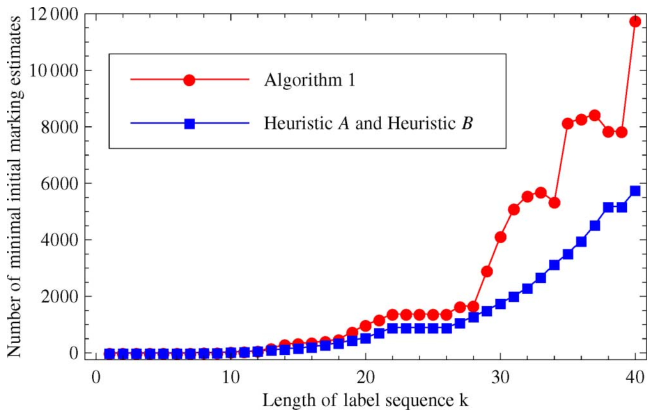

Fig. 5. Number of MIM estimates against the length of label sequence.

$Minimum - A_1$	$Minimum - H_A$	$Minimum - H_B$
$[7\ 6\ 0\ 1\ 1\ 0\ 3\ 0\ 0\ 0]^T$	$[7\ 6\ 0\ 1\ 1\ 0\ 3\ 0\ 0\ 0]^T$	$[7\ 6\ 0\ 2\ 0\ 0\ 2\ 1\ 0\ 0]^T$
$[7\ 6\ 0\ 2\ 0\ 0\ 2\ 1\ 0\ 0]^T$	$[7\ 6\ 0\ 2\ 0\ 0\ 2\ 1\ 0\ 0]^T$	$[7\ 6\ 0\ 2\ 0\ 0\ 3\ 0\ 0\ 0]^T$
$[7\ 6\ 0\ 2\ 0\ 0\ 3\ 0\ 0\ 0]^T$	$[7\ 6\ 0\ 2\ 0\ 0\ 3\ 0\ 0\ 0]^T$	$[7\ 6\ 0\ 2\ 0\ 1\ 2\ 0\ 0\ 0]^T$
$[7\ 6\ 0\ 2\ 0\ 1\ 2\ 0\ 0\ 0]^T$		
$[7\ 6\ 1\ 1\ 0\ 0\ 3\ 0\ 0\ 0]^T$		

Fig. 6. MIMs obtained by Algorithm 1 and Heuristics A and B.

The MIMs obtained by Algorithm 1, and Heuristics A and B following the observation of the sequence of labels ω are provided in Fig. 6 below (denoted by $Minimum - A_1$, $Minimum - H_A$, and $Minimum - H_B$, respectively). We observe that both Heuristics A and B are not able to find the complete set of MIMs, but are able to provide a reasonable approximation (they happen to provide a subset of the MIMs that are possible in this particular case) with faster running time. The minimum initial marking estimates shown in Fig. 6 can be viewed as the minimum number of resources required at initialization to accomplish the specific sequence of tasks (captured by ω).

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Stability of Positive Differential Systems With Delay

Pham Huu Anh Ngoc

***Abstract*—We first prove an explicit criterion for positive linear *time-varying* differential systems with distributed delay. Then some simple criteria for exponential stability of positive linear time-invariant differential systems with delay are presented. Finally, we extend obtained results to linear differential systems with *time-varying delay* and to *nonlinear* differential systems with delay. The results given in this technical note are extensions of some recent results in [7], [11], and [14].**

***Index Terms*—Exponential stability, positive system, time delay system.**

I. Introduction and Preliminaries

Delay differential equations have numerous applications in science and engineering. They are used as models for a variety of phenomena in the life sciences, physics and technology, chemistry, and economics, see, e.g., [5], [13].

A dynamical system is called *positive* if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative. In particular, a dynamical system with state space $\mathbb{R}^n$ is positive if any trajectory of the system starting at an initial state in $\mathbb{R}^n_+$ remains forever in $\mathbb{R}^n_+$ for all nonnegative inputs. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative. This model class is used in many areas such as economics, populations dynamics and ecology, see, e.g., [6], [8].

Recently, problems of stability of (positive) differential systems with delay have attracted much attention from researchers, see, e.g., [2], [7], [8], [10], [11]–[14], [16] and references therein. In this technical note, by exploiting positivity, we present some explicit criteria for exponential stability of (linear, nonlinear) differential systems with (time-varying) delays.

Let $\mathbb{N}$ be the set of all natural numbers. For given $m \in \mathbb{N}$, let us denote $\underline{m} := \{1, 2, \ldots, m\}$ and $\underline{m}_0 := \{0, 1, 2, \ldots, m\}$. Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ where $\mathbb{C}$ and $\mathbb{R}$ denote the sets of all complex and all real numbers, respectively. Set $\mathbb{C}_+ := \{z \in \mathbb{C} : \Re z \geq 0\}$. For integers $l, q \geq 1$, $\mathbb{K}^l$ denotes the $l-$dimensional vector space over $\mathbb{K}$ and $\mathbb{K}^{l \times q}$ stands for the set of all $l \times q$-matrices with entries in $\mathbb{K}$. Inequalities

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between real matrices or vectors will be understood componentwise, i.e. for two real matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{R}^{l\times q}$, we write $A \geq B$ iff $a_{ij} \geq b_{ij}$ for $i = 1,\cdots,l$, $j = 1,\cdots,q$. We denote by $\mathbb{R}_+^{l\times q}$ the set of all nonnegative matrices $A \geq 0$. Similar notations are adopted for vectors. For $x \in \mathbb{K}^n$ and $P \in \mathbb{K}^{l\times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$. A norm $\|\cdot\|$ on $\mathbb{K}^n$ is said to be *monotonic* if $\|x\| \leq \|y\|$ whenever $x, y \in \mathbb{K}^n, |x| \leq |y|$. Every p-norm on $\mathbb{K}^n$ ($\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}$, $1 \leq p < \infty$ and $\|x\|_\infty = \max_{i=1,2,\ldots,n}|x_i|$), is monotonic. Throughout the technical note, if otherwise not stated, the norm of a matrix $P \in \mathbb{K}^{l\times q}$ is understood as its operator norm associated with a given pair of monotonic vector norms on $\mathbb{K}^l$ and $\mathbb{K}^q$, that is $\|P\| = \max\{\|Py\| : \|y\| = 1\}$. Note that $|P\| \leq \||P|\|| \leq \|Q\|$, provided $P \in \mathbb{K}^{l\times q}$, $Q \in \mathbb{R}_+^{l\times q}$, $|P| \leq Q$, see e.g. [17]. For any matrix $M \in \mathbb{R}^{n\times n}$ the *spectral abscissa* of M is denoted by $\mu(M) = \max\{\Re\lambda : \lambda \in \sigma(M)\}$, where $\sigma(M) := \{z \in \mathbb{C} : \det(zI_n - M) = 0\}$ is the spectrum of M.

A matrix $M \in \mathbb{R}^{n\times n}$ is called a *Metzler matrix* if all off-diagonal elements of M are nonnegative. We now summarize in the following theorem some properties of Metzler matrices which will be used in what follows.

Theorem I.1. ([1], [17]): Suppose that $M \in \mathbb{R}^{n\times n}$ is a Metzler matrix. Then

(i) (*Perron-Frobenius*) $\mu(M)$ is an eigenvalue of M and there exists a nonnegative eigenvector $x \neq 0$ such that $Mx = \mu(M)x$.
(ii) Given $\alpha \in \mathbb{R}$, there exists a nonzero vector $x \geq 0$ such that $Mx \geq \alpha x$ if and only if $\mu(M) \geq \alpha$.
(iii) $(tI_n - M)^{-1}$ exists and is nonnegative if and only if $t > \mu(M)$.
(iv) Given $B \in \mathbb{R}_+^{n\times n}$, $C \in \mathbb{C}^{n\times n}$. If $|C| \leq B$ then $\mu(M + C) \leq \mu(M + B)$.

The following is immediate from Theorem I.1.

Theorem I.2: Let $M \in \mathbb{R}^{n\times n}$ be a Metzler matrix. The following statements are equivalent.

(i) $\mu(M) < 0$;
(ii) $Mp \ll 0$ for some $p \in \mathbb{R}_+^n$;
(iii) M is invertible and $M^{-1} \leq 0$;
(iv) For given $b \in \mathbb{R}^n$, $b \gg 0$, there exists $x \in \mathbb{R}_+^n$, such that $Mx + b = 0$;
(v) For any $x \in \mathbb{R}_+^n \setminus \{0\}$, the row vector $x^T M$ has at least one negative entry.

Let $\mathbb{K}^n$ be endowed with the norm $\|\cdot\|$ and let $C([\alpha,\beta],\mathbb{K}^n)$ be the Banach space of all continuous functions on $[\alpha,\beta]$ with values in $\mathbb{K}^n$ normed by the maximum norm $\|\phi\| = \max_{\theta\in[\alpha,\beta]}\|\phi(\theta)\|$. Let J be an interval of $\mathbb{R}$. For a matrix-valued function $\phi(\cdot) : J \to \mathbb{R}^{m\times n}$, we say that ϕ is nonnegative and write $\phi \geq 0$ if $\phi(\theta) \geq 0$ for all $\theta \in J$.

II. Explicit Criteria for Positive Linear Differential Systems With Delay

Consider a linear time-varying differential system with delay of the form

$$\dot{x}(t) = A_0(t)x(t) + \sum_{i=1}^{m} A_i(t)x(t-h_i) + \int_{-h}^{0} B(t,s)x(t+s)ds, \qquad t \geq \sigma \quad (1)$$

where $A_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ ($i \in \underline{m}_0$) and $B(\cdot,\cdot) : \mathbb{R}_+ \times [-h,0] \to \mathbb{R}^n$ are given continuous matrix-valued functions and $0 < h_1 < h_2 < \ldots < h_m \leq h$ are given real numbers.

It is well known that for fixed $\sigma \geq 0$ and given $\phi \in C([-h,0],\mathbb{R}^n)$, (1) has a unique solution satisfying the initial value condition

$$x(s+\sigma) = \phi(s), \quad s \in [-h,0] \quad (2)$$

see, e.g., [9]. This solution is denoted by $x(\cdot;\sigma,\phi)$. Especially, we write $x(\cdot;\phi)$ instead of $x(\cdot;0,\phi)$.

Definition II.1: The system (1) is said to be positive if

$$\forall\sigma \geq 0, \quad \forall\phi \in C([-h,0],\mathbb{R}^n), \quad \phi \geq 0 \implies \forall t \geq \sigma : x(t;\sigma,\phi) \geq 0.$$

We are now in the position to state the main result of this section.

Theorem II.2: The system (1) is positive if, and only if:

(i) $A_0(t)$ is a Metzler matrix for each $t \in \mathbb{R}_+$ and
(ii) $A_i(t)$ is nonnegative for each $i \in \underline{m}$ and each $t \in \mathbb{R}_+$ and
(iii) $B(t,s)$ is nonnegative for each $(t,s) \in \mathbb{R}_+ \times [-h,0]$.

In particular, when A_i ($i \in \underline{m}_0$) are constant matrices and $B(\cdot,\cdot) \equiv 0$, Theorem II.2 reduces to a well-known criterion for positive linear differential systems with discrete delays

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} A_ix(t-h_i), \quad t \geq 0 \quad (3)$$

see e.g. [7]. However, to the best of our knowledge, Theorem II.2 is new at least to the case of time-varying systems. The proof of Theorem II.2 is given in Appendix.

III. Explicit Criteria for Exponential Stability of Positive Linear Time-Invariant Systems With Delay

In this section, we deal with the problem of exponential stability of linear differential systems with delay of the form

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} A_ix(t-h_i) + \int_{-h}^{0} B(s)x(t+s)ds, \quad t \geq 0 \quad (4)$$

where A_i ($i \in \underline{m}_0$) are given matrix and $B(\cdot) : [-h,0] \to \mathbb{R}^{n\times n}$ is a given continuous matrix-valued function and $0 < h_1 < h_2 < \ldots < h_m \leq h$, are given real numbers.

Let $\phi \in C([-h,0],\mathbb{R}^n)$ be given and let $x(\cdot;\phi)$ be the solution of (4) satisfying the initial condition $x(s) = \phi(s)$, $s \in [-h,0]$. Then (4) is said to be exponentially stable if, and only if, there are positive numbers α, M such that

$$\forall t \geq 0, \quad \forall\phi \in C([-h,0],\mathbb{R}^n) : \|x(t;\phi)\| \leq Me^{-\alpha t}\|\phi\|. \quad (5)$$

Furthermore, it is well known that (4) is exponentially stable if, and only if,

$$\det\left(zI_n - A_0 - \sum_{i=1}^{m} A_ie^{-h_iz} - \int_{-h}^{0} e^{zs}B(s)ds\right) \neq 0, \quad \forall z \in \mathbb{C}_+,$$

see e.g. [9, Ch. 7].

We are now in the position to prove the main result of this section.

Theorem III.1: Let (4) be positive. Then the following statements are equivalent

(i) (4) is exponentially stable;
(ii) $\mu(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds) < 0$;
(iii) $(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds)p \ll 0$ for some $p \in \mathbb{R}_+^n$;
(iv) $A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds$, is invertible and $(\sum_{i=0}^m A_i + \int_{-h}^0 B(s)ds)^{-1} \leq 0$;
(v) For given $r \in \mathbb{R}_+^n$, $r \gg 0$, there exists $p \in \mathbb{R}_+^n$ such that

$$\left(A_0 + \sum_{i=1}^{m} A_i + \int_{-h}^{0} B(s)ds\right)p + r = 0;$$

(vi) For any $x \in \mathbb{R}^n_+ \setminus \{0\}$, the row vector $x^T(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds)$ has at least one negative entry.

In particular, when $B(\cdot) \equiv 0$, Theorem III.1 (iv) reduces to a well-known stability criterion for positive linear differential systems with discrete delays (3), see e.g. [7], [14]. Our proof given below is based on spectral properties of Metzler matrices while the Lyapunov function method was utilized in [7].

Proof: We first show that (i) ⇔ (ii). Let $\mu(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds) < 0$. Assume on the contrary that (4) is not exponentially stable. Thus, $\det(z_0 I_n - A_0 - \sum_{i=1}^m A_i e^{-h_i z_0} - \int_{-h}^0 e^{z_0 s} B(s)ds) = 0$, for some $z_0 \in \mathbb{C}_+$. This implies $0 \le \Re z_0 \le \mu(A_0 + \sum_{i=1}^m A_i e^{-h_i z_0} + \int_{-h}^0 e^{z_0 s} B(s)ds)$. Since (4) is positive, A_0 is a Metzler matrix and $A_i \ge 0, i \in \underline{m}$ and $B(s) \ge 0$ for any $s \in [-h, 0]$. It follows that $|A_i e^{-h_i z_0}| \le A_i$, for any $i \in \underline{m}$, and $|\int_{-h}^0 e^{z_0 s} B(s)ds| \le \int_{-h}^0 |e^{z_0 s} B(s)|ds \le \int_{-h}^0 B(s)ds$. Then Theorem I.1 (iv) gives $0 \le \mu(A_0 + \sum_{i=1}^m A_i e^{-h_i z_0} + \int_{-h}^0 e^{z_0 s} B(s)ds) \le \mu(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds)$, which is a contradiction.

Conversely, suppose (4) is exponentially stable. Assume on the contrary that $\mu(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds) \ge 0$. Consider the continuous function $f(t) := t - \mu(A_0 + \sum_{i=1}^m e^{-th_i} A_i + \int_{-h}^0 e^{ts} B(s)ds)$, $t \ge 0$. Since $f(0) = -\mu(A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds) \le 0$ and $\lim_{t\to\infty} f(t) = \infty$, it follows that $f(t_0) = 0$ for some $t_0 \ge 0$. That is, $t_0 = \mu(A_0 + \sum_{i=1}^m e^{-t_0 h_i} A_i + \int_{-h}^0 e^{t_0 s} B(s)ds)$. As A_0 is a Metzler matrix and $A_i \ge 0, i \in \underline{m}$ and $B(s) \ge 0$ for any $s \in [-h, 0]$, $A_0 + \sum_{i=1}^m e^{-t_0 h_i} A_i + \int_{-h}^0 e^{t_0 s} B(s)ds$ is a Metzler matrix. By Theorem I.1 (i), $\det(t_0 I_n - A_0 - \sum_{i=1}^m A_i e^{-h_i t_0} - \int_{-h}^0 e^{t_0 s} B(s)ds) = 0$. Thus, (4) is not exponentially stable. This is a contradiction which completes the proof of (i) ⇔ (ii).

Finally, since $A_0 + \sum_{i=1}^m A_i + \int_{-h}^0 B(s)ds$ is a Metzler matrix, the implications (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) ⇔ (vi), follow directly from Theorem I.2. This completes the proof.

IV. Extensions

In this section, we give some extensions of Theorem III.1 to linear differential systems with *time-varying delay* and to *nonlinear differential systems with delay*.

A. Stability of Linear Differential Systems With Time-Varying Delay

Consider a linear differential system with time-varying delay of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^m A_k x(t - h_k(t)) + \int_{-h(t)}^0 B(s)x(t+s)ds \quad (6)$$

where $t \ge 0$ and $h(\cdot), h_k(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ $(k \in \underline{m})$ are given continuous functions such that

$$0 < h(t) \le h, \quad 0 < h_k(t) \le h_k, \quad \forall t \ge 0, \quad \forall k \in \underline{m}, \quad (7)$$

for some positive numbers h, h_k $(k \in \underline{m})$, $h \ge \max_{k\in\underline{m}}\{h_k\}$ and $A_k \in \mathbb{R}^{n\times n}$ $(k \in \underline{m}_0)$ are given matrices and $B(\cdot) : [-h, 0] \to \mathbb{R}^{n\times n}$ is a given continuous matrix-valued function.

Recall that by definition, (6) is exponentially stable if, and only if, (5) holds where $x(\cdot, \phi)$ is the unique solution of (6) satisfying the initial condition $x(s) = \phi(s)$, $s \in [-h, 0]$.

Theorem IV.1: Suppose that A_0 is a Metzler matrix and $A_k \in \mathbb{R}^{n\times n}_+$ for each $k \in \underline{m}$ and $B(s) \in \mathbb{R}^{n\times n}_+$ for each $s \in [-h, 0]$. Then (6) is exponentially stable for any delays satisfying (7) if, and only if, one of the equivalent conditions (i)–(vi) stated in Theorem III.1 holds.

In particular, when $B(\cdot) \equiv 0$, Theorem IV.1 reduces to a very recent result in [14, Theorem 1]. The proof of Theorem IV.1 given below is different from that of Theorem 1 in [14]. More precisely, the proof given in [14] is quite complicated. Here we give a much more simple proof which also works for some different (general) classes of differential systems with delay.

Proof: Since A_0 is a Metzler matrix and $A_k \ge 0, k \in \underline{m}$ and $B(s) \ge 0, s \in [-h, 0]$, $A_0 + \sum_{k=1}^m A_k + \int_{-h}^0 B(s)ds$ is a Metzler matrix. So any two of conditions (i)–(vi) stated in Theorem III.1 are equivalent. It remains to show that (6) is exponentially stable for any delays satisfying (7) if, and only if, (iv) of Theorem III.1 holds.

Let $\phi \ge 0$ and let $x(\cdot;\phi)$ be the solution of (6) satisfying the initial value condition $x(t) = \phi(t), t \in [-h, 0]$. A similar argument as in the proof of the sufficient condition of Theorem II.2 shows that $x(t;\phi) \ge 0$, $\forall t \ge 0$. Suppose

$$\left(A_0 + \sum_{k=1}^m A_k + \int_{-h}^0 B(s)ds\right) p \ll 0, \quad (8)$$

for some $p \in \mathbb{R}^n_+$. By continuity, (8) holds for some $p \in \mathbb{R}^n_+$, $p \gg 0$. Let $p := (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, $\alpha_i > 0, \forall i \in \underline{n}$. Furthermore, (8) implies that

$$\left(A_0 + \sum_{k=1}^m e^{\beta h} A_k + \int_{-h}^0 e^{-\beta s} B(s)ds\right) p \ll -\beta(\alpha_1, \ldots, \alpha_n)^T, \quad (9)$$

for some $\beta > 0$ sufficiently small.

Let $\phi \in C([-h, 0], \mathbb{R}^n)$, $\phi \ge 0$ and $\|\phi\| \le 1$. Then there exists $K > 0$ such that $\phi(t) \ll Ke^{-\beta t}p$ for any $t \in [-h, 0]$ and for any $\phi \ge 0$, $\|\phi\| \le 1$. Define $u(t) := Ke^{-\beta t}p$, $t \in [-h, \infty)$. We claim that $x(t) := x(t;\phi) \le u(t)$ for all $t > 0$. Assume on the contrary that there exists $t_0 > 0$ such that $x(t_0) \not\le u(t_0)$. Set $t_1 := \inf\{t > 0 : x(t) \not\le u(t)\}$. By continuity, $t_1 > 0$ and there is $i_0 \in \underline{n}$ such that

$$\begin{aligned} &x(t) \le u(t), \quad \forall t \in [0, t_1); \quad x_{i_0}(t_1) = u_{i_0}(t_1), \\ &x_{i_0}(t) > u_{i_0}(t), \quad \forall t \in (t_1, t_1 + \epsilon) \end{aligned} \quad (10)$$

for some $\epsilon > 0$ sufficiently small. On the other hand, we have for every $i \in \underline{n}$

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij}^{(0)} x_j(t) + \sum_{k=1}^m \sum_{j=1}^n a_{ij}^{(k)} x_j(t - h_k(t)) + \sum_{j=1}^n \int_{-h(t)}^0 b_{ij}(s) x_j(t+s)ds$$

where $A_0 := (a_{ij}^{(0)})$, $A_k := (a_{ij}^{(k)})$, $k \in \underline{m}$, and $B(\cdot) := (b_{ij}(\cdot))$. Since A_0 is a Metzler matrix and $A_k \geq 0$ for each $k \in \underline{m}$ and $B(s) \geq 0$, $s \in [-h, 0]$, the first inequality and the equality in (10) imply that

$$\begin{aligned}\dot{x}_{i_0}(t_1) &\overset{(7)}{\leq} \sum_{j=1}^{n} a_{i_0 j}^{(0)} (K e^{-\beta t_1} \alpha_j) \\ &\quad + \sum_{k=1}^{m} \sum_{j=1}^{n} a_{i_0 j}^{(k)} (K e^{-\beta t_1} \alpha_j) e^{h\beta} \\ &\quad + \sum_{j=1}^{n} \int_{-h}^{0} b_{i_0 j}(s) (K e^{-\beta t_1} \alpha_j e^{-\beta s}) ds \\ &= K e^{-\beta t_1} \left(\sum_{j=1}^{n} a_{i_0 j}^{(0)} \alpha_j + \sum_{k=1}^{m} \sum_{j=1}^{n} a_{i_0 j}^{(k)} \alpha_j e^{h\beta} \right. \\ &\qquad \left. + \sum_{j=1}^{n} \int_{-h}^{0} b_{i_0 j}(s) \alpha_j e^{-\beta s} ds \right) \\ &\overset{(9)}{<} K e^{-\beta t_1} (-\beta \alpha_{i_0}) = \dot{u}_{i_0}(t_1).\end{aligned}$$

However, this conflicts with (10). Thus, for any $\phi \in C([-h, 0], \mathbb{R}^n)$, $\|\phi\| \leq 1$, $\phi \geq 0$, we have $0 \leq x(t;\phi) \leq K e^{-\beta t} p, \forall t \geq 0$. Therefore, there exists $K_1 > 0$ such that $\|x(t;\phi)\| \leq K_1 e^{-\beta t}$, $\forall t \geq 0$, for any $\phi \in C([-h, 0], \mathbb{R}^n)$, $\|\phi\| \leq 1$, $\phi \geq 0$. By the linearity of (6), $\|x(t;\phi)\| \leq K_1 e^{-\beta t} \|\phi\|$, $\forall t \geq 0$, $\forall \phi \in C([-h, 0], \mathbb{R}^n)$, $\phi \geq 0$. For given $\phi \in C([-h, 0], \mathbb{R}^n)$, it can be decomposed as $\phi = \phi^+ - \phi^-$ where $\phi^+, \phi^- \in C([-h, 0], \mathbb{R}^n)$ and $\phi^+ \geq 0$, $\phi^- \geq 0$. Finally, we have $\|x(t;\phi)\| \leq K_2 e^{-\beta t} \|\phi\|$, $\forall t \geq 0$, $\forall \phi \in C([-h, 0], \mathbb{R}^n)$, for some $K_2 > 0$, by the linearity of (6). Hence (6) is exponentially stable.

Conversely, if (6) is exponentially stable for any delays satisfying (7) then in particular it is exponentially stable with $h_k(t) := h$, $t \geq 0$ for each $k \in \underline{m}$. Then, (iv) holds by Theorem III.1. This completes the proof.

We now extend Theorems III.1, IV.1 to the case in which systems are not necessarily positive.

Theorem IV.2: Let $A_0 = (a_{ij}^{(0)}) \in \mathbb{R}^{n \times n}$ and let

$$\begin{aligned}M &:= diag\left(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)}\right) \\ &\quad + \left| A_0 - diag\left(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)}\right) \right| + \sum_{k=1}^{m} |A_k| + \int_{-h}^{0} |B(s)| \, ds.\end{aligned}$$

If M satisfies one of the equivalent conditions (i)–(v) of Theorem I.2 then (6) is exponentially stable for any delays satisfying (7).

Proof: The proof is almost the same as that of Theorem IV.1. Since M is a Metzler matrix, any two of (i)–(v) of Theorem I.2 are equivalent. As showed in the proof of Theorem IV.1, there exist $p := (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \in \mathbb{R}_+^n$, $p \gg 0$ and $\beta > 0$ such that

$$\begin{aligned}&\left(diag\left(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)}\right) + \left| A_0 - diag\left(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)}\right) \right| \right. \\ &\qquad \left. + \sum_{k=1}^{m} e^{\beta h} |A_k| + \int_{-h}^{0} e^{-\beta s} \, |B(s)| \, ds \right) p \ll -\beta (\alpha_1, \ldots, \alpha_n)^T.\end{aligned}$$

Let $\phi \in C([-h, 0]; \mathbb{R}^n)$, $\phi \geq 0$ and $\|\phi\| \leq 1$. Then there exists $K > 0$ such that $|\phi(t)| \ll K e^{-\beta t} p$ for any $t \in [-h, 0]$ and for any $\phi \geq 0$, $\|\phi\| \leq 1$. Define $u(t) := K e^{-\beta t} p$, $t \in [-h, \infty)$. Let $x(t) := x(t;\phi)$, $t \in \mathbb{R}_+$. We claim that $|x(t)| \leq u(t)$ for all $t > 0$. For every $i \in \underline{n}$, we have

$$\begin{aligned}\frac{d}{dt} |x_i(t)| &= sgn\,(x_i(t))\, \dot{x}_i(t) \leq a_{ii}^{(0)} \, |x_i(t)| \\ &\quad + \sum_{j=1, j \neq i}^{n} \left| a_{ij}^{(0)} \right| |x_j(t)| \\ &\quad + \sum_{k=1}^{m} \sum_{j=1}^{n} \left| a_{ij}^{(k)} \right| |x_j\,(t - h_k(t))| \\ &\quad + \sum_{j=1}^{n} \int_{-h}^{0} |b_{ij}(s)| \, |x_j(t+s)| \, ds,\end{aligned}$$

for almost any $t \in \mathbb{R}_+$. Therefore,

$$\begin{aligned}D^+ |x_i(t)| &= \limsup_{h \to 0^+} \frac{|x_i(t+h)| - |x_i(t)|}{h} \\ &= \limsup_{h \to 0^+} \frac{1}{h} \int_{t}^{t+h} \frac{d}{ds} |x_i(s)| \, ds \leq a_{ii}^{(0)} \, |x_i(t)| \\ &\quad + \sum_{j=1, j \neq i}^{n} \left| a_{ij}^{(0)} \right| |x_j(t)| \\ &\quad + \sum_{k=1}^{m} \sum_{j=1}^{n} \left| a_{ij}^{(k)} \right| |x_j\,(t - h_k(t))| \\ &\quad + \sum_{j=1}^{n} \int_{-h}^{0} |b_{ij}(s)| \, |x_j(t+s)| \, ds,\end{aligned}$$

where D^+ denotes the Dini upper-right derivative. By the same arguments as in the proof of Theorem IV.1 with A_0, A_i, $B(\cdot)$ and $x(\cdot)$ being replaced by $diag(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)}) + |A_0 - diag(a_{11}^{(0)}, a_{22}^{(0)}, \ldots, a_{nn}^{(0)})|$, $|A_i|$, $|B(\cdot)|$ and $|x(\cdot)|$, respectively, (11) yields $D^+ |x_{i_0}(t_1)| < D^+ u_{i_0}(t_1)$. However, this conflicts with (10). The remainder of the proof is the same as that of Theorem IV.1. This completes the proof.

We illustrate Theorem IV.2 by a simple example.

Example IV.3: Consider a linear differential system with time-varying delay in $\mathbb{R}^2$ given by

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h_1(t)) + \int_{-h(t)}^{0} B(\tau) x(t+\tau) d\tau, \quad t \geq 0, \quad (12)$$

where

$$A_0 = \begin{pmatrix} -3 & -\frac{1}{2} \\ 0 & -3 \end{pmatrix}; \qquad A_1 = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix};$$
$$B(s) = \begin{pmatrix} s & 0 \\ -2s & s \end{pmatrix}, \qquad s \leq 0$$

and $h_1(\cdot), h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ are given continuous functions. For $h > 0$, let us define

$$\begin{aligned} M &:= diag(-3,-3) + |A_0 - diag(-3,-3)| + |A_1| \\ &\quad + \int_{-h}^{0} |B(s)|ds \\ &= \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{h^2}{2} & 0 \\ h^2 & \frac{h^2}{2} \end{pmatrix} \\ &= \begin{pmatrix} -3 + \frac{h^2}{2} & \frac{5}{2} \\ h^2 & -3 + \frac{h^2}{2} \end{pmatrix}. \end{aligned}$$

It follows that $\mu(M) = (1/2)h^2 + (\sqrt{10}/2)h - 3$. Thus (12) is exponentially stable provided $\mu(M) < 0$, $0 < h_1(t) \leq h$, $0 < h(t) \leq h$, $\forall t \geq 0$, or equivalently, $0 < h < ((\sqrt{34} - \sqrt{10})/2)$, $0 < h_1(t) \leq h$, $0 < h(t) \leq h$, $\forall t \geq 0$, by Theorem IV.2.

B. Stability of Nonlinear Time-Delay Differential Systems

Consider a nonlinear differential system with time-varying delays of the form

$$\dot{x}(t) = Ax(t) + F\left(t; x(t - h_1(t)), \ldots, x(t - h_m(t)), \int_{-h(t)}^{0} B(s)x(t+s)ds\right), \quad t \geq \sigma \geq 0, \quad (13)$$

where

(i) $h_k(\cdot), h(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, $k \in \underline{m}$, are given continuous functions such that $0 \leq h_k(t) \leq h_k$, $0 < h(t) \leq h$, $h \geq h_k$ $\forall k \in \underline{m}$, for some positive numbers h, h_k, $k \in \underline{m}$;

(ii) $A \in \mathbb{R}^{n \times n}$ is given and $B(\cdot) : [-h, 0] \to \mathbb{R}^{n \times n}$ are given continuous functions.

(iii) $F(\cdot\ \ \cdot, \ldots, \cdot) : \mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}^{(m+1)\ times} \to \mathbb{R}^n$, is a given continuous function such that $F(t; 0, \ldots, 0) = 0$, $\forall t \geq 0$ and $F(t; u_1, u_2, \ldots, u_{m+1})$ is (locally) Lipschitz continuous with respect to $u_1, u_2, \ldots, u_{m+1}$ in each compact subset of $\mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}^{(m+1)\ times}$.

Then (i), (ii) and (iii) imply that for a fixed $\sigma \geq 0$ and a given $\phi \in C([-h, 0], \mathbb{R}^n)$, there exists a unique local solution of (13) satisfying the initial condition

$$x(s + \sigma) = \phi(s), \quad s \in [-h, 0]. \quad (14)$$

This solution is continuous on $[\sigma - h, \gamma)$ for some $\gamma > \sigma$ and satisfies (13) for every $t \in [\sigma, \gamma)$ see e.g. [4], [9]. It is denoted by $x(\cdot; \sigma, \phi)$. Furthermore, if the interval $[-h, \gamma)$ is the maximum interval of existence of the solution $x(\cdot; \sigma, \phi)$ then $x(\cdot; \sigma, \phi)$ is said to be *noncontinuable*. The existence of a noncontinuable solution follows from Zorn's lemma and the maximum interval of existence must be open.

Definition IV.4:

(i) The zero solution of (13) is said to be locally exponentially stable if there exist positive numbers r, K, β such that for each $\sigma \in \mathbb{R}_+$ and each $\phi \in C([-h, 0], \mathbb{R}^n)$ with $\|\phi\| \leq r$, the solution $x(\cdot; \sigma, \phi)$ of (13), (14) exists on $[\sigma - h, \infty)$ and furthermore satisfies
$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}, \quad \forall t \geq \sigma.$$

(ii) The zero solution of (13) is said to be globally exponentially stable if there exist positive numbers K, β such that for each $\sigma \in \mathbb{R}_+$ and each $\phi \in C([-h, 0], \mathbb{R}^n)$, the solution $x(\cdot; \sigma, \phi)$ of (13), (14) exists on $[\sigma - h, \infty)$ and furthermore satisfies
$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}\|\phi\|, \quad \forall t \geq \sigma.$$

When the zero solution of (13) is locally exponentially stable, globally exponentially stable then we also say that (13) is locally exponentially stable, globally exponentially stable, respectively.

Theorem IV.5: Suppose there exist $A_1, A_2, \ldots, A_{m+1} \in \mathbb{R}_+^{n \times n}$ so that

$$|F(t; u_1, \ldots, u_{m+1})| \leq \sum_{k=1}^{m+1} A_k |u_k|, \quad (15)$$

for all $t \geq 0$, $u_1, \ldots, u_{m+1} \in \mathbb{R}^n$. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and let

$$M := diag(a_{11}, a_{22}, \ldots, a_{nn}) + |A - diag(a_{11}, a_{22}, \ldots, a_{nn})| + \sum_{k=1}^{m} A_k + \int_{-h}^{0} A_{m+1}\, |B(s)|\, ds.$$

If M satisfies one of the equivalent conditions (i)–(v) of Theorem I.2 then (13) is locally exponentially stable. In addition, if the function F is positive homogeneous of degree one with respect to $u_1, u_2, \ldots, u_{m+1}$, that is, $F(t; \alpha u_1, \ldots, \alpha u_{m+1}) = \alpha F(t; u_1, \ldots, u_{m+1})$, for any $\alpha \geq 0$, $t \geq 0$, $u_1, u_2, \ldots, u_{m+1} \in \mathbb{R}^n$, then (13) is globally exponentially stable.

Remark IV.6: It is important to note that if $F(t; u_1, u_2, \ldots, u_{m+1})$ is (globally) Lipschitz continuous with respect to $u_1, u_2, \ldots, u_{m+1}$ in $\mathbb{R}_+ \times \overbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}^{(m+1)\ times}$ and $F(t; 0, 0, \ldots, 0) = 0$, $\forall t \geq 0$, then (15) holds automatically for some $A_k \in \mathbb{R}_+^{n \times n}$, $k \in \underline{m+1}$.

Proof: Since M is a Metzler matrix, any two of (i)–(v) of Theorem I.2 are equivalent. We first show that (13) is locally exponentially stable provided (iv) of Theorem I.2 holds. Let $\phi \in C([-h, 0], \mathbb{R}^n)$ be given and let $x(t) := x(t; \sigma, \phi)$, $t \in [\sigma - h, \gamma)$ be a noncontinuable solution of (13), (14). Taking (15) into account, by a similar argument as in the proof of Theorem IV.2, we can show that there exists $\beta > 0$ such that for any $\sigma \geq 0$ and any $r > 0$ and any $\phi \in C([-h, 0], \mathbb{R}^n)$ with $\|\phi\| \leq r$,

$$\|x(t; \sigma, \phi)\| \leq Ke^{-\beta(t-\sigma)}, \quad \forall t \in [\sigma, \gamma), \quad (16)$$

where $K > 0$ only depends on β, r. We claim that $\gamma = \infty$ and thus (13) is locally exponentially stable. Seeking a contradiction, we assume that $\gamma < \infty$. Then it follows from (16) that $x(\cdot) := x(\cdot; \sigma, \phi)$ is bounded on $[\sigma, \gamma)$. Furthermore, this together with (13) and (15) imply that $\dot{x}(\cdot)$ is bounded on $[\sigma, \gamma)$. Thus $x(\cdot)$ is uniformly continuous on $[\sigma, \gamma)$. Therefore, $\lim_{t \to \gamma^-} x(t)$ exists and $x(\cdot)$ can be extended to a continuous function on $[\sigma, \gamma]$. Moreover, the closure of $\{x_t : t \in [\sigma, \gamma)\}$ is a compact set in $C([-h, 0], \mathbb{R}^n)$, by Arzéla-Ascoli theorem [3]. Note that $\{(t, x_t) : t \in [\sigma, \gamma)\} \subset [\sigma, \gamma] \times$ the closure of $\{x_t : t \in [\sigma, \gamma)\}$. Thus, the closure of $\{(t, x_t) : t \in [\sigma, \gamma)\}$ is a compact set in $\mathbb{R}_+ \times C([-h, 0], \mathbb{R}^n)$. Since (γ, x_γ) belongs to this compact set, one can find a solution of (13) through this point to the right of γ. This contradicts the noncontinuability hypothesis on $x(\cdot)$. Thus γ must be equal to ∞.

Finally, we show that (13) is globally exponentially stable provided F is positive homogeneous of degree one with respect to $u_1, u_2 \ldots, u_{m+1}$. Let $\phi \in C([-h, 0], \mathbb{R}^n)$ be given. Since F is positive homogeneous of degree one with respect to $u_1, u_2 \ldots, u_{m+1}$, it follows that $(1/\|\phi\|)x(\cdot;\sigma,\phi)$ is the unique solution of (13) satisfying the initial condition $x(t+\sigma) = (1/\|\phi\|)\phi(t), t \in [-h,0]$. Since $(1/\|\phi\|)\phi \in C([-h,0],\mathbb{R}^n)$ and $\|(1/\|\phi\|)\phi\| = 1$, we have $\|(1/\|\phi\|)x(t;\sigma,\phi)\| \leq Ke^{-\beta(t-\sigma)}$, $\forall t \geq \sigma$, or equivalently, $\|x(t;\sigma,\phi)\| \leq Ke^{-\beta(t-\sigma)}\|\phi\|$, $\forall t \geq \sigma$. Here K,β are independent with σ,ϕ and thus (13) is globally exponentially stable. This completes the proof.

APPENDIX

Proof of Theorem II.2: Assume that (1) is positive. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Fix $k \in \mathbb{N}$ and $j \in \underline{n}$. Define

$$\phi_k(s) = \begin{cases} e_j, & \text{if } s = 0; \\ (ks+1)e_j, & \text{if } s \in (-1/k, 0); \\ 0, & \text{if } s \in [-h, -1/k]. \end{cases}$$

Clearly $\phi_k(\cdot) \in C([-h,0],\mathbb{R}^n)$ and $\phi_k \geq 0$. For any $\sigma \geq 0$, $x_k(\cdot) := x(\cdot;\sigma,\phi_k) \geq 0$, because of the positivity of (1). Furthermore, $x_k(\cdot)$ satisfies

$$\begin{aligned}\dot{x}_k(\sigma) &= A_0(\sigma)x_k(\sigma) + \sum_{i=1}^{m} A_i(\sigma)x_k(\sigma - h_i) \\ &\quad + \int_{-h}^{0} B(\sigma,s)x_k(\sigma+s)ds \\ &= A_0(\sigma)\phi_k(0) + \sum_{i=1}^{m} A_i(\sigma)\phi_k(-h_i) \\ &\quad + \int_{-h}^{0} B(\sigma,s)\phi_k(s)ds \\ &= A_0(\sigma)e_j + \int_{-h}^{0} B(\sigma,s)\phi_k(s)ds,\end{aligned}$$

for $k \in \mathbb{N}$ sufficiently large. Thus, $\lim_{k\to\infty} \dot{x}_k(\sigma) = A_0(\sigma)e_j$, by the Lebesgue's dominated convergence theorem. On the other hand, we have

$$\begin{aligned}e_r^T \dot{x}_k(\sigma) &= e_r^T \lim_{\epsilon\to 0^+} \frac{x_k(\sigma+\epsilon) - x_k(\sigma)}{\epsilon} \\ &= e_r^T \lim_{\epsilon\to 0^+} \frac{x_k(\sigma+\epsilon) - e_j}{\epsilon} \\ &= \lim_{\epsilon\to 0^+} \frac{e_r^T x_k(\sigma+\epsilon)}{\epsilon} \geq 0\end{aligned}$$

for any $r \in \underline{n}$, $r \neq j$. Hence $\lim_{k\to\infty} e_r^T \dot{x}_k(\sigma) = e_r^T A_0(\sigma)e_j \geq 0$, for any $r, j \in \underline{n}$, $r \neq j$. Therefore, $A_0(\sigma)$ is a Metzler matrix for any $\sigma \geq 0$.

Fix $i \in \underline{m}$. We now show that $A_i(\sigma) \geq 0$ for a fixed $\sigma \geq 0$. Fix $j \in \underline{n}$ and consider two separate cases:

(a) $m > 1$ and $i < m$: Define

$$\phi_k(s) = \begin{cases} 0 & \text{if } s \in [-h, -h_i - 1/k] \\ (ks + 1 + kh_i)e_j & \text{if } s \in (-h_i - 1/k, -h_i] \\ (-ks + 1 - kh_i)e_j & \text{if } s \in (-h_i, -h_i + 1/k] \\ 0 & \text{if } s \in (-h_i + 1/k, 0] \end{cases}$$

By a similar way as in the preceding paragraph, we get $\lim_{k\to\infty} e_r^T \dot{x}_k(\sigma) = e_r^T A_i(\sigma)e_j \geq 0$, for any $r, j \in \underline{n}$. Thus $A_i(\sigma) \geq 0$, for each $\sigma \geq 0$.

(b) $m \geq 1$ and $i = m$: The proof is similar to the above case and it is omitted here.

Finally, we show that $B(t,s) \geq 0$ for all $(t,s) \in \mathbb{R}_+ \times [-h,0]$. Let $\phi \in C([-h_1,0],\mathbb{R}^n)$ with $\phi(0) = \phi(-h_1) = 0$ and $\phi \geq 0$. We extend ϕ onto $[-h,0]$ by setting $\phi(s) = 0$, $s \in [-h,-h_1]$. For any $\sigma \geq 0$, we have $x(\cdot) := x(\cdot;\sigma,\phi) \geq 0$. By definition, $\dot{x}(\sigma) = \lim_{\epsilon\to 0^+}((x(\sigma+\epsilon) - x(\sigma))/\epsilon) = \lim_{\epsilon\to 0^+}(x(\sigma+\epsilon)/\epsilon) \geq 0$. On the other hand, since $x(\cdot)$ satisfies (1), $\dot{x}(\sigma) = \int_{-h_1}^{0} B(\sigma,s)\phi(s)ds$. Thus, $\int_{-h_1}^{0} B(\sigma,s)\phi(s)ds \geq 0$, for any $\phi \in C([-h_1,0],\mathbb{R}^n)$, $\phi \geq 0$, $\phi(0) = \phi(-h_1) = 0$. By [15, Lemma 3.4], $B(\sigma,s) \geq 0$ for any $\sigma \geq 0$ and any $s \in [-h_1,0]$. By a similar way, we can show that $B(\sigma,s) \geq 0$ for any $\sigma \geq 0$ and any $s \in [-h_{i+1},-h_i]$ for any $i \in \{1,2,\ldots,m-1\}$.

Conversely, assume that $A_0(t)$ is a Metzler matrix for all $t \geq 0$, $A_i(t) \geq 0$ for all $t \geq 0$ and all $i \in \underline{m}$ and $B(t,s) \geq 0$ for all $(t,s) \in \mathbb{R}_+ \times [-h,0]$. Let $\sigma = 0$ and fix $\phi \in C([-h,0],\mathbb{R}^n)$, $\phi \geq 0$. We prove that $x(t;\phi) \geq 0$ for all $t \geq 0$. Fix $T > 0$. Since $A_0(\cdot)$ is continuous on $[0,T]$ and $A_0(t)$ is a Metzler matrix for every $t \geq 0$, we may choose $r > 0$ such that $rI_n + A_0(t) \geq 0$ for all $t \in [0,T]$. Consider $z : [-h,T] \to \mathbb{R}^n$, $t \mapsto z(t) := e^{rt}x(t;\phi)$. Then z satisfies

$$\begin{aligned}\dot{z}(t) &= (A_0(t) + rI_n)\, z(t) + \sum_{i=1}^{m} e^{rh_i} A_i(t) z(t-h_i) \\ &\quad + \int_{-h}^{0} e^{-rs} B(t,s) z(t+s) ds \quad \forall t \in [0,T]. \end{aligned} \tag{17}$$

It remains to consider two cases:

(i) Assume $\phi(0) \gg 0$. We show that $z(t) \geq 0$ for all $t \in [0,T]$. Seeking a contradiction, suppose $t_0 = \inf\{t \in [0,T] |\, z(t) \ngeq 0\} \in [0,T]$. Then by continuity $z(t_0) \geq 0$ and so (17) yields

$$\begin{aligned}z(t_0) &= z(0) + \int_{0}^{t_0} \dot{z}(\tau)d\tau \\ &= \phi(0) + \int_{0}^{t_0} \Bigg((A_0(\tau) + rI_n)\, z(\tau) \\ &\qquad + \sum_{i=1}^{m} e^{rh_i} A_i(\tau) z(\tau - h_i) \\ &\qquad + \int_{-h}^{0} e^{-rs} B(\tau,s) z(\tau+s) ds \Bigg) d\tau \\ &\geq \phi(0) \gg 0.\end{aligned}$$

By continuity, there exists $\epsilon > 0$ such that $z(t) \gg 0$ for all $t \in [t_0, t_0+\epsilon)$. However, this contradicts the definition of t_0; whence $z(t) \geq 0$ for all $t \in [0,T]$. Since $T \geq 0$ is arbitrary, we have $z(t) \geq 0$ for all $t \geq 0$ and therefore, $x(t) \geq 0$ for all $t \geq 0$.

(ii) Assume $\phi(0) \geq 0$. Then $\phi_k := \phi + (1/k)e$, where $e = (1,1,\ldots,1)^T \in \mathbb{R}^n$ and $k \in \mathbb{N}$, yields $\phi_k(0) \gg 0$. Now the continuous dependence of solutions of (1) on initial functions ([9, Th. 2.2]) together with Part (i) gives $\lim_{k\to\infty} x(t;\phi_k) = x(t;\phi) \geq 0$, $\forall t \geq 0$.

In the case of $\sigma > 0$, the proof is similar to the above and it is omitted here. This completes the proof of the theorem.

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Almost Sure Stability of Markov Jump Linear Systems With Deterministic Switching

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***Abstract*—The technical note studies a class of linear systems whose piecewise-constant dynamic matrix is subject to both stochastic jumps, governed by a Markov chain, and deterministic switches. These systems will be dubbed switching dynamics Markov jump linear systems (SD-MJLS). Sufficient conditions for exponential almost sure stability (EAS-stability) are established under either hard or average constraints on the dwell-time between switching instants. The proof relies on easy-to-check norm contractivity conditions and the ergodic law of large numbers.**

***Index Terms*—Almost sure stability, dwell-time, Markov jump systems.**

I. Introduction

There are a number of applications in diverse fields that motivate the study of hybrid systems, characterized by the interconnection of logical and continuous dynamics [1], [2]. Within the probabilistic paradigm, hybrid systems can be described by Markov jump systems, where the jumps of the logical state follow a Markov chain model. These systems are a particular case of random systems, see [3], and there exists a wide literature dealing with the derivation of sufficient and/or necessary conditions for different types of stability (mean square, δ-moment, and almost sure) of Markov jump linear systems (MJLS), [4]–[9]. For deterministic switching systems, it is common to consider signals satisfying some form of dwell-time constraint [10], [11], meaning that a minimum dwell-time between two consecutive switching instants is assumed.

In the present technical note, hybrid systems subject to both stochastic jumps and deterministic switches are considered. As discussed in [12], a possible motivation is the stability analysis of fault-prone systems managed by a supervisor whose actions are represented by deterministic switches, while random jumps model faults and unexpected events. Another example arises in networked control with deterministically switching control laws and stochastic jumps in the level of network congestion. The first results on stochastic stability of switching MJLS [12] were restricted to the derivation of sufficient conditions for mean square stability. However, as well known in the literature on stochastic stability [3], almost sure stability is more useful than mean square stability in practical applications. Assuming that the deterministic switches occur between MJLS sharing the same underlying Markov chain, a first contribution of the present technical note is the

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