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Criteria for exponential dichotomy for triangular systems

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We study the exponential dichotomy properties of nonautonomous systems of linear differential equations. Any such system is kinematically similar to a triangular system. Since it is easy to determine whether or not a diagonal system has an exponential dichotomy, it is important to study the relation between the exponential dichotomy properties of the triangular system and its diagonal part. Without loss of generality, we consider block upper triangular systems and study the relation with their diagonal parts. Our study addresses both bounded and unbounded systems, on the whole line and a half line. We conclude with a study of the spectral properties of bounded systems, focusing on the relation between the whole line spectrum and the half line spectra.

1. Introduction

In this article we study the exponential dichotomy properties of nonautonomous systems of linear differential equations. It is well known that any such system can be transformed by a kinematic similarity into an upper triangular system. Since exponential dichotomy is preserved by kinematic similarity, this means that there is no loss of generality in studying this notion purely in the context of upper triangular systems. Indeed Dieci and Van Vleck [3] study Sacker–Sell spectra in this context. It turns out that the diagonal entries play a key role. So it is important to study the relation between the properties of a triangular system and properties of its diagonal part.

Exponential dichotomy is essentially the same as hyperbolicity. The main difference is that, at least originally, exponential dichotomy pertained to single differential or difference equations whereas hyperbolicity pertained to invariant sets of diffeomorphisms. Now they are essentially different names for the same idea.

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Definition. We say the linear system $\dot{u} = D(t)u$, where $D(t)$ is piecewise continuous, has an *exponential dichotomy* on an interval J (usually $\mathbb{R}$, $\mathbb{R}_+$ or $\mathbb{R}_-$) with projections $P(t)$, constant $k \geq 1$ and exponents $\alpha, \beta > 0$ if the transition matrix $U(t, s)$ satisfies the invariance property

$$P(t)U(t,s) = U(t,s)P(s) \quad \text{if } s,t \in J$$

and the inequalities

$$\begin{aligned} \|U(t,s)P(s)\| &\leq ke^{-\alpha(t-s)} \qquad \text{if } s \leq t \in J \\ \|U(t,s)[\mathbf{I} - P(s)]\| &\leq ke^{\beta(t-s)} \text{ if } t \leq s \in J. \end{aligned}$$

Note that it follows from the invariance that $P(t) = U(t, 0)P(0)U(0, t)$. Sometimes we say that the equation has a dichotomy with projection P where $P = P(0)$. If $P = \mathbf{I}$ we only have the first inequality and when $P = 0$ only the second inequality.

Exponential dichotomy generalizes the idea of uniform asymptotic stability to the conditionally stable case. It is an idea going back to Perron [7]. An autonomous system $\dot{x} = Dx$ has an exponential dichotomy if and only if all the eigenvalues of D have nonzero real parts; a periodic system $\dot{x} = D(t)x$ has an exponential dichotomy if and only if all the Floquet exponents have nonzero real parts. A scalar equation $\dot{x} = a(t)x$ has an exponential dichotomy if and only if

$$\limsup_{t-s\to\infty} \frac{1}{t-s} \int_s^t a(u)du < 0 \quad \text{or} \quad \liminf_{t-s\to\infty} \frac{1}{t-s} \int_s^t a(u)du > 0.$$

A diagonal system has an exponential dichotomy if and only if each component scalar equation has. For a general time-dependent system, the eigenvalues of $D(t)$ tell us nothing about the system's dichotomy properties. However if the absolute values of the real parts of the eigenvalues are bounded below by a positive number and $D(t)$ is slowly varying, then the system does have an exponential dichotomy (Coppel [2]). Another criterion is that of diagonal dominance (Lazer [4], Coppel [2]): if $D(t)$ is diagonally dominant in a uniform way, then $\dot{x} = D(t)x$ has an exponential dichotomy. There are also other criteria for exponential dichotomy in terms of the existence of solutions of inhomogeneous equations in certain spaces (Massera and Schäffer [5], Coppel [2]) or in terms of the existence of a Lyapunov function (Coppel [2]) but these are not practical criteria to determine whether or not a given system has a dichotomy. If $\dot{x} = D(t)x$ has an exponential dichotomy, then a small perturbation $\dot{x} = [D(t) + E(t)]x$ has an exponential dichotomy (Coppel [2]). It should be mentioned here that some of these results have been extended to not necessarily invertible difference equations and to equations in Banach spaces (Pötzsche [8]); also the definition has been weakened in certain ways such as nonuniform dichotomies (Barreira and Valls [1]). Our focus here is on exponential dichotomies for differential equations in finite-dimensional spaces; however let us just mention that it seems that our results could be extended quite easily to invertible difference equations in finite-dimensional spaces. (Note that the references given here represent only a small sample of the significant contributions to this field.)

So we study nonautonomous systems of linear differential equations which are in block triangular form (which we may assume without loss of generality are upper triangular). More precisely, we study the relation between the dichotomy and spectral properties of the system and its block diagonal part, both on half lines and on the whole line. What is known already is that a bounded block triangular system has an exponential dichotomy on a half line if and only if its diagonal part has. For the whole line $\mathbb{R}$, when the off diagonal part is bounded, it is known that if the block diagonal part of the system has an exponential dichotomy on $\mathbb{R}$, then the block triangular system has an exponential dichotomy on $\mathbb{R}$ (Palmer [6]).

In Section 2, we consider systems on a half line and, in Theorem 1, show that if a not necessarily bounded block triangular system has an exponential dichotomy on a half line, then so does its block diagonal part but that the converse is in general only true when the off diagonal part is bounded. Using Theorem 1 we deduce in Theorem 3 that if a not necessarily bounded upper triangular system has an exponential dichotomy on a half line, then each of the scalar systems along its diagonal has one also but that the converse is in general only true when the off diagonal entries are bounded.

In Section 3 we consider systems on the whole line $\mathbb{R}$. In Corollary 1, we recall the result that when the off diagonal part is bounded and the block diagonal part of the system has an exponential dichotomy on $\mathbb{R}$, then the block triangular system has an exponential dichotomy on $\mathbb{R}$. However the converse is not true as we show by example. Still Theorem 1 shows it is necessary that the diagonal part have an exponential dichotomy on both half lines. In Propositions 4 and 5, we derive exactly what other conditions are required to guarantee that the block triangular system has an exponential dichotomy on $\mathbb{R}$. Here an important role is played by what we call the *linking operators*.

Lastly in Section 4, we study spectral properties of bounded block triangular systems. Then it is known that on a half line, the Sacker–Sell spectrum of such a system coincides with the spectrum of its diagonal part. However the whole line spectrum lies between the union of its two half line spectra and the whole line spectrum of the diagonal part. We give a procedure to determine the whole line spectrum. Recently we became aware of Christian Pötzsche's preprint [9] where, using different methods, he proves similar results about spectral properties of nonautonomous systems of linear difference equations.

2. Block upper triangular systems on a half line

In this section we show that if a block upper triangular system has a dichotomy on a half line, then so also has its diagonal part. This appears to be a new result when the coefficient matrix is unbounded. When the off diagonal part is bounded, then the converse holds and we show by example that the assumption of boundedness cannot be omitted. Thus we recover the known result that for bounded systems on a half line, a block upper triangular system has a dichotomy if and only if its diagonal part has.

Consider the block upper triangular system

$$\begin{cases} \dot{x} = A(t)x + C(t)y \\ \dot{y} = B(t)y \end{cases} \tag{1}$$

with $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{n-d}$, $1 \le d < n$. We take $\|(x, y)\| = \max\{\|x\|, \|y\|\}$, where we use the Euclidean norm in $\mathbb{R}^d$ and $\mathbb{R}^{n-d}$. We first prove the following result.

Theorem 1. *Suppose that (1) has an exponential dichotomy on $J = \mathbb{R}_+$ or $\mathbb{R}_-$ with projection $P(t)$ and exponents $\alpha, \beta > 0$. Then both linear systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have an exponential dichotomy on J with the same exponents. Moreover the projection P of the dichotomy for (1) can be taken in the block upper triangular form:*

$$\begin{pmatrix} P^A & LP^B \\ 0 & P^B \end{pmatrix} \quad (J = \mathbb{R}_+), \quad \begin{pmatrix} P^A & L(\mathbf{I}_{n-d} - P^B) \\ 0 & P^B \end{pmatrix} \quad (J = \mathbb{R}_-) \tag{2}$$

where $P^A : \mathbb{R}^d \to \mathbb{R}^d$, $P^B : \mathbb{R}^{n-d} \to \mathbb{R}^{n-d}$ are projections for the dichotomy on J of $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ respectively and $L : \mathcal{R}P^B \to (\mathcal{R}P^A)^\perp$ $(J = \mathbb{R}_+)$, $L : \mathcal{N}P^B \to (\mathcal{N}P^A)^\perp$ $(J = \mathbb{R}_-)$ is a linear mapping, which we call the linking operator. In particular, this means that

$$\text{rank}P = \text{rank}P^A + \text{rank}P^B. \tag{3}$$

Proof. *Step 1:* We first show $\dot{x} = A(t)x$ has an exponential dichotomy on J. To this end, let $V_1 \subset \mathbb{R}^d$ be the subspace of initial conditions (at $t = 0$) for which the solution of $\dot{x} = A(t)x$ is bounded on J and takes V_2 as any fixed complement of V_1 in $\mathbb{R}^d$.

Next let $W_1 \subset \mathbb{R}^{n-d}$ be the subspace of initial conditions η (at $t = 0$) for which $Y(t, 0)\eta$ is bounded on J and the equation

$$\dot{x} = A(t)x + C(t)Y(t, 0)\eta \tag{4}$$

has a bounded solution on J and takes any fixed complement W_2 of W_1 in $\mathbb{R}^{n-d}$. If $\eta \in W_1$, we let $x_b(t)$ be the unique solution of (4) bounded on J such that $x_b(0) \in V_2$. If we write $x_b(0) = L\eta$, clearly $L : W_1 \mapsto V_2$ is a linear mapping. We see that (ξ, η) is the initial value of a solution of (1) bounded on J if and only if $\eta \in W_1$ and $\xi = \xi_0 + L\eta$ for some $\xi_0 \in V_1$.

Next when $J = \mathbb{R}_+$, let $P^A : \mathbb{R}^d \to \mathbb{R}^d$ be the projection such that $\mathcal{R}P^A = V_1$ and $\mathcal{N}P^A = V_2$ and $Q : \mathbb{R}^{n-d} \to \mathbb{R}^{n-d}$ the projection such that $\mathcal{R}Q = W_1$ and $\mathcal{N}Q = W_2$; when $J = \mathbb{R}_-$, we switch the range and nullspace in both projections. Then if we define the projection P as

$$\begin{pmatrix} P^A & LQ \\ 0 & Q \end{pmatrix} \quad (J = \mathbb{R}_+), \quad \begin{pmatrix} P^A & L(\mathbf{I}_{n-d} - Q) \\ 0 & Q \end{pmatrix} \quad (J = \mathbb{R}_-), \tag{5}$$

we see that the range of P when $J = \mathbb{R}_+$ (the nullspace when $J = \mathbb{R}_-$) consists of all vectors $(\xi_0 + L\eta, \eta)$ for some $\xi_0 \in V_1$ and $\eta \in W_1$ and hence coincides with the subspace of initial values of solutions of (1) bounded on J. It follows (see p. 16 in [2]) that (1) has an exponential dichotomy on J with exponents α, β and some constant k, and projection $P(t) = U(t, 0)PU(0, t)$, where the transition matrix of (1) is

$$U(t, s) = \begin{pmatrix} X(t, s) & W(t, s) \\ 0 & Y(t, s) \end{pmatrix},$$

$X(t, s)$, $Y(t, s)$ being the transition matrices of $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ resp. and

$$W(t, s) = \int_s^t X(t, \tau)C(\tau)Y(\tau, s)d\tau.$$

Note that in the rest of the argument in this step, we only use the fact that P and hence $P(t)$ are in upper triangular form. We see that for any $\xi \in \mathbb{R}^d$ and $s \in J$, we have:

$$P(s) \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} P^A(s)\xi \\ 0 \end{pmatrix},$$

where $P^A(t) = X(t, 0)P^A X(0, t)$, and

$$(\mathbf{I}_n - P(s)) \begin{pmatrix} \xi \\ 0 \end{pmatrix} = \begin{pmatrix} \xi \\ 0 \end{pmatrix} - \begin{pmatrix} P^A(s)\xi \\ 0 \end{pmatrix} = \begin{pmatrix} (\mathbf{I}_d - P^A(s))\xi \\ 0 \end{pmatrix}.$$

As a consequence $\|P^A(s)\| \le \|P(s)\|$ and, for $\xi \in \mathbb{R}^d$ and $s \le t$ in J, we get:

$$\begin{aligned} \|X(t, s)P^A(s)\xi\| &= \left\| U(t, s) \begin{pmatrix} P^A(s)\xi \\ 0 \end{pmatrix} \right\| = \left\| U(t, s)P(s) \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\| \\ &\le ke^{-\alpha(t-s)} \left\| \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\| = ke^{-\alpha(t-s)}\|\xi\| \end{aligned}$$

and, similarly, for any $\xi \in \mathbb{R}^d$ and $t \le s$ in J:

$$\|X(t,s)(\mathbf{I}_d - P^A(s))\xi\| = \left\| U(t,s)\begin{pmatrix}(\mathbf{I}_d - P^A(s))\xi \\ 0\end{pmatrix}\right\|$$
$$= \left\| U(t,s)(\mathbf{I}_n - P(s))\begin{pmatrix}\xi \\ 0\end{pmatrix}\right\| \le ke^{\beta(t-s)}\left\|\begin{pmatrix}\xi \\ 0\end{pmatrix}\right\| = ke^{-\beta(s-t)}\|\xi\|.$$

This completes the proof of the exponential dichotomy on J of the linear system $\dot{x} = A(t)x$ with projection P^A.

Step 2: Next we prove that $\dot{y} = B(t)y$ has an exponential dichotomy on J. To this end, consider the adjoint system

$$\begin{cases} \dot{x} = -A^*(t)x \\ \dot{y} = -C^*(t)x - B^*(t)y, \end{cases} \tag{6}$$

which is block lower triangular but can be considered as the block upper triangular system

$$\begin{cases} \dot{u} = -B^*(t)u - C^*(t)v \\ \dot{v} = -A^*(t)v \end{cases} \tag{7}$$

with $u = y$ and $v = x$. The lower triangular system (6) has an exponential dichotomy on J with projection $\mathbf{I}_n - P^*$, constant k and exponents α, β, where P is as in (5), and so (7) has an exponential dichotomy on J with projection

$$\begin{pmatrix} \mathbf{I}_{n-d} - Q^* & -Q^*L^* \\ 0 & \mathbf{I}_d - (P^A)^* \end{pmatrix}, \quad \begin{pmatrix} \mathbf{I}_{n-d} - Q^* & (\mathbf{I}_{n-d} - Q^*)L^* \\ 0 & \mathbf{I}_d - (P^A)^* \end{pmatrix}$$

respectively for $J = \mathbb{R}_+$ and $J = \mathbb{R}_-$, and with constant k and exponents α, β. Since this projection is in block upper triangular form, it follows by the argument in Step 1 that $\dot{y} = -B^*(t)y$ has an exponential dichotomy on J with projection $\mathbf{I}_{n-d} - Q^*$ and the same constant and exponents, and hence so does $\dot{y} = B(t)y$ with Q as the projection. So we may take $P^B = Q$ which means that $W_1 = \mathcal{R}P^B$ and also that P has the form (2) as stated in the theorem.

Step 3: Here we derive the rank conditions. With P as defined in the statement of the theorem, we observe that $(\xi, \eta) \in \mathcal{N}P$ if and only if $P^A\xi = 0$ and $P^B\eta = 0$ so that $\dim \mathcal{N}P = \dim \mathcal{N}P^A + \dim \mathcal{N}Q$. Hence (3) follows, thus completing the proof of the theorem. □

Remark 1. In particular we have proved that if the block upper triangular system (1) has an exponential dichotomy on $J = \mathbb{R}_+$ or $\mathbb{R}_-$, then for any bounded solution $y(t)$ of $\dot{y} = B(t)y$ there exists a bounded solution of the linear inhomogeneous system $\dot{x} = A(t)x + C(t)y(t)$ and this holds even if $C(t)$ is not bounded. However the fact that the upper triangular system has an exponential dichotomy imposes some restrictions on the choice of $C(t)$. For example, the proof of Theorem 1 does not work for the system

$$\begin{cases} \dot{x} = -x + e^{3t}y \\ \dot{y} = -y. \end{cases}$$

This system has $\mathcal{R}P^B = \mathbb{R}$ but we see that $W_1 = \{0\}$ since $\dot{x} = -x + e^{2t}$ has no solution bounded on $\mathbb{R}_+$. This means that $W_1 \ne \mathcal{R}P^B$. So the system cannot have an exponential dichotomy on $\mathbb{R}_+$.

Remark 2. We can derive formulas for the linking operators. From the proof of Theorem 1 we see that for any $\eta \in \mathcal{R}P^B$, the system

$$\dot{x} = A(t)x + C(t)Y(t,0)\eta$$

has a unique solution $x_b(t)$ bounded on J such that $P^A x(0) = 0$. By standard methods it is proved that when $J = \mathbb{R}_+$

$$x_b(t) = \int_0^t X(t,s)P^A(s)C(s)Y(s,0)\eta ds - \int_t^\infty X(t,s)(\mathbf{I}_d - P^A(s))C(s)Y(s,0)\eta ds$$

and so the linking operator on $\mathbb{R}_+$ is given by

$$L\eta\,(= x_b(0)) = -\int_0^\infty X(0,s)(\mathbf{I}_d - P^A(s))C(s)Y(s,0)\eta ds. \tag{8}$$

When $J = \mathbb{R}_-$

$$x_b(t) = \int_{-\infty}^t X(t,s)P^A(s)C(s)Y(s,0)\eta ds - \int_t^0 X(t,s)(\mathbf{I}_d - P^A(s))C(s)Y(s,0)\eta ds$$

and then the linking operator on $\mathbb{R}_-$ is given by

$$L\eta\,(= x_b(0)) = \int_{-\infty}^0 X(0,s)P^A(s)C(s)Y(s,0)\eta ds. \tag{9}$$

Remark 3. Note that it follows from the argument in Step 1 of the proof of Theorem 1 that if the projection for the dichotomy of (1) is taken in block upper triangular form, the A and B equations have dichotomies with the same constants and exponents.

The next result is a partial converse of Theorem 1, i.e. if the two systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ (or $\mathbb{R}_-$) then so also has system (1), provided $|C(t)|$ is bounded. This is a known result which is usually proved using roughness (see p. 187 in [6]). Here we give a direct proof based on the constructions in the previous theorem.

Theorem 2. *Assume that $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $J = \mathbb{R}_+$ or $\mathbb{R}_-$ with constants k_1, k_2 and exponents α_1, β_1 and α_2, β_2 respectively and projections P^A and P^B. Let $\alpha = \min\{\alpha_1, \alpha_2\}$ if $\alpha_1 \neq \alpha_2$, or any positive number strictly less than α if $\alpha_1 = \alpha_2$; similarly let $\beta = \min\{\beta_1, \beta_2\}$ if $\beta_1 \neq \beta_2$, or any positive number strictly less than β if $\beta_1 = \beta_2$; and suppose that $\sup_{t\geq 0} |C(t)| < +\infty$. Then the block upper triangular system (1) has an exponential dichotomy on J with exponents α, β.*

Proof. The transition matrix of (1) is

$$U(t,s) = \begin{pmatrix} X(t,s) & W(t,s) \\ 0 & Y(t,s) \end{pmatrix},$$

where $X(t,s)$, $Y(t,s)$ are the transition matrices of $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ resp. and

$$W(t,s) = \int_s^t X(t,\tau)C(\tau)Y(\tau,s)d\tau.$$

If system (1) has an exponential dichotomy on J, then Theorem 1 tells us we can take the projection P to be in the form (2), where from (8) and (9), the linking operator L is defined as follows:

$$L\eta = -\int_0^\infty (\mathbf{I}_d - P^A)X(0,t)C(t)Y(t,0)\eta dt, \quad \eta \in \mathcal{R}P^B$$

when $J = \mathbb{R}_+$, and

$$L\eta = \int_{-\infty}^0 P^A X(0,t)C(t)Y(t,0)\eta dt, \quad \eta \in \mathcal{N}P^B$$

when $J = \mathbb{R}_-$. Note that

$$P(t) = U(t,0)PU(0,t) = \begin{pmatrix} P^A(t) & R(t) \\ 0 & P^B(t) \end{pmatrix},$$

where $P^A(t) = X(t,0)P^A X(0,t)$, $P^B(t) = Y(t,0)P^B Y(0,t)$ and $R(t)$ solves

$$\dot{R} = A(t)R - RB(t) + C(t)P^B(t) - P^A(t)C(t)$$

with $R(0) = LP^B$ if $J = \mathbb{R}_+$ and $R(0) = L(\mathbf{I}_{n-d} - P^B)$ if $J = \mathbb{R}_-$. By variation of constants

$$R(t) = X(t,0)R(0)Y(0,t) + \int_0^t X(t,\tau)[C(\tau)P^B(\tau) - P^A(\tau)C(\tau)]Y(\tau,t)d\tau.$$

Then from the definition of L, we find after some rearrangement that

$$R(t) = -\int_0^t X(t,\tau)P^A(\tau)C(\tau)(\mathbf{I}_{n-d} - P^B(\tau))Y(\tau,t)d\tau$$

$$-\int_t^\infty X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)P^B(\tau)Y(\tau,t)d\tau$$

when $J = \mathbb{R}_+$; when $J = \mathbb{R}_-$, it is given by the same formula with upper limits ∞ replaced by 0 and lower limits 0 replaced by $-\infty$.

We now prove that (1) has an exponential dichotomy on J with projection P. Let $\|C\| = \sup\{|C(t)| \mid t \in J\}$. First we prove that

- $\|U(t,s)P(s)\| \le ce^{-\alpha(t-s)}$, *for* $s \le t$ *in* J, *and some* $c \ge 1$.
  We have

$$U(t,s)P(s) = \begin{pmatrix} X(t,s)P^A(s) & W(t,s)P^B(s) + X(t,s)R(s) \\ 0 & Y(t,s)P^B(s) \end{pmatrix}.$$

Note that we can write

$$W(t,s)P^B(s) = \int_s^t X(t,\tau)P^A(\tau)C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

$$+ \int_s^t X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

and, when $J = \mathbb{R}_+$,

$$X(t,s)R(s) = -\int_0^s X(t,\tau)P^A(\tau)C(\tau)(\mathbf{I}_{n-d} - P^B(\tau))Y(\tau,s)d\tau$$

$$- \int_s^\infty X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

so that after some rearrangement

$$W(t,s)P^B(s) + X(t,s)R(s) = \int_s^t X(t,\tau)P^A(\tau)C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

$$- \int_t^\infty X(t,\tau)[\mathbf{I}_d - P^A(\tau)]C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

$$- \int_0^s X(t,\tau)P^A(\tau)C(\tau)[\mathbf{I}_{n-d} - P^B(\tau)]Y(\tau,s)d\tau.$$

When $J = \mathbb{R}_-$, we replace lower limits 0 by $-\infty$ and upper limits ∞ by 0. Then if $s \le t$ are in J and $\alpha_1 \neq \alpha_2$:

$$\begin{aligned}
&\|W(t,s)P^B(s) + X(t,s)R(s)\| \\
&\le k_1k_2\|C\| \left[\frac{e^{-\alpha_2(t-s)} - e^{-\alpha_1(t-s)}}{\alpha_1 - \alpha_2} + \frac{e^{-\alpha_2(t-s)}}{\alpha_2 + \beta_1} + \frac{e^{-\alpha_1(t-s)}}{\alpha_1 + \beta_2}\right] \\
&\le k_1k_2\|C\| \left[\frac{1}{|\alpha_1 - \alpha_2|} + \frac{1}{\alpha_2 + \beta_1} + \frac{1}{\alpha_1 + \beta_2}\right] e^{-\alpha(t-s)}.
\end{aligned}$$

Note that, if $\alpha_1 = \alpha_2$ then the last two terms in the brackets do not change but the first is replaced by:

$$\int_s^t e^{-\alpha_1(t-\tau)}e^{-\alpha_1(\tau-s)}d\tau = \int_s^t e^{-\alpha_1(t-s)}d\tau = (t-s)e^{-\alpha_1(t-s)} \le c_\alpha e^{-\alpha(t-s)}$$

so that $\frac{1}{|\alpha_1-\alpha_2|}$ in the last inequality is replaced by c_α. Furthermore, note that if $t = s$, we do not have the first of the three terms in the inequality above so that we get

$$\|R(t)\| \le k_1k_2\|C\| \left[\frac{1}{\alpha_2 + \beta_1} + \frac{1}{\alpha_1 + \beta_2}\right].$$

Next we prove that

- *$\|U(t,s)[\mathbf{I}_n - P(s)]\| \le ce^{\beta(t-s)}$, for $t \le s$ in J, and some $c \ge 1$.*
  Here $U(t,s)[\mathbf{I}_n - P(s)] = [\mathbf{I}_n - P(t)]U(t,s)$, which equals

$$\begin{pmatrix} (\mathbf{I}_d - P^A(t))X(t,s) & (\mathbf{I}_d - P^A(t))W(t,s) - R(t)Y(t,s) \\ 0 & [\mathbf{I}_{n-d} - P^B(t)]Y(t,s) \end{pmatrix}.$$

Now when $J = \mathbb{R}_+$ (when $J = \mathbb{R}_-$, replace lower limits 0 by $-\infty$ and upper limits ∞ by 0)

$$R(t)Y(t,s) = -\int_0^t X(t,\tau)P^A(\tau)C(\tau)(\mathbf{I}_{n-d} - P^B(\tau))Y(\tau,s)d\tau$$
$$-\int_t^\infty X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)P^B(\tau)Y(\tau,s)d\tau$$

and

$$(\mathbf{I}_d - P^A(t))W(t,s) = \int_s^t X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)Y(\tau,s)d\tau.$$

So after some rearrangement

$$\begin{aligned}
&(\mathbf{I}_d - P^A(t))W(t,s) - R(t)Y(t,s) \\
&= \int_s^\infty X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)P^B(\tau)Y(\tau,s)d\tau \\
&- \int_t^s X(t,\tau)(\mathbf{I}_d - P^A(\tau))C(\tau)(\mathbf{I}_{n-d} - P^B(\tau))Y(\tau,s)d\tau \\
&+ \int_0^t X(t,\tau)P^A(\tau)C(\tau)(\mathbf{I}_{n-d} - P^B(\tau))Y(\tau,s)d\tau.
\end{aligned}$$

Then for $t \le s$ in J:

$$\begin{aligned}
&\|[\mathbf{I}_d - P^A(t)]W(t,s) - R(t)Y(t,s)\| \\
&\le k_1k_2\|C\| \left[\frac{e^{\beta_1(t-s)}}{\alpha_2 + \beta_1} + \frac{e^{\beta_1(t-s)} - e^{\beta_2(t-s)}}{\beta_2 - \beta_1} + \frac{e^{\beta_2(t-s)}}{\alpha_1 + \beta_2} \right] \\
&\le k_1k_2\|C\| \left[\frac{1}{\alpha_2 + \beta_1} + \frac{1}{|\beta_2 - \beta_1|} + \frac{1}{\alpha_1 + \beta_2} \right] e^{\beta(t-s)}
\end{aligned}$$

(this estimate holds when $\beta_1 \ne \beta_2$; when $\beta_1 = \beta_2$ a remark similar to that of the previous point applies).

The proof is complete. □

Example. We show that in Theorem 2 the boundedness of the matrix $C(t)$ cannot be easily relaxed. Consider the system:

$$\begin{cases} \dot{x} = x + ye^{\mu t} \\ \dot{y} = -y \end{cases} \tag{10}$$

with $\mu > 0$. Both linear equations $\dot{x} = x$ and $\dot{y} = -y$ have an exponential dichotomy on $\mathbb{R}_+$ with respective projections $P^A = 0$ and $P^B = 1$. We show that (10) does not have an exponential dichotomy on $\mathbb{R}_+$.

According to Theorem 1, if (10) has an exponential dichotomy the projection P has rank 1 and hence (10) has a nontrivial solution bounded on $\mathbb{R}_+$. However the general solution of (10) is

$$x(t) = \left[x(0) - \frac{y(0}{\mu - 2}\right] e^t + y(0)\frac{e^{(\mu-1)t}}{\mu - 2}, \quad y(t) = y(0)e^{-t},$$

if $\mu \neq 2$, and if $\mu = 2$, it is

$$x(t) = x(0)e^t + y(0)te^t, \quad y(t) = y(0)e^{-t}.$$

We see that a nontrivial bounded solution exists if and only if $\mu \leq 1$ and the subspace of initial values of such solutions is spanned by the vector $(1, \mu - 2)$. However it also follows from Theorem 1 that the projection can be taken as:

$$P = \begin{pmatrix} 0 & \ell \\ 0 & 1 \end{pmatrix}$$

with $\ell \in \mathbb{R}$. Now if $\mu \leq 1$, the fundamental matrix of Eq. (10) is:

$$U(t) = \begin{pmatrix} e^t & \frac{e^t - e^{(\mu-1)t}}{2-\mu} \\ 0 & e^{-t} \end{pmatrix}.$$

Then

$$U(t)PU(t)^{-1} = \begin{pmatrix} 0 & \frac{[(2-\mu)\ell+1]e^{2t} - e^{\mu t}}{2-\mu} \\ 0 & 1 \end{pmatrix}$$

which is not bounded on $\mathbb{R}_+$ if $\mu > 0$. So, for $\mu > 0$, Eq. (10) does not have an exponential dichotomy on $\mathbb{R}_+$.

As another example, consider the system:

$$\begin{cases} \dot{x} = -x + ye^t \\ \dot{y} = -y. \end{cases} \tag{11}$$

Both linear equations $\dot{x} = -x$ and $\dot{y} = -y$ have an exponential dichotomy on $\mathbb{R}_+$ with projections $P^A = P^B = 1$. However we see that $(1, e^{-t})$ is a solution which is bounded for $t \geq 0$ but does not approach 0 as $t \to \infty$. Hence Eq. (11) does not have an exponential dichotomy on $\mathbb{R}_+$.

Remark 4. Despite the previous example, there are unbounded $C(t)$ for which we do get a dichotomy. Take a bounded continuous real function $h(t)$ such as $\sin(t^2)$ for which $h(t)$ is bounded but $h'(t)$ is unbounded. Then define $c(t) = h'(t)+2h(t)$, which is unbounded. However we see that the kinematic similarity $x_1 = y_1+h(t)y_2$, $x_2 = y_2$ takes the upper triangular system

$$\dot{x} = \begin{pmatrix} -1 & c(t) \\ 0 & 1 \end{pmatrix} x$$

into the diagonal system

$$\dot{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} y$$

so that the upper triangular system does have an exponential dichotomy.

2.1. Upper triangular systems

We apply the results obtained for the upper block triangular system (1) to the triangular system

$$\dot{z} = L(t)z, \tag{12}$$

where

$$L(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \dots & a_{1n}(t) \\ 0 & a_{22}(t) & a_{23}(t) & \dots & a_{2n}(t) \\ 0 & 0 & a_{33}(t) & \dots & a_{3n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn}(t) \end{pmatrix}.$$

We are not assuming $L(t)$ is bounded. Write

$$A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) & \dots & a_{1,n-1}(t) \\ 0 & a_{22}(t) & a_{23}(t) & \dots & a_{2,n-1}(t) \\ 0 & 0 & a_{33}(t) & \dots & a_{3,n-1}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{n-1,n-1}(t) \end{pmatrix}, \quad C(t) = \begin{pmatrix} a_{1n}(t) \\ a_{2n}(t) \\ a_{3n}(t) \\ \vdots \\ a_{n-1,n}(t) \end{pmatrix}$$

and $B(t) = a_{nn}(t)$. From Theorem 1 we deduce the following result, which appears to be new when $L(t)$ is not bounded.

Theorem 3. *Suppose the linear system (12) has an exponential dichotomy on $J = \mathbb{R}_+$ or $\mathbb{R}_-$. Then, for any $j = 1\dots, n$, the scalar equation*

$$\dot{x}_j = a_{jj}(t)x_j$$

has an exponent dichotomy on J with the same exponents.

Proof. We use induction on n. When $n = 1$ there is nothing to prove. When $n = 2$ the result is Theorem 1. So suppose the result is true when the triangular system is $n-1$ dimensional. Since $\dot{z} = L(t)z$ has the form (1) with $A(t)$, $B(t)$ and $C(t)$ as above, it follows from Theorem 1 that $\dot{y} = B(t)y = a_{nn}(t)y$ and $\dot{x} = A(t)x$ have exponential dichotomies on J with the same exponents. From the inductive hypothesis it follows that all scalar equations $\dot{x}_j = a_{jj}(t)x_j$ have exponential dichotomies on J with the same exponents. $\square$

Remark 5. When the off diagonal entries $a_{ij}(t)$ are bounded, this theorem has a converse which is well known (see p. 187 in [6]).

3. Block upper triangular systems on the whole line

In the next part of the paper we study the block upper triangular system (1) with no boundedness restriction unless explicitly stated. The question we are concerned with is when the system has a dichotomy on the whole line. We are particularly interested in answering this question under the assumption that the block upper triangular system has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$.

Definition. The *index* of a linear system $\dot{u} = D(t)u$, $u \in \mathbb{R}^n$, with exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ is given by

$$\imath(D) = \dim \mathcal{R}P_+ + \dim \mathcal{N}P_- - n,$$

where P_+ is the projection for the dichotomy on $\mathbb{R}_+$ and P_- that for $\mathbb{R}_-$.

The following proposition is well known (see [2]). We omit its proof.

Proposition 1. *$\dot{u} = D(t)u$ has an exponential dichotomy on $\mathbb{R}$ if and only if it has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, no nontrivial bounded solution and $i(D) = 0$.*

From it we deduce the following proposition.

Proposition 2. *The block upper triangular system (1) has an exponential dichotomy on $\mathbb{R}$ if and only if it has an exponential dichotomy separately on $\mathbb{R}_+$ and $\mathbb{R}_-$, no nontrivial bounded solutions on $\mathbb{R}$ and $\imath(A) + \imath(B) = 0$.*

Proof. If (1) has exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$, then from Theorem 1, the projections P_+ and P_- of the dichotomies satisfy respectively

$$\dim \mathcal{R}P_+ = \dim \mathcal{R}P_+^A + \dim \mathcal{R}P_+^B$$

and

$$\dim \mathcal{N}P_- = \dim \mathcal{N}P_-^A + \dim \mathcal{N}P_-^B.$$

Here P_+^A is the projection for the dichotomy of $\dot{x} = A(t)x$ on $\mathbb{R}_+$ and similarly for the others. As a consequence:

$$\dim \mathcal{R}P_+ + \dim \mathcal{N}P_- = \dim \mathcal{R}P_+^A + \dim \mathcal{N}P_-^A + \dim \mathcal{R}P_+^B + \dim \mathcal{N}P_-^B \tag{13}$$

and hence

$$\imath \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \dim \mathcal{R}P_+ + \dim \mathcal{N}P_- - n = \imath(A) + \imath(B). \tag{14}$$

Then the proposition follows from Proposition 1. $\square$

Next we recover the following well known result (see p. 187 in [6] for the half line case; the proof is the same in the whole line case):

Corollary 1. *Assume the linear systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}$. Then if $C(t)$ is a piecewise continuous and bounded $d \times (n-d)$ matrix, system (1) has an exponential dichotomy on $\mathbb{R}$.*

Proof. By Theorem 2, (1) has an exponential dichotomy on both $\mathbb{R}_+$ and $\mathbb{R}_-$. Next as $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ do not have nontrivial bounded solutions on $\mathbb{R}$, it follows at once that (1) has no nontrivial bounded solution on $\mathbb{R}$. Finally from Proposition 1, it follows that $\imath(A) = 0$ and $\imath(B) = 0$. Then the corollary follows from Proposition 2. $\square$

However, even when $C(t)$ is bounded, if (1) has an exponential dichotomy on $\mathbb{R}$, it is not necessary that $\dot{x} = A(t)x$ or $\dot{y} = B(t)y$ have an exponential dichotomy on $\mathbb{R}$ (unlike the half line case).

Example. Here we give an example of an upper triangular system having an exponential dichotomy on $\mathbb{R}$ but such that the diagonal part does not. Consider the system:

$$\begin{cases} \dot{x} = x \operatorname{sgn}(t) + \delta y \\ \dot{y} = -y \operatorname{sgn}(t) \end{cases} \tag{15}$$

with $\delta \neq 0$. Both $\dot{x} = A(t)x = x \operatorname{sgn}(t)$ and $\dot{y} = B(t)y = -y \operatorname{sgn}(t)$ have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections satisfying

$$P_+^B = P_-^A = 1, \qquad P_-^B = P_+^A = 0.$$

It follows from Theorem 2 that (15) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ also. Note also that

$$\imath(A) + \imath(B) = -1 + 1 = 0.$$

Next the solution of (15) with initial value (x_0, y_0) is $(x_0 e^{|t|} + \delta y_0 \sinh t, y_0 e^{-|t|})$ which is only bounded if $(x_0, y_0) = (0, 0)$. So (15) has no nontrivial bounded solution. Thus by Theorem 2, system (15) has an exponential dichotomy on $\mathbb{R}$. However clearly neither of the two systems $\dot{x} = x \operatorname{sgn}(t)$ and $\dot{y} = -y \operatorname{sgn}(t)$ has an exponential dichotomy on $\mathbb{R}$.

In the following proposition, we show that if a block upper triangular system has an exponential dichotomy on $\mathbb{R}$, then either both diagonal systems have an exponential dichotomy on $\mathbb{R}$ or neither has (as in the previous example).

Proposition 3. *Suppose the block upper triangular system (1) has an exponential dichotomy on $\mathbb{R}$. Then the following conditions are equivalent:*

i) *$\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}$,*
ii) *$\dot{y} = B(t)y$ has an exponential dichotomy on $\mathbb{R}$,*
iii) *$\imath(A) = 0$,*
iv) *$\imath(B) = 0$.*

Proof. Suppose the projection for the dichotomy of (1) on $\mathbb{R}$ is P.

i) $\Leftrightarrow$ *ii)*: If $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}$ with projection P^A, we have

$$\mathbb{R}^n = \mathcal{R}P \oplus \mathcal{N}P, \quad \mathbb{R}^d = \mathcal{R}P^A \oplus \mathcal{N}P^A.$$

Let $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections P_+^B and P_-^B respectively. From (13) it follows that $\dim \mathcal{R}P_+^B + \dim \mathcal{N}P_-^B = n - d$ so that $\imath(B) = 0$. Now let $y(t)$ be a bounded solution of $\dot{y} = B(t)y$. Then it follows from the proof of Theorem 1 (see also Remark 1) that there are a unique solution $x_+(t)$ of

$$\dot{x} = A(t)x + C(t)y(t) \tag{16}$$

which is bounded on $\mathbb{R}_+$ with $x_+(0) \in \mathcal{N}P^A$, and another solution $x_-(t)$ which is bounded on $\mathbb{R}_-$ with $x_-(0) \in \mathcal{R}P^A$. Then, if $X(t,s)$ is the transition matrix for $\dot{x} = A(t)x$,

$$x(t) = \begin{cases} x_+(t) + X(t,0)x_-(0) & t \geq 0 \\ x_-(t) + X(t,0)x_+(0) & t \leq 0 \end{cases}$$

is a solution of (16) bounded on $\mathbb{R}$. This means that $(x(t), y(t))$ is a bounded solution of (1) and so must be zero. It follows that $y(t) = 0$ and hence $\dot{y} = B(t)y$ has no nontrivial bounded solution and therefore, by Proposition 1, has an exponential dichotomy on $\mathbb{R}$.

Conversely, if $\dot{y} = B(t)y$ has an exponential dichotomy on $\mathbb{R}$, then we change (x, y) to (y, x) and apply what we have just proved to the adjoint system:

$$\begin{aligned} \dot{x} &= -B^*(t)x - C^*(t)y \\ \dot{y} &= -A^*(t)y. \end{aligned}$$

iii) $\Leftrightarrow$ *iv)*: Since (1) has an exponential dichotomy on $\mathbb{R}$ its index is 0 or, using (14), $\imath(A) + \imath(B) = 0$ from which the equivalence *iii)* $\Leftrightarrow$ *iv)* follows immediately.

i) $\Leftrightarrow$ *iii)*: Since (1) has an exponential dichotomy on $\mathbb{R}$ it follows that it has no nontrivial bounded solutions and hence $\dot{x} = A(t)x$ also has no nontrivial bounded solutions. Then it follows that $\imath(A) = 0$ if and only if $\dot{x} = A(t)x$ has an exponential dichotomy on $\mathbb{R}$. $\square$

As observed in Proposition 1, a necessary condition for a dichotomy on $\mathbb{R}$ is that the system have no nontrivial bounded solution. When (1) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, we give a necessary and sufficient condition for this.

Proposition 4. *Suppose (1) has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$. Then (1) has no nontrivial bounded solution on $\mathbb{R}$ if and only if $\dot{x} = A(t)x$ has no nontrivial bounded solution and for all nontrivial bounded solutions $y(t)$ of $\dot{y} = B(t)y$, there exists a bounded solution $\psi(t)$ of $\dot{x} = -A^*(t)x$ such that $\int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt \neq 0$.*

Proof. By Theorem 1, the linear systems $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ with respective projections P_+^A, P_+^B and on $\mathbb{R}_-$ with respective projections P_-^A, P_-^B. Then, from the proof of Theorem 1, the subspace of initial values of solutions of (1) which are bounded on $\mathbb{R}_+$ is given by $(\xi_1 + L_+\eta_1, \eta_1)$, where $\xi_1 \in \mathcal{R}P_+^A$ and $\eta_1 \in \mathcal{R}P_+^B$; the subspace of initial values of solutions of (1) which are bounded on $\mathbb{R}_-$ is given by $(\xi_2 + L_-\eta_2, \eta_2)$, where $\xi_2 \in \mathcal{N}P_-^A$ and $\eta_2 \in \mathcal{N}P_-^B$; here L_+ and L_- are the linking operators on $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively and are given by formulas (8) and (9). Then $(\xi_1 + L_+\eta_1, \eta_1) = (\xi_2 + L_-\eta_2, \eta_2)$ if and only if $\eta_1 = \eta_2 = \eta \in N^B = \mathcal{R}P_+^B \cap \mathcal{N}P_-^B$ and $\xi_1 + L_+\eta = \xi_2 + L_-\eta$, that is, $(L_+ - L_-)\eta = \xi_2 - \xi_1$. So there exists a nontrivial bounded solution of (1) if and only if either (i) $N^A = \mathcal{R}P_+^A \cap \mathcal{N}P_-^A \neq \{0\}$ or (ii) there exists $\eta \neq 0$ in N^B such that $(L_+ - L_-)\eta \in S^A = \mathcal{R}P_+^A + \mathcal{N}P_-^A$. Condition (i) says that $\dot{x} = A(t)x$ has a nontrivial bounded solution. Now $(L_+ - L_-)\eta \in S^A$ if and only if $\psi^*(L_+ - L_-)\eta = 0$ for all $\psi \in (S^A)^\perp$, where the latter holds if and only if the solution $\psi(t)$ of $\dot{x} = -A^*(t)x$ with $\psi(0) = \psi$ is bounded; also we can write

$$\psi^*(L_+ - L_-)\eta = \int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt,$$

where $y(t)$ is the solution of $\dot{y} = B(t)y$ with $y(0) = \eta$. Thus condition (ii) says that there exists a nontrivial bounded solution $y(t)$ of $\dot{y} = B(t)y$ such that $\int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt = 0$ for all bounded solutions $\psi(t)$ of $\dot{x} = -A^*(t)x$. So the proof is complete. □

When $|C(t)|$ is bounded we use the last proposition to give a necessary and sufficient condition that a block upper triangular system have a dichotomy on the whole line.

Proposition 5. *Suppose $C(t)$ is bounded. Then (1) has an exponential dichotomy on $\mathbb{R}$ if and only if both $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, $i(A) + i(B) = 0$ and (1) has no nontrivial bounded solution or, equivalently, $\dot{x} = A(t)x$ has no nontrivial bounded solution, and for all nonzero bounded solutions $y(t)$ of $\dot{y} = B(t)y$, there exists a bounded solution $\psi(t)$ of $\dot{x} = -A^*(t)x$ such that $\int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt \neq 0$.*

Proof. First suppose (1) has an exponential dichotomy on $\mathbb{R}$. Then it follows from Theorem 1 that both $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$. Next it follows from Proposition 2 that $i(A) + i(B) = 0$ and that (1) has no nontrivial bounded solution.

On the other hand, suppose both $\dot{x} = A(t)x$ and $\dot{y} = B(t)y$ have exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$, $i(A) + i(B) = 0$ and (1) has no nontrivial bounded solution. Then it follows from Theorem 2 that (1) has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$. Next, using the other conditions, it follows from Proposition 2 that (1) has an exponential dichotomy on $\mathbb{R}$. □

4. The spectrum of block upper triangular systems

Definition. Given a linear system $\dot{u} = D(t)u$, $t \in J \subset \mathbb{R}$, we say that λ is in the *spectrum* of the equation on the interval J if

$$\dot{u} = [D(t) - \lambda \mathbf{I}]u$$

does not have an exponential dichotomy on J.

The spectrum of a linear system $\dot{u} = D(t)u$ on $J = \mathbb{R}_\pm$ will be denoted by $\sigma_\pm(D)$ and the spectrum on $\mathbb{R}$ by $\sigma(D)$. If $u \in \mathbb{R}^n$ and $|D(t)|$ is bounded (as we assume in this section), it is known that the spectrum consists of the union of at most n disjoint compact intervals.

For a general system $\dot{z} = D(t)z$, it follows from Proposition 1 that $\lambda \notin \sigma(D)$ if and only if

(i) $\lambda \notin \sigma_+(D) \cup \sigma_-(D)$,
(ii) $i(D - \lambda \mathbf{I}_n) = 0$, and
(iii) $\dot{z} = (D(t) - \lambda \mathbf{I}_n)z$ has no nontrivial bounded solution.

(i) implies that $\sigma_+(D) \cup \sigma_-(D) \subset \sigma(D)$ but they are in general not equal. Now $\sigma_+(D) \cup \sigma_-(D)$ consists of a union of a finite number of disjoint closed intervals. Its complement consists of a union of open intervals. First we observe that if U is the infinite interval on the right $\dot{x} = (D(t) - \lambda \mathbf{I}_n)x$ is uniformly asymptotically stable on $\mathbb{R}_+$ and $\mathbb{R}_-$ and hence on $\mathbb{R}$ so that $U \subset \sigma(D)^c$. Similarly the infinite interval on the left is in $\sigma(D)^c$.

In the following theorem, we show that the whole of each of the other intervals is in the spectrum of D or not in the spectrum.

Theorem 4. *Let $D(t)$ be bounded on $\mathbb{R}$. Then if I is a bounded connected component of the complement of $\sigma_+(D) \cup \sigma_-(D)$, each of the two conditions*

(a) $i(D - \lambda \mathbf{I}_n) = 0$,
(b) $\dot{z} = (D(t) - \lambda \mathbf{I}_n)z$ *has no nontrivial bounded solution,*

either holds for all $\lambda \in I$ *or for none. So* $I \subset \sigma(D)^c$ *if and only if (a) and (b) both hold. Otherwise* $I \subset \sigma(D)$.

Proof. Suppose for a fixed λ, $\dot{u} = (D(t) - \lambda \mathbf{I}_n)u$ has exponential dichotomies on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections P_+ and P_- respectively. If $|E(t)| \leq \delta$ and δ is sufficiently small, then $\dot{z} = [D(t) - \lambda \mathbf{I}_n + E(t)]z$ also has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projections Q_+ and Q_- close to P_+ and P_- and, in particular, with the same rank. It follows that

$$\imath(D - \lambda \mathbf{I}_n + E) = \dim \mathcal{R}Q_+ + \dim \mathcal{N}Q_- - n = \dim \mathcal{R}P_+ + \dim \mathcal{N}P_- - n = \imath(D - \lambda \mathbf{I}_n).$$

Now suppose $I \subset \mathbb{R}$ is a component, and hence an open interval, of the complement of $\sigma_+(D) \cup \sigma_-(D)$. Then from the above result, $i(D - \lambda \mathbf{I}_n)$ is a continuous integer-valued function of λ. It follows that if it is 0 for some λ in I, it must be 0 throughout I.

Next if $\lambda \in I$, since $I \subset \sigma_+(D)^c \cap \sigma_-(D)^c$, it is known from spectral theory (see [10] and [11]) that the range of the projection P_+ and the kernel of P_- for the dichotomy of $\dot{u} = (D(t) - \lambda \mathbf{I}_n)u$ on $\mathbb{R}_+$ and $\mathbb{R}_-$ do not vary with λ. Hence the subspace $\mathcal{R}P_+ \cap \mathcal{N}P_-$ of initial values of bounded solutions of $\dot{u} = (D(t) - \lambda \mathbf{I}_n)u$ does not change as λ varies through I. So if it is $\{0\}$ for some λ, it is $\{0\}$ for all λ. $\square$

Remark 6. It follows from the remarks before Theorem 4 that $\sigma(D) \subset [a, b]$, where $[a, b]$ is the smallest closed interval containing $\sigma_+(D) \cup \sigma_-(D)$.

Now suppose $\dot{u} = D(t)u$ is the block upper triangular system (1). We show how to determine the spectrum of D from information about the spectra of A and B.

For half lines, it is simple. Since

$$D(t) - \lambda \mathbf{I}_n = \begin{pmatrix} A(t) - \lambda \mathbf{I}_d & C(t) \\ 0 & B(t) - \lambda \mathbf{I}_{n-d} \end{pmatrix},$$

from Theorems 1 and 2, it immediately follows that

$$\lambda \notin \sigma_\pm(D) \Leftrightarrow \lambda \notin \sigma_\pm(A) \cup \sigma_\pm(B)$$

or, equivalently:

$$\sigma_\pm(D) = \sigma_\pm(A) \cup \sigma_\pm(B).$$

In the whole line case, it is more complicated to determine $\sigma(D)$ and we give a procedure for it. First it follows from Corollary 1 that:

$$\lambda \notin \sigma(A) \cup \sigma(B) \Rightarrow \lambda \notin \sigma(D)$$

or, equivalently:

$$\sigma(D) \subset \sigma(A) \cup \sigma(B).$$

We also know that

$$[\sigma(A) \cap \sigma(B)^c] \cup [\sigma(B) \cap \sigma(A)^c] \subset \sigma(D),$$

since from Proposition 3, if $\lambda \in \sigma(D)^c$, then λ is in both $\sigma(A)^c$ and $\sigma(B)^c$ or in neither; we also know that

$$\sigma_+(D) \cup \sigma_-(D) = \sigma_+(A) \cup \sigma_-(A) \cup \sigma_+(B) \cup \sigma_-(B) \subset \sigma(D).$$

The question remains: which part of $[\sigma(A) \cap \sigma(B)] \setminus [\sigma_+(A) \cup \sigma_-(A) \cup \sigma_+(B) \cup \sigma_-(B)]$ is in $\sigma(D)$?

With a view to answering this question, let I be a component of $[\sigma(A) \cap \sigma(B)] \setminus [\sigma_+(A) \cup \sigma_-(A) \cup \sigma_+(B) \cup \sigma_-(B)]$. If $\lambda \in I$, $\dot{x} = (A(t) - \lambda \mathbf{I}_d)x$ has exponential dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ but not on $\mathbb{R}$ and similarly $\dot{y} = (B(t) - \lambda \mathbf{I}_{n-d})y$. We divide the discussion into three cases.

Case 1: $\dot{x} = (A(t) - \lambda \mathbf{I}_d)x$ has a nontrivial bounded solution. Then $\dot{u} = (D(t) - \lambda \mathbf{I}_n)u$ has a nontrivial bounded solution and so $\lambda \in \sigma(D)$.

Case 2: $\dot{x} = (A(t) - \lambda \mathbf{I}_d)x$ has no nontrivial bounded solutions and $\imath(A - \lambda \mathbf{I}_d) + \imath(B - \lambda \mathbf{I}_{n-d}) \neq 0$. Then $\lambda \in \sigma(D)$.

Case 3: $\dot{x} = (A(t) - \lambda \mathbf{I}_d)x$ has no nontrivial bounded solutions but $\imath(A - \lambda \mathbf{I}_d) + \imath(B - \lambda \mathbf{I}_{n-d}) = 0$. Then $\lambda \in \sigma(D)$ if and only if there exists a nonzero bounded solution of $\dot{y} = (B(t) - \lambda \mathbf{I}_{n-d})y$ such that $\int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt = 0$ for all bounded solutions $\psi(t)$ of $\dot{x} = -(A^*(t) - \lambda \mathbf{I}_d)x$. Note that this is the only situation where $C(t)$ plays a role.

Observe that I is contained in the complements of $\sigma_+(A)$ and $\sigma_-(A)$. It follows that the subspaces of bounded solutions of $\dot{x} = (A - \lambda \mathbf{I}_d)x$ and $\dot{x} = -(A^* - \lambda \mathbf{I}_d)x$ do not vary as λ as varies through I. Similarly, the subspace of bounded solutions of $\dot{y} = (B - \lambda \mathbf{I}_{n-d})y$ does not vary. It follows that the conditions in the three cases above are the same for all $\lambda \in I$.

We illustrate the procedure with two examples.

Example. Consider the system:

$$\begin{cases} \dot{x} = x \operatorname{sgn}(t) + c(t)y \\ \dot{y} = -y \operatorname{sgn}(t) \end{cases}$$

Here $\sigma_+(A) = \{1\} = \sigma_-(B)$, $\sigma_-(A) = \{-1\} = \sigma_+(B)$ and $\sigma(A) = \sigma(B) = [-1, 1]$. Then the only component of $[\sigma(A) \cap \sigma(B)] \setminus [\sigma_+(A) \cup \sigma_-(A) \cup \sigma_+(B) \cup \sigma_-(B)]$ is $I = (-1, 1)$.

If $\lambda \in I$, $\dot{x} = (A - \lambda)x$ has no nontrivial bounded solution, $\imath(A - \lambda) = 0 + 0 - 1 \neq 0$ and $\imath(A - \lambda) + \imath(B - \lambda) = -1 + (1 + 1 - 1) = 0$. So we are in Case 3. We may take $\lambda = 0$. Then, up to a scalar multiple, $\psi(t) = e^{-|t|}$ is the unique bounded solution of $\dot{x} = -A^*(t)x$ and, up to a scalar multiple, $y(t) = \psi(t)$ is the unique bounded solution of $\dot{y} = B(t)y$. Then

$$\int_{-\infty}^{\infty} \psi^*(t)C(t)y(t)dt = \int_{-\infty}^{\infty} e^{-2|t|}c(t)dt.$$

We conclude that $\sigma(D) = [-1, 1]$ if $\int_{-\infty}^{\infty} e^{-2|t|}c(t)dt = 0$; otherwise it is $\{-1, 1\}$.

Example. Suppose system (1) is given by

$$A(t) = \begin{cases} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix} & t \geq 0 \\ \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix} & t \leq 0, \end{cases} \qquad C(t) = \begin{pmatrix} c(t) \\ d(t) \end{pmatrix} \quad \text{and} \quad B(t) = \begin{cases} \gamma_1 & t \geq 0 \\ \gamma_2 & t \leq 0, \end{cases}$$

where

$$\beta_2 < \gamma_1 < \alpha_1 < \alpha_2 < \gamma_2 < \beta_1.$$

Note that

$$\sigma_+(A) = \{\alpha_1, \beta_1\}, \quad \sigma_-(A) = \{\alpha_2, \beta_2\}, \quad \sigma_+(B) = \{\gamma_1\}, \quad \sigma_-(B) = \{\gamma_2\}.$$

We determine $\sigma(A)$. We know $\sigma(A) \subset [\beta_2, \beta_1]$. We find that

$$\dim \mathcal{R}P_+^{A-\lambda I_2} = \begin{cases} 1 & \alpha_1 < \lambda < \beta_1 \\ 0 & \beta_2 < \lambda < \alpha_1, \end{cases} \quad \dim \mathcal{N}P_-^{A-\lambda I_2} = \begin{cases} 0 & \beta_1 > \lambda > \alpha_2 \\ 1 & \beta_2 < \lambda < \alpha_2 \end{cases}$$

so that

$$\imath(A - \lambda \mathbf{I}_2) = \begin{cases} -1 & \alpha_2 < \lambda < \beta_1 \\ 0 & \alpha_1 < \lambda < \alpha_2 \\ -1 & \beta_2 < \lambda < \alpha_1. \end{cases}$$

We find that $\dot{x} = (A - \lambda \mathbf{I}_2)x$ has a nontrivial bounded solution when $\lambda \in (\alpha_1, \alpha_2)$ but none when λ in (β_2, α_1) or (α_2, β_1). The conclusion is that $\sigma(A) = [\beta_2, \beta_1]$.

Next it is easy to see that $\sigma(B) = [\gamma_1, \gamma_2]$. So $\sigma(A) \cap \sigma(B) = [\gamma_1, \gamma_2]$. We know that $[\beta_2, \gamma_1) \cup (\gamma_2, \beta_1] = \sigma(A) \setminus \sigma(B)$ and hence both intervals are in $\sigma(D)$. The three components of $[\sigma(A) \cap \sigma(B)] \setminus [\sigma_+(A) \cup \sigma_-(A) \cup \sigma_+(B) \cup \sigma_-(B)]$ are $I_1 = (\gamma_1, \alpha_1)$, $I_2 = (\alpha_1, \alpha_2)$ and $I_3 = (\alpha_2, \gamma_2)$.

When $\lambda \in I_1$, $\dot{x} = (A(t) - \lambda \mathbf{I}_2)x$ has no nontrivial bounded solution and $\imath(A - \lambda \mathbf{I}_2) + \imath(B - \lambda) = -1 + 1 = 0$. So we are in Case 3. Up to a scalar multiple, the unique bounded solution of $\dot{x} = -(A^*(t) - \lambda \mathbf{I}_2)x$ is

$$\psi(t) = \begin{cases} \begin{bmatrix} 0 \\ e^{-(\beta_2 - \lambda)t} \end{bmatrix} & t \le 0 \\ \begin{bmatrix} 0 \\ e^{-(\beta_1 - \lambda)t} \end{bmatrix} & t \ge 0 \end{cases} \tag{17}$$

and the unique bounded solution of $\dot{y} = (B(t) - \lambda)y$ is

$$y(t) = \begin{cases} e^{(\gamma_2 - \lambda)t} & t \le 0 \\ e^{(\gamma_1 - \lambda)t} & t \ge 0. \end{cases} \tag{18}$$

Then $I_1 \subset \sigma(D)$ if and only if

$$\int_{-\infty}^{\infty} \psi^*(t) C(t) y(t) dt = \int_{-\infty}^{0} e^{(\gamma_2 - \beta_2)t} d(t) dt + \int_{0}^{\infty} e^{(\gamma_1 - \beta_1)t} d(t) dt = 0. \tag{19}$$

When $\lambda \in I_2$, $\dot{x} = (A - \lambda \mathbf{I}_2)x$ has a nontrivial bounded solution and so $I_2 \subset \sigma(D)$.

When $\lambda \in I_3$, $\dot{x} = (A - \lambda \mathbf{I}_2)x$ has no nontrivial bounded solution and $\imath(A - \lambda \mathbf{I}_2) + \imath(B - \lambda) = -1 + 1 = 0$. So we are in Case 3 again. Up to a scalar multiple, the unique bounded solution $\psi(t)$ of $\dot{x} = -(A^*(t) - \lambda \mathbf{I}_2)x$ is as in (17) and the unique bounded solution $y(t)$ of $\dot{y} = (B(t) - \lambda)y$ is as in (18). Then $I_3 \subset \sigma(D)$ if and only if (19) holds.

The conclusion is that $\sigma(D) = [\beta_2, \beta_1]$ if (19) holds; otherwise $\sigma(D) = [\beta_2, \gamma_1] \cup [\alpha_1, \alpha_2] \cup [\gamma_2, \beta_1]$.

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