Generalized Fractional Differential and Difference Equations: Stability Properties and Modelling Issues

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Abstract

In the last decades, fractional differential equations have become more and more popular among scientists and daring engineers in order to model various stable physical phenomena with anomalous decay, say that are not of exponential type. Moreover in discrete-time series analysis, so-called fractional ARMA models have been proposed in the literature in order to model stochastic processes, the autocorrelation of which also exhibits an anomalous decay. Both types of models stem from a common property of complex variable functions: namely, multivalued functions and their behaviour in the neighborhood of the branching point, and asymptotic expansions performed along the cut between branching points. This more abstract point of view proves very much useful in order to extend these models by changing the location of the classical branching points ($s = 0$ for continuous-time systems, or $z = 1$ for discrete-time systems). Hence, stability properties of and modelling issues by generalized fractional differential and difference systems will be considered in the present paper.

1 Context and Motivation

In the fields of continuous-time modelling, fractional derivatives have proved useful in linear viscoelasticity, acoustics, rheology, polymeric chemistry... For a treatment of so-called fractional differential equations (FDEs), we refer to [19, chap. 8, sec. 42], [15, chap. 5 & 6] and [8]. There has been some recent advances in control theory of such systems (see e.g. [9] for *stability* questions, [12] for *controllability* and *observability* considerations and [13] for observer-based controller design), together with interesting applications. Turning to the infinite dimension (i.e. dealing with FPDEs) has been motivated by the example of a wave equation in viscothermal medium (see [11], [7] and [10]). Moreover, an interesting idea of generalized fractional differential systems appeared in [21] in a stochastic framework; in this approach however, new branching points are definitely poles.

In the fields of discrete-time modelling, the famous paper by Granger and Joyeux [4] has provided a wide variety of so-called long-memory models for describing the autocorrelation of discrete-time stochastic processes in finance and econometrics, mostly. An interesting idea of generalized fractional difference systems appeared in [22]; but once again, it must be noted that here new branching points were confused with poles. In our extension, the two notions will be naturally disconnected, thus allowing for a wider range of dynamics.

In both cases, it is the very *multivalued* nature of the transfer function that gives its richness to the model; but changing the branching point and studying the consequences both from spectral and time-domain point of views proves useful when one is interested in extending the models. The techniques involved in this work are of analytical nature: we use mostly *distributions theory* in the sense of Schwartz (see e.g. [20, chap. II & III], [5, chap. 1, sec. 3.2 & 5.5]) for the continuous-time domain, *complex variable theory*, *asymptotic expansions* (see [2, chap. 2]) and *special functions* (see e.g. [16], [1], [3]).

The paper is organized as follows: in section 2, we deal with continuous-time systems; after recalling the main features of fractional differential systems, we consider generalized basic elements in the Laplace domain in § 2.1, and generalized systems in § 2.2; stability properties are established from asymptotic results; other generalized systems are also presented and compared with in § 2.3. In section 3, we propose a straightforward discretization which proves useful in order to preserve stability properties: we consider generalized basic elements in z-plane in § 3.1, examine generalized systems in § 3.2 and give stability properties from asymptotic results; other generalized basic elements and their corresponding systems are presented and compared with in § 3.3. Finally in

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section 4, we conclude with the main features of our generalized systems and cite a few questions that are, as far as we know, left open.

2 Generalized fractional differential systems

Let us recall some now classical results on fractional differential equations: a transfer function $\mathcal{H}(s) = R(s^\alpha)$ for $\Re e(s) > a$, where $R = Q/P$ stands for a rational function with P and Q two coprime polynomials, and $0 < \alpha < 1$ is the fractional order of derivatives, has the main property of (see [8, 9]):

$$\text{BIBO stability} \quad \Longleftrightarrow \quad |\arg\sigma| > \alpha\frac{\pi}{2}, \quad \forall\sigma,\ P(\sigma) = 0$$

In this latter case, the impulse response h has the following asymptotics:

$$h(t) \sim K\,t^{-1-\alpha} \text{ as } t \to +\infty$$

2.1 Generalized basic elements

Considering $s_0 \in \mathbf{C}$ instead of 0 as branching point leads us to define the following basic element in the Laplace domain: $((s-s_0)^\alpha - \lambda)^{-1}$ for $\Re e(s) > a_{\lambda,s_0}$ (a cut being performed along the half-line $]-\infty + i\Im m(s_0),\ s_0]$), which proves to be the *fundamental solution* of the generalized fractional differential operator $D^\alpha_{s_0} - \lambda$ in the functional space of causal tempered distributions $\mathcal{S}'_+$, with

$$D^\alpha_{s_0} T = \exp(s_0 t) D^\alpha_0 (\exp(-s_0 t) T)$$

We can state the following stability results for these basic elements:

- when $\Re e(s_0) > 0$, these generalized basic elements are *all* unstable, whatever the location of λ in the complex plane,
- when $s_0 = i\omega_0$, the region of stability (namely $|\arg\sigma| > \alpha\pi/2$) is fully preserved, but the stable dynamics now behave like:

  $$K_\lambda\,t^{-1-\alpha}\,e^{i\omega_0 t} \text{ as } t \to +\infty$$

- when $\Re e(s_0) < 0$, the region of instability in $\mathbf{C}$ is shrunk (namely, the interior of the limiting curve

  $$|\sigma|^{1/\alpha}\cos((\arg\sigma)/\alpha) = -|s_0|\cos(\arg s_0)$$

  in the formerly unstable sector $|\arg\sigma| \le \alpha\pi/2$); the stable dynamics then behave like:

  $$K_\lambda\,t^{-1-\alpha}\,e^{s_0 t} \text{ as } t \to +\infty$$

Note that when $\lambda = 0$, the asymptotics is in:

$$K_0\,t^{-\alpha}\,e^{s_0 t}$$

in this case *only*, the branching point s_0 is also a pole, somehow ($\lim_{s\to s_0}|(s-s_0)^{-\alpha}| = +\infty$).

From a modelling point of view, the case $\Re e(s_0) = 0$ is the most interesting, because a new variety of anomalous decays are being captured.

2.2 Generalized systems

From the careful asymptotic analysis of the previously defined generalized basic elements, we can now define generalized (single-input single-output) systems in a straightforward way, namely:

$$P(D^\alpha_{s_0})\,y(t) = Q(D^\alpha_{s_0})\,u(t) \tag{1}$$

with input u and output y.

It is now clear that such a system is BIBO-stable iff all the poles of the rational function R in the σ-plane lie in the open stability region defined in § 2.1. In this latter case, the impulse response of the generalized system behaves like:

$$K\,t^{-1-\alpha}\,e^{s_0 t} \text{ as } t \to +\infty$$

2.3 Other generalized systems

We examined the possibility of having a pole that is disconnected from the branching point, and then we put these elementary systems in cascade, all of them sharing the same α and the same s_0; in fact, this amounts to considering rational functions R in the variable $\sigma = (s-s_0)^\alpha$, hence providing a much better structured setting which allows for a fully algebraic treatment of such systems.

But other generalized systems can be defined as well: the α and s_0 can vary from one system to the other; we still get nice properties if only series of such systems are considered. Otherwise the analysis of the whole system is not straightforward: in [21] for example, cascade of elements with different α and s_0 but same $\lambda = 0$ are considered.

3 Generalized fractional difference systems

Turning to discrete-time is not so easy, and different dicrete-time "equivalents" can be proposed; as far as stability is concerned however, the choice of the *bilinear* transform (namely: $z \mapsto s(z) = \frac{1-z^{-1}}{1+z^{-1}}$) proves very much useful, because it fully preserves the respective domains of stability between s-plane and z-plane. Thus, we define a causal fractional difference operator Δ^α_1 and the basic elements related to it (in other words the *fundamental solution* of $\Delta^\alpha_1 - \lambda$) by $\left((\frac{1-z^{-1}}{1+z^{-1}})^\alpha - \lambda\right)^{-1}$ for $|z| > \rho_{\lambda,1}$ (a cut being performed along the segment $[-1, 1]$). The qualitative stability results are directly translated from those in continuous-time (namely $|\arg\lambda| > \alpha\pi/2$), but are slightly different from a quantitative point of view; when stable, the basic elements now behave like:

$$[K_\lambda^+ + (-1)^n K_\lambda^-]\, n^{-1-\alpha} \text{ as } n \to +\infty$$

3.1 Generalized basic elements

Considering $\{z_0, -z_0\} \in \mathbf{C}$ instead of $\{1, -1\}$ as branching points leads us to define the following generalized basic element in z-domain: $\left((\frac{1-z_0 z^{-1}}{1+z_0 z^{-1}})^\alpha - \lambda\right)^{-1}$ for $|z| > \rho_{\lambda, z_0}$ (a cut being performed along the segment $[-z_0, z_0]$), which proves to be the fundamental solution of the causal difference operator $\Delta_{z_0}^\alpha - \lambda$, with

$$\Delta_{z_0}^\alpha T = {z_0}^n \Delta_1^\alpha ({z_0}^{-n} T)$$

We can state the following stability results for these basic elements:

- when $|z_0| > 1$, these generalized basic elements are *all* unstable, whatever the location of λ in the complex plane,

- when $z_0 = e^{i\theta_0}$, the region of stability is fully preserved, but the stable dynamics now behave like:

$$[K_\lambda^+ + (-1)^n K_\lambda^-]\, n^{-1-\alpha}\, e^{i\theta_0 n} \text{ as } n \to +\infty$$

- when $|z_0| < 1$, the region of instability in $\mathbf{C}$ is shrunk; the stable dynamics then behave like:

$$[K_\lambda^+ + (-1)^n K_\lambda^-]\, n^{-1-\alpha}\, {z_0}^n \text{ as } n \to +\infty$$

Note that when $\lambda = 0$, the asymptotics is in:

$$[K_0^+ + (-1)^n K_0^-]\, n^{-\alpha}\, {z_0}^n$$

in this case *only*, the branching point z_0 is also a pole, somehow ($\lim_{z \to z_0} \left|(\frac{1-z_0 z^{-1}}{1+z_0 z^{-1}})^{-\alpha}\right| = +\infty$).

From a modelling point of view, the case $|z_0| = 1$ is the most interesting, because a new variety of anomalous decays are being captured.

3.2 Generalized systems

From the careful asymptotic analysis of the previously defined generalized basic elements, we can now define generalized (single-input single-output) systems in a straightforward way, namely:

$$P(\Delta_{z_0}^\alpha)\, y_n = Q(\Delta_{z_0}^\alpha)\, u_n \tag{2}$$

with input u and output y.

It is now clear that such a system is BIBO-stable iff all the poles of the rational function R in the σ-plane lie in the open stability region defined in § 3.1. In this latter case, the impulse response of the generalized system behaves like:

$$[K^+ + (-1)^n K^-]\, n^{-1-\alpha}\, {z_0}^n \text{ as } n \to +\infty$$

3.3 Other generalized basic elements and generalized systems

We decided to use the bilinear transform to derive equivalent systems in discrete-time: in the language of numerical analysis, this is equivalent to applying a trapezoidal rule as quadrature formula; it is also possible to use other numerical schemes, such as the simpler backward Euler scheme, which results in $z \mapsto s(z) = 1 - z^{-1}$. This choice, extended with a suitable change of location of the branching points $\{0, z_0\}$, leads to *other* generalized basic elements, and also to other generalized systems for which stability considerations are not qualitatively staightforward; quantitatively, the stable dynamics then behave like:

$$K\, n^{-1-\alpha}\, {z_0}^n \text{ as } n \to +\infty$$

with *no* alternate sequence $(-1)^n$.

As for continuous-time, taking different α and z_0 for the same $\lambda = 0$, then putting these generalized basic elements in cascade has been examined in [22] for example.

4 Conclusion

The generalized fractional differential and difference systems that we investigated here extend many features of fractional differential and difference systems and provide a new variety of anomalous dynamics; moreover they prove to be better structured both from algebraic and analytic point of views, thus particularly suitable for modelling purposes.

Some prospects are in view, and will be considered in the near future:

- compare the frequency behaviour of the basic elements $(s^\alpha - \lambda)^{-1}$ and $(s - \lambda^{1/\alpha})^{-\alpha}$ on the imaginary axis $s = i\omega$ and their generalized versions with $s_0 \neq 0$,

- same comparison in discrete-time on the unit circle $z = e^{i\theta}$, once the bilinear transform has been performed, and study of their generalized versions when $z_0 \neq 1$,

- to what extent can such discrete-time models stand for good numerical approximations of the continuous-time models?

- to what extent can these generalized models be reformulated in the very promising context of so-called *diffusive realisations* introduced in [18] and used since then in e.g. [14, 17, 6]?

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