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On stability, Bohl exponent and Bohl-Perron theorem for implicit dynamic equations

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Abstract
In this paper, we develop a stability theory for implicit dynamic equations which is a general form of differential-algebraic equations and implicit difference equations. We derive some results about robust stability of these equations subjected to Lipschitz perturbations. After that the so-called Bohl-Perron type stability theorems, which are known in the literature for regular explicit difference equations, are extended for implicit dynamic equations. Finally, the notion of Bohl exponent is introduced and we characterize the relation between the exponential stability and the Bohl exponent. Then, it is investigated that how the Bohl exponent with respect to dynamic perturbations and two-side perturbations depends on the system data.

KEYWORDS
Implicit dynamic equation, Stability, Bohl-Perron theorem, Bohl exponent, Robustness.

1. Introduction

In a variety of applications, there is a frequently arising question, namely, how robust is a characteristic qualitative property of a system (e.g., stability) when the system comes under the effect of uncertain perturbations (see, e.g. (Hinrichsen & Pritchard, 2005)). The designers want to have operation systems working stably under small perturbations. Basically, assume that a given nominal system $\dot{x}(t) = Ax(t), t \geq 0$, be exponentially stable, then its stability robustness can be understood that there exists a number $\delta > 0$ such that the perturbed systems $\dot{x}(t) = Ax(t) + \Sigma x(t)$ are all exponentially stable whenever $\|\Sigma\| < \delta$ (with some given matrix norm $\|\cdot\|$). Therefore, the problem of finding the conditions for which the system is robust stable plays an important role in both theory and practice and has received as well significant attention in system and control theory during the last decades. However, it is not easy to deal with these conditions. Therefore, to measure the robust stability, one proceeds a test and expects that if with rather good input, then the output will satisfy

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some desired properties and our system is stable/exponentially stable. For example, the Bohl-Perron theorem says that, for a differential equation, if the bounded input implies the bounded output then the system must be exponentially stable (see, e.g. (Daleckii & Krein, 1974)).

The aim of this paper is to study the above problems. We desire to study the robust stability of time-varying systems of implicit dynamic equations (IDEs) in a general context. That is the implicit dynamic equation of the form

$$E_\sigma(t)x^\Delta(t) = A(t)x(t), \quad t \geq a, t \in \mathbb{T}, \tag{1.1}$$

where $E_\sigma(t) = E(\sigma(t))$ with $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $A(t)$ are continuous matrix functions defined on $\mathbb{T} \cap [a, \infty)$, valued in $\mathbb{R}^{n\times n}$. The leading term $E_\sigma(t)$ (rd-continuous) is supposed to be singular for all $t \geq a$. If system (1.1) is subject to an outer force q, then it becomes

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + q(t), \quad t \geq a, t \in \mathbb{T}. \tag{1.2}$$

If $\mathbb{T} = \mathbb{R}$ then the IDE (1.2) becomes a time-varying differential-algebraic equation

$$E(t)x'(t) = A(t)x(t) + q(t), \quad t \geq a, t \in \mathbb{T}, \tag{1.3}$$

and if $\mathbb{T} = \mathbb{Z}$ then the IDE (1.2) is a time-varying implicit difference equation

$$E_{n+1}x_{n+1} = A_n x_n + q_n, \quad n \geq n_0. \tag{1.4}$$

Differential-algebraic equations (DAEs) are the mathematical models arising in various applications, such as multibody mechanics, electrical circuits, prescribed path control, chemical engineering, etc., see (Bracke, 2000; Kunkel & Mehrmann, 2006; Lamour, März, & Tischendorf, 2013). Similarly, implicit difference equations also occur in different fields, such as population dynamics, economics, systems and control theory, and numerical analysis, etc., see (Linh, Nga, & Thuan, 2018; Luenberger, 1977, 1986). Therefore, it is very meaningful to have an equation combining these equations. In fact, this can be done by the theory of dynamic systems on an arbitrary time scale, which is a nonempty closed subset of the real numbers. This theory has been found promising because it demonstrates the interplay between the theories of continuous time and discrete time systems, see, e.g. (Bartosiewicz & Piotrowska, 2013; DaCunha & Davis, 2011; Taousser, Defoort, & Djemai, 2014). By using this theory, the IDE (1.2) can be considered as a unified and connected form between the time-varying DAE (1.3) and the time-varying implicit difference equation (1.4). Thus, they play an important role in mathematical modeling with many applications (see (Thuan, Nguyen, Ha, & Du, 2019)).

On the basis of the above discussion, it is worth considering the robust stability of these equations. To study that, the index notion, which plays a key role in the qualitative theory and in the numerical analysis of IDEs, should be taken into consideration in the robust stability analysis, (see (Griepentrog & März, 1986; März, 1998)). For the stability theory of time-varying linear DAEs, a few contributions are available (see (Berger, 2012; Berger & Ilchmann, 2013; Shcheglova & Chistyakov, 2004)). Some works concerning the Bohl exponents of discrete time-varying linear systems and continuous time-varying DAEs are derived in (Babiarz, Czornik, & Niezabitowski, 2015; Babiarz et al., 2017; Berger, 2014; Chyan, Du, & Linh, 2008; Du, Linh, & Nga, 2016).

To develop from DAEs to IDEs, the first results of the paper are to derive some characterizations for the robust stability of IDEs subject to Lipschitz perturbations. We will define Bohl exponent and investigate the robustness of Bohl exponents for these equations. The final result of this paper is to extend some well-known Bohl-Perron type stability theorems from regular explicit difference equations and implicit difference equations to IDEs. At the beginning of the twentieth century, Bohl, and later Perron, proved that the bounded input-bounded state property (also called Perron property) of a nonhomogeneous ordinary differential equation (under some assumptions on the coefficient) implies the exponential stability of the corresponding homogeneous equation and vice versa. The Bohl-Perron type stability theorems were formulated for differential and difference equations in Banach spaces in (Aulbach & Minh, 1996; Daleckii & Krein, 1974; Pituk, 2004), for difference equations with delay in (Berezansky & Braverman, 2005; Braverman & Karabash, 2012), for implicit difference equations in (Du et al., 2016; Linh & Nga, 2018).

The paper is organized as follows. In the next section some basic notions and preliminary results on time scales and the solvability of IDEs are recalled. In Section 3, we prove that if IDEs are exponentially stable, then under small Lipschitz perturbations they are still exponentially stable. The main result in this section is Theorem 3.3. Section 4 presents the famous Bohl-Perron theorem for IDEs in Theorem 4.2. Finally, in Section 5, we derive the definition of the Bohl exponent for IDEs and get some results about the robustness of this exponent with respect to dynamic perturbations. The main results in this section are Theorem 5.4 and Theorem 5.7.

2. Preliminary

2.1. Some basic notions

A time scale is an arbitrary, nonempty, closed subset of the set of real numbers $\mathbb{R}$, denoted by $\mathbb{T}$, enclosed with the topology inherited from the standard topology on $\mathbb{R}$.

Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ be the *forward operator*, and then $\mu(t) = \sigma(t) - t$ be called the *graininess*; $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ be the *backward operator*, and $\upsilon(t) = t - \rho(t)$. We supplement $\sup \emptyset = \inf \mathbb{T}, \inf \emptyset = \sup \mathbb{T}$.

For all $x, y \in \mathbb{T}$, we define some basic calculations:

(1) the *circle plus* $\oplus$: $x \oplus y := x + y + \mu(t)xy$;

(2) for all $x \in \mathbb{T}$, the element $\ominus x$ is defined: $\ominus x := \dfrac{-x}{1+\mu(t)x}$;

(3) the *circle minus* $\ominus$: $x \ominus y := \dfrac{x-y}{1+\mu(t)y}$.

A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered.

A function f defined on $\mathbb{T}$ valuated in $\mathbb{R}$ is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point.

A regulated function f is called *rd-continuous* if it is continuous at every right-dense point, and *ld-continuous* if it is continuous at every left-dense point. It is easy to see that a function is continuous if and only if it is both rd-continuous and ld-continuous. The set of rd-continuous functions defined on the interval J valued in X will be denoted by $\mathrm{C}_{\mathrm{rd}}(J, X)$.

A function f from $\mathbb{T}$ to $\mathbb{R}$ is *regressive* (resp., *positively regressive*) if for every $t \in \mathbb{T}$, then $1 + \mu(t)f(t) \neq 0$ (resp., $1 + \mu(t)f(t) > 0$). We denote by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of (resp., positively regressive) regressive functions, and $\mathrm{C_{rd}}\mathcal{R}(\mathbb{T}, \mathbb{R})$ (resp., $\mathrm{C_{rd}}\mathcal{R}^+(\mathbb{T}, \mathbb{R})$) the set of rd-continuous (resp., positively regressive) regressive functions from $\mathbb{T}$ to $\mathbb{R}$.

It is easy to verify that, for all $p, q \in \mathcal{R}$, $p \oplus q, p \ominus q, \ominus p, \ominus q \in \mathcal{R}$. Element $(\ominus q)(\cdot)$ is called the inverse element of element $q(\cdot) \in \mathcal{R}$. Hence, the set $\mathcal{R}(\mathbb{T}, \mathbb{R})$ with the calculation $\oplus$ forms an Abelian group.

Definition 2.1 (Delta derivative)**.** A function $f : \mathbb{T} \to \mathbb{R}^d$ is called *delta differentiable* at t if there exists a vector $f^\Delta(t)$ such that for all $\varepsilon > 0$,

$$\|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The vector $f^\Delta(t)$ is called the *delta derivative* of f at t.

Theorem 2.2 (see (Bohner & Peterson, 2001))**.** *If p is regressive and $t_0 \in \mathbb{T}$, then there exists a unique solution $e_p(\cdot, t_0), t \geq t_0$ of the initial value problem*

$$y^\Delta(t) = p(t)y(t),\ y(t_0) = 1,\ t \geq t_0. \tag{2.1}$$

We call the unique solution of (2.1) the exponential function. In fact, there is an explicit formula for $e_p(t, t_0)$, using the so-called cylinder transformation

$$\xi_h(z) = \begin{cases} z \text{ if } h = 0 \\ \frac{\mathrm{Ln(1+hz)}}{h} \text{ if } h \neq 0. \end{cases}$$

The formula reads

$$e_p(t, t_0) = \exp\left(\int_{t_0}^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right).$$

It is easy to verify that if $p(\cdot) \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, then the semigroup property is satisfied, i.e.,

$$e_p(t, r)e_p(r, s) = e_p(t, s)$$

for all $r, s, t \in \mathbb{T}$. Moverover, we have

Theorem 2.3 ((Bohner & Peterson, 2001), page 62)**.** *Given $p(\cdot), q(\cdot) \in \mathcal{R}(\mathbb{T}, \mathbb{R})$, for all $s, t \in \mathbb{T}$, we have*

i) $e_p(t, t) = 1,\ e_0(t, s) = 1;$

ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s),$ *and* *if* $1 + \mu(t)p(t) < 0,$ *then* $e_p(t, s)e_p(\sigma(t), s) < 0;$

iii) $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s);$

iv) $\dfrac{e_p(t, s)}{e_q(t, s)} = e_{p\ominus q}(t, s),$ *and* $e_p(t, s) = \dfrac{1}{e_p(s, t)} = e_{\ominus p}(s, t);$

v) *If* $p(\cdot) \in \mathcal{R}^+$ *then* $e_p(t, s) > 0;$

vi) *If* $p(\cdot), q(\cdot) \in \mathcal{R}^+$, $p \leq q$ *then* $0 < e_p(t, s) \leq e_q(t, s)$, *for all* $t \geq s;$

vii) $[e_p(\cdot, s)]^\Delta(t) = p(t)e_p(t, s),\ [e_p(t, \cdot)]^\Delta(s) = \ominus p(t)e_p(t, s);$

viii) $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\Delta}(t) = -\dfrac{p(t)}{e_p(\sigma(t),s)}$.

Let $\mathbb{T}$ be time scale that is unbounded above. For any $a,b \in \mathbb{R}$, the notation $[a,b]$ or (a,b) means the segment on $\mathbb{T}$, that is $[a,b] = \{t \in \mathbb{T} : a \le t \le b\}$ or $(a,b) = \{t \in \mathbb{T} : a < t < b\}$ and $\mathbb{T}_a = \{t \ge a : t \in \mathbb{T}\}$. We can define a measure $\Delta_{\mathbb{T}}$ on $\mathbb{T}$ by considering the Caratheodory construction of measures when we put $\Delta_{\mathbb{T}}[a,b) = b - a$. The Lebesgue integral of a measurable function f with respect to $\Delta_{\mathbb{T}}$ is denoted by $\int_a^b f(s)\Delta_{\mathbb{T}} s$ (see (Guseinov, 2003)).

The Gronwall-Bellman inequality will be introduced and applied in this paper.

Lemma 2.4. *Let $\tau \in \mathbb{T}$, $a \in \mathcal{R}^+$ and $u, b \in C_{rd}(\mathbb{T},\mathbb{R})$. Then*

$$u^{\Delta}(t) \ge -a(t)u_{\sigma}(t) + b(t) \quad \textit{for all} \quad t \ge \tau$$

implies

$$u(t) \ge u(\tau)e_{\ominus a}(t,\tau) + \int_{\tau}^{t} b(s)e_{\ominus a}(t,s)\Delta s \quad \textit{for all} \quad t \ge \tau.$$

Proof. The proof is similar to Theorem 3.5 in (Akin-Bohner, Bohner, & Akin, 2005). □

Lemma 2.5. *Let $\tau \in \mathbb{T}, u, b \in C_{rd}, u_0 \in \mathbb{R}$ and $b(t) \ge 0$ for all $t \ge \tau$. Then,*

$$u(t) \le u_0 + \int_{\tau}^{t} b(s)u(s)\Delta s \quad \textit{for all} \quad t \ge \tau$$

implies

$$u(t) \le u_0 e_b(t,\tau) \quad \textit{for all} \quad t \ge \tau.$$

Proof. See (Akin-Bohner et al., 2005, Corollary 2.10). □

In the whole paper, the time variable t will be omitted for brevity, if it does not cause misunderstanding. For any function $g(t)$ defined on the time scale $\mathbb{T}$, we write $g_{\sigma}(t)$ for $g(\sigma(t))$.

2.2. Solvability of the linear IDEs

Let $a \in \mathbb{T}$ be a fixed point. We consider the time-varying linear IDE on time scale $\mathbb{T}$,

$$E_{\sigma}(t)x^{\Delta}(t) = A(t)x(t) + q(t), \ t \ge a, \tag{2.2}$$

where E, A are continuous matrix functions, i.e., rank $E(t) = r$, $1 \le r < n$, for all $t \in \mathbb{T}_a$, and $q(\cdot)$ is a continuous function defined on $\mathbb{T}_a$, valued in $\mathbb{R}^n$. Assume that ker $E(t)$ is smooth in the sense that there exists a projector $Q(t)$ onto ker $E(t)$ such that $Q(t)$ is continuously differentiable for all $t > a$ and continuous on $\mathbb{T}_a$. Set $P(t) = I - Q(t)$. It is clear that $P(t)$ is a projector along ker $E(t), P^2 = P$, and

$EP = E$. Then, Equation (2.2) is rewritten into the form

$$E_\sigma(t)(Px)^\Delta(t) = \overline{A}(t)x(t) + q(t), \tag{2.3}$$

where $\overline{A} := A + E_\sigma P^\Delta \in L^{\mathrm{loc}}_\infty(\mathbb{T}; \mathbb{R}^{n\times n})$.

Let H be a continuous function defined on $\mathbb{T}_a$, taking values in $\mathrm{Gl}(\mathbb{R}^n)$ such that $H|_{\ker\, E_\sigma}$ is an isomorphism between $\ker\, E_\sigma$ and $\ker\, E$. We define the matrix

$$G := E_\sigma - \overline{A}HQ_\sigma,$$

and the set

$$S := \{x : Ax \in \mathrm{im}\; E_\sigma\}.$$

Lemma 2.6. *The following assertions are equivalent,*

i) $S \cap \ker\, E = \{0\}$;
ii) G *is nonsingular*;
iii) $\mathbb{R}^n = S \oplus \ker\, E$.

Proof. See (Du, Duy, & Viet, 2007, Lemma 2.1). □

Suppose that the matrix G is nonsingular, we have the following lemma.

Lemma 2.7. *There hold the following relations:*

i) $P_\sigma = G^{-1}E_\sigma$;
ii) $G^{-1}\overline{A}HQ_\sigma = -Q_\sigma$;
iii) $\widetilde{Q} := -HQ_\sigma G^{-1}\overline{A}$ *is the projector onto* $\ker\, E$ *along to* S*. We call* $\widetilde{Q}$ *the canonical projection*;
iv) If $\widehat{Q}$ *is a projector onto* $\ker\, E$ *then*

$$P_\sigma G^{-1}\overline{A} = P_\sigma G^{-1}\overline{A}\widehat{P},$$
$$Q_\sigma G^{-1}\overline{A} = Q_\sigma G^{-1}\overline{A}\widehat{P} - H^{-1}\widehat{Q}$$

with $\widehat{P} = I - \widehat{Q}$;
v) $P_\sigma G^{-1}, HQ_\sigma G^{-1}$ *does not depend on the choice of* H *and* Q.

Proof. See (Du et al., 2007, Lemma 2.2). □

Definition 2.8. The IDE (2.2) is said to be index-1 tractable on $\mathbb{T}$ if $G(t)$ is invertible for all $t \in \mathbb{T}$.

According to Lemma 2.6, the index-1 property does not depend on the choice of the projector Q and the isomorphism H (see also (Griepentrog & März, 1986; März, 1992)).

Let $J \subset \mathbb{T}$ be an interval. We denote

$$\mathrm{C}^1(J, \mathbb{R}^n) := \{x(\cdot) \in \mathrm{C}_{rd}(J, \mathbb{R}^n) : P(t)x(t) \text{ is delta differentiable for all } t \in J\}.$$

Note that we look for solutions $x(\cdot)$ of the index-1 equation (2.2) from elements of $\mathrm{C}^1(J, \mathbb{R}^n)$. So $x(\cdot)$ is not necessarily delta differentiable. Therefore, we agree to use the expression $E_\sigma x^\Delta$ in the meaning of $E_\sigma((Px)^\Delta - P^\Delta x)$.

Now, multiplying both sides of (2.3) by $P_\sigma G^{-1}$, $Q_\sigma G^{-1}$ respectively, we can decouple the index-1 equation (2.2) into the system

$$\begin{cases}(Px)^\Delta = (P^\Delta + P_\sigma G^{-1}\overline{A})Px + P_\sigma G^{-1}q,\\ Qx = HQ_\sigma G^{-1}\overline{A}Px + HQ_\sigma G^{-1}q.\end{cases}$$

Set $u = Px$ and $v = Qx$, we get

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\overline{A})u + P_\sigma G^{-1}q, \tag{2.4}$$

$$v = HQ_\sigma G^{-1}\overline{A}u + HQ_\sigma G^{-1}q. \tag{2.5}$$

Equation (2.2) is decomposed into two parts, a delta differential part (2.4) and an algebraic one (2.5). Thus, we can solve u from (2.4), after using (2.5) to compute v. Finally, set $x = u + v$. Therefore, we only need to address the initial value condition to the differential component (2.4) $u(t_0) = P(t_0)x_0$, $t_0 \geq a$, or equivalent to

$$P(t_0)(x(t_0) - x_0) = 0,\ x_0 \in \mathbb{R}^n. \tag{2.6}$$

Lemma 2.9. *Every solution of* (2.4) *starting in* im $P(t_0)$ *remains in* im $P(t)$ *for all* $t \in \mathbb{T}_{t_0}$.

Proof. Multiplying both sides of Equation (2.4) by Q_σ yields $Q_\sigma u^\Delta = Q_\sigma P^\Delta u$, which implies that $(Qu)^\Delta = Q^\Delta Qu$. Thus, if $Q(t_0)u(t_0) = 0$ then $Q(t)u(t) = 0$ for all $t \in \mathbb{T}_{t_0}$. This means that $u(t) = P(t)u(t)$ or $u(t) \in \operatorname{im} P(t)$. The proof is complete. □

In case $q(t) = 0$, we have the equation

$$E_\sigma(t)x^\Delta(t) = A(t)x(t). \tag{2.7}$$

Let $\Phi(t,s)$ denote the Cauchy operator generated by (2.7), i.e.,

$$\begin{cases}E_\sigma(t)\Phi^\Delta(t,s) = A(t)\Phi(t,s),\\ P(s)(\Phi(s,s) - I) = 0,\end{cases} \qquad t \geq s \geq a.$$

We can find $\Phi(t,s)$ by the following way. Firstly, consider the Cauchy operator $\Phi_0(t,s)$ of Equation (2.4)

$$\begin{cases}\Phi_0^\Delta(t,s) = (P^\Delta(t) + P_\sigma(t)G^{-1}(t)\overline{A}(t))\Phi_0(t,s),\\ \Phi_0(s,s) = I.\end{cases} \tag{2.8}$$

Then, the Cauchy operator of (2.7) can be given as follows

$$\Phi(t,s) = \widetilde{P}(t)\Phi_0(t,s)P(s), \tag{2.9}$$

where $\widetilde{P} = I + HQ_\sigma G^{-1}\overline{A}$. From this formula and Lemma 2.7 we see that

$$P(t)\Phi(t,s) = P(t)\widetilde{P}(t)\Phi_0(t,s)P(s) = P(t)\Phi_0(t,s)P(s) = \Phi_0(t,s)P(s). \tag{2.10}$$

Denote by $x(\cdot, t_0, x_0)$ the unique solution of the IDE (2.2) with the initial condition

$$P(x(t_0, t_0, x_0) - x_0) = 0. \tag{2.11}$$

We write shortly $x(t)$ for $x(t, t_0, x_0)$ if there is no confusion. By the variation of constants formula, we have the unique solution of Equation (2.4) defined by

$$u(t) = \Phi_0(t, t_0)u(t_0) + \int_{t_0}^{t} \Phi_0(t, \sigma(s))P_\sigma(s)G^{-1}(s)q(s)\Delta s, \tag{2.12}$$

and the unique solution of the IDE (2.2) satisfies

$$x(t) = \Phi(t, t_0)P(t_0)x_0 \ + \int_{t_0}^{t} \Phi(t, \sigma(s))P_\sigma(s)G^{-1}(s)q(s)\Delta s + H(t)Q_\sigma(t)G^{-1}(t)q(t). \tag{2.13}$$

In what follows, we suppose that

Assumption 1. There exists a bounded differential projector $Q(t)$ onto $\ker E(t)$. Let $P = I - Q$ and $K_0 = \sup_{t \geq a} \|P(t)\|$.

3. Stability of IDEs under small perturbations

Consider the perturbation under the form $q(t) = f(t, x(t))$ where f is a certain function defined on $\mathbb{T}_a \times \mathbb{R}^n$ such that $f(t, 0) = 0$ for all $t \in \mathbb{T}_a$. Then, Equation (2.2) becomes

$$E_\sigma(t)x^\Delta(t) = A(t)x(t) + f\big(t, x(t)\big), \ \ t \geq a. \tag{3.1}$$

Since $f(t, 0) = 0$, Equation (3.1) has the trivial solution $x(t) \equiv 0$. As before, denoting $u = Px$ and $v = Qx$ comes to

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\overline{A})u + P_\sigma G^{-1} f(t, u + v), \tag{3.2}$$

$$v = HQ_\sigma G^{-1}\overline{A}u + HQ_\sigma G^{-1} f(t, u + v). \tag{3.3}$$

Assume that $HQ_\sigma G^{-1} f(t, \cdot)$ is Lipschitz continuous with the Lipschitz coefficient $\gamma_t < 1$, i.e.,

$$\|HQ_\sigma G^{-1} f(t, x) - HQ_\sigma G^{-1} f(t, y)\| \leq \gamma_t \|x - y\| \quad \text{for all} \ \ t \geq a.$$

Since $HQ_\sigma G^{-1}$ does not depend on the choice of H and Q, so does the Lipschitz property of $HQ_\sigma G^{-1} f(t, \cdot)$.

Fix $u \in \mathbb{R}^m$ and choose $t \in \mathbb{T}_a$, we consider a mapping $\Gamma_t : \operatorname{im} Q(t) \to \operatorname{im} Q(t)$ defined by

$$\Gamma_t(v) := H(t)Q_\sigma(t)G^{-1}(t)\overline{A}(t)u + H(t)Q_\sigma(t)G^{-1}(t)f(t, u + v).$$

It is easy to see that

$$\|\Gamma_t(v) - \Gamma_t(v')\| \leq \gamma_t \|v - v'\|$$

for any $v, v' \in \text{im } Q(t)$. Since $\gamma_t < 1$, Γ_t is a contractive mapping. Hence, by the fixed point theorem, there exists a mapping $g_t : \text{im } P(t) \to \text{im } Q(t)$ satisfying

$$g_t(u) = H(t)Q_\sigma(t)G^{-1}(t)\overline{A}(t)u + H(t)Q_\sigma(t)G^{-1}(t)f\big(t, u + g_t(u)\big). \tag{3.4}$$

We have

$$\|g_t(u) - g_t(u')\| \le \beta_t\|u - u'\| + \gamma_t\big(\|u - u'\| + \|g_t(u) - g_t(u')\|\big),$$

where $\beta_t = \|H(t)Q_\sigma(t)G^{-1}(t)\overline{A}(t)\|$. This deduces

$$\|g_t(u) - g_t(u')\| \le \frac{\gamma_t + \beta_t}{1 - \gamma_t}\|u - u'\|.$$

Thus, g_t is Lipschitz continuous with the Lipschitz coefficient $L_t = \dfrac{\gamma_t + \beta_t}{1 - \gamma_t}$. Substituting $v = g_t(u)$ into (3.2) obtains

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\overline{A})u + P_\sigma G^{-1} f\big(t, u + g_t(u)\big). \tag{3.5}$$

Suppose that (3.5) is solvable. Then we find $u(t)$ from Equation (3.5) and get the solution of (3.1) by

$$x(t) = u(t) + g_t(u(t)), \ t \in \mathbb{T}_a. \tag{3.6}$$

Definition 3.1. The IDE (2.7) is said to be exponentially stable if there exist numbers $M_0 > 0, \alpha > 0$ such that $-\alpha \in \mathcal{R}^+$ and

$$\|x(t, t_0, x_0)\| \le M_0 e_{-\alpha}(t, t_0)\|P(t_0)x_0\|, \, t \ge t_0 \ge a, \ x_0 \in \mathbb{R}^n.$$

By the classical way, we see that the exponential stability of (2.7) are characterized by the Cauchy operator $\Phi(t, s)$ as follows (similar in (Du & Liem, 2013)):

Theorem 3.2. *The IDE* (2.7) *is exponentially stable if and only if there exist two numbers* $M_0 > 0, \alpha > 0$ *such that* $-\alpha \in \mathcal{R}_+$ *and*

$$\|\Phi(t, s)\| \le M_0 e_{-\alpha}(t, s), \, t \ge s \ge a. \tag{3.7}$$

From relation (2.10) we have

$$\|\Phi_0(t, s)P(s)\| = \|P(t)\Phi(t, s)\| \le \|P(t)\|\|\Phi(t, s)\| \le K_0\|\Phi(t, s)\|.$$

Thus, from the inequality (3.7), there is a constant $M > 0$ such that

$$\|\Phi_0(t, s)P(s)\| \le Me_{-\alpha}(t, s), \, t \ge s \ge a. \tag{3.8}$$

We now are in the position to consider the robust stability of IDEs under small perturbations. The following theorem shows that the exponential stability is also preserved under some integrable perturbations or small enough Lipschitz perturbations.

Theorem 3.3. *If the IDE* (2.7) *is index-1, exponentially stable and*

i) $L = \sup_{t\in\mathbb{T}_a} L_t < \infty$.

ii) *the function* $P_\sigma(t)G^{-1}(t)f(t,x)$ *is Lipschitz continuous with the Lipschitz coefficient* k_t *such that one of the following conditions hold*

(a) $\quad N = \displaystyle\int_a^\infty \frac{k_t}{1-\alpha\mu(t)}\Delta t < \infty,$

(b) $\quad \limsup_{t\to\infty} k_t(1+L_t) = \delta < \frac{\alpha}{LM}$, *with* α, M *in Definition 3.1.*

Then, there exist a constant $K > 0$ *and the positively regressive* $-\alpha_1$ *such that*

$$\|x(t)\| \le Ke_{-\alpha_1}(t,s)\|P(s)x(s)\| \quad \text{for all } \; t \ge s \ge a,$$

where $x(\cdot)$ *is the solution of* (3.1)*. That is, the perturbed equation* (3.1) *preserves the exponential stability.*

Proof. First, we prove this theorem with the condition (a). By using the variation of constants formula (2.12), for all $t > s \ge a$, we have

$$u(t) = \Phi_0(t,s)u(s) + \int_s^t \Phi_0(t,\sigma(\tau))P_\sigma G^{-1}f(\tau, u(\tau) + g_\tau(u(\tau))\Delta\tau. \tag{3.9}$$

By estimate (3.8), we get

$$\begin{aligned}\|u(t)\| &\le \|\Phi_0(t,s)u(s)\| + \int_s^t \|\Phi_0(t,\sigma(\tau))P_\sigma\|\|P_\sigma G^{-1}f(\tau, u(\tau) + g_\tau(u(\tau)))\|\Delta\tau \\ &\le Me_{-\alpha}(t,s)\|u(s)\| + M\int_s^t e_{-\alpha}(t,\sigma(\tau))k_\tau(1+L_\tau)\|u(\tau)\|\Delta\tau. \end{aligned}\tag{3.10}$$

Multiplying both sides of the above inequality with $\frac{1}{e_{-\alpha}(t,s)}$ yields

$$\frac{\|u(t)\|}{e_{-\alpha}(t,s)} \le M\|u(s)\| + M\int_s^t \frac{k_\tau(1+L_\tau)\|u(\tau)\|}{(1-\alpha\mu(\tau))e_{-\alpha}(\tau,s)}\Delta\tau. \tag{3.11}$$

By using the Gronwall-Bellman inequality we get

$$\frac{\|u(t)\|}{e_{-\alpha}(t,s)} \le M\|u(s)\|e_{\frac{M(1+L)k.}{1-\alpha\mu(.)}}(t,s).$$

Since $\frac{M(1+L)k.}{1-\alpha\mu(.)}$ is positive, by the definition of the exponential function it follows that

$$e_{\frac{M(1+L)k.}{1-\alpha\mu(.)}}(t,s) \le \exp\Big\{\int_s^t \frac{M(1+L)k_\tau}{1-\alpha\mu(\tau)}\Delta\tau\Big\} \le \exp\Big\{\int_s^\infty \frac{M(1+L)k_\tau}{1-\alpha\mu(\tau)}\Delta\tau\Big\} \le e^{MN(1+L)}.$$

Thus, there exists $M_1 > 0$ such that

$$\|u(t)\| \le M_1 e_{-\alpha}(t,s)\|u(s)\|.$$

By (3.6), it follows that

$$\|x(t)\| \le \|u(t)\| + \|g_t(u(t))\| \le (1+L)\|u(t)\| \le (1+L)M_1 e_{-\alpha}(t,s)\|u(s)\|, \quad (3.12)$$

or, $\|x(t)\| \le K_0 e_{-\alpha}(t,s)\|P(s)x(s)\|$, for all $t > s \ge a$ with $K_0 := (1+L)M_1$. We have the proof in the first case.

In the case condition (b) is satisfied, let ε_0 be a positive number such that $\delta + \varepsilon_0 \le \dfrac{\alpha}{LM}$. Then, by the second assumption, there exists an element $T_0 > a$ such that

$$k_t(1+L_t) < \delta + \varepsilon_0, \quad \text{for all } t > T_0. \quad (3.13)$$

By the continuity of solution of (3.5) on the initial condition we can find a positive constant M_{T_0}, depending only on T_0 such that

$$\|u(t)\| \le M_{T_0}\|u(s)\| \text{ for all } a \le s < t \le T_0. \quad (3.14)$$

First, we consider the case $t > T_0 > s \ge a$. In the same way as (3.10) and (3.11), it follows that

$$\frac{\|u(t)\|}{e_{-\alpha}(t,T_0)} \le M\|u(T_0)\| e_{\frac{Mk_{.}(1+L_{.})}{1-\alpha\mu(.)}}(t,T_0) \le M\|u(T_0)\| e_{\frac{M(\delta+\varepsilon_0)}{1-\alpha\mu(.)}}(t,T_0),$$

or equivalently,

$$\|u(t)\| \le M\|u(T_0)\| e_{-\alpha \oplus \frac{M(\delta+\varepsilon_0)}{1-\alpha\mu(.)}}(t,T_0) = M\|u(T_0)\| e_{-\alpha + M(\delta+\varepsilon_0)}(t,T_0).$$

Put $\alpha_1 := \alpha - M(\delta+\varepsilon_0) > 0$ (because $L > 1$). Since $-\alpha$ is positively regressive, so is $-\alpha_1$. Therefore,

$$\begin{aligned}\|u(t)\| &\le \frac{M\|u(T_0)\|}{e_{-\alpha_1}(T_0,s)} e_{-\alpha_1}(t,s) \le M e_{\ominus(-\alpha_1)}(T_0,t_0) e_{-\alpha_1}(t,s)\|u(T_0)\| \\ &\overset{(3.14)}{\le} M M_{T_0} e_{\ominus(-\alpha_1)}(T_0,t_0) e_{-\alpha_1}(t,s)\|u(s)\|.\end{aligned}$$

Thus, with $K_1 = MM_{T_0} e_{\ominus(-\alpha_1)}(T_0,t_0)$, we have

$$\|u(t)\| \le K_1 e_{-\alpha_1}(t,s)\|u(s)\|, \quad t > T_0 > s \ge a.$$

In case $t > s \ge T_0$, we have the estimate

$$\|P_\sigma G^{-1} f(\tau, u + g_\tau(u))\| \le (\delta+\varepsilon_0)\|u\|$$

for all $\tau \ge s$. Therefore, by a similar way as above we get

$$\|u(t)\| \le K_2\|u(s)\| e_{-\alpha_1}(t,s).$$

For the remaining case $a \le s < t \le T_0$, with $\alpha_1 > 0$ defined above, we have

$$\|u(t)\| \le M_{T_0}\|u(s)\| \le M_{T_0} e_{\alpha_1}(T_0,t_0) e_{-\alpha_1}(t,s)\|u(s)\|.$$

Put $K_3 = \max\{K_1, K_2, M_{T_0} e_{\alpha_1}(T_0, t_0)\}$ we imply that

$$\|u(t)\| \le K_3 e_{-\alpha_1}(t, s)\|u(s)\|.$$

Paying attention to (3.6) obtains

$$\|x(t)\| \le K e_{-\alpha_1}(t, s)\|P(s)x(s)\| \text{ for all } t \ge s \ge a,$$

where $K = (1 + L)K_3$. The proof is complete. □

Remark 1. If E_σ is the identity matrix then from Theorem 3.3 we can obtain results about robust stability of the dynamic systems on time scales $x^\Delta(t) = A(t)x(t) + f(t, x)$ in (Du & Tien, 2007).

Remark 2. Assume that the perturbation $f(t, x)$ is linear, i.e. $f(t, x) = \Sigma(t)x$ with $\Sigma(t) \in \mathbb{R}^{n\times n}$. Then the perturbed equation (3.1) has the form $E_\sigma x^\Delta = (A(t) + \Sigma(t))x(t)$. In this case it is easy to see that $\gamma(t) = \|H(t)Q_\sigma(t)G^{-1}(t)\Sigma(t)\| < 1$ if $\Sigma(t)$ is small enough and $k_t = \|P_\sigma(t)G^{-1}(t)\Sigma(t)\|$. By Theorem 3.3, we can derive bounds for the perturbation $\Sigma(t)$ such that the perturbed equation (3.1) is still exponentially stable. This can be used to evaluate the robust stability of DAEs, respectively $\mathbb{T} = \mathbb{R}$, and implicit difference equations, respectively $\mathbb{T} = \mathbb{Z}$, which arise in many applications, see (Kunkel & Mehrmann, 2006; Lamour et al., 2013; Linh et al., 2018; Luenberger, 1986).

4. Bohl-Perron theorem for IDEs

The aim of this section is to prove the Bohl-Perron theorem for the linear IDEs. That is, we investigate the relation between the boundedness of solutions of the nonhomogenous equation (2.2) and the exponential stability of the IDE (2.7).

In solving Equation (2.2), we see that the function q is split into two components $P_\sigma G^{-1}q$ and $HQ_\sigma G^{-1}q$. Therefore, for any $t_0 \in \mathbb{T}_a$ we consider q as an element of the set

$$L(t_0) = \left\{ \begin{array}{r} q \in C([t_0, \infty], \mathbb{R}^n) : \sup_{t\ge t_0} \|H(t)Q_\sigma(t)G^{-1}(t)q(t)\| < \infty \\ \text{and } \sup_{t\ge t_0} \|P_\sigma(t)G^{-1}(t)q(t)\| < \infty \end{array} \right\}.$$

It is easy to see that $L(t_0)$ is a Banach space with the norm

$$\|q\| = \sup_{t\ge t_0} \left(\|P_\sigma(t)G^{-1}(t)q(t)\| + \|H(t)Q_\sigma(t)G^{-1}(t)q(t)\|\right).$$

Denote by $x(t, s, q)$ the solution, associated with q, of (2.2) with the initial condition $P(s)x(s, s) = 0$. We write simply $x(t, s)$ or $x(t)$ for $x(t, s, q)$ if there is no confusion.

Lemma 4.1. *If for every function $q(\cdot) \in L(t_0)$, the solution $x(\cdot, t_0)$ of Cauchy problem (2.2) with the initial condition $P(t_0)x(t_0, t_0) = 0$ is bounded, then for all $t_1 \ge t_0$, there is a constant $k > 0$, independent of t_1, such that*

$$\sup_{t\ge t_1} \|x(t, t_1)\| \le k\|q\|. \tag{4.1}$$

Proof. Define a family of the operators $\{V_t\}_{t\geq t_0}$ as follows:

$$\begin{aligned} V_t : L(t_0) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto V_t(q) = x(t,t_0). \end{aligned}$$

From the assumption, we have $\sup_{t\geq t_0} \|V_t q\| < \infty$ for any $q \in L(t_0)$. Using the uniform boundedness principle, there exists a constant $k > 0$ such that

$$\sup_{t\geq t_0} \|x(t,t_0)\| = \|V_t q\| \leq k\|q\|, \quad \text{for all} \quad t \geq t_0. \tag{4.2}$$

Let q be an arbitrary function in $L(t_1)$. Define $\overline{q}$ in $L(t_0)$ as follows: if $t < t_1$ then $\overline{q}_t = 0$, else $\overline{q}(t) = q(t)$. By the variation of constants formula, for any $t \geq t_1$ we have

$$\begin{aligned} x(t,t_0,\overline{q}) &= \int_{t_0}^{t} \Phi(t,\sigma(\tau))P_\sigma(\tau)G^{-1}(\tau)\overline{q}(\tau)d\tau + H(t)Q_\sigma(t)G^{-1}(t)\overline{q}(t) \\ &= \int_{t_1}^{t} \Phi(t,\sigma(\tau))P_\sigma(\tau)G^{-1}(\tau)q(\tau)d\tau + H(t)Q_\sigma(t)G^{-1}(t)q(t) \end{aligned}$$

This means that $x(t,t_0,\overline{q}) = x(t,t_1,q)$ for all $t \geq t_1$. Therefore, from (4.2) we get

$$\sup_{t\geq t_1} \|x(t,t_1,q)\| = \sup_{t\geq t_0} \|x(t,t_0,\overline{q})\| \leq k\|\overline{q}\| = k\|q\|.$$

The proof is complete. $\square$

Now, we derive the main result in this section which is the Bohl-Perron theorem for IDEs.

Theorem 4.2. *All solutions of Cauchy problem* (2.2) *with the initial condition* $P(t_0)x(t_0) = 0$, *associated with an arbitrary* q *in* $L(t_0)$, *are bounded if and only if the index-1 IDE* (2.7) *is exponentially stable.*

Proof. The proof is divided into two parts.

Necessity. First, we prove that if all solutions of Equation (2.2) with the initial condition $P(t_0)x(t_0) = 0$, associated with $q \in L(t_0)$, are bounded then the IDE (2.7) is exponentially stable.

With an arbitrary number $t_1 \geq t_0$, let $\chi(t) = \|\Phi(\sigma(t),t_1)\|$, $t \geq t_1$. For any $y \in \mathbb{R}^n$, we consider the function

$$q(t) = \frac{E_\sigma(t)\Phi(\sigma(t),t_1)y}{\chi(t)}, \ t \geq t_1.$$

It is obvious that

$$\begin{aligned} \|P_\sigma(t)G^{-1}(t)q(t)\| &= \left\| P_\sigma(t)G^{-1}(t)\frac{E_\sigma(t)\Phi(\sigma(t),t_1)y}{\chi(t)} \right\| \\ &= \left\| P_\sigma(t)\frac{\Phi(\sigma(t),t_1)}{\chi(t)}y \right\| \leq K_0\|y\|, \\ \|H(t)Q_\sigma(t)G^{-1}(t)q(t)\| &= \left\| H(t)Q_\sigma(t)G^{-1}(t)\frac{E_\sigma(t)\Phi(\sigma(t),t_1)y}{\chi(t)} \right\| = 0. \end{aligned}$$

Thus, $q \in L(t_1)$ and

$$\|q\| = \sup_{t \geq t_1} \left(\|P_\sigma(t)G^{-1}(t)q(t)\| + \|H(t)Q_\sigma(t)G^{-1}(t)q(t)\| \right) \leq K_0\|y\|.$$

Moreover,

$$\begin{aligned} x(t,t_1) &= \int_{t_1}^{t} \Phi(t,\sigma(\tau))P_\sigma(\tau)G^{-1}(\tau)q(\tau)\Delta\tau + H(t)Q_\sigma(t)G^{-1}(t)q(t) \\ &= \int_{t_1}^{t} \Phi(t,\sigma(\tau))P_\sigma(\tau)\frac{\Phi(\sigma(\tau),t_1)y}{\chi(\tau)}\Delta\tau = \int_{t_1}^{t} \frac{\Phi(t,t_1)y}{\chi(\tau)}\Delta\tau. \end{aligned}$$

Put $\Psi(t) = \displaystyle\int_{t_1}^{t} \frac{1}{\chi(\tau)}\Delta\tau > 0$, we have

$$x(t,t_1) = \Phi(t,t_1)\Psi(t)y. \tag{4.3}$$

From Lemma 4.1, we obtain

$$\|x(t)\| = \|\Phi(t,t_1)\Psi(t)y\| = \|\Phi(t,t_1)y\|\Psi(t) \leq k\|q\| \leq kK_0\|y\|,$$

which implies

$$\|\Phi(t,t_1)\| \leq \frac{\overline{k}}{\Psi(t)}, \tag{4.4}$$

where $\overline{k} = kK_0$. On the other hand,

$$\frac{1}{\Psi^\Delta(t)} = \chi(t) = \|\Phi(\sigma(t),t_1)\| \leq \frac{\overline{k}}{\Psi(\sigma(t))} \Longrightarrow \Psi^\Delta(t) \geq \frac{1}{\overline{k}}\Psi(\sigma(t)).$$

Using Lemma 2.4, we get

$$\Psi(t) \geq \Psi(c)e_{\ominus(-\frac{1}{\overline{k}})}(t,c),$$

for every $t \geq c$. Hence, by (4.4) we have

$$\|\Phi(\sigma(t),t_1)\| \leq \frac{\overline{k}}{\Psi(c)}e_{-\frac{1}{\overline{k}}}(\sigma(t),c),$$

for all $t \geq c$. This estimate leads to

$$\|\Phi(t,t_1)\| \leq \frac{\overline{k}}{\Psi(c)}e_{-\frac{1}{\overline{k}}}(t,c) = \frac{\overline{k}}{\Psi(c)e_{-\frac{1}{\overline{k}}}(c,t_1)}e_{-\frac{1}{\overline{k}}}(t,t_1),\ t > c.$$

Set $\alpha = \dfrac{1}{\overline{k}}$, $N_1 = \dfrac{\overline{k}}{\Psi(c)e_{-\frac{1}{\overline{k}}}(c,t_1)}$ and $N = \max\left\{N_1, \max_{t_1 \leq t \leq c} \dfrac{\|\Phi(t,t_1)\|}{e_{-\alpha}(t,t_1)}\right\}$, we obtain desired estimate

$$\|\Phi(t,t_1)\| \le Ne_{-\alpha}(t,t_1) \quad \text{for all } t \ge t_1.$$

Sufficiency. To complete the proof, we will show that if (2.7) is exponentially stable then all solutions of the Cauchy problem (2.2) with the initial condition $P(t_0)x(t_0) = 0$, associated with q in $L(t_0)$ are bounded.

Let $q \in L(t_0)$, suppose that

$$\sup_{t\ge t_0} \|P_\sigma(t)G^{-1}(t)q(t)\| = C_1, \quad \sup_{t\ge t_0} \|H(t)Q_\sigma(t)G^{-1}(t)q(t)\| = C_2.$$

Using the formula (2.13) again

$$\begin{aligned}\|x(t)\| &\le \int_{t_0}^t \|\Phi(t,\sigma(\tau))P_\sigma G^{-1}q(\tau)\|\Delta\tau + \|HQ_\sigma G^{-1}q(t)\| \\ &\le MC_1 \int_{t_0}^t e_{-\alpha}(t,\sigma(\tau))\Delta\tau + C_2 \\ &= MC_1 e_{-\alpha}(t,t_0)\int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau + C_2.\end{aligned}$$

By using the L'Hôspital rule (see (Bohner & Peterson, 2001, Theorem 1.119)) we have

$$\begin{aligned}\lim_{t\to\infty} e_{-\alpha}(t,t_0)\int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau &= \lim_{t\to\infty} \frac{\int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau),t_0)\Delta\tau}{e_{\ominus(-\alpha)}(t,t_0)} \\ &= \lim_{t\to\infty} \frac{e_{\ominus(-\alpha)}(\sigma(t),t_0)}{\ominus(-\alpha)e_{\ominus(-\alpha)}(t,t_0)} = \frac{1}{\alpha}.\end{aligned}$$

Thus, $\sup_{t\ge t_0}\int_{t_0}^t e_{-\alpha}(t,\sigma(\tau))\Delta\tau < \infty$, which implies that the solutions of (2.2) associated with q are bounded. The proof is complete. □

Remark 3. The above result has extended the Bohl-Perron type stability theorem with bounded input/output for differential and difference equations (see (Aulbach & Minh, 1996; Daleckii & Krein, 1974; Pituk, 2004)), for differential-algebraic equations (see (Ha, 2018)) and for implicit difference equations (see (Du et al., 2016; Linh & Nga, 2018)), correspondingly to the case of $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$, for dynamic systems on time scales (see (Du & Tien, 2007)).

Example 4.3. Consider the model of a simple circuit on time scales consists of a voltage source $v_V = v(t)$, a resistor with conductance R and a capacitor with capacitance with $C > 0$, see Figure 1. As in (Tischendorf, 2000), this model can be written in the form

$$E_\sigma x^\Delta = Ax + f,$$

with

$$E_\sigma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix}, A = \begin{pmatrix} -R & R & 1 \\ R & -R & 0 \\ 1 & 0 & 0 \end{pmatrix}, x = \begin{pmatrix} e_1 \\ e_2 \\ i_v \end{pmatrix}, f = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}.$$

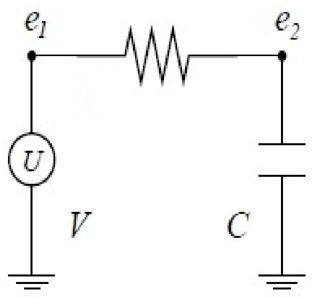

Figure 1. *A simple circuit*

In this case, it is easy to see that

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad H = I, \quad \overline{A} = A.$$

Therefore,

$$G = E_\sigma - \overline{A} H Q_\sigma = \begin{pmatrix} R & 0 & -1 \\ -R & C & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \frac{1}{C} & \frac{-R}{C} \\ -1 & 0 & -R \end{pmatrix}.$$

This implies that $\|q\| = \sqrt{1 + \frac{R^2(C^2+1)}{C^2}}\|v\|$. On the other hand, the spectral set

$$\sigma(E_\sigma, A) = \{\lambda : \det(A - \lambda E_\sigma) = 0\} = \left\{\frac{-R}{C}\right\}.$$

Assume that $1 - \frac{\mu(t)R}{C} > 0$, or equivalently $\frac{-R}{C} \in \mathcal{R}^+$. Then the homogenous equation $E_\sigma x^\Delta = Ax$ is exponentially stable. By Theorem 4.2, if v is bounded then e_1, e_2, i_v are bounded.

5. Bohl exponent for IDEs

In this section, we investigate the Bohl exponent of the linear IDEs and the robustness of the Bohl exponent when these equations are subject to dynamic perturbations. This will generalize and unify some results about the Bohl exponent for DAEs in (Berger, 2012; Chyan et al., 2008) and for implict difference equations in (Du et al., 2016).

5.1. Bohl exponent and their basic properties

We extend the Bohl exponent notion to the case of linear IDEs.

Definition 5.1 (Bohl exponents for IDEs)**.** Let the IDE (2.7) have index-1, $\Phi(t,s)$ be its Cauchy operator. Then, the (upper) Bohl exponent of IDE (2.7) is given by

$$\kappa_{\mathcal{B}}(E,A) = \inf\{\alpha \in \mathbb{R}; \exists M_\alpha > 0 : \|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s),\ \forall t \ge s \ge t_0\}.$$

In case $\mathbb{T} = \mathbb{R}$ (or resp. $\mathbb{T} = h\mathbb{Z}$), we come to the classical definition of Bohl exponent. When $\kappa_{\mathcal{B}}(E,A) = -\frac{1}{\mu^*}$ or $\kappa_{\mathcal{B}}(E,A) = +\infty$ we say that the Bohl exponent of IDE (2.7) is extreme. Further,

Proposition 5.2. *If $\alpha = \kappa_{\mathcal{B}}(E,A)$ is not extreme then for any $\epsilon > 0$, we have*

(1) $$\lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\epsilon}(t,s)} = 0,$$

(2) $$\limsup_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\ominus\epsilon}(t,s)} = \infty.$$

Proof. Since $\alpha > -\frac{1}{\mu^*}$, we can choose $\delta > 0$ such that $\alpha + 2\delta \le \alpha + \epsilon + \alpha\epsilon\mu(t) = \alpha \oplus \epsilon$ for any $t \in \mathbb{T}$. By definition, there is $K_\delta > 0$ such that

$$\|\Phi(t,s)\| \le K_\delta e_{\alpha+\delta}(t,s)$$

for any $t \ge s \ge t_0$. Hence,

$$\lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{\|\Phi(t,s)\|}{e_{\alpha\oplus\epsilon}(t,s)} \le K_\delta \lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{e_{\alpha+\delta}(t,s)}{e_{\alpha\oplus\epsilon}(t,s)} \le K_\delta \lim_{\substack{t-s\to\infty\\ s\to\infty}} \frac{e_{\alpha+\delta}(t,s)}{e_{\alpha+2\delta}(t,s)} = 0.$$

Thus we get (1). To prove (2) we choose $\delta > 0$ such that $1 + \alpha\mu(t) > \delta(1 + \epsilon\mu(t))/\epsilon$ which is equivalent $\alpha - \delta \ge \dfrac{\alpha - \epsilon}{1 + \epsilon\mu(t)} = \alpha \ominus \epsilon$, $\forall t \in \mathbb{T}$. By definition, there exists a sequence $\{t_n, s_n\}$ such that $s_n \to \infty$ and $t_n - s_n \to \infty$ and

$$\lim_{n\to\infty} \frac{\|\Phi(t_n,s_n)\|}{e_{\alpha-\delta}(t_n,s_n)} = \infty.$$

Further,

$$\frac{e_{\alpha-\delta}(t_n,s_n)}{e_{\alpha\ominus\epsilon}(t_n,s_n)} \ge 1,\ \forall\, n.$$

Therefore,

$$\lim_{n\to\infty} \frac{\|\Phi(t_n,s_n)\|}{e_{\alpha\ominus\epsilon}(t_n,s_n)} = \lim_{n\to\infty} \frac{\|\Phi(t_n,s_n)\|}{e_{\alpha-\delta}(t_n,s_n)} \frac{e_{\alpha-\delta}(t_n,s_n)}{e_{\alpha\ominus\epsilon}(t_n,s_n)} \ge \lim_{n\to\infty} \frac{\|\Phi(t_n,s_n)\|}{e_{\alpha-\delta}(t_n,s_n)} = \infty.$$

The proof is complete. □

From this proposition, it is easy to see that

(1) If $\mathbb{T} = \mathbb{R}$ then

$$\kappa_{\mathcal{B}}(E, A) = \limsup_{s, t-s\to\infty} \frac{\ln \|\Phi(t,s)\|}{t-s}.$$

(2) If $\mathbb{T} = h\mathbb{Z}$ then

$$\kappa_{\mathcal{B}}(E, A) = \frac{1}{h}\left(\limsup_{s, t-s\to\infty} \|\Phi(t,s)\|^{\frac{h}{t-s}} - 1\right).$$

Example 5.3. Consider Equation (2.7) with $E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $A(t) = \begin{bmatrix} p(t) & p(t) & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on time scale $\mathbb{T} = \bigcup\limits_{k=0}^{\infty}\{3k\} \bigcup\limits_{k=0}^{\infty}[3k+1, 3k+2]$, where

$$p(t) = \begin{cases} -1/4 \text{ if } t = 3k, \\ -1/2 \text{ if } t \in [3k+1, 3k+2]. \end{cases} \tag{5.1}$$

In this case, it is easy to compute that

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \widetilde{P} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = I, \quad G^{-1} = \begin{bmatrix} 1/2 & -1 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Simple calculations yield that the transition matrices of (E, A) is given by

$$\Phi_0(t,s) = \begin{bmatrix} \frac{e_p(t,s)+1}{2} & \frac{e_p(t,s)-1}{2} & 0 \\ \frac{e_p(t,s)-1}{2} & \frac{e_p(t,s)+1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Phi(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ e_p(t,s) & e_p(t,s) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By the definition of exponential function, it is easy to see that for $m, n \in \mathbb{N} : m < n$,

$$e_p(3n, 3m) = \left(1 - \frac{1}{4}\right)^{n-m}\left(1 - \frac{1}{2}\right)^{n-m} e^{-\frac{n-m}{2}} = e^{\left(-\frac{1}{2} + \ln\frac{3}{8}\right)(n-m)}.$$

Let $0 < \alpha < 1$ be solution of the equation

$$2\ln(1-\alpha) - \alpha = -\frac{1}{2} + \ln\frac{3}{8}.$$

Then, $e^{\left(-\frac{1}{2} + \ln\frac{3}{8}\right)(n-m)} = e_{-\alpha}(3n, 3m)$. Thus, by definition, we obtain $\kappa_{\mathcal{B}}(E, A) = -\alpha$.

Remark 4. Since $\widetilde{P}(t) = \Phi(t,t)$, it is easy to see that if the Bohl exponent of (2.7) is finite, then the canonical projector $\widetilde{P}$ is necessarily bounded.

Assumption 2. The term $P_\sigma G^{-1}$ is bounded above by constant K_3 and $HQ_\sigma G^{-1}$ is bounded above by a constant K_4 on $\mathbb{T}_{t_0}$.

Remark 5. According to the Lemma 2.7, the boundedness of $HQ_\sigma G^{-1}$, and $P_\sigma G^{-1}$ does not depend on the choice of the operators H, Q.

The main following result proves a relationship among the exponential stability, the Bohl exponent of (2.7) and the solutions of the Cauchy problem (2.2).

Theorem 5.4. *The following statements are equivalent:*

i) *The IDE* (2.7) *is exponentially stable.*
ii) *The Bohl exponent* $\kappa_{\mathcal{B}}(E, A)$ *is negative.*
iii) *The Bohl exponent* $\kappa_{\mathcal{B}}(E, A)$ *is finite and for any* $p > 0$*, there exists a positive constant* C_p *such that*

$$\int_s^\infty \|\Phi(t, s)\|^p \Delta t \leq C_p, \ \forall t \geq s \geq t_0.$$

iv) *All solutions of Cauchy problem* (2.2) *with the initial condition* $P(t_0)x(t_0) = 0$*, associated with* q *in* $L(t_0)$ *are bounded.*

Proof. *i)* $\iff$ *ii)*: By Theorem 3.2.
ii) $\Longrightarrow$ *iii)*: Let $\kappa_{\mathcal{B}}(E, A) = -2\alpha < 0$. Then exists a constant $M_\alpha > 0$ such that

$$\|\Phi(t, s)\| \leq M_\alpha e_{-\alpha}(t, s), \ \forall t \geq s \geq t_0.$$

We define the functions $\beta(p) = \min\{p, 1\}$ and $\gamma(p) = \min\{1, 2^{p-1}\}$. By Bernoulli inequality we see that if $0 \leq \alpha\mu^* \leq \frac{1}{2}$ then

$$1 - \gamma(p)\alpha\mu(t) \leq (1 - \alpha\mu(t))^p \leq 1 - \beta(p)\alpha\mu(t) \quad \forall \ t \in \mathbb{T}_a,$$

which implies that

$$e_{-\alpha\gamma(p)}(t, s) \leq e^p_{-\alpha}(t, s) \leq e_{-\alpha\beta(p)}(t, s). \tag{5.2}$$

Therefore,

$$\int_s^\infty \|\Phi(t, s)\|^p \Delta t \leq M^p_\alpha \int_s^\infty e^p_{-\alpha}(t, s)\Delta t \leq M^p_\alpha \int_s^\infty e_{-\alpha\beta(p)}(t, s)\Delta t = \frac{M^p_\alpha}{\alpha\beta(p)}.$$

Thus, we obtain *iii)*.
iii) $\Longrightarrow$ *ii)*: From the first inequality of (5.2) we see that

$$\int_s^{s+T} e^p_{-\alpha}(t, s)\Delta t \geq \int_s^{s+T} e_{-\alpha\gamma(p)}(t, s)\Delta t = \frac{1 - e_{-\alpha\gamma(p)}(s + T, s)}{\alpha\gamma(p)}.$$

Hence, we can choose $\alpha, T > 0$ such that

$$C_p < \inf_{s > t_0} \int_s^{s+T} e^p_{-\alpha}(t, s)\Delta t.$$

For $s = s_0$, since

$$\int_{s_0}^{s_0+T} \|\Phi(t,s)\|^p \Delta t \le C_p < \int_{s_0}^{s_0+T} e^p_{-\alpha}(t,s_0)\Delta t,$$

we can define

$$s_1 = \max\{t : s_0 < t \le s_0 + T, \|\Phi(t,s_0)\| \le e_{-\alpha}(t,s_0)\}.$$

Similarly, we define a sequence $\{s_k\}$

$$s_{k+1} = \max\{t : s_k < t \le s_k + T, \|\Phi(t,s_k)\| \le e_{-\alpha}(t,s_k)\}.$$

It is easy to see that $s_{k+1} > s_k$ and $s_{k+1} > s_{k-1} + T$ and $s_{k+1} - s_k \le T$ for all $k \in \mathbb{N}$. Therefore,

$$\inf_k e_{-\alpha}(s_{k+1}, s_k) := \alpha_1 > 0.$$

For $t \in [s_k, s_{k+1})$, we have

$$\begin{aligned}\|\Phi(t,s)\| =& \|\Phi(t,s_0)\| \le \|\Phi(t,s_k)\|\|\Phi(s_k,s_0)\| \\ &\le \left(\sup_{s_k \le t < s_k+T} \|\Phi(t,s_k)\|\right)\|\Phi(s_k,s_{k-1})\|\ldots\|\Phi(s_1,s_0)\|.\end{aligned}$$

Since $\kappa_{\mathcal{B}}(E,A) < \infty$, it is easy to show that there exists a constant $M_1 > 0$ such that

$$\|\Phi(t,s)\| \le M_1, \quad \forall\, t - s < T.$$

Therefore,

$$\|\Phi(t,s)\| \le M_1 e_{-\alpha}(s_k,s_0) \le \frac{M_1}{e_{-\alpha}(s_{k+1},s_k)} e_{-\alpha}(t,s) \le \frac{M_1}{\alpha_1} e_{-\alpha}(t,s).$$

This implies $\kappa_{\mathcal{B}}(E,A) < 0$.
$i) \iff iv)$: By Theorem 4.2. The proof is complete. □

Remark 6. When $\mathbb{T} = \mathbb{R}$ we obtain Theorem 4.12 in (Chyan et al., 2008). For an arbitrary time scale, we need to use some new techniques in the above proof.

Returning to equation (2.7), by using the variable change $x(t) = U(t)z(t)$ and scaling equation (2.7) by the matrix V, we come to a new equation

$$\widehat{E}_\sigma(t)z^\Delta(t) = \widehat{A}(t)z(t), \quad \forall t \ge t_0, \tag{5.3}$$

where $\widehat{E}_\sigma = VE_\sigma U_\sigma, \widehat{A} = V(AU - E_\sigma U^\Delta)$, and $U \in \mathrm{C}^1(\mathbb{T}, \mathbb{R}^{n\times n}), V \in \mathrm{C}(\mathbb{T}, \mathbb{R}^{n\times n})$ are the nonsingular matrix functions.

It is not difficult to prove that

$$\widehat{Q} = U^{-1}QU,\ \widehat{P} = U^{-1}PU,\ \widehat{T} = U^{-1}TU_\sigma,\ \widehat{G} = VGU_\sigma,$$

Hence, the Cauchy operator, $\widehat{\Phi}(t,s)$, of (5.3) satisfies

$$\widehat{\Phi}(t,s) = U^{-1}(t)\Phi(t,s)U(s),\ \forall t \geq s \geq t_0. \tag{5.4}$$

Definition 5.5. The transformation with the matrix functions $U \in \mathrm{C}^1(\mathbb{R}, \mathbb{R}^{n\times n})$, and $V \in \mathrm{C}(\mathbb{R}, \mathbb{R}^{n\times n})$ is said to be a Bohl transformation if

$$\inf\{\varepsilon \in \mathbb{R}; \exists M_\varepsilon > 0 : \|U^{-1}(t)\|\|U(s)\| \leq M_\varepsilon e_\varepsilon(t,s),\ \forall t, s \geq t_0\} = 0.$$

From above definition and the relationship between the Cauchy operators of (2.7) and (5.3) in (5.4), it is easy to prove that

Proposition 5.6.

i) *The set of Bohl transformations forms a group with respect to point multiplication.*
ii) *The Bohl exponent is invariant with respect to Bohl transformations.*

5.2. Robustness of Bohl exponent

Suppose that $\Sigma(\cdot)$ is a continuous $\mathbb{R}^{n\times n}$-matrix function. We consider the perturbed equation

$$E_\sigma(t)x^\Delta(t) = \big(A(t) + \Sigma(t)\big)x(t),\ \forall t \geq t_0. \tag{5.5}$$

This equation is equivalent to

$$E_\sigma(t)(Px)^\Delta(t) = \big(\overline{A}(t) + \Sigma(t)\big)x(t),\ \forall t \geq t_0. \tag{5.6}$$

Equation (5.5) is a special case of (3.1) with $f(t,x) = \Sigma(t)x$. Let the perturbation Σ be sufficiently small such that

$$\sup_{t\geq t_0} \|\Sigma(t)\| < \left(\sup_{t\geq t_0} \|HQ_\sigma G^{-1}(t)\|\right)^{-1}. \tag{5.7}$$

With $\mathbf{G}^* = (\overline{A} + \Sigma)HQ_\sigma$, by using (5.7) and the relation

$$(I - \Sigma HQ_\sigma G^{-1})^{-1}\mathbf{G}^* = G.$$

we see that $\mathbf{G}^*$ is invertible if only if so is G. This means that Equation (2.2) is index-1 if and only if Equation (5.6) is index-1. By the same argument as in Section 3, we can solve Equation (5.6) as follows: Since the function $HQ_\sigma G^{-1}\Sigma(t)x$ is Lipschitz continuous with the Lipschitz coefficient $\gamma_t = \|HQ_\sigma G^{-1}\Sigma(t)\| < 1$, the function g_t defined by (3.4) becomes

$$g_t(u) = \big(I - HQ_\sigma G^{-1}\Sigma(t)\big)^{-1} HQ_\sigma G^{-1}(\overline{A} + \Sigma)(t)u.$$

Then the solution of (5.6) is given by

$$x(t,s) = u(t,s) + g_t(u(t,s),$$

where $u(t,s)$ is the solution of the equation

$$u^\Delta = (P^\Delta + P_\sigma G^{-1}\overline{A})u + P_\sigma G^{-1}\Sigma(u + g_t(u)), \quad u(s,s) = P(s)x_0.$$

The main following result proves the robustness of Bohl exponent for IDEs under dynamic perturbations.

Theorem 5.7. *Let Assumption 2 hold. Then, for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that the inequality*

$$\limsup_{t\to\infty} \|\Sigma(t)\| \le \delta$$

implies

$$\kappa_{\mathcal{B}}(E, A+\Sigma) \le \kappa_{\mathcal{B}}(E, A) + \varepsilon. \tag{5.8}$$

Proof. If $\kappa_{\mathcal{B}}(E,A) = \infty$ then (5.8) is evident. So, we suppose that $\kappa_{\mathcal{B}}(E,A) < \infty$. Denote by $\Psi(t,s)$ the Cauchy operator of Equation (5.6). From the relation (2.10) and the variation of constants formula (2.13), it follows that

$$\Psi(t,s) = \Phi(t,s)P(s) + \int_s^t \Phi(t,\sigma(\tau))P_\sigma G^{-1}\Sigma\Psi(t,\tau)\Delta\tau + HQ_\sigma G^{-1}\Sigma\Psi(t,s).$$

Hence,

$$\Psi(t,s) = (I - HQ_\sigma G^{-1}\Sigma(t))^{-1}\left[\Phi(t,s)P(s) + \int_s^t \Phi(t,\sigma(\tau))P_\sigma G^{-1}\Sigma\Psi(t,\tau)\Delta\tau\right]. \tag{5.9}$$

Since $HQ_\sigma G^{-1}$ is bounded on $\mathbb{T}_{t_0}$, we can choose $\delta_0 > 0$ such that

$$\|(I - HQ_\sigma G^{-1}\Sigma(t))^{-1}\| \le 2 \quad \text{for all } t \in \mathbb{T}_a \text{ if } \sup_{t>t_0}\|\Sigma(t)\| < \delta_0. \tag{5.10}$$

For any $\varepsilon > 0$, put $\alpha = \kappa_{\mathcal{B}} + \frac{\varepsilon}{2}$. By Definition 5.1, there is a number $M_\alpha > 0$ such that

$$\|\Phi(t,s)\| \le M_\alpha e_\alpha(t,s), \ \forall t \ge s \ge t_0. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11) obtains

$$\|\Psi(t,s)\| \le 2M_\alpha e_\alpha(t,s) + 2K_3 M_\alpha \int_s^t e_\alpha(t,\sigma(\tau))\|\Sigma(\tau)\|\|\Psi(\tau,s)\|\Delta\tau,$$

or equivalently,

$$\begin{aligned} e_{\ominus\alpha}(t,s)\|\Psi(t,s)\| &\le 2M_\alpha + 2K_3M_\alpha \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Sigma(\tau)\|\|\Psi(\tau,s)\|\Delta\tau \\ &\le 2M_\alpha + 2K_3M_\alpha \sup_{\tau\ge t_0}\|\Sigma(\tau)\| \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Psi(\tau,s)\|\Delta\tau. \end{aligned}$$

Let $\delta = \min\{\delta_0, (2K_3M_\alpha)^{-1}\varepsilon\}$. When $\sup_{t\geq t_0} \|\Sigma(t)\| < \delta$ we have

$$e_{\ominus\alpha}(t,s)\|\Psi)(t,s)\| \leq 2M_\alpha + \varepsilon \int_s^t \frac{1}{1+\mu(\tau)\alpha} e_{\ominus\alpha}(\tau,s)\|\Psi(\tau,s)\|\Delta\tau.$$

Using the Gronwall-Bellman inequality, we obtain

$$e_{\ominus\alpha}(t,s)\|\Psi(t,s)\| \leq 2M_\alpha e_{\frac{\varepsilon}{1+\mu(\cdot)\alpha}}(t,s), \quad \forall t \geq s \geq t_0.$$

Thus,

$$\|\Psi(t,s)\| \leq 2M_\alpha e_{\frac{\varepsilon}{1-\mu(\cdot)\alpha}}(t,s) e_\alpha(t,s) = 2M_\alpha e_{\alpha \oplus \frac{\varepsilon}{1+\mu(\cdot)\alpha}}(t,s) = 2M_\alpha e_{\alpha+\varepsilon}(t,s)$$

for all $t \geq s \geq t_0$. This means that

$$\kappa_{\mathcal{B}}(E, A+\Sigma) \leq \kappa_{\mathcal{B}}(E,A) + \varepsilon.$$

The proof is complete. $\square$

Now we consider equation (1.1) subject to two-side perturbations

$$\big(E_\sigma(t) + F_\sigma(t)\big)x^\Delta(t) = \big(A(t) + \Sigma(t)\big)x(t), \ \forall t \geq t_0, \tag{5.12}$$

where $F_\sigma(t)$ and $\Sigma(t)$ are perturbation matrices. It is already known from the analysis in (Berger, 2014; Linh et al., 2018) that for the differential-algebraic equations and the implicit difference equation it is necessary to restrict the perturbation structure in the matrix E_σ in order to get a meaningful problem of the robust stability. The main result is under infinitesimally small perturbations, the solvability and/or the stability may be lost, which usually happens with a change in the index/regularity of the equations. Therefore, we assume that $F_\sigma(t)$ is the allowable structured perturbations, i.e. ker $(E_\sigma(t) + F_\sigma(t)) =$ ker $E_\sigma(t)$ for all $t \in \mathbb{T}$. Let us define

$$\overline{\mathbf{G}} = E_\sigma + F_\sigma - (\overline{A} + \Sigma)HQ_\sigma = \mathbf{G}^* + F_\sigma.$$

It implies that if F and Σ are small enough then Equation (5.12) is index-1 and

$$\overline{\mathbf{G}}^{-1} = \mathbf{G}^{*-1} - \mathbf{G}^{*-1}F_\sigma(\mathbf{G}^* + F_\sigma)^{-1}.$$

Multiplying both sides of (5.12) by $P_\sigma\overline{\mathbf{G}}^{-1}$ and $Q_\sigma\overline{\mathbf{G}}^{-1}$, we decouple Equation (5.12) into the system

$$\begin{aligned}(Px)^\Delta &= \Big(P^\Delta + P_\sigma\big(\mathbf{G}^{*-1} - \mathbf{G}^{*-1}F_\sigma(\mathbf{G}^* + F_\sigma)^{-1}\big)\big(\overline{A} + \Sigma + F_\sigma P^\Delta\big)\Big)Px,\\ Qx &= HQ_\sigma\big(\mathbf{G}^{*-1} - \mathbf{G}^{*-1}F_\sigma(\mathbf{G}^* + F_\sigma)^{-1}\big)\big(\overline{A} + \Sigma + F_\sigma P^\Delta\big)Px.\end{aligned}$$

Let us define

$$\Gamma = P_\sigma\mathbf{G}^{*-1}F_\sigma P^\Delta P - P_\sigma\mathbf{G}^{*-1}F_\sigma(\mathbf{G}^* + F_\sigma)^{-1})(\overline{A} + \Sigma + F_\sigma P^\Delta)P, \quad \overline{\Sigma} = \Sigma + \Gamma.$$

Then, this system becomes

$$(Px)^{\Delta} = (P^{\Delta} + P_{\sigma}\mathbf{G}^{*-1}(\overline{A} + \Sigma))Px + P_{\sigma}\mathbf{G}^{*-1}\Gamma x,$$
$$Qx = HQ_{\sigma}\mathbf{G}^{*-1}(\overline{A} + \Sigma)Px + HQ_{\sigma}\mathbf{G}^{*-1}\Gamma x.$$

Thus, Equation (5.12) is equivalent to

$$E_{\sigma}(t)x^{\Delta}(t) = \big(A(t) + \overline{\Sigma}(t)\big)x(t), \ \forall t \geq t_0. \tag{5.13}$$

From above argument and Theorem 5.7, we get the following theorem.

Theorem 5.8. *Let Assumption 2 hold. Then, for any $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that the inequality*

$$\limsup_{t\to\infty} \|\overline{\Sigma}(t)\| \leq \delta$$

implies

$$\kappa_{\mathcal{B}}(E + F, A + \overline{\Sigma}) \leq \kappa_{\mathcal{B}}(E, A) + \varepsilon.$$

6. Conclusion

In this paper we have investigated the robust stability, the Bohl exponent and the Bohl-Perron theorem for the linear time-varying implicit dynamic equations. Some characterizations for robust stabiliy of IDEs subject to Lipschitz perturbations are derived. The Bohl-Perron type stability theorems are extended for these equations. The notion of Bohl exponent is introduced and it is investigated that how the Bohl exponent with respect to dynamic perturbations and two-side perturbations depends on the system data. Many previous results for robust stability of the time-varying ordinary differential and difference equations, the time-varying differential-algebraic equations and the time-varying implicit difference equations are also unified and extended.

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