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Stability of ψ-Hilfer impulsive fractional differential equations

J. Vanterler da C. Sousa, [1]Kishor D. Kucche, E. Capelas de Oliveira

Department of Applied Mathematics, Imecc-Unicamp, 13083-859, Campinas, SP, Brazil

[1] *Department of Mathematics, Shivaji University, Kolhapur 416 004, Maharashtra, India*

Abstract

In this paper, we investigate the sufficient conditions for existence and uniqueness of solutions and δ-Ulam-Hyers-Rassias stability of an impulsive fractional differential equation involving ψ-Hilfer fractional derivative. Fixed point approach is used to obtain our main results. **Finally, we present examples are given to illustrate the results.**

Keywords: Impulsive fractional differential equation, δ-Ulam-Hyers-Rassias, ψ-Hilfer fractional derivative, Banach fixed point.

1. Introduction

Impulsive differential equations are used to describe the evolutionary processes that abruptly change their state at a certain moment. This subject received great importance and remarkable attention from the researchers because of its rich theory [1] and applicability in various branches of science and technology. Wang and Zhang [2] investigated the existence and uniqueness of solutions to differential equations with not instantaneous impulses in a $P\beta$-normed space.

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[*]Corresponding author

Email address: ra160908@ime.unicamp.br, kdkucche@gmail.com, capelas@ime.unicamp.br (J. Vanterler da C. Sousa, [1]Kishor D. Kucche, E. Capelas de Oliveira)

Wang et al. in [3], considering the nonlinear differential equation with not instantaneous impulses obtained existence and uniqueness of solutions and introduced an interesting concept of stability, generalized β-Ulam-Hyers–Rassias.

With the expansion of the fractional calculus [4, 5, 6], the impulsive fractional differential equations gained a much attention and began to be studied, mainly due to the variety of results, from the stability study, existence to uniqueness and of impulsive differential equations in the Banach spaces [7, 8, 9, 10]. More discussions on concept of solutions, existence and uniqueness of impulsive fractional differential equations see [11].

In this paper, we apply fixed point approach to study stability of the modified impulsive fractional differential equations

$$\left\{ \begin{array}{ccc} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}x\left(t\right) & = & f\left(t,x\left(t\right)\right),\ t\in\left(s_{i},t_{i+1}\right],\ i=0,1,...,m \\ x\left(t\right) & = & g_{i}\left(t,x\left(t_{i}^{+}\right)\right),\ t\in\left(t_{i},s_{i}\right],\ i=1,2,...,m \end{array} \right. \tag{1}$$

where ${}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}\left(\cdot\right)$ is the ψ-Hilfer fractional derivative with $0<\alpha\leq 1$, $0\leq\beta\leq 1$ and $0=t_{0}=s_{0}<t_{1}\leq s_{1}\leq t_{2}<\cdots<t_{m}\leq s_{m}<t_{m+1}=T$ are prefixed numbers, $f:[0,T]\times\mathbb{R}\rightarrow\mathbb{R}$ is continuous and $g_{i}:[t_{i},s_{i}]\times\mathbb{R}\rightarrow\mathbb{R}$ is continuous for all $i=1,2,...,m$ which is not instantaneous impulses.

The motivation for the elaboration of this paper is the contribution in the stability of fractional differential equations, in particular of the impulsive type. In this sense, as the main purpose of this paper, we investigated the δ-Ulam-Hyers-Rassias stability of the impulsive fractional differential equation by employing the fixed point approach. For the properties of ψ-Hilfer fractional derivative and the basic theory of fractional differential equation involving ψ-Hilfer fractional derivative, we refer the readers to the papers of Sousa and Oliveira [6, 12].

This paper is divided as follows: in Section 2, we present the concepts of weighted and piecewise weighted function spaces. We also recall the definitions of ψ-Riemann-Liouville fractional integral, ψ-Hilfer fractional derivative and define the concept of generalized δ-Ulam-Hyers-Rassias stability. In Section 3, we investigate through the Theorem 2 the generalized δ-Ulam-Hyers-Rassias stability of the fractional differential equation. **In the section 4, we present**

examples are given to illustrate the results.

2. Preliminaries

Let $J = [0, T]$, $J' = (0, T]$ and $C(J, \mathbb{R})$ the space of continuous functions.

The piecewise weighted space $PC_{1-\gamma;\psi}(J, \mathbb{R})$ of functions x on $C_{1-\gamma;\psi}((t_k, t_{k+1}], \mathbb{R})$ is defined by

$$PC_{1-\gamma;\psi}(J, \mathbb{R}) = \left\{ \begin{array}{l} (\psi(t) - \psi(t_k) - \psi(0))^{1-\gamma} x(t) \in C_{1-\gamma;\psi}((t_k, t_{k+1}], \mathbb{R}) \text{ and} \\ \lim\limits_{t \to t_k} (\psi(t) - \psi(t_k) - \psi(0))^{1-\gamma} x(t), \text{exists for } k = 1, 2, ..., m \end{array} \right\}$$

with norm

$$\|x\|_{PC_{1-\gamma;\psi},\delta} := \max_{k=1,2,\ldots,m} \left\{ \sup_{t \in (t_k, t_{k+1}]} (\psi(t) - \psi(t_k) - \psi(0))^{1-\gamma} \|x(t)\|_\delta \right\}$$

and there exists $x(t_k^-)$ and $x(t_k^+)$, $k = 1, 2, ..., m$ with $x(t_k^-) = x(t_k^+)$. $C_{1-\gamma;\psi}(J, \mathbb{R})$ is the weighted space functions x on J and $\|\cdot\|_\delta$ is the δ-norm [3, 6]. The space $PC_{1-\gamma;\psi}(J, \mathbb{R})$ is also Banach space.

Let $n - 1 < \alpha \leq n$ with $n \in \mathbb{N}$ and let $f, \psi \in C^n([0, T], \mathbb{R})$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The ψ-Hilfer fractional derivative given by [6]

$${}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} y(t) = I_{0+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{0+}^{(1-\beta)(n-\alpha);\psi} y(t),$$

where $I_{0+}^{\xi;\psi}(\cdot)$ $(0 < \xi \leq 1)$ is the ψ-Riemann-Liouville fractional integral [6].

Theorem 1. [3, 13] *Let (X, d) be a generalized complete metric space. Assume that $\Omega : X \to X$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(\Omega^{k+1}, \Omega^k) < \infty$ for some $x \in X$, then the following are true:*

1. *The sequence $\{\Omega^k x\}$ converges to a fixed point x^* of Ω;*
2. *x^* is the unique fixed point of Ω in $\Omega^* = \{y \in X / d(\Omega^k x, y) < \infty\}$;*
3. *If $y \in X^*$, then $d(y, x^*) \leq \dfrac{1}{1-L} d(\Omega y, y)$.*

Let the space of piecewise weighted space continuous functions

$$X=\left\{g:J\rightarrow\mathbb{R}/g\in PC_{1-\gamma;\psi}\left(J,\mathbb{R}\right),\ 0\leq\gamma<1\right\},$$

with the generalized metric on X given by

$$d\left(g,h\right)=\inf\left\{C_1+C_2\in[0,\infty]\,/\left|g\left(t\right)-h\left(t\right)\right|^{\delta}\leq\left(C_1+C_2\right)\left(\varphi^{\delta}\left(t\right)+\varepsilon^{\delta}\right),\ \text{for all } t\in J\right\} \tag{2}$$

where $C_1\in\left\{C\in[0,\infty]\,/\left|g\left(t\right)-h\left(t\right)\right|^{\delta}\leq C\varphi^{\delta}\left(t\right),\ \text{for all } t\in\left(s_i,t_{i+1}\right],\ i=0,1,...,m\right\}$ and $C_2\in\left\{C\in[0,\infty]\,/\left|g\left(t\right)-h\left(t\right)\right|^{\delta}\leq C\varepsilon^{\delta},\ \text{for all } t\in\left(t_i,s_i\right],\ i=1,...,m\right\}$.

The function $x\in\widetilde{U}:=PC_{1-\gamma;\psi}\left(J,\mathbb{R}\right)\bigcap\limits_{i=0}^{m}C^1_{1-\gamma;\psi}\left(\left(s_i,t_{i+1}\right],\mathbb{R}\right)$ is a solution of the impulsive fractional differential equations

$$\left\{\begin{array}{rcl}{}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{0+}x\left(t\right)&=&f\left(t,x\left(t\right)\right),\ t\in\left(s_i,t_{i+1}\right],\ i=0,1,...,m\\ x\left(t\right)&=&g_i\left(t,x\left(t_i^+\right)\right),\ t\in\left(t_i,s_i\right]\ ,\ i=1,2,...,m\\ I^{1-\gamma;\psi}_{0+}x\left(0\right)&=&x_0\in\mathbb{R}\end{array}\right.$$

where $\gamma=\alpha+\beta(1-\alpha)$, if x satisfies $I^{1-\gamma;\psi}_{0+}x\left(0\right)=x_0,\ x\left(t\right)=g_i\left(t,x\left(t_i^+\right)\right)$, $t\in\left(t_i,s_i\right]$, $i=1,2,...,m$ and

$$\left\{\begin{array}{rcl}x\left(t\right)&=&\Psi^{\gamma}(t,0)x_0+\dfrac{1}{\Gamma\left(\alpha\right)}\displaystyle\int_0^t N^{\alpha}_{\psi}\left(t,s\right)f\left(s,x\left(s\right)\right)ds,\ t\in\left[0,t_1\right],\\ x\left(t\right)&=&g_i\left(s_i,x\left(t_i^+\right)\right)+\dfrac{1}{\Gamma\left(\alpha\right)}\displaystyle\int_{s_i}^t N^{\alpha}_{\psi}\left(t,s\right)f\left(s,x\left(s\right)\right)ds,\ t\in\left(s_i,t_{i+1}\right],\ i=1,2,...,m,\end{array}\right.$$

with $N^{\alpha}_{\psi}\left(s,t\right):=\psi'\left(s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right)^{\alpha-1}$ and $\Psi^{\gamma}(t,0)=\dfrac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}$.

Let $0<\delta\leq1$, $\xi\geq0$, $\varphi\in PC_{1-\gamma;\psi}\left(J,\mathbb{R}_+\right)$ is nondecreasing and

$$\left\{\begin{array}{rcl}\left|{}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{0+}y\left(t\right)-f\left(t,y\left(t\right)\right)\right|&\leq&\varphi\left(t\right),\ t\in\left(s_i,t_{i+1}\right],\ i=0,1,...,m,\\ \left|y\left(t\right)-g_i\left(t,y\left(t_i^+\right)\right)\right|&\leq&\xi,\ t\left(t_i,s_i\right],i=1,2,....,m.\end{array}\right. \tag{3}$$

Definition 1. *The* Eq.(1) *is generalized δ-Ulam-Hyers-Rassias stable with respect to (φ,ξ) if there exists $C_{f,\delta,g_i,\varphi}>0$ such that for each solution $y\in\widetilde{U}$ of the inequality* (3) *there exists a solution $x\in\widetilde{U}$ of the* Eq.(1) *with*

$$\left|y\left(t\right)-x\left(t\right)\right|^{\delta}\leq C_{f,\delta,g_i,\varphi}\left(\xi^{\delta}+\varphi^{\delta}\left(t\right)\right),\ t\in J.$$

A function $y \in \widetilde{U}$ is a solution of the inequality (3) if, and only if, there is $G \in \bigcap_{i=0}^{m} C^{1}_{1-\gamma;\psi}\left(\left(s_i, t_{i+1}\right], \mathbb{R}\right)$ and $g \in \bigcap_{i=0}^{m} C_{1-\gamma;\psi}\left(\left[t_i, s_i\right], \mathbb{R}\right)$, such that:

(a) $|G(t)| \leq \varphi(t)$, $t \in \bigcup_{i=0}^{m} (s_i, t_{i+1}]$ and $|g(t)| \leq \xi$, $t \in \bigcup_{i=0}^{m} (t_i, s_i]$;

(b) ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{0+} y(t) = f(t, y(t)) + G(t)$, $t \in (s_i, t_{i+1}]$, $i = 0, 1, ..., m$;

(c) $y(t) = g_i\left(t, y\left(t_i^{+}\right)\right) + g(t)$, $t \in (t_i, s_i]$, $i = 1, 2, ..., m$.

Remark 1. *If $y \in \widetilde{U}$ is a solution of the inequality (3) then y is a solution of the fractional integral inequality system:*

$$\begin{cases} \left|y(t) - g_i\left(t, y\left(t_i^{+}\right)\right)\right| \leq \xi, \ t \in (t_i, s_i], \ i = 0, 1, ..., m \\ \left|y(t) - \Psi^{\gamma}(t,0) y(0) - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t N^{\alpha}_{\psi}(t,s) f(s, y(s)) \, ds\right| \\ \leq \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t N^{\alpha}_{\psi}(t,s) \varphi(s) \, ds, \ t \in [0, t_1] \\ \left|y(t) - g_i\left(s_i, y\left(t_i^{+}\right)\right) - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_i}^t N^{\alpha}_{\psi}(t,s) f(s, y(s)) \, ds\right| \\ \leq \xi + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_i}^t N^{\alpha}_{\psi}(t,s) \varphi(s) \, ds, \ t \in (s_i, t_{i+1}], \ i = 1, 2, ..., m. \end{cases} \tag{4}$$

3. δ-Ulam-Hyers-Rassias stability

In this section, we present the main result of this paper, the stability of the type generalized $\delta-$Ulam-Hyers-Rassias for the impulsive fractional differential equation Eq.(1), by means of the Banach's the fixed point theorem.

Before investigating the main result in this paper, we list some essential conditions for proof of the theorem:

(H1) $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$;

(H2) There exists a positive constant L_f such that

$$|f(t, u_1) - f(t, u_2)| \leq L_f |u_1 - u_2|, \text{ for } t \in J \text{ and } u_1, u_2 \in \mathbb{R};$$

(H3) $g_i \in C_{1-\gamma;\psi}([t_i, s_i] \times \mathbb{R}, \mathbb{R})$ and there are positive constants L_{g_i}, $i = 1, 2, ..., m$ such that

$$|g_i(t, u_1) - g_2(t, u_2)| \leq L_{g_i} |u_1 - u_2|, \text{ for } \ t \in [t_i, s_i] \text{ and } u_1, u_2 \in \mathbb{R};$$

(H4) Let $\varphi \in C_{1-\gamma;\psi}(J, \mathbb{R}_+)$ be a nondecreasing function. There exists $C_\varphi > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t N_\psi^\alpha(t,s)\,\varphi(s)\,ds \leq c_\varphi \varphi(t), \quad \text{for } t \in J.$$

Theorem 2. *Assume the conditions* (H1)-(H4) *be satisfied. If there exists a function $y \in \widetilde{U}$ satisfying* Eq.(3)*, then there exists a unique solution $y_0 : J \to \mathbb{R}$ such that*

$$y_0(t) = \begin{cases} \Psi^\gamma(t,0)x(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t N_\psi^\alpha(t,s)\, f(s, y_0(s))\,ds, \ t \in [0, t_1] \\ g_i\left(t, y_0\left(t_i^+\right)\right), \ t \in (t_i, s_i], \ i = 1, 2, ..., m \\ g_i\left(s_i, y_0\left(t_i^+\right)\right) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_i}^t N_\psi^\alpha(t,s)\, f(s, y_0(s))\,ds, \ t \in (s_i, t_{i+1}] \\ i = 1, 2, ..., m, \end{cases} \tag{5}$$

and

$$|y(t) - y_0(t)|^\delta \leq \frac{\left(1 + C_\varphi^\delta\right)\left(\varphi^\delta(t) + \xi^\delta\right)}{1 - \Phi}, \ t \in J \tag{6}$$

where

$$\Phi := \max_{i=1,2,\ldots,m} \left\{ L_{g_i}^\delta + L_f^\delta C_\varphi^\delta \right\}. \tag{7}$$

Proof. For prove of this result, consider the operator $\Omega : X \to X$ given by

$$\Omega x(t) = \begin{cases} \Psi^\gamma(t,0)x(0) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t N_\psi^\alpha(t,s)\, f(s, x_0(s))\,ds, \ t \in [0, t_1] \\ g_i\left(t, x_0\left(t_i^+\right)\right), \ t \in (t_i, s_i], \ i = 1, 2, ..., m \\ g_i\left(s_i, x_0\left(t_i^+\right)\right) + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{s_i}^t N_\psi^\alpha(t,s)\, f(s, x_0(s))\,ds, \\ \ t \in (s_i, t_{i+1}], \ i = 1, 2, ..., m. \end{cases} \tag{8}$$

for all $x \in X$ and $t \in [0, T]$. Note that, Ω is a well defined according with (H1).

In order to use Banach's fixed point theorem, we prove that Ω is strictly contractive on X, in three cases. Note that

$$|g(t) - h(t)|^\delta \leq \begin{cases} C_1 \varphi^\delta(t), \ t \in (s_i, t_{i+1}], \ i = 0, 1, ..., m \\ C_2 \xi^\delta, \ t \in (t_i, s_i], \ i = 1, ..., m \end{cases}$$

is equivalent to

$$|g(t)-h(t)| \le \begin{cases} C_1^{\delta}\varphi(t), \ t \in (s_i, t_{i+1}], \ i = 0, 1, ..., m \\ C_2^{\delta}\xi, \ t \in (t_i, s_i], \ i = 1, ..., m \end{cases} \tag{9}$$

Using the definition of Ω in Eq.(8), (H2), (H3) and Eq.(9), we have the following cases:

Case 1: For $t \in [0, t_1]$, and by the hypothesis (H2), (H4) and Eq.(9), we get

$$\begin{aligned} |\Omega g(t) - \Omega h(t)|^{\delta} &\le \left| \frac{1}{\Gamma(\alpha)} \int_0^t N_{\psi}^{\alpha}(t,s) f(s, g(s))\, ds - \frac{1}{\Gamma(\alpha)} \int_0^t N_{\psi}^{\alpha}(t,s) f(s, h(s))\, ds \right|^{\delta} \\ &\le L_f^{\delta} \left(\frac{1}{\Gamma(\alpha)} \int_0^t N_{\psi}^{\alpha}(t,s) |g(s) - h(s)|\, ds \right)^{\delta} \le L_f^{\delta} C_1 c_{\varphi}^{\delta} \varphi(t)^{\delta}. \end{aligned}$$

Case 2: For $t \in (t_i, s_i]$, and by the hypothesis (H3) and Eq.(8), we obtain

$$|\Omega g(t) - \Omega h(t)|^{\delta} \le \left| g_i\left(t, g\left(t_i^+\right)\right) - g_i\left(t, h\left(t_i^+\right)\right) \right|^{\delta} \le \left(L_{g_i} \left| g\left(t_i^+\right) - h\left(t_i^+\right) \right| \right)^{\delta} \le L_{g_i}^{\delta} C_2 \xi^{\delta}.$$

Case 3: For $t \in (s_i, t_{i+1}]$ and using (H1), (H2), (H3) and Eq.(8), we have

$$\begin{aligned} &|\Omega g(t) - \Omega h(t)|^{\delta} \\ \le\ & \left| g_i\left(s_i, g\left(t_i^+\right)\right) - g_i\left(s_i, h\left(t_i^+\right)\right) \right|^{\delta} + \left| \frac{1}{\Gamma(\alpha)} \int_{s_i}^t N_{\psi}^{\alpha}(t,s) \left(f(s, g(s)) - f(s, h(s))\right) ds \right|^{\delta} \\ \le\ & L_{g_i}^{\delta} C_2 \xi^{\delta} + L_f^{\delta} C_1 \left(\frac{1}{\Gamma(\alpha)} \int_0^t N_{\psi}^{\alpha}(t,s) \varphi(s)\, ds \right)^{\delta} \le \left(L_{g_i}^{\delta} + L_f^{\delta} c_{\varphi}^{\delta} \right) (C_1 + C_2) \left(\varphi^{\delta}(t) + \xi^{\delta} \right). \end{aligned}$$

Then, we obtain

$$\begin{aligned} |\Omega g(t) - \Omega h(t)|^{\delta} &\le \max_{i=1,2,\dots,m} \left(L_{g_i}^{\delta} + L_f^{\delta} C_{\varphi}^{\delta} \right) (C_1 + C_2) \left(\varphi^{\delta}(t) + \xi^{\delta} \right) \\ &= \Phi (C_1 + C_2) \left(\varphi^{\delta}(t) + \xi^{\delta} \right), \ t \in J. \end{aligned}$$

Hence, we get $d(\Omega g, \Omega h) \le \Phi d(g, h)$ for any $g, h \in X$ and with the condition Eq.(7).

Now, we take $g_0 \in X$ and from the piecewise continuous property of g_0 and Ωg_0, then there exists a constant $0 < G_1 < \infty$ so that

$$\begin{aligned} |\Omega g_0(t) - g_0(t)|^{\delta} &= \left| \Psi^{\gamma}(t, 0) x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t N_{\psi}^{\alpha}(t,s) f(s, g_0(s))\, ds - g_0(t) \right|^{\delta} \\ &\le G_1 \varphi^{\delta}(t) \le G_1 \left(\varphi^{\delta}(t) + \xi^{\delta} \right), \ t \in [0, t_1]. \end{aligned}$$

On the other hand, also G_2 and G_3 with $0 < G_2 < \infty$ and $0 < G_3 < \infty$, such that,

$$\begin{aligned} \left|\Omega g_0(t) - g_0(t)\right|^{\delta} &= \left|g_i\left(t, g_0\left(t_i^{+}\right)\right) - g_0(t)\right|^{\delta} \\ &\leq G_2\xi^{\delta} \leq G_2\left(\varphi^{\delta}(t) + \xi^{\delta}\right),\ t \in (t_i, s_i],\ i = 1, 2, ..., m \end{aligned}$$

and

$$\begin{aligned} \left|\Omega g_0(t) - g_0(t)\right|^{\delta} &= \left|g_i\left(s_i, g_0\left(t_i^{+}\right)\right) + \frac{1}{\Gamma(\alpha)}\int_{s_i}^{t} N_{\psi}^{\alpha}(t,s)\, f(s, g_0(s))\, ds - g_0(t)\right|^{\delta} \\ &\leq G_3\left(\varphi^{\delta}(t) + \xi^{\delta}\right),\ t \in (s_i, t_{i+1}],\ i = 1, 2, ..., m \end{aligned}$$

since f, g_i and g_0 are bounded on J and $\varphi(\cdot) + \xi^{\delta} > 0$. In this sense, Eq.(2) implies that $d(\Omega g_0, g_0) < \infty$.

Note that, exists a continuous function $y_0 : J \to \mathbb{R}$ such that $\Omega^n y_0 \to y_0$ in (X, d) as $n \to \infty$ and $\Omega y_0 = y_0$, that is y_0 satisfies Eq.(8) for every $t \in J$ (Theorem 1).

For finally the proof this theorem, we check that $0 < C_g < \infty$ such that

$$\left|g_0(t) - g(t)\right|^{\delta} \leq C_g\left(\varphi^{\delta}(t) + \xi^{\delta}\right),\ \text{for any } t \in J$$

and assuming that g, g_0 are bounded on J and $\min\limits_{t\in J}\left(\varphi^{\delta}(t) + \xi^{\delta}\right) > 0$.

Then, we get $d(g_0, g) < \infty$ for all $g \in X$, that is $X = \{g \in X / d(g_0, g) < \infty\}$. Therefore, y_0 is the unique solution continuous with the property Eq.(8).

On the other hand, using (H1)-(H4) and Eq.(4) it follows that

$$d(y, \Omega y) \leq 1 + C_{\varphi}^{\delta}. \tag{10}$$

Thus, from Eq.(10), we have

$$d(y, y_0) \leq \frac{d(\Omega y, y)}{1 - \Phi} \leq \frac{1 + C_{\varphi}^{\delta}}{1 - \Phi},$$

which means that Eq.(6) is true for $t \in J$. $\square$

4. Examples

In this section, we will present two examples for illustrate the result obtained before.

Consider

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}x(t) & = & \dfrac{|x(t)|}{8+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha\right]},\ t\in(0,1] \\ x(t) & = & \dfrac{|x(1+)|}{\left(3+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha-1\right]\right)(1+|x(1+)|)},\ t\in(1,2] \end{cases} \tag{11}$$

$$\begin{cases} \left|{}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}y(t)-\dfrac{|y(t)|}{8+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha\right]}\right| \le \mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha\right] & ,\ t\in(0,1] \\ \left|y(t)-\dfrac{|y(1+)|}{\left(3+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha-1\right]\right)(1+|y(1+)|)}\right| \le 1 & ,\ t\in(1,2] \end{cases}$$

where ${}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative and $\mathbb{E}_\alpha(\cdot)$ is a Mittag-Leffler function.

Now, consider $\delta=1/2$, $J=[0,2]$ and $0=t_0=s_0<t_1=1<s_1=2$. Denote

$$f(t,x(t))=\frac{|x(t)|}{8+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha\right]},\ \text{with } L_f=1/9 \text{ for } t\in[0,1]$$

and

$$g_1(t,x(t))=\frac{|x(1+)|}{\left(3+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha-1\right]\right)(1+|x(1+)|)},\ \text{with } L_{g_1}=1/4 \text{ for } t\in(1,2].$$

We put $\varphi(t)=\mathbb{E}_\alpha(\psi(t)-\psi(0))^\alpha$ and $\varepsilon=1$. Then, we choose $c_\varphi=1$ satisfying the condition $I_{0+}^{\alpha;\psi}\left(\mathbb{E}_\alpha(\psi(t)-\psi(0))^\alpha\right)\le\mathbb{E}_\alpha(\psi(t)-\psi(0))^\alpha$. Moreover, we have $L_{g_1}^\delta+L_f^\delta c_\varphi^\delta=\dfrac{5}{6}<1$.

Now, all assumptions of Theorem 2 are satisfying. Then, Eq.(11) has a unique solution $y_0:[0,2]\to\mathbb{R}$ such that

$$y_0(t)=\begin{cases} \Psi^\gamma(y,0)\,x(0)+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t N_\psi^\alpha(t,s)\frac{|y_0(s)|}{8+\mathbb{E}_\alpha\left[(\psi(s)-\psi(0))^\alpha\right]}ds,\ t\in[0,1] \\ \dfrac{|y(1+)|}{\left(3+\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha-1\right]\right)(1+|y(1+)|)},\ t\in(1,2] \end{cases}$$

and

$$|y(t)-y_0(t)|\le 12\left\{\left(\mathbb{E}_\alpha\left[(\psi(t)-\psi(0))^\alpha\right]\right)^{1/2}+1\right\},\ \forall t\in[0,2].$$

Now, let's give an example in which $f(t,x(t))\neq x(t)$, i.e., doesn't involve the function $x(t)$. Then, choosing $\psi(t)=t$ and taking the

limit $\beta \to 0$ on both sides of Eq.(11), we have the impulsive fractional differential equation with respect to the Caputo fractional derivative.

Consider

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) & = & \dfrac{1}{8+\mathbb{E}_{\alpha}(t^{\alpha})},\ t\in(0,1] \\ x(t) & = & \dfrac{1}{6+2\mathbb{E}_{\alpha}(t^{\alpha}-1)},\ t\in(1,2] \end{cases} \tag{12}$$

$$\begin{cases} \left|{}^{C}D_{0+}^{\alpha}y(t)-\dfrac{1}{8+\mathbb{E}_{\alpha}(t^{\alpha})}\right| \leq \mathbb{E}_{\alpha}(t^{\alpha}) & ,\ t\in(0,1] \\ \left|y(t)-\dfrac{1}{6+2\mathbb{E}_{\alpha}(t^{\alpha}-1)}\right| \leq 1 & ,\ t\in(1,2] \end{cases}$$

where ${}^{C}D_{0+}^{\alpha}(\cdot)$ is the Caputo fractional derivative.

Let $\delta = 1/2$, $J=[0,2]$ and $0=t_0=s_0<t_1=1<s_1=2$. Denote

$$f(t,x(t)) = \frac{1}{8+\mathbb{E}_{\alpha}(t^{\alpha})},\ \text{with } L_f = \frac{1}{\sqrt{2}} \text{ for } t\in[0,1]$$

and

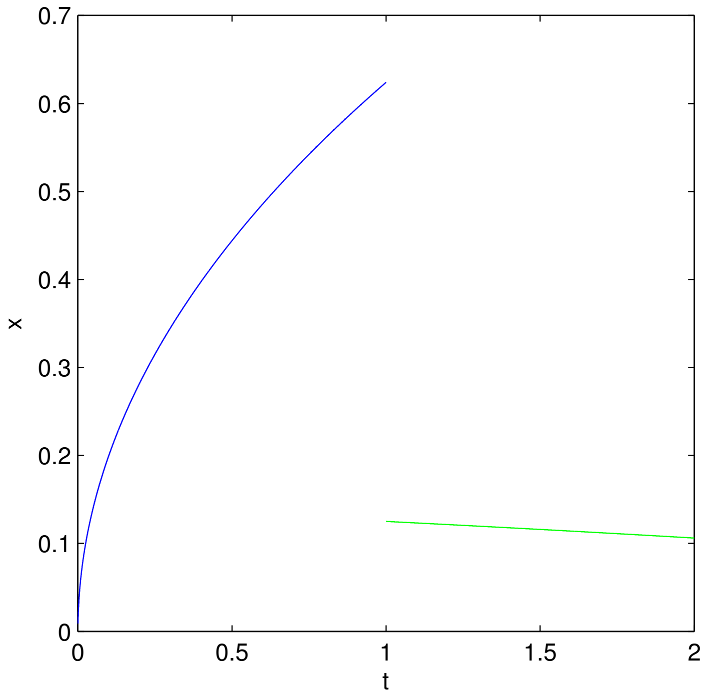
$$g_1(t,x(t)) = \frac{1}{6+2\mathbb{E}_{\alpha}(t^{\alpha}-1)},\ \text{with } L_{g_1} = \frac{1}{\sqrt{3}} \text{ for } t\in(1,2].$$

115 We put $\varphi(t)=\mathbb{E}_{\alpha}(t^{\alpha})$ and $\varepsilon = 1$. Then, we choose $c_{\varphi}=1$ satisfying the condition $I_{0+}^{\alpha;\psi}(\mathbb{E}_{\alpha}t^{\alpha}) \leq \mathbb{E}_{\alpha}(t^{\alpha})$. Moreover, $L_{g_1}^{\delta}+L_f^{\delta}c_{\varphi}^{\delta} \approx 0.67821 < 1$.

Now, all assumptions of Theorem 2 are satisfied. Then, Eq.(12) has a unique solution $y_0:[0,2]\to\mathbb{R}$ such that

$$y_0(t) = \begin{cases} \dfrac{t^{1/2}}{\sqrt{\pi}}x(0) + \dfrac{1}{\sqrt{\pi}}\displaystyle\int_0^t \dfrac{(t-s)^{-1/2}}{8+\mathbb{E}_{1/2}\left(s^{1/2}\right)}ds,\ t\in[0,1] \\ \dfrac{1}{6+2\mathbb{E}_{1/2}\left(t^{1/2}-1\right)},\ t\in(1,2] \end{cases}.$$

Figure 1: The numerical solution of Eq.(12).



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