

# Analysis, observation and control of time-delay systems

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# Introduction and References

This course has been mainly written thanks to:

the PhD dissertations of [Fattouh, 2000, Briat, 2008].

the author's work since 1991

interesting books cited below some of them I have reviewed [Sename, 2005, 2008]

Some interesting books:

L. Dugard and E. I. Verriest , (*Eds*) *Stability and control of time-delay systems*, ser. LNCIS. Springer Verlag, 1998, vol. 228.

S. Niculescu and K. Gu , *Eds.*, *Advances in Time-Delay Systems*, ser. LNCSE. Springer-Verlag, 2004, vol. 38.

J. Chiasson and J. Loiseau , *Eds.*, *Applications of Time-Delay Systems*, ser. LNCIS. Springer-Verlag, 2007, vol. 352.

S.-I. Niculescu , *Delay effects on stability. A robust control approach*. Springer-Verlag: Heidelberg, 2001, vol. 269.

K. Gu, V. Kharitonov, and J. Chen , *Stability of Time-Delay Systems*. Birkhäuser, 2003.

Q.C. Zhong , *Robust Control of Time-Delay Systems*, Springer-Verlag, 2006, London, UK.

**A survey paper:** JP. Richard, "Time delay systems : an overview of some recent advances and open problems", *Automatica* 39 (2003), 1667 - 1694.

# Many applications

Robotics

Automotive: Combustion model (ignition delay), electromechanical brakes (actuator delay)

Nuclear

Heat exchanger: distributed delay due to conduction in a tube

Hydraulic networks: the transport phenomenon of water is modeled as a varying time-delay

Electrical networks

Intelligent building: time-delay due to wireless transmission of sensor datas

Marine robotics: transport delay due to sonar measurement of depth

Population dynamics: Predator - prey model based on Volterra model with predator ( $y$ ) and prey ( $x$ ) populations ( $\tau$  is the time-life of prey):

$$\begin{aligned}\dot{x} &= rx\left(1 - \frac{x(t-\tau)}{K}\right) - \alpha xy \\ \dot{y} &= -cy + \beta xy\end{aligned}$$

BUT delays may induce:

Poor performances

Instability

Difficulties in control design

## About complexity

Let us consider system

$$\dot{x}(t) = Ax(t) + A_h x(t-h) \quad (1)$$

where  $h$  is the delay and  $x$  is the system state.

The calculation of the state evolution of (1) requires the knowledge of  $x(t)$  over  $t \in [-h, 0]$ .

The initial condition is then a function, and the system is therefore an infinite dimensional system.

The system state is then  $x_t(\theta) \in \mathcal{C}_0([-h, 0]; \mathbb{R}^n)$  defined by

$$x_t(\theta) = x(t + \theta), \quad \forall \theta \in [-h, 0]$$

Three different representations are commonly used for modeling time-delay systems:

- Differential equation with coefficients in a ring of operators

- Differential equation on an infinite dimensional abstract linear space

- Functional Differential equation

## Model over a ring

This framework has been developed early to study time-delay systems in [Morse, 1976, Kamen, 1978, Conte and Perdon, 1995a, Sename et al., 1995a].

A linear time-delay system is governed by a following linear differential equation with coefficient in a module, e.g.

$$\dot{x}(t) = A(\nabla)x(t) \quad (2)$$

where in the general case  $\nabla = \text{col}_i(\nabla_i)$  is the vector of delay operators such that  $x(t - h_i) = \nabla_i x(t)$ .

**$A$  is a polynomial matrix in the variable  $\nabla$**

Since the inverse of  $\nabla$  (the predictive operator  $x(t + h_i) = \nabla_i^{-1}x(t)$ ) is undefined from a causality point of view, the operators  $\nabla$  of the matrix  $A$  belong, indeed, to a ring

The "state space" is then the module  $\mathbb{R}^n[\nabla]$

Many studies have been devoted to such systems in the framework of time-delay systems. The studies concern:

Structural analysis [Malabre and Rabah, 1993, Sename et al., 2001b].

The disturbance decoupling problem [Conte and Perdon, 1995b]

The row-by-row decoupling problem [Sename and Lafay, 1997b, Sename et al., 1995b, Conte and Perdon, 1995a, Conte et al., 1997].

the pole placement problem [Sename and Lafay, 1997a, Sename et al., 1995a]

Geometric approach for analysis [Conte and Perdon, 1998a,b].

## Infinite dimensional model

This type of representation stems from the application of infinite dimensional systems theory to the case of time-delay system.

Let us consider system

$$\dot{x}(t) = Ax(t) + A_h x(t-h) \quad (3)$$

where  $h$  is the delay and  $x$  is the system state.

This system is completely characterized by the state

$$\tilde{x} = \begin{bmatrix} x(t) \\ x_t(s) \end{bmatrix}$$

for all  $s \in [-h, 0]$  and  $x_t(s) = x(t+s)$ . The state-space is then the Hilbert space

$$\mathbb{R}^n \times \mathcal{L}_2([-h, 0], \mathbb{R}^n)$$

One can easily see that the state of the system contains a point in an Euclidian space  $x(t)$  and a function of bounded energy  $x_t(s)$ , the latter belonging to an infinite dimensional linear space.

This motivates the denomination of 'Infinite Dimensional Abstract Linear Space' [Manitius and Triggiani, 1978, Meinsma and Zwart, 2000].

# Retarded Functional Differential Equations (RFDE)

The most spread representation is by the mean of functional differential equations [Bellman and Cooke, 1963, Kolmanovskii and Myshkis, 1962, K.Hale and Lunel, 1991].

Several types of time-delay systems:

System with discrete delay acting on the state  $x$ , inputs  $u$

$$\dot{x}(t) = Ax(t) + A_h x(t - h_x) + Bu(t) + B_h u(t - h_u)$$

where  $h_x, h_u$  are the state and input delays.

Distributed delay systems

$$\dot{x}(t) = Ax(t) + \int_{-h_x}^0 A_h(\theta) x(t + \theta) d\theta + Bu(t)$$

Neutral delay systems where the delay acts on the higher-order state-derivative, e.g.

$$\dot{x}(t) - F \dot{x}(t - h) = Ax(t)$$



## Systems with discrete delays

See Teleoperation systems and Wind tunnel model (above)

**Glucose-insulin model** [Palumbo et al., 2009]

$G(t)$ ,  $I(t)$  plasma glycemia and insulinemia

$$\begin{aligned}\dot{G}(t) &= K_{xgi}G(t)I(t) + \frac{T_{gh}}{V_G} \\ \dot{I}(t) &= -K_{xi}I(t) + \frac{T_{iGmax}}{V_I}f(G(t - \tau_g))\end{aligned}$$

where:

$K_{xgi}$ : rate of glucose uptake by tissues (insulin-dependent) per pM of plasma insulin concentration

$T_{gh}$ : net balance between hepatic glucose output and insulin-independent zero-order glucose tissue uptake (mainly by the brain)

$V_G$ : apparent distribution volume for glucose

$K_{xi}$ : apparent first-order disappearance rate constant for insulin

$T_{iGmax}$ : maximal rate of second-phase insulin release

$V_I$ : apparent distribution volume for insulin

$\tau_g$ : apparent delay with which the pancreas varies secondary insulin release in response to varying plasma glucose concentrations

$f$ : nonlinear function that models the Insulin Delivery Rate

## Distributed delay Systems

The delay has not a local effect as in pointwise delay systems but in a distributed fashion over a whole interval. For instance, consider the following SIR-model used in epidemiology [Briat and Verriest, 2008]

$$\begin{aligned}
 \dot{S}(t) &= -\beta S(t)I(t) \\
 \dot{I}(t) &= \beta S(t)I(t) - \beta \int_h^\infty \gamma(\tau) S(t-\tau) I(t-\tau) d\tau \\
 \dot{R}(t) &= \beta \int_h^\infty \gamma(\tau) S(t-\tau) I(t-\tau) d\tau
 \end{aligned} \tag{4}$$

where  $S$  is the number of susceptible people,  $I$  the number of infectious people and  $R$  the number of recovered people.

The distributed delay here taking value over  $[h, +\infty]$  is the time spent by infectious people before recovering from the disease. This delay may be different from a individual to another but obeys a probability density represented by  $\gamma(\tau)$  which tends to 0 at infinity and whose integral over  $[-\infty, -h]$  equals 1.

## Neutral Delay Systems

Arise for instance in the analysis of the coupling between transmission lines and population dynamics: evolution of forests. The model is based on a refinement of the delay-free logistic (or Pearl-Verhulst equation) where effects as soil depletion and erosion have been introduced

$$\dot{x}(t) = rx(t) \left[ 1 - \frac{x(t - \tau) + c\dot{x}(t - \tau)}{K} \right] \quad (5)$$

where  $x$  is the population,  $r$  is the intrinsic growth rate and  $K$  the environmental carrying capacity. See [Pielou, 1977, Gopalsamy and Zhang, 1988, Verriest and Pepe, 2007] for more details.

## About Delays

It is important to define the different categories or types of delays as :

constant or time-varying delay

commensurate or non commensurate delay

distributed delay

delays as a function of state [Verriest, 2002]

### Constant delays:

**Commensurate**  $h_i \in \mathbb{R}$ ,  $i \in \mathbb{N}$ . Are commensurate if  $h_j/h_i$  is rational, which corresponds to find a minimal delay  $h$  the others being multiple of  $h$

**Non commensurate** :  $h_i$ ,  $i \in \mathbb{N}$  are not rationally dependent

### Time-varying delays:

**Bounded delay**:  $0 < \tau_1 < \tau(t) < \tau_2$

**Derivative bounded delays**:  $\dot{\tau}(t) \geq d < 1$ . Means that  $f(t) = t - \tau(t)$  is monotonic.

**Arbitrary varying delays**:  $\tau$  and  $\dot{\tau}$  are not bounded

**'Quenching'** : it is possible to find systems which are stable for constant delay  $h \in [h_1, h_2]$  but unstable for time-varying delay belonging to the same interval.

In such a phenomenon, the bound on the delay derivative plays an important role.

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# Introduction

Many of the results given in this lecture come from Fattouh [2000]'s thesis and related works Sename [2001], Sename [1997], Fattouh et al. [1999a], Sename et al. [2001a], Fattouh et al. [1998, 1999b,c, 2000c,b], Fattouh et al. [2000a], Fattouh and Sename [2004], Sename and Briat [2006b].

The study will concern continuous-time linear Time-Invariant systems with delays (TDS) :

$$\begin{cases} \dot{x}(t) &= \sum_{i=0}^N A_i x(t - ih) + \sum_{i=0}^N B_i u(t - ih) \\ y(t) &= \sum_{i=0}^N C_i x(t - ih) \end{cases} \quad (6)$$

where

$x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  the known input vector,  $y(t) \in \mathbb{R}^p$  the output vector  
 $h \in \mathbb{R}^+$  is the delay and  $N$  represents the maximal delay in state, input and output variable  
 matrices  $A_i, B_i, C_i$  ( $i = 0, \dots, N$ ) are real matrices of appropriate dimensions

$x(t) = \varphi(t), t \in [-Nh, 0]$  is the functional initial condition of the time-delay system.

that can be written as the **ring model**:

$$\Sigma \begin{cases} \dot{x}(t) = A(\nabla)x(t) + B(\nabla)u(t)(t) \\ y(t) = C(\nabla)x(t) \end{cases} \quad (7)$$

# Introduction

Controllability of time-delay systems is rather more difficult than their finite dimensional counterpart: several nonequivalent controllability/observability properties can be defined and depend on the type of representation used to model time-delay systems.

Some references: Morse [1976], Sontag [1976], Kamen [1978], Lee and Olbrot [1981], Eising [1982], Picard [1996], Fattouh [2000], Sename [2001].

Example: let the system be

$$\begin{cases} \dot{x}_1(t) = u(t) \\ \dot{x}_2(t) = x_1(t-h) \end{cases}, \text{ with } x(t) = u(t) = 0, \text{ for } t < 0$$

Interval  $[0, h]$  :

choice of  $u(t)$  = control of  $x_1(t)$

reachable state  $X = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}$  at  $t = \varepsilon, \forall \varepsilon > 0 \Rightarrow$  only a state component can be controlled.

Interval  $[h, 2h]$  :

choice of  $u(t), t \in [h, 2h]$  : control of  $x_1(t)$

choice of  $u(t), t \in [0, h]$  : control of  $x_2(t)$ . Reachable state  $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$  at  $t = h + \varepsilon, \forall \varepsilon > 0$

Therefore time plays two roles:

with  $\varepsilon > 0$  (as fast as possible)

constraint linked to the delays

# Strong controllability

Morse [1976], Sontag [1976]

## Definition

The system (7) is said to be controllable over the ring  $\mathfrak{R}[\nabla]$  if :  
 $Im < A(\nabla)/B(\nabla) > = \mathfrak{R}^n[\nabla]$ .

where the **controllability matrix** is:

$$< A(\nabla)/B(\nabla) > = \begin{bmatrix} B(\nabla) & A(\nabla)B(\nabla) & \dots & A^{n-1}(\nabla)B(\nabla) \end{bmatrix}$$

→ possibility to reach, for any  $x_0 \in \mathfrak{R}^n[\nabla]$ , any element of the state module  $\mathfrak{R}^n[\nabla]$  using a 'polynomial' control law.

## Time-domain interpretation

Any  $x_1 \in \mathfrak{R}^n$  can be reached, from  $x(0) \in \mathfrak{R}^n$ , and at a given time  $T$ , with  $T$  as small as possible (and with an "ad hoc" control input  $u(t)$ ,  $t \in [0, T]$ ).



Referred to as the **Strong controllability** ~ delay free system



# Weak controllability

Morse [1976]

## Definition

*The system (7) is said to be controllable over the field  $\mathfrak{R}(\nabla)$  if :  
 $\text{rank} < A(\nabla)/B(\nabla) > = n$ .*

Exemple  $A(\nabla) = \begin{bmatrix} 0 & 0 \\ \nabla & 0 \end{bmatrix} = A_0 + \nabla A_1$  and  $B(\nabla) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = B_0$

controllable over the field  $\mathfrak{R}(\nabla)$

On the interval  $[0, h]$  : only  $B_0$  acts.

every state is  $\mathfrak{R}^2$  reachable , but only from  $t = h$ .

## Time-domain interpretation

It is not possible to reach from  $x(0)$ , any state  $x_1$  at a given time  $T$  as small as possible .

⇒ A **minimal time** will be necessary to reach  $x_1$

## Spectral controllability

The next notion corresponds to the extension of the Hautus condition to time-delay systems .

### Definition

*The system (7) is spectrally controllable if it satisfies the following characterization:*

$$\text{rank} \begin{bmatrix} sI_n - A(e^{-sh}) & B(e^{-sh}) \end{bmatrix} = n, \quad \forall s \in \mathbb{C}.$$

Spectral controllability is a necessary and sufficient condition for stabilizability of a time-delay system. However the control law to be used for stabilisation must include distributed time-delay.

## Towards observability

**LTI case:** A linear system ( $\dot{x} = Ax + Bu; y = Cx$ ) is completely observable if, given the control and the output over the interval  $t_0 \leq t \leq T$ , one can determine any initial state  $x(t_0)$ .

Note that this is equivalent to the state reconstruction of  $x(T)$  and observability is characterized through its unobservable subspace i. e.  $\bigcap_{i=1}^n \text{Ker} CA^{i-1} = \{0\}$ .

**For a TDS:**

the basic extension is the *initial observability*, i. e.: *any initial state  $(x(0), \varphi(t), t \in [-Nh, 0])$  is observable if the output of the autonomous system is not identically zero on  $[0, \infty)$ .*

However: knowing the initial condition is not necessary for control purpose: what is important is to be able to reconstruct  $x(t)$  at any time  $t$ .

By opposition to linear systems without delay, the notion of initial observability is not equivalent to the reconstructibility of the state variables: indeed time-delay may have finite duration transient behaviour.

Defining the observability matrix of (7) as:

$$\left\langle \frac{C(\nabla)}{A(\nabla)} \right\rangle = \begin{bmatrix} C(\nabla) \\ C(\nabla)A(\nabla) \\ \vdots \\ C(\nabla)A^{n-1}(\nabla) \end{bmatrix}.$$

the definition of strong, weak and spectral observability are straightforward by duality.

## Illustrative example

Let us consider the following time-delay system:

$$\begin{cases} \dot{x}(t) &= 0 \\ y(t) &= x(t) - x(t-h) \end{cases}$$

Therefore

$$A(\nabla) = \begin{bmatrix} 0 \end{bmatrix}, \quad C(\nabla) = \begin{bmatrix} 1 - \nabla \end{bmatrix}.$$

Let us apply the previous criteria of observability.

The observability matrix is given by:

$$\left\langle \frac{C(\nabla)}{A(\nabla)} \right\rangle = 1 - \nabla$$

- Strong observability: it is straightforward that the observability matrix has no left inverse over  $\mathbb{R}[\nabla]$ . Thus the system is not strongly observable.
- Spectral observability:

$$\begin{bmatrix} sI_n - A(z) \\ C(z) \end{bmatrix} = \begin{bmatrix} s \\ 1 - e^{-sh} \end{bmatrix}$$

which is of rank 1  $\forall s \in \mathbb{C}$  except for  $s = 0$ . Therefore the system is not spectrally observable.

- Weak observability: it is straightforward that the observability matrix has a left inverse over  $\mathbb{R}(\nabla)$ . Thus the system is weakly observable.

Note that taking  $C(\nabla) = 1 + \nabla$  will give a spectrally observable system and then a weakly observable one.

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# Introduction

The stability analysis of time-delay systems is a very studied problem and has led to lots of approaches which can be classified in two main framework: the frequency-domain and time-domain analysis [Niculescu \[2001\]](#), [Gu et al. \[2003\]](#). While the first one deals with characteristic quasipolynomial of the system, the second one considers directly the state-space domain and matrices.

Let us consider the system

$$\dot{x}(t) = Ax(t) + A_h x(t-h) \quad (8)$$

where  $h$  is the delay and  $x$  is the system state.

## Definition

*If a time-delay system is stable for any delay values belonging to  $\mathbb{R}_+$ , the system is said to be delay-independent stable.*

## Definition

*If a time-delay system is stable for all delay values belonging to a subspace  $D \subsetneq \mathbb{R}_+$  then the system is said to be delay-dependent stable.*

# Illustrative examples

## Example

A **delay-independent** stable time-delay system with constant time-delay is given by

$$\dot{x}(t) = \begin{bmatrix} -5 & 1 \\ 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} x(t-h) \quad (9)$$

It seems obvious that if a system is delay-independent stable, then it must be stable for  $h = 0$  and  $h \rightarrow +\infty$  which means that  $A$  and  $A + A_h$  must be Hurwitz stable (all the eigenvalues lie in the open left-half plane). Here

$$\lambda(A) = \{-5, -5\} \text{ and } \lambda(A + A_h) = \left\{ \frac{-13 \pm \sqrt{3}}{2} \right\}$$

On the second hand, for any value of  $h$  from 0 to  $+\infty$ , the system must be stable too. We will see that a supplementary sufficient condition is given by

$$\sup_{\omega \in \mathbb{R}} \bar{\rho}[(j\omega - A)^{-1} A_h] < 1, \text{ (here } \sim 0.4739\text{)}$$

where  $\bar{\rho}(\cdot)$  denotes the spectral radius (i.e.  $\max_i |\lambda_i(\cdot)|$ ).

The system is confirmed to be delay-independent stable.

The term 'delay-independent stable' has been introduced for the first time in [Kamen et al. \[1985\]](#) and has become commonly used in the time-delay community.

# Illustrative examples

## Example

A well-known system being **delay-dependent** stable **Gouaisbaut and Peaucelle [2006]** is given by

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t-h) \quad (10)$$

It is stable for any constant delay belonging to  $[0, 6.17]$ .

## Example

Analysis of uncertain (time-varying) delays around a fixed constant one. **Kharitonov and Niculescu [2003]**

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix} x(t-h(t)) \quad (11)$$

is stable for a delay equal to 1.

The time-varying delay is written as  $h(t) = h_0 + \eta(t)$ , it is shown that the stability is preserved for every  $|\eta(t)| \leq \eta_0$  and  $|\dot{\eta}(t)| \leq \dot{\eta}_0$  s.t.

$$\eta_0 < \frac{1}{640} \mu_0 \quad \dot{\eta}_0 < 1 - 8\mu_0$$

with  $\mu_0 \in (0, 1/40)$ . From these inequalities we can see that the larger  $\dot{\eta}_0$  is, the smaller  $\eta_0$  must be to preserve stability.



# A simple frequency-Domain analysis

Note  $L(s) = G(s)K(s)$  where  $K$  is the controller. Define:

$$L(s) = e^{-sh} \frac{N(s)}{D(s)}$$

The Nyquist criterion remains valid for a pure time-delay system, since  $e^{-sh}$  does not introduce any additional poles and zeros Closed loop system:

$$T(s) = \frac{N(s)e^{-sh}}{D(s) + N(s)e^{-sh}}$$

The characteristic **quasi-polynomial** is :

$$p(s, h) = D(s) + N(s)e^{-sh}$$

## Definition

$p(s, h)$  is said to be stable if

$$p(s, h) \neq 0, \quad \forall s \in \mathbb{C}_+$$

Usually, we denote  $z = e^{-sh}$ , and the characteristic polynomial is written:

$$p(s, z) = \sum_{k=0}^k p_k(s) z^k$$

# Frequency-seeping tests

Objective: Find the zero crossing frequencies directly based on the conjugate symmetry property of the quasipolynomial.

## Proposition (Tsyarkin method)

*If  $D(s)$  is a stable polynomial the the closed-loop system :*

$$T(s) = \frac{N(s)e^{-sh}}{D(s) + N(s)e^{-sh}}$$

*is stable for any value of the delay  $h$  if and only if*

$$|D(j\omega)| > |N(j\omega)|, \quad \forall \omega \in \mathbb{R}$$

## Delay margin

If the system is stable for  $h = 0$ , then **the delay margin** is :

$$\bar{h} = \min\{h \geq 0 | p(j\omega, e^{-jh\omega}) = 0, \text{ for some } \omega \in \mathbb{R}\}$$

Frequency-sweeping test: find the critical delay value at which the characteristic roots intersect the stability boundary, i.e. the imaginary axis, thus rendering the system unstable.

# Frequency-Domain Delay independent stability: robust stability based analysis

## Theorem

*The time-delay system  $\dot{x}(t) = Ax(t) + A_h x(t-h)$  is delay independent stable if and only if*

- $A$  is stable*
- $A + A_h$  is stable*

$$\rho[(j\omega - A)^{-1}A_h] < 1, \quad \forall \omega \geq 0$$

*where  $\rho(\cdot)$  denotes the spectral radius of a matrix*

Considering the delay operator  $\nabla = e^{-sh}$  as an uncertainty, leads to the small gain theorem result:

## Theorem

*A sufficient condition for delay independent stability is*

$$\| (sI - A)^{-1}A_h \|_{\infty} < 1$$

This results is valid for non commensurate delays, assuming  $\nabla_i = e^{-sh_i}, \forall i$ , and collecting all the delay operators in  $\Delta = \text{diag}(\nabla_1, \dots, \nabla_k)$ . The condition becomes  $\| M(s) \|_{\infty} < 1$ , where

$$M(s) = \begin{bmatrix} I \\ \vdots \end{bmatrix} (sI - A)^{-1} \begin{bmatrix} A_1 & \cdots & A_k \end{bmatrix}$$

# Towards Lyapunov analysis

For simplicity: let us consider the single delayed system

$$\begin{aligned}\dot{x}(t) &= f(x_t, t) \\ x_{t_0} &= \phi\end{aligned}\tag{12}$$

where  $x_t(\theta) = x(t + \theta)$  and  $\phi \in \mathcal{C}([-h, 0], \mathbb{R})$  is the **functional initial condition**. We also assume that  $x(t) = 0$  identically is a solution to (12), that will be referred to as the trivial solution. The system 'state' is therefore the value of  $x(\theta)$  in the interval  $\theta \in [t - h, t]$  (i.e.  $x_t$ ).

## Norm of a function

All the definitions of stability of finite-dimensional systems can be generalized to time-delay systems by introducing the continuous norm  $\|\cdot\|_c$  defined by

$$\|\phi\|_c := \max_{a \leq \theta \leq b} \|\phi(\theta)\|_2\tag{13}$$

where  $\phi \in \mathcal{C}([a, b], \mathbb{R}^n)$ .

The corresponding Lyapunov function should then be a **functional**  $V(t, x_t)$  depending on  $x_t$ , which is referred to as the **Lyapunov-Krasovskii functional**.

# Lyapunov-Krasovskii Stability Theorem

## Theorem (Lyapunov-Krasovskii Stability Theorem)

*Suppose  $f : \mathbb{R} \times \mathcal{C}_{[-h, 0]} \rightarrow \mathbb{R}^n$  in (12) maps  $\mathbb{R} \times (\text{bounded sets of } \mathcal{C}_{[-h, 0]})$  into bounded sets of  $\mathbb{R}^n$ , and  $u, v, w : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$  are continuous nondecreasing functions,  $u(s)$  and  $v(s)$  are positive for  $s > 0$ , and  $u(0) = v(0) = 0$ . If there exists a continuous differentiable functional  $V : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}$  such that*

$$u(\|\phi(0)\|) \leq V(t, \phi) \leq v(\|\phi\|_c)$$

*and*

$$\dot{V}(t, \phi) \leq -w(\|\phi(0)\|)$$

*then the trivial solution of (12) is uniformly stable. If  $w(s) > 0$  for  $s > 0$ , then it is uniformly asymptotically stable. If, in addition,  $\lim_{s \rightarrow +\infty} u(s) = +\infty$ , then it is globally uniformly asymptotically stable.*

# Lyapunov-Razumikhin Stability Theorem

Idea : use a **function**  $V(x)$  representative of the size of  $x(t)$ :

$$\bar{V}(x_t) = \max_{\theta \in [-h, 0]} V(x(t + \theta)) \quad (14)$$

serves to measure the size of  $x_t$ .

## Theorem (Lyapunov-Razumikhin Stability Theorem)

Suppose  $f : \mathbb{R} \times \mathcal{C}_{[-h, 0]} \rightarrow \mathbb{R}^n$  in (12), and  $u, v, w : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$  are continuous nondecreasing functions, such that  $u(s), v(s)$  and  $w(s)$  are positive for  $s > 0$ , and  $u(0) = v(0) = 0$ .

If there exists a continuously differentiable function  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\begin{aligned} u(\|x\|) &\leq V(t, x) \leq v(\|x\|), \quad \text{for } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n \\ \dot{V}(t, x(t)) &\leq -w(\|x(t)\|) \text{ whenever } V(t + \theta, x(t + \theta)) \leq V(t, x(t)) \text{ for } \theta \in [-h, 0] \end{aligned}$$

then the system (12) is uniformly stable.

If, in addition, there exists a continuous nondecreasing function  $p(s) > s$  for  $s > 0$  s.t:

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \text{ if } V(t + \theta, x(t + \theta)) \leq p(V(t, x(t))), \text{ for } \theta \in [-h, 0] \quad (15)$$

then the system (12) is uniformly asymptotically stable.

If in addition  $\lim_{s \rightarrow +\infty} u(s) = +\infty$ , then the system (12) is globally uniformly asymptotically stable.

## Model transformations

Model-transformations have been introduced early in the stability analysis of time-delay systems. They allow to turn a time-delay system into a new system, which is referred to as a *comparison system*.

The stability of the original system is determined through the stability analysis of the comparison model (which could be a time-delay or a delay-free system)

They are generally used to remove annoying terms in the equations or to turn the expression of the system in a more convenient form.

let us consider the linear time-delay system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h x(t-h) \\ x_0 &= \phi\end{aligned}\tag{16}$$

where  $A, A_h$  are given  $n \times n$  real matrices and  $\phi$  is the functional initial condition.

### Euler (or Leibniz) formula

The Euler formula is the oldest model transformation which has been introduced and is still in use for different purposes:

$$x(t-h) = x(t) - \int_{t-h}^t \dot{x}(\theta) d\theta\tag{17}$$

It allows to turn the time-delay system with discrete delay (16) into the following system with distributed delay:

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^t [Ax(s) + A_h x(s-h)] ds\tag{18}$$

## Additional Dynamics

Stability tests obtained from comparison systems are, in most of the cases, outer approximations of the original system only. This means that if the comparison model is stable then the original system is stable too but the converse does not necessary hold.

For instance the simpler model transformation (i.e. the Euler formula) leads to the comparison system

$$\dot{x}(t) = (A + A_h)x(t) - A_h \int_{t-h}^t [Ax(s) + A_h x(s-h)] ds \quad (19)$$

The characteristic polynomial of the latter comparison system is then given by

$$\begin{aligned} \Delta_c(s) &:= \det(s^2 I - (A + A_h)s + A_h A(1 - e^{-sh}) + A_h^2 e^{-sh}(1 - e^{-sh})) \\ &= \det(sI - A - A_h e^{-sh}) \times \det\left(I - \frac{1 - e^{-sh}}{s} A_h\right) \\ \Delta_c(s) &:= \det(s^2 I - (A + A_h)s + A_h A(1 - e^{-sh}) + A_h^2 e^{-sh}(1 - e^{-sh})) \end{aligned} \quad (20)$$

→ not equivalent to the original system.



## Delay-independent stability test via Lyapunov-Razumikhin theorem

Let us consider here a general linear time-delay system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h(t-h) \\ x(t+\theta) &= \phi(\theta), \quad \theta \in [-h, 0]\end{aligned}\quad (21)$$

A simple test on delay-independent stability using quadratic Lyapunov-Razumikhin function is provided here

$$V(x(t)) = x(t)^T P x(t) \quad (22)$$

The time-derivative of  $V$  along the trajectories solutions of system (21) is given by

$$\dot{V}(x(t)) = \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PA_h \\ A_h^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}$$

Applying the Lyapunov-Razumikhin theorem 5:

### Theorem

*System (21) is asymptotically stable independent of delay if there exists  $P = P^T \succ 0$  and a scalar  $\tau > 0$  such that*

$$\begin{bmatrix} A^T P + PA + \tau P & PA_h \\ A_h^T P & -\tau P \end{bmatrix} \prec 0 \quad (23)$$

it is worth noting, that (23) is not a LMI due to bilinear term  $\tau P$ . Nevertheless, the problem is quasi-convex since if  $\tau$  is fixed, then (23) becomes a LMI. This means that a suitable value for  $\tau$  can be found using an iterative line search.

## Delay-dependent stability test via Lyapunov-Razumikhin theorem

The following result is based on the Euler model transformation.

### Theorem

*System (21) is delay-dependent asymptotically stable if there exists  $P = P^T \succ 0$  and scalars  $\alpha, \alpha_0, \alpha_1 > 0$  such that*

$$\begin{bmatrix} M & P(\alpha I - A_h)A & P(\alpha I - A_h)A_h \\ \star & -\alpha_0 P - \alpha h A_0^T P A_0 & -\alpha h A^T P A_h \\ \star & \star & -\alpha_1 P - \alpha h A_h^T P A_h \end{bmatrix} \prec 0$$

*holds with  $M = \frac{1}{r} [P(A + A_h) + (A + A_h)^T P] + (\alpha_0 + \alpha_1)P$*

A discussion on the choice of scalars  $\alpha, \alpha_i, i = 0, 1$  is provided in [Gu et al., 2003]. As previously, the computation of  $P, \alpha, \alpha_i, i = 0, 1$  is not an easy task since the resulting condition is not a LMI. The problem is quasi-convex and an iterative procedure should be performed in order to find suitable values for  $\alpha, \alpha_i, i = 0, 1$ . However, this iterative procedure is more difficult than in the delay-independent case since the search has to be performed over a three-dimensional space (instead of a one-dimensional), which is more involved from an algorithmic and computational point of view.

# Stability test via Lyapunov-Krasovskii theorem

Lyapunov-Razumikhin functions: Simple functions but lead to nonlinear matrix inequalities and to conservative results due to the use of non-equivalent model transformations.

Using Lyapunov-Krasovskii functionals: more and more accurate LMI results do exist now by applying either more accurate model transformations (as of [Fridman, 2001]), more precise bounding techniques of cross-terms (as of [Park et al., 1998, Park, 1999]), or also other methods without any model transformations (see for instance [Han, 2005, Xu et al., 2006, Gouaisbaut and Peaucelle, 2006, Xu and Lam, 2007, Briat et al., 2009]).

## Delay-independent stability test via Lyapunov-Krasovskii theorem

Consider the Lyapunov-Krasovskii functional given by

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T Q x(\theta) d\theta \quad (24)$$

where  $P, Q \in \mathbb{S}_{++}^n$  are constant decision matrices.

Computing the derivative of the Lyapunov-Krasovskii functional  $V(x_t)$  along the trajectories solutions of system (16) yields

$$\begin{aligned} \dot{V}(x_t) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) + x(t)^T Q x(t) - x(t-h)^T Q x(t-h) \\ &= [Ax(t) + A_h x(t-h)]^T P x(t) + x(t)^T P [Ax(t) + A_h x(t-h)] + x(t)^T Q x(t) - x(t-h)^T Q x(t-h) \\ &= \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} A^T P + PA + Q & PA_h \\ A_h^T P & -Q \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \end{aligned}$$

We then obtain the following result which was first proved in [Verriest and Ivanov, 1991]:

### Theorem

*System (21) is asymptotically stable for any delay if there exist matrices  $P = P^T \succ 0$  and  $Q = Q^T \succ 0$  such that*

$$\begin{bmatrix} A^T P + PA + Q & PA_h \\ \star & -Q \end{bmatrix} \prec 0 \quad (25)$$

*holds.*

Note that the Lyapunov-Krasovskii based test includes the Lyapunov-Razumikhin test as a particular case.

## Delay-dependent stability test via LK theorem

Many studies have dealt with the problem of determination of the delay-margin for time-delay systems.

Park's Bounding Method [Park et al., 1998]: idea based on a more accurate bounding of cross terms in the derivative of the Lyapunov-Krasovskii functional

Descriptor Model Transformation [Fridman, 2001, Fridman and Shaked, 2001]: another model transformation (less restrictive)

Free Weighting Matrices [He et al., 2004]: consists in injecting additional constraints into the LMI, with free variables adding extra-degree of freedom, in order to tackle relations between signals involved in the system:

Jensen's inequality [Gouaisbaut and Peaucelle, 2006, Han, 2005]: allows to avoid the bounding of cross-terms and any use of model transformation

### Key issues:

Central role of the model transformations

difficulties: additional dynamics but also cross-terms in the the mathematical development of the stability condition (we should overcome)

## A delay-dependent stability test via LK theorem (cont.)

Let us consider the following Lyapunov-Krasovskii functional

$$\begin{aligned}
 V(x_t, \dot{x}_t) &= V_1(x_t) + V_2(x_t) + V_3(x_t, \dot{x}_t) \\
 V_1(x_t) &= x(t)^T P x(t) \\
 V_2(x_t) &= \int_{t-h}^t x(\theta)^T Q x(\theta) d\theta \\
 V_3(x_t, \dot{x}_t) &= \int_{-h}^0 \int_{t+\theta}^t \dot{x}(\eta)^T Z \dot{x}(\eta) d\eta d\theta
 \end{aligned} \tag{26}$$

with  $P = P^T, Q = Q^T, Z = Z^T \succ 0$ .

Using the Euler transformation and some technics to eliminate some cross-terms (bounding), we get (see [Briat, 2008] )

### Theorem

*System (16) is delay-dependent stable with delay margin  $h$  if there exists symmetric positive definite matrices  $P, Q, Z$  such that the LMI*

$$\begin{bmatrix}
 (A + Ah)^T P + P(A + Ah) + Q + hA^T Z A & hA^T Z A_h & +hPA_h \\
 \star & -Q + hA_h^T Z A_h & 0 \\
 \star & \star & -hZ
 \end{bmatrix} \prec 0 \tag{27}$$

*holds.*

## (Scaled) Small-Gain Theorem method

It is possible to provide delay-independent and delay-dependent stability tests based on the use of the small-gain theorem as emphasized for instance in [Zhang et al., 2001], showing some correspondence between small-gain and Lyapunov-Krasovskii based results .

Let us consider here the following operators:

$$\begin{aligned}\mathcal{D}_h &: x(t) \rightarrow x(t-h) \\ \mathcal{S}_h &: x(t) \rightarrow \int_{t-h}^t x(s)ds\end{aligned}$$

First of all, system (16) must be rewritten as an interconnection of two subsystems (i.e. a linear finite dimensional systems and the delay operator  $\mathcal{D}_h$ ) according to the framework of small-gain theorem.

$$\begin{aligned}\dot{x}(t) &= Ax(t) + A_h w(t) \\ z(t) &= x(t) \\ w(t) &= \mathcal{D}_h(z(t))\end{aligned}\tag{28}$$

It has been shown that the operator  $\mathcal{D}_h(\cdot)$  is asymptotically stable and therefore has finite  $\mathcal{H}_\infty$ -norm.

## Small-Gain Theorem method for Delay-Independent stability

Finally we obtain the following theorem:

### Theorem

*System (21) is delay-independent asymptotically stable if there exist matrices  $P = P^T \succ 0$  and  $L = L^T \succ 0$  such that the LMI*

$$\begin{bmatrix} A^T P + PA + L & PA_h \\ \star & -L \end{bmatrix} \prec 0 \quad (29)$$

*holds.*

It is easy to recognize the LMI obtained by application of the Lyapunov-Krasovskii theorem with Lyapunov-Krasovskii functional

$$V(x_t) = x(t)^T P x(t) + \int_{t-h}^t x(\theta)^T L x(\theta) d\theta \quad (30)$$



## Small-Gain Theorem method for Delay-Dependent stability

According to operator  $\mathcal{S}_h(\cdot)$ , system (16) is rewritten as

$$\begin{aligned}\dot{x}(t) &= (A + A_h)x(t) - A_h w(t) \\ z(t) &= (A + A_h)x(t) - A_h w(t) \\ w(t) &= \mathcal{S}_h(z(t))\end{aligned}\tag{31}$$

This reformulation is identical to the Euler model transformation and then adds additional dynamics. Hence systems (16) and (31) are not equivalent. The operator  $\mathcal{S}_h$  is LTI and it has been shown that it is stable; therefore it has finite  $\mathcal{H}_\infty$  norm. First, note that the corresponding transfer function is given by

$$\hat{\mathcal{S}}_h(s) = \frac{1 - e^{-sh}}{s}\tag{32}$$

Its  $\mathcal{H}_\infty$  norm is  $\gamma_\infty = \bar{h}$ .

A connection between Lyapunov-Krasovskii functionals and small-gain results has also been provided in [Zhang et al., 2001] in the delay-dependent framework.

## About time-varying delays

Time-varying delays induce complexity due to the need to handle the time-derivative of the delays. At a first sight, the Laplace transform of a time-varying delay does not exist, which means that we cannot write:

$$e^{-sh(t)}$$

which has no sense.

A solution is to use delay operators as in [Gu et al., 2003] valid for time-varying delays (see [Briat, 2008]) as for instance:

$$\begin{aligned}\mathcal{D}_h^s : w(t) &\rightarrow \frac{h_m}{h_M} \int_{t-h(t)}^t h(\eta)^{-1} w(\eta) d\eta \\ \mathcal{D}_h^c : w(t) &\rightarrow \int_{t-h(t)}^t \frac{1}{h(\xi) + h_{max} - h_{min}} w(\xi) d\xi\end{aligned}$$

Concerning the time-domain approach let us consider the Lyapunov-Krasovskii functional:

$$V(x_t) = x(t)^T P x(t) + \int_{t-h(t)}^t x(\theta)^T Q x(\theta) d\theta \quad (33)$$

where  $P, Q \in \mathbb{S}_{++}^n$  are constant decision matrices.

Compute the time-derivative of  $V(x_t)$ .

## About time-varying delays

A very intensive research area in order to reduce the conservatism of the methods.  
Most of the approaches presented before can deal with time-varying delays but need more complex tools:

LK approach: [He et al., 2007, Fridman, 2006]

Comparison system [Michiels et al., 2005]

Wellposedness [Ariba and Gouaisbaut, 2007]

Small gain theorem [Briat et al., 2007, 2008]: use of different operators

IQC [Kao and Rantzer, 2007]

# Outline

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## Polytopic uncertainties

Polytopic systems are really spread in robust analysis and robust control. A time-varying polytopic system is a system governed by the following expressions

$$\dot{x}(t) = A(\lambda(t))x(t) + A_h(\lambda(t))x(t - h(t)) + E(\lambda(t))w(t) \quad (34)$$

where

$$\begin{bmatrix} A(\lambda) & A_h(\lambda) & E(\lambda) \end{bmatrix} = \sum_{i=1}^N \lambda_i(t) \begin{bmatrix} A_i & A_h^i & E_i \end{bmatrix} \quad (35)$$

and  $\sum_{i=1}^N \lambda_i(t) = 1$ ,  $\lambda_i(t) \geq 0$ .

The term polytopic comes from the fact that the vector  $\lambda(t)$  evolves over the unit simplex (which is a polytope) defined by

$$\Gamma := \left\{ \underset{i}{\text{col}}(\lambda_i(t)) : \sum_{i=1}^N \lambda_i(t) = 1, \lambda_i(t) \geq 0 \right\} \quad (36)$$

See for instance [He et al., 2004, Gu et al., 2003]

## Norm-bounded uncertainties

In this case the matrices of the system:

$$\dot{x}(t) = Ax(t) + A_h x(t - h(t)) + E(w(t))$$

are written as:

$$\begin{aligned} A &= A_n + \Delta A \\ A_h &= A_{hn} + \Delta A_h \\ E &= E_n + \Delta E \end{aligned}$$

The uncertain matrices are given as:

$$\begin{bmatrix} \Delta A & \Delta A_h & \Delta E \end{bmatrix} = F.G. \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

where  $F$ ,  $H_1$ ,  $H_2$  and  $H_3$  are known and  $G$  is an uncertain matrix satisfying:

$$\|G\| \leq 1$$

They can also be given through the more general LFT formulation as :

$$\begin{bmatrix} \Delta A & \Delta A_h & \Delta E \end{bmatrix} = F.(I - GD)^{-1}G. \begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}$$

with  $(I - D^T D) > 0$  for well posedness.

See for instance [Gu et al., 2003]

# Introduction to uncertain delays

Stability with respect to delay uncertainty is an important problem which is still not really investigated. Some papers are devoted to or use results on robust stability analysis with respect to delay uncertainty [Verriest et al., 2002, Kharitonov and Niculescu, 2003, Michiels et al., 2005, Sename and Briat, 2006a].

Assuming that the stability of the system is known for a nominal delay value  $h_0$  the maximal deviation  $\delta$  from this nominal value for which the system remains stable is sought. Therefore the system will be shown to be stable for any delay belonging to  $[h_0 - \delta^+, h_0 - \delta^-]$ . In the case of a time-varying delay, the bound on the derivative of the variation  $\eta$  can also be considered.

*Remark: many authors consider this framework to deal with time-varying delay, assuming that only the uncertainty  $\delta(t)$  is time-varying.*

$$\dot{x}(t) = Ax(t) + A_h(x - h_0 + \theta(t)), \quad \theta(t) \in [\delta^-, \delta^+], \quad |\dot{\theta}| < \eta \quad (37)$$

# Stabilization of the Uncertain System

Let the **actual** system be given by the model

$$\dot{x}(t) = f_{\text{lin}}(x_t) + p_{\text{lin}}(x_t) + g_{\text{lin}}(u_t) + r_{\text{lin}}(u_t), \quad (38)$$

and

$$y(t) = h_{\text{lin}}(x_t) + q_{\text{lin}}(x_t) \quad (39)$$

where the terms  $p_{\text{lin}}$ ,  $q_{\text{lin}}$  and  $r_{\text{lin}}$  denote the **perturbation** away from the nominal system given previously.

**Assumption:** we shall assume that the perturbation  $p_{\text{lin}}$  is such that its Laplace transform may be expressed as

$$\mathcal{L}[p_{\text{lin}}(x_t)] = P(s, \theta)X(s) \quad (40)$$

where  $P(s, \theta) = P_1(s, 0)\theta + P_2(s, 0)\theta^2 + \dots$ , for some small parameter  $\theta$ . ( $\theta = 0$  retrieves the nominal system).

A similar expression is assumed to hold for  $q_{\text{lin}}$  and  $r_{\text{lin}}$ .

**Example:** let the nominal system be

$$\dot{x}(t) = Ax(t) + Bx(t - \tau_0)$$

and the actual system as above, but with  $\tau_0$  replaced by  $\tau$ , then  $p(x_t) = B[x(t - \tau) - x(t - \tau_0)]$ , and thus, with  $\theta = \tau - \tau_0$ , we get

$$P(s, \theta) = B[e^{-s\tau} - e^{-s\tau_0}] = Be^{-s\tau_0}[e^{-s\theta} - 1].$$



## Effect of Mismatching

The observer/controller is designed with the knowledge of the nominal system only. Hence,

$$\dot{\hat{x}}(t) = f_{\text{lin}}(\hat{x}_t) + g_{\text{lin}}(u_t) + l_{\text{lin}}(\varepsilon_t) \quad (41)$$

$$\varepsilon(t) = y(t) - h_{\text{lin}}(\hat{x}_t) \quad (42)$$

$$u(t) = -k_{\text{lin}}(\hat{x}_t). \quad (43)$$

The error dynamics is now coupled to the controlled system. Indeed, eliminating  $\hat{x}$ ,  $y$  and  $\varepsilon$  from the above set of equations, one obtains (denoting  $\tilde{x} = x - \hat{x}$ ):

$$\dot{\tilde{x}}(t) = \{[f_{\text{lin}} - l_{\text{lin}}h_{\text{lin}}] + r_{\text{lin}}k_{\text{lin}}\}(\tilde{x}_t) + [p_{\text{lin}} - l_{\text{lin}}q_{\text{lin}} - r_{\text{lin}}k_{\text{lin}}](x_t), \quad (44)$$

$$\dot{x}(t) = \{[f_{\text{lin}} - g_{\text{lin}}k_{\text{lin}}] + [p_{\text{lin}} - r_{\text{lin}}k_{\text{lin}}]\}(x_t) + \{g_{\text{lin}}k_{\text{lin}} + r_{\text{lin}}k_{\text{lin}}\}(\tilde{x}_t). \quad (45)$$

The **characteristic equation** is then:

$$\det \begin{bmatrix} sI - F + LH - RK & LQ + RK - P \\ -GK - RK & sI - F + GK - P + RK \end{bmatrix} = \det[sI - F + LH] \det[sI - F + GK] \det[I + \Delta^{-1}\Theta(s; \theta)].$$

where we set

$$\Delta(s) = \begin{bmatrix} sI - F + LH & 0 \\ -GK & sI - F + GK \end{bmatrix}.$$

The matrix

$$\Theta(s; \theta) = \left\{ \begin{bmatrix} R \\ R \end{bmatrix} [-K \ K] + \begin{bmatrix} L \\ 0 \end{bmatrix} [Q \ 0] + \begin{bmatrix} P \\ P \end{bmatrix} [0 \ -I] \right\}$$

collects the **perturbation terms** (reparametrized by  $\theta$ ).

# Rouché's Theorem

The Rouché's Theorem, a celebrated result of complex analysis [Levinson and Redheffer, 1970] allows to compute a bound on the variation on the delay for systems of the form (??). It provides a sufficient condition only but a bound can be easily computed from the computation of norms of operators. It has been employed in [Dugard and Verriest, 1998, Verriest et al., 2002, Sename and Briat, 2006a].

The Rouché's Theorem :

## Theorem

*Given two functions  $f$  and  $g$  analytic (holomorphic) inside and on a contour  $\gamma$ . If  $|g(z)| < |f(z)|$  for all  $z$  on  $\gamma$ , then  $f$  and  $f + g$  have the same number of roots inside  $\gamma$ .*

## Application to uncertain closed-loop TDS

In the previous case, if the closed loop nominal system is stable for a proper choice of  $k_{lin}$  and  $l_{lin}$ , then *the perturbed closed loop system remains stable if  $\rho(s; \theta) = \det[I + \Delta(s)^{-1} \Theta(s; \theta)]$  does not change sign when  $s$  sweeps the imaginary axis.*

# Actuator uncertainty

Here  $p_{\text{lin}} = q_{\text{lin}} = 0$ . We can prove that

$$\rho_r(s; \theta) = \det \left\{ I + [-K \ K] \Delta^{-1} \begin{bmatrix} R \\ R \end{bmatrix} \right\} := \det[I - Q_r(s)R(s; \theta)]$$

where  $R = R(s; \theta)$  is a perturbation and

$Q_r(s) = K[-(sI - F + LH)^{-1} + (sI - F + GK)^{-1}GK.(sI - F + LH)^{-1} + (sI - F + GK)^{-1}]$ , a quantity which only depends on the nominal system and observer/controller parameters.

Invoking Rouché's theorem, it follows that the [condition for stability](#) is

$$\|Q_r(s)R(s; \theta)\|_{\infty} < 1. \quad (46)$$

Note that this is the norm of a *scalar* quantity if one is dealing with a *single input* system, thus reducing the test to a simple frequency sweep.

**Conclusion:** If one can determine

$$\theta_+ = \min \{ \theta > 0 \mid \|Q_r(s)R(s; \theta)\|_{\infty} = 1 \},$$

$$\theta_- = \max \{ \theta < 0 \mid \|Q_r(s)R(s; \theta)\|_{\infty} = 1 \},$$

then for all  $\theta \in (\theta_-, \theta_+)$ , the determinant has a fixed sign, implying the absence of zero crossings, and henceforth the stability of the perturbed system (provided the nominal one is stable).

# Plant and Sensor uncertainties

In the same way, we get:

**Plant Uncertainty**  $\rho_p(s; \theta) = \det[I - Q_p(s)P(s; \theta)]$  where

$Q_p(s) = (sI - F + GK)^{-1}GK(sI - F + LH)^{-1} + (sI - F + GK)^{-1}$  The condition for stability is

$$\|Q_p(s)P(s; \theta)\|_{\infty} < 1. \quad (47)$$

In the case where  $P(s, \theta) = B[e^{-s\tau} - e^{-s\tau_0}]$ , the following bound can be deduced for the allowed delay uncertainty:

$$\theta_{max} < 1/(\|Q_p(s)Be^{-s\tau_0}s\|_{\infty}) \quad (48)$$

**Sensor Uncertainty**  $\rho_q(s; \theta) = \det[I + Q(s; \theta)Q_q(s)]$  where  $Q_q(s) = (sI - F + LH)^{-1}L$ .

The condition for stability is

$$\|Q(s; \theta)Q_q(s)\|_{\infty} < 1. \quad (49)$$

# Outline

- 1 Why TDS are of interest (but complex ...)?
- 2 TDS Analysis: controllability, observability
  - Controllability
  - Observability
- 3 Stability and Performance analysis
  - Frequency-Domain
  - Time-Domain (Lyapunov)
  - Lyapunov-Razumikhin Functions
  - Lyapunov-Krasovskii Functionals
  - Small-Gain Theorem
  - About time-varying delays
- 4 Towards robustness
  - Robust stability of time-delay systems
  - How to account for uncertain delays?
- 5 **Observation**
  - Observability - Observers**
  - $H_\infty$  observer design for uncertain time-delay systems**
- 6 **Control**
  - "Smith" predictors
  - $H_\infty$  state feedback control
  - $H_\infty$  dynamic output feedback control

## Strongly observable systems

No specific difficulty. Strong observability = dual to strong controllability for which pole placement (with polynomial poles) and coefficient assignment (with polynomial coefficients) can be solved.

for time-delay systems over a ring, coefficient assignment imply pole placement but the converse is not true as a characteristic polynomial may have non polynomial roots. For systems over a field (classical linear systems), these problems are equivalent.

Some references:

In [Lee and Olbrot, 1981] the dualization of Morse's result is presented to ensure the existence of some asymptotic observer.

Pourboghrat and Chyung [1986] have given a procedure to obtain a finite-time observer (exact) which is unusable when some parameter changes are considered as well as some external disturbances.

Emre and Khargonekar [1982] have proposed a polynomial approach to design observers by assigning the coefficient of the characteristic polynomial. The observer is constructed by solving some Bezout equations.

# Spectrally observable systems: finite spectrum assignment

[Watanabe, 1986]: spectral controllability is equivalent to finite spectrum assignment for multivariable systems with delays.

Use to design an observer that includes distributed time-delays as:

$$\begin{aligned}\dot{\hat{x}}(t) = & A(\nabla)\hat{x}(t) + B(\nabla)u(t) + K_0(\nabla)[C(\nabla)\hat{x}(t) - y(t)] \\ & + K_x(\nabla)[C_i(\nabla)\hat{x}(t) - y_i(t)] \\ & + \int_{-N_3h}^0 \phi(\tau)[C_i(\nabla)\hat{x}(t+\tau) - y_i(t+\tau)] d\tau + \int_{-N_4h}^0 \Psi(\tau)v_0(t+\tau) d\tau\end{aligned}$$

where  $i$  is a preassigned integer,  $C_i$  is the  $i$ th row of  $C$  and  $y_i$  the  $i$ th element of  $y$ . Moreover  $K_0(\nabla) \in \mathbb{R}^{n \times r}[\nabla]$ ,  $K_x(\nabla) \in \mathbb{R}^n[\nabla]$ ,  $\phi(\cdot) \in L_2([-N_3h, 0], \mathbb{R}^n)$ ,  $\Psi(\cdot) \in L_2([-N_4h, 0], \mathbb{R}^n)$ ,  $N_3$  and  $N_4$  are appropriate positive integers.

In this case the characteristic equation is:

$$\det \begin{bmatrix} sI_n - A^T(z) - C^T(z)K_0^T(z) & -C_i^T(z) \\ -K_a^T(s, z) & (1 - K_b(s)) \end{bmatrix} \quad (50)$$

with  $K_a(s, e^{-sh}) = K_x(e^{-sh}) + \int_{-N_3h}^0 \phi(\tau)e^{s\tau} d\tau$  and  $K_b(s) = \int_{-N_4h}^0 \Psi(\tau)e^{s\tau} d\tau$ .

## Weakly observable systems

Very few result since this assumption cannot guarantee to get a stable spectrum assignment in closed-loop.

[Sename et al., 1995a] : the coefficient of the characteristic polynomial cannot be assigned arbitrarily. In fact the coefficients must satisfy some constraints in terms of their polynomial form

[Picard et al., 1996] : have defined the constructible state submodule  $\mathcal{H}(\nabla)$  from the Smith form of the observability matrix, which is shown to correspond to some finite-time observer. Their result points out that the best that can be done without anticipation (i. e. without using  $y(t + \varepsilon)$ ) is to construct  $\mathcal{H}(\nabla)x(t)$ . This state reconstruction is then interpreted using observability indices relative to weak observability.



# Introduction to $H_\infty$ observer design

Lots of studies for TDS: Dugard and Verriest [1998], Niculescu [2001], Gu et al. [2003]

$H_\infty$  observers: Fattouh et al. [1999c, 1998], Choi and Chung [1996]

→ modified algebraic Riccati equations, no robustness

Robust filtering for time-delay systems: Fridman et al. [2003], Wu et al. [2006]

→ pbs when uncertainties are to be considered

Robust observer design:

Choi and Chung [1997], Wang et al. [1999b]: no delay uncertainties, involved calculations

Fattouh and Sename [2004]: parameterization of observers where the effects of *unstructured uncertainties* on the estimated states is minimized. But some "hand made" procedure for calculations.

In what follows

Presentation of the results developed in [Sename and Briat, 2007]

**Extension of the class of considered systems:** they can be submitted to uncertainties and external unknown disturbance as well.

The proposed methodology allows to **tackle all uncertainty types** while in [Fattouh et al., 2000a] only additive and multiplicative input uncertainties were considered

The observation error stability and the attenuation of uncertainties and disturbance effects are developed within a **delay dependent  $H_\infty$  framework**, allowing to reduce the conservatism of the approach in [Fattouh et al., 2000a].

The minimisation of both uncertainty and disturbance effects is shown within the concept of **Pareto optimality**.

# Modelling

$$G \begin{cases} \dot{x}(t) &= A_0 x(t) + A_1 x(t-h) + B u_o(t) + E d(t) \\ y_o(t) &= C_0 x(t) + C_1 x(t-h) \end{cases} \quad (51)$$

$x(t) \in \mathbb{R}^n$  : state,

$u_o(t)$ : nominal control input,

$d(t)$  unknown  $L_2$ -bounded disturbance input,

$y_o(t)$ : nominal output.

Input/output transfer matrix :

$$G(s, z) = C(z)(sI_n - A(z))^{-1} B(z) \quad (52)$$

where  $z = e^{-sh}$ ,  $A(z) = A_0 + A_1 z$  and  $C(z) = C_0 + C_1 z$ .

# Uncertainties

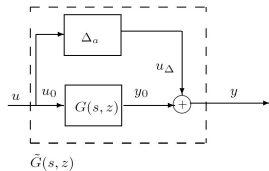


Figure: Additive.

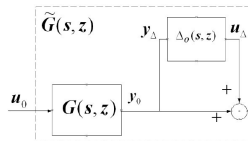


Figure: Mult. output

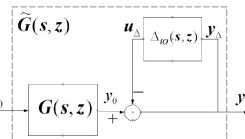


Figure: Inv. mult. output

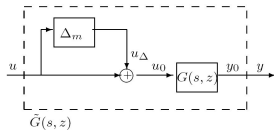


Figure: Mult. input

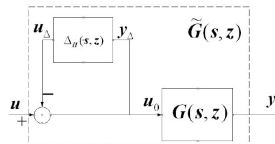


Figure: Inv. mult. input

## Uncertainties (2)

### Definition

Let  $G(s, z)$  be the nominal model (51) of the real plant  $\tilde{G}(s, z) = f(G(s, z), \Delta(s, z))$ , and  $\Delta(s, z)$  a variable stable transfer matrix s.t.  $\|\Delta(s, z)\|_\infty \leq \delta$ , which can be s.t.:

$$\begin{aligned}
 \text{Fig.1 : } \quad \tilde{G} &= G + \Delta_a, & \text{with } \|\Delta_a\|_\infty &\leq \delta_a \\
 \text{Fig.2 : } \quad \tilde{G} &= (I_p + \Delta_O)G, & \text{with } \|\Delta_O\|_\infty &\leq \delta_O \\
 \text{Fig.59 : } \quad \tilde{G} &= (I_p + \Delta_{iO})^{-1}G, & \text{with } \|\Delta_{iO}\|_\infty &\leq \delta_{iO} \\
 \text{Fig.4 : } \quad \tilde{G} &= G(I_r + \Delta_I), & \text{with } \|\Delta_I\|_\infty &\leq \delta_I \\
 \text{Fig.5 : } \quad \tilde{G} &= G(I_r + \Delta_{iI})^{-1}, & \text{with } \|\Delta_{iI}\|_\infty &\leq \delta_{iI}
 \end{aligned} \tag{53}$$

↪ Can include delay uncertainties

**Assumption 1:** The uncertainties are assumed to be  $L_2$ -bounded, i.e.  $\|u - u_o\|_2$  is assumed to be bounded (or  $\|y - y_o\|_2$  according to the uncertainty types).

## Observer form

the observer equations become:

$$\begin{cases} \dot{\hat{x}} &= A_0\hat{x} + A_1\hat{x}(t-h) + Bu + L(y - \hat{y}) \\ \hat{y} &= C_0\hat{x} + C_1\hat{x}(t-h) \\ \hat{r} &= D\hat{x} \end{cases} \quad (54)$$

where  $u$  and  $y$  do correspond to the real plant  $\tilde{G}$ , and take different forms according to the different uncertainty cases.

### Definition

Given a time-delay plant  $\tilde{G}(s, z)$  (53) with unstructured uncertainties.  $\hat{r}(t)$  is said to be a robust  $H_\infty$  estimation of  $r(t) = Dx(t)$  if:

1.  $\lim_{t \rightarrow +\infty} (r(t) - \hat{r}(t)) = 0$  when  $\Delta(s) \equiv 0$ ,
2. Under zero initial conditions, there exists some positive scalars  $\gamma_1$  and  $\gamma_2$  s.t., under  $\|\Delta(s)\| \leq \delta$ :

$$\frac{\|\hat{r}(s) - r(s)\|_2}{\|u_\Delta(s)\|_2} \leq \frac{\gamma_1}{\delta}, \quad \frac{\|\hat{r}(s) - r(s)\|_2}{\|d(s)\|_2} \leq \gamma_2$$

# Observer design

## Analysis of Condition 1:

Assume first that  $\Delta(s) \equiv 0$ .

The nominal estimation error is then such that:

$$\dot{e} = (A_0 - LC_0)e + (A_1 - LC_1)e(t-h) \quad (55)$$

If  $L$  is designed to ensure the stability of the previous time-delay system, then we have:

$$\lim_{t \rightarrow \infty} e_r(t) = \lim_{t \rightarrow \infty} (r(t) - \hat{r}(t)) = \lim_{t \rightarrow \infty} De(t) = 0$$

and Condition 1 in Definition 8 is satisfied.

## Observer design (2)

**Analysis of Condition 2:** When applying the observer on the real plant, we get (for instance) for direct multiplicative input uncertainties.

$$\dot{e} = (A_0 - LC_0)e + (A_1 - LC_1)e(t-h) + Ed + Bu_\Delta, \quad (56)$$

For all uncertainty types,  $e_r(t)$  satisfies:

$$\|e_r(s)\|_2 = \|T_a(s, z)\|_\infty \|u_\Delta(s)\|_2 + \|T_d(s, z)\|_\infty \|d(s)\|_2 \quad (57)$$

$$\text{or } \|e_r(s)\|_2 = \|T_I(s, z)\|_\infty \|u_\Delta(s)\|_2 + \|T_d(s, z)\|_\infty \|d(s)\|_2 \quad (58)$$

$$\text{with } \begin{cases} T_a(s, z) &= D(sI_n - A(z) + LC(z))^{-1}L \\ T_I(s, z) &= D(sI_n - A(z) + LC(z))^{-1}B \\ T_d(s, z) &= D(sI_n - A(z) + LC(z))^{-1}E \end{cases} \quad (59)$$

Then: if  $u_\Delta$  is bounded,  $e_r(t)$  will remain bounded as transfer matrices  $T_I(s, z)$ ,  $T_a(s, z)$  and  $T_d(s, z)$  are stable.

**Objective :** Minimize the  $H_\infty$  norm of the transfer matrices  $T_a(s, z)$  (or  $T_I(s, z)$ ) and  $T_d(s, z)$ .

Therefore Condition 2 in Definition 8 will be satisfied.

# Robust observer design

## Method

give a Bounded Real Lemma in the case of  $L_2$  unstructured uncertainties and disturbance input

Extend the results of [Wu et al. \[2004\]](#) (which concerns systems with time-varying delays) to the  $H_\infty$  framework. Developed in LK context.

apply the stability result to the estimation error system to get the observer gain.

Need of simplification : use of  $Z = \alpha P$ ,  $\alpha \in \mathbb{R}$ , as in [\[Jiang and Han, 2005\]](#)



# Robust observer design: main results

## Theorem

Given scalars  $\delta$ ,  $\alpha$ ,  $\gamma_1$  and  $\gamma_2$ ,  $\exists$  an  $H_\infty$  observer if there exist  $P = P^T > 0$ ,  $Q = Q^T > 0$ ,

$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \end{bmatrix} \geq 0$ , and matrices  $Y$ ,  $T$  and  $Z$  s.t:

$$\Phi < 0 \text{ and } \Psi \geq 0 \quad (60)$$

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \tau_{\max} \alpha A_0^T P & PB_\Delta & PE \\ * & \Phi_{22} & \tau_{\max} \alpha A_h^T P^T & 0 & 0 \\ * & * & -\tau_{\max} \alpha P & 0 & 0 \\ * & * & * & -(\frac{\gamma_1}{\delta})^2 I_r & 0 \\ * & * & * & * & * \end{bmatrix}, \quad \Psi = \begin{bmatrix} X_{11} & X_{12} & Y \\ * & X_{22} & T \\ * & * & \alpha P \end{bmatrix},$$

$$\begin{aligned} \Phi_{11} &= PA_0^* + A_0^T P + K C^T + K^T C^T + Y + Y^T + Q + \tau_{\max} X_{11} + D^T D \\ \Phi_{12} &= PA_h + K C_h - Y + T^T + \tau_{\max} X_{12} \\ \Phi_{22} &= -T - T^T - (1 - \nu)Q + \tau_{\max} X_{22} \end{aligned}$$

where  $PB_\Delta = -K$  or  $P = B$  according to the uncertainty type, and the observer gain is:  $L = -P^{-1}K$

# Robust observer design: optimisation

## Optimisation

To get the **minimal attenuation bound** (for the uncertainties subject to fixed disturbance attenuation level) :

$$\gamma_{min}^2 = \min_{P,Q,X,Y,T,Z} \gamma_1^2 \quad (61)$$

s.t.  $\Phi < 0, \Psi \geq 0, P > 0, Q > 0, X > 0, \gamma_2 > 0$

## Remark

The above result also ensures the stability of the nominal observer for time-varying delays (due to Wu et al. [2004]).

However the frequency-based uncertainty description cannot handle here time-varying delays. Then the  $\mathcal{L}_2$  properties is only true for systems with constant but *uncertain* delays.

## Example

Consider now the time-delay system which originates from [Wu et al., 2004] and has been modified by adding control and disturbance inputs:

$$\begin{cases} \dot{x} &= (A_0 + \Delta A_0)x + (A_1 + \Delta A_1)x(t-h) + Bu_o + Ed \\ y_o &= C_0x + C_1x(t-h) \end{cases} \quad (62)$$

with  $\Delta A_0 = HF(t)E_0$  and  $\Delta A_1 = HF(t)E_1$ ,  $F^T(t)F(t) \leq I_n \forall t$ .

where  $A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$ ,  $A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$ ,  $E_0 = \text{diag}(1.6; 0.05)$ ,  $E_1 = \text{diag}(0.1; 0.3)$ ,  $H = I_2$ ,

$B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$  and  $D = I_2$ .

The delay is assumed to be varying in the interval  $h \in [0; 1]$ , with a nominal value  $h = 1$ , and such that  $\dot{h}(t) \leq 0.9$ .

# Uncertainty modelling

## Uncertainty choice: additive uncertainties

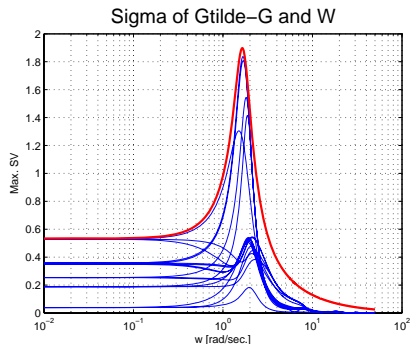
the transfer matrix can be represented as:

$\tilde{G}(s, z) = \{G(s, z) + W(s)\Delta(s, z) : \|\Delta\|_\infty \leq 1\}$ , with

$$G(s, z) = \frac{1}{s^2 + s - z}$$

$$W(s) = \frac{s + 1.2}{0.8s^2 + 0.65s + 2.25}$$

Note that  $\Delta_a = W\Delta$  is such that  $\|\Delta_a\|_\infty \leq \delta_a := 1.9$ .



Here the matrix  $F(t)$  has been fixed for six cases such that  $F^T(t)F(t) \leq I_n$  and  $h = 0, 0.5$  and  $1$ .

# Optimisation

$$\gamma_{1_{min}}^2 = \min \gamma_1^2 \text{ s.t. } \gamma_2 > 0 \text{ and } \gamma_{2_{min}}^2 = \min \gamma_2^2 \text{ s.t. } \gamma_1 > 0$$

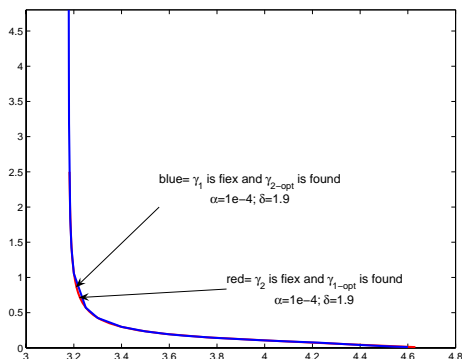


Figure: Pareto optimality, xaxis= $\gamma_1$ , yaxis =  $\gamma_2$

The Pareto limit in figure 6 emphasizes the usual performance/robustness trade-off, as the minimisation of  $\gamma_1$  will lead to an increase of  $\gamma_2$ .

# Frequency analysis

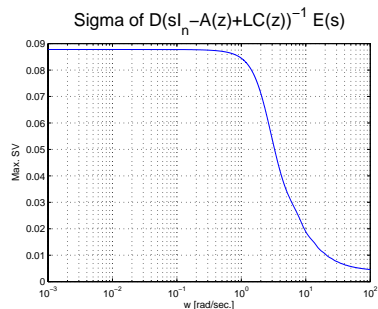
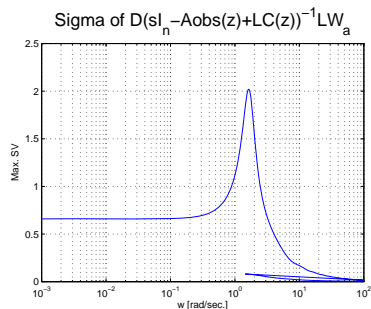
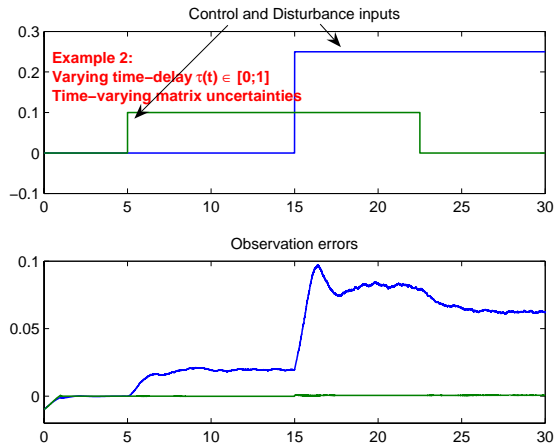


Figure: Effects of the uncertainty on the estimated error. Figure: Effect of disturbance on the estimation error

It shows that  $\|T_a(s, z)\|_\infty < \gamma_1 = 3.17$  and  $\|T_d(s, z)\|_\infty < \gamma_2 = 0.25$ .

# Simulation

Choice  $(\gamma_1 = \gamma_{unc}, \gamma_2 = \gamma_{dist}) = (3.47, 0.25) \Rightarrow L = \begin{bmatrix} 261.26 & 321.48 \end{bmatrix}^T$ .



The effects of the uncertainties and disturbance have been attenuated although the uncertainty attenuation level is quite high ( $\gamma_1 = 3.7$ ) which proves the performance and robustness of the observer.

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  - $H_\infty$  state feedback control
  - $H_\infty$  dynamic output feedback control



# Introduction

This lecture has been mainly written thanks to the results of:

Zhong [2006]

Witrant et al. [2005]

Lee et al. [1994, 2004]

Sename [2007]

Fattouh et al. [2001]

## Smith predictors

**Objective:** try to reduce the "presence" of the delay in the closed-loop system, as if the delay were shifted outside the feedback loop.

Consider

$$G(s) = e^{-sh}P(s)$$

Denote  $C_0(s)$  the controller designed using  $P(s)$  only

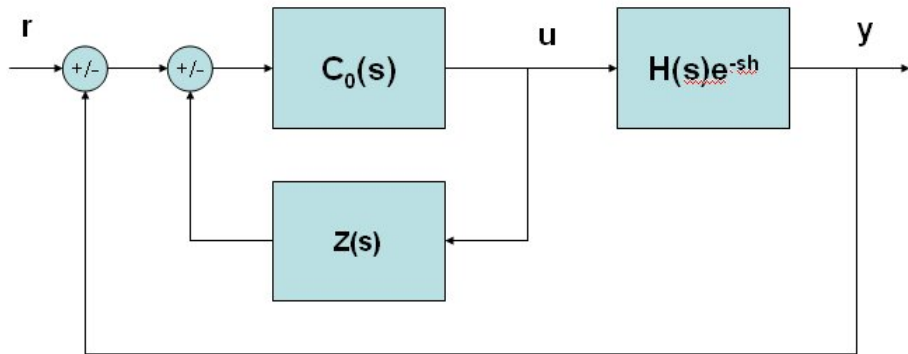
Define  $Z(s) = P(s) - e^{-sh}P(s)$  the predictor and the controller:

$$C(s) = \frac{C_0(s)}{1 + C_0(s)Z(s)}$$

The closed-loop system is then:

$$T(s) = \frac{C_0(s)P(s)}{1 + C_0(s)P(s)} e^{-sh}$$

## Smith predictors (cont.)



Calculate the sensitivity functions of this control scheme.

## About robustness

Problem : The model  $P$  of the plant differs from the real one  $\tilde{P}$ .  
For instance consider the multiplicative uncertainties  $\Delta$  s.t.

$$\tilde{P}(s)e^{-s\tilde{h}} = P(s)e^{-sh}(1 + \Delta(s))$$

The calculation of the new closed-loop system leads to:

$$\tilde{T}(s) = \frac{T(s)(1 + \Delta(s))}{1 + T(s)\Delta(s)e^{-sh}}e^{-sh}$$

where  $T$  is the nominal CL system.

Use of the small gain theorem. Robust stability is ensured if :

$$|T(j\omega)\Delta(j\omega)| < 1, \quad \forall \omega \geq 0$$

## Disturbance rejection

Calculate the input sensitivity function is the transfer function from  $d_u$  to  $y$ .

This leads to :

$$SG = \frac{P}{1 + C_0 P} \cdot e^{-sh} + \frac{C_0 P}{1 + C_0 P} \cdot Z \cdot e^{-sh}$$

$SG$  includes the poles of  $Z$ . If  $P$  is unstable ..... so is  $SG$ .

↪ the SP is applicable to stable plants only.

On the other hand  $SG(0) = 0$  only if:

$$\lim_{s \rightarrow 0} \frac{P}{1 + C_0 P} = 0$$

which is reached when  $C$  and  $CP$  incorporate an integrator.

**Examples:** analysis of the SP effect for :

$$G(s) = e^{-2s} \frac{1}{s + 1}$$

and for System:

$$G(s) = e^{-2s} \frac{1}{s - 1}$$

## Modified SP

To overcome the previous problem and allow for handling unstable system, the predictor is modified as follows:

Consider a state space realization of  $P$  denoted by:

$$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and note

$$\hat{P} = \begin{bmatrix} A & B \\ Ce^{-Ah} & D \end{bmatrix}$$

The predictor is then considered as:

$$Z = \hat{P} - Pe^{-sh} = Ce^{-Ah} \int_0^h e^{-(sI-A)\zeta} Bu(t-\zeta) d\zeta$$

Problem: there might be a unstable pole-zero cancelation in  $Z$  since  $P$  is unstable, which makes the implementation not trivial.

# Finite spectrum assignment

See [Manitius and Olbrot, 1979, Witrant et al., 2005, Wang et al., 1999a, Zhong, 2006].

Consider the system:

$$\begin{cases} \dot{x} &= Ax + Bu(t-h) \\ y &= Cx \end{cases} \quad (63)$$

The FSA adopts the state feedback control law:

$$u(t) = Fx_p(t)$$

where the predictive state is:

$$x_p(t) = e^{Ah}x(t) + \int_0^h e^{A\zeta} d\zeta B$$

and the CL system has a finite spectrum of  $(A + BF)$ .

Some new methods in the robust framework can be found in [Bresch-Pietri and Krstic, 2009, Bresch-Pietri et al., 2012]

# Stabilization by $H_\infty$ state feedback control

$$\begin{cases} \dot{x}(t) &= A_0x(t) + A_1x(t-h) + Bu(t) + Ew(t) \\ z(t) &= Dx(t) \end{cases} \quad (64)$$

with

$$u(t) = -Kx(t)$$

## Definition

*Let us consider a time-delay system (66). The  $H_\infty$  control problem consists in finding a state feedback control law  $u(t) = -Kx(t)$  such that, when  $w(t) \equiv 0$ ,  $x(t)$  converges to zero asymptotically and, under zero initial condition, the  $H_\infty$  norm of the transfer function between the disturbance  $w$  and the controlled output  $z$  is bounded, that is*

$$\lim_{t \rightarrow \infty} x(t) \rightarrow 0 \quad \text{for } w(t) \equiv 0 \quad \text{and} \quad \|T_{zw}(s)\|_\infty \leq \gamma$$

*where  $\|\cdot\|_\infty$  is the  $H_\infty$ -norm and  $\gamma > 0$  is a disturbance attenuation level.*



## $H_\infty$ state feedback control: a generic framework

Let us consider  $\dot{x}(t) = A_0x(t) + A_1x(t-h) + Bu(t)$ , with some controlled output  $z = Dx(t)$  and the control input  $u(t) = -Kx(t)$ .

In order to solve an  $H_\infty$  control problem we should ensure  $\|z\|_2 \leq \|w\|_2$ , i.e

$$J = \int_0^\infty (z^T z - \gamma^2 w^T w) \leq 0$$

Now, if  $V(\cdot)$  is a Lyapunov LR function (or LK functional), then,

$$J = \int_0^\infty (z^T z - \gamma^2 w^T w + \dot{V}(\cdot)) + V(\cdot)_{t=0} - V(\cdot)_{t=\infty} \leq 0$$

The method then consists in writing  $(z^T z - \gamma^2 w^T w + \dot{V})$  as a quadratic form, often like:

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \\ w(t) \end{bmatrix}^T [\mathcal{L}] \begin{bmatrix} x(t) \\ \dot{x}(t) \\ x(t-h) \\ w(t) \end{bmatrix}$$

Now, under 0 initial conditions and , and since  $V$  is positive, a sufficient condition to solve the  $H_\infty$  control problem is:

$$\mathcal{L} \leq 0$$

## $H_\infty$ state feedback control

Many works have been devoted to the stabilization problem (DeSouza, Niculescu, Park, Han, He, Fridman .....).

The problem is different for state delayed systems and for input-delayed ones.

One of the main problem is to use the "good" relaxation in order to reduce the conservatism of the method.

Thus "good" methods (i.e. less conservative ones) for stability analysis are not necessarily good ones for control design.

The extension of the BRL to Time-Delay systems can be found in [Fridman and Shaked, 2001]  
Interesting papers among others :

State delayed system [Wu et al., 2004, Jiang and Han, 2005]: delay-dependent robust stability for systems with time-varying structured uncertainties and time-varying delays. Use of free weighting matrices for reducing the conservatism.

Input delay system [Yue and Han, 2005]: delayed feedback control design for uncertain systems with time-varying input delay. Use of a reduction method, some relaxation matrices and turning parameters.

## Stabilization by $H_\infty$ dynamic output feedback control

In this case the objective is to find a controller such that

$$\begin{cases} \dot{x}_c(t) &= A_{c0}x_c(t) + A_{c1}x_c(t-h) + \dots + B_c y(t) \\ u(t) &= C_c x(t) + \dots + D_c y(t) \end{cases} \quad (65)$$

that solves the  $H_\infty$  control problem.

The main problem is that the coupling between controller matrices and decision variables ('Lyapunov' functions) is much more important than for the state feedback case and than from the LTI case.

# Towards observer-based control: are observer-controllers interesting ?

→  $H_\infty$  robust observer-controller (delay independent framework)

Contributions:

an LMI solution to the  $H_\infty$  observer-controller design, much simpler and powerful than previous results.

Provides an  $H_\infty$  disturbance attenuation property for the controlled output and for the state estimation errors as well

When the observer delay is different from the system one, a robustness analysis (w.r.t delay uncertainties) gives an evaluation of the maximum allowable delay uncertainty.

Comparison (on an illustrative example) with the method by [Choi and Chung, 1996]

For simplicity we consider:

$$\begin{cases} \dot{x} &= A_0x + A_1x_h + Bu + Ew \\ y &= Cx + Fw \\ z &= Dx \\ x(\theta) &= \phi(\theta); \quad \theta \in [-h, 0] \end{cases} \quad (66)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $x_h = x(t-h)$ ,  $u \in \mathbb{R}^r$  is the control input,  $y \in \mathbb{R}^p$  is the output measurement vector,  $z \in \mathbb{R}^m$  is the controlled output,  $w \in \mathbb{R}^q$  is the square-integrable disturbance vector,  $\phi(\theta) \in \mathbb{C}[-h, 0]$  is the functional initial condition and  $h \in \mathbb{R}^+$  is the delay.

## Background: Choi and Chung [1996] method

Consider system (66) (with  $F = I_p$ ) and the observer-controller:

$$\begin{cases} \dot{\hat{x}} &= (A_0 + EG)\hat{x} + A_1\hat{x}_h + Bu(t) - L(C\hat{x} - y) \\ u &= K\hat{x} \end{cases} \quad (67)$$

Suppose that the control parameters are:  $K = -\frac{1}{\varepsilon_c} B^T P_c$ ,  $G = \frac{1}{\gamma^2 \varepsilon_c} E^T P_c$ ,  $L = \frac{1}{\varepsilon_o} P_o C^T$  where  $\varepsilon_c > 0$ ,  $\varepsilon_o > 0$ , and  $P_c$  and  $P_o$  are positive definite solution matrices to the following Riccati-like equations for some  $\delta_c > 0$  and  $\delta_o > 0$  and some positive-definite weighting matrices  $Q_c$  and  $Q_o$ :

$$\begin{aligned} A_0^T P_c + P_c A_0 - \frac{1}{\varepsilon_c} P_c (BB^T - \frac{1}{\delta_c} A_1 A_1^T - \frac{1}{\gamma^2} EE^T) P_c + \varepsilon_c (\delta_c I_n + D^T D + Q_c) &= 0 \\ (A_0 + EG) P_o - \frac{1}{\varepsilon_o} P_o (C^T C - \frac{1}{\gamma^2} K^T K - \frac{\delta_o}{\gamma^2} I_n) P_o + P_o (A_0 + EG)^T + \varepsilon_o (\frac{\gamma^2}{\delta_o} A_1 A_1^T + EE^T + Q_o) &= 0 \end{aligned}$$

Then, for all  $h$ , the closed-loop system is asymptotically stable and such that :  $\|T_{zw}\|_\infty \leq \gamma$ .

## A preliminary approach: Independent design

Independent design of the observer and of the control.

Assumption: the observer has been designed using the results of Fattouh et al. [1999c] while the control has been obtained using Lee et al. [1994].

$H_\infty$  observer-based control using Luenberger-type observer as:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_0\hat{x}(t) + A_1\hat{x}(t-h) + Bu(t) - L(C\hat{x}(t) - y(t)) \\ u(t) &= K\hat{x}(t)\end{aligned}\tag{68}$$

where  $\hat{x}(t) \in \mathbb{R}^n$  is the estimated state of  $x(t)$  and  $L$  (resp.  $K$ ) is the  $n \times p$  (resp.  $r \times n$ ) constant observer (resp. controller) gain matrix.

Noting  $e(t) := x(t) - \hat{x}(t)$  and  $x_e(t) = \begin{bmatrix} x(t) & e(t) \end{bmatrix}^T$ , the closed-loop system with observer and control (68) is:

$$\begin{aligned}\dot{x}_e(t) &= \begin{bmatrix} A_0 + BK & -BK \\ 0 & A_0 - LC \end{bmatrix} x_e(t) + \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} x_e(t-h) \\ &\quad + \begin{bmatrix} E \\ E - LF \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} D & 0 \end{bmatrix} x_e(t)\end{aligned}\tag{69}$$

The characteristic polynomial of the extended system is:

$$\det(sI_n - A - BK - A_1e^{-sh}) \times \det(sI_n - A + LC - A_1e^{-sh})$$

# Independent design

**Stability:** the separation principle holds, then the closed-loop system with the dynamic measurement feedback is stable

**Performance:** the  $H_\infty$ -norm of the transfer function between the disturbance and the controlled output of (69),  $\tilde{T}_{zw}$ , is bounded as follows

$$\|\tilde{T}_{zw}\|_\infty \leq \|T_{zw}\|_\infty + \|H\|_\infty \|T_{ew}\|_\infty \leq \gamma_c + \gamma_o \|H\|_\infty \quad (70)$$

where  $T_{zw}(s) = D[sI_n - A_0 - BK - A_1 e^{-sh}]^{-1} E$ ,  
 $T_{ew}(s) = [sI_n - A_0 + LC - A_1 e^{-sh}]^{-1} (E - LF)$ ,  $H(s) = D[sI_n - A_0 - BK - A_1 e^{-sh}]^{-1} BK$ .

## Remark

*The obtained disturbance attenuation level for the controlled output is known only a posteriori by (70). Nevertheless, thanks to the triangular form of the extended system (69), the attenuation bound for the estimation error does not change when the system is in closed-loop.*

## Background: $H_\infty$ stability criterion

### Lemma

[Gu et al., 2003, Kokame et al., 1998] Consider system (66). Given a positive scalar  $\gamma$ , if there exist some positive definite matrices  $P = P^T$  and  $S$  such that

$$\mathcal{L} = \begin{bmatrix} A_0'P + PA_0 + D'D + S & PA_1 & PE \\ A_1'P & -S & 0 \\ E'P & 0 & -\gamma^2 I_q \end{bmatrix} < 0,$$

then the trivial solution of (66) with  $w \equiv 0$ ,  $u \equiv 0$ , is asymptotically stable for any delay, and  $\|T_{zw}(s)\|_\infty \leq \gamma$ , for zero initial condition and  $\gamma > 0$ .

$\gamma_{min}$  can be found by solving the convex optimization problem:

$$\begin{aligned} \gamma_{min}^2 &= \min_{P,S} \gamma^2 \\ \text{s.t. } \mathcal{L} &< 0, \quad P > 0, \quad S > 0, \end{aligned} \tag{71}$$



## Proposed observer-controller structure

Observer-controller

$$\begin{cases} \dot{\hat{x}} &= A_0 \hat{x} + A_1 \hat{x}_h + Bu - L(C\hat{x} - y) + G\hat{x} \\ u &= K\hat{x} \end{cases} \quad (72)$$

Noting  $e := x - \hat{x}$ , and the extended state  $x_e = \begin{bmatrix} x & e \end{bmatrix}^T$ , the extended closed-loop system with observer and control (72) is:

$$\begin{aligned} \dot{x}_e &= \begin{bmatrix} A_0 + BK & -BK \\ -G & A_0 - LC + G \end{bmatrix} x_e + \\ &\quad \begin{bmatrix} A_1 & 0 \\ 0 & A_1 \end{bmatrix} (x_e)_h + \begin{bmatrix} E \\ E - LF \end{bmatrix} w \\ &= \mathcal{A}_0 x_e + \mathcal{A}_1 (x_e)_h + \mathcal{E} w \end{aligned} \quad (73)$$

Objectives: provide results that ensure  $H_\infty$  stability for the controlled output  $z$  and the estimation error  $e$ . Two cases:

obtain an  $H_\infty$  stabilization of a **new controlled output, combining  $z$  and  $e$**  (with a unique attenuation bound  $\gamma$ )

get the  $H_\infty$  stabilization of the controlled output  $z = Dx$  and of the estimation error  $e$  (with **two different attenuation bounds**,  $\gamma_c$  and  $\gamma_o$  resp.).

# Problem definition

## Definition

*The system (72) is said to be an  $H_\infty$  observer-controller for system (66) if the trivial solution of (73) with  $w(t) \equiv 0$ , is asymptotically stable, and*

$$\|T_{z_e w}(s)\|_\infty \leq \gamma \text{ for zero initial condition and some positive scalar } \gamma, \text{ with } z_e = \begin{bmatrix} D & 0 \\ 0 & I_n \end{bmatrix} x_e = \mathcal{D}x_e, \text{ or}$$

$$\|T_{z_c w}(s)\|_\infty \leq \gamma_c \text{ and } \|T_{z_o w}(s)\|_\infty \leq \gamma_o, z_c = \begin{bmatrix} D & 0 \end{bmatrix} x_e \text{ and } z_o = \begin{bmatrix} 0 & I_n \end{bmatrix} x_e, \text{ for zero initial condition and some positive scalars } \gamma_c \text{ and } \gamma_o.$$

# Solution to case 1

## Theorem

Consider system (66) and observer-controller (72). Given  $\gamma > 0$ , if there exist positive definite  $P_c = P_c^T$ ,  $P_o = P_o^T$ ,  $S_c$  and  $S_o$ , and  $X \in \mathbb{R}^{m \times n}$ ,  $Y \in \mathbb{R}^{n \times p}$  satisfying:

$$\mathcal{L}_{oc} = \begin{bmatrix} M_c & 0 & A_1 P_c & 0 & E & P_c D^T \\ * & M_o & 0 & P_o A_1 & P_o E - LF & 0 \\ * & * & -S_c & 0 & 0 & 0 \\ * & * & * & -S_o & 0 & 0 \\ * & * & * & * & -\gamma^2 I_q & 0 \\ * & * & * & * & * & -I_n \end{bmatrix} < 0, \quad (74)$$

where  $*$  means the symmetric element

$$M_c = A_0 P_c + P_c A_0^T + B X + X^T B^T + S_c$$

$$M_o = A^T P_o + P_o A - P_c^{-1} (X B + B^T X^T) P_c^{-1} + C^T Y Y^T + Y C + I_n + S_o,$$

then (72) is an  $H_\infty$  observer-controller with the disturbance attenuation level  $\gamma$  and the observer-controller gains:

$$L = -P_o^{-1} Y, \quad K = X P_c^{-1}, \quad G = -K^T B^T P_c^{-1} P_o^{-1}$$

## Relaxation

If the minimal attenuation bound is to be searched, then the following optimization problem has to be solved:

$$\gamma_{min}^2 = \min_{P_c, P_o, X, Y, S_c, S_o} \gamma^2 \quad (75)$$

$$\text{s.t. } \mathcal{L}_{oc} < 0, P_c > 0, P_o > 0, \quad (76)$$

$$S_c > 0, S_o > 0,$$

Of course the problem to be solved is not convex due to the term  $P_c^{-1}(XB + B^T X^T)P_c^{-1}$  in  $M_o$ . Note that this can be rewritten as:

$$M_o = A^T P_o + P_o A + C^T Y^Y + Y C + I_n + S_o - Z^T - Z \quad (77)$$

with

$$Z = -G^T P_o = P_c^{-1} B K = P_c^{-1} B X P_c^{-1}$$

A first attempt (iterative procedure) has been used to solve this non convex problem

## Relaxation procedure (2)

Initialisation: solve the LMI problem (75) with  $M_o$  of the form (77),  $Z$  being unknown and get  $Z$ ,  $\gamma^1 = \gamma_{min}^1$ . Set  $test = 1$ ,  $tol = 1e-3$ ,  $i = 1$  and  $Niter = 50$ .

- a) while ( $test == 1$ ) and ( $i < Niter$ ),
- b) Solve the LMI problem (75) with  $M_o$  of the form (77) ( $Z$  being the one obtained at step 1) and get  $P_c$ ,  $P_o$ ,  $X$ ,  $Y$ ,  $S_c$ ,  $S_o$ ,  $\gamma_{min}$
- c) Calculate  $Z = P_c^{-1} B X$  and set  $\gamma^i = \gamma_{min}^i$
- d)  $test = (\frac{\gamma^i - \gamma^{i-1}}{\gamma^i} > tol)$
- e)  $i = i + 1$ ,  $\gamma^{i-1} = \gamma^i$
- f) end

If  $test = 0$ , then  $\gamma_{min} = \gamma_i$ . calculate  $L = -P_o^{-1} Y$ ,  $K = X P_c^{-1}$  and  $G = -K^T B^T P_c^{-1} P_o^{-1}$

Check the  $H_\infty$  closed-loop stability by solving the optimisation problem (71) (following Lemma 11) on the extended system (73) with  $L$ ,  $K$  and  $G$  obtained at step 3. This allows to **get a posteriori the minimal attenuation bound**.

## Solution to case 2

In that case the result of lemma 11 is applied to the extended system (73) in both configurations:

$$z_c = \begin{bmatrix} D & 0 \end{bmatrix} x_e = \mathcal{D}_c x_e \text{ (requiring an attenuation bound } \gamma_c)$$

$$z_o = \begin{bmatrix} 0 & I_n \end{bmatrix} x_e = \mathcal{D}_o x_e \text{ (requiring an attenuation bound } \gamma_o)$$

## Solution to case 2

**Theorem :** Consider the time-delay system (66) and the observer-controller (72). Given some positive scalars  $\gamma_c$  and  $\gamma_o$ , if there exist positive definite matrices  $P_c = P_c^T$ ,  $P_o = P_o^T$ ,  $S_c$  and  $S_o$ , and some matrices  $X \in \mathbb{R}^{m \times n}$ ,  $Y \in \mathbb{R}^{n \times p}$  satisfying the following matrix inequalities:

$$\mathcal{L}_c = \begin{bmatrix} M_c & 0 & A_1 P_c & 0 & E & P_c D^T \\ * & M_o^c & 0 & P_o A_1 & P_o E + YF & 0 \\ * & * & -S_c & 0 & 0 & 0 \\ * & * & * & -S_o & 0 & 0 \\ * & * & * & * & -\gamma_c^2 I_q & 0 \\ * & * & * & * & * & -I_n \end{bmatrix} < 0, \quad (78)$$

$$\mathcal{L}_o = \begin{bmatrix} M_c & 0 & A_1 P_c & 0 & E \\ * & M_o^o & 0 & P_o A_1 & P_o E + YF \\ * & * & -S_c & 0 & 0 \\ * & * & * & -S_o & 0 \\ * & * & * & * & -\gamma_o^2 I_q \end{bmatrix} < 0, \quad (79)$$

where  $*$  means the symmetric element, and

$$\begin{aligned} M_c &= A_0 P_c + P_c A_0^T + BX + X^T B^T + S_c, \\ M_o^c &= A^T P_o + P_o A - P_c^{-1} (BX + X^T B^T) P_c^{-1} + C^T Y^Y + YC + S_o, \\ M_o^o &= A^T P_o + P_o A - P_c^{-1} (BX + X^T B^T) P_c^{-1} + C^T Y^Y + YC + I_n + S_o \end{aligned}$$

## Solution to case 2

then the system (72) is an  $H_\infty$  observer-controller according to Definition 10, with the disturbance attenuation levels  $\gamma_c, \gamma_o$  (respectively for the controlled output  $z = Dx$  and for the state estimation error  $e$ ) and the observer-controller gains:

$$L = -P_o^{-1}Y, \quad K = XP_c^{-1}, \quad G = -P_o^{-1}K^TB^TP_c^{-1} \quad (80)$$

If a minimal attenuation bound is to be searched, a solution is to solve the following optimization problem:

$$\begin{aligned} \gamma_{min} &= \min_{P_c, P_o, X, Y, S_c, S_o} \frac{1}{2}(\gamma_c + \gamma_o) \\ \text{s.t. } &\mathcal{L}_c < 0, \mathcal{L}_o < 0, P_c > 0, P_o > 0, S_c > 0, S_o > 0, \end{aligned} \quad (81)$$

Finally the intuitive procedure given above to overcome the nonlinearity in the matrix inequality can be directly applied to this case, with the new optimisation problem (81).



## Application to a wind tunnel model

A simplified mathematical model of the Mach number dynamic response to guide vane changes [Manitius, 1984].

delay : transportation time between the guide vanes of the fan and the test section of the tunnel.

Treated in [Germani et al., 2000], where an approximation approach is used to design a LQG control, i.e. in the presence of Gaussian noise, and assuming the exact knowledge of the delay.

# Wind tunnel model

Linear approximated model:

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -0.5091 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -36 & -9.6 \end{bmatrix} x + \begin{bmatrix} 0 & -0.005956 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_h \\ &+ \begin{bmatrix} 0 \\ 0 \\ 36 \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} w \\ y &= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x + w\end{aligned}$$

and  $h = 0.33\text{sec.}$ ,  $x_1$  is the Mach number,  $x_2$  is the guide vane angle and  $x_3 = \dot{x}_2$ . The disturbance is a resistant torque on the input motor.

Controlled outputs (Mach number, vane angle):

$$z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x \quad (82)$$

## Separated design

Applying the results in Fattouh et al. [1999c] and Lee et al. [1994], a solution can be obtained with  $\gamma_o = 2.5$  and  $\gamma_c = 0.3$ , which gives:

$$L = \begin{bmatrix} 0 & 0.2048 & 0.000655 \end{bmatrix}^T \quad K = \begin{bmatrix} 0 & -0.14247 & -0.0898 \end{bmatrix}$$

A fixed output disturbance attenuation level is guaranteed (here  $\|\tilde{T}_{zw}\|_\infty = 0.289$ ) as we can see in figure 9.

We can prove that  $\|\tilde{T}_{zw}\|_\infty$  satisfies (see section 3):

$$\|\tilde{T}_{zw}\|_\infty \leq \|T_{zw}\|_\infty + \|H\|_\infty \|T_{ew}\|_\infty \quad (83)$$

$$\leq \gamma_c + \|H\|_\infty \gamma_o \text{ (i.e. } 0.3 + 2.5 \times 0.147 = 0.735) \quad (84)$$

# Separated design

Moreover, in Fig. 10 we can see that the attenuation property of  $T_{ew}(jw)$  is less than the prespecified bound (1.04 instead of 2.5).

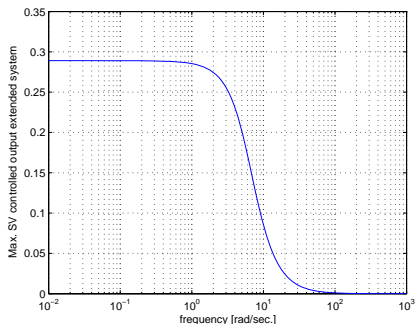


Figure:  $\sigma_{\max}(T_{zw}(jw))$ -Separated design

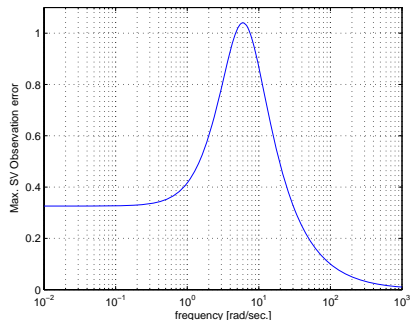


Figure:  $\sigma_{\max}(T_{ew}(jw))$  -Separated design

# Separated design

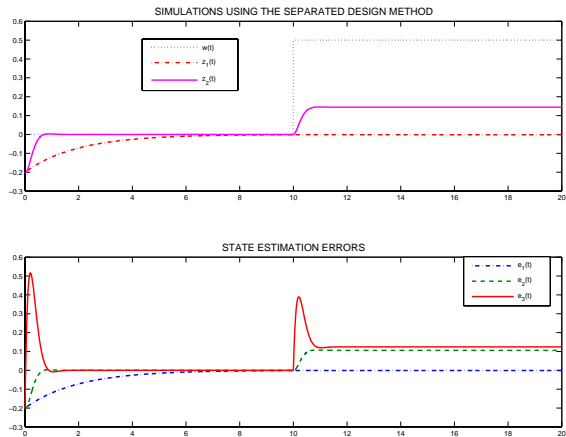


Figure: Controlled output and estimated errors -Separated design

## Choi and Chung [1996] method

First note that due to the 7 parameters to be set, the procedure is quite involved and not systematic. A solution can be obtained for  $\gamma = 1.4$  (for the closed-loop system). Results are shown in figure 12.

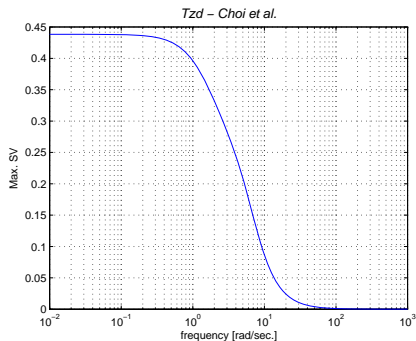


Figure:  $\sigma_{\max}(T_{zw}(jw))$

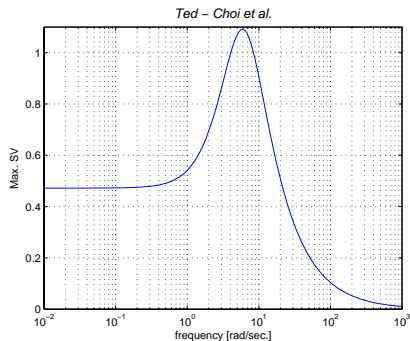
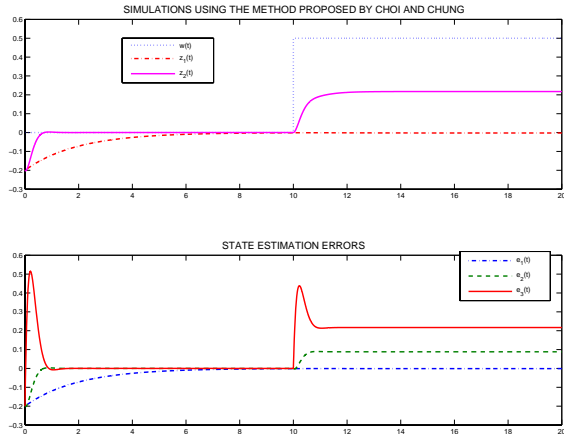


Figure:  $\sigma_{\max}(T_{ew}(jw))$

# Choi and Chung [1996] method (2)



No attenuation property is guaranteed in theory for the observer

Robustness analysis :  $\theta_{max} = 118.7 \text{ sec.}$

To conclude : **No real interest of this mixed design** (results are not so far different from what could be obtained using a separated design).

## Proposed method- case 1

The iterative procedure is solved in 4 iterations with  $\gamma_{min} = 0.01$  and

$$G \simeq 0_{3 \times 3}, \quad K = \begin{bmatrix} 0 \\ -4.482310^6 \\ -29315 \end{bmatrix}^T, \quad L = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

Now, when applying the step 4 of procedure, we obtain

$$\gamma_{min}^{step4} = 1.55 \cdot 10^{-6},$$

which, as we can see in figures 14 and 15 is a good estimation of the real obtained disturbance attenuation level.

Of course we can note that some gains are very large. This is due to the fact that the minimal attenuation bound is required which, in this case, is near 0. If one aims to solve a sub optimal problem only (i.e. with  $\gamma$  given a priori) the gain will be much less large.



# Results : frequency-domain

## Frequency-domain analysis

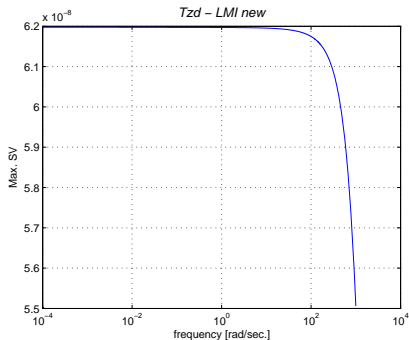


Figure:  $\sigma_{\max}(T_{zd}(j\omega))$

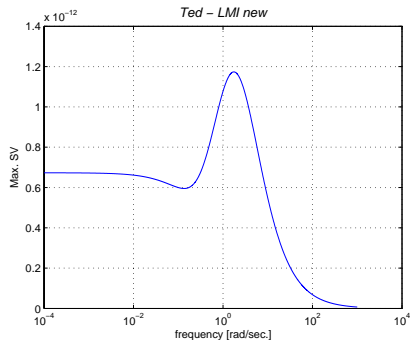
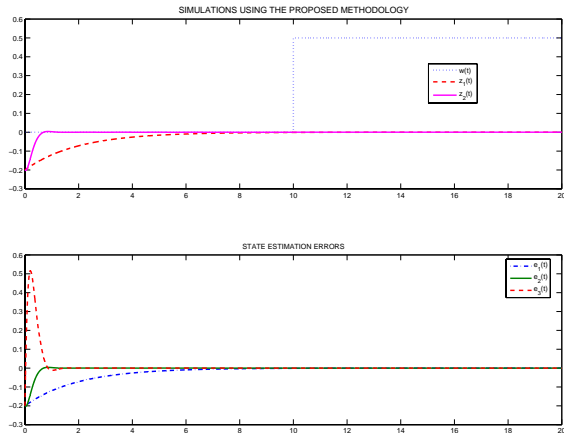


Figure:  $\sigma_{\max}(T_{ed}(j\omega))$

# Results : time domain



The observer controller is not affected by the disturbance input

Robustness : maximal delay uncertainty  $\theta_{max} = 118.7 \text{ sec.}$

To conclude: efficiency of the LMI formulation and  $H_\infty$  disturbance attenuation ensured for both  $z$  and  $e$ .

## Concluding remarks

A new solution to observer-controller design for TDS: simple design due to LMI formulation. It allows to get a closed-loop system and an observer which both satisfy an  $H_\infty$  attenuation property

Great flexibility : it is worth noting that one can design (if possible) the best observer-controller w.r.t a disturbance attenuation property but could also design an observer-controller scheme with attenuation levels specified a priori for the estimation errors and for the controlled outputs.

could be directly extended to the case of multiple time-delay and also for time-varying delays. Further study: delay-dependent  $H_\infty$  observer-controller.

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