SIAM J. MATH. ANAL.
Vol. 25, No. 2, pp. 720–732, March 1994

030

DISCRETE AND CONTINUOUS LIOUVILLE–GREEN–OLVER APPROXIMATIONS: A UNIFIED TREATMENT VIA VOLTERRA–STIELTJES INTEGRAL EQUATIONS*

RENATO SPIGLER† AND MARCO VIANELLO‡

Abstract. A unified treatment of the Liouville–Green–Olver approximation theory for linear second-order differential *and* difference equations is presented. This is based on reduction to Volterra–Stieltjes integral equations with respect to *complex* measures. The present approach embodies and improves several previous results. Moreover, error bounds are obtained for *recessive* solutions of certain difference equations, for which only qualitative results were known. The theory can be applied, for instance, to the asymptotics of certain families of orthogonal polynomials.

Key words. Volterra–Stieltjes integral equations, complex-valued measures, Liouville–Green (WKBJ) approximation, linear difference equations, linear differential equations

AMS subject classifications. 39A12, 34E20, 45D05, 28A10

1. Introduction.

The well-known and widely used Liouville–Green (or WKBJ) approximations for solutions to linear second-order *differential* equations, to which F. W. J. Olver has been able to associate precise estimates for the error terms since 1961 [10], have been recently extended to the case of *difference* equations [4], [12], [13]. Olver's analysis was based on integral equations of the Volterra type satisfied by the relevant error terms. In this paper we proceed similarly, using Volterra integral equations with respect to *complex* Lebesgue–Stieltjes measures to treat both differential equations (absolutely continuous measures) *and* difference equations (discrete measures) at the same time.

As an application, *error bounds* are obtained for *recessive* solutions to a class of linear difference equations of the second order. Only qualitative results were known for this class. Our results can be applied, for instance, to the asymptotics of certain orthogonal polynomials off their essential spectrum.

The paper is organized as follows. In §2 we prove some theorems on existence, uniqueness, and estimates for the solutions of certain types of Volterra–Stieltjes integral equations with respect to complex-valued measures. In §3 these results are related to the Liouville–Green approximation for the solutions of both differential *and* difference equations and several examples and applications are presented.

2. The main theorems.

In this section we prove some theorems yielding existence, uniqueness, and estimates for the solutions to certain linear Volterra integral equations with respect to Lebesgue–Stieltjes complex-valued measures (for generalities on complex-valued measures, we refer the reader to [16, Chap. 11]). These results will be related to the asymptotic theory of *differential* as well as *difference* equations. Throughout the paper we stipulate, for convenience, that

$$\int_x^{+\infty} f\,d\mu \equiv \int_{[x,+\infty)} f\,d\mu$$

* Received by the editors May 18, 1992; accepted for publication (in revised form) April 13, 1993. This work was supported by MURST Mathematical Analysis funds, MURST Numerical Analysis funds, the GNFM-CNR, and the Istituto Nazionale di Alta Matematica F. Severi.

† Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Padova, via Belzoni 7, 35131, Padova, Italy.

‡ Dipartimento di Matematica Pura e Applicata, Università di Padova, via Belzoni 7, 35131, Padova, Italy.

for the general (complex) measure μ.

THEOREM 2.1. *Consider the linear integral equation*

$$\varepsilon(x) = \int_x^{+\infty} K(x,t)[\phi(x,t) + \varepsilon(t)]d\mu, \qquad x \in [a, +\infty), \tag{1}$$

of the Volterra type, where $a \in (-\infty, +\infty)$ and μ is a complex (finite) measure. Suppose that for each fixed $x \in [a, +\infty)$

(i) *$K(x,\cdot)$, $\phi(x,\cdot)$ are μ-measurable complex-valued functions;*

(ii) *$|K(x,t)|$, $|K(x,t)\phi(x,t)| \le h(x,t)$ $\quad |\mu|$—almost everywhere for $t \ge x$, where $h(x,\cdot) \in L^1([x,+\infty);\mu)$, and, moreover,*

$$V(x) := \int_x^{+\infty} h(x,t)d|\mu| \tag{2}$$

is nonincreasing and $\lim_{x\to+\infty} V(x) < 1$.

Then there exists a unique solution $\varepsilon(x)$ of (1) for $x > x_1$, where

$$x_1 := \inf\{x : x \ge a,\ V(x) < 1\}, \tag{3}$$

and the estimate

$$|\varepsilon(x)| \le \frac{V(x)}{1 - V(x)}, \qquad x > x_1 \tag{4}$$

holds.

Proof. Consider the sequence

$$\begin{aligned} &\varepsilon_0(x) \equiv 0, \\ &\varepsilon_{s+1}(x) = \int_x^{+\infty} K(x,t)[\phi(x,t) + \varepsilon_s(t)]d\mu, \qquad s = 0,1,2,\ldots, \end{aligned} \tag{5}$$

that is well defined since it is easily proved, by induction on s, that

$$|\varepsilon_s(x)| \le C_s, \quad C_0 = 0, \quad C_s = (C_{s-1}+1)V(a), \quad s = 1,2,3,\ldots. \tag{6}$$

Define, formally,

$$\varepsilon(x) := \sum_{s=0}^{\infty} [\varepsilon_{s+1}(x) - \varepsilon_s(x)]. \tag{7}$$

Now,

$$|\varepsilon_1(x)| \le V(x) \tag{8}$$

holds, and, assuming as an inductive hypothesis that

$$|\varepsilon_s(x) - \varepsilon_{s-1}(x)| \le [V(x)]^s, \tag{9}$$

we obtain

$$|\varepsilon_{s+1}(x) - \varepsilon_s(x)| \le \int_x^{+\infty} |K(x,t)||[V(t)]^s d|\mu| \le [V(x)]^s \int_x^{+\infty} |K(x,t)|d|\mu|$$

$$\leq [V(x)]^s \int_x^{+\infty} h(x,t)d|\mu| = [V(x)]^{s+1}. \tag{10}$$

From (8), (9) then (4) follows, where x_1 is given in (3). The estimates (8) and (9) also show that the series in (7) converges uniformly with respect to x for $x \geq \xi$ and for every fixed $\xi > x_1$. Therefore,

$$\begin{aligned} \varepsilon(x) &= \varepsilon_1(x) + \sum_{s=1}^{\infty} \int_x^{+\infty} K(x,t)[\varepsilon_s(t) - \varepsilon_{s-1}(t)]d\mu \\ &= \varepsilon_1(x) + \int_x^{+\infty} K(x,t) \sum_{s=1}^{\infty} [\varepsilon_s(t) - \varepsilon_{s-1}(t)]d\mu, \end{aligned} \tag{11}$$

and hence $\varepsilon(x)$ given by (7) solves (1) for $x > x_1$. It is finally immediately proved that uniqueness holds for $x \geq \xi$, ξ being any fixed number with $\xi > x_1$. □

Remark 2.2. When $V(x_1) < 1$, i.e., "inf" can be replaced by "min" in (3), all results hold up to and include x_1. When h in (2) is *continuous* as a function of x *and* μ is *absolutely continuous*, then if $x_1 > a$, it follows that $V(x_1) = 1$ (and thus, if $V(x_1) < 1$, it is necessary that $x_1 = a$), $V(x)$ being a *continuous* function in this case.

When μ is absolutely continuous, a variant of Theorem 2.1 that has some interest for differential equations can be proved.

THEOREM 2.3. *Suppose that in Theorem* 2.1 μ *is absolutely continuous with density* $p(x)$ *and condition* (ii) *is replaced by*

(ii′) $|K(x,t)|$, $|K(x,t)\phi(x,t)| \leq M_0(t)N_0(x)$ *almost everywhere for* $t \geq x$, *where* $M_0 \in L^1((a,+\infty);\mu)$, *and* N_0 *is* nondecreasing *for* $x \geq a$.

Then, if we set

$$U_0(x) := \int_x^{+\infty} M_0(t)|p(t)|dt, \tag{12}$$

there exists a unique bounded solution $\varepsilon(x)$ *of* (1) *for* $x \geq a$, *with*

$$|\varepsilon(x)| \leq \exp\{N_0(x)U_0(x)\} - 1. \tag{13}$$

Proof. The proof follows the lines of the previous one, the only difference being that (9) and (10) are replaced with

$$|\varepsilon_s(x) - \varepsilon_{s-1}(x)| \leq \frac{[N_0(x)U_0(x)]^s}{s!}, \qquad s = 1, 2, \ldots, \tag{14}$$

and

$$\begin{aligned} |\varepsilon_{s+1}(x) - \varepsilon_s(x)| &\leq N_0(x) \int_x^{+\infty} M_0(t) \frac{[N_0(t)U_0(t)]^s}{s!} |p(t)|dt \\ &\leq -[N_0(x)]^{s+1} \int_x^{+\infty} \frac{[U_0(t)]^s}{s!} U_0'(t)dt = \frac{[N_0(x)U_0(x)]^{s+1}}{(s+1)!}, \end{aligned} \tag{15}$$

where (12) has been used. Therefore, (7) leads to the exponential estimate in (13). Moreover, there is a *unique* solution to (1). In fact, suppose that $\varepsilon(x)$ and $\eta(x)$ are two such solutions. Then, if we define $\delta(x) := \varepsilon(x) - \eta(x)$, there is a constant C such that $|\delta(x)| \le C$ for $x \ge a$, and $\delta(x)$ can be estimated as

$$
\begin{aligned}
|\delta(x)| &\le \int_x^{+\infty} |K(x,t)|\,|\delta(t)|\,|p(t)|dt \\
&\le CN_0(x) \int_x^{+\infty} M_0(t)|p(t)|dt = CN_0(x)U_0(x),
\end{aligned} \tag{16}
$$

and again, by successive resubstitutions (see [11, p. 141]),

$$
|\delta(x)| \le C\frac{[N_0(x)U_0(x)]^s}{s!} \quad \text{for } s \ge 1 \text{ and } x \ge a, \tag{17}
$$

and hence $\delta(x) \equiv 0$ for $x \ge a$. □

3. Applications to differential and difference equations.

As is well known, the measure μ appearing in Theorem 2.1, being finite and complex-valued on $\mathbf{R}$ (and thus Lebesgue–Stieltjes), can be represented as the sum of *three* (complex) measures, the first being *absolutely continuous*, the second being *discrete*, and the third being *singular*. When μ reduces merely to the first one, one is led to integral equations that can be related to linear *differential equations*; when μ reduces to the second one, one is led to integral equations that can be related to linear *difference equations*. In this section we analyze these two special cases in detail.

3.1. Differential equations.

In this case some regularity results are needed for the solution $\varepsilon(x)$. These can be estabilished by requiring some additional properties on $K(x,t)$, $\phi(x,t)$ and on the density of μ.

THEOREM 3.1. *Suppose that* (1) *is given, the measure μ being absolutely continuous with density $p(x)$. Moreover, assume that for $r \in \mathbf{N}$*

(i) *K, $\phi \in C^r(T)$, T being the sector $x \in [a, +\infty)$, $t \ge x$;*

(ii) *$p \in C^{r-1}([a,+\infty))$ when $r > 0$;*

(iii) *there exist $2r+2$ nonnegative functions $M_j(t)$, $N_j(x)$, $j = 0, 1, \dots, r$, with $M_j \in L^1((a,+\infty);\mu)$, $N_j \in C^0([a,+\infty))$, and*

$$
\left|\frac{\partial^j K}{\partial x^j}\right|, \quad \left|\frac{\partial^j (K\phi)}{\partial x^j}\right| \le M_j(t)N_j(x), \quad j = 0, 1, \dots, r, \quad \text{a.e. for } t \ge x. \tag{18}
$$

Then the solution $\varepsilon(x)$ to equation (1) *is of class $C^r([a,+\infty))$.*

Proof. We notice first that (18) with $j = 0$ ensures that there is *a unique bounded* solution to (1) by Theorem 2.3.

It is then easy to derive from (1) the representation

$$
\begin{aligned}
\varepsilon^{(r)}(x) &= \int_x^{+\infty} \left[\frac{\partial^r (K\phi)}{\partial x^r}(x,t) + \frac{\partial^r K}{\partial x^r}(x,t)\varepsilon(t)\right] p(t)dt \\
&\quad - \sum_{j=0}^{r-1} \frac{d^j}{dx^j}\left\{\left[\frac{\partial^{r-1-j}(K\phi)}{\partial x^{r-1-j}}(x,x) + \varepsilon(x)\frac{\partial^{r-1-j} K}{\partial x^{r-1-j}}(x,x)\right] p(x)\right\}
\end{aligned} \tag{19}
$$

for the rth derivative of $\varepsilon(x)$. The proof of this can be based essentially on legitimate differentiations under the sign of integral, in view of the dominated convergence theorem, which also shows that $\varepsilon \in C^r$. Details are left to the reader. □

Remark 3.2. Observe that Theorem 3.1 for $r = 0$ ensures uniqueness of solutions to (1) *without* requiring their boundedness a priori. In fact, all solutions would be continuous and thus locally bounded while uniqueness holds in a neighborhood of $+\infty$ since by (12) the integral operator in (1) is a contraction for x sufficiently large.

The following variant of Theorem 3.1 may be useful.

THEOREM 3.3. *Theorem* 3.1 *holds if assumption* (ii) *is replaced by*

$$\text{(ii}')\qquad \frac{\partial^j K}{\partial x^j}(x,x) \equiv 0 \quad \textit{for } j = 1,2,\ldots,r,$$

when $r > 0$. Moreover, if $N_0(x)$ is also nondecreasing for $x \geq a$, the following estimates hold for the derivatives of $\varepsilon(x)$:

$$|\varepsilon^{(j)}(x)| \leq N_j(x)U_j(x)\exp\{N_0(x)U_0(x)\}, \qquad x \geq a, \tag{20}$$

$$U_j(x) := \int_x^{+\infty} M_j(t)|p(t)|dt, \qquad j = 1,2,\ldots,r. \tag{21}$$

Proof. The first part of the theorem can be proved immediately by observing that (19) still holds true with all terms in the sum equal to zero. As for the second part, similarly to (19), from (5) the following is obtained:

$$\begin{gathered} \varepsilon_0(x) \equiv 0, \\ \varepsilon_{s+1}^{(j)}(x) = \int_x^{+\infty} \left[\frac{\partial^j (K\phi)}{\partial x^j}(x,t) + \frac{\partial^j K}{\partial x^j}(x,t)\varepsilon_s(t)\right] p(t)dt, \qquad s = 0,1,2,\ldots. \end{gathered} \tag{22}$$

Therefore, by using the monotonicity of $U_0(x)$ (and of $N_0(x)$) it can be proved by induction on s and (14) that

$$|\varepsilon_{s+1}^{(j)}(x) - \varepsilon_s^{(j)}(x)| \leq N_j(x)U_j(x)\frac{[N_0(x)U_0(x)]^s}{s!} \quad \text{for } s = 0,1,2,\ldots, \tag{23}$$

and thus (20) follows. In fact, exchanging series and derivatives in (7) is permissible because (23) shows the *uniform* convergence of $\sum_{s=0}^{\infty}[\varepsilon_{s+1}^{(j)}(x) - \varepsilon_s^{(j)}(x)]$ for each j since $N_0(x)U_0(x) \leq N_0(a)U_0(a)$ for $x \geq a$. □

Moreover, we have the following theorem.

THEOREM 3.4. *Under all hypotheses of Theorem* 3.1, *with condition* (ii$'$) *replacing* (ii), *and the additional estimates*

$$\left|\frac{\partial^j K}{\partial x^j}\right|, \left|\frac{\partial^j (K\phi)}{\partial x^j}\right| \leq P_j(x,t), \qquad j = 1,2,\ldots,r, \quad \textit{a.e. for } t \geq x, \tag{24}$$

where $P_j(x,\cdot) \in L^1((x,+\infty);\mu)$ and

$$W_j(x) := \int_x^{+\infty} P_j(x,t)|p(t)|dt \tag{25}$$

is nonincreasing, then

(I) *if the assumptions of Theorem* 2.1 *are also satisfied, we obtain*

$$|\varepsilon^{(j)}(x)| \leq \frac{W_j(x)}{1 - V(x)} \quad \textit{for } x > x_1, \tag{26}$$

$V(x)$ being defined in (2) *and x_1 in* (3), *whereas,*

(II) *if $N_0(x)$ (see Theorem* 3.1) *is nonincreasing, we obtain*

$$|\varepsilon^{(j)}(x)| \le W_j(x) \exp\{V(x)\} \quad \textit{for } x \ge a. \tag{27}$$

Proof. The proof is similar to that of Theorem 3.3, with inequality (23) being replaced by

$$|\varepsilon^{(j)}_{s+1}(x) - \varepsilon^{(j)}_s(x)| \le W_j(x)[V(x)]^s, \qquad s = 0, 1, 2, \dots \tag{28}$$

in case (I) and by

$$|\varepsilon^{(j)}_{s+1}(x) - \varepsilon^{(j)}_s(x)| \le W_j(x)\frac{[V(x)]^s}{s!}, \qquad s = 0, 1, 2, \dots \tag{29}$$

in case (II). □

Theorems 3.1, 3.3, and 3.4 turn out to be useful in the framework of linear differential equations. Here we consider linear second-order differential equations like

$$y'' + [f(x) + g(x)]y = 0, \qquad x \ge 1, \quad f, g \in C^0([1, +\infty)), \tag{30}$$

where f is real valued, g is complex valued, and

(a) $f(x) \ne 0$ in $[1, +\infty)$, $f \in C^2$, and

$$\mathcal{V}(x) = \int_x^{+\infty} |f^{-1/4}(f^{-1/4})'' - gf^{-1/2}|dt < \infty, \tag{31}$$

or

(b) $f(x) \equiv 0$ in $[1, +\infty)$ and

$$\mathcal{M}_k(x) = \int_x^{+\infty} t^k|g(t)|dt < \infty \quad \text{for } k = 1 \text{ or } 2. \tag{32}$$

Case (a) is the typical case considered by F. W. J. Olver in connection with the Liouville–Green approximation theory (see [11, Chap. 6]). If $f(x) > 0$ (oscillatory case), when we look for solutions to (30) of the form

$$y_j(x) = f^{-1/4}(x) \exp\left\{(-1)^j i \int^x f^{1/2}(t)dt\right\}[1 + \varepsilon_j(x)], \qquad j = 1, 2, \tag{33}$$

it turns out that the error term, $\varepsilon_j(x)$, must be a C^2-solution to an integral equation like (1), with $K(x,t) = (1/2i)[1 - \exp\{(-1)^j 2i \int_x^t f^{1/2}(s)ds\}]$, $\phi(x,t) \equiv 1$, $d\mu = [f^{-1/4}(f^{-1/4})'' - gf^{-1/2}]dt$. Choosing in (ii′) of Theorem 2.3, $M_0(t) \equiv 1$, $N_0(x) \equiv 1$, one gets $U_0(x) = \mathcal{V}(x)$ and hence Olver's result [11, Thm. 2.1, Chap. 6],

$$|\varepsilon_j(x)| \le \exp\{\mathcal{V}(x)\} - 1, \tag{34}$$

holds. As for the derivatives, one can choose, in Theorem 3.3, $M_1(t) \equiv 1$, $N_1(x) = f^{1/2}(x)$, and thus $U_1(x) = U_0(x)$ and $|\varepsilon_j'(x)| \le f^{1/2}(x)U_0(x)\exp\{U_0(x)\}$. The latter implies that $f^{-1/2}(x)\varepsilon_j'(x) = O(U_0(x))$, as in Olver's theorem [11, Thm. 2.1, Chap. 6].

Alternatively, one could apply Theorem 2.1 (and Theorem 3.4). In this case the estimates hold, in general, only for $x > x_1$ (x_1 being defined in (3)). However, $\varepsilon_j(x) = O(V(x))$ and $V(x) \le U_0(x)$ when $h = |K| = |K\phi| = |\sin(\int_x^t f^{1/2}(s)ds)|$ is chosen. Moreover, the geometric estimate (4) *may* provide better estimates for fixed x, with respect to certain parameters. This fact can be related to the *double asymptotic* nature of the Liouville–Green approximations with respect to both the independent variable and the parameters entering $f+g$. Here is a simple example, for the purpose of illustration, in which all calculations can be carried out explicitly. Let $f(x) \equiv 1$, $g(x) = x^{-\gamma}$, $\gamma > 2$ on $x \geq 1$. Then $|K| = |K\phi| = |\sin(t-x)|$ and $|p| = x^{-\gamma}$. Choosing $h(x,t) = t-x$ and $M_0(t) \equiv 1$, $N_0(x) \equiv 1$, we get $V(x) = x^{2-\gamma}/(\gamma-1)(\gamma-2)$ and $U_0(x) = x^{1-\gamma}/(\gamma-1)$. Now, the fact that the geometric estimate (4) yields an error of order of $O(\gamma^{-2})$ for fixed x while the exponential estimate (13) gives an order of $O(\gamma^{-1})$ suggests that the former may be better. Indeed, if we compare the two estimates for x such that, e.g., $V(x) \leq \frac{1}{2}$, clearly $V/(1-V) \leq 2V < U_0 < \exp\{U_0\}-1$ as long as $[2/(\gamma-1)(\gamma-2)]^{1/(\gamma-2)} \leq x < \gamma/2 - 1$, and thus the geometric estimate performs better. Case (a) with $f(x) < 0$ (nonoscillatory case) can be handled in a similar way, and Olver's results are recovered again.

Case (b) has been studied, for instance, in [6], [13]. If (32) holds for $k=1$, when one looks for a (recessive) solution of the form

$$y_1(x) = 1 + \varepsilon_1(x), \tag{35}$$

$\varepsilon_1(x)$ turns out to be a C^2-solution of (1) with $K(x,t) = t-x$, $\phi(x,t) \equiv 1$, $d\mu = g\,dt$. In this case it is known that a second (dominant) solution $y_2(x) \sim x$ as $x \to +\infty$ exists. If (18) holds for $k=2$, in addition, when one looks for a (dominant) solution like

$$y_2(x) = x + \varepsilon_2(x), \tag{36}$$

one finds that $\varepsilon_2(x)$ must be a C^2-solution of (1) with $K(x,t) = t-x$, $\phi(x,t) = t$, and $d\mu$ as before. As in case (a), choosing in (ii′) of Theorem 2.3 $M_0(t) = t^k$, $N_0(x) \equiv 1$ (see [6]), we obtain $U_0(x) = \mathcal{M}_k(x)$. Concerning the derivatives, Theorem 3.3 can be applied with $M_1(t) = t^{k-1}$, $N_1(x) \equiv 1$.

It is also possible to apply Theorems 2.1 and 3.4 with $h(x,t) = t^{k-1}(t-x)$, $P_1(x,t) = t^{k-1}$ (see [13]). The advantage obtainable by using the geometric versus the exponential estimates is more pronounced here. In fact, for $g(x) = x^{-\gamma}$, $\gamma > 2$, we get (for $k=1$) $V(x) = U_0(x)/(\gamma-1)$ for all x. Therefore, comparing the two estimates, (4) and (13), for x such that, e.g., $V(x) \leq \frac{1}{2}$, we find that the geometric estimate is certainly sharper for $\gamma > 3$. Such an estimate is of the order of $O(\gamma^{-2})$, whereas the exponential one is of the order of $O(\gamma^{-1})$, as in case (a), but this holds now without any further limitation on x.

3.2. Difference equations. When the measure μ in Theorem 2.1 is merely discrete, with discrete support $\mathbf{Z}_\nu := \{n \in \mathbf{Z} : n \geq \nu\}$, ν being a given integer, the integral equation (1) plays a role in studying linear difference equations. Hereafter we shall be concerned with the asymptotic solution of second-order difference equations like

$$\Delta^2 y_n + (\alpha + g_n) y_n = 0, \qquad n \in \mathbf{Z}_\nu, \tag{37}$$

$\Delta y_n := y_{n+1} - y_n$, where we consider the following.

(A) $\alpha > 0$ and $\sum_{n=\nu}^{\infty} |g_n| < \infty$;

(B) $\alpha = 0$ and $\sum_{n=\nu}^{\infty} n^k |g_n| < \infty$, $k = 1$ or 2;

(C) $\alpha < 0$, $\alpha \neq -1$, and $\sum_{n=\nu}^{\infty} |g_n| < \infty$.

It is worth noting that the importance of equations of the type

$$\Delta^2 y_n + q_n y_n = 0 \tag{38}$$

stems from the fact that they represent a kind of canonical form for all difference equations like

$$Y_{n+2} + A_n Y_{n+1} + B_n Y_n = 0. \tag{39}$$

Indeed, the transformation

$$Y_n = \alpha_n y_n, \quad \alpha_n := \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(-\frac{A_k}{2}\right), \quad n \geq \nu + 2, \tag{40}$$

α_ν and $\alpha_{\nu+1} \neq 0$ being arbitrary constants, takes (39) into (38) with a suitable q_n, provided that $A_k \neq 0$ (at least for k sufficiently large); see [12], [13].

Case (A), in which all real solutions turn out to be *oscillatory*, was studied in some detail in [12]. If one looks for a solution of the form $y_n = \lambda^n(1 + \varepsilon_n)$, λ being one of the roots of the characteristic equation of (37) with $g_n \equiv 0$, $\lambda = 1 \pm i\sqrt{\alpha}$, the "integral equation"

$$\varepsilon_n = \frac{1}{2\lambda(\lambda - 1)} \sum_{k=n}^{\infty} \left(1 - (\lambda/\overline{\lambda})^{k-n+1}\right) g_k(1 + \varepsilon_k) \tag{41}$$

for the error term ε_n is obtained. In [12] it was proved *directly* that such an equation has a solution estimated as

$$|\varepsilon_n| \leq \frac{V_n}{1 - V_n}, \tag{42}$$

$$V_n := \frac{1}{[\alpha(\alpha + 1)]^{1/2}} \sum_{k=n}^{\infty} |g_k|. \tag{43}$$

Both the existence of the solution and the estimate (42), (43) hold for $n \geq n_1 := \min\{n \in \mathbf{Z}_\nu : V_n < 1\}$. This result represents an extension to the discrete domain of the Liouville–Green–Olver approximation theorem for oscillatory-type differential equations (see [11, Thm. 2.2, p. 196]).

Note the formal analogy of (37) with (30) when we take $f(x) \equiv \text{const.} > 0$ (case (a), §3.1). In fact, the unified treatment presented in this paper leads to (42), (43) when we choose in Theorem 2.1 $\phi(x,t) \equiv 1$, $K(x,t) = (1/2\lambda(\lambda - 1))(1 - (\lambda/\overline{\lambda})^{t-x+1})$, and $\mu = \sum_{k=\nu}^{\infty} g_k \delta_k$, where δ_k is the Dirac measure centered in $t = k$. The function $h(x,t)$ estimating K, $K\phi$ in (ii) of Theorem 2.1 can be chosen equal to the constant $1/|\lambda(\lambda - 1)| = [\alpha(\alpha + 1)]^{-1/2}$. It is easily seen that $V(x)$ is left-continuous and piecewise constant and that $x_1 = n_1$; also, $a = \nu$, $V_n = V(n)$. Finally, observe that the solution to (1) restricted to the integers $n \geq n_1$ also solves (41) and that the estimate (42), (43) holds.

Case (B) represents the discrete analogue of case (b) of §3.1 and was studied in [13]. In [13] a unified approach was followed; it was based, however, on a special *subclass* of integral equations like those in (1). Indeed, when one looks for solutions to (37) with $\alpha = 0$ of the form $y_n = n^{k-1} + \varepsilon_n$, with $k = 1$ or 2, an "integral equation" of type (1) is obtained, with $K(x,t) = x - t + 1$, $\phi(x,t) = t^{k-1}$, and $\mu = \sum_{k=\nu}^{\infty} g_k \delta_k$ ($\nu = 1$). If one takes $h(x,t) = t^{k-1}(t - x + 1)$, estimates are obtained for the error terms corresponding to the *recessive* and the *dominant* solution ($k = 1$ *or* 2 also refers to the finiteness of the first *or* the second moment of $|g_n|$: When $\sum^{\infty} n^2 |g_n| = +\infty$ but $\sum^{\infty} n|g_n| < \infty$ the estimate for the recessive solution still holds true).

A Liouville–Green–Olver approximation result for case (C) seems to be missing so far. Note that, unlike in the previous cases, *qualitative* asymptotics for the solutions to (37) with $\alpha < 0$ could be obtained here by Poincaré's or Perron's theorems [9]. Deriving *precise error bounds*, however, is our goal, in the spirit of Olver's approach. This problem represents a discrete analogue of the differential case with solutions of the exponential type in [11, Thm. 2.1, p. 193]. We state our result as a theorem.

THEOREM 3.5. *Suppose that* (37) *is given with* $\alpha = -\beta < 0$, $\beta \neq 1$, *and* $\sum^{\infty} |g_n| < \infty$ (*see case* (C)). *Then there exist* $n_1 \in \mathbf{Z}_\nu$ *and two linearly independent solutions to* (37), $y_n^{\pm}$, *such that*

$$y_n^- = (\lambda_-)^n [1 + \varepsilon_n], \quad n \geq n_1; \qquad y_n^+ \sim (\lambda_+)^n, \quad n \to \infty, \tag{44}$$

where $\lambda_\pm = 1 \pm \sqrt{\beta}$ *are the roots of the characteristic equation associated to* (37) *with* $g_n \equiv 0$. *For the error term* ε_n *the estimate*

$$|\varepsilon_n| \leq \frac{V_n}{1 - V_n}, \quad V_n := \frac{1}{2\sqrt{\beta}(\sqrt{\beta} - 1)} \sum_{k=n}^{\infty} |g_k|, \quad n \geq n_1 \tag{45}$$

holds and

$$n_1 = \min\{n \in \mathbf{Z}_\nu : V_n < 1\}. \tag{46}$$

Moreover, when g_n *is real,* $y_n^{\pm}$ *are real.*

Remark 3.6. In view of (45), y_n^- and y_n^+ are *recessive* and *dominant* solutions, respectively. Note that, unlike the corresponding case for differential equations, while $y_n^+ \sim (1 + \sqrt{\beta})^n$ grows exponentially, $y_n^- \sim (1 - \sqrt{\beta})^n$ and thus decays exponentially when $0 < \beta < 1$, but it exhibits *oscillations* exponentially growing (when $\beta > 4$) *or* exponentially decreasing (when $1 < \beta < 4$); when $\beta = 4$, $y_n^- \sim (-1)^n$. The case $\beta = 1$ is pathological in that the unperturbed equation (see (37) with $g_n \equiv 0$) degenerates, having the lowest-order coefficient vanishing (see [8] and [9]), and *cannot* be treated by the present approach.

Remark 3.7. The qualitative behavior $y_n^{\pm} \sim (\lambda_\pm)^n$, $n \to \infty$, *cannot* be obtained directly from Poincaré's or Perron's theorems, despite the fact that such theorems can be applied, since $\lambda_+ \neq \lambda_-$ [8, §5.3, p. 221].

Remark 3.8. Observe even here the *double asymptotic nature* of the Liouville–Green–Olver approximations with respect to both n and β (as $n \to \infty$, and as $\beta \to +\infty$); see (45). In particular, $V_n = O(\beta^{-1})$, whereas the corresponding quantity for the analogous differential equation, $y'' + (-\beta + g(x))y = 0$, $g \in L^1(1, +\infty)$, is $O(\beta^{-1/2})$; see [11] and [12].

Proof of Theorem 3.5. Looking for a solution of the form $y_n^- = (\lambda_-)^n [1 + \varepsilon_n]$, one is led to the difference equation

$$(\lambda_-)^2 \Delta^2 \varepsilon_n + 2\lambda_-(\lambda_- - 1)\Delta \varepsilon_n + g_n(1 + \varepsilon_n) = 0 \tag{47}$$

for the error term. It is then easily proved that any solution of the linear discrete Volterra-type "integral" equation

$$\varepsilon_n = \sum_{k=n}^{\infty} (1 - \rho^{k-n+1}) g_k \sigma (1 + \varepsilon_k), \tag{48}$$

$$\rho = \frac{\lambda_-}{\lambda_+} = \frac{1-\sqrt{\beta}}{1+\sqrt{\beta}}, \qquad \sigma = \frac{1}{2\lambda_-(\lambda_- - 1)} = \frac{1}{2\sqrt{\beta}(\sqrt{\beta}-1)}, \tag{49}$$

also solves (47). The easy but lengthy verification is left to the reader; see [12]. At this point, consider equation (1), with $\phi(x,t) \equiv 1$, $K(x,t) = \sigma(1-\rho^{t-x+1})$, $x \geq a = \nu$, $\mu = \sum_{k=\nu}^{\infty} g_k \delta_k$, and choose $h(x,t) \equiv |\sigma|$, since $|\rho| < 1$. Then we use Theorem 2.1 to obtain (45) since it is clear that the solution to (1), restricted to the integers $\geq n_1$, also solves (48) and $V_n = V(n)$. Note that uniqueness of solutions to (48) does *not* follow immediately from Theorem 2.1 but can be proved directly for $n \geq n_1$.

As for the second (dominant) solution, we consider a solution of the form

$$z_n := y_n^- \sum_{k=n^*}^{n-1} \frac{C_k}{y_k^- y_{k+1}^-} \tag{50}$$

(see [8, §3.5, Thm. 3.9, p. 94]), n^* being the smallest integer $\geq n_1$ such that $y_n^- \neq 0$ for $n \geq n^*$, and C_k denoting the *Casoratian* of y_k^- and z_k. The latter is [8, §3.5]

$$C_k = C_{n^*} \prod_{j=n^*}^{k-1} (1 - \beta + g_j), \tag{51}$$

where C_{n^*} is a nonzero constant that we shall choose. Now,

$$\frac{z_n}{(\lambda_+)^n} \sim \left(\frac{\lambda_+}{\lambda_-}\right)^{-n} \sum_{k=n^*}^{n-1} \frac{C_k}{(\lambda_-)^{2k+1}[1+o(1)]}, \qquad n \to \infty. \tag{52}$$

When $0 < \beta < 1$, using *Cesaro's* theorem, we get

(53)
$$\frac{z_n}{(\lambda_+)^n} \sim \frac{C_n/(\lambda_-)^{2n+1}}{(\lambda_+/\lambda_-)^{n+1} - (\lambda_+/\lambda_-)^n} = \frac{(1-\beta)^{-n^*}}{2\sqrt{\beta}} C_{n^*} \prod_{j=n^*}^{n-1} \left(1 + \frac{g_j}{1-\beta}\right), \quad n \to \infty.$$

In fact, $(\lambda_+/\lambda_-)^n \uparrow +\infty$ as $n \to \infty$, and the product in (53) converges in view of the fact that $\sum^{\infty} |g_j| < \infty$. When $\beta > 1$, the same condition holds true, although the classical Cesaro's theorem cannot be invoked. Such a generalized version of Cesaro's theorem is contained in [14]. Finally, choosing the constant C_{n^*} in such a way that the limit of the right-hand side of (53) is equal to 1, one obtains $z_n = y_n^+$. □

Remark 3.9. When one looks for a dominant solution of the form $y_n^+ = (\lambda_+)^n(1+\eta_n)$, an error equation for η_n similar to (48) is obtained, with $\rho = \lambda_+/\lambda_-$, $\sigma = 1/2\lambda_+(\lambda_+ - 1)$. However, $|\rho| > 1$ prevents proving, in general, the existence itself of such a solution, as was done for ε_n. A noteworthy case, however, occurs when $g_n = \rho^{-n} u_n$, with $\sum^{\infty} |u_n| < \infty$, e.g., $g_n = c^{-n}$, with $|c| > |\rho|$. Here Theorem 2.1 can be applied with $\phi(x,t) \equiv 1$, $K(x,t) = \rho^{-t}(1-\rho^{t-x+1})$, $\mu = \sum_{k=\nu}^{\infty} u_k \delta_k$, and $h(x,t) \equiv 1$ for $x \geq 1$. The estimate

(54)
$$|\eta_n| \leq \frac{W_n}{1-W_n}, \qquad n \geq n_2 = \min\{n \in \mathbf{Z}_\nu : W_n < 1\},$$
$$W_n := \frac{1}{2\sqrt{\beta}|\sqrt{\beta}-1|} \sum_{k=n}^{\infty} |u_k|$$

is then obtained.

It may be useful, in closing, to reformulate the hypotheses of Theorem 3.5 in terms of the coefficients of (39). Since in (38) one obtains

$$q_n = -1 + \frac{4B_n}{A_n A_{n-1}} \tag{55}$$

for $n \geq \nu + 1$ (see [12] and [13]), they become

$$\lim_{n\to\infty} \frac{B_n}{A_n A_{n-1}} =: L, \qquad \text{with } L < \frac{1}{4}, \quad L \neq 0, \tag{56}$$

$$\sum_{k=\nu+1}^{\infty} \left| \frac{B_n}{A_n A_{n-1}} - L \right| < \infty. \tag{57}$$

We conclude with two simple applications.

Example 3.10 (perturbed Fibonacci equation). Consider

$$f_{n+2} - (1+\sigma_n) f_{n+1} - (1+\tau_n) f_n = 0, \qquad n \geq 0, \tag{58}$$

where

$$\sum_{n=0}^{\infty} (|\sigma_n| + |\tau_n|) < \infty, \tag{59}$$

and $\sigma_n \neq -1$ for $n \geq 0$. The latter restriction can be removed by confining ourselves to $n \geq \nu$ with ν sufficiently large. Then, setting $f_n = \alpha_n y_n$, with

$$\alpha_n = \prod_{k=0}^{n-2} \left(\frac{1+\sigma_k}{2} \right) \tag{60}$$

(cf. (40)) leads to (38) with

$$q_n = -1 - 4\frac{1+\tau_n}{(1+\sigma_n)(1+\sigma_{n-1})}. \tag{61}$$

Now, $q_n \to -5$ as $n \to \infty$, and then (56) and (57) are satisfied with $L = -1$ and owing to (59). Since $\beta = 5$,

$$g_n = 4\frac{(\sigma_n + \sigma_{n-1} + \sigma_n \sigma_{n-1} + \tau_n)}{(1+\sigma_n)(1+\sigma_{n-1})} \tag{62}$$

and $\lambda_\pm = 1 \pm \sqrt{5}$, we obtain a *recessive* solution to (58) like

$$f_n^- = \left(\frac{1-\sqrt{5}}{2} \right)^n \left(\prod_{k=0}^{n-2} (1+\sigma_k) \right) (1+\varepsilon_n) \quad \text{for } n \geq n_1, \tag{63}$$

with

(64)

$$|\varepsilon_n| \leq \frac{V_n}{1-V_n}, \qquad V_n := \frac{2}{\sqrt{5}(\sqrt{5}-1)} \sum_{k=n}^{\infty} \left| \frac{\sigma_n + \sigma_{n-1} + \sigma_n \sigma_{n-1} + \tau_n}{(1+\sigma_n)(1+\sigma_{n-1})} \right| \quad \text{for } n \geq n_1,$$

with n_1 defined in (46) (cf. Theorem 3.5). Moreover, a *dominant* solution f_n^+ exists, with

$$f_n^+ \sim \left(\frac{1+\sqrt{5}}{2}\right)^n \prod_{k=0}^{\infty}(1+\sigma_k), \qquad n \to \infty. \tag{65}$$

Note that the convergence of the infinite product in (65) is guaranteed by (59).

Example 3.11 (orthogonal polynomials). A field in which asymptotic representations of solutions to three-term recurrent equations is important is that of orthogonal polynomials. In [5] an application is made of the discrete WKB theory developed in [4] to a class of orthogonal polynomials obeying a recurrence with regularly and slowly varying coefficients. Here we show that Theorem 3.5 can be applied to obtain qualitative asymptotics for a well-known class of orthogonal polynomials, on the real line off their essential spectrum. Consider, in fact, the linear recurrence

$$P_{n+2}(x) - (x-\gamma_n)P_{n+1}(x) + \delta_n P_n(x) = 0, \tag{66}$$

with γ_n real, $\delta_n > 0$, $\gamma_n \to \gamma$ and $\delta_n \to \delta$, γ and δ both finite, which defines along with the initial conditions $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$, a family of orthogonal polynomials having as essential spectrum the interval $[\gamma - 2\sqrt{\delta}, \gamma + 2\sqrt{\delta}]$; see, e.g., [2]. In this case we obtain from (56)

$$L = \frac{\delta}{(x-\gamma)^2}, \tag{67}$$

and thus $L < 1/4$ for x *off* the essential spectrum. Note then that $x \neq \gamma$. In view of the fact that $\gamma_n \to \gamma$ we see that also $x - \gamma_n \neq 0$ for all n sufficiently large ($n \geq \nu(x)$). This ensures that the transformation in (40),

$$\alpha_n(x) = \prod_{k=\nu(x)}^{n-2}\left(\frac{x-\gamma_k}{2}\right), \tag{68}$$

is applicable and the previous theory can be used. Notice also that a *uniform* lower bound for $\nu(x)$ can be found for x off the essential spectrum. Condition (47) becomes

$$\sum_{n=\nu(x)+1}^{\infty}\left|\frac{\delta_n}{(x-\gamma_n)(x-\gamma_{n-1})} - \frac{\delta}{(x-\gamma)^2}\right| < \infty. \tag{69}$$

Observe that the assumption $\sum^\infty (|\gamma_n - \gamma| + |\delta_n - \delta|) < \infty$, which, incidentally, ensures the orthogonality measure to be absolutely continuous in the essential spectrum, *implies* (69) and appears, e.g., in [7] as a key condition for the asymptotic analysis of the linear recurrence (66). Such a condition also appears, for instance, in [15]. Apart from having a weaker condition in (69), asymptotic results obtainable through Theorem 3.5 off the essential spectrum *coincide* with those reported in [7], where completely different techniques were adopted. Our theory, however, yields a precise *error estimate* for a *recessive* solution. This occurrence may be useful in connection to certain well-known numerical algorithms (see [3]).

REFERENCES

[1] F. V. ATKINSON, *Discrete and Continuous Boundary Value Problems*, Academic Press, New York, 1964.

[2] T. S. CHIHARA, *The three term recurrence relation and spectral properties of orthogonal polynomials*, in Orthogonal Polynomials: Theory and Practice, P. G. Nevai, ed., NATO ASI Series, Kluwer, Dordrecht, The Netherlands, 1990, pp. 99–114.

[3] W. GAUTSCHI, *Minimal solutions of three-term recurrence relations and orthogonal polynomials*, Math. Comp., 36 (1981), pp. 547–554.

[4] J. S. GERONIMO AND D. T. SMITH, WKB (*Liouville–Green*) *analysis of second order difference equations and applications*, J. Approx. Theory, 69 (1992), pp. 269–301.

[5] J. S. GERONIMO, D. SMITH, AND W. VAN ASSCHE, *Strong asymptotics for orthogonal polynomials with regularly and slowly varying recurrence coefficients*, J. Approx. Theory, 72 (1993), pp. 141–158.

[6] E. HILLE, *Non-oscillation theorems*, Trans. Amer. Math. Soc., 64 (1948), pp. 234–252.

[7] M. E. H. ISMAIL, D. R. MASSON, AND E. B. SAFF, *A minimal solution approach to polynomial asymptotics*, in Orthogonal Polynomials and Their Applications, C. Brezinski, L. Gori, and A. Ronveaux, eds., IMACS, Baltzer, Basel, Switzerland, 1991, pp. 299–303.

[8] W. G. KELLEY AND A. C. PETERSON, *Difference Equations. An Introduction with Applications*, Academic Press, San Diego, CA, 1991.

[9] V. LAKSHMIKANTHAM AND D. TRIGIANTE, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, San Diego, CA, 1988.

[10] F. W. J. OLVER, *Error bounds for the Liouville–Green* (*or* WKB) *approximation*, Proc. Cambridge Philos. Soc., 57 (1961), pp. 790–810.

[11] ———, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[12] R. SPIGLER AND M. VIANELLO, *Liouville–Green approximations for a class of linear oscillatory difference equations of the second order*, J. Comput. Appl. Math., 41 (1992), pp. 105–116.

[13] ———, WKBJ-*type approximation for finite moments perturbations of the differential equation* $y''=0$ *and the analogous difference equation*, J. Math. Anal. Appl., 169 (1992), pp. 437–452.

[14] ———, *Cesaro's theorems for complex sequences*, J. Math. Anal. Appl., 179 (1993), to appear.

[15] W. VAN ASSCHE, *Asymptotics for orthogonal polynomials and three-term recurrences*, in Orthogonal Polynomials: Theory and Practice, P. G. Nevai, ed., NATO ASI Series, Kluwer, Dordrecht, The Netherlands, 1990, pp. 435–462.

[16] A. C. ZAANEN, *Integration*, North-Holland, Amsterdam, 1967.