

Computation of non-smooth local centre manifolds

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An iterative Lyapunov–Perron algorithm for the computation of inertial manifolds is adapted for centre manifolds and applied to two test problems. The first application is to compute a known non-smooth manifold (once, but not twice differentiable), where a Taylor expansion is not possible. The second is to a smooth manifold arising in a porous medium problem, where rigorous error estimates are compared to both the correction at each iteration and the addition of each coefficient in a Taylor expansion. While in each case the manifold is 1D, the algorithm is well-suited for higher dimensional manifolds. In fact, the computational complexity of the algorithm is independent of the dimension, as it computes individual points on the manifold independently by discretising the solution through them. Summations in the algorithm are reformulated to be recursive. This acceleration applies to the special case of inertial manifolds as well.

Keywords: computation of invariant manifolds; centre manifolds.

1. Introduction

Local centre manifolds are fundamental tools for the study of the nature of a flow near an invariant object, such as a steady state. In the most general sense, a centre manifold is associated with a middle range for the real part of the spectrum; the resulting upper and lower ranges may in fact be both negative or both positive. One of these two ranges may be empty. In this general sense, the notion of a centre manifold includes those of stable and unstable manifolds. These latter special cases are of interest not just locally in phase space but globally. Stable and unstable manifolds form connecting orbits between invariant objects and trigger dramatic global bifurcations (Glendinning, 1988; Johnson *et al.*, 2001). For many dissipative systems, unstable manifolds form the backbone of the global attractor, the largest bounded invariant set (see Hale, 1988; Temam, 1997). Stable manifolds are the boundaries between the basins of attraction.

There is a great variety of algorithms for the computation of global (un)stable manifolds, which evolve (forward for unstable, backward in time for stable manifolds) a discrete approximation of the boundary of a local manifold (see Guckenheimer & Worfolk, 1993; Johnson *et al.*, 1997; Krauskopf &

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Osinga, 1999, 2003; Guckenheimer & Vladimirovsky, 2004; Henderson, 2004). While the error from the local approximation will typically decay as the global manifold is evolved, improvement in the computation of local centre manifolds will lead to more accurate global manifolds.

For the local analysis of low-dimensional centre manifolds of smooth vector fields, it can be highly effective to expand the manifolds in power series. The lower order terms are often enough to determine local bifurcations by the analysis of normal forms, which are the restrictions of the flow to polynomial approximations of the manifold; see Guckenheimer & Holmes (1983), and Carr (1981). In such an application, the local centre manifold is the graph of a function over an eigenspace associated with eigenvalues of the linearised vector field that have zero real part.

When the manifold is not analytic or the series representation converges slowly, other algorithms may be more suitable. We present here an algorithm which converges even for non-smooth manifolds. All that is needed for the existence of a local centre manifold are gaps between the ranges in the real part of the spectrum of the linearised vector field. For any such gap, there is a small enough neighbourhood in which the centre manifold is found in terms of the fixed point of a contraction mapping on a function space. The algorithm presented and tested in this paper is based on the discretisation of such a mapping. It is generalised from an algorithm introduced in Rosa (1995) and implemented in Jolly *et al.* (2000) to compute inertial manifolds (see Foias *et al.*, 1988; Temam, 1997).

That an algorithm for inertial manifolds can be extended to local centre manifolds is natural since the former is really a special case of a global centre manifold. For certain PDEs, rather than consider a small enough neighbourhood, one has the luxury (or the curse) of choosing a large enough gap, since the eigenvalues grow with wave number at a non-linear rate.

An algorithm to compute centre manifolds is presented in Ma & Küpper (1994), which is also based on the discretisation of a contraction mapping. Our approach differs in several fundamental ways. Here the function space on which the mapping acts consists of functions of time, rather than of the independent variable for the function describing the local manifold. Unlike the approaches in Ma & Küpper (1994), Dellnitz & Hohmann (1997) and the global manifold constructions described above, we do not compute the manifold in its entirety. The advantage is that we discretise a 1D object, making the complexity independent of the dimension of the manifold. Our algorithm is particularly well-suited for computing solutions on the centre manifold. In this sense, it is akin to the algorithm for maps in Homburg *et al.* (1995), which also computes the manifold pointwise. That method is based on foliations around the fixed point and starts with a large number of points in phase space that is decreased with each iteration of a contraction mapping. In contrast, we begin with a small representation of the distinguished solution in time and refine it with each iteration. Iterations in our method are progressively more expensive, but the accuracy is not determined in advance; it can be monitored and improved with more iterations. This is another way in which our algorithm differs from that in Ma & Küpper (1994), which chooses a fixed finite-dimensional subspace of the function space.

An extensive review of methods for computing stable manifolds is given in Moore & Hubert (1999). Methods are called *indirect* if a difference scheme is first applied to replace the original system by a discrete dynamical system and *direct* if the manifold is constructed for the original continuous system generated by the differential equation. A unified treatment of direct methods is given in Stuart (1994). Within both the direct and indirect categories in Moore & Hubert (1999) are methods for computing the manifold in its entirety and others for computing the manifold at a single point. Included is a number of variations on the Lyapunov–Perron method. In Moore & Hubert (1999), the recommended direct approach for computing the manifold at a single point is to solve an infinite interval boundary value problem by truncation and collocation, while our approach is to iterate the Lyapunov–Perron contraction mapping on piecewise constant functions of time.

2. Centre manifold

We consider a differential equation of the form

$$u' + Au = f(u) \quad (2.1)$$

in \mathbb{R}^n , or more generally in a Banach space E with norm denoted by $|\cdot|$. It is assumed that the non-linear term f is globally Lipschitz continuous in E , with $f(0) = 0$, and that $-A$ is a closed operator in E generating a strongly continuous group $\{e^{-tA}\}_{t \in \mathbb{R}}$ of bounded linear operators in E (this automatically holds when $E = \mathbb{R}^n$).

For the existence of a centre manifold, a trichotomy condition is assumed for the group $\{e^{-tA}\}_{t \in \mathbb{R}}$. More precisely, it is assumed that there exist projectors $P, R, Q: E \rightarrow E$ with $I = P \oplus R \oplus Q$, real numbers $\lambda_- < \lambda_- \leq \lambda_+ < \lambda_+$ and constants K_P, K_R, K_Q such that

$$\begin{aligned} \|e^{-tA}P\|_{\mathcal{L}(E)} &\leq K_P e^{-\lambda_- t} \quad \forall t \leq 0, \\ \|e^{-tA}R\|_{\mathcal{L}(E)} &\leq K_R e^{-\lambda_+ t} \quad \forall t \leq 0, \\ \|e^{-tA}R\|_{\mathcal{L}(E)} &\leq K_R e^{-\lambda_- t} \quad \forall t \geq 0, \\ \|e^{-tA}Q\|_{\mathcal{L}(E)} &\leq K_Q e^{-\lambda_+ t} \quad \forall t \geq 0, \end{aligned}$$

where $\|\cdot\|_{\mathcal{L}(E)}$ denotes the operator norm in E . The projectors P, R, Q are spectral projectors in the sense that they commute with A and, hence, with e^{-tA} , $t \in \mathbb{R}$.

We consider an equivalent norm $|\cdot|_v$ in E given by

$$|u|_v = \max\{|Pu|, |Ru|, |Qu|\}.$$

The constants in the trichotomy remain unchanged with respect to this new norm. We decompose f into three components with the Lipschitz conditions written as

$$\begin{aligned} |Pf(u) - Pf(v)| &\leq M_{P,v}|u - v|_v, \\ |Rf(u) - Rf(v)| &\leq M_{R,v}|u - v|_v, \\ |Qf(u) - Qf(v)| &\leq M_{Q,v}|u - v|_v, \end{aligned}$$

for all $u, v \in E$.

If we write an element u of the phase space E in the form $u = p + r + q$, according to the decomposition above, the centre manifold is found as a graph $p + q = \Phi(r)$ over RE , with $\Phi: RE \rightarrow PE \oplus QE$.

There are several methods for obtaining the function Φ whose graph is the centre manifold, some with more or less geometrical flavour than others. We follow here the Lyapunov–Perron trajectory method Perron (1929), Hale & Perelló (1964), Chow & Lu (1988) and Castañeda & Rosa (1996). The manifold is obtained at each base point r_0 independently in terms of a global trajectory $\{\varphi(r_0)(t), t \in \mathbb{R}\}$ within the manifold, through the relation $r_0 + \Phi(r_0) = \varphi(r_0)(0)$, or alternatively $\Phi(r_0) = (P + Q)\varphi(r_0)(0)$.

Each global trajectory $\varphi = \varphi(r_0)$, $r_0 \in RE$, is found as a fixed point in the space of functions

$$\mathcal{F}_\sigma = \left\{ \varphi \in \mathcal{C}(\mathbb{R}, E), \sup_{t \in \mathbb{R}} (e^{-\sigma(t)} |\varphi(t)|_v) < \infty \right\},$$

where

$$\sigma(t) = \begin{cases} -\sigma_- t, & t \geq 0, \\ -\sigma_+ t, & t \leq 0, \end{cases}$$

and $\sigma = (\sigma_-, \sigma_+) \in \mathbb{R}^2$ satisfies

$$\Lambda_- < \sigma_- < \lambda_- \leq \lambda_+ < \sigma_+ < \Lambda_+.$$

For a norm in \mathcal{F}_σ , we take

$$\|\varphi\|_{\sigma,v} = \sup_{t \in \mathbb{R}} (e^{-\sigma(t)} |\varphi(t)|_v).$$

The fixed-point map is $\mathcal{T}(\cdot, r_0)$, where $\mathcal{T}: \mathcal{F}_\sigma \times RE \rightarrow \mathcal{F}_\sigma$ is defined by

$$\begin{aligned} \mathcal{T}(\varphi, r_0)(t) = & - \int_t^\infty e^{-(t-\tau)A} Pf(\varphi(\tau)) d\tau \\ & + e^{-tA} r_0 + \int_0^t e^{-(t-\tau)A} Rf(\varphi(\tau)) d\tau + \int_{-\infty}^t e^{-(t-\tau)A} Qf(\varphi(\tau)) d\tau, \end{aligned}$$

for $r_0 \in RE$ and $\varphi \in \mathcal{F}_\sigma$.

A fixed point of $\mathcal{T}(\cdot, r_0)$ in \mathcal{F}_σ is a trajectory in the centre manifold of the differential equation, and the existence of such a fixed point reflects the fact that there exists a unique trajectory $\varphi(r_0)$ that at time $t = 0$ has r_0 as its R -component, and that does not grow exponentially at a rate larger than or equal to $-\Lambda_-$, as $t \rightarrow \infty$, and at a rate larger than or equal to Λ_+ , as $t \rightarrow -\infty$. If the Lipschitz constants are sufficiently small relative to the spectral gaps $\Lambda_+ - \lambda_+$ and $\lambda_- - \Lambda_-$, one can show that such a trajectory indeed exists, and hence so does the centre manifold. This is achieved by showing that the map $\mathcal{T}(\cdot, r_0)$ is a strict contraction in \mathcal{F}_σ .

One can show that $\mathcal{T}(\cdot, r_0)$ is Lipschitz continuous in \mathcal{F}_σ with Lipschitz constant bounded by $\kappa_{\sigma,v}$ where

$$\kappa_{\sigma,v} = \max \left\{ \frac{K_P M_{P,v}}{\sigma_- - \Lambda_-}, \frac{K_R M_{R,v}}{\lambda_- - \sigma_-}, \frac{K_R M_{R,v}}{\sigma_+ - \lambda_+}, \frac{K_Q M_{Q,v}}{\Lambda_+ - \sigma_+} \right\}. \quad (2.2)$$

The gap conditions are

$$\begin{aligned} \lambda_- - \Lambda_- &> K_P M_{P,v} + K_R M_{R,v}, \\ \Lambda_+ - \lambda_+ &> K_R M_{R,v} + K_Q M_{Q,v}. \end{aligned} \quad (2.3)$$

With such gaps, one can choose $\sigma = (\sigma_-, \sigma_+) \in \mathbb{R}^2$ satisfying

$$\begin{aligned} \Lambda_- + K_P M_{P,v} &< \sigma_- < \lambda_- - K_R M_{R,v}, \\ \lambda_+ + K_R M_{R,v} &< \sigma_+ < \Lambda_+ - K_Q M_{Q,v}, \end{aligned} \quad (2.4)$$

for which $\kappa_{\sigma,v} < 1$ (see Section 7), guaranteeing that $\mathcal{T}(\cdot, r_0)$ is a strict contraction in \mathcal{F}_σ . The function $\Phi: RE \rightarrow RE \oplus QE$ so obtained can further be proved to be globally Lipschitz continuous. Under these spectral gap conditions, the following existence result holds (see Section 7).

THEOREM 2.1 Assume the spectral gap conditions (2.3) hold. Then, there is an invariant manifold for (2.1) given as the graph over RE of a globally Lipschitz function $\Phi: RE \rightarrow PE \oplus QE$.

REMARK 2.2 In fact, under the same spectral gap conditions (2.3), one can show following Castañeda & Rosa (1996) that there exist an unstable manifold $r + q = \Phi^u(p)$ and a stable manifold $p + r = \Phi^s(q)$, and that the flow associated with (2.1) is conjugated with the flow associated with the uncoupled system

$$\begin{cases} p' + Ap = Pf(p + \Phi^u(p)), \\ r' + Ar = Rf(r + \Phi(r)), \\ q' + Aq = Qf(q + \Phi^s(q)). \end{cases}$$

Moreover, the centre manifold can be obtained as the intersection between a centre-stable manifold $p = \Phi^{\text{cu}}(r + q)$ and a centre-unstable manifold $q = \Phi^{\text{cu}}(p + r)$.

Note that the nomenclature is not so strict here since we are not assuming that $\lambda_- \leq 0 \leq \lambda_+$; they can both be positive or negative, as long as $\lambda_- \leq \lambda_+$ and the spectral gaps in (2.3) hold.

3. Approximation of the centre manifold

An approximation of the centre manifold is obtained by replacing the functions in \mathcal{F}_σ by piecewise constant functions and by iterating a corresponding discretised version of the map $\mathcal{T}(\cdot, r_0)$. Under suitable conditions, this yields a convergent method of linear order.

Consider the space

$$\tilde{\mathcal{F}} = \{\psi: \mathbb{R} \rightarrow E; \psi \text{ is piecewise constant with a finite number of discontinuities}\}.$$

We define a discrete version \mathcal{T}_h^N of \mathcal{T} as follows. We consider $h > 0$ and a non-negative integer N . It is straightforward to see that the original map \mathcal{T} is well defined on $\tilde{\mathcal{F}}$ despite the functions being discontinuous. Then, for r_0 in RE , ψ in $\tilde{\mathcal{F}}$, and $k = -N, \dots, N$, we set

$$\mathcal{T}_h^N(\psi, r_0)(t) = \mathcal{T}(\psi, r_0)(kh), \quad (3.1)$$

for

$$t \in \begin{cases} (-\infty, -Nh], & \text{if } k = -N, \\ ((k-1)h, kh], & \text{if } k = -(N-1), \dots, 0, \\ [kh, (k+1)h], & \text{if } k = 0, \dots, N-1, \\ [Nh, \infty), & \text{if } k = N. \end{cases} \quad (3.2)$$

The sequence of approximate manifolds is initiated by taking $\psi = \varphi^0(r_0)$ identically equal to r_0 , i.e. $\varphi^0(r_0)(t) = r_0$, for all $t \in \mathbb{R}$. Then, by choosing a sequence $h_j > 0$ and non-negative integers N_j , $j = 1, 2, \dots$, we define

$$\varphi^j(r_0) = \mathcal{T}_{h_j}^{N_j}(\varphi^{j-1}, r_0), \quad j = 1, 2, \dots \quad (3.3)$$

It is natural to require that h_j be decreasing and $N_j h_j$ be increasing. For notational purposes, we consider $h_0 > 0$ and $N_0 = 0$, but their values are not relevant for the approximating sequence.

Then, for each j , we have a manifold given as the graph over RE of the map

$$\Phi_j(r_0) = (P + Q)\varphi^j(r_0)(0),$$

which serves as an approximation for the exact centre manifold. Under the spectral gap conditions (2.3) and for suitable choices of h_j and N_j , this sequence converges to the exact manifold. In particular, we have the following result, which is proved in Section 8.

THEOREM 3.1 Assume that the spectral gap conditions (2.3) hold, and consider $\sigma = (\sigma_-, \sigma_+) \in \mathbb{R}^2$ satisfying (2.4), so that $\kappa_{\sigma, \nu} < 1$. Suppose the time steps h_j decrease exponentially, and the time intervals $N_j h_j$ increase linearly according to

$$0 < h_j \leq c_1 \gamma^j, \quad N_j h_j \geq c_2 j, \quad (3.4)$$

for some constants $c_1, c_2 > 0$ and for some $0 < \gamma < 1$. Then, there exist constants $C_1, C_2 = C_2(\sigma)$ and $\eta = \eta(\sigma)$, with $C_1, C_2 > 0$ and $\gamma \leq \eta < 1$, such that

$$\frac{|\Phi(r_0) - \Phi_j(r_0)|_v}{|r_0|} \leq \frac{\|\varphi(r_0) - \varphi^j(r_0)\|_{\sigma,v}}{|r_0|} \leq C_1 \kappa_{\sigma,v}^j + C_2 j \eta^j, \quad (3.5)$$

for all $r_0 \neq 0$ and all $j = 0, 1, 2, \dots$.

This shows the exponential convergence of the approximate centre manifolds, which makes it a method of linear order according to the numerical analysis literature. The explicit expressions for C_1, C_2 and η are given in Section 8 and are useful for indicating how to choose the constants c_1 and c_2 related to the time discretisation.

In applications, the non-linear term f is usually not globally Lipschitz continuous and a truncation of this term must be considered for the application of the above method. One then has a choice of how to perform this truncation, and the choice of the truncation will determine the Lipschitz constants $M_{P,v}, M_{Q,v}$ and $M_{R,v}$. One may then attempt to choose the truncation in order to make the spectral gap condition less stringent. This fact is illustrated in the example of Section 4.1, in which the choice of the truncation is optimised among a certain family of truncations. In the example of Section 4.2, on the other hand, we wanted to preserve the smoothness of the non-linear term after truncation, which limited our choices, and for this reason only one truncation is considered.

In Sections 5 and 6, we present more explicit expressions for the numerical implementation of the above sequence of approximate centre manifolds. Before that, we consider some applications of the method.

4. Test cases

4.1 Approximation of a non-smooth centre manifold

We first apply the algorithm to a system of the form

$$\begin{aligned} \frac{dp}{dt} + \lambda_1 p &= f_1(r) := \lambda_1 g_1(r) + g'_1(r)(f_2(r) - \lambda_2 r), \\ \frac{dr}{dt} + \lambda_2 r &= f_2(r), \\ \frac{dq}{dt} + \lambda_3 q &= f_3(r) := \lambda_3 g_3(r) + g'_3(r)(f_2(r) - \lambda_2 r), \end{aligned} \quad (4.1)$$

where the eigenvalues satisfy $\lambda_1 < \lambda_2 < \lambda_3$. This system is conjugate to

$$\begin{aligned} \frac{d\tilde{p}}{dt} + \lambda_1 \tilde{p} &= 0, \\ \frac{d\tilde{r}}{dt} + \lambda_2 \tilde{r} &= f_2(\tilde{r}), \\ \frac{d\tilde{q}}{dt} + \lambda_3 \tilde{q} &= 0, \end{aligned} \quad (4.2)$$

through the change of variables $\tilde{p} = p - g_1(r)$, $\tilde{r} = r$, $\tilde{q} = q - g_3(r)$. Hence, the centre manifold of the original system is given by $p = g_1(r)$, $q = g_3(r)$.

The functions g_1 , f_2 and g_3 are chosen so that one component of the manifold (the p -component) is not smooth, and hence cannot be approximated by a Taylor expansion:

$$g_1(r) = M\varepsilon^2 \sin(r/\varepsilon) |\sin(r/\varepsilon)|, \quad f_2(r) = -r^3 \quad \text{and} \quad g_3(r) = r^3,$$

with $M, \varepsilon > 0$. With this choice g_1 is continuously differentiable with derivative

$$g_1'(r) = 2M\varepsilon |\sin(r/\varepsilon)| \cos(r/\varepsilon),$$

but not twice differentiable.

We consider the classical case of the centre manifold by choosing

$$\lambda_1 = -10, \quad \lambda_2 = 0, \quad \lambda_3 = 10.$$

We then assign $Pu = p$, $Ru = r$ and $Qu = q$, $A_- = \lambda_1$, $\lambda_- = \lambda_+ = \lambda_2$ and $A_+ = \lambda_3$. The constants in the trichotomy may be taken to be unity: $K_P = K_R = K_Q = 1$.

The non-linear term $f(p, r, q) = f(r) = (f_1(r), f_2(r), f_3(r))$ is not globally Lipschitz. In fact, the R -component of the solution to (4.1) is easily seen to blow up backward in time. We truncate the non-linear term by means of a piecewise linear function θ with

$$\theta(x) = \begin{cases} 1, & 0 \leq x \leq 1, \\ 1 - (x - 1), & 1 \leq x \leq 2, \\ 0, & x \geq 2. \end{cases} \quad (4.3)$$

Then, for a radius $\rho > 0$, and power $w > 0$ still to be chosen, we set

$$f_\rho(r) = \theta\left(\frac{|r|^w}{\rho^w}\right) f(r),$$

for all $r \in \mathbb{R}$. Note that since the non-linear term $f(r)$ is not smooth, there is no clear advantage in considering a smooth truncation function. In fact, any smooth alternative to θ satisfying (4.3) would have $\sup \theta' > 1$, which would inflate the Lipschitz constant of f_ρ . The blow-up of the original system will make the effect of the truncation appear within just a handful of iterations of the method.

Straightforward calculation leads to the following estimates for the Lipschitz constants for each component of f_ρ .

$$M_{P,v} = M \left(2w2^{-1/w} |\lambda_1| \frac{\varepsilon^2}{\rho} + 2|\lambda_1|\varepsilon + 2(2w+3)2^{2/w} \varepsilon \rho^2 + 2^{3/w} 4\rho^3 \right),$$

$$M_{R,v} = (2w+3)2^{2/w} \rho^2,$$

$$M_{Q,v} = (2w+3)2^{2/w} \lambda_3 \rho^3 + 6(w+2)2^{4/w} \rho^4.$$

Notice that we have used $\lambda_2 = 0$. With the choices $\lambda_1 = -10$ and $\lambda_3 = 10$, the spectral gap conditions in (2.3) read as follows:

$$M \left(40w2^{-1/w} \frac{\varepsilon^2}{\rho} + 20\varepsilon + 2(2w+3)2^{2/w} \varepsilon \rho^2 + 2^{3/w} 4\rho^3 \right) + (2w+3)2^{2/w} \rho^2 < 10, \quad (4.4)$$

$$(2w+3)2^{2/w} (\rho^2 + 10\rho^3) + 6(w+2)2^{4/w} \rho^4 < 10. \quad (4.5)$$

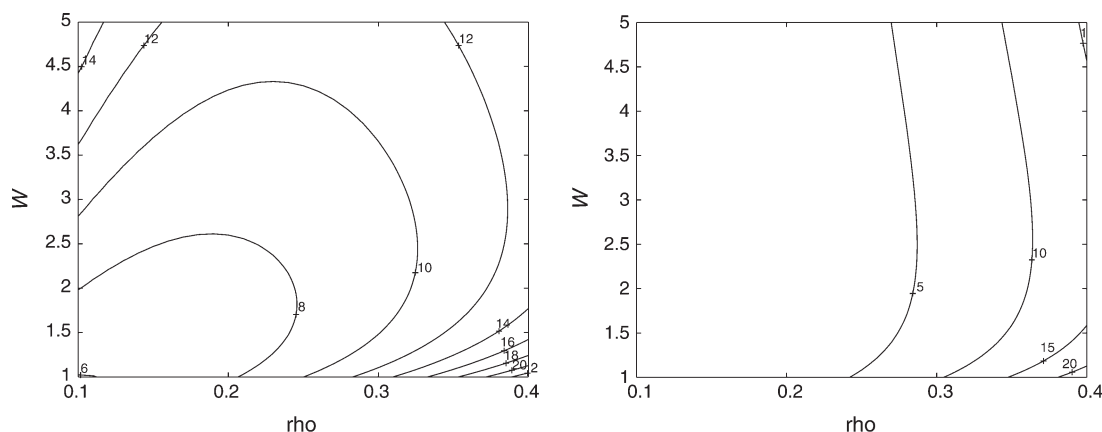


FIG. 1. Contour plots of the left hand side of (4.4) and (4.5), respectively.

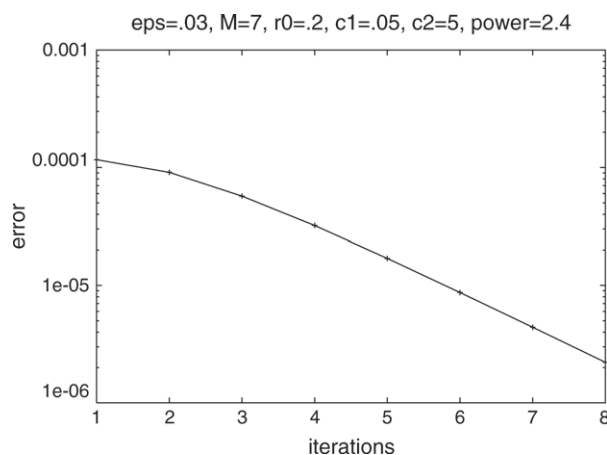


FIG. 2. Error from exact manifold of (4.1).

It is necessary that $M\varepsilon < 1/2$, otherwise no choice of ρ and w would satisfy the first condition. In fact, we want ε small, so that the non-smoothness of the centre manifold becomes a significant issue near the origin.

We take $\varepsilon = 0.03$ and $M = 7$. The corresponding contour plots for (4.4) and (4.5) are shown in Fig. 1. From those plots, one can see that the choice $w = 2.4$ yields approximately the largest possible choice for ρ , which we approximate to $\rho = 0.32$. Using these parameter settings, together with $c_1 = 0.05$ and $c_2 = 5$, we ultimately achieve the sort of convergence one expects from a contraction mapping (see Fig. 2). The time series plots for each component of the system (4.4) as computed by the mapping through the first seven iterations are plotted in Figs 3–5, along with the exact solution. Though not labelled, each iterate can be identified by its duration. The exact solution is plotted up to the point (backward in time) where the ball of truncation is reached. Recall that the desired quantities are at $t = 0$, so that the time series for the eighth and final iteration, which would be the most costly

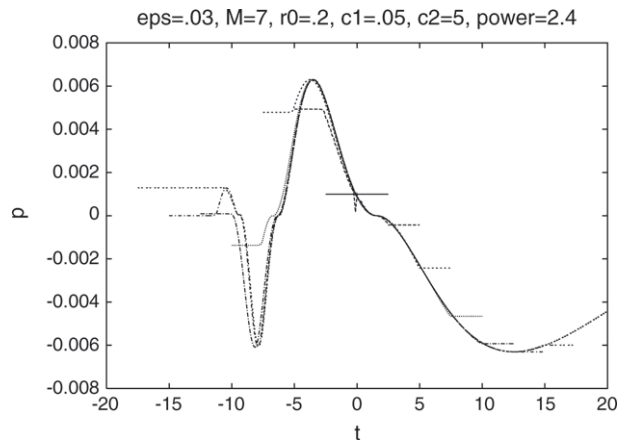


FIG. 3. Time series for P -component of solution to (4.1) as approximated by iterates 1 through 7.

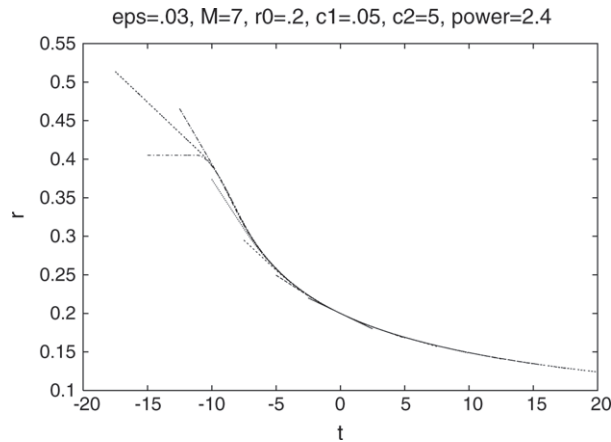


FIG. 4. Time series for R -component of iterates solution to (4.1) as approximated by 1 through 7.

of all, does not need to be computed. Note that for these choices of c_1 and c_2 , from the fourth iteration on, the error is reduced by roughly a factor of two. The depressed reduction factors for the first several iterations are consistent with the blip in the second iterate's time series for the P - and Q - components, which is progressively ironed out by subsequent iterations.

4.2 Application to an example from porous medium

Our second application is to a centre-stable manifold for a system of two ordinary differential equations

$$\begin{aligned} \frac{d\mu}{dt} &= f_1(\mu, \xi) = \mu\xi, \\ \frac{d\xi}{dt} &= f_2(\mu, \xi) = \frac{1}{n} \left(\frac{1}{a} \xi + \mu - \xi^2 - a\mu^2 - b\xi\mu \right). \end{aligned} \quad (4.6)$$

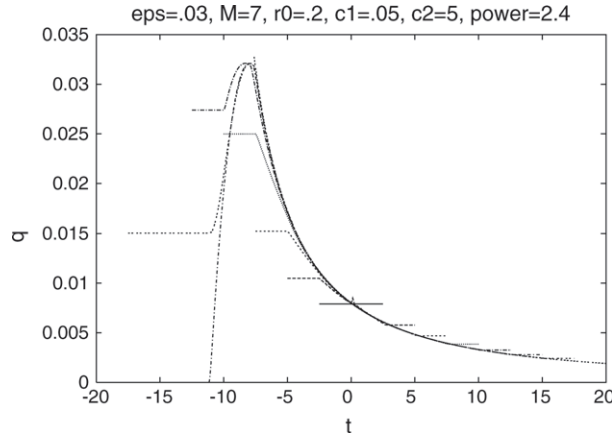


FIG. 5. Time series for Q -component of solution to (4.1) as approximated by iterates 1 through 7.

The coefficients a and b are simple expressions of n and d ;

$$a = 2(nd + 2) \quad \text{and} \quad b = n(d + 2) + 4.$$

At $\alpha = \alpha^*$, part of this manifold is a connecting orbit from the rest point at the origin to that at $(0, 1/\alpha)$. The critical value α^* determines the exponent in the interface between a gas and vacuum in the focusing problem of porous medium; see Aronson & Graveleau (1993). The parameter d is the spatial dimension of the porous medium model, and $n + 1$ is the power in the non-linear Laplacian. The asymptotic behaviour of $\alpha^* = \alpha^*(n, d)$ both as $n \rightarrow 0$ and as $n \rightarrow \infty$ has been studied in Aronson & Graveleau (1993). While we will note how the gap condition varies with n , we will not recompute α^* . Rather, we will compute just a single point on the centre-stable manifold for $n = d = 2$, at the critical value $\alpha = \alpha^*$ reported in Aronson & Graveleau (1993) and computed by the methods in Kevrekidis (1987). The purpose of this example is to demonstrate the error estimate for the mapping. The manifold, being smooth, can be computed much more effectively by means of a series, as we demonstrate below. Without some sort of estimate on the partial derivatives, however, there is no error estimate for the partial sum.

The series expansion of the manifold is straightforward. One sets

$$\zeta = \Phi(\mu) = \phi_0 + \phi_1\mu + \phi_2\mu^2 + \cdots,$$

and differentiates to obtain

$$f_2(\mu, \Phi) = \Phi'(\mu)f_1(\mu, \Phi(\mu)).$$

Expanding in the series gives

$$\frac{1}{n} \left[\frac{1}{\alpha} \sum \phi_j \mu^j + \mu - \left(\sum \phi_j \mu^j \right)^2 - a\mu^2 - b \left(\sum \phi_j \mu^j \right) \mu \right] = \left(\sum j \phi_j \mu^{j-1} \right) \mu \left(\sum \phi_j \mu^j \right),$$

which we rewrite as

$$\frac{1}{n} \left[\frac{1}{\alpha} \sum_{j=0}^{\infty} \phi_j \mu^j + \mu - \sum_{j=0}^{\infty} \left(\sum_{k=0}^j \phi_k \phi_{j-k} \right) \mu^j - a\mu^2 - b \sum_{j=1}^{\infty} \phi_{j-1} \mu^j \right] = \sum_{j=1}^{\infty} \left(\sum_{k=0}^j k \phi_k \phi_{j-k} \right) \mu^j.$$

Gathering the constant terms ($j = 0$) amounts to

$$\frac{1}{n} \left[\frac{1}{\alpha} \phi_0 - \phi_0^2 \right] = 0,$$

from which we select $\phi_0 = 0$. Solving similarly and successively for $j = 1, 2, 3 \dots$, we find

$$\phi_1 = -\alpha, \quad \phi_2 = \alpha[(n+1)\phi_1^2 + b\phi_1 + a],$$

and, for $j \geq 3$,

$$\phi_j = \alpha \left[\sum_{k=1}^{j-1} (1+nk) \phi_k \phi_{j-k} + b\phi_{j-1} \right].$$

We begin our treatment of (4.6) by diagonalising the linear part through the change of variables

$$r = -\mu \quad \text{and} \quad p = \alpha\mu + \xi,$$

to rewrite the system as

$$\begin{aligned} \frac{dp}{dt} &= \frac{1}{n\alpha} p - \tilde{a}r^2 - \tilde{b}rp - \frac{1}{n}p^2 \\ \frac{dr}{dt} &= \alpha r^2 + rp, \end{aligned} \tag{4.7}$$

where

$$\tilde{a} = \frac{1}{n} (\alpha^2 + a - ab) + \alpha^2 \quad \text{and} \quad \tilde{b} = \frac{1}{n} (2\alpha - b) + \alpha.$$

In the form of (2.1), we have for $u = (p, r)^\top$

$$A = \begin{bmatrix} \frac{1}{n\alpha} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f(u) = \begin{pmatrix} -\tilde{a}r^2 - \tilde{b}rp - \frac{1}{n}p^2 \\ \alpha r^2 + rp \end{pmatrix}. \tag{4.8}$$

The first ingredient of a rigorous error estimate are the local Lipschitz constants. We note, that for $|u_i|_v \leq \rho$,

$$\begin{aligned} |Rf(u_1) - Rf(u_2)| &\leq \alpha(|r_1| + |r_2|)|r_1 - r_2| + |r_1||p_1 - p_2| + |p_2||r_1 - r_2| \\ &\leq M_R(\rho)|u_1 - u_2|_v, \end{aligned}$$

for $M_R(\rho) = (2\alpha + 2)\rho$. It follows immediately from $f(0) = 0$ that

$$|Rf(u)| \leq M_R(\rho)|u|_v \leq \rho M_R(\rho), \quad \text{for all } |u|_v \leq \rho.$$

Since the manifold for this system is smooth, we truncate the non-linearity according to

$$f_\rho(u) = \tilde{\theta} \left(\frac{|u|_v}{\rho} \right),$$

where $\tilde{\theta}$ is the cubic spline function

$$\tilde{\theta}(x) = \begin{cases} 1, & \text{if } 0 \leq x < 1, \\ 2(x-1)^3 - 3(x-1)^2 + 1, & \text{if } 1 \leq x < 2, \\ 0, & \text{if } 2 \leq x. \end{cases}$$

For fixed ρ , the global Lipschitz constant for the R -component of the truncated non-linearity is estimated by

$$\begin{aligned} |Rf_\rho(u_1) - Rf_\rho(u_2)| &= \left| \tilde{\theta}\left(\frac{|u_1|_v}{\rho}\right) Rf(u_1) - \tilde{\theta}\left(\frac{|u_2|_v}{\rho}\right) Rf(u_2) \right|_v \\ &= \left| \left[\tilde{\theta}\left(\frac{|u_1|_v}{\rho}\right) - \tilde{\theta}\left(\frac{|u_2|_v}{\rho}\right) \right] f(u_1) - \tilde{\theta}\left(\frac{|u_2|_v}{\rho}\right) [f(u_1) - f(u_2)] \right|_v \\ &\leq \tilde{\theta}'(h_0) \left| \frac{|u_1|_v - |u_2|_v}{\rho} \right| |Rf(u_1)|_v + |f(u_1) - f(u_2)|_v \\ &\leq \frac{3}{2\rho} ||u_1|_v - |u_2|_v| \rho M_R(\rho) + M_R(\rho) |u_1 - u_2|_v \\ &\leq \frac{5}{2} M_R(\rho) |u_1 - u_2|_v \\ &= M_{R,\rho} |u_1 - u_2|_v, \end{aligned}$$

for all u_1, u_2 , where

$$M_{R,\rho} = \frac{5}{2} M_R(\rho) = 5(\alpha + 1)\rho.$$

Similarly, we derive

$$|Pf(u_1) - Pf(u_2)| \leq 2 \left(|\tilde{a}| + |\tilde{b}| + \frac{1}{n} \right) \rho |u_1 - u_2|_v,$$

and ultimately

$$|Pf_\rho(u_1) - Pf_\rho(u_2)| \leq M_{P,\rho} |u_1 - u_2|_v,$$

for all u_1, u_2 , where

$$M_{P,\rho} = 5 \left(|\tilde{a}| + |\tilde{b}| + \frac{1}{n} \right).$$

Since there is no Q -component, we have

$$\kappa_{\sigma,v} = \max \left(\frac{M_{P,\rho}}{\sigma_- - \lambda_-}, \frac{M_{R,\rho}}{-\sigma_-} \right).$$

Setting the two arguments in the maximum function equal, we arrive at

$$\sigma_- = \frac{M_{R,\rho} \lambda_-}{M_{R,\rho} + M_{P,\rho}}.$$

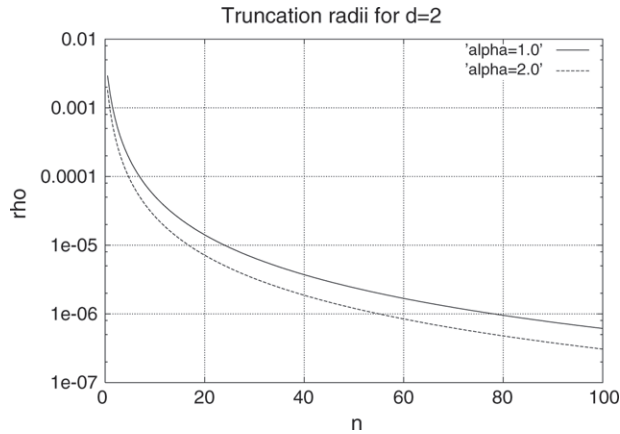


FIG. 6. Truncation radii from (4.9).

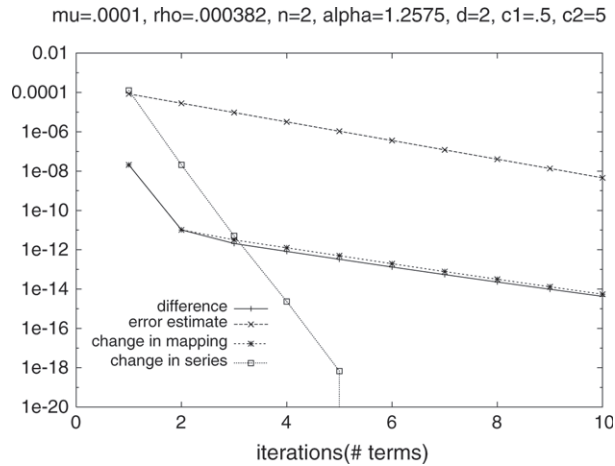


FIG. 7. Convergence to centre manifold for (4.6).

For this choice of σ_- , we have

$$\kappa_{\sigma,v} = \frac{M_{R,\rho} + M_{P,\rho}}{-\Lambda_-} = 5an \left(\alpha + 1 + |\tilde{a}| + |\tilde{b}| + \frac{1}{n} \right) \rho.$$

To achieve an asymptotic contraction rate of $1/2$, we take

$$\rho = \left[10an \left(\alpha + 1 + |\tilde{a}| + |\tilde{b}| + \frac{1}{n} \right) \right]^{-1}. \quad (4.9)$$

The truncation radii for $\alpha = 1.0$ and $\alpha = 2.0$ are plotted in Fig. 6.

The critical value, which in general satisfies $1 \leq \alpha^* < 2$, was computed by I. G. Kevrekidis to be 1.25750 for $n = 2$, Aronson & Graveleau (1993). The convergence of both the partial sums of the series and the iterates of the mapping are compared in Fig. 7. By ‘change from mapping’ we refer to

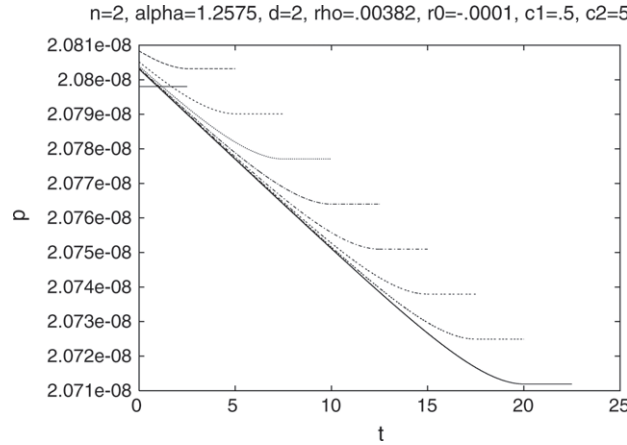


FIG. 8. Time series for the P -component of solution to (4.7) as approximated by iterates 1 through 9.

(the absolute value of) the difference in successive iterations of the computed ξ value. By ‘change in series’ we mean, of course, the (absolute value of the) final term in the partial sum. By ‘difference’ we refer to (the absolute value of) the difference between the P -component as computed by the series and by the mapping. For this smooth case, the series converges very rapidly, and is then, for all practical reasons, the preferred approach. The error estimate, while providing some piece of mind for the mapping, appears to be overly pessimistic and really more of theoretical interest.

The time series plots for the P -component of the system (4.7) as computed by the mapping through the first nine iterations are plotted in Fig. 8.

5. The basic algorithm

We rewrite the contraction mapping in the following form, depending on whether time t is positive or not. For $t > 0$, we use

$$\begin{aligned} \mathcal{T}(\varphi, r_0)(t) = & e^{-tA}r_0 - \int_t^\infty e^{-(t-\tau)A}Pf(\varphi(\tau))d\tau \\ & + \int_0^t e^{-(t-\tau)A}(R+Q)f(\varphi(\tau))d\tau + \int_{-\infty}^0 e^{-(t-\tau)A}Qf(\varphi(\tau))d\tau, \end{aligned} \quad (5.1)$$

and for $t \leq 0$

$$\begin{aligned} \mathcal{T}(\varphi, r_0)(t) = & e^{-tA}r_0 - \int_0^\infty e^{-(t-\tau)A}Pf(\varphi(\tau))d\tau \\ & - \int_t^0 e^{-(t-\tau)A}(P+R)f(\varphi(\tau))d\tau + \int_{-\infty}^t e^{-(t-\tau)A}Qf(\varphi(\tau))d\tau. \end{aligned} \quad (5.2)$$

Each ‘approximate trajectory’ $\varphi^j(r_0)$ is piecewise constant and, hence, can be represented by its values $\varphi_k^j = \varphi^j(kh_j)$ at the grid points kh_j , $k = -N_j, \dots, N_j$. Similarly, we set, for simplicity, the evaluation of the non-linear term $f_k^j = f(\varphi_k^j)$ for all such k .

In order to advance the sequence of approximate trajectories to the step $j + 1$, we define, for $k = 1, 2, \dots, N_{j+1}$, the quantity $\ell_k = \ell_k^j$ as the non-negative integer satisfying

$$\begin{cases} \ell_k h_j \leq kh_{j+1} < (\ell_k + 1)h_j, & \text{if } kh_{j+1} < N_j h_j, \\ \ell_k = N_j, & \text{otherwise.} \end{cases} \quad (5.3)$$

In the former case of (5.3), in which $0 < t = kh_{j+1}$ is situated somewhere within the data at the j th iterate, the mapping $\mathcal{T}_{h_j}^{N_j}(\varphi^j, r_0)$ may be written as

$$\begin{aligned} \varphi_k^{j+1} = & - \int_{N_j h_j}^{\infty} P f_{N_j}^j d\mu - \sum_{\ell=\ell_k+1}^{N_j-1} \int_{\ell h_j}^{(\ell+1)h_j} P f_{\ell}^j d\mu - \int_{kh_{j+1}}^{(\ell_k+1)h_j} P f_{\ell_k}^j d\mu \\ & + e^{-kh_{j+1}A} r_0 + \int_{\ell_k h_j}^{kh_{j+1}} (R + Q) f_{\ell_k}^j d\mu + \sum_{\ell=0}^{\ell_k-1} \int_{\ell h_j}^{(\ell+1)h_j} (R + Q) f_{\ell}^j d\mu \\ & + \sum_{\ell=-N_j+1}^0 \int_{(\ell-1)h_j}^{\ell h_j} Q f_{\ell}^j d\mu + \int_{-\infty}^{-N_j h_j} Q f_{-N_j}^j d\mu, \end{aligned}$$

where

$$d\mu = d\mu(\tau) = e^{-(kh_{j+1}-\tau)A} d\tau.$$

We split R into three projectors $R = R_- \oplus R_0 \oplus R_+$, where R_0 is the projector onto the null space of A , and

$$R_- = \text{proj. onto } \{v | Av = \lambda v, \operatorname{Re}(\lambda) < 0\} \quad R_+ = \text{proj. onto } \{v | Av = \lambda v, \operatorname{Re}(\lambda) > 0\}.$$

Note that in the most general setting, one may need to split instead P or Q . Evaluating the elementary integrals leads to the final expression

$$\begin{aligned} \varphi_k^{j+1} = & e^{-kh_{j+1}A} r_0 + E_{N_j} P f_{N_j}^j - \sum_{\ell=\ell_k+1}^{N_j-1} (E_{\ell+1} - E_{\ell}) P f_{\ell}^j - (E_{\ell_k+1} - A^{-1}) P f_{\ell_k}^j \\ & + (A^{-1} - E_{\ell_k})(R_{\pm} + Q) f_{\ell_k}^j + \sum_{\ell=0}^{\ell_k-1} (E_{\ell+1} - E_{\ell})(R_{\pm} + Q) f_{\ell}^j \\ & + (kh_{j+1} - \ell_k h_j) R_0 f_{\ell_k}^j + \sum_{\ell=0}^{\ell_k-1} h_j R_0 f_{\ell}^j + \sum_{\ell=-N_j+1}^0 (E_{\ell} - E_{\ell-1}) Q f_{\ell}^j + E_{-N_j} Q f_{-N_j}^j, \end{aligned} \quad (5.4)$$

where

$$E_{\ell} = E_{\ell}^{j,k} = A^{-1} e^{-(kh_{j+1}-\ell h_j)A}.$$

In the simpler case, where $0 < t = kh_{j+1}$ is beyond the data at the j th iterate, we may express the mapping as

$$\begin{aligned}\phi_k^{j+1} &= e^{-kh_{j+1}A}r_0 - \int_{kh_{j+1}}^{\infty} P f_{N_j}^j d\mu + \int_{N_j h_j}^{kh_{j+1}} (R + Q) f_{N_j}^j d\mu \\ &+ \sum_{\ell=0}^{N_j-1} \int_{\ell h_j}^{(\ell+1)h_j} (R + Q) f_{\ell}^j d\mu + \sum_{\ell=-N_j+1}^0 \int_{(\ell-1)h_j}^{\ell h_j} Q f_{\ell}^j d\mu \\ &+ \int_{-\infty}^{-N_j h_j} Q f_{-N_j}^j d\mu,\end{aligned}\quad (5.5)$$

and, after integrating, as

$$\begin{aligned}\phi_k^{j+1} &= e^{-kh_{j+1}A}r_0 + A^{-1} P f_{N_j}^j + (A^{-1} - E_{N_j})(R_{\pm} + Q) f_{N_j}^j \\ &+ \sum_{\ell=0}^{N_j-1} (E_{\ell+1} - E_{\ell})(R_{\pm} + Q) f_{\ell}^j + (kh_{j+1} - N_j h_j) R_0 f_{N_j}^j \\ &+ \sum_{\ell=0}^{N_j-1} h_j R_0 f_{\ell}^j + \sum_{\ell=-N_j+1}^0 (E_{\ell} - E_{\ell-1}) Q f_{\ell}^j + E_{-N_j} Q f_{-N_j}^j.\end{aligned}\quad (5.6)$$

For $k = -N_{j+1}, \dots, -2, -1$, we set $\ell_k = \ell_{-k}$ and $\ell_0 = 0$, so that a similar treatment of the case in which $-N_j h_j < kh_{j+1} \leq 0$ leads us to

$$\begin{aligned}\phi_k^{j+1} &= e^{-kh_{j+1}A}r_0 + E_{N_j} P f_{N_j}^j - \sum_{\ell=0}^{N_j-1} (E_{\ell+1} - E_{\ell}) P f_{\ell}^j \\ &- \sum_{\ell=-\ell_k+1}^0 (E_{\ell} - E_{\ell-1})(P + R_{\pm}) f_{\ell}^j - (E_{-\ell_k} - A^{-1})(P + R_{\pm}) f_{-\ell_k}^j \\ &- \sum_{\ell=-\ell_k+1}^0 h_j R_0 f_{\ell}^j - (-\ell_k h_j - kh_{j+1}) R_0 f_{-\ell_k}^j + (A^{-1} - E_{-(\ell_k+1)}) Q f_{-\ell_k}^j \\ &+ \sum_{\ell=-N_j+1}^{-\ell_k-1} (E_{\ell} - E_{\ell-1}) Q f_{\ell}^j + E_{-N_j} Q f_{-N_j}^j.\end{aligned}\quad (5.7)$$

For $kh_{j+1} \leq -N_j h_j < 0$, we have

$$\begin{aligned} \varphi_k^{j+1} &= e^{-kh_{j+1}A} r_0 + E_{N_j} P f_{N_j}^j - \sum_{\ell=0}^{N_j-1} (E_{\ell+1} - E_\ell) P f_\ell^j \\ &\quad - \sum_{\ell=-N_j+1}^0 (E_\ell - E_{\ell-1})(P + R_\pm) f_\ell^j - (E_{-N_j} - A^{-1})(P + R_\pm) f_{-N_j}^j \\ &\quad - \sum_{\ell=-N_j+1}^0 h_j R_0 f_\ell^j - (N_j h_j - kh_{j+1}) R_0 f_{N_j}^j + A^{-1} Q f_{-N_j}^j. \end{aligned} \quad (5.8)$$

In order to simplify the coding, we express both cases $k \leq 0$ and $k > 0$ by a unified formula. To do so, we set

$$s = s(k) = \begin{cases} 1, & \text{if } k > 0, \\ -1, & \text{if } k \leq 0, \end{cases} \quad (5.9)$$

and define projectors

$$\mathcal{P}_{ic} = \begin{cases} Q, & \text{if } s > 0, \\ P, & \text{if } s \leq 0, \end{cases} \quad \mathcal{P}_c = \begin{cases} (R_\pm \oplus Q), & \text{if } s > 0, \\ (P \oplus R_\pm), & \text{if } s \leq 0, \end{cases} \quad \mathcal{P}_t = \begin{cases} P, & \text{if } s > 0, \\ Q, & \text{if } s \leq 0. \end{cases} \quad (5.10)$$

Mainly to conform to programming languages, we also introduce the summation notation

$$\sum_{\ell=\ell_1, \ell_2, s} a_\ell = \begin{cases} a_{\ell_1} + a_{\ell_1+1} + \cdots + a_{\ell_2}, & \text{if } s > 0, \\ a_{\ell_1} + a_{\ell_1-1} + \cdots + a_{\ell_2}, & \text{if } s \leq 0. \end{cases} \quad (5.11)$$

For the cases in which t is within the data at the j th iterate, we have

$$\begin{aligned} \varphi_k^{j+1} &= e^{-kh_{j+1}A} r_0 + E_{N_j} P f_{N_j}^j + \sum_{\ell=0, -s \cdot (N_j-1), -s} (E_\ell - E_{\ell-s}) \mathcal{P}_{ic} f_\ell^j \\ &\quad + (A^{-1} - E_{s \cdot \ell_k}) \mathcal{P}_c f_{s \cdot \ell_k}^j + \sum_{\ell=0, s \cdot (\ell_k-1), s} (E_{\ell+s} - E_\ell) \mathcal{P}_c f_\ell^j \\ &\quad + (kh_{j+1} - s \cdot \ell_k h_j) R_0 f_{s \cdot \ell_k}^j + \sum_{\ell=0, s \cdot (\ell_k-1), s} s \cdot h_j R_0 f_\ell^j \\ &\quad + (A^{-1} - E_{s \cdot (\ell_k+1)}) \mathcal{P}_t f_{s \cdot \ell_k}^j + \sum_{\ell=s \cdot (\ell_k+1), s \cdot (N_j-1), s} (E_\ell - E_{\ell+s}) \mathcal{P}_t f_\ell^j \\ &\quad + E_{-N_j} Q f_{-N_j}^j, \end{aligned} \quad (5.12)$$

and for those in which t is beyond the previous data

$$\begin{aligned}\varphi_k^{j+1} = & e^{-kh_{j+1}A}r_0 + E_{-s \cdot N_j} \mathcal{P}_{ic} f_{-s \cdot N_j}^j + \sum_{\ell=0, -s \cdot (N_j-1), -s} (E_\ell - E_{\ell-s}) \mathcal{P}_{ic} f_\ell^j \\ & + \sum_{\ell=0, s \cdot (N_j-1), s}^{N_j-1} (E_{\ell+s} - E_\ell) \mathcal{P}_c f_\ell^j + (A^{-1} - E_{s \cdot N_j}) \mathcal{P}_c f_{s \cdot N_j}^j \\ & + \sum_{\ell=0}^{N_j-1} s \cdot h_j R_0 f_\ell^j + (kh_{j+1} - N_j h_j) R_0 f_{N_j}^j + A^{-1} \mathcal{P}_t f_{s \cdot N_j}^j.\end{aligned}\quad (5.13)$$

6. An accelerated algorithm

The computation of the centre manifold can be accelerated by noting that at each step, the components of the approximate solution φ_k^j can be computed recursively in k , avoiding the convolution involved in the variation of constants formula. The direction of recursion depends on the projection: the P -component goes backward in time, the Q -component goes forward in time and the R -component goes forward for positive time and backward for negative time. This can be seen more easily as follows. The map $\varphi = \mathcal{T}(\psi, r_0)$ can be characterised as yielding the function φ in \mathcal{F}_σ which solves the inhomogeneous linear equation

$$\varphi' + A\varphi = f(\psi), \quad R\varphi(0) = r_0. \quad (6.1)$$

From the definitions (3.1) and (3.3) and the characterisation above, we find (under the assumption that $N_j h_j$ does not decrease) that

$$Q\varphi_{-N_j}^j = \int_{-\infty}^{-N_j h_j} e^{-(N_j h_j - \tau)A} Qf(\varphi_{-N_{j-1}}^{j-1}) d\tau = (AQ)^{-1} Qf(\varphi_{-N_{j-1}}^{j-1}),$$

and

$$P\varphi_{N_j}^j = - \int_{N_j h_j}^{\infty} e^{-(N_j h_j - \tau)A} Pf(\varphi_{N_{j-1}}^{j-1}) d\tau = (AP)^{-1} Pf(\varphi_{N_{j-1}}^{j-1}).$$

Hence, at the j th step, we are given the initial conditions for each component of φ^j , but at different times: that for the P -component is given at $t = N_j h_j$, that for the R -component is given at $t = 0$ and that for the Q -component is given at $t = -N_j h_j$. Each component of the approximate solution φ^j can then be found by a recursive scheme in the proper direction in time.

Now, note that by (3.1) and (3.3), each φ_k^j is the exact solution at time kh_j of the inhomogeneous equation (6.1) with $\psi = \varphi^{j-1}$. In the case of the Q -component, for instance, which goes forward in time, the value of $Q\varphi_k^j$ can be obtained from $Q\varphi_{k-1}^j$ by the variation of constants formula

$$Q\varphi_k^j = e^{-h_j A} Q\varphi_{k-1}^j + \int_{(k-1)h_j}^{kh_j} e^{-(kh_j - \tau)A} Qf(\varphi^{j-1}(\tau)) d\tau, \quad k = -N_j + 1, \dots, N_j.$$

Since the previous approximate solution $\varphi^{j-1} = \varphi^{j-1}(\tau)$ is piecewise constant and the time step h_j is not supposed to increase with j , the integral above can be broken down into at most two explicit terms, depending on whether for some integer i the time ih_{j-1} is strictly between $(k-1)h_j$ and kh_j or is equal to one of these two values.

Another simplification comes from the fact that within the intervals $(-N_j h_j, -N_{j-1} h_{j-1}]$ and $[N_{j-1} h_{j-1}, N_j h_j)$, the previous approximation is assumed constant, equal to $\varphi_{-N_{j-1}}^{j-1}$ and $\varphi_{N_{j-1}}^{j-1}$, respectively. Then, from the definition of the map \mathcal{T} , one can see that the Q -component of φ^j is constant in the interval $(-N_j h_j, -N_{j-1} h_{j-1}]$ and equal to $\varphi_{-N_j}^j$, while the P -component of φ^j is constant in the interval $[N_{j-1} h_{j-1}, N_j h_j)$ and equal to $\varphi_{N_j}^j$. Thus, we may start the recursion at the smallest k for which $-N_{j-1} h_{j-1} < k h_j$, for the Q -component, and at the largest k for which $k h_j < N_{j-1} h_{j-1}$, for the P -component.

Then, we write

$$Q\varphi_k^j = \begin{cases} (AQ)^{-1} Qf(\varphi_{-N_{j-1}}^{j-1}), & \text{if } i_{-k}^j = -N_{j-1}, \\ e^{-h_j A} Q\varphi_{k-1}^j & \text{if } 0 \leq i_k^j h_{j-1} \leq (k-1)h_j, \\ + (AQ)^{-1} (Q - e^{-h_j A} Q)f(\varphi_{i_k^j}^{j-1}), & \text{or } (i_k^j - 1)h_{j-1} \leq (k-1)h_j < 0, \\ e^{-h_j A} Q\varphi_{k-1}^j & \text{otherwise,} \\ + (AQ)^{-1} (e^{-(kh_j - i_k^j h_{j-1})A} Q - e^{-h_j A} Q)f(\varphi_{i_k^j - 1}^{j-1}) \\ + (AQ)^{-1} (Q - e^{-(kh_j - i_k^j h_{j-1})A} Q)f(\varphi_{i_k^j}^{j-1}), \end{cases}$$

where

$$i_k^j = \begin{cases} \text{smallest integer } i \text{ such that } -N_{j-1} \leq i \leq 0 \text{ and } i h_{j-1} \geq k h_j, & \text{for } k = -N_j, \dots, 0, \\ \text{largest integer } i \text{ such that } 0 \leq i \leq N_{j-1} \text{ and } i h_{j-1} < k h_j, & \text{for } k = 1, \dots, N_j. \end{cases}$$

Similarly the P -component goes backward in time, and can be computed by

$$P\varphi_k^j = e^{h_j A} P\varphi_{k+1}^j - \int_{k h_j}^{(k+1)h_j} e^{-(kh_j - \tau)A} P f(\varphi^{j-1}(\tau)) d\tau, \quad k = N_j - 1, \dots, -N_j,$$

which can be written as

$$P\varphi_k^j = \begin{cases} (AP)^{-1} P f(\varphi_{N_{j-1}}^{j-1}), & \text{if } -i_{-k}^j = N_{j-1}, \\ e^{h_j A} P\varphi_{k+1}^j & \text{if } 0 \geq -i_{-k}^j h_{j-1} \geq (k+1)h_j, \\ - (AP)^{-1} (e^{h_j A} P - P)f(\varphi_{-i_{-k}^j}^{j-1}), & \text{or } (-i_{-k}^j + 1)h_{j-1} \geq (k+1)h_j > 0, \\ e^{h_j A} P\varphi_{k+1}^j & \text{otherwise.} \\ - (AP)^{-1} (e^{-(kh_j + i_{-k}^j h_{j-1})A} P - P)f(\varphi_{-i_{-k}^j}^{j-1}) \\ - (AP)^{-1} (e^{h_j A} P - e^{-(kh_j + i_{-k}^j h_{j-1})A} P)f(\varphi_{-i_{-k}^j + 1}^{j-1}), \end{cases}$$

The R -component must be broken down into two recursions, one starting from $k = 0$ and going up to $k = N_j$, and the other, from $k = 0$ down to $k = -N_j$. We omit the details in this case since it is a combination of the two above, except for the fact that the R -component of the linear operator A may not be invertible.

The algorithm given above computes exactly the same approximate manifolds as the previous one, except for the round-off errors due to the difference in implementation. The above algorithm could indeed be found from the previous algorithm by looking for the appropriate recursion formula. The same acceleration can be applied to the original algorithm for inertial manifolds in Rosa (1995) and Jolly *et al.* (2000).

Our last remark concerns the computation of the terms associated with $\exp(-(kh_j - i_{k+1}^j h_{j-1})A)$. In case N_j and h_j are defined according to equalities in (3.4), i.e.

$$h_j = c_1 \gamma^j \quad \text{and} \quad N_j = c_2 j / h_j,$$

with a rational parameter $0 < \gamma < 1$ written in the form $\gamma = p/q$, one can verify that there are only p operators of the form $\exp(-(kh_j - i_{k+1}^j h_{j-1})A)$. Such operators can be computed in advance in order to optimise the recursive scheme above.

7. Existence of the centre manifold

In this section, we prove the existence of a fixed point of each of the maps $\mathcal{T}(\cdot, r_0)$, with $r_0 \in RE$, thereby proving the existence of the centre manifold, which is formed by the collection of these fixed points. We also prove the Lipschitz continuity of the manifold. The proof of existence of the fixed points is similar to that in Chow & Lu (1988) and Rosa & Temam (1996), so we only present a sketch of the proof. The Lipschitz continuity, however, is given in more detail since the Lipschitz constant that we obtain is sharper under the given spectral gap conditions.

The strategy is to look first for a Lipschitz constant $\kappa_{\sigma,v}$ such that

$$\|\mathcal{T}(\varphi_1, r_0) - \mathcal{T}(\varphi_2, r_0)\|_{\sigma,v} \leq \kappa_{\sigma,v} \|\varphi_1 - \varphi_2\|_{\sigma,v}. \quad (7.1)$$

Then we look for the spectral gap conditions which guarantee that $\kappa_{\sigma,v} < 1$. Since $f(0) = 0$, it is easy to see that $\mathcal{T}(0, r_0) \in \mathcal{F}_\sigma$, so that from the Lipschitz continuity of \mathcal{T} , it follows that indeed $\mathcal{T}(\varphi, r_0)$ belongs to \mathcal{F}_σ for all $\varphi \in \mathcal{F}_\sigma$, justifying the calculations.

Then, one finds that for all $t \in \mathbb{R}$,

$$\begin{aligned} e^{-\sigma(t)} |P\mathcal{T}(\varphi_1, r_0)(t) - P\mathcal{T}(\varphi_2, r_0)(t)| &\leq \frac{K_P M_{P,v}}{\sigma_- - \lambda_-} \|\varphi_1 - \varphi_2\|_{\sigma,v}, \\ e^{-\sigma(t)} |R\mathcal{T}(\varphi_1, r_0)(t) - R\mathcal{T}(\varphi_2, r_0)(t)| &\leq \max \left\{ \frac{K_R M_{R,v}}{\lambda_- - \sigma_-}, \frac{K_R M_{R,v}}{\sigma_+ - \lambda_+} \right\} \|\varphi_1 - \varphi_2\|_{\sigma,v}, \\ e^{-\sigma(t)} |Q\mathcal{T}(\varphi_1, r_0)(t) - Q\mathcal{T}(\varphi_2, r_0)(t)| &\leq \frac{K_Q M_{Q,v}}{\lambda_+ - \sigma_+} \|\varphi_1 - \varphi_2\|_{\sigma,v}. \end{aligned}$$

Thus (7.1) follows with $\kappa_{\sigma,v}$ as in (2.2). Hence, for $\kappa_{\sigma,v} < 1$, we need that

$$\frac{K_P M_{P,v}}{\sigma_- - \lambda_-}, \frac{K_R M_{R,v}}{\lambda_- - \sigma_-}, \frac{K_R M_{R,v}}{\sigma_+ - \lambda_+}, \frac{K_Q M_{Q,v}}{\lambda_+ - \sigma_+} < 1,$$

i.e.

$$\begin{aligned} K_P M_{P,v} &< \sigma_- - \Lambda_-, \\ K_R M_{R,v} &< \lambda_- - \sigma_-, \\ K_R M_{R,v} &< \sigma_+ - \lambda_+, \\ K_Q M_{Q,v} &< \Lambda_+ - \sigma_+. \end{aligned}$$

These are equivalent to

$$\begin{aligned} \Lambda_- + K_P M_{P,v} &< \sigma_- < \lambda_- - K_R M_{R,v}, \\ \lambda_+ + K_R M_{R,v} &< \sigma_+ < \Lambda_+ - K_Q M_{Q,v}. \end{aligned} \quad (7.2)$$

Thus, the existence of σ_- , σ_+ satisfying (7.2) is equivalent to the gap conditions (2.3). Hence, once the gap conditions are satisfied, an invariant manifold is obtained.

For the Lipschitz continuity of the manifold, we can state more precisely the following lemma.

LEMMA 7.1 We have

$$\text{Lip}_v \Phi \leq K_R,$$

and, more generally,

$$\|\varphi(r_1) - \varphi(r_2)\|_{\sigma,v} \leq K_R |r_1 - r_2|_v,$$

for all $r_1, r_2 \in RE$.

The proof of this lemma is given in the Appendix. With it, the proof of Theorem 2.1 is complete.

8. Convergence estimates

A proof of the convergence of the approximate centre manifolds to the exact centre manifold can be done in much the same way as that for the convergence of approximate inertial manifolds given in Rosa (1995), so that here we present just an outline of the proof. But we actually end up improving those estimates by using the sharper Lipschitz constant of the centre manifold, as given in Lemma 7.1.

First, we need an estimate for a solution on the centre manifold. This is given by the following lemma, which is proved in the Appendix.

LEMMA 8.1 For either $t \leq \tau \leq 0$ or $0 \leq \tau \leq t$, we have

$$|\varphi(r_0)(t) - \varphi(r_0)(\tau)|_v \leq |t - \tau| \beta e^{\sigma(t)} \|\varphi(r_0)\|_{\sigma,v},$$

where

$$\beta = K_R (\|AR\| + M_{R,v}).$$

With this estimate at hand, we obtain the following estimate pertaining to the discretised map \mathcal{T}_h^N , which is used to build the sequence of approximate manifolds.

LEMMA 8.2 Let $\sigma = (\sigma_+, \sigma_-)$ and $\tilde{\sigma} = (\tilde{\sigma}_+, \tilde{\sigma}_-)$ satisfy

$$\Lambda_- + K_P M_{P,v} < \sigma_- < \tilde{\sigma}_- < \lambda_- - K_R M_{R,v}, \quad (8.1)$$

and

$$\lambda_+ + K_R M_{R,v} < \tilde{\sigma}_+ < \sigma_+ < \Lambda_+ - K_Q M_{Q,v}. \quad (8.2)$$

For all $r_0 \in RE$ and all $\psi \in \tilde{\mathcal{F}}$, we have

$$\begin{aligned} & \|\varphi(r_0) - \mathcal{T}_h^N(\psi, r_0)\|_{\sigma, v} \\ & \leq \kappa_{\sigma, v} \|\varphi(r_0) - \psi\|_{\sigma, v} \\ & \quad + \beta \max \left\{ h \|\varphi(r_0)\|_{\sigma, v}, \frac{e^{-1-(\sigma_+ - \tilde{\sigma}_+)Nh}}{\sigma_+ - \tilde{\sigma}_+} \|\varphi(r_0)\|_{\tilde{\sigma}, v}, \frac{e^{-1-(\tilde{\sigma}_- - \sigma_-)Nh}}{\tilde{\sigma}_- - \sigma_-} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \right\}. \end{aligned}$$

The proof of this lemma is also left to the Appendix. By recursively applying these estimates to the sequence $\varphi^j(r_0)$, we obtain

$$\begin{aligned} \|\varphi(r_0) - \varphi^j(r_0)\|_{\sigma, v} & \leq \kappa_{\sigma, v}^j \|\varphi(r_0) - \varphi^0(r_0)\|_{\sigma, v} \\ & \quad + \beta \sum_{\ell=0}^{j-1} \kappa_{\sigma, v}^\ell \max \left\{ h_{j-\ell} \|\varphi(r_0)\|_{\sigma, v}, \frac{e^{-1-(\sigma_+ - \tilde{\sigma}_+)N_{j-\ell}h_{j-\ell}}}{\sigma_+ - \tilde{\sigma}_+} \|\varphi(r_0)\|_{\tilde{\sigma}, v}, \right. \\ & \quad \left. \frac{e^{-1-(\tilde{\sigma}_- - \sigma_-)N_{j-\ell}h_{j-\ell}}}{\tilde{\sigma}_- - \sigma_-} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \right\}, \end{aligned}$$

for all j .

In order to bound the terms on the right-hand side of the above estimate, we use the following lemma.

LEMMA 8.3 The following estimates hold for σ satisfying (7.2) and for all $r_0 \in RE$:

$$\|\varphi(r_0)\|_{\sigma, v} \leq K_R |r_0|, \quad \|\varphi^0(r_0)\|_{\sigma, v} \leq K_R |r_0|.$$

This lemma is applied with both σ and $\tilde{\sigma}$. Then, we obtain the estimate

$$\begin{aligned} & \|\varphi(r_0) - \varphi^j(r_0)\|_{\sigma, v} \\ & \leq 2K_R \kappa_{\sigma, v}^j |r_0| + \beta K_R |r_0| \sum_{\ell=0}^{j-1} \kappa_{\sigma, v}^\ell \max \left\{ h_{j-\ell}, \frac{e^{-1-(\sigma_+ - \tilde{\sigma}_+)N_{j-\ell}h_{j-\ell}}}{\sigma_+ - \tilde{\sigma}_+}, \frac{e^{-1-(\tilde{\sigma}_- - \sigma_-)N_{j-\ell}h_{j-\ell}}}{\tilde{\sigma}_- - \sigma_-} \right\}, \end{aligned} \quad (8.3)$$

for all j . We may now minimise the right-hand side above over $\tilde{\sigma}$ satisfying (8.1) and (8.2) to find

$$\begin{aligned} & \|\varphi(r_0) - \varphi^j(r_0)\|_{\sigma, v} \leq 2K_R \kappa_{\sigma, v}^j |r_0| \\ & \quad + \beta K_R |r_0| \sum_{\ell=0}^{j-1} \kappa_{\sigma, v}^\ell \max \left\{ h_{j-\ell}, \frac{e^{-1-(\sigma_+ - \lambda_+ - K_R M_{R, v})N_{j-\ell}h_{j-\ell}}}{\sigma_+ - \lambda_+ - K_R M_{R, v}}, \right. \\ & \quad \left. \frac{e^{-1-(\lambda_- - K_R M_{R, v} - \sigma_-)N_{j-\ell}h_{j-\ell}}}{\lambda_- - K_R M_{R, v} - \sigma_-} \right\}. \end{aligned} \quad (8.4)$$

By choosing appropriately the sequences $\{h_j\}_j$ and $\{N_j\}_j$, one obtains the convergence of $\varphi^j(r_0)$ to $\varphi(r_0)$ with respect to the norm $\|\cdot\|_{\sigma, v}$, with the convergence being uniform for r_0 bounded. More precisely, assume that time steps h_j decrease exponentially and that the time intervals $N_j h_j$ increase linearly according to

$$0 < h_j \leq c_1 \gamma^j, \quad N_j h_j \geq c_2 j, \quad (8.5)$$

for some constants $c_1, c_2 > 0$ and for some $0 < \gamma < 1$. Then, we find

$$\frac{\|\varphi(r_0) - \varphi^j(r_0)\|_{\sigma,v}}{|r_0|} \leq 2K_R \kappa_{\sigma,v}^j + \beta K_R C_3 j \eta^j, \quad (8.6)$$

where

$$\eta = \max\{\kappa_{\sigma,v}, \gamma, e^{-c_2(\sigma_+ - \lambda_+ - K_R M_{R,v})}, e^{-c_2(\lambda_- - K_R M_{R,v} - \sigma_-)}\} < 1, \quad (8.7)$$

$$C_3 = \max\left\{c_1, \frac{e^{-1}}{\sigma_+ - \lambda_+ - K_R M_{R,v}}, \frac{e^{-1}}{\lambda_- - K_R M_{R,v} - \sigma_-}\right\}. \quad (8.8)$$

This proves Theorem 3.1, with $C_1 = 2K_R$ and $C_2 = \beta K_R C_3$.

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Appendix. Proof of the lemmas

Proof of Lemma 7.1. In order to avoid products of the constants K_P , K_R and K_Q artificially inflating the estimate for the Lipschitz constant of Φ in the $|\cdot|_V$ norm, we consider the following equivalent norm for the space E :

$$|u|_\mu = \max \left\{ \sup_{s \geq 0} e^{A-s} |e^{-sA} Pu|, \sup_{s \in \mathbb{R}} e^{\lambda(s)} |e^{-sA} Ru|, \sup_{s \geq 0} e^{A+s} |e^{-sA} Qu| \right\}, \quad (\text{A.1})$$

where we have set $\lambda(s)$ to be $\lambda_+ s$, for negative s , and $\lambda_- s$, otherwise. Under this norm, the trichotomy holds with the multiplying constants normalised to 1, while the Lipschitz constants become $K_i M_i$, $i = P, R, Q$. Moreover, since

$$|u|_V \leq |u|_\mu \quad \text{and} \quad |Ru|_\mu \leq K_R |Ru| = K_R |Ru|_V,$$

it suffices to show that the Lipschitz constant of Φ with respect to this new norm is at most one.

The first step is to prove the following cone-invariance property.

Claim 1. If u_1 and u_2 are two solutions of (2.1) such that

$$|(R + Q)(u_1(t_0) - u_2(t_0))|_\mu < |P(u_1(t_0) - u_2(t_0))|_\mu,$$

at some time $t_0 \in \mathbb{R}$, then

$$|(R + Q)(u_1(t) - u_2(t))|_\mu < |P(u_1(t) - u_2(t))|_\mu,$$

for all time $t \geq t_0$.

By the continuity of the solutions, the interval of times $t \geq t_0$ up to which the strict inequality above holds is open. Hence, we just need to show that it is also closed. For that purpose, assume that for some $t_1 > t_0$, we have strict inequality for t such that $t_0 \leq t < t_1$. Let us show that we still have strict inequality for $t = t_1$. For $t_0 \leq t \leq t_1$,

$$\begin{aligned} & |P(u_1(t) - u_2(t))|_\mu \\ & \leq e^{-\Lambda_-(t-t_1)} |P(u_1(t_1) - u_2(t_1))|_\mu + K_P M_{P,v} \int_{t_0}^{t_1} e^{-\Lambda_-(t-s)} |u_1(s) - u_2(s)|_\mu ds \\ & \leq e^{-\Lambda_-(t-t_1)} |P(u_1(t_1) - u_2(t_1))|_\mu + K_P M_{P,v} \int_{t_0}^{t_1} e^{-\Lambda_-(t-s)} |P(u_1(s) - u_2(s))|_\mu ds. \end{aligned}$$

An application of the Gronwall inequality yields

$$|P(u_1(t) - u_2(t))|_\mu \leq |P(u_1(t_1) - u_2(t_1))|_\mu e^{-(\Lambda_- + K_P M_{P,v})(t-t_1)}, \quad \text{for } t_0 \leq t \leq t_1. \quad (\text{A.2})$$

Then,

$$\begin{aligned} & |R(u_1(t_1) - u_2(t_1))|_\mu \leq e^{-\lambda_-(t_1-t_0)} |R(u_1(t_0) - u_2(t_0))|_\mu \\ & \quad + K_R M_{R,v} \int_{t_0}^{t_1} e^{-\lambda_-(t_1-s)} |P(u_1(s) - u_2(s))|_\mu ds \\ & \leq e^{-\lambda_-(t_1-t_0)} |R(u_1(t_0) - u_2(t_0))|_\mu \\ & \quad + K_R M_{R,v} \int_{t_0}^{t_1} e^{-\lambda_-(t_1-s)} |P(u_1(t_1) - u_2(t_1))|_\mu e^{-(\Lambda_- + K_P M_{P,v})(s-t_1)} ds \\ & \leq e^{-\lambda_-(t_1-t_0)} |R(u_1(t_0) - u_2(t_0))|_\mu \\ & \quad + K_R M_{R,v} |P(u_1(t_1) - u_2(t_1))|_\mu \int_{t_0}^{t_1} e^{-(\Lambda_- - \lambda_- + K_P M_{P,v})(s-t_1)} ds \\ & \leq e^{-\lambda_-(t_1-t_0)} |R(u_1(t_0) - u_2(t_0))|_\mu + \theta |P(u_1(t_1) - u_2(t_1))|_\mu, \end{aligned}$$

where

$$\theta = \frac{K_R M_{R,v} (1 - e^{-(\Lambda_- - \lambda_- + K_P M_{P,v})(t_0-t_1)})}{-(\Lambda_- - \lambda_- + K_P M_{P,v})}.$$

Note that $-(\Lambda_- - \lambda_- + K_P M_{P,v}) > 0$ and that $0 < \theta < 1$. In order to find a contradiction, suppose that

$$|R(u_1(t_1) - u_2(t_1))|_\mu = |P(u_1(t_1) - u_2(t_1))|_\mu.$$

Then,

$$|P(u_1(t_1) - u_2(t_1))|_\mu \leq e^{-\lambda_-(t_1-t_0)} |R(u_1(t_0) - u_2(t_0))|_\mu + \theta |P(u_1(t_1) - u_2(t_1))|_\mu,$$

so that

$$|P(u_1(t_1) - u_2(t_1))|_\mu \leq \frac{e^{-\lambda_-(t_1-t_0)}}{1 - \theta} |R(u_1(t_0) - u_2(t_0))|_\mu.$$

Then, using (A.2) with $t = t_0$, we would have

$$\begin{aligned} & |P(u_1(t_0) - u_2(t_0))|_\mu \leq |P(u_1(t_1) - u_2(t_1))|_\mu e^{-(\Lambda_- + K_P M_{P,v})(t_0-t_1)} \\ & \leq \frac{e^{-(\Lambda_- - \lambda_- + K_P M_{P,v})(t_0-t_1)}}{1 - \theta} |R(u_1(t_0) - u_2(t_0))|_\mu. \end{aligned}$$

Finally, since at $t = t_0$ the R -component of the difference of the solutions is smaller than the P -component, we find

$$|R(u_1(t_0) - u_2(t_0))|_\mu \leq \frac{e^{-(\Lambda_- - \lambda_- + K_P M_{P,v})(t_0 - t_1)}}{1 - \theta} |R(u_1(t_0) - u_2(t_0))|_\mu.$$

However, one may now verify from the spectral gap condition (2.3) and from $t_0 < t_1$ that

$$\frac{e^{-(\Lambda_- - \lambda_- + K_P M_{P,v})(t_0 - t_1)}}{1 - \theta} < 1,$$

which gives a contradiction. Hence, we must have

$$|R(u_1(t_1) - u_2(t_1))|_\mu < |P(u_1(t_1) - u_2(t_1))|_\mu.$$

Similarly, one finds that

$$|Q(u_1(t_1) - u_2(t_1))|_\mu < |P(u_1(t_1) - u_2(t_1))|.$$

Thus,

$$|(R + Q)(u_1(t_1) - u_2(t_1))|_\mu < |P(u_1(t_1) - u_2(t_1))|,$$

and this completes the proof of the cone-invariance property. Similarly, one can prove the backward cone-invariance property.

Claim 2. If u_1 and u_2 are two solutions of (2.1) such that

$$|(P + R)(u_1(t_0) - u_2(t_0))|_\mu < |Q(u_1(t_0) - u_2(t_0))|_\mu,$$

at some time $t_0 \in \mathbb{R}$, then

$$|(P + R)(u_1(t) - u_2(t))|_\mu < |Q(u_1(t) - u_2(t))|_\mu,$$

for all time $t \leq t_0$.

From both cone-invariance properties, we can deduce that for any $r_1, r_2 \in RE$, and for all $t \in \mathbb{R}$,

$$|P(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |(R + Q)(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu, \quad (\text{A.3})$$

$$|Q(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |(P + R)(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu. \quad (\text{A.4})$$

Let us prove the first inequality, the second one is similar. If (A.3) were not true, we would have the strict inequality in the opposite direction for some $t_0 \in \mathbb{R}$. From the cone-invariance property, this strict opposite inequality would hold for all $t \geq t_0$. Then, just as we proceeded for (A.2), we would find

$$|P(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |P(\varphi(r_1)(t_1) - \varphi(r_2)(t_1))|_\mu e^{-(\Lambda_- + K_P M_{P,v})(t - t_1)},$$

for all t and t_1 with $t_0 \leq t \leq t_1$. Now, the solutions $\varphi(r_i)$, $i = 1, 2$, belong to the space \mathcal{F}_σ , for $\sigma = (\sigma_-, \sigma)$ satisfying (7.2). Thus,

$$|P(\varphi(r_1)(t_1) - \varphi(r_2)(t_1))|_\mu \leq (\|\varphi(r_1)\|_{\sigma,v} + \|\varphi(r_2)\|_{\sigma,v}) e^{-\sigma - t_1}.$$

Thus,

$$|P(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq (\|\varphi(r_1)\|_{\sigma,v} + \|\varphi(r_2)\|_{\sigma,v}) e^{-(A_- + K_P M_{P,v})t} e^{-(\sigma_- - A_- - K_P M_{P,v})t_1}.$$

Let t_1 tend to infinity to find that

$$|P(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu = 0,$$

which contradicts the strict inequality.

Now, from (A.3) and (A.4), we obtain directly

$$|(P + Q)(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |R(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu, \quad \text{for all } t \in \mathbb{R}. \quad (\text{A.5})$$

Hence, for $t \geq 0$,

$$\begin{aligned} |R(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu &\leq e^{-\lambda t} |R(\varphi(r_1)(0) - \varphi(r_2)(0))|_\mu \\ &\quad + K_R M_{R,v} \int_0^t e^{-\lambda - (t-s)} |R(\varphi(r_1)(s) - \varphi(r_2)(s))|_\mu ds. \end{aligned}$$

By the Gronwall inequality,

$$|R(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |r_1 - r_2|_\mu e^{-(\lambda - K_R M_{R,v})t},$$

for $t \geq 0$. Similarly, for $t \leq 0$,

$$|R(\varphi(r_1)(t) - \varphi(r_2)(t))|_\mu \leq |r_1 - r_2|_\mu e^{-(\lambda + K_R M_{R,v})t}.$$

Thus, using (A.5),

$$e^{-\sigma(t)} \|\varphi(r_1)(t) - \varphi(r_2)(t)\|_\mu \leq |r_1 - r_2|_\mu.$$

This implies the second inequality claimed in the lemma. Then, taking in particular $t = 0$, we find

$$|\Phi(r_1) - \Phi(r_2)|_\mu \leq |r_1 - r_2|_\mu,$$

which completes the proof. □

Proof of Lemma 8.1. Fix $r_0 \in RE$ and let $r(t) = R\varphi(r_0)(t)$, for $t \in \mathbb{R}$. Hence,

$$r' + Ar = Rf(\varphi(r_0)(t)),$$

for $t \in \mathbb{R}$. Then, using that $f(0) = 0$,

$$\begin{aligned} |r'(s)| &\leq |Ar| + |Rf(\varphi(r_0)(s))| \\ &\leq \|AR\| |r| + M_{R,v} |\varphi(r_0)(s)|_v \\ &\leq (\|AR\| + M_{R,v}) |\varphi(r_0)(s)|_v \\ &\leq (\|AR\| + M_{R,v}) e^{\sigma(s)} \|\varphi(r_0)\|_{\sigma,v}. \end{aligned}$$

Since the maximum of $\sigma(s)$ for either $t \leq s \leq \tau \leq 0$ or $0 \leq \tau \leq s \leq t$ is attained at $s = t$, we find

$$\begin{aligned} |\varphi(r_0)(t) - \varphi(r_0)(\tau)|_v &= |r(t) + \Phi(r(t)) - r(\tau) - \Phi(r(\tau))|_v \\ &\leq \max \{1, \text{Lip}_v \Phi\} |r(t) - r(\tau)|_v \\ &\leq \max \{1, \text{Lip}_v \Phi\} |t - \tau| (\|AR\| + M_{R,v}) e^{\sigma(t)} \|\varphi(r_0)\|_{\sigma,v}. \end{aligned}$$

From Lemma 7.1, we have $\text{Lip}_v \Phi \leq K_R$, and since $K_R \geq 1$ (just take $t = 0$ in the trichotomy), the proof is complete. \square

Proof of Lemma 8.2. Let $t \in \mathbb{R}$ and let k be as in (3.2). Then,

$$\begin{aligned} & e^{-\sigma(t)} |\varphi(r_0)(t) - \mathcal{T}_h^N(\psi, r_0)(t)|_v \\ & \leq e^{-\sigma(t)} |\varphi(r_0)(t) - \varphi(r_0)(-kh)|_v + e^{-\sigma(t)} |\varphi(r_0)(-kh) - \mathcal{T}_h^N(\psi, r_0)(-kh)|_v. \end{aligned}$$

For the second term, note that formally we have $\varphi(r_0)(-kh) = \mathcal{T}_h^N(\varphi(r_0), r_0)(-kh)$, so that as for the original map, one can show that

$$e^{-\sigma(kh)} |\varphi(r_0)(kh) - \mathcal{T}_h^N(\psi, r_0)(-kh)|_v \leq \kappa_{\sigma, v} \|\varphi(r_0) - \psi\|_{\sigma, v}.$$

Therefore, since $\sigma(t) \geq \sigma(kh)$ for all t , we can bound the second term as

$$e^{-\sigma(t)} |\varphi(r_0)(-kh) - \mathcal{T}_h^N(\psi, r_0)(-kh)|_v \leq \kappa_{\sigma, v} \|\varphi(r_0) - \psi\|_{\sigma, v}.$$

For the first term, let us start by considering the case $-Nh \leq t \leq Nh$. In this case, we apply Lemma 8.1 with $\tau = kh$, which is always less than or equal to t in absolute value:

$$e^{-\sigma(t)} |\varphi(r_0)(t) - \varphi(r_0)(kh)|_v \leq e^{-\sigma(t)} |t - kh| \beta e^{\sigma(t)} \|\varphi(r_0)\|_{\sigma, v} \leq h \beta \|\varphi(r_0)\|_{\sigma, v}.$$

For $t \leq -Nh$, we apply Lemma 8.1 with $\tilde{\sigma}$ instead of σ :

$$\begin{aligned} e^{-\sigma(t)} |\varphi(r_0)(t) - \varphi(r_0)(-Nh)|_v & \leq e^{-\sigma(t)} |t + Nh| \beta e^{\tilde{\sigma}(t)} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & = e^{\sigma+t} |t + Nh| \beta e^{-\tilde{\sigma}+t} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & = |t + Nh| e^{-(\sigma_+ - \tilde{\sigma}_+)|t + Nh|} e^{-(\sigma_+ - \tilde{\sigma}_+)Nh} \beta \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & \leq \frac{e^{-1}}{(\sigma_+ - \tilde{\sigma}_+)} e^{-(\sigma_+ - \tilde{\sigma}_+)Nh} \beta \|\varphi(r_0)\|_{\tilde{\sigma}, v}. \end{aligned}$$

For $t \geq Nh$, we have similarly

$$\begin{aligned} e^{-\sigma(t)} |\varphi(r_0)(t) - \varphi(r_0)(Nh)|_v & \leq e^{-\sigma(t)} (t - Nh) \beta e^{\tilde{\sigma}(t)} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & = e^{\sigma-t} (t - Nh) \beta e^{-\tilde{\sigma}-t} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & = (t - Nh) e^{-(\tilde{\sigma}_- - \sigma_-)(t - Nh)} e^{-(\tilde{\sigma}_- - \sigma_-)Nh} \beta \|\varphi(r_0)\|_{\tilde{\sigma}, v} \\ & \leq \frac{e^{-1}}{(\tilde{\sigma}_- - \sigma_-)} e^{-(\tilde{\sigma}_- - \sigma_-)Nh} \beta \|\varphi(r_0)\|_{\tilde{\sigma}, v}. \end{aligned}$$

Therefore,

$$\begin{aligned} e^{-\sigma(t)} |\varphi(r_0)(t) - \mathcal{T}_h^N(\psi, r_0)(t)|_v & \leq \kappa_{\sigma, v} \|\varphi(r_0) - \psi\|_{\sigma, v} \\ & + \beta \max \left\{ h \|\varphi(r_0)\|_{\sigma, v}, \frac{e^{-1 - (\sigma_+ - \tilde{\sigma}_+)Nh}}{\sigma_+ - \tilde{\sigma}_+} \|\varphi(r_0)\|_{\tilde{\sigma}, v}, \right. \\ & \quad \left. \frac{e^{-1 - (\tilde{\sigma}_- - \sigma_-)Nh}}{\tilde{\sigma}_- - \sigma_-} \|\varphi(r_0)\|_{\tilde{\sigma}, v} \right\}, \end{aligned}$$

for all $t \in \mathbb{R}$, which proves the lemma. \square

Proof of Lemma 8.3. Since $f(0) = 0$, we have $\varphi(0) \equiv 0$, thus the first estimate follows from Lemma 7.1. The second estimate follows trivially from the definition $\varphi^0(r_0) = r_0$ and the fact that $K_R \geq 1$. \square