Linear Algebra and its Applications 487 (2015) 146–157

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Robust finite-time stability of linear differential-algebraic delay equations

V.N. Phat [a,*], N.H. Muoi [a], M.V. Bulatov [b]

[a] *Institute of Mathematics, VAST, 18 Hoang Quoc Viet Road, Hanoi 10307, Viet Nam*
[b] *Institute for System Dynamics and Control Theory, Siberian Branch of RAS, Irkutsk 664033, Russia*

ARTICLE INFO

Article history:
Received 13 May 2014
Accepted 26 August 2015
Available online 18 September 2015
Submitted by V. Mehrmann

MSC:
34A09
34D20
93D05

Keywords:
Differential-algebraic equation
Finite-time stability
Robustness
Time delay
Linear matrix inequality

ABSTRACT

In this paper, we develop finite-time stability theory for linear differential-algebraic equations with delay. Based on the Lyapunov-like functional method, new delay-dependent sufficient conditions such that the system is regular, impulse-free and robustly finite-time stable are developed in terms of solutions of some linear matrix inequalities. A numerical example is given to illustrate the effectiveness of the proposed result.

1. Introduction

Linear differential-algebraic equations (LDAEs) play an important role in mathematical modeling of real-life problems arising in a wide range of applications, as for instance,

* Corresponding author.
E-mail address: vnphat@math.ac.vn (V.N. Phat).

http://dx.doi.org/10.1016/j.laa.2015.08.036

multibody mechanics, prescribed path control, electrical design, chemically reacting systems and biomedicine. LDAEs with state delays arise naturally in many practical systems because of transmission of the measurement information [1,2]. The existence of these delays can affect profoundly the behavior of the system. The finite-time stability introduced by Dorato [3] plays an important role in stability theory of differential equations. A system is said to be finite-time stable if its state does not exceed a certain threshold during a specified time interval. Compared with the Lyapunov stability, finite-time stability concerns the boundedness of system during a fixed finite-time interval. While the Lyapunov stability theory for linear systems with delay has been extensively developed for decades (see, e.g., [4–7] and the references therein), there are few results on the finite-time stability. Some interesting results on finite-time stability and stabilization are obtained in [8–10] for linear continuous systems with constant delay. The use of Lyapunov functionals is certainly the main approach for solving the finite-time stability. So far, however, to the best of our knowledge, there has no research on finite-time stability for LDAEs with delay studied in the literature. Most of the existing results are focused on linear singular systems without delays. For linear delay differential-algebraic equations, some new asymptotic stability conditions have been derived in [11] in the sense of Lyapunov stability.

In this paper, we consider problem of robust finite-time stability for linear differential-algebraic equations subject to time-delay in the state variables. The main contribution of this paper is to propose robust finite-time stability conditions for the system. Based on the Lyapunov function method and new bounding estimation technique developed for linear delay systems, new finite-time stability conditions are derived via linear matrix inequalities (LMIs).

The paper is organized as follows. In Section 2, we present definitions and some auxiliary results which will be used in the proof of our main result. Section 3 addresses delay-dependent robust finite-time stability conditions and an illustrative numerical example.

Notation: $\mathbb{R}^+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $x^\top y$; $\mathbb{R}^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimension. $A^\top$ denotes the transpose of A; I_r denotes the identity matrix in $\mathbb{R}^r$; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\lambda_{\min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}$; $C([-h, 0], \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[-h, 0]$; The symmetric terms in a matrix are denoted by $*$. A matrix A is semi-positive definite, write $A \geq 0$, if $x^\top Ax \geq 0$ for all $x \in \mathbb{R}^n$; A is positive definite, write $A > 0$, if $x^\top Ax > 0$ for all $x \in \mathbb{R}^n, x \neq 0$; $A \geq B$ means that $A - B \geq 0$. The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-h, 0]\}$ with the norm $\|x_t\| = \sup_{s\in[-h,0]} \|x(t + s)\|$.

2. Preliminaries

Consider the following linear differential-algebraic equation with delay

$$\begin{cases} E\dot{x}(t) = Ax(t) + Dx(t-h) + Bw(t), \quad t \geq 0, \\ x(t) = \psi(t), \quad t \in [-h, 0], \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n, A, D \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $E \in \mathbb{R}^{n\times n}$ is a singular matrix: $rank\, E = r < n$; We assume that $\psi(t) \in C([-h,0], \mathbb{R}^n)$ and the disturbance $w(t)$ is a continuous function satisfying

$$\exists d > 0: \qquad w^\top(t)w(t) \leq d, \quad \forall t \geq 0. \tag{2.2}$$

Definition 2.1. (See [12].)

(i) System (2.1) is regular if $det(sE - A)$ is not identical zero.
(ii) System (2.1) is impulse-free if $deg(det(sE - A)) = r = rank(E)$.

The singular system (2.1) may have an impulsive solution, however, the regularity and the absence of impulses of the pair (E, A) ensure the existence and uniqueness of an impulse-free solution to this system, which is shown in [12].

Definition 2.2. Given positive numbers $T > 0$, $c_1, c_2, c_1 < c_2$, and a symmetric positive matrix R, the system (2.1) is said to be robustly finite-time stable w.r.t. $[c_1, c_2, T, R]$ if

$$\sup_{t\in[-h,0]} \psi^\top(t)R\psi(t) \leq c_1 \;\Rightarrow\; x^\top(t)Rx(t) < c_2, \quad t \in [0, T],$$

for all disturbances $w(.)$ satisfying (2.2).

We introduce the following technical well-known propositions for the proof of the main result.

Proposition 2.1 *(Schur complement lemma). (See [13].) Given matrices X, Y, Z with appropriate dimensions satisfying $Y = Y^\top > 0$, $X = X^\top$, we have*

$$\begin{pmatrix} X & Z \\ Z^\top & -Y \end{pmatrix} < 0 \;\Leftrightarrow\; X < 0, \quad X + ZY^{-1}Z^\top < 0.$$

Proposition 2.2 *(Cauchy matrix inequality). (See [13].) For any positive definite matrix $N \in \mathbb{R}^{n\times n}$, we have*

$$2y^\top x \leq x^\top Nx + y^\top N^{-1}y, \quad \forall x, y \in \mathbb{R}^n.$$

3. Main result

Consider system (2.1), since $rank(E) = r < n$, then there are two nonsingular matrices $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$, G such that $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = MEG$. It is easy to check $\begin{pmatrix} 0 \\ M_2 \end{pmatrix} E = 0$. Let

$$MAG = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \begin{pmatrix} 0 \\ M_2 \end{pmatrix} AG = \begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix},$$
$$MDG = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \quad MB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}.$$

Under coordinate transformation $y = G^{-1}x(t) = [y_1^\top(t)\ y_2^\top(t)]$, $y_1(t) \in \mathbb{R}^r$, $y_2(t) \in \mathbb{R}^{n-r}$, the systems (2.1) is reduced to the system

$$\begin{cases} \dot{y}_1(t) = A_{11}y_1(t) + A_{12}y_2(t) + D_{11}y_1(t-h) + D_{12}y_2(t-h) + B_1\omega(t), \\ 0 = A_{21}y_1(t) + A_{22}y_2(t) + D_{21}y_1(t-h) + D_{22}y_2(t-h) + B_2\omega(t), \\ y(t) = G^{-1}\psi(t), \quad t \in [-h, 0]. \end{cases} \tag{3.1}$$

Let us denote

$$\begin{aligned}
&W_1 = PA + A^\top P^\top + Q_1 + Q_2\bar{M}A + A^\top \bar{M}^\top Q_2^\top - \eta PE, \\
&W_2 = PD + Q_2\bar{M}D, \\
&\bar{M} = \begin{pmatrix} 0 \\ M_2 \end{pmatrix}, \quad G^\top PM^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, \\
&M^{-1^\top} RM^{-1} = \begin{pmatrix} R_{11} & R_{12} \\ * & R_{22} \end{pmatrix}, \quad b = \|A_{22}^{-1}B_2\|\sqrt{d}, \\
&\alpha_1 = \frac{\lambda_{\min}(P_{11})}{\lambda_{\max}(R_{11})}, \quad \alpha_2 = \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top RG)} + h\frac{\lambda_{\max}(Q_1)}{\lambda_{\min}(R)}, \\
&\alpha_3 = \frac{\alpha_2 c_1 + 2Td}{\alpha_1}, \quad \alpha_4 = \sum_{i=0}^{[\frac{T}{h}]-1} \|A_{22}^{-1}D_{22}\|^i, \\
&\alpha_5 = b\alpha_4 + \delta\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}}, \quad \delta = \max_{i=1,..[\frac{T}{h}]-1} \|A_{22}^{-1}D_{22}\|^i \\
&\gamma = \|A_{22}^{-1}D_{21} + A_{22}^{-1}D_{22}\|, \quad \beta = \lambda_{\max}(G^\top RG).
\end{aligned}$$

It is worth noting that to prove the stability of singular systems, one usually assumes the regularity and impulse-free (equivalently, the existence of A_{22}^{-1}). In the following theorem, we give a LMI condition for guaranteeing the regularity and impulse-free of system (2.1).

Theorem 3.1. *The system (2.1) is regular and impulse-free if there exist a symmetric positive definite matrix $Q_1 \in \mathbb{R}^{n\times n}$, nonsingular matrices $P \in \mathbb{R}^{n\times n}$, any matrix $Q_2 \in \mathbb{R}^{n\times n}$ and a number $\eta > 0$ such that:*

$$PE = E^\top P^\top \geq 0, \tag{3.2}$$

$$\begin{pmatrix} W_1 & W_2 & PB & Q_2\bar{M}B \\ * & -Q_1 & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{pmatrix} < 0. \tag{3.3}$$

Moreover, the system (2.1) is robustly finite-time stable w.r.t. $[c_1, c_2, T, R]$ if the following additional condition holds:

$$\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})} + \left(\alpha_5 + \gamma\alpha_4\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}}\right)^2 \leq \frac{c_2}{\beta}. \tag{3.4}$$

Proof. We first prove that the system (2.1) is regular and impulse-free. For this, we note that

$$G^\top PEG = G^\top PM^{-1}MEG = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix} \geq 0,$$
$$G^\top E^\top P^\top G = G^\top E^\top M^\top M^{-\top} P^\top G = \begin{pmatrix} P_{11}^\top & P_{21}^\top \\ 0 & 0 \end{pmatrix} \geq 0.$$

In view of (3.2), one gets $P_{21} = 0$, $P_{11} = P_{11}^\top \geq 0$. Since P is nonsingular, then $G^\top PM^{-1} = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$ is nonsingular, we have $det(P_{11}) \neq 0$ and hence $P_{11} > 0$. From (3.3) it follows that

$$PA + A^\top P^\top + \bar{M}Q_2A + A^\top Q_2^\top \bar{M}^\top - \eta PE < 0.$$

Since G is nonsingular, we obtain

$$\begin{aligned}
0 &> G^\top(PA + A^\top P^\top + Q_2\bar{M}A + A^\top\bar{M}^\top Q_2^\top - \eta PE)G \\
&= G^\top PM^{-1}MAG + G^\top A^\top M^\top M^{-\top}P^\top G \\
&\quad + G^\top Q_2\bar{M}AG + G^\top A^\top\bar{M}^\top Q_2^\top G - \eta G^\top PM^{-1}MEG \\
&= \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^\top & A_{21}^\top \\ A_{12}^\top & A_{22}^\top \end{pmatrix}\begin{pmatrix} P_{11}^\top & 0 \\ P_{12}^\top & P_{22}^\top \end{pmatrix} \\
&\quad + \begin{pmatrix} \hat{Q}_{211} & \hat{Q}_{212} \\ \hat{Q}_{221} & \hat{Q}_{222} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ A_{21} & A_{22} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix} 0 & A_{21}^\top \\ 0 & A_{22}^\top \end{pmatrix}\begin{pmatrix} \hat{Q}_{211}^\top & \hat{Q}_{221}^\top \\ \hat{Q}_{212}^\top & \hat{Q}_{222}^\top \end{pmatrix} - \eta \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} P_{11}A_{11} + P_{12}A_{21} & P_{11}A_{12} + P_{12}A_{22} \\ P_{22}A_{21} & P_{22}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^\top P_{11}^\top + A_{21}^\top P_{12}^\top & A_{21}^\top P_{22}^\top \\ A_{12}^\top P_{11}^\top + A_{22}^\top P_{12}^\top & A_{22}^\top P_{22}^\top \end{pmatrix} \\
&+ \begin{pmatrix} \hat{Q}_{212}A_{21} & \hat{Q}_{212}A_{22} \\ \hat{Q}_{222}A_{21} & \hat{Q}_{222}A_{22} \end{pmatrix} + \begin{pmatrix} \hat{Q}_{212}A_{21} & \hat{Q}_{212}A_{22} \\ \hat{Q}_{222}A_{21} & \hat{Q}_{222}A_{22} \end{pmatrix}^\top - \eta \begin{pmatrix} P_{11} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}
$$

which gives $A_{22}^\top(P_{22}^\top + \hat{Q}_{222}^\top) + (P_{22} + \hat{Q}_{222})A_{22} < 0$. Therefore, $det(A_{22}) \neq 0$ and the system, as shown in [12], is regular and impulse-free.

We now prove the robust finite-time stability of system (2.1). For this, we consider the following non-negative quadratic functional

$$V(t, x_t) = x^\top(t)PEx(t) + \int_{t-h}^{t} x^\top(s)Q_1x(s)ds.$$

Note that

$$
\begin{aligned}
G^\top E^\top REG &= G^\top E^\top M^\top M^{-\top} RM^{-1}MEG \\
&= \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R_{11} & 0 \\ 0 & 0 \end{pmatrix}, \\
G^\top PEG &= G^\top PM^{-1}MEG \\
&= \begin{pmatrix} P_{11} & P_{21} \\ 0 & P_{22} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} P_{11} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.5}
$$

Since $M^{-\top}RM^{-1} > 0$, by Silvester's criterion the matrix R_{11} is positive definite, we then have

$$
\begin{aligned}
\langle G^\top E^\top REGx, x\rangle &\geq \lambda_{\min}(R_{11}) \sum_{i=1}^{r} \|x_i\|^2, \quad \forall x \in \mathbb{R}^n, \\
\langle G^\top RGx, x\rangle &\geq \lambda_{\min}(G^\top RG) \sum_{i=1}^{n} \|x_i\|^2, \quad \forall x \in \mathbb{R}^n, \\
\langle G^\top PEGx, x\rangle &\leq \lambda_{\max}(P_{11}) \sum_{i=1}^{r} \|x_i\|^2, \quad \forall x \in \mathbb{R}^n.
\end{aligned}
$$

Moreover, we have

$$\langle [\frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top RG)} G^\top RG - G^\top PEG]x, x\rangle \geq 0, \quad \forall x \in \mathbb{R}^n,$$

which implies

$$G^\top(PE - \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top RG))}R)G \le 0.$$

Since G is a nonsingular matrix, it follows $PE \le \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top RG)}R$ and then

$$x^\top(0)PEx(0) \le \frac{\lambda_{\max}(P_{11})}{\lambda_{\min}(G^\top RG)} \sup_{t\in[-h,0]} \psi^\top(t)R\psi(t). \tag{3.6}$$

On the other hand, since

$$x^\top(t)Q_1x(t) \le \frac{\lambda_{\max}(Q_1)}{\lambda_{\min}(R)}x^\top(t)Rx(t),$$

we have

$$\int_{-h}^{0} x^\top(s)Q_1x(s)ds \le h\frac{\lambda_{\max}(Q_1)}{\lambda_{\min}(R)} \sup_{t\in[-h,0]} \psi^\top(t)R\psi(t). \tag{3.7}$$

Combining the conditions (3.6), (3.7) gives

$$V(0,x_0) \le \alpha_2 \sup_{t\in[-h,0]} \psi^\top(t)R\psi(t). \tag{3.8}$$

We now show that

$$\alpha_1 x^\top(t)E^\top REx(t) \le V(t,x_t), \quad \forall t \in [0,T]. \tag{3.9}$$

Indeed, from (3.5) it follows that

$$\langle G^\top E^\top REGx, x\rangle \le \lambda_{\max}(R_{11})\sum_{i=1}^{r}\|x_i\|^2, \quad \langle G^\top PEGx, x\rangle \ge \lambda_{\min}(P_{11})\sum_{i=1}^{r}\|x_i\|^2,$$

hence

$$\langle[\alpha_1 GE^\top REG - G^\top PEG]x, x\rangle \le 0,$$

which gives

$$G^\top(PE - \alpha_1 E^\top RE)G \ge 0.$$

Since G is a nonsingular matrix, we obtain $PE \ge \alpha_1 E^\top RE$, and hence

$$V(t,x_t) \ge x^\top(t)PEx(t) \ge \alpha_1 x^\top(t)E^\top REx(t), \quad \forall x \in \mathbb{R}^n,$$

as desired. Taking the derivative of $V(t,x_t)$ along the solution of system (2.1) one gets

$$\dot{V}(t,x_t) = x^\top(t)(PA + A^\top P^\top)x(t) + 2x^\top(t)PDx(t-h) + 2x^\top(t)PBw(t) \\ + x^\top(t)Q_1x(t) - x^\top(t-h)Q_1x(t-h). \tag{3.10}$$

Moreover, multiplying both sides of (2.1) by $2x^\top(t)Q_2\bar{M}$ from the right, we obtain:

$$2x^\top(t)Q_2\bar{M}Ax(t) + 2x^\top(t)Q_2\bar{M}Dx(t-h) + 2x^\top(t)Q_2\bar{M}Bw(t) = 0. \tag{3.11}$$

Combining the conditions (3.10)–(3.11) gives

$$\begin{aligned}\dot{V}(t,x_t) - \eta V(t,x_t) \le\ & x^\top(t)(PA + A^\top P^\top - \eta PE)x(t) + 2x^\top(t)PDx(t-h) \\ & + 2x^\top(t)PBw(t) + x^\top(t)Q_1x(t) \\ & - x^\top(t-h)Q_1x(t-h) + 2x^\top(t)Q_2\bar{M}Ax(t) \\ & + 2x^\top(t)Q_2\bar{M}Dx(t-h) + 2x^\top(t)Q_2\bar{M}Bw(t).\end{aligned} \tag{3.12}$$

Using Proposition 2.2 for the following inequalities

$$2x^\top(t)PBw(t) - w^\top(t)w(t) \le x^\top(t)PBB^\top P^\top x(t),$$
$$2x^\top(t)Q_2\bar{M}Bw(t) - w^\top(t)w(t) \le x^\top(t)Q_2\bar{M}BB^\top\bar{M}^\top Q_2^\top x(t),$$

we obtain from (3.12) that

$$\dot{V}(t,x_t) - \eta V(t,x_t) \le \xi^\top(t)\Phi\xi(t) + 2w^\top(t)w(t), \quad \forall t \in [0,T], \tag{3.13}$$

where $\xi(t) = [x(t), x(t-h)]^\top$, and

$$\begin{aligned}\Phi &= \begin{pmatrix} \Phi_{11} & PD + Q_2\bar{M}D \\ * & \Phi_{22} \end{pmatrix}, \\ \Phi_{11} &= PA + A^\top P^\top + PBB^\top P^\top + Q_2\bar{M}A + A^\top\bar{M}^\top Q_2^\top + Q_2\bar{M}BB^\top\bar{M}^\top Q_2^\top \\ &\quad + Q_1 - \eta PE, \\ \Phi_{22} &= -Q_1.\end{aligned}$$

If the LMI (3.3) holds, then by Schur complement lemma (Proposition 2.1), we have $\Phi < 0$. Therefore, in view of (3.13) we have

$$\dot{V}(t,x_t) - \eta V(t,x_t) < 2w^\top(t)w(t), \quad \forall t \in [0,T]. \tag{3.14}$$

Multiplying both sides of (3.14) with $e^{-\eta t}$ and integrating between 0 and t, we obtain

$$e^{-\eta t}V(t,x_t) - V(0,x_0) < 2\int_0^t e^{-\eta s}w^\top(s)w(s)ds < 2\int_0^t w^\top(s)w(s)ds, \quad \forall t \in [0,T],$$

and hence

$$V(t, x_t) < e^{\eta t}[V(0, x_0) + 2td], \quad \forall t \in [0, T]. \tag{3.15}$$

Combining the conditions (3.8), (3.9), (3.15) gives

$$x^\top(t)E^\top REx(t) < e^{\eta T}\frac{\alpha_2 c_1 + 2Td}{\alpha_1} = e^{\eta T}\alpha_3, \quad \forall t \in [0, T],$$

provided $\sup_{t\in[-h,0]} \psi^\top(t)R\psi(t) < c_1$. We have

$$\begin{aligned} x^\top(t)E^\top REx(t) &= y^\top(t)G^\top E^\top M^\top M^{-T}RM^{-1}MEGy(t) \\ &= y_1^\top(t)R_{11}y_1(t) \le e^{\eta T}\alpha_3, \end{aligned}$$

which implies

$$\|y_1(t)\| \le \sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}}, \quad \forall t \in [0, T]. \tag{3.16}$$

Let us denote

$$p(t) = -A_{22}^{-1}A_{21}y_1(t) - A_{22}^{-1}D_{21}y_1(t-h).$$

We have

$$\begin{aligned} \|p(t)\| &\le \|A_{22}^{-1}A_{21}\|\|y_1(t)\| + \|A_{22}^{-1}D_{21}\|\|y_1(t-h)\| \\ &\le (\|A_{22}^{-1}A_{21}\| + \|A_{22}^{-1}D_{21}\|)\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}}. \end{aligned}$$

Moreover, from the second equation of (3.1) we have

$$y_2(t) = p(t) - A_{22}^{-1}D_{22}y_2(t-h) - A_{22}^{-1}B_2\omega(t).$$

Therefore,

$$\|y_2(t)\| \le \|p(t)\| + \|A_{22}^{-1}D_{22}\|\|y_2(t-h)\| + \|A_{22}^{-1}B_2\omega(t)\|, \quad \forall t \ge 0. \tag{3.17}$$

If $t \in [0, h]$ then $t - h \in [-h, 0]$. We have

$$\begin{aligned} \|y_2(t)\| &\le \|p(t)\| + \|A_{22}^{-1}D_{22}\|\|y_2(t-h)\| + \|A_{22}^{-1}B_2\omega(t)\| \\ &\le \|p(t)\| + \|A_{22}^{-1}D_{22}\|\|G^{-1}\|\|\psi(t)\| + b \\ &\le (\|A_{22}^{-1}A_{21}\| + \|A_{22}^{-1}D_{21}\|)\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}} + \|A_{22}^{-1}D_{22}\|\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}} + b \end{aligned}$$

$$\leq \gamma\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}} + \|A_{22}^{-1}D_{22}\|\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}} + b. \tag{3.18}$$

If $t \in [h, 2h]$ then $t - h \in [0, h]$. Form (3.17), (3.18) we get

$$\begin{aligned}\|y_2(t)\| \leq (\|A_{22}^{-1}D_{22}\| + 1)\gamma\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}} + \|A_{22}^{-1}D_{22}\|^2\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}} \\ + \|A_{22}^{-1}D_{22}\|b + b.\end{aligned}$$

Thus, for all $t \in [(k-1)h, kh]$, we obtain

$$\begin{aligned}\|y_2(t)\| \leq b\sum_{i=0}^{k-1}\|A_{22}^{-1}D_{22}\|^i + \|A_{22}^{-1}D_{22}\|^{k-1}\|G^{-1}\|\sqrt{\frac{c_1}{\lambda_{\min}(R)}} \\ + \gamma\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}}\sum_{i=0}^{k-1}\|A_{22}^{-1}D_{22}\|^i.\end{aligned}$$

Therefore, for all $t \in [0, T]$, we have

$$\|y_2(t)\| \leq \alpha_5 + \gamma\alpha_4\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}}. \tag{3.19}$$

Finally, taking (3.4), (3.16) and (3.19) into account, we obtain

$$\begin{aligned}x^\top(t)Rx(t) = y^\top(t)G^\top RGy(t) &\leq \lambda_{\max}(G^\top RG)(\|y_1(t)\|^2 + \|y_2(t)\|^2) \\ &\leq \lambda_{\max}(G^\top RG)\Big[\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})} + (\alpha_5 + \gamma\alpha_4\sqrt{\frac{e^{\eta T}\alpha_3}{\lambda_{\min}(R_{11})}})^2\Big] \\ &\leq c_2,\end{aligned}$$

which concludes the proof of the theorem.

Remark 3.1. Theorem 3.1 includes three conditions (3.2)–(3.4) guaranteeing the regularity, impulse-free and stability of the system. We use the first two LMI conditions (3.2), (3.3) to prove the existence of A_{22}^{-1} included in the condition (3.4), we then use additional condition (3.4) to show the stability of the system.

Remark 3.2. We note that the condition (3.4) is not a LMI with respect to η, since η appears in a nonlinear term. However, the conditions (3.2)–(3.3) are LMIs, so we first find the scalar η from LMIs (3.2)–(3.3), and then check the condition (3.4). If the problem is feasible, then it solves the finite-time stability problem of the system.

Remark 3.3. Note that Theorem 3.1 extends some existing results for linear regular systems ($E = I$) reported in [9,15]. In this case, the finite-time stability problem can be solved without using state-space decomposition method. Moreover, in Theorem 3.1 we construct the Lyapunov-like functionals different from the ones in [6,9,10] and then we estimate the derivative of $V(\cdot)$ by using a generalized Cauchy matrix inequality. This approach not only leads to less conservative LMI conditions, but also reduces numerical complexity. Moreover, the conditions are derived in terms of LMIs, which can be easily determined by utilizing MATLABs LMI Control Toolbox [14].

Example 3.1. Consider system (2.1) where

$$E = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0.2 & 0.3 \\ 1 & 1.8 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0.2 \\ 0.1 & 0.5 \end{pmatrix},$$
$$B = \begin{pmatrix} 0.1 & 0.2 \\ 0.2 & 0.5 \end{pmatrix}, \quad R = \begin{pmatrix} 0.9 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\psi(t) = [t^2, t^3], \quad \forall t \in [-0.4; 0], \quad d = 10^{-2}, \quad h = 0.4, \quad c_1 = 0.3, \quad c_2 = 60, \quad T = 10.$$

We have $\psi^\top(t)R\psi(t) = 0.9t^4 + t^6$, it is a decreasing function on $[-0.4; 0]$ and its maximal value is 0.271 at $t = -0.4$, implies $\psi^\top(t)R\psi(t) < c_1$ for all $t \in [-0.4; 0]$. By simple algebraic computation, we can find

$$M = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0.9 \\ -1 & -2 \end{pmatrix},$$
$$MAG = \begin{pmatrix} -0.1 & -0.4 \\ 0.7 & 2.2 \end{pmatrix}, \quad MDG = \begin{pmatrix} -0.2 & -0.4 \\ 0.2 & 0.5 \end{pmatrix}.$$

Using LMI toolbox of Matlab, the LMIs (3.2), (3.3) and the condition (3.4) are feasible with $\eta = 0.29$ and

$$P = \begin{pmatrix} 9.0239 & -3.0692 \\ 5.4689 & -2.4916 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1.5981 & 2.2658 \\ 2.2658 & 4.4685 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1.0990 \\ 0 & -0.4074 \end{pmatrix},$$
$$b = 0.0144, \quad P_{11} = 2.9773, \quad \alpha_1 = 1.567, \quad \alpha_2 = 4.1072,$$
$$\alpha_3 = 0.9139, \quad \alpha_4 = 1.2941, \quad \delta = 0.2273, \quad \alpha_5 = 0.2309, \quad \gamma = 0.2448.$$

By Theorem 3.1, the system is robustly finite-time stable w.r.t. $[0.3, 60, 10, R]$.

4. Conclusions

In this paper, the robust finite-time stability of linear differential-algebraic equations with delay has been studied. By introducing a simple set of Lyapunov–Krasovskii functionals, we have proposed new delay-dependent criteria for robust finite-time stability in terms of solutions of some matrix inequalities. A numerical example is given to illustrate the effectiveness of the result.

Acknowledgements

This work was supported by the National Foundation for Science and Technology Development, Viet Nam (grant No. 101.01.2014.35) and the joint research project from RFBR, grants 13-01-93002-Viet and VAST.HTQT.NGA-01/13-14. The authors would like to thank anonymous reviewers for their valuable comments and suggestions, which allowed us to improve the paper.

References

[1] L.F. Shampine, P. Gahinet, Delay-differential-algebraic equations in control theory, Appl. Numer. Math. 56 (2006) 574–588.
[2] G. Gu, V. Kharitonov, J. Chen, Stability of Time-Delay Systems, Birkhäuser, Berlin, 2003.
[3] P. Dorato, Short time stability in linear time-varying systems, in: Proc. IRE Int. Convention Record, Part 4, 1961, pp. 83–87.
[4] V.N. Phat, Y. Khongtham, K. Ratchagit, LMI approach to exponential stability of linear systems with interval time-varying delays, Linear Algebra Appl. 436 (2012) 243–251.
[5] S. Xu, P. Van Dooren, S. Radu, J. Lam, Robust stability and stabilization for singular systems with state delay and parameter uncertainty, IEEE Trans. Automat. Control 47 (2002) 1122–1128.
[6] A. Haidar, E.K. Boukas, Exponential stability of singular systems with multiple time-varying delays, Automatica 45 (2009) 539–545.
[7] E. Virnik, Stability analysis of positive descriptor systems, Linear Algebra Appl. 429 (2009) 2640–2659.
[8] F. Amato, M. Ariola, C. Cosentino, Finite-time stabilization via dynamic output feedback, Automatica 42 (2006) 337–342.
[9] E. Moulay, M. Dambrine, N. Yeganefar, W. Perruquetti, Finite-time stability and stabilization of time-delay systems, Systems Control Lett. 57 (2008) 561–566.
[10] F. Amato, R. Ambrosino, C. Cosentino, G. De Tommasi, Finite-time stabilization of impulsive dynamical linear systems, Nonlinear Anal. Hybrid Syst. 5 (2011) 89–101.
[11] S.L. Campbell, V.H. Linh, Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions, Appl. Math. Comput. 208 (2009) 397–415.
[12] L. Dai, Singular Control Systems, Lecture Notes in Control and Inform. Sci., Springer-Verlag, Berlin, 1989.
[13] S. Boyd, L. El Ghaoui, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.
[14] P. Gahinet, A. Nemirovskii, A.J. Laub, M. Chilali, LMI Control Toolbox for Use with MATLAB, The MathWorks, Inc., 1995.
[15] S.B. Stojanovic, D.Lj. Debeljkovic, D.S. Antic, Finite-time stability and stabilization of linear time-delay systems, Facta Univ. 11 (2012) 25–36.