Optimal control for linear port-Hamiltonian descriptor systems

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Outline

General descriptor system

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t), x(t_0) = x^0 \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
E, A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{p\times n}, D \in \mathbb{C}^{p\times m} \\
u : \mathbb{R} \to \mathbb{C}^m, x : \mathbb{R} \to \mathbb{C}^n, y : \mathbb{R} \to \mathbb{C}^m
\end{aligned}
$$

Port-Hamiltonian descriptor system

$$
\begin{aligned}
\tilde{E}\dot{x} &= (\tilde{J} - \tilde{R})\tilde{Q}x + (\tilde{B} + \tilde{P})u, \quad x(t_0) = x^0 \\
y &= (\tilde{B} - \tilde{P})^T \tilde{Q}x + (\tilde{S} + \tilde{N})u
\end{aligned}
\tag{2}
$$

$\tilde{E}, \tilde{J}, \tilde{R}, \tilde{Q} \in \mathbb{C}^{n\times n}, \tilde{B}, \tilde{P} \in \mathbb{C}^{n\times m}, \tilde{S}, \tilde{N} \in \mathbb{C}^{p\times m}$

$\tilde{J} = -\tilde{J}^T, \tilde{N} = -\tilde{N}^T, \tilde{R} = \tilde{R}^T \succeq 0, \tilde{S} = \tilde{S}^T \succeq 0, \tilde{Q} = \tilde{Q}^T \succeq 0$

$\begin{bmatrix} \tilde{R} & \tilde{P} \\ \tilde{P}^T & \tilde{S} \end{bmatrix} \succeq 0$

Port-Hamiltonian descriptor system

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Properties

- passivity
- consistency
- regularity
- stability
- stabilizability
- detectability
- controllability
- observability

Passivity

Definition

We call a control problem *passive*, if there exists a state-dependant storage-function $\mathcal{H} \geq 0$, such that $\frac{d}{dt}\mathcal{H} \leq u^T y$.

[Beattie, Mehrmann, Xu, Zwart 2018]

$$\frac{d}{dt}\mathcal{H} = u^T y - \begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{P} \\ \tilde{P}^T & \tilde{S} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

If $\begin{bmatrix} \tilde{R} & \tilde{P} \\ \tilde{P}^T & \tilde{S} \end{bmatrix} \equiv 0$, then the system is called *lossless*.

Consistency and Regularity

Definition

We call a control problem *consistent*, if there exists an input function u for which $\dot{x} = Ax + Bu, x(t_0) = x^0$ has a solution

Definition

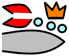
We call a control problem *regular*, if it has a unique solution for every initial value that is consistent for the system with input u.

Theorem

If (E, A) is regular, then the control problem is consistent and regular.

Consistency and Regularity

Definition



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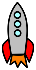

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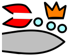

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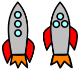

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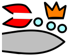

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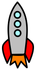

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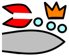

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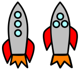

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Stability

Definition

We call a control problem *stable*, if one (and therefore every) solution to $\dot{x} = Ax$ is bounded on one (and therefore every) interval (t_0, ∞).
We call it *asymptotically stable*, if additionally $\lim\limits_{x\to\infty} \|x(t)\| = 0$.

Lemma

asymptotically stable $\Leftrightarrow$ *(E, A) regular, $\lambda E - A \prec 0$, all infinite Eigenvalues semisimple*

Lemma

stable $\Leftrightarrow$ *(E, A) regular, $\lambda E - A \preceq 0$, all purely imaginary Eigenvalues semisimple*

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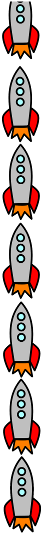
Stability and Stabilizability



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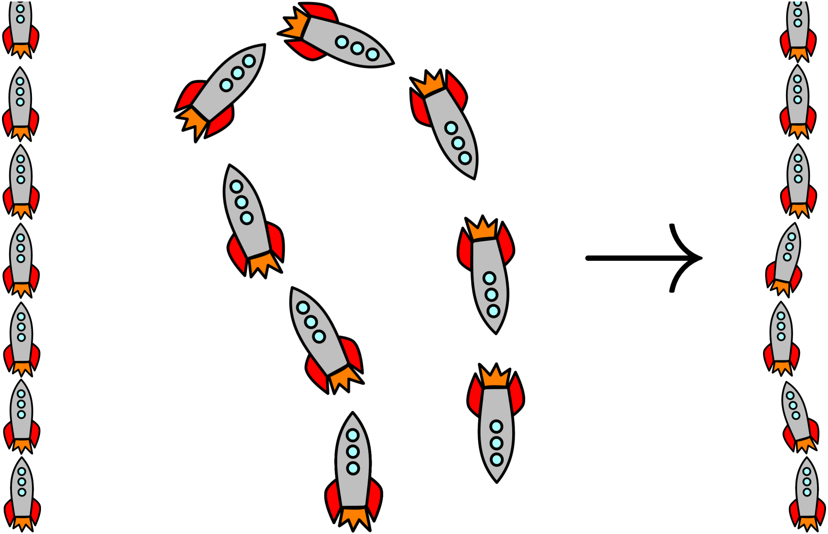

Stabilizability and Detectability

Definition

We call a control problem (asymptotically) *stabilizable*, if for each (t_0, x^0), there exists a piecewise continuous control function u(t), which is defined for all $t \geq t_0$, s.t. with this u we have $\lim_{x \to \infty} \|x(t)\| = 0$.

stabilizable

λ with $Re(\lambda) \geq 0$ Eigenvalue of A with corresponding Eigenvektor $x \neq 0$ $\Rightarrow x^T B \neq 0$

detectable

λ with $Re(\lambda) \geq 0$ Eigenvalue of A with corresponding Eigenvektor $x \neq 0$ $\Rightarrow Cx \neq 0$

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Which transformations are allowed

$$
\begin{aligned}
\tilde{E}\dot{x} &= (\tilde{J} - \tilde{R})x + (\tilde{B} - \tilde{P})u \\
y &= (\tilde{B} + \tilde{P})^T x + (\tilde{S} + \tilde{N})u
\end{aligned}
$$

Which transformations are allowed

$$
\begin{aligned}
Q\tilde{E}\dot{x} &= Q(\tilde{J} - \tilde{R})x + Q(\tilde{B} - \tilde{P})u \\
y &= (\tilde{B} + \tilde{P})^T x + (\tilde{S} + \tilde{N})u
\end{aligned}
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Which transformations are allowed

$$
\begin{aligned}
\tilde{E} Q^{-1} Q \dot{x} &= (\tilde{J} - \tilde{R}) Q^{-1} Q x + (\tilde{B} - \tilde{P}) u \\
y &= (\tilde{B} + \tilde{P})^T Q^{-1} Q x + (\tilde{S} + \tilde{N}) u
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Which transformations are allowed

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Q\tilde{E}Q^{-1}Q\dot{x} &= Q(\tilde{J}-\tilde{R})Q^{-1}Qx + Q(\tilde{B}-\tilde{P})u \\
y &= (\tilde{B}+\tilde{P})^{T}Q^{-1}Qx + (\tilde{S}+\tilde{N})u
\end{aligned}
$$

Split the system into its different parts

[Beattie, Gugercin, Mehrmann 2019]: There exists an orthogonal basis transformation, such that the system is of the form

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \\ \dot{\hat{x}}_4 \\ \dot{\hat{x}}_5 \end{bmatrix} = \begin{bmatrix} J_{11} - R_{11} & J_{12} - R_{12} & J_{13} & J_{14} & 0 \\ J_{21} - R_{21} & J_{22} - R_{22} & J_{23} & J_{24} & 0 \\ J_{31} & J_{32} & J_{33} & 0 & 0 \\ J_{41} & J_{42} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \end{bmatrix}$$

$$+ \begin{bmatrix} B_1 - P_1 \\ B_2 - P_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} u \qquad y = \begin{bmatrix} B_1 + P_1 \\ B_2 + P_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}^T \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \hat{x}_5 \end{bmatrix} + (S + N)u$$

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Develop needed lemmata

Lemma

For a quadratic invertible skew-symmetric matrix A its inverse A^{-1} is again skew-symmetric.

Lemma

For a symmetric matrix B the matrix ABA^T is again symmetric.

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For matrices $R = R^T \succeq 0, J = -J^T \in \mathbb{C}^{n \times n}$ *the inverse of the matrix* $(J - R)$ *can be written as*

$$\hat{J} - \hat{R} := (J - R)^{-1}$$

where $\hat{R} = \hat{R}^T \succeq 0, \hat{J} = -\hat{J}^T$.

Lemma

Given two matrices $X, Y \in \mathbb{C}^{n \times m}$.
There exist matrices $A, B \in \mathbb{C}^{n \times m}$ *with* $X = A - B, Y = A + B$.

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Given two matrices $X, Y \in \mathbb{C}^{n\times m}$.
There exist matrices $A, B \in \mathbb{C}^{n\times m}$ *with* $X = A - B, Y = A + B$.
Namely these are

$$A := \frac{1}{2}(X + Y), \quad B := \frac{1}{2}(Y - X)$$

Decompose $\begin{bmatrix} J_{41} & J_{42} \end{bmatrix}$

$\begin{bmatrix} J_{41} & J_{42} \end{bmatrix} \in \mathbb{C}^{x \times (y+z)}$ has full row rank. We get two two cases:

$y \geq x$

$\begin{bmatrix} J_{41} & J_{42} \end{bmatrix} = \begin{bmatrix} R_1 & 0 & 0 \end{bmatrix} \hat{Q}$ with $R_1 \in \mathbb{C}^{x \times x}$

$y < x$

$\begin{bmatrix} J_{41} & J_{42} \end{bmatrix} = \begin{bmatrix} * & R_1 & 0 \\ R_2 & 0 & 0 \end{bmatrix} \hat{Q}$ with $R_2 \in \mathbb{C}^{y \times y}$

After the transformation

In the first case we get the system

$$\begin{aligned} \dot{x}_1 &= (J_{11} - R_{11})x_1 + (B_1 - P_1)u \\ y &= (B_1 + P_1)^T x + (S + N)u \end{aligned} \tag{3}$$

In the second case we get the system

$$\begin{aligned} 0 &= (J_{11} - R_{11})x_1 + (B_1 - P_1)u \\ y &= (B_1 + P_1)^T x + (S + N)u \end{aligned} \tag{4}$$

After the transformation

We have $R_{11} \succ 0$.

consistency and regularity

$(I, J_{11} - R_{11})$ is regular and $(0, J_{11} - R_{11})$ is regular $\Rightarrow$ The system is consistent and regular.

stability

$J_{11} - R_{11} \prec 0 \Rightarrow$ The system is asymptotically stable.

We only need to look at the symmetric part

Note that $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

Lemma

A quadratic matrix A is positive definite if and only if its symmetric part is positive definite.

$$A \succ 0 \Leftrightarrow \frac{1}{2}(A + A^T) \succ 0$$

The same holds for positive semidefinite, negative definite and negative semidefinite.

Stabilizable $\Leftrightarrow$ Detectable

Let λ_0 be a purely imaginary Eigenvalue of $\tilde{J} - \tilde{R}$ with corresponding Eigenvektor x. Then x lies in the kernel of $\tilde{R}$.
[Mehl, Mehrmann, Wojtylak 2020]

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$$0 \leq \begin{bmatrix} x^T & y^T \end{bmatrix} \begin{bmatrix} \tilde{R} & \tilde{P} \\ \tilde{P}^T & \tilde{S} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T \tilde{R} x + 2x^T \tilde{P} y + y^T \tilde{S} y$$
$$= 2x^T \tilde{P} y + y^T \tilde{S} y \quad \forall \quad y \in \mathbb{C}^m \text{ and multiples of x}$$

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&= 2x^T \tilde{P} y + y^T \tilde{S} y \quad \forall \quad y \in \mathbb{C}^m \text{ and multiples of x} \\
&\Rightarrow x^T \tilde{P} = 0 \quad \wedge \quad \tilde{P}^T x = 0
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\end{aligned}
$$

$$
x^T(\tilde{B} - \tilde{P}) = x^T \tilde{B} = (\tilde{B}^T x)^T = ((\tilde{B} + \tilde{P})^T x)^T
$$

Controllability

C0 $rank[\alpha E - \beta A, B] = n \quad \forall \quad (\alpha, \beta) \in \mathbb{C}^2$

C1 $rank[\lambda E - A, B] = n \quad \forall \quad \lambda \in \mathbb{C}$

C2 $rank[E, AS_\infty(E), B] = n$

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We call a problem *completely controllable* if *C0* holds.
We call a problem *strongly controllable* if *C1* and *C2* hold.
C0 $\Rightarrow$ *C1*, *C2*
C0 $\Leftrightarrow$ *C1* and $rank \begin{bmatrix} E & B \end{bmatrix} = n$

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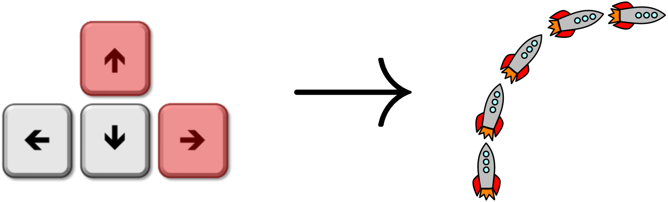
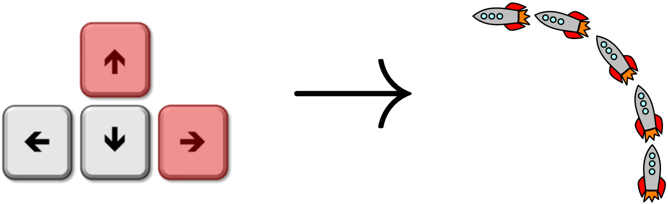
C0 $\Leftrightarrow$ *C1* and $rank \begin{bmatrix} E & B \end{bmatrix} = n$

$C1 \Leftrightarrow$ $\lambda \in \mathbb{C}$ Eigenvalue of A with corresponding Eigenvektor $x \neq 0$ $\Rightarrow x^T B \neq 0$

Observability

Take output $y \hat{=}$ *height above the ground*





Observability

O0 $rank \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} = n \quad \forall \quad (\alpha, \beta) \in \mathbb{C}^2$

O1 $rank \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \forall \quad \lambda \in \mathbb{C}$

O2 $rank \begin{bmatrix} E \\ T_\infty(E)^T A \\ C \end{bmatrix} = n$

Observability

O0 $rank \begin{bmatrix} \alpha E - \beta A \\ C \end{bmatrix} = n \quad \forall \quad (\alpha, \beta) \in \mathbb{C}^2$

O1 $rank \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n \quad \forall \quad \lambda \in \mathbb{C}$

O2 $rank \begin{bmatrix} E \\ T_\infty(E)^T A \\ C \end{bmatrix} = n$

We call a problem *completely observable* if *O0* holds.

We call a problem *strongly observable* if *O1* and *O2* hold.

O0 $\Rightarrow$ *O1*, *O2*

O0 $\Leftrightarrow$ *O1* and $rank \begin{bmatrix} E \\ C \end{bmatrix} = n$

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We call a problem *completely observable* if *O0* holds.
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$O0 \Rightarrow O1, O2$

$O0 \Leftrightarrow O1$ and $rank \begin{bmatrix} E \\ C \end{bmatrix} = n$

$O1 \Leftrightarrow$ $\lambda \in \mathbb{C}$ Eigenvalue of A with corresponding Eigenvektor $x \neq 0$ $\Rightarrow Cx \neq 0$

Transform the system to remove S+N

$$\begin{aligned} E\dot{x} &= (\tilde{J} - \tilde{R})x + (\tilde{B} + \tilde{P})u \\ y &= (\tilde{B} - \tilde{P})^T x + (\tilde{S} + \tilde{N})u \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \hat{E}\dot{x} &= (\hat{J} - \hat{R})x + (\hat{B} + \hat{P})u \\ y &= (\hat{B} - \hat{P})^T x \end{aligned}$$

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Decompose $S+N = XY$ and use the transferfunction

$$\begin{aligned} H(s) &= (\tilde{B}+\tilde{P})^T(s\tilde{E}-(\tilde{J}-\tilde{R}))^{-1}(\tilde{B}-\tilde{P}) + (S+N) \\ &= \begin{bmatrix} (B+P)^T & X \end{bmatrix} \begin{bmatrix} s\tilde{E}-(\tilde{J}-\tilde{R}) & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} (B-P) \\ Y \end{bmatrix} \end{aligned}$$

Transform the system to remove S+N

$$\begin{aligned} E\dot{x} &= (\tilde{J} - \tilde{R})x + (\tilde{B} + \tilde{P})u \\ y &= (\tilde{B} - \tilde{P})^T x + (\tilde{S} + \tilde{N})u \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \hat{E}\dot{x} &= (\hat{J} - \hat{R})x + (\hat{B} + \hat{P})u \\ y &= (\hat{B} - \hat{P})^T x \end{aligned}$$

Decompose $S + N = XY$ and use the transferfunction

$$\begin{aligned} H(s) &= (\tilde{B} + \tilde{P})^T (s\tilde{E} - (\tilde{J} - \tilde{R}))^{-1} (\tilde{B} - \tilde{P}) + (S + N) \\ &= \begin{bmatrix} (B + P)^T & X \end{bmatrix} \begin{bmatrix} s\tilde{E} - (\tilde{J} - \tilde{R}) & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} (B - P) \\ Y \end{bmatrix} \end{aligned}$$

$$\hat{B} := \begin{bmatrix} \tilde{B} \\ \frac{1}{2}(X + Y) \end{bmatrix}, \hat{P} := \begin{bmatrix} \tilde{P} \\ \frac{1}{2}(X - Y) \end{bmatrix}, \hat{E} := \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\hat{J} := \begin{bmatrix} \tilde{J} & 0 \\ 0 & 0 \end{bmatrix}, \hat{R} := \begin{bmatrix} \tilde{R} & 0 \\ 0 & I \end{bmatrix}$$

Transform the system to remove P

Lemma

The system

$$\begin{aligned} E\dot{x} &= (J-R)x + (B+P)u \\ y &= (B-P)^T x \end{aligned}$$

can be reformulated to

$$\begin{aligned} \hat{E}\dot{x} &= (\hat{J}-\hat{R})x + \hat{B}u \\ y &= \hat{B}^T x \end{aligned}$$

Controllable $\Leftrightarrow$ Observable

controllable

$\lambda \in \mathbb{C}$ Eigenvalue of $\tilde{J} - \tilde{R}$ with corresponding Eigenvektor $x \neq 0$ $\Rightarrow x^T(\tilde{B} - \tilde{P}) \neq 0$

observable

$\lambda \in \mathbb{C}$ Eigenvalue of $\tilde{J} - \tilde{R}$ with corresponding Eigenvektor $x \neq 0$ $\Rightarrow (\tilde{B} + \tilde{P})^T x \neq 0$

Controllable $\Leftrightarrow$ Observable

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$\lambda \in \mathbb{C}$ Eigenvalue of $\tilde{J} - \tilde{R}$ with corresponding Eigenvektor $x \neq 0$ $\Rightarrow x^T \tilde{B} \neq 0$

observable

$\lambda \in \mathbb{C}$ Eigenvalue of $\tilde{J} - \tilde{R}$ with corresponding Eigenvektor $x \neq 0$ $\Rightarrow \tilde{B}^T x \neq 0$

What to do if R is not positive semidefinite?

$$\dot{x} = (\tilde{J} - \tilde{R})x + \tilde{B}u$$
$$y = \tilde{B}^T x$$

Schurform

$$\begin{bmatrix} R_{11} & 0 \\ 0 & 0 \end{bmatrix} = URU^T \text{ where } R_{11} \in \mathbb{R}^{l,l}$$
$$\hat{B} := U\tilde{B}$$

What to do if R is not positive semidefinite?

Apply outputfeedback $u = Ky, K = K^T$ blockdiagonal

$$\dot{x} = (\tilde{J} - \tilde{R})x + \tilde{B}u = (\tilde{J} - \tilde{R})x + \tilde{B}K\tilde{B}^T x = (\tilde{J} - \tilde{R} + \tilde{B}K\tilde{B}^T)x$$

What we want

$$\begin{aligned}
0 &\overset{!}{\prec} U(\tilde{R} - \tilde{B}K\tilde{B}^T)U^T \\
&= \begin{bmatrix} R_{11} - \hat{B}_{11}K_{11}\hat{B}_{11}^T - \hat{B}_{12}K_{22}\hat{B}_{12}^T & -\hat{B}_{11}K_{11}\hat{B}_{21}^T - \hat{B}_{12}K_{22}\hat{B}_{22}^T \\ -\hat{B}_{21}K_{11}\hat{B}_{11}^T - \hat{B}_{22}K_{22}\hat{B}_{12}^T & -\hat{B}_{21}K_{11}\hat{B}_{21}^T - \hat{B}_{22}K_{22}\hat{B}_{22}^T \end{bmatrix} \\
&:= \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}
\end{aligned}$$

$$\Leftrightarrow 0 \overset{!}{\prec} X_{11} \quad \wedge \quad 0 \overset{!}{\prec} X_{22} - X_{21}X_{11}^{-1}X_{12}$$

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&:= \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \\
&\Leftrightarrow 0 \overset{!}{\prec} X_{11} \quad \wedge \quad 0 \overset{!}{\prec} X_{22} - X_{21}X_{11}^{-1}X_{12}
\end{aligned}$$

Outline

Cost functional - state feedback

$$
\begin{aligned}
&\mathcal{S}(x(t), u(t)) \\
&= \frac{1}{2}\Big(x(t_f)^T M x(t_f) + \int_{t_0}^{t_f} \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt\Big) \\
&= \frac{1}{2}\Big(x(t_f)^T M x(t_f) \\
&+ \int_{t_0}^{t_f} x(t)^T Q x(t) + u(t)^T R u(t) + x(t)^T S u(t) + u(t) S^T x(t) dt\Big) \\
&t_0 < t_f \leq \infty, M = M^T, Q = Q^T, R = R^T
\end{aligned}
\tag{5}
$$

KKT-System

$$\begin{bmatrix} 0 & A & B \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix}$$
$$x(t_0) = x^0, \mu(t_f) = Mx(t_f) \tag{6}$$

Hamiltonian Matrix

Solve the third line after u and insert into the other two lines

$$\begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} := \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ -(Q - SR^{-1}S^T) & -A^T + SR^{-1}B^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} \quad (7)$$

$$x(t_0) = x^0, \mu(t_f) = Mx(t_f)$$

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$$x(t_0) = x^0, \mu(t_f) = Mx(t_f)$$

Ansatz: $\mu(t) = X(t)x(t) \Rightarrow X(t_f) = M$

$\Rightarrow \dot{\mu} = X(t)\dot{x} + \dot{X}(t)x$

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Solve the third line after u and insert into the other two lines

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$$x(t_0) = x^0, \mu(t_f) = Mx(t_f)$$

Ansatz: $\mu(t) = X(t)x(t) \Rightarrow X(t_f) = M$

$\Rightarrow \dot{\mu} = X(t)\dot{x} + \dot{X}(t)x$

$\Rightarrow (\dot{X}(t) - H + F^T X(t) + X(t)F + X(t)GX(t))x = 0$

Riccati Equations

Differential equation

$$\begin{aligned} \dot{X}(t) &= H - F^T X(t) - X(t)F - X(t)GX(t) \\ X(t_f) &= M \end{aligned} \tag{8}$$

Algebraic equation $t_f = \infty, M = 0, (A, B)$ stabilizable

$$0 = H - F^T X - XF - XGX \tag{9}$$

State feedback

$$u(t) = -R^{-1}(S^T + B^T X(t))x(t)$$
$$u(t) = -R^{-1}(S^T + B^T X)x(t)$$

Cost functional - output feedback

$$
\begin{aligned}
&\mathcal{S}(y(t), u(t)) \\
&= \frac{1}{2}\Big(y(t_f)^T M y(t_f) + \int_{t_0}^{t_f} \begin{bmatrix} y(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} dt\Big) \\
&= \frac{1}{2}\Big(y(t_f)^T M y(t_f) \\
&+ \int_{t_0}^{t_f} y(t)^T B(t) Q B(t)^T y(t) + u(t)^T R u(t) + y(t)^T B(t) S u(t) \\
&+ u(t) S^T B(t)^T y(t) dt\Big) \\
&t_0 < t_f \leq \infty
\end{aligned}
$$

Insert y

$$\hat{Q} := (\tilde{B} + \tilde{P})Q(\tilde{B} + \tilde{P})^T$$
$$\hat{R} := (\tilde{S} + \tilde{N})^T Q(\tilde{S} + \tilde{N}) + S^T(\tilde{S} + \tilde{N}) + (\tilde{S} + \tilde{N})^T S + R$$
$$\hat{S} := (\tilde{B} + \tilde{P})Q(\tilde{S} + \tilde{N}) + (\tilde{B} + \tilde{P})S$$

We now get the cost functional

$$\begin{aligned}
&\mathcal{S}(y(t), u(t)) \\
&= \frac{1}{2}\Big(y(t_f)^T M y(t_f) + \int_{t_0}^{t_f} \begin{bmatrix} x(t)^T & u(t)^T \end{bmatrix} \begin{bmatrix} \hat{Q} & \hat{S} \\ \hat{S}^T & \hat{R} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt\Big) \\
&= \frac{1}{2}\Big(y(t_f)^T M y(t_f) \\
&+ \int_{t_0}^{t_f} x(t)^T \hat{Q} x(t) + u(t)^T \hat{R} u(t) + x(t)^T \hat{S} u(t) + u(t) \hat{S}^T x(t) dt\Big) \\
&t_0 < t_f \leq \infty
\end{aligned}$$

Ideas to improve finding optimal control via pH structure

Insert pH matrices and see if something cancels out

$$0 = H - F^T X - XF - XGX$$

Ideas to improve finding optimal control via pH structure

Insert pH matrices and see if something cancels out

$$
\begin{aligned}
0 =& H - F^T X - XF - XGX \\
=& [-Q + SR^{-1}S^T] - [A^T - SR^{-1}B^T]X \\
& - X[A - BR^{-1}S^T] - X[-BR^{-1}B^T]X
\end{aligned}
$$

Ideas to improve finding optimal control via pH structure

Insert pH matrices and see if something cancels out

$$
\begin{aligned}
0 &= H - F^T X - XF - XGX \\
&= [-Q + SR^{-1}S^T] - [A^T - SR^{-1}B^T]X \\
&\quad - X[A - BR^{-1}S^T] - X[-BR^{-1}B^T]X \\
&= [-Q + SR^{-1}S^T] - [(\tilde{J} - \tilde{R})^T - SR^{-1}(\tilde{B} - \tilde{P})^T]X \\
&\quad - X[(\tilde{J} - \tilde{R}) - (\tilde{B} - \tilde{P})R^{-1}S^T] - X[-(\tilde{B} - \tilde{P})R^{-1}(\tilde{B} - \tilde{P})^T]X
\end{aligned}
$$

Ideas to improve finding optimal control via pH structure

Transform the system with Z

$$Z := \begin{bmatrix} \frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I & 0 \\ -\frac{1}{\sqrt{2}} I & \frac{1}{\sqrt{2}} I & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} I \end{bmatrix}$$

$$Z^T \begin{bmatrix} 0 & A & B \\ A^T & Q & S \\ B^T & S^T & R \end{bmatrix} ZZ^T \begin{bmatrix} \mu \\ x \\ u \end{bmatrix} = Z^T \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} ZZ^T \begin{bmatrix} \dot{\mu} \\ \dot{x} \\ \dot{u} \end{bmatrix}$$

$$x(t_0) = x^0, \mu(t_f) = Mx(t_f)$$

Ideas to improve finding optimal control via pH structure

Transform the system with Z

$$0 = H - F^T X - XF - XGX$$

Ideas to improve finding optimal control via pH structure

Transform the system with Z

$$
\begin{aligned}
0 =& H - F^T X - XF - XGX \\
=& [A^T + A - Q + (-B + S)R^{-1}(-B^T + S^T)] \\
& - [A^T - A + Q - (-B + S)R^{-1}(B^T + S^T)]X \\
& - X[-A^T + A + Q - (B - S)R^{-1}(-B^T + S^T)] \\
& - X[A^T + A + Q - (B + S)R^{-1}(B^T + S^T)]X
\end{aligned}
$$

Ideas to improve finding optimal control via pH structure

Transform the system with Z

$$
\begin{aligned}
0 =& H - F^T X - XF - XGX \\
=& [-2\tilde{R} - Q + (-B + S)R^{-1}(-B^T + S^T)] \\
&- [-2\tilde{J} + Q - (-B + S)R^{-1}(B^T + S^T)]X \\
&- X[2\tilde{J} + Q - (B - S)R^{-1}(-B^T + S^T)] \\
&- X[-2\tilde{R} + Q - (B + S)R^{-1}(B^T + S^T)]X
\end{aligned}
$$

Ideas to improve finding optimal control via pH structure

Choose Q, S, R as $\tilde{R}, \tilde{P}, \tilde{S}$

$$
\begin{aligned}
0 =& H - F^T X - XF - XGX \\
=& [-\tilde{R} + \tilde{P}\tilde{S}^{-1}\tilde{P}^T] \\
& - [(\tilde{J} - \tilde{R})^T - \tilde{P}\tilde{S}^{-1}(\tilde{B} - \tilde{P})^T]X \\
& - X[(\tilde{J} - \tilde{R}) - (\tilde{B} - \tilde{P})\tilde{S}^{-1}\tilde{P}^T] \\
& + X[(\tilde{B} - \tilde{P})\tilde{S}^{-1}(\tilde{B} - \tilde{P})^T]X
\end{aligned}
$$

Ideas to improve finding optimal control via pH structure

Choose Q, S, R as $\tilde{R}, \tilde{P}, \tilde{S}$

$$
\begin{aligned}
0 =& H - F^T X - XF - XGX \\
=& -\tilde{R} - (\tilde{J} - \tilde{R})^T X - X(\tilde{J} - \tilde{R}) + X\tilde{B}\tilde{S}^{-1}\tilde{B}X
\end{aligned}
$$

Conclusions and Outlook

- pH structure helps us with stabilizing control systems
- We get detectability for free
- If a pH system is controllable, we get observability for free
- Not too much profit from the pH structure regarding optimal control
- Can we improve the algorithms with pH structure?

Thanks

Thank you very much for your attention

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