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Brief paper

On reachable set estimation of singular systems[☆]

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ABSTRACT

In this paper, the problem of reachable set estimation of singular systems is investigated. Based on the Lyapunov method, a sufficient condition is established in terms of a linear matrix inequality (LMI) to guarantee that the reachable set of singular system is bounded by the intersection of ellipsoids. Then the result is extended to the problem for singular systems with time-varying delay by utilizing the reciprocally convex approach. The effectiveness of the obtained results in this paper is illustrated by numerical examples.

1. Introduction

Reachable set estimation of dynamic systems is to derive some closed bounded set to bound the set of all the states from the origin by inputs with bounded peak value. It is not only an important problem in robust control theory (Fridman & Shaked, 2003; Zuo, Ho, & Wang, 2010b), but also in practical engineering when safe operation is required through synthesizing controllers to avoid undesirable (or unsafe) regions in the state space (Hwang, Stipanovic, & Tomlin, 2003; Lygeros, Tomlin, & Sastry, 1999). In the latter context, the system is regarded safe if its reachable set does not contain any undesirable state. For example, suppose the velocity and the steering angle form the state of a vehicle and the state should be bounded in a set to avoid the vehicle from drifting and rolling over. Therefore, when the state lies within this set the vehicle can be operated safely, otherwise may be unsafe. Reachable set estimation of dynamic systems has various applications in peak-to-peak gain minimization (Abedor, Nagpal, & Poola, 1996), control systems with actuator saturation (Hu, Teel, & Zaccarian, 2006) and aircraft collision avoidance (Hwang et al., 2003). By using the S-procedure, the problem is investigated in Boyd, El Ghaoui, Feron, and Balakrishnan (1994) with the result derived in terms of linear matrix inequality (LMI) for the linear systems.

Time-delay, often attributed as one of the main causes of instability and performance degradation of a control system, has been extensively incorporated in models of many practical engineering systems, such as networked control systems, teleoperation and aircraft (Chiasson & Loiseau, 2007). For the reachable set estimation problem of time-delay systems, it is first solved in Fridman and Shaked (2003) based on the Lyapunov–Razumikhin method. The applications of reachable set estimation to disturbance rejection of time-delay systems (Cai, Huang, & Liu, 2010) and regional control of time-delay systems with saturating actuators (Fridman, Pila, & Shaked, 2003) are reported, respectively. An improved result is proposed in Kim (2008) by using the modified Lyapunov–Krasovskii type functional. By utilizing convex-hull properties in Kwon, Lee, and Park (2011) and constructing the maximal Lyapunov–Krasovskii functional in Zuo et al. (2010b), respectively, both results further improve that in Kim (2008). Very recently, the authors in Nam and Pathirana (2011) presented an improved bound of the reachable set using the delay partitioning method. When discrete and distributed delay appear simultaneously, the reachable set estimation problem is considered in Zuo, Fu, and Wang (2012).

Singular systems can better describe the behavior of some physical systems than state-space ones (Fridman, 2002; Lu, Ho, & Zhou, 2011; Zuo, Ho, & Wang, 2010a). Singular systems have been widely found in many practical systems, such as chemical processes, circuit systems, economic systems and aircraft modeling. Apart from their practical significance, they are of theoretical importance and have received a great deal of attention in recent

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years owing to their fundamental differences from state-space systems. Many fundamental concepts and results based on the theory of state-space systems have been successfully extended to singular systems, such as stability and stabilization (Xu & Lam, 2004; Zhu, Zhang, & Feng, 2007), H_∞ control (Zhang, Xia, & Shi, 2008), model reduction (Xu & Lam, 2003), guaranteed cost control (Ren & Zhang, 2012) and dissipativity analysis (Wu, Park, Shu, & Chu, 2011). For nonlinear singular systems, there are several contributions on the reachable set estimation problem by employing different approaches. To mention a few, there are the level set method in Cross and Mitchell (2008), the Taylor models in Hoefkens, Berz, and Makino (2003) and Rauh, Brill, and Gunther (2009) and the differential inequalities in Scott and Barton (2013a,b). However, the conditions obtained in these works are difficult to solve or compute. On the other hand, time-varying delay is not considered in these nonlinear singular systems. Linear matrix inequalities (LMIs) can be solved efficiently via the Matlab LMI toolbox, Sedumi or Yalmip and LMI technique has been a powerful design tool in control theory and its applications (Boyd et al., 1994). For linear singular systems, some preliminary results about reachable set analysis are given in Feng and Lam (2014). For a class of nonlinearly affine singular systems, a state-feedback control approach is proposed in Azhmyakov, Poznyak, and Juarez (2013) by utilizing the LMI technique such that the state of closed-loop systems initiated in the ellipsoid remains inside the ellipsoid at all time instant.

In this paper, we extend the reachable set estimation result to singular systems. By using the Lyapunov–Krasovskii method, sufficient conditions are proposed in terms of LMIs and the intersection of ellipsoids is obtained to bound all states set of singular systems starting from the origin with a bounded input. Then the result is extended to singular systems with time-varying delay by utilizing reciprocally convex method. Finally, numerical examples are given to illustrate the effectiveness of the proposed results. There are major differences between this paper and (Azhmyakov et al., 2013). Firstly, the aim of Azhmyakov et al. (2013) is to find an ellipsoid (as small as possible) to bound the trajectory while our paper establishes a set (as small as possible but not necessarily an ellipsoid, it is the intersection of ellipsoids) to bound the state. Secondly, the system in Azhmyakov et al. (2013) is a nonlinear affine one while it is a linear one in this paper. The matrix before the exogenous disturbance in Azhmyakov et al. (2013) is the identity while it is a general matrix B in this paper. Furthermore, the singular system with time-varying delay is also considered in this paper. Thirdly, the regularity condition of matrix pair (E, A) is not included in the main results and regularity is assumed in Azhmyakov et al. (2013). However, the feasibility of results obtained in our paper can guarantee the regularity of the matrix pair.

The rest of this paper is briefly outlined as follows. In Section 2, the reachable set estimation problem of singular systems is formulated and solved. The reachable set bound of singular systems with time-varying delay is established in Section 3. Three illustrative examples are provided in Section 4 to show the effectiveness of our results. We conclude the paper in Section 5.

Notation: The notation used throughout the paper is standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $P > 0$ (≥ 0) means that P is real symmetric and positive definite (semi-definite); I and 0 refer to the identity matrix and zero matrix with compatible dimensions; $\star$ stands for the symmetric terms in a symmetric matrix and sym(A) is defined as $A+A^T$; $(M)_{m\times m}$ is the matrix composed of elements of first m rows and m columns of matrix M; $\|\cdot\|$ refers to the Euclidean vector norm and $x_t(\theta) = x(t+\theta)$ $(\theta \in [-\tau_M, 0])$. Matrices are assumed to be compatible for algebraic operations if their dimensions are not explicitly stated.

2. Reachable set estimation of singular system

Consider a class of linear continuous-time singular systems described by

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bw(t) \\ x(0) \equiv 0, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; matrices E, A and B are constant matrices with appropriate dimensions and rank $(E) = n_1$; $w(t) \in \mathbb{R}^l$ represents a disturbance which satisfies

$$w^T(t)w(t) \leq \bar{w}^2 \tag{2}$$

where $\bar{w}$ is a real constant.

Before moving on, we give some definitions and lemmas which will be used in deriving the main results.

Definition 1 (*Xu & Lam, 2006*).

(1) The matrix pair (E, A) is said to be regular if $\det(sE - A)$ is not identically zero.
(2) The matrix pair (E, A) is said to be impulse free if $\deg\{\det(sE - A)\} = \operatorname{rank} E$.
(3) The matrix pair (E, A) is said to be stable if all the roots of $\det(sE - A)$ have negative real parts.
(4) The singular system in (1) is said to be admissible if it is regular ((E, A) is regular), impulse free ((E, A) is impulse free) and stable ((E, A) is stable).

Lemma 1 (*Xu & Lam, 2006*). *The matrix pair (E, A) is admissible if and only if there exists a matrix P such that*

$$E^TP = P^TE \geq 0, \qquad P^TA + A^TP < 0.$$

Lemma 2 (*Xu & Lam, 2006*). *If system (1) is regular and impulse free, there exist two non-singular matrices M and N such that*

$$MEN = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad MAN = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix}.$$

A method to determine the transformation matrices M and N can be found in Algorithm 3.1 in Duan (2010). Let $\tilde{x}(t) = N^{-1}x(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$, where $\tilde{x}_1(t) \in \mathbb{R}^{n_1}$ and $\tilde{x}_2(t) \in \mathbb{R}^{n-n_1}$. Denote $MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$. Then system (1) is restricted system equivalent to the following one:

$$\dot{\tilde{x}}_1(t) = A_1\tilde{x}_1(t) + B_1w(t) \tag{3}$$

$$0 = \tilde{x}_2(t) + B_2w(t). \tag{4}$$

Lemma 3 (*Boyd et al., 1994*). *Let $V(x(t))$ be a Lyapunov function for system (1)–(2) and $V(x(0)) = 0$. If $\dot{V} + \alpha V - \frac{\alpha}{\bar{w}^2}w^T(t)w(t) \leq 0$ with a scalar $\alpha > 0$, then $V(x(T)) \leq 1$ for $T \geq 0$.*

Proof. Denote

$$\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{\bar{w}^2}w^T(t)w(t) \leq 0. \tag{5}$$

Multiplying both sides of the inequality in (5) with $e^{\alpha t}$ yields

$$\begin{aligned} e^{\alpha t}\dot{V}(x(t)) + \alpha e^{\alpha t}V(x(t)) &= \frac{d}{dt}(e^{\alpha t}V(x(t))) \\ &\leq \frac{\alpha}{\bar{w}^2}e^{\alpha t}w^T(t)w(t). \end{aligned} \tag{6}$$

Then performing the integral of (6) from 0 to $T > 0$, we have

$$e^{\alpha T}V(x(T)) \le \int_0^T \frac{\alpha}{\bar{w}^2} e^{\alpha t} w^T(t)w(t)dt \le \int_0^T \alpha e^{\alpha t}dt = e^{\alpha T} - 1. \tag{7}$$

Then it reaches $V(x(T)) \le 1$ for any $T \ge 0$. $\square$

Our main objective is to find a set as small as possible to bound the reachable set defined as

$$R_x \triangleq \{x(t) \mid x(t) \text{ and } w(t) \text{ satisfy (1) and (2)}, \ t \ge 0\}. \tag{8}$$

Theorem 1. *If there exist matrices $P > 0, Y, H, U$ and a scalar $\alpha > 0$ such that the following LMI holds:*

$$\Lambda = \begin{bmatrix} \alpha E^T PE + \text{sym}(Y^T A) & Z & Y^T B \\ \star & -\text{sym}(H) & H^T B \\ \star & \star & -\frac{\alpha}{\bar{w}^2} I \end{bmatrix} < 0 \tag{9}$$

where $Z = (PE + E_0 U)^T - Y^T + A^T H$ and E_0 is a full column rank matrix satisfying $E^T E_0 = 0$. Then the system in (1) is admissible and the reachable set of system (1)–(2) is bounded by the intersection of ellipsoids

$$\mathscr{B}(\varepsilon) = \{x \in \mathbb{R}^n \mid x^T \breve{P} x \le 1\} \tag{10}$$

where $\breve{P} = N^{-T} \begin{bmatrix} \varepsilon\tilde{P} & 0 \\ 0 & \frac{1-\varepsilon}{(\|B_2\|\bar{w})^2} I \end{bmatrix} N^{-1}$ with $\varepsilon \in (0, 1)$, $\tilde{P} = (M^{-T}PM^{-1})_{n_1\times n_1}$, M, N and B_2 are defined in Lemma 2.

Proof. Firstly, the regularity and non-impulsiveness characteristics of system (1) are to be established. From the LMI in (9), we can obtain

$$\bar{E}^T \bar{P} = \bar{P}^T \bar{E} = \begin{bmatrix} E^T PE & 0 \\ 0 & 0 \end{bmatrix} \ge 0 \tag{11}$$

$$\text{sym}(\bar{A}^T \bar{P}) + \bar{Q} = \begin{bmatrix} \alpha E^T PE + \text{sym}(Y^T A) & Z \\ \star & -\text{sym}(H) \end{bmatrix} < 0 \tag{12}$$

where

$$Z = (PE + E_0 U)^T - Y^T + A^T H, \qquad \bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{P} = \begin{bmatrix} PE + E_0 U & 0 \\ Y & H \end{bmatrix}, \qquad \bar{A} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} \alpha E^T PE & 0 \\ 0 & 0 \end{bmatrix}.$$

It yields from (12) that $\text{sym}(\bar{A}^T \bar{P}) < 0$ which implies the matrix pair $(\bar{E}, \bar{A})$ to be admissible combining (11) based on Lemma 1. Noting that $\det(sE - A) = \det(s\bar{E} - \bar{A})$, $\deg(\det(sE - A)) = \deg(\det(s\bar{E} - \bar{A}))$, it is easy to see the system in (1) is admissible.

Now, we use Lemma 3 to prove that the set provided in Theorem 1 can bound the reachable set of the system in (1). To this end, we construct the following Lyapunov function $V(x(t)) = x^T(t)E^T PEx(t)$. Calculating the derivative of $V(x)$, we have

$$\dot{V}(x(t)) = 2\dot{x}^T(t)E^T(PE + E_0 U)x(t). \tag{13}$$

Introducing the free weighting matrices Y and H, we have

$$2[x^T(t)Y^T + (E\dot{x}(t))^T H^T][-E\dot{x}(t) + Ax(t) + Bw(t)] = 0. \tag{14}$$

Then, combining (13)–(14) and denoting the augmented system variable as $\xi(t) = \begin{bmatrix} x^T(t) & (E\dot{x}(t))^T & w^T(t) \end{bmatrix}^T$ yields $\dot{V}(x(t)) + \alpha V(x(t)) - \frac{\alpha}{\bar{w}^2} w^T(t)w(t) = \xi^T(t)\Lambda\xi(t) < 0$. By Lemma 3, we have $x^T(t)E^T PEx(t) \le 1$, which implies $\tilde{x}^T(t)N^T E^T M^T M^{-T} PM^{-1} MEN\tilde{x}(t) \le 1$, that is,

$$\tilde{x}_1^T(t)\tilde{P}\tilde{x}_1(t) \le 1 \tag{15}$$

with $\tilde{P} = (M^{-T}PM^{-1})_{n_1\times n_1}$. On the other hand, it follows from (4) that $\|\tilde{x}_2(t)\| = \|B_2 w(t)\|$ which leads to

$$\tilde{x}_2^T(t)\frac{1}{(\|B_2\|\bar{w})^2}\tilde{x}_2(t) \le 1. \tag{16}$$

Adding (15) times ε and (16) times $(1 - \varepsilon)$ yields

$$\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}^T \begin{bmatrix} \varepsilon\tilde{P} & 0 \\ 0 & \frac{1-\varepsilon}{(\|B_2\|\bar{w})^2} I \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \le 1$$

which is $x^T \breve{P} x \le 1$. $\square$

Remark 1. In order to seek the 'smallest' possible ellipsoid, the following additional requirement is added $\delta I \le \breve{P}$ which is equivalent to

$$\begin{bmatrix} \bar{\delta} I & I \\ I & \breve{P} \end{bmatrix} = \begin{bmatrix} \bar{\delta} I & I \\ I & N^{-T}\mathscr{P}N^{-1} \end{bmatrix} \ge 0 \tag{17}$$

where $\bar{\delta} = \delta^{-1}$ will be minimized and

$$\mathscr{P} = \begin{bmatrix} \varepsilon(M^{-T})_{n_1\times n} P (M^{-1})_{n\times n_1} & 0 \\ 0 & \frac{1-\varepsilon}{(\|B_2\|\bar{w})^2} I \end{bmatrix}.$$

Moreover, the intersection of ellipsoids in (10) can be obtained when choosing different values of ε.

Remark 2. Under zero initial conditions and impulsive free condition, if $B_2 w(0) \ne 0$, there will be an initial state jump at time 0, that is, $\tilde{x}_2(0^+) = B_2 w(0)$ which also belongs to the set described in inequality (16).

3. Reachable set estimation of singular system with delay

Consider a class of linear continuous-time singular systems with time-varying delay described by

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_\tau x(t - \tau(t)) + Bw(t) \\ x(t) \equiv 0, \quad t \in [-\tau_M, 0] \end{cases} \tag{18}$$

where $x(t) \in \mathbb{R}^n$ is the state vector; matrices A, A_τ and B are constant matrices with appropriate dimensions; $\tau(t)$ is the time-varying delay satisfying

$$0 \le \tau_m \le \tau(t) \le \tau_M, \qquad \dot{\tau}(t) < \mu, \quad \mu > 0 \tag{19}$$

and $w(t) \in \mathbb{R}^l$ represents a disturbance which satisfies

$$w^T(t)w(t) \le \bar{w}^2 \tag{20}$$

where $\bar{w}$ is a real constant.

Before moving on, we give some definitions and lemmas which will be used in deriving the main results.

Definition 2 (*Xu & Lam, 2006*).

(1) The singular system in (18) is said to be regular if the matrix pair (E, A) is regular.
(2) The singular system in (18) is said to be impulse free if the matrix pair (E, A) is impulse free.

(3) The singular system in (18) is said to be stable, if for any $\varepsilon > 0$, there exists a scalar $\delta(\varepsilon) > 0$, such that for any compatible initial conditions x_0 satisfying $\|x_0\| \le \delta(\varepsilon)$, the solution $x(t)$ of (18) satisfies $\|x(t)\| \le \varepsilon$ for $t \ge 0$; furthermore, $x(t) \to 0$, when $t \to \infty$.

(4) The singular system in (18) is said to be admissible if it is regular, impulse free and stable.

Lemma 4 (Park, Ko, & Jeong, 2011)**.** *Let $f_1, f_2, \ldots, f_N\colon \mathbb{R}^m \to \mathbb{R}$ have positive values in an open subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t)$$

subject to

$$\left\{ g_{i,j} : \mathbb{R}^m \to \mathbb{R},\ g_{j,i}(t) = g_{i,j}(t),\ \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & f_j(t) \end{bmatrix} \ge 0 \right\}.$$

When time-delay appears, the Lyapunov function is chosen as $V(x_t)$ where x_t denotes $x(t)$ over $[t-\tau_M, t]$ and the zero initial condition means $x_0 = 0$ which implies $V(x_0) = 0$. Then following the similar lines as that in the proof of Lemma 3, Lemma 5 can be obtained.

Lemma 5. *Let $V(x_t)$ be a Lyapunov function for system (18)–(19) and $V(x_0) = 0$. If $\dot{V} + \alpha V - \frac{\alpha}{\bar{w}^2} w^T(t) w(t) \le 0$ with a scalar $\alpha > 0$, then $V(x_T) \le 1$ for $T \ge 0$.*

Lemma 6 (Li & Zhang, 2012)**.** *Let $0 \le \tau_m < \tau_M$, $0 < \lambda < 1$, $Q \ge 0$, $x(t)$ be a continuous vector-valued function on $[-\tau_M, +\infty)$, and $\tau(t)$ be a mapping from $[0, +\infty)$ into $[\tau_m, \tau_M]$. If $\|x(t)\| \le \lambda\|x(t-\tau(t))\| + Q$, $t \ge 0$, then $\|x(t)\| \le \sup_{-\tau_M \le t \le 0} \|x(t)\| + \frac{Q}{1-\lambda}$.*

Lemma 7 (Xu & Lam, 2006)**.** *If system (18) is regular and impulse free, there exist two non-singular matrices M and N such that*

$$MEN = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad MAN = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & I \end{bmatrix}.$$

Let $\tilde{x}(t) = N^{-1}x(t) = \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}$, where $\tilde{x}_1(t) \in \mathbb{R}^{n_1}$ and $\tilde{x}_2(t) \in \mathbb{R}^{n-n_1}$. Denote $MA_\tau N = \begin{bmatrix} \tilde{A}_{\tau 11} & \tilde{A}_{\tau 12} \\ \tilde{A}_{\tau 21} & \tilde{A}_{\tau 22} \end{bmatrix}$, $MB = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$. Then system (18) is restricted system equivalent to the following one:

$$\dot{\tilde{x}}_1(t) = \tilde{A}_1 \tilde{x}_1(t) + \tilde{A}_{\tau 11}\tilde{x}_1(t-\tau(t)) + \tilde{A}_{\tau 12}\tilde{x}_2(t-\tau(t)) + \tilde{B}_1 w(t)$$
$$0 = \tilde{x}_2(t) + \tilde{A}_{\tau 21}\tilde{x}_1(t-\tau(t)) + \tilde{A}_{\tau 22}\tilde{x}_2(t-\tau(t)) + \tilde{B}_2 w(t).$$

Assumption 1. The matrix pair (E, A_τ) is regular and $\|\tilde{A}_{\tau 22}\| < 1$.

The aim in this section is to find the intersection of ellipsoids $\mathscr{B}(\varepsilon)$ to bound the reachable set defined as

$$R_x \triangleq \{x(t) \mid x(t) \text{ and } w(t) \text{ satisfy (18)–(20)},\ t \ge 0\}. \tag{21}$$

In this section, a sufficient condition is derived in terms of LMIs such that the reachable set of system (18) is bounded by the intersection of ellipsoids. The result of reachable set estimation for system (18) is presented in the following theorem.

Theorem 2. *If there exist matrices $P > 0$, $Q > 0$, $R > 0$, $S > 0$, $Z_1 > 0$, $Z_2 > 0$, X, $H = \begin{bmatrix} H_1 & H_2 & H_3 & H_4 & H_5 & H_6 \end{bmatrix}$ and a scalar $\alpha > 0$ such that the following LMIs holds for*

$$\begin{bmatrix} Z_1 & X \\ X^T & Z_1 \end{bmatrix} > 0$$

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & A^T H_4 & \Omega_{15} & H_1^T B - A^T H_6 \\ \star & \Omega_{22} & \Omega_{23} & \Omega_{24} & -H_2^T & H_2^T B \\ \star & \star & \Omega_{33} & \Omega_{34} & \Omega_{35} & H_3^T B + A_\tau^T H_6 \\ \star & \star & \star & \Omega_{44} & -H_4^T & H_4^T B \\ \star & \star & \star & \star & \Omega_{55} & H_5^T B - H_6 \\ \star & \star & \star & \star & \star & -\frac{\alpha}{\bar{w}^2} I + \mathrm{sym}(B^T H_6) \end{bmatrix} < 0 \tag{22}$$

where

$$\Omega_{11} = \alpha E^T P E + Q + R + S + \mathrm{sym}(H_1^T A) - e^{-\alpha\tau_m} E^T Z_2 E$$
$$\Omega_{12} = e^{-\alpha\tau_m} E^T Z_2 E + A^T H_2, \qquad \Omega_{13} = H_1^T A_\tau + A^T H_3$$
$$\Omega_{15} = (PE + E_0 U)^T - H_1^T + A^T H_5$$
$$\Omega_{22} = -e^{-\alpha\tau_m} Q - e^{-\alpha\tau_m} E^T Z_2 E - e^{-\alpha\tau_M} E^T Z_1 E$$
$$\Omega_{23} = e^{-\alpha\tau_M} E^T Z_1 E - e^{-\alpha\tau_M} E^T X E + H_2^T A_\tau$$
$$\Omega_{24} = e^{-\alpha\tau_M} E^T X E$$
$$\Omega_{33} = -2e^{-\alpha\tau_M} E^T Z_1 E + \mathrm{sym}(e^{-\alpha\tau_M} E^T X E + A_\tau^T H_3) - (1-\mu) e^{-\alpha\tau_M} S$$
$$\Omega_{34} = e^{-\alpha\tau_M} E^T Z_1 E - e^{-\alpha\tau_M} E^T X E + A_\tau^T H_4$$
$$\Omega_{35} = A_\tau^T H_5 - H_3^T, \qquad \Omega_{44} = -e^{-\alpha\tau_M} R - e^{-\alpha\tau_M} E^T Z_1 E$$
$$\Omega_{55} = \tau_m^2 Z_2 + (\tau_M - \tau_m)^2 Z_1 - \mathrm{sym}(H_5).$$

Then the system in (18) is admissible and the reachable sets of system (18)–(20) is bounded by the intersection of ellipsoids

$$\mathscr{B}(\varepsilon) = \{x \in \mathbb{R}^n \mid x^T \breve{P} x \le 1\} \tag{23}$$

where $\breve{P} = N^{-T} \begin{bmatrix} \varepsilon\tilde{P} & 0 \\ 0 & \frac{1-\varepsilon}{r_2^2} I \end{bmatrix} N^{-1}$, $r_1 = \frac{1}{\sqrt{\lambda_{\min}(\tilde{P})}}$, $r_2 = \frac{\|\tilde{A}_{\tau 21}\| r_1 + \|\tilde{B}_2\| \bar{w}}{1 - \|\tilde{A}_{\tau 22}\|}$ with $\tilde{P} = (M^{-T} P M^{-1})_{n_1 \times n_1}$, M, N, $\tilde{A}_{\tau 21}$, $\tilde{A}_{\tau 22}$ and $\tilde{B}_2$ are defined in Lemma 7.

Proof. Firstly, the regularity and non-impulsiveness characteristics of system (18) are to be established. From the LMI in (22), we can obtain

$$\bar{E}^T \bar{P} = \bar{P}^T \bar{E} = \begin{bmatrix} E^T P E & 0 \\ 0 & 0 \end{bmatrix} \ge 0 \tag{24}$$

$$\mathrm{sym}(\bar{A}^T \bar{P}) + \bar{Q} - \bar{E}^T \bar{Z} \bar{E} = \begin{bmatrix} \Omega_{11} & \Omega_{15} \\ \star & \Omega_{55} \end{bmatrix} < 0 \tag{25}$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \qquad \bar{P} = \begin{bmatrix} PE + E_0 U & 0 \\ H_1 & H_5 \end{bmatrix},$$
$$\bar{A} = \begin{bmatrix} 0 & I \\ A & -I \end{bmatrix}$$
$$\bar{Q} = \begin{bmatrix} Q + R + S & 0 \\ 0 & \tau_m^2 Z_2 + (\tau_M - \tau_m)^2 Z_1 \end{bmatrix},$$
$$\bar{Z} = \begin{bmatrix} e^{-\alpha\tau_m} Z_2 & 0 \\ 0 & 0 \end{bmatrix}$$

and Ω_{11}, Ω_{15} and Ω_{55} are defined in Theorem 2.

It yields from (25) that

$$\mathrm{sym}(\bar{A}^T \bar{P}) - \bar{E}^T \bar{Z} \bar{E} < 0. \tag{26}$$

Since rank $(\bar{E})$ = rank (E) = $n_1 \le n$, there exist nonsingular matrices $\bar{M}$ and $\bar{N}$ such that $\tilde{E} = \bar{M}\bar{E}\bar{N} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Denote

$$\tilde{A} = \bar{M}\bar{A}\bar{N} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad \tilde{P} = \bar{M}^{-T}\bar{P}\bar{N} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}.$$

Considering (24), we obtain that $P_{12} = 0$ and $P_{11} > 0$. Then pre-multiplying and post-multiplying (26) by $\bar{N}^T$ and $\bar{N}$, respectively, it yields $\mathrm{sym}(A_{22}^T P_{22}) < 0$ which implies that A_{22} is nonsingular. Therefore, the pair $(\bar{E}, \bar{A})$ is regular and impulse free. Noting that $\det(sE - A) = \det(s\bar{E} - \bar{A})$, $\deg(\det(sE - A)) = \deg(\det(s\bar{E} - \bar{A}))$, it is easy to see the system in (18) is regular and impulse free.

Now, we use Lemma 5 to prove that the intersection of ellipsoids provided in Theorem 2 can bound the reachable sets of the system in (18). To this end, we construct the following Lyapunov functional:

$$V(x_t) = V_1(x) + V_2(x_t) + V_3(x_t) + V_4(x_t)$$

with

$$\begin{aligned} V_1(x) &= x^T(t)E^T P E x(t) \\ V_2(x_t) &= \int_{t-\tau_m}^{t} e^{\alpha(s-t)} x^T(s) Q x(s) ds \\ &\quad + \int_{t-\tau_M}^{t} e^{\alpha(s-t)} x^T(s) R x(s) ds \\ &\quad + \int_{t-\tau(t)}^{t} e^{\alpha(s-t)} x^T(s) S x(s) ds \\ V_3(x_t) &= (\tau_M - \tau_m) \int_{-\tau_M}^{-\tau_m} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) \\ &\quad \times E^T Z_1 E \dot{x}(s) ds d\theta \\ V_4(x_t) &= \tau_m \int_{-\tau_m}^{0} \int_{t+\theta}^{t} e^{\alpha(s-t)} \dot{x}^T(s) E^T Z_2 E \dot{x}(s) ds d\theta. \end{aligned}$$

Calculating the derivative of $V_i(x_t)$, we have

$$\dot{V}_1(x) = 2\dot{x}^T(t)E^T(PE + E_0 U)x(t) \tag{27}$$

$$\begin{aligned} \dot{V}_2(x_t) &= -\alpha V_2 + x^T(t)(Q + R + S)x(t) \\ &\quad - e^{-\alpha\tau_m} x^T(t-\tau_m) Q x(t-\tau_m) \\ &\quad - e^{-\alpha\tau_M} x^T(t-\tau_M) R x(t-\tau_M) \\ &\quad - (1-\dot{\tau}(t)) e^{-\alpha\tau(t)} x^T(t-\tau(t)) S x(t-\tau(t)) \\ &\le -\alpha V_2 + x^T(Q + R + S)x(t) \\ &\quad - e^{-\alpha\tau_m} x^T(t-\tau_m) Q x(t-\tau_m) \\ &\quad - e^{-\alpha\tau_M} x^T(t-\tau_M) R x(t-\tau_M) \\ &\quad - (1-\mu) e^{-\alpha\tau_M} x^T(t-\tau(t)) S x(t-\tau(t)) \end{aligned} \tag{28}$$

$$\begin{aligned} \dot{V}_3(x_t) &= -\alpha V_3 + (\tau_M - \tau_m)^2 \dot{x}^T(t) E^T Z_1 E \dot{x}(t) \\ &\quad - (\tau_M - \tau_m) \int_{t-\tau_M}^{t-\tau_m} e^{\alpha(s-t)} \dot{x}^T(s) E^T Z_1 E \dot{x}(s) ds \\ &\le -\alpha V_3 + (\tau_M - \tau_m)^2 \dot{x}^T(t) E^T Z_1 E \dot{x}(t) \\ &\quad - (\tau_M - \tau_m) e^{-\alpha\tau_M} \int_{t-\tau_M}^{t-\tau_m} \dot{x}^T(s) E^T Z_1 E \dot{x}(s) ds \\ \dot{V}_4(x_t) &= -\alpha V_4 + \tau_m^2 \dot{x}^T(t) E^T Z_2 E \dot{x}(t) \\ &\quad - \tau_m \int_{t-\tau_m}^{t} e^{\alpha(s-t)} \dot{x}^T(s) E^T Z_2 E \dot{x}(s) ds \\ &\le -\alpha V_4 + \tau_m^2 \dot{x}^T(t) E^T Z_2 E \dot{x}(t) \\ &\quad - \tau_m e^{-\alpha\tau_m} \int_{t-\tau_m}^{t} \dot{x}^T(s) E^T Z_2 E \dot{x}(s) ds. \end{aligned}$$

By utilizing Lemma 4 and the Jensen inequality of Gu, Kharitonov, and Chen (2003), there exists a matrix X satisfying $\begin{bmatrix} Z_1 & X \\ X^T & Z_1 \end{bmatrix} > 0$ such that for $\tau_m < \tau(t) < \tau_M$

$$\begin{aligned} \dot{V}_3(x_t) &\le -\alpha V_3 + (\tau_M - \tau_m)^2 \dot{x}^T(t) E^T Z_1 E \dot{x}(t) \\ &\quad - \frac{\tau_M - \tau_m}{\tau(t) - \tau_m} e^{-\alpha\tau_M} \chi_1^T(t) E^T Z_1 E \chi_1(t) \\ &\quad - \frac{\tau_M - \tau_m}{\tau_M - \tau(t)} e^{-\alpha\tau_M} \chi_2^T(t) E^T Z_1 E \chi_2(t) \\ &\le -\alpha V_3 + (\tau_M - \tau_m)^2 \dot{x}^T(t) E^T Z_1 E \dot{x}(t) \\ &\quad - e^{-\alpha\tau_M} \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix}^T \begin{bmatrix} E^T Z_1 E & E^T X E \\ \star & E^T Z_1 E \end{bmatrix} \begin{bmatrix} \chi_1(t) \\ \chi_2(t) \end{bmatrix} \end{aligned} \tag{29}$$

where $\chi_1(t) = x(t-\tau_m) - x(t-\tau(t))$ and $\chi_2(t) = x(t-\tau(t)) - x(t-\tau_M)$. When $\tau(t) = \tau_m$ or $\tau(t) = \tau_M$, it yields that $\chi_1(t) = 0$ or $\chi_2(t) = 0$ and the inequality in (29) still holds. Based on Lemma 4, the following inequality holds:

$$\begin{aligned} \dot{V}_4(x_t) &\le -\alpha V_4 + \tau_m^2 \dot{x}^T(t) E^T Z_2 E \dot{x}(t) \\ &\quad - e^{-\alpha\tau_m} (x(t) - x(t-\tau_m))^T E^T Z_2 E (x(t) \\ &\quad - x(t-\tau_m)). \end{aligned} \tag{30}$$

Introducing the free weighting matrix H, we have

$$2\xi^T(t) H^T [-E\dot{x}(t) + Ax(t) + A_\tau x(t-\tau(t)) + Bw(t)] = 0 \tag{31}$$

where

$$\begin{aligned} \xi(t) = [&x^T(t) \quad x^T(t-\tau_m) \quad x^T(t-\tau(t)) \\ &x^T(t-\tau_M) \quad (E\dot{x}(t))^T \quad w^T(t)]^T. \end{aligned}$$

Then, combining (22) and (27)–(31) yields $\dot{V}(x_t) + \alpha V(x_t) - \frac{\alpha}{\bar{w}^2} w^T(t)w(t) \le \xi^T(t)\Omega\xi(t) < 0$. For $w(t) = 0$, it yields $\dot{V}(x_t) < 0$ which implies the stability of the singular system in (18) (Wu & Zheng, 2009; Zhong & Yang, 2006). On the other hand, by Lemma 5, we have $x^T(t)E^T P E x(t) \le 1$, which implies $\tilde{x}^T(t) N^T E^T M^T M^{-T} P M^{-1} M E N \tilde{x}(t) \le 1$, that is,

$$\tilde{x}_1^T(t) \tilde{P} \tilde{x}_1(t) \le 1 \tag{32}$$

with $\tilde{P} = (M^{-T} P M^{-1})_{n_1 \times n_1}$. Therefore, the following inequality holds $\|\tilde{x}_1(t)\| \le r_1$ with $r_1 = \frac{1}{\sqrt{\lambda_{\min}(\tilde{P})}}$. Therefore, $\|\tilde{x}_1(t)\|$ is bounded.

On the other hand, based on Assumption 1, it follows from Lemma 7 that $\|\tilde{x}_2(t)\|$ is bounded and

$$\begin{aligned} \|\tilde{x}_2(t)\| &= \|\tilde{A}_{\tau 21}\tilde{x}_1(t-\tau(t)) + \tilde{A}_{\tau 22}\tilde{x}_2(t-\tau(t)) + \tilde{B}_2 w(t)\| \\ &\le \|\tilde{A}_{\tau 22}\| \, \|\tilde{x}_2(t-\tau(t))\| + \|\tilde{A}_{\tau 21}\| r_1 + \|\tilde{B}_2\| \bar{w}. \end{aligned}$$

By utilizing Lemma 6, it yields $\|\tilde{x}_2(t)\| \le r_2$, that is,

$$\tilde{x}_2^T(t) \frac{1}{r_2^2} \tilde{x}_2(t) \le 1 \tag{33}$$

where $r_2 = \frac{\|\tilde{A}_{\tau 21}\| r_1 + \|\tilde{B}_2\| \bar{w}}{1 - \|\tilde{A}_{\tau 22}\|}$.

Then adding the inequality in (32) times ε and the inequality in (33) times $1 - \varepsilon$, we have

$$\begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix}^T \begin{bmatrix} \varepsilon\tilde{P} & 0 \\ 0 & \dfrac{1-\varepsilon}{r_2^2} I \end{bmatrix} \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \le 1$$

which is $x^T \breve{P} x \le 1$. □

Remark 3. Existing results about admissibility of singular systems with time-varying delay mostly consider only the slow time-varying case (that is, $\mu < 1$) which may be more conservative (Haidar & Boukas, 2009; Haidar, Boukas, Xu, & Lam, 2009; Wu

Table 1
Different values of $\bar{\delta}$ and $\breve{P}$ by choosing different ε.

ε	$\bar{\delta}$	$\breve{P}$
0.3	1.3093	$\begin{bmatrix} 1.1998 & -1.1998 \\ -1.1998 & 3.9998 \end{bmatrix}$
0.5	1.322707	$\begin{bmatrix} 1.9987 & -1.9987 \\ -1.9987 & 3.9987 \end{bmatrix}$
0.8	2.6667	$\begin{bmatrix} 3.1874 & -3.1874 \\ -3.1874 & 3.9874 \end{bmatrix}$

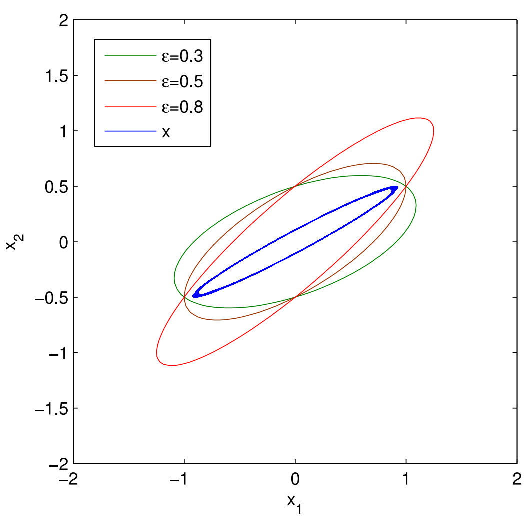

Fig. 1. Reachable set with $w(t) = \sin(t)$ and the bounding ellipsoids.

et al., 2011). The result in Theorem 2 is applicable for either fast or slow time-varying delay singular systems because of the term $\mathrm{sym}(A_\tau^T H_3)$ that appears in Ω_{33}. When the derivative of delay $\tau(t)$ is unknown, our result is also applicable by setting $S = 0$ in (22).

When $\tilde{A}_{\tau_{21}} = 0$, the 'smallest' ellipsoids in Theorem 2 can be obtained by the same way as that proposed in Remark 1. While when $\tilde{A}_{\tau_{21}} \neq 0$, it is not easy to find the 'smallest' ellipsoids in Theorem 2 by using the direct method in Remark 1 and the following iterative algorithm will be utilized.

Iterative Algorithm:

(1) Set $i = 1$, solve the LMI in (22), get a feasible solution P_i and find the largest value of δ_i satisfying $\delta_i I \leq \breve{P}_i$, where

$$\breve{P}_i = N^{-T} \begin{bmatrix} \varepsilon \tilde{P}_i & 0 \\ 0 & \dfrac{1-\varepsilon}{r_{2i}^2} I \end{bmatrix} N^{-1}$$

with

$$r_{1i} = \frac{1}{\sqrt{\lambda_{\min}(\tilde{P}_i)}}, \qquad r_{2i} = \frac{\|\tilde{A}_{\tau 21}\| r_{1i} + \|\tilde{B}_2\| \bar{w}}{1 - \|\tilde{A}_{\tau 22}\|}$$

with $\tilde{P}_i = (M^{-T} P_i M^{-1})_{n_1 \times n_1}$.

(2) Give a small scalar $\beta > 0$, set $\delta_{i+1} = \delta_i + \beta$ and solve the LMI in (22) with the decision matrix $P = P_{i+1}$ and $\delta_{i+1} I \leq \hat{P}_i$, where

$$\hat{P}_i = N^{-T} \begin{bmatrix} \varepsilon \tilde{P}_{i+1} & 0 \\ 0 & \dfrac{1-\varepsilon}{r_{2i}^2} I \end{bmatrix} N^{-1}.$$

Table 2
Different values of $\bar{\delta}$ and $\breve{P}$ by choosing different ε.

ε	$\bar{\delta}$	$\breve{P}$
0.3	0.5582	$\begin{bmatrix} 2.2136 & 2.2136 \\ 2.2136 & 13.4136 \end{bmatrix}$
0.5	0.4459	$\begin{bmatrix} 3.6822 & 3.6822 \\ 3.6822 & 11.6822 \end{bmatrix}$
0.8	0.7227	$\begin{bmatrix} 5.8326 & 5.8326 \\ 5.8326 & 9.0326 \end{bmatrix}$

(3) If the LMIs in Step 2 are feasible and $\delta_{i+1} I \leq \breve{P}_{i+1}$, then set $\breve{P} = \breve{P}_{i+1}$ and $i = i + 1$. Else set $\breve{P} = \breve{P}_i$ and stop.

4. Illustrative examples

In this section, examples are provided to illustrate the effectiveness of the proposed approach.

Example 1. Consider a singular system in (1) with following parameters:

$$E = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
$$E_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.5 \end{bmatrix}.$$

The system is restricted system equivalent to

$$\dot{\tilde{x}}_1(t) = -2\tilde{x}_1(t) - w(t)$$
$$0 = \tilde{x}_2(t) + w(t).$$

Then we have $B_2 = 1$. By solving the LMIs in (9) and (17), the different minimized values of $\bar{\delta}$ and the corresponding matrices are obtained which are listed in Table 1.

The bounding ellipsoids are given in Fig. 1 for different values of ε and we can see that the intersection of these ellipsoids is very close to the reachable set of the system with $w(t) = \sin(t)$.

Example 2. Consider a singular system in (18) with following parameters:

$$E = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -0.7 & -0.5 \\ -0.7 & -0.7 \end{bmatrix}$$
$$B = \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0.5 & 0 \\ -0.5 & 0.5 \end{bmatrix}$$
$$N = \begin{bmatrix} 2 & -2 \\ 0 & 2 \end{bmatrix}, \quad \tau(t) = 0.3 + 0.2\sin(0.5t).$$

The system is restricted system equivalent to

$$\dot{\tilde{x}}_1(t) = -\tilde{x}_1(t) - 0.7\tilde{x}_1(t - \tau(t)) + 0.2\tilde{x}_2(t - \tau(t)) + 0.25w(t)$$
$$0 = \tilde{x}_2(t) - 0.2\tilde{x}_2(t - \tau(t)) - 0.1w(t).$$

By using the same method of finding the 'smallest' ellipsoids as that in Example 1, we get different ellipsoids when the parameter ε is different. By solving the LMIs in (17) and Theorem 2, the different minimized values of $\bar{\delta}$ and the corresponding matrices are obtained which are listed in Table 2.

Fig. 2 is depicted the reachable sets corresponding to input with $w(t) = \sin(4t)$ and the bounding ellipsoids.

Table 3
Different values of $\check{P}$ by choosing different ε.

ε	0.3	0.5	0.8	0.95
$\check{P}$	$\begin{bmatrix} 0.2888 & 0 \\ 0 & 0.9467 \end{bmatrix}$	$\begin{bmatrix} 0.4813 & 0 \\ 0 & 0.6762 \end{bmatrix}$	$\begin{bmatrix} 0.7703 & 0 \\ 0 & 0.2705 \end{bmatrix}$	$\begin{bmatrix} 0.9142 & 0 \\ 0 & 0.0676 \end{bmatrix}$

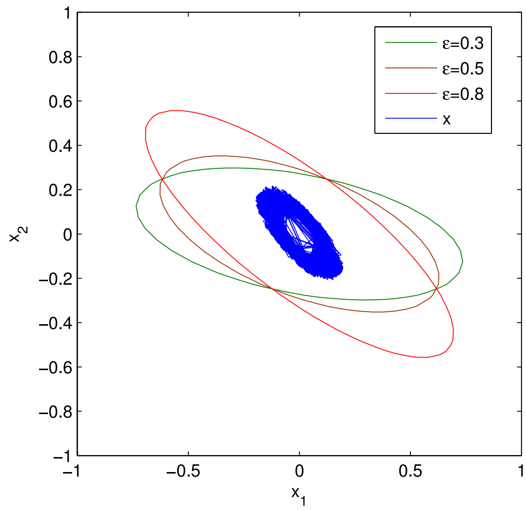

Fig. 2. Reachable set with $w(t) = \sin(4t)$ and the bounding ellipsoids.

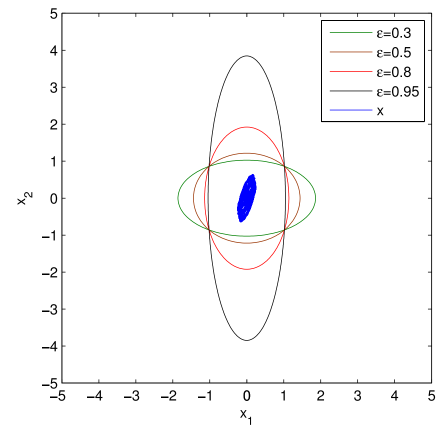

Fig. 3. Reachable set with $w(t) = \sin(10t)$ and the bounding ellipsoids.

Example 3. Without loss of generality, we consider a singular system consisting of the following slow and fast subsystems form:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -15 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -0.2 & -1 \\ 0.1 & -0.3 \end{bmatrix}$$
$$B = \begin{bmatrix} 4 \\ -0.5 \end{bmatrix}, \quad E_0 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau(t) = 0.75 + 0.25\sin(0.8t).$$

Due to $\tilde{A}_{\tau 21} \neq 0$, the Iterative Algorithm is employed. Different matrices $\check{P}$ can be obtained by setting different values of parameter ε, which are listed in Table 3. Fig. 3 is depicted the reachable set corresponding to input with $w(t) = \sin(10t)$ and the bounding ellipsoids.

5. Conclusions

The problem of reachable set estimation of continuous-time singular systems in cases without and with time-delay has been studied for the first time in this paper. A sufficient condition in terms of LMIs has been proposed for guaranteeing the reachable set of a delay-free singular system to be bounded by the intersection of ellipsoids. Based on reciprocally convex method, the result is extended to the singular systems with time-varying delay. The results presented in this paper are in terms of strict LMIs which make the conditions more tractable. Finally, numerical examples are given to demonstrate the effectiveness of our methods. The paper has presented a basic result which extends the LMI method to the reachable set estimation problem of singular system. As illustrated in the numerical examples, the steps taken to reduce the conservatism of the estimated reachable sets are reasonably effective. More useful methods will be tried in our future work, such as non-uniform delay-partitioning method, triple integral inequality and Wirtinger-based inequality.

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