported by the National Natural Science Foundation of China under Grant (61603180), the Natural Science Foundation of Jiangsu Province under Grant (BK20160810), and the Fundamental Research Funds for the Central Universities (HEUCFJ170401).

Yukang Cui and Yong Chen are with the Department of Mechanical Engineering, The University of Hong Kong, Hong Kong (e-mail: cuiyukang@gmail.com; cyong.hit@gmail.com).

Jun Shen is with the College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, 211106, China (e-mail: junshen2009@gmail.com).

Zhiguang Feng is with the College of Automation, Harbin Engineering University, Harbin 150001, China and the College of Engineering and Science, Victoria University, Melbourne, VIC 3000, Australia (e-mail: congdian@gmail.com).

Due to the strong application background, positive time-delay systems have attracted an ever-increasing interest over the past decades [9], [10], [11] and [12]. Singular systems concerned about delays are more complicated to study than the standard state-space systems as one needs to consider not only stability, but also regularity and non-impulsiveness (continuous-time systems) or causality (discrete-time systems) [7], [8], [13], [14]. To simplify the analysis, various assumptions have been imposed in the investigation of singular time-delay systems. With the assumption that some of the system matrices are positive, the internal positivity conditions of continuous singular time-delay systems were proposed in [15]. The positivity and exponential stability conditions for continuous-time singular systems with constant delays were established in [16]. Inspired by the result in [16], counterpart conditions for the discrete-time cases were proposed in [17]. However, both of them dealt with constant time-delays and required the existence of a monomial matrix to obtain the positivity and stability conditions for singular time-delay systems. On the other hand, numerous valuable properties of standard positive systems with time-varying delays have been explored. By investigating the monotonic property of the system trajectory, [2], [18] showed that the stability conditions of constant time-delay positive systems could be extended to time-varying delay ones. By constraining each state of the time-varying delay system by that of constant time-delay case, the asymptotic stability of both discrete-time and continuous-time positive systems with time-varying delays was shown insensitive to the magnitude of delays in [2], [18]. Motivated by the preceding discussion, we attempt to prove that the stability of the positive singular system with time-varying delays is not sensitive to the magnitude of delays and is fully determined by the system matrices.

In this paper, we aim at developing positive and stability conditions for positive singular systems with time-varying delays. By using the method used in [19] and [20], we transform the original system into an augmented system that is employed in the explicit solution formulas for singular systems. There exists a close relationship between the positivity of the original system and that of the transformed one. By resorting to the transformed system, we obtain a necessary and sufficient condition for the positivity of the singular system with time-varying delays. By showing the relationship between the constant delay system and the time-varying delay case, a necessary and sufficient stability criterion for the singular system with time-varying delays is proposed.

Notation: Let $\mathbb{R}^n$ be n-dimensional Euclidean space over

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the reals and $\mathbb{R}^{n\times m}$ denotes set of $n\times m$ real matrices. $C_{n,d} := C([-d,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-d,0]$ into $\mathbb{R}^n$, $x_t := x(t+\theta)$, $\theta\in[-d,0]$, $t\geq 0$ denotes the function family defined on $[-d,0]$. $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm and $\|\phi\|_c = \sup_{\theta\in[-d,0]}\|\phi(t)\|$ stands for the norm of a function $\phi\in C_{n,d}$. a_{ij} denotes the (i,j)th entry of matrix A. The ∞-norm of a column vector $x\in\mathbb{R}^n$ is $\|x\|_\infty = \max_{1\leq i\leq n}|x_i|$, which coincides with the ∞-norm of a matrix. A real matrix $A\in\mathbb{R}^{m\times n}$ with all of its entries nonnegative (respectively, positive) is called nonnegative (respectively, positive) matrix and is denoted by $A\succeq 0$ (respectively, $A\succ 0$) and $A\in\bar{\mathbb{R}}_+^{n\times m}$ (respectively, $A\in\mathbb{R}_+^{n\times m}$). For two matrices $A,B\in\mathbb{R}^{m\times n}$, $A\succeq B$ means that $A-B$ is a nonnegative matrix, or equivalently, $a_{ij}\geq b_{ij}$ for $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

II. Problem Formulation

Consider the following singular system with time-varying delays:

$$(\mathscr{S}):\ E\dot{x}(t) = Ax(t)+A_d x(t-d(t)), \quad (1)$$
$$x(s) = \phi(s),\ s\in[-\bar{d},0], \quad (2)$$

where $x(t)\in\mathbb{R}^n$ is the state vector and A, $A_d\in\mathbb{R}^{n\times n}$ are known constant real matrices with appropriate dimensions. The delay is assumed to be bounded, that is, $0<\underline{d}\leq d(t)\leq\bar{d}$ for $t\geq 0$. The matrix $E\in\mathbb{R}^{n\times n}$ is assumed to be singular, that is, $\mathrm{rank}(E)=r<n$; $\phi(\cdot)$ is the compatible initial condition.

Several definitions and lemmas are employed in the proof of the main results.

Definition 1: [21]

1) The pair (E,A) is said to be regular if $\det(sE-A)$ is not identically zero.
2) The pair (E,A) is said to be impulse-free if $\deg\{\det(sE-A)\}=\mathrm{rank}(E)$.

Definition 2: [21]

1) System $\mathscr{S}$ is said to be regular and impulse-free if the pair (E,A) is regular and impulse-free.
2) System $\mathscr{S}$ is said to be asymptotically stable, if for any $\varepsilon>0$, a scalar $\delta(\varepsilon)>0$ exists such that for any compatible initial condition $\phi(t)$ satisfying $\sup_{-\bar{d}\leq t\leq 0}\|\phi(t)\|\leq\delta(\varepsilon)$, the solution $x(t)$ of $\mathscr{S}$ satisfies $\|x(t)\|\leq\varepsilon$ for $t\geq 0$ and, furthermore, $x(t)\to 0$, when $t\to\infty$.

Definition 3: [22] System $\mathscr{S}$ is positive if for any admissible initial condition $\phi(\cdot)$ satisfying $\phi(s)\succeq 0$, $s\in[-\bar{d},0]$, the corresponding trajectory $x(t)\succeq 0$ for all $t\geq 0$.

Definition 4: [19] For any matrix $E\in\mathbb{R}^{n\times n}$, a unique matrix E^D, called the Drazin inverse of E, always exists such that

$$EE^D = E^DE,\ E^DEE^D = E^D,\ E^DE^{\nu+1} = E^\nu,$$

where ν is the smallest nonnegative integer such that $\mathrm{rank}(E^\nu)=\mathrm{rank}(E^{\nu+1})$ and is denoted by $\nu=\mathrm{ind}(E)$.

Lemma 1: [23] Let E, $A\in\mathbb{R}^{n\times n}$ with the pair (E,A) regular. Let $\beta\in\mathbb{R}$ be chosen such that the matrix $\beta E-A$ is nonsingular. Then, the matrices

$$\hat{E} := (\beta E-A)^{-1}E,\ \hat{A} := (\beta E-A)^{-1}A$$

commute.

Lemma 2: [24] Let E, $A\in\mathbb{R}^{n\times n}$ with the pair (E,A) regular. The pair (E,A) is impulse-free for singular system if and only if $\mathrm{rank}(\hat{E})=\mathrm{rank}(\hat{E}^2)$, that is, $\nu=1$.

Lemma 3: [19] Let E, $A\in\mathbb{R}^{n\times n}$ with $EA=AE$. Then,

$$EA^D = A^DE,\ E^DA = AE^D,\ E^DA^D = A^DE^D.$$

Lemma 4: [19] Let E, $A\in\mathbb{R}^{n\times n}$ with $EA=AE$ and suppose that (E,A) is regular. Then,

$$(I-E^DE)A^DA = (I-E^DE).$$

Inspired by the method developed in [19], we introduce a technical proposition that is useful in the following proof.

Lemma 5: Suppose that the pair (E,A) is regular, impulse-free and define $x_1(t)=Mx(t)$, $x_2(t)=(I-M)x(t)$ with $M=\hat{E}^D\hat{E}$. Then, $x_1(t)$ and $x_2(t)$ satisfy

$$\dot{x}_1(t) = A_1x_1(t)+A_{d1}[x_1(t-d(t))+x_2(t-d(t))], \quad (3)$$
$$0 = -x_2(t)+A_{d2}[x_1(t-d(t))+x_2(t-d(t))], \quad (4)$$

for $t\geq 0$, where

$$A_1 = \hat{E}^D\hat{A},\ A_{d1}=\hat{E}^D\hat{A}_d,\ A_{d2}=(M-I)\hat{A}^D\hat{A}_d,$$
$$\hat{A} = (\beta E-A)^{-1}A,\ \hat{A}_d=(\beta E-A)^{-1}A_d,$$
$$\hat{E} = (\beta E-A)^{-1}E,$$

with any $\beta\in\mathbb{R}$ such that $(\beta E-A)^{-1}$ exists.

Proof. By using Lemma 1, pre-multiplying (1) by $(\beta E-A)^{-1}$ presents

$$\hat{E}\dot{x}(t) = \hat{A}x(t)+\hat{A}_dx(t-d(t)),$$

where $\hat{E}$, $\hat{A}$ commute. (1) can be rewritten as

$$\hat{E}\{\dot{x}_1(t)+\dot{x}_2(t)\} = \hat{A}\{x_1(t)+x_2(t)\}+\hat{A}_dx(t-d(t)). \quad (5)$$

As $\hat{E}\dot{x}_2(t)=0$, it follows that

$$\hat{E}\dot{x}_1(t) = \hat{A}x_1(t)+\hat{A}x_2(t)+\hat{A}_dx(t-d(t)). \quad (6)$$

Left multiply (6) by $\hat{E}^D\hat{E}$ and it follows by Lemma 3 that

$$\hat{E}^D\hat{E}\hat{E}\dot{x}_1(t) = \hat{E}^D\hat{E}\hat{A}x_1(t)+\hat{E}^D\hat{E}\hat{A}_dx(t-d(t)).$$

It follows by Definition 4 and Lemma 2 that $\hat{E}^D\hat{E}\hat{E}=\hat{E}$ which implies that

$$\hat{E}\dot{x}_1(t) = \hat{E}^D\hat{E}\hat{A}x_1(t)+\hat{E}^D\hat{E}\hat{A}_dx(t-d(t)). \quad (7)$$

Since $\hat{E}^D\hat{E}\hat{E}^D=\hat{E}^D$ and $\hat{E}^D\hat{E}\dot{x}_1(t)=\dot{x}_1(t)$, pre-multiplying (7) by $\hat{E}^D$ results in

$$\dot{x}_1(t) = \hat{E}^D\hat{A}x_1(t)+\hat{E}^D\hat{A}_dx(t-d(t)),$$

which is (3). Subtracting (7) from (6) leads to

$$0 = \hat{A}x_2(t)+(I-\hat{E}^D\hat{E})\hat{A}_dx(t-d(t)). \quad (8)$$

According to Lemma 4, pre-multiplying (8) by $(\hat{E}^D\hat{E}-I)\hat{A}^D$ gives

$$0 = (\hat{E}^D\hat{E}-I)x_2(t)+(\hat{E}^D\hat{E}-I)\hat{A}^D\hat{A}_dx(t-d(t)). \quad (9)$$

For $\hat{E}^D\hat{E}x_2=0$, we have

$$0 = -x_2(t)+(M-I)\hat{A}^D\hat{A}_dx(t-d(t)).$$

This completes the proof. □

Based on Lemma 5, the admissible initial condition of system $\mathscr{S}$ is proposed, which is similar to, for example [8, Theorem 2.2], [23, Theorem 3.1.3], and [19, Corollary 2.30].

Lemma 6: Suppose that the pair (E,A) is regular and impulse-free. System $\mathscr{S}$ has a unique solution $x(t)=x_1(t)+x_2(t)$ as

$$\begin{cases} x_1(t) &= e^{A_1t}M\upsilon+\int_0^t e^{A_1(t-\tau)}A_{d1}x(\tau-d(\tau))d\tau, \\ x_2(t) &= A_{d2}x(t-d(t)), \end{cases}$$

with some $\upsilon\in\mathbb{R}^n$ for $t\geq 0$ if and only if $\phi(0)$ has the form

$$\phi(0)=x_1(0)+x_2(0)=M\upsilon+A_{d2}\phi(-d(0)). \tag{10}$$

In the following, initial condition ϕ, satisfying condition (10) is called admissible initial condition. Some helpful properties of the matrices $M=\hat{E}^D\hat{E}$, $A_1=\hat{E}^D\hat{A}$, and $A_{d1}=\hat{E}^D\hat{A}_d$ are presented in the following lemma.

Lemma 7: The following properties hold true.

(i) $(M)^2=M$.

(ii) $MA_1=A_1M=A_1$, $MA_{d1}=A_{d1}$.

(iii) $Mx_1(t)=x_1(t)$, $(I-M)x_2(t)=x_2(t)$.

Proof. **(i)** $(M)^2=M$ is obtained directly from the definition of the Drazin inverse.

(ii) Based on Definition 4 and Lemma 3, we have $MA_1=\hat{E}^D\hat{E}\hat{E}^D\hat{A}=\hat{E}^D\hat{A}\hat{E}^D\hat{E}=A_1M$ and $MA_{d1}=\hat{E}^D\hat{E}\hat{E}^D\hat{A}_d=\hat{E}^D\hat{A}_d=A_{d1}$.

(iii) As $x_1(t)=Mx(t)$, we find that $Mx_1(t)=MMx(t)=Mx(t)=x_1(t)$. Similarly, we have $(I-M)x_2(t)=(I-M)^2x(t)=(I-2M+M)x(t)=x_2(t)$. □

III. Positivity Analysis

In this section, we propose a positivity condition for singular time-delay system $\mathscr{S}$. Based on the assumption that the pair (E,A) is regular and impulse-free, the following system is used to replace the original system $\mathscr{S}$ by Lemma 5,

$$(\mathscr{S}'):\ \begin{aligned} \dot{x}_1(t) &= A_1x_1(t)+A_{d1}[x_1(t-d(t))+x_2(t-d(t))], \\ 0 &= -x_2(t)+A_{d2}[x_1(t-d(t))+x_2(t-d(t))], \end{aligned}$$

where $x_1(s)+x_2(s)=\phi(s)$ for $s\in[-\bar{d},0]$. By Lemma 5, we have $x_1(t)=Mx(t)$, $x_2(t)=(I-M)x(t)$ for $t\geq 0$.

The following lemma about the positive invariance of systems is necessary. It can be derived from the results of [25] and [26] which are based on the famous Farkas' lemma.

Lemma 8: Let $F\in\mathbb{R}^{p\times n}$, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times q}$ and consider the linear system $\dot{z}(t)=Az(t)+Bw(t)$. For $\bar{t}\in\mathbb{R}_+$,

$$\forall z(0),\ Fz(0)\succeq 0,\ \forall w(t)\succeq 0 \Rightarrow Fz(t)\succeq 0,\ t\in[0,\bar{t}), \tag{11}$$

if and only if there exist a Metzler matrix $H\in\mathbb{R}^{n\times n}$ and a matrix $K\in\mathbb{R}^{n\times q}, K\succeq 0$ such that

$$\begin{aligned} FA &= HF, \\ FB &= K. \end{aligned}$$

Then, we are present a necessary and sufficient condition for the positivity of system $\mathscr{S}'$.

Theorem 1: With the nonnegative initial condition, system $\mathscr{S}'$ is positive for all delays $d(t)$ satisfying $\underline{d}\leq d(t)\leq\bar{d}$ if and only if $A_{d1}\succeq 0$, $A_{d2}\succeq 0$ and a Metzler matrix H exists such that $A_1=HM$.

Proof. **Necessity**: Let $d(t)=\bar{d}$, $x_1(0)=M\upsilon$ for $\upsilon\in\mathbb{R}^n$ and $\phi(\cdot)\succeq 0$. By Lemma 6 and statement (ii) of Lemma 7, it follows that

$$\begin{aligned} x_1(t) &= e^{A_1t}M\upsilon+\int_0^t e^{A_1(t-\tau)}A_{d1}x(\tau-\bar{d})d\tau \\ &= e^{A_1t}M\upsilon+\int_0^t e^{A_1(t-\tau)}MA_{d1}x(\tau-\bar{d})d\tau. \end{aligned}$$

Applying statement (ii) of Lemma 7, it follows that

$$e^{A_1t}M=(M+tA_1M+\frac{t^2}{2}A_1^2M+\dots)=Me^{A_1t}.$$

Similarly, we have $e^{A_1(t-\tau)}M=Me^{A_1(t-\tau)}$. Thus, $x_1(t)$ can be treated as $x_1(t)=Mz(t)$ and $z(t)$ is the solution of linear system $\dot{z}(t)=A_1z(t)+A_{d1}\phi(t-\bar{d})$, $t\in[0,\bar{d})$ with initial condition $z(0)=\upsilon\in\mathbb{R}^n$. Therefore, for any initial condition $z(0)=\upsilon$ satisfying $Mz(0)=M\upsilon\succeq 0$ and any $\phi(\cdot)\succeq 0$, it follows that $x_1(t)\succeq 0$ for $t\in[0,\bar{d})$, which is equivalent to the fact that

$$\forall z(0),\ x_1(0)=Mz(0)\succeq 0\Rightarrow x_1(t)=Mz(t)\succeq 0,\ t\in[0,\bar{d}). \tag{12}$$

By using Lemma 8 with $F=M$, $A=A_1$ and $B=A_{d1}$, implication (12) amounts to the existence of a Metzler matrix $H\in\mathbb{R}^{n\times n}$ and a matrix $K\in\bar{\mathbb{R}}_+^{n\times n}$ such that

$$MA_1=HM,\ MA_{d1}=K.$$

By statement (ii) of Lemma 7, we obtain

$$A_1=HM,\ A_{d1}\succeq 0.$$

On the other hand, for any $\phi(\cdot)\succeq 0$, the positivity of $x_2(t)=A_{d2}\phi(t-\bar{d})$, $t\in[0,\bar{d}]$ implies the matrix $A_{d2}\succeq 0$.

Sufficiency: The solution of system $\mathscr{S}'$ is given as

$$\begin{aligned} x_1(t) &= e^{A_1t}M\upsilon+\int_0^t e^{A_1(t-\tau)}A_{d1}x(\tau-d(\tau))d\tau \\ &= e^{HMt}M\upsilon+\int_0^t e^{HM(t-\tau)}A_{d1}x(\tau-d(\tau))d\tau. \end{aligned}$$

By applying statement (i) of Lemma 7, it follows that

$$e^{HMt}M=(M+tHMM+\frac{t^2}{2}(HM)^2M+\dots)=e^{Ht}M.$$

As H is a Metzler matrix, it satisfies $e^{Ht}\succeq 0$. Under the conditions that $x_1(0)=M\upsilon\succeq 0$, $A_{d1}\succeq 0$ and $A_{d2}\succeq 0$, if $x(t-d(t))\succeq 0$ for $t\in[0,t]$, we have

$$\begin{cases} x_1(t) &= e^{Ht}x_1(0)+\int_0^t e^{H(t-\tau)}A_{d1}x(\tau-d(\tau))d\tau\succeq 0, \\ x_2(t) &= A_{d2}x(t-d(t))\succeq 0. \end{cases} \tag{13}$$

The time-varying delay $d(t)$ has a lower bound $\underline{d}$. As $x(s)=\phi(s)\succeq 0$ for $s\in[-\bar{d},0]$, it follows that $x(t-d(t))\succeq 0$, for $t\in[0,\underline{d}]$. It follows from (13) that $x_1(t)\succeq 0$, $x_2(t)\succeq 0$, $x(t)=x_1(t)+x_2(t)\succeq 0$, for $t\in[0,\underline{d}]$. For $t\in[\underline{d},2\underline{d}]$, due to $x(t)\succeq 0$ for $t\in[-\bar{d},\underline{d}]$, it is easy to obtain that $x_1(t)\succeq 0$ and $x_2(t)\succeq 0$. By mathematical induction, one can conclude that $x_1(t)\succeq 0$ and $x_2(t)\succeq 0$ for $t\geq 0$. This completes the proof. □

Assumption 1: The initial condition $x_1(0)=Mx(0)=M\upsilon\succeq 0$.

We are now in the position to present a necessary and sufficient condition on the positivity of singular system $\mathscr{S}$.

Theorem 2: Suppose the pair (E,A) is regular and impulse-free. Under Assumption 1 and the nonnegative initial condition, system $\mathscr{S}$ is positive for all delays $d(t)$ satisfying $\underline{d} \le d(t) \le \bar{d}$ if and only if $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists such that $A_1 = HM$.

Proof. **Necessity**: Assume $d(t) = \bar{d}$ and $\phi(s)$ is constant for $s \in [-\bar{d}, 0)$. The system $\mathscr{S}'$ can be rewritten in the augmented form

$$\begin{bmatrix} \dot{z}(t) \\ \dot{\phi}(t-\bar{d}) \end{bmatrix} = \begin{bmatrix} A_1 & A_{d1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \phi(t-\bar{d}) \end{bmatrix}, \ t \in [0,\bar{d}). \quad (14)$$

We have shown in the proof of Theorem 1 that $z(t)$ is the solution of linear system $\dot{z}(t) = Az(0) + A_{d1}\phi(t-\bar{d})$, $t \in [0,\bar{d})$ with initial condition $z(0) = \upsilon \in \mathbb{R}^n$. For system $\mathscr{S}$ is positive, with an initial condition $z(0)$ and $\phi(-\bar{d})$ satisfying $x(0) = Mz(0) + A_{d2}\phi(-\bar{d}) \succeq 0$, it follows that

$$\begin{aligned} x(t) &= x_1(t) + x_2(t) = Mz(t) + A_{d2}\phi(t-\bar{d}) \\ &= \begin{bmatrix} M & A_{d2} \end{bmatrix} \begin{bmatrix} z(t) \\ \phi(t-\bar{d}) \end{bmatrix} \succeq 0, \ t \in [0,\bar{d}). \end{aligned}$$

Consequently, based on Lemma 8, a Metzler matrix H must exist such that

$$H \begin{bmatrix} M & A_{d2} \end{bmatrix} = \begin{bmatrix} M & A_{d2} \end{bmatrix} \begin{bmatrix} A_1 & A_{d1} \\ 0 & 0 \end{bmatrix},$$

where $H \in \mathbb{R}^{n\times n}$. By statement (ii) of Lemma 7, simple algebraic manipulations give

$$A_1 = HM. \quad (15)$$

For system $\mathscr{S}$, let $\upsilon = 0$, and it follows that $x(0) = A_{d2}\phi(-\bar{d}) \succeq 0$, for any $\phi(\cdot) \succeq 0$, which implies $A_{d2} \succeq 0$.

Suppose A_{d1} is not nonnegative, that is, (i,j) exists such that $Ad_{(i,j)} \prec 0$ and let $\upsilon = 0$, $\phi(s) = \delta(s+\bar{d})e_j$ for $s \in [-\bar{d}, 0)$, where $\delta(\cdot)$ denotes the Dirac delta function and $e_j \in \mathbb{R}^n$ is a vector of zeros with one in the Jth entry. Then, for $t \in (0,\bar{d}]$, we have

$$x(t) = \int_0^t e^{H(t-\tau)} A_{d1}\phi(\tau-\bar{d})d\tau = e^{Ht}A_{d1}e_j.$$

By making t sufficiently small, it follows that $x(t)$ is not nonnegative, which is a contradiction and implies $A_{d1} \succeq 0$.

Sufficiency: By Assumption 1 and Theorem 1, we have both $x_1(t) \succeq 0$ and $x_2(t) \succeq 0$. As $x(t) = x_1(t) + x_2(t) \succeq 0$ for all $t \ge 0$, the statement directly follows. □

By Assumption 1 and Theorem 2, we prove that when $x(t) \succeq 0$ for $t \ge 0$, it follows that both $x_1(t) \succeq 0$ and $x_2(t) \succeq 0$, which plays a key role in the following stability analysis.

IV. Stability Analysis

In this section, we study the stability of singular system $\mathscr{S}$ with time-varying delays. In what follows, we first consider the following augmented system with constant delay $d(t) = \bar{d}$.

$$(\mathscr{S}''): \begin{aligned} E\dot{y}(t) &= Ay(t) + A_d y(t-\bar{d}), \\ y(s) &= \phi(s), \ s \in [-\bar{d}, 0]. \end{aligned} \quad (16)$$

Introducing the augmented state $\bar{y}(t) = \begin{bmatrix} y_1^T(t) & y_2^T(t) \end{bmatrix}^T$, we obtain that

$$\bar{E}\dot{\bar{y}}(t) = \bar{A}\bar{y}(t) + \bar{A}_d\bar{y}(t-\bar{d}), \quad (17)$$

where

$$\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{A} = \begin{bmatrix} A_1 & 0 \\ 0 & -I \end{bmatrix}, \ \bar{A}_d = \begin{bmatrix} A_{d1} & A_{d1} \\ A_{d2} & A_{d2} \end{bmatrix}.$$

where $y_1(s) + y_2(s) = \phi(s)$ for $s \in [-\bar{d}, 0]$. By Lemma 5, we have $y_1(t) = My(t)$, $y_2(t) = (I-M)y(t)$ for $t \ge 0$, where $y(t)$ is the state of system $\mathscr{S}$ with $d(t) = \bar{d}$. Before analyzing the stability of system $\mathscr{S}''$, we introduce the following lemmas.

Lemma 9: [8] A Metzler matrix A is Hurwitz if and only if $\gamma \in \mathbb{R}_+^n$ exists such that $\gamma^T A \prec 0$ or $\eta \in \mathbb{R}_+^n$ exists such that $A\eta \prec 0$.

Lemma 10: [2] Let A be a nonnegative matrix. Then, the following conditions are equivalent.

(i) A is Schur or all the eigenvalues of A lie inside the unit circle on the complex plane.

(ii) $\gamma \in \mathbb{R}_+^n$ exists such that $\gamma^T(A-I) \prec 0$.

Define the difference operator $\mathscr{D} : C_{n,\bar{d}} \to \mathbb{R}^n$

$$\mathscr{D}(y_2(t)) = -y_2(t) + A_{d2}y_2(t-\bar{d}),$$

then, along a similar line as in the proof of [27, Lemma 1] and [28, Lemma 2], we have the following lemma for positive singular system.

Lemma 11: If the operator $\mathscr{D}$ is stable and there exist positive numbers a, b, p and a continuous functional $V : C_{n,\bar{d}} \to \mathbb{R}$ such that

$$\begin{aligned} a\|y_1(t)\| \le V(\bar{y}_t) \le b\|\bar{y}_t\|_c, \\ \dot{V}(\bar{y}_t) \le -p\|\bar{y}(t)\|, \end{aligned}$$

and the function $\bar{V}(t) = V(\bar{y}_t)$ is absolutely continuous for $\bar{y}_t$ satisfying (17), then system (17) is asymptotically stable.

Then, a stability condition is proposed for system $\mathscr{S}''$.

Theorem 3: Suppose the pair (E,A) is regular and impulse-free. Under Assumption 1, system $\mathscr{S}''$ is positive and asymptotically stable with any admissible initial condition, if $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists, thereby satisfying

$$\begin{cases} A_1 = HM, \\ \bar{H} \text{ is Hurwitz,} \end{cases}$$

where

$$\bar{H} = \begin{bmatrix} H + A_{d1} & A_{d1} \\ A_{d2} & A_{d2} - I \end{bmatrix}.$$

Proof. Let $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and there exists a Metzler matrix H satisfying $A_1 = HM$. By Theorem 1, the system (17) is positive. Then, a Lyapunov-Krasovskii functional is constructed with any $\bar{\gamma}^T = \begin{bmatrix} \gamma_1^T & \gamma_2^T \end{bmatrix}$, $\gamma_1 \in \mathbb{R}_+^n$ and $\gamma_2 \in \mathbb{R}_+^n$ as:

$$\begin{aligned} V(\bar{y}(t)) &= \bar{\gamma}^T\bar{E}\bar{y}(t) + \int_{t-\bar{d}}^t \bar{\gamma}^T\bar{A}_d\bar{y}(\tau)d\tau \\ &= \bar{\gamma}^T \begin{bmatrix} y_1(t) \\ 0 \end{bmatrix} + \int_{t-\bar{d}}^t \bar{\gamma}^T \begin{bmatrix} A_{d1}y(\tau) \\ A_{d2}y(\tau) \end{bmatrix} d\tau, \end{aligned}$$

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which is nonnegative with $y(t) \succeq 0$, $t \in [-\bar{d}, \infty)$ and $y_1(t) \succeq 0$, $t \in [0, \infty)$ and $\bar{\gamma} \succ 0$. Then, we get

$$\min_{i=1,\ldots,2n}\{\bar{\gamma}_i\}\|y_1(t)\| \le V(\bar{y}_t) \le \max_{i=1,\ldots,2n}\{\bar{\gamma}_i\}(1+\bar{d}\|\bar{A}_d\|)\|\bar{y}_t\|_c, \tag{18}$$

where $\|\bar{y}_t\|_c = \sup_{\theta\in[-\bar{d},0]}\|\bar{y}(t+\theta)\|$. We calculate the derivative along the solution of the system and we have

$$\begin{aligned}\dot{V}(y(t)) &= \bar{\gamma}^T\bar{E}\dot{\bar{y}}(t) + \bar{\gamma}^T\bar{A}_d\bar{y}(t) - \bar{\gamma}^T\bar{A}_d\bar{y}(t-\bar{d}) \\ &= \bar{\gamma}^T\bar{A}\bar{y}(t) + \bar{\gamma}^T\bar{A}_d\bar{y}(t) \\ &= \bar{\gamma}^T(\bar{A}+\bar{A}_d)\bar{y}(t) \\ &= \bar{\gamma}^T\begin{bmatrix} A_1+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\bar{y}(t).\end{aligned}$$

By $A_1 = HM$ and $My_1(t) = y_1(t)$, it follows that

$$\begin{aligned}\dot{V}(y(t)) &= \bar{\gamma}^T\begin{bmatrix} H+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\bar{y}(t) \\ &= \bar{\gamma}^T\bar{H}\bar{y}(t).\end{aligned}$$

By Lemma 9, if $\bar{H}$ is both Metzler and Hurwitz, $\bar{\gamma} \succ 0$ exists such that $\bar{\gamma}^T\bar{H} \prec 0$ which implies $\dot{V}(\bar{y}(t)) < 0$ when $\bar{y}(t) \succeq 0$ and $\bar{y}(t) \neq 0$. Then, it follows that

$$\dot{V}(\bar{y}_t) \le -\min_{i=1,\ldots,2n}\{\bar{\gamma}_i\}\|\bar{H}\|\|\bar{y}(t)\|. \tag{19}$$

It follows by $\bar{\gamma}^T\bar{H} \prec 0$ that $(\gamma_1^T+\gamma_2^T)(A_{d2}-I) \prec 0$, which implies that $\rho(A_{d2}) < 1$ using Lemma 10. By virtue of [27, Lemma 2], we conclude that operator $\mathscr{D}$ is stable.

Therefore, from the stability of $\mathscr{D}$, (18), (19) and using Lemma 11, it follows that system (17) is asymptotically stable, which completes the proof. □

A necessary condition for the stability of positive singular systems with constant delays is established in the following theorem.

Theorem 4: Suppose that positive system $\mathscr{S}''$ is positive and asymptotically stable with any admissible initial condition. Then, $\bar{\lambda} = \begin{bmatrix}\lambda_1^T & \lambda_2^T\end{bmatrix}^T$ exists such $\lambda_1 \in \bar{\mathbb{R}}_+^n$, $\lambda_2 \in \bar{\mathbb{R}}_+^n$ satisfying $\bar{\Pi}\bar{\lambda} \preceq 0$, where

$$\bar{\Pi} = \begin{bmatrix} A_1+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}.$$

Moreover, an admissible initial condition for system $\mathscr{S}''$ can be given by $\varphi(s) \equiv \lambda_1+\lambda_2$, $-\bar{d} \le t < 0$, and $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1+\lambda_2)$.

Proof. By integrating system $\mathscr{S}''$ from 0 to $T > 0$, we have

$$y_1(T) - y_1(0) = A_1\int_0^T y_1(t)dt + A_{d1}\int_0^T y(t-\bar{d})dt.$$

The above equality can be rewritten in the form

$$\begin{aligned}& y_1(T) - y_1(0) + A_{d1}\int_{T-\bar{d}}^T y(t)dt - A_{d1}\int_{-\bar{d}}^0 y(t)dt \\ =\;& (A_1+A_{d1})\int_0^T y_1(t)dt + A_{d1}\int_0^T y_2(t)dt.\end{aligned} \tag{20}$$

Based on $y_2(t) = A_{d2}\{y_1(t-\bar{d}) + y_2(t-\bar{d})\} = A_{d2}y(t-\bar{d})$, the following equation is obtained:

$$\begin{aligned}& A_{d2}\int_0^T y_1(t)dt + (A_{d2}-I)\int_0^T y_2(t)dt \\ =\;& A_{d2}\int_0^T y(t)dt - \int_0^T y_2(t)dt \\ =\;& \int_{\bar{d}}^{T+\bar{d}} y_2(t)dt - \int_0^T y_2(t)dt \\ =\;& \int_T^{T+\bar{d}} y_2(t)dt - \int_0^{\bar{d}} y_2(t)dt \\ =\;& A_{d2}\int_{T-\bar{d}}^T y(t)dt - A_{d2}\int_{-\bar{d}}^0 y(t)dt.\end{aligned} \tag{21}$$

Combining (20) and (21) yields

$$\begin{aligned}&\begin{bmatrix} y_1(T) - y_1(0) + A_{d1}\int_{T-\bar{d}}^T y(t)dt - A_{d1}\int_{-\bar{d}}^0 y(t)dt \\ A_{d2}\int_{T-\bar{d}}^T y(t)dt - A_{d2}\int_{-\bar{d}}^0 y(t)dt\end{bmatrix} \\ =\;& \begin{bmatrix} A_1+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\int_0^T\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}dt.\end{aligned} \tag{22}$$

By statement (i) that the system is asymptotically stable, it follows that $y(t) \to 0$ and $y_1(t) = My(t) \to 0$ with $t \to \infty$. By appropriately choosing $\upsilon \in \mathbb{R}^n$, we can set $y_1(0) = M\upsilon \succeq 0$ with a maximum number of non-zero elements. With initial condition $y_1(0) \succeq 0$ and $\phi(s) \succ 0$, $t = [-\bar{d}, 0)$, a sufficiently large q must exist such that

$$\begin{aligned} y_1(q) - y_1(0) + A_{d1}\int_{q-\bar{d}}^q y(t)dt - A_{d1}\int_{-\bar{d}}^0 \phi(t)dt &\preceq 0, \\ A_{d2}\int_{T-\bar{d}}^T y(t)dt - A_{d2}\int_{-\bar{d}}^0 \phi(t)dt &\preceq 0,\end{aligned}$$

which implies

$$\begin{bmatrix} A_1+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\int_0^q\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}dt \preceq 0. \tag{23}$$

Let $\lambda_1 = \int_0^q y_1(t)dt$, $\lambda_2 = \int_0^q y_2(t)dt$ and it follows $\bar{\Pi}\bar{\lambda} \preceq 0$. Furthermore, one can obtain $\varphi(s) \equiv \lambda_1+\lambda_2$, $-\bar{d} \le s < 0$ and $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1+\lambda_2)$, which completes the proof. □

Under strictly positive initial conditions, this stability condition is not only sufficient but also necessary.

Theorem 5: Suppose that the pair (E, A) is regular and impulse-free and $\upsilon \in \mathbb{R}^n$ and $\mu \in \bar{\mathbb{R}}_+^n$ exist such that $M\upsilon \succ 0$ and $A_{d2}\mu \succ 0$. Under Assumption 1, system $\mathscr{S}''$ is positive and asymptotically stable with any admissible initial condition, if and only if $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists, thereby satisfying

$$\begin{cases} A_1 = HM, \\ \bar{H} \text{ is Hurwitz.}\end{cases}$$

Proof. By equation (22) in the proof of Theorem 3 and Theorem 2, we have

$$\begin{aligned}&\begin{bmatrix} y_1(T) - y_1(0) + A_{d1}\int_{T-\bar{d}}^T y(t)dt - A_{d1}\int_{-\bar{d}}^0 y(t)dt \\ \int_T^{T+\bar{d}} y(t)dt - \int_0^{\bar{d}} y_2(t)dt\end{bmatrix} \\ =\;& \begin{bmatrix} HM+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\int_0^T\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}dt \\ =\;& \begin{bmatrix} H+A_{d1} & A_{d1} \\ A_{d2} & A_{d2}-I\end{bmatrix}\int_0^T\begin{bmatrix} y_1(t) \\ y_2(t)\end{bmatrix}dt.\end{aligned}$$

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When $y_1(0) = M\upsilon \succ 0$, $y_2(0) = (M-I)A^DA_d\mu \succ 0$ and the continuity of $y_1(t)$, $y_2(t)$, it follows that $\int_0^{\bar{d}} y_1(t)dt \succ 0$ and $\int_0^{\bar{d}} y_2(t)dt \succ 0$. Therefore, a sufficiently large q must exist such that

$$\begin{aligned} y_1(q) - y_1(0) + A_{d1}\int_{q-\bar{d}}^{q} y(t)dt - A_{d1}\int_{-\bar{d}}^{0} y(t)dt &\prec 0, \\ \int_{q}^{q+\bar{d}} y(t)dt - \int_{0}^{\bar{d}} y(t)dt &\prec 0. \end{aligned}$$

As $\int_0^q \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} dt \succ 0$, one can conclude that $\bar{H}$ is Hurwitz. □

The following lemma will also be useful in stability analysis for singular systems with time-varying delays.

Lemma 12: Suppose that $\bar{\lambda} = \begin{bmatrix} \lambda_1^T & \lambda_2^T \end{bmatrix}^T$ with $\lambda_1 \in \bar{\mathbb{R}}_+^n$, $\lambda_2 \in \bar{\mathbb{R}}_+^n$, which satisfies $\bar{\Pi}\bar{\lambda} \preceq 0$. With the initial condition $\varphi(s) \equiv \lambda_1 + \lambda_2$, $-\bar{d} \le s < 0$ and $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1 + \lambda_2)$, the trajectory of system $\mathscr{S}$ and $\mathscr{S}''$ satisfies that

(i) $y_1(t) \preceq \lambda_1$, $y_2(t) \preceq \lambda_2$ and $y(t) \preceq \lambda_1 + \lambda_2$ for any $t \ge 0$.

(ii) $y_1(t_1) \succeq y_1(t_2)$, $y_2(t_1) \succeq y_2(t_2)$ and $y(t_1) \succeq y(t_2)$ for any $t_2 > t_1 \ge 0$.

(iii) $y(t) \succeq x(t)$ for any $t \ge 0$.

Proof. (i) Define $e_1(t) \triangleq \lambda_1 - y_1(t)$, $e_2(t) \triangleq \lambda_2 - y_2(t)$, and $e(t) \triangleq \lambda_1 + \lambda_2 - y(t)$. Then, the error system satisfies

$$\begin{aligned} \dot{e}_1(t) &= A_1e_1(t) + A_{d1}e(t-\bar{d}) - A_1\lambda_1 - A_{d1}(\lambda_1+\lambda_2), \\ 0 &= -e_2(t) + A_{d2}e(t-\bar{d}) - A_{d2}(\lambda_1+\lambda_2) + \lambda_2, \end{aligned}$$

where $e_1(0) = 0$ and $e(s) = 0$, for $s \in [-\bar{d}, 0)$. As $\bar{\Pi}\bar{\lambda} \preceq 0$, it follows that

$$-A_1\lambda_1 - A_{d1}(\lambda_1+\lambda_2) \succeq 0, \ -A_{d2}(\lambda_1+\lambda_2) + \lambda_2 \succeq 0.$$

Correspondingly, we have the following solutions of $e_1(t)$ and $e_2(t)$,

$$\begin{aligned} e_1(t) &= \int_0^t e^{A_1(t-\tau)}A_{d1}e(\tau-\bar{d})d\tau \\ &\quad + \int_0^t e^{A_1(t-\tau)}(-A_1\lambda_1 - A_{d1}(\lambda_1+\lambda_2))d\tau, \\ e_2(t) &= A_{d2}e(t-\bar{d}) - A_{d2}(\lambda_1+\lambda_2) + \lambda_2. \end{aligned}$$

Due to the positivity of system $\mathscr{S}''$, it follows that $A_1 = HM$, $A_{d1} \succeq 0$ and $A_{d2} \succeq 0$ by Theorem 1. Similar to the proof of Theorem 1, $e_1(t) \succeq 0$ and $e_2(t) \succeq 0$ for $0 \le t \le \underline{d}$, which also leads to $e(t) \succeq 0$ for $0 \le t \le \underline{d}$. For $\underline{d} \le t \le 2\underline{d}$, it is still valid by following the same manner. Therefore, repeating this procedure iteratively, one can conclude that $e_1(t) \succeq 0$, $e_2(t) \succeq 0$ and $e(t) \succeq 0$ for all $t \ge 0$, which directly indicates that $y_1(t) \preceq \lambda_1$ and $y_2(t) \preceq \lambda_2$ and $y(t) = y_1(t) + y_2(t) \preceq \lambda_1 + \lambda_2$ for all $t \ge 0$.

(ii) Randomly choose constant $\Delta t > 0$ and define augmented error state

$$\bar{e}(t) \triangleq \bar{y}(t) - \bar{y}(t+\Delta t) = \begin{bmatrix} y_1(t) - y_1(t+\Delta t) \\ y_2(t) - y_2(t+\Delta t) \end{bmatrix}.$$

Then, $\bar{e}(t)$ satisfies the following condition:

$$\bar{E}\dot{\bar{e}}(t) = \bar{A}\bar{e}(t) + \bar{A}_d\bar{e}(t-\bar{d}). \tag{24}$$

With statement (i), the initial condition of the error system (24) satisfies that $\bar{e}(s) = \bar{y}(s) - \bar{y}(s+\Delta t) = \bar{\lambda} - \bar{y}(s+\Delta t) \succeq 0$ for $s \in [-d, 0)$ where $\bar{\lambda} = \begin{bmatrix} \lambda_1^T & \lambda_2^T \end{bmatrix}^T$. For $s = 0$, we have

$$\bar{e}(0) = \begin{bmatrix} \lambda_1 - y_1(\Delta t) \\ A_{d2}(\lambda_1 + \lambda_2 - y(-\bar{d}+\Delta t)) \end{bmatrix} \succeq 0.$$

As the initial condition $\bar{e}(s) \succeq 0$ for $s \in [-\bar{d}, 0]$ and system is positive, it follows that $\bar{e}(t) \succeq 0$ holds for all $t > 0$, which implies that $y_1(t_1) \succeq y_1(t_2)$, $y_2(t_1) \succeq y_2(t_2)$ and $y(t_1) \succeq y(t_2)$ for any $t_2 > t_1 \ge 0$.

(iii) Define augmented error state $\bar{e}(t) \triangleq \bar{y}(t) - \bar{x}(t) = \begin{bmatrix} y_1^T(t) - x_1^T(t) & y_2^T(t) - x_2^T(t) \end{bmatrix}^T$. Then, the error system satisfies that

$$\bar{E}\dot{\bar{e}}(t) = \bar{A}\bar{e}(t) + \bar{A}_d\bar{e}(t-d(t)) + A_d(\bar{y}(t-\bar{d}) - \bar{y}(t-d(t))). \tag{25}$$

Note that $\bar{e}(s) = 0$ for $s \in [-\bar{d}, 0]$ and the statement (ii) indicates that $\bar{y}(t-\bar{d}) - \bar{y}(t-d(t)) \succeq 0$ as $d(t) \le \bar{d}$. Following the same analysis aforementioned, the solution of this error system is positive as statement (i) shows, which implies that $\bar{y}(t) \succeq \bar{x}(t)$ and $y(t) = y_1(t) + y_2(t) \succeq x(t) = x_1(t) + x_2(t)$ for all $t \ge 0$. □

Remark 1: The statement (ii) of Lemma 12 shows that the solution to system $\mathscr{S}''$ is monotonically nonincreasing, which plays a key role in the following proof of stability condition for time-varying delay cases.

Before analyzing the stability of system $\mathscr{S}$, we present the following lemma related to the initial condition.

Lemma 13: Suppose that $x(t;\phi_1)$ and $x(t;\phi_2)$ are the trajectories of system $\mathscr{S}$ with the admissible initial condition $\phi_1(s)$ and $\phi_2(s)$, respectively. Then, if $\phi_1(s) \preceq \phi_2(s)$ for $s \in [-d, 0]$ implies that $x(t;\phi_1) \preceq x(t;\phi_2)$ for $t \ge 0$.

Proof. Based on the positivity and linearity of system $\mathscr{S}$, with the initial condition $\tilde{\phi} = \phi_2 - \phi_1$, the solution $\tilde{e}(t) = x(t;\phi_2) - x(t;\phi_1) \succeq 0$ for $t \ge 0$. □

In the following discussion, we show that an $\alpha \in \mathbb{R}$ exists such that $\alpha\varphi(\cdot) \succeq \phi(\cdot)$, where $\phi(\cdot)$ is an arbitrary initial condition and $\varphi(s) \equiv \lambda_1 + \lambda_2$, $-\bar{d} \le t < 0$ and $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1 + \lambda_2)$ mentioned in Theorem 3. However, for the initial condition $\varphi(\cdot)$, the parameter α may not exist when it is not strictly positive.

Therefore, a new initial condition $\tilde{\phi}(\cdot)$ is introduced to replace the original $\phi(\cdot)$. $\tilde{\phi}(\cdot)$ is the solution of system $\mathscr{S}$ under $\phi(\cdot)$ for $0 \le t \le \bar{d}$. It means $\tilde{\phi}(t) = x(t+\bar{d})$ for $-\bar{d} \le t \le 0$ and new solution $x(t;\tilde{\phi})$ under $\tilde{\phi}(\cdot)$ satisfying $x(t;\tilde{\phi}) = x(t+\bar{d};\phi)$. Without loss of generality, $\tilde{\phi}(\cdot)$ can be set as the initial condition for the system instead of $\phi(\cdot)$.

With initial condition $\hat{\phi}(s) \equiv \max_{s\in[-\bar{d},0)} \phi(s)$ for $-\bar{d} \le s < 0$ and $\hat{\phi}(0) = \phi(0)$ and Lemma 13, we have $y(t|\hat{\phi}) = x(t|\hat{\phi}) \succeq x(t|\phi) = \tilde{\phi}(t-\bar{d})$ for $0 \le t \le \bar{d}$. From the proof of Theorem 3, we have

$$\lambda_1 + \lambda_2 = \int_0^T y_1(t|\hat{\phi})dt + \int_0^T y_2(t|\hat{\phi})dt = \int_0^T y(t|\hat{\phi})dt.$$

For $T > \bar{d}$ and each element $y_i(t) \neq 0$, it follows that $(\lambda_1 + \lambda_2)_i \succ 0$ for $i = 1, \ldots, n$, which implies that $\alpha_1 = \max_{t\in[0,\bar{d})} \|y(t)\|_\infty / \min_{j=1,2,\ldots,n}\{\lambda_j\}, \lambda_j \neq 0$ exists to ensure

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that $\alpha_1(\lambda_1+\lambda_2) \succeq y(t)$ for $0 \le t \le \bar{d}$. In the same way, we can obtain $\alpha_2\lambda_1 \succeq y_1(\bar{d})$.

Therefore, we can always find $\alpha\varphi(\cdot) \succeq \tilde{\phi}(\cdot)$ where $\varphi(s) \equiv (\lambda_1+\lambda_2)$, $-\bar{d} \le s < 0$, $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1+\lambda_2)$, and $\alpha = \max\{\alpha_1, \alpha_2\}$. By virtue of the preceding analysis, we propose the following stability conditions.

Theorem 6: Suppose that the pair (E,A) is regular and impulse-free. Under Assumption 1, system $\mathscr{S}$ is positive and asymptotically stable if $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists, satisfying

$$\begin{cases} A_1 = HM, \\ \bar{H} \text{ is Hurwitz.} \end{cases}$$

Proof. Assume that $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists such that $A_1 = HM$ and $\bar{H}$ is Hurwitz. Therefore, system $\mathscr{S}$ is positive and system $\mathscr{S}''$ is positive and stable. By Theorem 4, one can obtain an admissible initial condition $\varphi(s) \equiv (\lambda_1+\lambda_2), -\bar{d} \le t < 0$, $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1+\lambda_2)$ and system $\mathscr{S}''$ is asymptotically stable. Suppose that $x(t;\varphi)$ and $y(t;\varphi)$ are the corresponding solutions of systems $\mathscr{S}$ and $\mathscr{S}''$ with the same initial condition $\varphi(\cdot)$.

Based on the definition of asymptotic stability, for any given $\varepsilon > 0$, a scalar $\delta > 0$ exists such that if $\|\varphi\| < \delta$, then the corresponding solution $y(t;\varphi)$ to the stable system $\mathscr{S}''$ satisfies that $\|y(t;\varphi)\| < \varepsilon$ for all $t \ge 0$, and $\lim_{t\to+\infty}\|y(t;\varphi)\| = 0$.

From the preceding analysis, we can find $\alpha\varphi(\cdot) \succeq \tilde{\phi}(\cdot)$ where $\varphi(s) \equiv (\lambda_1+\lambda_2)$, $-\bar{d} \le t < 0$, $\varphi(0) = \lambda_1 + A_{d2}(\lambda_1+\lambda_2)$. By Lemma 13, for any initial condition $\tilde{\phi} \preceq \alpha\varphi$, one can obtain that $x(t;\tilde{\phi}) \preceq x(t;\alpha\varphi)$ for all $t \ge 0$. Furthermore, by Lemma 12, the corresponding solutions $x(t;\alpha\varphi)$ and $y(t;\alpha\varphi)$ follow that $x(t;\alpha\varphi) \preceq y(t;\alpha\varphi)$ for all $t \ge 0$.

For arbitrarily given $\varepsilon > 0$, one can find $\delta_1 = \|\alpha\varphi\| > 0$ such that $\|\tilde{\phi}\| < \delta_1$. Then, we can deduce that $\|x(t;\tilde{\phi})\| \le \|x(t;\alpha\varphi)\| \le \|y(t;\alpha\varphi)\| < \varepsilon$ for all $t \ge 0$. Based on $\|x(t;\tilde{\phi})\| \le \|y(t;\alpha\varphi)\| < \varepsilon$ for all $t \ge 0$ and $\lim_{t\to+\infty}\|y(t;\alpha\varphi)\| = 0$, it follows that

$$\lim_{t\to+\infty}\|x(t;\phi)\| = \lim_{t\to+\infty}\|x(t;\tilde{\phi})\| = 0,$$

which implies system $\mathscr{S}$ is stable. □

Theorem 7: Suppose that the pair (E,A) is regular and impulse-free and $\upsilon \in \mathbb{R}^n$ and $\mu \in \bar{\mathbb{R}}^n_+$ exist such that $M\upsilon \succ 0$ and $A_{d2}\mu \succ 0$. Under Assumption 1, system $\mathscr{S}$ is positive and asymptotically stable if and only if $A_{d1} \succeq 0$, $A_{d2} \succeq 0$ and a Metzler matrix H exists, satisfying

$$\begin{cases} A_1 = HM, \\ \bar{H} \text{ is Hurwitz.} \end{cases}$$

Proof. The result directly follows from Theorem 5 and Theorem 6. □

V. Illustrative Example

Example: Consider the singular system $\mathscr{S}$ with the following parameters:

$$E = \begin{bmatrix} 1.00 & -11.00 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.61 & 0.2 \\ 0.60 & -1.00 \end{bmatrix},$$
$$A_d = \begin{bmatrix} -0.2 & -1 \\ -0.01 & -0.80 \end{bmatrix}.$$

We have $\hat{E}$ and $\hat{A}$ commute with $\beta = 3$.

$$\hat{E} = \begin{bmatrix} -0.057 & 0.627 \\ -0.034 & 0.376 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} -1.171 & 1.882 \\ -0.102 & 0.129 \end{bmatrix},$$
$$\hat{A}_d = \begin{bmatrix} 0.030 & 1.572 \\ 0.008 & 0.143 \end{bmatrix}.$$

The widely used method investigating positive singular time-delay systems is slow-fast decomposition, for example, as [16], [29] and [30]. Note that the regularity and the absence of impulses of the pair (E,A) imply that there exist two invertible matrices P, Q such that

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix}.$$

When $Q \succ 0$ is a monomial matrix, it follows that $Q^{-1} \succ 0$, which implies that $x(t) \succeq 0$ if and only if $z(t) \succeq 0$, where $z(t) = Q^{-1}x(t) = [z_1(t), z_2(t)]$. Then, the researchers in [16], [29] and [30] established the positivity and stability conditions of $z(t)$ instead of $x(t)$.

However, the assumption that Q is a monomial matrix is clearly restrictive, which means the positivity and stability of some cases cannot be verified by their theory, such as this example. We solve this problem by using a different method in Lemma 5 to transform the original singular system, which needs no requirement for Q and can be applied to any singular matrix E as long as the pair (E,A) is regular and impulse-free. By Theorem 2 and 6, the positivity and stability can be checked as follows:

$$\hat{E}^D\hat{E} = \begin{bmatrix} -0.178 & 1.964 \\ -0.107 & 1.178 \end{bmatrix},$$
$$\hat{E}^D\hat{A} = \begin{bmatrix} 0.023 & -0.256 \\ 0.014 & -0.153 \end{bmatrix},$$
$$\hat{E}^D\hat{A}_d = \begin{bmatrix} 0.033 & 0.002 \\ 0.020 & 0.001 \end{bmatrix},$$
$$(\hat{E}^D\hat{E} - I)\hat{A}^D\hat{A}_d = \begin{bmatrix} 0.019 & 1.571 \\ 0.001 & 0.142 \end{bmatrix}.$$

By solving the equality in Theorem 2 by linear programming, we obtain

$$H = \begin{bmatrix} -0.130 & 0.000 \\ 0.000 & -0.130 \end{bmatrix}.$$

Obviously, the matrix H is a Metzler matrix. Therefore, with $\hat{E}^D\hat{A}_d \succeq 0$ and $(\hat{E}^D\hat{E} - I)\hat{A}^D\hat{A}_d \succeq 0$, one can conclude that the system is positive, which implies that our findings are less conservative.

For stability analysis, we note that there exists υ such that $\hat{E}^D\hat{E}\upsilon \succ 0$. By Theorem 7, we calculate the spectrum of $\bar{H}$,

$$\sigma(\bar{H}) = \{-0.054, -0.130, -0.879, -1.000\},$$

which indicates that $\bar{H}$ is Hurwitz and the system $\mathscr{S}$ is positive and stable, as shown in Figure 1. If the system matrix A is changed to $A' = \begin{bmatrix} 0.10 & 0.2 \\ 0.60 & -1.00 \end{bmatrix}$, it is similarly shown that system $\mathscr{S}$ is also positive as

$$H = \begin{bmatrix} -0.039 & 0.000 \\ 0.000 & -0.039 \end{bmatrix}$$

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TAC.2017.2749524, IEEE Transactions on Automatic Control

is Metzler and $\hat{E}^D\hat{A}_d \succeq 0$, $(\hat{E}^D\hat{E}-I)\hat{A}^D\hat{A}_d \succeq 0$. The spectrum of $\bar{H}$ is given as

$$\sigma(\bar{H}) = \{0.131, -0.039-0.886, -1.000\}.$$

$\bar{H}$ is not Hurwitz and this positive singular system is not stable, as shown in Figure 2. In Figures 1 and 2, the admissible initial conditions are given as $\hat{E}^D\hat{E}\upsilon = \begin{bmatrix} 19.107 \\ 11.464 \end{bmatrix}$, $\upsilon = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$ and $\phi(s) = \hat{E}^D\hat{E}\upsilon$, for $s \in [0, -\bar{d}]$ and the delay $d(t)$ are chosen as $d(t) = \arctan t$.

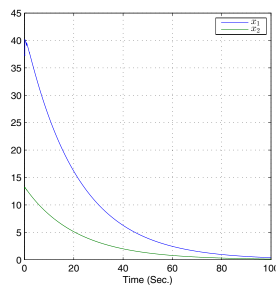

Fig. 1. Evolution of system state with A

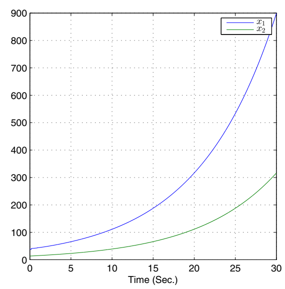

Fig. 2. Evolution of system state with A'

VI. Conclusion

This paper has studied the asymptotic stability of positive continuous-time singular systems with bounded time-varying delays. A necessary and sufficient condition has been proposed to ensure the positivity of singular time-delay systems. Then, a sufficient stability condition for continuous-time linear singular systems with time-varying delays has been established. In addition, we have shown that this stability condition is also necessary when the initial condition can be strictly positive.

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