An Approach for Pole Assignment in Singular Systems

AMIT AILON

Abstract—**An approach for pole assignment in a singular system $E\dot{x} = Ax + Bu$ is developed. It is shown that the problem of assigning the roots of det $(sE - (A + BF))$ by applying a proportional feedback $u = Fx + r$ in a given singular system is equivalent to the problem of pole assignment in an appropriate regular system. An immediate application of the presented approach is that procedures and computational algorithms that were originally developed for assigning eigenvalues in regular systems become useful tools for pole assignment in singular systems. The present approach provides a useful tool with regard to the combined problem of eliminating impulsive behavior and stabilizing a singular system.**

I. Introduction

In this note we consider the problem of pole assignment in a continuous-time singular system

$$E\dot{x}(t) = Ax(t) + Bu(t),\ x(0-) = x_0 \tag{1.1}$$

where $x \in R^n$, $u \in R^m$; $E, A \in R^{n\times n}$ are possible singular matrices, $B \in R^{n\times m}$, and $(sE - A)^{-1}$ exists for almost all s, i.e., the pencil $(sE - A)$ is regular, (or (1.1) is solvable).

In [1] it was first demonstrated that under certain controllability conditions the poles of the slow subsystem of (1.1) may be shifted arbitrarily and impulsive motion of the fast subsystem may be eliminated, and in [2] it was shown that controllability of the infinite eigenvalues is equivalent to the existence of a state feedback map which assigns those eigenvalues to specified complex numbers. A numerically robust algorithm for the computation of a required feedback for eigenstructure assignment has been presented in [3]. By applying a geometric approach, extensions of previous results with regard to eigenstructure assignment in (1.1) has been demonstrated in [4]. Pole assignment and compensator design have been considered in [5] for strongly controllable singular systems. Proportional and derivative feedback has been applied to singular systems [6] in order to obtain pole assignment and the application of modified proportional and derivative (MFD) feedback for the improvement of the system performance by shifting the poles, has been considered in [7].

This study supplements previous results, and presents a new approach regarding the problem of pole assignment in singular systems by applying admissible linear state feedback

$$u = Fx + r, \qquad F \in R^{m\times n}. \tag{1.2}$$

(F is admissible if the pair $\{E, (A + BF)\}$ is solvable.) It will be shown that for any given singular system (not necessarily controllable) there exists a regular state-space system

$$\dot{\zeta} = \alpha\zeta + \beta v, \qquad \zeta \in R^{n-j},\ V \in R^m \tag{1.3}$$

that satisfies the following. For any linear feedback $v = \Phi\zeta + f$, there exists an admissible linear feedback (1.2) that satisfies det $(\lambda_i E - (A + BF)) = 0 \Leftrightarrow$ det $(\lambda_i I - (\alpha + \beta\Phi)) = 0$. Moreover, deg [det $(sE - (A + BF))$] $= (n - j) \geq$ deg [det $(sE - (A + BF'))$], for any admissible $F' \neq F$, where deg [det $(\cdot)$] is the degree of the polynomial $[\cdot]$. In the opposite direction, the following exists. If, for admissible F, the set $\sigma(E, (A + BF)) = \{\lambda_i$:det $(\lambda_i E - (A + BF)) = 0,\ i \leq n - j\}$ is not empty, there exists $v = \Phi\zeta + f$ such that det $(\lambda_i I - (\alpha + \beta\Phi)) = 0$ if and only if $\lambda_i \in \sigma(E, (A + BF))$. In the special case, if for any admissible F, det $(sE - (A + BF)) \neq 0$, $\forall s$, then $n - j = 0$. Procedures for the derivation of α, β, Φ, and F are given by constructive proofs. Hence, by using the present approach, an admissible state feedback can be synthesized using standard procedures borrowed from the regular system theory.

It has been demonstrated in [10] that under certain conditions (which are in fact milder than complete controllability of the system) one can select a linear feedback that removes impulsive motion without destabilizing the system or destroying its regularity. This study shows that impulsive motion can be eliminated by admissible feedback if and only if rank $(E) = n - j$. If this condition holds, then the combined problem of eliminating impulsive modes and stabilizing a given singular system is associated with the problem of stabilizing the system (1.3) by state feedback, i.e., if (1.3) is stabilizable, then there exists an admissible feedback that stabilizes (1.1) and eliminates impulsive motions.

Manuscript received August 4, 1987; revised March 29, 1988 and September 6, 1988.
The author is with the Department of Electrical and Computer Engineering, Faculty of Engineering Sciences, Ben Gurion University of the Negev, Beer Sheva 84105, Israel.
IEEE Log Number 8928699.

II. Derivation of Preliminary Results

Let U denote the linear space of sufficiently differentiable controls: $t \to u(t) \in R^m$, defined for $t \geq 0$; and denote by $\psi(t;\ x_0,\ u)$ the corresponding solution of (1.1) with $x(0-) = x_0$. (For our purpose $u(\cdot)$ is sufficiently differentiable if it is $k - 1$ times differential where k is the degree of nilpotency of the fast subsystem in the standard canonical decomposition of (1.1), [1], [12].)

Definition 2.1: Assume rank $(B) = m = n - r$. Let C^T be a matrix whose columns are the basis of $\mathcal{R} = \ker B^T$; that is, $C \in R^{r\times n}$ is a maximal rank subject to

$$CB = 0. \tag{2.1}$$

The sets of admissible and reachable states have been defined in [12], and will be denoted, respectively, by R_0 and R, where $R = \cup R(x)$ ($R(x)$ is the set of states reachable from $x \in R_0$) and

$$x \in R_0$$

from [12] $R_0 = R$. Thus, in the sequel R_0 will denote the admissible as well as the reachable set.

The concept of admissible (reachable) set plays an important role in this study. Therefore, in the rest of this section we will describe precisely the structure of this set in terms of the original system (1.1), rather than in terms of its standard canonical form, as done in previous papers (see, for example, [12]). We start with the following result, which is obtained from [1].

Theorem 2.1: $R_0 = R^n$ if and only if rank $[E\ B] = n$.

As for the case rank $[E\ B] < n$, the following algorithm and the subsequent corollary describe the complete set of the admissible (reachable) states.

Algorithm 2.1—Step 1: By definition, for any $u \in U$ and any $t > 0$, $x(t) = \psi(t;\ x_0,\ u) \in R_0$ and hence from (1.1), and (2.1)

$$CE\dot{x}(t) = CAx(t); \qquad t > 0. \tag{2.2}$$

Since rank $([E\ B]) < n$, rank $(CE) = r - p(1) < r$. Define $C_1 = C$. Let $D_1 \in R^{p(1)\times r}$ be a maximal rank matrix, subject to $D_1C_1E = 0$. Then $D_1C_1A \in R^{p(1)\times n}$ is surjective, since if not there exists a surjective matrix $S \in R^{q\times p(1)}$ that satisfies

$$SD_1C_1Es - SD_1C_1A = SD_1C_1(sE - A) = 0 \tag{2.3}$$

but $(sE - A)$ is invertible, i.e., $SD_1C_1 = 0$ which is impossible since S, D_1, and C_1 are surjective.

From (2.2) for any $u \in U$, $x(t)$ satisfies

$$D_1C_1Ax(t) = 0; \qquad t > 0. \tag{2.4}$$

Since $D_1C_1A \in R^{p(1)\times n}$ is surjective, (2.4) implies the following: there exists a nonsingular matrix $P_1 \in R^{n\times n}$, which is obtained by changing the order of the columns of the identity matrix I_n, and a matrix $Q_1 \in R^{p(1)\times(n-p(1))}$ such that

$$[w_1^T\ z_1^T]^T = P_1x;\ z_1 = Q_1w_1 \tag{2.5}$$

where $z_1 \in R^{p(1)}$, $w_1 \in R^{n-p(1)}$. Following the last observation we may assume without loss of generality that $w_1 = [x_1\ x_2 \cdots x_{n-p(1)}]^T$, $z_1 = [x_{n-p(1)+1}\ x_{n-p(1)+2} \cdots x_n]^T$, i.e., $x = [w_1^T\ (Q_1 w_1)^T]^T$ which is equivalent to

$$x = T_1 w_1 \tag{2.6a}$$

$$T_1 = \begin{bmatrix} I_{n-p(1)} \\ Q_1 \end{bmatrix} \in R^{n \times (n-p(1))} \tag{2.6b}$$

where T_1 is maximal rank subject to $D_1 C_1 A T_1 = 0$. Using (2.5), (2.6), and (1.1), for any $t > 0$

$$ET_1 \dot{w}_1(t) = AT_1 w_1(t) + Bu(t);\ z_1(t) = Q_1 w_1(t). \tag{2.7}$$

Let us now define the following augmented matrix:

$$S_1^\# = [ET_1,\ AT_1,\ B]. \tag{2.8}$$

Since $D_1 C_1$ is maximal rank subject to $D_1 C_1 S_1^\# = 0$, only $n - p(1)$ equations in the first equation in (2.7) are linearly independent. Without loss of generality we may assume that the first $n - p(1)$ rows of $S_1^\#$ are linearly independent. Define $p_1 = [1\ 0\ 0 \cdots 0]^T$, $p_2 = [0\ 1\ 0\ 0 \cdots 0]^T$, $\cdots$, $p_{n-p(1)} = [0\ 0 \cdots 1\ 0 \cdots 0]^T$, and

$$S_1 = [p_1\ p_2 \cdots p_{n-p(1)}]^T S_1^\# = [E_1,\ A_1,\ B_1] \tag{2.9a}$$

$$E_1 \dot{w}_1(t) = A_1 w_1(t) + B_1 u(t);\ z_1(t) = Q_1 w_1(t); \qquad t > 0 \tag{2.9b}$$

where E_1, $A_1 \in R^{(n-p(1)) \times (n-p(1))}$, $B_1 \in R^{(n-p(1)) \times m}$.

If rank $([E_1 B_1]) = n - p(1)$, (i.e., $[E_1 B_1]$ is surjective), we set $e = p(1)$, $E^\blacklozenge = E_1$, $A^\blacklozenge = A_1$, $B^\blacklozenge = B_1$, $P^\blacklozenge = P_1$, $Q^\blacklozenge = Q_1$, $w = w_1$, and $z = z_1$, and we stop the process here. In case $[E_1 B_1]$ is not surjective we proceed from (2.9) to the next step.

Step 2: Consider (2.9b) and assume that rank $([E_1 B_1]) < n - p(1)$. Let C_2 and D_2 be maximal rank matrices subject to $C_2 B_1 = 0$, $D_2 C_2 E_1 = 0$. This implies $D_2 C_2 A_1 w_1(t) = 0$, $t > 0$ where $D_2 C_2 A_1$ is surjective [note that the pair $\{E_1, A_1\}$ is solvable and see (2.3)]. Hence, using previous observations we may define a nonsingular matrix $P_2^\# \in R^{(n-p(1)) \times (n-p(1))}$ which is obtained by changing the order of the columns of $I_{n-p(1)}$, and a matrix $Q_2^\#$, such that [compare to (2.5)]

$$[w_2^T\ z_2^T]^T = P_2^\# w_1;\ z_2 = Q_2^\# w_2. \tag{2.10}$$

Define

$$P_2 = \begin{bmatrix} P_2^\# & 0 \\ 0 & I_{p(1)} \end{bmatrix} \in R^{n \times n}. \tag{2.11}$$

From (2.5), and (2.10), (2.11) one has

$$[w_2^T\ z_2^T z_1^T]^T = \begin{bmatrix} P_2^\# & 0 \\ 0 & I_{p(1)} \end{bmatrix} P_1 x = P_2 x;\ [z_2^T\ z_1^T]^T = Q_2 w_2 \tag{2.12a}$$

where, again, the nonsingular matrix P_2 is obtained by changing the order of the columns of I_n.

Similarly to the previous step [see (2.6)] we define a maximal rank matrix T_2, subject to $D_2 C_2 A_1 T_2 = 0$, and $w_1 = T_2 w_2$. Again an augmented matrix $S_2^\#$ which plays the same role as $S_1^\#$ in Step 1 is defined. Next, the matrix $S_2 = [E_2, A_2, B_2]$ is obtained from $S_2^\#$ [see (2.9a)], and we define the following solvable system:

$$E_2 w_2(t) = A_2 w_2(t) + B_2 u(t);\ [z_2(t)^T\ z_1(t)^T]^T = Q_2 w_2(t); \qquad t > 0. \tag{2.12b}$$

If rank $([E_2 B_2)] = n - p(1) - p(2)$ (i.e., $[E_2 B_2]$ is surjective), we set $e = p(1) + p(2)$, $E^\blacklozenge = E_2$, $A^\blacklozenge = A_2$, $B^\blacklozenge = B_2$, $P^\blacklozenge = P_2$, $Q^\blacklozenge = Q_2$, $w = w_2$, and $z = [z_2^T z_1^T]^T$, and we stop the process. If rank $([E_2 B_2)] < n - p(1) - p(2)$, the algorithm is proceeded from (2.12b) to Steps 3, 4 $\cdots$.

Since in all steps along this algorithm rank $(B_i) = n - r$, one must obtain after $i = q \le r$ steps: $w = w_q$, $z = [z_q^T\ z_{q-1}^T \cdots z_1^T]^T$, and

$$E^\blacklozenge = E_q,\ A^\blacklozenge = A_q,\ B^\blacklozenge = B_q, P^\blacklozenge = P_q,\ Q^\blacklozenge = Q_q; \tag{2.13a}$$

$$[w^T\ z^T]^T = P^\blacklozenge x;\ z = Q^\blacklozenge w \tag{2.13b}$$

where $P^\blacklozenge$ is obtained by changing the order the columns of I_n, and $[E^\blacklozenge\ B^\blacklozenge]$ is surjective with

$$\text{rank}\ ([E^\blacklozenge\ B^\blacklozenge]) = n - (p(1) + p(2) + \cdots + p(q)) = n - e. \tag{2.13c}$$

The algorithm has been completed with this result. □

Thus, we have proved the following.

Corollary 2.1: Assume rank $([E\ B]) < n$, and consider Algorithm 2.1, especially (2.13a) through (2.13c). Let $x_0 \in R_0$, and define $[w^T\ z^T]^T = P^\blacklozenge x$ and

$$E^\blacklozenge \dot{w}(t) = A^\blacklozenge w(t) + B^\blacklozenge u(t); \qquad w(0-) = w_0;\ w \in R^{n-e} \tag{2.14a}$$

$$z(t) = Q^\blacklozenge w(t); \qquad z(0-) = z_0;\ z \in R^e \tag{2.14b}$$

where $H^\blacklozenge = [E^\blacklozenge\ B^\blacklozenge]$ is surjective. Then, for any $u \in U$ and $t \ge 0$ there exists

$$\psi^\blacklozenge(t;\ P^\blacklozenge x_0,\ u) = P^\blacklozenge \psi(t;\ x_0,\ u) \tag{2.15}$$

where $\psi(t;\ x_0,\ u)$ and $\psi^\blacklozenge(t;\ P^\blacklozenge x_0,\ u)$ are, respectively, the solutions of (1.1) and (2.14). In addition, any w in (2.14a) is admissible and R_0 is the column space of the matrix

$$M = (P^\blacklozenge)^{-1} \begin{bmatrix} H^\blacklozenge \\ Q^\blacklozenge H^\blacklozenge \end{bmatrix} \tag{2.16}$$

and since $H^\blacklozenge$ is surjective, dim $(R_0) = n - e$.

Remark: (a) If $x_0 \notin R_0$, (2.15) holds for any $t > 0$. (b) rank $(E^\blacklozenge) \le$ rank (E).

Theorem 2.2: Assume rank $([E\ B]) < n$. Then the following relation holds:

$$\det(\lambda_i E - A) = 0 \Leftrightarrow \det(\lambda_i E^\blacklozenge - A^\blacklozenge) = 0 \tag{2.17}$$

which implies: $\sigma(E,\ A) = \sigma(E^\blacklozenge,\ A^\blacklozenge)$, where $\sigma(E^\blacklozenge,\ A^\blacklozenge)$ is the spectrum of $E^\blacklozenge$ and $A^\blacklozenge$ in (2.14a).

Proof: Since T_1 in (2.6) is surjective, we may select column vectors $t_i, t_{i+1}, \cdots, t_n$, where $i = n - p(1) + 1$, such that the following matrix:

$$T^1 = [T_1\ t_i \cdots t_n] \in R^{n \times n} \tag{2.18}$$

is nonsingular. Clearly,

$$[ET^1,\ AT^1] = [ET_1,\ E[t_i\ t_{i+1} \cdots t_n],\ AT_1,\ A[t_i\ t_{i+1} \cdots t_n]] \tag{2.19}$$

and $\det(\lambda_i ET^1 - AT^1) = 0 \Leftrightarrow \det(\lambda_i E - A) = 0$. Now the matrix

$$P^1 = [p_1\ p_2 \cdots p_j [D_1 C_1]^T]^T \in R^{n \times n}, \qquad j = n - p(1) \tag{2.20}$$

where $D_1 C_1$ is given in (2.4) and p_i in (2.9a), is nonsingular. (Otherwise, S_1 in (2.9a) is not surjective.) Using (2.18)–(2.20) the following is satisfied (note that $D_1 C_1 [ET^1\ AT_1] = 0$):

$$P^1[ET^1\ AT^1] = \begin{bmatrix} E_1 & E_1^+ & A_1 & A_1^+ \\ 0 & 0 & 0 & A_{L1} \end{bmatrix} \tag{2.21}$$

where $[\cdot]_1^+ = [\cdot][t_i t_{i+1} \cdots t_n]$, and E_1, A_1 are given in (2.9b). Hence,

$$P^1[sE - A]T^1 = \begin{bmatrix} sE_1 - A_1 & E_1^+ - A_1^+ \\ 0 & -A_{L1} \end{bmatrix}. \tag{2.22}$$

Following the solvability condition A_{L1} is invertible, and from (2.22)

$$\det(\lambda_i E - A) = 0 \Leftrightarrow \det(\lambda_i E_1 - A_1) = 0. \tag{2.23}$$

Using the same arguments, and by observing Step 2 in Algorithm 2.1 [especially (2.9b) and (2.12b)] one can show that $\det(\lambda_i E_1 - A_1) = 0 \Leftrightarrow \det(\lambda_i E_2 - A_2) = 0$, and using (2.23)

$$\det(\lambda_i E - A) = 0 \Leftrightarrow \det(\lambda_i E_2 - A_2) = 0. \tag{2.24}$$

By continuing this process through Steps 3, 4, $\cdots$, in Algorithm 2.1 we establish the theorem. □

Theorem 2.3: Assume rank ($[\boldsymbol{E}\,\boldsymbol{B}]$) $< n$ and consider (2.14a). For any admissible $\boldsymbol{F}^{\blacklozenge} \in R^{m\times(n-e)}$ in

$$\boldsymbol{E}^{\blacklozenge}\dot{\boldsymbol{w}}(t) = (\boldsymbol{A}^{\blacklozenge} + \boldsymbol{B}^{\blacklozenge}\boldsymbol{F}^{\blacklozenge})\boldsymbol{w}(t) + \boldsymbol{B}^{\blacklozenge}\boldsymbol{r}(t),\ \boldsymbol{w} \in R^{n-e} \tag{2.25}$$

there exists an admissible feedback matrix $\boldsymbol{F} \in R^{m\times n}$ such that

$$\boldsymbol{E}\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{BF})\boldsymbol{x}(t) + \boldsymbol{Br}(t) \tag{2.26}$$

$$\det(\lambda_i\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF})) = 0 \Leftrightarrow \det(\lambda_i\boldsymbol{E}^{\blacklozenge} - (\boldsymbol{A}^{\blacklozenge} + \boldsymbol{B}^{\blacklozenge}\boldsymbol{F}^{\blacklozenge})) = 0. \tag{2.27}$$

Conversely, for any admissible $\boldsymbol{F}$, there exists an admissible $\boldsymbol{F}^{\blacklozenge}$ such that (2.27) holds.

Proof: Firstly, consider the singular systems (1.1) and (2.9b), and assume that $\boldsymbol{F}_1 \in R^{m\times(n-p(1))}$ is admissible. It will be shown that there exists an admissible $\boldsymbol{F} \in R^{m\times n}$ such that

$$\det(\lambda_i\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF})) = 0 \Leftrightarrow \det(\lambda_i\boldsymbol{E}_1 - (\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{F}_1)) = 0. \tag{2.28}$$

Assume $(s\boldsymbol{E}_1 - (\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{F}_1))$ is invertible, and define matrices $\boldsymbol{F}_a$ and $\boldsymbol{F}$ as follows:

$$\boldsymbol{F}_a = [\boldsymbol{F}_1\ \boldsymbol{0}],\ \boldsymbol{F}_a \in R^{m\times n},\ \boldsymbol{F}_1 \in R^{m\times(n-p(1))} \tag{2.29}$$

$$\boldsymbol{F} = \boldsymbol{F}_a(\boldsymbol{T}^1)^{-1} \tag{2.30}$$

where the nonsingular matrix $\boldsymbol{T}^1$ is given by (2.18). Then one clearly has

$$\boldsymbol{P}^1\{\boldsymbol{ET}^1, (\boldsymbol{A} + \boldsymbol{BF})\boldsymbol{T}^1] = \boldsymbol{P}^1[\boldsymbol{ET}^1, \boldsymbol{AT}^1 + \boldsymbol{BF}_a]. \tag{2.31}$$

Following (2.20), (2.29), using the obvious result $\boldsymbol{D}_1\boldsymbol{C}_1\boldsymbol{BF}_a = \boldsymbol{0}$ (note that $\boldsymbol{C}_1 = \boldsymbol{C}$), and applying (2.30), equation (2.22) becomes

$$\boldsymbol{P}^1[s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF})]\boldsymbol{T}^1 = \begin{bmatrix} s\boldsymbol{E}_1 - (\boldsymbol{A}_1 + \boldsymbol{BF}_1) & s\boldsymbol{E}_2^+ - \boldsymbol{A}_2^+ \\ \boldsymbol{0} & -\boldsymbol{A}_{L1} \end{bmatrix}. \tag{2.32}$$

Since $\boldsymbol{F}_1$ is admissible, (2.32) implies that $\boldsymbol{F}$ is admissible as well and (2.28) follows. If $q = 1$ in (2.13) we complete this part of the proof. If $q > 1$ we continue as follows: using previous arguments one can show that for a given admissible $\boldsymbol{F}_2$, there exist $\boldsymbol{F}_1$ and $\boldsymbol{F}$ that satisfy

$$\begin{aligned}\det(\lambda_i\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF})) = 0 &\Leftrightarrow \det(\lambda_i\boldsymbol{E}_1 - (\boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{F}_1)) \\ = 0 &\Leftrightarrow \det(\lambda_i\boldsymbol{E}_2 - (\boldsymbol{A}_2 + \boldsymbol{B}_2\boldsymbol{F}_2)) = 0\end{aligned} \tag{2.33}$$

where $\boldsymbol{E}_2$, $\boldsymbol{A}_2$, and $\boldsymbol{B}_2$ are given by (2.12b). This procedure completes this part of the proof.

Conversely, assume $\boldsymbol{F}$ is admissible. Using (2.22) and the obvious result $\boldsymbol{D}_1\boldsymbol{C}_1\boldsymbol{BF} = \boldsymbol{0}$, we have

$$\boldsymbol{P}^1[s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF})]\boldsymbol{T}^1 = \begin{bmatrix} s\boldsymbol{E}_1 - (\boldsymbol{A} + \boldsymbol{BF})_1 & s\boldsymbol{E}_1^+ - (\boldsymbol{A} + \boldsymbol{BF})_1^+ \\ \boldsymbol{0} & -\boldsymbol{A}_{L1} \end{bmatrix}. \tag{2.34}$$

From (2.8) and (2.9a) $[\boldsymbol{p}_1\boldsymbol{p}_2 \cdots \boldsymbol{p}_{n-p(1)}]^T(\boldsymbol{A} + \boldsymbol{BF})\boldsymbol{T}_1 = \boldsymbol{A}_1 + \boldsymbol{B}_1\boldsymbol{FT}_1$. Using (2.18)-(2.20) and defining $\boldsymbol{F}_1 = \boldsymbol{FT}_1$, (2.34) implies (2.28). Similarly, an admissible $\boldsymbol{F}_2$ can be defined for the next step, such that (2.33) holds. By continuing this procedure, the proof is completed. □

The terms *C*-controllability (CC) and *R*-controllability (RC) [12] are given by the following.

Definition 2.2: The system (1.1) is completely controllable if one can reach any state from any initial state.

Definition 2.3: The system (1.1) is controllable within the set of reachable states (*R*-controllable) if one can reach any state in the set of reachable states from any admissible initial state.

III. Main Results

In this section we study the following question. We are given a set $\Lambda_\alpha = \{\lambda_1, \lambda_2, \cdots, \lambda_\alpha\}$ of complex numbers (with possibly nondistinct λ_i) subject to complex conjugacy. Under what conditions does a real admissible matrix $\boldsymbol{F} \in R^{m\times n}$ exist, and how can this matrix be determined such that the set of roots of the polynomial $\det(s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF}))$ is precisely Λ_α. It is clear that α (the number of elements in Λ_α), satisfies $\alpha \leq \operatorname{rank}(\boldsymbol{E})$, and thus the value of the integer α is intrinsically part of the problem. Therefore, the following definition is intuitively appealing.

Definitions 3.1: Let $\boldsymbol{F}_s$ be a matrix that satisfies the following:

$$\deg[\det(s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF}_s))] = \alpha \geq \deg[\det(s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF}_r))],\quad \forall \boldsymbol{F}_r \neq \boldsymbol{F}_s. \tag{3.1}$$

The system (1.1) is completely pole assignable (CPA) if and only if for any self-conjugate set $\Lambda_\alpha = \{\lambda_1, \lambda_2, \cdots, \lambda_\alpha\}$, there exists an admissible feedback (1.2), such that the set of the roots of $\det(s\boldsymbol{E} - (\boldsymbol{A} + \boldsymbol{BF}))$ is precisely Λ_α.

Let $\boldsymbol{P}_f\boldsymbol{P}_r \in R^{n\times n}$ be regular matrices. Premultiplying of (1.1) by $\boldsymbol{P}_f$ and using $\boldsymbol{y} = \boldsymbol{P}_r^{-1}\boldsymbol{x}$ yields

$$\boldsymbol{P}_f\boldsymbol{EP}_r\dot{\boldsymbol{y}} = \boldsymbol{P}_f\boldsymbol{AP}_r\boldsymbol{y} + \boldsymbol{P}_f\boldsymbol{Bu}. \tag{3.2}$$

Clearly, $\det(\lambda_i\boldsymbol{P}_f\boldsymbol{EP}_r - \boldsymbol{P}_f\boldsymbol{AP}_r) = 0 \Leftrightarrow \det(\lambda_i\boldsymbol{E} - \boldsymbol{A}) = 0$. We also note that (1.1) is CC, (RC), if and only if (3.2) is CC, (RC), and from (2.1) $\boldsymbol{CP}_f^{-1}$ is a maximal rank matrix subject to $\boldsymbol{CP}_f^{-1}(\boldsymbol{P}_f\boldsymbol{B}) = \boldsymbol{0}$. Hence, for the present purpose we may assume without loss of generality that in (1.1), $\boldsymbol{E} = \boldsymbol{E}^*$, $\boldsymbol{A} = \boldsymbol{A}^*$, and $\boldsymbol{B} = \boldsymbol{B}^*$, where

$$\boldsymbol{E}^* = \boldsymbol{P}_f\boldsymbol{EP}_r = \begin{bmatrix} \boldsymbol{I}_{n-k} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix},\ \boldsymbol{A}^* = \boldsymbol{P}_f\boldsymbol{AP}_r,\ \boldsymbol{B}^* = \boldsymbol{P}_f\boldsymbol{B}. \tag{3.3}$$

Theorem 3.1: Consider a solvable singular system

$$\boldsymbol{E}^{\bullet}\dot{\boldsymbol{x}} = \boldsymbol{A}^{\bullet}\boldsymbol{x} + \boldsymbol{B}^{\bullet}\boldsymbol{u}, \tag{3.4a}$$

$$\boldsymbol{E}^{\bullet} = \boldsymbol{E}^*,\ \boldsymbol{A}^{\bullet} = \begin{bmatrix} \boldsymbol{A}_1^{\bullet} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_k \end{bmatrix} \begin{matrix} \}n-k \\ \}k \end{matrix},\ \boldsymbol{B}^{\bullet} = \begin{bmatrix} \boldsymbol{B}_1^{\bullet} \\ \boldsymbol{B}_2^{\bullet} \end{bmatrix} \begin{matrix} \}n-k \\ \}k \end{matrix} \tag{3.4b}$$

and assume that rank ($[\boldsymbol{E}^*\boldsymbol{B}^{\bullet}]$) $= n$, and rank $(\boldsymbol{E}^*) = n - k > 0$. Then the system (3.4) is CC if and only if the following regular system is controllable:

$$\dot{\boldsymbol{\zeta}} = \boldsymbol{\alpha\zeta} + \boldsymbol{\beta u},\ \boldsymbol{\zeta} \in R^{n-k};\qquad \boldsymbol{\alpha} = \boldsymbol{A}_1^{\bullet},\ \boldsymbol{\beta} = \boldsymbol{B}_1^{\bullet}. \tag{3.5}$$

Proof: Since $[\boldsymbol{E}^{\bullet}\boldsymbol{B}^{\bullet}]$ is surjective, (3.4) is CC [12] if and only if $\boldsymbol{T}(s) = [s\boldsymbol{E}^{\bullet} - \boldsymbol{A}^{\bullet}, \boldsymbol{B}^{\bullet}]$ is surjective for any finite s. Similarly, (3.5) is controllable if and only if $\boldsymbol{\tau}(s) = [s\boldsymbol{I} - \boldsymbol{A}_1^{\bullet}, \boldsymbol{B}_1^{\bullet}]$ is surjective for any finite s. Since $\boldsymbol{E}^{\bullet} = \boldsymbol{E}^*$, one can write

$$\boldsymbol{T}(s) = \left[\begin{bmatrix} s\boldsymbol{I}_{n-k} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}_1^{\bullet} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{I}_k \end{bmatrix}, \begin{bmatrix} \boldsymbol{B}_1^{\bullet} \\ \boldsymbol{B}_2^{\bullet} \end{bmatrix}\right]. \tag{3.6}$$

The application of a sequence of elementary column operations on the matrix $\left[\begin{smallmatrix} \boldsymbol{0} & \boldsymbol{B}_1^{\bullet} \\ -\boldsymbol{I}_k & \boldsymbol{B}_2^{\bullet} \end{smallmatrix}\right]$ implies

$$\boldsymbol{T}(s) \approx \left[\begin{bmatrix} s\boldsymbol{I}_{n-k} \\ \boldsymbol{0} \end{bmatrix} - \begin{bmatrix} \boldsymbol{A}_1^{\bullet} \\ \boldsymbol{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{I}_k \end{bmatrix}, \begin{bmatrix} \boldsymbol{B}_1^{\bullet} \\ \boldsymbol{0} \end{bmatrix}\right] \tag{3.7}$$

and thus $\boldsymbol{T}(s)$ is surjective if and only if $[s\boldsymbol{I}_{n-k} - \boldsymbol{A}_1^{\bullet}, \boldsymbol{B}_1^{\bullet}] = \boldsymbol{\tau}(s)$ is surjective. □

Theorem 3.2: a) Assume rank ($[\boldsymbol{E}\,\boldsymbol{B}]$) $= n$. Then the system (1.1) is completely pole assignable (CPA) if and only if the system is CC. b) Assume rank ($[\boldsymbol{E}\,\boldsymbol{B}]$) $< n$. Then the system (1.1) is CPA if and only if the system is RC.

Proof: It is sufficient to prove the theorem for the case [see (3.3)], $\boldsymbol{E} = \boldsymbol{E}^*$, $\boldsymbol{A} = \boldsymbol{A}^*$, and $\boldsymbol{B} = \boldsymbol{B}^*$. a) Assume rank ($[\boldsymbol{E}\,\boldsymbol{B}]$) $= n$ and let us consider first the case rank $(\boldsymbol{E}^*) = n - k > 0$. Partitions of $\boldsymbol{A}^*$ and $\boldsymbol{B}^*$ are given by

$$\boldsymbol{A}^* = \begin{bmatrix} \boldsymbol{A}_1^* & \boldsymbol{A}_2^* \\ \boldsymbol{A}_3^* & \boldsymbol{A}_4^* \end{bmatrix},\ \boldsymbol{B}^* = \begin{bmatrix} \boldsymbol{B}_1^* \\ \boldsymbol{B}_2^* \end{bmatrix} \begin{matrix} \}n-k \\ \}k \end{matrix}. \tag{3.8a}$$

Now, rank ($[\boldsymbol{E}^*\,\boldsymbol{B}^*]$) $= n$ implies [see (3.3)] rank $(\boldsymbol{B}_2^*) = k$. Hence, there exists a matrix $\boldsymbol{F}_1 = [\boldsymbol{F}_{11}\boldsymbol{F}_{12}] \in R^{m\times n}$ satisfies

$$[\boldsymbol{A}_3^*\ \boldsymbol{A}_4^*] + \boldsymbol{B}_2^*\boldsymbol{F}_1 = [\boldsymbol{0}\ \boldsymbol{I}_k]. \tag{3.8b}$$

Substituting $\boldsymbol{u} = \boldsymbol{F}_1\boldsymbol{x} + \boldsymbol{v}_1$, we have

$$\boldsymbol{E}^*\dot{\boldsymbol{x}} = (\boldsymbol{A}^* + \boldsymbol{B}^*\boldsymbol{F}_1)\boldsymbol{x} + \boldsymbol{B}^*\boldsymbol{v}_1, \tag{3.9a}$$

or

$$\begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1^* + B_1^* F_{11} & A_2^* + B_1^* F_{12} \\ 0 & I_k \end{bmatrix} x + \begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} v_1. \tag{3.9b}$$

Obviously the system (3.9) is solvable and moreover deg [det $(sE^* - (A^* + B^*F_1))$] $= n - k$. The system (1.1) is CC [11], if and only if either rank $(B) = n$, or CE is surjective and $(sCE - CA)$ is surjective for any finite s. But $(sCE - CA) = (sCE - C(A + BF))$, and thus (3.9) is CC if and only if (1.1) is CC.

Define a nonsingular matrix $P_d \in R^{n \times n}$ as follows:

$$P_d = \begin{bmatrix} I_{n-k} & -(A_2^* + B_1^* F_{12}) \\ 0 & I_k \end{bmatrix}.$$

Premultiplying of (3.9) by P_d yields

$$\begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} A_1^* + B_1^* F_{11} & 0 \\ 0 & I_k \end{bmatrix} x + \begin{bmatrix} B_1^* - (A_2^* + B_1^* F_{12}) B_2^* \\ B_2^* \end{bmatrix} v_1 \tag{3.10a}$$

or, equivalently [see (3.4)]

$$E^{\bullet} \dot{x} = A^{\bullet} x + B^{\bullet} v_1. \tag{3.10b}$$

Thus far we can state that the singular system (1.1) is CPA if and only if (3.10) is CPA.

Define a matrix $F_2 \in R^{m \times m}$ as follows:

$$F_2 = [F_{21} \ 0], \ F_{21} \in R^{m \times (n-k)}. \tag{3.11}$$

The application of a state feedback $v_1 = F_2 x + v_2$ in (3.10) yields, using (3.11)

$$sE^{\bullet} - (A^{\bullet} + B^{\bullet} F_1 + B^{\bullet} F_2) = \begin{bmatrix} sI_{n-k} - A_1^* - B_1^* F_{11} - (B_1^* - (A_2^* + B_1^* F_{12}) B_2^*) F_{21} & 0 \\ -B_2^* F_{21} & -I_k \end{bmatrix}.$$

Hence,

$$\det (sE^{\bullet} - (A^{\bullet} + B^{\bullet} F_1 + B^{\bullet} F_2)) = -\det (sI_{n-k} - A_1^* - B_1^* F_{11} - (B_1^* - (A_2^* + B_1^* F_{12}) B_2^*) F_{21}). \tag{3.12}$$

Assume (1.1) is CC. Then from Theorem 3.1 the pair $\{\alpha, \beta\} \equiv \{A_1^* + B_1^* F_{11}, B_1^* - (A_2^* + B_1^* F_{12}) B_2^*\}$ is controllable and, from regular system theory, the $n - k$ poles of the system (3.5) can arbitrarily be assigned by a linear state feedback. By recalling Definition 3.1 and noting that rank $(E^*) = n - k$, we complete this part of the theorem.

Conversely, assume (1.1) is not CC. Then the pair $\{\alpha, \beta\} \equiv \{A_1^* + B_1^* F_{11}, B_1^* - (A_2^* + B_1^* F_{12}) B_2^*\}$ is not controllable and there exists a nonsingular matrix $T \in R^{(n-k) \times (n-k)}$, such that the matrix $T^{\bullet} = [\begin{smallmatrix} T & 0 \\ 0 & I_k \end{smallmatrix}]\begin{smallmatrix} \}n-k \\ \}k \end{smallmatrix}$ satisfies the following. If $y = T^{\bullet} x$, we have from (3.10) and the definition of α and β: $T^{\bullet} E^{\bullet} (T^{\bullet})^{-1} \dot{y} = T^{\bullet} A^{\bullet} (T^{\bullet})^{-1} y + T^{\bullet} B^{\bullet} v_1$, where

$$T^{\bullet} E^{\bullet} (T^{\bullet})^{-1} = \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \}n-k \\ \}k \end{matrix},$$

$$T^{\bullet} A^{\bullet} (T^{\bullet})^{-1} = \begin{bmatrix} \alpha_1 & \alpha_2 & 0 \\ 0 & \alpha_4 & 0 \\ 0 & 0 & I_k \end{bmatrix} \begin{matrix} \Big\}n-k \\ \}k \end{matrix} \quad T^{\bullet} B^{\bullet} = \begin{bmatrix} \beta_1 \\ 0 \\ B_2^* \end{bmatrix} \begin{matrix} \Big\}n-k \\ \}k \end{matrix},$$

$\alpha_1 \in R^{j \times j}$, $\alpha_4 \in R^{(n-k-j) \times (n-k-j)}$, $n - k - j > 0$, $\beta_1 \in R^{j \times m}$, and the pair $\{\alpha_1, \beta_1\}$ is controllable. Now, for any admissible $F^{\bullet}$

$$\det (sT^{\bullet} E^{\bullet} (T^{\bullet})^{-1} - (T^{\bullet} A^{\bullet} (T^{\bullet})^{-1} + T^{\bullet} B^{\bullet} F^{\bullet})) = \det (sI_{(n-k)-j} - \alpha_4) h^{\bullet}(s)$$

where deg $[h^{\bullet}(s)] \le j$. (For $F^{\bullet} = [F_1^{\bullet} \ 0]$, deg $[h^{\bullet}(s)] = j$, and since the pair $\{\alpha_1, \beta_1\}$ is controllable, exactly j eigenvalues are relocatable.) Since the eigenvalues of α_4 are not affected by any state feedback, we complete the proof for this case.

If $n - k = 0$ and rank $([E \ B]) = n$, there exist rank $(B) = n$, $\alpha = 0$ [see (3.1)], and Λ_α is null, and the statement "the system is CPA" in the present case is consistent with Definition 3.1. Thus, we establish part a) of the theorem.

b) Assume rank $([E \ B])$ = rank $([E^* \ B^*]) < n$, and let rank $(E^{\blacklozenge}) = n - e - k^{\blacklozenge} > 0$ in (2.14a). By definition the system (1.1) is RC if and only if the system (2.14a) is CC. Hence, this part of the proof follows directly from part a) of this theorem, and from Theorems 2.2 and 2.3.□

IV. Applications

The following corollary concludes the practical applications of previous results.

Corollary 4.1:

a) Assume rank $([E \ B]) = n$, rank $(E) = n - k$, and consider the system

$$\dot{\zeta} = \alpha \zeta + \beta v, \ \zeta \in R^{n-k}, \tag{4.1a}$$

$$\alpha = A_1^* + B_1^* F_{11} \in R^{(n-k) \times (n-k)},$$

$$\beta = B_1^* - (A_2^* + B_1^* F_{12}) B_2^* \in R^{(n-k) \times m} \tag{4.1b}$$

where $[\cdot]_1^*$ and $[\cdot]_2^*$ are given by (3.8a), and $F_1 = [F_{11} \ F_{12}]$ satisfies (3.8b). The system (1.1) is CPA if and only if (4.1) is controllable. Moreover, any eigenvalue λ_i of α is relocatable by use of a state feedback in (4.1) if and only if λ_i is a relocatable root of det $(sE - (A + BF))$ in (1.1). If $n - k = 0$, then for any solvable pair $\{E, (A + BF)\}$, det $(sE - (A + BF)) \ne 0$, $\forall s$.

b) Assume rank $([E \ B]) < n$, and let [see (2.14a)] rank $(E^{\blacklozenge}) = n - e - k^{\blacklozenge}$. Further, assume the terms on the right-hand side of (4.1b) are derived from (2.14a) rather than from (1.1) [i.e., the matrices $[\cdot]^*$ in (3.3) and (3.8a) are determined now by replacing, respectively E, A, B in (3.3) with $E^{\blacklozenge}$, $A^{\blacklozenge}$, $B^{\blacklozenge}$ in (2.14a)], and the terms α and β on the left-hand side of (4.1b) are replaced by $\alpha^{\blacklozenge}$ and $\beta^{\blacklozenge}$. Consider now the regular system

$$\dot{\zeta} = \alpha^{\blacklozenge} \zeta + \beta^{\blacklozenge} v, \ \zeta \in R^{n-e-k^{\blacklozenge}}. \tag{4.2}$$

Then, the system (1.1) is CPA if and only if (4.2) is controllable. Moreover, any eigenvalue λ_i of $\alpha^{\blacklozenge}$ is relocatable by use of a state feedback in (4.2) if and only if λ_i is a relocatable root of det $(sE - (A + BF))$ in (1.1). If $n - e - k^{\blacklozenge} = 0$, then for any admissible F, det $(sE - (A + BF)) \ne 0$, $\forall s$.

Proof: a) The proof of this part along with the algorithm for the derivation of α, β, and the desired state feedback $u = Fx + r$ follows directly from the proofs of Theorems 3.1 and 3.2. b) By replacing (1.1) with (2.14a), and applying the result of Theorem 4, this part follows from part a). □

We are turning now to the combined problem of stabilizing a singular system, and eliminating impulsive motions. Assume rank $(E) = n - k$, and deg [det $(sE - A)$] $= f$, with $f \le n - k$. Then the free response of the singular system exhibits f exponential modes, and $(n - k) - f$ impulsive modes [13]. Regarding the stability problem, it has been proved rigorously [14] that a singular system is asymptotically stable if and only if all of its finite eigenvalues have negative real parts. In light of these facts we present the following results.

Theorem 4.1: Assume rank $([E \ B]) < n$, and consider the system (2.14a). There exists a linear feedback $u = Fx + r$ that eliminates impulsive motion if and only if rank $(E^{\blacklozenge})$ = rank (E).

Proof: Assume first that in (2.13) $q = 1$, i.e., [see (2.9b)] $E^{\blacklozenge} = E_1$, $A^{\blacklozenge} = A_1$, $B^{\blacklozenge} = B_1$. Premultiplication of (1.1) by P^1 in (2.20) and substitution of $x = T^1 y$ imply using (2.21)

$$\begin{bmatrix} E^{\blacklozenge} & E_1^+ \\ 0 & 0 \end{bmatrix} \dot{y} = \begin{bmatrix} A^{\blacklozenge} & A_1^+ \\ 0 & A_{L1} \end{bmatrix} y + \begin{bmatrix} B^{\blacklozenge} \\ 0 \end{bmatrix} u \tag{4.3}$$

where $A_{L1} \in R^{p(1) \times p(1)}$ is a nonsingular matrix. Assume rank $(E^{\blacklozenge})$ = rank (E). Since $[E^{\blacklozenge} \ B^{\blacklozenge}]$ is surjective, from previous results (see the proof of part b) of Theorem 3.2), there exists a matrix $F^{\blacklozenge}$ that satisfies deg [det $(sE^\blacklozenge - (A^\blacklozenge + B^\blacklozenge F^\blacklozenge))$] = rank $(E^\blacklozenge)$ = rank (E). This result implies that for a feedback matrix $F = [F^\blacklozenge \; F^+]$, we have from (4.3) the following:

$$\deg\left[\det\begin{bmatrix} sE^\blacklozenge-(A^\blacklozenge+B^\blacklozenge F^\blacklozenge) & sE_1^+-(A_1^++B^\blacklozenge F^+) \\ 0 & -A_{L1}\end{bmatrix}\right] = \text{rank}\,(E^\blacklozenge)=\text{rank}\,(E) \quad (4.4)$$

and thus for this case we obtain $(n - k) - f = 0$.

Assume rank $(E^\blacklozenge)$ < rank (E). Then, for any $F = [F^\blacklozenge \; F^+]$ (4.4), (2.20), and (2.21) imply

$$\deg\,[\det\,(sP^1ET^1-(P^1AT^1+P^1BF))] = \text{rank}\,(E^\blacklozenge)<\text{rank}\,(P^1ET^1)=\text{rank}\,(E) \quad (4.5)$$

and since $(n - k) - f > 0$, the proof is completed for this case.

Assume $q = 2$ in (2.13), i.e., [see (2.12b) and (2.14a)] $E^\blacklozenge = E_2$, $A^\blacklozenge = A_2$, $B^\blacklozenge = B_2$. In this case one may define nonsingular matrices

$$P^2=\begin{bmatrix} P_1^2 & 0 \\ 0 & I_{p(1)+p(2)}\end{bmatrix},\quad T^2=\begin{bmatrix} T_1^2 & 0 \\ 0 & I_{p(1)+p(2)}\end{bmatrix}$$

where $P_1^2,\ T_1^2 \in R^{(n-(p(1)+p(2)))\times(n-(p(1)+p(2)))}$ that satisfy, following (2.22),

$$P^2P^1[sE-A]T^1T^2=\begin{bmatrix} sE_2-A_2 & * & * \\ 0 & -A_{L2} & * \\ 0 & 0 & -A_{L1}\end{bmatrix} \quad (4.6)$$

where (due to the solvability condition), A_{Li}, $i = 1, 2$, are nonsingular matrices. [A_{L1} is given in (4.4)], $A_{L2} \in R^{p(2)\times p(2)}$, and * represent matrices with an appropriate dimension. By noting that det $(P^2P^1[sE - A]T^1T^2)$ = det $(sE_2 - A_2)$ det (A_{L2}) det (A_{L1}) and using previous arguments we complete the proof of this case ($q = 2$). The continuation of the proof is now obvious. □

Corollary 4.2: a) Assume rank ($[E\ B]$) = n. Then there exists an admissible feedback (1.2) that stabilizes (1.1) and eliminates impulsive modes if and only if there exists a feedback $v = \Phi\zeta + f$ that stabilizes the regular system (4.1). b) Assume rank ($[E\ B]$) < n. Then there exists an admissible feedback that eliminates impulsive motion and stabilizes the (1.1) if and only if rank $(E^\blacklozenge)$ = rank (E) and there exists a feedback $v = \Phi^\blacklozenge\zeta + f$ that stabilizes the system (4.2).

Proof: The corollary is an immediate result of Corollary 4.1 and Theorem 4.1. □

V. Conclusions

This note presents a complete solution to the problem of pole assignment in singular systems. We did not impose any restriction on the system that may reduce the generality of the results. The main result was obtained from the original system rather than its standard canonical form. Therefore, the underlying approach provides a convenient framework for actual computation without performing change of variables, and preserves the original physical significance of the various variables. In fact, the results of this study were obtained either directly from the original system (1.1) (in case rank $[E\ B] = n$), or from (2.14) (in case rank $[E\ B] < n$), which is equivalent to (1.1) with respect to the admissible (reachable) subspace. In the last case Corollary 2.1 shows [observe the structure of $P^\blacklozenge$ in (2.15)] that the (generalized) state variables of (2.14), i.e., the components of $[w^T\ z^T]^T$, are (in different order) the state variables x_i of (1.1).

In the present approach the use of a transformation group enables us to synthesize, using standard procedures borrowed from the regular system theory, a state feedback in a regular rather than in a singular system, and then to change the system back without changing the spectra of the system to the original singular one, in which the required admissible feedback is incorporated. From the computational point of view the significance of this approach is the following: after determining the regular system (4.1), [or (4.2)], one may apply reliable and computational algorithms [8], [9], in order to design a proportional feedback that ensures a desired location of the poles of the singular system.

The combined problem of pole assignment and elimination of impulsive motion has been considered. In this regard the contribution of this note is associated with Corollary 4.2, where it has been established that no conflict exists between the two problems. In other words, if there exist admissible state feedback that removes impulsive motions, and feedback that stabilizes the system, then there exists an admissible feedback that simultaneously eliminates impulses and stabilizes the singular system. A computational algorithm for deriving an appropriate feedback is obtained from the constructive proofs.

References

[1] D. Cobb, "Feedback and pole placement in descriptor variable systems," *Int. J. Contr.*, vol. 33, pp. 1135–1146, 1981.

[2] V. A. Armentano, "Eigenvalue placement for generalized linear systems," *Syst. Contr. Lett.*, vol. 4, pp. 199–202, 1984.

[3] L. R. Fletcher, J. Kautsky, and N. K. Nichols, "Eigenstructure assignment in descriptor systems," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 1138–1141, 1986.

[4] K. Ozcaldiran and F. Lewis, "A geometric approach to eigenstructure assignment for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 629–632, 1987.

[5] Y.-Y. Wang, S.-J. Shi, and Z.-J. Zhang, "Pole placement and compensator design of generalized systems," *Syst. Contr. Lett.*, vol. 8, pp. 205–209, 1987.

[6] R. Mukundan and W. Dayawansa, "Feedback control of singular system—Proportional and derivative feedback of the state," *Int. J. Syst. Sci.*, vol. 14, pp. 615–632, 1983.

[7] Z. Zhou, M. A. Shayman, and T.-J. Tarn, "Singular systems: A new approach in the time domain," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 42–50, 1987.

[8] P. HR. Petkov, N. D. Christov, and M. M. Konstantinov, "A computational algorithm for pole assignment of linear multiinput systems," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 1044–1047, 1986.

[9] K. Furuta and S. B. Kim, "Pole assignment in a specific disk," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 423–427, 1987.

[10] K. Ozcaldiran and F. L. Lewis, "A result on the placement of infinite eigenvalues in descriptor systems," in *Proc. ACC*, San Diego, CA, 1984, pp. 366–371.

[11] A. Ailon, "Controllability of generalized linear time-invariant systems," *IEEE Trans. Automat. Contr.*, vol. AC-32, pp. 429–432, 1987.

[12] E. L. Yip and R. F. Sincovec, "Solvability, controllability, and observability of continuous descriptor systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 702–707, 1981.

[13] G. C. Verghese, B. Lévy, and T. Kailath, "A generalized state-space for singular systems," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 811–831, 1981.

[14] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," *J. Optimiz. Theory Appl.*, vol. 30, pp. 601–620, 1980.

Minimal Realization of Transfer Function Matrices Via One Orthogonal Transformation

CONSTANTINE P. THERAPOS

***Abstract*—The minimal realization of a given arbitrary transfer function matrix $G(s)$ is obtained by applying one orthogonal similarity transformation to the controllable realization of $G(s)$. The similarity transformation is derived by computing the QR or the singular value decomposition of a matrix constructed from the coefficients of $G(s)$.**

I. Introduction

Many problems arising in control engineering are solved considerably easier if the system under consideration is described by a proper model. Therefore, it is very frequently convenient to represent a linear dynamical

Manuscript received January 4, 1988; revised July 11, 1988.

The author is with the Department of Electrical Engineering, National Technical University of Athens, Athens, Greece.

IEEE Log Number 8928715.

0018-9286/89/0800-0893$01.00