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*–Brief Paper–*

# ASYMPTOTIC STABILITY AND STABILIZATION FOR A CLASS OF NONLINEAR DESCRIPTOR SYSTEMS WITH DELAY

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**ABSTRACT**

This paper discusses asymptotic stability and stabilization for a class of nonlinear descriptor systems with delay. The nonlinearity of the system is a continuous function of the time and system state, and the Jacobi matrix of the function is norm-bounded. A sufficient condition for the existence and uniqueness of the solution to the descriptor system is proposed by a linear matrix inequality (LMI) approach. Under the condition, using nonlinear methods, the asymptotic stability for the system is obtained. In addition, to stabilize the descriptor system, a parameterized representation of the state feedback controller is given in terms of a solution to an LMI. Finally, the effectiveness of the approach is illustrated by numerical examples.

***Key Words:*** Nonlinear descriptor system, delay, asymptotic stability and stabilization, linear matrix inequality, state feedback.

## I. INTRODUCTION

Time-delay commonly appears in many practical systems [1–3]. To a given descriptor system with delay (DSD), the existence and uniqueness of a solution is not always guaranteed and the system may also have undesired impulsive behavior [1, 3]. It has been pointed out that such a system could be destabilized by a small delay in feedback [4]. These characteristics make DSD remarkably different from non-singular systems. The stability issue of DSD is, therefore, of theoretical and practical value.

Recently, many important results have been reported on linear DSD. The Lyapunov second method, for example, has been developed for the system [1], and some sufficient conditions for delay-dependent or delay-independent stability are given in terms of the Lyapunov method and linear matrix inequality (LMI) approach [1, 5, 6]. The stability for these systems with time-varying delay, robust stability, and stabilization for some uncertain descriptor systems have also been discussed [3, 7, 8] using an LMI approach. Unfortunately, few projects have dealt with descriptor systems with both delay and nonlinearity in terms of an LMI approach. In [9], the generalized quadratic stabilization for discrete-time descriptor systems with delay and nonlinear perturbation is discussed and the state feedback control gain is determined by a complex nonlinear matrix inequality that is difficult to solve. In [10], when the nonlinearity satisfies a sector condition, an absolute stability criterion for nonlinear DSD is given. To the best of the authors' knowledge, the issue of continuous nonlinear DSD has not been fully investigated. Many results for linear systems have not been extended yet to nonlinear systems. The issue, therefore, remains important and challenging.

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In this paper, we consider a class of nonlinear DSD, in which the nonlinearity is a continuous function of time and system state. A sufficient condition for the existence and uniqueness of the solution to the system is proposed by an LMI approach. Then, the Lyapunov stability theory is developed by nonlinear methods found in [11, 12]. Based on the methods, asymptotic stability for the system is obtained. Furthermore, to stabilize the system, a parameterized representation of a state feedback controller is given. Finally, the effectiveness of the approaches is illustrated by numerical examples.

**Notations**. Let $R^n$ be the Euclidean space with vector norm $\|\cdot\|$ and $C[a,b]$ be the space of continuous vector functions with the supremum norm $\|\boldsymbol{x}(t)\| = \max_{a\le x\le b}\|\boldsymbol{x}(t)\|$. The notation $\boldsymbol{P}>0$ for the $n\times n$ matrix means that $\boldsymbol{P}$ is symmetric and positive definite. Symmetric terms in symmetric matrices are denoted by $*$; *e.g.*

$$\begin{bmatrix}\boldsymbol{A} & *\\ \boldsymbol{B}^T & \boldsymbol{C}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ * & \boldsymbol{C}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{B}^T & \boldsymbol{C}\end{bmatrix}.$$

## II. PRELIMINARIES

Consider the following nonlinear DSD:

$$\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}_d\boldsymbol{x}_d + \boldsymbol{f}(t,\boldsymbol{x}), \tag{1}$$

$$\boldsymbol{x}(t) = \boldsymbol{b}(t), \quad t\in[-d,0], \tag{2}$$

where $\boldsymbol{E},\boldsymbol{A},\boldsymbol{A}_d\in R^{n\times n}$ are constant matrices, $0<\text{rank}\boldsymbol{E} = r<n$, $\boldsymbol{x}(t)\in R^n$ is the system state, $\boldsymbol{b}(t)\in R^n$ is a continuous and compatible initial state [1, 13], $\boldsymbol{x}_d = \boldsymbol{x}(t-d)$, $d>0$ is delay, and $f(t,\boldsymbol{x}(t))\in R^n$ is a nonlinear function.

We introduce the following hypothesis:

(H) The function $\boldsymbol{f}(t,\boldsymbol{x}(t))$ with $\boldsymbol{f}(t,0)=0$ is continuous with respect to $t\in[0,+\infty)$, is differentiable with respect to $\boldsymbol{x}\in R^n$, and there exists a constant matrix $\boldsymbol{F}$ such that:

$$\boldsymbol{J}^T\boldsymbol{J} \le \alpha^2\boldsymbol{F}^T\boldsymbol{F},\, 0\le t<+\infty, \boldsymbol{x}(t)\in R^n, \tag{3}$$

where $\mathbf{J}=\mathbf{J}(t,\mathbf{x}(t))$is the Jacobi matrix of$\mathbf{f}(t,\mathbf{x}(t))$ with respect to$\mathbf{x}$.

**Remark II.1.** Via the differential and integral theorem of vector function [11], the hypothesis (H) implies that for $0\le t<+\infty$, $\boldsymbol{x}_1, \boldsymbol{x}_2\in R^n$

$$\|\boldsymbol{f}(t,\boldsymbol{x}_1)-\boldsymbol{f}(t,\boldsymbol{x}_2)\| \le \alpha\|\boldsymbol{F}(\boldsymbol{x}_1-\boldsymbol{x}_2)\|. \tag{4}$$

Thus, $\boldsymbol{f}(t,\boldsymbol{x}(t))$ is Lipschitz continuous and:

$$\boldsymbol{f}(t,\boldsymbol{x})^T\boldsymbol{f}(t,\boldsymbol{x}) \le \alpha^2\boldsymbol{x}^T\boldsymbol{F}^T\boldsymbol{F}\boldsymbol{x},$$

$$0\le t<+\infty, \boldsymbol{x}\in R^n \tag{5}$$

The following definitions and lemmas will be used in this paper.

**Definition II.1.** The pair $(\boldsymbol{E},\boldsymbol{A})$ is said to be regular if $\det(s\boldsymbol{E}-\boldsymbol{A})$ is not zero; the pair $(\boldsymbol{E},\boldsymbol{A})$ is said to be impulse free if $\deg(\det(s\boldsymbol{E}-\mathbf{A})) = \text{rank}(\boldsymbol{E})$.

**Definition II.2.** Nonlinear DSD (1) is said to be asymptotically stable if $\lim_{t\to+\infty}\boldsymbol{x}(t)=0$ for an arbitrary solution $\boldsymbol{x}(t)$ to System (1).

**Lemma II.1** (*Barbalat's Lemma [1]*)**.** If $h(t)$ is uniformly continuous on $[0,+\infty)$ and $\int_0^{+\infty}|h(t)|dt<+\infty$, then $\lim_{t\to+\infty}h(t)=0$.

**Lemma II.2.** Let $\boldsymbol{h}(\boldsymbol{x})\in R^n$ be differentiable, $\boldsymbol{T}\boldsymbol{x}=\boldsymbol{x}+\boldsymbol{h}(\boldsymbol{x}), \boldsymbol{x}\in R^n$. If there exists a constant $0<q_1<1$ such that

$$\|\boldsymbol{h}(\boldsymbol{x}_1)-\boldsymbol{h}(\boldsymbol{x}_2)\| \le q\|\boldsymbol{x}_1-\boldsymbol{x}_2\|, \quad \forall \boldsymbol{x}_1,\boldsymbol{x}_2\in R^n, \tag{6}$$

then $\boldsymbol{T}$ is a homeomorphism mapping.

The proof of Lemma II.2 is omitted.

**Lemma II.3** (See the proof of Lemma II.2 and Lemma II in [13])**.** If there exists a matrix $\boldsymbol{P}\in R^{n\times n}$ that satisfies $\boldsymbol{E}^T\boldsymbol{P}=\boldsymbol{P}^T\boldsymbol{E}\ge 0$, $\boldsymbol{P}^T\boldsymbol{A}+\boldsymbol{A}^T\boldsymbol{P}<0$, then the pair $(\boldsymbol{E},\boldsymbol{A})$ is regular, impulse free and there are two nonsingular matrices $\boldsymbol{M}^T=(\boldsymbol{M}_1^T,\boldsymbol{M}_2^T)$, $\boldsymbol{N}=(\boldsymbol{N}_1,\boldsymbol{N}_2)$, $\boldsymbol{M}_1\in R^{r\times n}$, $\boldsymbol{M}_2\in R^{(n-r)\times n}$, $\boldsymbol{M}_2\boldsymbol{M}_2^T=\boldsymbol{I}_{n-r}, \boldsymbol{N}_1\in R^{n\times r}$, $\boldsymbol{N}_2\in R^{n\times(n-r)}$ such that $\boldsymbol{MEN}=\text{diag}(\boldsymbol{I}_r 0)$, $\boldsymbol{MAN}=\text{diag}(\boldsymbol{A}_r\boldsymbol{I}_{n-r}), \boldsymbol{A}_r\in R^{r\times r}$.

Let $\boldsymbol{M}^{-T}\boldsymbol{P}\boldsymbol{N}=\begin{pmatrix}\boldsymbol{P}_1 & \boldsymbol{P}_2\\ \boldsymbol{P}_3 & \boldsymbol{P}_4\end{pmatrix}$. Moreover, we have $\boldsymbol{P}_1>0$.

## III. EXISTENCE AND UNIQUENESS OF SOLUTION

**Theorem III.1.** Under (H) if there exist matrices $\boldsymbol{P}$, $\boldsymbol{Q}>0$ and a constant $\delta>0$ such that

$$\boldsymbol{E}^T\boldsymbol{P}=\boldsymbol{P}^T\boldsymbol{E}\ge 0 \tag{7}$$

$$\boldsymbol{A}^T\boldsymbol{P}+\boldsymbol{P}^T\boldsymbol{A}+\boldsymbol{Q}+\delta^{-1}\boldsymbol{P}^T\boldsymbol{P}+\delta\alpha^2\boldsymbol{F}^T\boldsymbol{F} + \boldsymbol{P}^T\boldsymbol{A}_d\boldsymbol{Q}^{-1}\boldsymbol{A}_d^T\boldsymbol{P}<0, \tag{8}$$

then there exists a unique continuous function $\boldsymbol{x}(t)\in C[0,+\infty)$ that satisfies (1) and (2).

**Proof.** LMI (8) implies $\boldsymbol{P^TA}+\boldsymbol{A^TP}<0$. It follows from [5, 13] that the pair $(\boldsymbol{E},\boldsymbol{A})$ is regular and impulse free. Furthermore, from Lemma II, there exist matrices $\boldsymbol{M},\boldsymbol{N}$ that satisfy all of the properties in Lemma II.3. Also, LMI (8) implies:

$$\boldsymbol{P^TA}+\boldsymbol{A^TP}+\delta^{-1}\boldsymbol{P^TP}+\delta\alpha^2\boldsymbol{F^TF}<0 \tag{9}$$

Thus, there exists a constant $0<q_1<1$ such that (see [13] Lemma II.2):

$$\alpha\|\boldsymbol{F}N_2\|<q_1<1. \tag{10}$$

Let $z$=col$(\boldsymbol{z_1}\ \boldsymbol{z_2})=\boldsymbol{N^{-1}x}$, $\boldsymbol{z_1}\in R^r$, $z_2\in R^{n-r}$, $\boldsymbol{z_d}=$ col$(\boldsymbol{z_{1d}}\ \boldsymbol{z_{2d}})$, $\boldsymbol{MA_d}N=\begin{pmatrix} A_{d11} & A_{d12}\\ A_{d21} & A_{d22}\end{pmatrix}$. Then, System (1) is equivalent to:

$$\begin{aligned}\dot{\boldsymbol{z}}_1 &= \boldsymbol{A_r z_1}+(\boldsymbol{A_{d11}}\quad \boldsymbol{A_{d12}})\boldsymbol{z_d}\\ &\quad +\boldsymbol{M_1 f}(t,\boldsymbol{N_1z_1}+\boldsymbol{N_2z_2}),\end{aligned} \tag{11}$$

$$\begin{aligned}0 &= \boldsymbol{z_2}+(\boldsymbol{A_{d21}}\quad \boldsymbol{A_{d22}})\boldsymbol{z_d}\\ &\quad +\boldsymbol{M_2 f}(t,\boldsymbol{N_1z_1}+\boldsymbol{N_2z_2}).\end{aligned} \tag{12}$$

Let $l>0$ be some integer, $t_0=ld$, the mapping $\boldsymbol{Tz_2}(\bar t)=\boldsymbol{z_2}(\bar t)+\boldsymbol{M_2f}(\bar t,\boldsymbol{N_1z_1}(\bar t)+\boldsymbol{N_2z_2}(\bar t))$, where $\boldsymbol{z_1}(t)$ is a given continuous function on $[t_0,t_0+d]$, $\bar t\in[t_0,t_0+d]$ is a fixed point. Suppose that there exists a unique continuous solution to (11), (12) on $[-d,t_0]$. We will demonstrate by induction that there exists a unique continuous solution to (11), (12) on $[t_0,t_0+d]$.

From (4), (10) and by Lemma II.2, $\boldsymbol{T}$ is a homeomorphism mapping. As $z_d(\bar t)$ has been discovered by induction, there exists a unique solution $\boldsymbol{z_2}(\bar t)$ that satisfies (12) for all fixed $\bar t\in[t_0,t_0+d]$. Furthermore, from (4) and (10), it can be shown that $z_2(t)$ is continuous on $[t_0,t_0+d]$ and $z_2(t)=\varphi(t,z_1(t))$, $\boldsymbol{f}(t,\boldsymbol{N_1z_1}+\boldsymbol{N_2z_2})$ are Lipschitz continuous with respect to $z_1(t)$. Thus, by the well-known Picard-Lindelof Theorem for the existence and uniqueness of a solution to ordinary differential equations (see [13] Lemma II.2), there exists a unique solution $z_1(t)$ to the differential Equation (11), which completes the proof. □

**Remark III.1.** The condition (8) is somewhat conservative Actually Theorem III.1 also holds under (9). So Theorem III.1 is an extension of the main results in [13].

**Remark III.2.** When $\boldsymbol{f}(t,\boldsymbol{x})=0$ from the proof of Theorem III.1, it can be shown that if the pair $(\boldsymbol{E},\boldsymbol{A})$ is regular and impulse free and if $\boldsymbol{b}(t)$ is continuous and compatible, then the conclusion of Theorem III.1 also holds. So Theorem III.1 is an extension of Lemma III in [3].

## IV. ASYMPTOTIC STABILITY FOR NONLINEAR DESCRIPTOR SYSTEM

**Lemma IV.1.** Let $\boldsymbol{x}(t)=$ col$(\boldsymbol{x_1}(t),\boldsymbol{x_2}(t))$ be the solution to System (1), $\boldsymbol{z}=\boldsymbol{N^{-1}x}$. Under the assumptions of Theorem III.1, $\|\boldsymbol{z_1}(t)\|$ and $\int_0^t\|z\|^2d\xi$ are bounded.

**Proof.** LMI (8) is equivalent to:

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{A^TP}+\boldsymbol{P^TA}+\boldsymbol{Q}+\delta^{-1}\boldsymbol{P^TP}+\delta\alpha^2\boldsymbol{F^TF} & \boldsymbol{P^TA_d}\\ * & -\boldsymbol{Q}\end{bmatrix}<0 \tag{13}$$

If there exist matrices $\boldsymbol{P}$, $\boldsymbol{Q}>0$ and a constant $\delta>0$ such that (7) and (8) hold, then we can choose the Lyapunov function candidate as follows:

$$V(\boldsymbol{x}(t))=\boldsymbol{x^TP^TEx}+\int_{t-d}^t \boldsymbol{x}^T(s)\boldsymbol{Qx}(s)ds. \tag{14}$$

The derivative of $V$ along the trajectory of System (1) yields:

$$\begin{aligned}\dot V &= 2\boldsymbol{x^TP^T}[\boldsymbol{Ax}+\boldsymbol{A_dx_d}+\boldsymbol{f}(t,\boldsymbol{x})]+\boldsymbol{x^TQx}-\boldsymbol{x_d^TQx_d}\\ &\le (\boldsymbol{x^T}\quad \boldsymbol{x_d^T})\boldsymbol{H}(\boldsymbol{x^T}\quad \boldsymbol{x_d^T})^T<0.\end{aligned}$$

Moreover, there is a constant $\lambda_0>0$ such that $\dot V\le -\lambda_0(\|z\|^2+\|z_d\|^2)<0$. Let $\boldsymbol{P_1}$ be determined by Lemma II, $\lambda_1=\lambda_{\min}(\boldsymbol{P}_1)>0$. It follows that:

$$\begin{aligned}\lambda_1\boldsymbol{z_1^Tz_1}-V(0)&\le V(\boldsymbol{x}(t))-V(0)\\ &=\int_0^t \dot V[\boldsymbol{x}(\xi)]d\xi\le-\lambda_0\int_0^t[\|z\|^2+\|\boldsymbol{z_d}\|^2]d\xi.\end{aligned}$$

Thus, for arbitrary $t>0$, we have:

$$\lambda_1\boldsymbol{z_1^T}(t)\boldsymbol{z_1}(t)+\lambda_0\int_0^t\|z\|^2d\xi\le V(0). \tag{15}$$

This completes the proof. □

**Lemma IV.2.** Given $\boldsymbol{z_1}(t)$, let mapping $\boldsymbol{T}$ be defined as Theorem III.1 Under the assumptions of Theorem III.1, we have:

(i) The inverse matrix of derivative for $\boldsymbol{T}$ satisfies $\|\boldsymbol{T'^{-1}}\|\le(1-q_1)^{-1}$, where $0<q_1<1$ is a constant in (10);

(ii) There exists a constant $0<q_2<1$ and a nonsingular matrix $\boldsymbol{S}\in R^{(n-r)\times(n-r)}$ such that

$$\|\boldsymbol{S^{-1}T'^{-1}A_{d22}S}\|\le q_2<1 \tag{16}$$

holds uniformly for arbitrary $\bar t>0$, $z\in R^n$.

**Proof.** Note that $\|M_2 J_f(t,z)N_2\| \le \alpha\|FN_2\| \le q_1 < 1$.

Via Neumann Lemma (see [11] Theorem 2.3.1), (i) can be shown easily.

The proof of (ii):

Matrix inequality (8) is equivalent to:

$$\begin{bmatrix} N^T(A^TP+P^TA+Q+\delta^{-1}P^TP+\delta\alpha^2F^TF)N & N^TP^TA_dN \\ * & -N^TQN \end{bmatrix} <0. \tag{17}$$

Let $N^TQN=\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22}\end{pmatrix}$. It follows from (17) that:

$$\begin{pmatrix} P_4^T+P_4+\delta^{-1}P_4^TP_4+\delta\alpha^2N_2^TF^TFN_2+Q_{22} & P_4^TA_{d22} \\ A_{d22}^TP_4 & -Q_{22}\end{pmatrix} <0. \tag{18}$$

Let $\Omega=P_4^T+P_4+\delta^{-1}P_4^TP_4+\delta\alpha^2N_2^T\ F^TFN_2+P_4^TA_{d22}\ Q_{22}^{-1}A_{d22}^TP_4\ +Q_{22}$. (18) is equivalent to $\Omega<0$. Thus, for arbitrary $0\le\bar{t}<+\infty, z\in R^n$, we have $P_4^TT'+T'^TP_4+Q_{22}+P_4^TA_{d22}\ Q_{22}^{-1}A_{d22}^TP_4\le\Omega<0$. This yields:

$$-\Omega+P_4^TA_{d22}Q_{22}^{-1}A_{d22}^TP_4<P_4^TT'Q_{22}^{-1}T'^TP_4. \tag{19}$$

Notice that $P_4^TA_{d22}Q_{22}^{-1}A_{d22}^TP_4\ge 0$ and $-\Omega>0$ are constant matrices (independent of $\bar{t}$,$z$). So, there exists a constant $\tau>0$ such that:

$$-\Omega-\tau P_4^TA_{d22}Q_{22}^{-1}A_{d22}^TP_4>0. \tag{20}$$

From (19) and (20), we have $q_2^{-2}T'^{-1}A_{d22}\ Q_{22}^{-1}\ A_{d22}^TT'^{-T}<Q_{22}^{-1}$, where $q_2=(1+\tau)^{-\frac{1}{2}}$. Let $Q_{22}^{-1}=SS^T$. We realize (16). This completes the proof of (ii). □

**Theorem IV.1.** Under the assumptions of Theorem III.1, the solution of System (1) is asymptotically stable.

**Proof.** By Theorem III.1 and Lemma IV.1, there exists a unique solution $z=N^{-1}x$ to (1) and (2), and the solution $z_1$ is bounded. We can show that $z_2$ is also bounded.

In fact, let $\bar{t}\in[0,+\infty)$ be a fixed point, $T$ be the homeomorphism mapping defined as Theorem III.1. For $0\le\lambda\le 1$, we can construct a single-parameter function [12]:

$$p(\lambda)=-\lambda[(A_{d21}A_{d22})z(\bar{t}-d)]+(1-\lambda)M_2f(\bar{t},N_1z_1) \tag{21}$$

Since $T$ is a homeomorphism mapping, for arbitrary $p(\lambda)\in R^{n-r}$ there exists a $q(\lambda)\in R^{n-r}$ such that $Tq(\lambda)=p(\lambda)$, $q(0)=0$, $q(1)=z_2(\bar{t})$ and $q'(\lambda)=T'^{-1}(q(\lambda))p'(\lambda)$. It follows that [12]:

$$\begin{aligned}\|S^{-1}z_2(\bar{t})| &= \|S^{-1}[q(1)-q(0)]\| \\ &\le \int_0^1 \|S^{-1}q'(\lambda)\|d\lambda \\ &\le \int_0^1 \|S^{-1}T'^{-1}(q(\lambda)) \\ &\quad \times[(A_{d21}\ \ A_{d22})z_d(\bar{t})+M_2f(\bar{t},N_1z_1(\bar{t}))]\|d\lambda.\end{aligned}$$

Let $W(\bar{t})=\|S^{-1}\|\|A_{d21}z_{1d}(\bar{t})+M_2\ f(\bar{t},N_1z_1(\bar{t}))\|(1-q_1)^{-1}$. We have:

$$\|S^{-1}z_2(\bar{t})\|\le W(\bar{t})+\int_0^1 \|S^{-1}T'^{-1}(q(\lambda)) \times A_{d22}SS^{-1}z_{2d}\|d\lambda. \tag{22}$$

From Lemma IV.1, $\|z_1(\bar{t})\|$ and $f(\bar{t},N_1z_1(\bar{t}))$ are bounded. So, there is a constant $c_1>0$ such that $W(\bar{t})\le c_1$. It follows by Lemma IV.2 that:

$$\|S^{-1}z_2(\bar{t})\|\le c_1+q_2\|S^{-1}z_2(\bar{t}-d)\|. \tag{23}$$

Using (23) repeatedly, we get:

$$\|S^{-1}z_2(\bar{t})\|<c_1(1-q_2)^{-1}+\|S^{-1}\varphi\|,\quad \forall\bar{t}\in[0,+\infty).$$

Therefore, $z_2(t)$ namely $z(t)$, is bounded.

When $z(t)$ is bounded, $f(t,Nz(t))$, $\dot{z}_1(t)$ and $(\|z_1(t)\|^2)'=2z_1^T(t)z_1'(t)$ are also bounded. So, $\|z_1(t)\|^2$ is uniformly continuous on $[0,+\infty)$. From Lemma IV.1 and Barbalat's Lemma, we have:

$$\lim_{t\to+\infty}\|z_1(t)\|=0,\ \lim_{t\to+\infty}f(t,N_1z_1(t))=0,$$
$$\lim_{t\to+\infty}W(t)=0.$$

Moreover, from (22), for an arbitrary positive integral $i$, we have:

$$\begin{aligned}\|S^{-1}z_2(t+id)\| &\le W(t+id)+q_2W \\ &\quad\times(t+(i-1)d)+\cdots+q_2^iW(t)+q_2^i\|S^{-1}z_2(t)\|.\end{aligned}$$

This implies $\lim_{t\to+\infty}\|S^{-1}z_2(t)\|=0$ by the boundedness of $z_2(t)$ and $\lim_{i\to\infty}q_2^i=0$. Thus,

$$\lim_{t\to+\infty}\|z_2(t)\|=0,$$
$$\lim_{t\to+\infty}\|x(t)\|=\lim_{t\to\infty}\|Nz(t)\|=0.$$

This completes the proof. □

**Remark IV.1.** If $r=n$ without loss of generality, taking $\boldsymbol{E}=\boldsymbol{I}$, then (7) becomes $\boldsymbol{P}\geq 0$. (8) implies that $\boldsymbol{P}$ is nonsingular, thus $\boldsymbol{P}>0$. So, by Theorem IV.1, if there exist matrices $\boldsymbol{P}>0$ and $\boldsymbol{Q}>0$ such that (8) holds, then the solution of the non-singular system (1) is asymptotically stable. This can be obtained directly by the standard Lyapunov stability theory.

Often, it is not convenient to judge the asymptotic stability of System (1) by (7) and (8). We will transform Inequalities (7) and (8) into a LMI that can be solved using Matlab toolboxes.

**Theorem IV.2.** Let $\Phi\in R^{(n-r)\times n}$ satisfy $\Phi\boldsymbol{E}=0$. Under (H) if there exist matrices $\boldsymbol{R}>0$, $\boldsymbol{Q}>0$, $\boldsymbol{Y}\in R^{(n-r)\times n}$ and a constant $\delta>0$ such that

$$\begin{bmatrix} \boldsymbol{A}^T(\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y})+(\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y})^T\boldsymbol{A}+\boldsymbol{Q}+\delta\boldsymbol{\alpha}^2\boldsymbol{F}^T\boldsymbol{F} & * & * \\ \boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y} & -\delta\boldsymbol{I} & * \\ \boldsymbol{A}_d^T(\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y}) & 0 & -\boldsymbol{Q} \end{bmatrix} <0 \tag{24}$$

then there exists a unique and asymptotically stable solution to System (1).

**Proof.** If (1) holds for some $\boldsymbol{R},\boldsymbol{Y}$, taking $\boldsymbol{P}=\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y}$, then it can be shown that $\boldsymbol{P}$ and $\boldsymbol{Q}>0$ satisfy (7) and (8). So, all of the conditions in Theorem IV.1 is satisfied, which completes the proof. □

**Remark IV.2.** If $A_d=0$, taking $\boldsymbol{Q}=0$, $\delta=1$, then (1) is equivalent to:

$$\begin{pmatrix} \boldsymbol{A}^T(\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y})+(\boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y})^T\boldsymbol{A}+\boldsymbol{\alpha}^2\boldsymbol{F}^T\boldsymbol{F} & * \\ \boldsymbol{RE}+\boldsymbol{\Phi}^T\boldsymbol{Y} & -I \end{pmatrix}<0.$$

Thus, Theorem IV.2 is an extension of Theorem 2.2 in [13].

**Example IV.1.** Consider System (1) with:

$$\boldsymbol{E}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{A}=\begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix},$$

$$\boldsymbol{A}_d=\begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad f=\alpha\begin{pmatrix} \sin(x_1+x_2) \\ \sin x_2 \end{pmatrix}.$$

When $\alpha=0$, by applying the LMI of [14], the result is that the system is asymptotically stable for $d\leq 0.14$, and by applying the LMI of Theorem 1 in [1], the result is that the system is asymptotically stable for all delays. When $\alpha=0.3$, taking $F^TF=\alpha^2\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, by applying LMI (1), we obtain that the nonlinear system is asymptotically stable for all delays.

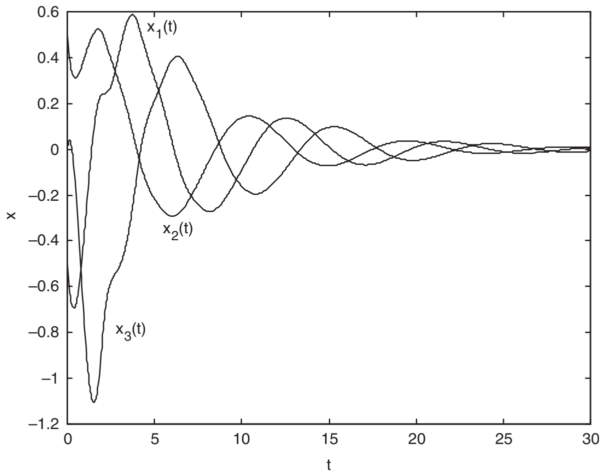


Fig. 1. The solutions to the nonlinear DSD of Example IV.2 when initial values are given.

**Example IV.2.** Consider System (1) with:

$$E=\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad A=\begin{pmatrix} -3 & -4 & -2 \\ 2 & 0 & -2 \\ -2 & 0 & -2 \end{pmatrix},$$

$$A_d=\begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0.5 \\ 0 & 0.5 & -1 \end{pmatrix}, \quad \boldsymbol{f}=0.6\begin{pmatrix} \sin(x_1+x_2) \\ 0 \\ \sin x_3 \end{pmatrix},$$

$$\boldsymbol{J}^T\boldsymbol{J}\leq \boldsymbol{F}^T\boldsymbol{F}=0.36\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using Matlab toolboxes, we can find out that $\boldsymbol{P}$ and $\boldsymbol{Q}$ satisfy (7) and (8). Thus, the system is asymptotically stable. Given compatible initial values $x_1(t)=t(t+d)-0.5$, $x_2(t)=\sin(\frac{\pi}{6}-\frac{4\pi}{3d}t)$, $x_3(t)=-2\sin(\frac{\pi}{4}t)$, $-2\leq t\leq 0$, $d=2$, the solutions to the nonlinear DSD are depicted in Fig. 1.

## V. STABILIZATION OF THE SYSTEM

In general, System (1) may not be stable. To stabilize System (1), we consider the following control system:

$$\boldsymbol{E}\dot{\boldsymbol{x}}=\boldsymbol{Ax}+\boldsymbol{A}_d\boldsymbol{x}_d+\boldsymbol{Bu}+\boldsymbol{f}(t,\boldsymbol{x}), \tag{25}$$

where $\boldsymbol{E}$, $\boldsymbol{A}$, $\boldsymbol{A}_d$, $\boldsymbol{x}$, $\boldsymbol{f}(t,\boldsymbol{x})$ are the same as those in System (1), $\boldsymbol{B}\in R^{n\times m}$ is constant matrix, $\boldsymbol{u}(t)\in R^m$ is the system input with the form $\boldsymbol{u}=\boldsymbol{K}\boldsymbol{x}, \boldsymbol{K}\in R^{m\times n}$. In this case, the closed-loop system of (25) is as follows:

$$\boldsymbol{E}\dot{\boldsymbol{x}}=(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K})\boldsymbol{x}+\boldsymbol{A_d}\boldsymbol{x_d}+\boldsymbol{f}(t,\boldsymbol{x}). \tag{26}$$

Our aim is to find a matrix $\boldsymbol{K}$ such that (26) displays asymptotic stability.

**Lemma V.1.** Matrix inequalities (7) and (8) are equivalent to the following matrix inequalities.

There exist $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{Q}}>0$ such that:

$$\boldsymbol{E}\bar{\boldsymbol{P}}^T=\bar{\boldsymbol{P}}\boldsymbol{E}^T\geq 0, \tag{27}$$

$$\begin{pmatrix} \boldsymbol{A}\bar{\boldsymbol{P}}^{\boldsymbol{T}}+\bar{\boldsymbol{P}}\boldsymbol{A}^{\boldsymbol{T}}+\bar{\boldsymbol{Q}}+\delta\boldsymbol{I} & \boldsymbol{A_d}\bar{\boldsymbol{P}}^{\boldsymbol{T}} & \boldsymbol{\alpha}\bar{\boldsymbol{P}}\boldsymbol{F}^{\boldsymbol{T}} \\ * & -\bar{\boldsymbol{Q}} & \boldsymbol{0} \\ * & * & -\delta\boldsymbol{I} \end{pmatrix}<0. \tag{28}$$

The proof is omitted.

**Theorem V.1.** Let $\Phi_1\in R^{n\times(n-r)}$ satisfy $\boldsymbol{E}\Phi_1=0$ and $\mathrm{rank}(\Phi_1)=n-r$. Under (H) if there exist positive definite matrices$\boldsymbol{R}$, $\bar{\boldsymbol{Q}}\in R^{n\times n}$, matrices $\boldsymbol{Z}\in R^{m\times n}$, $\boldsymbol{Y}\in R^{(n-r)\times n}$ and a constant $\delta>0$ such that

$$\begin{bmatrix} \boldsymbol{S} & A_d(\boldsymbol{R}\boldsymbol{E}^T+\boldsymbol{\Phi}\boldsymbol{Y}^T) & \alpha(\boldsymbol{R}\boldsymbol{E}+\boldsymbol{\Phi}^{\boldsymbol{T}}\boldsymbol{Y})^{\boldsymbol{T}}\boldsymbol{F}^{\boldsymbol{T}} \\ * & -\bar{\boldsymbol{Q}} & 0 \\ * & * & -\delta\boldsymbol{I} \end{bmatrix}<0, \tag{29}$$

where $\boldsymbol{S}=\boldsymbol{A}(\boldsymbol{R}\boldsymbol{E}^T+\boldsymbol{\Phi}\boldsymbol{Y}^T)+(\boldsymbol{R}\boldsymbol{E}^T+\boldsymbol{\Phi}\boldsymbol{Y}^T)^T\boldsymbol{A}^{\boldsymbol{T}}+\boldsymbol{B}\boldsymbol{Z}+\boldsymbol{Z}^{\boldsymbol{T}}\boldsymbol{B}^{\boldsymbol{T}}+\bar{\boldsymbol{Q}}+\delta\boldsymbol{I}$, then the closed-loop system (26) is asymptotically stable. In this case, the parameterized representation of state feedback controller is $\boldsymbol{K}=\boldsymbol{Z}(\boldsymbol{S_0}\boldsymbol{E}^{\boldsymbol{T}}+\boldsymbol{\Phi}\boldsymbol{Y}^{\boldsymbol{T}})^{-\boldsymbol{1}}$.

**Proof.** Let matrices $\boldsymbol{R}>0$, $\bar{\boldsymbol{Q}}>0$, $\boldsymbol{Z}$, $\boldsymbol{Y}$ and constant $\delta>0$ satisfy the conditions of Theorem V.1, $\bar{\boldsymbol{P}}=(\boldsymbol{R}\boldsymbol{E}^{\boldsymbol{T}}+\boldsymbol{\Phi}_1\boldsymbol{Y}^{\boldsymbol{T}})^T$, $\boldsymbol{K}=\boldsymbol{Z}\boldsymbol{P}^{-1}=\boldsymbol{Z}(\boldsymbol{S_0}\boldsymbol{E}^{\boldsymbol{T}}+\boldsymbol{\Phi}\boldsymbol{Y}^{\boldsymbol{T}})^{-\boldsymbol{1}}$, $\boldsymbol{A_K}=\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}$, then $\bar{\boldsymbol{P}}$ and $\bar{\boldsymbol{Q}}$ satisfy (28) and

$$\begin{pmatrix} \boldsymbol{A}_K\bar{\boldsymbol{P}}^{\boldsymbol{T}}+\bar{\boldsymbol{P}}\boldsymbol{A}_K^T+\bar{\boldsymbol{Q}}+\delta\boldsymbol{I} & \boldsymbol{A_d}\bar{\boldsymbol{P}}^{\boldsymbol{T}} & \boldsymbol{\alpha}\bar{\boldsymbol{P}}\boldsymbol{F}^{\boldsymbol{T}} \\ * & -\bar{\boldsymbol{Q}} & \boldsymbol{0} \\ * & * & -\delta\boldsymbol{I} \end{pmatrix}<0. \tag{30}$$

Thus, the conclusion of Theorem V.1 can be shown by Lemma V.1 and Theorem IV.1.

**Remark V.1.** If $\boldsymbol{f}(t,\boldsymbol{x})=0$, taking $\boldsymbol{F}=0, \delta=1$, then (31) is equivalent to:

$$\boldsymbol{A}_K\bar{\boldsymbol{P}}^{\boldsymbol{T}}+\bar{\boldsymbol{P}}\boldsymbol{A}_K^T+\bar{\boldsymbol{Q}}+\boldsymbol{A_d}\bar{\boldsymbol{P}}^{\boldsymbol{T}}\bar{\boldsymbol{Q}}^{-\boldsymbol{1}}\bar{\boldsymbol{P}}\boldsymbol{A_d^T}+\boldsymbol{I}<0. \tag{31}$$

(27) and (31) are equivalent to the following inequality: There exist matrices $\tilde{\boldsymbol{P}}$, $\tilde{\boldsymbol{Q}}>0$ such that

$$\boldsymbol{E}\tilde{\boldsymbol{P}}^T=\tilde{\boldsymbol{P}}\boldsymbol{E}^T\geq 0, \tag{32}$$

$$\boldsymbol{A}_K\tilde{P}^{\boldsymbol{T}}+\tilde{\boldsymbol{P}}\boldsymbol{A}_K^T+\tilde{\boldsymbol{Q}}+\boldsymbol{A_d}\tilde{P}^{\boldsymbol{T}}\tilde{\boldsymbol{Q}}^{-\boldsymbol{1}}\tilde{\boldsymbol{P}}\boldsymbol{A_d^T}<0. \tag{33}$$

(32) and (33) are stable criterion for linear closed-loop systems with delay in [5]. So, Theorem V.1 is an extension of the main result in [5].

**Example V.1.** Consider the control system (25) with

$$E=\begin{pmatrix}1&1&0\\1&-1&1\\2&0&1\end{pmatrix},\quad A=\begin{pmatrix}1.5&0.5&1\\-1&0&1\\0.5&0&1\end{pmatrix},$$

$$A_d=\begin{pmatrix}-1&0&-1\\1&-1&0.5\\0.3&0.5&-1\end{pmatrix},\quad B=\begin{pmatrix}1&1\\1&0\\0&1\end{pmatrix}.$$

$$f=\alpha\begin{pmatrix}\sin(x_1+x_2)\\0\\\sin x_3\end{pmatrix},\quad F=\alpha\begin{pmatrix}1&1&0\\0&0&0\\0&0&1\end{pmatrix},$$

$$\Phi=\begin{pmatrix}-1\\1\\2\end{pmatrix}.$$

When $\alpha=0$, the state feedback controller can be found in [5], thus, the closed-loop system is asymptotically stable. When $\alpha=0.4$, by solving inequality (31),

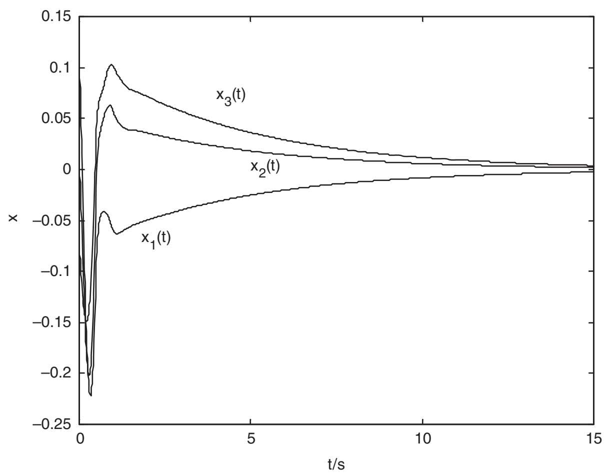


Fig. 2. The solutions to the closed-loop system of Example V.1.

we can find matrix $K=\begin{pmatrix} -32.23 & 49.46 & -47.99 \\ 78.67 & -172.21 & 142.32 \end{pmatrix}$ and matrices $\boldsymbol{R}>0$, $\bar{\boldsymbol{Q}}>0$, $\boldsymbol{Z}$, $\boldsymbol{Y}$. Thus, the closed-loop system is asymptotically stable. Given initial values

$$x_1(t)=0.1\cos 4\pi t, x_2(t)=5\sin 2\pi t, x_3(t)=-2\sin 3\pi t,$$

$-d\le t\le 0$, $d=0.5$, the solutions to the nonlinear DSD are depicted in Fig. 2.

## VI. CONCLUSION

Adopting an LMI approach combined with nonlinear analysis, an asymptotically stable condition for a nonlinear descriptor system with delay is obtained. A complete proof is given by the Lyapunov stability theory and nonlinear methods. The approach presented in this paper generalizes the results and techniques in the literature. Based on the approach, some sufficient conditions for delay-dependent stability can also be investigated.

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