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Controllability and observability of impulsive fractional linear time-invariant system☆

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ABSTRACT

In this paper, we deal with the controllability and observability of impulsive fractional linear time-invariant (IF-LTI for short) system. Our main purpose is to built some necessary and sufficient conditions of controllability and observability for the IF-LTI system. At the same time, we establish some conclusions of controllability and observability for a continuous fractional LTI system, which is a special case of the IF-LTI system. Examples are given to illustrate our results.

1. Introduction

Although most dynamical systems are analyzed in either the continuous- or discrete-time domain, many real systems in physics, chemistry, biology, engineering, and information science, may experience abrupt changes as certain instants during the continuous dynamical processes. This kind of impulsive behaviors can be modeled by impulsive systems. The study of impulsive system has seen a rapid development in the past few years. For the basic theory on impulsive system, the reader can refer to the monographs of Deo et al. [1], Bainov et al. [2], Sandberg [3] and Lakshmikantham et al. [4].

Recently, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in the fluid dynamic traffic model. Applications of fractional differential equations to different areas were considered by many authors and some basic results on fractional differential equations have been obtained; see for example, [5–10]. Actually, fractional differential equations are considered as an alternative model to integer differential equations. For more details on fractional calculus theory, one can see the monographs of Diethelm [11], Kilbas et al. [12], Lakshmikantham et al. [13], Miller and Ross [14], Michalski [15] and Tarasov [16]. Fractional differential equations involving the Riemann–Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attention (see for example [17–24]).

On the other hand, the study of controllability and observability plays an important role in the control theory and engineering [25,26]. In recent years, the study of impulsive control system has aroused great interest. For impulsive control system with integer derivative, Leela et al. [27] investigated the controllability of a class of time-invariant impulsive systems with the assumption that the impulses of impulsive control are regulated at discontinuous points. Lakshmikantham and Deo [28] made some further improvement over [27]. The controllability and observability also have been studied in [29,30]. Sufficient and necessary conditions for controllability and observability are established and their applications to time-invariant impulsive control systems are also discussed.

To our knowledge, the impulsive fractional control systems have not been studied very extensively. The controllability of continuous fractional dynamical systems have been investigated in [31], but its condition is difficult for computation. The

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observability, as we know, still has no paper to investigate it. Motivated by Balachandran et al. [31] and Fečkan et al. [32], in this paper, we consider the controllability and observability of the following IF-LTI systems

$$\begin{cases} {}^cD_{0,t}^q x(t) := {}^cD_t^q x(t) = Ax(t) + Bu(t), \quad t \in J' := J \setminus \{t_1, t_2, \ldots, t_k\},\ J := [0, T], \\ \Delta x(t_i) := x(t_i^+) - x(t_i^-) = I_i(t_i, x(t_i)), \\ y(t) = Cx(t) + Du(t), \\ x(0) = x_0, \end{cases} \tag{1}$$

and the following continuous fractional LTI system

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + Bu(t), \quad t \in J', \\ y(t) = Cx(t) + Du(t), \\ x(0) = x_0, \end{cases} \tag{2}$$

which can be seen as a special case of IF-LTI system (1) (in view of $I_i(t_i, x(t_i)) = 0$), where ${}^cD_t^q$ is the Caputo derivative, $0 < q < 1$, A, B, C and D are the known constant matrices, $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, $I_i : \Omega \to \mathbb{R}^n$, $\Omega \subset J \times \mathbb{R}^n$, $T < +\infty$, $i = 1, 2, \ldots, k$,

$$x(t_i^+) = \lim_{\epsilon \to 0^+} x(t_i + \epsilon)$$

$$x(t_i^-) = \lim_{\epsilon \to 0^-} x(t_i + \epsilon)$$

represent the right and left limits of $x(t)$ at $t = t_i$, and the discontinuity points

$$t_1 < t_2 < \cdots < t_i < \cdots < t_k,$$

where $0 = t_0 < t_1$, $t_k < t_{k+1} = T$ and $x(t_i) = x(t_i^-)$ which implies that the solution of system (1) is left continuous at t_i.

This paper is organized as follows. In Section 2, we give some notations and recall some concepts and preparation results. In Sections 3 and 4, we consider the controllability and observability of IF-LTI system (1) respectively. Some necessary and sufficient conditions of controllability and observability for system (1) are given. Especially, we give some conditions of controllability and observability for continuous fractional LTI system (2). At last, some examples are given to illustrate our results.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts. Throughout this paper, let $C(J, \mathbb{R}^n)$ be the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm $\|u\|_C := \sup\{\|u(t)\| : t \in J\}$ for $u \in C(J, \mathbb{R}^n)$. We also introduce the Banach space $PC(J, \mathbb{R}^n) = \{u : J \to \mathbb{R}^n | u \in C((t_k, t_{k+1}], \mathbb{R}^n), k = 0, \ldots, m$ and there exist $u(t_k^-)$ and $u(t_k^+)$, $k = 1, \ldots, m$, with $u(t_k^-) = u(t_k)\}$ with the norm $\|u\|_{PC} := \sup\{\|u(t)\| : t \in J\}$.

Let us recall the following known definitions. For more details, see [10,12].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0,\ \gamma > 0,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^LD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}} ds, \quad t > 0,\ n-1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ for a function $f : [0, \infty) \to \mathbb{R}$ can be written as

$${}^cD_t^\gamma f(t) = {}^LD_t^\gamma \left[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right], \quad t > 0,\ n-1 < \gamma < n.$$

Remark 2.4. (i) If $f(t) \in C^n[0, \infty)$, then

$${}^cD_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\gamma+1-n}} ds = I_t^{n-\gamma} f^{(n)}(t), \quad t > 0,\ n-1 < \gamma < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

Definition 2.5. The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{C},$$

where $\alpha>0$, $\beta>0$, $\mathbb{C}$ denotes the complex plane.

Remark 2.6. (i) For $\beta=1$,

$$E_{\alpha,1}(\lambda z^\alpha)=E_\alpha(\lambda z^\alpha)=\sum_{k=0}^{\infty}\frac{\lambda^k z^{k\alpha}}{\Gamma(\alpha k+1)},\quad \lambda,z\in\mathbb{C}.$$

(ii) For $\beta=1$, the matrix extension of the aforementioned Mittag-Leffler function has the following representation:

$$E_\alpha(At^\alpha)=\sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(\alpha k+1)},$$

with the property ${}^cD_t^\alpha E_\alpha(At^\alpha)=AE_\alpha(At^\alpha)$.

Definition 2.7. A function $x\in PC(J,\mathbb{R}^n)$ is said to be a solution of problem (1) if $x(t)=x_k(t)$ for $t\in(t_k,t_{k+1})$ and $x_k\in C([0,t_{k+1}],\mathbb{R}^n)$, $k=0,1,2,\ldots,m$ satisfies the equation ${}^cD_t^q x_k(t)=Ax_k(t)+Bu(t)$ a.e. on $(0,t_{k+1})$ with the restriction of $x_k(t)$ on $[0,t_k)$ is just $x_{k-1}(t)$, and the conditions $\Delta x(t_k)=I_k(t_k,x(t_k^-))$, $k=1,2,\ldots,m$, and $x(0)=x_0$.

We adopt the idea used in [33] and apply the Laplace transform for IF-LTI system (1); then the solution of (1) is given by

$$x(t)=\begin{cases}E_q(At^q)x_0+\displaystyle\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds, & t\in[0,t_1]\\ E_q(At^q)x_0+\displaystyle\sum_{j=1}^{i}E_q(A(t-t_j)^q)I_j(t_j,x(t_j)) & \\ \quad+\displaystyle\int_0^t(t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds, & t\in(t_i,t_{i+1}]\ i=1,2,\ldots,k.\end{cases}\tag{3}$$

3. Controllability

In this section, we discuss the controllability of IF-LTI system (1) and continuous fractional LTI system (2). At first, we give the definition of controllability for IF-LTI system (1) (continuous fractional LTI system (2)).

Definition 3.1. IF-LTI system (1) (continuous fractional LTI system (2)) is called state controllable on $[0,t_f]$ ($t_f\in(0,T]$) (or simply controllable if no confusion arises) if given any state $x_0,x_{t_f}\in\mathbb{R}^n$, there exists a control $u(t):[0,t_f]\to\mathbb{R}^m$ such that the corresponding solution of (1) satisfies $x(t_f)=x_{t_f}$.

Next, we give a necessary and sufficient condition of controllability for IF-LTI system (1).

Theorem 3.2. *IF-LTI system (1) is controllable on $[0,t_f]$ if and only if the controllability Gramian matrix*

$$W_c[0,t_f]=\int_0^{t_f}(t_f-s)^{q-1}[E_{q,q}(A(t_f-s)^q)B][E_{q,q}(A(t_f-s)^q)B]^*ds$$

is non-singular, for some $t_f\in(0,T]$. Here $*$ denotes the matrix transpose.

Proof. If $W_c[0,t_f]$ is non-singular, then its inverse is well-defined. For $t_f\in[0,t_1]$, define the control function as

$$u(t)=B^*E_{q,q}(A^*(t_f-t)^q)W_c^{-1}[0,t_f][x_{t_f}-E_q(A(t_f)^q)x_0].\tag{4}$$

Substituting $t=t_f$ in (3) and inserting (4), we have

$$\begin{aligned}x(t_f)&=E_q(At_f^q)x_0+\int_0^{t_f}(t_f-s)^{q-1}E_{q,q}(A(t_f-s)^q)BB^*E_{q,q}(A^*(t_f-s)^q)W_c^{-1}[0,t_f][x_{t_f}-E_q(A(t_f)^q)x_0]ds\\&=E_q(At_f^q)x_0+W_c[0,t_f]W_c^{-1}[0,t_f][x_{t_f}-E_q(A(t_f)^q)x_0]\\&=x_{t_f}.\end{aligned}$$

Thus (1) is controllable on $[0,t_f]$, $t_f\in[0,t_1]$.

For $t_f \in (t_1, t_2]$, define the control function as

$$u(t) = B^* E_{q,q}(A^*(t_f - t)^q) W_c^{-1}[0, t_f][x_{t_f} - E_q(A(t_f)^q)x_0 - E_q(A(t_f - t_1)^q) I_1(t_1, x(t_1))]. \tag{5}$$

Substituting $t = t_f$ in (3) and inserting (5), we have

$$\begin{aligned} x(t_f) &= E_q(At_f^q)x_0 + E_q(A(t_f - t_1)^q) I_1(t_1, x(t_1)) + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q) B \\ &\quad \times B^* E_{q,q}(A^*(t_f - s)^q) W_c^{-1}[0, t_f][x_{t_f} - E_q(A(t_f)^q)x_0 - E_q(A(t_f - t_1)^q) I_1(t_1, x(t_1))] ds \\ &= E_q(At_f^q)x_0 + E_q(A(t_f - t_1)^q) I_1(t_1, x(t_1)) \\ &\quad + W_c[0, t_f] W_c^{-1}[0, t_f][x_{t_f} - E_q(A(t_f)^q)x_0 - E_q(A(t_f - t_1)^q) I_1(t_1, x(t_1))] \\ &= x_{t_f}. \end{aligned}$$

Thus (1) is controllable on $[0, t_f]$, $t_f \in (t_1, t_2]$.

Moreover, for $t_f \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$, define the control function as

$$u(t) = B^* E_{q,q}(A^*(t_f - t)^q) W_c^{-1}[0, t_f] \left[x_{t_f} - E_q(A(t_f)^q)x_0 - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) \right]. \tag{6}$$

Substituting $t = t_f$ in (3) and inserting (6), we have

$$\begin{aligned} x(t_f) &= E_q(At_f^q)x_0 + \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q) B \\ &\quad \times B^* E_{q,q}(A^*(t_f - s)^q) W_c^{-1}[0, t_f] \left[x_{t_f} - E_q(A(t_f)^q)x_0 - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) \right] ds \\ &= E_q(At_f^q)x_0 + \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) \\ &\quad + W_c[0, t_f] W_c^{-1}[0, t_f] \left[x_{t_f} - E_q(A(t_f)^q)x_0 - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_i(t_j, x(t_j)) \right] \\ &= x_{t_f}. \end{aligned}$$

Thus IF-LTI system (1) is controllable on $[0, t_f]$.

On the other hand, if $W_c[0, t_f]$ is singular, without loss of generality, for $t_f \in (t_i, t_{i+1}]$, there exists a nonzero z such that

$$z^* W_c[0, t_f] z = 0,$$

that is,

$$\int_0^{t_f} z^*(t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q) B B^* E_{q,q}(A^*(t_f - s)^q) z ds = 0;$$

it yields

$$z^* E_{q,q}(A(t_f - s)^q) B = 0, \quad \text{on } s \in [0, t_f].$$

Let $x_0 = [E_q(At_f^q)]^{-1}[z - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j))]$. By the assumption, there exists an input u such that it steers x_0 to the origin in the interval $[0, t_f]$. It follows that

$$\begin{aligned} x(t_f) &= E_q(At_f^q)[E_q(At_f^q)]^{-1} \left[z - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) \right] \\ &\quad + \sum_{j=1}^{i} E_q(A(t_f - t_j)^q) I_j(t_j, x(t_j)) + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q) B u(s) ds \\ &= z + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q) B u(s) ds \\ &= 0. \end{aligned}$$

Then,

$$z^*z + \int_0^{t_f} (t_f - s)^{q-1} z^* E_{q,q}(A(t_f - s)^q)Bu(s)ds = 0.$$

The second term is zero, leading to the conclusion $z^*z = 0$. This contraction therefore completes the proof. □

Theorem 3.2 is a geometric type condition; by simple transformation, we can get an algebraic type condition.

Theorem 3.3. *IF-LTI system (1) is controllable on* $[0, t_f]$ *if and only if*

$$\text{rank}Q_c = \text{rank}(B \mid AB \mid \cdots \mid A^{n-1}B) = n.$$

Proof. With Cayley–Hamilton theorem, $t^{q-1}E_{q,q}(At^q)$ can be written in the following form:

$$\begin{aligned} t^{q-1}E_{q,q}(At^q) &= \sum_{k=0}^{\infty} \frac{t^{kq+q-1}}{\Gamma(kq+q)} A^k \\ &= \sum_{k=0}^{n-1} c_k(t)A^k. \end{aligned}$$

For $t_f \in [0, t_1]$,

$$x_{t_f} = E_q(At_f^q)x_0 + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q)Bu(s)ds.$$

Hence, we have

$$x_{t_f} - E_q(At_f^q)x_0 = \sum_{k=0}^{n-1} A^k B \int_0^{t_f} c_k(t_f - s)u(s)ds.$$

In matrix form, the above equation becomes

$$x_{t_f} - E_q(At_f^q)x_0 = (B \mid AB \mid \cdots \mid A^{n-1}B) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix},$$

where $d_k = \int_0^{t_f} c_k(t_f - s)u(s)ds$. Note that, since x_{t_f}, x_0 are arbitrary, to have a unique solution of $u(t)$, the necessary and sufficient condition is clearly that $\text{rank}Q_c = \text{rank}(B \mid AB \mid \cdots \mid A^{n-1}B) = n$.

For $t_f \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$,

$$x_{t_f} = E_q(At_f^q)x_0 + \sum_{j=1}^{i} E_q(A(t_f - t_j)^q)I_j(t_j, x(t_j)) + \int_0^{t_f} (t_f - s)^{q-1} E_{q,q}(A(t_f - s)^q)Bu(s)ds.$$

Hence, we have

$$x_{t_f} - E_q(At_f^q)x_0 - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q)I_j(t_j, x(t_j)) = \sum_{k=0}^{n-1} A^k B \int_0^{t_f} c_k(t_f - s)u(s)ds.$$

In matrix form, the above equation becomes

$$x_{t_f} - E_q(At_f^q)x_0 - \sum_{j=1}^{i} E_q(A(t_f - t_j)^q)I_j(t_j, x(t_j)) = (B \mid AB \mid \cdots \mid A^{n-1}B) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}.$$

Note that, since x_{t_f}, x_0 are arbitrary, to have a unique solution of $u(t)$, the necessary and sufficient condition is clearly that $\text{rank}Q_c = \text{rank}(B \mid AB \mid \cdots \mid A^{n-1}B) = n$. □

Corollary 3.4. *The continuous fractional LTI system (2) is controllable on* $[0, t_f]$ *if and only if the matrix*

$$\text{rank}Q_c = \text{rank}(B \mid AB \mid \cdots \mid A^{n-1}B) = n.$$

Proof. Note that system (2) is a special case of IF-LTI system (1) (in view of $I_i(t_i, x(t_i)) = 0$). Since the proof is standard, we omit it here. □

Remark 3.5. Corollary 3.4 is equivalent to the main result of [31], but the condition of Corollary 3.4 is more easy to calculate.

Remark 3.6. Our ideas of Theorems 3.2 and 3.3 can be adopted to the IF-LTI system with distributed delay in control, that is,

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + \int_{-h}^{0} d_\tau B_\tau(\tau, t)u(\tau, t), \quad t \in J' \\ \Delta x(t_i) := x(t_i^+) - x(t_i^-) = I_i(t_i, x(t_i)), \\ y(t) = Cx(t) + Du(t), \\ x(0) = x_0. \end{cases}$$

Remark 3.7. Our ideas of Theorems 3.2 and 3.3 also can be used to the IF-LTI system with independent delay in control, that is,

$$\begin{cases} {}^cD_t^q x(t) = Ax(t) + Bu(t) + B_1u(t-\tau), \quad t \in J' \\ \Delta x(t_i) := x(t_i^+) - x(t_i^-) = I_i(t_i, x(t_i)), \\ y(t) = Cx(t) + Du(t), \\ x(0) = x_0 \end{cases}$$

where $\tau > 0$ is a given constant.

4. Observability

In this section, we built some necessary and sufficient conditions of observability for IF-LTI system (1) and continuous fractional LTI system (2). At first, the definition of observability for IF-LTI system (continuous fractional LTI system) is given below.

Definition 4.1. IF-LTI system (1) (continuous fractional LTI system (2)) is called state observable on $[0, t_f](t_f \in (0, T])$ (or simply observable if no confusion arises) if any initial state $x(0) = x_0 \in \mathbb{R}^n$ is unique determined by the corresponding system input $u(t)$ and system output $y(t)$, for $t \in [0, t_f]$.

Theorem 4.2. *The continuous fractional LTI system (2) is observable on $[0, t_f]$ if and only if the observability Gramian matrix*

$$W_o[0, t_f] = \int_0^{t_f} E_q(A^*t^q)C^*CE_q(At^q)dt$$

is non-singular, for some $t_f > 0$.

Proof. The existence and uniqueness of solution for system (2) have been proved in [34]. The unique solution can be written as

$$x(t) = E_q(At^q)x_0 + \int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds.$$

Thus the output of system (2) has the following expression:

$$y(t) = CE_q(At^q)x_0 + C\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds + Du(t). \tag{7}$$

Let

$$\bar{y}(t) = y(t) - C\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds - Du(t);$$

then we have

$$\bar{y}(t) = CE_q(At^q)x_0.$$

It is obvious that the observability of system (2) is equivalent to the estimation of x_0 from $\bar{y}(t)$. Since $\bar{y}(t)$ and x_0 are arbitrary, this in turn is equivalent to the estimation of x_0 from $y(t)$ given by

$$y(t) = CE_q(At^q)x_0$$

as $u(t) \equiv 0$.

If $W_o[0, t_f]$ is non-singular, then $W_o^{-1}[0, t_f]$ is well-defined. Hence, for arbitrary $y(t)$, for $t_f > 0$, we can construct

$$\begin{aligned} W_o^{-1}[0, t_f]\int_0^{t_f} E_q(A^*t^q)C^*y(t)dt &= W_o^{-1}[0, t_f]\int_0^{t_f} E_q(A^*t^q)C^*CE_q(At^q)dtx_0 \\ &= W_o^{-1}[0, t_f]W_o[0, t_f]x_0 = x_0. \end{aligned} \tag{8}$$

The left side of (8) depends on $y(t)$, $t \in [0, t_f]$, and (8) is a linear algebraic equation of x_0. Since $W_o[0, t_f]$ is invertible, then the initial state $x(0) = x_0$ is unique determined by the corresponding system output $y(t)$, for $t \in [0, t_f]$.

On the other hand, if the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f > 0$, there exists a nonzero x_α such that

$$x_\alpha^* W_o[0, t_f] x_\alpha = 0.$$

Choose $x_0 = x_\alpha$; then we have

$$\begin{aligned}\int_0^{t_f} y^*(t)y(t)dt &= x_0^* \int_0^{t_f} E_q(A^*t^q)C^*CE_q(At^q)dtx_0 \\ &= x_\alpha^* W_o[0, t_f]x_\alpha = 0.\end{aligned}$$

Namely, $\int_0^{t_f} \|y(t)\|^2 dt = 0$, thus

$$0 = y(t) = CE_q(At^q)x_0,$$

which implies from Definition 4.1 that the fractional continuous LTI system (2) is not observable on $[0, t_f]$, $t_f > 0$. This contraction therefore completes the proof. □

Next, we give a necessary and sufficient condition of observability for IF-LTI system (1).

We need the following assumption.

[H1] : $I_i(t_i, x(t_i)) = a_i x(t_i)$, where a_i are given constants and $a_0 = 0$.

Briefly, denote by

$$\begin{aligned}I_i &= I_i(t_i, x(t_i)),\\ M(t) &= E_q(At^q),\\ M_i(t) &= C\Bigg[M(t) + \sum_{j=1}^{i-1} a_j M(t-t_j)M(t_j) + \sum_{1\le j<p\le i-1} a_j a_p M(t-t_p)M(t_p-t_j)M(t_j) \\ &\quad + \cdots + \left(\prod_{j=1}^{i-1} a_j\right) M(t-t_{i-1})M(t_{i-1}-t_{i-2})\ldots M(t_2-t_1)M(t_1)\Bigg], \quad i = 1, 2, \ldots, k+1.\end{aligned}$$

Theorem 4.3. *Assume that condition* [H1] *holds; then IF-LTI system* (1) *is observable on* $[0, t_f]$ *if and only if the observability Gramian matrix*

$$W_o[0, t_f] = \begin{cases} \displaystyle\int_0^{t_f} M_1^*(t)M_1(t)dt, & t_f \in [0, t_1],\\ \displaystyle\int_0^{t_1} M_1^*(t)M_1(t)dt + \sum_{j=1}^{i-1}\int_{t_j}^{t_{j+1}} M_{j+1}^*(t)M_{j+1}(t)dt & \\ \displaystyle\quad + \int_{t_i}^{t_f} M_{i+1}^*(t)M_{i+1}(t)dt, & t_f \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, k \end{cases}$$

is non-singular, for some $t_f \in (0, T]$.

Proof. It follows from (3) that the output of system (1) has the following expression:

$$y(t) = \begin{cases} CE_q(At^q)x_0 + C\displaystyle\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds + Du(t), & t \in [0, t_1]\\ CE_q(At^q)x_0 + C\displaystyle\sum_{j=1}^{i} E_q(A(t-t_j)^q)I_j(t_j, x(t_j)) & \\ \quad + C\displaystyle\int_0^t (t-s)^{q-1}E_{q,q}(A(t-s)^q)Bu(s)ds + Du(t), & t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, k. \end{cases} \tag{9}$$

Similar to the proof of Theorem 4.2, one can obtain that the observability of IF-LTI system (1) is equivalent to the estimation of x_0 from $y(t)$ given by

$$y(t) = \begin{cases} CE_q(At^q)x_0, & t \in [0, t_1],\\ CE_q(At^q)x_0 + C\displaystyle\sum_{j=1}^{i} E_q(A(t-t_j)^q)I_j(t_j, x(t_j)), & t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, k, \end{cases}$$

as $u(t) \equiv 0$.

If $W_o[0,t_f]$ is non-singular, then $W_o^{-1}[0,t_f]$ is well-defined. Hence, for arbitrary $y(t)$ and $t_f \in [0,t_1]$, we can construct

$$\begin{aligned} W_o^{-1}[0,t_f]\int_0^{t_f} M_1^*(t)y(t)dt &= W_o^{-1}[0,t_f]\int_0^{t_f} M_1^*(t)CM(t)dtx_0 \\ &= W_o^{-1}[0,t_f]\int_0^{t_f} M_1^*(t)M_1(t)dtx_0 \\ &= W_o^{-1}[0,t_f]W_o[0,t_f]x_0 = x_0. \end{aligned}$$

For $t_f \in (t_1,t_2]$,

$$\begin{aligned} &W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t) + \int_{t_1}^{t_f} M_2^*(t)\right\} y(t)dt \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dtx_0 + \int_{t_1}^{t_f} M_2^*(t)[CM(t)x_0 + CM(t-t_1)I_1]dt\right\} \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dt + \int_{t_1}^{t_f} M_2^*(t)[CM(t) + a_1CM(t-t_1)M(t_1)]dt\right\} x_0 \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)M_1(t)dt + \int_{t_1}^{t_f} M_2^*(t)M_2(t)dt\right\} x_0 \\ &= W_o^{-1}[0,t_f]W_o[0,t_f]x_0 = x_0. \end{aligned}$$

For $t_f \in (t_2,t_3]$,

$$\begin{aligned} &W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t) + \int_{t_1}^{t_2} M_2^*(t) + \int_{t_2}^{t_f} M_3^*(t)\right\} y(t)dt \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dtx_0 + \int_{t_1}^{t_2} M_2^*(t)[CM(t)x_0 + CM(t-t_1)I_1]dt\right. \\ &\quad \left. + \int_{t_2}^{t_f} M_3^*(t)[CM(t)x_0 + CM(t-t_1)I_1 + CM(t-t_2)I_2]dt\right\} \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dt + \int_{t_1}^{t_2} M_2^*(t)[CM(t) + a_1CM(t-t_1)M(t_1)]dt\right. \\ &\quad \left. + \int_{t_2}^{t_f} M_3^*(t)\left[CM(t) + \sum_{i=1}^{2} a_iCM(t-t_i)M(t_i) + a_1a_2CM(t-t_2)M(t_2-t_1)M(t_1)\right]dt\right\} x_0 \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)M_1(t)dt + \int_{t_1}^{t_2} M_2^*(t)M_2(t)dt + \int_{t_2}^{t_f} M_3^*(t)M_3(t)dt\right\} x_0 \\ &= W_o^{-1}[0,t_f]W_o[0,t_f]x_0 = x_0. \end{aligned}$$

Moreover, for $t \in (t_l,t_{l+1}]$, $l = 1,2,\ldots,k$,

$$\begin{aligned} &W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t) + \sum_{i=1}^{l-1}\int_{t_i}^{t_{i+1}} M_{i+1}^*(t) + \int_{t_l}^{t_f} M_{l+1}^*(t)\right\} y(t)dt \\ &= W_o^{-1}[0,t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dtx_0 + \sum_{i=1}^{l-1}\int_{t_i}^{t_{i+1}} M_{i+1}^*(t)\left[CM(t)x_0 + C\sum_{j=1}^{i} M(t-t_j)I_j\right]dt\right. \\ &\quad \left. + \int_{t_l}^{t_f} M_{l+1}^*(t)\left[CM(t)x_0 + C\sum_{j=1}^{l} M(t-t_j)I_j\right]dt\right\} \end{aligned}$$

$$
\begin{aligned}
&= W_o^{-1}[0, t_f]\left\{\int_0^{t_1} M_1^*(t)CM(t)dt + \sum_{i=1}^{l-1}\int_{t_i}^{t_{i+1}} M_{i+1}^*(t)C\left[M(t) + \sum_{j=1}^{i} a_jM(t-t_j)M(t_j)\right.\right.\\
&\quad \left. + \sum_{1\le j<k\le i} a_ja_kM(t-t_k)M(t_k-t_j)M(t_j) + \cdots + \left(\prod_{j=1}^{i} a_j\right) M(t-t_i)M(t_i-t_{i-1})\ldots M(t_2-t_1)M(t_1)\right]dt\\
&\quad + \int_{t_l}^{t_f} M_{l+1}^*(t)C\left[M(t) + \sum_{j=1}^{l} a_jM(t-t_j)M(t_j) + \sum_{1\le j<k\le l} a_ja_kM(t-t_k)M(t_k-t_j)M(t_j)\right.\\
&\quad \left.\left. + \cdots + \left(\prod_{j=1}^{l} a_j\right) M(t-t_l)M(t_l-t_{l-1})\ldots M(t_2-t_1)M(t_1)\right]dt\right\} x_0\\
&= W_o^{-1}[0, t_f]\left\{\int_0^{t_1} M_1^*(t)M_1(t)dt + \sum_{i=1}^{l-1}\int_{t_i}^{t_{i+1}} M_{i+1}^*(t)M_{i+1}(t)dt + \int_{t_l}^{t_f} M_{l+1}^*(t)M_{l+1}(t)dt\right\} x_0\\
&= W_o^{-1}[0, t_f]W_o[0, t_f]x_0 = x_0.
\end{aligned} \tag{10}
$$

The left side of (10) depends on $y(t)$, $t \in [0, t_f]$, and (10) is a linear algebraic equation of x_0. Since $W_o[0, t_f]$ is invertible, then the initial state $x(0) = x_0$ is unique determined by the corresponding system output $y(t)$, for $t \in [0, t_f]$.

Conversely, if the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f \in [0, t_1]$, there exists a nonzero x_α such that

$$x_\alpha^* W_o[0, t_f]x_\alpha = 0.$$

Choose $x_0 = x_\alpha$; then we have

$$
\begin{aligned}
\int_0^{t_f} y^*(t)y(t)dt &= x_0^*\int_0^{t_f} E_q(A^*t^q)C^*CE_q(At^q)dtx_0\\
&= x_0^*\int_0^{t_f} M_1^*(t)M_1(t)dtx_0\\
&= x_\alpha^* W_o[0, t_f]x_\alpha = 0.
\end{aligned}
$$

Namely, $\int_0^{t_f} \|y(t)\|^2dt = 0$, thus

$$0 = y(t) = CE_q(At^q)x_0 = M_1(t)x_0, \quad t \in [0, t_f]$$

which implies from Definition 4.1 that IF-LTI system (1) is not observable on $[0, t_f]$, $t_f \in [0, t_1]$. This contraction with system (1) is observable.

If the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f \in (t_1, t_2]$, there exists a nonzero x_α such that

$$x_\alpha^* W_o[0, t_f]x_\alpha = 0.$$

Choose $x_0 = x_\alpha$; then we have

$$
\begin{aligned}
\int_0^{t_f} y^*(t)y(t)dt &= x_0^*\int_0^{t_1} [CE_q(At^q)]^*CE_q(At^q)dtx_0\\
&\quad + \int_{t_1}^{t_f} [CE_q(At^q)x_0 + CE_q(A(t-t_1)^q)I_1]^*[CE_q(At^q)x_0 + CE_q(A(t-t_1)^q)I_1]dt\\
&= x_0^*\left\{\int_0^{t_1} [CM(t)]^*CM(t)dt\right.\\
&\quad \left. + \int_{t_1}^{t_f} [CM(t) + a_1CM(t-t_1)M(t_1)]^*[CM(t) + a_1CM(t-t_1)M(t_1)]dt\right\} x_0\\
&= x_0^*\left\{\int_0^{t_1} M_1^*(t)M_1(t)dt + \int_{t_1}^{t_f} M_2^*(t)M_2(t)dt\right\} x_0\\
&= x_\alpha^* W_o[0, t_f]x_\alpha = 0.
\end{aligned}
$$

Namely, $\int_0^{t_f} \|y(t)\|^2 dt = 0$, thus

$$0 = y(t) = \begin{cases} CE_q(At^q)x_0 = M_1(t)x_0, & t \in [0, t_1], \\ CE_q(At^q)x_0 + a_1 CE_q(A(t-t_1)^q)E_q(At_1^q)x_0 = M_2(t)x_0, & t \in (t_1, t_f], \end{cases}$$

which implies from Definition 4.1 that IF-LTI system (1) is not observable on $[0, t_f]$, $t_f \in (t_1, t_2]$. This contraction with system (1) is observable.

Similarly, if the Gramian matrix $W_o[0, t_f]$ is singular for some $t_f \in (t_l, t_{l+1}]$, there exists a nonzero x_α such that

$$x_\alpha^* W_o[0, t_f]x_\alpha = 0.$$

Choose $x_0 = x_\alpha$; then we have

$$\begin{aligned} \int_0^{t_f} y^*(t)y(t)dt &= \int_0^{t_1} [CE_q(At^q)x_0]^*[CE_q(At^q)x_0]dt \\ &\quad + \sum_{i=1}^{l-1} \int_{t_i}^{t_{i+1}} \left[CE_q(At^q)x_0 + C\sum_{j=1}^{i} E_q(A(t-t_j)^q)I_j \right]^* \left[CE_q(At^q)x_0 + C\sum_{j=1}^{i} E_q(A(t-t_j)^q)I_j \right] dt \\ &\quad + \int_{t_l}^{t_f} \left[CE_q(At^q)x_0 + C\sum_{j=1}^{l} E_q(A(t-t_j)^q)I_j \right]^* \left[CE_q(At^q)x_0 + C\sum_{j=1}^{l} E_q(A(t-t_j)^q)I_j \right] dt \\ &= x_0^* \left\{ \int_0^{t_1} M_1^*(t)M_1(t)dt + \sum_{i=1}^{l-1} \int_{t_i}^{t_{i+1}} M_{i+1}^*(t)M_{i+1}(t)dt + \int_{t_l}^{t_f} M_{l+1}^*(t)M_{l+1}(t)dt \right\} x_0 \\ &= x_\alpha^* W_o[0, t_f]x_\alpha = 0. \end{aligned}$$

Namely, $\int_0^{t_f} \|y(t)\|^2 dt = 0$, thus

$$0 = y(t) = \begin{cases} CE_q(At^q)x_0 = M_1(t)x_0, & t \in [0, t_1], \\ CE_q(At^q)x_0 + CE_q(A(t-t_1)^q)I_1 = M_2(t)x_0, & t \in (t_1, t_2], \\ \vdots \\ CE_q(At^q)x_0 + C\sum_{j=1}^{l} E_q(A(t-t_j)^q)I_j = M_{l+1}(t)x_0, & t \in (t_l, t_f] \end{cases}$$

which implies from Definition 4.1 that IF-LTI system (1) is not observable on $[0, t_f]$, $t_f \in (t_l, t_{l+1}]$. This contraction therefore completes the proof. □

Theorem 4.3 is a geometric type condition; by simple transformation, we can get an algebraic type condition.

Theorem 4.4. *Assume that condition* [H1] *holds; then IF-LTI system* (1) *is observable on* $[0, t_f]$ *if and only if*

$$\text{rank}Q_o = \text{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

Proof. From the proof of Theorem 4.3, one can obtain

$$y(t) = \begin{cases} M_1(t)x_0, & t \in [0, t_1], \\ M_{i+1}(t)x_0, & t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, k \end{cases}$$

x_0 is unique determined by $y(t)$ if and only if $M_i(t)$, $i = 1, 2, \ldots, k+1$ is nonsingular.

For $t_f \in [0, t_1]$, with Cayley–Hamilton theorem, $M_1(t_f)$ can be written in the following form:

$$\begin{aligned} M_1(t_f) &= CE_q(At_f^q) = C\sum_{k=0}^{\infty} \frac{A^k t_f^{kq}}{\Gamma(kq+1)} = C\sum_{k=0}^{n-1} A^k \beta_k(t_f) \\ &= (\beta_0(t_f), \beta_1(t_f), \ldots, \beta_{n-1}(t_f)) \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}. \end{aligned}$$

$M_1(t_f)$ is nonsingular if and only if $\mathrm{rank}Q_o = n$. By Definition 4.1, IF-LTI system (1) is observable on $[0, t_1]$ if and only if $\mathrm{rank}Q_o = n$.

For $t_f \in (t_l, t_{l+1}]$, with Cayley–Hamilton theorem, $M_{l+1}(t_f)$ can be written in the following form:

$$\begin{aligned} M_{l+1}(t_f) &= C\Bigg[M(t_f) + \sum_{j=1}^{l} a_j M(t_f - t_j)M(t_j) + \sum_{1\le j<p\le l} a_j a_p M(t_f - t_p)M(t_p - t_j)M(t_j) \\ &\quad + \cdots + \left(\prod_{j=1}^{l} a_j\right) M(t_f - t_l)M(t_l - t_{l-1})\ldots M(t_2 - t_1)M(t_1)\Bigg] \\ &= (\overline{\beta}_0(t_f), \overline{\beta}_1(t_f), \ldots, \overline{\beta}_{n-1}(t_f)) \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}. \end{aligned}$$

$M_{l+1}(t_f)$ is nonsingular if and only if $\mathrm{rank}Q_o = n$. By Definition 4.1, IF-LTI system (1) is observable on $[0, t_f]$, $t_f \in (t_l, t_{l+1}]$ if and only if $\mathrm{rank}Q_o = n$. □

Corollary 4.5. *The continuous fractional LTI system* (2) *is observable on* $[0, t_f]$, $t_f > 0$ *if and only if*

$$\mathrm{rank}Q_o = \mathrm{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n.$$

5. Illustrate examples

In this section, we give two examples to demonstrate how to utilize our results.

Example 1. Consider the following 3-dimensional IF-LTI system

$$\begin{cases} {}^cD_t^{\frac{1}{2}}x(t) = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} u(t), \quad t \in [0, 4] \setminus \{1, 2, 3\}, \\ \Delta x(t_i) = \dfrac{1}{4} x(t_i^-), \quad t_i = i,\ i = 1, 2, 3, \\ x(0) = 0. \end{cases} \tag{11}$$

Now, we try to use our criteria to investigate the controllability on $[0, 4]$ of system (11). Denote by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then, one can obtain

$$\begin{aligned} \mathrm{rank}Q_c &= \mathrm{rank}(B \mid AB \mid \cdots \mid A^{n-1}B) \\ &= \mathrm{rank}\begin{pmatrix} 1 & 0 & 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 4 & 2 \end{pmatrix} = 3. \end{aligned}$$

By Theorem 3.3, IF-LTI system (11) is controllable on $[0, 4]$.

Example 2. Consider the following 3-dimensional IF-LTI system

$$\begin{cases} {}^cD_t^{q}x(t) = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix} x(t), \quad t \in [0, 5] \setminus \{1, 2, 3, 4\}, \\ \Delta x(t_i) = \dfrac{1}{5} x(t_i^-), \quad t_i = i,\ i = 1, 2, 3, 4, \\ y(t) = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} x(t) \\ x(0) = 0. \end{cases} \tag{12}$$

System (12) satisfies condition [H1] in the case of $a_i = \frac{1}{5}$. Now, we try to use our criteria to investigate the observability on $[0, 5]$ of system (12). Denote by

$$A = \begin{pmatrix} -1 & -4 & -2 \\ 0 & 6 & 1 \\ 1 & 7 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix};$$

then, one can obtain

$$\begin{aligned} \mathrm{rank} Q_o &= \mathrm{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \\ &= \mathrm{rank} \begin{pmatrix} 0 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 19 & -3 \\ * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = 3. \end{aligned}$$

By Theorem 4.4, IF-LTI system (12) is observable on $[0, 5]$.

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