Asian Journal of Control, Vol. 15, No. 4, pp. 1–14, July 2013
Published online in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.579

DESCRIPTOR FRACTIONAL LINEAR SYSTEMS WITH REGULAR PENCILS

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ABSTRACT

New classes of descriptor fractional continuous-time and discrete-time linear systems with regular pencils are introduced. Electrical circuits are an example of descriptor fractional continuous-time systems. Using the Caputo definition of the fractional derivative, the Weierstrass regular pencil decomposition and Laplace transformation the solution to the state equation of descriptor fractional linear systems is derived. It is shown that every electrical circuit is a descriptor fractional systems if it contains at least one mesh consisting of branches with only ideal supercondensators and voltage sources, or at least one node with branches containing supercoils. Using the Weierstrass regular pencil decomposition the solution to the state equation of descriptor fractional discrete-time linear systems is derived. A method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts is proposed. The considerations are illustrated by numerical examples.

Key Words: fractional, descriptor, linear circuit, regular pencil, solution, supercondensator, supercoil.

I. INTRODUCTION

Descriptor (singular) linear systems have been addressed in many papers and books [1–4– 8–10– 13, 15, 16, 19]. The eigenvalues and invariants assigned by state and output feedbacks have been investigated in [1–4– 8, 9] and the realization problem for singular positive continuous-time systems with delays in [13]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [19]. A delay-dependent criterion for a class of descriptor system with a delay varying in an interval has been proposed in [2].

Fractional positive continuous-time linear systems have been addressed in [7] and positive linear systems with different fractional orders in [11]. A new concept of the practical stability of the positive fractional 2D systems has been proposed in [12]. An analysis of fractional linear electrical circuits has been presented in [6] and some selected problems in theory of fractional linear systems in the monograph [14]. Convergence speed of a fractional order consensus algorithm over undirected scale-free networks is analyzed in [18]. The convergence conditions of the iterative process for fractional-order systems is analyzed in [20].

In this paper a new class of descriptor fractional linear system and electrical circuit will be introduced and the solution of their state equations will be derived. A method for decomposition of the descriptor fractional linear systems with regular pencils into dynamic and static parts will be proposed.

The paper is organized as follows. In Section II the Caputo definition of the fractional derivative and the solution to the state equation of the fractional linear system are recalled. The solution of the state equation of descriptor fractional linear system is derived using the Weierstrass pencil decomposition and the Laplace transform in Section III. Descriptor fractional linear electrical circuits are introduced in Section IV. In Section V the fractional descriptor discrete-time linear systems are introduced and Weierstrass regular pencil decomposition is recalled. The solution of the state equation of descriptor fractional linear discrete-time system is derived using the Weierstrass pencil decomposition in Section VI. Illustrative numerical examples are given in Section VI. Decomposition of the descriptor fractional continuous-time and discrete-time linear systems into dynamic and static parts is given in Section VIII and IX, respectively. Concluding remarks are given in Section X.

To the best of the author's knowledge descriptor fractional linear systems and descriptor fractional electrical circuits have not been considered yet.

The following notation will be used in the paper. The set of $n \times m$ real matrices will be denoted by $\Re^{n\times m}$ and $\Re^n := \Re^{n\times 1}$ The set of $m \times n$ real matrices with nonnegative entries will be denoted by $\Re_+^{m\times n}$ and $\Re_+^n := \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n \times n$ identity matrix by I_n.

Manuscript received October 5, 2011; revised February 9, 2012; accepted June 3, 2012.
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This work was supported by National Science Centre in Poland under work No. N N514 6389 40.

II. FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

The following Caputo definition of the fractional derivative will be used [17, 14]:

$$\frac{d^\alpha}{dt^\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau,\quad n-1<\alpha\le n\in\{1,2,\ldots\} \tag{2.1}$$

where $\alpha\in\Re$ is the order of fractional derivative, $f^{(n)}(\tau)=\frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt$ is the gamma function.

Consider the continuous-time fractional linear system described by the state equation

$$\frac{d^\alpha}{dt^\alpha}x(t)=Ax(t)+Bu(t),\quad 0<\alpha\le 1 \tag{2.2}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times m}$, $B\in\Re^{n\times m}$.

Theorem 2.1. The solution of (2.2) is given by

$$x(t)=\Phi_0(t)x_0+\int_0^t\Phi(t-\tau)Bu(\tau)d\tau,\quad x(0)=x_0 \tag{2.3}$$

where

$$\Phi_0(t)=\sum_{k=0}^\infty\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{2.4}$$

$$\Phi(t)=\sum_{k=0}^\infty\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.5}$$

Proof is given in [14].

Remark 2.1. From (2.4) and (2.5) for $\alpha=1$ we have $\Phi_0(t)=\Phi(t)=\sum_{k=0}^\infty\frac{(At)^k}{\Gamma(k+1)}=e^{At}$.

III. DESCRIPTOR FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

Consider descriptor fractional linear system described by the state equations

$$E\frac{d^\alpha}{dt^\alpha}x(t)=Ax(t)+Bu(t),\quad 0<\alpha\le 1 \tag{3.1a}$$

$$y(t)=Cx(t)+Du(t) \tag{3.1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and E, $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$.

The initial condition for (3.1a) is given by

$$x(0)=x_0 \tag{3.1c}$$

It is assumed that $\det E=0$ but the pencil is regular, *i.e.*

$$\det[Es-A]\ne 0 \tag{3.2}$$

for some $z\in C$ (the field of complex numbers). It is well known [5, 10 p. 92] that if the pencil is regular then there exists a pair of nonsingular matrices P, $Q\in\Re^{n\times n}$ such that

$$P[Es-A]Q=\begin{bmatrix}I_{n_1}&0\\0&N\end{bmatrix}s-\begin{bmatrix}A_1&0\\0&I_{n_2}\end{bmatrix} \tag{3.3}$$

where n_1 is equal to degree of the polynomial $\det[Es-A]$, $A_1\in\Re^{n_1\times n_1}$, $N\in\Re^{n_2\times n_2}$ is a nilpotent matrix with the index μ (*i.e.* $N^\mu=0$ and $N^{\mu-1}\ne 0$) and $n_1+n_2=n$.

Applying the one-sided Laplace transform ($\mathcal{L}$) to (3.1a) with zero initial conditions $x_0=0$ we obtain

$$[Es^\alpha-A]X(s)=BU(s) \tag{3.4}$$

where $X(s)=\mathcal{L}[x(t)]=\int_0^\infty x(t)e^{-st}dt$ and $U(s)=\mathcal{L}[u(t)]$. By assumption (3.2) the pencil $[Es^\alpha-A]$ is regular and we may apply the decomposition (3.3) to (3.1a).

Premultiplying (3.1a) by the matrix $P\in\Re^{n\times n}$ and introducing the new state vector

$$\bar{x}(t)=Q^{-1}x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix},\ x_1(t)\in\Re^{n_1},\ x_2(t)\in\Re^{n_2} \tag{3.5}$$

we obtain

$$\frac{d^\alpha}{dt^\alpha}x_1(t)=A_1x_1(t)+B_1u(t) \tag{3.6a}$$

$$N\frac{d^\alpha}{dt^\alpha}x_2(t)=x_2(t)+B_2u(t) \tag{3.6b}$$

where

$$PB=\begin{bmatrix}B_1\\B_2\end{bmatrix},\ B_1\in\Re^{n_1\times m},\ B_2\in\Re^{n_2\times m} \tag{3.6c}$$

Using (2.3) we obtain the solution to the equation (3.6a) in the form

$$x_1(t)=\Phi_{10}(t)x_{10}+\int_0^t\Phi_{11}(t-\tau)B_1u(\tau)d\tau \tag{3.7a}$$

where

$$\Phi_{10}(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{3.7b}$$

$$\Phi_{11}(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{3.7c}$$

and $x_{10}\in\Re^{n_1}$ is the initial condition for (3.6a) defined by

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}=Q^{-1}x_0,\ x_0=x(0). \tag{3.7d}$$

To find the solution of (3.6b) we apply the Laplace transform to the equation and we obtain

$$Ns^{\alpha}X_2(s)-Ns^{\alpha-1}x_{20}=X_2(s)+B_2U(s) \tag{3.8a}$$

since [14] for $0<\alpha<1$

$$\mathcal{L}\left[\frac{d^{\alpha}}{dt^{\alpha}}x_2(t)\right]=s^{\alpha}X_2(s)-s^{\alpha-1}x_{20} \tag{3.8b}$$

where $X_2(s)=\mathcal{L}[x_2(t)]$. From (3.8) we have

$$X_2(s)=[Ns^{\alpha}-I_{n_2}]^{-1}(B_2U(s)+Ns^{\alpha-1}x_{20}) \tag{3.9}$$

It can be verified that

$$[Ns^{\alpha}-I_{n_2}]^{-1}=-\sum_{i=0}^{\mu-1}N^i s^{i\alpha} \tag{3.10}$$

since

$$[Ns^{\alpha}-I_{n_2}]\left(-\sum_{i=0}^{\mu-1}N^i s^{i\alpha}\right)=I_{n_2} \tag{3.11}$$

and $N^i=0$ for $i=\mu,\mu+1,\dots$

Substitution of (3.10) into (3.9) yields

$$X_2(s)=-B_2U(s)-\frac{Nx_{20}}{s^{1-\alpha}} -\sum_{i=1}^{\mu-1}[N^iB_2s^{i\alpha}U(s)+N^{i+1}s^{(i+1)\alpha-1}x_{20}] \tag{3.12}$$

Using inverse Laplace transform ($\mathcal{L}^{-1}$) to (3.12) and the convolution theorem we obtain for $1-\alpha>0$

$$x_2(t)=\mathcal{L}^{-1}[X_2(s)]=-B_2u(t)-Nx_{20}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} -\sum_{i=1}^{\mu-1}\left[N^iB_2\frac{d^{i\alpha}}{dt^{i\alpha}}u(t)+N^{i+1}\frac{d^{(i+1)\alpha-1}}{dt^{(i+1)\alpha-1}}x_{20}\right] \tag{3.13}$$

since $\mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right]=\frac{t^{\alpha}}{\Gamma(1+\alpha)}$ for $\alpha+1>0$.

Therefore, the following theorem has been proved.

Theorem 3.1. The solution to (3.1a) with the initial condition (3.1c) has the form

$$x(t)=Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3.14}$$

where $x_1(t)$ and $x_2(t)$ are given by (3.7) and (3.13), respectively.

Knowing the solution (3.14) we can find the output $y(t)$ of the system using the formula

$$y(t)=CQ\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}+Du(t) \tag{3.15}$$

IV. DESCRIPTOR FRACTIONAL ELECTRICAL CIRCUITS

Let the current $i_C(t)$ in the supercondensator with the capacity C be the α order derivative of its charge $q(t)$

$$i_C(t)=\frac{d^{\alpha}q(t)}{dt^{\alpha}} \tag{4.1}$$

Taking into account that $q(t)=Cu_C(t)$ we obtain

$$i_C(t)=C\frac{d^{\alpha}u_C(t)}{dt^{\alpha}} \tag{4.2}$$

where $u_C(t)$ is the voltage on the supercondensator.

Similarly, let the voltage $u_L(t)$ on the supercoil (inductor) with the inductance L be the β order derivative of its magnetic flux $\Psi(t)$

$$u_L(t)=\frac{d^{\beta}\Psi(t)}{dt^{\beta}}. \tag{4.3}$$

Taking into account that $\psi(t)=Li_L(t)$ we obtain

$$u_L(t)=L\frac{d^{\beta}i_L(t)}{dt^{\beta}} \tag{4.4}$$

where $i_L(t)$ is the current in the supercoil.

Example 4.1. Consider the electrical circuit shown in Fig. 1 with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.

Using the Kirchhoff's laws we can write the following equations for the electrical circuit:

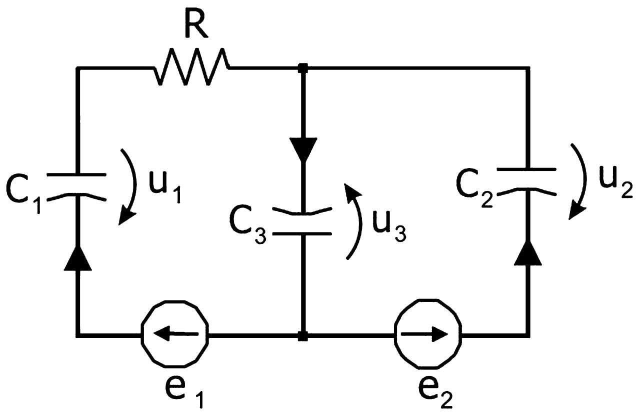

Fig. 1. Electrical circuit.

$$\begin{aligned} e_1 &= RC_1\frac{d^\alpha u_1}{dt^\alpha} + u_1 + u_3 \\ C_1\frac{d^\alpha u_1}{dt^\alpha} &+ C_2\frac{d^\alpha u_2}{dt^\alpha} - C_3\frac{d^\alpha u_3}{dt^\alpha} = 0 \\ e_2 &= u_2 + u_3 \end{aligned} \tag{4.5}$$

The equations (4.5) can be written in the form

$$\begin{aligned} &\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \end{aligned} \tag{4.6}$$

In this case we have

$$E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.7}$$

Note that the matrix E is singular ($\det E = 0$) but

$$\begin{aligned} \det[Es^\alpha - A] &= \begin{vmatrix} RC_1 s^\alpha + 1 & 0 & 1 \\ C_1 s^\alpha & C_2 s^\alpha & -C_3 s^\alpha \\ 0 & 1 & 1 \end{vmatrix} \\ &= (RC_1 s^\alpha + 1)(C_2 + C_3)s^\alpha + C_1 s^\alpha \end{aligned} \tag{4.8}$$

and the condition (3.2) is met. This means that the pencil is regular and the electrical circuit is a descriptor fractional linear system.

Remark 4.1. If the electrical circuit contains at least one mesh consisting of branches with only ideal supercondensators and voltage sources then its matrix E is singular since the row corresponding to this mesh is zero row. This follows from the fact that the equation written by the use of the voltage Kirchhoff's law is an algebraic one.

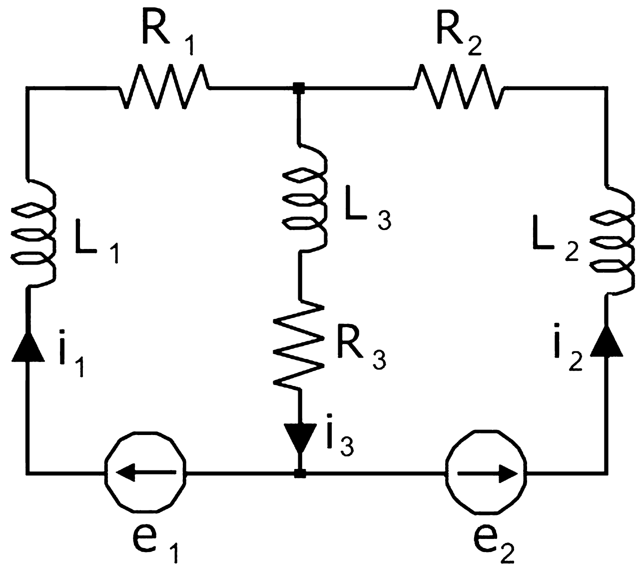

Fig. 2. Electrical circuit.

Example 4.2. Consider the electrical circuit shown in Fig. 2 with given resistances R_1, R_2, R_3 inductances L_1, L_2, L_3 and source voltages e_1 and e_2.

Using Kirchhoff's laws we can write the following equations for the electrical circuit:

$$\begin{aligned} e_1 &= R_1 i_1 + L_1\frac{d^\beta i_1}{dt^\beta} + R_3 i_3 + L_3\frac{d^\beta i_3}{dt^\beta} \\ e_2 &= R_2 i_2 + L_2\frac{d^\beta i_2}{dt^\beta} + R_3 i_3 + L_3\frac{d^\beta i_3}{dt^\beta} \\ i_1 &+ i_2 - i_3 = 0 \end{aligned} \tag{4.9}$$

The equations (4.9) can be written in the form

$$\begin{aligned} &\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}\frac{d^\beta}{dt^\beta}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \\ &= \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \end{aligned} \tag{4.10}$$

In this case we have

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.11}$$

Note that the matrix E is singular but

$$\det[Es^{\beta}-A]=\begin{vmatrix} L_1s^{\beta}+R_1 & 0 & L_3s^{\beta}+R_3 \\ 0 & L_2s^{\beta}+R_2 & L_3s^{\beta}+R_3 \\ -1 & -1 & 1 \end{vmatrix}$$
$$\begin{aligned} &=[L_1(L_2+L_3)+L_2L_3]s^{2\beta} \\ &\quad+[(L_2+L_3)R_1+(L_1+L_3)R_2 \\ &\quad+(L_1+L_2)R_3]s^{\beta} \\ &\quad+R_1(R_2+R_3)+R_2R_3 \end{aligned} \tag{4.12}$$

Equation (3.2) holds and the pencil is regular. Therefore, the electrical circuit is a descriptor fractional linear system.

Remark 4.2. If the electrical circuit contains at least one node with branches with supercoils then its matrix E is singular since it has at least one zero row. This follows from the fact that the equation written using the current Kirchhoff's law for this node is an algebraic one.

In the general case we have the following theorem.

Theorem 4.1. Every electrical circuit is a descriptor fractional system if it contains at least one mesh consisting of branches with only ideal supercondensators and voltage source or at least one node with branches with supercoils.

Proof. By Remark 4.1 the matrix E of the system is singular if the electrical circuit contains at least one mesh consisting of branches with only ideal supercondensators and a voltage source. Similarly according to Remark 4.2 the matrix E is singular if the electrical circuit contains at least one node with branches with supercoils. □

Using the solution (3.14) of (3.1a) we find the voltages on the supercondensators and currents in the supercoils in the transient states of the descriptor fractional linear electrical circuits. Knowing the voltages and currents and using (3.15) we may find also any currents and voltages in the descriptor fractional linear electrical circuits.

Example 4.3 (a continuation of Example 4.1)**.** Using one of the well-known methods [17, 3, 10] we can find the following matrices for the pencil of the pair (E, A):

$$P=\begin{bmatrix} \frac{1}{RC_1} & 0 & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{1}{C_2+C_3} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & 0 & -1 \end{bmatrix},\quad Q=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{C_3}{C_2+C_3} \\ 0 & -1 & \frac{C_2}{C_2+C_3} \end{bmatrix} \tag{4.13}$$

which transform it to the canonical form (3.3) with

$$A_1=\begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{R(C_2+C_3)} & -\frac{1}{R(C_2+C_3)} \end{bmatrix},\quad N=[0],\quad n_1=2,\quad n_2=1 \tag{4.14}$$

Using the matrix B given by (4.7), (4.13) and (3.6c) we obtain

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}=PB=\begin{bmatrix} \frac{1}{RC_1} & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & -1 \end{bmatrix} \tag{4.15}$$

and from (3.7) we have

$$x_1(t)=\Phi_{10}(t)x_{10}+\int_0^t \Phi_{11}(t-\tau)B_1u(\tau)d\tau \tag{4.16}$$

for any given initial condition $x_{10}\in\Re^{n_1}$ and input $u(t)$, where

$$\Phi_{10}(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi_{11}(t)=\sum_{k=0}^{\infty}\frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]},\quad 0<\alpha<1.$$

In this case using (3.13) we obtain

$$x_2(t)=-B_2u(t) \tag{4.17}$$

since $N=[0]$.

In a similar way we may find currents in the supercoils of the descriptor fractional electrical circuit shown on Fig. 2.

V. FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor fractional discrete-time linear system described by the state equation

$$E\Delta^{\alpha}x_{i+1}=Ax_i+Bu_i,\ i\in Z_+=\{0,1,\dots\} \tag{5.1}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors, $A\in\Re^{n\times n}$, $E\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, and the fractional difference of the order α is defined by

$$\Delta^{\alpha}x_i=\sum_{k=0}^{i}(-1)^k\binom{\alpha}{k}x_{i-k},\ 0<\alpha<1 \tag{5.2}$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,\ldots \end{cases} \tag{5.3}$$

It is assumed that

$$\det E = 0 \tag{5.4a}$$

and

$$\det[Ez-A] \neq 0 \tag{5.4b}$$

for some $z \in C$ (the field of complex numbers).

Lemma 5.1. [5, 10 p. 92] If (5.4) holds then there exist nonsingular matrices $P, Q \in \Re^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{5.5}$$

where $N \in \Re^{n_2\times n_2}$ is a nilpotent matrix with the index μ (*i.e.* $N^{\mu} = 0$ and $N^{\mu-1} \neq 0$), $A_1 \in \Re^{n_1\times n_1}$, n_1 is equal to degree of the polynomial

$$\det[Es-A] = a_{n_1} z^{n_1} + \ldots + a_1 z + a_0 \tag{5.6}$$

and $n_1 + n_2 = n$.

A method for computation of the matrices P and Q has been given in [19].

Using Lemma 5.1 we shall derive the solution x_i to (5.1) for a given initial condition x_0 and an input vector u_i, $i \in Z_+$.

VI. SOLUTION OF THE DESCRIPTOR FRACTIONAL DISCRETE-TIME LINEAR SYSTEMS

Premultiplying (5.1) by the matrix $P \in \Re^{n\times n}$ and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} = Q^{-1}x_i, \; \bar{x}_i^{(1)} \in \Re^{n_1}, \; \bar{x}_i^{(2)} \in \Re^{n_2}, \; i \in Z_+ \tag{6.1}$$

we obtain

$$PEQQ^{-1}\Delta^{\alpha}x_{i+1} = PEQ\Delta^{\alpha}Q^{-1}x_{i+1} = PAQQ^{-1}x_i + PBu_i \tag{6.2}$$

and after using (5.5) and (6.1)

$$\begin{aligned} &\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \Delta^{\alpha} \begin{bmatrix} \bar{x}_{i+1}^{(1)} \\ \bar{x}_{i+1}^{(2)} \end{bmatrix} \\ &\quad = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, \; i \in Z_+ \end{aligned} \tag{6.3}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, \; B_1 \in \Re^{n_1\times m}, \; B_2 \in \Re^{n_2\times m}. \tag{6.4}$$

Taking into account (5.2) from (6.3) we obtain

$$\begin{aligned} \bar{x}_{i+1}^{(1)} &= -\sum_{k=1}^{i+1} (-1)^k \binom{\alpha}{k} \bar{x}_{i-k+1}^{(1)} + A_1\bar{x}_i^{(1)} + B_1 u_i \\ &= A_{1\alpha}\bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{\alpha}{k} \bar{x}_{i-k+1}^{(1)} + B_1 u_i \end{aligned} \tag{6.5}$$

and

$$N\left[\bar{x}_{i+1}^{(2)} + \sum_{k=1}^{i+1} (-1)^k \binom{\alpha}{k} \bar{x}_{i-k+1}^{(2)}\right] = \bar{x}_i^{(2)} + B_2 u_i \tag{6.6}$$

where $A_{1\alpha} = A_1 + I_{n_1}\alpha$.

The solution $\bar{x}_i^{(1)}$ to the equation (6.5) is well-known [14] and it is given by the theorem.

Theorem 6.1. The solution $\bar{x}_i^{(1)}$ of the equation (6.5) is given by the formula

$$\bar{x}_i^{(1)} = \Phi_i \bar{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, \; i \in Z_+ \tag{6.7}$$

where the matrices Φ_i are determined by the equation

$$\Phi_{i+1} = \Phi_i A_{1\alpha} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{\alpha}{k} \Phi_{i-k+1}, \quad \Phi_0 = I_{n_1}. \tag{6.8}$$

To find the solution $\bar{x}_i^{(2)}$ of (6.6) for $N \neq 0$ it is assumed that

$$N = \begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \in \Re^{n_2\times n_2}. \tag{6.9}$$

For (6.9) the equation (6.6) can be written in the form

$$\begin{aligned} &\begin{bmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1} (-1)^j \binom{\alpha}{j} \begin{bmatrix} \bar{x}_{i-j+1}^{(21)} \\ \bar{x}_{i-j+1}^{(22)} \\ \vdots \\ \bar{x}_{i-j+1}^{(2,n_2)} \end{bmatrix} \right) \\ &\quad = \begin{bmatrix} \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \\ \vdots \\ \bar{x}_i^{(2,n_2)} \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2,n_2} \end{bmatrix} u_i, \; i \in Z_+ \end{aligned} \tag{6.10}$$

From (6.10) we have

$$\begin{aligned}
\bar{x}_i^{(21)} &= -B_{21}u_i \\
\bar{x}_i^{(22)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\bar{x}_{i-j+1}^{(21)} - B_{22}u_i \\
&= -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}B_{21}u_{i-j+1} - B_{22}u_i \\
\bar{x}_i^{(23)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\bar{x}_{i-j+1}^{(22)} - B_{23}u_i \\
&= -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\sum_{k=0}^{i-j+2}(-1)^k\binom{\alpha}{k}B_{21}u_{i-j-k+2} \\
&\quad -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}B_{22}u_{i-j+1} - B_{23}u_i \\
&\vdots \\
\bar{x}_i^{(2,n_2)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\bar{x}_{i-j+1}^{(2,n_2-1)} - B_{2,n_2}u_i
\end{aligned} \tag{6.11}$$

If $N = 0$, then from (6.6) we have

$$\bar{x}_i^{(2)} = -B_2u_i,\ i \in Z_+ \tag{6.12}$$

This approach can be extended for

$$N = blockdiag\,[N_1 \quad N_2 \quad \dots \quad N_h] \tag{6.13}$$

where $N_k \in \Re^{p_k \times p_k}$ has the form (6.9) and $\sum_{k=1}^{h} p_k = n_2$.

If the matrix N has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \Re^{n_2 \times n_2} \tag{6.13'}$$

the considerations are similar (dual).

Note that the matrices (6.9) and (6.13′) are related by

$$N = S\bar{N}S \quad \text{where} \quad S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Knowing $\bar{x}_i^{(1)}$ and $\bar{x}_i^{(2)}$ we can find the desired solution of the (5.1) from (6.1)

$$x_i = Q\begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix},\ i \in Z_+ \tag{6.14}$$

VII. EXAMPLES OF DESCRIPTOR FRACTIONAL DISCRETE-TIME SYSTEMS

Example 7.1. Find the solution x_i of the descriptor fractional linear system (5.1) with the matrices

$$E = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix},\ A = \begin{bmatrix} 0.8 & 1.7 & 2.8 \\ 0.4 & 0.8 & 1.4 \\ 2.2 & 4.6 & 2.2 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{7.1}$$

for $\alpha = 0.5$, $u_i = u$, $i \in Z_+$ and $x_0 = [1 \quad 2 \quad -1]^T$ (T denotes the transpose).

It is easy to check that the matrices (7.1) satisfy the assumptions (5.4). In this case the matrices P and Q have the forms

$$P = \frac{1}{11}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 4 & 3 & -2 \end{bmatrix},\ Q = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.2}$$

and

$$\begin{aligned}
\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} &= PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} &= PAQ = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} &= \frac{1}{11}\begin{bmatrix} -4 \\ -3 \\ 6 \end{bmatrix}, \\
A_{1\alpha} = A_1 + I_{n_1}\alpha &= \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix},\ (n_1 = 2,\ n_2 = 1).
\end{aligned} \tag{7.3}$$

The equations (6.5) and (6.6) have the forms

$$\begin{aligned}
\bar{x}_{i+1}^{(1)} &= \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix}\bar{x}_i^{(1)} + \sum_{k=2}^{i+1}(-1)^{k-1}\binom{0.5}{k}\bar{x}_{i-k+1}^{(1)} \\
&\quad -\frac{1}{11}\begin{bmatrix} 4 \\ 3 \end{bmatrix}u_i,\ i \in Z_+
\end{aligned} \tag{7.4}$$

and

$$\bar{x}_i^{(2)} = -B_2u_i = -\frac{6}{11}u_i,\ i \in Z_+. \tag{7.5}$$

The solution $\bar{x}_i^{(1)}$ of (7.4) has the form

$$\bar{x}_i^{(1)} = \Phi_i \bar{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, i \in Z_+ \tag{7.6}$$

where

$$\Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Phi_1 = A_{1\alpha} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix},$$
$$\Phi_2 = A_{1\alpha}^2 - I_{m_1} \frac{\alpha(\alpha-1)}{2!} = \begin{bmatrix} 0.485 & 1.300 \\ 0 & 0.615 \end{bmatrix}, \dots \tag{7.7}$$

and

$$\bar{x}_0 = Q^{-1} x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix},$$
$$\bar{x}_0^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \bar{x}_0^{(2)} = [-1] \tag{7.8}$$

The derived solution of the descriptor fractional system (5.1) with (7.1) is given by

$$x_i = Q\bar{x}_i = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} \tag{7.9}$$

where $\bar{x}_i^{(1)}$ and $\bar{x}_i^{(2)}$ are determined by (7.6) and (7.5), respectively.

Example 7.2. Find the solution x_i of the descriptor fractional linear system (5.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \tag{7.10}$$

for $\alpha = 0.8$, arbitrary u_i, $i \in Z_+$ and $x_0 = [1 \quad 1 \quad 1]^T$.

It is easy to check that the matrices (7.10) satisfy the assumptions (5.4). In this case the matrices P and Q have the forms

$$P = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{7.11}$$

and

$$\begin{bmatrix} I_{m_1} & 0 \\ 0 & N \end{bmatrix} = PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = PAQ = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix},$$
$$A_{1\alpha} = A_1 + I_{m_1}\alpha = [1], \quad (n_1 = 1, n_2 = 2). \tag{7.12}$$

In this case the equations (6.5) and (6.6) have the forms

$$\bar{x}_{i+1}^{(1)} = \bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{0.8}{k} \bar{x}_{i-k+1}^{(1)} + [1 \quad 0] u_i, i \in Z_+ \tag{7.13}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} \begin{bmatrix} \bar{x}_{i-j+1}^{(21)} \\ \bar{x}_{i-j+1}^{(22)} \end{bmatrix} \right)$$
$$= \begin{bmatrix} \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} u_i, i \in Z_+ \tag{7.14}$$

and

$$\bar{x}_0 = Q^{-1} x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix},$$
$$\bar{x}_0^{(1)} = [1], \quad \bar{x}_0^{(2)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \tag{7.15}$$

The solution $\bar{x}_i^{(1)}$ of the equation (7.13) with $\bar{x}_0^{(1)} = 1$ can be easily found using (6.7) and (6.8).

From (7.14) we have

$$\bar{x}_i^{(21)} = [0 \quad -1] u_i, \quad i \in Z_+$$
$$\bar{x}_i^{(22)} = \sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} \bar{x}_{i-j+1}^{(21)} + [1 \quad -1] u_i, \quad i \in Z_+ \tag{7.16}$$

The total solution of the descriptor fractional system with (7.10) is given by

$$x_i = Q\bar{x}_i = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix} \tag{7.17}$$

where $\bar{x}_i^{(1)}$, $\bar{x}_i^{(21)}$ and $\bar{x}_i^{(22)}$ are determined by (7.13) and (7.16), respectively.

VIII. DECOMPOSITION OF DISCRETE-TIME LINEAR SYSTEMS

Consider the descriptor fractional linear system (5.1). Substituting (5.2) into (5.1) we obtain

$$\sum_{k=0}^{i+1} Ec_k x_{i-k+1} = Ax_i + Bu_i,\ i \in Z_+ \tag{8.1}$$

where

$$c_k = (-1)^k \binom{\alpha}{k} \tag{8.2}$$

The following elementary row operations will be used:

1. Multiplication of the i-th row by a real number c. This operation will be denoted by $L[i \times c]$.
2. Addition to the i-th row of the j-th row multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$.
3. Interchange of the i-th and j-th rows. This operation will be denoted by $L[i, j]$.

The elementary column operations denoted by R are defined In a similar way.

Applying the row elementary operations to (8.1) we obtain

$$\sum_{k=0}^{i+1} \begin{bmatrix} E_1 \\ 0 \end{bmatrix} c_k x_{i-k+1} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i,\ i \in Z_+ \tag{8.3}$$

where $E_1 \in \Re^{n_1 \times n}$ is full row rank and $A_1 \in \Re^{n_1 \times n}$, $A_2 \in \Re^{(n-n_1)\times n}$, $B_1 \in \Re^{n_1 \times m}$, $B_2 \in \Re^{(n-n_1)\times m}$. The equation (8.3) can be rewritten as

$$\sum_{k=0}^{i+1} E_1 c_k x_{i-k+1} = A_1 x_i + B_1 u_i \tag{8.4a}$$

and

$$0 = A_2 x_i + B_2 u_i \tag{8.4b}$$

Substituting i by $i + 1$ in (8.4b) we obtain

$$A_2 x_{i+1} = -B_2 u_{i+1} \tag{8.5}$$

The equations (8.4a) and (8.5) can be written in the form

$$\begin{aligned} \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_1 - c_1 E_1 \\ 0 \end{bmatrix} x_i - \begin{bmatrix} c_2 E_1 \\ 0 \end{bmatrix} x_{i-1} \\ &- \ldots - \begin{bmatrix} c_{i+1} E_1 \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1} \end{aligned} \tag{8.6}$$

If the matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \tag{8.7}$$

is singular then applying the row operations to (8.6) we obtain

$$\begin{aligned} \begin{bmatrix} E_2 \\ 0 \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_{20} \\ \bar{A}_{20} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ \bar{A}_{21} \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} A_{2,i} \\ \bar{A}_{2,i} \end{bmatrix} x_0 \\ &+ \begin{bmatrix} B_{20} \\ \bar{B}_{20} \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ \bar{B}_{21} \end{bmatrix} u_{i+1} \end{aligned} \tag{8.8}$$

where $E_2 \in \Re^{n_2 \times n}$ is full row rank with $n_2 \ge n_1$ and $A_{2,j} \in \Re^{n_2 \times n}$, $\bar{A}_{2,j} \in \Re^{(n-n_2)\times n}$, $j = 0, 1, \ldots, i$ $B_{2,k} \in \Re^{n_2 \times m}$, $\bar{B}_{2,k} \in \Re^{(n-n_2)\times m}$, $k = 0, 1$.

Note that the array

$$\begin{matrix} E_1 & A_1 - c_1 E_1 & -c_2 E_1 & \ldots & -c_{i+1} E_1 & B_1 & 0 \\ A_2 & 0 & 0 & \ldots & 0 & 0 & -B_2 \end{matrix} \tag{8.9}$$

corresponding to (8.6) can be obtained from

$$\begin{matrix} E_1 & A_1 - c_1 E_1 & -c_2 E_1 & \ldots & -c_{i+1} E_1 & B_1 \\ 0 & A_2 & 0 & \ldots & 0 & B_2 \end{matrix} \tag{8.10}$$

by the shuffle of A_2.

From (8.8) we have

$$0 = \bar{A}_{20} x_i + \bar{A}_{21} x_{i-1} + \ldots + \bar{A}_{2,i} x_0 + \bar{B}_{20} u_i + \bar{B}_{21} u_{i+1} \tag{8.11}$$

Substituting i by $i + 1$ (in state vector x and in input u) in (8.11) we obtain

$$\bar{A}_{20} x_{i+1} = -\bar{A}_{21} x_i - \ldots - \bar{A}_{2,i} x_1 - \bar{B}_{20} u_{i+1} - \bar{B}_{21} u_{i+2}. \tag{8.12}$$

From (8.8) and (8.12) we have

$$\begin{aligned} \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ -\bar{A}_{22} \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix} x_0 \\ &+ \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix} u_{i+2} \end{aligned} \tag{8.13}$$

If the matrix

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} \tag{8.14}$$

is singular then we repeat the procedure. Continuing this procedure after a finite number of steps p we obtain

$$\begin{aligned} \begin{bmatrix} E_p \\ 0 \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_{p,0} \\ \bar{A}_{p,0} \end{bmatrix} x_i + \begin{bmatrix} A_{p,1} \\ \bar{A}_{p,2} \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} A_{pi} \\ \bar{A}_{pi} \end{bmatrix} x_0 \\ &+ \begin{bmatrix} B_{p,0} \\ \bar{B}_{p,0} \end{bmatrix} u_i + \begin{bmatrix} B_{p,1} \\ \bar{B}_{p,1} \end{bmatrix} u_{i+1} + \ldots + \begin{bmatrix} B_{p,p-1} \\ \bar{B}_{p,p-1} \end{bmatrix} u_{i+p-1} \end{aligned} \tag{8.15}$$

where $E_p \in \Re^{n_p \times n}$ is full row rank, $A_{pj} \in \Re^{n_p \times n}$, $\bar{A}_{pj} \in \Re^{(n-n_p)\times n}$, $j = 0, 1, \ldots, p$ and $B_{pk} \in \Re^{n_p \times m}$, $\bar{B}_{pk} \in \Re^{(n-n_p)\times m}$, $k = 0, 1, \ldots, p-1$ with nonsingular matrix

$$\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix} \in \Re^{n\times n} \tag{8.16}$$

Using elementary column operations we may reduce the matrix (8.16) to the form

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}, \quad A_{21} \in \Re^{(n-n_p)\times n_p}. \tag{8.17}$$

Performing the same elementary operations on the matrix I_n we can find the matrix $Q \in \Re^{n\times n}$ such that

$$\begin{bmatrix} E_p \\ \bar{A}_{p,0} \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}. \tag{8.18}$$

Taking into account (8.18) and defining the new state vector

$$\tilde{x}_i = Q^{-1}x_i = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \tilde{x}_i^{(1)} \in \Re^{n_p}, \tilde{x}_i^{(2)} \in \Re^{n-n_p}, i \in Z_+ \tag{8.19}$$

from (8.15) we obtain

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= E_p x_{i+1} = E_p QQ^{-1}x_{i+1} \\ &= A_{p,0}QQ^{-1}x_i + A_{p,1}QQ^{-1}x_{i-1} + \ldots + A_{pi}QQ^{-1}x_0 \\ &\quad + B_{p,0}u_i + B_{p,1}u_{i+1} + \ldots + B_{p,p-1}u_{i+p-1} \\ &= [A_{p,0}^{(1)} \quad A_{p,0}^{(2)}]\begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix} + [A_{p,1}^{(1)} \quad A_{p,1}^{(2)}]\begin{bmatrix} \tilde{x}_{i-1}^{(1)} \\ \tilde{x}_{i-1}^{(2)} \end{bmatrix} \\ &\quad + \ldots + [A_{pi}^{(1)} \quad A_{pi}^{(2)}]\begin{bmatrix} \tilde{x}_0^{(1)} \\ \tilde{x}_0^{(2)} \end{bmatrix} \\ &\quad + B_{p,0}u_i + B_{p,1}u_{i+1} + \ldots + B_{p,p-1}u_{i+p-1} \\ &= A_{p,0}^{(1)}\tilde{x}_i^{(1)} + A_{p,0}^{(2)}\tilde{x}_i^{(2)} + \ldots + A_{pi}^{(1)}\tilde{x}_0^{(1)} + A_{pi}^{(2)}\tilde{x}_0^{(2)} \\ &\quad + B_{p,0}u_i + B_{p,1}u_{i+1} + \ldots + B_{p,p-1}u_{i+p-1}, \quad i \in Z_+ \end{aligned} \tag{8.20}$$

and

$$\begin{aligned} \tilde{x}_i^{(2)} &= -A_{21}\tilde{x}_i^{(1)} - \bar{A}_{p,1}^{(1)}\tilde{x}_{i-1}^{(1)} - \bar{A}_{p,1}^{(2)}\tilde{x}_{i-1}^{(2)} - \ldots - \bar{A}_{pi}^{(1)}\tilde{x}_0^{(1)} \\ &\quad - \bar{A}_{pi}^{(2)}\tilde{x}_0^{(2)} - \bar{B}_{p,0}u_i - \ldots - \bar{B}_{p,p-1}u_{i+p-1}, \quad i \in Z_+ \end{aligned} \tag{8.21}$$

where

$$\begin{aligned} &A_{pj}Q = [A_{pj}^{(1)} \quad A_{pj}^{(2)}], \quad \bar{A}_{pj} = [\bar{A}_{pj}^{(1)} \quad \bar{A}_{pj}^{(2)}], \\ &j = 0, 1, \ldots, i \end{aligned} \tag{8.22}$$

Substitution of (8.21) into (8.20) yields

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= \tilde{A}_{p,0}\tilde{x}_i^{(1)} + \ldots + \tilde{A}_{pi}\tilde{x}_0^{(1)} + \tilde{B}_{p,0}u_i \\ &\quad + \ldots + \tilde{B}_{p,p-1}u_{i+p-1}, \quad i \in Z_+ \end{aligned} \tag{8.23}$$

where

$$\begin{aligned} &\tilde{A}_{p,0} = A_{p,0}^{(1)} - A_{p,0}^{(2)}A_{21}, \ldots, \tilde{A}_{pi} = A_{pi}^{(1)} - A_{p,0}^{(2)}\bar{A}_{pi}^{(1)} \\ &\tilde{B}_{p,0} = B_{p,0} - A_{p,0}^{(2)}\bar{B}_{p,0}, \ldots, \tilde{B}_{p,p-1} = B_{p,p-1} - A_{p,0}^{(2)}\bar{B}_{p,p-1} \end{aligned} \tag{8.24}$$

The standard system described by (8.23) is called the dynamic part of system (8.1) and the system described by (8.21) is called the static part of system (8.1).

The procedure can be justified as follows. The elementary row operations do not change the rank of the matrix $[Ez - A]$. The substitution of i by $i + 1$ in (8.4b) and (8.11) also does not change the rank of the matrix $[Ez - A]$ since it is equivalent to multiplication of its lower rows by z and by assumption (5.4b) holds. Therefore, the following theorem has been proved.

Theorem 8.1. The descriptor fractional discrete-time linear system (8.1) satisfying the assumption (5.4) can be decomposed into the dynamic part (8.23) and static part (8.21).

Example 8.1. Consider the descriptor fractional linear system (5.1) for $\alpha = 0.5$ with

$$E = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix} \tag{8.25}$$

In this case the conditions (5.4) are satisfied since

$$\det E = 0 \text{ and}$$

$$\det[Ez - A] = \begin{vmatrix} 5z - 0.2 & -2 & 2z + 2 \\ 2z - 2 & -1 & z \\ z + 1.8 & 0 & 1 \end{vmatrix} = z - 0.2$$

Applying to the matrices (8.25) the following elementary row operations $L[1 + 2 \times (-2)]$, $L[3 + 1 \times (-1)]$ we obtain

$$\begin{aligned} [E \quad A \quad B] &= \begin{bmatrix} 5 & 0 & 2 & 0.2 & 2 & -2 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 1 & 0 & 0 & -1.8 & 0 & -1 & 2 & -1 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 0 & 0 & -3.8 & 0 & -2 & 3 & -2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} \end{aligned} \tag{8.26}$$

and the equations (8.4) have the form

$$\begin{aligned} &\sum_{k=0}^{i+1} c_k \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} x_{i-k+1} \\ &= \begin{bmatrix} -3.8 & 0 & -2 \\ 2 & 1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i \end{aligned} \tag{8.27a}$$

and

$$0=[2 \quad 0 \quad 1]x_i+[-1 \quad 1]u_i \tag{8.27b}$$

Using (8.2) we obtain $c_1=-\binom{\alpha}{1}=-\alpha=-0.5$, $c_2=(-1)^2\binom{\alpha}{2}=\frac{\alpha(\alpha-1)}{2!}=-\frac{1}{8}$, ..., $c_{i+1}=(-1)^{i-1}\frac{\alpha(\alpha-1)\dots(\alpha-i)}{(i+1)!}\Big|_{\alpha=0.5}$ and the equation (8.6) has the form

$$\begin{aligned}\begin{bmatrix}1&0&0\\2&0&1\\2&0&1\end{bmatrix}x_{i+1}&=\begin{bmatrix}-3.3&0&-2\\3&1&0.5\\0&0&0\end{bmatrix}x_i+\frac{1}{8}\begin{bmatrix}1&0&0\\2&0&1\\0&0&0\end{bmatrix}x_{i-1}\\&-\dots-c_{i+1}\begin{bmatrix}1&0&0\\2&0&1\\0&0&0\end{bmatrix}x_0\\&+\begin{bmatrix}3&-2\\-1&2\\0&0\end{bmatrix}u_i+\begin{bmatrix}0&0\\0&0\\1&-1\end{bmatrix}u_{i+1}\end{aligned} \tag{8.28}$$

The matrix $\begin{bmatrix}1&0&0\\2&0&1\\2&0&1\end{bmatrix}$ is singular.

Performing the elementary row operation $L[3+2\times(-1)]$ on (8.28) we obtain the following

$$\begin{aligned}\begin{bmatrix}1&0&0\\2&0&1\\0&0&0\end{bmatrix}x_{i+1}&=\begin{bmatrix}-3.3&0&-2\\3&1&0.5\\-3&-1&-0.5\end{bmatrix}x_i\\&+\frac{1}{8}\begin{bmatrix}1&0&0\\2&0&1\\-2&0&-1\end{bmatrix}x_{i-1}\\&-\dots-c_{i+1}\begin{bmatrix}1&0&0\\2&0&1\\-2&0&-1\end{bmatrix}x_0\\&+\begin{bmatrix}3&-2\\-1&2\\1&-2\end{bmatrix}u_i+\begin{bmatrix}0&0\\0&0\\1&-1\end{bmatrix}u_{i+1}\end{aligned} \tag{8.29}$$

The matrix

$$\begin{bmatrix}E_2\\ \bar{A}_{20}\end{bmatrix}=\begin{bmatrix}1&0&0\\2&0&1\\-3&-1&-0.5\end{bmatrix} \tag{8.30}$$

is nonsingular and to reduce this matrix to the form (8.17) we perform the elementary column operations $R[1+3\times(-2)]$, $R[2\times(-1)]$, $R[2, 3]$. The matrix Q has the form

$$Q=\begin{bmatrix}1&0&0\\0&0&-1\\-2&1&0\end{bmatrix}$$

and

$$\begin{aligned}\begin{bmatrix}E_2\\ \bar{A}_{20}\end{bmatrix}Q&=\begin{bmatrix}1&0&0\\2&0&1\\-3&-1&-0.5\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&-1\\-2&1&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\0&1&0\\-2&-0.5&1\end{bmatrix},\\ A_{21}&=[-2 \quad -0.5], \quad n_2=2\end{aligned}$$

The new state vector (8.19) is

$$\begin{aligned}\tilde{x}_i&=Q^{-1}x_i=\begin{bmatrix}1&0&0\\2&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}x_{1,i}\\x_{2,i}\\x_{3,i}\end{bmatrix}=\begin{bmatrix}\tilde{x}_i^{(1)}\\ \tilde{x}_i^{(2)}\end{bmatrix},\\ \tilde{x}_i^{(1)}&=\begin{bmatrix}x_{1,i}\\2x_{1,i}+x_{3,i}\end{bmatrix}, \quad \tilde{x}_i^{(2)}=-x_{2,i}\end{aligned} \tag{8.31}$$

In this case (8.20) and (8.21) have the forms

$$\begin{aligned}\tilde{x}_{i+1}^{(1)}&=\begin{bmatrix}0.7&-2\\2&0.5\end{bmatrix}\tilde{x}_i^{(1)}+\begin{bmatrix}0\\-1\end{bmatrix}\tilde{x}_i^{(2)}+\frac{1}{8}\tilde{x}_{i-1}^{(1)}\\&-\dots-c_{i+1}\tilde{x}_0^{(1)}+\begin{bmatrix}3&-2\\-1&2\end{bmatrix}u_i\end{aligned} \tag{8.32}$$

and

$$\begin{aligned}\tilde{x}_i^{(2)}&=[2 \quad 0.5]\tilde{x}_i^{(1)}+[0.25 \quad 0]\tilde{x}_{i-1}^{(1)}\\&+\dots+c_{i+1}[-2 \quad 0]\tilde{x}_0^{(1)}-[1 \quad -2]u_i-[1 \quad -1]u_{i+1}\end{aligned} \tag{8.33}$$

Substituting (8.33) into (8.32) we obtain

$$\begin{aligned}\tilde{x}_{i+1}^{(1)}&=\begin{bmatrix}0.7&-2\\0&0\end{bmatrix}\tilde{x}_i^{(1)}+\frac{1}{8}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\tilde{x}_{i-1}^{(1)}\\&-\dots-c_{i+1}\begin{bmatrix}1&0\\-2&1\end{bmatrix}\tilde{x}_0^{(1)}+\begin{bmatrix}3&-2\\0&0\end{bmatrix}u_i\\&+\begin{bmatrix}0&0\\1&-1\end{bmatrix}u_{i+1}\end{aligned} \tag{8.34}$$

The dynamic part of the system is described by (8.34) and the static part by (8.33).

IX. DECOMPOSITION OF CONTINUOUS-TIME LINEAR SYSTEMS

Consider the descriptor fractional continuous-time linear system (3.1a).

It is assumed that det $E = 0$ and (3.2) holds. Performing elementary row operations on the array

$$\begin{matrix} E & A & B \end{matrix} \tag{9.1}$$

(or equivalently (3.1a)) we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} \tag{9.2}$$

and

$$E_1 \frac{d^\alpha}{dt^\alpha} x(t) = A_1 x(t) + B_1 u(t) \tag{9.3a}$$

$$0 = A_2 x(t) + B_2 u(t) \tag{9.3b}$$

where $E_1 \in \Re^{r_1 \times n}$ has full row rank. Differentiation of (9.3b) with respect to time yields

$$A_2 \frac{d^\alpha}{dt^\alpha} x(t) = -B_2 \frac{d^\alpha}{dt^\alpha} u(t) \tag{9.4}$$

The equations (9.3a) and (9.4) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} x(t) = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} u(t) \tag{9.5}$$

The array

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{matrix} \tag{9.6}$$

can be obtained from (9.2) by the shuffling of A_2. If matrix $\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ is singular then we repeat the step of the procedure for (9.5) and after finite numbers of p steps (in a similar way as for discrete-time systems) we obtain

$$\begin{aligned} \begin{bmatrix} E_p \\ 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} x(t) &= \begin{bmatrix} A_p \\ \bar{A}_p \end{bmatrix} x(t) + \begin{bmatrix} B_{p0} \\ \bar{B}_{p0} \end{bmatrix} u(t) \\ &+ \begin{bmatrix} B_{p1} \\ \bar{B}_{p1} \end{bmatrix} \frac{d^\alpha}{dt^\alpha} u(t) \\ &+ \ldots + \begin{bmatrix} B_{p,p-1} \\ \bar{B}_{p,p-1} \end{bmatrix} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \end{aligned} \tag{9.7}$$

where $E_p \in \Re^{r_p \times n}$ has full row rank and the matrix

$$\begin{bmatrix} E_p \\ \bar{A}_p \end{bmatrix} \tag{9.8}$$

is nonsingular.

Using the elementary column operations we may reduce the matrix (9.8) to the form

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix} \in \Re^{(n-n_p) \times n_p} \tag{9.9}$$

and find the matrix $Q \in \Re^{n \times n}$ such that

$$\begin{bmatrix} E_p \\ \bar{A}_p \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}. \tag{9.10}$$

Defining the new state vector

$$\bar{x}(t) = Q^{-1} x(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}, \bar{x}_1(t) \in \Re^{n_p}, \bar{x}_2(t) \in \Re^{n-n_p} \tag{9.11}$$

from (9.7) we obtain

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} \bar{x}_1(t) &= A_p Q Q^{-1} x(t) + B_{p0} u(t) + B_{p1} \frac{d^\alpha}{dt^\alpha} u(t) \\ &\quad + \ldots + B_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \\ &= [A_{p1} \quad A_{p2}] \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + B_{p0} u(t) + B_{p1} \frac{d^\alpha}{dt^\alpha} u(t) \\ &\quad + \ldots + B_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \end{aligned} \tag{9.12a}$$

and

$$\begin{aligned} \bar{x}_2(t) &= -\bar{A}_{21} \bar{x}_1(t) - \bar{B}_{p0} u(t) - \bar{B}_{p1} \frac{d^\alpha}{dt^\alpha} u(t) \\ &\quad - \ldots - \bar{B}_{p,p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \end{aligned} \tag{9.12b}$$

where

$$[A_{p1} \quad A_{p2}] = A_p Q, \ A_{p1} \in \Re^{n_p \times n_p}, \ A_{p2} \in \Re^{n_p \times (n-n_p)}.$$

Substitution of (9.12b) into (9.12a) yields

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} \bar{x}_1(t) &= \bar{A}_1 \bar{x}_1(t) + \bar{B}_0 u(t) + \bar{B}_1 \frac{d^\alpha}{dt^\alpha} u(t) \\ &\quad + \ldots + \bar{B}_{p-1} \frac{d^{(p-1)\alpha}}{dt^{(p-1)\alpha}} u(t) \end{aligned} \tag{9.13a}$$

where

$$\begin{aligned} &\bar{A}_1 = A_{p1} - A_{p2} \bar{A}_{21}, \quad \bar{B}_0 = B_{p0} - A_{p2} \bar{B}_{p0}, \\ &\bar{B}_1 = B_{p1} - A_{p2} \bar{B}_{p1}, \ldots, \bar{B}_{p-1} = B_{p,p-1} - A_{p2} \bar{B}_{p,p-1}. \end{aligned} \tag{9.13b}$$

The standard system described by the equation (9.13a) is called the dynamic part of the system (3.1a) and the system described by the equation (9.12b) is called the static part of the system (3.1a). The procedure can be justified in a similar way as for the discrete-time systems.

Therefore, the following theorem has been proved.

Theorem 9.1. The descriptor fractional continuous-time linear system (3.1a), satisfying the assumption (3.2), can be decomposed into the dynamic part (9.13a) and the static part (9.12b).

Example 9.1. Consider the descriptor fractional linear system (3.1a) with matrices

$$E=\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix},\quad A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\quad B=\begin{bmatrix}1\\-1\\2\end{bmatrix} \tag{9.14}$$

The matrices (9.14) satisfy the condition (3.2) since

$$\det[Es-A]=\begin{vmatrix}s&-1&s\\-1&s&0\\0&0&-1\end{vmatrix}=-s^2+1 \tag{9.15}$$

From (9.14) we have

$$E_1\frac{d^\alpha}{dt^\alpha}x(t)=A_1x(t)+B_1u(t) \tag{9.16a}$$

$$0=A_2x(t)+B_2u(t) \tag{9.16b}$$

where

$$E_1=\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix},\quad A_1=\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix},\quad B_1=\begin{bmatrix}1\\-1\end{bmatrix},$$
$$A_2=[0\quad 0\quad 1],\quad B_2=[2].$$

Differentiation with respect to time of (9.16b) yields

$$A_2\frac{d^\alpha}{dt^\alpha}x(t)=-B_2\frac{d^\alpha}{dt^\alpha}u(t) \tag{9.17}$$

The equations (9.16a) and (9.17) can be written in the form

$$\begin{bmatrix}E_1\\A_2\end{bmatrix}\frac{d^\alpha}{dt^\alpha}x(t)=\begin{bmatrix}A_1\\0\end{bmatrix}x(t)+\begin{bmatrix}B_1\\0\end{bmatrix}u(t)+\begin{bmatrix}0\\-B_2\end{bmatrix}\frac{d^\alpha}{dt^\alpha}u(t) \tag{9.18}$$

The matrix

$$\begin{bmatrix}E_1\\A_2\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \tag{9.19}$$

is nonsingular. Performing the elementary column operation $R[3+1\times(-1)]$ on (9.19) we obtain the identity matrix I_3 and

$$Q=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix} \tag{9.20}$$

such that

$$\begin{bmatrix}E_1\\A_2\end{bmatrix}Q=I_3. \tag{9.21}$$

Defining the new state vector

$$\bar{x}(t)=Q^{-1}x(t)=\begin{bmatrix}\bar{x}_1(t)\\\bar{x}_2(t)\end{bmatrix},\ \bar{x}_1(t)=\begin{bmatrix}x_1(t)+x_3(t)\\x_2(t)\end{bmatrix},\quad \bar{x}_2(t)=x_3(t) \tag{9.22}$$

from (9.18) we obtain

$$\begin{aligned}\frac{d^\alpha}{dt^\alpha}\bar{x}_1(t)&=E_1Q\bar{x}(t)=A_1\bar{x}(t)+B_1u(t)\\&=\begin{bmatrix}0&1\\1&0\end{bmatrix}\bar{x}_1(t)+\begin{bmatrix}1\\-1\end{bmatrix}u(t)\end{aligned} \tag{9.23a}$$

$$\bar{x}_2(t)=x_3(t)=-2u(t) \tag{9.23b}$$

The dynamic part of the system is described by the equation (9.23a) and the static part by the equation (9.23b).

X. CONCLUDING REMARKS

The descriptor fractional linear systems and electrical circuits have been introduced. Using the Caputo definition of the fractional derivative, the Weierstrass regular pencil decomposition and the Laplace transform the solution to the state equation of descriptor fractional linear system has been derived (Theorem 3.1). Descriptor fractional linear electrical circuits have been analyzed. It has been shown that every electrical circuit is a descriptor fractional system if it contains at least one mesh consisting of branches with only ideal supercondensators and voltage sources or at least one node with branches containing supercoils (Theorem 4.1). The descriptor fractional linear discrete-time systems have been introduced. Using the Weierstrass regular pencil decomposition the solution to the state equation of descriptor fractional linear discrete-time system has been derived. The method of finding of the solution to the descriptor fractional systems has been illustrated by two examples. The considerations have been illustrated by descriptor linear electrical circuits. A method for decomposition of the descriptor fractional discrete-time and continuous-time linear systems with regular pencils into dynamic and static parts has been proposed. Those considerations can be extended for descriptor fractional linear systems with singular pencils. Open problems are extension of these considerations for positive descriptor fractional linear systems and for descriptor positive linear systems with different fractional order. The linear systems with different fractional orders are described by the equation [11]

$$\begin{bmatrix} \dfrac{d^\alpha x_1}{dt^\alpha} \\ \dfrac{d^\beta x_2}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \tag{10.1}$$
$$p-1<\alpha<p; \quad q-1<\beta<q; \quad p,q \in N$$

where $x_1 \in \Re^{n_1}$, $x_2 \in \Re^{n_2}$ are the state vectors and $A_{ij} \in \Re^{n_i \times n_j}$, $B_i \in \Re^{n_i \times m}$, $i, j = 1,2$; and $u \in \Re^m$ is the input vector. Initial conditions for (10.1) have the form $x_2(0) = x_{10}$ and $x_2(0) = x_{20}$.

REFERENCES

1. Dodig, M. and M. Stosic, "Singular systems state feedbacks problems," *Lin. Alg. Applicat.*, Vol. 431, No. 8, pp. 1267–1292 (2009).
2. Wang, Cuihong, "New delay-dependent stability criteria for descriptor systems with interval time delay," *Asian J. Control*, Vol. 14, No. 1, pp. 197–206 (2012).
3. Dai, L., *Singular control systems*, Lectures Notes in Control and Information Sciences, Springer-Verlag, Berlin (1989).
4. Fahmy, M.H and J. O'Reill, "Matrix pencil of closed-loop descriptor systems: infinite-eigenvalues assignment," *Int. J. Control*, Vol. 49, No. 4, pp. 1421–1431 (1989).
5. Gantmacher, F. R., *The theory of Matrices*, Chelsea Publishing Co., New York (1960).
6. Kaczorek, T., "Positive linear systems consisting of n subsystems with different fractional orders," *IEEE Trans. Circ. Syst.*, Vol. 58, No. 7, pp. 1203–1210 (2011).
7. Kaczorek, T., "Fractional positive continuous-time linear systems and their reachability," *Int. J. Appl. Math. Comput. Sci.* Vol. 18, No. 2, pp. 223–228 (2008).
8. Kaczorek, T., "Infinite eigenvalue assignment by output-feedbacks for singular systems," *Int. J. Appl. Math. Comput. Sci.* Vol. 14, No. 1, pp. 19–23 (2004,).
9. Kaczorek, T., *Linear control systems*, Vol. 1, Research Studies Press J. Wiley, New York (1992).
10. Kaczorek, T., *Polynomial and rational matrices. Applications in dynamical systems theory*, Springer-Verlag, London (2007).
11. Kaczorek, T., "Positive linear systems with different fractional orders," *Bull. Pol. Ac. Sci. Techn.* Vol. 58, No. 3, pp. 453–458 (2010).
12. Kaczorek, T., "Practical stability and asymptotic stability of positive fractional 2D linear systems," *Asian J. Control*, Vol. 12, No. 2, pp. 200–207 (2010).
13. Kaczorek, T., "Realization problem for singular positive continuous-time systems with delays," *Control Cybern.*, Vol. 36, No. 1, pp. 47–57 (2007).
14. Kaczorek, T., *Selected Problems of Fractional System Theory*, Springer-Verlag, Berlin (2011).
15. Kucera, V. and P. Zagalak, "Fundamental theorem of state feedback for singular systems," *Automatica* Vol. 24, No. 5, pp. 653–658 (1988).
16. Luenberger, D.G., "Time-invariant descriptor systems," *Automatica*, Vol.14, pp. 473–480 (1978).
17. Podlubny, I., *Fractional differential equations*, Academic Press, New York (1999).
18. Wei, S., L. Yan, L. Changpin, and C. YangQuan, "Convergence speed of a fractional order consensus algorithm over undirected scale-free networks," *Asian J. Control*, Vol. 13, No. 6, pp. 936–946 (2011).
19. Van Dooren, P., "The computation of Kronecker's canonical form of a singular pencil," *Lin. Alg. Applicat.*, Vol. 27, pp. 103–140 (1979).
20. Yan, L., C. YangQuan, and A. Hyo-Sung, "Fractional-order iterative learning control for fractional-order systems," *Asian J. Control*, Vol. 13, No. 1, pp. 54–63 (2011).

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