EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE

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Abstract. We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line $\mathbb{R}_+$. Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform φ-Lipschitz condition, in which the Lipschitz function φ belongs to wide classes of admissible function spaces such as L_p, $1 \leqslant p \leqslant \infty$, $L_{p,q}$, etc.

1. Introduction and Preliminaries

The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear non-autonomous differential-algebraic equations (DAEs) of the form

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$$\begin{array}{l} d \text{ rows} \\ a \text{ rows} \end{array} \quad \underbrace{\begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}}_{E(t)} \dot{x}(t) = \underbrace{\begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}}_{A(t)} x(t) + \underbrace{\begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}}_{f(t,x(t))}, \quad t \in \mathbb{R}_+ := [0, +\infty). \tag{1.1}$$

To do that, we start by investigating the exponential dichotomy of the associated linear system

linDAE

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \tag{1.2}$$

Here $E = \begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}$ are matrix-valued functions acting on $\mathbb{R}_+$ to $\mathbb{R}^{n,n}$, $x : \mathbb{R}_+ \to \mathbb{R}^n$, and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Furthermore, we assume that for all t, the matrices $\mathbf{E}_1(t)$, $\mathbf{A}_2(t)$ have full row rank.

DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuits, chemical engineering, and many other applications. Due to the rank-deficiency of $E(t)$, the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the reader to recent monographs [2, 12–14] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [12, Chapter 2]), there are only few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [15] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [17] for Lyapunov and other spectra for linear DAEs, to [4, 8] for the robustness of exponential stability and Bohl exponents.

On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is characterized, the next important question in the qualitative theory of DAEs is to study the existence of integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1) [3, 6]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps, this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1).

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Our method is based on the classical "Lyapunov-Perron method" ([6, 25]) and the admissibility of function spaces ([10, 11]).

The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for later use, including the notion of the exponential dichotomy and its properties, as well as some important features of admissible function spaces. In Section 2 we gives a characterization for the existence of exponential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations, and we conclude this research by a summary and some open problems.

1.1. **Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By $(\mathbb{R}^n, \|\cdot\|)$ we denote the n-dimensional real vector space equipped with the Euclidean norm. For any matrix V, by V^T we denote its transpose. For any $p \in \mathbb{N}$, by $C^p([0,\infty),\mathbb{R}^n)$ we denote the space of p-times continuously differentiable functions acting on $[0,\infty)$ with values in $\mathbb{R}^n$. By $C_b([0,\infty),\mathbb{R}^n)$ we denote the space of continuous and bounded functions mapping from $[0,\infty)$ into $\mathbb{R}^n$. This space is a Banach space with the *ess* sup-norm $\|f\|_\infty := \sup\{\|f(t)\|,\ t \geqslant 0\}$.

It is well-known (e.g. [3]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t),\ t \geqslant s \geqslant 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

is well-posed, then there exists a pointwise nonsingular matrix-valued function $(t,s) \mapsto X(t,s) \in \mathbb{R}^{n,n}$ such that the solution of (1.3) is given by $x(t) = X(t,s)x_s$. This fact motivates the existence of an evolution family $(X(t,s))_{t\geqslant s\geqslant 0}$ associated with the matrix function $A(t)$. This family satisfies the condition $X(t,t) = Id$ and the so-called *semi-group property*

$$X(t,r)X(r,s) = X(t,s), \text{ for all } t \geqslant r \geqslant s \geqslant 0. \tag{1.4}$$

Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t,x(t)), \quad \text{for all } t \geqslant s \geqslant 0,$$

also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t,s)x(s) + \int_s^t X(t,\tau)f(\tau,x(\tau))d\tau, \quad \text{for all } t \geqslant s \geqslant 0. \tag{1.5}$$

For more details on the notion and discussion on properties and applications of evolution families we refer the readers to Pazy [21].

Definition 1.1. A given evolution family $\{X(t,s)\}_{t\geqslant s\geqslant 0}$ of the ODE (1.3) is said to have an *exponential dichotomy* on the half-line if there exist a family of projection matrices $\{P(t)\}_{t\geqslant 0}$ and two positive constants N, ν such that the following conditions are satisfied.

i) $P(t)X(t,s) = X(t,s)P(s)$ for all $t \geqslant s \geqslant 0$,

ii) for all $t \geqslant s \geqslant 0$, the restriction $X(t,s)_| : \ker P(s) \to \ker P(t)$ is an isomorphism, and we denote its inverse by $X(s,t)_|$,

iii) $\|X(t,s)P(s)x\| \leqslant Ne^{-\nu(t-s)}\|P(s)x\|$, for all $t \geqslant s \geqslant 0$, $x \in \mathbb{R}^n$,

iv) $\|X(t,s)_|\ (I-P(s))x\| \leqslant Ne^{\nu(t-s)}\|(I-P(s))x\|$, for all $s \geqslant t \geqslant 0$, $x \in \mathbb{R}^n$.

Here $\{P(t)\}_{t\geqslant 0}$ (reps. N, ν) are called *dichotomy projections* (resp. *dichotomy constants*).

The concept exponential dichotomy means that the state space $\mathbb{R}^n$ has been splitted into the (exponentially) stable subspace ($\mathrm{Im}(P(t))$) and the (exponentially) unstable subspace ($\ker(P(t))$).

1.2. **A short review of DAE solvability theory.** Linear DAEs of the form 1.2 have been extensively studied in the last thirty years, see [13] and the references therein. In order to understand the solution behavior and to obtain numerical solutions, the necessary information about derivatives of equations has to be utilized. This has led to the concept of the strangeness index, which allows to use the DAE and (some of) its derivatives to be reformulated as a system with the same solution that is *strangeness-free*, i.e., for which the algebraic and differential part of the system are easily separated. In this paper we restrict ourselves to regular DAEs with sufficiently smooth coefficients, i.e., we require that (1.2) (or the nonlinear DAE (1.1) locally) has a unique solution for appropriately chosen (consistent) initial conditions, see [13] for a discussion of more general nonregular DAEs. With the theory and appropriate numerical methods available, then throughout this paper, for regular DAEs we may assume that the homogeneous DAE 1.2 in consideration fulfills the following presumption.

Assumption 1.2. Assume that the function pair (E, A) in the DAEs (1.1), (1.2) is *strangeness-free*, i.e.,

$$\operatorname{rank}\begin{bmatrix}\mathbf{E}_1(t)\\ \mathbf{A}_2(t)\end{bmatrix} = n,$$

for all $t \geqslant 0$. Furthermore, we assume that $E \in C^1([0,\infty),\mathbb{R}^{n,n})$ and $A \in C^0([0,\infty),\mathbb{R}^{n,n})$.

Definition 1.3. The DAE

eq7

$$\tilde{E}(t)\dot{y}(t) = \tilde{A}(t)x(t) + \tilde{f}(t, y(t)) \tag{1.6}$$

is called *orthogonally equivalent* to the DAE (1.1) if there exist pointwise-orthogonal matrix-valued functions $U \in C^0([0,\infty),\mathbb{R}^{n,n})$ and $V \in C^1([0,\infty),\mathbb{R}^{n,n})$, such that after changing variable $x(t) = V(t)y(t)$, and scaling (1.2) with $U(t)$, we obtain exactly (1.6). In details, this means that the following identities hold true.

$$\tilde{E} = UEV,\ \tilde{A} = UAV - UE\dot{V},\ \tilde{f}(t, y(t)) = U(t)f(t, Vy(t)),\ \text{for all } t \geqslant 0. \tag{1.7}$$

We denote this orthogonal equivalence by $(E, A, f) \overset{o}{\sim} (\tilde{E}, \tilde{A}, \tilde{f})$ and omit the terms f, $\tilde{f}$ if the homogeneous system (1.2) is considered.

Indeed, one can directly verify that this orthogonal equivalence concept is an equivalent relation, i.e., it fulfills three properties: reflexivity, symmetry and transitivity. We omit the detailed proof here in order to keep the brevity of this work. By making use of some smooth factorizations, for example QR or SVD ([7] or [13, Thm 3.9]), we can decouple the differential and algebraic parts of the DAE (1.2) in the following lemma.

Lemma 1.4. *Consider the DAE* (1.2) *and assume that it satisfies Assumption 1.2. Then, there exist pointwise-orthogonal matrix-valued functions* $U \in C^0([0,\infty),\mathbb{R}^{n,n})$ *and* $V \in C^1([0,\infty),\mathbb{R}^{n,n})$*, such that by changing variable* $x(t) = V(t)y(t)$*, and scaling* (1.2) *with* $U(t)$*, we obtain the so-called* semi-explicit system

semi-explicit system

$$\begin{bmatrix}\Sigma(t) & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{y}_1(t)\\ \dot{y}_2(t)\end{bmatrix} = \begin{bmatrix}A_1(t) & A_2(t)\\ A_3(t) & A_4(t)\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}, \tag{1.8}$$

with pointwise nonsingular matrix-valued functions $\Sigma(t) \in \mathbb{R}^{d,d}$ *and* $A_4(t) \in \mathbb{R}^{a,a}$.

Proof. Applying an SVD factorization for $\mathbf{E}_1(t)$ we can find pointwise-orthogonal matrix functions $U_1(t) \in C^1([0,\infty),\mathbb{R}^{d,d})$ and $V \in C^1([0,\infty),\mathbb{R}^{n,n})$ such that $U_1(t)\mathbf{E}_1(t)V(t) = \begin{bmatrix}\Sigma(t) & 0\end{bmatrix}$, where $\Sigma(t)$ is a continuous, pointwise nonsingular function with values in $\mathbb{R}^{d,d}$. Changing the variable $x(t) = V(t)y(t)$ and scaling (1.2) with $U(t) := \begin{bmatrix}U_1(t) & 0\\ 0 & I_a\end{bmatrix}$, we obtain a new system exactly of the form (1.8). Furthermore, notice that

$$\begin{bmatrix}\Sigma(t) & 0\\ A_3(t) & A_4(t)\end{bmatrix} = \begin{bmatrix}U_1(t) & 0\\ 0 & I_a\end{bmatrix}\begin{bmatrix}\mathbf{E}_1(t)\\ \mathbf{A}_2(t)\end{bmatrix}V(t),$$

then Assumption 1.2 yields that both Σ and A_4 are nonsingular. This completes the proof. $\square$

Let $\hat{A}_3 := -A_4^{-1}A_3$, $\hat{A}_1 := \Sigma^{-1}\left(A_1 - A_2A_4^{-1}A_3\right)$, we rewrite system (1.8) as

eq10.1

eq10.2

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t), && \text{(1.9a)} \\ y_2(t) &= \hat{A}_3(t)y_1(t). && \text{(1.9b)} \end{aligned}$$

Since $V(t)$ is orthogonal for all $t \geqslant 0$, we see that all important qualitative properties of $x(t)$, such as boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function $y(t)$ without any difficulty. Clearly, we see that (1.9b) gives an *algebraic constraint* that the solution to (1.8) must obey, while (1.9a) gives the dynamic of (1.8). For this reason, we call it *an underlying ODE* to (1.8).

Definition 1.5. (i) Consider the DAE (1.2). A matrix function $X \in C([0,\infty),\mathbb{R}^{n,k})$, $d \leqslant k \leqslant n$, is called *a fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and rank $X(t) = d$, for all $t \geqslant 0$.
(ii) A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively. A maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

projected intial condition

$$E(0)\left(X(0) - Id\right) = 0. \tag{1.10}$$

We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of full rank. Furthermore, each fundamental solution matrix has exactly d-linear independent columns, and a minimal fundamental solution matrix can be made maximal by adding $n-d$ zero columns. This is the major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been proposed and carefully discussed in [17]. We recall it below, and notice that this concept is equivalent to the one proposed by Lentini and März in [15] within the context of the matrix chains approach and tractability index.

Let $\{\hat{Y}_1(t,s)\}_{t\geqslant s\geqslant 0}$ be the evolution family associated with (1.9a), then we can define the corresponding evolution families for two DAEs (1.8), (1.2) consecutively as follows.

eq11

$$\hat{Y}(t,s) := \begin{bmatrix} \hat{Y}_1(t,s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t,s) & 0 \end{bmatrix},\ \hat{X}(t,s) := V(t)\hat{Y}(t,s)V^T(s),\ \text{for all } t \geqslant s \geqslant 0. \tag{1.11}$$

Nevertheless, since $X(t,s)$ is not invertible, we will define the *reflexive generalized inverse matrix function* as in [17] by

eq12

$$\hat{Y}^-(t,s) := \begin{bmatrix} \hat{Y}_1^{-1}(t,s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t,s) & 0 \end{bmatrix},\ \hat{X}^-(t,s) := V(s)\hat{Y}^-(t,s)V^T(t),\ \text{for all } t \geqslant s \geqslant 0. \tag{1.12}$$

Then, we can directly verify the semigroup properties, i.e.

$$\begin{aligned} \hat{X}(t,r) &= \hat{X}(t,s)\hat{X}(s,r),\ \text{for all } t \geqslant s \geqslant r \geqslant 0, \\ \hat{X}(t,s) &= \hat{X}(t,0)\hat{X}^-(s,0),\ \text{for all } t \geqslant s \geqslant 0. \end{aligned}$$

Furthermore, Lemmas 1.6 and 1.7 below show that the family $\{\hat{X}(t,s)\}_{t\geqslant s\geqslant 0}$ does not depend on the choice of orthogonal transformations, and it plays the same role as the evolution family $\{X(t,s)\}_{t\geqslant s\geqslant 0}$, in comparison to (1.5).

Lemma 1.6. *The families* $\{X(t,s)\}_{t\geqslant s\geqslant 0}$, $\{X^-(t,s)\}_{t\geqslant s\geqslant 0}$ *defined by* (1.11), (1.12) *do not depend on the choice of orthogonal transformations.*

Proof. We will prove this claim only for the first family $\{X(t,s)\}_{t\geqslant s\geqslant 0}$, since for the second family the proof is essentially the same. Let us assume that we have two semi-explicit forms of system (1.2) obtained under orthogonal transformations, i.e.,

$$(E,A) \overset{o}{\sim} \left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) , \tag{1.13a}$$

$$(E,A) \overset{o}{\sim} \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right) . \tag{1.13b}$$

Now we will prove that the two corresponding systems have the same evolution family $\{\hat{X}(t,s)\}_{t\geqslant s\geqslant 0}$. Without loss of generality, we assume that (E,A) is already in the form of the right hand side of (1.13a), so U and V in Lemma 1.4 are identity matrices and $\hat{X}(t,s) = \hat{Y}(t,s)$ for all $t \geqslant s \geqslant 0$. The corresponding system to the right hand side of (1.13b) reads

$$\dot{\tilde{y}}_1(t) = \hat{\mathcal{A}}_1(t)\tilde{y}_1(t), \tag{1.14a}$$

$$\tilde{y}_2(t) = \hat{\mathcal{A}}_3(t)\tilde{y}_1(t). \tag{1.14b}$$

where $\hat{\mathcal{A}}_3 := -\tilde{A}_4^{-1}\tilde{A}_3$, $\hat{\mathcal{A}}_1 := \tilde{\Sigma}^{-1}\left(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3\right)$.

The transitivity applied to (1.13) implies that there exist pointwise-orthogonal matrix-valued functions $S = \left[\begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix}\right] \in C^0([0,\infty),\mathbb{R}^{n,n})$ and $T = \left[\begin{smallmatrix} T_1 & T_2 \\ T_3 & T_4 \end{smallmatrix}\right] \in C^1([0,\infty),\mathbb{R}^{n,n})$, such that $y(t) = T(t)\tilde{y}(t)$ and

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \tag{1.15a}$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T}_1 & \dot{T}_2 \\ \dot{T}_3 & \dot{T}_4 \end{bmatrix} \right) . \tag{1.15b}$$

Let $\{\hat{\mathcal{Y}}_1(t,s)\}_{t\geqslant s\geqslant 0}$ be the evolution family associated with (1.14a), then the evolution family associated with system (1.14) is

$$\hat{\mathcal{Y}}(t,s) = \begin{bmatrix} \hat{\mathcal{Y}}_1(t,s) & 0 \\ \hat{\mathcal{A}}_3(t)\hat{\mathcal{Y}}_1(t,s) & 0 \end{bmatrix}, \ \hat{\mathcal{X}}(t,s) := T(t)\hat{\mathcal{Y}}(t,s)T^T(s) \text{ for all } t \geqslant s \geqslant 0 . \tag{1.16}$$

Thus, we need to prove that $\hat{\mathcal{X}}(t,s) = \hat{X}(t,s)$.
From (1.15a), it implies that $S_3\Sigma\begin{bmatrix} T_1 & T_2 \end{bmatrix} = 0$. Thus, we have

$$\begin{bmatrix} S_3 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

and hence, this follows that $S_3 = 0$. Thus, $S = \left[\begin{smallmatrix} S_1 & S_2 \\ 0 & S_4 \end{smallmatrix}\right]$, and then, due to the orthogonality of S, S_1 is nonsingular and S_4 is orthogonal. Also from (1.15a), we see that $S_1\Sigma T_2 = 0$, which yields that $T_2 = 0$. Moreover, due to the orthogonality of S and T, from (1.14a) we have

$$\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^T & S_3^T \\ S_2^T & S_4^T \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^T & T_3^T \\ T_2^T & T_4^T \end{bmatrix}.$$

Therefore, using similar arguments as above, we can prove that $S_2 = 0$ and $T_3 = 0$.
Consequently, by inserting $S_3 = T_3 = 0$ and $S_2 = T_2 = 0$ into (1.15a) and (1.15b) we obtain

$$\tilde{\Sigma} = S_1\Sigma T_1, \tag{1.17a}$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1\left(A_1T_1 - \Sigma\dot{T}_1\right) & S_1A_2T_4 \\ S_4A_3T_1 & S_4A_4T_4 \end{bmatrix} , \tag{1.17b}$$

where the matrix-valued function S_i, T_i ($i = 1, 4$) are pointwise orthogonal. Thus, we have

$$\hat{A}_3 = -\tilde{A}_4^{-1}\tilde{A}_3 = -\left(S_4 A_4 T_4\right)^{-1} S_4 A_3 T_1 = T_4^{-1}\hat{A}_3 T_1 \ , \tag{1.18a}$$

$$\hat{A}_1 = \tilde{\Sigma}^{-1}\left(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3\right) = T_1^{-1}\left(\hat{A}_1 T_1 - \dot{T}_1\right) \ . \tag{1.18b}$$

Furthermore, since $y = T\tilde{y}$, the structure of T implies that $y_1 = T_1\tilde{y}_1$ and $y_4 = T_4\tilde{y}_4$. Therefore, the underlying ODE (1.14a) is directly obtained from (1.9a) by applying the variable transformation $\tilde{y}_1(t) = T_1(t)y_1(t)$ and scaling the system with T_1^{-1}. So, we have that $\hat{\mathcal{Y}}_1(t,s) = T_1^{-1}\hat{Y}_1(t,s)T_1(s)$. Making use of (1.18), we can deduce the evolution family $\{\hat{\mathcal{X}}(t,s)\}_{t\geqslant s\geqslant 0}$ as follows

$$\hat{\mathcal{X}}(t,s) = \begin{bmatrix} T_1(t) & 0 \\ 0 & T_4(t) \end{bmatrix} \begin{bmatrix} \hat{\mathcal{Y}}_1(t,s) & 0 \\ \hat{\mathcal{A}}_3\hat{\mathcal{Y}}_1(t,s) & 0 \end{bmatrix} \begin{bmatrix} T_1^T(s) & 0 \\ 0 & T_4^T(s) \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t,s) & 0 \\ \hat{A}_3\hat{Y}_1(t,s) & 0 \end{bmatrix},$$

and hence, this completes the proof. $\square$

Lemma 1.7. *Consider the DAE* (1.1) *and the evolution family* $(X(t,s))_{t\geqslant s\geqslant 0}$ *defined by* (1.11). *Furthermore, we also consider the pointwise-orthogonal matrix-valued functions* U, V *defined in Lemma 1.7. Then, the solution to* (1.1), *if exists, also satisfies the so-called* mild equation

$$\begin{aligned}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} =& \hat{X}(t,s)\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t,\tau)\begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau)f(\tau,x(\tau))d\tau \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t)f(t,x(t)),\end{aligned}$$

for all $t \geqslant s \geqslant 0$.

Proof. The proof can be obtained directly by using Lemma 1.4. Thus, in order to keep the brevity we will omit the details here. $\square$

In the following, for ease of notation, we will use the abbreviation $\hat{X}(t) := \hat{X}(t,0)$, $\hat{X}^-(t) := \hat{X}^-(t,0)$, $\hat{Y}(t) := \hat{Y}(t,0)$ and $\hat{Y}^-(t) := \hat{Y}^-(t,0)$. The concept of exponential dichotomy for the DAE (1.8) is given as below.

Definition 1.8. ([17]) The DAE (1.8) is said to have an *exponential dichotomy* if there exist a family of projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ in $\mathbb{R}^{d,d}$ and positive constants N, ν such that

$$\begin{aligned}&\left\|\hat{Y}(t)\begin{bmatrix} P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}\hat{Y}^-(s)\right\| \leqslant Ne^{-\nu(t-s)}, \text{ for all } t \geqslant s \geqslant 0, \\ &\left\|\hat{Y}(t)\begin{bmatrix} I_d - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}\hat{Y}^-(s)\right\| \leqslant Ne^{\nu(t-s)}, \text{ for all } s \geqslant t \geqslant 0,\end{aligned} \tag{1.19}$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.11) and (1.19) we have

$$\left\|\hat{X}(t)V^T(0)\begin{bmatrix} P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}V(0)\hat{X}^-(s)\right\| \leqslant Ne^{-\nu(t-s)}, \text{ for all } t \geqslant s \geqslant 0,$$

and

$$\left\|\hat{X}(t)V^T(0)\begin{bmatrix} Id - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}V(0)\hat{X}^-(s)\right\| \leqslant Ne^{\nu(t-s)}, \text{ for all } s \geqslant t \geqslant 0.$$

In addition, since $V^T(0)\begin{bmatrix} Id - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}V(0)$ is also a projection matrix for any $t \geqslant 0$, we can interpret the exponential dichotomy of (1.2) as the one of (1.8).

1.3. **Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces that play a fundamental role in the study of differential equations and refer to Nguyen [10], Massera and Schäffer [18, Chap. 2] and Räbiger and Schnaubelt [22, §1] for various applications.

Let E (endowed with the norm $\|\cdot\|_E$) be Banach function space of real-valued functions defined as in [10]. We then recall the Banach space corresponding to the space E as follows.

Definition 1.9 ([10])**.** Consider the Banach space $(\mathbb{R}^n, \|\cdot\|)$. For a Banach function space E we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \to \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

endowed with the norm $\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_E$. Thus, one can directly see that $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space. We call it *the Banach space corresponding to E.*

We now introduce the notion of admissibility in the following definition.

Definition 1.10 ([10])**.** The Banach function space E is called *admissible* if for any $\varphi \in E$ the following conditions hold.

(i) There exists a constant $M \geqslant 1$ such that for every compact interval $[a,b] \subset \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)|dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \text{ for all } \varphi \in E, \tag{1.20}$$

where $\chi_{[a,b]}$ is the indicator function of $[a,b]$.

(ii) The function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1}\varphi(\tau)d\tau$ belongs to E.

(iii) For any $\tau \geqslant 0$, the space E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined as

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geqslant \tau \geqslant 0, \\ 0 & \text{for } 0 \leqslant t \leqslant \tau, \end{cases} \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \text{ for } t \geqslant 0. \end{aligned} \tag{1.21}$$

Furthermore, there exist constants N_1, N_2 such that $\|T_\tau^+\|_E \leqslant N_1$, $\|T_\tau^-\|_E \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 1.11. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t\geqslant 0}\int_t^{t+\alpha}|h(\tau)|d\tau < \infty\},$$

(for any fixed $\alpha > 0$), endowed with the norm $\|h\|_{\mathbf{M}_\alpha} := \sup_{t\geqslant 0}\int_t^{t+\alpha}|h(\tau)|d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}, 1 < p < \infty, 1 \leqslant q < \infty$ (see [3], [24]) and, more general, the class of rearrangement invariant function spaces (see [16]) are admissible.

Remark 1.12. Following directly from Definition 1.10 we have that

$$\sup_{t\geqslant 0}\int_t^{t+1}|\varphi(\tau)|d\tau \leqslant \frac{M}{\inf_{t\geqslant 0}\|\chi_{[t,t+1]}\|_E}\|\varphi\|_E,$$

and hence, $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$. Furthermore, $C_b(\mathbb{R}^+)$, the Banach space of bounded, continuous function from $\mathbb{R}_+$ to $\mathbb{R}^n$, is dense in $\mathbf{M}_1$.

We present here some important features of admissible spaces in the following proposition (see [10, Proposition 2.6] and originally in [18, 23.V.(1)]).

Proposition 1.13 ([10]). Let E be an admissible Banach function space. Then the following assertions hold.
a) Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where, Λ_1 is defined as in definition 1.10 (ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by

$$\Lambda'_\sigma\varphi(t) := \int_0^t e^{-\sigma(t-s)}\varphi(s)ds,$$
$$\Lambda''_\sigma\varphi(t) := \int_t^\infty e^{-\sigma(s-t)}\varphi(s)ds.$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup\limits_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see remark 1.12)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for $ess\,\sup$-norm, we have

$$\|\Lambda'_\sigma\varphi\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma\varphi\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \tag{1.22}$$

for operator T_1^+ and constants N_1, N_2 defined as in Definition 1.10.
b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for any constant $\alpha > 0$.
c) E does not contain exponentially growing functions $f(t) := e^{bt}$ for any constant $b > 0$.

2. Exponential Dichotomy for linear DAEs

In the qualitative analysis of ODEs, one of the central topics is to find necessary and sufficient conditions such that the considered system admits an exponential dichotomy. Many researches have been devoted to this topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces (e.g. [6, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

Proposition 2.1. ([17]) The DAE (1.8) has an exponential dichotomy if and only if the corresponding underlying ODE (1.9a) also has exponential dichotomy and the matrix function $\hat{A}_3(t)$ is bounded. Moreover, the existence of an exponential dichotomy implies that $\sup\limits_{t\geqslant 0}\|P_{\mathrm{y}}(t)\| < \infty$.

Notice that, Proposition 2.1 is only valid for finite-dimensional but it is very hard to generalize for infinite dimensional DAE systems. For this reason, we aim at another approach, motivated from one classical result stated below.

Proposition 2.2. ([5, Chap. 3]) The ODE (1.3) has an exponential dichotomy if and only if one of the following conditions is satisfied.
i) For any function $g \in \mathbf{M}_1(\mathbb{R}_+)$ there exists a continuous, bounded solution $x(t)$ to the inhomogeneous system

$$\dot{x}(t) = A(t)x(t) + g(t)\ . \tag{2.1}$$

ii) For any function $g \in C_\infty(\mathbb{R}_+)$, there exists a continuous, bounded solution $x(t)$ to the inhomogeneous system (2.1), provided that the ODE (1.3) has bounded growth.

Comparable results to Proposition 2.2 have not been achieved for DAEs, and hence, this will be our main aim in this section. Together with (1.2), let us consider the following system

$$E(t)\dot{x}(t) = A(t)x(t) + g(t)\ . \tag{2.2}$$

The following example shows that Proposition 2.2 could not be directly applied to the DAE (2.2).

Example 2.3. Consider the system (2.2) with $E = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$, $A = \left[\begin{smallmatrix}-1 & 0\\ 0 & e^{-t}\end{smallmatrix}\right]$, $f = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$. Clearly, f is bounded. On the other hand, the homogeneous system clearly has an exponential dichotomy. Nevertheless, the explicit solution $x(t) = \left[\begin{smallmatrix}e^{-t}x_1(0)\\ e^t\end{smallmatrix}\right]$ is unbounded no matter how an initial condition $x(0)$ is chosen.

We define the linear space $C_b^{sys}(\mathbb{R}_+)$ associated with the system (1.2) as follows.

$$C_b^{sys}(\mathbb{R}_+) := \left\{ g \in C(\mathbb{R}_+) \;\middle|\; \sup_{t \geqslant 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} U(t)g(t) \right\| \right\} < +\infty \right\}, \tag{2.3}$$

Lemma 2.4. *The space $C_b^{sys}(\mathbb{R}_+)$ is invariant with respect to system orthogonal transformations.*

Proof. Let us consider two orthogonally equivalent systems

$$\begin{aligned} E(t)\dot{x}(t) &= A(t)x(t) + g(t), \\ \tilde{E}(t)\dot{\tilde{x}}(t) &= \tilde{A}(t)\tilde{x}(t) + \tilde{g}(t), \end{aligned}$$

where $(E, A, g) \overset{o}{\sim} (\tilde{E}, \tilde{A}, \tilde{g})$. Without loss of generality, in analogous to Lemma 1.6, we may assume that

$$(E, A) = \left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \text{ and } (\tilde{E}, \tilde{A}) = \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

Under this assumption, $U(t) = \tilde{U}(t) = I_n$ for all $t \geqslant 0$. As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued functions S_i, T_i $(i = 1, 4)$. Thus, we see that

$$\begin{aligned} \begin{bmatrix} \tilde{\Sigma}^{-1} & -\tilde{\Sigma}^{-1}\tilde{A}_2\tilde{A}_4^{-1} \\ 0 & \tilde{A}_4^{-1} \end{bmatrix} \tilde{g} &= \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} \begin{bmatrix} S_1^T & 0 \\ 0 & S_4^T \end{bmatrix} Sg \\ &= \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & -\Sigma^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} g. \end{aligned}$$

Since Euclidean norm is preserved under orthogonal transformation, this identity completes the proof. $\square$

The main result of this section is to prove a characterization of the exponential dichotomy for DAEs. Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping $\mathcal{L} := E\dfrac{d}{dt} - A$ is surjective on the space $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$. We formulate our main result in this section as follows.

Theorem 2.5. *Consider the linear, strangeness-free DAE* (1.2) *and the associated inhomogeneous DAE* (2.2)*. Then the following assertions hold.*

(i) If the DAE (1.2) *admits an exponential dichotomy then for any function $g \in C_b^{sys}(\mathbb{R}_+)$, there exists a continuous, bounded solution $x(t)$ to the DAE* (2.2)*.*

(ii) If the matrix function $\hat{A}_3(t)$ is bounded, then the converse of assertion (i) holds true.

Proof. Firstly, we notice that, since $\hat{g}(t) = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t)g(t)$, so the fact $g \in C_b^{sys}(\mathbb{R}_+)$ implies the boundedness of $\hat{g}$. Recall that the semi-explicit system (1.8) reads

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{g}_1(t), \tag{2.4a}$$
$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{g}_2(t). \tag{2.4b}$$

(i) Assuming that the DAE (1.2) admits an exponential dichotomy, then (1.8) also has an exponential dichotomy. Proposition 2.1 implies that equation (2.4a) has an exponential dichotomy, and the function $\hat{A}_3$ is bounded. Therefore, Proposition 2.2 implies that y_1 is bounded, and consequently, y_2 is also bounded.

(ii) Notice that the mapping

$$\begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) \;: \mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n) \to C_b(\mathbb{R}_+, \mathbb{R}^n)$$

is surjective, so $g_1(t)$ can be freely chosen in the space $C_b(\mathbb{R}_+, \mathbb{R}^n)$. Proposition 2.2 applied to system (2.4a) follows that (2.4a) has exponential dichotomy. On the other hand, the boundedness of $\hat{A}_3$ implies that (1.2) admits exponential dichotomy. ☐

3. Local Stable manifolds for semi-linear DAEs

In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout this section we assume that the evolution family $(X(t,s))_{t \geqslant s \geqslant 0}$ associated with the linear, homogeneous DAE (1.2) admits an exponential dichotomy on $\mathbb{R}_+$.

From Lemma 1.4, by using orthogonal transformation $x(t) = V(t)y(t)$, where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$ we can transform (1.1) to the coupled system

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), \tag{3.1}$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), \tag{3.2}$$

where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) \begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}. \tag{3.3}$$

Notice that, unlike the DAEs (1.2) and (2.2), equation (3.2) only gives an implicit algebraic constraint in terms of y_1 and y_2. In order to guarantee the strangeness-free of system (1.1), we need the following assumption.

Assumption 3.1. Assume that for some $\rho > 0$, the function $A_4^{-1}(t)f_2(t,x)$ is a contraction mapping in the ball $B_\rho := \{x \in \mathbb{R}^n | \ \|x\| \leqslant \rho\}$ (uniformly in time), i.e.,

$$\|A_4^{-1}(t)\left(f_2(t,x) - f_2(t,\tilde{x})\right)\| \leqslant L\|x - \tilde{x}\|,$$

for a.e. $t \in \mathbb{R}_+$, and for all $x, \tilde{x} \in B_\rho$ where the Lipschitz constant L satisfies that $L < 1$.

Lemma 3.2. *Under Assumption 3.1 and given $y_1 \in B_\rho$, there exists a unique function $y_2 \in \mathcal{B}_\rho$ satisfying* (3.2).

Proof. Firstly, notice that Assumption 3.1 implies that $\hat{f}_2(t,y)$ is also Lipschitz in y with the same constant L. Then, the desired claim is obtained directly by making use of [19, Lem. 2.7]. ☐

Remark 3.3. Lemma 3.2 leads to one important fact, that under Assumption 3.1, the coupled system (3.1)-(3.2) is still strangeness-free, as defined in [13, Chap. 4]. Therefore, in analogue to the linear case, (3.2) is called *an algebraic constraint*, whereas (3.1) is called *an underlying ODE*.

To obtain the existence of a stable manifold we need the following property of the nonlinear part f_1 defined as follows.

Definition 3.4. Let φ be a positive function belonging to an admissible Banach function space E. A function $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to the class (M, φ, ρ) for some positive constant M, ρ if h satisfies

(i) $\|h(t,x)\| \leqslant M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $x \in B_\rho$,
(ii) $\|h(t,x) - h(t,\tilde{x})\| \leqslant \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$, for all $x, \tilde{x} \in B_\rho$.

Assumption 3.5. Assume that the function $t \mapsto \Sigma^{-1}(t)\begin{bmatrix} I_d & -A_2(t)A_4^{-1}(t) \end{bmatrix} f(t, x(t))$ belongs to class (M, φ, ρ) for some positive constants M, ρ and a positive function $\varphi \in E$.

The following lemma shows that Assumptions 3.1, 3.5 are invariant with respect to system orthogonal transformations.

Lemma 3.6. *Assumptions 3.1, 3.5 are also invariant with respect to system orthogonal transformations.*

Proof. Let us consider two orthogonally equivalent systems

$$\left(\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\right) \overset{o}{\sim} \left(\begin{bmatrix}\tilde{\Sigma} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\tilde{A}_1 & \tilde{A}_2\\ \tilde{A}_3 & \tilde{A}_4\end{bmatrix}\right).$$

As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued function S_i, T_i $(i = 1, 4)$. Therefore, we have that

$$\|\tilde{A}_4^{-1}(t)\left(\tilde{f}_2(t,x) - \tilde{f}_2(t,\tilde{x})\right)\| = \|T_4^{-1}(t)A_4^{-1}(t)S_4^{-1}(t)S_4(t)\left(f_2(t,x) - f_2(t,\tilde{x})\right)\| \leqslant L\|x - \tilde{x}\|,$$

Then, due to the orthogonality of T_4, the desired claim is directly followed. □

The following proposition gives one sufficient condition for examining Assumptions 3.1, 3.5.

Proposition 3.7. Consider the semi-linear DAE (1.1). Furthermore, assume that all three functions Σ^{-1}, A_4^{-1}, $\Sigma^{-1}A_2A_4^{-1}$ are bounded. If the function $f = \begin{bmatrix} f_1 \\ f_2\end{bmatrix}$ belongs to the class (M, φ, ρ) then the following claims hold true.
i) $\hat{f}_1$ belongs to the class (M, φ, ρ), and
ii) f_2 is Lipschitz with the Lipschitz constant $\varphi \sup\limits_{t\geqslant 0} \|A_4^{-1}\|$.

We notice that a sufficient condition for Assumption (3.1) is that

Lipschitz

$$\|f_2(t,x) - f_2(t,\tilde{x})\| \leqslant \frac{L}{\|A_4^{-1}(t)\|}\|x - \tilde{x}\|\ . \tag{3.4}$$

For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.1)-(3.2). Moreover, we consider the mild/integral-algebraic system which reads

mild equation

$$\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix} = \hat{Y}(t,s)\begin{bmatrix} y_1(s)\\ y_2(s)\end{bmatrix} + \int_s^t \hat{Y}(t,\tau)\begin{bmatrix}\hat{f}_1(\tau, y(\tau))\\ 0\end{bmatrix} d\tau + \begin{bmatrix} 0\\ \hat{f}_2(t, y(t))\end{bmatrix}, \tag{3.5}$$

for all $t \geqslant s \geqslant 0$.

Lemma 3.8. *Let Assumptions 3.1 and 3.5 hold true. Then, for all $y, \tilde{y} \in B_\rho$ the following assertions hold.*
(i) $\|\hat{f}_1(t,y)\| \leqslant M\varphi(t)$ *for a.e.* $t \in \mathbb{R}_+$,
(ii) $\|\hat{f}_1(t,y) - \hat{f}_1(t,\tilde{y})\| \leqslant \varphi(t)\|y - \tilde{y}\|$ *for a.e.* $t \in \mathbb{R}_+$,
(iii) $\|\hat{f}_2(t,y) - \hat{f}_2(t,\tilde{y})\| \leqslant L\|y - \tilde{y}\|$ *for a.e.* $t \in \mathbb{R}_+$.

Proof. The proof is trivially followed from Assumptions 3.1 and 3.5 due to the fact that $\|y\| = \|Qy\|$ for any orthogonal matrix V. □

Let $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N, \nu > 0$ as in Definition 1.8. Furthermore, as in Proposition 2.1, let us denote by $H_1 := \sup\limits_{t\geqslant 0}\|\hat{A}_3(t)\|$ and $H_2 := \sup\limits_{t\geqslant 0}\|P_{\mathrm{y}}(t)\|$. Then, we can define the Green function on the half-line as follows

4.2

$$G(t,\tau) := \begin{cases} \hat{Y}(t,\tau)\begin{bmatrix} P_{\mathrm{y}}(\tau) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\hat{Y}_1(t,\tau)P_{\mathrm{y}}(\tau) & 0\\ \hat{A}_3(t)\hat{Y}_1(t,\tau)P_{\mathrm{y}}(\tau) & 0\end{bmatrix}, & \text{for all } t \geqslant \tau \geqslant 0,\\ -\hat{Y}(t,\tau)\begin{bmatrix} I_d - P_{\mathrm{y}}(\tau) & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau)) & 0\\ \hat{A}_3(\tau)\hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau)) & 0\end{bmatrix}, & \text{for all } 0 \leqslant t < \tau.\end{cases} \tag{3.6}$$

Then, we have

4.3

$$\|G(t,\tau)\| \leqslant (1 + H_1)(1 + H_2)\ Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geqslant 0. \tag{3.7}$$

In the following lemma, we give an explicit form for bounded solutions to system (3.5).

Lemma 3.9. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system (1.8) have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumptions 3.1, 3.5 hold true. Let $y(t)$ be any solution to (3.5) such that $\operatorname{ess\,sup}_{t\geqslant t_0}\|y(t)\|\leqslant\rho$ for fixed $t_0\geqslant 0$ and some $\rho>0$. Then, for $t\geqslant t_0\geqslant 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix} = Y(t,t_0)\begin{bmatrix} v_0\\ 0\end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix} d\tau + \begin{bmatrix} 0\\ \hat{f}_2(t,y(t))\end{bmatrix}, \tag{3.8}$$

for some $v_0\in \mathrm{Im}P_{\mathrm{y}}(t_0)$, where $G(t,\tau)$ is the Green function defined by (3.6).

Proof. Put

$$z(t)=\begin{bmatrix} z_1(t)\\ z_2(t)\end{bmatrix} := \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix} d\tau + \begin{bmatrix} 0\\ \hat{f}_2(t,y(t))\end{bmatrix}.$$

By direct computation, we can verify that z satisfies the integral equation

$$z(t)=\hat{Y}(t,t_0)\begin{bmatrix} z_1(t_0)\\ z_2(t_0)\end{bmatrix} + \int_{t_0}^{t}\hat{Y}(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix} d\tau \ + \begin{bmatrix} 0\\ \hat{f}_2(t,y(t))\end{bmatrix},$$

for all $t\geqslant t_0$. Now let us estimate $\|z(t)\|$. Making use of Lemma 3.8 and (3.7), we see that

$$\|z(t)\|\leqslant \int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\ M\varphi(\tau)\ d\tau \ + L\rho,$$

and then, from (1.22) it follows that

$$\|z(t)\|\leqslant M\ (1+H_1)(1+H_2)\ \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\ + L\rho\ ,$$

for all $t\geqslant t_0$. Thus, $z(t)-y(t)$ is also bounded. Moreover, since

$$z(t)-y(t)=\hat{Y}(t,t_0)\,(z(t_0)-y(t_0)) = \begin{bmatrix}\hat{Y}_1(t,t_0)\ (z_1(t_0)-y_1(t_0))\\ \hat{A}_3(t)\hat{Y}_1(t,t_0)\ (z_1(t_0)-y_1(t_0))\end{bmatrix},$$

we see that $v_0:=z_1(t_0)-y_1(t_0)\in\mathrm{Im}P_{\mathrm{y}}(t_0)$. Finally, since $z(t)=y(t)+\hat{Y}(t,t_0)\begin{bmatrix} v_0\\ 0\end{bmatrix}$ for all $t\geqslant t_0$, equality (3.8) follows. □

Remark 3.10. By computing directly, we can see that the converse of Lemma 3.9 is also true. It means, that all solutions to (3.8) also satisfy equation (3.5) for all $t\geqslant t_0$.

Let us denote by

$$H_3 := (1+H_1)(1+H_2)\ \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\ \text{ and }\ \tilde{\rho} := \frac{1-L}{2N(1+H_1)}\rho\ . \tag{3.9}$$

Lemma 3.11. *Under the assumptions of Lemma 3.9, let $y(t)$, $\tilde{y}(t)$ be any two functions lying in the ball B_ρ and satisfy (3.8) for v_0, $\tilde{v}_0\in\mathrm{Im}P_{\mathrm{y}}(t_0)$. If H_3 defined as in (3.9) satisfies $H_3+L<1$ then the following estimate holds true:*

$$\|y-\tilde{y}\|_\infty\leqslant \frac{N}{1-H_3-L}\ \|v_0-\tilde{v}_0\|. \tag{3.10}$$

Proof. Using the same arguments as in the proof of Lemma 3.8, we see that

$$\begin{aligned}\|y(t)-\tilde{y}(t)\| &\leqslant N\|v_0-\tilde{v}_0\| + \int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\ \varphi(\tau)\|y(\tau)-\tilde{y}(\tau)\|\ d\tau\ + L\|y(t)-\tilde{y}(t)\|,\\ &\leqslant N\|v_0-\tilde{v}_0\| + (1+H_1)(1+H_2)\ \frac{N}{1-e^{-\nu}}\left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\ \|y-\tilde{y}\|_\infty + L\|y(t)-\tilde{y}(t)\|,\\ &\leqslant N\|v_0-\tilde{v}_0\| + (H_3+L)\ \|y-\tilde{y}\|_\infty,\end{aligned}$$

which directly implies (3.10). □

In the following theorem, we exploit the local structure of bounded solutions to (3.5).

Theorem 3.12. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system (1.8) have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumptions 3.1, 3.5 hold true, and constant H_3 defined as in (3.9). Then, the following assertions hold true.*

(i) If

$$H_3 < \min\left\{1-L,\ \frac{(1-L)\rho}{2M}\right\}, \tag{3.11}$$

then there corresponds to each $v_0 \in B_{\tilde{\rho}} \cap \mathrm{Im}P_{\mathrm{y}}(t_0)$ one and only one solution $y(t)$ to (3.5) on $[t_0,\infty)$ satisfying $P_{\mathrm{y}}(t_0)y_1(t_0)=v_0$ and $\operatorname{ess\,sup}_{t\geqslant t_0}\|y(t)\|\leqslant\rho$.

(ii) Moreover, any two solutions $y(t)$, $\tilde{y}(t)$ corresponding to different v_0, $\tilde{v}_0$ in $B_{\tilde{\rho}} \cap \mathrm{Im}P_{\mathrm{y}}(t_0)$ attract each other exponentially, i.e.,

$$\|y(t)-\tilde{y}(t)\| \leqslant H_4\ e^{-\mu(t-t_0)}\ \|v_0-\tilde{v}_0\| \quad \text{for all } t\geqslant t_0, \tag{3.12}$$

for some positive constants H_4, μ.

Proof. (i) Consider in the space $L_\infty(\mathbb{R}_+,\mathbb{R}^n)$ the ball $\mathcal{B}_\rho := \{y\in L_\infty(\mathbb{R}_+,\mathbb{R}^n):\ \|y(\cdot)\|_\infty := \underset{t\geqslant 0}{ess\sup}\|y(t)\|\leqslant\rho\}$. For each fixed $v_0\in B_{\tilde{\rho}}$ we will prove the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t,t_0)\begin{bmatrix} v_0\\ 0\end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix} d\tau + \begin{bmatrix} 0\\ \hat{f}_2(t,y(t))\end{bmatrix} & \text{for all } t\geqslant t_0,\\ 0 & \text{for all } t<t_0,\end{cases} \tag{3.13}$$

is a contraction mapping from $\mathcal{B}_\rho$ to itself. Using the same argument as in the proof of Lemma 3.8, we see that

$$\begin{aligned}\|(Ty)(t)\| &\leqslant (1+H_1)Ne^{-\nu(t-t_0)}\|v_0\|\ + M\ (1+H_1)(1+H_2)\ \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\ + L\rho,\\ &\leqslant (1+H_1)N\|v_0\|\ + MH_3 + L\rho \text{ for all } t\geqslant 0,\end{aligned}$$

and by (3.11) we see that

$$\|(Ty)(t)\| \leqslant (1+H_1)N\tilde{\rho}\ + \frac{(1-L)\rho}{2} + L\rho = \rho \text{ for all } t\geqslant 0.$$

Therefore, T is a mapping from $\mathcal{B}_\rho$ to itself. Now we prove its contraction property. Indeed, making use of (3.7), we obtain the following estimate:

$$\begin{aligned}\|Ty(t)-T\tilde{y}(t)\| &\leqslant \int_{t_0}^{\infty}\|G(t,\tau)\|\,\|\hat{f}_1(\tau,y(\tau))-\hat{f}_1(\tau,\tilde{y}(\tau))\|d\tau\ +\|\hat{f}_2(t,y(t))-\hat{f}_2(t,\tilde{y}(t))\|,\\ &\leqslant \int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau\ +L\ \|y(t)-\tilde{y}(t)\|,\\ &\leqslant (H_3+L)\ \|y(\cdot)-\tilde{y}(\cdot)\|_\infty \text{ for all } t\geqslant 0.\end{aligned}$$

Consequently, due to (3.11), we see that T is a contraction mapping with the contraction constant H_3+L. Thus, there exist a unique function $y\in\mathcal{B}_\rho$ such that $y=Ty$, and hence, due to the definition of T, y is the solution to the mild/integral-algebraic system (3.5).

(ii) The proof of the estimate (3.12) can be done in a similar way as in [11, Thm 3.7]. We present here for seek of completeness. Let $y(t)$ and $\tilde{y}(t)$ be two essentially bounded solutions of (3.5) corresponding to different values $v_0,\,\tilde{v}_0\in B_{\tilde{\rho}}\cap \mathrm{Im}P_{\mathrm{y}}(t_0)$. Then, we have that

$$\begin{aligned}\|y(t)-\tilde{y}(t)\| &\leqslant Y(t,t_0)\|v_0-\tilde{v}_0\|+\int_{t_0}^{\infty}\|G(t,\tau)\|\,\|\hat{f}_1(\tau,y(\tau))-\hat{f}_1(\tau,\tilde{y}(\tau))\|d\tau\ +\|\hat{f}_2(t,y(t))-\hat{f}_2(t,\tilde{y}(t))\|,\\ &\leqslant (1+H_1)Ne^{-\nu(t-t_0)}+\int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau\ +L\ \|y(t)-\tilde{y}(t)\|,\end{aligned}$$

and hence,

$$\|y(t)-\tilde{y}(t)\|\leqslant \frac{1+H_1}{1-L}Ne^{-\nu(t-t_0)}\ +\int_{t_0}^{\infty}\frac{(1+H_1)(1+H_2)}{1-L}\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau\quad .$$

Then, due to the Cone Inequality, [6, Theorem 1.9.3], in analogue to [20, Theorem 3.7], we obtain the estimation (3.12) with H_4, μ are given by

$$0<\mu<\nu+\ln\left(1-\frac{H_3(1-e^{-\nu})}{1-L}\right),\qquad H_4:=\frac{(1+H_1)N}{1-L-\dfrac{H_3(1-e^{-\nu})}{1-e^{\mu-\nu}}}\ .$$

Furthermore, notice that from (3.11) it follows that $\mu<\nu$ implying the positivity of H_4. This completes the proof. ☐

Under Assumption 3.1, we then define the so-called *constrained manifold*, which all solutions to (3.1)-(3.2) must belong to

contraint manifold

$$\mathbb{L}(t,y):=\{(t,y_1,y_2)\in\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^a\mid y_2=\hat{A}_3(t)y_1+\hat{f}_2(t,y_1,y_2)\}\ . \tag{3.14}$$

We further notice that this manifold is of dimension d, which is the degree of freedom to the DAE (3.5). Now, we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic system (3.5).

Definition 3.13. A subset $\mathbb{M}$ of the constrained manifold $\mathbb{L}(t,y)$ is said to be a *local stable manifold* for solutions to (3.5) if for every $t\in\mathbb{R}_+$ the phase subspace $\mathbb{R}^d$ splits into a direct sum $\mathbb{R}^d=W_1(t)\oplus W_2(t)$ such that

$$\inf_{t\in\mathbb{R}_+}Sn(W_1(t),W_2(t)):=\inf_{t\in\mathbb{R}_+}\ \inf\{\|w_1+w_2\|,\ w_i\in W_i(t),\ \|w_i\|=1,\ i=0,1\}>0,$$

and if there exist positive constants ρ, ρ_1, ρ_2 and a family of Lipschitz continuous mappings

$$g_t: B_{\rho_1}\cap W_1(t)\to B_{\rho_2}\cap W_2(t),\quad t\in\mathbb{R}_+,$$

with a common Lipschitz constant independent of t such that

(i) $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$, and we denote by $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$,

(ii) $\mathbb{M}_t$ is homeomorphic to $B_{\rho_1} \cap W_1(t)$ for all $t \geqslant 0$,

(iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and $ess\sup_{t\geqslant t_0} \|y(t)\| \leqslant \rho$.

We now state and prove our main result on the existence of a local stable manifold for DAEs.

Theorem 3.14. *Let the evolution family* $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ *of system* (1.8) *have an exponential dichotomy with the corresponding projection matrices* $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ *and the dichotomy constants* $N, \nu > 0$. *Furthermore, assume that Assumptions 3.1, 3.5 hold true. If*

$$H_3 < \min\left\{1 - L,\ \frac{(1-L)(1+H_1)\rho}{2M},\ \frac{(1-L)(1+H_1)(1+H_2)}{N + (1+H_1)(1+H_2)}\right\},$$

then there exists a local stable manifold for the solutions of (3.5). *Moreover, every two solutions* $y(t)$, $\tilde{y}(t)$ *on the manifold* $\mathbb{M}$ *attract each other exponentially in the sense that there exist positive constants* H_4 *and* μ *independent of* $t_0 \geqslant 0$ *such that*

$$\|y(t) - \tilde{y}(t)\| \leqslant H_4\, e^{-\mu(t-t_0)}\, \|P(t_0)y_1(t_0) - P(t_0)y_2(t_0)\|, \quad \textit{for all } t \geqslant t_0\ . \tag{3.15}$$

Proof. First we notice that the phase subspace $\mathbb{R}^d$ splits into the direct sum $\mathbb{R}^d = \mathrm{Im}P_{\mathrm{y}}(t) \oplus \mathrm{kernel}\, P_{\mathrm{y}}(t)$ for all $t \geqslant 0$. We set $W_1(t) := \mathrm{Im}P_{\mathrm{y}}(t)$ and $W_2(t) := \mathrm{kernel}\, P_{\mathrm{y}}(t)$, then due to Proposition 2.1, we see that $\sup\limits_{t\geqslant 0} \|P_{\mathrm{y}}(t)\| < \infty$, and hence, $\inf_{t\in\mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$.

For any $\rho > 0$ defined as in Assumptions 3.1, 3.5, let $\rho_1 := \tilde{\rho} = \dfrac{1-L}{2N(1+H_1)}\rho$ and $\rho_2 := \dfrac{(1-L)\rho}{2}$. For each $t \geqslant 0$ we define the mapping g_t acting on $B_{\rho_1} \cap W_1(t)$ as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is uniquely defined via Theorem 3.12 i). Clearly, $g_t(w_1) \in \ker P_{\mathrm{y}}(t) = W_2(t)$.

Now, we prove that $\|g_t(w_1)\| \leqslant \rho_2$. Due to Theorem 3.12 (i) and Lemma 3.8 (i), we have that $\|y(t)\| \leqslant \rho$ and $\|f_1(\tau, y(\tau))\| \leqslant M\varphi(\tau)$ for a.e. $t \geqslant 0$. Therefore,

$$\begin{aligned}
\|g_t(w_1)\| &\leqslant \int_t^\infty N\, e^{-\nu(\tau-t)}\, \|f_1(\tau, y(\tau))\| d\tau \leqslant \int_t^\infty N\, e^{-\nu(\tau-t)}\, M\varphi(\tau)d\tau, \\
&\leqslant M\,(1+H_2)\, \frac{N}{1-e^{-\nu}}\, \left(\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right) = \frac{MH_3}{1+H_1} \leqslant \frac{(1-L)\rho}{2}\ ,
\end{aligned}$$

and hence, $g_t : B_{\rho_1} \cap W_1(t) \to B_{\rho_2} \cap W_2(t)$.

Notice that both part (iii) in Definition 3.13 and estimation (3.15) are followed directly from Theorem 3.12. We now only need to prove that $\mathbb{M}_t$ is homeomorphic to $B_{\rho_1} \cap W_1(t)$. We first prove that g_t is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned}
\|g_t(w_1) - g_t(\tilde{w}_1)\| &\leqslant \int_t^\infty N\, e^{-\nu(\tau-t)}\, \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leqslant \int_t^\infty N\, e^{-\nu(\tau-t)}\, \varphi(\tau)\, \|y(\tau) - \tilde{y}(\tau)\| d\tau\ , \\
&\leqslant \frac{N}{1-e^{-\nu}}\, \left(\|\Lambda_1 T_1^+\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\, \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)}\, \|y - \tilde{y}\|_\infty\ ,
\end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leqslant \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

Finally, $H_3 < \dfrac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\dfrac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a contraction mapping for all $t \geqslant 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous mappings ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \to B_{\rho_1} \cap W_1(t)$ is a homeomorphism. This implies the condition (ii) of Definition 3.13 finishing the proof. □

4. Global Invariant stable manifolds for semi-linear DAEs

In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1). We begin with the concept of φ-Lipschitz functions.

Definition 4.1. Let E be an admissible Banach function space and $\varphi \in E$ be a positive function. A function $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d$ is said to be *φ-Lipschitz* if the following conditions hold true.

(i) $\|h(t,0)\| = 0$ for a.e. $t \in \mathbb{R}_+$,

(ii) $\|h(t,x) - h(t,\tilde{x})\| \leqslant \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$ and all $x, \tilde{x} \in \mathbb{R}^n$.

In comparability to Assumptions 3.1, 3.5, we also need some global properties of the nonlinear term f.

Assumption 4.2. Assume that the following hypotheses hold true.

(i) The function $\Sigma^{-1}(t)\ f_1(t,x(t)) - \Sigma^{-1}(t)A_2(t)A_4^{-1}(t)\ f_2(t,x(t))$ is φ-Lipschitz.

(ii) The function $A_4^{-1}(t)\ f_2(t,x(t))$ is a contraction mapping with the Lipschitz constant $L < 1$ for all $(t,x(t))$ lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t,x) := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid A_2(t)x + f_2(t,x) = 0\}.$$

We can directly verify that orthogonal transformations of the form $x = Vy$ preserves the φ-Lipschitz property, and hence, function $\hat{f}_1$ in (3.1) is also φ-Lipschitz. Besides that, function $\hat{f}_2$ in (3.2) is also a contraction mapping with the Lipschitz constant $L < 1$. For notational simplicity, now we will study the transformed system (1.8) and the integral-algebraic system (3.5).

Definition 4.3. A subset $\mathbb{M}$ of the constrained manifold $\mathbb{L}(t,y)$ is said to be a *global, invariant stable manifold* for solutions to (3.5) if for every $t \in \mathbb{R}_+$ the phase subspace $\mathbb{R}^d$ splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t\in\mathbb{R}_+} Sn(W_1(t),W_2(t)) := \inf_{t\in\mathbb{R}_+} \inf\{\|w_1+w_2\|,\ w_i \in W_i(t),\ \|w_i\| = 1,\ i = 0,1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \to W_2(t), \quad t \in \mathbb{R}_+,$$

with the Lipschitz constants independent of t such that

(i) $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$, and we denote by $\mathbb{M}_t := \{(y_1,y_2) \mid (t,y_1,y_2) \in \mathbb{M}\}$,

(ii) $\mathbb{M}_t$ is homeomorphic to $W_1(t)$ for all $t \geqslant 0$,

(iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and $ess\sup_{t\geqslant t_0}\|y(t)\| < \infty$,

(iv) $\mathbb{M}$ is invariant under system (3.5), i.e., if y is a solution to (3.5), and $ess\sup_{t\geqslant t_0}\|y(t)\| < \infty$, then $y(s) \in \mathbb{M}_s$ for all $s \geqslant t_0$.

Analogously to Lemma 3.9, we give the explicit form of bounded solutions to system (3.5) as below.

Lemma 4.4. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system* (1.8) *have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumption 4.2 holds true. Let $y(t)$ be any solution to* (3.5) *such that* $\operatorname{ess\,sup}_{t\geqslant t_0}\|y(t)\|<\infty$ *for a fixed $t_0\geqslant 0$. Then, for all $t\geqslant t_0\geqslant 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t)\\ y_2(t)\end{bmatrix} = Y(t,t_0)\begin{bmatrix} v_0\\ 0\end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix} d\tau + \begin{bmatrix} 0\\ \hat{f}_2(t,y(t))\end{bmatrix}, \tag{4.1}$$

for some $v_0\in \mathrm{Im}P_{\mathrm{y}}(t_0)$, where $G(t,\tau)$ is the Green function defined by (3.6).

Proof. The proof can be done by using similar arguments as in the proof of Lemma 3.2. □

In the following two theorems, we present the global versions of Theorems 3.12 and 3.14, where we construct the structure of bounded solutions to (3.5) and prove the existence of a global, stable manifold, respectively.

Theorem 4.5. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system* (1.8) *have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumption 4.2 holds true.*

(i) For any fixed $t_0\geqslant 0$, if $H_3<1-L$ then there corresponds to each $v_0\in\mathrm{Im}P_{\mathrm{y}}(t_0)$ one and only one solution $y(t)$ to (3.5) *on $[t_0,\infty)$ satisfying $P_{\mathrm{y}}(t_0)y_1(t_0)=v_0$ and* $\operatorname{ess\,sup}_{t\geqslant t_0}\|y(t)\|<\infty$.

(ii) Any two solutions $y(t)$, $\tilde{y}(t)$ corresponding to different initial conditions v_0, $\tilde{v}_0$ in $\mathrm{Im}P_{\mathrm{y}}(t_0)$, are exponentially attracted to each other, i.e.,

$$\|y(t)-\tilde{y}(t)\|\leqslant H_4\, e^{-\mu(t-t_0)}\,\|v_0-\tilde{v}_0\| \quad \text{for all } t\geqslant t_0,$$

with some positive constants H_4, μ satisfying

$$0<\mu<\nu+\ln\left(1-\frac{H_3(1-e^{-\nu})}{1-L}\right),\qquad H_4:=\frac{(1+H_1)N}{1-L-\dfrac{H_3(1-e^{-\nu})}{1-e^{\mu-\nu}}}.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 3.12. The only change is, that instead of considering the ball B_ρ we will work with the space $L_\infty(\mathbb{R}_+,\mathbb{R}^n)$ itself. Then, we can prove (without any difficulty) that for each fixed $v_0\in\mathrm{Im}P_{\mathrm{y}}(t_0)$, the transformation T defined by

$$(Ty)(t)=\begin{cases} Y(t,t_0)\begin{bmatrix} v_0\\ 0\end{bmatrix}+\int_{t_0}^{\infty}G(t,\tau)\begin{bmatrix}\hat{f}_1(\tau,y(\tau))\\ 0\end{bmatrix}d\tau+\begin{bmatrix}0\\ \hat{f}_2(t,y(t))\end{bmatrix}, & \text{for all } t\geqslant t_0,\\ 0, & \text{for all } t<t_0,\end{cases}$$

is a contraction mapping, and therefore, all the assertions of the theorem follows. □

Theorem 4.6. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system* (1.8) *have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumption 4.2 holds true. If*

$$H_3<\min\left\{1-L,\ \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}\right\},$$

then there exists a global invariant stable manifold for the solutions of (3.5). *Moreover, every two solutions $y(t)$, $\tilde{y}(t)$ on the manifold $\mathbb{M}$ attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0\geqslant 0$ such that*

$$\|y(t)-\tilde{y}(t)\|\leqslant H_4\, e^{-\mu(t-t_0)}\,\|P(t_0)y_1(t_0)-P(t_0)y_2(t_0)\| \quad \text{for all } t\geqslant t_0.$$

Proof. Analogous to the proof of Theorem 3.14, we consider the decomposition $\mathbb{R}^d = \mathrm{Im}P_{\mathrm{y}}(t) \oplus \mathrm{kernel}\, P_{\mathrm{y}}(t)$ and set $W_1(t) := \mathrm{Im}P_{\mathrm{y}}(t)$ and $W_2(t) := \mathrm{kernel}\, P_{\mathrm{y}}(t)$. Thus, we see that $\inf_{t\in\mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$. Now we define the family of mappings $(g_t)_{t\geqslant 0}$ acting on W_1 as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is bounded and be uniquely defined via Theorem 4.5 i). Clearly, $g_t(w_1) \in \ker P_{\mathrm{y}}(t) = W_2(t)$. To verify the Lipschitz property of g_t, let us consider two arbitrary elements w_1 and $\tilde{w}_1$ in W_1 and let y and $\tilde{y}$ be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned}\|g_t(w_1) - g_t(\tilde{w}_1)\| &\leqslant \int_t^\infty N\ e^{-\nu(\tau-t)}\ \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leqslant \int_t^\infty N\ e^{-\nu(\tau-t)}\ \varphi(\tau)\ \|y(\tau) - \tilde{y}(\tau)\| d\tau\ , \\ &\leqslant \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty\right)\ \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)}\ \|y - \tilde{y}\|_\infty,\end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leqslant \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)}\ \|w_1 - \tilde{w}_1\|.$$

Finally, $H_3 < \dfrac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\dfrac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a contraction mapping for all $t \geqslant 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous mapping ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \to W_1(t)$ is a homeomorphism. This implies the condition ii) of Definition 3.13, and hence, the proof is finished. ☐

Now let us illustrate our results by the following examples.

Example 4.7. The dynamical behavior of a system in fluid mechanics and turbulence modeling is often described by the incompressible Navier-Stokes equation on an open, bounded domain $\Omega \subset \mathbb{R}^k$, $k = 2$ or 3, of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nu\Delta u - \nabla p - (u\cdot\nabla)u + f(t,u,p), \\ \nabla\cdot u &= 0, \\ u\,|_{\partial\Omega} &= 0, \\ u\,|_{t=0} &= u_0,\end{aligned}$$

where $\nu > 0$ is the viscosity, $u = u(t,\xi)$ is the velocity field which is a function of the time t and the position ξ, p is the pressure, f is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [9], one obtains a differential-algebraic system of the following form.

$$\begin{aligned}&M\dot{U} = (K + N(U))\ U - CP + F(t,U,P), \\ &C^T U = 0,\end{aligned}$$

where $U(t)$, $P(t)$ approximate the velocity $u(t,\xi)$ and the pressure $p(t,\xi)$, respectively. Here the leading matrix M is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix $C^T M^{-1}\left(C - \dfrac{\partial F}{\partial P}\right)$ is nonsingular. We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free, so Assumption 1.2 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned} M\dot{U} &= -(K+N(U))\,U - CP + F(t,U,P), \\ 0 &= C^T M^{-1} C\, P - C^T M^{-1} \left(F - (K+N(U))\,U\right) \ . \end{aligned} \tag{4.2}$$

Clearly, we still need to linearize (4.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.8)

$$\begin{aligned} M\dot{U} &= A_1(t)U + A_2(t)P + g_1(t,U,P), \\ 0 &= C^T M^{-1} \left(C - \frac{\partial F}{\partial P}\right) P - C^T M^{-1} \left(\frac{\partial F}{\partial U} - K\right) U + C^T M^{-1} g_2(t,U,P) \ . \end{aligned} \tag{4.3}$$

We further notice that since $C^T M^{-1}\left(C - \frac{\partial F}{\partial P}\right)$ is nonsingular, from the second equation we can uniquely determine P in term of U, and hence, system (4.2) is indeed strangeness-free. Let

$$A_3(t) := -C^T M^{-1}\left(\frac{\partial F}{\partial U} - K\right), \ A_4(t) := C^T M^{-1}\left(C - \frac{\partial F}{\partial P}\right)$$

Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

admits an exponential dichotomy, and g_1 satisfies the φ-Lipschitz condition, and g_2 is a contraction mapping (uniformly in time), then there exists a stable manifold for the solution to (4.2).

Example 4.8. Consider the nonlinear electrical circuit with Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation $i_2 = I_0 \sin(k\phi_2)$, where I_0 and k are positive constants depend on the device itself. Moreover, the resistance R, inductance L and conductance G are positive. Furthermore, i_1 is the current going through the inductance, v_1 and v_2 are voltage drops across the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead

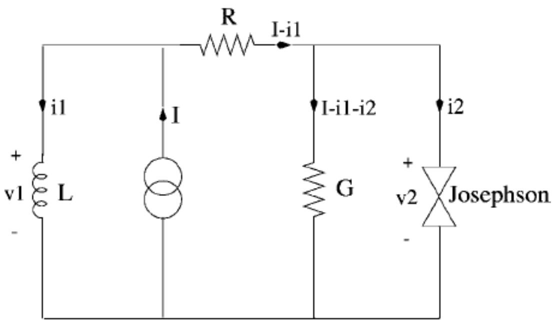

FIGURE 1. Electric circuit with Josephson junction, [23]

of linear resistance, inductance and conductance as in [23], and hence, we see that for the inductance $i_1 = i_L(L,\phi_1)$, for the resistance $v_R = v_R(R,i_1)$, and for the conductance $i_G = i_G(G,v_2)$. Therefore, we obtain the following system, which completely describes the behavior of this circuit.

$$\dot{\phi}_1 = v_1, \tag{4.4a}$$
$$\dot{\phi}_2 = v_2, \tag{4.4b}$$
$$i_1 = i_L(L, \phi_1), \tag{4.4c}$$
$$i_2 = I_0 \sin(k\phi_2), \tag{4.4d}$$
$$0 = v_1 - v_R(R, i_1) + v_2, \tag{4.4e}$$
$$0 = -i_G(G, v_2) + I - i_1 - i_2. \tag{4.4f}$$

From (4.4c)-(4.4f) we obtain an explicit form of v_1 in terms of ϕ_1, i_1 and v_2, so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \tag{4.5a}$$
$$\dot{\phi}_2 = v_2, \tag{4.5b}$$
$$i_1 = i_L(L, \phi_1), \tag{4.5c}$$
$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \tag{4.5d}$$

The linearized version of this system along equilibrium points defined by $v_2 = 0$, $i_1 = I$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, reads

$$\begin{aligned}
\dot{\phi}_1 &= RI - (R/L)\phi_1 + v_2, \\
\dot{\phi}_2 &= v_2, \\
i_1 &= \phi_1/L, \\
0 &= -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2),
\end{aligned}$$

will have a positive eigenvalue and a negative one (e.g. [23]). Hence, it admits exponential dichotomy for any odd number n. Thus, for φ-Lipschitz function v_R and contraction mapping i_G, we obtain a stable manifold for (4.5).

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