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The Index of an Infinite Dimensional Implicit System

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ABSTRACT

The idea of the index of a differential algebraic equation (DAE) (or implicit differential equation) has played a fundamental role in both the analysis of DAEs and the development of numerical algorithms for DAEs. DAEs frequently arise as partial discretizations of partial differential equations (PDEs). In order to relate properties of the PDE to those of the resulting DAE it is necessary to have a concept of the index of a possibly constrained PDE. Using the finite dimensional theory as motivation, this paper will examine what one appropriate analogue is for infinite dimensional systems. A general definition approach will be given motivated by the desire to consider numerical methods. Specific examples illustrating several kinds of behavior will be considered in some detail. It is seen that our definition differs from purely algebraic definitions. Numerical solutions, and simulation difficulties, can be misinterpreted if this index information is missing.

Keywords: numerical analysis, ordinary differential equations on manifolds, overdetermined systems, partial differential equations on manifolds (AMS Classification).

1 INTRODUCTION

Many physical problems are most easily initially modeled as a nonlinear implicit system of differential and algebraic equations (DAEs),

$$F(t, y, y') = 0, \tag{1}$$

with $F_{y'} = \partial F/\partial y'$ identically singular [2]. DAEs are also referred to as singular systems or descriptor systems depending on the context. Over the last decade there has been a substantial amount of effort expended in the numerical analysis community in developing numerical methods for integrating DAEs [2, 19, 20]. The potential advantages of considering the implicit models provided by DAEs include: quicker modeling, less necessity for model reduction and simplification, exploiting system structure and sparsity, and reduced time between modeling and simulation. In addition, many problems have one DAE model but would require several explicit models.

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In the years ahead the success with finite dimensional DAEs can be expected to impact on the analysis of infinite dimensional systems in several ways. First, the same advantages given above would apply to being able to directly consider implicit infinite dimensional systems. This is already common in chemical process simulation [3, 17, 25, 28]. There an effort is underway to develop numerical simulation based on the method of lines (MOL) solution of implicit systems of PDEs. However, to date most of the analysis in this effort has been restricted to the integration and analysis of the finite dimensional DAE that results from the approximation process.

Correct choice of numerical software requires understanding the relationship between the properties of the infinite dimensional system, the approximation process, and the properties of the finite dimensional system that results. However, much of the study of implicit infinite dimensional systems has been reduced to the studying of specific systems (note however, [12, 13, 23, 27] or [11, 21]). The ideas of the index of a DAE has played a fundamental role in the theory and numerical solution of DAEs. In order to compare the properties of the PDE with those of a finite dimensional DAE we need a concept of index for the PDE. In this paper we will examine one approach to defining the index of an infinite dimensional system. Some consequences for MOL are discussed in [1, 6, 9]. There it is shown that the MOL often acts as a regularization in that the finite dimensional DAE being integrated may appear to have better smoothness properties then the PDE does. However, this appearance is misleading and the numerical solution of the finite dimensional problem will try to approximate the less smooth behavior. This in turn leads to trouble for numerical methods unless they are prepared to deal with this behavior. The DAE integrator can experience difficulty, or the user can misinterpret the solution if the index of the infinite dimensional system is ignored or underestimated.

There are many equivalent definitions of the index of a DAE for linear time invariant finite dimensional DAEs. For nonlinear or time varying DAEs there are several nonequivalent definitions of the index [8]. In this paper we will show that the situation is more complex for infinite dimensional systems. The different approaches to index definitions differ even in the linear time invariant case. Furthermore, the most common approaches to estimating or computing the index for linear time invariant systems may not be appropriate in the infinite dimensional case. Thus the choice of definition in the infinite dimensional case will need to be made with the particular systems and questions of interest in mind. Also, the user needs to be careful about just applying finite dimensional ideas to infinite dimensional DAEs or numerical approximations can be misinterpreted.

There is an increasing interest in discrete systems and in hybrid systems which model systems composed of both discrete and continuous systems. While we will not directly consider these systems in this paper, our approach to defining the index could be applied to these types of systems also.

In Section 2 we briefly review some basic theory from the finite dimensional case to motivate what we do with the infinite dimensional case. Section 3 gives our philosophical approach for the definition of the index. In Section 4 we consider a class of linear time invariant PDEs. We also present two alternative definitions of the index which appear to be natural extensions of the finite dimensional linear time invariant theory that are popular in the engineering literature. Examples are given to show that these traditional approaches can give indices that are much less than our definition. We shall see that even for linear time invariant problems there are several kinds of behavior in the infinite dimensional case that did not occur in the finite dimensional case. Our concern here is with the PDE. An interesting discussion of the approximation process connecting a PDE and a DAE is given for one example in [1]. The examples in Section 4 are academic examples designed to simply illustrate different points. Section 5 gives examples of PDAEs arising in applications.

2 FINITE DIMENSIONAL DAES

In order to motivate a portion of what we need to consider in the infinite dimensional case, we briefly review the finite dimensional linear time invariant theory. This material can be found in several places [2, 4, 5, 10].

Suppose that A, B are $n \times n$ matrices and there exists a number s_0 such that $s_0 A + B$ is invertible. Then $sA + B$ is called a *regular pencil* (of matrices). There then exists invertible constant matrices P, Q such that

$$P(sA + B)Q = \begin{bmatrix} sI + C & 0 \\ 0 & sN + I \end{bmatrix}, \tag{2}$$

where $N^\nu = 0$, $N^{\nu-1} \neq 0$. The integer ν is called the *index* of the pencil $sA + B$. If $sA + B$ is regular, then ν has another algebraic interpretation as the smallest integer such that all entries of $s^{-\nu}(sA + B)^{-1}$ are strictly proper.

For the DAE

$$Ay'(t) + By(t) = f(t), \tag{3}$$

where $sA + B$ is a regular pencil, we suppose that the coordinate changes P, Q have been performed. Then the solutions of (3) are described by

$$z_1' = -Cz_1 + f_1(t) \tag{4a}$$

$$z_2 = \sum_{i=0}^{\nu-1} (-N)^i f_2^{(i)}(t), \tag{4b}$$

so that another interpretation of ν is the amount of smoothness required of f for z to be differentiable.

ν is called the *index* of the DAE (3). Note that only some initial conditions are consistent for a given f if we ignore distributional solutions. Also the solutions form a submanifold.

The DAE (3) is called *solvable* if for any sufficiently smooth f, there is at least one solution y and solutions are uniquely determined by their value at any time. For linear time invariant discrete and continuous DAEs, solvability is equivalent to $sA+B$ being a regular pencil.

The time varying and nonlinear cases are more complicated. It is known that the solutions of a DAE like $F(t, y, y', \delta(t)) = 0$ can depend on derivatives both of the forcing term δ and the function F defining the system. For example, the solution of

$$y' = x + \delta_1(t) \tag{5a}$$

$$0 = \beta(t)y + \delta_2(t), \tag{5b}$$

is $x = -[\beta^{-1}(t)\delta_2(t)]' - \delta_1(t)$, $y = -\beta^{-1}(t)\delta_2(t)$ which involves derivatives of both β and δ. Note also that not all initial data $(x(0), y(0))$ will be consistent for (5) if we restrict ourselves to continuously differentiable solutions.

There are several definitions of the index of a DAE like (1). For our purposes the most important are as follows. If (1) is differentiated k times with respect to t, we get the $(k+1)n$ *derivative array equations* [2]

$$\begin{bmatrix} F(t, y, y') \\ F_t(t, y, y') + F_y(t, y, y')y' + F_{y'}(t, y, y')y'' \\ \vdots \\ \dfrac{d^k}{dt^k}F(t, y, y') \end{bmatrix} = F_k(t, y, y', w) = 0, \tag{6}$$

where $w = [y^{(2)}, \ldots, y^{(k+1)}]$. Also let $v = y'$. Frequently in particular applications, different equations in $F = 0$ are differentiated a different number of times. This has no affect on our discussion.

Consideration of (6) means that we must also consider open sets Γ^e in (t, y, v, w) space. Define the projection map π by $\pi(\Gamma^e) = \{(t, y, v) : \ (t, y, v, w) \in \Gamma^e$ for some $w\}$. Given a consistent value (t, y), (6) viewed as an algebraic equation, will generally have a set of solutions for (y', w). Solvability includes both that (t_0, y_0) consistent implies that y is uniquely determined and continuity of y in terms of (t_0, y_0). A detailed discussion and references are in [8].

Definition 1 *Suppose that $F(t, y, y') = 0$ is a solvable DAE on the open set Γ. If v is uniquely determined by $F_k(t, y, v, w) = 0$ for all consistent values (t, y, y') in Γ and ν_d is the least such integer k that this holds for, we call ν_d the* **differentiation index** *of the DAE.*

Note that the definition of the differentiation index also assumes the specification of an open set Γ^e where $\pi(\Gamma^e) = \Gamma$. The DAE is *higher index* if $\nu_d \geq 2$. The differentiation index can also be defined with respect to a solution, or any other invariant manifold, but we will not do so.

There are several variants of the differentiation index. They are discussed in [8, 18]. However, when these indices are defined, they are equivalent when sufficient smoothness is present and certain constant rank assumptions are made.

A different type of index was defined in [20] in order to analyze implicit Runge-Kutta methods for DAEs. The usual Euclidean norm on $\boldsymbol{R}^n$ is $\|\cdot\|$. For integers $m \geq 1$, let $\mathcal{C}^m$ be the space of m times continuously differentiable $\boldsymbol{R}^n$-valued functions on the finite interval $\mathcal{I}$. Let $\mathcal{C}^0$ be the space of continuous $\boldsymbol{R}^n$-valued functions on the finite interval $\mathcal{I}$. For $m \geq 0$ we give $\mathcal{C}^m$ the norm $\|g\|_m = \sum_{i=0}^m \|g^{(i)}\|_\infty$ where $\|h\|_\infty = \sup_{t\in\mathcal{I}} \|h(t)\|$. For notational convenience let $\|h\|_{-1} = \int_{\mathcal{I}} \|h(t)\| \, dt$. Let $\|f\|_p^t$ be $\|f\|_p$ on the interval $[0,\ t]$ for $p \geq -1$.

Definition 2 *The DAE $F(t, y, y') = 0$ has* **perturbation index** ν_p *along a solution* $\overline{y}$ *on the interval $\mathcal{I} = [0,\ T]$ if ν_p is the smallest integer such that if*

$$F(t, y, y') = \delta(t), \tag{7}$$

for sufficiently smooth δ, then there is an estimate

$$\|\overline{y}(t) - y(t)\| \leq C\left(\|\overline{y}(0) - y(0)\| + \|\delta\|_{\nu_p-1}^t\right), \tag{8}$$

for sufficiently small δ in the $\|\cdot\|_{\nu_p-1}$ norm. C is a constant that depends on F and the length of the interval and the solution $\overline{y}$.

We will always have $\nu_p \geq 1$ since we assume that $F_{y'}$ is always singular.

Note that ν_d for linear time invariant systems has inherent in it a concept of a forcing function. In (4b) the dependence on $\nu - 1$ derivatives is clear. However, if we consider the same problem with $f_2 = 0$, then $z_2 = 0$ and we do not see any of the dependence on derivatives. Thus the concept of differentiation (or perturbation) index should carry some idea of forcing. These indices are referred to as the uniform indices in [8] where continuity of the solutions of (7) at $\delta = 0$ are strengthened to continuity in δ for small δ. This leads to the maximum perturbation, ν_{MP}, and maximum differentiation, ν_{MD}, indices. Whereas ν_d and ν_p can vary greatly for a given DAE, ν_{MP} and ν_{MD} differ by at most one [8]. ν_P^∞ is essentially the same as the index of [16] where the definition is based on differential geometric ideas applied to (7). Computation of ν_{MD} is discussed in [7]. ν_{MP} is (8) holding over an open set of $\overline{y}$.

The index is closely related to certain ideas in control theory. Consider an input-output system with control z

$$x' = f(x, z) \tag{9a}$$

$$y = g(x, z). \tag{9b}$$

Inversion is determining the input z, $z = q(x, y, y', \ldots, y^{(r)})$. For our purposes we can consider r to be the *relative degree* of (9). In this context (9) is a DAE in (x, u) with y a known (or independent) function. In practice, the index of the DAE is usually one more than the relative degree. However, one can give examples where the zero dynamics are a high index DAE and the relative degree is much less than the index. Whether this ever occurs in applications is unclear.

3 GENERAL INFINITE DIMENSIONAL SYSTEMS

Infinite dimensional systems are more complex because of the greater number of ways that functions can enter the equations. In this section we shall give a "philosophical definition" or outline of our approach. In Section 4 we consider a specific class of PDEs.

From the finite dimensional theory we have seen several key aspects of the definition of an index for an infinite dimensional system. First there are some types of free variables. These were the forcing functions f. There is some additional data, which were the initial conditions. While there were solutions for all forcing functions, only some data is consistent in that it goes along with a solution. For (4) the index measured continuity of the solution in terms of the forcing functions. We will see with infinite dimensional DAEs that we will also need to deal with the continuity in terms of data. One can consider some requirements that have to hold, such as boundary conditions for a PDE, as part of the underlying space, while other boundary conditions might be considered as free variables, or data. Choices in these matters are based on what the particular problem is and what questions are of interest.

The following definition is general enough to include many systems of interest but specific enough for us to give a fairly rigorous definition. In this paper we will think of the D_i as various differential operators and the L_i as integral operators. In applications other than those discussed in this paper, one of the D_i might be a delay operator in which case L_i would be a forward shift.

We have a set $\Omega = \Pi_{i=1}^{r}\Omega_i$ of subsets Ω_i in $\boldsymbol{R}^{n_i}$ and a set Σ of functions u defined on Ω. We take the u to be vector valued. However, the dimension of the vectors can vary with i. In greater generality the Ω_i, and the ranges of the u, could be infinite dimensional but we will only consider functions with finite dimensional domains. Σ will be the *states* although they should perhaps more accurately be called generalized states or semistates. There is another set $\mathcal{F}$ of functions f which we will call *forcing functions* or *inputs*. They are defined on a subset $\Omega_{\mathcal{F}}$ of Ω. Note that the forcing functions need not have the same domain as the states. This allows one to consider, for example, boundary control and various kinds of output. There is a third set $\mathcal{D}$ of functions g, also defined on a subset $\Omega_{\mathcal{D}}$ of Ω called the *(initial) data*. There is finite number of linear operators D_i with right inverses L_i. The D_i may be defined on data, inputs, or states. Let P denote a polynomial in various L_i, D_i where we assume, of course, that it makes sense to add, or multiply the given operators in P.

A generalized DAE (or GDAE) consists of two relationships:

$$F(P_1 u, P_2 u, \dots, P_r u, P_{r+1} f, \dots, P_m f) = 0 \tag{10a}$$

$$B(P_{m+1} u, \dots, P_s u, P_{s+1} g, \dots, P_N g) = 0. \tag{10b}$$

The relationship (10a) is the actual DAE. The second relationship (10b) describes how the data and the state are to be related.

Data are those conditions on solutions of (10a) that we would like to hold. The data is the generalization of the idea of an initial condition for a finite dimensional DAE. Those conditions which must hold are incorporated into the definitions of the different sets $\Sigma, \mathcal{F}, \mathcal{D}$. Examples might include smoothness assumptions or boundary conditions for a PDE. This distinction of which are and are not data must be made by the user and has important consequences for the interpretation of the solution.

There are different ways to proceed at this point. In some formulations the operators and functions are treated as purely algebraic objects. This is the case with the differential algebra approach. This approach can be very useful and provides considerable insight [14, 15]. Another algebraic/geometric development appears in [22]. That approach also ignores the effect of initial and boundary conditions. However, [22] also applies to overdetermined systems. In the next section, for comparison purposes, we give a specific example of a purely algebraic index. However, our interest is in the direction of numerical analysis. In these settings the various sets and spaces are equipped with topologies and it makes sense to talk of continuous operators. Properties of the numerical solution of a constrained PDE can be dependent not only on the algebraic properties of the operator but also on quantities such as the domain which do not play an obvious part in an algebraic theory. For concreteness we will assume that the topologies are given by some type of norm. Our terminology is adapted from the finite dimensional cases but there are certain differences we shall point out. For a given f, u is a *solution* of (10a) if it satisfies the equation (10a) in the specified sense. Typically this is pointwise or almost everywhere. A data g is *consistent (for a given f)* if there is a solution $u \in \Sigma$ which satisfies the data, that is (10b) holds for this f, g. Consistent with the finite dimensional case we say that the GDAE is *solvable* if for every forcing function f there are solutions of the GDAE for some data. These solutions are uniquely determined by consistent data.

The solutions u depend on the data and the forcing functions. Our definition of the index generalizes that of the perturbation index. We say that the *infinite pointwise perturbation index* at $\widehat{u}$, which satisfies (10) with $\widehat{f}, \widehat{g}$, is ν_p^∞ if a solution u of (10) with f and consistent g satisfies

$$\|u - \widehat{u}\| \le \sum_{j=0}^{r_1} M_j \|\widetilde{Q}_j (f - \widehat{f})\|_j + \sum_{j=0}^{r_2} B_j \|Q_j (g - \widehat{g})\|_j \tag{11}$$

where B_j, M_j are nonzero constants, $\|\cdot\|_j$, $\|\cdot\|$ are norms, $\widetilde{Q}_j$, Q_j are fixed polynomials in the D_i, and ν_p^∞ is one more than the largest *combined power* of the D_i that occurs. Combined power is the sum of the powers of the D_i in any one product in a $\widetilde{Q}_j$ or P_j. The *(maximum) perturbation index* at $\widehat{u}$, denoted ν_p^∞, is the maximum of ν_p^∞ in a u neighborhood of $\widehat{u}$.

In comparing (11) to (8) the first thing that one notices, besides the greater complexity, is that (11) includes not only derivatives of f but also the data g. Also the index depends on the particular choice of the basic operators D_i and polynomials $\widetilde{Q}_j$, Q_j. One of the advantages of implicit models is that they are similar to the behavioral approach to systems theory [29] so they make it easy to change ones mind about what is input and what is output and what is data. However, the index is dependent on making a particular choice.

Having set up this general framework, we turn to a particular class of PDEs and a number of examples. Some of these examples appear in [9] in a discussion of constrained PDEs and the method of lines (MOL). Here we discuss these examples more carefully with respect to what their index as a PDE is and show how the way a problem is posed can affect what we call the index. We also show that the modal index introduced in [9], like the algebraic index, can underestimate ν_p^∞. A key thing to note is that some of the obvious algebraic definitions are no longer the same in the infinite case. The relationship between the finite dimensional DAE and a PDE is discussed in detail for one example in [1].

4 LINEAR CONSTRAINED PDES

The development of the finite dimensional theory first required a thorough understanding of the linear time invariant theory. Similarly for PDEs we must first understand the linear time invariant case. Some work has been done [12, 13, 23, 27]. However, a general theory is lacking even in fairly basic cases. In this section we will consider a constrained partial differential algebraic equation or PDAE as they are called in the chemical engineering literature [25]. In order to be specific and simplify the discussion we will primarily consider:

$$Au_t + Bu_{xx} + Cu_x + Du = f(x,t) \tag{12a}$$

$$0 \le x \le L, \qquad 0 \le t \le T \tag{12b}$$

$$u(0,t) = 0 \qquad u(L,t) = 0, \tag{12c}$$

and

$$u(x,0) = u_0(x). \tag{13}$$

The value of the terminal time T in (12b) is not particularly important here. We have included T since it makes it easier to define some of the norms we use later. Other boundary conditions are of interest but (12c) allows us to be more specific. We shall see that (12), (13) already illustrates a variety of interesting behavior.

There are several ways to interpret (12), (13). depending on what is chosen as the state and what is free. We consider (12a) in the usual situation where f is free and u is the state. We take $\Omega = \Omega_{\mathcal{F}} = [0, 1] \times [0, T]$, $\Omega_{\mathcal{D}} = [0, 1] \times \{0\}$. We include in the definitions of $\Sigma, \mathcal{F}$ that the boundary conditions (12c) are defined. Then (12a) is (10a) and (13) is (10b).

We first define the index of interest to us, ν_P^∞. Then we define two other indices ν_A^∞ and ν_M^∞ in order to show how other traditional approaches can underestimate ν_P^∞ even for linear time invariant problems.

4.1 Perturbation Index

We now turn to carefully examining (12) with the boundary conditions (12c). One option explored in [23, 27] when $C = D = 0$ is the use of eigenfunctions for the operator $\partial^2/\partial x^2$. Here we also consider the use of an orthogonal basis. There are, of course, questions of convergence that must be considered. However, at this point we shall keep our calculations somewhat formal. Let $\psi_j(x) = \sin(j\pi x/L)/\sqrt{L}, \lambda_j = -(j\pi/L)^2$. We can then consider the series

$$u(x,t) = \sum_{j=1}^{\infty} \psi_j(x) u_j(t), \quad f(x,t) = \sum_{j=1}^{\infty} \psi_j(x) f_j(t), \quad u_0(x) = \sum_{j=1}^{\infty} u_{0j} \psi_j(x). \tag{14}$$

In general, $c_j(t)$ is the jth coefficient of the function $c(x, t)$ with respect to $\psi_j(x)$. We need to define some norms in order to be precise about the perturbation index. Let $\|\cdot\|$ be the usual Euclidean norm on $\boldsymbol{R}^n$. For a function $c(x, t)$ we define $\|c\|_\infty$ to be

$$\|c\|_\infty = \max_{0\le t\le T} \left(\int_0^L \|c(x,t)\|^2 dx \right)^{1/2} = \max_{0\le t\le T} \|c(x,t)\|_2,$$

where $\|\cdot\|_2$ is the usual L_2 norm in the x variable. Equivalently,

$$\|c\|_\infty = \max_{0\le t\le T} \left(\sum_{j=1}^{\infty} \|c_j(t)\|^2 \right)^{1/2}. \tag{15}$$

Finally, we define

$$\|c\|_{(p,q)} = \sum_{i=0}^{p} \sum_{k=0}^{q} \max_{0\le t\le T} \left(\sum_{j=1}^{\infty} \|c_j^{(i)}(t)\|^2 \left(\frac{j\pi}{L} \right)^{2k} \right)^{1/2} = \sum_{i=0}^{p} \sum_{k=0}^{q} \| \frac{\partial^{i+k}}{\partial t^i \partial x^k} c \|_\infty. \tag{16}$$

Note that $\|c\|_{(p,q)}$ being finite implies that one can take p t-derivatives and q x-derivatives of c term by term.

The estimate (11) then takes the form

$$\|u - \widehat{u}\| \leq C_1 \| f - \widehat{f}\|_{(p_1,q_1)} + C_2 \|g - \widehat{g}\|_{(0,q_2)}. \tag{17}$$

We take the perturbation index ν_p^∞ of (12), (13) to be :

$$\nu_p^\infty = 1 + \min\left\{\max\{p_1 + q_1, q_2\} : (17) \text{ holds for } (p_1, q_1, q_2)\right\}. \tag{18}$$

Again ν_P^∞ is the maximum of ν_p^∞ over a neighborhood of $\widehat{u}$. In the notation of (11) we have taken $\mathrm{D}_1 = \partial/\partial t$, $\widetilde{Q}_j(z) = z^j$, $\mathrm{D}_2 = \partial/\partial x$, $Q_i(z) = z^i$. This is the most natural choice in terms of the usual concept of smoothness. If $C = 0$, the x partials occur an even number of times and we could have taken $\mathrm{D}_2 = \partial^2/\partial x^2$. This would have altered the value of the index for some problems.

4.2 An Algebraic Index

An algebraic approach to linear time invariant DAEs has been very popular especially in the control and electrical engineering literature. There are a variety of algebraic approaches for time invariant PDEs and ODEs. They all have the characteristic that they tend to be independent to some extent of boundary conditions. We define one such algebraic index in order to compare it with ν_P^∞. We shall see that the algebraic index can underestimate the perturbation index. However, the algebraic index is often easier to compute and, as we will see in the examples, can often warn of possible parameter values where higher index behavior can occur.

Unfortunately, in order to be precise we need some terminology. Suppose that $r(s, z)$ is a fraction of two real polynomials in the real variables s, z. We say that r is *s-proper* if $\lim_{|s|\to\infty} r(s, z) = 0$ for almost all z. *z-proper* is defined the same way. We say that r is *proper* if it is both s and z proper. A matrix has a given properness property if every one of its entries has that property.

We call (12a) *regular* if $\det(sA + z^2B + Cz + D) \not\equiv 0$. Regularity is necessary for solvability. To see this suppose that we do not have regularity. Then there is a matrix polynomial $E(s, z)$ such that $(sA + z^2B + Cz + D)E(s, z) = 0$. Let b be any function of (x, t) for which enough derivatives of b satisfy the boundary conditions and a zero initial condition. Then $E(\frac{d}{dt}, \frac{d}{dx})b$ is a solution of the associated homogeneous equation (12a), (12b) and the PDAE is not solvable.

Assuming we have regularity, let

$$R(s, z) = (sA + z^2B + Cz + D)^{-1}. \tag{19}$$

We define the *algebraic t-index* ν_t of (12a) to be the smallest integer n_1 such that $s^{-n_1}R(s, z)$ is s-proper. The *algebraic x-index* ν_x of (12a) is the smallest n_2 such that $z^{-n_2}R(s, z)$ is z-proper. We want the algebraic index to capture, among other things, the highest degree of smoothness required of f. This leads to the following definition. Let $R_{i,j}$ be the i, j entry of R. Then we define the *algebraic index* of (12a) to be

$$\nu_A^\infty = \max_{i,j} \left\{ \min_{n_1,n_2 \geq 0} \left\{ n_1 + n_2 : \; z^{-n_1} s^{-n_2} R_{i,j}(s,z) \;\text{ is proper } \right\} \right\}. \tag{20}$$

Before considering (12) in some detail we note that regularity is no longer sufficient in general to guarantee solvability.

Proposition 1 *Suppose in (12) that* $C = D = 0$. *Suppose that the boundary conditions (12c) are replaced by linear homogeneous boundary conditions for which* 0 *is an eigenvalue of* $\frac{\partial^2}{\partial x^2}$. *Then (12) is not solvable for any* A, B *if* A *is singular.*

Proof. Let $\gamma(x)$ be a scalar eigenfunction for the eigenvalue 0 of $\frac{\partial^2}{\partial x^2}$. Thus γ satisfies the boundary conditions. Let $Av = 0$ and $\psi(t)$ be any smooth scalar function. Then $u = \gamma(x)\psi(t)v$ is a solution of the homogeneous PDE which satisfies the boundary conditions. The data (13) determines only $\psi(0)$ and not $\psi(t)$. ■

A simple example of the above phenomena is gotten by taking the boundary conditions to be $u_x(0,t) = u_x(L,t) = 0$.

4.3 Modal Index

Modal approximations are a modeling technique which is widely used with PDEs. As with any approximation process it is important to understand the relationship between the original and the approximate problem. In [9] we used a different concept of index based on a modal approximation which we call here the modal index. While it was a natural index to consider, we will show here that the modal index can also fail to correctly capture the information in ν_P^∞. Assume that $C = 0$ in (12a). Under the assumptions that u and f are smooth enough in t and x we may substitute the series (14) for u, f into the PDE (12a) and get the *modal DAE equations*

$$Au_j'(t) + \left(\lambda_j B + D\right) u_j(t) \; = \; f_j(t), \quad j \geq 1 \tag{21a}$$

$$u_j(0) \; = \; u_{0j}. \tag{21b}$$

The solutions of the *modal DAEs* are determined by the parameterized family of matrix pencils

$$\mathcal{P}_{\lambda_j} = \left\{A, \; \lambda_j B + D\right\}. \tag{22}$$

It is important to note the different roles played by the different components of (22). The matrices A, B, D come from the PDAE (12a). However, the λ_j are discrete numbers depending on the length L of the spatial interval and j. These numbers could be positive or negative or both. However, for the specific boundary conditions (12c) we can say more.

We shall say that (12) is *modal solvable* if (21) is solvable for every j. This means that we incorporate some smoothness assumptions into our spaces Σ. Let ν_j be the index of (21) if (22) is a regular pencil. Then the *modal index* is

$$\nu_M^\infty = \max_j \left\{ \nu_j \right\}. \tag{23}$$

Modal solvability is not a purely algebraic property. It can also depend on the length of the interval (domain) as shown by the next proposition.

Proposition 2 *Suppose that $\mathcal{P}_\alpha$ is a regular pencil for every $\alpha < 0$, then (12) is a modal solvable PDAE for any L. However, if there exists a number $\widehat{\alpha} < 0$ for which $\mathcal{P}_{\widehat{\alpha}}$ is not regular and a number $\overline{\alpha}$ for which $\mathcal{P}_{\overline{\alpha}}$ is regular, then (12) is modal solvable for all L except for a countable number of L where it is not solvable.*

Proof. The first statement is clear. So suppose that $\mathcal{P}_{\overline{\alpha}}$ is regular. Note that $\det(\lambda A + \alpha B + D)$ is a polynomial in two variables. Thus (22) is regular for α near $\overline{\alpha}$. On the other hand (22) is not regular if $L = n\pi(-\widehat{\alpha})^{1/2}$. ■

The $\alpha < 0$ condition arises because for these boundary conditions all the eigenvalues are negative. If the boundary conditions allow for some positive eigenvalues, then the proposition has to be modified.

Example 1 *Let A be singular and $B = D = I$. Then the conditions of Proposition 2 are met with $\widehat{\alpha} = -1$ and $\overline{\alpha} \neq -1$.*

The modal index can also vary with L.

Example 2 *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{24}$$

For most values of L we get that all the DAEs (21) are index one. For sufficiently smooth f the solution will be continuous in f. However, for L equal say π we have that (21), (24) will be index two for $j = 2$ and index 1 for $j \neq 2$. Thus ν_M^∞ is 1 or 2 depending on L.

4.4 Comparison of the Indices

We now will compute the algebraic and perturbation indices for Example 2. An easy calculation gives

$$R(s,z) = 4\begin{bmatrix} \dfrac{4+z^2}{16s+4sz^2-z^4} & \dfrac{z^2}{16s+4sz^2-z^4} \\ \dfrac{z^2}{16s+4sz^2-z^4} & \dfrac{4s}{16s+4sz^2-z^4} \end{bmatrix}. \tag{25}$$

Only the $R_{2,2}$ term is of interest since all the other terms are proper. We have $\nu_x = 0$, $\nu_t = 1$, and $\nu_A^\infty = 1$.

The modal DAE is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_j' + \begin{bmatrix} 0 & \gamma \\ \gamma & 1-\gamma \end{bmatrix} u_j = f_j, \tag{26}$$

where $\gamma = \frac{j^2\pi^2}{L^2 4}$. Note that $\lim_{j\to\infty} \gamma = +\infty$. For a given value of L there will be at most one exceptional case, namely when $\gamma = 1$. Other than for this case, if it exists, simple algebra gives (for $\gamma \neq 1$) that (26) is the same as

$$u_{1j}'(t) + \frac{-\gamma^2}{1-\gamma} u_{1j} = \frac{\gamma}{1-\gamma} f_{2j} + f_{1j} \tag{27a}$$

$$u_{2j} = \frac{1}{1-\gamma}\left[f_{2j} - \gamma u_{1,j} \right], \tag{27b}$$

The key point to note from the index one system (27a) is that the solutions look like bounded functions of f_j as j goes to infinity since $-\gamma^2/(1-\gamma)$ is positive for large j. Thus we have that with respect to the norms given that:

Proposition 3 *If the length L is such that $\gamma \neq 1$ for any j, then all three indices $\nu_M^\infty, \nu_P^\infty, \nu_A^\infty$ of the PDAE (24) are one. On the other hand if $\gamma = 1$ for some value of j, then $\nu_A^\infty = 1$, $\nu_M^\infty = \nu_P^\infty = 2$.*

This example illustrates an important point, namely that the algebraic and the perturbation index can differ. One might suppose that Example 2 is somehow special. However, this is not the case as the next proposition shows.

Proposition 4 *Suppose that A is a singular square matrix. Then there is an open set of matrices B, D such that*

(i) *$sA + z^2 B + D$ is regular*

(ii) *The modal DAEs are all index one for most L. In this case $\nu_M^\infty = 1$. If the modal DAEs have negative eigenvalues, then $\nu_P^\infty = 1$ also.*

(iii) *There is a sequence of L, with no accumulation point, for which at least one of the modal DAEs has ν_P, ν_M greater than one.*

Proof. Without loss of generality, by performing coordinate changes we may assume that $A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Then $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, $D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$. The modal DAE will be index one precisely when $\gamma B_4 + D_4$ is invertible. Also $\det(\gamma B_4 + D_4)$ is a polynomial in γ whose coefficients are continuous in the entries of B_4, D_4. The proof is completed by noting that there is an open set of B_4, D_4 such that $\gamma B_4 + D_4$ is regular and $\det(\gamma B_4 + D_4) = 0$ for some $\gamma < 0$. ■

To help understand what is happening in this example, note that a given entry of $R(s, z)$ has the form $R_{i,j}(s, z) = \left(\sum_{i=0}^{m_1} p_i(z^2)s^i\right) / \left(\sum_{i=0}^{m_2} q_i(z^2)s^i\right)$ where the p_i, q_i are polynomials. However, z^2 is really the operator $\partial^2/\partial x^2$. If α is an eigenvalue of $\partial^2/\partial x^2$ with corresponding eigenvector ϕ, then on that mode the system looks like

$$R_{i,j}(s, z)\Big|_{z^2=\alpha} = \frac{\sum_{i=0}^{m_1} p_i(\alpha)s^i}{\sum_{i=0}^{m_2} q_i(\alpha)s^i}\phi.$$

If $q_{m_2}(\alpha) = 0$, then it is possible for $R_{i,j}(s, \sqrt{\alpha})$ to have higher index in s then $R_{i,j}(s, z)$ does. For example, for (25) suppose that we have that -4 is an eigenvalue for $\partial^2/\partial x^2$ so that z^2 would be replaced by -4. Then we get that

$$R = \begin{bmatrix} 0 & 1 \\ 1 & -s. \end{bmatrix} \tag{28}$$

Note that (25) had t-index of 1 but (28) has t-index of 2 because of the reduction of order in terms of s of the denominator in $R_{2,2}$. Of course it is possible that an alternative type of algebraic index might better estimate ν_P^∞. However, as we have just seen, for some PDAEs, ν_P^∞ varies with L so how to do so without taking L into account is not clear.

In this example we have taken the boundary conditions as part of the definition and the initial conditions (13) as the data which may or may not be consistent. However, there are other options. For example we could view (12c) and (13) as data. In this case we are allowing for them to be consistent for some solutions and not consistent for other solutions.

Our next example illustrates why we had to modify the definition of the perturbation index to include derivatives of the data. This PDAE is not in the form of (12a).

Given a function $f(z, w)$ of several variables, let L_z be the operator of antidifferentiation with respect to z.

$$L_z(f(z, w)) = \int_0^z f(s, w)ds.$$

Example 3 *Consider the PDAE*

$$\begin{aligned} u_t + v_{xx} &= f_1(x, t) && \text{(29a)} \\ v_{tx} &= f_2(x, t) && \text{(29b)} \\ u(x, 0) &= g_1(x) && \text{(29c)} \\ v(x, 0) &= g_2(x) && \text{(29d)} \\ v(0, t) &= g_3(t) && \text{(29e)} \\ 0 \le x \le 1, &\qquad 0 \le t \le T. && \text{(29f)} \end{aligned}$$

Here we choose to treat (29c)– (29e) as data but that is not essential. The solution of the PDAE (29a) and (29b) is

$$u = L_t f_1 - tc_2''(x) - L_t L_t \frac{\partial f_2}{\partial x} + c_3(x) \tag{30a}$$
$$v = L_t L_x f_2 + L_t c_1(t) + c_2(x). \tag{30b}$$

Applying the data to (30a) and (30b) we get that

$$c_3(x) = g_1(x) \tag{31}$$
$$c_2(x) = g_2(x) \tag{32}$$
$$L_t c_1(t) = g_3(t). \tag{33}$$

From the solution we thus see that u, v depend on first x-derivatives of f but second x-derivatives of the data g. In this case we would say that $\nu_P^\infty = 3$. We have not yet defined the modal index, ν_M^∞, for this system since the g_j defined earlier are not eigenfunctions of $\partial^2/\partial x^2$ and $\partial/\partial x$.

On the other hand, for this example $R(s,z) = \frac{1}{s^2 z}\begin{bmatrix} sz & -z^2 \\ 0 & s \end{bmatrix}$ so that $\nu_P^\infty = 3$ while $\nu_A^\infty = 2$. Note that ν_A^∞ in this example captures the dependence on the first x-partial of f_2. However, it misses the dependence on the second x-partial of g_1. This dependence on derivatives of data is a new type of behavior not present with DAEs.

One way in which PDAEs differ significantly from ODEs is that there is more than one direction in which the system can exhibit higher index. In fact, an explicit PDE can have index greater than zero. The reason for this is that we have designated one of the variables t as special since there is an initial condition. However, the PDAE can be higher index because of the other variable.

Example 4 *Consider the explicit PDE in the form (12)*

$$v_t + u_{xx} = f_1(x,t) \tag{34a}$$
$$u_t = f_2(x,t), \tag{34b}$$

so that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $C = D = 0$.

We see in this example that all the modal DAEs are index zero so that $\nu_M^\infty = 0$. The solutions of (34) are

$$u = L_t f_2 + c_1(x) \tag{35a}$$
$$v = L_t \left(f_1 - L_t \frac{\partial^2 f_2}{\partial x^2} - c_1''(x) \right) + c_2(x). \tag{35b}$$

Here $\nu_P^\infty = 3$ because the solution depends on second derivatives with respect to x of a t-integral of f_2 and also second derivatives of $c_1 = u(x,0)$. Thus in general, we can not just use the indices of the modal equations themselves to estimate the index of a PDAE. For this example, we have $R(s,z) = \frac{1}{s^2}\begin{bmatrix} s & -z^2 \\ 0 & s \end{bmatrix}$. The algebraic t-index is zero, the algebraic x-index is 3, and $\nu_A^\infty = 3$. Unlike the previous Example 3 where $\nu_A^\infty < \nu_P^\infty$, we have here that $\nu_M^\infty < \nu_A^\infty = \nu_P^\infty$.

As an additional illustration suppose (12a) has the form

$$\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}\begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix}. \quad (36)$$

Notice that $B_3 v_{xx} + D_3 v = f_2(x,t)$ is a DAE in the spatial derivative $\partial/\partial x$.

The next theorem gives a general statement that covers this example.

Theorem 1 *Consider the PDAE $Au_t + Bu_{xx} + Du = f(x,t)$ with boundary conditions (12). If $\lambda B + D$ is a regular pencil with index ν, then $\nu_P^\infty \geq 2\nu - 1$.*

Proof. We take an $f(x)$ whose first ν derivatives also satisfy the boundary conditions and look for a steady state solution $u(x)$. But then $Bu'' + Du = f$. Since $\lambda B + D$ is regular of index ν we know from (4b) that u depends on $2(\nu - 1)$ derivatives of f. ■

In the modal DAE equations, a spatial PDAE gets converted to an algebraic equation.

Example 5 *Consider the PDAE*

$$\begin{bmatrix} N^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_t \\ v_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}\begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + \begin{bmatrix} I & I \\ 0 & I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = f(x,t). \quad (37)$$

where $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. The solutions are

$$u_1 = f_1 - f_3 + f_{4xx} \quad (38)$$
$$u_2 = f_2 - f_4 - f_{1t} + f_{3t} - f_{4txx} \quad (39)$$
$$v_1 = f_3 - f_{4xx} \quad (40)$$
$$v_2 = f_4. \quad (41)$$

This example has t-index 2, x-index three, and $\nu_P^\infty = \nu_A^\infty = 4$, but $\nu_M^\infty = 2$.

The next example generalizes Example 4. It is not covered by Theorem 1.

Example 6 *Let N be an $r \times r$ upper triangular nilpotent Jordan block and consider the PDE*

$$u_t + N u_{xx} = f. \tag{42}$$

The modal equations are all index zero so that $\nu_M^\infty = 0$. However,

$$u = \sum_{i=0}^{r-1} \left(-N L_t \frac{\partial^2}{\partial x^2} \right)^i f + \sum_{i=0}^{r-1} \frac{t^i}{i!} \left(-N \frac{\partial^2}{\partial x^2} \right)^i c(x),$$

where $c(x)$ is an arbitrary function of x. For this example, $\nu_M^\infty = 0$ but $\nu_P^\infty = \nu_A^\infty = 2(r-1)+1$.

Example 7 *Let N in Example 5 be an $r \times r$ upper triangular nilpotent Jordan block. Then $\nu_M^\infty = r$ but $\nu_P^\infty = \nu_A^\infty = 2r$.*

Another way that PDAEs differ from DAEs is that the index can depend on the particular way we present the initial conditions. In particular, the consistent initial conditions can satisfy differential equations themselves. $\mathcal{D}$ can then be chosen to be different projections of the initial values.

Example 8 *Consider the PDAE in the form (12)*

$$u_t + v_{xx} = f_1(x,t) \tag{43a}$$
$$v_{xx} - w = f_2(x,t) \tag{43b}$$
$$w_t = f_3(x,t). \tag{43c}$$

along with the homogeneous boundary conditions (12c) and the two different initial conditions

$$v(x,0) = v_0(x), \tag{44}$$

or

$$w(x,0) = w_0(x). \tag{45}$$

The general solution of the PDE (43) can be written

$$u = L_t\,(f_1 - f_2 - L_t f_3 - c_1(x)) + c_4(x) \tag{46a}$$
$$v = L_x L_x (f_2 + L_t f_3 + c_1(x)) + x c_2(t) + c_3(t) \tag{46b}$$
$$w = L_t f_3 + c_1(x). \tag{46c}$$

We assume that f also satisfies (12c). Then (12c) implies that c_1, c_2 are determined by the boundary conditions. Also

$$u(x,0) = c_4(x). \tag{47}$$

If we use the initial condition (45), (47), then $w(x,0) = c_1(x) = w_0(x)$ and $\nu_P^\infty = 1$. But if we use the initial condition (44), (47) we have that $v_0(x) =$

$L_x L_x\,(f_2(x,0)+c_1(x)) + xc_2(0) + c_3(0)$. *Thus* $c_1(x) = v_0''(x)$ *and* $\nu_P^\infty = 3$. *On the other hand, for this example we have that*

$$R(s,z) = \begin{bmatrix} s & z^2 & 0 \\ 0 & z^2 & -1 \\ 0 & 0 & s \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s} & -\frac{1}{s} & -\frac{1}{s^2} \\ 0 & \frac{1}{z^2} & \frac{1}{z^2 s} \\ 0 & 0 & \frac{1}{s} \end{bmatrix}.$$

Thus $\nu_A^\infty = 1$. *Also* $\nu_M^\infty = 1$.

One of the things that $\nu_M^\infty, \nu_A^\infty$ fail to capture is the need for convergence of the appropriate series. The next example illustrates this.

Example 9 *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *The modal DAE is*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_j' + \begin{bmatrix} 0 & \gamma \\ -\gamma & 1-\gamma \end{bmatrix} u_j = f_j \tag{48}$$

where $\gamma = \frac{j^2\pi^2}{L^2 4}$. *For a given value of* L *there will be at most one exceptional case, namely when* $\gamma = 1$. *Other than for this case, if it exists, simple algebra gives (for* $\gamma \neq 1$*) that (48) is the same as*

$$u_{1j}'(t) + \frac{\gamma^2}{1-\gamma} u_{1j} = \frac{\gamma}{1-\gamma} f_{2j} + f_{1j} \tag{49a}$$

$$u_{2j} = \frac{1}{1-\gamma}\left[f_{2,j} + \gamma u_{1j}\right], \tag{49b}$$

Let $f = 0$ *for the moment. Then we have that*

$$u_{1j} = e^{-t\gamma^2/(1-\gamma)} u_{1j}(0). \tag{50}$$

But $\gamma \to \infty$ *as* $j \to \infty$. *Thus* ν_P^∞ *is not even defined for* $t > 0$ *unless we place strong assumptions on the convergence of the Fourier coefficients of both the initial conditions and on* f.

Since the exponential also appears in the solution for $f \neq 0$,

$$\int_0^t e^{-(s-t)\gamma^2/(1-\gamma)} f(s)ds$$

we get that a data that may be acceptable on one time interval may not be acceptable on a longer time interval. In this case we could have a dependence of ν_P^∞ *on the length of the time interval.*

We now will compute the algebraic index for Example 9. An easy calculation gives

$$R(s,z) = 4\begin{bmatrix} \dfrac{4+z^2}{16s+4sz^2+z^4} & \dfrac{z^2}{16s+4sz^2+z^4} \\ \dfrac{-z^2}{16s+4sz^2+z^4} & \dfrac{4s}{16s+4sz^2+z^4} \end{bmatrix}. \tag{51}$$

Only the $R_{2,2}$ term is of interest since all the other terms are proper. We have $\nu_x = 0$, $\nu_t = 1$, and $\nu_A^\infty = 1$.

5 EXAMPLES FROM APPLICATIONS

In the preceding sections we have described the perturbation index of a PDE. We have also shown that some algebraic approaches can sometimes underestimate the perturbation index. Algebraic approaches can detect some parameter values where higher index behavior will occur. The examples given so far were designed to be as simple as possible and still show the desired behavior. They were not motivated by any particular type of application. In this section we mention some examples of PDAEs that arise in applications. Our intention is not to carefully analyze these PDAEs. That would take several additional papers. Rather it is to illustrate how PDAEs of the types discussed earlier occur. To simplify the exposition we denote various model constants by c_i. Definitions of the various parameters and variables can be found in the references.

PDAEs arise in many places in chemical engineering. Frequently they are solved by the method of lines (MOL) which results in a finite dimensional DAE to be solved. Our first example is the equations modeling the ignition of a single-component nonreacting gas in a closed cylindrical vessel [2].

Example 10 (Non-reacting Gas Ignition). *The system is*

$$T_t - \frac{1}{\rho c_1} p_t = \frac{1}{c_1}\frac{\partial}{\partial \psi}\left(\rho r^2 c_2 \frac{\partial T}{\partial \psi}\right) \tag{52a}$$

$$0 = r_\psi - \frac{1}{\rho r} \tag{52b}$$

$$0 = p_\psi \tag{52c}$$

$$0 = p - \rho c_2 T. \tag{52d}$$

Let $u = (T, r, p, \rho)$. The linearized equations have the form

$$\begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u_t + \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u_{\psi\psi} + \begin{bmatrix} * & * & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u_\psi + \begin{bmatrix} 0 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & 0 & 0 & 0 \\ * & 0 & 1 & * \end{bmatrix} u = 0.$$

Note that all the coefficient matrices are singular. For most values of $*$ we have $\nu_A = 1$.

The next example is a model of an adiabatic tubular reactor where there is startup of an exothermic reaction [25] where h is specific enthropy, T is temperature, and C is concentration.

Example 11 (Tubular Reactor)

$$C_t = c_1 C_{zz} + c_2 C_z + c_3 C \tag{53a}$$

$$h_t = c_4 T_{zz} + c_5 h_z + c_6 C \tag{53b}$$

$$0 = -h + c_7 T + c_8 T^2 + c_9 T^3 + c_{10} T^4. \tag{53c}$$

Note that while it is theoretically possible to solve (53c) for T it is not practical to do so and the problem is best handled as a DAE. The constraint (53c) is nonlinear. Let $u = [C, h, T]^T$. If we linearize we get locally

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_t - \begin{bmatrix} c_1 & 0 & 0 \\ 0 & 0 & c_4 \\ 0 & 0 & 0 \end{bmatrix} u_{zz} - \begin{bmatrix} c_2 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & 0 \end{bmatrix} u_z - \begin{bmatrix} c_3 & 0 & 0 \\ c_6 & 0 & 0 \\ 0 & -1 & c_{11} \end{bmatrix} u = 0. \tag{54}$$

None of the coefficients are invertible. As noted earlier, the algebraic index will sometimes indicate parameter values resulting in higher index behavior. Let $\Delta_1 = s - c_2 z - c_3 - z^2 c_1$ and $\Delta_2 = s c_{11} - c_{11} c_5 z - c_4 z^2$. Then

$$R(s, z) = \frac{1}{\Delta_1 \Delta_2} \begin{bmatrix} \Delta_2 & 0 & 0 \\ c_6 c_{11} & c_{11}\Delta_1 & -c_4 z^2 \Delta_1 \\ c_6 & \Delta_1 & (s - c_5 z)\Delta_1 \end{bmatrix}.$$

For most values of the c_i we get algebraic index 1 in t and x. However, if $c_{11} = 0$ but $c_4 \neq 0$, then the (3,3) entry has t-index two. The perturbation index is also 2 for some boundary conditions if $c_{11} = 0$, and $c_4 \neq 0$ since we then have that

$$u_2 = f_3 \tag{55a}$$

$$u_3 = \frac{1}{c_4} L_z L_z \left(\frac{\partial f_3}{\partial t} - c_5 \frac{\partial f_3}{\partial z} - c_6 u_1 - f_2 \right) + \phi_1(t) z + \phi_2(t), \tag{55b}$$

with u_1 being the solution of the first equation in (54).

Having c_{11} small but nonzero corresponds to a situation where the DAE that results from the method of lines is index one but the Jacobian of the constraint is ill conditioned and numerical difficulties are possible since the index one solution is trying to approximate the index two solution when $c_{11} = 0$.

Example 12 (Superconductive Coil) *The equations governing the coil transients of a superconductive coil can take the form [23]*

$$u_{xx} - \begin{bmatrix} 0 & 0 \\ -LC/l^2 & L/D \end{bmatrix} u_{tt} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} u = 0. \tag{56}$$

Letting $v_1 = u$, $v_2 = u_t$ we get

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ c_1 & c_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} v_t + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} v_{xx} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} v = 0. \tag{57}$$

System (57) is in the form of Theorem 1 with $\lambda B + D$ having $\nu = 2$. Thus ν_P^∞ will have index at least 3 for some boundary conditions.

Example 13 (Navier-Stokes Equations) *The incompressible Navier-Stokes equations take the form*

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \gamma \nabla^2 u = 0 \tag{58a}$$

$$\nabla \cdot u = 0. \tag{58b}$$

Here u is a 3-vector and p is a scalar function of 3 variables. For our purposes we shall give the 1-D form of these equations. Let $u = [u_1, p]$. Then we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_t - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_{xx} + \begin{bmatrix} u_1 & 1 \\ 1 & 0 \end{bmatrix} u_x = 0. \tag{59}$$

The linearized form is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_t - \gamma \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} u_{xx} + \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix} u_x = f, \tag{60}$$

with $f = 0$. For this system

$$R(s,z) = \begin{bmatrix} s + \alpha z - \gamma z^2 & z \\ z & 0 \end{bmatrix}^{-1} = \frac{1}{z^2} \begin{bmatrix} 0 & z \\ z & \gamma z^2 - \alpha z - s \end{bmatrix},$$

so that the t-index is 2 and the x-index is one. Letting f be nonzero in (60) and solving we get that

$$.u_1 = L_x f_2 + \phi(t) \tag{61a}$$

$$u_2 = -L_x L_x \frac{\partial f_2}{\partial t} - \phi'(t)x + \psi(t) - \alpha L_x f_2 + \gamma f_2 + f_1 \tag{61b}$$

where $\phi(t)$, $\psi(t)$ are arbitrary functions that are determined by the boundary conditions. The presence of $\phi'(t)$ and $(f_2)_x$ shows the perturbation index is two for some boundary conditions. This reflects the well known fact that the MOL solution of (58) leads to an index two DAE [2].

As discussed in the opening section of this paper, in some applications one gets mixed systems of PDEs, DAEs, and constraints. These systems occur, for example, in chemical engineering and in flexible mechanics. We give here the example of a planar slider-crank mechanism with one flexible link [26].

Example 14 (Flexible Slider Crank - A Mixed System) *This is a two link mechanism with a torque applied at the beginning of the first link which is rigid. The end of the second link is constrained to move in a straight line. We ignore actuator dynamics and take the angle of the first link ω as an input. $\phi(t)$ is the angle between the two links. The second link is assumed flexible. Let $u(x, y, t)$ denote the amount of deformation of point (x, y) on the link at time t measured in the links coordinate frame. The state is then $[\phi(t), u(x, y, t)^T]$. Let x_c, y_c denote the joint coordinates of the connecting joint, l_i the length of the ith link, ρ the constant mass density, $r = l_1[\cos\omega, \sin\omega]^T$, and $\tilde{I} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Note that r is known. Let $\overline{u} = x + u$. Then we have for the connecting rod, omitting the gravity, the PDAE*

$$\rho \begin{bmatrix} \overline{u}^T\overline{u} & \overline{u}^T\tilde{I} \\ \tilde{I}^T\overline{u} & I \end{bmatrix} \begin{bmatrix} \ddot{\phi} \\ \ddot{u} \end{bmatrix} = \begin{bmatrix} -2\rho\dot{\phi}\overline{u}^T\dot{u} - \rho\overline{u}^T A'^T\ddot{r} \\ 2\rho\dot{\phi}\tilde{I}\dot{u} + \rho\dot{\phi}^2\overline{u} + div\,\nabla\varphi\Sigma - \rho A^T\ddot{r} \end{bmatrix} \tag{62a}$$

$$0 = l_1 \sin\omega + (l_2 + u_1(x_c, y_c, t)) \sin\phi + u_2(x_c, y_c, t)\cos\phi \tag{62b}$$

subject to the boundary conditions

$$u(0, t) = 0, \quad u_x(0, t) = 0, \quad u_y(0, t) = 0. \tag{63}$$

Here φ is the actual shape of the second link in the coordinates of the second link, $A(\phi)$ is the rotation matrix of angle ϕ, $\Sigma = \Lambda(trace(E))I + 2\mu E$ where Λ, μ are constants, and E is the strain tensor $E(\nabla\varphi) = \frac{1}{2}(\nabla\varphi^T\nabla\varphi - I) = \frac{1}{2}(\nabla u^T + \nabla u + \nabla u^T\nabla u)$.

Note that $Q = \begin{bmatrix} -I & \overline{u}^T\tilde{I} \end{bmatrix}$ is in the left null space of the leading coefficient of (62b). Multiplying by Q gives a new constraint involving first t derivatives of the state *but second t derivatives of r. Thus we expect the solution, whatever it is, to involve at least first derivatives of the forcing terms r. This suggests the perturbation index would be at least two. This is consistent with the observation that modal approximations of constrained flexible mechanical systems are often the same index as the inflexible DAE systems which are known to usually have index two or three.*

6 CONCLUSION

The simulation of finite dimensional approximations can not be interpreted without taking into account properties of the infinite dimensional model. We have discussed one approach to defining the index of an infinite dimensional system. Applying this approach to a class of liner time invariant PDAEs we have defined an index, ν_P^∞ which describes the continuity of the solution in terms of forcing functions and the data. Examples show that even for linear time invariant PDAEs, this index can be different than the index of the modal equations or the index of the generalized resolvent $R(s, z)$ of the PDAE. Examples have also been given to show that the index ν_P^∞ can depend on the domain of the PDAE and on the particular choice of how the initial conditions are specified.

ν_P^∞ is a scalar quantity. A vector valued expression may be more useful in some cases particularly if different types of approximations are being used in the different variables which is usually the case with MOL.

Several differences between the finite dimensional and the infinite dimensional theory have been pointed out.

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