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Existence, uniqueness and conditional stability of periodic solutions to evolution equations

Nguyen Thieu Huy [a,*], Ngo Quy Dang [b]

[a] *School of Applied Mathematics and Informatics, Hanoi University of Science and Technology, Vien Toan ung dung va Tin hoc, Dai hoc Bach khoa Hanoi, 1 Dai Co Viet, Hanoi, Viet Nam*
[b] *Thai Binh Teacher Training College, Cao Dang Su Pham Thai Binh, Chu Van An, Quang Trung, Thai Binh, Viet Nam*

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ABSTRACT

Using an ergodic approach, we investigate the condition for existence and uniqueness of periodic solutions to linear evolution equation $\dot{u} = A(t)u + f(t)$, $t \geq 0$, and to semi-linear evolution equations of the form $\dot{u} = A(t)u + g(u)(t)$, where the operator-valued function $t \mapsto A(t)$ and the vector-valued function $f(t)$ are T-periodic, and Nemytskii's operator g is locally Lipschitz and maps T-periodic functions to T-periodic functions. We then apply the results to study the existence, uniqueness, and conditional stability of periodic solutions to the above semi-linear equation in the case that the family $(A(t))_{t\geq 0}$ generates an evolution family having an exponential dichotomy.

1. Introduction

Consider the abstract semi-linear evolution equation

$$\dot{u} = A(t)u + g(u)(t), \quad t \in \mathbb{R}_+, \tag{1.1}$$

where for each $t \in \mathbb{R}_+$, $A(t)$ is a possibly unbounded operator on a Banach space X such that $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ on X, and the so-called Nemytskii's operator g is locally Lipschitz and acts on some function space of vector-valued functions such as $C_b(\mathbb{R}_+, X)$. One of the important research directions related to the asymptotic behavior of the solutions to the above equation is to find conditions for the existence of a periodic solution to that equation in case g maps T-periodic functions to T-periodic functions. Beside some approaches which seem to be suitable only for specific equations like Tikhonov's fixed point method [12] or the Lyapunov functionals [14], the most popular approaches for proving the

* Corresponding author.
E-mail addresses: huy.nguyenthieu@hust.edu.vn (N.T. Huy), ngodangquy@gmail.com (N.Q. Dang).

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existence of a periodic solution are the ultimate boundedness of solutions and the compactness of Poincaré map realized through some compact embeddings (see [1,6,11–14] and the references therein).

However, in some concrete applications, e.g., to partial differential equations in unbounded domains or to equations that have unbounded solutions, such compact embeddings are no longer valid, and the existence of bounded solutions is not easy to obtain since one has to carefully choose an appropriate initial vector (or conditions) to guarantee the boundedness of the solution emanating from that vector.

Therefore, in the present paper, we propose another approach toward the existence and uniqueness of the periodic solution to the abstract evolution equation (1.1), so that it can help to overcome such difficulties. Namely, we start with the linear equation

$$\dot{u} = A(t)u + f(t), \quad t \geq 0 \tag{1.2}$$

and use an ergodic approach (see [15] for the origin of the approach) to prove the existence of a periodic solution through the existence of a bounded solution whose sup-norm can be controlled by the sup-norm of the input function f. We refer the reader to [5] for an extension of such an approach to the case of periodic solutions to Stokes and Navier–Stokes equations around rotating obstacles.

We then use the fixed point argument to prove such results for the abstract semi-linear evolution equation (1.1). Our approach invokes somehow the folklore methodology of Massera [7] for periodic solutions to Ordinary Differential Equations (which roughly said that if an ODE has a bounded solution then it has a periodic one) to the level of general Banach spaces.

It is worth noting that our framework fits perfectly to the situation of exponentially dichotomic linear parts, i.e., the case when family $(A(t))_{t\geq 0}$ generates an evolution family $(U(t,s))_{t\geq s\geq 0}$ having an exponential dichotomy (see Definition 4.1 below), since in this case we can choose the initial vector from that emanates a bounded solution. We can also prove the conditional stability of periodic solutions in this case.

Our main results are contained in Theorems 2.3 and 3.1. The applications of our abstract results to semi-linear equations with the exponentially dichotomic linear parts are given in Section 4.

2. Bounded and periodic solutions to linear evolution equations

Given a function f taking values in a Banach space X having a separable predual Y (i.e., $X = Y'$ for a separable Banach space Y) we consider the non-homogeneous linear problem for the unknown function $u(t)$

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + f(t) \text{ for } t > 0 \\ u(0) = u_0 \in X, \end{cases} \tag{2.1}$$

where the family of partial differential operators $(A(t))_{t\geq 0}$ is given such that the homogeneous Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) \text{ for } t > s \geq 0 \\ u(s) = u_s \in X \end{cases} \tag{2.2}$$

is well-posed. By this we mean that there exists an evolution family $(U(t,s))_{t\geq s\geq 0}$ such that the solution of the Cauchy problem (2.2) is given by $u(t) = U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [10] (see also Nagel and Nickel [9] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$). We next give the precise concept of an evolution family in the following definition

Definition 2.1. A family of bounded linear operators $(U(t,s))_{t\geq s\geq 0}$ on a Banach space X is a *(strongly continuous, exponentially bounded) evolution family* if

(i) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s \geq 0$,
(ii) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$, where $(t,s) \in \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\}$,
(iii) there are constants $K, \alpha \geq 0$ such that $\|U(t,s)x\| \leqslant Ke^{\alpha(t-s)}\|x\|$ for all $t \geq s \geq 0$ and $x \in X$.

The existence of the evolution family $(U(t,s))_{t\geq s\geq 0}$ allows us to define a notion of mild solutions as follows. By the *mild solution* to (2.1) we mean a function u satisfying the following integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)d\tau \text{ for all } t \geq 0. \tag{2.3}$$

We refer the reader to Pazy [10] for more detailed treatments on the relations between classical and mild solutions of evolution equations of the form (2.1).

We now state an assumption that will be used in the rest of the paper.

Assumption 2.2. We assume that $A(t)$ is T-periodic, i.e., $A(t+T) = A(t)$ for a fixed constant $T > 0$ and all $t \in \mathbb{R}_+$. Then $(U(t,s))_{t\geq s\geq 0}$ becomes T-periodic in the sense that

$$U(t+T, s+T) = U(t,s) \text{ for all } t \geq s \geq 0. \tag{2.4}$$

We also assume that the space Y considered as a subspace of Y'' (through the canonical embedding) is invariant under the operator $U'(T,0)$ which is the dual of $U(T,0)$.

To show the existence and uniqueness of the periodic mild solution to (2.1) we need the following space of bounded continuous functions with values in a Banach space X (with norm $\|\cdot\|$) defined as

$$C_b(\mathbb{R}_+, X) := \{v : \mathbb{R}_+ \to X | v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\| < \infty\} \tag{2.5}$$

endowed with the norm

$$\|v\|_{C_b} := \sup_{t\in\mathbb{R}_+} \|v(t)\|.$$

Theorem 2.3. *For the Banach spaces X and Y (with $X = Y'$ where Y is separable) let the following condition hold true: For $f \in C_b(\mathbb{R}_+, X)$ there exists $u_0 \in X$ such that the mild solution u of (2.1) with $u(0) = u_0$ (i.e., $u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds$) satisfies $u \in C_b(\mathbb{R}_+, X)$ and*

$$\|u\|_{C_b} \leqslant M\|f\|_{C_b}. \tag{2.6}$$

Then, under Assumption 2.2, if f is T-periodic, then Equation (2.1) has a T-periodic mild solution $\hat{u}$ satisfying:

$$\|\hat{u}\|_{C_b} \leqslant (M+T)Ke^{\alpha T}\|f\|_{C_b}. \tag{2.7}$$

Furthermore, if the evolution family $U(t,s)_{t\geq s\geq 0}$ satisfies:

$$\lim_{t\to\infty} \|U(t,0)x\| = 0 \text{ for } x \in X \text{ such that } U(t,0)x \text{ is bounded in } \mathbb{R}_+, \tag{2.8}$$

then the T-periodic mild solution of (2.1) is unique.

Proof. The existence of a T-periodic solution is clearly equivalent to the fact that there exists $\hat{x} \in X$ such that $\hat{x} = U(T,0)\hat{x} + \int_0^T U(T,s)f(s)ds$. To prove the existence of such an $\hat{x}$ we consider the function $u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)ds$ which belongs to $C_b(\mathbb{R}_+, X)$ by the hypothesis of the theorem. For a T-periodic function f and an evolution family $(U(t,s))_{t\geq s\geq 0}$ satisfying (2.4), we first prove by induction that

$$u((k+1)T) = U(T,0)u(kT) + \int_0^T U(T,s)f(s)ds \quad \text{for all } k \in \mathbb{N}. \tag{2.9}$$

Clearly, the formula (2.9) holds true for $k = 0$. Assuming that it is true for $k-1$, i.e.,

$$u(kT) = U(T,0)u((k-1)T) + \int_0^T U(T,s)f(s)ds,$$

then we prove that it holds true for k. To do this, from definition of u it follows that

$$\begin{aligned}
u((k+1)T) &= U((k+1)T,0)u_0 + \int_0^{(k+1)T} U((k+1)T,s)f(s)ds \\
&= U((k+1)T,0)u_0 + \int_0^{kT} U((k+1)T,s)f(s)ds + \int_{kT}^{(k+1)T} U((k+1)T,s)f(s)ds \\
&= U((k+1)T,kT)U(kT,0)u_0 + \int_0^{kT} U((k+1)T,kT)U(kT,s)f(s)ds \\
&+ \int_0^T U((k+1)T,kT+s)f(s)ds \\
&= U(T,0)U(kT,0)u_0 + \int_0^{kT} U(T,0)U(kT,s)f(s)ds + \int_0^T U(T,s)f(s)ds \\
&= U(T,0)u(kT) + \int_0^T U(T,s)f(s)ds.
\end{aligned}$$

Therefore, we obtain that the formula (2.9) holds true for all $k \in \mathbb{N}$.

Next, for each $n \in \mathbb{N}$ we define the Cesàro sum x_n by

$$x_n := \frac{1}{n}\sum_{k=1}^{n} u(kT). \tag{2.10}$$

Now, the inequality (2.6) implies

$$\sup_{k\in\mathbb{N}} \|u(kT)\| \leqslant M\|f\|_{C_b}. \tag{2.11}$$

Hence, $\{x_n\}_{n\in\mathbb{N}}$ is also bounded in X and by (2.11) we have

$$\sup_{n\in\mathbb{N}} \|x_n\| \leqslant M\|f\|_{C_b}. \tag{2.12}$$

Since the space $X = Y'$ and Y is separable, by Banach–Alaoglu's Theorem there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\{x_{n_k}\} \xrightarrow{weak^*} \hat{x} \text{ with } \|\hat{x}\| \leqslant M\|f\|_{C_b}. \tag{2.13}$$

A straightforward calculation using formula (2.10) yields

$$U(T,0)x_n + \int_0^T U(T,s)f(s)ds - x_n = \frac{1}{n}\big(u((n+1)T) - u(T)\big).$$

Since the sequence $\{u(nT)\}_{n\in\mathbb{N}}$ is bounded in X, we obtain that

$$\lim_{n\to\infty}(U(T,0)x_n + \int_0^T U(T,s)f(s)ds - x_n) = \lim_{n\to\infty}\frac{1}{n}\big(u((n+1)T) - u(T)\big) = 0 \text{ strongly in } X.$$

This particularly implies that for the subsequence $\{x_{n_k}\}$ from (2.13) we have

$$U(T,0)x_{n_k} + \int_0^T U(T,s)f(s)ds - x_{n_k} \xrightarrow{weak^*} 0. \tag{2.14}$$

Combining (2.13) and (2.14) we obtain that

$$U(T,0)x_{n_k} + \int_0^T U(T,s)f(s)ds \xrightarrow{weak^*} \hat{x} \in X. \tag{2.15}$$

We will prove that $U(T,0)\hat{x} + \int_0^T U(T,s)f(s)ds = \hat{x}$. To do this, denoting by $\langle\cdot,\cdot\rangle$ the dual pair between Y and Y' and using the fact that $U'(T,0)$ leaves Y invariant (see Assumption 2.2), for all $h \in Y$ we have

$$\begin{aligned}
\left\langle U(T,0)x_{n_k} + \int_0^T U(T,s)f(s)ds, h\right\rangle &= \langle U(T,0)x_{n_k}, h\rangle + \left\langle \int_0^T U(T,s)f(s)ds, h\right\rangle \\
&= \langle x_{n_k}, U'(T,0)h\rangle + \left\langle \int_0^T U(T,s)f(s)ds, h\right\rangle \\
&\xrightarrow{n_k\to\infty} \langle \hat{x}, U'(T,0)h\rangle + \left\langle \int_0^T U(T,s)f(s)ds, h\right\rangle \\
&= \langle U(T,0)\hat{x}, h\rangle + \left\langle \int_0^T U(T,s)f(s)ds, h\right\rangle \\
&= \left\langle U(T,0)\hat{x} + \int_0^T U(T,s)f(s)ds, h\right\rangle.
\end{aligned}$$

This yields

$$U(T,0)x_{n_k} + \int_0^T U(T,s)f(s)ds \xrightarrow{weak^*} U(T,0)\hat{x} + \int_0^T U(T,s)f(s)ds \in X. \tag{2.16}$$

It now follows from (2.15) and (2.16) that

$$U(T,0)\hat{x} + \int_0^T U(T,s)f(s)ds = \hat{x}. \tag{2.17}$$

The mild solution $\hat{u} \in C_b(\mathbb{R}_+, X)$ of (2.2) with the initial value $\hat{u}(0) = \hat{x}$ is clearly a T-periodic mild solution. The inequality (2.7) now follows from inequalities (2.6) and (2.13).

We now prove that if the evolution family $(U(t,s))_{t\geq s\geq 0}$ satisfies (2.8), then the periodic mild solution is unique. Indeed, let $\hat{u}_1$ and $\hat{u}_2$ be two T-periodic mild solutions to Equation (2.1). Then, putting $v = \hat{u}_1 - \hat{u}_2$ we have that v is T-periodic and, by formula (2.3),

$$v(t) = U(t,0)(\hat{u}_1(0) - \hat{u}_2(0)) \text{ for } t \geq 0. \tag{2.18}$$

Since $v(\cdot)$ is bounded on $\mathbb{R}_+$, the inequality (2.8) then implies that

$$\lim_{t\to\infty} \|v(t)\| = 0. \tag{2.19}$$

This fact, together with the periodicity of v, implies that $v(t) = 0$ for all $t \geqslant 0$. This yields $\hat{u}_1 = \hat{u}_2$. □

3. Bounded and periodic solutions to semi-linear problems

For Banach spaces X and Y (with $X = Y'$ where Y is separable) as in the previous section, we now consider the following semi-linear evolution equation

$$\begin{cases} \dfrac{du}{dt} = A(t)u(t) + g(u)(t) \\ u(0) = u_0 \in X, \end{cases} \tag{3.1}$$

where the operators $A(t)$ satisfy the hypotheses of Theorem 2.3, and Nemytskii's operator $g : C_b(\mathbb{R}_+, X) \to C_b(\mathbb{R}_+, X)$ satisfies:

(1) $\|g(0)\|_{C_b} \leqslant \gamma$ where γ is a non-negative constant,

(2) g maps T-periodic functions to T-periodic functions,

(3) there exist positive constants ρ and L such that

$$\|g(v_1) - g(v_2)\|_{C_b} \leqslant L\|v_1 - v_2\|_{C_b} \text{ for all } v_1, v_2 \in C_b(\mathbb{R}_+, X) \text{ with } \|v_1\|_{C_b}, \|v_2\|_{C_b} \leqslant \rho. \tag{3.2}$$

Furthermore, by the *mild solution* to (3.1) we mean the function u satisfying the following equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)g(u)(\tau)d\tau \text{ for all } t \geq 0. \tag{3.3}$$

We then come to our next result on the existence and uniqueness of the periodic mild solution to Equation (3.1).

Theorem 3.1. *Let the hypotheses of Theorem 2.3 be satisfied, and let g satisfy the conditions in (3.2). Then, if L and γ are small enough, equation (3.1) has one and only one mild T-periodic solution $\hat{u}$ on a small ball of $C_b(\mathbb{R}_+, X)$.*

Proof. Consider the following closed set $B_\rho^T \subset C_b(\mathbb{R}_+, X)$ defined by

$$B_\rho^T := \{v \in C_b(\mathbb{R}_+, X) : v \text{ is T-periodic and } \|v\|_{C_b} \leqslant \rho\}. \tag{3.4}$$

We then define the following transformation Φ given as follows: Consider the equation

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(v)(\tau)d\tau \text{ for } t \geq 0. \tag{3.5}$$

Then, for $v \in B_\rho^T$ we set $\Phi(v) := u$ where $u \in C_b(\mathbb{R}_+, X)$ is the unique T-periodic mild solution to (3.5) (the existence and uniqueness of such a u is guaranteed by Theorem 2.3). We will prove that if L and γ are small enough, then the transformation Φ acts from B_ρ^T into itself and is a contraction. To do this, taking any $v \in B_\rho^T$, by the properties of g given in (3.2) we have

$$\|g(v)\|_{C_b} \leqslant \|g(v) - g(0)\|_{C_b} + \|g(0)\|_{C_b} \leqslant L\|v\|_{C_b} + \gamma \leqslant L\rho + \gamma \text{ for all } t \geq 0. \tag{3.6}$$

Applying Theorem 2.3 for the right-hand side $g(v)$ instead of f we obtain that for $v \in B_\rho^T$ there exists a unique T-periodic mild solution u to (3.5) satisfying

$$\|u\|_{C_b} \leqslant (M+T)Ke^{\alpha T}\|g(v)\|_{C_b} \leqslant (M+T)Ke^{\alpha T}(L\rho + \gamma). \tag{3.7}$$

Therefore, we obtain that, if L and γ are small enough, then the map Φ acts from B_ρ^T into itself. Next, by formula (2.3) we have the following representation of Φ

$$\Phi(v)(t) = U(t,0)u(0) + \int_0^t U(t,\tau)g(v)(\tau)d\tau \text{ for } \Phi(v) = u. \tag{3.8}$$

Furthermore, for $v_1, v_2 \in B_\rho^T$ and $u_1 = \Phi(v_1)$, $u_2 = \Phi(v_2)$ by the representation (3.8) we obtain that $u = \Phi(v_1) - \Phi(v_2)$ is the unique T-periodic mild solution to the equation

$$u(t) = U(t,0)u(0) + \int_0^t U(t,\tau)(g(v_1)(\tau) - g(v_2)(\tau))d\tau \text{ for all } t \geq 0.$$

Thus, from Theorem 2.3 and the property (3) of g given in (3.2) we arrive at

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{C_b} &\leqslant (M+T)Ke^{\alpha T}\|g(v_1) - g(v_2)\|_{C_b} \\ &\leqslant L(M+T)Ke^{\alpha T}\|v_1 - v_2\|_{C_b}. \end{aligned} \tag{3.9}$$

We thus obtain that if L and γ are small enough, then $\Phi : B_\rho^T \to B_\rho^T$ is a contraction. Therefore, for these values of L and γ there exists a unique fixed point $\hat{u}$ of B_ρ^T, and by definition of Φ, this function $\hat{u}$ is the unique T-periodic mild solution to Equation (3.1). $\square$

4. The existence, uniqueness, and conditional stability of periodic solutions for dichotomic evolution families

In this section, we will consider equations (2.3) and (3.3) in the case that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy. In this case, the existence of bounded solutions to (2.3) (i.e., bounded mild solutions to (2.1)) is convenient to prove. Therefore, the existence and uniqueness of periodic solutions to (2.3) and hence to (3.3) easily follow. Moreover, using the Gronwall-typed inequalities, we will show the conditional stability of such periodic solutions. To do so, we start with the definition of exponential dichotomy and stability of an evolution family.

Definition 4.1. Let $\mathcal{U} := (U(t,s))_{t\geq s\geq 0}$ be an evolution family on Banach space X.

(1) The evolution family $\mathcal{U}$ is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t \geq 0$, on X and positive constants N, ν such that
  (a) $U(t,s)P(s) = P(t)U(t,s)$, $t \geq s \geq 0$,
  (b) the restriction $U(t,s)_| : \mathrm{Ker}P(s) \to \mathrm{Ker}P(t)$, $t \geq s \geq 0$, is an isomorphism, and we denote its inverse by $U(s,t)_| := (U(t,s)_|)^{-1}$, $0 \leqslant s \leqslant t$,
  (c) $\|U(t,s)x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geq s \geq 0$,
  (d) $\|U(s,t)_| x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in \mathrm{Ker}P(t)$, $t \geq s \geq 0$.
  The projections $P(t)$, $t \geq 0$, are called the *dichotomy projections*, and the constants N, ν – the *dichotomy constants*.
(2) The evolution family $\mathcal{U}$ is called *exponentially stable* if it has an exponential dichotomy with the dichotomy projections $P(t) = Id$ for all $t \geq 0$. In other words, $\mathcal{U}$ is exponentially stable if there exist positive constants N and ν such that

$$\|U(t,s)\| \leqslant Ne^{-\nu(t-s)} \text{ for all } t \geq s \geq 0. \tag{4.1}$$

We remark that properties (a)–(d) of dichotomy projections $P(t)$ already imply that

i) $H := \sup_{t\geq 0} \|P(t)\| < \infty$,
ii) $t \mapsto P(t)$ is strongly continuous

(see [8, Lemma 4.2]). We refer the reader to [4] for characterizations of exponential dichotomies of evolution families in general admissible spaces.

If $(U(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with dichotomy projections $(P(t))_{t\geq 0}$ and constants $N, \nu > 0$, then we can define the Green's function on a half-line as follows:

$$\mathcal{G}(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t > \tau \geq 0, \\ -U(t,\tau)_|(I - P(\tau)) & \text{for } 0 \leqslant t < \tau. \end{cases} \tag{4.2}$$

Also, $\mathcal{G}(t,\tau)$ satisfies the estimate

$$\|\mathcal{G}(t,\tau)\| \leqslant (1+H)Ne^{-\nu|t-\tau|} \text{ for } t \neq \tau \ \geq 0. \tag{4.3}$$

The following lemma gives the form of bounded solutions of equations (2.3) and (3.3).

Lemma 4.2. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geq 0}$ and dichotomy constants $N, \nu > 0$. Let $f \in C_b(\mathbb{R}_+, X)$ and let $g : C_b(\mathbb{R}_+, X) \to C_b(\mathbb{R}_+, X)$ satisfy conditions in (3.2). Then, the following assertions hold true.*

(a) *Let* $v \in C_b(\mathbb{R}_+, X)$ *be the solution to equation* (2.3). *Then,* v *can be rewritten in the form*

$$v(t) = U(t,0)\zeta_0 + \int_0^\infty \mathcal{G}(t,\tau)f(\tau)d\tau \text{ for some } \zeta_0 \in X_0 := P(0)X, \tag{4.4}$$

where $\mathcal{G}(t,\tau)$ *is the Green's function defined by equality* (4.2).

(b) *Let* $u \in C_b(\mathbb{R}_+, X)$ *be a solution to equation* (3.3) *such that* $\sup_{t\geq 0}\|u(t)\| \leqslant \rho$ *for a fixed* $\rho > 0$. *Then, for* $t \geq 0$ *this function* $u(t)$ *can be rewritten in the form*

$$u(t) = U(t,0)v_0 + \int_0^\infty \mathcal{G}(t,\tau)g(u)(\tau)d\tau \text{ for some } v_0 \in X_0, \tag{4.5}$$

where $\mathcal{G}$ *and* X_0 *are determined as in Item (a).*

Proof. (a): Put $y(t) := \int_0^\infty \mathcal{G}(t,\tau)f(\tau)d\tau$ for $t \geq 0$. Since $f \in C_b(\mathbb{R}_+, X)$, using estimate (4.3) we obtain that

$$\|y(t)\| \leqslant (1+H)N\|f\|_{C_b}\int_0^\infty e^{-\nu|t-\tau|}d\tau \leqslant \frac{2(1+H)N\|f\|_{C_b}}{\nu} \text{ for all } t \geq 0.$$

Moreover, it is straightforward to see that $y(\cdot)$ satisfies the equation

$$y(t) = U(t,0)y(0) + \int_0^t U(t,\tau)f(\tau)d\tau \text{ for } t \geq 0.$$

Since $v(t)$ is a solution of equation (2.3) we obtain that $v(t) - y(t) = U(t,0)(v(0) - y(0))$ for $t \geq 0$. Put now $\zeta_0 = v(0) - y(0)$. The boundedness of $v(\cdot)$ and $y(\cdot)$ on $[0,\infty)$ implies that $\zeta_0 \in X_0$. Finally, since $v(t) = U(t,0)\zeta_0 + y(t)$ for $t \geq 0$, the equality (4.4) follows.

(b): Similarly as in Item (a) we put $y(t) := \int_0^\infty \mathcal{G}(t,\tau)g(u)(\tau)d\tau$ for $t \geq 0$. Since g satisfies the conditions in (3.2) and using estimate (4.3) we obtain that

$$\begin{aligned}\|y(t)\| &\leqslant (1+H)N\int_0^\infty e^{-\nu|t-\tau|}(\|g(u)(\tau) - g(0)(\tau)\| + \|g(0)(\tau)\|)d\tau \\ &\leqslant (1+H)N(L\rho+\gamma)\int_0^\infty e^{-\nu|t-\tau|}d\tau \\ &\leqslant \frac{2(1+H)N(L\rho+\gamma)}{\nu} \text{ for } t \geq 0.\end{aligned}$$

Also, it is straightforward to see that $y(\cdot)$ satisfies the equation

$$y(t) = U(t,0)y(0) + \int_0^t U(t,\tau)g(u)(\tau)d\tau \text{ for } t \geq 0.$$

Since $u(t)$ is a solution of equation (3.3) we obtain that $u(t) - y(t) = U(t,0)(u(0) - y(0))$ for $t \geq 0$. Put now $v_0 = u(0) - y(0)$. The boundedness of $u(\cdot)$ and $y(\cdot)$ on $\mathbb{R}_+$ implies that $v_0 \in X_0$. Finally, the relation $u(t) = U(t,0)v_0 + y(t)$ for $t \geq 0$ yields the equality (4.5). □

Remark 4.3. By straightforward computations we can prove that the converses of statements (a) and (b) are also true, i.e., a solution of Equation (4.4) satisfies Equation (2.3) for $t \geq 0$, and that of Equation (4.5) satisfies Equation (3.3) for $t \geq 0$.

We next prove the existence of bounded solutions to Equations (2.3) and (3.3) (i.e., bounded mild solutions to (2.1) and (3.1)) and hence that of periodic solutions in the following theorem.

Theorem 4.4. *Consider equations (2.3) and (3.3). Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ satisfy (2.4) and have an exponential dichotomy with the dichotomy projections $P(t)$, $t \geq 0$, and constants N, ν. Suppose that $f \in C_b(\mathbb{R}_+, X)$ is T-periodic and that $g : C_b(\mathbb{R}_+, X) \to C_b(\mathbb{R}_+, X)$ satisfies the conditions in (3.2) with given constants ρ, L, γ. Then, the following assertions hold true.*

(a) *Equation (2.3) has a unique T-periodic solution.*
(b) *For sufficiently small L, γ equation (3.3) has unique T-periodic solutions.*

Proof. (a): For a given $f \in C_b(\mathbb{R}_+, X)$ taking $\zeta_0 = 0 \in X_0$ in (4.4) we have that equation (2.3) has a bounded solution

$$u(t) = \int_0^\infty \mathcal{G}(t,\tau) f(\tau) d\tau, \tag{4.6}$$

and this solution can be estimated using the inequality (4.3) by

$$\|u\|_{C_b} \leqslant \frac{2N(H+1)}{\nu} \|f\|_{C_b}. \tag{4.7}$$

Applying Theorem 2.3 we obtain that for T-periodic function $f \in C_b(\mathbb{R}_+, X)$ there exists a T-periodic solution $\hat{u}$ of (2.3) (i.e., a T-periodic mild solution of (2.1)) satisfying

$$\|\hat{u}\|_{C_b} \leqslant \left(\frac{2N(H+1)}{\nu} + T \right) K e^{\alpha T} \|f\|_{C_b}. \tag{4.8}$$

The uniqueness of the T-periodic solution follows from the fact that for two T-periodic (hence bounded on $\mathbb{R}_+$) solutions $\hat{u}$ and $\hat{v}$ we obtain by using the form for bounded solutions (4.4) that $\|\hat{u}(t) - \hat{v}(t)\| = \|U(t,0)(u_0 - v_0)\| \leqslant N e^{-\nu t} \|u_0 - v_0\| \to 0$ as $t \to \infty$ since $u_0, v_0 \in X_0$. This, together with the periodicity, implies $\hat{u}(t) = \hat{v}(t)$ for all $t \geq 0$, finishing the proof of assertion (a).

(b): By assertion (a), for each T-periodic input function f, the linear problem (2.3) has a unique T-periodic solution $\hat{u}$ satisfying inequality (4.8). Therefore, the assertion (b) then follows from Theorem 3.1. □

We now prove the conditional stability of periodic solutions to (3.3).

Theorem 4.5. *Let the assumptions of Theorem 4.4 hold, and let $\hat{u}$ be the T-periodic solution of (3.3) obtained in assertion (b) of Theorem 4.4. Denote by $B_r(x)$ (by $\mathcal{B}_r(v)$) the ball in X (in $C_b(\mathbb{R}_+, X)$) centered at x (at v) with radius r. Let $\mathcal{B}_\rho(0)$ be the ball containing $\hat{u}$ as in assertion (b) of Theorem 4.4. Suppose further that there exists a positive constant L_1 such that $\|g(v_1) - g(v_2)\|_{C_b} \leqslant L_1 \|v_1 - v_2\|_{C_b}$ for all $v_1, v_2 \in \mathcal{B}_{2\rho}(0)$. Then, if L_1 is small enough, then there corresponds to each $v_0 \in B_{\frac{\rho}{2N}}(P(0)\hat{u}(0)) \cap P(0)X$ one and only one solution $u(t)$ of equation (3.3) on $\mathbb{R}_+$ satisfying the conditions $P(0)u(0) = v_0$ and $u \in \mathcal{B}_\rho(\hat{u})$. Moreover, the following estimate is valid for $u(t)$ and $\hat{u}(t)$:*

$$\|u(t) - \hat{u}(t)\| \leqslant C e^{-\mu t} \|P(0)u(0) - P(0)\hat{u}(0)\| \text{ for } t \geq 0, \tag{4.9}$$

for some positive constants C and μ independent of u and $\hat{u}$.

Proof. For $v_0 \in B_{\frac{\rho}{2N}}(P(0)\hat{u}(0)) \cap P(0)X$ we will prove that the transformation F defined by

$$(Fw)(t) = U(t,0)v_0 + \int_0^\infty \mathcal{G}(t,\tau)(g(w)(\tau))d\tau \text{ for } t \geq 0$$

acts from $\mathcal{B}_\rho(\hat{u})$ into $\mathcal{B}_\rho(\hat{u})$ and is a contraction.

In fact, for $w(\cdot) \in \mathcal{B}_\rho(\hat{u})$ we have that

$$\|w\|_{C_b} \leqslant \|w - \hat{u}\|_{C_b} + \|\hat{u}\|_{C_b} \leqslant 2\rho \tag{4.10}$$

and $\|g(w) - g(\hat{u})\|_{C_b} \leqslant L_1\|w - \hat{u}\|_{C_b} \leqslant L_1\rho$. Therefore, putting

$$y(t) := (Fw)(t) = U(t,0)v_0 + \int_0^\infty \mathcal{G}(t,\tau)(g(w)(\tau))d\tau \text{ for } t \geq 0$$

we obtain

$$\begin{aligned}\|y(t) - \hat{u}(t)\| &\leqslant Ne^{-\nu t}\|v_0 - P(0)\hat{u}(0)\| + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}d\tau\|g(w) - g(\hat{u})\|_{C_b} \\ &\leqslant N\|v_0 - P(0)\hat{u}(0)\| + \frac{2(1+H)NL_1\rho}{\nu}\end{aligned}$$

for all $t \geq 0$. Therefore,

$$\|Fw - \hat{u}\|_{C_b} \leqslant N\|v_0 - P(0)\hat{u}(0)\| + \frac{2(1+H)NL_1\rho}{\nu}.$$

Using now the fact that $\|v_0 - P(0)\hat{u}(0)\| \leqslant \frac{\rho}{2N}$ we obtain that if L_1 is small enough, then the transformation F acts form $\mathcal{B}_\rho(\hat{u})$ into $\mathcal{B}_\rho(\hat{u})$.

Now, for $x, z \in \mathcal{B}_\rho(\hat{u})$ (similarly as in (4.10) we have $\|x\|_{C_b}, \|z\|_{C_b} \leqslant 2\rho$) we estimate

$$\begin{aligned}\|(Fx)(t) - (Fz)(t)\| &\leqslant \int_0^\infty \|\mathcal{G}(t,\tau)\|\|g(x)(\tau) - g(z)(\tau))\|d\tau \\ &\leqslant (1+H)N\int_0^\infty e^{-\nu|t-\tau|}d\tau\|g(x) - g(z)\|_{C_b} \text{ for all } t \geq 0.\end{aligned}$$

Therefore,

$$\|Fx - Fz\|_{C_b} \leqslant \frac{2(1+H)NL_1}{\nu}\|x(\cdot) - z(\cdot)\|_{C_b}.$$

Using now the fact that $\frac{2(1+H)NL_1}{\nu} < 1$ we obtain that $F : \mathcal{B}_\rho(\hat{u}) \to \mathcal{B}_\rho(\hat{u})$ is a contraction. Thus, there exists a unique $u \in \mathcal{B}_\rho(\hat{u})$ such that $Fu = u$. By definition of F we have that u is the unique solution in $\mathcal{B}_\rho(\hat{u})$ of equation (4.5) for $t \geq 0$. By Lemma 4.2 and Remark 4.3 we have that u is the unique solution in $\mathcal{B}_\rho(\hat{u})$ of equation (3.3).

Finally, we prove the estimate (4.9). To do this, since both $\hat{u}$ and u are bounded on $\mathbb{R}_+$, we can use the formula (4.5) to write

$$u(t) - \hat{u}(t) = U(t,0)(P(0)u(0) - P(0)\hat{u}(0)) + \int_0^\infty \mathcal{G}(t,\tau)(g(u)(\tau) - g(\hat{u})(\tau))d\tau.$$

Therefore,

$$\begin{aligned}\|u(t) - \hat{u}(t)\| &\leqslant Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| + (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\|g(u)(\tau) - g(\hat{u})(\tau)\|d\tau \\ &\leqslant Ne^{-\nu t}\|P(0)u(0) - P(0)\hat{u}(0)\| + (1+H)NL_1\int_0^\infty e^{-\nu|t-\tau|}\|u(\tau) - \hat{u}(\tau)\|d\tau.\end{aligned}$$

Applying now a Gronwall-typed inequality [2, Corollary III.2.3] we obtain for $\beta := (1+H)NL_1 < \frac{\nu}{2}$ that

$$\|u(t) - \hat{u}(t)\| \leqslant Ce^{-\mu t}\|P(0)u(0) - P(0)\hat{u}(0)\| \text{ for } \mu := \sqrt{\nu^2 - 2\nu\beta},\ C := \frac{2N\nu}{\nu + \sqrt{\nu^2 - 2\nu\beta}}.$$

The proof is complete. □

Remark 4.6. The assertion of the above theorem shows us the *conditional stability* of the periodic solution $\hat{u}$ in the sense that for any other solution u such that $P(0)u(0) \in B_{\frac{\rho}{2N}}(P(0)\hat{u}(0)) \cap P(0)X$ and u being in a small ball $\mathcal{B}_\rho(\hat{u})$ we have $\|u(t) - \hat{u}(t)\| \to 0$ exponentially as $t \to \infty$ (see inequality (4.9)).

For an exponentially stable evolution family (see Definition 4.1 (2)) we have the following corollary which is a direct consequence of Theorem 4.5.

Corollary 4.7. *Let the assumptions of Theorem 4.4 hold, and let $\hat{u}$ be the periodic solution of (3.3) obtained in assertion (b) of Theorem 4.4. Let further the evolution family $(U(t,s))_{t\geq s\geq 0}$ be exponentially stable. Then, the periodic solution $\hat{u}$ is exponentially stable in the sense that for any other solution $u \in C_b(\mathbb{R}_+, X)$ of (3.3) such that $\|u(0) - \hat{u}(0)\|$ is small enough we have*

$$\|u(t) - \hat{u}(t)\| \leqslant Ce^{-\mu t}\|u(0) - \hat{u}(0)\| \textit{ for all } t \geq 0 \tag{4.11}$$

for some positive constants C and μ independent of u and $\hat{u}$.

Proof. We just apply Theorem 4.5 for $P(t) = Id$ for all $t \geq 0$ to obtain the assertion of the theorem. □

We finally illustrate our results by the following example.

Example 4.8. We consider the problem

$$\begin{cases} w_t(t,x) = a(t)[w_{xx}(t,x) + \delta w(x,t)] + |w|^{k-1}w(x,t) + h(x,t), \text{ for } 0 < x < \pi,\ t \geq 0, \\ w(0,t) = w(\pi,t) = 0,\ t \geq 0. \end{cases} \tag{4.12}$$

Here, $\delta \in \mathbb{R}$ and $\delta \neq n^2$ for all $n \in \mathbb{N}$; the function $a(t) \in L_{1,loc}(\mathbb{R}_+)$ is T-periodic and satisfies the condition $0 < \gamma_0 \leqslant a(t) \leqslant \gamma_1$ for fixed γ_0, γ_1; the exponent $k \in \mathbb{N}$, $k > 1$; the function $h : [0,\pi] \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $[0,\pi] \times \mathbb{R}_+$ and T-periodic with respect to t.

We next put $X := L_2[0,\pi]$, and let $A : X \supset D(A) \to X$ be defined by $Ay = y'' + \delta y$, with the domain

$$D(A) = \{y \in X : y \text{ and } y' \text{ are absolutely continuous, } y'' \in X, y(0) = y(\pi) = 0\}.$$

It can be seen (see [3]) that A is the generator of an analytic semigroup $(\mathbb{T}(t))_{t\geq 0}$. Since $\sigma(A) = \{-n^2+\delta : n = 1,2,3,\dots\}$ applying the spectral mapping theorem for analytic semigroups we get

$$\sigma(\mathbb{T}(t)) = e^{t\sigma(A)} = \{e^{t(-n^2+\delta)} : n = 1,2,3,\dots\} \text{ and hence } \sigma(\mathbb{T}(t)) \cap \Gamma = \emptyset \text{ for all } t > 0, \tag{4.13}$$

where $\Gamma := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Putting now $A(t) := a(t)A$ we have that $A(t)$ is the T-periodic, and the family $(A(t))_{t\geq 0}$ generates a T-periodic evolution family $U(t,s)_{t\geq s\geq 0}$ which is defined by the formula $U(t,s) = \mathbb{T}(\int_s^t a(\tau)d\tau)$.

By (4.13) we have that the analytic semigroup $(\mathbb{T}(t))_{t\geq 0}$ is hyperbolic (or has an exponential dichotomy) with the projection P satisfying

i) $\|\mathbb{T}(t)x\| \leqslant Ne^{-\beta t}\|x\|$ for $x \in PX$, $t \geq 0$
ii) $\|\mathbb{T}(-t)_{|}x\| = \|(\mathbb{T}(t)_{|})^{-1}x\| \leqslant Ne^{-\beta t}\|x\|$ for $x \in \mathrm{Ker}P$, $t \geq 0$, where the invertible operator $\mathbb{T}(t)_{|}$ is the restriction of $T(t)$ to $\mathrm{Ker}P$, and N, β are positive constants.

Using the hyperbolicity of $(\mathbb{T}(t))_{t\geq 0}$ it is straightforward to check that the evolution family $U(t,s)_{t\geq s\geq 0}$ has an exponential dichotomy with the projection $P(t) = P$ for all $t \geq 0$ and the dichotomy constants N and $\nu := \beta\gamma_0$ by the following estimates:

$$\|U(t,s)x\| \leqslant Ne^{-\nu(t-s)}\|x\| \text{ for } x \in PX,\ t \geq s \geq 0,$$
$$\|U(s,t)_{|}x\| \leqslant Ne^{-\nu(t-s)}\|x\| \text{ for } x \in \mathrm{Ker}P,\ t \geq s \geq 0.$$

We then define the function $g : C_b(\mathbb{R}_+, X) \to C_b(\mathbb{R}_+, X)$ by $g(u)(t) := |u|^{k-1}u(t) + h(\cdot,t)$.

Equation (4.12) can now be rewritten as

$$\frac{du}{dt} = A(t)u(t) + g(u)(t) \text{ for } u(t)(\cdot) = w(\cdot,t),$$

where $\|g(0)\|_{C_b} \leqslant \gamma$ for $\gamma := \sup_{t\in[0,\pi]}(\int_0^\pi |h(x,t)|^2dx)^{1/2}$. Since $h(x,t)$ is T-periodic with respect to t, it follows that g maps T-periodic functions to T-periodic functions. Moreover, for all $v_1, v_2 \in C_b(\mathbb{R}_+, X)$ we have

$$\begin{aligned}
\|g(v_1) - g(v_2)\|_{C_b} &= \big\||v_1|^{k-1}v_1 - |v_2|^{k-1}v_2\big\|_{C_b} \\
&= \||v_1|^{k-1}v_1 - |v_1|^{k-1}v_2 + |v_1|^{k-1}v_2 - |v_1|^{k-2}|v_2|v_2 + \cdots + |v_1||v_2|^{k-2}v_2 - |v_2|^{k-1}v_2\|_{C_b} \\
&\leqslant \sum_{j=0}^{k-1} \big\||v_1 - v_2||v_1|^j|v_2|^{k-1-j}\big\|_{C_b} \\
&\leqslant k\|v_1 - v_2\|_{C_b} r^{k-1} \quad \text{ for all } \ v_1, v_2 \in \mathcal{B}_r(0).
\end{aligned}$$

Therefore, g satisfies the hypotheses of Theorems 4.4 and 4.5 with $\rho = r$, $L = k\rho^{k-1}$ and $L_1 = k(2\rho)^{k-1}$.

By Theorems 4.4 and 4.5 we obtain that, if ρ (therefore L and L_1) and γ are small enough, then equation (4.12) has one and only one mild T-periodic solution $\hat{u} \in \mathcal{B}_\rho(0)$ and this solution $\hat{u}$ is conditionally stable in the sense of Remark 4.6.

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