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Robust Pole Assignment in Descriptor Second-order Dynamical Systems

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Abstract This paper considers the eigenvalue assignment with minimum sensitivity in descriptor second-order dynamical systems *via* proportional plus derivative state feedback. Based on a result for eigenvalue sensitivities and a complete parametric eigenstructure assignment approach, the robust pole assignment problem is converted into an independent minimization problem. The closed-loop eigenvalues may be easily taken as a part of the design parameters and optimized within certain desired fields on the complex plane to improve robustness. An example is worked out. Both the indices and the numerical robustness test demonstrate the effect of the proposed approach.

Key words Descriptor second-order dynamical systems, eigenstructure assignment, eigenvalue sensitivities, robust pole assignment

1 Introduction

Many practical systems can be represented by a second-order dynamical system[1~5]. Robust pole assignment (RPA) of normal second-order systems *via* proportional plus derivative state feedback has been studied in [1], and two methods have been proposed. The first one is a modification of the singular value decomposition-based method proposed in [6]; the second is an extension of the recent non-modal approach proposed in [7] for feedback stabilization of second-order systems. In descriptor second-order dynamical systems, the problem has been studied by [2], in which the control law is composed of proportional-derivative plus partial second-derivative state, and $2n$ (n is the system dimension) finite relative eigenvalues are assigned to the system. We examine the RPA of descriptor second-order systems *via* proportional plus derivative state feedback, which differs from those of [1, 2].

In this paper, the eigenstructure assignment approach in descriptor second-order dynamical systems *via* proportional plus derivative state feedback proposed in [3] is adopted. This approach assigns the maximum number of finite closed-loop eigenvalues, guarantees closed-loop regularity, and provides the complete parametric expressions of both the closed-loop eigenvectors and feedback gains. The design freedom provided by this method is composed of three parts, namely, the finite close-loop eigenvalues, the group of parameter vectors, and the parameter matrix. Based on the eigenstructure assignment result and eigenvalue sensitivities measures, the RPA problem is converted into a minimization problem. Due to the advantages of the eigenstructure assignment approach used, the approach proposed for the RPA problem possesses the following features.

1) The procedures for solution of the proposed RPA problem are in sequential order, and no "going back" procedures are needed.

2) The finite closed-loop eigenvalues are also included in the design parameters and are optimized within certain desired fields on the complex plane, thus a closed-loop system with better robustness and desired transient performance can be obtained.

3) The optimality of the solution to the whole RPA problem is solely depend on the optimality of the solution to the minimization problem converted.

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2 Formulation of the problem

Consider the following descriptor second-order dynamical system

$$M\ddot{\boldsymbol{x}} + D\dot{\boldsymbol{x}} + K\boldsymbol{x} = B\boldsymbol{u} \tag{1}$$

where $\boldsymbol{x} \in \mathbf{R}^n$ and $\boldsymbol{u} \in \mathbf{R}^r$ are the descriptor state vector and input vector, respectively; M, D, K, and B are system matrices of proper dimension and satisfy the following assumptions:

Assumption 1. $\mathrm{rank}(M) = n_0,\ 0 < n_0 < n, \mathrm{rank}(B) = r$;

Assumption 2. $\mathrm{rank}\begin{bmatrix} s^2M + sD + K & B \end{bmatrix} = n$, for all $s \in \mathbf{C}$.

When the following proportional plus derivative state feedback controller

$$\boldsymbol{u} = F_0\boldsymbol{x} + F_1\dot{\boldsymbol{x}} \tag{2}$$

is applied to system (1), the closed-loop system is obtained as

$$M\ddot{\boldsymbol{x}} + \overline{D}\dot{\boldsymbol{x}} + \overline{K}\boldsymbol{x} = 0 \tag{3}$$

where

$$\overline{D} = D - BF_1,\ \overline{K} = K - BF_0$$

System (1) and (3) can be written in the first-order state-space model

$$E_e\dot{\boldsymbol{z}} = A_e\boldsymbol{z} + B_e\boldsymbol{u} \tag{4}$$

$$E_{ec}\dot{\boldsymbol{z}} = A_{ec}\boldsymbol{z} \tag{5}$$

where

$$E_e = E_{ec} = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix},\quad A_e = \begin{bmatrix} 0 & I_n \\ -K & -D \end{bmatrix}$$
$$A_{ec} = \begin{bmatrix} 0 & I_n \\ -\overline{K} & -\overline{D} \end{bmatrix},\quad B_e = \begin{bmatrix} 0 \\ B \end{bmatrix},\quad \boldsymbol{z} = \begin{bmatrix} \boldsymbol{x} \\ \dot{\boldsymbol{x}} \end{bmatrix} \tag{6}$$

Definition 1. The second-order descriptor dynamical system (3) is called regular if and only if the corresponding extended first-order state-space representation (5)~(6) is regular.

Lemma 1. The second-order dynamical system (3) is regular if and only if there exists a constant scalar $s \in \mathbf{C}$ such that

$$\det(s^2M + s\overline{D} + \overline{K}) \neq 0 \tag{7}$$

Proof. By the regular criterion in first-order descriptor linear systems and Definition 1, we need only to show that (7) is equivalent to

$$\det(A_{ec} - sE_{ec}) \neq 0$$

where E_{ec} and A_{ec} are given by (6).

Since

$$\begin{aligned}
&\det(A_{ec} - sE_{ec}) \\
&= \det \begin{bmatrix} -sI_n & I_n \\ -\overline{K} & -\overline{D} - sM \end{bmatrix} \\
&= \det \begin{bmatrix} 0 & I_n \\ -\overline{K} - s\overline{D} - s^2 M & -\overline{D} - sM \end{bmatrix} \\
&= \det(s^2 M + s\overline{D} + \overline{K})
\end{aligned}$$

The conclusion clearly follows.

Recall the fact that a nondefective matrix possesses eigenvalues, which are less sensitive to the parameter perturbations in the matrix. Here the closed-loop matrix pair $(E_{ec},\ A_{ec})$ is required to be nondefective, that is, the Jordan form of the matrix pair $(E_{ec},\ A_{ec})$ possesses a diagonal form. Further, following the pole assignment theory for first-order descriptor linear system, when the closed-loop system (5)∼(6) is regular, the maximum number of finite relative eigenvalues that can be assigned is $n + n_0$. The problem of RPA to be solved in the paper can be stated as follows.

Problem RPA. Given system (1) satisfying Assumptions 1 and 2, and a set of regions $\Omega_i,\ i = 1,\ 2,\ \cdots,\ n+n_0$ on the complex plane, seek a feedback in the form of (2) such that the following requirements are met:

1) The closed-loop system (3) is regular, that is, $\det(s^2 M + s\overline{D} + \overline{K}) \neq 0$, for some $s \in \mathbf{C}$.

2) The closed-loop system (3) has $n + n_0$ finite relative eigenvalues $s_i,\ i = 1,\ 2,\ \cdots,\ n+n_0$, satisfying $s_i \in \Omega_i,\ i = 1,\ 2,\ \cdots,\ n + n_0$.

3) The finite closed-loop eigenvalues $s_i,\ i = 1,\ 2,\ \cdots,\ n + n_0$, are as insensitive as possible to parameter perturbations in the closed-loop system matrices M, $\overline{D}$, and $\overline{K}$.

Remark 1. The requirement $s_i \in \Omega_i,\ i = 1,\ 2,\ \cdots,\ n+n_0$, in the above problem represents the requirement on the closed-loop stability and performance property. For a real closed-loop eigenvalue s_i, the region Ω_i may be chosen to be an interval $[a_i\ b_i]$. For a pair of complex eigenvalues s_i and s_l, the regions Ω_i and Ω_l may often be chosen as

$$\Omega_i = \{\, s_i = \sigma_i + \sigma_l j \,|\, \sigma_i \in [a_i\ b_i]\ ,\sigma_l \in [a_l\ b_l]\}$$

and

$$\Omega_l = \{\, s_l = \sigma_i - \sigma_l j \,|\, \sigma_i \in [a_i\ b_i]\ ,\sigma_l \in [a_l\ b_l]\}$$

3 Preliminaries

In this section, we first state a result on eigenstructure assignment in descriptor second-order dynamical systems *via* proportional plus derivative state feedback, and then present the sensitivity measures of the finite eigenvalues of the closed-loop system (3) to solve the problem RPA.

3.1 Closed-loop eigenstructure assignment

For the closed-loop system (3), the right normal eigenvector matrix V_{ec}^{∞} associated with the infinite eigenvalues is denoted by $V_{ec}^{\infty} = \begin{bmatrix} 0 \\ V_{\infty} \end{bmatrix}$, where V_{∞} satisfies

$$MV_{\infty} = 0, \quad \mathrm{rank}(V_{\infty}) = n - n_0 \tag{8}$$

It is shown that under Assumption 2, there exists a pair of real coefficient right coprime polynomial matrices $N(s) \in \mathbf{R}^{n\times r}[s]$ and $D(s) \in \mathbf{R}^{r\times r}[s]$ satisfying the following right coprime factorization

$$(s^2 M + sD + K)^{-1} B = N(s)D^{-1}(s) \tag{9}$$

Lemma 2[3]**.** Given system (1) satisfying Assumptions 1 and 2, let $V_{\infty} \in \mathbf{R}^{n\times(n-n_0)}$ be matrix satisfying (8), $N(s)$ and $D(s)$ be a pair of polynomial matrices satisfying the coprime factorization (9). Then

1) There exist a group of complex numbers $s_i,\ i = 1,\ 2,\ \cdots,\ n + n_0$, a matrix $V \in \mathbf{R}^{n\times(n+n_0)}$, and two real matrices $F_0,\ F_1 \in \mathbf{R}^{r\times n}$, such that closed-loop system (3) is regular, at the same time,

$$MV\Lambda^2 + (D - BF_1)V\Lambda + (K - BF_0)V = 0 \tag{10}$$

and

$$\det(V_{ec}) = \det \begin{bmatrix} V_{ec}^{f} & V_{ec}^{\infty} \end{bmatrix} = \det \begin{bmatrix} V & 0 \\ V\Lambda & V_{\infty} \end{bmatrix} \neq 0 \tag{11}$$

hold for

$$\Lambda = \mathrm{diag}(s_1,\ s_2,\ \cdots,\ s_{n+n_0}) \tag{12}$$

if and only if there exist a group of parameters vector $\boldsymbol{f}_i \in \mathbf{C}^r,\ i = 1,\ 2,\ \cdots,\ n + n_0$, satisfying the following constraints:

Constrain 1. $\boldsymbol{f}_i = \overline{\boldsymbol{f}_j}$ if $s_i = \overline{s_j}$;

Constrain 2. $\det(s^2 M + s\overline{D} + \overline{K}) \neq 0$, for some $s \in \mathbf{C}$;

Constrain 3. $\det(V_{ec}) \neq 0$

where

$$V_{ec} = \begin{bmatrix} N(s_1)\boldsymbol{f}_1 & N(s_2)\boldsymbol{f}_2 & \cdots & N(s_{n+n_0})\boldsymbol{f}_{n+n_0} & 0 \\ s_1 N(s_1)\boldsymbol{f}_1 & s_2 N(s_2)\boldsymbol{f}_2 & \cdots & s_{n+n_0} N(s_{n+n_0})\boldsymbol{f}_{n+n_0} & V_{\infty} \end{bmatrix} \tag{13}$$

2) When the above condition is met, the matrices V and W are given by

$$V = \begin{bmatrix} N(s_1)\boldsymbol{f}_1 & N(s_2)\boldsymbol{f}_2 & \cdots & N(s_{n+n_0})\boldsymbol{f}_{n+n_0} \end{bmatrix} \tag{14}$$

$$W = \begin{bmatrix} D(s_1)\boldsymbol{f}_1 & D(s_2)\boldsymbol{f}_2 & \cdots & D(s_{n+n_0})\boldsymbol{f}_{n+n_0} & W_{\infty} \end{bmatrix} \tag{15}$$

and the corresponding feedback gains are given by

$$[F_0 \quad F_1] = W V_{ec}^{-1} \tag{16}$$

where $\boldsymbol{f}_i \in \mathbf{C}^r,\ i = 1,\ 2,\ \cdots,\ n + n_0$, are arbitrary parameter vectors satisfying Constrains 1 ∼ 3, and $W_{\infty} \in \mathbf{R}^{n\times(n-n_0)}$ is an arbitrary parameter matrix.

3.2 Closed-loop eigenvalue sensitivity measures

In order to solve the RPA problem formulated in Section 2, proper sensitivity measures for the closed-loop eigenvalues need to be established.

Lemma 3[8]**.** Let F_0, F_1, and V be the general solutions obtained in Lemma 2, and $V_{ec}^{f},\ V_{ec}^{\infty}$, and W_{∞} be defined as previously. Then, the left eigenvector matrix T of the closed-loop system associated with the finite closed-loop eigenvalues, which form a normalized pair with the right eigenvector matrix V_{ec}^f, is unique with respect to parameters W_∞ and $\boldsymbol{f}_i$, and is given by

$$T^{\mathrm{T}} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} E_e V_{ec}^f & A_e V_{ec}^\infty + B_e W_\infty \end{bmatrix}^{-1} \tag{17}$$

According to the form of A_{ec}, E_{ec}, V_{ec}^f, and V_{ec}^∞, it is easy to know that T^{T} can be written, using the original system parameters, as

$$\begin{bmatrix} I_{n+n_0} & 0 \end{bmatrix} \begin{bmatrix} V & V_\infty \\ MV\Lambda & -DV_\infty + BW_\infty \end{bmatrix}^{-1}$$

Lemma 4[9]**.** Let T and V_{ec}^f be a pair of normalized left and right eigenvector matrices, associated with the finite closed-loop eigenvalues, of the non-defective matrix pair $(A_{ec},\ E_{ec})$. Then, the condition numbers corresponding to the closed-loop finite eigenvalues $s_i,\ i=1,\ 2,\ \cdots,\ n+n_0$, are given as follows

$$c_i = \frac{\|\boldsymbol{t}_i\|_2 \cdot \left\| [V_{ec}^f]_i \right\|_2}{(1+|s_i|^2)^{1/2}}, \quad i=1,\ 2,\ \cdots,\ n+n_0 \tag{18}$$

where $\boldsymbol{t}_i$ is the i-th column of the matrix T.

4 Main results

It follows from Lemma 2 that the design freedom existing in the close-loop eigenstructure assignment actually consists of three parts.

1) The finite close-loop eigenvalues $s_i,\ i=1,\ 2,\ \cdots,\ n+n_0$.

2) The group of parameter vectors $\boldsymbol{f}_i,\ i=1,\ 2,\ \cdots,\ n+n_0$.

3) The parameter matrices $V_\infty,\ W_\infty$.

It is clear that none of the above design parameters is generally unique, and thus can be properly chosen to satisfy the three requirements in our problem RPA stated in Section 2.

The first requirement in problem RPA can be ensured by Constrain 2. The second can be ensured by restricting $s_i \in \Omega_i,\ i=1,\ 2,\ \cdots,\ n+n_0$. The third can be realized by minimizing the closed-loop eigenvalue sensitivities $c_i,\ i=1,\ 2,\ \cdots,\ n+n_0$. However, note that minimizing the c_i may lead to matrices $F_0,\ F_1$ with large magnitude, we may define an objective, which takes the magnitudes of the feedback gains F_0, F_1 into consideration, as

$$J = J(s_i,\ f_i,\ V_\infty,\ W_\infty,\ i=1,\ 2,\ \cdots,\ n+n_0) = \sum_{i=1}^{n+n_0} \alpha_i c_i + \beta_0 \|F_0\|_F + \beta_1 \|F_1\|_F \tag{19}$$

where $c_i,\ i=1,\ 2,\ \cdots,\ n+n_0$, are given by (18), and $\alpha_i,\ i=1,\ 2,\ \cdots,\ n+n_0$, and $\beta_i,\ i=1,\ 2$, are positive scalars representing the weighting factors. Therefore, our problem RPA can be converted into the following minimization problem

$$\begin{cases} \min J(s_i,\ f_i,\ V_\infty,\ W_\infty, i=1,\ 2,\ \cdots,\ n+n_0) \\ \text{s.t.} \quad s_i \in \Omega_i,\ i=1,\ 2,\ \cdots,\ n+n_0 \\ \qquad \text{Constraints} \quad 1 \sim 3 \end{cases} \tag{20}$$

Based on the above analysis, an algorithm for solving problem RPA can be given as follows.

Algorithm RPA.

1) Solve the right coprime matrix polynomial matrices $N(s)$ and $D(s)$ satisfying (9), and construct the parametric expressions for matrix V_{ec}^f by (11) and (14).

2) Obtain the optimal design parameters $s_i,\ \boldsymbol{f}_i,\ i=1,\ 2,\ \cdots,\ n+n_0$, V_∞ and W_∞ by solving the minimization problem (20) by some optimization algorithm.

3) Calculate the matrices V_{ec} and W by (13) and (15) based on parameters $s_i, \boldsymbol{f}_i,\ i=1,\ 2,\ \cdots,\ n+n_0$, V_∞ and W_∞ obtained in 2).

4) Solve the proportional plus derivative state feedback gains $F_0,\ F_1$ by (16).

Obviously the above algorithm is in a sequential order, while no "going back" procedures are involved. Further, because of the completeness of the eigenstructure assignment approach used, the optimality of the solution to the RPA problem obtained through Algorithm RPA is totally dependent on the solution to the optimization problem (20).

5 Example

Consider a system in the form of (1) with the following system parameters[3]

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

5.1 Solutions

By a method given by [3], we obtain

$$N(s) = \begin{bmatrix} 2s+20 & 0 \\ 0 & 2s+20 \\ -0.5s-5 & s^2+2.5s+25 \end{bmatrix}$$

$$D(s) = \begin{bmatrix} 2s^3+25s^2+70s+200 & -s^2-20s-100 \\ -s^2-20s-100 & 2s^3+21s^2+20s+100 \end{bmatrix}$$

Let

$$\boldsymbol{f}_i = \begin{bmatrix} f_{i1} \\ f_{i2} \end{bmatrix},\ i=1,\ 2,\ \cdots,\ 5$$

Then, we have

$$\boldsymbol{v}_i = \begin{bmatrix} 2s_i f_{i1} + 20 f_{i1} \\ 2s_i f_{i2} + 20 f_{i2} \\ s_i^2 f_{i2} + s_i(2.5 f_{i2} - 0.5 f_{i1}) + 25 f_{i2} - 5 f_{i1} \end{bmatrix}, \quad i=1,\ 2,\ \cdots,\ 5$$

$$\boldsymbol{w}_i = \begin{bmatrix} 2s_i^3 f_{i1} + s_i^2(25 f_{i1} - f_{i2}) + s_i(70 f_{i1} - 20 f_{i2}) + 200 f_{i1} - 100 f_{i2} \\ 2s_i^3 f_{i2} + s_i^2(21 f_{i2} - f_{i1}) + s_i(20 f_{i2} - 20 f_{i1}) + 100 f_{i2} - 100 f_{i1} \end{bmatrix}, \quad i=1,\ 2,\ \cdots,\ 5$$

For this example, three cases are considered and the Matlab command fmincon is used for the optimization problem involved in obtaining the solutions.

Solution 1. The closed-loop eigenvalues are chosen as $s_i = -i,\ i=1,\ 2,\ \cdots,\ 5$, the parameters $f_{i1}\ , f_{i2},\ i=1,\ 2,\ \cdots,\ 5$, V_∞ and W_∞ are chosen as

$$\begin{array}{cccccc} f_{i1}: & 1 & 0 & 1 & -1 & 1 \\ f_{i2}: & 0 & 1 & -1 & 1 & 2 \end{array}$$

$$V_\infty = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}},\ W_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

The corresponding proportional plus derivative state feedback gains are

$$F_0 = \begin{bmatrix} -1.7091 & -3.1818 & -21.3818 \\ -1.4091 & -12.6818 & 13.8182 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -4.8636 & -15.6273 & 0.0000 \\ -0.1136 & -1.4773 & 1.0000 \end{bmatrix}$$

Solution 2. As in Solution 1, the closed-loop eigenvalues are taken as $s_i = -i$, $i = 1, 2, \cdots, 5$. The weighting factor are taken as $\alpha_i = 1$, $i = 1, 2, \cdots, 5$, $\beta_1 = \beta_2 = 1$. The parameters f_{i1}, f_{i2}, $i = 1, 2, \cdots, 5$, V_∞ and W_∞ are found through minimizing (20) as the following

f_{i1}:	19.5216	39.1878	7.1268	16.2472	5.2734
f_{i2}:	52.7704	−8.2749	18.8037	−1.2829	22.5896

$$V_\infty = \begin{bmatrix} 0 & 0 & 41.3337 \end{bmatrix}^{\mathrm{T}}, \ W_\infty = [-22.1128 \ \ -29.0762]^{\mathrm{T}}$$

With these parameters, the proportional plus derivative state feedback gains can be computed as

$$F_0 = \begin{bmatrix} 1.7659 & -1.2441 & -1.4847 \\ -4.3817 & 1.0414 & 1.4679 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -3.5848 & 0.9094 & -0.5108 \\ -0.6000 & -1.5588 & -0.7035 \end{bmatrix}$$

Solution 3. Both the closed-loop eigenvalues s_i, $i = 1, 2, \cdots, 5$, and the parameters f_{i1}, f_{i2}, $i = 1, 2, \cdots, 5$, V_∞ and W_∞ are chosen to minimize (20), and the closed-loop eigenvalues are restricted within the regions

$$\Omega_1 = \begin{bmatrix} -1.5 & -0.5 \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} -2.5 & -1.5 \end{bmatrix}$$
$$\Omega_3 = \begin{bmatrix} -3.5 & -2.5 \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} -4.5 & -3.5 \end{bmatrix}$$
$$\Omega_5 = \begin{bmatrix} -5.5 & -4.5 \end{bmatrix}$$

The solution to the minimization problem is given as

s_i:	−0.5000	−1.6445	−2.6396	−4.5000	−5.5000
f_{i1}:	17.1509	40.4523	5.1463	15.2612	5.2691
f_{i2}:	50.0645	−8.6288	19.8348	−1.9289	29.4087

$$V_\infty = \begin{bmatrix} 0 \ 0 \ 40.3105 \end{bmatrix}^{\mathrm{T}}, \ W_\infty = \begin{bmatrix} -19.0212 \ -24.8678 \end{bmatrix}^{\mathrm{T}}$$

Based on these parameters, the proportional plus derivative state feedback gains can be computed as

$$F_0 = \begin{bmatrix} 2.4175 & -1.1220 & -1.5280 \\ -4.1486 & 1.3811 & 2.0395 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -3.6931 & 0.9684 & -0.4719 \\ -0.5301 & -1.1146 & -0.6169 \end{bmatrix}$$

5.2 Analysis of solutions

In this subsection, we will summarize some of the values related to the obtained solutions, and also carry out a numerical robustness test.

In Table 1, the magnitudes of the feedback gains are listed. It is clear that the magnitudes of the feedback gains in Solutions 2 and 3 are smaller than those in Solution 1.

Table 1 Magnitudes of solutions

Solutions	$\|F_0\|_2$	$\|F_0\|_F$	$\|F_1\|_2$	$\|F_1\|_F$
1	25.9097	28.7051	16.4305	16.4640
2	5.3202	5.4138	3.7483	4.1501
3	5.7097	5.7198	3.8607	4.0870

Table 2 lists the closed-loop eigenvalue sensitivity measures c_i, $i = 1, 2, \cdots, 5$, and the spectral norm of the condition number vector

$$c = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 \end{bmatrix}$$

Table 2 Eigenvalue sensitivities

Solutions	c_1	c_2	c_3	c_4	c_5	$\|c\|_2$
1	9.5246	23.6875	68.5873	51.2665	9.8020	89.8909
2	2.4617	2.3752	5.2059	1.2812	4.7578	7.9423
3	1.9030	1.8338	3.5212	0.8180	3.3893	5.6160

In order to test the robustness of the solutions, we add the following perturbations to the matrices M, D, K, and B:

$$\Delta M(1, 1) = 0.0015k \cdot \mathrm{randn}(\mathrm{size}(1))$$

$$\Delta M(2, 2) = 0.0015k \cdot \mathrm{randn}(\mathrm{size}(1))$$

$$\Delta \mathrm{D}_k = 0.0015k \cdot \mathrm{randn}(\mathrm{size}(\mathrm{D}))$$

$$\Delta K_k = 0.0015k \cdot \mathrm{randn}(\mathrm{size}(K))$$

$$\Delta B_k = 0.0015k \cdot \mathrm{randn}(\mathrm{size}(B))$$

where randn is a command in Matlab. randn(size(A)) generates a random matrix of the same size as matrix A whose entries are chosen from a norm distribution with mean zero and standard deviation one. Applying each of our solutions to the perturbed system, then corresponding to each k, we can obtain the eigenvalues $s_i(k)$, $i = 1, 2, \cdots, 5$, of the perturbed closed-loop system. Corresponding to each solution, we have calculated the values of

$$d_i = \frac{\left[\sum\limits_{k=1}^{100} [\mathrm{abs}(s_i(k)) - \mathrm{abs}(s_i)]^2\right]^{\frac{1}{2}}}{100}, \quad i = 1, 2, \cdots, 5$$

and

$$d = \frac{\left[\sum\limits_{i=1}^{5} d_i^2\right]^{\frac{1}{2}}}{5}, \quad i = 1, 2, \cdots, 5$$

and list them in Table 3.

Table 3 The robustness test results

Solutions	d_1	d_2	d_3	d_4	d_5	d
1	0.4930	0.7707	1.3118	1.4950	14.8915	3.0103
2	0.0943	0.2136	0.5470	0.4539	0.7359	0.2099
3	0.0825	0.1356	0.6694	0.4673	0.3779	0.1827

In order to give a more intuitive picture, the values of

$$d(k) = \frac{\left[\sum_{i=1}^{5} [\mathrm{abs}(s_i(k)) - \mathrm{abs}(s_i)]^2\right]^{\frac{1}{2}}}{5},$$

$$k = 1,\ 2,\ \cdots,\ 100$$

corresponding to each solution are shown in Fig. 1.

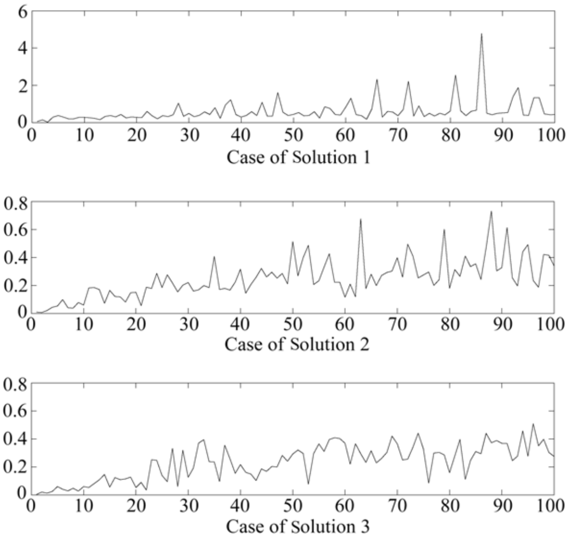

Fig. 1 Results of the numerical robustness test for Solutions 1∼3

Remark 2. Since the perturbation in the open-loop system matrices are considered to be random matrices, each time we carry out this robustness test, a different table and figure are obtained. However, the relative relations among the values in Table 3, and the relative height of the curves in the subplots in Fig. 1 remain almost the same.

From Tables 1∼3 and Fig. 1, we have the following observations.

1) Solution 1, which is obtained by a simple choice of the design parameters, has the biggest $\|c\|_2$ and $\|F\|_2$. Solutions 2 and 3 are obtained by minimizing (20). Moreover, Fig. 1 also shows that Solutions 2 and 3 have much better robustness than Solution 1. Such a fact states that the values of (20) can be treated as closed-loop eigenvalue sensitivity measures for descriptor second-order dynamical systems.

2) It can be seen from Table 2 for Solutions 2 and 3 that inclusion of closed-loop eigenvalues into the design parameters reduces the optimization indices.

3) It is interesting to note from Tables 2 and 3 that again the results come out to be almost consistent with the theory, that is, for solution with smaller eigenvalue sensitivity measures $\|c\|_2$, the corresponding overall drift magnitudes of the close-loop eigenvalues measured by d_i and d are also smaller.

6 Conclusion

This paper proposes a simple approach for RPA in descriptor second-order linear system *via* proportional plus derivative state feedback. The approach is based on a result for eigenvalue sensitivities and a complete parametric eigenstructure assignment approach for descriptor second-order linear system *via* proportional plus derivative state feedback. Due to the features of the eigenstructure assignment result, the procedures for solution of the proposed RPA problem are in a sequential order, and no "going back" procedures are needed; the finite close-loop eigenvalues are included in the design parameters and are optimized within certain desired fields on the complex plane, thus a close-loop system with better robustness and desired transient performance can be obtained; the optimality of the solution to the whole RPA problem is solely dependent on the optimality of the solution to an independent minimization problem.

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