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On the Stability Analysis of linear, time-delayed Hessenberg Differential-Algebraic Equations ⋆

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Abstract In this paper we discuss the stability analysis for linear Hessenberg Differential-Algebraic Equations with time delay. First we discuss the classification of these systems, which is followed by the stability analysis for not only non-advanced but also for *weakly-advanced* systems. The idea is to transform a given system to an equivalent regular, impulse-free system via an *index reduction procedure*, which preserves the spectrum of the original system. Then, we introduce a new concept of C^p-weak exponential stability and study it via the the spectral method. Numerical examples are presented to illustrate the advantages of the proposed results.

Keywords: Singular systems; Delay; Spectral.
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Nomenclature

$\mathbb{N}$ ($\mathbb{N}_0$)	the set of natural numbers (including 0)
$\mathbb{R}$ ($\mathbb{R}_+$)	the set of real (non-negative real) numbers
$\mathbb{C}$ ($\mathbb{C}_-$)	the set of complex numbers (the set $\{\lambda \in \mathbb{C} \mid Re\lambda < 0\}$)
I (I_n)	the identity matrix (of size $n \times n$)
$x^{(j)}$	the j-th derivative of a function x
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of p-times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$ (for $0 \leqslant p < \infty$)
$\\| \cdot \\|_p$	the p-norm of the Banach space $C^p([-\tau, 0], \mathbb{R}^n)$, i.e. $\\|f\\|_p := \sum_{j=1}^{p} \sup\limits_{t \in [-\tau,0]} \left\\| f^{(p)}(t) \right\\|$
$\\| \cdot \\|_\infty$	the sup-norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$
$A(i, :)$	the i-th row of matrix A (in MATLAB notation)
$A(i : j, :)$	the rows of A, ranging from the i-th row to the j-th row (for $i \leqslant j$)
Row(i)	the i-th block row equation of a system
Row($i : j$)	the block row equations, ranging from the i-th row to the j-th row (for $i \leqslant j$)
$\mathit{\Delta}$	the shift backward operator $\mathit{\Delta} : x(t) \mapsto x(t - \tau)$

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1 Introduction and Preliminaries

In the present paper we study the stability analysis of linear, time invariant *delay differential-algebraic equations (DDAEs)* of the following form

$$E\dot{x}(t) = A^{(0)}x(t) + A^{(1)}x(t-\tau), \tag{1.1}$$

for all $t \in [0,\infty)$, where the matrix coefficients belong to $\mathbb{R}^{n,n}$, $x : [-\tau,\infty) \to \mathbb{R}^n$, and $\tau > 0$ is a constant delay. However, we do not aim at the stability of general system, but only for a class of Hessenberg system, where the matrix coefficients have the special structure as follows

$$E = \begin{bmatrix} I & & & & \\ & I & & & \\ & & \ddots & & \\ & & & I & \\ & & & & 0 \end{bmatrix}, A^{(0)} = \begin{bmatrix} A^{(0)}_{11} & A^{(0)}_{12} & \dots & A^{(0)}_{1,k-1} & A^{(0)}_{1,k} \\ A^{(0)}_{21} & A^{(0)}_{22} & \dots & A^{(0)}_{2,k-1} & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & A^{(0)}_{k-1,k-1} & 0 \\ & & & A^{(0)}_{k,k-1} & 0 \end{bmatrix}, A^{(1)} = \begin{bmatrix} A^{(1)}_{11} & A^{(1)}_{12} & \dots & A^{(1)}_{1,k-1} & A^{(1)}_{1,k} \\ A^{(1)}_{21} & A^{(1)}_{22} & \dots & A^{(1)}_{2,k-1} & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & A^{(1)}_{k-1,k-1} & 0 \\ & & & A^{(1)}_{k,k-1} & 0 \end{bmatrix}, \tag{1.2}$$

and the matrix product

$$A^{(0)}_{k,k-1}A^{(0)}_{k-1,k-2}\cdots A^{(0)}_{2,1}A^{(0)}_{1,k}$$

is nonsingular. Here $k \geqslant 2$ and we say that the Hessenberg DDAE (1.1) has an index k. On the other hand, index-1 (Hessenberg) systems take the form

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}^{(0)}_{11} & \hat{A}^{(0)}_{12} \\ \hat{A}^{(0)}_{21} & \hat{A}^{(0)}_{22} \end{bmatrix} x(t) + \begin{bmatrix} \hat{A}^{(1)}_{11} & \hat{A}^{(1)}_{12} \\ \hat{A}^{(1)}_{21} & \hat{A}^{(1)}_{22} \end{bmatrix} x(t-\tau), \tag{1.3}$$

where $\hat{A}^{(0)}_{22}$ is nonsingular.

write some more ... Hessenberg differential-algebraic equations arises from ...
In the following example we demonstrate some difficulties that may arise in the stability analysis of DDAEs.

To achieve uniqueness of solutions for DDAEs of the form (1.1) one typically has to prescribe an initial function, which takes the form

$$x|_{[-\tau,0]} = \varphi : [-\tau,0] \to \mathbb{R}^n. \tag{1.4}$$

Throughout this paper, we use the following solution concept.

Definition 1 i) A function $x : [-\tau,\infty) \to \mathbb{R}^n$ is called a *piecewise differentiable solution* of (1.1), if Ex is piecewise continuously differentiable, x is continuous and satisfies (1.1) at every $t \in [0,\infty) \setminus \bigcup_{j\in\mathbb{N}_0} \{j\tau\}$.
ii) An initial function φ is called *consistent* with (1.1) if the associated initial value problem (IVP) (1.1), (1.4) has at least one solution.
iii) System (1.1) is called *solvable* (resp. *regular*) if for every consistent initial function φ, the associated IVP (1.1), (1.4) has a solution (resp. has a unique solution).
iv) The set $\sigma(E,\hat{A}^{(0)},\hat{A}^{(1)}) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - \hat{A}^{(0)} - e^{-\lambda\tau}\hat{A}^{(1)}) = 0\}$ is called the *spectrum* of (1.1).

2 Main Results

2.1 The case of index $k = 2, 3$

In this part we demonstrate the index reduction strategy for Hessenberg DDAEs of index $k \leqslant 3$ and its consequences to the solvability and stability analysis of system (1.1). For $k = 2$, we rewrite the system as follows

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ A_{21}^{(0)} & 0 \end{bmatrix} x(t) + \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & 0 \end{bmatrix} x(t-\tau), \text{ for all } t \geqslant 0, \tag{2.1}$$

where $A_{21}^{(0)} A_{12}^{(0)}$ is nonsingular.
We transform the system to the Hessenberg index 1 form by replacing the second block row equation, denoted by Row(2), by the new one $\text{Row}(2)^{new}$ defined by

$$\text{Row}(2)^{new} = \frac{d}{dt} \text{ Row}(2) + A_{21}^{(0)} \text{ Row}(1) + A_{21}^{(1)} \Delta \text{ Row}(1), \tag{2.2}$$

where Δ is the shift backward operator, which maps $x(t)$ to $x(t-\tau)$. Here by Δ Row(2) we mean that the whole block row equation has been shifted backward, i.e.

$$0 = A_{21}^{(0)} x(t-\tau) + A_{21}^{(1)} x(t-2\tau), \text{ for all } t \geqslant \tau.$$

Consequently, the equation (2.2) becomes

$$\begin{aligned} 0 =& A_{21}^{(0)} \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \end{bmatrix} x(t) + \left(A_{21}^{(0)} \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \end{bmatrix} + A_{21}^{(1)} \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \end{bmatrix} \right) x(t-\tau) \\ &+ A_{21}^{(1)} \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \end{bmatrix} x(t-2\tau), \text{ for all } t \geqslant \tau. \end{aligned} \tag{2.3}$$

Thus, combining this equation with the first equation of (2.1) gives us the system

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ A_{21}^{(0)} A_{11}^{(0)} & A_{21}^{(0)} A_{12}^{(0)} \end{bmatrix} x(t) + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ * & * \end{bmatrix} x(t-\tau) \\ + \begin{bmatrix} 0 & 0 \\ A_{21}^{(1)} A_{11}^{(1)} & A_{21}^{(1)} A_{12}^{(1)} \end{bmatrix} x(t-2\tau), \text{ for all } t \geqslant \tau, \end{aligned} \tag{2.4}$$

which is clearly an index 1 system, since $A_{21}^{(0)} A_{12}^{(0)}$ is nonsingular.

Remark 1 We notice, that if we rewrite system (2.1) in the operator form

$$0 = \mathcal{P}(\frac{d}{dt}, \Delta)x := \left(-\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} \\ A_{21}^{(0)} & 0 \end{bmatrix} + \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & 0 \end{bmatrix} \Delta \right) x(t), \tag{2.5}$$

then system (2.4) is obtained by simply acting the operator $\begin{bmatrix} I & 0 \\ A_{21}^{(0)} + A_{21}^{(1)} \Delta & \frac{d}{dt} \end{bmatrix}$ on (2.1). This leads to a consequence, that *an index reduction step*, which transforming the index-2 system (2.1) to the index-1 system (2.4), does not alter the non-zero eigenvalues. This is very important, in particular to study the stability analysis, as we will see later in Section 2.3.

Remark 2 From the numerical viewpoint, in fact we can simplify the index reduction step above by transforming the matrix coefficient in the second row as follows.

$$\begin{aligned} A^{(0)}(2,:) &:= A^{(0)}(2,:)\ A^{(0)}, \\ A^{(1)}(2,:) &:= A^{(0)}(2,:)\ A^{(1)} + A^{(1)}(2,:)\ A^{(0)}, \end{aligned}$$

and introduce a new matrix coefficient $A^{(2)}$ associated with $x(t-2\tau)$ via

$$A^{(2)} := \begin{bmatrix} 0 \\ A^{(1)}(2,:)\ A^{(1)} \end{bmatrix} .$$

Now let us consider the case of index-3 Hessenberg DDAEs (i.e., $k = 3$) of the form

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} \\ A_{21}^{(0)} & A_{22}^{(0)} & 0 \\ 0 & A_{32}^{(0)} & 0 \end{bmatrix} x(t) + \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} & 0 \\ 0 & A_{32}^{(1)} & 0 \end{bmatrix} x(t-\tau), \quad (2.6)$$

for all $t \geqslant 0$, where the matrix product $A_{32}^{(0)} A_{21}^{(0)} A_{13}^{(0)}$ is nonsingular. Our index reduction procedure consists of two steps: Step 1: reduce an index from $k = 3$ to $k = 2$; and Step 2: reduce an index from $k = 2$ to $k = 1$ as above. Similarly to (2.2), Step 1 is done by performing a transformation on the last row only, i.e.,

$$\mathrm{Row}(3) \mapsto \mathrm{Row}(3)^{new} := \frac{d}{dt}\ \mathrm{Row}(3) + A_{32}^{(0)}\ \mathrm{Row}(2) + A_{32}^{(1)} \Delta\ \mathrm{Row}(2). \quad (2.7)$$

The new system now takes the from

$$0 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & A_{32}^{(0)} + A_{32}^{(1)} \Delta & \frac{d}{dt} \end{bmatrix} \mathcal{P}(\frac{d}{dt}, \Delta) x \ . \quad (2.8)$$

Continue performing Step 2, as in the case $k = 2$, we obtain an index-1 Hessenberg DDAE of the form

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = & - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} A_{11}^{(0)} & A_{12}^{(0)} & A_{13}^{(0)} \\ A_{21}^{(0)} & A_{22}^{(0)} & 0 \\ * & * & A_{32}^{(0)} A_{21}^{(0)} A_{13}^{(0)} \end{bmatrix} x(t) + \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} \\ A_{11}^{(1)} & A_{12}^{(1)} & 0 \\ * & * & * \end{bmatrix} x(t-\tau) \\ & + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} x(t-2\tau) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{bmatrix} x(t-3\tau), \text{ for all } t \geqslant \tau. \quad (2.9) \end{aligned}$$

Here $*$ stands for an arbitrary matrix. Here we notice again, that the index reduction procedure does not alter the non-zero eigenvalues of system (2.6).

2.2 The general case

The index reduction procedure presented above can ge directly generalized to index-k Hessenberg DDAEs in the following algorithm.

Algorithm 1 Index reduction procedure of the index-k Hessenberg DDAE (1.1)

Input: The system coefficients E, $A^{(0)}$, $A^{(1)}$.
Output: The system coefficients E, $A^{(0)}, \dots, A^{(k)}$ of the new system.

```
for j = 1: k-1 do
    Update the last row of matrices A^(0), ..., A^(j) by
        A^(0)(k,:) = A^(0)(k,:) A^(0),
        A^(ℓ)(k,:) = A^(0)(k,:) A^(ℓ) + A^(1)(k,:) A^(ℓ-1).
    Introduce a new matrix A^(j+1) := [ 0 ; A^(j)(k,:) A^(1) ] ∈ R^{n,n}.
end for
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$$A^{(0)}(k,:) = A^{(0)}(k,:)A^{(0)},$$
$$A^{(\ell)}(k,:) = A^{(0)}(k,:)A^{(\ell)} + A^{(1)}(k,:)A^{(\ell-1)}.$$

$$A^{(j+1)} := \begin{bmatrix} 0 \\ A^{(j)}(k,:)A^{(1)} \end{bmatrix} \in \mathbb{R}^{n,n}.$$

Theorem 1 *Consider the index-k Hessenberg DDAE* (1.1) *and assume that it is uniquely solvable for all consistent and sufficiently smooth initial function* ϕ. *Provided that the solution* x *is alrealdy known/computed on the interval* $[0,(k-1)\tau]$, *then system* (1.1) *has exactly the same solution* x *on* $[(k-1)\tau,\infty)$ *as the index-1 system of the form*

$$\left[\begin{array}{cccc} I & & & \\ & I & & \\ & & \ddots & \\ & & & I \\ \hline & & & 0 \end{array}\right] \dot{x}(t) = \left[\begin{array}{ccccc} A^{(0)}_{11} & A^{(0)}_{12} & \dots & A^{(0)}_{1,k-1} & A^{(0)}_{1,k} \\ A^{(0)}_{21} & A^{(0)}_{22} & \dots & A^{(0)}_{2,k-1} & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & A^{(0)}_{k-1,k-1} & 0 \\ \hline * & * & * & * & \hat{\mathbf{A}}^{(\mathbf{0})}_{\mathbf{k},\mathbf{k}} \end{array}\right] x(t)$$
$$+ \left[\begin{array}{ccccc} A^{(1)}_{11} & A^{(1)}_{12} & \dots & A^{(1)}_{1,k-1} & A^{(1)}_{1,k} \\ A^{(1)}_{21} & A^{(1)}_{22} & \dots & A^{(1)}_{2,k-1} & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & A^{(1)}_{k-1,k-1} & 0 \\ \hline * & * & * & * & * \end{array}\right] x(t-\tau) + \sum_{j=2}^{k} \left[\begin{array}{c} 0\ 0 \dots\ 0\ 0 \\ 0\ 0 \dots\ 0\ 0 \\ 0\ 0 \dots\ 0\ 0 \\ \hline *\ * \dots\ *\ * \end{array}\right] x(t-j\tau), \tag{2.10}$$

for all $t \geqslant \tau$, *where* $*$ *stands for an arbitrary matrix, and the matrix*

$$\hat{\mathbf{A}}^{(\mathbf{0})}_{\mathbf{k},\mathbf{k}} := A^{(0)}_{k,k-1} A^{(0)}_{k-1,k-2} \cdots A^{(0)}_{2,1} A^{(0)}_{1,k}$$

is nonsingular. Furthermore, the transformed system (2.10) *preserves all non-zero eigenvalues of system* (1.1).

Corollary 1 *Consider the index-k Hessenberg DDAE* (1.1) *and assume that it is uniquely solvable for all consistent and sufficiently smooth initial function* ϕ. *In order to have a continuous, piecewise differentiable solution* $x|_{[0,\infty)}$, *the smoothness requirement for* ϕ *is upper-bounded by* $(k-1)^2$.

2.3 Stability analysis

We recall the stability concept for DDAEs as follows.

Definition 2 ([6, 7]) The null solution $x = 0$ of the DDAE (1.1) is called *exponentially stable* if there exist positive constants δ and γ such that for any consistent initial function $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, the solution $x = x(t, \varphi)$ of the corresponding IVP to (1.1) satisfies

$$\|x(t)\| \leqslant \delta e^{-\gamma t} \|\varphi\|_\infty, \quad \text{for every} \quad t \geqslant 0.$$

Definition 3 The DDAE (1.1) is called
i) *non-advanced* (or *impulse-free*) if for any consistent $\varphi \in C([-\tau, 0], \mathbb{R}^n)$, there exists a unique solution x to the corresponding IVP for (1.1).
ii) C^k*-weakly advanced* if for any consistent $\varphi \in C^p([-\tau, 0], \mathbb{R}^n)$, there exists a unique solution x to the corresponding IVP for (1.1).

Example 1 It is well-known, see e.g. [1, 2, 8], that for index-2 Hessenberg system (2.1), the system is non-advanced if $A_{21}^{(1)} = 0$. For index-2 Hessenberg system (2.1), the system is non-advanced if $A_{32}^{(1)} = 0$ and $A_{21}^{(1)} = 0$. From our discussion in Section 2.1, we see that system (2.1) is C^2-weakly advanced if $A_{21}^{(1)} \left[A_{11}^{(1)} \; A_{12}^{(1)} \right] \neq 0$. In case of the index-3 system (2.6), it is C^4-weakly advanced if $A_{32}^{(1)} \left[A_{21}^{(1)} \; A_{22}^{(1)} \right] \neq 0$. Furthermore, it is C^2-weakly advanced if $A_{32}^{(1)} \left[A_{21}^{(1)} \; A_{22}^{(1)} \right] = 0$. Due to Corollary 1, we see that the index-k Hessenberg system (1.1) is $C^{(k-1)^2}$-weakly advanced.

The characterization for exponential stability of the index-1 DDAE (1.3) is given in the following proposition.

Proposition 1 *([3, 6]) The index-*1 *DDAE* (1.3) *is exponentially stable if and only if the spectrum* $\sigma(E, A^{(0)}, A^{(1)})$ *lies entirely on the left half plane and is bounded away from the imaginary axis.*

Definition 4 The null solution $x = 0$ of the DDAE (1.1) is called C^p*-weakly exponentially stable (*C^p*-w.e.s)* if there exist an integer $0 \leqslant p < \infty$ and positive constants δ and γ such that for any consistent initial function $\varphi \in C^p([-\tau, 0], \mathbb{R}^n)$, the solution $x = x(t, \varphi)$ of the corresponding IVP for (1.1) satisfies

$$\|x(t)\| \leqslant \delta e^{-\gamma t} \|\varphi\|_p, \quad \text{for all } t \geqslant 0.$$

Notice that the (classical) exponential stability is exactly C^0-w.e.s.. Furthermore, even though C^p-w.e.s. has been considered for ODEs and PDEs as well, till now there are very few reference for DDAEs, see [4, 5].

Theorem 2 *Consider the index-k Hessenberg DDAE* (1.1) *and assume that it is uniquely solvable for all consistent and sufficiently smooth initial function* ϕ*. We also consider the transformed system* (2.10) *obtained by applying Algorithm 1 to system* (1.1)*. Furthermore, we assume that* $0 \notin \sigma(E, A^{(0)}, A^{(1)})$*. Then system* (1.1) *is* $C^{(k-1)^2}$*-weakly exponentially stable, provided that the spectrum* $\sigma(E, A^{(0)}, A^{(1)})$ *lies entirely on the left half plane and is bounded away from the imaginary axis.*

Corollary 2 *Stability condition*

Example 2 Numerical test

3 Conclusion and Outlook

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References

1. U. M. Ascher and L. R. Petzold. The numerical solution of delay-differential algebraic equations of retarded and neutral type. *SIAM J. Numer. Anal.*, 32:1635–1657, 1995.
2. S. L. Campbell and V. H. Linh. Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions. *Appl. Math Comput.*, 208(2):397 – 415, 2009.
3. N. H. Du, V. H. Linh, V. Mehrmann, and D. D. Thuan. Stability and robust stability of linear time-invariant delay differential-algebraic equations. *SIAM J. Matr. Anal. Appl.*, 34(4):1631–1654, 2013.
4. P. Ha. On the stability analysis of delay differential-algebraic equations. *VNU Journal of Science: Mathematics - Physics*, 34(2), 2018.
5. P. Ha. Spectral characterizations of solvability and stability for delay differential-algebraic equations. *Acta Mathematica Vietnamica*, 43:715–735, 2018.
6. W. Michiels. Spectrum-based stability analysis and stabilisation of systems described by delay differential algebraic equations. *IET Control Theory Appl.*, 5(16):1829–1842, 2011.
7. S. Xu, P. Van Dooren, R. Ştefan, and J. Lam. Robust stability and stabilization for singular systems with state delay and parameter uncertainty. *IEEE Trans. Automat. Control*, 47(7):1122–1128, 2002.
8. W. Zhu and L. R. Petzold. Asymptotic stability of Hessenberg delay differential-algebraic equations of retarded or neutral type. *Appl. Numer. Math.*, 27(3):309 – 325, 1998.