additional outputs gives more access to system states. It is well known that optimal LQ regulators with complete state feedback have large stability margins. It is also noted that additional sensors may introduce new undesirable effects.

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Stability of the Second-Order Matrix Polynomial

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Abstract—**This note summarizes some existing stability and instability conditions of the second-order matrix polynomial which arises in the formulation of classical mechanics, aerodynamics, and robotic systems. Also, some sufficient conditions for stability or instability of the second-order matrix polynomial are newly developed via the relevant linear matrix equation obtained from the Lyapunov theory.**

I. INTRODUCTION

Many dynamical systems [1]–[6] can be modeled by a second-order vector differential equation as follows:

$$A\ddot{q}(t)+B\dot{q}(t)+Cq(t)=0_{m\times 1}; \qquad q(0)=\alpha_1,\ \dot{q}(0)=\alpha_2 \tag{1}$$

where $q(t) \in R^m$, $A \in R^{m\times m}$, $B \in R^{m\times m}$, $C \in R^{m\times m}$ and $0_{m\times 1}$ denotes an $m \times 1$ null vector. The constant coefficient matrices A, B, and C are known as the inertia, damping, and stiffness matrices, respectively, usually having special structures; for example, symmetric positive definite, and/or negative definite matrices, etc. The second-order characteristic matrix polynomial $D(s)$ becomes

$$D(s)=As^2+Bs+C=0_m \tag{2a}$$

where s is a complex variable and 0_m is an $m \times m$ null matrix. An alternative matrix polynomial $\bar{D}(s)$ can be constructed as follows:

$$\bar{D}(s)=D(s)|_{s=1/s}=Cs^2+Bs+A=0_m. \tag{2b}$$

Notice that the stability of $D(s)$ and $\bar{D}(s)$ is invariant.

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A primary concern in the design of the dynamical system is the stability problem. The stability of the system in (1) can be determined by either directly computing the roots of the scalar characteristic polynomial obtained from det $[D(s)]$ or det $[\bar{D}(s)]$, or indirectly applying the Routh criterion [7] or Jury's inner theory [8] to the scalar polynomial. However, determination of the characteristic polynomial often involves many large-dimensional matrix operations which are usually numerically difficult. When the characteristic matrix polynomial is formulated in (2a), direct determination of the stability from the matrix polynomial in (2a) is more natural than indirect determination of the stability from the characteristic polynomial. In this note, we determine the stability in (1) by directly testing the properties of the coefficient matrices A, B, and C in (2).

Let a matrix N be an $m \times m$ real matrix, and let $N > 0$, $N \geq 0$, $N < 0$, $N \leq 0$, and $N \gtreqless 0$ denote symmetric positive definite, symmetric positive semidefinite, symmetric negative definite, symmetric negative semidefinite, and symmetric indefinite matrices, respectively. Also, define $\sigma(N)$ as the spectrum of the matrix N. Some existing stability and instability conditions of the matrix polynomial in (2a) are summarized as follows.

Lemma 1: The system described in (1) is asymptotically stable

i) [9], [10] if $A > 0$, $B > 0$, and $C > 0$; or

ii) [11] if $AB^{-1} > 0$ and $BC^{-1} > 0$; or

iii) [12] if $B^{-1}A > 0$ and $C^{-1}B > 0$; or

iv) [12] if $B > 0$, $\sigma(A)$ and $\sigma(C)$ are positive real, $AB = (AB)^T$, and $CB = (CB)^T$.

Lemma 2 [13]: The system described in (1) is stable if $A > 0$ and if there exists a matrix S_1 such that $S_1 > 0$, $S_1A^{-1}C > 0$, and $S_1C^{-1}B \geq 0$.

Lemma 3: The system in (1) is unstable

i) [12] if $B^{-1}A = (B^{-1}A)^T$, $C^{-1}B = (C^{-1}B)^T$, and any of them is negative definite or indefinite; or

ii) [12] if $A = I_m$ (an $m \times m$ identity matrix) and trace $(B) < 0$; or

iii) [12] if det $(A) > 0$ and det $(C) < 0$; or

iv) [12] if det $(A) < 0$ and det $(C) > 0$; or

v) [13] if $A > 0$ and if there exists a matrix S_2 such that $S_2 > 0$, $S_2A^{-1}B = (S_2A^{-1}B)^T$ and $S_2A^{-1}C < 0$.

In [14], some stability conditions have been derived from the matrix fraction descriptions (MFD's) instead of a matrix polynomial. For illustration, we express a simple right MFD into a matrix continued fraction description, as follows:

$$\begin{aligned} G_r(s) &= [B_{2,1}s+B_{2,2}][B_{1,1}s^2+B_{1,2}s+I_m]^{-1} \\ &= [M_1s+[M_2+[M_3s+[M_4]^{-1}]^{-1}]^{-1}]^{-1} \end{aligned} \tag{3a}$$

where $B_{i,j} \in R^{m\times m}$ for $i = 1, 2$ and $j = 1, 2$ and the matrix quotients M_i for $i = 1, \cdots, 4$ can be determined from the following right matrix Routh algorithm:

$$B_{i,j}=B_{i-2,j+1}-M_{i-2}B_{i-1,j-1};$$

$$M_i=B_{i,1}B^{-1}_{i+1,1};\ B_{1,3}=I_m \text{ for } j=1, 2, \cdots;\ i=1, \cdots, 4.$$

The corresponding state equation for the matrix continued fraction description in (3a) can be written as follows:

$$\dot{\hat{z}}=\hat{G}\hat{z}+\hat{H}u;\ y=\hat{E}\hat{z} \tag{3b}$$

where

$$\hat{G}=\begin{pmatrix} -M_1^{-1}M_2^{-1} & M_1^{-1}M_2^{-1} \\ M_3^{-1}M_2^{-1} & -M_3^{-1}(M_2^{-1}+M_4^{-1}) \end{pmatrix},$$

$$\hat{H}=\begin{pmatrix} M_1^{-1} \\ 0_m \end{pmatrix},\ \hat{E}=(I_m\ 0_m).$$

If the matrix quotients M_1 and M_3 are symmetric matrices, then the

Lyapunov matrix equation [15] can be written as:

$$\hat{P}\hat{G}+\hat{G}^T\hat{P}=-\hat{Q} \tag{3c}$$

where the symmetric matrix $\hat{P}$ is chosen as $\hat{P}$ = block diag (M_1, M_3) and

$$\hat{Q}=\begin{pmatrix} \bar{M}_2 & -\bar{M}_2 \\ -\bar{M}_2 & (\bar{M}_2+\bar{M}_4) \end{pmatrix}=\begin{pmatrix} I_m & 0_m \\ -I_m & I_m \end{pmatrix}\begin{pmatrix} \bar{M}_2 & -\bar{M}_2 \\ 0_m & \bar{M}_4 \end{pmatrix}$$

with $\bar{M}_2 \triangleq M_2^{-1} + M_2^{-T}$ and $\bar{M}_4 \triangleq M_4^{-1} + M_4^{-T}$. Thus, we have the following results.

Lemma 4 [14]: The system described in (3a) is asymptotically stable

i) if $M_1 > 0$, $M_3 > 0$, $M_2 + M_2^T > 0$ and $M_4 + M_4^T > 0$; or

ii) if $M_2 > 0$, $M_4 > 0$, $M_1 + M_1^T > 0$, and $M_3 + M_3^T > 0$.

Lemma 4 ii) can be derived in the same manner.

Lemma 5 [14]: The system described in (3a) is unstable

i) if $M_2 + M_2^T > 0$, $M_4 + M_4^T > 0$, M_1 and M_3 are symmetric, and any of M_1 and M_3 are negative definite or indefinite; or

ii) if $M_1 + M_1^T > 0$, $M_3 + M_3^T > 0$, M_2 and M_4 are symmetric, and any of M_2 and M_4 are negative definite or indefinite.

II. MAIN RESULTS

Some existing results in Lemmas 1–5 can be extended as follows. Equation (2a) can be written in six different forms as shown below:

$$I_m s^2+A^{-1}Bs+A^{-1}C=0_m \tag{4a}$$

$$I_m s^2+BA^{-1}s+CA^{-1}=0_m \tag{4b}$$

$$B^{-1}As^2+I_m s+B^{-1}C=0_m \tag{4c}$$

$$AB^{-1}s^2+I_m s+CB^{-1}=0_m \tag{4d}$$

$$C^{-1}As^2+C^{-1}Bs+I_m=0_m \tag{4e}$$

$$AC^{-1}s^2+BC^{-1}s+I_m=0_m. \tag{4f}$$

Applying Lemma 1 i) [9] to various equations in (4) yields the following stability conditions.

Theorem 1: The system described in (1) is asymptotically stable

i) if $A^{-1}B > 0$ and $A^{-1}C > 0$; or

ii) if $BA^{-1} > 0$ and $CA^{-1} > 0$; or

iii) if $B^{-1}A > 0$ and $B^{-1}C > 0$; or

iv) if $AB^{-1} > 0$ and $CB^{-1} > 0$; or

v) if $C^{-1}A > 0$ and $C^{-1}B > 0$; or

vi) if $AC^{-1} > 0$ and $BC^{-1} > 0$.

Notice that the stability conditions developed in Theorem 1 iii) and iv) are equivalent to the conditions developed in Lemma 1 iii) and ii), respectively.

The results in Lemma 4 can be simplified and extended as follows. Let a matrix polynomial in (2) be chosen as the denominator of the MFD in (3a) and a special matrix polynomial, constructed from the coefficient matrices of the matrix polynomial in (2), be chosen as the numerator of the MFD. The resulting right MFD's become

$$\begin{aligned} G_{r1}(s)&=[As+(B-A)][As^2+Bs+C]^{-1} \\ &=[AC^{-1}s+(B-A)C^{-1}][AC^{-1}s^2+BC^{-1}s+I_m]^{-1} \end{aligned} \tag{5a}$$

and

$$\begin{aligned} G_{r2}(s)&=[Cs+(B-C)][Cs^2+Bs+A]^{-1} \\ &=[CA^{-1}s+(B-C)A^{-1}][CA^{-1}s^2+BA^{-1}s+I_m]^{-1}. \end{aligned} \tag{5b}$$

Also, left MFD's can be constructed as follows:

$$\begin{aligned} G_{l1}(s)&=[As^2+Bs+C]^{-1}[As+(B-A)] \\ &=[C^{-1}As^2+C^{-1}Bs+I_m]^{-1}[C^{-1}As+C^{-1}(B-A)] \end{aligned} \tag{5c}$$

and

$$\begin{aligned} G_{l2}(s)&=[Cs^2+Bs+A]^{-1}[Cs+(B-C)] \\ &=[A^{-1}Cs^2+A^{-1}Bs+I_m]^{-1}[A^{-1}Cs+A^{-1}(B-C)]. \end{aligned} \tag{5d}$$

Following the results developed in [14] and Lemma 4 in this note, we can determine some simple stability conditions as follows.

Theorem 2: The system described in (1) is asymptotically stable

i) if $A(B-A-C)^{-1} > 0$ and $(B-A-C)C^{-1} + [(B-A-C)C^{-1}]^T > 0$; or

ii) if $(B-A-C)C^{-1} > 0$ and $A(B-A-C)^{-1} + [A(B-A-C)^{-1}]^T > 0$; or

iii) if $C(B-C-A)^{-1} > 0$ and $(B-C-A)A^{-1} + [(B-C-A)A^{-1}]^T > 0$; or

iv) if $(B-C-A)A^{-1} > 0$ and $C(B-C-A)^{-1} + [C(B-C-A)^{-1}]^T > 0$; or

v) if $(B-A-C)^{-1}A > 0$ and $C^{-1}(B-A-C) + [C^{-1}(B-A-C)]^T > 0$; or

vi) if $C^{-1}(B-A-C) > 0$ and $(B-A-C)^{-1}A + [(B-A-C)^{-1}A]^T > 0$; or

vii) if $(B-C-A)^{-1}C > 0$ and $A^{-1}(B-C-A) + [A^{-1}(B-C-A)]^T > 0$; or

viii) if $A^{-1}(B-C-A) > 0$ and $(B-C-A)^{-1}C + [(B-C-A)^{-1}C]^T > 0$.

Example 1: Let

$$A=\begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix},\ B=\begin{pmatrix} 10 & -5 \\ 6 & 3 \end{pmatrix},\ C=\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$$

in (2). Utilizing Theorem 2 i) yields $A(B-A-C)^{-1} = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right) > 0$ and $(B-A-C)C^{-1} + [(B-A-C)C^{-1}]^T = \left(\begin{smallmatrix} 6 & 3 \\ 3 & 4 \end{smallmatrix}\right) > 0$. Therefore, the system in (1) is asymptotically stable. It is interesting to note that the roots of det $(As^2 + Bs + C)$ are -4.0768, -1.5144, -0.24007, and -0.16866.

Employing Lyapunov theory [6], [15] to various linear systems in (4), we can derive additional stability conditions as follows:

Theorem 3:

i) The system in (1) is stable if $\sigma(A^{-1}C)$ is positive real distinct and $T_1(A^{-1}B)T_1^{-1} + [T_1(A^{-1}B)T_1^{-1}]^T \geq 0$, where T_1 is a modal matrix of $A^{-1}C$; and it is asymptotically stable if $\sigma(A^{-1}C)$ is positive real distinct and $T_1(A^{-1}B)T_1^{-1} + [T_1(A^{-1}B)T_1^{-1}]^T > 0$; or

ii) The system in (1) is stable if $\sigma(CA^{-1})$ is positive real distinct and $T_2(BA^{-1})T_2^{-1} + [T_2(BA^{-1})T_2^{-1}]^T \geq 0$, where T_2 is a modal matrix of CA^{-1}; and it is asymptotically stable if $\sigma(CA^{-1})$ is positive real distinct and $T_2(BA^{-1})T_2^{-1} + [T_2(BA^{-1})T_2^{-1}]^T > 0$; or

iii) The system in (1) is stable if $\sigma(C^{-1}A)$ is positive real distinct and $T_3(C^{-1}B)T_3^{-1} + [T_3(C^{-1}B)T_3^{-1}]^T \geq 0$, where T_3 is a modal matrix of $C^{-1}A$; and it is asymptotically stable if $\sigma(C^{-1}A)$ is positive real distinct and $T_3(C^{-1}B)T_3^{-1} + [T_3(C^{-1}B)T_3^{-1}]^T > 0$; or

iv) The system in (1) is stable if $\sigma(AC^{-1})$ is positive real distinct and $T_4(BC^{-1})T_4^{-1} + [T_4(BC^{-1})T_4^{-1}]^T \geq 0$, where T_4 is a modal matrix of AC^{-1}; and it is asymptotically stable if $\sigma(AC^{-1})$ is positive real distinct and $T_4(BC^{-1})T_4^{-1} + [T_4(BC^{-1})T_4^{-1}]^T > 0$.

Proof: To prove Theorem 3 i), we modify (4a) as

$$I_m s^2+T_1(A^{-1}B)T_1^{-1}s+T_1(A^{-1}C)T_1^{-1}=I_m s^2+K_1 s+\Lambda_2=0_m \tag{6a}$$

where T_1 is a modal matrix of $A^{-1}C$, $K_1 \triangleq T_1(A^{-1}B)T_1^{-1}$, and the $\Lambda_2(\triangleq T_1(A^{-1}C)T_1^{-1})$ is a diagonal matrix with positive real elements if $\sigma(A^{-1}C)$ is positive real and distinct. The corresponding state equation in a block companion form is

$$\dot{z}=Fz=\begin{pmatrix} 0_m & I_m \\ -\Lambda_2 & -K_1 \end{pmatrix}z. \tag{6b}$$

The Lyapunov matrix equation is

$$\bar{P}F+F^T\bar{P}=-\bar{Q} \tag{6c}$$

where the symmetric matrix $\bar{P}$ is chosen as $\bar{P}$ = block diag $[\Lambda_2, I_m] > 0$

and the matrix $\bar{Q}$ becomes $\bar{Q}$ = block diag $[0_m, K_1 + K_1^T]$. Hence, by the Lyapunov linear stability theory [6], [15], the system in (6b) is stable if $K_1 + K_1^T = T_1(A^{-1}B)T_1^{-1} + [T_1(A^{-1}B)T_1^{-1}]^T \geq 0$, and it is asymptotically stable if $K_1 + K_1^T > 0$ because the pair $(F, \bar{Q})$ is observable. The controllability and observability of linear matrix-second-order systems have been discussed in [16], [17]. Theorem 3 ii) through iv) can be proven in a similar manner.

Additional stability conditions can be developed as follows:

Theorem 4: The system described in (1) is asymptotically stable

i) if $A^{-1}(A - B + C) > 0$ and either $A^{-1}(B - A) + [A^{-1}(B - A)]^T \geq 0$ or $A^{-1}(B - A) + [A^{-1}(B - A)]^T > 0$; or

ii) if $(A - B + C)A^{-1} > 0$ and either $(B - A)A^{-1} + [(B - A)A^{-1}]^T \geq 0$ or $(B - A)A^{-1} + [(B - A)A^{-1}]^T > 0$; or

iii) if $C^{-1}(A - B + C) > 0$ and either $C^{-1}(B - C) + [C^{-1}(B - C)]^T \geq 0$ or $C^{-1}(B - C) + [C^{-1}(B - C)]^T > 0$; or

iv) if $(A - B + C)C^{-1} > 0$ and either $(B - C)C^{-1} + [(B - C)C^{-1}]^T \geq 0$ or $(B - C)C^{-1} + [(B - C)C^{-1}]^T > 0$.

Proof: Consider the system in (4a) with $\bar{B} \triangleq A^{-1}B$ and $\bar{C} \triangleq A^{-1}C$. The corresponding state equation in a block companion form is

$$\dot{w}_1 = G_1 w_1 = \begin{pmatrix} -\bar{B} & -\bar{C} \\ I_m & 0_m \end{pmatrix} w_1. \tag{7a}$$

Carrying out the similarity transformation $w_1 = Tw_2$, where

$$T = \begin{pmatrix} -I_m & X \\ 0_m & -X \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} -I_m & -I_m \\ 0_m & -X^{-1} \end{pmatrix}$$

and the X is a nonsingular matrix, gives

$$\dot{w}_2 = T^{-1}G_1 T w_2 = G_2 w_2 = \begin{pmatrix} -(\bar{B} - I_m) & (\bar{B} - \bar{C} - I_m)X \\ X^{-1} & -I_m \end{pmatrix} w_2. \tag{7b}$$

The Lyapunov matrix equation is

$$PG_2 + G_2^T P = -Q \tag{7c}$$

where the matrix P is chosen as P = block diag $[I_m, I_m] > 0$. Hence, the matrix Q becomes

$$Q = \begin{pmatrix} \bar{B} + \bar{B}^T - 2I_m & -X^{-T} - (\bar{B} - C - I_m)X \\ -X^{-1} - X^T(\bar{B} - \bar{C} - I_m)^T & 2I_m \end{pmatrix}. \tag{7d}$$

To derive the stability conditions, let the off-diagonal terms equal zero, i.e., $X^{-T} + (\bar{B} - \bar{C} - I_m)X = 0_m$, or $(XX^T)^{-1} = I_m - \bar{B} + \bar{C}$. If $I_m - \bar{B} + \bar{C} > 0$, then a nonsingular X exists, and the similarity transformation matrix T exists. Thus, Q = block diag $[(\bar{B} + \bar{B}^T - 2I_m), 2I_m]$. Using the Lyapunov stability theory [6], [15], we conclude that the system in (1) is asymptotically stable if $\bar{B} + \bar{B}^T - 2I_m > 0$. Since the pair (G_2, Q) is observable, the system in (1) is still asymptotically stable even if $\bar{B} + \bar{B}^T - 2I_m \geq 0$. Note that $I_m - \bar{B} + \bar{C} = A^{-1}(A - B + C)$ and $\bar{B} + \bar{B}^T - 2I_m = A^{-1}(B - A) + [A^{-1}(B - A)]^T$. Thus, we prove Theorem 4 i). Other parts can be proven in the same manner.

Example 2: Let

$$A = \begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 11 & -2 \\ -7 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 18 & -9 \\ -12 & 7 \end{pmatrix}$$

in (2). To apply Theorem 4 i), we compute $A^{-1}B = \left(\begin{smallmatrix} 5 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ and $A^{-1}C = \left(\begin{smallmatrix} 6 & 1 \\ 0 & 3 \end{smallmatrix}\right)$. Since $A^{-1}(A - B + C) = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) > 0$ and $A^{-1}(B - A) + [A^{-1}(B - A)]^T = \left(\begin{smallmatrix} 8 & 3 \\ 3 & 2 \end{smallmatrix}\right) > 0$; therefore, the system in (1) is asymptotically stable. It is interesting to note that the roots of det $(As^2 + Bs + C)$ are -4.17041, -1.34781, and $-0.74088 \pm j1.628923$.

To apply Theorem 3 i), we compute the modal matrix T_1 of $A^{-1}C$ and the diagonal matrix Λ_2 as $T_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & -3 \end{smallmatrix}\right)$ and $\Lambda_2 = T_1(A^{-1}C)T_1^{-1} = \left(\begin{smallmatrix} 6 & 0 \\ 0 & 3 \end{smallmatrix}\right)$. Since $\sigma(\Lambda_2) = \sigma(A^{-1}C)$ is positive real and distinct, and $T_1(A^{-1}B)T_1^{-1} + [T_1(A^{-1}B)T_1^{-1}]^T = 1/3\left(\begin{smallmatrix} 32 & -9 \\ -9 & 10 \end{smallmatrix}\right) > 0$; hence, the system in (1) is asymptotically stable.

Sufficient conditions for the instability of the system in (1) can be deduced from the results in Theorem 2 and the Lyapunov stability theory as follows.

Theorem 5: The system described in (1) is unstable

i) if $(B - A - C)C^{-1} + [(B - A - C)C^{-1}]^T > 0$ and either $A(B - A - C)^{-1} < 0$ or $A(B - A - C)^{-1} \gtreqless 0$; or

ii) if $A(B - A - C)^{-1} + [A(B - A - C)^{-1}]^T > 0$ and either $(B - A - C)C^{-1} < 0$ or $(B - A - C)C^{-1} \gtreqless 0$; or

iii) if $(B - C - A)A^{-1} + [(B - C - A)A^{-1}]^T > 0$ and either $C(B - C - A)^{-1} < 0$ or $C(B - C - A)^{-1} \gtreqless 0$; or

iv) if $C(B - C - A)^{-1} + [C(B - C - A)^{-1}]^T > 0$ and either $(B - C - A)A^{-1} < 0$ or $(B - C - A)A^{-1} \gtreqless 0$; or

v) if $C^{-1}(B - A - C) + [C^{-1}(B - A - C)]^T > 0$ and either $(B - A - C)^{-1}A < 0$ or $(B - A - C)^{-1}A \gtreqless 0$; or

vi) if $(B - A - C)^{-1}A + [(B - A - C)^{-1}A]^T > 0$ and either $C^{-1}(B - A - C) < 0$ or $C^{-1}(B - A - C) \gtreqless 0$; or

vii) if $A^{-1}(B - C - A) + [A^{-1}(B - C - A)]^T > 0$ and either $(B - C - A)^{-1}C < 0$ or $(B - C - A)^{-1}C \gtreqless 0$; or

viii) if $(B - C - A)^{-1}C + [(B - C - A)^{-1}C]^T > 0$ and either $A^{-1}(B - C - A) < 0$ or $A^{-1}(B - C - A) \gtreqless 0$.

III. Conclusion

The objective of this note has been to review and develop some sufficient conditions for stability or instability of a second-order matrix polynomial. Using the relevant linear matrix equation obtained from Lyapunov theory, the structures of various second-order matrix polynomials allow the derivation of various sufficient conditions for stability or instability of the second-order matrix polynomials.

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