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Nontrivial solutions of boundary value problems of second-order difference equations ☆

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Abstract

We employ the critical point theory to establish the existence of nontrivial solutions for some boundary value problems of second-order difference equations.

Keywords: Nontrivial solution; Difference equation; Boundary value problem; Critical point; Palais–Smale condition

1. Introduction

Let $T > 1$ be a fixed positive integer. For $a, b \in \mathbf{N}$ with $a < b$, define $[a, b] = \{a, a+1, \dots, b-1, b\}$. We consider the following boundary value problem of second-order difference equation:

$$\begin{aligned} &\Delta^2 u_{k-1} + \lambda h(k, u_k) = 0, \quad k = 1, 2, \dots, T, \\ &u_0 = u_{T+1} = 0, \end{aligned} \tag{1.1}$$

where $\Delta u_{k-1} = u_k - u_{k-1}$, $\Delta^2 u_{k-1} = \Delta(\Delta u_{k-1})$, $\lambda > 0$ is a positive parameter. We assume that

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(H1) $h : [0, T + 1] \times \mathbf{R} \to \mathbf{R}$ is continuous;
(H2) there exists an $\alpha > 0$ such that $h(t, \alpha) = 0$ and $h(t, u) > 0$ for $u \in (0, \alpha)$;
(H3) $h(t, u)$ is odd in u.

Due to the wide applications in many fields such as computer science, economics, neural network, ecology, cybernetics, etc., the theory of nonlinear difference equations has been widely studied since 70's of last century. See, for example, [1–5]. Also, in recent years, there were much literature on the boundary value problems of difference equations. We refer reader to [6–11] and the references therein.

As we know that the critical point theory (including minimax theory, geometrical index theory and Morse theory) is a very powerful tool to deal with the existence of solutions for the boundary value problems of differential equations. See, for example, Rabinowitz [12]. However, there are relatively rare results of the existence of solutions for the boundary value problems of difference equations by using of the critical point theory. In the paper, we apply a version of Clark' Theorem [12, Theorem 9.1] to (1.1) and study the existence of solutions of (1.1).

We state our main result as following:

Theorem 1.1. *Let* (H1)–(H3) *be satisfied. Then there exists a* $\lambda^* > 0$ *such that if* $\lambda > \lambda^*$, (1.1) *has at least* T *distinct pairs of nontrivial solutions. Furthermore, each nontrivial solution* u *satisfies that* $|u_k| \leqslant \alpha$, $k \in [0, T + 1]$.

We note that if u is a solution of (1.1), then $-u$ also solves (1.1) and we say that $(u, -u)$ is a pair of solutions of (1.1).

2. Preliminaries

Suppose that E is a real Banach space. Let $C^1(E, \mathbf{R})$ denote the set of functionals that are Fréchet differentiable and whose Fréchet derivatives are continuous on E. For $I \in C^1(E, \mathbf{R})$, we say I satisfies the Palais–Smale condition (henceforth denoted by P.S.) if any sequence $\{u_m\} \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent subsequence.

Let θ denote the zero element of Banach space E. Let Σ denote the family of sets $A \subset E \setminus \{\theta\}$ such that A is closed in E and symmetric with respect to θ, i.e. $u \in A$ implies $-u \in A$.

The following theorem [12] is crucial in the proof of Theorem 1.1.

Theorem 2.1. *Let* E *be a real Banach space,* $I \in C^1(E, \mathbf{R})$ *with* I *even, bounded from below, and satisfying P.S. condition. Suppose* $I(\theta) = 0$, *there is a set* $K \subset \Sigma$ *such that* K *is a homeomorphic to* S^{j-1} ($j - 1$ *dimension unit sphere*) *by an odd map, and* $\sup_K I < 0$. *Then* I *has at least* j *distinct pairs of nonzero critical points.*

3. Proof of Theorem 1.1

Let $\|u\|_1 = \max\{|u_1|, \ldots, |u_T|\}$ for $u = (u_1, \ldots, u_T) \in \mathbf{R}^T$.

Define

$$h_1(t, s) = \begin{cases} h(t, \alpha), & s > \alpha, \\ h(t, s), & |s| \leqslant \alpha, \\ h(t, -\alpha), & s < -\alpha. \end{cases}$$

We claim that if $u:[0, T+1] \to \mathbf{R}$ satisfies

$$\begin{aligned} &\Delta^2 u_{k-1} + \lambda h_1(k, u_k) = 0, \quad k = 1, 2, \ldots, T, \\ &u_0 = u_{T+1} = 0. \end{aligned} \tag{3.1}$$

Then $\|u\|_1 \leqslant \alpha$ and consequently u is a solution of (1.1). Otherwise, there is a k_0: $1 \leqslant k_0 \leqslant T$ such that $|u_{k_0}| > \alpha$ and $|u_k| \leqslant \alpha$ for $k \in \{1, 2, \ldots, k_0 - 1\}$.

If $u_{k_0} > \alpha$, then $h_1(t, u_{k_0}) = h(t, \alpha) = 0$. It follows that

$$\Delta^2 u_{k_0-1} = 0,$$

or

$$u_{k_0+1} = 2u_{k_0} - u_{k_0-1},$$

which implies from $u_{k_0} > \alpha$ and $|u_{k_0-1}| < \alpha$ that $u_{k_0+1} > \alpha$ and consequently that $h_1(t, u_{k_0+1}) = h(t, \alpha) = 0$. Thus

$$\Delta^2 u_{k_0} = 0,$$

or

$$u_{k_0+2} = 2u_{k_0+1} - u_{k_0} = 3u_{k_0} - 2u_{k_0-1},$$

which again implies that $u_{k_0+2} > \alpha$.

Repeat the above progresses and we have that

$$u_i > \alpha, \quad i = k_0, k_0 + 1, \ldots, T,$$

and

$$u_{k_0+m} = (m+1)u_{k_0} - m u_{k_0-1}, \quad m = 1, 2, \ldots, T + 1 - k_0.$$

Especially,

$$0 = u_{T+1} = (T + 2 - k_0)u_{k_0} - (T + 1 - k_0)u_{k_0-1} > \alpha,$$

which is a contradiction.

If $u_{k_0} < -\alpha$, we can similarly get a contradiction.

Let

$$E = \{u:[0, T+1] \to \mathbf{R} \mid u_0 = u_{T+1} = 0\}.$$

Define the inner product on E as

$$\langle u, v\rangle = \sum_{k=0}^{T} \Delta u_k \Delta v_k, \quad \forall u, v \in E,$$

by which norm $\|\cdot\|$ is induced by

$$\|u\| = \left(\sum_{k=0}^{T} |\Delta u_k|^2\right)^{1/2}, \quad \forall u \in E.$$

Define functional I on E as

$$I(u) = \sum_{k=0}^{T} \left[\frac{1}{2}(\Delta u_k)^2 - \lambda H(k, u_k)\right], \quad \forall u \in E,$$

where $H(t, z) = \int_0^z h_1(t, s)\, ds$.

It is easy see that $I \in C^1(E, \mathbf{R})$ by the continuity of h, and that $u \in E$ is a critical point of I, i.e. $I'(u) = 0$, if and only if u is a solution of (3.1).

Now we check that I satisfies the conditions of Theorem 2.1.

Since $h_1 : [0, T+1] \times \mathbf{R} \to \mathbf{R}$ is a continuous, bounded and odd function, we know that $I \in C^1(E, \mathbf{R})$ is even and $I(\theta) = 0$. It follows from $h_1(t, s) = 0$ for $|s| \geqslant \alpha$ that

$$\sum_{k=0}^{T} H(k, u_k) = \sum_{k=0}^{T} \int_0^{u_k} h_1(k, s)\, ds \leqslant \sum_{k=0}^{T} \int_{-\alpha}^{\alpha} \big|h_1(k, s)\big|\, ds =: C, \quad \forall u \in E.$$

It implies that

$$I(u) \geqslant \sum_{k=0}^{T} \frac{1}{2}[\Delta u_k]^2 - \lambda C \geqslant -\lambda C, \quad \forall u \in E.$$

Thus I is bounded from below.

The P.S. condition is verified as following.

Let $\{u^{(m)}\}_{m=1}^{\infty} \subset E$ be an any sequence such that $\{I(u^{(m)})\}_{m=1}^{\infty}$ is bounded and $I'(u^{(m)}) \to 0$ as $m \to \infty$. Let $c_1 \leqslant I(u^{(m)}) \leqslant c_2$, $m = 1, 2, \ldots$. By the above arguments we know that

$$I\big(u^{(m)}\big) \geqslant \sum_{k=0}^{T} \frac{1}{2}\big[\Delta u_k^{(m)}\big]^2 - \lambda C.$$

Then

$$\sum_{k=0}^{T} \big[\Delta u_k^{(m)}\big]^2 \leqslant 2(c_2 + \lambda C),$$

which implies that $\|u^{(m)}\| \leqslant (2(c_2 + \lambda C))^{1/2}$, $m = 1, 2, \ldots$. Therefore, $\{u^{(m)}\}_{m=1}^{\infty}$ is a bounded sequence in finite-dimensional space E and so has a convergent subsequence in E.

Now we take $\{\upsilon^i\}_{i=1}^{T}$ is a base of E with $\|\upsilon^i\| = 1$, $i = 1, 2, \ldots, T$. Define

$$K(r) = \left\{ \sum_{i=1}^{T} \beta^i \upsilon^i \;\middle|\; \sum_{i=1}^{T} \big|\beta^i\big|^2 = r^2 \right\}, \quad r > 0.$$

One can find that $\theta \notin K(r)$, and $K(r)$ is closed in E and symmetric with respect to θ. It is clear that $K(r)$ is homeomorphic to S^{T-1} by an odd map for any $r > 0$.

Since $\dim E < \infty$, there exists a $c_0 > 0$ such that $\|u\|_1 \leqslant c_0 \|u\|$ for all $u \in E$. Now we take r: $0 < r < \frac{\alpha}{c_0 \sqrt{T}}$.

For $u \in K(r)$, we have that

$$\|u\|^2 = \sum_{k=0}^{T} \left| \sum_{i=1}^{T} \beta^i \Delta \upsilon_k^i \right|^2 \leqslant \sum_{k=0}^{T} \left(\sum_{i=1}^{T} \big|\beta^i\big|^2 \sum_{i=1}^{T} \big|\Delta \upsilon_k^i\big|^2 \right) = r^2 T \big\|\upsilon^i\big\|^2 = r^2 T,$$

so

$$\|u\|_1 \leqslant c_0 \|u\| \leqslant c_0 r \sqrt{T} < \alpha.$$

It follows that $h(t,u_k)=h_1(t,u_k)$ for all $u\in K(r)$. Thus, from condition (H2), we have that for $u\in K(r)$,

$$H(k,u_k)=\int_0^{u_k} h(k,s)\,ds>0$$

if $u_k\neq 0$. In fact, if $u_k>0$, it is clear that the above inequality holds. If $u_k<0$, we still have

$$H(k,u_k)=\int_0^{u_k} h(k,s)\,ds=\int_0^{-u_k} h(k,-t)\,d(-t)=\int_0^{-u_k} h(k,s)\,ds>0.$$

Therefore, from $\theta\notin K(r)$, we have that for all $u\in K(r)$,

$$\sum_{k=0}^{T} H(k,u_k)=\sum_{k=0}^{T}\int_0^{u_k} h(k,s)\,ds>0.$$

Set $\tau=\inf_{u\in K(r)}\sum_{k=0}^{T} H(k,u_k)$. It is easy to see that $\tau>0$. Choose $\lambda^*=\frac{\alpha^2}{2\tau c_0^2}$. Then if $\lambda>\lambda^*$, we have that for $u\in K(r)$,

$$\begin{aligned}
I(u)&=\sum_{k=0}^{T}\left\{\frac{1}{2}\left(\sum_{i=1}^{T}\beta^i\,\Delta\upsilon_k^i\right)^2-\lambda H(k,u_k)\right\}\\
&\leqslant\sum_{k=0}^{T}\frac{1}{2}\left(\sum_{i=1}^{T}\beta^i\,\Delta\upsilon_k^i\right)^2-\lambda\tau\\
&\leqslant\frac{r^2}{2}\sum_{k=0}^{T}\left(\sum_{i=1}^{T}\left|\Delta\upsilon_k^i\right|^2\right)-\lambda\tau\\
&=\frac{r^2}{2}T-\lambda\tau\\
&<\frac{\alpha^2}{2c_0^2}-\lambda\tau\\
&<0.
\end{aligned}$$

Thus all the conditions of Theorem 2.1 are satisfied, and I has at least T distinct pairs of nonzero critical points. Consequently, (1.1) has at least T distinct pairs of nontrivial solutions. The proof of Theorem 1.1 is completed.

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