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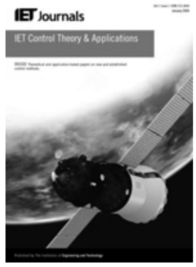

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Optimal pole assignment for discrete-time systems via Stein equations

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Abstract: This study is concerned with designing a feedback gain to minimise a quadratic performance index with guaranteed pole locations for closed-loop discrete-time linear systems. Firstly, a method that shifts the open-loop poles to desired locations by using a parametric linear Stein equation is presented. Then a recursive approach is proposed to shift every eigenvalue of a discrete-time linear system separately without mode decomposition in each step. By using such method, it is required to solve a linear Stein matrix equation of low order in each step. The presented method yields a solution which is optimal with respect to a quadratic performance index that can be obtained explicitly. The attractive feature of this method comparing with existing results is that it enables solutions to complex problems to be easily found without solving any non-linear algebraic Riccati equations. Moreover, analytical solutions can be obtained which may have advantages in some design problems. Numerical examples illustrate the proposed approach.

1 Introduction

The problem of designing a feedback gain that shifts the poles of a given linear multivariable system to some prescribed positions and simultaneously minimises a quadratic cost function has been widely studied in the past several decades, especially in the case of multi-input system (see [1–4] and the references therein). The motivation for studying this problem is that the resulting control law does not only have the advantages provided by linear quadratic (LQ) control (such as at least a 6 dB gain margin and a 60° phase margin for single input linear system [5]) but also can provide gratifying transient response quality determined by closed-loop poles [6].

There are several solutions in the literature to this problem. Solheim [7] suggested the successive optimal pole placement approach. By this method, full-order non-linear Riccati equation for modifying one real pole or two complex conjugated poles in each step would be required to be solved, which may be rather expensive for large-scale system. Solheim's method was extended and modified in several directions by many authors (see, for example [8, 9] and the references therein). Sebakhy and Sorial [10] proposed another approach to this problem in the optimisation point of view. The freedom in pole assignment design for multiple input system is utilised to minimise a quadratic function. Because they used gradient-based search techniques, these implicit algorithms may impose unacceptable computation burdens for high-order systems. Another approach partly solves this problem is to modify the performance criterion in LQ design [11, 12]. Such LQ re-design method can modify the real parts and modulus of closed-loop poles in continuous-time case and discrete-time case, respectively, which can improve the state response performances of the closed-loop system. However, minimal and maximal solutions to some discrete-time algebraic Riccati matrix equations (DARE) are required to be obtained. Besides exact pole locations requirement in this problem, region pole requirement, which is a relaxed constraint, is also considered in the literature (see [13–15] and the references therein). Recently, the robust optimal pole shifting was also considered in the literature [16]. For more related papers and applications of this method, see [17–20] and the references therein.

Amin [3] presented another very simple approach to solve this problem in continuous-time case. In this recursive method, only real parts of the open-loop system are shifted while the imaginary parts are preserved without a change. The advantage of such method is that only reduced-order Lyapunov equation is required to be solved in each step. Amin's method has been generalised by many authors (for example, [7, 21]) and has been found applications in practice (e.g. [22]). Unfortunately, unlike Amin's results in continuous-time case, non-linear DARE should be solved in each step in this method for discrete-time system given in [21]. Moreover, the feedback gain may not be optimal for the closed-loop system with respect to the resulting quadratic function.

Our work in this paper is a generalisation of that in [3, 21]. We first study a type of DARE and prove that such DARE has a positive definite solution (also unique) if and only if a corresponding Stein matrix equation parameterised in a scalar has a positive definite solution. Necessary and sufficient condition guaranteeing the existence of a positive definite solution is also presented. Pole locations of the closed-loop system by using state feedback constructed by the positive definite solution to the DARE are clarified by using the circle-symmetric principle. These results generate several aspects with respect to existing results. We then apply this method recursively on the system to shift the undesired open-loop poles to desired locations. In each step, only low-order linear Stein equation needs to be solved. A quadratic performance index that is minimised by the resulting feedback gain is obtained. Also, unlike the method given in [3, 21], we need not to do mode decomposition in each step but only once, which may need less computation.

Finally, we should point out that the optimal pole shift problem certainly has relationship with the so-called inverse optimal control problem which was firstly considered by Kalman [23] and latterly studied by many researchers (see, e.g. [2, 24–26]). The discrete-time version of the inverse optimal control problem is stated as follows: given a linear system $x_{k+1} = Ax_k + Bu_k$ and a state feedback control law $u_k = -Fx_k$, find necessary and sufficient conditions on A, B and F such that the control law F is 'optimal' in the sense that $u_k = -Fx_k$ minimises the cost $J(u) = \Sigma_{i=0}^{\infty}(x_i^{\mathrm{T}}Qx_i + u_i^{\mathrm{T}}Ru_i)$ for some semi-positive definite matrix Q and positive definite matrix R. It is well-known that the inverse optimal control problem has a solution not for arbitrary feedback matrix F. Namely, for some given feedback matrix F that assigns the eigenvalues of $A + BF$ at certain prescribed locations, the inverse optimal control problem may have no solution. In this paper, the optimal pole assigned problem is similar to the inverse optimal control problem stated above in the sense that the closed-loop eigenvalue locations are guaranteed and the weighting matrices Q and R are found simultaneously. But on the other hand, in our optimal pole assignment problem, the feedback matrix F is not arbitrary chosen but is determined by the locations of the closed-loop eigenvalues which are also not arbitrary assigned but should satisfy some restrictions. Therefore the feedback matrix F and the weighting matrices Q and R are required to be obtained at the same time in our method which is different from the inverse optimal control problem. Owing to the complexity of the inverse optimal control problem, our approach clearly indicates a trade-off between arbitrary pole assignment and arbitrary choice of the weighting matrices Q and R in the optimal control problem.

The remainder of this paper is organised as follows. In Section 2, the basic development for pole shift via Stein equation is proposed. In Section 3, such developed method is applied on the system recursively to achieve more precise pole re-location. Two examples are given in Section 4 to illustrate the proposed results. Section 5 concludes the paper.

Notations: Throughout this paper, we use A^{T}, $\sigma(A)$, $\lambda(A)$, $|\lambda(A)|_{\min}$ and $|\lambda(A)|_{\max}$ to denote the transpose, the eigenvalue set, the eigenvalue, the minimal modulus of the eigenvalues and the maximal modulus of the eigenvalues of matrix A, respectively. The symbol $\mathbb{R}^{m\times n}$ denotes the set consisting of all $m \times n$ real matrices. For a complex number α, we use $\bar{\alpha}$ and $\arg(\alpha)$ to denote its conjugate and angle, respectively. The symbol i refers to the unit imaginary number, that is, $i = \sqrt{-1}$. Finally, $A > (\geq)0$ means that A is a symmetric and positive (semi-positive) definite matrix.

2 Pole shift via parametric Stein equation

We first recall the following well-known matrix inverse formulation that can be found in some matrix analysis text books (e.g. [27]).

Lemma 1: Let A, B, C and D be some matrices with appropriate dimensions, A, C, $A + BCD$ and $C^{-1} + DA^{-1}B$ be all non-singular. Then

$$A(A + BCD)^{-1}A = A - B(C^{-1} + DA^{-1}B)^{-1}D \tag{1}$$

We recall that a set $\mathbb{F}$ is symmetric with respect to real axis if $\alpha \in \mathbb{F}$ implies $\bar{\alpha} \in \mathbb{F}$.

Definition 1: Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be two sets that are symmetric with respect to real axis. Then $\mathbb{F}_1$ and $\mathbb{F}_2$ are called circle-symmetric with respect to the circle $|z|^2 = r$ if for every element $\alpha \in \mathbb{F}_1$, there exists an element $\beta \in \mathbb{F}_2$ such that $\alpha\bar{\beta} = r$, and for every element $\beta \in \mathbb{F}_2$, there exists an element $\alpha \in \mathbb{F}_1$ such that $\alpha\bar{\beta} = r$. The element β (or α) is called the mirror of α (or β) with respect to the circle $|z|^2 = r$.

Remark 1: By definition, an element α in $\mathbb{F}_1$ and its mirror element β in $\mathbb{F}_2$ have the same argument, that is $\arg(\alpha) = \arg(\beta)$. Also we have $|\alpha|\ |\beta| = r$.

Consider a discrete time-invariant linear system

$$x_{k+1} = Ax_k + Bu_k \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrix and input matrix, respectively. The optimal control problem is

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to find the control sequence such that the following objective function

$$J = \sum_{i=0}^{\infty} (x_i^{\mathrm{T}} Q x_i + u_i^{\mathrm{T}} R u_i), \quad Q \geq 0, \quad R > 0 \tag{3}$$

is minimised. It is well-known that the solution is a time-invariant state feedback law given by $u_k = Fx_k$ where

$$F = -(R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A \tag{4}$$

and $P > 0$ is the positive definite solution to the following DARE

$$P = A^{\mathrm{T}} P A + Q - A^{\mathrm{T}} P B (R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A \tag{5}$$

In this paper, we are interested in a special case that

$$Q = \theta P$$

where θ is some real scalar that will be specified later. In this case, the DARE (5) becomes

$$(1 - \theta) P = A^{\mathrm{T}} P A - A^{\mathrm{T}} P B (R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A \tag{6}$$

Regarding solutions to the DARE (6), we have the following result that generalises the results in [28].

Theorem 1: Consider the DARE. Then we have the following statements:

1. The DARE (6) has a unique positive definite solution $P = P(\theta) > 0$ if and only if A is non-singular, (A, B) is controllable, and

$$1 - |\lambda(A)|_{\min}^2 < \theta < 1 \tag{7}$$

Moreover, the unique positive definite solution to (6) when (7) is satisfied is given by $P(\theta) = S^{-1}(\theta)$, where $S(\theta)$ is the unique positive definite solution to the following parametric Stein matrix equation

$$S - \left(\frac{A}{\sqrt{1-\theta}}\right) S \left(\frac{A}{\sqrt{1-\theta}}\right)^{\mathrm{T}} = -BR^{-1}B^{\mathrm{T}} \tag{8}$$

2. Assume that $P(\theta) > 0$ exists. Denote the closed-loop system matrix $A_{\mathrm{c}}(\theta)$ as

$$A_{\mathrm{c}}(\theta) = A - B(R + B^{\mathrm{T}} P(\theta) B)^{-1} B^{\mathrm{T}} P(\theta) A \tag{9}$$

Then the eigenvalue set of the matrix $A_{\mathrm{c}}(\theta)$ and the eigenvalue set of the matrix A are circle-symmetric with respect to the circle $|z|^2 = 1 - \theta$. Moreover, $A_{\mathrm{c}}(\theta)$ is Schur stable if and only if

$$1 - |\lambda(A)|_{\min} < \theta \tag{10}$$

Proof: (1) (*Only if*). Assume that the DARE (6) admits a positive definite solution P. Using the identity (1), we can obtain

$$P^{-1} - B(R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} = P^{-1} (P^{-1} + BR^{-1} B^{\mathrm{T}})^{-1} P^{-1} \tag{11}$$

Substituting (11) into the DARE (6) gives

$$(1 - \theta) P = A^{\mathrm{T}} (P^{-1} + BR^{-1} B^{\mathrm{T}})^{-1} A \tag{12}$$

If $\theta = 1$, it follows from (12) that

$$A^{\mathrm{T}} (P^{-1} + BR^{-1} B^{\mathrm{T}})^{-1} A = 0$$

Since $(P^{-1} + BR^{-1}B^{\mathrm{T}})^{-1}$ is positive definite, the above equation implies that $A = 0$. Consequently, the DARE (6) becomes

$$(1 - \theta) P = 0$$

which has infinite number of positive definite solutions. A contradiction. Therefore we must have $\theta \neq 1$.

If $\theta \neq 1$, as P is positive definite, it follows from (12) that A is non-singular. Then taking inverse on both sides of (12) and rearranging gives

$$A^{-1} P^{-1} A^{-\mathrm{T}} + \frac{1}{\theta - 1} P^{-1} = -A^{-1} B R^{-1} B^{\mathrm{T}} A^{-\mathrm{T}} \tag{13}$$

Note that if $\theta > 1$, the left-hand side of the above equation is strictly positive definite whereas the right-hand side is semi-negative definite. This is also impossible. Therefore a positive definite solution may exist only when $\theta < 1$. In this case, the equation in (13) can be rewritten as

$$A_\theta^{-1} P^{-1} A_\theta^{-\mathrm{T}} - P^{-1} = -A_\theta^{-1} B R^{-1} B^{\mathrm{T}} A_\theta^{-\mathrm{T}} \tag{14}$$

where $A_\theta = (1/\sqrt{1-\theta})A$. Note that (14) is a standard Stein matrix equation. We will show that (A, B) is controllable.

Assume that $(A_\theta^{-1}, A_\theta^{-1}B)$ is not controllable, then there exists an eigenvalue λ of A_θ^{-1} and its corresponding eigenvector z^{H} such that

$$z^{\mathrm{H}} A_\theta^{-1} = \lambda z^{\mathrm{H}}, \quad z^{\mathrm{H}} A_\theta^{-1} B = 0$$

Multiplying (14) from left by z^{H} and from right by z gives

$$(|\lambda|^2 - 1) z^{\mathrm{H}} P^{-1} z = 0 \tag{15}$$

which, as P is positive definite, implies that $|\lambda| = 1$. This contradicts with that (14) has a unique positive definite solution. Consequently, we know that $(A_\theta^{-1}, A_\theta^{-1}B)$ is controllable, which is further equivalent to that (A, B) is controllable.

Since (A, B) is controllable and $P > 0$, it follows from the standard Lyapunov stability theory that the matrix A_θ^{-1} is Schur stable. Note that

$$|\lambda(A_\theta^{-1})|_{\max} = \left|\lambda\left(A^{-1}\sqrt{1-\theta}\right)\right|_{\max} = \frac{\sqrt{1-\theta}}{|\lambda(A)|_{\min}} \tag{16}$$

Therefore the matrix A_θ^{-1} is Schur stable if and only if $\theta > 1 - |\lambda(A)|^2_{\min}$. This inequality, together with $\theta < 1$, is just condition (7).

(*If*) Assume that θ satisfies the inequality (7), (A, B) is controllable and A is non-singular. We show the DARE (6) has a unique positive definite solution. It follows from (16) that A_θ^{-1} is Schur stable. Consequently, as $\left(A_\theta^{-1}, A_\theta^{-1}BR^{-1/2}\right)$ is controllable, the following Stein matrix equation

$$A_\theta^{-1}WA_\theta^{-\mathrm{T}} - W = -A_\theta^{-1}BR^{-1}B^{\mathrm{T}}A_\theta^{-\mathrm{T}} \tag{17}$$

has a unique positive definite solution W. By using the arguments used in deriving (14), we can obtain the following equation from (17)

$$\begin{aligned}(1-\theta)W^{-1} = {} & A^{\mathrm{T}}W^{-1}A - A^{\mathrm{T}}W^{-1}B(R \\ & + B^{\mathrm{T}}W^{-1}B)^{-1}B^{\mathrm{T}}W^{-1}A\end{aligned} \tag{18}$$

Comparing (18) with (6) we clearly see that the DARE (6) has a positive definite solution $P = W^{-1}$. Therefore we need only to show that the positive solution to the DARE (6) is unique. Assume that the DARE (6) has two positive definite solutions P_1 and P_2. Then by using the arguments used in deriving (14), we obtain

$$A_\theta^{-1}P_1^{-1}A_\theta^{-\mathrm{T}} - P_1^{-1} = -A_\theta^{-1}BR^{-1}B^{\mathrm{T}}A_\theta^{-\mathrm{T}}$$
$$A_\theta^{-1}P_2^{-1}A_\theta^{-\mathrm{T}} - P_2^{-1} = -A_\theta^{-1}BR^{-1}B^{\mathrm{T}}A_\theta^{-\mathrm{T}}$$

from which we have

$$A_\theta^{-1}(P_1^{-1} - P_2^{-1})A_\theta^{-\mathrm{T}} - (P_1^{-1} - P_2^{-1}) = 0 \tag{19}$$

Since A_θ^{-1} is Schur stable, (19) clearly implies $P_1^{-1} - P_2^{-1} = 0$, that is $P_1 = P_2$.

(2) Using the DARE (6), we have

$$(1-\theta)P(\theta) = A^{\mathrm{T}}P(\theta)(A - B(R + B^{\mathrm{T}}P(\theta)B)^{-1}B^{\mathrm{T}}P(\theta)A)$$

Since A is non-singular, the above equation is equivalent to

$$(1-\theta)A^{-\mathrm{T}}P(\theta) = P(\theta)A_{\mathrm{c}}(\theta) \tag{20}$$

Because P is positive definite, (20) in turn implies

$$\lambda(A_{\mathrm{c}}(\theta)) = (1-\theta)\lambda(A^{-\mathrm{T}}) = \frac{1-\theta}{\lambda(A)} \tag{21}$$

Since both $A_{\mathrm{c}}(\theta)$ and A are real matrices, it follows from (21) that for arbitrary $\mu \in \sigma(A_{\mathrm{c}}(\theta))$, there exists an eigenvalue $\vartheta \in \sigma(A)$ such that $\mu\vartheta = 1 - \theta$, and vice versa. Accordingly, $\sigma(A_{\mathrm{c}}(\theta))$ and $\sigma(A)$ are circle-symmetric with respect to the circle $|z|^2 = 1 - \theta$ by definition. Moreover, $A_{\mathrm{c}}(\theta)$ is Schur stable if and only if $|\lambda(A_{\mathrm{c}}(\theta))|_{\max} < 1$ which is equivalent to (10) by virtue of (21). With this, we complete the proof. □

If $A_{\mathrm{c}}(\theta)$ defined as (9) is Schur stable, the positive definite solution $P(\theta) = S^{-1}(\theta)$ is called a stabilising solution. Obviously, based upon Theorem 1, a stabilising solution exists if and only if

$$\mathrm{cond}(A) < \theta < 1 \tag{22}$$

where

$$\mathrm{cond}(A) = \max\{1 - |\lambda(A)|^2_{\min}, 1 - |\lambda(A)|_{\min}\}$$

Denote the pole shift operator

$$\mathfrak{M}(\theta): \sigma(A) \to \sigma(A - B(R + B^{\mathrm{T}}P(\theta)B)^{-1}B^{\mathrm{T}}P(\theta)A)$$

where θ satisfies (22). The shift operator $\mathfrak{M}(\theta)$ shifts all the eigenvalues of A that locating outside the circle $|z|^2 = 1 - \theta$ into the unit circle. The maximal circle $\mathcal{C}_{\max}$ that the eigenvalues of A can be shifted into by $\mathfrak{M}(\theta)$ is quite different in the case $|\lambda(A)|_{\min} \le 1$ and in the case $|\lambda(A)|_{\min} > 1$. If $|\lambda(A)|_{\min} \le 1$, the maximal circle $\mathcal{C}_{\max} = \{z : |z| = 1 - |\lambda(A)|_{\min}\}$ which is strictly inside the unit circle. If $|\lambda(A)|_{\min} > 1$, the maximal circle $\mathcal{C}_{\max}$ is exactly the unit circle. Whenever (22) is satisfied, Theorem 1 provides a simple and novel approach to stabilise a discrete-time linear system by solving a linear matrix equation (8). Such results are directly applicable to initialising certain iterative methods that find steady-state gains for the discrete optimal regulator (see e.g. [29]).

To complete this section, several remarks are in order concerning Theorem 1.

Remark 2: Although θ can be a negative scalar if $|\lambda(A)|_{\min} > 1$, the corresponding quadratic performance index (3) with $Q = \theta P$ is no longer positive if $\theta < 0$ which may loose system meaning in linear quadratic design. For this reason, we should restrict us to the case $\theta \ge 0$. In this case, the condition in (22) can be written as

$$1 > \theta > \begin{cases} 0, & |\lambda(A)|_{\min} \ge 1 \\ 1 - |\lambda(A)|^2_{\min}, & |\lambda(A)|_{\min} < 1 \end{cases} \tag{23}$$

Remark 3: In [28], the controllability of (A, B) and the non-singularity of A are assumed firstly without discussion. In Theorem 1, we further show that these two conditions are necessary for guaranteeing that the DARE (6) has a unique positive definite solution.

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Remark 4: The result of item 2 of Theorem 1 can be viewed as a generalisation of the result given in [30], where it was shown that when A is Schur anti-stable, that is all the eigenvalues of A have modulus strictly bigger than 1, and (A, B) is controllable, then the eigenvalue set of $A - B(I + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA$ with P the unique positive definite solution to

$$P = A^{\mathrm{T}}PA - A^{\mathrm{T}}PB(I + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA$$

and the eigenvalue set of A are circle-symmetric with respect to the unit circle $|z| = 1$. This result is a discrete-time version of that given in [31].

Remark 5: Theorem 1 extends several issues of the results in [21], where the following DARE

$$P = \alpha A^{\mathrm{T}}\left(P - PB\left(I + B^{\mathrm{T}}PB\right)^{-1}B^{\mathrm{T}}P\right)\alpha A \qquad (24)$$

is used to shift the poles of matrix A. Note that (24) is equivalent to (6) with $\alpha = (1/\sqrt{1-\theta})$. Firstly, it was claimed in [21] that a positive definite solution exists provided $\alpha^2|\lambda(A)| > 1$. However, this statement is incorrect according to Theorem 1. Secondly, we use a linear Stein equation but not a non-linear DARE to shift the poles, which has the advantage that the linear Stein equation can be solved by some numerically stable algorithm (see e.g. [32]). Thirdly, the situations for different cases for pole shift are clarified by introducing the circle-symmetry principle.

Remark 6: Unlike obtaining solution to the quadratic equation (24), the closed form of the solution to the parametric Stein (8) can be readily obtained by using the Matlab symbolic computation toolbox. Furthermore, it is easy to see that $P(\theta) = S^{-1}(\theta)$ is a rational function of θ.

Remark 7: The DARE (6) can be viewed as a DARE in the form of (5) by setting $Q = 0$. We note that similar discussion was made in [33]. However, a recursive type of DARE of system order are required to be solved to realise pole shift until the solution is zero, which may lead to rather expensive computation.

3 Successive optimal pole shift

Although Theorem 1 provides a simple method to stabilise a system with guaranteed minimisation of certain quadratic performance index and pole locations of closed-loop system, all the closed-loop eigenvalues are determined by a single parameter θ. This is not enough in most cases. Moreover, Theorem 1 requires that A is a non-singular matrix which is generally not satisfied in practice. In this section, we develop a successive approach to shift the poles more precisely, which also allows us to avoid the restriction that A should be a non-singular matrix. The following result is classical and can be found in, for example, [34].

Lemma 2: Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ be two constant matrices. Then the following Sylvester matrix equation

$$AX - XB = C$$

has a unique solution for arbitrary $C \in \mathbb{R}^{n\times m}$ if and only if

$$\sigma(A) \cap \sigma(B) = \emptyset$$

To go further, we first prove the following lemma.

Lemma 3: Let $A \in \mathbb{R}^{n\times n}$ be a given matrix and

$$C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

be a non-singular matrix such that

$$C^{\mathrm{T}}AC^{-\mathrm{T}} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix} \triangleq \mathcal{A}, \quad C^{\mathrm{T}}B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \triangleq \mathcal{B} \qquad (25)$$

where $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m}$, $C_i \in \mathbb{R}^{n\times n_i}$, $i = 1, 2, 3$ and $n_1 + n_2 + n_3 = n$. Let P_2 satisfy the following reduced-order DARE

$$P_2 = A_2^{\mathrm{T}}P_2A_2 + Q_2 - A_2^{\mathrm{T}}P_2B_2(R + B_2^{\mathrm{T}}P_2B_2)^{-1}B_2^{\mathrm{T}}P_2A_2 \qquad (26)$$

where $Q_2 \geq 0$ and $R > 0$. Using matrix P_2 to construct the following series of matrices

$$\begin{aligned} F_2 &= -(R + B_2^{\mathrm{T}}P_2B_2)^{-1}B_2^{\mathrm{T}}P_2A_2 \\ A_{c2} &= A_2 + B_2F_2 \\ P &= C_2P_2C_2^{\mathrm{T}} \\ F &= -(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA \end{aligned} \qquad (27)$$

If the following conditions

$$\begin{aligned} \sigma(A_{c2}) \cap \sigma(A_1) &= \emptyset \\ \sigma(A_3) \cap \sigma(A_{c2}) &= \emptyset \end{aligned} \qquad (28)$$

are satisfied, then

$$A + BF = A + BF_2C_2^{\mathrm{T}} = C_c^{-\mathrm{T}}\begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_{c2} & 0 \\ 0 & 0 & A_3 \end{bmatrix}C_c^{\mathrm{T}} \qquad (29)$$

where

$$C_c = \begin{bmatrix} C_1 + C_2X_1^{\mathrm{T}} & C_2 & C_3 + C_2X_3^{\mathrm{T}} \end{bmatrix} \qquad (30)$$

with X_1 and X_3 the solutions to the following Sylvester equations

$$\begin{aligned} A_1X_1 - X_1A_{c2} &= B_1F_2 \\ A_3X_3 - X_3A_{c2} &= B_3F_2 \end{aligned} \tag{31}$$

Proof: Note that the reduced order DARE (26) can be equivalently rewritten as

$$\mathcal{P} = \mathcal{A}^{\mathrm{T}}\mathcal{P}\mathcal{A} + \mathcal{Q} - \mathcal{A}^{\mathrm{T}}\mathcal{P}\mathcal{B}\left(R + \mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B}\right)^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{A} \tag{32}$$

by denoting

$$\mathcal{P} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{33}$$

Then straightforward manipulation gives

$$\begin{aligned} \mathcal{A}_c &= \mathcal{A} - \mathcal{B}(R + \mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{B})^{-1}\mathcal{B}^{\mathrm{T}}\mathcal{P}\mathcal{A} \\ &= \begin{bmatrix} A_1 & B_1F_2 & 0 \\ 0 & A_2 + B_2F_2 & 0 \\ 0 & B_3F_2 & A_3 \end{bmatrix} \end{aligned} \tag{34}$$

On the other hand, using relations (25) and (33) gives

$$\mathcal{A}_c = C^{\mathrm{T}}(A - B(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA)C^{-\mathrm{T}} \tag{35}$$

where we have used the substitution

$$P = C\mathcal{P}C^{\mathrm{T}} = C_2P_2C_2^{\mathrm{T}}$$

Using (31), (34) and (35), we obtain

$$\begin{aligned} &\begin{bmatrix} I & X_1 & 0 \\ 0 & I & 0 \\ 0 & X_3 & I \end{bmatrix} C^{\mathrm{T}}(A + BF) \\ &= \begin{bmatrix} I & X_1 & 0 \\ 0 & I & 0 \\ 0 & X_3 & I \end{bmatrix} \begin{bmatrix} A_1 & B_1F_2 & 0 \\ 0 & A_2 + B_2F_2 & 0 \\ 0 & B_3F_2 & A_3 \end{bmatrix} C^{\mathrm{T}} \\ &= \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_{c2} & 0 \\ 0 & 0 & A_3 \end{bmatrix} \begin{bmatrix} I & X_1 & 0 \\ 0 & I & 0 \\ 0 & X_3 & I \end{bmatrix} C^{\mathrm{T}} \end{aligned} \tag{36}$$

Therefore if we denote

$$C_c^{\mathrm{T}} = \begin{bmatrix} I & X_1 & 0 \\ 0 & I & 0 \\ 0 & X_3 & I \end{bmatrix} C^{\mathrm{T}} = \begin{bmatrix} C_1^{\mathrm{T}} + X_1C_2^{\mathrm{T}} \\ C_2^{\mathrm{T}} \\ C_3^{\mathrm{T}} + X_3C_2^{\mathrm{T}} \end{bmatrix} \tag{37}$$

then (36) becomes (29). Note that (37) is equivalent to (30) and

$$\begin{aligned} F &= (R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA \\ &= (R + B_2^{\mathrm{T}}P_2B_2)^{-1}B_2^{\mathrm{T}}P_2A_2C_2^{\mathrm{T}} \\ &= -F_2C_2^{\mathrm{T}} \end{aligned}$$

Finally, by virtue of (28), it follows from Lemma 2 that the two Sylvester matrix equations in (31) have unique solutions. We thus complete the proof. □

Remark 8: Equations (29) and (30) mean that the transformation matrix C_c for the closed-loop system matrix $A + BF$ can be obtained via an update on the open-loop transformation matrix C. Such a property is useful in the recursive design described later.

We use the notation $S = \mathrm{Stein}(A, B, R, \theta)$ to denote the solution to the Stein equation (8) with coefficient matrices A, B, R and parameter θ. By applying Theorem 1 on the reduced-order DARE (26) and denoting

$$Q_2 = \theta_2P_2 \tag{38}$$

we can obtain the following corollary.

Corollary 1: Let A and C be defined as in Lemma 3. Assume that

$$\mathrm{cond}(A_2) < \theta_2 < 1$$

Then the reduced-order DARE (26) has a unique positive definite solution $P_2 = S_2^{-1}$ where

$$S_2 = \mathrm{Stein}(A_2, B_2, R, \theta_2)$$

Furthermore, the closed-loop system (29) has eigenvalue set

$$\sigma(A + BF) = \sigma(A_1) \cup \frac{1 - \theta_2}{\sigma(A_2)} \cup \sigma(A_3) \tag{39}$$

where F is defined in (27), and the quadratic performance index function

$$J(u) = \sum_{k=0}^{\infty} (\theta_2x_k^{\mathrm{T}}C_2P_2C_2^{\mathrm{T}}x_k + u_k^{\mathrm{T}}Ru_k) \tag{40}$$

is minimised via the state feedback

$$u_k = Fx_k$$

Proof: Substituting (25) into the DARE (32) gives

$$P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA \tag{41}$$

where $P = C\mathcal{P}C^{\mathrm{T}}$ and $Q = C\mathcal{Q}C^{\mathrm{T}}$. In view of (33) and (38),

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we obtain

$$P = C\mathcal{P}C^{\mathrm{T}} = C_2 P_2 C_2^{\mathrm{T}}$$
$$Q = C\mathcal{Q}C^{\mathrm{T}} = C_2 Q_2 C_2^{\mathrm{T}} = \theta_2 C_2 P_2 C_2^{\mathrm{T}}$$

which together with (41) and (27) imply that $u_k = Fx_k$ is the solution to the linear quadratic control problem with performance index given by (40). With the help of (26) and (38), it follows from Theorem 1 that

$$\sigma(A_{c2}) = \frac{1-\theta_2}{\sigma(A_2)} \tag{42}$$

Therefore (39) follows from (29) and (42). The proof is done. □

Let the eigenvalue set of matrix A be

$$\sigma(A) = \{s_1, \ldots, s_{r_1}, \lambda_1 \pm \omega_1 i, \ldots, \lambda_{r_2} \pm \omega_{r_2} i\}$$
$$\omega_k > 0, \quad k = 1, 2, \ldots, r_2$$

where $s_i \neq s_j$, $\forall i \neq j$ and $\lambda_i + \omega_i i \neq \lambda_k + \omega_k i$, $\forall i \neq k$. Assume that the geometric multiplicities of s_i be g_i, $i = 1, 2, \ldots, r_1$ and the geometric multiplicities of $\lambda_i + i\omega_i$ be h_i, $i = 1, 2, \ldots, r_2$, respectively. Let

$$r_{\max} = \sum_{i=1}^{r_1} g_i + \sum_{i=1}^{r_2} h_i \tag{43}$$

Then for arbitrary positive integer

$$r + 1 \leq r_{\max} \tag{44}$$

there exists a non-singular matrix

$$C = \begin{bmatrix} C_1 & C_2 & \cdots & C_r & C_0 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

where $C_i \in \mathbb{R}^{n\times n_i}$, $\Sigma_{i=0}^{r} n_i = n$, such that

$$C^{\mathrm{T}}AC^{-\mathrm{T}} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_r & 0 \\ 0 & 0 & \cdots & 0 & A_0 \end{bmatrix}, \quad C^{\mathrm{T}}B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_r \\ B_0 \end{bmatrix} \tag{45}$$

where $A_i \in \mathbb{R}^{n_i\times n_i}$, $B_i \in \mathbb{R}^{n_i\times m}$, $i = 0, 1, \ldots, r$ and $\Sigma_{i=0}^{r} n_i = n$.

Remark 9: The purpose of using such special coordinates form in (45) is that the eigenvalues of A_i, $i = 1, 2, \ldots, r$ are expected to be shifted while the eigenvalues of A_0 are kept invariance. Indeed, by using Corollary 1 recursively, the eigenvalue sets of A_i, $i = 1, 2, \ldots, r$ are shifted to the new eigenvalue sets $(1-\theta_i)/\sigma(A_i)$, $i = 1, 2, \ldots, r$ while the eigenvalue set of A_0 keeps the same. This property has the following two advantages. On the one hand, the special form (45) allows us to handle the case that the matrix A is singular. In this case, we let all the zero eigenvalues of A_0 coincide with all the zero eigenvalues of A. Therefore all the matrices A_i, $i = 1, 2, \ldots, r$ are non-singular and thus we can apply Corollary 1 to shift the eigenvalues of A_i, $i = 1, 2, \ldots, r$ recursively. On the other hand, if there are some eigenvalues that we do not want to shift, then they can also be contained in the eigenvalue set of A_0.

Remark 10: There are many methods existing in the literature that can be used to compute the matrix C and A_i, $i = 0, 1, \ldots, r$ in the special coordinates form (45) (see [35–38] and the references therein). For example, in [35], a numerical method is developed to find non-singular matrix T such that a linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

is transformed into a block form

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} B_3 \\ B_4 \end{bmatrix} u$$

via the transformation $y = Tx$. Clearly, applying this method recursively on the system (2), we can obtain the special coordinates from (25).

Let (A, B) be controllable, $R > 0$, $Q \geq 0$ and (A,Q) be observable. Then the DARE (5) has a unique positive definite solution. Denote such solution P and the resulting optimal feedback gain F defined in (4) as

$$P = \mathrm{DARE}(A, B, R, Q) \quad F = \mathrm{Optf}(A, B, R, Q)$$

To proceed, we recall the following interesting result.

Lemma 4 [39]: Given a controllable pair (A, B) and a matrix $R > 0$. Assume that

$$P_i = \mathrm{DARE}(A_i, B, R_i, Q_i), \quad F_i = \mathrm{Optf}(A_i, B, R_i, Q_i)$$

with $Q_i \geq 0$ some arbitrary semi-positive definite matrix such that (A_i, Q_i) is observable, and

$$A_{i+1} = A_i + BF_i, \quad A_1 = A$$
$$R_{i+1} = R_i + B^{\mathrm{T}}P_iB, \quad R_1 = R$$

for $i = 1, 2, \ldots, r$. Then

$$\sum_{i=1}^{r} P_i = \text{DARE}\left(A, B, R, \sum_{i=1}^{r} Q_i\right)$$
$$\sum_{i=1}^{r} F_i = \text{Optf}\left(A, B, R, \sum_{i=1}^{r} Q_i\right)$$

By applying Corollary 1 on the system (45) recursively, we can obtain the following successive pole shift algorithm.

Algorithm 1: Assume that (A, B) is controllable and admits the decomposition (45) where A_i, $i = 1, 2, \ldots, r$ are non-singular.

1. (Initialisation) Set $\mathcal{C}_j = C_j$, $\mathcal{B}_j = B_j$, $j = 1, 2, \ldots, r$ and $R_1 = R > 0$. Set $i = 1$.

2. (Solve Stein equation) Assume that $\text{cond}(A_i) < \theta_i < 1$. Let $S_i = \text{Stein}(A_i, \mathcal{B}_i, R_i, \theta_i) > 0$. Denote a series of matrices by using S_i as follows

$$\mathcal{F}_i = -\left(R_i + \mathcal{B}_i^{\mathrm{T}} S_i^{-1} \mathcal{B}_i\right)^{-1} \mathcal{B}_i^{\mathrm{T}} S_i^{-1} A_i$$
$$F_i = \mathcal{F}_i \mathcal{C}_i^{\mathrm{T}}$$
$$P_i = \mathcal{C}_i S_i^{-1} \mathcal{C}_i^{\mathrm{T}}$$
$$A_{ci} = A_i + \mathcal{B}_i \mathcal{F}_i$$

3. (Update matrices) The matrices R_k and $\mathcal{C}_k$ are updated as

$$R_{i+1} = R_i + \mathcal{B}_i^{\mathrm{T}} S_i^{-1} \mathcal{B}_i$$
$$\mathcal{C}_k = \mathcal{C}_k + \mathcal{C}_i X_k^{\mathrm{T}}, \quad k = i+1, i+2, \ldots, r \tag{46}$$

where X_k solves the following Sylvester matrix equation

$$\mathcal{B}_k \mathcal{F}_i = A_k X_k - X_k A_{ci}, \quad k = i+1, i+2, \ldots, r \tag{47}$$

Furthermore, matrix $\mathcal{B}_k$ is updated as

$$\mathcal{B}_k = \mathcal{C}_k^{\mathrm{T}} B, \quad k = i+1, i+2, \ldots, r \tag{48}$$

4. If $i = r$ then end. Else set $i = i + 1$ and go to Step 2.

Regarding the optimality of the resulting feedback gain for the closed-loop system, we can obtain the following theorem by virtue of Corollary 1 and Lemma 4.

Theorem 2: Assume that (A, B) is controllable and satisfies the decomposition (45), and P_i, F_i, $i = 1, 2, \ldots, r$, are given by Algorithm 1. Moreover, all the matrices A_i, $i = 1, 2, \ldots, r$ are non-singular. Then

$$P = \text{DARE}(A, B, R, Q), \quad F = \text{Optf}(A, B, R, Q) \tag{49}$$

and

$$\sigma(A + BF) = \bigcup_{i=1}^{r} \frac{1 - \theta_i}{\sigma(A_i)} \bigcup \sigma(A_0) \tag{50}$$

where

$$P = \sum_{i=1}^{r} P_i, \quad Q = \sum_{i=1}^{r} \theta_i P_i, \quad F = \sum_{i=1}^{r} F_i \tag{51}$$

Remark 11: It follows from (46) and (48) that in step i, we need only to update matrices $\mathcal{C}_k$ and $\mathcal{B}_k$ for $k > i$. Accordingly, there are $(r(r-1))/2$ equations in the form of (47) that need to be solved in the algorithm. Furthermore, we note that (47) has a unique solution X_k if and only if A_k and A_{ci} have no common eigenvalues which can always be guaranteed by choosing θ_i suitably.

Remark 12: Note that real poles and complex conjugate pairs of poles are only shifted radially in the complex plane. Hence, this method does not allow one to shift a complex conjugate pair of poles of the open-loop system matrix A to real locations, so that unwanted oscillations can only be damped, but not removed. Also, it follows from (23) and (50) that if $|\lambda(A_s)|_{\min} < 1$, for some $1 \le s \le r$, then the closed-loop eigenvalue set associated with A_{cs} is restricted as $1 - |\lambda_s(A)|^2_{\min} < \theta_s < 1$. These facts certainly indicate practical limitations of the approach.

Remark 13: Algorithm 1 also allows us to handle the case that the matrix A has eigenvalue with multiplicity larger than 1. For example, let λ be a real eigenvalue of A with associated Jordan form as

$$J_\lambda = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

Then we can choose $A_s = J_\lambda$ for some $1 \le s \le r$. In this case, the eigenvalue set of the closed-loop system is $\{(1-\theta)/\lambda, (1-\theta)/\lambda, (1-\theta)/\lambda\}$. Moreover, if λ be an complex eigenvalue, then we can choose A_s as the real Jordan form associated with the eigenvalue λ.

4 Illustrative examples

Example 1: We consider a discrete-time linear system with parameters

$$A = \begin{bmatrix} -1 & \frac{1}{2} & 0 \\ -1 & -1 & 1 \\ 0 & -\frac{1}{2} & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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Straightforward manipulation gives

$$C^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathcal{A}_1 = -1, \quad \mathcal{A}_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

It follows that this system is controllable and has the open-loop eigenvalue set $\{-1, -1 \pm i\}$. We need to find the gain matrix F that ensures desired closed-loop eigenvalues and minimises a quadratic function of the form (3). It follows from Theorem 2 that the eigenvalues can be shifted to the stable eigenvalue set $\{\theta_1 - 1, ((-1 \pm i)/2)(1 - \theta_2)\}$ where $\theta_1 \in (0, 1)$, $\theta_2 \in (0, 1)$. We accomplish the requirement in two steps according to Algorithm 1. Set $R = I_2$. In the first step, we obtain $S_1 = (2/\theta_1) - 2$ and

$$F_1 = \frac{1}{2}\theta_1 \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_1 = \frac{\theta_1}{2 - 2\theta_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The matrices R_2, $\mathcal{C}_2$ and $\mathcal{B}_2$ are updated according to (46) as follows

$$R_2 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2(1-\theta_1)} & -\frac{1}{2} + \frac{1}{2(1-\theta_1)} \\ \frac{1}{2} + \frac{1}{2(1-\theta_1)} & \frac{1}{2} + \frac{1}{2(1-\theta_1)} \end{bmatrix}$$

$$\mathcal{C}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{B}_2 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$

In the second step, similarly, we obtain

$$S_2 = \begin{bmatrix} \frac{(-2 + 2\theta_2)(\theta_2 - 3)}{\theta_2^3 - \theta_2^2 + 3\theta_2 + 5} & \frac{2 - 2\theta_2}{\theta_2^2 - 2\theta_2 + 5} \\ \frac{2 - 2\theta_2}{\theta_2^2 - 2\theta_2 + 5} & \frac{(2 - 2\theta_2)(\theta_2^2 - \theta_2 + 2)}{\theta_2^3 - \theta_2^2 + 3\theta_2 + 5} \end{bmatrix}$$

$$F_2 = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\theta_2 & \frac{1}{4}\theta_2^2 - \frac{1}{4} & -\frac{1}{2} - \frac{1}{2}\theta_2 \\ -\frac{1}{2} - \frac{1}{2}\theta_2 & -\frac{1}{4}\theta_2^2 + \frac{1}{4} & \frac{1}{2} + \frac{1}{2}\theta_2 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \frac{(1+\theta_2)(3-\theta_2)}{2(\theta_2-1)^2} & \frac{(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(\theta_2-3)}{2(\theta_2-1)^2} \\ \frac{(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)({\theta_2}^2-\theta_2+2)}{2(\theta_2-1)^2} & \frac{-(1+\theta_2)^2}{2(\theta_2-1)^2} \\ \frac{(1+\theta_2)(\theta_2-3)}{2(\theta_2-1)^2} & \frac{-(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(3-\theta_2)}{2(\theta_2-1)^2} \end{bmatrix}$$

According to (51), the resulting matrices P, Q and feedback gain matrix F are, respectively, given by the equation shown at the bottom of the page.

It is easy to verify that the relation (49) is satisfied and the closed-loop system has desired eigenvalues.

$$P = \begin{bmatrix} \frac{(1+\theta_2)(3-\theta_2)}{2(\theta_2-1)^2} + \frac{\theta_1}{2-2\theta_1} & \frac{(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(\theta_2-3)}{2(\theta_2-1)^2} + \frac{\theta_1}{2-2\theta_1} \\ \frac{(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(\theta_2^2-\theta_2+2)}{2(\theta_2-1)^2} & \frac{-(1+\theta_2)^2}{2(\theta_2-1)^2} \\ \frac{(1+\theta_2)(\theta_2-3)}{2(\theta_2-1)^2} + \frac{\theta_1}{2-2\theta_1} & \frac{-(1+\theta_2)^2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(3-\theta_2)}{2(\theta_2-1)^2} + \frac{\theta_1}{2-2\theta_1} \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{(1+\theta_2)(3-\theta_2)\theta_2}{2(\theta_2-1)^2} + \frac{\theta_1^2}{2-2\theta_1} & \frac{(1+\theta_2)^2\theta_2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(\theta_2-3)\theta_2}{2(\theta_2-1)^2} + \frac{\theta^2}{2-2\theta_2} \\ \frac{(1+\theta_2)^2\theta_2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(\theta_2^2-\theta_2+2)}{2(\theta_2-1)^2} & \frac{-(1+\theta_2)^2\theta_2}{2(\theta_2-1)^2} \\ \frac{(1+\theta_2)(\theta_2-3)\theta_2}{2(\theta_2-1)^2} + \frac{\theta_1^2}{2-2\theta_1} & \frac{-(1+\theta_2)^2\theta_2}{2(\theta_2-1)^2} & \frac{(1+\theta_2)(3-\theta_2)\theta_2}{2(\theta_2-1)^2} + \frac{\theta_1^2}{2-2\theta_1} \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{1}{2}(\theta_2+\theta_1) + \frac{1}{2} & \frac{1}{4}(\theta_2^2 - 1) & -\frac{1}{2} + \frac{1}{2}(\theta_1 - \theta_2) \\ -\frac{1}{2} + \frac{1}{2}(\theta_1 - \theta_2) & \frac{1}{4}(1 - \theta_2^2) & \frac{1}{2}(\theta_2 + \theta_1) + \frac{1}{2} \end{bmatrix}$$

Example 2: We consider a six-order discrete-time linear system in the form of (2) with

$$A=\begin{bmatrix} 1.061 & -1.082 & 1.585 & 0.0784 & 0.4410 & -1.355 \\ 0.7218 & 0.1957 & 0.7262 & -0.0802 & 0.7373 & -0.7827 \\ -0.6980 & 0.1014 & 0.2161 & -0.1113 & -0.7330 & -0.0826 \\ 0.1161 & -0.4283 & 1.366 & 0.8102 & 0.1224 & -0.5440 \\ -0.4412 & 1.283 & -1.972 & -0.2005 & 0.0370 & 2.194 \\ 0.0431 & 0.1985 & -0.3289 & 0.0391 & -0.1049 & 1.193 \end{bmatrix}$$

$$B^{\mathrm{T}}=\begin{bmatrix} 0.0280 & 0.069 & 0.4873 & 0.2886 & 0.1787 & -0.0451 \\ 0.1142 & 0.3146 & 0.2450 & 0.3301 & -0.0736 & -0.3212 \\ -0.1292 & -0.3832 & -0.0382 & 0.1678 & 0.2756 & -0.1664 \end{bmatrix}$$

Note that A is singular since $\sigma(A) = \{1.106 \pm 0.3426i, 0.6507 \pm 0.2649i, 0, 0\}$. Hence according to Theorem 2, we can shift the open-loop poles to the following stable set

$$\{(1-\theta_1)(0.8252 \pm 0.2557i), (1-\theta_2)(1.3183 \pm 0.5368i), 0, 0\} \tag{52}$$

where $0 < \theta_1 < 1$ and $0.5065 < \theta_2 < 1$. We choose $\theta_1 = 0.3$ and $\theta_2 = 0.75$ in this example. A possible block representation of A is given by

$$A_1=\begin{bmatrix} 1.501 & -0.3001 \\ 0.9124 & 0.7102 \end{bmatrix}, \quad A_2=\begin{bmatrix} 0.5012 & 0.2201 \\ -0.4204 & 0.8001 \end{bmatrix}$$

$$A_0=\begin{bmatrix} 0.6250 & 1.250 \\ -0.3125 & -0.6250 \end{bmatrix}$$

and the associated transformation matrix can be chosen as

$$C=\begin{bmatrix} -1.687 & 0.3589 & 1.444 & 1.939 & -1.385 & -0.7251 \\ -0.9528 & -1.575 & 1.189 & -0.9871 & -0.1006 & 0.4821 \\ 1.295 & 1.768 & 1.205 & 0.2196 & 1.189 & -1.327 \\ 0.3772 & 0.5885 & -0.4152 & -1.727 & -0.6885 & -0.0113 \\ -1.742 & -1.253 & 0.9112 & 0.1360 & -1.353 & -0.8379 \\ -1.668 & 1.961 & 1.205 & 0.3254 & 1.921 & 0.2844 \end{bmatrix}$$

Consequently, following the procedures in Algorithm 1, we finally obtain

$$P=\begin{bmatrix} 37.67 & 24.70 & -18.09 & -0.7187 & 37.89 & 49.31 \\ 24.70 & 20.37 & -15.72 & 0.1002 & 28.54 & 32.15 \\ -18.09 & -15.72 & 13.97 & 0.2401 & -22.15 & -22.78 \\ -0.7187 & 0.1002 & 0.2401 & 0.4965 & -0.3214 & -0.2868 \\ 37.89 & 28.54 & -22.15 & -0.3214 & 41.55 & 49.25 \\ 49.31 & 32.15 & -22.78 & -0.2868 & 49.25 & 65.70 \end{bmatrix}$$

$$Q=\begin{bmatrix} 24.13 & 16.96 & -11.28 & 0.0946 & 24.86 & 32.22 \\ 16.96 & 14.22 & -10.39 & 0.4901 & 19.63 & 22.73 \\ -11.28 & -10.39 & 8.595 & -0.3719 & -14.15 & -14.92 \\ 0.0946 & 0.4901 & -0.3719 & 0.2092 & 0.4653 & 0.3595 \\ 24.86 & 19.63 & -14.15 & 0.4653 & 27.71 & 33.21 \\ 32.22 & 22.73 & -14.92 & 0.3595 & 33.21 & 43.39 \end{bmatrix}$$

and

$$F=\begin{bmatrix} 0.9557 & 0.8433 & -1.143 & -0.1623 & 1.300 & 0.9456 \\ 1.789 & 0.4150 & -0.4962 & -0.1124 & 1.245 & 2.415 \\ 2.349 & 1.071 & -0.7434 & -0.1082 & 1.964 & 3.098 \end{bmatrix}$$

The closed-loop eigenvalue set can then be computed as $\sigma(A + BF) = \{0.5776 \pm 0.1790i, 0.3296 \pm 0.1342i, 0, 0\}$ which coincides with (52) by setting $\theta_1 = 0.3$ and $\theta_2 = 0.75$.

5 Conclusion

The problem of shifting the poles of a discrete-time linear system to desired location is considered. It is shown that the eigenvalue shift can be achieved by an optimal feedback control law

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minimising a quadratic performance index without solving any non-linear DARE but a linear Stein equation parameterised by a free scalar. The proposed approach can be applied on large-scale system successively by a aforehand mode decomposition and therefore only requires the solution of low-order Stein equations. The merit of the proposed approach is its cheap computation cost and the permission of obtaining analytical solutions to optimal feedback design problem.

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