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CONDITIONING AND DICHOTOMY IN DIFFERENTIAL ALGEBRAIC EQUATIONS*

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Abstract. The relationship between the dichotomy of the solution space for a differential algebraic system and the well conditioning of associated boundary value problems is investigated. This work extends results that are known for regular explicit ordinary differential equations and it follows the presentation by de Hoog and Mattheij [*SIAM J. Numer. Anal.*, 24 (1987), pp. 89-105].

Key words. differential algebraic equations, boundary value problems, dichotomy, conditioning

AMS(MOS) subject classifications. 65L05, 34A50

1. Introduction. The recent literature on boundary value problems for explicit regular ordinary differential equations has shown a great interest in and development of the question of characterizing well-conditioned problems (see for example [2], [3], [5], [7]). In particular, de Hoog and Mattheij have shown in [3] that the concepts of the dichotomy of the solution space and well conditioning of associated boundary value problems are equivalent in some sense.

In our previous paper [6] we started the study of well-conditioned boundary value problems in transferable differential algebraic equations. There we constructed a Green function well suited for the conditioning analysis and we obtained some equivalence results for equations in canonical normal form and general linear constant coefficient equations.

In [1] Ascher studied the conditioning for transferable differential algebraic equations (DAEs) using a transformation that separated the algebraic from the differential part, at least theoretically. We prefer to avoid such a transformation because it might be difficult and expensive to implement numerically.

This paper is organized as follows. In § 2 we present some notation and preliminary results for differential algebraic equations. In § 3 we discuss the quantities (bounds) involved in the inequalities which characterize the conditioning. In § 4 we introduce the concept of dichotomy for differential algebraic equations, and we show some results on the relationship between well conditioning and dichotomy.

2. Preliminaries. Let us consider the linear boundary value problem for a transferable differential algebraic system

$$A(t)x'(t)+B(t)x(t)=q(t), \qquad t_0 \leqq t \leqq T, \tag{2.1}$$

$$D_1x(t_0)+D_2x(T)=b, \tag{2.2}$$

where

— The function matrices $A(\cdot)$, $B(\cdot):[t_0, T]\to L(\mathbb{R}^m)$ are continuous;

— There exist continuously differentiable projector functions $P(\cdot)$, $Q(\cdot):[t_0, T]\to L(\mathbb{R}^m)$ so that $P(t)=I-Q(t)$,

$$\operatorname{im}(Q(t))=\ker(A(t)) \quad \text{for all } t\in[t_0, T];$$

— $K(t):=A(t)+B(t)Q(t)$ is nonsingular for all $t\in[t_0, T]$.

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The matrices

$$Q_s(t) := Q(t)K(t)^{-1}B(t) \quad \text{and} \quad P_s(t) := I - Q_s(t)$$

are called canonical projectors. They are related to the canonical subspaces $\ker(A(t))$ and $\{z \in \mathbb{R}^m \colon B(t)z \in \operatorname{im}(A(t))\}$ (cf. [4]). We seek solutions of (2.1) which belong to the function space

$$\mathscr{C} := \{x \in C([t_0, T], \mathbb{R}^m) \colon Px \in C^1([t_0, T], \mathbb{R}^m)\}.$$

The fundamental solution matrix $X(\cdot)$, whose columns belong to $\mathscr{C}$, can be uniquely determined by the initial value problem

$$AX' + BX = 0, \qquad P(t_0)(X(t_0) - C) = 0 \tag{2.3}$$

where C is any nonsingular matrix. Note that

$$X(t) = P_s(t)X(t) \quad \text{holds} \quad \text{(see [4], [6])}.$$

We observe that the singularity of $A(t)$ implies that $X(t)$ is also singular. Moreover,

$$\ker(X(t)) = C^{-1} \ker(A(t_0)). \tag{2.4}$$

The following theorem is known.

THEOREM 2.5 (Griepentrog and März [4]). *The boundary value problem* (2.1), (2.2) *has a unique solution, for any* $q \in C([t_0, T], \mathbb{R}^m)$ *and* $b \in \operatorname{im}((D_1 | D_2))$, *if and only if the matrix*

$$S := D_1 X(t_0) + D_2 X(T)$$

satisfies

$$\begin{aligned} &\ker(S) = C^{-1} \ker(A(t_0)), \\ &\operatorname{im}(S) = \operatorname{im}((D_1 | D_2)). \end{aligned} \tag{2.6}$$

□

In the case of unique solvability we can formally write the solution $x(t)$ as (see Lentini and März [6])

$$x(t) = X(t)S^{-}\tilde{b} + \int_{t_0}^{T} \mathscr{G}(t, s)f(s)\, ds + h(t) \tag{2.7}$$

where

— S^{-} is a generalized inverse of S which satisfies

$$SS^{-}S = S, \quad S^{-}SS^{-} = S^{-}, \quad S^{-}S = C^{-1}P(t_0)C,$$

— $f(t) := (P(t) + P'(t)Q(t))K(t)^{-1}q(t),$

$h(t) := Q(t)K(t)^{-1}q(t),$

$\tilde{b} := b - D_1 h(t_0) - D_2 h(T),$

and the Green function $\mathscr{G}(t, s)$ is defined by

$$\mathscr{G}(t, s) := \begin{cases} X(t)S^{-}D_1 X(t_0)\bar{X}(s), & s < t, \\ -X(t)S^{-}D_2 X(T)\bar{X}(s), & t < s \end{cases} \tag{2.8}$$

with $\bar{X}(t)$ the reflexive generalized inverse of $X(t)$ that satisfies

$$\bar{X}(t)X(t) = C^{-1}P(t_0)C \quad \text{and} \quad X(t)\bar{X}(t) = P_s(t). \tag{2.9}$$

We denote

$$r := m - \dim(\ker(A(t_0))) = \operatorname{rank}(A(t_0)). \tag{2.10}$$

Furthermore, in the rest of this paper we will assume, without loss of generality, that

$$P(t_0) = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \operatorname{im}((D_1 | D_2)) = \operatorname{im}(P(t_0)). \tag{2.11}$$

Since $\operatorname{rank}(S) = r$, $P_s(t)X(t) = X(t)$, and

$$S = (D_1 P_s(t_0) | D_2 P_s(T)) \begin{bmatrix} X(t_0) \\ X(T) \end{bmatrix}$$

we immediately get

$$\operatorname{rank}(D_1 P_s(t_0) | D_2 P_s(T)) = r.$$

LEMMA 2.12 (extension of Lemma 3.88 of [2, p. 114]). *Let $(B_1 | B_2)$ be an $m \times 2m$ matrix of rank $r \leqq m$. Then there exist a number k, $0 \leqq k \leqq r$, a nonsingular matrix R, orthogonal matrices Q_1, Q_2, and diagonal matrices Δ_1, Δ_2, all in $L(\mathbb{R}^m)$, such that*

$$B_1 = R\Delta_1 Q_1, \qquad B_2 = R\Delta_2 Q_2 \tag{2.13}$$

and

$$\Delta_1 = \begin{bmatrix} \Delta_{11} & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Delta_2 = \begin{bmatrix} I_{r-k} & 0 & 0 \\ 0 & \Delta_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.14}$$

where Δ_{11} and Δ_{22} have their elements between zero and one.

Proof. Since $(B_1 | B_2)$ has rank $r \leqq m$ it is possible to find orthogonal matrices $Q \in L(\mathbb{R}^m)$, $\tilde{Q} \in L(\mathbb{R}^{2m})$ and a nonsingular matrix $U \in L(\mathbb{R}^r)$ such that

$$\begin{bmatrix} U & 0 \\ 0 & I_{m-r} \end{bmatrix} Q(B_1 | B_2) = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tilde{Q} =: \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \\ 0 & 0 \end{bmatrix},$$

i.e., the rows of $(\tilde{D}_1 | \tilde{D}_2) \in L(\mathbb{R}^{2m}, \mathbb{R}^r)$ are orthogonal. Hence

$$\tilde{D}_1 \tilde{D}_1^t + \tilde{D}_2 \tilde{D}_2^t = I_r$$

holds. Thus $\tilde{D}_1 \tilde{D}_1^t$ and $\tilde{D}_2 \tilde{D}_2^t$ have the same eigenvectors. This induces the singular value decompositions

$$\tilde{D}_1^t = Q_1^t \Sigma_1 V, \qquad \tilde{D}_2^t = Q_2^t \Sigma_2 V$$

where the columns of V represent r orthonormal eigenvectors of $\tilde{D}_1 \tilde{D}_1^t$. From the row orthogonality of $(\tilde{D}_1 | \tilde{D}_2)$ we also get

$$(\sigma_i^1)^2 + (\sigma_i^2)^2 = 1, \qquad i = 1, \cdots, r$$

for the diagonal elements σ_i^1, σ_i^2 of Σ_1 and Σ_2, respectively. Clearly, $\sigma_1^j \geqq \sigma_2^j \geqq \cdots \geqq \sigma_r^j \geqq 0$ is also given. From this we conclude the lemma. □

Applying Lemma 2.12 to the matrix

$$(B_1 | B_2) := (D_1 P_s(t_0) | D_2 P_s(T)),$$

we know that premultiplying the boundary condition (2.2) by R^{-1} leads to

$$R^{-1} S = \Delta_1 Q_1 X(t_0) + \Delta_2 Q_2 X(T).$$

Trivially, (2.11), (2.14) imply that $\operatorname{im}((\Delta_1 | \Delta_2)) = \operatorname{im}(P(t_0))$.

In what follows we assume that the original boundary conditions are normalized, i.e., the matrices D_1 and D_2 in (2.2) satisfy

$$D_1P_s(t_0)=\Delta_1Q_1 \quad \text{and} \quad D_2P_s(T)=\Delta_2Q_2. \tag{2.15}$$

3. Conditioning. In our previous paper [6], it was found that the well conditioning of the boundary value problem (2.1), (2.2) depends on the condition numbers

$$\begin{aligned}\kappa_1&:=\max\{|X(t)S^-|: t\in[t_0,T]\},\\ \kappa_2&:=(T-T_0)\sup\{|\mathcal{G}(t,s)|: s,t\in[t_0,T]\}\end{aligned} \tag{3.1}$$

as well as the quantities

$$\begin{aligned}\kappa_3&:=\max\{|P(t)K(t)^{-1}|: t\in[t_0,T]\},\\ \kappa_4&:=\max\{|Q(t)K(t)^{-1}|: t\in[t_0,T]\},\\ \kappa_5&:=\max\{|P'(t)|: t\in[t_0,T]\},\end{aligned} \tag{3.2}$$

which are related to the well regularity, the well transferability, and the speed of movement of the nullspace of $A(t)$, respectively. The solution representation (2.7) leads to the inequality

$$\|x\|_\infty\leqq\kappa_1|b|+(\kappa_2(\kappa_3+\kappa_5\kappa_4)+\kappa_4+2\kappa_1\kappa_4)\|q\|_\infty, \tag{3.3}$$

if (2.2) is scaled so that $|D_i|\leqq 1$, $i=1,2$. Clearly, instead of κ_5 we could use

$$\max\{|P'(t)Q(t)|: t\in[t_0,T]\}.$$

Note that using the Euclidean and spectral norms as well as the orthoprojectors $Q(t)$, $P(t)$, we obtain

$$\kappa_6\leqq\kappa_3+\kappa_4,\quad \kappa_3\leqq\kappa_6,\quad \kappa_4\leqq\kappa_6, \tag{3.4}$$

where

$$\kappa_6:=\max\{|K(t)^{-1}|_2: t\in[t_0,T]\}. \tag{3.5}$$

Hence in (3.3) we may replace the constants κ_3 and κ_4 by κ_6. This yields

$$\|x\|_\infty\leqq\kappa_1|b|+(\kappa_2(1+\kappa_5)+1+2\kappa_1)\kappa_6\|q\|_\infty. \tag{3.6}$$

In particular, if the DAE (2.1) has the semi-explicit form

$$\begin{aligned}&A_{11}(t)u'(t)+B_{11}(t)u(t)+B_{12}(t)v(t)=q_1(t),\\ &B_{21}(t)u(t)+B_{22}(t)v(t)=q_2(t),\end{aligned} \tag{3.7}$$

then we can take

$$Q(t)=\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix},\qquad K(t)^{-1}=\begin{bmatrix}A_{11}(t)^{-1} & -A_{11}(t)^{-1}B_{12}(t)B_{22}(t)^{-1}\\ 0 & B_{22}(t)^{-1}\end{bmatrix}.$$

Moreover, if (2.1) is in canonical normal form then $K(t)^{-1}=I$.

On the other hand, when the boundary conditions have been taken to affect only the components of Px, i.e., if $D_1Q(t_0)=0$ and $D_2Q(T)=0$, we derive from (2.7) the somehow simpler inequality

$$\|x\|_\infty\leqq\kappa_1|b|+(\kappa_2(\kappa_3+\kappa_5\kappa_4)+\kappa_4)\|q\|_\infty.$$

In what follows we will bound κ_1 in terms of κ_2. Introduce

$$\phi(t):=X(t)S^-P(t_0) \tag{3.8}$$

so that (cf. (2.11))

$$D_1\phi(t_0)+D_2\phi(T)=SS^-P(t_0)=P(t_0) \tag{3.9}$$

is valid. Clearly, $\phi(t)$ is also a fundamental solution matrix. It corresponds to the choice $C=S_I^-+Q(t_0)$, where $S_I := D_1X_I(t_0)+D_2X_I(T)$ and $X_I(t)$ is a fundamental matrix that satisfies $P(t_0)(X_I(t_0)-I)=0$. We have

$$\phi(t)=X_I(t)(S_I^-+Q(t_0))=X_I(t)S_I^-. \tag{3.10}$$

THEOREM 3.11. *It holds that*

$$|\phi(t)|_2 \leqq 2\sup\{|\mathcal{G}(t,s)|_2 : s\in[t_0,T]\}.$$

Proof. Using (2.8) we get

$$\mathcal{G}(t,t_0)=\phi(t)D_1\phi(t_0)\bar{\phi}(t_0)=\phi(t)D_1P_s(t_0)$$

and

$$\mathcal{G}(t,T)=-\phi(t)D_2P_s(T).$$

On the other hand, using (2.15), we can compute

$$\begin{aligned}|\phi(t)|_2 &= |\phi(t)P(t_0)|_2 \leqq |\phi(t)(\Delta_1+\Delta_2)|_2\\ &\leqq |\phi(t)\Delta_1|_2+|\phi(t)\Delta_2|_2\\ &= |\phi(t)D_1P_s(t_0)|_2+|\phi(t)D_2P_s(T)|_2\\ &= |\mathcal{G}(t,t_0)|_2+|\mathcal{G}(t,T)|_2.\end{aligned}$$ □

COROLLARY 3.12. *If κ_1 and κ_2 were defined using the Euclidean and spectral norms, then*

$$(T-t_0)\kappa_1 \leqq 2\kappa_2.$$

If κ_1 and κ_2 were defined using max-norms, then

$$(T-t_0)\kappa_1 \leqq 2m\kappa_2.$$

It is now clear that κ_2, κ_5, and κ_6 determine the conditioning of the boundary value problem (2.1), (2.2). In this context, it becomes important to identify more precisely the class of boundary value problems which have κ_2 of moderate size.

In the next section we will characterize conditions under which the Green function $\mathcal{G}(t,s)$ can be bounded by a constant of moderate size.

4. Dichotomy for transferable differential algebraic equations.

DEFINITION 4.1. The transferable differential algebraic equation (2.1) is dichotomic if there exist a projection matrix $\Pi \in L(\mathbb{R}^m)$ and constants $\alpha>0$, $\lambda \geqq 0$, $\mu \geqq 0$ such that

$$|\phi(t)\Pi\bar{\phi}(s)| \leqq \alpha\exp(\lambda(s-t)) \quad \text{for } s<t,$$

$$|\phi(t)(P(t_0)-\Pi)\bar{\phi}(s)| \leqq \alpha\exp(\mu(t-s)) \quad \text{for } t<s$$

with

$$P(t_0)\Pi=\Pi P(t_0)=\Pi.$$

DEFINITION 4.2. The transferable equation (2.1) is exponentially dichotomic if it is dichotomic with $\lambda>0$, $\mu>0$.

As for regular explicit ordinary differential equations, on finite intervals, any transferable DAE is dichotomic (exponentially dichotomic) but from a practical point of view it is only interesting to obtain bounds for α that are moderate, i.e., $O(\max_{t_0 \leqq t \leqq T} |M(t)|)$ with

$$M(t) := P'(t)P_s(t) - P(t)K(t)^{-1}B(t)$$

(see [6] for the state variable form of (2.1)). Denote by

$$S(t) := \{\phi(t)d \colon d \in \mathbb{R}^m\} \tag{4.3}$$

the solution space related to (2.1).

If (2.1) is dichotomic we can also introduce

$$\begin{aligned} S_1(t) &:= \{\phi(t)\Pi d \colon d \in \mathbb{R}^m\}, \\ S_2(t) &:= \{\phi(t)(P(t_0) - \Pi)d \colon d \in \mathbb{R}^m\}, \end{aligned} \tag{4.4}$$

and use the splitting

$$S(t) = S_1(t) \oplus S_2(t). \tag{4.5}$$

Let us agree on the condensed writing "$y \in S$" for "$y \in \mathscr{C}$, $y(t) \in S(t)$ for all $t \in [t_0, T]$," and similarly "$y \in S_1$," "$y \in S_2$."

LEMMA 4.6. *Let* (2.1) *be dichotomic. Then there exists a constant k such that*

$$y \in S_1 \backslash \{0\} \textit{ implies } \left|\frac{y(t)}{y(s)}\right| \leqq k \quad \textit{for } t > s, \textit{ and}$$

$$y \in S_2 \backslash \{0\} \textit{ implies } \left|\frac{y(t)}{y(s)}\right| \leqq k \quad \textit{for } s > t,$$

i.e., S_1 is the subspace of nonincreasing solutions and S_2 is the subspace of nondecreasing ones.

Proof. For $y \in S_1 \backslash \{0\}$ and $t > s$ we have

$$\begin{aligned} \frac{|y(t)|}{|y(s)|} &= \frac{|\phi(t)\Pi d|}{|\phi(s)\Pi d|} = \frac{|\phi(t)\Pi P(t_0)\Pi d|}{|\phi(s)\Pi d|} \\ &= \frac{|\phi(t)\Pi\bar{\phi}(s)\phi(s)\Pi d|}{|\phi(s)\Pi d|} \leqq |\phi(t)\Pi\bar{\phi}(s)| \\ &\leqq \alpha \exp(\lambda(s-t)). \end{aligned}$$

Thereby, we used the fact that $\bar{\phi}(s)\phi(s) = P(t_0)$.

The result for $y \in S_2$ and $s > t$ follows in a similar way. □

THEOREM 4.7. *Let the* DAE (2.1) *have a dichotomy and assume that the boundary conditions* (2.2) *are such that*

$$\begin{aligned} D_1\phi(t_0)(P(t_0) - \Pi) &= 0, \\ D_2\phi(T)\Pi &= 0, \end{aligned} \tag{4.8}$$

i.e., the left (*right*) *boundary conditions determine the nonincreasing* (*nondecreasing*) *components of the solution. Then*

$$|\mathscr{G}(t,s)| \leqq \alpha \exp(\lambda(s-t)) \quad \textit{for } s < t,$$

$$|\mathscr{G}(t,s)| \leqq \alpha \exp(\mu(t-s)) \quad \textit{for } t < s.$$

Proof. For $s<t$, we have

$$\begin{aligned}\mathcal{G}(t,s)&=\phi(t)P(t_0)D_1\phi(t_0)\bar{\phi}(s)\\&=\phi(t)P(t_0)D_1\phi(t_0)\Pi\bar{\phi}(s).\end{aligned}$$

On the other hand,

$$\begin{aligned}P(t_0)D_2\phi(T)&=P(t_0)D_2\phi(T)(P(t_0)-\Pi)\\&=(P(t_0)-P(t_0)D_1\phi(t_0))(P(t_0)-\Pi)\\&=P(t_0)D_2\phi(T)-\Pi+P(t_0)D_1\phi(t_0)\Pi,\end{aligned}$$

hence

$$\Pi=P(t_0)D_1\phi(t_0)\Pi. \tag{4.9}$$

This yields

$$\mathcal{G}(t,s)=\phi(t)\Pi\phi(s)\quad\text{for } s<t. \tag{4.10}$$

Similarly, it is found that

$$\mathcal{G}(t,s)=\phi(t)(P(t_0)-\Pi)\bar{\phi}(s)\quad\text{for } t<s.$$ □

Obviously, Theorem 4.7 characterizes a class of problems (2.1), (2.2) having condition numbers κ_2 of moderate size.

LEMMA 4.11. *Let the boundary conditions* (2.2) *be separable, i.e.,*

$$\operatorname{rank}(D_1)=p,\quad \operatorname{rank}(D_2)=q,\quad p+q=r.$$

Then the matrix

$$\Pi := P(t_0)D_1\phi(t_0) \tag{4.12}$$

is a projector.

Proof. It is sufficient to show that

$$D_1\phi(t_0))^2=D_1\phi(t_0).$$

First, recall (2.11), (2.14), and (2.15). Let E be an orthogonal matrix that reduces the last $m-q$ rows of D_2 to zero. E can be taken to have the form

$$E=\begin{bmatrix}E_{11} & 0\\ 0 & I_{m-r}\end{bmatrix}.$$

This leads to

$$ED_1\phi(t_0)E^t+ED_2\phi(T)E^t=EP(t_0)E^t=P(t_0),$$

and the matrix $H:=ED_1\phi(t_0)E^t$ has the form

$$H=\begin{bmatrix}H_{11} & H_{12} & 0\\ 0 & I_p & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Since $\operatorname{rank}(H)=p$ we can conclude that $H_{11}=0$. Hence, we have $H^2=H$, and the result follows immediately. □

THEOREM 4.13. *Let the boundary conditions be separable. Assume that the Green function satisfies*

$$|\mathcal{G}(t,s)| = \begin{cases} \alpha \exp(\lambda(s-t)) & \text{for } s < t, \\ \alpha \exp(\mu(t-s)) & \text{for } t < s \end{cases}$$

with some constants $\lambda \geqq 0$, $\mu \geqq 0$, $\alpha > 0$. *Then* (2.1) *is dichotomic with these constants and with the projection matrix defined in* (4.12). □

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