Pergamon

0005-1098(94)00095-6

Automatica, Vol. 31, No. 2, pp. 315–319, 1995
Elsevier Science Ltd
Printed in Great Britain
0005-1098/95 \$9.50 + 0.00

Brief Paper

Non-pathological Sampling for Generalized Sampled-data Hold Functions*†

RICK MIDDLETON‡ and JIM FREUDENBERG§

Key Words—Control systems analysis; digital control; discrete-time systems; sampled-data systems.

Abstract—Given a generalized sampled-data hold function (GSHF), we define a frequency dependent response function having many properties analogous to those of a transfer function. In particular, the zeros of the response function have transmission blocking properties. We then study the problem of *non-pathological sampling* with a GSHF. Sampling is said to be *pathological* if the discretized version of a stabilizable/detectable continuous time plant is not itself stabilizable and detectable. Sufficient conditions for non-pathological sampling with a zero order hold have long been known. We extend these to the case of a GSHF, and describe the role in non-pathological sampling played by right half plane zeros of the response function. The results are presented for square multivariable linear systems and include a generalization to allow for a time delay.

1. *Introduction*

We consider a linear, time invariant, strictly proper continuous time system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^v$.

Our standing assumption is that (A, B) is a stabilizable pair, and (A, C) is a detectable pair. Clearly, therefore, continuous time controllers which exponentially stabilize (1)–(2) can be designed.

To apply digital control, the output of the continuous time system (1)–(2) is converted into a discrete sequence $\{y_k\}$ by sampling with period T and angular sampling frequency $\omega_s \doteq 2\pi/T$. The discrete sequence is then processed digitally to yield a discrete control sequence $\{u_k\}$. The control sequence is converted to a continuous time control input, $u(\cdot)$, through use of a hold function. Most commonly used is the Zero Order Hold (ZOH) defined by

$$u(t) = u_k, \quad t \in [kT, (k+1)T). \tag{3}$$

Using a ZOH, the continuous time system (1)–(2) is discretized to yield

$$x_{k+1} = \tilde{A}x_k + \tilde{B}u_k \tag{4}$$

$$y_k = \tilde{C}x_k, \tag{5}$$

where $\tilde{A} \doteq e^{AT}$, $\tilde{B} \doteq \int_0^T e^{A(t-\tau)}B\,d\tau$, and $\tilde{C} \doteq C$.

A crucial question in this context is whether or not discrete time control of (4), (5) can exponentially stabilize the continuous time system (1)–(2). As in Francis and Georgiou (1988) this question may be divided into the following two questions:

(a) Can we exponentially stabilize the discrete time system (4), (5)?
(b) Let $\{\zeta_k\}$ denote the state of a digital controller and suppose $\{x_k\}$ and $\{\zeta_k\} \to 0$ exponentially. Does this imply that $x(t) \to 0$ exponentially?

Fortunately, the answer to (b) is easily shown to be affirmative (see for example Francis and Georgiou, 1988, Theorem 4).

Also, in answer to (a), sufficient conditions for stabilizability and detectability of (4), (5) have long been known (Kalman *et al.*, 1963) when a ZOH is used (see Lemma 1.1). Let the spectrum of A be denoted $\sigma(A)$, and denote the closed right half of the complex plane by CRHP.

Lemma 1.1. (Kalman *et al.*, 1963.) Suppose that (4), (5) is obtained by discretizing (1), (2) with a ZOH (3). Assume that (i) (1), (2) is stabilizable and detectable and (ii) for any $\lambda_i, \lambda_k \in \sigma(A) \cap \text{CRHP}$,

$$\lambda_i \neq \lambda_k + jl\omega_s; \quad l = \pm 1, \pm 2, \ldots, \tag{6}$$

then the discretized system (4), (5) is stabilizable and detectable. (In the SISO case, the above conditions are necessary and sufficient.) ∇∇∇

Recently, (see for example Kabamba, 1987; Yan *et al.*, 1989; Ortega and Kreisselmeier, 1990; Araki, 1993) there has been a great deal of interest in alternatives to the ZOH (3), known as generalized sampled-data hold functions (GSHF). These functions may be defined by the following mapping from a discrete time control sequence, $\{u_k\}$, to the continuous time plant input, $u(t)$:

$$u(t) = h(t - kT)u_k \quad \text{for } t \in [kT, (k+1)T). \tag{7}$$

We consider the case in which the sequence $\{u_k\}$ takes values in $\mathbb{R}^p$, and thus $h(t)$ takes values‖ in $\mathbb{R}^{p\times p}$. Define $\|h(t)\| = \max\{|h_{ij}(t)|; i, j = 1, \ldots, p\}$. We shall assume throughout the paper that the GSHF $h(t)$ satisfies the following technical conditions:

Assumption 1.1.
(a)

$$\|h(t)\| \text{ is bounded on } [0, T). \tag{8}$$

(b) $h(t)$ has finitely many points of discontinuity

$$\tau_k \in [0, T), \quad k = 1, \ldots, N. \tag{9}$$

(c) Define $\tau_0 = 0$, $\tau_{N+1} = T$, and $h'(t) = dh(t)/dt$. Then $\exists M_k > \infty$ such that

$$\sup_{t\in(\tau_k, \tau_{k+1})} \|h'(t)\| \le M_k, \quad k = 0, 1, \ldots, N. \tag{10}$$

∇∇∇

* Received 3 June 1993; revised 15 March 1994; received in final form 10 June 1994. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor J. M. Dion under the direction of Editor Huibert Kwakernaak. Corresponding author Professor R. Middleton. Tel. +61 49 216026; Fax +61 49 216993; E-mail rick@ee.newcastle.edu.au.

† Work supported by the Australian Research Council under grant number A49030040.

‡ Department of Electrical and Computer Engineering, The University of Newcastle, Newcastle, NSW 2308, Australia.

§ Department of Electrical Engineering and Computer Science, The University of Michigan, Ann Arbor, MI 48109-2122, U.S.A.

‖ It is possible to extend the subsequent analysis to non-square hold functions at the expense of notational complexity.

By appropriate choice of the GSHF various attractive properties such as simultaneous stabilization (Kabamba 1987), improved gain margins (Yan *et al.*, 1989), and zero placement (Kabamba, 1987; Åström and Wittenmark, 1990) may be achieved.

In this paper, we examine questions of stabilizability/detectability for GSHF discrete time systems. We begin by defining a frequency dependent response function for a GSHF, and examining properties of this function. In particular, we define the zeros of the response function. Following this, we show how to compute a discrete time model of the form (4), (5) for a continuous system (1), (2) with GSHF (7). Our main result is a set of sufficient conditions for non-pathological sampling of an analog system with GSHF input.

2. *Response function of a GSHF*

For a GSHF described by (7), we define the associated response function

$$H(s) \doteq \int_0^T e^{-st} h(t)\,\mathrm{d}t. \tag{11}$$

Suppose that the discrete input to the hold function is the discrete unit pulse sequence $\{u_k\} = e_j\{\delta_k\}$ where $\delta_k = 1$ for $k = 0$ and 0 for all other k, and e_j is the jth elementary basis vector. Then the jth column of the response function (11) is precisely the Laplace transform of the output of the hold function:

$$H_j(s) = \mathcal{L}\{u(t)|_{\{u_k\}=e_j\{\delta_k\}}\}. \tag{12}$$

$H(s)$ has many properties analogous to those of a transfer function. Of particular interest are the *zeros* of a response function. As we shall see, these have transmission blocking properties and also can affect the stabilizability properties of the discretized system.

Before defining the zeros of the response function, note that $H(s)$ is a square matrix valued function of the complex variable, s, taking values in $\mathbb{C}^{p\times p}$. Furthermore, since the GSHF $h(\cdot)$ is bounded on the interval $[0, T)$, then the associated response function $H(s)$ has no finite poles.*

Definition 2.1. (Zeros of a response function.) Consider a response function defined by (11) and suppose that $\det(H(\cdot)) \not\equiv 0$. Then the zeros of $H(s)$ are those values of $s \in \mathbb{C}$ for which $H(s)$ has less than full rank. ∇∇∇

We shall now develop an expression for the response function in terms of the coefficients in the Fourier series expansion of the GSHF. Let

$$h(t) = \sum_{k=-\infty}^{\infty} \beta_k e^{jk\omega_s t}; \quad t \in (0, T), \tag{13}$$

where β_k are the complex Fourier coefficients

$$\beta_k \doteq \frac{1}{T}\int_0^T h(t) e^{-jk\omega_s t}\,\mathrm{d}t. \tag{14}$$

Then we have the result below.

Lemma 2.1. (Response function from Fourier coefficients.) Suppose a GSHF has Fourier coefficients β_k as in (13). Then the response function of the GSHF can be written as:

$$H(s) = (1 - e^{-sT}) \sum_{k=-\infty}^{\infty} \frac{\beta_k}{s - jk\omega_s}. \tag{15}$$

Proof. Immediate from (11) and (13). ∇∇∇

The next lemma gives a formula for the response function when the GSHF is generated by a finite dimensional linear time invariant (FDLTI) system. This covers, for example, the case of the GSHF suggested in Kabamba (1987).

Lemma 2.2. (Response function for FDLTI GSHFs.) Consider a GSHF defined as:

$$h(t) = K e^{L(T-t)} M \quad \text{for } t \in [0, T), \tag{16}$$

(for suitably dimensioned matrices K, L and M). Then the response function is:

$$H(s) = K(sI + L)^{-1}(e^{LT} - e^{-sT} I)M. \tag{17}$$

Proof. Immediate from (11) and (16). ∇∇∇

Yet another class of interesting GSHFs are the piecewise constant GSHFs, defined as:

$$H(t) = \begin{Bmatrix} a_0 & \text{for} & t \in [0, T/N) \\ a_1 & \text{for} & t \in [T/N, 2T/N) \\ \vdots & \vdots & \vdots \\ a_{N-1} & \text{for} & t \in [(N-1)T/N, T) \end{Bmatrix}. \tag{18}$$

It is clear (see for example, Yan *et al.*, 1989) that these types of GSHFs can arbitrarily closely approximate any GSHF of the form (16) by taking N large. They are also the only type of GSHFs that can be readily implemented with existing computer hardware. The following result, which follows directly from the above definitions, shows how we may compute the response function of such a GSHF.

Lemma 2.3. (Response function of piecewise constant GSHFs.) The response function of the GSHF defined by (18) is:

$$H(s) = \left(\frac{1 - e^{-sT/N}}{s}\right) \sum_{k=0}^{N-1} (a_k e^{-skT/N}). \tag{19}$$

∇∇∇

Note that GSHFs described by either (16) or (18) satisfy the technical conditions of Assumption 1.1.

We now describe a *transmission blocking* property of the zeros of $H(s)$. First, we note that one may use $H(s)$ to calculate the Laplace transform of the output of the hold device given the Z-transform of the input sequence (see Åström and Wittenmark, 1990, for example):

Lemma 2.4. Consider a GSHF $h(t)$ given by (7) and the associated response function $H(s)$ (11). Denote the Z-transform of the input sequence $\{u_k\}$ by $\tilde{U}(z)$, and the Laplace transform of the output $u(t)$ by $U(s)$. Then

$$U(s) = H(s)\tilde{U}(z)|_{z=e^{sT}}. \tag{20}$$

Our result on the transmission blocking properties of zeros of the response function will use a frequency response interpretation. It will prove convenient to allow $\{u_k\}$ and $u(t)$ to take values in $\mathbb{C}^p$. We first note that the response of the hold function to an input sequence given by samples of a complex exponential $e^{\zeta t}$, $\zeta \in \mathbb{C}$, is an infinite sum of exponentials at frequencies differing from ζ by integer multiples of the sampling frequency.

Lemma 2.5. Let the discrete input sequence $\{u_k\}$ be given by

$$\begin{aligned} u_k &= 0, & k &< 0 \\ u_k &= \tilde{u} e^{k\zeta T}, & k &\geq 0, \end{aligned} \tag{21}$$

where $\tilde{u} \in \mathbb{C}^p$ is constant. Then, for $t \geq 0$

$$u(t) = \frac{1}{T} \sum_{l=-\infty}^{\infty} H(\zeta + jl\omega_s)\tilde{u}\, e^{(\zeta + jl\omega_s)t}. \tag{22}$$

∇∇∇

The proof follows from Assumption 1.1 and a straightforward but tedious application of contour integration to invert

*This follows since, by Assumption 1.1 and Definition 2.1, $H(s)$ is a finite integral of a bounded function. The traditional expression for a ZOH includes a pole-zero cancellation when $s = 0$, and in fact has no poles.

the Laplace transform of $u(t)$ (see for example Middleton and Freudenberg, 1993). The following corollary is immediate:

Corollary 2.1. Suppose that $H(s)$ has a zero, ξ. Let v and w be non-zero vectors such that $H(\xi)v = 0$ and $w^T H(\xi) = 0$. Consider an input sequence given by (21) with $\zeta = \xi$.
(i) If $\bar{u} = v$, then for $t \geq 0$,

$$u(t) = \frac{1}{T}\sum_{l=-\infty}^{\infty} H(\xi + jl\omega_s)v e^{(\xi + jl\omega_s)t}. \tag{23}$$

(ii) Let $\bar{u}$ be arbitrary. Then for $t \geq 0$

$$w^T u(t) = \frac{1}{T}\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} w^T H(\xi + jl\omega_s)\bar{u}\, e^{(\xi + jl\omega_s)t}. \tag{24}$$

∇∇∇

It follows from (i) that inputs of the same frequency and 'input direction' as the zero of $H(s)$ yield zero output *at the frequency of the zero*. Such inputs *will* produce harmonics that appear in the output; this phenomenon has no counterpart for continuous time systems, where the output is completely zero. Also, from (ii), the hold cannot produce a steady state output with the same frequency and 'output direction' of the zero. Harmonics of the zero frequency may still appear in this output direction. These transmission blocking properties suggest that stabilizability may be lost in our system if a zero of the hold response function coincides with the plant pole in the CRHP. This intuition will be further investigated in the following section:

3. *Models for the discretized plant*

The situation we wish to consider is depicted in Fig. 1. We will denote the action of the sampler mapping a continuous signal, $y(t)$ to a discrete sequence, $\{y_k\}$, via the (obvious) definition

$$\{y_k\} = \mathcal{S}_T(y(\cdot)), \tag{25}$$

where $y_k \doteq y(kT)$.

The plant is described by the transfer function matrix, $P(s)$. Thus the Laplace transform, $Y(s)$, of $y(t)$ is:

$$Y(s) = P(s)U(s), \tag{26}$$

where $U(s)$ is the Laplace transform of $u(t)$. For a given hold function, we define the corresponding discretized plant transfer function matrix, $(PH)_d(z)$, as follows. The jth column of $(PH)_d(z)$ is the Z-transform of the sampled response, $\{y_k\}$, to a unit pulse in the jth input, $\{u_k\} = e_j\{\delta_k\}$. By the definition (12) of the response function we have (under the above conditions):

$$U(s) = H_j(s). \tag{27}$$

Combining (25)–(27), and using (12) it is clear that:

$$(PH)_d(z) = \mathcal{Z}\{\mathcal{S}_T\{\mathcal{L}^{-1}\{P(s)H(s)\}\}\}. \tag{28}$$

As noted previously, $H(s)$ is an entire* function, and we can thus simplify (28) in the case where $P(s)$ has only simple poles.

Lemma 3.1. (Discretized plant by partial fractions expansion.) Suppose $P(s)$ has only simple poles, and thus let:

$$P(s) = \sum_{i=1}^{n} \frac{R_i}{s - p_i}. \tag{29}$$

Then

$$(PH)_d(z) = \sum_{i=1}^{n} \left\{R_i H(p_i)\left(\frac{e^{p_i T}}{z - e^{p_i T}}\right)\right\}. \tag{30}$$

Proof. Following (28), note that

$$\mathcal{L}^{-1}\{P(s)H(s)\} = \mathcal{L}^{-1}\left\{\sum_{i=1}^{n} \frac{R_i H(s)}{(s - p_i)}\right\}$$
$$= \sum_{i=0}^{n} R_i H(p_i) e^{p_i t} \quad \text{for } t > 0 \tag{31}$$

(since $H(s)$ is entire). So

$$(PH)_d(z) = \mathcal{Z}\left\{\mathcal{S}_T\left\{\sum_{i=1}^{n} R_i H(p_i) e^{p_i^t}\right\}\right\}$$
$$= \sum_{i=1}^{n}\left(R_i H(p_i)\frac{e^{p_i T}}{z - e^{p_i T}}\right) \tag{32}$$

as required. ∇∇∇

The above lemma indicates that in the case of simple poles, we may lose stabilizability and/or detectability in our discretized plant if p_i is a CRHP zero of $H(s)$. The following state space results (which follow directly from solving the state space equations) will be used later to make a more general, precise statement:

Lemma 3.2. (State space formula for discretized plant.) Consider a continuous time system as described in (1), (2), with input, $u(t)$ generated as in (7). Let x_k denote $x(kT)$, then

$$x_{k+1} = e^{AT}x_k + \left(\int_0^T e^{A(T-t)}Bh(t)\,dt\right)u_k \tag{33}$$

and

$$y_k = Cx_k. \tag{34}$$

∇∇∇

Corollary 3.1. Consider a continuous time system with time delay $\Delta < T$, described by:

$$\dot{x}(t) = Ax(t) + Bu(t - \Delta) \tag{35}$$
$$y(t) = Cx(t), \tag{36}$$

where the input, $u(\cdot)$ is generated by a GSHF, (7). Then, if we let $\bar{x}_k^T \doteq [x^T(kT)\ u_{k-1}^T]$ a discrete time state space model is:

$$\bar{x}_{k+1} = \begin{bmatrix} e^{AT} & \bar{B}_1 \\ 0 & 0 \end{bmatrix}\bar{x}_k + \begin{bmatrix} \bar{B}_2 \\ I \end{bmatrix}u_k \tag{37}$$

$$y_k = [C \quad 0]\bar{x}_k, \tag{38}$$

where

$$\bar{B}_1 \doteq \int_0^{\Delta} e^{A(T-\tau)}Bh(T + \tau - \Delta)\,d\tau \tag{39}$$

* An 'Entire' function is a complex function that is analytic everywhere in the finite complex plane.

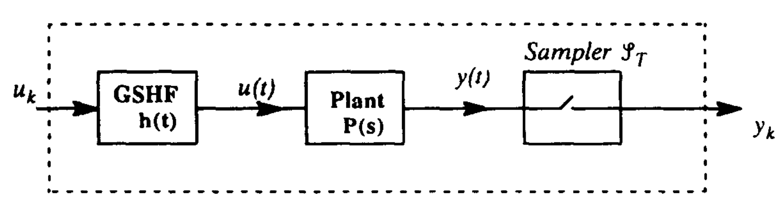

Fig. 1. Discrete plant configuration.

and

$$\bar{B}_2 \doteq \int_{\Delta}^{T} e^{A(T-\tau)} B h(\tau - \Delta)\, d\tau. \tag{40}$$

∇∇∇

Corollary 3.2. Consider a continuous time system with time delay ($lT + \Delta$) ($\Delta < T, l \in \mathbb{N}^+$) described by:

$$\dot{x}(t) = Ax(t) + Bu(t - lT - \Delta) \tag{41}$$

$$y(t) = Cx(t). \tag{42}$$

Suppose that the input, $u(t)$, is generated by a GSHF, (7) and let

$$\bar{x}_k^T \doteq [x^T(kT) \quad u_{k-l-1}^T \quad u_{k-l}^T \cdots u_{k-1}^T]. \tag{43}$$

Then a discrete time state space model is:

$$\bar{x}_{k+1} = \begin{bmatrix} e^{AT} & \bar{B}_1 & \bar{B}_2 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ 0 & 0 & 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & I \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \bar{x}_k + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ I \end{bmatrix} u_k \tag{44}$$

$$y_k = [C \quad 0 \quad 0 \quad 0 \quad \cdots \quad 0]\bar{x}_k, \tag{45}$$

where $\bar{B}_1$ and $\bar{B}_2$ are as defined in (40) and (39). ∇∇∇

In the next section, we develop a set of sufficient conditions for stabilizability/detectability of the discretized plant, using the state space models developed above.

4. *Stabilizability and detectability*

Let A (as in (1)) have 'r' distinct closed right half plane (CRHP) eigenvalues, $\lambda_1 \cdots \lambda_r$, of geometric multiplicities, $g_1 \cdots g_r$ (respectively). We denote the right null space of $(\lambda_i I - A^T)$ by W_i, and a basis for W_i by the columns of $W_i \in \mathbb{C}^{n \times g_i}$. By definition of the geometric multiplicities, $\dim(W_i) = g_i$. For the remainder of this section we assume that whenever an eigenvalue, λ_i, of A is mentioned, that we are only considering (CRHP) eigenvalues. Similarly, whenever we consider eigenvalues, $\tilde{\lambda}_i$, of a discrete time matrix, $\tilde{A}$, then we are only considering eigenvalues in the region:

$$\tilde{\lambda}_i \in D^c \doteq \{z \in \mathbb{C} : |z| \geq 1\}. \tag{46}$$

We then have the following Lemma:

Lemma 4.1. (Continuous time stabilizability.) (A, B) is stabilizable if and only if for all λ_i in CRHP:

$$\operatorname{rank}[W_i^T B] = g_i. \tag{47}$$

Proof. By the PBH test (e.g. Kailath, 1980, §2.43) (A, B) is stabilizable if and only if

$$\operatorname{rank}[\lambda_i I - A \quad B] = n. \tag{48}$$

Equation (48) is equivalent to saying that there does not exist $w_i \in \operatorname{Span}\{W_i\}$ such that

$$w_i^T B = 0. \tag{49}$$

This condition is in turn equivalent to saying that there does not exist a non-trivial $v_i \in \mathbb{C}^{g_i}$ such that

$$v_i^T W_i^T B = 0, \tag{50}$$

which is precisely the same as the condition

$$\operatorname{rank}[W_i^T B] = g_i \tag{51}$$

as required. ∇∇∇

Following the above proof steps, we may easily derive the following corollary:

Corollary 4.1. (Discrete time stabilizability.) Consider a discrete time system given by matrices $\tilde{A}$, $\tilde{B}$; with $\tilde{r}$ distinct D^c eigenvalues, $\tilde{\lambda}_1 \cdots \tilde{\lambda}_{\tilde{r}}$ of geometric multiplicities $\tilde{g}_1 \cdots \tilde{g}_{\tilde{r}}$, respectively. Denote a basis for the right null-space of $(\tilde{\lambda}_i I - \tilde{A}^T)$ by the columns of $\tilde{W}_i \in \mathbb{C}^{n \times \tilde{g}_i}$. Then $(\tilde{A}, \tilde{B})$ is stabilizable if and only if

$$\operatorname{rank}[\tilde{W}_i^T \tilde{B}] = \tilde{g}_i. \tag{52}$$

∇∇∇

We will now begin to relate stabilizability/detectability for discrete and continuous systems. To do this, we will make use of the following condition [see also Lemma 1.1 condition (ii)]:

Condition (C1): for all $\lambda_i, \lambda_k \in$ CRHP: $\lambda_i - \lambda_k \neq jl\omega_S$ for any non-zero integer l. ∇∇∇

We then have the following result:

Lemma 4.2. (Equivalence of eigenstructure.) Let $\tilde{A} = e^{AT}$ (as indicated by Lemma 3.2). Then under condition (C1), the distinct eigenvalues of $\tilde{A}$ in D^c are given by

$$\tilde{\lambda}_i = e^{\lambda_i T} \tag{53}$$

with geometric multiplicities

$$\tilde{g}_i = g_i. \tag{54}$$

Furthermore, we may choose

$$\tilde{W}_i = W_i. \tag{55}$$

Proof. Immediate on noting that $\tilde{\lambda}_i$ are distinct under (C1) and, therefore, the eigenstructure is preserved. ∇∇∇

Corollary 4.2. Under condition (C1), $(\tilde{A}, \tilde{B})$ is discrete time stabilizable if and only if

$$\operatorname{rank}[W_i^T \tilde{B}] = g_i \quad \text{for all } i. \tag{56}$$

Proof. Immediate from Lemma 4.2. ∇∇∇

We now proceed to our main result on non-pathological sampling, building on the previous corollaries and Lemmas:

Theorem 4.1. The following three conditions are sufficient for stabilizability/detectability of the discretized plant, (33), (34):
(i) Condition (C1).
(ii) The continuous plant, (1), (2), is stabilizable and detectable.
(iii) $H(s)$ has no zeros at $s = \lambda_i$.
(In the single input, single output case, these conditions are in fact necessary and sufficient.)

Proof. The discretized plant can be represented by state space matrices, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ following the obvious definitions from Lemma 3.2. Conditions (i) and (ii) are sufficient (using the dual of Corollary 4.2) to establish detectability of $\tilde{A}$, $\tilde{C}$.

Consider then $W_i^T \tilde{B}$:

$$W_i^T \tilde{B} = \int_0^T W_i^T e^{A(T-t)} B h(t)\, dt = \int_0^T W_i^T B e^{\lambda_i (T-t)} h(t)\, dt \tag{57}$$

(since W_i is an invariant subspace of A^T). From (57) we have

$$W_i^T \tilde{B} = (W_i^T B) e^{\lambda_i T} \int_0^T e^{-\lambda_i t} h(t)\, dt = (e^{\lambda_i T})(W_i^T B) H(\lambda_i). \tag{58}$$

Thus sufficient conditions for stabilizability are that: (i) $\tilde{W}_i = W_i$ (so that $\tilde{W}_i^T \tilde{B} = W_i^T \tilde{B}$); (ii) $(W_i^T B)$ is full rank; and (iii) $H(\lambda_i)$ is full rank. ∇∇∇

Note that the extra condition here, compared with the ZOH case, arises to ensure there are not unstable pole-zero cancellations between the plant and the GSHF response function. In the case of a ZOH, condition (C1) is sufficient to guarantee that there are no such pole zero cancellations. The following corollary extends Theorem 4.1 to allow for time delays in the plant:

Corollary 4.3. Sufficient conditions for stabilizability/detectability of the discretized plant with time delay, (37)–(39) or (44), (45) are:
(i) Condition (C1).
(ii) The continuous plant without time delay (1), (2) is stabilizable and detectable.
(iii) $H(s)$ has no zeros at $s = \lambda_i$.

Proof. We first consider the case of a plant described by (37)–(39).

Note that the only eigenvalues of the discretized A matrix lying in D^c are $e^{\lambda_i T}$ (as in Lemma 4.2). The discretized eigenstructure in this case can be selected as [under condition (C1)]:

$$\tilde{W}_i = \begin{pmatrix} W_i \\ e^{-\lambda_i T}\bar{B}_1^T W_i \end{pmatrix}. \tag{59}$$

We now consider $\tilde{W}_i^T\begin{bmatrix} \bar{B}_2 \\ I \end{bmatrix}$. Using (59) we have:

$$\begin{aligned}\tilde{W}_i^T\begin{bmatrix} \bar{B}_2 \\ I \end{bmatrix} &= W_i^T\bar{B}_2 + e^{-\lambda_i T}W_i^T\bar{B}_1 \\ &= e^{-\lambda_i T}\int_0^{\Delta} W_i^T e^{A(T-\tau)}Bh(T+\tau-\Delta)\,d\tau \\ &\quad + \int_{\Delta}^{T} W_i^T e^{A(T-\tau)}Bh(T-\Delta)\,d\tau.\end{aligned} \tag{60}$$

Since the columns of W_i form a basis for the right nullspace of $(\lambda_i I - A^T)$, we have

$$\begin{aligned}\tilde{W}_i^T\begin{pmatrix} \bar{B}_2 \\ I \end{pmatrix} &= \int_0^{\Delta} W_i^T B e^{\lambda_i(T-\tau)}h(T+\tau-\Delta)\,d\tau \\ &\quad + \int_{\Delta}^{T} W_i^T B e^{-\lambda_i T}e^{\lambda_i(T-\tau)}h(\tau-\Delta)\,d\tau \\ &= (W_i^T B)\left(\int_{T-\Delta}^{T} e^{\lambda_i(T-\Delta-\sigma)}h(\sigma)\,d\sigma \right. \\ &\quad \left. + \int_0^{T-\Delta} e^{\lambda_i(T-\Delta-\sigma)}h(\sigma)\,d\sigma\right) \\ &= (W_i^T B)e^{\lambda_i(T-\Delta)}H(\lambda_i).\end{aligned} \tag{61}$$

The remainder of the proof follows that of Theorem 4.1.

The proof for the case of a plant with a time delay of greater than one sample (as in Corollary 3.2) follows similar lines to that above. Firstly, we note that in this case, a suitable matrix of basis vectors for the right eigenvectors of $\tilde{A}^T$ with eigenvalue λ_i is:

$$\tilde{W}_i = \begin{pmatrix} W_i \\ e^{-\lambda_i T}\bar{B}_1^T W_i \\ e^{-\lambda_i T}(e^{-\lambda_i T}\bar{B}_1^T W_i + \bar{B}_2^T W_i) \\ \vdots \\ e^{-l\lambda_i T}(e^{-\lambda_i T}\bar{B}_1^T W_i + \bar{B}_2^T W_i) \end{pmatrix}. \tag{62}$$

In this case, $\tilde{W}_i^T\tilde{B}$ becomes

$$\begin{aligned}\tilde{W}_i^T\tilde{B} &= e^{-l\lambda_i T}(e^{-\lambda_i T}W_i^T\bar{B}_1 + W_i^T\bar{B}_2) \\ &= (W_i^T B)e^{-l\lambda_i T}e^{\lambda_i(T-\Delta)}H(\lambda_i)\end{aligned} \tag{63}$$

and the remainder of the proof is as for Theorem 4.1. ∇∇∇

5. *Conclusions*

Given a GSHF, we have defined an associated frequency response function having many properties analogous to those of a transfer function. In particular, the zeros of the response function have transmission blocking properties. Furthermore, if an unstable plant is discretized with a hold function having a zero at the location of the unstable pole, then the resulting discretized system may be unstabilizable and/or undetectable. (In the scalar case this is guaranteed to happen.) It follows, therefore, that design procedures using GSHFs must take care to ensure that such unstable pole-zero cancellations do not occur. The procedures of Kabamba (1978) implicitly achieve this by utilising information about the plant in constructing the GSHF.

By analogy with the continuous time case, one might expect that *approximate* cancellations between a non-minimum phase zero of the hold and an unstable plant pole would yield sensitivity and robustness difficulties. The implications of such difficulties, if they indeed exist, remain a topic for future research.

Acknowledgement—The authors are grateful to Julio Braslavsky for helpful comments and suggestions.

References

Araki, M. (1993). Recent developments in digital control theory. In *Proc. 12th IFAC World Congress,* Sydney, Australia.

Åström, K. J. and B. Wittenmark (1990). *Computer-Controlled Systems, Theory and Design,* 2nd Ed. Prentice-Hall, Englewood Cliffs, New Jersey.

Francis, B. A. and T. T. Georgiou (1988). Stability theory for liner time-invariant plants with periodic digital controllers. *IEEE Trans. Autom. Contr.,* 820–832.

Kabamba, P. T. (1987). Control of linear systems using generalised sampled-data hold functions. *IEEE Trans. Autom. Contr.,* 772–783.

Kailath, T. (1980). *Linear Systems.* Prentice-Hall, Englewood Cliffs, New Jersey.

Kalman, R., B. L. Ho and K. Narendra (1963). Controllability of linear dynamical systems. In *Contributions to Differential Equations* Vol. 1. Interscience, New York.

Middleton, R. H. and J. Freudenberg (1993). Zeros and Non-Pathological Sampling for Generalised Sampled-Data Hold Functions. Technical Report EE9318, Department of Electrical and Computer Engineering, University of Newcastle, 2308, Australia.

Ortega, R. and G. Kreisselmeier (1990). Discrete-time model reference adaptive control for continuous-time systems using generalised sampled-data hold functions. *IEEE Trans. Autom. Contr.,* 334–338.

Yan, W., B. D. O. Anderson and R. R. Bitmead (1989). On the Gain Margin Improvement Using Dynamic Compensation Based on Generalized Sampled-Data Hold Functions. Technical Report, Department of Systems Engineering, Australian National University.