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Reduction of Second Order Unilateral Singular Systems. Applications in Mechanics

The aim of this paper is to discuss the mathematical strategies permitting the treatment of second order unilateral systems involving singular mass, damping, and stiffness matrices. Reduction methods are used here to transform second order differential inclusions in first order ones and classical results on differential inclusions are considered in order to obtain solutions. Friction and impact problems arising in Unilateral Mechanics are studied so as to illustrate the theoretical approach.

Key words: unilateral problems in mechanics, friction Coulomb's law, unilateral constraints, impact laws, differential inclusions, variational inequalities, reduction methods, matrix analysis, difference methods for differential inclusions, nonsmooth damped spring-mass systems

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1. Introduction

Many mechanisms consist of parts that can be considered as perfectly rigid bodies. Some of these parts may come into contact or separate from each other, however they do not penetrate each other. That means that forces of constraints or reaction need to be included in the formulation of mathematical models for such processes. The use of forces of constraints to specify the contact phenomena introduces serious mathematical difficulties into these models. This is the reason why the place of unilateral constraints received in publications in classical mechanics is very modest in comparison with the abundance of unilateral constraints in engineering systems.

Using modern tools of convex analysis, MOREAU [16], [17] has recently proposed rigorous mathematical expressions of normal contact laws, Coulomb's friction law, shock laws, etc., which lead to differential inclusions.

A second order differential inclusion model can be formulated as follows:

Find $q\colon [0,T] \longrightarrow \mathbb{R}^N$, $t \longmapsto q(t)$ such that

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t)) , \quad \text{a.e.} \quad t \in (0,T) , \tag{1.1}$$

where M is the mass matrix, C is the damping matrix, K is the stiffness matrix, $f\colon [0,T] \longrightarrow \mathbb{R}^N$ is a vector-valued function related to the given forces acting on the system, and $F\colon [0,T] \times \mathbb{R}^N \longrightarrow \mathcal{P}(\mathbb{R}^N)$ is a set-valued function, i.e. a function from $[0,T] \times \mathbb{R}^N$ onto the set $\mathcal{P}(\mathbb{R}^N)$ of all subsets of $\mathbb{R}^N$, that defines, for each $t \in [0,T]$, a graph in $\mathbb{R}^N$ used to express the unilateral reaction forces. Usual initial conditions such as

$$q(0) = q_0 , \qquad \dot{q}(0) = q_1 ,$$

and impact laws (provided that the system under consideration involves rigid body collisions) are generally introduced to complete the formulation of the model.

Until now only some special cases of second order differential inclusions have been studied. See for example the works of CHOLET [5], FRÉMOND [10], MARQUES [15], and MOREAU [16], [17]. However, various problems formulated as in (1.1) cannot be studied using the current theoretical results in the mathematical literature. In particular, if the matrices M, K, and C involved in the model (1.1) are singular, then most of the known results do not apply.

However, singularities occur frequently in models of the dynamics of multi-body systems. Indeed, most problems in Mechanics are formulated in terms of parameters $q_1, \ldots, q_N$ making the element q appearing in (1.1). The formulation of usual damped spring-mass systems lead to the so-called mass matrices, damping matrices, and stiffness matrices that are in parts determined by the physical characteristics of the rigid bodies, springs, and dampers involved in the system. This part of the matrix formulation of the whole model is rarely the cause of singularities. However, a whole model may encompass the matrix formulation of bilateral constraints whose geometric or kinematical effects are expressed by equalities of the form

$$A_1\dot{q} + A_2 q = b ,$$

with A_1, $A_2 \in \mathbb{R}^{M\times N}$ $(M < N)$, $b \in \mathbb{R}^M$. This last first order system introduces a zero matrix block in the whole mass matrix that can be the cause of the singularities of this last one (see e.g. [3]).

The use of zero-mass points for example to denote a connection between two springs or two dampers may also be the cause of singularities in the resulting models (see e.g. [1]). The treatment of some forces like reaction forces or friction forces as unknowns of the problem may also introduce singularities in the whole model. Problems of the form (1.1) with singular matrices M, C, and K, and $F \equiv 0$ occur also in formulating the dynamic of various problems in Economics.

The aim of this paper is to discuss the mathematical strategies permitting the treatment of the model (1.1) for matrices M, C, and K allowed to be singular. In particular, we develop reduction techniques in such a way that the resulting reduced model can be studied by means of standard arguments. More precisely, the main goal of this paper is to develop reduction model strategies so as to rewrite second order singular differential unilateral systems in the following first order form:

$$\begin{aligned} &\dot{y}(t) \in \Phi(t, y(t)), \quad \text{a.e.} \quad t \in (0, T), \\ &y(0) = y_0, \end{aligned} \tag{1.2}$$

where $y_0 \in \mathbb{R}^n$ and Φ is a map from $[0, T] \times \mathbb{R}^n$ into the set of all subsets of $\mathbb{R}^n$.

Differential inclusions of this type have been the subject of many papers and, for more details, we refer the reader to the book of Filippov [9], the survey of Dontchev and Lempio [7] and the paper of Lempio and Veliov [14]. The following theorem (see [7], [19], and [9]) gives sufficient conditions for existence.

Theorem 1.1: *Suppose that Φ satisfies the conditions*

(i) *Φ is nonempty, compact, and convex-valued on $[0, T] \times \mathbb{R}^n$,*
(ii) *$\Phi(t, \cdot)$ is upper semicontinuous, for all $t \in [0, T]$,*
(iii) *$\Phi(\cdot\,, x)$ is measurable, for all $x \in \mathbb{R}^n$,*
(iv) *there exist constants k_1 and k_2 such that*

$$\|z\| \le k_1 \|x\| + k_2, \qquad \forall z \in \Phi(t, x), \qquad x \in \mathbb{R}^n, \qquad t \in [0, T].$$

Then Problem (1.2) *has at least one solution, i.e. an absolutely continuous function y that satisfies* (1.2).

If, in addition, the map Φ possesses the decomposition

$$\Phi(t, x) = \theta(t, x) - \beta(x),$$

where $\theta\colon [0, T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a single-valued function satisfying the one-sided Lipschitz condition

$$(\theta(t, x_1) - \theta(t, x_2))^T \, (x_1 - x_2) \le L \|x_1 - x_2\|^2,$$

uniformly for all $t \in [0, T]$ and $\beta\colon \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ is a monotone set-valued mapping, then the solution of Problem (1.2) *is necessarily unique.*

Models as the one formulated in (1.2) can be solved by means of appropriate set-valued version of classical difference methods like Euler's method and 4-stage Runge-Kutta method.

The Euler Method is here outlined. For $q \in N \backslash \{0\}$, a grid $0 = t_0 < t_1 < \ldots < t_q = T$ is chosen with stepsize $h = (T - t_0)/q = t_j - t_{j-1}$ $(j = 1, \ldots, q)$. Let

$$\eta_0 = y_0,$$

and, for $j = 0, \ldots, q - 1$, the vector η_{j+1} is computed by the formula

$$\eta_{j+1} \in \eta_j + h\Phi(t_j, \eta_j). \tag{1.3}$$

Then one sets

$$\eta^q(t) = \eta_j + \frac{1}{h}\,(t - t_j)\,(\eta_{j+1} - \eta_j), \tag{1.4}$$

for $t_j \le t \le t_{j+1}$, $j = 0, \ldots, q - 1$. The piecewise linear function η^q yields an approximation of the solution of Problem (1.2).

A 4-Stage Runge-Kutta scheme can be outlined in a similar way:

$$\begin{aligned} &\eta_0 = y_0, \\ &\eta_{j+1} = \eta_j + \frac{h}{6}\,(k_{j1} + 2k_{j2} + 2k_{j3} + k_{j4}), \end{aligned} \tag{1.5}$$

with

$$\begin{aligned} &k_{j1} \in \Phi(t_j, \eta_j), \qquad k_{j2} \in \Phi\left(t_j + \frac{h}{2}, \eta_j + \frac{k_{j1}}{2}\right), \qquad k_{j3} \in \Phi\left(t_j + \frac{h}{2}, \eta_j + \frac{k_{j2}}{2}\right), \\ &k_{j4} \in \Phi(t_j + h, \eta_j + k_{j3}). \end{aligned} \tag{1.6}$$

The convergence properties of these schemes are overviewed in [7]. Various problems in Unilateral Mechanics have been recently treated in [19].

Theorem 1.1 can be applied to study a great variety of models of unilateral phenomena like dry friction, debonding effects and delamination effects. However, the assumptions required on Φ are too strong to encompass frictionless normal contact laws expressing non-penetration constraints and reactions. Indeed, simple dynamic models or reduced dynamic models involving such unilateral constraints are generally governed by a system of differential inclusions of the type

$$\ddot{q} \in f(t, q, \dot{q}) + \partial\psi_K(q)\,, \tag{1.6}$$

where $f\colon [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a single-valued function and $\partial\psi_K$ denotes the convex subdifferential of the indicator function of some nonempty closed convex set $K \subset \mathbb{R}^N$ defined by the geometric constraints imposed on q. Impact laws are also usually considered so as to complete the formulation of the problem in consideration. In practical situations, the set K can also be nonconvex, but the theory we use here applies only in the convex case.

The differential inclusion (1.6) reduces to (1.2) by setting $y = (q \;\; \dot{q})^T$, $n = 2N$, and

$$\boldsymbol{\Phi}(t,y) = (y_2 \;\; f(t,y_1,y_2) + \partial\psi_K(y_1))^T\,.$$

It is clear that $\boldsymbol{\Phi}$ does not satisfy the sublinear growth condition (iv) in Theorem 1.1. We have indeed

$$\partial\psi_K(x) = N_K(x)\,,$$

where $N_K(x)$ denotes the normal cone of K at x, that is

$$N_K(x) = \{w \in \mathbb{R}^N : w^T z \le 0,\ \forall z \in T_K(x)\}\,,$$

where

$$T_K(x) = \overline{\bigcup_{\lambda>0} \lambda(K-x)}\,.$$

In the following, we remind a result proved by PAOLI and SCHATZMAN [18]. It gives a weak solution to Problem (1.6) coupled with the impact law

$$\dot{q}(t_+) = -e\dot{q}_N(t_-) + \dot{q}_T(t_-)\,, \qquad \forall t \in [0,T] : q(t) \in \partial K\,,$$

where $\dot{q}_N$ and $\dot{q}_T$ denote the projection of $\dot{q}$ onto $\mathbb{R}\nu(q(t))$ and $(\mathbb{R}\nu(q(t)))^\perp$, respectively. The parameter e is called the recovery coefficient.

Theorem 1.2: *Let K be a closed convex subset of $\mathbb{R}^N$ with nonempty interior and a regular boundary ∂K of class C^2 in the sense that there exists a unique mapping $\nu\colon \partial K \longrightarrow \mathbb{R}^N$ of class C^1 such that*

$$N_K(x) = \mathbb{R}_+\nu(x)\,, \qquad \forall x \in \partial K\,.$$

Let

(i) *$f\colon [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ continuous,*
(ii) *$f(t,\cdot,\cdot)$ Lipschitz continuous for all $t \in [0,T]$.*

Let also $q_0 \in K$, $q_1 \in T_K(q_0)$, and $e \in (0,1]$ be given. Then there exists $q\colon [0,T] \longrightarrow \mathbb{R}^N$ Lipschitz continuous such that

(a) *$\dot{q}$ has bounded variations,*
(b) *$q(0) = q_0$,*
(c) *$\dot{q}(0_+) = q_1$,*
(d) *$q(t) \in K$, $\forall t \in [0,T]$,*
(e) *$\langle \ddot{q} - f(t,q,\dot{q}),\, \varphi - q\rangle \ge 0$, $\forall \varphi \in C^0([0,T];K)$,*
(f) *$\dot{q}(t_+) = -e\dot{q}_N(t_-) + \dot{q}_T(t_-)$, $\forall t \in [0,T]$ such that $q(t) \in \partial K$.*

Note that the expression $\langle \ddot{q} - f(t,q,\dot{q}),\, \varphi - q\rangle \ge 0$ (considered in the sense of distributions) constitutes a weak formulation for (1.6).

The solution outlined in Theorem 1.2 is obtained as the limit (in $W^{1,p}(0,T;\mathbb{R}^N)$, for all $p \in [1,+\infty[$) when $\lambda \to 0$ of a subsequence of the solutions of the system

$$\begin{aligned}
&\ddot{q}_\lambda + \frac{2}{\sqrt{\lambda}}\left(\frac{\ln(1/e)}{\sqrt{\pi^2 + (\ln(e))^2}}\right) G(q_\lambda - P_K(q_\lambda), \dot{q}_\lambda) + \frac{q_\lambda - P_K(q_\lambda)}{\lambda} = f(t, q_\lambda, \dot{q}_\lambda)\,,\\
&q_\lambda(0) = q_0\,,\\
&\dot{q}_\lambda(0) = q_1\,,
\end{aligned} \tag{1.7}$$

where

$$G(u,v) = \begin{cases} (u^T v)\; u/|u|^2 & \text{if} \quad u \ne 0\,,\\ 0 & \text{if} \quad u = 0\,,\end{cases}$$

and P_K is the projection map on K. Note also here that the operator $(u - P_K u)/\lambda$ is the Yosida approximant of the operator $\partial\psi_K$.

In the case of elastic impact, i.e. $e = 1$, the first equation in (1.7) reduces to

$$\ddot{q}_\lambda + \frac{q_\lambda - P_K(q_\lambda)}{\lambda} = f(t, q_\lambda, \dot{q}_\lambda) \,. \tag{1.8}$$

This case will be considered later in this paper.

In this paper, three interesting engineering problems are revisited by means of advanced mathematical tools in unilateral analysis. We show how matrix-reduction methods can be used together with Theorem 1.1, Theorem 1.2, and the difference methods outlined in this introduction to provide a complete analysis of these problems.

Jordan's reduction method for first order differential inclusions is outlined in Section 2 and used in Section 5 to discuss a problem in biomechanics. In Section 3, we discuss a mathematical condition that is required to specialize the result of Section 2 to the second order differential inclusions. Model reduction techniques are studied in Section 4. The results proved in [1] are established here in a more general framework. Symmetry and positive semi-definite properties assumed in [1] on the involved matrices have here been relaxed. The generalized results developed in Section 4 are of particular interest to reduce problems in Economics because they often do not present the "nice" properties encountered in most problems of Mechanics (see [8]). Limitated in space, problems in Economics are not discussed in this paper. The theoretical results discussed here are however illustrated in this sense in [8]. Note also that not-necessarily symmetric matrices appear in the mathematical formulation of electric power systems [2]. In Section 5, we present some examples of mechanical systems involving rigid bodies subject to unilateral constraints. These examples illustrate the methodology developed in this paper to study unilateral problems in Mechanics.

2. Jordan's reduction of first order differential inclusions

Let us first discuss the first order model:

Find

$$x\colon [0,T] \longrightarrow \mathbb{R}^n \,, \qquad t \longmapsto x(t) \,,$$

such that

$$E\dot{x}(t) \in Ax(t) + H(t) + G(t, x(t)) \,, \quad \text{a.e.} \quad t \in (0,T) \,, \tag{2.1}$$

where E, $A \in \mathbb{R}^{n\times n}$ are singular matrices, $H\colon [0,T] \longrightarrow \mathbb{R}^n$ is a given vector-valued function and $G\colon [0,T] \times \mathbb{R}^n \longrightarrow \mathcal{P}(\mathbb{R}^n)$ denotes a set-valued function. Together with (2.1), we may consider some initial condition like

$$x(0) = c \,. \tag{2.2}$$

Definition 2.1: One says that the pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ forms a regular matrix pencil provided that there exists $\lambda \in \mathbb{R}$ such that

$$\operatorname{rank}(\lambda E - A) = n \,.$$

Remark 2.2: Assume that $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is a regular matrix pencil. If

$$G(t, y) = -\partial_y \Psi(t, y) \,, \qquad \forall t \in [0,T] \,, \qquad y \in \mathbb{R}^n \,,$$

where $\Psi\colon [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous functional in the second variable for any $t \in [0,T]$ and ∂_y denotes the convex subdifferential operator with respect to the second variable, then (2.1) is equivalent to the variational inequality

$$\left(E\dot{x}(t) - Ax(t) - H(t)\right)^T (v - x(t)) + \Psi(t, v) - \Psi(t, x(t)) \geq 0 \,, \qquad \forall v \in \mathbb{R}^n \,.$$

Setting $x = \mathrm{e}^{\lambda t}\, u$, we see that (2.1)–(2.2) reduces to

$$\begin{aligned} &E\dot{u}(t) \in (A - \lambda E)\, u(t) + \mathrm{e}^{-\lambda t}\, H(t) + \mathrm{e}^{-\lambda t}\, G(t, \mathrm{e}^{\lambda t}\, u(t)) \,, \\ &u(0) = c \,. \end{aligned} \tag{2.3}$$

The matrix $\lambda E - A$ is regular and the matrix $\hat{E}_\lambda = (\lambda E - A)^{-1}\, E$ is well defined. The relation (2.3) may be formulated as

$$\begin{aligned} &-\hat{E}_\lambda \dot{u}(t) = u(t) + \bar{h}(t) + \bar{g}(t) \,, \quad \text{a.e.} \quad t \in (0,T) \,, \\ &u(0) = c \,, \end{aligned} \tag{2.4}$$

where

$$\bar{h}(t) = (A - \lambda E)^{-1}\, \mathrm{e}^{-\lambda t}\, H(t) \,,$$

and

$$\bar{g}(t) \in (A - \lambda E)^{-1} \, \mathrm{e}^{-\lambda t} \, G(t, \mathrm{e}^{\lambda t} \, u(t)) \, .$$

Set $p := \dim(\ker(\hat{E}_\lambda))$. The Jordan form J of the singular matrix $\hat{E}_\lambda = TJT^{-1}$ has the structure

$$J = \begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix},$$

where $W \in \mathbb{R}^{(n-p)\times(n-p)}$ contains all the Jordan blocks corresponding to the nonzero eigenvalues of $\hat{E}_\lambda$ and $N \in \mathbb{R}^{p\times p}$ is nilpotent of order $k \le p$ (see e.g. [4]). From (2.4) we deduce that

$$\begin{aligned} &-JT^{-1}\dot{u} = T^{-1}u + T^{-1}\bar{h} + T^{-1}\bar{g} \, , \\ &u(0) = c \, . \end{aligned}$$

Setting $v = T^{-1}u$, we get

$$\begin{aligned} &-J\dot{v} = v + h + g \, , \\ &v(0) = T^{-1}c \, , \end{aligned}$$

with

$$g(t) \in T^{-1}(A - \lambda E)^{-1} \, \mathrm{e}^{-\lambda t} \, G(t, \mathrm{e}^{\lambda t} \, Tv(t)) \, ,$$

and

$$h(t) = T^{-1}(A - \lambda E)^{-1} \, \mathrm{e}^{-\lambda t} \, H(t) \, .$$

Let us now define the rectangular block matrices

$$\begin{aligned} &\Lambda_1 = (I_{(n-p)\times(n-p)} \mid 0_{(n-p)\times p}) \, , \\ &\Lambda_2 = (0_{p\times(n-p)} \mid I_{p\times p}) \, , \end{aligned}$$

and set

$$v_i = \Lambda_i v \, , \qquad h_i = \Lambda_i h \, , \qquad g_i = \Lambda_i g \quad \text{and} \quad c_i = \Lambda_i T^{-1} c \, , \qquad i = 1, 2 \, .$$

We see that

$$\begin{aligned} &-W\dot{v}_1 = v_1 + h_1 + g_1 \, , \qquad -N\dot{v}_2 = v_2 + h_2 + g_2 \, , \\ &v_1(0) = c_1 \, , \qquad v_2(0) = c_2 \, . \end{aligned} \tag{2.5}$$

System (2.5) is coupled through the relation

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in T^{-1}(A - \lambda E)^{-1} \, \mathrm{e}^{-\lambda t} \, G\left(t, \mathrm{e}^{\lambda t} \, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right).$$

If $G \equiv 0$ then System (2.5) is decoupled since in this case, it reduces to

$$\begin{aligned} &\dot{v}_1 = -W^{-1}v_1 - W^{-1}h_1 \, , \\ &v_1(0) = c_1 \, , \end{aligned} \tag{2.6}$$

$$\begin{aligned} &N\dot{v}_2 = -v_2 - h_2 \, , \\ &v_2(0) = c_2 \, . \end{aligned} \tag{2.7}$$

System (2.6) can be solved by means of standard methods. Remind that

$$v_1(t) = \mathrm{e}^{-W^{-1}t} \, c_1 - \int_0^t \mathrm{e}^{-W^{-1}(t-\tau)} \, W^{-1} h_1(\tau) \, d\tau \, ,$$

provided that h_1 is continuous on $[0, T]$. Under the additional smoothness assumption, that h_2 is of class C^{k-1} on $[0, T]$, the system (2.7) is solved by

$$v_2(t) = -h_2(t) + N\dot{h}_2(t) - N^2\ddot{h}_2(t) + \ldots + (-1)^k \, N^{k-1} h_2^{(k-1)}(t) \, .$$

That means that the initial condition of the initial value c cannot be chosen arbitrarily since $c_2 = \Lambda_2 c$ is determined by $h_2 = \Lambda_2 h$ and its derivatives $\dot{h}_2, \ldots, h_2^{(k-1)}$. The initial condition needs indeed to satisfy the consistency condition

$$\Lambda_2 T^{-1} c = -h_2(0) + N\dot{h}_2(0) + \ldots + (-1)^k \, N^{k-1} h_2^{(k-1)}(0) \, . \tag{2.8}$$

Remark 2.3: If $h_2 = 0$, then it is clear that if $c \in \mathrm{R}(\hat{E}_\lambda^k)$ then the consistency condition (2.8) is satisfied. Indeed

$$\hat{E}_\lambda^k = T \begin{pmatrix} W^k & 0 \\ 0 & 0 \end{pmatrix} T^{-1},$$

and if $c \in \mathrm{R}(\hat{E}_\lambda^k)$, then $c = \hat{E}_\lambda^k x$ for some $x \in \mathbb{R}^n$ and clearly

$$\Lambda_2 T^{-1} c = \Lambda_2 \begin{pmatrix} W^k & 0 \\ 0 & 0 \end{pmatrix} T^{-1} x = 0\,.$$

Assume now that $T^{-1}(A - \lambda E)^{-1}\, \mathrm{e}^{-\lambda t}\, G\left(t, \mathrm{e}^{\lambda t}\, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right)$ is decoupled as

$$T^{-1}(A - \lambda E)^{-1}\, \mathrm{e}^{-\lambda t}\, G\left(t, \mathrm{e}^{\lambda t}\, T \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} \Phi(t, v_1) \\ 0_{k \times k} \end{pmatrix},$$

where $\Phi\colon [0, T] \times \mathbb{R}^{n-k} \longrightarrow \mathcal{P}(\mathbb{R}^{n-k})$ is some set-valued function. Such reduction may be expected for various problems in mechanics since the unilateral forces are introduced in the formulation of the usual equations of motions that are in fact not the cause of the singularities occurring in the whole model. Then the systems (2.6) and (2.7) reduce to

$$\begin{aligned} &\dot{v}_1 \in -W^{-1} v_1 - W^{-1} h_1 - W^{-1} \Phi(t, v_1)\,, \\ &v_1(0) = c_1\,, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &N \dot{v}_2 = -v_2 - h_2\,, \\ &v_2(0) = c_2\,. \end{aligned} \tag{2.10}$$

The differential inclusion (2.9) reduces now to a standard one that has been studied in the mathematical literature while (2.10) can be solved as described here-above.

3. Second order differential inclusions with singular matrices

We consider in this section a second order system:

Find $q\colon [0, T] \longrightarrow \mathbb{R}^N$ such that

$$\begin{aligned} &M \ddot{q}(t) + C \dot{q}(t) + K q(t) \in f(t) + F(t, q(t), \dot{q}(t))\,, \quad \text{a.e} \quad t \in (0, T)\,, \\ &q(0) = q_0\,, \\ &\dot{q}(0) = q_1\,, \end{aligned} \tag{3.1}$$

where M, C, $K \in \mathbb{R}^{N \times N}$ denote mass, damping, and stiffness matrices, respectively, and $f\colon [0, T] \longrightarrow \mathbb{R}^N$ denotes a vector-valued function while $F\colon [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathcal{P}(\mathbb{R}^N)$ is a set-valued function.

This system can be rewritten in the first order form (2.1) by defining

$$\begin{aligned} &x = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}, \qquad E = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I \\ -K & -C \end{pmatrix}, \\ &H(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \qquad G(t, x) = \begin{pmatrix} 0 \\ F(t, x_1(t), x_2(t)) \end{pmatrix}, \qquad c = \begin{pmatrix} q_0 \\ q_1 \end{pmatrix}. \end{aligned} \tag{3.2}$$

Note that if the matrix M is singular then E is singular, too. On the other hand, if K is singular then A is also singular.

Let us define by

$$P(\lambda) = \det\left(\lambda^2 M + \lambda C + K\right)$$

the characteristic matrix polynomial associated to the second order system in (3.1).

It is clear that the approach stated in the previous section can be specialized to the second order differential inclusion (3.1) as soon as the matrices E and A in (3.2) form a regular matrix pencil. The following theorem provides a general result in this sense.

Theorem 3.1: *There exists $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$ if and only if* $\mathrm{rank}(\lambda E - A) = 2N$, *i.e., the matrices E and A form a regular matrix pencil.*

Proof: Suppose that there exists $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. If $\lambda \neq 0$ then

$$\det(\lambda E - A) = \det\begin{pmatrix} \lambda I & -I \\ K & \lambda M + C \end{pmatrix} = \det\begin{pmatrix} \lambda I & -I \\ \lambda \dfrac{K}{\lambda} & \lambda M + C \end{pmatrix}$$

$$= \lambda^N \det\begin{pmatrix} I & -I \\ \dfrac{K}{\lambda} & \lambda M + C \end{pmatrix} = \lambda^N \det\Big(\lambda M + C + \frac{K}{\lambda}\Big) = \det\big(\lambda^2 M + \lambda C + K\big)\,.$$

It results that there exists $\lambda \in \mathbb{R}\backslash\{0\}$ such that $P(\lambda) \neq 0$ if and only if $\det(\lambda E - A) \neq 0$.

If $\lambda = 0$ we have

$$\det(-A) = \det\begin{pmatrix} 0 & -I \\ K & C \end{pmatrix} = P(0)\,.$$

Thus $P(0) \neq 0$ if and only if K is regular. □

Remark 3.2: Condition $P(\lambda) \neq 0$ is more general than the one (discussed in [1]) requiring the invertibility of $M + C + K$. For example, if

$$M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}, \qquad K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we see that

$$M + C + K = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

is singular, while

$$4M + 2C + K = \begin{pmatrix} 5 & -6 \\ -6 & 6 \end{pmatrix}$$

is regular, that is $P(2) \neq 0$.

We define $\hat{M}_\lambda = \lambda^2 M + \lambda C + K$ and $\hat{E}_\lambda = (\lambda E - A)^{-1}\, E$. Note that

$$\hat{E}_\lambda = \begin{pmatrix} \hat{M}_\lambda^{-1}(\lambda M + C) & \hat{M}_\lambda^{-1} M \\ -\hat{M}_\lambda^{-1} K & \lambda \hat{M}_\lambda^{-1} M \end{pmatrix}.$$

Indeed, assume that $\lambda \neq 0$. Then we check that

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} \lambda I & -I \\ K & \lambda M + C \end{pmatrix} \begin{pmatrix} \hat{M}_\lambda^{-1}(\lambda M + C) & \hat{M}_\lambda^{-1} M \\ -\hat{M}_\lambda^{-1} K & \lambda \hat{M}_\lambda^{-1} M \end{pmatrix}.$$

Only the relation $0 = K\hat{M}_\lambda^{-1}(\lambda M + C) - (\lambda M + C)\, \hat{M}_\lambda^{-1} K$ needs some attention. We have indeed

$$K\hat{M}_\lambda^{-1}(\lambda M + C) - (\lambda M + C)\, \hat{M}_\lambda^{-1} K = K\hat{M}_\lambda^{-1}\, \Big(\lambda M + C + \frac{K}{\lambda}\Big) - K\hat{M}_\lambda^{-1}\, \frac{K}{\lambda}$$

$$- \Big(\lambda M + C + \frac{K}{\lambda}\Big)\, \hat{M}_\lambda^{-1} K + K\hat{M}_\lambda^{-1}\, \frac{K}{\lambda} = K\hat{M}_\lambda^{-1}\, \frac{\hat{M}_\lambda}{\lambda} - \frac{\hat{M}_\lambda}{\lambda}\, \hat{M}_\lambda^{-1} K = 0\,.$$

If $\lambda = 0$ then K is regular and we check that

$$\begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} 0 & -I \\ K & C \end{pmatrix} \begin{pmatrix} K^{-1}C & K^{-1}M \\ -I & 0 \end{pmatrix}.$$

The following two propositions give few properties of the matrix $\hat{E}_\lambda$ in a general framework. The approach used here is similar to the one developed in [1].

Proposition 3.3: *Let $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. Then the matrix $\hat{E}_\lambda$ satisfies the following properties:*

(i) $\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix}$,

(ii) $\operatorname{ind}(\hat{E}_\lambda) \leq 1$ *if and only if* $\operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N$,

(iii) $\operatorname{ind}(\hat{E}_\lambda) = 0$ *if and only if M is invertible.*

Proof: (i) Let

$$y_k = \begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix},$$

and $y_{k+1} = \hat{E}_\lambda y_k$ where y_{k1}, $y_{k2} \in \mathbb{R}^N$ for $k = 1, 2$. We suppose that

$$\hat{E}_\lambda^2 y_1 = 0 .$$

Using the notation above, we may write

$$y_3 = \hat{E}_\lambda y_2 = \hat{E}_\lambda^2 y_1 = 0 .$$

Let us first check that $\ker(\hat{E}_\lambda^2) \subset \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix}$. Reminding the definition of $\hat{E}_\lambda$, we obtain $Ey_2 = (\lambda E - A)\, y_3 = 0$ and $E\hat{E}_\lambda y_1 = Ey_2$. It results that

$$y_{21} = \hat{M}_\lambda^{-1}[(\lambda M + C)\, y_{11} + My_{12}] = 0\,, \tag{3.3}$$

and

$$My_{22} = M\hat{M}_\lambda^{-1}[-Ky_{11} + \lambda My_{12}] = 0\,. \tag{3.4}$$

Then, using (3.3) and (3.4), we see that

$$\lambda My_{21} - My_{22} = My_{11}\,,$$

and, since $\lambda My_{21} - My_{22} = 0$, we deduce that $My_{11} = 0$. Then from (3.3), we obtain

$$Cy_{11} + My_{12} = 0\,.$$

Consequently, we have

$$\begin{pmatrix} C & M \\ M & 0 \end{pmatrix} y_1 = \begin{pmatrix} C & M \\ M & 0 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix} = 0\,.$$

Conversely, assume that $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix}$, we have

$$\hat{E}_\lambda y = \begin{pmatrix} \hat{M}_\lambda^{-1}[(\lambda M + C)\, y_1 + My_2] \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Using the fact that $Cy_1 + My_2 = 0$ and $My_1 = 0$, we deduce

$$\hat{E}_\lambda y = \begin{pmatrix} 0 \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Then we obtain

$$\hat{E}_\lambda^2 y = \hat{E}_\lambda \begin{pmatrix} 0 \\ \hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix} = \begin{pmatrix} \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \\ \lambda\hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) \end{pmatrix}.$$

Using the fact that $-K = \lambda^2 M + \lambda C - \hat{M}_\lambda$, we obtain

$$\begin{aligned} \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) &= \hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}((\lambda^2 M + \lambda C)\, y_1 + \lambda My_2 - \hat{M}_\lambda y_1)\,, \\ &= \hat{M}_\lambda^{-1} M(\hat{M}_\lambda^{-1}(\lambda^2 My_1 + \lambda(Cy_1 + My_2))) - \hat{M}_\lambda^{-1} My_1\,. \end{aligned}$$

Reminding that $Cy_1 + My_2 = 0$ and $My_1 = 0$, we see that

$$\hat{M}_\lambda^{-1} M\hat{M}_\lambda^{-1}(-Ky_1 + \lambda My_2) = 0\,,$$

which implies that

$$\hat{E}_\lambda^2 y = 0\,.$$

Finally, we deduce that

$$\ker(\hat{E}_\lambda^2) = \ker\begin{pmatrix} C & M \\ M & 0 \end{pmatrix},$$

and

$$\operatorname{rank}(\hat{E}_\lambda^2) = \operatorname{rank}(M) + \operatorname{rank}\begin{pmatrix} C \\ M \end{pmatrix}.$$

(ii) Recall that $\mathrm{ind}(\hat{E}_\lambda) \leq 1$ if and only if $\mathrm{rank}(\hat{E}_\lambda) = \mathrm{rank}(\hat{E}_\lambda^2)$. Since

$$\mathrm{rank}(\hat{E}_\lambda) = \mathrm{rank}(E) = N + \mathrm{rank}(M)\,,$$

and then $\mathrm{rank}(\hat{E}_\lambda) = \mathrm{rank}(\hat{E}_\lambda^2)$ if and only if

$$\mathrm{rank}(M) + \mathrm{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N + \mathrm{rank}(M)\,,$$

that is

$$\mathrm{rank}\begin{pmatrix} C \\ M \end{pmatrix} = N\,,$$

which implies the desired result.

(iii) Finally, $\mathrm{ind}(\hat{E}_\lambda) = 0$ if and only if $\mathrm{rank}(\hat{E}_\lambda) = 2N$. Since

$$\mathrm{rank}(\hat{E}_\lambda) = \mathrm{rank}(E) = N + \mathrm{rank}(M)\,,$$

it follows that $\mathrm{rank}(\hat{E}_\lambda) = 2N$ if and only if M is invertible. □

Definition 3.4: We say that a matrix $A \in \mathbb{R}^{N\times N}$ has Property (K) if

$$\{x \in \mathbb{R}^N : x^T A x = 0\} \subset \ker(A)\,.$$

The next proposition gives further information on the matrix $\hat{E}_\lambda$.

Proposition 3.5: *Let $\lambda \in \mathbb{R}$ such that $P(\lambda) \neq 0$. Then the matrix $\hat{E}_\lambda$ satisfies the following properties*:

(i) $\ker(\lambda M + C) \subset \ker(M)$, *then*

$$\mathrm{rank}(\hat{E}_\lambda^2) = \mathrm{rank}(M) + \mathrm{rank}(\lambda M + C)\,.$$

(ii) *If*
(a) $\hat{M}_\lambda$ *has Property* (K),
(b) C *has Property* (K),
(c) $\ker(C) = \ker(C^T)$,
(d) $\ker(M) = \ker(M^T)$,

then

$$\mathrm{ind}(\hat{E}_\lambda) \leq 2\,.$$

Proof: (i) It is clear that

$$\ker\begin{pmatrix} C \\ M \end{pmatrix} = \ker(M) \cap \ker(C) \subset \ker(\lambda M + C)\,.$$

Then, if we suppose $x \in \ker(\lambda M + C) \subset \ker(M)$, it follows immediately that $x \in \ker(C)$, which implies $\ker(\lambda M + C) \subset \ker(M) \cap \ker(C)$. Hence

$$\ker\begin{pmatrix} C \\ M \end{pmatrix} = \ker(\lambda M + C)\,.$$

We also deduce

$$\mathrm{rank}\begin{pmatrix} C \\ M \end{pmatrix} = \mathrm{rank}(\lambda M + C)\,,$$

and thus by (i) in Proposition 3.3, we get

$$\mathrm{rank}(\hat{E}_\lambda^2) = \mathrm{rank}(M) + \mathrm{rank}(\lambda M + C)\,.$$

(ii) We know that if $\ker(\hat{E}_\lambda^3) \subset \ker(\hat{E}_\lambda^2)$ then $\mathrm{ind}(\hat{E}_\lambda^2) \leq 2$. Let

$$y_k = \begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix},$$

and $y_{k+1} = \hat{E}_\lambda y_k$, where $y_{k1},\, y_{k2} \in \mathbb{R}^N$ for $k = 0,\, 1$. We suppose that

$$\hat{E}_\lambda^3 y_0 = 0\,.$$

Since $y_1 = \hat{E}_\lambda y_0$, we have

$$\begin{aligned} y_{11} &= \hat{M}_\lambda^{-1}((\lambda M + C)\, y_{01} + M y_{02})\,, \\ y_{12} &= \hat{M}_\lambda^{-1}(-K y_{01} + \lambda M y_{02})\,. \end{aligned} \tag{3.5}$$

Moreover, y_1 belongs to $\ker(\hat{E}_\lambda^2)$ and thus

$$\begin{aligned} C y_{11} + M y_{12} &= 0\,, \\ M y_{11} &= 0\,. \end{aligned} \tag{3.6}$$

From $(3.6)_1$, we get

$$y_{11}^T C y_{11} + y_{11}^T M y_{12} = 0\,.$$

Since $\ker(M) = \ker(M^T)$ and C has Property (K) we obtain $y_{11} \in \ker(C) = \ker(C^T)$ and then by (3.6) it results that $M y_{12} = 0$. Using (3.5), we get

$$y_{11}^T \hat{M}_\lambda y_{11} = y_{11}^T (\lambda M + C)\, y_{01} + y_{11}^T M y_{02}\,,$$

and we obtain

$$y_{11}^T \hat{M}_\lambda y_{11} = 0\,.$$

Assumption (ii)$-$(a) together with the regularity of $\hat{M}_\lambda$ yields

$$y_{11} = 0\,.$$

Then, using the fact that $M y_{12} = 0$, we deduce

$$\lambda M y_{11} - M y_{12} = M \hat{M}_\lambda^{-1} \big(\lambda^2 M + \lambda C + K\big)\, y_{01} = M y_{01} = 0\,,$$

and, from $(3.5)_1$, we obtain

$$C y_{01} + M y_{02} = 0\,.$$

Finally, we have

$$\begin{aligned} C y_{01} + M y_{02} &= 0\,, \\ M y_{01} &= 0\,, \end{aligned}$$

which yields $y_0 \in \ker(\hat{E}_\lambda^2)$. □

Remark 3.6: i) If C is positive semi-definite and M is symmetric and positive semi-definite then

$$\ker(\lambda M + C) \subset \ker(M)\,, \qquad \forall \lambda > 0\,.$$

Indeed, if $x \in \ker(\lambda M + C)$ then $x^T(\lambda M + C)\, x = 0$. However $x^T M x \geq 0$, $x^T C x \geq 0$ and thus $x^T M x = x^T C x = 0$. The matrix M being symmetric, it results that $x \in \ker(M)$.

ii) More generally, if C is positive semi-definite and M is cocoercive (see e.g. [12]), i.e., there exists $\alpha > 0$ such that $x^T M x \geq \alpha \|Mx\|^2$, then

$$\ker(\lambda M + C) \subset \ker(M)\,, \qquad \forall \lambda > 0\,.$$

As above, we see that if $x \in \ker(\lambda M + C)$, then in particular $x^T M x = 0$. It results that $\|Mx\| = 0$ and then $x \in \ker(M)$.

4. Model reduction

The aim of this section is to present a mathematical approach allowing the reduction of the second order system (3.1).

Define $\varGamma = \{\lambda \in \mathbb{R} : P(\lambda) \neq 0\}$, $N_1(\lambda) := \operatorname{def}(\lambda M + C)$, $N_2(\lambda) := \operatorname{def}(M) - N_1(\lambda)$ for all $\lambda \in \varGamma$, and $N_3 = \operatorname{rank}(M)$. Here, for a matrix A, $\operatorname{def}(A) := \dim(\ker(A))$. It is clear that $N_1(\lambda) + N_2(\lambda) + N_3 = N$. We assume that M is singular but non-zero. In the rest of this section, we note for simplicity $N_1 = N_1(\lambda)$ and $N_2 = N_2(\lambda)$.

Theorem 4.1: *Let the following assumptions hold*:

$$\ker(M) = \ker(M^T)\,, \tag{4.1}$$

There exists $\lambda \in \varGamma$ *such that*

$$\begin{aligned} &\text{(i)}\ \ \ker(\lambda M + C) \subset \ker(M)\,, \\ &\text{(ii)}\ \ \ker(\lambda M + C) = \ker(\lambda M + C^T)\,, \\ &\text{(iii)}\ \ N_1 > 0\,, \qquad N_2 > 0\,. \end{aligned} \tag{4.2}$$

Let $U = \{x_1, \ldots, x_N\}$ be an orthonormal basis for $\mathbb{R}^N$ such that $\{x_1, \ldots, x_{N_2+N_1}\}$ is an orthonormal basis for $\ker(M)$ *and $\{x_1, \ldots, x_{N_1}\}$ is an orthonormal basis for* $\ker(\lambda M + C)$. *Then we have*

$$U^T M U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_{33} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix},$$

where $M_{33} \in \mathbb{R}^{N_3 \times N_3}$ is regular and $C_{ij} \in \mathbb{R}^{N_i \times N_j}$, $i, j = 2, 3$.

Proof: Under the previous notation, we get the scheme

$$\underbrace{\overbrace{\underbrace{x_1, x_2, \ldots, x_{N_1}}_{\ker(\lambda M+C)}, x_{N_1+1}, \ldots, x_{N_1+N_2}}_{\ker(M)}, x_{N_1+N_2+1}, \ldots, x_N}^{\mathbb{R}^N} .$$

We have

$$x_i^T M x_j = 0\,, \qquad 1 \le i \le N\,, \qquad 1 \le j \le N_1 + N_2\,,$$

and

$$x_i^T C x_j = x_i^T (\lambda M + C)\, x_j = 0\,, \qquad 1 \le i \le N\,, \qquad 1 \le j \le N_1\,.$$

Thus, we may write

$$U^T M U = \begin{pmatrix} 0 & 0 & M_{13} \\ 0 & 0 & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & C_{32} & C_{33} \end{pmatrix}, \qquad U^T K U = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}.$$

Moreover, using assumption (4.1), we obtain

$$x_i^T M x_j = (M^T x_i)^T\, x_j = 0\,, \qquad 1 \le i \le N_1 + N_2\,, \qquad N_1 + N_2 + 1 \le j \le N\,,$$

so that

$$M_{13} = M_{23} = 0\,.$$

Consequently, we deduce that the matrix M_{33} is regular since $\operatorname{rank}(M_{33}) = \operatorname{rank}(M) = N_3$. On the other hand, using assumption (4.2), we see that

$$x_i^T C x_j = (C^T x_i)^T\, x_j = ((\lambda M + C^T)\, x_i)^T\, x_j = 0\,, \qquad 1 \le i \le N_1\,, \qquad N_1 + 1 \le j \le N\,,$$

and thus

$$C_{12} = C_{13} = 0\,. \qquad \square$$

Remark 4.2: i) Condition $\ker(M) = \ker(M^T)$ is for instance satisfied in the following cases: a) M symmetric, b) M skew-symmetric, c) M positive semi-definite.

ii) Condition $\ker(\lambda M + C) = \ker(\lambda M + C^T)$ is for instance satisfied in the following cases: a) C symmetric, b) M symmetric and $\lambda M + C$ positive semi-definite.

To go further, we suppose now that the matrix $U^T \hat{M}_\lambda U$ has all leading principal minors non-zero. We will refer to this assumption as Condition (H_U). Note that this property is for example ensured as soon as we suppose that $\hat{M}_\lambda$ is positive definite.

Theorem 4.3: *Let assumptions* (4.1) *and* (4.2) *hold together with*

$$\begin{aligned} &\ker(\lambda^2 M + K) \perp \ker(\lambda M + C)\,, \\ &\ker(\lambda^2 M + K) \oplus \ker(\lambda M + C) = \ker(M)\,, \end{aligned} \tag{4.3}$$

$$\ker(\lambda^2 M + K) = \ker(\lambda^2 M + K^T)\,. \tag{4.4}$$

Then

$$U^T K U = \begin{pmatrix} K_{11} & 0 & K_{13} \\ 0 & 0 & 0 \\ K_{31} & 0 & K_{33} \end{pmatrix}.$$

Moreover, if $\hat{M}_\lambda$ satisfies Condition (H_U), then $K_{11} \in \mathbb{R}^{N_1 \times N_1}$ and $C_{22} \in \mathbb{R}^{N_2 \times N_2}$ are regular.

P r o o f: Assumption (4.3) yields $\ker(\lambda^2 M + K) \subset \ker(M)$ and thus

$$Kx_j = (\lambda^2 M + K)\, x_j = 0\,, \qquad N_1 + 1 \le j \le N_1 + N_2\,.$$

Then it results that

$$K_{12} = K_{22} = K_{32} = 0\,.$$

On the other hand, we have

$$x_i^T K x_j = (K^T x_i)^T\, x_j = ((\lambda^2 M + K^T)\, x_i)^T\, x_j\,, \qquad N_1 + 1 \le i \le N_1 + N_2\,, \qquad 1 \le j \le N\,,$$

and thus, by assumption (4.4), we get

$$x_i^T K x_j = 0\,, \qquad N_1 + 1 \le i \le N_1 + N_2\,, \qquad 1 \le j \le N\,.$$

Then it results that

$$K_{21} = K_{23} = 0\,.$$

Moreover, $\begin{pmatrix} K_{11} & 0 \\ 0 & \lambda C_{22} \end{pmatrix}$ is a principal submatrix of $U^T \hat{M}_\lambda U$ so that K_{11} and C_{22} are regular. □

The special case $N_2 = 0$, i.e. $\ker(M) = \ker(\lambda M + C)$, is now treated.

Theorem 4.4: *Let the following assumptions hold*:

$$\ker(M) = \ker(M^T)\,, \tag{4.5}$$

$$\textit{There exists } \lambda \in \Gamma \textit{ such that } \begin{aligned} &\text{(i) } \ker(\lambda M + C) \subset \ker(M)\,, \\ &\text{(ii) } \ker(\lambda M + C) = \ker(\lambda M + C^T)\,, \\ &\text{(iii) } N_1 = N_1 + N_2 > 0\,. \end{aligned} \tag{4.6}$$

Let $U = \{x_1, \ldots, x_N\}$ be an orthonormal basis for $\mathbb{R}^N$ such that $\{x_1, \ldots, x_{N_1}\}$ is an orthonormal basis for $\ker(M)$. *Then we have*

$$U^T M U = \begin{pmatrix} 0 & 0 \\ 0 & M_{22} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & 0 \\ 0 & C_{22} \end{pmatrix},$$

where $M_{22} \in \mathbb{R}^{N_3 \times N_3}$ is regular, $C_{22} \in \mathbb{R}^{N_3 \times N_3}$. If, in addition,

$$\hat{M}_\lambda \textit{ satisfies Condition } (H_U)\,, \tag{4.7}$$

then $K_{11} \in \mathbb{R}^{N_1 \times N_1}$ is regular.

P r o o f: Under the previous notation, we get the scheme

$$\overbrace{\underbrace{x_1,\, x_2, \ldots,\, x_{N_1}}_{\substack{\ker(M) \\ = \\ \ker(\lambda M + C)}},\, x_{N_1+1}, \ldots,\, x_N}^{\mathbb{R}^N}\,.$$

As in Theorem 4.1, we check that

$$U^T M U = \begin{pmatrix} 0 & M_{12} \\ 0 & M_{22} \end{pmatrix}, \qquad U^T C U = \begin{pmatrix} 0 & C_{12} \\ 0 & C_{22} \end{pmatrix}, \qquad U^T K U = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}.$$

Moreover, using assumption (4.5), we obtain

$$x_i^T M x_j = (M^T x_i)^T\, x_j = 0\,, \qquad 1 \le i \le N_1\,, \qquad N_1 + 1 \le j \le N\,.$$

Then it results that $M_{12} = 0$. Consequently, we deduce that the matrix M_{22} is regular since $\operatorname{rank}(M_{22}) = \operatorname{rank}(M) = N_3$.

On the other hand, using assumption (4.6), we get

$$x_i^T C x_j = (C^T x_i)^T\, x_j = ((\lambda M + C^T)\, x_i)^T\, x_j = 0\,, \qquad 1 \le i \le N_1\,, \qquad N_1 + 1 \le j \le N\,,$$

and thus $C_{12} = 0$.

If condition (4.7) holds then K_{11} is regular as a principal submatrix of $U^T \hat{M}_\lambda U$. □

The following corollary shows that under assumptions of Theorem 4.3 System (3.1) can be reduced to a regular second order system of differential inclusions.

Corollary 4.5: *Assume that assumptions* (4.1)–(4.5) *hold and let $S \in \mathbb{R}^{N \times N_3}$ be the matrix defined by*

$$S = U \begin{pmatrix} -K_{11}^{-1} K_{13} \\ -C_{22}^{-1} C_{23} \\ I \end{pmatrix}.$$

Then the matrix $S^T MS$ is regular.

Proof: Using the results of Theorem 4.3, it easy to check that

$$S^T MS = M_{33}\,, \qquad S^T CS = C_{33} - C_{32}C_{22}^{-1}C_{23}\,, \qquad S^T KS = K_{33} - K_{31}K_{11}^{-1}K_{13}\,.$$

Consequently, using the transformation $q = S\tilde{q}$, System (3.1) reduces to

$$M_{33}\ddot{\tilde{q}} + (C_{33} - C_{32}C_{22}^{-1}C_{23})\,\dot{\tilde{q}} + (K_{33} - K_{31}K_{11}^{-1}K_{13})\,\tilde{q} \in S^T f + S^T F(t, S\tilde{q}, S\dot{\tilde{q}})\,. \qquad \square$$

The following corollary which follows from Theorem 4.4 gives another case in which a reduction in the number of degrees of freedom can be achieved.

Corollary 4.6: *Assume that assumptions* (4.5)–(4.6) *hold and let $S \in \mathbb{R}^{N\times N_3}$ be the matrix defined by*

$$S = U\begin{pmatrix} -K_{11}^{-1}K_{12} \\ I \end{pmatrix}.$$

Then the matrix $S^T MS$ is regular.

Proof: Using the results of Theorem 4.4, it easy to check that

$$S^T MS = M_{22}\,, \qquad S^T CS = C_{22}\,, \qquad S^T KS = K_{22} - K_{21}K_{11}^{-1}K_{12}\,.$$

Consequently, using the transformation $q = S\tilde{q}$, system (3.1) reduces to

$$M_{22}\ddot{\tilde{q}} + C_{22}\dot{\tilde{q}} + (K_{22} - K_{21}K_{11}^{-1}K_{12})\,\tilde{q} \in S^T f + S^T F(t, S\tilde{q}, S\dot{\tilde{q}})\,. \qquad \square$$

5. Applications

We present in this section some mechanical applications of the theoretical results given previously. Examples 5.1 and 5.3 are dedicated to illustrate the results of Section 4 while Example 5.4 illustrates the Jordan reduction technique presented in Section 2.

Example 5.1: Consider the spring mass system of Fig. 5.1. The mass m is constrained to move only in the vertical direction. The design of the system invokes two linear springs with positive spring constants k_1 and k_2 and two

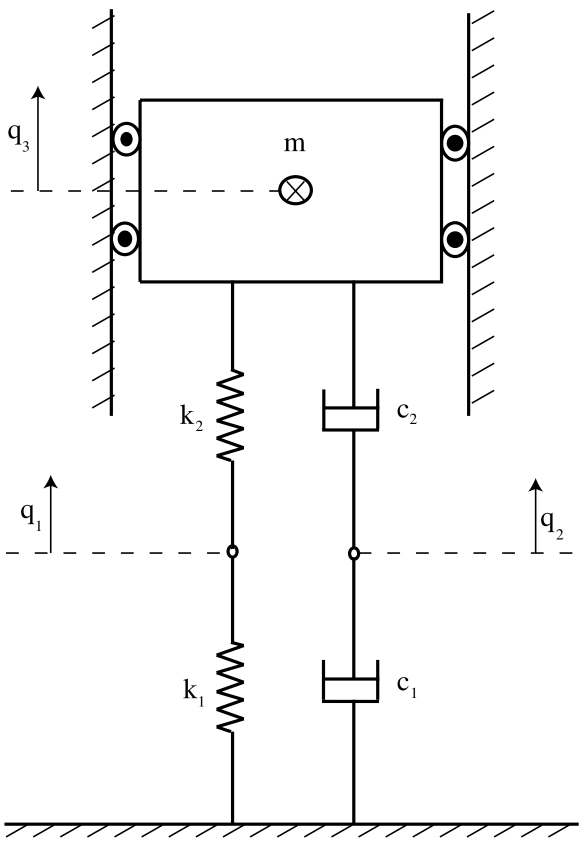

Fig. 5.1

linear viscous dampers with positive damping coefficients c_1 and c_2. The mass m has displacement q_3, the massless joint between the springs has a displacement q_1 while the massless joint between the dampers has a displacement q_2. In such machinery, the rigid body of mass m denotes a machine while the rest of the system models a vibration absorber that is installed between the machine and the supporting ground in order to reduce the effect of vibrations induced for example by a force of excitation $E_0 \sin w_0 t$ (E_0, $w_0 \in \mathbb{R}$). The machine slide yields a total friction force $\tau(\dot{q}_3)$. Here we postulate the following relation between τ and the velocity $\dot{q}_3$:

$$\begin{aligned} &\text{if} \quad \dot{q}_3 > 0 \quad \text{then} \quad \tau = -a\,,\\ &\text{if} \quad \dot{q}_3 < 0 \quad \text{then} \quad \tau = b\,,\\ &\text{if} \quad \dot{q}_3 = 0 \quad \text{then} \quad \tau \in [-a, b]\,, \end{aligned}$$

with a, $b > 0$. Equivalently, we write

$$\tau \in \Gamma(-\dot{q}_3)\,,$$

where Γ is the set-valued function

$$u \longmapsto \Gamma(u) = \begin{cases} -a & \text{if} \quad u < 0\,,\\ [-a, b] & \text{if} \quad u = 0\,,\\ b & \text{if} \quad u > 0\,. \end{cases}$$

The equations of motion are

$$\begin{aligned} 0 &= -k_1 q_1 + k_2(q_3 - q_1)\,,\\ 0 &= -c_1 \dot{q}_2 + c_2(\dot{q}_3 - \dot{q}_2)\,,\\ m\ddot{q}_3 &= -k_2(q_3 - q_1) - c_2(\dot{q}_3 - \dot{q}_2) - mg + E_0 \sin w_0 t + \tau(\dot{q}_3)\,. \end{aligned}$$

This system can be written under the form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) \in f(t) + F(t, q(t), \dot{q}(t))\,,$$

with

$$M = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & m \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 0\\ 0 & c_1 + c_2 & -c_2\\ 0 & -c_2 & c_2 \end{pmatrix}, \qquad K = \begin{pmatrix} k_1 + k_2 & 0 & -k_2\\ 0 & 0 & 0\\ -k_2 & 0 & k_2 \end{pmatrix},$$

$$q = \begin{pmatrix} q_1\\ q_2\\ q_3 \end{pmatrix}, \qquad f(t) = \begin{pmatrix} 0\\ 0\\ E_0 \sin w_0 t - mg \end{pmatrix}, \qquad F(t, q(t), \dot{q}(t)) = \begin{pmatrix} 0\\ 0\\ \Gamma(-\dot{q}_3(t)) \end{pmatrix}.$$

The matrix $M + C + K$ is symmetric and positive definite. Moreover the matrices M, C, and K satisfy the assumptions of Corollary 4.5. Indeed, it is easy to remark that

$$\ker(M) = \ker(M^T)\,, \qquad \ker(M + C) = \ker(M + C^T)\,, \qquad \ker(M + K) = \ker(M + K^T)\,,$$

since the matrices M, $M + C$, and $M + K$ are symmetric. Moreover, it is clear that

$$\operatorname{def}(M) > \operatorname{def}(M + C) > 0 \quad \text{with} \quad \ker(M + C) \subset \ker(M)\,.$$

On the other hand, we have

$$\ker(M + C) \oplus \ker(M + K) = \ker(M) \quad \text{with} \quad \ker(M + C) \perp \ker(M + K)\,,$$

$$\ker(M + C) = \operatorname{span}\left\{ \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix} \right\}, \qquad \ker(M + K) = \operatorname{span}\left\{ \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix} \right\},$$

and

$$\ker(M) = \operatorname{span}\left\{ \begin{pmatrix} 1\\ 0\\ 0 \end{pmatrix}, \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix} \right\}.$$

Hence, we apply Corollary 4.5 to construct the orthogonal matrix $U \in \mathbb{R}^{3\times 3}$ and the matrix $S \in \mathbb{R}^{3\times 1}$ as follows:

$$U = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}, \qquad S = U \begin{pmatrix} -K_{11}^{-1} K_{13}\\ -C_{22}^{-1} C_{23}\\ I \end{pmatrix},$$

respectively, where $K_{11} = k_1 + k_2$, $K_{13} = -k_2$, $C_{22} = c_1 + c_2$, and $C_{23} = -c_2$. This implies that

$$S = \begin{pmatrix} k_2/(k_1+k_2) \\ c_2/(c_1+c_2) \\ 1 \end{pmatrix}.$$

Then we obtain $S^T MS = m$, $S^T CS = c_1c_2/(c_1+c_2)$, and $S^T KS = k_1k_2/(k_1+k_2)$. Finally, setting $q = S\tilde{q}$, we see that $\tilde{q}$ $(= q_3)$ is the solution of

$$m\ddot{\tilde{q}}(t) + \frac{c_1c_2}{c_1+c_2}\,\dot{\tilde{q}}(t) + \frac{k_1k_2}{k_1+k_2}\,\tilde{q}(t) \in E_0 \sin w_0 t - mg + \Gamma(-\dot{\tilde{q}}(t))\,. \tag{5.1}$$

Remark 5.2: From the mechanical point of view, the system described in this example can be immediately reduced to (5.1) by setting $1/k := 1/k_1 + 1/k_2$ and $1/c := 1/c_1 + 1/c_2$. But here the goal is just to illustrate simply the mathematical reduction technique presented in Section 4. A more complicated and realistic example (Example 5.3) is given below.

Setting $y = (\tilde{q}, \dot{\tilde{q}})$, we obtain from (5.1) the first order system

$$\dot{y}(t) \in \boldsymbol{\Phi}(t, y(t))\,,$$

where

$$\boldsymbol{\Phi}(t, y(t)) = \begin{pmatrix} y_2 \\ -\frac{1}{m}\,\frac{c_1c_2}{c_1+c_2}\,y_2 - \frac{1}{m}\,\frac{k_1k_2}{k_1+k_2}\,y_1 + \frac{1}{m}\,E_0 \sin\omega_0 t - g + \frac{1}{m}\,\Gamma(-y_2(t)) \end{pmatrix}.$$

The system is now studied on the time interval $[0, T]$, with the initial conditions $y(0) = 0$, i.e. $\tilde{q}(0) = \dot{\tilde{q}}(0) = 0$. The existence of a solution follows from Theorem 1.1. Moreover,

$$\Gamma(x) = \partial h_{a,b}(x)\,,$$

where $h_{a,b}$ is the convex function

$$h_{a,b}(x) = \begin{cases} -ax & \text{if} \quad x < 0\,, \\ bx & \text{if} \quad x \geq 0\,. \end{cases}$$

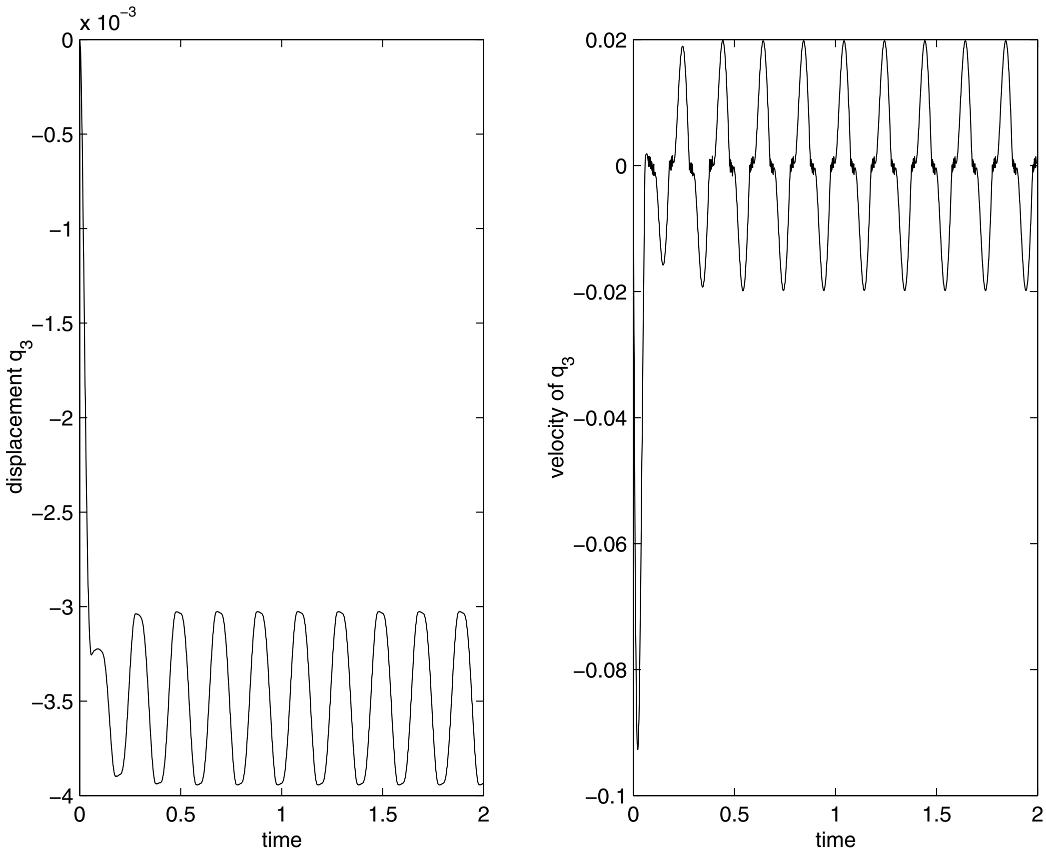

Fig. 5.2

It results that Γ is monotone. We have

$$\Phi(t, y(t)) = \theta(t, y(t)) - \beta(y(t)),$$

with

$$\theta(t, y(t)) = \begin{pmatrix} y_2 \\ -\frac{1}{m} \frac{c_1 c_2}{c_1 + c_2} y_2 - \frac{1}{m} \frac{k_1 k_2}{k_1 + k_2} y_1 + \frac{1}{m} E_0 \sin \omega_0 t - g \end{pmatrix},$$

and

$$\beta(y(t)) = \begin{pmatrix} 0 \\ -(1/m)\, \Gamma(-y_2(t)) \end{pmatrix}.$$

The application θ is Lipschitz continuous uniformly for all $t \in [0, T]$ and β is monotone. The monotonicity of β follows from the one of Γ. We have indeed

$$\begin{aligned} (\beta(z) - \beta(w))^T (z - w) &= (1/m)\, (((-\Gamma(-z_2)) - (-\Gamma(-w_2)))^T (z_2 - w_2), \\ &= (1/m)\, ((\Gamma(-z_2) - \Gamma(-w_2))^T (-z_2 - (-w_2)), \\ &\geq 0. \end{aligned}$$

The uniqueness of the solution follows. The Euler method was applied to the problem with the following data:

m	k_1	k_2	c_1	c_2	E_0	ω_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
25	25×10^4	10×10^4	2500	2000	50	10π	29.43	22.075	0	0

The displacement q_3 of the machine and its velocity $\dot{q}_3$ are depicted in Fig. 5.2. Stick-slip phenomena appear clearly.

In Figs. 5.3, 5.4 we further illustrate the model by means of some additional numerical simulations. We present, in each case, the data chosen and the graph representing the displacement q_3 as well as the phase portrait.

m	k_1	k_2	c_1	c_2	E_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
1	100	25	6	4	0	1	1	2	0

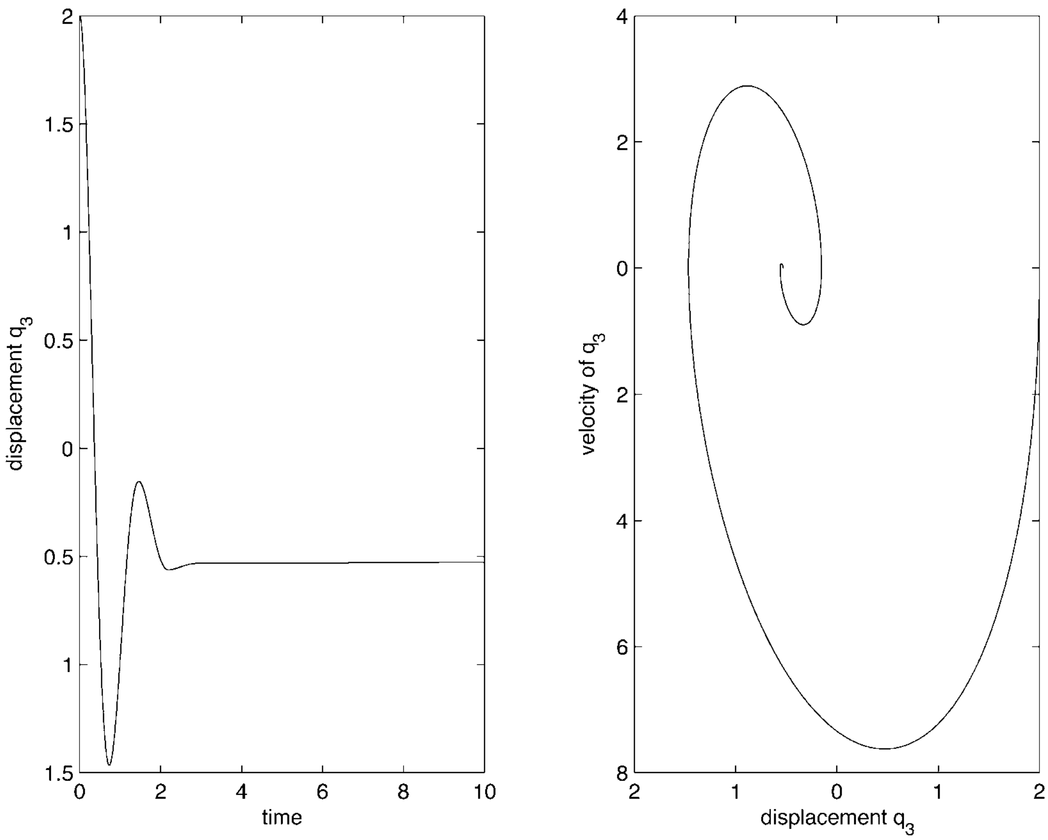

Fig. 5.3

m	k_1	k_2	c_1	c_2	E_0	a	b	$q_3(0)$	$\dot{q}_3(0)$
1	100	25	0	4	0	12	12	2	0

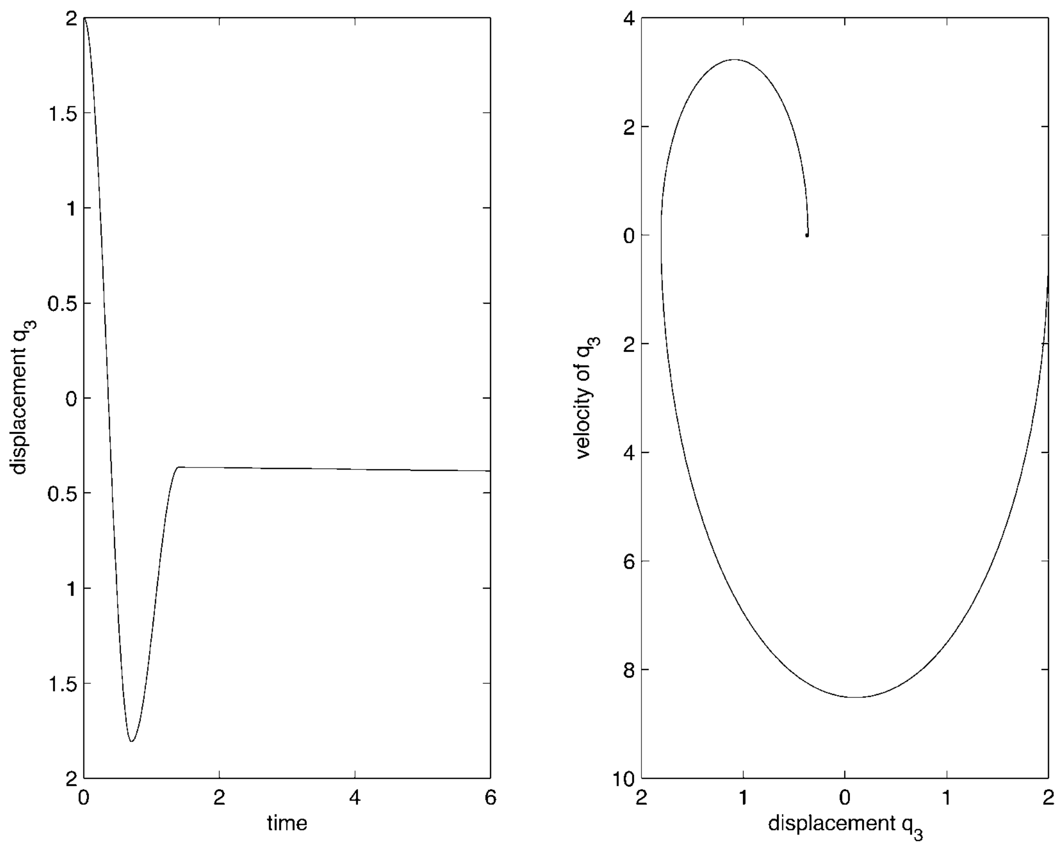

Fig. 5.4

Example 5.3: Fig. 5.5. depicts a model of a shock absorber supported vehicle traveling over a road. The shock absorber involves a linear spring with a positive spring constant k_1 and a linear viscous damper with a positive damping coefficient c_1. The tire is modeled through a mass m. The rigid body of mass M denotes the vehicle. The mass M is topped by a system constituted by a chain of springs and dampers that can be used to model the spinal column of a driver. Here three parts of the column are considered. It is clear that a complete system of vertebrae could be formulatedin a similar way. Here B denotes the mass supported by the spinal column. The road is modeled through the function h. The displacement coordinates are q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7. Let us denote by $\mathcal{N}$ the normal reaction force occurring as soon as the tire is in contact with the road. It is clear that $\mathcal{N} \geq 0$. On the other hand, we must impose the nonpenetration condition $q_0 \geq h$. If $q_0 > h$, then the tire and the road are not in contact so that $\mathcal{N} = 0$. Otherwise, if $\mathcal{N} > 0$ then the tire and the road are in contact and, consequently $q_0 = h$. This normal contact force-displacement relation is depicted in Fig. 5.6.

It is known that the relations

$$q_0 \geq h\,, \qquad \mathcal{N} \geq 0\,,$$
$$q_0 > h \Longrightarrow \mathcal{N} = 0\,,$$
$$\mathcal{N} > 0 \Longrightarrow q_0 = h$$

are equivalent to the set-valued relation

$$\mathcal{N} \in -\partial \Psi_{C(t)}(q_0)\,,$$

where $\Psi_{C(t)}$ denotes the indicator function of the convex set

$$C(t) = \{v \in \mathbb{R} : v \geq h(t)\}\,,$$

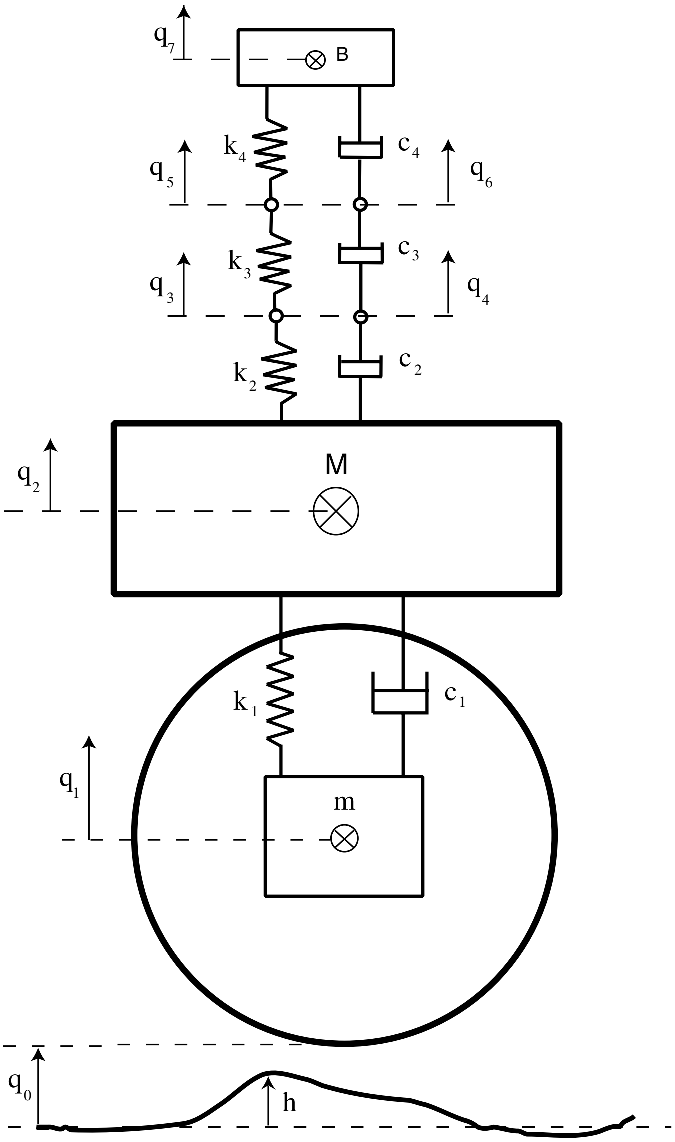

Fig. 5.5

that is

$$\Psi_{C(t)}(z) = \begin{cases} 0 & \text{if} \quad z \in C(t)\,, \\ +\infty & \text{otherwise} \ . \end{cases}$$

The deformation of the tire is here neglected, so that $q_0 = q_1$. Consequently, we may assume that the transmitted part of the normal reaction $\mathcal{N}$ through the tire spring which is applied on the mass m is equal to $\mathcal{N}$. Therefore, the equa-

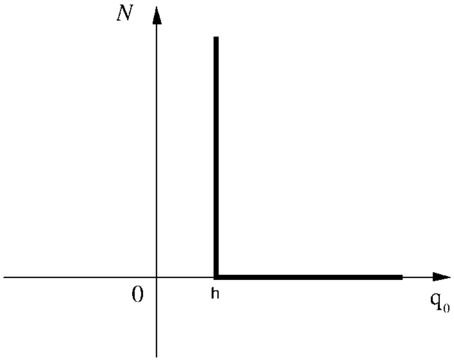

Fig. 5.6. The normal contact force-displacement graph

tions of motion are

$$\begin{aligned}
m\ddot{q}_1 &= k_1(q_2-q_1)+c_1(\dot{q}_2-\dot{q}_1)+\mathcal{N}-mg\,,\\
M\ddot{q}_2 &= -k_1(q_2-q_1)-c_1(\dot{q}_2-\dot{q}_1)+k_2(q_3-q_2)+c_2(\dot{q}_4-\dot{q}_2)-Mg\,,\\
0 &= -k_2(q_3-q_2)+k_3(q_5-q_3)\,,\\
0 &= -c_2(\dot{q}_4-\dot{q}_2)+c_3(\dot{q}_6-\dot{q}_4)\,,\\
0 &= -k_3(q_5-q_3)+k_4(q_7-q_5)\,,\\
0 &= -c_3(\dot{q}_6-\dot{q}_4)+c_4(\dot{q}_7-\dot{q}_6)\,,\\
B\ddot{q}_7 &= -k_4(q_7-q_5)-c_4(\dot{q}_7-\dot{q}_6)-Bg\,.
\end{aligned}$$

We get the system

$$M\ddot{q}(t)+C\dot{q}(t)+Kq(t)\in f(t)+F(t,q(t),\dot{q}(t))\,,$$

where

$$M=\begin{pmatrix} m&0&0&0&0&0&0\\ 0&M&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&B \end{pmatrix},\qquad C=\begin{pmatrix} c_1&-c_1&0&0&0&0&0\\ -c_1&c_1+c_2&0&-c_2&0&0&0\\ 0&0&0&0&0&0&0\\ 0&-c_2&0&c_2+c_3&0&-c_3&0\\ 0&0&0&0&0&0&0\\ 0&0&0&-c_3&0&c_3+c_4&-c_4\\ 0&0&0&0&0&-c_4&c_4 \end{pmatrix},$$

$$K=\begin{pmatrix} k_1&-k_1&0&0&0&0&0\\ -k_1&k_1+k_2&-k_2&0&0&0&0\\ 0&-k_2&k_2+k_3&0&-k_3&0&0\\ 0&0&0&0&0&0&0\\ 0&0&-k_3&0&k_3+k_4&0&-k_4\\ 0&0&0&0&0&0&0\\ 0&0&0&0&-k_4&0&k_4 \end{pmatrix},$$

and

$$q=\begin{pmatrix} q_1\\ q_2\\ q_3\\ q_4\\ q_5\\ q_6\\ q_7 \end{pmatrix},\qquad f(t)=\begin{pmatrix} -mg\\ -Mg\\ 0\\ 0\\ 0\\ 0\\ -Bg \end{pmatrix},\qquad F(t,q(t),\dot{q}(t))=\begin{pmatrix} -\partial\Psi_{C(t)}(q_1)\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{pmatrix}.$$

The matrix $M+C+K$ is symmetric and positive definite. Moreover the matrices M, C, and K satisfy the conditions of Corollary 4.5. Indeed, it is easy to remark that

$$\ker(M)=\ker(M^T)\,,\qquad \ker(M+C)=\ker(M+C^T)\,,\qquad \ker(M+K)=\ker(M+K^T)\,,$$

since the matrices M, $M+C$, and $M+K$ are symmetric. Moreover, it is clear that

$$\operatorname{def}(M)>\operatorname{def}(M+C)>0 \quad \text{with} \quad \ker(M+C)\subset\ker(M)\,.$$

On the other hand, we have

$$\ker(M+C)\oplus\ker(M+K)=\ker(M) \quad \text{with} \quad \ker(M+C)\perp\ker(M+K)\,,$$

$$\ker(M+C)=\operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}\right\},\qquad \ker(M+K)=\operatorname{span}\left\{\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}\right\},$$

and

$$\ker(M) = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\end{pmatrix}\right\}.$$

Here the matrix $U \in \mathbb{R}^{7\times 7}$ is given by

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we obtain

$$U^T M U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & m & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & M & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & B \end{pmatrix},$$

$$U^T C U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & c_2+c_3 & -c_3 & 0 & -c_2 & 0\\ 0 & 0 & -c_3 & c_3+c_4 & 0 & 0 & -c_4\\ 0 & 0 & 0 & 0 & c_1 & -c_1 & 0\\ 0 & 0 & -c_2 & 0 & -c_1 & c_1+c_2 & 0\\ 0 & 0 & 0 & -c_4 & 0 & 0 & c_4 \end{pmatrix},$$

$$U^T K U = \begin{pmatrix} k_2+k_3 & -k_3 & 0 & 0 & 0 & -k_2 & 0\\ -k_3 & k_3+k_4 & 0 & 0 & 0 & 0 & -k_4\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & k_1 & -k_1 & 0\\ -k_2 & 0 & 0 & 0 & -k_1 & k_1+k_2 & 0\\ 0 & -k_4 & 0 & 0 & 0 & 0 & k_4 \end{pmatrix}.$$

Moreover, the matrix $S \in \mathbb{R}^{7\times 3}$ is given by

$$S = U\begin{pmatrix} -K_{11}^{-1}K_{13}\\ -C_{22}^{-1}C_{23}\\ I_{33} \end{pmatrix},$$

where

$$K_{11} = \begin{pmatrix} k_2+k_3 & -k_3\\ -k_3 & k_4+k_3 \end{pmatrix}, \qquad K_{13} = \begin{pmatrix} 0 & -k_2 & 0\\ 0 & 0 & -k_4 \end{pmatrix},$$

$$C_{22} = \begin{pmatrix} c_2+c_3 & -c_3\\ -c_3 & c_3+c_4 \end{pmatrix}, \qquad C_{23} = \begin{pmatrix} 0 & -c_2 & 0\\ 0 & 0 & -c_4 \end{pmatrix}.$$

This implies that

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \dfrac{(k_4+k_3)\,k_2}{k_4k_2+k_4k_3+k_3k_2} & \dfrac{k_3k_4}{k_4k_2+k_4k_3+k_3k_2} \\ 0 & \dfrac{(c_3+c_4)\,c_2}{c_2c_3+c_2c_4+c_3c_4} & \dfrac{c_3c_4}{c_2c_3+c_2c_4+c_3c_4} \\ 0 & \dfrac{k_3k_2}{k_4k_2+k_4k_3+k_3k_2} & \dfrac{(k_2+k_3)\,k_4}{k_4k_2+k_4k_3+k_3k_2} \\ 0 & \dfrac{c_3c_2}{c_2c_3+c_2c_4+c_3c_4} & \dfrac{(c_2+c_3)\,c_4}{c_2c_3+c_2c_4+c_3c_4} \\ 0 & 0 & 1 \end{pmatrix}.$$

It results that

$$S^TMS = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & B \end{pmatrix},$$

$$S^TCS = \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \\ 0 & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \end{pmatrix},$$

$$S^TKS = \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \\ 0 & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \end{pmatrix}.$$

Finally, setting $q = S\tilde{q}$, we see that $\tilde{q}$ is the solution of

$$\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & B \end{pmatrix} \ddot{\tilde{q}} + \begin{pmatrix} c_1 & -c_1 & 0 \\ -c_1 & c_1 + \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \\ 0 & -\dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} & \dfrac{c_4c_2c_3}{c_4c_2+c_2c_3+c_3c_4} \end{pmatrix} \dot{\tilde{q}}$$

$$+ \begin{pmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \\ 0 & -\dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} & \dfrac{k_4k_2k_3}{k_4k_2+k_2k_3+k_3k_4} \end{pmatrix} \tilde{q} \in \begin{pmatrix} -mg \\ -Mg \\ -Bg \end{pmatrix} + \begin{pmatrix} -\partial\Psi_{C(t)}(\tilde{q}_1) \\ 0 \\ 0 \end{pmatrix}.$$

Note here that S^TCS and S^TKS are singular and that $\tilde{q}_1 = q_1 = q_0$, $\tilde{q}_2 = q_2$, and $\tilde{q}_3 = q_7$. The contact is supposed elastic so that the recovery coefficient takes value 1 and q_1 satisfies the impact law

$$\dot{q}_1(t_+) - \dot{h}(t_+) = -(\dot{q}_1(t_-) - \dot{h}(t_-)),$$

for all $t \in [0,T]$ such that $q_1 = 0$ (see for instance [18]). Setting now

$$\bar{q} = \begin{pmatrix} \tilde{q}_1 - h \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix},$$

and assuming that h is Lipschitz continuous and $\dot h$ has bounded variations, we obtain the system

$$\ddot{\bar q} + \begin{pmatrix} \dfrac{c_1}{m} & -\dfrac{c_1}{m} & 0 \\ -\dfrac{c_1}{M} & \dfrac{c_1}{M} + \dfrac{c_4c_2c_3}{M(c_4c_2+c_2c_3+c_3c_4)} & -\dfrac{c_4c_2c_3}{M(c_4c_2+c_2c_3+c_3c_4)} \\ 0 & -\dfrac{c_4c_2c_3}{B(c_4c_2+c_2c_3+c_3c_4)} & \dfrac{c_4c_2c_3}{B(c_4c_2+c_2c_3+c_3c_4)} \end{pmatrix} \dot{\bar q}$$

$$+ \begin{pmatrix} \dfrac{k_1}{m} & -\dfrac{k_1}{m} & 0 \\ -\dfrac{k_1}{M} & \dfrac{k_1}{M} + \dfrac{k_4k_2k_3}{M(k_4k_2+k_2k_3+k_3k_4)} & -\dfrac{k_4k_2k_3}{M(k_4k_2+k_2k_3+k_3k_4)} \\ 0 & -\dfrac{k_4k_2k_3}{B(k_4k_2+k_2k_3+k_3k_4)} & \dfrac{k_4k_2k_3}{B(k_4k_2+k_2k_3+k_3k_4)} \end{pmatrix} \bar q$$

$$+ \begin{pmatrix} \ddot h + \dfrac{c_1}{m}\,\dot h + \dfrac{k_1}{m}\,h + g \\ -\dfrac{c_1}{M}\,\dot h - \dfrac{k_1}{M}\,h + g \\ g \end{pmatrix} \in -\partial\psi_K(\bar q)\,, \tag{5.2}$$

where $K = \mathbb{R}_+ \times \mathbb{R}$. Note that

$$\partial\psi_K(\bar q) = \begin{pmatrix} \partial\psi_{\mathbb{R}_+}(\bar q_1) \\ 0 \\ 0 \end{pmatrix}.$$

System (5.2) is considered with the initial conditions

$$\bar q(0) = \bar q_0 \in \mathbb{R}_+\,, \qquad \dot{\bar q}(0_+) = \bar q_1 \in T_{\mathbb{R}_+}(\bar q_0)\,,$$

and the impact law, as used in [18],

$$\dot{\bar q}(t_+) = -\dot{\bar q}_N(t_-) + \dot{\bar q}_T(t_-)\,,$$

for all $t \in [0,T]$ such that $\bar q(t) \in \partial K$, i.e. $\bar q_1(t) = 0$. The last formulation of the impact law follows from the fact that here

$$\dot{\bar q}_N = \begin{pmatrix} \dot{\bar q}_1 - \dot h \\ 0 \\ 0 \end{pmatrix}, \qquad \dot{\bar q}_T = \begin{pmatrix} 0 \\ \dot{\bar q}_2 \\ \dot{\bar q}_3 \end{pmatrix}.$$

It is now easy to remark that all assumptions required in Theorem 1.2 are here satisfied. Therefore, the existence of a solution in the sense of Theorem 1.2 is ensured. To solve Problem (5.2) a Yosida approximant $(i_d - P_{C(t)})/\lambda$ of $\partial\psi_{C(t)}$ with λ small has been considered (see also (1.8)). Note here that we have

$$\frac{x - P_{C(t)}x}{\lambda} = \frac{x - \max\{x, h(t)\}}{\lambda}\,.$$

The following technical data were considered:

M	m	B	k_1	k_2	k_3	k_4	c_1	c_2	c_3	c_4
1460	35	25	95×10^5	20×10^5	15×10^5	10^6	21700	20000	15000	10000

and displacements q_1, q_2, and q_7 have been simulated for the following type of road:

$$\text{Figs. 5.7, 5.8:} \begin{cases} \max\{0,\, 0.05\,\sin(10\pi t)\} & \text{if} \quad 0.2 < t < 0.3\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

The difference between q_2 and q_7 cannot be really distinguished for the previous data. For this aim, let us end this section with data leading to displacements q_2 and q_7 whose relative changes are appreciable (see Fig. 5.9.). The data that we consider are (cf. the table on the top of p. 242).

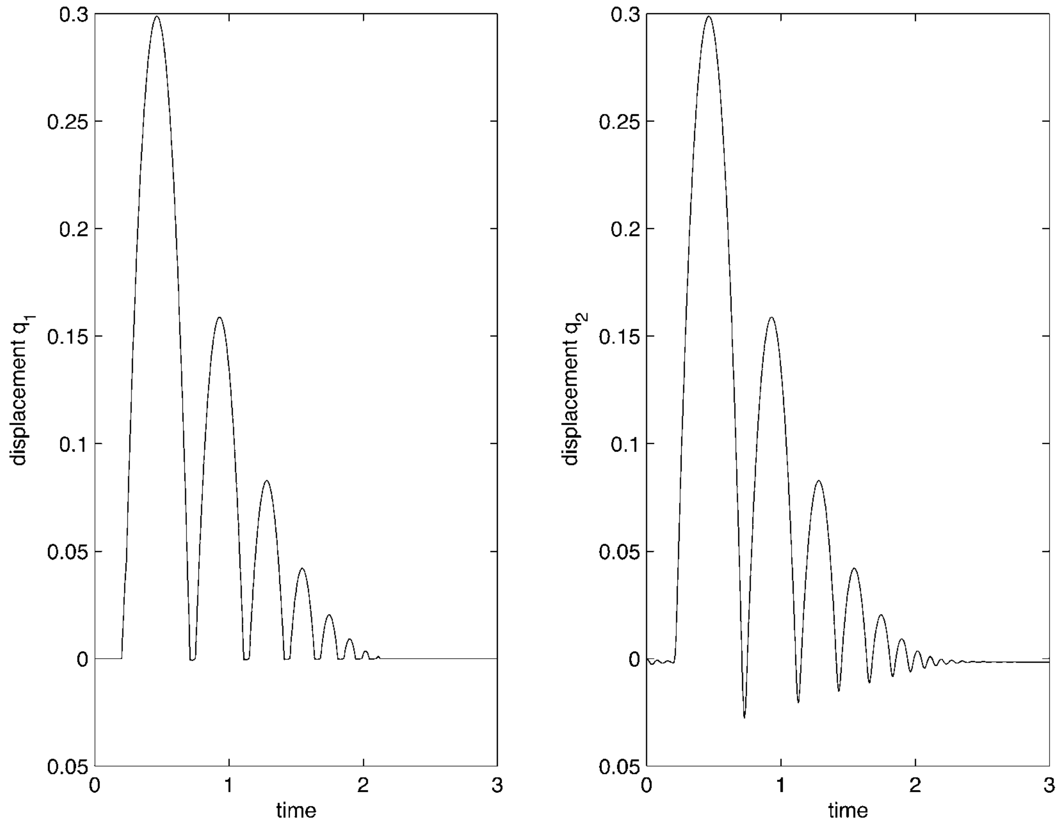
displacement q_1
displacement q_2
time
time

Fig. 5.7

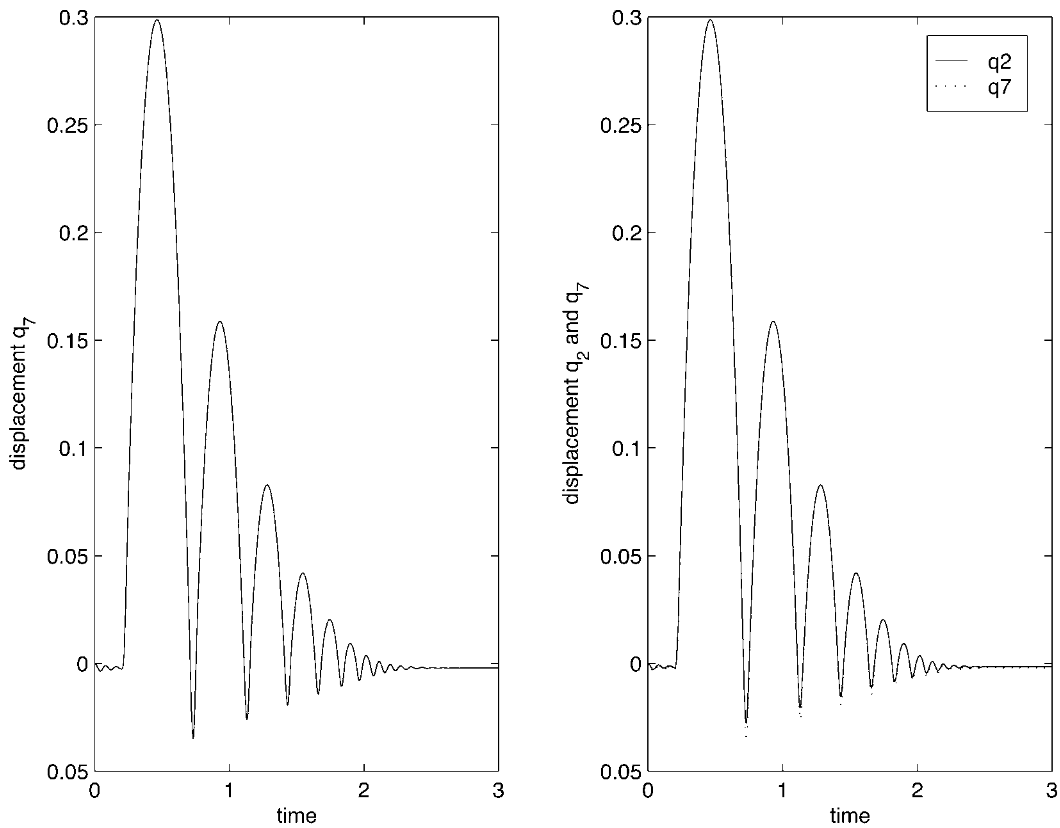
displacement q_7
displacement q_2 and q_7
q2
q7
time
time

Fig. 5.8

M	m	B	k_1	k_2	k_3	k_4	c_1	c_2	c_3	c_4
1460	35	25	35×10^5	20000	15000	10000	21700	2000	1500	1000

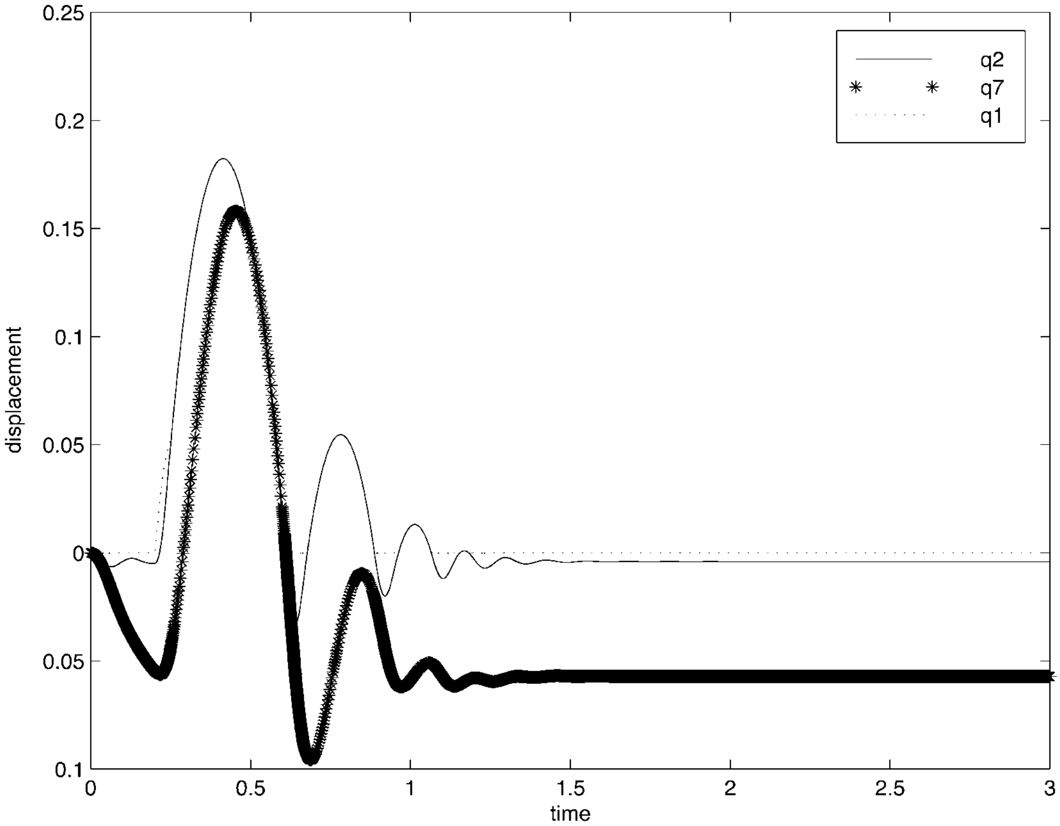

Fig. 5.9

Example 5.4: Reduction methods are of great interest in the mathematical treatment of models of skeletal muscle. The approach discussed in Section 2 is illustrated here. The system displayed in Fig. 5.8. is formed by a model of a fiber of some skeletal muscle and a device of mass m used to simulate the fiber response to external forces. The symbols T_0 and T_N denote the tendinous fiber parts. A linear model for the force across the tendinous part is $(i = 0, \ldots, N)$

$$F_{T_i} = k_i \delta_i + c_i \dot{\delta}_i \,, \tag{5.3}$$

where δ_i denotes the relative elongation of the element T_i and k_i, c_i are positive constants. The symbols T_i $(i = 1, \ldots, N-1)$ denote the Z-disks marking the boundaries of the muscle fiber. The linear model (5.3) is also considered for $i = 1, \ldots, N-1$. The symbols F_i $(i = 1, \ldots, N-1)$ are used to represent the fibril substructure. The springs of constant K_i $(i = 1, \ldots, N-1)$ are introduced to model the i-th sarcomere within the fiber. The symbols B_i $(i = 1, \ldots, N-1)$ denote the cross-bridges. The force across the cross-bridges is defined by the relation $(i = 1, \ldots, N-1)$

$$F_{B_i} = \gamma_i \lambda_i + \theta_i \dot{\lambda}_i \,,$$

where λ_i is the relative elongation of the element B_i and γ_i, θ_i are positive constants. The "boxes" C_i $(i = 1, \ldots, N-1)$ are pure contractile elements. These elements are active components in the fiber model. The force output of the i-th-contractile machinery is denoted by $\delta_{C_i}(\theta_i)$, where θ_i is the relative elongation of the element C_i. For further details about the functional characteristics and models of muscle elements, we refer the reader to [13]. The machine slide yields a total friction force that can be written as follows:

$$\tau \in \Gamma(-\dot{q}_N) \,,$$

where Γ is defined as in example 5.1.

For example, in the case $N = 2$, the dynamic of the model is described by the system

$$M\ddot{Q}(t) + C\dot{Q}(t) + KQ(t) \in F(Q(t)) \,,$$

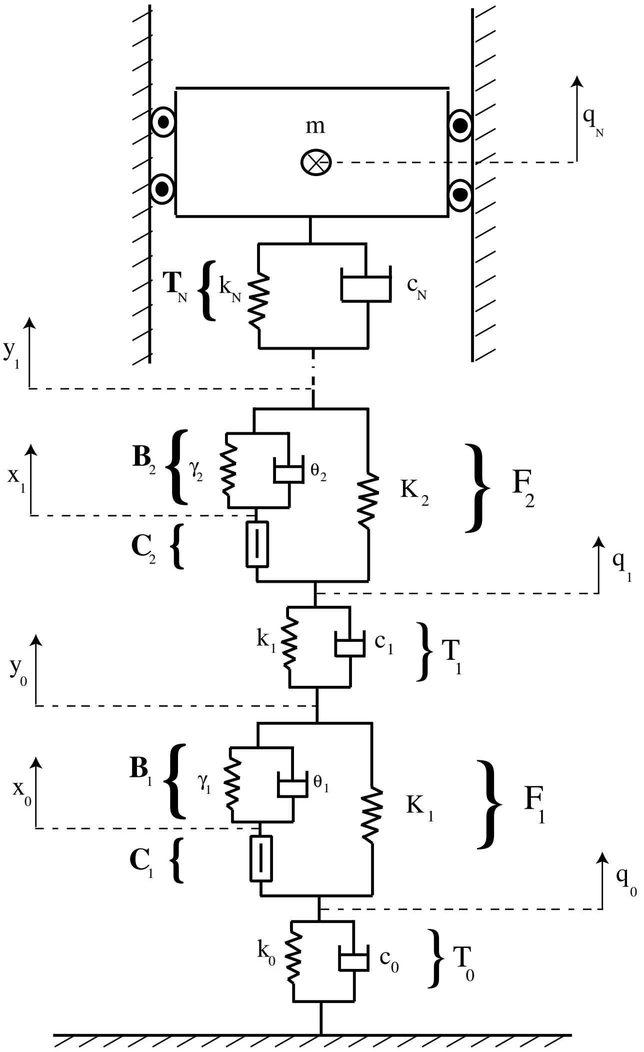

Fig. 5.10

where

$$M = \begin{pmatrix} 0_{6\times 6} & 0_{6\times 1} \\ 0_{1\times 6} & m \end{pmatrix},$$

$$K = \begin{pmatrix} k_0+K_1 & 0 & -K_1 & 0 & 0 & 0 & 0 \\ 0 & \gamma_1 & -\gamma_1 & 0 & 0 & 0 & 0 \\ 0 & -\gamma_1-K_1 & \gamma_1+k_1+K_1 & -k_1 & 0 & 0 & 0 \\ 0 & 0 & -k_1 & k_1+K_2 & 0 & -K_2 & 0 \\ 0 & 0 & 0 & 0 & \gamma_2 & -\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & -\gamma_2-K_2 & \gamma_2+K_2+k_2 & -k_2 \\ 0 & 0 & 0 & 0 & 0 & -k_2 & k_2 \end{pmatrix},$$

$$C = \begin{pmatrix} c_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & -\theta_1 & 0 & 0 & 0 & 0 \\ 0 & -\theta_1 & \theta_1+c_1 & -c_1 & 0 & 0 & 0 \\ 0 & 0 & -c_1 & c_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_2 & -\theta_2 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 & \theta_2+c_2 & -c_2 \\ 0 & 0 & 0 & 0 & 0 & -c_2 & c_2 \end{pmatrix}, \quad Q = \begin{pmatrix} q_0 \\ x_0 \\ y_0 \\ q_1 \\ x_1 \\ y_1 \\ q_2 \end{pmatrix}, \quad F(Q) = \begin{pmatrix} F_{C_1}(x_0-q_0) \\ -F_{C_1}(x_0-q_0) \\ 0 \\ F_{C_2}(x_1-q_1) \\ -F_{C_2}(x_1-q_1) \\ 0 \\ \Gamma(-\dot{q}_2) \end{pmatrix}.$$

Setting $X_1 = q_0$, $X_2 = x_0$, $X_3 = y_0$, $X_4 = q_1$, $X_5 = x_1$, $X_6 = y_1$, $X_7 = q_2$, $X_8 = \dot{q}_0$, $X_9 = \dot{x}_0$, $X_{10} = \dot{y}_0$, $X_{11} = \dot{q}_1$, $X_{12} = \dot{x}_1$, $X_{13} = \dot{y}_1$, $X_{14} = \dot{q}_2$, we rewrite the system as the first order differential inclusion

$$E\dot{X} \in AX + G(X),$$

where

$$E = \begin{pmatrix} I_{7\times 7} & 0_{7\times 6} & 0_{7\times 1} \\ 0_{6\times 7} & 0_{6\times 6} & 0_{6\times 1} \\ 0_{1\times 7} & 0_{1\times 6} & m \end{pmatrix}, \qquad A = \begin{pmatrix} 0_{7\times 7} & I_{7\times 7} \\ -K & -C \end{pmatrix}, \qquad G = \begin{pmatrix} 0_{7\times 1} \\ F_{C_1}(X_2 - X_1) \\ -F_{C_1}(X_2 - X_1) \\ 0 \\ F_{C_2}(X_5 - X_4) \\ -F_{C_2}(X_5 - X_4) \\ 0 \\ \Gamma(-X_{14}) \end{pmatrix}.$$

Let us consider now our problem with the following data:

m	k_0	c_0	K_1	γ_1	θ_1	k_1	c_1	K_2	γ_2	θ_2	k_N	c_N
1	2	2	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1

Applying the method discussed in Section 2 with $\lambda = 1$, we check that $\hat{E}_1 := (E - A)^{-1} E$ has a normal Jordan form $\hat{E}_1 = TJT^{-1}$ where

$$J = \begin{pmatrix} W & 0 \\ 0 & N \end{pmatrix}, \quad W = \begin{pmatrix} \frac{2}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the notation of Section 2, we see that

$$g = T^{-1}(A - E)^{-1}\, \mathrm{e}^{-t}\, G(\mathrm{e}^t Tv(t))\,.$$

Let us recall that $v := T^{-1}u$ and $u = \mathrm{e}^{-t}Q$. The explicit computation of g shows that the first system in (2.5) consists in two differential inclusions and five ordinary differential equations. On the other hand, the second system in (2.5) is formed by one differential equation and six algebraic equations.

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