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Novel Mittag-Leffler stability of linear fractional delay difference equations with impulse

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A B S T R A C T

In this letter we propose a class of linear fractional difference equations with discrete-time delay and impulse effects. The exact solutions are obtained by use of a discrete Mittag-Leffler function with delay and impulse. Besides, we provide comparison principle, stability results and numerical illustration.

1. Introduction

Stability theory has intensive applications in control systems. Since fractional calculus holds memory effects and history dependence, we can use fully all of the past information for control and decision. Hence, fractional control has captured much attentions. It is meaningful and crucial to investigate the stability conditions.

As is well known, the linear differential equation

$$\frac{du}{dt} = \lambda u(t), u(a) = x_0 \tag{1}$$

has an exact solution $u_0 e^{\lambda(t-a)}$ and the system is exponentially stable for $\lambda < 0$.

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In the seminal work [1], the Lyapunov direct method is proposed by use of

$$ {}^{C}D_a^{\alpha}u(t) = \lambda u(t), 0 < \alpha \le 1,\ t \ge a, u(a) = u_0 \tag{2} $$

where ${}^{C}D_a^{\alpha}u(t)$ is the Caputo fractional derivative. Eq. (2) has the exact solution in form of the famous Mittag-Leffler function

$$ u(t) = u_0 E_{\alpha}(\lambda, t-a). \tag{3} $$

The system is Mittag-Leffler stable for $\lambda < 0$.

Considering the stability of the following fractional difference equations with delay and impulse

$$ \begin{cases} {}^{C}\Delta_a^{\nu}u(t) = \lambda u(t+\nu-1), 0 < \nu \le 1, t \in \mathbb{N}_{a+1-\nu},\ t \ne a+n_j+1-\nu, \\ u_{n_j+1} = u_{n_j+1}^{-} + q_j u_{n_j+1}^{-}, -1 < q_j < 0, t = a+n_j+1-\nu, j \in \mathbb{N}_1, \\ u(a) = u_0 = \eta, \end{cases} \tag{4} $$

where ${}^{C}\Delta_a^{\nu}u(t)$ is the Caputo difference of $u(t)$, $0 = n_0 < n_1 < \cdots < n_j < n_{j+1} < \cdots$ and $1 < n_{j+1} - n_j$, naturally, whether there is a Mittag-Leffler-like stability and what is the asymptotic stability condition for this concept? In this letter, we obtain an impulsive solution of Eq. (4) and propose a generalized Mittag-Leffler stability.

2. Preliminaries

2.1. Discrete fractional calculus

Definition 2.1 (*Fractional Sum [2,3]*)**.** Let x: $\mathbb{N}_a \to \mathbb{R}$ and $0 < \nu$. The fractional order sum is given by

$$ \Delta_a^{-\nu}x(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} K_{\nu}(t,s)x(s), \quad \sigma(s) = s+1, a \in \mathbb{R},\ t \in \mathbb{N}_{a+\nu}. \tag{5} $$

where $K_{\nu}(t,s) = (t-\sigma(s))^{(\nu-1)}$ and $t^{(\nu)}$ is the discrete factorial function defined by

$$ t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}. $$

Definition 2.2 (*Riemann–Liouville Difference [2,3]*)**.** Let x: $\mathbb{N}_a \to \mathbb{R}$ and $0 < \nu < 1$. The Riemann–Liouville (R–L) difference is given by

$$ \Delta_a^{\nu}x(t) := \frac{1}{\Gamma(-\nu)} \sum_{s=a}^{t+\nu} K_{-\nu}(t,s)x(s),\ t \in \mathbb{N}_{a+1-\nu}. \tag{6} $$

Definition 2.3 (*Caputo Difference [4,5]*)**.** For $x(t)$ defined on $\mathbb{N}_a$ and $0 < \nu < 1$, the Caputo difference is defined by

$$ {}^{C}\Delta_a^{\nu}x(t) := \Delta_a^{-(1-\nu)}\Delta x(t),\ t \in \mathbb{N}_{a+1-\nu}, \tag{7} $$

where $\Delta x(t) = x(t+1) - x(t)$. For $\nu = 1$, ${}^{C}\Delta_a^{\nu}x(t) = \Delta x(t)$.

2.2. Fractional delay difference equations with impulse

Inspired by the technique in the continuous fractional calculus [6], we need the following lemma to establish equivalent sum equations of fractional order. The proof is trivial and omitted here.

Lemma 2.4. *$x(t) : \mathbb{N}_a \to \mathbb{R}$ is a solution of the sum equation*

$$x(t) = x(t^*) - \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t^*-\nu} K_\nu(t^*, s)F(s+\nu-1) + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} K_\nu(t,s)F(s+\nu-1),\ t \in \mathbb{N}_{a+1} \tag{8}$$

if and only if $x(t)$ solves the following Cauchy problem:

$$^{C}\Delta_a^\nu x(t) = F(t+\nu-1), t \in \mathbb{N}_{a+1-\nu},\ 0 < \nu \leq 1$$

subject to

$$x(a) = x(t^*) - \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t^*-\nu} K_\nu(t^*, s)F(s+\nu-1). \tag{9}$$

Furthermore, we get the following sum equation.

Lemma 2.5. *A function $x(t) : \mathbb{N}_a \to \mathbb{R}$ is a solution of (9) if and only if $x(t)$ is a solution of the following fractional sum equation*

$$x(t) = \begin{cases} x_0 + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1), t \in \{a+1, \ldots, a+n_1\}, \\ \vdots \\ x_0 + \sum_{i=1}^{j} c_i + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1),\ t \in \{a+n_j+1, \ldots, a+n_{j+1}\}, j = 1, 2, \ldots, \\ \vdots \\ x_0 + \sum_{i=1}^{N} c_i + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1),\ t \in \{a+n_N+1, \ldots\},\ N \to \infty. \end{cases} \tag{10}$$

where $c_i = q_i x_{n_i+1}^-$.

Proof. For $t \in \{a, a+1, \ldots, a+n_1\}$, we get

$$x(t) = x_0 + \frac{1}{\Gamma(\nu)} \sum_{s=a+1-\nu}^{t-\nu} K_\nu(t,s)F(s+\nu-1), \tag{11}$$

and

$$x_{n_1+1}^- = x_0 + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1)|_{t=a+n_1+1}. \tag{12}$$

For $t \in \{a+n_1+1, \ldots, a+n_2\}$, we use Lemma 2.4 and Eq. (12) to derive

$$\begin{aligned} x(t) &= x_{n_1+1} - \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1)|_{t=a+n_1+1} + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1) \\ &= c_1 + x_{n_1+1}^- - \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1)|_{t=a+n_1+1} + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1) \\ &= x_0 + c_1 + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1). \end{aligned} \tag{13}$$

Let $t \in \{a+n_k+1, \ldots, a+n_{k+1}\}$. We have

$$x(t) = x_0 + \sum_{i=1}^{k} c_i + \Delta_{a+1-\nu}^{-\nu} F(t+\nu-1). \tag{14}$$

Then for $t \in \{a+n_{k+1}+1,\ldots,a+n_{k+2}\}$, it follows that

$$\begin{aligned} x(t) &= x_{n_{k+1}+1} - \Delta_{a+1-\nu}^{-\nu}F(t+\nu-1)|_{t=n_{k+1}+1} + \Delta_{a+1-\nu}^{-\nu}F(t+\nu-1) \\ &= c_{k+1} + x^{-}_{n_{k+1}+1} - \Delta_{a+1-\nu}^{-\nu}F(t+\nu-1)|_{t=a+n_{k+1}+1} + \Delta_{a+1-\nu}^{-\nu}F(t+\nu-1) \\ &= x_0 + \sum_{i=1}^{k+1} c_i + \Delta_{a+1-\nu}^{-\nu}F(t+\nu-1). \end{aligned} \tag{15}$$

By mathematical induction, this completes the proof for $t \in \{a+n_j+1,\ldots,a+n_{j+1}\}$. □

Theorem 2.6. *The fractional difference equation with impulse*

$$\begin{cases} {}^{C}\Delta_a^{\nu}x(t) = \lambda x(t+\nu-1),\ t\in\mathbb{N}_{a+1-\nu},\ t\neq a+n_j+1-\nu,\ 0<\nu<1, \\ x_{n_j+1} = x^{-}_{n_j+1} + q_j x^{-}_{n_j+1}, t=a+n_j+1-\nu, -1<q_j<0, j\in\mathbb{N}_1, \\ x(a)=x_0=\eta \end{cases} \tag{16}$$

has a unique solution

$$x(t) = \begin{cases} x_0\bar{e}_{\nu}(\lambda,t-a), t\in\{a+n_0+1,\ldots,a+n_1\}, \\ \vdots \\ x_0\prod_{i=1}^{j}(1+q_i\bar{e}_{\nu}(\lambda,n_i+1))\bar{e}_{\nu}(\lambda,t-a), t\in\{a+n_j+1,\ldots,a+n_{j+1}\}, j=1,2,\ldots,N-1, \\ \vdots \\ x_0\prod_{i=1}^{N}(1+q_i\bar{e}_{\nu}(\lambda,n_i+1))\bar{e}_{\nu}(\lambda,t-a), t\in\{a+n_N,\ldots\}, N\to\infty \end{cases} \tag{17}$$

where $\bar{e}_{\nu}(\lambda,t-a)$ is the discrete Mittag-Leffler function with delay [5]

$$\bar{e}_{\nu}(\lambda,t-a) = 1 + \sum_{k=1}^{\infty}\frac{\lambda^k(t-a+k(\nu-1))^{(k\nu)}}{\Gamma(k\nu+1)}.$$

Proof. We adopt the idea [7] to derive the exact solution. Let $t \in \{a+n_0+1,\ldots,a+n_1\}$. By Lemma 2.5, we have

$$x(t) = x_0 + \lambda\Delta_{a+1-\nu}^{-\nu}x(t+\nu-1) \tag{18}$$

and its exact solution reads

$$x(t) = x_0\bar{e}_{\nu}(\lambda,t-a).$$

Hence, we have

$$x^{-}_{n_1+1} = x_0\bar{e}_{\nu}(\lambda,n_1+1).$$

For $t \in \{a+n_1+1,\ldots,a+n_2\}$, we use Lemma 2.4 to derive

$$\begin{aligned} x(t) &= x_{n_1+1} - \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1)|_{t=a+n_1+1} + \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1) \\ &= (1+q_1)x^{-}_{n_1+1} - \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1)|_{t=a+n_1+1} + \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1) \\ &= x_0 + q_1x^{-}_{n_1+1} + \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1) \\ &= x_0(1+q_1\bar{e}_{\nu}(\lambda,n_1+1)) + \Delta_{a+1-\nu}^{-\nu}x(t+\nu-1) \end{aligned} \tag{19}$$

which leads to the solution

$$x(t) = x_0(1 + q_1\bar{e}_\nu(\lambda, n_1 + 1))\bar{e}_\nu(\lambda, t - a)$$

and

$$x^-_{n_1+2} = x_0(1 + q_1\bar{e}_\nu(\lambda, n_1 + 1))\bar{e}_\nu(\lambda, n_2 + 1).$$

By mathematical induction, we can obtain

$$x(t) = x_0\hat{e}_\nu(\lambda, t - a), t \in \{a + n_j + 1, \ldots, a + n_{j+1}\}, j = 1, \ldots, N, \ldots$$

where $\hat{e}_\nu(\lambda, t - a)$ denotes an impulsive discrete Mittag-Leffler function by

$$\hat{e}_\nu(\lambda, t - a) = \begin{cases} \bar{e}_\nu(\lambda, t - a), t \in \{a + n_0 + 1, \ldots, a + n_1\}, \\ \qquad \vdots \\ \prod_{i=1}^{j}(1 + q_i\bar{e}_\nu(\lambda, n_i + 1))\bar{e}_\nu(\lambda, t - a), t \in \{a + n_j + 1, \ldots, a + n_{j+1}\}, \\ \qquad\qquad j = 1, 2, \ldots, N - 1, \\ \qquad \vdots \\ \prod_{i=1}^{N}(1 + q_i\bar{e}_\nu(\lambda, n_i + 1))\bar{e}_\nu(\lambda, t - a), t \in \{a + n_N, \ldots\}, N \to \infty. \end{cases} \tag{20}$$

Remark 2.7. The asymptotic stability of $\bar{e}_\nu(\lambda, (t - a))$ was given in [8] as

$$-(z + 1)^\nu z^{\nu-1} < \lambda < 0,\ 0 < z < 1.$$

On the other hand, in order to guarantee the positivity, we should select $-\nu < \lambda$. Hence, we conclude that $-\nu < \lambda < 0$. □

3. Generalized Mittag-Leffler stability

Let $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_l(t)), t \in \mathbb{N}_a$ and $m(\mathbf{x}(t))$ is locally Lipschitz on the discrete domain $D \in \mathbb{R}^l$, $m(0) = 0$ and $m(\mathbf{x}(t)) \geq 0$. We assume that the solutions exist here and $0 < \nu \leq 1$.

Let $\mathbf{x} = 0$ be an equilibrium point of the fractional difference equation:

$$\begin{cases} {}^C\Delta_a^\nu \mathbf{x}(t) = f(t + \nu - 1, \mathbf{x}(t + \nu - 1)),\ t \in \mathbb{N}_{a+1-\nu},\ t \neq a + n_j + 1 - \nu, \\ \mathbf{x}_{n_j+1} = \mathbf{x}^-_{n_j+1} + q_j\mathbf{x}^-_{n_j+1}, -1 < q_j < 0,\ t \neq a + n_j + 1 - \nu,\ j \in \mathbb{N}_1, \\ \mathbf{x}(a) = \mathbf{x}_0 = \eta. \end{cases} \tag{21}$$

Definition 3.1 (*Generalized Mittag-Leffler Stability*)**.** The fractional difference equation or its zero solution is said to be generalized Mittag-Leffler stable if $\mathbf{x}(t)$ holds.

$$\|\mathbf{x}(t)\| \leq m(\mathbf{x}(t_0))\hat{e}_\nu(\lambda, (t - a)),\ t \in \mathbb{N}_{a+1}, -\nu < \lambda < 0$$

Example 3.2. Consider the following linear impulsive fractional difference equation:

$$\begin{cases} {}^C\Delta_a^\nu x(t) = \lambda x(t + \nu - 1),\ \ -\nu < \lambda < 0,\ t \in \mathbb{N}_{a+1-\nu}, t \neq a + n_j + 1 - \nu, \\ x_{n_j+1} = x^-_{n_j+1} + q_jx^-_{n_j+1},\ \ -1 < q_j < 0,\ t = a + n_j + 1 - \nu,\ j \in \mathbb{N}_1, \\ x(a) = x_0 > 0. \end{cases} \tag{22}$$

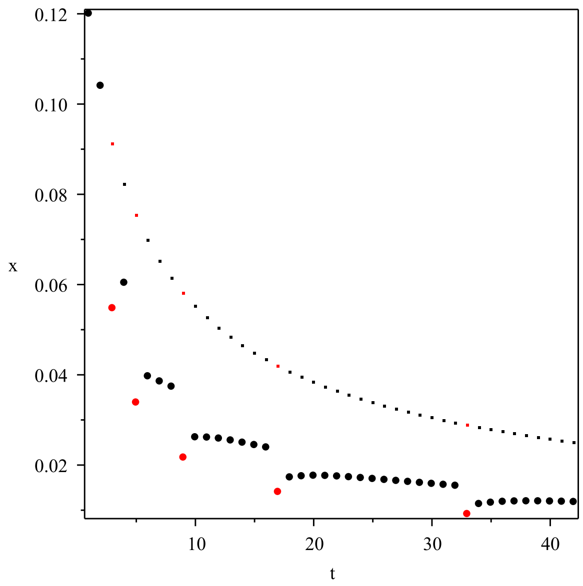

Fig. 1. $\nu = 0.6$ and $q_j = -0.4$.

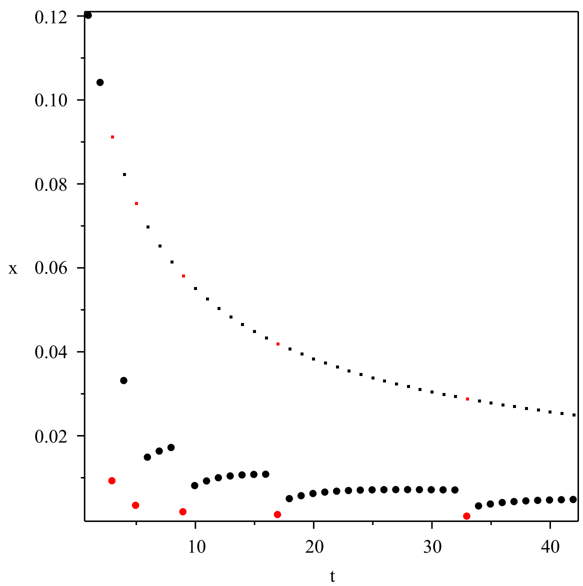

Fig. 2. $\nu = 0.6$ and $q_j = -0.9$.

For the conditions $-\nu < \lambda < 0$ and $-1 < q_j < 0$, we can see the system is impulsive Mittag-Leffler stable. We consider the numerical solution here by Lemma 2.4. We choose fractional order $\nu = 0.6$, $\lambda = -0.4$ and $x_0 = 0.2$, the impulse points $n_j = 2^j,\ j = 1, \ldots, N$ and $N = 5$.

The asymptotical behavior of the solution is illustrated in Figs. 1 and 2, respectively. We give numerical comparison of the impulse (solid circle) and the non-impulsive case (point). We can observe that with the decrease in $q_j \in (-1,\ 0)$, better results are obtained for the asymptotic stability.

Theorem 3.3 (*Discrete Fractional Comparison Principle*)**.** *Let $x(t)$ and $y(t)$ satisfy*

$$\begin{cases} {}^{C}\Delta_a^{\nu} x(t) = \lambda x(t+\nu-1),\ t \in \mathbb{N}_{a+1-\nu},\ t \neq a+n_j+1-\nu, \\ x_{n_j+1} = x^{-}_{n_{j+1}} + q_j x^{-}_{n_{j+1}},\ \ -1 < q_j < 0,\ t = a+n_j+1-\nu,\ j \in \mathbb{N}_1, \\ x_0 = \eta > 0. \end{cases} \tag{23}$$

and the inequality

$$\begin{cases} {}^{C}\Delta_a^{\nu} y(t) \leq \lambda y(t+\nu-1),\ t \in \mathbb{N}_{a+1-\nu}, t \neq n_j+1, \\ y_{n_j+1} = y^{-}_{n_{j+1}} + q_j y^{-}_{n_{j+1}},\ t = a+n_j+1-\nu,\ j \in \mathbb{N}_1, \\ y_0 = \eta, \end{cases} \tag{24}$$

respectively. If $-\nu < \lambda$, then $y(t) \leq x(t)$ for all $t \in \mathbb{N}_{a+1}$.

Proof. For the impulse case (23) and (24), we can derive the fractional sum equations of Volterra type as follows:

$$x(t) = \begin{cases} x_0 + \lambda \Delta^{-\nu}_{a+1-\nu} x(t+\nu-1),\ t \in \{a+1,\ldots,a+n_1\}, \\ \vdots \\ x_0 + \sum_{i=1}^{j} c_i + \lambda \Delta^{-\nu}_{a+1-\nu} x(t+\nu-1),\ t \in \{a+n_j+1,\ldots,a+n_{j+1}\}, j = 1,2,\ldots,N-1, \\ \vdots \\ x_0 + \sum_{i=1}^{N} c_i + \lambda \Delta^{-\nu}_{a+1-\nu} x(t+\nu-1),\ t \in \{a+n_N+1,\ldots\} \end{cases} \tag{25}$$

and

$$y(t) \leq \begin{cases} y_0 + \lambda \Delta^{-\nu}_{a+1-\nu} y(t+\nu-1),\ t \in \{a+1,\ldots,a+n_1\}, \\ \vdots \\ y_0 + \sum_{i=1}^{j} c_i + \lambda \Delta^{-\nu}_{a+1-\nu} y(t+\nu-1),\ t \in \{a+n_j+1,\ldots,a+n_{j+1}\}, j = 1,2,\ldots,N-1, \\ \vdots \\ y_0 + \sum_{i=1}^{N} c_i + \lambda \Delta^{-\nu}_{a+1-\nu} y(t+\nu-1),\ t \in \{a+n_N+1,\ldots\}. \end{cases} \tag{26}$$

Considering the relationship between the Caputo and the Riemann–Liouville differences, we rewrite Eqs. (23) and (24) as

$$\Delta_a^{\nu} x(t) = \frac{x(a)(t-a)^{(-\nu)}}{\Gamma(1-\nu)} + \lambda x(t+\nu-1) \tag{27}$$

and

$$\Delta_a^{\nu} y(t) \leq \frac{y(a)(t-a)^{(-\nu)}}{\Gamma(1-\nu)} + \lambda y(t+\nu-1), \tag{28}$$

respectively.

Let $t = a+n-\nu,\ 1 \leq n \leq n_1$. We get

$$\frac{1}{\Gamma(-\nu)} \sum_{j=0}^{n} \frac{\Gamma(n-j-\nu)}{\Gamma(n-j+1)} x_j = \frac{x_0 \Gamma(n-\nu+1)}{\Gamma(1-\nu)\Gamma(n+1)} + \lambda x_{n-1},$$

and

$$x_n = \frac{x_0 \Gamma(n-\nu+1)}{\Gamma(1-\nu)\Gamma(n+1)} + (\lambda+\nu)x_{n-1} - \frac{1}{\Gamma(-\nu)} \sum_{j=0}^{n-2} \frac{\Gamma(n-j-\nu)}{\Gamma(n-j+1)} x_j,$$

$$y_n \le \frac{y_0 \Gamma(n-\nu+1)}{\Gamma(1-\nu)\Gamma(n+1)} + (\lambda+\nu)y_{n-1} - \frac{1}{\Gamma(-\nu)} \sum_{j=0}^{n-2} \frac{\Gamma(n-j-\nu)}{\Gamma(n-j+1)} y_j,$$

respectively. Due to $-\nu < \lambda$, it is clear that $y_1 < x_1$.

If $n = k$, the assumption holds $y_j \le x_j,\ j = 2, \ldots, k, n_0 + 1 \le k \le n_1 - 1$;

For $n = k + 1$, we can derive that

$$y_{k+1} - x_{k+1} \le \frac{(y_0 - x_0)\Gamma(k+2-\nu)}{\Gamma(1-\nu)\Gamma(k+2)} + (\lambda+\nu)(y_k - x_k) - \frac{1}{\Gamma(-\nu)} \sum_{j=0}^{k-1} \frac{\Gamma(k-j+1-\nu)}{\Gamma(k-j+2)} (y_j - x_j).$$

It arrives at $y_{k+1} \le x_{k+1}$ from $y_0 \le x_0$, ..., $y_k \le x_k$.

Let $t = a + n - \nu,\ n_1 + 1 \le n \le n_2$. Since $y^-_{n_1+1} \le x^-_{n_1+1}$ and $0 < 1 + q_1 < 1$, we have $y(t) \le x(t)$ at the impulsive point $t = a + n_1 + 1 - \nu$. Hence, on the each set $\{a + n_j + 1, \ldots, a + n_{j+1}\},\ j \ge 0,\ n_0 = 0$, this also holds and completes the proof. □

Finally, we obtain the theorem for Gronwall-type inequality by the comparison principle.

Theorem 3.4. *Let $u(t) : \mathbb{N}_a \to \mathbb{R}^+$ and suppose that the fractional sum inequality with impulse holds.*

$$\begin{cases} u(t) \le u_0 + \lambda \Delta^{-\nu}_{a+1-\nu} u(t+\nu-1), u_0 = u(a) > 0, -\nu < \lambda < 0,\ t \in \mathbb{N}_{a+1}, t \ne a + n_j + 1, \\ u_{n_j+1} = u^-_{n_j+1} + q_j u^-_{n_j+1}, -1 < q_j < 0, t = a + n_j + 1, j \in \mathbb{N}_1, \end{cases} \tag{29}$$

then $u(t)$ is generalized Mittag-Leffler stable.

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