Available online at www.sciencedirect.com

ScienceDirect

Applied Mathematics and Computation 190 (2007) 382–401

APPLIED MATHEMATICS AND COMPUTATION

www.elsevier.com/locate/amc

On positivity of singular regular linear time-delay time-invariant systems subject to multiple internal and external incommensurate point delays

M. De la Sen

Departamento de Electricidad y Electrónica, Instituto de Investigación y Desarrollo de Procesos, Campus of Leioa (Bizkaia), Aptdo. 644, Bilbao 48080, Bilbao, Spain

Abstract

This paper deals with the positivity properties of singular regular linear time-delay time-invariant systems subject to multiple internal and external incommensurate constant point delays. The main idea behind the investigation is that its main body is performed based on the construction of the whole state-space trajectory solution without using as usual equivalence or similarity transformations on the matrix of dynamics in order to split the state-trajectory solution into two parts, one being typically associated with a nilpotent matrix. In that way, the whole state trajectory solution contains impulsive terms associated with the initial conditions and inputs. Some extensions concerning positivity aspects are given for a special canonical form which separates the dynamics associated with the nilpotent matrix obtained from an equivalence transformation on the singular matrix of the dynamic system.

Keywords: Positive systems; Singular systems; Time-delay systems

1. Introduction

Time-delay systems are very common in practical situations in nature like, for instance, transmission problems, communications, population growth models and other biological systems or war/peace models, etc., [1–13]. Delays may be discrete (or point delays) (see, for instance, [1–4,13]), distributed, [9–13], and internal (i.e. in the state) or external (i.e. in the input or output). On the other hand, positivity is a very important property related to the fact that the relevant signals concerning a dynamic system evolution are always in the first real closed orthant [14–25] or, in general, in a relevant cone [24]. The positivity property is relevant in some systems where the state variables, controls and outputs cannot be negative at any time instant as it is commonly the case in problems concerning population dynamics, some biological problems of components which are essentially nonnegative, some classes of Markov's models [23–25], etc. This paper is devoted to the study of the positivity properties of singular linear time-delay systems subject to multiple internal (i.e. in the state vector)

E-mail address: wepdepam@lg.ehu.es

0096-3003/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2007.01.053

and external (i.e. in the input or output vectors) constant, in general incommensurate point delays (i.e. they are not all integer multiple of a base delay). The singular systems dealt with are assumed to be regular; i.e. they satisfy a consistency condition ensuring the existence of at least a solution for each given admissible function of initial conditions and control input [23,26]. The paper is organized as follows. A notation section is introduced at the end of this introductory section. Section 2 is devoted to the system structure and the formulation of the consistency condition for the system subject to point delays to be regular in the time-invariant case. Several equivalent conditions of regularity are given and their mutual implications are formulated. Some of those conditions are easily testable with numerical or algebraic tests. Section 3 is devoted to the formulation of the state-trajectory solution in a general explicit way without using similarity or equivalence transformations to reduce the singular matrix to a diagonal form consisting of a nonsingular matrix and a nilpotent one. Several evolution operators are used to construct algebraically equivalent forms of the state-space trajectory solution based on those of auxiliary unforced systems. The main idea is to construct the state-space trajectory solution by direct superposition of the unforced response and the forced one which may include some derivative/impulsive terms (in the so-called impulsive case [23,26]) associated with the index of the singular matrix of the dynamic system which makes it a singular system and which coincides with the nilpotence index of the nilpotent matrix resulting in the block diagonal similar matrix to the original singular one in the case that such a matrix is nonzero [27–29]. The obtained expressions are useful to discuss the state-trajectory solution form in both the impulse-free case and the impulsive one. Section 4 discusses the positivity, respectively external positivity, in terms of all the state and output components, respectively the output components, being nonnegative for all time for any given function of initial conditions being nonnegative and any given control of nonnegative components for all time. Since the positivity properties in the impulsive case cannot be guaranteed for any nonnegative control and any nonnegative function of initial conditions, the concepts of "weak positivity" and "weak external positivity" are introduced. Those concepts are applicable to both impulse-free and impulsive-regular systems are related to pre-positivity properties, which are applicable for sufficiently smooth control and initial conditions functions with the existing time-derivatives being all nonnegative for all time, the number of time-derivatives requested to exist being related to the index of the singular matrix. Section 5 is devoted to an extension concerned with a particular canonical state-space realization of the system (the so-called Weierstrass canonical form [12,26]) where the nilpotent matrix associated with an equivalence transformation for the singular matrix appears explicitly as related to the dynamics of a system substate. The positivity properties of such a realization are investigated. Finally, Section 6 discusses two examples.

1.1. Notation summary

The acronym "iff" stands for "if and only if" as usual. The subset of the positive integers $\bar{n} := \{1, 2, \ldots, n\}$ stands for any integer $n \geqslant 1$.

$\mathbf{R}$, $\mathbf{C}$ and $\mathbf{Z}$ are the sets of real, complex and integer numbers, respectively. $\mathbf{R}_+ := \{z \in \mathbf{R} : z \geqslant 0\}$, $\mathbf{Z}_+ := \{z \in \mathbf{Z} : z \geqslant 0\}$ and $\mathbf{C}_+ := \{z \in \mathbf{C} : \mathrm{Re}\, z \geqslant 0\}$ denote, respectively, the sets of nonnegative real and integer numbers and the set of complex numbers of nonnegative real part. Also, $\mathbf{R}^n_+ := \{z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}} \in \mathbf{R}^n : z_i \geqslant 0,\ \forall i \in \bar{n}\}$ is the first closed orthant of $\mathbf{R}^n$, with the superscript T denoting transposition, $\mathbf{R}^n_- := (\mathbf{R}^n / \mathbf{R}^n_+) \cup \{0 \in \mathbf{R}^\mathbf{n}\}$ is its complement in $\mathbf{R}^n$ adding zero and corresponding definitions follow directly for $\mathbf{Z}^n_+$ and $\mathbf{C}^n_+$.

I_n and O_n denotes the nth identity matrix and the zero matrix in $\mathbf{R}^{n\times n}$ or $\mathbf{C}^{n\times n}$ and $O_{n\times m}$ denotes the zero matrix in $\mathbf{R}^{n\times m}$ or $\mathbf{C}^{n\times m}$.

$C^{(\mu)}(V, \mathbf{R}^n)$ is the set of μ-continuously differentiable real n-vector functions $f : V \to \mathbf{R}^n$; i.e. those which possess μ continuous derivatives everywhere in its open definition domain V.

$AC(V, \mathbf{R}^n)$ is the set of absolutely continuous real n-vector functions $f : V \to \mathbf{R}^n$.

$B(V, \mathbf{R}^n)$ is the set of bounded real n-vector functions $f : V \to \mathbf{R}^n$ and $B_z(V, \mathbf{R}^n) \subset B(V, \mathbf{R}^n)$ is the set of bounded real n-vector functions with support of zero measure in its definition domain V.

$PC(V, \mathbf{R}^n)$ is the set of piecewise continuous real n-vector functions from V to $\mathbf{R}^n$

$$\hat{f}(s) := \mathrm{Lap}(f(t)) = \int_{-\infty}^{\infty} f(\tau) \mathrm{e}^{-s\tau}\, \mathrm{d}\tau$$

and $\mathrm{Lap}_+(f(t)) = \int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau = \mathrm{Lap}(f(t)\mathbf{U}(t))$ denote, respectively, the two-sided and right Laplace transforms of the real function or real vector function provided they exist and $\mathbf{U}(t)$ is the unit step (Heaviside) function: i.e. it is unity for $t \geqslant 0$ and zero otherwise. $f(t) := \mathrm{Lap}^{-1}(\hat{f}(s))$ is a function obtained as an inverse Laplace transform if such a transform and its inverse exist.

$\delta(i,0)$ is the Kronecker delta and $\delta^{(i)}(t)$ for $i \geqslant 1$ is the ith order distributional derivative of the Dirac impulse $\delta(t) = \delta^{(0)}(t)$, $\delta(t)$ being the Dirac-delta.

The notations $A = (A_{ij}) \in \mathbf{R}^{p\times m}$ and $v = (v_i) \in \mathbf{R}^p$ are standard entry-by-entry notations for a matrix A and a vector v. A vector $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}} \in \mathbf{R}_+^n$ is nonnegative (denoted by $v \geqslant 0$) if $v_i \geqslant 0,\ \forall i \in \bar{n}$ so that any $v \in \mathbf{R}_+^n$ is nonnegative. It is positive (denoted by $v > 0$) if $v_i \geqslant 0,\ \forall i \in \bar{n}$ with $v_j > 0$ for at least one $j \in \bar{n}$ and it is strictly positive if $v_i > 0,\ \forall i \in \bar{n}$ (denoted by $v \gg 0$). A matrix $M \in \mathbf{R}_+^{n\times m}$ of entries M_{ij} is said to be nonnegative. If $M_{ij} > 0$ for at least one $(i,j) \in \bar{n}\times\bar{m}$ then M is positive denoted by $M > 0$. If $M_{ij} > 0,\ \forall (i,j) \in \bar{n}\times\bar{m}$ then M is strictly positive denoted by $M \gg 0$. The symbol $\|\,\|_2$ stands for the ℓ_2 (or spectral) vector and induced matrix norms.

A matrix $A \in \mathbf{R}^{n\times n}$ is a Metzler matrix (denoted by $A \in M_E^{n\times n}$) if $A_{ij} \geqslant 0,\ \forall i, j(\neq i) \in \bar{n}$. A matrix $A \in \mathbf{R}^{n\times n}$ is a M-matrix (denoted by $A \in M_M^{n\times n}$) if $A_{ii} \geqslant 0$ and $A_{ij} \leqslant 0,\ \forall i, j(\neq i) \in \bar{n}$ (equivalently if all its eigenvalues have negative real parts).

2. Singular system and consistency condition

Consider the following singular linear time-invariant system with multiple internal and external incommensurate constant point delays:

$$S:\quad E\dot{x}(t) = \sum_{j=0}^{q} A_j x(t-h_j) + \sum_{j=0}^{q'} B_j u(t-h'_j), \tag{1}$$

$$y(t) = Cx(t) + Du(t), \tag{2}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^p$ are the state, control input and output vectors, respectively, and $A_j\ (j \in \bar{q} \cup \{0\})$, $\bar{q} = \{1, 2, \ldots., q\}$, $B_j(j \in \bar{q}' \cup \{0\})$, $\bar{q}' = \{1, 2, \ldots., q'\}$, C and D are real matrices of compatible orders with the dimensions of those vectors, and $0 = h_0 \leqslant h_i \leqslant h_{i+1} \leqslant h\ (i \in \overline{q-1})$ and $0 = h'_0 \leqslant h'_i \leqslant h'_{i+1} \leqslant h'\ (i \in \overline{q'-1})$ are, respectively, the q internal and q' external point delays. The zero delays $h_0 = h'_0 = 0$ corresponding to the delay-free dynamics and current delay-free input are added for notational simplification convenience. The singular matrix $E \in \mathbf{R}^{n\times n}$ with $\mathrm{rank}(E) = r < n$ gives the singular character to the system S compared to the cases $E = I_n$ (standard system) and $E(\neq I_n)$ being nonsingular so that being reducible to the standard form (see, for instance [23]). The initial conditions of (1) are defined by $x(t) \equiv \varphi(t)$ where $\varphi : [-h, 0] \to R^n$ fulfills $\varphi(0) = x(0) = x_0$ with $h = \mathrm{Max}_{1\leqslant j\leqslant q}(h_j)$ while having the general form $\varphi(t) = \varphi_1(t) + \varphi_2(t),\ \forall t \in [-h, 0]$, where $\varphi_1 \in AC([-h,0], \mathbf{R}^n)$ and $\varphi_2 \in B_z([-h,0], \mathbf{R}^n)$; i.e. the vector function φ is almost everywhere absolutely continuous and includes a finite set of bounded isolated discontinuities. Such a function φ is said to belong to the set of admissible initial conditions $IC([-h,0], \mathbf{R}^n)$. The maximum delays $h = \mathrm{Max}_{1\leqslant j\leqslant q}(h_j)$ and $h' = \mathrm{Max}_{1\leqslant j\leqslant q'}(h'_j)$ any be infinite when results independent of the delays are investigated. The following definition is useful to discuss a wide class of the so-called singular regular systems:

Definition 1. The system S is said to be regular if the consistency condition $\mathrm{Det}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) \neq 0$ holds.

Note that if E is nonsingular then $\mathrm{Det}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) = \mathrm{Det}(E)\mathrm{Det}\left(sI_n - \sum_{j=0}^{q} E^{-1}A_J \mathrm{e}^{-h_j s}\right) \neq 0$ almost everywhere in $\mathbf{C}$. Note also that the operator $\mathbf{O}_S \equiv \left(sI_n - \sum_{j=0}^{q} E^{-1}A_j \mathrm{e}^{-h_j s}\right) : \mathbf{C} \to \mathbf{C}^{n\times n}$, associated with the unforced generalized standard system S, has a finite spectrum (the cardinal of $Sp(\mathbf{O}_s)$) if $A_j = 0,\ \forall j \in \bar{q}$, and a countable one otherwise and it is invertible on its resolvent set $\mathbf{C}/Sp(\mathbf{O}_S)$. If S is singular regular (Definition 1) then $\mathbf{O}_S$ is invertible at least for one complex s and it has at least a state-trajectory solution for each $\varphi \in IC([-h,0], \mathbf{R}^n)$ [3]. It is then proved that there exist infinitely many values of that argument for which the operator is invertible. Also, the forced system S has a solution on $\mathbf{R}_+$ for each $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$. Definition 1, concerning regularity, is not easy to test as stated. Some alternative characterizations of regularity are now given. First, the generic rank (g.r.) in $\mathbf{C}$ of a complex matrix $M(s)$ is defined as $\text{g.r.}(M(s)) = \text{Max}_{s\in\mathbf{C}}(\text{rank}[M(s)])$. A similar definition of the generic rank in $\mathbf{R}$ or in any appropriate subset of $\mathbf{C}$ or $\mathbf{R}$ would follow "mutatis-mutandis". The subsequent result formulates equivalent conditions for regularity of the system S to that given explicitly in Definition 1.

Theorem 1. *The following propositions hold*:

(i) $\text{Det}\left(sE - \sum_{j=0}^{q} A_j e^{-h_j s}\right) \neq 0$ *for some* $s \in \mathbf{C}$ *and any set of finite delays* $h_i \in [0, \infty)$, $i \in \bar{q}$ *iff* $\text{rank}[E, \overline{A}_d] = n$ *where* $\overline{A}_d = [A_0, \overline{A}_{d1}]$ *with* $\overline{A}_{d1} = [A_1, A_2, \ldots, A_q]$.

(ii) $\text{Det}\left(sE - \sum_{j=0}^{q} A_j e^{-h_j s}\right) \neq 0$ *for some* $s \in \mathbf{R}$ (*and also for some* $s \in \mathbf{R}_+/\{0\}$, *for some* $s \in \mathbf{R}_-$, *for some* $s \in \mathbf{R}_-/\{0\}$) *and any set of finite delays* $h_i \in [0, \infty)$, $i \in \bar{q}$ *iff* $\text{rank}[E, \overline{A}_d] = n$.

(iii) *The system S is regular independent of the delays iff for any given* $s_0 \in \mathbf{C}$ *and any complex neighborhood* $B(s_0, r)$ *of center* s_0 *and radius* $r > 0$, *there are infinitely many points* $s \in B(s_0, r)$ *such that* $\text{Det}\left(sE - \sum_{j=0}^{q} A_j e^{-h_j s}\right) \neq 0$.

(iv) *The system S is regular independent of the delays iff for any given* $s_0 = \sigma_0 \in \mathbf{R}$, *there are infinitely many real numbers* $\sigma \in B(\sigma_0, r_0)$ *such that* $\text{Det}\left(\sigma E - \sum_{j=0}^{q} A_j e^{-h_j s}\right) \neq 0$ *for some* $B(\sigma_0, r_0) = (\sigma_0 - r_0, \sigma_0 + r_0)$ *being a real bounded neighborhood of center* σ_0 *and length* $2r_0$.

(v) *The system S is regular independent of the delays if any of the two auxiliary delay-free unforced systems is regular*

$$S_{a1}: \quad E\dot{x}(t) = A_0 x(t); \quad y(t) = Cx(t), \tag{3}$$

$$S_{a2}: \quad E\dot{x}(t) = \left(\sum_{j=0}^{q} A_j\right) x(t); \quad y(t) = Cx(t). \tag{4}$$

Proof

(i) Direct calculation yields

$$\begin{aligned} sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s} &= \left([sI_n, -I_n, -\mathrm{e}^{-h_1 s} I_n, \ldots, -\mathrm{e}^{-h_q s} I_n] \begin{bmatrix} E^{\mathrm{T}} \\ \overline{A}_d^{\mathrm{T}} \end{bmatrix}\right)^{\mathrm{T}} \\ &= [E, \overline{A}_d] \lfloor sI_n, -I_n, -\mathrm{e}^{-h_1 s} I_n, \ldots, -\mathrm{e}^{-h_q s} I_n \rfloor^{\mathrm{T}}. \end{aligned} \tag{5}$$

Thus, Eq. (5) $\Rightarrow \exists s \in \mathbf{C} : \text{Det}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) \neq 0 \iff \text{rank}[E, \overline{A}_d] = n$. Also, since $\text{rank}\lfloor sI_n, -I_n, -\mathrm{e}^{-h_1 s} I_n, \ldots, -\mathrm{e}^{-h_q s} I_n \rfloor = n$, $\forall s \in \mathbf{C}$ then $\text{g.r.}_{s\in\mathbf{C}}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) = \text{rank}\lfloor E, \overline{A}_d \rfloor$ from (5). Thus, $\text{rank}[E, \overline{A}_d] = n \Rightarrow \text{g.r.}_{s\in\mathbf{C}}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) = n \Rightarrow \text{Det}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) \neq 0$ for $s \in \mathbf{C}$ and Property (i) has been fully proved.

(ii) Note that

$$\underset{s\in\mathbf{C}}{\text{g.r.}}\left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right) = \underset{\sigma,\omega\in\mathbf{R}}{\text{Max}}\left(\underset{\sigma,\omega\in\mathbf{R}}{\text{g.r.}}\left(\sigma E - \sum_{j=0}^{q} A_j \mathrm{e}^{-\sigma h_j}\cos(\omega h_j)\right), \underset{\sigma,\omega\in\mathbf{R}}{\text{g.r.}}\left(\omega E - \sum_{j=0}^{q} A_j \mathrm{e}^{-\sigma h_j}\sin(\omega h_j)\right)\right) \tag{6}$$

since the rank of a complex matrix is the maximum of the ranks of its real and imaginary parts. Note that since the linear independence of a set of column vectors holds iff it holds for its associate normalized vectors by any nonzero real scalar it is sufficient to test the first and second maximum rank conditions in (6) for $\omega = 0 \Rightarrow \cos(\omega h_j) = 1 (j \in \bar{q} \cup \{0\})$ and $\pm\sigma \in \mathbf{R}$; and $\sigma = 0 \Rightarrow \mathrm{e}^{-\sigma h_j} = 1 (j \in \bar{q} \cup \{0\})$ and $\pm\omega \in \mathbf{R}$, respectively. Then,

$$\underset{s\in\mathbf{C}}{\text{g.r.}}\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right)=\mathrm{Max}\left(\underset{\sigma\in\mathbf{R}}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\cos(\omega h_j)\right),\underset{\omega\in\mathbf{R}}{\text{g.r.}}\left(\omega E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\sin(\omega h_j)\right)\right)$$
$$=\underset{\pm\sigma\in\mathbf{R}_+}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right)=\underset{\sigma\in\mathbf{R}_S}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right) \tag{7}$$

with $\mathbf{R}_S$ being any of the sets $\mathbf{R}_+$, $\mathbf{R}_-$, $\mathbf{R}_+/\{0\}$, $\mathbf{R}_-/\{0\}$ since

$$\underset{\sigma\in\mathbf{R}_+}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right)=\underset{\sigma\in\mathbf{R}_+/\{0\}}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right) \tag{8}$$

unless

$$\mathrm{rank}\left(\sum_{j=0}^{q}A_j\right)=\lim_{\sigma\to 0^+}\left[\mathrm{rank}\left(\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right)\right]>\underset{\sigma\in\mathbf{R}_+/\{0\}}{\text{g.r.}}\left(\sigma E-\sum_{j=0}^{q}A_j\mathrm{e}^{-\sigma h_j}\right) \tag{9}$$

and, in the same way, the generic ranks on $\mathbf{R}_-$ and $\mathbf{R}_-/\{0\}$ are identical. Otherwise, $s=0$ is an isolated point in $\mathbf{C}$ which is not in the resolvent set of the operator $\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right)$ from $\mathbf{C}^n$ to $\mathbf{C}^n$ what would imply that the resolvent set is not open so that the spectrum is not closed what is impossible. As a result

$$\mathrm{Det}\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right)\neq 0\quad \text{for some } s\in\mathbf{R}_S \Longleftrightarrow \underset{s\in\mathbf{R}_S}{\text{g.r.}}\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right)=n\Longleftrightarrow \mathrm{rank}\left[E,\overline{A}_d\right]=n.$$

(iii) The "If Part" follows since $\mathrm{Det}\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right)\neq 0$ for some $s\in\mathbf{C}$ if the system S is regular from Definition 1. The "Only if Part" is now proved proceeding by contradiction. Define $g(s):=\mathrm{Det}\left(sE-\sum_{j=0}^{q}A_j\mathrm{e}^{-h_js}\right):\mathbf{C}\to\mathbf{C}$ and assume that $g(s_0)=0$ for some $s_0\in\mathbf{C}$. Thus, either s_0 is an isolated zero (of any finite multiplicity) so that $g(s)\neq 0$ for all $s(\neq s_0)\in B(s_0,r_0)$, $B(s_0,r_0)$ being some neighborhood of s_0 in $\mathbf{C}$, centered at s_0, of finite radius $r_0>0$, or $g(s)\equiv 0$ on $B(s_0,r_0)$. In the first case, the property holds (since $g(s)\neq 0$ for some $s\in\mathbf{C}$) and it then suffices to consider the case $g(s)\equiv 0$ on $B(s_0,r_0)$. Consider any open connected subset V_C of $\mathbf{C}$ containing properly $B(s_0,r_0)$. Then, by analytic continuation $g(s)\equiv 0$ on V_C what may be extended to $g(s)\equiv 0$ on $\mathbf{C}$ by taking $V_C=\mathbf{C}$ which is the largest open connected subset V_C of $\mathbf{C}$ containing properly $B(s_0,r_0)$. Now, consider the extended complex plane $\mathbf{C}_e:=\mathbf{C}\cup\{\infty\}$ by including the infinity point defined as the limit of the projections of all sequences of points of arbitrarily large modules located at a tangent plane to the south pole of the Riemann sphere on its north pole. From Liouville's theorem, either the entire function $g:\mathbf{C}_e\to\mathbf{C}_e$ is constant, so that $g(s)\equiv 0$ on $\mathbf{C}_e$, or it diverges as $|s|\to\infty$. But the last possibility fails since for any arbitrary sufficiently large finite $r>0$, $g(s)=0$ for all $s\in\mathbf{C}$ exists with $|s|>r$. As a result, $g(s)\equiv 0$ on $\mathbf{C}_e$ and then the system S is not regular from Definition 1. Thus, either each $s_0\in\mathbf{C}$ such that $g(s_0)=0$ is isolated and then $g(s)\neq 0$ at infinitely many points $s(\neq s_0)\in\mathbf{C}$, or the system S is not regular. Thus, the "If Part" has been proved.

(iv) The "If Part" follows directly from Definition 1. The "Only If Part" is proved by contradiction. Assume that S is regular and $g(\sigma)\equiv 0$ for $\sigma\in(\sigma_0-r_0,\sigma_0+r_0)$ and arbitrary $\sigma_0, r_0(>0)\in\mathbf{R}$, the function g being defined as in the proof of (iii). Since there are no isolated zeros of g there is a complex neighborhood $B(\sigma_0,r)\supset(\sigma_0-r_0,\sigma_0+r_0)$. As in the proof of (iii), $B(\sigma_0,r)$ may be extended by analytic continuation to any connected open set V_C of $\mathbf{C}$, to $\mathbf{C}$ itself and, finally to $\mathbf{C}_e$, concluding that $g(\sigma)\equiv 0$ for $\sigma\in(\sigma_0-r_0,\sigma_0+r_0)\Rightarrow g(s)\equiv 0$ on $\mathbf{C}_e$ and then the system S is not regular. Thus, either $g(\sigma)\neq 0$ at infinitely many points $\sigma\in\mathbf{R}$, or the system S is not regular. The "If Part" has been proved.

(v) The system S_{a1} is regular $\Longleftrightarrow \mathrm{rank}[E,A_0]=n$ from (i) $\Rightarrow \mathrm{rank}[E,\overline{A}_d]=n\Rightarrow$ the system S is regular independent of the delays from (i). Also, if the system S_{a2} is regular then from (i),

$$n=\mathrm{rank}\left[E,\sum_{j=0}^{q}A_j\right]=\mathrm{rank}\left[E,\sum_{j=0}^{q}A_j,\overline{A}_{d1}\right]=\mathrm{rank}\left[E,A_0+\sum_{j=1}^{q}A_j,\overline{A}_{d1}\right]=\mathrm{rank}[E,A_0,\overline{A}_{d1}]$$

the last rank identity being obtained by subtracting in the second block matrix the sum of all the columns of $\overline{A}_d$ from $\sum_{j=0}^{q} A_j$. Since $\mathrm{rank}[E, \overline{A}_d] = n$, the system S is regular independent of the delays from (i).

Alternative proof of property (*v*): *Part* 1

If the system S_{a1} is regular then from (iii) there exist infinitely many points $s \in \mathbf{C}$ such that $(sE - A_0)^{-1}$ exists. Thus, by construction, there is a sequence $\Sigma_S := \{s_j\}_0^{\infty}$ of points $s_j \in \mathbf{C}$ fulfilling $Re(s_j) > 0$ and $|s_{j+1}| > |s_j|$ and $s_j \to \infty$, such that:

(a) $(s_jE - A_0)^{-1}$ exists for all $s_j \in \Sigma_S$ what follows from (ii) since $\mathrm{Det}(s_jE - A_0) \neq 0$ for infinitely many point s_j if S_{a1} is regular then Σ_S might be chosen "ad doc" to fulfill that property.

(b) $\|(s_jE - A_0)^{-1}\| \to 0$ for $s_j \in \Sigma_S$ as $j \to \infty$ for any matrix norm since the entries of $(s_jE - A_0)^{-1}$ are all meromorphic and defined by quotients of two polynomials, the common denominator being $\mathrm{Det}(s_jE - A_0)$ with largest degree than any of the numerator polynomials of modulus and all tending to zero as $|s_j| \to \infty$.

(c) $|\mathrm{e}^{-h_is_j}| \to 0$ for any $h_i \in (0, \infty)$, $\forall i \in \bar{q}$, as $|s_j| \to \infty$ since $Re(s_j) > 0$ for $j \geqslant 1$ by construction of Σ_S and the choice of $s_j \in \Sigma_S$ and $|\mathrm{e}^{-h_is_j}| = 1$ if $h_i = 0$ for all $s_j \in \Sigma_S$. Then, there is a sufficiently large positive integer j_0, depending on the matrices A_i and the maximum allowable delays $\bar{h}_i \geqslant 0$; i.e. $h_i \in [0, \bar{h}_i]$ $(i \in \bar{q})$, $\bar{h}_0 = 0$ such that $(s_jE - A_0)^{-1}$ exists, and

$$\|(s_jE - A_0)^{-1}\|_2 \left\| \sum_{i=1}^{q} A_i \mathrm{e}^{-h_is_j} \right\|_2 < 1 \tag{10}$$

for any $h_i \in [0, \bar{h}_i)$, $\forall i \in \bar{q}$, and any $s_j \in \Sigma_S$ with $j \geqslant j_0$. As a result, $\left(s_jE - \sum_{i=1}^{q} A_i \mathrm{e}^{-h_is_j}\right)^{-1}$ exists for all $h_i \in [0, \bar{h}_i)$, $\forall i \in \bar{q}$ and the factorization

$$\left(s_jE - \sum_{i=0}^{q} A_i \mathrm{e}^{-h_is_j}\right) = (s_jE - A_0)\left(I_n - (s_jE - A_0)^{-1} \sum_{i=1}^{q} A_i \mathrm{e}^{-h_is_j}\right) \tag{11}$$

is well-posed for any $s_j \in \Sigma_S$ fulfilling $j \geqslant j_0$ from Banach's Perturbation Lemma [2,39]. Thus, $\mathrm{Det}\left(s_jE - \sum_{i=0}^{q} A_i \mathrm{e}^{-h_is_j}\right) \neq 0$ and then the system S is regular for all $h_i \in [0, \bar{h}_i)$, $\forall i \in \bar{q}$ and $j \geqslant j_0$. Since $\bar{h}_i$ $(i \in \bar{q})$ may be chosen arbitrarily large and the sequence Σ_S can be re-chosen accordingly and then j_0 recalculated if necessary, the system S is regular independent of the delays if the system S_{a1} is regular.

Part 2

If the system S_{a2} is regular then the sequence $\Sigma_S := \{s_j\}_0^{\infty}$ of points $s_j \in \mathbf{C}$ fulfilling $\mathrm{Re}(s_j) > 0$ and $|s_{j+1}| > |s_j|$ and $s_j \to \infty$ may be modified from Part 1 of the proof such that:

(d) $\left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1}$ exists for all $s_j \in \Sigma_S$ what follows from (ii) since $\mathrm{Det}\left(s_jE - \sum_{i=0}^{q} A_i\right) \neq 0$ for infinitely many point s_j if S_{a2} is regular then Σ_S might be chosen "ad doc" to fulfill that property.

(e) $\left\|\left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1}\right\| \to 0$ for $s_j \in \Sigma_S$ as $j \to \infty$ for any matrix norm since the entries of $\left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1}$ are all meromorphic and rational defined by quotients of two polynomials, the common denominator being $\mathrm{Det}\left(s_jE - \sum_{i=0}^{q} A_i\right)$ with largest degree than any of the numerator polynomials and all tending to zero as $|s_j| \to \infty$.

(f) $|1 - \mathrm{e}^{-h_is_j}| \leqslant 2$ for any $h_i \in (0, \infty)$, $\forall i \in \bar{q}$ since $Re(s_j) > 0$ for $j \geqslant 1$ by construction of Σ_S.

Then, there is a sufficiently large positive integer j_0, depending on the matrices A_i, such that $\left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1}$ exists and $\left\|\left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1} \sum_{i=1}^{q} A_i(1 - \mathrm{e}^{-h_is_j})\right\|_2 < 1$.

As a result, $\left(s_jE - \sum_{i=1}^{q} A_i \mathrm{e}^{-h_is_j}\right)^{-1}$ exists for all $h_i \in [0, \bar{h}_i)$, $\forall i \in \bar{q}$ and $j \geqslant j_0$ and the factorization

$$\left(s_jE - \sum_{i=0}^{q} A_i e^{-h_i s_j}\right) = (s_jE - A_0)\left(I_n - \left(s_jE - \sum_{i=0}^{q} A_i\right)^{-1} \sum_{i=1}^{q} A_i(1 - e^{-h_i s_j})\right)$$

is well-posed for any $s_j \in \Sigma_S$ fulfilling $j \geqslant j_0$ from Banach's Perturbation Lemma. Thus, $\mathrm{Det}\left(s_jE - \sum_{i=0}^{q} A_i e^{-h_i s_j}\right) \neq 0$ and then the system S is regular for all $h_i \in [0, \bar{h}_i)$, $\forall i \in \bar{q}$. Since $\bar{h}_i (i \in \bar{q})$ may be chosen arbitrarily large and the sequence Σ_S can be re-chosen accordingly and thus j_0 potentially modified, the system S is regular independent of the delays if the system S_{a2} is regular. □

Note that an important feature associated with Theorem 1 is that the regularity condition may be tested in arbitrary small neighborhoods of any chosen real or complex numbers and it can also be tested through rank tests over real matrices. Another important feature is that regularity under infinite delays and zero delays (tested via the auxiliary delay-free systems) guarantees regularity independent of the delays (Theorem 1(v)). These features are emphasized in the next result which is also concerned with stronger properties to those referred to in Theorem 1(v) including some extra necessary-type conditions.

Theorem 2. *The following properties hold*:

(i) *The system S is regular for zero delays iff the system S_{a2} is regular (i.e. iff $\mathrm{Det}\left(sE - \sum_{j=0}^{q} A_j\right) \neq 0$ for some $s \in \mathbf{C}$ and iff $\mathrm{rank}\left[E, \sum_{j=0}^{q} A_j\right] = n$). If the system S_{a2} is regular then the system S is regular for any given set of finite delays $0 \leqslant h_i \leqslant h_{i+1} < \infty$ $(i \in \overline{q-1})$.*

(ii) *The system S is regular independent of the delays iff $\mathrm{rank}[E, A_0] = \mathrm{rank}\left[E, \sum_{j=0}^{q} A_j\right] = n$ or, equivalently, iff there exist $s, s' \in \mathbf{C}$ such that $\mathrm{Det}(sE - A_0) \neq 0$, $\mathrm{Det}\left(s'E - \sum_{j=0}^{q} A_j\right) \neq 0$ (i.e., iff both systems S_{a1} and S_{a2} are regular). If the system S_{a2} is regular then the system S is regular independent of the delays).*

Proof

(i) The first part follows directly from Theorem 1(i) (and also from Theorem 1(v)) since the unforced systems S and S_{a2} are identical for $h_i \equiv 0 (i \in \bar{q})$. The second part of Property (i) is a part of Theorem 1(v).

(ii) The system S is regular as $h_i \to \infty$ $(i \in \bar{q})$ iff $\mathrm{rank}[E, A_0] = n$ from Theorem 1(i). The system S is regular for $h_i = 0$ $(i \in \bar{q})$ from Theorem 1(ii) iff $\mathrm{rank}\left[E, \sum_{j=0}^{q} A_j\right] = n$ (full rank) $\Rightarrow \mathrm{rank}[E, \overline{A}_d] = n$ (full rank) from direct evaluation of both identical full rank conditions) (see also, Property (i)). This implies still from Theorem 1(i) that S is regular independent of the delays for any finite set of bounded delays $0 \leqslant h_i \leqslant h_{i+1}$, $i \in \overline{q-1}$. As a result, the system S is regular independent of the delays iff $\mathrm{rank}[E, A_0] = \mathrm{rank}\left[E, \sum_{j=0}^{q} A_j\right] = n \Longleftrightarrow \exists s,\ s' \in \mathbf{C}$ such that $\mathrm{Det}(sE - A_0) \neq 0$, $\mathrm{Det}\left(s'E - \sum_{j=0}^{q} A_j\right) \neq 0$. □

Note from Theorem 2 that the regularity test of Definition 1 may be also alternatively tested, with no loss in generality, on the real axis and on the nonnegative, negative, nonpositive or positive real semi-axes and also for complex numbers in arbitrary neighborhoods or arbitrary prefixed complex or real numbers.

3. State-trajectory solution

Since $\mathrm{rank}(E) = r < n$, it exists $T \in \mathbf{R}^{n \times n}$ such that $E = T^{-1}\,\mathrm{BlockDiag}(E_0, E_1)T$ matrix similar to $\mathrm{BlockDiag}(E_0, E_1)$, provided that E is not nilpotent, where $E_0 \in \mathbf{R}^{r_0 \times r_0}$ $(1 \leqslant r_0 = r - \mathrm{rank}(E_1) \leqslant r)$ is nonsingular and $E_1 \in \mathbf{R}^{(n-r_0)\times(n-r_0)}$ is nilpotent so that the Drazin inverse of E is $E^{(D)} = T^{-1}\,\mathrm{BlockDiag}(E_0^{-1}, 0)T$, [27–29]. Thus,

$$E^{(D)}E = EE^{(D)} = T^{-1}\,\mathrm{BlockDiag}(I_{r_0}, 0_{(n-r_0)\times(n-r_0)})^{\mathrm{T}}.$$

The nilpotence index of E_1 (abbreviated as $\mathrm{nind}(E_1)$), μ, verifies

$$\begin{aligned}\mu &:= \mathrm{nindex}(E_1) = \mathrm{ind}(E)\ \text{(the index of } E)\\ &= \mathrm{Min}(k \in \mathbf{Z} : \mathrm{Ker}(E^{(k)}) = \mathrm{Ker}(E^{k+1})) \leqslant \mathrm{rank}(E) - \deg(\det(sE - A_0)) + 1 = r - \deg(\det(sE - A_0)) + 1.\end{aligned} \tag{12}$$

The state trajectory solution of (1) is obtained for any $\varphi \in IC([-h,0],\mathbf{R}^n)$ by using Laplace transforms in the state dynamics of (1). Let $\hat{v}(s) = \mathrm{Lap}[v(t)] = \int_{-\infty}^{\infty} \mathrm{e}^{-sT} v(t)\,\mathrm{d}t$ the (two-sided) Laplace transform of $v : \mathbf{R} \to \mathbf{R}^n$ provided that such a transform exists. If $v(t) = 0$ for $t \in \mathbf{R}/\mathbf{R}_+$ (i.e. v is an original function) then $\hat{v}(s) = \mathrm{Lap}_+[v(t)] = \int_0^{\infty} \mathrm{e}^{-st} v(t)\,\mathrm{d}t$ coincides with the right Laplace transform of $v(t)$. In all the sequel, the following assumption is made unless another specification be made.

Assumption 1. The system S is regular independent of the delays.

The subsequent result holds:

Theorem 3. *The following properties hold*:

(i) *The subsequent identities are true*

$$\hat{x}(s) = \hat{\Phi}(s)\left(\sum_{j=1}^{q} A_j \mathrm{e}^{-h_j s}\hat{x}(s) + Ex_0 + \hat{i}(s) + \sum_{j=1}^{q} B_j \mathrm{e}^{-h'_j s}\hat{u}(s)\right), \tag{13}$$

$$= \hat{\Psi}(s)\left[Ex_0 + \hat{i}(s) + \left(\sum_{j=0}^{q'} B_j \mathrm{e}^{-h'_j s}\right)\hat{u}(s)\right], \tag{14}$$

$$= \left(I_n - \hat{\Phi}(s)\left(\sum_{j=1}^{q} A_j \mathrm{e}^{-h_j s}\right)\right)^{-1}\hat{\Phi}(s)\left(Ex_0 + \hat{i}(s) + \sum_{j=1}^{q} B_j \mathrm{e}^{-h'_j s}\hat{u}(s)\right), \tag{15}$$

where

$$\hat{\Phi}(s) := \mathrm{Lap}[\Phi(t) := \mathrm{e}^{A_0 t}] = (sE - A_0)^{-1} \tag{16}$$

$$\hat{\Psi}(s) := \mathrm{Lap}[\Psi(t)] = \left(sE - \sum_{j=0}^{q} A_j \mathrm{e}^{-h_j s}\right)^{-1} = \left(I_n - \hat{\Phi}(s)\left(\sum_{j=1}^{q} A_j \mathrm{e}^{-h_j s}\right)\right)^{-1}\hat{\Phi}(s)$$

$$= \hat{\Phi}(s)\left(I_n - \left(\sum_{j=1}^{q} A_j \mathrm{e}^{-h_j s}\right)\hat{\Phi}(s)\right)^{-1} \tag{17}$$

are compact operators in $\mathbf{C}^n$ *where they exist* (*i.e. at the resolvent subset of their definition domains*), *and*

$$\hat{i}(s) = \mathrm{Lap}\left[\sum_{j=1}^{q} A_j \varphi_{ej}(t)\right] = \sum_{j=1}^{q} A_j \hat{\varphi}_{0j}(s), \tag{18}$$

where

$$\hat{\varphi}_{ej}(s) := \mathrm{Lap}[\varphi_{ej}(t)]; \quad \varphi_{ej}(t) = \begin{cases} \varphi(t) & \textit{for } t \in [-h_j, 0],\ \forall j \in \bar{q}, \\ 0 & \textit{otherwise}. \end{cases} \tag{19}$$

(ii) *The state equation* (1) *of the system S may be rewritten equivalently as*

$$\dot{x}(t) = \Phi_0\left(\sum_{j=0}^{q} A_j x(t-h_j) + \sum_{j=0}^{q'} B_j u(t-h'_j)\right)$$
$$+ \sum_{i=1}^{\mu} \Phi_{-i}\left(Ex_0\delta^{(i)}(t) + \sum_{j=1}^{q} A_j x^{(i)}(t-h_j) + \sum_{j=1}^{q'} B_j u^{(i)}(t-h'_j)\right), \tag{20}$$

where

$$\mu = \begin{cases} \mathrm{nind}(E_1) = \mathrm{nind}(E) \geqslant 1 & \textit{if } \mathrm{rank}(E) = r < n, \\ 0 & \textit{if } \mathrm{rank}(E) = r = n \end{cases}$$

(*note that* $\mu = 0$ *iff the system S is nonsingular*) *and* $\Phi_i \in \mathbf{R}^{n\times n}$; $i \in [-\mu, \infty) \cap \mathbf{Z}$ *are* (*in general nonunique*) *solutions the algebraic linear system*

$$\Phi_i E + \Phi_{i-1} A_0 = E\Phi_i - A_0\Phi_{i-1} = \delta(i,0) I_n, \quad i \in [-\mu, \infty) \cap \mathbf{Z}, \tag{21a}$$

$$\Phi_{-\mu} E = E\Phi_{-\mu} = \Phi_{-\mu-1} A_0 = A_0 \Phi_{-\mu-1} = 0, \tag{21b}$$

where $\delta(i,0)$ *is the Kronecker delta defined by* $\delta(0,0) = 1$ *and* $\delta(i,0) = 0$ *for* $i \neq 0$, $v^{(i)}(t) = \frac{\mathrm{d}^i v(t)}{\mathrm{d}t^i}$ *if the ith time derivative of the real function, or real vector function,* $v(t)$ *provided that it exists and* $\delta^{(i)}(t)$ *is the ith order distributional derivative of the Dirac impulse* $\delta(t) = \delta^{(0)}(t)$, $\delta(t)$ *being the Dirac delta. If* $v^{(\ell)}(t) = \frac{\mathrm{d}^\ell v(t)}{\mathrm{d}t^\ell}$ *exists but it is not differentiable then* $v^{(i)}(t) = \delta^{(i-\ell)}(t)$ *for* $i > \ell$. *The* (*in general nonunique*) *solution of* (20) *for each given* $\varphi \in IC([-h, 0], \mathbf{R}^n)$ *and* $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$ *is*

$$x(t) = e^{\Phi_0 A_0 t}\Phi_0\left(Ex_0 + \sum_{j=1}^{q}\int_0^t \mathrm{e}^{-\Phi_0 A_0 \tau} A_j x(\tau - h_j)\,\mathrm{d}\tau + \sum_{j=0}^{q'}\int_0^t \mathrm{e}^{-\Phi_0 A_0 \tau} B_j u(\tau - h'_j)\,\mathrm{d}\tau\right) + \sum_{i=1}^{\mu}\Phi_{-i}\left(Ex_0\delta^{(i-1)}(t) + \sum_{j=1}^{q} A_j x^{(i-1)}(t - h_j) + \sum_{j=1}^{q'} B_j u^{(i-1)}(t - h'_j)\right). \tag{22}$$

(iii) *The subsequent identity holds*

$$\hat{\Phi}(s) = (sI_n - A_0)^{-1}\Phi_0 + \sum_{i=1}^{\mu}\Phi_{-i} s^{i-1} \tag{23}$$

and the solution (22) *is equivalent to*

$$\begin{aligned} x(t) &= z(t) + \mathrm{e}^{\Phi_0 A_0 t}\left[\sum_{j=1}^{q}\int_{h_j}^{t} \mathrm{e}^{-\Phi_0 A_0 \tau}\Phi_0 A_j x(\tau - h_j)\,\mathrm{d}\tau\right] + \sum_{i=1}^{\mu}\sum_{j=1}^{q}\Phi_{-i} A_j x^{(i-1)}(t - h_j),\\ z(t) &= \mathrm{e}^{\Phi_0 A_0 t}\Phi_0\left[Ex_0 + \sum_{j=1}^{q}\int_{-h_j}^{0} \mathrm{e}^{-\Phi_0 A_0(\tau + h_j)} A_j \varphi(\tau)\,\mathrm{d}\tau + \sum_{j=1}^{q'}\int_0^t \mathrm{e}^{-\Phi_0 A_0(\tau + h_j)} B_j u(\tau - h'_j)\,\mathrm{d}\tau\right]\\ &\quad + \sum_{i=1}^{\mu}\Phi_{-i}\left(Ex_0\delta^{(i-1)}(t) + \sum_{j=1}^{q'} B_j u^{(i-1)}(t - h'_j)\right). \end{aligned} \tag{24}$$

(iv) (1) *If* $\mu = 0$ *then* $E = E_0$, $\Phi_0 = E^{-1}$ *and* $\Phi_{-i} = 0$ *so that there are no impulsive terms in* $\dot{x}(t)$ *and* $x(t)$, *and the system S is either standard or generalized standard* (*i.e. nonsingular*) *with a unique state-trajectory solution* $x : \mathbf{R}_+/\{0\} \to \mathbf{R}^n$ *for each given* $\varphi \in IC \in ([-h, 0], \mathbf{R}^n)$ *and* $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$.

(2) *If* $\mu = \mathrm{ind}(E) = \mathrm{nind}(E_1) = 1$ *then* $E_1 = 0$, $r_0 = r > 0$, *and E is similar to* $\mathrm{BlockDiag}(E_0, 0_{(n-r)\times(n-r)})$ *with* $E_0 \in \mathbf{R}^{r\times r}$ *being nonsingular. There exist impulsive terms in* $\dot{x}(t)$ *but not in* $x(t)$ *and the system S is singular regular with a unique state-trajectory solution* $x : \mathbf{R}_+/\{0\} \to \mathbf{R}^n$ *for each given* $\varphi \in IC \in ([-h, 0], \mathbf{R}^n)$ *and* $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$.

(3) *If* $\mu \geqslant 2$ *then* $n = r_0 + r_1 \geqslant r_0 + \mathrm{rank}(E_1) \geqslant r_0 + \mu \geqslant r_0 + 2 \geqslant 2$. *There exist impulsive terms in both* $\dot{x}(t)$ *and* $x(t)$ *and the system S is singular regular* (*since Assumption* 1 *holds*) *with nonunique state-trajectory solution* $x : \mathbf{R}_+/\{0\} \to \mathbf{R}^n$ *for each given* $\varphi \in IC([-h, 0], \mathbf{R}^n)$ *and* $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$.

Proof. (i) Eqs. (13)–(15) follow directly by taking Laplace transforms in (1) subject to (16)–(19).

(ii) Expanding $(sE - A_0)^{-1}$ in powers of s^{-i} yields

$$\hat{\Phi}(s) = (sE - A_0)^{-1} = \sum_{i=-\mu}^{\infty}\Phi_i s^{-(i+1)} \tag{25}$$

with $\mu = \mathrm{ind}(E)$ and $\{\Phi_i\}_{i=-\mu}^{\infty}$ being a sequence of square n-real matrices fulfilling

$$(sE - A_0)^{-1}(sE - A_0) = (sE - A_0)\left(\sum_{i=-\mu}^{\infty}\Phi_i s^{-(i+1)}\right) = \sum_{i=-\mu}^{\infty}(E\Phi_i - A_0\Phi_{i-1})s^{-i} = \sum_{i=-\mu}^{\infty}(\Phi_i E - \Phi_{i-1}A_0)s^{-i} = I_n, \tag{26}$$

which holds irrespective of the complex indeterminate s so that the coefficient of $s^0 = 1$ should be the nth identity matrix and the remaining ones should be zero; i.e. the sequence of square n-real matrices $\{\Phi_i\}_{i=-\mu}^{\infty}$ has to be a solution to (21) provided that a solution exists. Eqs. (21) may be compactly rewritten as

$$\overline{Q}_\ell \overline{\Phi}_\ell = \overline{J}_\ell, \tag{27}$$

where $\overline{Q}_{-(\mu+1)} = 0$ and for each integer $\ell \geqslant -\mu$

$$\begin{aligned} &\overline{Q}_\ell = [Q_\ell^{\mathrm{T}}, Q_{\ell-1}^{\mathrm{T}}, \ldots, Q_{-\mu+1}^{\mathrm{T}}]^{\mathrm{T}}; \quad \overline{\Phi}_\ell = [\Phi_\ell^{\mathrm{T}}, \Phi_{\ell-1}^{\mathrm{T}}, \ldots, \Phi_{-\mu-1}^{\mathrm{T}}]^{\mathrm{T}}, \\ &Q_i = \left[0_{n\times n}, \overbrace{\ldots}^{\ell-i}, 0_{n\times n}, E, -A_0, 0_{n\times n}, \overbrace{\ldots}^{\mu+i}, 0_{n\times n}\right] \quad \text{for } -\mu+1 \leqslant i \leqslant \ell, \\ &\overline{J}_\ell = \left[0_{n\times n}, \overbrace{\ldots}^{\ell}, 0_{n\times n}, I_n, 0_{n\times n}, \overbrace{\ldots}^{\mu+1}, 0_{n\times n}\right]^{\mathrm{T}} \quad \text{for } \ell \geqslant -\mu. \end{aligned} \tag{28}$$

Since the system S is regular independent of the delays $\mathrm{rank}[E, A_0] = n$ from Theorem 2(ii), it follows from Eqs. (27) and (28) that $\mathrm{rank}[\overline{Q}_\ell] = \mathrm{rank}[\overline{Q}_\ell, \overline{J}_\ell] = n$ for any integer $\ell \geqslant -\mu$ so that a (in general, nonunique) solution $\overline{\Phi}_\ell$ exists for Eq. (27), subject to Eq. (28), for each integer $\ell \geqslant -\mu$ from Rouché–Froebenius theorem from Linear Algebra. Equivalently the algebraic system of linear equations (21) has a solution. Now, the next auxiliary identity is proved

$$(sE - A_0)^{-1} = \sum_{i=-\mu}^{\infty} \Phi_i s^{-(i+1)} = (sI_n - A_0\Phi_0)^{-1}\Phi_0 + \sum_{i=1}^{\mu} \Phi_{-i} s^{i-1}, \tag{29}$$

which holds iff

$$(sI_n - A_0\Phi_0)^{-1}\Phi_0 = \sum_{i=0}^{\infty} \Phi_i s^{-(i+1)}. \tag{30}$$

It is now proved that (30) holds which implies and it is implied by (29) to hold. Direct calculations with (30) yield directly

$$\begin{aligned} \Phi_0 &= (sI_n - A_0\Phi_0)\left((sE - A_0)^{-1} - \sum_{i=1}^{\mu} \Phi_{-i} s^{i-1}\right), \\ \Phi_0(sE - A_0) &= (sI_n - A_0\Phi_0)\left(I_n - \sum_{i=1}^{\mu} \Phi_{-i}(sE - A_0) s^{i-1}\right), \end{aligned}$$

and then

$$\begin{aligned} sI_n - A_0\Phi_0 &= \Phi_0(sE - A_0) + \sum_{i=1}^{\mu} \Phi_{-i}(sE - A_0)s^i - \sum_{i=1}^{\mu} A_0\Phi_0\Phi_{-i}(sE - A_0)s^{i-1} \\ &= \Phi_0 A_0(\Phi_{-1}A_0 - I_n) + E\Phi_{-\mu}s^{\mu+1} + \sum_{i=0}^{\mu}[E\Phi_{-i+1} - A_0\Phi_{-i} + A_0\Phi_0(E\Phi_{-i} - A_0\Phi_{-i-1})]s^i \\ &= sI_n - A_0\Phi_0 - \Phi_0 A_0 \Phi_{-1} A_0 \end{aligned} \tag{31}$$

so that (31) is true and then identical to (30) and, furthermore

$$\Phi_0 A_0 \Phi_{-1} A_0 = \Phi_0 A_0(E\Phi_0 - I_n) = 0.$$

Then (30) and, equivalently (29) hold and thus for any original $g : \mathbf{R} \to \mathbf{R}^n$ (i.e. $g(t) = 0$ for $t < 0$), $(sI_n - A_0\Phi_0)^{-1}\Phi_0\hat{g}(s) = \mathrm{Lap}\left(\int_0^t \mathrm{e}^{A_0\Phi_0\tau}\Phi_0 g(t-\tau)\,\mathrm{d}\tau\right)$ from the Laplace convolution theorem since $(sI_n - A_0\Phi_0)^{-1}\Phi_0 = \mathrm{Lap}(\mathrm{e}^{A_0\Phi_0 t}\Phi_0)$. Using these identities and (29) into (13) yield (20) with (nonunique) state trajectory solution for any (22) since a solution to the algebraic system (21) exists if the system S is regular. Property (ii) has been fully proved.

(iii) The identity (23) is fully equivalent to the proved identity (29). Then, the equivalent state-trajectory solution Eqs. (24) follows directly from the relationships in Laplace transforms obtained from (15) and (29)

$$\left[I_n - (sI_n - A_0\Phi_0)^{-1}\Phi_0 + \sum_{i=1}^{\mu}\Phi_{-i}s^{i-1}\left(\sum_{j=1}^{q}A_j\mathrm{e}^{-\mathrm{sh}_j}\right)\right]\hat{x}(s) = \hat{z}(s)$$

$$:= \left[(sI_n - A_0\Phi_0)^{-1}\Phi_0 + \sum_{i=1}^{\mu}\Phi_{-i}s^{i-1}\right] \times \left[Ex_0 + \hat{i}(s) + \sum_{j=0}^{q'}B_j\mathrm{e}^{-h'_j s}\hat{u}(s)\right].$$

(iv) It follows directly by considering the particular cases $\mu = 0, 1$ and $\mu \geqslant 2$ together with the constraints (12) and $r_0 = r - \mathrm{rank}(E_1) \leqslant r \leqslant n$ with $r < n$ if E is singular. $\square$

Remarks

1. Note that for $\mu = 0$, the system S is either in standard form or in generalized standard form so that its solution is unique on $\mathbf{R}_+$. For $\mu = 1$, the system S is singular but impulsive-free so that the solutions has no impulses while it is still unique on $\mathbf{R}_+$. This result is apparent from (24) and Theorem 3(iv).
2. The state-trajectory solution expressions given in Theorem 3 may be generalized directly to the case when the control is impulsive. In that case, for any time $t \in \mathbf{R}_+$, the control may be decomposed as $u(t^\pm) = u_{pc}(t^\pm) + \sum_{t_i \in I_p(t^\pm)} u(t_i)\delta(t - t_i)$ for the right and left limits of any time instant $t \in \mathbf{R}_+$ where $u_{pc} \in PC(\mathbf{R}_+, \mathbf{R}^m)$, $t_i \in I_0 := \{t_j\}_1^\chi$ with $\chi \leqslant \chi_0$ denoting the cardinal of the sequence I_0 of time instants where control impulses take place. I_0 might be either finite of cardinal χ or infinite many of cardinal χ_0 (infinity numerable). The sequence $\{u(t_i), t_i \in I_0\}$ is assumed to be uniformly bounded. If two discrete subsets of I_0 are defined to the left/right of any $t \in \mathbf{R}_+$, respectively, as $I_p(t^-) := [0, t) \cap I_0 = \{t_j\}_1^{\mathrm{Max}(t_j \in \mathrm{I}_0 : t_j < t)}$, and $I_p(t^+) := [0, t] \cap I_0 = \{t_j\}_1^{\mathrm{Max}(t_j \in \mathrm{I}_0 : t_j \leqslant t)}$. Thus, $I_p(t^+) = I_p(t^-)$ if $t \notin I_0$ and $I_p(t^+) = I_p(t^-) \cup \{t\}$ if $t \in I_0$. For this class of controls, the state-trajectory solution of the system S obtained in Theorem 3 hold with the extra additive terms so that $\tilde{x}(t^+) = \tilde{x}(t^-) + \Phi_0 u(t)$ if $t \in I_0$ and $\tilde{x}(t^+) = \tilde{x}(t^-) = \sum_{t_j \in I_p(t^-)} \mathrm{e}^{A_0\Phi_0(t-t_j)}\Phi_0 u(t_j)$ if $t \notin I_0$ or, written in a compact form, $\tilde{x}(t^\pm) = \sum_{t_j \in I_p(t^\pm)} \mathrm{e}^{A_0\Phi_0(t-t_j)}\Phi_0 u(t_j)$.

4. Positivity properties

The subsequent positivity definitions will be used in the following.

Definition 2. The system S is said to be internally positive (or simply positive) if $x(t) \in \mathbf{R}_+^n$ and $y(t) \in \mathbf{R}_+^p$ for any $t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$. The system S is said to be externally positive iff $y(t) \in \mathbf{R}_+^p$ for any $t \in \mathbf{R}_+$ for any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ provided that $\varphi \equiv 0$ on its definition domain.

Definition 3. The system S is said to be weakly positive if

$$\left(x(t) - \sum_{i=1}^{\mu}\Phi_{-i}\left(Ex_0\delta^{(i-1)}(t) + \sum_{j=1}^{q}A_j x^{(i-1)}(t - h_j) + \sum_{j=1}^{q'}B_j u^{(i-1)}(t - h'_j)\right)\right) \in \mathbf{R}_+^n$$

and

$$\left(C\left[x(t) - \sum_{i=1}^{\mu}\Phi_{-i}\left(Ex_0\delta^{(i-1)}(t) + \sum_{j=1}^{q}A_j x^{(i-1)}(t - h_j) + \sum_{j=1}^{q'}B_j u^{(i-1)}(t - h'_j)\right)\right] + Du(t)\right) \in \mathbf{R}_+^p$$

for any $t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$. The system S is said to be weakly externally positive if

$$\left(C\left[x(t) - \sum_{i=1}^{\mu}\Phi_{-i}\left(\sum_{j=1}^{q}A_j x^{(i-1)}(t - h_j) + \sum_{j=1}^{q'}B_j u^{(i-1)}(t - h'_j)\right)\right] + Du(t)\right) \in \mathbf{R}_+^p$$

for any $t \in \mathbf{R}_+$ for any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ provided that $\varphi \equiv 0$ on its definition domain.

Note that Definitions 3 are equivalent to the system S being positive, respectively, externally positive iff $\Phi_{-i} = 0$ for $i \in \bar{\mu}$; i.e. if μ is zeroed. Note also that weak positivity and weak external positivity are equivalent to positivity and external positivity in the nonsingular case and therefore those concepts are not introduced in the background literature for systems in standard and generalized standard forms. The following result holds.

Theorem 4. *The subsequent properties stand.*

(i) *If the system S is weakly positive (respectively, positive) then it is weakly externally positive (respectively, weakly positive).*

(ii) *The system S is weakly positive iff*

$$x(t) \geqslant \gamma(t) := \sum_{i=1}^{\mu} \Phi_{-i}\left(Ex_0\delta^{(i-1)}(t) + \sum_{j=1}^{q} A_j x^{(i-1)}(t-h_j) + \sum_{j=1}^{q'} B_j u^{(i-1)}(t-h'_j)\right); \quad \forall t \in \mathbf{R}_+$$

and $y(t) \geqslant C\gamma(t) + Du(t)$; $\forall t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h,0],\mathbf{R}_+^n)$ and any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$.

(iii) *The system S is weakly externally positive iff $y(t) \geqslant C\gamma_0(t) + Du(t)$ for any $t \in \mathbf{R}_+$ for any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$ provided that $\varphi \equiv 0$ on its definition domain, where*

$$\gamma_0(t) := \sum_{i=1}^{\mu} \Phi_{-i}\left(\sum_{j=1}^{q} A_j x^{(i-1)}(t-h_j) + \sum_{j=1}^{q'} B_j u^{(i-1)}(t-h'_j)\right).$$

The above condition implies that S is weakly externally positive if $x(t) \geqslant \gamma_0(t)$ for any $t \in \mathbf{R}_+$ for any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$ provided that $\varphi \equiv 0$.

(iv) *If $\mu = 0$; i.e. if the system S is standard or generalized standard then it is weakly positive (respectively, externally weakly positive) iff it is positive (respectively, externally positive).If $\mu = 1$ then the system S is positive if it is weakly positive and $\Phi_{-1} \geqslant 0$. The system S is externally positive if it is weakly externally positive and, furthermore, $C\Phi_{-1}A_j \geqslant 0 (j \in \bar{q})$ and $C\Phi_{-1}B_j \geqslant 0$ $(j \in \bar{q}')$.*

Proof. Property (i) follows from the fact that positivity, respectively, weak positivity imply that $y(t)$, respectively, $C\gamma_0(t) + Du(t)$, are nonnegative for all $t \in \mathbf{R}_+$ for any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$ provided that $\varphi \equiv 0$ (Definitions 2 and 3). Properties (ii) and (iii) follows directly from Definitions 2 and 3.

(iv) The case $\mu = 0$ follows directly from the equivalences in-between positivity/external positivity and their weak counterparts for the standard/generalized standard cases.

For the case $\mu = 1$, note that:

(a) $\gamma(t) \geqslant 0 \Rightarrow x(t) \geqslant 0$ for any $t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h,0],\mathbf{R}_+^n)$ and any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$ if $\Phi_{-1} \geqslant 0$.

(b) Furthermore, if the system is either weakly positive (or positive in the nonsingular case) then $C \geqslant 0$, since, otherwise, it would be possible to find some admissible initial conditions and some controls such that some component of the output is negative at some time even if the state is a nonnegative vector (see, for instance, [23,25]). Then $C\Phi_{-1} \geqslant 0$ if $\Phi_{-1} \geqslant 0$ and then

$$C\Phi_{-1} \geqslant 0 \wedge (C\gamma_0(t) + Du(t)) \in \mathbf{R}_+^p, \quad \forall t \in \mathbf{R}_+ \Rightarrow y(t) \in \mathbf{R}_+^p, \ \forall t \in \mathbf{R}_+$$

for any given $\varphi \in IC([-h,0],\mathbf{R}^n)$ and any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$.

As a result of (a) and (b), if S is weakly positive and $\Phi_{-1} \geqslant 0$ then S is positive.

On the other hand, $C\Phi_{-1}A_j \geqslant 0$ $(j \in \bar{q})$ and $C\Phi_{-1}B_j \geqslant 0$ $(j \in \bar{q}')$ imply $C\gamma_0(t) \in \mathbf{R}_+^p$, $\forall t \in \mathbf{R}_+$ any given $u \in PC(\mathbf{R}_+,\mathbf{R}_+^m)$ and zero initial conditions. Also, weakly external positivity (and also positivity in the nonsingular case) requires $D \geqslant 0$ since otherwise it would be possible to find nonnegative inputs implying negativity at certain times of some output components. As a result, S being weakly externally positive and the conditions $C\Phi_{-1}A_j \geqslant 0$ $(j \in \bar{q})$ and $C\Phi_{-1}B_j \geqslant 0$ $(j \in \bar{q}')$ imply that S is externally positive. □

It is well-known the fact that the C_0-semigroup generated by an infinitesimal generator $A \in \mathbf{R}^{n\times n}$, $e^{At} \in \mathbf{R}_+^{n\times n}$ iff $A \in M_E^{n\times n}$, [23]. This property is used in the subsequent result.

Theorem 5. *The following properties hold*:

(i) *Assume that* $\operatorname{rank}[E, A_0] = n$ *or*, *equivalently*, *the system S is regular as* $h_i \to \infty$ $(i \in \bar{q})$. *Then the system S is weakly positive independent of the delays*, *and then also weakly externally positive independent of the delays*, *if the sets of constraints below hold*:

(C.1) $\Phi_0 A_0 \in M_E^{n\times n}$, $\Phi_0 E \in \mathbf{R}_+^{n\times n}$, $\Phi_0 B_j \in \mathbf{R}_+^{n\times n}$ $(j \in \bar{q}' \cup \{0\})$, $C \in \mathbf{R}_+^{p\times n}$, $D \in \mathbf{R}_+^{p\times m}$.
(C.2) $\Phi_0 A_j \in \mathbf{R}_+^{n\times n} (j \in \bar{q})$.

The constraints (*C*.1) *are also all necessary. The constraint* (*C*.2) *is not necessary* (*except for* $q = 1$), *a necessary condition being* $\sum_{j=1}^{q} \Phi_0 A_j \in \mathbf{R}_+^{n\times n}$.

(ii) *Assume that* $\operatorname{rank}[E, \overline{A}_d] = n$. *Then*, *Property* (*i*) *applies "mutatis-mutandis" for any set of finite delays* $h_i \geqslant 0$ $(i \in \bar{q})$.

(iii) *If* $\operatorname{rank}[E, A_0] = n$ *then the system S is externally positive independent of the delays if*

$$Ce^{\Phi_0 A_0 t}\Phi_0 A_j \in \mathbf{R}_+^{n\times n}(j \in \bar{q}); \quad Ce^{\Phi_0 A_0 t}\Phi_0 B_j \in \mathbf{R}_+^{n\times n}(j \in \bar{q}' \cup \{0\}) \quad \forall t \in \mathbf{R}_+$$
$$CD \in \mathbf{R}_+^{n\times n}.$$

(iv) *If* $\operatorname{rank}[E, \overline{A}_d] = n$ *then Property* (*iii*) *applies "mutatis-mutandis" for any set of finite delays* $h_i \geqslant 0$ $(i \in \bar{q})$.

Sketch of proof

(i) If $\operatorname{rank}[E, A_0] = n \Rightarrow \operatorname{rank}[E, \overline{A}_d] = \operatorname{rank}[E, A_0] = n$ then the system S is regular independent of the delays (i.e. for all finite delays and as delays tend to infinity) from Theorems 1(i) and 2(i); i.e. Assumption 1 holds. Thus, if $\Phi_{-i} = 0$ for $i \in \bar{\mu}$ then $x(t) \in \mathbf{R}_+^n$, $y(t) \in \mathbf{R}_+^p$, $\forall t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and for any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ from (22) provided that the Constraints C.1 and C.2 hold since $x(t'^-) \in \mathbf{R}_+^n \Rightarrow x(t^+) \in \mathbf{R}_+^n$ for any $t \in \mathbf{R}_+$. Then, the system S is weakly internally positive. The last part follows that if the initial conditions are zero, since $\varphi(\equiv 0) \in IC([-h, 0], \mathbf{R}_+^n)$, $x(t) \in \mathbf{R}_+^n$, $\forall t \in \mathbf{R}_+$ so that the system S is weakly externally positive. All the sufficiency proofs of Properties (ii)–(iv) follow from direct inspection of (22) which guarantees, provided that $\Phi_{-i} = 0$ for $i \in \bar{\mu}$, that:

$x(t) \in \mathbf{R}_+^n$ and $y(t) \in \mathbf{R}_+^p$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and for any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ implying weak positivity.
$y(t) \in \mathbf{R}_+^p$ for identically zero $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and for any given $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ implying weak external positivity.

All the necessity parts might be proven by contradiction as, for instance,

Necessity part of (ii)

Assume that $\Phi_{-i} = 0$, $\forall i \in \bar{\mu}$, and $(\Phi_0 A_1)_{11} < 0$ (i.e. Constraint C.2 fails for $q = 1$ but the remaining Constraint C.2; i.e. $(\Phi_0 A_i)_{jk} \geqslant 0$ except for $i = j = k = 1$ and Constraint C.1 hold) with $u \equiv 0$, $\varphi_1(t) = K_1 > 0$ for $t \in [-h_1, h_2)$, $\varphi_1(t) = 0$ for $t \in [-h, 0)/[-h_1, h_2)$, $\varphi_i(t) = 0$ for $t \in [-h, 0)$, $\forall i \in \bar{q}/\{1\}$. Since $e^{A_0 \Phi_0 t} > 0$, since $A_0 \Phi_0 \in M_E^{n\times n}$, it follows that $x_1(0^+) < 0$ and then the system is not weakly internally positive. A close proof is valid for weak external positivity and also quite close proofs might be developed to prove the remaining necessary conditions in (ii)–(iv) by considering each isolated failure of one of the parts of the two constraints and a contradiction procedure.

Note that the distribution δ is a linear functional from a space (commonly a Schwartz space) or that of all smooth functions of compact support so that the high-order Dirac distribution fulfill $\delta^{(i)}(t) = (-1)^i i! x^{-i}\delta(t)$ and $x^{(i)}(t - h_j)$ always exist everywhere in $[h_j, \infty)$. It also exists in $[-h, 0) \cup \mathbf{R}_+$ from (22) if $\varphi \in C^{(i)}([-h, 0), \mathbf{R}_+^n)$. The following result follows from Theorem 4(iv) and Theorem 5. Note, in particular, that for $\mu = 0$, $\Phi_{-i} \equiv 0$, $\forall i \in \bar{\mu}$ and that, for $\mu = 1$, $\Phi_{-i} \equiv 0$, $\forall i \in \bar{\mu}/\{1\}$ while $\delta^{(i-1)}(t) \equiv \delta^{(0)}(t)$ and $u^{(i-1)}(t) \equiv u^{(0)}(t)$ are removed from (22).

Theorem 6. *Assume that $\mu = 0$ and Constraint C.1 of Theorem 5 is modified by replacing $\Phi_0 E \geqslant 0$ with $E > 0$ if the system S is generalized standard and deleting $\Phi_0 E \geqslant 0$ if the system S is standard (since $E = \Phi_0 = I_n$). Then, Theorem 5 holds under the strongest form of S being positive, respectively externally positive. If $\mu = 1$ then Theorem 5 holds under the strongest form of S being positive, respectively externally positive.*

Theorem 7. *Assume that $\mu \geqslant 2$. Thus, Theorem 5 holds by replacing S being weakly positive with S being positive and S being weakly externally positive with S being externally positive with the constraints below:*

(C.3) $\Phi_{-1} \in \mathbf{R}_+^{n\times n}$; $(E_\ell) \in \mathrm{Ker}(\Phi_{-i})$ for $2 \leqslant i \leqslant \mu$, $\ell \in \bar{n}$, $j \in \bar{q}$; $(B_j) \in\in \mathrm{Ker}(\Phi_{-i})$ for $2 \leqslant i \leqslant \mu$, $\ell \in \bar{m}$, $j \in \bar{q}'$

being added to Constraints C.1 and C.2 of Theorem 5 (i).

Proof. It is direct since all the impulsive terms in the right-hand-side of (22) are zeroed and the remaining ones are nonnegative. □

Note that Theorem 7 is not extendable to any parameterizations of S since the high-order Dirac distributions have alternate signs as their order increase and the successive time-derivatives of the state-trajectory and input might have positive or negative signs if nonzero either in the classical sense or in the distributional one. The positiveness of the matrices $\Phi_{-i};\ i \in \bar{\mu} \cup \{0\}$ can be tested from the solvable linear algebraic system (27) and (28) (see Theorem 2). In particular, if $\ell = 1$, the resulting algebraic system is

$$[E_\alpha, -A_{0\alpha}, E_\beta, -A_{0\beta}]\left[\Phi_{0\alpha}^{\mathrm{T}}, \Phi_{-1\alpha}^{\mathrm{T}}, \Phi_{0\beta}^{\mathrm{T}}, \Phi_{-1\beta}^{\mathrm{T}}\right]^{\mathrm{T}} = I_n, \tag{32}$$

where $[E_\alpha, -A_{0\alpha}] \in \mathbf{R}^{n\times n}$ and is formed by r columns of E (since $\mathrm{rank}[E] = r$) and $(n-r)$ of A_0. Such a matrix $[E_\alpha, -A_{0\alpha}]$ being nonsingular exists since $\mathrm{rank}[E, A_0] = n$ from Theorem 1 since the system S is regular. Direct calculation yields a unique solution for $\Phi_{0\alpha}$, $\Phi_{-1\alpha}$ given by

$$[\Phi_{0\alpha}^{\mathrm{T}}, \Phi_{-1\alpha}^{\mathrm{T}}]^{\mathrm{T}} = [E_\alpha, -A_{0\alpha}]^{-1}[I_n + A_{0\beta}\Phi_{-1\beta} - E_\beta\Phi_{0\beta}] \tag{33}$$

An infinite set of complete solutions $(\Phi_{0\alpha}, \Phi_{-1\alpha}, \Phi_{0\beta}, \Phi_{-1\beta})$ may be obtained nonnegative by appropriately prefixing $\Phi_{0\beta} \geqslant 0$ and then calculating the triple $(\Phi_{0\alpha}, \Phi_{-1\alpha}, \Phi_{-1\beta})$. Another set of complete solutions may be obtained by prefixing $\Phi_{-1\beta} \geqslant 0$ and then calculating the triple $(\Phi_{0\alpha}, \Phi_{-1\alpha}, \Phi_{0\beta})$. If $[E_\alpha, -A_{0\alpha}] \in M_M^{n\times n} \iff [E_\alpha, -A_{0\alpha}]^{-1} \in \mathbf{R}_+^{n\times n}$, [23], then it suffices to fix $\Phi_{0\beta}$ and $\Phi_{1\beta}$ so that $[I_n + A_{0\beta}\Phi_{-1\beta} - E_\beta\Phi_{0\beta}] \in \mathbf{R}_+^{n\times n}$ to guarantee that $[\Phi_{0\alpha}^{\mathrm{T}}, \Phi_{-1\alpha}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{R}_+^{n\times n}$. Necessary conditions for weak positivity and positivity independent of the delays as well as necessary and sufficient conditions for weak external positivity of the regular system S are discussed in the subsequent two results.

Proposition 1. *The system S may be weakly positive and positive only if $\left(\sum_{j=0}^{q} A_j\right) \in M_E^{n\times n}$ (what holds if $A_0 \in M_E^{n\times n}$ and $\left(\sum_{j=1}^{q} A_j\right) \in \mathbf{R}_+^{n\times n}$) and only if $\left(\sum_{j=i}^{q} A_j\right) \in \mathbf{R}_+^{n\times n}$, $\forall i \in \bar{q} \cup \{0\}$.*

Proof. First note that $A_0 \in M_E^{n\times n} \wedge \left(\sum_{j=1}^{q} A_j\right) \in \mathbf{R}_+^{n\times n}$ implies that any off-diagonal entry of $\left(\sum_{j=0}^{q} A_j\right)$ is nonnegative so that $\left(\sum_{j=0}^{q} A_j\right) \in M_E^{n\times n}$. Now, proceed by contradiction by assuming that $\left(\sum_{j=0}^{q} A_j\right) \notin M_E^{n\times n}$. Then, the system S is not weakly positive nor positive for $h_i = 0$ $(i \in q)$ so that it is not weakly positive or positive independent of the delays. The first necessary condition has been proved. Now, assume that $\left(\sum_{j=i}^{q} A_j\right) \notin \mathbf{R}_+^{n\times n}$ for some $i \in \bar{q} \cup \{0\}$. Then, if $h_j = \hat{h} > 0$, for some $j \in [i, q]$ and $h_j < \hat{h}$, $\forall j \in \overline{i-1}$. Since $\left(\sum_{j=i}^{q} A_j\right) \notin \mathbf{R}_+^{n\times n}$, it exists a negative (k,ℓ)-entry of $\left(\sum_{j=i}^{q} A_j\right)$. Choose $\varphi_{e\ell}(t) = -K < 0$ for $t \in [-\hat{h}, 0]$, $\varphi_{ej}(t) = 0$, $\forall j(\neq \ell) \in \bar{n}$, $\forall t \in [-\hat{h}, 0]$ and $u \equiv 0$ on $\mathbf{R}_+$. Since $\left(\sum_{j=i}^{q} A_j\right)_{k\ell} < 0$, it follows that $x_k(0^+) < 0$ so that the system S is not weakly positive/positive independent of the delays. The second necessary condition has been proved. □

Proposition 2. *The system S is weakly externally positive for any given set of internal delays $h_i \geqslant 0$ $(i \in \bar{q})$ and any given set of external delays $h'_i \geqslant 0$ $(i \in \bar{q}')$ iff the impulse response matrix $H(t) := \mathrm{Lap}^{-1}[\widehat{H}(s)] = \mathrm{Lap}^{-1}[\hat{y}(s)]_{\varphi_e \equiv 0, u(t)=\boldsymbol{\delta}(t)} \in \mathbf{R}_+^{p\times m}$, $\forall t \in \mathbf{R}_+$ under zero initial conditions, where $\boldsymbol{\delta}(t) = (\delta_i(t)) \in \mathbf{R}^m$ (i.e. all its components are Dirac impulses at $t = 0$)*

$$\widehat{H}(s) = (\widehat{H}_{ij}(s)) := \left[\frac{\hat{y}_i(s)}{\hat{u}_j(s)}\right]_{\varphi\equiv 0} = \left[C\left(sE - \sum_{j=0}^{q} A_j e^{-h_j s}\right)^{-1}\left(\sum_{j=0}^{q'} B_j e^{-h'_j s}\right) + D\right]$$
$$= \left[C\left(I_n - (sE - A_0)^{-1}\sum_{j=1}^{q} A_j e^{-h_j s}\right)^{-1}(sE - A_0)^{-1}\left(\sum_{j=0}^{q'} B_j e^{-h'_j s}\right) + D\right]$$

A necessary and sufficient condition for the system S to be weakly externally positive under infinity delays is that $(C(sE - A_0)^{-1}B_0 + D) \in \mathbf{R}_+^{p\times m}$. A necessary and sufficient condition for the system S to be weakly externally positive under zero delays is that $\left(C\left(sE - \sum_{j=0}^{q} A_j\right)^{-1}\left(\sum_{j=0}^{q'} B_j\right) + D\right) \in \mathbf{R}_+^{p\times m}$. Both conditions are necessary for the system S to be weakly externally positive independent of the delays.

The above result is a direct consequence of the property of external positivity of standard and generalized standard systems formulated in terms of their impulse responses since external positivity of a generalized standard system translates directly into weak external positivity of any singular regular system possessing the same parameterization, the matrix E excepted which is singular in this case.

5. Positivity properties of a special state-space realization

In this section, the subsequent Assumption is introduced on the system S:

Assumption 2. $\mathrm{rank}(E) = \deg(\mathrm{Det}(sE - A_0)) = r$.

Note from (22) the subsequent immediate result.

Proposition 3. *If $r \leqslant n$ and Assumptions 1 and 2 hold, then the system S is singular regular and its state-trajectory solution is unique and impulse-free for any given $\varphi \in IC([-h, 0], \mathbf{R}^n)$, $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$, $h_i \in [0, \infty)$, $h'_j \in [0, \infty)$, $\forall i \in \bar{q}$, $\forall j \in \bar{q}'$ with $h_0 = h'_0 = 0$.*

Note that Proposition 3 also holds trivially for $r = n$ if E is nonsingular (i.e. for the system S being in standard or generalized standard forms). The absence of impulses in the solution motivates that if Assumption 2 holds then the pair (E, A_0) is said to be impulse-free. It may be proved (see, for instance, [12,26]) that, under Assumption 2, an algebraic equivalent state-space form of the system S is the so-called canonical Weierstrass realization S_W defined by

$$\begin{aligned} &S_W: \\ &\dot{z}_1(t) = R_0 z_1(t) + \sum_{j=1}^{q}(R_{j11}z_1(t-h_j) + R_{j12}z_2(t-h_j)) + \sum_{j=0}^{q'}(P_{j1}u(t-h'_j)), \\ &N\dot{z}_2(t) = z_2(t) + \sum_{j=1}^{q}(R_{j21}z_1(t-h_j) + R_{j22}z_2(t-h_j)) + \sum_{j=0}^{q'}(P_{j2}u(t-h'_j)), \\ &y(t) = \chi z(t) + Du(t), \end{aligned} \tag{34}$$

with $z_1(t) \in \mathbf{R}^r$, $z_2(t) \in \mathbf{R}^{n-r}$, $N \in \mathbf{R}^{r\times r}$ is nilpotent with $r = \mathrm{nind}(N) = \mathrm{nind}(E_1) = \mathrm{ind}(E) = \mu$; and $G, H \in \mathbf{R}^{n\times n}$ are nonsingular matrices, which exist under Assumption 2, such that

$$GEH = \mathrm{BlockDiag}(I_r, N),$$
$$R_j = GA_jH = \mathrm{Block\ Matrix}(R_{j\ell k}; \ell, k = 1, 2) := \begin{bmatrix} R_{j11} & R_{j12} \\ R_{j21} & R_{j22} \end{bmatrix}; \quad \forall j \in \bar{q} \cup \{0\},$$

$$R_{011} = R_0 \in \mathbf{R}^{r\times r}; \quad R_{022} = I_{n-r}; \quad R_{012} = O_{r\times(n-r)}; \quad R_{021} = O_{(n-r)\times r},$$
$$P_i = (P_{i1}^{\mathrm{T}}, P_{i2}^{\mathrm{T}})^{\mathrm{T}} = GB_i; \quad \forall i \in \bar{q}' \cup \{0\}; \quad \chi = CH = C[H_1, H_2]. \tag{35}$$

It follows from (35) that:

$$R_d := G\overline{A}_{d1}\overline{H} = G(A_1, A_2, \ldots, A_d)\mathrm{BlockDiag}(H, H, \ldots, H)$$

Thus, the state transformation $H : S_X \to S_{WZ}$ defined by the matrix H from the state space S_X into the state space S_{WZ} associated with (1), (2) and (34), respectively, so that $z(t) = (z_1^{\mathrm{T}}(t), z_2^{\mathrm{T}}(t))^{\mathrm{T}} = H^{-1}x(t)$. The state-trajectory solution and output are

$$z_1(t) = \mathrm{e}^{R_0 t}\left\{z_1(0) + \sum_{j=1}^{q}\int_0^t \mathrm{e}^{-R_0\tau}\left[(R_{j11}z_1(\tau - h_j) + R_{j12}z_2(\tau - h_j)) + \sum_{j=0}^{q'} P_{j1}u(\tau - h'_j)\right]\mathrm{d}\tau\right\}$$
$$z_2(t) = -\sum_{i=0}^{\mu-1} N^i\left[\sum_{j=1}^{q}\left[(R_{j21}z_1^{[i]}(t - h_j) + R_{j22}z_2^{[i]}(t - h_j)) + \sum_{j=0}^{q'} P_{j2}u^{(i)}(t - h'_j)\right]\right] \tag{36}$$
$$y(t) = C(H_1 z_1(t) + H_2 z_2(t)) + Du(t)$$

for any given $\varphi \in IC([-h, 0], \mathbf{R}^n)$, $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$, $h_i \in [0, \infty)$, $h'_j \in [0, \infty)$, $\forall i \in \bar{q}$, $\forall j \in \bar{q}'$ with $h_0 = h'_0 = 0$. For $\mu = 0$ (the standard/generalized standard forms), $z(t) = z_1(t)$. For $\mu = 1$, $z(t) = z_1(t)$ is impulse-free and then the canonical state-space realization and then any other realization being algebraically equivalent to it is also impulse-free. For $\mu > 1$, $z(t)$ is still impulse-free for $u \in C^{(\mu-1)}(\mathbf{R}_+, \mathbf{R}_+^m)$ since Assumption 2 holds. However, for general controls $u \in PC(\mathbf{R}_+, \mathbf{R}^m)$, response impulses can occur trough the last right-hand-side term of $z_2(t)$ at times where some $j(\in \overline{\mu - 1})$th time-derivative of the control does not exist. The following result concerning weak positivity and positivity of the Weierstrass canonical form holds:

Theorem 8. *The subsequent properties hold*:

(i) *Suppose that Assumptions 1 and 2 hold. Then, S_W is weakly positive independent of the delays iff $R_0 \in M_E^{(n-r)\times(n-r)}$, $R_{j11} \in \mathbf{R}_+^{(n-r)\times(n-r)}$, $R_{j12} \in \mathbf{R}_+^{(n-r)\times r}(j \in \bar{q})$, $P_{k1} \in \mathbf{R}_+^{(n-r)\times m}(k \in \bar{q}' \cup \{0\})$ and $CH_1 \in R_+^{p\times(n-r)}$.*
(ii) *Suppose that Assumptions 1 and 2 hold and that either $\mu = 0, 1$ or NR_{j21}, $NR_{j22}(j \in \bar{q})$ and NP_{j2} $(j \in \bar{q}' \cup \{0\})$ are zero if $\mu \geqslant 2$. Then, S_W is both weakly positive independent of the delays and positive independent of the delays iff $R_0 \in M_E^{(n-r)\times(n-r)}$, $R_{j11} \in \mathbf{R}_+^{(n-r)\times(n-r)}$, $R_{j12} \in \mathbf{R}_+^{(n-r)\times r}(j \in \bar{q})$, $P_{k1} \in \mathbf{R}_+^{(n-r)\times m}(k \in \bar{q}' \cup \{0\})$ and $CH_1 \in R_+^{p\times(n-r)}$.*

Proof. (i) [If Part]: If $N = 0$, then $z_2 \equiv 0$ for $t \in \mathbf{R}_+$ for any $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ then S_W is weakly positive independent of the delays since $z(t) \in \mathbf{R}_+^n$, $y(t) \in \mathbf{R}_+^p$ for all $t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$. (ii) [If Part]: If N is nonzero but $\mu = 0, 1$ or NR_{j21}, $NR_{j22}(j \in \bar{q})$ and NP_{j2} $(j \in \bar{q}' \cup \{0\})$ are zero if $\mu \geqslant 2$ then S_W is positive since $z(t) \in \mathbf{R}_+^n$ (with $z_2 \equiv 0$), $y(t) \in \mathbf{R}_+^p$ for all $t \in \mathbf{R}_+$ for any given $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ and $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$. The "*Only If Parts*" of (i) and (ii) follow with the same contradiction techniques as those used in Theorems 6 and 7 since by deleting any of the given conditions, either function of initial conditions $\varphi \in IC([-h, 0], \mathbf{R}_+^n)$ or control $u \in PC(\mathbf{R}_+, \mathbf{R}_+^m)$ may be found such that at least one of the components of the state/output is negative at some time. □

Parallel results for weak external positivity and external positivity follow directly by discussing the impulse response matrix.

6. Examples

Example 1. It is taken into account through the following example the fact that the discrete system $x_{k+1} = Ax_k + Bu_k$ is positive iff $A \geqslant 0$ (what replaces the continuous-time counterpart of A being a Metzler matrix) and $B \geqslant 0$. The so-called Leontieff input–output models (following Wassily Leontieff – Nobel prize in

1973) are useful in Economy (see, for instance, [30]). A typical time-invariant delay-free (except for the one-step discrete delay inherent to discretization) discrete Leontieff model is

$$Ex_{k+1} = (I_n - A + E)x_k - Bu_k, \tag{37}$$

where f_k denotes $f(kT)$ for the kth sample at time $t = kT$ with sampling period T, and $E \in \mathbf{R}^{n\times n}$ is the square matrix of fixed capital coefficients.

$A \in \mathbf{R}^{n\times n}$ is the material input–output matrix (inclusive of wear and tear of fixed capital goods or rather depreciation).

$(A - E) \in \mathbf{R}^{n\times n}$ is the consumption matrix (which coincides with A if $E = 0$). A consumption matrix C is called productive if $\exists(I_n - C)^{-1} > 0$.

u_k is the mth demand vector (or vector of final deliveries) excluding fixed capital investment at time $t = kT$.

x_k is the nth production vector at time $t = kT$.

The above model is endogenous if the control matrix B is zero and exogenous, otherwise.

Note that the discrete Leontieff model has a continuous-time counterpart. For instance, for a sampling period normalized to unity, the continuous-time counterpart of the discrete model is

$$E\dot{x}(t) = (I_n - A)x(t) - Bu(t). \tag{38}$$

Several particular cases being of interest can occur, namely

Case 1: $E = 0$ and B is a nth vector with all components being unity and the consumption matrix is productive. Then, $\sum_{i=1}^{n} A_{ij} < 1,\ \forall j \in \bar{n} \Rightarrow \exists(I_n - A)^{-1} > 0$. Thus, the continuous-time and discrete-time dynamic models (37) and (38) have the same structures $x_k = (I_n - A)^{-1}u_k$, respectively, $x(t) = (I_n - A)^{-1}u(t)$ and are both positive so that the production vector is always positive for any positive control input. The result still holds for any $B \geqslant 0$.

Case 2: E is nonsingular (standard form if $E = I_n$ and generalized standard form otherwise). The dynamic discrete Leontieff model (37) is

$$x_{k+1} = (E^{-1}(I_n - A) + I_n)x_k - E^{-1}Bu_k$$

The system is positive iff $(E^{-1}(I_n - A) + I_n) \geqslant 0$ and $E^{-1}B \leqslant 0$. The static model becomes $x = (I_n - A)^{-1}Bu$ provided that $\exists(I_n - A)^{-1}$ for constant $x_k = x$ and any given $u_k = u$. Note that $(I_n - A)^{-1} > 0$ if $\sum_{i=1}^{n} A_{ij} < 1,\ \forall j \in \bar{n}$ and iff $(I_n - A) \in M_M^{n\times n}$ and nonsingular. Then, the static discrete model in generalized standard is positive if $\sum_{i=1}^{n} A_{ij} < 1,\ \forall j \in \bar{n}$ and $B \geqslant 0$ or, if $(I_n - A)^{-1}B \geqslant 0$, what is irrespective of E (provided it is nonsingular). However, note that positivity for the dynamic model requires constraints on E. The dynamic continuous-time Leontieff model (38) becomes when E is nonsingular

$$\dot{x}(t) = (E^{-1}(I_n - A) + I_n)x(t) - E^{-1}Bu(t),$$

which is positive iff $(E^{-1}(I_n - A) + I_n) \in M_E^{n\times n}$ and $E^{-1}B \leqslant 0$.

Case 3: (backward-in-time discrete model): Eq. (37) becomes

$$x_k = (I_n - A + E)^{-1}(Ex_{k+1} + Bu_k)$$

provided that $(I_n - A + E)^{-1}$ exists. The backward-in-time discrete model is positive iff $(I_n - A + E)^{-1}E \geqslant 0$ and $(I_n - A + E)^{-1}B \geqslant 0$ what is guaranteed if $(I_n - A + E) \in M_M^{n\times n}$ and nonsingular, $E \geqslant 0$ and $B \geqslant 0$.

Case 4: The matrix E is singular but the system is regular. The continuous-time model Eq. (38) is weakly positive iff $\Phi_0 E \geqslant 0$, $(I_n - A + E)\Phi_0 \in M_E^{n\times n}$ where Φ_{-i} satisfies $E\Phi_{-i} - (I_n - A + E)\Phi_{-i-1} = \delta(i,0)I_n$, $\forall i \in \{0\} \cup \overline{i-1}$. If point discrete internal and external delays are added in the right-hand-side of (38) then the system is still weakly positive independent of the delays iff the above constraints hold for the replacement $A \to A_0$ and, in addition, $A_i \geqslant 0$, $B_j \geqslant 0$, $\forall i \in \bar{q}$, $\forall j \in \bar{q}'$. Furthermore, if $\mu = 1$, the system is positive independent of the delays iff $\Phi_0 E \geqslant 0$, $(I_n - A_0 + E)\Phi_0 \in M_E^{n\times n}$, $\Phi_{-1}E \geqslant 0$, $\Phi_{-1}A_j \geqslant 0$ $(\forall j \in \bar{q})$, $\Phi_{-1}B_k \geqslant 0$ $(\forall k \in \bar{q}')$.

Example 2. Consider the singular regular third-order continuous-time model where E is given by $E = T^{-1}\text{BlockDiag}(E_0, E_1)T$ where T is a nth square nonsingular real matrix, and E_0 is r-square real nonsingular and E_1 is nilpotent of nilpotence index μ which is the index of E. Thus, $E^{(D)} = T^{-1}\text{BlockDiag}(E_0^{-1}, 0)T$ is the Drazin inverse of E. Assume that $n = 3$, $E_0 = \alpha \neq 0$ and $E_1 = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ with $\beta \neq 0$, $r = 1$ and $\mu = 2$ since $E_1^2 = 0$. Now, assume that T is subject to the constraint $T = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with the entries a, b, c and d satisfying the constraint $\text{Det}(T) = ad - bc \neq 0$. Thus, the matrix E is parameterized as follows:

$$E(a,b,c,d,\alpha,\beta) = \frac{1}{ad - bc}\begin{bmatrix} \alpha ad & \alpha bd & -\beta d \\ -\alpha ac & -\alpha bc & \beta a \\ 0 & 0 & 0 \end{bmatrix}$$

Case 1: Assume $ad \neq 0$ and let k and $\lambda_{1,2}$ be real constants such that $b = -\lambda_1 a$, $c = -\lambda_2 d = -k\lambda_1 d$. Thus, E is re-parameterized as follows:

$$E(\lambda_1, \lambda_2, k, a, \alpha, \beta) = \begin{bmatrix} \frac{\alpha}{1+k\lambda_1^2} & -\frac{\alpha\lambda_1 a}{1+k\lambda_1^2} & \frac{\lambda_1\beta}{1+k\lambda_1^2} \\ \frac{\alpha\lambda_2}{1+k\lambda_1^2} & \frac{k\alpha\lambda_1^2}{1+k\lambda_1^2} & \frac{\beta}{1+k\lambda_1^2} \\ 0 & 0 & 0 \end{bmatrix}$$

Since $\text{Det}(T) = ad - bc = (1 - \lambda_1\lambda_2)ad \neq 0$, it is required $\lambda_1\lambda_2 \neq 1$. Thus, the system is weakly positive independent of the delays iff the subsequent set of constraints holds: $k \neq \frac{1}{\lambda_1^2}$ (to guarantee $\lambda_1\lambda_2 \neq 1$), $\alpha \neq 0$, $ad \neq 0$. For the set of real n-matrices Φ_{-i} satisfying (21) for all $i \in \bar{\mu} \cup \{0\}$, $\Phi_0 E \geqslant 0$, $A_0\Phi_0 \in M_E^{n\times n}$, $A_j \geqslant 0$ $(\forall j \in \bar{q})$, $B_k \geqslant 0$ $(\forall k \in \bar{q}')$. If, in particular, $\beta = 0$ then $\mu = 1$ and the system is, furthermore, positive iff the above constraints hold (i.e., it is weakly positive) and, furthermore, $\Phi_{-1}E \geqslant 0$, $\Phi_{-1}A_j \geqslant 0$ $(\forall j \in \bar{q})$, $\Phi_{-1}B_k \geqslant 0$ $(\forall k \in \bar{q}')$. If E is a generic real square nonsingular n-matrix, then the system is either in standard or in generalized standard forms. It is both weakly positive and positive independent of the delays iff $E^{-1}A_0 \in M_E^{n\times n}$, $E^{-1}A_j \geqslant 0$ $(\forall j \in \bar{q})$, $E^{-1}B_k \geqslant 0$ $(\forall k \in \bar{q}')$.

Case 2: Assume $ad = 0$ so that $\text{Det}(T) \neq 0 \Rightarrow b \neq 0$ and $c \neq 0$. The generic form of the singular matrix E for the given transformation matrix T and E_0 (nonsingular) and E_1 (nilpotent) as in Case 1 is $E(a,b,c,d,\alpha,\beta) = \begin{bmatrix} 0 & -\frac{\alpha d}{c} & \frac{\beta d}{c} \\ \frac{\alpha a}{b} & \alpha & -\frac{\beta a}{bc} \\ 0 & 0 & 0 \end{bmatrix}$. Then, the system is weakly positive iff Φ_{-i} satisfying (21) for all $i \in \bar{\mu} \cup \{0\}$, $\Phi_0 E \geqslant 0$, $A_0\Phi_0 \in M_E^{n\times n}$, $A_j \geqslant 0$ $(\forall j \in \bar{q})$, $B_k \geqslant 0$ $(\forall k \in \bar{q}')$.

Note that in the single-input single-output case, positive realness of a transfer function of a linear time-invariant case implies the external positivity of the associated dynamic system while the converse is not true in general since, in particular, external positivity does not imply stability (which always holds under positive realness) [6,31,32]. Note also that the various concepts of positivity would be potentially extendable to both hybrid, which possess coupled digital and continuous-time substates, and compartmental models . They are also potentially extendable to the synthesis of a wide class of stabilizing controllers which be able to guarantee simultaneous stability (internal stability) and external positivity (positivity) closed-loop positivity [33–38] including the case of hybrid dynamic systems which include continuous-time and digital coupled subsystems [39,40]. These extensions are of a significant potential interest in Control Theory.

Acknowledgement

The author is very grateful to MEC by its partial support through grant DPI2006/00714.

References

[1] M. De la Sen, S. Alonso, Adaptive control of time-invariant systems with discrete delays subject to multiestimation, Discrete Dynamics in Nature and Society, Art. No. 41973, doi:10.1155/DDNS/2006/41973.
[2] M. De la Sen, Stabilization criteria for continuous-time linear time-invariant systems with constant lags, Discrete Dynamics in Nature and Society, Art. No. 87062, doi:10.1155/DDNS/2006/87062.
[3] M. de la Sen, Stability of impulsive time-varying systems and compactness of the operators mapping the input space into the state and output spaces, Journal of Mathematical Analysis and Applications 321 (2) (2006) 621–650.
[4] M. de la Sen, Absolute stability of feedback systems independent of the internal point delays, IEE Proceedings – Control Theory and Applications 152 (5) (2005) 567–574.
[5] V.M. Marchenko, O.N. Poddubnaya, Z. Zackiewicz, On the observability of linear differential-algebraic systems with delays, IEEE Transactions on Automatic Control 51 (8) (2006) 1387–1392.
[6] M. de la Sen, On positivity and stability of a class of time-delay systems, Nonlinear Analysis-Theory Methods & Applications, doi:10.1016/j.norwa.2006.03.002, in press.
[7] T. Kaczorek, Realization problem for positive linear systems with time delay, Mathematical Problems in Engineering 2005 (4) (2005) 455–463, doi:10.1155/MPE.2005.455.
[8] J.R. Yan, A.M. Zhao, Q.X. Zhang, Oscillation properties of nonlinear impulsive delay differential equations and applications to population models, Journal of Mathematical Analysis and Applications 322 (1) (2006) 359–370.
[9] X.Y. Liao, H.Y. Zhu, F.D. Chen, On asymptotic stability of delay-difference systems, Applied Mathematics and Computation 176 (2) (2006) 775–784.
[10] J.A.D. Appleby, I. Gyori, D.W. Reynolds, On exact rates of decay of solutions of linear systems of Volterra equations with delay, Journal of Mathematical Analysis and Applications 320 (1) (2006) 56–77.
[11] B.X. Dai, N. Zhang, Stability and global attractivity for a class of nonlinear delay difference equations, Discrete Dynamics in Nature and Society 2005 (3) (2005) 227–234.
[12] W. Jiang, W.Z. Shong, Controllability of singular systems with control delay, Automatica 37 (11) (2001) 1873–1877.
[13] J.P. Richard, Time-delay systems: an overview of some recent advances and open problems, Automatica 39 (10) (2003) 1667–1694.
[14] L.H. Ding, W.X. Li, Stability and bifurcations of numerical discretization Nicholson blowflies equation with delay, Discrete Dynamics in Nature and Society, Art. No. 19413, doi:10.1155/DDNS/2006/19413.
[15] Z.D. Yang, Existence of explosive positive solutions of quasilinear elliptic equations, Applied Mathematics and Computation 177 (2) (2006) 581–588.
[16] H.H. Yin, Z.D. Yang, Some new results on the existence of bounded positive entire solutions for quasilinear elliptic equations, Applied Mathematics and Computation 177 (2) (2006) 606–613.
[17] Z.L. Yang, Positive solutions of a second-order integral boundary value problem, Journal of Mathematical Analysis and Applications 321 (2) (2006) 751–765.
[18] X.P. Li, T.H. Liu, .J. Zhou, The existence of positive solutions and bounded oscillation for even-order neutral differential equations with unstable type, Applied Mathematics and Computation 176 (2) (2006) 632–641.
[19] Z.G. Wang, L.S. Liu, Y.H. Wu, Multiple positive solutions of Sturm–Liouville equations with singularities, Discrete Dynamics in Nature and Society, Art. No. 32018 , doi:10.1155/DDNS/2006/32018.
[20] Z.M. He, X.M. Jiang, Triple positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales, Journal of Mathematical Analysis and Applications 321 (2) (2006) 911–920.
[21] G.A. Afrouzi, S. Khademioo, A numerical method to find positive solution of semilinear elliptic Dirichlet problems, Applied Mathematics and Computation 174 (2) (2006) 1408–1415.
[22] H.F. Huo, W.T. Li, Permanence and global stability of positive solutions of nonautonomous discrete ratio-dependent predator–prey model, Discrete Dynamics in Nature and Society 2005 (2) (2005) 35–144.
[23] T. Kaczorek, Positive 1D and 2D systems, Communications and Control Engineering Series, Springer-Verlag, London, 2002.
[24] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, SIAM, Philadelphia, PA, 1994.
[25] L. Farina, S. Rinaldi, Positive linear systems. Theory and applications, Series on Pure and Applied Mathematics, John Wiley & Sons, New York, 2000.
[26] D. Hinrichsen, W. Manthey, U. Helmke, Minimal partial realizations by descriptor systems, Linear Algebra and its Applications 326 (1–3) (2001) 45–84.
[27] H.I. Dao, Structured perturbations of Drazin inverse, Applied Mathematics and Computation 158 (2) (2004) 419–432.
[28] N. Castro-González, E. Dopazo, J. Robles, Formulas for the Drazin inverse of special block matrices, Applied Mathematics and Computation 174 (1) (2006) 252–270.
[29] T.G. Lei, Y.M. Wei, C.W. Woo, Condition numbers and structured perturbation of the W-weighted Drazin inverse, Applied Mathematics and Computation 165 (1) (2005) 185–194.
[30] H.D. Kurz, N. Salvadori, The dynamic Leontief model and the theory of endogenous growth, in: Twelfth International Conference on Input–Output Techniques, New York 18–22 May 1998 (Special Session 11).
[31] M. De la Sen, Asymptotic hyperstability under unstructured and structured modelling deviations from the linear behaviour, Non Linear Analysis-Real World Applications 7 (2) (2006) 248–264.
[32] M. De la Sen, Some conceptual links between dynamic physical systems and operator theory issues concerning energy balances and stability, Informatica 16 (3) (2005) 395–406.

[33] M. de la Sen, On the adaptive control of a class of SISO dynamic hybrid systems, Applied Numerical Mathematics 56 (5) (2006) 618–647.
[34] M. de la Sen, Robust stable pole-placement adaptive control of linear systems with multiestimation, Applied Mathematics and Computation 172 (2) (2006) 1145–1174.
[35] H. Saadaoui, N. Manamanni, M. Djemai, J.P. Barbot, T. Floquet, Exact differentiation and sliding mode observers for switched Lagrangian systems, Nonlinear Analysis – Theory Methods & Applications 65 (5) (2006) 1050–1069.
[36] A.K. Agrawal, Z. Xu, W.L. He, Ground motion pulse-based active control of a linear base-isolated benchmark building, Structural Control & Health Monitoring 13 (2–3) (2006) 792–808.
[37] F. Pozo, F. Ikhouane, G. Pujol, J. Rodellar, Adaptive backstepping control of hysteric base-isolated structures, Journal of Vibration and Control 12 (4) (2006) 373–394.
[38] W.M. Haddad, V. Chellaboina, S.G. Nersesov, Hybrid nonnegative and compartmental dynamic systems, Mathematical Problems in Engineering 8 (6) (2002) 493–515.
[39] M. De la Sen, Sufficient conditions for Lyapunov's global stability of time-varying hybrid linear systems '', International Journal of Control 72 (2) (1999) 107–114.
[40] M. De la Sen, About positivity of a class of hybrid dynamic linear systems, Applied Mathematics and Computation, doi:10.1016/j.amc.2006.11.182.