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Algorithms to Verify Generic Causality and Controllability of Descriptor Systems

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Abstract—**Graph-theoretic algorithms are developed for determining generic causality and controllability of structured descriptor systems (SDS's). For the case of nonsingular E, a completely graph-theoretic characterization of generic controllability of SDS's is developed. The case of singular E is partially resolved, with graph-theoretic characterization of necessary conditions and sufficient conditions for generic causality and controllability developed separately. Although not stated explicitly, these results can be easily translated into a graph-theoretic algorithm to test for generic observability of SDS's.**

I. INTRODUCTION

LIN [5] made a remarkable contribution to the theory of dynamical systems by his work on structural controllability. His and later results on this topic proved the following fact: controllability of linear time-invariant systems is essentially determined by the "structure" of the system, almost independently from specific values of parameters of the model. In addition, a completely graph-theoretic algorithm is now known for determining whether a system is structurally controllable.

However, the standard state-space representation of a dynamical system

$$x(t+1)=Ax(t)+Bu(t) \qquad t=0,\ 1,\ \cdots$$

is often inconvenient for modeling practical problems (see, e.g., [6] and [7]). A more flexible alternative is the descriptor system representation of the form

$$Ex(t+1)=Ax(t)+Bu(t) \qquad t=0,\ 1,\ \cdots.$$

Indeed, the descriptor representation is standard in the literature of econometrics [10]. In a companion paper [11], we investigated the problem of generic controllability of structured descriptor systems (SDS's), where the matrix (E, A, B) is a structured matrix whose elements are either fixed at zero or mutually independent free parameters. Two different but related concepts of controllability, C- and R-controllability, were introduced and characterized in Yamada and Luenberger [11]. A fundamental assumption for these concepts of controllability was causality.

This paper complements the above-mentioned paper by presenting graph-theoretic algorithms to verify generic causality and controllability of SDS's. In Section II, the earlier work is summarized. Section III presents a graphical algorithm for structural matrix inversion. Section IV develops the conditions and graphical algorithms for verifying generic causality of SDS's. Sections V and VI are devoted to the derivation and development of the conditions and algorithms for verifying generic controllability of SDS's.

II. SUMMARY OF PREVIOUS RESULTS

Consider the SDS of the form

$$Ex(t+1)=Ax(t)+Bu(t) \qquad t=0,\ 1,\ \cdots \tag{1}$$

where $x(t)$ and $u(t)$ are $\bar{n}$ and m-vectors representing the descriptor vector and input at time t, respectively, and (E, A, B) is a structured matrix with consistent dimensions. Without loss of generality, we can assume, by appropriate permutation of the columns and rows of E if necessary, that

$$g-\text{rank}\ (E)=g-\text{rank}\ (E_{11})=n\ (\leq \bar{n}),$$

$$E=\begin{array}{cc} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & 0 \end{bmatrix} & \begin{matrix} n \\ d \end{matrix} \\ \begin{matrix} n \quad & \quad d \end{matrix} & \end{array}$$

$$\bar{n}=n+d.$$

Here, g-rank $(\cdot)$ denotes the generic rank[1] of the matrix $(\cdot)$.

[1] See [3] or [8] for definition.

Manuscript received April 17, 1984; revised December 8, 1984. Paper recommended by Past Associate Editor, W. A. Wolovich. This work was supported in part by the National Science Foundation under Grant ECS-8214254 and by the National Defense Academy, Japan.
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Let

$$x(t)=\begin{bmatrix} I & -E_{11}^{-1}E_{12} \\ 0 & I \end{bmatrix}\xi(t)$$

then (1) can be transformed into

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}\xi(t+1)=\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\xi(t)+\begin{bmatrix} G_1 \\ G_2 \end{bmatrix}u(t) \qquad (2)$$

where

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}\triangleq\begin{bmatrix} I & 0 \\ -E_{21}E_{11}^{-1} & I \end{bmatrix}A\begin{bmatrix} I & -E_{11}^{-1}E_{12} \\ 0 & I \end{bmatrix}.$$

Luenberger [6] proved that system (2) is *causal*[2] if and only if

$$\begin{bmatrix} E_{11} & 0 \\ F_{21} & F_{22} \end{bmatrix} \text{ is nonsingular.}^3$$

As a straightforward extension, we have the following.

Theorem 1 (Generic Causality): SDS (1) is generically causal if and only if g-rank$(F_{22}) = d$ (full), where

$$F_{22}\triangleq[-E_{21}E_{11}^{-1}\,|I]A\left[\frac{-E_{11}^{-1}E_{12}}{I}\right]. \qquad (3)$$

Under the assumption of generic causality, the SDS (1) can be transformed into the following.

Canonical Form:[4]

$$\begin{cases} y(t+1)=\bar{A}y(t)+\bar{B}_1u(t) & (4a)\\ z(t)=-\bar{B}_2u(t) & (4b)\end{cases}$$

where

$$\bar{A}\triangleq D\left[\frac{A_{11}}{A_{21}}\right] \qquad (5a)$$

$$\bar{B}_1\triangleq DB,\ \bar{B}_2\triangleq F_{22}^{-1}[-E_{21}E_{11}^{-1}|I]B \qquad (5b)$$

$$D\triangleq E_{11}^{-1}[I|-F_{12}F_{22}^{-1}]\begin{bmatrix} I & 0 \\ -E_{21}E_{11}^{-1} & I \end{bmatrix} \qquad (5c)$$

$$x(t)=\begin{bmatrix} I & -E_{11}^{-1}E_{12} \\ 0 & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -F_{22}^{-1}F_{21} & I \end{bmatrix}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix}. \qquad (5d)$$

Before stating the theorems on generic controllability, we need the following.

Definition 1 (Irreducibility): Let A and B be matrices of respective dimensions $n \times n$ and $n \times m$. (A, B) is *reducible* if for some permutation matrix P,

$$(PAP',\ PB)=\left[\begin{array}{cc|c} * & 0 & 0 \\ \hline * & * & * \end{array}\right]$$

holds. It is *irreducible* if it is not reducible.

[2] A descriptor system (1) is said to be causal if $x(t)$ is independent of $u(t+1), u(t+2), \cdots$ for $t = 0, 1, 2, \cdots$. Generic causality is a trivial extension of this concept to structured descriptor systems.

[3] See [6, Theorem 7.1] although the concept of causality is referred to as "regularity" in that paper.

[4] See [11] for more details.

Then, the following theorems characterize generic controllability of SDS's. The proof of these theorems can be found in [11].

A. Case of Nonsingular E

Theorem 2 (Generic Controllability): The SDS (1) with generically nonsingular E is generically controllable if and only if:

i) g-rank$(A, B) = \bar{n}$ (full),

and

ii) $(E^{-1}A, E^{-1}B)$ is irreducible.

B. Case of Singular E

The following necessary condition is easily proved.

Theorem 3 (Necessary Conditions for Generic Controllability): The causal SDS (1) is:

a) generically R-controllable[5] only if

i) g-rank$(A, B) = \bar{n}$ (full),

and

ii) $(\bar{A}, \bar{B}_1)$ is irreducible

b) generically C-controllable only if in addition to i) and ii), the following holds:

iii) g-rank(E, B) $= \bar{n}$ (full).

To state the sufficient conditions, we introduce the concept of a subsystem S_γ. A square matrix M is said to be p-diagonal if for some permutation matrices P and Q, PMQ is a diagonal matrix.

Definition 2 (Subsystem S_γ): An SDS

$$E_\gamma x(t+1)=A_\gamma x(t)+B_\gamma u(t)$$

is a subsystem S_γ of the SDS (1) if:

i) S_γ is obtained from SDS (1) by fixing some parameters at zero,

and

ii) $E_{11\gamma}$ is nonsingular diagonal and $F_{22\gamma}$ is nonsingular p-diagonal.

Here $E_{11\gamma}$ and $F_{22\gamma}$ denote the counterpart in S_γ of the matrices E_{11} and F_{22} in SDS (1).

Theorem 4 (Sufficient Condition for Generic Controllability): The SDS (1) is generically causal and:

a) generically R-controllable if there exists a subsystem S_γ such that

i) g-rank$(A_\gamma, B_\gamma) = \bar{n}$ (full),

and

ii) $(\bar{A}_\gamma, \bar{B}_{1\gamma})$ is irreducible

b) generically C-controllable if in addition to i) and ii), the following holds:

iii) g-rank$(E, B) = \bar{n}$ (full).

III. Inversion of Structured Matrices

In the following sections, Theorems 1–4 are restated in terms of the "structure," or equivalently the graph representation, of the SDS (1). To develop these characterizations, we will need to determine the locations of all fixed zeros and nonzero elements of the inverse of a structured matrix. This section describes an algorithm for doing this, based on a graph-theoretic representation of a structured matrix.

However, it should be noted that the inverse of a structured matrix is not structured in the same sense: while there may be fixed zeros, the nonzero terms can be mutually dependent. This causes a difficulty (in Sections IV and VI) in translating Theorems 1–4 into purely graph-theoretic tests to verify generic causality and controllability.

Definition 3 (Bipartite Graph Representation of a Structured Matrix): Let M be an $m \times n$ structured matrix. Then,

[5] C-controllability refers to the complete reachability in the entire n-dimensional space, while R-controllability requires reachability within the n-dimensional subspace of admissible vectors. See [12] for more details (on continuous-time descriptor systems).

$H(M)$ is defined to be the directed bipartite graph [1] with the set of nodes $N = \{1, 2, \cdots, n\} \cup \{1', 2', \cdots, m'\}$ and the set of arcs $A \subseteq N \times N$, where $(i, j') \in A$ if and only if the (j, i)th element of M is not fixed at zero.

Now, consider an $n \times n$ generically nonsingular structured matrix W. Then, there exists a *complete matching* [1] in $H(W)$ (see also [9]). The algorithm for constructing the graph of W^{-1} is given by the following.

Algorithm 1 (Structured Matrix Inversion):

Step 1: Construct $H(W)$ with the set of originating and terminating nodes $X_W = \{1, 2, \cdots, n\}$ $Y_W = \{1', 2', \cdots, n'\}$.

Step 2: Find a maximal matching in $H(W)$. If this is a complete matching, let it be

$$M = \{(i, k_i') | i \in X_w, k_i' \in Y_w \quad (i = 1, 2, \cdots, n)\}.$$

If this is not a complete matching, W is generically singular.

Step 3: Draw an arrow from node $k_i' \in Y_W$ to $i \in X_W$ for each $i = 1, 2, \cdots, n$. Denote the resulting graph by $\hat{H}(W)$.

Step 4: Construct the graph $H(W^{-1})$ by the following rule. Let $X_{W^{-1}} = \{1', 2', \cdots, n'\}$ and $Y_{W^{-1}} = \{1, 2, \cdots, n\}$ be the sets of originating and terminating nodes of $H(W^{-1})$. Draw an arrow from $j' \in X_{W^{-1}}$ to $i \in Y_{W^{-1}}$ if and only if there exists a path in $\hat{H}(W)$ going from $j' \in Y_W$ to $i \in X_W$ for all pairs (i, j').

Note that this algorithm can be efficiently carried out using the *Hungarian method* or solving a maximal network flow problem to find a maximal matching in Step 2. (See [2] for these topics.)

Example 1: Consider the structured matrix of the form

$$W = \begin{bmatrix} [*] & 0 & 0 & 0 \\ * & [*] & 0 & 0 \\ 0 & * & * & [*] \\ 0 & 0 & [*] & * \end{bmatrix}.$$

By Cramer's formula, the nonzero elements of W^{-1} are located as

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & * \\ * & * & * & * \end{bmatrix}.$$

The graphs $H(W)$ and $\hat{H}(W)$ are illustrated in Fig. 1 (a) and (b). Although $H(W)$ is a directed graph, direction of each arc is not explicitly shown, since it is always from left to right. This is the rule in all subsequent examples. Instead, the arrows in Fig. 1 are used to indicate a complete matching which is also shown in the matrix W by brackets. The graph $H(W^{-1})$ constructed by the algorithm is shown in Fig. 1(c), and it agrees with the result found by Cramer's formula.

The above algorithm is justified by the following.

Theorem 5 (Structured Matrix Inversion): Let W be an $n \times n$ structured matrix which is generically nonsingular. Then, $H(W^{-1})$ gives the bipartite graph representation of W^{-1}.

Proof: i) Case of identity matching $M = \{(i, i') | i = 1, 2, \cdots, n\}$. Let $W = (W_{ij})$ be defined by

$$w_{ij} \triangleq \begin{cases} 1-\epsilon & : \text{diagonal elements} \\ -\epsilon & : \text{off-diagonal nonzero elements} \\ 0 & : \text{fixed zeros} \end{cases}$$

and

$$\bar{W} \triangleq I - W.$$

Then, for sufficiently small $\epsilon > 0$, we have

$$W^{-1} = I + \bar{W} + \bar{W}^2 + \cdots.$$

Clearly the following statements are equivalent:

a) the (i, j)th element of $W^{-1} \neq 0$.

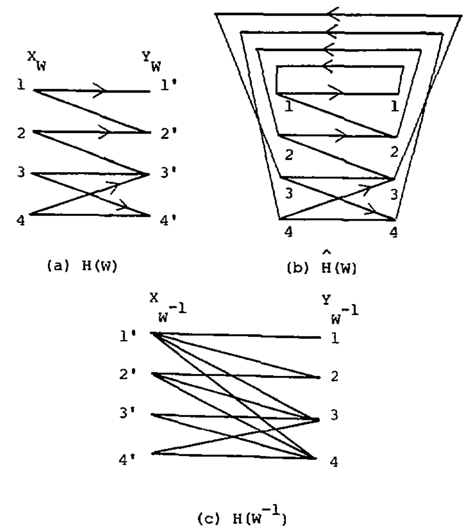


Fig. 1. Structured matrix inversion algorithm.

b) the (i, j)th element of $\bar{W}^p \neq 0$ for some $p \geq 0$.

c) there exists a path in $\hat{H}(W)$ from $j' \in Y_W$ to $i \in X_W$.

By identifying X_W and Y_W with $Y_{W^{-1}}$ and $X_{W^{-1}}$, respectively, the proof is complete.

ii) General case. Let $M = \{(i, k_i') | i = 1, 2, \cdots, n\}$ be a maximal matching. Consider the permutation matrix such that $W_0 = PW$ has $M_0 = \{(i, i') | i = 1, 2, \cdots, n\}$ as its complete matching. Then, by i), the nonzero elements of W_0^{-1} can be found by the above algorithm. Since $W^{-1} = (P'W_0)^{-1} = W_0^{-1}P$, $H(W^{-1})$ can be found simply by connecting $H(P)$ to $H(W_0^{-1})$ from the left, which is equivalent to connecting $k_i' \in Y_W$ to $i \in X_W$ in $H(W)$ for each $i = 1, 2, \cdots, n$. Q.E.D.

It should be noted that this algorithm is not invertible. That is, applying the algorithm to $H(W^{-1})$ does not necessarily give $H(W)$. This is due to lack of independence among the nonzero elements of W^{-1}.

IV. The Conditions and Algorithms for Generic Causality

This section presents a graph-theoretic characterization of generic causality for SDS (1). By Theorem 1, this amounts to testing for generic full-rankness of F_{22} defined by (3). The idea is to extend the one-to-one correspondence between the generic rank of a structured matrix and a maximal matching in its graph representation [9] to the case of F_{22}.

However, due to the lack of independence among the elements of F_{22} (specifically, in E_{11}^{-1}), we could only obtain a necessary condition and a sufficient condition (Corollary 1 and Corollary 2, respectively) separately, by weakening the requirements of Theorem 1 (which was necessary and sufficient) in such a way as to permit graph-theoretic treatments.

A. Graph $H(F_{22})$

The graph $H(F_{22})$ is constructed by combining the component graphs corresponding to the matrices E_{11}^{-1}, E_{12}, E_{21}, A and I, consistently to the definition (3). The following example illustrates how this is done.

Example 2: Let E and A be as follows:

$$E = \left[\begin{array}{ccc|cc} * & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * \\ 0 & * & * & 0 & * \\ \hline * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{array}\right] \quad A = \left[\begin{array}{ccc|cc} * & 0 & * & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * \\ \hline * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \end{array}\right].$$

Note that g-rank(E) = g-rank(E_{11}) = 3. Graphs of the component matrices are shown in Fig. 2 (a)–(d). Using Algorithm 1, the graph $H(E_{11}^{-1})$ is obtained as Fig. 2 (e). Combining these graphs according to (3), we obtain the graph $H(F_{22})$, as illustrated in Fig. 2 (f). Here, the set of left and right side nodes are denoted by X and Y, respectively.

B. Linking on $H(F_{22})$

Now, the concept of matching is extended to the graph $H(F_{22})$.

Definition 4 [Linking in $H(F_{22})$]:

i) $\pi = \{a_1, a_2, \cdots, a_k\}$ is a *path* in $H(F_{22})$ if each $a_i (i = 1, 2, \cdots, k)$ is an arc in $H(F_{22})$, a_i and a_{i+1} are adjacent, a_1 originates in X, and a_k terminates in Y.

ii) A set of paths $\{\pi_1, \pi_2, \cdots, \pi_h\}$ is said to be a *linking* in $H(F_{22})$ if no two paths in this set share a node in common. A matching is a *complete linking* if it consists of d different paths, where $d = |X| = |Y|$.

Parallel to the case of a structured matrix, it might be conjectured that a necessary and sufficient condition for F_{22} to be generically nonsingular is that there exists a complete linking in $H(F_{22})$. This conjecture is not valid, however, as shown by the following counterexample.

Example 3: Let E and A be

$$E = \left[\begin{array}{ccccc|cc} * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] A = \left[\begin{array}{ccccc|cc} 0 & 0 & 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

After row operations to sweep out the nonzero element of the seventh row of E, we have

$$(E, A) \simeq \left[\begin{array}{ccccc|cc|ccccc|cc} * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} E_1 & D_1 \\ \hline 0 & D_2 \end{array}\right].$$

Thus, we have

$$g-\text{rank}\begin{bmatrix} E_1 \\ D_2 \end{bmatrix} = g-\text{rank}\left[\begin{array}{ccccccc} * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & * & 0 \\ 0 & 0 & * & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 \end{array}\right] < 7,$$

since the last three rows are not independent. Therefore, this system is not generically causal (see [6]). However, there exists a complete linking in $H(F_{22})$ as illustrated in Fig. 3 by arrows.

This lack of correspondence between the generic rank and complete linking is due to the dependence of the elements in F_{22}, or more specifically, in E_{11}^{-1}. Indeed, in Example 3, the submatrix of E_{11}^{-1} associated with the complete linking (consisting of columns 3, 4 and rows 4, 5, corresponding to the nodes lying on

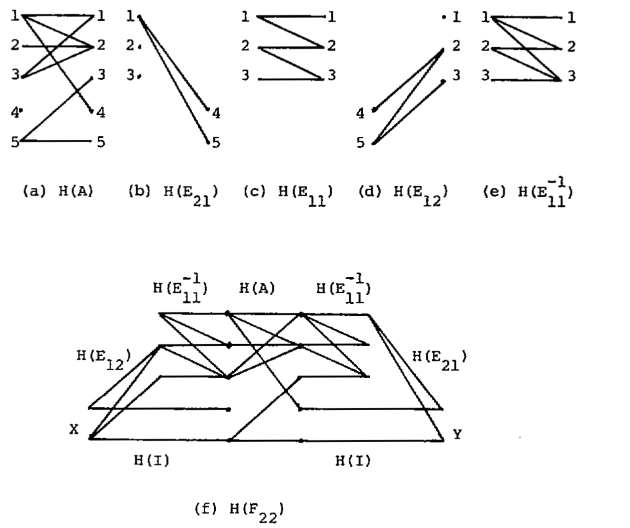


Fig. 2. Graph $H(F_{22})$.

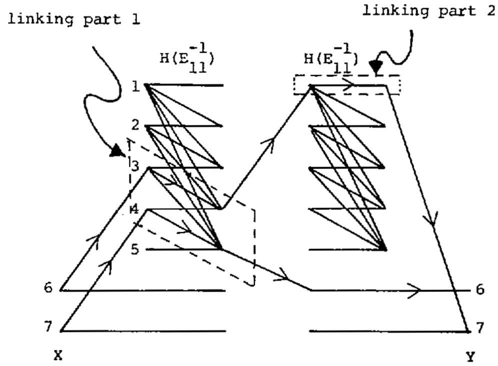


Fig. 3. Maximal linking and linking parts in $H(F_{22})$.

the linking paths) is generically singular. These observations motivate the following.

Definition 5 (Linking Part): Consider the graph $H(F_{22})$ and a linking $M = \{\pi_1, \pi_2, \cdots, \pi_h\}$ on this graph. Let H be the (left or right side) graph $H(E_{11}^{-1})$ included in $H(F_{22})$, and define a subgraph $\mathring{H}$ by $\mathring{H} \triangleq M \cap H$. The submatrix of E_{11}^{-1} corresponding to $\mathring{H}$ is said to be the *linking part* of E_{11}^{-1} with respect to the linking M. There may be zero, one, or two linking parts. A matching without linking parts or with generically nonsingular linking parts only is said to be a *nondegenerate linking.*

Example 4: In Fig. 3, the submatrix of E_{11}^{-1} consisting of columns 3, 4 and rows 4, 5 is a linking part with respect to the linking defined in Example 3. Also, the submatrix consisting of the (1, 1)th element of E_{11}^{-1} is another linking part. This linking is not nondegenerate, since the first linking part is singular.

C. A Necessary and Sufficient Condition for Generic Causality

With the above definitions, generic causality of SDS (1) is characterized by the following.

Theorem 6 (Generic Causality): SDS (1) is generically causal if and only if there exists a complete nondegenerate linking in $H(F_{22})$.

Proof: (Sufficiency) Take the parameter values of E_{11} such that if linking parts exist, they are all nonsingular. Set the free parameters of E_{12}, E_{21}, and A in such a way that they are 1 if the corresponding arc lies on a linking path, and 0 otherwise. Then,

$\det(F_{22}) = \pm 1 \times \delta_1 \times \delta_2 \neq 0$, where

$$\delta_i \triangleq \begin{cases} \det(i\text{th linking part}), & \text{if it exists.} \\ 1, & \text{otherwise.} \end{cases}$$

(Necessity) Let f_{ij} be the (i, j)th element of F_{22}. Remember the construction of $H(F_{22})$ described in Section IV-A. Each arc in this graph is associated with an element of the corresponding matrix. Then, for a path π from $j \in X$ to $i \in Y$ in $H(F_{22})$, we can define its *value* $v(\pi)$ as the product of the matrix entries associated with each arc along the path π. Let P_{ij} be the set of paths in $H(F_{22})$ connecting nodes $j \in X$ to $i \in Y$. Then,

$$f_{ij} = \sum_{\pi \in P_{ij}} v(\pi).$$

So, we have

$$\det(F_{22}) = \sum_{(i_1, i_2, \cdots, i_d)} \operatorname{sgn}(i_1, i_2, \cdots, i_d) f_{1i_1} f_{2i_2} \cdots f_{di_d}$$
$$= \sum_{\pi_1, \pi_2, \cdots, \pi_d} \operatorname{sgn}(\pi_1, \pi_2, \cdots, \pi_d) v(\pi_1) v(\pi_2) \cdots v(\pi_d)$$

where summation ranges over all possible permutations of $(1, 2, \cdots, d)$.

If π_1 and π_2 are two paths in $H(F_{22})$, which meet at some node, we can write these paths as $\pi_1 = \pi_1'\pi_1''$ and $\pi_2 = \pi_2'\pi_2''$. Consider two other paths defined by $\hat{\pi}_1 \triangleq \pi_1'\pi_2''$ and $\hat{\pi}_2 \triangleq \pi_2'\pi_1''$ as shown in Fig. 4.

Then, since

$$\operatorname{sgn}(\pi_1, \pi_2, \cdots, \pi_d) v(\pi_1) v(\pi_2) \cdots v(\pi_d)$$
$$+ \operatorname{sgn}(\hat{\pi}_1, \hat{\pi}_2, \cdots, \pi_d) v(\hat{\pi}_1) v(\hat{\pi}_2) \cdots v(\pi_d) = 0$$

summing over all permutations, the contribution of all such intersecting paths cancel each other, leaving only the terms associated with complete linkings.

Furthermore, if a linking part associated with a complete linking is generically singular, we can again eliminate that linking, since when summed over all permutations the terms cancel each other. Thus, we finally obtain

$$\det(F_{22}) = \sum_{\substack{\text{complete nondegenerate} \\ \text{linking} \\ \{\pi_1, \pi_2, \cdots, \pi_d\}}} \operatorname{sgn}(\pi_1, \pi_2, \cdots, \pi_d) v(\pi_1) v(\pi_2) \cdots v(\pi_d).$$

Then, $\det(F_{22}) \neq 0$ implies the existence of at least one complete nondegenerate linking, which completes the proof. Q.E.D

D. Structural Conditions for Generic Causality

Theorem 6 gives a complete characterization of generic causality for descriptor systems. However, checking the nondegeneracy condition only by graph-theoretic methods seems intractable, since the entries of the linking parts of E_{11}^{-1} are usually interdependent. Therefore, some weaker but graph-theoretic conditions are presented in the remainder of this section.

From the proof of Theorem 6, the following is obvious.

Corollary 1 (Necessary Condition for Generic Causality): SDS (1) is generically causal only if there exists a complete matching in $H(F_{22})$.

There are several conditions which guarantee generic causality for SDS's. These conditions are obtained by requiring the existence of a complete matching of some special form which necessarily satisfies the nondegeneracy condition.

Corollary 2 (Sufficient Condition for Generic Causality): If either one of the following conditions holds, the SDS (1) is generically causal.

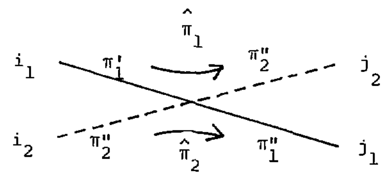
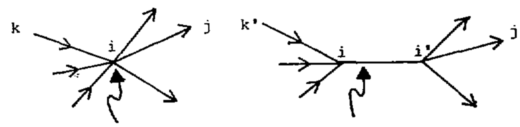


Fig. 4. Intersecting paths.



Node constraint Arc constraint

(a) Original Network (b) Modified Network

Fig. 5. Modifying a network.

i) $g\text{-rank}\ (E_{21}, I)A \begin{bmatrix} E_{21} \\ I \end{bmatrix} = d$ (full).

ii) $g\text{-rank} \begin{bmatrix} E_{11} & E_{12} \\ A_{21} & A_{22} \end{bmatrix} = \bar{n}$ (full).

Proof: i) Without loss of generality (by appropriate permutations), we can assume that the diagonal elements of E_{11} are nonzero. Select parameter values for E_{21}, A and E_{12} such that the matrix in i) is of full rank, and set $E_{11} = I$. Then, $F_{22} = (E_{12}, I)A\left[\begin{smallmatrix} E_{21} \\ I \end{smallmatrix}\right]$ is nonsingular.

ii) Select parameter values such that the matrix in ii) is of full rank, and set $E_{21} = 0$. Then, the descriptor system (E, A) is causal, since ii) is exactly the condition for causality (see [6]). This is sufficient for the generic causality of (1). Q.E.D.

E. An Algorithm for Verifying Generic Causality

The test of Corollary 1 can be implemented with graph-theoretic methods. First, construct the graph $H(F_{22})$. The question of the existence of a complete linking in this graph can be examined by solving the following maximal network flow problem [2] on $H(F_{22})$.

Maximal Network Flow Problem: Let all arc and node capacities be 1 in the network $H(F_{22})$. Find the maximal flow through this network (originating in X and terminating in Y in Fig. 2).

Network flow problems with node capacity constraints are not entirely standard in the literature. However, this problem can be easily *standardized* by the node splitting technique illustrated in Fig. 5.

The maximal network flow problem can be efficiently solved used a *labeling method* (see [2]), and clearly the solution gives a maximal linking in $H(F_{22})$, and vice versa.

Similarly, Corollary 2 can be implemented by solving the maximal network flow problem on the networks $H(F_{22}^*)$ and $H(F_{22}^\#)$, where

$$F_{22}^* \triangleq (E_{21}, I)A \begin{bmatrix} E_{12} \\ I \end{bmatrix} \text{ and } F_{22}^\# \triangleq \begin{bmatrix} E_{11} & E_{12} \\ A_{21} & A_{11} \end{bmatrix}.$$

These procedures give a graph-theoretic algorithm to test for generic causality of descriptor systems.

V. Structural Controllability of SDS's: Nonsingular E

Consider the SDS (1) with generically nonsingular E. The system can be equivalently written as

$$x(t+1) = \bar{A}x(t) + \bar{B}u(t)$$

where

$$(\bar{A},\ \bar{B}) \triangleq E^{-1}(A,\ B).$$

The graph $H(\bar{A}, \bar{B})$ can be obtained simply by connecting the graphs $H(A, B)$ and $H(E^{-1})$ serially. Let us define the graph $\bar{H}(\bar{A}, \bar{B})$ as the graph obtained from $H(\bar{A}, \bar{B})$ by identifying the left and right side nodes of the same indexes. This is shown in Fig. 6(e), where the broken lines indicate identifying arcs.

Example 5: Let the SDS be

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} x(t+1) = \begin{bmatrix} 0 & 0 & * \\ * & 0 & 0 \\ 0 & 0 & * \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ * \end{bmatrix} u(t).$$

The graphs $H(A, B)$ and $H(E)$ are illustrated in Fig. 6(a), (b). By Algorithm 1, we obtain $H(E^{-1})$, and combining this with $H(A, B)$, the graphs $H(\bar{A}, \bar{B})$ and $\bar{H}(\bar{A}, \bar{B})$ are obtained as in Fig. 6(d), (e).

The following theorem completely characterizes generic controllability of SDS's with nonsingular E in graph-theoretic terms, and hence may be appropriately called as the structural controllability theorem for SDS's.

Theorem 7 (Structural Controllability): SDS (1) with generically nonsingular E is generically controllable if and only if

i) there exists a complete matching in $H(A, B)$, and

ii) each node of $\bar{H}(\bar{A}, \bar{B})$ is input-reachable, i.e., each node is accessible from some input-node corresponding to a column of B.

Proof: Note that i) is equivalent to the full-rank condition of Theorem 2. We prove that ii) is equivalent to the irreducibility of $(\bar{A}, \bar{B})$. If this is proved, Theorem 2 implies Theorem 7.

If, on the contrary, there exist some nodes in $\bar{H}(\bar{A}, \bar{B})$ which are not accessible from any input nodes, we can classify the nodes of $\bar{H}(\bar{A}, \bar{B})$ into two nonempty sets

$$X_1 = \{\text{nodes which are not input-reachable in } \bar{H}(\bar{A},\ \bar{B})\}$$

$$X_2 = \{\text{nodes which are input-reachable in } \bar{H}(\bar{A},\ \bar{B})\}.$$

Note, by definition, that there can be no arc in $H(\bar{A}, \bar{B})$ connecting an input-node or a node in X_2 to a node in X_1, since otherwise the latter is input-reachable in $\bar{H}(\bar{A}, \bar{B})$. This implies that $(\bar{A}, \bar{B})$ is reducible.

Conversely, assume that $(\bar{A}, \bar{B})$ is reducible. Without loss of generality, it can be assumed that

$$(\bar{A},\ \bar{B}) = \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & * \end{array}\right]$$

$$X_1 X_2 U \cdots \text{ set of nodes in } \bar{H}(\bar{A},\ \bar{B}).$$

If there exists a path connecting a node in $H(\bar{A}, \bar{B})$ to another, then the corresponding entry of $(\bar{A}, \bar{B})$ is generically nonzero, since there can be no other paths to cancel mutually in this graph where all arcs can be assumed to be positive. Then, the above form of $(\bar{A}, \bar{B})$ implies that nodes in X_1 are not input-reachable in $\bar{H}(\bar{A}, \bar{B})$. Q.E.D.

Example 6: Consider the same SDS as in Example 5. Condition i) is satisfied, since there exists a complete matching in $H(A, B)$ as shown in Fig. 6 (a) with arrows. Since all nodes in $\bar{H}(\bar{A}, \bar{B})$ are input-reachable in Fig. 6 (e), condition ii) is also satisfied. Thus, this system is generically controllable.

VI. Structural Conditions for Generic Controllability: General Case

The necessary and sufficient conditions (Theorems 3 and 4) consist of two requirements: generic full-rankness of (A, B) and irreducibility of $(\bar{A}, \bar{B}_1)$. The first condition can be easily examined by attempting to find a maximal matching in $H(A, B)$. In this section, we develop a graph-theoretic algorithm for examining irreducibility of $(\bar{A}, \bar{B}_1)$.

Again, due to the lack of independence among the elements of $(\bar{A}, \bar{B}_1)$ (specifically in E_{11}^{-1}), exact graph-theoretic restatements of Theorems 3 and 4 are difficult to obtain. We could only obtain weaker conditions (Theorems 8 and 9) by relaxing some of the requirements. This is achieved by constructing the graph $\bar{H}^*(\bar{A}, \bar{B}_1)$, which in some sense approximates the true graph $\bar{H}(\bar{A}, \bar{B}_1)$, and examining its input-reachability.

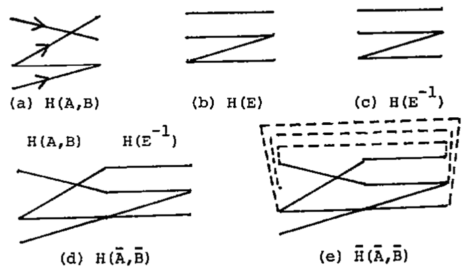


Fig. 6. Testing for structural controllability.

A. Graph $\bar{H}^*(\bar{A}, \bar{B}_1)$

This graph is, in a sense, an approximation of $\bar{H}(\bar{A}, \bar{B}_1)$. The latter is difficult to construct, since it involves inversion of F_{22}. If we apply Algorithm 1 to $H(F_{22})$, the result would be a graph $H^*(F_{22}^{-1})$, which is usually different from the actual graph $H(F_{22}^{-1})$. Using $H^*(F_{22}^{-1})$ in place of $H(F_{22}^{-1})$, and combining the component graphs according to the definition of $(\bar{A}, \bar{B}_1)$ given by (5a) and (5b), we obtain the graph $H^*(\bar{A}, \bar{B}_1)$, and identifying the left and right side nodes, we obtain $\bar{H}^*(\bar{A}, \bar{B}_1)$. This is explained in terms of the following example.

Example 7: Consider the SDS defined by

$$E = \begin{bmatrix} * & 0 & * & 0 & * \\ * & 0 & 0 & * & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} * & * & 0 & 0 & * \\ 0 & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ * & 0 & * & 0 & * \\ 0 & * & 0 & * & 0 \end{bmatrix} \quad B = \begin{bmatrix} * & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & * \\ * & 0 \end{bmatrix}$$

The graphs $H(A, B)$ and $H(E)$ are shown in Fig. 7 (a), (b). $H(E_{11}^{-1})$ is obtained using Algorithm 1, and it is depicted in Fig. 7 (c). Then, $H(F_{22})$ is constructed by the procedure of Section IV-A, as given in Fig. 7 (d). This may be simply depicted as Fig. 7 (e). Applying Algorithm 1 to this graph yields $H^*(F_{22}^{-1})$ of Fig. 7 (f), which is different from the actual inverse $H(F_{22}^{-1})$, which may be as Fig. 7 (g). Now, combine $H^*(F_{22}^{-1})$ with other component graphs, we obtain the graphs $H^*(\bar{A}, \bar{B}_1)$ and $\bar{H}^*(\bar{A}, \bar{B}_1)$ as shown in Fig. 7 (h), (i).

B. A Necessary Condition and a Sufficient Condition

Using the graphs defined above, we can state the conditions for generic controllability of SDS's.

Theorem 8 (Necessary Condition for Generic Controllability): The SDS (1) is

a) generically R-controllable only if

i) there exists a complete matching in $H(A, B)$, and

ii)each node in $\bar{H}^*(\bar{A}, \bar{B}_1)$ is input-reachable

b) generically C-controllable only if

in addition to i) and ii), the following holds:

iii) there exists a complete matching in $H(E, B)$.

Proof: i) follows immediately from i) of Theorem 3. From ii) of Theorem 3 and the argument in the proof of Theorem 7, we conclude input-reachability in $\bar{H}(\bar{A}, \bar{B}_1)$, and therefore in $H^*(\bar{A}, \bar{B}_1)$. iii) is also immediate from iii) of Theorem 3. Q.E.D

Theorem 9 (Sufficient Condition for Generic Controllability): The SDS (1) is generically causal and:

a) generically R-controllable if there exists a subsystem S_γ such that

i) there exists a complete matching in $H(A_\gamma, B_\gamma)$,

and

ii) each node in $H^*(\bar{A}_\gamma, \bar{B}_{1\gamma})$ is input-reachable

b) generically C-controllable if in addition to i) and ii), the following holds:

iii) there exists a complete matching in $H(E, B)$.

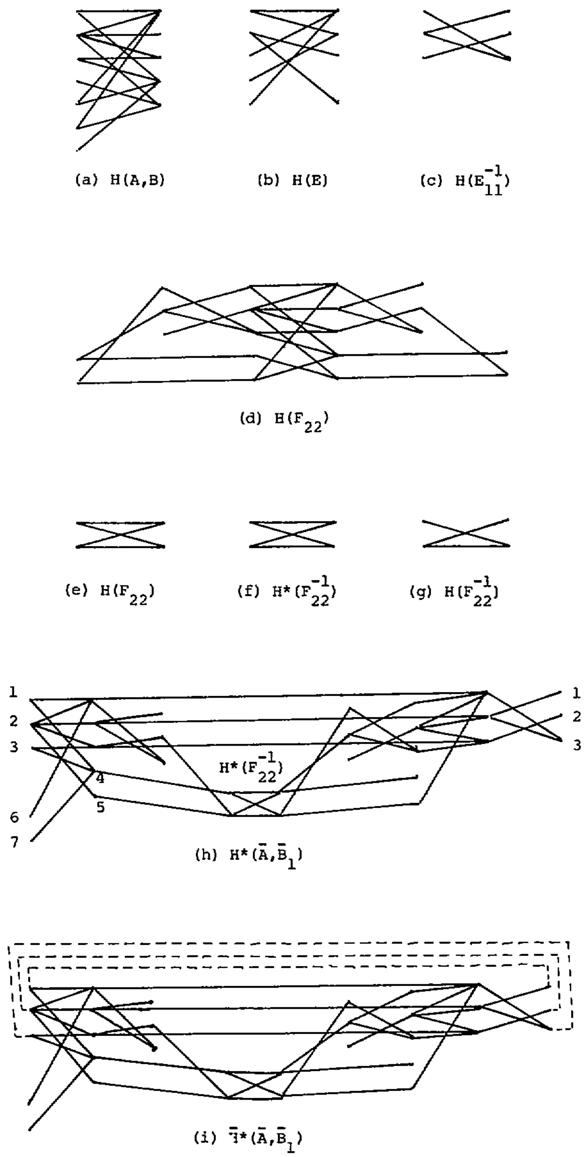


Fig. 7. Graphs $H^*(\bar{A}, \bar{B}_1)$ and $\bar{H}^*(\bar{A}, \bar{B}_1)$.

Proof: Since $F_{22\gamma}$ is p-diagonal, $H^*(F_{22\gamma}^{-1}) = H(F_{22\gamma}^{-1})$. Furthermore, since $E_{11\gamma}$ is diagonal, each arc in $\bar{H}(\bar{A}_\gamma, \bar{B}_{1\gamma})$ can be taken to be positive. Thus, no canceling of paths occurs in $\bar{H}(\bar{A}_\gamma, \bar{B}_{1\gamma})$, and therefore the existence of a path in $\bar{H}^*(\bar{A}_\gamma, \bar{B}_{1\gamma})$ implies generic nonzeroness of the corresponding element of $(\bar{A}_\gamma, \bar{B}_{1\gamma})$. By similar arguments as in the proof of Theorem 7, we can prove that ii) implies irreducibility of $(\bar{A}_\gamma, \bar{B}_{1\gamma})$. Theorem 4 then proves sufficiency of a). b) is immediate, since iii) is equivalent to generic full-rankness of (E, B), which is iii) of Theorem 4.

Q.E.D.

C. Algorithm to Verify Generic Controllability of SDS's

Theorems 8 and 9 can be implemented into a graph-theoretic algorithm to test for generic controllability of SDS's. Two types of tests are used throughout.

Test 1: examines existence of a complete matching in $H(A, B)$ and $H(E, B)$.

Test 2: examines input-reachability of $H^*(\bar{A}, \bar{B}_1)$.

These tests can be carried out efficiently using, e.g., the Hungarian method for Test 1, and a technique like dynamic programming for Test 2. Fig. 8 summarizes these tests into an algorithm for examining generic controllability of SDS's. Due to the gap remaining between the necessary condition and the sufficient condition, however, this algorithm is not yet complete. The problem is only partially solved if E is singular.

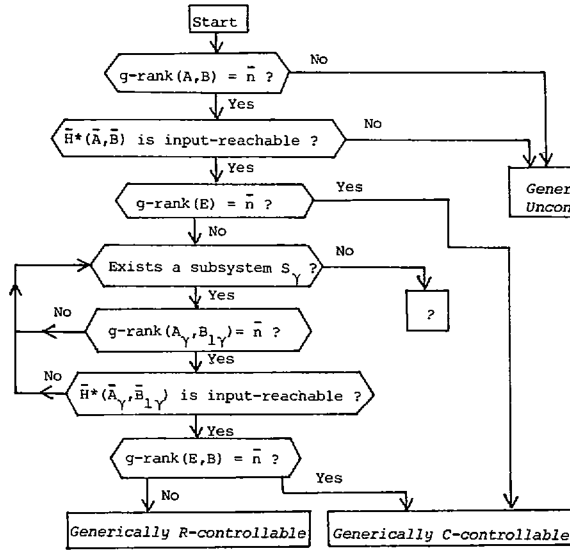


Fig. 8. Algorithms for generic controllability.

VII. Conclusion

Graph-theoretic algorithms have been developed for determining generic causality and controllability of SDS's. In the case of a nonsingular E, the problem has been completely solved, with the algorithm for determining generic controllability being based only on structural information. The problem was only partially solved for the case of singular E. Although not stated explicitly in this paper, the obtained results can be easily translated to the theorems and algorithms to determine generic observability of SDS's.

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Takeo Yamada (M'83), for a photograph and biography, see p. 152 of the February 1985 issue of this TRANSACTIONS.

David G. Luenberger (S'57-M'64-SM'71-F'75), for a photograph and biography, see p. 152 of the February 1985 issue of this TRANSACTIONS.