



Partial Eigenvalue Assignment for LTI Systems with \mathbb{D} -Stability and LMI

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Abstract

Partial eigenvalue assignment (PEVA) is a milestone in the field of structural dynamics control, for which the system models based on finite-element analysis have larger dimensions. The solution for PEVA has early seminal contributions, but today, other relevant advances have been offered by researchers in this interdisciplinary field. This work aims to demonstrate the construction of a new method to solve PEVA in linear time-invariant systems to guarantee the regional stability of the reassigned eigenvalues. The theoretical review of PEVA and \mathbb{D} -stability are shown to support the development of an algorithm that involves linear matrix inequalities and left-eigenvectors parametrization. To verify the efficiency of the algorithm, tests were performed on numerical examples borrowed from the literature. Furthermore, the solution's quality is illustrated through location plots and eigenvalue comparison tables.

Keywords EVA · PEVA · LMI · LTI · Stability

1 Introduction

The eigenvalue assignment problem (EVA) in linear time-invariant system (LTI) is one of the primary issues for control design that requires both stability and meeting the transient response specifications. The state feedback control structure is the most popular control technique used to solve this problem (Krokavec et al. 2015). Several works show different theoretical and computational methods to compute feedback gain, and the Krokavec et al. (2015) show some of them using: (1) explicit parametric form, (2) Sylvester's equation, (3) singular matrix decomposition properties, (4) the minimal condition number of eigenvector matrix, (5) Moore's parametric form, and (6) linear matrix inequalities (LMIs).

The control design using partial eigenvalue assignment (PEVA) is very profitable for applications, e.g., in structural

dynamics control or largely distributed parameters electrical networks. In such models, the dimension can be of hundreds or thousands, due to the nature of finite-element modeling (FEM). The conventional eigenvalue assignment (EVA) methods (e.g., QR iterations and real Schur form) for many of these control designs whose systems are large or sparse (as in electrical networks, power systems, and computer networks) can fail due to the spillover phenomenon: small changes in the system parameters can lead to significant variations in the real eigenvalues' locations (Datta et al. 2002). The primary goal of the control design in such systems to assign only a small number of eigenvalues using state feedback, which is generally responsible for instability or undesirable phenomena, as resonance or flutter, kept the remaining of the spectrum unchanged. The PEVA problem was revisited in recent work by Araújo (2018) by applying state-derivative feedback.

In light of these facts, the issue of PEVA has gained greater relevance in the engineering and scientific fields. Recent studies are being carried out to solve the PEVA for second-order and high-order systems with or without delay control; thus, some articles recommended for reading that briefly are the developed works by Belotti et al. (2018); Wang and Zhang (2017). Meanwhile, works focused on methods for PEVA with the objective of spectrum assignment (eigenvalues) to predefined regions in the complex plane are less frequent. The article written by Ou et al. (2014) creates an optimization

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algorithm to solve a particular PEVA: make a strict partial assignment for $n - m$ poles and a non-strict assignment for m poles. Unfortunately, the application of an algorithm written by Ou et al. (2014) is restricted to a single-input and single-output (SISO) system.

It is known from the theory of linear systems that the position of eigenvalues influences the transient response behavior, for instance, if its a SISO system: rise time, settling time, and maximum overshoot can be predicted by knowing the real and imaginary part of the poles while using specific formulations. In the multivariable context, thanks to the linear properties, the transfer function can be analyzed as a superposition of several first and second-order functions, each of them with its respective transient response characteristics. An interesting application of the EVA methods is in the monotonic tracking control problem, where it is necessary to adjust the controller to ensure rapid transient stabilization avoiding overshoot and undershoot phenomena. Relevant applications for this problem are in heating/cooling systems, satellite and elevator positioning, and automobile cruise control (Ntogramatzidis et al. 2014, 2016). Nevertheless, these works approach only strict EVA, and the all eigenvalues that are not invariant must be reassigned, in contrast with PEVA methodologies.

The ability to stabilize LTI systems to let the eigenvalues in specified regions is supplied by the theorem of \mathbb{D} -stability and uses LMI that can be solved by interior point optimization algorithms (Ostertag 2011). Some approaches to solving EVA with LMI for continuous and discrete system with \mathbb{D} -stability can be read in Duan and Yu (2013).

Thus, when specifying a \mathbb{D} -region in PEVA to a controllable system, it is desirable to establish limits and ranges for transient response measures, consequently this flexibility will allow PEVA methods to be applied in multiobjective control design, and furtherly it uses fewer equation and variables than regional EVA methods. This approach is relevant for design specifications in sparse and large-scale systems, especially in cases where it is desired to allocate fewer eigenvalues, but techniques capable of solving this type of problem are not reported in the literature to the best of the knowledge of the authors. To provide a method to solve PEVA for regional assignment, equations from PEVA solutions with strict assignment and regional EVA with LMI are intuitively merged to obtain the algorithm proposed in this work. The rest of the paper is organized as follows: in Sect. 2, a classical parametric solution for the PEVA is presented, as well as the statement of the problems involving EVA and PEVA. In Sect. 3, a summary of \mathbb{D} -stability for LTI systems is lined, the proposed approach is described in Sects. 4 and 5, and some examples based on real-world data are offered to show the effectiveness of the proposed approach.

2 Review of Datta's Parametric Solution to PEVA for LTI System

Some useful definitions for formulating the PEVA problem are given:

Definition 1 The equations to define a control system LTI MIMO with state feedback are:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m} \quad (1)$$

$$\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t), \quad \mathbf{F} \in \mathbb{R}^{m \times n} \quad (2)$$

where $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ are, respectively, the state vector and its time derivative; $\mathbf{u}(t)$ is input vector; \mathbf{A} , \mathbf{B} , and \mathbf{F} are, respectively, state matrix, control or influence matrix, and state feedback gain. \square

The spectrum of a matrix and the open- and closed-loop system modes are useful for describing the problems involving EVA and PEVA methods:

Definition 2 The spectrum of matrix \mathbf{A} symbolized by function $\sigma(\mathbf{A})$ is the set of its eigenvalues. \square

Definition 3 $\lambda_i \in \mathbb{C}$ will be defined as an open-loop LTI system mode if $\lambda_i \in \sigma(\mathbf{A})$. $\mu_i \in \mathbb{C}$ will be defined as a closed-loop LTI system mode if $\mu_i \in \sigma(\mathbf{A} - \mathbf{B}\mathbf{F})$. \square

The control problems involving the EVA and PEVA methods are described below:

Problem 1 The EVA problem consists in determining the matrix \mathbf{F} of the control system such that sets \mathbb{S}_{OL} and \mathbb{S}_{CL} :

$$\mathbb{S}_{OL} = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{C} \quad (3)$$

$$\mathbb{S}_{CL} = \{\mu_1, \dots, \mu_n\} \subset \mathbb{C} \quad (4)$$

as the set of open- and closed-loop system modes, respectively. \square

Problem 2 The PEVA problem consists in determining the matrix \mathbf{F} of the control system such that sets \mathbb{S}_{OL} and \mathbb{S}_{CL} :

$$\mathbb{S}_{OL} = \{\lambda_1, \dots, \lambda_n\} \quad (5)$$

$$\mathbb{S}_{CL} = \{\mu_1, \dots, \mu_p, \lambda_{p+1}, \dots, \lambda_n\}, \quad p < n \quad (6)$$

as the set of open- and closed-loop system modes, respectively. \square

The fundamental difference between Problems 2 and 3 is the amount of system modes closed-loop specified: in the PEVA only p desired modes are needed while in the EVA all modes. The theorem of parametric solution for PEVA developed by Datta et al. (2002) is built based on a set of theorems, definitions and assumptions:

Theorem 1 (Datta et al. (2002)) *LTI system is defined by Eq. (1), and the LTI system in open-loop mode λ_i is controllable if the left eigenvector associated \mathbf{y} satisfies the equation:*

$$\mathbf{y}^H \mathbf{B} \neq \mathbf{0}_{1 \times m} \quad (7)$$

where \mathbf{y}^H is the Hermitian transpose of \mathbf{y} and $\mathbf{0}_{1 \times m}$ is null matrix with m -columns. \square

Definition 4 For a LTI system defined by equation (1), set of p open-loop system modes defined by:

$$\mathbb{S}_{pOL} = \{\lambda_1, \dots, \lambda_p\} \subset \mathbb{S}_{OL}, \quad n < p, \quad (8)$$

the system will be partially controllable through the modes of set \mathbb{S}_{pOL} if all modes of \mathbb{S}_{pOL} are controllable. \square

Theorem 2 (Datta et al. (2002)) *Problem 2 is solvable for any choice of \mathbb{S}_{CL} , Eq. (6), if the system is partially controllable with respect to the set \mathbb{S}_{pOL} , Eq. (8). That is, whenever a solution exists, there will be endless many others.* \square

Assumption 1 The system in problem (2) is partially controllable with respect to the \mathbb{S}_{pOL} describe in Eq. (8). \square

Assumption 2 The sets \mathbb{S}_{OL} described by Problem 2 and:

$$\mathbb{S}_{pCL} = \{\mu_1, \dots, \mu_p\} \subset \mathbb{S}_{CL}, \quad p < n \quad (9)$$

should be closed under complex conjugation and also disjoint. \square

Assumption 3 The control matrix \mathbf{B} has full column rank. \square

Based on these assumptions, theorems and definition obtain a parametric solution theorem for the PEVA:

Theorem 3 (Datta et al. (2002)) *Let \mathbf{y}_i be the left eigenvector associated with the eigenvalue λ_i , and let \mathbf{Y}_1 be the matrix of linearly independent eigenvectors of the spectrum to be assigned:*

$$\mathbf{Y}_1 = [\mathbf{y}_1 \cdots \mathbf{y}_p] \quad (10)$$

Let γ_i be an arbitrary m -row vector having the constraints:

$$\gamma_i = \bar{\gamma}_k \Leftrightarrow \mu_i = \bar{\mu}_k \quad (11)$$

Let the matrices be:

$$\mathbf{\Gamma} = [\gamma_1 \cdots \gamma_p] \quad (12)$$

$$\mathbf{\Lambda}_1 = \text{blkdiag} \{ \lambda_1, \dots, \lambda_p \} \quad (13)$$

$$\mathbf{\Lambda}_{c1} = \text{blkdiag} \{ \mu_1, \dots, \mu_p \} \quad (14)$$

that is, $\mathbf{\Gamma} \in \mathbb{R}^{m \times p}$ is an arbitrary, column self-conjugated matrix as defined by (11); and $\text{blkdiag} \{ \}$ is an operator

that builds up a block diagonal matrix with the elements in it argument. The parametric solution for PEVA is the solution for the equations:

$$\mathbf{\Lambda}_1 \mathbf{Z}_1 - \mathbf{Z}_1 \mathbf{\Lambda}_{c1} = \mathbf{Y}_1^H \mathbf{B} \mathbf{\Gamma} \quad (15)$$

$$\mathbf{\Phi} \mathbf{Z}_1 = \mathbf{\Gamma} \quad (16)$$

$$\mathbf{F} = \mathbf{\Phi} \mathbf{Y}_1^H \quad (17)$$

the matrix \mathbf{Z}_1 is a non-singular matrix, solution of the Sylvester's equation described by (15), and \mathbf{Y}_1^H is the conjugated transpose of \mathbf{Y}_1

Proof See in Theorem 4.1 of Datta et al. (2002). \square

Remark 1 The choice of the matrix $\mathbf{\Gamma}$ has been extensively discussed in several works to design of the feedback matrix using Sylvester type equations - see, e.g., Araújo et al. (2009), Chen (2013). All these works show that for the single-input case, the (arbitrary) choice of $\mathbf{\Gamma}$ has no relevance for the solution, since it is unique and invariant with respect to $\mathbf{\Gamma}$. However, for the multi-input case, for each choice of $\mathbf{\Gamma}$, a different solution is obtained, and it gives the designer the possibility of to explore EVA or PEVA design with additional control requisites, e.g., the minimum norm of the feedback matrix \mathbf{F} (Brahma and Datta 2009).

Remark 2 In numerical methods to solve EVA problems, it is estimated that a quantity of FLOPS (floating point operations) in function to n^3 is required, since Theorem 3, depending on the numerical method used, requires only an amount in function to p^3 . In the case of control designs that need to allocate a number of poles $p \ll n$, the preference for numerical methods applied to Theorem 3 is a wise decision. Some mitigable disadvantage of applying Theorem 3 are (1) search for the appropriate choice of the numerical method to solve Sylvester's equation; (2) and the requirement that $\mathbb{S}_{pCL} \cap \mathbb{S}_{OL} = \emptyset$.

Remark 3 Substituting the Gamma expression of (16) into (15) has the following equation:

$$(\mathbf{\Lambda}_1 - \mathbf{Y}_1^H \mathbf{B} \mathbf{\Phi}) \mathbf{Z}_1 = \mathbf{Z}_1 \mathbf{\Lambda}_{c1} \quad (18)$$

which shows that \mathbf{Z}_1 act as a matrix of eigenvectors and $\mathbf{\Lambda}_{c1}$ is the spectrum matrix of $\mathbf{\Lambda}_1 - \mathbf{Y}_1^H \mathbf{B} \mathbf{\Phi}$. \square

Remark 3 shows that the matrix \mathbf{Z}_1 is important in the sensitivity of the $\mathbf{A} - \mathbf{B} \mathbf{F}$ spectrum, so the numerical method to solve the Sylvester's equation directly influences quality of allocation.

According to the author, the parametric algorithm is numerically viable and computationally feasible for large and sparse systems; for some other issues, see Datta et al. (2002).

3 Review of \mathbb{D} -Stability for LTI System

An important concept for \mathbb{D} -stability is the definition of \mathbb{D} -region, according to Duan and Yu (2013):

Definition 5 Consider the complex plane whose coordinates are governed by the variable $s \in \mathbb{C}$, a region of spectrum \mathbb{D} will be described by LMI if there is a characteristic function $\mathbf{G}(s)$ describing:

$$\mathbb{D} = \{s \in \mathbb{C} | \mathbf{G}(s) \prec 0\} \quad (19)$$

$$\mathbf{G}(s) = \mathbf{L} + s\mathbf{V} + \bar{s}\mathbf{V}^H \quad (20)$$

$$\mathbf{L} \in \mathbb{R}_S^r \quad (21)$$

$$\mathbf{V} \in \mathbb{R}^{r \times r} \quad (22)$$

\mathbb{R}_S^r is the set of symmetric matrices of order r and $\mathbf{X} \prec 0$ says that matrix \mathbf{X} is definite-negative. \square

In the literature, there are some types of regions already defined, among them some are described in the book written by Duan and Yu (2013):

- Strip whose eigenvalues have a real part limited by $-\beta < \text{Re}\{s\} < -\alpha < 0$;
- Disk of radius r and center by $s = -q \in \mathbb{R}$.
- Sector that is an isosceles triangle of vertex in the origin and bisector coincident with real axis of negative values with the base angle base θ .

Remark 4 If the designer wants to define a region that reflects the non-empty intersection of several regions as (19) through (22), that is, $\mathbb{D} = \bigcap_1^k \mathbb{D}_k \neq \emptyset$, the characteristic matrices (23) and (24) are given as Duan and Yu (2013):

$$\mathbf{L} = \text{blkdiag} \{\mathbf{L}_1, \dots, \mathbf{L}_k\} \quad (23)$$

$$\mathbf{V} = \text{blkdiag} \{\mathbf{V}_1, \dots, \mathbf{V}_k\} \quad (24)$$

Table 1 shows some typical characteristic functions and the related matrices. An important definition is the Kronecker product that helps to formulate the Problem of \mathbb{D} -stability by LMI, see Duan and Yu (2013):

Table 1 Characteristic functions and related matrices

Type	Identification	\mathbf{L}	\mathbf{V}
Strip	\mathbb{H}_β^α	$2 \begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Disc	\mathbb{D}_q^r	$\begin{bmatrix} -r & q \\ q & -r \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
Sector	\mathbb{H}_θ	$\mathbf{0}_{2 \times 2}$	$\begin{bmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{bmatrix}$

Definition 6 Consider the matrices $\tilde{\mathbf{A}} = [a_{ij}] \in \mathbb{C}^{n \times m}$ and $\tilde{\mathbf{B}} \in \mathbb{C}^{l \times q}$, be $\tilde{\mathbf{a}}_{ik}$, the Kronecker product between the matrix $\tilde{\mathbf{A}}$, and $\tilde{\mathbf{B}}$ will be:

$$\tilde{\mathbf{A}} \otimes \tilde{\mathbf{B}} = \begin{bmatrix} a_{11}\tilde{\mathbf{B}} & a_{12}\tilde{\mathbf{B}} & \cdots & a_{1m}\tilde{\mathbf{B}} \\ a_{21}\tilde{\mathbf{B}} & a_{22}\tilde{\mathbf{B}} & \cdots & a_{2m}\tilde{\mathbf{B}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\tilde{\mathbf{B}} & a_{n2}\tilde{\mathbf{B}} & \cdots & a_{nm}\tilde{\mathbf{B}} \end{bmatrix} \quad (25)$$

\square

To determine if the eigenvalues of an LTI system are contained in a stable region specified by the set \mathbb{D} , it is useful to define \mathbb{D} -stability:

Theorem 4 Be an LTI system in state space governed by:

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}\mathbf{x}(t) + \tilde{\mathbf{B}}\mathbf{u}(t) \Leftrightarrow \tilde{\mathbf{A}} \in \mathbb{R}^{n \times n}, \tilde{\mathbf{B}} \in \mathbb{R}^{n \times m} \quad (26)$$

$$\mathbf{u}(t) = -\tilde{\mathbf{F}}\mathbf{x}(t) \Leftrightarrow \tilde{\mathbf{F}} \in \mathbb{R}^{m \times n} \quad (27)$$

the matrix $\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\tilde{\mathbf{F}}$ will be \mathbb{D} -stable, having eigenvalues in set \mathbb{D} , if there is a positive definite symmetric matrix \mathbf{P} and a real \mathbf{W} satisfying the LMI:

$$\mathbf{L} \otimes \mathbf{P} + \text{He} \left\{ \mathbf{V}, \tilde{\mathbf{A}}\mathbf{P} \right\} + \text{He} \left\{ \mathbf{V}, \tilde{\mathbf{B}}\mathbf{W} \right\} \prec 0 \quad (28)$$

$$\text{He} \{ \mathbf{X}, \mathbf{Y} \} = \mathbf{X} \otimes \mathbf{Y} + \mathbf{X}^H \otimes \mathbf{Y}^H \quad (29)$$

$$\tilde{\mathbf{F}} = \mathbf{W}\mathbf{P}^{-1} \quad (30)$$

\square

4 PEVA with \mathbb{D} -Stability for LTI System

In order for \mathbf{F} to be a solution for PEVA with $\mathbb{S}_{\text{pCL}} \subset \mathbb{D}$, a new method based on Eqs. (17), (18), (28) and (30) is sought to endeavor, thus resulting in the following problem:

Problem 3 Consider the LTI system:

$$\dot{\mathbf{z}}(t) = \mathbf{\Lambda}_1 \mathbf{z}(t) + \mathbf{Y}_1^H \mathbf{B}(t) \quad (31)$$

$$\psi(t) = -\Phi \mathbf{z}(t) \quad (32)$$

The matrices $\mathbf{\Lambda}_1$, \mathbf{Y}_1^H , Φ , and \mathbf{B} are the same as those in Eq. (15). The objective is to solve the PEVA for this system in order to ensure that the spectrum \mathbb{S}_{pCL} belongs to the set \mathbb{D} . \square

Problem 3 is based on Remark (1), since Eq. (18) explicitly describes the fundamental relation between matrix of eigenvalues $\mathbf{\Lambda}_{c1}$ and eigenvectors \mathbf{Z}_1 of the system written by (31) and (32).

Hypothesis 1 The matrix Φ of Problem 3 is obtained by solving Theorem 4, while replacing matrices $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{F}}$ by:

$$\tilde{\mathbf{A}} = \mathbf{A}_1, \tilde{\mathbf{B}} = \mathbf{Y}_1^H \mathbf{B}, \tilde{\mathbf{F}} = \Phi \quad (33)$$

□

The rationale for the formulation of Hypothesis 1 is that the \mathbb{D} -stability exposed in Definition 2 is applicable to the LTI system of Problem 3; consequently, there are results depending on the system structure and the quality of the algorithm used.

Hypothesis 2 If Hypothesis 1 is true, then the matrix that guarantees PEVA with \mathbb{D} -stability can be determined by Eq. (17).

To avoid working with complex matrices \mathbf{A}_1 and \mathbf{Y}_1^H , it is decided to use similar matrices with real representation. The ones that will be used are based on Zhang et al. (2015). First, rewrite the eigenvalue and eigenvector to separate the real and imaginary parts:

$$\lambda_i = \alpha_i + j\beta_i \quad (34)$$

$$\mathbf{y}_i = \mathbf{y}_{iR} + j\mathbf{y}_{iI} \quad (35)$$

Suppose there are l -conjugate pairs complex in \mathbb{S}_{OL} , so the \mathbf{A}_1 matrix can be written in the real similar Jordan's form:

$$\tilde{\mathbf{A}}_1 = \text{blkdiag} \left\{ \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}, \alpha_{l+1}, \dots, \alpha_p \right\} \quad (36)$$

A possible similar representation for the matrix \mathbf{Y}_1 is given by:

$$\tilde{\mathbf{Y}}_1 = [\mathbf{y}_{1R} \ \mathbf{y}_{1I} \ \dots \ \mathbf{y}_{pR} \ \mathbf{y}_{pI}] \quad (37)$$

Let the matrices be:

$$\mathbf{T}_j = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \quad (38)$$

$$\mathbf{T} = \text{blkdiag} \{ \mathbf{T}_j, \dots, \mathbf{T}_j, 1, \dots, 1 \} \quad (39)$$

being \mathbf{T}_j repeated l times and number 1 $p - 2l$ times. Therefore, the relation between \mathbf{Y}_1 and $\tilde{\mathbf{Y}}_1$ by

$$\tilde{\mathbf{Y}}_1 = \frac{1}{\sqrt{2}} \mathbf{Y}_1 \mathbf{T} \quad (40)$$

Through all the hypotheses and equations obtained in this section, a desired algorithm is created by the following procedure:

Procedure 1 Consider a classical PEVA as specified by Definition 1 and a \mathbb{D} -stable target region for spectrum assignment \mathbb{S}_c . To determine the matrix \mathbf{F} , the steps are followed:

1. Obtain the expression Eq. (1);
2. Determine \mathbb{S}_{POL} ;
3. Specify \mathbf{L} and \mathbf{V} of the characteristic function;
4. Get the matrices \mathbf{A}_1 , \mathbf{Y}_1 , and \mathbf{T} ;
5. Compute $\tilde{\mathbf{A}}_1$, and $\tilde{\mathbf{Y}}_1$;
6. Solve the LMI:

$$\mathbf{L} \otimes \mathbf{P} + \text{He} \{ \mathbf{V}, \tilde{\mathbf{A}}_1 \mathbf{P} \} + \text{He} \{ \mathbf{V}, \tilde{\mathbf{Y}}_1^H \mathbf{B} \mathbf{W} \} < 0$$

7. Compute $\Phi = \mathbf{W} \mathbf{P}^{-1}$;
8. Compute $\mathbf{F} = \Phi \tilde{\mathbf{Y}}_1^H$.

□

The applying of Procedure 1 consists of transforming the PEVA problem to an LMI. In practice, problems formulated by this mathematical structure have potential applications in multiobjective control design, which can be solved through convex or quasi-convex optimization algorithms (Boyd et al. 1994; Boyd and Vandenberghe 2004).

5 Numerical Examples

The methodology focuses on numerical software simulations for testing the proposed algorithm; these tests are summarily into seven steps:

1. Three models borrowed from the literature were selected to show the effectiveness of the proposal;
2. Rewrite second-order system to an LTI system as (38);
3. Specify three \mathbb{D} -regions for each model: a strip, a sector, or a disk;
4. Execute Procedure 1 for each model with the respective \mathbb{D} -region;
5. Check the location of eigenvalues in open and closed loop.
6. Compare the system spectrum in open and closed loop through tables.
7. Analyze whether partial assignment exists and if it meets the specifications of each problem using the tables and graphs.

In the two last examples, a comparison with the EVA is presented, in order to confront advantages or drawbacks. The software used for simulations is MATLAB 2014a, which was executed on a Windows 7 Ultimate 64-bit operating system on a 2.5-GHz Intel (R) Core (TM) i5-3210M processor. This MATLAB has a package called Control System Toolbox that provides algorithms and applications for analyzing, design-

ing, and adjusting linear control systems. Also, it has the Robust Control Toolbox, which is a package that offers commands to solve LMI elaborated in theoretical discussion.

Although the proposed methodology can be applied in state-space models in general, the focused case studies are of a second-order dynamic type, which is very common for modeling structural and flexible vibrating systems. The number of degrees of freedom of the specific second-order model can be arbitrarily large, as in the case of finite-element models derived from the discretization of distributed parameter models. The time-invariant multivariable second-order model with a control input class can be written according to the differential equation in the matrix form:

$$\mathbf{M}\ddot{\mathbf{h}}(t) + \mathbf{D}\dot{\mathbf{h}}(t) + \mathbf{K}\mathbf{h}(t) = \mathbf{N}\mathbf{u}(t) \quad (41)$$

$$\mathbf{u}(t) = \mathbf{F}_d\dot{\mathbf{h}}(t) + \mathbf{F}_p\mathbf{h}(t)$$

$$\mathbf{N} \in \mathbb{R}^{n \times m}; \quad \mathbf{M}, \mathbf{D}, \mathbf{K} \in \mathbb{R}^{n \times n}; \quad \mathbf{F}_p, \mathbf{F}_d \in \mathbb{R}^{m \times n} \quad (42)$$

where \mathbf{M} , \mathbf{D} and \mathbf{K} are, respectively, the mass, damping, and stiffness matrices of the system; \mathbf{N} is the influence matrix (actuators); and the vector $\mathbf{u}(t)$ is the control signal. \mathbf{F}_d and \mathbf{F}_p are, respectively, the derivative and proportional feedback matrices; and $\mathbf{h}(t)$, $\dot{\mathbf{h}}(t)$, and $\ddot{\mathbf{h}}(t)$ are, respectively, the vectors of displacement, velocity, and acceleration variables of the system.

The representation of the second-order system for the first order in the controllable canonical form is written according to the equations:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (43)$$

$$\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t) \quad (44)$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{0}_{n \times n} & \mathbf{I}_{n \times n} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \quad (45)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0}_{n \times m} \\ \mathbf{M}^{-1}\mathbf{N} \end{bmatrix} \quad (46)$$

$$\mathbf{F} = [\mathbf{F}_p \quad \mathbf{F}_d] \quad (47)$$

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{h}(t) \\ \dot{\mathbf{h}}(t) \end{bmatrix} \quad (48)$$

where the matrix $\mathbf{I}_{n \times n} \in \mathbb{R}^{n \times n}$ is an identity matrix.

5.1 Model I

The work of Shapiro (2005) presents a dynamic linearized model of a grasping robot. Its matrices are described by:

$$\mathbf{M} = \begin{bmatrix} 10 & 0 \\ 0 & 11 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 4 & 1 \\ 1 & 5 \end{bmatrix}, \mathbf{K} = \begin{bmatrix} 8 & \eta \\ -\eta & 9 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In this model, it is considered $\eta = 4$ to obtain the instability condition. \mathbb{D} -stable regions used for assignment are

Table 2 \mathbb{D} -stable regions for the assignment of the system described by Model I

\mathbb{D} -region	Values
Disc	$r = 2 \text{ e } q = 1$
Strip	$\alpha = 0.1 \text{ e } \beta = 0.3$
Sector	$\theta = 100^\circ$

Table 3 Model I eigenvalues to be assigned by the specified D-region.

Strip	Sector	Disc
$0.0039 \pm 0.9001j$	$0.0039 \pm 0.9001j$	$0.0039 \pm 0.9001j$

described in Table 2, and whether these specifications should allocate the eigenvalue is presented in Table 3.

The feedback matrices calculated by the algorithm are

$$\mathbf{F}_{\beta}^{\mathbb{H}\alpha} = [0.2104 \quad 2.1223 \quad -2.1679 \quad 1.5249]$$

$$\mathbf{F}_{\theta}^{\mathbb{H}\alpha} = [0.2281 \quad 8.7828 \quad -9.1237 \quad 5.7802]$$

$$\mathbf{F}_q^{\mathbb{D}'} = [-10.6054 \quad 24.7646 \quad -28.3866 \quad 7.0186]$$

In Fig. 1, spectrum's migration to the desired region occurs to the targeted unwanted eigenvalues that must to be reassigned. Table 4 confirms that the others eigenvalues remain unchanged with a precision of four digits.

5.2 Model II

A model of the aircraft wing under an air stream is explored by works as Henrion et al. (2005), Abdelaziz (2015), and Araújo et al. (2016):

$$\mathbf{M} = \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.60 & 0.756 & 0.658 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The \mathbb{D} -stable regions used for assignment are described in Table 5, and the location of eigenvalues in the complex plane is described in Fig. 2. With the specified eigenvalues presented in Table 6, they should be then assigned.

The feedback matrices calculated by the algorithm are:

$$\mathbf{F}_p^{\mathbb{H}\alpha} = [-14.2486 \quad -2.1054 \quad -2.6478]$$

$$\mathbf{F}_d^{\mathbb{H}\alpha} = [-2.6682 \quad -0.9657 \quad -0.6635]$$

$$\mathbf{F}_p^{\mathbb{H}\theta} = [-6.3282 \quad 1.2251 \quad -1.4776]$$

$$\mathbf{F}_d^{\mathbb{H}\theta} = [-9.7790 \quad -1.3312 \quad -5.7153]$$

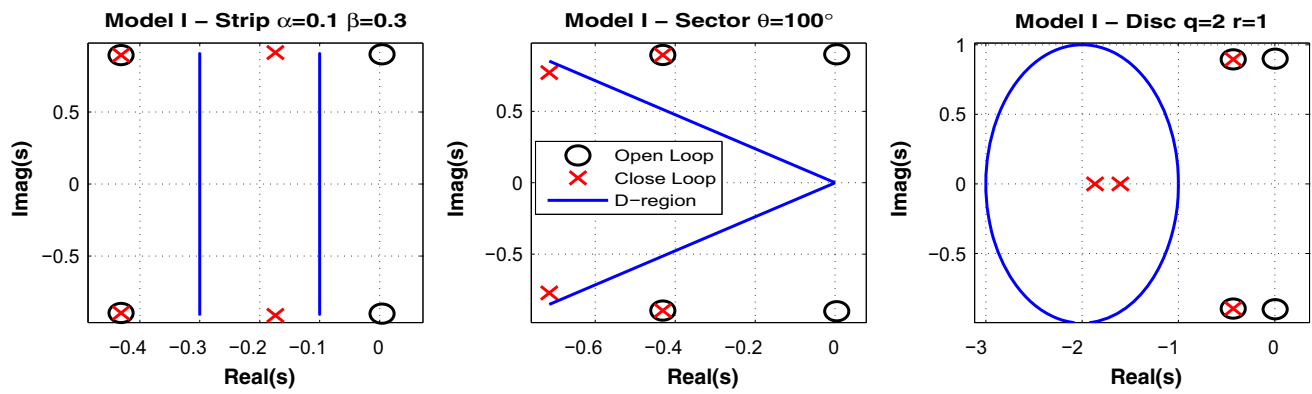


Fig. 1 Location of eigenvalues before and after control by feedback of states for Model I

Table 4 Eigenvalues before and after feedback state control Model I

	Open Loop	Closed-loop		
		Strip	Sector	Disc
1	$-0.4312 \pm 0.8953j$	$-0.4312 \pm 0.8953j$	$-0.4312 \pm 0.8953j$	$-0.4312 \pm 0.8953j$
2	$0.0039 \pm 0.9001j$	$-0.1738 \pm 0.9126j$	$-0.7150 \pm 0.7714j$	-1.8651
3				-1.6038

Table 5 \mathbb{D} -stable regions for the assignment of the system described by Model III

\mathbb{D} -region	Values
Disc	$r = 3$ e $q = 1$
Strip	$\alpha = 0.4$ e $\beta = 0.6$
Sector	$\theta = 110^\circ$

Table 6 Model II eigenvalues to be assigned by specified \mathbb{D} -region

Strip	$0.0947 \pm 2.5229j$
Sector and Disc	$0.0947 \pm 2.5229j$
	$-0.8848 \pm 8.4415j$

$$\mathbf{F}_p^{\mathbb{D}^q} = [-2.9185 \quad 1.5559 \quad 7.0511]$$

$$\mathbf{F}_d^{\mathbb{D}^q} = [-5.4921 \quad -0.4791 \quad -4.1253]$$

Rows 2–4 of Table 7 show the open-loop modes to be allocated; the final positions of closed-loop modes belonging to \mathbb{S}_{pCL} in the specified \mathbb{D} -regions can be confirmed in Fig. 2.

Another numerical experiment was performed to compare the proposed approach with the full-spectrum assignment EVA in a given \mathbb{D} -region. For this experiment, we adopted a disk \mathbb{D} -region with $r = 0.5$, $q = 1.5$. The unstable eigenpair $0.0947 \pm 2.5229j$ was reassigned to the desired region in the PEVA case. Then, the stiffness matrix \mathbf{K} was perturbed by 2% of its nominal value. The performance of the EVA and PEVA

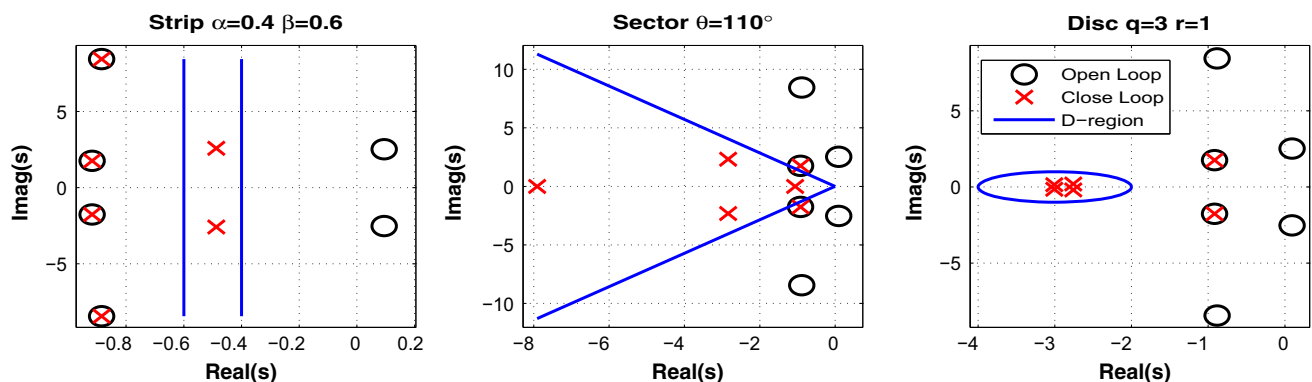
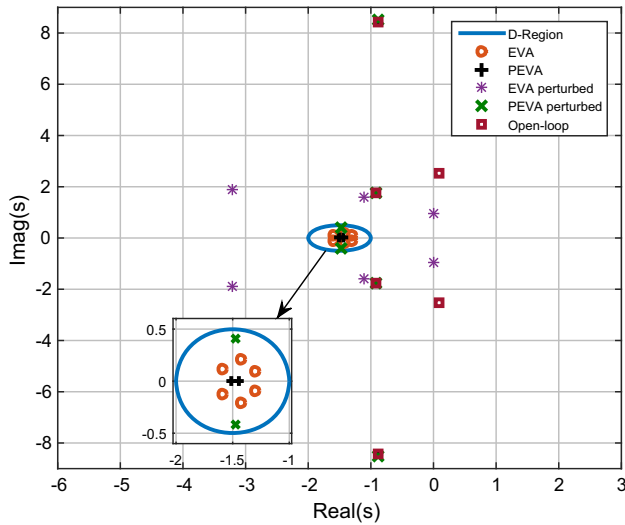


Fig. 2 Location of eigenvalues before and after the control by feedback of states for Model II

Table 7 Eigenvalues before and after feedback state control Model II

	Open Loop	Closed-loop		
		Strip	Sector	Disc
1	$-0.9180 \pm 1.7606j$	$-0.9180 \pm 1.7606j$	$-0.9180 \pm 1.7606j$	$-0.9180 \pm 1.7606i$
2	$-0.8848 \pm 8.4415j$	$-0.8848 \pm 8.4415j$	$-2.8410 \pm 2.3202j$	$-2.8410 \pm 2.3202j$
3	$0.0947 \pm 2.5229j$	$-0.4873 \pm 2.5866j$	-7.9106	$-3.0071 \pm 0.1502i$
4			-1.0607	

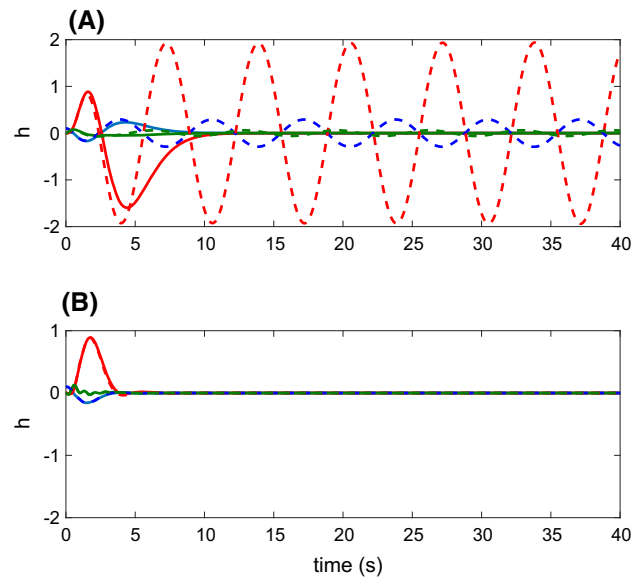
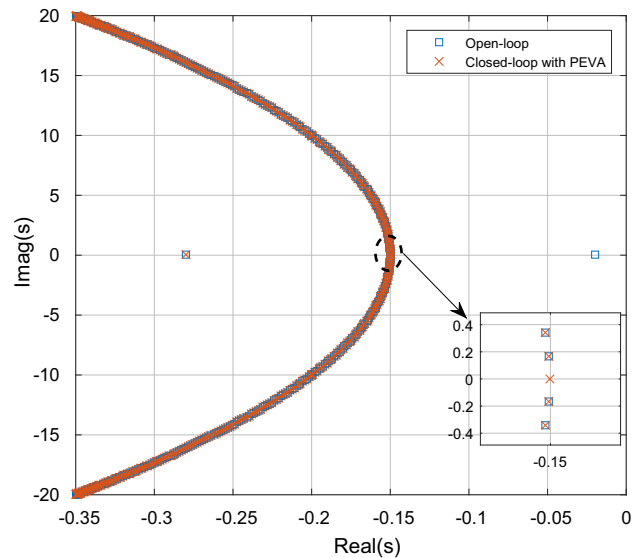
**Fig. 3** Location of eigenvalues for open loop, closed loop with EVA, and closed loop with PEVA for a wing in an airstream flutter suppression design (Model II)

designs is displayed in Fig. 3. It is evident that the presented approach outperforms the EVA in a closed loop, with the spillover phenomenon creating a serious threat for the EVA design. The time evolution for an initial displacement $\mathbf{h}(0) = [0.1 \ 0 \ 0]^T$ in a closed loop for the nominal and perturbed systems is displayed in Fig. 4.

5.3 Model III

This model is borrowed and adapted from Benner et al. (1995), and is an adequate finite-element model to particular structures, like turbine blades and cantilevers. This is a moderately large dimension system, with the displacement vector with dimension 211. The system matrices are:

$$\mathbf{K} = \begin{bmatrix} 100 & -100 & & & \\ -100 & 200 & -100 & & \\ & -100 & 200 & -100 & \\ & & \ddots & \ddots & \ddots \\ & & & -100 & 200 & -100 \\ & & & & -100 & 200 \end{bmatrix}_{211 \times 211},$$

**Fig. 4** Time response of displacements in the flutter suppression design with nominal matrices (solid) and 2%-perturbed stiffness matrix (dashed): **a** is for the EVA design and **b** is for the proposed approach (PEVA)**Fig. 5** Spectrum distribution for the Model III, with emphasis on the reassigned open-loop dominant eigenvalue

$$\mathbf{M} = \mathbf{I}_{211}$$

$$\mathbf{D} = \begin{bmatrix} 0.4 & -0.1 & & & \\ -0.1 & 0.5 & -0.1 & & \\ & -0.1 & 0.5 & -0.1 & \\ & & \ddots & \ddots & \ddots \\ & & & -0.1 & 0.5 & -0.1 \\ & & & & -0.1 & 0.4 \end{bmatrix}_{211 \times 211},$$

$$\mathbf{B} = [1 \ 0 \ 0 \ \dots \ 0 \ 0]^T_{1 \times 211}$$

This system has a real negative, dominant eigenvalue close to the origin, namely $\lambda_1 = -0.0199$, that can cause a slow transient response of the structure. Then, the PEVA design is conducted to reassign this eigenvalue in a strip with $\alpha = 0.1$, $\beta = 0.2$. The undesired eigenvalue is correctly reassigned into the \mathbb{D} region, more precisely at the new position $\mu_1 = -0.1500$, that is, the center of the strip, as depicted in Fig. 5. For the EVA design, the gain cannot be computed with the hardware utilized, and MATLAB showed an “of out of memory” event message.

6 Conclusions

An approach to PEVA in linear time-invariant systems using LMIs and a left-eigenvector parametrization was presented. The numerical examples show the effectiveness of the method, with the target eigenvalues being assigned to the given \mathbb{D} -region. Given these results, it is possible to conclude that the algorithm proposed can solve PEVA with \mathbb{D} -stability. The algorithm's ability, in the process, to maintain eigenvalues of partial assignment with good accuracy allows it to be said that the method guarantees no spillover, a feature desired in the control design developed on the subject of PEVA, mainly in a second-order system. This method uses partial assignment, which does not depend on knowledge of the entire system eigenstructure (eigenvectors and eigenvalues). Future studies focused on numerical analysis of the algorithm may help to understand the limitations and potential applications in multiobjective control.

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