Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 18, NUMBER 2 (2015)
(Print) ISSN 1311-0454
(Electronic) ISSN 1314-2224

RESEARCH PAPER

SOLVING FRACTIONAL DELAY DIFFERENTIAL EQUATIONS: A NEW APPROACH

Varsha Daftardar-Gejji [1], Yogita Sukale [2], Sachin Bhalekar [3]

Abstract

A new method to solve non-linear fractional-order differential equations involving delay has been presented. Applications to a variety of problems demonstrate that the proposed method is more accurate and time efficient compared to existing methods. A detailed error analysis has also been given.

MSC 2010: Primary 34A08, 26A33; Secondary 65J15, 4725

Key Words and Phrases: delay differential equations, fractional nonlinear differential equations, Adam's method, new iterative method, predictor-corrector method

1. Introduction

The analysis and applications of fractional differential equations (FDEs) has currently become an active area of research owing to their applications to a variety of problems in Science and Engineering, see e.g. [14, 16]. In last two decades a number of differential equations have been generalized using fractional order derivatives to model non-local phenomena. Developing suitable methods to solve FDEs has thus been of growing interest in the recent years. Transform methods used for ordinary differential equations (ODEs) are extended for deriving solutions of linear FDEs [16]. Nonetheless solving non-linear FDEs is relatively difficult. Pursuance to this, analytical

pp. 400–418 , DOI: 10.1515/fca-2015-0026 DE GRUYTER

methods for example, Adomian decomposition, homotopy perturbation and New iterative methods, have widely been employed in the recent literature [1, 11, 18, 5, 6, 2]. On the other hand the works of Diethelm *et al* [9, 10] extending standard numerical methods such as Adams-Bashforth method, have widely been employed for numerical integration of FDEs.

Inclusion of delay in the FDEs seems to be opening new vistas especially in the field of bioengineering [3]. Fractional order delay differential equations (FDDEs) are also finding applications in all disciplines including chemistry, physics, and finance [19]. Analysis of FDDEs has been carried out [12, 15]. With this motif Adams-Bashforth method is extended to solve non-linear FDDEs [4, 17].

Recently Daftardar-Gejji, Sukale and Bhalekar have introduced a new predictor-corrector method [7], based on the earlier developed New iterative method proposed by Daftardar-Gejji and Jafari [6], to numerically solve FDEs. The method has shown to be accurate and time efficient compared to other methods. Fractional derivatives are non-local and incorporate memory effects due to which integrating FDEs is relatively difficult and time consuming. In such a situation this method [7] deserves attention. In the present paper we extend the new predictor-corrector method for solving FDDEs and carry out related error analysis.

The paper is organized as follows. Notations, basic definitions and properties of fractional integral/derivative are given in Section 2. In Section 3 fractional Adams method for fractional delay differential equations is outlined. In Section 4 new iterative method (NIM) is described and further a new predictor corrector method for fractional delay differential equations has been presented followed by the error analysis of the newly developed method in Section 5. Illustrative examples are given in Section 6 and conclusions are summarized in Section 7.

2. **Fractional integral/derivative**

In the present section basic preliminaries and definitions are given. Let $f(x) \in C[0,a]$ and $\alpha > 0$, then the (left sided) Riemann-Liouville integral of fractional order α is defined as [16]

$$I_0^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad 0 < x < a.$$

The (left sided) Caputo fractional derivative of $f(x)$, $f \in C^m$, $m \in \mathbb{N}$ of order α is defined as follows:

$${}^cD_0^\alpha f(x) = \begin{cases} I_0^{m-\alpha} f^{(m)}(x) & \text{if} \quad m-1 < \alpha < m; \\ \dfrac{d^m}{dx^m} f(x) & \text{if} \qquad \alpha = m. \end{cases}$$

Note that:

1. $I_0^\alpha[{}^cD_0^\alpha f(x)] = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0)\frac{x^k}{k!}$, $m = \lceil\alpha\rceil$, where $\lceil\alpha\rceil$ is greatest integer less than or equal to α.
2. $I_0^\alpha x^\nu = \dfrac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)}\, x^{\alpha+\nu}$.

In the following sections, ${}^cD^\alpha$ will denote ${}^cD_0^\alpha$ for brevity of notation.

3. Fractional Adams method for FDE with delay

Consider the following initial value problem (IVP) for fractional delay differential equation:

$$\begin{aligned} {}^cD_0^\alpha y(t) &= f_1(t, y(t), y(t-\tau)),\ \ t \geq 0,\ \ \tau > 0,\ \ m-1 < \alpha \leq m, \\ y(t) &= \phi(t), \quad -\tau \leq t \leq 0, \end{aligned} \tag{3.1}$$

where ${}^cD_0^\alpha$, denotes the Caputo derivative of order α. It is well known that the initial value problem (IVP) (3.1) is equivalent to the following Volterra integral equation [8]:

$$y(t) = \sum_{k=0}^{\lceil\alpha\rceil-1} \phi(t)\frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f_1(s, y(s), y(s-\tau))ds. \tag{3.2}$$

For solving Eq. (3.2) on $[0, T]$, the interval $[-\tau, T]$ is divided into l subintervals. Let $h = \frac{T}{l}, t_n = nh, n = 0, 1, 2, \ldots, l \in \mathbb{Z}^+$. The discretized version of (3.2) is given as [9]:

$$y(t_{n+1}) = \sum_{k=0}^{\lceil\alpha\rceil-1} \phi_k(t_{n+1})\frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f_1(s, y(s), y(s-\tau))ds. \tag{3.3}$$

Bhalekar and Daftardar-Gejji [4] have extended fractional Adams method [9] for solving the FDEs involving constant delay. Further Wang [17] has extended this method for solving the FDEs involving variable delay wherein the following product trapezoidal quadrature formula has been used.

$$y_{n+1} = \sum_{k=0}^{\lceil\alpha\rceil-1} \phi_k(t)\frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n+1} a_{j,n+1} f_1(t_j, y_j, y(t_j-\tau)), \tag{3.4}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & \text{if} \quad j=0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1} & \text{if} \quad 1 \leq j \leq n, \\ 1 & \text{if} \quad j = n+1. \end{cases}$$

Eq. (3.4) can be written as

$$\begin{aligned} y_{n+1} &= \sum_{k=0}^{\lceil\alpha\rceil-1} \phi_k(0)\frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, y(t_j-\tau)) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}, y(t_{n+1}-\tau)), \end{aligned} \tag{3.5}$$

where y_{n+1} and $y(t_{n+1})$ denote approximate and exact solutions respectively. Eq. (3.5) is of the form $u = f + N(u)$, and can be solved by a New Iterative Method (NIM), [2, 6]. In the present paper we develop a new predictor-corrector method which is based on the NIM. The details regarding NIM are furnished in Section 4.

3.1. **Approximation of the delay term.** The approximation of the delay term in Eq. (3.5) is denoted by ν_n.

Case 1. τ is constant.

When τ is constant, $(t_n - \tau)$ may not be a grid point t_n for any n. Suppose $(m+\delta)h = \tau, m \in \mathbb{N}$ and $0 \le \delta < 1$. If $\delta = 0$, $y(t_n - \tau)$ is approximated as

$$y(t_n - \tau) \approx \nu_n = \begin{cases} y_{n-m} & \text{if } n \ge m; \\ \phi(t_n - \tau) & \text{if } n < m. \end{cases}$$

Note that $(m-1)h < \tau < mh$. Now if $0 < \delta < 1$ and $m > 1$ we interpolate ν_{n+1} by the two nearest points,

$$y(t_{n+1} - \tau) \approx \nu_{n+1} = \begin{cases} \delta y_{n-m+2} + (1-\delta) y_{n-m+1} & \text{if } n+1 \ge m, \\ \delta y_{n-m+2} + (1-\delta)\phi(t_{n+1} - \tau) & \text{if } n+1 < m. \end{cases}$$

If $m = 1$ and $\delta \ne 0$, $\nu_{n+1} = \delta y_{n+1} + (1-\delta) y_n$. Hence we need to predict y_{n+1} that appears on RHS of the above equation. Predictor to the term y_{n+1} we take as the three term approximate solution $u_0 + u_1 + u_2$ obtained by new predictor corrector method which is discussed further in Section 4.2.

Case 2. $\tau = \tau(t)$. In this case approximation to $y(t_{n+1} - \tau) \approx \nu_{n+1}$ can be written as

$$\nu_{n+1} = \begin{cases} \delta_{n+1} y_{n-m_{n+1}+2} + (1-\delta_{n+1}) y_{n-m_{n+1}+1} & \text{if } n+1 \ge m, \\ \delta_{n+1} y_{n-m_{n+1}+2} + (1-\delta_{n+1})\phi(t_{n+1} - \tau) & \text{if } n+1 < m, \end{cases}$$

where $\tau(t_{n+1}) = (m_{n+1} + \delta_{n+1})h$, m_{n+1} is a positive integer and $\delta_{n+1} \in [0, 1)$.

4. **New iterative method (NIM)**

Daftardar-Gejji and Jafari [6] have proposed a new iterative method for solving linear/non-linear functional equations of the form

$$u = f + N(u), \tag{4.1}$$

where f is a known function and N a non linear operator. Equation (4.1) represents integral equations, ordinary differential equations, fractional order differential equations and so on. Solutions obtained by this method are in the form of rapidly converging infinite series which can be effectively approximated by calculating only first few terms.

In this method non linear operator N is decomposed as

$$N(u) = N(u_0) + [N(u_0+u_1) - N(u_0)] + [N(u_0+u_1+u_2) - N(u_0+u_1)] + \dots$$

Denote by $G_0 = N(u_0)$, $G_i = N(\sum_{n=0}^{i} u_n) - N(\sum_{n=0}^{i-1} u_n), i = 1, 2, 3, \dots$. Observe that $N(u) = \sum_{i=0}^{\infty} G_i$. Put $u_0 = f$, and $u_n = G_{n-1}, n = 1, 2, 3, \dots$ Note that

$$u = u_0 + u_1 + u_2 + \cdots = f + N(u_0) + [N(u_0+u_1) - N(u_0)] + \cdots = f + N(u).$$

Hence u satisfies the functional Eq. (4.1). In the present work NIM is used to solve Eq. (3.5).

4.1. **New algorithm.** It should be noted that Eq. (3.5) is of the form $y_{n+1} = f + N(y_{n+1})$, where

$$f = \sum_{k=0}^{\lceil\alpha\rceil-1} \phi_k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, v_j) \tag{4.2}$$

and

$$N(y_{n+1}) = \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}, \nu_{n+1}). \tag{4.3}$$

Thus we can employ NIM for getting approximate value of y_{n+1}.

4.2. **New predictor-corrector formula.** The three term approximation of the NIM scheme gives the following two-step predictor-corrector formula:

$$u_0 = y_{n+1}^p = \sum_{k=0}^{\lceil\alpha\rceil-1} \phi_k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, \nu_j), \tag{4.4}$$

$$u_1 = N(u_0) = z_{n+1}^p = \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}^p, \nu_{n+1}), \tag{4.5}$$

$$u_2 = N(u_0+u_1) - N(u_0), \tag{4.6}$$

where ν_{n+1} is given in Section 3.1. Hence the three term approximate solution is $u = u_0 + u_1 + u_2 = u_0 + N(u_0 + u_1)$,

$$y_{n+1}^{c} = y_{n+1}^{p} + \frac{h^{\alpha}}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}^{p} + z_{n+1}^{p}, \nu_{n+1}). \tag{4.7}$$

Here y_{n+1}^{p} and z_{n+1}^{p} are called as predictors and y_{n+1}^{c} is the corrector, and y_j denotes the approximate value of the solution.

5. Error analysis of the new algorithm

5.1. Preliminaries.

THEOREM 5.1. ([10, 13]) *(a) Let $y(t) \in C^2[0,T]$, then there exists a constant C_α depending on α such that*

$$\left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} y(t)dt - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} y(t_j) \right| \le C_\alpha t_{n+1}^{\alpha} h^2.$$

(b) Let $y \in C^1[0,T]$ and assume that y' fulfills Lipschitz condition of order μ for some $\mu \in (0,1)$. Then there exist positive constants $B_{\alpha,\mu}^{Tr}$ and $M(y,\mu)$ such that

$$\left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} y(t)dt - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} y(t_j) \right| \le B_{\alpha,\mu}^{Tr} M(y,\mu) t_{n+1}^{\alpha} h^{1+\mu}.$$

(c) Let $y(t) = t^p; p \in (0,2)$ and $Q = \min(2, p+1)$, then

$$\left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} y(t)dt - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1} y(t_j) \right| \le c_\alpha^{Tr} t_{n+1}^{\alpha+p-Q} h^{Q}.$$

LEMMA 5.1. ([13]) *Let $\psi \in C^1[0,T]$ and $0 < \alpha < 1$, then*

$$F_{n+1}\left[\int_0^{t} (t-\tau)^{-\alpha} \psi'(\tau) d\tau \right] \le \frac{A}{(1-\alpha)} ||\psi'||_\infty t_{n+1}^{\alpha} h^{1-\alpha},$$

where A is a constant independent of k and h. Here F_n is defined as:

$$F_{n+1}[g(t)] = \left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} g(t)dt - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n+1} a_{j,n+1}\, g(t_j) \right|.$$

For proof of the above lemma, see [13, Lemma 3.3].

5.2. **Main results.** Let f_1 be a function defined on a suitable set G. In this section we prove some results regarding error bounds under different conditions on the function f_1 and solution $y(t)$. We assume that f_1 satisfies Lipschitz conditions with respect to its variables as follows:

$$|f_1(t, y_1, u) - f_1(t, y_2, u)| \leq L_1|y_1 - y_2|,$$

$$|f_1(t, y, u_1) - f_1(t, y, u_2)| \leq L_2|u_1 - u_2|,$$

where L_1 and L_2 are positive constants.

THEOREM 5.2. *Suppose the solution $y(t) \in C^2[0, T]$ of the IVP (3.1) satisfies the following condition:*

$$\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1}\, {}^cD_0^\alpha y(t)dt - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, {}^cD_0^\alpha y(t_j) \right| \leq c t_{n+1}^\gamma h^\delta \tag{5.1}$$

for some $\gamma \geq 0$ and $\delta > 0$ and f_1 satisfies Lipschitz condition in variables y and μ with Lipschitz constants L_1 and L_2 respectively. Then for some suitably chosen $T > 0$, we have

$$\max_{0\leq j\leq l} |y(t_j) - y_j| \leq kh^\delta,$$

where $l = \frac{T}{h}$, and k is a positive constant. Here $y(t_j)$ denotes exact solution of the IVP and y_j denotes the approximate solution of IVP obtained by the new predictor-corrector method.

P r o o f. We prove this result using the principle of mathematical induction. Suppose that the conclusion is true for $j = 0, 1, 2, \ldots, n$. Note that

$$\begin{aligned}
&|f_1(t_j, y(t_j), y(t_j - \tau)) - f_1(t_j, y_j, \nu_j)| \\
&= |f_1(t_j, y(t_j), y(t_j - \tau)) + f_1(t_j, y_j, y(t_j - \tau)) \\
&- f_1(t_j, y_j, y(t_j - \tau)) - f_1(t_j, y_j, \nu_j)| \\
&\leq |f_1(t_j, y(t_j), y(t_j - \tau)) - f_1(t_j, y_j, y(t_j - \tau))| \\
&+ |f_1(t_j, y_j, y(t_j - \tau)) - f_1(t_j, y_j, \nu_j)| \\
&\leq L_1|y(t_j) - y_j| + L_2|y(t_j - \tau) - \nu_j| \leq (L_1 + L_2)h^\delta.
\end{aligned} \tag{5.2}$$

By induction hypothesis Eq. (5.1) is true for $j = 1, 2, \ldots, n$ and we have to prove that the inequality holds for $j = n + 1$.

At $(n+1)^{st}$ step $|y(t_{n+1}) - y_{n+1}^p|$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)} \Big| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} f_1(t, y(t), y(t-\tau))dt \\
&\quad - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, v_j) \Big| \\
&\le \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1}\, {}^cD_0^\alpha y(t)dt - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, {}^cD_0^\alpha y(t_j) \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1} |f_1(t_j, y(t_j), y(t_j-\tau)) - f_1(t_j, y_j, \nu_j)| \\
&\le \frac{c t_{n+1}^\gamma h^\delta}{\Gamma(\alpha)} + \frac{(L_1+L_2) h^{\alpha+\delta} T^\alpha}{\alpha \Gamma(\alpha+2)}\,, \qquad (5.3)
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{j=0}^{n} a_{j,n+1} &\le \sum_{j=0}^{n} \big[(n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}\big] \\
&= \sum_{j=0}^{n} \big[(n-j+2)^{\alpha+1} - (n-j+1)^{\alpha+1} - (n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}\big] \\
&= \Big[\int_0^{t_{n+1}} (t_{n+2}-t)^\alpha dt - \int_0^{t_{n+1}} (t_{n+1}-t)^\alpha dt \Big] \\
&= \frac{1}{\alpha} \int_0^{t_{n+1}} [(t_{n+1}-t)^\alpha]'(t) dt = \frac{1}{\alpha} t_{n+1}^\alpha \le \frac{T^\alpha}{\alpha}.
\end{aligned}
$$

Thus

$$
|y(t_{n+1}) - y_{n+1}^p| \le \frac{c t_{n+1}^\gamma h^\delta}{\Gamma(\alpha)} + \frac{h^{\alpha+\delta}(L_1+L_2)T^\alpha}{\alpha\Gamma(\alpha+2)}\,. \qquad (5.4)
$$

Now consider

$$
\begin{aligned}
&\big| y(t_{n+1}) - [y_{n+1}^p + z_{n+1}^p] \big| \\
&\quad = \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} f_1(t, y(t), y(t-\tau))dt \\
&\quad - \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f_1(t_j, y_j, \nu_j) - \frac{h^\alpha}{\Gamma(\alpha+2)} f_1(t_{n+1}, y_{n+1}^p, \nu_{n+1}) \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\Bigg\{\Bigg|\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}f_1(t,y(t),y(t-\tau))dt \\
&-\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n+1}a_{j,n+1}f_1(t_j,y(t_j),y(t_j-\tau))\Bigg| \\
&+\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n}a_{j,n+1}\left|f_1(t_j,y_j,\nu_j)-f_1(t_j,y(t_j),y(t_j-\tau))\right| \\
&+\frac{h^\alpha}{\alpha(\alpha+1)}\left|f_1(t_{n+1},y(t_{n+1}),y(t_{n+1}-\tau))-f_1(t_{n+1},y^p_{n+1},\nu_{n+1})\right|\Bigg\} \\
&\le \frac{1}{\Gamma(\alpha)}\Bigg[ct^\gamma_{n+1}h^\delta+\frac{(L_1+L_2)h^{\alpha+\delta}T^\alpha}{\alpha^2(\alpha+1)}+\frac{h^\alpha}{\alpha(\alpha+1)}\Bigg(L_1[\frac{ct^\gamma_{n+1}h^\delta}{\Gamma(\alpha)} \\
&\qquad\qquad +\frac{h^{\alpha+\delta}}{\alpha\Gamma(\alpha+2)}(L_1+L_2)T^\alpha]+L_2h^\delta\Bigg)\Bigg] \\
&= \frac{ct^\gamma_{n+1}h^\delta}{\Gamma(\alpha)}+\frac{(L_1+L_2)T^\alpha h^{\alpha+\delta}}{\alpha\Gamma(\alpha+2)}+\frac{L_1ct^\gamma_{n+1}h^{\alpha+\delta}}{\Gamma(\alpha)\Gamma(\alpha+2)} \\
&\qquad +\frac{L_1(L_1+L_2)T^\alpha h^{2\alpha+\delta}}{\alpha^2(\alpha+1)\Gamma(\alpha)(\Gamma(\alpha+2))}+\frac{L_2h^{\alpha+\delta}}{\Gamma(\alpha+2)}. \qquad (5.5)
\end{aligned}
$$

Using Eq. (5.2), Eq. (5.4), Eq. (5.5) we find a bound for a difference between actual solution and three term approximate solution,

$$
\begin{aligned}
&\left|y(t_{n+1})-\left[y^p_{n+1}+\frac{h^\alpha}{\Gamma(\alpha+2)}f_1(t_{n+1},y^p_{n+1}+z^p_{n+1},\nu_{n+1})\right]\right| \\
&= \Bigg|\sum_{k=0}^{\lceil\alpha\rceil-1}\frac{\phi_k t^k_{n+1}}{k!}+\frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}f_1(t,y(t),y(t-\tau))dt \\
&-\left[y^p_{n+1}+\frac{h^\alpha}{\Gamma(\alpha+2)}f_1(t_{n+1},y^p_{n+1}+z^p_{n+1},\nu_{n+1})\right]\Bigg| \\
&\le \frac{1}{\Gamma(\alpha)}\Bigg\{\Bigg|\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}f_1(t,y(t),y(t-\tau))dt \\
&-\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n+1}a_{j,n+1}f_1(t_j,y(t_j),y(t_j-\tau))\Bigg| \\
&+\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n}a_{j,n+1}\left|f_1(t_j,y_j,\nu_j)-f_1(t_j,y(t_j),y(t_j-\tau))\right| \\
&+\frac{h^\alpha}{\alpha(\alpha+1)}\left|f_1(t_{n+1},y(t_{n+1}),y(t_{n+1}-\tau))-f_1(t_{n+1},y^p_{n+1}+z^p_{n+1},\nu_{n+1})\right|\Bigg\}
\end{aligned}
$$

$$= \frac{ct_{n+1}^{\gamma}h^{\delta}}{\Gamma(\alpha)} + \frac{(L_1+L_2)T^{\alpha}h^{\alpha+\delta}}{\alpha\Gamma(\alpha+2)} + \frac{L_1ct_{n+1}^{\gamma}h^{\alpha+\delta}}{\Gamma(\alpha)\Gamma(\alpha+2)} + \frac{L_1(L_1+L_2)T^{\alpha}h^{2\alpha+\delta}}{\alpha\Gamma(\alpha+2)}$$
$$+ \frac{L_1^2ct_{n+1}^{\gamma}h^{2\alpha+\delta}}{\Gamma(\alpha)(\Gamma(\alpha+2))^2} + \frac{L_1^2(L_1+L_2)T^{\alpha}h^{3\alpha+\delta}}{\alpha(\Gamma(\alpha+2))^3} + \frac{L_1L_2h^{2\alpha+\delta}}{(\Gamma(\alpha+2))^2} + \frac{L_2h^{\alpha+\delta}}{\Gamma(\alpha+2)}$$
$$\leq kh^{\delta}.$$

□

THEOREM 5.3. *If* $0 < \alpha < 1$ *and* $f_1 \in C^2(G)$, *then* $\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = O(h^{1-\alpha})$, *for every* $\epsilon > 0$ *and* $\max_{0 \leq j \leq N} |y(t_j) - y_j| = O(1)$.

P r o o f. If $f_1 \in C^2(G)$ then by [13, Theorem 2.1(a)] there exists a function $\psi(t) \in C^1$ such that $y(t) = \sum_{\nu=1}^{\nu_1} c_\nu t^{\alpha\nu} + \psi(t)$ where $c_i \in \mathbb{R}$ and $\nu_1 = \lceil \frac{1}{\alpha} \rceil - 1$. Therefore,

$$^{c}D_0^{\alpha}y(t) = \sum_{\nu=1}^{\nu_1} \frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)} c_\nu t^{\nu\alpha-\alpha} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\psi'(\tau)d\tau. \tag{5.6}$$

Let $g(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha}\psi'(\tau)d\tau$. Using Lemma 5.1 we get

$$\left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} g(t)dt - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, g(t_j) \right|$$
$$= \frac{1}{\Gamma(1-\alpha)} \left| \int_0^{t_{n+1}} (t_{n+1}-t)^{\alpha-1} \int_0^t (t-\tau)^{-\alpha}\psi'(\tau)d\tau dt \right.$$
$$\left. - \frac{h^{\alpha}}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1} \int_0^{t_j} (t_j-\tau)^{-\alpha}\psi'(\tau)d\tau \right|$$
$$\leq \frac{A}{1-\alpha} ||\psi'||_{\infty} t_{n+1}^{\alpha} h^{1-\alpha}. \tag{5.7}$$

Further using Theorem 5.1(c),

$$\Bigg|\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}\sum_{\nu=2}^{\nu_1}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}c_\nu t^{\nu\alpha-\alpha}dt$$

$$-\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n}a_{j,n+1}\sum_{\nu=2}^{\nu_1}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}c_\nu t_j^{\nu\alpha-\alpha}\Bigg|$$

$$\leq\sum_{\nu=2}^{\nu_1}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|\Bigg|\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}\,t^{\nu\alpha-\alpha}dt$$

$$-\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n}a_{j,n+1}\,t_j^{\nu\alpha-\alpha}\Bigg|$$

$$\leq\sum_{\nu=2}^{\nu_1}\frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)}|c_\nu|c^{Tr}_{\alpha,\nu\alpha-\alpha}t_{n+1}^{\alpha+\nu\alpha-\alpha-\sigma_\nu}h^{\sigma_\nu}\leq c^{Tr}t_{n+1}^{\alpha-1}h^{1+\alpha}, \qquad (5.8)$$

where $\sigma_\nu=\min(2,\nu\alpha-\alpha+1)$. Hence by Eq. (5.6), Eq. (5.7), Eq. (5.8)

$$\Bigg|\int_0^{t_{n+1}}(t_{n+1}-t)^{\alpha-1}\,{}^cD_0^\alpha y(t)dt-\frac{h^\alpha}{\alpha(\alpha+1)}\sum_{j=0}^{n}a_{j,n+1}\,{}^cD_0^\alpha y(t_j)\Bigg|$$

$$\leq 2\max\,\{\frac{A}{1-\alpha}||\psi'||_\infty t_{n+1}^\alpha h^{1-\alpha},c^{Tr}t_{n+1}^{\alpha-1}h^{1+\alpha}\}\leq c_3t_{n+1}^{\alpha-1}h^{1-\alpha}.$$

So if $\epsilon\in(0,T]$ and $t_j\in[\epsilon,T]$, then $|y(t_j)-y_j|\leq ct_j^{\alpha-1}h^{1-\alpha}\leq c\epsilon^{\alpha-1}h^{1-\alpha}$. Hence by Theorem 5.2, $\max\limits_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=O(h^{1-\alpha})$ for every $\epsilon>0$, and $\max\limits_{0\leq j\leq N}|y(t_j)-y_j|=O(1)$. □

THEOREM 5.4. *Let $f_1\in C^3(G)$ and $0<\alpha<1$, then*

$$\max_{0\leq j\leq N}|y(t_j)-y_j|=\begin{cases}O(h^{2\alpha}) & \text{if}\ \ 0<\alpha<0.5,\\ O(h) & \text{if}\ \ 0.5\leq\alpha<1,\end{cases}$$

and

$$\max_{t_j\in[\epsilon,T]}|y(t_j)-y_j|=\begin{cases}O(h^{1+\alpha}) & \text{if}\ \ 0<\alpha<0.5,\\ O(h^{2-\alpha}) & \text{if}\ \ 0.5\leq\alpha<1,\end{cases}$$

for every $\epsilon>0$.

The proof of the above theorem can be given on the similar lines as that of Theorem 5.7 of [7].

THEOREM 5.5. *Suppose the solution $y(t)$ of IVP(3.1) is of the form $y(t) = \sum_{\nu=0}^{\nu_0} c_\nu t^{\alpha\nu}$ in which $\nu_0 \in \mathbb{Z}^+$ and $c_1, c_2, \ldots, c_{\nu_0} \in \mathbb{R}$, then*

$$\max_{0 \le j \le N} |y(t_j) - y_j| = \begin{cases} O(h^{2\alpha}) & \text{if} \quad \alpha > 1, \\ O(h^{2\alpha}) & \text{if} \quad 0 < \alpha < 1, \end{cases}$$

and

$$\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = \begin{cases} O(h^2) & \text{if} \quad \alpha > 1, \\ O(h^{1+\alpha}) & \text{if} \quad 0 < \alpha < 1. \end{cases}$$

P r o o f. As $y(t) = \sum_{\nu=0}^{\nu_0} c_\nu t^{\alpha\nu}$,

$${}^cD_0^\alpha y(t) = \sum_{\nu=0}^{\nu_0} c_\nu \frac{\Gamma(\nu\alpha + 1)}{\Gamma(\nu\alpha + 1 - \alpha)} t^{\alpha\nu - \alpha}.$$

Case 1: If $\alpha > 1$, by Theorem 5.1(c):

$$\begin{aligned} &\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1}\, {}^cD_0^\alpha y(t) dt - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, {}^cD_0^\alpha y(t_j) \right| \\ &\le \sum_{\nu=1}^{\nu_0} \frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)} |c_\nu| \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1}\, t^{\nu\alpha-\alpha} dt \right. \\ &\left. - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, t_j^{\nu\alpha-\alpha} \right| \le c_4 t_{n+1}^{2(\alpha-1)} h^2. \end{aligned}$$

So if $\epsilon \in (0, T]$ and $t_j \in [\epsilon, T]$, then $|y(t_j) - y_j| \le c t_j^{2(\alpha-1)} h^2 \le c\epsilon^{2(\alpha-1)} h^2$. Hence by Theorem 5.2, $\max_{t_j \in [\epsilon, T]} |y(t_j) - y_j| = O(h^2)$ and $\max_{0 \le j \le N} |y(t_j) - y_j| = O(h^{2\alpha})$.

Case 2: If $0 < \alpha < 1$,

$$\begin{aligned} &\left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1}\, {}^cD_0^\alpha y(t) dt - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, {}^cD_0^\alpha y(t_j) \right| \\ &\le \sum_{\nu=1}^{\nu_0} \frac{\Gamma(\nu\alpha+1)}{\Gamma(\nu\alpha+1-\alpha)} |c_\nu| \left| \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1}\, t^{\nu\alpha-\alpha} dt \right. \\ &\left. - \frac{h^\alpha}{\alpha(\alpha+1)} \sum_{j=0}^{n} a_{j,n+1}\, t_j^{\nu\alpha-\alpha} \right| \le c_5 t_{n+1}^{\alpha-1} h^{\alpha+1}. \end{aligned}$$

So if $\epsilon \in (0,T]$ and $t_j \in [\epsilon, T]$, then $|y(t_j) - y_j| \le c t_j^{(\alpha-1)} h^{\alpha+1} \le c\epsilon^{(\alpha-1)} h^{\alpha+1}$. Hence by Theorem 5.2, $\max\limits_{t_j \in [\epsilon,T]} |y(t_j) - y_j| = O(h^{\alpha+1})$ and $\max\limits_{0\le j\le N} |y(t_j) - y_j| = O(h^{2\alpha})$. □

THEOREM 5.6. *Suppose the solution $y(t)$ of IVP (3.1) can be expressed in the form*

$$y(t) = \sum_{\nu=0}^{\nu_0} c_{0,\nu} t^{\alpha\nu} + \sum_{\nu=1}^{\nu_1} c_{1,\nu} t^{\alpha\nu} + \ldots + \sum_{\nu=1}^{\nu_l} c_{l,\nu} t^{\alpha\nu+l}$$

in which $\nu_0, \nu_1, \ldots, \nu_l \in \mathbb{Z}^+$ and $c_{0,\nu}, c_{1,\nu}, \ldots, c_{l,\nu} \in \mathbb{R}$, then

$$\max_{0\le j\le N} |y(t_j) - y_j| = \begin{cases} O(h^{2\alpha}) & \text{if} \quad \alpha > 1, \\ O(h^{2\alpha}) & \text{if} \quad 0 < \alpha < 1, \end{cases}$$

and

$$\max_{t_j\in[\epsilon,T]} |y(t_j) - y_j| = \begin{cases} O(h^{2}) & \text{if} \quad \alpha > 1, \\ O(h^{1+\alpha}) & \text{if} \quad 0 < \alpha < 1. \end{cases}$$

The proof of this theorem is on similar lines as that of Theorem 5.5 and is omitted here.

6. Illustrative examples

We present some illustrative examples which are solved using *Mathematica 10.*

EXAMPLE 6.1. Consider the fractional order equation with delay

$$\begin{aligned} {}^cD^{0.9}y(x) &= \frac{2x^{1.1}}{\Gamma(2.1)} - \frac{x^{0.1}}{\Gamma(1.1)} + y(x-0.1) - y(x) + 0.2x - 0.11, \\ y(x) &= 0, \quad x \le 0. \end{aligned} \tag{6.1}$$

In Fig. 1(a), we compare the graphs of approximate solutions of Eq. (6.1) by new predictor-corrector method and fractional Adams method (FAM) with the exact solution $y(x) = x^2 - x$. It is observed that the approximate solutions are in agreement with exact solution. The absolute errors in these methods are shown in Fig. 1(b) where solid and dashed lines show errors in the new method and the FAM respectively. Further, the CPU time required to solve this example using new method and FAM are 88.296875 seconds and 178.671875 seconds respectively. Thus, the new method is more accurate and time efficient as compared with FAM.

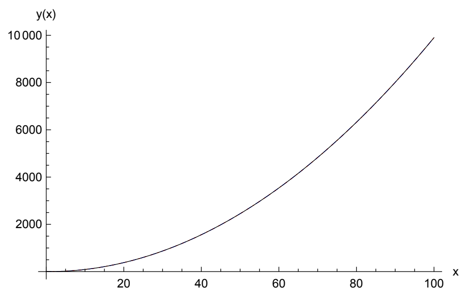

Fig. 1(a)
Solution of Eq. (6.1)

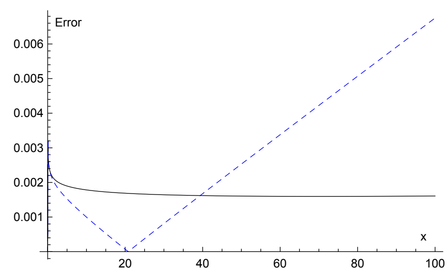

Fig. 1(b)
Error in approx solutions of (6.1)

EXAMPLE 6.2. Consider the following equation for $0 < \alpha \leq 1$ and $\tau > 0$:

$$\begin{aligned} {}^cD^\alpha y(x) &= \frac{2y(x)^{1-\alpha/2}}{\Gamma(3-\alpha)} + y(x-\tau) - y(x) + 2\tau\sqrt{y(x)} - \tau^2 \\ y(x) &= 0, \quad x \leq 0. \end{aligned} \tag{6.2}$$

The solutions of the Eq. (6.2) obtained using FAM and the new method are in good agreement with the exact solution $y(x) = x^2$. The errors in both numerical methods (for $\alpha = 0.6, \tau = 0.3$ and the step size $h = 0.001$) are compared in Table 1. The time required for the New method is 104.343750 seconds whereas the FAM has taken 215.031250 seconds for completing the same task.

x	Error (new method)	Error (FAM)
0.2	0.0781197	0.078155
0.4	0.129928	0.129978
0.6	0.190687	0.19076
0.8	0.248601	0.248694
1.0	0.307649	0.307763
1.2	0.366427	0.366563
1.4	0.425322	0.425479
1.6	0.484208	0.484387
1.8	0.54311	0.543312
2.0	0.602019	0.602243

Table 1: Errors in numerical solutions of Eq.(6.2)

Example 6.3. Consider the fractional order model of population of lemmings [4], for $0 < \alpha \leq 1$:

$$\begin{aligned} {}^cD^\alpha y(x) &= 3.5y(x)\left(1 - \frac{y\,(x-\tau)}{19}\right), \quad y(0) = 19.00001, \\ y(x) &= 19, \quad x < 0. \end{aligned} \tag{6.3}$$

The results obtained with new method are matching with those from FAM. The phase portraits shown in Figs. (2a)-2(d) clearly point out that the new method is more time efficient than FAM.

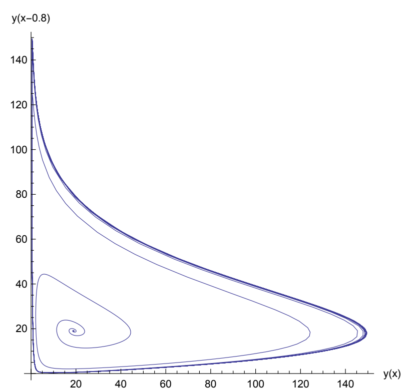

Fig. 2(a) $\alpha = 0.95$, $\tau = 0.80$
(Using new method, CPU Time = 31.42 S)

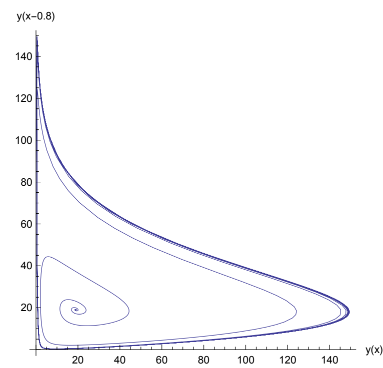

Fig. 2(b) $\alpha = 0.95$, $\tau = 0.80$
(Using FAM, CPU Time 62.25 S)

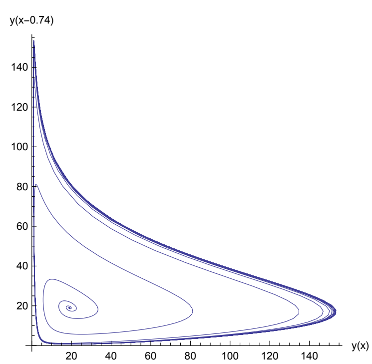

Fig. 2(c): $\alpha = 0.90$, $\tau = 0.74$
(Using new method, CPU Time = 43.89 S)

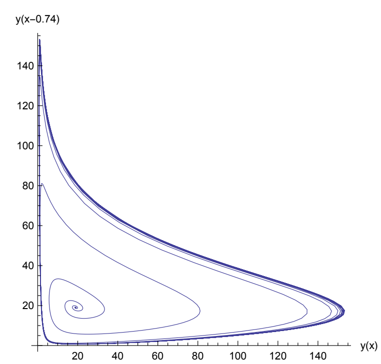

Fig. 2(d): $\alpha = 0.90$, $\tau = 0.74$
(Using FAM, CPU Time 90.69 S)

EXAMPLE 6.4. Consider the chaotic fractional order delay differential equation discussed in [4]:

$$\begin{aligned} {}^{c}D^{\alpha}y(x) &= \frac{2y\,(x-2)}{1+y(x-2)^{9.65}} - y(x), \\ y(x) &= 0.5, \quad x \le 0. \end{aligned} \tag{6.4}$$

Eq. (6.4) was solved for the fractional orders $\alpha = 0.98$ and $\alpha = 0.84$. The chaotic phase portraits and the periodic limit cycles obtained from both the methods are displayed in Figs. 3(a)-3(d).

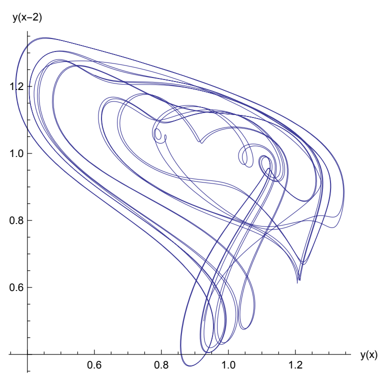

Fig. 3(a): Chaotic Solution, $\alpha = 0.98$
(Using new method, CPU Time 87.72 S)

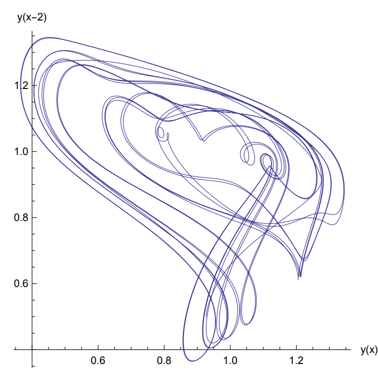

Fig. 3(b): Chaotic Solution, $\alpha = 0.98$
(Using FAM, CPU Time 177.47 S)

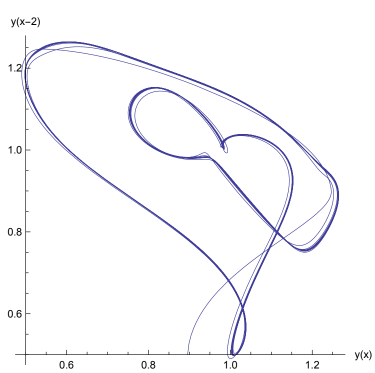

Fig. 3(c), Limit Cycle $\alpha = 0.84$
(Using new method, CPU Time 134.64 S)

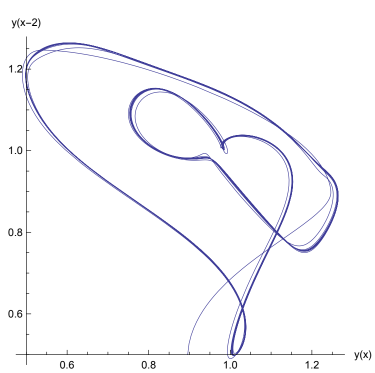

Fig. 3(d), Limit Cycle $\alpha = 0.84$
(Using FAM, CPU Time 244.17 S)

7. Conclusions

In this paper a new predictor corrector method had been developed to solve fractional delay differential equations (FDDEs). Related error analysis of the method has been presented and the error bounds obtained show that this method is as accurate as the existing methods. A variety of illustrative examples including fractional order model of population of lemmings and some chaotic systems are solved by fractional Adams method as well as the new predictor corrector method and the CPU time required for both the methods is compared. It turns out that the proposed method is more time efficient over the existing fractional Adams method used for solving FDDEs.

Acknowledgements

V. Daftardar-Gejji acknowledges Savitribai Phule Pune University for the travel grant to attend ICFDA14, Catania, Italy. S. Bhalekar acknowledges the National Board for Higher Mathematics, Mumbai, India for the Research Grant (Ref. 2/48(6)/2013/NBHM(R.P.)/R&DII/689).

References

[1] G. Adomian, A review of decomposition method in applied mathematics. *J. Math. Anal. Appl.* **135** (1988), 501–544.

[2] S. Bhalekar, V. Daftardar-Gejji, Convergence of the new iterative method. *International J. of Differ. Equ.* **2011** (2011), Article ID 989065, 10 p.; doi:10.1155/2011/989065.

[3] S. Bhalekar, V. Daftardar-Gejji, D. Baleanu and R. Magin, Generalized fractional order Bloch equation with extended delay. *International J. of Bifurcation and Chaos* **22**, No 4 (2012), 1–15.

[4] S. Bhalekar, V. Daftardar-Gejji, A predictor-corrector scheme for solving non-linear delay differential equations of fractional order. *J. Fractional Calculus and its Applications* **1**, No 5 (2011), 1–8.

[5] V. Daftardar-Gejji, H. Jafari, Adomian decomposition: a tool for solving a system of fractional differential equations. *J. Math. Anal. Appl.* **301**, No 2 (2005), 506–518.

[6] V. Daftardar-Gejji, H. Jafari, An iterative method for solving non linear functional equations. *J. Math. Anal. Appl.* **316** (2006), 753–763.

[7] V. Daftardar-Gejji, Y. Sukale, S. Bhalekar, A new predictor-corrector method for fractional differential equations. *Appl. Math. Comput.* **244** (2014), 158–182.

[8] V. Daftardar-Gejji, H. Jafari, Analysis of a system of non autonomous fractional differential equations involving Caputo derivatives. *J. Math. Anal. Appl.* **328** (2007), 1026–1033.

[9] K. Diethelm, N.J. Ford, A.D. Freed, A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics* **29** (2002), 3–22.

[10] K. Diethelm, N.J. Ford, A.D. Freed, Detailed error analysis for a fractional Adams method. *Numerical Algorithms* **36** (2004), 31–52.

[11] A. Hemeda, Homotopy perturbation method of solving systems of nonlinear coupled equations. Appl. Math. Sci. **6** (2012), 4787–4800.

[12] V. Lakshmikantham, Theory of fractional functional differential equations. *Nonlinear Analysis* **69** (2008), 3337–3343.

[13] C. Li, C. Tao, On fractional Adams method. *Comp. Math. Appl.* **58** (2009), 1573–1588.

[14] F. Mainardi, *Fractional Calculus and Waves in Linear Visco-elasticity: An Introduction to Mathematical Models*. Imperial College Press, London (2010).

[15] M.L. Morgado, N.J. Ford, P.M. Lima, Analysis and numerical methods for fractional differential equations with delay. *J. of Comp. and Appl. Math.* (2013), 159–168.

[16] I. Podlubny, *Fractional Differential Equations*. Academic Press, San Diego (1999).

[17] Z. Wang, A numerical method for delayed fractional-order differential equations. *J. of Appl. Math.* **256071** (2013), Article II.

[18] A. Wazwaz, A reliable modification in Adomian decomposition method. *Appl. Math. Comput.* **102** (1999), 77–86.

[19] W. Zhen, H. Xia, and S. Guodong, Analysis of nonlinear dynamics and chaos in a fractional order financial system with time delay. *Computers and Mathematics with Applications* **62** (2011), 1531–1539.

[1] *Department of Mathematics*
Savitribai Phule Pune University
Ganeshkhind, Pune – 411007, INDIA
Received: September 7, 2014
e-mail: vsgejji@math.unipune.ac.in , vsgejji@gmail.com

[2] *Department of Mathematics*
Savitribai Phule Pune University
Ganeshkhind, Pune – 411007, INDIA

and

Department of Mathematics
College of Engineering Pune,
Pune – 411005, INDIA
e-mail: yvs.maths@coep.ac.in

[3] *Department of Mathematics,*
Shivaji University
Kolhapur – 416004, INDIA
e-mail: sbb_maths@unishivaji.ac.in , sachin.math@yahoo.co.in

Please cite to this paper as published in:
Fract. Calc. Appl. Anal., Vol. **18**, No 2 (2015), pp. 400–418,
DOI: 10.1515/fca-2015-0026