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Massera-type theorem for the existence of $C^{(n)}$-almost-periodic solutions for partial functional differential equations with infinite delay

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Abstract

In this paper, we study the existence of $C^{(n)}$-almost-periodic solutions for partial functional differential equations with infinite delay. We assume that the undelayed part is not necessarily densely defined and satisfies the Hille–Yosida condition. We use the reduction principle developed recently in [M. Adimy, K. Ezzinbi, A. Ouhinou, Variation of constants formula and almost-periodic solutions for some partial functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications 317 (2006) 668–689] to prove the existence of a $C^{(n)}$-almost- periodic solution when there is at least one bounded solution in $\mathbb{R}^+$. We give an application to the Lotka–Volterra model with diffusion.

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1. Introduction

The aim of this work is to study the existence of $C^{(n)}$-almost-periodic solutions for the following partial functional differential equation with infinite delay

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where $A : D(A) \to X$ is a not necessarily densely defined linear operator on a Banach space X, for every $t \in \mathbb{R}$, the history function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t+\theta) \quad \text{for } \theta \in (-\infty, 0],$$

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where $\mathcal{B}$ is a normed linear space of functions mapping $(-\infty, 0]$ to X and satisfying some fundamental axioms given by Hale and Kato in [7]. L is a bounded linear operator from $\mathcal{B}$ to X and f is an almost-periodic X-valued function on $\mathbb{R}$. They assume that the undelayed part A satisfies the Hille–Yosida condition

(H_0) there exist $M_0 \geq 1$, $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$|(\lambda I - A)^{-n}| \leq \frac{M_0}{(\lambda - \omega_0)^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega_0,$$

where $\rho(A)$ is the resolvent set of A. Recall in [10], Massera proposed a new approach to show the existence of periodic solutions of ordinary differential equations in finite dimensional spaces. For some kind of ordinary differential equations, the author proved the existence of periodic solutions under a minimal condition, namely, the existence of a bounded solution on $\mathbb{R}^+$ is enough to get periodic solutions. Many authors used Massera's approach to prove the existence of periodic, almost periodic or $C^{(n)}$-almost-periodic solutions in the context of differential equations. In [2], the authors proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost-periodic solution. Firstly, they established a new variation of constants formula for Eq. (1.1). Secondly, they used the spectral decomposition of the phase space to get a (new) reduction principle of Eq. (1.1) to a finite dimensional space when $\mathcal{B}$ is a uniform fading memory space. More precisely, they established a relationship between the bounded solutions of Eq. (1.1) on $\mathbb{R}^+$ with the bounded solutions on $\mathbb{R}$ for an ordinary differential equation in a finite dimensional space. In this work, we propose to use the reduction principle established in [2] and the Massera's approach to show the existence of $C^{(n)}$-almost-periodic solutions of Eq. (1.1).

$C^{(n)}$-almost-periodic functions are functions such that the ith derivative are almost periodic for $i = 1, \ldots, n$; they have many applications in dynamical systems. For more details, we refer to [1] and references therein. In [4], the authors discussed some properties of $C^{(n)}$-almost-periodic functions taking values in Banach spaces.

In [3], the authors proved the existence of a $C^{(n)}$-almost-periodic solution for the following nonautonomous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = \mathcal{A}(t)x(t) + \theta(t) \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where $\mathcal{A}(t)$ generates an exponentially stable family in a Banach space, they showed that when θ is $C^{(n)}$-almost periodic, then the only bounded solution of Eq. (1.2) is also C^n-almost periodic. In [9], the authors proved the existence of $C^{(n)}$-almost-periodic solutions for some ordinary differential equations by using the exponential dichotomy approach.

In this work, we first discuss the existence of $C^{(n)}$-almost- periodic solution for the following ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Gx(t) + e(t) \quad \text{for } t \in \mathbb{R} \tag{1.3}$$

where G is a constant $n \times n$-matrix and $e : \mathbb{R} \to \mathbb{R}^n$ is $C^{(n)}$-almost periodic. We then prove the Massera-type theorem for the existence of $C^{(n)}$-almost-periodic solution, more precisely, we prove that the existence of a bounded solution in $\mathbb{R}^+$ implies the existence of a $C^{(n)}$-almost-periodic solution. Moreover, we show that every bounded solution in $\mathbb{R}$ is $C^{(n+1)}$-almost periodic. We use the reduction principle developed recently in [2] to obtain a Massera-type theorem for the existence of $C^{(n)}$-almost-periodic solution of Eq. (1.1).

This work is organized as follows, in Section 2, we recall some results on the existence of solutions of Eq. (1.1) and we give the variation of constants formula that will be used in this work. In Section 3, we give the reduction principle of Eq. (1.1) to a finite dimensional ordinary differential equation. Section 4 is devoted to state some results on $C^{(n)}$-almost-periodic functions. In Section 5, we use the reduction principle to show that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an $C^{(n)}$-almost-periodic solution of Eq. (1.1). In Section 6, we prove the existence and uniqueness of an $C^{(n)}$-almost-periodic solution of Eq. (1.1) where the solution semigroup of Eq. (1.1) with $f = 0$ has an exponential dichotomy. Finally, for illustration, we propose to study the existence of a $C^{(n)}$-almost-periodic solution for a Lotka–Volterra model with diffusion.

2. Integral solutions and variation of constants formula

We use the (classical) axiomatic approach of Hale and Kato [7] for the phase space $\mathcal{B}$. We assume that $(\mathcal{B}, \|\cdot\|)$ is a normed space of functions mapping $(-\infty, 0]$ into a Banach space X and satisfying the following fundamental axioms:

(A) there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,

(ii) $t \to x_t$ is continuous with respect to $\|\cdot\|$ on $[\sigma, a]$,

(iii) $N\,|x(t)| \le \|x_t\| \le K(t-\sigma) \sup\limits_{\sigma \le s \le t} |x(s)| + M(t-\sigma)\,\|x_\sigma\|$.

(B) $\mathcal{B}$ is a Banach space.

The following lemma is well known.

Lemma 2.1 ([2, p. 140]). *Assume that* (H_0) *holds. Let A_0 be the part of the operator A in $D(A)$, which is defined by*

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax. \end{cases}$$

Then A_0 generates a C_0-semigroup $(T_0(t))_{t\ge 0}$ on $\overline{D(A)}$.

To Eq. (1.1), we associate the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \ge \sigma, \\ x_\sigma = \phi \in \mathcal{B}. \end{cases} \tag{2.1}$$

The following results are taken from [2].

Definition 2.2 ([2]). Let $\phi \in \mathcal{B}$. A function $u : \mathbb{R} \to X$ is called an integral solution of Eq. (2.1) on $\mathbb{R}$ if the following conditions hold:

(i) u is continuous on $[\sigma, \infty)$,

(ii) $u_\sigma = \phi$,

(iii) $\int_\sigma^t u(s)\mathrm{d}s \in D(A)$ for $t \ge \sigma$,

(iv) $u(t) = \phi(0) + A\int_\sigma^t u(s)\mathrm{d}s + \int_\sigma^t L(u_s)\mathrm{d}s + \int_\sigma^t f(s)\mathrm{d}s$ for $t \ge \sigma$.

For simplicity, integral solutions will be called solutions in this work.

Theorem 2.3 ([2]). *Assume that* (H_0), (A) *and* (B) *hold. Then for all $\phi \in \mathcal{B}$ such that $\phi(0) \in \overline{D(A)}$, Eq.* (2.1) *has a unique solution $u = u(\cdot, \phi, L, f)$ on $\mathbb{R}$ which is given by*

$$u(t) = \begin{cases} T_0(t-\sigma)\phi(0) + \lim\limits_{\lambda \to +\infty} \displaystyle\int_\sigma^t T_0(t-s)\lambda R(\lambda, A)\,[L(u_s) + f(s)]\,\mathrm{d}s, & \text{for } t \ge \sigma, \\ \phi(t) \quad \text{for } t \le \sigma. \end{cases}$$

Let $\mathcal{B}_A = \{\phi \in \mathcal{B} : \phi(0) \in \overline{D(A)}\}$ be the phase space corresponding to Eq. (2.1). For $t \ge 0$, we define the operator $U(t)$ for $\phi \in \mathcal{B}_A$, by

$$U(t)\phi = u_t(\cdot, \phi, L, 0),$$

where $u(\cdot, \phi, L, 0)$ is the solution of Eq. (2.1) with $f = 0$ and $\sigma = 0$.

Theorem 2.4 ([2]). *Assume that* (H_0), (A) *and* (B) *hold. Then $(U(t))_{t\ge 0}$ is a C_0-semigroup on $\mathcal{B}_A$. That is*

(i) $U(0) = Id$,

(ii) $U(t+s) = U(t)U(s)$ *for* $t, s \ge 0$,

(iii) *for all* $\phi \in \mathcal{B}_A$, $t \mapsto U(t)\phi$ *is continuous from* $[0, \infty)$ *to* $\mathcal{B}_A$.

Moreover, $(U(t))_{t\geq 0}$ *satisfies, for* $t \geq 0, \phi \in \mathcal{B}_A$*, the translation property*

$$(U(t)\phi)(\theta) = \begin{cases} (U(t+\theta)\phi)(0), & \text{for } t+\theta \geq 0 \\ \phi(t+\theta), & \text{for } t+\theta \leq 0. \end{cases}$$

Due to the relationship between the semigroup and its generator, it is fundamental to compute the infinitesimal generator of $(U(t))_{t\geq 0}$; to this end, we assume furthermore that
(D_1) If $(\phi_n)_n$ is a sequence in $\mathcal{B}$ such that $\phi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then for all $\theta \leq 0$, $\phi_n(\theta) \to 0$ in X as $n \to +\infty$.
(D_2) $\mathcal{B} \subset C((-\infty, 0]; X)$, where $C((-\infty, 0]; X)$ is the space of continuous functions from $(-\infty, 0]$ into X.
(D_3) There exists $\lambda_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > \lambda_0$ and $x \in X$, we have $e^{\lambda \cdot} x \in \mathcal{B}$ and

$$K_0 = \sup_{\substack{\operatorname{Re} \lambda > \lambda_0, x \in X \\ x \neq 0}} \frac{\left\| e^{\lambda \cdot} x \right\|}{|x|} < \infty,$$

where

$$(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x \quad \text{for } \theta \in (-\infty, 0] \text{ and } x \in X.$$

The aim of the following result is the computation of the infinitesimal generator of $(U(t))_{t\geq 0}$.

Lemma 2.5 (*[2]*). *Assume that* (H_0), (A), (B), (D_1) *and* (D_2) *hold. Then the infinitesimal generator* A_U *of* $(U(t))_{t\geq 0}$ *is given by*

$$\begin{cases} D(A_U) = \{\phi \in C^1((-\infty, 0]; X) \cap \mathcal{B}_A : \phi' \in \mathcal{B}_A, \phi(0) \in D(A) \text{ and } \phi'(0) = A\phi(0) + L(\phi)\}, \\ A_U \phi = \phi'. \end{cases}$$

Recently in [2], a variation of constants formula for Eq. (2.1) has been established. In order to recall this formula, we need to give some preliminary results. Firstly, we consider the space $\mathfrak{X} := \mathcal{B}_A \oplus \langle X_0 \rangle$, where

$$\langle X_0 \rangle = \{X_0 x : x \in X\}$$

and $X_0 x$ is the discontinuous function defined by

$$(X_0 x)(\theta) = \begin{cases} 0 & \text{for } \theta \in (-\infty, 0) \\ x & \text{for } \theta = 0. \end{cases}$$

The space $\mathfrak{X}$ endowed with the norm

$$\|\phi + X_0 x\| = \|\phi\| + |x|$$

is a Banach space.

According to Axiom (D_3), we define for each complex number λ such that $\operatorname{Re} \lambda > \lambda_0$, the linear operator $\Delta(\lambda) : D(A) \to X$ by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I),$$

where $L(e^{\lambda \cdot} I)$ is a bounded linear operator on X, which is defined by

$$L(e^{\lambda \cdot} I)(x) = L(e^{\lambda \cdot} x) \quad \text{for } x \in X.$$

Theorem 2.6 (*[2]*). *Assume that* (H_0), (A), (B), (D_1)–(D_3) *hold. Then the extension* $\widetilde{A_U}$ *of the operator* A_U *defined on* $\mathfrak{X}$ *by*

$$\begin{cases} D(\widetilde{A_U}) = \{\phi \in \mathcal{B}_A : \phi' \in \mathcal{B}_A \text{ and } \phi(0) \in D(A)\}, \\ \widetilde{A_U} \phi = \phi' + X_0(A\phi(0) + L\phi - \phi'(0)), \end{cases}$$

satisfies the Hille–Yosida condition on $\mathfrak{X}$*. More precisely, there exists* $\overline{\omega} \in \mathbb{R}$ *such that* $(\overline{\omega}, \infty) \subset \rho\left(\widetilde{A_U}\right)$ *and for* $\lambda > \overline{\omega}, \phi \in \mathcal{B}_A, x \in X, n \in \mathbb{N}^*$*, one has*

$$\left(\lambda - \widetilde{A_U}\right)^{-n} (\phi + X_0 x) = (\lambda - A_U)^{-n} \phi + (\lambda - A_U)^{n-1} (e^{\lambda \cdot} \Delta(\lambda)^{-1} x).$$

Theorem 2.7 (*[2]*). *Assume that* (H_0), (A), (B), (D_1)–(D_3) *hold. Let* $\phi \in \mathcal{B}_A$. *Then the corresponding solution u of Eq.* (2.1) *is given by the following variation of constants formula*

$$u_t = U(t-\sigma)\phi + \lim_{n\to+\infty} \int_\sigma^t U(t-s)\widetilde{B_n}\,(X_0 f(s))\,\mathrm{d}s \quad \textit{for } t \geq \sigma, \tag{2.2}$$

where $\widetilde{B_n} = n(nI - \widetilde{A_U})^{-1}$ *for n large enough.*

3. Reduction principle in uniform fading memory spaces

In this work, we assume that $\mathcal{B}$ satisfies Axioms (A), (B), (D_1)–(D_3). Let C_{00} be the space of all X-valued continuous functions on $(-\infty, 0]$ with compact support. We suppose the following axiom: (C) if a uniformly bounded sequence $(\varphi_n)_n$ in C_{00} converges to a function φ compactly in $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $\|\varphi_n - \varphi\| \to 0$ as $n \to \infty$.

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0 = \{\phi \in \mathcal{B} : \phi(0) = 0\}$$

by

$$(S_0(t)\phi)(\theta) = \begin{cases} \phi(t+\theta) & \text{for } t+\theta \leq 0 \\ 0 & \text{for } t+\theta \geq 0. \end{cases}$$

Definition 3.1. Assume that the space $\mathcal{B}$ satisfies Axioms (A)–(C). $\mathcal{B}$ is said to be a fading memory space if for all $\phi \in \mathcal{B}_0$,

$$S_0(t)\phi \underset{t\to\infty}{\longrightarrow} 0 \quad \text{in } \mathcal{B}.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$\|S_0(t)\| \underset{t\to\infty}{\longrightarrow} 0.$$

The following results give some properties of fading memory spaces.

Lemma 3.2 (*[8, p. 190]*). *The following statements hold*

(i) *If* $\mathcal{B}$ *is a fading memory space, then the functions* $K(\cdot)$ *and* $M(\cdot)$ *in Axiom* (A) *can be chosen to be constants.*
(ii) *If* $\mathcal{B}$ *is a uniform fading memory space, then the functions* $K(\cdot)$ *and* $M(\cdot)$ *can be chosen such that* $K(\cdot)$ *is constant and* $M(t) \to 0$ *as* $t \to \infty$.

Proposition 3.3 (*[8]*). *If* $\mathcal{B}$ *is a fading memory space, then the space* $\mathcal{BC}((-\infty, 0]; X)$ *of all bounded and continuous* X*-valued functions on* $(-\infty, 0]$*, endowed with the uniform norm topology, is continuously embedding in* $\mathcal{B}$.

By Proposition 3.3, one can observe that if $\mathcal{B}$ is a fading memory space then (D_3) is satisfied with $\lambda_0 \geq 0$.

In order to study the qualitative behavior of the semigroup $(U(t))_{t\geq 0}$, we suppose the following assumption that (H_1) $T_0(t)$ is compact on $\overline{D(A)}$, for each $t > 0$.

Let V be a bounded subset of a Banach space Y. The Kuratowskii measure of noncompactness $\alpha(V)$ of V is defined by

$$\alpha(V) = \inf\Big\{ d > 0 \text{ such that there exists a finite number of sets } V_1, \ldots, V_n \text{ with } \operatorname{diam}(V_i) \leq d \text{ such that } V \subseteq \bigcup_{i=1}^{n} V_i \Big\}.$$

Moreover, for a bounded linear operator P on Y, we define $|P|_\alpha$ by

$$|P|_\alpha = \inf\{k > 0 : \alpha(P(V)) \leq k\alpha(V) \text{ for any bounded set } V \text{ of } Y\}.$$

For the semigroup $(U(t))_{t\geq 0}$, we define the essential growth bound $\omega_{\text{ess}}(U)$ by

$$\omega_{\text{ess}}(U) = \lim_{t\to\infty} \frac{1}{t}\log |U(t)|_{\alpha}.$$

We have the following fundamental result.

Theorem 3.4 ([5]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then*

$$\omega_{\text{ess}}(U) < 0.$$

Definition 3.5. Let $\mathcal{C}$ be a densely defined operator on Y. The essential spectrum of $\mathcal{C}$ denoted by $\sigma_{\text{ess}}(\mathcal{C})$ is the set of $\lambda \in \sigma(\mathcal{C})$ such that one of the following conditions holds:

(i) $\operatorname{Im}(\lambda I - \mathcal{C})$ is not closed,
(ii) the generalized eigenspace $M_\lambda(\mathcal{C}) = \bigcup_{k\geq 1} \operatorname{Ker}(\lambda I - \mathcal{C})^k$ is of infinite dimension,
(iii) λ is a limit point of $\sigma(\mathcal{C}) \setminus \{\lambda\}$.

The essential radius of any bounded operator $\mathcal{T}$ is defined by

$$r_{\text{ess}}(\mathcal{T}) = \sup\{|\lambda| : \lambda \in \sigma_{\text{ess}}(\mathcal{C})\}.$$

Theorem 3.6 ([2]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then* $\sigma^+(A_U) = \{\lambda \in \sigma(A_U) : \mathcal{R}e(\lambda) \geq 0\}$ *is a finite set of the eigenvalues of* A_U *which is not in the essential spectrum. More precisely,* $\lambda \in \sigma^+(A_U)$ *if and only if there exists* $x \in D(A)\backslash\{0\}$ *which solves the following characteristic equation*

$$\Delta(\lambda)x = \lambda x - Ax - L(\mathrm{e}^{\lambda\cdot}x) = 0.$$

We have the following spectral decomposition result.

Theorem 3.7 ([2]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then the phase space* $\mathcal{B}_A$ *is decomposed as*

$$\mathcal{B}_A = \mathcal{S} \oplus \mathcal{V},$$

where $\mathcal{S}$, $\mathcal{V}$ *are two closed subspaces of* $\mathcal{B}_A$ *which are invariant under the semigroup* $(U(t))_{t\geq 0}$. *Let* $U^{\mathcal{S}}(t)$ *be the restriction of* $U(t)$ *on* $\mathcal{S}$. *Then there exist positive constants* N *and* μ *such that*

$$\left\| U^{\mathcal{S}}(t)\phi \right\| \leq N\mathrm{e}^{-\mu t}\,\|\phi\| \quad \text{for } \phi \in \mathcal{S}.$$

On the other hand, $\mathcal{V}$ *is a finite dimensional space. Then the restriction* $U^{\mathcal{V}}(t)$ *of* $U(t)$ *on* $\mathcal{V}$ *becomes a group.*

Let $\Pi^{\mathcal{S}}$ and $\Pi^{\mathcal{V}}$ denote the projections on $\mathcal{S}$ and $\mathcal{V}$ respectively and $d = \dim \mathcal{V}$. Take a basis $\{\phi_1, \ldots, \phi_d\}$ in $\mathcal{V}$. Then there exist d-elements $\{\psi_1, \ldots, \psi_d\}$ in the dual space $\mathcal{B}_A^*$ of $\mathcal{B}_A$ such that $\langle \psi_i, \phi_j \rangle = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\langle \psi_i, \phi \rangle = 0$, for $\phi \in \mathcal{S}$ and $i = 1, \ldots, d$, with $\langle \cdot, \cdot \rangle$ being the canonical pairing between the dual space and the original space. Denote by $\Phi = (\phi_1, \ldots, \phi_d)$ and by Ψ the transpose of $(\psi_1, \ldots, \psi_d)$. One has

$$\langle \Psi, \Phi \rangle = I_{\mathbb{R}^d},$$

where $I_{\mathbb{R}^d}$ is the identity $d \times d$-matrix. For each $\phi \in \mathcal{B}_A$, $\Pi^{\mathcal{V}}\phi$ is computed as

$$\begin{aligned} \Pi^{\mathcal{V}}\phi &= \Phi\langle \Psi, \phi \rangle, \\ &= \sum_{i=1}^{d} \langle \psi_i, \phi \rangle \phi_i. \end{aligned}$$

Let u be the solution of Eq. (2.1) and $\zeta(t) = (\zeta_1(t), \ldots, \zeta_d(t))$ be the components of $\Pi^{\mathcal{V}} u_t$ in the basis Φ. Then

$$\Pi^{\mathcal{V}} u_t = \Phi\zeta(t) \quad \text{and} \quad \zeta(t) = \langle \Psi, u_t \rangle.$$

Since $\left(U^{\mathcal{V}}(t)\right)_{t\geq 0}$ is a group on the finite dimensional space $\mathcal{V}$, then there exists a $d \times d$-matrix G such that

$$U^{\mathcal{V}}(t)\phi = \Phi \mathrm{e}^{Gt} \langle \Psi, \phi \rangle \quad \text{for } t \in \mathbb{R} \text{ and } \phi \in \mathcal{V}.$$

This means that

$$U^{\mathcal{V}}(t)\Phi = \Phi \mathrm{e}^{Gt} \quad \text{for } t \in \mathbb{R}.$$

Let $n_0 \in \mathbb{N}$ such that $n_0 > \overline{\omega}$. We define, for $n \in \mathbb{N}$ such that $n \geq n_0$ and $i \in \{1, \ldots, d\}$, the functional x_n^{*i} by

$$\langle x_n^{*i}, x \rangle = \langle \psi_i, \widetilde{B_n}(X_0 x) \rangle \quad \text{for all } x \in X.$$

By Theorem 2.6, we have $\widetilde{B_n}\,(X_0 x) = n\mathrm{e}^{n\cdot}\Delta^{-1}(n)x$, for $n \geq n_0$. Then, we choose n_0 large enough such that

$$\left|\widetilde{B_n}\,(X_0 x)\right| \leq M_0\,|x| \quad \text{for all } x \in X \text{ and } n \geq n_0.$$

This implies that x_n^{*i} is a bounded linear operator on X with $\left|x_n^{*i}\right| \leq M_0\,|\psi_i|$. Define the d-columns vector x_n^* as an element of $\mathcal{L}(X, \mathbb{R}^d)$ given by the transpose of $\left(x_n^{*1}, \ldots, x_n^{*d}\right)$. Then, for all $n \geq n_0$ and $x \in X$, we have

$$\langle x_n^*, x \rangle = \langle \Psi, \widetilde{B_n}(X_0 x) \rangle \quad \text{and} \quad \sup_{n\geq n_0} \left|x_n^*\right| \leq M_0 \sup_{i=1,\ldots,d} |\psi_i| < \infty.$$

We have the following important result.

Theorem 3.8 ([2]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then the sequence* $\left(x_n^*\right)_{n\geq n_0}$ *converges weakly in* $\mathcal{L}(X, \mathbb{R}^d)$*, in the sense that there exists* $x^* \in \mathcal{L}(X, \mathbb{R}^d)$ *such that*

$$\langle x_n^*, x \rangle \underset{n\to\infty}{\longrightarrow} \langle x^*, x \rangle \quad \text{for } x \in X.$$

Corollary 3.9 ([2]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Then for any continuous function* $h : [\sigma, T] \to X$*, we have for all* $t \in [\sigma, T]$

$$\lim_{n\to\infty} \int_\sigma^t U^{\mathcal{V}}(t-s)\Pi^{\mathcal{V}}\left(\widetilde{B_n}\,(X_0 h(s))\right)\mathrm{d}s = \Phi \int_\sigma^t \mathrm{e}^{(t-s)G} \langle x^*, h(s) \rangle \mathrm{d}s.$$

In the next theorem, we state a finite dimensional reduction principle of Eq. (1.1).

Theorem 3.10 ([2]). *Assume that* (H_0), (H_1) *hold and* $\mathcal{B}$ *is a uniform fading memory space. Let* u *be a solution of Eq. (1.1) on* $\mathbb{R}$*. Then* $\zeta(t) = \langle \Psi, u_t \rangle$ *for* $t \in \mathbb{R}$*, is a solution of the following ordinary differential equation*

$$\dot{\zeta}(t) = G\zeta(t) + \langle x^*, f(t) \rangle \quad \text{for } t \in \mathbb{R}. \tag{3.1}$$

Conversely, if f *is bounded and* ζ *is a solution of (3.1), then the function*

$$\left(\Phi\zeta(t) + \lim_{n\to+\infty} \int_{-\infty}^t U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B_n}\,(X_0 f(s))\right)\mathrm{d}s\right)(0) \tag{3.2}$$

is a solution of Eq. (1.1) on $\mathbb{R}$.

4. $C^{(n)}$-almost-periodic functions

We recall some properties about $C^{(n)}$-almost-periodic functions. Let $\mathcal{BC}(\mathbb{R}, X)$ be the space of all bounded and continuous functions from $\mathbb{R}$ to X, equipped with the uniform norm topology. Let $h \in \mathcal{BC}(\mathbb{R}, X)$ and $\tau \in \mathbb{R}$; we define the function h_τ by

$$h_\tau(s) = h(\tau + s) \quad \text{for } s \in \mathbb{R}.$$

Let $C^{(n)}(\mathbb{R}, X)$ be the space of all continuous functions which have a continuous nth derivative on $\mathbb{R}$ and $C_b^n(\mathbb{R}, X)$ be the subspace of $C^{(n)}(\mathbb{R}, X)$ of functions satisfying

$$\sup_{t\in\mathbb{R}} \sum_{i=0}^{i=n} \left| f^{(i)}(t) \right| < \infty,$$

$f^{(i)}$ denotes the ith derivative of f. Then $C_b^n(\mathbb{R}, X)$ is a Banach space equipped with the following norm

$$|f|_n = \sup_{t\in\mathbb{R}} \sum_{i=0}^{i=n} \left| f^{(i)}(t) \right|.$$

Definition 4.1 (*[6]*). A bounded continuous function $h : \mathbb{R} \to X$ is said to be almost periodic if

$$\{h_\tau : \tau \in \mathbb{R}\} \quad \text{is relatively compact in } \mathcal{BC}(\mathbb{R}, X).$$

Definition 4.2 (*[3]*). Let $\varepsilon > 0$ and $f \in C_b^n(\mathbb{R}, X)$. A number $\tau \in \mathbb{R}$ is said to be a $|.|_n - \varepsilon$ almost periodic of the function f if

$$|f_\tau - f|_n < \varepsilon.$$

The set of all $|.|_n - \varepsilon$ almost periodic of the function f is denoted by $E^{(n)}(\varepsilon, f)$.

Definition 4.3 (*[3]*). A function $f \in C_b^n(\mathbb{R}, X)$ is said to be a $C^{(n)}$-almost-periodic function if for every $\varepsilon > 0$, the set $E^{(n)}(\varepsilon, f)$ is relatively dense in $\mathbb{R}$.

Definition 4.4. $AP^{(n)}(\mathbb{R}, X)$ denotes the space of the all $C^{(n)}$-almost-periodic functions $\mathbb{R} \to X$.

Since it is well known that for any almost-periodic functions f and g and $\varepsilon > 0$, there exists a relatively dense set of their common ε almost periods, we get the following result.

Proposition 4.5. *$f \in AP^{(n)}(\mathbb{R}, X)$ if and only if $f^{(i)} \in AP(\mathbb{R}, X)$ for $i = 1, \ldots, n$.*

Proposition 4.6 (*[4]*). *$AP^{(n)}(\mathbb{R}, X)$ provided with the norm $|.|_n$ is a Banach space.*

The following example of a $C^{(n)}$-almost-periodic function has been given in [4].

Example. Let $g(t) = \sin(\alpha t) + \sin(\beta t)$, where $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Then the function $f(t) = \mathrm{e}^{g(t)}$ is a $C^{(n)}$-almost-periodic function for any $n \geq 1$.

In [4], one can find example of a function which is $C^{(n)}$-almost periodic but not $C^{(n+1)}$-almost periodic.

The following theorem provides the sufficient and necessary condition for the existence of $C^{(n)}$-almost-periodic solutions of Eq. (1.3).

Theorem 4.7. *Assume that e is a $C^{(n)}$-almost-periodic function. If Eq. (1.3) has a bounded solution on $\mathbb{R}^+$, then it has an $C^{(n+1)}$-almost-periodic solution. Moreover, every bounded solution of Eq. (1.3) on $\mathbb{R}$ is $C^{(n+1)}$-almost periodic.*

Proof. We assume that Eq. (1.3) has a bounded solution χ in $\mathbb{R}^+$. Since e is an almost-periodic solution, then there exists a sequence $(t_n)_n$, $t_n \to \infty$ such that

$$e(t + t_n) \to e(t), \quad \text{uniformly in } t \in \mathbb{R}.$$

Using the diagonal extraction process, we show that the sequence $\chi(t + t_n)$ has a subsequence which converges compactly to the solution of Eq. (1.3) which is bounded and defined in $\mathbb{R}$. For $m = 0$, it has been proved in [6], that if e is an almost-periodic function then every bounded solution of equation in $\mathbb{R}$ is almost periodic. Let x be a bounded solution of Eq. (1.3), then x is almost periodic Gx is also almost periodic and

$$x'(t) = Gx(t) + e(t) \quad \text{for } t \in \mathbb{R},$$

it follows that x' is almost periodic. Since the function f is $C^{(n)}$-almost periodic and for $i = 1, \dots, n$, we have the formula,

$$x^{(i)}(t) = Gx^{(i-1)}(t) + \mathrm{e}^{(i-1)}(t) \quad \text{for } t \in \mathbb{R}.$$

Consequently, we deduce that $x^{(i)}$ is almost periodic for $i = 1, \dots, n$ and by Proposition 4.5, we deduce that x is $C^{(n)}$-almost periodic. Moreover

$$x^{(n+1)}(t) = Gx^{(n)}(t) + \mathrm{e}^{(n)}(t) \quad \text{for } t \in \mathbb{R},$$

which implies that $x^{(n+1)}$ is almost periodic and x is $C^{(n+1)}$-almost periodic. $\square$

5. $C^{(n)}$-almost-periodic solutions

In the following, we assume that
(H_2) f is a $C^{(n)}$-almost-periodic function.

Theorem 5.1. *Assume that* (H_0), (H_1), (H_2) *hold and $\mathcal{B}$ is a uniform fading memory space. If there is at least $\phi \in \mathcal{B}$ such that Eq. (2.1) has a bounded solution on $\mathbb{R}^+$, then Eq. (1.1) has a $C^{(n)}$-almost-periodic solution. Moreover every bounded solution on the whole line is a $C^{(n)}$-almost-periodic solution.*

Proof. Let u be a bounded solution of Eq. (1.1) on $\mathbb{R}^+$. By Theorem 3.10, the function $z(t) = \langle \Psi, u_t\rangle$, for $t \geq 0$, is a solution of the ordinary differential equation (3.1) and z is bounded on $\mathbb{R}^+$. Moreover, the function

$$\varrho(t) = \langle x^*, f(t)\rangle \quad \text{for } t \in \mathbb{R},$$

is $C^{(n)}$-almost periodic from $\mathbb{R}$ to $\mathbb{R}^d$. By Theorem 4.7, we get that the reduced system (3.1) has a $C^{(n)}$-almost-periodic solution $\widetilde{z}$ and $\Phi\widetilde{z}(.)$ is a $C^{(n)}$-almost-periodic function on $\mathbb{R}$. From Theorem 3.10, we know that the function $u(t) = v(t)(0)$, where

$$v(t) = \Phi\widetilde{z}(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s \quad \text{for } t \in \mathbb{R},$$

is a solution of Eq. (1.1) on $\mathbb{R}$. We claim that v is $C^{(n)}$-almost periodic. In fact, let y be defined by

$$y(t) = \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s \quad \text{for } t \in \mathbb{R}.$$

Then $y \in C_b^n(\mathbb{R}, X)$. Since f is $C^{(n)}$-almost periodic. Let $\varepsilon > 0$ and τ be a $|.|_n - \varepsilon$ almost periodic of the function f that is

$$|f_\tau - f|_n < \varepsilon.$$

It follows that

$$y(t+\tau) - y(t) = \lim_{n\to+\infty} \int_{0}^{\infty} U^{\mathcal{S}}(s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 \left(f(t+\tau-s) - f(t-s)\right)\right) \mathrm{d}s$$

and we get for some positive constant γ that

$$|y_\tau - y|_n \leq \gamma\varepsilon\, |f_\tau - f|_n\,,$$

which implies that y is $C^{(n)}$-almost periodic.

Let v be a bounded solution on the whole line then v is given by the

$$v(t) = \Phi z(t) + \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s \quad \text{for } t \in \mathbb{R},$$

where

$$z(t) = \langle \Psi, u_t\rangle \quad \text{for } t \in \mathbb{R}$$

is a solution of the reduced system (3.1), which is $C^{(n)}$-almost periodic by Theorem 4.7, and arguing as above, one can prove that the function

$$t \to \lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s \quad \text{for } t \in \mathbb{R},$$

is also $C^{(n)}$-almost periodic. □

6. Exponential dichotomy

Definition 6.1. The semigroup $(U(t))_{t\geq 0}$ is said to have an exponential dichotomy if

$$\sigma(A_U) \cap \mathrm{i}\mathbb{R} = \emptyset.$$

Since $\omega_{\mathrm{ess}}(U) < 0$, then we get the following result on the spectral decomposition of the phase space $\mathcal{B}_A$.

Theorem 6.2 *([2]). Assume that* (H_0), (H_1) *hold and $\mathcal{B}$ is a uniform fading memory space. If the semigroup $(U(t))_{t\geq 0}$ has an exponential dichotomy, then the space $\mathcal{B}_A$ is decomposed as a direct sum $\mathcal{B}_A = \mathcal{S} \oplus \mathcal{U}$ of two $U(t)$ invariant closed subspaces $\mathcal{S}$ and $\mathcal{U}$ such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants N_0 and ε_0 such that*

$$|U(t)\varphi| \leq N_0 \mathrm{e}^{-\varepsilon_0 t} |\varphi| \quad \textit{for } t \geq 0 \textit{ and } \varphi \in \mathcal{S}$$
$$|U(t)\varphi| \leq N_0 \mathrm{e}^{\varepsilon_0 t} |\varphi| \quad \textit{for } t \leq 0 \textit{ and } \varphi \in \mathcal{U}.$$

As a consequence of the exponential dichotomy, we get the following result on the uniqueness of the bounded solution of Eq. (1.1).

Theorem 6.3. *Assume that* (H_0), (H_1) *hold and $\mathcal{B}$ is a uniform fading memory space. If the semigroup $(U(t))_{t\geq 0}$ has an exponential dichotomy, then for any bounded continuous function f on $\mathbb{R}$, Eq. (1.1) has a unique bounded solution on $\mathbb{R}$. Moreover, this solution is $C^{(n)}$-almost periodic if f is C^n-almost periodic.*

Proof. Since the semigroup $(U(t))_{t\geq 0}$ has an exponential dichotomy, then Eq. (1.1) has one and only one bounded solution on $\mathbb{R}$ which is given for $t \in \mathbb{R}$ by the following formula

$$\left(\lim_{n\to+\infty} \int_{-\infty}^{t} U^{\mathcal{S}}(t-s)\Pi^{\mathcal{S}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s + \lim_{n\to+\infty} \int_{+\infty}^{t} U^{\mathcal{U}}(t-s)\Pi^{\mathcal{U}}\left(\widetilde{B}_n X_0 f(s)\right) \mathrm{d}s\right)(0).$$

By Theorem 3.10, we conclude that this solution is $C^{(n)}$-almost periodic when f is $C^{(n)}$-almost periodic. □

7. Application

To illustrate the previous results, we consider the following Lotka–Volterra model with diffusion

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,\xi) = \dfrac{\partial^2}{\partial \xi^2} v(t,\xi) + \displaystyle\int_{-\infty}^{0} \eta(\theta) v(t+\theta,\xi) \mathrm{d}\theta + \rho(t) F(\xi) \quad \text{for } t \in \mathbb{R} \text{ and } 0 \leq \xi \leq \pi, \\ v(t,0) = v(t,\pi) = 0 \quad \text{for } t \in \mathbb{R} \end{cases} \tag{7.1}$$

where η is a positive function on $(-\infty, 0]$, $\rho : \mathbb{R} \to \mathbb{R}$ is C^n-almost periodic, for example, we consider

$$g(t) = \sin(\alpha t) + \sin(\beta t) \quad \text{for } t \in \mathbb{R}$$

where $\frac{\alpha}{\beta} \notin \mathbb{Q}$. Then the function $\rho(t) = \mathrm{e}^{g(t)}$. $F : [0,\pi] \to \mathbb{R}$ is a continuous function. Let $X = C([0,\pi]; \mathbb{R})$ be the space of all continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. Consider the operator $A : D(A) \subset X \to X$ defined by

$$\begin{cases} D(A) = \{z \in C^2([0,\pi]; \mathbb{R}) : z(0) = z(\pi) = 0\}, \\ Az = z''. \end{cases}$$

Lemma 7.1 *([2]). A satisfies the Hille–Yosida condition on X.*

On the other hand, one can see that

$$\overline{D(A)} = \{\psi \in C([0, \pi]; \mathbb{R}) : \psi(0) = \psi(\pi) = 0\}.$$

We introduce the following space $\mathcal{B} = C_\gamma$, $\gamma > 0$, where

$$C_\gamma = \left\{\phi \in C((-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\phi(\theta) \text{ exists in } X\right\}$$

provided with the following norm

$$\|\phi\|_\gamma = \sup_{-\infty < \theta \le 0} e^{\gamma\theta} |\phi(\theta)| \quad \text{for } \phi \in C_\gamma.$$

Lemma 7.2 *([8, p. 15]). The space C_γ, for $\gamma > 0$, is a uniform fading memory space satisfying* (D_1) *and* (D_2).

We make the assumption
(E_1) $\eta(\cdot)e^{-\gamma\cdot}$ is integrable on $(-\infty, 0]$.
Define

$$\begin{cases} (L(\phi))(\xi) = \displaystyle\int_{-\infty}^{0} \eta(\theta)\phi(\theta)(\xi)d\theta & \text{for } \xi \in [0, \pi] \text{ and } \phi \in \mathcal{B}, \\ f(t)(\xi) = \rho(t)F(\xi) & \text{for } t \in \mathbb{R} \text{ and } \xi \in [0, \pi]. \end{cases}$$

By assumption (E_1), one can see that L is a bounded linear operator from $\mathcal{B}$ to X and $f : \mathbb{R} \to X$ is C^n-almost periodic. We put

$$x(t)(\xi) = v(t, \xi) \quad \text{for } t \in \mathbb{R} \text{ and } \xi \in [0, \pi].$$

Then Eq. (7.1) takes the abstract form

$$\frac{d}{dt}x(t) = Ax(t) + L(x_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{7.2}$$

The part A_0 of A in $\overline{D(A)}$ is given by

$$\begin{cases} D(A_0) = \{z \in C^2([0, \pi]; \mathbb{R}) : z(0) = z(\pi) = z''(0) = z''(\pi) = 0\}, \\ A_0 z = z''. \end{cases}$$

Then it is well known that A_0 generates a compact C_0-semigroup on $\overline{D(A)}$. In order to prove the existence and uniqueness of a $C^{(n)}$-almost- periodic solution of Eq. (7.2), we make the assumption
(E_2) $\int_{-\infty}^{0} \eta(\theta)d\theta < 1$.

Theorem 7.3. *If* (E_1) *and* (E_2) *hold. Then the semigroup solution associated to (7.2) with $f = 0$ has an exponential dichotomy.*

Proof. By Theorem 3.6, it suffices to show that $\sigma^+(A_U) = \emptyset$. In fact, we proceed by contradiction and assume that there exists $\lambda \in \sigma^+(A_U)$. Then there exists $\vartheta \in D(A)\backslash\{0\}$ such that $\Delta(\lambda)\vartheta = 0$, which is equivalent to

$$\left(\lambda - A - \int_{-\infty}^{0} \eta(\theta)e^{\lambda\theta}d\theta\right)\vartheta = 0. \tag{7.3}$$

The spectrum $\sigma(A)$ is reduced to the point spectrum $\sigma_p(A)$ and $\sigma_p(A) = \{-n^2 : n \in \mathbb{N}^*\}$. Then λ is a solution of the characteristic equation (7.3) with $\mathcal{R}e(\lambda) \ge 0$ if and only if λ satisfies

$$\lambda - \int_{-\infty}^{0} \eta(\theta)e^{\lambda\theta}d\theta = -n^2 \quad \text{for some } n \in \mathbb{N}^*. \tag{7.4}$$

It follows that

$$\begin{aligned}\mathcal{R}e(\lambda) &= \int_{-\infty}^{0} \eta(\theta)\mathrm{e}^{\mathcal{R}e(\lambda)\theta} \cos(\mathrm{Im}(\lambda)\theta)\mathrm{d}\theta - n^2 \\ &\le \int_{-\infty}^{0} \eta(\theta)\mathrm{d}\theta - n^2.\end{aligned}$$

Since $\int_{-\infty}^{0} \eta(\theta)\mathrm{d}\theta < 1$, then a contradiction is obtained with the fact that $\mathcal{R}e(\lambda) \ge 0$. Consequently, the semigroup solution associated to (7.2) with $f = 0$ has an exponential dichotomy. □

By Theorem 6.2, we obtain the following existence and uniqueness result on $C^{(n)}$-almost-periodic solution of Eq. (7.2).

Corollary 7.4. *Assume that* (E_1) *and* (E_2) *hold. Then Eq.* (7.2) *has a unique* $C^{(n)}$*-almost-periodic solution.*

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