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Asymptotic Stabilization of Linear Time-Varying Systems with Input Delays via Delayed Static Output Feedback

Frédéric Mazenc Silviu-Iulian Niculescu Nikolaos Bekiaris-Liberis

***Abstract*— A family of linear time-varying systems with an input, an output and delays in the input is considered. It is shown that, under a stabilizability and a detectability assumption, systems of this family can be exponentially stabilized through a time-varying feedback depending on past values of the output and the input.**

Key Words: Delay, output, time-varying, asymptotic stabilization

I. Introduction

The problem of the stabilization of dynamical systems when only some components of the state variable are measured is one of the most fundamental problems in control theory. The most classical approach to this problem is based on the design of dynamic extensions called observers (see for instance [9], [6], [2], [10], [7]). It leads to dynamic output feedbacks. This technique is widely applied in engineering applications and many results on this topic, for both linear and nonlinear systems, are available in the literature.

However, in particular cases, another technique, based on the introduction of delays, makes it possible to achieve output-feedback stabilization without using observers (see for instance [18]), which is an advantage since the construction of extra dynamics may be a difficult task. In addition, this technique applies in cases where a delay is imposed on the input. In the contributions [17], [8] and [13], it is shown, for continuous-time chains of integrators and oscillators, that, in some cases, the introduction of several appropriately chosen pointwise delays in the input can be used to design asymptotically stabilizing static output feedbacks. These papers consider time-invariant systems only and present proofs which do not seem to extend to time-varying systems. But the family of the time-varying systems, notably the linear one, is of fundamental interest. Indeed, the problem of asymptotically stabilizing the origin of linear time-varying systems naturally arises when one wishes to locally asymptotically track a trajectory of a nonlinear system; for more explanations, see for instance [1], [15], [11], [12]. Notice that these systems are in general exponentially unstable and thereby cannot be transformed through a change of coordinates into a cascade of chains of integrators and oscillators. Finally, it is worth mentioning that delays in the feedbacks or in the measurements are present in many engineering applications (e.g. chemical, mechanical and communication systems) and their presence can be source of performance degradation and instability, which implies that they should be taken into account when a control law is designed.

All these remarks motivate the main results of the present work. We complement the papers [17], [8], [13] in several directions. We show that systems belonging to a family of single-input single-output time-varying systems, (which encompasses the class of time-invariant, detectable and stabilizable linear systems) with pointwise, constant and known delays in the input or in the output, can be stabilized by a linear feedback depending on the past values of the output and the input. This result is based on the determination of a representation of the solutions as a delayed function of the input and output, which is of interest for its own sake: we presume that it can be used in other contexts, those for example of sampled or bounded feedbacks. The approach we propose is a *prediction-based approach* in the sense that the current value of the state (and hence, using the variation of constants formula, also the predictor of the state in the case of a single, pointwise input delay) is expressed as a function of past values of the input and the output and applies to the case where pointwise delays in the input are present. It relies on tools fundamentally different from those of the frequency-domain approach that are used in particular in [17], [8], [16] and make it possible to cope with systems that are exponentially unstable when the input is set to zero. We also use the key ideas of the reduction model approach [3], [12] to handle the case where several delays are present. Notice that the present paper also complements the contribution [5], where a prediction-based approach for linear systems of triangular structure with several delays on the input and the state is presented and the contributions [3] and [4] which present full-state predictor-based control laws for multi-input systems with multiple distributed delays.

Finally, we point out that, by contrast to the results presented in [12] none of the results we propose relies on an assumption of rapidness or a slowness of the varying matrices, but the knowledge of the fundamental solution of certain time-varying systems is assumed.

The paper is organized as follows. In Section II, we state and prove a general predictive result, which plays a central role in Section III, where a stabilization result is established. An example in Section IV illustrates the main results. Concluding remarks are given in Section V.

Mazenc and Niculescu are with EPI DISCO INRIA-Saclay, the Laboratoire des Signaux et Systèmes (L2S, UMR CNRS 8506), CNRS, CentraleSupélec, Université Paris-Sud, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France. Frederic.Mazenc@lss.supelec.fr and NICULESCU@lss.supelec.fr. Mazenc was supported by the research project iCODE.

N. Bekiaris-Liberis is with the Department of Civil and Environmental Engineering, University of California, Berkeley, CA 94704 USA. bekiaris-liberis@berkeley.edu

A. Notation and preliminary results

1) Notation: Throughout the paper, we simplify the notations and omit arguments of functions when the arguments are clear from the context.

The usual Euclidean norm of vectors, and the induced norm of matrices, of any dimensions are denoted by $|\cdot|$, and I is the identity matrix in the dimension under consideration.

Let C^1 denote the set of all continuously differentiable functions, and C^0 denote the set of all continuous functions, when the domains and ranges of the functions are clear from the context.

Given any constant $\tau > 0$, we let $C([-\tau, 0], \mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions that are defined on $[-\tau, 0]$. We call it the set of (all) initial functions. We often use the simplifying notation $C_{in} = C([-\tau, 0], \mathbb{R}^n)$. For any continuous function $\varphi : [-\tau, +\infty) \to \mathbb{R}^n$ and any $t \geq 0$, we define the function φ_t by $\varphi_t(\theta) = \varphi(t+\theta)$ for all $\theta \in [-\tau, 0]$.

Let us recall Young's inequality for vectors and matrices. Consider two vectors $V_1 \in \mathbb{R}^n$, $V_2 \in \mathbb{R}^n$. Then, for all $c > 0$, the inequality

$$V_1^\top V_2 \leq \frac{c}{2} V_1^\top V_1 + \frac{1}{2c} V_2^\top V_2$$

is satisfied. Consider two matrices $M_1 \in \mathbb{R}^{n\times n}$ and $M_2 \in \mathbb{R}^{n\times n}$. Then for all vector $V \in \mathbb{R}^n$ and all real number $c > 0$, the inequality

$$V^\top M_1^\top M_2 V \leq \frac{c}{2} V^\top M_1^\top M_1 V + \frac{1}{2c} V^\top M_2^\top M_2 V$$

is satisfied.

2) Preliminary results: In Appendix , we prove the following result:

Lemma 1: Let $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^n$ be such that the pair (A, C) is detectable. Then there is a real number $\tau_\dagger > 0$, such that for all $\tau \in (0, \tau_\dagger)$, the matrix

$$F = \begin{bmatrix} C \\ Ce^{-A\tau} \\ \vdots \\ Ce^{-(n-1)A\tau} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{1}$$

is invertible.

II. New representation of the solutions of a linear system

We consider the linear time-varying system

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + Z(t) \\ y(t) & = & C(t)x(t), \end{array} \tag{2}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$ is the output, $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $C : \mathbb{R} \to \mathbb{R}^n$ are continuous and $Z : [0, +\infty) \to \mathbb{R}^n$ is a continuous function.

Let Φ be the fundamental solution associated to A, i.e. the function such that for all $t_0 \in \mathbb{R}$, $\Phi(t_0, t_0) = I$ and, for all $t \in \mathbb{R}$,

$$\frac{\partial \Phi}{\partial t}(t, t_0) = A(t)\Phi(t, t_0). \tag{3}$$

Remark. Notice that Z is the abstract representation of a function depending on values of time. So it may represent a function with several delays: for instance, a possible Z is $Z(t) = u(t-h) + 3\cos(t)u(t-2h) + \int_{t-h-1}^{t-h} u(\ell)d\ell$, where u is an input and $h > 0$.

In this section, we show that, after a finite time instant, any solution of the system (2) is equal to a function of past and present values of y and Z.

We introduce an assumption:

Assumption A1. *There are n real numbers τ_i satisfying $\tau_n > ... > \tau_2 > \tau_1 \geq 0$ and such that for all $t \in \mathbb{R}$, the matrix*

$$M(t) = \begin{bmatrix} C(t-\tau_1)\Phi(t-\tau_1, t) \\ C(t-\tau_2)\Phi(t-\tau_2, t) \\ \vdots \\ C(t-\tau_n)\Phi(t-\tau_n, t) \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{4}$$

is invertible and $M^{-1} : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a continuous function.

Remark. In the time-invariant case, detectability of the pair (A, C) ensures that Assumption A1 is satisfied. Indeed, this property implies that, for any $h \in \mathbb{R}$, the pair (A, Ce^{-hA}) is detectable. According to Lemma 1, it follows that there are n real numbers τ_i, $\tau_n > ... > \tau_2 > \tau_1 \geq 0$ such that the matrix M is invertible. No similar result can be proved in the time-varying case. However, in many cases, determining constants τ_i so that $M(t)$ is invertible for all t is not a difficult task. The main difficulty in verifying Assumption A1 resides in general the determination of the fundamental solution Φ. However, some results in [11] make it possible, for particular triangular systems, to construct it.

Before stating and proving the main result of the section, we introduce the operators $\mathcal{Y} : \mathbb{R} \times C_{in} \to \mathbb{R}^n$,

$$\mathcal{Y}(t, \phi_x) = \begin{pmatrix} C(t-\tau_1)\phi_x(-\tau_1) \\ C(t-\tau_2)\phi_x(-\tau_2) \\ \vdots \\ C(t-\tau_n)\phi_x(-\tau_n) \end{pmatrix} \tag{5}$$

and $\mathcal{P} : \mathbb{R} \times C_{in} \to \mathbb{R}^n$,

$$\mathcal{P}(t, \phi_Z) = \begin{pmatrix} C(t-\tau_1)\int_{-\tau_1}^{0} \Phi(t-\tau_1, t+\ell)\phi_Z(\ell)d\ell \\ C(t-\tau_2)\int_{-\tau_2}^{0} \Phi(t-\tau_2, t+\ell)\phi_Z(\ell)d\ell \\ \vdots \\ C(t-\tau_n)\int_{-\tau_n}^{0} \Phi(t-\tau_n, t+\ell)\phi_Z(\ell)d\ell \end{pmatrix} \tag{6}$$

and, for any solution of (2), the function $\mathcal{Y}_R : [\tau_n - \tau_1, +\infty) \to \mathbb{R}^n$,

$$\mathcal{Y}_R(t) = \begin{pmatrix} y(t) \\ y(t+\tau_1-\tau_2) \\ \vdots \\ y(t+\tau_1-\tau_n) \end{pmatrix}. \tag{7}$$

Let us notice that, along the trajectories, for all $t \geq \tau_n$,

$$\mathcal{Y}(t, x_t) = \mathcal{Y}_R(t - \tau_1) \tag{8}$$

and

$$\mathcal{P}(t, Z_t) = \begin{pmatrix} C(t-\tau_1)\int_{t-\tau_1}^{t} \Phi(t-\tau_1, \ell)Z(\ell)d\ell \\ C(t-\tau_2)\int_{t-\tau_2}^{t} \Phi(t-\tau_2, \ell)Z(\ell)d\ell \\ \vdots \\ C(t-\tau_n)\int_{t-\tau_n}^{t} \Phi(t-\tau_n, \ell)Z(\ell)d\ell \end{pmatrix}. \tag{9}$$

We are ready to state and prove the following result:

Theorem 1: Let the system (2) satisfy Assumption A1. Then any solution $x(t)$ of (2) is such that, for all $t \geq \tau_n$,

$$x(t) = M(t)^{-1}\mathcal{Y}_R(t-\tau_1) + M(t)^{-1}\mathcal{P}(t, Z_t), \tag{10}$$

where $M(t)$ is the function defined in (4).

Discussion of Theorem 1.

(i) Consider a system (2) such that Assumption A1 is satisfied. Let $\tau_* \in [-\tau_1, +\infty)$. Then for all $t \in \mathbb{R}$, the matrix

$$M(t-\tau_*) = \begin{bmatrix} C(t-\tau_1-\tau_*)\Phi(t-\tau_1-\tau_*, t-\tau_*) \\ C(t-\tau_2-\tau_*)\Phi(t-\tau_2-\tau_*, t-\tau_*) \\ \vdots \\ C(t-\tau_n-\tau_*)\Phi(t-\tau_n-\tau_*, t-\tau_*) \end{bmatrix} \tag{11}$$

is invertible. On the other hand, the matrix $\Phi(t-\tau_*, t)$ is invertible for all $t \in \mathbb{R}$. It follows that $M(t-\tau_*)\Phi(t-\tau_*, t)$ is invertible for all $t \in \mathbb{R}$. Consequently, the matrix

$$\begin{bmatrix} C(t-\tau_1-\tau_*)\Phi(t-\tau_1-\tau_*, t) \\ C(t-\tau_2-\tau_*)\Phi(t-\tau_2-\tau_*, t) \\ \vdots \\ C(t-\tau_n-\tau_*)\Phi(t-\tau_n-\tau_*, t) \end{bmatrix} \tag{12}$$

is invertible for all $t \in \mathbb{R}$. We deduce that when Assumption A1 is satisfied, then, for any constant $\tau_\triangle \geq 0$, this assumption is satisfied with a set of constants τ_i such that $\tau_1 = \tau_\triangle$. Therefore if there is a constant pointwise delay in the output y, a formula of prediction similar to (10) can be proved by choosing τ_1 equal (or larger) than the delay present in y.

(ii) For the sake of simplicity, we restricted ourselves to the case where y is in $\mathbb{R}$. But extensions to the case where y is in $\mathbb{R}^p$ with $p > 1$ can be obtained.

Proof. Let $x(t)$ be a solution of the system (2), defined for all $t \in [0, +\infty)$. It is well-known that, for all $s_1 \in [0, +\infty)$ and $s_2 \in [0, +\infty)$, the equality

$$x(s_1) = \Phi(s_1, s_2)x(s_2) + \int_{s_2}^{s_1} \Phi(s_1, m)Z(m)dm \tag{13}$$

is satisfied. Therefore for any constants $t_0 \in [0, +\infty)$ and $t \geq t_0$, the equality

$$\begin{array}{rcl} x(t-t_0) & = & \Phi(t-t_0, t)x(t) \\ & & -\int_{t-t_0}^{t} \Phi(t-t_0, m)Z(m)dm \end{array} \tag{14}$$

holds. We deduce that

$$\mathcal{Y}(t, x_t) = M(t)x(t) - \mathcal{P}(t, Z_t). \tag{15}$$

Since the matrix $M(t)$ is invertible for all $t \geq 0$, we deduce from (8) that the equality (10) is satisfied for all $t \geq \tau_n$.

III. Asymptotic stabilization results

In this section, the objective we pursue is the exponential stabilization of the systems with delays in the input through a static time-varying output feedback.

The result we present applies to systems that can be exponentially unstable when the input is identically equal to zero, and with several pointwise delays. Feedbacks whose expression includes distributed terms of the past values of the input are proposed.

We consider the following system

$$\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + \sum_{i=1}^{k} B_i(t-r_i)u(t-r_i) \\ y(t) & = & C(t)x(t), \end{array} \tag{16}$$

where k is a positive integer, $u \in \mathbb{R}^q$ is the input, $y \in \mathbb{R}$ is the output, the constants r_i are known and such that $0 = r_1 < ... < r_k$ and the functions $B_i : \mathbb{R} \to \mathbb{R}^{n\times q}$ and $C : [0, +\infty) \to \mathbb{R}^n$ are continuous. We let x and u have initial conditions over the interval $[-\mathcal{T}, 0]$, where $\mathcal{T} \geq r_k$. Notice that we can select $\mathcal{T}$ arbitrarily large.

We introduce an assumption of stabilizability:

Assumption A2. *There exists a continuous function $K : [0, +\infty) \to \mathbb{R}^{q\times n}$ such that the system*

$$\dot{\mathcal{X}}(t) = [A(t) + G(t)K(t)]\,\mathcal{X}(t) \tag{17}$$

with

$$G(t) = \sum_{i=1}^{k} \Phi(t, t+r_i)B_i(t), \tag{18}$$

where Φ is the function defined as in (3) associated to A in (16), is globally uniformly exponentially stable. Moreover, there is a constant $a_* > 0$ such that for all $t \in \mathbb{R}$,

$$|A(t)| + \sum_{i=1}^{k} |B_i(t)| + |C(t)| + |K(t)| \leq a_*. \tag{19}$$

Assuming that Assumption A1 is satisfied with A and C in (16), we introduce two operators $\mathcal{G}$ and $\mathcal{H}$, defined, along the trajectories of (16) by

$$\mathcal{G}(t, u_t) = \begin{pmatrix} C(t-\tau_1)\int_{t-\tau_1}^{t} \Phi(t-\tau_1, \ell)\mathcal{S}(\ell, u_\ell) \\ C(t-\tau_2)\int_{t-\tau_2}^{t} \Phi(t-\tau_2, \ell)\mathcal{S}(\ell, u_\ell) \\ \vdots \\ C(t-\tau_n)\int_{t-\tau_n}^{t} \Phi(t-\tau_n, \ell)\mathcal{S}(\ell, u_\ell) \end{pmatrix} \tag{20}$$

with

$$\mathcal{S}(\ell, u_\ell) = \sum_{i=1}^{k} B_i(\ell - r_i)u(\ell - r_i)d\ell \tag{21}$$

and

$$\mathcal{H}(t,u_t)=\sum_{i=1}^{k}\int_{t-r_i}^{t}\Phi(t,\ell+r_i)B_i(\ell)u(\ell)d\ell. \quad (22)$$

We state and prove the main result of the section.

Theorem 2: Let the system (16) satisfy Assumptions A1 and A2 with $\mathcal{T}=\tau_n+r_k$. Then this system in closed-loop with the feedback defined, for all $t\geq 0$, by

$$\begin{array}{rcl} u(t) & = & K(t)\left[M(t)^{-1}\mathcal{Y}_R(t-\tau_1)\right. \\ & & \left.+M(t)^{-1}\mathcal{G}(t,u_t)+\mathcal{H}(t,u_t)\right], \end{array} \quad (23)$$

with $\mathcal{Y}_R$ defined as in (7), admits the origin as a globally uniformly exponentially stable equilibrium point.

Discussion of Theorem 2.

(i) In the time invariant case, Assumption A2 is satisfied if and only if the pair (A,G) is stabilizable.

(ii) Theorem 2 can be easily adapted to other types of delays in the input, and in particular to distributed delays.

(iii) The proof of Theorem 2 relies on Theorem 1 and the reduction model approach for time-varying systems (see for instance [3], [12]). Moreover, combining the proof below and the key ideas of [14], a Lyapunov functional with a negative definite time derivative along the trajectories of the closed-loop system after a finite time-interval can be constructed.

(iv) When A, B, C and K are periodic functions, then the condition (19) is automatically satisfied.

Proof. First, let us observe that since $\mathcal{Y}_R(t-\tau_1)$ depends on values of y at instants smaller than $t-\tau_1$ the feedback (23) is well-defined.

Second, let us apply Theorem 1 to the system (16) with

$$Z(t)=\sum_{i=1}^{k}B_i(t-r_i)u(t-r_i).$$

From this theorem, we deduce easily that, for all $t\geq\tau_n$,

$$x(t)=M(t)^{-1}\mathcal{Y}_R(t-\tau_1)+M(t)^{-1}\mathcal{G}(t,u_t), \quad (24)$$

with $\mathcal{G}$ defined in (20).

Therefore (23) becomes, for all $t\geq\tau_n$,

$$u(t)=K(t)\left[x(t)+\mathcal{H}(t,u_t)\right]. \quad (25)$$

Next, we introduce the operator ζ, defined along the trajectories of the system (16), by

$$\zeta(t)=x(t)+\mathcal{H}(t,u_t), \quad (26)$$

which is the one used when one applies the reduction model approach. One can check readily that, for all $t\geq\mathcal{T}$,

$$\begin{array}{l} \dot{\zeta}(t)=A(t)x(t)+\displaystyle\sum_{i=1}^{k}B_i(t-r_i)u(t-r_i) \\ +A(t)\mathcal{H}(t,u_t) \\ +\displaystyle\sum_{i=1}^{k}[\Phi(t,t+r_i)B_i(t)u(t)-B_i(t-r_i)u(t-r_i)] \\ =A\zeta(t)+G(t)u(t), \end{array} \quad (27)$$

where G is the function defined in (18).

Combining (25) and (26), we obtain, for all $t\geq\tau_n$,

$$u(t)=K(t)\zeta(t). \quad (28)$$

Then (27) and (28) give for all $t\geq\tau_n$,

$$\dot{\zeta}(t)=[A(t)+G(t)K(t)]\zeta(t). \quad (29)$$

From Assumption A2, it follows that ζ is solution of a globally uniformly exponentially stable system. It follows that there are $k_a>0$ and $k_b>0$ such that for all $t_1\geq\tau_n$ and $t\geq t_1$, the inequality

$$|\zeta(t)|\leq k_a e^{-k_b(t-t_1)}|\zeta(t_1)| \quad (30)$$

is satisfied. Now, observe that (26) and (28) give, for all $t\geq\tau_n+r_k$,

$$\begin{array}{rcl} |x(t)| & \leq & |\zeta(t)|+|\mathcal{H}(t,u_t)| \\ & = & |\zeta(t)| \\ & & +\left|\displaystyle\sum_{i=1}^{k}\int_{t-r_i}^{t}\Phi(t,\ell+r_i)B_i(\ell)K(\ell)\zeta(\ell)d\ell\right|. \end{array} \quad (31)$$

From Assumption A2 we deduce that for all $t\geq 0$, $r\geq 0$, the inequality $|\Phi(t,t+r)|\leq e^{ra_*}$ is satisfied. It follows that, for all $t\geq\tau_n+r_k$,

$$|x(t)|\leq|\zeta(t)|+kr_ke^{r_ka_*}a_*^2\sum_{i=1}^{k}\int_{t-r_i}^{t}|\zeta(\ell)|\,d\ell. \quad (32)$$

This inequality and (30) allows us to conclude.

IV. Example

In this section, we illustrate Theorem 2.

We consider the system

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t)-\frac{\sin(t)^2}{5}u(t-\pi) \\ \dot{x}_2(t) & = & -x_1(t)+u\left(t-\frac{\pi}{2}\right), \end{array}\right. \quad (33)$$

with $x\in\mathbb{R}^2$ and the output $y(t)=x_1(t)$. It belongs to the family of the systems (16).

Remark. The paper [13] provides with globally exponentially stabilizing static output feedbacks for the systems

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & x_2(t) \\ \dot{x}_2(t) & = & -x_1(t)+u(t-h), \end{array}\right. \quad (34)$$

where h is an arbitrary positive real number and the output is x_1. But the extension of the main result of [13] to the systems (33) is an open problem.

Let us check that the assumptions of Theorem 2 are satisfied by the system (33). We use the notation of the previous section and observe that $r_1=\frac{\pi}{2}$, $r_2=\pi$,

$$A=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\ B_1=\begin{bmatrix}0\\ 1\end{bmatrix},\ B_2(t)=\begin{bmatrix}-\frac{\sin(t)^2}{5}\\ 0\end{bmatrix},$$

$$C=[1\quad 0]\ ,\quad e^{-Am}=\begin{bmatrix}\cos(m) & -\sin(m)\\ \sin(m) & \cos(m)\end{bmatrix}$$

and $Ce^{-Am} = [\cos(m) \quad -\sin(m)]$. Choosing $\tau_1 = \frac{\pi}{2}$, $\tau_2 = \pi$ and $P = I$, we obtain the matrix

$$M(t) = \begin{bmatrix} Ce^{-A\frac{\pi}{2}} \\ Ce^{-A\pi} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

which is invertible. Thus Assumption A1 is satisfied.

Now, we have $\Phi(t, t + r_i) = e^{-Ar_i}$. Thus,

$$\begin{aligned} G(t) &= e^{-A\frac{\pi}{2}}B_1 + e^{-A\pi}B_2(t) \\ &= \begin{bmatrix} -1 + \frac{\sin(t)^2}{5} \\ 0 \end{bmatrix}. \end{aligned}$$

Then, choosing $K = [1 \quad 0]$, the matrix

$$A + G(t)K = \begin{bmatrix} -1 + \frac{\sin(t)^2}{5} & 1 \\ -1 & 0 \end{bmatrix}$$

is obtained. Next, one can prove easily that Assumption A2 is satisfied. It follows that Theorem 2 applies.

We also have

$$\mathcal{G}(t, u_t) = \begin{pmatrix} \int_{t-\frac{\pi}{2}}^{t} \varrho_1(t,\ell)d\ell \\ \int_{t-\pi}^{t} \varrho_2(t,\ell)d\ell \end{pmatrix}. \tag{35}$$

with

$$\begin{aligned} \varrho_1(t,\ell) &= -\cos(\ell - t)u\left(\ell - \tfrac{\pi}{2}\right) \\ &\quad + \tfrac{\sin(\ell-\pi)^2}{5}\sin(\ell - t)u(\ell - \pi), \end{aligned} \tag{36}$$

$$\begin{aligned} \varrho_2(t,\ell) &= \sin(\ell - t)u\left(\ell - \tfrac{\pi}{2}\right) \\ &\quad + \tfrac{\sin(\ell-\pi)^2}{5}\cos(\ell - t)u(\ell - \pi) \end{aligned} \tag{37}$$

and

$$\begin{aligned} \mathcal{H}(t, u_t) &= \int_{t-\frac{\pi}{2}}^{t} \begin{bmatrix} -\cos(\ell - t) \\ \sin(\ell - t) \end{bmatrix} u(\ell)d\ell \\ &\quad + \tfrac{1}{5}\int_{t-\pi}^{t} \sin(\ell)^2 \begin{bmatrix} \cos(\ell - t) \\ \sin(\ell - t) \end{bmatrix} u(\ell)d\ell. \end{aligned} \tag{38}$$

As an immediate consequence, we have

$$\begin{aligned} K\mathcal{H}(t, u_t) &= [1 \quad 0]\mathcal{H}(t, u_t) \\ &= -\int_{t-\frac{\pi}{2}}^{t} \cos(\ell - t)\, u(\ell)d\ell \\ &\quad + \tfrac{1}{5}\int_{t-\pi}^{t} \sin(\ell)^2 \cos(\ell - t)u(\ell)d\ell \end{aligned} \tag{39}$$

and $KM(t)^{-1} = [0 \quad -1]$. It follows from Theorem 2 that the following feedback:

$$\begin{aligned} u(t) &= -x_1(t - \pi) - \int_{t-\frac{\pi}{2}}^{t} \sin(\ell - t)u(\ell - \tfrac{\pi}{2})d\ell \\ &\quad - \tfrac{1}{5}\int_{t-\pi}^{t} \sin(\ell)^2 \cos(\ell - t)u(\ell - \pi)d\ell \\ &\quad - \int_{t-\frac{\pi}{2}}^{t} \cos(\ell - t)\, u(\ell)d\ell \\ &\quad + \tfrac{1}{5}\int_{t-\pi}^{t} \sin(\ell)^2 \cos(\ell - t)u(\ell)d\ell, \end{aligned} \tag{40}$$

which can be simplified as

$$\begin{aligned} u(t) &= -x_1(t - \pi) - \int_{t-\pi}^{t} \cos(\ell - t)u(\ell)d\ell \\ &\quad + \tfrac{1}{5}\int_{t-2\pi}^{t} \sin(\ell)^2 \cos(\ell - t)u(\ell)d\ell, \end{aligned} \tag{41}$$

globally exponentially stabilizes the system (33). Simulations below illustrate the convergence of the solutions and the control of the closed-loop system. The selected initial conditions are, for all $t \in [-2\pi, 0]$, $(x_1(t), x_2(t), u(t)) = (5, 6, 0)$.

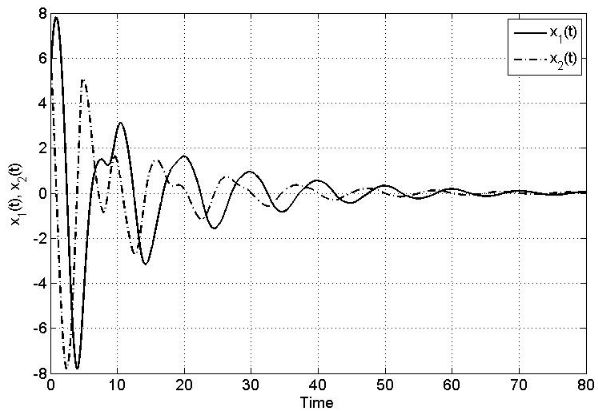

Fig. 1. Trajectories of x_1 and x_2

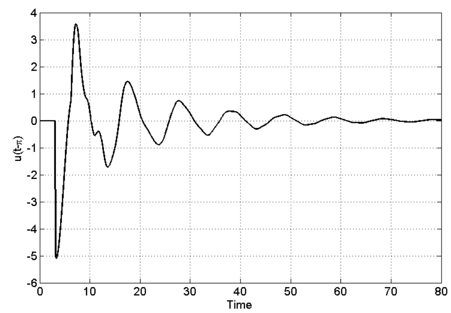

Fig. 2. Evolution of the control u

V. Conclusion

We have presented a stabilization result for time-varying linear systems under pointwise input delays. The feedbacks we proposed are static output feedbacks. We developed a new design of control laws which combines prediction and the reduction model approach. The design relies on a new representation of the solutions of the considered systems as a function or past values of the input and the output.

Much work remains to be done. We conjecture that the averaging ideas used in [12] can be also used to extend the results of our paper to families of systems for which the fundamental matrix cannot be explicitly determined. The adaptation of our result to the case of sampled input will be the subject of further studies too.

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APPENDIX

Proof of Lemma 1

We give a sketch of the proof.

We introduce the matrix

$$E = \begin{bmatrix} C \\ C\left(e^{-A\tau} - I\right) \\ \vdots \\ C\left(e^{-A\tau} - I\right)^{n-1} \end{bmatrix} \in \mathbb{R}^{n\times n} \tag{42}$$

and denote the rows of F defined in (1) and E by respectively F_j and E_j, i.e. $F_j = Ce^{-jA\tau}$ and $E_j = C\left(e^{-A\tau} - I\right)^j$.

One can prove by induction that, for all $j \in \{0, ..., n_1\}$, there are real numbers $\alpha_{i,j}$, $i \in \{0, ..., j-1\}$ such that

$$E_j = F_j + \sum_{i=0}^{\max\{0,j-1\}} \alpha_{i,j} F_i. \tag{43}$$

Thus we can demonstrate that

$$E = \begin{bmatrix} F_0 \\ F_1 + \alpha_{0,1}F_0 \\ \vdots \\ F_{n-1} + \sum_{i=0}^{n-2} \alpha_{i,n-1}F_i \end{bmatrix}. \tag{44}$$

We can determine an invertible matrix R such that $E = RF$. It follows that F is invertible if and only if E is invertible.

Now, observe that E is invertible if and only if

$$\tilde{E} = \begin{bmatrix} C \\ C\frac{e^{-A\tau}-I}{\tau} \\ \vdots \\ C\left(\frac{e^{-A\tau}-I}{\tau}\right)^{n-1} \end{bmatrix} \tag{45}$$

is invertible. Now, observe that

$$\begin{aligned} \frac{e^{-A\tau}-I}{\tau} &= \frac{1}{\tau}\int_0^\tau (-A)e^{-A\ell}d\ell \\ &= -A + \frac{1}{\tau}\int_0^\tau (-A)\left(e^{-A\ell} - I\right)d\ell \\ &= -A + \frac{1}{\tau}\int_0^\tau (-A)^2 \int_0^\ell e^{-Am}dmd\ell. \end{aligned} \tag{46}$$

Therefore

$$\tilde{E} = \begin{bmatrix} C \\ C\left[-A + \tau(-A)\psi(\tau)\right] \\ \vdots \\ C\left[-A + \tau(-A)\psi(\tau)\right]^{n-1} \end{bmatrix} \tag{47}$$

with $\psi(\tau) = \frac{(-A)}{\tau^2}\int_0^\tau \int_0^\ell e^{-Am}dmd\ell$. Therefore

$$\tilde{E} = \begin{bmatrix} C \\ C(-A)\left(I + \tau\psi(\tau)\right) \\ \vdots \\ C(-A)^{n-1}\left(I + \tau\psi(\tau)\right)^{n-1} \end{bmatrix}. \tag{48}$$

Let

$$Z = \begin{bmatrix} C \\ -CA \\ \vdots \\ (-1)^{n-1}CA^{n-1} \end{bmatrix}.$$

Then

$$\tilde{E} = Z + \begin{bmatrix} 0 \\ C(-A)\tau\psi(\tau) \\ \vdots \\ C(-A)^{n-1}\left[\left(I + \tau\psi(\tau)\right)^{n-1} - I\right] \end{bmatrix}. \tag{49}$$

Therefore there is a nonnegative continuous function $\Psi_\triangle : [0, +\infty) \to [0, +\infty)$ such that

$$\det(\tilde{E}) = \det Z + \tau\Psi_\triangle(\tau).$$

The detectability of (A, C) implies that Z is invertible. Therefore the constant $\det Z$ is different from zero. It follows that there is $\tau_* > 0$ such that, for all $\tau \in (0, \tau_*]$, $\det(\tilde{E}) \neq 0$. This allows us to conclude the proof.