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An Extended Predictor-Corrector Algorithm for Variable-order Fractional Delay Differential Equations

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ABSTRACT

This article presents a numerical method based on the Adams-Bashforth-Moulton scheme to solve Variable-order Fractional Delay Differential Equations. In these equations, the variable order fractional derivatives are described in the Caputo sense. The existence and uniqueness of the solutions are proved under Lipschitz condition. Numerical examples are presented showing the applicability and efficiency of the novel method.

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1 Introduction

In the last decades fractional differential equations became a major topic of research. Recent studies in science and engineering demonstrated that the dynamics of many systems may be described more accurately by means of differential equations of non-integer order. In fact, several phenomena in engineering, physics and other applied sciences can be modeled successfully using mathematical tools inspired in the fractional calculus (FC), that is, in the theory of derivatives and integrals of non-integer order [1-8]. Due to these reasons, fractional differential equations have emerged as an interdisciplinary area of research in the recent years.

In real world systems, delays can be recognized everywhere and there has been widespread interest in the study of delay differential equations. Fractional delay differential equations (FDDEs) are dynamical systems involving non-integer order derivatives as well as time delays [9-13]. Due to its non-local nature, a fractional derivative is capable of modeling memory and hereditary properties [14-17]. On the other hand, delay introduces also information from the past. Hence, models including both fractional derivatives and time delays are presently an important area of study.

Both the analytical formulation and the numerical analysis of FDDEs have received attention. However, there are only a few equations for which closed-form analytical solutions are available and therefore, numerical methods have to be applied. There are different techniques for solving FDDEs, such as the Adams-Bashforth-Morton [18-20], finite difference [21-23], fractional backward difference [24] and Hermite wavelet [25] methods.

Recently, the concept of variable-order (VO) fractional integration and differentiation became a novel and relevant topic [26-32]. It is intutive to model some classes of processes and systems with memory properties by means of this tool and,

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consequently, important research and results are to be expected in the future. The VO fractional operator was first suggested by S. Samko and B. Ross [33]. In their study, the authors investigated the properties of VO differentiation and integration operators in the sense of the Riemann-Liouville definition. From this point of view, the order of operator is enabled to alternate as a function either of the independent variable, or of some other variables.

The VO operator definitions recently proposed in the literature include the Riemann-Liouvile, Caputo, Marchaud, Coimbra and Grünwald [15,34-36] definitions. In this paper we focus attention on Caputo's definition. The pros and cons of the distinct definitions of fractional derivatives is a long standing discussion and is out of the scope of this paper. Therefore, we will simply assume that the Caputo formulation is useful in the perspective of applications since it can be coupled with initial conditions having a clear physical meaning.

Having these ideas in mind, this paper is organized as follows. In section 2, we introduce the main definitions of FC. In section 3, we prove the uniqueness and existence of solutions of VO fractional delay differential equations (VFDDEs) with the preshape function. In section 4, we generalize the fractional Adams-Bashforth-Moulton method and we use it for solving VFDDEs. The adaptation of a variety of differential equations in the mathematical modeling process of difference applications will be considered. In section 5, we analyze several models, namely, the problem of a semi-dynamic model which is related to chemistry and biology sciences [37], the impact of a specific factor on the changes of population [38], and the problem of the business cycle systems [39]. The interesting point is that the fractional versions of such models describe more closely real world phenomena. Finally, in section 6 we outline the main conclusions.

2 Fractional Variable Order Calculus

In this section, we present definitions of VO fractional integrals and derivatives.

Definition 1 Let $f \in C(0,T)$, where $C(0,T)$ denotes the space of all continuous functions that are defined on a closed interval $[0,T]$ and $\alpha(t) \geq 0$, then the expression [33,34]

$$ {}_0^{V1}I_t^{\alpha(t)} f(t) = \int_0^t \frac{1}{\Gamma(\alpha(t))} (t-\tau)^{\alpha(t)-1} f(\tau) d\tau, \qquad t > 0, \tag{1} $$

where $\Gamma(\cdot)$ denotes the Gamma function, is called the left Riemann-Liouville integral type 1 (V1) of VO fractional $\alpha(t)$. The expression

$$ {}_0^{V2}I_t^{\alpha(t)} f(t) = \int_0^t \frac{1}{\Gamma(\alpha(\tau))} (\tau-t)^{\alpha(\tau)-1} f(\tau) d\tau, \qquad t > 0, \tag{2} $$

is called the left Riemann-Liouville integral type 2 (V2) of VO fractional $\alpha(t)$.

There are mainly two types of VO fractional derivative definitions introduced by researchers [15,34,35].

Definition 2 The definition of the VO derivative type 1 $(V1)$ can be formulated as follows [15,16,40]

$$ {}_{V1}^{C}D_{0+}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{f^{'}(\tau) \mathrm{d}\tau}{(t-\tau)^{\alpha(t)}}, \qquad 0 < \alpha(t) < 1. \tag{3} $$

In this definition, if $\alpha(t)$ is a constant, then it reduces to the constant-order fractional derivative in Caputo sense.

Definition 3 The definition of the VO derivative type 2 $(V2)$ can be expressed as follows [15,16,40]

$$ {}_{V2}^{C}D_{0+}^{\alpha(t)} f(t) = \int_0^t \frac{1}{\Gamma(1-\alpha(\tau))} \frac{f^{'}(\tau) \mathrm{d}\tau}{(t-\tau)^{\alpha(\tau)}}, \qquad 0 < \alpha(t) < 1. \tag{4} $$

Both definitions have been adopted by researchers. For example, Coimbra et al. [15,16] used type $V1$ in the modeling of a viscous-viscoelastic oscillator, while Ingman and Suzdalnitsky [17] employed type $V2$ in the modeling of a viscoelastic deformation process. However, the differences between $V1$ and $V2$ in applications are still not clear [41]. The main difference between equations Eq. (3) and Eq. (4) is that type $V2$ contains history memory of the derivative order itself. Hence, type $V2$ VO derivative has a special feature of memory contains memory [42].

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3 Existence and Uniqueness of the Solution

In this section, we prove existence and uniqueness theorem pertaining to the system of general problem

$$\begin{cases} {}^{C}D_{0+}^{\alpha(t)}y(t) = f(t, y(t), y(t-\delta)), & 0 \le t \le T, \\ y(t) = \phi(t), & -\delta \le t \le 0, \end{cases} \tag{5}$$

where $T \in R^+$ and $0 < \alpha(t) < 1$. Here, $\phi(t)$ and the coefficients of $y(t)$ and $y(t-\delta)$ represent smooth functions, $\delta \in R^+$ denotes the delay, and ${}^{C}D_{0+}^{\alpha(t)}$ denotes the VO derivative type 2 (V2). Moreover, $f(t,u,v) \in C([0,T] \times R^N \times R^N, R^N)$ is continuous with respect to t and globally Lipschitz continuous with respect to u and v in a given norm $\|.\|$ of R^N, that is

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \le L\|u_1 - u_2\| + M\|v_1 - v_2\|, \tag{6}$$

for all $t \in [0,T]$, $u_1, u_2, v_1, v_2 \in R^N$ and for some Lipschitz constants $L > 0$ and $M > 0$ [43-45].

Lemma 1. *If the function f is continuous, then problem (5) can be rewritten in the form of the Volterra integral equation (VIE):*

$$y(t) = \sum_{k=0}^{\lceil \alpha(t) \rceil - 1} \phi^{(k)}(0)\frac{t^k}{k!} + \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s, y(s), y(s-\delta))\mathrm{d}s. \tag{7}$$

In other words, every continuous solution of (7) is also a solution of original problem (5) and vice versa.

Proof The result follows immediately from the fractional VO integral and derivative definition.

Theorem 2. *Let $a \in (0,T]$, and $y(a) = y_a$ be constant, $0 < \alpha(t) < 1$. For equation*

$$y(t) = y_a + \int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s, y(s), y(s-\delta))\mathrm{d}s - \int_0^a \frac{(a-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s, y(s), y(s-\delta))\mathrm{d}s, \tag{8}$$

we assume the following conditions hold:

1. *Function f satisfies a Lipschitz condition with Lipschitz constants $L, M > 0$ with respect to its second and third arguments, respectively.*
2. *Constants follow the relation $L, M < \frac{\alpha^* a^{-\alpha^*}}{2}$, where $\alpha^* = \inf\{\alpha(s); 0 < s < a\}$.*

Then problem (8) has a unique solution y(t) for $0 \le t \le a$.

Proof We rewrite (8) as the Fredholm equation:

$$y(t) = y_a + \int_0^a \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} \chi_{[0,t]}(s) f(s, y(s), y(s-\delta))\mathrm{d}s - \int_0^a \frac{(a-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s, y(s), y(s-\delta))\mathrm{d}s, \tag{9}$$

in which χ is the indicator function of the interval $[0,t]$, defined by:

$$\chi_{[0,t]}(s) = \begin{cases} 1, & if \quad s \in [0,t], \\ 0, & if \quad s \notin [0,t]. \end{cases}$$

As usual, we set up the recurrence

$$\widehat{y}_0(t) = y_a,$$

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$$\widehat{y}_n = y_a + \int_0^a \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))}\chi_{[0,t]}(s) f(s,\widehat{y}_{n-1},\widehat{y}_{n-m-1})\mathrm{d}s - \int_0^a \frac{(a-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))} f(s,\widehat{y}_{n-1},\widehat{y}_{n-m-1})\mathrm{d}s,$$

and it follows that

$$\begin{aligned}\|\widehat{y}_{n+1}-\widehat{y}_n\| \le [L\|\widehat{y}_n-\widehat{y}_{n-1}\| + M\|\widehat{y}_{n-m}-\widehat{y}_{n-m-1}\|]|\textstyle\int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))}\mathrm{d}s| \\ +[L\|\widehat{y}_n-\widehat{y}_{n-1}\| + M\|\widehat{y}_{n-m}-\widehat{y}_{n-m-1}\|]|\textstyle\int_0^a \frac{(a-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))}\mathrm{d}s|.\end{aligned} \tag{10}$$

Assume that $\alpha^* = \inf\{\alpha(s); 0 < s < a\}$ and, by using the fact that $\sup\{\alpha(s); 0 < s < a\} = 1$, we have

$$|\int_0^t \frac{(t-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))}\mathrm{d}s| \le |\int_0^t \frac{(t-s)^{\inf_{0<s<a}\alpha(s)-1}}{\Gamma(\sup_{0<s<a}\alpha(s))}\mathrm{d}s| = |\int_0^t (t-s)^{\alpha^*-1}\mathrm{d}s| = |\frac{1}{\alpha^*}t^{\alpha^*}| \le \frac{1}{\alpha^*}a^{\alpha^*}. \tag{11}$$

In a similar way we get

$$|\int_0^a \frac{(a-s)^{\alpha(s)-1}}{\Gamma(\alpha(s))}\mathrm{d}s| \le \frac{1}{\alpha^*}a^{\alpha^*}. \tag{12}$$

Putting (11)-(12) in (10), we obtain

$$\|\widehat{y}_{n+1}-\widehat{y}_n\| \le \frac{2}{\alpha^*}a^{\alpha^*}[L\|\widehat{y}_n-\widehat{y}_{n-1}\| + M\|\widehat{y}_{n-m}-\widehat{y}_{n-m-1}\|], \tag{13}$$

for $t \in [0,a]$. So, the sequence $\{\widehat{y}_n\}$ converges absolutely and uniformly to a solution of Eq. (9) and, hence, of Eq. (8). Also, we can prove the uniqueness of solution of Eq. (8) in the usual way.

4 Numerical Method

In this section, we present a numerical method for solving the boundary value problems. In general, we consider Eq. (5) with VO derivative type 1 (V1). If $y(t)$ is a smooth solution in problem Eq. (5), then it should be verified in the boundary problems of Eq. (5). Moreover, $y(t)$ should be consistent in the interval $[0,T]$ and differential consistent in the interval $(0,T)$. This numerical method includes the finite difference operator on a specific consistent mesh.

Consider a uniform grid $\{t_j = jh : j = -m, -m+1, \ldots, -1, 0, 1, \ldots, n\}$, where m and n are integers such that $m = \frac{\delta}{h}$ and $n = \frac{T}{h}$. The numerical algorithm for VFDDEs is

$$y_{n+1} = \sum_{k=0}^{\lceil\alpha(t_{n+1})\rceil-1} \phi^{(k)}(0)\frac{t_{n+1}^k}{k!} + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)} f(t_{n+1}, y_{n+1}^p, y_{n+1-m}) + \frac{h^{\alpha(t_{n+1})}}{\Gamma(\alpha(t_{n+1})+2)}\sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j, y_{j-m}), \tag{14}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha(t_{n+1})+1} - (n-\alpha(t_{n+1}))(n+1)^{\alpha(t_{n+1})}, & j = 0, \\ (n-j+2)^{\alpha(t_{n+1})+1} - 2(n-j+1)^{\alpha(t_{n+1})+1} + (n-j)^{\alpha(t_{n+1})+1}, & 1 \le j \le n, \\ 1, & j = n+1. \end{cases}$$

The preliminary approximation y_{n+1}^p is called predictor and is determined by

$$y_{n+1}^p = \sum_{k=0}^{\lceil\alpha(t_{n+1})\rceil-1} \phi^{(k)}(0)\frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha(t_{n+1}))}\sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j, y_{j-m}), \tag{15}$$

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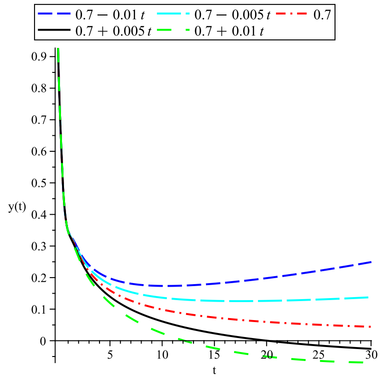

Fig. 1. The numerical solution of Model 1 for various values of $\alpha(t)$ and $\delta = 0.5$, with step size $h = \frac{1}{64}$.

where

$$b_{j,n+1} = \frac{h^{\alpha(t_{n+1})}}{\alpha(t_{n+1})}\left((n-j+1)^{\alpha(t_{n+1})} - (n-j)^{\alpha(t_{n+1})}\right), \qquad 0 \le j \le n.$$

5 Numerical Results

In this section, three examples are considered and solved by means of the proposed method. The graphs and tables show the result for various values of $\alpha(t)$. We analyze the accuracy and efficiency of the new method, in the view point of the maximum absolute error (MAE) defined as

$$\text{MAE} = \max\{|y_i^N - y_{2i}^{2N}|; 0 \le i \le N\}.$$

It is interesting to know not only that a numerical result is convergent, but also how quickly it converges. We describe this by estimating convergence order and rate of maximum absolute error in the Models. Moreover, we employ the formula $Convergent\ order = \frac{\log(MAE)}{\log(step\ size)}$ for estimating the convergent order.

Model 1. This model has applications in chemistry and biological sciences [37]. The following FDDE can describe a semi-dynamical model as:

$$\begin{cases} {}^{C}_{V1}D^{\alpha(t)}_{0+}y(t) = -0.75y(t-\delta), & 0 < t \le 30, \\ y(t) = 1, & -\delta \le t \le 0, \end{cases} \tag{16}$$

where $y(t)$ encodes the neuron's activity level at time t.
In Model 1, we investigate the effect of allowing the value of $\alpha(t)$ to vary in the neighborhood of the fractional order $\alpha(t) = 0.7$. We calculate the approximate value for $y(30)$ using step sizes of $h = \frac{1}{8}$, $h = \frac{1}{16}$, $h = \frac{1}{32}$ and $h = \frac{1}{64}$ for each value of $\alpha(t)$. Furthermore, we applied the proposed method to evaluate the numerical solutions of Model 1 for various values of $\alpha(t)$ with $\delta = 0.5$ and $h = \frac{1}{64}$. The results are shown in Table 1 and Fig. 1. For $\alpha(t) = |sin(t)|$ and $\alpha(t) = 0.8 - 0.02t$, the MAE, convergent order and Rate of MAEs for delay parameter $\delta = 0.5$ are listed in Table 2. The approximate solution for VO fractional derivative $\alpha(t) = |sin(t)|$ is shown in Fig. 2. Moreover, in Fig. 3 we depict the phase plane of approximate solutions of Model 1 for VO fractional derivative $\alpha(t) = |sin(t)|$ and $\delta = 0.5$. In Fig. 4 we adopt logarithmic scale to have a better comparison of approximate solutions. The results reveal show that the numerical errors decay rapidly as h decrease for all values of $\alpha(t)$.

Model 2. This model describes the growth of a population and consist of the Verhulst-Pearl differential equation given by:

$$\begin{cases} {}^{C}_{V1}D^{\alpha(t)}_{0+}y(t) = 0.3y(t) - 0.3y(t)y(t-\delta), & 0 < t \le T, \\ y(t) = 0.1, & -\delta \le t \le 0, \end{cases} \tag{17}$$

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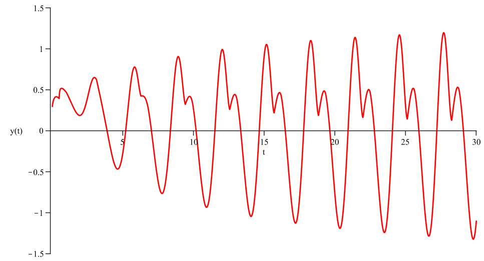

Fig. 2. The numerical solution of Model 1 for $\alpha(t) = |\sin(t)|$ and $\delta = 0.5$, with step size $h = \frac{1}{64}$.

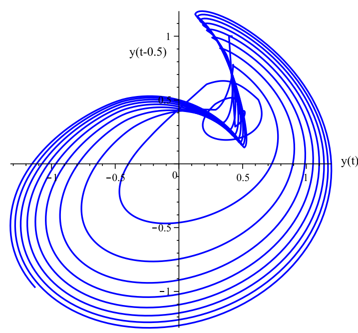

Fig. 3. The numerical solution of Model 1 for $\alpha(t) = |\sin(t)|$, $\delta = 0.5$ and $T = 30$, with step size $h = \frac{1}{64}$, in the phase plane.

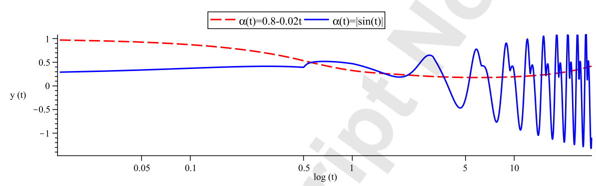

Fig. 4. The numerical solution of Model 1 for various values of $\alpha(t)$ and $\delta = 0.5$, with step size $h = \frac{1}{64}$, in logarithmic scale.

Table 1. Model 1: Approximate value of $y(30)$ evaluated with varying step sizes and various VO with $\delta = 0.25$.

$\alpha(t) \downarrow$ *Step sizes* $\longrightarrow$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
$0.7 - 0.01t$	0.2498371501	0.2494667163	0.2492839569	0.2490988406
$0.7 - 0.005t$	0.1378440443	0.1377464535	0.1376944893	0.1375944236
0.7	0.0440416065	0.0441111314	0.0441447141	0.0441813900
$0.7 + 0.005t$	−0.025900352	−0.025787891	−0.025736136	−0.025677819
$0.7 + 0.01t$	−0.069360188	−0.069335612	−0.069323131	−0.069294940

Table 2. Model 1: MAE, convergent and Rate of MAEs order for $\delta = 0.5$, $T = 30$, with different step sizes h and different $\alpha(t)$.

	$\alpha(t) = \lvert sin(t) \rvert$			$\alpha(t) = 0.8 - 0.02t$		
Step sizes	*MAE*	*Convergent order*	*Rate of MAE*	*MAE*	*Convergent order*	*Rate of MAE*
$h = \frac{1}{8}$	0.0462941608	1.477675319	–	0.0029798130	2.796854162	–
$h = \frac{1}{16}$	0.0361111646	1.197852810	1.281990246	0.0008579960	2.546685364	3.472991716
$h = \frac{1}{32}$	0.0278522864	1.033212085	1.296524245	0.0002761067	2.364497297	3.107479826
$h = \frac{1}{64}$	0.0210698442	0.928112757	1.321902817	0.0000918416	2.235082116	3.006335909

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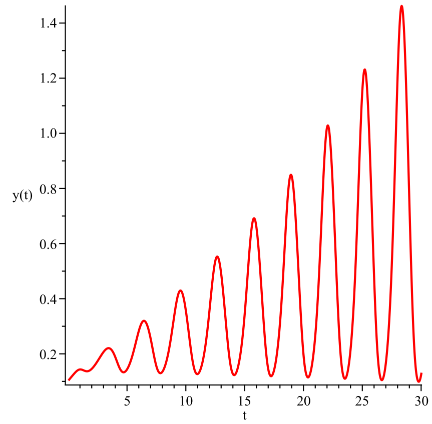

Fig. 5. The numerical solution of Model 2 for $\alpha(t) = \frac{1+\cos(2t)}{3}$ and $\delta = 1$, with step size $h = \frac{1}{64}$.

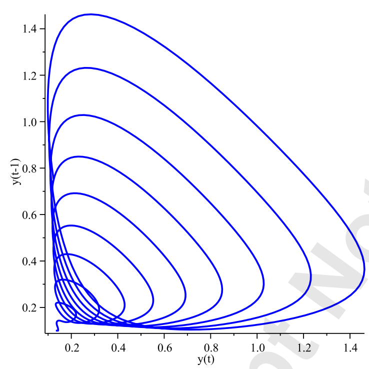

Fig. 6. The numerical solution of Model 2 for $\alpha(t) = \frac{1+\cos(2t)}{3}$, $\delta = 1$ and $T = 30$, with step size $h = \frac{1}{64}$, in the phase plane.

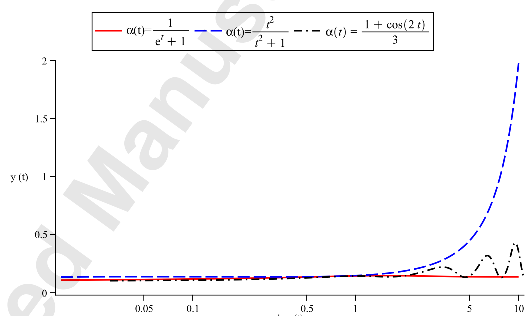

Fig. 7. The numerical solution of Model 2 for various values of $\alpha(t)$, $\delta = 1$ and $T = 10$, with step size $h = \frac{1}{64}$, in logarithmic scale.

where $y(t)$ is the size of the population at time t [38].

Fig. 5 shows the approximate solution $y(t)$ of system Eq. (17) for $\alpha(t) = \frac{1+\cos(2t)}{3}$ and $\delta = 1$. Moreover, Fig. 6 depicts the phase portrait that is $y(t)$ versus $y(t-1)$, for $\alpha(t) = \frac{1+\cos(2t)}{3}$. It may be observed from these figures that the system shows a stable behavior.

In Tables 3, we list MAEs, convergent order and Rate of MAEs using the proposed method for several values of T and $\alpha(t) = \frac{1+\cos(2t)}{3}$. Several choices of h show that decreasing the step size increases the accuracy of the results. Moreover, for several values of $\alpha(t)$ and $T = 30$, the results are presented in Table 4. In Fig. 7, we depict approximate solutions of Model 2 for various values of variable fractional derivative $\alpha(t)$ in logarithmic scale.

Model 3. This model represents the continuous-time Kalecki's business cycle system [39] and is formulated as follows:

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Table 3. Model 2: MAE, convergent and Rate of MAEs order for $\delta = 1$ and $\alpha(t) = \frac{1+\cos(2t)}{3}$, with different T, step sizes h and $\alpha(t)$.

	$T = 10$			$T = 30$		
Step sizes	*MAE*	*Convergent order*	*Rate of MAE*	*MAE*	*Convergent order*	*Rate of MAE*
$h = \frac{1}{8}$	0.0144142087	2.038788183	–	0.105496734	1.081576586	–
$h = \frac{1}{16}$	0.0108875866	1.630292999	1.329121974	0.081159404	0.9057744800	1.299870733
$h = \frac{1}{32}$	0.0085137388	1.375198291	1.278825537	0.062505571	0.7999742819	1.298434726
$h = \frac{1}{64}$	0.0066980229	1.203674830	1.271082366	0.0486958036	0.6828794718	1.28359255

Table 4. Model 2: MAE, convergent and Rate of MAEs order for $\delta = 1$, $T = 10$, with different step sizes h and different $\alpha(t)$.

	$\alpha(t) = \frac{1}{e^t+2}$			$\alpha(t) = \frac{t^2}{t^2+1}$		
Step sizes	*MAE*	*Convergent order*	*Rate of MAE*	*MAE*	*Convergent order*	*Rate of MAE*
$h = \frac{1}{8}$	0.0019351177	3.004454322	–	0.096021414	1.126833336	–
$h = \frac{1}{16}$	0.0018601324	2.267594744	1.040311808	0.054678853	1.048218303	1.756097810
$h = \frac{1}{32}$	0.0278522864	0.0017879364	1.825497774	0.030720328	1.004932514	1.779891575
$h = \frac{1}{64}$	0.0017283427	1.529399163	1.034480257	0.017575036	0.9717214243	1.747952494

$$\begin{array}{l} a)\ Y(t) = C(t) + I(t), \\ b)\ C(t) = cY(t), \\ c)\ I(t) = \frac{1}{\delta}\int_{t-\delta}^{t} B(t)dt, \\ d)\ B(t) = \lambda(\upsilon Y(t) - K(t)), \\ e)\ {}_{V1}^{C}D_{0+}^{\alpha(t)}K(t) = B(t-\delta), \end{array} \tag{18}$$

In Eq. (18.a), the aggregate demand of consumption C and the investment outlays I equals the total revenue Y. In Eq. (18.b), the consumption is proportional to the total revenue by means of constant $C > 0$. In Eq. (18.c), the investment orders depend on the past investment decisions B with a fixed delay of δ. In Eq. (18.d), the determination of the investment decisions B, is proportional to the gap between the desired level of equipment $\upsilon Y(t)$ and the existing capital stock K, where $\upsilon > 0$. Finally, according to Eq. (18.e), a fixed delay separates the deliveries of capital goods from the orders. After some algebraic manipulations, the system may be reduced to a linear FDDE with constant coefficients, as follows:

$$\begin{cases} {}_{V1}^{C}D_{0+}^{\alpha(t)}K(t) = aK(t) - bK(t-\delta), 0 < t \leq T \\ \qquad K(t) = 1, \qquad -\delta \leq t \leq 0, \end{cases} \tag{19}$$

where $a = \frac{\lambda\nu}{\delta(1-c)}$, $b = \lambda(1+\frac{\upsilon}{\delta(1-c)})$, $\lambda = \frac{2}{5}$, $\delta = 1$, $c = \frac{3}{4}$ and $\nu = \frac{1}{2}$. Therefore, we obtain $a = 0.8$ and $b = 1.2$.
To demonstrate the performance of the proposed method, we list in Table 5, the MAEs, convergent order and Rate of MAEs order for VO fractional derivatives $\alpha(t) = tanh(t+1)$ and $\alpha(t) = F(t) = \begin{cases} 0.75, & t < \frac{T}{2}, \\ 0.85, & t \geq \frac{T}{2}, \end{cases}$, for $T = 30$.
We observe in the tables that the results are accurate for small values of h. Fig. 8 plots the approximate solution for VO fractional derivative $\alpha(t) = tanh(t+1)$, for $\delta = 1$ and $T = 30$. In addition, in Fig. 9, we plot the phase plane of the approximate solution of Model 3, showing a stable behavior for VO fractional derivative $\alpha(t) = tanh(t+1)$ and $\delta = 1$. In Fig. 10, we depict approximate solutions of Model 3 for various values of VO fractional derivative $\alpha(t)$ in the logarithmic scale.

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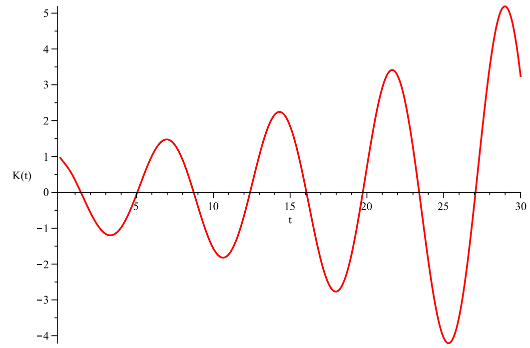

Fig. 8. The numerical solution of Model 3 for value of $\alpha(t) = tanh(t+1)$, $\delta = 1$ with step size $h = \frac{1}{64}$.

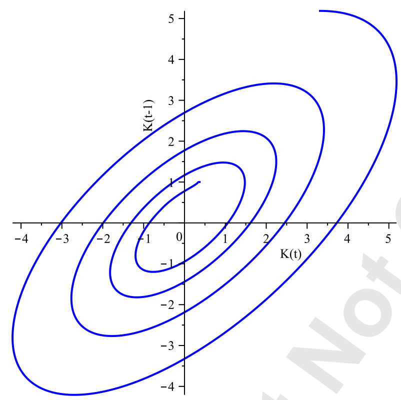

Fig. 9. The numerical solution of Model 3 for value of $\alpha(t) = tanh(t+1)$, $\delta = 1$ and $T = 30$ with step size $h = \frac{1}{64}$ in the phase plane.

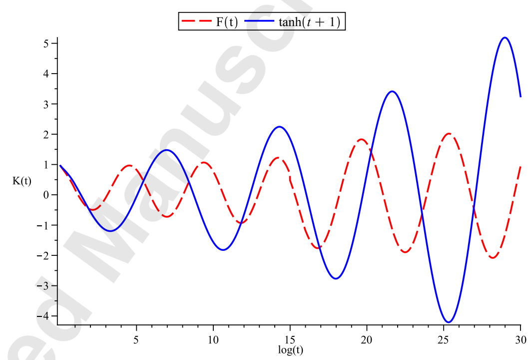

Fig. 10. The numerical solution of Model 3 for various values of $\alpha(t)$ and $\delta = 0.5$ with step size $h = \frac{1}{64}$ in logarithmic scale.

Table 5. Model 3: MAE, convergent and Rate of MAEs order for $\delta = 1$, $T = 30$, with different step sizes h and different $\alpha(t)$.

	$\alpha(t) = F(t)$			$\alpha(t) = \tanh(t+1)$		
Step sizes	*MAE*	*Convergent order*	*Rate of MAE*	*MAE*	*Convergent order*	*Rate of MAE*
$h = \frac{1}{8}$	0.6954985625	0.1746268540	–	0.8942126880	0.05377002616	–
$h = \frac{1}{16}$	0.2643599701	0.4798560891	2.855087913	0.2402965672	0.5142780141	3.721287817
$h = \frac{1}{32}$	0.0796388182	0.7300768755	3.319486351	0.061993159	0.8023494338	3.876178776
$h = \frac{1}{64}$	0.0207659050	0.9316065738	2.879304177	0.015706112	0.9987550166	3.947072261

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6 Conclusions

In this work we investigated the solution of nonlinear VFDDEs. The proposed method is based on Adams-Bashforth-Moulton scheme. This algorithm is applied to obtain the approximate the numerical solution of nonlinear VFDDEs. The numerical experiments demonstrated performance of the new method. From the figures and Tables, we conclude that for all chosen parameters and orders of the fractional derivatives the numerical errors decay quickly by reducing the integration step size. We observe also that the algorithm provides accurate and stable numerical results.

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