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Non-linear circuit modelling via nodal methods

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SUMMARY

We discuss in this paper several interrelated nodal methods for setting up the equations of non-linear, lumped electrical circuits. A rather exhaustive framework is presented, aimed at surveying different approaches and terminologies in a comprehensive manner. This framework includes charge-oriented, conventional, and hybrid systems. Special attention is paid to so-called augmented node analysis (ANA) models, which somehow articulate the tableau and modified node analysis (MNA) approaches to non-linear circuit modelling. We use a differential–algebraic formalism and, extending previous results proved in the MNA context, we provide index-1 conditions for augmented systems, which are shown to be transferred to tableau models. This approach gives, in particular, precise conditions for the feasibility of certain state-space reductions. We work with very general assumptions on device characteristics; in particular, our approach comprises a wide range of resistive devices, going beyond voltage-controlled ones. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: differential–algebraic equation; index; graph; circuit model; tableau approach; modified node analysis; augmented node analysis; state-space equation

1. INTRODUCTION

A major problem in lumped circuit modelling is how to set up circuit equations. Within the non-linear context, this is done in the time domain, and much effort have been directed to the derivation of *state-space* models based on ordinary differential equations (ODEs) [1–7]. These models allow for the application of many analytical, qualitative and numerical tools coming

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from the ODE framework and dynamical systems theory. However, state models display two important drawbacks: for several circuit configurations an explicit state-space equation may not exist, and even if it exists, its formulation may be hardly automatable. The latter is extremely important in very large-scale integration systems.

These limitations have led, in the last three decades, to the formulation of *semistate* models, based on differential–algebraic equations (DAEs) [8–21]. In particular, tableau formulations and modified node analysis (MNA) models have received much recent attention from a DAE viewpoint [10–13, 21]. But the term *nodal analysis* refers to a variety of different techniques, which include also those labelled as *basic* and *augmented* , and are sometimes combined with branch replacement approaches [3]. Actually, there seems to exist in the literature several apparently different strategies which in fact only differ significantly in the language or thread of thinking: compare, e.g. the use of the terms 'augmented' and 'modified' in References [3, 7, 12, 21], to name a few works.

In this direction, we survey in the present paper different nodal methods for setting up the network equations of non-linear lumped circuits, using a differential–algebraic formalism. The notion of the *index* plays a key role in this framework; index-1 models are of particular importance because they allow for a direct application of efficient qualitative and numerical tools developed in the DAE setting [22–25]. Special attention will be paid to so-called *augmented node analysis* (ANA) models. These models have received much less attention in the non-linear setting than tableau or MNA systems, but from the authors' point of view ANA models seem to somehow articulate or link the tableau/MNA families, in the following sense. On the one hand, ANA models are obtained as a reduction of tableau systems, having index one if and only if the corresponding tableau model does. On the other hand, MNA models can be easily obtained from ANA, but the semiexplicit structure of augmented systems simplifies the formulation of index-1 conditions and allows for just non-singular reactances, in contrast to MNA, where positive definiteness is required in the index analysis [21]. Additionally, ANA displays a symmetry in the topologies which are precluded for index-1, again in contrast to MNA.

From a different point of view, we aim to illustrate how the state-space formulation problem can be better handled in this DAE framework. Specifically, the state-space model of Reference [3] can be seen as a reduction of ANA systems, emphasizing the key position of these within the above-mentioned hierarchy of nodal methods. We will provide precise circuit-theoretic conditions under which this state-space reduction is actually feasible, also improving the approach of Reference [3] on how to handle in non-linear problems reactive variables (through the distinction between charge-oriented, conventional, and hybrid models) and also resistive devices.

The results rely upon linearization and are local for non-linear problems; their application to linear cases is straightforward and applies globally. From a mathematical point of view, we combine DAE theory with results coming from graph theory and linear algebra. Schur complements [26] will play an important role in the analysis. Mathematical background is compiled in Section 2.

The structure of Sections 3–6 is based on Figure 1.Tableau models are presented in Section 3. We discuss conditions which make it possible to derive conventional (voltage-oriented) and hybrid models from the most general charge-oriented system. Actually, index-1 conditions will read the same for the three families of systems, as shown in Theorem 1.

In the remainder of the paper we will focus, for the sake of brevity, on the conventional family displayed on the left of Figure 1, characterized by the use of capacitor voltages and

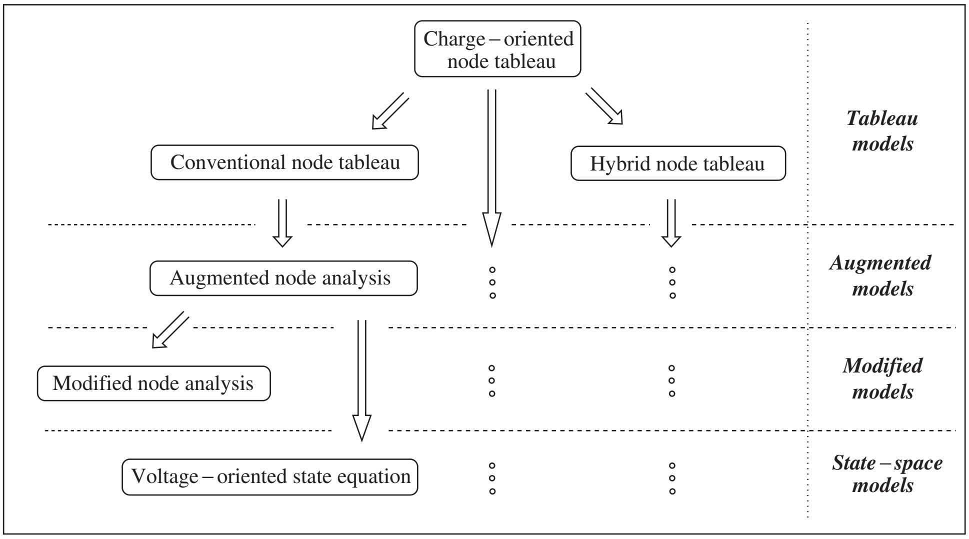

Figure 1. A hierarchy of nodal methods.

inductor currents as dynamic variables. We consider in Section 4 augmented node analysis (ANA) systems. These models have index-1 if and only if the corresponding tableau system does (Theorem 2), and bifurcate towards two different directions. The first one leads to modified node analysis (MNA). The index of MNA models has been carefully analysed in References [11, 21], and we simply state in Theorem 3 sufficient index-1 conditions taken from Reference [21].

The second branch emanating from ANA leads to state models and will be considered in Sections 5 and 6. Whereas Sections 3 and 4 are mostly descriptive in nature, we provide in Section 5 explicit index-1 conditions for ANA systems (Theorem 4). The importance of this result is two-fold. First, in virtue of the above-mentioned Theorems 1 and 2, these index-1 conditions are automatically transferred to tableau systems, as acknowledged in Corollary 1. Second, it provides explicit and precise assumptions for the state-space analysis of Reference [3] to be feasible, as asserted in Corollary 2. The main methodological difference with Reference [3] is that we formulate here this state-space equation as a reduction of differential–algebraic systems.

Theorem 4 assumes voltage-control in the resistors, what makes it easily comparable to other results in the same direction (such as the above-mentioned MNA Theorem 3, taken from Reference [21]). However, examples 1 and 2 will motivate the analysis of circuits under less restrictive assumptions on resistors; this will be performed in Section 6, the main result in this direction being Theorem 5.

2. BACKGROUND

2.1. *Differential–algebraic equations*

Comprehensive surveys on differential–algebraic equations (DAEs) can be found in References [22–25, 27]. We present here some background on autonomous semiexplicit index-1 systems. The extension of the discussion to problems with (non-autonomous) excitation terms is straightforward. Semiexplicit systems are defined by an equation of the form

$$x' = f(x, y) \tag{1a}$$

$$0 = g(x, y) \tag{1b}$$

where $x \in \mathbb{R}^r$ denotes the *differential* or *dynamic* variables, $y \in \mathbb{R}^p$ stands for the *algebraic* ones, $f \in C^1(\mathbb{R}^{r+p}, \mathbb{R}^r)$, and $g \in C^1(\mathbb{R}^{r+p}, \mathbb{R}^p)$.

Let (x^*, y^*) satisfy $g(x^*, y^*) = 0$. If the derivative $g_y(x^*, y^*)$ defines an invertible matrix, then Equation (1) is said to have differential index one around (x^*, y^*) [22]. Under the index-one assumption, Equation (1b) locally defines a C^1 r-dimensional *solution manifold* $\mathscr{M}_1$ where the solutions of the DAE lie. From the implicit function theorem it follows that there exists a local C^1 map α such that $g(x, y) = 0 \Leftrightarrow y = \alpha(x)$. The dynamical behaviour on $\mathscr{M}_1$ may be described, in the local parameterization defined by x, through a *reduced* or *state-space equation*

$$x' = f(x, \alpha(x)) \tag{2}$$

Note that this reduced description is not unique, and that other subsets (different from x) of the *semistate* variables (x, y) might as well qualify as state ones; this will the case for conventional and hybrid systems in Section 3.

Assume that, for some reason, the non-singularity condition on $g_y(x^*, y^*)$ is difficult to check in practice. Let us also suppose that y and g are split as indicated in Equations (3b) and (3c) below:

$$x' = f(x, y_1, y_2) \tag{3a}$$

$$0 = g_1(x, y_1, y_2) \tag{3b}$$

$$0 = g_2(x, y_1, y_2) \tag{3c}$$

and that the non-singularity of the derivative $g_{2_{y_2}}$ is easier to assert at (x^*, y_1^*, y_2^*). Then, there will exist a local map β such that $g_2(x, y_1, y_2) = 0$ if and only if $y_2 = \beta(x, y_1)$; inserting this into Equations (3a) and (3b), we get

$$x' = f(x, y_1, \beta(x, y_1)) \tag{4a}$$

$$0 = g_1(x, y_1, \beta(x, y_1)) \equiv h(x, y_1) \tag{4b}$$

The importance of this kind of reduction (which will be called a *Schur reduction*) is that, provided that $g_{2_{y_2}}(x^*, y_1^*, y_2^*)$ is non-singular, then $g_y(x^*, y^*)$ is non-singular if and only if so it is $h_{y_1}(x^*, y_1^*)$ in Equation (4). Equivalently, the index-1 condition may be studied in the simpler model (4). This is a consequence of the implicit function theorem and the below-stated result.

Lemma 1
Let

$$M = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} \tag{5}$$

denote a matrix in $\mathbb{R}^{n\times n}$, with $E_{11}\in\mathbb{R}^{r\times r}$, $E_{12}\in\mathbb{R}^{r\times p}$, $E_{21}\in\mathbb{R}^{p\times r}$, $E_{22}\in\mathbb{R}^{p\times p}$, $n=r+p$. Assume that E_{22} is non-singular. Then M is non-singular if and only if the *Schur complement* of E_{22},

$$S(E_{22}) = E_{11} - E_{12}E_{22}^{-1}E_{21} \tag{6}$$

is non-singular.

This can be found in Reference [26, 0.8.5], or can be directly obtained from the identity

$$\begin{pmatrix} I & -E_{12}E_{22}^{-1} \\ 0 & E_{22}^{-1} \end{pmatrix} \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} E_{11} - E_{12}E_{22}^{-1}E_{21} & 0 \\ E_{22}^{-1}E_{21} & I \end{pmatrix} \tag{7}$$

noting that the first matrix is always non-singular and that the last one is non-singular if and only if so it is $E_{11} - E_{12}E_{22}^{-1}E_{21}$.

2.2. Graphs

Kirchhoff current and voltage laws [2] can be easily described as $Ai=0$, $v=A^{\mathrm{T}}e$ in terms of the *reduced incidence matrix* $A=(a_{ij})\in\mathbb{R}^{(n-1)\times b}$ (n and b being the number of nodes and branches in the electrical circuit, respectively), defined as

$$a_{ij} = \begin{cases} 1 & \text{if branch } j \text{ leaves node } i \\ -1 & \text{if branch } j \text{ enters node } i \\ 0 & \text{if branch } j \text{ is not incident with node } i \end{cases}$$

The existence of pathological configurations in the graph associated with the circuit can be characterized in terms of this matrix. We compile below some results in this direction which follow from References [28, 3.13, 3.16, 3.17]; [29, 2.29].

If $\mathscr{K}$ is a subset of the set of branches of a connected graph $\mathscr{G}$, we denote as $A_{\mathscr{K}}$ (resp. $A_{\mathscr{G}-\mathscr{K}}$) the submatrix of A formed by the columns corresponding to the branches in $\mathscr{K}$ (resp. not in $\mathscr{K}$).

Lemma 2
Let $\mathscr{K}$ be a subset of the set of branches of a connected graph $\mathscr{G}$. $\mathscr{K}$ does not contain loops if and only if $A_{\mathscr{K}}y=0 \Rightarrow y=0$.

A subset $\mathscr{K}$ of the set of branches of a connected graph is a *cutset* if the deletion of $\mathscr{K}$ results in a disconnected graph, and it is minimal with respect to this property (i.e. the deletion of any proper subset of $\mathscr{K}$ does not disconnect the graph).

Lemma 3

Let $\mathscr{K}$ be a subset of the set of branches of a connected graph $\mathscr{G}$. $\mathscr{K}$ does not contain cutsets if and only if $A^{\mathrm{T}}_{\mathscr{G}-\mathscr{K}}x = 0 \Rightarrow x = 0$.

3. NODE TABLEAU ANALYSIS

Roughly speaking, basic node analysis of lumped circuits begins with the formulation of Kirchhoff current law in the form $Ai = 0$. Branch currents of voltage controlled elements are expressed as far as possible in terms of branch voltages, and these are in turn written in terms of node voltages using Kirchhoff voltage law $A^{\mathrm{T}}e = v$. For instance, if the circuit has only current sources $j(t)$ and voltage-controlled resistors ($i_r = \gamma(v_r)$), KCL reads $A_R i_r = -A_I j$ and can be rewritten as $A_R \gamma(A_R^{\mathrm{T}} e) = -A_I j$ after using resistors' characteristics and KVL. The fact that unknowns in this equation are all node voltages explains the term 'nodal analysis'.

When non-voltage controlled resistors, voltage sources, or dynamic (reactive) elements are present, some additional equations and variables must be included in the circuit model. This leads to *modified* or *augmented* methods, which can be found in the literature under several different names, sometimes the same model being called with different names and even the same name applying to different methods, depending on the context (see e.g. References [12, 15, 3, 21, 7]). Instead of this bottom–up approach, we propose a top–down hierarchy, beginning with the most-general general charge-oriented tableau model [14] and checking the conditions needed to eliminate certain variables and formulate reduced models.

3.1. Charge-oriented node tableau analysis (qNTA)

The charge-oriented node tableau model is the most general one of those considered in this paper. It can be written out for lumped RLC circuits without imposing specific hypotheses on the characteristics of (possibly non-linear) capacitors, inductors, and resistors. Assuming that the circuit is time-invariant, and that coupling is possible only within these three sets of devices, the charge-oriented node tableau analysis (qNTA) model reads:

$$q' = i_c \tag{8a}$$

$$\phi' = v_l \tag{8b}$$

$$0 = g_c(q, v_c) \tag{8c}$$

$$0 = g_l(\phi, i_l) \tag{8d}$$

$$0 = i_j - j(t) \tag{8e}$$

$$0 = v_u - u(t) \tag{8f}$$

$$0 = g_r(i_r, v_r) \tag{8g}$$

$$0 = Ai \tag{8h}$$

$$0 = v - A^{\mathrm{T}}e \tag{8i}$$

Here, i and v stand for branch currents and voltages, whereas e represents node voltages. Capacitor charges and inductor fluxes are represented by q and ϕ, respectively, Equations (8a) and (8b) describing standard electromagnetic relations for them. The mappings g_c, g_l and g_r describe the characteristics of capacitors, inductors and resistors, and are assumed to be C^1. The subindices r, l, c, u and j in branch variables stand for resistors, inductors, capacitors, voltage sources and current sources. The C^0 functions $j(t)$, $u(t)$ represent currents and voltages in the (assumed independent) corresponding sources, and would need to be changed by a function of some branch variables to accommodate controlled sources. Note that Equations (8h) and (8i) describe Kirchhoff laws. From the DAE point of view, differential variables of this model are q, ϕ, whereas e, v, i are algebraic ones.

Several reductions of this model will be analysed in subsequent sections. Most of the discussion in this one will be descriptive and will concern only the four Equations (8a)–(8d), regarding which reactive variables qualify as differential ones in other (so called conventional and hybrid) models. In particular, this will clarify the discussion of Reference [3, Section 6.1] in this regard.

Assume that (q^*, v_c^*) satisfies Equation (8c), and that it is non-critical in the sense that $\operatorname{rk} g_c'(q^*, v_c^*)$ is maximum. As shown below, under this assumption a straightforward application of the implicit function theorem characterizes different sets of capacitive variables which are qualified as dynamic variables for other models. The same will happen with inductive devices. It is more subtle the fact that the same set of conditions characterizes index one in all these models, as proved in Theorem 1.

3.2. *Conventional node tableau analysis (vNTA)*

Assume that $g_c(q^*, v_c^*) = 0$, $g_l(\phi^*, i_l^*) = 0$, and that

$$\frac{\partial g_c}{\partial q}(q^*, v_c^*) \quad \text{and} \quad \frac{\partial g_l}{\partial \phi}(\phi^*, i_l^*) \text{ are non-singular} \tag{9}$$

The implicit function theorem implies that capacitors and inductors are locally voltage/current controlled through certain relations

$$q = \psi(v_c) \tag{10a}$$

$$\phi = \varphi(i_l) \tag{10b}$$

respectively. Define the incremental capacitance and inductance matrices as

$$C(v_c) = \psi'(v_c) = -\left(\frac{\partial g_c}{\partial q}\right)^{-1}\left(\frac{\partial g_c}{\partial v_c}\right) \tag{11a}$$

$$L(i_l) = \varphi'(i_l) = -\left(\frac{\partial g_l}{\partial \phi}\right)^{-1}\left(\frac{\partial g_l}{\partial i_l}\right) \tag{11b}$$

Under assumptions (9), the resulting conventional (voltage-oriented) node tableau analysis (vNTA) model can be written as the following quasilinear DAE [2, 12]:

$$C(v_c)v_c' = i_c \tag{12a}$$

$$L(i_l)i_l' = v_l \tag{12b}$$

$$0 = i_j - j(t) \tag{12c}$$

$$0 = v_u - u(t) \tag{12d}$$

$$0 = g_r(i_r, v_r) \tag{12e}$$

$$0 = Ai \tag{12f}$$

$$0 = v - A^{\mathrm{T}}e \tag{12g}$$

where the (uncoupled) relations (10) which replace Equations (8c) and (8d) can now be seen as output functions and are not included. With regard to Equation (8), note that in Equation (12) v_c and i_l are no longer algebraic variables but differential ones (replacing q, ϕ).

3.3. Hybrid node tableau analysis (hNTA)

Let the b_c capacitive branches be split in two sets (to be represented with $\wedge$ and $\sim$, so that $q = (\hat{q}, \tilde{q})$, $v_c = (\hat{v}_c, \tilde{v}_c)$) in a way such that

$$\begin{pmatrix} \dfrac{\partial g_c}{\partial \hat{q}} & \dfrac{\partial g_c}{\partial \tilde{v}_c} \end{pmatrix} (q^*, v_c^*) \quad \text{is non-singular} \tag{13}$$

Such a splitting always exists under the above-mentioned non-critical assumption $\operatorname{rk} g_c'(q^*, v_c^*) = b_c$. Again, a straightforward application of the implicit function theorem shows that the capacitors locally admit a hybrid representation of the form

$$\hat{q} = \eta(\tilde{q}, \hat{v}_c) \tag{14a}$$

$$\tilde{v}_c = \xi(\tilde{q}, \hat{v}_c) \tag{14b}$$

From the former, we get

$$\hat{q}' = \frac{\partial \eta}{\partial \tilde{q}}(\tilde{q}, \hat{v}_c)\tilde{q}' + \frac{\partial \eta}{\partial \hat{v}_c}(\tilde{q}, \hat{v}_c)\hat{v}_c'$$

Denote

$$H_1 = \begin{pmatrix} \dfrac{\partial \eta}{\partial \tilde{q}} & \dfrac{\partial \eta}{\partial \hat{v}_c} \\ I & 0 \end{pmatrix} \tag{15}$$

Proceed analogously with inductors, assuming a splitting such that

$$\begin{pmatrix} \dfrac{\partial g_l}{\partial \hat{\phi}} & \dfrac{\partial g_l}{\partial \tilde{i}_l} \end{pmatrix} (\phi^*, i_l^*) \quad \text{is non-singular} \tag{16}$$

This yields a local hybrid description

$$\hat{\phi} = \zeta(\tilde{\phi}, \hat{i}_l) \tag{17a}$$

$$\tilde{i}_l = \kappa(\tilde{\phi}, \hat{i}_l) \tag{17b}$$

where, obviously, the numbers of $\wedge$ and $\sim$ inductors have nothing to do with those of $\wedge$ and $\sim$ capacitors. Let

$$H_2 = \begin{pmatrix} \dfrac{\partial \zeta}{\partial \tilde{\phi}} & \dfrac{\partial \zeta}{\partial \hat{i}_l} \\ I & 0 \end{pmatrix} \tag{18}$$

From the charge-oriented NTA model we then get the following quasilinear hybrid tableau (hNTA) system:

$$H_1(\tilde{q}, \hat{v}_c) \begin{pmatrix} \tilde{q}' \\ \hat{v}_c{}' \end{pmatrix} = i_c \tag{19a}$$

$$H_2(\tilde{\phi}, \hat{i}_l) \begin{pmatrix} \tilde{\phi}' \\ \hat{i}_l{}' \end{pmatrix} = v_l \tag{19b}$$

$$0 = i_j - j(t) \tag{19c}$$

$$0 = v_u - u(t) \tag{19d}$$

$$0 = g_r(i_r, v_r) \tag{19e}$$

$$0 = A_I i_j + A_{\hat{L}} \hat{i}_l + A_{\tilde{L}} \kappa(\tilde{\phi}, \hat{i}_l) + A_V i_u + A_C i_c + A_R i_r \tag{19f}$$

$$0 = \xi(\tilde{q}, \hat{v}_c) - A_{\tilde{c}}^{\mathrm{T}} e \tag{19g}$$

$$0 = v_* - A_*^{\mathrm{T}} e \tag{19h}$$

For notational simplicity, in Equations (19g) and (19h), we have split A as $(A_{\tilde{c}}\ A_*)$, the latter comprising all branches except those in the set of $\sim$ capacitive branches. Branch voltages are split in the same way. The splitting of A appearing in Equation (19f) should be self-explanatory. Note that again, the (uncoupled) output relations (14) and (17) are not included. As in the conventional model (12), algebraic variables are node voltages, branch voltages except capacitive ones, and branch currents except inductive ones. Differential variables are now $\tilde{q}, \hat{v}_c, \tilde{\phi}, \hat{i}_l$.

3.4. On the index of tableau models

Theorem 1
Consider the charge-oriented NTA system (8), and assume that the non-critical assumptions (9) (resp. Equations (13), (16)) supporting the conventional (resp. hybrid) NTA model (12) (resp. Equation (19)) are satisfied. Then

1. The matrix $C(v_c)$ (resp. $H_1(\tilde{q}, \hat{v}_c)$) is non-singular if and only if so it is $\partial g_c/\partial v_c(q, v_c)$.
2. The matrix $L(i_l)$ (resp. $H_2(\tilde{\phi}, \hat{i}_l)$) is non-singular if and only if so it is $\partial g_l/\partial i_l(\phi, i_l)$.

If $\partial g_c/\partial v_c$ and $\partial g_l/\partial i_l$ are indeed non-singular, the conventional (resp. hybrid) tableau system has index 1 if and only if the charge-oriented model (8) has index 1.

Proof
The equivalence for the conventional case, involving $C(v_c)$ and $L(i_l)$, is trivial due to the expression depicted in Equation (11). Let us then focus on the hybrid case, restricting the proof to the simultaneous non-singular nature of $H_1(\tilde{q}, \hat{v}_c)$ and $\partial g_c/\partial v_c(q, v_c)$ since the result for $H_2(\tilde{\phi}, \hat{i}_l)$ and $\partial g_l/\partial i_l(\phi, i_l)$ is entirely analogous.

In the light of Equation (15), the matrix H_1 is non-singular if and only if so it is $\partial\eta/\partial\hat{v}_c$. From Equation (13) and the implicit function theorem, this derivative reads

$$\frac{\partial \eta}{\partial \hat{v}_c} = -\,(I_{\hat{b}_c}\ \ 0)\begin{pmatrix} \dfrac{\partial g_c}{\partial \hat{q}} & \dfrac{\partial g_c}{\partial \tilde{v}_c} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial g_c}{\partial \hat{v}_c} \end{pmatrix}$$

In turn, this is the Schur complement of $E_{22} = \begin{pmatrix} \frac{\partial g_c}{\partial \hat{q}} & \frac{\partial g_c}{\partial \tilde{v}_c} \end{pmatrix}$ in

$$M = \begin{pmatrix} 0 & I_{\hat{b}_c} & 0 \\ \dfrac{\partial g_c}{\partial \hat{v}_c} & \dfrac{\partial g_c}{\partial \hat{q}} & \dfrac{\partial g_c}{\partial \tilde{v}_c} \end{pmatrix}$$

with $E_{11} = 0$, $E_{12} = (I_{\hat{b}_c}\ 0)$, $E_{21} = \partial g_c/\partial \hat{v}_c$, with the notation of Lemma 1. Therefore, $\partial\eta/\partial\hat{v}_c$ is non-singular if and only if so it is M, but the latter is non-singular if and only if so it is $(\partial g_c/\partial \hat{v}_c\ \partial g_c/\partial \tilde{v}_c) = (\partial g_c/\partial v_c)$.

Finally, the equivalence of index-1 conditions can be seen as follows. Reorder, in the charge-oriented model (8), algebraic variables as $v_c, i_l, i_j, v_u, e, i_u, i_c, i_r, v_r, v_l, v_j$. The derivative of the algebraic restrictions (8c)–(8i) with respect to these co-ordinates has the form

$$\begin{pmatrix} \dfrac{\partial g_c}{\partial v_c} & 0 & 0 \\ 0 & \dfrac{\partial g_l}{\partial i_l} & 0 \\ * & * & J \end{pmatrix} \tag{20}$$

with

$$J = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{\partial g_r}{\partial i_r} & \dfrac{\partial g_r}{\partial v_r} & 0 & 0 \\ A_I & 0 & 0 & A_V & A_C & A_R & 0 & 0 & 0 \\ 0 & I & -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_C^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_R^{\mathrm{T}} & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -A_L^{\mathrm{T}} & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & -A_I^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{21}$$

Therefore, provided that $\partial g_c/\partial v_c$ and $\partial g_l/\partial i_l$ are non-singular, the charge-oriented tableau system (8) has index 1 if and only if J is non-singular. But it is a simple matter to check that J is the derivative of the algebraic restrictions (12c)–(12g) and (19c)–(19h) of the conventional (12) and hybrid (19) models, respectively, with respect to their algebraic variables (which do not include v_c, i_l), so that non-singularity of J characterizes index-1 in conventional and hybrid models. □

Note that this approach clarifies the sets of reactive variables which eventually qualify for state-space descriptions, including circuits with coupling among capacitors or inductors. Compare e.g. with Reference [3, Section 6.1], where all reactive devices are implicitly assumed to be uncoupled and locally flux- or charge-controlled, in the light of the inverse matrices appearing in Equations (6.3) and (6.5) there. This particular case would amount in the discussion above to the non-singularity of $\partial g_c/\partial v_c$ and $\partial g_l/\partial i_l$. It is worth emphasizing that these non-singular conditions are not needed for the derivation of conventional (12) or hybrid (19) models, which in fact rely on Equations (9) or (13), (16). The non-singularity of $\partial g_c/\partial v_c$ and $\partial g_l/\partial i_l$ just makes it possible to reduce these systems from quasilinear to semiexplicit form. Note that a singular $\partial g_c/\partial v_c$ or $\partial g_l/\partial i_l$ yield singular $C(v_c)$, $H_1(\tilde{q}, \hat{v}_c)$, or $L(i_l)$, $H_2(\tilde{\phi}, \hat{i}_l)$, respectively, and may lead to impasse points and related phenomena [8, 9, 17, 30].

A correspondence between the index of conventional and charge-oriented MNA models can be found in Reference [21, Theorem 7]. Note, however, that in the aforementioned paper global voltage- and current-controlled representations are assumed for the capacitor and inductors, respectively; namely, equations $g_c = 0$, $g_l = 0$ are assumed to have globally the form $q = \psi(v_c)$, $\phi = \varphi(i_l)$. Note in contrast that the general setting defined by Equations (8c) and (8d) makes it possible to extend the result to the more complicated hybrid case.

4. AUGMENTED NODE ANALYSIS (ANA)

In the sequel we assume for simplicity that the non-singularity conditions (9) supporting the existence of the conventional model (12) do hold at the operating point, and that $\partial g_c/\partial v_c$ and $\partial g_l/\partial i_l$ or equivalently $C(v_c)$ and $L(i_l)$ are non-singular, so that this conventional model admits a semiexplicit form. It is straightforward to extend the results below to charge-oriented or hybrid models, due to the index-1 equivalence shown in Theorem 1.

For better readability and comparative purposes, we also assume in this and the next section that resistors are locally voltage-controlled by a relation of the form $i_r = \gamma(v_r)$. In Section 6, we show how to extend the results to problems without this restriction.

Splitting the incidence matrix A as $(A_R\ A_L\ A_C\ A_V\ A_I)$, the conventional NTA system with voltage-controlled resistors can be written as

$$C(v_c)v_c' = i_c \tag{22a}$$

$$L(i_l)i_l' = v_l \tag{22b}$$

$$0 = i_j - j(t) \tag{22c}$$

$$0 = v_u - u(t) \tag{22d}$$

$$0 = i_r - \gamma(v_r) \tag{22e}$$

$$0 = A_R i_r + A_L i_l + A_C i_c + A_V i_v + A_I i_j \tag{22f}$$

$$0 = v_r - A_R^{\mathrm{T}} e \tag{22g}$$

$$0 = v_l - A_L^{\mathrm{T}} e \tag{22h}$$

$$0 = v_c - A_C^{\mathrm{T}} e \tag{22i}$$

$$0 = v_u - A_V^{\mathrm{T}} e \tag{22j}$$

$$0 = v_i - A_I^{\mathrm{T}} e \tag{22k}$$

Differential variables are $x = (i_l, v_c)$, whereas $y = (i_r, i_c, i_v, v_r, v_l, v_i, e)$ are algebraic ones. Let us eliminate resistive currents and voltages using Equations (22e) and (22g). Inductive voltages will be substituted by means of Equation (22h), and current and voltage variables in the corresponding sources can be trivially eliminated using Equations (22c) and (22d). Finally, Equation (22k) will be considered as an output equation giving voltages in current source branches and will therefore be removed from the model. This way we get

$$C(v_c)v_c' = i_c \tag{23a}$$

$$L(i_l)i_l' = A_L^{\mathrm{T}} e \tag{23b}$$

$$0 = A_R\gamma(A_R^{\mathrm{T}} e) + A_L i_l + A_C i_c + A_V i_v + A_I j(t) \tag{23c}$$

$$0 = v_c - A_C^{\mathrm{T}} e \tag{23d}$$

$$0 = u(t) - A_V^{\mathrm{T}} e \tag{23e}$$

Equations (23c)–(23e) can be understood as a time-domain analog of [3, Equation (2.2)], and the method yielding this system will be therefore called augmented node analysis (ANA), here formulated without the need for branch replacements. The differential relations in the form (23a)–(23b) are those used in Reference [3, Section 2, step 4].

The additional interest of system (23) stems from the fact that it provides an intermediate formulation between NTA and several different methods, having index-1 if and only if so it has NTA, as shown in Theorem 2 below. It can be understood as the result of eliminating 'superfluous' variables from NTA. Using Equation (23), the key step in the state-space formulation of Li and Woo [3, Section 2, step 3] may be seen as an index-1 assumption on this differential–algebraic system. Such an index-1 condition can be rephrased in circuit-theoretic terms: this will be performed in Section 5. Furthermore, MNA can be seen as a (non-Schur) reduction of Equation (23), as shown at the end of this section.

Theorem 2
If the capacitance and inductance matrices $C(v_c)$, $L(i_l)$ are non-singular, the conventional node tableau analysis (NTA) system (22) has index 1 if and only if the augmented node analysis (ANA) system (23) has index 1.

Proof
Let G stand for the incremental conductance γ'. The derivative of the algebraic restrictions of NTA (22) with respect to algebraic variables $(i_j, v_u, e, i_v, i_c, i_r, v_r, v_l, v_i)$ reads

$$\begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & -G & 0 & 0 \\ A_I & 0 & 0 & A_V & A_C & A_R & 0 & 0 & 0 \\ 0 & I & -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_C^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -A_R^{\mathrm{T}} & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -A_L^{\mathrm{T}} & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & -A_I^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{24}$$

This matrix is non-singular if and only if so it is

$$\begin{pmatrix} 0 & 0 & 0 & I & -G \\ 0 & A_V & A_C & A_R & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -A_C^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -A_R^{\mathrm{T}} & 0 & 0 & 0 & I \end{pmatrix} \tag{25}$$

For better clarity regarding the application of Schur lemma, let us reorder rows in the matrix above, to get

$$\begin{pmatrix} 0 & A_V & A_C & A_R & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ -A_C^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & -G \\ -A_R^{\mathrm{T}} & 0 & 0 & 0 & I \end{pmatrix} \tag{26}$$

Writing

$$E_{11} = \begin{pmatrix} 0 & A_V & A_C \\ -A_V^{\mathrm{T}} & 0 & 0 \\ -A_C^{\mathrm{T}} & 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} A_R & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ -A_R^{\mathrm{T}} & 0 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} I & -G \\ 0 & I \end{pmatrix}$$

the Schur complement of E_{22} may be easily checked to read

$$\begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_V & A_C \\ -A_V^{\mathrm{T}} & 0 & 0 \\ -A_C^{\mathrm{T}} & 0 & 0 \end{pmatrix} \tag{27}$$

which is the derivative of the algebraic relations in ANA (23) with respect to the algebraic variables (e, i_v, i_c). □

MNA: Modified node analysis (MNA) [10–13, 15, 21] is easily obtained from ANA by eliminating capacitive currents and voltages. This is done via Equations (23a) and (23d), and yields

$$L(i_l)i_l' = A_L^{\mathrm{T}} e \tag{28a}$$

$$A_C C(A_C^{\mathrm{T}} e) A_C^{\mathrm{T}} e' = -A_R \gamma(A_R^{\mathrm{T}} e) - A_L i_l - A_V i_v - A_I j(t) \tag{28b}$$

$$0 = u(t) - A_V^{\mathrm{T}} e \tag{28c}$$

This is a quasilinear system, for which the matrix $A_C C(A_C^{\mathrm{T}} e) A_C^{\mathrm{T}}$ will typically be singular. Its analysis requires more sophisticated techniques; specifically, index-1 and index-2 conditions have been obtained for these systems using projector methods and the tractability index framework [11, 21].

For later comparison, we state here [21, Theorem 4], providing index-1 conditions under positive definiteness assumptions in all matrices involved. A square matrix F will be said to be positive definite if $x^{\mathrm{T}}Fx > 0$ for all non-vanishing x; we do not assume it to be symmetric. Additionally, a V–C loop (resp. an I–L cutset) is a loop (resp. a cutset) which only consists of voltage sources and/or capacitors (resp. current sources and/or inductors). Note that it is implicitly assumed that, in the absence of voltage sources, there are no capacitive trees, since otherwise the MNA system would have index-0 [31].

Theorem 3 (Tischendorf [21])
Assume that the capacitance, inductance, and conductance matrices are positive definite. If the network contains neither V–C loops (except for C-loops) nor I–L cutsets, then the MNA system (28) has index 1.

5. INDEX-1 CONDITIONS FOR ANA/NTA AND STATE-SPACE REDUCTION

A result analogous to Theorem 3 can be stated for ANA systems. Note that, below, we do not need to restrict the analysis to problems with positive definite reactances. We also emphasize that the reasoning in this case is much easier due to the semiexplicit form of the ANA system, *versus* the quasilinear one of MNA.

Theorem 4
Assume that the conductance matrix G is (positive or negative) definite, and that the local capacitance and inductance matrices C, L are non-singular. Then, the ANA system (23) has index-1 if and only if there are neither V–C loops nor I–L cutsets in the circuit.

Proof
Note that the derivative of the algebraic relations (23c)–(23e) with respect to the algebraic variables e, i_c, i_v reads

$$J = \begin{pmatrix} A_R G A_R^{\mathrm{T}} & A_C & A_V \\ -A_C^{\mathrm{T}} & 0 & 0 \\ -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix}$$

Non-singularity of this matrix is equivalent to index-1 in the ANA system. Such non-singularity condition holds if and only if the homogeneous linear system

$$A_R G A_R^{\mathrm{T}} x + A_C y + A_V z = 0 \tag{29a}$$

$$-A_C^{\mathrm{T}} x = 0 \tag{29b}$$

$$-A_V^{\mathrm{T}} x = 0 \tag{29c}$$

has only the zero solution. If we premultiply Equation (29a) by x^{T} and use Equations (29b) and (29c), we get

$$x^{\mathrm{T}} A_R G A_R^{\mathrm{T}} x = 0 \tag{30}$$

which implies

$$A_R^{\mathrm{T}}x = 0 \tag{31}$$

because of the definiteness of G, whereas Equation (29a) amounts to

$$A_C y + A_V z = 0 \tag{32}$$

The existence of a non-vanishing solution holds simultaneously for Equation (29) and for Equations (29b), (29c), (31), (32) altogether. Due to Lemmas 2 and 3, this means that index-1 in the ANA model is equivalent to the absence of V–C loops and I–L cutsets. □

Corollary 1
If the local conductance matrix G is (positive or negative) definite, and the local capacitance and inductance matrices C, L are non-singular, then the NTA system (22) has index-1 if and only if there are neither V–C loops nor I–L cutsets in the circuit. The same is true for charge-oriented tableau systems and, provided that the supporting conditions (13) and (16) are satisfied, also for hybrid tableau models.

Corollary 2
If the local conductance matrix G is (positive or negative) definite, and there are neither V–C loops nor I–L cutsets in the circuit, then Equations (23c)–(23e) yield $i_c = \psi_1(i_l, v_c, j(t), u(t))$, $e = \psi_2(i_l, v_c, j(t), u(t))$, and an output equation $i_v = \psi_3(i_l, v_c, j(t), u(t))$, for locally well-defined mappings ψ_1, ψ_2, ψ_3. Inserting these into Equations (23a) and (23b), we get the state-space equation

$$C(v_c)v_c' = \psi_1(i_l, v_c, j(t), u(t)) \tag{33a}$$

$$L(i_l)i_l' = A_L^{\mathrm{T}}\psi_2(i_l, v_c, j(t), u(t)) \tag{33b}$$

This system trivially amounts to an explicit ODE if, additionally, the local capacitance and inductance matrices C, L are non-singular.

Corollary 2 follows immediately from the Implicit Function Theorem, and provides precise assumptions under which the state-space formulation of Reference [3] is feasible. More precisely, the above-stated index-1 condition guarantees that the matrix $\mathbf{Y}_n'$ in Reference [3, Section 2, step 3] is invertible.

5.1. Example 1

Let us consider the series RLC circuit displayed in Figure 2.

The capacitor and the inductor are linear. We assume that the non-linear resistor is voltage-controlled through a characteristic $i = \gamma(v_r)$. An example of this is shown in Figure 3, where the maximum and the minimum define the boundaries of the so-called tunnel-effect region in a tunnel diode.

The ANA system (23) may be easily shown to read for this circuit

$$Cv_c' = i_c \tag{34a}$$

$$Li_l' = -e_3 \tag{34b}$$

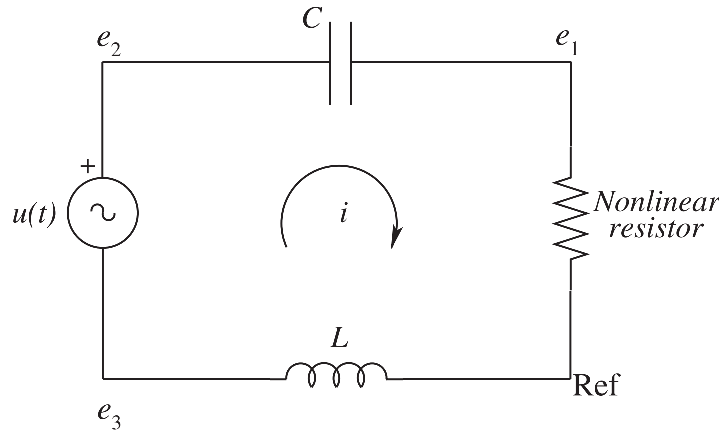

Figure 2. Non-linear series RLC circuit.

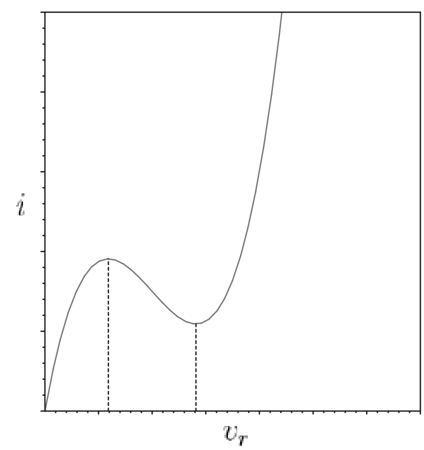

Figure 3. Voltage-controlled resistor (tunnel diode).

$$0 = i_v - i_c \tag{34c}$$

$$0 = i_c - \gamma(e_1) \tag{34d}$$

$$0 = \gamma(e_1) - i_l \tag{34e}$$

$$0 = v_c - (e_2 - e_1) \tag{34f}$$

$$0 = u(t) - (e_2 - e_3) \tag{34g}$$

Since there are neither V–C loops nor I–L cutsets in the circuit, Theorem 4 predicts that this DAE has index-1 in the regions where the conductance matrix is definite, what amounts in this case to the condition $\gamma' \neq 0$. This is actually the case, as can be easily seen if we eliminate i_c, e_2 and e_3 to get

$$Cv_c' = i_l \tag{35a}$$

$$Li_l' = -v_c - e_1 + u(t) \tag{35b}$$

$$0 = \gamma(e_1) - i_l \tag{35c}$$

This form makes it obvious that the DAE has index 1 if (and only if) $\gamma'(e_1) \neq 0$, that is, everywhere except at the extrema of Figure 3.

Note finally that a state equation

$$Cv_c' = i_l \tag{36a}$$

$$Li_l' = -v_c - \gamma^{-1}(i_l) + u(t) \tag{36b}$$

can be locally formulated in regions where the inverse γ^{-1} is well-defined, that is, in the three index-1 intervals delimited by the extrema of the resistor characteristic.

A reader might conjecture that such critical points in the characteristic should always prevent the model from being index-1, regardless of the topology. This is not at all the case, as illustrated by example 2 below. In fact, the analysis of these non-definite cases requires more careful topological conditions, which will be considered in Theorem 5 in next section.

Another natural question is to what extent this result depends on the assumed voltage-controllability of resistors. This is a common hypothesis in nodal methods, since it avoids using explicitly current variables in resistors. But, as shown in example 3 in the next section, current-controlled resistors may also be modelled in an index-1 way, even in the presence of critical points. These problems are also accommodated in the setting of Theorem 5.

5.2. Example 2

Consider now the parallel RLC circuit displayed in Figure 4, in which the capacitor and the inductor are linear. As in example 1, the non-linear resistor is voltage-controlled through a characteristic $i = \gamma(v_r)$, such as the one depicted in Figure 3.

ANA equations (23) now read

$$Cv_c' = i_c \tag{37a}$$

$$Li_l' = e \tag{37b}$$

$$0 = j(t) - \gamma(e) - i_c - i_l \tag{37c}$$

$$0 = v_c - e \tag{37d}$$

Since there are neither V–C loops nor I–L cutsets in the circuit, Theorem 4 predicts again that this DAE has index-1 in the regions where the conductance matrix is definite, what amounts as

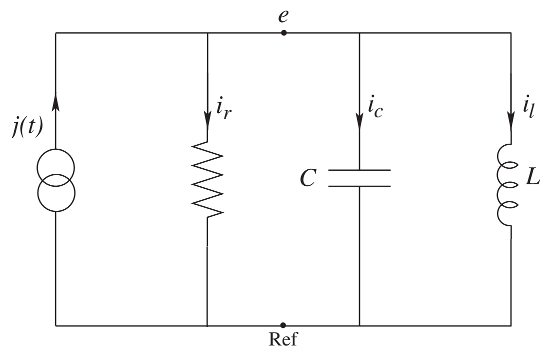

Figure 4. Non-linear parallel RLC circuit.

before to the condition $\gamma' \neq 0$. But, in contrast to Example 1, Equation (37) makes it possible to check that the index-1 nature holds actually regardless of the (sufficient, but not necessary) condition $\gamma' \neq 0$. This means that the model has globally index-1; indeed, the following global state-space equation is easily derived from Equation (37):

$$Cv_c' = -\gamma(v_c) - i_l + j(t) \tag{38a}$$

$$Li_l' = v_c \tag{38b}$$

The different behaviour of both examples concerning critical points will be explained in the light of Theorem 5.

6. CURRENT-CONTROLLED AND SEMIDEFINITE RESISTORS

As indicated above, in nodal methods it is common to assume voltage-controlled descriptions (see e.g. Reference [21]). Nevertheless, it is of interest to extend the analysis in order to accommodate current-controlled resistors, as well as non-definite problems, allowing, e.g. for the inclusion of open- or short-circuits without the need to change the graph by removing or contracting the corresponding branch. Local open- and short-circuits can be also handled this way; an example of a local open circuit is given by the points where $\gamma' = 0$ in examples 1 and 2.

Let us then assume that, instead of the voltage-controlled representation (22e), the general equation $g_r(i_r, v_r) = 0$ appearing in tableau models splits into four different subequations, describing four uncoupled groups (some of which might be empty) with characteristics

$$v_{r1} = \rho_1(i_{r1}) \tag{39a}$$

$$v_{r2} = \rho_2(i_{r2}) \tag{39b}$$

$$i_{r3} = \gamma_1(v_{r3}) \tag{39c}$$

$$i_{r4} = \gamma_2(v_{r4}) \tag{39d}$$

The first two are current-controlled resistors, and both groups are distinguished by the fact that, at the operating point, we will assume that:

(i) $R_1 = \rho_1'(i_{r1}^*)$ is definite, whereas
(ii) $R_2 = \rho_2'(i_{r2}^*)$ is symmetric and semidefinite.

For the last two (which are voltage-controlled), we assume that

(iii) $G_1 = \gamma_1'(v_{r3}^*)$ is definite, and
(iv) $G_2 = \gamma_2'(v_{r4}^*)$ is symmetric and semidefinite.

Additionally,
(v) All matrices are assumed simultaneously either positive or negative (semi)definite,

that is, (semi)definiteness holds with the same sign for all matrices. Note that the setting of Theorem 4 would correspond to cases in which all resistors are in the γ_1 group. In fact, the proof of Theorem 4 may be particularized from that of Theorem 5; we believe the former to be worth an independent statement, not only because of the 'if and only if' condition there, but also for clarity and comparative purposes.

Focusing, as in previous sections, on conventional models, the ANA system would read in this context:

$$C(v_c)v_c' = i_c \tag{40a}$$

$$L(i_l)i_l' = A_L^{\mathrm{T}} e \tag{40b}$$

$$\begin{aligned} 0 &= A_{R_1} i_{r1} + A_{R_2} i_{r2} + A_{G_1}\gamma_1(A_{G_1}^{\mathrm{T}} e) + A_{G_2}\gamma_2(A_{G_2}^{\mathrm{T}} e) \\ &\quad + A_L i_l + A_C i_c + A_V i_v + A_I j(t) \end{aligned} \tag{40c}$$

$$0 = \rho_1(i_{r1}) - A_{R_1}^{\mathrm{T}} e \tag{40d}$$

$$0 = \rho_2(i_{r2}) - A_{R_2}^{\mathrm{T}} e \tag{40e}$$

$$0 = v_c - A_C^{\mathrm{T}} e \tag{40f}$$

$$0 = u(t) - A_V^{\mathrm{T}} e \tag{40g}$$

where we have split the previous A_R in an obvious manner. Note that this model includes as new variables the currents i_{r1}, i_{r2} of current-controlled resistors.

Index-1 conditions, as well as the consequences discussed in Corollaries 1 and 2, now require a more delicate interplay between the above-indicated assumptions on resistive devices and the topology of the circuit. We provide below sufficient conditions for index-1.

Theorem 5
Assume that the local capacitance and inductance matrices C, L are non-singular, and that the above-indicated assumptions (i)–(v) on R_1, R_2, G_1, G_2 hold. Then, the ANA system (40) has index-1 if there are neither V–C–R_2 loops nor I–L–G_2 cutsets in the circuit.

Proof
The derivative of the algebraic relations (40c)–(40g) in the general ANA system with respect to the algebraic variables $(i_{r1}, i_{r2}, e, i_v, i_c)$ reads

$$\begin{pmatrix} A_{R_1} & A_{R_2} & A_{G_1}G_1A_{G_1}^{\mathrm{T}} + A_{G_2}G_2A_{G_2}^{\mathrm{T}} & A_C & A_V \\ R_1 & 0 & -A_{R_1}^{\mathrm{T}} & 0 & 0 \\ 0 & R_2 & -A_{R_2}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & -A_C^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & -A_V^{\mathrm{T}} & 0 & 0 \end{pmatrix} \tag{41}$$

The problem has index 1 if and only if this matrix is non-singular, what is in turn equivalent to the condition that the following homogeneous linear system has only the trivial solution:

$$A_{R_1}s + A_{R_2}w + (A_{G_1}G_1A_{G_1}^{\mathrm{T}} + A_{G_2}G_2A_{G_2}^{\mathrm{T}})x + A_C y + A_V z = 0 \tag{42a}$$

$$R_1 s - A_{R_1}^{\mathrm{T}}x = 0 \tag{42b}$$

$$R_2 w - A_{R_2}^{\mathrm{T}}x = 0 \tag{42c}$$

$$-A_C^{\mathrm{T}}x = 0 \tag{42d}$$

$$-A_V^{\mathrm{T}}x = 0 \tag{42e}$$

Suppose then that there is a non-vanishing solution of Equation (42). Premultiplication of Equation (42a) by x^{T} yields, using Equations (42d) and (42e),

$$x^{\mathrm{T}}A_{R_1}s + x^{\mathrm{T}}A_{R_2}w + x^{\mathrm{T}}A_{G_1}G_1A_{G_1}^{\mathrm{T}}x + x^{\mathrm{T}}A_{G_2}G_2A_{G_2}^{\mathrm{T}}x = 0 \tag{43}$$

which can be rewritten, using Equations (42b) and (42c), as

$$s^{\mathrm{T}}R_1^{\mathrm{T}}s + w^{\mathrm{T}}R_2 w + x^{\mathrm{T}}A_{G_1}G_1A_{G_1}^{\mathrm{T}}x + x^{\mathrm{T}}A_{G_2}G_2A_{G_2}^{\mathrm{T}}x = 0 \tag{44}$$

Now, due to the simultaneous semidefiniteness of all matrices in this equation, all terms must vanish. Since R_1 and G_1 are definite, this means that actually it must be $s=0$, $A_{G_1}^{\mathrm{T}}x=0$. In turn, since R_2 and G_2 are symmetric semidefinite, using congruent diagonalization it is not difficult to check that it must be $R_2w=0$, $G_2A_{G_2}^{\mathrm{T}}x=0$. These relations transform system (42) into

$$A_{R_2}w + A_C y + A_V z = 0 \tag{45a}$$

$$-A_{G_1}^{\mathrm{T}}x = 0 \tag{45b}$$

$$-A_{R_1}^{\mathrm{T}}x = 0 \tag{45c}$$

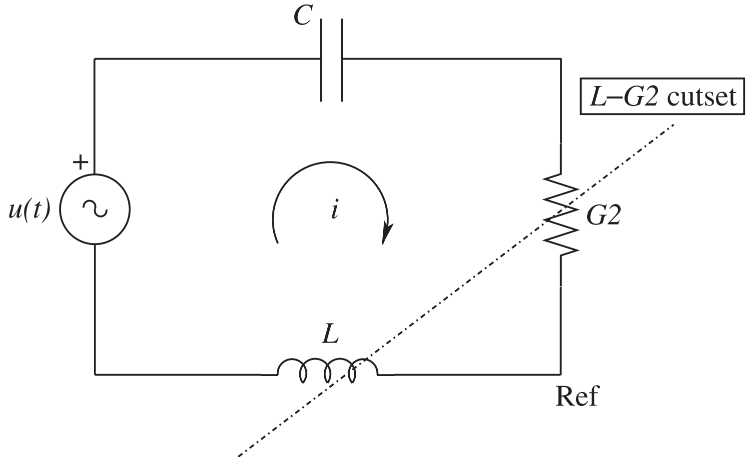

Figure 5. Pathological cutset in Example 1.

$$-A_{R_2}^{\mathrm{T}} x = 0 \tag{45d}$$

$$-A_C^{\mathrm{T}} x = 0 \tag{45e}$$

$$-A_V^{\mathrm{T}} x = 0 \tag{45f}$$

and, since $s = 0$, the original non-trivial solution of Equation (42) leads to a non-vanishing solution of Equation (45), that is, to the existence of at least one V–C–R_2 loop or a I–L–G_2 cutset, in the light of Lemmas 2 and 3. Equivalently, the absence of these configurations guarantees index-1 in the model. □

Now the reason for the different behaviour displayed at critical points by the circuits in Examples 1 and 2 becomes clear. To consider the local behaviour at critical points (extrema of Figure 3), we must frame the resistor in the group G_2. In Example 1 (Figure 2), there appears a pathological cutset, as shown in Figure 5, and this explains why the index-1 condition is not met at critical points. Note that, away from critical points, the resistor can be included in the G_1 group and therefore the cutset causes no problem. On the contrary, in Example 2 (Figure 4) there are no pathological configurations and hence the model has index 1 even at critical points. Note that, in particular, the C–G_2 loop does not cause any difficulty.

6.1. Example 3

We illustrate further the scope of Theorem 5 by considering again the series circuit displayed in Figure 2, but now assuming that the resistor is current-controlled by $v_r = \rho(i_r)$. This is the case, for instance, in Van der Pol circuit. We assume that there may be critical points $\rho' = 0$ in this characteristic.

Including the new variable i_r in the model, the ANA system (40) reads

$$Cv_c' = i_c \tag{46a}$$

$$Li_l' = -e_3 \tag{46b}$$

$$0 = i_v - i_c \tag{46c}$$

$$0 = i_c - i_r \tag{46d}$$

$$0 = i_r - i_l \tag{46e}$$

$$0 = \rho(i_r) - e_1 \tag{46f}$$

$$0 = v_c - (e_2 - e_1) \tag{46g}$$

$$0 = u(t) - (e_2 - e_3) \tag{46h}$$

The resistor can be locally framed in the group R_1 (if $\rho' \neq 0$) or in R_2 (if $\rho' = 0$). In any case, since the pathological topologies appearing in Theorem 5 are not displayed, we conclude that the circuit has index-1, what may be easily checked through Equation (46). Actually, we may derive a valid state-space equation regardless of the actual shape of $\rho(i_r)$, namely:

$$Cv_c' = i_l \tag{47a}$$

$$Li_l' = -v_c - \rho(i_r) + u(t) \tag{47b}$$

Note that the scope of the previous Theorem 4 would be restricted to points where $\rho' \neq 0$, for which we might take a local inverse $i_r = \rho^{-1}(v_r)$. With the improved version displayed in Theorem 5, we do not only provide a global description, but also accommodate critical points.

7. CONCLUDING REMARKS

Several interrelations between tableau, augmented, and modified models for nodal analysis of non-linear circuits have been addressed in this paper within a differential–algebraic context. Besides the well-known tableau and modified node analysis (MNA) systems, the augmented node analysis (ANA) approach has been extensively discussed. This approach has been shown to link the tableau and MNA formalisms, additionally yielding semiexplicit models for which index conditions can be easily examined.

The feasibility of certain state-space reductions has been also discussed in this framework, again using a differential–algebraic setting. DAEs are nowadays pervasive in non-linear circuit modelling, and numerical methods for them have been exhaustively discussed in the last years, not only for general systems [22, 23, 25] but also specifically in the circuit context (see e.g. Reference [31] and references therein). Circuit simulation programs such as SPICE or TITAN employ MNA models; it is worth emphasizing in this direction that, with respect to MNA, the additional capacitive variables arising in ANA might be a drawback in the numerical simulation of VLSI circuits, specially in the presence of a large number of parasitic capacitances.

The analytical results here discussed hold for models based on different sets of reactive variables; the distinction between charge-oriented, conventional (voltage-oriented) and hybrid models clarifies the reactive variables which qualify as dynamic ones, and index-1 conditions have been proved to read the same for all these models as far as they are well-defined. These results accommodate both definite and semidefinite conductance/resistance matrices, going beyond systems with just voltage-controlled resistors.

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