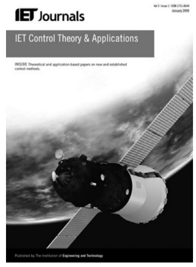
Published in IET Control Theory and Applications
Received on 10 December 2013
Accepted on 4 April 2014
doi: 10.1049/iet-cta.2014.0231



ISSN 1751-8644

Brief Paper

Exponential stability of impulsive positive systems with mixed time-varying delays

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Abstract: This study addresses the problem of exponential stability for a class of impulsive positive systems with mixed time-varying delays. A delayed impulsive positive system model is introduced for the first time and a necessary and sufficient condition guaranteeing the positivity of this kind of system is proposed. By using a copositive Lyapunov–Krasovskii functional and the average impulsive interval method, a sufficient criterion of global exponential stability for delayed impulsive positive systems is established in terms of linear programming problems. A numerical example is given to show the effectiveness of the proposed method.

1 Introduction

In many practical systems, there is a special kind of systems, namely, the positive systems, whose state variables and outputs are always positive (at least non-negative) whenever the initial states and the inputs are positive [1, 2]. Positive systems play a key role in many areas such as economics [3], biology [4], communication networks [5] and synchronisation/consensus problems [6]. Positivity of the system state for all times will bring about many new issues, which cannot be solved in general via the well-established methods for general systems. The main reason is that the states of positive systems are defined on cones rather than in the whole space. Therefore in recent years, positive systems have drawn considerable research interest in the control community and a large number of theoretical contributions to this field have appeared [7–15].

Time-delay, which is the inherent feature of many physical processes, is an important source of instability and poor performance even makes systems out of control. Therefore considerable attention has been devoted to the study of different issues related to time-delay systems. Recently, there are many researchers considering the problem related to time-delayed positive systems and some valuable results have been established [16–25]. To name a few, in [20], by using a multiple copositive type Lyapunov–Krasovskii functional, the stability problem for a class of switched positive systems in the presence of time delay was studied; in [23], the authors investigated the problem of exponential stability analysis and static output feedback stabilisation for discrete-time and continuous-time positive systems with bounded time-varying delays.

On the other hand, many dynamic systems in engineering, physics, biology, economics, chemistry and information science have impulsive dynamical behaviours because of abrupt jumps at certain instants during the dynamical processes. These complex dynamic behaviours can be modelled by impulsive systems [26, 27]. Impulsive dynamical systems can be viewed as a certain class of hybrid systems and consist of three elements; namely, continuous evolution, which is typically described by ordinary differential equations; impulse effects, which is described by a difference equation and also referred to as state jumps; and a rule for determining when the states of the system are to be jump. Impulsive systems have been investigated extensively over the last decade and plenty of results have been reported [28–35].

Note that all the positive systems are modelled without impulsive effects in previous literature. However, in practice, some dynamical positive system models are required to consider not only time delay but also impulsive effects because of the various jumping parameters and changing environmental factors at some instants. For example, in the dynamic portfolio management, the dynamical behaviour of stock value for a particular investor can be described by a delayed positive system. More specifically, we can see that when a certain amount of stock is purchased or sold at certain instants, the stock value changes instantaneously to a new value. This situation can be described by impulsive effects exactly. Another example of the integrated pest management, the population dynamics of a certain pest can also be described by a delayed positive system. We know that the number of pests will be reduced immediately when spraying pesticide at certain times, this situation can be viewed as impulsive control.

A delayed positive system can be called a positive system with time delay, if the delayed positive system is modelled with impulsive effect, which will increase complexity. Therefore, it is important and, in fact, necessary to study such systems. To the best our knowledge, no results about impulsive positive systems with time delay have been reported up to now. This has motivated our research.

In this paper, we will first introduce a delayed impulsive positive system model, and give a criterion (necessary and sufficient) to ensure positivity of this kind of system. A sufficient criterion of exponential stability analysis of impulsive positive systems with mixed time-varying delays is obtained by virtue of a copositive Lyapunov–Krasovskii functional and the average impulsive interval method. The theoretical result is proposed in the form of linear programming problems, which can easily be solved by using certain available software. Finally, a simulation is presented to demonstrate the effectiveness of the proposed approach.

The remainder of this paper is organised as follows. In Section 2, some preliminaries and definitions are given so as to formulate the main problems. The main results are presented in Section 3, where a necessary and sufficient condition is proposed for guaranteeing positivity of delayed impulsive positive systems, and a sufficient criterion of global exponential stability is established for impulsive positive systems with mixed time-varying delays. A numerical example is presented in Section 4 to illustrate the effectiveness of the proposed theoretical results. The conclusions are finally drawn in Section 5.

Notations: The notations used in this paper are fairly standard. Let $\Re$ denotes the set of real numbers, $\Re_+$ is the set of non-negative real numbers. Let N denotes the set of positive integers, that is, $N = \{1, 2, \ldots\}$. $\Re^n$ and $\Re^{n\times n}$ denote the n-dimensional Euclidean space and the space of $n \times n$-dimensional matrices with real entries; $\Re^n_+$ stands for the non-negative orthant in $\Re^n$. The superscript 'T' stands for matrix transposition, the notation $\|\bullet\|$ refers to the vector norm. For a matrix $A \in \Re^{n\times n}$, $A \succ 0$ ($\succeq 0$) means that all elements of the matrix A are positive (non-negative). For a vector $v \in \Re^n$, $v \succ 0$ ($\succeq 0, \prec 0, \preceq 0$) means that all elements of the vector v are positive (non-negative, negative and non-positive), $\max\{v\}$ and $\min\{v\}$ denote the maximum and the minimum element of the vector, respectively. $\mathbf{M}$ denotes the set of Metzler matrices whose off-diagonal entries are non-negative. I denotes an identity matrix with appropriate dimension. For $d > 0$, let $PC([-d, 0], \Re^n)$ denotes the set of piecewise right continuous function with the norm defined by $\|\varphi\|_d = \sup_{-d\leqslant\theta\leqslant 0}\|\varphi(\theta)\|$ where $\varphi \in PC([-d, 0], \Re^n)$. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2 Problem formulation and preliminaries

Consider the following impulsive system with mixed time-varying delays (see (1))

$$\begin{cases} \dot{x}(t) = Ax(t) + A_1x(t-\tau(t)) + A_2\int_{t-h(t)}^{t} x(s)\,\mathrm{d}s, & t \neq t_k,\ t \geqslant t_0 \\ \Delta x = Bx(t), & t = t_k,\ k \in N \\ x(t_0+\theta) = \varphi(\theta) \succeq 0, & t_0 = 0,\ \theta \in [-d, 0] \end{cases} \tag{1}$$

where $x(t) \in \Re^n$ is the state variable, A, A_1, A_2 and $B \in \Re^{n\times n}$, where $B \neq 0$, $\Delta x = x(t_k^+) - x(t_k^-)$, where $\lim_{h\to 0^+} x(t_k + h) = x(t_k^+)$, $\lim_{h\to 0^-} x(t_k + h) = x(t_k^-)$, $\tau(t)$ denotes the time-varying discrete delay which is everywhere time-differentiable and satisfies $0 < \tau(t) \leqslant \tau$ and $\dot{\tau}(t) \leqslant d_\tau$, where τ and d_τ are known constants, $h(t)$ denotes the time-varying distributed delay which satisfies $0 < h(t) \leqslant h$ for known constant h, $\varphi(\bullet) \in PC([-d, 0], \Re^n)$ is a vector-valued initial continuous function, where $d = \max\{\tau, h\}$.

Consider a discrete time set $\{t_k\}_{k=1}^{\infty}$ of impulsive jump instants which satisfy $0 < t_1 < t_2 < \cdots < t_{k-1} < t_k < \cdots$ and $\lim_{k\to\infty} t_k \to \infty$, where $t_1 > t_0$. Without loss of generality, we assume $x(t_k^+) = x(t_k)$, which implies that the solution of system (1) is right-continuous at t_k.

We always assume that (1) has a unique solution with respect to the initial condition. For simplicity, we denote $x(t, t_0, \varphi)$ by $x(t)$ for any $\varphi(\bullet) \in PC([-d, 0], \Re^n)$ in this paper.

Remark 1: Compared with the general delayed impulsive system, the delayed impulsive positive system (1) must possess positivity, that is, $x(t) \in \Re^n_+$ for all $t \geqslant t_0$, which will be more challenging for stability analysis of such system.

Before proceeding, we need to introduce some definitions to develop our theories and results in what follows.

Definition 1: The system (1) is said to be an impulsive positive system with time delay if for any initial condition $\varphi(\bullet) \succeq 0$, the corresponding trajectory $x(t) \succeq 0$ holds for all $t \geqslant t_0$.

Remark 2: Definition 1 follows the general positivity definition of a positive system, which means that the states and outputs are non-negative whenever the initial conditions and inputs are non-negative [1, 2].

Definition 2 [36]: The average impulsive interval of the impulsive sequence $\sigma = \{t_1, t_2, \ldots\}$ is equal to T_a if there exist a positive integer N_0 (N_0 is called the chatter bound), and a positive number T_a, such that

$$\frac{t-t_0}{T_a} - N_0 \leqslant N_\sigma(t, t_0) \leqslant \frac{t-t_0}{T_a} + N_0, \quad \forall t \geqslant t_0 \geqslant 0 \tag{2}$$

where $N_\sigma(t, t_0)$ denotes the number of impulsive times of the impulsive sequence σ on the interval $[t_0, t)$.

Definition 3: The system (1) is said to be globally exponentially stable if there exist some constants $\zeta > 0$ and $\lambda > 0$ such that for any initial condition $\varphi(\bullet)$

$$\|x(t)\| \leqslant \zeta e^{-\lambda(t-t_0)}\|\varphi\|_d, \quad t \geqslant t_0$$

Definition 4 [33]: The function $V : \Re_+ \times \Re^n \to \Re_+$ belongs to class ν_0 if

(1) the function V is continuous in each of the sets $[t_{k-1}, t_k) \times \Re^n$ and for each $x, y \in \Re^n$, $t \in [t_{k-1}, t_k)$, $k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists.
(2) $V(t, x(t))$ is locally Lipschitzian in all $x \in \Re^n$, and for all $t \geqslant t_0$, $V(t, 0) \equiv 0$.

doi: 10.1049/iet-cta.2014.0231

Definition 5: Given a function $V \in \nu_0$, the upper right-hand derivative of V along the solution $x(t)$ of system (1) is defined by

$$D^+V(t,x(t)) = \lim_{\varepsilon\to 0^+} \sup \frac{1}{\varepsilon}[V(t+\varepsilon, x(t+\varepsilon)) - V(t,x(t))]$$

We shall make the following remark.

Remark 3: Note that in the above Definition 3, the vector norm $\|x\|$ can be any type of the norm of x. However, to make the subsequent proof easy to understand, we use the following norm

$$\|x\| = \sum_{i=1}^{n} |x_i|$$

where x_i is the ith element of $x \in \Re^n$.

We end this section by introducing a lemma which will be used in the subsequent proof.

Lemma 1 [37]: Let $A \in \mathbf{M}$, then the following holds true:

(1) $A \in \mathbf{M} \Leftrightarrow e^{At} \succeq 0$ for $t \geqslant 0$;
(2) if a vector $v \succeq 0$, then $e^{At}v \succeq 0$ for $t \geqslant 0$.

3 Main results

In this section, a necessary and sufficient condition of the positivity with respect to the system (1) is first given, and then, a sufficient criterion of global exponential stability for the system (1) is proposed.

Proposition 1: The system (1) is an impulsive positive system with mixed time-varying delays if and only if $A \in \mathbf{M}$, $A_1 \succeq 0$, $A_2 \succeq 0$ and $(I+B) \succeq 0$.

Proof (sufficiency): By system (1), any initial condition $\varphi(\bullet) \succeq 0$, for $t \in [t_0, t_1)$, we have

$$\begin{aligned} &e^{-At}[\dot{x}(t) - Ax(t)] \\ &\quad = e^{-At}\left[A_1 x(t-\tau(t)) + A_2 \int_{t-h(t)}^{t} x(\varsigma)\,\mathrm{d}\varsigma\right] \end{aligned}$$

By integrating the above equality, it holds that

$$\begin{aligned} e^{-At}x(t) = \int_{t_0}^{t} e^{-As}\Bigg[&A_1 x(s-\tau(s)) \\ &+ A_2 \int_{s-h(s)}^{s} x(\varsigma)\,\mathrm{d}\varsigma\Bigg]\mathrm{d}s + x(t_0) \end{aligned} \tag{3}$$

It then follows from (3) that (see (4))

$$\begin{aligned} x(t) &= e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-s)}\left[A_1 x(s-\tau(s)) + A_2 \int_{s-h(s)}^{t} x(\varsigma)\,\mathrm{d}\varsigma\right]\mathrm{d}s \\ &= e^{At}x(t_0) + \int_{t_0}^{t} e^{A(t-s)} A_1 x(s-\tau(s))\,\mathrm{d}s + \int_{t_0}^{t} e^{A(t-s)} A_2 \int_{s-h(s)}^{s} x(\varsigma)\,\mathrm{d}\varsigma\,\mathrm{d}s \end{aligned} \tag{4}$$

If $A \in \mathbf{M}$, $A_1 \succeq 0$ and $A_2 \succeq 0$, by Lemma 1, it is straightforward to obtain $x(t) \succeq 0$ for all $t \in [t_0, t_1)$.

When $t = t_1$, $x(t_1) = (I+B)x(t_1^-)$, since $x(t_1^-) \succeq 0$ from (4), if $(I+B) \succeq 0$, we have $x(t_1) \succeq 0$.

Now, similar to the above processes, one has that, for $t \in [t_1, t_2)$

$$\begin{aligned} x(t) = e^{At}x(t_1) &+ \int_{t_1}^{t} e^{A(t-s)} A_1 x(s-\tau(s))\,\mathrm{d}s \\ &+ \int_{t_1}^{t} e^{A(t-s)} A_2 \int_{s-h(s)}^{s} x(\varsigma)\,\mathrm{d}\varsigma\,\mathrm{d}s \succeq 0 \end{aligned}$$

and when $t = t_2$, $x(t_2) = (I+B)x(t_2^-) \succeq 0$.

By repeating the same procedure, we can easily obtain that $x(t) \succeq 0$ for all $t \geqslant t_0$. By Definition 1, the system (1) is an impulsive positive system with mixed time-varying delays.

(Necessity): Since the system (1) is an impulsive positive system with mixed time-varying delays for any initial condition $\varphi(\bullet) \succeq 0$, we know that

$$x(t) \succeq 0, \text{ for all } t \geqslant t_0 \tag{5}$$

We first suppose that there exists an element $a_{ij} < 0$, $1 \leqslant i,j \leqslant n$, $i \neq j$, where a_{ij} is the (i,j)th component of A.

When $t \neq t_k$, let $x(t) = [x_1(t)x_2(t)\ldots x_i(t)\ldots x_n(t)]^{\mathrm{T}}$, we can obtain

$$\begin{aligned} \dot{x}_i(t) = &\sum_{l=1, l\neq i, l\neq j}^{n} a_{il}x_l(t) + a_{ii}x_i(t) + a_{ij}x_j(t) \\ &+ \sum_{l=1}^{n} a_{1il}x_l(t-\tau(t)) + \sum_{l=1}^{n} a_{2il}\int_{t-h(t)}^{t} x(\varsigma)\,\mathrm{d}\varsigma \end{aligned}$$

where a_{1il} is the (i,l)th component of A_1 and a_{2il} is the (i,l)th component of A_2.

Then, if $x_j(t) \neq 0$, we can conclude that $\dot{x}_i(t) < 0$ is possible whenever $x_i(t) = 0$, which implies that $x_i(t^+) < 0$. This is contradictory with (5). Consequently, there must be $a_{ij} \geqslant 0$, $1 \leqslant i,j \leqslant n$, $i \neq j$, that is, $A \in \mathbf{M}$.

Next, we assume that A_1 has an element $a_{1ij} < 0$, $1 \leqslant i,j \leqslant n$, we have

$$\begin{aligned} \dot{x}_i(t) = &\sum_{l=1, l\neq i}^{n} a_{il}x_l(t) + a_{ii}x_i(t) + \sum_{l=1, l\neq j}^{n} a_{1il}x_l(t-\tau(t)) \\ &+ a_{1ij}x_j(t-\tau(t)) + \sum_{l=1}^{n} a_{2il}\int_{t-h(t)}^{t} x(\varsigma)\,\mathrm{d}\varsigma \end{aligned}$$

Then, if $x_j(t-\tau(t)) \neq 0$, we can conclude that $\dot{x}_i(t) < 0$ is possible whenever $x_i(t) = 0$, which implies that $x_i(t^+) < 0$. This is contradictory with (5). Therefore, there must be $a_{1ij} \geqslant 0$, $1 \leqslant i,\ j \leqslant n$, that is, $A_1 \succeq 0$.

Similarly, if A_2 has an element $a_{2ij} < 0$, one can also obtain a contradiction. Therefore, there must be $A_2 \succeq 0$.

When $t = t_1$, $x(t_1) = (I+B)x(t_1^-)$.

Suppose $P = I + B$ dissatisfies $P \succeq 0$, without loss of generality, we assume that P has an element $p_{11} < 0$, where

p_{11} is the $(1,1)$th component of P. By choosing the system matrix and an initial condition, we can obtain that $x(t_1^-) = [\mu\ \ 0\cdots 0]^{\mathrm{T}} \succeq 0$, where $\mu > 0$, thus, one has the $(1,1)$th component of $x(t_1)$ be negative, that is, $x(t_1)$ dissatisfies $x(t_1) \succeq 0$. This is contradictory with (5). Therefore, there must be $(I+B) \succeq 0$.

This completes the whole proof. □

In what follows, a sufficient criterion of global exponential stability for the system (1) is presented.

Theorem 1: Consider the system (1) with $A \in \mathbf{M}$, $A_1 \succeq 0$, $A_2 \succeq 0$ and $(I+B) \succeq 0$, let $\alpha \neq 0$, $\beta_1 > 0$, $\beta_2 > 0$ and $\beta_3 > 0$ be given constants, let $\eta_1 = \min\{e^{-\alpha\tau}, 1\}$, $\eta_2 = \max\{e^{-\alpha\tau}, 1\}$, $\gamma_1 = \min\{e^{-\alpha d}, 1\}$, $\gamma_2 = \max\{e^{-\alpha d}, 1\}$, if there exist vectors $u \succ 0$, $v \succ 0$, $w \succ 0$ and ξ such that the following inequalities hold

$$A^{\mathrm{T}}u + \alpha u + v + 2\,\mathrm{d}w + \xi \preceq 0 \tag{6}$$

$$A_1{}^{\mathrm{T}}u + (d_\tau\eta_2 - \eta_1)v - \xi \preceq 0 \tag{7}$$

$$A_2{}^{\mathrm{T}}u - \gamma_1 w \preceq 0 \tag{8}$$

$$-A^{\mathrm{T}}\xi - \gamma_1 w \preceq 0 \tag{9}$$

$$-A_1{}^{\mathrm{T}}\xi \preceq 0 \tag{10}$$

$$-A_2{}^{\mathrm{T}}\xi \preceq 0 \tag{11}$$

$$[(I+B)^{\mathrm{T}} - \beta_1 I]u \preceq 0 \tag{12}$$

$$[(I+B)^{\mathrm{T}} - \beta_2 I]v \preceq 0 \tag{13}$$

$$[(I+B)^{\mathrm{T}} - \beta_3 I]w \preceq 0 \tag{14}$$

and an average impulsive interval of the impulsive sequence $\sigma = \{t_1, t_2, \ldots\}$ is equal to T_a, then, if the following condition is satisfied

$$\lambda \triangleq \alpha - \frac{\ln\beta_4}{T_a} > 0 \tag{15}$$

where $\beta_4 = \max\{\beta_1, \beta_2, \beta_3\}$, the delayed impulsive positive system (1) is globally exponentially stable with convergence rate λ.

Remark 4: In Theorem 1, the sign of α is no restricted since the impulse is either harmful or beneficial for global exponential stability of the system (1). When selecting $0 < \beta_4 < 1$, we can choose $\alpha > 0$ or $\alpha < 0$ to guarantee the inequalities (6)–(14) hold; when selecting $\beta_4 \geqslant 1$, we must only choose $\alpha > 0$ to ensure the inequalities (6)–(14) satisfy.

Proof: From Definition 1, we have $x^{\mathrm{T}}(t) \succeq 0$, and construct a copositive Lyapunov–Krasovskii functional $V \in \nu_0$ in the form of

$$V(t,x(t)) = V_1(t,x(t)) + V_2(t,x(t)) + V_3(t,x(t)) \tag{16}$$

where

$$V_1(t,x(t)) = x^{\mathrm{T}}(t)u$$

$$V_2(t,x(t)) = \int_{t-\tau(t)}^{t} e^{\alpha(-t+s)}x^{\mathrm{T}}(s)v\,\mathrm{d}s$$

$$V_3(t,x(t)) = 2\int_{-d}^{0}\int_{t+\theta}^{t} e^{\alpha(-t+s)}x^{\mathrm{T}}(s)w\,\mathrm{d}s\,\mathrm{d}\theta$$

and $u \succ 0$, $v \succ 0$, $w \succ 0$, are to be determined, $\alpha \neq 0$ is a given constant.

For simplicity, $V(t,x(t))$ is written as $V(t)$ in subsequent proof.

When $t \in [t_{k-1}, t_k)$ $(k \in N)$, by calculating the upper right-hand derivative of $V(t)$ along the solution of (1), we can obtain

$$\begin{aligned}
\dot V_1(t) &= \dot x^{\mathrm{T}}(t)u = x^{\mathrm{T}}(t)A^{\mathrm{T}}u + x^{\mathrm{T}}(t-\tau(t))A_1{}^{\mathrm{T}}u \\
&\quad + \left[\int_{t-h(t)}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\right]A_2{}^{\mathrm{T}}u \\
\dot V_2(t) &= -\alpha V_2 + x^{\mathrm{T}}(t)v - (1-\dot\tau(t))e^{-\alpha\tau(t)}x^{\mathrm{T}}(t-\tau(t))v \\
&\leqslant -\alpha V_2 + x^{\mathrm{T}}(t)v + d_\tau\eta_2 x^{\mathrm{T}}(t-\tau(t))v \\
&\quad - \eta_1 x^{\mathrm{T}}(t-\tau(t))v \\
&= -\alpha V_2 + x^{\mathrm{T}}(t)v + (d_\tau\eta_2 - \eta_1)x^{\mathrm{T}}(t-\tau(t))v \\
\dot V_3(t) &= -\alpha V_3(t) + 2dx^{\mathrm{T}}(t)w - 2\int_{-d}^{0} e^{\alpha\theta}x^{\mathrm{T}}(t+\theta)w\,\mathrm{d}\theta \\
&\leqslant -\alpha V_3(t) + 2dx^{\mathrm{T}}(t)w - 2\gamma_1\int_{-d}^{0} x^{\mathrm{T}}(t+\theta)w\,\mathrm{d}\theta \\
&\leqslant -\alpha V_3(t) + 2dx^{\mathrm{T}}(t)w - \gamma_1\int_{t-h(t)}^{t} x^{\mathrm{T}}(s)w\,\mathrm{d}s \\
&\quad - \gamma_1\int_{t-\tau(t)}^{t} x^{\mathrm{T}}(s)w\,\mathrm{d}s
\end{aligned}$$

Using Leibniz–Newton formula, one can obtain

$$\int_{t-\tau(t)}^{t} \dot x(s)\,\mathrm{d}s = x(t) - x(t-\tau(t)) \tag{17}$$

From the system (1), we obtain

$$\int_{t-\tau(t)}^{t} \dot x(s)\,\mathrm{d}s = \int_{t-\tau(t)}^{t}\left[Ax(s) + A_1x(s-\tau(s)) + A_2\int_{s-h(s)}^{s} x(\theta)\,\mathrm{d}\theta\right]\mathrm{d}s \tag{18}$$

Moreover, we have the following equality for any vector ξ

$$\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t}\left(Ax(s) + A_1x(s-\tau(s)) + A_2\int_{s-h(s)}^{s} x(\theta)\,\mathrm{d}\theta\right)\mathrm{d}s\right]^{\mathrm{T}}\xi = 0 \tag{19}$$

From (19) and the obtained derivative terms above, we can obtain (see equation at the bottom of the page)

$$\begin{aligned}
\dot V(t) + \alpha V(t) &\leqslant x^{\mathrm{T}}(t)(A^{\mathrm{T}}u + v + \alpha u + 2\,\mathrm{d}w + \xi) + x^{\mathrm{T}}(t-\tau(t))(A_1{}^{\mathrm{T}}u + d_\tau\eta_2 v - \eta_1 v - \xi) + \left[\int_{t-h(t)}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\right](A_2{}^{\mathrm{T}}u - \gamma_1 w) \\
&\quad - \int_{t-\tau(t)}^{t}\left[x^{\mathrm{T}}(s)(A^{\mathrm{T}}\xi + \gamma_1 w) + x^{\mathrm{T}}(s-\tau(s))A_1{}^{\mathrm{T}}\xi + \left(\int_{s-h(s)}^{s} x^{\mathrm{T}}(\theta)\,\mathrm{d}\theta\right)A_2{}^{\mathrm{T}}\xi\right]\mathrm{d}s
\end{aligned}$$

IET Control Theory Appl., 2014, Vol. 8, Iss. 15, pp. 1537–1542
doi: 10.1049/iet-cta.2014.0231

From (6) to (11), one has $\dot{V}(t) + \alpha V(t) \leqslant 0$, further, we can derive

$$V(t) \leqslant V(t_{k-1})e^{-\alpha(t-t_{k-1})}, \quad t \in [t_{k-1}, t_k) \tag{20}$$

When $t = t_k$ $(k \in N)$, from (12) to (14), we can obtain

$$\begin{aligned} V(t_k) &= x^{\mathrm{T}}(t_k^-)(I+B)^{\mathrm{T}}u \\ &\quad + \int_{t_k^- - \tau(t_k^-)}^{t_k^-} e^{\alpha(-t_k^- + s)} x^{\mathrm{T}}(s)(I+B)^{\mathrm{T}} v \,\mathrm{d}s \\ &\quad + 2\int_{-d}^{0} \int_{t_k^- + \theta}^{t_k^-} e^{\alpha(-t_k^- + s)} x^{\mathrm{T}}(s)(I+B)^{\mathrm{T}} w \,\mathrm{d}s\,\mathrm{d}\theta \\ &\leqslant \beta_1 x^{\mathrm{T}}(t_k^-)u + \beta_2 \int_{t_k^- - \tau(t_k^-)}^{t_k^-} e^{\alpha(-t_k^- + s)} x^{\mathrm{T}}(s) v \,\mathrm{d}s \\ &\quad + 2\beta_3 \int_{-d}^{0} \int_{t_k^- + \theta}^{t_k^-} e^{\alpha(-t_k^- + s)} x^{\mathrm{T}}(s) w \,\mathrm{d}s\,\mathrm{d}\theta \\ &\leqslant \beta_4 V(t_k^-) \end{aligned} \tag{21}$$

Combining (20) and (21), for $t \in [t_0, t_1)$, $V(t) \leqslant e^{-\alpha(t-t_0)} V(t_0)$, which leads to

$$V(t_1^-) \leqslant e^{-\alpha(t_1-t_0)} V(t_0)$$

and

$$V(t_1) \leqslant \beta_4 e^{-\alpha(t_1-t_0)} V(t_0)$$

Similarly, for $t \in [t_1, t_2)$, one has

$$V(t) \leqslant e^{-\alpha(t-t_1)} V(t_1) \leqslant \beta_4 e^{-\alpha(t-t_0)} V(t_0)$$

In general, for $t \in [t_k, t_{k+1})$ $(k \in N)$, we have

$$V(t) \leqslant {\beta_4}^k e^{-\alpha(t-t_0)} V(t_0)$$

Considering the definition of $V(t)$, and letting $\rho_1 = \min\{u\}$, $\rho_2 = \max\{u\}$, $\rho_3 = \max\{v\}$ and $\rho_4 = \max\{w\}$, we can obtain the following two equalities

$$V(t) \geqslant \rho_1 \|x(t)\| \tag{22}$$

$$\begin{aligned} V(t_0) &\leqslant \rho_2 \|x(t_0)\| + (\eta_2\rho_3\tau + 2\gamma_2\rho_4 h^2)\|\varphi\|_d \\ &\leqslant (\rho_2 + \eta_2\rho_3\tau + 2\gamma_2\rho_4 h^2)\|\varphi\|_d \end{aligned} \tag{23}$$

According to (22) and (23), we can conclude

$$\|x(t)\| \leqslant c{\beta_4}^k e^{-\alpha(t-t_0)} \|\varphi\|_d \tag{24}$$

where $c = (\rho_2 + \eta_2\rho_3\tau + 2\gamma_2\rho_4 h^2)/\rho_1 > 0$.

Let $N_\sigma(t, t_0)$ be the number of impulsive times of the impulsive sequence σ on the interval $[t_0, t)$. From (24), we have

$$\|x(t)\| \leqslant c{\beta_4}^{N_\sigma(t,t_0)} e^{-\alpha(t-t_0)} \|\varphi\|_d \tag{25}$$

When $0 < \beta_4 < 1$, it follows from (2), (15) and (25) that

$$\begin{aligned} \|x(t)\| &\leqslant c{\beta_4}^{\left(\frac{t-t_0}{T_a} - N_0\right)} e^{-\alpha(t-t_0)} \|\varphi\|_d \\ &= c{\beta_4}^{-N_0} e^{\frac{\ln\beta_4}{T_a}(t-t_0)} e^{-\alpha(t-t_0)} \|\varphi\|_d \\ &= c{\beta_4}^{-N_0} e^{\left(\frac{\ln\beta_4}{T_a} - \alpha\right)(t-t_0)} \|\varphi\|_d \\ &= c{\beta_4}^{-N_0} e^{-\lambda(t-t_0)} \|\varphi\|_d \end{aligned} \tag{26}$$

Similarly, when $\beta_4 \geqslant 1$, it follows from (2) and (25) that

$$\|x(t)\| \leqslant c{\beta_4}^{N_0} e^{-\lambda(t-t_0)} \|\varphi\|_d \tag{27}$$

Summarising (26) and (27) gives that there exists constant $\zeta = \max\{c{\beta_4}^{-N_0},\ c{\beta_4}^{N_0}\}$ such that

$$\|x(t)\| \leqslant \zeta e^{-\lambda(t-t_0)} \|\varphi\|_d$$

Since $\zeta > 0$ and $\lambda > 0$, according to Definition 3, the delayed impulsive positive system (1) is globally exponentially stable with convergence rate λ.

This has completed the proof of Theorem 1. □

4 Numerical example

In this section, a numerical example will be presented to demonstrate the applicability and validity of our theoretical results.

Consider the system (1) described by

$$A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.9 & 0.1 \\ 0.1 & -0.9 \end{bmatrix}$$

Let $\tau(t) = 0.2 + 0.1\sin t$, $h(t) = 0.3e^{-t}$ and the average impulsive interval be $T_a = 1$, and give the impulsive sequence in Fig. 1.

It is obvious that $A \in \mathbf{M}$, $A_1 \succeq 0$, $A_2 \succeq 0$ and $(I+B) \succeq 0$.

Letting $\alpha = -1$, $\beta_1 = 0.25$, $\beta_2 = 0.25$ and $\beta_3 = 0.35$. Solving inequalities (6)–(14) gives

$$u = [0.9543 \quad 1.0832]^{\mathrm{T}}, \quad v = [0.1551 \quad 0.1572]^{\mathrm{T}}$$
$$w = [0.3633 \quad 0.3334]^{\mathrm{T}}, \quad \xi = [0.2402 \quad 0.2378]^{\mathrm{T}}$$

This implies that the inequalities (6)–(14) hold.

Furthermore, $\lambda \triangleq \alpha - [\ln\beta_4/T_a] = 0.0498 > 0$.

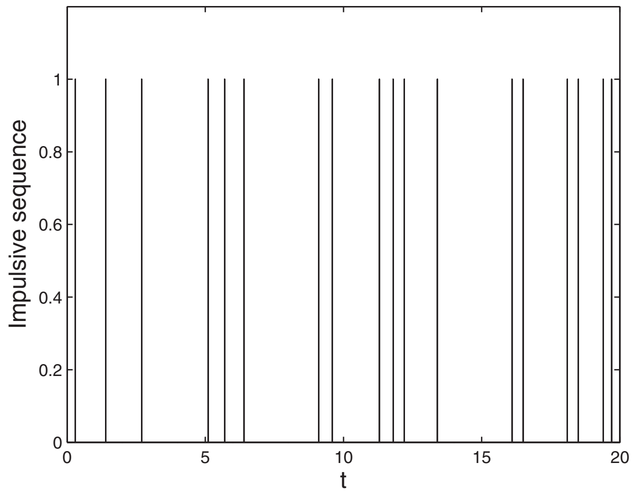

Fig. 1 *Impulsive sequence*

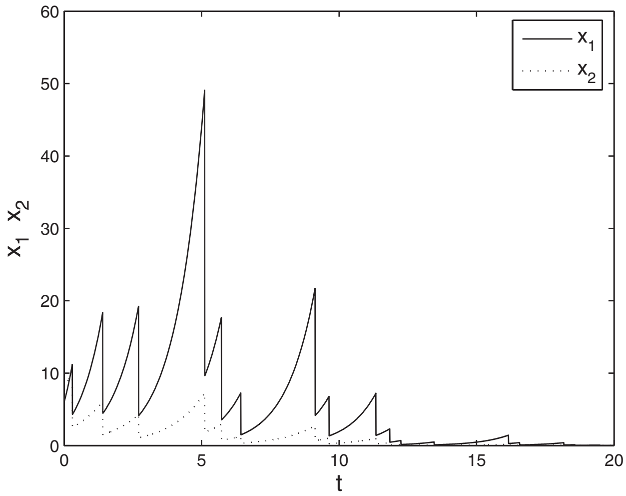

Fig. 2 *Evolution of the states $x(t)$ under $T_a = 1$*

Based on Theorem 1, we can concluded that the system is globally exponentially stable with convergence rate $\lambda = 0.0498$.

Evolution of the states $x(t)$ of the system under $T_a = 1$ is shown in Fig. 2.

From Fig. 2, it is shown that the system is globally exponentially stable under $T_a = 1$.

5 Conclusions

In this paper, a delayed impulsive positive system model has been put forward for the first time. A sufficient criterion for global exponential stability of impulsive positive systems with mixed time-varying delays has been derived on the basis of the average impulsive interval method and a copositive Lyapunov–Krasovskii functional. A numerical example has been presented to illustrate the theoretical results. In our future work, the stability problem of impulsive positive systems with multiple time-delays will be considered.

6 Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grants nos. 61374171 and 61074124, and the Program for Changjiang Scholars and Innovative Research Team in University [IRT1245].

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doi: 10.1049/iet-cta.2014.0231