Applied Mathematics and Computation 269 (2015) 664–673

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

bao của tập đạt được

On backwards and forwards reachable sets bounding for perturbed time-delay systems

H. Trinh [a], Phan T. Nam [b,*], Pubudu N. Pathirana [a], H.P. Le [b]

[a] *School of Engineering, Deakin University, Geelong, VIC 3217, Australia*
[b] *Department of Mathematics, Quynhon University, Vietnam*

A R T I C L E   I N F O

Keywords:
Forwards reachable set
Backwards reachable set
Perturbed time-delay systems
Lyapunov–Krasovskii method

A B S T R A C T

Linear systems with interval time-varying delay and unknown-but-bounded disturbances are considered in this paper. We study the problem of finding outer bound of forwards reachable sets and inter bound of backwards reachable sets of the system. Firstly, two definitions on forwards and backwards reachable sets, where initial state vectors are not necessary to be equal to zero, are introduced. Then, by using the Lyapunov–Krasovskii method, two sufficient conditions for the existence of: (i) the smallest possible outer bound of forwards reachable sets; and (ii) the largest possible inter bound of backwards reachable sets, are derived. These conditions are presented in terms of linear matrix inequalities with two parameters need to tuned, which therefore can be efficiently solved by combining existing convex optimization algorithms with a two-dimensional search method to obtain optimal bounds. Lastly, the obtained results are illustrated by four numerical examples.

1. Introduction

Within recent years, there has been an increasing interest in the problem of reachable set bounding for time-delay systems perturbed by unknown-but-bounded disturbances [1–15]. Most of the existing results on this topic have been based on the definition introduced in [16] that "Reachable set of dynamic systems perturbed by bounded inputs (disturbances) is the set of all the states that are reachable from the origin, in finite time, by inputs with peak value".

In many practical applications, for example, in safety verification, model checking [17–20], one usually requires to find a bound of a set of all the states that are reachable from a given set (not only from the origin point). A general definition on reachable set (forwards reachable set) to express a requirement like this, and a converse definition on backwards reachable set, have been given for perturbed systems without time-delay [17]. These notions have been widely applied to safety verification, model checking, state bounding observers, etc. (see [17–20] and the references therein). Forwards reachable set with respect to a given initial set of a perturbed dynamic system is the set of all the states starting from this given initial set. Backwards reachable set with respect to a given set in state vector space (called a target set) is the set of initial state such that the set of all the states starting from this initial set is covered by the given target set. It is easy to see that reachable set defined in [16] is a special case of forwards reachable set when the given initial set contains only the origin point. Most of existing results on forwards and backwards reachable sets bounding reported on perturbed systems without time-delay and, to our knowledge, there has not been any research attention reported on an extension to perturbed time-delay systems. Motivated by this, in this paper, we study

* Corresponding author. Tel.: +84907416946.
E-mail address: phanthanhnam@qnu.edu.vn (P.T. Nam).

http://dx.doi.org/10.1016/j.amc.2015.07.116

a new problem of finding the smallest possible outer bound of forwards reachable sets with respect to a given set of initial state for perturbed time-delay systems and a converse problem of finding the largest possible inner bound of backwards reachable sets with respect to a given target set.

With regard to the problem of reachable set bounding for perturbed linear time-delay systems, whose matrices are constant, the widely used approach is based on the Lyapunov method and linear matrix inequalities [1–13,16]. Based on the Lyapunov Razumikhin method, Fridman and Shaked [16] first reported a result on reachable set bounding for perturbed linear time-delay systems. Later, based on modified Lyapunov–Krasovskii functionals, improved and extended results were reported for linear systems [1,3,5–7,11], neutral systems [2], systems with distributed delay [4,10], discrete-time systems [8,9,13], neural networks systems [12]. It is worthwhile to note that there is another approach based on linear positive systems and was proposed recently for linear time-varying systems [14] or nonlinear systems [15]. In this paper, we also use the Lyapunov–Krasovskii method to study the problem of forwards and backwards reachable set bounding for perturbed linear time-delay systems. Firstly, we propose an extended Lyapunov–Krasovskii functional (LKF) in which a delay-dependent matrix is incorporated. Note that the time-delay dependent matrix technique is first introduced in [21]. This technique allows one to exploit information of the upper and lower bound of derivative of time-varying delay and to reduce requirement of the existence of common matrix variables. Hence, the condition obtained by using this technique is expected to be less conservative. Next, we use the Wirtinger-based integral inequality [22,23] and the reciprocally convex technique [24] in estimating the derivative of the proposed LKF. The Wirtinger-based inequality, which encompasses the well-known Jensen inequality was introduced in [22,23] and further developed in [25–29]. Also note that this inequality in combination with the reciprocally convex technique [24] gives a more effective estimation of the derivative of the proposed LKF. As a result, we obtain two delay-derivative-dependent sufficient conditions for the existence of the smallest possible ball which outer bounds forwards reachable sets and the largest possible ball which inter bounds backwards reachable sets of the system. These new conditions are given in terms of linear matrix inequalities with two parameters need to tuned, which therefore can be efficiently solved by using existing convex optimization algorithms combining with a two-dimensional search method and allow us to obtain optimal possible bounds. To further optimize the obtained bounds, the technique on optimization on each axis with different exponential rates [7,8] is also used in the derivation of our results. Lastly, the feasibility and the effectiveness of the obtained results are illustrated by four numerical examples.

This paper is organized as follows. After the introduction, the problem statement and preliminaries are introduced in Section 2. The main results are given in Section 3. Four numerical examples are given in Section 4. Finally, a conclusion is drawn in Section 5.

2. Problem statement and preliminaries

Consider the following system

$$\begin{aligned}&\dot{x}(t) = Ax(t) + A_1x(t-\tau(t)) + B\omega(t), \quad t \ge 0,\\ &x(s) \equiv \varphi(s), \quad s \in [-\tau_M, 0],\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$, $A_1 \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times p}$ are known constant matrices. The time-varying delay is assumed to be differentiable and satisfying

$$0 \le \tau_m \le \tau(t) \le \tau_M, \quad d_m \le \dot{\tau}(t) \le d_M \le 1. \tag{2}$$

The function $\varphi(s) \in C_1([-\tau_M, 0], \mathbb{R}^n)$ is the initial condition function satisfying

$$\sup_{s\in[-\tau_M,0]} \dot{\varphi}^T(s)\dot{\varphi}(s) \le \mu^2. \tag{3}$$

The disturbance vector, $\omega(t) \in \mathbb{R}^p$, is unknown but it is assumed to be bounded

$$\omega^T(t)\omega(t) \le \omega^2, \quad \forall t \ge 0. \tag{4}$$

Here, $\tau_m, \tau_M, \mu, \omega$ are given non-negative scalars, d_m and d_M are given scalars.

Definition 1 (Forwards and backwards reachable sets)**.** (i) Given a closed convex set, which contains the origin point, $\Omega_0 \in \mathbb{R}^n$ (called an initial set). A set $\Omega \in \mathbb{R}^n$ is called *a forwards reachable set with respect to the given initial set* Ω_0 of system (1) with conditions (2), (3) and (4) if for all initial condition function $\varphi(s) \in \Omega_0, \forall s \in [-\tau_M, 0]$, the solution $x(t, \varphi, \omega(t)) \in \Omega, \forall t \ge 0$.

(ii) Given a closed convex set, which contains the origin point, $\Lambda \in \mathbb{R}^n$ (called a target set). A set $\Lambda_0 \in \mathbb{R}^n$ is called *a backwards reachable set with respect to the given target set* Λ of system (1) with conditions (2), (3) and (4) if for all initial condition function $\varphi(s) \in \Lambda_0, \forall s \in [-\tau_M, 0]$, the solution $x(t, \varphi, \omega(t)) \in \Lambda, \forall t \ge 0$.

The objective of this paper is to find the smallest possible outer bound of forwards reachable sets with respect to a given initial set; and the largest possible inner bound of backwards reachable sets with respect to a given target set.

The following lemmas are useful for our main results.

Lemma 1. *For a given positive scalar* α, *let* V *be a Lyapunov–Krasovskii-like function for system (1)–(4). If* $\dot{V}(t) + \alpha V(t) - \frac{\alpha}{\omega^2}\omega^T(t)\omega(t) \le 0, \forall t \ge 0$ *then*

$$V(t) \le \max\{1, V(0)\}, \quad \forall t \ge 0.$$

Proof. Putting $v(t) = e^{\alpha t}V(t)$ and taking the derivative of $v(t)$, we have

$$\dot{v} = e^{\alpha t}\left(\dot{V} + \alpha V - \frac{\alpha}{\omega^2}\omega^T(t)\omega(t)\right) + \frac{\alpha}{\omega^2}\omega^T(t)\omega(t)e^{\alpha t} \\ \leqslant \alpha e^{\alpha t}.$$

Integrating from 0 to t both sides of the above inequality, we obtain $v(t) - v(0) \le e^{\alpha t} - 1$, which implies

$$V(t) \le 1 + e^{-\alpha t}(V(0) - 1) \le \max\{1, V(0)\}, \quad \forall t \ge 0.$$

The proof of Lemma 1 is completed. □

The reciprocally convex combination inequality provided in [24] is used in this paper. This inequality has been reformulated in [23] and is stated in Lemma 2 below.

Lemma 2. *For given positive integers n, m, a scalar $\alpha \in (0, 1)$, a $n \times n$-matrix $R > 0$, two $n \times m$-matrices W_1, W_2. Define, for all vector $\xi \in \mathbb{R}^m$, the function $\Theta(\alpha, R)$ given by:*

$$\Theta(\alpha, R) = \frac{1}{\alpha}\xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha}\xi^T W_2^T R W_2 \xi.$$

If there is a matrix $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} R & X \\ \star & R \end{bmatrix} > 0,$$

then the following inequality holds

$$\min_{\alpha\in(0,1)} \Theta(\alpha, R) \ge \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}^T \begin{bmatrix} R & X \\ \star & R \end{bmatrix} \begin{bmatrix} W_1\xi \\ W_2\xi \end{bmatrix}.$$

The following Wirtinger-based inequality was first introduced in [22] and further developed in [23].

Lemma 3. *For a given $n \times n$-matrix $R > 0$, any differentiable function $\varphi : [a, b] \to \mathbb{R}^n$, then the following inequality holds*

$$\int_a^b \dot{\varphi}(u)R\dot{\varphi}(u)du \ge \frac{1}{b-a}(\varphi(b) - \varphi(a))^T R(\varphi(b) - \varphi(a)) + \frac{12}{b-a}\Omega^T R\Omega,$$

where $\Omega = \frac{\varphi(b)+\varphi(a)}{2} - \frac{1}{b-a}\int_a^b \varphi(u)du$.

Lemma 4. *Given a function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \longrightarrow \mathbb{R}, (x, y) \longmapsto f(x, y)$. Assume that $\frac{\partial^2 f}{\partial x^2} \ge 0, \forall (x, y) \in [a, b] \times [c, d]$ and $\frac{\partial^2 f}{\partial y^2} \ge 0, \forall (x, y) \in [a, b] \times [c, d]$. Then, we have*

$$\max_{(x,y)\in[a,b]\times[c,d]} f(x, y) = \max\{f(a, c), f(a, d), f(b, c), f(b, d)\}. \tag{5}$$

Proof. Assume that $(x_0, y_0) \in [a, b] \times [c, d]$ is an optimal point of function $f(.,.)$, i.e.,

$$\max_{(x,y)\in[a,b]\times[c,d]} f(x, y) = f(x_0, y_0).$$

Since $f(x_0, y)$ is convex with respect to y, we have

$$f(x_0, y_0) \le \max\{f(x_0, c), f(x_0, d)\}.$$

Similarly, we also have

$$f(x_0, c) \le \max\{f(a, c), f(b, c)\} \text{ and } f(x_0, d) \le \max\{f(a, d), f(b, d)\}.$$

Note that

$$\max\{f(a, c), f(a, d), f(b, c), f(b, d)\} \le f(x_0, y_0).$$

From the above, we obtain Eq. (5). The proof of Lemma 4 is completed. □

The following notations are used in our development:

$$\zeta^T(t) = \left[x^T(t) \int_{t-\tau_m}^t x^T(s)ds \int_{t-\tau(t)}^{t-\tau_m} x^T(s)ds \int_{t-\tau_M}^{t-\tau(t)} x^T(s)ds\right],$$

$$\xi^T(t) = \Big[x^T(t)x^T(t-\tau(t))x^T(t-\tau_m)x^T(t-\tau_M)\frac{1}{\tau_m}\int_{t-\tau_m}^t x^T(s)ds \frac{1}{\tau(t)-\tau_m}\int_{t-\tau(t)}^{t-\tau_m} x^T(s)ds \\ \frac{1}{\tau_M - \tau(t)}\int_{t-\tau_M}^{t-\tau(t)} x^T(s)ds \ \ \omega^T(t)\Big],$$

$$e_i = [0_{n\times(i-1)n} I_n 0_{n\times(7-i)n} 0_{n\times p}]^T, i = 1, 2, \dots 7.$$

$$e_8 = [0_{p\times 7n} I_p]^T, \mathcal{A}_c^T = [AA_1 0_{n\times 5n} B] \text{ and } He(.) = \frac{(.) + (.)^T}{2}.$$

3. Main results

Firstly, we derive a sufficient condition for the existence of an outer bound of forwards reachable sets with respect to a given ellipsoidal initial set, $\Omega_0 = \{\varphi(s) \in C_1([-\tau_M, 0], \mathbb{R}^n) : \varphi^T(s)E\varphi(s) \le 1, \forall s \in [-\tau_M, 0]\}$, where E is a given positive-definite matrix, of system (1)–(4) and state in the following theorem.

Theorem 1. *Assume that there exist 11 positive scalars $\alpha, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, q_1, q_2, q_3, r_1, r_2$, five $n \times n$ positive-definite matrices Q_1, Q_2, Q_3, R_1, R_2, three $2n \times 2n$ positive-definite matrices P_{11}^1, P_{11}^2, P_{22} and two $2n \times 2n$-matrices X, P_{12}, such that the following matrix inequalities hold*

$$Q_1 \le q_1 I_n,\ Q_2 \le q_2 I_n,\ Q_3 \le q_3 I_n,\ R_1 \le r_1 I_n,\ R_2 \le r_1 I_n, \tag{6}$$

$$\begin{bmatrix} (\tau_M - \tau_m)P_{11}^i & P_{12} \\ \star & P_{22} \end{bmatrix} < \mathrm{diag}\{\beta_1 I_n, \ldots, \beta_4 I_n\}, \quad i = 1, 2, \tag{7}$$

$$\begin{bmatrix} (\tau_M - \tau_m)P_{11}^i & P_{12} \\ \star & P_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta_0^2} I_n & 0_{n\times 3n} \\ \star & 0_{3n\times 3n} \end{bmatrix} > 0, \quad i = 1, 2, \tag{8}$$

$$\Theta = \begin{bmatrix} \widetilde{R}_2 & X \\ \star & \widetilde{R}_2 \end{bmatrix} > 0, \tag{9}$$

$$\Sigma(\tau, \dot{\tau}) - \frac{\alpha}{\omega^2} e_8 e_8^T < 0, \quad \forall (\tau, \dot{\tau}) \in \{\tau_m, \tau_M\} \times \{d_m, d_M\}, \tag{10}$$

$$\mathcal{K}_1 \mu_0^2 + \mathcal{K}_2 \mu^2 \le 1, \tag{11}$$

where $\mu_0 = \lambda_{min}(E)$,

$$\begin{aligned}
\Sigma(\tau, \dot{\tau}) = {} & He\big(2[e_1 \tau_m e_5 (\tau(t) - \tau_m)e_6 (\tau_M - \tau(t))e_7] P(t) [\mathcal{A}_c e_1 - e_3 e_3 - (1 - \dot{\tau}(t))e_2\ (1 - \dot{\tau}(t))e_2 - e_4]^T\big) \\
& + [e_1\ \ \tau_m e_5\ \ (\tau(t) - \tau_m)e_6 (\tau_M - \tau(t))e_7] \left(\dot{\tau}(t) \begin{bmatrix} -P_{11}^1 + P_{11}^2 & 0 \\ \star & 0 \end{bmatrix} + \alpha P(t) \right) \\
& \times [e_1\ \ \tau_m e_5\ \ (\tau(t) - \tau_m)e_6\ \ (\tau_M - \tau(t))e_7]^T + [e_1](Q_1 + Q_2 + Q_3)[e_1]^T \\
& - [e_3] Q_1 e^{-\alpha \tau_m} [e_3]^T - (1 - \dot{\tau}(t))[e_2] e^{-\alpha \tau_M} Q_2 [e_2]^T - [e_4] e^{-\alpha \tau_M} Q_3 [e_4]^T \\
& + [\mathcal{A}_c][\tau_m^2 R_1 + (\tau_M - \tau_m)^2 R_2][\mathcal{A}_c]^T - e^{-\alpha \tau_m} \Gamma_1 \widetilde{R}_1 \Gamma_1^T - e^{-\alpha \tau_M} \Gamma_2 \Theta \Gamma_2^T,
\end{aligned}$$

$$P(t) = \begin{bmatrix} (\tau_M - \tau(t))P_{11}^1 + (\tau(t) - \tau_m)P_{11}^2 & P_{12} \\ \star & P_{22} \end{bmatrix},$$

$$\begin{aligned}
\widetilde{R}_1 &= \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}, \quad \widetilde{R}_2 = \begin{bmatrix} R_2 & 0 \\ 0 & R_2 \end{bmatrix}, \\
\Gamma_1 &= [e_1 - e_3\quad \sqrt{3}(e_1 + e_3 - 2e_5)], \\
\Gamma_2 &= [e_2 - e_4\quad \sqrt{3}(e_2 + e_4 - 2e_7)\quad e_3 - e_2\quad \sqrt{3}(e_3 + e_2 - 2e_6)], \\
\mathcal{K}_1 &= \beta_1 + \tau_m^2 \beta_2 + (\tau_M - \tau_m)^2 (\beta_3 + \beta_4) + \frac{q_1(1 - e^{-\alpha \tau_m}) + (q_2 + q_3)(1 - e^{-\alpha \tau_M})}{\alpha}, \\
\mathcal{K}_2 &= \frac{1}{\alpha^2} (r_1 \tau_m (\tau_m \alpha + e^{-\alpha \tau_m} - 1) + r_2 (\tau_M - \tau_m)((\tau_M - \tau_m)\alpha + e^{-\alpha \tau_M} - e^{-\alpha \tau_m}))
\end{aligned}$$

Then all the forwards reachable sets of system (1) are bounded by the ball $\mathcal{B}(0, \beta_0) = \{x \in \mathbb{R}^n : ||x|| \le \beta_0\}$.

Proof. Consider the following extended Lyapunov–Krasovskii functional which contains a time-delay dependent matrix $P(t)$:

$$V = V_1 + V_2 + V_3, \tag{12}$$

where

$$\begin{aligned}
V_1 &= \zeta(t)^T P(t) \zeta(t), \\
V_2 &= \int_{t-\tau_m}^{t} e^{\alpha(s-t)} x^T(s) Q_1 x(s) ds + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)} x^T(s) Q_2 x(s) ds + \int_{t-\tau_M}^{t} e^{\alpha(s-t)} x^T(s) Q_3 x(s) ds, \\
V_3 &= \tau_m \int_{-\tau_m}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta + (\tau_M - \tau_m) \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta.
\end{aligned}$$

From the definition of $P(t)$ as given in Theorem 1, we can re-express $P(t)$ as follows:

$$P(t) = \frac{1}{\tau_M - \tau_m}\left\{(\tau_M - \tau(t))\begin{bmatrix}(\tau_M - \tau_m)P_{11}^1 & P_{12}\\ \star & P_{22}\end{bmatrix} + (\tau(t) - \tau_m)\begin{bmatrix}(\tau_M - \tau_m)P_{11}^2 & P_{12}\\ \star & P_{22}\end{bmatrix}\right\}.$$

Due to (8), the following inequality holds

$$V \geq \frac{||x(t)||^2}{\beta_0^2}, \quad \forall t \geq 0. \tag{13}$$

Taking the derivatives of $V_i,\ i = 1, 2, 3$ in t, we have

$$\begin{aligned}\dot{V}_1 + \alpha V_1 &= 2\zeta^T(t)P(t)\dot{\zeta}(t) + \zeta^T(t)\dot{P}(t)\zeta(t) + \alpha\zeta^T(t)P(t)\zeta(t)\\ &= \xi^T(t)\Big\{2[e_1\ \ \tau_m e_5\ \ (\tau(t) - \tau_m)e_6\ \ (\tau_M - \tau(t))e_7]P(t)[\mathcal{A}_c\ \ e_1 - e_3\ \ e_3 - (1 - \dot{\tau}(t))e_2\ \ (1 - \dot{\tau}(t))e_2 - e_4]^T\\ &\quad + [e_1\ \ \tau_m e_5\ \ (\tau(t) - \tau_m)e_6\ \ (\tau_M - \tau(t))e_7]\left(\dot{\tau}(t)\begin{bmatrix}-P_{11}^1 + P_{11}^2 & 0\\ \star & 0\end{bmatrix} + \alpha P(t)\right)\\ &\quad \times [e_1 \tau_m e_5\ \ (\tau(t) - \tau_m)e_6(\tau_M - \tau(t))e_7]^T\Big\}\xi(t),\end{aligned} \tag{14}$$

$$\begin{aligned}\dot{V}_2 + \alpha V_2 =& x^T(t)(Q_1 + Q_2 + Q_3)x(t) - e^{-\alpha\tau_m}x^T(t - \tau_m)Q_1x(t - \tau_m)\\ &- e^{-\alpha\tau(t)}(1 - \dot{\tau}(t))x^T(t - \tau(t))Q_2x(t - \tau(t))\\ &- e^{-\alpha\tau_M}x^T(t - \tau_M)Q_3x(t - \tau_M)\\ \leq& \xi^T(t)\Big\{[e_1](Q_1 + Q_2 + Q_3)[e_1]^T - [e_3]e^{-\alpha\tau_m}Q_1[e_3]^T\\ &- (1 - \dot{\tau}(t))[e_2]e^{-\alpha\tau_M}Q_2][e_2]^T - [e_4]e^{-\alpha\tau_M}Q_3[e_4]^T\Big\}\xi(t),\end{aligned} \tag{15}$$

$$\begin{aligned}\dot{V}_3 + \alpha V_3 =& \dot{x}^T(t)\left(\tau_m^2R_1 + (\tau_M - \tau_m)^2R_2\right)\dot{x}(t) - \tau_m\int_{t-\tau_m}^{t} e^{\alpha(s-t)}\dot{x}^T(s)R_1\dot{x}(s)ds\\ &- (\tau_M - \tau_m)\int_{t-\tau_M}^{t-\tau_m} e^{\alpha(s-t)}\dot{x}^T(s)R_2\dot{x}(s)ds\\ \leq& \xi^T(t)\Big\{[\mathcal{A}_c][\tau_m^2R_1 + (\tau_M - \tau_m)^2R_2][\mathcal{A}_c]^T\Big\}\xi(t)\\ &- \tau_m e^{-\alpha\tau_m}\int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds\\ &- (\tau_M - \tau_m)e^{-\alpha\tau_M}\int_{t-\tau_M}^{t-\tau_m}\dot{x}^T(s)R_2\dot{x}(s)ds.\end{aligned} \tag{16}$$

Using Lemma 3, we have

$$\begin{aligned}-\int_{t-\tau_m}^{t}\dot{x}^T(s)R_1\dot{x}(s)ds \leq& -\frac{1}{\tau_m}\Big(x(t) - x(t - \tau_m)\Big)^T R_1\Big(x(t) - x(t - \tau_m)\Big)\\ &- \frac{12}{\tau_m}\left(\frac{x(t)}{2} + \frac{x(t - \tau_m)}{2} - \frac{1}{\tau_m}\int_{t-\tau_m}^{t} x(s)ds\right)^T R_1\\ &\times\left(\frac{x(t)}{2} + \frac{x(t - \tau_m)}{2} - \frac{1}{\tau_m}\int_{t-\tau_m}^{t} x(s)ds\right)\\ =& -\xi^T(t)\frac{1}{\tau_m}\Big\{(e_1 - e_3)R_1(e_1 - e_3)^T + 3(e_1 + e_3 - 2e_5)R_1\\ &\times(e_1 + e_3 - 2e_5)^T\Big\}\xi(t)\\ =& -\xi^T(t)\frac{1}{\tau_m}\Gamma_1\widetilde{R}_1\Gamma_1^T\xi(t)\end{aligned} \tag{17}$$

and

$$
\begin{aligned}
-\int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)R_2\dot{x}(s)ds &= -\int_{t-\tau_M}^{t-\tau(t)} \dot{x}^T(s)R_2\dot{x}(s)ds - \int_{t-\tau(t)}^{t-\tau_m} \dot{x}^T(s)R_2\dot{x}(s)ds \\
&\le -\xi^T(t)\Bigg\{\frac{1}{\tau_M-\tau(t)}\Big[(e_2-e_4)R_2(e_2-e_4)^T \\
&\quad + 3(e_2+e_4-2e_7)R_2(e_2+e_4-2e_7)^T\Big] \\
&\quad + \frac{1}{\tau(t)-\tau_m}\Big[(e_3-e_2)R_2(e_3-e_2)^T \\
&\quad + 3(e_3+e_2-2e_6)R_2(e_3+e_2-2e_6)^T\Big]\Bigg\}\xi(t) \\
&= -\xi^T(t)\Bigg\{\frac{1}{\tau_M-\tau(t)}[e_2-e_4 \quad \sqrt{3}(e_2+e_4-2e_7)] \\
&\quad \times \widetilde{R}_2[e_2-e_4 \quad \sqrt{3}(e_2+e_4-2e_7)]^T \\
&\quad + \frac{1}{\tau(t)-\tau_m}[e_3-e_2 \quad \sqrt{3}(e_3+e_2-2e_6)] \\
&\quad \times \widetilde{R}_2[e_3-e_2 \quad \sqrt{3}(e_3+e_2-2e_6)]^T\Bigg\}\xi(t).
\end{aligned} \tag{18}
$$

From (9), (18) and Lemma 2, we have

$$
-\int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s)R_2\dot{x}(s)ds \le -\xi^T(t)\frac{1}{\tau_M-\tau_m}\Gamma_2\Theta\Gamma_2^T\xi(t). \tag{19}
$$

From (14)–(17) and (19), we obtain

$$
\dot{V}(t) + \alpha V - \frac{\alpha}{\omega^2}\omega^T(t)\omega(t) \le \xi^T(t)\Big[\Sigma(\tau,\dot{\tau}) - \frac{\alpha}{\omega^2}e_8e_8^T\Big]\xi(t). \tag{20}
$$

On the other hand, by some simple computations, we can check easily that

$$
\Big[\frac{\partial^2}{\partial\tau^2}(\Sigma(\tau,\dot{\tau}) - \frac{\alpha}{\omega^2}e_8e_8^T)\Big] = 0 \text{ and } \Big[\frac{\partial^2}{\partial\dot{\tau}^2}(\Sigma(\tau,\dot{\tau}) - \frac{\alpha}{\omega^2}e_8e_8^T)\Big] = 0.
$$

Combining with Lemma 4 and (10), we obtain $\dot{V}(t) + \alpha V - \frac{\alpha}{\omega^2}\omega^T(t)\omega(t) \le 0$. By (6), (7), (11) and some computations, we have $V(0) \le \mathcal{K}_1\mu_0^2 + \mathcal{K}_2\mu^2 \le 1$. This implies that $V \le 1$ due to Lemma 1. Using inequality (13), we obtain

$$
||x(t)|| \le \beta_0, \quad \forall t \ge 0.
$$

The proof of Theorem 1 is completed. □

Remark 1. To solve matrix inequalities (6)–(11) with 11 parameters $\alpha, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, q_1, q_2, q_3, r_1, r_2$, we first simplify them into linear matrix inequalities (LMIs) with only two parameters α and β_0. For matrix inequality $Q_1 \le q_1 I_n$, by denoting $\overline{Q}_1 = \mathrm{diag}\{q_1, q_1, \ldots, q_1\}$, then inequality $Q_1 \le q_1 I_n$ is represented into linear matrix inequality $Q_1 \le \overline{Q}_1$ where $\overline{Q}_1$ is defined as a matrix variable which has a prescribed structure and is dependent on the decision variable q_1. Similarly, matrix inequalities (6) and (7) are also represented into linear matrix inequalities. For matrix inequalities (8), (10) and (11), by fixing parameters α and β_0, then (8), (10) and (11) are reduced to linear matrix inequalities. Thus, matrix inequalities (6)–(11) can be represented into linear matrix inequalities with only two parameters α and β_0. Therefore, we can combine a two-dimensional search method with a convex optimization algorithm such as Matlab's LMI Toolbox to solve matrix inequalities (6)–(11). Accordingly, the following optimization problem gives the smallest possible bound of β_0:

$$
\begin{aligned}
(OP_1):\ & \min\ \beta_0 \\
& \text{subject to (6)–(11)}.
\end{aligned}
$$

Also note that the two parameters α and β_0 are independent and hence in practice one can use parallel computing to find the two feasible parameters. Furthermore, since parameter $\alpha > 0$ is the exponential rate, it belongs to a bounded interval. Also since $\mathcal{B}(0, \beta_0)$ is the smallest possible upper bounds of forwards reachable sets and $\beta_0 \ge \mu_0$, parameter β_0 also belongs to a bounded interval. These facts therefore help to reduce partly the difficulty in searching for the two feasible parameters α and β_0.

Remark 2. Inspired by the technique [7,8], we replace condition (8) by the less strict condition

$$
\begin{bmatrix} (\tau_M-\tau_m)P_{11}^i & P_{12} \\ \star & P_{22} \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta_{0h}^2}G_h & 0_{n\times 3n} \\ \star & 0_{3n\times 3n} \end{bmatrix} > 0, \quad i = 1, 2, \tag{21}
$$

where $G_h = [g_{lk}]_{l,k=1,\ldots,n}$, $g_{hh} = 1, g_{lk} = 0, \forall (l,k) \ne (h,h)$. Then, Theorem 1 gives the smallest possible outer bounds $\beta_{0h}, h = 1, \ldots, n$ of forwards reachable sets with respect to each hth axis, i.e., for every $h = 1, \ldots, n$, set $\{x = [x_1, \ldots, x_h, \ldots, x_n]^T \in \mathbb{R}^n : |x_h| \le \beta_{0h}\}$ covers all forwards reachable sets of system (1). Consequently, the boxes $\{x \in \mathbb{R}^n : |x_h| \le \beta_{0h}, \forall h = 1, \ldots, n\}$ are outer bounds of forwards reachable sets of system (1). By taking the intersection of these boxes with the ball $\mathcal{B}(0, \beta_0)$, then a smaller bound is obtained.

Remark 3. Theorem 1 gives a sufficient condition for the existence of a ball, $\mathcal{B}(0, \beta_0)$, which outer bounds forwards reachable sets with respect to the ellipsoidal initial set $\varepsilon(E, 1) = \{x \in R^n : x^T Ex \le 1\}$. In the general case where the initial set is an ellipsoid $\varepsilon(E', 1) = \{x \in R^n : (x-a)^T E'(x-a) \le 1\}$, then by taking the state transformation $z = x - a$, extending Lemma 1 and doing the same lines as in Theorem 1, we can obtain a sufficient condition for the existence of a ball, $\mathcal{B}(a, \beta_0')$, which outer bounds forwards reachable sets with respect to the given initial set $\varepsilon(E', 1)$.

Remark 4. To be able to compute the radius β_0 of a ball $\mathcal{B}(0, \beta_0)$ which is an outer bound of forwards reachable sets with respect to the ellipsoidal initial set $\varepsilon(E, 1) = \{x \in R^n : x^T Ex \le 1\}$, the value of the Lyapunov–Krasovskii functional V at $t = 0$ needs to be bounded by 1, i.e. $V(0) \le 1$. We have derived three conditions (6), (7) and (11) to guarantee this requirement. As shown in Remark 1, these three conditions can be simplified into linear matrix inequalities with only two parameters and hence the derived conditions in Theorem 1 can be solved by using existing convex optimization algorithms to obtain the smallest possible radius β_0. Note that in the case where the initial set contains only the origin point, i.e., $\mu_0 = \mu = 0$, then $V(0) = 0$ without needing conditions (6), (7) and (11). Hence, we also obtain a delay-derivative-dependent sufficient condition for the existence of an outer bound of reachable sets as defined in [16] and it is stated in the following corollary.

Corollary 1. *Assume that there exist two positive scalars α, β_0 five $n \times n$ positive-definite matrices Q_1, Q_2, Q_3, R_1, R_2, three $2n \times 2n$ positive-definite matrices $P_{11}^1, P_{11}^2, P_{22}$ and two $2n \times 2n$-matrices X, P_{12}, such that (8), (9) and (10) hold. Then all reachable sets of system (1) are outer bounded by the ball $\mathcal{B}(0, \beta_0) = \{x \in \mathbb{R}^n : ||x|| \le \beta_0\}$.*

Remark 5. Similar to Remark 2, by replacing (8) with (21) in Corollary 1, we also obtain the smallest possible bounds of each partial state vector as well as a box which bounds the reachable sets of system (1).

Remark 6. In the case where the disturbance is not present, i.e., $\omega(t) = 0, \forall t \ge 0$, then the result of Theorem 1 is reduced to an exponential stability condition for system (1). This is stated in the following corollary.

Corollary 2. *System (1) is exponential stable with a given exponential rate $\alpha > 0$ if there exist six $n \times n$ positive-definite matrices $P, Q_1, Q_2, Q_3, R_1, R_2$, three $2n \times 2n$ positive-definite matrices $P_{11}^1, P_{11}^2, P_{22}$ and two $2n \times 2n$-matrices X, P_{12}, such that (9), and two following linear matrix inequalities hold*

$$\begin{bmatrix} (\tau_M - \tau_m)P_{11}^i & P_{12} \\ \star & P_{22} \end{bmatrix} > 0, \quad i = 1, 2, \tag{22}$$

$$\Sigma(\tau, \dot\tau) \le 0, \quad \forall (\tau, \dot\tau) \in \{\tau_m, \tau_M\} \times \{d_m, d_M\}. \tag{23}$$

Remark 7. Different from the Lyapunov–Krasovskii functional used in [7,23,25], the functional V_1 in formula (12) is extended and incorporated a delay-dependent matrix $P(t)$. This allows us to reduce the requirement of the existence of a common matrix variable and to exploit more information of the upper and lower bound of the derivative of the time-varying delay in our derived condition. Hence, for the case where initial set contains only the origin point, the conditions derived by using this technique (Corollaries 1 and 2) will be less conservative than the ones obtained in [7,23,25]. Note that when matrix $P(t)$ is replaced by a constant matrix then Corollaries 1 and 2 are reduced to the results [7,23,25].

Next, we consider the second reachable set bounding problem. We will derive a sufficient condition for the existence of a ball $\mathcal{B}(0, \mu_0) = \{x \in \mathbb{R}^n : ||x|| \le \mu_0\}$ which inter bounds all backwards reachable sets of system (1)–(4) with respect to a given ellipsoidal target set, $\Lambda = \{x \in \mathbb{R}^n : x^T Fx \le 1\}$. To guarantee all forwards reachable sets with respect to an initial set $\mathcal{B}(0, \mu_0)$ is outer bounded by the given ellipsoidal target set, Λ, we replace condition (8) in Theorem 1 by the following condition:

$$\begin{bmatrix} (\tau_M - \tau_m)P_{11}^i & P_{12} \\ \star & P_{22} \end{bmatrix} - \begin{bmatrix} F & 0_{n\times 3n} \\ \star & 0_{3n\times 3n} \end{bmatrix} > 0, \quad i = 1, 2. \tag{24}$$

Similar to Theorem 1, we also obtain a sufficient condition for the existence of an inter bound of backwards reachable sets of system (1)–(4) and it is stated in the following theorem.

Theorem 2. *Assume that there exist 11 positive scalars $\alpha, \mu_0, \beta_1, \beta_2, \beta_3, \beta_4, q_1, q_2, q_3, r_1, r_2$, five $n \times n$ positive-definite matrices Q_1, Q_2, Q_3, R_1, R_2, three $2n \times 2n$ positive-definite matrices $P_{11}^1, P_{11}^2, P_{22}$ and two $2n \times 2n$-matrices X, P_{12}, such that matrix inequalities (6), (7), (9)–(11) and (24) hold. Then all backwards reachable sets of system (1) are inter bounded by the ball $\mathcal{B}(0, \mu_0) = \{x \in \mathbb{R}^n : ||x|| \le \mu_0\}$.*

Remark 8. Similar to Remark 1, the following optimization problem gives the largest possible bound of μ_0:

$$\begin{aligned} (OP_2): \quad & \max \ \mu_0 \\ & \text{subject to (6), (7), (9)–(11) and (24).} \end{aligned}$$

Note that the optimization problem (OP_2) can be effectively solved by combining a two-dimensional search method (α, μ_0) with a convex optimization algorithm such as Matlab's LMI Toolbox.

Table 1
The smallest possible bounds β_0 for different cases of τ_m and d_M of Example 2.

$\tau_m \backslash d_M$	Method	0.1	0.5	1
0.0	Theorem 1	1.680	1.705	1.705
0.3	Theorem 1	1.630	1.644	1.644

Table 2
The largest possible bounds μ_0 for different cases of τ_m and d_M of Example 2.

$\tau_m \backslash d_M$	Method	0.1	0.5	1
0.0	Theorem 2	1.367	1.334	1.334
0.3	Theorem 2	1.402	1.398	1.398

4. Numerical examples

In this section, we consider four numerical examples and show the feasibility and the improvement of our results. Examples 1 and 2 study the problem of forwards and backwards reachable set bounding for time-delay systems with bounded disturbances to illustrate the feasibility of Theorems 1 and 2. Example 3 studies the problem of reachable set bounding, i.e., finding a bound of forwards reachable set with respect to the initial set which contains only the origin point and Example 4 studies stability of time-delay systems without any disturbances to illustrate the improvement of the derived Lyapunov–Krasovskii functional approach.

Example 1 (Forwards reachable set bounding)**.** Consider the satellite system which was considered in [1], with

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.09 & 0.09 & -0.004 & 0.004 \\ 0.09 & -0.09 & 0.004 & -0.004 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3.3092 & -0.7443 & -2.5909 & -8.0395 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and $||\omega(t)|| \le 1$. The time-varying delay is differentiable and satisfies that $\tau(t) \in [0.2, 0.3]$, $d_M = -d_m = 0.1$. Given $\mu = 0.1$ and an initial set $\Omega_0 \equiv \mathcal{B}(0, 0.1)$. By using Theorem 1 combining with Remark 1, we found the smallest possible ball $\mathcal{B}(0, 2.86)$, which outer bounds all forwards reachable sets with respect to the given initial set $\mathcal{B}(0, 0.1)$.

Example 2 (Forwards and backwards reachable set bounding)**.** Consider a perturbed time-delay system (1) which was considered in [7], with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0.1 \\ -1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},$$

and $||\omega(t)|| \le 1$. The time-varying delay is differentiable and satisfies that $\tau(t) \in [\tau_m, 0.7]$, $d_M = -d_m$. In this example, we find bounds for the following cases:

(a) Given $\mu = 0.5$. Find the smallest possible outer bound of forwards reachable sets with respect to a given initial set $\Omega_0 \equiv \mathcal{B}(0, 1)$.

(b) Given $\mu = 0.5$. Find the largest possible inter bound of backwards reachable sets with respect to a given target set $\Lambda \equiv \mathcal{B}(0, 2)$.

Solution: (a) We consider six different cases of τ_m and d_M. By using Theorem 1 combining with Remark 1, the smallest possible balls $\mathcal{B}(0, \beta_0)$, which outer bounds all forwards reachable sets with respect to a given initial set $\Omega_0 \equiv \mathcal{B}(0, 1)$, are found and the values of β_0 for different cases of τ_m and d_M are listed in Table 1. For a visual illustration, we choose: time-varying delay for the case $\tau(t) \in [0.3, 0.7]$ and $d_M = -d_m = 1$, $\tau(t) \in \{0.7, 0.3 + 0.4\sin^2(\frac{10t}{4})\}$; disturbances $\omega(t) \in \{a\sin(t),\ a = -1, -0.9, \ldots, 0.9, 1\} \cup \{-1, -0.9, \ldots, 0.9, 1\}$; initial condition functions $(x_1(t), x_2(t)) \in \{(b\sin(t/2), \pm\sqrt{1-b^2}\cos(t/2)),\ b = -1, -0.9, \ldots, 0.9, 1\} \cup \{(b, \pm\sqrt{1-b^2}),\ b = -1, -0.9, \ldots, 0.9, 1\}$. Fig. 1 shows that all trajectories starting in the ball $\mathcal{B}(0, 1)$ of system (1) are outer bounded by the ball $\mathcal{B}(0, 1.644)$.

(b) By using Theorem 2 and Remark 8, we found the largest possible balls, $\mathcal{B}(0, \mu_0)$, which inter bound all backwards reachable sets with respect to the given target set $\Lambda \equiv \mathcal{B}(0, 2)$. Table 2 lists the values of μ_0 for six different cases of τ_m and d_M. For a visual illustration, we choose: time-varying delay for the case $\tau(t) \in [0.3, 0.7]$ and $d_M = -d_m = 1$, $\tau(t) \in \{0.7, 0.3 +$

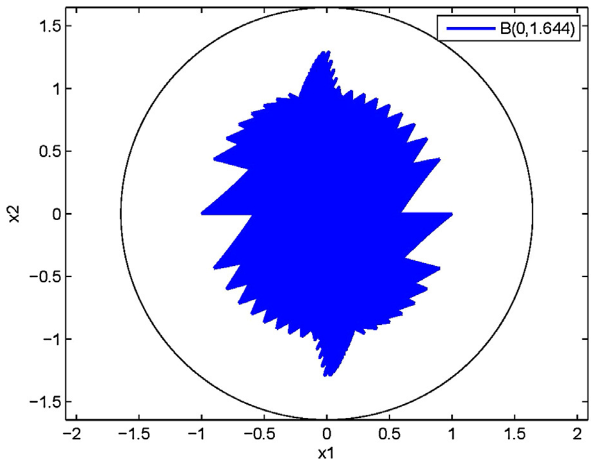

Fig. 1. The ball $\mathcal{B}(0, 1.644)$ outer bounds all forwards reachable sets with respect to a given initial set $\Omega_0 \equiv \mathcal{B}(0, 1)$.

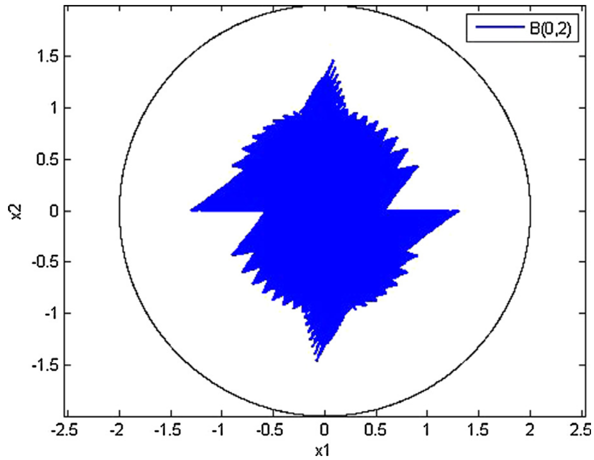

Fig. 2. The ball $\mathcal{B}(0, 2)$ outer bounds all state starting from a ball $\mathcal{B}(0, 1.398)$.

$0.4\sin^2(\frac{10t}{4})\}$; disturbances $\omega(t) \in \{-1, -0.9, \ldots, 0.9, 1\}$; initial condition functions $(x_1(t), x_2(t)) \in \{(b, \pm\sqrt{1-b^2}),\ b = -1.3, -1.2, \ldots, 1.2, 1.3\}$. Fig. 2 shows that all trajectories starting in the ball $\mathcal{B}(0, 1.398)$ of system (1) are outer bounded by the ball $\mathcal{B}(0, 2)$.

Example 3 (Reachable set bounding)**.** Consider system (1), which was considered in [11], with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix},$$

and $||\omega(t)|| \le 1$. The time-varying delay is differentiable and satisfies $\dot{\tau}(t) \le 0.3$, $0.1 \le \tau(t) \le 0.7$. The smallest radius β_0, which is computed by the method [11], such that ball $\mathcal{B}(0, \beta_0)$ outer bounds all forwards reachable sets with respect to the initial set $\Omega_0 \equiv 0$, is 1.288. Since our method depends on the lower bound of the derivative of the time-varying delay, we make an assumption that $-100,000 \le \dot{\tau}(t)$. By using Corollary 1, we found a smaller bound $\beta_0 = 1.160$.

Example 4 (Stability condition)**.** Consider system (1) without any disturbance, which was considered in [25], with $d_M = -d_m$, $\tau(t) = 0.9 + \Delta\tau(t)$, where $|\Delta\tau(t)| \le \overline{\Delta}$ and

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

In this example, the maximal allowable values for $\overline{\Delta}$, which are computed by using Corollary 2 for $\alpha = 0.00001$, are listed in Table 3. From Table 3, it can be seen that Corollary 2 provides upper bounds which are larger than the ones reported in [23]. This demonstrates the effectiveness of the used Lyapunov–Krasovskii functional (12).

Table 3
The maximal allowable values $\overline{\Delta}$ for Example 4.

Methods\ $d_M(=-dm)$	0.1	0.2	0.5	0.8	1
Seuret et al. [23,25]	0.528	0.493	0.278	0.189	0.161
Corollary 2	0.676	0.626	0.572	0.540	0.520

5. Conclusion

This paper has studied the problem of finding outer bound of forwards reachable sets and inter bound of backwards reachable sets for time-delay systems perturbed by bounded disturbances. Two sufficient conditions for the existence of these bounds and two respective optimization problems to find optimal bounds have been derived. In the case where disturbance is absent, as a consequence, an improved delay-derivative-dependent stability criterion has been obtained. Four numerical examples have been considered to illustrate the feasibility and effectiveness of the obtained results.

Acknowledgments

This work is supported by the National Foundation for Science and Technology Development, Vietnam under grant 101.01-2014.35 and the Australian Research Council under the Discovery grant DP130101532.

References

[1] O.M. Kwon, S.M. Lee, J.H. Park, On the reachable set bounding of uncertain dynamic systems with time-varying delays and disturbances, Inform. Sci. 181 (17) (2011) 3735–3748.
[2] C. Shen, S. Zhong, The ellipsoidal bound of reachable sets for linear neutral systems with disturbances, J. Franklin Inst. 348 (9) (2011) 2570–2585.
[3] P.T. Nam, P.N. Pathirana, Further result on reachable set bounding for linear uncertain polytopic systems with interval time-varying delays, Automatica 47 (8) (2011) 1838–1841.
[4] Z. Zuo, Y. Fu, Y. Wang, Results on reachable set estimation for linear systems with both discrete and distributed delays, IET Control Theory Appl. 6 (14) (2012) 2346–2350.
[5] Z. Zuo, Y. Fu, Y. Chen, Y. Wang, A new method of reachable set estimation for time delay systems with polytopic uncertainties, Appl. Math. Comput. 221 (15) (2013) 639–647.
[6] Z. Zuo, Y. Chen, Y. Wang, D.W.C. Ho, M.Z.Q. Chen, H. Li, A note on reachable set bounding for delayed systems with polytopic uncertainties, J. Franklin Inst. 350 (7) (2013) 1827–1835.
[7] P.T. Nam, P.N. Pathirana, H. Trinh, Linear functional state bounding for perturbed time-delay systems and its application, IMA J. Math. Control Inform. 32 (2) (2015) 245–255, doi:10.1093/imamci/dnt039.
[8] N.D. That, P.T. Nam, Q.P. Ha, Reachable set bounding for linear discrete-time systems with delays and bounded disturbances, J. Optim. Theory Appl. 157 (1) (2013) 96–107.
[9] J. Lam, B. Zhang, Y. Chen, S. Xu, Reachable set estimation for discrete-time linear systems with time delays, Int. J. Robust Nonlin. Control 25 (2) (2015) 269–281.
[10] B. Zhang, J. Lam, S. Xu, Reachable set estimation and controller design for distributed delay systems with bounded disturbances, J. Franklin Inst. 351 (6) (2014) 3068–3088.
[11] Z. Feng, J. Lam, An improved result on reachable set estimation and synthesis of time-delay systems, Appl. Math. Comput. 249 (2014) 89–97.
[12] Z. Zuo, Z. Wang, Y. Chen, Y. Wang, A non-ellipsoidal reachable set estimation for uncertain neural networks with time-varying delay, Commun. Nonlin. Sci. Numer. Simul. 19 (4) (2014) 1097–1106.
[13] L.V. Hien, N.T. An, H. Trinh, New results on state bounding for discrete-time systems with interval time-varying delay and bounded disturbance inputs, IET Control Theory Appl. 8 (14) (2014) 1405–1414.
[14] L.V. Hien, H.M. Trinh, A new approach to state bounding for linear time-varying systems with delay and bounded disturbances, Automatica 50 (6) (2014) 1735–1738.
[15] P.T. Nam, P.N. Pathirana, H. Trinh, Reachable set bounding for nonlinear perturbed time-delay systems: the smallest bound, Appl. Math. Lett. 43 (2015) 68–71.
[16] E. Fridman, U. Shaked, On reachable sets for linear systems with delay and bounded peak inputs, Automatica 39 (11) (2003) 2005–2010.
[17] I.M. Mitchell, C.J. Tomlin, Over approximating reachable sets by Hamilton-Jacobi projections, J. Sci. Comput. 19 (1–3) (2003) 323–346.
[18] C.J. Tomlin, I.M. Mitchell, A.M. Bayen, M. Oishi, Computational techniques for the verification of hybrid systems, Proc. IEEE 91 (7) (2003) 986–1001.
[19] I.M. Mitchell, A.M. Bayen, C.J. Tomlin, A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games, IEEE Trans. Autom. Control 50 (7) (2005) 947–957.
[20] M. Althoff, Reachability analysis and its application to the safety assessment of autonomous cars (Dissertation), Technische Universitat Munchen, 2010.
[21] E. Fridman, U. Shaked, K. Liu, New conditions for delay-derivative-dependent stability, Automatica 45 (11) (2009) 2723–2727.
[22] K. Liu, E. Fridman, Wirtinger's inequality and Lyapunov-based sampled-data stabilization, Automatica 48 (1) (2012) 102–108.
[23] A. Seuret, F. Gouaisbaut, Wirtinger-based integral inequality: application to time-delay systems, Automatica 49 (9) (2013) 2860–2866.
[24] P.G. Park, J.W. Ko, C. Jeong, Reciprocally convex approach to stability of systems with time-varying delays, Automatica 47 (1) (2011) 235–238.
[25] A. Seuret, F. Gouaisbaut, Integral inequality for time-varying delay systems, Proc. Eur. Control Conf. (2013) 3360–3365.
[26] O.M. Kwon, M.J. Park, J.H. Park, S.M. Lee, E.J. Cha, Improved results on stability of linear systems with time-varying delays via Wirtinger-based integral inequality, J. Franklin Inst. 351 (12) (2014) 5386–5398.
[27] T.H. Lee, M.J. Park, J.H. Park, O.M. Kwon, S.M. Lee, Extended dissipative analysis for neural networks with time-varying delays, IEEE Trans. Neural Netw. Learn. Syst. 25 (10) (2014) 1936–1941.
[28] T.H. Lee, J.H. Park, H.Y. Jung, O.M. Kwon, S.M. Lee, Improved results on stability of time-delay systems using Wirtinger-based inequality, in: Proceedings of World Congress of the IFAC (2014) 6826–6830.
[29] P.G. Park, W.I. Lee, S.Y. Lee, Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems, J. Franklin Inst. 352 (4) (2015) 1378–1396.