

1 **EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR**
2 **DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE**

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ABSTRACT. We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line \mathbb{R}_+ . Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform φ -Lipschitz condition, in which the Lipschitz function φ belongs to wide classes of admissible function spaces such as L_p , $1 \leq p \leq \infty$, $L_{p,q}$, etc.

4 1. INTRODUCTION AND PRELIMINARIES

5 The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear
6 non-autonomous differential-algebraic equations (DAEs) of the form

$$\begin{array}{ll} d \text{ rows} & \left[\begin{array}{c} \mathbf{E}_1(t) \\ 0 \end{array} \right] \dot{x}(t) = \underbrace{\left[\begin{array}{c} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{array} \right]}_{A(t)} x(t) + \underbrace{\left[\begin{array}{c} f_1(t, x(t)) \\ f_2(t, x(t)) \end{array} \right]}_{f(t, x(t))}, & t \in \mathbb{R}_+ := [0, +\infty). \end{array} \quad \begin{matrix} \text{semi linDAE} \\ (1.1) \end{matrix}$$

7 To do that, we start by investigating the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \quad \begin{matrix} \text{linDAE} \\ (1.2) \end{matrix}$$

8 Here $E = \begin{bmatrix} \mathbf{E}_1(t) \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} \mathbf{A}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix}$ are matrix-valued functions acting on \mathbb{R}_+ to $\mathbb{R}^{n,n}$, $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$, and
9 $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Furthermore, we assume that for all t , the matrices $\mathbf{E}_1(t)$, $\mathbf{A}_2(t)$ have full row rank.

10 DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuits, chemical engineering, and many other applications. Due to the rank-deficiency of $E(t)$, the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the reader to recent monographs [2, 12–14] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [12, Chapter 2]), there are only few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [15] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [17] for Lyapunov and other spectra for linear DAEs, to [4, 8] for the robustness of exponential stability and Bohl exponents.

19 On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is
20 characterized, the next important question in the qualitative theory of DAEs is to study the existence of
21 integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1)
22 [3, 6]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps,
23 this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1).

Date: May 2, 2021.

Key words and phrases. Exponential dichotomy, semilinear, differential-algebraic equation, admissibility of function spaces, stable manifold.

24 Our method is based on the classical "Lyapunov-Perron method" ([6, 25]) and the admissibility of function
 25 spaces ([10, 11]).

26 The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for
 27 later use, including the notion of the exponential dichotomy and its properties, as well as some important
 28 features of admissible function spaces. In Section 2 we give a characterization for the existence of expo-
 29 nential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of
 30 local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in
 31 Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations,
 32 and we conclude this research by a summary and some open problems.
 33

34 **1.1. Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By $(\mathbb{R}^n,$
 35 $\|\cdot\|)$ we denote the n -dimensional real vector space equipped with the Euclidean norm. For any matrix
 36 V , by V^T we denote its transpose. For any $p \in \mathbb{N}$, by $C^p([0, \infty), \mathbb{R}^n)$ we denote the space of p -times
 37 continuously differentiable functions acting on $[0, \infty)$ with values in \mathbb{R}^n . By $C_b([0, \infty), \mathbb{R}^n)$ we denote the
 38 space of continuous and bounded functions mapping from $[0, \infty)$ into \mathbb{R}^n . This space is a Banach space with
 39 the *ess sup*-norm $\|f\|_\infty := \sup\{\|f(t)\|, t \geq 0\}$.

It is well-known (e.g. [3]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \geq s \geq 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

40 is well-posed, then there exists a pointwise nonsingular matrix-valued function $(t, s) \mapsto X(t, s) \in \mathbb{R}^{n,n}$ such
 41 that the solution of (1.3) is given by $x(t) = X(t, s)x_s$. This fact motivates the existence of an evolution family
 42 $(X(t, s))_{t \geq s \geq 0}$ associated with the matrix function $A(t)$. This family satisfies the condition $X(t, t) = Id$ and
 43 the so-called *semi-group property*

$$X(t, r)X(r, s) = X(t, s), \quad \text{for all } t \geq r \geq s \geq 0. \tag{1.4}$$

44 Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), \quad \text{for all } t \geq s \geq 0,$$

45 also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \tau)f(\tau, x(\tau))d\tau, \quad \text{for all } t \geq s \geq 0. \tag{1.5}$$

46 For more details on the notion and discussion on properties and applications of evolution families we refer
 47 the readers to Pazy [21].

48 **Definition 1.1.** A given evolution family $\{X(t, s)\}_{t \geq s \geq 0}$ of the ODE (1.3) is said to have an *exponential*
 49 *dichotomy* on the half-line if there exist a family of projection matrices $\{P(t)\}_{t \geq 0}$ and two positive constants
 50 N, ν such that the following conditions are satisfied.

- 51 i) $P(t)X(t, s) = X(t, s)P(s)$ for all $t \geq s \geq 0$,
 - 52 ii) for all $t \geq s \geq 0$, the restriction $X(t, s)| : \ker P(s) \rightarrow \ker P(t)$ is an isomorphism, and we denote its
 53 inverse by $X(s, t)|$,
 - 54 iii) $\|X(t, s)P(s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$, for all $t \geq s \geq 0$, $x \in \mathbb{R}^n$,
 - 55 iv) $\|X(t, s)|(I - P(s))x\| \leq Ne^{\nu(t-s)}\|(I - P(s))x\|$, for all $s \geq t \geq 0$, $x \in \mathbb{R}^n$.
- 56 Here $\{P(t)\}_{t \geq 0}$ (reps. N, ν) are called *dichotomy projections* (resp. *dichotomy constants*).

57 The concept exponential dichotomy means that the state space \mathbb{R}^n has been splitted into the (exponentially) stable subspace ($\text{Im}(P(t))$) and the (exponentially) unstable subspace ($\ker(P(t))$).

59 **1.2. A short review of DAE solvability theory.** Linear DAEs of the form 1.2 have been extensively
60 studied in the last thirty years, see [13] and the references therein. In order to understand the solution
61 behavior and to obtain numerical solutions, the necessary information about derivatives of equations has to
62 be utilized. This has led to the concept of the strangeness index, which allows to use the DAE and (some of)
63 its derivatives to be reformulated as a system with the same solution that is *strangeness-free*, i.e., for which
64 the algebraic and differential part of the system are easily separated. In this paper we restrict ourselves to
65 regular DAEs with sufficiently smooth coefficients, i.e., we require that (1.2) (or the nonlinear DAE (1.1)
66 locally) has a unique solution for appropriately chosen (consistent) initial conditions, see [13] for a discussion
67 of more general nonregular DAEs. With the theory and appropriate numerical methods available, then
68 throughout this paper, for regular DAEs we may assume that the homogeneous DAE 1.2 in consideration
69 fulfills the following presumption.

70 **Assumption 1.2.** Assume that the function pair (E, A) in the DAEs (1.1), (1.2) is *strangeness-free*, i.e.,

$$\text{rank} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} = n,$$

71 for all $t \geq 0$. Furthermore, we assume that $E \in C^1([0, \infty), \mathbb{R}^{n,n})$ and $A \in C^0([0, \infty), \mathbb{R}^{n,n})$.

72 **Definition 1.3.** The DAE

$$\tilde{E}(t)\dot{y}(t) = \tilde{A}(t)x(t) + \tilde{f}(t, y(t)) \quad (\text{eq7})$$

73 is called *orthogonally equivalent* to the DAE (1.1) if there exist pointwise-orthogonal matrix-valued functions
74 $U \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that after changing variable $x(t) = V(t)y(t)$, and scaling
75 (1.2) with $U(t)$, we obtain exactly (1.19). In details, this means that the following identities hold true.

$$\tilde{E} = UEV, \quad \tilde{A} = UAV - U\dot{E}V, \quad \tilde{f}(t, y(t)) = U(t)f(t, Vy(t)), \quad \text{for all } t \geq 0. \quad (1.7)$$

76 We denote this orthogonal equivalence by $(E, A, f) \stackrel{\sim}{\sim} (\tilde{E}, \tilde{A}, \tilde{f})$ and omit the terms f, \tilde{f} if the homogeneous
77 system (1.2) is considered.

78 Indeed, one can directly verify that this orthogonal equivalence concept is an equivalent relation, i.e., it
79 fulfills three properties: reflexivity, symmetry and transitivity. We omit the detailed proof here in order to
80 keep the brevity of this work. By making use of some smooth factorizations, for example QR or SVD ([7] or
81 [13, Thm 3.9]), we can decouple the differential and algebraic parts of the DAE (1.2) in the following lemma.

82 **Lemma 1.4.** Consider the DAE (1.2) and assume that it satisfies Assumption 1.2. Then, there exist
83 pointwise-orthogonal matrix-valued functions $U \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that by
84 changing variable $x(t) = V(t)y(t)$, and scaling (1.2) with $U(t)$, we obtain the so-called semi-explicit system

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{semi-explicit system} \quad (1.8)$$

85 with pointwise nonsingular matrix-valued functions $\Sigma(t) \in \mathbb{R}^{d,d}$ and $A_4(t) \in \mathbb{R}^{a,a}$.

86 *Proof.* Applying an SVD factorization for $\mathbf{E}_1(t)$ we can find pointwise-orthogonal matrix functions $U_1(t) \in$
87 $C^1([0, \infty), \mathbb{R}^{d,d})$ and $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ such that $U_1(t)\mathbf{E}_1(t)V(t) = [\Sigma(t) \ 0]$, where $\Sigma(t)$ is a continuous,
88 pointwise nonsingular function with values in $\mathbb{R}^{d,d}$. Changing the variable $x(t) = V(t)y(t)$ and scaling (1.2)
89 with $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$, we obtain a new system exactly of the form (1.8). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ A_3(t) & A_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} \mathbf{E}_1(t) \\ \mathbf{A}_2(t) \end{bmatrix} V(t),$$

90 then Assumption 1.2 yields that both Σ and A_4 are nonsingular. This completes the proof. \square

Let $\hat{A}_3 := -A_4^{-1}A_3$, $\hat{A}_1 := \Sigma^{-1}(A_1 - A_2A_4^{-1}A_3)$, we rewrite system (1.8) as

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t), \quad (1.9a)$$

$$y_2(t) = \hat{A}_3(t)y_1(t). \quad (1.9b)$$

91 Since $V(t)$ is orthogonal for all $t \geq 0$, we see that all important qualitative properties of $x(t)$, such as
92 boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function $y(t)$
93 without any difficulty. Clearly, we see that (1.9b) gives an *algebraic constraint* that the solution to (1.8)
94 must obey, while (1.9a) gives the dynamic of (1.8). For this reason, we call it *an underlying ODE* to (1.8).

95 **Definition 1.5.** (i) Consider the DAE (1.2). A matrix function $X \in C([0, \infty), \mathbb{R}^{n,k})$, $d \leq k \leq n$, is called a
96 *fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and $\text{rank } X(t) = d$, for all
97 $t \geq 0$.

98 (ii) A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively. A
99 maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)(X(0) - Id) = 0. \quad \text{projected intial conditon} \quad (1.10)$$

100 We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of
101 full rank. Furthermore, each fundamental solution matrix has exactly d -linear independent columns, and
102 a minimal fundamental solution matrix can be made maximal by adding $n - d$ zero columns. This is the
103 major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a
104 DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been
105 proposed and carefully discussed in [17]. We recall it below, and notice that this concept is equivalent to the
106 one proposed by Lentini and März in [15] within the context of the matrix chains approach and tractability
107 index.

108 Let $\{\hat{Y}_1(t, s)\}_{t \geq s \geq 0}$ be the evolution family associated with (1.9a), then we can define the corresponding
109 evolution families for two DAEs (1.8), (1.2) consecutively as follows.

$$\hat{Y}(t, s) := \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, s) & 0 \end{bmatrix}, \quad \hat{X}(t, s) := V(t)\hat{Y}(t, s)V^T(s), \quad \text{for all } t \geq s \geq 0. \quad (1.11)$$

110 Nevertheless, since $X(t, s)$ is not invertible, we will define the *reflexive generalized inverse matrix function*
111 as in [17] by

$$\hat{Y}^-(t, s) := \begin{bmatrix} \hat{Y}_1^{-1}(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t, s) & 0 \end{bmatrix}, \quad \hat{X}^-(t, s) := V(s)\hat{Y}^-(t, s)V^T(t), \quad \text{for all } t \geq s \geq 0. \quad (1.12)$$

Then, we can directly verify the semigroup properties, i.e.

$$\hat{X}(t, r) = \hat{X}(t, s)\hat{X}(s, r), \quad \text{for all } t \geq s \geq r \geq 0,$$

$$\hat{X}(t, s) = \hat{X}(t, 0)\hat{X}^-(s, 0), \quad \text{for all } t \geq s \geq 0.$$

112 Furthermore, Lemmas 1.6 and 1.7 below show that the family $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$ does not depend on the
113 choice of orthogonal transformations, and it plays the same role as the evolution family $\{X(t, s)\}_{t \geq s \geq 0}$, in
114 comparison to (1.5).

115 **Lemma 1.6.** *The families $\{X(t, s)\}_{t \geq s \geq 0}$, $\{X^-(t, s)\}_{t \geq s \geq 0}$ defined by (1.11), (1.12) do not depend on the
116 choice of orthogonal transformations.*

Proof. We will prove this claim only for the first family $\{X(t, s)\}_{t \geq s \geq 0}$, since for the second family the proof is essentially the same. Let us assume that we have two semi-explicit forms of system (1.2) obtained under orthogonal transformations, i.e.,

$$(E, A) \sim \left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right), \quad (\text{eq13.1})$$

$$(E, A) \sim \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right). \quad (\text{eq13.2})$$

Now we will prove that the two corresponding systems have the same evolution family $\{\hat{X}(t, s)\}_{t \geq s \geq 0}$. Without loss of generality, we assume that (E, A) is already in the form of the right hand side of (1.13a), so $\hat{X}(t, s) = \hat{Y}(t, s)$ for all $t \geq s \geq 0$. The corresponding system to the right hand side of (1.13b) reads

$$\dot{\tilde{y}}_1(t) = \hat{A}_1(t)\tilde{y}_1(t), \quad (\text{eq14.1})$$

$$\dot{\tilde{y}}_2(t) = \hat{A}_3(t)\tilde{y}_1(t). \quad (\text{eq14.2})$$

where $\hat{A}_3 := -\tilde{A}_4^{-1}\tilde{A}_3$, $\hat{A}_1 := \tilde{\Sigma}^{-1}(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3)$.

The transitivity applied to (1.13) implies that there exist pointwise-orthogonal matrix-valued functions $S = [S_1 \ S_2 \ S_3 \ S_4] \in C^0([0, \infty), \mathbb{R}^{n,n})$ and $T = [T_1 \ T_2 \ T_3 \ T_4] \in C^1([0, \infty), \mathbb{R}^{n,n})$, such that $y(t) = T(t)\tilde{y}(t)$ and

$$\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}, \quad (\text{15.1})$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} \left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} - \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{T}_1 & \dot{T}_2 \\ \dot{T}_3 & \dot{T}_4 \end{bmatrix} \right). \quad (\text{15.2})$$

Let $\{\hat{\mathcal{Y}}_1(t, s)\}_{t \geq s \geq 0}$ be the evolution family associated with (1.14a), then the evolution family associated with system (1.14) is

$$\hat{\mathcal{Y}}(t, s) = \begin{bmatrix} \hat{\mathcal{Y}}_1(t, s) & 0 \\ \hat{A}_3(t)\hat{\mathcal{Y}}_1(t, s) & 0 \end{bmatrix}, \quad \hat{\mathcal{X}}(t, s) := T(t)\hat{\mathcal{Y}}(t, s)T^T(s) \text{ for all } t \geq s \geq 0. \quad (\text{eq16})$$

Thus, we need to prove that $\hat{\mathcal{X}}(t, s) = \hat{X}(t, s)$.

From (1.15a), it implies that $S_3\Sigma[T_1 \ T_2] = 0$. Thus, we have

$$[S_3 \ 0] \begin{bmatrix} \Sigma & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} = [0 \ 0],$$

and hence, this follows that $S_3 = 0$. Thus, $S = [S_1 \ S_2 \ 0 \ S_4]$, and then, due to the orthogonality of S , we see that both S_1 is nonsingular and S_4 is also orthogonal. Also from (1.15a), we see that $S_1\Sigma T_2 = 0$, which yields that $T_2 = 0$. Moreover, due to the orthogonality of S and T , from (1.14a) we see that

$$\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} S_1^T & S_3^T \\ S_2^T & S_4^T \end{bmatrix} \begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_1^T & T_3^T \\ T_2^T & T_4^T \end{bmatrix}.$$

Therefore, using similar arguments as above, we can prove that $S_2 = 0$ and $T_3 = 0$.

Consequently, by inserting $S_3 = T_3 = 0$ and $S_2 = T_2 = 0$ into (1.15a) and (1.15b) we obtain

$$\tilde{\Sigma} = S_1\Sigma T_1, \quad (\text{17a})$$

$$\begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} = \begin{bmatrix} S_1(A_1T_1 - \Sigma\dot{T}_1) & S_1A_2T_4 \\ S_4A_3T_1 & S_4A_4T_4 \end{bmatrix}, \quad (\text{17b})$$

where the matrix-valued function S_i , T_i ($i = 1, 4$) are pointwise orthogonal. Thus, we see that

$$\hat{A}_3 = -\tilde{A}_4^{-1}\tilde{A}_3 = -(S_4A_4T_4)^{-1}S_4A_3T_1 = T_4^{-1}\hat{A}_3T_1, \quad (1.18a)$$

$$\hat{A}_1 = \tilde{\Sigma}^{-1}(\tilde{A}_1 - \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3) = T_1^{-1}(\hat{A}_1T_1 - \dot{T}_1). \quad (1.18b)$$

124 Furthermore, since $y = T\tilde{y}$, the structure of T implies that $y_1 = T_1\tilde{y}_1$ and $y_4 = T_4\tilde{y}_4$. Therefore, the
125 underlying ODE (1.14a) is directly obtained from (1.9a) by applying the variable transformation $\tilde{y}_1(t) =$
126 $T_1(t)y_1(t)$ and scaling the system with T_1^{-1} . So, we have that $\hat{Y}_1(t, s) = T_1^{-1}\hat{Y}_1(t, s)T_1(s)$. Making use of
127 (1.18), we can deduce the evolution family $\{\hat{\mathcal{X}}(t, s)\}_{t \geq s \geq 0}$ as follows

$$\hat{\mathcal{X}}(t, s) = \begin{bmatrix} T_1(t) & 0 \\ 0 & T_4(t) \end{bmatrix} \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3\hat{Y}_1(t, s) & 0 \end{bmatrix} \begin{bmatrix} T_1^T(s) & 0 \\ 0 & T_4^T(s) \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3\hat{Y}_1(t, s) & 0 \end{bmatrix},$$

128 and hence, this completes the proof. \square

Lemma 1.7. *Consider the DAE (1.1) and the evolution family $(X(t, s))_{t \geq s \geq 0}$ defined by (1.11). Furthermore, we also consider the pointwise-orthogonal matrix-valued functions U, V defined in Lemma 1.7. Then, the solution to (1.1), if exists, also satisfies the so-called mild equation*

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \hat{X}(t, s) \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t, \tau) \begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau)f(\tau, x(\tau))d\tau + \begin{bmatrix} 0 & 0 \\ 0 & -A_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t)),$$

129 for all $t \geq s \geq 0$.

130 *Proof.* The proof can be obtained directly by using Lemma 1.4. Thus, in order to keep the brevity we will
131 omit the details here. \square

132 In the following, for ease of notation, we will use the abbreviation $\hat{X}(t) := \hat{X}(t, 0)$, $\hat{X}^-(t) := \hat{X}^-(t, 0)$,
133 $\hat{Y}(t) := \hat{Y}(t, 0)$ and $\hat{Y}^-(t) := \hat{Y}^-(t, 0)$. The concept of exponential dichotomy for the DAE (1.8) is given as
134 below.

Definition 1.8. ([17]) The DAE (1.8) is said to have an *exponential dichotomy* if there exist a family of projection matrices $\{P_y(t)\}_{t \geq 0}$ in $\mathbb{R}^{d,d}$ and positive constants N, ν such that

$$\begin{aligned} \left\| \hat{Y}(t) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0, \\ \left\| \hat{Y}(t) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0, \end{aligned} \quad (1.19)$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.11)-(1.19) we see that

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0.$$

and

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^-(s) \right\| \leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0.$$

135 In addition, since $V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$ is also a projection matrix for any $t \geq 0$, we can interpret the
136 exponential dichotomy of (1.2) as the one of (1.8).

¹³⁷ **1.3. Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces
¹³⁸ that play a fundamental role in the study of differential equations and refer to Nguyen [10], Massera and
¹³⁹ Schäffer [18, Chap. 2] and Räbiger and Schnaubelt [22, §1] for various applications.

¹⁴⁰ Let E (endowed with the norm $\|\cdot\|_E$) be Banach function space of real-valued functions defined as in [10].
¹⁴¹ We then recall the Banach space corresponding to the space E as follows.

¹⁴² **Definition 1.9** ([10]). Consider the Banach space $(\mathbb{R}^n, \|\cdot\|)$. For a Banach function space E we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

¹⁴³ endowed with the norm $\|f\|_{\mathcal{E}} := \|f(\cdot)\|_E$. Thus, one can directly see that $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space. We
¹⁴⁴ call it *the Banach space corresponding to E* .

¹⁴⁵ We now introduce the notion of admissibility in the following definition.

¹⁴⁶ **Definition 1.10** ([10]). The Banach function space E is called *admissible* if for any $\varphi \in E$ the following
¹⁴⁷ conditions hold.

¹⁴⁸ (i) There exists a constant $M \geq 1$ such that for every compact interval $[a, b] \subset \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \quad (1.20)$$

¹⁴⁹ where $\chi_{[a,b]}$ is the indicator function of $[a, b]$.

¹⁵⁰ (ii) The function $\Lambda_1 \varphi$ defined by $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$ belongs to E .

¹⁵¹ (iii) For any $\tau \geq 0$, the space E is T_τ^+ -invariant and T_τ^- -invariant, where T_τ^+ and T_τ^- are defined as

$$T_\tau^+ \varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \quad (1.21)$$

$$T_\tau^- \varphi(t) := \varphi(t+\tau) \text{ for } t \geq 0.$$

¹⁵² Furthermore, there exist constants N_1, N_2 such that $\|T_\tau^+\|_E \leq N_1, \|T_\tau^-\|_E \leq N_2$ for all $\tau \in \mathbb{R}_+$.

¹⁵³ **Example 1.11.** Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leq p \leq \infty$, and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau < \infty\},$$

¹⁵⁴ (for any fixed $\alpha > 0$), endowed with the norm $\|h\|_{\mathbf{M}_\alpha} := \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau$, many other function spaces
¹⁵⁵ occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty, 1 \leq q < \infty$ (see [3], [24]) and,
¹⁵⁶ more general, the class of rearrangement invariant function spaces (see [16]) are admissible.

¹⁵⁷ *Remark 1.12.* Following directly from Definition 1.10 we have that

$$\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau \leq \frac{M}{\inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

¹⁵⁸ and hence, $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$. Furthermore, $C_b(\mathbb{R}^+)$ is dense in \mathbf{M}_1 .

¹⁵⁹ We present here some important features of admissible spaces in the following proposition (see [10, Propo-
¹⁶⁰ sition 2.6] and originally in [18, 23.V.(1)]).

Proposition 1.13 ([10]). Let E be an admissible Banach function space. Then the following assertions hold.

a) Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1 \varphi \in E$, where, Λ_1 is defined as in definition 1.10 (ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ by

$$\begin{aligned}\Lambda'_\sigma \varphi(t) &:= \int_0^t e^{-\sigma(t-s)} \varphi(s) ds, \\ \Lambda''_\sigma \varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds.\end{aligned}$$

161 Then, $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ belong to E . In particular, if $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see
162 remark 1.12)) then $\Lambda'_\sigma \varphi$ and $\Lambda''_\sigma \varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for *ess sup*-norm, we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty \quad (\text{eq22})$$

163 for operator T_1^+ and constants N_1, N_2 defined as in Definition 1.10.

164 b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for any constant $\alpha > 0$.

165 c) E does not contain exponentially growing functions $f(t) := e^{bt}$ for any constant $b > 0$.

2. EXPONENTIAL DICHOTOMY FOR LINEAR DAEs

167 In the qualitative analysis of ODEs, one of the central topics is to find necessary and sufficient conditions
168 such that the considered system admits an exponential dichotomy. Many researches have been devoted to
169 this topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces
170 (e.g. [6, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

171 **Proposition 2.1.** ([17]) The DAE (1.8) has an exponential dichotomy if and only if the corresponding
172 underlying ODE (1.9a) also has exponential dichotomy and the matrix function $\hat{A}_3(t)$ is bounded. Moreover,
173 the existence of an exponential dichotomy implies that $\sup_{t \geq 0} \|P_y(t)\| < \infty$.

174 Notice that, Proposition 2.1 is only valid for finite-dimensional but it is very hard to generalize for infinite
175 dimensional DAE systems. For this reason, we aim at another approach, motivated from one classical result
176 stated below.

177 **Proposition 2.2.** ([5, Chap. 3]) The ODE (1.3) has an exponential dichotomy if and only if one of the
178 following conditions is satisfied.

179 i) For any function $g \in \mathbf{M}_1(\mathbb{R}_+)$ there exists a continuous, bounded solution $x(t)$ to the inhomogeneous
180 system

$$\dot{x}(t) = A(t)x(t) + g(t). \quad (\text{inho_ODE})$$

181 ii) For any function $g \in C_\infty(\mathbb{R}_+)$, there exists a continuous, bounded solution $x(t)$ to the inhomogeneous
182 system (2.1), provided that the ODE (1.3) has bounded growth.

183 Comparable results to Proposition 2.2 have not been achieved for DAEs, and hence, this will be our main
184 aim in this section. Together with (1.2), let us consider the following system

$$E(t)\dot{x}(t) = A(t)x(t) + g(t). \quad (\text{eq2.1})$$

185 The following example shows that Proposition 2.2 could not be directly applied to the DAE (2.2).

186 **Example 2.3.** Consider the system (2.2) with $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 0 \\ 0 & e^{-t} \end{bmatrix}$, $f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly, f is bounded.

187 The explicit solution $x(t) = \begin{bmatrix} e^{-t}x_1(0) \\ e^t \end{bmatrix}$ is clearly unbounded no matter how the initial condition $x(0)$ is
188 chosen. On the other hand, the homogeneous system clearly has an exponential dichotomy.

¹⁸⁹ **Definition 2.4.** Consider the matrix functions E , A in system (1.2). Then, a function $g : \mathbb{R}_+ \rightarrow \mathbb{R}^n$
¹⁹⁰ satisfying the condition

$$\sup_{t \geq 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} g(t) \right\| \right\} < +\infty, \quad \text{eq2.2}$$
(2.3)

¹⁹¹ is called (E, A) -bounded. We denote the set of all continuous and (E, A) -bounded functions by $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$.

¹⁹² **Lemma 2.5.** *The (E, A) -boundedness is invariant with respect to orthogonal transformations.*

¹⁹³ *Proof.* Let us consider two orthogonally equivalent systems

$$\left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \sim \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued function S_i, T_i ($i = 1, 4$). Thus, we see that

$$\begin{bmatrix} \tilde{\Sigma}^{-1}(t) & -\tilde{\Sigma}^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & \tilde{A}_4^{-1}(t) \end{bmatrix} g(t) = \begin{bmatrix} T_1^T & 0 \\ 0 & T_4^T \end{bmatrix} \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & A_4^{-1}(t) \end{bmatrix} \begin{bmatrix} S_1^T & 0 \\ 0 & S_4^T \end{bmatrix} g(t).$$

¹⁹⁴ Since Euclidean norm is preserved under orthogonal transformation, this identity completes the proof. \square

¹⁹⁵ The main result of this section is to prove a characterization of the exponential dichotomy for DAEs.

¹⁹⁶ Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping $\mathcal{L} := E \frac{d}{dt} - A$
¹⁹⁷ is surjective on the space $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$. We formulate our main result in this section as follows.

¹⁹⁸ **Theorem 2.6.** *Consider the linear, strangeness-free DAE (1.2) and the associated inhomogeneous DAE
¹⁹⁹ (2.2). Then the following assertions hold.*

- ²⁰⁰ (i) *If the DAE (1.2) admits an exponential dichotomy then for any continuous, (E, A) -bounded function
²⁰¹ $g(t)$ on $[0, \infty)$, there exists a continuous, bounded solution $x(t)$ to the DAE (2.2).*
- ²⁰² (ii) *If the matrix function $\hat{A}_3(t)$ is bounded, then the converse of assertion (i) holds true.*

Proof. Firstly, we notice that, since $\hat{g} = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t)g(t)$, where U is orthogonal,
so the (E, A) -boundedness of g is equivalent to the boundedness of \hat{g} . Recall that the semi-explicit system
(1.8) reads

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{g}_1(t), \quad \text{eq3.10a}$$
(2.4a)

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{g}_2(t). \quad \text{eq3.10b}$$
(2.4b)

²⁰³ (i) Assuming that the DAE (1.2) admits an exponential dichotomy, then (1.8) also has an exponential
²⁰⁴ dichotomy. Proposition 2.1 implies that equation (2.4a) has an exponential dichotomy, and the function \hat{A}_3
²⁰⁵ is bounded. Therefore, Proposition 2.2 implies that y_1 is bounded, and consequently, y_2 is also bounded.
²⁰⁶ (ii) Notice that the mapping

$$\begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) : \mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n) \rightarrow C_\infty(\mathbb{R}_+, \mathbb{R}^n)$$

²⁰⁷ is surjective, so $g_1(t)$ can be freely chosen in the space $C_\infty(\mathbb{R}_+, \mathbb{R}^n)$. Proposition 2.2 applied to system (2.4a)
²⁰⁸ follows that (2.4a) has exponential dichotomy. On the other hand, the boundedness of \hat{A}_3 implies that (1.2)
²⁰⁹ admits exponential dichotomy. \square

210 3. LOCAL STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

211 In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout
 212 this section we assume that the evolution family $(X(t, s))_{t \geq s \geq 0}$ associated with the linear, homogeneous DAE
 213 (1.2) admits an exponential dichotomy on \mathbb{R}_+ .

From Lemma 1.4, by using orthogonal transformation $x(t) = V(t)y(t)$, where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$ we can transform (1.1) to the coupled system

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), & \text{eq4_1a} \\ y_2(t) &= \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), & \text{eq4_1b} \end{aligned} \quad (3.1) \quad (3.2)$$

214 where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)A_2(t)A_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t) \begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}. \quad \text{eq4_2}$$

215 Notice that, unlike the DAEs (1.2) and (2.2), equation (3.2) only gives an implicit algebraic constraint
 216 in terms of y_1 and y_2 . In order to guarantee the strangeness-free of system (1.1), we need the following
 217 assumption.

218 **Assumption 3.1.** Assume that for some $\rho > 0$, the function $A_4^{-1}(t)f_2(t, x)$ is a contraction mapping in the
 219 ball $B_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$ (uniformly in time), i.e.,

$$\|A_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

220 for a.e. $t \in \mathbb{R}_+$, and for all $x, \tilde{x} \in B_\rho$ where the Lipschitz constant L satisfies that $L < 1$.

221 **Lemma 3.2.** Under Assumption 3.1 and given $y_1 \in B_\rho$, there exists a unique function $y_2 \in \mathcal{B}_\rho$ satisfying
 222 (3.2).

223 *Proof.* Firstly, notice that Assumption 3.1 implies that $\hat{f}_2(t, y)$ is also Lipschitz in y with the same constant
 224 L . Then, the desired claim is obtained directly by making use of [19, Lem. 2.7]. \square

225 **Remark 3.3.** Lemma 3.2 leads to one important fact, that under Assumption 3.1, the coupled system (3.1)-
 226 (3.2) is still strangeness-free, as defined in [13, Chap. 4]. Therefore, in analogue to the linear case, (3.2) is
 227 called *an algebraic constraint*, whereas (3.1) is called *an underlying ODE*.

228 To obtain the existence of a stable manifold we need the following property of the nonlinear part f_1
 229 defined as follows.

230 **Definition 3.4.** Let φ be a positive function belonging to an admissible Banach function space E . A
 231 function $h : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to belong to the class (M, φ, ρ) for some positive constant M , ρ if h
 232 satisfies

- 233 (i) $\|h(t, x)\| \leq M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $x \in B_\rho$,
- 234 (ii) $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$, for all $x, \tilde{x} \in B_\rho$.

235 **Assumption 3.5.** Assume that the function $t \mapsto \Sigma^{-1}(t)[I_d - A_2(t)A_4^{-1}(t)]f(t, x(t))$ belongs to class
 236 (M, φ, ρ) for some positive constants M , ρ and a positive function $\varphi \in E$.

237 The following lemma shows that Assumptions 3.1, 3.5 are invariant with respect to orthogonal transfor-
 238 mations.

239 **Lemma 3.6.** Assumptions 3.1, 3.5 are also invariant with respect to orthogonal transformations.

240 *Proof.* Let us consider two orthogonally equivalent systems

$$\left(\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \right) \sim \left(\begin{bmatrix} \tilde{\Sigma} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ \tilde{A}_3 & \tilde{A}_4 \end{bmatrix} \right).$$

As shown in the proof of Lemma 1.6, the identities (1.17) hold true for some pointwise orthogonal matrix-valued function S_i , T_i ($i = 1, 4$). Therefore, we have that

$$\|\tilde{A}_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| = \|T_4^{-1}(t)A_4^{-1}(t)S_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

241 Then, the desired claim is directly followed. \square

242 The following proposition gives one sufficient condition for examining Assumptions 3.1, 3.5.

243 **Proposition 3.7.** Consider the semi-linear DAE (1.1). Furthermore, assume that all three functions Σ^{-1} ,
244 A_4^{-1} , $\Sigma^{-1}A_2A_4^{-1}$ are bounded. If the function $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ belongs to the class (M, φ, ρ) then the following
245 claims hold true.

- 246 i) \hat{f}_1 belongs to the class (M, φ, ρ) , and
247 ii) f_2 is Lipschitz with the Lipschitz constant $\varphi \sup_{t \geq 0} \|A_4^{-1}\|$.

248 We notice that a sufficient condition for Assumption (3.1) is that

$$\|f_2(t, x) - f_2(t, \tilde{x})\| \leq \frac{L}{\|A_4^{-1}(t)\|} \|x - \tilde{x}\|. \quad \text{Lipschitz (3.4)}$$

249 For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.1)-
250 (3.2). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad \text{mild equation (3.5)}$$

251 for all $t \geq s \geq 0$.

252 **Lemma 3.8.** Let Assumptions 3.1 and 3.5 hold true. Then, for all $y, \tilde{y} \in B_\rho$ the following assertions hold.

- 253 (i) $\|\hat{f}_1(t, y)\| \leq M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$,
254 (ii) $\|\hat{f}_1(t, y) - \hat{f}_1(t, \tilde{y})\| \leq \varphi(t)\|y - \tilde{y}\|$ for a.e. $t \in \mathbb{R}_+$,
255 (iii) $\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| \leq L\|y - \tilde{y}\|$ for a.e. $t \in \mathbb{R}_+$.

256 *Proof.* The proof is trivially followed from Assumptions 3.1 and 3.5 due to the fact that $\|y\| = \|Qy\|$ for any
257 orthogonal matrix V . \square

258 Let $(\hat{Y}(t, s))_{t \geq s \geq 0}$ has an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$
259 and the dichotomy constants $N, \nu > 0$ as in Definition 1.8. Furthermore, as in Proposition 2.1, let us denote
260 by $H_1 := \sup_{t \geq 0} \|\hat{A}_3(t)\|$ and $H_2 := \sup_{t \geq 0} \|P_y(t)\|$. Then, we can define the Green function on the half-line as
261 follows

$$G(t, \tau) := \begin{cases} \hat{Y}(t, \tau) \begin{bmatrix} P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)P_y(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, \tau)P_y(\tau) & 0 \end{bmatrix}, & \text{for all } t \geq \tau \geq 0, \\ -\hat{Y}(t, \tau) \begin{bmatrix} I_d - P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \end{bmatrix}, & \text{for all } 0 \leq t < \tau. \end{cases} \quad \text{(4.2)}$$

262 Then, we have

$$\|G(t, \tau)\| \leq (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geq 0. \quad \text{(4.3)}$$

263 In the following lemma, we give an explicit form for bounded solutions to system (3.5).

264 **Lemma 3.9.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy with the
265 corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
266 that Assumptions 3.1, 3.5 hold true. Let $y(t)$ be any solution to (3.5) such that $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ for
267 fixed $t_0 \geq 0$ and some $\rho > 0$. Then, for $t \geq t_0 \geq 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (3.8)$$

268 for some $v_0 \in \text{Im}P_y(t_0)$, where $G(t, \tau)$ is the Green function defined by (3.6).

Proof. Put

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}.$$

269 By direct computation, we can verify that z satisfies the integral equation

$$z(t) = \hat{Y}(t, t_0) \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix},$$

270 for all $t \geq t_0$. Now let us estimate $\|z(t)\|$. Making use of Lemma 3.8 and (3.7), we see that

$$\|z(t)\| \leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} M \varphi(\tau) d\tau + L\rho,$$

271 and then, from (1.22) it follows that

$$\|z(t)\| \leq M (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho,$$

272 for all $t \geq t_0$. Thus, $z(t) - y(t)$ is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t, t_0) (z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t) \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

273 we see that $v_0 := z_1(t_0) - y_1(t_0) \in \text{Im}P_y(t_0)$. Finally, since $z(t) = y(t) + \hat{Y}(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$ for all $t \geq t_0$, equality
274 (3.8) follows. \square

275 **Remark 3.10.** By computing directly, we can see that the converse of Lemma 3.9 is also true. It means, that
276 all solutions to (3.8) also satisfy equation (3.5) for all $t \geq t_0$.

277 Let us denote by

$$H_3 := (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)} \rho. \quad (3.9)$$

278 **Lemma 3.11.** *Under the assumptions of Lemma 3.9, let $y(t), \tilde{y}(t)$ be any two functions lying in the ball
279 B_ρ and satisfy (3.8) for $v_0, \tilde{v}_0 \in \text{Im}P_y(t_0)$. If H_3 defined as in (3.9) satisfies $H_3 + L < 1$ then the following
280 estimate holds true:*

$$\|y - \tilde{y}\|_\infty \leq \frac{N}{1 - H_3 - L} \|v_0 - \tilde{v}_0\|. \quad (3.10)$$

Proof. Using the same arguments as in the proof of Lemma 3.8, we see that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq N \|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq N \|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty + L \|y(t) - \tilde{y}(t)\|, \\ &\leq N \|v_0 - \tilde{v}_0\| + (H_3 + L) \|y - \tilde{y}\|_\infty, \end{aligned}$$

281 which directly implies (3.10). \square

282 In the following theorem, we exploit the local structure of bounded solutions to (3.5).

283 **Theorem 3.12.** *Let the evolution family $(\hat{Y}(t,s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy
284 with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore,
285 assume that Assumptions 3.1, 3.5 hold true, and constant H_3 defined as in (3.9). Then, the following
286 assertions hold true.*

287 (i) If

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)\rho}{2M} \right\}, \quad (3.11)$$

288 then there corresponds to each $v_0 \in B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$ one and only one solution $y(t)$ to (3.5) on $[t_0, \infty)$ satisfying
289 $P_y(t_0)y_1(t_0) = v_0$ and $\text{esssup}_{t \geq t_0} \|y(t)\| \leq \rho$.

290 (ii) Moreover, any two solutions $y(t), \tilde{y}(t)$ corresponding to different v_0, \tilde{v}_0 in $B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$ attract each
291 other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0, \quad (3.12)$$

292 for some positive constants H_4, μ .

293 *Proof.* (i) Consider in the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ the ball $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) : \|y(\cdot)\|_\infty := \text{esssup}_{t \geq 0} \|y(t)\| \leq \rho\}$.

294 For each fixed $v_0 \in B_{\tilde{\rho}}$ we will prove the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix} & \text{for all } t \geq t_0, \\ 0 & \text{for all } t < t_0, \end{cases} \quad (3.13)$$

295 is a contraction mapping from \mathcal{B}_ρ to itself. Using the same argument as in the proof of Lemma 3.8, we see
296 that

$$\begin{aligned} \|Ty(t)\| &\leq (1 + H_1)N e^{-\nu(t-t_0)} \|v_0\| + M(1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho, \\ &\leq (1 + H_1)N \|v_0\| + MH_3 + L\rho \quad \text{for all } t \geq 0, \end{aligned}$$

297 and by (3.11) we see that

$$\|(Ty)(t)\| \leq (1 + H_1)N \tilde{\rho} + \frac{(1 - L)\rho}{2} + L\rho = \rho \quad \text{for all } t \geq 0.$$

Therefore, T is a mapping from \mathcal{B}_ρ to itself. Now we prove its contraction property. Indeed, making use of
(3.7), we obtain the following estimate:

$$\begin{aligned} \|Ty(t) - T\tilde{y}(t)\| &\leq \int_{t_0}^\infty \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq \int_{t_0}^\infty (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq (H_3 + L) \|y(\cdot) - \tilde{y}(\cdot)\|_\infty \quad \text{for all } t \geq 0. \end{aligned}$$

298 Consequently, due to (3.11), we see that T is a contraction mapping with the contraction constant $H_3 + L$.
299 Thus, there exist a unique function $y \in \mathcal{B}_\rho$ such that $y = Ty$, and hence, due to the definition of T , y is the
300 solution to the mild/integral-algebraic system (3.5).

(ii) The proof of the estimate (3.12) can be done in a similar way as in [11, Thm 3.7]. We present here for seek of completeness. Let $y(t)$ and $\tilde{y}(t)$ be two essentially bounded solutions of (3.5) corresponding to different values $v_0, \tilde{v}_0 \in B_{\bar{\rho}} \cap \text{Im}P_y(t_0)$. Then, we have that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq Y(t, t_0) \|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq (1 + H_1) N e^{-\nu(t-t_0)} + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} |\varphi(\tau)| \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \end{aligned}$$

301 and hence,

$$\|y(t) - \tilde{y}(t)\| \leq \frac{1 + H_1}{1 - L} N e^{-\nu(t-t_0)} + \int_{t_0}^{\infty} \frac{(1 + H_1)(1 + H_2)}{1 - L} N e^{-\nu|t-\tau|} |\varphi(\tau)| \|y(\tau) - \tilde{y}(\tau)\| d\tau.$$

Then, due to the Cone Inequality, [6, Theorem 1.9.3], in analogue to [20, Theorem 3.7], we obtain the estimation (3.12) with H_4, μ are given by

$$0 < \mu < \nu + \ln \left(1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu - \nu}}}.$$

302 Furthermore, notice that from (3.11) it follows that $\mu < \nu$ implying the positivity of H_4 . This completes the
303 proof. \square

304 Under Assumption 3.1, we then define the so-called *constrained manifold*, which all solutions to (3.1)-(3.2)
305 must belong to

$$\mathbb{L}(t, y) := \{(t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t, y_1, y_2)\}. \quad \text{constraint manifold} \quad (3.14)$$

306 We further notice that this manifold is of dimension d , which is the degree of freedom to the DAE (3.5).
307 Now, we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic
308 system (3.5).

Definition 3.13. A subset \mathbb{M} of the constrained manifold $\mathbb{L}(t, y)$ is said to be a *local stable manifold* for
solutions to (3.5) if for every $t \in \mathbb{R}_+$ the phase subspace \mathbb{R}^d splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such
that

$$\inf_{t \in \mathbb{R}_+} S_n(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exist positive constants ρ, ρ_1, ρ_2 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t), \quad t \in \mathbb{R}_+,$$

309 with a common Lipschitz constant independent of t such that

- 310 (i) $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$, and we denote by
311 $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\},$
- 312 (ii) \mathbb{M}_t is homeomorphic to $B_{\rho_1} \cap W_1(t)$ for all $t \geq 0$,
- 313 (iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and
314 $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$.

315 We now state and prove our main result on the existence of a local stable manifold for DAEs.

Theorem 3.14. *Let the evolution family $(\hat{Y}(t,s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumptions 3.1, 3.5 hold true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)\rho}{2M}, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a local stable manifold for the solutions of (3.5). Moreover, every two solutions $y(t), \tilde{y}(t)$ on the manifold \mathbb{M} attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0 \geq 0$ such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\|, \quad \text{for all } t \geq t_0. \quad (\text{eq4.12})$$

Proof. First we notice that the phase subspace \mathbb{R}^d splits into the direct sum $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{ker}P_y(t)$ for all $t \geq 0$. We set $W_1(t) := \text{Im}P_y(t)$ and $W_2(t) := \text{ker}P_y(t)$, then due to Proposition 2.1, we see that $\sup_{t \geq 0} \|P_y(t)\| < \infty$, and hence, $\inf_{t \in \mathbb{R}_+} \text{Sn}(W_1(t), W_2(t)) > 0$.

For any $\rho > 0$ defined as in Assumptions 3.1, 3.5, let $\rho_1 := \tilde{\rho} = \frac{1 - L}{2N(1 + H_1)}\rho$ and $\rho_2 := \frac{(1 - L)\rho}{2}$. For each $t \geq 0$ we define the mapping g_t acting on $B_{\rho_1} \cap W_1(t)$ as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is uniquely defined via Theorem 3.12 i). Clearly, $g_t(w_1) \in \text{ker}P_y(t) = W_2(t)$.

Now, we prove that $\|g_t(w_1)\| \leq \rho_2$. Due to Theorem 3.12 (i) and Lemma 3.8 (i), we have that $\|y(t)\| \leq \rho$ and $\|f_1(\tau, y(\tau))\| \leq M\varphi(\tau)$ for a.e. $t \geq 0$. Therefore,

$$\begin{aligned} \|g_t(w_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} M\varphi(\tau) d\tau, \\ &\leq M (1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) = \frac{MH_3}{1 + H_1} \leq \frac{(1 - L)\rho}{2}, \end{aligned}$$

and hence, $g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t)$.

Notice that both part (iii) in Definition 3.13 and estimation (3.15) are followed directly from Theorem 3.12. We now only need to prove that \mathbb{M}_t is homeomorphic to $B_{\rho_1} \cap W_1(t)$. We first prove that g_t is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1 + H_1)(1 + H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1 + H_1)(1 + H_2)(1 - H_3 - L)} \|w_1 - \tilde{w}_1\|.$$

Finally, $H_3 < \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)}$ yields that $\frac{NH_3}{(1 + H_1)(1 + H_2)(1 - H_3 - L)} < 1$, and hence, g_t is a contraction mapping for all $t \geq 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous mappings ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \rightarrow B_{\rho_1} \cap W_1(t)$ is a homeomorphism. This implies the condition (ii) of Definition 3.13 finishing the proof. \square

336 4. GLOBAL INVARIANT STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

337 In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1).
 338 We begin with the concept of φ -Lipschitz functions.

339 **Definition 4.1.** Let E be an admissible Banach function space and $\varphi \in E$ be a positive function. A function
 340 $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^d$ is said to be φ -Lipschitz if the following conditions hold true.

- 341 (i) $\|h(t, 0)\| = 0$ for a.e. $t \in \mathbb{R}_+$,
- 342 (ii) $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$ and all $x, \tilde{x} \in \mathbb{R}^n$.

343 In comparability to Assumptions 3.1, 3.5, we also need some global properties of the nonlinear term f .

344 **Assumption 4.2.** Assume that the following hypotheses hold true.

- 345 (i) The function $\Sigma^{-1}(t) f_1(t, x(t)) - \Sigma^{-1}(t) A_2(t) A_4^{-1}(t) f_2(t, x(t))$ is φ -Lipschitz.
- (ii) The function $A_4^{-1}(t) f_2(t, x(t))$ is a contraction mapping with the Lipschitz constant $L < 1$ for all
 $(t, x(t))$ lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid A_2(t)x + f_2(t, x) = 0\}.$$

346 We can directly verify that orthogonal transformations of the form $x = Vy$ preserves the φ -Lipschitz
 347 property, and hence, function \hat{f}_1 in (3.1) is also φ -Lipschitz. Besides that, function \hat{f}_2 in (3.2) is also a
 348 contraction mapping with the Lipschitz constant $L < 1$. For notational simplicity, now we will study the
 349 transformed system (1.8) and the integral-algebraic system (3.5).

Definition 4.3. A subset \mathbb{M} of the constrained manifold $\mathbb{L}(t, y)$ is said to be a *global, invariant stable manifold* for solutions to (3.5) if for every $t \in \mathbb{R}_+$ the phase subspace \mathbb{R}^d splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t \in \mathbb{R}_+} S_n(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \rightarrow W_2(t), \quad t \in \mathbb{R}_+,$$

350 with the Lipschitz constants independent of t such that

- 351 (i) $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$, and we denote by
 $\mathbb{M}_t := \{(y_1, y_2) \mid (t, y_1, y_2) \in \mathbb{M}\},$
- 353 (ii) \mathbb{M}_t is homeomorphic to $W_1(t)$ for all $t \geq 0$,
- 354 (iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (3.5) satisfying $y_1(t_0) = \tilde{w}$ and
 $ess\sup_{t \geq t_0} \|y(t)\| < \infty$,
- 356 (iv) \mathbb{M} is invariant under system (3.5), i.e., if y is a solution to (3.5), and $ess\sup_{t \geq t_0} \|y(t)\| < \infty$, then
 $y(s) \in \mathbb{M}_s$ for all $s \geq t_0$.

358 Analogously to Lemma 3.9, we give the explicit form of bounded solutions to system (3.5) as below.

359 **Lemma 4.4.** Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy with the
 360 corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
 361 that Assumption 4.2 holds true. Let $y(t)$ be any solution to (3.5) such that $ess\sup_{t \geq t_0} \|y(t)\| < \infty$ for a fixed
 362 $t_0 \geq 0$. Then, for all $t \geq t_0 \geq 0$, we can rewrite $y(t)$ in the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (4.1)$$

363 for some $v_0 \in \text{Im}P_y(t_0)$, where $G(t, \tau)$ is the Green function defined by (3.6).

364 *Proof.* The proof can be done by using similar arguments as in the proof of Lemma 3.2. \square

365 In the following two theorems, we present the global versions of Theorems 3.12 and 3.14, where we
366 construct the structure of bounded solutions to (3.5) and prove the existence of a global, stable manifold,
367 respectively.

368 **Theorem 4.5.** *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy with the
369 corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume
370 that Assumption 4.2 holds true.*

- 371 (i) *For any fixed $t_0 \geq 0$, if $H_3 < 1 - L$ then there corresponds to each $v_0 \in \text{Im}P_y(t_0)$ one and only one
372 solution $y(t)$ to (3.5) on $[t_0, \infty)$ satisfying $P_y(t_0)y_1(t_0) = v_0$ and $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$.*
(iii) *Any two solutions $y(t), \tilde{y}(t)$ corresponding to different initial conditions v_0, \tilde{v}_0 in $\text{Im}P_y(t_0)$, are exponentially attracted to each other, i.e.,*

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0,$$

with some positive constants H_4, μ satisfying

$$0 < \mu < \nu + \ln \left(1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu - \nu}}}.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 3.12. The only change is, that instead of considering the ball B_ρ we will work with the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ itself. Then, we can prove (without any difficulty) that for each fixed $v_0 \in \text{Im}P_y(t_0)$, the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geq t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

373 is a contraction mapping, and therefore, all the assertions of the theorem follows. \square

Theorem 4.6. *Let the evolution family $(\hat{Y}(t, s))_{t \geq s \geq 0}$ of system (1.8) have an exponential dichotomy with the corresponding projection matrices $\{P_y(t)\}_{t \geq 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumption 4.2 holds true. If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a global invariant stable manifold for the solutions of (3.5). Moreover, every two solutions $y(t), \tilde{y}(t)$ on the manifold \mathbb{M} attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0 \geq 0$ such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\| \quad \text{for all } t \geq t_0.$$

374 *Proof.* Analogous to the proof of Theorem 3.14, we consider the decomposition $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel } P_y(t)$
375 and set $W_1(t) := \text{Im}P_y(t)$ and $W_2(t) := \text{kernel } P_y(t)$. Thus, we see that $\inf_{t \in \mathbb{R}_+} \text{Sn}(W_1(t), W_2(t)) > 0$.
376 Now we define the family of mappings $(g_t)_{t \geq 0}$ acting on W_1 as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function $y(t)$ is bounded and be uniquely defined via Theorem 4.5 i). Clearly, $g_t(w_1) \in \text{ker } P_y(t) = W_2(t)$. To verify the Lipschitz property of g_t , let us consider two arbitrary elements w_1 and \tilde{w}_1 in W_1 and

let y and \tilde{y} be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau , \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty , \end{aligned}$$

377 and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

378 Finally, $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a
379 contraction mapping for all $t \geq 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous
380 mapping ([19, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \rightarrow W_1(t)$ is a homeomorphism. This
381 implies the condition ii) of Definition 3.13, and hence, the proof is finished. \square

382 Now let us illustrate our results by the following examples.

383 **Example 4.7.** The dynamical behavior of a system in fluid mechanics and turbulence modeling is often
384 described by the incompressible Navier-Stokes equation on an open, bounded domain $\Omega \subset \mathbb{R}^k$, $k = 2$ or 3 ,
385 of the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p - (u \cdot \nabla) u + f(t, u, p), \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned}$$

where $\nu > 0$ is the viscosity, $u = u(t, \xi)$ is the velocity field which is a function of the time t and the position ξ , p is the pressure, f is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [9], one obtains a differential-algebraic system of the following form.

$$\begin{aligned} M \dot{U} &= (K + N(U)) U - CP + F(t, U, P), \\ C^T U &= 0, \end{aligned}$$

where $U(t)$, $P(t)$ approximate the velocity $u(t, \xi)$ and the pressure $p(t, \xi)$, respectively. Here the leading matrix M is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix $C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$ is nonsingular.

We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free, so Assumption 1.2 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned} M \dot{U} &= -(K + N(U)) U - CP + F(t, U, P), \\ 0 &= C^T M^{-1} C P - C^T M^{-1} (F - (K + N(U)) U) . \end{aligned} \tag{4.2)3}$$

Clearly, we still need to linearize (4.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.8)

$$\begin{aligned} M \dot{U} &= A_1(t)U + A_2(t)P + g_1(t, U, P), \\ 0 &= C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right) P - C^T M^{-1} \left(\frac{\partial F}{\partial U} - K \right) U + C^T M^{-1} g_2(t, U, P) . \end{aligned} \tag{4.3)4}$$

³⁸⁶ We further notice that since $C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$ is nonsingular, from the second equation we can uniquely
³⁸⁷ determine P in term of U , and hence, system (4.2) is indeed strangeness-free. Let

$$A_3(t) := -C^T M^{-1} \left(\frac{\partial F}{\partial U} - K \right), \quad A_4(t) := C^T M^{-1} \left(C - \frac{\partial F}{\partial P} \right)$$

³⁸⁸ Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

³⁸⁹ admits an exponential dichotomy, and g_1 satisfies the φ -Lipschitz condition, and g_2 is a contraction mapping
³⁹⁰ (uniformly in time), then there exists a stable manifold for the solution to (4.2).

Example 4.8. Consider the nonlinear electrical circuit with Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation $i_2 = I_0 \sin(k\phi_2)$, where I_0 and k are positive constants depend on the device itself. Moreover, the resistance R , inductance L and conductance G are positive. Furthermore, i_1 is the current going through the inductance, v_1 and v_2 are voltage drops across the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead

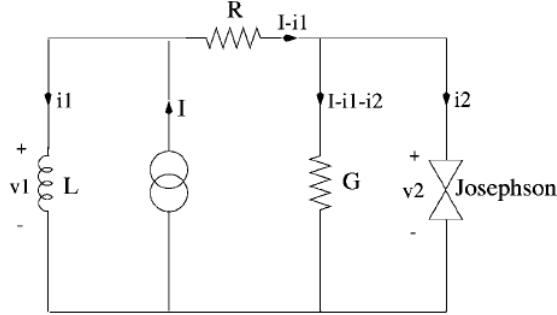


FIGURE 1. Electric circuit with Josephson junction, [23]

of linear resistance, inductance and conductance as in [23], and hence, we see that for the inductance $i_1 = i_L(L, \phi_1)$, for the resistance $v_R = v_R(R, i_1)$, and for the conductance $i_G = i_G(G, v_2)$. Therefore, we obtain the following system, which completely describes the behavior of this circuit.

$$\dot{\phi}_1 = v_1, \tag{4.4a}$$

$$\dot{\phi}_2 = v_2, \tag{4.4b}$$

$$i_1 = i_L(L, \phi_1), \tag{4.4c}$$

$$i_2 = I_0 \sin(k\phi_2), \tag{4.4d}$$

$$0 = v_1 - v_R(R, i_1) + v_2, \tag{4.4e}$$

$$0 = -i_G(G, v_2) + I - i_1 - i_2. \tag{4.4f}$$

From (4.4c)-(4.4f) we obtain an explicit form of v_1 in terms of ϕ_1 , i_1 and v_2 , so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \quad (4.5a)$$

$$\dot{\phi}_2 = v_2, \quad (4.5b)$$

$$i_1 = i_L(L, \phi_1), \quad (4.5c)$$

$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \quad (4.5d)$$

The linearized version of this system along equilibrium points defined by $v_2 = 0$, $i_1 = I$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, reads

$$\begin{aligned} \dot{\phi}_1 &= RI - (R/L)\phi_1 + v_2, \\ \dot{\phi}_2 &= v_2, \\ i_1 &= \phi_1/L, \\ 0 &= -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2), \end{aligned}$$

391 will have a positive eigenvalue and a negative one (e.g. [23]). Hence, it admits exponential dichotomy for any
 392 odd number n . Thus, for φ -Lipschitz function v_R and contraction mapping i_G , we obtain a stable manifold
 393 for (4.5).

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