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DESCRIPTOR SYSTEMS WITHOUT CONTROLLABILITY AT INFINITY*

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Abstract. This paper concerns the structure that can be achieved by feedback in descriptor systems that lack controllability at infinity. Staircase and double staircase condensed forms obtained through a sequence of orthogonal state transformations display when and how feedback can be used to achieve minimal index. Furthermore, they reveal that the modes that are uncontrollable at infinity have a fixed minimal index that cannot be reduced by feedback. However, this fixed higher index part of the control system is constrained to be zero in an appropriate coordinate system, provided the initial conditions are consistent. The remainder is a reduced order system that is controllable at infinity that can be made to have index one by feedback.

Key words. descriptor system, controllability, numerical methods, impulse

AMS subject classifications. 93B11, 93B52, 93B05, 93B27, 93B40

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1. Introduction.

Consider the linear, time-invariant descriptor system

$$E\dot{x} = Ax + Bu, \qquad Ex(0) = Ex^0, \qquad y = Cx, \tag{1}$$

with system matrices $E \in \mathbf{C}^{n\times n}$, $A \in \mathbf{C}^{n\times n}$, $B \in \mathbf{C}^{n\times m}$, $C \in \mathbf{C}^{p\times n}$, state $x = x(t) \in \mathbf{C}^n$, input $u = u(t) \in \mathbf{C}^m$, and output $y = y(t) \in \mathbf{C}^p$. Descriptor systems arise naturally in circuit design, mechanical multibody systems, and a variety of other applications [25, 32, 33]. They have recently attracted the attention of many authors to all aspects of control, including pole placement, filtering, stabilization, controllability, observability, optimal control problems, invertibility, duality, realization, etc. See, for example, [6, 14, 13, 27] and the references therein.

In contrast to standard systems in which $E = I$, continuous inputs to a descriptor system can give rise to discontinuities or impulsive modes in the state trajectories. A detailed analysis of solvability aspects of the distributional version of (1) is given in [14]. A simple example is the single input system determined by

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2}$$

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The input $u(t)$ induces the state $x(t)$:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \dot{u}(t) \\ u(t) \end{bmatrix}.$$

If $x^0 = x(0^-) = \left[\begin{smallmatrix} x_1^0 \\ x_2^0 \end{smallmatrix}\right]$ and u is smooth but $x_2^0 \neq u(0^+)$, then x_1 will still exhibit a pulse $[u(0^+) - x_2^0]\delta$, where δ is the Dirac delta distribution [14]. (If $u(0^+) = x_2^0$ but $\dot{u}(0^+) \neq x_1^0$, i.e., $x_1(0^+) \neq x_1(0^-)$, then x_1 will still be impulse free [14, 40].)

If (1) is controllable and observable at infinity, i.e., rank$[E,\ AS_\infty,\ B] = n$, where

$$\operatorname{Im}(S_\infty) = \ker(E) \quad \text{and} \quad \operatorname{rank} \begin{bmatrix} E \\ T_\infty^H A \\ C \end{bmatrix} = n,$$

where $\operatorname{Im}(T_\infty) = \ker(E^H)$, then the problem of impulses can be avoided (or at least disguised) by using an appropriate feedback [4, 3]. Here, $\operatorname{Im}(\cdot)$ denotes the image (or range) and $\ker(\cdot)$ is the kernel (or null space).

For example, using the feedback control $u = FCx + v$, where $F = \begin{bmatrix} -1 & 0 \end{bmatrix}$ in (2) gives the closed loop system matrices

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3}$$

Here the inputs $v(t)$ and the resulting state trajectories exhibit the same impulsive behavior, and thus x will be impulse free if v is. In addition if v is q-times continuously differentiable for $t > 0$, then x is as well.

Notice that the closed loop system matrices (3) have a stronger robustness property than the ones in (2). If the closed loop system is perturbed by some unmodeled dynamic forcing function $f(t)$ giving

$$E\dot{x} = (A + BFC)x + Bv + f(t),$$

then the resulting state $x(t)$ still has as many derivatives as $\left[\begin{smallmatrix} f(t) \\ v(t) \end{smallmatrix}\right]$, even if $x(0^-) = x^0$ is not consistent. Using distributions, we get

$$x_1 = -x_2 + v - f_2,$$
$$x_2 = (\delta^{(1)} + \delta)^{*^{-1}} [x_2(0^-)\delta + v + f_1 - f_2].$$

In time domain, if v and f are functions, we have

$$x_1(t) = -x_2(t) + v(t) - f_2(t),$$
$$x_2(t) = e^{-t} x_2(0^-) + \int_0^t e^{-(t-s)} [v(s) + f_1(s) - f_2(s)] ds.$$

Hence, $x = \left[\begin{smallmatrix} x_1(t) \\ x_2(t) \end{smallmatrix}\right]$ has indeed as many derivatives for $t > 0$ as $\left[\begin{smallmatrix} f \\ v \end{smallmatrix}\right]$, even if $x(0^-) \neq x(0^+)$.

We call this property *index-one robustness* because it is shared by regular descriptor systems of index at most one. (Regularity and index are defined in the next section.) Even smooth perturbations f in (2) will in general give rise to extra pulses in the solution, whereas this cannot happen in systems that are index-one robust. It is implicit in the results of [4, 3] that systems that are controllable and observable at infinity can be made to be index-one robust by feedback.

In several applications including mechanical multibody systems [19, 29, 28, 33, 34] the assumptions of controllability and/or observability at infinity do not hold. Consider for example the planar model of a three-link manipulator introduced in [19],

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & I & 0 \\ -K_0 & -D_0 & F_0^T \\ F_0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ S_0 \\ 0 \end{bmatrix}.$$

This system is not controllable at infinity. With output $y = \begin{bmatrix} C_1 & C_2 & 0 \end{bmatrix}$ it is not observable at infinity either. Due to the special structure of this mechanical multibody system, however, the part of the system that is characterized by the uncontrollable modes at infinity can be neglected. The remaining system can be made index-one robust [28]. If this simplification is not carried out, then undesired phenomena as in the following example may occur. Let

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = I, \quad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad C = I. \tag{4}$$

In this case, the first two components of the state obey (2), while the second two components of the state are confined to be zero (assuming consistent initial conditions). Choosing an appropriate feedback causes the first two components of the state to obey (3) while the second two remain zero. However, regardless of what feedback is chosen, the closed loop system does not have index-one robustness. If the forcing function $f(t) = \begin{bmatrix} f_1(t), & f_2(t), & f_3(t), & f_4(t) \end{bmatrix}^T$ were added to the closed loop system, then the third state component is $x_3(t) = -f_3(t) - \dot{f}_4$. In the third component of the state, we will obtain an impulse of the form $(-f_4(0^+) - x_{04})\delta$, where $x_{04} = x_4(0^-)$. Lack of differentiability in $f_4(t)$ may translate into lack of continuity in x_3. A jump discontinuity in $f(t)$ may cause an impulse.

This paper concerns the properties that can be achieved without controllability and observability at infinity, including when and how feedback can be used to achieve minimal index by numerically stable methods. All these properties are displayed by the Kronecker-like feedback canonical form introduced in [24]. Extracting this canonical form may, however, require ill-conditioned transformations which are sensitive to rounding errors. For this reason, following the approaches of [4, 3, 37, 36], we derive condensed staircase and double staircase forms through a sequence of unitary state space transformations. They display when and how feedback can be used to achieve minimal index. They also reveal that (4) is typical of systems which lack controllability at infinity. The parts of the state which are uncontrollable at infinity are constrained to be zero in an appropriate coordinate system, provided the initial conditions are consistent and may be decoupled from the rest of the system. This leaves a reduced order system that is controllable at infinity to which the work of [4, 3] applies. A similar argument applies to parts of the state which are not observable at infinity. By choosing an appropriate basis, these parts can be decoupled from the rest, and since they cannot be observed, they can be removed without changing the dynamics of the system.

2. Definitions and lemmas. The control system (1) and the associated matrix pencil $\lambda E - A$ are said to be *regular* if the characteristic polynomial $\det(\lambda E - A)$ is not identically zero. If the pencil $\lambda E - A$ is not regular, then the system of differential algebraic equations

$$E\dot{x} = Ax + f(t) \tag{5}$$

is underdetermined in the sense that consistent initial conditions do not uniquely determine solutions [12]. If the pencil $\lambda E - A$ is regular, then the roots of the characteristic polynomial are the finite eigenvalues of the pencil and include the poles of the transfer function of (1). In addition, if E is singular, the pencil is said to have infinite eigenvalues which may be identified as the zero eigenvalues of the inverse pencil $E - \lambda A$.

The eigenstructure of regular pencils is displayed by the Weierstraß canonical form (WCF).

THEOREM 2.1 (Weierstraß canonical form [12]). *If $\lambda E - A$ is regular, then there exist nonsingular matrices $X = [X_r, X_\infty] \in \mathbf{C}^{n\times n}$ and $Y = [Y_r, Y_\infty] \in \mathbf{C}^{n\times n}$ for which*

$$Y^H E X = \begin{bmatrix} Y_r^H \\ Y_\infty^H \end{bmatrix} E \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \tag{6}$$

and

$$Y^H A X = \begin{bmatrix} Y_r^H \\ Y_\infty^H \end{bmatrix} A \begin{bmatrix} X_r & X_\infty \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \tag{7}$$

where J is a matrix in Jordan form whose diagonal elements are the finite eigenvalues and N is a nilpotent matrix also in Jordan form. J and N are unique up to permutation of Jordan blocks. □

The *index* of the pencil $\lambda E - A$ and of the descriptor system (1) is the index of nilpotency of the nilpotent block N in the WCF; i.e., the index of the pencil is μ if and only if $N^{\mu-1} \neq 0$ and $N^\mu = 0$. By convention, if E is nonsingular, then the pencil is said to have index zero. We denote the index of the pencil $\lambda E - A$ by $\mathrm{index}(\lambda E - A)$. If E is a nilpotent matrix and A nonsingular, then we write $\mathrm{index}(E)$ instead of $\mathrm{index}(\lambda E - A)$.

Most of the information displayed by the WCF is also easily obtained from triangular pencils or block triangular pencils. It often simplifies derivations to use triangular or block triangular pencils. Furthermore, numerical algorithms that transform pencils to triangular form are usually more reliable than those that reduce to the WCF (6)–(7) [7, 8, 20].

LEMMA 2.2. *The eigenvalues of the block triangular pencil*

$$\lambda \begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{8}$$

are the union of the eigenvalues of the diagonal blocks

$$\lambda E_{11} - A_{11}, \tag{9}$$

$$\lambda E_{22} - A_{22}. \tag{10}$$

In particular, (8) *is regular if and only if* (9) *and* (10) *are regular.*

Moreover, if (9) *and* (10) *have disjoint eigenvalues, then the Jordan and nilpotent parts of the WCF of* (8) *are the union of the Jordan and nilpotent parts of the WCFs of* (9) *and* (10). □

The next lemma gives a useful characterization of regular, index-one pencils.

LEMMA 2.3 (see [21]). *The pencil $\lambda E - A$ is regular and has index at most one if and only if*

$$\operatorname{rank}\left(\begin{bmatrix} E \\ T_\infty^H A \end{bmatrix}\right) = \operatorname{rank}\left(E + T_\infty T_\infty^H A\right) = n,$$

where the columns of T_∞ span the null space of E^H. Equivalently, the pencil $\lambda E - A$ is regular and of index less than or equal to one if and only if

$$\operatorname{rank}\left([E,\ AS_\infty]\right) = \operatorname{rank}\left(E + AS_\infty S_\infty^H\right) = n,$$

where the columns of S_∞ span the null space of E. □

Similar statements for arbitrary linear systems are given in [14].

If $\lambda E - A$ is regular, then in terms of the WCF (6)–(7), the solutions of (5) take the form

$$x(t) = X_r z_1(t) + X_\infty z_2(t),$$

where

$$\begin{aligned} z_1(t) &= e^{tJ} z_1(0) + \int_0^t e^{(t-s)J} Y_r^H f(s)\,ds, \\ z_2(t) &= -\sum_{i=0}^{\mu-1} \frac{d^i}{dt^i}\left(N^i Y_\infty^H f(t)\right). \end{aligned} \tag{11}$$

From this we see that in order to have a smooth solution $x(t)$, the initial condition $x(0^-)$ must be a member of the set of *admissible* initial conditions

$$\left\{ X_r z_1 + X_\infty z_2 \;\middle|\; z_1 \in \mathbf{C}^r,\ z_2 = -\sum_{i=0}^{\mu-1}\left(N^i Y_\infty^H f^{(i)}(0)\right) \right\}.$$

It may be worthwhile to use feedback to minimize the index of a control system even when it cannot be reduced to index one in order to minimize the effect of discontinuities in the derivatives of unmodeled or perturbing forcing functions. One of the goals of this study is to determine what is the minimal index that can be achieved and to determine a feedback that achieves it. It turns out that according to the linear model (1), the modes that cannot be made to be index one by feedback are constrained to be zero in an appropriate coordinate system, provided the initial conditions are consistent. The remaining active modes may be made to have index one.

We now introduce some further definitions and notation. A system of the form (1) is *regularizable by state feedback*, if there exists a feedback $F \in \mathbf{C}^{m\times n}$ such that the pencil $\lambda E - (A + BF)$ is regular [3, 31]. Similarly, it is *regularizable by output feedback* if there exists $G \in \mathbf{C}^{m\times p}$ such that the pencil $\lambda E - (A + BGC)$ is regular. A system (1) is *controllable at infinity or impulse controllable* if $\operatorname{rank}[E,\ AS_\infty,\ B] = n$, where $\operatorname{Im}(S_\infty) = \ker(E)$. It is called *observable at infinity or impulse observable* if

$$\operatorname{rank}\begin{bmatrix} E \\ T_\infty^H A \\ C \end{bmatrix} = n,$$

where $\mathrm{Im}(T_\infty) = \ker(E^H)$ [5, 40]. In geometric terms, controllability at infinity is equivalent to

$$\mathrm{Im}(E) + A\ker(E) + \mathrm{Im}(B) = \mathbf{C}^n.$$

(See [5, 14, 17, 40].)

A regular descriptor system with index of at most one is a fortiori controllable at infinity. Systems that are controllable at infinity admit a state feedback control which makes the closed loop system be regular and have index of at most one [4, 3]. Moreover, the system transformations may be chosen to minimize the effects of rounding error [3, 10].

Let $P \in \mathbf{C}^{n\times n}$, $Q \in \mathbf{C}^{n\times n}$, $R \in \mathbf{C}^{m\times m}$, and $S \in \mathbf{C}^{p\times p}$ be nonsingular. If

$$\begin{aligned} \tilde{E} = PEQ\tilde{A} = PAQ\tilde{B} &= PBR, \\ \tilde{C} &= SCQ, \end{aligned} \tag{12}$$

then the descriptor system

$$\begin{aligned} \tilde{E}\dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \\ \tilde{y} &= \tilde{C}\tilde{x} \end{aligned}$$

is equivalent to (1) in the sense that

$$\begin{aligned} x &= Q\tilde{x}, \\ u &= R\tilde{u}, \\ y &= S\tilde{y}. \end{aligned}$$

The transformation (12) is a *generalized state transformation*. Such transformations establish an equivalence relation among descriptor systems. Controllability at infinity, observability at infinity, regularity, eigenvalues, and index are preserved by generalized state transformations. Canonical forms under these and other state transformations are discussed in [24, 31]. However, these canonical forms are not easily computed, because modeling errors, measurement errors, or rounding errors may sometimes change them completely. In the next section we use a sequence of state transformations via unitary matrices to bring $\tilde{E}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ into a staircase-like form in the style of [4, 3, 37, 36]. Although our canonical forms display less information than those of [24, 31], they are less sensitive to data perturbations and rounding errors.

The proofs of the staircase-like form in this paper are constructive and form the basis of a numerically stable algorithm for computing the factorization. The basic operations are *row compressions* and *column compressions*. A row compression of a matrix $M \in \mathbf{C}^{h\times k}$ of rank r is the factorization

$$UM = \begin{array}{c} \\ r \\ h-r \end{array} \begin{array}{c} \begin{array}{cc} r & k-r \end{array} \\ \begin{bmatrix} M_1 & M_2 \\ 0 & 0 \end{bmatrix} \end{array},$$

where $U \in \mathbf{C}^{h\times h}$ is unitary and $\begin{bmatrix} M_1, & M_2 \end{bmatrix}$ has full row rank r. The unitary matrix U may be obtained from a QR factorization or the singular value decomposition (SVD) of M or a combination of both [18]. If necessary, U may be chosen so that M_1 is upper triangular. Excellent software for computing the SVD and QR factorizations is widely available [1, 9, 35]. A column compression is a row compression of M^H.

3. Reduction to condensed form. In this section we construct a unitary state transformation that reduces the system matrices of (1) to a staircase and double staircase form similar to those constructed in [2, 4, 3, 37, 36, 38, 39]. A feedback that minimizes the index can be constructed from this staircase form. In addition, the staircase form reveals which modes are uncontrollable at infinity and cannot be reduced to index one by feedback. According to the linear model (1) these modes are not excited and play no role in the system dynamics.

In what follows, it is convenient to allow partitioned matrices which in some special cases may have submatrices with no rows or no columns. In this case, of course, those submatrices are vacuous and simply do not appear. By convention, "0−by−0 matrices" are nonsingular.

LEMMA 3.1. *There exists a state transformation of* (1) *by unitary matrices* $P \in \mathbf{C}^{n\times n}$ *and* $Q \in \mathbf{C}^{n\times n}$ *such that*

$$PEQ = \begin{array}{c} \\ r \\ s \\ q \end{array} \begin{array}{c} \begin{array}{ccc} r & s & q \end{array} \\ \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{13}$$

$$PAQ = \begin{array}{c} \\ r \\ s \\ q \end{array} \begin{array}{c} \begin{array}{ccc} r & s & q \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \end{array}, \tag{14}$$

$$PB = \begin{array}{c} \\ r \\ s \\ q \end{array} \begin{array}{c} m \\ \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \end{array}, \tag{15}$$

$$CQ = p \begin{array}{c} \begin{array}{ccc} r & s & q \end{array} \\ \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \end{array}, \tag{16}$$

where $r = \operatorname{rank}(E)$*,* $s = \operatorname{rank}(B_2)$*, and* $q = n - r - s$*.*

Proof. The proof is by construction. First choose a row compression of the augmented matrix $\left[\ E, B, A\ \right]$,

$$P\left[\ E, B, A\ \right] = \begin{array}{c} \\ r \\ s \\ q \end{array} \begin{array}{c} \begin{array}{ccccccc} r & s & q & m & r & s & q \end{array} \\ \begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} & B_1 & \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & 0 & 0 & B_2 & \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & 0 & 0 & \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \end{array}, \tag{17}$$

where $r = \operatorname{rank}(E)$, $s = \operatorname{rank}(B_2)$, and $q = n - r - s$. If the column space of B is contained in the column space of E, then B_2 is vacuous and $s = 0$. Now, choose a column compression of the permuted submatrix

$$\begin{array}{c} \\ q \\ r \end{array} \begin{array}{c} \begin{array}{ccc} q & r & s \end{array} \\ \begin{bmatrix} \tilde{A}_{33} & \tilde{A}_{31} & \tilde{A}_{32} \\ \tilde{E}_{13} & \tilde{E}_{11} & \tilde{E}_{12} \end{bmatrix} \end{array}$$

to get

$$\begin{array}{c} \\ q \\ r \end{array} \begin{array}{c} \begin{array}{ccc} q & r & s \end{array} \\ \begin{bmatrix} \tilde{A}_{33} & \tilde{A}_{31} & \tilde{A}_{32} \\ \tilde{E}_{13} & \tilde{E}_{11} & \tilde{E}_{12} \end{bmatrix} \end{array} \tilde{Q} = \begin{array}{c} \\ q \\ r \end{array} \begin{array}{c} \begin{array}{ccc} q & r & s \end{array} \\ \begin{bmatrix} A_{33} & 0 & 0 \\ E_{13} & E_{11} & 0 \end{bmatrix} \end{array}.$$

If $K \in \mathbf{C}^{n\times n}$ is the permutation matrix

$$K = \begin{array}{c} \\ r \\ s \\ q \end{array} \!\!\begin{array}{c} \begin{array}{ccc} q & r & s \end{array} \\ \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ I & 0 & 0 \end{bmatrix} \end{array},$$

then P as in (17) and $Q = K\tilde{Q}K^T$ satisfy the statement of the lemma. $\square$

Note that E_{11} in (13) is not necessarily of full row rank. To achieve this we apply the lemma recursively to construct the following staircase-like condensed form, which generalizes the staircase form in [37] to three and four matrices.

THEOREM 3.2. *There exists a state transformation of* (1) *by unitary matrices* $P \in \mathbf{C}^{n\times n}$ *and* $Q \in \mathbf{C}^{n\times n}$ *which puts the system pencil in the form*

$$PEQ = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \end{array}, \tag{18}$$

$$PAQ = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \end{array}, \tag{19}$$

$$PB = \begin{array}{c} \\ t_1 \\ t_2 \\ t_3 \end{array} \!\!\begin{array}{c} m \\ \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} \end{array}, \tag{20}$$

$$CQ = \begin{array}{c} \\ p \end{array} \!\!\begin{array}{c} \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \end{array}, \tag{21}$$

where

(1) $\operatorname{rank}(E_{11}) = t_1$,
(2) $\operatorname{rank}(B_2) = t_2$,
(3) A_{33} *is block upper triangular, and*
(4) E_{33} *is block upper triangular, has zero diagonal blocks, and is partitioned conformally with* A_{33}.

Proof. The proof uses Lemma 3.1 inductively. Initially, apply Lemma 3.1 to get unitary matrices $P^{(1)}$ and $Q^{(1)}$ such that

$$P^{(1)}EQ^{(1)} = \begin{bmatrix} E_{11}^{(1)} & 0 & E_{13}^{(1)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$P^{(1)}AQ^{(1)} = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & A_{13}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} & A_{23}^{(1)} \\ 0 & 0 & A_{33}^{(1)} \end{bmatrix},$$

$$P^{(1)}B = \begin{bmatrix} B_1^{(1)} \\ B_2^{(1)} \\ 0 \end{bmatrix},$$

$$CQ^{(1)} = \begin{bmatrix} C_1^{(1)} & C_2^{(1)} & C_3^{(1)} \end{bmatrix}.$$

For the inductive step, assume that we have constructed a unitary state transformation $P^{(k)}$ and $Q^{(k)}$ such that the transformed system is in the form of (18)–(21) with the exception that $\operatorname{rank}(E_{11}^{(k)}) < t_1$. Apply Lemma 3.1 to the subsystem

$$\tilde{E} = \begin{bmatrix} E_{11}^{(k)} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1^{(k)} \\ B_2^{(k)} \end{bmatrix},$$
$$\tilde{C} = \begin{bmatrix} C_1^{(k)} & C_2^{(k)} \end{bmatrix}$$

to obtain a state transformation of the subsystem $\tilde{P}$ and $\tilde{Q}$ which brings the subsystem into the form of (13)–(16). Embed this transformation by defining

$$P^{(k+1)} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & I \end{bmatrix} P^{(k)},$$
$$Q^{(k+1)} = Q^{(k)} \begin{bmatrix} \tilde{Q} & 0 \\ 0 & I \end{bmatrix}.$$

If the $(1,1)$ block of $P^{(k+1)}EQ^{(k+1)}$ is nonsingular, then the pencil is in the required form. Otherwise, Lemma 3.1 may be applied again to further refine the block structure. Each application of Lemma 3.1 reduces t_1 by at least one. After at most n steps either the $(1,1)$ block of the transformed E is nonsingular or $t_1 = 0$. In either case, the pencil reaches the required form in at most n steps. ∎

Theorem 3.2 essentially separates the uncontrollable infinite modes from the others. We have the following corollary.

COROLLARY 3.3. *In Theorem* 3.2*, the subsystem obtained from the first two block rows and columns of* (18)–(21)*,*

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u,$$

is controllable at infinity.

Proof. The proof follows directly from the definition. ∎

To understand the output feedback case, it is helpful to condense (18)–(21) somewhat further to a double staircase form that decouples both the uncontrollable at infinity modes and the unobservable at infinity modes. (See also [6].)

THEOREM 3.4. *There exists a state transformation of* (1) *by unitary matrices* $\tilde{P} \in \mathbf{C}^{n\times n}$ *and* $\tilde{Q} \in \mathbf{C}^{n\times n}$ *which puts the system pencil in the form*

$$\tilde{P}E\tilde{Q} = \begin{array}{c} \\ \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \\ \tilde{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \end{array} \\ \begin{bmatrix} \tilde{E}_{11} & 0 & 0 & \tilde{E}_{14} \\ 0 & 0 & 0 & \tilde{E}_{24} \\ \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} & \tilde{E}_{34} \\ 0 & 0 & 0 & \tilde{E}_{44} \end{bmatrix} \end{array}, \tag{22}$$

$$\tilde{P}A\tilde{Q} = \begin{array}{c} \\ \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \\ \tilde{t}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \end{array} \\ \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 & \tilde{A}_{14} \\ \tilde{A}_{21} & \tilde{A}_{22} & 0 & \tilde{A}_{24} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & 0 & \tilde{A}_{44} \end{bmatrix} \end{array}, \tag{23}$$

$$\tilde{P}B = \begin{array}{c} \\ \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \\ \tilde{t}_4 \end{array} \begin{array}{c} m \\ \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ 0 \end{bmatrix} \end{array}, \tag{24}$$

$$C\tilde{Q} = p \begin{array}{c} \begin{array}{cccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 & \tilde{t}_4 \end{array} \\ \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 & 0 & \tilde{C}_4 \end{bmatrix} \end{array}, \tag{25}$$

with the following properties.

(1) $\operatorname{rank}(\tilde{E}_{11}) = \tilde{t}_1$*;*

(2) $\operatorname{rank}(\tilde{C}_2) = \tilde{t}_2$*;*

(3) $\tilde{A}_{33}$ *is block lower triangular;*

(4) $\tilde{E}_{33}$ *is block lower triangular with zero diagonal blocks, partitioned conformally with* $\tilde{A}_{33}$*;*

(5) $\tilde{A}_{44}$ *is block upper triangular;*

(6) $\tilde{E}_{44}$ *is block upper triangular with zero diagonal blocks, partitioned conformally with* $\tilde{A}_{44}$*;*

(7) *the subsystem obtained by deleting the last block row and column in* (22)–(25) *is controllable at infinity.*

Proof. Apply Theorem 3.2 to system (1) to get the state transformations P_1 and Q_1 and partitioning of (18)–(21). Apply Theorem 3.2 again to the transposed subsystem given by

$$\hat{E} = \left(\begin{array}{cc} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \end{array} & \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{array} \right)^H,$$

$$\hat{A} = \left(\begin{array}{cc} & \begin{array}{cc} t_1 & t_2 \end{array} \\ \begin{array}{c} t_1 \\ t_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array} \right)^H,$$

$$\hat{B} = \left(\begin{array}{cc} & \begin{array}{cc} t_1 & t_2 \end{array} \\ p & \begin{bmatrix} C_1 & C_2 \end{bmatrix} \end{array} \right)^H,$$

$$\hat{C} = \left(\begin{array}{cc} & m \\ \begin{array}{c} t_1 \\ t_2 \end{array} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array} \right)^H$$

to get orthogonal matrices $\hat{P}$, $\hat{Q} \in \mathbf{C}^{(t_1+t_2)\times(t_1+t_2)}$ that reduce the subsystem to the form of (18)–(21). Define P_2 and Q_2 by

$$P_2 = \begin{bmatrix} \hat{Q}^H & 0 \\ 0 & I_{t_3} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} \hat{P}^H & 0 \\ 0 & I_{t_3} \end{bmatrix}.$$

The state transformation given by $\tilde{P} = P_2P_1$ and $\tilde{Q} = Q_1Q_2$ achieves the condensed form of (22)–(25).

Properties (1)–(4) come directly from Theorem 3.2 applied to the subsystem. Properties (5) and (6) also follow from Theorem 3.2 because when the second state transformation P_2, Q_2 is applied to (18)–(21), E_{33} and A_{33} are unchanged; i.e., $\tilde{A}_{44}$ in (23) is just A_{33} in (19) and $\tilde{E}_{44}$ in (22) is just E_{33} in (19). Property (7) follows because the first three block rows and columns in (22)–(25) are a state transformation by $\hat{P}$ and $\hat{Q}$ of the first two block rows and columns of (18)–(21). ∎

We have the following corollary.

COROLLARY 3.5. *The subsystem obtained by deleting the last two block rows and columns from* (22)–(25) *is controllable and observable at infinity.*

Proof. It is clear that the subsystem is observable at infinity by construction. By Theorem 3.4, we have that the subsystem given by

$$\begin{array}{c c} & \begin{array}{ccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \end{array} \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{array} & \begin{bmatrix} \tilde{E}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{E}_{31} & \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix} \end{array}, \quad \begin{array}{c c} & \begin{array}{ccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \end{array} \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{array} & \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & 0 \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \end{array}, \quad \begin{array}{c c} & m \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \\ \tilde{t}_3 \end{array} & \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix} \end{array},$$

$$\begin{array}{c c} & \begin{array}{ccc} \tilde{t}_1 & \tilde{t}_2 & \tilde{t}_3 \end{array} \\ p & \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 & 0 \end{bmatrix} \end{array}$$

is controllable at infinity. This directly implies that the subsystem given by

$$\begin{array}{c c} & \begin{array}{cc} \tilde{t}_1 & \tilde{t}_2 \end{array} \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \end{array} & \begin{bmatrix} \tilde{E}_{11} & 0 \\ 0 & 0 \end{bmatrix} \end{array}, \quad \begin{array}{c c} & \begin{array}{cc} \tilde{t}_1 & \tilde{t}_2 \end{array} \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \end{array} & \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \end{array}, \quad \begin{array}{c c} & m \\ \begin{array}{c} \tilde{t}_1 \\ \tilde{t}_2 \end{array} & \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} \end{array},$$

$$\begin{array}{c c} & \begin{array}{cc} \tilde{t}_1 & \tilde{t}_2 \end{array} \\ p & \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} \end{array}$$

is also controllable at infinity. ∎

4. Regularization and index minimization by state feedback.

The following theorem answers the question of when state feedback may be used to make a descriptor system regular.

THEOREM 4.1. *If system* (1) *is in the form of Theorem* 3.2*, then the system is regularizable by state feedback; i.e., there exists a state feedback gain matrix* $F \in \mathbf{C}^{m\times n}$ *such that the pencil* $\lambda E - (A + BF)$ *is regular if and only if* A_{33} *is nonsingular.*

Proof. Let $F \in \mathbf{C}^{m\times n}$ be partitioned as

$$F = \begin{array}{c c} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ m & \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \end{array}.$$

The pencil $\lambda E - (A + BF)$ is block upper triangular, so its characteristic polynomial is

$$\begin{aligned} &\det(\lambda E - (A + BF)) \\ &\quad = \det\left(\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \left(\begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ A_{21} + B_2F_1 & A_{22} + B_2F_2 \end{bmatrix}\right)\right) \det(\lambda E_{33} - A_{33}) \\ &\quad = \det\left(\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \left(\begin{bmatrix} A_{11} + B_1F_1 & A_{12} + B_1F_2 \\ A_{21} + B_2F_1 & A_{22} + B_2F_2 \end{bmatrix}\right)\right) \det(-A_{33}). \end{aligned}$$

(26)

The last equality follows because $\lambda E_{33} - A_{33}$ is block triangular with diagonal blocks that are independent of λE_{33}.

If A_{33} is singular, then (26) is zero independent of λ, and the pencil is not regular.

Suppose that A_{33} is nonsingular. Because B_2 has full row rank, there exists a matrix $F_2 \in \mathbf{C}^{m\times t_2}$ such that $A_{22} + B_2F_2$ is nonsingular. Let

$$F = \begin{array}{c c} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ m & \begin{bmatrix} 0 & F_2 & 0 \end{bmatrix} \end{array}.$$

The first factor on the right-hand side of (26) is the characteristic polynomial of the subpencil

$$\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} 0 & F_2 \end{bmatrix} \right). \tag{27}$$

Since $A_{22} + B_2F_2$ is nonsingular, the pencil in (27) is equivalent to a pencil of the form

$$\lambda \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & A_{22} + B_2F_2 \end{bmatrix}. \tag{28}$$

It then follows from the nonsingularity of E_{11} and Lemma 2.3 that (28) is regular and has index one. Hence, neither factor in (26) is identically zero and the pencil is regular. ◻

The next theorem shows what index can be achieved by state feedback.

THEOREM 4.2. *If system* (1) *is in the form of Theorem* 3.2 *and* A_{33} *is nonsingular, then there exists a state feedback gain matrix* $F \in \mathbf{C}^{m\times n}$ *such that* $\lambda E - (A + BF)$ *is regular and*

$$\operatorname{index}(\lambda E - (A + BF)) = \operatorname{index}\left(\begin{array}{c c} & \begin{array}{cc} t_2 & t_3 \end{array} \\ \begin{array}{c} t_2 \\ t_3 \end{array} & \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} \end{array} \right).$$

Proof. Choose $F_1 \in \mathbf{C}^{m\times t_1}$ so that $A_{21} + B_2F_1 = 0$ and choose $F_2 \in \mathbf{C}^{m\times t_2}$ so that $A_{22} + B_2F_2$ is nonsingular. Both F_1 and F_2 exist, because B_2 has full row rank. Define $F \in \mathbf{C}^{m\times n}$ by

$$F = \begin{array}{c c} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ m & \begin{bmatrix} F_1 & F_2 & 0 \end{bmatrix} \end{array}.$$

The pencil $\lambda E - (A + BF)$ is block upper triangular with diagonal blocks

$$\lambda E_{11} - (A_{11} + B_1F_1), \tag{29}$$

$$\lambda \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} - \begin{bmatrix} A_{22} + B_2F_2 & A_{32} \\ 0 & A_{33} \end{bmatrix}. \tag{30}$$

Pencil (29) has only finite eigenvalues because E_{11} is nonsingular. Pencil (30) has only infinite eigenvalues, because the left-hand side is nilpotent and the right-hand side is nonsingular. Lemma 2.2 implies that

$$\begin{aligned} \operatorname{index}(\lambda E - (A + BF)) &= \operatorname{index}\left(\lambda \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} - \begin{bmatrix} A_{22} + B_2F_2 & A_{32} \\ 0 & A_{33} \end{bmatrix} \right) \\ &= \operatorname{index}\left(\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} \right). \end{aligned}$$

Here we have used properties 3 and 4 of Theorem 3.2. ◻

The index of nilpotency of $(\begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix})$ can be displayed by applying another staircase algorithm.

The Kronecker-like feedback canonical form of [24] also displays this minimal index, but this canonical form is not suitable for numerical computation.

An obvious consequence of Theorem 4.2 is that the index of nilpotency of E_{33} differs from the minimal obtainable index by at most one.

THEOREM 4.3. *Suppose that system* (1) *is in the form of Theorem* 3.2. *If* $F \in \mathbf{C}^{m\times n}$ *is a state feedback gain matrix for which* $\lambda E - (A + BF)$ *is regular, then*

$$\mathrm{index}(E_{33}) + 1 \geq \mathrm{index}(\lambda E - (A + BF)) \geq \mathrm{index}(E_{33}).$$

Proof. The proof is an immediate consequence of Theorem 4.2. $\square$

If (1) is in the form of Theorem 3.2, then the subsystem $E_{33}\dot{z} = A_{33}z$ represents the uncontrollable infinite eigenvalue modes. Being uncontrollable, as Theorem 4.3 shows, there is nothing that feedback can do to lower $\mathrm{index}(\lambda E_{33} - A_{33})$. However, the next theorem shows that, regardless of initial condition, $z(t) = 0$ for $t > 0$ and $z(0^+) = 0$.

If $z(0^-) = 0$, then z is impulse free and constrained to be zero. Hence, the uncontrollable, infinite modes are not involved in the dynamics! To prove this result we use the trivial fact that if $N \in \mathbf{C}^{n\times n}$ is nilpotent, then the only smooth function $x = x(t) \in \mathbf{C}^n$ satisfying $N\dot{x} = x$ is $x = 0$. It follows that any distributional solution of $N\dot{x} = x$ is purely impulsive and for $t > 0$, $x(t) = 0$. The distributional solution is impulse free if and only if $Nx(0^-) = 0$. Moreover, $x(0^-) = x(0^+)$ if and only if $x(0^-) = x(0^+) = 0$.

THEOREM 4.4. *If the descriptor system in the form of Theorem* 3.2*,*

$$\begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} u, \tag{31}$$

is regularizable, then for $t > 0$*,* $x_3(t) = 0$ *independent of the control* u*. If* $E_{33}x_3(0^-) = 0$*, then* x_3 *is impulse free.*

Proof. The third equation of (31) is $E_{33}\dot{x}_3 = A_{33}x_3$. Theorem 4.1 implies that A_{33} is nonsingular, so this is equivalent to $A_{33}^{-1}E_{33}\dot{x}_3 = x_3$ and $A_{33}^{-1}E_{33}x_3(0^-) = 0$. By hypothesis, properties 3 and 4 of Theorem 3.2 hold, so A_{33} and E_{33} are block upper triangular and $A_{33}^{-1}E_{33}$ is nilpotent. The theorem follows. $\square$

It follows from Theorems 4.1, 4.2, and 4.4 that a regularizable system decouples into the uncontrollable infinite modes and the subsystem

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u. \tag{32}$$

The infinite uncontrollable modes play no role in the system dynamics. With consistent initial conditions, they are constrained to be zero. Only the modes involved in (32) are active. The subsystem (32) is controllable at infinity, so the results of [4, 3] apply. It follows that those modes constrained to be zero may be eliminated from the system. In this way, all regularizable descriptor systems may be made to be controllable at infinity. Hence, the methods designed for linear quadratic control, pole assignment, stabilization, etc., under the assumption of controllability at infinity, may be used [27].

A similar result in the context of linear quadratic control of a particular mechanical multibody system was obtained by explicit transformation in [28].

5. Geometric proofs. In this section we will provide *geometric* proofs for the results in the previous sections.

Let $\mathcal{I}_s$ denote the *largest* subspace $\mathcal{L}$ that satisfies

$$\mathcal{L} \subset A^{(-1)}(E\mathcal{L} + \mathrm{Im}(B)).$$

For a discussion of this space see [13, 23, 26]. The subspace $\mathcal{I}_s = A^{(-1)}(E\mathcal{L}+\mathrm{Im}(B))$ is called the *consistent* subspace, since every point in $\mathcal{I}_s$ is consistent; i.e., for every point $x_0 \in \mathcal{I}_s$ there exists a smooth input $u(t)$ and an associated smooth state trajectory $x(t)$ of system (1) satisfying $x(0) = x_0$ [13].

Let $\mathcal{X}_1$ be such that $\mathcal{X}_1 \oplus (\mathcal{I}_s \cap \ker(E)) = \mathcal{I}_s$; let $\mathcal{X}_3$ be such that $\mathcal{I}_s \oplus \mathcal{X}_3 = \mathbf{C}^n$; and let $\mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$ be spaces chosen such that $E\mathcal{I}_s \oplus \mathcal{Y}_2 = E\mathcal{I}_s + \mathrm{Im}(B)$, $(E\mathcal{I}_s + \mathrm{Im}(B)) \oplus \mathcal{Y}_3 = E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A)$, and $(E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A)) \oplus \mathcal{Y}_4 = \mathbf{C}^n$. Choose $\mathcal{U}_2$ so that $B^{(-1)}(E\mathcal{I}_s) \oplus \mathcal{U}_2 = \mathbf{C}^m$. With respect to suitably chosen bases, (1) transforms to

$$\begin{bmatrix} E_{11} & 0 & E_{13} \\ 0 & 0 & E_{23} \\ 0 & 0 & E_{33} \\ 0 & 0 & E_{43} \end{bmatrix} \dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t). \tag{33}$$

By construction, $\ker(B_{22}) = 0$, B_{22} is right invertible, E_{11} is invertible, and

$$M(\lambda) = \begin{bmatrix} \lambda \begin{bmatrix} E_{23} \\ E_{33} \\ E_{43} \end{bmatrix} - \begin{bmatrix} A_{23} \\ A_{33} \\ 0 \end{bmatrix}, & \begin{bmatrix} B_{22} \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}$$

has full column rank for all $\lambda \in \mathbf{C}$ [15, Appendix, Lemma 1]. In addition, we can show that A_{33} is square and therefore nonsingular.

LEMMA 5.1. *If the system* (1) *is in the form* (33)*, then* A_{33} *is invertible.*

Proof. We have that $A\mathcal{I}_s = (E\mathcal{I}_s + \mathrm{Im}(B)) \cap \mathrm{Im}(A)$. (See, for example, [16].) In addition [30]

$$\begin{aligned} \dim(A\mathcal{I}_s) &= \dim(\mathcal{I}_s) - \dim(\mathcal{I}_s \cap \ker(A)) \\ &= \dim(\mathcal{I}_s) - \dim(\ker(A)) \\ &= \dim(\mathcal{I}_s) - n + \mathrm{rank}(A). \end{aligned}$$

Thus,

$$\begin{aligned} &\dim(E\mathcal{I}_s + \mathrm{Im}(B)) + \dim(\mathrm{Im}(A)) \\ &= \dim(E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A)) + \dim(E\mathcal{I}_s + \mathrm{Im}(B)) \cap \mathrm{Im}(A)) \\ &= \dim(E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A)) + \dim(\mathcal{I}_s) - n + \dim(\mathrm{Im}(A)) \end{aligned}$$

and $\dim(E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A)) = n - \dim(\mathcal{I}_s)$. Therefore, A_{33} is square and hence invertible. ☐

The next step is to relate the condensed form (33) to the regularizability of the system. We have the following well-known result [11, 31].

THEOREM 5.2. *The following are equivalent.*

(i) $E\mathcal{I}_s + \mathrm{Im}(B) + \mathrm{Im}(A) = \mathbf{C}^n$.

(ii) $\begin{bmatrix} \lambda E - A, & B \end{bmatrix}$ *is right invertible as a rational matrix.*

(iii) *The system* (1) *is regularizable by proportional state feedback.*

Proof. For completeness we give a short proof of this result. Statement (i) implies that $\mathcal{Y}_4 = \{0\}$. Hence, the last block row in (33) does not occur. From Lemma 5.1, it follows that A_{33} is invertible. By choosing feedback $u = Fx + v$ with

$$F = \begin{bmatrix} 0 & 0 & 0 \\ -B_{22}^{-1}A_{21} & B_{22}^{-1}(I - A_{22}) & 0 \end{bmatrix}, \tag{34}$$

we obtain the closed loop system

$$E\dot{x} = (A + BF)x + Bv \tag{35}$$

for which $\begin{bmatrix} \lambda E - (A + BF), & B \end{bmatrix}$ is right invertible. This implies (ii).

Conversely (ii) implies that $M(\lambda)$ is right invertible. Hence, $\mathcal{Y}_4 = \{0\}$ and we have (i).

Statement (i) implies statement (iii) because the feedback (34) makes the pencil (35) regular. The converse is clear. If (i) did not hold, then for every feedback F, the pencil of the closed loop system would be singular, which contradicts (iii). ∎

From this we see that for regularizable systems the condensed form (18)–(21), which is constructible in a numerically stable way, coincides with the form (33).

6. Derivative and output feedback. In this section we give a few results about derivative and output feedback.

If we use state derivative feedback, the minimal attainable index is $\mathrm{index}(E_{33})$ in (18).

THEOREM 6.1. *If* (1) *is in the form of Theorem* 3.2 *and* A_{33} *is nonsingular, then then there exists a derivative feedback gain matrix* $G \in \mathbf{C}^{m\times n}$ *such that the pencil* $\lambda(E + BG) - A$ *is regular with* $t_1 + t_2$ *finite eigenvalues and* $\mathrm{index}(\lambda(E + BG) - A = \mathrm{index}(E_{33})$.

Proof. Let $G_2 \in \mathbf{C}^{m\times t_2}$ be chosen so that B_2G_2 is nonsingular. Define G by

$$G = \begin{array}{c c} & \begin{array}{ccc} t_1 & t_2 & t_3 \end{array} \\ m & \begin{bmatrix} 0 & G_2 & 0 \end{bmatrix} \end{array}.$$

Then

$$\lambda(E + BG) - A = \lambda \begin{bmatrix} E_{11} & B_1G_2 & E_{13} \\ 0 & B_2G_2 & E_{23} \\ 0 & 0 & E_{33} \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}.$$

This is a block triangular pencil with diagonal blocks

$$\lambda \begin{bmatrix} E_{11} & B_1G_2 \\ 0 & B_2G_2 \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$
$$\lambda E_{33} - A_{33}.$$

By Lemma 2.2, the infinite eigenvalues are the eigenvalues of $\lambda E_{33} - A_{33}$ and

$$\begin{aligned} \mathrm{index}(\lambda(E + BG) - A) &= \mathrm{index}(\lambda E_{33} - A_{33}) \\ &= \mathrm{index}(E_{33}). \qquad ∎ \end{aligned}$$

The conclusion of Theorem 6.1 also holds if both state and derivative feedback are used.

An extra hypothesis is needed to obtain the same results in the output feedback case.

THEOREM 6.2. *If equation* (1) *is in the form of Theorem* 3.2*, then the system is regularizable by output feedback; i.e., there exists an output feedback gain matrix* F *such that* $\lambda E - (A + BFC)$ *is regular and*

$$\operatorname{index}(\lambda E - (A + BFC)) = \operatorname{index}\left(\begin{array}{cc} & \begin{array}{cc} t_2 & t_3 \end{array} \\ \begin{array}{c} t_2 \\ t_3 \end{array} & \begin{bmatrix} 0 & E_{23} \\ 0 & E_{33} \end{bmatrix} \end{array}\right)$$

if and only if A_{33} *is nonsingular and* $\left[\begin{smallmatrix} A_{22} \\ C_2 \end{smallmatrix}\right]$ *has full column rank.*

Proof. Since B_2 has full row rank there is a matrix F such that $A_{22} + B_2FC_2$ is nonsingular if and only if $\left[\begin{smallmatrix} A_{22} \\ C_2 \end{smallmatrix}\right]$ has full column rank. Applying the argument in the proof of Theorem 4.1, the result follows. ∎

It follows immediately from Theorem 3.4 that the part of the system which is unobservable at infinity can be completely decoupled from the rest of the system. This part of the system can be removed because, according to the linear model, it does not influence the dynamics of the system and the possible impulsive behavior cannot be observed.

7. Discrete time systems and linear quadratic control. So far, we have discussed continuous time systems only. It should be noted that Lemma 3.1, Corollary 3.5, and Theorems 3.2, 3.4, 4.1, 4.2, 4.3, and 6.1 are independent of the origin of the matrices and thus also hold for discrete time systems. There exists an easily formulated, analogous discrete time version of Theorem 4.4.

The results apply to linear quadratic optimal control problems of the following form:

minimize the cost functional

$$J(x, u) := \int_{t_0}^{t_1} (x^T Qx + u^T Ru)\, dt$$

subject to the descriptor system (1).

After transforming to the reduced form of Theorem 3.2, we may just omit the components which are uncontrollable at infinity from the cost functional and the constraint to obtain a reduced order problem which is controllable at infinity. For such systems, the methods described in [27] apply. For a detailed analysis of general linear quadratic optimal control problems for descriptor systems see [15].

8. Conclusions. According to the linear model (1), problems associated with uncontrollable infinite modes in a regularizable system do not occur. With consistent initial conditions, they are constrained to be zero. The only active dynamics in (1) are controllable at infinity. The active dynamics may be made to be index one by state feedback and the entire system may be treated as if it were controllable at infinity as in [4, 3]. However, the resulting system is not index-one robust. If there is an unmodeled forcing function that excites modes that are uncontrollable at infinity, then it may generate impulses.[1]

[1]The situation is similar to one described in a nonsense verse [22]:

Yesterday, upon the stair
I saw a man who wasn't there.
He wasn't there again today.
Gee, I wish he'd go away!

The uncontrollable high-index infinite modes are the man upon the stair. He is not there, but he is disturbing nevertheless.

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