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Brief paper

Two equivalent sets: Application to singular systems[☆]

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ABSTRACT

In this paper, we consider the problem of state feedback control design for continuous singular systems by applying equivalent sets technique. A new formulation of dissipativity condition is proposed. Based on this condition, the desired state feedback controller is designed such that the closed-loop system is admissible and dissipative. For singular Markovian systems, necessary and sufficient conditions are proposed for the system to be admissible and for the state feedback control design. For time-delay singular Markovian systems, a new bounded real lemma is proposed and the corresponding H_∞ control problem is studied. Numerical examples are given to illustrate the effectiveness of the theoretic results developed.

1. Introduction

Singular systems which are formulated as a set of coupled differential and algebraic equations (Xu & Lam, 2006) can be employed to describe a variety of practical systems such as electrical circuits (Dai, 1989) and economic systems (Luenberger & Arbel, 1977). Therefore, many control and filtering problems for singular systems have attracted extensively attention (Masubuchi, Kamitane, Ohara, & Suda, 1997; Xu & Yang, 1999). Dissipativity introduced in Willems (1972) has played an important role in system and control theory and gives strong links among physics, system theory and control engineering (Lozano, Maschke, Brogliato, & Egeland, 2000). The theory of dissipativity generalizes the H_∞ performance and passivity and considers both gain and phase properties. Therefore, it can provide an appropriate framework for less conservative robust controller design and many results have been reported in literature. Necessary and sufficient dissipativity conditions for continuous-time systems are proposed in Xie, Xie, and De Souza (1998). The results have been extended to continuous-time singular systems in Masubuchi (2006). However, the necessary and sufficient conditions in Masubuchi (2006) contain an equality constraint and a semi-definite matrix inequality, that is, $E^TX = X^TE \geq 0$, which may cause trouble in checking the condition numerically. Therefore, a natural question is how to obtain necessary and sufficient dissipativity and dissipative control conditions without these constraints.

On the other hand, parameters in practical systems may experience random abrupt changes either because of failures, repair, or environmental changes of modification operating points. Markovian jump systems have been widely employed to model these dynamic systems (Mariton, 1990; Shi, Boukas, & Agarwal, 1999). The necessary and sufficient stability conditions of Markovian jump systems are given in terms of linear matrix inequalities (LMIs) in de Farias, Geromel, do Val, and Costa (2000). Based on this work, many important stabilization results are obtained in Wu, Shi, Shu, Su, and Lu (2016), Zhang, Leng, and Colaneri (2016). For singular Markovian jump systems, a necessary and sufficient stability condition is proposed in terms of LMIs in Xu and Lam (2006) which provides fundamental results for further studying other control and filtering problems. However, there exists equality constraint which makes the numerical computation fragile and synthesis of state feedback controller more difficult. To remove the equality constraint, an excellent work is given in the form of strict LMIs in Xia, Boukas, Shi, and Zhang (2009). But the state feedback control result in Xia et al. (2009) is only a necessary condition. Hence, a meaningful research problem is raised naturally, that is, how to establish a necessary and sufficient

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stability condition for singular Markovian jump systems which would make the state feedback control design easier.

In another research front line, a great deal of attention has been devoted to the study of time-delay singular Markovian jump systems (Wang, Wang, Xue, & Lu, 2013; Wang & Xu, 2015). More details about the development of time-delay singular Markovian jump systems can refer to Wang, Zhang, and Yan (2015). H_∞ performance whose analysis result is known as bounded real lemma is one of the most popular and important performance specifications. A bounded real lemma in terms of LMIs for time-delay singular Markovian jump systems is proposed in Wu, Su, and Chu (2009) for the first time. However, the equality constraint $E^TP_i = P_i^TE$ is involved which makes the numerical computation fragile. A new bounded real lemma in terms of strict LMIs is established in Wang et al. (2013). However, when the time-delay is time-varying, the controller design method will not work. The result in Wu, Su, and Shi (2012) is not only without equality constraint but also can be used for time-varying delay case. It should be pointed out that there are some restrictions on matrix E and free-weighting matrix W. Moreover, when the state feedback control is studied, the matrix $\mathscr{P}_iE$ is enlarged as $\mathscr{P}_iE + \sigma\mathscr{L}$ and the inverse of $\mathscr{P}_iE + \sigma\mathscr{L}$ is used. This will introduce conservatism and the inverse of $\mathscr{P}_iE + \sigma\mathscr{L}$ may not exist when matrix E is in a general form. Based on the motivations given earlier, a natural research problem is how to establish a less conservative H_∞ control condition in terms of strict LMIs without any constraint on matrix E for time-delay singular Markovian jump system which also can be used for time-varying delay case.

In this paper, a new equivalent sets approach is proposed to investigate the problems of admissibilization and dissipative control of singular systems. By employing an equivalent parametrization of the constrained sets, a novel necessary and sufficient dissipativity condition of singular systems is presented without the equality constraint. Based on the criterion, a necessary and sufficient condition for the existence of a state feedback controller is established to render the closed-loop system to be admissible and dissipative. For singular Markovian systems, necessary and sufficient conditions of admissibility analysis and state feedback control are obtained in terms of LMIs. For singular Markovian systems with time-delay, a new bounded real lemma and an H_∞ control method are provided. Numerical examples are given to demonstrate the effectiveness of the obtained results.

Notation: The notation used throughout the paper is standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices; $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite); I and 0 denote the identity matrix and zero matrix, respectively, with compatible dimensions; $\star$ stands for the symmetric terms in a symmetric matrix and sym(A) is defined as $A + A^T$; the notation diag(A, B) denotes the block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

2. Preliminaries

In this section, some useful lemmas and the equivalent sets are given which will be used to develop our main results in sequel.

Lemma 1 (*Xia et al., 2009*). *For matrix $E \in \mathbb{R}^{n\times n}$ with* rank$(E) = r \leq n$*, denote E_L and E_R are full column rank with $E = E_LE_R^T$,* rank$(E_L) =$ rank$(E_R) = r$ *and let $P = P^T$ such that $E_L^TPE_L > 0$, and Q is nonsingular. U with full row rank and Λ with full column rank are the left and right null matrices of matrix E, respectively, that is $UE = 0$ and $E\Lambda = 0$. Then, $PE + U^TQ\Lambda^T$ is nonsingular and its inverse is expressed as $(PE + U^TQ\Lambda^T)^{-1} = \bar{P}E^T + \Lambda\bar{Q}U$, where $\bar{P} = \bar{P}^T$ and $\bar{Q}$ is nonsingular such that $E_R^T\bar{P}E_R = (E_L^TPE_L)^{-1}$, $\bar{Q} = (\Lambda^T\Lambda)^{-1}Q^{-1}(UU^T)^{-1}$.*

Lemma 2. *The following sets are equivalent:*

$$\mathbb{A} = \left\{X \in \mathbb{R}^{n\times n} : E^TX = X^TE \geq 0,\ X \text{is nonsingular}\right\},$$

$$\mathbb{B} = \left\{X = PE + U^T\Phi\Lambda^T : P = P^T \in \mathbb{R}^{n\times n},\ E_L^TPE_L > 0,\right.$$
$$\left.\Phi \in \mathbb{R}^{(n-r)\times(n-r)} \text{is nonsingular}\right\},$$

where E_L, E_R, $U^T \in \mathbb{R}^{n\times(n-r)}$ and $\Lambda \in \mathbb{R}^{n\times(n-r)}$ are defined in Lemma 1, respectively.

Proof (*Sufficiency*). Let $X = PE + U^T\Phi\Lambda^T$, we have $E^TX = X^TE = E_RE_L^TPE_LE_R^T \geq 0$ and X is nonsingular based on Lemma 1.

(Necessity) Without loss of generality, denote $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$, $E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $X_{11} \in \mathbb{R}^{r\times r}$ and $X_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$, then we have $E_L = E_R = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$ and $U^T = \Lambda = \begin{bmatrix} 0 \\ I_{n-r} \end{bmatrix}$. By using $E^TX = X^TE$, we have $X_{12} = 0$, $X_{11} \geq 0$. Due to rank$(E^TX) =$ rank$(X_{11}) = r$, we arrive at $X_{11} > 0$. Recalling X is nonsingular, it yields that X_{22} is nonsingular. Then, we can find a matrix $P = P^T = \begin{bmatrix} X_{11} & X_{21}^T \\ X_{21} & I_{n-r} \end{bmatrix}$ and $\Phi = X_{22}$ such that $E_L^TPE_L = X_{11} > 0$ and $X = PE + U^T\Phi\Lambda^T$. □

Remark 3. It should be noted that Lemma 2 is different from that in Feng and Lam (2013) because the set X_1 in Feng and Lam (2013) and the set $\mathbb{A}$ in this paper are different. On the other hand, the matrices E_L and E_R satisfying $E = E_L^TE_R$ are not used in Feng and Lam (2013). The system considered in Feng and Lam (2013) is discrete-time singular system and the synthesis results in Feng and Lam (2013) are just sufficient conditions while the system considered in this paper is continuous-time singular systems and the synthesis results in terms of strict LMIs are necessary and sufficient conditions for delay free singular systems.

3. Dissipative control of singular systems

In this section, the problems of dissipativity analysis and dissipative control of singular systems are addressed by employing the two equivalent sets defined in Lemma 2.

Consider a class of linear continuous singular systems described by

$$\begin{cases} E\dot{x}(t) = Ax(t) + B_ww(t), & x(0) = x_0 \\ z(t) = Cx(t) + D_ww(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; x_0 is the initial condition; $w(t) \in \mathbb{R}^p$ represents the exogenous input which includes disturbances to be rejected, and $z(t) \in \mathbb{R}^q$ is the controlled output; A, B_w, C and D_w are constant matrices with appropriate dimensions. In contrast with standard linear systems with $E = I$, the matrix $E \in \mathbb{R}^{n\times n}$ has $0 <$ rank$(E) = r < n$. First, we give some definitions and lemmas on unforced system (1):

Definition 4 (*Dai, 1989*).

(1) The singular system in (1) is said to be regular if $\det(sE - A)$ is not identically zero.
(2) The singular system in (1) is said to be impulse-free if $\deg\{\det(sE - A)\} =$ rank(E).
(3) The singular system in (1) is said to be asymptotically stable, if all the finite roots of $\det(sE - A) = 0$ have negative real parts.
(4) The singular system in (1) or the pair (E, A) is said to be admissible if the system is regular, impulse-free and asymptotically stable.

For a supply rate $s(w, z) = \begin{bmatrix} w \\ z \end{bmatrix}^T S \begin{bmatrix} w \\ z \end{bmatrix}$ with $S \in \mathbb{R}^{(p+q)\times(p+q)}$, the definition of dissipativity is given as follows:

Definition 5. The singular system (1) is said to be strictly dissipative with respect to the supply rate $s(w, z)$, under zero initial condition, if for any $t_1 \geq 0$ and for any $w \in L_2[0, t_1]$, the following inequality holds:

$$\int_0^{t_1} s(w(t), z(t))dt < 0.$$

Denote $S = \begin{bmatrix} S_{11} & S_{12} \\ \star & S_{22} \end{bmatrix}, M = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}$ as that in Masubuchi (2006), the following lemma gives a necessary and sufficient dissipativity condition for singular system (1).

Lemma 6 (Masubuchi, 2006)**.** *Consider partition of M as $M = \begin{bmatrix} M_{11} & M_{12} \\ \star & M_{22} \end{bmatrix}, M_{11} \in \mathbb{R}^{n\times n}$ and suppose that $M_{11} \geq 0$. Then, the singular system in (1) is admissible and strictly dissipative if and only if there exists a matrix X such that the following matrix equality and inequalities hold:*

$$E^T X = X^T E \geq 0 \tag{2}$$

$$\begin{bmatrix} \text{sym}(A^T X) + M_{11} & X^T B_w + M_{12} + A^T W \\ \star & \text{sym}(W^T B_w) + M_{22} \end{bmatrix} < 0 \tag{3}$$

where $W \in \mathbb{R}^{n\times p}$ is a matrix satisfying $E^T W = 0$.

Based on above, we present a new necessary and sufficient dissipativity condition below.

Theorem 7. *Suppose that $M_{11} \geq 0$. Then, the singular system (1) is admissible and strictly dissipative if and only if there exist a symmetric matrix P and a nonsingular matrix Φ such that the following LMIs hold:*

$$E_L^T P E_L > 0 \tag{4}$$

$$\begin{bmatrix} \text{sym}(A^T X) + M_{11} & X^T B_w + M_{12} + A^T W \\ \star & \text{sym}(W^T B_w) + M_{22} \end{bmatrix} < 0 \tag{5}$$

where $X = PE + U^T \Phi \Lambda^T$, $M_{11} = C^T S_{22} C$, $M_{12} = C^T S_{12}^T + C^T S_{22} D_w$, $M_{22} = S_{11} + \text{sym}(S_{12} D_w) + D_w^T S_{22} D_w$. E_L, U and Λ are defined in Lemma 2 and $W \in \mathbb{R}^{n\times p}$ is a matrix satisfying $E^T W = 0$.

Proof. From the inequality in (3), we can see that $\text{sym}(A^T X) + M_{11} < 0$. Then, it yields $\text{sym}(A^T X) < 0$ from $M_{11} \geq 0$ which implies that matrix X is nonsingular. Therefore, together with the condition in (2), the matrix X in Lemma 6 satisfies the set $\mathbb{A}$ in Lemma 2. By the two equivalent sets in Lemma 2, we have the desired result. □

Remark 8. The result in Theorem 1 is more general because it covers several important performance analysis criteria as special cases. When $S = \begin{bmatrix} -\mathscr{R} & -\mathscr{S} \\ \star & -\mathscr{Q} \end{bmatrix}$, the result in Theorem 1 changes to $(\mathscr{Q}, \mathscr{S}, \mathscr{R})$-dissipativity criterion (Meisami-Azad, Mohammadpour, & Grigoriadis, 2009; Xie et al., 1998). When $S = \begin{bmatrix} -\gamma^2 I & 0 \\ \star & I \end{bmatrix}$, the condition in Theorem 1 will be a new bounded real lemma (Masubuchi et al., 1997). A new formulation of positive real lemma can be obtained by setting $S = \begin{bmatrix} 0 & -I \\ \star & 0 \end{bmatrix}$ (Zhou, Hu, & Duan, 2010).

Remark 9. It can be seen that the equality constraint $E^T X = X^T E$ and non-strict linear matrix inequality $E^T X \geq 0$ in Lemma 6 are removed in the dissipativity criterion in Theorem 7, which makes the condition easier to check.

Next we will study the dissipative control problem by using state feedback. The aim is to design the following controller:

$$u(t) = Kx(t) + Jw(t) \tag{6}$$

for the open-loop system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t), \quad x(0) = x_0 \\ z(t) = Cx(t) + Du(t) + D_w w(t) \end{cases} \tag{7}$$

with $u(t) \in \mathbb{R}^m$, B and D denoting constant matrices with appropriate dimensions, such that the closed-loop system

$$\begin{cases} E\dot{x}(t) = (A + BK)x(t) + (B_w + BJ)w(t) \\ z(t) = (C + DK)x(t) + (D_w + DJ)w(t) \end{cases} \tag{8}$$

is admissible and strictly dissipative. A necessary and sufficient dissipative control method is given in the following theorem.

Theorem 10. *Suppose that $S_{22} = N^T N \geq 0$. There exists a state feedback controller in form of (6) such that the closed-loop system in (8) is admissible and strictly dissipative if and only if there exist a symmetric matrix $\bar{P}$, a nonsingular matrix $\bar{\Phi}$, matrices H and G such that the following LMIs hold:*

$$E_R^T \bar{P} E_R > 0 \tag{9}$$

$$\begin{bmatrix} \text{sym}(AY^T + BH) & \Xi_{12} & (YC^T + H^T D^T)N^T \\ \star & \Xi_{22} & (VC^T + D_w^T + G^T D^T)N^T \\ \star & \star & -I \end{bmatrix} < 0 \tag{10}$$

where $Y = (\bar{P}E^T + \Lambda\bar{\Phi}U)^T$, $\Xi_{12} = AV^T + BG + B_w + YC^T S_{12}^T + H^T D^T S_{12}^T$, $\Xi_{22} = \text{sym}(S_{12}CV^T + S_{12}D_w + S_{12}DG) + S_{11}$. E_R, U and Λ are defined in Lemma 2 and $V \in \mathbb{R}^{p\times n}$ is a matrix satisfying $EV^T = 0$. Then, a desired controller can be obtained by $K = HY^{-T}, J = G - KV^T$.

Proof. Based on Theorem 1, the closed-loop system is admissible and strictly dissipative if and only if the following inequalities hold:

$$E_L^T P E_L > 0 \tag{11}$$

$$\Pi = \begin{bmatrix} \text{sym}(A_c^T X) + C_c^T S_{22} C_c & \Pi_{12} \\ \star & \Pi_{22} \end{bmatrix} < 0 \tag{12}$$

where $\Pi_{12} = X^T B_{wc} + C_c^T (S_{12}^T + S_{22} D_{wc}) + A_c^T W, A_c = A + BK$, $B_{wc} = B_w + BJ, C_c = C + DK, D_{wc} = D_w + DJ, \Pi_{22} = \text{sym}(W^T B_{wc}) + S_{11} + \text{sym}(S_{12} D_{wc}) + D_{wc}^T S_{22} D_{wc}$.

Considering $S_{22} = N^T N \geq 0$ and employing Schur complement equivalence, (12) is equivalent to

$$\begin{bmatrix} \text{sym}\left(\bar{\Pi}\right) + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} & \begin{bmatrix} C_c^T \\ D_{wc}^T \end{bmatrix} N^T \\ \star & -I \end{bmatrix} < 0 \tag{13}$$

where $\bar{\Pi} = \begin{bmatrix} X^T & 0 \\ W^T & I \end{bmatrix} \begin{bmatrix} A_c & B_{wc} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ S_{12} \end{bmatrix} \begin{bmatrix} C_c & D_{wc} \end{bmatrix}$. Performing the congruence transformation to (13) by $\text{diag}\left(\begin{bmatrix} Y^T & V^T \\ 0 & I \end{bmatrix}, I\right)$, with $Y = (PE + U^T \Phi \Lambda^T)^{-T}$, $V = -W^T X^{-T}$, and noting that $X = PE + U^T \Phi \Lambda^T, YX^T = I, VX^T + W^T = 0$, we have

$$\begin{bmatrix} \text{sym}\left(\tilde{\Pi}\right) + \begin{bmatrix} 0 & 0 \\ 0 & S_{11} \end{bmatrix} & \begin{bmatrix} Y & 0 \\ V & I \end{bmatrix} \begin{bmatrix} C_c^T \\ D_{wc}^T \end{bmatrix} N^T \\ \star & -I \end{bmatrix} < 0 \tag{14}$$

where $\tilde{\Pi} = \begin{bmatrix} A_c & B_{wc} \\ S_{12} C_c & S_{12} D_{wc} \end{bmatrix} \begin{bmatrix} Y^T & V^T \\ 0 & I \end{bmatrix}$. Then setting $H = KY^T$, $G = KV^T + J$ in (14), we obtain (10). Noting that $E_L^T P E_L = (E_R^T \bar{P} E_R)^{-1}$, (11) is equivalent to (9). □

Remark 11. It should be remarked that a necessary and sufficient dissipative control criterion is proposed in Theorem 10. Compared with the controller design method in Masubuchi (2006), the equality and non-strict LMI constraints $EY^T = YE^T \geq 0$ have been moved, which makes the numerical computations more tractable and reliable.

4. Admissibility of singular Markovian jump systems

In this section, the equivalent sets are used to deal with the admissibility and the state feedback control problems of singular Markovian jump systems.

Consider the following singular Markovian jump systems:

$$E\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t) \tag{15}$$

where $x(t)$ and $u(t)$ are defined in (7). The matrices $A(\cdot)$ and $B(\cdot)$ which are functions of r_t are known real matrices with appropriate dimensions. $\{r_t, t \geq 0\}$ is a continuous-time Markov process which takes values in a finite set $\mathscr{N} = \{1, 2, \ldots, N\}$ and describes the evolution of the mode at time t. For notational simplicity, for each $r_t = i, i \in \mathscr{N}$, matrix $A(r_t)$ will be denoted by A_i. The Markov process describes the switching between different modes and its evolution is governed by the following probability transitions

$$\Pr\{r_{t+\delta} = j | r_t = i\} = \begin{cases} \pi_{ij}\delta + o(\delta) & i \neq j \\ 1 + \pi_{ii}\delta + o(\delta) & i = j \end{cases} \tag{16}$$

where $\delta > 0$ and $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$, and $\pi_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i at time t to mode j at time $t + \delta$, which satisfies $\pi_{ii} = -\sum_{j=1, i\neq j}^{N} \pi_{ij}$.

We recall the following lemma.

Lemma 12 (Xu & Lam, 2006). *System (15) with $u(t) = 0$ is stochastically admissible if and only if there exist matrices $X_i, i \in \mathscr{N}$, such that the following coupled LMIs hold for each $i \in \mathscr{N}$:*

$$E^T X_i = {X_i}^T E \geq 0 \tag{17}$$

$${X_i}^T A_i + A_i^T X_i + \sum_{j=1}^{N} \pi_{ij} E^T X_j < 0. \tag{18}$$

Remark 13. The matrices $X_i, i \in \mathscr{N}$, in Lemma 12 are nonsingular. If they are singular, then there exist nonzero vectors $\xi_i, i \in \mathscr{N}$, such that $X_i \xi_i = 0$. Then for the nonzero vector ξ_i, we have the following inequality from (18):

$$\begin{aligned} &\sum_{j=1, j\neq i}^{N} \pi_{ij} {\xi_i}^T E^T X_j \xi_i + \pi_{ii} {\xi_i}^T E^T X_i \xi_i + {\xi_i}^T {X_i}^T A_i \xi_i + \xi_i^T A_i^T X_i \xi_i \\ &\quad = \sum_{j=1, j\neq i}^{N} \pi_{ij} {\xi_i}^T E^T X_j \xi_i < 0. \end{aligned} \tag{19}$$

As $E^T X_j \geq 0$ and $\pi_{ij} \geq 0,\ i \neq j$, the inequality in (19) cannot hold which implies the inequality in (18) does not hold. Based on the discussions, matrices $X_i,\ i \in \mathscr{N}$, are nonsingular.

Then combining Lemma 2 with Remark 13, a new necessary and sufficient admissibility condition of singular Markovian jump systems can be obtained from Lemma 12, which is given in the following theorem.

Theorem 14. *System (15) with $u(t) = 0$ is stochastically admissible if and only if there exist symmetric matrices $P_i,\ i \in \mathscr{N}$, and nonsingular matrices $\Phi_i,\ i \in \mathscr{N}$, such that the following coupled LMIs hold for each $i \in \mathscr{N}$:*

$$E_L^T P_i E_L > 0 \tag{20}$$

$$\mathrm{sym}(A_i^T (P_i E + U^T \Phi_i \Lambda^T)) + \sum_{j=1}^{N} \pi_{ij} E^T P_j E < 0 \tag{21}$$

where E_L, U and Λ are defined in Lemma 2.

Now we consider the state feedback control problem for system (15) with $u(t) = K(r_t)x(t)$ such that the closed-loop system

$$E\dot{x}(t) = \bar{A}(r_t)x(t) = (A(r_t) + B(r_t)K(r_t))x(t) \tag{22}$$

is stochastically admissible.

A necessary and sufficient state feedback controller design condition is proposed in the following result based on Theorem 14.

Theorem 15. *There exists a state feedback controller such that the closed-loop system in (22) is stochastically admissible if and only if there exist symmetric matrices $\bar{P}_i,\ i \in \mathscr{N}$, nonsingular matrices $\bar{\Phi}_i,\ i \in \mathscr{N}$, and matrices $H_i,\ i \in \mathscr{N}$, such that the following LMIs hold for each $i \in \mathscr{N}$:*

$$E_R^T \bar{P}_i E_R > 0 \tag{23}$$

$$\begin{bmatrix} \mathrm{sym}(A_i Y_i + B_i H_i) + \pi_{ii} E \bar{P}_i E^T & Y_i^T \Omega_i \\ \star & -\Psi_i \end{bmatrix} < 0 \tag{24}$$

where $Y_i = \bar{P}_i E^T + \Lambda \bar{\Phi}_i U$, $\Omega_i = \big[\sqrt{\pi_{i1}} E_R \ \ \sqrt{\pi_{i2}} E_R \ \cdots \ \sqrt{\pi_{i(i-1)}} E_R \ \ \sqrt{\pi_{i(i+1)}} E_R \ \cdots \ \sqrt{\pi_{iN}} E_R\big]$, $\Psi_i = \mathrm{diag}\{E_R^T \bar{P}_1 E_R, E_R^T \bar{P}_2 E_R, \ldots, E_R^T \bar{P}_{i-1} E_R, E_R^T \bar{P}_{i+1} E_R, \ldots, E_R^T \bar{P}_N E_R\}$. E_R, U and Λ are defined in Lemma 2. Then, the desired controller can be obtained by $K_i = H_i Y_i^{-1}$.

Proof. Based on Theorem 14, the closed-loop system (22) is stochastically admissible if and only if there exist symmetric matrices $P_i,\ i \in \mathscr{N}$ and nonsingular matrices $\Phi_i,\ i \in \mathscr{N}$ such that the following coupled LMIs hold for each $i \in \mathscr{N}$:

$$E_L^T P_i E_L > 0 \tag{25}$$

$$\mathrm{sym}((A_i + B_i K_i)^T (P_i E + U^T \Phi_i \Lambda^T)) + \sum_{j=1}^{N} \pi_{ij} E^T P_j E < 0. \tag{26}$$

Considering $\pi_{ij} > 0,\ i \neq j,\ E_L^T P_i E_L > 0$, and using Schur complement equivalence, (26) is equivalent to

$$\begin{bmatrix} \Lambda + \pi_{ii} E^T (P_i E + U^T \Phi_i \Lambda^T) & \Omega_i \\ \star & -\Psi_i \end{bmatrix} < 0$$

where $\Lambda = \mathrm{sym}((A_i + B_i K_i)^T (P_i E + U^T \Phi_i \Lambda^T))$. Then performing congruence transformation to above inequality with matrix $\begin{bmatrix} Y_i & 0 \\ 0 & I \end{bmatrix}$ with $Y_i = \bar{P}_i E^T + \Lambda \bar{\Phi}_i U = (P_i E + U^T \Phi_i \Lambda^T)^{-1}$ and setting $H_i = K_i Y_i$, the inequality in (24) is obtained. Because of $E_R^T \bar{P}_i E_R = (E_L^T P_i E_L)^{-1}$, the inequalities in (23) and (25) are equivalent. □

Remark 16. It should be mentioned that Theorem 15 provides a necessary and sufficient admissibility condition of state feedback control of singular Markovian systems while the admissibilization condition proposed in Xia et al. (2009) is only a necessary condition. On the other hand, in order to obtain a feedback control method in terms of linear matrix inequalities, the term $\pi_{ii} Y_i^T E_R (E_R^T \bar{P}_i E_R)^{-1} E_R^T Y_i$ is enlarged as $\pi_{ii}(Y_i^T E^T + E Y_i - E \bar{P}_i E^T)$ and a condition is obtained in Xia et al. (2009). Because $E\Lambda = E_L E_R^T \Lambda = 0$ and E_L is of full column rank, it yields that $E_R^T \Lambda = 0$. Moreover, noting that $\pi_{ii} Y_i^T E_R (E_R^T \bar{P}_i E_R)^{-1} E_R^T Y_i = \pi_{ii} E_L E_R^T \bar{P}_i E_R (E_R^T \bar{P}_i E_R)^{-1} E_R^T \bar{P}_i E_R E_L^T = \pi_{ii} E \bar{P}_i E^T$, we have the necessary and sufficient state feedback control result in Theorem 15.

5. H_∞ control of singular Markovian systems with time-delay

Consider a class of singular Markovian jump systems with time-delay described by

$$\begin{cases} E\dot{x}(t) = A(r_t)x(t) + A_d(r_t)x(t-d) + B(r_t)u(t) \\ \qquad\qquad + B_w(r_t)w(t) \\ z(t) = C(r_t)x(t) + C_d(r_t)x(t-d) + D(r_t)u(t) \\ \qquad\qquad + D_w(r_t)w(t) \\ x(t) = \varphi(t), \quad t \in [-d, 0] \end{cases} \tag{27}$$

where $x(t), u(t), E, A(r_t)$, and $B(r_t)$ are defined in system (15); $w(t)$ and $z(t)$ are defined in (1); $A_d(r_t), C_d(r_t), B_w(r_t), C(r_t), D(r_t)$ and $D_w(r_t)$ denote known real matrices with appropriate dimensions; d is the constant time-delay and $\varphi(t)$ represents initial condition.

Following a similar line as that in Wu and Zheng (2009) and Wu et al. (2012), a new bounded real lemma for time-delay singular Markovian jump system (27) is proposed in the following lemma:

Lemma 17. *Given a scalar $\gamma > 0$, the singular Markovian jump system in (27) with $u(t) = 0$ is stochastically admissible with an H_∞ performance γ, if there exist nonsingular matrices X_i, $i \in \mathscr{N}$, matrices $Q > 0$, $R > 0$ and W_i, $i \in \mathscr{N}$ such that the following LMIs hold for each $i \in \mathscr{N}$:*

$$E^T X_i = X_i^T E \geq 0 \qquad (28)$$

$$\begin{bmatrix} \Psi_{11i} & X_i^T A_{di} - W_i E & X_i^T B_{wi} & dW_i E & dA_i^T R & C_i^T \\ \star & -Q & 0 & 0 & dA_{di}^T R & C_{di}^T \\ \star & \star & -\gamma^2 I & 0 & dB_{wi}^T R & D_{wi}^T \\ \star & \star & \star & -dR & 0 & 0 \\ \star & \star & \star & \star & -dR & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0 \qquad (29)$$

where $\Psi_{11i} = X_i^T A_i + A_i^T X_i + Q + W_i E + E^T W_i^T + \sum_{j=1}^N \pi_{ij} E^T X_j$.

Proof. The result can be obtained by choosing the following Lyapunov function and free weighting equation:

$$V(x_t) = x^T(t) E^T X_i x(t) + \int_{t-d}^{t} x(\tau)^T Q x(\tau) d\tau + \int_{-d}^{0}\int_{t+\theta}^{t} \dot{x}(\tau)^T E^T R E \dot{x}(\tau) d\tau d\theta$$

$$0 = 2x(t)^T W_i E \left[x(t) - x(t-d) - \int_{t-d}^{t} \dot{x}(\tau) d\tau \right].$$

To prove the invertibility of matrices X_i, $i \in \mathscr{N}$, the following inequality will be used

$$\begin{bmatrix} \Psi_{11i} & X_i^T A_{di} - W_i E \\ \star & -Q \end{bmatrix} < 0 \qquad (30)$$

which can be derived from (29). Then pre- and post-multiplying (30) by $[I, I]$ and its transpose, respectively, we obtain $\mathrm{sym}(X_i^T(A_i + A_{di})) + \sum_{j=1}^N \pi_{ij} E^T X_j < 0$, which implies the matrices X_i, $i \in \mathscr{N}$, are nonsingular based on discussions in Remark 13. □

Noting that the invertibility of matrices X_i, $i \in \mathscr{N}$, in Lemma 17, a new bounded real lemma for singular system (27) is presented below in terms of strict LMIs based on Lemma 2.

Theorem 18. *Given a scalar $\gamma > 0$, the singular Markovian jump system in (27) is stochastically admissible with an H_∞ performance γ if there exist symmetric matrices P_i, $i \in \mathscr{N}$, and nonsingular matrices Φ_i, $i \in \mathscr{N}$, such that the following LMIs hold for each $i \in \mathscr{N}$:*

$$E_L^T P_i E_L > 0 \qquad (31)$$

$$\begin{bmatrix} \bar{\Psi}_{11i} & X_i^T A_{di} - W_i E & X_i^T B_{wi} & dW_i E & dA_i^T R & C_i^T \\ \star & -Q & 0 & 0 & dA_{di}^T R & C_{di}^T \\ \star & \star & -\gamma^2 I & 0 & dB_{wi}^T R & D_{wi}^T \\ \star & \star & \star & -dR & 0 & 0 \\ \star & \star & \star & \star & -dR & 0 \\ \star & \star & \star & \star & \star & -I \end{bmatrix} < 0 \qquad (32)$$

where $\bar{\Psi}_{11i} = X_i^T A_i + A_i^T X_i + Q_i + W_i E + E^T W_i^T + \sum_{j=1}^N \pi_{ij} E^T P_j E$, $X_i = P_i E + U^T \Phi_i \Lambda^T$. E_L, U and Λ are defined in Lemma 2.

Remark 19. A bounded real lemma for singular Markovian systems with time-delay is proposed in Lemma 1 of Wu et al. (2012). However, the matrix E is restricted to be $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ and the free weighting matrix W_i needs to satisfy $W_i E = W_i$ in the process of deriving the bounded real lemma. In Theorem 18, these constraints are both removed. From this point of view, the result in Theorem 18 is more general and less conservative.

Based on Theorem 18, the state feedback controller $u(t) = K_i x(t)$ is designed in the following theorem.

Theorem 20. *Given a scalar $\gamma > 0$, the singular system in (27) is stochastically admissible with an H_∞ performance γ if there exist symmetric matrices $\bar{P}_i$, $i \in \mathscr{N}$, nonsingular matrices $\bar{\Phi}_i$, $i \in \mathscr{N}$, matrices $\bar{Q} > 0$, $\bar{R} > 0$, $\bar{W}_i$ and H_i, $i \in \mathscr{N}$ such that the following LMIs hold for each $i \in \mathscr{N}$:*

$$E_R^T \bar{P}_i E_R > 0 \qquad (33)$$

$$\begin{bmatrix} \tilde{\Psi}_{11i} & \tilde{\Psi}_{12i} & B_{wi} & d\bar{W}_i E^T & \tilde{\Psi}_{15i} & \tilde{\Psi}_{16i} & Y_i^T & Y_i^T \Omega_i \\ \star & \tilde{\Psi}_{22i} & 0 & 0 & \tilde{\Psi}_{25i} & \tilde{\Psi}_{26i} & 0 & 0 \\ \star & \star & -\gamma^2 I & 0 & dB_{wi}^T & D_{wi}^T & 0 & 0 \\ \star & \star & \star & \tilde{\Psi}_{44i} & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -d\bar{R} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\bar{Q} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\Psi_i \end{bmatrix} < 0 \qquad (34)$$

where

$$\tilde{\Psi}_{11i} = \mathrm{sym}(A_i Y_i + B_i H_i + \bar{W}_i E^T) + \pi_{ii} E \bar{P}_i E^T$$

$$\tilde{\Psi}_{12i} = A_{di} Y_i - \bar{W}_i E^T, \qquad \tilde{\Psi}_{15i} = d(A_i Y_i + B_i H_i)^T$$

$$\tilde{\Psi}_{22i} = -Y_i^T - Y_i + \bar{Q}, \qquad \tilde{\Psi}_{44i} = -dY_i^T - dY_i + d\bar{R}$$

$$\tilde{\Psi}_{25i} = d(A_{di} Y_i)^T, \qquad \tilde{\Psi}_{26i} = (C_{di} Y_i)^T$$

$$Y_i = \bar{P}_i E^T + \Lambda \bar{\Phi}_i U, \qquad \tilde{\Psi}_{16i} = (C_i Y_i + D_i H_i)^T$$

E_R, U and Λ are defined in Lemma 2. Moreover, if the above LMIs are feasible then the state feedback controller K_i can be given by $K_i = H_i Y_i^{-1}$.

Proof. Based on Theorem 18 and using Schur complement equivalence, for closed-loop system, it is stochastically admissible with an H_∞ performance if the following inequalities hold:

$$E_L^T P_i E_L > 0 \qquad (35)$$

$$\begin{bmatrix} \hat{\Psi}_{11i} & \hat{\Psi}_{12i} & X_i^T B_{wi} & dW_i E & d\bar{A}_i^T R & \bar{C}_i^T & I & \Omega_i \\ \star & -Q & 0 & 0 & dA_{di}^T R & C_{di}^T & 0 & 0 \\ \star & \star & -\gamma^2 I & 0 & dB_{wi}^T R & D_{wi}^T & 0 & 0 \\ \star & \star & \star & -dR & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -dR & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -Q^{-1} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\Psi_i \end{bmatrix} < 0 \qquad (36)$$

where $\hat{\Psi}_{11i} = \mathrm{sym}(X_i^T \bar{A}_i + W_i E) + \pi_{ii} E^T P_i E$, $\hat{\Psi}_{12i} = X_i^T A_{di} - W_i E$, $\bar{A}_i = A_i + B_i K_i$, $\bar{C}_i = C_i + D_i K_i$, Ω_i and Ψ_i are defined in Theorem 15.

Defining $Y_i = X_i^{-1}$, $\bar{Q} = Q^{-1}$, $\bar{R} = R^{-1}$, $H_i = K_i Y_i$ and $\bar{W}_i = Y_i^T W_i Y_i^T$, pre- and post-multiplying the inequality in (36) by $\mathrm{diag}\{Y_i^T, Y_i^T, I, Y_i^T, R^{-1}, I, Y_i^T, I\}$ and its transpose, and noticing

$EY_i = Y_i^T E^T = E\bar{P}_i E^T$, we have

$$\begin{bmatrix} \tilde{\Psi}_{11i} & \tilde{\Psi}_{12i} & B_{wi}X_i & d\bar{W}_i E^T & \tilde{\Psi}_{15i} & \tilde{\Psi}_{16i} & Y_i^T & Y_i^T \Omega_i \\ \star & -Y_i^T Q Y_i & 0 & 0 & \tilde{\Psi}_{25i} & \tilde{\Psi}_{26i} & 0 & 0 \\ \star & \star & -\gamma^2 I & 0 & dB_{wi}^T & D_{wi}^T & 0 & 0 \\ \star & \star & \star & -dY_i^T R Y_i & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -d\bar{R} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\bar{Q} & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\Psi_i \end{bmatrix} < 0. \quad (37)$$

Considering $-Y_i^T Q Y_i \le -Y_i^T - Y_i + \bar{Q}$, and $-Y_i^T R Y_i \le -Y_i^T - Y_i + \bar{R}$, the inequality in (34) implies that the inequality in (37) holds. On the other hand, the inequality in (35) is equivalent to the inequality in (33). Therefore, the proof is completed. □

Remark 21. Note that the H_∞ control problem is also investigated in Wu et al. (2012), where the invertibility of matrix $\mathscr{P}_j E + \sigma \mathscr{N}$ with $\mathscr{N} = \begin{bmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ \star & I_{(n-r)\times(n-r)} \end{bmatrix}$ is required, that is, $(\mathscr{P}_j E + \sigma \mathscr{N})^{-1} = \mathscr{P}_j^{-1} E + \sigma \mathscr{N}$. The above constraints have the following two disadvantages. One is that the inverse of matrix $\mathscr{P}_j E + \sigma \mathscr{N}$ may not exist when matrix E is not in the form of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. On the other hand, applying the parameter σ to enlarge matrix $\mathscr{P}_j E$, that is $\mathscr{P}_j E \le \mathscr{P}_j E + \sigma \mathscr{N}$, will derive conservatism. In Theorem 20, the invertibility of matrix $X_i = P_i E + U^T \Phi_i \Lambda^T$ is needed. From Lemma 2, we can see the invertibility of matrix X_i can be guaranteed regardless of the form of matrix E and without any enlargement.

Remark 22. In Wang et al. (2013), based on the equivalence of the H_∞ norms of transfer functions of original system

$$\begin{cases} E\dot{x}(t) = (A_i + B_i K_i)x(t) + A_{di}x(t-d) + B_{wi}\omega(t) \\ z(t) = (C_i x(t) + D_i K_i)x(t) \end{cases} \quad (38)$$

and its dual system

$$\begin{cases} E^T\dot{x}(t) = (A_i + B_i K_i)^T x(t) + A_{di}^T x(t-d) + (C_i + D_i K_i)^T \omega(t) \\ z(t) = B_{wi}^T x(t) \end{cases} \quad (39)$$

an H_∞ control method is given in Theorem 2 there. However, the method cannot be used when the delay is time-varying because the transfer function of a system with time-varying delay cannot be expressed explicitly as in Wang et al. (2013). The two equivalent sets method without using transfer function in this paper can be extended to systems with time-varying delay.

6. Examples

In this section, numerical examples are provided to show the advantages on numerical computations and the state feedback control of the equivalent sets approach.

Example 1. Consider the following singular system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t)$$

$$z(t) = \begin{bmatrix} 2 & 5 & -2 \end{bmatrix} x(t) + 3w(t)$$

We choose $E_L = E_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$, $U^T = \Lambda = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Based on Theorem 7, the following three performances are discussed respectively:

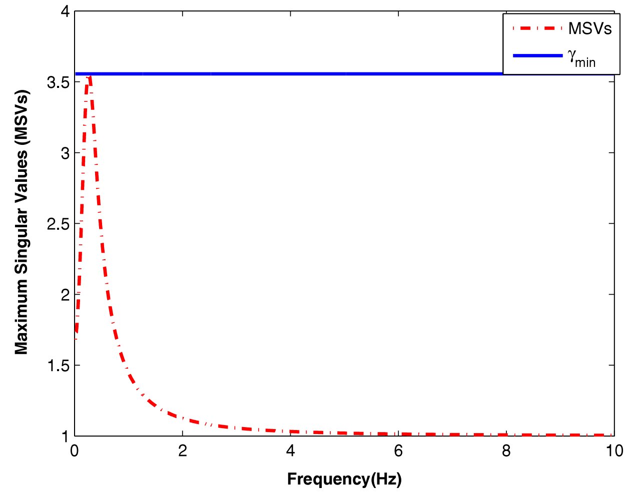

Fig. 1. Maximal singular values and $\gamma_{\min}$.

- H_∞ case: $S_{11} = -\gamma^2 I$, $S_{12} = 0$, $S_{22} = I$. By solving the LMIs in (4)–(5), the minimal value of γ is $\gamma_{\min} = 3.5551$ and the corresponding decision variables P and Q are computed to be

$$P = \begin{bmatrix} 40.8318 & 0.7772 & -6.0000 \\ 0.7772 & 18.2773 & -14.9999 \\ -6.0000 & -14.9999 & 0 \end{bmatrix}, \quad Q = 10.6386$$

  To verify the effectiveness of the result, the maximal singular values (MSVs) of the considered system are depicted in Fig. 1. From the figure, we can see that the value of $\gamma_{\min}$ is very near the supremum of MSVs which implies that the system satisfies the prescribed H_∞ performance.
- General dissipative case: $S_{11} = -1.6$, $S_{12} = -0.5$, $S_{22} = 0.4$. Similarly, to verify the dissipativity of the considered system, we can verify whether the LMIs in Theorem 7 are feasible. By solving the LMIs, feasible matrices P and Q are computed to be

$$P = \begin{bmatrix} 10.8800 & 0.2980 & -1.4001 \\ 0.2980 & 4.8022 & -3.4971 \\ -1.4001 & -3.4971 & 0 \end{bmatrix}, \quad Q = 3.3810$$

  According to the definition of $(\mathscr{Q}, \mathscr{N}, \mathscr{R})$-dissipativity in Xie et al. (1998), we can see the considered system is $(-0.4, 0.5, 1.6)$-dissipativity.
- Positive realness case: $S_{11} = 0$, $S_{12} = -I$, $S_{22} = 0$. By solving the LMIs in Theorem 7, the conditions in (4) and (5) are satisfied with the following solution:

$$P = \begin{bmatrix} 14.5494 & 1.8136 & 1.9068 \\ 1.8136 & 4.0456 & 4.5228 \\ 1.9068 & 4.5228 & 0 \end{bmatrix}, \quad Q = 3.0400$$

  Therefore, the system is positive real.

Example 2. Consider a singular system with the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}, B_w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}, D_w = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Choose $E_L = E_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^T$, $U^T = \Lambda = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. To illustrate the effectiveness of Theorem 10, we consider the H_∞ case, that is $S_{11} = -\gamma^2 I$, $S_{12} = 0$, $S_{22} = I$. By solving the LMIs in

Theorem 10, the minimal value of γ is obtained as $\gamma_{\min} = 2.2472$ and the corresponding solutions of decision variables are

$$\bar{P} = \begin{bmatrix} 0.0459 & -0.0011 & 0.0425 \\ -0.0011 & 0.1079 & -0.2776 \\ 0.0425 & -0.2776 & 0 \end{bmatrix}, \qquad \bar{\Phi} = 0.8001$$

$$H = \begin{bmatrix} 0.0000 & -0.1099 & -0.0007 \end{bmatrix}, \qquad G = -0.0030.$$

Then, the feedback controller is

$$K = \begin{bmatrix} -0.0244 & -1.0215 & -0.0009 \end{bmatrix}, \qquad J = -0.0021.$$

Example 3. Consider the singular Markovian jump system in (15) with two operating modes, that is, $N = 2$ and the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1.3 & 0.8 & 1.0 \\ 0.7 & 0.8 & 0.9 \\ 0.4 & 0.2 & -0.7 \end{bmatrix}, B_1 = \begin{bmatrix} 1.5 & -1.5 \\ 0.9 & 0.6 \\ 1.1 & 5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.7 & 0.9 & 0.3 \\ 1.1 & 1.4 & -0.7 \\ 0.5 & 0.3 & 1.6 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0.9 \\ -2 & 0.6 \\ 0.7 & 1 \end{bmatrix}, E_L = E_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & -0.7 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 \\ 0 \\ 0.8 \end{bmatrix}, \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.9 & 0.9 \\ 0.5 & -0.5 \end{bmatrix}.$$

By solving the characteristic equation $sE - A_1 = 0$, two finite characteristic foots are 2.6825 and 0.24604, respectively. Therefore, the open-loop system is not admissible. The controller design method in Theorem 6 and Theorem 7 in Xia et al. (2009) cannot be used. The purpose is to design a state feedback controller to guarantee the closed-loop system to be stochastically admissible. By solving the LMIs in Theorem 15, we have $K_1 = \begin{bmatrix} 34.8763 & 28.5998 & 0.6234 \\ 37.9459 & 28.4043 & 0.0044 \end{bmatrix}$, $K_2 = \begin{bmatrix} -102.5126 & -18.2368 & -0.5429 \\ -490.4724 & -90.5122 & -1.2175 \end{bmatrix}$.

Example 4. A Direct Current (DC) motor has been modelled as a singular Markovian jump system in Sakthivel, Joby, Mathiyalagan, and Santra (2015), Wang and Bo (2016), Wang et al. (2015) and Zhang, Cao, and Wang (2014). The singular Markovian model is given as follows:

$$E\dot{x}(t) = A(r_t)x(t) + B(r_t)u(t). \tag{40}$$

The matrix parameters are borrowed from Example 2 in Wang and Bo (2016):

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ -20 & -3 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 9.8 & 0 & 1 \\ -20 & -3 & -0.5 \end{bmatrix}, B_2 = \begin{bmatrix} -1.2 \\ 0.5 \\ -0.2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.4 \\ -0.2 \\ 0.5 \end{bmatrix}, \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.6 & 0.6 \\ 0.2 & -0.2 \end{bmatrix}$$

$$E_L = E_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & 0 & -0.7 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 \\ 0 \\ 0.8 \end{bmatrix}.$$

By solving the condition in Theorem 15, the state-feedback controllers are obtained $K_1 = \begin{bmatrix} 186.3514 & 38.5301 & 2.0267 \end{bmatrix}$, $K_2 = \begin{bmatrix} 188.1730 & 21.4539 & -2.3464 \end{bmatrix}$. Then the closed-loop system is $E\dot{x}(t) = \bar{A}(r_t)x(t)$ with $\bar{A}(r_t) = A(r_t) + B(r_t)K(r_t)$.

In order to testify the effectiveness of the state-feedback control method, Theorem 2.2 in Wang et al. (2015), Theorem 15 in this paper and Theorem 4 in Xia et al. (2009) are utilized. By solving the LMIs in these theorems, we note that all the conditions are feasible which implies the state-feedback controller is applicable.

Example 5. Consider the following singular Markovian jump time-delay system with two operating modes ($N = 2$) and the following parameters:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 0.5023 & 2.0125 & 0.0150 \\ 0.3025 & 0.4004 & -4.0020 \\ -0.1002 & 0.3002 & -3.5001 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.5005 & 0.5052 & -0.1002 \\ 0.1256 & -0.0552 & 0.3003 \\ 0.1033 & 1.0015 & -2.0045 \end{bmatrix}, B_{w1} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.4 \end{bmatrix}$$

$$C_{d1} = \begin{bmatrix} 0.2486 & 0.1025 & -0.0410 \end{bmatrix}, D_1 = D_2 = 0.$$

$$A_{d1} = \begin{bmatrix} -0.1669 & 0.0802 & 1.6820 \\ -0.8162 & -0.9373 & 0.5936 \\ 2.0941 & 0.6357 & 0.7902 \end{bmatrix}, B_{w2} = \begin{bmatrix} -0.6 \\ 0.5 \\ 0.8 \end{bmatrix}$$

$$A_{d2} = \begin{bmatrix} 0.1053 & -0.1948 & -0.6855 \\ 0.1586 & 0.0755 & -0.2684 \\ 0.7709 & -0.5266 & -1.1883 \end{bmatrix}, B_1 = \begin{bmatrix} 0.9 \\ 1.8 \\ 1.4 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1.5 \\ 0.9 \\ 1.1 \end{bmatrix}, C_1 = \begin{bmatrix} 0.8 & 0.3 & 0.9 \end{bmatrix}, C_2 = \begin{bmatrix} -0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$C_{d2} = \begin{bmatrix} -2.2476 & -0.5108 & 0.2492 \end{bmatrix}, D_{w1} = 0.2, D_{w2} = 0.5$$

$$U = \begin{bmatrix} 0 & 0 & -0.7 \end{bmatrix}, \Lambda = \begin{bmatrix} 0 & 0 & 0.8 \end{bmatrix}, E_R = E_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$d = 0.2, \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \begin{bmatrix} -0.3 & 0.3 \\ 0.5 & -0.5 \end{bmatrix}.$$

The purpose is to design a state feedback controller such that the closed-loop system is stochastically admissible with an H_∞ performance γ. By solving LMIs in Theorem 20, the minimal value of γ is obtained as $\gamma_{\min} = 2.43$. When $\gamma = 2.43$, the state feedback controllers are obtained as $K_1 = \begin{bmatrix} -4.4763 & -2.2181 & -0.1445 \end{bmatrix}$, $K_2 = \begin{bmatrix} -5.3581 & -0.4309 & -1.1303 \end{bmatrix}$. By using the method in Theorem 1 of Wu et al. (2012), the minimal value of γ is solved as $\gamma = 2.68$ which illustrates the reduced conservatism of out result in Theorem 20.

On the other hand, when some system parameters change to

$$E = \begin{bmatrix} 2 & \sqrt{2} & 0 \\ \sqrt{2} & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad E_R = E_L = \begin{bmatrix} 1 & 1 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix},$$

$$D_1 = 0.1, \qquad D_2 = -0.5$$

the condition in Theorem 1 of Wu et al. (2012) cannot be applied. By solving the conditions with $\gamma = 5.60$ in Theorem 20 in this note, the corresponding state feedback controller is given as $K_1 = \begin{bmatrix} -4.7783 & -0.8063 & -0.9794 \end{bmatrix}$, $K_2 = \begin{bmatrix} -4.4531 & 0.3942 & -4.0966 \end{bmatrix}$.

7. Conclusions

In this paper, two equivalent sets are proposed which provides a uniform method for analysis and control problems of singular systems. Necessary and sufficient conditions have been established for problems of dissipativity analysis and dissipative control of the systems. Moreover, the application of the equivalent sets is extended to admissibility and corresponding state feedback control of singular Markovian jump systems, which gives a novel admissibility condition and an effective state feedback control method. H_∞ control of time-delay singular Markovian jump systems is also addressed. A new bounded real lemma and improved H_∞ control result are given. The effectiveness and improvement of the presented method have been illustrated by numerical examples. Compared with existing results, the main contributions of this manuscript are not only presenting the new two equivalent sets, but also developing some improved results as follows: (a) Novel dissipativity and dissipative control criteria in terms of strict LMIs are proposed which are tractable and reliable in numerical computations compared with results in Masubuchi (2006) where the equality and a semi-definite matrix inequality constraints are involved; (b) New necessary and sufficient admissibilization condition is obtained for singular Markovian jump systems; (c) For singular Markovian jump systems with time-delay, a new bounded real lemma is given in terms of strict LMIs without any constraint on system matrix E.

For future works, by employing the equivalent sets technique, the observer-based sliding mode control problem for singular stochastic systems will be studied. Moreover, the method proposed in Zhang, Zhuang, and Braatz (2016) will be combined with the equivalent sets technique to investigate the switched model predictive control for singular switched systems.

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