On the Controllability of a Sampled-Data System under Nonuniform Sampling

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***Abstract*—In this paper, controllability of a sampled-data system under nonuniform sampling is investigated. The problem is motivated with the widespread networked and embedded control systems in which sampling periods may be varying and exact sampling instants are unknown. The continuous-time system is assumed to be controllable and a condition is derived for sampling intervals under which the resulting sampled-data system will also be controllable. The proposed condition requires lesser number of samples to make sampled system controllable as compared with the existing results.**

I. Introduction

Nowadays most of the controllers are implemented using computers because of the numerous advantages. Using computers, complex control algorithms can be implemented and modified quite easily. This motivates us to study properties of systems that involve digital controllers. Conventionally, sampling periods in such systems are assumed to be constant [1]. In recent applications like networked and embedded control systems, sampling period may not be constant and sampling instants are uncertain[2].

In this paper, we investigate how nonuniform sampling affect the controllability of the sampled-data system. Controllability is a sufficient condition for stabilization of the sampled-data system. For sampled-data systems with uniform sampling, system matrices are constant and there exist algebraic conditions based on which we can infer about their controllability. But in non-uniformly sampled systems, exact sampling intervals are unknown and time-varying, resulting in uncertain and time-varying discrete-time systems. This makes the analysis of non-uniformly sampled systems much more difficult.

A sampled-data system with uniform sampling may loose controllability if the sampling frequency is pathological. This issue was investigated in [3] wherein, the authors demonstrated that if the continuous-time system is controllable and sampling is done at uniform rate relative to eigenvalues of A matrix, that is for any two eigenvalues of A with equal real part and with the difference of imaginary parts as the integer multiple of sampling frequency in radians, then the resulting discrete-time system may loose controllability. Since then, many studies are conducted to propose techniques to preserve the controllability and/or observability of sampled-data systems [4], [5], [6], [7]. In [4], it is shown that if the continuous-time system is controllable and a sample taken at an irrational time instant is included with the samples taken at rational time instants such that the ratio of the irrational to rational sampling interval is not a rational number, then the resulting discrete-time system will also be controllable for any choice of sampling period. For periodic nonuniform sampling, it has been shown in [5] that the discrete-time system will be controllable if the frame sampling frequency is non-pathological. For uncertain and time-varying sampling periods, as is the case with networked and embedded control systems, a condition for observability is derived in [6] in which it is proved that for a given time interval, if the number of samples are greater than a threshold, the resulting discrete-time system will be observable. The minimum number of samples required in a given time interval for the discrete time system to be controllable depends on the imaginary parts of the eigenvalues and order of the system.

In our study we show that if a system is sampled non-uniformly and the number of samples are greater than a threshold, then the discrete-time system will be controllable. The major contribution of this paper is that the derived condition for the threshold number of samples depends solely on the imaginary part of the eigenvalues of the system. Because of this, the required number of samples for controllable/observable discrete-time system for the proposed condition will be lesser as compared to [6]. Consequently, the proposed condition is less conservative.

The rest of this paper is organized as follows: In Section II, problem is formulated. Controllability analysis is done in Section III. Section IV gives different numerical result for the above condition. Finally, Section V gives the conclusion of our study.

II. Problem Formulation

Consider a continuous-time system given below:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{1}$$

where $x(t) \in R^n$ denotes the state of the system, $u(t) \in R^m$ and $y(t) \in R^p$ are the inputs and outputs of the system. Matrices A, B, C and D are constant matrices of appropriate dimensions.

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Fig. 1: Sampling Pattern

Let $t_0 < t_1 < \cdots < t_n$ be a monotonically increasing sequence of sampling instants and $T_1 \neq T_2 \neq T_3 \cdots \neq T_n$ be non-uniformly spaced sampling intervals as shown in Fig. 1. The state of the system given in (1) between any two sampling instants evolves as:

$$x(t) = e^{A(t-t_k)}x(t_k) + \int_{t_k}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \quad \forall t \in [t_k, t_{k+1}). \tag{2}$$

Define $T_k = t_k - t_{k-1}$ as the k-th sampling interval, the state of the system over k-th sampling interval will be governed by

$$x(t_{k+1}) = e^{AT_{k+1}}x(t_k) + \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)}Bu(\tau)d\tau,$$

which may be written as

$$x(t_{k+1}) = G(T_{k+1})x(t_k) + H(T_{k+1})u(t_k), \tag{3}$$

where $G(T_{k+1}) = e^{AT_{k+1}}$ and $H(T_{k+1}) = A^{-1}(e^{AT_{k+1}} - I)B$, for an invertible matrix A. We note that the matrices $G(T_{k+1})$ and $H(T_{k+1})$ depend on the uncertain and time-varying sampling period T_k, making the resultant discrete-time system time-varying and uncertain. If the sampling period is uniform and constant, the discrete-time system in (3) will be controllable if the associated controllability matrix is full rank [1]. However, if the sampling period is time-varying and unknown, it becomes very difficult and challenging to prove that the controllability matrix associated with the discrete-time system in (3) is full rank. The goal of this paper is to derive conditions for the controllability of the discrete-time system in (3), which in turn, reduces to proving that the controllability matrix is full rank.

In order to verify the controllability of the system in (3), we express the state of system in terms of initial condition $x(t_0)$ and input vector $u(t)$.

$$x(t_1) = e^{AT_1}x(t_0) + A^{-1}(e^{AT_1} - I)Bu(t_0), \tag{4}$$

$$x(t_1) = e^{A(t_1-t_0)}x(t_0) + A^{-1}(e^{A(t_1-t_0)} - I)Bu(t_0).$$

Similarly $x(t_2)$ can be written as

$$x(t_2) = e^{AT_2}x(t_1) + A^{-1}(e^{AT_2} - I)Bu(t_1).$$

Inserting the expression of $x(t_1)$ in above equation, we obtain

$$\begin{aligned} x(t_2) = e^{A(t_2-t_0)}x(t_0) + A^{-1}[(e^{A(t_2-t_0)} \\ - e^{A(t_2-t_1)})Bu(t_0) + (e^{A(t_2-t_1)} - I)Bu(t_1)]. \end{aligned}$$

Similarly, $x(t_n)$ can be written as

$$\begin{aligned} x(t_n) &= e^{A(t_n-t_0)}x(t_0) \\ &+ A^{-1}\Big[\Big(e^{A(t_n-t_0)} - e^{A(t_n-t_1)}\Big)Bu(t_0) \\ &+ \Big(e^{A(t_n-t_1)} - e^{A(t_n-t_2)}\Big)Bu(t_1) \\ &+ \cdots + \Big(e^{A(t_n-t_{n-1})} - I\Big)Bu(t_{n-1})\Big]. \end{aligned}$$

Rewriting the above equation in matrix form, it is implicit to obtain

$$x(t_n) - e^{A(t_n-t_0)}x(t_0) = A^{-1}M \begin{bmatrix} u(t_0) \\ u(t_1) \\ u(t_2) \\ \vdots \\ u(t_{n-1}) \end{bmatrix},$$

where

$$\begin{aligned} M = \Big[\Big(e^{A(t_n-t_0)} - e^{A(t_n-t_1)}\Big)B \quad \Big(e^{A(t_n-t_1)} - e^{A(t_n-t_2)}\Big)B \\ \cdots \quad \Big(e^{A(t_n-t_{n-1})} - I\Big)B\Big]. \end{aligned}$$

Where as M depends on the sampling instants $t_0, t_1 \cdots t_n$. If M is full rank then we can find an input vector that can drive the system from any initial state to a desired state in a finite time.

III. Controllability Analysis

In this section, we investigate that under what conditions M will not be full rank and propose some sampling patterns that will ensure M to be a full rank matrix and, consequently, conclude that system will remain controllable.

To write the matrix M in a compact form, define a new variable τ_k as

$$\tau_k = t_n - t_{n-k} = \sum_{i=1}^{k} T_{n+1-i}. \tag{5}$$

The matrix M can then be written as

$$M = \begin{bmatrix} e^{A\tau_1} - I & e^{A\tau_2} - e^{A\tau_1} & \cdots & e^{A\tau_n} - e^{A\tau_{n-1}} \end{bmatrix} B. \tag{6}$$

It is clear that τ_k represents the time variable as its value increases with the increasing index. Then M can also be rearranged as

$$M = Q \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

where

$$Q = \begin{bmatrix} B & e^{A\tau_1}B & e^{A\tau_2}B & \cdots & e^{A\tau_{n-1}}B & e^{A\tau_n}B \end{bmatrix}. \tag{7}$$

The constant matrix is full rank. The matrix Q depends on $\tau_1, \tau_2 \cdots \tau_n$ and problem reduces to ensure that Q is full rank. Assume that the system is single input, presence of multiple inputs will improve controllability, therefore, the matrix B will contain a single column. For single input system with n states, the size of Q matrix will be $n \times (n+1)$. If the matrix A is diagonalizable, then it can be written as

$$A = PDP^{-1},$$

where P is a nonsingular transformation matrix and D is a diagonal matrix having eigenvalues of A as its diagonal entries, given by

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Now for given P and D, $e^{A\tau}$ is given as

$$e^{A\tau} = Pe^{D\tau}P^{-1},$$

where

$$e^{D\tau} = \begin{bmatrix} e^{\lambda_1\tau} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2\tau} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n\tau} \end{bmatrix}.$$

For a continous time controllable system, it is well-known that [8].

$$P^{-1}B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix},$$

where $c_i \neq 0, i = 1, \cdots, n$. Using the expression $e^{A\tau} = Pe^{D\tau}P^{-1}$, the matrix in equation (7) can be partitioned as

$$Q = P \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} R,$$

where the matrix R is

$$R = \begin{bmatrix} 1 & e^{\lambda_1\tau_1} & e^{\lambda_1\tau_2} & \cdots & e^{\lambda_1\tau_n} \\ 1 & e^{\lambda_2\tau_1} & e^{\lambda_2\tau_2} & \cdots & e^{\lambda_2\tau_n} \\ 1 & e^{\lambda_3\tau_1} & e^{\lambda_3\tau_2} & \cdots & e^{\lambda_3\tau_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\lambda_n\tau_1} & e^{\lambda_n\tau_2} & \cdots & e^{\lambda_n\tau_n} \end{bmatrix}. \tag{8}$$

Now, P and the matrix of constant $c's$ are also full rank. So problem reduces to proving R as a full rank matrix. We consider two cases of sampling in the subsequent paragraphs.

A. Uniform Sampling Case

Lemma 1 ([1]). *If sampling is uniform, the discrete-time system will be controllable if sampling interval T is non-pathological. That is for any two eigenvalues λ_i and λ_j having equal real parts, Im$(\lambda_i - \lambda_j) \neq 2n\pi/T$ where n is an integer.*

Proof: The effect of uniform sampling on controllability has been investigated and consequently validated non-pathological sampling condition. In uniform sampling, let T be the sampling period, then

$$\tau_1 = T, \tau_2 = 2T, \cdots, \tau_n = nT.$$

The condition for pathological sampling in which controllability of the system will be lost is

$$\mathrm{Im}(\lambda_i - \lambda_j) = 2n\pi/T \quad \text{or} \quad \lambda_i = \lambda_j + j2n\pi/T.$$

So

$$\begin{aligned} e^{\lambda_i\tau_1} &= e^{\lambda_i T} = e^{(\lambda_j + j2n\pi/T)T} = e^{\lambda_j T + j2n\pi} \\ &= e^{\lambda_j T} = e^{\lambda_j \tau_1}. \end{aligned}$$

Consequently, $e^{\lambda_i\tau_2} = e^{\lambda_j\tau_2}$, $e^{\lambda_i\tau_3} = e^{\lambda_j\tau_3}$,and $e^{\lambda_i\tau_n} = e^{\lambda_j\tau_n}$. Then, two rows of matrix R will be equal, thus the matrix will not be full rank and, hence, the system will not be controllable. Therefore, for the sampled-data system to be controllable, the sampling frequency should not be pathological. ■

B. Nonuniform Sampling Case

As discussed earlier that using uniform sampling, pathological sampling may arise in which two rows of the matrix R given in equation (8) will become equal resulting in loss of controllability. If uniform sampling is present then we can assure the controllability of discrete-time system by forcefully making one sampling period irrational to original sampling period after n sampling instants [4] such that ratio of sampling period of irrational sample with uniform sampling period should not be an integer.

In general, if exact sampling times are not known and sampling period varies between an upper and a lower bound, then the condition for controllability is given in the following theorem.

Theorem 2. *For a non-uniformly sampled system, if exact sampling instants are not known, then the controllability will be preserved if for a given time interval $[0, \Gamma]$, the number of samples N satisfy $N \geq 1 + \Gamma\delta/2\pi$, where $\delta = max[Im(\lambda_i - \lambda_j)]$.*

Proof: In the following, we prove that if the condition specified in Theorem 2 is satisfied, matrix R in equation (8) has full rank and the discrete-time system will be controllable. It is well known that controllability of sampled system will be affected by the nature of the eigenvalues of the system. For real systems, eigenvalues can be real or complex conjugate. We considered the effect of different types of eigenvalues on the rank of matrix given in equation (8).

1) If eigenvalues are real, then it can be inferred from Lemma 1 that if eigenvalues are unique and real then for $\lambda_1, \lambda_2 \cdots\cdots \lambda_k$ to be real, the k rows of matrix R will be independent and controllability will not be affected by these eigenvalues. Thus discrete-time system will also be controllable for any arbitrary sampling sequence.
2) If eigenvalues are complex and have different real parts, then resulting rows corresponding to these eigenvalues in matrix R will be independent due to different real parts. But if eigenvalues have same real part but different imaginary parts as the case for complex conjugate, then as given in Lemma 1 that for a given sampling period T, two rows of matrix R in (8) will become equal that results in loss of rank of matrix and system will be uncontrollable. If we use $N = \Gamma\delta/2\pi$, where $\delta = \max[\text{Im}(\lambda_i - \lambda_j)]$, then this N corresponds to maximum number of samples in a given time interval $[0, \Gamma]$ that can cause pathological sampling. If we take N samples for the given time Γ, then we have sampled all the possible points causing pathological sampling, so adding 1 more sample will result in independent rows of the matrix due to unequal value of additional sampled point. Thus we conclude that if number of samples are $N \geq 1 + \Gamma\delta/2\pi$, where $\delta = \max[\text{Im}(\lambda_i - \lambda_j)]$, then system will remain controllable for any possible sampling sequence.

■

IV. Numerical Example

Consider an example system with eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1 + j\pi$ and $\lambda_3 = -1 - j\pi$. For this system, we will check the rank of matrix M for different sampling sequences:

1) $\tau_1 = 2s, \tau_2 = 4s, \tau_3 = 6s$. Rank of resultant matrix will be 1. If last sample is 5.4 instead of 6; rank will be 2. But if this sequence will be repeated then after 6_{th} sample rank will be 3.
2) $\tau_1 = 1s, \tau_2 = 2s, \tau_3 = 3s$. Rank of resultant matrix will be 2. If last sample is at $3.5s$ instead of $3s$; resultant matrix will be full rank. In this case condition in 2 is not satisfied yet the resulting system will be controllable.
3) Applying the condition $\delta = 2\pi$, $\Gamma = 3$, this will result in $N \geq 4$. So taking any 4 sampling instants in the interval $[0, 3]$ will give the resultant matrix of rank 3 and the system will be controllable.

Remark 1. *Now let consider another example with eigenvalues $\lambda_1 = -1, \lambda_2 = -2$ and $\lambda_3 = 1$. For this system, rank of matrix M for above given sampling sequences is 3. So this systems with real eigenvalues did not loose controllability due to sampling.*

Theorem 2 is applicable to the case when the system matrix is diagonalizable, i.e., the matrix A has distinct eigenvalues. If there are repeated eigenvalues in the system matrix, Theorem 2 may not be used.

We repeat the above example when the complex eigenvalues has multiplicity 2 and check for the rank of matrix. The data for this example will be $\lambda_1 = -1$, $\lambda_2 = -1 + j\pi$ repeated twice and $\lambda_3 = -1 - j\pi$ also repeated twice.

1) $\tau_1 = 1s, \tau_2 = 2s, \tau_3 = 3s$. Rank of resultant matrix will be 3. If last sample is at $3.5s$ instead of $3s$; resultant matrix will be of rank 4 and another irrational sample is required, let that sample be at $4.2s$, then system will be controllable. In this case condition in Theorem 2 is not satisfied yet the resulting system will be controllable.
2) Applying the condition $\delta = 2\pi$ and $\Gamma = 5$, this would result in $N \geq 6$, So taking any 6 sampling instants between 0 and 5 sec will give the resultant matrix of rank 5. But, when we take the value of τ as $\tau_1 = 1s, \tau_2 = 2s, \tau_3 = 3s, \tau_4 = 4s, \tau_5 = 4.5s$ and $\tau_6 = 5s$. These values of τ satisfy the condition given in Theorem2 but the rank of resultant matrix is not 5, so we need more samples for the case of repeated eigenvalues.

V. Conclusion

In this paper, we study the effect of nonuniform sampling on the controllability of a sampled-data system. By using the definition of controllability, we find out a matrix which can be used to verify the controllability of the discrete-time system. It becomes clear that controllability will be affected due to complex eigenvalues of the system and sampling period that can cause loss of controllability will be dependent on the imaginary part of the eigenvalues. Based on this a condition has been proposed for sampling instants that will preserve the controllability of the resultant discrete-time system under any arbitrary sampling instants.

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