Finite-time stability and stabilization of nonlinear singular time-delay systems via Hamiltonian method[☆]

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Abstract

This paper investigates the finite-time stability (FTS) and finite-time stabilization for a class of nonlinear singular time-delay Hamiltonian systems, and proposes a number of new results on these issues. Firstly, an equivalent form is obtained for the nonlinear singular time-delay Hamiltonian systems by the singular matrix decomposition method, based on which some delay-independent and delay-dependent conditions on the FTS are derived for the systems by constructing a kind of novel Lyapunov function. Secondly, we use the equivalent form as well as the energy shaping plus damping injection technique to investigate the finite-time stabilization problem for a class of nonlinear singular port-controlled Hamiltonian (PCH) systems with time delay, and present a specific control design procedure for the systems. Finally, we give several illustrative examples to show the effectiveness of the results obtained in this paper.
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1. Introduction

Singular system, which is called as descriptor system, implicit system, generalized state-space system, differential-algebraic system, or semistate system, has been extensively studied since it can better describe some real physical systems than normal one [4,13]. In the past years, many results have been extended to the singular systems based on the theories of normal ones [11,14,15,24,26,30,31,42,48] and references therein. In addition to singularity, time delay is also often encountered in many practical control systems [7,16], and thus the singular time-delay systems can be closer to the real systems [6,48]. However, because singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study for such systems is much more complicated than that for normal time-delay systems or singular systems without delay [48]. Up to now, it is still a challenging issue for the study of general nonlinear singular time-delay systems due to the lack of effective research tools.

As well known, the port-controlled Hamiltonian (PCH) system is a kind of important nonlinear system because such system is widely present in many real fields such as physical sciences, life sciences and engineering sciences, and many models in classical mechanics, celestial mechanics, bioengineering and power systems have appeared in their Hamilton forms. Moreover, the Hamiltonian function in a PCH system is considered as the total energy and can be used as a candidate of Lyapunov function in many real systems [37,38]. In view of these, many scholars devoted to study the systems and presented some important results [37]. In particular, since some Hamiltonian realization methods were proposed in Reference [38] for general nonlinear systems, it becomes an effective tool for studying general nonlinear systems, and has successfully solved many nonlinear problems [3,19,27,28,31,40]. Recently, the Hamiltonian approach has been also extended to study the analysis and synthesis of both nonlinear time-delay systems and nonlinear singular systems without delay, and some nice results have been developed in [21,28–30,43].

However, it is well worth pointing out that the results mentioned are of asymptotic ones. In general, an asymptotically stable controller cannot guarantee that the system under study achieves the control performance of fast convergence. While the finite-time controller possesses fast convergence, and better robustness and disturbance attenuation properties [8,9]. Therefore, it is particularly suitable for studying many real systems and has always attracted considerable attention of many scholars and many results on the issue have been obtained for nonlinear (or nonlinear time-delay) systems [2,9,10,12,17,20,41,44–48]. The papers [2,9,10,12,17,20,44,45] and [48] studied the finite-time stability and stabilization problems on nonlinear systems and nonlinear time-delay systems. In [46,47], the authors obtained several novel finite-time control results for a class of complicated switched nonlinear systems by developing some new techniques. Recently, the finite-time problem on nonlinear singular system without delay has been investigated in [31]. Based on Hamiltonian and state decomposition method, the authors of [31] have studied the finite-time control problem, and designed the finite-time H_∞ controller for the singular systems without delay. However, it should be pointed out that the method used in [31] cannot be extended to study the finite-time problem of nonlinear singular time-delay systems. In fact, on one hand, since there exist simultaneously delay and nonlinear terms, one cannot apply the index-one condition and the special matrix form required in [31], and on the other hand, there is no the finite-time stability criterion for nonlinear singular time-delay system similar to the normal one and singular one without delay. Therefore, to the authors' best knowledge, there are few works on the finite-time analysis

and synthesis for nonlinear singular time-delay systems in the existing literature. This paper attempts to fill the gap.

In this paper, motivated by the derivative feedback control method [1,4,5,18,23], we investigate the FTS and finite-time stabilization problem for a class of nonlinear singular time-delay systems via Hamiltonian method and propose a number of new results. The main contributions of this paper are as follows: 1) Different from the systems studied and methods used in existing papers [4,26,30,31,35,39,42,48], we investigate a class of nonlinear singular time-delay systems via Hamiltonian method in the paper. Based on the derivative feedback technique and the singular matrix decomposition method, the paper presents an equivalent form of the original system. Thanks to the equivalence system, the paper can study the finite-time problem of nonlinear singular time-delay systems. It should be pointed out that since there exist nonlinear and delay terms in the system, and there is no a finite-time stability criterion for the nonlinear singular time-delay system, the traditional method of studying singular system without delay and with delay cannot be used. 2) In the paper, by constructing a novel Lyapunov function, we propose several delay-dependent results on the FTS and finite-time stabilization problem. Note that it is a very difficult task to obtain the delay-dependent condition with respect to the finite-time problem by applying the Lyapunov method because one cannot use the model transformation method for the systems with special requirements(Further details, please see Remark 7 below of the paper). 3) Different from the recent work on the finite-time problem for singular Hamiltonian system without delay [31], the paper studies the finite-time stability and stabilization problem simultaneously for a class of nonlinear singular time-delay systems and designs a practical controller for a class of real circuit systems.

The remainder of the paper is organized as follows. Section 2 is the problem formulation and preliminaries. In Section 3, the FTS is studied for a class of nonlinear singular Hamiltonian systems with time delay, and the finite-time stabilization problem is discussed in Section 4. Section 5 gives an illustrative example to support our new result, which is followed by the conclusion in Section 6.

Notation: Throughout this paper, the notes are standard.

2. Problem formulation and preliminaries

Consider the following nonlinear singular time-delay Hamiltonian system:

$$\begin{cases} E\dot{x}(t) = R(x)\nabla_x H(x) + T(x,\tilde{x})\nabla_{\tilde{x}} H(\tilde{x}), \\ x(\tau) = \phi(\tau), \quad \forall \tau \in [-h, \ 0], \end{cases} \tag{1}$$

where $x(t) \in \Omega \subseteq \mathbb{R}^n$ is the state, Ω is some bounded closed convex set of the origin, $R(x)$ ($\in \mathbb{R}^{n \times n}$) satisfies $R(x) + R^T(x) < 0$ for all $x \in \Omega$, $T(x, \tilde{x}) \in \mathbb{R}^{n \times n}$ with $T(0, \tilde{x}) = 0$, $\tilde{x} := x(t - h)$, $h > 0$ is a constant time delay, E ($\in \mathbb{R}^{n \times n}$) is a singular matrix satisfying rank$(E) = r < n$, $\phi(\tau)$ is a vector-valued initial-condition function, and $H(x)$ is the Hamiltonian function of the system (1) in the form of

$$H(x) = \sum_{i=1}^{n} (x_i^2)^{\frac{\alpha}{2\alpha-1}}, \quad \alpha > 1. \tag{2}$$

Throughout this paper, similar to [17,39], we assume that the system (1) has a unique continuous solution without impulses in the forward time for any admissible initial conditions

ϕ. Furthermore, we assume that

$$E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{3}$$

Remark 1. If E is an arbitrary matrix satisfying $rank(E) = r < n$, then it is easy to see from the matrix theory [4] that there exist nonsingular matrices $\bar{P}$ and $\bar{Q}$ such that $\tilde{E} = \bar{Q}E\bar{P} = \text{Diag}\{I_r, 0\}$. Therefore, in the paper, we always assume that the matrix E is in the form of Eq. (3).

Remark 2. In the section, to obtain the FTS result of the singular time-delay system, the paper first studies a kind of special Hamiltonian system (1) with (2). The main reasons are as follows: on one hand, a system with general Hamiltonian form may not be FTS ([2,44,45]), and on the other hand, if the FTS result is developed for the special system (1) and (2), then one can study easily the finite-time stabilization problem (or other control problems) for general one by designing suitable controller (Further details, also see Sections 4 and 5 below). Moreover, it should be pointed out that although the paper studies the finite-time problem of a class of singular time-delay Hamiltonian systems (1), the methods used in the paper can be extended to investigate general nonlinear singular time-delay systems by applying Hamiltonian realization methods [37,38]. In fact, from [37], it is easy to know that any system can be converted into its Hamiltonian form by applying the orthogonal decomposition method, which implies that the method proposed in the paper can also be used to study the general singular time-delay one(Further details, also see [37,38]).

In the following, we give some preliminaries. Consider the following normal delay system:

$$\begin{cases} \dot{x}(t) = f(x_t), \\ x_{t0}(\tau) = \phi(\tau), & \forall \tau \in [-h, \ 0], \end{cases} \tag{4}$$

where $x_t = x_t(\tau) := x(t + \tau), \tau \in [-h, 0], f(x_t)(\in R^n)$ is a continuous vector field satisfying $f(0) = 0$.

Definition 1 ([17]). Assume that the system (4) possesses uniqueness of the solution in forward time. The system (4) is finite-time stable if the system (4) is stable, and there exists $\delta > 0$ such that, for any $\phi \in C_\delta$, there exists $0 \leq T(\phi) < +\infty$ for which $x(t, \phi) = 0$ for all $t \geq T(\phi)$, where $C_\delta := \{\phi \in \pounds_h^n : \|\phi\|_{\pounds_h^n} < \delta\}$. $T_0(\phi) = \inf\{T(\phi) \geq 0 : x(t, \phi) = 0, \forall t \geq T(\phi)\}$ is a functional called the settling time of the system (4).

Definition 2. Given $F(x)(x \in R^n)$ and $G(x)(x \in R^n)$, if $F(x)$ and $G(x)$ are two homogeneous functions ([10]) and the order of each variable in the function $G(x)$ is less than the one of the related variable (If it exists) in the function $F(x)$, then $F(x)$ is called as the high-order term of $G(x)$.

Lemma 1 ([7]). *Consider the system (4). Assume that $u, v, w : R^+ \to R^+$ with u, v being two class $\mathcal{K}$ functions and w being a continuous non-decreasing function. If there exists a continuously differentiable function $V : [0, +\infty) \times R^n \to R^1$, such that*

1). $u(\|x\|) \leq V(t, x) \leq v(\|x\|)$, *and*
2). $\dot{V}(t, x) \leq -w(\|x\|)$

hold along all the trajectories of the system for $t \in [0, \ +\infty)$ whenever $V(t + \tau, x(t + \tau)) \leq V(t, x(t))$ ($\tau \in [-h, 0]$), then the system (4) is uniformly stable. Furthermore, if $w(s) > 0, s >$

0 *and there exists a continuous non-decreasing function $p(s) > s$, $s > 0$ such that $\dot{V}(t, x) \leq -w(\|x\|)$ holds whenever $V(t + \tau, x(t + \tau)) \leq p(V(t, x(t)))$ $(\tau \in [-h, 0])$, then the system (4) is uniformly asymptotically stable.*

Lemma 2 ([2]). *Consider the system (4). If there exists $\beta > 1$ and a C^1 radially unbounded Lyapunov function $V(x)$ such that*

$$\dot{V} \leq -kV^{\frac{1}{\beta}}(x), k > 0 \tag{5}$$

holds along the trajectories of the system starting from any initial value in R^n, then the origin is a globally finite-time stable.

Lemma 3 ([9]). *If the equilibrium of the system (4) is globally asymptotically stable, and meantime locally finite-time stable, then the system is globally finite-time stable.*

Lemma 4. *For any real matrices Σ_1, Σ_2, $0 < \Sigma_3 = \Sigma_3^T$ of appropriate dimensions and a scalar $\epsilon > 0$, the inequality holds: $\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \leq \epsilon \Sigma_1^T \Sigma_3 \Sigma_1 + \epsilon^{-1} \Sigma_2^T \Sigma_3^{-1} \Sigma_2$.*

Lemma 5 ([43]). *For any given constant number $p \geq 1$, we have $n^{\frac{p-1}{p}} (\sum_{i=1}^{n} |x_i|)^{\frac{1}{p}} \geq \sum_{i=1}^{n} |x_i|^{\frac{1}{p}} \geq (\sum_{i=1}^{n} |x_i|)^{\frac{1}{p}}$, where $|\cdot|$ denotes the absolute value function.*

Remark 3. From Lemma 5, it is obvious that, for any real numbers $a \geq 0$, $b \geq 0$ and a given $\alpha > 1$, the following inequality holds.

$$a^{\frac{1}{\alpha}} + b^{\frac{1}{\alpha}} \geq (a + b)^{\frac{1}{\alpha}}. \tag{6}$$

3. Finite-time stability

This section studies the FTS of the system (1) by using Lemmas 1, 2 and 3. In this section, the main difficulty is how to deal with the singular matrix and construct a suitable Lyapunov function satisfying the FTS conditions.

Consider the system (1). Motivated by the derivative feedback control method ([1,4,5,18,23]), we decompose E as follows

$$E := I_n - E_1, \tag{7}$$

where $E_1 = \text{Diag}\{0, I_{n-r}\}$. Obviously,

$$E_1 E_1 = E_1. \tag{8}$$

Thus, based on Eq. (7), the system (1) can be expressed as

$$\begin{cases} \dot{x}(t) = R(x)\nabla_x H(x) + T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + y(t), \\ x(\tau) = \phi(\tau), \quad \forall \tau \in [-h, \ 0], \end{cases} \tag{9}$$

where $y(t) := E_1 \dot{x}(t)$. From Eq. (8), it is easy to obtain that

$$E_1 y = E_1 \dot{x}(t), \tag{10}$$

with which Eq. (9) is rewritten as

$$\begin{cases} \dot{x}(t) = R(x)\nabla_x H(x) + T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + E_1 y(t), \\ x(\tau) = \phi(\tau), \quad \forall \tau \in [-h, \ 0]. \end{cases} \tag{11}$$

Obviously, the system (11) is equivalent to the system (1).

Remark 4. Noting that there is not a similar Lyapunov theory to Lemma 1 for singular delay system. Moreover, it is very difficult to generalize the results of both linear time-delay systems and nonlinear systems [39,42] to the system (1) by using the condition of index one since there exists nonlinear time-delay term in the system (1). Therefore, motivated by the derivative feedback control method ([1,4,5,18,23]), we have introduced the equivalent form Eq. (11) to study the FTS of the system (1).

Now, we give the main results.

Theorem 1. *Consider the system (1) with the Hamiltonian function in the form of Eq. (2). Assume that $\nabla_x^T H(x(t))E_1\dot{x}(t)$ is continuous, $\nabla_x^T H(x(t))E_1\dot{x}(t)$ is the high-order term of $H(Ex)$, $\lambda_{\max}\{T^T(x,\tilde{x})T(x,\tilde{x})\}$ is the high-order term of $H(x)$ respectively. Furthermore, assume that there exist constant matrices L, S with appropriate dimensions such that $R^T(x) + R(x) \leq L$, $T(x,\tilde{x})T^T(x,\tilde{x}) \leq S$, $m := \lambda_{\max}(L+S) + n^{\frac{\alpha-1}{\alpha}} < 0$ hold on Ω, where n is the state's dimension and $\lambda_{\max}(*)$ denotes the maximum eigenvalue of $*$. Then, the system (1) is locally finite-time stable.*

Proof. Note that the system (1) is equivalent to the system (11). In the following, we need to prove the FTS of the system (11).

Since $\nabla_x^T H(x(t))E_1\dot{x}(t)$ is continuous, it is easy to obtain that there exists a constant number $\nu > 0$ such that

$$\left|\nabla_x^T H(x(t))E_1\dot{x}(t)\right| \leq \nu, \quad x \in \Omega. \tag{12}$$

In addition, noting that $\nabla_x^T H(x(t))E_1\dot{x}(t)$ is a high order term of $H(Ex)$, it is obvious that there exists some small constant number ς ($\nu \geq \varsigma > 0$) satisfying

$$\|\nabla_x^T H(x(t))E_1\dot{x}(t)\| \leq H(Ex) \leq H(x) \tag{13}$$

for $0 \leq H(x) \leq \varsigma$.

Construct a Lyapunov function candidate as follows:

If $\nabla_x^T H(x(t))E_1\dot{x}(t) \nrightarrow 0$ for $t \to \infty$, let

$$V(t, x(t)) = 2G(t)H(x(t)), \tag{14}$$

and if $\nabla_x^T H(x(t))E_1\dot{x}(t) \to 0$ for $t \to \infty$, set

$$V(t, x(t)) = 2H(Ex(t)), \tag{15}$$

where $G(t) := e^{-\int_\varrho^t \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+\nu)ds}$ with $\varrho > 0$ being a constant number.

Next, under the case: $\nabla_x^T H(x(t))E_1\dot{x}(t) \nrightarrow 0$ ($t \to \infty$), we prove the result is true.

First, we prove that the system (11) is asymptotically stable under the conditions of the theorem by applying Lemma 1. To do this, we need to prove that there exist two positive constant numbers κ_1 and κ_2 such that

$$\kappa_2 \geq e^{-\int_\varrho^t \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+\nu)ds} \geq \kappa_1 \tag{16}$$

holds on Ω for all $t > 0$, which is considered in the following two cases.

***Case 1* : $t \geq \varrho$.**

In the case, based on Eq. (12), it is easy to see that $\int_\varrho^t \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s) + \nu)ds \geq 0$, namely

$$1 \geq e^{-\int_\varrho^t \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+\nu)ds} \tag{17}$$

holds on Ω. On the other hand, using Eq. (12) and $0 \leq H(E_1 x) \leq H(x)$, we have

$$
\begin{aligned}
&\int_\varrho^t \frac{1}{H(x(s)) + \varsigma} (\nabla_x^T H(x(s)) E_1 \dot{x}(s) + v) ds \\
&\leq \int_\varrho^t \frac{1}{H(E_1 x(s)) + \varsigma} (\nabla_x^T H(x(s)) E_1 \dot{x}(s) + v) ds \\
&= \int_\varrho^t \frac{1}{H(E_1 x(s)) + \varsigma} d(H(E_1 x(s)) + \varsigma) + \int_\varrho^t \frac{v}{H(E_1 x(s)) + \varsigma} ds \\
&= ln\left(\frac{H(E_1 x(t) + \varsigma)}{H(E_1 x(\varrho)) + \varsigma} \right) + v \int_\varrho^t \frac{H(E_1 x(s)) + \varsigma}{(H(E_1 x(s)) + \varsigma)^2} ds.
\end{aligned}
\tag{18}
$$

Note that there exists a constant number $l_1 > 0$ such that

$$
H(E_1 x(s)) + \varsigma \leq l_1
\tag{19}
$$

holds on Ω.

For the term $\int_\varrho^t \frac{1}{[H(E_1 x(s)) + \varsigma]^2} ds$, we have

$$
\begin{aligned}
\int_\varrho^t \frac{1}{[H(E_1 x(s)) + \varsigma]^2} ds &= \int_\varrho^t \frac{1}{[H(E_1 x(s)) - H(E_1 x(0)) + \varsigma + H(E_1 x(0))]^2} ds \\
&= \int_\varrho^t \frac{1}{[s\frac{d(H(E_1 x(\bar{\xi})))}{ds} + \varsigma + H(E_1 x(0))]^2} ds = \int_\varrho^t \frac{1}{s^2[\frac{dH(E_1 x(\bar{\xi}))}{ds} + \frac{H(E_1 x(0)) + \varsigma}{s}]^2} ds,
\end{aligned}
\tag{20}
$$

where $\bar{\xi} \in [0, s]$.

Since $[H(E_1 x(s)) + \varsigma]^2 \geq \varsigma^2$, it is easy to obtain that

$$
s^2 \left(\frac{dH(E_1 x(\bar{\xi}))}{ds} + \frac{H(E_1 x(0)) + \varsigma}{s} \right)^2 \geq \varsigma^2 > 0
\tag{21}
$$

holds for arbitrary $s \in [\varrho, t]$.

Obviously, if s is a finite number, then from Eq. (21), it is easy to obtain that there exists a constant number $\varphi > 0$ such that $(\frac{dH(E_1 x(\bar{\xi}))}{ds} + \frac{H(E_1 x(0)) + \varsigma}{s})^2 \geq \varphi$. On the other hand, if $s \to \infty$, using $\nabla_x^T H(x(s)) E_1 \dot{x}(s) = \frac{dH(E_1 x(s))}{ds} \nrightarrow 0$ $(s \to \infty)$ and $\frac{H(E_1 x(0)) + \varsigma}{s} \to 0$, one can obtain that there exists a positive real number $\tilde{l}_2$ such that $(\frac{dH(E_1 x(\bar{\xi}))}{ds} + \frac{H(E_1 0) + \varsigma}{s})^2 \geq \tilde{l}_2$.

From the above, we have

$$
\frac{1}{\left(\frac{dH(E_1 x(\bar{\xi}))}{ds} + \frac{H(E_1 0) + \varsigma}{s} \right)^2} \leq \frac{1}{l_2},
\tag{22}
$$

where $l_2 = max\{\varphi, \tilde{l}_2\}$.

Substituting Eqs. (19), (20) and (22) into Eq. (18), we have

$$
\begin{aligned}
\int_\varrho^t \frac{1}{H(x(s)) + \varsigma} \left(\nabla_x^T H(x(s)) E_1 \dot{x}(s) + v \right) ds &\leq ln\left(\frac{H(E_1 x(t)) + \varsigma}{H(E_1 x(\varrho)) + \varsigma} \right) + \frac{v l_1}{l_2} \int_\varrho^t \frac{1}{s^2} ds \\
&\leq ln\left(\frac{H(E_1 x(t)) + \varsigma}{H(E_1 x(\varrho)) + \varsigma} \right) + \frac{v l_1}{l_2} \frac{1}{\varrho} := \iota,
\end{aligned}
\tag{23}
$$

which implies that

$$e^{-\int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v)ds} \geq e^{-\iota}. \tag{24}$$

***Case** 2:* $0 < t < \varrho$.

In the case, since $\nabla_x^T H(x(s))E_1\dot{x}(s)$ is continuous, it is easy to obtain that there exists a constant number $l_3 < 0$ such that $\int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v)ds = l_3$, from which it follows that $e^{-\int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v)ds} = e^{-l_3}$ holds on Ω. Summarizing Cases 1 and 2, Eq. (16) holds, where $\kappa_1 := \min\{e^{-\iota},\ e^{-l_3}\}$ and $\kappa_2 := e^{-l_3}$.

Therefore, from Eq. (16), one can obtain that

$$2\kappa_2 H(x) \geq V(t,x) \geq 2\kappa_1 H(x), \tag{25}$$

from which and the fact that $H(x)$ is positive definite, we obtain that $V(t, x)$ is positive definite on Ω.

To prove Condition 2) of Lemma 1, based on the Razumikhin method(R-condition): $V(t, x(t+\tau)) \leq pV(t, x(t))$, we show that

$$H(x(t+\tau)) \leq pH(x(t)), \quad \tau \in [-h, 0] \tag{26}$$

holds, where $V(t, x(t))$ is given in Eq. (14). (**Note:** In the paper, similar to [7], let $p = 1 + \varepsilon$ with ε being a sufficiently small positive number.)

In fact, to obtain Eq. (26), we only need to show

$$e^{-\int_{\varrho}^{t+\tau} \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v)ds} \geq e^{-\int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v)ds}, \tag{27}$$

holds, which can be divided into the following two cases.

***Case** 1 :* $t + \tau \geq \varrho$.

In this case, since $\varrho \leq t + \tau \leq t$ and $\frac{1}{H(x(s))+\varsigma}(\nabla_x^T H(x(s))E_1\dot{x}(s)+v) \geq 0$, we have

$$\int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}\left(\nabla_x^T H(x(s))E_1\dot{x}(s)+v\right)ds$$

$$\geq \int_{\varrho}^{t+\tau} \frac{1}{H(x(s))+\varsigma}\left(\nabla_x^T H(x(s))E_1\dot{x}(s)+v\right)ds \geq 0,$$

from which it is easy to obtain that Eq. (27) holds.

***Case** 2 :* $t + \tau \leq \varrho$.

In this case, we need to consider two sub-cases: (i) $t \geq \varrho$ or (ii) $t < \varrho$.

If (i) holds, since $t + \tau \leq \varrho \leq t$, we obtain

$$\int_{\varrho}^{t+\tau} \frac{1}{H(x(s))+\varsigma}\left(\nabla_x^T H(x(s))E_1\dot{x}(s)+v\right)ds$$

$$\leq 0 \leq \int_{\varrho}^{t} \frac{1}{H(x(s))+\varsigma}\left(\nabla_x^T H(x(s))E_1\dot{x}(s)+v\right)ds,$$

that is,

$$-\int_{\varrho}^{t} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds$$

$$\leq -\int_{\varrho}^{t+\tau} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds,$$

which implies that Eq. (27) holds.

If (ii) is true, since $t + \tau < t < \varrho$, we have

$$\int_{\varrho}^{t+\tau} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds$$

$$\leq 0, \quad \int_{\varrho}^{t} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds \leq 0,$$

and

$$\int_{t+\tau}^{\varrho} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds$$

$$\geq \int_{t}^{\varrho} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds \geq 0,$$

that is,

$$-\int_{\varrho}^{t} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds$$

$$\leq -\int_{\varrho}^{t+\tau} \frac{1}{H(x(s)) + \varsigma}\Big(\nabla_x^T H(x(s))E_1\dot{x}(s) + v\Big)ds,$$

which implies that Eq. (27) holds.

Therefore, from Eq. (27) and the R-condition, one can obtain Eq. (26).

Now, we show that the condition 2) of Lemma 1 holds on Ω. To do this, we prove that

$$\frac{\varsigma}{H(x) + \varsigma}\nabla_x^T H(x)E_1\dot{x} - \frac{H(x)}{H(x) + \varsigma}v \leq 0. \tag{28}$$

Obviously, if $H(x) = 0$, then $\nabla_x^T H(x)E_1\dot{x} = 0$, and thus Eq. (28) holds. If $0 < H(x) \leq \varsigma$, using Eq. (13) and $v \geq \varsigma > 0$, one can obtain that Eq. (28) holds. For $H(x) > \varsigma$, noting that $\frac{\varsigma}{H(x)+\varsigma}$ is a decreasing function and $\frac{H(x)}{H(x)+\varsigma}$ is a increasing function on $H(x)$, we have $\frac{\varsigma}{H(x)+\varsigma} < \frac{1}{2}$ and $\frac{H(x)}{H(x)+\varsigma} > \frac{1}{2}$, with which and Eq. (12), it is easy to obtain that Eq. (28) holds. From the above, we obtain that Eq. (28) holds.

Computing the derivative of $V(t, x)$ along the trajectory of the system (11) and noting that Eq. (28) yield

$$\dot{V}(t, x) = G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + 2\nabla_x^T H(x)[T(x, \tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) + E_1 y(t)]$$

$$-2\frac{H(x)}{H(x) + \varsigma}[\nabla_x^T H(x)E_1\dot{x} + v]\Big\}$$

$$= G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x, \tilde{x})\nabla_{\tilde{x}}H(\tilde{x})$$

$$
\begin{aligned}
&+2\frac{\varsigma}{H(x)+\varsigma}\nabla_x^T H(x)E_1\dot{x} - 2\frac{H(x)}{H(x)+\varsigma}v\Bigg\} \\
&\leq G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x,\tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\Big\} \\
&\leq G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + \nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) + \nabla_x^T H(x)T(x,\tilde{x})T^T(x,\tilde{x})\nabla_x H(x)\Big\} \\
&= G(t)\Bigg\{\nabla_x^T H(x)[R(x)+R^T(x)+T(x,\tilde{x})T^T(x,\tilde{x})]\nabla_x H(x) + \Big(\frac{2\alpha}{2\alpha-1}\Big)^2\sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{1}{2\alpha-1}}\Bigg\} \\
&\leq G(t)\Bigg\{(\lambda_{\max}(L+S)\nabla_x^T H(x)\nabla_x H(x) + \Big(\frac{2\alpha}{2\alpha-1}\Big)^2\sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{1}{2\alpha-1}}\Bigg\} \\
&= G(t)\Bigg\{(\lambda_{\max}(L+S)\Big(\frac{2\alpha}{2\alpha-1}\Big)^2\sum_{i=1}^{n}(x_i^2)^{\frac{1}{2\alpha-1}} + \Big(\frac{2\alpha}{2\alpha-1}\Big)^2\sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{1}{2\alpha-1}}\Bigg\}. \quad (29)
\end{aligned}
$$

Using Lemma 5, and Eq. (26), one can obtain that

$$
\sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{1}{2\alpha-1}} = \sum_{i=1}^{n}[(\tilde{x}_i^2)^{\frac{\alpha}{2\alpha-1}}]^{\frac{1}{\alpha}} \leq n^{\frac{\alpha-1}{\alpha}}\Bigg[\sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{\alpha}{2\alpha-1}}\Bigg]^{\frac{1}{\alpha}} = n^{\frac{\alpha-1}{\alpha}}(H(\tilde{x}))^{\frac{1}{\alpha}} \leq n^{\frac{\alpha-1}{\alpha}}p^{\frac{1}{\alpha}}(H(x))^{\frac{1}{\alpha}}, \quad (30)
$$

$$
\sum_{i=1}^{n}(x_i^2)^{\frac{1}{2\alpha-1}} = \sum_{i=1}^{n}[(x_i^2)^{\frac{\alpha}{2\alpha-1}}]^{\frac{1}{\alpha}} \geq \Bigg[\sum_{i=1}^{n}(x_i^2)^{\frac{\alpha}{2\alpha-1}}\Bigg]^{\frac{1}{\alpha}} = (H(x))^{\frac{1}{\alpha}}. \quad (31)
$$

Substituting Eq. (30) and (31) into Eq. (29) and noting that $\lambda_{\max}(L+S) < 0$, we have

$$
\begin{aligned}
\dot{V}(t,x) &\leq G(t)\Bigg\{\lambda_{\max}(L+S)\Big(\frac{2\alpha}{2\alpha-1}\Big)^2(H(x))^{\frac{1}{\alpha}} + n^{\frac{\alpha-1}{\alpha}}\Big(\frac{2\alpha}{2\alpha-1}\Big)^2 p^{\frac{1}{\alpha}}\Big(H(x)\Big)^{\frac{1}{\alpha}}\Bigg\} \\
&= G(t)\Bigg\{\Big(\lambda_{\max}(L+S) + n^{\frac{\alpha-1}{\alpha}}p^{\frac{1}{\alpha}}\Big)\Big(\frac{2\alpha}{2\alpha-1}\Big)^2(H(x))^{\frac{1}{\alpha}}\Bigg\} \\
&=: m_1 G(t)\Big(\frac{2\alpha}{2\alpha-1}\Big)^2(H(x))^{\frac{1}{\alpha}}. \quad (32)
\end{aligned}
$$

Using Eq. (6), we obtain $p^{\frac{1}{\alpha}} = (1+\varepsilon)^{\frac{1}{\alpha}} \leq 1+\varepsilon^{\frac{1}{\alpha}}$, with which, $m < 0$ and ε being a sufficiently small positive number, it is easy to obtain that $m_1 \leq \lambda_{\max}(L+S) + n^{\frac{\alpha-1}{\alpha}}(1+\varepsilon^{\frac{1}{\alpha}}) < 0$. From Lemma 1, we know that the system is locally asymptotically stable.

Second, we prove the system (1) is finite-time stable by applying Lemmas 3 and 2. That to say, it needs to prove there exists some small neighborhood $\bar{\Omega} \subseteq \Omega$ of the origin such that the system (1) is finite-time stable.

In fact, since $\lambda_{\max}\{T^T(x,\tilde{x})T(x,\tilde{x})\}$ is a high order term of $H(x)$ and $T(0,\tilde{x}) = 0$, one can obtain that $\lim_{\|x\|\to 0}\frac{\lambda_{\max}\{T^T(x,\tilde{x})T(x,\tilde{x})\}}{H(x)} = 0$, which implies that, for some sufficiently small positive number c, there always exists a real number $\mu > 0$ such that

$$
\frac{\lambda_{\max}\{T^T(x,\tilde{x})T(x,\tilde{x})\}}{H(x)} < c \quad \text{if} \quad 0 \neq \|x\| \leq \mu. \quad (33)
$$

Let $\Omega_1 := \{x: \|x\| \leq \mu\}$ and take $\bar{\Omega} := \Omega_1 \cap \Omega$, then we have $\bar{\Omega} \subseteq \Omega$.

Next, we show the conditions of Lemma 2 hold for $x \in \bar{\Omega}$.

To do this, choose the same Lyapunov function candidate as Eq. (14). Computing the derivative of $V(t, x)$ along the trajectory of the system (1), similar to the above proof and using Eq. (33), we have

$$
\begin{aligned}
\dot{V}(t, x) &\leq G(t)\left\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\} \\
&\leq G(t)\left\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + \nabla_{\tilde{x}}^T H(\tilde{x})T^T(x, \tilde{x})T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + \nabla_x^T H(x)\nabla_x H(x)\right\} \\
&\leq G(t)\left\{\nabla_x^T H(x)[R(x) + R^T(x) + I_n]\nabla_x H(x)\right. \\
&\quad\left. + \lambda_{\max}\{T^T(x, \tilde{x})T(x, \tilde{x})\}\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\} \\
&\leq G(t)\left\{\nabla_x^T H(x)[R(x) + R^T(x) + I_n]\nabla_x H(x) + cH(x)\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\} \\
&\leq G(t)\left\{\lambda_{\max}\{L + I_n\}\nabla_x^T H(x)\nabla_x H(x) + cH(x)\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\} \\
&= G(t)\left\{\lambda_{\max}\{L + I_n\}\left(\frac{2\alpha}{2\alpha - 1}\right)^2 H^{\frac{1}{\alpha}}(x) + cH(x)\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\}.
\end{aligned}
\tag{34}
$$

For $H(x)\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})$, using Lemma 5, we obtain

$$
\begin{aligned}
H(x) &= \nabla_x^T H(x)N(x)\nabla_x H(x) \leq \lambda_{\max}\{N(x)\}\nabla_x^T H(x)\nabla_x H(x) \\
&= \left(\frac{2\alpha}{2\alpha - 1}\right)^2 \lambda_{\max}\{N(x)\}\sum_{i=1}^n \left(x_i^2\right)^{\frac{1}{2\alpha - 1}} \leq \left(\frac{2\alpha}{2\alpha - 1}\right)^2 \lambda_{\max}\{N(x)\}n^{\frac{\alpha - 1}{\alpha}}(H(x))^{\frac{1}{\alpha}},
\end{aligned}
$$

where $N(x) := (\frac{2\alpha - 1}{2\alpha})^2 \text{Diag}\{x_1^{\frac{2\alpha - 2}{2\alpha - 1}}, x_2^{\frac{2\alpha - 2}{2\alpha - 1}}, \cdots, x_n^{\frac{2\alpha - 2}{2\alpha - 1}}\}$, from which and Eq. (34), one can obtain that

$$
\dot{V}(t, x) \leq G(t)\left(\frac{2\alpha}{2\alpha - 1}\right)^2 [H(x)]^{\frac{1}{\alpha}}\left\{\lambda_{\max}\{L + I_n\} + c\lambda_{\max}\{N(x)\}n^{\frac{\alpha - 1}{\alpha}}\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})\right\}.
\tag{35}
$$

Since the system is locally asymptotically stable and c is a sufficiently small positive number, we obtain that $c\lambda_{\max}\{N(x)\}n^{\frac{\alpha - 1}{\alpha}}\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x})$ is a sufficiently small positive number. Thus, using $\lambda_{\max}\{L + I_n\} < 0$, one can obtain that $m_2 := \lambda_{\max}\{L + I_n\} + c\lambda_{\max}\{N(x)\}n^{\frac{\alpha - 1}{\alpha}}\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) < 0$ holds on $\bar{\Omega}$.

Based on Eq. (35), we have

$$
\dot{V}(t, x) \leq m_2 G(t)\left(\frac{2\alpha}{2\alpha - 1}\right)^2 [(H(x))]^{\frac{1}{\alpha}} = \frac{m_2}{2^{\frac{1}{\alpha}}}\left(\frac{2\alpha}{2\alpha - 1}\right)^2 (V(t, x))^{\frac{1}{\alpha}}(G(t))^{(1 - \frac{1}{\alpha})}.
\tag{36}
$$

Noting that $m_2 < 0$ and Eq. (16) holds, it is easy to obtain that $\dot{V}(t, x) \leq \frac{m_2}{2^{\frac{1}{\alpha}}}(\frac{2\alpha}{2\alpha - 1})^2 (V(t, x))^{\frac{1}{\alpha}}\kappa_1^{1 - \frac{1}{\alpha}}$, which implies that the condition (5) of Lemma 2 holds on $\bar{\Omega}$.

From Lemma 2, we know that the theorem follows under the case $\nabla_x^T H(x(t))E_1\dot{x}(t) \nrightarrow 0$ $(t \to \infty)$. In addition, we still need to prove that the theorem is true for the case $\nabla_x^T H(x(t))E_1\dot{x}(t) \to 0$ $(t \to \infty)$.

Noting that since $\nabla_x^T H(x(t))E_1\dot{x}(t) \to 0$ $(t \to \infty)$, one can obtain that the states x_j $(j = r + 1, \ldots, n)$ asymptotically tend zero. Therefore, if we prove that the states x_i $(i = 1, \ldots, r)$

are asymptotically stable and finite-time stable under the conditions of the theorem, then from Eq. (13), we obtain that the states x_j $(j = r + 1, \ldots, n)$ are also finite-time stable, which implies that all states of the system are finite-time stable.

In fact, if the states x_i $(i = 1, \ldots, r)$ are finite-time stable, then there exists some finite-time T_1 such that $H(Ex(t)) = 0$ for all $t \geq T_1$. In addition, noting that all states of the system asymptotically tend zero, then one can always find a moment T_2 such that $H(x(t)) \leq \varsigma$ for all $t \geq T_2$. Letting $T = \max\{T_1, \; T_2\}$ and using Eq. (13), we obtain $\|\nabla_x^T H(x(t))E_1 \dot{x}(t)\| = 0$ for all $t \geq T$, namely $\frac{dH(E_1 x)}{dt} = 0$ $(t \geq T)$ holds, from which it follows that $\|H(E_1 x(t)) - H(E_1 x(T))\| = \frac{dH(E_1 x(\tilde{\xi}))}{dt} = 0$ with $\tilde{\xi} \in [T, \; t]$. Thus, we obtain that $H(E_1 x(t)) = H(E_1 x(T))$ always holds for all $t \geq T$. On the other hand, noted that $H(E_1 x(t)) \to 0$ if $t \to +\infty$, then one can obtain $\lim_{t \to +\infty} H(E_1 x(t)) = H(E_1 x(T)) = 0$, from which and $\|H(E_1 x(t)) - H(E_1 x(T))\| = 0$ $(t \geq T)$, it is easy to obtain $H(E_1 x(t)) = 0$ for all $t \geq T$. Therefore, the states x_i $(i = r + 1, \ldots, n)$ are finite-time stable.

From the above analysis, we need to show that the states x_i $(i = 1, \ldots, r)$ are finite-time stable. This only needs to prove the following normal system is asymptotically stable and finite-time stable under the conditions of the theorem: $E\dot{x}(t) = R(x)\nabla_x H(Ex) + T(x, \tilde{x})\nabla_{\tilde{x}} H(E\tilde{x})$.

To do this, construct a Lyapunov function candidate as Eq. (15), and similar to the above proofs, one can obtain easily the conclusion, and thus is omitted. $\square$

Remark 5. To obtain the FTS result of nonlinear singular time-delay Hamiltonian system (1), we have presented the conditions: $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ is continuous, $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ is the high-order term of $H(Ex)$ and $\lambda_{\max}\{T^T(x, \tilde{x})T(x, \tilde{x})\}$ is the high-order term of $H(x)$ respectively. It is well worth pointing out that the conditions are restrictive, which implies that they may not hold, or are not easy to be checked for some singular systems. However, in the section, we only establish a theoretical result on the FTS of the system (1). Once the FTS result has been obtained, the finite-time stabilization (or other control) problems can be done for general singular Hamiltonian systems by designing a suitable controller (Specific details, also see Section 4 below).

Remark 6. In the proof of Theorem 1, we have constructed a novel Lyapunov function (14). From the proof of Theorem 1, it is easy to see that the biggest advantage of choosing the Lyapunov function (14) is that it can eliminate the term $2\nabla_x^T H(x)E_1 y(t)$. It is this handling that provides a way to study the FTS of the system (1) by using Lemmas 1, 3 and 2.

In the following, we propose a delay-dependent result on the FTS of the system (1).

Theorem 2. *Consider the system* (1) *with the Hamiltonian function in the form of Eq.* (2). *Assume that* $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ *is continuous,* $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ *and* $\lambda_{\max}\{T^T(x, \tilde{x})T(x, \tilde{x})\}$ *are the high-order terms of* $H(Ex)$ *and* $H(x)$ *respectively, and*

(i) *there exist constant matrices* $L < 0$, $Z > 0$ *and* $S > 0$ *with appropriate dimensions such that*

$$R^T(x) + R(x) \leq L, \;\; T(x, \tilde{x})T^T(x, \tilde{x}) \leq S, \;\; N(x) \leq Z, \;\; (x \in \Omega); \tag{37}$$

(ii) *there exists a positive constant number* a *such that*

$$L + a^{-1}S \leq 0, \tag{38}$$

$$m := \lambda_{\max}(L + S + 2Z) + (1 + ha)n^{\frac{\alpha - 1}{\alpha}} < 0, \tag{39}$$

where n and $\lambda_{\max}(*)$ are the same those as Theorem 1, and $N(x) = (\frac{2\alpha-1}{2\alpha})^2$ Diag $\{x_1^{\frac{2(\alpha-1)}{2\alpha-1}}, x_2^{\frac{2(\alpha-1)}{2\alpha-1}}, \cdots, x_n^{\frac{2(\alpha-1)}{2\alpha-1}}\}$. Then, the system (1) is locally finite-time stable.

Proof. Consider the system (11) and choose $V(t, x)$ given in Theorem 1 as a Lyapunov function candidate. Computing the derivative of $V(t, x)$ along the trajectory of the system (11), similar to the proof of Theorem 1, we have

$$\dot{V}(t, x) \leq G(t)\left\{2\nabla_x^T H(x) R(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x, \tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\right\}. \tag{40}$$

Using Eqs. (1) and (26), it is easy to obtain that

$$
\begin{aligned}
0 &= \int_{t-h}^{t} \nabla_x^T H(x(s))\left\{R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s)) - E\dot{x}(s)\right\}ds \\
&\leq \int_{t-h}^{t} \nabla_x^T H(x(s))\left\{R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\right\}ds + H(Ex(t-h)) \\
&\leq \int_{t-h}^{t} \nabla_x^T H(x(s))\left\{R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\right\}ds + pH(x) \\
&= \int_{t-h}^{t} \nabla_x^T H(x(s))\left\{R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\right\}ds \\
&\quad + p\nabla_x^T H(x))N(x)\nabla_x H(x). \tag{41}
\end{aligned}
$$

Substituting Eqs. (41) into (40), and similar to the proof of Theorem 1, we obtain

$$
\begin{aligned}
\dot{V}(t, x) &\leq G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x, \tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) + 2p\nabla_x^T H(x)N(x)\nabla_x H(x) \\
&\quad + 2\int_{t-h}^{t} \nabla_x^T H(x(s\big(R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\big)ds\Big\} \\
&\leq G(t)\Big\{2\nabla_x^T H(x)R(x)\nabla_x H(x) + \nabla_x^T H(x)T(x, \tilde{x})T^T(x, \tilde{x})\nabla_x H(x) + \nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) \\
&\quad + 2p\nabla_x^T H(x)N(x)\nabla_x H(x) + 2\int_{t-h}^{t} \nabla_x^T H(x(s))\big(R(x(s))\nabla_x H(x(s)) \\
&\quad + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\big)ds\Big\} \tag{42}
\end{aligned}
$$

$$
\begin{aligned}
&= G(t)\Big\{\nabla_x^T H(x)[R(x) + R^T(x) + T(x, \tilde{x})T^T(x, \tilde{x}) + 2pN(x)]\nabla_x H(x) \\
&\quad + \left(\frac{2\alpha}{2\alpha-1}\right)^2 \sum_{i=1}^{n}(\tilde{x}_i^2)^{\frac{1}{2\alpha-1}} \\
&\quad + 2\int_{t-h}^{t} \nabla_x^T H(x(s))\big(R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\big)ds\Big\} \\
&\leq G(t)\Big\{(\lambda_{\max}(L + S + 2pZ)\left(\frac{2\alpha}{2\alpha-1}\right)^2 (H(x))^{\frac{1}{\alpha}} + n^{\frac{\alpha-1}{\alpha}}\left(\frac{2\alpha}{2\alpha-1}\right)^2 (pH(x))^{\frac{1}{\alpha}} \\
&\quad + 2\int_{t-h}^{t} \nabla_x^T H(x(s))\big(R(x(s))\nabla_x H(x(s)) + T(x(s), \tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\big)ds\Big\}. \tag{43}
\end{aligned}
$$

Based on the *R*-condition and Lemma 5, one can obtain

$$a \int_{t-h}^{t} \nabla_{\tilde{x}}^{T} H(\tilde{x}(s)) \nabla_{\tilde{x}} H(\tilde{x}(s)) ds \leq a n^{\frac{\alpha-1}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} \int_{t-h}^{t} (H(\tilde{x}(s)))^{\frac{1}{\alpha}} ds$$

$$\leq a h n^{\frac{\alpha-1}{\alpha}} p^{\frac{2}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}},$$

from which it follows that

$$0 \leq -a \int_{t-h}^{t} \nabla_{\tilde{x}}^{T} H(\tilde{x}(s)) \nabla_{\tilde{x}} H(\tilde{x}(s)) ds + a h n^{\frac{\alpha-1}{\alpha}} p^{\frac{2}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}}. \tag{44}$$

Substituting Eq. (44) into Eq. (43), we have $\dot{V}(t, x)$

$$\leq G(t) \left\{ (\lambda_{\max}(L + S + 2pZ) \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} + (ahp^{\frac{1}{\alpha}} + 1) p^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} \right.$$

$$+ \int_{t-h}^{t} \left(2\nabla_{x}^{T} H(x(s)) R(x(s)) \nabla_{x} H(x(s)) + 2\nabla_{x}^{T} H(x(s)) T(x(s), \tilde{x}(s)) \nabla_{\tilde{x}} H(\tilde{x}(s)) \right.$$

$$\left. -a\nabla_{\tilde{x}}^{T} H(\tilde{x}(s)) \nabla_{\tilde{x}} H(\tilde{x}(s)) \right) ds \right\}$$

$$= G(t) \left\{ (\lambda_{\max}(L + S + 2pZ) \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} + (ahp^{\frac{1}{\alpha}} + 1) p^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} \right.$$

$$+ \int_{t-h}^{t} \xi^{T} \Upsilon \xi ds \Bigg\},$$

where $\xi := [\nabla_{x}^{T} H(x(s)), \nabla_{\tilde{x}}^{T} H(\tilde{x}(s))]^{T}$ and

$$\Upsilon := \begin{bmatrix} R(x(s)) + R^{T}(x(s)) & T(x(s), \tilde{x}(s)) \\ T^{T}(x(s), \tilde{x}(s)) & -aI_{n} \end{bmatrix}.$$

According to Schur complement lemma and Eq. (38), we have $R(x(s)) + R^{T}(x(s)) + a^{-1}T(x(s))T^{T}(x(s)) \leq 0$, which implies that $\Upsilon \leq 0$ on Ω.

On the other hand, based on Eq. (39), it is easy to obtain that

$$\dot{V}(t, x) \leq G(t) \left\{ \lambda_{\max}(L + S + 2pZ) \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} \right.$$

$$\left. + (1 + ahp^{\frac{1}{\alpha}}) p^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} \right\}$$

$$= G(t) \left\{ m_{1} \left(\frac{2\alpha}{2\alpha-1} \right)^{2} (H(x))^{\frac{1}{\alpha}} \right\},$$

where $m_{1} := \lambda_{\max}(L + S + 2pZ) + (1 + ahp^{\frac{1}{\alpha}}) p^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}}$.

The remainder of this proof is similar to that of Theorem 1, and thus is omitted. $\square$

Remark 7. Note that it is very difficult to present a delay-dependent result by using the R-approach, mainly because it needs to use the model transformation form of the system (1) (Further details, see [7]). However, in the paper, since $\text{Hess}(H(x)) = \frac{\partial^{2} H(x)}{\partial x^{2}}$ is not continuous, one cannot use the equation $\int_{t-h}^{t} \frac{\partial^{2} H(x)}{\partial x^{2}} \dot{x}(s) ds = \nabla_{x} H(x) - \nabla_{\tilde{x}} H(\tilde{x})$ to obtain the model

transformation form of the system (1), which is the reason that we have introduced the inequality (41) in the proof of Theorem 2. In addition, noting that $Z > 0$ and $ha > 0$ in Condition (ii) of Theorem 2, and thus from Theorem 2, one can obtain easily Theorem 1, which implies that Theorem 2 is more restrictive than Theorem 1. However, compared with Theorem 1, Theorem 2 is more meaningful for small delay systems. In fact, because the effect of time delay on system behavior is very large for small time-delay systems, it is very important to find a time-delay upper bound to ensure system stability. Further details, please see Example 3 below of the paper and [7].

Remark 8. In the section, a kind of specific form of Lyapunov function is constructed to obtain the FTS results of the singular time-delay Hamiltonian systems. From the proofs of Theorems 1 and 2, it can be seen that one can apply Lemmas 1, 2 and the Lyapunov function to easily study the singular time-delay Hamiltonian system and obtain its derivative condition on the FTS, which are some advantages of the method proposed in this paper.

4. Finite-time stabilization

In this section, we use the results obtained in Section 3 to investigate the finite-time stabilization problem for a class of nonlinear singular time-delay PCH systems.

Consider the following nonlinear singular time-delay PCH system:

$$\begin{cases} E\dot{x} = R_1(x)\nabla_x H_1(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_1(\tilde{x}) + g_1(x)u, \\ x(\tau) = \phi(\tau), \ \forall \tau \in [-h, \ 0], \end{cases} \tag{45}$$

where $x(t) \in \Omega \subseteq \mathbb{R}^n$ is the state vector, E, h and $\phi(\tau)$ are the same as Section 3, $u \in \mathbb{R}^m$ is the control input, $g_1(x)$ is a weighted matrix with appropriate dimension and has full column rank [41], and $H_1(x)$ is the smooth Hamiltonian function satisfying $H_1(0) = 0$.

Remark 9. In the section, to make the results obtained has a wider range of applications, we assume that $H_1(x)$ is smooth rather than the special form as Eq. (2).

To facilitate the analysis, let

$$R_1(x) = \begin{bmatrix} R_{11}(x) & R_{12}(x) \\ R_{21}(x) & R_{22}(x) \end{bmatrix}, \ T_1(x, \tilde{x}) = \begin{bmatrix} T_{11}(x, \tilde{x}) & T_{12}(x, \tilde{x}) \\ T_{21}(x, \tilde{x}) & T_{22}(x, \tilde{x}) \end{bmatrix},$$

where $R_{11}(x) \in \mathbb{R}^{r \times r}$, $R_{12}(x) \in \mathbb{R}^{r \times (n-r)}$, $R_{21}(x) \in \mathbb{R}^{(n-r) \times r}$, $R_{22}(x) \in \mathbb{R}^{(n-r) \times (n-r)}$, $T_{11}(x, \tilde{x}) \in \mathbb{R}^{r \times r}$, $T_{12}(x, \tilde{x}) \in \mathbb{R}^{r \times (n-r)}$, $T_{21}(x, \tilde{x}) \in \mathbb{R}^{(n-r) \times r}$ and $T_{22}(x, \tilde{x}) \in \mathbb{R}^{(n-r) \times (n-r)}$.

Our aim is to design a controller u such that the closed-loop system consisting of the system (45) and the controller u is locally finite-time stable.

By the "energy shaping plus damping injection" technique, the designed control law u of the system (45) should satisfy the following equation:

$$g_1(x)u = R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x}) + R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + g_1 v \tag{46}$$

where v is a new input, $H(x) = \sum_{i=1}^{n}(x_i^2)^{\frac{\alpha}{2\alpha-1}}$, (**Note**: from Section 3, we known the results obtained are also true for all $\alpha > 1$, and thus, without loss of generality, let $\alpha > 1 + \frac{\sqrt{2}}{2} > 1$ in the section.),

$$H_a(x) := H(x) - H_1(x), \tag{47}$$

$$R_a(x) = \begin{bmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ R_{(n-r)\times r}(x) - R_{21}(x) & -\bar{K} - R_{22}(x) \end{bmatrix},$$

$$T_a(x, \tilde{x}) = \begin{bmatrix} 0_{r\times r} & 0_{r\times(n-r)} \\ T_{(n-r)\times r}(x, \tilde{x}) - T_{21}(x, \tilde{x}) & -T_{22}(x, \tilde{x}) \end{bmatrix}$$

with $\bar{K} \in \mathbb{R}^{(n-r)\times(n-r)}$ satisfying $\bar{K} := \text{Diag}\{k_1, k_2, \overset{\cdots}{\cdot}, k_{n-r}\}$ $(k_i > 0)$, and $R_{(n-r)\times r}(x)$ and $T_{(n-r)\times r}(x, \tilde{x})$ are given as follows.

For $n - r < r$, we design

$$R_{(n-r)\times r}(x) = \begin{bmatrix} x_1^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & x_{n-r}^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 \end{bmatrix},$$

$$T_{(n-r)\times r}(x, \tilde{x}) = \begin{bmatrix} x_1\tilde{x}_1^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & x_{n-r}\tilde{x}_{n-r}^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 \end{bmatrix},$$

and for $n - r > r$, we let

$$R_{(n-r)\times r}(x) = \begin{bmatrix} x_1^{\frac{2\alpha-2}{2\alpha-1}} & 0 & \cdots & 0 \\ 0 & x_2^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_r^{\frac{2\alpha-2}{2\alpha-1}} \\ x_1^{\frac{2\alpha-2}{2\alpha-1}} & 0 & \cdots & 0 \\ 0 & x_2^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix},$$

$$T_{(n-r)\times r}(x, \tilde{x}) = \begin{bmatrix} x_1\tilde{x}_1^{\frac{2\alpha-2}{2\alpha-1}} & 0 & \cdots & 0 \\ 0 & x_2\tilde{x}_2^{\frac{2\alpha-2}{2\alpha-1}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_r\tilde{x}_r^{\frac{2\alpha-2}{2\alpha-1}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

In addition, if $n - r = r$, then

$$R_{(n-r)\times r}(x) = \text{Diag}\{x_1^{\frac{2\alpha-2}{2\alpha-1}}, \ x_2^{\frac{2\alpha-2}{2\alpha-1}}, \ \cdots, \ x_r^{\frac{2\alpha-2}{2\alpha-1}}\},$$

$$T_{(n-r)\times r}(x, \tilde{x}) = \text{Diag}\{x_1\tilde{x}_1^{\frac{2\alpha-2}{2\alpha-1}}, \ x_2\tilde{x}_2^{\frac{2\alpha-2}{2\alpha-1}}, \ \cdots, \ x_r\tilde{x}_r^{\frac{2\alpha-2}{2\alpha-1}}\}.$$

The controller u satisfying Eq. (46) is called the derived controller. In the following, to facilitate the expression, we denote the derived controller as u_1, that is,

$$g_1(x)u_1(x, \tilde{x}) = R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x}) + g_1 v + R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}). \tag{48}$$

Remark 10. Noting that $g_1(x)$ is full column rank, from Eq. (48), one can obtain $u_1(x, \tilde{x})$ by solving some algebraic equations (Further details, please see Remark 12 below of the paper and [41]).

Substituting Eq. (46) into the system (45), we have

$$E\dot{x}(t) = R(x)\nabla_x H(x) + M(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + g_1 v(t), \tag{49}$$

where $R(x) := R_1(x) + R_a(x)$ and $M(x, \tilde{x}) := T_1(x, \tilde{x}) + T_a(x, \tilde{x})$.

Considering the system (49), we design a control law v as follows:

$$g_1(x)v = -EKg_1(x)g_1^T(x)E\nabla_x H(x) + E[\iota\|x\|I_n - M(x, \tilde{x})]\nabla_{\tilde{x}} H(\tilde{x}), \tag{50}$$

based on which, we obtain

$$E\dot{x} = F_1(x)\nabla_x H(x) + T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}). \tag{51}$$

where K is a constant matrix, ι is a constant number to be determined later, and $F_1(x) := R(x) - EKg_1(x)g_1^T(x)E$,

$$T(x, \tilde{x}) := M(x, \tilde{x}) + E[\iota\|x\|I_n - M(x, \tilde{x})] = \begin{bmatrix} \iota\|x\|I_r & 0 \\ T_{n-r \times r}(x, \tilde{x}) & 0 \end{bmatrix} \tag{52}$$

For the finite-time stabilization problem of the system (45), we have the following result.

Theorem 3. *Consider the system (45), assume that*

(i). *there exist constant matrices $L < 0$, $Z > 0$ and $S > 0$ with appropriate dimensions such that*

$$F_1^T(x) + F_1(x) \leq L, \quad T(x, \tilde{x})T^T(x, \tilde{x}) \leq S, \quad N(x) \leq Z \quad (x \in \Omega); \tag{53}$$

(ii). *there exist positive real numbers a and b such that*

$$L + a^{-1}S \leq 0,$$
$$m := \lambda_{\max}\{L + b^{-1}S + 2Z\} + (b + ah)n^{\frac{\alpha-1}{\alpha}} < 0. \tag{54}$$

Then, a finite-time control law of the system (45) can be designed as

$$u = u_1(x, \tilde{x}), \tag{55}$$

where $u_1(x, \tilde{x})$ is given by Eqs. (48) and (50), and $N(x)$ is the same as that in Theorem 2.

Proof. Substituting Eq. (55) into the system (45) leads to Eq. (51).

Similar to Section 3, one can express the system (51) into the form

$$\dot{x} = F_1(x)\nabla_x H(x) + T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) + E_1 y(t). \tag{56}$$

• First, based on Eq. (51), we show that $\nabla_x^T H(x)E_1\dot{x}$ is continuous and $\nabla_x^T H(x)E_1\dot{x}$ is a high order term of $H(Ex)$. Noting that

$$\nabla_x^T H(x)E_1\dot{x} = \frac{2\alpha}{2\alpha - 1}(x_{r+1}^{\frac{1}{2\alpha-1}}\dot{x}_{r+1} + \cdots + x_n^{\frac{1}{2\alpha-1}}\dot{x}_n). \tag{57}$$

we need to prove that all $\dot{x}_i$ $(i = r + 1, \ldots, n)$ are continuous. To do this, we give the expressions of $\dot{x}_i$ $(i = r + 1, \ldots, n)$.

Without loss of generality, let $n - r < r$. Then, from Eq. (51), it is easy to obtain the following equations:

$$\begin{cases} 0 = \frac{2\alpha}{2\alpha-1}\left[x_1 - k_1 x_{r+1}^{\frac{1}{2\alpha-1}} + x_1\tilde{x}_1\right], \\ 0 = \frac{2\alpha}{2\alpha-1}\left[x_2 - k_2 x_{r+2}^{\frac{1}{2\alpha-1}} + x_2\tilde{x}_2\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ 0 = \frac{2\alpha}{2\alpha-1}\left[x_{n-r} - k_{n-r} x_n^{\frac{1}{2\alpha-1}} + x_{n-r}\tilde{x}_{n-r}\right], \end{cases}$$

from which it follows that

$$\begin{cases} x_{r+1} = \left(\frac{x_1(1+\tilde{x}_1)}{k_1}\right)^{2\alpha-1}, \\ x_{r+2} = \left(\frac{x_2(1+\tilde{x}_2)}{k_2}\right)^{2\alpha-1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ x_n = \left(\frac{x_{n-r}(1+\tilde{x}_{n-r})}{k_{n-r}}\right)^{2\alpha-1}. \end{cases}$$

Thus, we have

$$\begin{cases} \dot{x}_{r+1} = \frac{(2\alpha-1)}{k_1}\left(\frac{x_1(1+\tilde{x}_1)}{k_1}\right)^{2\alpha-2}(\dot{x}_1(1+\tilde{x}_1) + x_1\dot{\tilde{x}}_1), \\ \dot{x}_{r+2} = \frac{(2\alpha-1)}{k_2}\left(\frac{x_2(1+\tilde{x}_2)}{k_2}\right)^{2\alpha-2}(\dot{x}_2(1+\tilde{x}_2) + x_2\dot{\tilde{x}}_2), \\ \cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_n = \frac{(2\alpha-1)}{k_{n-r}}\left(\frac{x_{n-r}(1+\tilde{x}_{n-r})}{k_{n-r}}\right)^{2\alpha-2}(\dot{x}_{n-r}(1+\tilde{x}_{n-r}) \\ \qquad\qquad + x_{n-r}\dot{\tilde{x}}_{n-r}). \end{cases} \tag{58}$$

Noting that $\alpha > 1$, $\dot{x}_i$ and $\dot{\tilde{x}}_i$ $(i = 1, \ldots, n - r)$ are known and continuous, it is easy to obtain that $\dot{x}_i$ $(i = r + 1, \ldots, n)$ are also continuous, which implies that $\nabla_x^T H(x)E_1\dot{x}$ is continuous. Similarly, one can prove that $\nabla_x^T H(x)E_1\dot{x}$ is continuous for $n - r \geq r$.

Now, we show that $\nabla_x^T H(x)E_1\dot{x}$ is a high order term of $H(Ex)$.

To do this, noting that $\nabla_x^T H(x)E_1\dot{x} = \frac{2\alpha}{2\alpha-1}\sum_{i=r+1}^n x_i^{\frac{1}{2\alpha-1}}\dot{x}_i$ and Eq. (58), we need to show that $x_i^{2\alpha-2}$ $(i = 1, \ldots, n - r)$ is the high term of $x_i^{\frac{2\alpha}{2\alpha-1}}$ $(i = 1, \ldots, n - r)$.

In fact, from $\alpha > 1 + \frac{\sqrt{2}}{2}$, one can obtain that $2\alpha - 2 - \frac{2\alpha}{2\alpha-1} = \frac{4\alpha^2 - 8\alpha + 2}{2\alpha-1} > 0$, namely, $\nabla_x^T H(x)E_1\dot{x}$ is a high order term for $H(Ex)$. In addition, we still need to prove that $T^T(x, \tilde{x})T(x, \tilde{x})$ is the high term of $H(x)$.

Using Eq. (52), we have $T^T(x, \tilde{x})T(x, \tilde{x}) = \text{Diag}\{d, 0\}$, where $d := \iota^2\|x\|^2 I_r + T_{n-r\times r}^T(x, \tilde{x})T_{n-r\times r}(x, \tilde{x})$. It is obvious that $T^T(x, \tilde{x})T(x, \tilde{x})$ is the high term of $H(x)$.

• Second, choose $V(t, x)$ given in Eq. (14) as a Lyapunov function candidate. Computing the derivative of $V(t, x)$ along the trajectory of the equivalent system (56) and similar to the

proof of Theorem 2 yields

$$\dot{V}(t,x) \le G(t)\Big\{2\nabla_x^T H(x)F_1(x)\nabla_x H(x) + 2\nabla_x^T H(x)T(x,\tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\Big\}. \tag{59}$$

Based on Lemma 4, it is easy to obtain that

$$2\nabla_x^T H(x)T(x,\tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) \le b^{-1}\nabla_x^T H(x)T(x,\tilde{x})T^T(x,\tilde{x})\nabla_x H(x) + b\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x}). \tag{60}$$

Substituting Eq. (60) into Eq. (59), and using Condition (i), one can obtain

$$\dot{V}(t,x) \le G(t)\Big\{\nabla_x^T H(x)[F_1(x) + F_1^T(x) + b^{-1}T(x,\tilde{x})T^T(x,\tilde{x})]\nabla_x H(x) + b\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\Big\}$$

$$\le G(t)\Big\{\nabla_x^T H(x)[L + b^{-1}S]\nabla_x H(x) + b\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\Big\}$$

$$\le G(t)\Big\{\lambda_{\max}\{L + b^{-1}S\}\nabla_x^T H(x)\nabla_x H(x) + b\nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x})\Big\}. \tag{61}$$

From Eqs. (30) and (31), we have

$$\nabla_x^T H(x)\nabla_x H(x) \ge \left(\frac{2\alpha}{2\alpha-1}\right)^2 (H(x))^{\frac{1}{\alpha}}, \quad \nabla_{\tilde{x}}^T H(\tilde{x})\nabla_{\tilde{x}}H(\tilde{x}) \le n^{\frac{\alpha-1}{\alpha}}\left(\frac{2\alpha}{2\alpha-1}\right)^2 (pH(x))^{\frac{1}{\alpha}},$$

based on which and $m < 0$, it is easy to obtain that

$$\dot{V}(t,x) \le G(t)\Big\{\lambda_{\max}\{L + b^{-1}S\}\left(\frac{2\alpha}{2\alpha-1}\right)^2 (H(x))^{\frac{1}{\alpha}} + bn^{\frac{\alpha-1}{\alpha}}\left(\frac{2\alpha}{2\alpha-1}\right)^2 (pH(x))^{\frac{1}{\alpha}}\Big\}$$

$$= G(t)\Big\{\Big[\lambda_{\max}\{L + b^{-1}S\} + p^{\frac{1}{\alpha}}bn^{\frac{\alpha-1}{\alpha}}\Big]\left(\frac{2\alpha}{2\alpha-1}\right)^2 (H(x))^{\frac{1}{\alpha}}\Big\}. \tag{62}$$

On the other hand, using the R-condition and Eq. (51), we have

$$0 = 2\int_{t-h}^{t}\nabla_{x(s)}^T H(x(s))\Big\{F_1(x(s))\nabla_{x(s)}H(x(s)) + T(x(s),\tilde{x}(s))\nabla_{\tilde{x}(s)}H(\tilde{x}(s)) - E\dot{x}(s)\Big\}ds$$

$$\le \int_{t-h}^{t}\Big\{2\nabla_{x(s)}^T H(x(s))F_1(x(s))\nabla_{x(s)}H(x(s)) + 2\nabla_{x(s)}^T H(x(s))T(x(s),\tilde{x}(s))\nabla_{\tilde{x}(s)}H(\tilde{x}(s))\Big\}ds$$

$$+ 2pH(x(t))$$

$$= \int_{t-h}^{t}\Big\{2\nabla_{x(s)}^T H(x(s))F_1(x(s))\nabla_{x(s)}H(x(s)) + 2\nabla_{x(s)}^T H(x(s))T(x(s),\tilde{x}(s))\nabla_{\tilde{x}(s)}H(\tilde{x}(s))\Big\}ds,$$

$$+ 2p\nabla_x^T H(x)N(x)\nabla_x H(x). \tag{63}$$

Thus, substituting Eqs. (44) and (63) into Eq. (62), we have

$$\dot{V}(t,x) \le G(t)\Big\{\Big[\lambda_{\max}\{L + b^{-1}S + 2pZ\} + (bp^{\frac{1}{\alpha}} + ahp^{\frac{2}{\alpha}})n^{\frac{\alpha-1}{\alpha}}\Big]\left(\frac{2\alpha}{2\alpha-1}\right)^2 (H(x))^{\frac{1}{\alpha}}\Big\}$$

$$+ G(t)\int_{t-h}^{t}\Big\{\nabla_{x(s)}^T H(x(s))\Big[F_1(x(s)) + F_1^T(x(s))\Big]\nabla_{x(s)}H(x(s)))$$

$$+ 2\nabla_{x(s)}^T H(x(s))T(x(s),\tilde{x}(s))\nabla_{\tilde{x}(s)}H(\tilde{x}(s)) - a\nabla_{\tilde{x}}^T H(\tilde{x}(s))\nabla_{\tilde{x}}H(\tilde{x}(s))\Big\}ds$$

$$\leq G(t) \left\{ \left[\lambda_{\max}\{L + b^{-1}S + 2pZ\} + (bp^{\frac{1}{\alpha}} + ahp^{\frac{2}{\alpha}})n^{\frac{\alpha-1}{\alpha}} \right] \left(\frac{2\alpha}{2\alpha - 1} \right)^2 (H(x))^{\frac{1}{\alpha}} \right.$$

$$\left. + \int_{t-h}^{t} \xi^T \Upsilon \xi \, ds \right\},$$

where $\xi := [\nabla_x^T H(x(s)), \nabla_{\tilde{x}}^T H(\tilde{x}(s))]^T$ and $\Upsilon := \begin{bmatrix} F_1(x(s)) + F_1^T(x(s)) & T(x(s), \tilde{x}(s)) \\ T^T(x(s), \tilde{x}(s)) & -a \end{bmatrix}$.

Based on Eq. (54) and Schur complement lemma, one can obtain $\Upsilon \leq 0$. The remainder of this proof is similar to Theorem 1, and thus is omitted. $\square$

Remark 11. Note that $g_1(x)u_1(x, \tilde{x})$ designed in Eq. (48) contains three parts, that is, $R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x})$, $R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x})$ and $-EKg_1(x)g_1^T(x)E\nabla_x H(x) + E[\iota\|x\|I_n - M(x, \tilde{x})]\nabla_{\tilde{x}} H(\tilde{x})$. The purpose of designing the three parts is as follows: on one hand, the controller can shape energy function into the special $H(x)$ and alter the structural matrix to the dissipative form, and on the other hand, it can ensure that the conditions hold on $\nabla_x H(x(t))E_1\dot{x}(t)$ and $T^T(x, \tilde{x})T(x, \tilde{x})$ in Theorem 2.

Remark 12. It should be pointed out that although we maintain the form $g_1(x)u_1(x, \tilde{x})$ designed in Eq. (48), it is easy to obtain $u_1(x, \tilde{x})$ from Eq. (48). In fact, since $g_1(x)$ has full column rank, one can obtain that $g_1^T(x)g_1(x)$ is non-singular, from which and Eq. (48), we have $u_1(x, \tilde{x}) = (g_1^T(x)g_1(x))^{-1} g_1^T(x)[R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x}) + g_1 v + R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x})]$. Thus, substituting Eq. (50) into the above formula, one can obtain that $u_1(x, \tilde{x}) = (g_1^T(x)g_1(x))^{-1} g_1^T(x)[R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x}) - EKg_1(x)g_1^T(x)E\nabla_x H(x) + E[\iota\|x\|I_n - M(x, \tilde{x})]\nabla_{\tilde{x}} H(\tilde{x}) + R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x})]$.

Remark 13. In the paper, based on the nonlinear matrix inequality technique, we have presented several finite-time stability and stabilization results on nonlinear singular time-delay systems. However, different from the ones obtained on nonlinear time-delay systems in [22,43] and therein references, these conditions given of the present paper are concise and easy to test. In fact, to verify the conditions: $R^T(x) + R(x) \leq L$, $T(x, \tilde{x})T^T(x, \tilde{x}) \leq S$, $N(x) \leq Z$ $(x \in \Omega)$, one only needs to find the upper bounds of these matrix inequalities instead of verifying each x and $\tilde{x}$(**Note**: Since $T(x, \tilde{x})T^T(x, \tilde{x})$ is a infinite-dimension condition, it is a challenging issue to test each x and $\tilde{x}$). In addition, for the conditions $L + a^{-1}S \leq 0$ and $m := \lambda_{\max}(L + S + 2Z) + (1 + ha)n^{\frac{\alpha-1}{\alpha}} < 0$ or $m = \lambda_{\max}(L + S) + n^{\frac{\alpha-1}{\alpha}} < 0$, one can easily test them by applying the linear matrix inequality Toolbox. Therefore, the method presented in the paper can effectively reduce the computational burden, which is an advantage of the paper.

5. Illustrative examples

In this section, we give several illustrative examples to show the effectiveness of the results.

Example 1. Consider the following nonlinear singular time-delay system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -\frac{8}{3}x_1^{\frac{1}{3}} + \frac{4}{3}x_1 x_2^{\frac{1}{3}} + \frac{4}{3}x_1^2 x_2 \tilde{x}_1^{\frac{1}{3}} \\ -\frac{4}{3}x_2^{\frac{1}{3}}(x_1^2 + 2) + \frac{4}{3}x_1 x_2^2 \tilde{x}_3 \tilde{x}_2^{\frac{1}{3}} \\ -\frac{8}{3}x_3^{\frac{1}{3}} + \frac{\frac{32}{300}x_2^{\frac{10}{3}}}{\frac{1}{100} + x_3^{\frac{2}{3}}} \end{bmatrix}. \tag{64}$$

To apply Theorem 2 to study the FTS of the system (64), we express the system as the following Hamiltonian form:

$$E\dot{x}(t) = R(x)\nabla_x H(x) + T(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}), \tag{65}$$

where $H(x) = x_1^{\frac{4}{3}} + x_2^{\frac{4}{3}} + x_3^{\frac{4}{3}}$ (**Note:** The Hamiltonian function satisfies the form of Eq. (2), where $\alpha = 2$.), and

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ R(x) = \begin{bmatrix} -2 & x_1 & 0 \\ 0 & -(x_1^2 + 2) & 0 \\ 0 & \frac{0.08 x_2^3}{0.01 + x_3^{\frac{2}{3}}} & -2 \end{bmatrix},$$

$$T(x, \tilde{x}) = \begin{bmatrix} x_1^2 x_2 & 0 & 0 \\ 0 & x_1 x_2^2 \tilde{x}_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now, we verify that the conditions of theorem 2 hold. To do this, let $x \in \Omega = \{(x_1, x_2, x_3) : |x_1| \leq 0.5, \ |x_2| \leq 0.5, \ |x_3| \leq 0.5\}$.

It is obvious that there exist constant symmetric matrices $L = -3I_3$, $S = 0.125I_3$ and $Z = 0.3544I_3$, and constant number $a = 0.1$ such that Conditions (i) and (ii) hold for $0 \leq h \leq 2.5$ on Ω.

In addition, we still need to show that the other conditions are true in Theorem 2.

Using the third equation in the system (64), one can obtain $0.02 x_3^{\frac{1}{3}} + 2x_3 = 0.08 x_2^{\frac{10}{3}}$. Deriving both sides of the equation, we have $[\frac{0.02}{3} x_3^{\frac{-2}{3}} + 2]\dot{x}_3 = \frac{0.8}{3} x_2^{\frac{7}{3}} \dot{x}_2$, from which it is easy to obtain that

$$\dot{x}_3 = \frac{\frac{0.8}{3} x_2^{\frac{7}{3}} \dot{x}_2}{\frac{0.02}{3} x_3^{\frac{-2}{3}} + 2} = \frac{0.4 x_2^{\frac{7}{3}} x_3^{\frac{2}{3}} \dot{x}_2}{3 x_3^{\frac{2}{3}} + 0.01}. \tag{66}$$

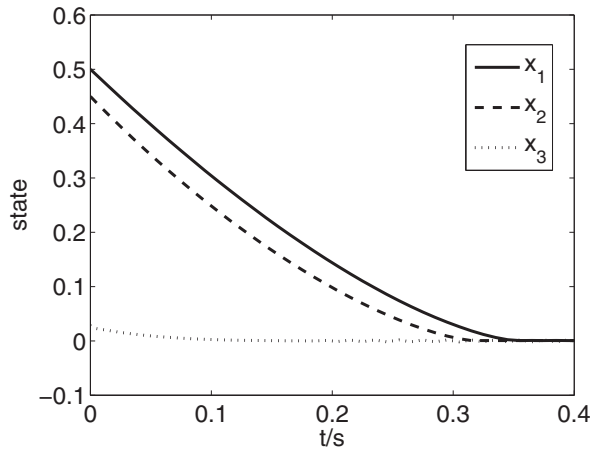Substituting the second equation of the system (64) into Eq. (66), one can obtain

$$\dot{x}_3 = \frac{0.4 x_2^{\frac{7}{3}} x_3^{\frac{2}{3}} \left(- \frac{4}{3} x_2^{\frac{1}{3}} (x_1^2 + 2) + \frac{4}{3} x_1 x_2^2 \tilde{x}_3 \tilde{x}_2^{\frac{1}{3}} \right)}{3 x_3^{\frac{2}{3}} + 0.01}, \tag{67}$$

with which and $\nabla_x^T H(x(t))E_1 \dot{x}(t) = \frac{4}{3} x_3^{\frac{1}{3}} \dot{x}_3$, we have

$$\nabla_x^T H(x(t))E_1 \dot{x}(t) = \frac{\frac{8}{15} x_2^{\frac{7}{3}} x_3 \left(- \frac{4}{3} x_2^{\frac{1}{3}} (x_1^2 + 2) + \frac{4}{3} x_1 x_2^2 \tilde{x}_3 \tilde{x}_2^{\frac{1}{3}} \right)}{3 x_3^{\frac{2}{3}} + 0.01}.$$

Therefore, $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ is continuous on Ω. In addition, since $H(Ex) = x_1^{\frac{4}{3}} + x_2^{\frac{4}{3}}$, it is obvious that $\nabla_x^T H(x(t))E_1 \dot{x}(t)$ is a high-order term of $H(Ex)$. Furthermore, noting that $H(x) = x_1^{\frac{4}{3}} + x_2^{\frac{4}{3}} + x_3^{\frac{4}{3}}$ and

$$\lambda_{\max}\{T^T(x, \tilde{x})T(x, \tilde{x})\} = \lambda_{\max}\left\{ \begin{bmatrix} x_1^4 x_2^2 & 0 & 0 \\ 0 & x_1^2 x_2^4 \tilde{x}_3^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = \lambda_{\max}\{x_1^4 x_2^2, \ x_1^2 x_2^4 \tilde{x}_3^2\},$$

Fig. 5.1. State Curves ($h = 2$).

it is easy to see that $\lambda_{\max}\{T^T(x, \tilde{x})T(x, \tilde{x})\}$ is the high-order terms of $H(x)$, and thus all conditions of theorem 2 are satisfied for the system. From Theorem 2, one can obtain that the system (64) is finite-time stability.

To show the effectiveness, we give a simulation result in Fig. 5.1 with the following choices: Initial condition: $E\phi(t) = (0.5, 0.45, 0)$; Time delay: $h = 2$. The simulation result is shown in Fig. 5.1, which is the state's responses in the system (64). From Fig. 5.1, it is easy to see that the system (64) is finite-time stability within 0.4 second.

Example 2. Consider a nonlinear singular time-delay circuit system as follows [30]

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \dot{y}(t) = \begin{bmatrix} -y_2 \\ y_1^3 + 0.5y_1^3(t-h) \\ y_2 + y_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} u(t), \tag{68}$$

where y_1 denotes the magnetic flux of the inductance, y_2 and y_3 mean the electric charge of the capacitance, respectively. Further details, please see [30].

To apply the results of the paper, we first transform the matrix E of the system (68) to the form Eq. (3). To do this, choosing $\bar{Q}$ and $\bar{P}$ as

$$\bar{Q} := \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{P} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

one can obtain $E = \text{Diag}\{1, 1, 0\}$.

Taking $x = \bar{P}^{-1}y$ as a coordinate transformation and pre-multiplying the Eq. (68) by $\bar{Q}$, Eq. (68) can be expressed as the following equivalent form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} u(t) + \begin{bmatrix} -x_2 - x_3 - 2x_1^3 - x_1^3(t-h) \\ x_1^3 + 0.5x_1^3(t-h) \\ x_2 + 2x_3 \end{bmatrix}, \tag{69}$$

from which it is easy to obtain the Hamiltonian form

$$E\dot{x}(t) = R_1(x)\nabla_x H_1(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}}H_1(\tilde{x}) + g_1(x)u, \tag{70}$$

where $H_1(x) = 0.25x_1^4 + 0.5x_2^2 + 0.5x_3^2$,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_1(x) = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix},$$

$$T_1(x, \tilde{x}) = \begin{bmatrix} -1 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad g_1(x) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that the state variables of the system (69) are bounded in the real application. Thus, in this example, we assume that the state satisfies $x \in \Omega = \{(x_1, x_2, x_3) : |x_1| \leq 0.5, |x_2| \leq 0.5, |x_3| \leq 0.5\}$.

Based on Theorem 3, the derived controller u of the system (69) satisfies the following condition:

$$g_1(x)u = R_1(x)\nabla_x H_a(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_a(\tilde{x}) + R_a(x)\nabla_x H(x) + T_a(x, \tilde{x})\nabla_{\tilde{x}} H(\tilde{x}) \\ - EKg_1(x)g_1^T(x)E\nabla_x H(x) + E[\iota\|x\|I_3 - M(x, \tilde{x})]\nabla_{\tilde{x}} H(\tilde{x}), \tag{71}$$

where $H(x) = x_1^{\frac{4}{3}} + x_2^{\frac{4}{3}} + x_3^{\frac{4}{3}}$ with $\alpha = 2$, $H_a(x) = H(x) - H_1(x)$, $R_a(x)$ and $T_a(x, \tilde{x})$ are given as

$$R_a(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1^{\frac{2}{3}} & -1 & -k_1 - 2 \end{bmatrix}, \quad T_a(x, \tilde{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_1\tilde{x}_1^{\frac{2}{3}} & 0 & 0 \end{bmatrix},$$

$$M(x, \tilde{x}) = T_1(x, \tilde{x}) + T_a(x, \tilde{x}) = \begin{bmatrix} -1 & 0 & 0 \\ 0.5 & 0 & 0 \\ x_1\tilde{x}_1^{\frac{2}{3}} & 0 & 0 \end{bmatrix}$$

with $k_1 > 0$.

Next, we prove that the controller u satisfying Eq. (71) is a finite-time controller of the system (69).

A straightforward computation shows that $R(x) = R_1(x) + R_a(x) = \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 0 \\ x_1^{\frac{2}{3}} & 0 & -k_1 \end{bmatrix}$.
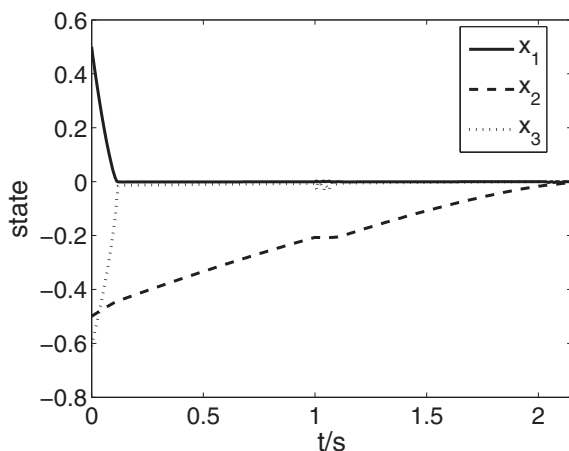
Select $\iota = 1$, $k_1 = 6$ and $K = \begin{bmatrix} -1 & -3 & -3 \\ 11 & 27 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then it is easy to obtain that

$$F_1(x) = R(x) - EKg_1(x)g_1^T(x)E = \begin{bmatrix} -6 & 0 & -1 \\ 0 & -5 & 0 \\ x_1^{\frac{2}{3}} & 0 & -6 \end{bmatrix},$$

$$T(x, \tilde{x}) = \begin{bmatrix} \iota\|x\| & 0 & 0 \\ 0 & \iota\|x\| & 0 \\ x_1\tilde{x}_1^{\frac{2}{3}} & 0 & 0 \end{bmatrix}.$$

Thus, we have $F_1^T(x) + F_1(x) \leq \text{Diag}\{-10, -10, -10\} := L$, $T(x, \tilde{x})T^T(x, \tilde{x}) \leq \text{Diag}\{2, 1, 2\} := S, N(x) = \frac{9}{16}\text{Diag}\{x_1^{\frac{2}{3}}, x_2^{\frac{2}{3}}, x_3^{\frac{2}{3}}\} \leq \frac{9}{16}I_3 := Z$ which implies that Condition (i) of Theorem 3 holds.

Now, we show that the condition (ii) of Theorem 3 holds. Choosing $a = 0.6$ and $b = 1$, it is easy to obtain that $m < 0$ for $0 < h \leq 4.9490$, which implies that Condition (ii) of Theorem 3 holds.

Fig. 5.2. State Curves with $h = 0.5$ and the controller (72).

Thus, all conditions of Theorem 3 are satisfied for this system when $0 < h \leq 4.9490$. From Theorem 3, the controller u satisfying Eq. (71) is a finite-time controller of the system (69). Therefore, from Eq. (71), one can obtain that the following finite-time controller of the system $u = [u_1, u_2]^T$ with Eq. (69):

$$u_1 = \frac{28}{9}x_1^{\frac{1}{3}} + \frac{2}{15}\tilde{x}_1^3 - \frac{8}{45}\tilde{x}_1^{\frac{1}{3}} + x_2 + \frac{38}{9}x_3^{\frac{1}{3}} - \frac{16}{9}x_2^{\frac{1}{3}} + \frac{11}{6}x_3 - \frac{10}{9}x_1 - \frac{10}{9}x_1^2\tilde{x}_1 + \frac{2}{9}\|x\|^2\tilde{x}_1^{\frac{1}{3}}$$
$$+ \frac{4}{9}\|x\|^2\tilde{x}_2^{\frac{1}{3}},$$

$$u_2 = \frac{52}{9}x_1^{\frac{1}{3}} - x_1^3 - \frac{11}{30}\tilde{x}_1^{\frac{1}{3}} - \frac{16}{9}x_2^{\frac{1}{3}} - \frac{8}{45}\tilde{x}_1^{\frac{1}{3}} + \frac{1}{3}x_3 + \frac{28}{9}x_3^{\frac{1}{3}} - \frac{4}{9}\|x\|^2\tilde{x}_1^{\frac{1}{3}} - \frac{4}{9}x_1^2\tilde{x}_1$$
$$+ \frac{4}{9}\|x\|^2\tilde{x}_2^{\frac{1}{3}} - \frac{4}{9}x_1, \tag{72}$$
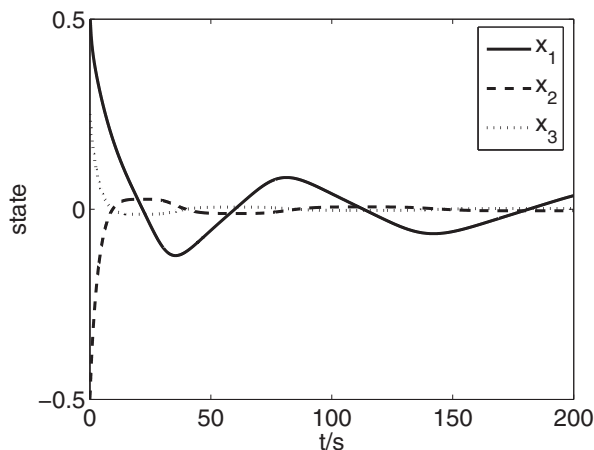
from which it follows that the finite-time controller of the system (68) is given as

$$u_1 = \frac{28}{9}y_1^{\frac{1}{3}} + \frac{2}{15}\tilde{y}_1^3 - \frac{8}{45}\tilde{y}_1^{\frac{1}{3}} + y_2 - y_3 + \frac{38}{9}y_3^{\frac{1}{3}} - \frac{16}{9}(y_2 - y_3)^{\frac{1}{3}} + \frac{11}{6}y_3 - \frac{10}{9}y_1 - \frac{10}{9}y_1^2\tilde{y}_1$$
$$+ \frac{2}{9}Y\tilde{y}_1^{\frac{1}{3}} + \frac{4}{9}Y(\tilde{y}_2 - \tilde{y}_3)^{\frac{1}{3}}, \tag{73}$$

$$u_2 = \frac{52}{9}y_1^{\frac{1}{3}} - y_1^3 - \frac{11}{30}\tilde{y}_1^{\frac{1}{3}} - \frac{16}{9}(y_2 - y_3)^{\frac{1}{3}} - \frac{8}{45}\tilde{y}_1^{\frac{1}{3}} + \frac{1}{3}y_3 + \frac{28}{9}y_3^{\frac{1}{3}} - \frac{4}{9}Y\tilde{y}_1^{\frac{1}{3}} - \frac{4}{9}y_1^2\tilde{y}_1$$
$$+ \frac{4}{9}Y(\tilde{y}_2 - \tilde{y}_3)^{\frac{1}{3}} - \frac{4}{9}y_1,$$
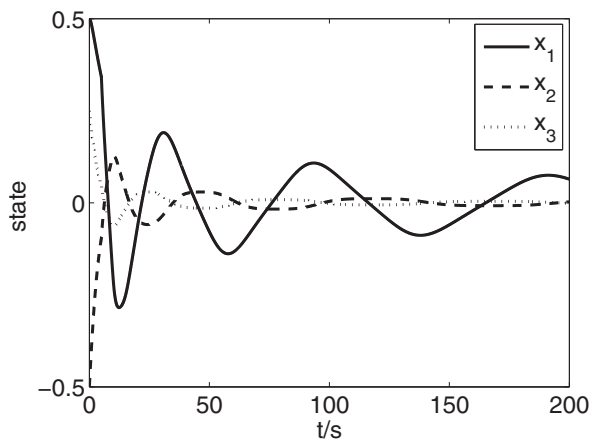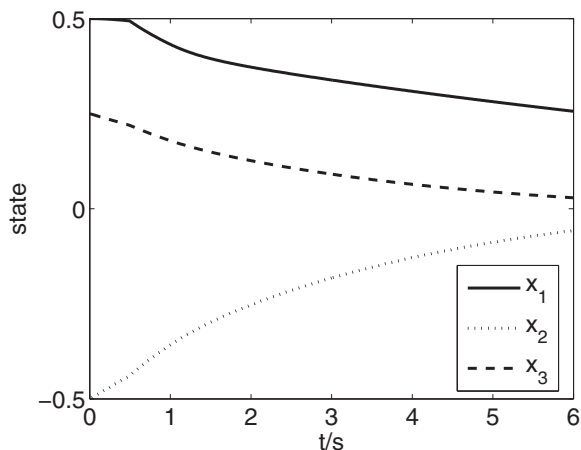
where $Y := y_1^2 + y_2^2 + 2y_3^2 - 2y_2y_3$.

To show the effectiveness of the designed controller, we carry out some simulation results. Noting that the system (68) is equivalent to the system (69), we give the simulation of the system (69) under the controller (72) with the following choices: Initial condition: $E\phi(t) = (0.5, -0.5, 0)$; Time delay: $h = 0.5$. The simulation results are shown in Figs. 5.2 and 5.3, which are the state's responses and the control signal u used in this example, respectively.

In addition, to show the effect of time delay and disturbance on system behavior, we give several simulation results for the free system in Figs. 5.4–5.7. Where Figs. 5.4 and 5.5 are

Fig. 5.3. Control Curves with $h = 0.5$.

Fig. 5.4. State Curves ($h = 0.5$, $u = 0$).

two simulation results under different delays and without disturbance. While in Fig. 5.7, we add a sudden disturbance $w = 6$ into the first equation of the system (69) in the time duration [2s ∼ 3s] for the free system. Furthermore, for the purpose of comparison, we also give two simulation results in 5.8 and 5.9, which are the responses of the state for the system (69) under $u = 0$, $w = 6$ and the controller (72), $w = 6$, respectively.

From Figs. 5.2 and 5.4, it is easy to see that the free system does not reach the equilibrium point until 200 seconds. While under the finite-time controller designed in the paper, the system (69) quickly converges to equilibrium within 2.5 seconds, which implies that the controller designed of the paper is very effective. Furthermore, from Figs. 5.4 and 5.5, it is obvious the effect of the size of the time delay on the behavior of the system is relatively large. Particularly, the response of the magnetic flux (namely, x_1) is affected obviously by the size of the time delay, which implies that it is very important to provide an estimate of the upper bound on time delay (namely, delay-dependent result). While from Figs. 5.6 and 5.7, one can know that the external disturbance has a greater impact on system's states. Compared

Fig. 5.5. State Curves ($h = 5$, $u = 0$).

Fig. 5.6. State Curves with $u = 0$ and $w = 0$.

with Figs. 5.7 and 5.8, Fig. 5.9 shows that, whether it is from convergence time or suppression of oscillation on time delay and disturbance, the controller (72) is very effective. That is to say, the finite-time control algorithm has a good robustness, which is the main reason why the paper develops the finite-time controller.

Example 3. Consider a singular time-delay system [35] as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix}$$

$$x(t) + \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} x(t-h) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t), \tag{74}$$

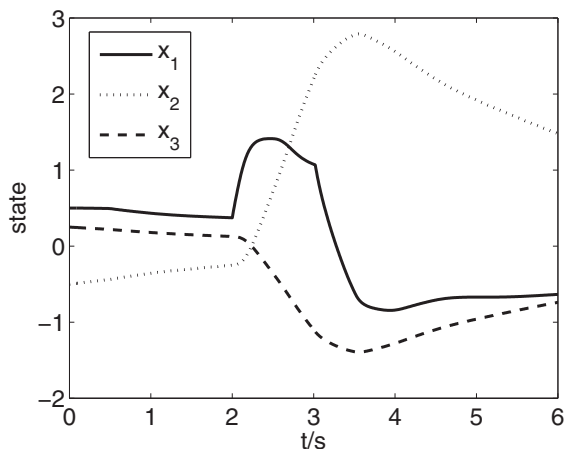where h is a constant delay.

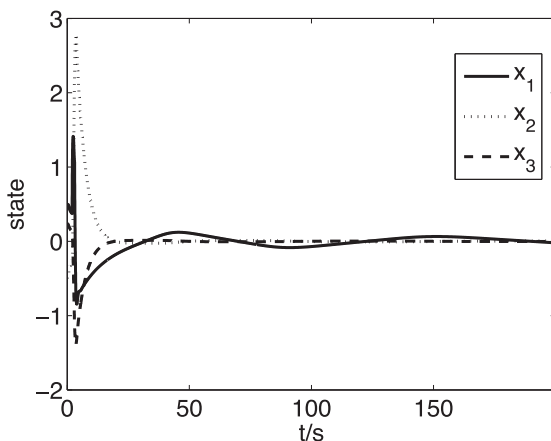Fig. 5.7. State Curves with $u = 0$ and $w = 6$.

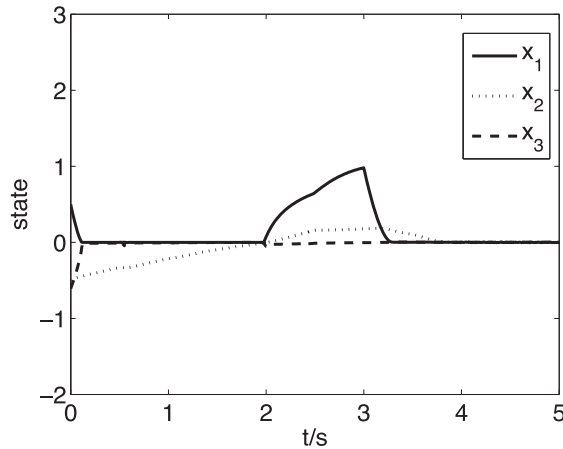Fig. 5.8. State Curves with $w = 6$ and $u = 0$.

Note that, for the example, the reference [35] designed a finite-time bounded stabilization controller (Further details, also see [35]). To compare the control effects of the two methods, in the following, we design its finite-time stabilization controller based on Theorem 3. Moreover, to apply the result of the paper, assume that $\Omega = \{(x_1, \ x_2, \ x_3) : |x_1| \le 0.5, \ |x_2| \le 0.5, \ |x_3| \le 0.5\}$.

First, we express the system (74) as the following Hamiltonian form:

$$E\dot{x} = R_1(x)\nabla_x H_1(x) + T_1(x, \tilde{x})\nabla_{\tilde{x}} H_1(\tilde{x}) + g_1(x)u, \tag{75}$$

where $H_1(x) = 0.5x_1^2 + 0.5x_2^2 + 0.5x_3^2$,

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ R_1(x) = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ -1 & 0 & -2 \end{bmatrix},$$

Fig. 5.9. State Curves with $w = 6$ and the controller (72).

$$T_1(x, \tilde{x}) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ g_1(x) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

By choosing $K = I_3$, $k_1 = 6$, $\iota = 0.1$, a direct calculation yields $N(x) \leq 0.3544 I_3 := Z$,

$$F_1(x) + F_1^T(x) = \begin{bmatrix} -6 & -1 & x_1^{\frac{2}{3}} \\ -1 & -6 & 0 \\ x_1^{\frac{2}{3}} & 0 & -12 \end{bmatrix} \leq -4.5 I_3 := L,$$

$$T(x, \tilde{x}) T^T(x, \tilde{x}) = \begin{bmatrix} \iota^2(x_1^2 + x_2^2 + x_3^2) & 0 & x_1 \tilde{x}_1^{\frac{2}{3}} \iota \sqrt{x_1^2 + x_2^2 + x_3^2} \\ 0 & \iota^2(x_1^2 + x_2^2 + x_3^2) & 0 \\ x_1 \tilde{x}_1^{\frac{2}{3}} \iota \sqrt{x_1^2 + x_2^2 + x_3^2} & 0 & x_1^2 \tilde{x}_1^{\frac{4}{3}} \end{bmatrix}$$

$$\leq I_3 := S.$$

Let $a = 0.5$ and $b = 1$, one can obtain that Condition (ii) holds for $h \leq 1.22$. Thus, a finite-time stabilization controller can be designed as

$$u = -\frac{16}{9} x_1^{\frac{1}{3}} - \frac{4}{3} x_2^{\frac{1}{3}} - \frac{8}{3} x_3^{\frac{1}{3}} + \frac{13}{9} x_1 + \frac{1}{3} x_2 + \frac{2}{3} x_3 + \frac{4}{9} x_1 \tilde{x}_1 + \frac{2}{45} \tilde{x}_1^{\frac{1}{3}} \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$+ \frac{2}{45} \tilde{x}_2^{\frac{1}{3}} \sqrt{x_1^2 + x_2^2 + x_3^2} - \tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3. \tag{76}$$

To show the effectiveness of the designed controller, we carry out some simulations for the system (75) under the controller (76) and the one designed in [35] with the following choices: Initial condition: $E\phi(t) = (0.1, -0.3, 0)$; Time delay: $h = 1$. The simulation results are shown in Figs. 5.10 and 5.11, which are the responses of the states under the controller (76) of the paper and the one designed in [35], respectively.
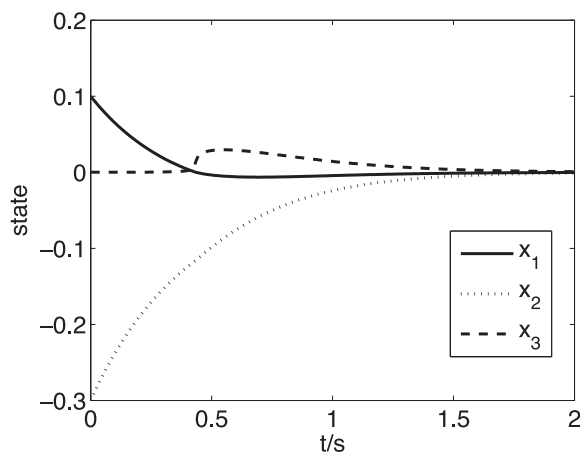
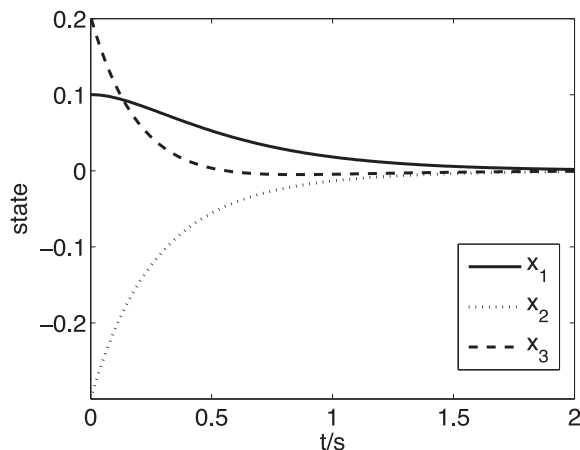Fig. 5.10. State response with Eq. (76) of the paper.

Fig. 5.11. State response with the controller in [35].

Compared with Figs. 5.10 and 5.11, it is easy to see that the response of the state with the controller (76) of the paper has a smaller amplitude of oscillation (also see x_1 and x_3), and the convergence speed is relatively faster. In addition, to show the advantage of the finite-time controller (**Note:** it has better robustness and disturbance attenuation properties), we add a sudden disturbance $w = 1$ into the third equation of the system (75) in the time duration [1.5s ∼ 2s]. The simulation results are shown in Figs. 5.12 and 5.13 under the same parameters, which are the norm's response on $x^T E^T E x$ for the systems (75) with the controller (76) designed of this paper and the controller designed in [35], respectively.

From Figs. 5.12 and 5.13, it is obvious that, under the same disturbance, the norm on $x^T E^T E x$ has smaller oscillation amplitude with the finite-time controller (76) designed of the paper than the one designed in [35], which implies that the closed-loop system under the finite-time controller has a strong ability to resist external interference.
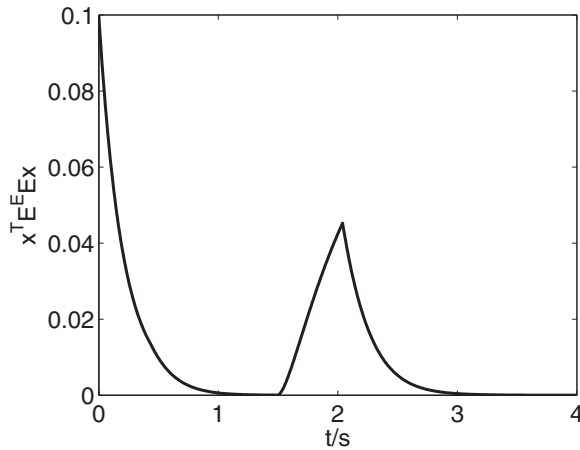
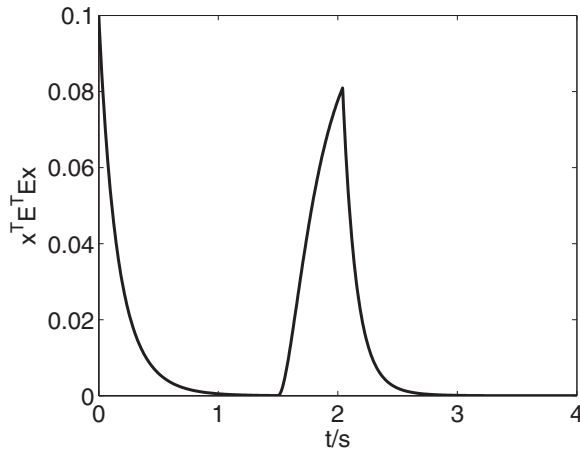Fig. 5.12. The norm $x^T E^T E x$ under Eq. (76) of the paper.

Fig. 5.13. The norm $x^T E^T E x$ with the controller in [35].

6. Conclusion

In this paper, we have investigated the FTS and finite-time stabilization problem for a class of nonlinear singular time-delay Hamiltonian systems, and obtained a number of new results on the problems for the systems. Different from the existing methods, we have presented an equivalent form for the system by decomposing singular matrix method and obtained a delay-dependent FTS results by constructing a kind of novel Lyapunov function. In addition, a finite-time control design procedure has also proposed for a class of nonlinear singular time-delay PCH systems. The study of several illustrative examples has shown that the results obtained in this paper work very well in the FTS and finite-time stabilization for some nonlinear singular time-delay (PCH) systems. In the future, we will apply the method proposed in the paper to further study more general systems such as switched systems [11,25,46,47], fuzzy systems [16,24] and chaotic neural network systems [32–34,36] and so on.

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