International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

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Z. Shu [a] & J. Lam [a]

[a] Department of Mechanical Engineering , The University of Hong Kong , Pokfulam Road, Hong Kong

Published online: 10 Feb 2011.

International Journal of Control
Vol. 81, No. 6, June 2008, 865–882

Taylor & Francis
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Z. SHU and J. LAM*

Department of Mechanical Engineering, The University of Hong Kong, Pokfulam Road, Hong Kong

(*Received 11 October 2006; in final form 5 February 2007*)

This paper is concerned with exponential estimates and stabilization for a class of uncertain singular systems with discrete and distributed delays. A sufficient condition, which does not only guarantee the exponential stability and admissibility but also gives the estimates of decay rate and decay coefficient, is established in terms of the linear matrix inequality (LMI) technique and a new Lyapunov–Krasovskii functional. The estimating procedure is implemented by solving a set of LMIs, which can be checked easily by effective algorithms. Under the proposed condition, the algebraic subsystems possess the same decay rate as the differential ones. Moreover, a state feedback stabilizing controller which makes the closed-loop system exponentially stable and admissible with a prescribed lower bound of the decay rate is designed. Numerical examples are provided to illustrate the effectiveness of the theoretical results.

1. Introduction

Over the past decades, lots of effort has been devoted to the study of time-delay systems, since time delay is encountered in various practical systems such as chemical processes, long transmission lines in pneumatic systems (Hale 1977), and is considered as a major cause for instability and poor performance of dynamic systems. Much attention has been paid to stability analysis of delay systems, and various stability conditions have been obtained (Dugard and Veriest 1998, Gu *et al*. 2003). Typical stability analysis is concerned with asymptotic stability, whereas from the standpoint of practical application, exponential stability is more significant since the transient process of a system can be characterized more clearly once the decay rate is determined. Unlike linear systems without time delay, eigenvalues of delay systems cannot be computed analytically owing to the existence of transcendental characteristic equations. This motivates researchers to study the exponential estimating problem of time-delay systems. A lot of results on the exponential estimates of time-delay systems have been reported in the literature. By using the Lyapunov–Razumikhin approach, some results on retarded delay systems have been obtained in Wu and Mizukami (1995) and Hou and Qian (1998). Based on the generalized Gronwall–Bellman Lemma and the matrix measure concept, the exponential estimates of retarded delay systems have been obtained in Mori *et al*. (1982), Wang *et al*. (1987) an Lehman and Shujaee (1994), respectively. Some improvements have been presented in Hmamed (1985) and Mori (1986). Recently, a new estimating approach for retarded delay systems, which is based on a new Lyapunov–Krasovskii functional and the LMI technique, has been presented in Mondié and Kharitonov (2005). Furthermore, an LMI-based approach has also been provided to establish the estimates of neutral delay systems in Kharitonov *et al*. (2005, 2006). Very recently, an improved estimating approach for retarded delay systems has been obtained in Xu *et al*. (2006). In general, LMI-based conditions are easier to check than other types of conditions. As for exponential stability analysis on the case with time-varying delay or with stochastic disturbance, we refer readers to Mao (2002) and Seurat *et al*. (2004, 2005) and references therein.

*Corresponding author. Email: james.lam@hku.hk

International Journal of Control
ISSN 0020–7179 print/ISSN 1366–5820 online © 2008 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/00207170701261986

Singular system models, which not only describe the dynamics of systems but also reveal the algebraic constraints existing in the state of systems, have been widely used to model different practical processes such as singular perturbations, electrical circuits, constrained control systems. Compared with state-space models, the study of singular systems is more arduous, since not only finite dynamic modes, but also impulsive modes and non-dynamic modes should be taken into account, and the latter two issues do not arise in the state-space case. Recently, much attention has been paid to singular systems with time delay. Stability of singular systems with time delay has been investigated in Fridman (2002) and Wei (2004) by employing the Lyapunov–Krasovskii functional method and the eigenvalue-based method, respectively. In Xu *et al*. (2002, 2003), generalized quadratic stabilization and robust $\mathcal{H}_\infty$ control problems have been investigated, respectively. Robust $\mathcal{H}_\infty$ filtering for singular systems with discrete and distributed delays has been discussed in Yue and Han (2004), and further results on robust $\mathcal{H}_\infty$ control for singular systems with time-varying delays have been reported in Yue and Han (2005). However, to the authors' knowledge, the problem of exponential estimates of singular systems with discrete and distributed delays remains open and unsolved. The difficulties lie, not only in the distributed delay, but also in the algebraic constraints in singular systems. Furthermore, the problem of robust stabilization with decay rate constraint, which requires the closed-loop system to satisfy a prescribed specification on lower bound of the decay rate, has not been fully studied. This forms the motivation of our study.

In this paper, we investigate the problems of exponential estimates and robust stabilization for uncertain singular systems with discrete and distributed delays. The parameter uncertainties appearing in both the state and input matrices are assumed to be time-varying and unknown but norm bounded. Unlike the Halanay inequality based approach and the conventional Razumikhin approach, a novel Lyapunov–Krasovskii functional based approach is introduced to prove the exponential stability. Special exponential terms are embedded appropriately into the constructed Lyapunov–Krasovskii functional. The obtained decay rate characterization is a free parameter, which can be selected according to different practical conditions, rather than a fixed value computed by solving a transcendental equation or a complicated function. In addition, slack matrix variables are used to reduce conservatism. The proposed condition is expressed in LMIs, which can be checked easily by effective algorithms such as the interior-point method, and the estimates of decay rate and decay coefficient are obtained through the parameters and solutions of LMIs. Based on the proposed estimating condition, a design approach of the state feedback controller, which not only stabilizes the system but also guarantees that the system possesses a prescribed lower bound of the decay rate, is given. It should be pointed out here that an exponential admissibility condition for singular systems with discrete and distributed delays has been proposed in Yue and Han (2004). However, the decay rate obtained by solving a transcendental equation is a fixed value, which cannot be adjusted to cater to design specifications, and the result cannot tell us whether a system can possess a larger decay rate than the computed value. Moreover, the bounding techniques used in the derivation of the result may cause conservatism. The contributions of this paper lie in the following.

(1) The establishment of exponential estimates for singular systems with distributed delay for the first time, and the results are also applicable to the state-space case.
(2) The technical improvement of showing the exponential stability for the algebraic subsystems with delay. The algebraic subsystems possess the same decay rate as the differential ones under our proposed condition. This allows us to control the transient process of differential and algebraic subsystems with a unified performance specification.
(3) The provision of an alternative Lyapunov-functional based approach to give exponential estimates for time-delay systems, and the approach could be extended to other types of dynamic systems in a straightforward manner.

Notation: Throughout this paper, for real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). I is the identity matrix with appropriate dimension, and the superscript "T" represents the transpose. $|\cdot|$ denotes the Euclidean norm for vectors and $\|\cdot\|$ denotes the spectral norm for matrices, while $\lambda_{\max}(M)$ ($\lambda_{\min}(M)$) denotes the maximal (minimal) eigenvalue of the real matrix M. The asterisk * is used to denote a matrix which will not be used in the development, and # is used to denote a matrix which can be inferred by symmetry. $\mathcal{C}([-\tau, 0], \mathbb{R}^n)$ denotes the family of continuous functions ϕ from $[-\tau, 0]$ to $\mathbb{R}^n$ with norm $|\phi|_\tau = \sup_{-\tau \leq s \leq 0} |\phi(s)|$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Preliminaries and problem formulation

Consider the following class of singular systems with discrete and distributed delays:

$$(\Sigma):\begin{cases} E\dot{x}(t) = A(t)x(t) + A_d(t)x(t-d) \\ \qquad + \displaystyle\int_{t-h}^{t} A_h(s)x(s)\mathrm{d}s + B(t)u(t) \\ x(s) = \phi(s), \quad s\epsilon[-\tau, 0], \quad \tau = \max\{\bar{d}, \bar{h}\}, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, and $A(t)$, $A_d(t)$, $A_h(t)$, $B(t)$ are the system matrices. The matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we shall assume that $\text{rank}E = r \le n$. The scalars $0 < d \le \bar{d}$ and $0 < h \le \bar{h}$ are the unknown constant discrete and distributed delays, respectively. The initial condition $\phi(t)$ is a vector-valued function satisfying $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Time-varying uncertainties may appear in the system matrices, that is,

$$\begin{aligned} A(t) &= A + \Delta A(t), \quad A_d(t) = A_d + \Delta A_d(t), \\ A_h(t) &= A_h + \Delta A_h(t), \quad B(t) = B + \Delta B(t), \end{aligned} \tag{2}$$

where A, A_d, A_h, B are constant matrices and $\Delta A(t)$, $\Delta A_d(t)$, $\Delta A_h(t)$, $\Delta B(t)$ are real-valued functions representing the time-varying parameter uncertainties. It is assumed that the uncertainties are norm-bounded and can be described as

$$\begin{aligned} &[\Delta A(t) \quad \Delta A_d(t) \quad \Delta A_h(t) \quad \Delta B(t)] \\ &\quad = DF(t)[H_1 \quad H_2 \quad H_3 \quad H_4], \end{aligned} \tag{3}$$

where D, H_1, H_2, H_3, H_4 are known constant matrices with compatible dimensions and $F(t)$ are unknown Lebesgue measurable matrix functions satisfying

$$F^T(t)F(t) \le I. \tag{4}$$

When $F(t) \equiv 0$, the system is referred to as a nominal system. Throughout the paper, we shall use the following definitions.

Definition 1:

(1) The pair (E, A) is said to be regular if $\det(sE - A)$ is not identically zero.
(2) The pair (E, A) is said to be impulse free if $\deg(\det(sE - A)) = \text{rank}\,E$.

Definition 2:

(1) The nominal system in (Σ) is said to be regular and impulse free if the pair (E, A) is regular and impulse free.
(2) The nominal system in (Σ) is said to be λ-exponentially admissible if, for all initial function $\phi \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the nominal system is regular, impulse free and exponentially stable with a decay rate which is not less than λ, that is,

$$|x(t)| \le \sigma e^{-\lambda t}|\phi|_\tau,$$

where λ and σ are called the decay rate and decay coefficient, respectively.
(3) The uncertain system (Σ) is said to be robustly λ-exponentially admissible if it is λ-exponentially admissible for all uncertainties $F(t) \le 1$.

For analysis, our goals are to establish a sufficient condition for robust λ-exponential admissibility and to give the estimates of λ and σ. As for synthesis, our goal is to design a static state-feedback controller

$$\left(\hat{G}_{\text{state}}\right): u(t) = Kx(t), \tag{5}$$

such that the resulting closed-loop system is robustly λ-exponential admissible. The following lemmas are essential for the proofs in the sequel.

Lemma 1: *Suppose that a positive continuous function $f(t)$ satisfies*

$$f(t) \le \zeta_1 \sup_{t-d\le s\le t} f(s) + \zeta_2 e^{-\lambda t}, \tag{6}$$

where

$$0 < \zeta_1 < 1, \quad \zeta_1 e^{\lambda d} < 1, \quad \lambda > 0, \quad \zeta_2 > 0,$$

then

$$f(t) \le \sup_{-d\le s\le 0} f(s)e^{-\lambda t} + \frac{\zeta_2}{1 - \zeta_1 e^{\lambda d}} e^{-\lambda t}.$$

Proof: See appendix. □

Lemma 2: *Suppose that $A(t) \in \mathbb{R}^{n\times n}$ is a matrix function with the form*

$$A(t) = A + DF(t)H,$$

where D and H are known constant matrices with compatible dimensions and $F(t)$ are unknown Lebesgue measurable matrix functions satisfying $F^T(t)F(t) \le I$. If there exist a matrix $P \in \mathbb{R}^{n\times n}$ and a scalar $\epsilon > 0$ such that

$$A^TP + P^TA + \epsilon H^TH + \frac{1}{\epsilon}P^TDD^TP < 0, \tag{7}$$

then

$$\|A^{-1}(t)\| \le \frac{2}{a}\|P\|,$$

where a is any scalar satisfying

$$0 < a < \left|\lambda_{\max}\left(A^TP + P^TA + \epsilon H^TH + \frac{1}{\epsilon}P^TDD^TP\right)\right|. \tag{8}$$

Proof: See appendix. □

Lemma 3 (Petersen 1987, Wang *et al.* 1992)**:** *Assume that H, D, E are real matrices with appropriate dimensions and $F(t)$ is a real matrix function satisfying $F^T(t)F(t) \le I$. Then,*

(1) $H + DF(t)E + (DF(t)E)^T < 0$ *holds if and only if there exists a scalar* $\epsilon > 0$ *satisfying* $H + \epsilon DD^T + \frac{1}{\epsilon}E^T E < 0$.
(2) *For* $\epsilon > 0$, *we have* $DF(t)E + (DF(t)E)^T \le \epsilon DD^T + \frac{1}{\epsilon}E^T E$.

Lemma 4: *For any real matrix $M > 0$, scalars $b > a$, vector function $x(\theta)$ integrable on $[a, b]$, we have*

$$(b-a)\int_a^b x^T(\theta)Mx(\theta)\,\mathrm{d}\theta \ge \left(\int_a^b x(\theta)\,\mathrm{d}\theta\right)^T M\left(\int_a^b x(\theta)\,\mathrm{d}\theta\right).$$

This lemma is a slightly modified version of Gu's inequality (Gu 2000). The proof can be conducted by following the same line as in Gu (2000), and is thus omitted here.

Remark 1: It should be pointed out that a lemma similar to Lemma 1 has appeared in Yue and Han (2004, 2005) and Yue and Lam (2005). However, the decay rate of $f(t)$ in their lemma is $\min\{\lambda, \xi\}$ where $0 < \xi < (-1/d)\ln \zeta_1$.

3. Delay-dependent exponential estimates

Theorem 1: *For prescribed scalars $\lambda > 0$, $\epsilon > 0$, if there exist matrices $Q_d > 0$, $Q_h > 0$, $Q_\tau > 0$, $Z > 0$, P, Y_1, Y_2, Y_3, Y_4 and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ such that the following LMIs hold*:

$$E^T P = P^T E \ge 0, \tag{9}$$

$$\begin{bmatrix} \Omega_{11} & \# & \# & \# & \# & \# & \# & \# \\ \Omega_{21} & \Omega_{22} & \# & \# & \# & \# & \# & \# \\ -Y_3^T E & Y_3^T E & -Q_\tau & \# & \# & \# & \# & \# \\ A_h^T P - Y_4^T E & Y_4^T E & 0 & -\frac{1}{\bar{h}}Q_h + \epsilon_2 H_3^T H_3 & \# & \# & \# & \# \\ Y_1 & Y_2 & Y_3 & Y_4 & -\frac{1}{\bar{d}}Z & \# & \# & \# \\ ZA & ZA_d & 0 & ZA_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1}Z & \# & \# \\ D^T P & 0 & 0 & 0 & 0 & D^T Z & -\epsilon_1 I & \# \\ D^T P & 0 & 0 & 0 & 0 & D^T Z & 0 & -\epsilon_2 I \end{bmatrix} < 0, \tag{10}$$

$$\bar{h}e^{2\lambda\bar{d}}Q_h - \epsilon Q_d < 0, \tag{11}$$

where

$$\begin{aligned}\Omega_{11} &= P^T A + A^T P + 2\lambda E^T P + (1+\epsilon)Q_d + \frac{e^{2\lambda\bar{h}}-1}{2\lambda}Q_h \\ &\quad + \left(\frac{e^{2\lambda\bar{h}} - 1 - 2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2}\right)Q_\tau \\ &\quad - E^T Y_1 - Y_1^T E + \epsilon_1 H_1^T H_1, \\ \Omega_{21} &= A_d^T P + E^T Y_1 - Y_2^T E + \epsilon_1 H_2^T H_1, \\ \Omega_{22} &= -e^{-2\lambda\bar{d}}Q_d + Y_2^T E + E^T Y_2 + \epsilon_1 H_2^T H_2,\end{aligned}$$

then uncertain system (Σ) *is robustly λ-exponentially admissible for any d and h satisfying $0 < d \le \bar{d}$, $0 < h \le \bar{h}$.*

Proof: The whole proof is divided into three parts. In the first part, regularity and non-implusiveness are proven. Exponential stability of the differential subsystem and the algebraic subsystem are shown in the second part and the third part, respectively.

(*Part 1*: Regularity and non-impulsiveness) Under the condition of the theorem, we shall first show that uncertain system (Σ) is regular and impulse free. It follows from (10), by Schur complement equivalence on the 7th row and column, that

$$\begin{aligned}&P^T A + A^T P + 2\lambda E^T P - E^T Y_1 - Y_1^T E + \epsilon_1 H_1^T H_1 \\ &\quad + \frac{1}{\epsilon_1}P^T DD^T P < 0.\end{aligned} \tag{12}$$

Now choose two non-singular matrices $M = [M_1^T \; M_2^T]^T$ and $N = [N_1 \; N_2]$ (Fridman 2002, Yue and Han 2004) such that

$$\begin{bmatrix} M_1 \\ M_2 \end{bmatrix} E \begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{13}$$

and write

$$MA(t)N = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \triangleq \bar{A}(t),$$
$$MA_d(t)N = \begin{bmatrix} A_{d1}(t) & A_{d2}(t) \\ A_{d3}(t) & A_{d4}(t) \end{bmatrix} \triangleq \bar{A}_d(t),$$
$$MA_h(t)N = \begin{bmatrix} A_{h1}(t) & A_{h2}(t) \\ A_{h3}(t) & A_{h4}(t) \end{bmatrix} \triangleq \bar{A}_h(t),$$
$$M^{-T}PN = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$
$$N^TQ_dN = \begin{bmatrix} Q_{d1} & Q_{d2} \\ Q_{d2}^T & Q_{d4} \end{bmatrix},$$
$$N^TQ_hN = \begin{bmatrix} Q_{h1} & Q_{h2} \\ Q_{h2}^T & Q_{h4} \end{bmatrix},$$
$$N^TQ_\tau N = \begin{bmatrix} Q_{\tau1} & Q_{\tau2} \\ Q_{\tau2}^T & Q_{\tau4} \end{bmatrix}, \tag{14}$$
$$M^{-T}Y_iN = \begin{bmatrix} Y_{1i} & Y_{2i} \\ Y_{3i} & Y_{4i} \end{bmatrix}, \quad i = 1, 2, 3,$$

where the partition is compatible with that of (13). Substituting the partition into (9) yields that

$$P_1 > 0, \quad P_2 = 0. \tag{15}$$

Pre- and post-multiplying (12) by N^T and N, respectively, and noticing (15), we obtain

$$\begin{bmatrix} * & * \\ * & \begin{matrix} A_4^TP_4 + P_4^TA_4 + \epsilon_1 N_2^TH_1^TH_1N_2 \\ + \dfrac{1}{\epsilon_1}N_2^TP^TDD^TPN_2 \end{matrix} \end{bmatrix} < 0. \tag{16}$$

From the partition in (14), we know that

$$P\begin{bmatrix} N_1 & N_2 \end{bmatrix} = \begin{bmatrix} M_1^T & M_2^T \end{bmatrix}\begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix},$$

which implies that

$$PN_2 = M_2^TP_4. \tag{17}$$

Combining (16) and (17) yields that

$$A_4^TP_4 + P_4^TA_4 + \epsilon_1 N_2^TH_1^TH_1N_2 + \frac{1}{\epsilon_1}P_4^TM_2DD^TM_2^TP_4 < 0.$$

Noticing

$$A_4(t) = A_4 + M_2DF(t)H_1N_2,$$

and using Lemma 2, we obtain that $A_4(t)$ is non-singular for all $t \geq 0$ and

$$\left\| A_4^{-1}(t) \right\| \leq \frac{2}{a}\|P_4\| \triangleq \delta_{p4}, \tag{18}$$

where

$$0 < a < \left| \lambda_{\max}\begin{pmatrix} A_4^TP_4 + P_4^TA_4 + \epsilon_1 N_2^TH_1^TH_1N_2 \\ + \dfrac{1}{\epsilon_1}P_4^TM_2DD^TM_2^TP_4 \end{pmatrix} \right|.$$

Therefore, uncertain system (Σ) is regular and impulse free.

(*Part 2*: Stability of the differential subsystem) By the transformation matrices M and N, uncertain system (Σ) can be re-written as

$$\begin{cases} \dot{\xi}_1(t) = A_1(t)\xi_1(t) + A_2(t)\xi_2(t) + A_{d1}(t)\xi_1(t-d) \\ \qquad + A_{d2}(t)\xi_2(t-d) + \displaystyle\int_{t-h}^{t} A_{h1}(s)\xi_1(s)\,\mathrm{d}s \\ \qquad + \displaystyle\int_{t-h}^{t} A_{h2}(s)\xi_2(s)\,\mathrm{d}s \\ 0 = A_3(t)\xi_1(t) + A_4(t)\xi_2(t) + A_{d3}(t)\xi_1(t-d) \\ \qquad + A_{d4}(t)\xi_2(t-d) + \displaystyle\int_{t-h}^{t} A_{h3}(s)\xi_1(s)\,\mathrm{d}s \\ \qquad + \displaystyle\int_{t-h}^{t} A_{h4}(s)\xi_2(s)\,\mathrm{d}s, \end{cases} \tag{19}$$

where

$$\begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} \triangleq \xi(t) = N^{-1}x(t).$$

Choose a Lyapunov–Krasovskii functional candidate as follows:

$$V(x_t, t) = e^{2\lambda t}V_1(x_t, t) + V_2(x_t, t) + V_3(x_t, t) + V_4(x_t, t) + V_5(x_t, t) + V_6(x_t, t) \tag{20}$$

where

$$x_t = x(t+m), \; -2d \leq m \leq 0,$$

and

$$V_1(x_t, t) = x^T(t)E^TPx(t),$$
$$V_2(x_t, t) = \int_{t-d}^{t} e^{2\lambda\alpha}x^T(\alpha)Q_dx(\alpha)\,\mathrm{d}\alpha,$$
$$V_3(x_t, t) = \int_{-h}^{0}\int_{t+\beta}^{t} e^{2\lambda(\alpha-\beta)}x^T(\alpha)Q_hx(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta,$$
$$V_4(x_t, t) = \int_{t-h}^{t} e^{2\lambda(s+h)}\left(\int_s^t x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_s^t x(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s,$$
$$V_5(x_t, t) = \int_0^h\int_{t-s}^{t} e^{2\lambda(\theta+2h)}(\theta - t + s)x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta\,\mathrm{d}s,$$
$$V_6(x_t, t) = \int_{-d}^{0}\int_{t+\beta}^{t} e^{2\lambda(\alpha-\beta)}\dot{x}^T(\alpha)E^TZE\dot{x}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta.$$

Then, we take the derivative of $V_i(x_t, t)$, $i = 1, 2, 3, 4, 5, 6$, along the trajectories of (Σ) for $t > d$

$$\dot{V}_1(x_t, t) = 2x^T(t)P^T\begin{bmatrix} A(t)x(t) + A_d(t)x(t-d) \\ + \int_{t-h}^{t} A_h(s)x(s)\,\mathrm{d}s \end{bmatrix} \tag{21}$$

$$\begin{aligned}\dot{V}_2(x_t, t) &= e^{2\lambda t}x^T(t)Q_d x(t) \\ &\quad - e^{2\lambda t}x^T(t-d)\big(e^{-2\lambda d}Q_d\big)x(t-d),\end{aligned} \tag{22}$$

$$\begin{aligned}\dot{V}_3(x_t, t) &= \int_{-h}^{0}\begin{bmatrix} e^{2\lambda(t-\beta)}x^T(t)Q_h x(t) \\ - e^{2\lambda t}x^T(t+\beta)Q_h x(t+\beta)\end{bmatrix}\mathrm{d}\beta \\ &= e^{2\lambda t}x^T(t)\left(\frac{e^{2\lambda h}-1}{2\lambda}Q_h\right)x(t) \\ &\quad - e^{2\lambda t}\int_{t-h}^{t} x^T(s)Q_h x(s)\,\mathrm{d}s,\end{aligned} \tag{23}$$

$$\begin{aligned}\dot{V}_4(x_t, t) &= -e^{2\lambda t}\left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_{t-h}^{t} x(\theta)\,\mathrm{d}\theta\right) \\ &\quad + 2\int_{t-h}^{t} e^{2\lambda(s+h)}x^T(t)Q_\tau\left(\int_{s}^{t} x(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\ &= -e^{2\lambda t}\left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_{t-h}^{t} x(\theta)\,\mathrm{d}\theta\right) \\ &\quad + \int_{t-h}^{t}\int_{t-h}^{\theta} 2e^{2\lambda(s+h)}x^T(t)Q_\tau x(\theta)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\le -e^{2\lambda t}\left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_{t-h}^{t} x(\theta)\,\mathrm{d}\theta\right) \\ &\quad + \int_{t-h}^{t}\int_{t-h}^{\theta} e^{2\lambda(s+h)}\begin{bmatrix} x^T(t)Q_\tau x(t) \\ + x^T(\theta)Q_\tau x(\theta)\end{bmatrix}\mathrm{d}s\,\mathrm{d}\theta \\ &= -e^{2\lambda t}\left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_{t-h}^{t} x(\theta)\,\mathrm{d}\theta\right) \\ &\quad + e^{2\lambda t}x^T(t)\left[\frac{e^{2\lambda h}-1-2\lambda h}{4\lambda^2}Q_\tau\right]x(t) \\ &\quad + \int_{t-h}^{t}\int_{t-h}^{\theta} e^{2\lambda(s+h)}x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}s\,\mathrm{d}\theta \\ &\le -e^{2\lambda t}\left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_{t-h}^{t} x(\theta)\,\mathrm{d}\theta\right) \\ &\quad + e^{2\lambda t}x^T(t)\left[\frac{e^{2\lambda h}-1-2\lambda h}{4\lambda^2}Q_\tau\right]x(t) \\ &\quad + \int_{t-h}^{t} e^{2\lambda(t+h)}(\theta - t + h)x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta,\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{V}_5(x_t, t) &= e^{2\lambda t}\int_{0}^{h} s x^T(t)\big(e^{4\lambda h}Q_\tau\big)x(t)\,\mathrm{d}s \\ &\quad - \int_{0}^{h}\int_{t-s}^{t} e^{2\lambda(\theta+2h)}x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta\,\mathrm{d}s \\ &= e^{2\lambda t}x^T(t)\left(\frac{h^2e^{4\lambda h}}{2}Q_\tau\right)x(t) \\ &\quad - \int_{t-h}^{t}\int_{t-\theta}^{h} e^{2\lambda(\theta+2h)}x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}s\,\mathrm{d}\theta \\ &= e^{2\lambda t}x^T(t)\left(\frac{h^2e^{4\lambda h}}{2}Q_\tau\right)x(t) \\ &\quad - \int_{t-h}^{t} e^{2\lambda(\theta+2h)}(\theta - t + h)x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta \\ &\le e^{2\lambda t}x^T(t)\left(\frac{h^2e^{4\lambda h}}{2}Q_\tau\right)x(t) \\ &\quad - \int_{t-h}^{t} e^{2\lambda(t+h)}(\theta - t + h)x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta,\end{aligned} \tag{25}$$

$$\begin{aligned}\dot{V}_6(x_t, t) &= \int_{-d}^{0}\begin{bmatrix} e^{2\lambda(t-\beta)}\dot{x}^T(t)E^TZE\dot{x}(t) \\ - e^{2\lambda t}\dot{x}^T(t+\beta)E^TZE\dot{x}(t+\beta)\end{bmatrix}\mathrm{d}\beta \\ &= e^{2\lambda t}\begin{bmatrix} A(t)x(t) + A_d(t)x(t-d) \\ + \int_{t-h}^{t} A_h(s)x(s)\,\mathrm{d}s\end{bmatrix}^T \\ &\quad \times \left(\frac{e^{2\lambda d}-1}{2\lambda}Z\right) \\ &\quad \times \begin{bmatrix} A(t)x(t) + A_d(t)x(t-d) \\ + \int_{t-h}^{t} A_h(s)x(s)\,\mathrm{d}s\end{bmatrix} \\ &\quad - e^{2\lambda t}\int_{t-d}^{t} \dot{x}^T(\alpha)E^TZE\dot{x}(\alpha)\,\mathrm{d}\alpha \\ &\le e^{2\lambda t}[A(t)x(t) + A_d(t)x(t-d)]^T\left(\frac{e^{2\lambda d}-1}{2\lambda}Z\right) \\ &\quad \times [A(t)x(t) + A_d(t)x(t-d)] \\ &\quad + 2e^{2\lambda t}[A(t)x(t) + A_d(t)x(t-d)]^T \\ &\quad \times \left(\frac{e^{2\lambda d}-1}{2\lambda}Z\right)\int_{t-h}^{t} A_h(s)x(s)\,\mathrm{d}s \\ &\quad + e^{2\lambda t}h\int_{t-h}^{t} x^T(s)A_h^T(s)\left(\frac{e^{2\lambda d}-1}{2\lambda}Z\right)A_h(s)x(s)\,\mathrm{d}s \\ &\quad - e^{2\lambda t}\int_{t-d}^{t} \dot{x}^T(\alpha)E^TZE\dot{x}(\alpha)\,\mathrm{d}\alpha,\end{aligned} \tag{26}$$

where Lemma 4 is used in $\dot{V}_6(x_t, t)$. It follows from the Newton–Leibniz formula that

$$2e^{2\lambda t}\begin{bmatrix} x^T(t)Y_1^T + x^T(t-d)Y_2^T + \left(\int_{t-h}^{t} x^T(\theta)\,\mathrm{d}\theta\right)Y_3^T \\ + \int_{t-h}^{t} x^T(s)Y_4^T\,\mathrm{d}s \end{bmatrix} \times \left[\int_{t-d}^{t} E\dot{x}(\alpha)\,\mathrm{d}\alpha - Ex(t) + Ex(t-d)\right] = 0. \quad (27)$$

From (21)–(27), the derivative of $V(x_t,\ t)$ for $t > d$ evaluated along the solution of (Σ) is obtained as

$$\begin{aligned}\dot{V}(x_t,t) &= e^{2\lambda t}\dot{V}_1(x_t,t) + 2\lambda e^{2\lambda t}V_1(x_t,t) + \dot{V}_2(x_t,t) \\ &\quad + \dot{V}_3(x_t,t) + \dot{V}_4(x_t,t) + \dot{V}_5(x_t,t) + \dot{V}_6(x_t,t) \\ &\quad + 2e^{2\lambda t}\begin{bmatrix} x^T(t)Y_1^T + x^T(t-d)Y_2^T \\ + \left(\int_{t-h}^{t} x^T(\theta)\mathrm{d}\theta\right)Y_3^T \\ + \int_{t-h}^{t} x^T(s)Y_4^T\,\mathrm{d}s \end{bmatrix} \\ &\quad \times \left[\int_{t-d}^{t} E\dot{x}(\alpha)\mathrm{d}\alpha - Ex(t) + Ex(t-d)\right] \\ &\le \frac{e^{2\lambda t}}{hd}\int_{t-h}^{t}\int_{t-d}^{t}\begin{pmatrix} \zeta^T(t,\alpha,s)\Delta(h,d)\Lambda(t,s,h,d) \\ \times\,\Delta(h,d)\zeta(t,\alpha,s)\end{pmatrix}\mathrm{d}\alpha\,\mathrm{d}s,\end{aligned}$$

where

Since $\Lambda(t,s,h,d) \le \Lambda(t,s,\bar{h},\bar{d})$, it follows from (10), by using Schur complement equivalence on the last two rows and columns and applying Lemma 3, that $\Lambda(t,\ s,\ h,\ d) < 0$. Hence,

$$\dot{V}(x_t,t) < 0, \quad t > d. \quad (28)$$

Since $A_4(t)$ is non-singular for all $t \ge 0$, we know from (19) that

$$\begin{aligned}\xi_2(t) = &-A_4^{-1}(t)A_3(t)\xi_1(t) - A_4^{-1}(t)A_{d3}(t)\xi_1(t-d) \\ &- A_4^{-1}(t)A_{d4}(t)\xi_2(t-d) \\ &- A_4^{-1}(t)\int_{t-h}^{t} A_{h3}(s)\xi_1(s)\,\mathrm{d}s \\ &- A_4^{-1}(t)\int_{t-h}^{t} A_{h4}(s)\xi_2(s)\,\mathrm{d}s. \quad (29)\end{aligned}$$

By (29) and integrating the first equation of (19), $\xi(t)$ can be re-written as

$$\begin{aligned}\xi(t) = &\bar{I}\xi(0) + \hat{A}(t)\xi(t) + \bar{A}_4(t)\bar{A}_d(t)\xi(t-d) \\ &+ \bar{A}_4(t)\int_{t-h}^{t} \bar{A}_h(s)\xi(s)\,\mathrm{d}s \\ &+ \int_0^t \begin{bmatrix} \bar{I}\bar{A}(\alpha)\xi(\alpha) + \bar{I}\bar{A}_d(\alpha)\xi(\alpha-d) \\ + \int_{\alpha-h}^{\alpha} \bar{I}\bar{A}_h(s)\xi(s)\,\mathrm{d}s \end{bmatrix}\mathrm{d}\alpha, \quad (30)\end{aligned}$$

$$\zeta(t,\alpha,s) = \begin{bmatrix} x^T(t) & x^T(t-d) & \int_{t-h}^{t} x^T(\theta)\mathrm{d}\theta & x(s) & \dot{x}^T(\alpha)E^T \end{bmatrix}^T,$$

$$\Delta(h,d) = \mathrm{diag}(I, I, I, hI, dI),$$

$$\Lambda(t,s,h,d) = \begin{bmatrix} \Phi(t,h,d) & P^TA_d(t) + Y_1^TE - E^TY_2 & -E^TY_3 & P^TA_h(s) - E^TY_4 & Y_1^T \\ A_d^T(t)P + E^TY_1 - Y_2^TE & -e^{-2\lambda d}Q_d + Y_2^TE + E^TY_2 & E^TY_3 & E^TY_4 & Y_2^T \\ -Y_3^TE & Y_3^TE & -Q_\tau & 0 & Y_3^T \\ A_h^T(s)P - Y_4^TE & Y_4^TE & 0 & -\frac{1}{h}Q_h & Y_4^T \\ Y_1 & Y_2 & Y_3 & Y_4 & -\frac{1}{d}Z \end{bmatrix}$$

$$+ \frac{e^{2\lambda d}-1}{2\lambda}\begin{bmatrix} A^T(t)Z \\ A_d^T(t)Z \\ 0 \\ A_h^T(s)Z \\ 0 \end{bmatrix} Z^{-1} \begin{bmatrix} A^T(t)Z \\ A_d^T(t)Z \\ 0 \\ A_h^T(s)Z \\ 0 \end{bmatrix}^T,$$

$$\begin{aligned}\Phi(t,h,d) = &P^TA(t) + A^T(t)P + 2\lambda E^TP + Q_d + \frac{e^{2\lambda h}-1}{2\lambda}Q_h + \left(\frac{e^{2\lambda h}-1-2\lambda h}{4\lambda^2} + \frac{h^2e^{4\lambda h}}{2}\right)Q_\tau \\ &- E^TY_1 - Y_1^TE.\end{aligned}$$

where $\bar{A}(\cdot)$, $\bar{A}_d(\cdot)$ and $\bar{A}_h(\cdot)$ are defined in (14) and

$$\bar{I} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$
$$\hat{A}(t) = \begin{bmatrix} 0 & 0 \\ -A_4^{-1}(t)A_3(t) & 0 \end{bmatrix},$$
$$\bar{A}_4(t) = \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix}.$$

Noting that

$$\left(I - \hat{A}(t)\right)^{-1} = \begin{bmatrix} I & 0 \\ -A_4^{-1}(t)A_3(t) & I \end{bmatrix} \triangleq \tilde{A}(t),$$

thus (30) can be transformed to

$$\begin{aligned} \xi(t) &= \tilde{A}(t)\bar{I}\xi(0) + \tilde{A}(t)\bar{A}_4(t)\bar{A}_d(t)\xi(t-d) \\ &\quad + \tilde{A}(t)\bar{A}_4(t)\int_{t-h}^{t}\bar{A}_h(s)\xi(s)\,\mathrm{d}s \\ &\quad + \tilde{A}(t)\int_0^t \begin{bmatrix} \bar{I}\bar{A}(\alpha)\xi(\alpha) + \bar{I}\bar{A}_d(\alpha)\xi(\alpha-d) \\ + \int_{\alpha-h}^{\alpha}\bar{I}\bar{A}_h(s)\xi(s)\,\mathrm{d}s \end{bmatrix}\mathrm{d}\alpha. \end{aligned} \tag{31}$$

From (3), (13), (14), and (18), the following relationship can be obtained immediately.

$$\begin{aligned} \|A_3(t)\| &\le \|M_2AN_1\| + \|M_2D\|\,\|H_1N_1\| \triangleq \delta_3, \\ \|\bar{A}(t)\| &\le \|MAN\| + \|MD\|\,\|H_1\ N\| \triangleq \delta, \\ \|\bar{A}_d(t)\| &\le \|MA_dN\| + \|MD\|\,\|H_2\ N\| \triangleq \delta_d, \\ \|\bar{A}_h(t)\| &\le \|MA_hN\| + \|MD\|\,\|H_3\ N\| \triangleq \delta_h, \\ \|\bar{A}_4(t)\| &\le \delta_{p4}, \\ \|\tilde{A}(t)\| &= \left\| \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -A_4^{-1}(t)A_3(t) & 0 \end{bmatrix} \right\| \\ &\le 1 + \delta_{p4}\delta_3 \triangleq \rho. \end{aligned}$$

Hence, for any $t \ge 0$,

$$\begin{aligned} |\xi(t)| &\le \rho|\xi(0)| + \rho\delta_{p4}\delta_d|\xi(t-d)| + \rho\delta_{p4}\delta_h\int_{t-h}^{t}|\xi(s)|\,\mathrm{d}s \\ &\quad + \rho\delta\int_0^t|\xi(\alpha)|\,\mathrm{d}\alpha + \rho\delta_d\int_0^t|\xi(\alpha-d)|\,\mathrm{d}\alpha \\ &\quad + \rho\delta_h\int_0^t\int_{\alpha-h}^{\alpha}|\xi(s)|\,\mathrm{d}s\,\mathrm{d}\alpha \\ &\le \rho|\xi(0)| + \rho\delta_{p4}\delta_d|\xi(t-d)| \\ &\quad + \rho\delta_{p4}\delta_h\int_{-h}^{0}|\xi(s)|\,\mathrm{d}s + \rho(\delta_{p4}\delta_h + \delta) \\ &\quad \times \int_0^t|\xi(\alpha)|\,\mathrm{d}\alpha + \rho\delta_d\int_0^t|\xi(\alpha-d)|\,\mathrm{d}\alpha \\ &\quad + \rho\delta_h\int_{-h}^{0}\int_0^t|\xi(s)|\,\mathrm{d}\alpha\,\mathrm{d}s + \rho\delta_h\int_0^t\int_s^t|\xi(s)|\,\mathrm{d}\alpha\,\mathrm{d}s. \end{aligned}$$

When $0 < t \le d$, the above inequality implies that

$$\begin{aligned} |\xi(t)| &\le \rho\|N^{-1}\|\,|\phi|_\tau + \rho\delta_{p4}\delta_d\|N^{-1}\|\,|\phi|_\tau + \rho\delta_{p4}\delta_h h\|N^{-1}\|\,|\phi|_\tau \\ &\quad + \rho(\delta_{p4}\delta_h + \delta)\int_0^t|\xi(\alpha)|\,\mathrm{d}\alpha + \rho\delta_d d\|N^{-1}\|\,|\phi|_\tau \\ &\quad + \rho\delta_h dh\|N^{-1}\|\,|\phi|_\tau + \rho\delta_h\int_0^t(d-s)|\xi(s)|\,\mathrm{d}s \\ &= \rho\|N^{-1}\|\big[1 + \big(\delta_{p4} + d\big)(\delta_d + \delta_h h)\big]|\phi|_\tau \\ &\quad + \int_0^t\rho\big(\delta_{p4}\delta_h + \delta + \delta_h d - \delta_h s\big)|\xi(s)|\,\mathrm{d}s, \end{aligned}$$

where $\|N^{-1}\|$ comes from the change of the system state. Applying Gronwall–Bellman Lemma to this inequality yields that, when $0 < t \le d$,

$$|\xi(t)| \le \rho\|N^{-1}\|\big[1 + \big(\delta_{p4} + d\big)(\delta_d + \delta_h h)\big]e^{\rho d(\delta_{p4}\delta_h + \delta + \delta_h d/2)}|\phi|_\tau,$$

which infers, when $0 < t \le d$,

$$\begin{aligned} |x(t)| &\le \rho\|N\|\,\|N^{-1}\|\big[1 + \big(\delta_{p4} + d\big)(\delta_d + \delta_h h)\big]e^{\rho d(\delta_{p4}\delta_h + \delta + \delta_h d/2)} \\ &\quad \times |\phi|_\tau \triangleq \gamma(d,h)|\phi|_\tau. \end{aligned}$$

Noting $\gamma(d, h) > 1$, we obtain that, when $-\tau \le t \le d$,

$$|x(t)| \le \max\{1, \gamma(d,h)\}|\phi|_\tau = \gamma(d,h)|\phi|_\tau. \tag{32}$$

It is easy to show that

$$\begin{aligned} \|A(t)\| &\le \|A\| + \|D\|\,\|H_1\| \triangleq \eta, \\ \|A_d(t)\| &\le \|A_d\| + \|D\|\,\|H_2\| \triangleq \eta_d, \\ \|A_h(s)\| &\le \|A_h\| + \|D\|\,\|H_3\| \triangleq \eta_h. \end{aligned}$$

thus, by (28) and (32), we obtain that, when $t > d$,

$$\begin{aligned} V(x_t,t) &\le V(x_d,d) \\ &= e^{2\lambda d}x^T(d)E^TPx(d) + \int_0^d e^{2\lambda\alpha}x^T(\alpha)Q_dx(\alpha)\,\mathrm{d}\alpha \\ &\quad + \int_{-h}^{0}\int_{d+\beta}^{d} e^{2\lambda(\alpha-\beta)}x^T(\alpha)Q_hx(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &\quad + \int_{d-h}^{d} e^{2\lambda(s+h)}\left(\int_s^d x^T(\theta)\,\mathrm{d}\theta\right)Q_\tau\left(\int_s^d x(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s \\ &\quad + \int_0^h\int_{d-s}^{d} e^{2\lambda(\theta+2h)}(\theta-d+s)x^T(\theta)Q_\tau x(\theta)\,\mathrm{d}\theta\,\mathrm{d}s \\ &\quad + \int_{-d}^{0}\int_{d+\beta}^{d} e^{2\lambda(\alpha-\beta)}\dot{x}^T(\alpha)E^TZE\dot{x}(\alpha)\,\mathrm{d}\alpha\,\mathrm{d}\beta \\ &\le e^{2\lambda d}\|E^TP\|\gamma^2(d,h)|\phi|_\tau^2 + \frac{e^{2\lambda d}-1}{2\lambda}\|Q_d\|\gamma^2(d,h)|\phi|_\tau^2 \\ &\quad + e^{2\lambda d}\frac{e^{2\lambda h}-1-2\lambda h}{4\lambda^2}\|Q_h\|\gamma^2(d,h)|\phi|_\tau^2 \\ &\quad + \frac{e^{2\lambda d}\big(e^{2\lambda h}-2\lambda^2h^2-2\lambda h-1\big)}{4\lambda^3}\|Q_\tau\|\gamma^2(d,h)|\phi|_\tau^2 \end{aligned}$$

$$+\frac{e^{2\lambda(d+2h)}\left(1-e^{-2\lambda h}-2\lambda h+2\lambda^2h^2\right)}{(2\lambda)^3}\|Q_\tau\|\gamma^2(d,h)|\phi|_\tau^2$$
$$+3e^{2\lambda d}\left(\eta^2+\eta_d^2+h^2\eta_h^2\right)\frac{e^{2\lambda d}-1-2\lambda d}{4\lambda^2}\|Z\|\gamma^2(d,h)|\phi|_\tau^2.$$

On the other hand,

$$\begin{aligned}V(x_t,t)&\geq e^{2\lambda t}x^T(t)E^TPx(t)\\&=e^{2\lambda t}x^T(t)N^{-T}(N^TE^TM^T)(M^{-T}PN)N^{-1}x(t)\\&\geq\frac{1}{\|P_1^{-1}\|}e^{2\lambda t}|\xi_1(t)|^2.\end{aligned}$$

Therefore, when $t>d$,

$$\begin{aligned}|\xi_1(t)|&\leq\sqrt{\begin{pmatrix}k_7[k_1(h,d)+k_2(h,d)+k_3(h,d)\\+k_4(h,d)+k_5(h,d)+k_6(h,d)]\end{pmatrix}}e^{-\lambda t}|\phi|_\tau\\&\triangleq K(h,d)e^{-\lambda t}|\phi|_\tau,\end{aligned}\tag{33}$$

where

$$\begin{aligned}k_1(h,d)&=e^{2\lambda d}\|E^TP\|\gamma^2(d,h),\\k_2(h,d)&=\frac{e^{2\lambda d}-1}{2\lambda}\|Q_d\|\gamma^2(d,h),\\k_3(h,d)&=e^{2\lambda d}\frac{e^{2\lambda h}-1-2\lambda h}{4\lambda^2}\|Q_h\|\gamma^2(d,h),\\k_4(h,d)&=\frac{e^{2\lambda d}\left(e^{2\lambda h}-2\lambda^2h^2-2\lambda h-1\right)}{4\lambda^3}\|Q_\tau\|\gamma^2(d,h),\\k_5(h,d)&=\frac{e^{2\lambda(d+2h)}\left(1-e^{-2\lambda h}-2\lambda h+2\lambda^2h^2\right)}{(2\lambda)^3}\|Q_\tau\|\gamma^2(d,h),\\k_6(h,d)&=3e^{2\lambda d}\left(\eta^2+\eta_d^2+h^2\eta_h^2\right)\frac{e^{2\lambda d}-1-2\lambda d}{4\lambda^2}\|Z\|\gamma^2(d,h),\\k_7&=\|P_1^{-1}\|.\end{aligned}\tag{34}$$

When $0<t\leq d$, we also have

$$\begin{aligned}\frac{1}{\|P_1^{-1}\|}e^{2\lambda t}|\xi_1(t)|^2&\leq e^{2\lambda t}x^T(t)E^TPx(t)\\&\leq e^{2\lambda d}\|E^TP\|\gamma^2(d,h)|\phi|_\tau^2.\end{aligned}$$

That is, when $0<t\leq d$,

$$|\xi_1(t)|\leq\sqrt{k_7k_1(h,d)}e^{-\lambda t}|\phi|_\tau.\tag{35}$$

It follows from (33) and (35) that, when $t>0$,

$$|\xi_1(t)|\leq\max\left\{\sqrt{k_7k_1(d)},K(h,d)\right\}e^{-\lambda t}|\phi|_\tau=K(h,d)e^{-\lambda t}|\phi|_\tau.\tag{36}$$

Therefore, the differential subsystem of (Σ) is exponentially stable with a decay rate which is not less than λ. The remaining task is to show the exponential stability of the algebraic subsystem.

(*Part 3*: Stability of the algebraic subsystem) Define an auxiliary variable $q(t)$ as follows:

$$q(t)=A_3(t)\xi_1(t)+A_{d3}(t)\xi_1(t-d)+\int_{t-h}^{t}A_{h3}(s)\xi_1(s)\,\mathrm{d}s.\tag{37}$$

From (14), we know that

$$\begin{aligned}\left\|A_{d3}(t)\right\|&\leq\|M_2A_dN_1\|+\|M_2D\|\,\|H_2N_1\|\triangleq\delta_{d3},\\\left\|A_{h3}(s)\right\|&\leq\|M_2A_hN_1\|+\|M_2D\|\,\|H_3N_1\|\triangleq\delta_{h3}.\end{aligned}$$

It follows from (36) that, when $t\geq-\tau$,

$$|\xi_1(t)|\leq\max\left\{\|N^{-1}\|,K(h,d)\right\}e^{-\lambda t}|\phi|_\tau.$$

Hence, when $t>0$,

$$|q(t)|^2\leq r(h,d)e^{-2\lambda t}|\phi|_\tau^2,$$

where

$$r(h,d)=3(\max\{\|N^{-1}\|,\ K(h,d)\})^2\times\left(\delta_3^2+e^{2\lambda d}\delta_{d3}^2+e^{2\lambda h}h^2\delta_{h3}^2\right).$$

To show the exponential stability of the algebraic subsystem, construct a function

$$J(t)=\xi_2^T(t)Q_{d4}\xi_2(t)-\xi_2^T(t-d)\left(e^{-2\lambda d}Q_{d4}\right)\xi_2(t-d)-\int_{t-h}^{t}\xi_2^T(s)Q_{h4}\xi_2(s)\,\mathrm{d}s.\tag{38}$$

By pre-multiplying the second equation of (19) with $\xi_2^T(t)P_4^T$, we obtain that

$$0=\xi_2^T(t)P_4^TA_4(t)\xi_2(t)+\xi_2^T(t)P_4^TA_{d4}(t)\xi_2(t-d)+\xi_2^T(t)P_4^T\int_{t-h}^{t}A_{h4}(s)\xi_2(s)\,\mathrm{d}s+\xi_2^T(t)P_4^Tq(t).\tag{39}$$

Adding (39) and its transpose to (38) yields that

$$\begin{aligned}J(t)=&\ \xi_2^T(t)\left[P_4^TA_4(t)+A_4^T(t)P_4+Q_{d4}\right]\xi_2(t)\\&+2\xi_2^T(t)P_4^TA_{d4}(t)\xi_2(t-d)\\&-\xi_2^T(t-d)(e^{-2\lambda d}Q_{d4})\xi_2(t-d)\\&+2\xi_2^T(t)P_4^T\int_{t-h}^{t}A_{h4}(s)\xi_2(s)\,\mathrm{d}s\\&-\int_{t-h}^{t}\xi_2^T(s)Q_{h4}\xi_2(s)\,\mathrm{d}s+2\xi_2^T(t)P_4^Tq(t).\end{aligned}$$

Applying Lemma 3 to this yields that, for some $\alpha > 0$,

$$
\begin{aligned}
J(t) &\le \xi_2^T(t)\left[P_4^T A_4(t) + A_4^T(t)P_4 + Q_{d4}\right]\xi_2(t) \\
&\quad + 2\xi_2^T(t)P_4^T A_{d4}(t)\xi_2(t-d) \\
&\quad - \xi_2^T(t-d)\left(e^{-2\lambda d}Q_{d4}\right)\xi_2(t-d) \\
&\quad + \int_{t-h}^{t} \xi_2^T(t)P_4^T A_{h4}(s)Q_{h4}^{-1}A_{h4}^T(s)P_4\xi_2(t)\,\mathrm{d}s \\
&\quad + \alpha\xi_2^T(t)\xi_2(t) + \frac{1}{\alpha}q^T(t)P_4P_4^T q(t) \\
&= \frac{1}{h}\int_{t-h}^{t}\begin{bmatrix}\xi_2^T(t) & \xi_2^T(t-d)\end{bmatrix}\Theta(t,s,h,d)\begin{bmatrix}\xi_2(t) \\ \xi_2(t-d)\end{bmatrix}\mathrm{d}s \\
&\quad + \alpha\xi_2^T(t)\xi_2(t) + \frac{1}{\alpha}q^T(t)P_4P_4^T q(t), \qquad (40)
\end{aligned}
$$

where

$$
\Theta(t,s,h,d) = \begin{bmatrix} P_4^T A_4(t) + A_4^T(t)P_4 + Q_{d4} + hP_4^T A_{h4}(s)Q_{h4}^{-1}A_{h4}^T(s)P_4 & P_4^T A_{d4}(t) \\ A_{d4}^T(t)P_4 & -e^{-2\lambda d}Q_{d4} \end{bmatrix}. \qquad (41)
$$

It follows from (10), by using Schur complement equivalence on the last two rows and columns and applying Lemma 3, that

$$
\begin{bmatrix}
\bar{\Omega}_{11} & \bar{\Omega}_{21}^T & * & P^T A_h(s) - E^T Y_4 \\
\bar{\Omega}_{21} & \bar{\Omega}_{22} & * & * \\
* & * & * & * \\
A_h^T(s)P - Y_4^T E & * & * & -\bar{h}^{-1}Q_h
\end{bmatrix} < 0, \qquad (42)
$$

where

$$
\begin{aligned}
\bar{\Omega}_{11} &= P^T A(t) + A^T(t)P + 2\lambda E^T P + (1+\epsilon)Q_d + \frac{e^{2\lambda\bar{h}} - 1}{2\lambda}Q_h \\
&\quad + \left(\frac{e^{2\lambda\bar{h}} - 1 - 2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2}\right)Q_\tau \\
&\quad - E^T Y_1 - Y_1^T E, \\
\bar{\Omega}_{21} &= A_d^T(t)P + E^T Y_1 - Y_2^T E, \\
\bar{\Omega}_{22} &= -e^{-2\lambda\bar{d}}Q_d + Y_2^T E + E^T Y_2.
\end{aligned}
$$

From (14) and (15), (42) can be re-written as

$$
\begin{bmatrix}
* & * & * & * & * & * \\
* & \begin{pmatrix} P_4^T A_4(t) + A_4^T(t)P_4 \\ +(1+\epsilon)Q_{d4} \\ +\Psi(\bar{h},\bar{d}) \end{pmatrix} & * & P_4^T A_{d4}(t) & * & P_4^T A_{h4}(s) \\
* & * & * & * & * & * \\
* & A_{d4}^T(t)P_4 & * & -e^{-2\lambda\bar{d}}Q_{d4} & * & * \\
* & * & * & * & * & * \\
* & A_{h4}^T(s)P_4 & * & * & * & -\bar{h}^{-1}Q_{h4}
\end{bmatrix} < 0, \qquad (43)
$$

where

$$
\Psi(\bar{h},\bar{d}) = \frac{e^{2\lambda\bar{h}} - 1}{2\lambda}Q_{h4} + \left(\frac{e^{2\lambda\bar{h}} - 1 - 2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2}\right)Q_{\tau 4}.
$$

By Schur complement equivalence, (43) implies that

$$
\begin{bmatrix}
P_4^T A_4(t) + A_4^T(t)P_4 + Q_{d4} + \bar{h}P_4^T A_{h4}(s)Q_{h4}^{-1}A_{h4}^T(s)P_4 + \Psi(\bar{h},\bar{d}) + \epsilon Q_{d4} & P_4^T A_{d4}(t) \\
A_{d4}^T(t)P_4 & -e^{-2\lambda\bar{d}}Q_{d4}
\end{bmatrix} < 0. \qquad (44)
$$

Noting that $\Theta(t,s,h,d)$ is a monotonic increasing function with respect to h and d, respectively, we obtain from (41) and (44) that

$$
\Theta(t,s,h,d) \le \Theta\left(t,s,\bar{h},\bar{d}\right) < \mathrm{diag}\left(-\Psi\left(\bar{h},\bar{d}\right) - \epsilon Q_{d4}, 0\right). \qquad (45)
$$

Therefore, from (40) and (45), we have that

$$
\begin{aligned}
&\xi_2^T(t)Q_{d4}\xi_2(t) - \xi_2^T(t-d)\left(e^{-2\lambda d}Q_{d4}\right)\xi_2(t-d) \\
&\quad - \int_{t-h}^{t}\xi_2^T(s)Q_{h4}\xi_2(s)\,\mathrm{d}s \\
&\quad = J(t) \\
&\quad \le \xi_2^T(t)\left[-\Psi\left(\bar{h},\bar{d}\right) - \epsilon Q_{d4} + \alpha I\right]\xi_2(t) \\
&\qquad + \frac{1}{\alpha}q^T(t)P_4P_4^T q(t),
\end{aligned}
$$

which can be re-written as

$$
\begin{aligned}
&e^{2\lambda d}\xi_2^T(t)\left[(1+\epsilon)Q_{d4} + \Psi\left(\bar{h},\bar{d}\right) - \alpha I\right]\xi_2(t) \\
&\quad \le \xi_2^T(t-d)Q_{d4}\xi_2(t-d) \\
&\qquad + e^{2\lambda d}\int_{t-h}^{t}\xi_2^T(s)Q_{h4}\xi_2(s)\,\mathrm{d}s + \frac{e^{2\lambda d}}{\alpha}q^T(t)P_4P_4^T q(t). \qquad (46)
\end{aligned}
$$

On one hand, we can choose a small enough α such that

$$
(1+\epsilon)Q_{d4} + \Psi\left(\bar{h},\bar{d}\right) - \alpha I \ge (1+\epsilon+\nu)Q_{d4}, \qquad (47)
$$

where $\nu > 0$ is a small enough scalar satisfying

$$
\Psi\left(\bar{h},\bar{d}\right) > \nu Q_{d4}.
$$

On the other hand, we obtain from (11) that

$$
e^{2\lambda d}hQ_h \le e^{2\lambda\bar{d}}\bar{h}Q_h < \epsilon Q_d,
$$

which implies that

$$
e^{2\lambda d}hQ_h - \epsilon Q_d < 0.
$$

Pre- and post-multiplying this inequality by N^T and N yields that

$$\begin{bmatrix} * & * \\ * & e^{2\lambda d} h Q_{h4} - \epsilon Q_{d4} \end{bmatrix} < 0,$$

which infers

$$Q_{h4} < \frac{\epsilon}{e^{2\lambda d} h} Q_{d4}. \tag{48}$$

It follows from (46), (47) and (48) that

$$\begin{aligned} \xi_2^T(t) Q_{d4} \xi_2(t) \leq{}& \frac{1+\epsilon}{e^{2\lambda d}(1+\epsilon+\nu)} \sup_{t-\tau \leq s \leq t} \left\{ \xi_2^T(s) Q_{d4} \xi_2(s) \right\} \\ &+ \frac{1}{\alpha(1+\epsilon+\nu)} q^T(t) P_4 P_4^T q(t). \end{aligned} \tag{49}$$

Define

$$\begin{aligned} f(t) &= \xi_2^T(t) Q_{d4} \xi_2(t), \\ \zeta_1 &= \frac{1+\epsilon}{e^{2\lambda d}(1+\epsilon+\nu)}, \\ \zeta_2 &= \frac{r(h,d) \| P_4 \|^2}{\alpha(1+\epsilon+\nu)} |\phi|_\tau^2, \end{aligned}$$

then, (49) infers

$$f(t) \leq \zeta_1 \sup_{t-\tau \leq s \leq t} f(s) + \zeta_2 e^{-2\lambda t}.$$

Applying Lemma 1 to the above inequality yields that

$$f(t) \leq \sup_{-\tau \leq s \leq 0} f(s) e^{-2\lambda t} + \frac{\zeta_2}{1 - \zeta_1 e^{2\lambda d}} e^{-2\lambda t},$$

which means that

$$\begin{aligned} |\xi_2(t)|^2 \leq{}& \| Q_{d4}^{-1} \| \, \| Q_{d4} \| e^{-2\lambda t} |\phi|_\tau^2 \\ &+ \alpha^{-1} \nu^{-1} \| Q_{d4}^{-1} \| r(h,d) \| P_4 \|^2 e^{-2\lambda t} |\phi|_\tau^2. \end{aligned} \tag{50}$$

Combining (36) and (50) yields that

$$\begin{aligned} |x(t)|^2 &= \xi^T(t) N^T N \xi(t) \\ &\leq \|N\|^2 \left(|\xi_1(t)|^2 + |\xi_2(t)|^2 \right) \\ &\leq \sigma^2(h,d) e^{-2\lambda t} |\phi|_\tau^2, \end{aligned}$$

where

$$\begin{cases} \sigma(h,d) = \|N\| \sqrt{\begin{pmatrix} K^2(h,d) + \| Q_{d4}^{-1} \| \, \| Q_{d4} \| \\ \quad + \alpha^{-1} \nu^{-1} \| Q_{d4}^{-1} \| r(h,d) \| P_4 \|^2 \end{pmatrix}}, \\ \Psi(\bar{h}, \bar{d}) - \alpha I \geq \nu Q_{d4}. \end{cases} \tag{51}$$

Noting that $\sigma(h, d)$ is a monotonic increasing function with respect to h and d, we finally have that

$$|x(t)| \leq \sigma(\bar{h}, \bar{d}) e^{-\lambda t} |\phi|_\tau.$$

This completes the proof. □

From the proof of Theorem 1, we can easily obtain the following corollary.

Corollary 1: *For prescribed scalars $\lambda > 0$, $\epsilon > 0$, if there exist matrices $Q_d > 0$, $Q_h > 0$, $Q_\tau > 0$, $Z > 0$, P, Y_1, Y_2, Y_3, Y_4 and scalars $\epsilon_1 > 0$, $\epsilon_2 > 0$ such that* (9), (10), *and* (11) *hold, then the estimate of decay coefficient for uncertain system* (Σ) *is given by $\sigma(\bar{h}, \bar{d})$, where $\sigma(\cdot,\cdot)$ is defined in* (51).

Remark 2: It should be stressed that the approach employed here is different from that used in Mondié and Kharitonov (2005) and Kharitonov *et al.* (2005) although their results are also expressed in LMIs. Our approach could be extended to other types of dynamic systems in an easy way. More importantly, distributed delay and algebraic constraints in system states are taken into account in our condition, and thus the result can be applied to a more general class of systems. As for Xu *et al.* (2006), the derived condition based on exponential scaling is not monotonic with respect to the decay rate. This may bring difficulties into optimizing the decay rate by feedback controllers. Additionally, positive scalars a and ν in Corollary 1 can be selected freely as long they satisfy $\Psi(\bar{h}, \bar{d}) - \alpha I \geq \nu Q_{d4}$.

Remark 3: In the derivation of Theorem 1, Y_1, Y_2, Y_3, Y_4 are introduced to reduce the conservatism. If the left side of (10) is re-written as

$$[*] + Y,$$

where

$$Y = \begin{bmatrix} -E^T Y_1 - Y_1^T E & \# & \# & \# & \# & \# \\ E^T Y_1 - Y_2^T E & Y_2^T E + E^T Y_2 & \# & \# & \# & \# \\ -Y_3^T E & Y_3^T E & 0 & \# & \# & \# \\ -Y_4^T E & Y_4^T E & 0 & 0 & \# & \# \\ Y_1 & Y_2 & Y_3 & Y_4 & 0 & \# \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then it can be easily verified that Y is an indefinite matrix. Therefore, the matrix variables do not affect the derivative of $V(x_t, t)$, but will provide more flexibility and freedom in LMI condition (10). Additionally, compared with the delay-independent results in Mondié and Kharitonov (2005) and Kharitonov *et al.* (2005), Theorem 1 is a delay-dependent condition with slack matrix variables, which may be less conservative and thus more desirable.

Remark 4: We improve the technique to show the exponential stability of singular systems with time delay, and the algebraic subsystems possess the same decay rate as the differential ones under the proposed condition. This allows us to control the transient process of the differential and the algebraic subsystems with a unified performance specification.

Remark 5: One may notice that equality constraint (9), which may lead to numerical problems when checking the LMIs, appears in Theorem 1. This computational difficulty can be overcome by employing a transformation on the matrix variable P (Xu *et al.* 2002, Yue and Han 2004).

Remark 6: Based on the Lyapunov–Krasovskii functional and the corresponding technique used in the proof of Theorem 1, it is easy to extend the result to singular systems with time-varying delay or multiple delays. For the case with time-varying delay, if

$$0 < d(t) \le d, \quad \dot{d}(t) \le \tau_d < 1,$$
$$0 < h(t) \le h, \quad \dot{h}(t) \le \tau_h < 1,$$

we only need to modify $V_2(x_t, t)$ and $V_4(x_t, t)$ to

$$V_2(x_t, t) = \int_{t-d(t)}^{t} e^{2\lambda\alpha} x^T(\alpha) Q_d x(\alpha)\,\mathrm{d}\alpha,$$

$$V_4(x_t, t) = \int_{t-h(t)}^{t} e^{2\lambda(s+h)} \left(\int_s^t x^T(\theta)\,\mathrm{d}\theta\right) Q_\tau \left(\int_s^t x(\theta)\,\mathrm{d}\theta\right) \mathrm{d}s.$$

Then following the same line as in the proof of Theorem 1, we can obtain similar results. For the case with multiple delays, it can be treated by adding the corresponding functionals for different delays.

In the sequel, we consider three special cases. The proofs of all the subsequent corollaries follow easily from Theorem 1, and are therefore omitted.

For the singular case with only discrete delay,

$$E\dot{x}(t) = (A + DF(t)H_1)x(t) + (A_d + DF(t)H_2)x(t-d), \tag{52}$$

we have the following result.

Corollary 2: *For a prescribed scalar* $\lambda > 0$, *if there exist matrices* $P > 0$, $Q_d > 0$, $Z > 0$, Y_1, Y_2 *and scalars* $\epsilon > 0$ *such that the following LMIs hold*:

$$E^T P = P^T E \ge 0,$$

$$\begin{bmatrix} \Omega_{11} & \# & \# & \# & \# \\ \Omega_{21} & \Omega_{22} & \# & \# & \# \\ Y_1 & Y_2 & -\frac{1}{\bar{d}} Z & \# & \# \\ ZA & ZA_d & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1} Z & \# \\ D^T P & 0 & 0 & D^T Z & -\epsilon I \end{bmatrix} < 0, \tag{53}$$

where

$$\Omega_{11} = P^T A + A^T P + 2\lambda E^T P + Q_d - E^T Y_1 - Y_1^T E + \epsilon H_1^T H_1,$$
$$\Omega_{21} = A_d^T P + E^T Y_1 - Y_2^T E + \epsilon H_2^T H_1,$$
$$\Omega_{22} = -e^{-2\lambda\bar{d}} Q_d + Y_2^T E + E^T Y_2 + \epsilon H_2^T H_2,$$

then uncertain system (52) *is robustly* λ*-exponentially admissible for any* d *satisfying* $0 < d \le \bar{d}$.

For the state-space case with discrete and distributed delays,

$$\dot{x}(t) = (A + DF(t)H_1)x(t) + (A_d + DF(t)H_2)x(t-d) + \int_{t-h}^{t} (A_h + DF(t)H_3)x(s)\,\mathrm{d}s, \tag{54}$$

the following result can be obtained from Theorem 1 and its proof. When $E = I$, Part 3 of the proof of Theorem 1 becomes redundant, and this leads to the removal of condition (11).

Corollary 3: *For a prescribed scalar* $\lambda > 0$, *if there exist matrices* $Q_d > 0$, $Q_h > 0$, $Q_\tau > 0$, $Z > 0$, $P > 0$, Y_1, Y_2, Y_3, Y_4 *and scalars* $\epsilon_1 > 0$, $\epsilon_2 > 0$ *such that the following LMI holds*:

$$\begin{bmatrix} \Omega_{11} & \# & \# & \# & \# & \# & \# & \# \\ \Omega_{21} & \Omega_{22} & \# & \# & \# & \# & \# & \# \\ -Y_3^T & Y_3^T & -Q_\tau & \# & \# & \# & \# & \# \\ A_h^T P - Y_4^T & Y_4^T & 0 & -\frac{1}{h} Q_h + \epsilon_2 H_3^T H_3 & \# & \# & \# & \# \\ Y_1 & Y_2 & Y_3 & Y_4 & -\frac{1}{\bar{d}} Z & \# & \# & \# \\ ZA & ZA_d & 0 & ZA_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1} Z & \# & \# \\ D^T P & 0 & 0 & 0 & 0 & D^T Z & -\epsilon_1 I & \# \\ D^T P & 0 & 0 & 0 & 0 & D^T Z & 0 & -\epsilon_2 I \end{bmatrix} < 0, \tag{55}$$

where

$$\begin{aligned}\Omega_{11} &= P^T A + A^T P + 2\lambda P + Q_d + \frac{e^{2\lambda\bar{h}} - 1}{2\lambda} Q_h \\ &\quad + \left(\frac{e^{2\lambda\bar{h}} - 1 - 2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2} \right) Q_\tau \\ &\quad - Y_1 - Y_1^T + \epsilon_1 H_1^T H_1, \\ \Omega_{21} &= A_d^T P + Y_1 - Y_2^T + \epsilon_1 H_2^T H_1, \\ \Omega_{22} &= -e^{-2\lambda\bar{d}} Q_d + Y_2^T + Y_2 + \epsilon_1 H_2^T H_2,\end{aligned}$$

then uncertain system (54) *is robustly λ-exponentially stable for any d and h satisfying* $0 < d \le \bar{d}$, $0 < h \le \bar{h}$.

For the state-space case with only discrete delay,

$$\dot{x}(t) = (A + DF(t)H_1)x(t) + (A_d + DF(t)H_2)x(t-d), \tag{56}$$

we similarly have the following result.

Corollary 4: *For a prescribed scalar $\lambda > 0$, if there exist matrices $Q_d > 0$, $Z > 0$, $P > 0$, Y_1, Y_2 and scalars $\epsilon > 0$ such that the following LMI holds*:

$$\begin{bmatrix} \Omega_{11} & \# & \# & \# & \# \\ \Omega_{21} & \Omega_{22} & \# & \# & \# \\ Y_1 & Y_2 & -\frac{1}{\bar{d}} & \# & \# \\ ZA & ZA_d & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}} - 1} Z & \# \\ D^T P & 0 & 0 & D^T Z & -\epsilon I \end{bmatrix} < 0, \tag{57}$$

where

$$\begin{aligned}\Omega_{11} &= P^T A + A^T P + 2\lambda P + Q_d - Y_1 - Y_1^T + \epsilon H_1^T H_1, \\ \Omega_{21} &= A_d^T P + Y_1 - Y_2^T + \epsilon H_2^T H_1, \\ \Omega_{22} &= -e^{-2\lambda\bar{d}} Q_d + Y_2^T + Y_2 + \epsilon H_2^T H_2,\end{aligned}$$

then uncertain system (56) *is robustly λ-exponentially stable for any d satisfying* $0 < d \le \bar{d}$.

Remark 7: Unlike other exponential stability conditions, the lower bound of the decay rate in Theorem 1 and Corollaries 2–4 is a free value equal to a prescribed constant λ which can be selected according to different practical conditions. This will introduce more flexibility in analysis and design of systems. It should be emphasized that the left side of (53), (55) and (57) are monotonic increasing functions with respect to λ. This allows us to compute the maximal lower bound of the decay rate by convex optimization algorithms.

4. Robust stabilization with decay rate constraint

In this section, we shall turn to consider the problem of robust stabilization for singular systems with discrete and distributed delays. When a control law in (5) is applied to system (1), the closed-loop system becomes

$$\begin{aligned}E\dot{x}(t) &= \left(\hat{A} + DF(t)\hat{H}_1\right)x(t) + (A_d + DF(t)H_2)x(t-d) \\ &\quad + \int_{t-h}^{t} (A_h + DF(t)H_3)x(s)\,\mathrm{d}s,\end{aligned} \tag{58}$$

where $\hat{A} = A + BK$ and $\hat{H}_1 = H_1 + H_4 K$. The designed closed-loop system is not only robustly exponentially admissible, but also decreases with a decay rate which is not less than a prescribed lower bound. This specification is very useful in practical applications since the transient process of a dynamic system can be controlled more accurately once the decay rate is determined.

Theorem 2: *For a prescribed $\lambda > 0$ and given scalars $\mu > 0$, a_i, $i = 1, 2, 3, 4$, if there exist matrices $R_d > 0$, $R_h > 0$, $R_\tau > 0$, $S > 0$, L, X and scalars $\mu_1 > 0$, $\mu_2 > 0$ such that the following LMIs hold*:

$$EX = (EX)^T \ge 0, \tag{59}$$

$$\begin{bmatrix} \bar{\Omega}_{11} & \# & \# & \# & \# & \# & \# & \# & \# & \# & \# \\ \bar{\Omega}_{21} & \bar{\Omega}_{22} & \# & \# & \# & \# & \# & \# & \# & \# & \# \\ -a_3 EX & a_3 ER_d & -R_\tau & \# & \# & \# & \# & \# & \# & \# & \# \\ R_h A_h^T - a_4 EX & a_4 ER_d & 0 & -\frac{1}{\bar{h}} R_h & \# & \# & \# & \# & \# & \# & \# \\ a_1\ S & a_2\ S & a_3\ S & a_4\ S & -\frac{1}{\bar{d}} S & \# & \# & \# & \# & \# & \# \\ AX & A_d R_d & 0 & A_h R_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}} - 1} S & \# & \# & \# & \# & \# \\ \mu_1 D^T & 0 & 0 & 0 & 0 & \mu_1 D^T & -\mu_1 I & \# & \# & \# & \# \\ \mu_2 D^T & 0 & 0 & 0 & 0 & \mu_2 D^T & 0 & -\mu_2 I & \# & \# & \# \\ H_1 X + H_4 L & H_2 R_d & 0 & 0 & 0 & 0 & 0 & 0 & -\mu_1 I & \# & \# \\ 0 & 0 & 0 & H_3 R_h & 0 & 0 & 0 & 0 & 0 & -\mu_2 I & \# \\ \Xi^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\Gamma \end{bmatrix} < 0, \tag{60}$$

$$R_h - \mu\bar{h}e^{2\lambda\bar{d}} R_d > 0, \tag{61}$$

where

$$\begin{aligned}\bar{\Omega}_{11} &= (AX+BL)+(AX+BL)^T+2\lambda X^TE^T\\&\quad - a_1X^TE^T - a_1EX,\\ \bar{\Omega}_{21} &= R_dA_d^T + a_1R_dE^T - a_2EX,\\ \bar{\Omega}_{22} &= -e^{-2\lambda\bar{d}}R_d + a_2ER_d + a_2R_dE^T,\\ \Xi &= \left[X^T, X^T, X^T, X^T, X^T\right],\\ \Gamma &= \mathrm{diag}\left(R_d, \mu R_d, \frac{2\lambda}{e^{2\lambda\bar{h}}-1}R_h,\right.\\ &\qquad\left.\frac{4\lambda^2}{e^{2\lambda\bar{h}}-1-2\lambda\bar{h}}R_\tau, \frac{2}{\bar{h}^2e^{4\lambda\bar{h}}}R_\tau\right),\end{aligned}$$

then a state feedback control law $u(t)=LX^{-1}x(t)$ *exists such that uncertain closed-loop system* (58) *is robustly* λ*-exponentially admissible for any* d *and* h *satisfying* $0<d\le\bar{d}$, $0<h\le\bar{h}$.

Proof: According to Theorem 1, the closed-loop system is robustly λ-exponentially admissible if, for a given scalar $\epsilon>0$, there exist matrices $Q_d>0$, $Q_h>0$, $Q_\tau>0$, $Z>0$, P, Y_1, Y_2, Y_3, Y_4 and scalars $\epsilon_1>0$, $\epsilon_2>0$ such that the following LMIs hold:

$$E^TP = P^TE \ge 0, \tag{62}$$

$$\begin{bmatrix} \hat{\Omega}_{11} & \# & \# & \# & \# & \# & \# & \# \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} & \# & \# & \# & \# & \# & \# \\ -Y_3^TE & Y_3^TE & -Q_\tau & \# & \# & \# & \# & \# \\ A_h^TP - Y_4^TE & Y_4^TE & 0 & -\frac{1}{\bar{h}}Q_h + \epsilon_2H_3^TH_3 & \# & \# & \# & \# \\ Y_1 & Y_2 & Y_3 & Y_4 & -\frac{1}{\bar{d}}Z & \# & \# & \# \\ ZA & ZA_d & 0 & ZA_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1}Z & \# & \# \\ D^TP & 0 & 0 & 0 & 0 & D^TZ & -\epsilon_1I & \# \\ D^TP & 0 & 0 & 0 & 0 & D^TZ & 0 & -\epsilon_2I \end{bmatrix} < 0, \tag{63}$$

$$\bar{h}e^{2\lambda\bar{d}}Q_h - \epsilon Q_d < 0, \tag{64}$$

where

$$\begin{aligned}\hat{\Omega}_{11} &= P^T\hat{A} + \hat{A}^TP + 2\lambda E^TP + (1+\epsilon)Q_d + \frac{e^{2\lambda\bar{h}}-1}{2\lambda}Q_h\\ &\quad + \left(\frac{e^{2\lambda\bar{h}}-1-2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2e^{4\lambda\bar{h}}}{2}\right)Q_\tau\\ &\quad - E^TY_1 - Y_1^TE + \epsilon_1\hat{H}_1^T\hat{H}_1,\\ \hat{\Omega}_{21} &= A_d^TP + E^TY_1 - Y_2^TE + \epsilon_1H_2^T\hat{H}_1,\\ \hat{\Omega}_{22} &= -e^{-2\lambda d}Q_d + Y_2^TE + E^TY_2 + \epsilon_1H_2^TH_2.\end{aligned}$$

It follows from (63) that

$$\begin{bmatrix} \hat{\Omega}_{11} & \# \\ \hat{\Omega}_{21} & \hat{\Omega}_{22} \end{bmatrix} < 0.$$

Pre- and post-multiplying the above inequality by [I, I] and its transpose yields that

$$\begin{aligned}&P^T\left(\hat{A}+A_d\right) + \left(\hat{A}^T + A_d^T\right)P + 2\lambda E^TP\\ &\quad + \left(1+\epsilon-e^{-2\lambda\bar{d}}\right)Q_d + \frac{e^{2\lambda\bar{h}}-1}{2\lambda}Q_h\\ &\quad + \left(\frac{e^{2\lambda\bar{h}}-1-2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2e^{4\lambda\bar{h}}}{2}\right)Q_\tau\\ &\quad + \epsilon_1\left(\hat{H}_1^T\hat{H}_1 + H_2^TH_2 + H_2^T\hat{H}_1 + \hat{H}_1^TH_2\right) < 0,\end{aligned}$$

which implies that P is invertible. Now, let $X=P^{-1}$, $R_d=Q_d^{-1}$, $R_h=Q_h^{-1}$, $R_\tau=Q_\tau^{-1}$, $S=Z^{-1}$. Pre- and post-multiplying (63) by diag(X^T, R_d, R_τ, R_h, S, S, I, I) and its transpose, respectively, we obtain that (63) is equivalent to

$$\begin{bmatrix} \breve{\Omega}_{11} & \# & \# & \# & \# & \# & \# & \# \\ \breve{\Omega}_{21} & \breve{\Omega}_{22} & \# & \# & \# & \# & \# & \# \\ -R_\tau Y_3^TEX & R_\tau Y_3^TER_d & -R_\tau & \# & \# & \# & \# & \# \\ R_hA_h^T - R_hY_4^TEX & R_hY_4^TER_d & 0 & -\frac{1}{\bar{h}}R_h + \epsilon_2R_hH_3^TH_3R_h & \# & \# & \# & \# \\ SY_1X & SY_2R_d & SY_3R_\tau & SY_4R_h & -\frac{1}{\bar{d}}S & \# & \# & \# \\ AX & A_dR_d & 0 & A_hR_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1}S & \# & \# \\ D^T & 0 & 0 & 0 & 0 & D^T & -\epsilon_1I & \# \\ D^T & 0 & 0 & 0 & 0 & D^T & 0 & -\epsilon_2I \end{bmatrix} < 0 \tag{65}$$

where

$$\breve{\Omega}_{11} = \hat{A}X + X^T\hat{A}^T + 2\lambda X^T E^T + (1+\epsilon)X^T Q_d X + \frac{e^{2\lambda\bar{h}}-1}{2\lambda}X^T Q_h X + \left(\frac{e^{2\lambda\bar{h}}-1-2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2}\right)X^T Q_\tau X - X^T E^T Y_1 X - X^T Y_1^T E X + \epsilon_1 X^T \hat{H}_1^T \hat{H}_1 X,$$

$$\breve{\Omega}_{21} = R_d A_d^T + R_d E^T Y_1 X - R_d Y_2^T E X + \epsilon_1 R_d H_2^T \hat{H}_1 X,$$

$$\breve{\Omega}_{22} = -e^{-2\lambda\bar{d}}R_d + R_d Y_2^T E R_d + R_d E^T Y_2 R_d + \epsilon_1 R_d H_2^T H_2 R_d.$$

Now let

$$Y_1 = a_1 P, \quad Y_2 = a_2 Q_d, \quad Y_3 = a_3 Q_\tau, \quad Y_4 = a_4 Q_h,$$

where a_i, i = 1, 2, 3, 4, are scalars. Thus, (65) holds if the following inequality holds:

$$\begin{bmatrix} \tilde{\Omega}_{11} & \# & \# & \# & \# & \# & \# & \# \\ \tilde{\Omega}_{21} & \tilde{\Omega}_{22} & \# & \# & \# & \# & \# & \# \\ -a_3 E X & a_3 E R_d & -R_\tau & \# & \# & \# & \# & \# \\ R_h A_h^T - a_4 E X & a_4 E R_d & 0 & -\frac{1}{\bar{h}}R_h + \epsilon_2 R_h H_3^T H_3 R_h & \# & \# & \# & \# \\ a_1 S & a_2 S & a_3 S & a_4 S & -\frac{1}{\bar{d}}S & \# & \# & \# \\ AX & A_d R_d & 0 & A_h R_h & 0 & -\frac{2\lambda}{e^{2\lambda\bar{d}}-1}S & \# & \# \\ D^T & 0 & 0 & 0 & 0 & D^T & -\epsilon_1 I & \# \\ D^T & 0 & 0 & 0 & 0 & D^T & 0 & -\epsilon_2 I \end{bmatrix} < 0, \qquad (66)$$

where

$$\tilde{\Omega}_{11} = \hat{A}X + X^T\hat{A}^T + 2\lambda X^T E^T + (1+\epsilon)X^T Q_d X + \frac{e^{2\lambda\bar{h}}-1}{2\lambda}X^T Q_h X + \left(\frac{e^{2\lambda\bar{h}}-1-2\lambda\bar{h}}{4\lambda^2} + \frac{\bar{h}^2 e^{4\lambda\bar{h}}}{2}\right)X^T Q_\tau X - a_1 X^T E^T - a_1 E X + \epsilon_1 X^T \hat{H}_1^T \hat{H}_1 X,$$

$$\tilde{\Omega}_{21} = R_d A_d^T + a_1 R_d E^T - a_2 E X + \epsilon_1 R_d H_2^T \hat{H}_1 X,$$

$$\tilde{\Omega}_{22} = -e^{-2\lambda\bar{d}}R_d + a_2 E R_d + a_2 R_d E^T + \epsilon_1 R_d H_2^T H_2 R_d,$$

By using Schur complement equivalence and letting $L = KX$, $\mu = 1/\epsilon$, $\mu_1 = 1/\epsilon_1$, $\mu_2 = 1/\epsilon_2$, (66) is equivalent to (60). In addition, it is easy to verify that conditions (62) and (64) are equivalent to (59) and (61). This completes the proof. □

Remark 8: Theorem 2 provides a sufficient condition for the existence of a robust stabilizing controller. A desired controller gain matrix for (1) can be constructed through the solutions of LMIs, which can be solved efficiently by effective algorithms such as the interior-point method (Boyd *et al.* 1994). It is worth mentioning that (63) is expressed in terms of the system matrices of (1). This means that the design procedure involves no system decomposition, which otherwise may give rise to some certain numerical problems, and thus making the design procedure reliable.

Remark 9: Scalars μ, a_1, a_2, a_3, a_4 in Theorem 2 are tuning parameters which need to be specified first. In fact, (60) and (61), for fixed $\lambda > 0$; are bilinear matrix inequalities (BMIs) regarding to these tuning parameters. If one can accept more intensive computation burden, better results can be obtained by directly solving the BMIs, which can be implemented by resorting to some existing algorithms (Goh *et al.* 1994).

5. Numerical examples

In this section, several examples are provided to show the effectiveness of the proposed conditions.

Example 1: Consider a free system in (1) with the following system matrices:

$$E = \begin{bmatrix} 1 & 0.2 \\ 0.5 & 0.1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0.5 \\ -0.8 & -1 \end{bmatrix},$$

$$A_d = \begin{bmatrix} -0.25 & 0 \\ 0 & -0.2 \end{bmatrix}, \quad A_h = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 & 0.05 \end{bmatrix},$$

$$H_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0.05 & 0.08 \end{bmatrix}.$$

It is assumed that the system has maximal time delays $\bar{d}=0.6$ and $\bar{h}=0.7$. For $\lambda=0.7$ and $\epsilon=1.95$, (9), (10) and (11) are feasible with the following solutions:

$$\epsilon_1 = 2.1521\times 10^4, \quad \epsilon_2 = 2.1543\times 10^4,$$
$$P=\begin{bmatrix}4.4714 & -1.0567\\ 2.6454 & 4.4310\end{bmatrix}\times 10^4,$$
$$Z=\begin{bmatrix}7.4336 & 0.1525\\ 0.1525 & 6.6972\end{bmatrix}\times 10^3,$$
$$Q_d=\begin{bmatrix}0.5977 & 0.0746\\ 0.0746 & 1.4230\end{bmatrix}\times 10^4,$$
$$Q_h=\begin{bmatrix}3.8042 & 0.7442\\ 0.7442 & 8.2454\end{bmatrix}\times 10^3,$$
$$Q_\tau=\begin{bmatrix}4.3215 & 1.0922\\ 1.0922 & 6.5460\end{bmatrix}\times 10^3.$$

Then an estimate σ can be computed by Corollary 1. Hence, the system is 0.7-exponentially admissible for any d and h satisfying $0 < d \le 0.6$, $0 < h \le 0.7$, and the solutions of the system satisfy

$$|x(t)| \le 74.3988 e^{-0.7t}|\phi|_\tau.$$

Example 2: In the existing literature (Fridman 2002, Fridman and Shaked 2002), to show stability of the algebraic subsystem, the following norm upper bound assumption is extensively used:

$$\left\|A_4^{-1}A_{d4}\right\| + \int_{-h}^{0}\left\|A_4^{-1}A_{h4}\right\|\,\mathrm{d}s < 1, \tag{67}$$

where A_4, A_{d4} and A_{h4} are defined in (14). To determine whether the assumption is satisfied, it is necessary to decompose the system matrices, which may lead to numerical problems and the complexity of the approach. More importantly, this assumption may introduce much restriction and conservatism. Consider a free nominal system in (1) with the following system matrices:

$$E=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\qquad A=\begin{bmatrix}-1.2&0.8&0.5\\0&-1&0\\0&0&-1\end{bmatrix}$$
$$A_d=\begin{bmatrix}0.1&-0.1&0.5\\0&-0.2&0.5\\0&0&-0.5\end{bmatrix},\qquad A_h=\begin{bmatrix}0&0.2&-0.2\\0&0.5&0\\0&0.5&-0.3\end{bmatrix},$$
$$d=0.2,\qquad h=0.5.$$

It is easily shown that assumption (67) is not satisfied, and the approach in Fridman (2002) and Fridman and Shaked (2002) fails to obtain any conclusion on the stability of the algebraic subsystem. However, by Theorem 1, we can obtain that, for $\epsilon=0.4$, the system is 0.3-exponentially admissible and the solutions of the system satisfy

$$|x(t)| \le 9.1600 e^{-0.3t}|\phi|_\tau.$$

Example 3: Consider a nominal system in (52) with the following system matrices:

$$E=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix},\qquad A=\begin{bmatrix}-0.8&0.4&0.5\\0&-6&0.4\\0.2&0&-0.6\end{bmatrix},$$
$$A_d=\begin{bmatrix}0.1&0.2&0.4\\0&-0.5&0.5\\0&-0.8&-0.5\end{bmatrix}.$$

Table 1 lists the comparison of the maximum allowed $\bar{d}$ by different methods. It is obvious that our condition gives better results.

Example 4: Consider a nominal system in (56) with the following system matrices:

$$A=\begin{bmatrix}-2.8&-2\\0.9&-0.2\end{bmatrix},\qquad A_d=\begin{bmatrix}0.6&0.2\\0.2&-0.1\end{bmatrix}$$

The upper bounds of the decay rate $\bar{\lambda}$ by Liu (2003) and Mondié and Kharitonov (2005), and Corollary 4 are given in table 2, from which it is obvious that Corollary 4 provides a larger upper bound than other two methods.

Table 1. Comparison of $\bar{d}$ in Example 3.

Methods	Maximum allowed $\bar{d}$
Xu *et al.* (2002)	Fail
Fridman (2002)	0.8572
Zhu *et al.* (2005)	1.5931
Corollary 2 with $\lambda=0.001$	1.6402

Table 2. Comparison of decay rates in Example 4.

$\bar{d}$	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
λ by Mondié *et al.* (2005)	0.6102	0.5638	0.5227	0.4863	0.4540	0.4253	0.3996	0.3766
λ by Liu (2003)	1.0524	0.8382	0.6326	0.4484	0.2906	0.1602	0.0552	0.0000
λ by Corollary 4	1.0880	0.9468	0.8281	0.7293	0.6469	0.5778	0.5196	0.4701

Example 5: Consider a forced system in (1) with maximal delays $\bar{d} = 0.5$ and $\bar{h} = 0.8$. The system matrices are given as follows.

$$E = \begin{bmatrix} 1 & 0.5 \\ 0.4 & 0.2 \end{bmatrix}, \quad A = \begin{bmatrix} 1.2 & 1.5 \\ -0.5 & 1 \end{bmatrix},$$
$$A_d = \begin{bmatrix} -0.1 & 0.15 \\ 0.1 & 0.2 \end{bmatrix}, \quad A_h = \begin{bmatrix} -0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$
$$B = \begin{bmatrix} 1.2 \\ 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.1 & 0.08 \end{bmatrix},$$
$$H_2 = \begin{bmatrix} 0.12 & 0.08 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \quad H_4 = [0.1].$$

It is aimed at designing a stabilizing controller such that the closed-loop system is robustly 1.2-exponentially admissible. To this end, we choose

$$\mu = 0.85, \quad a_1 = 0.3, \quad a_2 = -0.5, \quad a_3 = 0.3, \quad a_4 = 0.05.$$

Then (59), (60) and (61) are feasible with the following solutions:

$$X = \begin{bmatrix} 3.3701 & 2.6972 \\ 0.5818 & -2.4656 \end{bmatrix} \times 10^3,$$

$$L = \begin{bmatrix} -3.5378 & -1.1040 \end{bmatrix} \times 10^4.$$

Therefore, the gain matrix of a stabilizing controller can be obtained as

$$K = \begin{bmatrix} -9.4803 & -5.8933 \end{bmatrix}.$$

The obtained controller not only stabilizes the original system, but also guarantees that the closed-loop system is exponentially stable with a decay rate no less than 1.2.

6. Conclusion

Exponential estimates and a sufficient condition on the robust exponential admissibility of singular systems with discrete and distributed delays are established in terms of a new Lyapunov–Krasovskii functional and the LMI technique. The estimates of the decay rate λ and the coefficient σ are obtained by solving a set of LMIs. Based on this, a design approach of stabilizing controller with decay rate constraint is proposed. The obtained controller not only stabilizes the original system but also guarantees the closed-loop system possesses a prescribed lower bound of the decay rate. In terms of the new Lyapunov–Krasovskii functional and the corresponding technique, it is not difficult to extend the results to singular systems with time-varying delay or multiple delays. Numerical examples show the effectiveness of the theoretical results.

Acknowledgement

This work was partially supported by RGC HKU 7031/06P.

Appendix

Proof of Lemma 1: We first prove that for any $\epsilon_0 > 0$,

$$f(t) < \sup_{-d \le s \le 0} f(s) e^{-\lambda t} < \frac{\zeta_2 e^{-\lambda t}}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0, \quad t \ge 0. \tag{68}$$

From the conditions of Lemma 1, we know that

$$f(0) \le \zeta_1 \sup_{-d \le s \le 0} f(s) + \zeta_2 < \sup_{-d \le s \le 0} f(s) + \frac{\zeta_2}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0.$$

If (68) does not hold, then these exists a $\bar{t}$ such that

$$f(\bar{t}) = \sup_{-d \le s \le 0} f(s) e^{-\lambda \bar{t}} + \frac{\zeta_2 e^{-\lambda \bar{t}}}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0, \tag{69}$$

and

$$f(t) < \sup_{-d \le s \le 0} f(s) e^{-\lambda t} + \frac{\zeta_2 e^{-\lambda t}}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0, \quad t < \bar{t}. \tag{70}$$

When $t \in [-d, 0]$, we have

$$f(t) \le \sup_{-d \le s \le 0} f(s) < \sup_{-d \le s \le 0} f(s) e^{-\lambda t} + \frac{\zeta_2 e^{-\lambda t}}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0.$$

Therefore, (70) holds for any $t \in [-d, \bar{t}]$. However, from (6) and (70), we can obtain that

$$\begin{aligned} f(\bar{t}) &\le \zeta_1 \sup_{\bar{t} - d \le s \le \bar{t}} f(s) + \zeta_2 e^{-\lambda \bar{t}} \\ &< \zeta_1 e^{\lambda d} \sup_{-d \le s \le 0} f(s) e^{-\lambda \bar{t}} + \zeta_1 e^{\lambda d} \frac{\zeta_2 e^{-\lambda \bar{t}}}{1 - \zeta_1 e^{\lambda d}} + \zeta_1 \epsilon_0 + \zeta_2 e^{-\lambda \bar{t}} \\ &< \sup_{-d \le s \le 0} f(s) e^{-\lambda \bar{t}} + \frac{\zeta_2 e^{-\lambda \bar{t}}}{1 - \zeta_1 e^{\lambda d}} + \epsilon_0, \end{aligned}$$

which contradicts (69). Hence (68) holds. By taking $\epsilon_0 \to 0$, we obtain the result. □

Proof of Lemma 2.: It follows from (7) and (8) that

$$A^T P + P^T A + \epsilon H^T H + \frac{1}{\epsilon} P^T D D^T P + aI < 0,$$

which, by Lemma 3, is equivalent to

$$A^T(t) P + P^T A(t) + aI < 0. \tag{71}$$

By (7) and Lemma 3, It is easy to show that $A^{-1}(t)$ exists for all t. Hence, we can let $y \in \mathbb{R}^n$ be given and $x = A^{-1}(t) y$, then it follows from (71) that

$$2x^T A^T(t) P x \le -a|x|^2,$$

which is equivalent to

$$2y^T PA^{-1}(t)y \leq -a|A^{-1}(t)y|^2.$$

By this, we can obtain that

$$|A^{-1}(t)y|^2 \leq -\frac{2}{a}y^T PA^{-1}(t)y \leq \frac{2}{a}|P^T y|\,|A^{-1}(t)y|,$$

which infers to

$$|A^{-1}(t)y|\left(|A^{-1}(t)y| - \frac{2}{a}|P^T y|\right) \leq 0.$$

Hence,

$$y^T A^{-T}(t)A^{-1}(t)y = |A^{-1}(t)y|^2 \leq \frac{4}{a^2}|P^T y|^2 = y^T\left(\frac{4}{a^2}PP^T\right)y.$$

Since y can take any value,

$$\|A^{-1}(t)\| \leq \frac{2}{a}\|P\|.$$

This completes the proof. □

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