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On implicit Euler for high-order high-index DAEs

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Abstract

The Implicit Euler method is seldom used to solve differential–algebraic equations (DAEs) of differential index $r \geqslant 3$, since the method in general fails to converge in the first $r-2$ steps after a change of stepsize. However, if the differential equation is of order $d = r-1 \geqslant 1$, an alternative variable-step version of the Euler method can be shown uniformly convergent. For $d = r-1$, this variable-step method is equivalent to the Implicit Euler except for the first $r-2$ steps after a change of stepsize. Generalization to DAEs with differential equations of order $d > r-1 \geqslant 1$, and to variable-order formulas is discussed.

Keywords: Linear multistep method; Backward Differentiation Formula; Differential–algebraic equation; Differential index; Initial value problem; Divided difference

1. Introduction

According to [3, p. 46], those systems of differential–algebraic equations (DAEs) which arise most commonly in applications are the index one systems, the semi-explicit index two systems and the index three systems in Hessenberg form. However, systems of arbitrarily high index may occur naturally in mathematical models [3, p. 150], and thus methods for such systems are of interest. In this paper we consider DAEs of order $d \geqslant r-1 \geqslant 1$, where r is the (differential) index.

Several codes for DAEs (e.g., DASSL [9], LSODI [7] and SPRINT [2]) have been based on the Backward Differentiation Formulas (BDFs), of which the first-order formula (Implicit Euler) plays a central role—at least in the beginning of the integration. However, it has been known for a long time (cf., e.g., [5,6,4]) that Implicit Euler in general fails to converge in the first $r-2$ steps after a change of stepsize, where the initial point may be regarded as one of the positions, where the stepsize is changed (from 0 to a positive value). In [1] an algorithm for correcting the numerical values after stepchanges was derived for $r = 3$. However, the algorithm assumes the DAE to depend linearly on the algebraic variables, and consecutive stepchanges seem to worsen the corrected values. In this paper we will derive

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Table 1
Comparison with results listed in Table 8.3 of [1]

Step no.	Stepsize $\times 10^3$	Value of alg. var.	Absolute error of approximation (Euler)	(Corrected)	(Alt. Euler)
1	1.000	−4.0080	**2.0080**	**0.0044**	0.0080
2	1.000	−4.0160	0.0080	0.0080	0.0120
3	0.200	−4.0176	**8.0303**	**0.0391**	0.0057
4	0.040	−4.0179	**8.0348**	**0.2121**	0.0012
5	0.008	−4.0180	**8.0357**	**1.0725**	0.0003
6	0.008	−4.0181	0.0001	0.0001	0.0001
7	0.016	−4.0182	**1.0047**	**0.0001**	0.0002
8	0.032	−4.0185	**1.0048**	**0.0002**	0.0004
9	0.064	−4.0190	**1.0052**	**0.0003**	0.0007
10	0.064	−4.0195	0.0006	0.0006	0.0008

an alternative variable-step version of the Implicit Euler method applicable to dth order DAEs of index $r \in [2, d+1]$, and for $(d, r) = (2, 3)$ we may compare the errors produced by this method to those of Implicit Euler with/without correction, listed in Table 8.3 of [1] (cf. Table 1 above).

Consider the initial value problem

$$y^{(d)} = f\big(t, y, y', \ldots, y^{(d-1)}, \lambda\big), \quad y^{(j)}(t_0) = \eta_j, \quad j = 0, 1, \ldots, d-1, \tag{1}$$

$$0 = g\big(t, y, y', \ldots, y^{(d+1-r)}\big), \quad r \in [2, d+1]. \tag{2}$$

Most often, high-order ordinary differential equations (ODEs) are solved by transforming the equation to a system of first-order ODEs, and then by applying some of the many methods for first-order ODEs, e.g., the Implicit Euler. It thus seems natural to consider the following 'Implicit Euler method' for producing the approximations $(y_{0,n}, \lambda_n)$ to the values $(y(t_n), \lambda(t_n))$, $n = 1, 2, 3, \ldots,$ of the DAE-solution:

$$\begin{aligned} &(y_{j,n} - y_{j,n-1})/(t_n - t_{n-1}) = y_{j+1,n}, \quad j = 0, 1, \ldots, d-2, \\ &(y_{d-1,n} - y_{d-1,n-1})/(t_n - t_{n-1}) = f(t_n, y_{0,n}, y_{1,n}, \ldots, y_{d-1,n}, \lambda_n), \\ &0 = g(t_n, y_{0,n}, y_{1,n}, \ldots, y_{d+1-r,n}), \end{aligned} \tag{3}$$

where $y_{j,0} = \eta_j$ for $j = 0, 1, \ldots, d-1$.

However, methods for systems of first-order ODEs are designed to estimate each component of the solution with the *same* order of accuracy, and for low-order methods (such as Implicit Euler) the accuracy of the $y(t_n)$-estimate is in general too low for producing reasonable estimates of the *derivatives* of y and thus of λ.

Another approach is to *exchange the* '*equation order reduction*' *and the discretization*! If we thus discretize the DAE (1), (2) by using divided differences, and then write the *discretized* equations as a system of equations, we obtain for the approximations $(y_{j,n}, \lambda_n) \approx (j!y[t_n, t_{n-1}, \ldots, t_{n-j}], \lambda(t_n))$

$$\begin{aligned} &(y_{j,n} - y_{j,n-1})/\big((t_n - t_{n-1-j})/(j+1)\big) = y_{j+1,n}, \quad j = 0, 1, \ldots, d-2, \\ &(y_{d-1,n} - y_{d-1,n-1})/\big((t_n - t_{n-d})/d\big) = f(t_n, y_{0,n}, y_{1,n}, \ldots, y_{d-1,n}, \lambda_n), \\ &0 = g(t_n, y_{0,n}, y_{1,n}, \ldots, y_{d+1-r,n}), \end{aligned} \tag{4}$$

where $y_{j,0} = \eta_j$ for $j = 0, 1, \ldots, d-1$, and t_m is interpreted as t_0 for m negative.

We notice that (4) only differs from (3) in $d-1$ steps after a change of stepsize, and for $r=d+1\geqslant 3$, this corresponds to the case where (3) fails to converge. Hence, one might expect (4) to remedy this lack of convergence. However, as seen in Example 1 below, (4) must be modified for $r\in[2,d]$, since the accuracy of the $y(t_n)$-estimate may then be affected by the lower accuracy of the estimates of the derivatives via the algebraic condition.

Example 1. Consider the following DAE of order $d=3$ and index $r=3$:

$$\begin{aligned}
&y^{(3)}(t)=\lambda(t), \qquad y^{(0)}(0)=y^{(1)}(0)=y^{(2)}(0)=1,\\
&0=a\exp(t)+by^{(0)}(t)-(a+b)y^{(1)}(t), \quad |a|+|b|>0,
\end{aligned}$$

for which the solution is $y(t)=\lambda(t)=\exp(t)$. Applying method (4), the approximations in the first grid point $t_1=h$ will satisfy the equations

$$\begin{aligned}
&6\left(y_{0,1}-1-h-\frac{1}{2}h^2\right)\Big/h^3=\lambda_1,\\
&0=a\exp(h)+by_{0,1}-(a+b)(y_{0,1}-1)/h.
\end{aligned}$$

Hence,

$$\lambda_1=\begin{cases}3!\exp[h,0,0,0] & \text{if } a+b=0,\\ 3h^{-1}+\mathcal{O}(1) & \text{otherwise,}\end{cases}$$

and we have no (uniform) convergence for $a+b\neq 0$.

On the other hand, if $y^{(1)}(h)$ *in the constraint* is approximated by using the third-order BDF formula

$$y^{(1)}(t_n)\approx y[t_n,t_{n-1}]+(t_n-t_{n-1})\big\{y[t_n,t_{n-1},t_{n-2}]+(t_n-t_{n-2})y[t_n,t_{n-1},\ldots,t_{n-3}]\big\},$$

$y_{0,n}$ will for $n\geqslant 3$ become a BDF3-solution of the ODE

$$(a+b)y'(t)=by(t)+a\exp(t),$$

and for *constant* stepsize h, λ_n will for $n\geqslant 6$ become a BDF3-solution of

$$(a+b)\lambda'(t)=b\lambda(t)+a(3!)\exp[t,t-h,t-2h,t-3h].$$

Hence, we obtain convergence for *fixed* stepsize, provided the starting values $y_{j,0}$ are chosen $\mathcal{O}(h^{4-j})$-accurate, as this implies $\mathcal{O}(h)$-accuracy of λ_3,λ_4 and λ_5.

For *variable* stepsize, however, third-order accuracy of $y_{0,n}$ does not necessarily imply first-order accuracy of λ_n, and one might think of using the BDF4-formula in the constraint, assuming that an $\mathcal{O}(H)$-accurate estimate of the initial value $\lambda(0)$ is known, as well as $\mathcal{O}(H^{4-j})$-estimates of $y^{(j)}(0)$, where H is a finite upper bound of the stepsizes. We will, however, leave this possibility for further research.

Example 1 indicates that method (4) should be modified for $r\in[2,d]$ in the following way:

$$\begin{aligned}
&(y_{j,n}-y_{j,n-1})/\big((t_n-t_{n-1-j})/(j+1)\big)=y_{j+1,n}, \quad j=0,1,\ldots,d-2,\\
&(y_{d-1,n}-y_{d-1,n-1})/\big((t_n-t_{n-d})/d\big)=f(t_n,y_{0,n},y_{1,n},\ldots,y_{d-1,n},\lambda_n),\\
&0=g\big(t_n,y_{0,n},p_n'(t_n),\ldots,p_n^{(d+1-r)}(t_n)\big),
\end{aligned}\tag{5}$$

where $y_{j,0} = \eta_j$ for $j = 0, 1, \ldots, d-1$, t_m is interpreted as t_0 for m negative, and p_n is an interpolation polynomial, which—in case the BDFd-formula is used for estimating $y'(t_n)$—reads

$$\sum_{i=0}^{d-1}\prod_{j=0}^{i-1}\left(\frac{t-t_{n-j}}{j+1}\right)y_{i,n} + \prod_{j=0}^{d-1}\left(\frac{t-t_{n-j}}{j+1}\right)f(t_n, y_{0,n}, y_{1,n}, \ldots, y_{d-1,n}, \lambda_n).$$

In Section 2 we list the assumptions on the DAE (1), (2), ensuring a unique local DAE-solution, and show that for fixed $n \geqslant 1$ (5) has a unique solution within a neighbourhood of the DAE-solution provided the previous numerical values $(y_{j,n-i}, \lambda_{n-i})$, $i \geqslant 1$, are sufficiently accurate, satisfying the algebraic condition to a certain accuracy, and the stepsizes are sufficiently small with bounded ratios. In Section 3 we restrict ourselves to the case $d = r-1 \geqslant 1$ and show that for sufficiently accurate starting values and small stepsizes with bounded ratios, the numerical values will remain accurate, since the method (5) is then (uniformly) convergent. In Section 4 we outline how method (5) may be generalized to variable-step variable-order methods based on the approach of discretization *prior* to any equation order reduction of the DAE. Experiments indicate that contrary to the BDFs (cf. [1]), the order of convergence of these new methods does not drop when the stepsize and/or order is changed in a proper way.

2. Existence and uniqueness of solutions to (1), (2) and (5)

Let $g^{(i)}$, $i = 0, 1, \ldots, r-1$, be formally defined as

$$g^{(i)}\big(t, y(t), y'(t), \ldots, y^{(d+1-r+i)}(t)\big) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{(i)} g\big(t, y(t), y'(t), \ldots, y^{(d+1-r)}(t)\big).$$

The assumptions on the DAE (1), (2) can then be written as follows.

Assumptions.

(1) f and $g^{(r-1)}$ are C^1-functions with bounded and Lipschitz-continuous partial derivatives on open sets Ω_1, Ω_2, containing $v_0 = (t_0, \eta_0, \ldots, \eta_{d-1}, \lambda_0)$ and $(t_0, \eta_0, \ldots, \eta_{d-1}, f(v_0))$, respectively, where λ_0 is the unique value of $\lambda(t_0)$ (cf., (3) below).

(2) The initial values $\eta_0, \ldots, \eta_{d-1}$ are consistent with the equations

$$0 = g^{(i)}(t_0, \eta_0, \ldots, \eta_{d+1-r+i}), \quad i = 0, 1, \ldots, r-2.$$

(3) There exists a unique solution $\lambda(t_0) = \lambda_0$ to the equation

$$0 = g^{(r-1)}\big(t_0, \eta_0, \ldots, \eta_{d-1}, f\big(t_0, \eta_0, \ldots, \eta_{d-1}, \lambda(t_0)\big)\big),$$

or a solution $\lambda(t_0) = \lambda_0$ is given as initial value.

(4) The matrix

$$\frac{\partial}{\partial\lambda(t)} g^{(r-1)}\big(t, y_0^{[2]}(t), \ldots, y_{d-1}^{[2]}(t), f\big(t, y_0^{[1]}(t), \ldots, y_{d-1}^{[1]}(t), \lambda(t)\big)\big)$$

is regular with bounded inverse for all $v(t) = (t, y_0^{[1]}(t), \ldots, y_{d-1}^{[1]}(t), \lambda(t)) \in \Omega_1$, $(t, y_0^{[2]}(t), \ldots, y_{d-1}^{[2]}(t), f(v(t))) \in \Omega_2$.

Since the low derivatives $y^{(i)}(t)$, $i = 0, 1, \dots, d-1$, may be formally expressed in terms of $y^{(d)}(t)$:

$$y^{(i)}(t) = \sum_{j=0}^{d-1-i} \frac{(t-t_0)^j}{j!}\,\eta_{i+j} + \int_{t_0}^{t} \frac{(t-s)^{d-1-i}}{(d-1-i)!}\, y^{(d)}(s)\,\mathrm{d}s, \tag{6}$$

the assumptions above are easily seen to ensure a unique local solution to the DAE by considering the following iteration for $k = 0, 1, \dots$

$$\begin{aligned}
&y_0^{(d)}(t) \equiv 0,\\
&y_{k+1}^{(d)}(t) = f\big(t, y_k(t), y_k'(t), \dots, y_k^{(d-1)}(t), \lambda_k(t)\big)\\
&0 = \frac{\mathrm{d}}{\mathrm{d}t}\, g^{(r-1)}\big(t, y_{k+1}(t), \dots, y_{k+1}^{(d-1)}(t), y_{k+1}^{(d)}(t)\big), \qquad \lambda_k(t_0) = \lambda_0,
\end{aligned}$$

where $y_m^{(i)}(t)$ denotes (6) with $y_m^{(d)}$ substituted for $y^{(d)}$, $m = k, k+1$.

As concerns the solution of (5), we note that

$$\begin{aligned}
&p_n(t) = p_{n-1}(t) + q_{n-1}(t)d!(p_n - p_{n-1})[t_n, \dots, t_{n-d}],\\
&q_{n-1}(t) = \prod_{j=0}^{d-1}\left(\frac{t - t_{n-1-j}}{j+1}\right).
\end{aligned} \tag{7}$$

Since $q_{n-1}(t) = 0$ for $t = t_{n-1}, t_{n-2}, \dots, t_{n-d}$, we thus have a discrete analogue to (6):

$$\begin{aligned}
i!p_n[t_n, \dots, t_{n-i}] = {}& \sum_{j=0}^{d-1-i} \prod_{k=i}^{i+j-1}\left(\frac{t_n - t_{n-1-k}}{k+1}\right)(i+j)!p_{n-1}[t_{n-1}, \dots, t_{n-1-i-j}]\\
&+ \prod_{k=i}^{d-1}\left(\frac{t_n - t_{n-1-k}}{k+1}\right)d!p_n[t_n, \dots, t_{n-d}],
\end{aligned} \tag{8}$$

and we try to find a solution of (5) by simple functional iteration:

$$\begin{aligned}
&p_{n,0}(t) \equiv p_{n-1}(t),\\
&d!p_{n,k+1}[t_n, \dots, t_{n-d}] = f\big(t_n, p_{n,k}[t_n], \dots, (d-1)!p_{n,k}[t_n, \dots, t_{n-d+1}], \lambda_{n,k}\big),\\
&0 = g\big(t_n, p_{n,k+1}(t_n), \dots, p_{n,k+1}^{(d+1-r)}(t_n)\big)/q_{n-1}^{(d+1-r)}(t_n), \quad k = 0, 1, \dots,
\end{aligned} \tag{9}$$

where $p_{n,k+1}^{(i)}(t)$ denotes the ith derivative of (7) with $p_{n,k+1}[t_n, \dots, t_{n-d}]$ substituted for $p_n[t_n, \dots, t_{n-d}]$, and the ith order divided difference is found from the dth order through (8).

Lemma 2. *Assume that the unique solution of* (1), (2), *ensured by our* Assumptions, *exists for* $t \in [t_0, t_{n-1} + H]$, *where* $H < \infty$ *is an upper bound of the stepsizes* $t_i - t_{i-1}$, $i \geqslant 1$, *and that the DAE-solution remains within* Ω_1, Ω_2.

If for $j = 0, 1, \dots, d-1$, $m = 1, 2, \dots, n-1$,

(i) $y_{j,0} = y^{(j)}(t_0) + \mathcal{O}(H)(t_1 - t_0)^{d-j}$,
(ii) $y_{j,m} = y^{(j)}(t_m) + \mathcal{O}(H)$, $\lambda_m = \lambda(t_m) + \mathcal{O}(H)$,

(iii) $g(t_m, p_m(t_m), \ldots, p_m^{(d+1-r)}(t_m))/q_{m-1}^{(d+1-r)}(t_m) = \mathcal{O}(H)$,
(iv) $(t_{m+1} - t_m)/(t_m - t_{m-1}) \in [\gamma, \Gamma]$ *for* $0 < \gamma \leqslant \Gamma < \infty$,

then the iteration (9) *converges for sufficiently small* H *to the solution of* (5) *satisfying* $\lambda_n = \lambda_{n-1} + \mathcal{O}(H)$.

Proof. First we prove that for sufficiently small H, a unique $\lambda_{n,0} = \lambda_{n-1} + \mathcal{O}(H)$ exists, and that $\|(p_{n,1} - p_{n-1})[t_n, \ldots, t_{n-d}]\|$ is $\mathcal{O}(H)$. Then we show, by induction in $k \geqslant 1$, the existence of a unique $\lambda_{n,k}$ satisfying $\|\lambda_{n,k} - \lambda_{n,k-1}\| = \mathcal{O}(H)\|(p_{n,k} - p_{n,k-1})[t_n, \ldots, t_{n-d}]\|$, and that $\|(p_{n,k+1} - p_{n,k})[t_n, \ldots, t_{n-d}]\| = \mathcal{O}(H)\|(p_{n,k} - p_{n,k-1})[t_n, \ldots, t_{n-d}]\|$. Hence, for sufficiently small H, the Cauchy sequence $(\lambda_{n,k}, p_{n,k+1}[t_n, \ldots, t_{n-d}])_k$ will converge to a fixpoint $(\lambda_n, p_n[t_n, \ldots, t_{n-d}])$ of (9), since f and g are continuous. That (9) has no other fixpoints with $\lambda_n = \lambda_{n-1} + \mathcal{O}(H)$ follows from the boundedness of $(\partial g^{(r-1)}/\partial\lambda)^{-1}$ and the partial derivatives of f, which is valid for sufficiently small H.

Let $k \geqslant 0$ and $p_{n,k}$ be given with $\|(p_{n,k} - p_{n-1})[t_n, \ldots, t_{n-d}]\|$ being $\mathcal{O}(H)$. In order to find $\lambda_{n,k} = \lambda_{n-1} + \mathcal{O}(H)$, we use the iterative scheme

$$\lambda_{n,k}^{[j+1]} = \lambda_{n,k}^{[j]} - \left[\frac{\partial G_{n,k}}{\partial\lambda}\left(\lambda_{n,k}^{[0]}\right)\right]^{-1} G_{n,k}\left(\lambda_{n,k}^{[j]}\right), \quad j = 0, 1, \ldots, \qquad \lambda_{n,k}^{[0]} = \lambda_{n-1}, \tag{10}$$

where

$$G_{n,k}(\lambda) = g\left(t_n, \left(p_{n-1}^{(i)}(t_n) + q_{n-1}^{(i)}(t_n)\Delta f_{n,k}(\lambda)\right)_{i=0}^{d+1-r}\right)/q_{n-1}^{(d+1-r)}(t_n),$$

and $\Delta f_{n,k}(\lambda)$ denotes

$$f\left(t_n, \left(s!p_{n,k}[t_n, \ldots, t_{n-s}]\right)_{s=0}^{d-1}, \lambda\right) - f\left(t_{n-1}, \left(s!p_{n-1}[t_{n-1}, \ldots, t_{n-1-s}]\right)_{s=0}^{d-1}, \lambda_{n-1}\right).$$

Since $p_{n,k}(t)$ is defined in (7) with $p_{n,k}[t_n, \ldots, t_{n-d}]$ substituted for $p_n[t_n, \ldots, t_{n-d}]$, we find, for $s = 0, 1, \ldots, d-1$, that $p_{n,k}[t_n, \ldots, t_{n-s}] - p_{n-1}[t_{n-1}, \ldots, t_{n-1-s}]$ equals

$$(t_n - t_{n-1-s})p_{n-1}[t_n, \ldots, t_{n-1-s}] + \prod_{i=s}^{d-1}(t_n - t_{n-1-i})\mathcal{O}(H) = \mathcal{O}(H). \tag{11}$$

Hence,

$$\frac{\partial G_{n,k}}{\partial\lambda}\left(\lambda_{n,k}^{[0]}\right) = \sum_{i=0}^{d+1-r} \frac{q_{n-1}^{(i)}(t_n)}{q_{n-1}^{(d+1-r)}(t_n)} M_{i,n-1}(t_n),$$

where

$$M_{i,n-1}(t_n) = \frac{\partial g}{\partial y^{(i)}}\left(t_n, \left(p_{n-1}^{(s)}(t_n) + \mathcal{O}(H)\right)_{s=0}^{d+1-r}\right)\frac{\partial f}{\partial\lambda}\left(t_n, \left(y_{s,n-1} + \mathcal{O}(H)\right)_{s=0}^{d-1}, \lambda_{n-1}\right),$$

and

$$\frac{q_{n-1}^{(i)}(t_n)}{q_{n-1}^{(d+1-r)}(t_n)} \leqslant \frac{(t_n - t_{n-1-i})\cdots(t_n - t_{n-d})d!/(d-i)!}{(t_n - t_{n-2-d+r})\cdots(t_n - t_{n-d})(d+1-r)!} = \mathcal{O}(H), \quad i = 0, 1, \ldots, d-r.$$

Due to (ii) in the lemma, and Assumption 4, we may thus assume that

$$\left\|\left[\frac{\partial G_{n,k}}{\partial\lambda}\left(\lambda_{n,k}^{[0]}\right)\right]^{-1}\right\| \leqslant M, \tag{12}$$

where M is a constant independent of k. Hence, if $G_{n,k}(\lambda_{n,k}^{[0]}) = \mathcal{O}(H)$ it will follow from the scheme (10) that $\lambda_{n,k}^{[1]} = \lambda_{n-1} + \mathcal{O}(H)$.

$$G_{n,k}\big(\lambda_{n,k}^{[0]}\big) = \tilde{g}_{n-1}(t_n)/q_{n-1}^{(d+1-r)}(t_n) + \mathcal{O}(H),$$

where

$$\tilde{g}_{n-1}(t) = g\big(t, p_{n-1}(t), \dots, p_{n-1}^{(d+1-r)}(t)\big),$$

and for $n \geqslant r+1$ we obtain from (ii), (iii) and (7) with $n = n-1$

$$\begin{aligned} \tilde{g}_{n-1}(t_{n-i}) &= g\big(t_{n-i}, p_{n-i}(t_{n-i}), \dots, p_{n-i}^{(d+1-r)}(t_{n-i})\big) + \mathcal{O}(H)\sum_{s=2}^{i} q_{n-s}^{(d+1-r)}(t_{n-i}) \\ &= \mathcal{O}(H) q_{n-r-1}^{(d+1-r)}(t_{n-1}), \quad i = 1, 2, \dots, r. \end{aligned}$$

Thus (iv) implies that the C^r-function $\tilde{g}_{n-1}(t)$ satisfies

$$\begin{aligned} \tilde{g}_{n-1}(t_n)/q_{n-1}^{(d+1-r)}(t_n) &= \Bigg[\sum_{i=1}^{r}\prod_{s=1}^{i-1}(t_n - t_{n-s}) g_{n-1}[t_{n-1}, \dots, t_{n-i}] \\ &\quad + \mathcal{O}\Bigg(\prod_{s=1}^{r}(t_n - t_{n-s})\Bigg)\Bigg]\Big/ q_{n-1}^{(d+1-r)}(t_n) = \mathcal{O}(H). \end{aligned}$$

Due to (i), the result above is also valid for $n \in [1, r]$, but we leave this as an exercise for the reader. Having proved that $\lambda_{n,k}^{[1]} = \lambda_{n-1} + \mathcal{O}(H)$, we may now conclude the existence of $\lambda_{n,k} = \lambda_{n-1} + \mathcal{O}(H)$ by showing that the iterative scheme (10) is strongly contractive. The uniqueness of $\lambda_{n,k}$ for small H follows from (12).

Subtracting the equation in (10) from the one with $j = j-1$, we have, by induction in $j \geqslant 1$, that

$$\begin{aligned} \big\|\lambda_{n,k}^{[j+1]} - \lambda_{n,k}^{[j]}\big\| &\leqslant M\left\|G_{n,k}\big(\lambda_{n,k}^{[j]}\big) - G_{n,k}\big(\lambda_{n,k}^{[j-1]}\big) - \left[\frac{\partial G_{n,k}}{\partial \lambda}(\lambda_{n-1})\right]\big(\lambda_{n,k}^{[j]} - \lambda_{n,k}^{[j-1]}\big)\right\| \\ &\leqslant \mathcal{O}(HM)\big\|\lambda_{n,k}^{[j]} - \lambda_{n,k}^{[j-1]}\big\|, \end{aligned}$$

since $G_{n,k}$ is a C^1-function, and $\lambda_{n,k}^{[j-1]}, \lambda_{n,k}^{[j]}$ stays within a certain neighbourhood of λ_{n-1}.

Returning to the outer iteration (9), we note that, for $k = 0$, the uniform Lipschitz continuity of the C^1-function f implies that $\|(p_{n,1} - p_{n-1})[t_n, \dots, t_{n-d}]\|$ is $\mathcal{O}(H)$. If H is sufficiently small, we may thus find a unique $\lambda_{n,1} = \lambda_{n-1} + \mathcal{O}(H)$, satisfying $G_{n,1}(\lambda) = 0$. Subtracting $G_{n,0}(\lambda_{n,0})$ from $G_{n,1}(\lambda_{n,1})$ we obtain

$$\begin{aligned} 0 &= \Big[g\big(t_n, \big(p_{n,2}^{(i)}(t_n)\big)_{i=0}^{d+1-r}\big) - g\big(t_n, \big(p_{n,1}^{(i)}(t_n)\big)_{i=0}^{d+1-r}\big)\Big]\Big/ q_{n-1}^{(d+1-r)}(t_n) \\ &= \left\{\sum_{i=0}^{d+1-r}\frac{q_{n-1}^{(i)}(t_n)}{q_{n-1}^{(d+1-r)}(t_n)}\int_0^1 \frac{\partial g}{\partial y^{(i)}}\big(t_n, \big(\big(\theta p_{n,2}^{(s)} + (1-\theta)p_{n,1}^{(s)}\big)(t_n)\big)_{s=0}^{d+1-r}\big)\,\mathrm{d}\theta\right\} \\ &\quad \times \Big\{f\big(t_n, \big(s!p_{n,1}[t_n, \dots, t_{n-s}]\big)_{s=0}^{d-1}, \lambda_{n,1}\big) - f\big(t_n, \big(s!p_{n,0}[t_n, \dots, t_{n-s}]\big)_{s=0}^{d-1}, \lambda_{n,0}\big)\Big\} \\ &= \left\{\mathcal{O}(H) + \int_0^1 \frac{\partial g}{\partial y^{(d+1-r)}}\big(t_n, \big(\big(\theta p_{n,2}^{(s)} + (1-\theta)p_{n,1}^{(s)}\big)(t_n)\big)_{s=0}^{d+1-r}\big)\,\mathrm{d}\theta\right\} \end{aligned}$$

$$\times \left\{ \mathcal{O}(H) \left\| (p_{n,1} - p_{n,0})[t_n, \dots, t_{n-d}] \right\| \right.$$
$$\left. + \int_0^1 \frac{\partial f}{\partial \lambda} \left(t_n, \left(s! p_{n,1}[t_n, \dots, t_{n-s}]\right)_{s=0}^{d-1}, \theta\lambda_{n,1} + (1-\theta)\lambda_{n,0}\right) \mathrm{d}\theta (\lambda_{n,1} - \lambda_{n,0}) \right\}.$$

Using Assumption 4 and the fact that g is a C^1-function, we thus have

$$\|\lambda_{n,1} - \lambda_{n,0}\| = \mathcal{O}(H) \left\| (p_{n,1} - p_{n,0})[t_n, t_{n-1}, \dots, t_{n-d}] \right\|. \tag{13}$$

From (9) and (8) with subscript n replaced by $n,1$ and $n,0$, it thus follows from the Lipschitz-continuity of f that

$$\left\| (p_{n,2} - p_{n,1})[t_n, t_{n-1}, \dots, t_{n-d}] \right\| = \mathcal{O}(H) \left\| (p_{n,1} - p_{n,0})[t_n, t_{n-1}, \dots, t_{n-d}] \right\|. \tag{14}$$

Hence, we may find a unique $\lambda_{n,2} = \lambda_{n-1} + \mathcal{O}(H)$ satisfying $G_{n,2}(\lambda_{n,2}) = 0$, and since (13),(14) can be generalized to all consecutive iterates, the lemma follows by induction. □

3. Uniform convergence of method (5) in case $r = d + 1$

Since the purpose of this section is to prove condition (ii) of Lemma 2 for all $m \geqslant 1$ (provided the solution remains within Ω_1, Ω_2), we may as well use a formulation similar to Lemma 2.

Theorem 3. *Consider the case* $r = d + 1$. *Assume that the unique solution of* (1), (2), *ensured by our* Assumptions, *exists for* $t \in [t_0, t_{N-1} + H]$, *where* $H < \infty$ *is an upper bound of the stepsizes* $t_i - t_{i-1}$, $i \geqslant 1$, *and that the DAE-solution remains within* Ω_1, Ω_2. *If*

(i) $y_{j,0} = y^{(j)}(t_0) + \mathcal{O}(H)(t_1 - t_0)^{d-j}$ *for* $j = 0, 1, \dots, d-1$,
(ii) $(t_{n+1} - t_n)/(t_n - t_{n-1}) \in [\gamma, \Gamma]$ *for* $0 < \gamma \leqslant \Gamma < \infty$, $n = 1, 2, \dots, N-1$.

then, for sufficiently small H, (5) *has a unique solution satisfying* $\lambda_n = \lambda(t_n) + \mathcal{O}(H)$ *for all* t_n, $n = 1, 2, \dots, N$, *and*

$$\begin{aligned} &\left\| y_{j,n} - j! y[t_n, t_{n-1}, \dots, t_{n-j}] \right\| \\ &\quad = \mathcal{O}(H)(H + t_n - t_0)^{d-j} \left[1 + K(H + t_n - t_0) \exp\left(\left(K + \mathcal{O}(H)\right)(t_n - t_0)\right)\right], \end{aligned}$$

for $j = 0, 1, \dots, d-1$. *The constant* $K = d + L_f(1 + ML_g(1 + L_f))$ *depends on the bounds* L_f, L_g *of the partial derivatives of* f *and* $g^{(d)}$ *and on the bound* M *of* $[\partial g^{(d)}/\partial\lambda(t)]^{-1}$ *on* Ω_1, Ω_2 *(cf. the* Assumptions).

The error bounds of $y_{j,n}$, $j = 0, 1, \dots, d-1$, *are also valid if the algebraic constraint is replaced by*

$$g(t_n, y_{0,n}) = \mathcal{O}(H) \prod_{j=1}^{d} (t_n - t_{n-j}). \tag{15}$$

Proof. The theorem is clearly valid for $n = 0$. Assume that it holds for $n \leqslant n-1$. Then according to Lemma 2 a unique $\lambda_n = \lambda(t_n) + \mathcal{O}(H)$ exists. Defining the errors

$$e_{j,n} = j!y[t_n, t_{n-1}, \dots, t_{n-j}] - y_{j,n}, \quad j = 0, 1, \dots, d-1,$$

we obtain from (5) the inequalities

$$\|e_{j,n}\| \leqslant \|e_{j,n-1}\| + \left(\frac{t_n - t_{n-1-j}}{j+1}\right)\|e_{j+1,n}\|, \quad j = 0, 1, \dots, d-2, \tag{16}$$

$$\begin{aligned}\|e_{d-1,n}\| &\leqslant \|e_{d-1,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\left\|d!y[t_n, \dots, t_{n-d}] - f\big(t_n, (y_{i,n})_{i=0}^{d-1}, \lambda_n\big)\right\| \\ &\leqslant \|e_{d-1,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\left(\mathcal{O}(H) + L_f\left(\sum_{i=0}^{d-1}\|e_{i,n}\| + \left\|\lambda(t_n) - \lambda_n\right\|\right)\right).\end{aligned} \tag{17}$$

Hence, since $(1-x)^{-1} = \exp(x + \mathcal{O}(x^2))$ for all small $x > 0$, we obtain by summation

$$\begin{aligned}\sum_{j=0}^{d-1}\|e_{j,n}\| &\leqslant \sum_{j=0}^{d-1}\|e_{j,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\left(\mathcal{O}(H) + (d + L_f)\sum_{j=0}^{d-1}\|e_{j,n}\| + L_f\|\lambda(t_n) - \lambda_n\|\right) \\ &\leqslant \exp\left(\big(d + L_f + \mathcal{O}(H)\big)\left(\frac{t_n - t_{n-d}}{d}\right)\right) \\ &\quad\times\left(\sum_{j=0}^{d-1}\|e_{j,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\big(\mathcal{O}(H) + L_f\left\|\lambda(t_n) - \lambda_n\right\|\big)\right).\end{aligned} \tag{18}$$

For sufficiently small H we may thus assume that the bounds L_f, L_g and M are applicable on the line from the DAE-solution to the numerical solution at t_n. We shall make use of this and prove that

$$\left\|\lambda(t_n) - \lambda_n\right\| \leqslant \mathcal{O}(H) + ML_g\big(1 + L_f + \mathcal{O}(H)\big)\sum_{j=0}^{d-1}\|e_{j,n}\|. \tag{19}$$

It will then follow from the first inequality of (18) that

$$\begin{aligned}\sum_{j=0}^{d-1}\|e_{j,n}\| &\leqslant \sum_{j=0}^{d-1}\|e_{j,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\left(\mathcal{O}(H) + \big(K + \mathcal{O}(H)\big)\sum_{j=0}^{d-1}\|e_{j,n}\|\right) \\ &\leqslant \exp\left(\big(K + \mathcal{O}(H)\big)\left(\frac{t_n - t_{n-d}}{d}\right)\right)\left(\sum_{j=0}^{d-1}\|e_{j,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\mathcal{O}(H)\right) \\ &\leqslant \exp\big(\big(K + \mathcal{O}(H)\big)(t_n - t_0)\big)(H + t_n - t_0)\mathcal{O}(H).\end{aligned}$$

Inserting this bound in (19) and (17) we obtain

$$\begin{aligned}\|e_{d-1,n}\| &\leqslant \|e_{d-1,n-1}\| + \left(\frac{t_n - t_{n-d}}{d}\right)\mathcal{O}(H)\big[1 + K(H + t_n - t_0)\exp\big(\big(K + \mathcal{O}(H)\big)(t_n - t_0)\big)\big] \\ &\leqslant \|e_{d-1,0}\| + \sum_{i=1}^{n}\left(\frac{t_i - t_{i-d}}{d}\right)\mathcal{O}(H)\big[1 + K(H + t_n - t_0)\exp\big(\big(K + \mathcal{O}(H)\big)(t_n - t_0)\big)\big] \\ &\leqslant \mathcal{O}(H)(H + t_n - t_0)\big[1 + K(H + t_n - t_0)\exp\big(\big(K + \mathcal{O}(H)\big)(t_n - t_0)\big)\big].\end{aligned}$$

For $j = d-2, d-3, \dots, 0$, we obtain the error bound of $y_{j,n}$ by a similar substitution into (16) of the error bound of $y_{j+1,n}$, and the theorem will thus follow from (19).

In order to prove (19) we consider the function

$$\tilde{g}_n(t) = g\bigl(t, p_n(t)\bigr),$$

where p_n is the polynomial defined in connection with (5). From (7) we know that $p_n(t_{n-j}) = p_{n-j}(t_{n-j})$ for $j = 0, 1, \dots, d$. Since the ratio between consecutive stepsizes are bounded, it thus follows from (15) (and (i) in case $n \leqslant d$) that

$$\tilde{g}_n[t_n, t_{n-1}, \dots, t_{n-d}] = \mathcal{O}(H).$$

Since $\tilde{g}_n$ is a C^{d+1}-function there exists a $t_n^* \in [t_{n-d}, t_n]$ such that

$$g^{(d)}\bigl(t_n^*, p_n\bigl(t_n^*\bigr), \dots, p_n^{(d)}\bigl(t_n^*\bigr)\bigr) = \mathcal{O}(H).$$

Let $r_{i,n}$ denote the polynomials

$$r_{i,n}(t) = \prod_{j=0}^{i-1}(t - t_{n-j})/i!, \quad i = 0, 1, \dots, d.$$

We may then write

$$\begin{aligned}
p_n^{(s)}\bigl(t_n^*\bigr) &= y^{(s)}\bigl(t_n^*\bigr) + \mathcal{O}(H) - \sum_{i=s}^{d-1} r_{i,n}^{(s)}\bigl(t_n^*\bigr)e_{i,n} \\
&\quad - r_{d,n}^{(s)}\bigl(t_n^*\bigr)\bigl[f\bigl(t_n, \bigl(j!y[t_n, \dots, t_{n-j}]\bigr)_{j=0}^{d-1}, \lambda(t_n)\bigr) - f\bigl(t_n, (y_{j,n})_{j=0}^{d-1}, \lambda_n\bigr)\bigr] \\
&= y^{(s)}\bigl(t_n^*\bigr) + \mathcal{O}(H) - e_{s,n} + \mathcal{O}\left(H\left(\sum_{i=0}^{d-1}\|e_{i,n}\| + \bigl\|\lambda(t_n) - \lambda_n\bigr\|\right)\right)
\end{aligned}$$

for $s = 0, 1, \dots, d-1$, whereas $p_n^{(d)}(t_n^*)$ is

$$y^{(d)}\bigl(t_n^*\bigr) + \mathcal{O}(H) - \bigl[f\bigl(t_n, \bigl(j!y[t_n, \dots, t_{n-j}]\bigr)_{j=0}^{d-1}, \lambda(t_n)\bigr) - f\bigl(t_n, (y_{j,n})_{j=0}^{d-1}, \lambda_n\bigr)\bigr].$$

Using the boundedness of $[\partial g^{(d)}/\partial\lambda(t)]^{-1}$ and the partial derivatives of f and $g^{(d)}$ on the line from the DAE-solution to the numerical solution at t_n, (19) and thus the theorem follows by induction in n. $\square$

4. Generalization to variable-step variable-order BDFs

It is outside the scope of this paper to extend the convergence result of Section 3 to a variable-step variable-order method. However, since several codes for DAEs have been based on the BDFs (cf. Section 1), it may be of interest to perform some experiments with a method similar to the BDFs, e.g. (for $d = r - 1 \geqslant 1$):

$$\begin{aligned}
&\sum_{i=1}^{k_n} \alpha_{i,n}^{[j+1]} y_j[t_n, t_{n-1}, \dots, t_{n-i}] = y_{j+1,n}, \quad j = 0, 1, \dots, d-1, \\
&y_{d,n} = f(t_n, y_{0,n}, y_{1,n}, \dots, y_{d-1,n}, \lambda_n), \\
&0 = g(t_n, y_{0,n}).
\end{aligned}$$

In order to avoid the drop in the order of convergence, seen when changing stepsize or order in the BDFs, the formula $y_{j+1,n} \approx y_0^{(j+1)}(t_n)$ should be exact for polynomials y_0 of degree $k_n + j$, $j = 0, 1, \ldots, d-1$. Hence, for $j = 0$ the coefficients $\alpha_{i,n}^{[j+1]}$ are those of the ordinary BDFk_n-formula, whereas for $j > 0$ the coefficients $\alpha_{i,n}^{[j+1]}$ depend on the formulas used in producing $y_{j,n-1}, y_{j,n-2}, \ldots, y_{j,n-k_n}$. For $k_n \in \{1, 2\}$, this type of method was derived in [10], where also experiments can be found (cf. http://www.diku.dk/research-groups for an electronic version of the report). However, even for moderate sizes of k_n and d ($\geqslant 2$), this approach leads to a large family of formulas, and therefore we in this paper follow the alternative approach of Section 1 (discretization *prior* to any equation order reduction) obtaining the following generalization of (5) for $d = r - 1 \geqslant 1$:

$$
\begin{aligned}
&p_{n,d-1+k_n}^{(d)}(t_n) = f\big(t_n, y_{0,n}, p'_{n,k_n}(t_n), p''_{n,1+k_n}(t_n), \ldots, p_{n,d-2+k_n}^{(d-1)}(t_n), \lambda_n\big),\\
&0 = g(t_n, y_{0,n}),\\
&p_{n,s}(t) = \sum_{i=0}^{s} \prod_{j=0}^{i-1} (t - t_{n-j}) y_0[t_n, t_{n-1}, \ldots, t_{n-i}], \quad s = k_n, \ldots, d-1+k_n.
\end{aligned}
\tag{20}
$$

For fixed stepsize and $k_n \leqslant 5$ these formulas are zero-stable for *all* $d = r - 1 \geqslant 1$, and we hope to generalize Theorem 3 to cover these formulas sometime. Let us end this paper by applying (20) to two index-3 DAEs, both satisfying the Assumptions in Section 2:

Example 4. The problem, which was solved by merely first-order formulas in Table 1 was introduced in [1] and it describes the position of a particle on a circular track. The problem reads

$$
\begin{aligned}
&\begin{pmatrix} x'' \\ y'' \end{pmatrix} = 2\begin{pmatrix} y \\ -x \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}(1) = \begin{pmatrix} \sin(1) & 2\cos(1) \\ \cos(1) & -2\sin(1) \end{pmatrix},\\
&0 = x^2 + y^2 - 1,
\end{aligned}
$$

and the solution is $(x(t), y(t), \lambda(t)) = (\sin(t^2), \cos(t^2), -4t^2)$.

Applying (20) with $k_n \in \{1, 2\}$ we may compare the results to those listed in Table 8.2 of [1]:

Table 2
Comparison with results in Table 8.2 of [1]. The results are errors in the estimated algebraic variable λ, and second-order formulas are used except for the first step

t_n	Absolute errors for stepsize 0.005			Absolute errors for stepsize 0.01		
	(BDF1&2)	(Corrected)	(20), $k_n \leqslant 2$	(BDF1&2)	(Corrected)	(20), $k_n \leqslant 2$
1.005	**2.0400**	**0.0403**	0.0402			
1.010	**4.0190**	**0.0190**	0.0010	**2.0810**	**0.0812**	0.0809
1.015	**1.0120**	**0.0119**	0.0010			
1.020	0.0012	0.0012	0.0010	**4.0350**	**0.0360**	0.0041
1.030	0.0013	0.0013	0.0009	**1.0280**	**0.0280**	0.0041
1.040	0.0013	0.0013	0.0010	0.0052	0.0052	0.0040
1.050	0.0014	0.0014	0.0010	0.0054	0.0054	0.0041

Example 5. Since Theorem 3 is valid also for problems where the algebraic variables λ appear non-linearly, it may be of interest to apply our generalization of (5) (i.e., (20)) to such a problem. Modifying a DAE with $d = r - 1 = 1$ introduced in [8] slightly, we obtained the following problem with $d = r - 1 = 2$:

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} xy^2\lambda^2 \\ y^2\left[x^2 - 3\lambda + 6(x')^2\right] \end{pmatrix}, \qquad \begin{pmatrix} x & x' \\ y & y' \end{pmatrix}(0) = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix},$$
$$0 = x^2y - 1.$$

This problem has two solutions, viz. $(x(t), y(t), \lambda(t)) = (e^t, e^{-2t}, e^{2t})$ and $(x(t), y(t), \lambda(t)) = (a(t), a(t)^{-2}, 0.5a(t)^2)$, where $a(t) = 0.5(3e^{t/2} - e^{-t/2})$. We wanted to find the same solution as in [8], and thus we assumed that $\lambda(0) = 1$ was given (approximately) as initial value (cf. (3) in the Assumptions, Section 2).

Since the existence of a real-valued numerical solution of (20) has not yet been proved, (20) was solved by applying Newton's method with linesearch to the following constrained least squares problem:

$$\min_{y_{0,n},\mu_n} \left\| \sigma_n\left(p''_{n,1+k_n}(t_n) - f\left(t_n, y_{0,n}, p'_{n,k_n}(t_n), \mu_n/\sigma_n\right)\right)\right\|_2,$$
$$\text{subject to } 0 = g(t_n, y_{0,n}),$$

where σ_n is the scaling factor $(t_n - t_{n-1})(t_n - t_{n-2})$ and μ_n is $\sigma_n\lambda_n$. Due to the scaling, only few Newton iterations were needed in each step, irrespective of the variation of order and stepsize.

Because of Theorem 3, we expected that the *local* errors of $y_{0,n}$ and λ_n were $\mathcal{O}(H^{k_n+2})$ and $\mathcal{O}(H^{k_n})$, respectively, whereas the *global* errors should be $\mathcal{O}(H^{k_n})$ for both components. However, especially for the high order formulas, extreme variation of order and stepsize will probably result in lower orders, and thus we started our runs with a relatively slow increase in the order k_n of the formulas:

> For $k = 1, 2, \ldots, 5$, $k + 2$ steps were computed by means of the kth order formula with stepsizes $h_n = \widetilde{H}^{(5+0.2\theta_n)/k}$, where θ_n was randomly chosen in $[0, 1]$ and $\widetilde{H}$ was a program parameter.

Varying $\widetilde{H}$, these 25 steps should render our assumptions on the *local* errors probable and result in *local* errors of order at least $\mathcal{O}(\widetilde{H}^7)$ and $\mathcal{O}(\widetilde{H}^5)$, despite the fact that the initial values $(x, y, x', y', \lambda)(0)$ were perturbed by $(\pm h_1^3, \pm h_1^3, \pm h_1^2, \pm h_1^2, \pm h_1)$ with randomly chosen signs.

The computations were ended by a somewhat faster return to the first-order formula, making certain that $\max\{h_{n-k_n-1}, \ldots, h_{n-1}, h_n\}$ was $\mathcal{O}(\widetilde{H}^{5/k_n})$ before reducing the order to k_n, i.e.,

> Five steps of size $\widetilde{H}^{(5+0.2\theta_n)/4}$, $\widetilde{H}^{(5+0.2\theta_n)/4}$, $\widetilde{H}^{(5+0.2\theta_n)/3}$, $\widetilde{H}^{(5+0.2\theta_n)/3}$, $\widetilde{H}^{(5+0.2\theta_n)/2}$ were computed using the 5th order formula. Then one step with $k_n = 4$ and $h_n = \widetilde{H}^{(5+0.2\theta_n)/2}$ was taken, followed by four steps of size $\widetilde{H}^{(5+0.2\theta_n)}$ and orders 3, 2, 1, 1.

In the first run we chose $\widetilde{H} = 0.1$ and computed the values at the points $t_1, t_2, \ldots, t_{35}$ described above. In the second run $\widetilde{H}$ was 0.01, and after the first 24 steps we continued using the 5th order formula with the variable stepsize $\widetilde{H}^{(1+0.04\theta_n)}$ until the point t_{25} of the *first* run was reached. Hence, the results should indicate, whether our assumptions on the local *and* global errors are true. The results are listed below and they seem to confirm our assumptions. Since the algebraic equation $0 = g(t_n, y_{0,n}) = x_n^2y_n - 1$ was solved exactly, the relative errors of y_n are not listed.

Table 3
Results for the variable-step variable-order formulas (20)

k_n	log(stepsize)/ log($\widetilde{H}$)	Rel. error of x_n, $\widetilde{H}=0.1$	Rel. error of λ_n, $\widetilde{H}=0.1$	No. of steps	$\log_{10}$[(Rel. error, $\widetilde{H}=0.1$)/ (Rel. error, $\widetilde{H}=0.01$)]	
	For $t \in [0, 1.000]$			For $t \in [0, 0.074]$		
1	$5+0.2\theta_1$	5.7×10^{-15}	8.2×10^{-5}	1	15.1	5.0
–	$5+0.2\theta_2$	2.8×10^{-14}	1.1×10^{-4}	1	15.5	5.4
–	$5+0.2\theta_3$	6.5×10^{-14}	1.0×10^{-4}	1	15.5	5.2
2	$(5+0.2\theta_4)/2$	5.4×10^{-10}	1.8×10^{-5}	1	10.3	5.0
–	$(5+0.2\theta_5)/2$	1.3×10^{-9}	5.9×10^{-5}	1	10.3	5.1
–	$(5+0.2\theta_6)/2$	1.6×10^{-9}	8.9×10^{-5}	1	10.4	5.1
–	$(5+0.2\theta_7)/2$	6.3×10^{-10}	1.0×10^{-4}	1	10.7	5.2
3	$(5+0.2\theta_8)/3$	4.6×10^{-8}	1.4×10^{-5}	1	8.4	5.2
–	$(5+0.2\theta_9)/3$	1.5×10^{-7}	4.8×10^{-5}	1	8.2	5.0
–	$(5+0.2\theta_{10})/3$	2.5×10^{-7}	1.3×10^{-4}	1	8.1	5.0
–	$(5+0.2\theta_{11})/3$	2.7×10^{-7}	2.2×10^{-4}	1	7.9	5.0
–	$(5+0.2\theta_{12})/3$	1.3×10^{-7}	2.7×10^{-4}	1	7.5	5.1
4	$(5+0.2\theta_{13})/4$	1.0×10^{-6}	7.3×10^{-5}	1	7.9	6.0
–	$(5+0.2\theta_{14})/4$	2.8×10^{-6}	2.8×10^{-4}	1	7.7	5.5
–	$(5+0.2\theta_{15})/4$	3.1×10^{-6}	7.3×10^{-4}	1	7.6	5.6
–	$(5+0.2\theta_{16})/4$	2.8×10^{-8}	9.5×10^{-4}	1	6.0	5.5
–	$(5+0.2\theta_{17})/4$	7.2×10^{-6}	9.0×10^{-4}	1	7.8	5.4
–	$(5+0.2\theta_{18})/4$	1.9×10^{-5}	6.2×10^{-4}	1	7.8	5.2
5	$1+0.04\theta_{19}$	4.1×10^{-5}	7.2×10^{-4}	1	7.5	6.0
–	$1+0.04\theta_{20}$	4.9×10^{-5}	1.8×10^{-3}	1	7.4	6.2
–	$1+0.04\theta_{21}$	2.9×10^{-5}	1.5×10^{-3}	1	7.2	6.5
–	$1+0.04\theta_{22}$	1.4×10^{-5}	1.2×10^{-3}	1	6.9	6.0
–	$1+0.04\theta_{23}$	3.5×10^{-5}	5.0×10^{-3}	1	7.2	6.2
–	$1+0.04\theta_{24}$	1.9×10^{-5}	6.5×10^{-3}	1	6.9	6.0
	For $t \in [1.095, 1.248]$			For $t \in [1.096, 1.102]$		
–	$1+0.04\theta_m$	1.5×10^{-4}	2.0×10^{-3}	111	4.6	5.2
–	$(5+0.2\theta_{m+1})/4$	2.4×10^{-4}	2.7×10^{-3}	1	4.8	5.5
–	$(5+0.2\theta_{m+2})/4$	3.0×10^{-4}	5.1×10^{-3}	1	4.9	5.8
–	$(5+0.2\theta_{m+3})/3$	3.2×10^{-4}	1.9×10^{-4}	1	4.9	5.4
–	$(5+0.2\theta_{m+4})/3$	3.4×10^{-4}	3.8×10^{-4}	1	5.0	4.7
–	$(5+0.2\theta_{m+5})/2$	3.4×10^{-4}	6.9×10^{-4}	1	5.0	5.0
4	$(5+0.2\theta_{m+6})/2$	3.4×10^{-4}	7.1×10^{-4}	1	5.0	5.0
3	$5+0.2\theta_{m+7}$	3.4×10^{-4}	6.8×10^{-4}	1	5.0	5.0
2	$5+0.2\theta_{m+8}$	3.4×10^{-4}	6.8×10^{-4}	1	5.0	5.0
1	$5+0.2\theta_{m+9}$	3.4×10^{-4}	5.9×10^{-4}	1	5.0	4.9
–	$5+0.2\theta_{m+10}$	3.4×10^{-4}	5.8×10^{-4}	1	5.0	4.9

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