Mathematics and Computers in Simulation 53 (2000) 287–291

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.nl/locate/matcom

A further index concept for linear PDAEs of hyperbolic type

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Accepted 28 July 2000

Abstract

For many technical systems the use of a refined network approach yields mathematical models given by initial-boundary value problems of partial differential algebraic equations (PDAEs). The boundary conditions of these systems are governed by time-dependent differential–algebraic equations (DAEs) that couple the PDAE system with the network elements that are modelled by DAEs in time only. As the numerical difficulties for a DAE can be classified by the index concept, it seems to be natural to generalize these ideas to the PDAE case. There already exist some approaches for parabolic and hyperbolic equations. Here, we will focus on a new kind of index, the characteristics index for hyperbolic equations, that does not depend on the elimination of one of the independent variables. It relies on the fact that hyperbolic PDEs can be regarded as ODEs along the characteristics.

Keywords: Partial differential algebraic equations (PDAEs); Differential–algebraic systems (DAEs); Characteristics index

1. Introduction

In technical simulation of time dependent processes most of today's industrial software is based on a network approach [4]. Only topology, and no spatial dimension is considered. However, if coupling and second order effects become more important or distributed elements have to be considered, the network approach has to be combined with corresponding models to cope with the spatial extension. In several applications [2,6,7], one has to deal with mixed systems of differential–algebraic systems (DAEs) in time only and hyperbolic systems of partial differential equations (PDEs) both in time and space. These systems are coupled by appropriate physical boundary conditions, connecting the network variables at the boundaries with the inner variables. For pure DAE systems, the index concept has turned out to give a deep insight into the solution properties, as well as in the numerical problems to be expected when solving these systems. Generalization of the index concept to linear PDAE systems has recently been proposed in [1,5] with a main focus on parabolic systems. Some of these concepts were transferred and extended to hyperbolic systems in [3]. A time and space index was defined and compared with the indices of the semidiscretized system. In this work, we focus on the characteristics index that is independent

PII: S0378-4754(00)00215-9

of special elimination methods of the time or space variable. Furthermore, we will compare with some examples the value of the characteristics index with a perturbation index, that measures the influence of small perturbations in the initial and boundary data on the numerical solution. Linear hyperbolic-type equations are defined in [3] by

$$Au_t + Bu_x + Cu = f(x,t), \quad u = u(x,t), \quad x \in [0,1], \quad t \in [0,T] \tag{1}$$

where $u,\ f \in C^1$. In order to guarantee the hyperbolicity we assume A regular and $A^{-1}B$ real diagonalizable. The initial conditions are given by $u(x,0) = g(x)$ and the boundary conditions by the linear DAE in time

$$R_1 \begin{pmatrix} u_t(0,t) \\ u_t(1,t) \\ \dot{z}(t) \end{pmatrix} + R_2 \begin{pmatrix} u(0,t) \\ u(1,t) \\ z(t) \end{pmatrix} - s(t) = 0. \tag{2}$$

Hereby z denotes the additional network variables. As we restrict ourselves to the analysis of linear systems of the types (1) and (2) we obtain smooth solutions, if $f(x, t)$, $g(x)$ and $s(t)$ are smooth enough.

2. The characteristics index

Considering the definitions of time and space index given in [3], we find that the t resp. x-variable has to be eliminated first with some special method. In order to avoid this, the hyperbolic PDAE is transformed to an ODE along the characteristics. The transformation of (1) to characteristic form

$$v_t + \Lambda v_x + \tilde{C} v = \tilde{f} \quad \text{with} \quad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_{n_p}), \tag{3}$$

with $\lambda_i > 0$ for $i = 1, \dots, l$, with $\lambda_i < 0$ for $i = l+1, \dots, r$ and with $\lambda_i = 0$ for $i = r+1, \dots, n_p$ gives n_p characteristic curves $x = k_i^a(t)$ for $i = 1, \dots, n_p$ with the parameter $a \in \mathbb{R}$. The functions $k_i^a : \mathbb{R} \to \mathbb{R}$ are linear. Thus, we get for the system of ODEs along the characteristics

$$\begin{aligned} \frac{\mathrm{d}v_1(k_1^a(t), t)}{\mathrm{d}t} &= \sum_{i=1}^{n_p} \tilde{c}_{1i} v_i(k_1^a(t), t) + \tilde{f}_1(k_1^a(t), t), \\ &\vdots \\ \frac{\mathrm{d}v_{n_p}(k_{n_p}^a(t), t)}{\mathrm{d}t} &= \sum_{i=1}^{n_p} \tilde{c}_{n_p i} v_i(k_{n_p}^a(t), t) + \tilde{f}_{n_p}(k_{n_p}^a(t), t). \end{aligned}$$

$\tilde{c}_{ji}$ are the entries in the coupling matrix $\tilde{C}$. The initial conditions for these ODEs are given by the initial resp. boundary conditions of the original PDAE. When integrating the system, they are inserted depending on the characteristics. The equations for $1 \le i \le l$ are

$$0 \le k_i^a(0) : \int_0^t \frac{\mathrm{d}v_i(k_i^a(\tau), \tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau = v_i(k_i^a(t), t) - g_i(k_i^a(0)),$$

$$0 > k_i^a(0) : \int_{\tilde{t}}^t \frac{\mathrm{d}v_i(k_i^a(\tau), \tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau = v_i(k_i^a(t), t) - v_i(0, \tilde{t}) \quad \text{with} \quad k_i^a(\tilde{t}) = 0,$$

and those for $l+1 \le i \le r$

$$1 \ge k_i^a(0): \int_0^t \frac{\mathrm{d}v_i(k_i^a(\tau),\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau = v_i(k_i^a(t),t) - g_i(k_i^a(0)),$$

$$1 < k_i^a(0): \int_{\tilde t}^t \frac{\mathrm{d}v_i(k_i^a(\tau),\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau = v_i(k_i^a(t),t) - v_i(1,\tilde t) \quad \text{with} \quad k_i^a(\tilde t) = 1.$$

The characteristic curves for $r+1 \le i \le n_p$ are constants, i.e. only the initial conditions influence the solution:

$$\int_0^t \frac{\mathrm{d}v_i(k_i^a,\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau = v_i(k_i^a,t) - g_i(k_i^a).$$

The values $v_i(0,\tilde t)$ resp. $v_i(1,\tilde t)$ are determined by the boundary conditions. If the solution components are coupled at the boundaries, these components have to be computed gradually along the other characteristics until the time layer $t=0$ is reached. Using this procedure we can define an index that is determined by the boundary values and the influence of the initial conditions. For the PDAE system itself only yields ODEs along the characteristics. In order to obtain a hill system for the solutions at the boundaries, we have to complete the system by the integrated ODEs.

Definition. Assume that all equations for the formal determination of the solutions at the boundaries are given, the boundary and initial conditions and the additional ODEs along the characteristics. The network variable z is eliminated. Then the *characteristics index*, ν_C denotes the differential index of this system.

Remark. The characteristics index does not depend on a special method to eliminate the space or time variable. However, the DAE system in time is set up with integrals along different characteristic curves. It is shown that the numerical problems are determined mainly by the boundary conditions but that also initial conditions play a role in solving PDAEs of hyperbolic type.

3. Examples

Example 1. Considering purely time-dependent algebraic boundary conditions without coupling or network components, the characteristics index is one, $\nu_C = 1$. This value coincides with the value of the perturbation index as defined in [3]. The characteristics and perturbation index are also the same, if additional network components appear — but then it is possible to obtain a higher index.

Example 2. For $l=1$, $r=n_p=2$ and $f \equiv 0$ the boundary values are coupled:

$$v_1(0,t) + v_2(0,t) = s_1(t), \quad v_1(1,t) - v_2(1,t) = s_2(t).$$

If we want to compute the solution v_1 at the marked point (x, t) in the following sketch, we get with $k_1^a(\tilde t) = 0$

$$\int_{\tilde t}^t \frac{\mathrm{d}v_1(k_1^a(\tau),\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau = v_1(k_1^a(t),t) - v_1(0,\tilde t) = \int_{\tilde t}^t \tilde c_{11} v_1(k_1^a(\tau),\tau) + \tilde c_{12} v_2(k_1^a(\tau),\tau)\,\mathrm{d}\tau.$$

Evaluation at the left boundary $v_1(0,\tilde{t}) = s_1(\tilde{t}) - v_2(0,\tilde{t})$ yields first the computation of $v_2(0,\tilde{t})$ with $k_2^{\tilde{a}}(\tilde{t}) = 0$ and $k_2^{\tilde{a}}(\bar{t}) = 1$. This is done by integration

$$\int_{\bar{t}}^{\tilde{t}} \frac{\mathrm{d}v_2(k_2^{\tilde{a}}(t),t)}{\mathrm{d}t}\,\mathrm{d}\tau = v_2(0,\tilde{t}) - v_2(1,\bar{t}) = \int_{\bar{t}}^{\tilde{t}} \tilde{c}_{21}v_1(k_2^{\tilde{a}}(\tau),\tau) + \tilde{c}_{22}v_2(k_1^{\tilde{a}}(\tau),\tau)\,\mathrm{d}\tau.$$

Thus, we reach the opposite boundary along the characteristic line where

$$v_2(1,\bar{t}) = v_1(1,\bar{t}) - s_2(\bar{t})$$

holds. Again $v_1(1,\bar{t})$ is not given, but with another integration along the second characteristic line we can insert directly the initial conditions, i.e. we get with $k_1^{\tilde{a}}(\bar{t}) = 1$

$$v_1(1,\bar{t}) = g_1(k_1^{\bar{a}}(0)) + \int_0^{\bar{t}} \ldots\,\mathrm{d}\tau.$$

Therefore, the solution can be expressed on an arbitrarily chosen point with integrals over the source terms along the characteristics.

Combining the above equations to a system for all unknowns, we obtain $\nu_{\mathrm{C}} = 1$ for the characteristics index. This is due to the fact, that the value at the left resp. at the right only influences one boundary condition at a time.

Example 3. We consider two uncoupled advection equations $v_t + v_x = 0$ and $w_t + w_x = 0$ with initial data $v(x,0) = g_1(x)$, $w(x,0) = g_2(x)$ and boundary conditions

$$v(0,t) + \dot{w}(0,t) + v(1,t) = s_1(t),$$

$$w(0,t) + w(1,t) = s_2(t),$$

where the solution at the right and at the left side appears in the same equation. This system yields for $x < t$ and $t < 1$ the analytical solution

$$v(x,t) = s_1(t-x) - g_1(1-t+x) - \dot{s}_2(t-x) + \dot{g}_2(1-t+x),$$

$$w(x,t) = s_2(t-x) - g_2(1-t+x).$$

Thus the derivatives of both boundary and initial data influences the solution. Measuring the sensitivity of the whole system with the perturbation index ν_{P}, we get $\nu_{\mathrm{P}} = 2$.

For the characteristics index we have to set up the system for the physically given boundary values and the solution at the other boundary due to the coupling in the boundary conditions. The latter is computed by integration of the ODEs along the characteristics. As illustrated in the sketch at the left, the whole system four equations in $\tilde{t}$ for four unknowns — thus reads for $x < t < 1$:

$$v(0,\tilde{t}) + \dot{w}(0,\tilde{t}) + v(1,\tilde{t}) = s_1(\tilde{t}),$$

$$w(0,\tilde{t}) + w(1,\tilde{t}) = s_2(\tilde{t}),$$

$$v(1,\tilde{t}) - g_1(\tilde{x}) = 0,$$

$$w(1,\tilde{t}) - g_2(\tilde{x}) = 0.$$

This system has differential index 2, $\nu_C = 2$, as we would expect from the exact solution and the perturbation index.

4. Summary

In this paper, we have presented a further index concept for linear PDAEs of hyperbolic type. Using the fact that the PDAE can be written as an ODE system along the characteristic curves, we can compute the differential index of this system together with the boundary functions. This approach yields for different examples the same index value as the perturbation index and thus reflects well the sensitivity of the PDAE with respect to initial and boundary conditions.

5. Further Reading

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