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ON THE REACHABILITY AND CONTROLLABILITY OF POSITIVE LINEAR TIME-INVARIANT DYNAMIC SYSTEMS WITH INTERNAL AND EXTERNAL INCOMMENSURATE POINT DELAYS

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ABSTRACT. This paper is devoted to the characterization of reachability and controllability properties of singular linear time-delay, time-invariant dynamic positive systems subject to constant known internal and external point delays. The research is performed on two different levels, namely, based on a general nonunique solution trajectory including discontinuities at points of the initialization time-interval and based on a particular solution with initial conditions in an appropriate subspace of the whole potential admissible set which avoids such discontinuities and guarantees uniqueness. In this second case, the positivity, reachability and controllability characterizations of the Weierstrass canonical state-space realization are also investigated.

I. Introduction. This paper investigates the general state-trajectory solution and the reachability and controllability properties of singular linear time-delay, time-invariant dynamic positive systems subject to a finite number of constant known internal and external point delays. The general problem statement is concerned with the possible presence of impulses at zero of the initial conditions in the regular impulsive case which results in the loss of uniqueness of the state-trajectory solution, [**11**]. A certain subset of initial conditions of the fast dynamics partial state guarantees the state-trajectory solution in both the natural state variables of the given problem and its counterpart in the Weierstrass canonical form. The state-trajectory of the regular (referred to as well as solvable [**1, 13, 15, 19**]) state impulse-free case where the nilpotent matrix of the general singular system becomes zero is also obtained

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as a particular case, [**1, 11, 16, 19**]. The weak internal and external positivity of the given general singular system are characterized as well as the internal and external positivity of the impulse-free situation. The reachability and controllability properties of the positive singular system, [**5, 10, 14**], with internal and external point delays are both formulated and characterized analytically, [**2, 5, 8, 10, 14, 17, 18**]. The paper is organized as follows. Section II is devoted to the general problem statement of the singular system and to obtaining explicit formulas for state-space solution trajectories for the general and impulse-free cases. Section III is devoted to the weak and standard positivity characterizations of the singular system and its reachability and controllability properties. Two illustrative examples are discussed in Section IV and, finally, conclusions end the paper.

I.1. Notation. I_n is the nth identity matrix; $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ are the sets of real numbers and the fields of the real and complex numbers, respectively; $\mathbf{R}_+^n := \{z = (z_1, z_2, \dots, z_n)^T \in \mathbf{R}^n : z_i \geq 0, \text{ for all } i \in \overline{n}\}$ is the first orthant of $\mathbf{R}^n$; $\mathbf{Z}_+ = \{z \in \mathbf{Z}_+ : z \geq 0\}$; $\mathbf{C}_+ := \{z \in \mathbf{C} : \operatorname{Re} z \geq 0\}$; $\mathbf{C}^{(l)}((-h, 0), \mathbf{R}^n)$ is the set of real n-vector continuously differentiable l times; $\mathrm{PC}\,(\mathbf{R}_+, \mathbf{R}^n)$ is the set of real n-vector piecewise continuous functions; $\delta(t)$ is the Dirac distribution such that $f(t)\delta(t - t_1) = f(t_1)$ for any real well-posed testing function whose definition domain contains t_1; and $\delta(i, j)$ is the Kronecker delta of value unity for all integers i, j with $i = j$ and zero for all integers i, j with $i \neq j$.

A real matrix M is said to be nonnegative $M = (M_{ij}) \in \mathbf{R}_+^{n \times m}$ if $M_{ij} \geq 0$ for all $i \in \overline{n}$, for all $j \in \overline{m}$. This may be abbreviated as $M \geq 0$. A nonnegative matrix M is said to be positive (abbreviated as $M > 0$ if $M_{ij} > 0$ for at least one $(i, j) \in \overline{n} \times \overline{m}$ and strictly positive ($M \gg 0$) if $M_{ij} > 0$ for all $(i, j) \in \overline{n} \times \overline{m}$. In the same way, $v = (v_1, v_2, \dots, v_n)^T \in \mathbf{R}_+^n$, i.e., is nonnegative, if $v_i \geq 0$ which is abbreviated as $v \geq 0$. $v \in \mathbf{R}_+^n$ is positive ($v > 0$) if $v_i > 0$ for some $i \in \overline{n}$ and strictly positive ($v \gg 0$) if $v_i > 0$ for all $i \in \overline{n}$. A matrix $M = (M_{ij}) \in \mathbf{R}^{n \times m}$ is said to be a Metzler matrix $M = (M_{ij}) \in \mathbf{M}_E^{n \times m}$ if $M_{ij} \geq 0$ for all $(i, j \neq i) \in \overline{n} \times \overline{m}$, i.e., if all its off-diagonal entries are nonnegative.

II. Problem statement and the state-space trajectory. Consider the singular linear time-delay system with time lags:

$$E\dot{x}(t) = \sum_{j=0}^{q} A_j(x)(t-h_j) + \sum_{j=0}^{q} B_j u(t-h_j)U(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$ are the state, input and output vector functions in the corresponding spaces, E (being rank defective), A , B and C are complex matrices of compatible orders with the above vectors and $h_0 = 0$, $h_j \geq 0$ for all $j \in \bar{q} : \{1, 2, \ldots, q\}$ are internal, in general, incommensurate delays (A_0 being the delay-free matrix of dynamics); and $h_0 = 0$, $h_j \geq 0$ for all $j \in \bar{q}$ are internal incommensurate delays (B_0 being the delay-free control matrix). $U(t)$ is the unit (Heaviside) function. The function of initial conditions $\varphi \in \mathrm{IC}\,([-h, 0], \mathbf{R}^n)$ with $h := \operatorname*{Max}_{j \in \bar{q}}(h_j)$ is almost everywhere absolutely continuous and can be impulsive at isolated points except for the impulse-free case, namely, $\operatorname{rank} E = \deg(\det(sE - A_0)) < n$. Extra conditions might be established for the admissible set of initial conditions $\mathrm{IC}\,([-h, 0], \mathbf{R}^n)$ to obtain some stronger results in several parts of the manuscript. Assume that $l = \operatorname{ind}(E)$, the index of E, namely,

$$l = \begin{cases} \geq 1 & \text{if } \operatorname{rank}(E) = r < n \\ 0 & \text{if } \operatorname{rank}(E) = r = n \end{cases}$$

(note that $l = 0$ if the system is nonsingular) and $\Phi_i \in \mathbf{R}^{n \times n}$; $i \in [-l, \infty) \cap \mathbf{Z}$ are (in general nonunique) solutions of the algebraic linear system:

(3.a)
$$\Phi_i E - \Phi_{i-1} A_0 = E\Phi_i - A_0 \Phi_{i-1} = \delta(i, 0) I_n, \quad i \in [-l, \infty) \cap \mathbf{Z}$$

(3.b)
$$\Phi_{-l} E = E\Phi_{-l} = \Phi_{-l-1} A_0 = A_0 \Phi_{-l-1} = 0,$$

where $\delta(i, 0)$ is the Kronecker delta defined by $\delta(0, 0) = 1$ and $\delta(i, 0) = 0$ for $i \neq 0$, $v^{(i)}(t) = d^i v(t)/dt^i$ if the ith time derivative of the real function, or real vector function, $v(t)$ provided that it exists and

$\delta^{(i)}(t)$ is the ith order distributional derivative of the Dirac impulse $\delta(t) = \delta^{(0)}(t)$, $\delta(t)$ being the Dirac delta. If $v^{(l)}(t) = d^l v(t)/dt^l$ exists but it is not differentiable, then $v^{(i)}(t) = \delta^{(i-l)}(t)$ for $i > l$. The general solution of (1) can be expressed in closed form as addressed in the subsequent result:

Theorem 1. *The following properties hold*:

(i) *Assume that the function of initial conditions and the control input satisfy* $\varphi \in C^{(l-1)}((-h,0),\mathbf{R}^n)$, *i.e.*, $\mathrm{IC}((-h,0),\mathbf{R}^n) \equiv \mathbf{C}^{(l-1)}((-h,0),\mathbf{R}^n)$ *and* $u \in C^{(l-1)}(r_+,\mathbf{R}^m)$, *respectively, and* $x_0^+ \in \Im$, *the* (*so-called*) *admissible set of point initial conditions, defined by*:

$$\Im := \Big\{ x_1 + x_2 : x_1 \in \operatorname{Im} E^D, x_2 = -(I_n - \widehat{E}E^D) \cdot \Big(\sum_{i=0}^{l-1} \widehat{E}^i \Big(\sum_{j=0}^{q} \widehat{A}_j \varphi^{(i)}(-h_j) + \widehat{B}_0 u^{(i)}(0) \Big) \Big) \Big\}. \tag{4}$$

Then, the general solution of (1) *may be expressed as follows*:

$$\begin{aligned} x(t) &= Z(t)\Big(\widehat{E}E^D \hat{x}_0 + \sum_{j=1}^{q} \int_{-h_j}^{0} Z(-\tau) E^D \widehat{A}_j \varphi(\tau)\, d\tau \\ &\qquad + \sum_{j=0}^{q} \int_{h'_j}^{t} Z(-\tau) E^D \widehat{B}_j u(\tau - h'_j)\, d\tau \Big) \\ &\quad - (I_n - \widehat{E}E^D)\Big(\sum_{i=1}^{l-1} \widehat{E}^i \Big(\sum_{j=0}^{q} \widehat{A}_j \varphi^{(i)}(-h_j) \\ &\qquad + \sum_{j=0}^{q'} \widehat{B}_j u^{(i)}(t - h'_j) U(t) \Big)\Big) \\ &= e^{E^D A_0 t}\Big(\widehat{E}E^D \hat{x}_0 + \sum_{j=1}^{q} \int_{0}^{h_j} e^{-A_0 E^D \tau} E^D \widehat{A}_j \varphi(\tau - h_j)\, d\tau \\ &\qquad + \sum_{j=0}^{q} \int_{h_j}^{t} e^{-A_0 E^D_\tau} E^D \widehat{A}_j x(\tau - h_j)\, d\tau \Big) \end{aligned} \tag{5.a}$$

$$
\begin{aligned}
&+ e^{E^D A_0 t} \sum_{j=0}^{q} \int_{h'_j}^{t} e^{-A_0 E^D \tau} E^D \widehat{B}_j u(\tau - h'_j)\, d\tau \\
\text{(5.b)}\qquad & \\
&- (I_n - \widehat{E} E^D) \Bigg(\sum_{i=0}^{l-1} \widehat{E}^i \Bigg(\sum_{j=0}^{q} \widehat{A}_j x^{(i)}(t - h_j) \\
&\qquad\qquad + \sum_{j=0}^{q'} \widehat{B}_j u^{(i)}(t - h'_j) U(t) \Bigg) \Bigg)
\end{aligned}
$$

for all $\widehat{x}_0 \in \mathbf{C}^n$, $x(t) \equiv \varphi(t)$ *for all* $t \in [-h, 0]$ *and* $\varphi(0) = x_0$, *implying that* $x(0^+) = x(0^-) = x_0 \in \Im$, *where* $E^D \to E^D = (E_1^{-1}, 0)$ *is the Drazin inverse of* E, *and*

$$
\text{(6)}\qquad Z(t) = \Bigg\{ e^{E^D A_0 t} \Bigg(I_n + \sum_{j=1}^{q} \int_{h_j}^{t} e^{-E^D A_0 \tau} A_j Z(\tau - h_j) U(\tau)\, d\tau \Bigg), \quad t \geq 0, \quad 0, t < 0 \Bigg\}
$$

is the evolution operator of differential system (1) *where* $U(t)$ *is a unit step* (*Heaviside*) *function, and*

$$
\begin{aligned}
\text{(7.a)}\qquad & \widehat{E} = (\lambda E + A_0)^{-1} E \\
& \widehat{A}_j = (\lambda E + A_0)^{-1} A_j, \quad j \in \bar{q} \cup \{0\}; \\
\text{(7.b)}\qquad & \widehat{B}_j = (\lambda E + A_0)^{-1} B_j, \quad j \in \bar{q} \cup \{0\}
\end{aligned}
$$

for any complex constant λ *such that* $\det(\lambda E - \sum_{j=0}^{q} A_j e^{-h_j \lambda})$. *If* $EA_0 = A_0 E$, *then the replacements* $\widehat{E} \to E$, $\widehat{A}_j \to A_j$ $(j \in \bar{q} \cup \{0\})$, $\widehat{B}_j \to B_j$ $(j \in \bar{q}' \cup \{0\})$ *are used in* (7).

(ii) *The* (*in general nonunique*) *solution of* (2) *for each given function of initial conditions* $\varphi \in IC([-h, 0], \mathbf{R}^n)$ *and control input* $u \in$

$PC(\mathbf{R}_+, \mathbf{R}^m)$ is

(8.a)

$$x(t) = \Psi(t)\Bigg(\Phi_0 E x_0^- + \sum_{j=1}^{q} \int_0^{h_j^-} \Psi(-\tau)\Phi_0 A_j \varphi(\tau - h_j)\, d\tau + \sum_{j=1}^{q} \int_{h_j^+}^{t} \Psi(-\tau)\Phi_0 A_j x(\tau - h_j)\, d\tau + \sum_{j=0}^{q'} \int_{h_j'^+}^{t} \Psi(-\tau)\Phi_0 B_j u(\tau - h_j')\, d\tau\Bigg) + \sum_{i=0}^{l-1} \Phi_{-i-1}\Bigg(E x_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(= h_j) + \sum_{j=0}^{q'} B_j u^{(i)}(t - h_j') U(t)\Bigg)$$

$$= e^{\Phi_0 A_0 t}\Bigg(\Phi_0 E x_0^- + \sum_{j=1}^{q} \int_0^{h_j^-} e^{-\Phi_0 A_0 \tau}\Phi_0 A_j \varphi(\tau - h_j)\, d\tau + \sum_{j=1}^{q} \int_{h_j^+}^{t} e^{-\Phi_0 A_0 \tau}\Phi_0 A_j x(\tau - h_j)\, d\tau + \sum_{j=0}^{q'} \int_{h_j'^+}^{t} e^{-\Phi_0 A_0 \tau}\Phi_0 B_j u(\tau - h_j')\, d\tau\Bigg)$$

(8.b)

$$+ \sum_{i=1}^{l-1} \Phi_{-i-1}\Bigg(E x_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(-h_j) + \sum_{j=0}^{q'} B_j u^{(i)}(t - h_j') U(t)\Bigg)$$

where the evolution operator satisfies

$$(9)\quad \Psi(t) = \begin{cases} e^{\Phi_0 A_0 t}\left(I_n + \sum_{j=1}^{q} \int_{h_j}^{t} e^{-\Phi_0 A_0 \tau} A_j \Psi(\tau - h_j) U(\tau)\, d\tau\right), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

where $\varphi^{(i)}(-h_j) = \varphi^{(j)}(-h_j)\delta^{(i-j-1)}(t-h_j)$, *i.e., either a classical or a distributional derivative of appropriate order provided that* $j \le i$ *is the largest integer less than* i *such that* $\varphi^{(j)}(-h_j)$ *exists; and similarly* $u^{(i)}(t-h'_j) = u^{(j)}(t-h'_j)\delta^{(i-j-1)}(t-h'_j)$ *for the control input provided that* $u^{(j)}(t-h'_j)$ *is the highest order input time-derivative at time* $t-h'_j$.

Outline of proof. The proof of (i) equations (5)–(7) is given in [**1**] for delay-free singular linear systems and a particular set of initial conditions, which result in being continuous at the origin, which guarantees uniqueness of the state-trajectory solution. Its extension to the case of multiple internal and external discrete delays is direct by considering the evolution operator (6) obtained in [**5**] and a set of particular initial conditions (4) which still guarantees the uniqueness of the solution. The proof of (ii) equations (8)–(9) is given in [**10**] for the delay-free case and extended in [**5**] for the case of multiple discrete constant internal and external delays. □

Remark 1. The Drazin inverse has the properties $\operatorname{Ker} E^D = \operatorname{Ker} EE^D$, $\operatorname{Im} E^D = \operatorname{Im} EE^D$ so that EE^D is the projection of $\mathbf{C}^n$ on $\operatorname{Im} E^D$ and $I_n - EE^D$ is the projection of $\mathbf{C}^n$ on $\operatorname{Ker} E^D$ so that:

$$\begin{aligned}\mathbf{C}^n &= \operatorname{Im} E^D \oplus \operatorname{Ker} E^D = \operatorname{Im} EE^D \oplus \operatorname{Ker} EE^D \\ &= \operatorname{Im} E^D E \oplus \operatorname{Ker} E^D E\end{aligned}$$

[**1, 9**] which is used in establishing the set $\Im$ of initial conditions at $t = 0^+$ for the proof of Theorem 1 (i). For such a set of vector functions of initial conditions, the uniqueness of the state-trajectory solution is guaranteed.

Remark 2. Theorem 1 (ii) applies for very general sets of initial conditions and piecewise continuous inputs which implies that the sate-trajectory solution is not unique, in general. However, linear time-invariant dynamic systems (1)–(2) in standard form, i.e., being nonsingular with E, $\Phi_0 = E^D = E^{-1}$, with E being nonsingular, and singular regular impulse-free systems, i.e., $n_1 = \operatorname{rank}(E) = \deg(\det(sE - A_0)) < n$ (implying $\Phi_0 \ne I_n$, $\Phi_{-i} = 0$; for all $i \ge 1$) exhibit a unique state-trajectory solution for all $\varphi \in \mathrm{IC}([-h,0],\mathbf{R}^n)$, for all $u \in \mathrm{PC}(\mathbf{R}_+,\mathbf{R}^m)$ since the polynomial part of the solution is identically zero. These two cases might be referred to as those having an index system of system (1), or simply as those having $\operatorname{ind}(E, A_0)$, the index of the pair (E, A_0), being zero and unity, respectively, by borrowing the concept from the delay-free case (see, for instance, [**13**]). Note from inspection of (8)–(9) that, for any index of E, the solution is still unique for the singular solvable impulsive case, i.e., $n > n_1 = \operatorname{rank}(E) > \deg(\det(sE - A_0))$ for all $\varphi \in \mathrm{IC}([-h, 0], \mathbf{R}^n)$ which satisfies $\varphi(0^-) = x_0^- \in \operatorname{Ker}(\sum_{i=1}^{l}(\Phi_{-i}E))$ (in particular, if $x_0^- \in \cap_{i\in\bar{l}}\operatorname{Ker}(\Phi_{-i}E))$ and all control functions $u \in \mathrm{PC}(\mathbf{R}_+, \mathbf{R}^m)$, provided that

$$\Phi_{-i}A_j = 0, \quad \Phi_{-i}B_k = 0, \text{ for all } i \in \bar{l}, \text{ for all } j \in \bar{q}, \text{ for all } k \in \bar{q}' \cup \{0\}$$

and also for any sufficiently regular functions of initial conditions and input so that the distributional Dirac time-derivatives are standard time-derivatives which coincide with the uniqueness result of Theorem 1 (i). □

It is well known for the delay-free case and the case with one single point input delay [**15, 16**] that, if system (1) is solvable, i.e., $\det(sE - A_0)$ is not identically zero, then it can be transformed to the Weierstrass canonical form by premultiplying and postmultiplying the state vector $x(t)$ in (1) by two nonsingular complex n-matrices G and H, respectively. A generalization for the case of multiple internal and external delays of the Weierstrass canonical form with the transformed state

$$x(t) = (x_1^T(t), x_2^T(t))^T = H\overline{x}(t) = H(\overline{x}_1^T(t), \overline{x}_2^T(t))^T$$

results in

$$\overline{E}\dot{\overline{x}}(t) = \sum_{j=0}^{q} \overline{A}_j \overline{x}(t - h_j) + \sum_{j=0}^{q} \overline{B}_j u(t - h_j')U(t) \tag{10}$$

$$y(t) = \overline{C}\overline{x}(t) \tag{11}$$

where

$$\overline{E} = GEH = \text{Block Diag}\,(I_{n_1}, N);$$
$$\overline{A}_0 = GA_0H = \text{Block Diag}\,(\overline{A}_{011}, I_{n_2})$$

$$
(12) \qquad
\begin{aligned}
&\overline{A}_j = GA_jH = \begin{bmatrix} \overline{A}_{j11} & \overline{A}_{j12} \\ \overline{A}_{j21} & \overline{A}_{j22} \end{bmatrix}, \text{ for all } j \in \bar{q};\\
&\overline{B}_j = GB_j = \begin{bmatrix} \overline{B}_{j1} \\ \overline{B}_{j2} \end{bmatrix}, \text{ for all } j \in \bar{q}' \cup \{0\}\\
&\overline{C} = CH
\end{aligned}
$$

with $n_1 = \operatorname{rank}(E) - \operatorname{rank}(N)$, N nilpotent with $n_2 = n - n_1$, and

$$
\begin{aligned}
l &:= \operatorname{ind}(E) = n\operatorname{ind}(N) \Longleftrightarrow l\\
&= \operatorname{Min}(k \in \mathbf{Z}_+ : \operatorname{rank}(E^k) = \operatorname{rank}(E^{k+1}))\\
&\Longleftrightarrow l = \operatorname{Min}(k \in \mathbf{Z}_+ : \operatorname{Im}(E^k) = \operatorname{Im}(E^{k+1}))\\
&\Longleftrightarrow l = \operatorname{Min}(k \in \mathbf{Z}_+ : \operatorname{Ker}(E^k) = \ker(E^{k+1}))\\
&\Longleftrightarrow l = \operatorname{Min}(k \in \mathbf{Z}_+ : N^k = 0).
\end{aligned}
$$

For $l = 0$, $\operatorname{rank}(E_1) = \operatorname{rank}(E) = n$, N is removed in the block matrix decomposition and system (1) is not singular. If $\operatorname{rank}(E_1) = \operatorname{rank}(E) = \deg(\det(sE - A_0)) = n_1 < n$, then $l = 1$ and $N = 0$ is square of order $(n - n_1)$ and trivially of nilpotency index unity. If $n > n_1 = \operatorname{rank}(E_1) < \operatorname{rank}(E) = \operatorname{rank}(E_1) + \operatorname{rank}(N) > \deg(\det(sE - A_0))$, then $N \neq 0$ and $l = \operatorname{ind}(E) = n\operatorname{ind}(N) \geq 2$. See, for instance, [**9**]. For the canonical Weierstrass form (10)–(12) of (1)–(2), the subsequent result extends Theorem 1 (i):

Theorem 2. *Assume that system* (1)–(2) *is solvable and assume also that the function of initial conditions and the control input satisfy* $\overline{\varphi} = H^{-1}\varphi = (\overline{\varphi}_1^T, \overline{\varphi}_2^T)^T \in \mathbf{C}^{(l-1)}((-h, 0), \mathbf{R}^n)$, $u \in \mathbf{C}^{(l-1)}(\mathbf{R}_+, \mathbf{R}^m)$, *respectively, and* $\overline{x}_0 = H^{-1}x_0 \in \Im_I$, *the* (*so-called*) *admissible set of point initial conditions of* (10)–(12), *defined by*:

Then, the unique solution of the Weierstrass canonical form (10)–(12) is:

(13.a)

$$
\begin{aligned}
\overline{x}_1(t) = Z_{w11}(t)\Bigg(&\overline{x}_{01} + \sum_{j=1}^{q} \int_{-h_j}^{0} Z_{w11}(-\tau)(\overline{A}_{j11}\overline{\varphi}_1(\tau) + \overline{A}_{j12}\overline{\varphi}_2(\tau))\, d\tau\\
&+ \sum_{j=0}^{q} \int_{h_j'}^{t} Z_{w11}(-\tau)\overline{B}_{j1}u(\tau - h_j')\, d\tau\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= e^{\overline{A}_{011}t}\bigg(\overline{x}_{01}\\
&\qquad + \sum_{j=1}^{q}\int_0^{h_j} e^{-\overline{A}_{011}\tau}(\overline{A}_{j11}\overline{\varphi}_1(\tau-h_j)+\overline{A}_{j12}\overline{\varphi}_2(\tau-h_j))\,d\tau\\
&\qquad + \sum_{j=0}^{q}\int_{h_j}^{t} e^{-\overline{A}_{011}\tau}(\overline{A}_{j11}\overline{x}_1(\tau-h_j)+\overline{A}_{j12}\overline{x}_2(\tau-h_j))\,d\tau
\end{aligned}
\tag{13.b}
$$

$$
+ \sum_{j=0}^{q}\int_{h'_j}^{t}\overline{B}_{j1}u(\tau-h'_j)\,d\tau\bigg)
$$

$$
\begin{aligned}
\overline{x}_2(t) = -\sum_{i=0}^{l-1}N^i\bigg(&\sum_{j=0}^{q}\overline{A}_{j21}\overline{x}_1^{(i)}(t-h_j)\\
&+\sum_{j=1}^{q}\overline{A_{j22}}\overline{x}_2^{(j)}(t-h_j)
\end{aligned}
\tag{13.c}
$$

$$
+\sum_{j=0}^{q}\overline{B}_{j2}u^{(i)}(t-h'_j)U(t)\bigg)
$$

with $x(t)\equiv\varphi(t)$ for all $t\in[-h,0]$ and $\varphi(0)=x_0$, where

$$
\begin{aligned}
Z_w(t) &= \begin{bmatrix} Z_{w11}(t) & Z_{w_{12}}(t)\\ Z_{w21}(t) & Z_{w22}(t)\end{bmatrix} = \begin{bmatrix} Z_{w1}(t)\\ Z_{w1}(t)\end{bmatrix}\\
&:= \begin{cases} e^{\overline{A}_0 t}(I_n+\sum_{j=1}^{q}\int_{h_j}^{t} e^{-\overline{A}_0\tau}\overline{A}_j Z_w(\tau-h_j)\,d\tau & t\ge 0\\ 0 & t<0,\end{cases}
\end{aligned}
\tag{14}
$$

for all $\overline{x}_0=\overline{x}(0)\in\Im_I$.

Outline of proof. It is a direct extension of a parallel result in [**16**] for linear systems free of internal delays with only one single external point delay by considering the initial condition function as a forcing term and using directly Theorem 1 (i). □

III. Positivity, reachability and controllability. The positivity (usual abbreviation for internal positivity) and external positivity properties, [**2, 4, 5, 10, 11**] do not stand for the general class of systems

(1)–(2) since the uniqueness of the solution becomes lost (Theorem 1 (ii)) and it suffices to take particular initial conditions or inputs to find neither positive state nor output trajectory solutions. However, we can refer to weak positivity/external positivity of the dynamic system for corresponding properties after removal of the polynomial part of the solution which coincide with classical related concepts for the two cases below:

(1) $\operatorname{ind}(E, A_0) = 0$, namely, nonsingular systems (1)–(2) in standard form, being always solvable implying $l = 0$, E nonsingular, $\Phi_0 = E^{-1}$.

(2) $\operatorname{ind}(E, A_0) = 1$, namely, (singular nonimpulsive systems (1)–(2)) provided they are solvable implying $l = 1$, E singular with $\operatorname{rank} E = \deg(\det(sE - A_0)) < n$, $\Phi_0 = E^D$ (then $\Phi_0 E = E\Phi_0$ and $N = 0$).

More precise positivity definitions follow.

Definitions. 1. (Weak positivity). The system (1)–(2) is weakly positive if for $D = I_n$ and $D = C$, the subsequent property holds:

$$D\Bigg(x(t) - \sum_{i=0}^{l-1} \Phi_{-i-1}\Bigg(Ex_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(-h_j) + \sum_{j=0}^{q'} B_j u^{(i)}(t - h'_j)U(t)\Bigg)\Bigg) \geq 0, \text{ i.e., } \in R_+^n$$

for all $t \in \mathbf{R}_+$ for all admissible $\varphi : [-h, 0] \to \mathbf{R}_+^n$ and all piecewise continuous $u : \mathbf{R}_+ \to \mathbf{R}_+^n$ satisfying $\varphi \in C^{(l-1)}((-h, 0), \mathbf{R}^n)$ and $u \in C^{(l-1)}(\mathbf{R}_+, \mathbf{R}^m)$.

2. (Weak external positivity). The system (1)–(2) is weakly externally positive if the subsequent property holds:

$$C\Bigg(x(t) - \sum_{i=0}^{l-1} \Phi_{-i-1}\Bigg(Ex_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(-h_j) + \sum_{j=0}^{q} B_j u^{(i)}(t - h'_j)U(t)\Bigg)\Bigg) \geq 0, \text{ i.e., } \in R_+^n$$

for all $t \in \mathbf{R}_+$ if for $\varphi \equiv 0 \in \mathbf{R}^n$ of domain $[-h, 0]$ and all $u : \mathbf{R}_+ \to \mathbf{R}_+^n$ satisfying $u \in C^{(l-1)}(\mathbf{R}_+, \mathbf{R}^m)$.

3. (Weak strong positivity). The system (1)–(2) is weakly strongly positive if, for $D = I_n$ and $D = C$, the subsequent property holds:

$$D\Bigg(x(t) - \sum_{i=2}^{l-1} \Phi_{-i-1}\Bigg(Ex_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(-h_j) + \sum_{j=0}^{q'} B_j u^{(i)}(t - h_j') U(t)\Bigg)\Bigg) \geq 0 \text{ i.e., } \in R_+^n,$$

for all $t \in \mathbf{R}_+$, for all admissible $\varphi : [-h, 0] \to \mathbf{R}_+^n$ and all piecewise continuous $u : \mathbf{R}_+ \to \mathbf{R}_+^n$ satisfying $\varphi \in \mathbf{C}^{(l-1)}((-h, 0), \mathbf{R}^n)$ and all $u : \mathbf{R}_+ \to \mathbf{R}_+^n$ satisfying $u \in \mathbf{C}^{(l-1)}(\mathbf{R}_+, \mathbf{R}^m)$.

4. (Weak strong external positivity). The system (1)–(2) is weakly strongly externally positive if the subsequent property holds:

$$C\Bigg(x(t) - \sum_{i=2}^{l-1} \Phi_{-i-1}\Bigg(Ex_0^- \delta^{(i)}(t) + \sum_{j=1}^{q} A_j \varphi^{(i)}(-h_j) + \sum_{j=0}^{q'} B_j u^{(i)}(t - h_j') U(t)\Bigg)\Bigg) \geq 0 \text{ i.e., } \in R_+^n,$$

for all $t \in \mathbf{R}_+$ if for $\varphi \equiv 0 \in \mathbf{R}^n$ of domain $[-h, 0]$ and all $u : \mathbf{R}_+ \to \mathbf{R}_+^n$ satisfying $u \in \mathbf{C}^{(l-1)}(\mathbf{R}_+, \mathbf{R}^m)$.

5. (Positivity). The system (1)–(2) is positive if $\operatorname{ind}(E, A_0) = 0, 1$; $x(t) \geq 0$ and $y(t) \geq 0$ for all $t \in \mathbf{R}_+$ for all admissible $\varphi : [-h, 0] \to \mathbf{R}_+^n$ and all $u \in \mathrm{PC}(\mathbf{R}_+, \mathbf{R}^m)$.

6. (External positivity). The system (1)–(2) is externally positive if $\operatorname{ind}(E, A_0) = 0, 1$ and $y(t) \geq 0$ for all $t \in \mathbf{R}_+$ if, for $\varphi \equiv 0 \in \mathbf{R}^n$ of domain $[-h, 0]$ and all $u \in \mathrm{PC}(\mathbf{R}_+, \mathbf{R}^m)$. □

Note that Definitions 1, 2 of weak positivity/external weak positivity are equivalent to positivity/external positivity if the polynomial parts of the state and output trajectory solutions are identically zero. Definitions 3, 4 of weak strong/external weak strong positivity are equivalent to positivity/external positivity for systems of index zero (i.e., nonsingular systems reducible to the standard form) or unity (i.e., singular regular, i.e., solvable and impulse-free systems). Note also that if the system (1), (2) is characterized as weakly (weakly strongly) positive, weakly (weakly strongly) externally positive, then properties like, for instance,

$$D\Bigg(x(t)-\sum_{i=0}^{l-1}\Phi_{-i-1}\Bigg(Ex_0^-\delta^{(i)}(t)+\sum_{j=1}^{q}A_j\delta^{(i)}(-h_j) \\ +\sum_{j=0}^{q'}B_j\delta^{(i)}(t-h_j')U(t)\Bigg)\Bigg)\geq 0,\text{i.e.,}\in R_+^n$$

(Definition 1), or any "ad-hoc" modifications for Definitions 2–4, also hold for distributional derivatives so for no differentiable functions up to order $(l-1)$ of initial conditions/controls. The following result follows directly from Theorems 1, 2.

Theorem 3. *The following properties hold*:

(i) *The system* (1)–(2) *is weakly positive if and only if* $\Phi_0A_0\in\mathbf{M}_E^{n\times n}$, $\Phi_0A_j\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times n}$), $\Phi_0B_k\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times m}$) *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$ *and* $\mathbf{C}\geq 0$. *It is weakly externally positive if and only if* $\mathbf{C}e^{\Phi_0A_0t}\Phi_0A_j\geq 0$ *and* $\mathbf{C}e^{\Phi_0A_0t}\Phi_0B_k\geq 0$ *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$, *for all* $t\in\mathbf{R}_+$.

(ii) *The system* (1)–(2) *is weakly strongly positive if and only if* $\Phi_0A_0\in\mathbf{M}_E^{n\times n}$, $\Phi_0A_j\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times n}$), $\Phi_0B_k\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times m}$) *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$, $i=0,1$ *and* $C\geq 0$. *It is weakly strongly externally positive if and only if and* $\mathbf{C}e^{\Phi_0A_0t}\Phi_{-i}A_j\geq 0$ *and* $\mathbf{C}e^{\Phi_0A_0t}\Phi_{-i}B_k\geq 0$ *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$, *for all* $t\in\mathbf{R}_+$, $i=0,1$.

(iii) *The system* (1)–(2) *is positive if and only if* $\Phi_0A_0\in\mathbf{M}_E^{n\times n}$, $\Phi_0E\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times n}$), $\Phi_0A_j\geq 0$, $\Phi_0B_k\geq 0$ (*i.e.*, $\in\mathbf{R}_+^{n\times m}$), *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$, $\mathbf{C}\geq 0$, *and* $\Phi_{-i}A_j=0$, $\Phi_{-i}k=0$, *for all* $j\in\overline{q}$, *for all* $k\in\overline{q}'\cup\{0\}$, *for all* $t\in\mathbf{R}_+$, $i\in\overline{l-1}\cup\{0\}$. *It is externally positive if and only if* $C\Phi_0E\geq 0$,

$\mathbf{C}e^{\Phi_0 A_0 t}\Phi_0 A_j \geq 0$, $\mathbf{C}e^{\Phi_0 A_0 t}\Phi_0 B_k \varepsilon 0$, *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$, *and* $\mathbf{C}e^{\Phi_0 A_0 t}\Phi_{-i} A_j = 0$, $\mathbf{C}e^{\Phi_0 A_0 t}\Phi_{-i} B_k = 0$, *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$, *for all* $t \in \mathbf{R}_+$, $i \in \overline{l-1} \cup \{0\}$ *for all* $t \in \mathbf{R}_+$. *The system* (1)–(2) *is guaranteed to be positive* (*externally positive*) *if and only if it is weakly strongly positive* (*weakly strongly externally*) *positive and, furthermore,* $\operatorname{ind}(E, A_0) = 0, 1$.

Proof. (i)–(ii). The "sufficiency parts" follow directly from the expressions of the state/output trajectory solutions in Theorem 1 (ii) and Definitions 1–4 since the polynomial parts of the trajectory solutions are removed from the characterizations and the evolution operator $\Psi : [-h, \infty) \to \mathbf{R}_+^{n\times n}$ and the $\mathbf{C}_0$-semigroup $\mathbf{e}^{A_0 \Phi_0 t} \geq 0$, for all $t \in \mathbf{R}_+$ since its infinitesimal generator is a Metzler matrix. The "necessity parts" follow by contradiction since, if any of the given conditions is violated, it is always possible to construct nonnegative functions of initial conditions or, respectively, controls of sufficiently large amplitudes such that the corresponding state/output trajectory becomes negative at some time instant. Property (iii) is a direct consequence of Properties (i)–(ii) and Definitions 5–6 for $\operatorname{ind}(E, A_0) \geq 2$. For the case $\operatorname{ind}(E, A_0) = 0, 1$, the proof is direct since $\Phi_{-i} = 0$, for all $i \geq 2$. □

Corollary 1. *All the particular solutions of solvable singular systems* (1)–(2) *with* ind,$(E, A_0) \geq 2$ *satisfying* $x_0^+ \in \Im$ *are nonnegative for all time and all nonnegative input and function of initial conditions if and only if* $E^D A_0 \in \mathbf{M}_E^{n\times n}$, $\widehat{E}E^D \geq 0$, $E^D A_j \geq 0$, $E^D B_k \geq 0$, $(I_n - \widehat{E}E^D)\widehat{E}^i \widehat{A}_j = 0$ *and* $(I_n - \widehat{E}E^D)\widehat{E}^i \widehat{B}_k = 0$ *for all* $j \in \overline{q} \cup \{0\}$, *for all* $k \in \overline{q}\,'$, *for all* $i \in \overline{l-1}$. *The last two conditions are removed if* $\operatorname{ind}(E, A_0) = 0, 1$.

Proof. It follows by direct inspection of the state/output-trajectory solutions from Theorem 1 (i) and (2). □

On the other hand, for the Weierstrass canonical form (10)–(12) of (1)–(2), all solutions satisfying are nonnegative under parallel conditions to those of Corollary 1 from Theorem 2 and (2). In this case, the positivity property is characterized as follows from Theorems 2–3 and Definitions 1–6:

Theorem 4. *Assume that the following hypotheses hold*:

(H1) *The system* (1)–(2) *is solvable.*

(H2) *The system* (1)–(2) *satisfies one of the subsequent constraints*:

1. $\operatorname{ind}(E, A_0) = 0$.

2. $\operatorname{ind}(E, A_0) = 1$ *and the nilpotent matrix of its Weierstrass canonical form satisfies* $(-N) \geq 0$ *or* $N\overline{A}_{j21} = 0$, $N\overline{B}_{k2} = 0$ *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$.

3. $l = \operatorname{ind}(E, A_0) \geq 2$ *and* $N^i\overline{A}_{j21} = 0$, $N^i\overline{B}_{k2} = 0$ *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$, *for all* $i \in \overline{l-1} \cup \{0\}$.

Then, any solution of initial conditions satisfying $\overline{\varphi} : [-h, 0] \to \mathbf{R}^n$ *satisfies* $\Im_I := \{\mathbf{R}_+^{n_1} \times \mathbf{0}_{n_2}\}$ *as well and any state-trajectory and output trajectories, subject to* $\overline{\varphi} : [-h, 0] \to \mathbf{R}_+^n$ *and* $\overline{x}_0 \in \Im_I$, *are both nonnegative for all time if and only if* $\overline{A}_{j11} \geq 0$, $\overline{A}_{j12} \geq 0$ *and* $\overline{B}_{k1} \geq 0$, *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$. *As a result, if* $\operatorname{ind}(E, A_0) = 0$, *the system in Weierstrass canonical form is positive for any* $\overline{\varphi} : [-h, 0] \to \mathbf{R}_+^n$ *satisfying* $\Im_I := \{\mathbf{R}_+^n \times \mathbf{0}_{n_2}\}$ *is positive if and only if* $\overline{A}_{011} \in \mathbf{M}_E^{n_1 \times n_1}$, $\overline{A}_{j11} \geq 0$, $\overline{A}_{j12} \geq 0$ *and* $\overline{B}_{k1} \geq 0$, *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$. *The same result arises if* $\operatorname{ind}(E, A_0) = 1$, $(-N) \geq 0$ *for such initial conditions if and only if* $\overline{A}_{011} \in \mathbf{M}_E^{n_1 \times n_1}$, $\overline{A}_j \geq 0$ *and* $\overline{B}_k \geq 0$ *for all* $j \in \overline{q}$, *for all* $k \in \overline{q}\,' \cup \{0\}$.

Proof. The "sufficiency parts" follow directly from Theorems 2–3 by inspecting (13.b). The "necessity parts" follow by establishing contradiction arguments as indicated in Theorem 3. □

The "so-called" weak reachability of a weakly positive singular system (Theorems 1–2) is now discussed in the sense that both properties should jointly hold. *Weak reachability* is defined as the ability of attaining any nonnegative state in some finite time from zero initial conditions while keeping the weak positivity property. Then, a weakly reachable, weakly positive system attains any finite nonnegative state in some finite time, from an identically zero function of initial conditions, with a control and through an intermediate state whose components never take negative values in the time interval of interest. A weakly reachable weakly positive system is reachable if it is positive (see Theorems 3–4). A less strong property is that of partial weak reachability where the reachability property is only guaranteed for the slow dynamics sub-state. In other words, the fast dynamics sub-state $\overline{x}_2(t)$ is only guaranteed to be kept nonnegative but it is not guaranteed to be driven to a prescribed value while the slow dynamics sub-state $\overline{x}_1(t)$ is driven to any prescribed nonnegative value. A more restrictive property than weak reachability is that of weak controllability implying similar characterizations except that the initial conditions are arbitrary admissible vector functions. For the nonsingular, delay-free case, reachability of a positive system requires a monomial controllability-like Grammian while controllability in finite time (asymptotically) requires, in addition, that the unforced dynamics evolution operator be nilpotent (a convergent matrix) in the discrete case. In the continuous-time case, the evolution operator cannot be nilpotent so that only asymptotic controllability (often referred to as asymptotic stabilizability) is feasible. Some related requirements, although more involved, are discussed in the following in the context of singular systems with delayed dynamics.

Definition 7 (Weak reachability of a weakly positive system, reachability of a positive system). A weakly positive system (1)–(2) is weakly reachable in finite time $T > 0$ if any prefixed state $x^* > 0$ is attainable in time T with some control $\mathbf{u} : [0, T) \to \mathbf{R}_+^m$ from zero initial conditions $\varphi : [-h, 0] \to 0 \in \mathbf{R}_+^m$ provided that $\widehat{x}_0 = 0 \in \mathbf{R}_+^n$ in Theorem 1, equation (5.a). If, furthermore, $\widehat{E}^i \widehat{A}_j = 0$ and $\widehat{E}^i \widehat{B}_k = 0$ for all $i \in \overline{l-1} \cup \{0\}$, for all $j \in \overline{q}$, for all $k \in \overline{q}\,' \cup \{0\}$ (which includes the case $\operatorname{ind}(E, \widehat{A}_0) = 0, 1$), then the weakly positive system is said to be reachable in finite time.

Remark 3. Note that, since positivity (Definition 5) is an extension of weak positivity for systems whose polynomial part of the state-trajectory solution is zero, reachability is a natural extension of weak reachability for such a class of weakly positive systems as addressed in Definition 7. In this way, weakly reachable systems of index zero or unity are reachable, and this also occurs for certain classes of systems with indexes exceeding unity whose polynomial part of the state-trajectory solution is zero. Note that both concepts are not generically equivalent except when $\widehat{E}^i \widehat{A}_j = 0$ and $\widehat{E}^i \widehat{B}_k = 0$ for all $i \in \overline{l-1} \cup \{0\}$, for all $j \in \overline{q}$, for all $k \in \overline{q}\,' \cup \{0\}$ (which includes the cases of systems of indexes zero or unity) since, otherwise, the input may have nonzero time-derivatives or may be impulsive for time instants $t > 0$, see equation (5.a).

Remark 4. Output reachability, respectively, weak/output reachability are concepts directly extendable from reachability, respectively, weak reachability, by substituting in Definition 7 positivity (standard or weak) by output positivity and the attainability of an arbitrary positive state by that of an arbitrary prefixed output value $y^* > 0$ at time T. In particular, above-mentioned constraints $\widehat{E}^i \widehat{A}_j = 0$ and $\widehat{E}^i \widehat{B}_k = 0$ for all $i \in \overline{l-1} \cup \{0\}$, for all $j \in \overline{q}$, for all $k \in \overline{q'} \cup \{0\}$ should now be replaced with $C\widehat{E}^i \widehat{A}_j = 0$ and $C\widehat{E}^i \widehat{B}_k = 0$ for all $i \in \overline{l-1} \cup \{0\}$, for all $j \in \overline{q}$, for all $k \in \overline{q}' \cup \{0\}$. □

Definition 8 (Weak controllability of a weakly positive system, controllability of a positive system). A weakly positive system (1)–(2) is weakly controllable in finite time $T > 0$ if any prefixed state $x^* > 0$ is attainable in time T with some control $u : [0, T) \to \mathbf{R}_+^m$ from zero initial conditions $\varphi : [-h, 0] \to 0 \in \mathbf{R}_+^m$ provided that $\widehat{x}_0 = 0 \in \mathbf{R}_+^n$ in Theorem 1, equation (5.a). If, furthermore, $\widehat{E}^i \widehat{A}_j = 0$ and $\widehat{E}^i \widehat{B}_k = 0$, for all $i \in \overline{l-1} \cup \{0\}$, for all $j \in \overline{q}$, for all $k \in \overline{q'} \cup \{0\}$ (which includes the case $\operatorname{ind}(E, \widehat{A}_0) = 0, 1$), then the weakly positive system is said to be controllable in finite time.

Definition 9 (Weak asymptotic controllability of a weakly positive system, asymptotic controllability of a positive system). They are direct extensions of the corresponding Definition 8 when the attainability time for the prefixed states tends to infinity.

It has to be pointed out that controllability is a stronger concept than reachability in the context of positive systems since the positivity property has to be maintained, namely, the corresponding property should be achievable under a nonnegative control (see, for instance, [**12**] for the discrete delay-free case) starting with a right nonzero initial condition at $t = 0$. This implies that, at time T, where the prefixed objective state is reached, the amount is $(x^* - Z(T)x_0^+) > 0$ for the given arbitrary $x^* > 0$. It is then shown that this requires that the controllability property for continuous-time systems is necessarily asymptotic, in general, contrarily to the reachability one, and that the unforced system is required to be asymptotically stable. In the delay-free, discrete-time case, controllability of positive systems is feasible in finite time if the matrix of dynamics is nilpotent, [**12**], which is impossible in the continuous-time case since such a matrix is a fundamental one of the differential system and then nonsingular for all finite time.

Note from Theorem 1 (i) that

$$\tag{15} \begin{aligned} \widetilde{x}(t) &= \sum_{j=0}^{q'} \int_{h'_j}^{t} Z(t-\tau)E^D\widehat{B}_j u(\tau - h'_j)\,d\tau, \\ \widetilde{x}(t) &= \sum_{j=1}^{q'} \int_{-h'_j}^{0} Z(t-\tau)E^D\widehat{B}_j u(\tau - h'_j)\,d\tau, \end{aligned}$$

where

$$\tag{16} \begin{aligned} \widetilde{x}(t) :=& \; x(t) - Z(t)\widehat{E}E^D\widehat{x}_0 - \sum_{j=1}^{q} \int_{-h_j}^{0} Z(t-\tau)E^D\widehat{A}_j\varphi(\tau)\,d\tau \\ &+ (I_n - \widehat{E}E^D)\bigg(\sum_{i=1}^{l-1}\widehat{E}^i\bigg(\sum_{j=0}^{q}\widehat{A}_j\varphi^{(i)}(-h_j) \\ &\qquad + \sum_{j=0}^{q'}\widehat{B}_j u^{(i)}(t-h'_j)U(t)\bigg)\bigg). \end{aligned}$$

The situation of interest for characterization of the reachability of positive singular systems is $\widetilde{x}(0^+) = x(0^+) \in \Im \cap \mathbf{R}_+^n$ and $\widehat{x}_0 = x(0^-) = \varphi(t) = 0$, for all $t \in [-h, 0)$. Under those conditions, $x(t) = \widetilde{x}(t)$ can be prefixed at some $x^* > 0$ for any $t > h$ by some nonnegative control $u : [0, t) \to \mathbf{R}_+^m$.

Theorem 5. *Assume that* (1)–(2) *is solvable and either positive* (*then implying* $\operatorname{ind}(E, A_0) = 0, 1$) *or solvable weakly positive with* $\operatorname{ind}(E, A_0) \geq 2$ *and, furthermore,* $E^D A_0 \in M_E^{n\times n}$, $\widehat{E}E^D \geq 0$, $E^D A_j \geq 0$, $E^D B_k \geq 0$, $(I_n - \widehat{E}E^D)\widehat{E}^i\widehat{A}_j = 0$ *and* $(I_n - \widehat{E}E^D)\widehat{E}^i\widehat{B}_k = 0$ *for all* $j \in \overline{q} \cup \{0\}$ *for all* $k \in \overline{q}$, *for all* $i \in \overline{l-1}$. *Then, the following properties hold*:

(i) *The system* (1)–(2) *is positive if* $\operatorname{ind}(E, A_0) = 0, 1$ *or if* $\operatorname{ind}(E, A_0) \geq 2$ *and Theorem* 1 (iii) *holds.*

(ii) *Assume with no loss of generality that the sizes of the external delays are strictly ordered according to* $0 = h'_0 < h'_1 < \cdots < h'_1$, *and consider some time instant* T *satisfying* $h'_q + h'_1 > T := h'_{q+1} h'_q$. *Then, the system* (1)–(2) *is weakly reachable at any finite time* $t \geq T$ *if and only if the controllability-like Grammian*:

$$\left(\int_0^{T-h'_{q'}} \left(\sum_{i=0}^{j} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right) \left(\sum_{i=0}^{j} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right)^T d\tau \right) \tag{17}$$

is monomial, i.e., positive nonsingular with only one nonzero entry per row. Furthermore, the system is reachable under the constraints given in Definition 7 *for reachability, in particular, if* $\operatorname{ind}(E, A_0) = 0, 1$.

Proof. Property (i) follows directly from Theorem 1 (i) for $\operatorname{ind}(E, A_0) = 0, 1$ and Corollary 1 for $\operatorname{ind}(E, A_0) \geq 2$ and $x_0^+ \in \Im$. In the case that $\operatorname{ind}(E, A_0) \geq 2$, the system is weakly positive, i.e., positive provided that the polynomial part of the state and output trajectories are zeroed. The proof of the "sufficiency part" of Property (ii) is direct from (15)–(16) with

$$\begin{aligned} \widetilde{x}(T) &= \sum_{j=0}^{q'} \int_{h'_j}^{h'_{q'+1}} Z(T-\tau) E^D \widehat{B}_j u(\tau - h'_j)\, d\tau \\ &= \sum_{j=0}^{q} \int_0^{h'_{j+1} - h'_j} \left(\sum_{i=0}^{j} Z(h'_{j+1} - h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right) \\ &\qquad \times u(\tau + h'_{q'} - h'_j)\, d\tau. \end{aligned} \tag{18}$$

Now, $x(T) = \widetilde{x}(T) = x^* > 0$ from (18) if

$$x^* = \int_0^{T-h'_{q'}} \left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right) u(\tau)\, d\tau$$

$$(19) \qquad = \left(\int_0^{T-h'_{q'}} \left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right) \times \left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right)^T d\tau \right) g$$

for some $g(> 0) \in \mathbf{R}_+^n$ by applying the following control input on $[0, T)$:

$$(20) \qquad u(\tau) = \begin{cases} \left[\left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right)^T d\tau \right] g; & \forall \tau \in [0, T - h'_q - \varepsilon) \\ 0; & \forall \tau \in [\varepsilon, T), \end{cases}$$

for some prefixed real constant $\varepsilon \in (0, T - h'_q)$. Note that (19) equalizes (18) for the control input (20) so that $x(T) = x^* > 0$ provided that

$$g = \left(\int_0^{T-h'_{q'}} \left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right) \times \left(\sum_{i=0}^{h'_q} Z(h'_{j+1} - h'_i - \tau) E^D \widehat{B}_i \right)^T d\tau \right)^{-1} x^* > 0$$

and $u : [0, T) \to \mathbf{R}_+^m$, since $x^* > 0$ and the above inverse matrix is also weakly positive since it is the inverse of a monomial matrix and the system is solvable (at least) weakly positive and there are no nonzero either derivative or impulsive terms in the solution associated with the input at time T, since it is identically zero in a neighborhood of T. There are also neither impulsive nor derivative terms associated with the initial conditions since the function of initial conditions is identically zero on $[-h, 0]$. For any finite time $t \geq T$, the same result applies if the input remains zero for all time $\tau \in [0, T]$. The *sufficiency part* has been proved. The *necessity part* of Property (ii) is now proved by contradiction. Note that, by construction,

$$\begin{aligned}
\operatorname{Im}\Bigg[\int_0^{T-h'_{q'}} &\left(\sum_{i=0}^{h'_q} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right) \\
&\left(\sum_{i=0}^{h'_q} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right)^T d\tau\Bigg] \\
&\equiv \operatorname{Im}\Bigg[\sum_{j=0}^{q'}\int_0^{h'_{j+1}-h'_j}\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right) \\
&\qquad\times\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right)^T d\tau\Bigg] \\
&\supseteq \operatorname{Im}\Bigg[\int_0^{T-h'_j}\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right) \\
&\qquad\times\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right)^T d\tau\Bigg],
\end{aligned}$$

for all $j \in \overline{q}' \cup \{0\}$. Then, weak reachability while keeping weak positivity requires that any $u(t) \in \mathbf{U} \subseteq \mathbf{R}_m^+$ ($\mathbf{U}$ being the input space) be achievable at any $t \in (0,T)$ and, furthermore,

$$\mathbf{R}_n^+ \equiv \operatorname{Im}\left[\sum_{j=0}^{q'}\int_0^{h'_{j+1}-h'_j}\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right)\right](u)$$

for such an input real vector function. Then, if $u : [0,T) \to \mathbf{R}_m^+$ and if

$$\mathbf{R}_n^+ \neq \operatorname{Im}\left[\sum_{j=0}^{q'}\int_0^{h'_{j+1}-h'_j}\left(\sum_{i=0}^{j} Z(h'_{j+1}-h'_i-\tau)E^D\widehat{B}_i\right)d\tau\right]\mathbf{U},$$

the weakly positive system is not weakly reachable. Note that both properties are weak positivity, respectively, reachability also implies positivity, respectively reachability, if $\operatorname{ind}(E, A_0) = 0, 1$ under identical proofs. □

Note that controllability of a weak positive system from any state in the first orthant to any strictly positive state in the first orthant is guaranteed under reachability provided that the unforced system is asymptotically stable. It is also guaranteed asymptotic controllability, i.e., asymptotic stabilizability, too. Thus, the subsequent result is immediate.

Theorem 6. *Assume that the weakly positive system is weakly reachable under Theorem 5. Then, system (1)–(2) is also weakly controllable from any state in $\mathbf{R}_+^n$ to any state $x^* \in \mathbf{R}_{+\varepsilon}^n := \{z(\in \mathbf{R}_+^n) = (z_1, z_2, \dots, z_n)^T : z_i > 0$, for all $i \in \overline{n}$, $\varepsilon(>0) \in \mathbf{R}_+\}$ in a finite time, dependent on ε, x_0^+ and x^* for any $\varphi : [-h, 0] \to 0 \in \mathbf{R}_{+\varepsilon}^n$ and $x_0^T(>0) \in \Im$, provided that it is asymptotically stable. It is then also asymptotically weakly controllable to $\mathbf{R}_{+\varepsilon}^n$ for any $\varepsilon(>0) \in \mathbf{R}_+$; i.e., asymptotically weakly controllable, or asymptotically weakly stabilizable, as time tends to infinity under a nonnegative control input when neglecting the polynomial part of the state trajectory solution. If* $\operatorname{ind}(E, A_0) = 0, 1$, *the above result becomes one of controllability of a positive system to $\mathbf{R}^n{+}\varepsilon$. The same result holds if* $\operatorname{ind}(E, A_0) \geq 2$ *Theorem* 1 (iii) *holds.*

Outline of Proof. It follows directly from Theorem 5 by considering any function of initial conditions $\varphi : [-h, 0) \to 0 \in \mathbf{R}^n$, $x_0^+(>0) \in \Im$ and a finite time $T = T(x_0^+, x^*, \varepsilon)$ such that $(x^* - \Psi(t)x_0^+) \geq 0$, for all $t \geq T$. □

Remark 5. Note that (spectral) reachability conditions of (1)–(2) are the following:

$$\operatorname{rank}\left(sE - A_0 - \sum_{i=1}^{q} A_i e^{-h_i s}, B_0 + \sum_{i=1}^{q'} B_i e^{-h_i' s}\right) = n,$$

for all $s \in \mathbf{C}$ (Popov-Belevitch-Hautus test)

$$\operatorname{rank}\left(sE - A_0 - \sum_{i=1}^{q} A_i \mu_i, B_0 + \sum_{i=1}^{q'} B_i \mu_{q+i}\right) < n$$

for $(s, \mu) \in \mathbf{C}^{q+q'+1} \Longleftrightarrow s \neq -(\ln \mu_i / h_i)$ for some $i \in \overline{q}$ or $s \neq -(\ln \mu_i / h_i')$ for some $i \in \overline{q}\,'$. Reachability independent of the external delays holds if and only if

$$\text{rank}\left(sE - A_0 - \sum_{i=1}^{q} A_i e^{-h_i s}, B\right) = n, \tag{3}$$

for all $s \in \mathbf{C}$ where $B := [B_0, B_1, \ldots, B_q]$. This follows by noting that

$$\sum_{i=0}^{q'} B_i u(t - h_i') = \sum_{i=0}^{q'} B_i v_i(t)$$

under the definition

$$v_i(t) = \begin{cases} u(t - h_i') & \text{for all } t \in [t - h_{i+1}', t - h_i') \\ 0 & \text{for all } t \in [0, t - h_{i+1}') \cup [t - h_i', t), \end{cases}$$

for all $i \in \overline{q}' \cup \{0\}$ after size reordering of the external delays as $0 = h_0' < h_1' < \cdots < h_{q'}'$, if necessary, with no loss in generality. As a result, the reachability independent of both internal and external delays (including the delay-free case) holds if and only if

$$\text{rank}\left(sE - A_0 - \sum_{i=1}^{q} A_i \mu_i, \overline{B}\right) < n,$$

some $(s, \mu) \in \mathbf{C}^{q+1} \Leftrightarrow \ln(\mu_i / s) \in \mathbf{R}_- := \mathbf{R} \backslash \mathbf{R}_+$, for all $i \in \overline{q}$. However, none of the above conditions guarantees weak reachability of a positive system (1)–(2) under any of its forms (including weak/strong positivity) since the control necessary to achieve reachability is not guaranteed to be nonnegative for all time. Then, the system is also weakly output reachable as a result. The same conclusion follows for controllability. In the case of a nonsingular system, the above conditions only guarantee reachability if, at the same time, the test for positivity for a system (1)–(2) with $\text{ind}\,(E, A_0) = 0, 1$ of Theorem 1 holds.

If the system (1)–(2) is represented in Weierstrass canonical form, then the reachability property reduces to test the ranks of two matrices related, respectively, to the slow and fast dynamics of the associate decomposition in subsystems.

Remark 6. Note from Remark 5 (3) that, since reachability is independent of the choice of any reachable realization, reachability independent of the external delays holds if and only if the Weierstrass canonical form (10)–(12) of (1)–(2) satisfies:

$$\operatorname{rank}\begin{bmatrix} sI_{n1}-\overline{A}_{011} & 0 & \sum_{i=1}^{q}\overline{A}_{i11}e^{-h_is} & \sum_{i=1}^{q}\overline{A}_{i12}e^{-h_is} & \overline{B}_1 \\ 0 & sN-I_{n_2} & \sum_{i=1}^{q}\overline{A}_{i21}e^{-h_is} & \sum_{i=1}^{q}\overline{A}_{i22}e^{-h_is} & \overline{B}_2 \end{bmatrix}=n,$$

for all $s\in\mathbf{C}$ where $\overline{B}_i=[\overline{B}_{i0},\overline{B}_{i1},\dots,\overline{B}_{iq}]$, $i=1,2$. □

In [**1**–**3**], the partial reachability for substates of slow dynamics of delay-free singular systems for admissible initial conditions and the state-trajectory evolving on the manifold defined by the fast dynamics sub-trajectory is referred to as *reachability* compared with the more stringent property of *complete controllability*, i.e., of the whole state. In the current approach in this manuscript, *reachability* is referred to as controllability of any state from zero and *controllability* is a related property from any initial state to any final state while keeping positivity or weak positivity of the system. This characterization was proposed in [**10, 17**] by having in mind the fact that the initial state is zero or nonzero is crucial to characterize reachability/controllability of positive systems (being understood as reachability/controllability under a nonnegative control effort). The last part of the subsequent result extends Theorem 5 concerning the Weierstrass canonical form refers to the reachability of the slow sub-state (referred to as *partial reachability* of such a substate) while keeping positivity:

Theorem 7. *The following properties hold*: (i) *The Weierstrass canonical state-space realization* (10)–(12) *of system* (1)–(2) *is reachable independent of external delays if and only if*

$$\operatorname{rank}\left[sI_{n1}-\overline{A}_{011},\sum_{i=1}^{q}\overline{A}_{i11}e^{-h_is},\sum_{i=1}^{q}\overline{A}_{i12}e^{-h_is},\overline{B}_1\right]=n_1,$$
$$\textit{for all } s\in\mathbf{C}$$
$$\operatorname{rank}\left[sN-I_{n_2},\sum_{i=1}^{q}\overline{A}_{i21}e^{-h_is},\sum_{i=1}^{q}\overline{A}_{i22}e^{-h_is},\overline{B}_2\right]=n_2=n-n_1,$$
$$\textit{for all } s\in\mathbf{C}.$$

(ii) *The Weierstrass canonical state-space realization is reachable independent of external delays if and only if both conditions below hold together*:

$$\operatorname{rank}\lfloor B_{e1}(\mu), \overline{A}_{011}\overline{B}_{e1}(\mu), \overline{A}_{011}^{n_1-1}\overline{B}_{e1}(\mu)\rfloor \leq n_1$$
$$\operatorname{rank}\lfloor B_{e2}(\mu), N\overline{B}_{e2}(\mu), N^{n_2-1}\overline{B}_{e2}(\mu)\rfloor \leq n_2,$$

with at least one of the two rank inequalities being strict for some

$$\mu \in \mathbf{C}^q \Longleftrightarrow \ln \frac{\mu_i}{s} \in \mathbf{R}_- := \mathbf{R} \setminus \mathbf{R}_+, \textit{ for all } i \in \overline{q}$$

where

$$\overline{B}_{ej}(\mu) := \left[\sum_{i=1}^{q} \overline{A}_{ij1}\mu_i, \sum_{i=1}^{q} \overline{A}_{ij2}\mu_i, \overline{B}_j\right] \textit{ for } j = 1, 2.$$

(iii) *Assume that the Weierstrass canonical state-space realization satisfies the following parameterization constraints*:

$$\overline{A}_{011} \in \mathbf{R}_+^{n_1 \times n_1}; \quad [\overline{A}_{j11} \quad \overline{A}_{j12}] \in \mathbf{R}_+^{n_1 \times n}, \textit{ for all } j \in \overline{q};$$
$$\overline{B}_{j1} \in \mathbf{R}_+^{n_1 \times m}, \textit{ for all } j \in \overline{q}' \cup \{0\};$$
$$\overline{C} \in \mathbf{R}_+^{p}; \quad (-N^i)\overline{A}_{j2l} \in \mathbf{R}_+^{n_2 \times n_l}; \quad (-N^i)\overline{B}_{j2} \in \mathbf{R}_+^{n_2 \times m};$$
$$\textit{for all } i \in \overline{l-1} \cup \{0\}, \textit{ for all } j \in \overline{q}' \cup \{0\}, \ l = 1, 2.$$

Then, the Weierstrass state-space realization (10)–(12) *is positive and the slow-dynamics reachable* (*and then the whole Weierstrass form is then weakly reachable*) *independent of external delays at any finite time* $t \geq T$, T *satisfying the constraint* $h'_q + h'_1 > T := h'_{q+1} > h'_q$ *for any initial conditions*, $\overline{\varphi} : [-h, 0] \to 0 \in \mathbf{R}^n$ *if the controllability-like Grammian*:

$$\left(\int_0^{T-h'_{q'}} \left(\sum_{i=0}^{j} Z_{w11}(h'_{j+1} - h'_i - \tau)\overline{B}_{i1}\right) \times \left(\sum_{i=0}^{j} Z_{w11}(h'_{j+1} - h'_i - \tau)\overline{B}_{i1}\right)^T d\tau\right)$$

is monomial with sizes of the external delays being strictly ordered according to $0 = h'_0 < h'_1 < \cdots < h'_q$ *with no loss in generality.*

Proof. Property (i) follows from Remark 6 since any complex block partitioned matrix with no less columns than rows

$$Q := \begin{bmatrix} S_1 & 0 & L_1 \\ 0 & S_2 & L_2 \end{bmatrix}$$

is full (row) rank if and only if the square matrix QQ^T is nonsingular. By defining as Q any of the two matrices in Property (i) the result follows directly. Property (ii) follows from Property (i) and the fact that $\lfloor sN - I_{n_2}, \Delta(\mu) \rfloor$ is full (row) rank for any complex nonzero s if and only if $[\Delta(\mu), \dots, N^{n_2 - 1}\Delta(\mu)]$ and also for $s = 0$ since rank $\lfloor -I_{n_2}, \Delta(\mu) \rfloor = n_2$.

Property (iii) follows from (12)–(13) and Theorem 5 since the current controllability-like Grammian is monomial. Thus, a control (20) with the corresponding parameterization and evolution operator redefinitions for the Weierstrass canonical form yield that all the relevant input and state derivatives in the righthand sides of (13) are nonnegative for all time from direct calculation with (20) together with the equations below arising from (13) $\overline{\varphi} : [-h, 0] \to 0 \in \mathbf{R}^n$,

$$\overline{x}_1(t) = \sum_{j=0}^{q'} \int_{h'_j}^{t} Z_{w11}(t-\tau)\overline{B}_{j1} u(\tau - h'_j)\, d\tau$$

$$\overline{x}_2(t) = -\sum_{i=0}^{l-1} N^i \Bigg(\sum_{j=0}^{q} \overline{A}_{j21} \overline{x}_1^{(i)}(t - h_j) + \sum_{j=1}^{q} \overline{A}_{j22} \overline{x}_2^{(i)}(t - h_j) + \sum_{j=0}^{q'} \overline{B}_{j2} u^{(i)}(t - h'_j) U(t) \Bigg)$$

and the evolution operator is positive. Under the remaining conditions given that the Weierstrass canonical realization is positive (both sub-states and output are nonnegative for all time under nonnegative controls for the given initial conditions), the slow sub-state is reachable. □

Note that Theorem 7 is not directly extendable to controllability under extra conditions in the same way as Theorem 6 extends Theorem 5. The reason is that the decoupling of slow and fast dynamics together with the requirement that the relevant control time-derivatives have to be all nonnegative for all time while the system is kept positive require that the dynamics of the slow dynamics not only be a Metzler matrix but also a positive one. Then, it is not a stability matrix and controllability of the slow dynamics to any arbitrary sub-state of the slow dynamics in open subsets of the first orthant that is not achievable in finite time if reachability holds.

IV. Examples.

Example 1. Consider the Weierstrass canonical state-space realization (10)–(12) of a singular solvable single-output system (1)–(2) with two inputs of fourth order being free of internal delay and with one single external delay $h_1' > 0$ whose parameterization is:

$$\overline{E} = \text{Block Diag}\,(I_2, N);\ \overline{A}_0 = \text{Block Diag}\,(\overline{A}_{011}, I_2);\ \overline{C} = \begin{pmatrix} c_1 & c_2 \end{pmatrix}^T$$

$$N = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \overline{A}_{011} = \begin{bmatrix} \lambda & g \\ 0 & \lambda \end{bmatrix}$$

$$\overline{B}_0 = \begin{pmatrix} \overline{B}_{01}^T & \overline{B}_{02}^T \end{pmatrix}^T, \qquad \overline{B}_1 = \begin{pmatrix} \overline{B}_{11}^T & \overline{B}_{12}^T \end{pmatrix}^T$$

$$\overline{B}_{01} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \overline{B}_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\overline{B}_{02} = \begin{pmatrix} -2 & 0 \\ -1 & 0 \end{pmatrix}, \qquad \overline{B}_{12} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix},$$

where $\lambda \in \mathbf{R}$, $c_i \geq 0$, $b_{ij} \geq 0$; $i, j \in \overline{2}$ with $\overline{C} \neq 0$ and $\overline{B}_{01} \neq 0$; and g is a nonnegative real constant being typically zero (namely, two Jordan blocks implying a diagonal matrix of dynamics associated with the repeated real eigenvalue λ) or unity (namely, one Jordan block associated with a repeated real eigenvalue λ). Note that $\operatorname{rank} \overline{E} = 3$ with the nilpotent matrix N being of unity rank and nilpotency index equal to 2. The algebraic multiplicity number of the impulsive modes is $l = \operatorname{rank} \overline{E} - \deg\left(\det\left(s\overline{E} - \overline{A}_0\right)\right) = 2$. The fast dynamics associated with the impulsive modes is positive for $\vec{x}_0 (> 0) \in \Im_I$

since then $\overline{x}_2 : \mathbf{R}_+ \to \mathbf{R}_+^{n_2}$ from Theorem 2, equation (13.c), since direct calculations yield: $-(N^i)\overline{B}_{j2} \geq 0$; $i, j = 0, 1$. Such a partial dynamics is also reachable independent of the external delay size since $\operatorname{rank}(N, \overline{B}_{02}, \overline{B}_{12}) = \operatorname{rank}(\overline{B}_{02}, \overline{B}_{12}, N\overline{B}_{02}, N\overline{B}_{12}) = 2$, see Theorem 7. For a finite time T, the discrete state-transition and control-transition matrices are given, respectively, by:

$$\Psi_g(\Gamma) = \begin{bmatrix} e^{\lambda T} & gTe^{\lambda T} \\ 0 & e^{\lambda T} \end{bmatrix}$$

$$\Gamma_g(T) = \\ \frac{1}{\lambda}\begin{bmatrix} b_{11}(e^{\lambda T}-1) + b_{21}g\left(e^{\lambda T}\left(T - \frac{1}{\lambda}\right) + \frac{1}{\lambda}\right) & b_{12}(e^{\lambda T}-1) + b_{22}g\left(e^{\lambda T}\left(T - \frac{1}{\lambda}\right) + \frac{1}{\lambda}\right) \\ e^{\lambda T} b_{21}(e^{\lambda T}-1) & e^{\lambda T} b_{22}(e^{\lambda T}-1) \end{bmatrix}.$$

In the case of two Jordan blocks, i.e., $g = 0$, the discrete system is both weakly positive and weakly strongly positive as well as weakly reachable through a constant nonnegative control vector function for any $\lambda \in (-\infty, -\varepsilon) \cup (\varepsilon, \infty)$, $T \in (\varepsilon, \infty)$, provided that $b_{ii} = 0$ for $i \in \overline{2}$ and $b_{ij} > 0$, $i, j(\neq i) \in \overline{2}$ and in the case that $b_{ii} > 0$ for $i \in \overline{2}$ and $b_{ij} = 0$, $i, j(\neq i) \in \overline{2}$. For the case $g = 1$, the system is weakly reachable if $b_{ii} > 0$ for $i \in \overline{2}$ and $b_{ij} = 0$ for $i, j(\neq i) \in \overline{2}$ since then the $(1,1)$ and $(2,2)$ entries of $\Gamma_1(T)$ are positive, its $(2,1)$ entry is zero while its $(1,2)$ entry is also zero provided that

$$\lambda = f(\lambda, T) := \frac{\mathbf{e}^{\lambda T} - 1}{T\mathbf{e}^{\lambda T}} : \mathbf{R} \times \mathbf{R}_+ \longrightarrow \mathbf{R}$$

since $g = 1$. If $\lambda > 0$, this always holds for some sufficiently large $T > 0$ since $0 < (\mathbf{e}^{\lambda T} - 1)/(T\mathbf{e}^{\lambda T}) \to 0$ as $T \to \infty$ so that there is a real solution to the constraint $\lambda = (\mathbf{e}^{\lambda T} - 1)/(T\mathbf{e}^{\lambda T})$. For $\lambda < 0$ such a finite time exists implying that the above constraint holds since $f(\lambda, 0) = 0$, $f(\lambda, \infty) := \lim_{T\to\infty} f(\lambda, \infty) = -\infty$ and $f(\lambda, T)$ is continuous on $\mathbf{R} \times \mathbf{R}_\varepsilon$ for all $\varepsilon \in \mathbf{R}_+$. In this last case, the discrete system is also asymptotically weakly controllable to any region $\mathbf{R}_\varepsilon^n := \{z \in \mathbf{R}_+^n : z_i \geq \varepsilon, \forall i \in \overline{n}\}$, for all $\varepsilon \in \mathbf{R}_+$. If, in addition to the above conditions, $\lambda < 0$, then the system is asymptotically weakly controllable to any region $\mathbf{R}_\varepsilon^n := \{z \in \mathbf{R}_+^n : z_i \geq \varepsilon, \forall i \in \overline{n}\}$, for all $\varepsilon \in \mathbf{R}_+$. If the parameterization changes to $A = \operatorname{Diag}(\lambda_1, \lambda_2)$, $\lambda_{1,2} \in \mathbf{R}$ (i.e., the two eigenvalues are real and distinct or they are equal with two Jordan blocks discussed above) then $\Psi_0(T) = \operatorname{Diag}((e^{\lambda_1 T}, e^{\lambda_2 T})$ and $\Gamma(T) = \Gamma_0(T)$, defined above with the replacements $\lambda \to \lambda_1$ and $\lambda \to \lambda_2$

in the first and second row vectors, respectively. The same conclusions about weak reachability and weak asymptotic controllability to $\mathbf{R}_{\varepsilon}^{n} := \{z \in \mathbf{R}_{+}^{n} : z_i \geq \varepsilon, \forall i \in \overline{n}\}$, for all $\varepsilon \in \mathbf{R}_{+}$, provided that $\lambda_i < 0$, $i \in \overline{2}$, as in the case $g = 0$.

Example 2 (Generalizations of Example 1). Assume that $\overline{B}_{11} = 0$ is replaced with any $\overline{B}_{11} > 0$. The system is still weakly positive and weakly reachable for any finite $T > h_1'$ which follows directly by considering the above nonnegative input which is zeroed for the time interval $[T - h_1', T)$ (see the proofs of Theorems 5 and 7). If the matrices G and H of the transformation to the canonical Weierstrass form are monomial, then their inverses are positive so that the original state-space realization (1)–(2) of the singular system also maintains the weak positivity property since $E = G^{-1}\overline{E}H^{-1} > 0$: $\mathbf{C} = \overline{\mathbf{C}}H^{-1} > 0$, etc., and also the weak reachability/weak controllability properties under similar conditions as in Example 1. Note that while reachability/controllability of the original state-space realization would hold for any nonsingular transformation matrices, i.e., for any equivalent realization, this is not the case for reachability/controllability of positive systems since both positivity and positivity maintaining reachability/controllability are dependent on the state-space realization, in general (see, for instance, [**10, 11, 18, 19**]).

V. Conclusion. This paper has dealt with the characterization of reachability and controllability properties of singular linear time-delay, time-invariant dynamic positive systems subject to constant known internal and external point delays. The research has been performed based either on a general nonunique solution trajectory including discontinuities at points of the initialization time-interval or based on a particular solution with initial conditions in an appropriate subspace of the whole potential admissible set which avoids such discontinuities and guarantees uniqueness. In the first case, the various properties obtained are characterized as weak (weak positivity, reachability, etc.) since they do not fully characterize singular systems of index strictly exceeding unity. In all of the manuscript, *reachability/controllability properties* refer to controllability from zero or among any given initial and final states, respectively, and furthermore, since applied on positive systems, they guarantee simultaneously some kind of state-trajectory positivity under nonnegative initial states and controls. In this second case, the positivity, reachability and controllability properties of the Weierstrass canonical state-space realization have also been investigated.

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