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A note on the stability of linear dynamical systems with time delay

Xiao-Yan Zhang[1,2] and Jian-Qiao Sun[3]

Abstract

This paper presents a stability study of linear time-invariant and periodic systems with time delay. The methods of semi-discretization, continuous time approximation and Lyapunov stability theory are used to study the stability of two benchmark systems. It is found that for linear time-invariant systems, the Lyapunov method is usually conservative leading to a much smaller domain of stability in a parameter space than the true solution, with the exception of the complete Lyapunov functional due to Gu, which gives highly accurate predictions with little conservatism. For periodic systems, it is difficult to find appropriate Lyapunov–Krasovskii functionals. Numerical methods such as semi-discretization and continuous time approximation are more appealing, and can compute geometrically complex stability boundaries in the parameter space with high accuracy.

Keywords

Dynamical systems, time delay, stability, Lyapunov theory

1. Introduction

The stability of dynamical systems with time delay is an important topic and has been the subject of many studies. The stability of time-delayed dynamical systems can be studied using different methods including the Lyapunov approach and numerical methods. This brief note presents a comparison of stability prediction by several popular methods. Two classes of system are considered: linear time invariant (LTI) and linear periodic systems with time delay.

The Lyapunov approach is a popular method for stability analysis (Wu and Mizukami, 1995; Kapila and Haddad, 1999). Cao et al. (2002), Kim (2008) and Fridman and Orlov (2009) analyzed the stability of linear systems with time delay using the Lyapunov approach. Fan et al. (2002) discussed asymptotic stability problems for a class of neutral systems with discrete and distributed delays via linear matrix inequalities. Han (2009) was concerned with the stability of linear time-delay systems of both retarded and neutral types, using time-independent and time-dependent Lyapunov–Krasovskii functionals. Kolmanovskii and Richard (1999), Ivanescu et al. (2000) and Zhang et al. (2002) investigated delay-dependent and delay-independent stability conditions. Shao (2008) provided improved delay-dependent stability criteria for systems with a varying delay in a range. The Lyapunov method is used in He et al. (2007) for the stability analysis of systems with time-varying delay with known lower and upper bounds. The Lyapunov function dependent on the known upper bound of uncertain state-delays is derived in the study of model predictive controls (MPCs) for constrained linear digital systems with uncertain state-delays (Hu and Chen, 2004). Kwon et al. (2008) investigated delay-dependent robust stability for neutral systems with the help of the Lyapunov method. An early study of the necessary and sufficient Lyapunov condition was done by Huang (1989). A much-improved version of the necessary and sufficient condition involving a complete quadratic form of the infinite dimensional state vector is due to Gu (2001)

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and Gu and Niculescu (2003). Although the stability prediction by the Lyapunov approach is often conservative, the necessary and sufficient condition for stability according to the Lyapunov–Krasovskii functionals by Gu is highly accurate and not conservative at all.

Semi-discretization (SD) is a well-established method in the literature and is widely used in structural and fluid mechanics (Pfeiffer and Marquardt, 1996; Leugering, 2000). The method has been applied to delayed deterministic dynamical systems by Insperger and Stepan (2001, 2002). Recently, Insperger and Stepan (2011) have published a comprehensive and general overview of the SD method. The method has been extended to control systems with delayed feedback (Sheng et al., 2004; Sheng and Sun, 2005). The continuous time approximation (CTA) method is an extension of the method of SD and provides an alternative to handle systems with multiple independent time delays (Sun, 2008; Song and Sun, 2011). The CTA method has been applied to study control problems of the time-delayed linear dynamical systems, and stochastic dynamical systems with time delay. Butcher and colleagues (Butcher and Bobrenkov, 2011; Bobrenkov et al., 2012a,b) developed the method of Chebyshev spectral CTA which uses a Chebyshev collocation grid to study the stability and control of linear periodic systems with time delay. Chebyshev spectral CTA provides the most accurate solution of time-delayed dynamical systems. The concept of CTA is based on the mathematical work on approximating the infinitesimal generator of delayed differential equations (DDEs) (Bellen and Maset, 2000; Bellen and Zennaro, 2003).

The remainder of the paper is organized as follows. Section 2 states the stability problem, reviews the SD and CTA methods for stability analysis, and presents two representative Lyapunov functionals with different levels of complexity. Section 3 presents the stability boundaries in a parameter space for a benchmark second-order LTI system with delayed feedback control. The Lyapunov stability conditions and CTA methods are applied to compute the stability boundaries and the results are compared with the results obtained from the characteristic equation. In Section 4, the CTA methods are applied to compute stability boundaries in a parameter space for the time-delayed linear periodic system and the results are compared with the SD methods. Section 5 concludes the paper.

2. The stability problem

Consider a time-delayed linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{A}_d(t)\mathbf{x}(t-\tau) \tag{1}$$

subject to an initial condition

$$\mathbf{x}(\theta) = \phi(\theta), \forall\theta \in [-\tau, 0] \tag{2}$$

where $\phi(\theta)$ is a given function of time. In this paper, we consider two cases: a) when $\mathbf{A}(t)$ and $\mathbf{A}_d(t)$ are constant matrices and b) when they are periodic functions of time. When the system is time-invariant, the characteristic equation of the system reads

$$\det(\mathbf{A} + \mathbf{A}_d e^{-\tau s} - \mathbf{I}s) = 0 \tag{3}$$

The stability of the system is determined by the roots of the characteristic equation. Stability boundaries in a parameter space are characterized by the roots of this equation with zero real parts. These results are used as the benchmark for comparison with the approximate solutions in the paper.

When the system is periodic, there is no longer a characteristic equation to determine the stability of the system. Floquet theory is often used to determine the stability.

In the following, we present the methods of SD and CTA for studying the stability of the time-delayed system.

2.1. The numerical methods

According to Sun (2008), we consider a mesh $\Omega_N = \{\tau_i, i = 0, 1, \ldots, N\}$ of $N+1$ points in [0, τ] such that $0 = \tau_0 < \tau_1 < \ldots < \tau_N = \tau$. We define an extended state vector $\mathbf{y}(t)$ as

$$\begin{aligned}\mathbf{y}(t) &= [\mathbf{x}(t), \mathbf{x}(t-\tau_1), \ldots, \mathbf{x}(t-\tau_N)]^T \\ &\equiv \left[\mathbf{y}_1(t), \mathbf{y}_2(t), \mathbf{y}_3(t), \ldots, \mathbf{y}_{N+1}(t)\right]^T\end{aligned} \tag{4}$$

Apply a polynomial interpolation to the sampled responses $\mathbf{x}(t-\tau_i)$ and denote $\mathbf{D}_N$ as the spectral derivative matrix of the interpolation. Let $\mathbf{D}_N^-$ be the matrix $\mathbf{D}_N$ with the first row removed. We have an equation of motion for the extended state vector as follows:

$$\dot{\mathbf{y}}(t) = \mathbf{A}_{CTA} \cdot \mathbf{y}(t) \tag{5}$$

with the matrix $\mathbf{A}_{CTA}$ in the following block form:

$$\mathbf{A}_{CTA} = \begin{bmatrix} \mathbf{A} & \ldots & \mathbf{A}_d \\ & \mathbf{D}_N^- & \end{bmatrix} \tag{6}$$

The spectral derivative matrix $\mathbf{D}_N$ can be obtained by different numerical schemes. The simplest one is the forward finite difference approximation (Sun, 2008):

$$\dot{x}(t-i\Delta\tau) = \frac{1}{\Delta\tau}[\mathbf{x}(t-(i-1)\Delta\tau) - \mathbf{x}(t-i\Delta\tau)], \quad 1 \le i \le N \tag{7}$$

The accuracy of the finite difference scheme is poor and improves slowly as N increases. A much more accurate approach involves Chebyshev collocation, which requires nonuniformly sampled points in the interval $[0, \tau]$ defined by

$$\tau_j = \frac{\tau}{2}\left[1-\cos\left(\frac{j\pi}{N-1}\right)\right], \quad j = 0, 1, 2, \ldots, N-1 \tag{8}$$

A Matlab program for computing the spectral derivative matrix $\mathbf{D}_N$ of Lagrange polynomials with Chebyshev collocation points is provided in Trefethen (2000). The resulting method is called Chebyshev spectral CTA or Chebyshev CTA for short (Butcher and Bobrenkov, 2011; Bobrenkov et al., 2012b).

Although it is the most accurate numerical method, Chebyshev collocation uses nonuniformly sampled time instances. Note that real-time controls are usually implemented with constant sampling frequency. Nonuniform sampling for the delayed control with Chebyshev collocation would require a variable sample frequency. This is difficult. To overcome this difficulty and to improve the accuracy of the finite difference scheme, a low-pass filter approximation was developed by Song and Sun (2011).

It is common to filter a measured signal before taking the derivative. For a first-order low-pass filter, it amounts to applying the following transfer function to the signal:

$$H(s) = \frac{p}{s+p}s, \quad p > 0 \tag{9}$$

Hence, $\dot{x}(s) \approx H(s)x(s)$ in the Laplace domain. With Tustin's approximation (Franklin et al., 1986), we obtain $\dot{x}(z) = H(z)x(z)$ in the z-domain. In the digital time domain, we have

$$\left(\frac{1}{2}+r\right)\dot{x}(n) + \left(\frac{1}{2}-r\right)\dot{x}(n-1) = \frac{1}{\Delta t}(x(n)-x(n-1)) \tag{10}$$

where $r = 1/(p\Delta t)$, Δt is the sample time of a digital system and p is the bandwidth of the low-pass filter. This relationship introduces an approximation of derivatives of the delayed response just like the spectral derivative of the Lagrange polynomial, and represents a combination of forward, backward and central finite differences. With this approximation and $\Delta t = \Delta\tau$, the system matrix in equation (5) reads

$$\mathbf{A}_{LPCTA} = \mathbf{H}^{-1}\mathbf{F} \tag{11}$$

where LPCTA stands for 'lowpass filter-based continuous-time approximation' and

$$\mathbf{H} = \begin{bmatrix} \mathbf{I} & 0 & \ldots & \ldots & 0 \\ (\frac{1}{2}+r)\mathbf{I} & (\frac{1}{2}-r)\mathbf{I} & 0 & \ldots & 0 \\ 0 & \ldots & \ddots & \ldots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \ldots & 0 & (\frac{1}{2}+r)\mathbf{I} & (\frac{1}{2}-r)\mathbf{I} \end{bmatrix} \tag{12}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{A} & 0 & \ldots & \ldots & \mathbf{A}_d \\ \frac{1}{\Delta\tau}\mathbf{I} & -\frac{1}{\Delta\tau}\mathbf{I} & 0 & \ldots & 0 \\ 0 & \ldots & \ddots & \ldots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \ldots & 0 & \frac{1}{\Delta\tau}\mathbf{I} & -\frac{1}{\Delta\tau}\mathbf{I} \end{bmatrix} \tag{13}$$

The stability of LTI systems is determined by the eigenvalue of the matrix $\mathbf{A}_{CTA}$ (or $\mathbf{A}_{LPCTA}$). That is to say,

$$\max_i \mathrm{Re}(\lambda_i(\mathbf{A}_{CTA})) \leq 0 \tag{14}$$

where $\lambda_i(\mathbf{A}_{CTA})$ denotes the i^{th} eigenvalue of the matrix.

For periodic systems, we apply Floquet theory to system (5). Recall that $\mathbf{A}_{CTA}(t) = \mathbf{A}_{CTA}(t+T)$ where T is the period. Let $\mathbf{\Phi}(t)$ be the fundamental matrix of the system such that

$$\frac{\mathrm{d}\mathbf{\Phi}(t)}{dt} = \mathbf{A}_{CTA}(t)\mathbf{\Phi}(t), \quad \mathbf{\Phi}(0) = \mathbf{I} \tag{15}$$

Since

$$\dot{\mathbf{\Phi}}(t+T) = \mathbf{A}(t+T)\mathbf{\Phi}(t+T) = \mathbf{A}(t)\mathbf{\Phi}(t+T) \tag{16}$$

we have $\mathbf{\Phi}(t+T) = \mathbf{\Phi}(t)C$. This defines a mapping over one period. The stability of the system is determined by the eigenvalue of the mapping $\mathbf{C}$ through the characteristic equation

$$|\mathbf{C} - \lambda\mathbf{I}| = 0 \tag{17}$$

If $\mathbf{\Phi}(0)$ is $\mathbf{I}$, then $\mathbf{C} = \mathbf{\Phi}(\mathrm{T})$. The characteristic equation of the system also reads

$$\left|\mathbf{\Phi}(T) - \lambda\mathbf{I}\right| = 0 \tag{18}$$

The matrix $\mathbf{\Phi}(T)$ defines a mapping of the extended state vector over a period:

$$\mathbf{y}_{j+1} = \mathbf{\Phi}(T)\mathbf{y}_j \tag{19}$$

The stability condition follows:

$$|\lambda_i(\mathbf{\Phi}(T))| \leq 1, \quad \forall i \tag{20}$$

where $\lambda_i(\mathbf{\Phi}(T))$ denotes the i^{th} eigenvalue of the matrix $\mathbf{\Phi}(T)$.

Another method to compute the mapping $\mathbf{\Phi}(T)$ for linear time-delayed systems is SD (Elbeyli and Sun, 2004). It should be noted that for LTI systems with a single time delay, the mapping time interval is often chosen to be equal to the time delay. We should point out that the method of SD has been well-studied and proven to be effective and reliable (Elbeyli and Sun, 2004), and has been found to be equivalent to the method of CTA.

Finally, we point out that Chebyshev CTA and LPCTA are merely two examples of approximation of the infinitesimal generator of DDEs (Bellen and Maset, 2000; Bellen and Zennaro, 2003).

2.2. Stability with Lyapunov–Krasovskii functionals

We present two Lyapunov–Krasovskii functionals for LTI systems with time delay. They are presented in order of complexity of the functional.

2.2.1. A mixed-delay independent and dependent condition.

Consider the Lyapunov–Krasovskii functional by He et al. (2007),

$$V_1(t) = \mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(s)\mathbf{Q}\mathbf{x}(s)\ \mathrm{d}s + \int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{\mathbf{x}}^T(s)\mathbf{Z}\dot{\mathbf{x}}(\mathbf{s})\ \mathrm{d}s\ \mathrm{d}\theta \qquad (21)$$

where $\mathbf{P}=\mathbf{P}^T>\mathbf{0}$, $\mathbf{Q}=\mathbf{Q}^T\geq\mathbf{0}$ and $\mathbf{Z}=\mathbf{Z}^T\geq\mathbf{0}$. The stability condition is given by

$$\dot{V}_1(t) = 2\mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t) + \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) - \mathbf{x}^T(t-\tau)\mathbf{Q}\mathbf{x}(t-\tau) + \tau\dot{\mathbf{x}}^T(t)\mathbf{Z}\dot{\mathbf{x}}(t) - \int_{t-\tau}^{t}\dot{\mathbf{x}}^T(s)\mathbf{Z}\dot{\mathbf{x}}(\mathbf{s})\ ds < 0 \qquad (22)$$

For the LTI system (1), this condition leads to a linear matrix inequality (LMI), which can be solved with the Matlab LMI toolbox. This stability condition is an improvement over the delay-independent one, and is determined by a finite number of matrices. For the complete expression of the LMI, the reader is referred to He et al. (2007).

2.2.2. Gu's functional.

The above Lyapunov stability condition is sufficient, but not necessary. Consider a Lyapunov–Krasovskii functional due to Gu (2001),

$$V_2(\phi) = \frac{1}{2}\phi^T(0)\mathbf{P}\phi(0) + \phi^T(0)\int_{-\tau}^{0}\mathbf{Q}(\zeta)\phi(\zeta)\ \mathrm{d}\zeta + \frac{1}{2}\int_{-\tau}^{0}\int_{-\tau}^{0}\phi^T(\zeta)\mathbf{R}(\zeta,\eta)\phi(\eta)\ \mathrm{d}\zeta\ \mathrm{d}\eta + \frac{1}{2}\int_{-\tau}^{0}\phi^T(\zeta)\mathbf{S}(\zeta)\phi(\zeta)\ \mathrm{d}\zeta \qquad (23)$$

where $\mathbf{P}=\mathbf{P}^T$, $\mathbf{Q}(\zeta)$, $\mathbf{R}(\zeta,\eta)=\mathbf{R}^T(\eta,\zeta)$, $\mathbf{S}(\zeta)=\mathbf{S}^T(\zeta)$ and $-\tau\leq\eta,\ \zeta\leq 0$. The system is stable if there exists $\epsilon>0$ such that the derivative of the Lyapunov functional along the response $\mathbf{x}_t(\theta)=\mathbf{x}(t+\theta)$ satisfies

$$V_2(\phi)\geq\epsilon\phi^T(0)\phi(0),\ \ \dot{V}_2(\phi)=\frac{\mathrm{d}}{\mathrm{d}t}V_2(\mathbf{x}_t)|_{\mathbf{x}_t=\phi} \leq -\epsilon\phi^T(0)\phi(0) \qquad (24)$$

This condition can also be expressed in terms of a matrix inequality (equation (46) in Gu, 2001). Note that the matrices $\mathbf{Q}(\zeta)$, $\mathbf{R}(\zeta,\eta)$ and $\mathbf{S}(\zeta)$ are functions of their arguments and have to be determined by discretizing the indices ζ and η over the time interval $[-\tau, 0]$. Gu (2001) has developed an algorithm for computing these matrices.

Since the matrices $\mathbf{Q}(\zeta)$, $\mathbf{R}(\zeta,\eta)$ and $\mathbf{S}(\zeta)$ are functions of real numbers defined in a finite interval, it implies that there are an infinite number of matrices involved in the Lyapunov–Krasovskii functional. Furthermore, $V_2(\phi)$ consists of a complete quadratic form of the infinite dimensional state vector $\mathbf{x}(t+\theta)$ where $\theta\in[-\tau,\ 0]$. These are the reasons that the Lyapunov–Krasovskii functional due to Gu leads to a necessary and sufficient condition for the stability of linear systems with time delay.

We also note that the stability boundary in the context of Lyapunov stability theory is determined by setting the time derivative of the Lyapunov functional to be zero along the response.

3. A benchmark second-order LTI system

We consider a second-order linear system with delayed PD feedback control,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ -k_p & -k_d \end{bmatrix}\mathbf{x}(t-\tau) \qquad (25)$$

where $\mathbf{x}=[x_1(t),\ x_2(t)]^T$, τ is the time delay and k_p and k_d are feedback gains. For given parameters k and c, we delineate the stability boundaries in the k_p-k_d gain space for different time delays. The stability boundaries of the system can be accurately obtained from equation (3) (Sheng et al., 2004). Numerical results of the stability boundaries by different methods are reported below and are compared with the solutions obtained from equation (3). In the numerical results, we have selected $k=4$ and $c=0.2$ with varying time delays.

3.1. Lyapunov approach

The LMI toolbox in Matlab is used to identify the stability of the system. The function `feasp` returns an index. When the index is negative, it indicates that the optimization scheme of the LMI package has found all

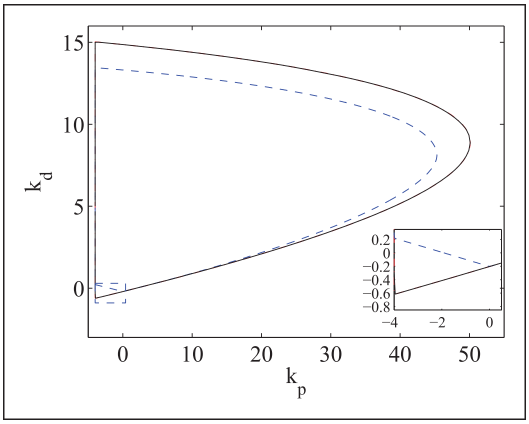

Figure 1. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/30$) by two Lyapunov functionals. V_1: $- \cdot - \cdot -$. V_2: $- - -$. Solution from equation (3): Solid line.

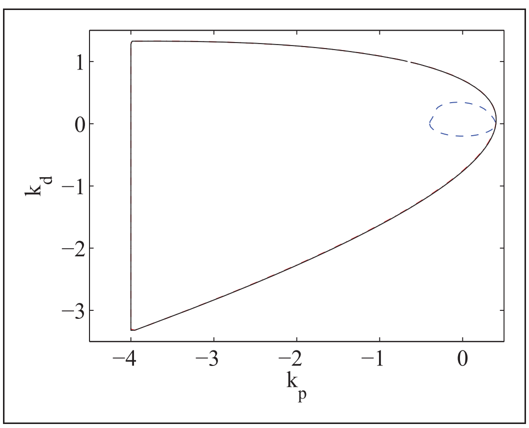

Figure 2. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/4$) by two Lyapunov functionals. V_1: $- \cdot - \cdot -$. V_2: $- - -$. Solution from equation (3): Solid line.

the matrices required to satisfy the matrix inequality. When the index is positive, it suggests that the LMI package cannot find the matrices to satisfy the matrix inequality. The stability boundaries are outlined with the index equal to zero. In other words, the stability boundary is determined by $\dot{V} = 0$.

Figures 1 to 3 show the comparison of the stability domains by the two Lyapunov–Krasovskii functionals and the stability boundaries obtained from equation (3) and reported by Sheng et al. (2004) with $\tau = \pi/30$, $\pi/4$ and $\pi/2$. Note that in Figures 1 and 2, the stability boundaries predicted with Gu's functionals overlap with the curves obtained from equation (3).

We have found the necessary and sufficient Lyapunov stability condition due to Gu produces the best prediction of the stability boundaries in the parameter space, which are closest to the boundaries

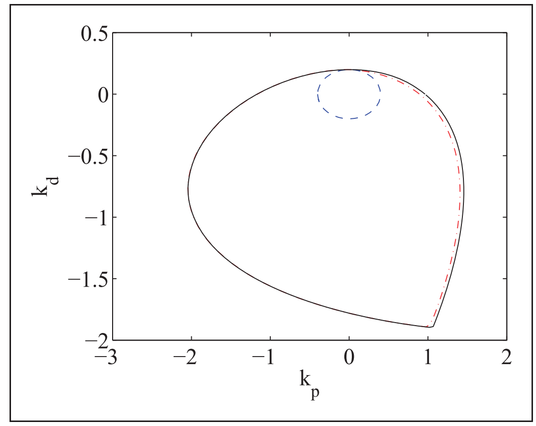

Figure 3. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/2$) by two Lyapunov functionals. V_1: $- \cdot - \cdot -$. V_2: $- - -$. Solution from equation (3): Solid line.

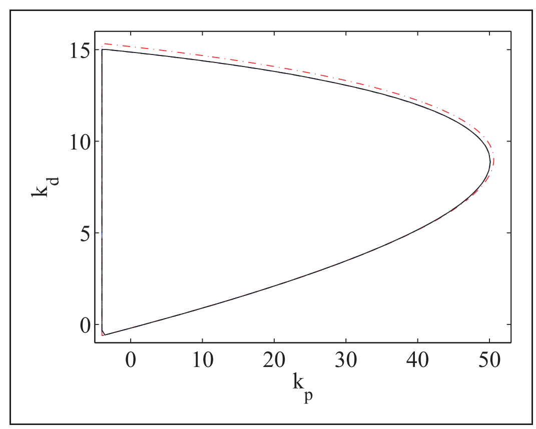

Figure 4. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/30$) by three CTA methods ($N = 2^6$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: .... Solution from equation (3): Solid line.

obtained from equation (3) and not conservative, among all the Lyapunov stability conditions we have examined. Other Lyapunov–Krasovskii functionals lead to conservative stability domains as is the case of V_1 shown in the figures.

3.2. CTA method

For all the CTA methods, we have selected the discretization level $N = 2^6$. We have found that this level of discretization provides adequately accurate predictions.

Figures 4 to 6 show the comparison of the stability domains by the CTA methods and the stability boundaries obtained from equation (3) with $\tau = \pi/30$, $\pi/4$ and $\pi/2$. We have found that the CTA method with Chebyshev nodes and LPCTA method both compute

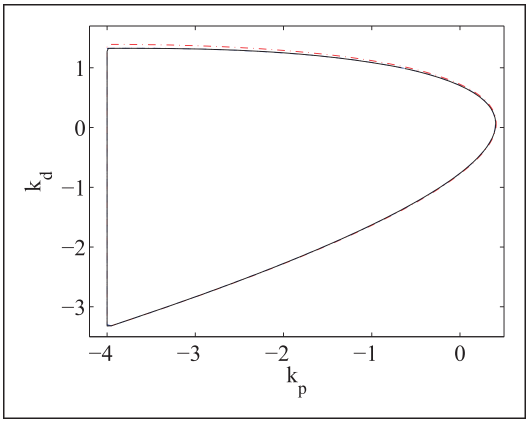

Figure 5. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/4$) by three CTA methods ($N = 2^6$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . Solution from equation (3): Solid line.

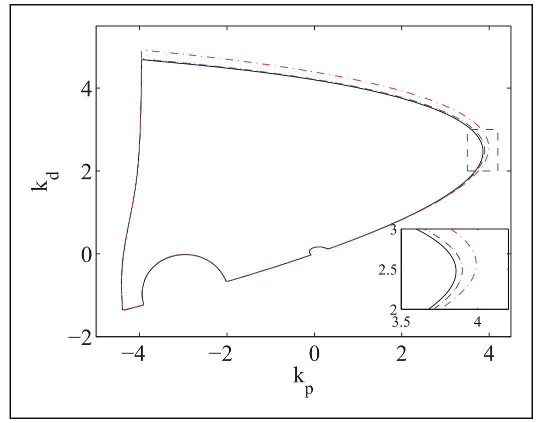

Figure 7. The stability boundaries of the periodic system in the $k_p - k_d$ gain space ($\tau = \pi/10$) by three CTA methods ($N = 2^5$). Forwards: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . SD solution: Solid line.

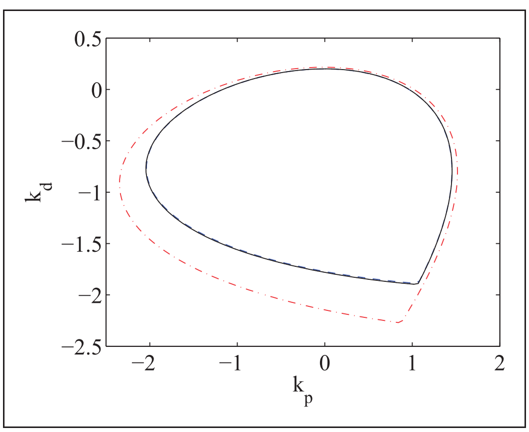

Figure 6. The stability boundaries of the second-order LTI system in the $k_p - k_d$ gain space ($\tau = \pi/2$) by three CTA methods ($N = 2^6$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . Solution from equation (3): Solid line.

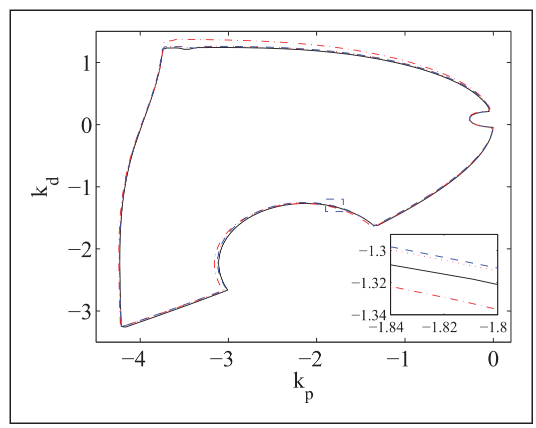

Figure 8. The stability boundaries of the periodic system in the $k_p - k_d$ gain space ($\tau = \pi/4$) by three CTA methods ($N = 2^5$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . SD solution: Solid line.

the eigenvalues of the linear time-delayed systems with high accuracy. The stability domains by the Chebyshev CTA and LPCTA methods are overlapped with the solutions obtained from equation (3). It is interesting to note that the stability domains by the CTA method with forward finite difference is bigger than the boundaries obtained from equation (3), implying that it is nonconservative.

4. Stability of a periodic system

Next, we consider the classical Mathieu equation with delayed feedback,

$$\ddot{x}(t) + (\delta + 2\varepsilon \cos 2t)x(t) = -k_d \dot{x}(t - \tau) - k_p x(t - \tau) \tag{26}$$

The period of the system is $T = \pi$. We have selected $\varepsilon = 1$, $\delta = 4$ and $N = 2^5$. The uncontrolled system is parametrically unstable.

Lyapunov–Krasovskii functionals for periodic systems are hard to find. We adopt the CTA methods combining Floquet theory to predict the stability boundaries of the system. Figures 7 to 10 show the comparison of the stability domains by the CTA methods and SD for $\tau = \pi/10$, $\pi/4$, $\pi/2$ and $3\pi/4$.

These examples show that the CTA method with Chebyshev collocation or low-pass-filter-based algorithm can yield solutions of periodic systems with high accuracy. The stability domains by the Chebyshev CTA and LPCTA methods are nearly overlapped with the boundaries obtained by the SD method. The stability boundary by the CTA method

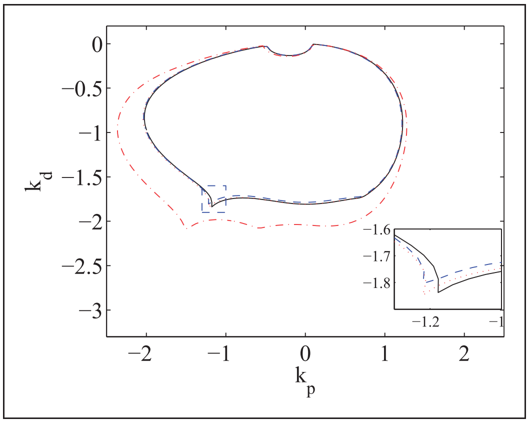

Figure 9. The stability boundaries of the periodic system in the $k_p - k_d$ gain space ($\tau = \pi/2$) by three CTA methods ($N = 2^5$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . SD solution: Solid line.

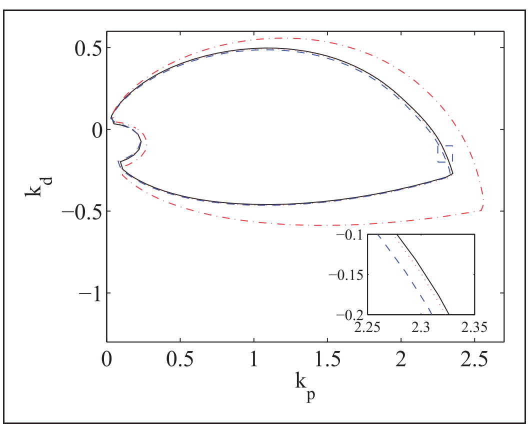

Figure 10. The stability boundaries of the periodic system in the $k_p - k_d$ gain space ($\tau = 3\pi/4$) by three CTA methods ($N = 2^5$). Forward finite difference: $- \cdot - \cdot -$. LPCTA: $- - -$. Chebyshev CTA: . . . . SD solution: Solid line.

with forward finite difference is again larger than the boundary by the SD method, implying that it is nonconservative.

When the CTA method with forward finite difference is used to design feedback controls for systems with time delay, we must pick the gains deep in the interior of the domain to avoid instability.

5. Concluding remarks

We have compared the stability results obtained by the Lyapunov, SD and CTA methods. From extensive numerical simulations, we have found that the Lyapunov method, whether it is delay-dependent or delay-independent, leads to conservative stability results compared with those obtained from equation (3), except for the necessary and sufficient stability condition due to Gu. Gu's Lyapunov–Krasovskii functional is not conservative, and produces the best prediction of the stability boundaries in the parameter space, which are closest to the boundaries obtained from equation (3), among all the Lyapunov stability conditions we have examined.

The CTA methods can be used to study both LTI and periodic systems. The stability domains by the Chebyshev CTA and LPCTA methods are highly accurate compared to the solutions obtained from equation (3) or by the SD method.

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