A NOTE ON DETECTABILITY, OBSERVABILITY AND STABILITY OF IMPLICIT LINEAR DISCRETE–TIME SYSTEMS [1]

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Abstract: The problem of detectability for a (not necessarily regular or square) implicit linear discrete-time system is considered. Some necessary and sufficient conditions of detectability of such a system are given. In addition some relationships between the concepts of detectability, observability and stability are discussed.

Résumé: On considère la détecibilité pour un système linéaire implicite discret, pas nécessairement régulier ou carré; en particulier, on donne quelques conditions nécessaires et suffisantes. De surcroît, on considère des relations entre les concepts de détectabilité, d'observabilité et de stabilité.

Keywords: Observability, Stability, Implicit systems, Singular systems, Discrete-time systems, Linear systems, Detectability, Matrix pencils.

1. INTRODUCTION

We will consider *observed implicit linear discrete-time systems*, i.e., systems given by

$$\boxed{\begin{aligned} Ex_{k+1} &= Fx_k, \\ y_k &= Hx_k; \end{aligned}} \qquad (*)$$

here $k \in \mathbb{Z}_0^+ := \{0, 1, \ldots\}$, E, $F \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$, $H \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{X}$, $\mathcal{Z}$, $\mathcal{Y}$ are arbitrary finite-dimensional linear spaces over $\mathbb{R}$. (By $\mathcal{L}(\cdot, \cdot)$ we mean the space of all linear mappings $\cdot \rightarrow \cdot$.) The above introduced observed implicit system will be denoted by $\mathfrak{S}(E, F, H)$. Let us note that such a system can be viewed as the set of all pairs of sequences $((x_k), (y_k))$ satisfying Equations (*). When we are only interested in the solutions of the following difference equation

$$\boxed{Ex_{k+1} = Fx_k,} \qquad (**)$$

we denote the resulting system by $\mathfrak{S}(E, F)$. The spaces $\mathcal{Z}$, $\mathcal{X}$ and $\mathcal{Y}$ are sometimes called the *external variable space, internal variable space* and *output space of* $\mathfrak{S}(E, F, H)$, respectively. An analogous terminology is also used for the system $\mathfrak{S}(E, F)$. For the system $\mathfrak{S}(E, F, H)$ we will study the problem of detectability. In the present note detectability is defined as the property which guarantees that when an output trajectory $(y_k) = (Hx_k)$ of an observed implicit system $\mathfrak{S}(E, F, H)$ vanishes then necessarily the corresponding internal variable trajectory (x_k) vanishes asymptotically. The so defined concept of detectability plays an important role when constructing an observer for implicit systems (see e.g. (Blanchini, 1989; Przyłuski, 1995a; Przyłuski, 1995b; Przyłuski and Sosnowski, 1996; Shafai and Carroll, 1987; Verhaegen and Van Dooren, 1986)). In the note we provide some necessary and sufficient conditions

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for detectability. We will also discuss the question in what way detectability is related to observability and stability. To explain in detail the relationship between detectability, observability and stability, a Kalman-type decomposition for observed implicit systems will be presented.

Let us emphasize that the considered systems are *not* assumed to be regular (or even 'square'). Such systems (or their continuous-time counterparts) arise naturally in various applications. (For more details, we refer the reader to (Banaszuk *et al.*, 1989a; Bernhard, 1982), and to the references cited therein.) To obtain the results summarized in the present note we are using both frequency-domain techniques (for their use in the theory of standard systems, see e.g. (Hautus, 1983; Wolovich, 1974)) and some ideas taken from the so called geometric theory of linear systems (for standard systems, see (Wonham, 1974)). Details of our approach as well as some examples and all proofs will be presented in a separate publication. The proofs given there are algebraic and do not require such tools like Kronecker's canonical form. It is interesting to note that almost the same proofs work for continuous-time implicit systems.

We end this introduction by noting that various results concerning non-regular implicit linear (continuous-time and discrete-time) systems are recorded in (Banaszuk *et al.*, 1990a; Banaszuk *et al.*, 1987; Bernhard, 1982; Özçaldiran, 1986; Wong, 1974), and also in other references.

2. BASIC DEFINITIONS

Let $\mathfrak{S}(E, F, H)$ be an observed system, with $\mathcal{Z}$, $\mathcal{X}$ and $\mathcal{Y}$ being its external, internal variable and output space, respectively. For any linear space $\mathcal{S}$, the symbol $s_0^+(\mathcal{S})$ will denote the linear space of all sequences $\mathbb{Z}_0^+ \to \mathcal{S}$.

The basic definitions of the note are as follows.

Definition 1. The system $\mathfrak{S}(E, F, H)$ is said to be *detectable* iff for every $((x_k), (y_k)) \in s_0^+(\mathcal{X}) \times s_0^+(\mathcal{Y})$ satisfying Equations (*), $(y_k) = 0$ implies $\lim_{k\to\infty} x_k = 0$.

Definition 2. The system $\mathfrak{S}(E, F, H)$ is said to be *very detectable* iff for every $((x_k), (y_k)) \in s_0^+(\mathcal{X}) \times s_0^+(\mathcal{Y})$ satisfying Equations (*), $\lim_{k\to\infty} y_k = 0$ implies $\lim_{k\to\infty} x_k = 0$.

Definition 3. The system $\mathfrak{S}(E, F, H)$ is said to be *observable* iff for every $((x_k), (y_k)) \in s_0^+(\mathcal{X}) \times s_0^+(\mathcal{Y})$ satisfying Equations (*), $(y_k) = 0$ implies $(x_k) = 0$.

Remark 1. In (Banaszuk *et al.*, 1996) observability is called 'initial observability' since (as one can easily show) *a system $\mathfrak{S}(E, F, H)$ is observable if and only if for every $((x_k), (y_k)) \in s_0^+(\mathcal{X}) \times s_0^+(\mathcal{Y})$ satisfying Equations (*), $(y_k) = 0$ implies $x_0 = 0$.*

Let us consider now a system $\mathfrak{S}(E, F)$. The following properties of such a system will be used in the note.

Definition 4. The system $\mathfrak{S}(E, F)$ is said to be *stable* iff for every $x \in \mathcal{X}$ for which there exists $(x_k) \in s_0^+(\mathcal{X})$ satisfying Equation (**) with $x_0 = x$, one can find $(\tilde{x}_k) \in s_0^+(\mathcal{X})$ satisfying Equation (**) with $\tilde{x}_0 = x$, such that $\lim_{k\to\infty} \tilde{x}_k = 0$.

Definition 5. The system $\mathfrak{S}(E, F)$ is said to be *strictly stable* iff for every $(x_k) \in s_0^+(\mathcal{X})$ satisfying Equation (**), $\lim_{k\to\infty} x_k = 0$.

Definition 6. The system $\mathfrak{S}(E, F)$ is said to *possesses the uniqueness property* iff for every $(x_k) \in s_0^+(\mathcal{X})$ satisfying Equation (**), $x_0 = 0$ implies $(x_k) = 0$.

Definition 7. The system $\mathfrak{S}(E, F)$ is said to *accepts absolutely all input sequences* iff for every $(z_k) \in s_0^+(\mathcal{Z})$ there exists $(x_k) \in s_0^+(\mathcal{X})$ such that the pair $((z_k), (x_k))$ satisfies $Ex_{k+1} = Fx_k + z_k$, for $k \in \mathbb{Z}_0^+$.

Remark 2. The concept of stability defined above is a special case of the concept of stabilizability considered in (Banaszuk and Przyluski, 1997a). The uniqueness property and also the property of acceptance are discussed in (Banaszuk *et al.*, 1990a; Banaszuk *et al.*, 1987). It is interesting to note that *a system accepts absolutely all input sequences and possesses the uniqueness property if and only if the system is regular*; let us recall that a system $\mathfrak{S}(E, F)$ is said to be *regular* iff the corresponding pencil $(\lambda E - F)$ is regular, i.e. $(\mu E - F)$ is invertible for at least one $\mu \in \mathbb{C}$. Of course, a necessary condition of regularity is the equality $\dim \mathcal{X} = \dim \mathcal{Z}$.

We say that an observed system $\mathfrak{S}(E, F, H)$ has one of the above introduced properties (i.e. it is stable, strictly stable or accepts absolutely all input sequences) iff the system $\mathfrak{S}(E, F)$ has this property.

We end this section with the following obvious result.

Proposition 1. $\mathfrak{S}(E, F)$ is strictly stable if and only if it is stable and possesses the uniqueness property.

3. GENERAL RESULTS ON DETECTABILITY

Let $\mathbb{D}_1$ denote the open unit disk of $\mathbb{C}$. The following theorem provides necessary and sufficient conditions of detectability.

Theorem 1. Let $\mathfrak{S}(E, F, H)$ be an observed system. Then the following conditions are equivalent:

(i) $\mathfrak{S}(E, F, H)$ is very detectable.
(ii) $\mathfrak{S}(E, F, H)$ is detectable.
(iii) The condition

$$\operatorname{rank}\begin{bmatrix}\mu E - F \\ H\end{bmatrix} = \dim \mathcal{X}$$

is satisfied, for all $\mu \in \mathbb{C} \setminus \mathbb{D}_1$.

A direct frequency-domain proof of the above theorem can be easily given. The most difficult step in such a proof is to show that Condition (iii) of Theorem 1 implies very detectability. A proof of this implication can be based on the fact that Condition (iii) of this theorem is equivalent with left invertibility of the polynomial

$$\begin{bmatrix}\lambda E - F \\ H\end{bmatrix}$$

over an appropriately defined principal ideal domain of stable rational functions.

The following result is an immediate consequence of Theorem 1.

Corollary 1. $\mathfrak{S}(E, F)$ is strictly stable if and only the condition

$$\operatorname{rank}\left[\mu E - F\right] = \dim \mathcal{X}$$

is satisfied, for all $\mu \in \mathbb{C} \setminus \mathbb{D}_1$.

At this point it is interesting to formulate the following necessary and sufficient condition of stability. (The condition is a special case of some more general results of (Banaszuk and Przyluski, 1997a).)

Proposition 2. $\mathfrak{S}(E, F)$ is stable if and only the condition

$$\max_{\mu\in\mathbb{C}} \operatorname{rank}\left[\mu E - F\right] = \min_{\mu\in\mathbb{C}\setminus\Omega} \operatorname{rank}\left[\mu E - F\right]$$

is satisfied.

Let us also record the following simple result.

Proposition 3. Let $\mathfrak{S}(E, F, H)$ be an observed system.

(a) Observability of $\mathfrak{S}(E, F, H)$ implies its detectability.
(b) Strict stability of $\mathfrak{S}(E, F, H)$ implies its detectability.
(c) Stability of $\mathfrak{S}(E, F, H)$ does not imply its detectability.

To prove the last statement of the above proposition it is enough to indicate an example of a *stable system which is not detectable*; for this it is sufficient to find a stable system $\mathfrak{S}(E, F, H)$ with $H = 0$ and without the uniqueness property. But this is an easy task.

As we have noted observability implies detectability. In this context it is interesting to recall (see e.g. (Przyłuski and Sosnowski, 1996)) the following necessary and sufficient condition of observability.

Proposition 4. $\mathfrak{S}(E, F, H)$ is observable if and only if the condition

$$\operatorname{rank}\begin{bmatrix}\mu E - F \\ H\end{bmatrix} = \dim \mathcal{X}$$

is satisfied, for all $\mu \in \mathbb{C}$.

To formulate our next result, let us denote by $ls(\mathcal{S})$ the linear space of all sequences $\mathbb{Z} \to \mathcal{S}$ which support is bounded on the left, i.e. such sequences (s_k) for which $s_k = 0$, for all k less than a number depending only on (s_k) ($\mathcal{S}$ is here an arbitrary linear space).

Theorem 2. Let $\mathfrak{S}(E, F, H)$ be an observed system. The system is detectable if and only if for every $((z_k), (x_k), (y_k)) \in ls(\mathcal{Z}) \times ls(\mathcal{X}) \times ls(\mathcal{Y})$ satisfying

$$\begin{cases} Ex_{k+1} &= Fx_k + z_k, \\ y_k &= Hx_k, \end{cases}$$

for $k \in \mathbb{Z}$, we have $\lim_{k\to\infty} x_k = 0$, whenever $\lim_{k\to\infty} z_k = 0$ and $\lim_{k\to\infty} y_k = 0$.

4. SUPPLEMENTARY RESULTS ON DETECTABILITY

We begin by defining the following subspace of the internal variable space $\mathcal{X}$:

$$\mathcal{V}(E, F)|\operatorname{Ker}H := \{x \in \mathcal{X} \mid \exists (x_k) \in s_0^+(\mathcal{X}) \text{ s.t. } \forall k \in \mathbb{Z}_0^+ : Ex_{k+1} = Fx_k,\ Hx_k = 0\ \&\ x = x_0\}.$$

The following result is obvious.

Proposition 5. Let $\mathfrak{S}(E, F, H)$ be an observed system. This system is observable if and only if $\mathcal{V}(E, F)|\operatorname{Ker}H = 0$.

For that reason the space is called the *unobservable space of the system* $\mathfrak{S}(E,F,H)$. Some properties of this space are studied in (Banaszuk *et al.*, 1996). With the aid of this space we can introduce a Kalman-type decomposition for an observed system (for a parallel theory for controlled systems, see (Banaszuk *et al.*, 1990c)). Then the *unobservable subsystem of* $\mathfrak{S}(E,F,H)$ can be defined and it can be identified with an appropriately defined system $\mathfrak{S}(E_n,F_n)$. It happens that then the following results holds true.

Proposition 6. $\mathfrak{S}(E,F,H)$ is detectable if and only if $\mathfrak{S}(E_n,F_n)$ is strictly stable.

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