International Journal of Bifurcation and Chaos, Vol. 22, No. 4 (2012) 1250084 (8 pages)

DOI: 10.1142/S0218127412500848

ON STABILITY OF COMMENSURATE FRACTIONAL ORDER SYSTEMS

JOCELYN SABATIER* and CHRISTOPHE FARGES†
University of Bordeaux, IMS, UMR 5218,
F-33400 Talence, France
**jocelyn.sabatier@ims-bordeaux.fr*
†christophe.farges@ims-bordeaux.fr

Received March 31, 2011; Revised June 21, 2011

This paper proposes a new proof of the Matignon's stability theorem. This theorem is the starting point of numerous results in the field of fractional order systems. However, in the original work, its proof is limited to a fractional order ν such that $0 < \nu < 1$. Moreover, it relies on Caputo's definition for fractional differentiation and the study of system trajectories for non-null initial conditions which is now questionable in regard of recent works. The new proof proposed here is based on a closed loop realization and the application of the Nyquist theorem. It does not rely on a peculiar definition of fractional differentiation and is valid for orders ν such that $1 < \nu < 2$.

Keywords: Fractional systems; Matignon's theorem; stability; Nyquist theorem.

1. Introduction

Matignon's theorem is a result frequently used to study commensurate fractional system stability. As shown in Sec. 2, this theorem permits to conclude the BIBO stability of a fractional system described by its pseudo-state space description, fractional order ν being such that $0 < \nu < 1$.

Proof of this theorem [Matignon, 1996] is based on the computation of the system response to nonzero initial conditions. This theorem is decidedly the starting point of many other results on fractional systems [Sabatier *et al.*, 2010b; Farges *et al.*, 2010, 2011; Trigeassou *et al.*, 2011]. However, given the progresses done in the field since 1996, its proof is questionable for the following reasons.

(1) Fractional differentiation definition used in the pseudo-state space representation is the Caputo's definition. Recently it was demonstrated by several authors [Lorenzo & Hartley, 2001; Achar *et al.*, 2003; Ortigueira & Coito, 2008; Sabatier *et al.*, 2010a, 2012], that, if it is used in a pseudo-state space description, Caputo's definition does not permit to take into account initial conditions in a convenient way (an initialization function must be added to the initial conditions produced by the Laplace transformation).
(2) The proof is limited to a fractional order such that $0 < \nu < 1$. An extension of Matignon's theorem to the case $1 < \nu < 2$ is given in [Malti *et al.*, 2002] and another proof is provided in [Moze *et al.*, 2005].

In this paper, a new proof of Matignon's theorem is proposed, valid for a fractional order such that $0 < \nu < 2$ and independent of the fractional derivative definition used.

2. Preliminary Results

Let a Single Input, Single Output commensurate fractional order system be described as in

[Matignon, 1996] by

$$\begin{cases} D^\nu x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad \text{with } x(t_0) = x_0. \tag{1}$$

Given the definition of initial conditions in (1), such a description, suggests that the Caputo's definition is used for fractional differentiation. Matignon's theorems stated in [Matignon, 1996] are the following.

Theorem 1 [Matignon, 1996]. *Autonomous system* $D^\nu x(t) = Ax(t)$, *with* $x(t_0) = x_0$ *and* $0 < \nu < 1$, *is asymptotically stable if and only if* $|\arg\{\mathrm{spec}(A)\}| > \nu\frac{\pi}{2}$, *where* $\mathrm{spec}(A)$ *is the set of all eigenvalues of* A. *Also, state vector* $x(t)$ *decays towards* 0 *and meets the following condition*: $\|x(t)\| < Nt^{-\nu}$, $t > 0$, $\nu > 0$, $N \in \mathbb{R}^+$.

For a minimal realization of (1), the following theorem is also demonstrated in [Matignon, 1996].

Theorem 2 [Matignon, 1996]. *If the triplet* (A, B, C) *is minimal, system* (1) *is BIBO stable if and only if* $|\arg\{\mathrm{spec}(A)\}| > \nu\frac{\pi}{2}$.

If the system is supposed at rest before time $t = 0$ (null initial conditions), it can be represented, whatever the fractional differentiation definition chosen, by the transfer function

$$H(s) = C(s^\nu I - A)^{-1}B. \tag{2}$$

Another important result on stability was also given in [Bonnet & Partington, 2000].

Theorem 3 [Bonnet & Partington, 2000]. *A fractional order system defined by transmittance*

$$H(s) = \frac{\sum_{i=0}^{n_b} b_i s^{\delta_i}}{\sum_{i=0}^{n_a} a_i s^{\nu_i}} \quad \textit{with } \nu_{n_a} > \delta_{n_b} \tag{3}$$

is BIBO stable if and only if $H(s)$ *has no pole in* $\mathcal{R}e(s) \geq 0$ (*in particular, no poles of fractional order at* $s = 0$).

In the next section and using Theorem 3, a new demonstration of Theorem 2 is proposed. It is based on a frequency analysis of equivalent closed loops.

3. Stability of Fractional Systems: A Frequency Interpretation

System (3) stability demonstration can be done in the frequency domain. As shown in the sequel, such a demonstration also holds for $1 < \nu < 2$. Such a demonstration also holds whatever the definition of fractional differentiation used given the definition of the frequency response of a linear system. It is indeed linked to the response of the system to a sinusoidal input as all the transient behaviors have vanished. Thus a linear system frequency response does not depend on initial conditions and on the way they are taken into account. Moreover, for the most used definition for fractional differentiation, the steady state response to a sinusoidal input is the same thus leading to the same frequency response (see Appendix A).

To start the demonstration, the following one pole fractional system is considered

$$H_1(s) = \frac{1}{s^\nu - \rho e^{i\theta}} \tag{4}$$

with $\rho \in \mathbb{R}^+$, $\theta \in\,]-\pi, \pi]$ and $0 < \nu < 2$. Here, the argument of s^ν belongs in $\theta \in\,]-\pi, \pi]$ (main part of s^ν only considered).

$H_1(s)$ is also a closed loop transmittance of the system described in Fig. 1 in which open loop transmittance is denoted as $B_1(s)$.

Nyquist theorem can thus be used to evaluate the internal stability of the loop and thus of the system described by transmittance $H_1(s)$. Given

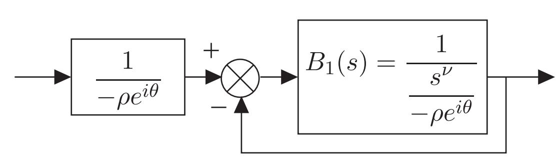

Fig. 1. Equivalent closed loop.

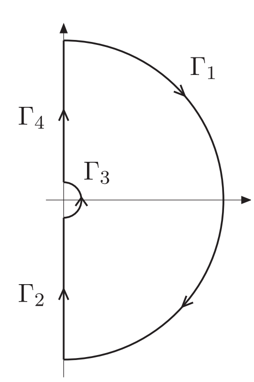

Fig. 2. Nyquist's path.

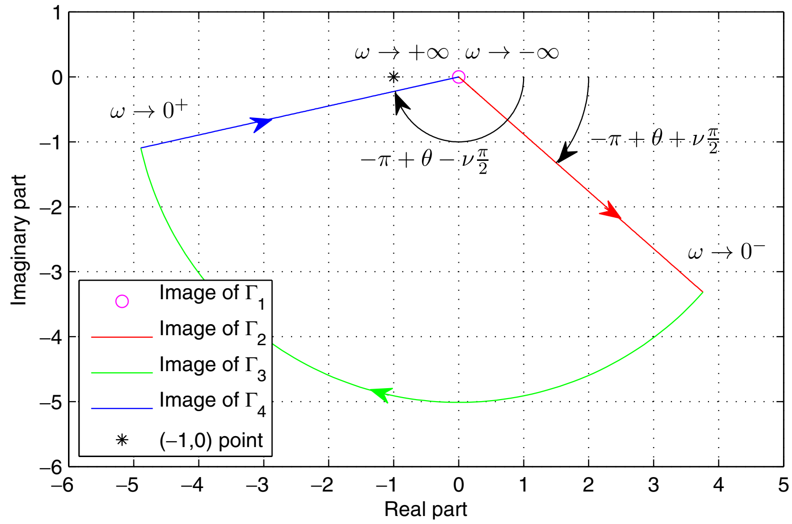

Fig. 3. Image of Γ if $\theta \in [\nu\frac{\pi}{2}, \pi]$ ($\nu = 0.7$, $\theta = 1.2\nu\frac{\pi}{2}$).

that the open loop system has no pole in the right half of the complex plane (only an infinity on the $\mathcal{R}$-axis), the closed loop system is stable if and only if the image of the Nyquist path Γ of Fig. 2 by the open loop transmittance, does not encircle the point $(-1, 0)$ of the complex plane [Nyquist, 1932].

Let Ω_1 be the image of path $\Gamma_1 = \rho_1 e^{i\alpha}$, $\rho_1 \to \infty$, $\alpha \in [\frac{\pi}{2}, -\frac{\pi}{2}]$ by B_1. Ω_1 is the origin of the complex plane.

Image of path $\Gamma_2 = \omega e^{-i\frac{\pi}{2}}$, $\omega \in \mathbb{R}^+$ is

$$\Omega_2 : \frac{\rho e^{i(\theta-\pi)}}{(\omega e^{-i\frac{\pi}{2}})^\nu} = \frac{\rho e^{i(\theta-\pi+\nu\frac{\pi}{2})}}{\omega^\nu}. \tag{5}$$

Ω_2 is thus a half straight line from the complex plane origin towards infinity and whose argument is equal to $-\pi + \theta + \nu\frac{\pi}{2}$.

Image of path $\Gamma_4 = \omega e^{i\frac{\pi}{2}}$, $\omega \in \mathbb{R}^+$ is

$$\Omega_4 : \frac{\rho e^{i(\theta-\pi)}}{(\omega e^{i\frac{\pi}{2}})^\nu} = \frac{\rho e^{i(\theta-\pi-\nu\frac{\pi}{2})}}{\omega^\nu}. \tag{6}$$

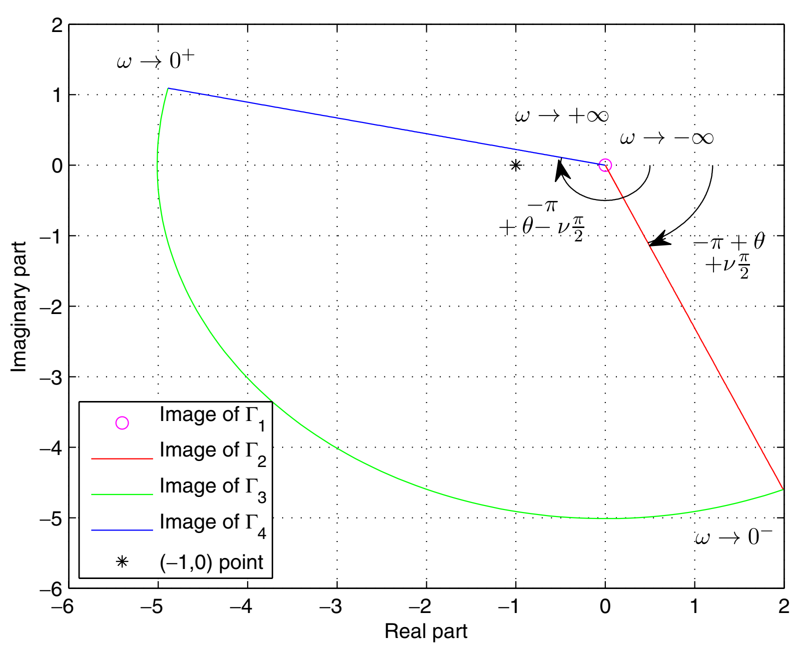

Fig. 4. Image of Γ if $\theta \in [0, \nu\frac{\pi}{2}]$ ($\nu = 0.7$, $\theta = 0.8\nu\frac{\pi}{2}$).

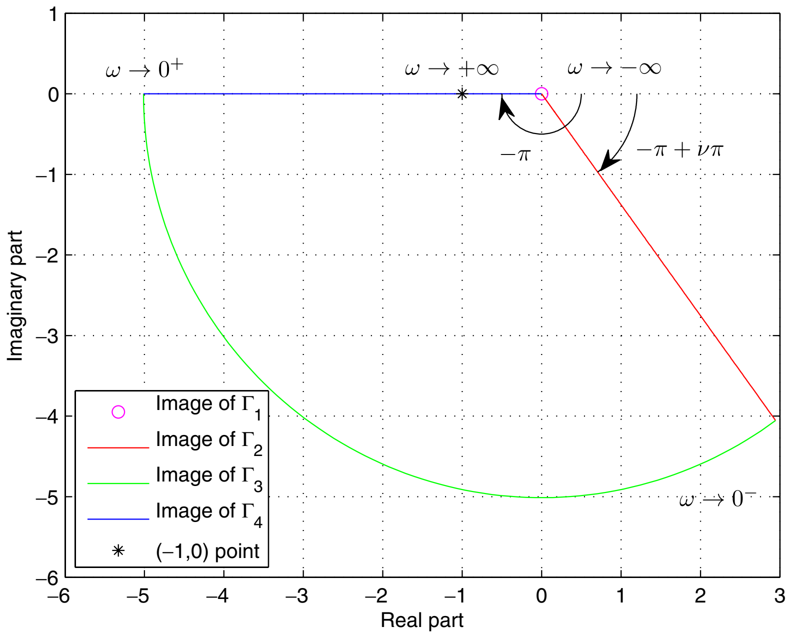

Fig. 5. Image of Γ if $\theta = \nu\frac{\pi}{2}$ ($\nu = 0.7$).

Ω_4 is thus a half straight line from the complex plane origin towards infinity and whose argument is equal to $-\pi + \theta - \nu\frac{\pi}{2}$.

Image of path $\Gamma_3 = \rho_1 e^{i\alpha}, \rho_1 \to 0, \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is

$$\Omega_3 : \frac{\rho e^{i(\theta-\pi)}}{(\rho_1 e^{i\alpha})^\nu} = \frac{\rho e^{i(\theta-\pi-\nu\alpha)}}{\rho_1{}^\nu}. \tag{7}$$

Ω_3 is thus an arc of circle with an infinite radius and whose center is the complex plane origin. The arc of circle goes from argument $-\pi + \theta + \nu\frac{\pi}{2}$ to $-\pi + \theta - \nu\frac{\pi}{2}$.

The image of path Γ by transmittance $B_1(s)$ with $\theta \in [\nu\frac{\pi}{2}, \pi]$, $\theta \in [0, \nu\frac{\pi}{2}]$, $\theta = \nu\frac{\pi}{2}$ and $0 < \nu < 1$ is represented in Figs. 3–5.

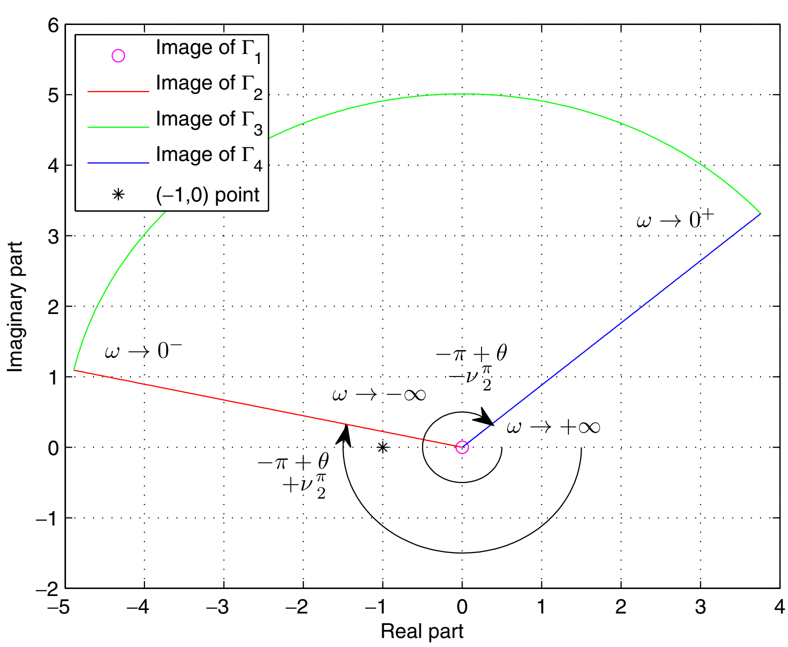

Fig. 6. Image of Γ if $\theta \in [-\pi, -\nu\frac{\pi}{2}]$ ($\nu = 0.7$, $\theta = -1.2\nu\frac{\pi}{2}$).

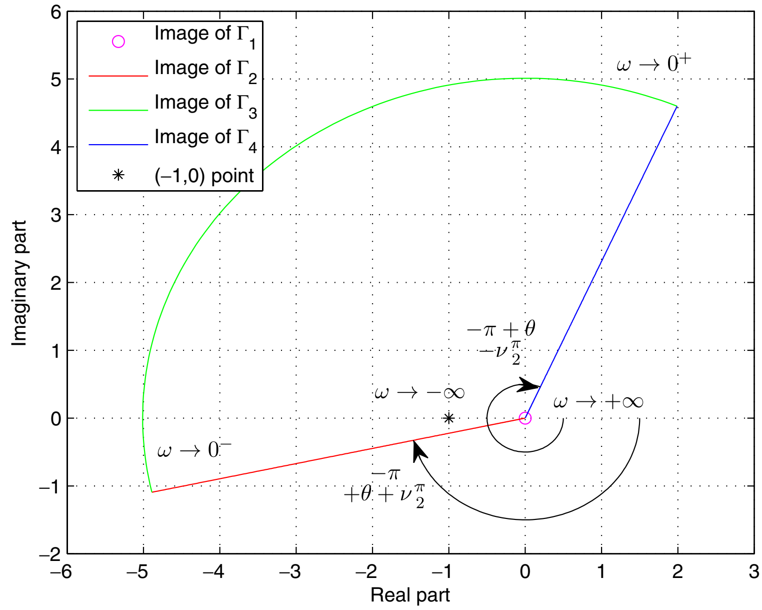

Fig. 7. Image of Γ if $\theta \in [-\nu\frac{\pi}{2}, 0]$ ($\nu = 0.7$, $\theta = -0.8\nu\frac{\pi}{2}$).

Remark 3.1. Figures 3–5 also give the analytical expressions of the argument of path Γ_2 and Γ_4 images.

The following conclusions can be drawn from Figs. 3–5:

(1) For $\theta \in]\nu\frac{\pi}{2}, \pi]$, Fig. 3 highlights that the point $(-1, 0)$ is not encircled by path Γ image if $\arg(\Gamma_4) = -\pi + \theta - \nu\frac{\pi}{2} > \pi$. The closed loop system is thus stable if $\theta > \nu\frac{\pi}{2}$.

(2) For $\theta \in [0, \nu\frac{\pi}{2}[$, Fig. 4 highlights that the point $(-1, 0)$ is encircled by path Γ image. The closed loop system is thus not stable.

(3) For $\theta = \nu\frac{\pi}{2}$, Fig. 5 shows the limit situation when the system becomes unstable.

Consequently, system of relation (4) is unstable if $\theta \in [0, \nu\frac{\pi}{2}]$ and is stable if $\theta \in]\nu\frac{\pi}{2}, \pi]$.

Similar figures can be drawn if $\theta \in]-\pi, 0]$. Thus Remark 3.1 can also be extended to the case $\theta \in]-\pi, 0]$ (see Figs. 6–8). The following conclusions

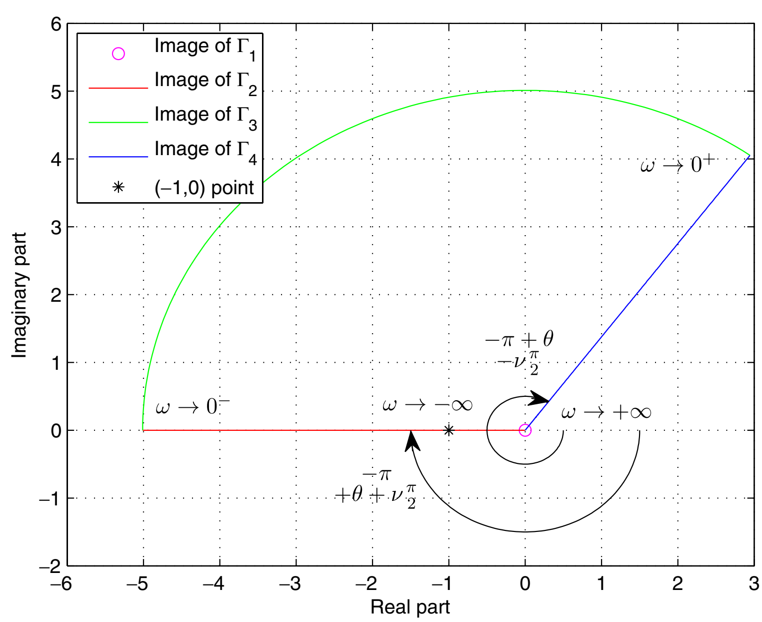

Fig. 8. Image of Γ if $\theta = -\nu\frac{\pi}{2}$ ($\nu = 0.7$).

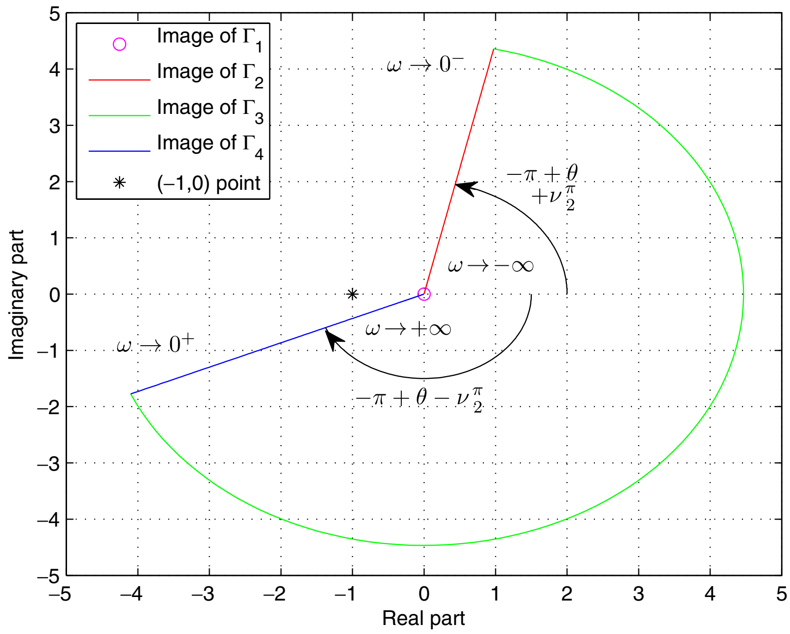

Fig. 9. Image of Γ if $\theta \in [\nu\frac{\pi}{2}, \pi]$ ($\nu = 1.3$, $\theta = 1.2\nu\frac{\pi}{2}$).

can be drawn from Figs. 6–8:

(1) For $\theta \in]-\pi, -\nu\frac{\pi}{2}]$, Fig. 6 highlights that the point $(-1, 0)$ is not encircled by path Γ image if $\arg(\Gamma_2) = -\pi + \theta + \nu\frac{\pi}{2} < \pi$. The closed loop system is thus stable if $\theta < -\nu\frac{\pi}{2}$.
(2) For $\theta \in [-\nu\frac{\pi}{2}, 0]$, Fig. 7 highlights that the point $(-1, 0)$ is encircled by path Γ image. The closed loop system is thus not stable.
(3) For $\theta = -\nu\frac{\pi}{2}$, Fig. 8 shows the limit situation when the system becomes unstable.

Consequently, system of relation (4) is unstable if $\theta \in [-\nu\frac{\pi}{2}, 0]$ and is stable if $\theta \in]-\pi, -\nu\frac{\pi}{2}]$.

A similar analysis can be done if $1 < \nu < 2$. Images of path Γ by transmittance $B_1(s)$ with $1 < \nu < 2$ are represented by Figs. 9–12. The same conclusions are then obtained.

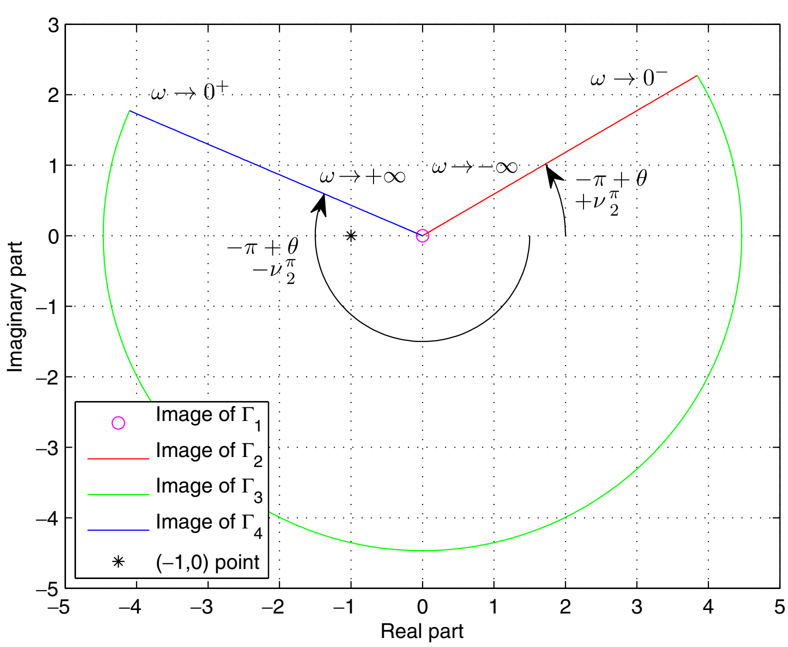

Fig. 10. Image of Γ if $\theta \in [0, \nu\frac{\pi}{2}, \pi]$ ($\nu = 1.3$, $\theta = 0.8\nu\frac{\pi}{2}$).

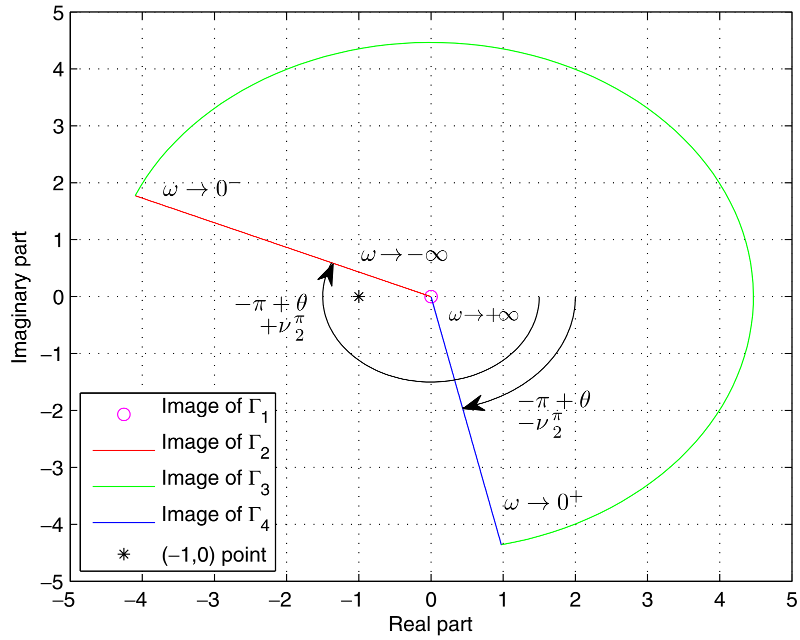

Fig. 11. Image of Γ if $\theta \in [-\pi, \nu\frac{\pi}{2}]$ ($\nu = 1.3$, $\theta = -1.2\nu\frac{\pi}{2}$).

In conclusion, Figs. 3–12 demonstrate that the one pole fractional system of Eq. (4) is stable if and only if $|\theta| \in [\nu\frac{\pi}{2}, \pi]$.

Given that any commensurate system such as system (1) can be represented as the sum and product of several systems (4) where poles $\rho e^{i\theta}$ are matrix A eigenvalues. The following theorem can thus be deduced.

Theorem 4. *If the triplet (A, B, C) is minimal, system (1) is BIBO stable if and only if closed loop system in Fig. 1 is stable for any system mode*

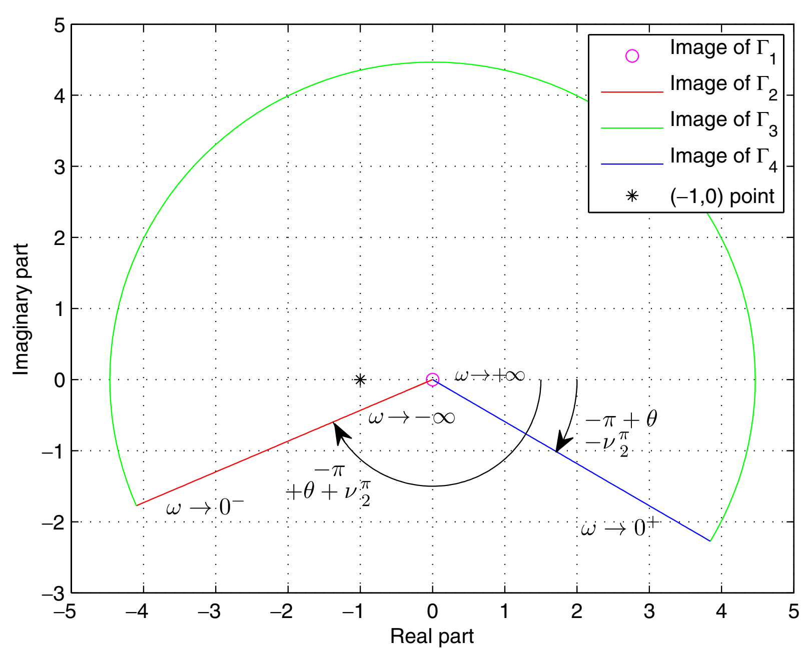

Fig. 12. Image of Γ if $\theta \in [-\nu\frac{\pi}{2}, 0]$ ($\nu = 1.3$, $\theta = -0.8\nu\frac{\pi}{2}$).

or equivalently as shown in Fig. 3 if and only if $|\arg\{\mathrm{spec}(A)\}| > \nu\frac{\pi}{2}$.

Proof. See comments before Theorem 4. ■

4. Multiinput, Multioutput Case

For a multiinput, multioutput system, all the system poles are generated by the polynomial

$$D(s) = \det(s^{\nu}I - A). \tag{8}$$

The analysis done in the previous section can also be applied on this polynomial whose partial fraction decomposition produces systems such system (4) in which $\rho e^{i\theta}$ denotes a matrix A eigenvalue. Theorem 4 can thus be extended to multiinput, multioutput systems.

5. Conclusions

This paper proposes a new proof for the BIBO stability theorem of Matignon and extends this theorem to fractional order ν such that $1 < \nu < 2$. Such an extension already exists in the literature but the proof proposed here is original and can be adapted to address noncommensurate fractional systems [Sabatier *et al.*, 2012]. The proof is based on the representation of the system modes by equivalent closed loop systems on which Nyquist theorem is then applied.

References

Achar, N., Lorenzo, C. F. & Hartley, T. T. [2003] "Initialization and the caputo fractional derivative," Tech. Rep., NASA John H. Glenn Research Center at Lewis Field.

Bonnet, C. & Partington, J. [2000] "Coprime factorizations and stability of fractional differential systems," *Syst. Contr. Lett.* **41**, 167–174.

Farges, C., Moze, M. & Sabatier, J. [2010] "Pseudo-state feedback stabilization of commensurate fractional order systems," *Automatica* **46**, 1730–1734.

Farges, C., Sabatier, J. & Moze, M. [2011] "Fractional order polytopic systems: Robust stability and stabilisation," *J. Diff. Eqs.* **35**, doi:10.1186/1687-1847-2011-35.

Lorenzo, D. & Hartley, T. [2001] "Initialization in fractional order systems," *European Control Conf.* (Porto, Portugal), pp. 1471–1476.

Malti, R., Cois, O., Levron, F. & Oustaloup, A. [2002] "Computing impulse response energy of fractional transfer function," *15th IFAC World Congress* (Barcelona, Spain).

Matignon, D. [1996] "Stability results on fractional differential equations with applications to control processing," *Computational Engineering in Systems Applications* (Lille, France), pp. 963–968.

Moze, M., Sabatier, J. & Oustaloup, A. [2005] "LMI tools for stability analysis of fractional systems," *20th ASME Int. Design Engineering Technical Conf/ Computers and Information in Engineering Technical Conference (ASME IDETC/CIE'05)* (Long Beach, California, USA).

Nyquist, H. [1932] "Regeneration theory," *Bell Syst. Tech. J.* **11**, 126–147.

Ortigueira, M. & Coito, F. [2008] "Initial conditions: What are we talking about?" *Third IFAC Workshop on Fractional Differentiation and Its Applications FDA'08* (Ankara, Turkey).

Sabatier, J., Merveillaut, M., Malti, R. & Oustaloup, A. [2010a] "How to impose physically coherent initial conditions to a fractional system?" *Commun. Nonlin. Sci. Numer. Simul.* **15**, 1318–1326.

Sabatier, J., Moze, M. & Farges, C. [2010b] "LMI stability conditions for fractional order systems," *Comput. Math. Appl.* **9**, 1594–1609.

Sabatier, J., Farges, C., Merveillaut, M. & L., F. [2012] "On observability and pseudo-state estimation of fractional order systems," *Eur. J. Contr.*, to appear.

Trigeassou, J., Maamri, N., Sabatier, J. & Oustaloup, A. [2011] "A Lyapunov approach to the stability of fractional differential equations," *Sign. Process.* **91**, 437–445.

Appendix A

This appendix demonstrates that whatever the definition used for fractional differentiation, a fractional order system response to a sinusoidal input is the same in steady state. If a fractional system of the form:

$$D^{\nu}x(t) = Ax(t) \tag{A.1}$$

is considered, a realization of this system is presented in Fig. 13.

This realization shows that only fractional integration is involved in the response of the system. Given that all the definitions coincide for integration, a unique response is obtained.

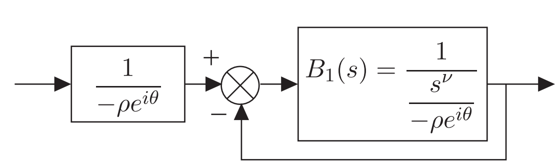

Fig. 13. Fractional system realization.