Port-Hamiltonian Differential-Algebraic Systems

A.J. van der Schaft

Abstract The basic starting point of port-Hamiltonian systems theory is *network modeling*; considering the overall physical system as the *interconnection* of simple subsystems, mutually influencing each other via energy flow. As a result of the interconnections *algebraic constraints* between the state variables commonly arise. This leads to the description of the system by *differential-algebraic equations* (DAEs), i.e., a combination of ordinary differential equations with algebraic constraints. The basic point of view put forward in this survey paper is that the differential-algebraic equations that arise are not just arbitrary, but are endowed with a special mathematical structure; in particular with an underlying geometric structure known as a Dirac structure. It will be discussed how this knowledge can be exploited for analysis and control.

Keywords Port-Hamiltonian systems · Passivity · Algebraic constraints · Kinematic constraints · Casimirs · Switching systems · Dirac structure · Interconnection

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1 Introduction to Port-Hamiltonian Differential-Algebraic Systems

The framework of port-Hamiltonian systems is intended to provide a systematic approach to the modeling, analysis, simulation and control of, possibly large-scale, multi-physics systems; see [9, 15, 19, 20, 24, 25, 29, 31–34, 38, 39] for some key references. Although the framework includes distributed-parameter systems as well, we will focus in this paper on lumped-parameter, i.e., finite-dimensional, systems.

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The basic starting point of port-Hamiltonian systems theory is (power-based) *network modeling*; considering the overall system as the *interconnection* of simple subsystems, mutually influencing each other via energy flow [27]. As a result of the interconnections *algebraic constraints* between the state variables commonly arise. This leads to the description of the system by *differential-algebraic equations* (DAEs), i.e., a combination of ordinary differential equations with algebraic constraints. However, the basic point of view put forward in this paper is that the differential-algebraic equations that arise *are not just arbitrary differential-algebraic equations*, but are endowed with a special mathematical structure, which may be fruitfully used for analysis, simulation and control.

As a motivating and guiding example for the theory surveyed in this paper we will start with the following example.

1.1 A Motivating Example

Consider an LC-circuit consisting of two capacitors and one inductor, all in parallel. Naturally this system can be seen as the interconnection of three subsystems, the two capacitors and the inductor, interconnected by Kirchhoff's current and voltage laws. The capacitors (first assumed to be linear) are described by the following dynamical equations:

$$\begin{aligned} \dot{Q}_i &= I_i, \\ V_i &= \frac{Q_i}{C_i}, \end{aligned} \qquad i = 1, 2. \tag{1.1}$$

Here I_i and V_i are the currents through, respectively the voltages across, the two capacitors, and C_i are their capacitances. Furthermore, Q_i are the *charges* stored at the capacitors; regarded as basic state variables.[1]

Similarly, the linear inductor is described by the dynamical equations

$$\begin{aligned} \dot{\varphi} &= V_L, \\ I_L &= \frac{\varphi}{L}, \end{aligned} \tag{1.2}$$

where I_L is the current through the inductor, and V_L is the voltage across the inductor. Here the (magnetic) *flux* φ is taken as the state variable of the inductor, and L denotes its inductance.

Parallel interconnection of these three subsystems by Kirchhoff's laws amounts to the interconnection equations

$$V_1 = V_2 = V_L, \qquad I_1 + I_2 + I_L = 0, \tag{1.3}$$

[1]In the port-Hamiltonian formulation there is a clear preference for taking the charges to be the state variables instead of the voltages V_i. Although this comes at the expense of the introduction of extra variables, it will turn out to be very advantageous from a geometric point of view.

where the equation $V_1 = V_2$ gives rise to the algebraic constraint

$$\frac{Q_1}{C_1} = \frac{Q_2}{C_2} \tag{1.4}$$

relating the two state variables Q_1, Q_2.

There are multiple ways to describe the total system. One is to regard either I_1 or I_2 as a *Lagrange multiplier* for the constraint $\frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0$. Indeed, by defining $\lambda = I_1$ one may write the total system as

$$\begin{aligned}
\begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q_1}{C_1} \\ \frac{Q_2}{C_2} \\ \frac{\varphi}{L} \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \lambda, \\
0 &= \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q_1}{C_1} \\ \frac{Q_2}{C_2} \\ \frac{\varphi}{L} \end{bmatrix},
\end{aligned} \tag{1.5}$$

where the algebraic constraint $\frac{Q_1}{C_1} - \frac{Q_2}{C_2} = 0$ represented in the last equation of (1.5) can be seen to give rise to a *constraint current* $[1 \ -1 \ 0]^T \lambda$, which is added to the first three ordinary differential equations in (1.5).

Next one may *eliminate* the Lagrange multiplier λ by pre-multiplying the differential equations by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Together with the algebraic constraint this yields the *differential-algebraic system*

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{Q_1}{C_1} \\ \frac{Q_2}{C_2} \\ \frac{\varphi}{L} \end{bmatrix}. \tag{1.6}$$

Equations (1.5) and (1.6) are different representations of the same *port-Hamiltonian system* defined by the LC-circuit, which is geometrically (i.e., coordinate-free) described by a Dirac structure and constitutive relations corresponding to energy-storage. In this example the Dirac structure is given by the linear space

$$\begin{aligned}
\mathcal{D} := \Bigg\{ (f, e) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \\
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = 0 \Bigg\}
\end{aligned} \tag{1.7}$$

having the characteristic property that $e^T f = 0$ for all $(f, e) \in \mathcal{D}$ (total power is zero), and moreover having maximal dimension with regard to this property (in this case $\dim \mathcal{D} = 3$). The two representations (1.5) and (1.6) correspond to two different representations of this same Dirac structure.

Furthermore, the constitutive relations of energy-storage are given by $f = [f_1\ f_2\ f_3]^T = -[\dot{Q}_1\ \dot{Q}_2\ \dot{\varphi}]$, and $e = [e_1\ e_2\ e_3]^T = [\frac{Q_1}{C_1}\ \frac{Q_2}{C_2}\ \frac{\varphi}{L}]^T$, where the last vector is the gradient vector of the total stored energy, or *Hamiltonian*

$$H(Q_1, Q_2, \varphi) := \frac{Q_1^2}{2C_1} + \frac{Q_2^2}{2C_2} + \frac{\varphi^2}{2L}. \tag{1.8}$$

We may easily replace the linear constitutive relations of the capacitors and the inductor by more general nonlinear ones, corresponding to a general non-quadratic Hamiltonian

$$H(Q_1, Q_2, \varphi) = H_1(Q_1) + H_2(Q_2) + H_3(\varphi) \tag{1.9}$$

with $H_i(Q_i), i = 1, 2$, denoting the electric energies of the two capacitors, and $H_3(\varphi)$ the magnetic energy of the inductor. Then the resulting dynamics are given by the *nonlinear* differential-algebraic equations

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{Q}_1 \\ \dot{Q}_2 \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{dH_1}{dQ_1}(Q_1) \\ \frac{dH_2}{dQ_2}(Q_2) \\ \frac{dH_3}{d\varphi}(\varphi) \end{bmatrix}. \tag{1.10}$$

In the port-Hamiltonian description there is thus a clear *separation*[2] between the constitutive relations of the elementary subsystems (captured by the Hamiltonian H), and the interconnection structure (formalized by the Dirac structure $\mathcal{D}$). This has several advantages in terms of flexibility and standardization (e.g., one may replace linear subsystems by nonlinear ones, without changing the interconnection structure), and will give rise to a completely *compositional* theory of network models of physical systems: the interconnection of port-Hamiltonian systems defines another port-Hamiltonian system, where the Hamiltonians are simply added and the new Dirac structure results from the composition of the Dirac structures of the interconnected individual physical systems.

From a DAE perspective it may be noted that the algebraic constraint $\frac{Q_1}{C_1} = \frac{Q_2}{C_2}$ is of *index one*. In fact, under reasonable assumptions on the Hamiltonian this will turn out to be a general property of port-Hamiltonian differential-algebraic systems.

[2] Note that this separation is already present in the geometric description of Hamiltonian dynamics in classical mechanics; see e.g. [1]. There the dynamics is defined with the use of the Hamiltonian and the *symplectic structure* on the phase space of the system. Dirac structures form a generalization of symplectic structures, and allow the inclusion of *algebraic constraints*. Note furthermore that the symplectic structure in classical mechanics is commonly determined by the geometry of the *configuration space*, while the Dirac structure of a port-Hamiltonian system captures its *network topology*.

In the above example, the subsystems are all energy-storing elements (capacitors, inductors), and thus the total energy (Hamiltonian) is *preserved* along solutions of the differential-algebraic equations. The framework, however, extends to energy-dissipating elements (such as resistors), in which case the Hamiltonian will decrease along solutions. Furthermore, in the above example there are no external inputs to the system (such as voltage or current sources). In the port-Hamiltonian framework these are, however, immediately incorporated, and are in fact essential to describe the interconnection of port-Hamiltonian systems. By including external ports in the system description it will follow that along system trajectories of the port-Hamiltonian differential-algebraic system $\frac{dH}{dt}$ is always less than or equal than the power supplied to the system through these external ports, i.e., *passivity*.

Finally, the port-Hamiltonian formalism emphasizes the *analogy* between physical system models. The same system of equations as in (1.5) or (1.6) also results from the modeling of a system of two rigidly coupled masses connected to a single spring. In this case, the rigid coupling between the two masses with kinetic energies

$$H_i(p_i) = \frac{p_i^2}{2m_i}, \quad i = 1, 2$$

(where p_1, p_2 denote the momenta of the masses m_1, m_2) is given by the (index one) algebraic constraint

$$\frac{p_1}{m_1} = v_1 = v_2 = \frac{p_2}{m_2} \tag{1.11}$$

with v_1, v_2 denoting the velocities of the masses. Note that this is different from formulating the rigid coupling between two masses by the (index two) algebraic constraint

$$q_1 = q_2 \tag{1.12}$$

in terms of the *positions* $q_i, i = 1, 2$, of the two masses. In fact, the constraint (1.11) results from differentiation of (1.12). Indeed, in the port-Hamiltonian approach there is a preference for modeling constraints in mechanical systems as *kinematic constraints* (which can be holonomic, as in this simple example, or nonholonomic). This will be discussed in more detail later in this paper.

The contents of Sects. 2, 3, 5, 6, 7 of the present paper are a thoroughly reworked version of material that appeared before in [34], emphasizing and expanding the differential-algebraic nature of port-Hamiltonian systems.

2 Definition of Port-Hamiltonian Systems

In this section we will provide the general geometric (coordinate-free) definition of a finite-dimensional port-Hamiltonian system, and discuss different examples and subclasses.

A port-Hamiltonian system can be represented as in Fig. 1. Central in the definition of a port-Hamiltonian system is the notion of a *Dirac structure*, denoted in

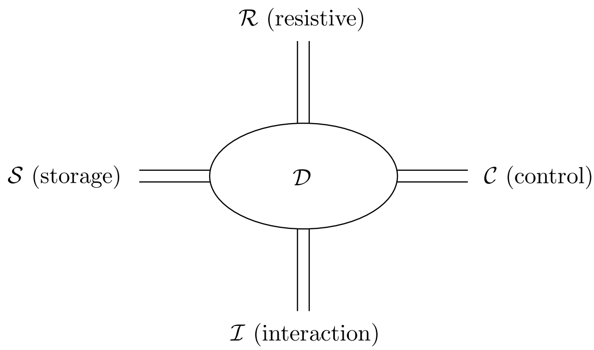


Fig. 1 Port-Hamiltonian system

Fig. 1 by $\mathcal{D}$. Basic property of a Dirac structure is *power-preservation*: the Dirac structure links the port variables in such a way that the total power associated with all the port-variables is zero.

The port variables entering the Dirac structure have been split in Fig. 1 in different parts. First, there are two *internal* ports. One, denoted by $\mathcal{S}$, corresponds to energy-storage and the other one, denoted by $\mathcal{R}$, corresponds to internal energy-dissipation (resistive elements). Second, two *external* ports are distinguished. The external port denoted by $\mathcal{C}$ is the port that is accessible for controller action. Also the presence of *sources* may be included in this port. Finally, the external port denoted by $\mathcal{I}$ is the interaction port, defining the interaction of the system with (the rest of) its environment.

2.1 Dirac Structures

We start with a finite-dimensional linear space of *flows* $\mathcal{F}$. The elements of $\mathcal{F}$ will be denoted by $f \in \mathcal{F}$, and are called *flow vectors*. The space of *efforts* is given by the *dual* linear space $\mathcal{E} := \mathcal{F}^*$, and its elements are denoted by $e \in \mathcal{E}$. In the case of $\mathcal{F} = \mathbb{R}^k$ the space of efforts is $\mathcal{E} = (\mathbb{R}^k)^*$, and as the elements $f \in \mathbb{R}^k$ are commonly written as *column* vectors the elements $e \in (\mathbb{R}^k)^*$ are appropriately represented as *row* vectors. Then the *total space* of flow and effort variables is $\mathcal{F} \times \mathcal{F}^*$, and will be called the space of *port variables*. On the total space of port variables, the *power* is defined by

$$P = \langle e \mid f \rangle, \quad (f, e) \in \mathcal{F} \times \mathcal{F}^*, \tag{2.1}$$

where $\langle e \mid f \rangle$ denotes the duality product, that is, the linear functional $e \in \mathcal{F}^*$ acting on $f \in \mathcal{F}$. Often we will write the flow f and effort e both as column vectors, in which case $\langle e \mid f \rangle = e^T f$.

Definition 2.1 A *Dirac structure* on $\mathcal{F} \times \mathcal{F}^*$ is a subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ such that

(i) $\langle e \mid f \rangle = 0$, for all $(f, e) \in \mathcal{D}$,
(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

Property (i) corresponds to *power-preservation*, and expresses the fact that the total power entering (or leaving) a Dirac structure is zero. It can be shown that the *maximal dimension* of any subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ satisfying property (i) is equal to $\dim \mathcal{F}$. Instead of proving this directly, we will give an equivalent definition of a Dirac structure from which this claim immediately follows. Furthermore, this equivalent definition of a Dirac structure has the advantage that it generalizes to the case of an *infinite-dimensional* linear space $\mathcal{F}$, leading to the definition of an infinite-dimensional Dirac structure. This will be instrumental in the definition of *distributed-parameter* port-Hamiltonian systems [39].

In order to give this equivalent characterization of a Dirac structure, let us look more closely at the geometric structure of the total space of flow and effort variables $\mathcal{F} \times \mathcal{F}^*$. Closely related to the definition of power, there exists a canonically defined *bilinear form* $\langle\langle\, , \rangle\rangle$ on the space $\mathcal{F} \times \mathcal{F}^*$, defined as

$$\langle\langle \big(f^a, e^a\big), \big(f^b, e^b\big) \rangle\rangle := \langle e^a \mid f^b \rangle + \langle e^b \mid f^a \rangle \tag{2.2}$$

with $(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*$. Note that this bilinear form is *indefinite*, that is, $\langle\langle (f, e), (f, e) \rangle\rangle$ may be positive or negative. However, it is *non-degenerate*, that is, $\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle = 0$ for all (f^b, e^b) implies that $(f^a, e^a) = 0$.

Proposition 2.1 ([8, 12]) *A* (*constant*) *Dirac structure on* $\mathcal{F} \times \mathcal{F}^*$ *is a subspace* $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ *such that*

$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}, \tag{2.3}$$

where $\perp\!\!\!\perp$ *denotes the orthogonal complement with respect to the bilinear form* $\langle\langle\, , \rangle\rangle$.

Proof Let $\mathcal{D}$ satisfy (2.3). Then for every $(f, e) \in \mathcal{D}$

$$0 = \langle\langle (f, e), (f, e) \rangle\rangle = \langle e \mid f \rangle + \langle e \mid f \rangle = 2\langle e \mid f \rangle.$$

By non-degeneracy of $\langle\langle\, , \rangle\rangle$

$$\dim \mathcal{D}^{\perp\!\!\!\perp} = \dim\big(\mathcal{F} \times \mathcal{F}^*\big) - \dim \mathcal{D} = 2 \dim \mathcal{F} - \dim \mathcal{D}$$

and hence property (2.3) implies $\dim \mathcal{D} = \dim \mathcal{F}$. Conversely, let $\mathcal{D}$ be a Dirac structure and thus satisfying properties (i) and (ii) of Definition 2.1. Let $(f^a, e^a), (f^b, e^b)$ be any vectors contained in $\mathcal{D}$. Then by linearity also $(f^a + f^b, e^a + e^b) \in \mathcal{D}$. Hence by property (i)

$$\begin{aligned} 0 &= \langle e^a + e^b \mid f^a + f^b \rangle \\ &= \langle e^a \mid f^b \rangle + \langle e^b \mid f^a \rangle + \langle e^a \mid f^a \rangle + \langle e^b \mid f^b \rangle \\ &= \langle e^a \mid f^b \rangle + \langle e^b \mid f^a \rangle = \langle\langle \big(f^a, e^a\big), \big(f^b, e^b\big) \rangle\rangle \end{aligned} \tag{2.4}$$

since by another application of property (i), $\langle e^a \mid f^a \rangle = \langle e^b \mid f^b \rangle = 0$. This implies that $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\!\perp}$. Furthermore, by property (ii) and $\dim \mathcal{D}^{\perp\!\!\!\perp} = 2 \dim \mathcal{F} - \dim \mathcal{D}$ it follows that $\dim \mathcal{D} = \dim \mathcal{D}^{\perp\!\!\!\perp}$, thus yielding $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$. □

Remark 2.1 Note that we have actually shown that property (i) implies $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$. Together with the fact that $\dim \mathcal{D}^{\perp\!\!\perp} = 2\dim \mathcal{F} - \dim \mathcal{D}$ this implies that any subspace $\mathcal{D}$ satisfying property (i) has the property that $\dim \mathcal{D} \leq \dim \mathcal{F}$. Thus, as claimed before, a Dirac structure is a linear subspace *of maximal dimension* satisfying property (i).

Remark 2.2 The property $\mathcal{D} = \mathcal{D}^{\perp\!\!\perp}$ can be regarded as a generalization of Tellegen's theorem in circuit theory, since it describes a constraint between two *different* realizations of the port variables, in contrast to property (i).

From a mathematical point of view, there are a number of direct examples of Dirac structures $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$. We leave the proofs as an exercise to the reader.

(i) Let $J : \mathcal{F}^* \to \mathcal{F}$ be a skew-symmetric linear mapping, that is, $J = -J^*$, where $J^* : \mathcal{F}^* \to (\mathcal{F})^{**} = \mathcal{F}$ is the adjoint mapping. Then

$$\operatorname{graph} J := \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je\right\}$$

is a Dirac structure.

(ii) Let $\omega : \mathcal{F} \to \mathcal{F}^*$ be a skew-symmetric linear mapping, then

$$\operatorname{graph} \omega := \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid e = \omega f\right\}$$

is a Dirac structure.

(iii) Let $\mathcal{G} \subset \mathcal{F}$ be any subspace. Define

$$\mathcal{G}^\perp = \left\{e \in \mathcal{F}^* \mid \langle e \mid f\rangle = 0 \text{ for all } f \in \mathcal{G}\right\}.$$

Then $\mathcal{G} \times \mathcal{G}^\perp \subset \mathcal{F} \times \mathcal{F}^*$ is a Dirac structure. Special cases of such a Dirac structure are *ideal constraints*. Indeed, the ideal effort constraint

$$\mathcal{D} := \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid e = 0\right\}$$

is defining a Dirac structure, and the same holds for the ideal flow constraint

$$\mathcal{D} := \left\{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = 0\right\}.$$

2.2 Energy Storage

The port variables associated with the internal storage port will be denoted by (f_S, e_S). They are interconnected to the energy storage of the system, which is defined by a finite-dimensional state space manifold $\mathcal{X}$ with coordinates x, together with a Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$ denoting the energy. The flow variables of the energy storage are given by the *rate* $\dot{x}$ of the energy variables x. Furthermore, the effort variables of the energy storage are given by the *co-energy* variables $\frac{\partial H}{\partial x}(x)$, resulting in the energy balance[3]

$$\frac{d}{dt}H = \left\langle \frac{\partial H}{\partial x}(x) \mid \dot{x} \right\rangle = \frac{\partial^T H}{\partial x}(x)\dot{x}. \tag{2.5}$$

The interconnection of the energy storing elements to the storage port of the Dirac structure is accomplished by setting

$$f_S = -\dot{x} \quad \text{and} \quad e_S = \frac{\partial H}{\partial x}(x). \tag{2.6}$$

Hence the energy balance (2.5) can be also written as

$$\frac{d}{dt}H = \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_S^T f_S. \tag{2.7}$$

2.3 Energy Dissipation

The second internal port corresponds to internal energy dissipation (due to friction, resistance, etc.), and its port variables are denoted by (f_R, e_R). These port variables are terminated on a static resistive relation $\mathcal{R}$. In general, a static resistive relation will be of the form

$$R(f_R, e_R) = 0 \tag{2.8}$$

with the property that for all (f_R, e_R) satisfying (2.8)

$$\langle e_R \mid f_R \rangle \leq 0. \tag{2.9}$$

A typical example of such a nonlinear resistive relation will be given in Example 4.4. In many cases we may restrict ourselves to *linear* resistive relations in which case (f_R, e_R) satisfy relations of the form

$$R_f f_R + R_e e_R = 0. \tag{2.10}$$

The inequality (2.9) then corresponds to the square matrices R_f and R_e satisfying the properties

$$R_f R_e^T = R_e R_f^T \geq 0, \tag{2.11}$$

together with the dimensionality condition

$$\operatorname{rank}[R_f \mid R_e] = \dim f_R. \tag{2.12}$$

[3]Throughout we adopt the convention that $\frac{\partial H}{\partial x}(x)$ denotes the *column* vector of partial derivatives of H.

Indeed, by the dimensionality condition (2.12) and the symmetry (2.11) we can equivalently rewrite the kernel representation (2.10) of $\mathcal{R}$ into an image representation

$$f_R = R_e^T \lambda \quad \text{and} \quad e_R = -R_f^T \lambda. \tag{2.13}$$

That is, any pair (f_R, e_R) satisfying (2.10) can be written into the form (2.13) for a certain λ, and conversely any (f_R, e_R) for which there exists λ such that (2.13) holds is satisfying (2.10). Hence by (2.11) all f_R, e_R satisfying the resistive relation are such that

$$e_R^T f_R = -\left(R_f^T \lambda\right)^T R_e^T \lambda = -\lambda^T R_f R_e^T \lambda \leq 0. \tag{2.14}$$

Without the presence of additional external ports, the Dirac structure of the port-Hamiltonian system satisfies the power balance

$$e_S^T f_S + e_R^T f_R = 0 \tag{2.15}$$

which leads by substitution of equations (2.7) and (2.14) to

$$\frac{d}{dt} H = -e_S^T f_S = e_R^T f_R \leq 0. \tag{2.16}$$

An important special case of resistive relations between $f_R \in \mathbb{R}^{m_r}$ and $e_R \in \mathbb{R}^{m_r}$ occurs when the resistive relations can be expressed as an *input–output* mapping

$$f_R = -F(e_R), \tag{2.17}$$

where the resistive characteristic[4] $F : \mathbb{R}^{m_r} \to \mathbb{R}^{m_r}$ satisfies

$$e_R^T F(e_R) \geq 0, \quad e_R \in \mathbb{R}^{m_r}. \tag{2.18}$$

For *linear* resistive elements, (2.17) specializes to

$$f_R = -\tilde{R} e_R \tag{2.19}$$

for some positive semi-definite symmetric matrix $\tilde{R} = \tilde{R}^T \geq 0$.

2.4 External Ports

Now, let us consider in more detail the *external* ports to the system. We shall distinguish between two types of external port. One is the *control port* $\mathcal{C}$, with port variables (f_C, e_C), which are the port variables which are accessible for controller

[4] In many cases, F will be derivable from a so-called *Rayleigh dissipation function* $\mathfrak{R} : \mathbb{R}^{m_r} \to \mathbb{R}$, in the sense that $F(e_R) = \frac{\partial \mathfrak{R}}{\partial e_R}(e_R)$.

action. The other type of external port is the *interaction port* $\mathcal{I}$, which denotes the interaction of the port-Hamiltonian system with its environment. The port variables corresponding to the interaction port are denoted by (f_I, e_I). Taking both the external ports into account the power-balance (2.15) extends to

$$e_S^T f_S + e_R^T f_R + e_C^T f_C + e_I^T f_I = 0 \tag{2.20}$$

whereby (2.16) extends to

$$\frac{d}{dt} H = e_R^T f_R + e_C^T f_C + e_I^T f_I. \tag{2.21}$$

2.5 Resulting Port-Hamiltonian Dynamics

The port-Hamiltonian system with state space $\mathcal{X}$, Hamiltonian H corresponding to the energy storage port $\mathcal{S}$, resistive port $\mathcal{R}$ with relations (2.8), control port $\mathcal{C}$, interconnection port $\mathcal{I}$, and total Dirac structure $\mathcal{D}$ will be succinctly denoted by $\Sigma = (\mathcal{X}, H, \mathcal{R}, \mathcal{C}, \mathcal{I}, \mathcal{D})$. The dynamics of the port-Hamiltonian system is specified by considering the constraints on the various port variables imposed by the Dirac structure, that is,

$$(f_S, e_S, f_R, e_R, f_C, e_C, f_I, e_I) \in \mathcal{D}$$

and to substitute in these relations the equalities $f_S = -\dot{x}$ and $e_S = \frac{\partial H}{\partial x}(x)$. This leads to the implicitly defined dynamics

$$\left(-\dot{x}(t), \frac{\partial H}{\partial x}\big(x(t)\big), f_R(t), e_R(t), f_C(t), e_C(t), f_I(t), e_I(t)\right) \in \mathcal{D} \tag{2.22}$$

with $f_R(t), e_R(t)$ satisfying for all t the resistive relation

$$R\big(f_R(t), e_R(t)\big) = 0. \tag{2.23}$$

In many cases of interest, Eqs. (2.22) will *constrain* the state x. Thus in a coordinate representation (as will be treated in detail in the next section), port-Hamiltonian systems generally will consist of a mixed set of *differential* and *algebraic* equations (DAEs).

Example 2.1 (General RLC-circuits) We start by showing how Kirchhoff's laws define a Dirac structure on the space of currents and voltages of any electrical circuit. Consider a circuit-graph with m edges and n vertices, where the current through the ith edge is denoted by I_i and the voltage across the ith edge is V_i. Collect the currents in a single column vector I (of dimension m) and the voltages in an m-dimensional column vector V. Then Kirchhoff's *current* laws can be written as

$$\mathcal{B}I = 0, \tag{2.24}$$

where $\mathcal{B}$ is the $n \times m$ incidence matrix of the graph. Dually, Kirchhoff's *voltage* laws can be written as follows: all allowed vectors of voltages V in the circuit are given as

$$V = \mathcal{B}^T \lambda, \quad \lambda \in \mathbb{R}^n. \tag{2.25}$$

It is immediately seen that the total space of currents and voltages allowed by Kirchhoff's current and voltage laws

$$\mathcal{D} = \left\{ (I, V) \mid \mathcal{B} I = 0, \exists \lambda \text{ s.t. } V = \mathcal{B}^T \lambda \right\} \tag{2.26}$$

defines a Dirac structure. In particular $(V^a)^T I^b + (V^b)^T I^a = 0$ for all pairs $(I^a, V^a), (I^b, V^b) \in \mathcal{D}$. By taking V^a, I^b equal to zero, we obtain $(V^b)^T I^a = 0$ for all I^a satisfying (2.24) and all V^b satisfying (2.25), which amounts to *Tellegen's theorem*. Hence for an arbitrary RLC-circuit Kirchhoff's current and voltage laws take the form [35]

$$\begin{aligned} B_L I_L + B_C I_C + B_R I_R = 0&, \\ V_L = B_L^T \lambda&, \\ V_C = B_C^T \lambda&, \\ V_P = B_P^T \lambda& \end{aligned} \tag{2.27}$$

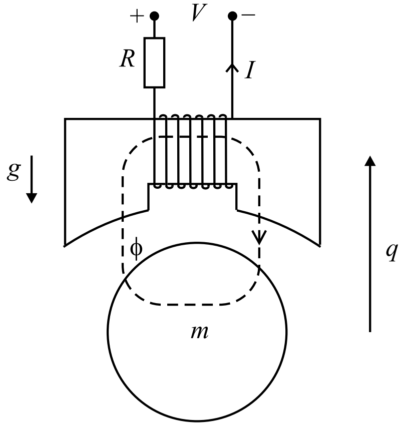
with $[B_L \;\; B_C \;\; B_R]$ denoting the incidence matrix of the circuit graph, where the edges have been ordered according to being associated to the inductors, capacitors, and resistors. Furthermore, I_L, I_C and I_R denote the currents through, respectively, the inductors, capacitors and resistors. Likewise, V_L, V_C and V_R denote the voltages across the inductors, capacitors and terminals. Kirchhoff's current and voltage laws define a Dirac structure $\mathcal{D}$ between the flows and efforts

$$\begin{aligned} f_S &= (I_C, V_L, I_R) = (-\dot{Q}, -\dot{\phi}, I_R), \\ e_S &= (V_C, I_L, V_R) = \left(\frac{\partial H}{\partial Q}, \frac{\partial H}{\partial \phi}, V_R \right) \end{aligned}$$

with Hamiltonian $H(Q, \phi)$ equal to the total energy. This leads to the port-Hamiltonian differential-algebraic system

$$\begin{aligned} -\dot{\phi} &= B_L^T \lambda, \\ \frac{\partial H}{\partial Q} &= B_C^T \lambda, \\ V_R &= B_R^T \lambda, \\ 0 &= B_L \frac{\partial H}{\partial \phi} - B_C \dot{Q} + B_R I_R, \\ V_R &= -R I_R \end{aligned}$$

Fig. 2 Magnetically levitated ball



with state vector $x = (Q, \phi)$, where R is a positive diagonal matrix (Ohm's law describing the linear resistors). The equations can be easily extended to cover voltage or current sources, external ports or terminals [40].

Example 2.2 (Electro-mechanical system) Consider the dynamics of an iron ball in the magnetic field of a controlled inductor, as shown in Fig. 2. The port-Hamiltonian description of this system (with q the height of the ball, p the vertical momentum, and φ the magnetic flux of the inductor) is given as

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -R \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial \varphi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \\ I &= \frac{\partial H}{\partial \varphi}. \end{aligned} \tag{2.28}$$

This is an example of a system where the *coupling* between two different physical domains (mechanical and magnetic) takes place via the Hamiltonian; in this case

$$H(q, p, \varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2k_1(1 - \frac{q}{k_2})},$$

where the last term depends both on the magnetic variable φ and the mechanical variable q.

2.6 *Port-Hamiltonian Systems and Passivity*

By the power-preserving property of the Dirac structure

$$e_S^T f_S + e_R^T f_R + e_C^T f_C + e_I^T f_I = 0.$$

Hence the port-Hamiltonian dynamics defined in (2.22) satisfies

$$\begin{aligned}\frac{dH}{dt} &= \frac{\partial^T H}{\partial x}(x)\dot{x} = -e_S^T f_S \\ &= e_R^T f_R + e_C^T f_C + e_I^T f_I \le e_C^T f_C + e_I^T f_I,\end{aligned} \tag{2.29}$$

where the last inequality follows from the energy-dissipating property (2.9) of the resistive relation between f_R and e_R. Thus, whenever H is bounded from below (and thus can be changed into a non-negative function by adding a constant), the port-Hamiltonian system is *passive*. Furthermore, notice that in fact we may relax the requirement of H being bounded from below on the whole state space $\mathcal{X}$ by requiring that H is bounded from below on the part of $\mathcal{X}$ satisfying the algebraic constraints present in the system.

2.7 Modulated Dirac Structures and Port-Hamiltonian Systems on Manifolds

For many systems, especially those with 3-D mechanical components, the Dirac structure is actually *modulated* by the state variables. Furthermore, the state space $\mathcal{X}$ is a *manifold* and the flow vector $f_S = -\dot{x}$ corresponding to energy-storage are in the tangent space $T_x\mathcal{X}$ at the state $x \in \mathcal{X}$, while the effort vector e_S is in the co-tangent space $T_x^*\mathcal{X}$. The modulation of the Dirac structure is often intimately related to the underlying geometry of the system.

Example 2.3 (Spinning rigid body) Consider a rigid body spinning around its center of mass in the absence of gravity. The energy variables are the three components of the body angular momentum p along the three principal axes: $p = (p_x, p_y, p_z)$, and the energy is the kinetic energy

$$H(p) = \frac{1}{2}\left(\frac{p_x^2}{I_x} + \frac{p_y^2}{I_y} + \frac{p_z^2}{I_z}\right),$$

where I_x, I_y, I_z are the principal moments of inertia. Euler's equations describing the dynamics are

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -p_z & p_y \\ p_z & 0 & -p_x \\ -p_y & p_x & 0 \end{bmatrix}}_{J(p)} \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_y} \\ \frac{\partial H}{\partial p_z} \end{bmatrix}. \tag{2.30}$$

The Dirac structure is given as the graph of the skew-symmetric matrix $J(p)$, and thus defines a subspace which is modulated by the state variables p.

This motivates to extend the definition of a *constant* Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ (with $\mathcal{F}$ a linear space) as given before in Proposition 2.1 to *Dirac structures on manifolds*. Simply put, a Dirac structure on a manifold $\mathcal{X}$ is pointwise (that is, for every $x \in \mathcal{X}$) a constant Dirac structure $\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$.

Definition 2.2 Let $\mathcal{X}$ be a manifold. A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is a vector sub-bundle of the Whitney sum[5] $T\mathcal{X} \oplus T^*\mathcal{X}$ such that

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X}$$

is for every $x \in \mathcal{X}$ a constant Dirac structure as before.

If, next to the energy storage port, there are additional ports (such as resistive, control or interaction ports) with port variables $f \in \mathcal{F}$ and $e \in \mathcal{F}^*$, then a modulated Dirac structure is pointwise specified by a constant Dirac structure

$$\mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F} \times \mathcal{F}^*. \tag{2.31}$$

2.7.1 Kinematic Constraints in Mechanics

Modulated Dirac structures often arise as a result of *ideal constraints* imposed on the generalized velocities of the mechanical system by its environment, called *kinematic constraints*. In many cases, these constraints will be configuration dependent, leading to a Dirac structure modulated by the configuration variables.

Consider a mechanical system with n degrees of freedom, locally described by n configuration variables $q = (q_1, \dots, q_n)$. Expressing the kinetic energy as $\frac{1}{2}\dot{q}^T M(q)\dot{q}$, with $M(q) > 0$ being the generalized mass matrix, we define in the usual way the Lagrangian function $L(q, \dot{q})$ as the *difference* of kinetic energy and potential energy $P(q)$, i.e.,

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - P(q). \tag{2.32}$$

Suppose now that there are constraints on the generalized velocities $\dot{q}$, described as

$$A^T(q)\dot{q} = 0 \tag{2.33}$$

with $A(q)$ an $n \times k$ matrix of rank k everywhere (that is, there are k independent kinematic constraints). Classically, the constraints (2.33) are called *holonomic* if it is possible to find new configuration coordinates $\overline{q} = (\overline{q}_1, \dots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0 \tag{2.34}$$

[5]The Whitney sum of two vector bundles with the same base space is defined as the vector bundle whose fiber above each element of this common base space is the product of the fibers of each individual vector bundle.

in which case one may eliminate the configuration variables $\overline{q}_{n-k+1}, \dots, \overline{q}_n$, since the kinematic constraints (2.34) are equivalent to the *geometric* constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \quad \dots, \quad \overline{q}_n = c_n \tag{2.35}$$

for certain constants $c_{n-k+1}, \dots, c_n$ determined by the initial conditions. Then the system reduces to an *unconstrained* system in the $(n-k)$ remaining configuration coordinates $(\overline{q}_1, \dots, \overline{q}_{n-k})$. If it is *not* possible to find coordinates $\overline{q}$ such that (2.34) holds (that is, if we are not able to *integrate* the kinematic constraints as above), then the constraints are called *nonholonomic*.

The equations of motion for a mechanical system with Lagrangian $L(q, \dot{q})$ and constraints (2.33) are given by the constrained Euler–Lagrange equations (derived from d'Alembert's principle) [22]

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} &= A(q)\lambda + B(q)u, \quad \lambda \in \mathbb{R}^k, u \in \mathbb{R}^m, \\ A^T(q)\dot{q} &= 0, \end{aligned} \tag{2.36}$$

where $B(q)u$ are the external forces (controls) applied to the system, for some $n \times m$ matrix $B(q)$, while $A(q)\lambda$ are the *constraint forces*. The Lagrange multipliers $\lambda(t)$ at any time t are uniquely determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all t. Note that (2.36) defines a set of second-order differential-algebraic equations in the configuration variables q. Defining the generalized momenta

$$p = \frac{\partial L}{\partial \dot{q}} = M(q)\dot{q} \tag{2.37}$$

the constrained Euler–Lagrange equations (2.36) transform into the *constrained Hamiltonian equations*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)u, \\ y &= B^T(q)\frac{\partial H}{\partial p}(q, p), \\ 0 &= A^T(q)\frac{\partial H}{\partial p}(q, p) \end{aligned} \tag{2.38}$$

with $H(q, p) = \frac{1}{2}p^T M^{-1}(q)p + P(q)$ the total energy. This defines a port-Hamiltonian differential-algebraic system with respect to the modulated Dirac structure

$$\begin{aligned} \mathcal{D} = \Big\{ &(f_S, e_S, f_C, e_C) \mid 0 = A^T(q)e_S,\ e_C = B^T(q)e_S, \exists \lambda \in \mathbb{R}^k \text{ s.t.} \\ &- f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c, \lambda \in \mathbb{R}^k \Big\}. \end{aligned} \tag{2.39}$$

Example 2.4 (Rolling euro) Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, φ denotes the heading angle, and θ the angle of Queen Beatrix' head.[6] With all constants set to unity, the constrained Euler–Lagrangian equations of motion are

$$\begin{aligned}\ddot{x} &= \lambda_1,\\ \ddot{y} &= \lambda_2,\\ \ddot{\theta} &= -\lambda_1\cos\varphi - \lambda_2\sin\varphi + u_1,\\ \ddot{\varphi} &= u_2\end{aligned} \tag{2.40}$$

with u_1 the control torque about the rolling axis, and u_2 the control torque about the vertical axis. The total energy is $H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$. The rolling constraints are the nonholonomic kinematic constraints $\dot{x} = \dot{\theta}\cos\varphi$ and $\dot{y} = \dot{\theta}\sin\varphi$, i.e., rolling without slipping, which can be written in the form (2.33) by defining

$$A^T(x, y, \theta, \phi) = \begin{bmatrix} 1 & 0 & -\cos\phi & 0 \\ 0 & 1 & -\sin\phi & 0 \end{bmatrix}.$$

2.8 Input–State–Output Port-Hamiltonian Systems

An important subclass of port-Hamiltonian systems is the class of *input–state–output port-Hamiltonian systems*, where there are no algebraic constraints on the state space variables, and the flow and effort variables of the resistive, control and interaction port can be split into conjugated input–output pairs.

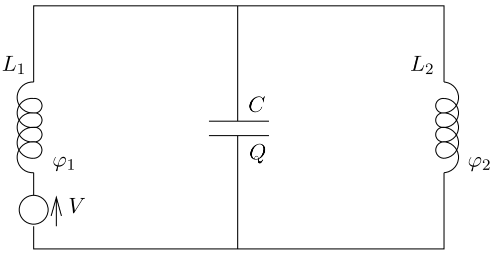
Input–state–output port-Hamiltonian systems are defined as dynamical systems of the following form:

$$\Sigma: \begin{aligned} \dot{x} &= \bigl[J(x) - R(x)\bigr]\frac{\partial H}{\partial x}(x) + g(x)u + k(x)d, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad x \in \mathcal{X}, \\ z &= k^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{2.41}$$

where (u, y) are the input–output pairs corresponding to the control port $\mathcal{C}$, while (d, z) denote the input–output pairs of the interaction port $\mathcal{I}$. Note that $y^T u$ and $z^T d$ equal the power corresponding to the control, respectively, interaction port. Here the matrix $J(x)$ is skew-symmetric, that is, $J(x) = -J^T(x)$. The matrix $R(x) = R^T(x) \geq 0$ specifies the resistive structure. From a resistive port point of view, it is given as $R(x) = g_R^T(x)\tilde{R}(x)g_R(x)$ for some linear resistive relation $f_R = -\tilde{R}e_R$

[6] On the Dutch version of the euro.

Fig. 3 Controlled LC-circuit



with $\tilde{R}(x) = \tilde{R}^T(x) \geq 0$ and g_R representing the input matrix corresponding to the resistive port.

The underlying Dirac structure of the system is then given by the graph of the skew-symmetric linear map

$$\begin{bmatrix} -J(x) & -g_R(x) & -g(x) & -k(x) \\ g_R^T(x) & 0 & 0 & 0 \\ g^T(x) & 0 & 0 & 0 \\ k^T(x) & 0 & 0 & 0 \end{bmatrix}. \tag{2.42}$$

In general, the Dirac structure defined as the graph of the mapping (2.42) is a *modulated Dirac structure* since the matrices J, g_R, g, and k may all depend on the energy variables x.

Example 2.5 (LC-circuit with independent storage elements) Consider a controlled LC-circuit (see Fig. 3) consisting of two inductors with magnetic energies $H_1(\varphi_1)$ and $H_2(\varphi_2)$ (φ_1 and φ_2 being the magnetic flux linkages), and a capacitor with electric energy $H_3(Q)$ (Q being the charge). If the elements are linear, then

$$H_1(\varphi_1) = \frac{1}{2L_1}\varphi_1^2, \qquad H_2(\varphi_2) = \frac{1}{2L_2}\varphi_2^2, \qquad H_3(Q) = \frac{1}{2C}Q^2.$$

Furthermore, let $V = u$ denote a voltage source. Using Kirchhoff's laws, one immediately arrives at the input–state–output port-Hamiltonian system

$$\begin{bmatrix} \dot{Q} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{J} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial \varphi_1} \\ \frac{\partial H}{\partial \varphi_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u,$$

$$y = \frac{\partial H}{\partial \varphi_1} \quad (= \text{current through first inductor})$$

with $H(Q, \varphi_1, \varphi_2) := H_1(\varphi_1) + H_2(\varphi_2) + H_3(Q)$ the total energy. Clearly the matrix J is skew-symmetric. In this way, cf. [20], every LC-circuit with *independent* storage elements can be modeled as an input–state–output port-Hamiltonian system (with respect to a constant Dirac structure).

Input–state–output port-Hamiltonian systems with additional *feed-through terms* are given as (for simplicity we do not take the interaction port into account) [13, 31]

$$\begin{aligned} \dot{x} &= \big[J(x) - R(x)\big]\frac{\partial H}{\partial x}(x) + \big[g(x) - P(x)\big]u, \\ y &= \big[g(x) + P(x)\big]^\top \frac{\partial H}{\partial x}(x) + \big[M(x) + S(x)\big]u \end{aligned} \tag{2.43}$$

with the matrices P, R, S satisfying

$$Z = \begin{bmatrix} R(x) & P(x) \\ P^\top(x) & S(x) \end{bmatrix} \geq 0. \tag{2.44}$$

The relation between u, y and the storage port variables f_S, e_S is in this case given as

$$\begin{bmatrix} f_S \\ y \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & M \end{bmatrix}\begin{bmatrix} e_S \\ u \end{bmatrix} + \begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix}\begin{bmatrix} e_S \\ u \end{bmatrix}. \tag{2.45}$$

It follows that

$$e_S^T f_S + u^T y = \begin{bmatrix} e_S^T & u^T \end{bmatrix}\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix}\begin{bmatrix} e_S \\ u \end{bmatrix} \geq 0$$

and thereby

$$\frac{d}{dt}H(x) = -e_S^T f_S = u^T y - \begin{bmatrix} e_S^T & u^T \end{bmatrix}\begin{bmatrix} R(x) & P(x) \\ P^T(x) & S(x) \end{bmatrix}\begin{bmatrix} e_S \\ u \end{bmatrix} \leq u^T y$$

thus recovering the basic energy balance for port-Hamiltonian systems. Port-Hamiltonian input–state–output systems with feed-through terms readily show up in the modeling of power converters [13], as well as in friction models (see e.g. [16] for a port-Hamiltonian description of the dynamic *LuGre* friction model).

Although the class of input–state–output port-Hamiltonian systems is a very important subclass, it is *not* closed under general power-preserving interconnections. Basically, only negative *feedback* interconnections of input–state–output port-Hamiltonian systems will result in another input–state–output port-Hamiltonian system, while otherwise algebraic constraints will arise, leading to port-Hamiltonian differential-algebraic systems. On the other hand, input–state–output port-Hamiltonian systems may arise from *solving the algebraic constraints* in a port-Hamiltonian differential-algebraic system. This will be discussed in Sect. 4.

3 Representations of Dirac Structures and Port-Hamiltonian Systems

In the preceding section, we have provided the geometric definition of a port-Hamiltonian system containing three main ingredients. First, the energy storage which is represented by a state space manifold $\mathcal{X}$ specifying the space of state variables together with a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$ defining the energy. Secondly, there are the static resistive elements, and thirdly there is the Dirac structure linking all the flows and efforts associated to the energy storage, resistive elements, and the external ports (e.g. control and interaction ports) in a power-conserving manner. This, together with the general formulation (2.2) of a Dirac structure, leads to a completely *coordinate-free* definition of a port-Hamiltonian system, because of three reasons: (a) we do not start with coordinates for the state space manifold $\mathcal{X}$, (b) we define the Dirac structure as a *subspace* instead of a set of equations, (c) the resistive relations are defined as a subspace constraining the port variables (f_R, e_R).

This geometric, coordinate-free, point of view has a number of advantages. It allows one to reason about port-Hamiltonian systems without the need to choose specific representations. For example, in Sect. 4 we will see that a number of properties of the port-Hamiltonian system, such as passivity, stability, existence of conserved quantities and algebraic constraints, can be analyzed without the need for choosing coordinates and equations. On the other hand, for many other purposes, including simulation, the need for a representation in coordinates of the port-Hamiltonian system is indispensable, in which case the emphasis shifts to finding the most convenient coordinate representation for the purpose at hand. The examples of the previous section have already been presented in this way. In this section, we will briefly discuss a number of possible representations of port-Hamiltonian systems. It will turn out that the key issue is the representation of the Dirac structure.

3.1 Representations of Dirac Structures

Dirac structures admit different *representations*. Here we just list a number of them. See e.g. [7–9, 15] for more information.

3.1.1 Kernel and Image Representation

Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented in *kernel representation* as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0 \right\} \tag{3.1}$$

for linear maps $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying

$$\begin{aligned} &\text{(i)} \quad EF^* + FE^* = 0, \\ &\text{(ii)} \quad \operatorname{rank}(F + E) = \dim \mathcal{F}, \end{aligned} \tag{3.2}$$

where $\mathcal{V}$ is a linear space with the same dimension as $\mathcal{F}$, and where $F^* : \mathcal{V}^* \to \mathcal{F}^*$ and $E^* : \mathcal{V}^* \to \mathcal{F}^{**} = \mathcal{F}$ are the adjoint maps of F and E, respectively.

It follows from (3.2) that $\mathcal{D}$ can be also written in *image representation* as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = E^*\lambda, e = F^*\lambda, \lambda \in \mathcal{V}^*\}. \tag{3.3}$$

Sometimes it will be useful to relax this choice of the linear mappings F and E by allowing $\mathcal{V}$ to be a linear space of dimension greater than the dimension of $\mathcal{F}$. In this case we shall speak of *relaxed* kernel and image representations.

Matrix kernel and image representations are obtained by choosing linear coordinates for $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{V}$. Indeed, take any basis $f_1, \dots, f_n$ for $\mathcal{F}$ and the *dual basis* $e_1 = f_1^*, \dots, e_n = f_n^*$ for $\mathcal{F}^*$, where $\dim \mathcal{F} = n$. Furthermore, take any set of linear coordinates for $\mathcal{V}$. Then the linear maps F and E are represented by $n \times n$ matrices F and E satisfying

$$\begin{aligned} &\text{(i)} \quad EF^T + FE^T = 0, \\ &\text{(ii)} \quad \operatorname{rank}[F \mid E] = \dim \mathcal{F}. \end{aligned} \tag{3.4}$$

In the case of a relaxed kernel and image representation F and E will be $n' \times n$ matrices with $n' > n$.

A (constructive) proof for the existence of matrix kernel and image representations can be given as follows. Consider a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ where we have chosen linear coordinates for $\mathcal{F}$, $\mathcal{F}^*$ and $\mathcal{V}$. In particular, choose any basis $f_1, \dots, f_n$ for $\mathcal{F}$ and the *dual basis* $e_1 = f_1^*, \dots, e_n = f_n^*$ for $\mathcal{F}^*$, where dim $\mathcal{F} = n$. Since $\mathcal{D}$ is a subspace of $\mathcal{F} \times \mathcal{F}^*$ it follows that there exist square $n \times n$ matrices F and E such that

$$\mathcal{D} = \operatorname{im} \begin{bmatrix} E^T \\ F^T \end{bmatrix},$$

where $\operatorname{rank}[F \mid E] = \dim \mathcal{F}$. Thus any element $(f, e) \in \mathcal{D}$ can be written as

$$f = E^T\lambda, \qquad e = F^T\lambda$$

for some $\lambda \in \mathbb{R}^n$. Since $e^T f = 0$ for every $(f, e) \in \mathcal{D}$ this implies that

$$\lambda^T F E^T \lambda = 0$$

for every λ, or equivalently, $EF^T + FE^T = 0$. Conversely, any subspace $\mathcal{D}$ given by (3.4) is a Dirac structure, since it satisfies $e^T f = 0$ for every $(f, e) \in \mathcal{D}$ and its dimension is equal to n.

Remark 3.1 A special type of kernel representation occurs if not only $EF^* + FE^* = 0$ but even more $FE^* = 0$. This implies that $\operatorname{im} E^* \subset \ker F$. Since it follows from the kernel/image representation of any Dirac structure that $\ker F \subset \operatorname{im} E^*$, we thus obtain $\operatorname{im} E^* = \ker F$. Hence the Dirac structure is the product of the subspace $\ker F \subset \mathcal{F}$ and the subspace $\ker F^\perp = \ker E \subset \mathcal{F}^*$. We have already encountered this special type of Dirac structure in the case of Kirchhoff's current and voltage laws.

3.1.2 Constrained Input–Output Representation

Every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda, G^T e = 0 \right\} \tag{3.5}$$

for a skew-symmetric mapping $J : \mathcal{F} \to \mathcal{F}^*$ and a linear mapping G such that $\operatorname{im} G = \{f \mid (f, 0) \in \mathcal{D}\}$. Furthermore, $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$.

The proof that (3.5) defines a Dirac structure is straightforward. Indeed, for any (f, e) given as in (3.5) we have

$$e^T f = e^T (Je + G\lambda) = e^T Je + e^T G\lambda = 0$$

by skew-symmetry of J and $G^T e = 0$. Furthermore, let rank $G = r \leq n$. If $r = 0$ (or equivalently $G = 0$) then the dimension of $\mathcal{D}$ is clearly n since in that case it is the graph of the mapping J. For $r \neq 0$ the freedom in e will be reduced by dimension r, while at the other hand the freedom in f will be increased by dimension r (because of the term $G\lambda$). For showing that every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented in this way we refer to [8].

3.1.3 Hybrid Input–Output Representation

Let $\mathcal{D}$ be given in matrix kernel representation by square matrices E and F as in 1. Suppose rank $F = m (\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \mid F_2]$ and correspondingly $E = [E_1 \mid E_2]$,

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{and} \quad e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Then it can be shown [4] that the matrix $[F_1 \mid E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \left(\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right) \middle| \begin{bmatrix} f_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \right\} \tag{3.6}$$

with $J := -[F_1 \mid E_2]^{-1} [F_2 \mid E_1]$ skew-symmetric.

It follows that any Dirac structure can be written as the graph of a skew-symmetric map. The vectors e_1, f_2 can be regarded as *input* vectors, while the complementary vectors f_1, e_2 can be seen as *output* vectors.[7]

[7] This is very much like the multi-port description of a passive linear circuit, where it is known that although it is *not* always possible to describe the port as an admittance or as an impedance, it *is* possible to describe it as a hybrid admittance/impedance transfer matrix, for a suitable selection of input voltages and currents and complementary output currents and voltages [3].

3.1.4 Canonical Coordinate Representation

For any constant Dirac structure there exist a basis for $\mathcal{F}$ and dual basis for $\mathcal{F}^*$, such that the vector (f, e), when partitioned as $(f, e) = (f_q, f_p, f_r, f_s, e_q, e_p, e_r, e_s)$, is in $\mathcal{D}$ if and only if

$$\begin{aligned} f_q &= -e_p, \\ f_p &= e_q, \\ f_r &= 0, \\ e_s &= 0. \end{aligned} \tag{3.7}$$

For a proof we refer to [8]. For a modulated Dirac structure the existence of canonical coordinates requires an additional *integrability condition*, for which we refer to Sect. 7. The representation of a Dirac structure by canonical coordinates is very close to the classical Hamiltonian equations of motion.

In [4, 9, 32] it is shown how one may convert any of the above representations into any other. An easy transformation that will be used frequently in the sequel is the transformation of the *constrained input–output* representation into the *kernel* representation. Consider the Dirac structure $\mathcal{D}$ given in constrained input–output representation by (3.5). Construct a linear mapping $G^\perp$ of maximal rank satisfying $G^\perp G = 0$. Then, pre-multiplying the first equation of (3.5) by $G^\perp$, one eliminates the Lagrange multipliers λ and obtains

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{F}^* \mid G^\perp f = G^\perp J e, G^T e = 0 \right\} \tag{3.8}$$

which is easily seen to lead to a kernel representation. Indeed,

$$F = \begin{bmatrix} -G^\perp \\ 0 \end{bmatrix}, \qquad E = \begin{bmatrix} G^\perp J \\ G^T \end{bmatrix}$$

defines a kernel representation.

3.2 Representations of Port-Hamiltonian Systems

Coordinate representations of the port-Hamiltonian system (2.22) are obtained by choosing a specific coordinate representation of the Dirac structure $\mathcal{D}$. For example, if $\mathcal{D}$ is given in matrix kernel representation

$$\mathcal{D} = \left\{ (f_S, e_S, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F} \times \mathcal{F}^* \mid F_S f_S + E_S e_S + F f + E e = 0 \right\} \tag{3.9}$$

with

$$\begin{aligned} &\text{(i)} \quad E_S F_S^T + F_S E_S^T + E F^T + F E^T = 0, \\ &\text{(ii)} \quad \operatorname{rank}[F_S \mid E_S \mid F \mid E] = \dim(\mathcal{X} \times \mathcal{F}) \end{aligned} \tag{3.10}$$

then the port-Hamiltonian system is given by the set of equations

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}\big(x(t)\big) + Ff(t) + Ee(t). \tag{3.11}$$

Note that, in general, (3.11) consists of differential equations *and* algebraic equations in the state variables x (DAEs), together with equations relating the state variables and their time-derivatives to the external port variables (f, e).

Example 3.1 (1-D mechanical systems) Consider a *spring* with elongation q and energy function $H_s(q)$, which for a linear spring is given as $H_s(q) = \frac{1}{2}kq^2$. Let (v_s, F_s) represent the external port through which energy can be exchanged with the spring, where v_s is equal to the rate of elongation (velocity) and F_s is equal to the elastic force. This port-Hamiltonian system (without dissipation) can be written in kernel representation as

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{q} \\ v_s \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} kq \\ F_s \end{bmatrix} = 0. \tag{3.12}$$

Similarly we can model a *moving mass* m with scalar momentum p and kinetic energy $H_m(p) = \frac{1}{2m}p^2$ as the port-Hamiltonian system

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{p} \\ F_m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{p}{m} \\ v_m \end{bmatrix} = 0, \tag{3.13}$$

where (F_m, v_m) are, respectively, the external force exerted on the mass and the velocity of the mass. The mass and the spring can be *interconnected* to each other using the symplectic gyrator

$$\begin{bmatrix} v_s \\ F_m \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} F_s \\ v_m \end{bmatrix}. \tag{3.14}$$

Collecting all equations we have obtained a port-Hamiltonian system with energy variables $x = (q, p)$, total energy $H(q, p) = H_s(q) + H_m(p)$ and with interconnected port variables (v_s, F_s, F_m, v_m). After elimination of the interconnection variables (v_s, F_s, F_m, v_m) one obtains the port-Hamiltonian system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{q} \\ -\dot{p} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} kq \\ \frac{p}{m} \end{bmatrix} = 0 \tag{3.15}$$

which is the ubiquitous mass–spring system. Note that the Dirac structure of this mass–spring system is derived from the Dirac structures of the spring system and the mass system together with their interconnection by means of the symplectic gyrator (which itself defines a Dirac structure). The systematic derivation of the resulting interconnected Dirac structure will be studied in Sect. 6.

In case of a Dirac structure modulated by the state variables x and the state space $\mathcal{X}$ being a manifold, the flows $f_S = -\dot{x}$ are elements of the tangent space $T_x\mathcal{X}$

at the state $x \in \mathcal{X}$, and the efforts e_S are elements of the co-tangent space $T_x^*\mathcal{X}$. Still, locally on $\mathcal{X}$, we obtain the kernel representation (3.11) for the resulting port-Hamiltonian system, but now the matrices F_S, E_S, F and E will depend on x.

The important special case of input–state–output port-Hamiltonian systems as treated before

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \qquad x \in \mathcal{X},$$

can be interpreted as arising from a hybrid input–output representation of the Dirac structure (from e_S, u to f_S, y). If the matrices J, g depend on the energy variables x, then this is again a modulated Dirac structure.

In general, by a combination of the hybrid representation and the constrained input–output representation, it can be shown [9] that, locally, any port-Hamiltonian system can be represented in the following way:

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)u + b(x)\lambda, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad x \in \mathcal{X}, \\ 0 &= b^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{3.16}$$

where $y^T u$ denotes the power at the external port, and $0 = b^T(x)\frac{\partial H}{\partial x}(x)$ represents the algebraic constraints.[8] Note that the Hamiltonian formulation of mechanical systems with kinematic constraints, as discussed in Sect. 2.7.1, leads to this form; see in particular the constrained Hamiltonian equations (2.38) and its Lagrangian counterpart (2.36).

Example 3.2 (Coupled masses—Internal constraints) Consider two point masses m_1 and m_2 that are rigidly linked to each other, moving in one dimension. When decoupled, the masses are described by the input–state–output port-Hamiltonian systems

$$\begin{aligned} \dot{p}_i &= F_i, \\ v_i &= \frac{p_i}{m_i}, \end{aligned} \qquad i = 1, 2 \tag{3.17}$$

with F_i denoting the force exerted on mass m_i. Rigid coupling of the two masses is achieved by setting

$$v_1 = v_2, \qquad F_1 = -F_2. \tag{3.18}$$

[8]The equality $0 = b^T(x)\frac{\partial H}{\partial x}(x)$ also has the interpretation (well-known in a mechanical system context) that the constraint input $b(x)\lambda$ is 'workless'; i.e., the evolution of the value of the Hamiltonian H is not affected by this term.

This leads to the port-Hamiltonian differential-algebraic system

$$
\begin{aligned}
\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \lambda, \\
0 &= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{p_1}{m_1} \\ \frac{p_2}{m_2} \end{bmatrix}
\end{aligned}
\tag{3.19}
$$

where $\lambda = F_1 = -F_2$ now denotes the *internal* constraint force. The resulting interconnected system no longer has external ports. On the other hand, external ports for the interconnected system can be included by either extending (3.17) to

$$
\begin{aligned}
\dot{p}_i &= F_i + F_i^{\text{ext}}, \\
v_i &= \frac{p_i}{m_i}, \qquad\qquad i = 1, 2 \\
v_i^{\text{ext}} &= \frac{p_i}{m_i},
\end{aligned}
\tag{3.20}
$$

with F_i^{ext} and v_i^{ext} denoting the external forces and velocities, or by modifying the interconnection constraints (3.18) to e.g. $F_1 + F_2 + F^{\text{ext}} = 0$ and $v_1 = v_2 = v^{\text{ext}}$, with F^{ext} and v^{ext} denoting the external force exerted on the coupled masses, respectively the velocity of the coupled masses.

Remark 3.2 Note that in the above port-Hamiltonian description of the two coupled masses the *position* variables $q_i, i = 1, 2$, of the two masses do not come into play, while the interconnection is *not* described by the alternative formulation $q_1 = q_2$, $F_1 = -F_2$, but instead by $v_1 = v_2$, $F_1 = -F_2$. The positions $q_i, i = 1, 2$, *can* be included, albeit somewhat redundantly, by extending the port-Hamiltonian descriptions $\dot{p}_i = F_i, v_i = \frac{p_i}{m_i}$ of the two masses to the input–state–output port-Hamiltonian systems

$$
\begin{aligned}
\dot{q}_i &= \frac{p_i}{m_i}, \\
\dot{p}_i &= F_i, \\
v_i &= \frac{p_i}{m_i}
\end{aligned}
$$

with Hamiltonians $H_i(q_i, p_i) = \frac{2m_i}{p_i^2}$ (not depending on q_i!). Imposing again the ‘port-Hamiltonian’ interconnection constraints $v_1 = v_2$, $F_1 = -F_2$ this leads to a total system having as *conserved quantity* (Casimir) $q_1 - q_2$. Thus the fact that $v_1 = v_2$ implies $q_1 = q_2$ only up to a constant (since this constant disappears in differentiation) is reflected in the initial condition of the extended total system.

Note furthermore that specifying constraints as constraints on the velocities is in line with the use of kinematic constraints in mechanical systems. In general, such kinematic constraints can be *integrated* to geometric (position) constraints if integrability conditions are satisfied (such as in the simple case $v_1 = v_2$); otherwise the kinematic constraints are called *nonholonomic*.

Since it is easy to eliminate the Lagrange multipliers in any constrained input–output representation of the Dirac structure, cf. (3.8), it is also relatively easy to eliminate the Lagrange multipliers in any port-Hamiltonian system. Indeed, consider the port-Hamiltonian system (3.16). The Lagrange multipliers λ are eliminated by constructing a matrix $b^{\perp}(x)$ of maximal rank such that

$$b^{\perp}(x)b(x) = 0.$$

Then, by pre-multiplication with this matrix $b^{\perp}(x)$, one obtains the equations

$$\begin{aligned} b^{\perp}(x)\dot{x} &= b^{\perp}(x)J(x)\frac{\partial H}{\partial x}(x) + b^{\perp}(x)g(x)u, \\ y &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad x \in \mathcal{X} \\ 0 &= b^T(x)\frac{\partial H}{\partial x}(x), \end{aligned} \tag{3.21}$$

without Lagrange multipliers. This is readily seen to be a kernel representation of the port-Hamiltonian differential-algebraic system.

Example 3.3 (Example 3.2, continued) Consider the system of two coupled masses in Example 3.2. Pre-multiplication of the dynamic equations by the row vector [1 1] yields the equations

$$\dot{p}_1 + \dot{p}_2 = 0, \qquad \frac{p_1}{m_1} - \frac{p_2}{m_2} = 0 \tag{3.22}$$

which constitutes a kernel representation of the port-Hamiltonian DAE system, with matrices

$$F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}. \tag{3.23}$$

Remark 3.3 Consider the representation (3.16) *without* the external port corresponding to the input and output variables u, y. Furthermore assume that $b(x)$ is given as

$$b(x) = -J(x)\frac{\partial \varphi}{\partial x}(x)$$

for a certain mapping $\varphi = (\varphi_1, \ldots, \varphi_m)^T$, where $m = \dim \lambda$, satisfying additionally

$$\frac{\partial \varphi_i}{\partial x}(x)J(x)\frac{\partial \varphi_j}{\partial x}(x) = 0, \quad i, j = 1, \ldots, m.$$

Then the constraints $b^T(x)\frac{\partial H}{\partial x}(x) = 0$ can be rewritten as

$$\frac{d\varphi}{dt} = 0.$$

Replacing the constraints $\frac{d\varphi}{dt} = 0$ by their *time-integrated* version $\varphi(x) = 0$, one obtains the constrained system

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) - J(x)\frac{\partial \varphi}{\partial x}(x)\lambda, \\ 0 &= \varphi(x). \end{aligned} \tag{3.24}$$

If additionally $J(x)$ is the standard symplectic form, then this is the starting point for the classical Dirac theory of Hamiltonian systems with constraints, leading to the concept of the *Dirac brackets* defined by $J(x)$, the Hamiltonian H, together with the constraint functions $\varphi_i(x), i = 1, \dots, m$; see [11, 28].

4 Analysis of Port-Hamiltonian DAEs

In this section we will analyse a number of key aspects of port-Hamiltonian differential-algebraic systems, and discuss how the specific structure yields advantages as compared to general differential-algebraic (DAE) systems.

First of all, we will study the *index* of port-Hamiltonian differential-algebraic systems, and the possibilities to solve the algebraic constraints. Then we will study the algebraic constraints from a geometric point of view, directly based on the Dirac structure of the system. This coordinate-free approach shows how different coordinate representations can be chosen to express the algebraic constraints; each with their own advantages and disadvantages. Next it will be shown how the geometric theory of algebraic constraints can be dualized to the study of Casimir functions (conserved quantities independent of the Hamiltonian). In the last section we will show how the port-Hamiltonian structure naturally leads to stability analysis, using the Hamiltonian (or a combination of the Hamiltonian and a Casimir function) as a Lyapunov function for the differential-algebraic system. Finally, we will provide some observations concerning the (lack of) well-posedness of port-Hamiltonian differential-algebraic systems in case of nonlinear resistive characteristics.

4.1 Analysis and Elimination of Algebraic Constraints

An important problem concerns the possibility to solve for the *algebraic constraints* of a port-Hamiltonian differential-algebraic system. In case of the representation (3.21), the algebraic constraints are explicitly given by

$$0 = b^T(x)\frac{\partial H}{\partial x}(x). \tag{4.1}$$

In general, these equations will constrain the state variables x. However, the precise way this takes place depends on the properties of the Hamiltonian H as well as of the matrix $b(x)$. For example, if the Hamiltonian H is such that its gradient $\frac{\partial H}{\partial x}(x)$ happens to be contained in the kernel of the matrix $b^T(x)$ for all x, then the algebraic constraints (4.1) are automatically satisfied, and actually the state variables are not constrained.

In general, under constant rank assumptions, the set

$$\mathcal{X}_c := \left\{ x \in \mathcal{X} \mid b^T(x) \frac{\partial H}{\partial x}(x) = 0 \right\}$$

will define a submanifold of the total state space $\mathcal{X}$, called the *constrained state space*. In order that this constrained state space qualifies as the state space for a port-Hamiltonian system *without* further algebraic constraints, one needs to be able to restrict the dynamics of the port-Hamiltonian system to the constrained state space. This is always possible under the condition that the matrix

$$b^T(x) \frac{\partial^2 H}{\partial x^2}(x) b(x) \tag{4.2}$$

has full rank. Indeed, under this condition, the differentiated constraint equation

$$0 = \frac{d}{dt}\left(b^T(x) \frac{\partial H}{\partial x}(x) \right) = * + b^T(x) \frac{\partial^2 H}{\partial x^2}(x) b(x) \lambda \tag{4.3}$$

(with $*$ denoting unspecified terms) can always be uniquely solved for λ, leading to a uniquely defined dynamics on the constrained state space $\mathcal{X}_c$. Hence the set of *consistent states* for the port-Hamiltonian differential-algebraic system (the set of initial conditions for which the system has a unique ordinary solution) is equal to the constrained state space $\mathcal{X}_c$. Using terminology from the theory of DAEs, the condition that the matrix in (4.2) has full rank ensures that the *index* of the DAEs specified by the port-Hamiltonian system is equal to one. This can be summarized as

Proposition 4.1 *Consider the port-Hamiltonian differential-algebraic system represented as in* (3.21), *with algebraic constraints* $b^T(x)\frac{\partial H}{\partial x}(x) = 0$. *Suppose that the matrix* $b^T(x)\frac{\partial^2 H}{\partial x^2}(x)b(x)$ *has full rank for all* $x \in \mathcal{X}_c$. *Then the system has index one*, *and the set of consistent states is equal to* $\mathcal{X}_c$.

Hence under the condition that $b^T(x)\frac{\partial^2 H}{\partial x^2}(x)b(x)$ has full rank, then the algebraic constraints $b^T(x)\frac{\partial H}{\partial x}(x) = 0$ can be eliminated, leading to a set of ordinary differential equations defined on the constrained state space $\mathcal{X}_c$. Of course, in the nonlinear case the *explicit* elimination of the algebraic constraints may be difficult, or even impossible.

If the matrix in (4.2) does not have full rank, then the index of the port-Hamiltonian differential-algebraic system will be larger than one, and it will be necessary to further constrain the space $\mathcal{X}_c$ by considering apart from the 'primary' algebraic constraints (4.1), also their (repeated) time-derivatives (sometimes called *secondary constraints*). We refer to [23, 28] for a detailed treatment and conditions for reducing the port-Hamiltonian DAE system to a system without algebraic constraints in case $J(x)$ corresponds to a symplectic structure.

4.1.1 The Linear Index One Case

In the *linear* case the explicit elimination of the algebraic constraints under the assumption that the matrix in (4.2) has full rank proceeds as follows.

Consider a linear port-Hamiltonian system, without energy-dissipation and external ports, given in constrained input–output representation,

$$\begin{aligned} \dot{x} &= JQx + G\lambda, \qquad J = -J^T, \; Q = Q^T, \\ 0 &= G^T Qx, \qquad\quad H(x) = \frac{1}{2}x^T Qx. \end{aligned} \tag{4.4}$$

As before, the constraint forces $G\lambda$ are eliminated by pre-multiplying the first equation by the annihilating matrix $G^\perp$, leading to the DAE system

$$\begin{bmatrix} G^\perp \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} G^\perp J \\ G^T \end{bmatrix} Qx.$$

The corresponding matrix pencil

$$s\begin{bmatrix} G^\perp \\ 0 \end{bmatrix} - \begin{bmatrix} G^\perp JQ \\ G^T Q \end{bmatrix} \tag{4.5}$$

is non-singular if $G^T QG$ has full rank, and in fact, the system has index one *if and only if* $G^T QG$ has full rank.

In this case the algebraic constraints $G^T Qx = 0$ are eliminated as follows. Assume throughout (without loss of generality) that G has full rank. Then define the linear coordinate transformation

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} G^\perp \\ G^T \end{bmatrix} x =: Vx \tag{4.6}$$

leading to the transformed system

$$\begin{aligned} \dot{z} &= \left(VJV^T\right)\left(V^{-T}QV^{-1}\right)(Vx) + VG\lambda = \tilde{J}\tilde{Q}z + \begin{bmatrix} 0 \\ G^T G \end{bmatrix}\lambda, \\ 0 &= G^T Qx = \begin{bmatrix} 0 & G^T G \end{bmatrix}\tilde{Q}z, \end{aligned} \tag{4.7}$$

where $\tilde{J} = VJV^T$, $\tilde{Q} = V^{-T}QV^{-1}$. Since $G^T G$ is assumed to have full rank, this means that the constraint $G^T Qx = 0$ amounts to $(\tilde{Q}z)_2 = 0$, where $\tilde{Q}z = \left[\begin{smallmatrix} (\tilde{Q}z)_1 \\ (\tilde{Q}z)_2 \end{smallmatrix}\right]$.

Hence the system reduces to the port-Hamiltonian system without algebraic constraints

$$\dot{z}_1 = J_{11}\big(\tilde{Q}_{11} - \tilde{Q}_{12}\tilde{Q}_{22}^{-1}\tilde{Q}_{21}\big)z_1, \tag{4.8}$$

where $z_1 = G^{\perp}x$ are coordinates for the constrained state space $\mathcal{X}_c = \{x \in \mathcal{X} \mid G^T Qx = 0\}$.

4.1.2 Elimination of Kinematic Constraints

An important example of differential-algebraic port-Hamiltonian systems are mechanical systems subject to kinematic constraints, as discussed in Sect. 2.7. The constrained Hamiltonian equations (2.38) define a port-Hamiltonian system with respect to the Dirac structure $\mathcal{D}$ (in constrained input–output representation)

$$\begin{aligned}\mathcal{D} = \bigg\{&(f_S, e_S, f_C, e_C) \mid 0 = \begin{bmatrix} 0 & A^T(q) \end{bmatrix} e_S, e_C = \begin{bmatrix} 0 & B^T(q) \end{bmatrix} e_S,\\ &- f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c, \lambda \in \mathbb{R}^k \bigg\}.\end{aligned} \tag{4.9}$$

The algebraic constraints on the state variables (q, p) are thus given as

$$0 = A^T(q)\frac{\partial H}{\partial p}(q, p). \tag{4.10}$$

The *constrained state space* is therefore given as the following subset of the phase space (q, p):

$$\mathcal{X}_c = \left\{(q, p) \mid A^T(q)\frac{\partial H}{\partial p}(q, p) = 0\right\}. \tag{4.11}$$

We may solve for the algebraic constraints and eliminate the resulting constraint forces $A(q)\lambda$ in the following way [37]. Since rank $A(q) = k$, there exists locally an $n \times (n-k)$ matrix $S(q)$ of rank $n-k$ such that

$$A^T(q)S(q) = 0. \tag{4.12}$$

Now define $\tilde{p} = (\tilde{p}^1, \tilde{p}^2) = (\tilde{p}_1, \ldots, \tilde{p}_{n-k}, \tilde{p}_{n-k+1}, \ldots, \tilde{p}_n)$ as

$$\begin{aligned}\tilde{p}^1 &:= S^T(q)p,\\ \tilde{p}^2 &:= A^T(q)p,\end{aligned} \qquad \tilde{p}^1 \in \mathbb{R}^{n-k}, \ \tilde{p}^2 \in \mathbb{R}^k. \tag{4.13}$$

It is readily checked that $(q, p) \mapsto (q, \tilde{p}^1, \tilde{p}^2)$ is a coordinate transformation. Indeed, by (4.12), the rows of $S^T(q)$ are orthogonal to the rows of $A^T(q)$. In the new coordinates, the constrained Hamiltonian system (2.38) takes the form (see [37] for details), $*$ denoting unspecified elements,

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \\ \dot{\tilde{p}}^2 \end{bmatrix} = \begin{bmatrix} 0_n & S(q) & * \\ -S^T(q) & (-p^T[S_i, S_j](q))_{i,j} & * \\ * & * & * \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial q} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^1} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ A^T(q)A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B_c(q) \\ \overline{B}(q) \end{bmatrix} u, \tag{4.14}$$

$$A^T(q)\frac{\partial H}{\partial p} = A^T(q)A(q)\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$$

with $\tilde{H}(q, \tilde{p})$ the Hamiltonian H expressed in the new coordinates $q, \tilde{p}$. Here, S_i denotes the ith column of $S(q)$, $i = 1, \dots, n-k$, and $[S_i, S_j]$ is the *Lie bracket* of S_i and S_j, in local coordinates q given as (see e.g. [1, 23])

$$[S_i, S_j](q) = \frac{\partial S_j}{\partial q}(q)S_i(q) - \frac{\partial S_i}{\partial q}S_j(q) \tag{4.15}$$

with $\frac{\partial S_j}{\partial q}$ and $\frac{\partial S_i}{\partial q}$ denoting the $n \times n$ Jacobian matrices. Since λ only influences the $\tilde{p}^2$-dynamics, and the constraints $A^T(q)\frac{\partial H}{\partial p}(q, p) = 0$ are equivalently given by $\frac{\partial \tilde{H}}{\partial \tilde{p}^2}(q, \tilde{p}) = 0$, the constrained dynamics is determined by the dynamics of q and $\tilde{p}^1$, which serve as coordinates for the constrained state space $\mathcal{X}_c$:

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \end{bmatrix} = J_c(q, \tilde{p}^1) \begin{bmatrix} \frac{\partial H_c}{\partial q}(q, \tilde{p}^1) \\ \frac{\partial H_c}{\partial \tilde{p}^1}(q, \tilde{p}^1) \end{bmatrix} + \begin{bmatrix} 0 \\ B_c(q) \end{bmatrix} u, \tag{4.16}$$

where $H_c(q, \tilde{p}^1)$ equals $\tilde{H}(q, \tilde{p})$ with $\tilde{p}^2$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$, and where the skew-symmetric matrix $J_c(q, \tilde{p}^1)$ is given as the left-upper part of the structure matrix in (4.14), that is,

$$J_c(q, \tilde{p}^1) = \begin{bmatrix} O_n & S(q) \\ -S^T(q) & (-p^T[S_i, S_j](q))_{i,j} \end{bmatrix}, \tag{4.17}$$

where p is expressed as function of $q, \tilde{p}$, with $\tilde{p}^2$ eliminated from $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. In fact, for the Hamiltonian $\tilde{H}$ given as

$$\tilde{H}(q, \tilde{p}) = \frac{1}{2}\tilde{p}^T \tilde{M}^{-1}(q)\tilde{p} + P(q)$$

with $\tilde{M}(q) =: N^{-1}(q)$ the transformed mass matrix for the pseudo-momenta, and $V(q)$ the potential energy, it follows that

$$\tilde{H}\big(q, \tilde{p}^1\big) = \frac{1}{2}\tilde{p}^{1T}\big(N_{11} - N_{12}(q)N_{22}^{-1}(q)N_{21}(q)\big)\tilde{p}^1 + P(q).$$

Furthermore, in the coordinates q, $\tilde{p}$, the output map is given in the form

$$y = \begin{bmatrix} B_c^T(q) & \overline{B}^T(q) \end{bmatrix} \begin{bmatrix} \frac{\partial \tilde{H}}{\partial \tilde{p}^1} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2} \end{bmatrix} \tag{4.18}$$

which reduces on the constrained state space $\mathcal{X}_c$ to

$$y = B_c^T(q)\frac{\partial \tilde{H}}{\partial \tilde{p}^1}\big(q, \tilde{p}^1\big). \tag{4.19}$$

Summarizing, (4.16) and (4.19) define an *input–state–output* port-Hamiltonian system on $\mathcal{X}_c$, with Hamiltonian H_c given by the constrained total energy, and with structure matrix J_c given by (4.17).

Example 4.1 (Example 2.4, continued) Define according to (4.13) new p-coordinates

$$\begin{aligned} p_1 &= p_\varphi, \\ p_2 &= p_\theta + p_x \cos\varphi + p_y \sin\varphi, \\ p_3 &= p_x - p_\theta \cos\varphi, \\ p_4 &= p_y - p_\theta \sin\varphi. \end{aligned} \tag{4.20}$$

The constrained state space $\mathcal{X}_c$ is given by $p_3 = p_4 = 0$, and the dynamics on $\mathcal{X}_c$ is computed as

$$\begin{aligned} &\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} & & & & 0 & \cos\varphi \\ & & & & 0 & \sin\varphi \\ & O_4 & & & 0 & 1 \\ & & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos\varphi & -\sin\varphi & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_c}{\partial x} \\ \frac{\partial H_c}{\partial y} \\ \frac{\partial H_c}{\partial \theta} \\ \frac{\partial H_c}{\partial \varphi} \\ \frac{\partial H_c}{\partial p_1} \\ \frac{\partial H_c}{\partial p_2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \\ &\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}p_2 \\ p_1 \end{bmatrix}, \end{aligned} \tag{4.21}$$

where $H_c(x, y, \theta, \varphi, p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{4}p_2^2$.

4.2 The Geometric Description of Algebraic Constraints of Port-Hamiltonian DAEs

We will start by considering port-Hamiltonian differential-algebraic systems without external and resistive ports, described by a Dirac structure $\mathcal{D}$ and a Hamiltonian H. Define for every $x \in \mathcal{X}$ the subspace

$$P_{\mathcal{D}}(x) := \{\alpha \in T_x^*\mathcal{X} \mid \exists X \in T_x\mathcal{X} \text{ such that } (\alpha, X) \in \mathcal{D}(x)\}. \tag{4.22}$$

This defines a *co-distribution* on the manifold $\mathcal{X}$. Then it follows from the definition of a port-Hamiltonian system that the algebraic constraints are given in coordinate-free form as

$$\frac{\partial H}{\partial x}(x) \in P_{\mathcal{D}}(x). \tag{4.23}$$

Thus from a Dirac structure point of view algebraic constraints may only arise if the Dirac structure $\mathcal{D}$ is such that its associated co-distribution $P_{\mathcal{D}}$ is *not equal* to the whole cotangent bundle $T^*\mathcal{X}$, that is, if $P_{\mathcal{D}}(x)$ is a strict subspace of $T_x^*\mathcal{X}$.

The particular equational representation of the algebraic constraints depends on the chosen representation of the Dirac structure. For example, if the Dirac structure and the port-Hamiltonian system is given in constrained input–output representation (3.21), then the algebraic constraints are, as discussed above, given by $b^T(x)\frac{\partial H}{\partial x}(x) = 0$. On the other hand, if the Dirac structure is given in image representation as

$$\mathcal{D}(x) = \{(X, \alpha) \in T_x\mathcal{X} \times T_x^*\mathcal{X} \mid X = E^T(x)\lambda, \alpha = F^T(x)\lambda\} \tag{4.24}$$

then the algebraic constraints amount to the satisfaction of

$$\frac{\partial H}{\partial x}(x) \in \operatorname{im} F^T(x). \tag{4.25}$$

In the case of external ports, the algebraic constraints on the state variables x may also depend on the external port variables. A special case arises for resistive ports. Consider a Dirac structure

$$\{(X, \alpha, f_R, e_R) \in \mathcal{D}(x) \subset T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_R^*\} \tag{4.26}$$

with the resistive flow and effort variables satisfying a relation $R(f_R, e_R) = 0$. Then the gradient of the Hamiltonian has to satisfy the condition

$$\begin{aligned}\frac{\partial H}{\partial x}(x) \in \{\alpha \in T_x^*\mathcal{X} \mid & \exists X, f_R, e_R \in T_x\mathcal{X} \times \mathcal{F}_R \times \mathcal{F}_R^* \\ & \text{such that } (X, \alpha, f_R, e_R) \in \mathcal{D}(x) \text{ and } R(f_R, e_R) = 0\}.\end{aligned}$$

Depending on the resistive relation $R(f_R, e_R) = 0$ this may again induce algebraic constraints on the state variables x.

4.2.1 Algebraic Constraints in the Canonical Coordinate Representation

A particular elegant representation of algebraic constraints arises from the *canonical coordinate representation*. We will only consider the case of a system without resistive and external ports. For a constant Dirac structure $\mathcal{D}$, there always exist (linear) canonical coordinates such that the Dirac structure is described by (3.7). If on the other hand $\mathcal{D}$ is a modulated Dirac structure on a manifold $\mathcal{X}$, then only if the Dirac structure $\mathcal{D}$ satisfies an additional *integrability condition*,[9] we can choose local coordinates $x = (q, p, r, s)$ for $\mathcal{X}$ (with $\dim q = \dim p$), such that, in the corresponding bases for (f_q, f_p, f_r, f_s) for $T_x\mathcal{X}$ and (e_q, e_p, e_r, e_s) for $T_x^*\mathcal{X}$, the Dirac structure on this coordinate neighborhood is still given by the relations (3.7).

Substituting in this case the flow and effort relations of the energy storage

$$
\begin{aligned}
f_q &= -\dot{q}, \qquad & e_q &= \frac{\partial H}{\partial q}, \\
f_p &= -\dot{p}, \qquad & e_p &= \frac{\partial H}{\partial p}, \\
f_r &= -\dot{r}, \qquad & e_r &= \frac{\partial H}{\partial r}, \\
f_s &= -\dot{s}, \qquad & e_s &= \frac{\partial H}{\partial s}
\end{aligned}
\tag{4.27}
$$

into the canonical coordinate representation (3.7) of the Dirac structure yields the following dynamics:

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q, p, r, s), \\
\dot{p} &= -\frac{\partial H}{\partial q}(q, p, r, s), \\
\dot{r} &= 0, \\
0 &= \frac{\partial H}{\partial s}(q, p, r, s).
\end{aligned}
\tag{4.28}
$$

The variables q, p are the canonical coordinates known from classical Hamiltonian dynamics, while the variables r have the interpretation of Casimirs (conserved quantities independent of the Hamiltonian), see Sect. 4.3. The last equations $\frac{\partial H}{\partial s} = 0$ specify the algebraic constraints present in the system; in a form that is reminiscent of the first-order condition for optimality in the Maximum principle in optimal control theory.

The condition that the matrix in (4.2) has full rank (implying that the system has index one; cf. Proposition 4.1) is in the canonical coordinate representation equivalent to the partial Hessian matrix $\frac{\partial^2 H}{\partial s^2}$ being invertible. Solving, by the Implicit

[9]For more details regarding the precise form of the integrability conditions see Sect. 7.

Function theorem, the algebraic constraints $\frac{\partial H}{\partial s} = 0$ for s as a function $s(q, p, r)$ reduces the DAEs (4.28) to the ODEs

$$\begin{aligned} \dot{q} &= \frac{\partial \bar{H}}{\partial p}(q, p, r), \\ \dot{p} &= -\frac{\partial \bar{H}}{\partial q}(q, p, r), \\ \dot{r} &= 0, \end{aligned} \tag{4.29}$$

where $\bar{H}(q, p, r) := H(q, p, r, s(q, p, r))$.

4.3 Casimirs of Port-Hamiltonian DAEs

Consider a port-Hamiltonian differential-algebraic system without external and resistive ports, with Dirac structure $\mathcal{D}$ involving f_S, e_S. Similarly to (4.22) we may define the following, smaller, co-distribution:

$$\bar{P}_{\mathcal{D}}(x) := \{\alpha \in T_x^* \mathcal{X} \mid (\alpha, 0) \in \mathcal{D}(x)\} \tag{4.30}$$

This co-distribution will characterize the *conserved quantities* that are *independent* of the Hamiltonian H. In fact, it can be seen that $\bar{P}_{\mathcal{D}} = G_{\mathcal{D}}^{\perp}$, where $G_{\mathcal{D}}$ is the distribution on $\mathcal{X}$ defined as

$$G_{\mathcal{D}}(x) := \{X \in T_x \mathcal{X} \mid \exists \alpha \in T_x^* \mathcal{X} \text{ such that } (\alpha, X) \in \mathcal{D}(x)\}. \tag{4.31}$$

Now, let $C : \mathcal{X} \to \mathbb{R}$ be a function such that $\frac{\partial^T C}{\partial x}(x) \in \bar{P}_{\mathcal{D}}(x)$. Then

$$\frac{d}{dt} C\big(x(t)\big) = \frac{\partial^T C}{\partial x}\big(x(t)\big)\dot{x}(t) = 0, \tag{4.32}$$

for all possible vectors $\dot{x}(t)$ occurring in the system equations; independently of the Hamiltonian H. Such functions $C : \mathcal{X} \to \mathbb{R}$ are called the *Casimirs* of the system, and are very important for the analysis of the system. Note that the existence of finding functions C such that $\frac{\partial^T C}{\partial x}(x) \in \bar{P}_{\mathcal{D}}(x), x \in \mathcal{X}$ is related to the *integrability* of the co-distribution $\bar{P}_{\mathcal{D}}$, and thus to the integrability of the Dirac structure $\mathcal{D}$, cf. Sect. 7. Thus the Casimirs are completely characterized by the Dirac structure of the port-Hamiltonian system.

Similarly, we define the Casimirs of a port-Hamiltonian differential-algebraic system with a resistive relation to be all functions $C : \mathcal{X} \to \mathbb{R}$ satisfying $(0, e = \frac{\partial^T C}{\partial x}, 0, 0) \in \mathcal{D}$. Indeed, this will imply that

$$\frac{d}{dt} C = \frac{\partial^T C}{\partial x}\big(x(t)\big)\dot{x}(t) = 0 \tag{4.33}$$

for every possible derivative vector $\dot{x}$ occurring in the system equations, independently of the Hamiltonian and of the resistive relations.[10]

Example 4.2 In the case of a spinning rigid body (Example 2.3) the well-known Casimir is the total angular momentum $p_x^2 + p_y^2 + p_z^2$ (whose vector of partial derivatives is indeed in the kernel of the matrix $J(p)$ in (2.30)).

Similarly, in the LC-circuit of Example 2.5 the total flux $\phi_1 + \phi_2$ is a Casimir for $u = 0$.

Example 4.3 Consider a mechanical system with kinematic constraints (2.38) and $u = 0$. Then $(0, e) \in \mathcal{D}$ if and only if

$$0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda, \qquad \begin{bmatrix} 0 & A^T(q) \end{bmatrix} e = 0.$$

Partioning $e = \left[\begin{smallmatrix} e_q \\ e_p \end{smallmatrix}\right]$ this means that $e_q = A(q)\lambda$, or equivalently, $e_q \in \operatorname{im} A(q)$. Since in general $A(q)$ is depending on q, finding Casimirs involves an additional *integrability condition*, see also Sect. 7. In fact, Casimirs correspond to vectors $e_q \in \operatorname{im} A(q)$ which additionally can be written as a vector of partial derivatives $\frac{\partial C}{\partial q}(q)$ for some function $C(q)$ (the Casimir). In the case of Example 4.1 it can be verified that this additional integrability condition is *not* satisfied, corresponding to the fact that the kinematic constraints in this example are completely *nonholonomic*.

In general it can be shown [37] that there exist as many independent Casimirs as the rank of the matrix $A(q)$ if and only if the kinematic constraints are holonomic, in which case the Casimirs are equal to the integrated kinematic constraints.

4.4 Stability Analysis of Port-Hamiltonian DAEs

As we have seen before, any port-Hamiltonian differential-algebraic system, without control and interaction ports, satisfies the energy balance (2.16), that is,

$$\frac{d}{dt} H = e_R^T f_R \leq 0. \tag{4.34}$$

This immediately follows from the power-conserving property of Dirac structures. As a consequence, the Hamiltonian H qualifies as a Lyapunov function if it is bounded from below.

Recently, the notion of Lyapunov functions for general nonlinear DAE systems was studied in depth in [18]; also providing a formal treatment of asymptotic stability. Let us show, again by exploiting the properties of Dirac structures, how the

[10] However it can be shown [26] that if (4.33) holds for *some* non-degenerate resistive relation then it has to hold for *all*.

Hamiltonian H of a port-Hamiltonian differential-algebraic system also defines a Lyapunov function in this set-up.[11] Consider a port-Hamiltonian system without control and interaction ports, whose Dirac structure $\mathcal{D}$ is given in *kernel representation* as, see (3.9),

$$\left\{(f_S, e_S, f_R, e_R) \mid F_S(x) f_S + E_S(x) e_S + F_R(x) f_R + E_R(x) e_R = 0\right\} \tag{4.35}$$

with

$$\begin{aligned}
&F_S(x) E_S^T(x) + E_S(x) F_S^T(x) + F_R(x) E_R^T(x) + E_R(x) F_R^T(x) = 0, \\
&\operatorname{rank}\left[\, F_S(x) \quad E_S(x) \quad F_R(x) \quad E_R(x) \,\right] = \dim f_S + \dim f_R.
\end{aligned} \tag{4.36}$$

It follows that $\mathcal{D}$ is equivalently given in *image representation* as

$$\begin{aligned}
f_S &= E_S^T(x)\lambda(x), \qquad & e_S &= F_S^T(x)\lambda(x), \\
f_R &= E_R^T(x)\lambda(x), \qquad & e_R &= F_R^T(x)\lambda(x).
\end{aligned} \tag{4.37}$$

The resulting port-Hamiltonian system for a Hamiltonian H is thus given by the equations

$$\begin{aligned}
\dot{x} &= -E_S^T(x)\lambda(x), \qquad & \frac{\partial H}{\partial x}(x) &= F_S^T(x)\lambda(x), \\
f_R &= E_R^T(x)\lambda(x), \qquad & e_R &= F_R^T(x)\lambda(x)
\end{aligned} \tag{4.38}$$

(together with energy-dissipating constitutive relations). In particular

$$\frac{\partial^T H}{\partial x}(x) z = F_S^T(x)\lambda^T(x) F_S(x) z \tag{4.39}$$

for all vectors z; in agreement with one of the requirements for H being a Lyapunov function as stated in [18]. It also follows from here that (using the first line of (4.36))

$$\begin{aligned}
\dot{H} &= -\lambda^T(x) F_S(x)\dot{x} = -\lambda^T(x) F_S(x) E_S^T(x)\lambda(x) \\
&= \lambda^T(x) F_R(x) E_R^T(x)\lambda(x) = e_R^T f_R \leq 0
\end{aligned} \tag{4.40}$$

being another condition in the formulation of [18].

Finally, if H does *not* have a minimum at a desired equilibrium x^*, then a well-known method in Hamiltonian dynamics, called the *Energy-Casimir method*, is to use in the Lyapunov analysis, next to the Hamiltonian function, additional *conserved quantities* of the system, in particular the Casimirs. Indeed, candidate Lyapunov functions can be sought within the class of *combinations* of the Hamiltonian H and the Casimirs. For more information we refer to e.g. [24, 25, 31]. Most of this literature is however on port-Hamiltonian systems *without* algebraic constraints, and the

[11] I thank Stephan Trenn for an enlightening discussion on this issue.

presence of algebraic constraints poses new questions. For example, the necessary conditions for a Lyapunov function only have to hold on the subset of the state space where the algebraic constraints are satisfied.

4.5 Ill-posedness Due to Nonlinear Resistive Characteristics

In the above we have largely confined ourselves to differential-algebraic port-Hamiltonian systems without resistive relations. The presence of such resistive relations, especially in the nonlinear case, may pose additional difficulties. In particular, well-posedness problems may arise for port-Hamiltonian systems where the flow variables of the resistive ports are input variables for the dynamics, while the resistive relation is *not* effort-controlled. We will not elaborate on this (difficult) topic, but confine ourselves to an example (taken from [10]) illustrating the problems which may arise.

Example 4.4 (Degenerate Van der Pol oscillator) Consider a degenerate form of the Van der Pol oscillator consisting of a unit capacitor

$$\dot{Q} = I, \qquad V = Q \tag{4.41}$$

in parallel with a nonlinear resistor given by the characteristic

$$\left\{(f_R, e_R) = (I, V) \mid V = -\frac{1}{3}I^3 + I\right\}. \tag{4.42}$$

This resistive characteristic is *not* voltage-controlled, but instead is current-controlled. As a consequence, Eqs. (4.41) and (4.42) define an implicitly defined dynamics on the one-dimensional constraint submanifold R in (I, V) space given by

$$R = \left\{(I, V) \mid V + \frac{1}{3}I^3 - I = 0\right\}.$$

Difficulties in the dynamical interpretation arise at the points $(-1, -\frac{2}{3})$ and $(1, \frac{2}{3})$. At these points $\dot{V}$ is negative, respectively positive (while the corresponding time-derivative of I at these points tends to plus or minus infinity, depending on the direction along which these points are approached). Hence, because of the form of the constraint manifold R it is not possible to "integrate" the dynamics from these points (sometimes called *impasse points*) in a continuous manner along R.

For a careful analysis of the dynamics of this system we refer to [10]. In particular, it has been suggested in [10] that a suitable interpretation of the dynamics from the impasse points is given by the following *jump rules*:

$$\left(-1, -\frac{2}{3}\right) \to \left(2, -\frac{2}{3}\right), \qquad \left(1, \frac{2}{3}\right) \to \left(-2, \frac{2}{3}\right). \tag{4.43}$$

The resultant trajectory (switching from the region $I \leq -1$ to the region $I \geq 1$) is a ‘limit cycle’ that is known as a *relaxation oscillation*. For related examples in the context of constrained mechanical systems we refer to [5].

Existence and uniqueness of solutions *is* guaranteed if the resistive relation is well-behaved and the DAEs are of index one as discussed in the previous Sect. 4.1. Indeed, consider again the case of a port-Hamiltonian system given in the constrained input–output representation

$$\begin{aligned}
\dot{x} &= \big[J(x) - R(x)\big]\frac{\partial H}{\partial x}(x) + g(x)u + b(x)\lambda, \\
y &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad x \in \mathcal{X}. \\
0 &= b^T(x)\frac{\partial H}{\partial x}(x),
\end{aligned} \tag{4.44}$$

Imposing the same condition as before in Sect. 4.1, Proposition 4.1, namely that the matrix

$$b^T(x)\frac{\partial^2 H}{\partial x^2}(x)b(x) \tag{4.45}$$

has full rank, it can be seen that there is a unique solution starting from every feasible initial condition $x_0 \in \mathcal{X}_c$. Furthermore, this solution will remain in the constrained state space $\mathcal{X}_c$ for all time.

Example 4.5 A simple, but illustrative, example of a case where multiple solutions arise from feasible initial conditions can be deduced from the example of a linear LC-circuit with standard Hamiltonian $H(Q,\phi) = \frac{1}{2C}Q^2 + \frac{1}{2L}\phi^2$, where the voltage across the capacitor is constrained to be zero:

$$\begin{aligned}
\dot{Q} &= \frac{1}{L}\phi + \lambda, \\
\dot{\phi} &= \frac{1}{C}Q, \\
0 &= \frac{1}{C}Q.
\end{aligned} \tag{4.46}$$

Here λ denotes the current through the external port whose voltage is set equal to zero. Since $b^T(x)\frac{\partial^2 H}{\partial x^2}(x)b(x)$ in this case reduces to $\frac{1}{C}$ it follows that there is a unique solution starting from every feasible initial condition. Indeed, the constrained state space $\mathcal{X}_c$ of the above port-Hamiltonian system is given by $\{(Q,\phi) \mid Q = 0\}$, while the Lagrange multiplier λ for any feasible initial condition $(0,\phi_0)$ is uniquely determined as $\lambda = -\frac{1}{L}\phi$. On the other hand, in the singular case where $C = \infty$ the Hamiltonian reduces to $H(Q,\phi) = \frac{1}{2L}\phi^2$ and the constraint equation $0 = \frac{1}{C}Q$ becomes vacuous, i.e., there are no constraints anymore. In this case the Lagrange

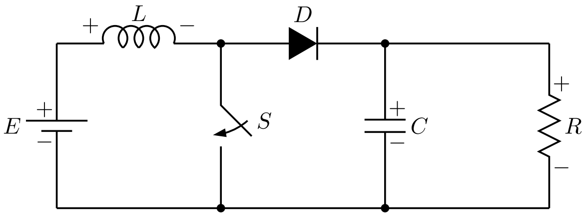


Fig. 4 Boost circuit with clamping diode

multiplier λ (the current through the external port) is not determined anymore, leading to multiple solutions $(Q(t), \phi(t))$ where $\phi(t)$ is constant (equal to the initial value ϕ_0) while $Q(t)$ is an *arbitrary* function of time.

5 Port-Hamiltonian Systems with Variable Topology

In many cases of interest it is useful to model fast transitions in physical systems as *instantaneous switches*. Examples include the description of elements like diodes and thyristors in electrical circuits, and impacts in mechanical systems. Within the port-Hamiltonian description, one obtains in all these cases an (idealized) model where the Dirac structure is not constant, but depends on the position of the switches. On the other hand, the Hamiltonian H and the resistive relations are usually *independent* of the position of the switches.

In both examples below, we thus obtain a *switching* port-Hamiltonian system, specified by a Dirac structure $\mathcal{D}_s$ depending on the switch position $s \in \{0, 1\}^n$ (here n denotes the number of independent switches), a Hamiltonian $H : \mathcal{X} \to \mathbb{R}$, and a resistive structure $\mathcal{R}$. Furthermore, every switching may be internally induced (like in the case of a diode in an electrical circuit or an impact in a mechanical system) or externally triggered (like an active switch in a circuit or mechanical system).

Example 5.1 (Boost converter) Consider the power converter in Fig. 4. The circuit consists of an inductor L with magnetic flux linkage ϕ_L, a capacitor C with electric charge q_C and a resistance load R, together with a diode D and an ideal switch S, with switch positions $s = 1$ (switch closed) and $s = 0$ (switch open). The diode is modeled as an ideal diode with voltage-current characteristic $v_D i_D = 0$, with $v_D \leq 0$ and $i_D \geq 0$.

The state variables are the electric charge Q_C and the magnetic flux linkage φ_L, and the stored energy (Hamiltonian) is the quadratic function $\frac{1}{2C} Q_C^2 + \frac{1}{2L}\varphi_L^2$. Note that there are four modes of operation of this system corresponding to the positions of the active switch (open or closed) and the diode (voltage- or current blocking). Two out of these four modes correspond to an algebraic constraint: namely $Q_C = 0$ if the switch is closed and the diode has $v_D = 0$, and $\varphi_L = 0$ if the switch is open and the diode has $i_D = 0$. (These two exceptional modes are sometimes called the *discontinuous modes* in the power converter literature.)

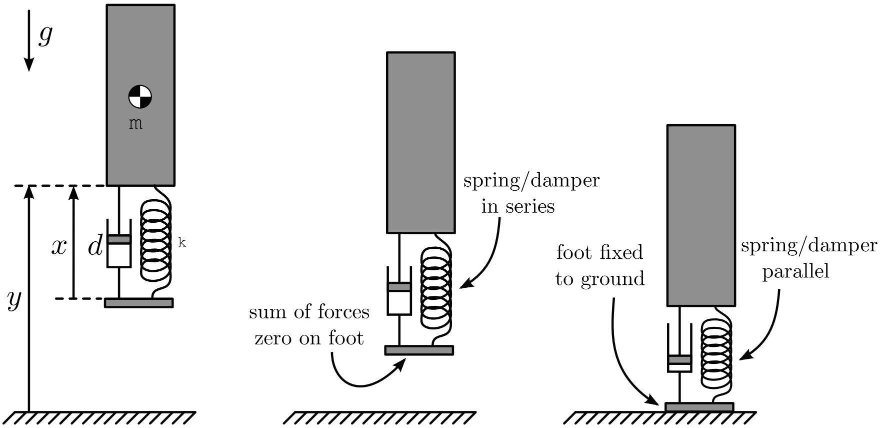


Fig. 5 Model of a bouncing pogo-stick: definition of the variables (*left*), situation without ground contact (*middle*), and situation with ground contact (*right*)

Example 5.2 (Bouncing pogo-stick) Consider the example of the vertically bouncing pogo-stick in Fig. 5: it consists of a mass m and a massless foot, interconnected by a linear spring (stiffness k and rest-length x_0) and a linear damper d. The mass can move vertically under the influence of gravity g until the foot touches the ground. The states of the system are taken as x (length of the spring), y (height of the bottom of the mass), and p (momentum of the mass, defined as $p := m\dot{y}$). Furthermore, the contact situation is described by a variable s with values $s = 0$ (no contact) and $s = 1$ (contact). The total energy (Hamiltonian) of the system equals

$$H(x, y, p) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}p^2, \tag{5.1}$$

where y_0 is the distance from the bottom of the mass to its center of mass.

When the foot is not in contact with the ground (middle figure), the total force on the foot is zero (since it is mass-less), which implies that the spring and damper forces must be equal but opposite. When the foot is in contact with the ground (right figure), the variables x and y remain equal, and hence also $\dot{x} = \dot{y}$. For $s = 0$ (no contact) the system can be described by the port-Hamiltonian system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} -\frac{1}{d} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix} \tag{5.2}$$

i.e. two independent systems (spring plus damper, and mass plus gravity), while for $s = 1$, the port-Hamiltonian description of the system is given as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -d \end{bmatrix}\begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}. \tag{5.3}$$

In this last case the resistive force $-d\dot{x}$ is added to the spring force and the gravitational force exerted on the mass, while for $s = 0$ the resistive force is equal to the spring force.

The two situations can be taken together into one port-Hamiltonian system with switching Dirac structure as follows

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} \frac{s-1}{d} & 0 & s \\ 0 & 0 & 1 \\ -s & -1 & -sd \end{bmatrix} \begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}. \tag{5.4}$$

In addition, the conditions for switching of the contact are functions of the states, namely as follows: contact is switched from off to on when $y - x$ crosses zero in the negative direction, and contact is switched from on to off when the velocity $\dot{y} - \dot{x}$ of the foot is positive in the no-contact situation, i.e. when $\frac{p}{m} + \frac{k}{d}(x - x_0) > 0$.

In the present modeling of the system no algebraic constraints arise in any of the two modes. It should be noted, however, that this is critically on the assumption of a massless foot. Indeed, if the mass of the foot is taken into the account then another state variable (namely the momentum of the foot) needs to be taken into account, while the contact situation would correspond to this extra state variable being constrained to zero.

We note that because the Hamiltonian function is *common* to all the modes of the switching port-Hamiltonian system it still can be employed for the stability analysis, see e.g. [6, 14]. Clearly this presents enormous advantages as compared to the stability analysis of general switched differential-algebraic systems [18]. The presence of algebraic constraints in (some of) the modes poses another question: the specification of the instantaneous reset of the state at the moment of switching in order to satisfy the algebraic constraints of the mode which is active immediately after the switching time. This involves the determination of consistent state *reset rules*. For a rather complete analysis in the context of switching electrical circuits we refer to the treatment in [6, 35]. The study of reset rules and mode selection is a classical subject in mechanical systems; see [5] and the references therein. For the related theory of *complementarity* hybrid systems we refer to [41, 42].

6 Interconnection of Port-Hamiltonian Systems and Composition of Dirac Structures

Crucial feature of network modeling, analysis and control is 'interconnectivity' or 'compositionality', meaning that complex systems can be built from simpler parts, and that the complex system can be studied in terms of its constituent parts and the way they are interconnected. The class of port-Hamiltonian systems completely fits within this paradigm, in the sense that the power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system. Furthermore, it will turn out that the Hamiltonian of the interconnected system is simply the sum of the

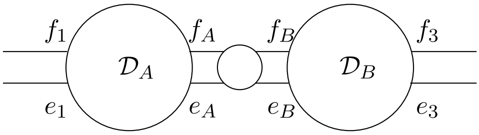


Fig. 6 The composition of $\mathcal{D}_A$ and $\mathcal{D}_B$

Hamiltonians of its parts, while the Dirac structure $\mathcal{D}$ of the interconnected system is solely determined by the Dirac structures of its components. This is clearly of immediate relevance for port-Hamiltonian differential-algebraic systems, since, as we have seen, the algebraic constraints are determined by the overall Dirac structure $\mathcal{D}$, in particular its co-distribution $P_{\mathcal{D}}$, and the overall Hamiltonian.

6.1 Composition of Dirac Structures

In this subsection, we investigate the *composition* or *interconnection* properties of Dirac structures. Physically it is clear that the composition of a number of power-conserving interconnections with partially shared variables should yield again a power-conserving interconnection. We show how this can be formalized within the framework of Dirac structures.

First, we consider the composition of *two* Dirac structures with partially shared variables. Once we have shown that the composition of two Dirac structures is again a Dirac structure, it is immediate that the power-conserving interconnection of any number of Dirac structures is again a Dirac structure.[12] Thus consider a Dirac structure $\mathcal{D}_A$ on a product space $\mathcal{F}_1 \times \mathcal{F}_2$ of two linear spaces $\mathcal{F}_1$ and $\mathcal{F}_2$, and another Dirac structure $\mathcal{D}_B$ on a product space $\mathcal{F}_2 \times \mathcal{F}_3$, with also $\mathcal{F}_3$ being a linear space. The linear space $\mathcal{F}_2$ is the space of shared flow variables, and $\mathcal{F}_2^*$ the space of shared effort variables; see 6.

In order to compose $\mathcal{D}_A$ and $\mathcal{D}_B$, a problem arises of *sign* convention for the power flow corresponding to the power variables $(f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^*$. Indeed, if $\langle e \mid f \rangle$ denotes *incoming* power (see the previous section), then for

$$(f_1, e_1, f_A, e_A) \in \mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$$

the term $\langle e_A \mid f_A \rangle$ denotes the incoming power in $\mathcal{D}_A$ due to the power variables $(f_A, e_A) \in \mathcal{F}_2 \times \mathcal{F}_2^*$, while for

$$(f_B, e_B, f_3, e_3) \in \mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$$

the term $\langle e_B \mid f_B \rangle$ denotes the incoming power *in* $\mathcal{D}_B$. Clearly, the *incoming* power in $\mathcal{D}_A$ due to the power variables in $\mathcal{F}_2 \times \mathcal{F}_2^*$ should equal the *outgoing* power from

[12] See [2] for a direct approach to the composition of multiple Dirac structures.

$\mathcal{D}_B$. Thus we cannot simply equate the flows f_A and f_B and the efforts e_A and e_B, but instead we define the interconnection constraints as

$$f_A = -f_B \in \mathcal{F}_2, \qquad e_A = e_B \in \mathcal{F}_2^*. \tag{6.1}$$

Therefore, the *composition* of the Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$, denoted $\mathcal{D}_A \parallel \mathcal{D}_B$, is defined as

$$\begin{aligned}\mathcal{D}_A \parallel \mathcal{D}_B := \bigl\{&(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^* \\ &\text{s.t. } (f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\bigr\}.\end{aligned} \tag{6.2}$$

The fact that the composition of two Dirac structures is again a Dirac structure has been proved in [9, 30]. Here we follow the simpler alternative proof provided in [7] (inspired by a result in [21]), which, among other things, allows one to obtain explicit representations of the composed Dirac structure.

Theorem 6.1 *Let $\mathcal{D}_A$ and $\mathcal{D}_B$ be Dirac structures (defined with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, respectively $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, and their bilinear forms). Then $\mathcal{D}_A \parallel \mathcal{D}_B$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.*

In the following theorem, an explicit expression is given for the composition of two Dirac structures in terms of a matrix kernel/image representation.

Theorem 6.2 *Let $\mathcal{F}_i, i = 1, 2, 3$ be finite-dimensional linear spaces with $\dim \mathcal{F}_i = n_i$. Consider Dirac structures $\mathcal{D}_A \subset \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, $n_A = \dim \mathcal{F}_1 \times \mathcal{F}_2 = n_1 + n_2$, $\mathcal{D}_B \subset \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$, $n_B = \dim \mathcal{F}_2 \times \mathcal{F}_3 = n_2 + n_3$, given by relaxed matrix kernel/image representations $(F_A, E_A) = ([F_1 \mid F_{2A}], [E_1 \mid E_{2A}])$, with F_A and E_A $n_A' \times n_A$ matrices, $n_A' \geq n_A$, respectively $(F_B, E_B) = ([F_{2B} \mid F_3], [E_{2B} \mid E_3])$, with F_B and E_B $n_B' \times n_B$ matrices, $n_B' \geq n_B$. Define the $(n_A' + n_B') \times 2n_2$ matrix*

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix} \tag{6.3}$$

and let L_A and L_B be $m \times n_A'$, respectively $m \times n_B'$, matrices ($m := \dim \ker M^T$), with

$$L = [L_A \mid L_B], \quad \ker L = \operatorname{im} M. \tag{6.4}$$

Then

$$F = [L_A F_1 \mid L_B F_3], \qquad E = [L_A E_1 \mid L_B E_3] \tag{6.5}$$

is a relaxed matrix kernel/image representation of $\mathcal{D}_A \parallel \mathcal{D}_B$.

Remark 6.1 The relaxed kernel/image representation (6.5) can be readily understood by pre-multiplying the equations characterizing the composition of $\mathcal{D}_A$

with $\mathcal{D}_B$

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0, \tag{6.6}$$

by the matrix $L := [L_A | L_B]$. Since $LM = 0$ this results in the relaxed kernel representation

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0 \tag{6.7}$$

corresponding to (6.5).

Instead of the canonical interconnection constraints $f_A = -f_B$, $e_A = e_B$ (cf. (6.1)), another standard power-conserving interconnection is the 'gyrative' interconnection

$$f_A = e_B, \qquad f_B = -e_A. \tag{6.8}$$

Composition of two Dirac structures $\mathcal{D}_A$ and $\mathcal{D}_B$ by this gyrative interconnection also results in a Dirac structure. In fact, the gyrative interconnection of $\mathcal{D}_A$ and $\mathcal{D}_B$ equals the interconnection $\mathcal{D}_A \parallel \mathcal{I} \parallel \mathcal{D}_B$, where $\mathcal{I}$ is the gyrative (or *symplectic*) Dirac structure

$$f_{IA} = -e_{IB}, \qquad f_{IB} = e_{IA} \tag{6.9}$$

interconnected to $\mathcal{D}_A$ and $\mathcal{D}_B$ via the canonical interconnections $f_{IA} = -f_A, e_{IA} = e_A$ and $f_{IB} = -f_B, e_{IB} = e_B$.

Example 6.1 (Feedback interconnection) The standard negative feedback interconnection of two input–state–output systems can be regarded as an example of a gyrative interconnection as above. Indeed, let us consider two input–state–output systems as in (2.41), for simplicity *without* external inputs d and external outputs z,

$$\Sigma_i : \begin{cases} \dot{x}_i = [J_i(x_i) - R_i(x_i)] \dfrac{\partial H_i}{\partial x_i}(x_i) + g_i(x_i) u_i, \\ y_i = g_i^T(x_i) \dfrac{\partial H_i}{\partial x_i}(x_i), \end{cases} \qquad x_i \in \mathcal{X}_i \tag{6.10}$$

for $i = 1, 2$. The standard feedback interconnection

$$u_1 = -y_2, \qquad u_2 = y_1 \tag{6.11}$$

is equal to the negative gyrative interconnection between the flows u_1, u_2 and the efforts y_1, y_2. The closed-loop system is the port-Hamiltonian system

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} J_1(x) - R_1(x_1) & -g_1(x_1)g_2^T(x_2) \\ g_2(x_2)g_1^T(x_1) & J_2(x_2) - R_2(x_2) \end{bmatrix} \begin{bmatrix} \frac{\partial H_1}{\partial x_1}(x_1) \\ \frac{\partial H_2}{\partial x_2}(x_2) \end{bmatrix}$$

with state space $\mathcal{X}_1 \times \mathcal{X}_2$ and Hamiltonian $H_1(x_1) + H(x_2)$. This once more emphasizes the close connections of port-Hamiltonian systems theory with *passivity* theory.

6.2 Interconnection of Port-Hamiltonian Systems

The result derived in Sect. 6.1 concerning the compositionality of Dirac structures immediately leads to the result that any power-conserving interconnection of port-Hamiltonian systems again defines a port-Hamiltonian system. This can be regarded as a fundamental building block in the theory of port-Hamiltonian systems. The result not only means that the theory of port-Hamiltonian systems is a completely modular theory for modeling, but it also serves as a starting point for design and control.

Consider k port-Hamiltonian systems $(\mathcal{X}_i, \mathcal{F}_i, \mathcal{D}_i, H_i)$, $i = 1, \ldots, k$, interconnected by a Dirac structure $\mathcal{D}_I$ on $\mathcal{F}_1 \times \cdots \times \mathcal{F}_k \times \mathcal{F}$, with $\mathcal{F}$ a linear space of flow port variables. This can be seen to define a port-Hamiltonian system $(\mathcal{X}, \mathcal{F}, \mathcal{D}, H)$, where $\mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$, $H := H_1 + \cdots + H_k$, and where the Dirac structure $\mathcal{D}$ on $\mathcal{X} \times \mathcal{F}$ is determined by $\mathcal{D}_1, \ldots, \mathcal{D}_k$ and $\mathcal{D}_I$. Indeed, consider the *product* of the Dirac structures $\mathcal{D}_1, \ldots, \mathcal{D}_k$ on $(\mathcal{X}_1 \times \mathcal{F}_1) \times (\mathcal{X}_2 \times \mathcal{F}_2) \times \cdots \times (\mathcal{X}_k \times \mathcal{F}_k)$, and compose this with the Dirac structure $\mathcal{D}_I$ on $(\mathcal{F}_1 \times \cdots \times \mathcal{F}_k) \times \mathcal{F}$. This yields a total Dirac structure $\mathcal{D}$ modulated by $x = (x_1, \ldots, x_k) \in \mathcal{X} = \mathcal{X}_1 \times \cdots \mathcal{X}_k$ which is point-wise given as

$$\mathcal{D}(x_1, \ldots, x_k) \subset T_{x_1}\mathcal{X}_1 \times T^*_{x_1}\mathcal{X}_1 \times \cdots \times T_{x_k}\mathcal{X}_k \times T^*_{x_k}\mathcal{X}_k \times \mathcal{F} \times \mathcal{F}^*.$$

Finally we mention that the theory of composition of Dirac structures and the interconnection of port-Hamiltonian systems can be also extended to *infinite-dimensional* Dirac structures and port-Hamiltonian systems [17, 26].

7 Integrability of Modulated Dirac Structures

A key issue in the case of modulated Dirac structures is that of *integrability*. Loosely speaking, a Dirac structure is *integrable* if it is possible to find local coordinates for the state space manifold such that, in these coordinates, the Dirac structure becomes a *constant* Dirac structure, that is, it is *not* modulated anymore by the state variables. As we have seen before, in particular in the context of the canonical coordinate representation (Sect. 3.1.4), this plays an important role in the representation of algebraic constraints (as well as in the existence of Casimirs).

First let us consider modulated Dirac structures which are given for every $x \in \mathcal{X}$ as the *graph* of a skew-symmetric mapping $J(x)$ from the co-tangent space $T_x^*\mathcal{X}$ to the tangent space $T_x\mathcal{X}$.

Integrability in this case means that the structure matrix J satisfies the conditions

$$\sum_{l=1}^{n}\left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x)\right] = 0, \quad i,j,k = 1,\dots,n. \tag{7.1}$$

In this case we may find, by Darboux's theorem (see e.g. [1]) around any point x_0 where the rank of the matrix $J(x)$ is constant, local coordinates $x = (q, p, r)$ in which the matrix $J(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & -I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.2}$$

Such coordinates are called *canonical*. A skew-symmetric matrix $J(x)$ satisfying (7.1) defines a *Poisson bracket* on $\mathcal{X}$, given for every $F, G : \mathcal{X} \to \mathbb{R}$ as

$$\{F, G\} = \frac{\partial^T F}{\partial x} J(x) \frac{\partial G}{\partial x}. \tag{7.3}$$

Indeed, by (7.1) the Poisson bracket satisfies the *Jacobi-identity*

$$\bigl\{F, \{G, K\}\bigr\} + \bigl\{G, \{K, F\}\bigr\} + \bigl\{K, \{F, G\}\bigr\} = 0 \tag{7.4}$$

for all functions F, G, K.

The choice of coordinates $x = (q, p, r)$ for the state space manifold also induces a basis for $T_x\mathcal{X}$ and a dual basis for $T_x^*\mathcal{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_r)$ and for the efforts by $e = (e_q, e_p, e_r)$, the Dirac structure defined by J in canonical coordinates is seen to be given by

$$\mathcal{D} = \bigl\{(f_q, f_p, f_r, e_q, e_p, e_r) \mid f_q = -e_p,\ f_p = e_q,\ f_r = 0\bigr\}. \tag{7.5}$$

A similar story can be told for the case of a Dirac structure given as the graph of a skew-symmetric mapping $\omega(x)$ from the tangent space $T_x\mathcal{X}$ to the co-tangent space $T_x^*\mathcal{X}$. In this case the integrability conditions take the (slightly simpler) form

$$\frac{\partial \omega_{ij}}{\partial x_k}(x) + \frac{\partial \omega_{ki}}{\partial x_j}(x) + \frac{\partial \omega_{jk}}{\partial x_i}(x) = 0, \quad i,j,k = 1,\dots,n. \tag{7.6}$$

The skew-symmetric matrix $\omega(x)$ can be regarded as the coordinate representation of a *differential two-form* ω on the manifold $\mathcal{X}$, that is, $\omega = \sum_{i=1,j=1}^{n} dx_i \wedge dx_j$, and the integrability condition (7.6) corresponds to the *closedness* of this two-form ($d\omega = 0$). The differential two-form ω is called a *pre-symplectic structure*, and a *symplectic structure* if the rank of $\omega(x)$ is equal to the dimension of $\mathcal{X}$. If (7.6) holds, then again by a version of Darboux's theorem we may find, around any point x_0 where the rank of the matrix $\omega(x)$ is constant, local coordinates $x = (q, p, s)$ in which the matrix $\omega(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.7}$$

Such coordinates are again called *canonical*. The choice of coordinates $x = (q, p, s)$ as before induces a basis for $T_x\mathcal{X}$ and a dual basis for $T_x^*\mathcal{X}$. Denoting the corresponding splitting for the flows by $f = (f_q, f_p, f_s)$ and for the efforts by $e = (e_q, e_p, e_s)$, the Dirac structure corresponding to ω in canonical coordinates is seen to be given by

$$\mathcal{D} = \left\{(f_q, f_p, f_s, e_q, e_p, e_s) \mid f_q = -e_p,\ f_p = e_q,\ e_s = 0\right\}. \tag{7.8}$$

In case of a symplectic structure the variables s are absent and the Dirac structure reduces to

$$\mathcal{D} = \left\{(f_q, f_p, e_q, e_p) \mid f_q = -e_p,\ f_p = e_q\right\} \tag{7.9}$$

which is the standard *symplectic gyrator*.

For general Dirac structures, integrability is defined in the following way.

Definition 7.1 ([12]) A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is *integrable* if for arbitrary pairs of smooth vector fields and differential one-forms $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$ we have

$$\langle L_{X_1}\alpha_2 \mid X_3\rangle + \langle L_{X_2}\alpha_3 \mid X_1\rangle + \langle L_{X_3}\alpha_1 \mid X_2\rangle = 0 \tag{7.10}$$

with L_{X_i} denoting the Lie-derivative.

Remark 7.1 (Pseudo-Dirac structures) In the usual definition of Dirac structures on manifolds (see [8, 12]), this *integrability* condition is *included* in the definition. Dirac structures that do *not* satisfy this integrability condition are therefore sometimes called *pseudo*-Dirac structures.

The above integrability condition for Dirac structures generalizes properly the closedness of symplectic forms and the Jacobi identity for Poisson brackets as discussed before. In particular, for Dirac structures given as the graph of a symplectic or Poisson structure, the notion of integrability is equivalent to the Jacobi-identity or closedness condition as discussed above (see e.g. [8, 9, 12] for details).

Note that a *constant* Dirac structure trivially satisfies the integrability condition. Conversely, a Dirac structure satisfying the integrability condition together with an additional constant rank condition can be represented *locally* as a *constant* Dirac structure. The precise form of the constant rank condition can be stated as follows. Recall that for any Dirac structure $\mathcal{D}$, we may define the distribution

$$G_{\mathcal{D}}(x) = \left\{X \in T_x\mathcal{X} \mid \exists\alpha \in T_x^*\mathcal{X} \text{ s.t. } (X,\alpha) \in D(x)\right\}$$

and the co-distribution

$$P_{\mathcal{D}}(x) = \left\{\alpha \in T_x^*\mathcal{X} \mid \exists X \in T_x\mathcal{X} \text{ s.t. } (X,\alpha) \in D(x)\right\}.$$

We call x_0 a *regular* point for the Dirac structure if both the distribution $G_{\mathcal{D}}$ and the co-distribution $P_{\mathcal{D}}$ have constant dimension around x_0.

If the Dirac structure is integrable and x_0 is a regular point, then, again by a version of Darboux's theorem, we can choose local coordinates $x = (q, p, r, s)$ for $\mathcal{X}$ (with $\dim q = \dim p$), such that, in the resulting bases for (f_q, f_p, f_r, f_s) for $T_x\mathcal{X}$ and (e_q, e_p, e_r, e_s) for $T_x^*\mathcal{X}$, the Dirac structure on this coordinate neighborhood is given as (see (3.7))

$$\begin{cases} f_q = -e_p, \\ f_p = e_q, \\ f_r = 0, \\ e_s = 0. \end{cases} \tag{7.11}$$

Coordinates $x = (q, p, r, s)$ as above are again called *canonical*. Note that the choice of canonical coordinates for a Dirac structure satisfying the integrability condition encompasses the choice of canonical coordinates for a Poisson structure and for a (pre-)symplectic structure as above.

Explicit conditions for integrability of a Dirac structure can be readily stated in terms of a kernel/image representation. Indeed, let

$$\begin{aligned}\mathcal{D} &= \left\{(f,e) \mid F(x)f + E(x)e = 0\right\} \\ &= \left\{(f,e) \mid f = E^T(x)\lambda,\ e = F^T(x)\lambda,\ \lambda \in \mathbb{R}^n\right\}.\end{aligned}$$

Denote the transpose of ith row of $E(x)$ by $Y_i(x)$ and the transpose of the ith row of $F(x)$ by $\beta_i(x)$. The vectors $Y_i(x)$ are naturally seen as coordinate representations of *vector fields* while the vectors $\beta_i(x)$ are coordinate representations of *differential forms*. Then integrability of the Dirac structure is equivalent to the condition

$$\langle L_{Y_i}\beta_j \mid Y_k\rangle + \langle L_{Y_j}\beta_k \mid Y_i\rangle + \langle L_{Y_k}\beta_i \mid Y_j\rangle = 0 \tag{7.12}$$

for all indices $i, j, k = 1, \dots, n$.

Another form of the integrability conditions can be obtained as follows. In [8, 9, 12] it has been shown that a Dirac structure on a manifold $\mathcal{X}$ is integrable if and only if, for all pairs of smooth vector fields and differential one-forms $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathcal{D}$, we have

$$\left([X_1, X_2], i_{X_1}d\alpha_2 - i_{X_2}d\alpha_1 + d\langle\alpha_2 \mid X_1\rangle\right) \in \mathcal{D}. \tag{7.13}$$

Using the definition of the vector fields Y_i and differential forms β_i, $i = 1, \dots, n$, as above, it follows that the Dirac structure is integrable if and only if

$$\big([Y_i, Y_j], i_{Y_i} d\beta_j - i_{Y_j} d\beta_i + d\langle \beta_j \mid Y_i \rangle\big) \in \mathcal{D} \tag{7.14}$$

for all $i, j = 1, \dots, n$. This can be more explicitly stated by requiring that

$$F(x)[Y_i, Y_j](x) + E(x)\big(i_{Y_i} d\beta_j(x) - i_{Y_j} d\beta_i(x) + d\langle \beta_j \mid Y_i \rangle(x)\big) = 0 \tag{7.15}$$

for all $i, j = 1, \dots, n$ and for all $x \in \mathcal{X}$. See for more details [9].

Example 7.1 (Kinematic constraints) Recall from the discussion in Sect. 2.7.1 that the modulated Dirac structure corresponding to an actuated mechanical system subject to kinematic constraints $A^T(q)\dot{q} = 0$ is given by

$$\mathcal{D} = \Big\{(f_S, e_S, f_C, e_C) \mid 0 = \begin{bmatrix} 0 & A^T(q) \end{bmatrix} e_S,\ e_C = \begin{bmatrix} 0 & B^T(q) \end{bmatrix} e_S,$$
$$-f_S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} e_S + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} f_c,\ \lambda \in \mathbb{R}^k \Big\}.$$

Complete necessary and sufficient conditions for integrability of this Dirac structure have been derived in [9]. Here we only state a slightly simplified version of this result, also detailed in [9]. We assume that the actuation matrix $B(q)$ has the special form (often encountered in examples) where every jth column ($j = 1, \dots, m$) is given as

$$\begin{bmatrix} 0 \\ \frac{\partial C_j}{\partial q}(q) \end{bmatrix}$$

for some function $C_j(q)$ only depending on the configuration variables q. In this case, the Dirac structure $\mathcal{D}$ is integrable *if and only if the kinematic constraints are holonomic*.

It has been shown in Sect. 2.7.1 that, after elimination of the Lagrange multipliers and the algebraic constraints, the constrained mechanical system reduces to a port-Hamiltonian system on the constrained submanifold defined with respect to a Poisson structure matrix J_c. As has been shown in [37], J_c satisfies the integrability condition (7.1) again if and only if the constraints (2.33) are *holonomic*. In fact, if the constraints are holonomic, then the coordinates s as in (3.7), (4.28) can be taken to be equal to the 'integrated constraint functions' $\overline{q}_{n-k+1}, \dots, \overline{q}_n$ of (2.35).

It can be verified that the structure matrix J_c obtained in 2.4, see (4.21), does not satisfy the integrability conditions, in accordance with the fact that the rolling constraints in this example are *nonholonomic*.

8 Conclusions

In this paper we have surveyed how the port-Hamiltonian formalism offers a systematic framework for modeling and control of large-scale multi-physics systems, emphasizing at the same time the network structure of the system (captured by its Dirac structure) and the energy-storage and energy-dissipation (formalized with the help of Hamiltonian functions and resistive relations). In many cases of interest this will lead to the description of the system dynamics by a mixed set of differential and algebraic equations (DAEs); however, endowed with a (generalized) Hamiltonian structure. We have shown how the identification of the underlying Hamiltonian structure offers additional insights and tools for analysis and control, as compared to general differential-algebraic systems.

In this paper we have confined ourselves to *lumped-parameter*, i.e., finite-dimensional, models. The port-Hamiltonian framework, however, has been successfully extended to distributed-parameter models (see e.g. [39]), corresponding to infinite-dimensional Dirac structures. Therefore an important venue for further research concerns the analysis within the port-Hamiltonian framework of mixed systems of differential, algebraic, as well as of *partial differential* equations.

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