Journal of Sound and Vibration ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

The decoupling of second-order linear systems with a singular mass matrix

Daniel T. Kawano [a,*], Matthias Morzfeld [b], Fai Ma [c]

[a] Department of Mechanical Engineering, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, USA
[b] Department of Mathematics, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA
[c] Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

ARTICLE INFO

Article history:
Received 12 July 2012
Received in revised form
31 July 2013
Accepted 5 August 2013
Handling Editor: S. Ilanko

ABSTRACT

It was demonstrated in earlier work that a nondefective, linear dynamical system with an invertible mass matrix in free or forced motion may be decoupled in the configuration space by a real and isospectral transformation. We extend this work by developing a procedure for decoupling a linear dynamical system with a singular mass matrix in the configuration space, transforming the original differential-algebraic system into decoupled sets of real, independent, first- and second-order differential equations. Numerical examples are provided to illustrate the application of the decoupling procedure.

1. Introduction

Coordinate coupling in viscously damped, linear dynamical systems presents challenges for computational efficiency and for understanding the underlying nature of the system response. Because of the difficulties arising from coupling in both practical and theoretical pursuits, much attention has been given to the problem of decoupling a linear dynamical system through simultaneous diagonalization of the coefficient matrices defining the system in terms of its inertia and viscoelasticity. The equation of motion governing the response of a second-order linear system with n degrees of freedom has the matrix-vector representation

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t), \tag{1}$$

where the order n matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real, symmetric, and represent the mass, damping, and stiffness properties, respectively. Generally, the system matrices are not diagonal, resulting in coupling among the coordinates. The n-dimensional column vector $\mathbf{x}(t)$ specifies the system coordinates, and $\mathbf{f}(t)$ denotes the applied forcing. We assume that system (1) is nondefective (i.e., there exists a complete set of system eigenvectors), and the damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are taken to be positive definite. In most cases, the mass matrix $\mathbf{M}$ is also positive definite, but under certain circumstances, it is possible to generate a mass matrix that is singular. For example, singular mass matrices arise in the analysis of constrained mechanical systems when more than the minimum number of required coordinates are used for modeling to simplify obtaining the governing equations (e.g., see [1]). Also, a lumped-mass approach to modeling may result in a singular mass matrix if certain degrees of freedom have no inertia associated with them (e.g., see [2]). In addition, when a mass matrix $\mathbf{M}$ is invertible but contains very small terms such that $\mathbf{M}$ is ill-conditioned, it is common practice to set these terms to zero, rendering $\mathbf{M}$ singular. Finally, as discussed by Balakrishnan in [3], modeling of smart structures (that is, beams

* Corresponding author. Tel.: +1 812 877 8813.
E-mail address: kawano@rose-hulman.edu (D.T. Kawano).

0022-460X/$ - see front matter © 2013 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.jsv.2013.08.005

Please cite this article as: D.T. Kawano, et al., The decoupling of second-order linear systems with a singular mass matrix, *Journal of Sound and Vibration* (2013), http://dx.doi.org/10.1016/j.jsv.2013.08.005

with self-straining materials, i.e., piezoelectric strips) may yield a differential-algebraic system of equations with a singular mass matrix. We shall focus our attention on the singular case when the mass matrix $\mathbf{M}$ for system (1) is positive semi-definite with rank $r<n$. Since $\mathbf{M}$ is singular, the equation of motion (1) is characterized as a differential-algebraic equation, which has certain properties that often make such a system a challenge to solve numerically (e.g., see [4]).

The purpose of this paper is to develop a well-defined decoupling procedure in the n-dimensional configuration space such that the coupled differential-algebraic system (1) is transformed into

$$\mathbf{A}_2\ddot{\mathbf{p}}(t)+\mathbf{A}_1\dot{\mathbf{p}}(t)+\mathbf{A}_0\mathbf{p}(t)=\mathbf{g}(t), \tag{2}$$

where the order n coefficient matrices $\mathbf{A}_i$ $(i=1,\ 2,\ 3)$ are real and diagonal, and the leading coefficient matrix $\mathbf{A}_2$ is singular with the same rank r as the mass matrix $\mathbf{M}$. For this to be the case, $\mathbf{A}_2$ must have $n-r$ zeros on its diagonal, and hence the decoupled system (2) is composed of a set of r real, independent, second-order differential equations and a set of $n-r$ real, independent, first-order differential equations, the latter corresponding to the zeros on the diagonal of $\mathbf{A}_2$. The coordinates and forcing associated with the decoupled equations are denoted by $\mathbf{p}(t)$ and $\mathbf{g}(t)$, respectively. A major advantage of the decoupling process is that it transforms the differential-algebraic system (1) into a set of independent differential equations that is far simpler to solve numerically. The work we present herein represents an extension of previous efforts [5–8] in decoupling nondefective systems with an invertible mass matrix $\mathbf{M}$.

The organization of this paper is as follows. In Section 2, we provide preliminary information about decoupling an undamped system (1) (i.e., when system (1) has no damping matrix $\mathbf{C}$), difficulties that arise when viscous damping effects are included, and the limitations and inadequacies associated with current techniques for analyzing a damped system (1). A review of decoupling nondefective systems with an invertible mass matrix $\mathbf{M}$ in free and forced motion is provided as background information for the decoupling methodology developed thereafter for systems with singular $\mathbf{M}$. We demonstrate in Section 3 how the homogeneous form of the coupled differential-algebraic equation (1) may be transformed into the unforced decoupled system (2) and develop a decoupling transformation in the n-dimensional configuration space that recovers the response of (1) from the solution of (2). We repeat this analysis for forced motion in Section 4. Examples are provided in Section 5 to illustrate the decoupling process. Finally, we summarize the major results of this paper in Section 6.

2. Preliminaries

We shall begin our treatment of systems with a singular mass matrix by first discussing how the undamped system may be decoupled and why including viscous damping introduces difficulties for decoupling. We will then briefly review the highlights of prior work on decoupling a nondefective system with an invertible mass matrix in free and forced motion since this information will be helpful when tackling the singular case. In the equations that follow, the identity matrix, zero matrix, and zero vector are denoted by $\mathbf{I}$, $\mathbf{O}$, and $\mathbf{0}$, respectively. When a subscript is omitted, it is implied that the identity and zero matrices are square and of size n, and the zero vector is of length n. Otherwise, as an example, $\mathbf{O}_\alpha$ is an order α zero matrix, while $\mathbf{O}_{\alpha\times\beta}$ denotes an $\alpha\times\beta$ matrix of zeros.

2.1. Decoupling of an undamped system with a singular mass matrix

Associated with the undamped form of the differential-algebraic system (1) (i.e., when the damping matrix $\mathbf{C}=\mathbf{O}$) is the generalized eigenvalue problem (e.g., see [9])

$$\lambda\mathbf{M}\mathbf{u}=\mathbf{K}\mathbf{u}. \tag{3}$$

Solution of the generalized eigenvalue problem (3) yields n eigenvalues λ_j $(j=1,2,\ldots,n)$, of which r are real and positive, and the remaining $n-r$ are infinite because the stiffness matrix $\mathbf{K}$ is positive definite and the mass matrix $\mathbf{M}$ is positive semi-definite with rank $r<n$. The real eigenvectors $\mathbf{u}_j$ associated with the r finite eigenvalues are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$. Make the substitution $\mu=1/\lambda$ in Eq. (3) to obtain

$$\mathbf{M}\mathbf{u}=\mu\mathbf{K}\mathbf{u}. \tag{4}$$

Thus, infinite eigenvalues $\lambda=\infty$ of the generalized eigenvalue problem (3) are zero eigenvalues $\mu=0$ of Eq. (4). Consequently, the eigenvectors $\mathbf{u}_j$ that correspond to the $n-r$ infinite eigenvalues are (linearly independent) vectors in the null space of $\mathbf{M}$. Physically, an infinite eigenvalue corresponds to a static mode of vibration in which all system components with nonzero mass are at rest, while only those components with zero mass may be in motion.

For a nondefective equation of motion (1), the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, because they are symmetric, may be diagonalized by a congruence transformation in the n-dimensional configuration space, where the corresponding transformation matrix $\mathbf{U}$ is constructed from the undamped system's eigenvectors $\mathbf{u}_j$ (see Theorem 6 in Chapter 12 of [10]). This procedure is conceptually similar to classical modal analysis for a system with a positive definite mass matrix $\mathbf{M}$ (e.g., see [11]). For the case when $\mathbf{M}$ is singular, suppose the eigenvectors of the r finite eigenvalues are normalized according to $\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_j=\delta_{ij}$ $(i,j=1,2,\ldots,r)$, where the superscript T is the transpose operator and δ_{ij} represents the Kronecker delta. Let the first r columns of $\mathbf{U}$ contain these eigenvectors, and hence the remaining $n-r$ columns consist of the eigenvectors for the infinite eigenvalues (i.e., those linearly independent vectors in the null space of $\mathbf{M}$). It is straightforward to verify that $\mathbf{M}$ is always diagonalized by congruence transformation via $\mathbf{U}$ such that $\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}=\mathbf{I}_r \oplus \mathbf{O}_{n-r}$. Unfortunately, not all choices of the eigenvectors for the infinite eigenvalues are such that $\mathbf{K}$ is diagonalized according to $\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}$. Once an appropriate transformation matrix $\mathbf{U}$ has been determined, the (undamped) coupled and decoupled systems are related by the real, linear, time-invariant coordinate transformation $\mathbf{x}(t)=\mathbf{U}\mathbf{p}(t)$:

$$\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}\ddot{\mathbf{p}}(t)+\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}\mathbf{p}(t)=\mathbf{U}^{\mathrm{T}}\mathbf{f}(t), \tag{5}$$

where it is clear that $\mathbf{A}_2=\mathbf{U}^{\mathrm{T}}\mathbf{M}\mathbf{U}$, $\mathbf{A}_0=\mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U}$, and $\mathbf{g}(t)=\mathbf{U}^{\mathrm{T}}\mathbf{f}(t)$ when Eq. (5) is compared to the decoupled form (2) with no damping. A closer inspection of Eq. (5) reveals that the decoupled system consists of r real, independent, second-order differential equations and $n-r$ real solutions that are linear combinations of the components of the applied forcing $\mathbf{f}(t)$.

2.2. Limitations and inadequacies

Including the effects of viscous damping introduces difficulties for decoupling the associated equation of motion. While it is always possible to diagonalize the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ by congruence transformation, it is generally the case that the damping matrix $\mathbf{C}$ is not diagonalized (i.e., $\mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}$ is not diagonal). When $\mathbf{M}$ is invertible, it is known that the system matrices must satisfy the commutativity relationship $\mathbf{C}\mathbf{M}^{-1}\mathbf{K}=\mathbf{K}\mathbf{M}^{-1}\mathbf{C}$ for the equation of motion to be decoupled by congruence transformation in the n-dimensional configuration space using the eigenvectors of the undamped system (i.e., by classical modal analysis) [12]. If satisfied, the system is said to be classically damped because it may be decoupled by classical modal analysis. Unfortunately, an analogous condition when $\mathbf{M}$ is singular for simultaneous diagonalization of all system matrices does not yet exist, and to be clear, the goal of this paper is not to attempt to provide such a condition. We instead choose to make knowledge of this condition irrelevant by extending the decoupling methodology in [5–8] for a nondefective system of the form (1) with an invertible mass matrix to one with a singular mass matrix so that it may be decoupled in the configuration space under any circumstance. However, we do hope that our efforts here would lead to discovering a necessary and sufficient condition for decoupling viscously damped systems with a singular mass matrix by congruence transformation.

Of course, one could consider decoupling the damped system (1) by transforming it into a first-order form in the $2n$-dimensional state space and applying a complex congruence transformation to the larger state equation. However, doing so requires greater computational effort than analysis in the configuration space. Moreover, the decoupled coordinates are generally complex, greatly diminishing insight into the nature of the solution since it is impossible to connect the $2n$ state variables with physical quantities such as displacements and velocities. An alternative approach developed by Bhat and Bernstein [13] extends Guyan reduction to viscously damped systems with a singular mass matrix and decomposes an unforced coupled system (1) into three subsystems: one that is algebraic, another that is a set of first-order differential equations, and the last being a set of second-order differential equations. Unfortunately, these subsystems are generally coupled, and even if they are independent of one another, there is no guarantee that the first- and second-order differential equations themselves will be decoupled. Furthermore, this reduction technique is limited to homogeneous systems. Similarly, Newland demonstrated that the original n-degree-of-freedom, differential-algebraic system (1) may be rewritten as a first-order differential equation through a reduction method (see [Chapter 6, 14]), but his approach is limited to homogenous systems with a mass matrix of rank $n-1$. Finally, Garvey et al. [15,16] have recently proposed decoupling through what they term "structure-preserving transformations." While their work focuses on the case when the mass matrix $\mathbf{M}$ is invertible, the authors do briefly mention how their decoupling methodology may be extended to include the case when $\mathbf{M}$ is singular. However, their procedure involves an iterative process by which a singular mass matrix $\mathbf{M}$ is replaced with an invertible matrix $\mathbf{M}+\epsilon\Delta\mathbf{M}$, where ϵ is a small parameter and $\Delta\mathbf{M}$ is a constant perturbation matrix. Diagonalizing transformations for decreasing values of ϵ are obtained, and the value of the transformation is deduced for $\epsilon=0$ from some visible pattern in the results. What we provide in this paper is an explicit form for this transformation.

2.3. Review of decoupling nondefective systems with an invertible mass matrix

Before we address how to decouple a nondefective system (1) with a singular mass matrix $\mathbf{M}$, it is instructive to briefly review the methodology detailed in [5,6,8] by which any nondefective system of the form (1) with invertible $\mathbf{M}$ is decoupled, as this information will form the basis for generalizing the decoupling procedure to the singular case. The interested reader may find additional details on decoupling nondefective systems with an invertible mass matrix in [7].

As demonstrated in [5,6,8], a process referred to as phase synchronization is able to decouple a nondefective system (1) with an invertible mass matrix $\mathbf{M}$ into system (2) given spectral data, which are obtained by solving the quadratic eigenvalue problem (e.g., see [17–19]) associated with system (1):

$$(\mathbf{M}\lambda^2+\mathbf{C}\lambda+\mathbf{K})\mathbf{v}=\mathbf{0}. \tag{6}$$

Because the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are real, of the $2n$ eigenvalues λ_j $(j=1,\ 2,\ \dots,\ 2n)$ and eigenvectors $\mathbf{v}_j$ generated by the quadratic eigenvalue problem (6), n of the eigenvalues λ_k $(k=1,\ 2,\ \dots,\ n)$ and their corresponding eigenvectors $\mathbf{v}_k$ may be paired with the remaining n eigenvalues $\hat{\lambda}_k$ and their associated eigenvectors $\hat{\mathbf{v}}_k$, where the ornamenting hat denotes a pairing of complex conjugate or distinct real eigensolutions, whichever is appropriate (see [6,8] for further details of pairing real eigensolutions). It was shown in [5,6,8] that, when the paired eigenvectors $\mathbf{v}_k$ and $\hat{\mathbf{v}}_k$ are normalized according to

$$2\lambda_k \mathbf{v}_k^{\mathrm{T}} \mathbf{M} \mathbf{v}_k + \mathbf{v}_k^{\mathrm{T}} \mathbf{C} \mathbf{v}_k = \lambda_k - \hat{\lambda}_k, \tag{7}$$

$$2\hat{\lambda}_k \hat{\mathbf{v}}_k^{\mathrm{T}} \mathbf{M} \hat{\mathbf{v}}_k + \hat{\mathbf{v}}_k^{\mathrm{T}} \mathbf{C} \hat{\mathbf{v}}_k = \hat{\lambda}_k - \lambda_k \tag{8}$$

(which reduces to normalization with respect to the mass matrix $\mathbf{M}$ when system (1) is undamped or classically damped [20]), application of phase synchronization yields the coefficient matrices

$$\mathbf{A}_2 = \mathbf{I}, \quad \mathbf{A}_1 = -(\boldsymbol{\Lambda} + \widehat{\boldsymbol{\Lambda}}), \quad \mathbf{A}_0 = \boldsymbol{\Lambda}\widehat{\boldsymbol{\Lambda}} \tag{9}$$

for the decoupled system (2), where $\boldsymbol{\Lambda}$ and $\widehat{\boldsymbol{\Lambda}}$ are order n diagonal matrices of the paired eigenvalues λ_k and $\hat{\lambda}_k$, respectively:

$$\boldsymbol{\Lambda} = \bigoplus_{k=1}^{n} \lambda_k, \quad \widehat{\boldsymbol{\Lambda}} = \bigoplus_{k=1}^{n} \hat{\lambda}_k. \tag{10}$$

In addition, the decoupled system's excitation $\mathbf{g}(t)$ is given by

$$\mathbf{g}(t) = \mathbf{T}_1^{\mathrm{T}} \mathbf{f}(t) + \mathbf{T}_2^{\mathrm{T}} \dot{\mathbf{f}}(t), \quad \mathbf{T}_1 = (\mathbf{V}\widehat{\boldsymbol{\Lambda}} - \widehat{\mathbf{V}}\boldsymbol{\Lambda})(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1}, \quad \mathbf{T}_2 = (\widehat{\mathbf{V}} - \mathbf{V})(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1}, \tag{11}$$

where $\mathbf{V}$ and $\widehat{\mathbf{V}}$ are order n matrices whose columns contain the paired eigenvectors $\mathbf{v}_k$ and $\hat{\mathbf{v}}_k$, respectively:

$$\mathbf{V} = [\mathbf{v}_1 \mid \cdots \mid \mathbf{v}_n], \quad \widehat{\mathbf{V}} = [\hat{\mathbf{v}}_1 \mid \cdots \mid \hat{\mathbf{v}}_n]. \tag{12}$$

When phase synchronization is formulated in the state space, the response $\mathbf{x}(t)$ of the coupled system (1) is related to the decoupled solution $\mathbf{p}(t)$ by the state transformation

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V}\boldsymbol{\Lambda} & \widehat{\mathbf{V}\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \widehat{\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) - \mathbf{T}_2^{\mathrm{T}} \mathbf{f}(t) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) - \mathbf{T}_2^{\mathrm{T}} \mathbf{f}(t) \end{bmatrix}, \tag{13}$$

which provides a convenient means for connecting the initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ for the decoupled system to the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ of the original system when evaluated at time $t=0$. Upon solving the decoupled system (2), the response $\mathbf{x}(t)$ of the coupled system (1) may be obtained exactly from the decoupled solution $\mathbf{p}(t)$ in the n-dimensional configuration space by

$$\mathbf{x}(t) = \mathbf{T}_1 \mathbf{p}(t) + \mathbf{T}_2 \dot{\mathbf{p}}(t) - \mathbf{T}_2 \mathbf{T}_2^{\mathrm{T}} \mathbf{f}(t). \tag{14}$$

To summarize, a coupled, nondefective, n-degree-of-freedom system (1) with an invertible mass matrix may be decoupled by phase synchronization into a set of n real, independent, second-order differential equations. All parameters required for decoupling are obtained through solution of the quadratic eigenvalue problem (6), and the decoupling transformation itself is isospectral because the eigenvalues of system (1) and their multiplicities are preserved upon transformation. Finally, in the event that system (1) is classically damped (i.e., when $\mathbf{CM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{C}$), the eigenvectors $\mathbf{v}_k = \hat{\mathbf{v}}_k$ correspond to the classical normal modes $\mathbf{u}_k$ of the undamped system if they have been normalized in accordance with Eqs. (7) and (8), reducing Eq. (14) to the classical modal transformation $\mathbf{x}(t) = \mathbf{U}\mathbf{p}(t)$ (e.g., see [9]).

2.4. Some additional preliminary details

A few additional details that were not presented in [5–8] will later prove useful in our treatment of decoupling nondefective systems with a singular mass matrix. First, the response $\mathbf{x}(t)$ of the forced system (1) may be written as (e.g., see [18])

$$\mathbf{x}(t) = \mathbf{V}_x e^{\mathbf{J}_x t} \left(\mathbf{c}_x + \int_0^t e^{-\mathbf{J}_x s} \mathbf{Y}_x \mathbf{f}(s)\, ds \right), \tag{15}$$

for which $\mathbf{J}_x$ is an order $2n$ Jordan matrix of the system eigenvalues on the diagonal, $\mathbf{V}_x$ is an $n \times 2n$ matrix of the associated (right) eigenvectors, $\mathbf{c}_x$ is a $2n$-dimensional vector of coefficients related to the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$, and $\mathbf{Y}_x$ is a $2n \times n$ matrix containing the system's left eigenvectors. Using the Leibniz integral rule and exploiting biorthogonality of the right and left eigevectors (e.g., see [11,19]), the derivative of Eq. (15) is

$$\dot{\mathbf{x}}(t) = \mathbf{V}_x \mathbf{J}_x e^{\mathbf{J}_x t} \left(\mathbf{c}_x + \int_0^t e^{\mathbf{J}_x s} \mathbf{Y}_x \mathbf{f}(s)\, ds \right). \tag{16}$$

Expressing Eqs. (15) and (16) as a state equation and evaluating it at time $t=0$ yields

$$\begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} \mathbf{c}_x = \mathbf{S}_x \mathbf{c}_x. \tag{17}$$

Analogous to Eq. (15), the decoupled forced response $\mathbf{p}(t)$ has the solution

$$\mathbf{p}(t) = \mathbf{V}_p e^{\mathbf{J}_p t} \left(\mathbf{c}_p + \int_0^t e^{-\mathbf{J}_p s} \mathbf{Y}_p \mathbf{g}(s)\, ds \right). \tag{18}$$

Here, $\mathbf{J}_p$ is an order $2n$ Jordan matrix containing the system eigenvalues, $\mathbf{V}_p$ is an $n\times 2n$ matrix of (right) eigenvectors that essentially pairs the eigenvalues according to a specified pairing scheme, the $2n$-long vector $\mathbf{c}_p$ of coefficients is determined by applying the decoupled system's initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$, and $\mathbf{Y}_p$ is a $2n\times n$ matrix of the corresponding left eigenvectors. Similar to Eq. (16), the decoupled response's derivative

$$\dot{\mathbf{p}}(t)=\mathbf{V}_p\mathbf{J}_p e^{\mathbf{J}_p t}\left(\mathbf{c}_p+\int_0^t e^{\mathbf{J}_p s}\mathbf{Y}_p\mathbf{g}(s)\,\mathrm{d}s\right). \tag{19}$$

Using Eqs. (18) and (19) to write the initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ in the form of a state equation,

$$\begin{bmatrix}\mathbf{p}(0)\\ \dot{\mathbf{p}}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_p\\ \mathbf{V}_p\mathbf{J}_p\end{bmatrix}\mathbf{c}_p=\mathbf{S}_p\mathbf{c}_p. \tag{20}$$

As discussed in Section 3.1 of [21], when system (1) is nondefective,

$$\mathbf{S}_x=\begin{bmatrix}\mathbf{V}_x\\ \mathbf{V}_x\mathbf{J}_x\end{bmatrix}=\begin{bmatrix}\mathbf{V} & \widehat{\mathbf{V}}\\ \mathbf{V}\boldsymbol{\Lambda} & \widehat{\mathbf{V}\boldsymbol{\Lambda}}\end{bmatrix},\quad \mathbf{S}_p=\begin{bmatrix}\mathbf{V}_p\\ \mathbf{V}_p\mathbf{J}_p\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{I}\\ \boldsymbol{\Lambda} & \widehat{\boldsymbol{\Lambda}}\end{bmatrix}. \tag{21}$$

Finally, evaluating the state transformation (13) at time $t=0$, substituting Eqs. (17) and (20) into the resulting state equation, and utilizing Eq. (21) reveal that

$$\mathbf{S}_p(\mathbf{c}_p-\mathbf{c}_x)-\begin{bmatrix}\mathbf{0}\\ \mathbf{T}_2^{\mathrm{T}}\mathbf{f}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}. \tag{22}$$

In our upcoming discussion of decoupling nondefective systems with a singular mass matrix, we shall make use of the result in Eq. (22). It is interesting to note that when no forcing is applied to system (1), Eq. (22) implies that the coefficient vectors $\mathbf{c}_x$ and $\mathbf{c}_p$ are identical since the order $2n$ matrix $\mathbf{S}_p$ is invertible. Moreover, it can be shown that, for the unforced case, the elements of $\mathbf{c}_x$ and $\mathbf{c}_p$ are, in fact, the paired eigensolution coefficients, which is consistent with the observation that phase synchronization does not alter the eigensolution coefficients upon decoupling an unforced system (1) (see [21]).

3. Decoupling of nondefective systems with a singular mass matrix in free motion

We shall now illustrate how the homogenous form of system (1) may be decoupled when the mass matrix $\mathbf{M}$ is singular. We begin with an explanation of the quadratic eigenvalue problem for a differential-algebraic system (1) and follow this with a detailed discussion of the methodology for decoupling this type of system when it is unforced.

3.1. The quadratic eigenvalue problem

Similar to the case when the mass matrix of system (1) is invertible, the solutions of the associated quadratic eigenvalue problem (6) are $2n$ generally complex eigenvalues λ_j ($j=1, 2, \ldots, 2n$) and their corresponding eigenvectors $\mathbf{v}_j$, where the complex eigensolutions necessarily occur in conjugate pairs because the mass matrix $\mathbf{M}$, damping matrix $\mathbf{C}$, and stiffness matrix $\mathbf{K}$ are real. Since $\mathbf{M}$ is singular with rank $r<n$ and $\mathbf{C}$ is positive definite, the eigenspectrum of system (1) contains $\epsilon=n-r$ infinite eigenvalues, while the remaining $\sigma=2n-\epsilon=n+r$ finite eigenvalues consist of some combination of complex conjugate pairs and pairs of distinct real eigenvalues (e.g., see [19]). Let $\mathbf{J}_{x,f}$ be an order σ Jordan matrix of the σ finite eigenvalues (that is diagonal because system (1) is nondefective), and take $\mathbf{V}_{x,f}$ to be an $n\times\sigma$ matrix containing the associated eigenvectors in its columns. The matrices $\mathbf{J}_{x,f}$ and $\mathbf{V}_{x,f}$ constitute a Jordan pair for $\mathbf{M}\lambda^2+\mathbf{C}\lambda+\mathbf{K}$ (e.g., see [18]) and satisfy the relation

$$\mathbf{M}\mathbf{V}_{x,f}\mathbf{J}_{x,f}^2+\mathbf{C}\mathbf{V}_{x,f}\mathbf{J}_{x,f}+\mathbf{K}\mathbf{V}_{x,f}=\mathbf{O}_{n\times\sigma}. \tag{23}$$

Multiplying the quadratic eigenvalue problem (6) through by $\mu^2=1/\lambda^2$ yields

$$(\mathbf{M}+\mathbf{C}\mu+\mathbf{K}\mu^2)\mathbf{v}=\mathbf{0}, \tag{24}$$

and hence, the ϵ infinite eigenvalues $\lambda=\infty$ of Eq. (6) correspond to the ϵ zero eigenvalues $\mu=0$ of Eq. (24), where the associated eigenvectors form a set of linearly independent vectors in the null space of $\mathbf{M}$. Analogous to Eq. (23), the ϵ zero eigenvalues of the related quadratic eigenvalue problem (24) and the corresponding eigenvectors satisfy

$$\mathbf{M}\mathbf{V}_{x,\infty}+\mathbf{C}\mathbf{V}_{x,\infty}\mathbf{J}_{x,\infty}+\mathbf{K}\mathbf{V}_{x,\infty}\mathbf{J}_{x,\infty}^2=\mathbf{O}_{n\times\epsilon}, \tag{25}$$

where $\mathbf{J}_{x,\infty}$ is an order ϵ Jordan matrix of the zero eigenvalues $\mu=0$, and $\mathbf{V}_{x,\infty}$ is an $n\times\epsilon$ matrix of the associated eigenvectors in its columns. Since system (1) is nondefective, the Jordan matrix $\mathbf{J}_{x,\infty}=\mathbf{O}_\epsilon$, which implies from Eq. (25) that $\mathbf{M}\mathbf{V}_{x,\infty}=\mathbf{O}_{n\times\epsilon}$, i.e., the eigenvectors that correspond to the infinite eigenvalues lie in the null space of $\mathbf{M}$. We may then construct an order $2n$

invertible matrix $\mathbf{S}_x$ from the spectral data obtained by solving the quadratic eigenvalue problem (e.g., see [19]):

$$\mathbf{S}_x = \begin{bmatrix} \mathbf{V}_{x,f} & \mathbf{V}_{x,\infty}\mathbf{J}_{x,\infty} \\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty} \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{x,f} & \mathbf{O}_{n\times\epsilon} \\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty} \end{bmatrix}. \tag{26}$$

3.2. State-space representation of the free response

The free response $\mathbf{x}(t)$ of the n-degree-of-freedom differential-algebraic system (1) has the general form (e.g., see [18])

$$\mathbf{x}(t) = \mathbf{V}_{x,f}\mathrm{e}^{\mathbf{J}_{x,f}t}\mathbf{c}, \tag{27}$$

where the $\sigma-$dimensional column vector $\mathbf{c}$ of eigensolution coefficients is determined by the consistent initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ (that is, those initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ that satisfy $\mathbf{M}\ddot{\mathbf{x}}(0)+\mathbf{C}\dot{\mathbf{x}}(0)+\mathbf{K}\mathbf{x}(0)=\mathbf{f}(0)$). Casting the solution (27) and its derivative as a state equation,

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{x,f} \\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} \end{bmatrix} \mathrm{e}^{\mathbf{J}_{x,f}t}\mathbf{c}, \tag{28}$$

which may be modified to incorporate the order $2n$ invertible matrix $\mathbf{S}_x$:

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_{x,f} & \mathbf{O}_{n\times\epsilon} \\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{J}_{x,f}t} & \mathbf{O}_{\sigma\times\epsilon} \\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{I}_\epsilon \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_\epsilon \end{bmatrix} = \mathbf{S}_x \begin{bmatrix} \mathrm{e}^{\mathbf{J}_{x,f}t} & \mathbf{O}_{\sigma\times\epsilon} \\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{I}_\epsilon \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_\epsilon \end{bmatrix}. \tag{29}$$

In principle, the order ϵ identity matrix $\mathbf{I}_\epsilon$ in Eq. (29) may be replaced by any matrix of the same dimensions since it is eventually eliminated by matrix-vector multiplication so that Eqs. (28) and (29) are equivalent. However, for reasons that will become clear later, we have deliberately chosen to use $\mathbf{I}_\epsilon$ so that the matrix product in Eq. (29) is invertible.

3.3. Mechanics of decoupling the free response

As mentioned earlier, since the mass matrix $\mathbf{M}$ of system (1) has rank $r<n$, the decoupled system (2) will be partitioned into two sets of real, independent differential equations, one of which contains r second-order equations, while the other consists of $n-r$ first-order equations. Moreover, we know from prior work [5,6,8] that these independent, second-order differential equations are obtained through phase synchronization of paired eigensolutions. Consequently, of the $\sigma=n+r$ finite eigenvalues of system (1), we must assign r pairs of complex conjugate or distinct real eigenvalues, while the remaining $n-r$ finite eigenvalues are to be left unpaired and correspond to the independent, first-order differential equations of the decoupled system (2). Of course, complex eigenvalues must always be paired, so the $n-r$ unpaired eigenvalues are always real, which is consistent with the observation that a real, first-order differential equation must always have a real eigenvalue. As a result, we may express the free response $\mathbf{x}(t)$ of system (1) in the expanded form

$$\mathbf{x}(t) = \sum_{i=1}^{r}(\mathbf{v}_i a_i \mathrm{e}^{\lambda_i t} + \hat{\mathbf{v}}_i \hat{a}_i \mathrm{e}^{\hat{\lambda}_i t}) + \sum_{k=1}^{n-r}\mathbf{w}_k b_k \mathrm{e}^{\xi_k t}. \tag{30}$$

In the first summation of Eq. (30), λ_i $(i=1,\ 2,\ \ldots,\ r)$ represent the r finite eigenvalues that constitute the r pairs of eigenvalues, while $\mathbf{v}_i$ and a_i denote the corresponding eigenvectors and eigensolution coefficients, respectively. As before, the ornamenting hat in Eq. (30) denotes a pairing of complex conjugate or distinct real eigensolutions, whichever is appropriate. Likewise, the second summation of Eq. (30) consists of the remaining $n-r$ unpaired eigenvalues ξ_k $(k=1,\ 2,\ \ldots,\ n-r)$ and their associated eigenvectors and eigensolution coefficients $\mathbf{w}_k$ and b_k, respectively. As in the case when the mass matrix $\mathbf{M}$ is invertible, we assume that the paired eigenvectors $\mathbf{v}_i$ and $\hat{\mathbf{v}}_i$ are normalized in accordance with

$$2\lambda_i \mathbf{v}_i^{\mathrm{T}}\mathbf{M}\mathbf{v}_i + \mathbf{v}_i^{\mathrm{T}}\mathbf{C}\mathbf{v}_i = \lambda_i - \hat{\lambda}_i, \tag{31}$$

$$2\hat{\lambda}_i \hat{\mathbf{v}}_i^{\mathrm{T}}\mathbf{M}\hat{\mathbf{v}}_i + \hat{\mathbf{v}}_i^{\mathrm{T}}\mathbf{C}\hat{\mathbf{v}}_i = \hat{\lambda}_i - \lambda_i. \tag{32}$$

In addition, inspired by Eqs. (31) and (32), we shall normalize the eigenvectors $\mathbf{w}_k$ associated with the unpaired eigenvalues according to

$$2\xi_k \mathbf{w}_k^{\mathrm{T}}\mathbf{M}\mathbf{w}_k + \mathbf{w}_k^{\mathrm{T}}\mathbf{C}\mathbf{w}_k = -\xi_k. \tag{33}$$

We can connect the matrix-vector representation (27) of the free response $\mathbf{x}(t)$ to the summation form (30) as follows. First, let $\mathbf{\Lambda}$ and $\widehat{\mathbf{\Lambda}}$ be order r diagonal matrices of the paired eigenvalues λ_i and $\hat{\lambda}_i$, respectively, and define a diagonal matrix $\mathbf{\Xi}$ consisting of the remaining $n-r$ unpaired eigenvalues ξ_k:

$$\mathbf{\Lambda} = \bigoplus_{i=1}^{r}\lambda_i, \quad \widehat{\mathbf{\Lambda}} = \bigoplus_{i=1}^{r}\hat{\lambda}_i, \quad \mathbf{\Xi} = \bigoplus_{k=1}^{n-r}\xi_k. \tag{34}$$

Next, arrange the paired eigenvectors $\mathbf{v}_i$ and $\hat{\mathbf{v}}_i$ in the columns of the $n\times r$ matrices $\mathbf{V}$ and $\widehat{\mathbf{V}}$, respectively, and let the $n\times(n-r)$ matrix $\mathbf{W}$ contain the eigenvectors $\mathbf{w}_k$ of the $n-r$ unpaired finite eigenvalues:

$$\mathbf{V}=[\mathbf{v}_1\mid\cdots\mid\mathbf{v}_r],\quad \widehat{\mathbf{V}}=[\hat{\mathbf{v}}_1\mid\cdots\mid\hat{\mathbf{v}}_r],\quad \mathbf{W}=[\mathbf{w}_1\mid\cdots\mid\mathbf{w}_{n-r}]. \tag{35}$$

Lastly, define r-dimensional column vectors $\mathbf{a}$ and $\hat{\mathbf{a}}$ consisting of the paired eigensolution coefficients a_i and $\hat{a}_i$, respectively, and let $\mathbf{b}$ be a column vector of length $n-r$ containing the unpaired eigensolution coefficients b_k:

$$\mathbf{a}=[a_1\ \cdots\ a_r]^{\mathrm{T}},\quad \hat{\mathbf{a}}=[\hat{a}_1\ \cdots\ \hat{a}_r]^{\mathrm{T}},\quad \mathbf{b}=[b_1\ \cdots\ b_{n-r}]^{\mathrm{T}}. \tag{36}$$

It follows that Eq. (30) may be written more compactly as

$$\mathbf{x}(t)=(\mathbf{V}\mathrm{e}^{\boldsymbol{\Lambda}t}\mathbf{a}+\widehat{\mathbf{V}}\mathrm{e}^{\widehat{\boldsymbol{\Lambda}}t}\hat{\mathbf{a}})+\mathbf{W}\mathrm{e}^{\boldsymbol{\Xi}t}\mathbf{b}, \tag{37}$$

and thus it must be the case that

$$\mathbf{J}_{x,f}=\boldsymbol{\Lambda}\oplus\widehat{\boldsymbol{\Lambda}}\oplus\boldsymbol{\Xi},\quad \mathbf{V}_{x,f}=[\mathbf{V}\mid\widehat{\mathbf{V}}\mid\mathbf{W}],\quad \mathbf{c}=[\mathbf{a}^{\mathrm{T}}\ \hat{\mathbf{a}}^{\mathrm{T}}\ \mathbf{b}^{\mathrm{T}}]^{\mathrm{T}} \tag{38}$$

by comparing Eqs. (27) and (37).

Now, suppose we partition the decoupled free response vector $\mathbf{p}(t)$ as

$$\mathbf{p}(t)=\begin{bmatrix}\mathbf{y}(t)\\ \mathbf{z}(t)\end{bmatrix},\quad \mathbf{y}(t)=[y_1(t)\ \cdots\ y_r(t)]^{\mathrm{T}},\quad \mathbf{z}(t)=[z_1(t)\ \cdots\ z_{n-r}(t)]^{\mathrm{T}}, \tag{39}$$

where $y_i(t)=p_i(t)$ and $z_k(t)=p_{r+k}(t)$ are the solutions to the r second-order and $n-r$ first-order, independent differential equations, respectively, that comprise the unforced decoupled system (2). From previous work on systems with an invertible mass matrix [5–8], we know that each of the r decoupled, second-order equations generated via phase synchronization of the r paired eigensolutions (i.e., the r summands in the first summation of Eq. (30)) is of the form

$$\ddot{y}_i(t)-(\lambda_i+\hat{\lambda}_i)\dot{y}_i(t)+\lambda_i\hat{\lambda}_i y_i(t)=0 \tag{40}$$

and has as its solution

$$y_i(t)=a_i\mathrm{e}^{\lambda_i t}+\hat{a}_i\mathrm{e}^{\hat{\lambda}_i t}. \tag{41}$$

Since phase synchronization relies on the pairing of eigensolutions, we must conclude that phase synchronization does not apply to the eigensolutions associated with the $n-r$ unpaired eigenvalues ξ_k (i.e., the $n-r$ terms in the second summation of Eq. (30)), and hence they remain unchanged during decoupling. Consequently,

$$z_k(t)=b_k\mathrm{e}^{\xi_k t} \tag{42}$$

represents the solution to each of the $n-r$ independent, first-order subsystems in the unforced decoupled system (2), where each subsystem is governed by the differential equation, say,

$$-\xi_k\dot{z}_k(t)+\xi_k^2 z_k(t)=0. \tag{43}$$

It should be noted that, as in the case when the mass matrix of system (1) is invertible, the eigensolution coefficients a_i, $\hat{a}_i$, and b_k are not affected by the decoupling procedure. Expressing the r second-order equations (40) and the $n-r$ first-order equations (43) in matrix-vector form,

$$\ddot{\mathbf{y}}(t)-(\boldsymbol{\Lambda}+\widehat{\boldsymbol{\Lambda}})\dot{\mathbf{y}}(t)+\boldsymbol{\Lambda}\widehat{\boldsymbol{\Lambda}}\mathbf{y}(t)=\mathbf{0}_r, \tag{44}$$

$$-\boldsymbol{\Xi}\dot{\mathbf{z}}(t)+\boldsymbol{\Xi}^2\mathbf{z}(t)=\mathbf{0}_{n-r}. \tag{45}$$

Based on the partitioning (39) of the decoupled free response $\mathbf{p}(t)$, combining Eqs. (44) and (45) yields the n-degree-of-freedom decoupled system (2) with the transformed forcing $\mathbf{g}(t)=\mathbf{0}$ and coefficient matrices given by

$$\mathbf{A}_2=\mathbf{I}_r\oplus\mathbf{O}_{n-r},\quad \mathbf{A}_1=-(\boldsymbol{\Lambda}+\widehat{\boldsymbol{\Lambda}})\oplus-\boldsymbol{\Xi},\quad \mathbf{A}_0=\boldsymbol{\Lambda}\widehat{\boldsymbol{\Lambda}}\oplus\boldsymbol{\Xi}^2. \tag{46}$$

It is straightforward to verify that solution of the quadratic eigenvalue problem associated with the decoupled system matrices (46) yields the same eigenvalues (and multiplicities) obtained for the coupled system (1), and thus the decoupling transformation that relates the two systems must be isospectral as expected. Note that if system (1) has more than one real finite eigenvalue, then the form of the decoupled system is not unique since different choices of pairing distinct real eigenvalues result in different forms of the coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_0$ (see [6,8] for more detail regarding this nonuniqueness due to pairing real eigenvalues).

3.4. Decoupling transformations in the state and configuration spaces

How are the free responses of the coupled and decoupled systems related? Since the eigensolution coefficients that comprise the column vector $\mathbf{c}$ are preserved upon transforming the homogeneous form of system (1) into its unforced decoupled form (2), we may cast the free response $\mathbf{p}(t)$ of the decoupled system in a matrix-vector form analogous to Eq. (27):

$$\mathbf{p}(t)=\mathbf{V}_{p,f}e^{\mathbf{J}_{p,f}t}\mathbf{c}. \tag{47}$$

Here, $\mathbf{J}_{p,f}$ is an order $\sigma=n+r$ Jordan matrix of the decoupled system's finite eigenvalues, and the $n\times\sigma$ matrix $\mathbf{V}_{p,f}$ essentially specifies if and how the finite eigenvalues are paired. Since the decoupling transformation is isospectral, it must be the case that the Jordan matrix $\mathbf{J}_{p,f}=\mathbf{J}_{x,f}$. To determine the form of $\mathbf{V}_{p,f}$, first note that the collection of scalar solutions (41) and (42) for the r second-order and $n-r$ first-order, independent subsystems may be written more compactly as $\mathbf{y}(t)=e^{\Lambda t}\mathbf{a}+e^{\widehat{\Lambda}t}\hat{\mathbf{a}}$ and $\mathbf{z}(t)=e^{\Xi t}\mathbf{b}$, respectively. Recalling how the response $\mathbf{p}(t)$ is partitioned, a comparison of these expressions and Eq. (47) reveals that

$$\mathbf{V}_{p,f}=[\mathbf{I}_r\mid\mathbf{I}_r]\oplus\mathbf{I}_{n-r}. \tag{48}$$

The matrices $\mathbf{J}_{p,f}$ and $\mathbf{V}_{p,f}$ constitute a Jordan pair for $\mathbf{A}_2\lambda^2+\mathbf{A}_1\lambda+\mathbf{A}_0$ and satisfy a relationship analogous to Eq. (23), albeit with the system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ replaced with $\mathbf{A}_2$, $\mathbf{A}_1$, and $\mathbf{A}_0$, respectively. Likewise, let $\mathbf{J}_{p,\infty}$ and $\mathbf{V}_{p,\infty}$ denote the Jordan pair associated with the $\epsilon=n-r$ infinite eigenvalues. Since system (1) is nondefective, the Jordan matrix $\mathbf{J}_{p,\infty}=\mathbf{O}_\epsilon$, and thus the matrix $\mathbf{V}_{p,\infty}$ is determined by the relation $\mathbf{A}_2\mathbf{V}_{p,\infty}=\mathbf{O}_\epsilon$ by analogy to Eq. (25). In other words, $\mathbf{V}_{p,\infty}$ contains linearly independent vectors in the null space of the leading coefficient matrix $\mathbf{A}_2$.

Similar to the state-space formulation of the free response $\mathbf{x}(t)$ in Section 3.2, express the decoupled solution $\mathbf{p}(t)$ and its derivative in the form of a state equation and rewrite it in terms of an invertible matrix $\mathbf{S}_p$ of size $2n$:

$$\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{p,f}\\ \mathbf{V}_{p,f}\mathbf{J}_{p,f}\end{bmatrix}e^{\mathbf{J}_{p,f}t}\mathbf{c}=\begin{bmatrix}\mathbf{V}_{p,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{p,f}\mathbf{J}_{p,f} & \mathbf{V}_{p,\infty}\end{bmatrix}\begin{bmatrix}e^{\mathbf{J}_{p,f}t} & \mathbf{O}_{\sigma\times\epsilon}\\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{I}_\epsilon\end{bmatrix}\begin{bmatrix}\mathbf{c}\\ \mathbf{0}_\epsilon\end{bmatrix}=\mathbf{S}_p\begin{bmatrix}e^{\mathbf{J}_{p,f}t} & \mathbf{O}_{\sigma\times\epsilon}\\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{I}_\epsilon\end{bmatrix}\begin{bmatrix}\mathbf{c}\\ \mathbf{0}_\epsilon\end{bmatrix}. \tag{49}$$

Next, use the state equation (49) to eliminate the coefficient vector $[\mathbf{c}^{\mathrm{T}}\ \mathbf{0}_\epsilon^{\mathrm{T}}]^{\mathrm{T}}$ from Eq. (29) to obtain

$$\begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix}=\mathbf{S}_x\begin{bmatrix}e^{\mathbf{J}_{x,f}t}e^{-\mathbf{J}_{p,f}t} & \mathbf{O}_{\sigma\times\epsilon}\\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{I}_\epsilon\end{bmatrix}\mathbf{S}_p^{-1}\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}. \tag{50}$$

Setting $\mathbf{J}_{p,f}=\mathbf{J}_{x,f}$ because system (1) is nondefective yields a transformation in the state space that relates the free responses of the coupled and decoupled systems:

$$\begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix}=\mathbf{S}_x\mathbf{S}_p^{-1}\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{x,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{p,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{p,f}\mathbf{J}_{p,f} & \mathbf{V}_{p,\infty}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}=\mathbf{S}\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)\end{bmatrix}. \tag{51}$$

Eq. (51) implies that the system free response $\mathbf{x}(t)$ is recovered from the decoupled solution $\mathbf{p}(t)$ by a linear time-invariant transformation in the state space. Moreover, while the components of $\mathbf{S}_x$ and $\mathbf{S}_p$ may be complex, the overall transformation $\mathbf{S}$ is real because the solutions $\mathbf{x}(t)$ and $\mathbf{p}(t)$ are real. Inverting the state equation (51) and evaluating it at time $t=0$ gives the relationship between the consistent initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ and the decoupled system's consistent initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$:

$$\begin{bmatrix}\mathbf{p}(0)\\ \dot{\mathbf{p}}(0)\end{bmatrix}=\mathbf{S}^{-1}\begin{bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{p,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{p,f}\mathbf{J}_{p,f} & \mathbf{V}_{p,\infty}\end{bmatrix}\begin{bmatrix}\mathbf{V}_{x,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{bmatrix}. \tag{52}$$

Lastly, it is possible to extract a decoupling transformation in the n-dimensional configuration space from Eq. (51) that recovers $\mathbf{x}(t)$ directly, making it unnecessary to evaluate the larger state equation. It is straightforward to show that isolating the upper half of the state equation (51) yields

$$\mathbf{x}(t)=\mathbf{T}_1\mathbf{p}(t)+\mathbf{T}_2\dot{\mathbf{p}}(t), \tag{53}$$

with the real transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ given by

$$\mathbf{T}_1=[(\mathbf{V}\widehat{\Lambda}-\widehat{\mathbf{V}}\Lambda)(\widehat{\Lambda}-\Lambda)^{-1}\mid\mathbf{W}],\quad \mathbf{T}_2=[(\widehat{\mathbf{V}}-\mathbf{V})(\widehat{\Lambda}-\Lambda)^{-1}\mid\mathbf{O}_{n\times\epsilon}], \tag{54}$$

where the matrices Λ, $\widehat{\Lambda}$, $\mathbf{V}$, $\widehat{\mathbf{V}}$, and $\mathbf{W}$ are as defined in Eqs. (34) and (35).

To summarize, we have illustrated how an unforced, nondefective, n-degree-of-freedom system (1) with a singular mass matrix $\mathbf{M}$ of rank $r<n$ may be decoupled into a set of r decoupled, real, second-order differential equations and a set of $n-r$ independent, real, first-order subsystems by phase synchronization of the r paired eigensolutions. The coefficient matrices of the decoupled system are given by Eq. (46), and its consistent initial conditions are determined from Eq. (52). The decoupled solution $\mathbf{p}(t)$ may be obtained through standard methods for ordinary differential equations since coupled algebraic constraints have been eliminated. The free response $\mathbf{p}(t)$ of the decoupled system and transformation (53) may then be used to recover the coupled system's free response $\mathbf{x}(t)$ in the configuration space. While the decoupled system's coefficient matrices $\mathbf{A}_1$ and $\mathbf{A}_0$ may not be unique because of the choice of pairing distinct real eigensolutions, the solution $\mathbf{x}(t)$ will always be unique (see [6,8]). The parameters required for decoupling are obtained through solving the quadratic eigenvalue problem (6) and by finding linearly independent vectors in the null spaces of the leading coefficient matrices $\mathbf{M}$ and $\mathbf{A}_2$. The decoupling process is isospectral, preserving the eigenvalues of system (1) and their multiplicities. In the event

that the mass matrix of system (1) is invertible, the results developed in this section reduce to their counterparts in Section 2.3 since all terms associated with infinite eigenvalues vanish and every decoupled subsystem is second order.

Finally, we wish to briefly elaborate on the properties of the order $2n$ transformation matrix $\mathbf{S}$ in Eq. (51). Suppose we express the coupled differential-algebraic system (1) in the symmetric state-space realization (e.g., see [6,15])

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t) \\ \mathbf{0} \end{bmatrix} \tag{55}$$

with the forcing $\mathbf{f}(t) = \mathbf{0}$, apply the coordinate transformation (51), and premutliply the resulting state equation by $\mathbf{S}^{\mathrm{T}}$. Doing so, we would find that

$$\mathbf{S}^{\mathrm{T}} \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \mathbf{S} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{O} \end{bmatrix}, \quad \mathbf{S}^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \mathbf{S} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{O} \\ \mathbf{O} & -\mathbf{A}_2 \end{bmatrix}. \tag{56}$$

While the symmetric structure of the state matrices in Eq. (56) is preserved under congruence transformation using $\mathbf{S}$, it can be shown that, in general,

$$\mathbf{S}^{\mathrm{T}} \begin{bmatrix} \mathbf{O} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \mathbf{S} \neq \begin{bmatrix} \mathbf{O} & \mathbf{A}_0 \\ \mathbf{A}_0 & \mathbf{A}_1 \end{bmatrix}. \tag{57}$$

Thus, strictly speaking, $\mathbf{S}$ alone does not generate a structure-preserving transformation as defined by Garvey et al. [15] since not every symmetric state-space formulation of system (1) in [15] has its symmetry preserved under transformation.

4. Decoupling of nondefective systems with a singular mass matrix in forced motion

Let us now extend the decoupling procedure to the case when the coupled differential-algebraic system (1) is forced. We begin by expressing the forced equation of motion (1) as the symmetric state equation (55) and, to ensure that the decoupling transformation for forced motion is also isospectral, by defining a linear, time-invariant change of coordinates based on the free response's state-space transformation (51):

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix}, \quad \mathbf{p}_1(t) = \begin{bmatrix} \mathbf{y}_1(t) \\ \mathbf{z}_1(t) \end{bmatrix}, \quad \mathbf{p}_2(t) = \begin{bmatrix} \mathbf{y}_2(t) \\ \mathbf{z}_2(t) \end{bmatrix}, \tag{58}$$

where $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are n-dimensional column vectors whose relationship to the decoupled forced response $\mathbf{p}(t)$ and its corresponding velocity $\dot{\mathbf{p}}(t)$ is to be determined. The column vectors $\mathbf{y}_j(t)$ $(j = 1,\ 2)$ and $\mathbf{z}_j(t)$ denote the partitioning of the decoupled solutions into the sets of r second-order and $n-r$ first-order, independent subsystems, respectively, similar to Eq. (39). We must also address how the decoupled system's excitation $\mathbf{g}(t)$ is related to the applied forcing $\mathbf{f}(t)$.

4.1. Transformation of the applied forcing

Apply the state transformation (58) to the symmetric state-space realization (55) and premultiply the resulting equation by $\mathbf{S}^{\mathrm{T}}$ to obtain

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1(t) \\ \dot{\mathbf{p}}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{A}_0 & \mathbf{O} \\ \mathbf{O} & -\mathbf{A}_2 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1^{\mathrm{T}} \\ \mathbf{T}_2^{\mathrm{T}} \end{bmatrix} \mathbf{f}(t), \tag{59}$$

where the coefficient matrices $\mathbf{A}_i$ $(i = 1,\ 2,\ 3)$ are still given by Eq. (46), and the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are the same as in Eq. (54), respectively. Separating the state equation (59) into its upper and lower halves yields, respectively,

$$\mathbf{A}_1 \dot{\mathbf{p}}_1(t) + \mathbf{A}_2 \dot{\mathbf{p}}_2(t) + \mathbf{A}_0 \mathbf{p}_1(t) = \mathbf{T}_1^{\mathrm{T}} \mathbf{f}(t), \tag{60}$$

$$\mathbf{A}_2 \dot{\mathbf{p}}_1(t) - \mathbf{A}_2 \mathbf{p}_2(t) = \mathbf{T}_2^{\mathrm{T}} \mathbf{f}(t). \tag{61}$$

To continue with our analysis, it is necessary to examine the structures of Eqs. (60) and (61) more closely. For convenience, let

$$\mathbf{T}_1^{\mathrm{T}} \mathbf{f}(t) = \begin{bmatrix} \mathbf{h}_1(t) \\ \mathbf{k}_1(t) \end{bmatrix}, \quad \mathbf{T}_2^{\mathrm{T}} \mathbf{f}(t) = \begin{bmatrix} \mathbf{h}_2(t) \\ \mathbf{k}_2(t) \end{bmatrix}, \tag{62}$$

where the column vectors $\mathbf{h}_j(t)$ and $\mathbf{k}_j(t)$ are of lengths r and $n-r$, respectively. Based on the form of the transformation matrix $\mathbf{T}_2$ in Eq. (54), it is straightforward to show that $\mathbf{k}_2(t)$ is identically zero: $\mathbf{k}_2(t) = \mathbf{0}_{n-r}$. In terms of $\mathbf{y}_j(t)$ and $\mathbf{z}_j(t)$, Eq. (60) contains the expressions

$$-(\boldsymbol{\Lambda} + \widehat{\boldsymbol{\Lambda}}) \dot{\mathbf{y}}_1(t) + \dot{\mathbf{y}}_2(t) + \boldsymbol{\Lambda} \widehat{\boldsymbol{\Lambda}} \mathbf{y}_1(t) = \mathbf{h}_1(t), \tag{63}$$

$$-\boldsymbol{\Xi} \dot{\mathbf{z}}_1(t) + \boldsymbol{\Xi}^2 \mathbf{z}_1(t) = \mathbf{k}_1(t). \tag{64}$$

Comparing Eq. (64) to its unforced counterpart (45), it becomes clear that $\mathbf{z}_1(t)$ corresponds to the forced response $\mathbf{z}(t)$ of the decoupled first-order subsystem. Furthermore, we deduce from the left-hand side of Eq. (64) that the last $n-r$ rows of the excitation $\mathbf{g}(t)$ for the decoupled system (2) are given by $\mathbf{k}_1(t)$. To eliminate $\dot{\mathbf{y}}_2(t)$ from Eq. (63), we will need to investigate how Eq. (61) is partitioned into $\mathbf{y}_j(t)$ and $\mathbf{z}_j(t)$:

$$\dot{\mathbf{y}}_1(t)-\mathbf{y}_2(t)=\mathbf{h}_2(t), \tag{65}$$

$$\mathbf{O}_{n-r}\dot{\mathbf{z}}_1(t)+\mathbf{O}_{n-r}\mathbf{z}_2(t)=\mathbf{0}_{n-r}. \tag{66}$$

Rearranging Eq. (65) to solve for $\mathbf{y}_2(t)$, differentiating this expression, and substituting in the result for $\dot{\mathbf{y}}_2(t)$ in Eq. (63) yields an equation of motion in $\mathbf{y}_1(t)$:

$$\ddot{\mathbf{y}}_1(t)-(\boldsymbol{\Lambda}+\widehat{\boldsymbol{\Lambda}})\dot{\mathbf{y}}_1(t)+\boldsymbol{\Lambda}\widehat{\boldsymbol{\Lambda}}\mathbf{y}_1(t)=\mathbf{h}_1(t)+\dot{\mathbf{h}}_2(t). \tag{67}$$

A comparison of Eqs. (67) and (44) reveals that $\mathbf{y}_1(t)$ must be the solution $\mathbf{y}(t)$ to the set of r real, independent, second-order differential equations that constitute the forced decoupled system (2). Consequently, we find from Eq. (65) that $\mathbf{y}_2(t)=\dot{\mathbf{y}}(t)-\mathbf{h}_2(t)$. Moreover, since we have determined that $\mathbf{y}_1(t)=\mathbf{y}(t)$ and $\mathbf{z}_1(t)=\mathbf{z}(t)$, it follows from the partitioning in Eq. (58) that $\mathbf{p}_1(t)$ is in fact the response $\mathbf{p}(t)$ of the decoupled equation of motion (2): $\mathbf{p}_1(t)=\mathbf{p}(t)$. Lastly, the left-hand side of Eq. (67) implies that the first r rows of the excitation $\mathbf{g}(t)$ may be written as $\mathbf{h}_1(t)+\dot{\mathbf{h}}_2(t)$. Thus, when Eqs. (67) and (64) are combined to yield the decoupled system (2), we observe that the associated excitation $\mathbf{g}(t)$ is related to the applied forcing $\mathbf{f}(t)$ according to

$$\mathbf{g}(t)=\begin{bmatrix}\mathbf{h}_1(t)+\dot{\mathbf{h}}_2(t)\\ \mathbf{k}_1(t)\end{bmatrix}=\begin{bmatrix}\mathbf{h}_1(t)\\ \mathbf{k}_1(t)\end{bmatrix}+\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{h}_2(t)\\ \mathbf{0}_{n-r}\end{bmatrix}=\mathbf{T}_1^{\mathrm{T}}\mathbf{f}(t)+\mathbf{T}_2^{\mathrm{T}}\dot{\mathbf{f}}(t). \tag{68}$$

4.2. *Connecting the consistent initial conditions*

How are the consistent initial conditions of the coupled and decoupled systems related? To answer this question, we shall start with the analytical solution for the forced response $\mathbf{x}(t)$ of system (1) (e.g., see [18]):

$$\mathbf{x}(t)=\mathbf{V}_{x,f}\mathrm{e}^{\mathbf{J}_{x,f}t}\left(\mathbf{c}_x+\int_0^t \mathrm{e}^{-\mathbf{J}_{x,f}s}\mathbf{Z}_{x,f}\mathbf{f}(s)\,\mathrm{d}s\right), \tag{69}$$

for which $\mathbf{c}_x$ is a $\sigma-$dimensional column vector of coefficients and the $\sigma\times n$ matrix $\mathbf{Z}_{x,f}$ is determined from

$$\mathbf{Z}_{x,f}=[\mathbf{I}_\sigma \mid \mathbf{O}_{\sigma\times\epsilon}]\begin{bmatrix}\mathbf{J}_{x,f} & \mathbf{O}_{\sigma\times\epsilon}\\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{O}_\epsilon\end{bmatrix}\begin{bmatrix}\mathbf{V}_{x,f} & \mathbf{V}_{x,\infty}\\ \mathbf{M}\mathbf{V}_{x,f}\mathbf{J}_{x,f} & -\mathbf{C}\mathbf{V}_{x,\infty}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{O}\\ \mathbf{I}\end{bmatrix}. \tag{70}$$

Utilizing the Leibniz integral rule to differentiate the solution (69),

$$\dot{\mathbf{x}}(t)=\mathbf{V}_{x,f}\mathbf{J}_{x,f}\mathrm{e}^{\mathbf{J}_{x,f}t}\left(\mathbf{c}_x+\int_0^t \mathrm{e}^{-\mathbf{J}_{x,f}s}\mathbf{Z}_{x,f}\mathbf{f}(s)\,\mathrm{d}s\right)+\mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(t), \tag{71}$$

where the last term $\mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(t)$ would vanish due to biorthogonality of the left and right eigenvectors in the event that the mass matrix $\mathbf{M}$ is invertible (e.g., see [19]). We may then evaluate Eqs. (69) and (71) at time $t=0$ and combine the results in the form of a state equation to obtain an expression for the consistent initial conditions for the coupled system (1):

$$\begin{bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{x,f}\\ \mathbf{V}_{x,f}\mathbf{J}_{x,f}\end{bmatrix}\mathbf{c}_x+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}. \tag{72}$$

Alternatively, writing Eq. (72) in terms of the order $2n$ invertible matrix $\mathbf{S}_x$,

$$\begin{bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{x,f} & \mathbf{O}_{n\times\epsilon}\\ \mathbf{V}_{x,f}\mathbf{J}_{x,f} & \mathbf{V}_{x,\infty}\end{bmatrix}\begin{bmatrix}\mathbf{c}_x\\ \mathbf{0}_\epsilon\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}=\mathbf{S}_x\begin{bmatrix}\mathbf{c}_x\\ \mathbf{0}_\epsilon\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}. \tag{73}$$

Analogous to Eq. (69), the forced response $\mathbf{p}(t)$ of the decoupled system (2) has the analytical solution

$$\mathbf{p}(t)=\mathbf{V}_{p,f}\mathrm{e}^{\mathbf{J}_{p,f}t}\left(\mathbf{c}_p+\int_0^t \mathrm{e}^{-\mathbf{J}_{p,f}s}\mathbf{Z}_{p,f}\mathbf{g}(s)\,\mathrm{d}s\right), \tag{74}$$

where $\mathbf{c}_p$ is a $\sigma-$long column vector of coefficients and the $\sigma\times n$ matrix $\mathbf{Z}_{p,f}$ is calculated as follows:

$$\mathbf{Z}_{p,f}=\begin{bmatrix}\mathbf{I}_\sigma & \mathbf{O}_{\sigma\times\epsilon}\end{bmatrix}\begin{bmatrix}\mathbf{J}_{p,f} & \mathbf{O}_{\sigma\times\epsilon}\\ \mathbf{O}_{\epsilon\times\sigma} & \mathbf{O}_\epsilon\end{bmatrix}\begin{bmatrix}\mathbf{V}_{p,f} & \mathbf{V}_{p,\infty}\\ \mathbf{A}_2\mathbf{V}_{p,f}\mathbf{J}_{p,f} & -\mathbf{A}_1\mathbf{V}_{p,\infty}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{O}\\ \mathbf{I}\end{bmatrix}. \tag{75}$$

The corresponding derivative of Eq. (74) is given by

$$\dot{\mathbf{p}}(t)=\mathbf{V}_{p,f}\mathbf{J}_{p,f}\mathrm{e}^{\mathbf{J}_{p,f}t}\left(\mathbf{c}_p+\int_0^t \mathrm{e}^{-\mathbf{J}_{p,f}s}\mathbf{Z}_{p,f}\mathbf{g}(s)\,\mathrm{d}s\right)+\mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(t). \tag{76}$$

By analogy to Eqs. (72) and (73), setting time $t=0$ in Eqs. (74) and (76) and writing the results as a state equation in terms of the invertible matrix $\mathbf{S}_p$ of size $2n$ yields an expression for the decoupled system's consistent initial conditions:

$$\begin{bmatrix}\mathbf{p}(0)\\ \dot{\mathbf{p}}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{p,f}\\ \mathbf{V}_{p,f}\mathbf{J}_{p,f}\end{bmatrix}\mathbf{c}_p+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{V}_{p,f} & \mathbf{0}_{n\times\epsilon}\\ \mathbf{V}_{p,f}\mathbf{J}_{x,f} & \mathbf{V}_{p,\infty}\end{bmatrix}\begin{bmatrix}\mathbf{c}_p\\ \mathbf{0}_\epsilon\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)\end{bmatrix}=\mathbf{S}_p\begin{bmatrix}\mathbf{c}_p\\ \mathbf{0}_\epsilon\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)\end{bmatrix}. \tag{77}$$

We could then connect the coupled and decoupled system's consistent initial conditions using the coordinate transformation (58) if we had expressions for the column vectors $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ in terms of the decoupled solution $\mathbf{p}(t)$ and its derivative $\dot{\mathbf{p}}(t)$. We found earlier that $\mathbf{p}_1(t)=\mathbf{p}(t)$, but it is not entirely clear how $\mathbf{p}_2(t)$ is related to $\mathbf{p}(t)$ and $\dot{\mathbf{p}}(t)$. We do have some information about $\mathbf{p}_2(t)$, namely that $\mathbf{y}_2(t)=\dot{\mathbf{y}}(t)-\mathbf{h}_2(t)$ by Eq. (65) since $\mathbf{y}_1(t)=\mathbf{y}(t)$. Unfortunately, the situation is not so clear for $\mathbf{z}_2(t)$ because Eq. (66) is identically satisfied and provides nothing meaningful. Based on the form of $\mathbf{y}_2(t)$, it would seem reasonable to assume that $\mathbf{z}_2(t)=\dot{\mathbf{z}}(t)-\mathbf{k}_2(t)$. However, since the forcing component $\mathbf{k}_2(t)=\mathbf{0}_{n-r}$, this statement implies that $\mathbf{z}_2(t)=\dot{\mathbf{z}}(t)$, and it can be shown that the resulting transformation of initial conditions returns erroneous values for the initial derivative $\dot{\mathbf{z}}(0)$. Fortunately, $\dot{\mathbf{z}}(0)$ is not actually needed to solve the first-order subsystem since only a consistent $\mathbf{z}(0)$ needs to be specified, but the described discrepancy is undesirable and we should make some attempt to determine the true form of $\mathbf{z}_2(t)$ to establish a proper transformation. Considering how $\mathbf{y}_2(t)=\dot{\mathbf{y}}(t)-\mathbf{h}_2(t)$, postulate that $\mathbf{z}_2(t)=\dot{\mathbf{z}}(t)-\mathbf{q}_2(t)$, where our goal is to determine the form of the unknown column vector $\mathbf{q}_2(t)$ that yields the correct transformation relating the responses of the coupled and decoupled systems. Given the assumed form of $\mathbf{z}_2(t)$, we may express $\mathbf{p}_2(t)$ as

$$\mathbf{p}_2(t)=\begin{bmatrix}\mathbf{y}_2(t)\\ \mathbf{z}_2(t)\end{bmatrix}=\begin{bmatrix}\dot{\mathbf{y}}(t)\\ \dot{\mathbf{z}}(t)\end{bmatrix}-\begin{bmatrix}\mathbf{h}_2(t)\\ \mathbf{0}_{n-r}\end{bmatrix}-\begin{bmatrix}\mathbf{0}_r\\ \mathbf{q}_2(t)\end{bmatrix}=\dot{\mathbf{p}}(t)-\mathbf{T}_2^{\mathrm{T}}\mathbf{f}(t)-\mathbf{q}(t). \tag{78}$$

By setting time $t=0$ in Eq. (78) and utilizing the state transformation (77) with $\mathbf{p}(0)=\mathbf{p}_1(0)$, the initial values $\mathbf{p}_1(0)$ and $\mathbf{p}_2(0)$ may be cast in the form of a state equation:

$$\begin{bmatrix}\mathbf{p}_1(0)\\ \mathbf{p}_2(0)\end{bmatrix}=\mathbf{S}_p\begin{bmatrix}\mathbf{c}_p\\ \mathbf{0}_\epsilon\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)-\mathbf{T}_2^{\mathrm{T}}\mathbf{f}(0)\end{bmatrix}-\begin{bmatrix}\mathbf{0}\\ \mathbf{q}(0)\end{bmatrix}. \tag{79}$$

Next, substitute Eqs. (73) and (79) into the coordinate transformation (58) evaluated at time $t=0$, premultiply the resulting equation by $\mathbf{S}^{-1}$, and rearrange terms to obtain

$$\mathbf{S}_p\begin{bmatrix}\mathbf{c}_p-\mathbf{c}_x\\ \mathbf{0}_\epsilon\end{bmatrix}-\begin{bmatrix}\mathbf{0}\\ \mathbf{T}_2^{\mathrm{T}}\mathbf{f}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{q}(0)\end{bmatrix}-\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)\end{bmatrix}+\mathbf{S}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}. \tag{80}$$

Due to the similarities we have observed thus far in decoupling a system (1) with an invertible mass matrix and one with a singular mass matrix, it seems reasonable to postulate that the left-hand side of Eq. (80) vanishes as it does in Eq. (22), and hence

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{q}(0)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)\end{bmatrix}-\mathbf{S}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}. \tag{81}$$

By inspection, it is clear that the upper half of the state equation (81) is identically satisfied. As for the bottom half of Eq. (81), direct numerical calculations show that the first r rows yield the expression $\mathbf{0}_r=\mathbf{0}_r$, while $\mathbf{q}_2(0)$ is generally nonzero. Thus, as desired, Eq. (81) affects only the initial value $\mathbf{z}_2(0)$. It follows from Eqs. (58), (78), and (81) that the consistent initial conditions for the decoupled system (2) are related to those of the coupled system (1) by the state equation

$$\begin{bmatrix}\mathbf{p}(0)\\ \dot{\mathbf{p}}(0)\end{bmatrix}=\mathbf{S}^{-1}\begin{bmatrix}\mathbf{x}(0)\\ \dot{\mathbf{x}}(0)-\mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(0)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(0)+\mathbf{T}_2^{\mathrm{T}}\mathbf{f}(0)\end{bmatrix}. \tag{82}$$

Numerical checks of Eq. (82) verify that the given transformation does indeed generate an initial derivative $\dot{\mathbf{z}}(0)$ that is consistent with the first-order subsystem's governing equation (64) evaluated at time $t=0$.

4.3. Decoupling transformations in the state and configuration spaces

We infer from Eq. (81) that $\mathbf{q}(t)$ satisfies the relationship

$$\begin{bmatrix}\mathbf{0}\\ \mathbf{q}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(t)\end{bmatrix}-\mathbf{S}^{-1}\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(t)\end{bmatrix}, \tag{83}$$

and hence Eqs. (58), (78), and (83) imply that the forced response $\mathbf{x}(t)$ of the coupled system (1) is recovered from the decoupled solution $\mathbf{p}(t)$ in the state space via

$$\begin{bmatrix}\mathbf{x}(t)\\ \dot{\mathbf{x}}(t)\end{bmatrix}=\mathbf{S}\begin{bmatrix}\mathbf{p}(t)\\ \dot{\mathbf{p}}(t)-\mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(t)-\mathbf{T}_2^{\mathrm{T}}\mathbf{f}(t)\end{bmatrix}+\begin{bmatrix}\mathbf{0}\\ \mathbf{V}_{x,f}\mathbf{Z}_{x,f}\mathbf{f}(t)\end{bmatrix}. \tag{84}$$

Please cite this article as: D.T. Kawano, et al., The decoupling of second-order linear systems with a singular mass matrix, *Journal of Sound and Vibration* (2013), http://dx.doi.org/10.1016/j.jsv.2013.08.005

Alternatively, we may solve for the response $\mathbf{x}(t)$ directly without needing to evaluate the entire state equation (84) by extracting the upper half. Doing so, we obtain

$$\mathbf{x}(t) = \mathbf{T}_1\mathbf{p}(t) + \mathbf{T}_2(\dot{\mathbf{p}}(t) - \mathbf{V}_{p,f}\mathbf{Z}_{p,f}\mathbf{g}(t) - \mathbf{T}_2^{\mathrm{T}}\mathbf{f}(t)), \tag{85}$$

where the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are still given by Eq. (54). It can be shown that the product $\mathbf{T}_2\mathbf{V}_{p,f}\mathbf{Z}_{p,f} = \mathbf{O}$, and hence the configuration-space transformation (85) reduces to the simpler form

$$\mathbf{x}(t) = \mathbf{T}_1\mathbf{p}(t) + \mathbf{T}_2\dot{\mathbf{p}}(t) - \mathbf{T}_2\mathbf{T}_2^{\mathrm{T}}\mathbf{f}(t). \tag{86}$$

In summary, we have presented an extension of the decoupling procedure in [6,8] to a forced differential-algebraic system (1). Its associated decoupled form (2) is defined by the same coefficient matrices (46) as the unforced system (and are based on the solutions of the quadratic eigenvalue problem (6)), and the decoupled system's excitation is related to the applied forcing by Eq. (68). The response of the decoupled system may then be obtained using the consistent initial conditions (82) and standard techniques for solving ordinary differential equations, as opposed to requiring a differential-algebraic equation solver. The forced response $\mathbf{x}(t)$ of system (1) is then directly recovered from the decoupled solution $\mathbf{p}(t)$ using the configuration-space transformation (86). Reduction of Eq. (86) and the initial conditions transformation (82) to their unforced counterparts (53) and (52) is obvious, and it should be clear that the procedure for decoupling a differential-algebraic system (1) is equivalent to

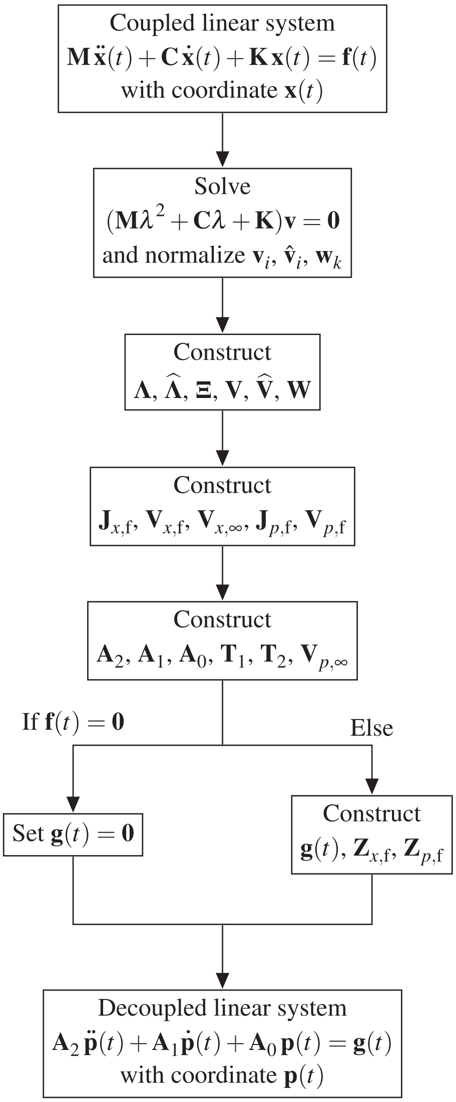

Fig. 1. Flowchart for decoupling a nondefective system (1) with a singular mass matrix $\mathbf{M}$ in free or forced motion. The consistent initial conditions for the decoupled system (2) are obtained from Eq. (52) for free motion and Eq. (82) for forced motion. The system response $\mathbf{x}(t)$ is recovered from the decoupled solution $\mathbf{p}(t)$ using transformation (53) for free motion and transformation (86) for forced motion.

Please cite this article as: D.T. Kawano, et al., The decoupling of second-order linear systems with a singular mass matrix, *Journal of Sound and Vibration* (2013), http://dx.doi.org/10.1016/j.jsv.2013.08.005

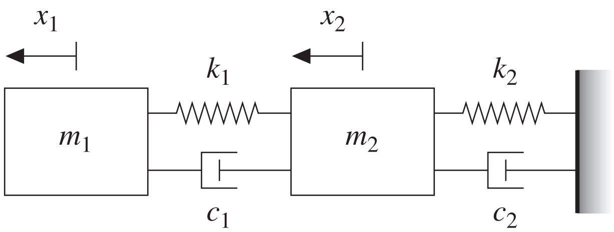

Fig. 2. Mass-spring-damper system of Example 1.

the methodology in [5,6,8] in the event that the mass matrix $\mathbf{M}$ is invertible. A flowchart outlining the process for decoupling system (1) in free or forced motion is illustrated in Fig. 1 for convenience.

5. Illustrative examples

We now provide a few examples that illustrate the decoupling methodology developed for systems of the form (1) with a singular mass matrix in free and forced motion.

Example 1. Consider the mass-spring-damper system depicted in Fig. 2, for which the system parameters $m_1=1$, $m_2=0$, $c_1=c_2=1$, $k_1=1$, and $k_2=2$ so that the equation of motion (1) has as its coefficient matrices

$$\mathbf{M}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \mathbf{C}=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix},\quad \mathbf{K}=\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}. \tag{87}$$

Let the system be unforced, in which case $\mathbf{x}(0)=[1,\ -1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0)=[2,\ 3]^{\mathrm{T}}$ constitute a set of consistent initial conditions. Because the rank of $\mathbf{M}$ is $r=1$, there is a single infinite eigenvalue, and we can take $\mathbf{V}_{x,\infty}=[0,\ 1]^{\mathrm{T}}$. Solving the associated quadratic eigenvalue problem, we find that the 3 finite eigenvalues consist of a pair of complex conjugates for which

$$\lambda=-0.28+\mathrm{i}0.78,\quad \mathbf{V}=\begin{bmatrix}1.02\mathrm{e}^{-\mathrm{i}0.66^{\circ}}\\ -0.37\mathrm{e}^{-\mathrm{i}165.95^{\circ}}\end{bmatrix}, \tag{88}$$

and the remaining real eigenvalue is such that

$$\Xi=-1.44,\quad \mathbf{W}=\begin{bmatrix}0.21\\ -0.77\end{bmatrix}. \tag{89}$$

The matrices $\mathbf{J}_{x,f}$ and $\mathbf{V}_{x,f}$ may then be constructed according to Eq. (38). From Eqs. (88), (89), and (46), the coefficient matrices for the decoupled system are given by

$$\mathbf{A}_2=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad \mathbf{A}_1=\begin{bmatrix}0.56 & 0\\ 0 & 1.44\end{bmatrix},\quad \mathbf{A}_0=\begin{bmatrix}0.69 & 0\\ 0 & 2.08\end{bmatrix}. \tag{90}$$

Since $\mathbf{A}_2=\mathbf{M}$, we can take $\mathbf{V}_{p,\infty}=\mathbf{V}_{x,\infty}$ for this example. By Eq. (48), $\mathbf{V}_{p,f}=[1|1]\oplus 1$ since $r=1$, and the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are calculated from Eq. (54):

$$\mathbf{T}_1=\begin{bmatrix}1.01 & 0.21\\ 0.39 & -0.77\end{bmatrix},\quad \mathbf{T}_2=\begin{bmatrix}-0.01 & 0\\ 0.12 & 0\end{bmatrix}. \tag{91}$$

The decoupled system's consistent initial conditions are calculated using Eq. (52): $\mathbf{p}(0)=[0.62,\ 1.98]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0)=[2.54,\ -2.86]^{\mathrm{T}}$. The decoupled solution $\mathbf{p}(t)$ is illustrated in Fig. 3(a), and the system response $\mathbf{x}(t)$ obtained via transformation (53) is shown in Fig. 3(b). It can be verified that the solution of the original system by direct numerical integration and that obtained by Eq. (53) are indeed the same.

We shall now compare the illustrated decoupling procedure with the iterative approach proposed by Garvey et al. [15]. First, replace the given singular mass matrix $\mathbf{M}$ in Eq. (87) with an invertible one of the form $\mathbf{M}+\epsilon\Delta\mathbf{M}$, where ϵ is a small parameter. A convenient choice of the perturbation matrix $\Delta\mathbf{M}$ is the identity matrix: $\Delta\mathbf{M}=\mathbf{I}_2$. Next, since the modified mass matrix is invertible, we may decouple the system using the procedure detailed in [5,6,8] and reviewed in Section 2.3 of this paper, from which we obtain the spectral data needed to construct the state transformation matrix $\mathbf{S}$ defined in Eq. (13) for a particular value of the small parameter ϵ. According to Garvey et al., by repeating this process for values of ϵ approaching zero, we should find that $\mathbf{S}$ converges (i.e., the magnitude of each element smoothly approaches a finite value as $\epsilon\to 0$), presumably to the numerical form of $\mathbf{S}$ obtained through the decoupling procedure for the case when $\mathbf{M}$ is singular (see Eq. (51)). For this example, decoupling the differential-algebraic system defined by the matrices in Eq. (87) yields the

Please cite this article as: D.T. Kawano, et al., The decoupling of second-order linear systems with a singular mass matrix, *Journal of Sound and Vibration* (2013), http://dx.doi.org/10.1016/j.jsv.2013.08.005

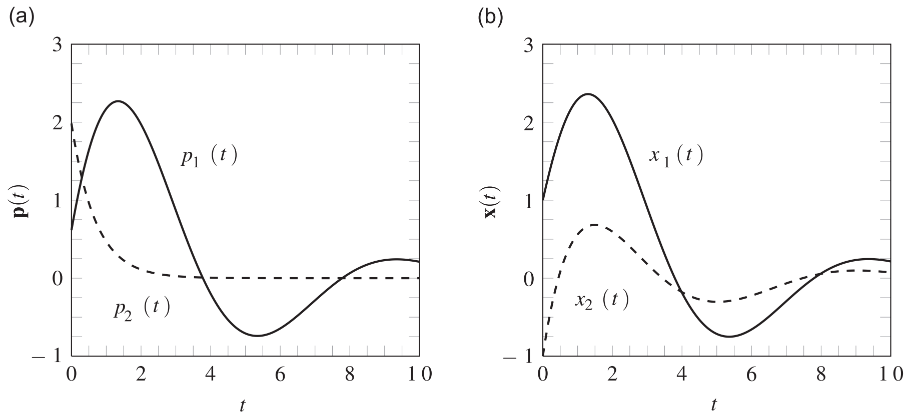

Fig. 3. Free response of Example 1. (a) Decoupled solutions $p_j(t)$ ($j=1, 2$) and (b) system responses $x_j(t)$.

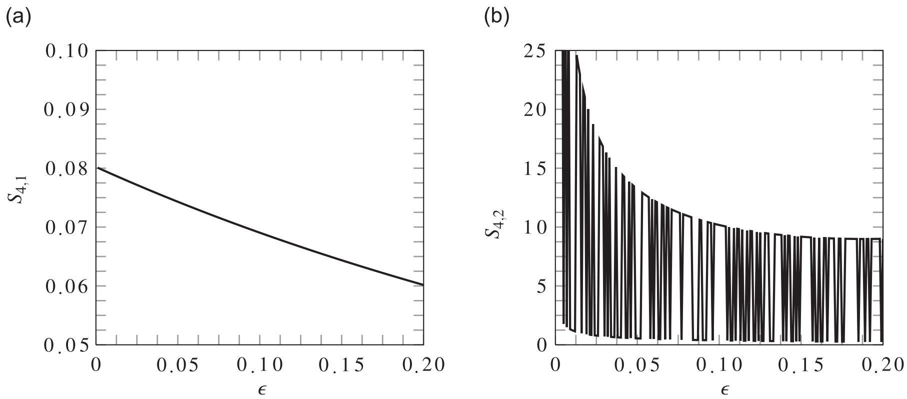

Fig. 4. Convergence of the iterative approach to decoupling a nondefective system with a singular mass matrix proposed by Garvey et al. [15] for Example 1. (a) The magnitude $S_{4,1}$ of the entry in the fourth row and first column of the state transformation matrix **S** converges as $\epsilon \to 0$ and (b) the magnitude $S_{4,2}$ of the entry in the fourth row and second column of **S** diverges as ϵ vanishes.

state transformation matrix

$$\mathbf{S} = \begin{bmatrix} 1.01 & 0.21 & -0.01 & 0 \\ 0.39 & -0.77 & 0.12 & 0 \\ 0.01 & -0.30 & 1.02 & 0 \\ -0.08 & 2.56 & 0.33 & 1 \end{bmatrix}, \tag{92}$$

and so for decreasing ϵ, we should notice that, say, $S_{4,1}$ (the magnitude of the entry in the fourth row and first column) and $S_{4,2}$ (the magnitude of the entry in the fourth row and second column) approach 0.08 and 2.56, respectively. Fig. 4 depicts the evolution of $S_{4,1}$ and $S_{4,2}$ as $\epsilon \to 0$ from the right (which ensures that the modified mass matrix is positive definite). From Fig. 4(a), it is clear that $S_{4,1}$ does indeed converge to 0.08 as ϵ is decreased. In fact, we find from this iterative approach that convergence occurs for all elements in the first and third columns. However, as seen in Fig. 4(b), $S_{4,2}$ oscillates wildly and grows without bound as $\epsilon \to 0$, and the same is true of the other elements in the second column. Thus, it appears that we are not able to deduce a unique and bounded form of the transformation matrix **S** by this type of iterative technique, implying that the exact structure of the decoupled system is indeterminate.

Example 2. Suppose the mass-spring-damper system illustrated in Fig. 5 has the following values for its systems parameters: $m_1 = 1$, $m_2 = m_3 = 0$, $k_1 = k_2 = k_3 = 1$, $c_1 = c_5 = 2$, and $c_2 = c_3 = c_4 = 1$. The coefficient matrices for the associated equation of motion (1) are given by

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \tag{93}$$

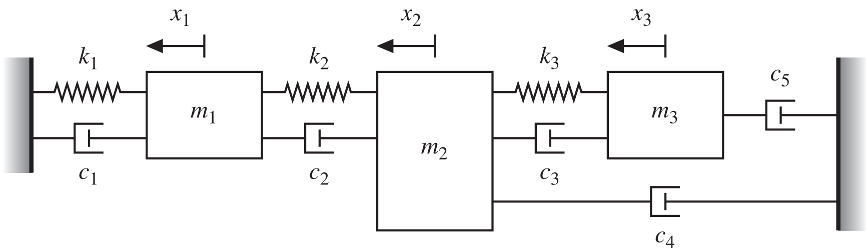

Fig. 5. Mass-spring-damper system of Example 2.

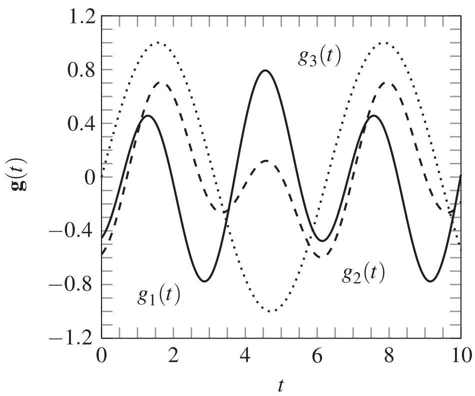

Fig. 6. Components $g_j(t)$ ($j=1, 2, 3$) of the forcing $\mathbf{g}(t)$ exerted on the decoupled system for Example 2.

The forcing $\mathbf{f}(t)=[\sin\, t,\ -\cos\, 2t,\ \cos\, t]^{\mathrm{T}}$ is applied to the system, and $\mathbf{x}(0)=[1,\ 0,\ -1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0)=[1,\ 0.25,\ 0.75]^{\mathrm{T}}$ form a corresponding set of consistent initial conditions. The rank of $\mathbf{M}$ is $r=1$, so there are 2 infinite eigenvalues, and we can assign

$$\mathbf{V}_{x,\infty}=\left[\begin{array}{c|c}0 & 0\\ 0 & 1\\ 1 & 1\end{array}\right]. \tag{94}$$

By solving the associated quadratic eigenvalue problem, we find that the 4 finite eigenvalues are all real: $\lambda_1=-1.76$, $\lambda_2=-1$, $\lambda_3=-0.63$, and $\lambda_4=-0.11$. Since $r=1$, we may pair, say, λ_1 and λ_4 such that

$$\mathbf{\Lambda}=-1.76,\quad \mathbf{V}=\begin{bmatrix}1.40\\ 0.34\\ 0.06\end{bmatrix},\quad \widehat{\mathbf{\Lambda}}=-0.11,\quad \widehat{\mathbf{V}}=\begin{bmatrix}0.29\\ 0.55\\ 0.74\end{bmatrix}, \tag{95}$$

while the remaining real eigenvalues, λ_2 and λ_3, must go unpaired:

$$\mathbf{\Xi}=-0.63\oplus-1,\quad \mathbf{W}=\left[\begin{array}{c|c}0.30 & 1\\ 0.40 & 0\\ -0.17 & 0\end{array}\right]. \tag{96}$$

We may then construct the matrices $\mathbf{J}_{x,f}$, $\mathbf{V}_{x,f}$, and $\mathbf{Z}_{x,f}$ using Eqs. (38) and (70), respectively. Eqs. (95), (96) and (46) imply that the coefficient matrices for the decoupled system are

$$\mathbf{A}_2=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{A}_1=\begin{bmatrix}1.87 & 0 & 0\\ 0 & 0.63 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \mathbf{A}_0=\begin{bmatrix}0.20 & 0 & 0\\ 0 & 0.39 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{97}$$

As $\mathbf{A}_2=\mathbf{M}$, we can take $\mathbf{V}_{p,\infty}=\mathbf{V}_{x,\infty}$ for this example. By Eq. (48), $\mathbf{V}_{p,f}=[1|1]\oplus\mathbf{I}_2$ because $r=1$, and $\mathbf{Z}_{p,f}$ may then be calculated from Eq. (75). According to Eq. (54), the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are

$$\mathbf{T}_1=\begin{bmatrix}0.22 & 0.30 & 1\\ 0.56 & 0.40 & 0\\ 0.78 & -0.17 & 0\end{bmatrix},\quad \mathbf{T}_2=\begin{bmatrix}-0.67 & 0 & 0\\ 0.13 & 0 & 0\\ 0.41 & 0 & 0\end{bmatrix}. \tag{98}$$

After determining the transformation matrix $\mathbf{S}$ and using Eq. (68) to obtain the excitation $\mathbf{g}(t)$, whose components are illustrated in Fig. 6, we can then solve for the decoupled system's consistent initial conditions from Eq. (82): $\mathbf{p}(0)=[-2.45,\ 2.39,\ 3]^{\mathrm{T}}$ and

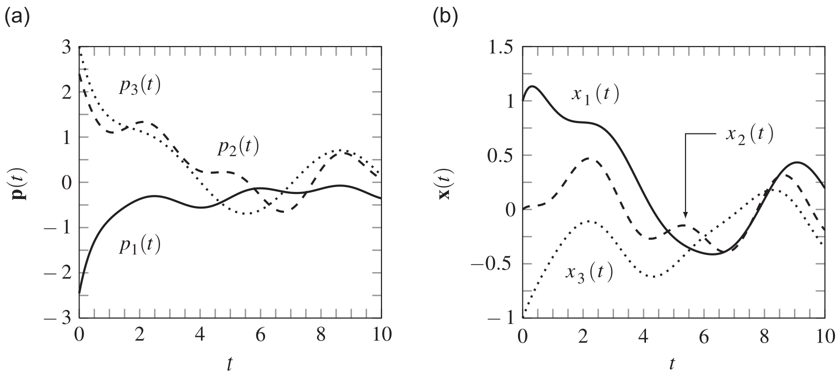

Fig. 7. Forced response of Example 2. (a) Decoupled solutions $p_j(t)$ ($j=1, 2, 3$) and (b) system responses $x_j(t)$.

$\dot{\mathbf{p}}(0)=[3.52,\ -2.42,\ -3]^{\mathrm{T}}$. Fig. 7(a) depicts the decoupled system's solution $\mathbf{p}(t)$, while the system response $\mathbf{x}(t)$ obtained via transformation (86) is shown in Fig. 7(b). It can be verified that direct numerical integration of the original system yields the illustrated solution $\mathbf{x}(t)$. Note that there are other ways to pair the real finite eigenvalues. For example, we could have chosen to pair λ_1 and λ_2 (and thus λ_3 and λ_4 are unpaired), in which case the decoupled system would take the form

$$\mathbf{A}_2=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad \mathbf{A}_1=\begin{bmatrix}2.76&0&0\\0&0.63&0\\0&0&0.11\end{bmatrix},\quad \mathbf{A}_0=\begin{bmatrix}1.76&0&0\\0&0.39&0\\0&0&0.01\end{bmatrix}. \tag{99}$$

Of course, analyzing the decoupled system defined by the coefficient matrices (99) results in the same response $\mathbf{x}(t)$ as depicted in Fig. 7(b).

Example 3. Balakrishnan [3] discussed a class of vibrating systems with singular mass matrices associated with modeling smart structures (i.e., beams with piezoelectric strips for rate feedback), and he presented and analyzed a simple continuum model of a smart beam by considering it to be a beam experiencing uniform torsion only (i.e., a "Saint-Venant shaft"), or what he terms a "smart string." An alternative approach to analyzing and simulating the response of such a "smart string" involves semidiscretization of the continuum model by, say, finite differences, and one possible formulation leads to a differential-algebraic system of the form (1). We shall illustrate how the methodology developed in this paper may be used to analyze this discrete, differential-algebraic representation of a simple smart beam model.

Based on [3], suppose we model a smart beam as a shaft in torsional motion that has a self-straining material along its entire length with rate feedback and no tip inertia. In dimensionless form for convenience, the equations governing the response behavior of this beam are given by

$$m\frac{\partial^2 x}{\partial t^2}+c\frac{\partial x}{\partial t}-k\frac{\partial^2 x}{\partial s^2}=0,\quad 0<s<1$$

$$x(t,0)=0,$$

$$k\frac{\partial x}{\partial s}\bigg|_{s=1}+\alpha\frac{\partial x}{\partial t}\bigg|_{s=1}=0, \tag{100}$$

where $x=x(t,s)$ is the torsion angle, $s\in[0,1]$ denotes the position along the beam, and t is the time. The parameters m and k represent the beam's inertia and stiffness, respectively, and $\alpha>0$ is the rate feedback gain. We also consider the effects of external damping, captured by the parameter c (which may be obtained empirically via experimental modal analysis). Dividing the beam into n segments of length $h=1/n$ and semidiscretizing the continuum model (100) by finite differences yields the discrete system of equations

$$m\ddot{x}_i+c\dot{x}_i+\frac{k}{h^2}(-x_{i-1}+2x_i-x_{i+1})=0,\quad 0<i<n$$

$$x_0=0,$$

$$\frac{k}{h^2}(-x_{n-1}+x_n)+\frac{\alpha}{h}\dot{x}_n=0, \tag{101}$$

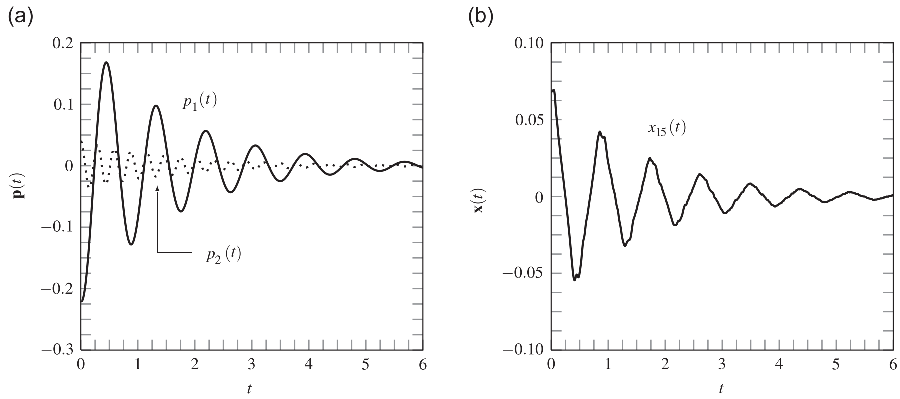

Fig. 8. Free response of Example 3. (a) Decoupled solutions $p_1(t)$ and $p_2(t)$ and (b) system response $x_{15}(t)$.

which may then be expressed as a homogenous matrix-vector equation of the form (1) with column vector $\mathbf{x}$, singular mass matrix $\mathbf{M}$, and damping matrix $\mathbf{C}$ given by, respectively,

$$\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix},\quad \mathbf{M}=\begin{bmatrix} m & 0 & 0 & \cdots & 0 \\ 0 & m & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & m & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix},\quad \mathbf{C}=\begin{bmatrix} c & 0 & 0 & \cdots & 0 \\ 0 & c & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & c & 0 \\ 0 & \cdots & 0 & 0 & \alpha/h \end{bmatrix}, \tag{102}$$

and a stiffness matrix $\mathbf{K}$ with the following structure:

$$\mathbf{K}=\frac{k}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & 0 & \cdots & 0 \\ & & & \ddots & & & \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 1 \end{bmatrix}. \tag{103}$$

If we specify an initial deflection pattern $x(0,s)=0.1[\exp(\ln(2)s)-1]$ and take as parameter values $m=1$, $k=20$, $c=0.01$, and $\alpha=0.6$, then we require $\dot{x}(0,1)=\dot{x}_n(0)=-4.54$ and $\dot{x}_1(0)=\dot{x}_2(0)=\cdots\dot{x}_{n-1}(0)=0$ for the initial conditions of the differential-algebraic system to be consistent. Upon choosing the desired number n of discretization segments, we may then analytically obtain the system response $\mathbf{x}(t)$ by the decoupling methodology presented herein and illustrated in Example 1, though on a much larger scale. For example, Fig. 8(a) depicts the first two decoupled solutions, $p_1(t)$ and $p_2(t)$, when $n=20$, and Fig. 8(b) shows the corresponding response $x_{15}(t)$ of a point three-quarters of the way down the beam from its fixed end. It can be verified that the same response is obtained by direct numerical integration of the original discrete system defined by Eqs. (102) and (103).

6. Conclusions

We have demonstrated how a nondefective, linear dynamical system of the form (1) with a singular mass matrix $\mathbf{M}$ may be decoupled into real, independent, first- and second-order differential equations in the configuration space. By formulating a general decoupling transformation that builds on the previous work [5–8], we have provided a complete solution to the problem of decoupling nondefective, linear dynamical systems in free or forced motion. While we have limited our attention to systems with a positive definite damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$, a further extension of the decoupling process reported herein to include differential-algebraic systems with nonsymmetric coefficient matrices is possible by incorporating the work given in [8]. In addition, decoupling a defective system (1) with a singular mass matrix appears feasible by utilizing the results in [21]. Major results presented in this paper are summarized in the following statements:

1. The configuration-space decoupling transformation for a linear, nondefective, differential-algebraic system (1) in free or forced motion is real and isospectral (i.e., the system eigenvalues and their multiplicities are preserved).
2. For an n-degree-of-freedom system (1) with a mass matrix $\mathbf{M}$ of rank $r<n$, r of the real, decoupled differential equations are second-order and generated by pairing $2r$ of the $n+r$ finite eigenvalues (complex conjugates or distinct real

eigenvalues). The remaining $n-r$ real, independent differential equations are first order and correspond to the $n-r$ unpaired finite eigenvalues.

3. All parameters required for generating the decoupled system (2) and evaluating the decoupling transformations (53) and (86) for free and forced motion, respectively, are obtained by solving the quadratic eigenvalue problem (6) and by determining linearly independent vectors in the null spaces of the leading coefficient matrices $\mathbf{M}$ and $\mathbf{A}_2$.
4. Since the decoupled system (2) consists of independent, first- and second-order differential equations, the decoupling procedure eliminates the numerical difficulties associated with solving the original differential-algebraic system (1).
5. The decoupling transformations (53) and (86) for free and forced motion, respectively, are direct generalizations of their counterparts in [5–8] for the case when the mass matrix is invertible. Should system (1) have an invertible mass matrix and be classically damped, the decoupling methodology presented herein reduces to classical modal analysis.

References

[1] F.E. Udwadia, P. Phohomsiri, Explicit equations of motion for constrained mechanical systems with singular mass matrices and applications to multi-body dynamics, *Proceedings of the Royal Society, Series A* 462 (2071) (2006) 2097–2117, http://dx.doi.org/10.1098/rspa.2006.1662.

[2] R.R. Craig, A. Kurdila, *Fundamentals of Structural Dynamics*, 2nd edition, John Wiley & Sons, Hoboken, New Jersey, 2006.

[3] A.V. Balakrishnan, Vibrating systems with singular mass-inertia matrices, in: S. Sivasundaram (Ed.), *Proceedings of First International Conference on Nonlinear Problems in Aviation and Aerospace*, Embry-Riddle Aeronautical University Press, 1996.

[4] L. Petzold, Differential/algebraic equations are not ODE's, *SIAM Journal on Scientific and Statistical Computing* 3 (3) (1982) 367–384, http://dx.doi.org/10.1137/0903023.

[5] F. Ma, A. Imam, M. Morzfeld, The decoupling of damped linear systems in oscillatory free vibration, *Journal of Sound and Vibration* 324 (2009) 408–428, http://dx.doi.org/10.1016/j.jsv.2009.02.005.

[6] F. Ma, M. Morzfeld, A. Imam, The decoupling of damped linear systems in free or forced vibration, *Journal of Sound and Vibration* 329 (2010) 3182–3202, http://dx.doi.org/10.1016/j.jsv.2010.02.017.

[7] M. Morzfeld, F. Ma, The decoupling of damped linear systems in configuration and state spaces, *Journal of Sound and Vibration* 330 (2011) 155–161, http://dx.doi.org/10.1016/j.jsv.2010.09.005.

[8] M. Morzfeld, F. Ma, B.N. Parlett, The transformation of second-order linear systems into independent systems, *SIAM Journal on Applied Mathematics* 71 (4) (2011) 1026–1043, http://dx.doi.org/10.1137/100818637.

[9] L. Meirovitch, *Analytical Methods in Vibrations*, Macmillan, New York, 1967.

[10] F.R. Gantmacher, *The Theory of Matrices*, Vol. 2., Chelsea, New York, 1959.

[11] L. Meirovitch, *Principles and Techniques of Vibrations*, Prentice Hall, Upper Saddle River, New Jersey, 1997.

[12] T.K. Caughey, M.E.J. O'Kelly, Classical normal modes in damped linear dynamic systems, *ASME Journal of Applied Mechanics* 32 (1965) 583–588.

[13] S.P. Bhat, D.S. Bernstein, Second-order systems with singular mass matrix and an extension of Guyan reduction, *SIAM Journal on Matrix Analysis and Applications* 17 (3) (1996) 649–657, http://dx.doi.org/10.1137/S0895479894268269.

[14] D.E. Newland, *Mechanical Vibration Analysis and Computation*, Longman Scientific & Technical, Essex, United Kingdom, 1989.

[15] S.D. Garvey, M.I. Friswell, U. Prells, Co-ordinate transformations for second order systems, part I: *general transformations*, *Journal of Sound and Vibration* 258 (5) (2002) 885–909, http://dx.doi.org/10.1006/jsvi.2002.5165.

[16] S.D. Garvey, M.I. Friswell, U. Prells, Co-ordinate transformations for second order systems, part II: *elementary structure-preserving transformations*, *Journal of Sound and Vibration* 258 (5) (2002) 911–930, http://dx.doi.org/10.1006/jsvi.2002.5166.

[17] P. Lancaster, *Lambda-Matrices and Vibrating Systems*, Pergamon Press, Oxford, United Kingdom, 1966.

[18] I. Gohberg, P. Lancaster, L. Rodman, *Matrix Polynomials*, Academic Press, New York, 1982.

[19] F. Tisseur, K. Meerbergen, The quadratic eigenvalue problem, *SIAM Review* 43 (2) (2001) 235–286, http://dx.doi.org/10.1137/S0036144500381988.

[20] A. Sestieri, S.R. Ibrahim, Analysis of errors and approximations in the use of modal co-ordinates, *Journal of Sound and Vibration* 177 (2) (1994) 145–157, http://dx.doi.org/10.1006/jsvi.1994.1424.

[21] D.T. Kawano, M. Morzfeld, F. Ma, The decoupling of defective linear dynamical systems in free motion, *Journal of Sound and Vibration* 330 (2011) 5165–5183, http://dx.doi.org/10.1016/j.jsv.2011.05.013.