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Duality, Observability, and Controllability for Linear Time-Varying Descriptor Systems*

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Abstract. A characterization of observability for linear time-varying descriptor systems $E(t)x'(t) + F(t)x(t) = B(t)u(t)$, $y(t) = C(t)x(t)$ was recently developed. Neither E nor C were required to have constant rank. This paper defines a dual system, and a type of controllability so that observability of the original system is equivalent to controllability of the dual system. Criteria for observability and controllability are given in terms of arrays of derivatives of the original coefficients. In addition, the duality results of this paper lead to an improvement on a previous fundamental structure result for solvable systems of the form $E(t)x'(t) + F(t)x(t) = f(t)$.

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1. Introduction

For linear systems, it is well known that

$$x' = Ax + Bu,$$
$$y = Cx,$$

is controllable (observable) if and only if the dual system

$$x' = -A^{\mathrm{T}}x + C^{\mathrm{T}}u,$$
$$y = B^{\mathrm{T}}x,$$

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is observable (controllable). Similar results hold for time-invariant descriptor systems

$$Ex' + Fx = Bu,$$
$$y = Cx,$$

and their dual

$$E^T x' - F^T x = C^T u,$$
$$y = B^T x,$$

given appropriate definitions [9], [17].

In [7] we studied the observability of the linear time-varying input–output system with descriptor plant

$$E(t)x' + F(t)x = B(t)u, \tag{1a}$$

$$y = B(t)x, \tag{1b}$$

and characterized observability in terms of algebraic conditions on the derivative arrays formed from the coefficients E, F, B, C using only differentiations and pointwise linear algebra. None of the coefficients are assumed to have constant rank.

In this paper we show that if (1a) is solvable (defined in Section 2), then (1) is observable in the sense of [7] if and only if the *dual system*

$$(E(t)^T x)' - F(t)^T x = C(t)^T u, \tag{2a}$$

$$y = B(t)^T x, \tag{2b}$$

of (1) is controllable in a sense to be defined shortly. All matrices are assumed to be time varying unless explicitly stated to be constant. We also assume the coefficient matrices are real. The theory is easily modified for the complex case. The independent variable t is real.

A key feature of our approach is that we try to derive our criteria so that they can be verified by *numerical* operations applied to arrays of derivatives of the *original* coefficients. No symbolic operations are required other than differentiation of the original coefficients. The resulting criteria are not only easier to compute but are more general since they require fewer constant rank assumptions than are usually made in the literature.

The remainder of this paper is organized as follows. Section 2 introduces some needed terminology, develops the key results linking the solvability and structure of (1) with those of its dual (2), and improves on the fundamental structure result of [5]. Section 3 reviews the definition of observability from [7], defines the appropriate concept of controllability for the dual problem, and develops the results relating them. Concluding comments are given in Section 4. A discussion of applications of descriptor systems and their numerical solution can be found in [1].

2. Dual systems

The descriptor system

$$E(t)x' + F(t)x = f(t) \tag{3}$$

is said to be *solvable* on the connected interval $\mathscr{I}$ if, for every sufficiently smooth f on $\mathscr{I}$, there is a smooth solution defined on all of $\mathscr{I}$. In addition, all solutions are defined on all of $\mathscr{I}$ and solutions are uniquely determined by their value of any $t_0 \in \mathscr{I}$ [5]. This definition of solvability does not require E to have constant rank nor for it to be possible to carry out the usual inversion algorithms involving coordinate changes and differentiations [5], [13]. To avoid technical problems dealing with various degrees of smoothness we assume that E, F, C, B, f in (1), (2) are infinitely differentiable. Unless stated otherwise, smooth means infinitely differentiable on $\mathscr{I}$. The interval $\mathscr{I}$ is often omitted from the statements of the results that follow.

For time-invariant descriptor systems solvability is equivalent to regularity of the pencil E, F so that solvability of $Ex' + Fx = f$ implies solvability of $E^{\mathrm{T}}x' + F^{\mathrm{T}}x = g$ and $E^{\mathrm{T}}x' - F^{\mathrm{T}}x = g$. This is no longer true for time-varying descriptor systems.

Example 1. Let

$$N(t) = \begin{bmatrix} t & -t^2 \\ 1 & -t \end{bmatrix}.$$

Then $Nx' + x = f$ is solvable but $N^{\mathrm{T}}z' + z = g$ is not solvable [4].

Example 2. Define H, G, N by

$$H'H^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad H(0) = I, \qquad G^{-1}G' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G(0) = I,$$

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$Ny' + y = f \tag{4}$$

is solvable. Let $y = G^{-1}x$ and premultiply (4) by H^{-1} to get

$$\bar{E}x' + \bar{F}x = \bar{f},$$

which is solvable. However,

$$\bar{E}^{\mathrm{T}}\bar{x}' - \bar{F}^{\mathrm{T}}\bar{x} = \bar{f} \tag{5}$$

is not solvable. To see this, let $\bar{x} = H^{\mathrm{T}}\bar{y}$ in (5) and premultiply by G^{T} to get $N^{\mathrm{T}}\bar{y}' = \bar{g}$ which is not solvable.

We define the *dual* (*descriptor*) *system* of (3) to be

$$(E^{\mathrm{T}}z)' - F^{\mathrm{T}}z = g. \tag{6}$$

Note that (3), (6) can be identified with (1), (2) when $C = B = I$.

Examples 1 and 2 show that in general solvability of (3) does not imply the solvability of a descriptor system using transposed matrices as coefficients. The next three results relate the solvability of a system (3) and its dual (6).

Lemma 1. *Suppose that the smooth time-varying coordinate changes $x = Qy$, Q nonsingular, and multiplication by nonsingular P, are applied to $Ex' + Fx = f$ to give*

$$\bar{E}y' + \bar{F}y = \bar{f}. \tag{7}$$

Then letting $z = P^{\mathrm{T}}w$ and multiplying by Q^{T} converts (6) *to*

$$(\bar{E}^{\mathrm{T}}w)' - \bar{F}^{\mathrm{T}}w = \bar{g},$$

which is the dual of (7).

Proof. Straightforward calculation. □

Lemma 1 shows that dual systems are transferred into dual systems by time-varying coordinate changes. This enables us to use the structural forms developed for infinitely differentiable [5] and for real analytic descriptor systems [6] to answer several questions about dual systems.

Suppose that (3) is solvable and that E, F are infinitely differentiable. Then there exists infinitely differentiable coordinate changes $x = Q(t)z$, premultiplication by $P(t)$, which change (3) to

$$z_1' + H(t)z_2' = f_1(t), \tag{8a}$$

$$N(t)z_2' + z_2 = f_2(t), \tag{8b}$$

where (8b) has exactly one solution for each smooth f_2 [5]. That is, (8b) is a *totally singular* solvable system. There is an open dense subset Ω of $\mathcal{I}$ such that N can be made strictly upper triangular on a neighborhood of any point in Ω, but N cannot always be made strictly upper or lower triangular on all of $\mathcal{I}$ [5].

Before proceeding, we need a lemma.

Lemma 2. *Suppose that E, F are infinitely differentiable on the interval $\mathcal{I}$ and consider the systems*

$$(E^{\mathrm{T}}x)' - F^{\mathrm{T}}x = f \tag{9}$$

and

$$z' - F^{\mathrm{T}}x = \tilde{f}, \tag{10a}$$

$$z - E^{\mathrm{T}}x = \tilde{g}. \tag{10b}$$

If (9) *is solvable on* $\mathscr{I}$, *then* (10) *is solvable on* $\mathscr{I}$. *If* (10) *is solvable on* $\mathscr{I}$, *then, for any infinitely differentiable* f, (9) *has an infintely differentiable solution defined on all of* $\mathscr{I}$.

Proof. Suppose that (9) is solvable. Take any infinitely differentiable $\tilde{f}$, $\tilde{g}$ and let $\bar{x}$ be a solution of (9) with $f = \tilde{f} - \tilde{g}'$. Let $\bar{z} = E^{\mathrm{T}}\bar{x} + \tilde{g}$. Then $(\bar{x}, \bar{z})$ is a solution of (10). Given $\tilde{f}$, $\tilde{g}$, $t_0 \in \mathscr{I}$, and $\bar{x}(t_0)$, we have that $\bar{x}$ is uniquely determined since (9) is solvable. Then (10b) uniquely determines $\bar{z}$ and we have that (10) is solvable.

Now suppose that (10) is solvable. Given f, take any infinitely differentiable $\tilde{f}$, $\tilde{g}$ such that $f = \tilde{f} - \tilde{g}'$ and let (x, z) be a solution of (10). Then x is the required solution of (9). □

Theorem 1. *If* E, F *are infinitely differentiable on an interval* $\mathscr{I} = [a, b]$, *then* $Ex' + Fx = f$ *is solvable if and only if the dual system* $(E^{\mathrm{T}}z)' - F^{\mathrm{T}}z = g$ *is solvable.*

Proof. It suffices to show that solvability of $Ex' + Fx = f$ implies solvability of the dual system since the dual of the dual is the original system. If E, F are real analytic, then the coordinate changes P, Q can be chosen so that $H(t) \equiv 0$ in (8a) and N is strictly upper triangular for all t in (8b). That is, N is structurally nilpotent [6]. But then the dual system of (8) is

$$z_1' = g_1(t), \tag{11a}$$

$$N(t)^{\mathrm{T}}z_2' + (N'(t)^{\mathrm{T}} - I)z_2 = g_2(t), \tag{11b}$$

since $(N^{\mathrm{T}}z_2)' - z_2 = N^{\mathrm{T}}z_2' + (N'^{\mathrm{T}} - I)z_2$. But (11b) is solvable since $(N'^{\mathrm{T}} - I)^{-1}N^{\mathrm{T}}$ is structurally nilpotent if N is structurally nilpotent.

Suppose then that E, F are only infinitely differentiable. The dual of (8) is

$$\left(\begin{bmatrix} I & H \\ 0 & N \end{bmatrix}^{\mathrm{T}} z\right)' - \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} z = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

or

$$\begin{bmatrix} I & 0 \\ H^{\mathrm{T}} & N^{\mathrm{T}} \end{bmatrix} z' + \begin{bmatrix} 0 & 0 \\ H'^{\mathrm{T}} & N'^{\mathrm{T}} - I \end{bmatrix} z = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}. \tag{12}$$

After minor simplification, (12) becomes

$$z_1' = g_1, \tag{13a}$$

$$N^{\mathrm{T}}z_2' + (N'^{\mathrm{T}} - I)z_2 = g_2 - (H^{\mathrm{T}}z_1)'. \tag{13b}$$

It is straightforward to show that (13) is solvable if (and only if)

$$N^{\mathrm{T}} z_2' + (N'^{\mathrm{T}} - I) z_2 = h$$

or equivalently

$$(N^{\mathrm{T}} z_2)' - z_2 = h$$

is solvable. Thus if suffices to prove Theorem 1 for the special case $Nx' + x = f$ of a totally singular solvable system.

In order to complete the proof we need to introduce some notation and terminology. Let $\mathscr{C}^\infty$ be the space of infintely differentiable functions on the interval $\mathscr{I} = [a, b]$, with values in n-dimensional Euclidean space, equipped with the topology of uniform convergence of each derivative. That is $f_n \to f$ if $\|f_n^{(i)} - f^{(i)}\|_\infty \to 0$ for $i = 0, 1, \ldots$, where $\|\cdot\|_\infty$ is the sup norm on $\mathscr{I}$. Let $\mathscr{C}_0^\infty = \{f \in \mathscr{C}^\infty : f^{(i)}(a) = f^{(i)}(b) = 0,\ i = 0, 1, 2, \ldots\}$. A *kth-order differential operator* V is an expression of the form

$$V = \sum_{i=0}^{k} V_i D^i = \sum_{i=0}^{k} V_i(t) \frac{d^i}{dt^i},$$

where the V_i are infinitely differentiable $n \times n$ matrix-valued functions and $V_k \neq 0$. A differential operator can be viewed as a continuous linear transformation of both $\mathscr{C}^\infty$ and $\mathscr{C}_0^\infty$ into themselves given by $Vf = \sum_{i=0}^{k} V_i(t) f^{(i)}$. Define the *dual* of V to be the differential operator

$$V^d = \sum_{i=0}^{k} (-1)^i D^i V_i^{\mathrm{T}}.$$

For $f, g \in \mathscr{C}^\infty$ define

$$\langle f, g \rangle = \int_a^b f(t)^{\mathrm{T}} g(t)\, dt.$$

Then integration by parts shows that

$$\langle Vf, g \rangle = \langle f, V^d g \rangle, \qquad f \in \mathscr{C}_0^\infty, \quad g \in \mathscr{C}^\infty.$$

Also if $g \in \mathscr{C}^\infty$ and $\langle f, g \rangle = 0$ for all $f \in \mathscr{C}_0^\infty$, then $g = 0$. Finally it is straightforward to show that if two differential operators V, R agree on $\mathscr{C}_0^\infty$, then they are identical on $\mathscr{C}^\infty$ and have identical expansions $V = \sum_{i=0}^{r} V_i D^i$. Thus we can show by the usual argument that

$$(VR)^d = R^d V^d.$$

Now let $L = ND + I$ and assume that $Lx = f$ is a totally singular solvable system. Then from [5] we have that the solution of $Lx = f$ is $x = \sum_{i=0}^{r} L_i f^{(i)}$. Thus L is an inverbible operator of $\mathscr{C}^\infty$, and $\mathscr{C}_0^\infty$, onto themselves and $L^{-1} = \sum_{i=0}^{r} L_i D^i$ is an rth-order differential operator. The fact that L^{-1} is a differential operator implies $(L^d)^{-1} = (L^{-1})^d$ so that L^d is an invertible differential operator on $\mathscr{C}^\infty$ and hence $L^d x = f$ is a solvable system. $\square$

It is important to keep in mind that we are interpreting V^d as simply another differential operator on $\mathscr{C}^\infty$ and not as an operator on the dual space of $\mathscr{C}^\infty$. Also $\langle \cdot, \cdot \rangle$ is merely a convenient bilinear map on $\mathscr{C}^\infty \times \mathscr{C}^\infty$ and not the usual bi- (or conjugate-) linear map from a space times its dual into the scalar field.

From Theorem 1 we get the following corollary which is useful in improving the fundamental structural form (8) and in discussing controllability.

Corollary 1. *Suppose that $Nx' + x = f$ is a totally singular solvable system on the interval $\mathscr{I}$ with infinitely differentiable N. Then, for every infinitely differentiable f, the system $-N^{\mathrm{T}}x' + x = f$ has a unique infinitely differentiable solution defined on all of $\mathscr{I}$.*

Proof. The solvability of $Nx' + x = f$ implies that the dual $(N^{\mathrm{T}}x)' - x = f$ is also solvable. Lemma 2 then yields

$$z' - x = h, \tag{14a}$$

$$z - N^{\mathrm{T}}x = g, \tag{14b}$$

is solvable. System (14) has smooth solutions for every smooth h, g. Adding $-N^{\mathrm{T}}$ times (14a) to (14b) gives that $-N^{\mathrm{T}}z' + z = \tilde{f}$ has solutions defined on all of $\mathscr{I}$ for every $\tilde{f} = g - N^{\mathrm{T}}h$. But then $\tilde{f}$ is an arbitrary infinitely differentiable function.

To show the uniqueness assume that $-N^{\mathrm{T}}x' + x = 0$. Let $z = x'$. Then (x, z) is a solution of

$$x' - z = 0, \tag{15a}$$

$$-N^{\mathrm{T}}z + x = 0. \tag{15b}$$

But (15) implies that z satisfies

$$(N^{\mathrm{T}}z)' - z = 0. \tag{16}$$

By the last line of the proof of Theorem 1, $(N^{\mathrm{T}}z)' - z = g$ is a totally singular solvable system since $Nx' + x = f$ was assumed to be one. Thus (16) implies that $z = 0$. Then (15b) gives $x = 0$ and the uniqueness of solutions of $-N^{\mathrm{T}}x' + x = f$ follows. □

Corollary 1 should be compared with Examples 1 and 2. Note that $\bar{F} \neq I$ in Example 2. Corollary 1 enables us to improve on the fundamental structural form (8) by eliminating the H.

Theorem 2. *Suppose that E, F are infinitely differentiable and that* (3) *is solvable. Then there exists infinitely differentiable coordinate changes*

$x = Q(t)z$, multiplication by $P(t)$, which transform (3) *to*

$$z_1' = g_1(t), \tag{17a}$$

$$N(t)z_2' + z_2 = g_2(t), \tag{17b}$$

where (17b) *is a totally singular solvable system.*

Proof. From [5] we know that we can get (8). We only need to show that it is possible to get $H = 0$. Letting

$$z = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} w$$

and premultiplying by

$$\begin{bmatrix} I & -X' \\ 0 & I \end{bmatrix}$$

transforms (8) to

$$\begin{bmatrix} I & X + H - X'N \\ 0 & N \end{bmatrix} w' + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} w = \tilde{g}(t).$$

Theorem 2 will follow if an X exists such that $-X'N + X = -H$. Taking transposes of this equation gives

$$-N^{\mathrm{T}}[X^{\mathrm{T}}]' + [X^{\mathrm{T}}] = -H^{\mathrm{T}}. \tag{18}$$

But Corollary 1 shows that (18) has a solution X^{T} and hence X exists. □

3. Observability and controllability

3.1. *Observability*

The most common classical type of observability is total observability.

Definition 1. The descriptor input–output system (1) is (*totally*) *observable* on the interval $\mathscr{I}$ if knowledge of the output y and the control u on any subinterval $\tilde{\mathscr{I}}$ of $\mathscr{I}$ uniquely determines *smooth* solutions x of (1a) on $\tilde{\mathscr{I}}$.

If C in (1b) is not full column rank on a dense set, then the additional information to determine x is obtained (at least theoretically) by differentiating (1b). Observability has been frequently discussed when $E(t) = I$ since the early work in [10], [14], [16] and in the descriptor case when E, F, B, C are constant matrices [8], [9], [11], [12]. Computational issues for the time-invariant case are covered in [15].

We assume that the controls u are sufficiently smooth and the initial conditions for the descriptor system consistent so that no impulsive behavior is present. In some problems a stronger type of observability is needed.

Definition 2. System (1) is *smoothly observable* (of order (k, j)) on the interval $\mathscr{I}$, if there exists smooth $K_i(t)$, $L_i(t)$ on $\mathscr{I}$ such that

$$x = \sum_{i=0}^{k} K_i(t) y^{(i)}(t) + \sum_{i=0}^{j} L_i(t)(Bu)^{(i)}(t).$$

Here smooth is taken to be either infinitely differentiable or real analytic depending on context. System (1) is *uniformly observable* if it is smoothly observable of order $(n-1, n-1)$. Uniform observability is usually defined differently [14]. Note that any portions of the solution x which are completely determined by the control u are automatically observable.

A system of algebraic equations, $Ax = b$,

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b \tag{19}$$

is 1-*full* [2], [3] with respect to x_1 if (19) uniquely determines x_1 for any consistent b. The matrix A being 1-full is equivalent to the row echelon form of A being

$$\begin{bmatrix} I_{s\times s} & 0 \\ 0 & * \end{bmatrix},$$

where $*$ is a possibly nonzero entry and $s = \dim x_1$.

Lemma 3. *System* (19) *is* 1-*full with respect to* x_1 *if and only if*

$$\operatorname{rank}\left[A^{\mathrm{T}} \;\middle|\; \begin{matrix} I_{s\times s} \\ 0 \end{matrix} \right] = \operatorname{rank} A^{\mathrm{T}}, \tag{20}$$

where $s = \dim x_1$. *Condition* (20) *is equivalent to*

$$\operatorname{rank}\begin{bmatrix} A \\ \hline I_{s\times s} \quad 0 \end{bmatrix} = \operatorname{rank} A. \tag{21}$$

Proof. Clearly (20) and (21) are equivalent. Let $\mathscr{R}(X)$, $\mathscr{N}(X)$ denote the range and nullspace of a matrix X. Define

$$\mathscr{M}_s = \mathscr{R}\left(\begin{bmatrix} I_{s\times s} \\ 0 \end{bmatrix}\right).$$

Then A is 1-full if and only if $\mathscr{M}_s \perp \mathscr{N}(A)$, or $\mathscr{M}_s \subset \mathscr{N}(A)^{\perp} = \mathscr{R}(A^{\mathrm{T}})$, which is (20). $\square$

For simplicity, let $b(t) = B(t)u(t)$ and assume that the descriptor system (1a) is solvable. Differentiating (1a) j times and (1b) k times gives the system of equations

$$[\mathscr{F}_j \quad \mathscr{E}_j]\begin{bmatrix} x \\ \mathbf{x}_j \end{bmatrix} = \mathbf{b}_j, \tag{22a}$$

$$\mathscr{C}_k\begin{bmatrix} x \\ \mathbf{x}_{k-1} \end{bmatrix} = \mathbf{y}_k, \tag{22b}$$

where

$$\mathscr{F}_j = \begin{bmatrix} F \\ F' \\ \vdots \\ F^{(j)} \end{bmatrix}, \qquad \mathbf{y}_k = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(k)} \end{bmatrix}, \qquad \mathbf{b}_j = \begin{bmatrix} b \\ b' \\ \vdots \\ b^{(j)} \end{bmatrix}, \qquad \mathbf{x}_j = \begin{bmatrix} x' \\ \vdots \\ x^{(j+1)} \end{bmatrix},$$

$$\mathscr{E}_j = \begin{bmatrix} E & 0 & \cdots & \cdots & 0 \\ E' + F & E & 0 & \cdots & \vdots \\ E'' + 2F' & 2E' + F & E & \ddots & \vdots \\ * & * & * & \ddots & 0 \\ E^{(j)} + jF^{(j-1)} & * & * & * & E \end{bmatrix},$$

$$\mathscr{C}_k = \left[\begin{array}{c|cccc} C & 0 & \cdots & \cdots & 0 \\ C' & C & 0 & \cdots & \vdots \\ C'' & 2C' & C & \ddots & \vdots \\ * & * & * & \ddots & 0 \\ C^{(k)} & * & * & * & C \end{array}\right] = [\tilde{\mathscr{C}}_k \mid \hat{\mathscr{C}}_k].$$

Then from [7] we have the following two results.

Proposition 1. *The descriptor system* (1) *is totally observable on the interval* $\mathscr{I}$ *if and only if there exist* j, k, *with* $k \leq j + 1$, *such that the matrix*

$$\mathscr{O}_{j,k} = \left[\begin{array}{c|c} \mathscr{F}_j & \mathscr{E}_j \\ \tilde{\mathscr{C}}_k & [\hat{\mathscr{C}}_k \quad 0_{(k+1)m \times (j+1-k)n}] \end{array}\right] \tag{23}$$

is 1-*full with respect to* x *on a dense subset of* $\mathscr{I}$.

Proposition 2. *If* $\mathscr{O}_{j,k}$ *is* 1-*full on a dense subset of* $\mathscr{I}$ *and has constant rank, then* (1) *is smoothly observable of order* (j, k).

Using Lemma 3 we have

Proposition 3. *If* $\mathscr{O}_{j,k}$ *has constant rank, then* (1) *is smoothly observable if and only if*

$$\operatorname{rank}\left[\begin{array}{cc} \multicolumn{2}{c}{\mathscr{O}_{j,k}} \\ \hline I_{n\times n} & 0 \end{array}\right] = \operatorname{rank} \mathscr{O}_{j,k} \tag{24}$$

on $\mathscr{I}$.

Propositions 1 and 2 are the types of results we are seeking in that the only symbolic operations that need to be performed are the differentiation of the given coefficients. Proposition 3 is interesting since it is similar to classical results relating observability to the constant ranks of certain matrices. Another useful result is [5]

Theorem 3. *Suppose that (1a) is solvable on the interval $\mathscr{I}$ and that E, F are $2n$-times differentiable. Then*

$$\mathscr{E}_i \text{ has constant rank on } \mathscr{I} \text{ for } i = n, \tag{25}$$

$$\mathscr{E}_i \text{ is 1-full with respect to } x' \text{ for } i = n, \tag{26}$$

$$[\mathscr{F}_i \quad \mathscr{E}_i] \text{ has full row rank for } 1 \leq i \leq n. \tag{27}$$

If the coefficients E, F are infinitely differentiable as we assume here, then Theorem 3 provides sufficient as well as necessary conditions for solvability. If (27) holds, then the smallest value of i that satisfies conditions (25) and (26) of Theorem 3 is called the *index* v of the descriptor system (1a). For time-invariant descriptor systems, the index is the same as the index of the pencil $\lambda E + F$. However, for time-varying solvable descriptor systems, the pencil $\lambda E + F$ need not be regular, and if the pencil is regular its index need not be that of the descriptor system.

Theorem 3 is important since it assures us that if the descriptor system (1a) is solvable, then $\mathscr{E}_j$ will have constant rank for some j even if E does not. Thus a computation concerning $\mathscr{E}_j$ can be well conditioned.

If we perform the time-varying coordinate changes

$$x = S(t)\bar{x}$$

and premultiplication by $T(t)$, the new derivative arrays are related to the old by

$$[\bar{\mathscr{F}}_j \quad \bar{\mathscr{E}}_j] = \mathscr{T}_j[\mathscr{F}_j \quad \mathscr{E}_j]\mathscr{S}_j,$$
$$[\bar{\mathscr{C}}_k \quad 0] = [\mathscr{C}_k\mathscr{S}_k \quad 0] = [\mathscr{C}_k \quad 0]\mathscr{S}_j,$$

where

$$\mathscr{X}_i = \begin{bmatrix} X & 0 & \cdots & \cdots & 0 \\ X' & X & 0 & \cdots & \vdots \\ X'' & 2X' & X & \ddots & \vdots \\ * & * & * & \ddots & 0 \\ X^{(i)} & * & * & * & X \end{bmatrix} \quad \text{for} \quad X = S, T.$$

Thus, for a given j, k, the 1-fullness of $\mathscr{O}_{j,k}$ is unchanged by time-varying coordinate changes. Hence we may assume by Theorem 2 that (1) has the form

$$x_1' = B_1(t)u, \tag{28a}$$

$$N(t)x_2' + x_2 = B_2(t)u, \tag{28b}$$

$$y = C_1(t)x_1 + C_2(t)x_2. \tag{28c}$$

As noted in the proof of Theorem 1 there exists smooth L_i such that $x_2 = \sum_{i=0}^{r} L_i(t)(B_2 u)^{(i)}$ so that x_2 is already observable independent of C. Thus observability of (1) reduces to considering (28a), (28c) with $C_2 x_2$ known, which is a classical nonsingular observability problem.

3.2. *Controllability*

Observability, as we have defined it, of (1) is observability of (28a), (28c). Thus the following definition is natural.

Definition 3. Suppose that $Ex' + Fx = f$ is solvable with infinitely differentiable coefficients. Then the descriptor control system (1a) is *R-controllable* if when transformed to the form (28) we have that (28a) is (totally) controllable.

R-controllable was originally defined in [17] for time-invariant systems. In order for R-controllability to be well defined, the property of being R-controllable must be invariant under time-varying coordinate changes on (3). To show this suppose that time-varying coordinate changes on (1a) also produce

$$z_1' = \tilde{B}_1 u, \tag{29a}$$

$$\tilde{N}(t) z_2' + z_2 = \tilde{B}_2 u. \tag{29b}$$

By assumption there exist Q, P nonsingular such that $x = Qz$, premultiplication by P, transform (28a), (28b) to (29). Thus

$$P\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Q = \begin{bmatrix} I & 0 \\ 0 & \tilde{N} \end{bmatrix} \tag{30}$$

and

$$P\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} Q' + P\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} Q = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \tag{31}$$

The time-varying transformation Q must send solutions of the associated homogenous equation for (29) onto solutions of the associated homogeneous equation for (28a), (28b). Thus $\mathscr{R}\left(\begin{bmatrix} \mathscr{I} \\ 0 \end{bmatrix}\right)$ is an invariant subspace of Q for every t. It follows that

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_4 \end{bmatrix}, \qquad P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix},$$

where Q_1, Q_4 are nonsingular. Then (30) implies that

$$P_1Q_1 = I, \tag{32}$$

$$P_1Q_2 + P_2NQ_4 = 0, \tag{33}$$

$$P_3Q_1 = 0, \tag{34}$$

$$P_3Q_2 + P_4NQ_4 = \tilde{N}. \tag{35}$$

Then Q_1 is nonsingular and $P_3 = 0$ from (34) so that (35) becomes

$$P_4NQ_4 = \tilde{N}. \tag{36}$$

Equation (31) implies that

$$P_1Q_1' = 0, \tag{37}$$

$$P_1Q_2' + P_2NQ_4' + P_2Q_4 = 0, \tag{38}$$

$$P_4NQ_4' + P_4Q_4 = I, \tag{39}$$

P_1 is nonsingular and (37) implies $Q_1' = 0$ and hence P_1, Q_1 are constants. We shall show that $Q_2 = 0$. Then (38) and (39) will imply that $P_2 = 0$ also by the nonsingularity of $NQ_4' + Q_4$. The only equations involving Q_2 are (33) and (38) where we have P_1Q_2. Thus for convenience we may assume that $P_1 = I$ (or, equivalently, let $\tilde{Q}_2 = P_1Q_2$). From (38) and (39) we have

$$Q_2' = -P_2(NQ_4' + Q_4) = -P_2P_4^{-1}$$

and from (33) and (36) we have

$$Q_2 = -P_2NQ_4 = -P_2P_4^{-1}P_4NQ_4 = -P_2P_4^{-1}\tilde{N}$$

so that

$$Q_2'\tilde{N} - Q_2 = 0$$

or

$$\tilde{N}^{\mathrm{T}}[Q_2^{\mathrm{T}}]' - [Q_2^{\mathrm{T}}] = 0.$$

But then $Q_2^{\mathrm{T}} = 0$ by Corollary 1 since $\tilde{N}x' + x = f$ is totally singular by assumption. Thus we have

$$P = \begin{bmatrix} Q_1^{-1} & 0 \\ 0 & P_4 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_4 \end{bmatrix},$$

where Q_1 is constant. Hence $\tilde{B}_1 = Q_1^{-1}B_1$ and R-controllability is well defined.

From our preceding discussions we have the following result.

Theorem 4. *Suppose that E, F, B, C are infinitely differentiable and $Ex' + Fx = f$ is solvable. Then the descriptor input–output system* (1) *is* (*totally*) *observable if and only if its dual* (2) *is R-controllable.*

It is natural to wonder what the dual of smooth observability is. In [14] it is shown that uniform observability is dual to uniform controllability. Uniform controllability means state transitions from x_0 to x_1 can be performed "instantly" if distributional controls are used. A similar interpretation is possible for the dual of smooth observability, except higher than nth-order distributions may be required for the controls.

We now give an explicit test for R-controllability. Let

$$\mathscr{E}_j^c = \begin{bmatrix} E & 2E' - F & 3E'' - 2F' & 4E''' - 3F'' & * & (j+1)E^{(j)} - jF^{(j-1)} \\ 0 & E & 3E' - F & 6E'' - 3F' & * & * \\ 0 & 0 & 0 & E & * & * \\ 0 & 0 & 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 0 & \cdots & E \end{bmatrix},$$

$$\mathscr{F}_j^c = [E' - F,\ E'' - F', \ldots, E^{(j+1)} - F^{(j)}],$$

and

$$\mathscr{O}_{j,k}^c = \begin{bmatrix} \mathscr{F}_j^c & \tilde{C}_k^{\mathrm{T}} \\ \mathscr{E}_j^c & \begin{bmatrix} \hat{C}_k^{\mathrm{T}} \\ 0 \end{bmatrix} \end{bmatrix}.$$

Then

Theorem 5. *Suppose that E, F, B, C are infinitely differentiable and* (3) *is solvable on $\mathscr{I}$. Then $Ex' + Fx = C^{\mathrm{T}}u$ is R-controllable if there is a dense subset Ω of $\mathscr{I}$ such that, on Ω:*

1. *$\mathscr{O}_{j,k}^c$ has constant rank.*
2. $\operatorname{rank}\left[\mathscr{O}_{j,k}^c \,\middle|\, \begin{matrix} I_{n\times n} \\ 0 \end{matrix}\right] = \operatorname{rank} \mathscr{O}_{j,k}^c$.

Proof. By Theorem 4, R-controllability is equivalent to observability of the dual system $(E^{\mathrm{T}}x)' - F^{\mathrm{T}}x = Bu$, $y = Cx$. $\mathscr{O}_{j,k}^c$ is the transpose of the $\mathscr{O}_{j,k}$ matrix for this dual system. Now apply Lemma 3. □

While conceptually nice, the results in this subsection are not quite as satisfying as the earlier results and those of [7] since the definition of R-controllability is in terms of a structural form and not in terms of the original derivative array. It is relatively easy to generate a smooth projection onto the solution manifold in terms of the derivative array (23). The difficulty is in generating directly from a derivative array a projection with the correct null space which is essential for a characterization of controllability since in (28) only B_1u is important and B_2u cannot be used to control x_1.

This paper has been developed under the assumption that E, F, B, C are infinitely differentiable. If this assumption is relaxed, then different parts of the state x and input f can be required to have different levels of smoothness on different parts of the t interval $\mathscr{I}$. The definition of a dual operator to be used as in the proof of Theorem 1 will require the definition of a space based on the decomposition (8). Additional assumptions, such as smoothness of the nullspace of $\mathscr{N}$ in (8b), will also probably be needed.

Finally, it would be of interest of examine the relationship between the ideas discussed in this paper and different types of feedback. This is a nontrivial problem since solvability is not invariant under arbitrary state or output feedback without assumptions on E, F, B, C.

Example 3. Consider the system

$$Ex' + x = u$$

with E singular. The state feedback $u = x + v$ gives $Ex' = v$ which is not solvable.

Positive results should be possible if the feedback is restricted to the dynamical part of the system, such as z_1 in (8a). Since these variables are generally smoother than the "algebraic" variables such as z_2 in (8b), there can be practical reasons for only considering this type of feedback.

4. Conclusion

In this paper we have introduced the dual of a linear time-varying descriptor input–output system, and a type of controllability that is the dual of total observability. Our assumptions do not require constant rank of the leading coefficient E nor of intermediate matrices as is usually the case. Although the theory was developed using time-varying coordinate changes we do not actually need to take time-varying coordinate changes to verify the conditions since observability can be determined without them.

In addition, we have improved on the fundamental structure result in [5] which should simplify the future study of linear time-varying descriptor systems whose coefficients are infinitely differentiable but not real analytic.

A major remaining open question is to find a better definition of R-controllability directly in terms of (1).

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