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Frequency Domain Methods and Decoupling of Linear Infinite Dimensional Differential Algebraic Systems

TIMO REIS and CAREN TISCHENDORF

Abstract. We discuss the analysis of linear constant coefficient differential algebraic equations $E\dot{x}(t) = Ax(t) + q(t)$ on infinite dimensional Hilbert spaces. We give solution concepts and discuss solvability criteria which are mainly based on Laplace transform. Furthermore, we investigate the decoupling of these systems motivated by the Kronecker normal form for the finite dimensional case. Applications are given by the analysis of mixed systems of ordinary differential, partial differential and differential algebraic equations.

1. Introduction

In today's engineering applications, there is an increasing interest in partial differential algebraic equations (PDAE's), which are mainly coupled systems of partial differential equations (PDE's) and differential algebraic equations (DAE's). This type appears e.g. in modeling of electrical circuits with further components which are modeled by PDE's. These can be parasitic like heat conduction or transmission lines [3, 17] as well as they could be the result of a more reliable modeling of complex components like semiconductor devices [4, 22, 23]. Moreover, PDAE's are the outcome of mathematical models of several mechanical systems like elastic multibody systems [9] or biomechanical systems like blood flow networks. In order to study these problems in a systematic way, we are led to differential algebraic systems $F(\dot{x}(t), x(t), t) = 0$ in an abstract setting, the so called abstract DAE's (ADAE's). The unknown function $x(\cdot)$ is now a path in an appropriate (mostly infinite dimensional) Hilbert space, and the Frechét derivative $\frac{\mathrm{d}}{\mathrm{d}\dot{x}} F(\dot{x}, x, t)$ has a nontrivial nullspace, in general. In this work, we focus on the linear constant coefficient case

$$E\dot{x}(t) = Ax(t) + q(t) \tag{1}$$

and we make use of that gaining additional structure. $E : X \to Z$ is now a bounded linear operator and X, Z are some separable Hilbert spaces. In many practical cases, A is

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often acting on some product spaces and it is a block operator containing differential and evaluation operators. Hence, it is natural to assume that it is unbounded in general and that it is defined on some proper subspace $D(A) \subset X$.

The aim of this work is a step-by-step generalization of the known theory for the finite dimensional version of (1), where E and A are square matrices. This kind of systems is well-studied and subject of various textbooks like [5, 8]. If the matrix pair (E, A) is *regular*, i.e. $\det(sE - A)$ does not vanish identically, it is known that a state space transform of (1) leads to the following decoupled differential equations

$$N\dot{x}_1(t) = x_1(t) + q_1(t) \tag{2a}$$
$$\dot{x}_2(t) = \bar{A}x_2(t) + q_2(t), \tag{2b}$$

where N is nilpotent and $\bar{A}$ is some square matrix. This representation is called *Kronecker normal form*. The nilpotency index $\nu \in \mathbb{N}$ of N is well-defined by the pair (E, A) and is called the *Kronecker index*. (2a) contains algebraic equations and some further hidden relations, being algebraic, when (2a) is differentiated. Thus it is called the (*hidden*) *algebraic constraints*. The second expression (2b) is nothing but an ordinary differential equation extracted from the DAE (1) and is therefore called the *inherent ODE*. Altogether, solutions of these equations are given by

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k q_1^{(k)}(t), \quad x_2(t) = e^{\bar{A}t}x_2(0) + \int_0^t e^{\bar{A}(t-\tau)}q_2(\tau)d\tau. \tag{3}$$

Algorithms for the computation of the Kronecker normal form are e.g. presented in [5, 8]. Further, (3) implies that the Kronecker index determines the maximal derivative order of $q(\cdot)$ entering the solution $x(\cdot)$.

Besides a solvability analysis, we will generalize the decoupling framework to infinite dimensional descriptor systems in this work. In order to forecast the results of this work, we will obtain a form as follows

$$\begin{pmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} I & K \\ 0 & \mathfrak{A} \\ 0 & R \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix}. \tag{4}$$

The second two rows, i.e. the system

$$\begin{aligned} \dot{x}_2(t) &= \mathfrak{U}x_2 + q_2(t) \\ 0 &= Rx_2(t) + q_3(t), \end{aligned} \tag{5}$$

takes the role of the inherent ODE for the finite dimensional case. This type of equations is called an *abstract boundary control system* (see e.g. [7]), since, in an abstract setting, PDE systems with boundary action can be written in this way. After solving (5) for $x_2(\cdot)$, we obtain for the first component of the state vector $x_1(t) = -\sum_{k=0}^{\nu-1} N^k(q_1^{(k)}(t) + Kx_2(t))$. An extraordinary role is taken by the coupling term K. In the finite dimensional case there can be always found a representation with $K = 0$ due to the existence of the Kronecker normal form. However, this is not true for infinite dimensional DAE's. It will turn out that K has to satisfy a certain boundedness condition in order to guarantee that it can be eliminated. The proof of the existence of the form (4) is constructive and requires some projector chain to be existent and stagnant. There, we lean against the results of [13]. Besides that E is bounded, we will make the assumption that the *generalized resolvent* $(sE - A)^{-1}$ is analytic on some complex half-plane and has there an at most polynomial growth. With the help of the Laplace transform, we will shift the problem into some frequency domain spaces, namely the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$. From that, we will derive some criteria for the solvability of ADAE's. Many examples of practical relevance fulfill the requirements, we make on E and A.

A possible application of this paper is the structural analysis of coupled systems. Especially, by the transformation into (4), we gain inside to the system behavior. A theory which could also benefit from this work is that of *consistent initialization* of PDAE's (see [18]). In our abstract formalism, this denotes that for a given inhomogeneity $q(\cdot)$, we determine the x_0, for which (1) possesses a solution with $x(0) = x_0$. It is clear that not every initial condition is allowed since even in the finite dimensional case, it has to fulfill the algebraic constraints and the hidden ones. In this work, we will always assume homogeneous initial condition. Another application is given by the perturbation analysis for these systems, i.e. the sensitivity of the solution with respect to the inhomogeneity. Outgoing from the decoupling theory of systems (1), the results of this work can be used for the control and observation of PDAE systems [18]. A generalization of the system theoretic concepts of controllability and observability to the infinite dimensional differential algebraic case can be performed with the help of the presented theory. For finite dimensional DAE systems, these concepts are well-known and subject of various works like e.g. [5, 8]. As well, there is an advanced theory about the control and observation of infinite dimensional ODE systems [7]. With the decoupling of ADAE's, we are able to link these two approaches.

This work is organized as follows: In the first section, the functional analytic framework is presented. The needed spaces are introduced. Thereafter, the solvability of abstract differential algebraic systems is analyzed in Section 2. Furthermore, the concept of perturbation index and its interpretation in both time and frequency domain is treated. In Section 3, we develop the decoupling theory for infinite dimensional differential algebraic systems and related topics. We introduce the ADAE index, a generalization of the Kronecker index. Before this work is concluded, an example from analytic circuit theory is given in the fourth section, where the presented theory is applied.

2. Preliminaries

In this section, we collect some necessary fundamentals. The non-negative real numbers are denoted by $\mathbb{R}^+$, the complex numbers whose real part exceeds $\omega \in \mathbb{R}$ by $\mathbb{C}^+_\omega$ and $\mathbb{C}^+ := \mathbb{C}^+_0$.

Let X and Z be complex and separable Hilbert spaces throughout this work. The space of bounded linear operators from X to Z is denoted by $L_b(X, Z)$ and associated with the usual operator norm $\|\cdot\|_{L_b(X,Z)}$. Further, we abbreviate $L_b(X) := L_b(X, X)$. For $E \in L_b(X, Z)$ and $A : D(A) \subset X \to Z$, the *generalized resolvent*, denoted by $\rho(E, A)$, consists of $\lambda \in \mathbb{C}$, for which $(\lambda E - A)^{-1} \in L_b(Z, X)$. In analogy to the finite dimensional case, we call the operator pair (E, A) *regular*, if the $\rho(E, A) \neq \emptyset$. It can be shown that the regularity of (E, A) implies the closedness of A with domain $D(A)$. In the case where $X = Z$ and E is the identity I, we speak of the *resolvent* $\rho(A) := \rho(I, A)$. Furthermore, a linear operator Q with $Q^2 = Q$ is called *projector onto* im Q *and along* ker Q. The *complementary projector* $P = I - Q$ satisfies im P = ker Q and ker P = im Q. In this work, the complementary projector of Q_i is always denoted by P_i for all subindices i.

Further, we introduce several Hilbert space valued function spaces needed in this work. More details about the definitions can be found in [1, 7, 10, 20]. We will sometimes speak of measurability and holomorphy in the following. It should be remarked that for the Hilbert space valued case, there are the notions of weak and strong measurability (holomorphy) (see [7]). However, these notions coincide if the function has values in a separable Hilbert space, which is one reason for this claim. Let I be an interval containing zero during this work. The space of distributions on I with values in X is denoted by $\mathcal{D}'(I, X)$ and the Lebesgue space consisting of measurable and square integrable functions mapping from I to X is denoted by $L^2(I, X)$. Outgoing from that, we introduce the weighted Lebesgue spaces $L^2_\omega(I, X) = \{f(\cdot) \in \mathcal{D}'(I, X) : \; e^{-\omega \cdot} f(\cdot) \in L^2(I, X)\}$ associated with the norm $\|f\|^2_{L^2_\omega(I,X)} = \|e^{-\omega\cdot} f(\cdot)\|_{L^2(I,X)}$.

For $k \in \mathbb{N}$, $f^{(k)}$ denotes the k-th distributional derivative of f and, moreover, $\dot{f} = f^{(1)}$. The *distributional derivative with homogeneous initial condition*, denoted by $\frac{\mathrm{d}}{\mathrm{d}t}|_0$, is defined to be the restriction of the differential operator to the domain

$$D\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right) := \{f(\cdot) \in L^2_\omega(I, X), \dot{f}(\cdot) \in L^2_\omega(I, X) \text{ and } f(0) = 0\}.$$

It can be seen that $\rho(\frac{\mathrm{d}}{\mathrm{d}t}|_0) = \mathbb{C}$, if the interval I is bounded and $\rho(\frac{\mathrm{d}}{\mathrm{d}t}|_0) = \mathbb{C}^+_\omega$ in the case where $I = \mathbb{R}^+$. With the help of that operator, we construct the *Sobolev space with homogeneous initial values* $H^k_{0,\omega}(I, X)$ which is supposed to be the domain of the k-th power of the distributional derivative with homogeneous initial condition. For $\lambda \in \rho(\frac{\mathrm{d}}{\mathrm{d}t}|_0)$, we can make $H^k_{0,\omega}(I, X)$ a Hilbert space by associating it with the norm

$$\|f(\cdot)\|_{H^k_{0,\omega}(I,X)} := \left\| \left(\lambda I - \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0\right)^k f(\cdot) \right\|_{L^2_\omega(I,X)}. \tag{6}$$

For negative k, the space $H^k_{0,\omega}(I, X)$ is defined to be the completion of $L^2_\omega(I, X)$ with the norm (6) (see [10]). The above construction is a convenient concept and available for general operators with non-empty resolvent. In [10], it this called *abstract Sobolev space*.

There it is also shown that the topology in the above defined space does not depend on the particular $\lambda \in \rho(.\frac{\mathrm{d}}{\mathrm{d}t}|_0)$. This is the reason why we despite on an additional subindex.

For $k \in \mathbb{Z}$, $T > 0$ and $f \in H^k_{0,\omega}(\mathbb{R}^+, X)$, it can be seen that the restriction of f to the interval $[0, T]$, denoted by $.f|_{[0,T]}$, satisfies $.f|_{[0,T]} \in H^k_0([0, T], X)$. For convenience, we shortly write $\|f\|_{H^k_0([0,T],X)}$ instead of $\|.f|_{[0,T]}\|_{H^k_0([0,T],X)}$.

The *Hardy space* $\mathcal{H}_2(\mathbb{C}^+_\omega, X)$ consists of all holomorphic functions $f : \mathbb{C}^+_\omega \to X$ with the property $\|f\|_{\mathcal{H}_2(\mathbb{C}^+_\omega,X)} := \sup_{\gamma>\omega} \|f(\gamma + i\cdot)\|_{L^2(\mathbb{R},X)} < \infty$. In a similar way as for the Sobolev spaces with homogeneous initial condition, we define the *polynomially weighted* $\mathcal{H}_2$ *spaces*. Let $p_\lambda(s) := \lambda + s$ for $\lambda \in \mathbb{C}^+_\omega$. The space $p^k_\lambda \cdot \mathcal{H}_2(\mathbb{C}^+_\omega, X)$ is associated with the norm

$$\|f\|_{p^k_\lambda \cdot \mathcal{H}_2(\mathbb{C}^+_\omega,X)} := \sup_{\gamma>\omega} \left\| \frac{1}{p_\lambda(\cdot)^k} f(\cdot) \right\|_{\mathcal{H}_2(\mathbb{C}^+_\omega,X)}.$$

In the following, we address the relation between the introduced Hardy and Sobolev spaces. The Laplace transform $\mathcal{L}(f)(s) := \int_0^\infty f(t)e^{-st}dt$ is known to be an isometry from $L^2_\omega(\mathbb{R}^+, X)$ to $\mathcal{H}_2(\mathbb{C}^+_\omega, X)$ [7]. For a function f with $f^{(l)} \in L^2_\omega(\mathbb{R}^+, X)$ for $l = 0, \ldots, k-1$ and $f(0) = \dot{f}(0) = f^{(k-1)}(0) = 0$, the differentiation rule of Laplace transform reads $\mathcal{L}(f^{(k)})(s) = s^k \mathcal{L}(f)(s)$. Hence, the Laplace transform restricts, respectively extends, to an isometry $\mathcal{L}_k$ mapping from $H^k_{0,\omega}(\mathbb{R}^+, X)$ to the space $p^{-k}_\lambda \cdot \mathcal{H}_2(\mathbb{C}^+_\omega, X)$. In terms of better overview, we skip the index and write $\mathcal{L}$ instead of $\mathcal{L}_k$ if it can be seen from the context.

In addition, the Hardy space $\mathcal{H}_\infty(\mathbb{C}^+_\omega, X, Z)$ consists of $L_b(X, Z)$-valued functions which are holomorphic and bounded on $\mathbb{C}^+_\omega$ and it is a Banach space associated with the norm $\|F\|_{\mathcal{H}_\infty(\mathbb{C}^+_\omega,X,Z)} := \sup_{s\in\mathbb{C}^+_\omega} \|F(s)\|_{L_b(X,Z)}$. Further, we define the polynomially weighted $\mathcal{H}_\infty$-spaces $p^k_\lambda \cdot \mathcal{H}_\infty(\mathbb{C}^+_\omega, X, Z)$ and, similar to the $p^k_\lambda \mathcal{H}_2$-spaces, the norm in that space reads

$$\|F\|_{p^k_\lambda \cdot \mathcal{H}_\infty(\mathbb{C}^+_\omega,X,Z)} := \left\| \frac{1}{p_\lambda(\cdot)^k} F(\cdot) \right\|_{\mathcal{H}_\infty(\mathbb{C}^+_\omega,X,Z)}.$$

$G \in \mathcal{H}_\infty(\mathbb{C}^+_\omega, X, Z)$ defines a bounded linear operator $\mathcal{H}_2(\mathbb{C}^+_\omega, X) \to \mathcal{H}_2(\mathbb{C}^+_\omega, Z)$ via pointwise multiplication, i.e. $(Gf)(s) := G(s)f(s)$. In [7] it is shown that

$$\|G\|_{L_b(\mathcal{H}_2(\mathbb{C}^+_\omega,X),\mathcal{H}_2(\mathbb{C}^+_\omega,Z))} = \|G\|_{\mathcal{H}_\infty(\mathbb{C}^+_\omega,X,Z)}. \tag{7}$$

Further, G defines a bounded map $\mathcal{F}$ from $L^2_\omega(\mathbb{R}^+, X)$ to $L^2_\omega(\mathbb{R}^+, Z)$ by

$$x \mapsto \mathcal{F}x = \mathcal{L}^{-1}(G(\cdot)\hat{x}(\cdot)). \tag{8}$$

The operator norms equal, i.e. $\|\mathcal{F}\|_{L^2_\omega(\mathbb{R}^+,X),\, L^2_\omega(\mathbb{R}^+,Z)} = \|G\|_{\mathcal{H}_\infty(\mathbb{C}^+_\omega,X,Z)}$. We repeat the following results from [24].

THEOREM 2.1. *Let* $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)$ *and the corresponding* $\mathcal{F}$ *be defined by* (8). *Then, for all* $T > 0$, $z := \mathcal{F}x$, *the restriction* $z|_{[0,T]} \in L^2([0,T], Z)$, *only depends on* $x|_{[0,T]} \in L^2([0,T], X)$. *Furthermore, for some constant* $c_T > 0$, *we have an estimate*

$$\|z\|_{L^2([0,T],Z)} \leq c_T \|x\|_{L^2([0,T],X)}. \tag{9}$$

Conversely, if $\mathcal{F} \in L_b(L^2_\omega(\mathbb{R}^+, X), L^2_\omega(\mathbb{R}^+, Z))$ *satisfies these properties, there exists a* $G \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)$, *such that* $\mathcal{F}x = \mathcal{L}^{-1}(G(\cdot)\hat{x}(\cdot))$.

We have a causal dependence of $z(\cdot)$ to $x(\cdot)$, i.e. $x(t_2)$ has no influence on $z(t_1)$ for $t_1 < t_2$. Subsequently, we formulate a generalization of the previous statements to the polynomially weighted case. It can be seen that a function $G \in p_\lambda^k \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)$ defines a bounded linear operator from $p_\lambda^l \mathcal{H}_2(\mathbb{C}_\omega^+, X)$ to $p_\lambda^{l+k} \mathcal{H}_2(\mathbb{C}_\omega^+, Z)$ by pointwise multiplication. Moreover, we have

$$\|G\|_{L_b(p_\lambda^l \mathcal{H}_2(\mathbb{C}_\omega^+, X), p_\lambda^{l+k} \mathcal{H}_2(\mathbb{C}_\omega^+, Z))} = \|G\|_{p_\lambda^k \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)} \text{ for all } l, k \in \mathbb{Z}, \tag{10}$$

and G defines a map $\mathcal{F} \in L_b(H^l_{0,\omega}(\mathbb{R}^+, X), H^{l+k}_{0,\omega}(\mathbb{R}^+, Z))$ by the relation (8). The corresponding operator norms of G and $\mathcal{F}$ coincide. As a conclusion of Theorem 2.1, we can formulate a version for Sobolev spaces with homogeneous initial condition, respectively polynomially weighted Hardy spaces.

COROLLARY 2.2. *Let* $k, l \in \mathbb{Z}$, $G \in p_\lambda^k \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)$ *and let the operator* $\mathcal{F}$ *be defined by* $z := \mathcal{F}x = \mathcal{L}^{-1}\left(G(\cdot)\hat{x}(\cdot)\right)$. *Then for all* $T > 0$, *the restriction* $z|_{[0,T]} \in H_0^{l+k}([0,T], Z)$ *only depends on* $x|_{[0,T]} \in H_0^l([0,T], X)$ *and the following uniform estimate holds for some constant* $c_T > 0$

$$\|z\|_{H_0^{l+k}([0,T],Z)} \leq c_T \|x\|_{H_0^l([0,T],X)}. \tag{11}$$

Conversely, for $\mathcal{F} \in L_b(H^l_{0,\omega}(\mathbb{R}^+, X), H^{l+k}_{0,\omega}(\mathbb{R}^+, Z))$ *with the above properties, there exists a* $G \in p_\lambda^k \mathcal{H}_\infty(\mathbb{C}_\omega^+, X, Z)$, *such that* $\mathcal{F}x = \mathcal{L}^{-1}(G(\cdot)\hat{x}(\cdot))$ *for all* $x \in H^l_{0,\omega}(\mathbb{R}^+, X)$.

Overall, by the Laplace transform, we discovered a duality between the defined Sobolev and Hardy spaces. Since the Laplace transform is closely related to Fourier transform, one often speaks of *frequency domain* if Hardy spaces are meant and, correspondingly, the *time domain* is identified with Sobolev (or Lebesgue) spaces in literature like e.g. [7].

Further relations between the defined spaces are that, by restriction of analytic functions on $\mathbb{C}_{\omega_2}^+$ to some smaller half-plane, we have dense inclusion $p_\lambda^{k_1} \mathcal{H}_2(\mathbb{C}_{\omega_1}^+, X) \subset p_\lambda^{k_2} \mathcal{H}_2(\mathbb{C}_{\omega_2}^+, X)$ for $\omega_1 \leq \omega_2$ and $k_1 \leq k_2$. Then the Laplace transform yields that $H^k_{0,\omega_1}(\mathbb{R}^+, X) \subset H^k_{0,\omega_2}(\mathbb{R}^+, X)$. For the $\mathcal{H}_\infty$-spaces, analogous inclusions are holding true. However, they are not dense anymore.

3. Solvability of Abstract Differential Algebraic Systems

The aim of this section is to derive conditions for ADAE's (1) to be solvable for $x(\cdot)$ in some sense. As in the introduction, we generally assume that X, Z are some separable Hilbert spaces. Further, we claim that $E \in L_b(X, Z)$ and $A : D(A) \subset X \to Z$ is densely defined. The inhomogeneity $q(\cdot)$ is a Z-valued function of sufficient smoothness. This will be specified throughout this section. Inspired by [2], we introduce the following solution concepts.

For a continuous $q(\cdot) : \mathbb{R}^+ \to Z$, we call the function $x(\cdot) : \mathbb{R}^+ \to Z$ a *classical solution*, if it is continuously differentiable, $x(t) \in D(A)$ for all $t \in \mathbb{R}^+$ and (1) is pointwisely fulfilled for all $t \in \mathbb{R}^+$. Furthermore for $q(\cdot) \in L^2_\omega(\mathbb{R}^+, Z)$, we call $x(\cdot)$ a *weak solution*, if $\frac{\mathrm{d}}{\mathrm{d}t}\langle x(\cdot), E^*z^*\rangle \in L^2_\omega(\mathbb{R}^+, \mathbb{R})$ for all $z^* \in D(A^*)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(\cdot), E^*z^*\rangle = \langle x(\cdot), A^*z^*\rangle + \langle q(\cdot), z^*\rangle. \tag{12}$$

The notion of classical solution is rather intuitive and it can be seen that a classical solution is also a weak one. We will give a criterion for the existence of a weak solution $x(\cdot)$ with $x(0) = 0$. For non-trivial initialization as well as classical solvability, we refer to [18].

In contrast to the finite dimensional case, the regularity of (E, A) alone does not suffice to guarantee the solvability of (1). This even holds if $X = Z$ and E is just the identity, where (1) stands for an abstract ordinary differential equation. There, the question of weak solvability is closely connected to the property of A being the generator of a strongly continuous semigroup (see [2]).

Subsequently, we give a frequency-domain criterion for the solvability of abstract differential algebraic systems and the smoothness of the solution $x(\cdot)$.

THEOREM 3.1. *Let $\bar\nu, \nu \in \mathbb{N}$, $\omega \in \mathbb{R}$ with $\bar\nu > \nu$ and*

$$(sE - A)^{-1} \in p_\lambda^{\bar\nu} \cdot \mathcal{H}_\infty(\mathbb{C}^+_\omega, X, Z). \tag{13}$$

Further, let $q(\cdot) \in H^\nu_{0,\omega}(\mathbb{R}^+, Z)$. Then, there exists a unique $x(\cdot) \in H^{\nu-\bar\nu}_{0,\omega}(\mathbb{R}^+, X)$ which weakly solves (1). *Moreover, for all $T > 0$, there exist constants $c, c_T > 0$ such that*

$$\|x\|_{H^{\nu-\bar\nu}_{0,\omega}(\mathbb{R}^+,X)} \le c\,\|q\|_{H^{\nu-\bar\nu}_{0,\omega}(\mathbb{R}^+,Z)} \tag{14a}$$

$$\|x\|_{H^{\nu-\bar\nu}_{0}([0,T],X)} \le c_T\,\|q\|_{H^{\nu-\bar\nu}_{0}([0,T],Z)}. \tag{14b}$$

Before we state the proof, we shortly mention that the homogeneous initial condition is included in the condition $x(\cdot) \in H^{\nu-\bar\nu}_{0,\omega}(\mathbb{R}^+, X)$. The continuity of that $x(\cdot)$ is implied by its weak differentiability together with the fact that X is separable (see [7]).

Proof. For the existence, we are giving a candidate for a weak solution of (1) by $x(\cdot)$ with $\mathcal{L}(x)(s) = (sE - A)^{-1}\mathcal{L}(q)(s)$, and show that it satisfies (12). Due to our assumptions, we have that $\widehat{x}(\cdot) \in p_\lambda^{-1}\mathcal{H}_2(\mathbb{C}_\omega^+, X)$, and hence $x(\cdot) \in H_0^1(\mathbb{R}^+, X)$. Then for all $s \in \mathbb{C}_\omega^+$, $z^* \in D(A^*)$, we obtain

$$\begin{aligned}
s\langle \mathcal{L}(x)(s), E^*z^*\rangle &= \langle sE\mathcal{L}(x)(s), z^*\rangle = \langle sE(sE - A)^{-1}\mathcal{L}(q)(s), z^*\rangle \\
&= \langle (I + A(sE - A)^{-1})\mathcal{L}(q)(s), z^*\rangle \\
&= \langle A(sE - A)^{-1})\mathcal{L}(q)(s), z^*\rangle + \langle \mathcal{L}(q)(s), z^*\rangle \\
&= \langle A\mathcal{L}(x)(s), z^*\rangle + \langle \mathcal{L}(q)(s), z^*\rangle \\
&= \langle \mathcal{L}(q)(s), A^*z^*\rangle + \langle \mathcal{L}(q)(s), z^*\rangle.
\end{aligned}$$

An inverse Laplace transform of the above relation yields (12).

In order to establish that the weak solution is unique, it is assumed that the problem with trivial inhomogeneity, i.e. $q \equiv 0$, is solved by $x(\cdot) \in H_{0,\omega}^{\nu-\bar{\nu}}(\mathbb{R}^+, X)$. By a Laplace transform of (1), we obtain $\langle \mathcal{L}(x)(s), (sE^* - A^*)z^*\rangle = 0$. Since the $(sE - A)$ is surjective for $s \in \mathbb{C}_\omega^+$, that holds for its adjoint $(sE^* - A^*)$. Hence $\mathcal{L}(x)(s)$ vanishes identically on $\mathbb{C}_\omega^+$. This implies that $x(\cdot)$ can only be the trivial function. $\square$

Due to Corollary 2.2, it can be seen that $(sE-A)^{-1} \in p_\lambda^{\bar{\nu}}\mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$ implies that the solution operator, which maps $q(\cdot) \in H_{0,\omega}^{k-\bar{\nu}}([0, T], Z)$ to the corresponding weak solution $x(\cdot) \in H_{0,\omega}^{k+1}([0, T], X)$ is bounded for all $T > 0$, $k \in \mathbb{Z}$. Hence, we have also an existence result for a solution on a finite time horizon. The uniqueness in that case is not directly implied by Theorem 3.1. However, it is shown in [18] that it is implied by the regularity of (E, A) detached from the stronger condition (13). The polynomial growth $\bar{\nu}$ of the resolvent is a quantity which expresses "how often a system differentiates the inhomogeneity". This measure is intimately connected to the so-called *perturbation index* which is often used when dealing with DAE's and is therefore subject of various textbooks (see e.g. [5, 12]).

4. Decoupling of Abstract Differential Algebraic Equations

As it is advertised at the beginning of this work, we will consider a generalization of the Kronecker normal form to the infinite dimensional case. The main result about the decoupling is followed.

THEOREM 4.1. *Let X, Z be Hilbert spaces and let (E, A) be a regular operator pair with $E : X \to Z$ be bounded and $A : D(A) \subset X \to Z$ be densely defined. Moreover, let* *the projector sequence* $Q_i \in L_b(X) \cap L_b(D(A))$, $P_i = I - Q_i$ *with*

$$\begin{aligned}
&E_0 := E,\\
&A_0 := A\\
&\operatorname{im} Q_i = \ker E_i,\ \sum_{j=0}^{i-1} \ker E_i \subset \ker Q_i\\
&E_{i+1} = E_i - A_i Q_i,\\
&A_{i+1} = A_i P_i,
\end{aligned}$$

be existent and stagnant, i.e. there exists a $\nu \in \mathbb{N}$, *such that* $\ker E_\nu = \{0\}$. *Further, let*

$$\operatorname{im} E + A\left(D(A) \cap \sum_{k=0}^{\nu-1} \ker E_k\right) \tag{15}$$

be closed. Then, there exist Hilbert spaces X_1, X_2, X_3 *and bounded mappings* $W \in L_b(Z, X_1 \times X_2 \times X_3)$, $T \in L_b(X_1 \times X_2, X)$, *where* T *is bijective and* W *is injective and has dense range, such that*

$$WET = \begin{pmatrix} N & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix} : X_1 \times X_2 \to X_1 \times X_2 \times X_3, \tag{16a}$$

$$WAT = \begin{pmatrix} I & K \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix} : X_1 \times D(K) \cap D(\mathfrak{U}) \cap D(R) \to X_1 \times X_2 \times X_3. \tag{16b}$$

In particular, $N \in L_b(X_1)$ *is a nilpotent operator with nilpotency index* ν.

The number ν is called *ADAE index*. This concept was first published in [13] as a generalization of the tractability index (see [15]) to infinite dimensions. Further, that theory is applied in [22, 4, 23]. Later on, we will discuss the benefit for ADAE's.

Theorem 4.1 is shown constructively by making use of the above defined projector chain. Hence, the above result not only represents a pure existence statement but the decoupling is computable in a numeric or symbolic way. In the known cases of practical relevance, the existence of the projector chain with the above requirements as well as the closedness of (15) is guaranteed. The fact that the projectors Q_i are not only bounded projectors on X but also bounded on the graph space $D(A)$ is needed for the well-definedness of the operators E_{i+1} and A_{i+1}, which are defined on $D(E_{i+1}) = D(E_i) \cap ((D(A_i) \cap \operatorname{im} Q_i) + \operatorname{im} P_i)$ and $D(A_{i+1}) = D(A_i) \cap \operatorname{im} P_i + \operatorname{im} Q_i$, respectively. In many cases of practical relevance, we even have $\ker E_i \subset D(A)$, which implies that projectors with the above postulated properties exist. However, this condition is stronger than ours. One might think about a coupled system where a poisson equation $\Delta x = q$ with some boundary conditions is involved. This counts as an algebraic equation in the abstract setting, since no derivative with respect to time is involved, and we do not have $\ker E \subset D(A)$ in that case. Nevertheless, one can also construct projectors for the decoupling in that case.

Proof. Let E_ν be injective. At first, we show that it possesses a bounded left inverse. As a consequence of the *inverse mapping theorem* (see e.g. [20]), this is holding true if $\operatorname{im} E_\nu$ is closed. By induction on i, we show that

$$\operatorname{im} E_i = \operatorname{im} E + A\left(D(A) \cap \sum_{k=0}^{i-1} \ker E_k\right) \quad \text{for } i = 0, \ldots, \nu.$$

From the claim that (15) is closed, we can deduce that $\operatorname{im} E_\nu$ is closed. Since $\operatorname{im} E = \operatorname{im} E + A(D(A) \cap \sum_{k=0}^{-1} \ker E_i)$ the induction start holds trivially. Further, for $i > 0$ and $x \in D(E_i)$, we obtain $E_i x = E_{i-1} P_{i-1} x + A_{i-1} Q_{i-1} x$, and hence

$$\begin{aligned}\operatorname{im} E_i &= \operatorname{im} E_{i-1} + \operatorname{im} A Q_{i-1} = \operatorname{im} E_{i-1} + A \ker E_{i-1} \\ &= \operatorname{im} E + A\left(\sum_{k=0}^{i-2} \ker E_k\right) + A \ker E_{i-1} = \operatorname{im} E + A\left(\sum_{k=0}^{i-1} \ker E_k\right).\end{aligned}$$

Therefore we can assume to have a $E_\nu^- \in L_b(Z, X)$ with $E_\nu^- E_\nu = I$. Further, we denote $W_\nu := I - E_\nu E_\nu^- \in L_b(Z)$ and it can be seen that it is a projector with $\ker W_\nu = \operatorname{im} E_\nu$. We define the Hilbert spaces $X_1 := \operatorname{im} Q_0 \times \cdots \times \operatorname{im} Q_{\nu-1}$, $X_2 := \operatorname{im} P_0 \cdots P_{\nu-1}$ and $X_3 := \operatorname{im} W_\nu$ and operators $W \in L_b(X_1 \times X_2 \times X_3, Z)$, $T \in L_b(X_1 \times X_2, X)$ by

$$Wz = \begin{pmatrix} \begin{pmatrix} -Q_0 P_1 \cdots P_{\nu-1} E_\nu^- z \\ -Q_1 P_2 \cdots P_{\nu-1} E_\nu^- z \\ \vdots \\ -Q_{\nu-1} E_\nu^- z \end{pmatrix} \\ \\ P_0 \cdots P_{\nu-1} E_\nu^- z \\ W_\nu z \end{pmatrix}, \quad T \cdot \begin{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\nu-1} \end{pmatrix} \\ \\ x_P \end{pmatrix} = x_P + \sum_{i=0}^{\nu-1} P_0 \cdots P_{i-1} Q_i x_i.$$

It can be seen that T is bijective and W is injective. The fact that the range of W is dense in $X_1 \times X_2 \times X_3$ is implied by $\overline{\operatorname{im} E_\nu^-} = X$. Now we determine the products WET and WAT and show that a decoupling form (16a-16b) is yielded. The subsequent equivalent transforms can be performed to E and A

$$\begin{aligned} E &= E_0P_0 = (E_0 - A_0Q_0)P_0 = E_1P_0 = E_1P_1P_0 = (E_1 - A_1Q_1)P_1P_0 \\ &= E_2P_1P_0 = \cdots = E_\nu P_{\nu-1}\cdots P_0 \\ &= E_\nu(I - Q_0 - Q_1 - \cdots - Q_{\nu-1}) \end{aligned} \tag{17a}$$

$$\begin{aligned} A &= A_0P_0 + A_0Q_0 = -(E_0 - A_0Q_0)Q_0 + A_0P_0 = -E_1Q_0 + A_1 \\ &= -(E_1 - A_1Q_1)(P_1Q_0 + Q_1) + A_1P_1 = -E_2(P_1Q_0 + Q_1) + A_2 \\ &= \cdots = -E_\nu(Q_0 + \cdots + Q_{\nu-1}) + A_\nu. \end{aligned} \tag{17b}$$

Hence, by premultiplying E and A with $-Q_0P_1\cdots P_{\nu-1}E_\nu^-, \ldots, -Q_{\nu-2}P_{\nu-1}E_\nu^-$ and $-Q_{\nu-1}E_\nu^-$, we obtain then the following for $i = 1, \ldots, \nu - 1$

$$\begin{aligned} &-Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-E \\ &\quad = -Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-E_\nu(I - Q_0 - \cdots - Q_{\nu-1}) \\ &\quad = Q_i - Q_iP_{i+1}\cdots P_{\nu-1} \end{aligned} \tag{18a}$$

$$\begin{aligned} &-Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A \\ &\quad = -Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-(-E_\nu(Q_0 + \cdots + Q_{\nu-1}) + A_\nu) \\ &\quad = Q_iP_{i+1}\cdots P_{\nu-1}(Q_0 + \cdots + Q_{\nu-1}) - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu \\ &\quad = Q_i - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu. \end{aligned} \tag{18b}$$

Further, we compute

$$\begin{aligned} &P_0\cdots P_{\nu-1}E_\nu^-E \\ &\quad = P_0\cdots P_{\nu-1}E_\nu^-E_\nu(I - Q_0 - \cdots - Q_{\nu-1}) \\ &\quad = P_0\cdots P_{\nu-1}(I - Q_0 - \cdots - Q_{\nu-1}) \\ &\quad = P_0\cdots P_{\nu-1} \end{aligned} \tag{18c}$$

$$\begin{aligned} &P_0\cdots P_{\nu-1}E_\nu^-A \\ &\quad = P_0\cdots P_{\nu-1}E_\nu^-(-E_\nu(Q_0 + \cdots + Q_{\nu-1}) + A_\nu) \\ &\quad = P_0\cdots P_{\nu-1}A_\nu, \end{aligned} \tag{18d}$$

and, moreover

$$W_\nu E = W_\nu E_\nu P_{\nu-1}\cdots P_0 = 0 \tag{18e}$$

$$W_\nu A = W_\nu(-E_\nu(P_{\nu-1}\cdots P_1Q_0 + \cdots + Q_{\nu-1}) + A_\nu) = W_\nu A_\nu. \tag{18f}$$

Altogether, the relations (18a–18f) imply that *WET* and *WAT* have the form (16a–16b) with particularly $\mathfrak{U} = P_0 \cdots P_{\nu-1} E_\nu^- A_\nu$, $R = W_\nu A_\nu$ and

$$N = \begin{pmatrix} 0 & Q_0 Q_1 & Q_0 P_1 Q_2 & Q_0 P_1 P_2 Q_3 & \cdots & Q_0 P_1 \cdots P_{\nu-2} Q_{\nu-1} \\ & \ddots & Q_1 Q_2 & Q_1 P_2 Q_3 & & \vdots \\ & & \ddots & \ddots & \ddots & \vdots \\ & & & \ddots & \ddots & Q_{\nu-3} P_{\nu-2} Q_{\nu-1} \\ & & & & \ddots & Q_{\nu-2} Q_{\nu-1} \\ & & & & & 0 \end{pmatrix}, \quad K = -\begin{pmatrix} Q_0 P_1 \cdots P_{\nu-1} E_\nu^- A_\nu \\ Q_1 P_2 \cdots P_{\nu-1} E_\nu^- A_\nu \\ \vdots \\ Q_{\nu-1} E_\nu^- A_\nu \end{pmatrix}.$$

In order to complete the proof, it remains to be shown that the nilpotency index of the above defined N equals ν. The fact that the nilpotency index of N does not exceed ν can be understood. Now we show that $N^{\nu-1} \neq 0$. We calculate

$$N^{\nu-1} = \begin{pmatrix} 0 & \cdots & 0 & Q_0 \cdots Q_{\nu-1} \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}$$

and assume that $Q_0 \cdots Q_{\nu-1} x = 0$, i.e. $Q_1 \cdots Q_{\nu-1} x \in \ker Q_0 \cap \operatorname{im} Q_1$. Then we have $0 = E_1 Q_1 \cdots Q_{\nu-1} x = (E_0 - A_0 Q_0) Q_1 \cdots Q_{\nu-1} x = E_0 Q_1 \cdots Q_{\nu-1} x$. This implies $Q_1 \cdots Q_{\nu-1} x \in \ker E_0 = \operatorname{im} Q_0$. Together with the assumption that $Q_1 \cdots Q_{\nu-1} x$ is in $\ker Q_0$, we get $Q_1 \cdots Q_{\nu-1} x = 0$. An iteration of this argumentation yields $Q_{\nu-1} x = 0$. Thus, we have $\ker Q_0 \cdots Q_{\nu-1} = \ker Q_{\nu-1} \neq \{0\}$, i.e. both $Q_0 \cdots Q_{\nu-1}$ and N are non-zero operators. □

An essential difference to the finite dimensional case is that the injectivity of E_ν does not imply its invertibility but only the existence of a left inverse E_ν^-, i.e. $E_\nu^- E_\nu = I$. By the construction of W and T, one can see that this causes the "third row" in the decoupling form, i.e. the operator R. In order to guarantee the boundedness of E_ν^-, we need the technical condition that the space (15) is closed. Although there is a freedom in the choice of the Q_i, it holds that $\sum_{k=0}^{\nu-1} \ker E_k$ is an invariant of the pair (E, A). This is shown in [15] for the matrix case but - since no topological but only arithmetical argumentations are involved - this holds true for operators as well. Thus we have no redundancies in our requirements. It remains to be an open question whether the regularity of (E, A) and the closedness of the range of E suffice to imply $E_\nu^- \in L_b(Z, X)$.

By a coordinate transform

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} := T^{-1} x(t), \qquad \begin{pmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{pmatrix} = W q(t),$$

the result of Theorem 4.1 leads to the analysis of systems of the type (4). The second and the third row are relations for $x_2(\cdot)$ alone, namely the abstract boundary control system (5). These systems are well-studied in a system theoretic framework. In [21] it is shown that (5) can be rewritten as

$$\dot{x}_2(t) = \bar{A}x_2(t) + q_2(t) + \bar{B}q_3(t), \tag{19}$$

where $\bar{A}$ is a restriction of $\mathfrak{U}$ to the space $D(\bar{A}) = \ker R \cap D(\mathfrak{U})$ and $\bar{B}$ is an operator whose range is in some larger space than X_2, namely the dual of $D(\bar{A}^*)$.

Hence, the dynamics of the system are mainly determined by the properties of $\bar{A}$. A "nice" property of $\bar{A}$ would be that it generates a strongly continuous semigroup $T_{\bar{A}}(\cdot)$. By the Hille-Yosida Theorem [10], $\bar{A}$ is the generator of a strongly continuous semigroup if there exist constants $M > 0$, $\gamma \in \mathbb{R}$ such that

$$\|(\lambda I - \bar{A})^{-k}\| \leq \frac{M}{(\lambda - \gamma)^k} \quad \text{for all } k \in \mathbb{N}, \lambda > \gamma.$$

If this is fulfilled, a solution of (4) can be found by first solving for $x_2(\cdot)$ by using the variation of constants formula (see [10]) and then, by a backsubstitution, we can compute $x_1(\cdot)$ analogously to (3)

$$x_1(t) = -\sum_{k=0}^{\nu-1} N^k(Kx_2^{(k)}(t) + q_1^{(k)}(t)), \tag{20a}$$

$$\text{where} \quad x_2(t) = \int_0^t T_{\bar{A}}(t-s)(q_2(s) + \bar{B}q_3(s)). \tag{20b}$$

Now we will investigate criteria on E and A for the property of $\bar{A}$ being the generator of a strongly continuous semigroup. At first, we take a closer look at the generalized resolvent of the decoupling form. For $\lambda \in \rho(E, A)$, we have

$$\begin{pmatrix} \lambda N - I & -K \\ 0 & \lambda I - \mathfrak{U} \\ 0 & -R \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} (\lambda N - I)^{-1}N & -(\lambda N - I)^{-1}K(\lambda I - \mathfrak{U}|_{\ker R})^{-1} & (\lambda N - I)^{-1}K\ R|_{\ker \lambda I - \mathfrak{U}}^{-1} \\ 0 & (\lambda I - \mathfrak{U}|_{\ker R})^{-1} & -R|_{\ker \lambda I - \mathfrak{U}}^{-1} \end{pmatrix},$$

whereas $z_1 = (\lambda I - \mathfrak{U}|_{\ker R})^{-1}x_1$ and $z_2 = R|_{\ker \lambda I - \mathfrak{U}}^{-1}x_2$ are defined to be the solutions of the equations

$$\begin{aligned} (\lambda I - \mathfrak{U})z_1 &= x_1 & \qquad (\lambda I - \mathfrak{U})z_2 &= 0 \\ Rz_1 &= 0 & Rz_2 &= x_2. \end{aligned}$$

It can be seen that the unique solvability of these equations is a consequence of $\lambda \in \rho(E, A)$. According to (20b) and a result of [24], we can alternatively express the above operators as follows

$$(\lambda I - \mathfrak{U}|_{\ker R})^{-1} = (\lambda I - \bar{A})^{-1}, \quad \mathbb{R}^{-1}|_{\ker \lambda I - \mathfrak{U}} = (\lambda I - \bar{A})^{-1}\bar{B}.$$

As a consequence, we have $\rho(E, A) = \rho(\bar{A})$. A necessary criterion for $\bar{A}$ generating a strongly continuous semigroup is $(sE - A)^{-1} \in p_\lambda^{\bar{\nu}} \cdot \mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$. However, the sufficiency is not guaranteed, it can only be concluded that $\bar{A}$ satisfies an estimate $\|(\lambda I - \bar{A})^{-1}\| \leq M(\gamma - \lambda)^{\tilde{\nu}}$ for some $\tilde{\nu} < \bar{\nu}$, $M, \gamma > 0$ and all $\lambda > \gamma$. Operators with this property generate a so-called *integrated semigroup*, or also called *distributive semigroup*. These semigroups are not continuous functions but operator-valued distributions. Integrated semigroups are e.g. treated in [16]. They are not subject of this paper.

Since $(sN - I)^{-1}$ is an operator-valued polynomial with degree $\nu - 1$, we have $\nu < \bar{\nu}$, if $(sE - A)^{-1} \in p_\lambda^{\bar{\nu}}\mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$ for some $\omega \in \mathbb{R}$. In contrast to the finite dimensional case, we do not have $\nu - 1 = \bar{\nu}$, if $\bar{\nu}$ with the above property is chosen to be minimal.

As for finite dimensions, the decoupling form is not unique. In the following we expose, how two such forms differ. Further, the question arises, if the property of $\bar{A}$ generating a strongly continuous semigroup depends on the particular choice of the decoupling form. Indeed, it can be shown that this property is an invariant of the pair (E, A). This will be a conclusion of the subsequent result.

THEOREM 4.2. *Let an operator pair (E, A) be given with the assumptions stated at the beginning of Section 3. Further, let $(W_1ET_1, W_1AT_1) = (\tilde{E}_1, \tilde{A}_1)$ and $(W_2ET_2, W_2AT_2) = (\tilde{E}_2, \tilde{A}_2)$ be two decoupling forms with particularly*

$$\tilde{E}_1 := \begin{pmatrix} N_1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \ \tilde{A}_1 := \begin{pmatrix} I & K_1 \\ 0 & \mathfrak{U}_1 \\ 0 & R_1 \end{pmatrix}, \ \tilde{E}_2 := \begin{pmatrix} N_2 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \ \tilde{A}_2 := \begin{pmatrix} I & K_2 \\ 0 & \mathfrak{U}_2 \\ 0 & R_2 \end{pmatrix}.$$

Then, N_1 and N_2 are similar, i.e. there exists a bounded and boundedly invertible $T_N \in L_b(X_{11}, X_{21})$ with $T_N^{-1} \in L_b(X_{21}, X_{11})$ such that $N_1 = T_N^{-1}N_2T_N$. Additionally, the operators $\bar{A}_1$ and $\bar{A}_2$ defined to be the restrictions of $\mathfrak{U}_1$ to the space $\ker R_1$ *and $\mathfrak{U}_2$ to* $\ker R_2$*, respectively, are similar.*

Proof. Let $\tilde{T} := T_1^{-1}T_2$ and $\tilde{W} := W_2W_1^{-1}$ be partitioned as

$$\tilde{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \quad \tilde{W} = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}.$$

For $\lambda \in \rho(E, A)$, we have

$$(\lambda\tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1 = \begin{pmatrix} (\lambda N_1 - I)^{-1}N_1 & -(\lambda N_1 - I)^{-1}K_1(\lambda I - \bar{A}_1)^{-1} \\ 0 & (\lambda I - \bar{A}_1)^{-1} \end{pmatrix},$$

and it can be seen that $(\lambda\tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2$ has the same structure. From the similarity relation $\tilde{T}^{-1}(\lambda\tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1\tilde{T} = (\lambda\tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2$, we deduce

$$\begin{aligned}\begin{pmatrix}0 & T_{11}K_{2*} + T_{12}(\lambda I - \bar{A}_2)^{-\nu}\\ 0 & T_{21}K_{2*} + T_{22}(\lambda I - \bar{A}_2)^{-\nu}\end{pmatrix} &= \tilde{T}((\lambda\tilde{E}_2 - \tilde{A}_2)^{-1}\tilde{E}_2)^{\nu}\\ &= ((\lambda\tilde{E}_1 - \tilde{A}_1)^{-1}\tilde{E}_1)^{\nu}\tilde{T}\\ &= \begin{pmatrix}K_{1*}T_{21} & K_{1*}T_{22}\\ (\lambda I - \bar{A}_1)^{-\nu}T_{21} & (\lambda I - \bar{A}_1)^{-\nu}T_{22}\end{pmatrix}\end{aligned}$$

for some bounded operators K_{1*}, K_{2*}. Hence, we can conclude $T_{21} = 0$. By an analogous argumentation, it is clear that $\bar{T}^{-1}$ has the same block structure as $\bar{T}$ and thus, both T_{11} and T_{22} are boundedly invertible. Moreover, the equality $A_2 = \bar{W}A_1\bar{T}$ implies

$$\begin{pmatrix}I & K_2\\ 0 & \mathfrak{U}_2\\ 0 & R_2\end{pmatrix} = \begin{pmatrix}W_{11}T_{11} & W_{11}(T_{12} + K_2T_{22}) + W_{12}\mathfrak{U}_2T_{22} + W_{13}R_2T_{22}\\ W_{21}T_{11} & W_{21}(T_{12} + K_2T_{22}) + W_{22}\mathfrak{U}_2T_{22} + W_{23}R_2T_{22}\\ W_{31}T_{11} & W_{31}(T_{12} + K_2T_{22}) + W_{32}\mathfrak{U}_2T_{22} + W_{33}R_2T_{22}\end{pmatrix},$$

and therefore $W_{21} = 0$, $W_{31} = 0$, $W_{11} = T_{11}^{-1}$. Together with $E_2 = \bar{W}E_1\bar{T}$ this implies

$$\begin{pmatrix}N_2 & 0\\ 0 & I\\ 0 & 0\end{pmatrix} = \begin{pmatrix}W_{11}N_1T_{11} & W_{11}N_1T_{12} + W_{12}T_{22}\\ 0 & W_{22}T_{22}\\ 0 & W_{32}T_{22}\end{pmatrix},$$

and hence $W_{32} = 0$, $W_{22} = T_{22}^{-1}$. Since $N_2 = T_{11}^{-1}N_1T_{11}$, N_1 and N_2 are similar. In addition, we have $\mathfrak{U}_2 = T_{22}^{-1}\mathfrak{U}_1T_{22} + W_{23}R_1T_{22}$ and $R_2 = W_{33}R_1T_{22}$. Hence, $\bar{A}_2$ is the restriction of $T_{22}^{-1}\mathfrak{U}_1T_{22} + W_{23}R_1T_{22}$ to the space $\ker W_{33}R_1T_{22}$, and therefore, it is the restriction of $T_{22}^{-1}\mathfrak{U}_1T_{22}$ to the space $\ker R_1T_{22}$. From that, we get $\bar{A}_2 = T_{22}^{-1}\bar{A}_1T_{22}$, which completes the proof. □

COROLLARY 4.3. *Let the same preliminaries hold as in Theorem* 4.2. *Then, if* $\bar{A}_1$ *generates a strongly continuous semigroup, the same holds for* $\bar{A}_2$.

Proof. This statement is immediately concluded by Theorem 4.2 since $\bar{A}_2 = T_{22}^{-1}\bar{A}_1T_{22}$ for some bounded and boundedly invertible operator T_{22}. Hence, the semigroup $T_{\bar{A}_2}(\cdot)$ generated by $\bar{A}_2$ is given by $T_{22}^{-1}T_{\bar{A}_1}(\cdot)T_{22}$, whereas $T_{\bar{A}_1}(\cdot)$ denotes the semigroup generated by $\bar{A}_1$. □

In analogy to the notion of inherent ODE for finite dimensional DAE's, we will call the semigroups $T_{\bar{A}_1}(\cdot)$ and $T_{\bar{A}_2}(\cdot)$ *inherent semigroups* of E and A. From the proof above, we can conclude that all inherent semigroups of E and A are similar. In the following, we give some a priori criteria on E and A possessing a strongly continuous inherent semigroup.

THEOREM 4.4. *Let a system* (1) *be given. Further, let* $Q_i : i = 0, \dots, \nu - 1$ *be the projector sequence of Theorem* 4.1 *and* $P_{\Sigma_\nu} = P_0 \cdots P_{\nu-1}$. *Then, the following three statements are equivalent:*

(i) *There exists a decoupling form with a strongly continuous inherent semigroup*
(ii) *All decoupling forms have a strongly continuous inherent semigroup*
(iii) *There exist $M, \gamma > 0$, such that for all $k \in \mathbb{N}, \lambda > \gamma$ holds*

$$\|P_{\Sigma_\nu}((\lambda E - A)^{-1}E)^k P_{\Sigma_\nu}\| \leq \frac{M}{(\lambda - \gamma)^k}. \tag{21}$$

Proof. We only have to show the equivalence between (i) and (iii). The remaining part is a consequence of Corollary 4.3. Let W, T be the transformations, such that the operator pair (WET, WAT) is in decoupling form (16a-16b). Since $P_{\Sigma\nu}$ is a projector along $\sum_{i=0}^{\nu-1} \operatorname{im} Q_i$ and that space does not depend on the particular choice of Q_i, we can conclude that

$$P_{\Sigma\nu} = T\begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} T^{-1}$$

for some $H \in L_b(X_1, X_2)$. Thus, for $\lambda \in \rho(E, A)$ we compute

$$\begin{aligned} &P_{\Sigma\nu}((\lambda E - A)^{-1}E)^k P_{\Sigma\nu} \\ &\quad = T\begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} \begin{pmatrix} (\lambda N - I)N & (\lambda N - I)NK(\lambda I - \bar{A})^{-1} \\ 0 & (\lambda I - \bar{A})^{-1} \end{pmatrix}^k \begin{pmatrix} 0 & H \\ 0 & I \end{pmatrix} T^{-1} \\ &\quad = T\begin{pmatrix} 0 & 0 \\ 0 & (\lambda I - \bar{A})^{-k} \end{pmatrix} T^{-1}. \end{aligned}$$

Hence, if $\bar{A}$ generates a strongly continuous semigroup, there exist constants $m > 0$ and $\gamma \in \mathbb{R}$, such that $\|(\lambda I - \bar{A})^{-k}\| \leq \frac{m}{(\gamma-\lambda)^k}$ for all $k \in \mathbb{N}$ and $\lambda > \gamma$. Thus, for $M = m\|T\|\|T^{-1}\|$, the relation (21) is valid. Conversely, if statement (iii) is fulfilled, we have $\|(\lambda I - \bar{A})^{-k}\| \leq \frac{m}{(\gamma-\lambda)^k}$ for the constant $m := M\|T\|\|T^{-1}\|$ and for all $k \in \mathbb{N}, \gamma > \lambda$. Therefore, the Hille-Yosida Theorem guarantees that $\bar{A}$ generates a strongly continuous semigroup. □

4.1. *Complete Decouplings*

Comparing the decoupling form with the Kronecker normal form, it was already mentioned that - besides the boundary term R - another essential difference is the appearance of a coupling operator K. We will now answer the question whether we can find transformations, such that the coupling term K vanishes, which we call a *complete decoupling* from now on. In fact, there are practically motivated examples of ADAE's, where a complete decoupling is not possible. We will confirm this with an example at the end of this section. The following theorem gives a sufficient criterion for the existence of a complete decoupling.

THEOREM 4.5. *Let* (E, A) *be an operator pair. Then, there exist transformations* W_1, T_1*, such that*

$$(W_1 E T_1, W_1 A T_1) = \left(\begin{pmatrix} N_1 & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & \mathfrak{U}_1 \\ 0 & R_1 \end{pmatrix} \right) \tag{22}$$

if and only if transformations W, T *exist such that* (WET, WAT) *has the decoupling form* (4)*, and, additionally*

$$K\mathfrak{U}^k \in L_b(X_2, X_1) \text{ for } k = 0, \ldots, \nu - 1. \tag{23}$$

For the proof, we use the projector approach of Theorem 4.1 as well. In order to achieve a complete decoupling, we have to choose the kernels of the decoupling projectors Q_i in a particular way. Our approach is inspired by the work [14], where these complete decouplings are realized for matrix pairs. There, the projectors yielding a complete decoupling were called *canonical*. The presented method was based on an iteration method leading to the canonical projectors in finitely many steps. The problem when generalizing this to infinite dimensions is that one has to pay attention for possible unboundednesses. Moreover, it can be seen that Theorem 4.5 goes with the case of regular matrix pairs, where, as a matter of course, (23) is fulfilled. Before we state the proof of Theorem 4.5, the following lemma is presented. Its proof is extensive and left to the Appendix.

LEMMA 4.6. *Let an operator pair* (E, A) *be given and let* $Q_0, \ldots, Q_{\nu-1}$ *the corresponding projector sequence for the decoupling according to Theorem* 4.1. *Further let* $\mathfrak{U} = P_{\Sigma\nu} E_\nu^- A_\nu$ *and*

$$K = - \begin{pmatrix} Q_0 P_1 \cdots P_{\nu-1} E_\nu^- A_\nu \\ Q_1 P_2 \cdots P_{\nu-1} E_\nu^- A_\nu \\ \vdots \\ Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^- A_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{24}$$

with $K(P_{\Sigma\nu} E_\nu^- A_\nu)^l \in L_b(X)$ *for* $l = 0, \ldots, k$*. Then, a decoupling with the projectors* $\bar{Q}_j := -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j$ *leads to decoupling of* (E, A)*, namely*

$$\left(\begin{pmatrix} \bar{N} & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} I & \bar{K} \\ 0 & \bar{\mathfrak{U}} \\ 0 & \bar{R} \end{pmatrix} \right)$$

with $\bar{\mathfrak{U}} = \bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu$ *and*

$$\bar{K} = -\begin{pmatrix} \bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \vdots \\ \bar{Q}_{i-1}\bar{P}_i\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{25}$$

Moreover, we have $\bar{K}\bar{\mathfrak{U}}^l \in L_b(X)$ *for* $l = 0, \ldots, k-1$.

The constructed $\bar{Q}_i$ are indeed projectors, since

$$\begin{aligned}(-Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_j)^2 &= Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_jQ_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_j \\ &= -Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-E_\nu Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_j \\ &= -Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_j.\end{aligned}$$

Their boundedness of is a consequence of $\bar{Q}_j = Q_j - Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_\nu$, which holds due to the relation (18b) and the fact that $Q_jP_{j+1}\cdots P_{\nu-1}E_\nu^-A_\nu$ is bounded. Now we show Theorem 4.5.

Proof. Since the $K = 0$ obviously satisfies (23), the existence of a form (22) clearly implies the second assertion. For the converse implication, let an operator pair in decoupling form (16a–16b) be given and assume that $K\mathfrak{U}^l$ is bounded for $l = 0, \ldots, \nu-1$. The decoupling procedure can be performed with projectors of the form

$$Q_i = \begin{pmatrix} \hat{Q}_i & 0 \\ 0 & 0 \end{pmatrix},$$

Then, we get

$$E_\nu^- = \begin{pmatrix} (N-I)^{-1} & 0 & 0 \\ 0 & I & 0 \end{pmatrix},\ A_\nu = \begin{pmatrix} 0 & K \\ 0 & \mathfrak{U} \\ 0 & R \end{pmatrix},\ P_{\Sigma\nu}E_\nu^-A_\nu = \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{U} \end{pmatrix},$$

we obtain K as in the proof of Theorem 4.1 with

$$Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu = \begin{pmatrix} 0 & \hat{Q}_i\hat{P}_{i+1}\cdots\hat{P}_{\nu-1}(N-I)^{-1}K \\ 0 & 0 \end{pmatrix}.$$

Therefore, for $l = 0, \ldots, \nu - 1$, we have the boundedness of the operator $Q_i P_{i+1} \cdots P_{\nu-1} E_\nu^- A_\nu (P_{\Sigma\nu} E_\nu^- A_\nu)^l$. Now, successively using Lemma 4.6., we get a bounded

$$\bar{K} = \begin{pmatrix} \bar{Q}_0 \bar{P}_1 \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- A_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

in the $\nu-1$-th iteration. This can be further eliminated by a new construction of the $\bar{Q}_i$. Finally, we get a representation, where the coupling term $\bar{K}$ vanishes. □

In order to justify that K cannot always be eliminated, we present the following example with

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 0 & -C_1 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{\partial}{\partial \xi} \\ 0 & 0 & C_0 \end{pmatrix}.$$

C_p denotes the evaluation operator which maps a continuous function f to its value at $h \in [0, 1]$, i.e. $C_p f = f(p)$. The spaces Z, X and $D(A)$ are given by $Z = \mathbb{R}^3 \times L_2([0, 1], \mathbb{R}) \times \mathbb{R}$, $X = \mathbb{R}^3 \times L^2([0, 1], \mathbb{R})$, and $D(A) = \mathbb{R}^3 \times H^1([0, 1], \mathbb{R})$, where $H^1([0, 1], \mathbb{R})$ denotes the following Sobolev space of absolutely continuous functions $f : [0, 1] \to \mathbb{R}$, whose distributional derivative is square integrable (see [1]). The operator pair (E, A) is regular, and the generalized inverse reads

$$(sE - A)^{-1} \begin{pmatrix} x_1 \\ x_2 \\ f \\ x_3 \end{pmatrix} = \begin{pmatrix} e^{-s} x_3 + \int_0^1 e^{-s(1-y)} f(y) dy \\ s e^{-s} x_3 + s \int_0^1 e^{-s(1-y)} f(y) dy - x_2 \\ e^{-sx} x_3 + \int_0^x e^{-s(x-y)} f(y) dy \end{pmatrix}$$

for all $s \in \mathbb{C}$. It can be seen that this generalized resolvent is located in the space $p_\lambda \mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$ for all $\omega \in \mathbb{R}, \lambda \in \mathbb{C}_\omega^+$. Moreover, the system is already in decoupling form, the ADAE index reads $\nu = 2$ and projectors are given by

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

If we now construct $\bar{Q}_0$, $\bar{Q}_1$ according to Lemma 4.6, we get

$$\bar{Q}_1 = -Q_1 E_2^- A_1 = \begin{pmatrix} 1 & 0 & C_1 \\ 1 & 0 & C_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have $\bar{Q}_1 \notin L_b(X)$, since it contains some evaluation operators and hence, a complete decoupling of this pair is not possible with the presented method. We can even show that a complete decoupling does not exist. Assume that there exists a complete decoupling, i.e. transformations W, T, such that (WET, WAT) is in decoupling form (16a–16b) with $K = 0$. The relations between W and T bringing one decoupling form into another have been investigated in the proof of Theorem 4.2, and hence, we get

$$W = \begin{pmatrix} T_{11}^{-1} & W_{12} & W_{13} \\ 0 & T_{22}^{-1} & W_{23} \\ 0 & 0 & W_{33} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where T, W are partitioned according to the block structure of (WET, WAT) and (E, A). Denoting $D_1(s) := (sI - \frac{\partial}{\partial\xi}|_{\ker C_0})^{-1}$ and $D_2(s) := C_0|_{\ker sI-\mathfrak{U}}^{-1}$ we can therefore derive

$$\begin{aligned} \begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1} &= \begin{pmatrix} sI - T_{22}^{-1}\frac{\partial}{\partial\xi}T_{22} + W_{23}C_0T_{22} \\ W_{33}C_0T_{22} \end{pmatrix}^{-1} \\ &= (T_{22}^{-1}D_1(s)T_{22}T_{22}^{-1}D_2(s)W_{33}^{-1} - T_{22}^{-1}D_1(s)W_{23}W_{33}^{-1}) \end{aligned}$$

From that and the fact that D_1, D_2 are in $\mathcal{H}_\infty$ of suitable spaces, we can conclude that this holds true for $(sI - \mathfrak{U}|_{\ker R})^{-1}$, $R|_{\ker sI-\mathfrak{U}}^{-1}$. Especially, if we multiply with a column operator $F = (0,\ 1,\ 0)$ and $H = (1\ 0\ 0)$, we can verify that $F(sE - A)^{-1}H = se^{-s}$. Now, let the operators $WH = (H_1,\ H_2,\ H_3)$, $FT = (F_1\ F_2)$ be partitioned according to the block structure of WET and WAT. Then, we obtain

$$\begin{aligned} se^{-s} &= F(sE - A)^{-1}H = FT(sWET - WAT)^{-1}WH \\ &= (F_1\ F_2)\begin{pmatrix} sN - I & 0 \\ 0 & sI - \mathfrak{U} \\ 0 & -R \end{pmatrix}^{-1}\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} \\ &= F_1(-sN - I)H_1 + F_2\begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1}\begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \end{aligned}$$

Thus we would be able to express the function se^{-s} as a sum of the polynomial $F_1(-sN - I)H_1$ and the function

$$F_2\begin{pmatrix} sI - \mathfrak{U} \\ -R \end{pmatrix}^{-1}\begin{pmatrix} H_2 \\ H_3 \end{pmatrix} \in \mathcal{H}_\infty(\mathbb{C}_\omega^+, \mathbb{R}, \mathbb{R}).$$

This is a contradiction, since $se^{-s} - p(s)$ is unbounded in any complex halfplane for any polynomial p. This argumentation yields that a complete decoupling is not possible in that case.

5. Example: An Electrical Circuit with a Transmission Line

We present a simple practical example to demonstrate the reliability of the discussed decoupling theory. Consider an electrical circuit containing a transmission line of unit length as below.

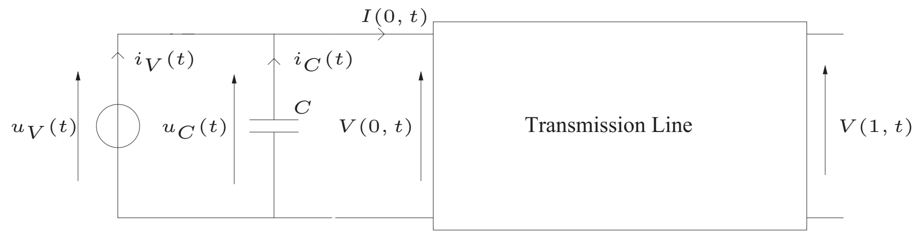

Figure 1 Electrical circuit with transmission line

The voltage and current courses $V(\xi,t)$, $I(\xi,t)$ along the transmission line satisfy the telegraph equations (see [19])

$$C_T\frac{\partial}{\partial t}V(\xi,t) = -G_T I(\xi,t) - \frac{\partial}{\partial \xi}V(\xi,t)$$

$$L_T\frac{\partial}{\partial t}I(\xi,t) = -\frac{\partial}{\partial \xi}I(\xi,t) - R_T V(\xi,t),$$

for some constants $G_T, R_T \geq 0$, $C_T, L_T > 0$. Further, due to element relations and the Kirchhoff laws [6], we get the equations

$$\begin{aligned} C\dot{u}_C(t) &= i_C(t), & 0 &= u_C(t) - u_V(t),\\ 0 &= -i_V(t) - i_C(t) + I(0,t), & 0 &= -u_C(t) + V(0,t),\\ 0 &= I(1,t). \end{aligned}$$

Equivalently, we model that system with an abstract differential algebraic system (1). The state $x(t)$ and the inhomogeneity $q(t)$ are chosen to be

$$x(t) = \begin{pmatrix} u_C(t)\\ i_C(t)\\ u_V(t)\\ V(t)\\ I(t)\end{pmatrix} \in X := \mathbb{R}^3 \times (L^2([0,1],\mathbb{R}))^2,$$

$$q(t) = \begin{pmatrix} 0 \\ -u_V(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in Z := \mathbb{R}^5 \times (L^2([0,1],\mathbb{R}))^2.$$

$V(t)$, $I(t)$ are the spacial distributions of the voltage and current along the transmission line, i.e. $(V(t))(\xi) := V(\xi, t)$ and $(I(t))(\xi) := I(\xi, t)$. The operators E and A are given by

$$E = \begin{pmatrix} C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & C_0 \\ -1 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial\xi} \\ 0 & 0 & 0 & -\frac{\partial}{\partial\xi} & -R_T \end{pmatrix},$$

and the domain of A reads $D(A) = \mathbb{R}^5 \times (H([0,1],\mathbb{R}))^2$. C_0 and C_1 are evaluation operators as in the previous section. It will turn out that this system has index 2 and operators of the operator chain in Theorem 4.1 read

$$Q_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_1 = \begin{pmatrix} C & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 \\ -1 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial\xi} \\ 0 & 0 & 0 & -\frac{\partial}{\partial\xi} & -R_T \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & 0 \\ -C & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} C & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_T & 0 \\ 0 & 0 & 0 & 0 & L_T \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_0 \\ 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & -G_T & -\frac{\partial}{\partial\xi} \\ 0 & 0 & 0 & -\frac{\partial}{\partial\xi} & -R_T \end{pmatrix},$$

It can be seen that E_1, E_2 are defined on X and $D(A) = D(A_1) = D(A_2)$. A left inverse of E_2 and, consequently, the projector $W_2 = I - E_2 E_2^-$ read

$$E_2^- = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & C & 0 & 0 & 0 & 0 & 0 \\ -1 & -C & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & C_T^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_T^{-1} \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the inherent abstract ordinary differential equation on the subspace im $P_0 P_1 = \{0\} \times (L^2[0,1])^2$ with boundary control $0 = W_2 A_2 x(t) = W_2 q(t)$ is the following

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{G_T}{C_T} & -\frac{1}{C_T}\frac{\partial}{\partial\xi} \\ 0 & 0 & 0 & -\frac{1}{L_T}\frac{\partial}{\partial\xi} & -\frac{R_T}{L_T} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & 0 & C_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ V(t) \\ I(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ -u_V(t) \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be shown that the operator A_T being a restriction of

$$\mathfrak{U}_T = \begin{pmatrix} -\frac{G_T}{C_T} & -\frac{1}{C_T}\frac{\partial}{\partial \xi} \\ -\frac{1}{L_T}\frac{\partial}{\partial \xi} & -\frac{R_T}{L_T} \end{pmatrix}$$

to the space $D(A_T) = (H^1([0,1],\mathbb{R}) \cap \ker C_0) \times (H^1([0,1],\mathbb{R}) \cap \ker C_1)$ generates a strongly continuous semigroup $T_T(\cdot)$. For the proof, we refer to [17, 18]. The computation of $T_T(\cdot)$ can e.g. be performed by an inverse Laplace transform of the resolvent $(sI - A_T)^{-1}$ (see [7]). As a consequence, $P_0P_1E_2^- A_2$ with domain $D(A) \cap \ker W_2A_2 \cap \operatorname{im} P_0P_1 = \{0\} \times D(A_T)$ is a generator of a strongly continuous semigroup on the space im P_0P_1. With the method of [21], we get according to formula (20b)

$$\begin{pmatrix} V(t) \\ I(t) \end{pmatrix} = \int_0^t T_T(t-s) \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix} u_V(s)ds,$$

where $\delta_0 \in \mathcal{D}'([0,1],\mathbb{R})$ is the Dirac delta distribution.

The (hidden) algebraic relations $Q_0Q_1\dot{x}(t) = Q_0x + Q_0P_1E_2^- A_2x(t) - Q_0P_1E_2^- q(t)$ and $0 = Q_1x(t) - Q_1E_2^- A_2x(t) - Q_1E_2^- q(t)$ read

$$\begin{pmatrix} 0 \\ C\dot{u}_C(t) \\ -C\dot{u}_C(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i_C(t) \\ i_V(t) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ C_0I(t) \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} u_C(t) \\ -Cu_C(t) \\ Cu_C(t) \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} u_V(t) \\ -Cu_V(t) \\ Cu_V(t) \\ 0 \\ 0 \end{pmatrix}.$$

The first relation implies $u_C(t) = u_V(t)$. Plugging that into the second one, we get

$$i_C(t) = C\dot{u}_V(t)$$
$$i_V(t) = -C\dot{u}_V(t) + I(0,t) = -C\dot{u}_V(t) + \begin{pmatrix} 0 & C_0 \end{pmatrix} \int_0^t T_T(t-s) \begin{pmatrix} \delta_0 \\ 0 \end{pmatrix} u_V(s)ds.$$

Hence, the decoupling of the abstract differential algebraic system which models the given circuit was helpful for the determination of the solution trajectory. However, by a more inspired way in writing down the circuit equations, a solution can be obtained without the decoupling procedure in this example. Nevertheless, for more complicated examples, the decoupling seems to be a reasonable method for getting inside the solvability of ADAE's and the structure of its solution. The ADAE index turned out to be 2. Moreover, it can be shown that the resolvent $(sE - A)^{-1}$ is in the space $p_\lambda \mathcal{H}_\infty(\mathbb{C}_\omega^+, Z, X)$ for all $\omega \in \mathbb{R}$. Since the coupling term K also contains some evaluations, a complete decoupling with the proposed method cannot be achieved in this example.

The example consisted of a PDE whose boundaries where coupled with some finite dimensional DAE's. This is the reason why E_i and A_i for $i = 0, 1, 2$ are preserving block structure with matrices at the upper left part, which are mainly responsible for the kernels of the E_i. Due to that fact, the projectors Q_i can be obtained by numerical computations, in principle.

6. Conclusions

In this work, we have developed a framework for analyzing linear abstract differential algebraic systems. Solvability criteria which are mainly based on Laplace transform methods have been presented. Motivated by the famous Kronecker normal form, we educed a method for decoupling of infinite dimensional differential algebraic systems. It turned out that a complete decoupling, i.e. a representation of the system, where the (hidden) algebraic and the differential conditions are totally independent, is not possible in each case of practical relevance. We established criteria for systems possessing such a complete decoupling. Another difference to the finite dimensional case is the appearance of a third relation, which has been interpreted as a boundary action in an abstract setting. The advantage of a decoupling is that one can filter out an inherent abstract ODE and the computation of a solution of the ADAE is led back to the determination of the semigroup, which the operator appearing in the inherent ODE generates. We exposed that this inherent semigroup is - up to similarity - an invariant of the system. Especially, the strong continuity of an inherent semigroup is a property of the system and not of the particular decoupling form, we choose. The main intention of the authors for developing this theory is to deal with coupled systems of partial differential and differential algebraic equations and analyze their properties.

Appendix

Before Lemma 4.6 is shown, we present three Lemmas which are essential for that proof. The first and the last lemma are proven in [14] for matrices. Since the proofs only involve symbolic matrix calculations, the results also cover the case of bounded operators and hence, we refer to that work for the proof. However, this work uses the slightly other notation $A\dot{x}(t) + Bx(t) = f(t)$ and consequently, the operator chain for the decoupling reads $A_{i+1} = A_i + B_i Q_i$, $B_{i+1} = B_i P_i$. We use the notation with E and A which is commonly employed in systems theory like e.g. [8].

LEMMA 6.1 ([14], Lemma A.1). *Let Q_j for $j = 0, \ldots, \nu - 1$ be a projector sequence according to Theorem 4.1. Define $\bar{Q}_j := -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j$ for $j = 0, \ldots, \nu - 1$.*

Then for $\bar{E}_0 = E$, $\bar{A}_0 = A$, $\bar{E}_i := \bar{E}_{i-1} - \bar{A}_{i-1}\bar{Q}_i$, $\bar{A}_i := \bar{A}_{i-1}\bar{P}_i$, *the projectors* $\bar{Q}_j$ *satisfy* $\operatorname{im} \bar{Q}_i = \ker \bar{E}_i$ *and* $\bar{Q}_i\bar{Q}_j = 0$ *for* $i > j$. *Moreover, for* $j = 0, \ldots, \nu - 1$, *we have* $\bar{E}_j = E_j F_j$, *where* $F_j := I + \bar{Q}_0 P_0 + \cdots + \bar{Q}_{j-1}P_{j-1}$. *Furthermore, it holds that*

$$F_j^{-1} := I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{j-1}P_{j-1} \tag{1}$$

$$\bar{Q}_i\bar{Q}_j = Q_i\bar{Q}_j \quad \text{for all} i < j. \tag{2}$$

LEMMA 6.2. *Let* $Q_0, \ldots, Q_{\nu-1}$ *be projectors for the decoupling of* (E, A) *and let* $Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$ *for* $j = k, \ldots, \nu - 1$. *Then, for* $j = k, \ldots, \nu - 1$, *we have* $Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j$.

Proof. For $j = \nu - 1$, we have $0 = Q_{\nu-1}E_\nu^- A_\nu = \bar{Q}_{\nu-1}P_{\nu-1}$. Hence, it holds that $\ker \bar{Q}_{\nu-1} = \ker Q_{\nu-1}$, which implies $\bar{Q}_{\nu-1} = Q_{\nu-1}$. Assuming that $\bar{Q}_j = Q_j$ for $j > k$ and $Q_k P_{k+1} \cdots P_{\nu-1} E_\nu^- A_\nu = 0$, we get

$$\begin{aligned}
0 &= \bar{Q}_k P_k P_{k+1} \cdots P_{\nu-1} \\
&= \bar{Q}_k P_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1} \\
&= \bar{Q}_k P_k - \bar{Q}_k (P_k \bar{Q}_{k+1} + P_k \bar{P}_{k+1}\bar{Q}_{k+2} + \cdots + P_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1}\bar{Q}_{\nu-1}) \\
&= \bar{Q}_k P_k - \bar{Q}_k (\bar{P}_k \bar{Q}_{k+1} + \bar{P}_k \bar{P}_{k+1}\bar{Q}_{k+2} + \cdots + \bar{P}_k \bar{P}_{k+1} \cdots \bar{P}_{\nu-1}\bar{Q}_{\nu-1}) \\
&= \bar{Q}_k P_k.
\end{aligned}$$

Now, by the same argumentation as above, we get $\bar{Q}_k = Q_k$. The second last equality holds due to the relation (2). □

LEMMA 6.3 ([14], Lemma A.2). *Let* $Q_0, \ldots, Q_{\nu-1}$ *be projectors for the decoupling of* (E, A). *Further, let* $Q_j = -Q_j P_{j+1} \cdots P_{\nu-1} E_\nu^- A_j$ *for* $j = k, \ldots, \nu - 1$. *Then we have* $Q_j = \bar{Q}_j = -\bar{Q}_j \bar{P}_{j+1} \cdots \bar{P}_{\nu-1} \bar{E}_\nu^- \bar{A}_j$ *for* $j = k, \ldots, \nu - 1$, *and, additionally* $\bar{Q}_{k-1} = -\bar{Q}_{k-1}\bar{P}_k \cdots \bar{P}_{\nu-1}\bar{E}_\nu^- \bar{A}_{k-1}$.

The remaining part of this section consists of the proof of Lemma 4.6.

Proof. Since, by Lemma 6.1, we have $\bar{E}_\nu = E_\nu F_\nu$ holds, we can choose a left inverse $\bar{E}_\nu^- = F_\nu^{-1} E_\nu^- = (I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\nu-1}P_{\nu-1})E_\nu^-$ and then, we calculate

$$\bar{P}_{\Sigma\nu}\bar{E}_\nu^- \;=\; \bar{P}_{\Sigma\nu}(I - \bar{Q}_0 P_0 - \cdots - \bar{Q}_{\nu-1}P_{\nu-1})E_\nu^- = \bar{P}_{\Sigma\nu}E_\nu^-.$$

Due to [15], Theorem 2.3, we have

$$\ker P_{\Sigma\nu} = \bigoplus_{i=0}^{\nu-1} \operatorname{im} Q_i = \bigoplus_{i=0}^{\nu-1} \operatorname{im} \bar{Q}_i = \ker \bar{P}_{\Sigma\nu}.$$

and thus, the relations $P_{\Sigma\nu}\bar{P}_{\Sigma\nu} = P_{\Sigma\nu}$ and $\bar{P}_{\Sigma\nu}P_{\Sigma\nu} = \bar{P}_{\Sigma\nu}$ are valid. Hence,

$$\begin{aligned}
&\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = \bar{P}_{\Sigma\nu}F_\nu^{-1}E_\nu^-A\bar{P}_{\Sigma\nu} \\
&\quad = \bar{P}_{\Sigma\nu}(I - \bar{Q}_0P_0 - \cdots - \bar{Q}_{j-1}P_{j-1})E_\nu^-A\bar{P}_{\Sigma\nu} \\
&\quad = \bar{P}_{\Sigma\nu}E_\nu^-A\bar{P}_{\Sigma\nu} \\
&\quad = \bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A)\bar{P}_{\Sigma\nu} \\
&\quad = \bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A_\nu)\bar{P}_{\Sigma\nu}.
\end{aligned}$$

For the last equality, (18d) was used. Using the relations above, we can write $(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l = \bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A_\nu)^l\bar{P}_{\Sigma\nu}$ for $l \in \mathbb{N}$. Now we show that $\bar{K}(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l$ is bounded for $l = 0, \ldots, k-1$. Using the relations $Q_i\bar{Q}_j = 0$ for $i > j$ and $Q_i\bar{Q}_j = \bar{Q}_i\bar{Q}_j$ for $i < j$ from Lemma 6.3, Furthermore, we need the relation (18b) for the fourth equality sign in the following calculations

$$\begin{aligned}
&-\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = -\bar{Q}_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = Q_iP_{\Sigma\nu}E_\nu^-A_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = Q_iP_{\Sigma\nu}E_\nu^-AP_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = (Q_i - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu})P_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = Q_iP_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}P_0\cdots P_{i-1}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = Q_i\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu \\
&\quad = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu - Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu.
\end{aligned}$$

Hence, we have the equation

$$2\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu.$$

Using that, we get

$$\begin{aligned}
&2\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^{l-1} \\
&\quad = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^{l-1} \\
&\quad = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l \\
&\quad = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-AP_{\Sigma\nu}\bar{P}_{\Sigma\nu}(P_{\Sigma\nu}E_\nu^-A_\nu)^l\bar{P}_{\Sigma\nu} \\
&\quad = Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu(P_{\Sigma\nu}E_\nu^-A_\nu)^l\bar{P}_{\Sigma\nu}.
\end{aligned}$$

By assumption, $Q_iP_{i+1}\cdots P_{\nu-1}E_\nu^-A_\nu(P_{\Sigma\nu}E_\nu^-A_\nu)^l$ is bounded for $l = 0, \ldots, k$. The above calculations imply that $\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu(\bar{P}_{\Sigma\nu}\bar{E}_\nu^-\bar{A}_\nu)^l$ is bounded for $l = 0, \ldots, k-1$.

Having K as in (24), we derive for $\bar{K}$

$$
-\begin{pmatrix} \bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \vdots \\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \bar{Q}_{i+1}\bar{P}_{i+2}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \vdots \\ \bar{Q}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \end{pmatrix} = -\begin{pmatrix} \bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \vdots \\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ Q_{i+1}P_{i+1}\cdots P_{\nu-1} \\ \vdots \\ Q_{\nu-1}P_{\nu-1} \end{pmatrix}
$$

$$
= -\begin{pmatrix} \bar{Q}_0\bar{P}_1\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \bar{Q}_1\bar{P}_2\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ \vdots \\ \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

The first equality holds, since, by Lemma 6.2., we get $\bar{Q}_j = Q_j$ for $j = i, \dots, \nu-1$. Then, Lemma 6. implies $\bar{Q}_i = \bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_i$, and therefore, we have $\bar{Q}_i\bar{P}_{i+1}\cdots\bar{P}_{\nu-1}\bar{E}_\nu^-\bar{A}_\nu = 0$. From that, we can conclude that (25) holds and the proof is complete. □

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REFERENCES

[1] ADAMS, R. A., *Sobolev Spaces*, Academic Press 1978.

[2] BALL, J. M., *Strongly Continuous semigroups, weak solutions and the variations of constants formula*, Proc. AMS, vol. *63* (1977), no. 2, 370–373.

[3] BARTEL, A., *First order thermal PDAE models in electric circuit design*, Troch, I., F. Breitenecker, F. (eds.) Proc. 4th Mathmod, Vienna 2003.

[4] BODESTEDT, M. and TISCHENDORF, C., *PDAE Models of Integrated Circuits and Index Analysis*, To appear in Math. Comput. Model. Dyn. Syst.

[5] CAMPBELL, S. L., *Singular Systems of Differential Equations I and II*, Pitman Advanced Publishing Program 1982.

[6] CHUA, L. O. and DESOER, C. A. and KUH, E. S., *Linear and Nonlinear Circuits*, McGraw-Hill 1987.

[7] CURTAIN, R. F. and ZWART, H. J., *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer 1995.

[8] DAI, L., *Singular Control Systems*, Springer 1988.

[9] DIAZ, J. and FUEHRER, C., *A wavelet semidiscretisation of elastic multibody systems*, ZAMM *83* (2003), no. 10, 677–689.

[10] ENGEL, K. J. and NAGEL, R., *One-parameter semigroups for linear evolution equations*, Springer 2000.

[11] FAVINI, A. and YAGI, A., *Degenerate Differential Equations in Banach Spaces*, Pure and Applied Mathematics, vol. *215*, Dekker 1999.

[12] GRIEPENTROG, E. and MAERZ, R., *Differential Algebraic Equations*, Springer 1988.

[13] LAMOUR, R., MAERZ, R. and TISCHENDORF, C., *PDAE's and Further Mixed Systems as Abstract Differential Algebraic Systems* Preprint, Humboldt Universität zu Berlin 2001.

[14] MAERZ, R., *Canonical Projectors for Linear Differential Algebraic Equations*, Comp. Math. Appl., vol. *31* (1996), no. 4/5, 121–135.

[15] MAERZ, R., *Solvability of linear differential algebraic equations with properly stated leading terms*. Preprint Humboldt Universität zu Berlin 2002.

[16] NEUBRANDER, F., *Integrated Semigroups and their Applications to the Abstract Cauchy Problem*, Pacific Journal of Mathematics, vol. *135* (1988), no. 1, 111–155.

[17] REIS, T., *An Infinite Dimensional Descriptor System Model for Electrical Circuits with Transmission Lines*, Proc. MTNS 2004, Leuven, Belgium 2004.

[18] REIS, T., *A System Theoretic Approach to Partial Differential Algebraic Equations and Applications to Electrical Circuits*, PhD-Thesis, in preparation 2005.

[19] RENARDY, M. and ROGERS, R. C., *An introduction to partial differential equations*, Springer 1996.

[20] RUDIN, W., *Functional Analysis*, McGraw-Hill 1991.

[21] SALAMON, D., *Infinite Dimensional Linear Systems with Unbounded Control and Observation: A Functional Analytic Approach*, Trans. AMS, vol. *300* (1987), no. 2, 383–430.

[22] SCHULZ, S., *Ein PDAE-Newtzwerkmodell als Abstraktes Differential-Algebraisches System*, Diploma Thesis, Humboldt Universität zu Berlin 2002.

[23] TISCHENDORF, C., *Coupled systems of differential algebraic and partial differential equations in circuit and device simulation. Modeling and numerical analysis*, Habilitation Thesis, Humboldt Universität zu Berlin 2003.

[24] WEISS, G., *Transfer functions of regular linear systems, part I: Characterizations of regularity*, Trans AMS, *342*, no. 2, 827–854.

Timo Reis
Fachbereich Mathematik
TU Kaiserslautern
reis@mathematik.uni-kl.de

Caren Tischendorf
Institut für Mathematik
TU Berlin
tischend@math.tu-berlin.de