Given the choice of p^*, by completing the squares, we have $(2\epsilon_1 x_1/p^* g) \leq \epsilon_1^2 + y^2$ and $(2\epsilon^T P^{-1}(t)\Phi(\zeta, y)/p^* g) \leq (1/2)\epsilon^T P^{-2}(t)\epsilon + 4\sum_{i=1}^n (p_{i1}^2 \phi_{i1}^2(|\zeta|) + p_{i2}^2 \phi_{i2}^2(|y|))$. A direct substitution leads to (11).

Appendix B
Proof of *Proposition* 2

From (31) with $i = n$, it can be verified that

$$\pi_n(\zeta, y) = \left(d_1(t) + 1 + \frac{1}{4\nu} g^2 p^{*2} p_{\max}^2\right) y^2 + y\phi_1(\zeta, y) + (n-1) g^2 \vartheta(y) y^2 + (n-1)\frac{\phi_1^2(\zeta, y)}{4p^*} + 4\sum_{i=1}^n \left(p_{i1}^2 \phi_{i1}^2(|\zeta|) + p_{i2}^2 \phi_{i2}^2(|y|)\right). \tag{49}$$

In view of (3), the following inequality holds:

$$y\phi_1(\zeta, y) \leq \frac{1}{2} y^2 + p_{11}^2 \phi_{11}^2(|\zeta|) + p_{12}^2 \phi_{12}^2(|y|)$$

and

$$(n-1)\frac{\phi_1^2(\zeta, y)}{4p^*} \leq \frac{n-1}{2}\phi_{11}^2(|\zeta|) + \frac{n-1}{2}\phi_{12}^2(|y|).$$

Therefore

$$\begin{aligned} 4\sum_{i=1}^n &\left(p_{i1}^2 \phi_{i1}^2(|\zeta|) + p_{i2}^2 \phi_{i2}^2(|y|)\right) + y\phi_1(\zeta, y) \\ &+ (n-1)\frac{\phi_1^2(\zeta, y)}{4p^*} \\ &\leq 4\sum_{i=1}^n \left(p_{i1}^2 \phi_{i1}^2(|\zeta|) + p_{i2}^2 \phi_{i2}^2(|y|)\right) \\ &\quad + \frac{1}{2} y^2 + p_{11}^2 \phi_{11}^2(|\zeta|) + p_{12}^2 \phi_{12}^2(|y|) + \frac{n-1}{2}\phi_{11}^2(|\zeta|) \\ &\quad + \frac{n-1}{2}\phi_{12}^2(|y|) \\ &\leq \chi_1 \sum_{i=1}^n \phi_{i1}^2(|\zeta|) + \chi_2 \sum_{i=1}^n \phi_{i2}^2(|y|) + \frac{1}{2} y^2 \end{aligned} \tag{50}$$

with $\chi_1 = \max\{5p_{11}^2 + ((n-1)/2), 4p_{i1}^2 (i = 2, \cdots, n)\}$ and $\chi_2 = \max\{5p_{12}^2 + ((n-1)/2), 4p_{i2}^2 (i = 2, \cdots, n)\}$. Let $\psi(t) = d_1(t) + 1 + (1/2) + (1/4\nu) g^2 {p^*}^2 p_{\max}^2$, with (49), (50), and we establish (39).

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Disturbance Impulse Controllability in Descriptor System

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***Abstract*—The descriptor system $E\dot{x} = Ax + Gu + Hv$ with external disturbance v is considered. To characterize the possibility to cancel the impulses produced by external disturbance v through using control u, the concept of disturbance impulse controllability is introduced.**

***Index Terms*—Distributional solution, H_∞ control of descriptor system, impulse controllability, reachability, singular system.**

I. Introduction

Recently, in order to attenuate the influence of external disturbance to control performance, some H_∞ control problems for descriptor system attract attention [1]–[6]. In such research area the descriptor system with control u and external disturbance v in the form

$$E\dot{x} = Ax + Gu + Hv \tag{1}$$

is considered, where x is the system state, and the square matrix E may be singular. The main difference from the standard case where $E = I$, the identity matrix, is in that we have to deal with impulse behavior in the descriptor case. Generally speaking, impulse behavior in

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descriptor system has been extensively studied [7]–[34], which is always related to either the inconsistency between initial value and input [7, p.18], or the discontinuity (or jump) in input and its derivatives [7, p.35]. However, a single formula expressing impulses in terms of both the inconsistency and the discontinuity appears missed in the literature. As a result, the distinction between the two kinds of impulses generated respectively by the inconsistency and the discontinuity gets not enough attention, and some notions, e.g., impulse reachability, some controllability treating of impulses due to external disturbance, are not satisfactorily developed yet. In the aspect of the H_∞ feedback control for descriptor system [1]–[6], only the feedback strategy for cancelling the impulses due to the inconsistency is used. The feedback strategy for exactly cancelling the impulses due to the discontinuity in the external disturbance is not considered.

The regularity of the pair (E, A) (i.e., the property $\det(sE-A) \neq 0$ for some $s \in \mathbb{C}$) guarantees the existence of the Weierstrass (or called slow-fast) decomposition to the system (1) [7, p.12], where only the fast part exhibits impulse behavior. Therefore as a preparation we first consider the state response of the fast descriptor system $N\dot{x} = x + Bu$, where the nilpotent matrix $N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, and the nilpotency index of N is denoted as q, i.e.

$$q := \min\{k : k \geq 1, N^k = 0\}. \tag{2}$$

We need to study generalized solution in the sense of distribution to the following *initial value problem* of the singular differential equation (or called differential-algebraic equation) system

$$N\dot{x}(t) = x(t) + Bu(t), t \geq 0; x(0) = x_0 \tag{3}$$

where the initial value x_0 is in $\mathbb{R}^n$, and the input u is a piecewise smooth function mapping $\mathbb{R}_+ = [0, +\infty)$ to $\mathbb{R}^r$. On distributional solution, many works have been reported; see, for example, [10], [11], [13], [17], [19], and [34]. To our knowledge, a unitive distributional solution that explicitly exhibits the aforementioned two kinds of impulses generated respectively by the inconsistency and the discontinuity is not found. As a special case, when the input u is sufficiently smooth (not piecewise smooth), such a formula is seen in, e. g., [33], [26, Eq. (3.32)], [23]. It is further written into a form to associate the inconsistency between initial value and input directly with impulse in [18] and [21]. As for another kind of impulses due to discontinuity in input and its derivatives, we refer to a formula in the seminal work [13] on impulse analysis for descriptor system in time domain (see [13, Eq. (3)]. It is derived from a *distributional differential equation* (denoted as θ_f^τ [13, p. 1077]) that directly models the relation between the impulse and the jump in the state but is not related to the original initial value problem (3).

In this note, we give an explicit distributional solution formula to the initial value problem (3); see (10) below. Compared with the smooth input case, it exhibits how the discontinuity excites impulse further when the input u and/or its derivatives do contain jumps. In [22], the authors established the concept of reachability for descriptor system, which treats of consistent initial value and smooth input. Now due to the distributional solution able to deal with both arbitrary initial value and piecewise smooth input, we make the idea "impulse reachability" precise as well. It turns out that two types of reachable impulses are characterized: the first excited by the inconsistency between initial value and input, and the second by the jump in input and its derivatives. Furthermore, we introduce the concept of disturbance impulse controllability, which occurs very naturally in view of the distinction between the two types of impulses. A descriptor system may be disturbance impulse controllable but not impulse controllable in the usual sense [13], [18]; see Example 1 below. The paper [27] considered feedback for descriptor system without controllability at infinity (equivalent to impulse controllability [10]). It was concerned with the achievable minimal index for the closed-loop systems, not directly involve the structure of the second type of impulses produced by external disturbance. Other papers, e. g., [28] and [29], considered feedback design for regularizing descriptor systems, which do not imply the concept of disturbance impulse controllability yet.

The rest of the technical note is organized as follows. Section II is devoted to distributional solution of the initial value problem (3), where a solution formula exhibits all impulses is given. Based on the formula, Section III classifies impulses and makes the idea of impulse reachability precise. Section IV defines disturbance impulse controllability as an open-loop dynamic property and establishes several test criteria for such controllability. Section V concludes the technical note.

II. Distributional Solution to Initial Value Problem

As a preparation, we collect some facts about Laplace transform. Let f be a function mapping $\mathbb{R}_+$ to $\mathbb{R}^l$. Its Laplace transform is defined as the Lebesgue integral $\mathcal{L}[f](s) = \int_0^{+\infty} f(t)e^{-st}dt$, if integrable, with the complex parameter s. Here the function f is only needed to be defined on $\mathbb{R}_+$ almost everywhere. If there exist real numbers α and $c > 0$ such that $\|f(t)\| \leq c\exp\{\alpha t\}$ for all $t \geq 0$, then the Laplace transform $\mathcal{L}[f](s)$ is defined for all complex s with $\mathrm{Re}(s) > \alpha$. For convenience we call such a function to be *exponentially bounded*.

Let f be an h times piecewise continuously differentiable function mapping $\mathbb{R}_+$ to $\mathbb{R}^l$ with jump points $0 < \tau_1 < \tau_2 < \cdots$ which are finitely many in any bounded interval [13]. Then $f^{(k)}(t)$ exists for $t \in \mathbb{R}_+, t \neq \tau_i, i = 0, 1, \ldots$ and $k = 0, \ldots, h$. One may additionally define, for example, $f^{(k)}(0) := f^{(k)}(0^+)$, $f^{(k)}(\tau_i) := f^{(k)}\left(\tau_i^+\right)$, $f^{(k)}(\tau_i) := f^{(k)}\left(\tau_i^-\right)$ (the right-hand and left-hand limits exist by assumption), or even simply, $f^{(k)}(\tau_i) := 0$ so that $f^{(k)}$ becomes a usual function, i.e., a map from $\mathbb{R}_+$ to $\mathbb{R}^l$. However, since the subset $\{0, \tau_1, \tau_2, \ldots\}$ of $\mathbb{R}$ is of Lebesgue measure zero, any complementary definition makes no difference to the Laplace transform of $f^{(k)}$, which is a Lebesgue integral. We emphasize that this function $f^{(k)}$, called *pointwise k-th order derivative* for convenience, as a map from $\mathbb{R}_+$ to $\mathbb{R}^l$ differs from the k-th order derivative of f in the distributional sense (see [35], [36]). We adopt the following notational convention:

Definition 1 (Function Space $\mathcal{K}_P^h(\mathbb{R}_+, \mathbb{R}^l)$): The notation $\mathcal{K}_P^h(\mathbb{R}_+, \mathbb{R}^l)$ denotes the set of all h times piecewise continuously differentiable functions mapping $\mathbb{R}_+$ to $\mathbb{R}^l$, whose pointwise k-th order derivatives, $k = 0, 1, \ldots, h$, are all exponentially bounded.

For a sufficiently smooth function f mapping $\mathbb{R}_+$ to $\mathbb{R}^l$, it is well known that

$$\mathcal{L}[f^{(k)}](s) = s^k\mathcal{L}[f](s) - \sum_{j=0}^{k-1} s^j f^{(k-1-j)}(0^+)$$

under exponentially bounded assumptions, where $f^{(k-1-j)}(0^+) = \lim_{t\downarrow 0} f^{(k-1-j)}(t)$. This formula is no longer valid when f is a piecewise smooth function. The following result explicitly exhibits the impact of the jump in f and its derivatives to the Laplace transforms of the pointwise derivatives.

Proposition 1: Let $f \in \mathcal{K}_P^k(\mathbb{R}_+, \mathbb{R}^r)$ with only one jump point $\tau > 0$. Then

$$\mathcal{L}[f^{(k)}](s) = s^k\mathcal{L}[f](s) - \sum_{j=0}^{k-1} s^j f^{(k-1-j)}(0^+) - e^{-s\tau}\sum_{j=0}^{k-1} s^j\left(\Delta_\tau f^{(k-1-j)}\right) \tag{4}$$

where $\Delta_\tau f^{(k-1-j)} = f^{(k-1-j)}(\tau^+) - f^{(k-1-j)}(\tau^-)$.

Proof: For $k \geq 1$

$$\begin{aligned}
&\mathcal{L}[f^{(k)}](s)\\
&= \int_0^{\tau} e^{-st} f^{(k)}(t)dt + \int_{\tau}^{+\infty} e^{-st} f^{(k)}(t)dt\\
&= e^{-st} f^{(k-1)}(t)|_{0^+}^{\tau^-} + s\int_0^{\tau} e^{-st} f^{(k-1)}(t)dt\\
&\quad + e^{-st} f^{(k-1)}(t)|_{\tau^+}^{\infty} + s\int_{\tau}^{+\infty} e^{-st} f^{(k-1)}(t)dt\\
&= s\mathcal{L}[f^{(k-1)}](s) - f^{(k-1)}(0^+)\\
&\quad - e^{-s\tau}(f^{(k-1)}(\tau^+) - f^{(k-1)}(\tau^-)).
\end{aligned}$$

Then the result follows from this computation and mathematical induction method. ∎

Proposition 1 can be extended to the cases of any finite and infinite number of jump points by using similar computation.

Now we consider the initial value problem (3). The *frequency domain version* (see [10]) of (3) is

$$(sN - I)X(s) = Nx_0 + BU(s) \tag{5}$$

where U is the Laplace transform of u, seen as a known complex variable function, and X is an unknown complex variable function to be found. Obviously, X can be solved out uniquely by

$$\begin{aligned}
X(s) &= (sN - I)^{-1}(Nx_0 + BU(s))\\
&= -\sum_{i=0}^{q-2} s^i N^{i+1} x_0 - \sum_{i=0}^{q-1} s^i N^i BU(s)
\end{aligned}\tag{6}$$

which is termed the frequency domain solution to (3) for convenience, as a function on the complex variable s.

The following Lemma shows the impact of jump in input and its derivatives to the frequency domain solution.

Lemma 1: Let $u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^r)$ with infinite number of jump points $0 < \tau_1 < \tau_2 < \cdots$. Then the frequency domain solution $X(s)$ can be further written into

$$\begin{aligned}
&X(s)\\
&= -\sum_{i=0}^{q-1} N^i B\mathcal{L}[u^{(i)}](s)\\
&\quad - \sum_{i=0}^{q-2} s^i N^{i+1}\left(x_0 + \sum_{l=0}^{q-1} N^l Bu^{(l)}(0^+)\right)\\
&\quad - \sum_{j=1}^{\infty}\sum_{i=0}^{q-2} e^{-\tau_j s} s^i N^{i+1}\left(\sum_{l=0}^{q-1} N^l B\left(\Delta_{\tau_j} u^{(l)}\right)\right)
\end{aligned}\tag{7}$$

where $\Delta_{\tau_j} u^{(l)} = u^{(l)}\left(\tau_j^+\right) - u^{(l)}\left(\tau_j^-\right)$.

Proof: For simplicity, we only consider the case that there exists only one jump point $\tau > 0$. By Proposition 1, we have

$$\begin{aligned}
s^i U(s) &= \mathcal{L}[u^{(i)}](s) + \sum_{j=0}^{i-1} s^j u^{(i-1-j)}(0^+)\\
&\quad + e^{-s\tau}\sum_{j=0}^{i-1} s^j\left(\Delta_\tau u^{(k-1-j)}\right)
\end{aligned}$$

Substituting it into (6) gives

$$\begin{aligned}
&X(s)\\
&= -\sum_{i=0}^{q-2} s^i N^{i+1} x_0 - \sum_{i=0}^{q-1} N^i B\mathcal{L}[u^{(i)}](s)\\
&\quad - \sum_{i=0}^{q-1} N^i B\sum_{j=0}^{i-1} s^j u^{(i-1-j)}(0^+)\\
&\quad - \sum_{i=0}^{q-1} N^i B\sum_{j=0}^{i-1} s^j\left(e^{-s\tau}\Delta_\tau u^{(k-1-j)}\right).
\end{aligned}$$

Exchanging the order of the sums and noting $N^q = 0$, we obtain

$$\begin{aligned}
&\sum_{i=0}^{q-1} N^i B\sum_{j=0}^{i-1} s^j u^{(i-1-j)}(0^+)\\
&\quad = \sum_{j=0}^{q-2} s^j N^{j+1}\sum_{l=0}^{q-1} N^l Bu^{(l)}(0^+)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=0}^{q-1} N^i B\sum_{j=0}^{i-1} s^j\left(e^{-s\tau}\Delta_\tau u^{(k-1-j)}\right)\\
&\quad = \sum_{j=0}^{q-2} e^{-s\tau} s^j N^{j+1}\sum_{l=0}^{q-1} N^l B\left(\Delta_\tau u^{(l)}\right)
\end{aligned}$$

by careful computation similar to [18]. These equations give the result immediately. ∎

Let $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^l)$ be the test function space, which is the space, with a topology on it, of all infinite times continuously differentiable functions with compact support from $\mathbb{R}$ to $\mathbb{R}^l$, and let $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^l)'$ be the distribution space, which is the dual space of $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R}^l)$. See [35] for detail. The distributions $\delta^{(k)}(t)$ and their shifts $\delta_a^{(k)}(t) = \delta^{(k)}(t-a)$ with $a > 0$, $k = 0, 1, \ldots$, as elements in $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})'$, are defined by $\langle \delta^{(k)}, h\rangle = (-1)^k h^{(k)}(0)$ and $\langle \delta_a^{(k)}, h\rangle = (-1)^k h^{(k)}(a)$, $\forall\, h \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{R})$, respectively. We use the following familiar formulas:

$$\mathcal{L}[\delta^{(k)}](s) = s^k, k = 0, 1, \ldots \tag{8}$$

which can be found in books involving Laplace transform, and

$$\mathcal{L}\left[\delta_a^{(k)}\right](s) = e^{-as} s^k, k = 0, 1, \ldots \tag{9}$$

which follows from (8) and the shifting property of Laplace transform.

Definition 2 (Distributional Solution): The inverse Laplace transform of the complex variable function $X(s)$ solved out from (5) is called the distributional solution to (3) in the sense of Laplace transform, or simply, the distributional solution to (3).

Theorem 1 (Distributional Solution Formula): Let $u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^r)$ with the jump points $0 < \tau_1 < \tau_2 < \cdots$. Then the distributional solution to the system (3) is given by

$$\begin{aligned}
&x(t)\\
&= -\sum_{i=0}^{q-1} N^i Bu^{(i)}(t)\\
&\quad - \sum_{i=0}^{q-2} \delta^{(i)}(t) N^{i+1}\left(x_0 + \sum_{l=0}^{q-1} N^l Bu^{(l)}(0^+)\right)\\
&\quad - \sum_{j=1}^{\infty}\sum_{i=0}^{q-2} \delta_{\tau_j}^{(i)}(t) N^{i+1}\left(\sum_{l=0}^{q-1} N^l B(\Delta_{\tau_j} u^{(l)})\right)
\end{aligned}\tag{10}$$

for $t \geq 0$, where $\Delta_{\tau_j} u^{(l)} = u^{(l)}(\tau_j^+) - u^{(l)}(\tau_j^-)$.

Proof: Follows from (8), (9) and Lemma 1 immediately. ∎

For convenience, we write the coefficient of $\delta_{\tau_j}^{(i)}$ into the following matrix form:

$$\begin{aligned}
&N^{i+1}\left(\sum_{l=0}^{q-1} N^l B\left(\Delta_{\tau_j} u^{(l)}\right)\right)\\
&\quad = N^i N \underbrace{\begin{bmatrix} B & NB & \cdots & N^{q-1}B\end{bmatrix}}_{=:\mathfrak{C}(N,B)} \begin{bmatrix} \Delta_{\tau_j} u\\ \Delta_{\tau_j} u^{(1)}\\ \vdots\\ \Delta_{\tau_j} u^{(q-1)}\end{bmatrix}.
\end{aligned}\tag{11}$$

Here the partitioned matrix $[B \quad NB \quad \cdots \quad N^{q-1}B]$, denoted by $\mathfrak{C}(N,B)$, is a shortened form of the usual controllability matrix $[B \quad NB \quad \cdots \quad N^{n-1}B]$ of the pair (N,B), noting that

$$[N^qB \quad \cdots \quad N^{n-1}B] = 0.$$

III. Classification of Impulses

It can be seen that there are two basic types of impulses in the state response: type (I) represented by

$$x_{\text{impulse(I)}}(x_0,u) = -\sum_{i=0}^{q-2}\delta^{(i)}N^{i+1}\left(x_0+\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+)\right) \tag{12}$$

which is produced by the inconsistency between initial value and input (i.e., the property $-\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+) \neq x_0$. For more detail, see [21]); and type (II) represented by

$$x_{\text{impulse(II)}}(u;\tau) = -\sum_{i=0}^{q-2}\delta_\tau^{(i)}N^{i+1}\left(\sum_{l=0}^{q-1}N^lB\left(\Delta_\tau u^{(l)}\right)\right) \tag{13}$$

which is produced by the jumps $\Delta_\tau u^{(l)}$, $l=0,1,\ldots,q-1$, of input and its derivatives at $\tau>0$, and has no relation to the initial value.

We stress that the type (I) differs from the type (II) essentially. Noting that

$$\sum_{l=0}^{q-1}N^lB\left(\Delta_\tau u^{(l)}\right) = -\sum_{i=0}^{q-1}N^iBu^{(i)}(\tau^-)+\sum_{l=0}^{q-1}N^lBu^{(l)}(\tau^+)$$

and comparing (13) with (12), one may want to interpret the type (I) by using the type (II): suppose that the input u has definition before the initial time point $t=0$; u and its derivatives has jumps at $t=0$; and the condition

$$-\sum_{i=0}^{q-1}N^iBu^{(i)}(0^-) = x_0 \tag{14}$$

happens to hold, where $u^{(i)}(0^-) = \lim_{t\uparrow 0}u^{(i)}(t)$. Obviously, only when $x_0 \in \operatorname{im}\mathfrak{C}(N,B)$, i.e., the initial value is consistent in the sense of [22], this interpretation can be available to guarantee the validity of (14). Here $\operatorname{im}(\cdot)$ means the image of matrix. The inconsistent initial value can not be interpreted in this manner.

Definition 3 (Two Types of Reachable Impulses Sets): Consider the system (3). The set

$$\mathfrak{I}_{(\text{I})}(N,B) = \left\{x_{\text{impulse(I)}}(x_0,u) : x_0 \in \mathbb{R}^n, u\in\mathcal{K}_p^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\right\}$$

of impulses is called the reachable impulse set of type (I); the set

$$\mathfrak{I}_{(\text{II})}(N,B,\tau) = \{x_{\text{impulse(II)}}(u;\tau) : u\in\mathcal{K}_p^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\}$$

of impulses is called the reachable impulse set of type (II) at $\tau>0$.

Note that the elements in the reachable impulse sets are distribution, not vector in $\mathbb{R}^n$.

Lemma 2:

1) $\left\{-\left(x_0+\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+)\right) : x_0 \in \mathbb{R}^n, u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\right\} = \mathbb{R}^n$.
2) $\left\{-\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+) : u\in\mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\right\} = \operatorname{im}\mathfrak{C}(N,B)$.
3) $\left\{-\sum_{l=0}^{q-1}N^lB\left(\Delta_\tau u^{(l)}\right) : u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\right\} = \operatorname{im}\mathfrak{C}(N,B)$ for $\tau>0$.

Proof:

1) Follows from the fact that $\{-(x_0+\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+)) : x_0 \in \mathbb{R}^n, u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\} \supseteq \{-(x_0+\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+)) : x_0\in\mathbb{R}^n, u=0\} = \mathbb{R}^n$.
2) The vector $-\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+)$ can be seen as a linear combination of the columns of $\mathfrak{C}(N,B)$; see (11). This implies that

$$\left\{-\sum_{l=0}^{q-1}N^lBu^{(l)}(0^+) : u\in\mathcal{K}_p^{q-1}(\mathbb{R}_+,\mathbb{R}^r)\right\} \subseteq \operatorname{im}\,\mathfrak{C}(N,B).$$

The inverse containing relation follows from that we can construct $u\in\mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)$ such that $u^{(l)}(0^+)$, $l=0,1,\ldots,q-1$ are a group of arbitrarily given vectors beforehand. ■

3) The proof is similar to that of 2).

Theorem 2 (Characterizing Reachable Impulses Sets):

1) $\mathfrak{I}_{(\text{I})}(N,B)$ is a linear subspace of $\mathcal{C}_C^\infty(\mathbb{R};\mathbb{R}^n)'$, which is isomorphic to $\operatorname{im}N$.
2) For $\tau>0$, $\mathfrak{I}_{(\text{II})}(N,B,\tau)$ is a linear subspace of $\mathcal{C}_C^\infty(\mathbb{R};\mathbb{R}^n)'$ as well, which is isomorphic to $\operatorname{im}(N\mathfrak{C}(N,B))$.

Proof:

1) $\mathfrak{I}_{(\text{I})}(N,B)$ can be seen as the image of the linear map

$$(x_0,u)\longmapsto x_{\text{impulse(I)}}(x_0,u)$$

from $\mathbb{R}^n\times\mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)$ to $\mathcal{C}_c^\infty(\mathbb{R};\mathbb{R}^n)'$, so it is a linear subspace of $\mathcal{C}_C^\infty(\mathbb{R};\mathbb{R}^n)'$. Further, it follows from Lemma 2 1) that

$$\mathfrak{I}_{(\text{I})}(N,B) = \left\{\sum_{i=0}^{q-2}\delta^{(i)}N^{i+1}z : z\in\mathbb{R}^n\right\}.$$

Any impulse $\sum_{i=0}^{q-2}\delta^{(i)}N^{i+1}z \in \mathfrak{I}_{(\text{I})}(N,B)$ is uniquely determined by the coefficient vector Nz of $\delta^{(0)}=\delta$, which is an element of $\operatorname{im}N$. This correspondence, which is from impulse to its coefficient vector of zero order component, is obviously a linear isomorphism between the linear spaces $\mathfrak{I}_{(\text{I})}(N,B)$ and $\operatorname{im}N$.
2) For $\tau>0$, $\mathfrak{I}_{(\text{II})}(N,B,\tau)$ can be seen as the image of the linear map

$$u\longmapsto x_{\text{impulse(II)}}(u;\tau)$$

from $\mathcal{K}_P^{q-1}(\mathbb{R}_+,\mathbb{R}^r)$ to $\mathcal{C}_C^\infty(\mathbb{R};\mathbb{R}^n)'$, so it is a linear subspace of $\mathcal{C}_C^\infty(\mathbb{R};\mathbb{R}^n)'$. Further, it follows from Lemma 2 3) that

$$I_{(\text{II})}(N,B) = \left\{\sum_{i=0}^{q-2}\delta_\tau^{(i)}N^{i+1}z : z\in\operatorname{im}\mathfrak{C}(N,B)\right\}.$$

The rest of the proof is similar to that of 1). ■

IV. Disturbance Impulse Controllability

To characterize the possibility to cancel the impulses produced by external disturbance through using control input, we introduce the concept of disturbance impulse controllability in following.

Consider the system model

$$N\dot{x}(t) = x(t) + Bu(t) + Fv(t), t \geq 0; x(0) = x_0 \quad (15)$$

where N, B are the same as in (3), $F \in \mathbb{R}^{n\times m}$, and v models the external disturbance input.

Definition 4 (Disturbance Impulse Controllability): The system (15) is called disturbance impulse controllable, if for any disturbance input $v \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^m)$, there exists a control input $u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^r)$ such that for any initial value $x_0 \in \mathbb{R}^n$, the state response $x(t)$ does not contain impulse at any time point $\tau > 0$.

Since the impulse at initial time point is transient, it is practically significant to only require impulse free at any time point $\tau > 0$, especially as we consider the long-term behavior. On the other hand, the controllability treating of the impulse of type (I) (at zero time point) has been established: the system (3) is called impulse controllable, if for arbitrary initial value $x_0 \in \mathbb{R}^n$, there exists a control input $u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^r)$ such that $x_{\text{impulse(I)}}(x_0, u) = 0$ [18]. So the Definition 4 focuses on the impulse of type (II) (at positive time point).

We use the following notations for a matrix $A \in \mathbb{R}^{p\times q}$: $\text{im}(A) := \{y : y = Ax, \exists x \in \mathbb{R}^q\}$; $\ker(A) := \{x : x \in \mathbb{R}^q \text{ and } Ax = 0\}$; $A(X) := \{y : y = Ax, \exists x \in X\}$ for $X \subseteq \mathbb{R}^q$; and $A^{-1}(Y) := \{x : x \in \mathbb{R}^q \text{ and } Ax \in Y\}$ for $Y \subseteq \mathbb{R}^p$. Thus we see $\text{im}(A) = A(\mathbb{R}^q)$ and $\ker(A) = A^{-1}(\{0\})$.

Lemma 3: For two matrices C and D of appropriate sizes, $C^{-1}(\text{im}(CD)) = \text{im}(D) + \ker(C)$.

Proof: Let $z \in C^{-1}(\text{im}(CD))$. Then $Cz \in \text{im}(CD)$, i.e., there exists x such that $Cz = CDx$. Thus $z - Dx \in \ker C$ and $z = Dx + (z - Dx) \in \text{im}(D) + \ker(C)$. Conversely, let $x_1 \in \text{im}(D)$, $x_2 \in \ker(C)$. Then $C(x_1 + x_2) = Cx_1 \in \text{im}(CD)$. This means $x_1 + x_2 \in C^{-1}(\text{im}(CD))$. ■

Theorem 3 (Disturbance Impulse Controllability Criteria): The following statements are equivalent:

1) The system (15) is disturbance impulse controllable.
2) $\mathfrak{I}_{(II)}(N, B, \tau) \supseteq \mathfrak{I}_{(II)}(N, F, \tau), \forall\, \tau > 0$.
3) $\text{im}(N\mathfrak{C}(N, B)) \supseteq \text{im}(N\mathfrak{C}(N, F))$.
4) $\ker N + \text{im}\mathfrak{C}(N, B) \supseteq \text{im}\mathfrak{C}(N, F)$.

Proof:

1) $\Longrightarrow$ 2): By Definition 4.
2) $\Longleftrightarrow$ 3): Follows from Lemma 2 and Theorem 2.
3) $\Longrightarrow$ 1): See below.
4) $\Longrightarrow$ 4): It follows from Lemma 3 that

$$N^{-1}(\text{im}\,(N\mathfrak{C}(N, B))) = \ker N + \text{im}\,\mathfrak{C}(N, B)$$
$$N^{-1}(\text{im}(N\mathfrak{C}(N, F))) = \ker N + \text{im}\mathfrak{C}(N, F).$$

Then the result follows from 3), which implies

$$N^{-1}(\text{im}(N\mathfrak{C}(N, B))) \supseteq N^{-1}(\text{im}(N\mathfrak{C}(N, F))).$$

5) $\Longrightarrow$ 3): Follows from:

$$\begin{aligned} N(\ker N + \text{im}\mathfrak{C}(N, B)) &= N(\ker N) + N(\text{im}\mathfrak{C}(N, B)) \\ &= \{0\} + \text{im}(N\mathfrak{C}(N, B)) \\ &= \text{im}(N\mathfrak{C}(N, B)) \end{aligned}$$

and $N(\text{im}\mathfrak{C}(N, F)) = \text{im}(N\mathfrak{C}(N, F))$. ■

Lemma 4: If $\text{im}(N\mathfrak{C}(N, B)) \supseteq \text{im}(N\mathfrak{C}(N, F))$, then there exists a matrix L of dimension $rq \times mq$ (recall the dimensions of N, B, and F) such that

$$N\mathfrak{C}(N, B)L = N\mathfrak{C}(N, F). \quad (16)$$

Proof: Since $\text{im}(N\mathfrak{C}(N, B)) \supseteq \text{im}(N\mathfrak{C}(N, F))$, we have $\text{rank}N\mathfrak{C}(N, B) = \text{rank}\,[\,N\mathfrak{C}(N, B) \quad N\mathfrak{C}(N, F)\,]$. This guarantees (16) as an equation in L having solution. ■

Lemma 5: Given real numbers a and b with $a < b$, and $z_n, w_n \in \mathbb{R}^l$, $n = 0, 1, \ldots$, then the following recursively constructed l-dimensional polynomial vectors:

$$g_0(t) = w_0 \frac{t-a}{b-a} + z_0 \frac{t-b}{a-b},$$
$$g_n(t) = \left(w_n \frac{t-a}{(b-a)^{n+1}} + z_n \frac{t-b}{(a-b)^{n+1}} \right) \cdot \frac{((t-a)(t-b))^n}{n!} + g_{n-1}(t), n \geq 1$$

satisfy:

1) $\deg(g_n(t)) \leq 2n + 1$;
2) $g_n^{(i)}(a) = z_i$, and $g_n^{(i)}(b) = w_i$, $i = 0, 1, \ldots, n$.

For convenience, we denote

$$g_n(t) =: g_n(a, b; z_{[0,n]}; w_{[0,n]}; t) \quad (17)$$

to explicitly mark the data used for the construction, where

$$z_{[0,n]} := \begin{bmatrix} z_0 \\ \vdots \\ z_n \end{bmatrix}, \text{ and } w_{[0,n]} := \begin{bmatrix} w_0 \\ \vdots \\ w_n \end{bmatrix}. \quad (18)$$

Proof: Repeatedly use the following Leibnitz formula on n-th order derivative of product $f_1(t)f_2(t)$:

$$(f_1(t)f_2(t))^{(n)} = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} f_1^{(i)}(t) f_2^{(n-i)}(t).$$

■

Proof of 3) $\Longrightarrow$ 1) *in Theorem 3:* We construct a required open-loop control strategy. Arbitrarily given a disturbance input $v \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^m)$ with infinite number of jump points $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots$ (in the case of finite number of jump points, using an evidently modified manner to construct the required control on the last one interval), we take

$$\begin{bmatrix} w_0^k \\ w_1^k \\ \vdots \\ w_{q-1}^k \end{bmatrix} = L \begin{bmatrix} v(\tau_k^-) \\ v^{(1)}(\tau_k^-) \\ \vdots \\ v^{(q-1)}(\tau_k^-) \end{bmatrix} =: w_{[0,q-1]}^k \quad (19)$$

and

$$\begin{bmatrix} z_0^{k-1} \\ z_1^{k-1} \\ \vdots \\ z_{q-1}^{k-1} \end{bmatrix} = L \begin{bmatrix} v(\tau_{k-1}^+) \\ v^{(1)}(\tau_{k-1}^+) \\ \vdots \\ v^{(q-1)}(\tau_{k-1}^+) \end{bmatrix} =: z_{[0,q-1]}^{k-1} \quad (20)$$

for $k = 1, 2, \ldots$ with the matrix L from Lemma 4. Now we define $u \in \mathcal{K}_P^{q-1}(\mathbb{R}_+, \mathbb{R}^r)$ as

$$u(t) = -g_{q-1}\left(\tau_{k-1}, \tau_k; z_{[0,q-1]}^{k-1}; w_{[0,q-1]}^k; t\right) \quad (21)$$

for $\tau_{k-1} \leq t < \tau_k$, $k = 1, 2, \ldots$, where the notation g_{q-1} follows Lemma 5. Then from Lemmas 4 and 5 we have:

$$N\mathfrak{C}(N, B) \begin{bmatrix} \Delta_{\tau_k} u \\ \Delta_{\tau_k} u^{(1)} \\ \vdots \\ \Delta_{\tau_k} u^{(q-1)} \end{bmatrix} = -N\mathfrak{C}(N, F) \begin{bmatrix} \Delta_{\tau_k} v \\ \Delta_{\tau_k} v^{(1)} \\ \vdots \\ \Delta_{\tau_k} v^{(q-1)} \end{bmatrix}$$

for $k = 1, 2, \ldots$. Therefore, for any initial value $x_0 \in \mathbb{R}^n$, the state response $x(t)$ of (15) does not contain impulse at any time point $\tau > 0$ by Theorem 1. ■

Remark 1: If $F = I$, the identity matrix, or $\mathfrak{C}(N, F)$ is of full row rank (i.e., the pair (N, F) is completely controllable [22]), then the criterion 3) becomes

$$\mathrm{im}(N\mathfrak{C}(N, B)) = \mathrm{im}(N) \tag{22}$$

which is a sufficient and necessary condition for the pair (N, B) to be impulse controllable [13], [18]. In [13, p.1078], the model (15) with $F = I$ is indeed used to interpret the impulse controllability introduced there. Although the impulse controllability and the disturbance impulse controllability with $F = I$ have a same mathematical criterion, they are concerned with different types of impulses.

Let (N, B) be impulse controllable. Since $\mathrm{im}(N) \supseteq (N\mathfrak{C}(N, F))$ always holds, it follows from Theorem 3 3) and (22) that the system (15) is disturbance impulse controllable. Simply speaking, impulse controllability implies disturbance impulse controllability.

Example 1: Let

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } F = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\mathrm{im}(N) \neq \mathrm{im}(N\mathfrak{C}(N, B)) \supset \mathrm{im}(N\mathfrak{C}(N, F))$. Therefore the system $N\dot{x} = x + Bu + Fv$ is not impulse controllable, but it is disturbance impulse controllable.

V. Conclusion

For descriptor system, the existing impulse controllability characterizes the possibility to cancel the impulse at initial time point due to initial value through selecting control input, which treats of the first type of impulses. Now as a comparison, the disturbance impulse controllability focuses on the second type: through selecting control input to cancel the impulse at a positive time point due to external disturbance.

We will prove elsewhere that the disturbance impulse controllability provides a sufficient and necessary condition for the existence of state feedback under which the closed-loop system does not have the second type of impulses. This will further confirm the significance of introducing the concept of disturbance impulse controllability from system design perspective.

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