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Zero dynamics and funnel control for linear electrical circuits ☆

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Abstract

We consider electrical circuits containing linear resistances, capacitances and inductances. The circuits can be described by differential-algebraic input–output systems, where the input consists of voltages of voltage sources and currents of current sources and the output consists of currents of voltage sources and voltages of current sources. We generalize a characterization of asymptotic stability of the circuit and give sufficient topological criteria for its invariant zeros being located in the open left half-plane. We show that asymptotic stability of the zero dynamics can be characterized by means of the interconnectivity of the circuit and that it implies that the circuit is high-gain stabilizable with any positive high-gain factor. Thereafter we consider the output regulation problem for electrical circuits by funnel control. We show that for circuits with asymptotically stable zero dynamics, the funnel controller achieves tracking of a class of reference signals within a pre-specified funnel; this means in particular that the transient behavior of the output error can be prescribed and the funnel controller does neither incorporate any internal model for the reference signals nor any identification mechanism, it is simple in its design. The results are illustrated by a simulation of a discretized transmission line.

1. Introduction

The concept of zero dynamics and its asymptotic stability is important in the theory and application of adaptive control. Especially systems governed by ordinary differential equations

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with asymptotically stable zero dynamics, relative degree one and positive definite high-gain matrix can be stabilized by high-gain output feedback [7,31]. In particular, this class allows the application of the *funnel controller* [18,20,21], a closed-loop control law of intriguing simplicity that guarantees that the output evolves inside a prescribed domain (the so-called *funnel*) around some given reference trajectory. Recently, funnel control was shown to be also feasible for systems governed by differential-algebraic equations with asymptotically stable zero dynamics [2–4] and some additional criterion that comprises systems with relative degree one as well as systems with transfer function having a proper inverse.

Asymptotic stability of the zero dynamics and relative degree are said to be *structural properties* in [16,17,19]. That is, for the feasibility of the funnel controller, the exact knowledge of the system parameters is not required.

The present paper is devoted to the analysis of zero dynamics and feasibility of the funnel controller for systems governed by electrical circuits with voltage and current sources and linear time-invariant resistances, capacitances and inductances. The input is composed of voltages of voltage sources and currents of current sources, whereas the output consists of currents of voltage sources and voltages of current sources. We show that the zero dynamics have a descriptive physical interpretation: they are the free dynamics of an artificial circuit which emerges from the to-be-considered circuit by replacing voltage sources by open circuits and current sources by short circuits. Based on this finding, we can show that several properties of the zero dynamics are *physically structural*: for instance, we show that autonomy of the zero dynamics (that is, its evolution is fully described by the initial value) is equivalent to the absence of loops of current sources and cutsets of voltage sources. Inspired by the results of Riaza and Tischendorf in [27] on asymptotic stability of circuits, we are able to characterize asymptotic stability of the zero dynamics by sufficient criteria on the circuit topology. In particular, the parameter values of resistances, capacitances and inductances need not to be known explicitly (only the physically reasonable assumption of positivity of these values is made).

We further analyze high-gain output stabilizability of circuits. We will see that an output feedback has the physical interpretation of replacing sources by resistances. Analysis of asymptotic stability of the closed-loop system therefore again leads to the problem of stability analysis of a certain replacement circuit.

Finally, we discuss funnel control for electrical circuits. It is shown that asymptotic stability of the zero dynamics is sufficient for feasibility of the funnel controller. Only some assumptions on smoothness of the reference trajectory and the funnel boundary are made. In particular, we will show that no further condition on the relative degree has to be imposed. These results are further extended to circuits with possibly non-autonomous zero dynamics: under the condition of the invariant zeros of the system being located in the open left half-plane (this condition is necessary for systems with asymptotically stable zero dynamics), we show that funnel control is feasible, provided that the reference trajectory evolves in a certain subspace. The latter will be shown to have the physical interpretation that the reference trajectory satisfies Kirchhoff's laws pointwise.

1.1. Nomenclature

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\mathbb{R}_{\geq 0}$, $(\mathbb{R}_{>0})$	$=[0,\infty)$, $((0,\infty))$
$\mathbb{C}_+(\mathbb{C}_-)$	open set of complex numbers with positive (negative) real part, resp.

$\mathbb{R}[s]$	the ring of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}(s)$	the quotient field of $\mathbb{R}[s]$
$R^{n,m}$	the set of $n\times m$ matrices with entries in a ring R
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n,n}$
$\mathcal{O}_n(\mathbb{R})$	the group of orthogonal matrices in $\mathbb{R}^{n,n}$
M^*	$=\overline{M}^\top$, the conjugate transpose of $M\in\mathbb{C}^{n,m}$
$\lVert x\rVert$	$=\sqrt{x^\top x}$, the Euclidean norm of $x\in\mathbb{R}^n$
$\lVert M\rVert$	$=\max\{\lVert Mx\rVert \mid x\in\mathbb{R}^m,\ \lVert x\rVert=1\}$, induced norm of $M\in\mathbb{R}^{n,m}$
$\mathcal{C}^\ell(\mathcal{T};\mathbb{R}^n)$	the set of ℓ-times continuously differentiable functions $f:\mathcal{T}\to\mathbb{R}^n$, $\ell\in\mathbb{N}_0\cup\{\infty\}$, $\mathcal{T}\subseteq\mathbb{R}$ an interval
$\mathcal{B}^\ell(\mathcal{T};\mathbb{R}^n)$	$=\{f\in\mathcal{C}^\ell(\mathcal{T};\mathbb{R}^n) \mid (\mathrm{d}^i/\mathrm{d}t^i)f$ is bounded for $i=0,\dots,\ell\}$, $\ell\in\mathbb{N}_0\cup\{\infty\}$, $\mathcal{T}\subseteq\mathbb{R}$ an interval

1.2. System class

We consider linear differential-algebraic systems of the form

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)&=Ax(t)+Bu(t)\\ y(t)&=Cx(t),\end{aligned}\tag{1}$$

where $E,A\in\mathbb{R}^{n,n}$, $B,C^\top\in\mathbb{R}^{n,m}$; the set of these square systems (i.e., same number of inputs and outputs) is denoted by $\Sigma_{n,m}$ and we write $[E,A,B,C]\in\Sigma_{n,m}$.

The functions $u,y:\mathbb{R}\to\mathbb{R}^m$ are called *input* and *output* of the system, respectively. A trajectory $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m$ is said to be a *solution* of Eq. (1) if it belongs to the *behavior* of Eq. (1):

$$\mathfrak{B}_{[E,A,B,C]}:=\left\{(x,u,y)\in\mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m)\;\middle|\;\begin{array}{l}Ex\in\mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R}^n)\text{ and }(x,u,y)\\ \text{solves Eq. (1) for all }t\geq 0\end{array}\right\}.$$

In the present paper, we are interested in systems of the form (1), which arise from modified nodal analysis (MNA) models of electrical circuits [15], i.e.,

$$sE-A=\begin{bmatrix}sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top+A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} & A_{\mathcal{V}}\\ -A_{\mathcal{L}}^\top & s\mathcal{L} & 0\\ -A_{\mathcal{V}}^\top & 0 & 0\end{bmatrix},\quad B=C^\top=\begin{bmatrix}-A_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & -I_{n_{\mathcal{V}}}\end{bmatrix},\tag{2}$$

$$x=(\eta^\top,i_{\mathcal{L}}^\top,i_{\mathcal{V}}^\top)^\top,\quad u=(i_{\mathcal{I}}^\top,v_{\mathcal{V}}^\top)^\top,\quad y=(-v_{\mathcal{I}}^\top,-i_{\mathcal{V}}^\top)^\top,\tag{3}$$

where

$$\left.\begin{array}{l}\mathcal{C}\in\mathbb{R}^{n_{\mathcal{C}},n_{\mathcal{C}}},\ \mathcal{G}\in\mathbb{R}^{n_{\mathcal{G}},n_{\mathcal{G}}},\ \mathcal{L}\in\mathbb{R}^{n_{\mathcal{L}},n_{\mathcal{L}}},\\ A_{\mathcal{C}}\in\mathbb{R}^{n_e,n_{\mathcal{C}}},\ A_{\mathcal{R}}\in\mathbb{R}^{n_e,n_{\mathcal{G}}},\ A_{\mathcal{L}}\in\mathbb{R}^{n_e,n_{\mathcal{L}}},\ A_{\mathcal{V}}\in\mathbb{R}^{n_e,n_{\mathcal{V}}},\ A_{\mathcal{I}}\in\mathbb{R}^{n_e,n_{\mathcal{I}}},\\ n=n_e+n_{\mathcal{L}}+n_{\mathcal{V}},\quad m=n_{\mathcal{I}}+n_{\mathcal{V}}.\end{array}\right\}\tag{4}$$

Here $A_{\mathcal{C}}$, $A_{\mathcal{R}}$, $A_{\mathcal{L}}$, $A_{\mathcal{V}}$ and $A_{\mathcal{I}}$ denote the element-related incidence matrices, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, respectively, $\eta(t)$ is the vector of node potentials, $i_{\mathcal{L}}(t)$, $i_{\mathcal{V}}(t)$, $i_{\mathcal{I}}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_{\mathcal{V}}(t)$, $v_{\mathcal{I}}(t)$ are the voltages of voltage and current sources, respectively.

1.3. Control objective

We consider output regulation for Eq. (1) by *funnel control*

$$\begin{aligned} &u(t) = -k(t)e(t), \quad \text{where } e(t) = y(t) - y_{\text{ref}}(t) \\ &k(t) = \frac{1}{1-\varphi(t)^2\|e(t)\|^2}, \end{aligned} \tag{5}$$

where it is desired that the reference signal y_{ref} is tracked by the output signal y within the pre-specified performance funnel

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \,|\, \varphi(t)\|e\| < 1\}, \tag{6}$$

where φ is a nonnegative bounded function with $\varphi(0) = 0$ and otherwise φ is bounded away from 0, see Section 7. Note that no exact tracking is pursued, but a tracking error evolving in $\mathcal{F}_\varphi$. In contrast to approximate tracking, funnel control achieves arbitrarily given transient behavior and the funnel boundary is not necessarily monotonically decreasing or even constant.

The concept of funnel control as a simple strategy for output regulation has been developed in [21] for ODEs, see also the survey [19] and the references therein. Funnel control for linear DAE systems has been investigated in the recent papers [2–4]; nonlinear DAEs have been considered in [5].

The funnel controller proved to be the appropriate tool for tracking problems in various "real world" applications, such as chemical reactor models [23], industrial servo-systems [14,22] and rigid and revolute joint robotic manipulators [13]. In the present paper we show that the funnel controller can be further applied to problems in signal processing. This is underlined in Section 8, where we provide a simulation of the funnel controller with a reference signal formed by a sawtooth wave signal.

In the present paper we aim to apply the funnel controller to MNA models of passive electrical circuits, see Section 7. To this end, the guiding research idea is that the controller design is independent of the system parameters, that means only structural assumptions are required to be satisfied.

As outlined in [18], for linear single-input single-output ODE systems, these structural assumptions are represented by the three properties (a) positive high-frequency gain, (b) relative degree one and the (c) minimum phase property. For DAEs, the situation is more involved, but essentially (a)–(c) together with (d) right-invertibility of the system are the crucial assumptions needed for funnel control [2]. In the present paper we show that (a), (b) and (d) are always satisfied for MNA models of electrical circuits, so it remains to find conditions for (c). Property (c) is equivalent to the so-called asymptotically stable zero dynamics (see the next subsection) and it is shown in Section 5 that the latter can be characterized by topological criteria on the circuit which qualifies this property as structural. The aforementioned statements do also explain the main difference compared to our earlier work [2–5].

1.4. Zero dynamics

As explained in the previous subsection, we place particular emphasis on the *zero dynamics* of Eq. (1); this concept has been introduced in [6]. The zero dynamics are, for $[E, A, B, C] \in \Sigma_{n,m}$, defined by

$$\mathcal{ZD}_{[E,A,B,C]} := \{(x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \,|\, y = 0\}.$$

By linearity of Eq. (1), $\mathcal{ZD}_{[E,A,B,C]}$ is a real vector space.

The zero dynamics of Eq. (1) are called *autonomous* if

$$\forall w_1, w_2 \in \mathcal{ZD}_{[E,A,B,C]} \, \forall I \subseteq \mathbb{R}_{\geq 0} \text{ open interval}: \; w_1|_I = w_2|_I \Longrightarrow w_1 = w_2; \tag{7}$$

and *asymptotically stable* if

$$\forall (x,u,y) \in \mathcal{ZD}_{[E,A,B,C]} : \lim_{t \to \infty} (x(t), u(t)) = 0.$$

Note that the above definitions are within the spirit of the *behavioral approach* [25] and take into account that the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are a linear behavior. In this framework the definition for autonomy of a general behavior was given in [25, Section 3.2] and the definition of asymptotic stability in [25, Definition 7.2.1]. (Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero).

In Section 5 we show that the zero dynamics of an MNA model of an electrical circuit can be interpreted as the free dynamics of a replacement circuit, where voltage sources are replaced with open circuits and current sources are replaced with short circuits. Therefore, stability investigations of the zero dynamics can be lead back to the respective considerations for the replacement circuit. In this regard, the approach from [27] can be used: it is clear that passive RLC circuits are always stable, however not necessarily asymptotically stable. The absence of eigenvalues on the imaginary axis for MNA models of electrical circuits with regular matrix pencil $sE - A$ has been characterized in [27] in terms of conditions on the network topology. In Section 4 we generalize this result to circuits which might contain cutsets of current sources and/or loops of voltage sources, i.e., where $sE - A$ is not necessarily regular.

The aforementioned Theorem 4.6 is the basis for the investigation of asymptotically stable zero dynamics and the location of invariant zeros in terms of topological criteria in Section 5. These results in turn allow for the application of high-gain output feedback $u = -ky$ for the stabilization of the circuit equation (1) in Section 6. It turns out that high-gain feedback has a practical interpretation as introduction of resistances: all current and voltage sources are replaced with resistances of values k^{-1} and k, resp.

2. Matrix pencils and rational functions

Let $sE - A \in \mathbb{R}[s]^{k,n}$ be a matrix pencil. Then $sE - A$ is called *regular* if $k = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

We introduce the following notation: for $k \in \mathbb{N}$, we define the matrices

$$N_k = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \;\; 0 \end{bmatrix} \in \mathbb{R}^{k,k}, \quad K_k = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \; L_k = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{k-1,k}.$$

Many properties of a matrix pencil can be characterized in terms of the *Kronecker canonical form (KCF)*.

Lemma 2.1 (*Kronecker canonical form [10]*). *For a matrix pencil $sE-A \in \mathbb{C}[s]^{k,n}$, there exist matrices $W \in \mathbf{Gl}_k(\mathbb{C})$, $T \in \mathbf{Gl}_n(\mathbb{C})$, such that*

$$W(sE-A)T = \operatorname{diag}(\mathcal{C}_1(\mathrm{s}), \ldots, \mathcal{C}_k(\mathrm{s})), \tag{8}$$

where each of the pencils $\mathcal{C}_j(s)$ is of one of the types presented in Table 1.

The numbers λ appearing in the blocks of type W1 are called the generalized eigenvalues of $sE-A$. A generalized eigenvalue is called semi-simple, if all blocks of type W1 corresponding to λ are of size 1×1.

The index $\nu \in \mathbb{N}_0$ of $sE-A$ is defined as

$$\nu := \max(\{k_j | \mathcal{C}_j(s) \text{ is of type W2 or W4},\ j=1,\ldots,k\} \cup \{0\}).$$

The following is immediate from the block structure of the KCF.

Corollary 2.2 (*Generalized eigenvalues*). *Let a pencil $sE-A \in \mathbb{R}[s]^{k,n}$ be given. Then $\lambda \in \mathbb{C}$ is a generalized eigenvalue of $sE-A$ if, and only if,*

$$\operatorname{rk}_{\mathbb{C}}(\lambda E - A) < \operatorname{rk}_{\mathbb{R}(s)}(sE-A).$$

It is shown in [10] that the KCF is unique up to permutation of the indices $j=1,\ldots,k$. Since each block of type W3 (W4) leads to an additional column (resp. row) rank deficiency of 1, the regularity of a pencil is equivalent to the absence of blocks of type W3 and W4 in its KCF.

In the following we collect some facts on rational matrix functions. These concepts and findings will play an important role for the analysis of an MNA model (1) and (2).

Definition 2.3 (*Positive real/proper rational function*). A rational matrix function $G(s) \in \mathbb{R}(s)^{m,m}$ is called *positive real* if $G(s)$ does not have any poles in $\mathbb{C}_+$ and, for all $\lambda \in \mathbb{C}_+$, we have

$$G(\lambda) + G^*(\lambda) \geq 0.$$

$G(s)$ is called *proper* if $\lim_{s\to\infty} G(s) \in \mathbb{R}^{m,m}$ exists.

Lemma 2.4 (*Properties of positive real functions [1, Section 5.1]*). *Let $G(s) \in \mathbb{R}(s)^{m,m}$ be positive real. Then there exist $\omega_1, \ldots, \omega_k \in \mathbb{R}$, Hermitian and positive semi-definite matrices $M_1, \ldots, M_k \in \mathbb{C}^{m,m}$, $M_0, M_\infty \in \mathbb{R}^{m,m}$ and some proper and positive real function $G_s(s) \in \mathbb{R}(s)^{m,m}$ which does not have any poles on $i\mathbb{R}$, such that*

$$G(s) = G_s(s) + sM_\infty + \frac{M_0}{s} + \sum_{j=1}^{k} \frac{M_j}{s - i\omega_j} + \frac{\overline{M}_j}{s + i\omega_j}.$$

Table 1
Block types in Kronecker canonical form.

Type	Size	$\mathcal{C}_j(s)$	Parameters
W1	$k_j \times k_j$	$(s-\lambda)I_{k_j} - N_{k_j}$	$k_j \in \mathbb{N}$, $\lambda \in \mathbb{C}$
W2	$k_j \times k_j$	$sN_{k_j} - I_{k_j}$	$k_j \in \mathbb{N}$
W3	$(k_j-1) \times k_j$	$sK_{k_j} - L_{k_j}$	$k_j \in \mathbb{N}$
W4	$k_j \times (k_j-1)$	$sK_{k_j}^\top - L_{k_j}^\top$	$k_j \in \mathbb{N}$

In particular, we may characterize the positive realness of matrix pencils $sE-A \in \mathbb{R}[s]^{n,n}$ by means of certain definiteness properties of the matrices $E, A \in \mathbb{R}^{n,n}$. This is a direct consequence of Lemma 2.4.

Corollary 2.5 (*Positive real matrix pencils*). *A matrix pencil $sE-A \in \mathbb{R}[s]^{n,n}$ is positive real if, and only if, $E=E^\top \geq 0$ and $A+A^\top \leq 0$.*

In the following we collect some further properties of positive real matrix pencils $sE-A$ with the additional assumption that the kernels of E and A intersect trivially. This in particular encompasses regular MNA models of passive electrical networks.

Lemma 2.6 (*Properties of positive real pencil*). *Let a positive real pencil $sE-A \in \mathbb{R}[s]^{n,n}$ be such that* $\ker E \cap \ker A = \{0\}$. *Then the following holds true*:

(i) *$sE-A$ is regular.*
(ii) *$(sE-A)^{-1} \in \mathbb{R}(s)^{n,n}$ is positive real.*
(iii) *All generalized eigenvalues of $sE-A$ have non-positive real part.*
(iv) *All generalized eigenvalues of $sE-A$ on the imaginary axis are semi-simple.*
(v) *The index of $sE-A$ is at most two.*

Proof. *Step* 1: To prove that (i) and (iii) hold true, we show that $\ker(\lambda E-A)=\{0\}$ for all $\lambda \in \mathbb{C}_+$. Seeking a contradiction, assume that $\lambda \in \mathbb{C}_+$ and $x \in \mathbb{C}^n \setminus \{0\}$ are such that $(\lambda E-A)x=0$. Then we obtain

$$0 = x^*((\lambda E-A)+(\lambda E-A)^*)x = 2\,\mathrm{Re}(\lambda)x^*Ex - x^*(A+A^\top)x.$$

Since, by Corollary 2.5, there holds $E \geq 0$, $A+A^\top \leq 0$ and $\mathrm{Re}(\lambda)>0$, we have $x^*Ex = x^*(A+A^\top)x=0$, whence, in particular, $Ex=0$. Therefore, the equation $(\lambda E-A)x=0$ gives also rise to $Ax=0$ and consequently, $x \in \ker E \cap \ker A = \{0\}$, a contradiction.

Step 2: (ii) follows from the fact the inverse of a positive real function is positive real as well [28].

Step 3: It remains to show that (iv) and (v) are valid: since $(sE-A)^{-1}$ is positive real by (ii), Lemma 2.4 gives rise to the fact that all poles on the imaginary axis are of order one and, moreover, $(sE-A)^{-1} = sM + G_p(s)$, where $G_p(s) \in \mathbb{R}[s]^{n,n}$ is proper and $M \in \mathbb{R}^{n,n}$. This in particular means that $s^{-1}(sE-A)^{-1}$ is proper. Let $W, T \in \mathbf{Gl}_n(\mathbb{C})$ be such that $W(sE-A)T$ is in KCF (8). Regularity of $sE-A$ then gives rise to

$$(sE-A)^{-1} = T^{-1}\mathrm{diag}(\mathcal{C}_1(\mathrm{s})^{-1}, \ldots, \mathcal{C}_k(\mathrm{s})^{-1})\mathrm{W}^{-1}. \tag{9}$$

Assuming that (iv) does not hold, i.e., there exists some $\omega \in \mathbb{R}$ such that $i\omega$ is a generalized eigenvalue of $sE-A$ which is not semi-simple. Then there exists some block $\mathcal{C}_j(s) = (s-i\omega)I_{k_j} - N_{k_j}$ with $k_j > 1$ in the KCF of $sE-A$. Hence, due to

$$\mathcal{C}_j(s)^{-1} = \sum_{l=0}^{k_j-1} \frac{1}{(s-i\omega)^{l+1}} N_{k_j}^l,$$

the formula (9) implies that $(sE-A)^{-1}$ has a pole of order greater than one on the imaginary axis, a contradiction.

Assume that (v) does not hold, i.e., the index of $sE-A$ exceeds two. Then there exists some block $\mathcal{C}_j(s)=sN_{k_j}-I_{k_j}$ with $k_j>2$ in the KCF of $sE-A$. Then

$$\mathcal{C}_j(s)^{-1}=-\sum_{l=0}^{k_j-1}s^lN_{k_j}^l,$$

and this contradicts properness of $s^{-1}(sE-A)^{-1}$. □

3. Graph theoretical preliminaries

In this section we introduce the graph theoretical concepts which are crucial for the modified nodal analysis of electrical circuits. We derive some characterizations for the absence of cutsets and loops in a given subgraph. These characterizations will be given in terms of algebraic properties of the incidence matrices.

Definition 3.1 (*Graph theoretical concepts*). A *graph* is a triple $\mathcal{G}=(V,E,\varphi)$ consisting of a *node set* V and a *branch set* E together with an *incidence map*

$$\varphi: E\to V\times V, \quad e\mapsto\varphi(e)=(\varphi_1(e),\varphi_2(e)),$$

where $\varphi_1(e)\neq\varphi_2(e)$ for all $e\in E$, i.e., the graph does not contain self-loops. If $\varphi(e)=(v_1,v_2)$, we call e to be *directed from v_1 to v_2*. v_1 is called the *initial node* and v_2 the *terminal node* of e. Two graphs $\mathcal{G}_a=(V_a,E_a,\varphi_a)$, $\mathcal{G}_b=(V_b,E_b,\varphi_b)$ are called *isomorphic*, if there exist bijective mappings $\iota_E: E_a\to E_b$, $\iota_V: V_a\to V_b$, such that $\varphi_{a,1}=\iota_V^{-1}\circ\varphi_{b,1}\circ\iota_E$ and $\varphi_{a,2}=\iota_V^{-1}\circ\varphi_{b,2}\circ\iota_E$.

Let $V'\subseteq V$ and let E' be a set of branches satisfying

$$E'\subseteq E|_{V'}:=\{e\in E|\varphi_1(e)\in V' \text{ and } \varphi_2(e)\in V'\}.$$

Further let $\varphi|_{E'}$ be the restriction of φ to E'. Then the triple $\mathcal{K}:=(V',E',\varphi|_{E'})$ is called *subgraph of* $\mathcal{G}$. In the case where $E'=E|_{V'}$, we call $\mathcal{K}$ the *induced subgraph on* V'. If $V'=V$, then $\mathcal{K}$ is called a *spanning subgraph*. A *proper subgraph* is one with $E\neq E'$.

$\mathcal{G}$ is called *finite*, if both the node and the branch set are finite.

For each branch e, define an additional branch $-e$ being directed from the terminal to the initial node of e, that is $\varphi(-e)=(\varphi_2(e),\varphi_1(e))$ for $e\in E$. Now define the set $\widetilde{E}=\{e|e\in E \textit{ or } -e\in E\}$. A tuple $w=(w_1,\dots,w_r)\in\widetilde{E}^r$, where for $i=1,\dots,r-1$,

$$v_0:=\varphi_1(v_1), \quad v_i:=\varphi_2(w_i)=\varphi_1(w_{i+1})$$

is called *path from v_0 to v_r*; w is called *elementary path*, if $v_1,\dots,v_r$ are distinct. A *loop* is an elementary path with $v_0=v_r$. Two nodes v,v' are called *connected*, if there exists a path from v to v'. The graph itself is called connected, if any two nodes are connected. A subgraph $\mathcal{K}=(V',E',\varphi|_{E'})$ is called *component of connectivity*, if it is connected and $\mathcal{K}^c:=(V\backslash V',E\backslash E',\varphi|_{E\backslash E'})$ is a subgraph.

A spanning subgraph $\mathcal{K}=(V,E',\varphi|_{E'})$ is called a *cutset* of $\mathcal{G}=(V,E,\varphi)$, if its branch set is non-empty, $\mathcal{G}-\mathcal{K}:=(V,E\backslash E',\varphi|_{E\backslash E'})$ is a disconnected subgraph and $\mathcal{G}-\mathcal{K}'$ is a connected subgraph for any proper spanning subgraph $\mathcal{K}'$ of $\mathcal{K}$.

For finite graphs we can set up special matrices which will be useful to describe Kirchhoff's laws.

Definition 3.2 (*Incidence matrix*). Let a finite graph $\mathcal{G}=(V,E,\varphi)$ with l branches $E=\{e_1,\dots,e_l\}$ and k nodes $V=\{v_1,\dots,v_k\}$ be given. Then the *all-node incidence matrix* of $\mathcal{G}$ is given by $A_0=(a_{ij})\in\mathbb{R}^{k,l}$, where

$$a_{ij}=\begin{cases}1 & \text{if } \varphi_1(e_j)=v_i,\\ -1 & \text{if } \varphi_2(e_j)=v_i,\\ 0 & \text{otherwise.}\end{cases}$$

Since the rows of A_0 sum up to the zero row vector, one might delete an arbitrary row of A_0 to obtain a matrix A having the same rank as A_0. We call A an *incidence matrix* of $\mathcal{G}$.

This section continues with some results on the relation between properties of subgraphs and linear algebraic properties of corresponding submatrices of incidence matrices. First we declare some manners of speaking.

Definition 3.3. Let $\mathcal{G}$ be a graph, $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, $\mathcal{L}$ be a subgraph of $\mathcal{G}$, and ℓ be a path of $\mathcal{G}$.

(i) $\mathcal{L}$ is called a $\mathcal{K}-$*cutset*, if $\mathcal{L}$ is a cutset of $\mathcal{K}$.
(ii) ℓ is called a $\mathcal{K}-$loop, if ℓ is a loop of $\mathcal{K}$.

A spanning subgraph $\mathcal{K}$ of the finite graph $\mathcal{G}$ has an incidence matrix $A_{\mathcal{K}}$ which is constructed by deleting columns of the incidence matrix A of $\mathcal{G}$ corresponding to the branches of the complementary spanning subgraph $\mathcal{G}-\mathcal{K}$. By a suitable reordering of the branches, the incidence matrix reads

$$A=[A_{\mathcal{K}}\;\; A_{\mathcal{G}-\mathcal{K}}]. \tag{10}$$

Lemma 3.4 (*Subgraphs and incidence matrices [27, Lemmas 2.1 and 2.3]*). *Let $\mathcal{G}$ be a connected graph with incidence matrix $A\in\mathbb{R}^{l-1,k}$. Further, let $\mathcal{K}$ be a spanning subgraph. Assume that the branches of $\mathcal{G}$ are sorted in a way that* Eq. (10) *is satisfied. Then the following holds true*:

(i) *The following two assertions are equivalent*:
 (a) *$\mathcal{G}$ does not contain $\mathcal{K}-$cutsets.*
 (b) $\ker A_{\mathcal{G}-\mathcal{K}}^{\top}=\{0\}$.
(ii) *The following two assertions are equivalent*:
 (a) *$\mathcal{G}$ does not contain $\mathcal{K}-$loops.*
 (b) $\ker A_{\mathcal{K}}=\{0\}$.

The following two auxiliary results are concerned with properties of subgraphs of subgraphs, and give some equivalent characterizations in terms of properties of their incidence matrices.

Lemma 3.5 (*Loops in subgraphs [27, Proposition 4.5]*). *Let $\mathcal{G}$ be a connected graph with incidence matrix $A\in\mathbb{R}^{k-1,l}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a* *spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that*

$$A=[A_{\mathcal{L}}\ \ A_{\mathcal{K}-\mathcal{L}}\ \ A_{\mathcal{G}-\mathcal{K}}] \quad \textit{and} \quad A_{\mathcal{K}}=[A_{\mathcal{L}}\ \ A_{\mathcal{K}-\mathcal{L}}].$$

Then the following two assertions are equivalent:

(a) *$\mathcal{G}$ does not contain $\mathcal{K}$−loops except for $\mathcal{L}$−loops.*
(b) $\ker A_{\mathcal{K}}=\ker A_{\mathcal{L}}\times\{0\}$.

Lemma 3.6 (*Cutsets in subgraphs [27, Proposition 4.4]*). *Let $\mathcal{G}$ be a connected graph with incidence matrix $A\in\mathbb{R}^{k-1,l}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that*

$$A=[A_{\mathcal{L}}\ \ A_{\mathcal{K}-\mathcal{L}}\ \ A_{\mathcal{G}-\mathcal{K}}] \quad \textit{and} \quad A_{\mathcal{G}-\mathcal{L}}=[A_{\mathcal{K}-\mathcal{L}}\ \ A_{\mathcal{G}-\mathcal{K}}].$$

Then the following two assertions are equivalent:

(a) *$\mathcal{G}$ does not contain $\mathcal{K}$−cutsets except for $\mathcal{L}$−cutsets.*
(b) $\ker A_{\mathcal{G}-\mathcal{K}}^{\top}=\ker A_{\mathcal{G}-\mathcal{L}}^{\top}$.

4. Circuit equations

It is well-known [15,8] that the graph underlying an electrical circuit can be described by an incidence matrix $A\in\mathbb{R}^{k-1,l}$, which can be decomposed into submatrices

$$A=[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}\ \ A_{\mathcal{I}}]$$

for the quantities in Eq. (4), where $n_e=k-1$ and $l=n_{\mathcal{C}}+n_{\mathcal{G}}+n_{\mathcal{L}}+n_{\mathcal{V}}+n_{\mathcal{I}}$. Each submatrix is the incidence matrix of a specific subgraph of the circuit graph. $A_{\mathcal{C}}$ is the incidence matrix of the subgraph consisting of all circuit nodes and all branches corresponding to capacitors. Similarly, $A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$ are the incidence matrices corresponding to the resistor, inductor, voltage source and current source subgraphs, respectively. Then using the standard MNA modeling procedure [15], which is just a clever arrangement of Kirchhoff's laws together with the characteristic equations of the devices, results in a differential-algebraic system (1) with (2)–(4). $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the constitutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_{\mathcal{L}}(t)$, $i_{\mathcal{V}}(t)$, $i_{\mathcal{I}}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_{\mathcal{V}}(t)$, $v_{\mathcal{I}}(t)$ are the voltages of voltage and current sources, respectively.

Definition 4.1 (*MNA model*). For a given linear electrical circuit, any differential-algebraic system (1) satisfying Eqs. (2)–(4), which arises from the MNA modeling procedure [15], is said to be an MNA model of the circuit.

It is a reasonable assumption that an electrical circuit is connected; otherwise, since the components of connectivity do not physically interact, one might consider them separately. Furthermore, in the present paper we consider circuits with *passive* devices. These assumptions lead to the following assumptions on an MNA model (2)–(4) of the circuit (compare Lemma 3.4):

(A1) $\operatorname{rk}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}\ \ A_{\mathcal{I}}]=n_e$,

(A2) $\mathcal{C}=\mathcal{C}^{\top}>0, \quad \mathcal{L}=\mathcal{L}^{\top}>0, \quad \mathcal{G}+\mathcal{G}^{\top}>0.$

It is possible that in the circuit equations (1) there are still redundant equations and superfluous variables, i.e., in general the pencil $sE-A$ arising from Eqs. (2) and (4) is not regular. In the following we show how this can be overcome by a simple transformation; the reduced circuit model is regular and positive real. This transformation is also important to show feasibility of funnel control in Section 7.

Remark 4.2. The above assumptions on the circuit imply that the system (1) is passive in the systems theoretic sense [29,30]. That is, the L_2-inner product of input and output of the trivially initialized system is always non-negative. This follows from the fact that the circuit equations imply

$$\int_{t_0}^{t_1} i_{\mathcal{V}}(t)^\top v_{\mathcal{V}}(t) + i_{\mathcal{I}}(t)^\top v_{\mathcal{I}}(t)\ \mathrm{d}t = \int_{t_0}^{t_1} \eta(t)^\top A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top \eta(t)\ \mathrm{d}t + \eta(t)^\top A_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top \eta(t)\Big|_{t=t_0}^{t=t_1} + i_{\mathcal{L}}(t)^\top \mathcal{L} i_{\mathcal{L}}(t)\Big|_{t=t_0}^{t=t_1}.$$

This means that the energy consumed by the circuit in the interval $[t_0, t_1]$ equals the sum of (a) the energy dissipated at the resistances, (b) the difference between the capacitive energies at t_0 and t_1, and (c) the difference between the inductive energies at t_0 and t_1.

Theorem 4.3 (*Reduction of circuit pencil*). *Let $sE-A \in \mathbb{R}[s]^{n,n}$ with E, A as in Eqs.* (2) and (4) *be given and suppose that* (**A1**) *and* (**A2**) *hold. Let $Z_{\mathcal{CRLV}}$, $Z'_{\mathcal{CRLV}}$, $\overline{Z}_{\mathcal{V}}$, $\overline{Z}'_{\mathcal{V}}$ be real matrices with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRLV}} = \ker\,[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}]^\top, \quad \operatorname{im} Z'_{\mathcal{CRLV}} = \operatorname{im}\,[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}],$$
$$\operatorname{im} \overline{Z}_{\mathcal{V}} = \ker A_{\mathcal{V}}, \quad \operatorname{im} \overline{Z}'_{\mathcal{V}} = \operatorname{im} A_{\mathcal{V}}^\top.$$

Then we have

$$T = \begin{bmatrix} Z'_{\mathcal{CRLV}} & 0 & 0 & Z_{\mathcal{CRLV}} & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & \overline{Z}'_{\mathcal{V}} & 0 & \overline{Z}_{\mathcal{V}} \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}), \tag{11}$$

and

$$T^\top (sE-A)T = \begin{bmatrix} s\widetilde{E}-\widetilde{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where the pencil

$$s\widetilde{E}-\widetilde{A} = \begin{bmatrix} (Z'_{\mathcal{CRLV}})^\top (sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top) Z'_{\mathcal{CRLV}} & (Z'_{\mathcal{CRLV}})^\top A_{\mathcal{L}} & (Z'_{\mathcal{CRLV}})^\top A_{\mathcal{V}}\overline{Z}'_{\mathcal{V}} \\ -A_{\mathcal{L}}^\top Z'_{\mathcal{CRLV}} & s\mathcal{L} & 0 \\ -\overline{Z}'_{\mathcal{V}}A_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} & 0 & 0 \end{bmatrix} \tag{12}$$

is regular and satisfies $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$, $\widetilde{E} = \widetilde{E}^\top \geq 0$ *and* $\widetilde{A} + \widetilde{A}^\top \leq 0$.

Proof. The invertibility of T is a consequence of $\operatorname{im} Z_{\mathcal{CRLV}} \oplus \operatorname{im} Z'_{\mathcal{CRLV}} = \mathbb{R}^{n_e}$ and $\operatorname{im} \overline{Z}_{\mathcal{V}} \oplus \operatorname{im} \overline{Z}'_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$. The properties $\widetilde{E} = \widetilde{E}^\top \geq 0$ and $\widetilde{A} + \widetilde{A}^\top \leq 0$ follow immediately from the construction of $\widetilde{E}$ and $\widetilde{A}$. To prove that $s\widetilde{E}-\widetilde{A}$ is regular, it suffices by Lemma 2.6 to show that $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$: Let $x \in \ker \widetilde{E} \cap \ker \widetilde{A}$. Partitioning according to the block structure of

$\widetilde{E}$ and $\widetilde{A}$, i.e., $x=(x_1^\top, x_2^\top, x_3^\top)^\top$, and using that, by (A2), $\mathcal{C}>0$, $\mathcal{L}>0$ and $\mathcal{G}+\mathcal{G}^\top>0$, we obtain from $x^\top \widetilde{E}x = x^\top(\widetilde{A}+\widetilde{A}^\top)x=0$ that $x_2=0$ and

$$\begin{bmatrix} A_\mathcal{C}^\top \\ A_\mathcal{R}^\top \end{bmatrix} Z'_{\mathcal{CRLV}} x_1 = 0. \tag{13}$$

Furthermore, $\widetilde{A}x=0$ gives rise to

$$\text{(a) } (\overline{Z}'_\mathcal{V})^\top A_\mathcal{V}^\top Z'_{\mathcal{CRLV}} x_1 = 0, \quad \text{(b) } A_\mathcal{L}^\top Z'_{\mathcal{CRLV}} x_1 = 0, \text{ and (c) } (Z'_{\mathcal{CRLV}})^\top A_\mathcal{V} \overline{Z}'_\mathcal{V} x_3 = 0.$$

(a) implies

$$A_\mathcal{V}^\top Z'_{\mathcal{CRLV}} x_1 \in \ker(\overline{Z}'_\mathcal{V})^\top = (\operatorname{im} \overline{Z}'_\mathcal{V})^\perp = (\operatorname{im} A_\mathcal{V}^\top)^\perp,$$

whence $A_\mathcal{V}^\top Z'_{\mathcal{CRLV}} x_1 = 0$. Together with Eq. (13) and (b) this yields

$$Z'_{\mathcal{CRLV}} x_1 \in \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top = \operatorname{im} Z_{\mathcal{CRLV}} = (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp,$$

and therefore $x_1=0$. By (c) we find

$$A_\mathcal{V}\overline{Z}'_\mathcal{V} x_3 \in \ker(Z'_{\mathcal{CRLV}})^\top = (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp = \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \subseteq \ker A_\mathcal{V}^\top = (\operatorname{im} A_\mathcal{V})^\perp,$$

and thus $A_\mathcal{V}\overline{Z}'_\mathcal{V} x_3 = 0$. From this, we obtain

$$\overline{Z}'_\mathcal{V} x_3 \in \ker A_\mathcal{V} = (\operatorname{im} A_\mathcal{V}^\top)^\perp = (\operatorname{im} \overline{Z}'_\mathcal{V})^\perp,$$

whence $x_3=0$. □

We may infer the following characterization of the presence of generalized eigenvalues from Theorem 4.3.

Corollary 4.4 (*Kernel and generalized eigenvalues*). *Let $sE-A\in\mathbb{R}[s]^{n,n}$ with E,A as in Eqs.* (2) and (4) *be given and suppose that* (**A1**) *and* (**A2**) *hold. Then*

$$\ker_{\mathbb{R}(s)} sE-A = \ker_{\mathbb{R}(s)} [A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_\mathcal{V}.$$

Furthermore, $\lambda\in\mathbb{C}$ is not a generalized eigenvalue of $sE-A$ if, and only if,

$$\ker_{\mathbb{C}} \lambda E-A = \ker_{\mathbb{C}} [A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \times \{0\} \times \ker_{\mathbb{C}} A_\mathcal{V}.$$

Proof. Using the transformation matrix T in Eq. (11) and accompanying notation from Theorem 4.3, we obtain (denoting the number of columns of $Z_{\mathcal{CRLV}}$ by k_1 and the number of columns of $\overline{Z}_\mathcal{V}$ by k_2) that

$$\begin{aligned}\ker_{\mathbb{R}(s)} sE-A &= T(\underbrace{\ker_{\mathbb{R}(s)} (s\widetilde{E}-\widetilde{A})}_{=\{0\}} \times \mathbb{R}(s)^{k_1+k_2}) \\ &= \operatorname{im}_{\mathbb{R}(s)} Z_{\mathcal{CRLV}} \times \{0\} \times \operatorname{im}_{\mathbb{R}(s)} \overline{Z}_\mathcal{V} = \ker_{\mathbb{R}(s)} [A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_\mathcal{V}.\end{aligned}$$

Now let $\lambda\in\mathbb{C}$ and observe that

$$\ker_\mathbb{C} \lambda E-A = T(\ker_\mathbb{C} \lambda\widetilde{E}-\widetilde{A} \times \mathbb{C}^{k_1+k_2}).$$

By Corollary 2.2, λ is not a generalized eigenvalue of $sE-A$ if, and only if, $\operatorname{rk}_\mathbb{C} \lambda E-A = \operatorname{rk}_{\mathbb{R}(s)} sE-A$ or, equivalently, $\dim\ker_\mathbb{C} \lambda E-A = \dim\ker_{\mathbb{R}(s)} sE-A$. Therefore, λ is not a generalized eigenvalue of $sE-A$ if, and only if, $\ker_\mathbb{C} \lambda\widetilde{E}-\widetilde{A}=\{0\}$ and this implies the last statement of the corollary. □

In the following we will use expressions like $\mathcal{VL}-$loop for a loop in the circuit graph whose branch set consists only of branches corresponding to voltage sources and/or inductors. Likewise, a $\mathcal{IC}-$cutset is a cutset in the circuit graph whose branch set consist only of branches corresponding to current sources and/or capacitors.

Corollary 4.5 (*Regularity of circuit pencil*). *Let $sE-A \in \mathbb{R}[s]^{n,n}$ with E, A as in Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Then the following statements are equivalent*:

(a) *$sE-A$ is regular.*
(b) $\ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top = \{0\}$ *and* $\ker A_\mathcal{V} = \{0\}$.
(c) *The circuit neither contains $\mathcal{V}-$loops nor $\mathcal{I}-$cutsets.*

Proof. The result follows immediately from Corollary 4.4 and Lemma 3.4. □

Next we give sufficient criteria for the absence of purely imaginary generalized eigenvalues of the pencil $sE-A$ as in Eqs. (2) and (4). This result can be seen as a generalization of the results in [27] to circuits which might contain $\mathcal{I}-$cutsets and/or $\mathcal{V}-$loops, i.e., where $sE-A$ is not necessarily regular.

Theorem 4.6 (*Absence of imaginary eigenvalues*). *Let $sE-A \in \mathbb{R}[s]^{n,n}$ with E, A as in Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Furthermore, suppose that at least one of the following two assertions holds*:

(i) *The circuit neither contains $\mathcal{VL}-$loops except for $\mathcal{V}-$loops, nor $\mathcal{ICL}-$cutsets except for $\mathcal{IL}-$cutsets; equivalently*

$$\ker[A_\mathcal{V}\ \ A_\mathcal{L}] = \ker A_\mathcal{V} \times \{0\} \quad \textit{and} \quad \ker[A_\mathcal{R}\ \ A_\mathcal{V}]^\top = \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{V}]^\top. \tag{14}$$

(ii) The circuit neither contains $\mathcal{IC}-$cutsets except for $\mathcal{I}-$cutsets, nor $\mathcal{VCL}-$loops except for $\mathcal{VC}-$loops; equivalently

$$\ker[A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top = \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \quad \textit{and} \quad \ker[A_\mathcal{V}\ \ A_\mathcal{C}\ \ A_\mathcal{L}] = \ker[A_\mathcal{V}\ \ A_\mathcal{C}] \times \{0\}. \tag{15}$$

Then all generalized eigenvalues of $sE-A$ are in contained $\mathbb{C}_-$.

Proof. The equivalent characterizations of the absence of certain loops or cutsets in the circuit graph, respectively, and kernel conditions on the element-related incidence matrices follow from Lemmas 3.5 and 3.6.

By Theorem 4.3 and Lemma 2.6 all generalized eigenvalues of $sE-A$ are contained in $\overline{\mathbb{C}}_-$. Then, using Corollary 4.4, we have to show that

$$\forall \omega \in \mathbb{R} : \ker_\mathbb{C}(i\omega E - A) = \ker_\mathbb{C}[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{V}]^\top \times \{0\} \times \ker_\mathbb{C} A_\mathcal{V}. \tag{16}$$

Since "$\supseteq$" does always hold true, we show "$\subseteq$". Let $\omega \in \mathbb{R}$ and $x_1 \in \mathbb{C}^{n_e}$, $x_2 \in \mathbb{C}^{n_\mathcal{L}}$ and $x_3 \in \mathbb{C}^{n_\mathcal{V}}$ be such that

$$x := (x_1^\top, x_2^\top, x_3^\top)^\top \in \ker_\mathbb{C}(i\omega E - A). \tag{17}$$

By the structure of $sE-A$ as in Eq. (2), relation (17) implies $A_{\mathcal{V}}^\top x_1 = 0$ and

$$0 = x^*((i\omega E-A)+(i\omega E-A)^*)x = -x^*(A+A^\top)x = -x_1^* A_{\mathcal{R}}(\mathcal{G}+\mathcal{G}^\top)A_{\mathcal{R}}^\top x_1,$$

hence $A_{\mathcal{R}}^\top x_1 = 0$ since $\mathcal{G}+\mathcal{G}^\top > 0$ by (A2).

We show that (i) implies Eq. (16): since $x_1 \in \ker_{\mathbb{C}}[A_{\mathcal{R}}\ \ A_{\mathcal{V}}]^\top$ we obtain from Eq. (14) that $x_1 \in \ker_{\mathbb{C}} A_{\mathcal{C}}^\top$. Then Eq. (17) implies $A_{\mathcal{L}}x_2 + A_{\mathcal{V}}x_3 = 0$ and by Eq. (14) we find $A_{\mathcal{V}}x_3 = 0$ and $x_2 = 0$. The latter implies that $x_1 \in \ker_{\mathbb{C}} A_{\mathcal{L}}^\top$. Altogether, we have that Eq. (16) is valid.

We show that (ii) implies Eq. (16): from Eq. (17) we have

$$A_{\mathcal{C}}(i\omega \mathcal{C} A_{\mathcal{C}}^\top x_1) + A_{\mathcal{L}}x_2 + A_{\mathcal{V}}x_3 = 0, \tag{18}$$

and by Eq. (15) we obtain $x_2 = 0$. This implies $A_{\mathcal{L}}^\top x_1 = 0$, hence $x_1 \in \ker_{\mathbb{C}}[A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}]^\top$ which by Eq. (15) yields

$$[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{V}}]^\top x_1 = 0.$$

Now, from Eq. (18) we have $A_{\mathcal{V}}x_3 = 0$ and Eq. (16) is shown. □

5. Zero dynamics and invariant zeros

In this section we derive topological characterizations of autonomous and asymptotically stable zero dynamics of the circuit system. The latter is done by an investigation of the invariant zeros of the system.

Using a simple transformation of the system, properties of the zero dynamics can be led back to properties of a circuit pencil where voltage sources are replaced with current sources, and vice versa. To this end, consider $[E, A, B, C] \in \Sigma_{n,m}$ with Eqs. (2) and (4) and define the matrices $W, T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ by

$$W = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & -A_{\mathcal{V}} \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_{\mathcal{I}}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \\ 0 & 0 & I_{n_{\mathcal{V}}} & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{\mathcal{V}}} & 0 \\ 0 & 0 & I_{n_{\mathcal{I}}} & 0 & 0 \\ -A_{\mathcal{V}}^\top & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}.$$

Then we obtain

$$W \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top & A_{\mathcal{L}} & A_{\mathcal{I}} & 0 & 0 \\ -A_{\mathcal{L}}^\top & s\mathcal{L} & 0 & 0 & 0 \\ -A_{\mathcal{I}}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{\mathcal{V}}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}. \tag{19}$$

As desired, the upper left part is a matrix pencil which is an MNA model of a circuit in which voltage sources are replaced with current sources, and vice versa. We may now derive the following important properties, which are immediate from Corollary 4.4 and Eq. (19).

Corollary 5.1 (*Kernel and generalized eigenvalues of system pencil*). *Let* $[E, A, B, C] \in \Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Then*

$$\ker_{\mathbb{R}(s)}\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1(s) \\ 0 \\ 0 \\ x_3(s) \\ -A_{\mathcal{V}}^\top x_1(s) \end{bmatrix} \middle| \begin{array}{l} x_1(s) \in \ker_{\mathbb{R}(s)}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top, \\ x_3(s) \in \ker_{\mathbb{R}(s)} A_{\mathcal{I}} \end{array} \right\}.$$

Furthermore, $\lambda \in \mathbb{C}$ is not a generalized eigenvalue of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ if, and only if,

$$\ker_{\mathbb{C}}\begin{bmatrix} \lambda E-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_3 \\ -A_{\mathcal{V}}^\top x_1 \end{bmatrix} \middle| \begin{array}{l} x_1 \in \ker_{\mathbb{C}}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top, \\ x_3 \in \ker_{\mathbb{C}} A_{\mathcal{I}} \end{array} \right\}.$$

We now aim to characterize autonomous zero dynamics. By considering the zero dynamics as the set of solutions of the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} z = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} z, \tag{20}$$

it is clear that this system is autonomous if, and only if, the matrix pencil $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ has full column rank over $\mathbb{R}(s)$, i.e., there are no blocks of type W3 in its KCF. An application of Corollary 5.1 and Lemma 3.4 then yields the following result.

Proposition 5.2 (*Autonomous zero dynamics*). *Let $[E,A,B,C] \in \Sigma_{n,m}$ with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Then the following statements are equivalent*:

(i) *The zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous.*
(ii) $\operatorname{rk}_{\mathbb{R}(s)}\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = n+m$.
(iii) $\ker[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top = \{0\}$ and $\ker A_{\mathcal{I}} = \{0\}$.
(iv) *The circuit neither contains $\mathcal{I}$–loops nor $\mathcal{V}$–cutsets.*

We also remark that autonomy of the zero dynamics implies left invertibility of the system $[E,A,B,C]$, but in general not vise versa, see [2]; equivalence holds true for systems with regular $sE-A$. By [11, Corollary 4.15], autonomous zero dynamics are furthermore equivalent to "left invertibility in the strong sense" as in [11, Definition 4.10], provided that $[E^\top, A^\top, C^\top]$ has full row rank.

In order to characterize asymptotic stability of the zero dynamics we need the concept of invariant zeros. An invariant zero of $[E,A,B,C] \in \Sigma_{n,m}$ is defined as a generalized eigenvalue of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$, see e.g. [24].

Definition 5.3 (*Invariant zeros*). Let $[E, A, B, C] \in \Sigma_{n,m}$. Then $\lambda \in \mathbb{C}$ is called *invariant zero of* $[E, A, B, C]$ if

$$\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}.$$

From Theorem 4.6 and Eq. (19) we get the following result on the location of invariant zeros.

Corollary 5.4 (*Location of invariant zeros*). *Let* $[E, A, B, C] \in \Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Furthermore, suppose that at least one of the following two assertions holds*:

(i) *The circuit neither contains* $\mathcal{IL}-$*loops except for* $\mathcal{I}-$*loops, nor* $\mathcal{VCL}-$*cutsets except for* $\mathcal{VL}-$*cutsets.*
(ii) *The circuit neither contains* $\mathcal{VC}-$*cutsets except for* $\mathcal{V}-$*cutsets, nor* $\mathcal{ICL}-$*loops except for* $\mathcal{IC}-$*loops.*

Then all invariant zeros of $[E, A, B, C]$ *are contained in* $\mathbb{C}_-$.

We are now in the position to characterize asymptotically stable zero dynamics. By verifying that the zero dynamics is the solution set of the DAE (20), it can be deduced that the zero dynamics are asymptotically stable if, and only if, the matrix pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank over $\mathbb{R}(s)$ and all its generalized eigenvalues are located in the open left half-plane. In other words, the matrix $\left[\begin{smallmatrix} \lambda E-A & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.

Theorem 5.5 (*Asymptotically stable zero dynamics*). *Let* $[E, A, B, C] \in \Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Then the zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable if, and only if,*

(a) $\mathcal{ZD}_{[E,A,B,C]}$ *are autonomous and*
(b) *all invariant zeros of* $[E, A, B, C]$ *are contained in* $\mathbb{C}_-$.

Furthermore, suppose that at least one of the following two assertions holds:

(i) *The circuit neither contains* $\mathcal{IL}-$*loops, nor* $\mathcal{VCL}-$*cutsets except for* $\mathcal{VL}-$*cutsets with at least one inductor.*
(ii) *The circuit neither contains* $\mathcal{VC}-$*cutsets, nor* $\mathcal{ICL}-$*loops except for* $\mathcal{IC}-$*loops with at least one capacitor.*

Then the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable.*

Proof. The equivalence between asymptotic stability of the zero dynamics and the validity of (a) and (b) has been proved in front of the statement of the present theorem. It remains to be shown that (i) or (ii) implies asymptotically stable zero dynamics. In particular, we have "The circuit neither contains $\mathcal{I}-$loops nor $\mathcal{V}-$cutsets" and hence Proposition 5.2 implies (*a*). Furthermore, (i) or (ii) from Corollary 5.4 holds true and therefore (*b*) is valid. This yields the assertion of the theorem. □

6. High-gain stabilization

In this section we consider high-gain output feedback for a system $[E,A,B,C]\in\Sigma_{n,m}$, i.e., system (1) together with the feedback equation $u(t)=-k\cdot y(t)$, where $k>0$. This gives rise to a differential-algebraic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t)=(A-kBC)x(t). \tag{21}$$

Usually (see e.g. [4, Definition 5.5]) a system is called *high-gain stabilizable* if the feedback interconnection leads to an asymptotically stable closed-loop system (21) (i.e., any solution tends to zero) for k large enough. In other words, there exists $\kappa>0$ such that for all $k\geq\kappa$ the pencil $sE-(A-kBC)$ is regular and all of its generalized eigenvalues are contained in $\mathbb{C}_-$.

We will show that for electrical circuits, i.e., $[E,A,B,C]$ with Eqs. (2) and (4), the high-gain need not be high; any positive k is sufficient. In order to achieve this note that we have

$$sE-(A-kBC)=\begin{bmatrix} sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top+A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top+kA_\mathcal{I}A_\mathcal{I}^\top & A_\mathcal{L} & A_\mathcal{V}\\ -A_\mathcal{L}^\top & s\mathcal{L} & 0\\ -A_\mathcal{V}^\top & 0 & kI_{n_\mathcal{V}}\end{bmatrix}. \tag{22}$$

Then, for

$$W=\begin{bmatrix} I_{n_e} & 0 & -k^{-1}A_\mathcal{V}\\ 0 & I_{n_\mathcal{L}} & 0\\ 0 & 0 & k^{-1}I_{n_\mathcal{V}}\end{bmatrix},\quad T=\begin{bmatrix} I_{n_e} & 0 & 0\\ 0 & I_{n_\mathcal{L}} & 0\\ k^{-1}A_\mathcal{V}^\top & 0 & I_{n_\mathcal{V}}\end{bmatrix},$$

we find that

$$W(sE-(A-kBC))T=\begin{bmatrix} sA_\mathcal{C}\mathcal{C}A_\mathcal{C}^\top+A_\mathcal{R}\mathcal{G}A_\mathcal{R}^\top+kA_\mathcal{I}A_\mathcal{I}^\top+k^{-1}A_\mathcal{V}A_\mathcal{V}^\top & A_\mathcal{L} & 0\\ -A_\mathcal{L}^\top & s\mathcal{L} & 0\\ 0 & 0 & I_{n_\mathcal{V}}\end{bmatrix}. \tag{23}$$

The upper left part is a matrix pencil which is an MNA model of a circuit in which all current and voltage sources are replaced with resistances of values k^{-1} and k, respectively. We may therefore conclude the following from Corollary 4.5.

Corollary 6.1 (*Closed-loop pencil is regular*). *Let* $[E,A,B,C]\in\Sigma_{n,m}$ *with Eqs.* (2) and (4) *be given and suppose that* (**A1**) *and* (**A2**) *hold true. Then, for all* $k>0$, *the pencil* $sE-(A-kBC)$ *is regular.*

As a consequence of Theorem 4.6, we can furthermore analyze the asymptotic stability of the closed-loop system.

Theorem 6.2 (*Asymptotic stability of closed-loop pencil*). *Let* $[E,A,B,C]\in\Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Furthermore, suppose that at least one of the following two assertions holds true*:

(i) *The circuit neither contains* $\mathcal{L}$*-loops, nor* $\mathcal{CL}$*-cutsets except for* $\mathcal{L}$*-cutsets.*
(ii) *The circuit neither contains* $\mathcal{C}$*-cutsets, nor* $\mathcal{CL}$*-loops except for* $\mathcal{C}$*-loops.*

Then, for any $k>0$, all generalized eigenvalues of $sE-(A-kBC)$ are contained in $\mathbb{C}_-$.

Remark 6.3 (*Asymptotically stable zero dynamics and high-gain*). Let $[E,A,B,C]\in\Sigma_{n,m}$ with Eqs. (2) and (4) be an MNA model of an electrical circuit and suppose that (**A1**) and (**A2**) hold. Then, under one of the assumptions (i) or (ii) from Theorem 5.5, the respective assumption from Theorem 6.2 holds true, but not vice versa. Therefore, the (topological condition for) asymptotic stability of the zero dynamics implies high-gain stabilizability, but in general not the other way round; this has already been observed for two important classes of DAEs in [3, Section 4].

7. Funnel control

In this section we consider funnel control for systems $[E,A,B,C]\in\Sigma_{n,m}$ with (2) and (4). The aim is to achieve tracking of a reference trajectory by the output signal with prescribed transient behavior. The funnel controller resolves several problems of other classical adaptive controllers (see e.g. the survey [19]): high-gain adaptive control requires an internal model, the gain is monotonically increasing, noise in the output measurement can lead to an unbounded gain, and transient behavior of the tracking error is not taken into account. Adaptive λ-tracking resolves some of these issues, but still the gain is monotonically increasing and the transient behavior is not addressed. Note that a comprehensive comparison of the aforementioned methods by means of several (also practically relevant) simulations has been undertaken in [12].

For any function φ belonging to

$$\Phi:=\left\{\varphi\in\mathcal{C}^\infty(\mathbb{R}_{\geq 0};\mathbb{R})\cap\mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R})\,\middle|\,\begin{array}{l}\varphi(0)=0,\varphi(s)>0\quad\text{for all } s>0\\ \text{and } \liminf_{s\to\infty}\varphi(s)>0\end{array}\right\}$$

we associate the *performance funnel* $\mathcal{F}_\varphi$ as in Eq. (6), see Fig. 1. The control objective is feedback control so that the tracking error $e(\cdot)=y(\cdot)-y_{\text{ref}}(\cdot)$, where $y_{\text{ref}}(\cdot)$ is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behavior is supposed to satisfy

$$\forall\ t>0:\ \|e(t)\|<1/\varphi(t),$$

and, moreover, if φ is chosen so that $\varphi(t)\geq 1/\lambda$ for all t sufficiently large, then the tracking error remains smaller than λ.

By choosing $\varphi(0)=0$ we ensure that the width of the funnel is infinity at $t=0$, see Fig. 1. In the following we only treat "infinite" funnels for technical reasons, since if the funnel is finite,

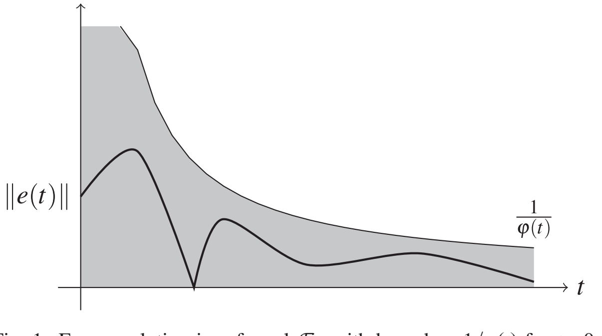

Fig. 1. Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$ for $t>0$.

that is $\varphi(0)>0$, then we need to assume that the initial error is within the funnel boundaries at $t=0$, i.e., $\varphi(0)\|Cx^0-y_{\mathrm{ref}}(0)\|<1$, and this assumption suffices.

As indicated in Fig. 1, we do not assume that the funnel boundary decreases monotonically. Certainly, in most situations it is convenient to choose a monotone funnel, however there are situations where widening the funnel at some later time might be beneficial, e.g., when it is known that the reference signal varies strongly.

To ensure error evolution within the funnel, we use the *funnel controller* (5). If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in Eq. (5). The control design (5) has two advantages: $k(\cdot)$ is non-monotone and Eq. (5) is a static time-varying proportional output feedback of striking simplicity.

Before we state and prove feasibility of funnel control for electrical circuits, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. We also define what "feasibility of funnel control" will mean.

Definition 7.1 (*Consistent initial value*). Let $[E,A,B,C]\in\Sigma_{n,m}$, $\varphi\in\Phi$ and $y_{\mathrm{ref}}\in\mathcal{B}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m)$. An initial value $x^0\in\mathbb{R}^n$ is called *consistent* for the closed-loop system (1) and (5) if there exists a solution of the initial value problem (1) and (5), $x(0)=x^0$, i.e., a function $x\in\mathcal{C}^1([0,\omega);\mathbb{R}^n)$ for some $\omega\in(0,\infty]$, such that $x(0)=x^0$ and x satisfies (1) and (5) for all $t\in[0,\omega)$.

Note that, in practice, consistency of the initial state of the "unknown" system should be satisfied as far as the DAE $[E,A,B,C]$ is the correct model.

In the following we define feasibility of funnel control for a system on a set of reference trajectories. For reference trajectories we allow signals in $\mathcal{B}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m)$, whereas in [2] signals in $\mathcal{B}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ are allowed and $\nu\in\mathbb{N}$ is a number which can be calculated out of a certain system decomposition. To avoid the details of this calculation we restrict ourselves to the case of $\mathcal{B}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m)$.

Definition 7.2 (*Feasibility of funnel control*). Let $[E,A,B,C]\in\Sigma_{n,m}$ and $\mathcal{S}\subseteq\mathcal{B}^\infty(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ be a set of reference trajectories. We say that *funnel control is feasible for* $[E,A,B,C]$ *on* $\mathcal{S}$, if for all $\varphi\in\Phi$, any reference signal $y_{\mathrm{ref}}\in\mathcal{S}$ and any consistent initial value $x^0\in\mathbb{R}^n$ the application of the funnel controller (5)– (1) yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,

(i) $x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot)=Cx(\cdot)-y_{\mathrm{ref}}(\cdot)$ evolves uniformly within the per- formance funnel $\mathcal{F}_\varphi$; more precisely,

$$\exists\varepsilon>0\,\forall t>0:\ \|e(t)\|\leq\varphi(t)^{-1}-\varepsilon. \tag{24}$$

(ii) the corresponding gain function $k(\cdot)$ given by Eq. (5) is bounded.

Remark 7.3 (*Bound for the gain*). If funnel control is feasible as stated in Definition 7.2, then the gain function k is bounded in a way that

$$\forall t_0>0:\ \sup_{t\geq t_0}|k(t)|\leq\frac{1}{1-(1-\varepsilon\lambda_{t_0})^2},$$

where ε is given in Eq. (24) and $\lambda_{t_0} := \inf_{t \geq t_0} \varphi(t) > 0$ for all $t_0 > 0$. A proof for this can be found in [2, Theorem 5.3].

In the following we show that funnel control for systems $[E, A, B, C] \in \Sigma_{n,m}$ with Eqs. (2) and (4) is feasible provided that the invariant zeros have negative real part and the reference signal is sufficiently smooth and evolves in a certain subspace, see Theorem 7.7. The former means that the autonomous part of the zero dynamics has to be asymptotically stable, but autonomy of the whole zero dynamics is not required. As a preliminary result we derive that, for positive real systems $[E, A, B, C] \in \Sigma_{n,m}$ with asymptotically stable zero dynamics, funnel control will be feasible for any sufficiently smooth reference signal.

Proposition 7.4 (*Funnel control for systems with stable zero dynamics*). *Let $[E, A, B, C] \in \Sigma_{n,m}$ be such that $E = E^\top \geq 0$, $A + A^\top \leq 0$, and $B = C^\top$. Further, assume that the zero dynamics of $[E, A, B, C]$ are asymptotically stable. Then funnel control is feasible for $[E, A, B, C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.*

Proof. We aim to apply [2, Theorem 5.3] for $\hat{k} = 1$ and to this end verify its assumptions.

Step 1: The zero dynamics of $[E, A, B, C]$ are asymptotically stable by assumption.

Step 2: We show that for the inverse $L(s)$ of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ over $\mathbb{R}(s)$ the matrix

$$\Gamma = -\lim_{s \to \infty} s^{-1} [0, I_m] L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{m,m}$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$. By Corollary 2.5, the pencil

$$\begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$$

is positive real. Then, for the inverse $L(s)$ of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ over $\mathbb{R}(s)$, $\widetilde{L}(s) := L(s) \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix}$ is the inverse of $\begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix}$, and we have

$$\widetilde{L}(\lambda) + \widetilde{L}(\lambda)^* = \widetilde{L}(\lambda) \left(\begin{bmatrix} \lambda E-A & -B \\ C & 0 \end{bmatrix}^* + \begin{bmatrix} \lambda E-A & -B \\ C & 0 \end{bmatrix} \right) \widetilde{L}(\lambda)^* \geq 0$$

for all $\lambda \in \mathbb{C}_+$. Furthermore, since $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ does not have any invariant zeros in $\mathbb{C}_+$, $\widetilde{L}(s)$ has no poles in $\mathbb{C}_+$. This shows that $\widetilde{L}(s)$ is positive real. Hence, $H(s) := [0, I_m] \widetilde{L}(s) [0, I_m]^\top$ is positive real and satisfies $H(s) = -[0, I_m] L(s) [0, I_m]^\top$. Now Lemma 2.4 yields that

$$\Gamma = \lim_{s \to \infty} s^{-1} H(s) \in \mathbb{R}^{m,m}$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$.

Step 3: We show that $[E, A, B, C]$ is right-invertible in the sense of [2, Definition 4.1]. Since the zero dynamics of $[E, A, B, C]$ are in particular autonomous it follows from Proposition 5.2 (ii) that $\operatorname{rk} C = m$ and hence right-invertibility can be concluded from [2, Remark 4.12].

Step 4: It remains to show that $\hat{k}$ in [2, Theorem 5.3] can be chosen as $\hat{k} = 1$ and funnel control is still feasible. A careful inspection of the proof of [2, Theorem 5.3] reveals that '$\hat{k}$ large enough' is needed

(a) in Step 1 of the proof of [2, Theorem 5.3] to show that the map $M(\cdot) = \hat{A}_{22} - k(I + G(\cdot))$, where $G(\cdot) = G(\cdot)^\top \geq 0$ pointwise, is well-defined (i.e., pointwise invertible) for $k \geq \hat{k}$,

(b) in Step 3 of the proof of [2, Theorem 5.3] to show that $\hat{A}_{22} - \hat{k} \cdot k(t) I_m$ is invertible for all $t \geq 0$.

It can be deduced that both (a) and (b) are satisfied with $\hat{k} = 1$ if $\hat{A}_{22} - kI_m$ is negative definite for all $k > 0$, where

$$\hat{A}_{22} = [0, I_{m-m_1}] V A_{22} V^\top \begin{bmatrix} 0 \\ I_{m-m_1} \end{bmatrix},$$

m_1 and the orthogonal matrix V have been defined in Step 1 of the proof of [2, Theorem 5.3], and

$$A_{22} = \lim_{s \to \infty} \left([0, I_m] L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right).$$

We may calculate that

$$A_{22} = \lim_{s \to \infty} (s\Gamma - H(s)) = -H_0 - \lim_{s \to \infty} H_{sp}(s)$$

where, since $H(s)$ is positive real, by Lemma 2.4 the rational function $H_0 + H_{sp}(s)$ is positive real and $\lim_{s \to \infty} H_{sp}(s) = 0$. Hence, it is easy to derive that $H_0 \geq 0$ (H_0 not necessarily symmetric) and hence

$$A_{22} - kI_m = -H_0 - kI_m < 0$$

for all $k > 0$ (again $A_{22} - kI_m$ not necessarily symmetric). This implies that $\hat{A}_{22} - kI_m$ is negative definite for all $k > 0$. □

Before we prove our main result we need to know how feasibility of funnel control behaves under transformation of the system.

Lemma 7.5 (*Funnel control under system transformation*). *Let $E, A \in \mathbb{R}^{n,n}$, $B, C^\top \in \mathbb{R}^{n,m}$ and $\mathcal{S} \subseteq \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. Further, let $W, T \in \mathbf{Gl}_n(\mathbb{R})$, $U \in \mathcal{O}_m(\mathbb{R})$, and define*

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then funnel control is feasible for $[E, A, B, C]$ on $\mathcal{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on $U^\top \mathcal{S}$.

Proof. Observe that $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$ and $y_{ref} \in \mathcal{S}$ if, and only if,

$$(\widetilde{x}, \widetilde{u}, \widetilde{y}) = (T^{-1}x, U^\top u, U^\top y) \in \mathfrak{B}_{[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]} \quad \wedge \quad U^\top y_{ref} \in U^\top \mathcal{S}.$$

Then the assertion follows from the observation that, for any $\varphi \in \Phi$, and tracking errors $e = y - y_{ref}$, $\widetilde{e} = \widetilde{y} - \widetilde{y}_{ref}$ we have, for all $t \geq 0$,

$$\frac{1}{1 - \varphi(t)^2 \|e(t)\|^2} = \frac{1}{1 - \varphi(t)^2 \|\widetilde{e}(t)\|^2}. \qquad \square$$

In the following, in order to show that funnel control is feasible for circuits where all invariant zeros are located in $\mathbb{C}_-$, but the zero dynamics are not necessarily autonomous, we derive a transformation of the circuit which decouples the "non-autonomous part" of the zero dynamics. This part, in particular, does not affect the input–output behavior of the system.

Proposition 7.6 (*Decomposition of circuit pencil*). *Let* $[E,A,B,C]\in\Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Let* $Z'_{\mathcal{CRLI}}\in\mathbb{R}^{n_e,k_1}$, $Z_{\mathcal{CRLI}}\in\mathbb{R}^{n_e,k_2}$ *with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRLI}}=\ker[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top \quad \text{and} \quad \operatorname{im} Z'_{\mathcal{CRLI}}=\operatorname{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}].$$

Further, let $Z_{\mathcal{V}-\mathcal{CRLI}}\in\mathbb{R}^{n_{\mathcal{V}},k_3}$, $Z'_{\mathcal{V}-\mathcal{CRLV}}\in\mathbb{R}^{n_{\mathcal{V}},k_4}$, $\overline{Z}_{\mathcal{I}}\in\mathbb{R}^{n_{\mathcal{I}},k_5}$, $\overline{Z}'_{\mathcal{I}}\in\mathbb{R}^{n_{\mathcal{I}},k_6}$ *with orthonormal columns such that*

$$\begin{aligned}&\operatorname{im} Z_{\mathcal{V}-\mathcal{CRLI}}=\ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}}, \quad \operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}}=\operatorname{im} A_{\mathcal{V}}^\top Z_{\mathcal{CRLI}},\\ &\operatorname{im} \overline{Z}_{\mathcal{I}}=\ker A_{\mathcal{I}}, \quad \operatorname{im} \overline{Z}'_{\mathcal{I}}=\operatorname{im} A_{\mathcal{I}}^\top.\end{aligned}$$

Then we have

$$W^\top := T := \begin{bmatrix} Z_{\mathcal{CRLI}} & Z'_{\mathcal{CRLI}} & 0 & 0 & 0\\ 0 & 0 & I_{n_{\mathcal{L}}} & 0 & 0\\ 0 & 0 & 0 & Z'_{\mathcal{V}-\mathcal{RCLI}} & Z_{\mathcal{V}-\mathcal{RCLI}}\end{bmatrix}\in \mathbf{Gl}_n(\mathbb{R}) \tag{25a}$$

and

$$U := \begin{bmatrix} 0 & \overline{Z}_{\mathcal{I}} & \overline{Z}'_{\mathcal{I}} & 0\\ Z_{\mathcal{V}-\mathcal{RCLI}}{}' & 0 & 0 & Z_{\mathcal{V}-\mathcal{RCLI}}\end{bmatrix}\in \mathcal{O}_m(\mathbb{R}), \tag{25b}$$

and

$$W(sE-A)T=\begin{bmatrix} 0 & 0 & Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRLI}}\\ 0 & s\widetilde{E}_r-\widetilde{A}_r & \begin{bmatrix}(Z'_{\mathcal{CRLI}})^\top A_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRLI}}\\ 0\\ 0\end{bmatrix}\\ -(Z'_{\mathcal{V}-\mathcal{CRLI}})^\top A_{\mathcal{V}}^\top Z_{\mathcal{CRLI}} & [-(Z'_{\mathcal{V}-\mathcal{CRLI}})^\top A_{\mathcal{V}}^\top Z'_{\mathcal{CRLI}},0,0] & 0\end{bmatrix} \tag{26}$$

and

$$WBU=(U^\top CT)^\top = \begin{bmatrix} 0 & 0\\ 0 & \widetilde{B}_r\\ [-I_{k_4},0] & 0\end{bmatrix}, \tag{27}$$

where

$$s\widetilde{E}_r-\widetilde{A}_r=\begin{bmatrix}(Z'_{\mathcal{CRLI}})^\top(sA_{\mathcal{C}}\mathcal{C}A_{\mathcal{C}}^\top + A_{\mathcal{R}}\mathcal{G}A_{\mathcal{R}}^\top)Z'_{\mathcal{CRLI}} & (Z'_{\mathcal{CRLI}})^\top A_{\mathcal{L}} & Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} Z_{\mathcal{V}-\mathcal{CRLI}}\\ -A_{\mathcal{L}}^\top Z'_{\mathcal{CRLI}} & s\mathcal{L} & 0\\ -Z_{\mathcal{V}-\mathcal{CRLI}}^\top A_{\mathcal{V}}^\top Z'_{\mathcal{CRLI}} & 0 & 0\end{bmatrix},$$

$$\widetilde{B}_r=\widetilde{C}_r^\top = \begin{bmatrix} -(Z'_{\mathcal{CRLI}})^\top A_{\mathcal{I}}\overline{Z}'_{\mathcal{I}} & 0\\ 0 & 0\\ 0 & -I_{k_3}\end{bmatrix} \tag{28}$$

Furthermore, the following holds true:

(a) $k_2=k_4$ *and* $Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRLI}}\in\mathbf{Gl}_{k_2}(\mathbb{R})$.
(b) *The zero dynamics of the system* $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ *are autonomous.*

(c) *$\lambda \in \mathbb{C}$ is an invariant zero of $[E,A,B,C]$ if, and only if, λ is an invariant zero of $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$.*

Proof. The invertibility of W,T and U is a consequence of

$$\begin{aligned}
&\operatorname{im} Z'_{\mathcal{CRLI}} \oplus \operatorname{im} Z_{\mathcal{CRLI}} = \operatorname{im}[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{I}] \oplus \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{I}]^\top = \mathbb{R}^{n_e},\\
&\operatorname{im} Z'_{\mathcal{V}-\mathcal{RCLI}} \oplus \operatorname{im} Z_{\mathcal{V}-\mathcal{RCLI}} = \operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}} \oplus \ \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} = \mathbb{R}^{n_\mathcal{V}},\\
&\operatorname{im} \overline{Z}'_\mathcal{I} \oplus \operatorname{im} \overline{Z}_\mathcal{I} = \operatorname{im} A_\mathcal{I}^\top \oplus \ \ker A_\mathcal{I} = \mathbb{R}^{n_\mathcal{I}}.
\end{aligned}$$

Furthermore, by choice of $Z_{\mathcal{V}-\mathcal{CRLI}}$, $Z'_{\mathcal{V}-\mathcal{CRLV}}$, $\overline{Z}_\mathcal{I}$ and $\overline{Z}'_\mathcal{I}$ the matrix U is orthogonal. The representation of the transformed system in Eqs. (26)–(28) is then a simple calculation.

We prove assertions (a)–(c).

(a) The assertion will be inferred from the fact that both matrices $Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}}$ and $(Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}})^\top$ have trivial kernels. To prove the first assertion, let $z \in \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}}$. Then

$$Z'_{\mathcal{V}-\mathcal{CRLI}} z \in \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} = (\operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}})^\perp = (\operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}})^\perp.$$

Therefore, $Z'_{\mathcal{V}-\mathcal{CRLI}} z = 0$, and the full column rank of $Z'_{\mathcal{V}-\mathcal{CRLI}}$ implies $z=0$. Now let $z \in \ker (Z'_{\mathcal{V}-\mathcal{CRLI}})^\top A_\mathcal{V}^\top Z_{\mathcal{CRLI}}$. Then

$$A_\mathcal{V}^\top Z_{\mathcal{CRLI}} z \in \ker\, (Z'_{\mathcal{V}-\mathcal{CRLI}})^\top = (\operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}})^\perp = (\operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}})^\perp.$$

Thus, $Z_{\mathcal{CRLI}} z \in \ker A_\mathcal{V}^\top$ and by choice of $Z_{\mathcal{CRLI}}$ we have

$$Z_{\mathcal{CRLI}} z \in \ker[A_\mathcal{C}\ \ A_\mathcal{R}\ \ A_\mathcal{L}\ \ A_\mathcal{I}]^\top \cap \ \ker A_\mathcal{V}^\top \overset{\textbf{(A1)}}{=} \{0\},$$

Hence, we obtain $z=0$ from the full column rank of $Z_{\mathcal{CRLI}}$.

(b) By Proposition 5.2 it is sufficient to show that the pencil

$$s\mathcal{E} - \mathcal{A} := \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & \widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} = \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is regular. Observing that $\mathcal{E} = \mathcal{E}^\top \geq 0$ and $\mathcal{A} + \mathcal{A}^\top \leq 0$, we can use Lemma 2.6 to further reduce the problem to showing that $\ker \mathcal{E} \cap \ \ker \mathcal{A} = \{0\}$:
Let $z = (z_1, z_2, z_3, z_4, z_5) \in \ker \mathcal{E} \cap \ \ker \mathcal{A}$ be suitably partitioned according to the block structure of $\widetilde{E}_r$, $\widetilde{A}_r$, $\widetilde{B}_r$ and $\widetilde{C}_r$ as in Eq. (28). Then, by (**A2**), the equation $z^\top \mathcal{E} z = z^\top (\mathcal{A} + \mathcal{A}^\top) z = 0$ gives rise to $z_2 = 0$ and

$$z_1 \in \ker[A_\mathcal{C}\ \ A_\mathcal{R}]^\top Z'_{\mathcal{CRLI}}.$$

The equation $\mathcal{A} z = 0$ further implies $z_3 = 0$ and

$$z_1 \in \ker A_\mathcal{L}^\top Z'_{\mathcal{CRLI}} \wedge z_1 \in \ker (\overline{Z}'_\mathcal{I})^\top A_\mathcal{I}^\top Z'_{\mathcal{CRLI}}.$$

The latter implies

$$A_\mathcal{I}^\top Z'_{\mathcal{CRLI}} z_1 \in \ker (\overline{Z}'_\mathcal{I})^\top = (\operatorname{im} \overline{Z}'_\mathcal{I})^\perp = (\operatorname{im} A_\mathcal{I}^\top)^\perp,$$

whence $z_1 \in \ker A_{\mathcal{I}}^\top Z'_{\mathcal{CRLI}}$. Altogether, we have

$$Z'_{\mathcal{CRLI}} z_1 \in \ker[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top = (\mathrm{im}[\mathrm{A}_{\mathcal{C}}\ \ \mathrm{A}_{\mathcal{R}}\ \ \mathrm{A}_{\mathcal{L}}\ \ \mathrm{A}_{\mathcal{I}}])^\perp = (\mathrm{im}\, \mathrm{Z}'_{\mathcal{CRLI}})^\perp.$$

The full column rank of $Z'_{\mathcal{CRLI}}$ now implies that $z_1 = 0$. Now using that $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$, we can infer from $\mathcal{A}z = 0$ that $z_5 = 0$ and $(Z'_{\mathcal{CRLI}})^\top A_{\mathcal{I}} \overline{Z}'_{\mathcal{I}} z_4 = 0$. Thus,

$$A_{\mathcal{I}} \overline{Z}'_{\mathcal{I}} z_4 \in \ker (Z'_{\mathcal{CRLI}})^\top = (\mathrm{im}\, Z'_{\mathcal{CRLI}})^\perp = (\mathrm{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}])^\perp \subseteq (\mathrm{im}\, A_{\mathcal{I}})^\perp.$$

Therefore, $A_{\mathcal{I}} \overline{Z}'_{\mathcal{I}} z_4 = 0$ or, equivalently,

$$\overline{Z}'_{\mathcal{I}} z_4 \in \ker A_{\mathcal{I}} = (\mathrm{im}\, A_{\mathcal{I}}^\top)^\perp = (\mathrm{im}\, \overline{Z}'_{\mathcal{I}})^\perp.$$

This implies $\overline{Z}'_{\mathcal{I}} z_4 = 0$, and since $\overline{Z}'_{\mathcal{I}}$ has full column rank, we have that $z_4 = 0$.

(c) It can be obtained from simple row and column operations that for all $\lambda \in \mathbb{C}$ we have

$$\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda WET - WAT & -WBU \\ -U^T CT & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda \widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4$$

and, similarly,

$$\mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4.$$

This implies that the generalized eigenvalues of $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ coincide with those of $\begin{bmatrix} s\widetilde{E} - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix}$ and hence the assertion is proved.

This concludes the proof of the proposition. □

We are now in the position to prove the main result of this section, which improves Proposition 7.4 by relaxing the assumption of asymptotically stable zero dynamics (which in particular requires autonomous zero dynamics) to the assumption of asymptotic stability of the autonomous part of the zero dynamics; the latter is characterized by the stability of the invariant zeros of the system. This leads to the following result, where we show that funnel control is feasible for circuits where all invariant zeros are located in $\mathbb{C}_-$. However, the drawback is that the reference trajectory must be restricted to a certain subspace in order to account for the non-autonomous part of the zero dynamics.

Theorem 7.7 (*Funnel control for circuits*). *Let* $[E, A, B, C] \in \Sigma_{n,m}$ *with Eqs.* (2) and (4) *be an MNA model of an electrical circuit and suppose that* (**A1**) *and* (**A2**) *hold. Assume that the system* $[E, A, B, C]$ *does not have any invariant zeros on the imaginary axis. Let* $Z_{\mathcal{CRLI}}$ *be a matrix with full column rank such that*

$$\mathrm{im}\, Z_{\mathcal{CRLI}} = \ker\, [A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top.$$

Then funnel control is feasible for $[E, A, B, C]$ *on*

$$\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathrm{im}\, A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}}).$$

Proof. *Step* 1: Use the notation from Proposition 7.6 and define

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then, by Lemma 7.5, it suffices to prove that funnel control is feasible for $[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]$ on

$$\mathcal{S}:=U^\top \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \text{im}\, A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}}).$$

Step 2: We show that $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ has asymptotically stable zero dynamics. By Proposition 7.6 (c), the zero dynamics of $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ are autonomous. Furthermore, by Proposition 7.6 (d) and the fact that the invariant zeros of $[E,A,B,C]$ all have negative real part, we obtain from Theorem 5.5 that the zero dynamics of $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ are asymptotically stable.

Step 3: We reduce the feasibility problem of funnel control to that of the system $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$. Let

$$(\widetilde{x},\widetilde{u},\widetilde{y}) \in \mathfrak{B}_{[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]} \quad \text{and} \quad \widetilde{y}_{\text{ref}} = U^\top \begin{pmatrix} y_{\text{ref},1} \\ y_{\text{ref},2} \end{pmatrix} \in \mathcal{S}.$$

Since

$$y_{\text{ref},1} \in \text{im}\, A_{\mathcal{I}}^\top = \text{im}\, \overline{Z}_{\mathcal{I}}' = (\text{im}\, \overline{Z}_{\mathcal{I}})^\perp = \ker \overline{Z}_{\mathcal{I}}^\top$$

and

$$y_{\text{ref},2} \in \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} = \text{im}\, Z_{\mathcal{V}-\mathcal{CRLI}} = (\text{im}\, Z_{\mathcal{V}-\mathcal{CRLI}}')^\perp = \ker(Z_{\mathcal{V}-\mathcal{CRLI}}')^\top$$

we obtain that

$$\widetilde{y}_{\text{ref}} = [0,\ \ 0,\ \ \widetilde{y}_{\text{ref},1},\ \ \widetilde{y}_{\text{ref},2}]^\top,$$

where $\widetilde{y}_{\text{ref},1} = (\overline{Z}_{\mathcal{I}}')^\top y_{\text{ref},1}$ and $\widetilde{y}_{\text{ref},2} = Z_{\mathcal{V}-\mathcal{CRLI}}^\top y_{\text{ref},2}$. By suitably partitioning

$$\widetilde{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix}, \quad \widetilde{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}, \quad \widetilde{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}$$

according to the block structure of $s\widetilde{E}-\widetilde{A}$ as in Eq. (26), and $\widetilde{B}$, $\widetilde{C}$ as in Eq. (27), we obtain $Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} Z_{\mathcal{V}-\mathcal{CRLI}}' x_5 = 0$, whence, by Proposition 7.6 (b), we have $x_5 = 0$, and thus also $y_1 = 0$. Moreover, $y_2 = 0$ and

$$x_1 = -(Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} Z_{\mathcal{V}-\mathcal{CRLI}}')^{-1} (Z_{\mathcal{V}-\mathcal{CRLI}}')^\top A_{\mathcal{V}}^\top Z_{\mathcal{CRLI}}' x_2 - u_1,$$

and, further

$$\widetilde{x}_r(t) = \begin{bmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \quad \widetilde{u}_r(t) = \begin{bmatrix} u_3(t) \\ u_4(t) \end{bmatrix}, \quad \widetilde{y}_r(t) = \begin{bmatrix} y_3(t) \\ y_4(t) \end{bmatrix}$$

satisfy

$$(\widetilde{x}_r,\widetilde{u}_r,\widetilde{y}_r) \in \mathfrak{B}_{[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]}.$$

Application of the funnel controller (5) then yields $\widetilde{u} = -k(\widetilde{y}-\widetilde{y}_{\text{ref}})$ and hence $u_1 = 0$ and $u_2 = 0$. Therefore, funnel control is feasible for $[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]$ on $\mathcal{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}_r,\widetilde{A}_r,\widetilde{B}_r,\widetilde{C}_r]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{k_3+k_6})$. The latter however follows from Step 2 and Proposition 7.4. This concludes the proof of the theorem. □

Remark 7.8 (*Topological criteria for funnel control*). We analyze the constraints on the reference trajectories in Theorem 7.7.

(a) The subspace restriction

$$y_{\text{ref}}(t) \in \operatorname{im} A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} \quad \forall\, t \geq 0 \tag{29}$$

on the reference signal can be interpreted as follows: since we do no longer assume autonomous zero dynamics it is, by Proposition 5.2, possible that the circuit contains $\mathcal{I}-$loops and $\mathcal{V}-$cutsets and we have to check the consequences of these two configurations. If the circuit contains a $\mathcal{V}-$cutset, then, by Kirchhoff's current law, the currents of the voltage sources in the $\mathcal{V}-$cutset sum up to zero. Since both are components of the output y this leads to an output constraint. Clearly, the reference trajectory has to satisfy this restriction as well if we want funnel control to be feasible. Likewise, if the circuit contains an $\mathcal{I}-$loop, then Kirchhoff's voltage law implies that the voltages of the current sources in the $\mathcal{I}-$loop sum up to zero which again leads to an output constraint. Condition (29) therefore means that, in a sense, the reference signal has to satisfy Kirchhoff's laws pointwise, see also Fig. 2.

(b) Invoking that

$$\ker Z_{\mathcal{CRLI}}^\top = (\operatorname{im} Z_{\mathcal{CRLI}})^\perp = (\ker[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top)^\perp = \operatorname{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}],$$

we find

$$\ker Z_{\mathcal{CRLI}}^T A_{\mathcal{V}} = \{x \in \mathbb{R}^{n_{\mathcal{V}}} \,|\, A_{\mathcal{V}} x \in \operatorname{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]\}.$$

In particular, this space is independent of the choice of the matrix $Z_{\mathcal{CRLI}}$ with $\operatorname{im} Z_{\mathcal{CRLI}} = \ker[A_{\mathcal{C}}\ A_{\mathcal{R}}\ A_{\mathcal{L}}\ A_{\mathcal{I}}]^\top$.

(c) We have that $\ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$ if, and only if,

$$\operatorname{im} A_{\mathcal{V}} \subseteq \ker Z_{\mathcal{CRLI}}^\top = (\operatorname{im} Z_{\mathcal{CRLI}})^\perp = (\ker\,[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}]^\top)^\perp = \operatorname{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}].$$

Hence, by (**A1**), $\ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$ is equivalent to

$$\operatorname{im}[A_{\mathcal{C}}\ \ A_{\mathcal{R}}\ \ A_{\mathcal{L}}\ \ A_{\mathcal{I}}] = \mathbb{R}^{n_e}.$$

The latter is, by Lemma 3.4, equivalent to the absence of $\mathcal{V}-$cutsets in the given electrical circuit.
Furthermore, $\operatorname{im} A_{\mathcal{I}}^\top = \mathbb{R}^{n_{\mathcal{I}}}$ if, and only if, $\{0\} = (\operatorname{im} A_{\mathcal{I}}^\top)^\perp = \ker A_{\mathcal{I}}$. By Lemma 3.4 the latter is equivalent to the absence of $\mathcal{I}-$loops in the given electrical circuit.

(d) By virtue of Theorem 7.7 and Corollary 5.4, we see that funnel control is feasible for passive and connected electrical circuits (on a suitable set of reference trajectories) provided that at least one of the following two properties is satisfied:
 (i) The circuit neither contains $\mathcal{IL}-$loops except for $\mathcal{I}-$loops, nor $\mathcal{VCL}-$cutsets except for $\mathcal{VL}-$cutsets.
 (ii) The circuit neither contains $\mathcal{VC}-$cutsets except for $\mathcal{V}-$cutsets, nor $\mathcal{ICL}-$loops except for $\mathcal{IC}-$loops.

By virtue of Proposition 7.4 and Theorem 5.5, we see that funnel control is feasible for

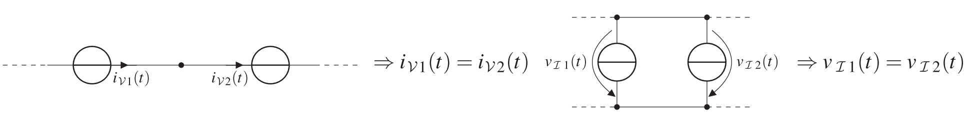

Fig. 2. Interpretation of condition (29) in terms of Kirchhoff's laws.

(e) passive and connected electrical circuits (on the set of *all* sufficiently smooth reference trajectories) provided that at least one of the following two properties is satisfied:
  (i) The circuit neither contains $\mathcal{IL}$−loops, nor $\mathcal{VCL}$−cutsets except for $\mathcal{VL}$−cutsets with at least one inductor.
  (ii) The circuit neither contains $\mathcal{VC}$−cutsets, nor $\mathcal{ICL}$−loops except for $\mathcal{IC}$−loops with at least one capacitor.

8. Simulations

In this section we demonstrate the applicability of the funnel controller. First, we consider an example of a discretized transmission line [9] and show that the funnel controller (5) achieves tracking of two different types of reference signals: a sinusoidal reference signal is considered in Section 8.1 and a sawtooth wave signal in Section 8.2. In each case the transient behavior of the tracking error is prescribed. We also like to emphasize again that funnel control does not need the exact knowledge of the system parameters. In particular, no pre-computation (such as, for instance, in flatness-based control [9]) has do be done.

In Section 8.3 we consider a circuit with unstable zero dynamics (and also unstable invariant zeros) in order to show that this assumption is essential. Although the funnel controller achieves tracking of a constant reference signal, the gain grows unboundedly.

8.1. Discretized transmission line - sinusoidal reference signal

We consider a discretized transmission line as depicted in Fig. 3, where n is the number of spacial discretization points.

The element related incidence matrices of this circuit can be calculated as

$$A_{\mathcal{C}} = \operatorname{diag}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \in \mathbb{R}^{2n+1,n},$$

$$A_{\mathcal{R}} = \left[\operatorname{diag}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right), A_{\mathcal{C}}\right] \in \mathbb{R}^{2n+1,2n},$$

$$A_{\mathcal{L}} = \operatorname{diag}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) \in \mathbb{R}^{2n+1,n},$$

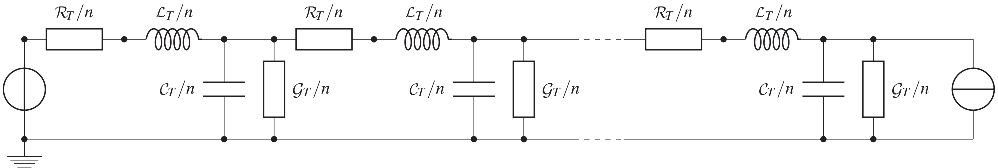

Fig. 3. Discretized transmission line.

$$A_{\mathcal{V}} = [1, 0, \ldots, 0]^\top \in \mathbb{R}^{2n+1,1},$$

$$A_{\mathcal{I}} = [0, \ldots, 0, 1]^\top \in \mathbb{R}^{2n+1,1}.$$

The matrices expressing the constitutive relations of capacitances, resistances (and conductances, resp.) and inductances are given by

$$\mathcal{C} = \frac{\mathcal{C}_T}{n} I_n, \quad \mathcal{G} = \operatorname{diag}\left(\frac{n}{\mathcal{R}_T} I_n, \frac{\mathcal{G}_T}{n} I_n\right), \quad \mathcal{L} = \frac{\mathcal{L}_T}{n} I_n.$$

The differential-algebraic system (1) describing the discretized transmission line is then given by $[E, A, B, C]$ for the matrices in Eq. (2).

The circuit in Fig. 3 does not contain any $\mathcal{IL}$−loops. Further, the only $\mathcal{VCL}$−cutset of the circuit is formed by the voltage source and the inductance of the left branch. We can therefore conclude from Theorem 5.5 that $[E, A, B, C]$ has asymptotically stable zero dynamics. Then, by Proposition 7.4, funnel control is feasible for $[E, A, B, C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$.

For the simulation we chose the parameters

$$n = 50, \quad \mathcal{C}_T = \mathcal{R}_T = \mathcal{G}_T = \mathcal{L}_T = 1, \tag{30}$$

and the (consistent) initial value for the closed-loop system $[E, A, B, C]$, Eq. (5) by

$$x^0 = \left(-1, -1.04, \underbrace{2, 1.96, \ldots, 2, 1.96}_{(n-1)-\text{times}}, \underbrace{2, \ldots, 2}_{(n+1)-\text{times}}, -2\right) \in \mathbb{R}^{3n+2}. \tag{31}$$

As reference signal we take the sinusoidal signal $y_{\text{ref}} = (\sin, \cos)^\top \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$. The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\, te^{-t} + 2 \arctan t. \tag{32}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Fig. 4d.

Note further that the asymptotic stability of the zero dynamics can also be verified by a numerical test which shows that all invariant zeros of $[E, A, B, C]$ have real part -1.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 2.3×10^{-14}, absolute tolerance: 10^{-10}). In Fig. 4 the simulation, over the time interval $[0, 10]$, of the funnel controller (5) with funnel boundary specified in Eq. (32) and reference signal $y_{\text{ref}} = (\sin, \cos)^\top$, applied to system $[E, A, B, C]$ with initial data (31) is depicted. Fig. 4a shows the output components y_1 and y_2 tracking the reference signal y_{ref} within the funnel shown in Fig. 4d. Note that an action of the input components u_1 and u_2 in Fig. 4c and the gain function k in Fig. 4b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 1$, the distance between error and funnel boundary gets larger and the gain drops accordingly. In particular we see that the gain function k is non-monotone.

8.2. *Discretized transmission line - sawtooth wave signal*

To highlight the applicability of the funnel controller, we provide another simulation for the discretized transmission line as discussed in Section 8.1 with the same system parameters (30)

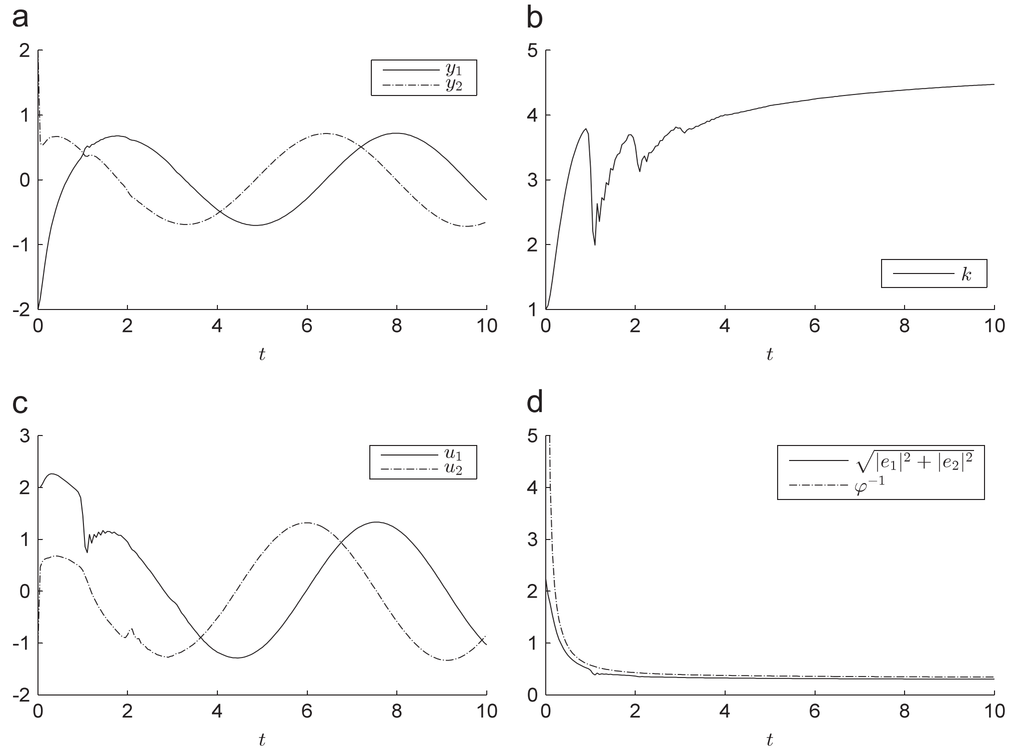

Fig. 4. Simulation of the funnel controller (5) with funnel boundary specified in Eq. (32) and reference signal $y_{\text{ref}} = (\sin, \cos)^\top$ applied to system $[E,A,B,C]$ with initial data (31). (a) Solution components y_1 and y_2, (b) Gain k, (c) Input components u_1 and u_2, (d) Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$.

and (consistent) initial value x^0 for the closed-loop system $[E,A,B,C]$, Eq. (5) given by

$$x^0 = \left(x_1^0, x_2^0, \underbrace{2, 1.96, \ldots, 2, 1.96}_{(n-1)-\text{times}}, \underbrace{2, \ldots, 2}_{(n+1)-\text{times}}, -2 \right) \in \mathbb{R}^{3n+2}, \tag{33}$$

where approximately $x_1^0 \approx 1.6366$ and $x_2^0 \approx 1.6766$.

As reference signal we take a sawtooth wave signal $y_{\text{ref}} = (s_1, s_2)^\top \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$ with period 1 and amplitude 1. The component s_1 has a phase shift of 0.3 and s_2 has a phase shift of 2.0. In order to meet the smoothness requirements on the reference trajectory, the signals s_1 and s_2 are smoothed by a Gaussian low-pass filter with standard deviation 0.2. The components of the reference signal y_{ref} are depicted in Fig. 5.

The funnel $\mathcal{F}_\varphi$ is again determined by Eq. (32).

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 2.3×10^{-14}, absolute tolerance: 10^{-10}). In Fig. 6 the simulation, over the time interval $[0, 10]$, of the funnel controller (5) with funnel boundary specified in Eq. (32) and reference signal $y_{\text{ref}} = (s_1, s_2)^\top$ as shown in Fig. 5, applied to system $[E,A,B,C]$ with initial data (33) is depicted. In Fig. 6a we see that the output components y_1 and y_2 can keep track of the vivid sawtooth wave y_{ref} within the funnel as shown in Fig. 6d. The input components u_1 and u_2 resemble sawtooth

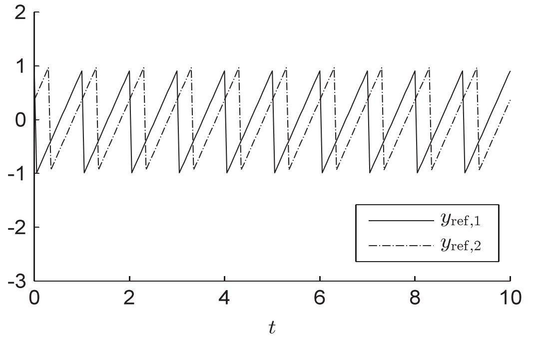

Fig. 5. Sawtooth wave reference signal y_{ref}.

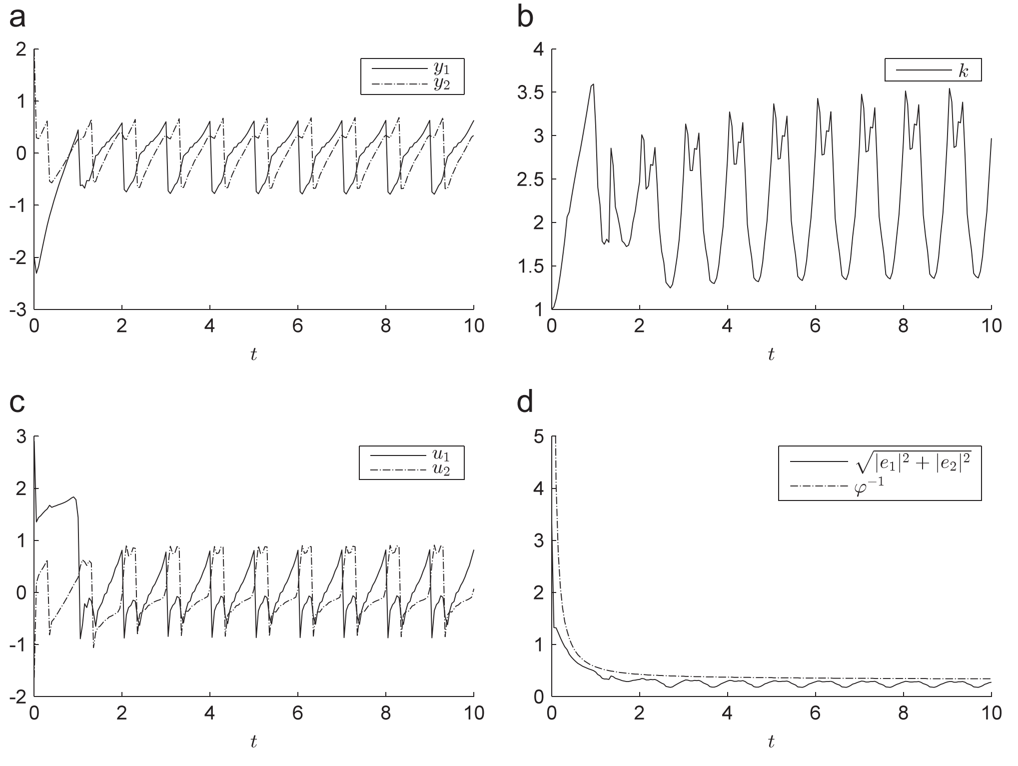

Fig. 6. Simulation of the funnel controller (5) with funnel boundary specified in Eq. (32) and reference signal $y_{\mathrm{ref}} = (s_1, s_2)^\top$ applied to system $[E, A, B, C]$ with initial data (33). (a) Solution components y_1 and y_2, (b) Gain k, (c) Input components u_1 and u_2, (d) Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$.

waves as well, see Fig. 6c. The gain function k in Fig. 6b has a strong non-monotonic character in this example and varies between 1.5 and 3.5 in the depicted time interval.

8.3. Example with unstable zero dynamics

To illustrate that asymptotic stability of the zero dynamics is an essential assumption for funnel control we consider the simple RC circuit depicted in Fig. 7.

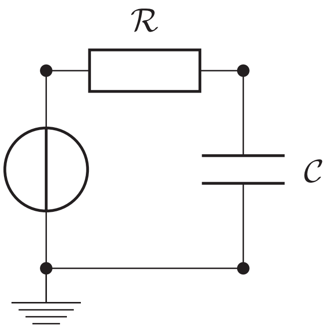

Fig. 7. Simple RC circuit.

The incidence matrices are given by

$$A_{\mathcal{R}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad A_{\mathcal{C}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A_{\mathcal{V}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The system is consequently described by $[E, A, B, C] \in \Sigma_{3,1}$ with

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathcal{C} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -\mathcal{R} & \mathcal{R} & -1 \\ \mathcal{R} & -\mathcal{R} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

To analyze the zero dynamics, let $x = (\eta_1, \eta_2, i_{\mathcal{V}})^\top$, $u = v_{\mathcal{V}}$, $y = -i_{\mathcal{V}}$ be such that $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$. The equation $Cx = 0$ implies $i_{\mathcal{V}} = 0$. Then $(\mathrm{d}/\mathrm{d}t)Ex(t) = Ax(t) + Bu(t)$ gives $\eta_1 = \eta_2$ and $(\mathrm{d}/\mathrm{d}t)\eta_2 = 0$, whence η_2 is constant. The relation $(\mathrm{d}/\mathrm{d}t)Ex(t) = Ax(t) + Bu(t)$ further gives rise to $u = -\eta_1$. On the other hand we obtain that a trajectory $(x, u, y) \equiv ((c, c, 0)^\top, c, 0)$ satisfies $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$. Thus we have

$$\mathcal{ZD}_{[E,A,B,C]} = \left\{ \left(\begin{pmatrix} c \\ c \\ 0 \end{pmatrix}, c, 0 \right) \,\middle|\, c \in \mathbb{R} \right\}.$$

Consequently, the zero dynamics are not asymptotically stable. Note that the transfer function of this system reads

$$G(s) = [0 \;\; 0 \;\; 1] \begin{bmatrix} \mathcal{R} & -\mathcal{R} & 1 \\ -\mathcal{R} & \mathcal{C}s + \mathcal{R} & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{\mathcal{C}s}{\mathcal{R} + \mathcal{C}s}.$$

For the simulation we chose the parameters $\mathcal{C} = \mathcal{R} = 1$ and the (consistent) initial value $x^0 = (1, 1, 0)^\top \in \mathbb{R}^3$ for the closed-loop system $[E, A, B, C]$, Eq. (5). As reference signal we take the constant function $y_{\mathrm{ref}} \equiv 1$ and the funnel $\mathcal{F}_\varphi$ is determined by

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto \frac{2t}{t+1}. \tag{34}$$

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 2.3×10^{-14}, absolute tolerance: 10^{-10}) and is depicted in Fig. 8 over the time interval $[0,25]$. In Fig. 8a we see that the output converges to 0.5 and thus stays away from the reference trajectory by at least 0.5. This can also be concluded from the tracking error shown in Fig. 8d. Since the zero dynamics are not asymptotically stable the error does not stay away from the funnel boundary uniformly, but converges to φ^{-1}. In Fig. 8b and c we see that the gain and input grow unboundedly. This is necessary for feasibility of tracking in this case.

Indeed we can show that no output signal evolving in the performance funnel can be generated by a bounded input: Assuming that $(x,u,y)\in\mathfrak{B}_{[E,A,B,C]}$ with $(t, y(t)-y_{\text{ref}}(t))\in\mathcal{F}_\phi$ for all $t\in\mathbb{R}_{\geq 0}$ we have

$$\forall t>0: |1-y(t)| = |y_{\text{ref}}(t)-y(t)| \leq \varphi(t)^{-1}.$$

Since $\varphi(t)^{-1}\leq\frac{3}{4}$ for all $t\geq 2$, we have in particular that $y(t)\geq\frac{1}{4}$ for $t\geq 2$. Then, by $y(t) = -i_{\mathcal{V}}(t) = \mathcal{R}\eta_1(t)-\mathcal{R}\eta_2(t) = \mathcal{C}\dot\eta_2(t)$, we obtain

$$\eta_2(t) = \eta_2(2)+\int_2^t \dot\eta_2(\tau)\,\mathrm{d}\tau = \eta_2(2)+\frac{1}{\mathcal{C}}\int_2^t y(\tau)\,\mathrm{d}\tau \geq \eta_2(2)+\frac{1}{4\mathcal{C}}(t-2) = \eta_2(2)-\frac{1}{2\mathcal{C}}+\frac{t}{4\mathcal{C}},\quad t\geq 2.$$

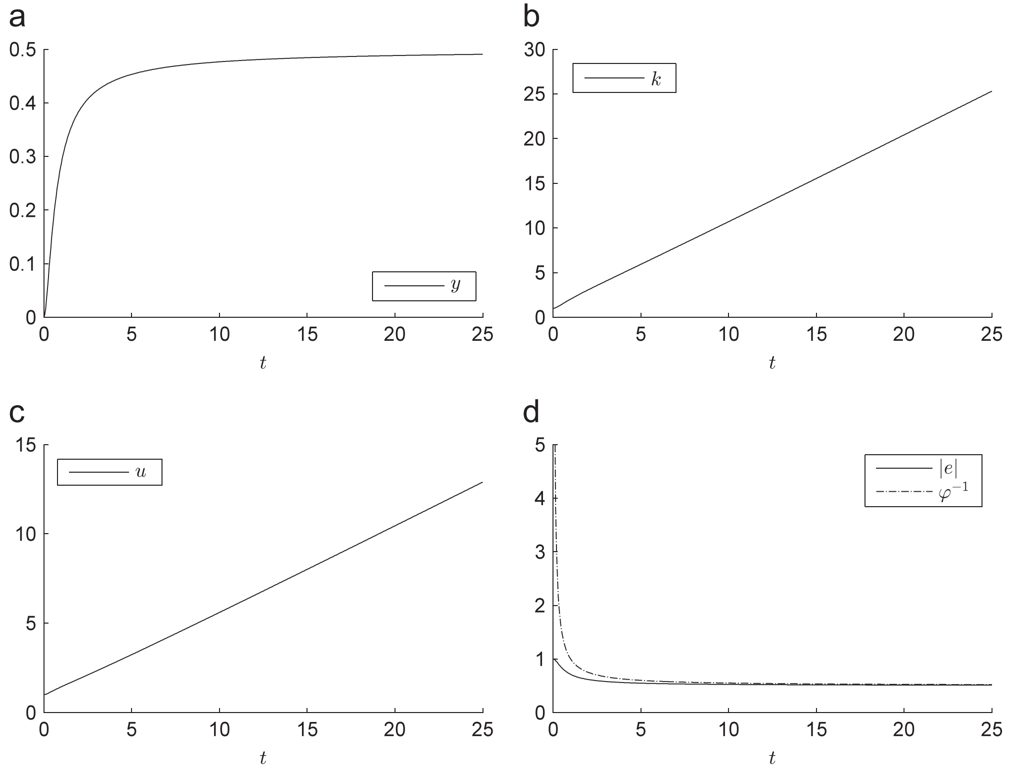

Fig. 8. Simulation of the funnel controller (5) with funnel boundary specified in Eq. (34) and reference signal $y_{\text{ref}}\equiv 1$ applied to system $[E,A,B,C]$ with initial data $x^0=(1,1,0)^\top$. (a) Solution y, (b) Gain k, (c) Input u, (d) Absolute value of error $|e(\cdot)|$ and funnel boundary $\varphi(\cdot)^{-1}$.

Therefore,

$$u(t) = \eta_1(t) = \mathcal{R}^{-1} y(t) + \eta_2(t) \geq \frac{1}{4\mathcal{R}} + \eta_2(2) - \frac{1}{2\mathcal{C}} + \frac{t}{4\mathcal{C}}, \quad t \geq 2.$$

This implies that the input is unbounded. Therefore, it seems that stability of the zero dynamics is a fundamental principle that is generally required for output tracking problems in order to achieve boundedness of the input.

9. Conclusion

We have investigated the zero dynamics of linear passive electrical circuits and proved that autonomy and asymptotic stability of the zero dynamics are physically structural properties. Furthermore, we have shown that the application of high-gain output feedback leads to a stability problem for a certain replacement circuit. In particular, it follows that funnel control is feasible only under physically structural assumptions on the circuit and some smoothness assumptions on the reference trajectory and the funnel boundary. The case of circuits with possibly non-autonomous zero dynamics can also be treated, provided that the reference trajectory is restricted to a certain subspace.

Finally, we like to stress that the findings of the present paper can directly be carried over to MLA (modified loop analysis) models of electrical circuits; for the latter see e.g. [26].

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