Eigenstructure Assignment by Displacement–Acceleration Feedback for Second-Order Systems

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This paper presents a new technique for controlling the dynamic response of second-order systems by means of combined displacement and acceleration feedback. The necessary conditions that guarantee the solvability for the problem are formulated. Parametric expressions for the displacement–acceleration gains and the eigenvector matrix are derived. The solution can be applied for the systems with nonsingular or singular mass matrices. Based on the simulation results, we can conclude that the proposed technique is effective. [DOI: 10.1115/1.4032877]

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1 Introduction

Consider the matrix second-order linear system

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{D}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) = \boldsymbol{C}\boldsymbol{u}(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0,\ \dot{\boldsymbol{x}}(0) = \dot{\boldsymbol{x}}_0 \quad (1)$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the displacement vector, $\boldsymbol{u}(t) \in \mathbb{R}^r$ is the control vector, $\boldsymbol{M}, \boldsymbol{D}, \boldsymbol{K} \in \mathbb{R}^{n\times n}$ are, respectively, the matrices of mass, damping, and stiffness, and $\boldsymbol{C} \in \mathbb{R}^{n\times r}$ is the control matrix. Second-order models arise naturally in the study of a wide variety of practical applications, such as control of multibody systems and vibration control. Maintaining the stability of second-order system via displacement–velocity feedback, $\boldsymbol{u}(t) = -\boldsymbol{F}_d\boldsymbol{x}(t) - \boldsymbol{F}_v\dot{\boldsymbol{x}}(t)$, has attracted growing interest in the last few decades; see Refs. [1–10]. Several algorithms have been established using eigenstructure assignment (ESA) in Refs. [1–8] and robust pole assignment in Refs. [9] and [10]. For second-order system, the displacement–velocity feedback is used to attain a desired performance. However, the velocities, for several applications, are not as easily and precisely obtainable as accelerations. Accelerometers have some good properties, such as being easy to install and high sensitivity, and are thus widely used in practice. Acceleration is often easier to measure than displacement or velocity, particularly when the structure is stiff [11]. So, the available signals for feedback are displacements and accelerations.

This paper presents the application of displacement–acceleration feedback for second-order models. There has been little work utilizing this feedback in the literature. Recently, the partial ESA for undamped vibration systems uses displacement–acceleration feedback is introduced in Ref. [12]. On the other hand, the velocity–acceleration feedback, $\boldsymbol{u}(t) = -\boldsymbol{F}_v\dot{\boldsymbol{x}}(t) - \boldsymbol{F}_a\ddot{\boldsymbol{x}}(t)$, has received attention in the last few years; see Refs. [13–16]. Moreover, the derivative feedback methodology that utilizes the velocity and acceleration variables for first-order state-space systems has received attention in the last decade, see Refs. [17,18]. Also, the proportional-derivative control for first-order systems is utilized in Ref. [19].

To this end, the displacement–acceleration feedback in a general form can be described as

$$\boldsymbol{u}(t) = -\boldsymbol{F}_d\boldsymbol{x}(t) - \boldsymbol{F}_a\ddot{\boldsymbol{x}}(t) \quad (2)$$

where $\boldsymbol{F}_d,\ \boldsymbol{F}_a \in \mathbb{R}^{r\times n}$ are, respectively, displacement and acceleration gains. So, the closed-loop system is

$$(\boldsymbol{M} + \boldsymbol{C}\boldsymbol{F}_a)\ddot{\boldsymbol{x}}(t) + \boldsymbol{D}\dot{\boldsymbol{x}}(t) + (\boldsymbol{K} + \boldsymbol{C}\boldsymbol{F}_d)\boldsymbol{x}(t) = 0 \quad (3)$$

The effect of the closed-loop control is therefore to modify the mass and stiffness parameters. Thus, this feedback can permit the treatment of systems involving singular mass matrices, see Refs. [13,14,20–26]. Descriptor second-order models arise naturally in several practical applications. The impulsive natural response is an important characteristic of descriptor models. In fact, impulsive terms may destroy the system and hence it is significant to eliminate the impulsive response using certain feedback. The use of zero-mass points, for example, to denote a connection between two dampers or two springs may be the cause of singularities in the system [21]. The treatment of some forces as unknowns of the problem may also introduce singularities in the system [21]. The mathematical strategies permitting the treatment of second-order systems involving singular mass matrices are discussed in Ref. [22]. The equation of motion for constrained mechanical systems with singular mass matrices is studied in Ref. [23]. In addition, when $\boldsymbol{M}$ is invertible but contains very small terms such that $\boldsymbol{M}$ is ill-conditioned, it is a common practice to set these terms to zero, rendering $\boldsymbol{M}$ singular [24]. The controllability and observability conditions for second-order descriptor systems are analyzed in Ref. [25]. The output regulation for descriptor system is presented in Ref. [26]. Recently, the robust pole assignment and ESA problems using velocity–acceleration feedback for second-order system with singular mass matrix are solved in Refs. [13] and [14].

The aim of this work is to present a new procedure for ESA controller design by displacement–acceleration feedback for second-order system. The necessary conditions that guarantee the solvability are formulated. Parametric expressions for gains and eigenvector matrix are derived. Both the cases of singular and nonsingular mass matrices are discussed.

2 Problem Formulation

For system (1), the corresponding matrix polynomial, $\boldsymbol{P}_o(\lambda) = \lambda^2\boldsymbol{M} + \lambda\boldsymbol{D} + \boldsymbol{K},\ \boldsymbol{P}_o(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$. System (1) is called regular if $\deg(\det(\boldsymbol{P}_o)) = 2n$. The characteristic frequencies are the zeroes of $\det(\boldsymbol{P}_o) = \alpha_{2n}\lambda^{2n} + \cdots + \alpha_1\lambda + \alpha_0 = 0,\ \alpha_{2n} = \det(\boldsymbol{M})$ and $\alpha_0 = \det(\boldsymbol{K})$. System (1) possesses infinite eigenvalues if $\det(\boldsymbol{M}) = 0$. Let n_f and n_∞ denote the finite eigenvalues and the eigenvalues at infinity, respectively, then $n_f + n_\infty = 2n$ and $n_f = \deg(\det(\boldsymbol{P}_o))$. Applying the Laplace transform to system (1), we can obtain the corresponding transfer function

$$\boldsymbol{H}_o(\lambda) = (\lambda^2\boldsymbol{M} + \lambda\boldsymbol{D} + \boldsymbol{K})^{-1}\boldsymbol{C}.$$

The behavior of closed-loop system (3) is determined by the corresponding quadratic polynomial $\boldsymbol{P}_c(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$

$$\boldsymbol{P}_c(\lambda) = \lambda^2(\boldsymbol{M} + \boldsymbol{C}\boldsymbol{F}_a) + \lambda\boldsymbol{D} + \boldsymbol{K} + \boldsymbol{C}\boldsymbol{F}_d \quad (4)$$

Let $\Gamma = \{\lambda_i \in \mathbb{C},\ i = 1, 2, \ldots, 2n\}$ be a set of prespecified, self-conjugate, eigenvalues. Further, denote the right eigenvector associated with λ_i by $\boldsymbol{v}_i \in \mathbb{C}^n,\ i = 1, 2, \ldots, 2n$, then the following relations hold:

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$$(\lambda_i^2(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)+\lambda_i\boldsymbol{D}+\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)\boldsymbol{v}_i=0,\ \ \boldsymbol{v}_i\neq 0,\ i=1,2,\ \ldots,2n \tag{5}$$

Denote

$$\boldsymbol{V}=[\boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots,\boldsymbol{v}_{2n}]\in\mathbb{C}^{n\times 2n},\ \ \boldsymbol{\Lambda}=\mathrm{diag}(\lambda_1,\lambda_2,\ \ldots,\lambda_{2n})\in\mathbb{C}^{2n\times 2n}$$

where the columns of $\boldsymbol{V}$ comprise the right eigenvectors of $\boldsymbol{P}_c$ and $\boldsymbol{\Lambda}$ is in Jordan canonical form with the eigenvalues of $\boldsymbol{P}_c$ on the diagonal. There exists matrix $\boldsymbol{V}$ that satisfying

$$(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)\boldsymbol{V}\boldsymbol{\Lambda}^2+\boldsymbol{D}\boldsymbol{V}\boldsymbol{\Lambda}+(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)\boldsymbol{V}=0 \tag{6}$$

We now formulate the problem as: Given system (1) and desired set Γ, find the real gains $\boldsymbol{F}_d$ and $\boldsymbol{F}_a$ such that the closed-loop system has admissible eigenvalues and associated eigenvectors.

3 Necessary Conditions for Solvability

This section is concerned with the necessary conditions that guarantee the solvability for the problem.

Definition 1. *System (1) is called regularizable via displacement–acceleration controller (2) if* $\deg(\det(\boldsymbol{P}_c))=2n$.

Matrix $\boldsymbol{P}_c$ is stable if $\det(\boldsymbol{P}_c)$ has all roots in $\mathbb{C}^-$. If $\boldsymbol{P}_c(\lambda=0)$ is nonsingular, then $\det(\boldsymbol{P}_c)$ has no roots at the origin. On the other hand, if $\boldsymbol{P}_c(\lambda=0)$ is singular, then $\det(\boldsymbol{P}_c(0))$ is zero, and $\boldsymbol{P}_c(\lambda)$ has a root at the origin so that $\boldsymbol{P}_c$ is unstable.

In the following, both the cases of singular and nonsingular mass matrices are discussed.

3.1 Case of Singular Mass Matrix.

First, the controllability of descriptor system (1) is presented, see Ref. [25].

Lemma 1. *A descriptor second-order linear system (1) is*

(i) $\Re2$-*controllable if and only if* $\mathrm{rank}([\lambda^2\boldsymbol{M}+\lambda\boldsymbol{D}+\boldsymbol{K},\ \ \boldsymbol{C}])=n,\ \forall\lambda\in\mathbb{C}$
(ii) *strongly C2-controllable if and only if* $\mathrm{rank}([\boldsymbol{M},\ \ \boldsymbol{C}])=n$

The following result can now be stated.

Lemma 2. *If a system (1) is controllable, then the resulting closed-loop system (3) is also controllable.*

Proof. For the resulting closed-loop system (3), one can obtain that

$$\begin{aligned}&[\lambda^2(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)+\lambda\boldsymbol{D}+\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d,\ \ \boldsymbol{C}]\\&=[\lambda^2\boldsymbol{M}+\lambda\boldsymbol{D}+\boldsymbol{K},\ \ \boldsymbol{C}]\begin{pmatrix}\boldsymbol{I}_n & 0_{n,r}\\ \lambda^2\boldsymbol{F}_a+\boldsymbol{F}_d & \boldsymbol{I}_r\end{pmatrix},\quad\forall\lambda\in\mathbb{C}\\&[\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a,\ \ \boldsymbol{C}]=[\boldsymbol{M},\ \boldsymbol{C}]\begin{pmatrix}\boldsymbol{I}_n & 0_{n,r}\\ \boldsymbol{F}_a & \boldsymbol{I}_r\end{pmatrix},\quad\forall\boldsymbol{F}_a\end{aligned}$$

This implies that

$$\begin{aligned}&\mathrm{rank}([\lambda^2(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)+\lambda\boldsymbol{D}+\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d,\ \ \boldsymbol{C}])\\&\quad=\mathrm{rank}([\lambda^2\boldsymbol{M}+\lambda\boldsymbol{D}+\boldsymbol{K},\ \ \boldsymbol{C}]),\ \forall\lambda\in\mathbb{C}\\&\mathrm{rank}([\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a,\ \ \boldsymbol{C}])=\mathrm{rank}([\boldsymbol{M},\ \boldsymbol{C}]),\ \forall\boldsymbol{F}_a\end{aligned} \tag{7}$$

Hence, the displacement–acceleration controller (2) does not change the controllability of descriptor system (1). ■

In the following theorem, we establish the necessary conditions that guarantee that system (3) is regular.

Theorem 1. *Consider a descriptor system (1) where* $0<\mathrm{rank}(\boldsymbol{M})=q<n,\ \ 0<\mathrm{rank}(\boldsymbol{K})=g<n$ *and* $\mathrm{rank}(\boldsymbol{C})=r$. *This system is regularizable via displacement–acceleration controller (2) if*

(1) *All eigenvalues are finite and nonzero.*
(2) $\mathrm{Rank}([\boldsymbol{M},\ \ \boldsymbol{C}])=n.$
(3) $\mathrm{Rank}([\boldsymbol{K},\ \ \boldsymbol{C}])=n.$

Then the regularizing acceleration gain and the stabilizing displacement gain are given by

$$\boldsymbol{F}_a=[0_{r,q},\ \ \boldsymbol{C}_2^{\mathrm{T}}]\boldsymbol{T}_2 \tag{8}$$

$$\boldsymbol{F}_d=[0_{r,g},\ \ \boldsymbol{C}_4^{\mathrm{T}}]\boldsymbol{T}_4 \tag{9}$$

where $\boldsymbol{T}_2,\ \boldsymbol{T}_4\in\mathbb{R}^{n\times n}$ are orthogonal while $\boldsymbol{C}_2\in\mathbb{R}^{(n-q)\times r}$ and $\boldsymbol{C}_4\in\mathbb{R}^{(n-g)\times r}$ are of full-row rank.

Proof. The characteristic polynomial of closed-loop system (3) can be expanded as

$$\det(\boldsymbol{P}_c(\lambda))=a_{2n}\lambda^{2n}+a_{2n-1}\lambda^{2n-1}+\cdots+a_1\lambda+a_0$$

where $a_{2n}=\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)$ and $a_0=\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)$. Observe that $\boldsymbol{P}_c$ has $2n$ finite eigenvalues as long as $\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)\neq 0$. Otherwise, if $(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)$ is singular, then $\deg(\det(\boldsymbol{P}_c))=d<2n$ (system has $2n$–d infinite eigenvalues). So, $\boldsymbol{F}_a$ is restricted to ensure that $\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)\neq 0$. It is easy to rewrite an equivalent form of $\det(\boldsymbol{P}_c)$ as a finite product as

$$\begin{aligned}\det(\boldsymbol{P}_c(\lambda))&=a_{2n}(\lambda-\lambda_1)(\lambda-\lambda_2)\ \cdots\ (\lambda-\lambda_{2n})\\&=a_{2n}\lambda^{2n}+\cdots+a_{2n}\prod_{i=1}^{2n}\lambda_i\end{aligned} \tag{10}$$

where $\lambda_1,\ \lambda_2,.,\ \lambda_{2n}$ are zeroes of $\det(\boldsymbol{P}_c)$. Remark that

$$\frac{a_0}{a_{2n}}=\prod_{i=1}^{2n}\lambda_i\ \ \text{or}\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)\prod_{i=1}^{2n}\lambda_i=\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)$$

To guarantee that the system (3) is regular, then

$$\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)=\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)\prod_{i=1}^{2n}\lambda_i^{-1}\neq 0 \tag{11}$$

or $\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)\neq 0$ and $\lambda_i\neq 0,\ \forall i$.

The singular value decomposition (SVD) of $\boldsymbol{M}$ can be obtained as

$$\boldsymbol{M}=\boldsymbol{T}_1\begin{pmatrix}\boldsymbol{\Sigma}_M & 0_{q,(n-q)}\\ 0_{(n-q),q} & 0_{(n-q),(n-q)}\end{pmatrix}\boldsymbol{T}_2 \tag{12}$$

where $\boldsymbol{T}_1,\ \boldsymbol{T}_2\in\mathbb{R}^{n\times n}$ are orthogonal, $\boldsymbol{\Sigma}_M=\mathrm{diag}\{\sigma_{M1},\ \ldots,\ \sigma_{Mq}\}\in\mathbb{R}^{q\times q}$ is the singular values matrix of $\boldsymbol{M}$ and $\sigma_{M1}\geq\cdots\geq\sigma_{Mq}>0$. Furthermore, define

$$\boldsymbol{T}_1^{\mathrm{T}}\boldsymbol{C}=\begin{pmatrix}\boldsymbol{C}_1\\ \boldsymbol{C}_2\end{pmatrix},\ \boldsymbol{C}_1\in\mathbb{R}^{q\times r}\text{ and }\boldsymbol{C}_2\in\mathbb{R}^{(n-q)\times r} \tag{13}$$

Utilizing Eqs. (12) and (13), one can obtain

$$\begin{aligned}\mathrm{rank}([\boldsymbol{M},\ \boldsymbol{C}])&=\mathrm{rank}\left\{\boldsymbol{T}_1^{\mathrm{T}}[\boldsymbol{M},\ \ \boldsymbol{C}]\begin{pmatrix}\boldsymbol{T}_2^{\mathrm{T}} & 0_{n,r}\\ 0_{r,n} & \boldsymbol{I}_r\end{pmatrix}\right\}\\&=\mathrm{rank}\left(\begin{array}{cc:c}\Sigma_M & 0_{q,(n-q)} & \boldsymbol{C}_1\\ 0_{(n-q),q} & 0_{(n-q),(n-q)} & \boldsymbol{C}_2\end{array}\right)\end{aligned}$$

Obviously, this matrix has full-row rank only if $\boldsymbol{C}_2$ is of full-row rank (system (1) is strongly C2-controllable). It is known that any right invertible matrix multiplied by its transpose is positive definite. Therefore, $\boldsymbol{C}_2\boldsymbol{C}_2^{\mathrm{T}}\in\mathbb{R}^{(n-q)\times(n-q)}$ is nonsingular and $1\leq n$–$q<n$. It is easily verified that

$$\begin{aligned}\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a) &= \det(\boldsymbol{M}+\boldsymbol{C}[0_{r,q},\ \boldsymbol{C}_2^{\mathrm{T}}]\boldsymbol{T}_2)\\ &= \det(\boldsymbol{T}_1)\det(\boldsymbol{T}_2)\det(\boldsymbol{T}_1^{\mathrm{T}}\boldsymbol{M}\boldsymbol{T}_2^{\mathrm{T}}+\boldsymbol{T}_1^{\mathrm{T}}\boldsymbol{C}[0_{r,q},\ \boldsymbol{C}_2^{\mathrm{T}}])\\ &= \det(\boldsymbol{T}_1)\det(\boldsymbol{T}_2)\det\begin{pmatrix}\boldsymbol{\Sigma}_M & \boldsymbol{C}_1\boldsymbol{C}_2^{\mathrm{T}}\\ 0_{(n-q),q} & \boldsymbol{C}_2\boldsymbol{C}_2^{\mathrm{T}}\end{pmatrix}\\ &= \det(\boldsymbol{C}_2\boldsymbol{C}_2^{\mathrm{T}})\prod_{i=1}^{q}\sigma_{Mi}\\ &\neq 0.\end{aligned}$$

Hence, Eq. (8) is indeed a regularizing acceleration controller.

Without loss of generality, there exist orthogonal matrices $\boldsymbol{T}_3,\ \boldsymbol{T}_4 \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{K} = \boldsymbol{T}_3\begin{pmatrix}\boldsymbol{\Sigma}_K & 0_{g,(n-g)}\\ 0_{(n-g),g} & 0_{(n-g),(n-g)}\end{pmatrix}\boldsymbol{T}_4 \tag{14}$$

where $\boldsymbol{\Sigma}_K = \mathrm{diag}\{\sigma_{K1},\ \ldots,\ \sigma_{Kg}\} \in \mathbb{R}^{g\times g}$ is the singular values matrix of $\boldsymbol{K}$ and $\sigma_{K1} \geq \cdots \geq \sigma_{Kg} > 0$. Further, let

$$\boldsymbol{T}_3^{\mathrm{T}}\boldsymbol{C} = \begin{pmatrix}\boldsymbol{C}_3\\ \boldsymbol{C}_4\end{pmatrix},\ \boldsymbol{C}_3 \in \mathbb{R}^{g\times r} \text{ and } \boldsymbol{C}_4 \in \mathbb{R}^{(n-g)\times r} \tag{15}$$

Then under the assumption that $\mathrm{rank}([\boldsymbol{K},\ \boldsymbol{C}]) = n$ and utilizing Eqs. (14) and (15), one can obtain

$$\begin{aligned}\mathrm{rank}([\boldsymbol{K},\ \boldsymbol{C}]) &= \mathrm{rank}\left\{\boldsymbol{T}_3^{\mathrm{T}}[\boldsymbol{K},\ \ \boldsymbol{C}]\begin{pmatrix}\boldsymbol{T}_4^{\mathrm{T}} & 0_{n,r}\\ 0_{r,n} & \boldsymbol{I}_r\end{pmatrix}\right\}\\ &= \mathrm{rank}\begin{pmatrix}\Sigma_K & 0_{g,(n-g)} & \boldsymbol{C}_3\\ 0_{(n-g),g} & 0_{(n-g),(n-g)} & \boldsymbol{C}_4\end{pmatrix}\end{aligned}$$

Then this matrix has full-row rank only if $\boldsymbol{C}_4$ is of full-row rank. Thus, $\boldsymbol{C}_4\boldsymbol{C}_4^{\mathrm{T}} \in \mathbb{R}^{(n-g)\times(n-g)}$ is positive definite and $1 \leq n–g < n$. It follows that

$$\begin{aligned}\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d) &= \det(\boldsymbol{K}+\boldsymbol{C}[0_{r,g},\ \boldsymbol{C}_4^{\mathrm{T}}]\boldsymbol{T}_4)\\ &= \det(\boldsymbol{T}_3)\det(\boldsymbol{T}_4)\det(\boldsymbol{T}_3^{\mathrm{T}}\boldsymbol{K}\boldsymbol{T}_4^{\mathrm{T}}+\boldsymbol{T}_3^{\mathrm{T}}\boldsymbol{C}[0_{r,g},\ \boldsymbol{C}_4^{\mathrm{T}}])\\ &= \det(\boldsymbol{T}_3)\det(\boldsymbol{T}_4)\det\begin{pmatrix}\boldsymbol{\Sigma}_K & \boldsymbol{C}_3\boldsymbol{C}_4^{\mathrm{T}}\\ 0_{(n-g),g} & \boldsymbol{C}_4\boldsymbol{C}_4^{\mathrm{T}}\end{pmatrix}\\ &= \det(\boldsymbol{C}_4\boldsymbol{C}_4^{\mathrm{T}})\prod_{i=1}^{g}\sigma_{Ki}\\ &\neq 0\end{aligned}$$

Thus, Eq. (9) is indeed a stabilizing displacement controller such that the system eigenvalues are nonzero. ∎

Now, the following assumptions are proposed for descriptor system (1).

Assumption 1. *rank(*$[\boldsymbol{M}, \boldsymbol{C}]$*) = n.*

Assumption 2. *rank(*$[\boldsymbol{K}, \boldsymbol{C}]$*) = n.*

Assumption 3. *rank(*$\boldsymbol{C}$*) = r.*

Assumption 4. $\lambda_i \neq 0$*, ∀i, and closed under complex conjugation.*

Example 1. Consider a two degrees-of-freedom (DOF) descriptor system

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\ddot{\boldsymbol{x}}(t) + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\dot{\boldsymbol{x}}(t) + \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\boldsymbol{x}(t) = \begin{pmatrix}0\\ 1\end{pmatrix}u(t)$$

This system is not regular ($\det(\boldsymbol{P}_o) = 0,\ \forall\lambda \in \mathbb{C}$) and satisfies Assumptions 1–3. Suppose that the control $u(t) = -\boldsymbol{F}_d\boldsymbol{x}(t) -\boldsymbol{F}_a\ddot{\boldsymbol{x}}(t),\ \boldsymbol{F}_d = [f_{d1},\ f_{d2}],\ \boldsymbol{F}_a = [f_{a1},\ f_{a2}]$. Thus, the closed-loop characteristic polynomial is $\det(\boldsymbol{P}_c) = f_{a2}\lambda^4 + (f_{d2}+f_{a2})\lambda^2 + f_{d2}$. The zeros of $\det(\boldsymbol{P}_c)$ are computed as $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \pm\sqrt{-f_{d2}/f_{a2}}$. Furthermore, $\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a) = f_{a2}$ and $\det(\boldsymbol{K}+\boldsymbol{C}\boldsymbol{F}_d)\prod_{i=1}^{2n}\lambda_i^{-1} = f_{a2}$. So, two cases can arise as follows:

(1) One can verify that $\deg(\det(\boldsymbol{P}_c)) = 2n$ provided that the gain element $f_{a2} \neq 0$. Obviously, the conditions in Theorem 1 are satisfied and the closed-loop system is regular ($\lambda_i \neq \infty,\ \forall i$).

(2) If $f_{a2} = 0$, then $\det(\boldsymbol{P}_c) = (\lambda^2+1)f_{d2}$ and $\deg(\det(\boldsymbol{P}_c)) = 2 < 2n$ whenever $f_{d2} \neq 0$. In this case, the eigenvalues of closed-loop system are located as: $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \infty$.

3.1.1 System With Model Uncertainty. In the presence of uncertainty in the system matrices, where the system is modeled as

$$(\boldsymbol{M}+\Delta\boldsymbol{M})\ddot{\boldsymbol{x}}(t) + (\boldsymbol{D}+\Delta\boldsymbol{D})\dot{\boldsymbol{x}}(t) + (\boldsymbol{K}+\Delta\boldsymbol{K})\boldsymbol{x}(t) = \boldsymbol{C}\boldsymbol{u}(t)$$

with the uncertainty matrices $\Delta\boldsymbol{M}$, $\Delta\boldsymbol{D}$, $\Delta\boldsymbol{K}$. Similar to the results of Theorem 1, the system is regularizable via displacement–acceleration controller (2) if

(1) $\mathrm{Rank}([\boldsymbol{M}+\Delta\boldsymbol{M}, \boldsymbol{C}]) = n$
(2) $\mathrm{Rank}([\boldsymbol{K}+\Delta\boldsymbol{K}, \boldsymbol{C}]) = n$.
(3) $\lambda_i \neq 0$, $\forall i$, and closed under complex conjugation.

3.2 Case of Nonsingular Mass Matrix. In this case, $\mathrm{rank}(\boldsymbol{M}) = n$. Then Assumption 1 is simplified to system (1) is $\Re 2$ -controllable and Assumptions 2–4 are essential to ensure the stability of system and the controllers are real.

4 Displacement–Acceleration Controllers

In this section, the parametric expressions for controller gains and eigenvector matrix are derived when the mass matrix of the system under consideration is singular or nonsingular.

First, the system dynamics (5) can be expressed as

$$(\lambda_i^2\boldsymbol{M}+\lambda_i\boldsymbol{D}+\boldsymbol{K})\boldsymbol{v}_i + \boldsymbol{C}(\lambda_i^2\boldsymbol{F}_a+\boldsymbol{F}_d)\boldsymbol{v}_i = 0,\ \ \boldsymbol{v}_i \neq 0,\ i = 1,\ \ldots,\ 2n$$

This equation can be rewritten as

$$\begin{pmatrix}\lambda_i^2\boldsymbol{M}+\lambda_i\boldsymbol{D}+\boldsymbol{K}, & \boldsymbol{C}\end{pmatrix}\begin{pmatrix}\boldsymbol{v}_i\\ \boldsymbol{w}_i\end{pmatrix} = 0,\quad i = 1,\ \ldots,\ 2n \tag{16}$$

where

$$\boldsymbol{w}_i = (\lambda_i^2\boldsymbol{F}_a+\boldsymbol{F}_d)\boldsymbol{v}_i,\ \ i = 1,\ \ldots,\ 2n \tag{17}$$

First, the SVD for $\begin{pmatrix}\lambda_i^2\boldsymbol{M}+\lambda_i\boldsymbol{D}+\boldsymbol{K}, & \boldsymbol{C}\end{pmatrix} \in \mathbb{C}^{n\times(n+r)}$, $\forall i$, can be obtained as

$$\begin{pmatrix}\lambda_i^2\boldsymbol{M}+\lambda_i\boldsymbol{D}+\boldsymbol{K}, & \boldsymbol{C}\end{pmatrix} = \boldsymbol{E}_i[\boldsymbol{\Sigma}_i,\ 0_{n,r}]\boldsymbol{Q}_i^{\mathrm{H}},\ i = 1,\ \ldots,\ 2n \tag{18}$$

where $\boldsymbol{E}_i \in \mathbb{C}^{n\times n}$ and $\boldsymbol{Q}_i \in \mathbb{C}^{(n+r)\times(n+r)}$ are orthogonal, $\boldsymbol{\Sigma}_i = \mathrm{diag}\{\sigma_{i1}, \ldots, \sigma_{in}\} \in \mathbb{C}^{n\times n}$ is nonsingular, $\sigma_{i1} \geq \ldots \geq \sigma_{in} > 0$ and superscript H denotes complex-conjugate and transpose. Further, partition $\boldsymbol{Q}_i$ as $\boldsymbol{Q}_i = \begin{pmatrix}\boldsymbol{Q}_{i,11} & \boldsymbol{Q}_{i,12}\\ \boldsymbol{Q}_{i,21} & \boldsymbol{Q}_{i,22}\end{pmatrix}$, $\boldsymbol{Q}_{i,12} \in \mathbb{C}^{n\times r}$ and $\boldsymbol{Q}_{i,22} \in \mathbb{C}^{r\times r}, \forall i$. Thus, the parametric solution for the problem is obtained.

Theorem 2. *Consider a system (1) and a set* Γ *satisfying Assumptions 1–4. The parametric expressions for* $\boldsymbol{v}_i$ *and* $\boldsymbol{w}_i$*,* $\forall i$*, in (16) are expressed by*

$$\begin{pmatrix}\boldsymbol{v}_i\\ \boldsymbol{w}_i\end{pmatrix} = \begin{pmatrix}\boldsymbol{Q}_{i,12}\\ \boldsymbol{Q}_{i,22}\end{pmatrix}\boldsymbol{f}_i,\ \ i = 1,\ 2, \ldots,\ 2n \tag{19}$$

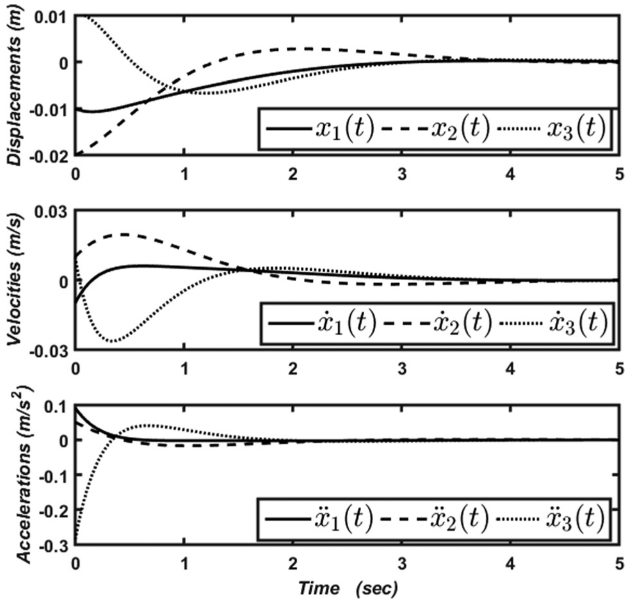

Fig. 1 Closed-loop system response of a 3DOF system for Solution 2

Accordingly, the parametric expression for gains can be expressed as

$$(\boldsymbol{F}_d, \quad \boldsymbol{F}_a) = (\boldsymbol{Q}_{1,22}\boldsymbol{f}_1, \boldsymbol{Q}_{2,22}\boldsymbol{f}_2, \ldots, \boldsymbol{Q}_{2n,22}\boldsymbol{f}_{2n}) \times \begin{pmatrix} \boldsymbol{Q}_{1,12}\boldsymbol{f}_1, & \boldsymbol{Q}_{2,12}\boldsymbol{f}_2, & \ldots, & \boldsymbol{Q}_{2n,12}\boldsymbol{f}_{2n} \\ \lambda_1^2\boldsymbol{Q}_{1,12}\boldsymbol{f}_1, & \lambda_2^2\boldsymbol{Q}_{2,12}\boldsymbol{f}_2, & \ldots, & \lambda_{2n}^2\boldsymbol{Q}_{2n,12}\boldsymbol{f}_{2n} \end{pmatrix}^{-1} \tag{20}$$

where $\boldsymbol{f}_i \in \mathbb{C}^r, \ \forall i$, are parameter vectors satisfying the following constraints:

Constraint 1: $\boldsymbol{f}_k = \boldsymbol{f}_i^*$ when $\lambda_k = \lambda_i^*, \ \forall i, k$.

Constraint 2: $\det\begin{pmatrix} \boldsymbol{Q}_{1,12}\boldsymbol{f}_1, & \boldsymbol{Q}_{2,12}\boldsymbol{f}_2 & \ldots, \boldsymbol{Q}_{2n,12}\boldsymbol{f}_{2n} \\ \lambda_1^2\boldsymbol{Q}_{1,12}\boldsymbol{f}_1, & \lambda_2^2\boldsymbol{Q}_{2,12}\boldsymbol{f}_2, & \ldots, \lambda_{2n}^2\boldsymbol{Q}_{2n,12}\boldsymbol{f}_{2n} \end{pmatrix} \neq 0$.

Proof. First, postmultiplying Eq. (18) by $\begin{pmatrix} \boldsymbol{Q}_{i,12} \\ \boldsymbol{Q}_{i,22} \end{pmatrix}$ yields

$$\begin{aligned} (\lambda_i^2\boldsymbol{M} + \lambda_i\boldsymbol{D} + \boldsymbol{K}, \quad \boldsymbol{C})\begin{pmatrix} \boldsymbol{Q}_{i,12} \\ \boldsymbol{Q}_{i,22} \end{pmatrix} &= \boldsymbol{E}_i\left[\sum\nolimits_i, \ 0_{n,r}\right]\boldsymbol{Q}_i^{\mathrm{H}}\begin{pmatrix} \boldsymbol{Q}_{i,12} \\ \boldsymbol{Q}_{i,22} \end{pmatrix} \\ &= \boldsymbol{E}_i\left[\sum\nolimits_i, \ 0_{n,r}\right]\begin{pmatrix} 0_{n,r} \\ \boldsymbol{I}_r \end{pmatrix} \\ &= 0, \qquad \forall i \end{aligned}$$

Then the columns of $\begin{pmatrix} \boldsymbol{Q}_{i,12} \\ \boldsymbol{Q}_{i,22} \end{pmatrix}$ form a set of orthogonal bases for $\ker(\lambda_i^2\boldsymbol{M} + \lambda_i\boldsymbol{D} + \boldsymbol{K}, \quad \boldsymbol{C}), \ \forall i$. Then, Eq. (19) holds. Denote $\boldsymbol{W} = [\boldsymbol{w}_1, \ \boldsymbol{w}_2, \ldots, \ \boldsymbol{w}_{2n}] \in \mathbb{C}^{r\times 2n}$, then it is straightforward to put Eq. (17) as

$$\boldsymbol{W} = \boldsymbol{F}_a\boldsymbol{V}\boldsymbol{\Lambda}^2 + \boldsymbol{F}_d\boldsymbol{V} = (\boldsymbol{F}_d, \quad \boldsymbol{F}_a)\begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{V}\boldsymbol{\Lambda}^2 \end{pmatrix}$$

Consequently, the gains are $(\boldsymbol{F}_d, \quad \boldsymbol{F}_a) = \boldsymbol{W}\begin{pmatrix} \boldsymbol{V} \\ \boldsymbol{V}\boldsymbol{\Lambda}^2 \end{pmatrix}^{-1}$.

Utilizing Eq. (19) and $\boldsymbol{\Lambda}^2 = \mathrm{diag}(\lambda_1^2, \lambda_2^2, \ \ldots, \ \lambda_{2n}^2) \in \mathbb{C}^{2n\times 2n}$, the parametric expression for gains can be computed as Eq. (20). To ensure that the gains are real, then for the complex-conjugate eigenvalues λ_i and λ_k the corresponding vectors $\boldsymbol{f}_i$ and $\boldsymbol{f}_k$ must also be complex-conjugates, i.e., $\boldsymbol{f}_k = \boldsymbol{f}_i^*$. ■

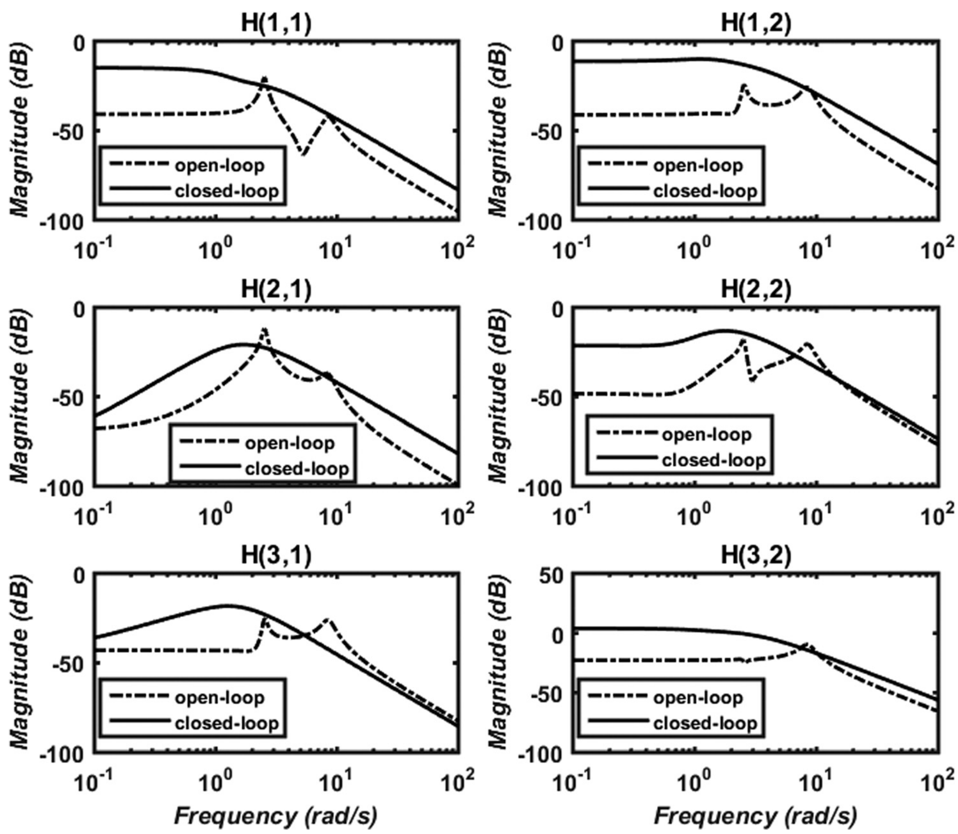

Fig. 2 Open- and closed-loop frequency response of a 3DOF system for Solution 2

Table 1 Simulation results for Example 2

Solutions	$\|V\|$	$\kappa(V)$	$\|F_d\|$	$\|F_a\|$	Eigenvalues of perturbed closed-loop system	Error norm in poles
1	1.0951	189.1249	118.8937	12.7744	$-0.9961 \pm 1.0065i$, $-1.9656 \pm 1.0276i$, $-3.0125 \pm 1.1451i$	0.2155
2	1.4485	77.7439	118.0446	12.1872	$-0.9719 \pm 1.0054i$, $-2.0197 \pm 1.0777i$, $-2.9931 \pm 1.0618i$	0.1491

Note that the design vectors $\boldsymbol{f}_i$, $\forall i$, represent the DOF offered by displacement–acceleration feedback.

Finally, we can present a numerical algorithm to compute the controller gains.

ALGORITHM

Given system (1) satisfying $\text{rank}([\boldsymbol{M}, \boldsymbol{C}]) = n$, $\text{rank}([\boldsymbol{K}, \boldsymbol{C}]) = n$ and $\text{rank}(\boldsymbol{C}) = r$ and a nonzero, self-conjugate, set Γ.

Step 1: Select the nonzero parameter vectors $\boldsymbol{f}_i$, $\forall i$, satisfying Constraints 1 and 2.

Step 2: Utilize the SVD to get matrices $\boldsymbol{Q}_{i,12}$ and $\boldsymbol{Q}_{i,22}$, $\forall i$, satisfying Eq. (18).

Step 3: Compute the gain controllers according to Eq. (20).

5 Simulation Results

In this section, numerical simulations are conducted to demonstrate and verify the previous results.

Example 2. Consider the analysis of the oscillations of a wing in an air stream in Ref. [10]. The dynamic system equations are given as Eq. (1), where the coefficient matrices are

$$\boldsymbol{M} = \begin{pmatrix} 17.600 & 1.280 & 2.890 \\ 1.280 & 0.824 & 0.413 \\ 2.890 & 0.413 & 0.725 \end{pmatrix}, \quad \boldsymbol{D} = \begin{pmatrix} 7.660 & 2.450 & 2.100 \\ 0.230 & 1.040 & 0.223 \\ 0.600 & 0.756 & 0.658 \end{pmatrix},$$

$$\boldsymbol{K} = \begin{pmatrix} 121.000 & 18.900 & 15.900 \\ 0.000 & 2.700 & 0.145 \\ 11.900 & 3.640 & 15.500 \end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The system eigenvalues are $-0.9180 \pm 1.7606i$, $-0.8848 \pm 8.4415i$, and $0.0947 \pm 2.5229i$. So, the system is unstable and $\det(\boldsymbol{M}) \neq 0$. The simulation will be undertaken for the eigenvalues $-1 \pm i$, $-2 \pm i$, $-3 \pm i$. Here, we have worked out two solutions.

Solution 1. In the first solution, the design vectors $\boldsymbol{f}_i$ can be selected satisfying Constraint 1 as

$$\boldsymbol{f}_1 = \boldsymbol{f}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \boldsymbol{f}_3 = \boldsymbol{f}_4 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{f}_5 = \boldsymbol{f}_6 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then according to Theorem 2, the controllers are computed as

$$\boldsymbol{F}_d = \begin{pmatrix} -116.0885 - 14.8717 - 14.3891 \\ -13.1129 - 2.8075 - 15.0614 \end{pmatrix},$$

$$\boldsymbol{F}_a = \begin{pmatrix} -12.5085 & 0.9440 - 1.4965 \\ -1.8564 & 0.2539 - 0.3210 \end{pmatrix}$$

One can verify, $\det(\boldsymbol{M} + \boldsymbol{C}\boldsymbol{F}_a) = 0.0946 \neq 0$ and $\det(\boldsymbol{K} + \boldsymbol{C}\boldsymbol{F}_d) = 9.4625 \neq 0$.

Solution 2. In this solution, the design vectors are taken as $\boldsymbol{f}_1 = \boldsymbol{f}_2 = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \boldsymbol{f}_3 = \boldsymbol{f}_4 = \begin{pmatrix} 15 \\ 2 \end{pmatrix}, \boldsymbol{f}_5 = \boldsymbol{f}_6 = \begin{pmatrix} -3 \\ 0.5 \end{pmatrix}$, so the gains are

$$\boldsymbol{F}_d = \begin{pmatrix} -115.4595 - 13.9116 - 14.6857 \\ -11.9276 - 2.1398 - 14.7951 \end{pmatrix},$$

$$\boldsymbol{F}_a = \begin{pmatrix} -11.9106 & 1.2451 - 1.2091 \\ -1.8761 & 0.2007 - 0.3385 \end{pmatrix}$$

Here, $\det(\boldsymbol{M} + \boldsymbol{C}\boldsymbol{F}_a) = 0.0941 \neq 0$ and $\det(\boldsymbol{K} + \boldsymbol{C}\boldsymbol{F}_d) = 9.4102 \neq 0$. The system response is displayed in Fig. 1 using the following initial conditions $\boldsymbol{x}_0 = [-0.01, -0.02, 0.01]^{\mathrm{T}}$ m and $\dot{\boldsymbol{x}}_0 = [-0.01, 0.01, 0.01]^{\mathrm{T}}$ m/s. Moreover, the frequency responses of the open-loop system (dotted lines) and closed-loop system (solid lines) are shown in Fig. 2.

It is significant to study the performance of the controllers with respect to perturbations in the system. Assume that the system matrices $\boldsymbol{M}$, $\boldsymbol{D}$, and $\boldsymbol{K}$ are perturbed and the perturbations $\Delta\boldsymbol{M}$, $\Delta\boldsymbol{D}$, and $\Delta\boldsymbol{K}$ are defined as

$\Delta\boldsymbol{M} = 0.001\boldsymbol{M}$, $\Delta\boldsymbol{D} = 0.001\boldsymbol{D}$, and $\Delta\boldsymbol{K} = 0.001\boldsymbol{K}$, satisfying $\text{rank}([\boldsymbol{M} + \Delta\boldsymbol{M}, \boldsymbol{C}]) = n$ and $\text{rank}([\boldsymbol{K} + \Delta\boldsymbol{K}, \boldsymbol{C}]) = n$. Table 1 summarizes the following results $\|\boldsymbol{V}\|$, $\kappa(\boldsymbol{V}) = \|\boldsymbol{V}\|\|\boldsymbol{V}^{-1}\|$, $\|\boldsymbol{F}_d\|$, $\|\boldsymbol{F}_a\|$, eigenvalues of perturbed system and the norm of errors in poles due to perturbation related to the computed solutions. From these results, it is clear that Solution 2 obtains better performance compared with Solution 1. The norm of errors in poles due to perturbations for Solution 2 is significantly reduced by 30.8113% compared with Solution 1.

Example 3. Consider a 4DOF mass-spring-damper with singular mass matrix in Ref. [14]. The coefficient system matrices are

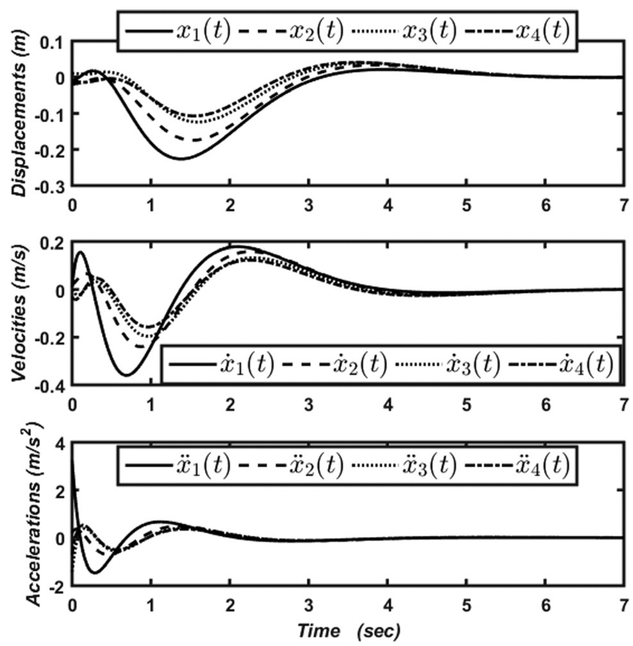

Fig. 3 Closed-loop system response for a 4DOF system

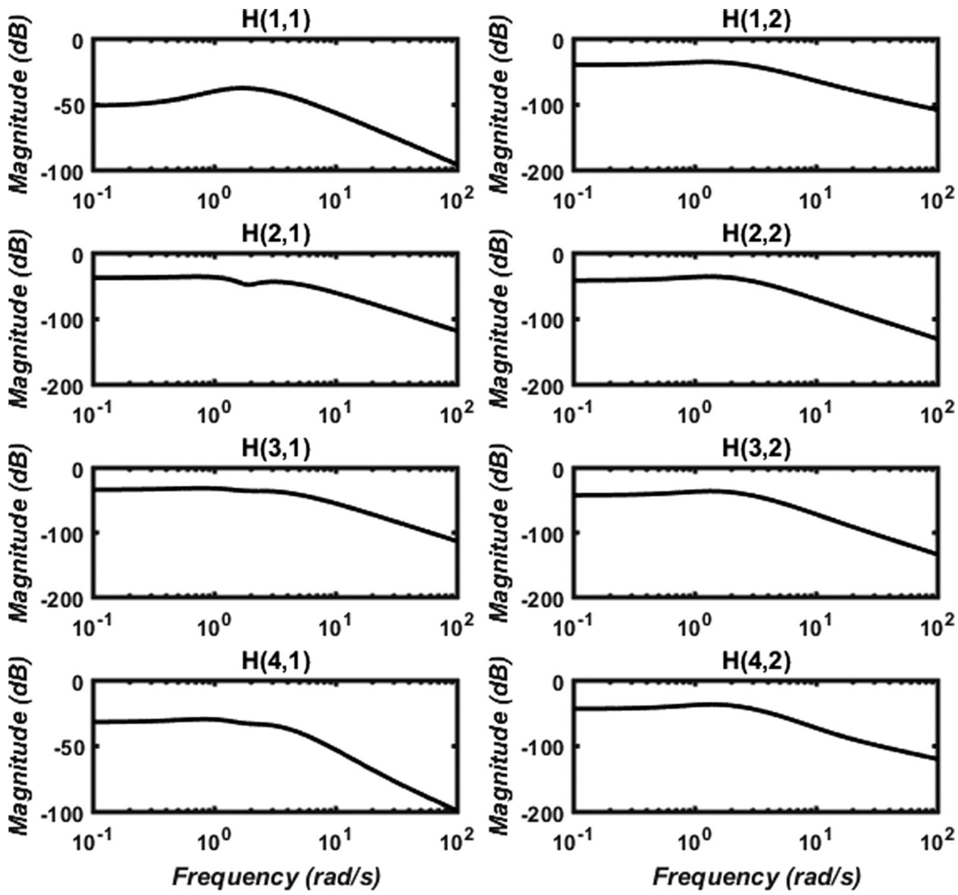

Fig. 4 Closed-loop frequency response for a 4DOF system

$$\boldsymbol{M}=\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&0\end{pmatrix},\quad \boldsymbol{D}=\begin{pmatrix}25&-15&0&0\\-15&35&-20&0\\0&-20&60&-40\\0&0&-40&40\end{pmatrix},$$

$$\boldsymbol{K}=\begin{pmatrix}15&-10&0&0\\-10&25&-15&0\\0&-15&35&-20\\0&0&-20&20\end{pmatrix},\quad \boldsymbol{C}=\begin{pmatrix}1&0\\0&0\\0&0\\0&1\end{pmatrix}$$

One can compute the system characteristic frequencies as

$$\infty, -25.1734, -8.1208, -0.5601\pm 0.5461i,\\ -0.7701, -0.6488, -0.5000$$

Accordingly, $n_f = 7$ and $n_\infty = 1$. Selecting the desired eigenvalues as

$$\lambda_{1,2}=-1\pm 1i,\ \lambda_{3,4}=-2\pm i,\ \lambda_{5,6}=-3\pm i,\ \lambda_{7,8}=-4\pm i$$

and the parameter vectors

$$\boldsymbol{f}_1=\boldsymbol{f}_2=\begin{pmatrix}2\\5\end{pmatrix},\quad \boldsymbol{f}_3=\boldsymbol{f}_4=\begin{pmatrix}5\\1\end{pmatrix},$$

$$\boldsymbol{f}_5=\boldsymbol{f}_6=\begin{pmatrix}3\\4\end{pmatrix},\quad \boldsymbol{f}_7=\boldsymbol{f}_8=\begin{pmatrix}5\\2\end{pmatrix}$$

Hence, the controller gains are

$$\boldsymbol{F}_d=\begin{pmatrix}-41.3767 & 42.2382 & 157.4361 & -177.3051\\ -413.1671 & 897.4563 & -725.9625 & 150.8834\end{pmatrix},$$

$$\boldsymbol{F}_a=\begin{pmatrix}0.8054 & -10.6981 & 16.9209 & -15.1425\\ -40.2050 & 85.8296 & -128.9817 & 61.7815\end{pmatrix}$$

Here, $\det(\boldsymbol{M}+\boldsymbol{C}\boldsymbol{F}_a)=-1494.8108\neq 0$. The simulation results of a system is presented in Fig. 3 using $\boldsymbol{x}_0=[-0.01, -0.02, 0.01, -0.01]^{\mathrm{T}}$ m and $\dot{\boldsymbol{x}}_0=[0.01, 0.02, 0.02, -0.03]^{\mathrm{T}}$ m/s. Furthermore, Fig. 4 depicts the frequency response of a system.

6 Conclusions

The paper has dealt with a technique to use acceleration and displacement measurements for second-order system. The availability of accelerometers makes the proposed control methodology favorable to several applications when the acceleration measurements are easier to obtain than the velocity ones. The necessary conditions that guarantee the solvability for the problem are formulated. All the parametric expressions for the controllers and the eigenvector matrix are derived. The solution can be applied for systems with nonsingular or singular mass matrices. Simulation results are provided to validate the effectiveness of the proposed technique.

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