Journal of Dynamics and Differential Equations, Vol. 14, No. 1, January 2002 (© 2002)

Dichotomy Spectrum for Nonautonomous Differential Equations

Stefan Siegmund[1,2]

Received May 5, 2000; revised October 30, 2000

For nonautonomous linear differential equations $\dot{x} = A(t)\,x$ with locally integrable $A: \mathbf{R} \to \mathbf{R}^{N \times N}$ the so-called dichotomy spectrum is investigated in this paper. As the closely related dichotomy spectrum for skew product flows with compact base (Sacker–Sell spectrum) our dichotomy spectrum for nonautonomous differential equations consists of at most N closed intervals, which in contrast to the Sacker–Sell spectrum may be unbounded. In the constant coefficients case these intervals reduce to the real parts of the eigenvalues of A. In any case the spectral intervals are associated with spectral manifolds comprising solutions with a common exponential growth rate. The main result of this paper is a spectral theorem which describes all possible forms of the dichotomy spectrum.

KEY WORDS: Dichotomy spectrum; nonautonomous differential equations.

1. INTRODUCTION

Consider a linear system of differential equations

$$\dot{x} = A(t)\,x \tag{1}$$

with $A \in L^1_{\mathrm{loc}} = L^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^{N \times N})$, the space of locally integrable matrix functions $A: \mathbf{R} \to \mathbf{R}^{N \times N}$, $N \in \mathbf{N}$. Let $\mathbf{\Phi}: \mathbf{R} \times \mathbf{R} \to \mathbf{R}^{N \times N}$, $(t, \tau) \mapsto \mathbf{\Phi}(t, \tau)$ denote its evolution operator, i.e., $\mathbf{\Phi}(\cdot, \tau)\,\xi$ solves the initial value problem (1), $x(\tau) = \xi$, for $\tau \in \mathbf{R}$, $\xi \in \mathbf{R}^N$.

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[1] This paper has been written while the author was a member of the "Graduiertenkolleg: Nichtlineare Probleme in Analysis, Geometrie und Physik" (GRK 283) financed by the Deutsche Forschungsgemeinschaft and the State of Bavaria.

[2] CDSNS, Georgia Institute of Technology, Atlanta, Georgia 30332-0190. E-mail: siegmund @math.gatech.edu

We introduce a spectral notion for system (1) which is adequate to show that the qualitative structure of system (1) carries over to weakly nonlinear perturbations of (1). Let all Lyapunov exponents for $t \to \infty$ of (1) be negative. Then (1) is asymptotically stable (even exponentially stable). But Hahn [8, Example 3, pp. 321–322] gives a nonlinearly perturbed system

$$\dot{x} = A(t)\,x + f(t, x), \qquad \|f(t, x)\| \leqslant C\,\|x\|^{1+\alpha} \quad \text{for} \quad \|x\| \leqslant h, \quad C, \alpha > 0 \tag{2}$$

with unstable zero solution. On the other hand a nonautonomous version of the theorem of linearized asymptotic stability shows that the zero solution of (2) inherits the asymptotic stability of (1) if the evolution operator satisfies the estimate

$$\|\Phi(t, s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s \tag{3}$$

with $K \geqslant 1$ and $\alpha > 0$ independent of t, s. If all Lyapunov exponents of (1) are negative then (3) also holds but now K depends on $s \in \mathbf{R}$. The estimate (3) is a special case of an exponential dichotomy for (1). Exponential dichotomies also play an important role in the theory of integral manifolds for systems (2) (see, e.g., Aulbach and Wanner [1]). We agree with Coppel [7] "that dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behaviour for nonautonomous differential equations."

For linear skew product flows with compact base Sacker and Sell [11–14] introduced and investigated a spectrum (called *Sacker–Sell spectrum, dynamical spectrum or dichotomy spectrum*) defined with exponential dichotomies. Since these classical papers a lot of research has been done to understand and extend this fruitful concept to various situations. For connections with other spectra see, e.g., Johnson, Palmer and Sell [9], Chicone and Latushkin [2] or Colonius and Kliemann [6]; generalizations to the infinite dimensional case can be found e.g., in Sacker and Sell [15] and Chow and Leiva [3–5], etc.

There is a well-known procedure to associate a linear skew product flow to a linear system (1) of differential equations (see, e.g., [14]). Therefore let $\sigma\colon \mathbf{R} \times L^1_{\text{loc}} \to L^1_{\text{loc}}$ denote the translation which maps every $A \in L^1_{\text{loc}}$ to its translate $\sigma(s, A) = A(\cdot + s)$. It is a continuous flow (G. R. Sell [16, Thm. III.11, p. 43]). Define $\phi(t, A)$ to be the solution of the matrix differential equation

$$\dot{X} = A(t)\,X, \qquad X(0) = I \in \mathbf{R}^{N \times N}$$

for arbitrary $A \in L^1_{\mathrm{loc}}$. Then $(t, A, x) \mapsto \phi(t, A)\, x$ is a cocycle in $\mathbf{R}^N$ over σ, i.e., it satisfies the cocycle property

$$\phi(0, A) = I \qquad \text{for all} \quad A \in L^1_{\mathrm{loc}}$$
$$\phi(t+s, A) = \phi(t, \sigma(s, A)) \circ \phi(s, A) \qquad \text{for all} \quad s, t \in \mathbf{R}, \quad A \in L^1_{\mathrm{loc}}$$

The cocycle $\phi(t, A)\, x$ is continuous in $(t, A, x) \in \mathbf{R} \times L^1_{\mathrm{loc}} \times \mathbf{R}^N$. Since this important result has not been proved in the literature yet, we indicate its easy proof here. Let $A_n \to A_0$ in L^1_{loc} for $n \to \infty$, i.e., $\int_{-T}^{T} \|A_n(s) - A_0(s)\|\, ds \to 0$ for every $T > 0$. We show only $\phi(t, A_n) \to \phi(t, A_0)$ uniformly in $t \in J = [-T, T]$ for arbitrary $T > 0$, the rest is clear. To estimate $u_n(t) = \|\phi(t, A_n) - \phi(t, A_0)\|$ note that the difference in the norm satisfies the differential equation $\dot{X} = A_n(t)\, X + (A_n(t) - A_0(t))\, \phi(t, A_0)$. Hence with the variation of constants formula

$$u_n(t) \leqslant \int_0^t \|\phi(t-s, A_n)\| \cdot \|A_n(s) - A_0(s)\| \cdot \|\phi(s, A_0)\|\, ds$$

and we are finished if $\sup_{s \in J} \|\phi(s, A_n)\| \leqslant M$ for all $n \in \mathbf{N}$ with some $M > 0$. Therefore apply the Gronwall inequality to $u_n(t) \leqslant v_n(t) + \int_0^t \|A_n(s)\| \cdot u_n(s)\, ds$ with $v_n(t) = \int_0^t \|A_n(s) - A_0(s)\| \cdot \|\phi(s, A_0)\|\, ds$ to get the estimate

$$u_n(t) \leqslant v_n(t) + \left| \int_0^t v_n(s) \cdot \|A_n(s)\| \cdot \exp\left(\left| \int_s^t \|A_n(\tau)\|\, d\tau \right| \right) ds \right|$$

Using the uniform boundedness of $\sup_{s \in J} v_n(s)$ and $\int_J \|A_n(\tau)\|\, d\tau$ in $n \in \mathbf{N}$ we get the result. The continuous mapping

$$\pi\colon \mathbf{R} \times L^1_{\mathrm{loc}} \times \mathbf{R}^N \to L^1_{\mathrm{loc}} \times \mathbf{R}^N, \qquad \pi(t, A, x) = (\sigma(t, A), \phi(t, A)\, x)$$

is a *skew product flow*. Its restriction $\pi\colon \mathbf{R} \times H(A) \times \mathbf{R}^N \to H(A) \times \mathbf{R}^N$ to the so-called *hull* $H(A) = \mathrm{cl}\{A(\cdot + s) : s \in \mathbf{R}\}$ of $A \in L^1_{\mathrm{loc}}$, is said to be the skew product flow which is associated to the linear system (1).

We stress the point that the existing results on dichotomy spectrum for linear skew product flows can not be applied to general linear systems (1), see also Remark 3.2. For this reason we give a direct treatment of the subject in this paper resulting in a Spectral Theorem describing all possible forms of the dichotomy spectrum for systems (1). A new phenomenon is the occurance of spectral manifolds which have an empty or unbounded associated spectrum. Applications of our Spectral Theorem can be found in the papers [18] and [19] on block diagonalization and normal forms for nonautonomous differential equations.

2. PRELIMINARIES

An *invariant projector* of (1) is defined to be a function $P: \mathbf{R} \to \mathbf{R}^{N \times N}$ of projections $P(t)$, $t \in \mathbf{R}$, such that

$$P(t)\, \Phi(t, s) = \Phi(t, s)\, P(s) \qquad \text{for} \quad t, s \in \mathbf{R} \tag{4}$$

Note that P is continuous due to the identity $P \equiv \Phi(\cdot, s)\, P(s)\, \Phi(s, \cdot)$. We shall say that (1) admits an *exponential dichotomy (ED)* if there is an invariant projector P and constants $K \geqslant 1$ and $\alpha > 0$ such that

$$\|\Phi(t, s)\, P(s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s$$

$$\|\Phi(t, s)[I - P(s)]\| \leqslant K e^{\alpha(t-s)} \qquad \text{for} \quad t \leqslant s$$

In the following the shifted system $\dot{x} = [A(t) - \gamma I]\, x$ for $\gamma \in \mathbf{R}$ will play an important role and one can easily see that $\Phi_\gamma(t, s) := e^{-\gamma(t-s)} \Phi(t, s)$ is its evolution operator. If $\dot{x} = [A(t) - \gamma I]\, x$ admits an ED then its invariant projector P is also invariant for $\dot{x} = A(t)\, x$, i.e., (4) holds. The dichotomy estimates are equivalent to

$$\|\Phi(t, s)\, P(s)\| \leqslant K e^{(\gamma-\alpha)(t-s)} \qquad \text{for} \quad t \geqslant s$$

$$\|\Phi(t, s)[I - P(s)]\| \leqslant K e^{(\gamma+\alpha)(t-s)} \qquad \text{for} \quad t \leqslant s$$

Remark 2.1. If $\dot{x} = [A(t) - \gamma I]\, x$ admits an ED with invariant projector $P \equiv I$ then $\dot{x} = [A(t) - \zeta I]\, x$ also admits an ED with the same projector for every $\zeta > \gamma$ resp. for every $\zeta < \gamma$ if $P \equiv 0$.

We follow Aulbach [1] and introduce a handy notion describing exponential growth of functions.

Definition 2.1. Let $\gamma \in \mathbf{R}$. A continuous function $g: \mathbf{R} \to \mathbf{R}^N$ is

(a) γ^+-*quasibounded* if $\sup_{t \geqslant 0} \|g(t)\|\, e^{-\gamma t} < \infty$,

(b) γ^--*quasibounded* if $\sup_{t \leqslant 0} \|g(t)\|\, e^{-\gamma t} < \infty$.

Obviously g is γ^+-quasibounded if and only if for an *arbitrary* $\tau \in \mathbf{R}$ there exists a positive constant C such that $\|g(t)\| \leqslant C e^{\gamma t}$ for all $t \in [\tau, \infty)$. The zero function is γ^+-and γ^--quasibounded for every $\gamma \in \mathbf{R}$.

Definition 2.2. We shall say that a nonempty set $\mathcal{W} \subset \mathbf{R} \times \mathbf{R}^N$ is a *linear integral manifold of* (1) if

(a) it is *invariant*, i.e., $(\tau, \xi) \in \mathcal{W} \Rightarrow (t, \Phi(t, \tau)\, \xi) \in \mathcal{W}$ for all $t \in \mathbf{R}$,

(b) for every $\tau \in \mathbf{R}$ the *fiber* $\mathcal{W}(\tau) = \{\xi \in \mathbf{R}^N : (\tau, \xi) \in \mathcal{W}\}$ is a linear subspace of $\mathbf{R}^N$.

The fibers of a linear integral manifold $\mathcal{W}$ have constant dimension. Let $\dim \mathcal{W} := \dim \mathcal{W}(\tau)$ denote the fiber dimension. The extended state space $\mathbf{R} \times \mathbf{R}^N$ and the graph $\mathbf{R} \times \{0\}$ of the zero solution are always linear integral manifolds. A linear integral manifold is a topological manifold in $\mathbf{R} \times \mathbf{R}^N$ and a vector bundle over $\mathbf{R}$. If $\mathcal{W}_1$ and $\mathcal{W}_2$ are linear integral manifolds of (1), then the *intersection* and the *sum*

$$\mathcal{W}_1 \cap \mathcal{W}_2 := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \xi \in \mathcal{W}_1(\tau) \cap \mathcal{W}_2(\tau)\}$$
$$\mathcal{W}_1 + \mathcal{W}_2 := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \xi \in \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau)\}$$

are also linear integral manifolds of (1). A sum $\mathcal{W}_1 + \cdots + \mathcal{W}_n$ of linear integral manifolds is said to be a *Whitney-sum* $\mathcal{W}_1 \oplus \cdots \oplus \mathcal{W}_n$ if $\mathcal{W}_i \cap \mathcal{W}_j = \mathbf{R} \times \{0\}$ for $i \neq j$. The image $\operatorname{im} P := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \xi \in \operatorname{im} P(\tau)\}$ and kernel $\ker P := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \xi \in \ker P(\tau)\}$ of an invariant projector P are linear integral manifolds of (1) with $\ker P \oplus \operatorname{im} P = \mathbf{R} \times \mathbf{R}^N$.

Example 2.1. The system $\dot{x}_1 = -x_1$, $\dot{x}_2 = x_2$ has the integral manifolds

$$\mathbf{R} \times \{0\} \times \{0\}, \qquad \mathcal{S} = \{(\tau, \xi_1, 0) \in \mathbf{R}^3\}, \qquad \mathcal{U} = \{(\tau, 0, \xi_2) \in \mathbf{R}^3\}, \qquad \mathbf{R} \times \mathbf{R}^2$$

The fibers are constant and the projections $\{0\} \times \{0\}$, $\mathbf{R} \times \{0\}$, $\{0\} \times \mathbf{R}$, $\mathbf{R}^2$ on the state space $\mathbf{R}^2$ are usually called *invariant manifolds*.

The eigenvalues of this example are -1 and 1. They describe the exponential growth of solutions and we get (note that $0 \in (-1, 1)$)

$$\mathcal{S} := \{(\tau, \xi_1, \xi_2) \in \mathbf{R} \times \mathbf{R}^2 : (e^{-(t-\tau)}\xi_1, e^{t-\tau}\xi_2) \text{ is } 0^+\text{-quasibounded}\}$$
$$\mathcal{U} := \{(\tau, \xi_1, \xi_2) \in \mathbf{R} \times \mathbf{R}^2 : (e^{-(t-\tau)}\xi_1, e^{t-\tau}\xi_2) \text{ is } 0^-\text{-quasibounded}\}$$

This motivates the definition of the following sets for system (1):

$$\mathcal{S}_\gamma := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \Phi(\cdot, \tau)\, \xi \text{ is } \gamma^+\text{-quasibounded}\}$$
$$\mathcal{U}_\gamma := \{(\tau, \xi) \in \mathbf{R} \times \mathbf{R}^N : \Phi(\cdot, \tau)\, \xi \text{ is } \gamma^-\text{-quasibounded}\}$$

It is easy to see that $\mathcal{S}_\gamma$ and $\mathcal{U}_\gamma$ are linear integral manifolds of (1) and that the following *monotonicity* holds

$$\gamma \leqslant \zeta \Rightarrow \mathcal{S}_\gamma \subset \mathcal{S}_\zeta \quad \text{and} \quad \mathcal{U}_\gamma \supset \mathcal{U}_\zeta$$

Lemma 2.1. *If $\dot{x} = [A(t) - \gamma I]\, x$ admits an ED with invariant projector P for a $\gamma \in \mathbf{R}$ then*

$$\mathscr{S}_\gamma = \operatorname{im} P, \qquad \mathscr{U}_\gamma = \ker P \qquad \textit{and} \qquad \mathscr{S}_\gamma \oplus \mathscr{U}_\gamma = \mathbf{R} \times \mathbf{R}^N$$

Proof. We show only $\mathscr{S}_\gamma = \operatorname{im} P$, since the rest is analogous.

($\subset$) Let $\tau \in \mathbf{R}$ and $\xi \in \mathscr{S}_\gamma(\tau)$, i.e., $\|\Phi(t, \tau)\, \xi\| \leqslant Ce^{\gamma t}$ for $t \geqslant \tau$ with some positive constant C. Therefore $\|\Phi_\gamma(t, \tau)\, \xi\| \leqslant Ce^{\gamma\tau}$ for $t \geqslant \tau$. Now write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \operatorname{im} P(\tau)$, $\xi_2 \in \ker P(\tau)$. We show $\xi_2 = 0$. The invariance of P implies for $t \in \mathbf{R}$ the identity

$$\xi_2 = \Phi_\gamma(\tau, t)\, \Phi_\gamma(t, \tau)[I - P(\tau)]\, \xi = \Phi_\gamma(\tau, t)[I - P(t)]\, \Phi_\gamma(t, \tau)\, \xi$$

and with the ED of $\dot{x} = [A(t) - \gamma I]\, x$ one has the estimate

$$\|\xi_2\| \leqslant Ke^{\alpha(\tau - t)}\, \|\Phi_\gamma(t, \tau)\, \xi\| \qquad \text{for} \quad t \geqslant \tau$$

Due to $\alpha > 0$ and the boundeness of $\|\Phi_\gamma(\cdot, \tau)\, \xi\|$ the right hand side converges to 0 for $t \to \infty$. It follows $\xi_2 = 0$.

($\supset$) Let $\tau \in \mathbf{R}$ and $\xi \in \operatorname{im} P(\tau)$, i.e., $P(\tau)\, \xi = \xi$. The ED implies

$$\|\Phi_\gamma(t, \tau)\, \xi\| \leqslant Ke^{-\alpha(t - \tau)}\, \|\xi\| \leqslant K\, \|\xi\| \qquad \text{for} \quad t \geqslant \tau$$

and therefore $\Phi(\cdot, \tau)\, \xi$ is γ^+-quasibounded and we get $\xi \in \mathscr{S}_\gamma(\tau)$. □

3. DICHOTOMY SPECTRUM

We continue our investigation of system (1).

Definition 3.1. The *dichotomy spectrum of* (1) is the set

$$\Sigma(A) = \{\gamma \in \mathbf{R} : \dot{x} = [A(t) - \gamma I]\, x \text{ admits no ED}\}$$

and the *resolvent set* $\rho(A) = \mathbf{R} \setminus \Sigma(A)$ is its complement.

Lemma 3.1. *The resolvent set is open, i.e., for every $\gamma \in \rho(A)$ exists a $\varepsilon = \varepsilon(\gamma) > 0$ such that $(\gamma - \varepsilon, \gamma + \varepsilon) \subset \rho(A)$ and moreover*

$$\mathscr{S}_\zeta = \mathscr{S}_\gamma \qquad \textit{and} \qquad \mathscr{U}_\zeta = \mathscr{U}_\gamma \qquad \textit{for} \quad \zeta \in (\gamma - \varepsilon, \gamma + \varepsilon)$$

Proof. Let $\gamma \in \rho(A)$. Then $\dot{x}[A(t)-\gamma I]\, x$ admits an ED, i.e.,

$$\|\Phi_\gamma(t, s)\, P(s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s$$
$$\|\Phi_\gamma(t, s)[I-P(s)]\| \leqslant K e^{\alpha(t-s)} \qquad \text{for} \quad t \leqslant s$$

with an invariant projector P and constants $K \geqslant 1$ and $\alpha > 0$. For $\varepsilon := \alpha/2$ and $\zeta \in (\gamma-\varepsilon, \gamma+\varepsilon)$ we have $\Phi_\zeta(t, s) = e^{(\gamma-\zeta)(t-s)} \Phi_\gamma(t, s)$. Now P is also an invariant projector for $\dot{x} = [A(t)-\zeta I]\, x$ and the estimates

$$\|\Phi_\zeta(t, s)\, P(s)\| \leqslant K e^{(\gamma-\zeta-\alpha)(t-s)} \leqslant K e^{-\varepsilon(t-s)} \qquad \text{for} \quad t \geqslant s$$
$$\|\Phi_\zeta(t, s)[I-P(s)]\| \leqslant K e^{(\gamma-\zeta+\alpha)(t-s)} \leqslant K e^{\varepsilon(t-s)} \qquad \text{for} \quad t \leqslant s$$

hold. Hence $\zeta \in \rho(A)$. Moreover, since the exponential dichotomies involve the same projector, Lemma 2.1 yields $\mathcal{S}_\zeta = \mathcal{S}_\gamma$ and $\mathcal{U}_\zeta = \mathcal{U}_\gamma$. □

Lemma 3.2. *Let $\gamma_1, \gamma_2 \in \rho(A)$ with $\gamma_1 < \gamma_2$. Then $\mathcal{F} = \mathcal{U}_{\gamma_1} \cap \mathcal{S}_{\gamma_2}$ is a linear integral manifold which satisfies exactly ore of the following two alternatives and the statements given in each alternative are equivalent:*

	Alternative I		*Alternative II*
(A)	$\mathcal{F} = \mathbf{R} \times \{0\}$	*(A′)*	$\mathcal{F} \neq \mathbf{R} \times \{0\}$
(B)	$[\gamma_1, \gamma_2] \subset \rho(A)$	*(B′)*	*There is a* $\zeta \in (\gamma_1, \gamma_2) \cap \Sigma(A)$
(C)	$\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ *and* $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$	*(C′)*	$\dim \mathcal{S}_{\gamma_1} < \dim \mathcal{S}_{\gamma_2}$
(D)	$\mathcal{S}_\gamma = \mathcal{S}_{\gamma_2}$ *and* $\mathcal{U}_\gamma = \mathcal{U}_{\gamma_2}$ *for* $\gamma \in [\gamma_1, \gamma_2]$	*(D′)*	$\dim \mathcal{U}_{\gamma_1} > \dim \mathcal{U}_{\gamma_2}$

Proof. (B) ⇒ (D). Arguing negatively, let us assume that there exists a $\gamma \in [\gamma_1, \gamma_2]$ such that $\mathcal{S}_\gamma \neq \mathcal{S}_{\gamma_2}$, or $\mathcal{U}_\gamma \neq \mathcal{U}_{\gamma_2}$, w.l.o.g. $\mathcal{S}_\gamma \neq \mathcal{S}_{\gamma_2}$. Define

$$\zeta_0 := \inf\{\zeta \in [\gamma, \gamma_2] : \mathcal{S}_\zeta = \mathcal{S}_{\gamma_2}\}$$

The inequality $\mathcal{S}_\gamma \neq \mathcal{S}_{\gamma_2}$ implies $\zeta_0 \in [\gamma, \gamma_2]$ and therefore $\zeta_0 \in \rho(A)$. There are two cases to consider: (i) $\mathcal{S}_{\zeta_0} = \mathcal{S}_{\gamma_2}$ or (ii) $\mathcal{S}_{\zeta_0} \neq \mathcal{S}_{\gamma_2}$. In case (i) Lemma 3.1 implies $\mathcal{S}_\zeta = \mathcal{S}_{\zeta_0}$ for $\zeta \in (\zeta_0 - \varepsilon, \zeta_0 + \varepsilon)$ with some $\varepsilon > 0$, which contradicts the definition of ζ_0. In case (ii) Lemma 3.1 implies $\mathcal{S}_\zeta \neq \mathcal{S}_{\zeta_0}$ for $\zeta \in (\zeta_0 - \varepsilon, \zeta_0 + \varepsilon)$, which also contradicts the definition of ζ_0.

(D) ⇒ (C). This is obvious.

(C) ⇒ (B). Both systems $\dot{x} = [A(t) - \gamma_i I]\, x$, $i = 1, 2$, admit ED with constants $K_i \geqslant 1$ and $\alpha_i > 0$. Since $\mathcal{S}_{\gamma_1} = \mathcal{S}_{\gamma_2}$ and $\mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_2}$, Lemma 2.1

implies that both ED involve the same invariant projector P. With $K := \max\{K_1, K_2\}$ and $\alpha := \min\{\alpha_1, \alpha_2\}$ we get

$$\|\Phi_{\gamma_i}(t, s)\, P(s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s$$
$$\|\Phi_{\gamma_i}(t, s)[I - P(s)]\| \leqslant K e^{\alpha(t-s)} \qquad \text{for} \quad t \leqslant s$$

for $i = 1, 2$. The first inequality for $i = 1$ and the second for $i = 2$ imply

$$\|\Phi_{\gamma}(t, s)\, P(s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s$$
$$\|\Phi_{\gamma}(t, s)[I - P(s)]\| \leqslant K e^{\alpha(t-s)} \qquad \text{for} \quad t \leqslant s$$

for every $\gamma \in [\gamma_1, \gamma_2]$ and therefore $[\gamma_1, \gamma_2] \subset \rho(A)$.

(C) $\Rightarrow$ (A). Lemma 2.1 implies $\mathscr{F} = \mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_2} = \mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_1} = \mathbf{R} \times \{0\}$. Thus we have (B) $\Leftrightarrow$ (C) $\Leftrightarrow$ (D) $\Rightarrow$ (A).

(C$'$) $\Leftrightarrow$ (D$'$). Lemma 2.1 implies $\dim \mathscr{S}_{\gamma_i} + \dim \mathscr{U}_{\gamma_i} = N$, $i = 1, 2$, hence $\dim \mathscr{S}_{\gamma_1} < \dim \mathscr{S}_{\gamma_2} \Leftrightarrow N - \dim \mathscr{U}_{\gamma_1} < N - \dim \mathscr{U}_{\gamma_2} \Leftrightarrow \dim \mathscr{U}_{\gamma_1} > \dim \mathscr{U}_{\gamma_2}$.

(B$'$) $\Rightarrow$ (C$'$), (D$'$). Since (B$'$) is the opposite of (B), the proved implication (C) $\Rightarrow$ (B) yields $\mathscr{S}_{\gamma_1} \neq \mathscr{S}_{\gamma_2}$ or $\mathscr{U}_{\gamma_1} \neq \mathscr{U}_{\gamma_2}$. Monotonicity implies $\mathscr{S}_{\gamma_1} \subsetneq \mathscr{S}_{\gamma_2}$ or $\mathscr{U}_{\gamma_1} \supsetneq \mathscr{U}_{\gamma_2}$, w.l.o.g. $\mathscr{S}_{\gamma_1} \subsetneq \mathscr{S}_{\gamma_2}$. Then there is a $\tau \in \mathbf{R}$ such that $\mathscr{S}_{\gamma_1}(\tau) \subsetneq \mathscr{S}_{\gamma_2}(\tau)$. For subspaces, however, this is possible only if $\dim \mathscr{S}_{\gamma_1}(\tau) < \dim \mathscr{S}_{\gamma_2}(\tau)$.

(C$'$), (D$'$) $\Rightarrow$ (A$'$). Using $\dim \mathscr{S}_{\gamma_1} < \dim \mathscr{S}_{\gamma_2}$ and $\dim \mathscr{S}_{\gamma_1} + \dim \mathscr{U}_{\gamma_1} = N$ we get

$$\begin{aligned} \dim \mathscr{F} &= \dim[\mathscr{U}_{\gamma_1} \cap \mathscr{S}_{\gamma_2}] \geqslant \dim \mathscr{U}_{\gamma_1} + \dim \mathscr{S}_{\gamma_2} - N \\ &> \dim \mathscr{U}_{\gamma_1} + \dim \mathscr{S}_{\gamma_1} - N = 0 \end{aligned}$$

and therefore $\mathscr{F}$ is not the trivial integral manifold which has dimension 0.

(A$'$) $\Rightarrow$ (B$'$). Since (A$'$) is the opposite of (A), the proved implication (B) $\Rightarrow$ (A) implies the opposite of (B) which is (B$'$).

Thus we have (A$'$) $\Leftrightarrow$ (B$'$) $\Leftrightarrow$ (C$'$) $\Leftrightarrow$ (D$'$).

(A) $\Rightarrow$ (B), (C), (D). The implication (A) $\Rightarrow$ (B) is equivalent to the proved implication (B$'$) $\Rightarrow$ (A$'$). $\square$

The following *algebraic lemma* is quite obvious: Let A, B and C be subspaces of a vector space X. If $A \supseteq C$ then

$$A \cap [B + C] = [A \cap B] + C$$

Spectral Theorem. *The dichotomy spectrum $\Sigma(A)$ of (1) is the disjoint union of n closed intervals (called spectral intervals) where $0 \leqslant n \leqslant N$, i.e., $\Sigma(A) = \emptyset$ or $\Sigma(A) = \mathbf{R}$ or one of the four cases*

$$\Sigma(A) = \begin{Bmatrix} [a_1, b_1] \\ \text{or} \\ (-\infty, b_1] \end{Bmatrix} \cup [a_2, b_2] \cup \cdots \cup [a_{n-1}, b_{n-1}] \cup \begin{Bmatrix} [a_n, b_n] \\ \text{or} \\ [a_n, \infty) \end{Bmatrix}$$

where $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_n \leqslant b_n$. Choose a

$$\gamma_0 \in \rho(A) \qquad \text{with} \quad (-\infty, \gamma_0) \subset \rho(A) \text{ if possible} \tag{5}$$

otherwise define $\mathscr{U}_{\gamma_0} := \mathbf{R} \times \mathbf{R}^N$, $\mathscr{S}_{\gamma_0} := \mathbf{R} \times \{0\}$. Choose a

$$\gamma_n \in \rho(A) \qquad \text{with} \quad (\gamma_n, \infty) \subset \rho(A) \text{ if possible} \tag{6}$$

otherwise define $\mathscr{U}_{\gamma_n} := \mathbf{R} \times \{0\}$, $\mathscr{S}_{\gamma_n} := \mathbf{R} \times \mathbf{R}^N$. Then the sets

$$\mathscr{W}_0 = \mathscr{S}_{\gamma_0} \qquad \text{and} \qquad \mathscr{W}_{n+1} = \mathscr{U}_{\gamma_n}$$

are linear integral manifolds of (1). For $n \geqslant 2$ choose $\gamma_i \in \rho(A)$ with

$$b_i < \gamma_i < a_{i+1} \qquad \text{for} \quad i = 1, \ldots, n-1 \tag{7}$$

Then for every $i = 1, \ldots, n$ the intersection

$$\mathscr{W}_i = \mathscr{U}_{\gamma_{i-1}} \cap \mathscr{S}_{\gamma_i}$$

is a linear integral manifold of (1) with $\dim \mathscr{W}_i \geqslant 1$. The linear integral manifolds $\mathscr{W}_i$, $i = 0, \ldots, n+1$, are called spectral manifolds and they are independent of the choice of γ_i in (5), (6) and (7). Moreover

$$\mathscr{W}_0 \oplus \cdots \oplus \mathscr{W}_{n+1} = \mathbf{R} \times \mathbf{R}^N \qquad \text{(Whitney sum)}$$

i.e., $\mathscr{W}_i \cap \mathscr{W}_j = \mathbf{R} \times \{0\}$ for $i \neq j$ and $\mathscr{W}_0 + \cdots + \mathscr{W}_{n+1} = \mathbf{R} \times \mathbf{R}^N$.

Proof. First, recall that the resolvent set $\rho(A)$ is open (Lemma 3.1) and therefore the dichotomy spectrum $\Sigma(A)$ is the disjoint union of closed intervals. Next we will show that $\Sigma(A)$ consists of at most N intervals. Indeed, if $\Sigma(A)$ contains $N+1$ components, then one can choose a collection of points $\zeta_1, \ldots, \zeta_N$ in $\rho(A)$ such that $\zeta_1 < \cdots < \zeta_N$ and each of the intervals $(-\infty, \zeta_1)$, $(\zeta_1, \zeta_2), \ldots, (\zeta_{N-1}, \zeta_N)$, (ζ_N, ∞) has nonempty intersection with the spectrum $\Sigma(A)$. Now alternative II of Lemma 3.2 implies

$$0 \leqslant \dim \mathcal{S}_{\zeta_1} < \cdots < \dim \mathcal{S}_{\zeta_N} \leqslant N$$

and therefore either $\dim \mathcal{S}_{\zeta_1} = 0$ or $\dim \mathcal{S}_{\zeta_N} = N$ or both, w.l.o.g. $\dim \mathcal{S}_{\zeta_N} = N$, i.e., $\mathcal{S}_{\zeta_N} = \mathbf{R} \times \mathbf{R}^N$. Because of Lemma 2.1 the projector P of the ED of $\dot{x} = [A(t) - \zeta_N I]\, x$ equals I and Remark 2.1 yields the contradiction $(\zeta_N, \infty) \subset \rho(A)$. This proves the alternatives for $\Sigma(A)$.

Obviously the sets $\mathcal{W}_0, \ldots, \mathcal{W}_{n+1}$ are linear integral manifolds. To prove now that $\dim \mathcal{W}_1 \geqslant 1, \ldots, \dim \mathcal{W}_n \geqslant 1$ for $n \geqslant 1$, let us assume that $\dim \mathcal{W}_1 = 0$, i.e., $\mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_1} = \mathbf{R} \times \{0\}$. If $(-\infty, b_1]$ is a spectral interval this implies that $\mathcal{S}_{\gamma_1} = \mathbf{R} \times \{0\}$. Then the projector of the ED of $\dot{x} = [A(t)\, x - \gamma_1 I]\, x$ equals 0 and Remark 2.1 yields the contradiction $(-\infty, \gamma_1) \subset \rho(A)$. If $[a_1, b_1]$ is a spectral interval then $[\gamma_0, \gamma_1] \cap \Sigma(A) \neq \varnothing$ and alternative II of Lemma 3.2 yields a contradiction. Therefore $\dim \mathcal{W}_1 \geqslant 1$ and similarly $\dim \mathcal{W}_n \geqslant 1$. Furthermore for $n \geqslant 3$ and $i = 2, \ldots, n-1$ one has $(\gamma_{i-1}, \gamma_i) \cap \Sigma(A) \neq \varnothing$ and again alternative II of Lemma 3.2 yields $\dim \mathcal{W}_i \geqslant 1$.

For $i < j$ we have $\mathcal{W}_i \subset \mathcal{S}_{\gamma_i}$ and $\mathcal{W}_j \subset \mathcal{U}_{\gamma_{j-1}} \subset \mathcal{U}_{\gamma_i}$ and with Lemma 2.1 this gives $\mathcal{W}_i \cap \mathcal{W}_j \subset \mathcal{S}_{\gamma_i} \cap \mathcal{U}_{\gamma_i} = \mathbf{R} \times \{0\}$, so $\mathcal{W}_i \cap \mathcal{W}_j = \mathbf{R} \times \{0\}$ for $i \neq j$.

To show that $\mathcal{W}_0 + \cdots + \mathcal{W}_{n+1} = \mathbf{R} \times \mathbf{R}^N$, choose and fix a $\tau \in \mathbf{R}$ and recall the monotonicity relations $\mathcal{S}_{\gamma_0}(\tau) \subset \cdots \subset \mathcal{S}_{\gamma_n}(\tau)$, $\mathcal{U}_{\gamma_0}(\tau) \supset \cdots \supset \mathcal{U}_{\gamma_n}(\tau)$ and the identities $\mathcal{S}_{\gamma_i}(\tau) + \mathcal{U}_{\gamma_i}(\tau) = \mathbf{R}^N$ for $i = 0, \ldots, n$. Therefore $\mathbf{R}^N = \mathcal{W}_0(\tau) + \mathcal{U}_{\gamma_0}(\tau)$. Now using the algebraic lemma for $n \geqslant 1$, one has

$$\begin{aligned} \mathbf{R}^N &= \mathcal{W}_0(\tau) + \mathcal{U}_{\gamma_0}(\tau) \cap [\mathcal{S}_{\gamma_1}(\tau) + \mathcal{U}_{\gamma_1}(\tau)] \\ &= \mathcal{W}_0(\tau) + [\mathcal{U}_{\gamma_0}(\tau) \cap \mathcal{S}_{\gamma_1}(\tau)] + \mathcal{U}_{\gamma_1}(\tau) \\ &= \mathcal{W}_0(\tau) + \mathcal{W}_1(\tau) + \mathcal{U}_{\gamma_1}(\tau) \end{aligned}$$

Doing the same for $\mathcal{U}_{\gamma_1}(\tau)$, we get

$$\begin{aligned} \mathbf{R}^N &= \mathcal{W}_0(\tau) + \mathcal{W}_1(\tau) + \mathcal{U}_{\gamma_1}(\tau) \cap [\mathcal{S}_{\gamma_2}(\tau) + \mathcal{U}_{\gamma_2}(\tau)] \\ &= \mathcal{W}_0(\tau) + \mathcal{W}_1(\tau) + [\mathcal{U}_{\gamma_1}(\tau) \cap \mathcal{S}_{\gamma_2}(\tau)] + \mathcal{U}_{\gamma_2}(\tau) \\ &= \mathcal{W}_0(\tau) + \mathcal{W}_1(\tau) + \mathcal{W}_2(\tau) + \mathcal{U}_{\gamma_2}(\tau) \end{aligned}$$

and mathematical induction yields $\mathbf{R}^N = \mathcal{W}_0(\tau) + \cdots + \mathcal{W}_{n+1}(\tau)$.

To finish the proof, let also $\hat{\gamma}_0, \ldots, \hat{\gamma}_n \in \rho(A)$ be given with properties (5), (6) and (7). Then alternative I of Lemma 3.2 implies

$$\mathcal{S}_{\gamma_i} = \mathcal{S}_{\hat{\gamma}_i} \qquad \text{and} \qquad \mathcal{U}_{\gamma_i} = \mathcal{U}_{\hat{\gamma}_i} \qquad \text{for} \quad i = 0, \ldots, n$$

and therefore the linear integral manifolds $\mathscr{W}_0, \ldots, \mathscr{W}_{n+1}$ are independent of the choice of $\gamma_0, \ldots, \gamma_n$ in (5), (6) and (7). $\square$

Remark 3.1 (Robustness). A well-known perturbation result (see, e.g., Coppel [7, Proposition 4.1, p. 34]) implies that for each sufficiently small $\varepsilon > 0$ and each γ in the resolvent set $\rho(A)$ there exists a $\delta = \delta(\varepsilon, \gamma) > 0$ such that for each locally integrable matrix function $B: \mathbf{R} \to \mathbf{R}^{N \times N}$ with $\sup_{t \in \mathbf{R}} \|B(t)\| < \delta$ the resolvent set $\rho(A+B)$ of the perturbed system $\dot{x} = [A(t) + B(t)]\, x$ also contains the value γ. For systems (1) with compact hull (see Remark 3.2) Pliss and Sell [10] prove that the dichotomy constants and the invariant projector depend continuously on the perturbation. Nevertheless in general perturbations can "melt together" two nearby spectral intervals as well as "break up" a spectral interval into two new intervals.

3.1. Bounded Growth

We follow Coppel [7] and will say that (1) has *bounded growth* if there exist constants $K \geqslant 1$ and $a \geqslant 0$ such that

$$\|\Phi(t, s)\| \leqslant K e^{a|t-s|} \qquad \text{for} \quad t, s \in \mathbf{R} \tag{8}$$

Note that (1) has bounded growth if and only if each solution of (1) depends *uniformly continuous* on initial conditions, i.e., for each $h > 0$ and $\varepsilon > 0$ there is a corresponding $\delta = \delta(h, \varepsilon) > 0$ such that for $\xi, \eta \in \mathbf{R}^N$

$$\|\xi - \eta\| < \delta \Rightarrow \|\Phi(t, \tau)\, \xi - \Phi(t, \tau)\, \eta\| < \varepsilon \quad \text{for all } t, \tau \in \mathbf{R} \quad \text{with } |t - \tau| < h$$

Theorem 3.1. *The following statements are equivalent:*

(A) The linear system (1) has bounded growth.

(B) The linear system (1) has a nonempty and compact dichotomy spectrum $\Sigma(A) = [a_1, b_1] \cup \cdots \cup [a_n, b_n]$ where $1 \leqslant n \leqslant N$ and the spectral manifolds $\mathscr{W}_0$ and $\mathscr{W}_{n+1}$ are trivial, i.e., $\mathscr{W}_1 \oplus \cdots \oplus \mathscr{W}_n = \mathbf{R} \times \mathbf{R}^N$.

Proof. (A) $\Rightarrow$ (B). Assume that (8) holds. Let $\gamma > a$. With the definition $\alpha := \gamma - a > 0$ the estimate (8) implies

$$\|\Phi_\gamma(t, s)\| \leqslant K e^{-\alpha(t-s)} \qquad \text{for} \quad t \geqslant s$$

and therefore $\dot{x} = [A(t) - \gamma I]\, x$ admits an ED with invariant projector $P = I$. We have $\gamma \in \rho(A)$ and similary for $\gamma < -a$, therefore $\Sigma(A) \subset [-a, a]$, i.e., the dichotomy spectrum is bounded. Additionally Lemma 2.1 implies

$$\mathcal{S}_\gamma = \mathbf{R} \times \mathbf{R}^N \quad \text{and} \quad \mathcal{U}_\gamma = \mathbf{R} \times \{0\} \quad \text{for} \quad \gamma > a$$
$$\mathcal{S}_\gamma = \mathbf{R} \times \{0\} \quad \text{and} \quad \mathcal{U}_\gamma = \mathbf{R} \times \mathbf{R}^N \quad \text{for} \quad \gamma < -a$$

i.e., $\mathcal{W}_0 = \mathcal{W}_{n+1} = \mathbf{R} \times \{0\}$. To show that $\Sigma(A)$ is nonempty, define

$$\gamma_0 := \inf\{\gamma \in \rho(A) : \mathcal{S}_\gamma = \mathbf{R} \times \mathbf{R}^N\}$$

it follows $\gamma_0 \in [-a, a]$. Arguing negatively, let us assume now that $\gamma_0 \in \rho(A)$. There are two cases to consider: (i) $\mathcal{S}_{\gamma_0} = \mathbf{R} \times \mathbf{R}^N$ or (ii) $\mathcal{S}_{\gamma_0} \neq \mathbf{R} \times \mathbf{R}^N$. In case (i) Lemma 3.1 implies $\mathcal{S}_\gamma = \mathbf{R} \times \mathbf{R}^N$ for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$ with some $\varepsilon > 0$, which contradicts the definition of γ_0. In case (ii) Lemma 3.1 implies $\mathcal{S}_\gamma \neq \mathbf{R} \times \mathbf{R}^N$ for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$, which also contradicts the definition of γ_0. Hence $\gamma_0 \in \Sigma(A) \neq \emptyset$.

(B) $\Rightarrow$ (A). Choose a collection of points $\gamma_0, \gamma_1, \ldots, \gamma_n \in \rho(A)$ such that

$$\gamma_0 < a_1 \leqslant b_1 < \gamma_1 < \cdots < \gamma_{n-1} < a_n \leqslant b_n < \gamma_n$$

Monotonicity implies the inclusion $\mathcal{W}_i = \mathcal{U}_{\gamma_{i-1}} \cap \mathcal{S}_{\gamma_i} \subset \mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_n}$ for $i = 1, \ldots, n$ and therefore $\mathcal{W}_1 + \cdots + \mathcal{W}_n \subset \mathcal{U}_{\gamma_0} \cap \mathcal{S}_{\gamma_n}$. Since by assumption $\mathcal{W}_1 + \cdots + \mathcal{W}_n = \mathbf{R} \times \mathbf{R}^N$, one has

$$\mathcal{U}_{\gamma_0}(\tau) = \mathbf{R}^N \quad \text{and} \quad \mathcal{S}_{\gamma_n}(\tau) = \mathbf{R}^N \quad \text{for all} \quad \tau \in \mathbf{R}$$

It then follows from Lemma 2.1 that $\dot{x} = [A(t) - \gamma_0 I]\, x$ admits an ED with constants $K_1 \geqslant 1$, $\alpha_1 > 0$ and invariant projector $P \equiv 0$, i.e., the dichotomy estimate

$$\|\Phi(t, s)\| \leqslant K_1 e^{(\gamma_0 + \alpha_1)(t-s)} \quad \text{for} \quad t \leqslant s$$

holds. Also $\dot{x} = [A(t) - \gamma_n I]\, x$ admits an ED with constants $K_2 \geqslant 1$, $\alpha_2 > 0$ and invariant projector $P \equiv I$ and one has

$$\|\Phi(t, s)\| \leqslant K_2 e^{(\gamma_n - \alpha_2)(t-s)} \quad \text{for} \quad t \geqslant s$$

Combining these two estimates with $K := \max\{K_1, K_2\}$ and $a := \max\{0, -\gamma_0 - \alpha_1, \gamma_n - \alpha_2\}$ we get $\|\Phi(t, s)\| \leqslant K e^{a|t-s|}$ for $t, s \in \mathbf{R}$. $\square$

Remark 3.2 (Dichotomy spectrum and Sacker–Sell spectrum). In a now famous paper [14] Sacker and Sell introduced a spectral theory for linear skew product flows with compact base. To apply the Sacker–Sell theory to the linear skew product flow

$$\pi\colon \mathbf{R}\times H(A)\times \mathbf{R}^N \to H(A)\times \mathbf{R}^N, \qquad \pi(t, A, x) = (\sigma(t, A), \phi(t, A)\, x)$$

which is associated to the linear system (1), the hull $H(A)$ has to be compact and this is the case if and only if (G. R. Sell [16, Thm. III.12, p. 44])

(i) there is a b in $\mathbf{R}$ such that $\int_0^1 |A(s+t)|\, ds \leqslant b$ for all $t \in \mathbf{R}$,

(ii) for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\int_0^1 |A(s+t+h) - A(s+t)|\, ds \leqslant \varepsilon$$

whenever $|h| \leqslant \delta$ and $t \in \mathbf{R}$.

If L^1_{loc} is replaced by the space of almost periodic A or bounded and uniformly continuous A or essentially bounded A then the hull is compact with the uniform topology or the compact-open topology (see [14]) or the weak*-topology (see [6]), respectively. At this point our Spectral Theorem shows that the search for the adequate topology of $H(A)$ is not necessary. On the other hand, if π is a linear skew product flow with compact base $H(A)$, then the Sacker–Sell spectrum and the dichotomy spectrum coincide and the Sacker–Sell theory implies the statement (B) of Lemma 3.1. Therefore the equivalent statement (A) is also valid, i.e., (1) has bounded growth. In other words: The Sacker–Sell theory is applicable to equations (1) with bounded growth, if one knows an adequate topology for $H(A)$. However, there are systems (1) with bounded growth for which such a topology does not exist, for example $\dot{x} = 2t\cos(t^2)\, x$ has bounded growth. Assume that π is the associated linear skew product flow with compact base $H(A)$. Then for $\sigma(s, A) = 2(\cdot + s)\cos(\cdot + s)^2 \in H(A)$ one has $\phi(t, \sigma(s, A))\, x = \exp(\sin (s+t)^2 - \sin s^2)\, x$ and by continuity of π and compactness of $H(A)$ there exists an $\varepsilon > 0$ such that

$$|\phi(t, \sigma(s, A))| < 2 \qquad \text{for} \quad |t| < \varepsilon \qquad \text{and all} \quad s \in \mathbf{R}$$

Choosing $k \in \mathbf{N}$ with $t := \sqrt{\pi/2 + 2k\pi} - \sqrt{-\pi/2 + 2k\pi} < \varepsilon$ and $s := \sqrt{-\pi/2 + 2k\pi}$ this yields the contradiction

$$\exp(2) = \exp(\sin(s+t)^2 - \sin s^2) < 2$$

We used the ideas of the Sacker–Sell theory to prove directly our Spectral Theorem for linear systems (1). Although the base $H(A) \subset L^1_{\text{loc}}$ of the associated linear skew product flow is in general not compact one can show along the lines of [11, proof of Thm. 5, p. 453] that $\Sigma(A)=\Sigma(B)$ for every $B\in H(A)$. This indicates a possible generalization of the Sacker–Sell theory to linear skew product flows where the base space is note compact, thereby yielding an analogous Spectral Theorem for the associated skew product flow π.

3.2. Scalar Differential Equations

For a scalar linear differential equation $\dot{x}=A(t)\,x$ with $A\colon \mathbf{R}\to\mathbf{R}$ locally integrable, the evolution operator can be given explicitely,

$$\Phi(t,s)=\exp\left(\int_s^t A(\tau)\,d\tau\right)\qquad\text{for}\quad t,s\in\mathbf{R}$$

Now we give examples to show that each alternative of the Spectral Theorem really occurs (we could also give C^∞ examples).

(a) $\Sigma(A)=\varnothing$ for $A(t)=|t|$. Choose and fix an arbitrary $\gamma\in\mathbf{R}$ and $\alpha>0$. With the compact set $M_{\gamma+\alpha}:=\{\tau\in\mathbf{R}:A(\tau)\leqslant\gamma+\alpha\}$ one has the following estimate for $t\leqslant s$:

$$\begin{aligned}&\int_s^t[A(\tau)-(\gamma+\alpha)]\,d\tau\\&\quad=\int_{[t,s]\cap M_{\gamma+\alpha}}[\gamma+\alpha-A(\tau)]\,d\tau+\int_{[t,s]\cap M^c_{\gamma+\alpha}}[\gamma+\alpha-A(\tau)]\,d\tau\\&\quad\leqslant\int_{M_{\gamma+\alpha}}[\gamma+\alpha-A(\tau)]\,d\tau=:c\end{aligned}$$

Since $e^{-(\gamma+\alpha)(t-s)}\Phi(t,s)=\exp(\int_s^t[A(\tau)-(\gamma+\alpha)]\,d\tau)$, it follows that

$$|\Phi(t,s)|\leqslant Ke^{(\gamma+\alpha)(t-s)}\qquad\text{for}\quad t\leqslant s$$

where $K=e^c$ and this shows that $\dot{x}=[A(t)-\gamma]\,x$ admits an ED with projector $P\equiv 0$, i.e., $\gamma\in\rho(A)$.

(b) $\Sigma(A)=\mathbf{R}$ for $A(t)=t$. Arguing negatively, let us assume that there exists a $\gamma\in\mathbf{R}$ such that $\dot{x}=[A(t)-\gamma]\,x$ admits an ED. Recall that the invariant projector P is continuous and $P(t)\in\mathbf{R}$ is a projection, so either $P\equiv 0$ or $P\equiv 1$. If $P\equiv 0$ the dichotomy estimate

$$e^{\frac{1}{2}t^2-\frac{1}{2}s^2}\leqslant Ke^{(\gamma+\alpha)(t-s)}\qquad\text{for}\quad t\leqslant s$$

yields a contradiction for $s=0$ and $t \to -\infty$ and analogously for $P \equiv 1$. With similar arguments the following dichotomy spectra are calculated.

(c) $\Sigma(A) = (-\infty, b]$ for $A(t) = \begin{cases} b+t & \text{for } t \leqslant 0 \\ b & \text{for } t \geqslant 0 \end{cases}$, where $b \in \mathbf{R}$

(d) $\Sigma(A) = [a, \infty)$ for $A(t) = \begin{cases} a & \text{for } t \leqslant 0 \\ a+t & \text{for } t \geqslant 0 \end{cases}$, where $a \in \mathbf{R}$

(e) $\Sigma(A) = [a, b]$ for $A(t) = \begin{cases} a & \text{for } t \leqslant 0 \\ b & \text{for } t > 0 \end{cases}$, where $a, b \in \mathbf{R}$, $a \leqslant b$

One can easily see that a scalar differential equation $\dot{x} = A(t)\, x$ has bounded growth if and only if

$$\int_s^t A(\tau)\, d\tau \leqslant c + d\,|t-s| \qquad \text{for} \quad t, s \in \mathbf{R}$$

with constants c, $d \geqslant 0$. An interesting unsolved problem in this context is to determine a characterization of bounded growth (in terms of the linear part A) in higher dimensions ($N \geqslant 2$). Note that the scalar equation $\dot{x} = 2t\cos(t^2)\, x$ has bounded growth and one-point spectrum $\Sigma(A) = \{0\}$ although the linear part $A(t) = 2t \cos t^2$ is unbounded. The class of scalar differential equations with bounded growth is stable under addition, i.e., if $\dot{x} = A_i(t)\, x$, $i = 1, 2$, are scalar differential equations with bounded growth then the equation $\dot{x} = [A_1(t) + A_2(t)]\, x$ has bounded growth as well. This is due to the fact, that for a scalar equation the evolution operator of the sum is the product of the evolution operators of the first and second system. It is an unanswered question if the class of higher dimensional systems (1) with bounded growth is stable under addition.

ACKNOWLEDGMENTS

The author thanks George R. Sell for his comments leading to an improvement of the paper.

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Printed in Belgium