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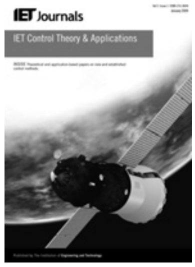

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Brief Paper

Controllability and observability at infinity of linear time-varying descriptor systems

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Abstract: Controllability and observability at infinity of linear time-varying descriptor systems are considered. New characterisations of controllability and observability at infinity are given. Based on the definitions of controllability and observability at infinity, necessary and sufficient conditions for these properties are obtained and presented in terms of original system parameters. The present framework is shown to overcome several difficulties inherent in other treatments of descriptor systems.

1 Introduction

In this paper, we consider controllability and observability of linear time-varying descriptor systems of the form

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1a}$$

$$y(t) = C(t)x(t) \tag{1b}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $y \in \mathbf{R}^m$ are, respectively, the state vector, the input vector and the output vector; $E, A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times r}$ and $C \in \mathbf{R}^{m\times n}$ are known matrix functions with E singular. A pioneering work in this aspect was because of Campbell and Terrell [1], who studied the issue of observability. Further, in [2], controllability and observability were treated. Controllability and observability were also treated in [3]. Different from [1, 2], the analysis in [3] includes the possible impulsive zero-input response. In [4], impulse controllability and impulse observability were examined. Alternative characterisations of impulse controllability and impulse observability presented in [4] were given in [5].

We look at the class of analytically solvable linear time-varying descriptor systems whose E and A are both real-valued, analytic matrix functions. According to Theorem 2 of [6], if (1a) is analytically solvable, then there exist analytic and invertible matrix functions $P(t)$ and $Q(t)$ such that the premultiplication of (1a) by $P(t)$ and the transformation $x = Q(t)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ decompose (1) into two subsystems: the slow subsystem

$$\dot{x}_1(t) = A_1(t)x_1(t) + B_1(t)u(t) \tag{2a}$$

$$y_1(t) = C_1(t)x_1(t) \tag{2b}$$

and the fast subsystem

$$N(t)\dot{x}_2(t) = x_2(t) + B_2(t)u(t) \tag{3a}$$

$$y_2(t) = C_2(t)x_2(t) \tag{3b}$$

with

$$y(t) = y_1(t) + y_2(t) \tag{4}$$

where $x_1 \in \mathbf{R}^{n_1}$, $x_2 \in \mathbf{R}^{n_2}$, $n_1 + n_2 = n$ and N is strictly upper triangular for all t. Assume that N, B_2 and C_2 are analytic.

Controllability and observability, impulse controllability and impulse observability of system (1) have been treated in [3, 4]. However, three difficulties exist for such treatments. (i) Controllability of the fast subsystem (3a) does not imply impulse controllability of the fast subsystem (3a). To see this, we give an example. Consider the fast subsystem in the form of (3a) with the following coefficient matrices

$$N(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_2(t) = \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \end{bmatrix}$$

where

$$\psi_1(t) = \psi_2(t) = \begin{cases} 0, & t < 1 \\ (t-1)^2, & t \geq 1 \end{cases}$$

According to Theorem 2 of [3], this subsystem is controllable. However, for every $t_0 < 1$, impulses excited from non-zero initial value $x_2(t_0^-)$ cannot be cancelled by choosing the input $u(t)$. The example shows that controllability of the fast subsystem (3a) does not imply impulse controllability of system (1a). This is clearly a very strange phenomenon since the impulsive behaviour of system (1a) is completely determined by the fast subsystem (3a). (ii) As the authors themselves point out, controllability and observability are not algebraically dual concepts. (iii) Necessary and sufficient conditions for controllability and observability are only presented in terms of subsystem parameters, but not in terms of original system parameters. In order to obtain these subsystem parameters, one needs to decompose system (1) into two subsystems (2) and (3) through analytically equivalent transformation. This is computationally expensive. Among all these three difficulties, we find that the first two difficulties are closely related to the definition of controllability of the fast subsystem (3a) in [3]. In fact, the idea of the definition is clearly borrowed from state-space system theory. However, the definition cannot characterise controllability of instantaneous state transitions (jumps) which are closely related to infinite modes of system (1a). Consequently, controllability of the fast subsystem (3a) say nothing to impulsive behaviour of system (1a).

Like in [3, 4], we also consider controllability and observability at infinity of system (1). The main contributions of this paper are as follows. (i) A solution with explicit impulse terms to the fast subsystem (3a) is given. This form of the solution is convenient for an analysis of problems involving impulse. (ii) It is pointed out that there exist difficulties for a generalisation of concept of controllability from state-space systems to descriptor systems by using a traditional approach (based on exploiting state transitions from initial time t_0 to some time t_1). To overcome the difficulties, new characterisations of controllability and observability at infinity for system (1) are given. Different from traditional characterisations, the new characterisations of controllability and observability at infinity are based on exploiting jumps which are closely related to infinite modes, and offer deep insights into the meaning of these concepts. (iii) Sufficient and necessary conditions for controllability and observability at infinity are derived and presented in terms of system parameters without any transformations being needed.

2 Preliminaries

First, we give a brief introduction to the theory of distributions (see e.g. [7, 8]). Let C^i be the i times continuously differentiable maps, and C_p^i be the i times piecewise continuously differentiable maps on $\mathbf{R}$ with range depending on context. Let K be the C^∞ functions $\phi : \mathbf{R} \to \mathbf{R}$ with bounded support and K' the space of distributions on $\mathbf{R}$. From [9], K_p' denotes the space of the piecewise continuous distributions. Also from [9], $f|[t_0, \infty)$ stands for the restriction of $f \in K_p'$ on the interval $[t_0, \infty)$, $f[t_0]$ the impulsive part of f at t_0 and $\Delta_{t_0} f$ the jump in f at t_0. The distribution $\delta_{t_0} \in K'$ is defined as $\langle \delta_{t_0}, \phi \rangle = \phi(t_0)$, $\forall \phi \in K$. The multiplication of $f \in K'$ by $g \in C^\infty$, $gf \in K'$, is defined as $\langle gf, \phi \rangle = \langle f, g\phi \rangle$, $\forall \phi \in K$. The distributional derivative of $f \in K'$, $\dot{f}$ is defined as $\langle \dot{f}, \phi \rangle = -\langle f, \mathrm{d}\phi/\mathrm{d}t \rangle$, $\forall \phi \in K$. Let D denote the distributional differentiation operator, and $f^{(i)}$ the ith distributional derivative of $f \in K'$.

It is pointed out in [3] that the product of a C^∞ function and $\delta_{t_0}^{(i)}$, $i \in \mathbf{Z}^+$, is a distribution with point support.

Proposition 1 [3]:

$$M(t)\delta_{t_0}^{(i)} = \sum_{k=0}^{i} (-1)^{i+k} \binom{i}{k} M^{(i-k)}(t_0)\delta_{t_0}^{(k)}$$

where $M(t) \in \mathbf{C}^\infty$, $i \in \mathbf{Z}^+$.

Next, we consider the solution of the fast subsystem (3).

Proposition 2 [9]: For any $f \in K_p'$

1. $(f|[t_0, \infty))^{\cdot} = \dot{f}|[t_0, \infty) + \delta_{t_0} f(t_0^-)$
2. $(f[t_0])^{\cdot} = \dot{f}[t_0] - \delta_{t_0}(\Delta_{t_0} f)$

Applying Proposition 2 to system (1), we have

$$E(t)(x|[t_0, \infty))^{\cdot} = A(t)x|[t_0, \infty) + B(t)u|[t_0, \infty) + E(t)\delta_{t_0} x(t_0^-) \tag{5a}$$

$$y|[t_0, \infty) = C(t)x|[t_0, \infty) \tag{5b}$$

and

$$E(t)(x[t_0])^{\cdot} = A(t)x[t_0] - E(t)\delta_{t_0}(\Delta_{t_0} x) \tag{6a}$$

$$y[t_0] = C(t)x[t_0] \tag{6b}$$

Further, by using $P(t)$, and transformations $x|[t_0, \infty) = Q(t)\begin{bmatrix} x_1|[t_0, \infty) \\ x_2|[t_0, \infty) \end{bmatrix}$ and $x[t_0] = Q(t)\begin{bmatrix} x_1[t_0] \\ x_2[t_0] \end{bmatrix}$, we can obtain the following proposition.

Proposition 3: Let $u \in C_p^{n_2 - 1}$, then

$$N(t)(x_2|[t_0, \infty))^{\cdot} = x_2|[t_0, \infty) + B_2(t)u|[t_0, \infty) + N(t)\delta_{t_0} x_2(t_0^-) \tag{7a}$$

$$y_2|[t_0, \infty) = C_2(t)x_2|[t_0, \infty) \tag{7b}$$

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and

$$N(t)(x_2\dot{[t_0]}) = x_2[t_0] - N(t)\delta_{t_0}(\Delta_{t_0} x_2) \quad (8a)$$

$$y_2[t_0] = C_2(t)x_2[t_0] \quad (8b)$$

which characterise, respectively, the response of the fast subsystem (3) for $t \geq t_0$ because of the initial condition $x_2(t_0^-)$ and the input $u|[t_0, \infty)$, and the impulsive behaviour of the fast subsystem at $t = t_0$.

According to Proposition 3, Theorem 2.1 of [10], and on noticing that $N(t_0)\delta_{t_0} = N(t)\delta_{t_0}$, the solutions for (7a) and (8a) are given by

$$\begin{aligned} x_2|[t_0, \infty) = &-(I - N(t)D)^{-1}(N(t_0)\delta_{t_0} x_2(t_0^-)) \\ &- (I - N(t)D)^{-1}(B_2(t)u|[t_0, \infty)) \end{aligned} \quad (9)$$

(see [3]) and

$$x_2[t_0] = (I - N(t)D)^{-1}(N(t_0)\delta_{t_0}(\Delta_{t_0} x_2)) \quad (10)$$

Since $N(t)D$ is a nilpotent operator, we can obtain

$$(I - N(t)D)^{-1} = I + N(t)D + (N(t)D)^2 + \cdots + (N(t)D)^{n_2 - 1}$$

Thus

$$(I - N(t)D)^{-1}\delta_{t_0} \quad (11)$$

can be expressed as

$$\sum_{i=1}^{n_2} G_i(t)\delta_{t_0}^{(i-1)} \quad (12)$$

Equating (11) and (12), we have

$$(I - N(t)D)\left(\sum_{i=1}^{n_2} G_i(t)\delta_{t_0}^{(i-1)}\right) = I\delta_{t_0} \quad (13)$$

From (13), we can obtain the following

$$\begin{aligned} &G_1(t) = I, \quad G_i(t) = (I - N(t)D)^{-1}(N(t)G_{i-1}(t)) \\ &\quad i = 2, 3, \ldots, n_2 \end{aligned} \quad (14)$$

Again, it can be seen from [3] that

$$(I - N(t)D)^{-1}(B_2(t)u|[t_0, \infty))$$

can be expressed as

$$\sum_{i=0}^{n_2 - 1} F_i(t)(u|[t_0, \infty))^{(i)}$$

where

$$F_0(t) = (I - N(t)D)^{-1}B_2(t) \quad (15)$$

and

$$F_i(t) = (I - N(t)D)^{-1}(N(t)F_{i-1}(t)), \quad i = 1, 2, \ldots, n_2 - 1 \quad (16)$$

From the above analysis and on noticing that $G_{n_2}(t)N(t_0) = 0$, we have the following theorem.

Theorem 1: Let $u \in C_p^{n_2 - 1}$, then

1. The solution of (7a) is given by

$$\begin{aligned} x_2|[t_0, \infty) = &-\sum_{i=1}^{n_2 - 1} G_i(t)\delta_{t_0}^{(i-1)}(N(t_0)x_2(t_0^-)) \\ &- \sum_{i=0}^{n_2 - 1} F_i(t)(u|[t_0, \infty))^{(i)} \end{aligned} \quad (17)$$

2. The solution of (8a) is given by

$$x_2[t_0] = \sum_{i=1}^{n_2 - 1} G_i(t)\delta_{t_0}^{(i-1)}(N(t_0)\Delta_{t_0} x_2) \quad (18)$$

where $G_i(t)$, $i = 1, 2, \ldots, n_2 - 1$, $F_i(t)$, $i = 0, 1, \ldots, n_2 - 1$ are given by (14)–(16).

Remark 1: Repeatedly using Proposition 2, part (1), we have

$$\begin{aligned} &(u|[t_0, \infty))^{(i)} = u^{(i)}|[t_0, \infty) + \sum_{k=1}^{i} u^{(i-k)}(t_0^+)\delta_{t_0}^{(k-1)} \\ &i = 1, 2, \ldots, n_2 - 1 \end{aligned}$$

Thus, solution (17) can be written as

$$\begin{aligned} x_2|[t_0, \infty) = &-\sum_{i=1}^{n_2 - 1} G_i(t)\delta_{t_0}^{(i-1)}(N(t_0)x_2(t_0^-)) \\ &- \sum_{i=1}^{n_2 - 1}\sum_{k=i}^{n_2 - 1} F_k(t)\delta_{t_0}^{(i-1)}u^{(k-i)}(t_0^+) \\ &- \sum_{i=0}^{n_2 - 1} F_i(t)u^{(i)}|[t_0, \infty) \end{aligned}$$

Obviously, the solution includes not only any impulses in x_2 at t_0 excited by the initial condition, but also those excited by the input u.

Remark 2: Since N, B_2 are analytic, from (17), we have

$$x_2(t_0^+) = -\sum_{i=0}^{n_2-1} F_i(t_0)u^{(i)}(t_0^+)$$

Then, $\Delta_{t_0}x_2$ in (18) is given by

$$\Delta_{t_0}x_2 = -\sum_{i=0}^{n_2-1} F_i(t_0)u^{(i)}(t_0^+) - x_2(t_0^-)$$

Thus, solutions (17) and (18) can also be written, respectively, as

$$\begin{aligned} x_2|[t_0, \infty) = &-\sum_{i=1}^{n_2-1} G_i(t)\delta_{t_0}^{(i-1)} \\ &\times \left(\sum_{i=0}^{n_2-1} N(t_0)F_i(t_0)u^{(i)}(t_0^+) + N(t_0)x_2(t_0^-)\right) \\ &- \sum_{i=0}^{n_2-1} F_i(t)u^{(i)}|[t_0, \infty) \end{aligned}$$

and

$$\begin{aligned} x_2[t_0] = &-\sum_{i=1}^{n_2-1} G_i(t)\delta_{t_0}^{(i-1)} \\ &\times \left(\sum_{i=0}^{n_2-1} N(t_0)F_i(t_0)u^{(i)}(t_0^+) + N(t_0)x_2(t_0^-)\right) \end{aligned}$$

3 Controllability and observability at infinity

Since jumps and impulses in the natural response are a feature unique to descriptor systems, which are clearly related to infinite modes of the systems, we shall treat controllability and observability of system (1) with respect to infinite modes by exploiting jumps and impulses.

Definition 1: System (1a) is called controllable at infinity at t_0 if for any $x(t_0^-) \in \mathbf{R}^n$, there exists $u \in C_p^{n_2-1}$ such that $\Delta_{t_0}x = 0$. If the system is controllable at infinity at every $t_0 \in \mathbf{R}$, then it is called controllable at infinity.

Remark 3: A traditional approach to characterise the controllability is based on exploiting state transitions from initial time t_0 to some time t_1. However, since such state transitions are 'slow', the approach is not suitable to characterise controllability of infinite modes. Owing to this reason, we characterise controllability of infinite modes by exploiting jumps (instantaneous state transitions that are 'fast').

Theorem 2: System (1a) is controllable at infinity if and only if

$$\text{rank}\begin{bmatrix} E(t) & B(t) \end{bmatrix} = n \tag{19}$$

for all $t \in \mathbf{R}$.

Proof: Since $u \in C_p^{n_2-1}$, $\Delta_{t_0}x_1 = 0$ so $\Delta_{t_0}x = Q(t_0)\begin{bmatrix} 0 \\ \Delta_{t_0}x_2 \end{bmatrix}$. From this, we know that for any $t_0 \in \mathbf{R}$ and $x(t_0^-) \in \mathbf{R}^n$, $\Delta_{t_0}x = 0$ is equivalent to $\Delta_{t_0}x_2 = 0$, that is

$$x_2(t_0^-) = -\sum_{i=0}^{n_2-1} F_i(t_0)u^{(i)}(t_0^+)$$

which is further equivalent to

$$\text{rank}\begin{bmatrix} F_0(t_0) & F_1(t_0) & \cdots & F_{n_2-1}(t_0) \end{bmatrix} = n_2 \tag{20}$$

since $x_2(t_0^-)$, $u(t)$ are arbitrary. It is known from [3] that (20) is equivalent to

$$\text{rank}\begin{bmatrix} N(t_0) & B_2(t_0) \end{bmatrix} = n_2$$

Owing to the arbitrariness of $t_0 \in \mathbf{R}$, we have shown that system (1a) is controllable at infinity if and only if

$$\text{rank}\begin{bmatrix} N(t) & B_2(t) \end{bmatrix} = n_2 \tag{21}$$

for all $t \in \mathbf{R}$. Note that

$$\begin{aligned} \text{rank}\begin{bmatrix} E(t) & B(t) \end{bmatrix} &= \text{rank}\begin{bmatrix} P(t)E(t)Q(t) & P(t)B(t) \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} I_{n_1} & 0 & B_1(t) \\ 0 & N(t) & B_2(t) \end{bmatrix} \\ &= n_1 + \text{rank}\begin{bmatrix} N(t) & B_2(t) \end{bmatrix} \end{aligned}$$

We have completed the proof. □

Remark 4: It is easy to see from the proof of Theorem 2 that controllability at infinity of system (1a) is equivalent to controllability of the fast subsystem (3a) (in our sense, i.e. controllability at infinity, but not in the sense of Wang [3]). Controllability at infinity here for linear time-varying descriptor systems is a generalisation of controllability at infinity in the sense of Rosenbrock [11] for linear time-invariant descriptor systems.

Remark 5: It follows from an extension (to matrix functions) of the well-known matrix algebra theorem that (19) holds if and only if there exists a matrix function $K(t) \in \mathbf{R}^{r\times n}$ such that $\det(E(t) + B(t)K(t)) \neq 0$ for all $t \in \mathbf{R}$. This clearly shows that system (1a) is controllable at infinity if and only if the system can be normalised by a

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state derivative feedback controller $u(t) = -K(t)\dot{x}(t)$ (the resulted closed-loop system is a state-space system).

Definition 2: System (1a) is called impulse controllable at t_0 if for any $x(t_0^-) \in \mathbf{R}^n$, there exists $u \in C_p^{m_2-1}$ such that $x[t_0] = 0$. If the system is impulse controllable at every $t_0 \in \mathbf{R}$, then it is called impulse controllable.

Remark 6: Since $u \in C_p^{m_2-1}$, $x_1[t_0] = 0$ so $x[t_0] = Q(t)\begin{bmatrix} 0 \\ x_2[t_0] \end{bmatrix}$. From this, we see that impulse controllability of system (1a) defined here is equivalent to impulse controllability of the fast subsystem (3a) defined in [4].

From Definitions 1 and 2, we see that controllability at infinity implies impulse controllability (Note that elimination of all jumps $\Delta_{t_0}x$ implies elimination of all impulses $x[t_0]$, but the converse is not true). It is easy to understand that system (1a) being controllable at infinity means that all its infinite modes (including infinite non-dynamical modes and infinite dynamical modes) are controllable, whereas system (1a) being impulse controllable means all its impulsive modes (infinite dynamical modes) are controllable.

Now we treat observability.

Definition 3: System (1) is called observable at infinity at t_0 if knowledge of $\Delta_{t_0} y$ and $y[t_0]$ is sufficient to determine $\Delta_{t_0} x$. If the system is observable at infinity at every $t_0 \in \mathbf{R}$, then it is called observable at infinity.

Theorem 3: System (1) is observable at infinity if and only if

$$\operatorname{rank}\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = n$$

for all $t \in \mathbf{R}$.

Proof: For any $t_0 \in \mathbf{R}$, since $\Delta_{t_0}y = \Delta_{t_0}y_2$, $y[t_0] = y_2[t_0]$ and $\Delta_{t_0}x = Q(t_0)\begin{bmatrix} 0 \\ \Delta_{t_0}x_2 \end{bmatrix}$, the statement that knowledge of $\Delta_{t_0}y$ and $y[t_0]$ is sufficient to determine $\Delta_{t_0}x$ is equivalent to the statement that knowledge of $\Delta_{t_0}y_2$ and $y_2[t_0]$ is sufficient to determine $\Delta_{t_0}x_2$. The latter statement is equivalent to observability of the fast subsystem (3) in the sense of [3] since $x_2(t_0^+)$ can always be computed from $u(t)$ for $t > t_0$, which is further equivalent to

$$\operatorname{rank}\begin{bmatrix} N(t_0) \\ C_2(t_0) \end{bmatrix} = n_2$$

(see [3]). Owing to the arbitrariness of $t_0 \in \mathbf{R}$, we have shown that system (1) is observable at infinity if and only if

$$\operatorname{rank}\begin{bmatrix} N(t) \\ C_2(t) \end{bmatrix} = n_2$$

for all $t \in \mathbf{R}$. Note that

$$\operatorname{rank}\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} P(t)E(t)Q(t) \\ C(t)Q(t) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} I_{n_1} & 0 \\ 0 & N(t) \\ C_1(t) & C_2(t) \end{bmatrix}$$
$$= n_1 + \operatorname{rank}\begin{bmatrix} N(t) \\ C_2(t) \end{bmatrix}$$

We have completed the proof. □

Definition 4: System (1) is called impulse observable at t_0 if knowledge of $y[t_0]$ is sufficient to determine $x[t_0]$. If the system is impulse observable at every $t_0 \in \mathbf{R}$, then it is called impulse observable.

Remark 7: Note that $y[t_0] = y_2[t_0]$ and $x[t_0] = Q(t)\begin{bmatrix} 0 \\ x_2[t_0] \end{bmatrix}$. We see that impulse observability of system (1) defined here is equivalent to impulse observability of the fast subsystem (3) defined in [4].

Theorems 2 and 3 indicate that controllability and observability at infinity are dual concepts.

It is worth pointing out that characterisations of controllability and observability of system (1) with respect to infinite modes at t_0 depend only on state x, input u and output y with time t in an arbitrary sufficient small neighbourhood of t_0, and have no relation to state x, input u and output y with time t outside the neighbourhood. This suggests that controllability and observability related to infinite modes should be treated by investigating jumps (and/or the corresponding impulses).

4 Example

Consider a linear time-varying descriptor system in the form of (1) with the following coefficient matrices

$$E(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sin t & \cos t \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} -\cos t & 0 & 0 & 0 \\ 0 & & 0 & 1 \\ 0 & & 1 & 0 & 0 \\ 0 & & 0 & 0 & 1 \end{bmatrix} \tag{22a}$$

$$B(t) = \begin{bmatrix} \cos t & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C(t) = \begin{bmatrix} 1 & 2 & t & (1+t) \end{bmatrix} \tag{22b}$$

For the example system it is easy to check that

$$\operatorname{rank}\begin{bmatrix} E(t) & B(t) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \sin t & \cos t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 4$$

$$\operatorname{rank}\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sin t & \cos t \\ 0 & 0 & 0 & 0 \\ 1 & 2 & t & (1+t) \end{bmatrix} = 3 < 4$$

for all $t \in \mathbf{R}$. According to Theorems 2 and 3, system (22) is controllable at infinity, but not observable at infinity. In the following we shall verify these.

It is easy to verify that under the transformation (P, Q) with

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = I_4$$

system (22) can be decomposed into the slow subsystem in the form of (2) with

$$A_1(t) = -\cos t, \quad B_1(t) = \begin{bmatrix} \cos t & 0 \end{bmatrix}, \quad C_1(t) = 1 \tag{23}$$

and the fast subsystem in the form of (3) with

$$N(t) = \begin{bmatrix} 0 & \sin t & \cos t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{24}$$
$$C_2(t) = \begin{bmatrix} 2 & t & (1+t) \end{bmatrix}$$

Since controllability and observability at infinity of system (22) is equivalent to controllability and observability of the fast subsystem (24), to verify that system (22) is controllable at infinity, we need only to consider the fast subsystem (24). For every $t_0 \in \mathbf{R}$, let

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \tag{25}$$

with

$$u_i(t) = \begin{cases} \varphi_i(t), & t \geq t_0 \\ 0, & t < t_0 \end{cases}, \quad i = 1, 2$$

where φ_i, $i = 1, 2$ are continuously differentiable functions.

From Theorem 1, we have

$$x_2(t_0^+) = \begin{bmatrix} -\sin t_0 \dot{\varphi}_1(t_0) - \cos t_0 \dot{\varphi}_2(t_0) \\ -\varphi_1(t_0) \\ -\varphi_2(t_0) \end{bmatrix}$$

For any $x_2(t_0^-) = [w_1\ w_2\ w_3]^{\mathrm{T}} \in \mathbf{R}^3$, letting $x_2(t_0^-) = x_2(t_0^+)$ gives

$$\begin{cases} -\sin t_0 \dot{\varphi}_1(t_0) - \cos t_0 \dot{\varphi}_2(t_0) = w_1 \\ -\varphi_1(t_0) = w_2 \\ -\varphi_2(t_0) = w_3 \end{cases} \tag{26}$$

If we choose

$$\begin{aligned} \varphi_1(t) &= -w_2 - w_1 \cos t_0 + w_1 \cos t, \\ \varphi_2(t) &= -w_3 + w_1 \sin t_0 - w_1 \sin t \end{aligned} \tag{27}$$

then the equations in (26) are satisfied. From Theorem 1, the solution of $N(t)\dot{x}_2(t) = x_2(t) + B_2(t)u(t)$ with the input $u(t)$ given by (25) and (27) is

$$x_2|[t_0, \infty) = \begin{bmatrix} w_1 \\ w_2 + w_1 \cos t_0 - w_1 \cos t \\ w_3 - w_1 \sin t_0 + w_1 \sin t \end{bmatrix}, \quad t \geq t_0$$

which is continuous on $[t_0, \infty)$ (no jump and/or impulse occur at t_0). This means that, for every $t_0 \in \mathbf{R}$, jumps and the corresponding impulses excited by any non-zero initial value $x_2(t_0^-)$ are cancelled by the input we choose. Hence, the fast subsystem (24) is controllable (in our sense, but not in the sense of Wang [3]). From this, we see that the example system (22) is indeed controllable at infinity.

Now we begin to verify that system (22) is not observable at infinity. This is equivalent to show that the fast subsystem (24) is not observable. For every $t_0 \in \mathbf{R}$, denote $\Delta_{t_0} x_2 = [v_1\ v_2\ v_3]^{\mathrm{T}}$. Then

$$\Delta_{t_0} y_2 = C_2(t_0)\Delta_{t_0} x_2 = 2v_1 + t_0 v_2 + (1 + t_0)v_3 \tag{28}$$

From Theorem 1, we have

$$x_2[t_0] = \begin{bmatrix} \sin t_0 v_2 + \cos t_0 v_3 \\ 0 \\ 0 \end{bmatrix} \delta_{t_0}$$

Then

$$y_2[t_0] = C_2(t)x_2[t_0] = 2(\sin t_0 v_2 + \cos t_0 v_3)\delta_{t_0} \tag{29}$$

Clearly, if $\Delta_{t_0} y_2$ and $y_2[t_0]$ are given, the unknowns v_1, v_2, v_3 cannot be solved uniquely from (28) and (29). This means that knowledge of $\Delta_{t_0} y_2$ and $y_2[t_0]$ does not imply knowledge of $\Delta_{t_0} x_2$, or that knowledge of $\Delta_{t_0} y_2$, $y_2[t_0]$ and $u|[t_0, \infty)$ does not imply knowledge of $x_2(t_0^-)$ (note that $\Delta_{t_0} x_2 = x_2(t_0^+) - x_2(t_0^-)$ and $x_2(t_0^+)$ can always be computed from $u|[t_0, \infty)$). Hence, the fast subsystem (24)

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is not observable. This shows that system (22) is indeed not observable at infinity.

5 Conclusion

New characterisations of controllability and observability at infinity of linear time-varying descriptor systems are given by exploiting jumps which are clearly related to infinite modes of the systems. Based on the new definitions of controllability and observability at infinity, necessary and sufficient conditions for these properties are obtained and presented in terms of system parameters. The present framework is shown to overcome several difficulties inherent in other treatments of descriptor systems.

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