ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 41, Number 5, 2011

GREEN'S FUNCTION FOR THIRD-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. Existence of positive periodic solutions of third-order differential equations is established using an explicit Green's function and a fixed-point theorems on cones.

1. Introduction. The study of periodic solutions plays a major role in the theory of differential equations, in particular the existence of positive periodic solutions of these equations. Currently there are plenty of existence results for periodic solutions (or positive periodic solutions) for second-order differential equations, but there are relatively few results on higher order differential equation. On the other hand, in the study of higher order differential equations, the naive idea to translate the equation into a first order system of differential equations by defining $x_1 = x$, $x_2 = x'$, $x_3 = x''$, $\dots$, which works well for showing existence of periodic solutions, does not obviously lead to existence proofs for positive periodic solutions, since the condition $x = x_1 \geq 0$ of positivity for the higher order equation is different from the natural positivity condition $(x_1, x_2, \dots) \geq 0$ for the corresponding system. An approach which is frequently used is to transform the higher-order equation into a corresponding integral equation and to establish the existence of positive periodic solutions based on a fixed point theorem in cones. Following this path one needs an explicit representation of the Green's function which is rather intricate to compute. In this paper, we provide the Green's function for third-order differential equations with constant coefficients. This should be helpful for further studies of this type of equations.

The remaining part of the paper is organized as follows. In Section 2, the Green's function for third-order constant-coefficient linear differential equation

$$u''' + au'' + bu' + cu = h(t), \tag{1.1}$$

The second author was supported by AvH Foundation of Germany and NNSF of China (10971202).
Received by the editors on February 2, 2009.
DOI:10.1216/RMJ-2011-41-5-1417

will be given. Here $a,\ b,\ c \in \mathbf{R}$ and $h \in C(\mathbf{R}, (0, +\infty))$ is an ω-periodic function with period $\omega > 0$. In Section 3, some useful properties for the Green's function are obtained. Finally, Section 4 is devoted to applications of the computed Green's function, we establish the existence of positive ω-periodic solutions for the following third-order nonlinear differential equation

$$u''' + au'' + bu' + cu = f(t, u(t)), \tag{1.2}$$

where $f \in C(\mathbf{R} \times [0, +\infty), [0, +\infty))$ and $f(t, u) > 0$, for $u > 0$. Our approach is based on the Guo-Krasnoselskii fixed point theorem [**1, 2**].

The associated homogeneous equation of (1.1) (or (1.2)) is

$$u''' + au'' + bu' + cu = 0. \tag{1.3}$$

Its characteristic equation is

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0, \tag{1.4}$$

and the roots $\lambda_1,\ \lambda_2,\ \lambda_3$ of (1.4) satisfy one of the four cases:

(i) $\lambda_1 \neq \lambda_2 \neq \lambda_3$,

(ii) $\lambda_1 = \lambda_2 \neq \lambda_3$,

(iii) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$,

(iv) $\lambda_1 = \alpha + i\beta,\ \lambda_2 = \alpha - i\beta,\ \lambda_3 = \lambda$,

where $\alpha,\ \beta,\ \lambda \in \mathbf{R}$. If $c = 0$, then at least one of the roots of (1.4) is 0, in this case we call equation (1.1) degenerate. This case will be discussed elsewhere. In this paper, we always assume $c \neq 0$.

2. Finding Green's functions.

Theorem 2.1. *If* $\lambda_1 \neq \lambda_2 \neq \lambda_3$, *then the equation* (1.1) *has a unique ω-periodic solution which is of the form*

$$u(t) = \int_t^{t+\omega} G_1(t, s) h(s)\, \mathrm{d}s, \tag{2.1}$$

where
(2.2)
$$G_1(t,s) = \frac{\exp(\lambda_1(t+\omega-s))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{\exp(\lambda_2(t+\omega-s))}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} + \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))}, \quad \textit{for } s \in [t, t+\omega].$$

Proof. If $\lambda_1 \neq \lambda_2 \neq \lambda_3$, it is easy to check that the associated homogeneous equation of (1.1) has the solution

$$u(t) = c_1 \exp(\lambda_1 t) + c_2 \exp(\lambda_2 t) + c_3 \exp(\lambda_3 t).$$

The only periodic solution of the associated homogeneous equation of (1.1) is the trivial solution, i.e., $c_1,\ c_2,\ c_3 = 0$.

Applying the method of variation of parameters, we get

$$c_1'(t) = \frac{h(t)\exp(-\lambda_1 t)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)},$$
$$c_2'(t) = \frac{h(t)\exp(-\lambda_2 t)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)},$$
$$c_3'(t) = \frac{h(t)\exp(-\lambda_3 t)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)},$$

and hence

$$c_1(t+\omega) = c_1(t) + \int_t^{t+\omega} \frac{\exp(-\lambda_1 s)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} h(s)\,\mathrm{d}s,$$
$$c_2(t+\omega) = c_2(t) + \int_t^{t+\omega} \frac{\exp(-\lambda_2 s)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)} h(s)\,\mathrm{d}s,$$
$$c_3(t+\omega) = c_3(t) + \int_t^{t+\omega} \frac{\exp(-\lambda_3 s)}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)} h(s)\,\mathrm{d}s.$$

Since we are looking for ω-periodic solutions of (1.1), i.e., $u(t+\omega) = u(t)$, $u'(t+\omega) = u'(t)$ and $u''(t+\omega) = u''(t)$, this implies

$$c_1(t) = \int_t^{t+w} \frac{h(s)\exp(\lambda_1(\omega-s))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))}\mathrm{d}s,$$

$$c_2(t) = \int_t^{t+w} \frac{h(s)\exp(\lambda_2(\omega - s))}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} \mathrm{d}s,$$
$$c_3(t) = \int_t^{t+w} \frac{h(s)\exp(\lambda_3(\omega - s))}{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(1-\exp(\lambda_3\omega))} \mathrm{d}s.$$

Therefore,

$$\begin{aligned} u(t) &= c_1(t)\exp(\lambda_1 t) + c_2(t)\exp(\lambda_2 t) + c_3(t)\exp(\lambda_3 t) \\ &= \int_t^{t+\omega} G_1(t,s)h(s)\mathrm{d}s, \end{aligned}$$

where $G_1(t,s)$ is as defined in (2.2).

On the other hand, since $G(t+r, s+r) = G(t,s)$ for any $r \in \mathbf{R}$, it follows that

$$\begin{aligned} u(t+\omega) &= \int_{t+\omega}^{t+2\omega} G_1(t+\omega, s)h(s)\,\mathrm{d}s \\ &= \int_t^{t+\omega} G_1(t+\omega, \theta+\omega)h(\theta+\omega)\,\mathrm{d}\theta \\ &= \int_t^{t+\omega} G_1(t,s)h(s)\,\mathrm{d}s = u(t), \end{aligned}$$

showing that $u(t)$ is an ω-periodic function. Assume that u_1 and u_2 are two ω-periodic solutions of (1.1). Then $v(t) = u_1(t) - u_2(t)$ is an ω-periodic solution of the associated homogeneous equation of (1.1) and hence $v(t) \equiv 0$, proving uniqueness of the ω-periodic solution u. □

Theorem 2.2. *If $\lambda_1 = \lambda_2 \neq \lambda_3$, then equation* (1.1) *has a unique ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s, \tag{2.3}$$

where

$$\begin{aligned} G_2(t,s) = &\frac{\exp(\lambda_1(t+\omega-s))[(1-\exp(\lambda_1\omega))((s-t)(\lambda_3-\lambda_1)-1)-(\lambda_3-\lambda_1)\omega]}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} \\ &+ \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \quad \text{for } s \in [t, t+\omega]. \end{aligned} \tag{2.4}$$

Proof. First it is easy to see that the associated homogeneous equation of (1.1) has solutions

$$u(t) = c_1 \exp(\lambda_1 t) + c_2 t \exp(\lambda_1 t) + c_3 \exp(\lambda_3 t).$$

Applying the method of variation of parameters, we get

$$\begin{aligned} c_1'(t) &= \frac{h(t)(\lambda_3 t - \lambda_1 t - 1)}{(\lambda_3 - \lambda_1)^2 \exp(\lambda_1 t)}, \\ c_2'(t) &= \frac{h(t)}{(\lambda_1 - \lambda_3) \exp(\lambda_1 t)}, \\ c_3'(t) &= \frac{h(t)}{(\lambda_3 - \lambda_1)^2 \exp(\lambda_3 t)}. \end{aligned}$$

Since $u(t),\ u'(t),\ u''(t)$ are supposed to be periodic functions, we get

$$\begin{aligned} c_1(t) &= \int_t^{t+w} \frac{h(s)\exp(\lambda_1(\omega-s))[(1-\exp(\lambda_1\omega))(\lambda_3 s-\lambda_1 s-1)-(\lambda_3-\lambda_1)\omega]}{(\lambda_3-\lambda_1)^2(1-\exp(\lambda_1\omega))^2}\,\mathrm{d}s, \\ c_2(t) &= \int_t^{t+w} \frac{h(s)\exp(\lambda_1(\omega - s))}{(\lambda_1 - \lambda_3)(1 - \exp(\lambda_1\omega))}\,\mathrm{d}s, \\ c_3(t) &= \int_t^{t+w} \frac{h(s)\exp(\lambda_3(\omega - s))}{(\lambda_1 - \lambda_3)^2(1 - \exp(\lambda_3\omega))}\,\mathrm{d}s. \end{aligned}$$

Therefore

$$\begin{aligned} u(t) &= c_1(t)\exp(\lambda_1 t) + c_2(t) t \exp(\lambda_1 t) + c_3(t)\exp(\lambda_3 t) \\ &= \int_t^{t+\omega} G_2(t,s) h(s) \mathrm{d}s, \end{aligned}$$

where $G_2(t,s)$ is defined as in (2.4).

Similar as in the proof of Theorem 2.1, we can prove the uniqueness and the periodicity of the ω-periodic solution $u(t)$. □

Theorem 2.3. *If* $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, *then equation* (1.1) *has a unique* ω*-periodic solution which is of the form*

$$u(t) = \int_t^{t+\omega} G_3(t,s) h(s)\,\mathrm{d}s, \tag{2.5}$$

where
(2.6)
$$G_3(t,s) = \frac{[(s-t)\exp(\lambda\omega)+\omega-s+t]^2+\omega^2\exp(\lambda\omega)}{2(1-\exp(\lambda\omega))^3}\exp(\lambda(t+\omega-s)),$$
$$\textit{for } s \in [t, t+\omega].$$

Proof. In this case, the associated homogeneous equation of (1.1) has solutions

$$u(t) = c_1\exp(\lambda t) + c_2 t\exp(\lambda t) + c_3 t^2\exp(\lambda t).$$

Analogously, by applying the method of variation of parameters we get

$$c_1'(t) = \frac{h(t)t^2}{2\exp(\lambda t)},$$
$$c_2'(t) = \frac{-h(t)t}{\exp(\lambda t)},$$
$$c_3'(t) = \frac{h(t)}{2\exp(\lambda t)}.$$

Noting that $u(t),\ u'(t),\ u''(t)$ are periodic functions, we get

$$c_1(t) = \int_t^{t+w}\frac{h(s)\exp(\lambda(\omega-s))[(s\exp(\lambda\omega)+\omega-s)^2+\omega^2\exp(\lambda\omega)]}{2(1-\exp(\lambda\omega))^3}\,\mathrm{d}s,$$
$$c_2(t) = \int_t^{t+w}\frac{h(s)\exp(\lambda(\omega-s))(\omega-s+s\exp(\lambda\omega))}{(1-\exp(\lambda\omega))^2}\,\mathrm{d}s,$$
$$c_3(t) = \int_t^{t+w}\frac{h(s)\exp(\lambda(\omega-s))}{2(1-\exp(\lambda\omega))}\,\mathrm{d}s.$$

Therefore,

$$\begin{aligned} u(t) &= c_1(t)\exp(\lambda t) + c_2(t)t\exp(\lambda t) + c_3(t)t^2\exp(\lambda t) \\ &= \int_t^{t+\omega} G_3(t,s)h(s)\mathrm{d}s, \end{aligned}$$

where $G_3(t,s)$ is as defined in (2.6). One can prove the uniqueness and the periodicity of the ω-periodic solution u in the same way as in the proof of Theorem 2.1. $\square$

For convenience, we define the abbreviations

$$\begin{aligned} l_1(t,s) &= \sin\beta(t+\omega-s), \qquad l_2(t,s) = \cos\beta(t+\omega-s).\\ A_1(t) &= \cos\beta t - \exp(\alpha\omega)\cos\beta(t+\omega),\\ A_2(t) &= \cos\beta(t+\omega-s) - \exp(\alpha\omega)\cos\beta(t-s),\\ B_1(t) &= \sin\beta t - \exp(\alpha\omega)\sin\beta(t+\omega),\\ B_2(t) &= \sin\beta(t+\omega-s) - \exp(\alpha\omega)\sin\beta(t-s). \end{aligned}$$

Theorem 2.4. *If $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\lambda_3 = \lambda$, then equation* (1.1) *has a unique ω-periodic solution which is of the form*

$$u(t) = \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s, \tag{2.7}$$

where

$$\begin{aligned} G_4(t,s) = {}& \frac{\exp(\alpha(t+\omega-s))[(\alpha-\lambda)B_2(t) - \beta A_2(t)]}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega) - 2\exp(\alpha\omega)\cos\beta\omega)}\\ &+ \frac{\exp(\lambda(t+\omega-s))}{(1-\exp(\lambda\omega))[(\alpha-\lambda)^2+\beta^2]},\\ &\textit{for } s \in [t, t+\omega]. \end{aligned} \tag{2.8}$$

Proof. It is easy to see that the associated homogeneous equation of (1.1) has solutions

$$u(t) = c_1\exp(\alpha t)\cos\beta t + c_2\exp(\alpha t)\sin\beta t + c_3\exp(\lambda t).$$

Applying the method of variation of parameters, we get

$$\begin{aligned} c_1'(t) &= -\frac{h(t)\exp(-\alpha t)[(\alpha-\lambda)\sin\beta t + \beta\cos\beta t]}{\beta[\beta^2+(\alpha-\lambda)^2]},\\ c_2'(t) &= \frac{h(t)\exp(-\alpha t)[(\alpha-\lambda)\cos\beta t - \beta\sin\beta t]}{\beta[\beta^2+(\alpha-\lambda)^2]},\\ c_3'(t) &= \frac{h(t)\beta\exp(-\lambda t)}{\beta[\beta^2+(\alpha-\lambda)^2]}. \end{aligned}$$

Noting that $u(t),\ u'(t),\ u''(t)$ are periodic functions, we obtain

$$c_1(t) = \int_t^{t+w} \frac{h(s)\exp(\alpha(\omega-s))\{[(\alpha-\lambda)A_1(t)-\beta B_1(t)]l_1(t,s)-[(\alpha-\lambda)B_1(t)+\beta A_1(t)]l_2(t,s)\}}{\beta(A_1^2(t)+B_1^2(t))[(\alpha-\lambda)^2+\beta^2]}\,\mathrm{d}s,$$

$$c_2(t) = \int_t^{t+w} \frac{h(s)\exp(\alpha(\omega-s))\{[(\alpha-\lambda)A_1(t)-\beta B_1(t)]l_2(t,s)+[(\alpha-\lambda)B_1(t)+\beta A_1(t)]l_1(t,s)\}}{\beta(A_1^2(t)+B_1^2(t))[(\alpha-\lambda)^2+\beta^2]}\,\mathrm{d}s,$$

$$c_3(t) = \int_t^{t+w} \frac{h(s)\exp(\lambda(\omega-s))}{(1-\exp(\lambda\omega))[(\alpha-\lambda)^2+\beta^2]}.$$

Therefore,

$$\begin{aligned} u(t) &= c_1(t)\exp(\alpha t)\cos\beta t + c_2(t)\exp(\alpha t)\sin\beta t + c_3(t)\exp(\lambda t) \\ &= \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s, \end{aligned}$$

where $G_4(t,s)$ is as defined in (2.8). Similarly as in the proof of Theorem 2.1, we get the uniqueness and the periodicity of $u(t)$. □

3. Properties of the Green's functions (2.2), (2.4), (2.6), (2.8). First, we denote

$$\begin{aligned} C_w^+ &= \{u \in C(\mathbf{R},(0,+\infty)) : u(t+\omega) = u(t)\}, \\ C_w^- &= \{u \in C(\mathbf{R},(-\infty,0)) : u(t+\omega) = u(t)\}. \end{aligned}$$

Case (i): $\lambda_1 \neq \lambda_2 \neq \lambda_3$. For the sake of convenience, we use the following abbreviations

$$\begin{aligned} g_1(t,s) &= \frac{\exp(\lambda_1(t+\omega-s))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))}, \\ g_2(t,s) &= \frac{\exp(\lambda_2(t+\omega-s))}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))}, \\ g_3(t,s) &= \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(1-\exp(\lambda_3\omega))}, \end{aligned}$$

$$
\begin{aligned}
q_1 &= (\lambda_1 - \lambda_3) + (\lambda_1 + \lambda_2 - 2\lambda_3)\exp((\lambda_2 + \lambda_3)\omega) \\
&\quad + (2\lambda_1 - \lambda_2 - \lambda_3)\exp((\lambda_1 + \lambda_2)\omega), \\
q_2 &= (\lambda_1 - \lambda_3) + (\lambda_1 - \lambda_2)\exp((\lambda_2 + \lambda_3)\omega) + (\lambda_2 - \lambda_3)\exp((\lambda_1 + \lambda_2)\omega) \\
&\quad + 2(\lambda_1 - \lambda_3)\exp((\lambda_1 + \lambda_3)\omega), \\
p_1 &= (\lambda_2 - \lambda_3)\exp(\lambda_3\omega) + 2(\lambda_1 - \lambda_3)\exp(\lambda_2\omega) + (\lambda_1 - \lambda_2)\exp(\lambda_1\omega) \\
&\quad + (\lambda_1 - \lambda_3)\exp((\lambda_1 + \lambda_2 + \lambda_3)\omega), \\
p_2 &= (\lambda_1 + \lambda_2 - 2\lambda_3)\exp(\lambda_1\omega) + (2\lambda_1 - \lambda_2 - \lambda_3)\exp(\lambda_3\omega) \\
&\quad + (\lambda_1 - \lambda_3)\exp((\lambda_1 + \lambda_2 + \lambda_3)\omega),
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= \frac{\exp(\lambda_1\omega)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{1}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)(1 - \exp(\lambda_2\omega))} \\
&\quad + \frac{\exp(\lambda_3\omega)}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(1 - \exp(\lambda_3\omega))}, \\
B_3 &= \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} + \frac{\exp(\lambda_2\omega)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)(1 - \exp(\lambda_2\omega))} \\
&\quad + \frac{1}{(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)(1 - \exp(\lambda_3\omega))}.
\end{aligned}
$$

Theorem 3.1.

$$\int_t^{t+\omega} G_1(t,s)\,\mathrm{d}s = -\frac{1}{\lambda_1\lambda_2\lambda_3}$$

for all $t \in [0,\omega]$ and $s \in [t, t+\omega]$.

Proof. By a simple calculation we have

$$\int_t^{t+\omega} g_1(t,s)\mathrm{d}s = -\frac{1}{\lambda_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}, \tag{3.1}$$

$$\int_t^{t+\omega} g_2(t,s)\,\mathrm{d}s = -\frac{1}{\lambda_2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}, \tag{3.2}$$

$$\int_t^{t+\omega} g_3(t,s)\,\mathrm{d}s = -\frac{1}{\lambda_3(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)},$$

and then

$$\begin{aligned}\int_t^{t+\omega} G_1(t,s)\,\mathrm{d}s = \int_t^{t+\omega} g_1(t,s)\mathrm{d}s + \int_t^{t+\omega} g_2(t,s)\,\mathrm{d}s \\ + \int_t^{t+\omega} g_3(t,s)\mathrm{d}s = -\frac{1}{\lambda_1\lambda_2\lambda_3}.\end{aligned} \tag{3.3}$$

□

Theorem 3.2. *If $p_1 < q_1$ and one of the following conditions*

(i) $\lambda_1 > \lambda_2 > \lambda_3 > 0$, *or*

(ii) $\lambda_1 > 0$ *and* $\lambda_3 < \lambda_2 < 0$, *is satisfied, then*

$$A_3 \le G_1(t,s) \le B_3 < 0.$$

Proof. If (i) or (ii) is satisfied, we have

$$\begin{aligned}\frac{\partial g_1(t,s)}{\partial s} &= \frac{-\lambda_1 \exp(\lambda_1(t+\omega-s))}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(1-\exp(\lambda_1\omega))} > 0,\\ \frac{\partial g_2(t,s)}{\partial s} &= \frac{-\lambda_2 \exp(\lambda_2(t+\omega-s))}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)(1-\exp(\lambda_2\omega))} < 0,\\ \frac{\partial g_3(t,s)}{\partial s} &= \frac{-\lambda_3 \exp(\lambda_3(t+\omega-s))}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(1-\exp(\lambda_3\omega))} > 0,\end{aligned}$$

which implies that $g_1(t,\cdot)$, $g_2(t,\cdot)$, $g_3(t,\cdot)$ have no extreme points. Hence, we have $g_1(t,t) < g_1(t,s) < g_1(t,t+\omega)$, $g_2(t,t+\omega) < g_2(t,s) < g_2(t,t)$, $g_3(t,t) < g_3(t,s) < g_3(t,t+\omega)$ for any $s \in [t,t+\omega]$ and, using these inequalities together with the assumption $p_1 < q_1$, one gets $A_3 \le G_1(t,s) \le B_3 < 0$. □

Corollary 3.1. *If $h \in C_\omega^-$, $p_1 < q_1$ and one of the following conditions is satisfied*

(i) $\lambda_1 > \lambda_2 > \lambda_3 > 0$, *or*

(ii) $\lambda_1 > 0$ *and* $\lambda_3 < \lambda_2 < 0$,

then equation (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_1(t,s)h(s)\,\mathrm{d}s.$$

Example 3.1. Consider the equation

$$u''' - 0.601u'' + 0.0506u' - 5 \cdot 10^4 u = h(t);$$

here, $h(t)$ is a given continuous and 2π-periodic function. The characteristic equation is $(\lambda - 0.5)(\lambda - 0.1)(\lambda - 0.01) = 0$, with roots: $\lambda_1 = 0.5$, $\lambda_2 = 0.1$, $\lambda_3 = 0.01$. We compute $p_1 = 33.8217 < q_1 = 40.2525$, and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\mathrm{d}s$, where

$$\begin{aligned} G(t,s) = {} & \exp(0.5(t-s))\frac{250\exp(\pi)}{49(1-\exp(\pi))} \\ & - \exp(0.1(t-s))\frac{250\exp(0.2\pi)}{9(1-\exp(0.2\pi))} \\ & + \exp(0.01(t-s))\frac{10^4 \cdot \exp(0.02\pi)}{441(1-\exp(0.02\pi))}, \quad s \in [t, t+2\pi]. \end{aligned}$$

with $\int_t^{t+2\pi} G(t,s)\,\mathrm{d}s = -2000$, $-345.9189 \le G(t,s) \le -290.3633 < 0$.

Example 3.2. Consider the equation

$$u''' + 0.49u'' + 0.055u' - 6 \cdot 10^4 u = h(t)$$

with a continuous, 2π-periodic function h. The characteristic equation is $(\lambda - 0.01)(\lambda + 0.2)(\lambda + 0.3) = 0$, with roots $\lambda_1 = 0.01$, $\lambda_2 = -0.2$, $\lambda_3 = -0.3$. We compute $p_1 = 0.4295 < q_1 = 0.4853$, and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) = {} & \exp(0.01(t-s))\frac{10000\exp(0.02\pi)}{651(1-\exp(0.02\pi))} \\ & - \exp(0.2(s-t))\frac{1000}{21(\exp(0.4\pi)-1)} \\ & + \exp(0.3(s-t))\frac{1000}{31(\exp(0.6\pi)-1)}, \quad s \in [t, t+2\pi]. \end{aligned}$$

with $\int_t^{t+2\pi} G(t,s)\mathrm{d}s = -5000/3$, $-313.0275 \le G(t,s) \le -217.7894 < 0$.

Analogously the following result holds.

Theorem 3.3. *If $p_2 > q_2$ and one of the following conditions is satisfied*

(i) $\lambda_3 < \lambda_2 < \lambda_1 < 0$, *or*

(ii) $\lambda_1 > \lambda_2 > 0$ *and* $\lambda_3 < 0$, *then*

$$0 < A_3 \le G_1(t,s) \le B_3.$$

Proof. Similarly as in the proof of Theorem 3.2, we get $A_3 \le G_1(t,s) \le B_3$ and, using the assumption $p_2 > q_2$, it follows that $0 < A_3$, proving that $0 < A_3 \le G_1(t,s) \le B_3$. □

Corollary 3.2. *If $h \in C_\omega^+$, $p_2 > q_2$ and one of the following conditions is satisfied*

(i) $\lambda_3 < \lambda_2 < \lambda_1 < 0$, *or*

(ii) $\lambda_1 > \lambda_2 > 0$ *and* $\lambda_3 < 0$, *then equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_1(t,s)h(s)\,\mathrm{d}s.$$

Example 3.3. Consider the equation

$$u''' + 0.1u'' + 2.3\cdot 10^{-3}u' + 1.4\cdot 10^{-5}u = h(t)$$

with a continuous and 2π-periodic function h. Its characteristic equation is $(\lambda+0.01)(\lambda+0.02)(\lambda+0.07) = 0$, has the roots: $\lambda_1 = -0.01$, $\lambda_2 = -0.02$, $\lambda_3 = -0.07$. A computation yields $p_2 = 0.1804 > q_2 = 0.1797$, and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$G(t,s) = \frac{5\cdot 10^3 \exp(0.01(s-t))}{3(\exp(0.02\pi)-1)} - \frac{2\cdot 10^3 \exp(0.02(s-t))}{\exp(0.04\pi)-1} + \frac{10^3 \exp(0.07(s-t))}{3(\exp(0.14\pi)-1)},$$

for $s \in [t, t+2\pi]$ with $\int_t^{t+2\pi} G(t,s)\,\mathrm{d}s = (5/7)\cdot 10^5$ and $0 < 9.3682\cdot 10^3 \le G(t,s) \le 1.3368 \cdot 10^4$.

Example 3.4. Consider the equation

$$u''' - 2.9u'' + 1.7u' + 0.2u = h(t),$$

with a continuous, 2π-periodic function h. The characteristic equation is $(\lambda-2)(\lambda-1)(\lambda+0.1) = 0$, with roots: $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -0.1$. We compute $p_2 = 1.7295 \cdot 10^8 > q_2 = 1.6955 \cdot 10^8$ and therefore the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) &= \exp(2(t-s))\frac{10\exp(4\pi)}{21(1-\exp(4\pi))} + \exp(t-s)\frac{10\exp(2\pi)}{11(\exp(2\pi)-1)} \\ &\quad + \exp(0.1(s-t))\frac{100}{231(\exp(0.2\pi)-1)}, \quad s \in [t, t+2\pi], \end{aligned}$$

with $\int_t^{t+2\pi} G(t,s)\mathrm{d}s = 5$, $0 < 0.0206 \le G(t,s) \le 1.8387$.

Case (ii): $\lambda_1 = \lambda_2 \ne \lambda_3$. For convenience, define the abbreviations

$$\begin{aligned} A_4 &= \tfrac{\exp(\lambda_1\omega)-1+(\lambda_1-\lambda_3)\omega}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{\exp(\lambda_3\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ A_5 &= \tfrac{\exp(2\lambda_1\omega)-\exp(\lambda_1\omega)+(\lambda_1-\lambda_3)\omega\exp(2\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{\exp(\lambda_3\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ A_6 &= \tfrac{\exp(\lambda_1\omega)-1+(\lambda_1-\lambda_3)\omega\exp(\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{\exp(\lambda_3\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ B_4 &= \tfrac{\exp(2\lambda_1\omega)-\exp(\lambda_1\omega)+(\lambda_1-\lambda_3)\omega\exp(2\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{1}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ B_5 &= \tfrac{\exp(\lambda_1\omega)-1+(\lambda_1-\lambda_3)\omega}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{1}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ B_6 &= \tfrac{\exp(2\lambda_1\omega)-\exp(\lambda_1\omega)+(\lambda_1-\lambda_3)\omega\exp(\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + \tfrac{1}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}, \\ p_3 &= \exp(\lambda_1\omega) + (\lambda_1-\lambda_3)\omega + (\exp(\lambda_1\omega)-3) \\ &\quad \times \exp((\lambda_1+\lambda_3)\omega) + (2+(\lambda_3-\lambda_1)\omega)\exp(\lambda_3\omega), \\ p_4 &= (3-(\lambda_1-\lambda_3)\omega)\exp(\lambda_1\omega) + ((\lambda_1-\lambda_3)\omega-1) \\ &\quad \times \exp((\lambda_1+\lambda_3)\omega) + (\exp(\lambda_3\omega)-2)\exp(2\lambda_1\omega). \end{aligned}$$

Theorem 3.4.

$$\int_t^{t+\omega} G_2(t,s)\,\mathrm{d}s = -\frac{1}{\lambda_1^2\lambda_3}$$

for all $t \in [0,\omega]$ *and* $s \in [t, t+\omega]$.

Proof. A direct computation yields

$$\int_t^{t+\omega} \exp(\lambda_1(t+\omega-s))\,\mathrm{d}s = \frac{\exp(\lambda_1\omega)-1}{\lambda_1}, \tag{3.4}$$

$$\int_t^{t+\omega} \exp(\lambda_3(t+\omega-s))\,\mathrm{d}s = \frac{\exp(\lambda_3\omega)-1}{\lambda_3}, \tag{3.5}$$

$$\int_t^{t+\omega} (s-t)\exp(\lambda_1(t+\omega-s))\,\mathrm{d}s = \frac{\exp(\lambda_1\omega)-\lambda_1\omega-1}{\lambda_1^2}. \tag{3.6}$$

On the other hand,

$$\begin{aligned}
&\int_t^{t+\omega} G_2(t,s)\,\mathrm{d}s \\
&= \int_t^{t+\omega} \frac{(1-\exp(\lambda_1\omega))((s-t)(\lambda_3-\lambda_1)-1)-(\lambda_3-\lambda_1)\omega}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} \\
&\qquad\qquad\qquad\qquad \times \exp(\lambda_1(t+\omega-s))\,\mathrm{d}s \\
&\quad + \int_t^{t+\omega} \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}\,\mathrm{d}s \\
&= \frac{\int_t^{t+\omega}(s-t)\exp(\lambda_1(t+\omega-s))\,\mathrm{d}s}{(\lambda_3-\lambda_1)(1-\exp(\lambda_1\omega))} - \frac{\int_t^{t+\omega}\exp(\lambda_1(t+\omega-s))\,\mathrm{d}s}{(\lambda_3-\lambda_1)^2(1-\exp(\lambda_1\omega))} \\
&\quad - \frac{\omega\int_t^{t+\omega}\exp(\lambda_1(t+\omega-s))\,\mathrm{d}s}{(\lambda_3-\lambda_1)(1-\exp(\lambda_1\omega))^2} + \frac{\int_t^{t+\omega}\exp(\lambda_3(t+\omega-s))\,\mathrm{d}s}{(\lambda_3-\lambda_1)^2(1-\exp(\lambda_3\omega))},
\end{aligned}$$

and by substituting (3.4), (3.5), (3.6) into the above equality, we get

$$\int_t^{t+\omega} G_2(t,s)\,\mathrm{d}s = -\frac{1}{\lambda_1^2\lambda_3}. \qquad \square$$

Theorem 3.5. *If* $\lambda_1 > 0, \lambda_3 < 0$*, then* $0 < A_4 \le G_2(t,s) \le B_4$.

Proof. For $t \in [0,\omega]$, $s \in [t, t+\omega]$, define

$$g_4(t,s) = (s-t)(1-\exp(\lambda_1\omega))(\lambda_3-\lambda_1) + (\exp(\lambda_1\omega)-1) + (\lambda_1-\lambda_3)\omega.$$

Then
$$\frac{\partial g_4(t,s)}{\partial s} = (\exp(\lambda_1\omega) - 1)(\lambda_1 - \lambda_3) > 0.$$
Hence, $g_4(t,\cdot)$ has no extreme point; moreover,
$$g_4(t,t) = \exp(\lambda_1\omega) - 1 + (\lambda_1 - \lambda_3)\omega,$$
$$g_4(t,t+\omega) = \exp(\lambda_1\omega) - 1 + (\lambda_1 - \lambda_3)\omega\exp(\lambda_1\omega),$$
and $0 < g_4(t,t) < g_4(t,t+\omega)$. We denote
$$H_1(t,s) = \frac{\exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}.$$
Then
$$\frac{\partial H_1(t,s)}{\partial s} = \frac{-\lambda_3\exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))} > 0.$$
Using the fact that
$$H_1(t,t) = \frac{\exp(\lambda_3\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))}$$
and
$$H_1(t,t+\omega) = \frac{1}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))},$$
we get $0 < H_1(t,t) \le H_1(t,s) \le H_1(t,t+\omega)$. For $\lambda_1 > 0$, we have $1 \le \exp(\lambda_1(t+\omega-s)) \le \exp(\lambda_1\omega)$, and hence
$$\begin{aligned} 0 &< \frac{g_4(t,t)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + H_1(t,t) \le G_2(t,s) \\ &\le \frac{g_4(t,t+\omega)\exp(\lambda_1\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + H_1(t,t+\omega), \end{aligned}$$
i.e., $0 < A_4 \le G_2(t,s) \le B_4$. □

Corollary 3.3. *If $\lambda_1 > 0, \lambda_3 < 0$ and $h \in C_\omega^+$, then equation* (1.1) *has a unique positive ω-periodic solution*
$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s.$$

Example 3.5. Consider the equation $u''' - u'' - u' + u = h(t)$, with a continuous, 2π-periodic h, i.e., $\omega = 2\pi$. The characteristic equation $(\lambda - 1)^2(\lambda + 1) = 0$ has the roots $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = -1$. Hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) = {} & \frac{\exp(2\pi)(t-s)\exp(t-s)}{2(1-\exp(2\pi))} - \frac{\exp(2\pi)\exp(t-s)}{4(1-\exp(2\pi))} \\ & + \frac{\pi\exp(2\pi)\exp(t-s)}{(1-\exp(2\pi))^2} + \frac{\exp(s-t)}{4(\exp(2\pi)-1)}, \end{aligned}$$

for $s \in [t, t+2\pi]$ and $0 < 9.4647 \cdot 10^{-4} \le G(t,s) \le 3.6543$.

Analogously, one can prove the following result.

Theorem 3.6. *If $\lambda_3 > 0$, $\lambda_1 < 0$, then $A_4 \le G_2(t,s) \le B_4 < 0$.*

Corollary 3.4. *If $\lambda_3 > 0$, $\lambda_1 < 0$ and $h \in C_\omega^-$, then the equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s.$$

Example 3.6. Consider the equation $u''' + u'' - u' + u = h(t)$ with continuous, 2π-periodic h. The characteristic equation is $(\lambda+1)^2(\lambda - 1) = 0$, with roots: $\lambda_1 = \lambda_2 = -1$, $\lambda_3 = 1$. Hence, the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) = {} & \frac{(s-t)\exp(s-t)}{2(\exp(2\pi)-1)} - \frac{\exp(s-t)}{4(\exp(2\pi)-1)} \\ & - \frac{\pi\exp(2\pi)\exp(s-t)}{(\exp(2\pi)-1)^2} + \frac{\exp(2\pi)\exp(t-s)}{4(1-\exp(2\pi))}, \end{aligned}$$

for $s \in [t, t+2\pi]$ and $0 < -3.6543 \le G(t,s) \le -9.4647 \cdot 10^{-4}$.

Theorem 3.7. *If $\lambda_3 > \lambda_1 > 0$, and $\exp(\lambda_1\omega) < 1 + (\lambda_3 - \lambda_1)\omega$, then $A_5 \le G_2(t,s) \le B_5 < 0$.*

Proof. Since

$$\frac{\partial g_4(t,s)}{\partial s} = (\exp(\lambda_1\omega) - 1)(\lambda_1 - \lambda_3) < 0,$$

$g_4(t, t+\omega) < g_4(t,t)$ and if $\exp(\lambda_1\omega) < 1 + (\lambda_3 - \lambda_1)\omega$, we get $g_4(t, t+\omega) < g_4(t,t) < 0$. Because of $\lambda_1 > 0$, we have $1 \leq \exp(\lambda_1(t+\omega-s)) \leq \exp(\lambda_1\omega)$, and

$$\frac{\partial H_1(t,s)}{\partial s} = \frac{-\lambda_3 \exp(\lambda_3(t+\omega-s))}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_3\omega))} > 0.$$

Hence $0 < H_1(t,t) \leq H_1(t,s) \leq H_1(t,t+\omega)$ and therefore

$$\begin{aligned}\frac{\exp(\lambda_1\omega)g_4(t,t+\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + H_1(t,t) &\leq G_2(t,s)\\ &\leq \frac{g_4(t,t)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + H_1(t,t+\omega).\end{aligned}$$

It follows that $A_5 \leq G_2(t,s) \leq B_5 < 0$. □

Corollary 3.5. *If $\lambda_3 > \lambda_1 > 0$, $\exp(\lambda_1\omega) < 1 + (\lambda_3 - \lambda_1)\omega$ and $h(t) \in C_\omega^-$, then equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s.$$

Example 3.7. Consider the equation $u''' - 5.2u'' + 1.01u' - 0.05u = h(t)$ with continuous, 2π-periodic continuous h. The characteristic equation $(\lambda - 0.1)^2(\lambda - 5) = 0$ has the roots $\lambda_1 = \lambda_2 = 0.1, \lambda_3 = 5$. Since $\exp(0.2\pi) = 1.8745 < 1 + 9.8\pi = 31.7876$. The equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned}G(t,s) = &\tfrac{10(s-t)\exp(0.1(t-s))\exp(0.2\pi)}{49(1-\exp(0.2\pi))} - \tfrac{100\exp(0.1(t-s))\exp(0.2\pi)}{2401(1-\exp(0.2\pi))}\\ &- \tfrac{20\pi\exp(0.2\pi)\exp(0.1(t-s))}{49(1-\exp(0.2\pi))^2} + \tfrac{100\exp(10\pi)\exp(5(t-s))}{2401(1-\exp(10\pi))},\end{aligned}$$

for $s \in [t, t+2\pi]$ and $-5.8443 \leq G(t,s) \leq -1.6293 < 0$.

Theorem 3.8. *If* $\lambda_1 < \lambda_3 < 0$*, and* $p_3 > 1$*, then* $0 < A_4 \le G_2(t,s) \le B_4$.

Proof. Similarly as in the proof of Theorem 3.5, we get

$$A_4 \le G_2(t,s) \le B_4,$$

and if $p_3 > 1$ then $A_4 > 0$ and hence $0 < A_4 \le G_2(t,s) \le B_4$. □

Corollary 3.6. *If* $\lambda_1 < \lambda_3 < 0$*,* $p_3 > 1$ *and* $h(t) \in C_\omega^+$*, then the equation* (1.1) *has a unique positive* ω*-periodic solution*

$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s.$$

Example 3.8. Consider the equation $u''' + 0.07u'' + 1.5 \cdot 10^{-3}u' + 9 \cdot 10^{-6}u = h(t)$ with continuous, 2π-periodic h. The characteristic equation is $(\lambda+0.03)^2(\lambda+0.01) = 0$ and has the roots $\lambda_1 = \lambda_2 = -0.03$, $\lambda_3 = -0.01$. We get $p_3 = 1.0096 > 1$ and hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) = {} & \frac{50(s-t)\exp(0.03(s-t))}{\exp(0.06\pi)-1} - \frac{2500\exp(0.03(s-t))}{\exp(0.06\pi)-1} \\ & - \frac{100\pi\exp(0.06\pi)\exp(0.03(s-t))}{(\exp(0.06\pi)-1)^2} + \frac{2500\exp(0.01(s-t))}{\exp(0.02\pi)-1}, \end{aligned}$$

for $s \in [t, t+2\pi]$, moreover, $1.3355 \cdot 10^4 \le G(t,s) \le 2.1698 \cdot 10^4$.

Theorem 3.9. *If* $\lambda_1 > \lambda_3 > 0$ *and* $p_4 < 1$*, then* $A_6 \le G_2(t,s) \le B_6 < 0$.

Proof. Define

$$g_5(t,s) = \exp(\lambda_1(t+\omega-s))[(1-\exp(\lambda_1\omega))((s-t)(\lambda_3-\lambda_1)-1)-(\lambda_3-\lambda_1)\omega],$$

Then

$$\begin{aligned} \frac{\partial g_5(t,s)}{\partial s} = {} & [(\lambda_1^2 - \lambda_1\lambda_3)(s-t) + \lambda_3]\exp(\lambda_1(t+\omega-s)) \\ & \times (1-\exp(\lambda_1\omega)) + (\lambda_1\lambda_3 - \lambda_1^2)\omega\exp(\lambda_1(t+\omega-s)) < 0. \end{aligned}$$

Moreover,

$$g_5(t,t) = \exp(\lambda_1\omega)(\exp(\lambda_1\omega) - 1 + (\lambda_1 - \lambda_3)\omega),$$
$$g_5(t,t+\omega) = \exp(\lambda_1\omega) - 1 + (\lambda_1 - \lambda_3)\omega\exp(\lambda_1\omega).$$

Hence $0 < g_5(t,t+\omega) < g_5(t,t)$. Similarly

$$\frac{\partial H_1(t,s)}{\partial s} > 0,$$

and $0 < H_1(t,t) \le H_1(t,s) \le H_1(t,t+\omega)$. Consequently,

$$\begin{aligned}\frac{g_5(t,t+\omega)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2} + H_1(t,t) \le G_2(t,s)& \\ \le \frac{g_5(t,t)}{(\lambda_1-\lambda_3)^2(1-\exp(\lambda_1\omega))^2}& \\ + H_1(t,t+\omega),&\end{aligned}$$

i.e., $A_6 \le G_2(t,s) \le B_6$. If $p_4 < 1$, then $B_6 < 0$ and therefore

$$A_6 \le G_2(t,s) \le B_6 < 0. \qquad \square$$

Corollary 3.7. *If $\lambda_1 > \lambda_3 > 0$, $p_4 < 1$ and $h \in C_\omega^-$, then equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_2(t,s)h(s)\,\mathrm{d}s.$$

Example 3.9. Consider the equation $u''' - 0.21u'' + 0.012u' - 0.0001u = h(t)$ with continuous, 2π-periodic h. The characteristic equation is $(\lambda - 0.1)^2(\lambda - 0.01) = 0$ which has the roots $\lambda_1 = \lambda_2 = 0.1$, $\lambda_3 = 0.01$. We get $p_4 = 0.4104 < 1$, and hence the equation has a unique 2π-periodic solution $x(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned}G(t,s) = {}& \frac{100\exp(0.2\pi)(t-s)\exp(0.1(t-s))}{9(1-\exp(0.2\pi))} - \frac{10000\exp(0.2\pi)\exp(0.1(t-s))}{81(1-\exp(0.2\pi))} \\ &+ \frac{200\pi\exp(0.2\pi)\exp(0.1(t-s))}{9(1-\exp(0.2\pi))^2} + \frac{10000\exp(0.02\pi)\exp(0.01(t-s))}{81(1-\exp(0.02\pi))},\end{aligned}$$

for $s \in [t, t+2\pi]$ and $-1.7149 \cdot 10^3 \le G(t,s) \le -1.4680 \cdot 10^3$.

Case (iii): $\lambda_1 = \lambda_2 = \lambda_3$**.** For convenience, define

$$A_7 = \frac{\omega^2 \exp(2\lambda\omega)(1+\exp(\lambda\omega))}{2(1-\exp(\lambda\omega))^3} \quad \text{and} \quad B_7 = \frac{\omega^2(1+\exp(\lambda\omega))}{2(1-\exp(\lambda\omega))^3}.$$

Theorem 3.10.

$$\int_t^{t+\omega} G_3(t,s)\,\mathrm{d}s = -\frac{1}{\lambda^3}$$

for all $t \in [0,\omega]$ *and* $s \in [t, t+\omega]$.

Proof. As a preparation we compute the following expressions

$$\int_t^{t+\omega} \exp(\lambda(t+\omega-s))\,\mathrm{d}s = \tfrac{\exp(\lambda\omega)-1}{\lambda}, \tag{3.7}$$

$$\int_t^{t+\omega} s\exp(\lambda(t+\omega-s))\,\mathrm{d}s = \tfrac{\lambda t+1}{\lambda^2}\exp(\lambda\omega) - \tfrac{\lambda(t+\omega)+1}{\lambda^2}, \tag{3.8}$$

$$\int_t^{t+\omega} s^2\exp(\lambda(t+\omega-s))\,\mathrm{d}s = -\tfrac{(\lambda(t+\omega)+1)^2+1}{\lambda^3} + \tfrac{(\lambda t+1)^2+1}{\lambda^3}\exp(\lambda\omega). \tag{3.9}$$

On the other hand,

$$\begin{aligned}
&\int_t^{t+\omega} G_3(t,s)\,\mathrm{d}s \\
&= \int_t^{t+\omega} \tfrac{\exp(\lambda(t+\omega-s))}{2(\exp(\lambda\omega)-1)^3}\{[(s-t)(\exp(\lambda\omega)-1)+\omega]^2 + \omega^2\exp(\lambda\omega)\}\,\mathrm{d}s \\
&= \frac{1}{2(\exp(\lambda\omega)-1)}\int_t^{t+\omega} s^2\exp(\lambda(t+\omega-s))\,\mathrm{d}s \\
&\quad + \left[\frac{-t}{\exp(\lambda\omega)-1} + \frac{\omega}{(\exp(\lambda\omega)-1)^2}\right]\int_t^{t+\omega} s\exp(\lambda(t+\omega-s))\,\mathrm{d}s \\
&\quad + \left[\frac{t^2}{2(\exp(\lambda\omega)-1)} - \frac{t\omega}{(\exp(\lambda\omega)-1)^2} + \frac{\omega^2(\exp(\lambda\omega)+1)}{2(\exp(\lambda\omega)-1)^3}\right] \\
&\quad \times \int_t^{t+\omega} \exp(\lambda(t+\omega-s))\,\mathrm{d}s.
\end{aligned}$$

Substituting (3.7), (3.8), (3.9) into the above equality, we get

$$\int_t^{t+\omega} G_3(t,s)\,\mathrm{d}s = -\frac{1}{\lambda^3}. \qquad \square$$

Theorem 3.11. *If $\lambda > 0$, then $A_7 \le G_3(t,s) \le B_7 < 0$.*

Proof. Define

$$g_6(t,s) = [(s-t)(\exp(\lambda\omega)-1)+\omega]^2 + \omega^2\exp(\lambda\omega)$$

for $s \in [t, t+\omega]$. Then

$$\frac{\partial g_6(t,s)}{\partial s} = 2(\exp(\lambda\omega)-1)[(s-t)(\exp(\lambda\omega)-1)+\omega] > 0,$$

i.e., $g(t,\cdot)$ has no extreme points. Since

$$g_6(t,t) = \omega^2(1+\exp(\lambda\omega)) > 0,$$
$$g_6(t,t+\omega) = \omega^2\exp(\lambda\omega)(1+\exp(\lambda\omega)) > 0,$$

$0 < g_6(t,t) < g_6(t,t+\omega)$. For

$$H_2(t,s) = \frac{\exp(\lambda(t+\omega-s))}{2(1-\exp(\lambda\omega))^3},$$

we get

$$\frac{\partial H_2(t,s)}{\partial s} = \frac{-\lambda\exp(\lambda(t+\omega-s))}{(1-\exp(\lambda\omega))^3} > 0,$$

and hence

$$H_2(t,t) = \tfrac{\exp(\lambda\omega)}{2(1-\exp(\lambda\omega))^3} \le H_2(t,s) \le H_2(t,t+\omega) = \tfrac{1}{2(1-\exp(\lambda\omega))^3} < 0.$$

We get $H_2(t,t)g_6(t,t+\omega) \le G_3(t,s) \le H_2(t,t+\omega)g_6(t,t) < 0$, and hence

$$A_7 \le G_3(t,s) \le B_7 < 0. \qquad \square$$

Corollary 3.8. *If $\lambda > 0$, and $h \in C_\omega^-$, then equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_3(t,s)h(s)\,\mathrm{d}s.$$

Example 3.10. Consider the equation $u''' - 3u'' + 3u' - u = h(t)$ with continuous, 2π-periodic h. The characteristic equation is $(\lambda - 1)^3 = 0$ and has the roots $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$G(t,s) = \frac{[(s-t)\exp(2\pi) + 2\pi - s + t]^2 + 4\pi^2\exp(2\pi)}{2(1-\exp(2\pi))^3}\exp(t - s + 2\pi),$$

for $s \in [t, t+\omega]$ and $-19.8873 \le G(t,s) \le -6.9354 \cdot 10^{-5}$.

Similarly we can prove the following result.

Theorem 3.12. *If $\lambda < 0$, then $0 < A_7 \le G_3(t,s) \le B_7$.*

Corollary 3.9. *If $\lambda < 0$, and $h \in C_\omega^+$, then equation* (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_3(t,s)h(s)\,\mathrm{d}s.$$

Example 3.11. Consider the equation $u''' + 3u'' + 3u' + u = h(t)$ with continuous, 2π-periodic function h. The characteristic equation is $(\lambda+1)^3 = 0$ and has the roots $\lambda_1 = \lambda_2 = \lambda_3 = -1$. Hence the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$G(t,s) = \frac{[(s-t)\exp(-2\pi) + 2\pi - s + t]^2 + 4\pi^2\exp(-2\pi)}{2(1-\exp(-2\pi))^3}\exp(s - t - 2\pi),$$

for $s \in [t, t+\omega]$ and $0 < 6.9354 \cdot 10^{-5} \le G(t,s) \le 19.8873$.

Case (iv): $\lambda_1 = \alpha + i\beta$, $\lambda_2 = \alpha - i\beta$, $\lambda_3 = \lambda$. For the sake of convenience, define

$$
\begin{aligned}
m &= (\alpha-\beta)(\cos(\beta\omega) - \exp(\alpha\omega)) + \beta\sin(\beta\omega),\\
n &= (\alpha-\beta)\sin(\beta\omega) - \beta(\cos(\beta\omega) - \exp(\alpha\omega)),\\
A_8 &= \frac{-\exp(\alpha\omega)}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2]}(1+\exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega))}\\
&\quad + \frac{\exp(\lambda\omega)}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))},\\
B_8 &= \frac{\exp(\alpha\omega)}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2]}(1+\exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega))}\\
&\quad + \frac{1}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))},\\
A_9 &= \frac{-1}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2]}(1+\exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega))}\\
&\quad + \frac{\exp(\lambda\omega)}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))},\\
B_9 &= \frac{1}{\beta\sqrt{[(\alpha-\lambda)^2+\beta^2]}(1+\exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega))}\\
&\quad + \frac{1}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}.
\end{aligned}
$$

Theorem 3.13.

$$\int_t^{t+\omega} G_4(t,s)\,\mathrm{d}s = -\frac{1}{\lambda(\alpha^2+\beta^2)}$$

for all $t \in [0,\omega]$ and $s \in [t, t+\omega]$.

Proof. A simple calculation yields

$$\int_t^{t+\omega} \frac{\exp(\lambda(t+\omega-s))}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}\,\mathrm{d}s = -\frac{1}{\lambda[(\alpha-\lambda)^2+\beta^2]}, \tag{3.10}$$

$$\begin{aligned}\int_t^{t+\omega} &\exp(\alpha(t+\omega-s))\sin(\beta(t-s))\,\mathrm{d}s\\ &= \frac{\beta\cos(\beta\omega)+\alpha\sin(\beta\omega)-\beta\exp(\alpha\omega)}{\alpha^2+\beta^2},\end{aligned} \tag{3.11}$$

$$\int_t^{t+\omega} \exp(\alpha(t+\omega-s))\cos(\beta(t-s))\,\mathrm{d}s = \tfrac{\beta\sin(\beta\omega)-\alpha\cos(\beta\omega)+\alpha\exp(\alpha\omega)}{\alpha^2+\beta^2}, \tag{3.12}$$

$$\int_t^{t+\omega} \exp(\alpha(t+\omega-s))\sin(\beta(t+\omega-s))\,\mathrm{d}s = \tfrac{\beta+\exp(\alpha\omega)(\alpha\sin(\beta\omega)-\beta\cos(\beta\omega))}{\alpha^2+\beta^2}, \tag{3.13}$$

$$\int_t^{t+\omega} \exp(\alpha(t+\omega-s))\cos(\beta(t+\omega-s))\,\mathrm{d}s = \tfrac{\exp(\alpha\omega)(\alpha\cos(\beta\omega)+\beta\sin(\beta\omega))-\alpha}{\alpha^2+\beta^2}. \tag{3.14}$$

On the other hand,

$$\begin{aligned}
&\int_t^{t+\omega} G_4(t,s)\,\mathrm{d}s \\
&\quad= \frac{(\alpha-\lambda)\int_t^{t+\omega}\exp(\alpha(t+\omega-s))\sin(\beta(t+\omega-s))\,\mathrm{d}s}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega))} \\
&\qquad- \frac{(\alpha-\lambda)\exp(\alpha\omega)\int_t^{t+\omega}\exp(\alpha(t+\omega-s))\sin(\beta(t-s))\,\mathrm{d}s}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega))} \\
&\qquad- \frac{\beta\int_t^{t+\omega}\exp(\alpha(t+\omega-s))\cos(\beta(t+\omega-s))\,\mathrm{d}s}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega))} \\
&\qquad+ \frac{\beta\exp(\alpha\omega)\int_t^{t+\omega}\exp(\alpha(t+\omega-s))\cos(\beta(t-s))\,\mathrm{d}s}{\beta[(\alpha-\lambda)^2+\beta^2](1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega))} \\
&\qquad+ \frac{\int_t^{t+\omega}\exp(\lambda(t+\omega-s))\,\mathrm{d}s}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))},
\end{aligned}$$

and substituting (3.10), (3.11), (3.12), (3.13), (3.14) into the above equation, we get

$$\int_t^{t+\omega} G_4(t,s)\,\mathrm{d}s = -\frac{1}{\lambda(\alpha^2+\beta^2)}. \qquad \square$$

Theorem 3.14. *If $\alpha > 0$, $\beta > 0$, $\lambda < 0$, and*

$$\frac{1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)},$$

then $0 < A_8 \le G_4(t,s) \le B_8$.

Proof. Define

$$\begin{aligned} H_3(t,s) &= (\alpha-\lambda)(\sin(\beta(t+\omega-s)) - \exp(\alpha\omega)\sin(\beta(t-s))) \\ &\quad - \beta(\cos(\beta(t+\omega-s)) - \exp(\alpha\omega)\cos(\beta(t-s))) \\ &= [(\alpha-\lambda)(\cos(\beta\omega)-\exp(\alpha\omega)) + \beta\sin(\beta\omega)]\sin(\beta(t-s)) \\ &+ [(\alpha-\lambda)\sin(\beta\omega) - \beta(\cos(\beta\omega)-\exp(\alpha\omega))]\cos(\beta(t-s)) \\ &= m\sin(\beta(t-s)) + n\cos(\beta(t-s)) \\ &= \sqrt{m^2+n^2}\sin(\beta(t-s)+\varphi), \quad \text{where } \tan\varphi = \frac{n}{m}. \end{aligned}$$

We get

$$-\sqrt{m^2+n^2} \le H_3(t,s) \le \sqrt{m^2+n^2}.$$

Because of $\alpha > 0$, $\lambda < 0$, we have

$$\begin{aligned} 1 &\le \exp(\alpha(t+\omega-s)) \\ &\le \exp(\alpha\omega), \frac{\exp(\lambda\omega)}{(1-\exp(\lambda\omega))[(\alpha-\lambda)^2+\beta^2]} \\ &\le \frac{\exp(\lambda(t+\omega-s))}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))} \\ &\le \frac{1}{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))}, \end{aligned}$$

and get $A_8 \le G_4(t,s) \le B_8$. If

$$\frac{1+\exp(2\alpha\omega)-2\exp(\alpha\omega)\cos(\beta\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2+\beta^2](1-\exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)}$$

then $0 < A_8 \le G_4(t,s) \le B_8$. ∎

Corollary 3.10. *If $\alpha > 0$, $\beta > 0$, $\lambda < 0$, and*

$$\frac{1 + \exp(2\alpha\omega) - 2\exp(\alpha\omega)\cos(\beta\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2 + \beta^2](1 - \exp(\lambda\omega))^2}{\beta^2 \exp(2\lambda\omega)},$$

and $h \in C_\omega^+$, then equation (1.1) *has a unique positive ω-periodic solution*

$$u(t) = \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s.$$

Example 3.12. The equation $u''' - 0.1u'' + 0.0525u' + 0.00725u = h(t)$ with continuous, 2π-periodic h. The characteristic equation is $(\lambda - 0.1 - 0.25i)(\lambda - 0.1 + 0.25i)(\lambda + 0.1) = 0$ which has the roots $\lambda_1 = 0.1 + 0.25i$, $\lambda_2 = 0.1 - 0.25i$, $\lambda_3 = -0.1$, and $\alpha = 0.1$, $\beta = 0.25$, $\lambda = -0.1$. Hence

$$\begin{aligned}\tfrac{1+\exp(0.4\pi)-2\exp(0.2\pi)\cos(0.5\pi)}{\exp(0.4\pi)} &= 1.2846 > \tfrac{0.1025(1-\exp(-0.2\pi))^2}{0.0625\exp(-0.4\pi)}\\ &= 1.2541,\end{aligned}$$

and therefore the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned}G(t,s) = {}& \tfrac{80\exp(0.2\pi)\exp(0.1(t-s))}{41(1+\exp(0.4\pi))}\\ &\times \tfrac{[(4+5\exp(0.2\pi))\cos(0.25(t-s))+(5-4\exp(0.2\pi))\sin(0.25(t-s))]}{41(1+\exp(0.4\pi))}\\ &+ \tfrac{400\exp(0.1(s-t))}{41(\exp(0.2\pi)-1)},\end{aligned}$$

for $s \in [t, t+2\pi]$ and $0 < 0.1334 \le G(t,s) \le 31.9362$.

Similarly we can prove the following result.

Theorem 3.15. *If $\alpha < 0$, $\lambda < 0$, $\beta > 0$, and*

$$1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega) > \frac{[(\alpha-\lambda)^2 + \beta^2](1 - \exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)},$$

then $0 < A_9 \le G_4(t,s) \le B_9$.

Corollary 3.11. *If* $\alpha < 0$, $\lambda < 0$, $\beta > 0$, *and*

$$1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega) > \frac{[(\alpha-\lambda)^2 + \beta^2](1-\exp(\lambda\omega))^2}{\beta^2\exp(2\lambda\omega)},$$

and $h \in C_\omega^+$, *then equation* (1.1) *has a unique positive* ω*-periodic solution*

$$u(t) = \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s.$$

Example 3.13. Consider the equation $u''' + 0.3u'' + 0.0925u' + 0.00725u = h(t)$ with a continuous and 2π-periodic function h. The characteristic equation is $(\lambda + 0.1 - 0.25i)(\lambda + 0.1 + 0.25i)(\lambda + 0.1) = 0$ has the roots $\lambda_1 = -0.1 + 0.25i$, $\lambda_2 = -0.1 - 0.25i$, $\lambda_3 = -0.1$ and, $\alpha = -0.1$, $\beta = 0.25$, $\lambda = -0.1$. In this condition,

$$1 + \exp(-0.4\pi) = 1.2846 > (\exp(0.2\pi) - 1)^2 = 0.7647.$$

So the equation has a unique 2π-periodic solution

$$x(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s,$$

where

$$\begin{aligned} G(t,s) = {} & \tfrac{16\exp(0.1(s-t))[\exp(0.2\pi)\sin(0.25(t-s)) + \cos(0.25(t-s))]}{\exp(0.4\pi)+1} \\ & + \tfrac{16\exp(0.1(s-t))}{\exp(0.2\pi)-1}, \end{aligned}$$

for $s \in [t, t+2\pi]$ and $0 < 4.1803 \le G(t,s) \le 48.4138$.

Theorem 3.16. *If* $\alpha > 0$, $\lambda > 0$, $\beta > 0$ *and*

$$\frac{1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2 + \beta^2](1-\exp(\lambda\omega))^2}{\beta^2},$$

then $A_8 \le G_4(t,s) \le B_8 < 0$.

Corollary 3.12. *If* $\alpha > 0$, $\lambda > 0$, $\beta > 0$ *and*

$$\frac{1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega)}{\exp(2\alpha\omega)} > \frac{[(\alpha-\lambda)^2 + \beta^2](1-\exp(\lambda\omega))^2}{\beta^2},$$

and $h \in C_\omega^-$, *then equation* (1.1) *has a unique positive* ω*-periodic solution*

$$u(t) = \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s.$$

Example 3.14. Consider the equation $u''' - 0.21u'' + 1.012u' - 0.0101u = h(t)$ with a continuous, 2π-periodic function h. The characteristic equation is $(\lambda - 0.1 - i)(\lambda - 0.1 + i)(\lambda - 0.01) = 0$, which has the roots $\lambda_1 = 0.1 + i$, $\lambda_2 = 0.1 - i$, $\lambda_3 = 0.01$ and $\alpha = 0.1$, $\beta = 1$, $\lambda = 0.01$. Consequently,

$$(1 - \exp(-0.2\pi))^2 = 0.2176 > 1.0081(\exp(0.02\pi) - 1)^2 = 0.0042.$$

Hence, the equation has a unique 2π-periodic solution

$$x(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{\exp(0.2\pi)\exp(0.1(t-s))[0.09\sin(t-s) - \cos(t-s)]}{1.0081(1-\exp(0.2\pi))} + \frac{\exp(0.02\pi)\exp(0.01(t-s))}{1.0081(1-\exp(0.02\pi))},$$

for $s \in [t, t+2\pi]$ and $-18.4237 \le G(t,s) \le -13.1619 < 0$.

Theorem 3.17. *If* $\alpha < 0$, $\lambda > 0$, $\beta > 0$ *and*

$$1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega) > \frac{[(\alpha-\lambda)^2 + \beta^2](\exp(\lambda\omega) - 1)^2}{\beta^2},$$

then

$$A_9 \le G_4(t,s) \le B_9 < 0.$$

Corollary 3.13. *If* $\alpha < 0$, $\lambda > 0$, $\beta > 0$ *and*

$$1 + \exp(2\alpha\omega) - 2\cos(\beta\omega)\exp(\alpha\omega) > \frac{[(\alpha-\lambda)^2 + \beta^2](\exp(\lambda\omega) - 1)^2}{\beta^2},$$

and $h \in C_\omega^-$, *then the equation* (1.1) *has a unique positive* ω*-periodic solution*

$$u(t) = \int_t^{t+\omega} G_4(t,s)h(s)\,\mathrm{d}s.$$

Example 3.15. Consider the equation $u''' + 0.1u'' + 0.0525u' - 0.00725u = h(t)$ with continuous, 2π-periodic function h. The characteristic equation is $(\lambda + 0.1 - 0.25i)(\lambda + 0.1 + 0.25i)(\lambda - 0.1) = 0$, which has the roots $\lambda_1 = -0.1 + 0.25i$, $\lambda_2 = -0.1 - 0.25i$, $\lambda_3 = 0.1$ and $\alpha = -0.1$, $\beta = 0.25$, $\lambda = 0.1$. Hence

$$1 + \exp(-0.4\pi) = 1.2846 > \frac{41(\exp(0.2\pi) - 1)^2}{25} = 1.2541,$$

and therefore the equation has a unique 2π-periodic solution $u(t) = \int_t^{t+2\pi} G(t,s)h(s)\,\mathrm{d}s$, where

$$\begin{aligned} G(t,s) &\\ = \frac{80\exp(0.1(s-t))[(5-4\exp(0.2\pi))\cos(0.25(t-s)) + (5\exp(0.2\pi)+4)\sin(0.25(t-s))]}{41(\exp(0.4\pi)+1)} &\\ + \frac{400\exp(0.2\pi)\exp(0.1(t-s))}{41(1-\exp(0.2\pi))}, & \end{aligned}$$

for $s \in [t, t + 2\pi]$ and $-31.9362 \leq G(t,s) \leq -0.1334 < 0$.

4. Applications.

In order to obtain the existence of positive periodic solutions of (1.2), we need the following Guo-Krasnoselskii fixed point theorem [**1, 2**].

Lemma 4.1. *Let X be a Banach space, and let $K \subset X$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K \tag{4.1}$$

be a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$;

or

(ii) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$;

then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Denote

$$\overline{f}_0 = \lim_{u\to 0^+} \sup_{t\in[0,\omega]} \frac{f(t,u)}{u}, \qquad \underline{f}_\infty = \lim_{u\to\infty} \inf_{t\in[0,\omega]} \frac{f(t,u)}{u}.$$

Theorem 4.2. *If $\lambda_3 < \lambda_2 < \lambda_1 < 0$, or $\lambda_1 > \lambda_2 > 0$, $\lambda_3 < 0$ and $p_2 < q_2$, then the equation (4.1) has at least one positive solution in the case*

(i) $\overline{f}_0 = 0$ *and* $\underline{f}_\infty = \infty$; *or*

(ii) $\overline{f}_0 = \infty$ *and* $\underline{f}_\infty = 0$.

Proof. Let $X = \{u \in C(\mathbf{R},\mathbf{R}) : u(t+\omega) = u(t),\ t\in\mathbf{R}\}$ with norm $\|u\| = \sup_{t\in[0,\omega]} |u(t)|$, then $(X, \|\cdot\|)$ is a Banach space. A cone K is defined by $K = \{u \in X :\ u(t) \geq A_3/B_3\|u\| \text{ for all } t\in[0,\omega]\}$.

For $u\in K$, we define

$$Tu(t) = \int_t^{t+\omega} G_1(t,s)f(s,u(s))\,\mathrm{d}s;$$

then it follows from Theorem 3.3 that

$$0 < Tu(t) = \int_t^{t+\omega} G_1(t,s)f(s,u(s))\,\mathrm{d}s \leq B_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s.$$

So $\|Tu\| \leq B_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s$. On the other hand,

$$Tu(t) = \int_t^{t+\omega} G_1(t,s)f(s,u(s))\,\mathrm{d}s \geq A_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s \geq \frac{A_3}{B_3}\|Tu\|,$$

which shows that $TK \subset K$. Moreover, it is easy to see that $T: K\to K$ is completely continuous and that a fixed point of T is a solution of (4.1).

(i) First, we consider the case: $\overline{f}_0 = 0$ and $\underline{f}_\infty = \infty$.

Since $\overline{f}_0 = 0$, we may choose $0 < r_1 < 1$, so that $f(t,u) \leq \varepsilon u$, for $0 \leq u \leq r_1$, $t\in[0,\omega]$, where $\varepsilon > 0$ satisfies $B_3\omega\varepsilon \leq 1$.

Thus, if $u \in K$, and $\|u\| = r_1$, then we have

$$
\begin{aligned}
Tu(t) &= \int_t^{t+\omega} G_1(t,s) f(s,u(s))\,\mathrm{d}s \\
&\le B_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s \\
&\le B_3 \omega \varepsilon \|u\| \le r_1.
\end{aligned} \tag{4.2}
$$

Now, if we set $\Omega_1 = \{u \in X : \|u\| < r_1\}$, then (4.2) shows that $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$.

On the other hand, since $\underline{f}_\infty = \infty$, there is an $r > r_1$ such that $f(t,u) \ge \eta u$, for $u \ge r$, $t \in [0,\omega]$ where $\eta > 0$ is chosen so that $(A_3^2 \omega \eta)/B_3 \ge 1$.

Let $r_2 = \max\{2r_1, (B_3 r/A_3)\}$, and $\Omega_2 = \{u \in X : \|u\| < r_2\}$; then $u \in K$ and $\|u\| = r_2$ implies that $u(t) \ge (A_3/B_3)\|u\| = (A_3/B_3) r_2 \ge r$. And so

$$
\begin{aligned}
Tu(t) &= \int_t^{t+\omega} G_1(t,s) f(s,u(s))\,\mathrm{d}s \\
&\ge A_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s \\
&\ge \frac{A_3^2 \omega \eta}{B_3} \|u\| \ge \|u\|.
\end{aligned} \tag{4.3}
$$

Hence, (4.3) shows that $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$.

Therefore, it follows from the first part of Lemma 4.1 that T has a fixed point $u^* \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Consequently, equation (4.1) has a positive ω-periodic solution $0 < r_1 < u(t) \le r_2$.

(ii) Next, we consider the other case: $\overline{f}_0 = \infty$ and $\underline{f}_\infty = 0$.

We first choose $r_3 > 0$ such that $f(u) \ge \lambda u$ for $0 \le u \le r_3$, where $\lambda > 0$ satisfies $(\lambda A_3^2 \omega)/B_3 \ge 1$.

Then for $u \in K$ and $\|u\| = r_3$, we have

$$
\begin{aligned}
Tu(t) &= \int_t^{t+\omega} G_1(t,s) f(s,u(s))\,\mathrm{d}s \\
&\ge A_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{d}s \ge \frac{\lambda A_3^2 \omega}{B_3} \|u\| \ge \|u\|.
\end{aligned} \tag{4.4}
$$

Thus, if we set $\Omega_3 = \{u \in X : \|u\| < r_3\}$, (4.4) shows that $\|Tu\| \geq \|u\|$, for $u \in K \cap \partial\Omega_3$.

Now, since $\underline{f}_\infty = 0$, there exists an $M > 0$, such that $f(t,u) \leq \xi u$ for $u \geq M$, where $\xi > 0$ satisfies $\xi B_3 \omega < 1$. We choose $r_4 = \max\{2r_3, (B_3 M/A_3)\}$; then $u \in K$ and $\|u\| = r_4$ implies that $u(t) \geq (A_3/B_3)\|u\| \geq M$, and so

$$\begin{aligned} Tu(t) &= \int_t^{t+\omega} G_1(t,s) f(s,u(s))\,\mathrm{d}s \\ &\leq B_3 \int_t^{t+\omega} f(s,u(s))\,\mathrm{rm\ d}s \\ &\leq B_3 \xi \int_t^{t+\omega} u(s)\,\mathrm{d}s \leq B_3 \omega \xi \|u\| \leq \|u\|. \end{aligned} \tag{4.5}$$

Therefore, we put $\Omega_4 = \{u \in X : \|u\| < r_4\}$, and for any $u \in K \cap \partial\Omega_4$, we have $\|Tu\| \leq \|u\|$.

By the second part of Lemma 4.1, we know that equation (4.1) has at least one positive solution $0 < r_3 < u(t) \leq r_4$. $\square$

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