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A matrix pencil approach to the local stability analysis of non-linear circuits

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SUMMARY

This paper addresses local stability issues in non-linear circuits via matrix pencil theory. The limitations of the state–space approach in circuit modelling have led to semistate formulations, currently framed within the context of differential-algebraic equations (DAEs). Stability results for these DAE models can be stated in terms of matrix pencils, avoiding the need for state–space reductions which are not advisable in actual circuit simulation problems. The stability results here presented are applied to electrical circuits containing non-linear devices such as Josephson junctions or MOS transistors. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: differential-algebraic equation; matrix pencil; non-linear circuit; semistate equation; stability

1. INTRODUCTION

The state–space approach for the time-domain analysis of non-linear lumped networks became popular in the late 1950s [1]. This approach is characterized by the formulation of circuit equations in terms of an explicit ordinary differential equation (ODE), which uses a minimal set of state variables taken from either capacitor voltages and inductor currents, or capacitor charges and inductor fluxes. Different qualitative issues can be tackled within this framework: a nice survey of the main results up to 1980 can be found in Reference [2].

However, a state–space equation does not always exist globally or is difficult to obtain in practice, and such a formulation presents several drawbacks from a computational point of view [3]. In the early 1980s, some of the limitations of the state–space approach led

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to *semistate* (also called *descriptor*, *constrained* and *generalized state*) formulations [4–6], which allow some redundancy between the variables appearing in the network equations. Modern schemes used in computer simulation of non-linear networks modelling electronic devices or integrated circuits employ the semistate approach: this is the case of modified nodal analysis (MNA), used in SPICE, or Tableau analysis (see References [3, 7–10] and the bibliography therein). These semistate formulations are now framed within the context of differential-algebraic equations (DAEs) [11–15], and the above-mentioned state–space equations can be seen, under certain assumptions, as explicit ODEs describing the *reduced* flow on the so-called solution manifold.

One of the main qualitative properties of autonomous non-linear circuits concerns the existence and stability of equilibrium points in the state–space equation. Usually, there exists a one-to-one correspondence between equilibria and DC operating points, that is, solutions of the resistive network obtained after short-circuiting inductors and open-circuiting capacitors. The precise relation between the notions of equilibrium and operating point, together with several illustrating examples, can be found in References [2, 16]. The existence and uniqueness of DC operating points in different families of dynamic circuits, as well as their practical computation, have received considerable recent attention [17–23].

Stability features of equilibria in non-linear circuits have also been the focus of considerable research in the last decades: see References [2, 16, 24–29] and the bibliography therein. This local stability analysis is commonly addressed via linearization techniques or Lyapunov functions, and often relies upon a state–space formulation. Nevertheless, as mentioned above, the state–space approach has several drawbacks and, additionally, the scope of these stability results is restricted by certain hypotheses which are needed to obtain such a state–space formulation (see e.g. References [28, 29]). It is therefore of interest to enunciate local stability results which apply to equilibrium points of semistate (differential-algebraic) equations, avoiding the need for explicit ODE reductions. This will be particularly useful in problems leading to higher-index DAEs, for which the actual computation of a state–space equation is usually a difficult task.

Several results concerning stability of equilibria in DAEs have been discussed in the last decade [30–35]. In this context, the main purpose of the present work is to show how asymptotic stability properties of equilibria in semistate-modelled non-linear circuits can be tackled via linearization techniques based on matrix pencil theory; this will provide an alternative to the methods presented in References [16, 25, 26]. Our results will be applied to electrical circuits containing non-linear devices such as Josephson junctions [7] or metal-oxide-semiconductor (MOS) transistors. In particular, the present approach will make it possible to relax some assumptions in the local stability analysis of circuits containing MOS transistors carried out in Reference [29]. It is worth remarking that our goal is to present a (matrix-pencil based) analytical framework for the stability analysis of DAE-modelled circuits: this generality will exclude specific circuit-theoretic characterizations such as those of References [28, 29]. Particular circuit-theoretic stability results for different families of non-linear networks described in terms of semistate equations are the focus of future research.

The paper is structured as follows. Section 2 presents some background on DAEs and DAE-modelling of non-linear circuits. Stability issues in the DAE setting are discussed in Section 3, which includes a brief introduction to matrix pencils. Section 4 addresses the application of these results to the local stability analysis of circuits containing Josephson junctions and, in more detail, MOS transistors. Concluding remarks are compiled in Section 5.

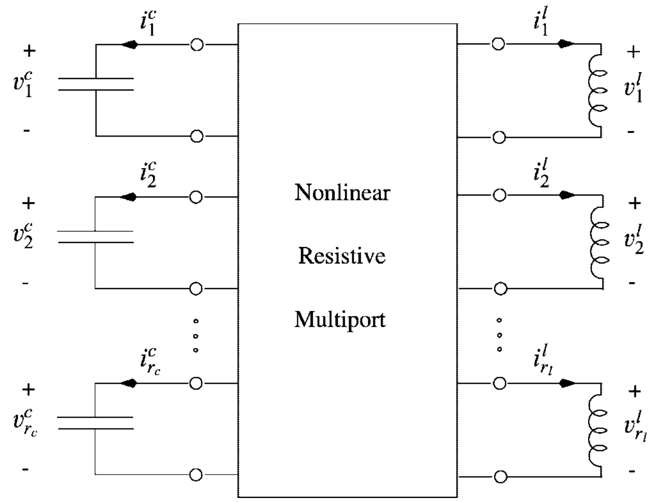

Figure 1. Nonlinear dynamic circuit.

2. BACKGROUND: DAE MODELLING OF NON-LINEAR CIRCUITS

The use of differential-algebraic models for non-linear networks has made it possible to introduce important concepts in the circuit framework, such as the *index* notion (see References [3, 9, 10, 36] and the bibliography therein), and has also led to a precise characterization of *singular* phenomena such as those occurring at impasse points [6, 37–39]. In this section, we present some general background on DAEs, as well as their use on circuit modelling.

We will focus on autonomous non-linear networks, which will be viewed as multiports terminated by capacitors and inductors (Figure 1), as it is done e.g. in References [2, 6]. Specifically, consider an autonomous non-linear circuit consisting of r_c (possibly coupled and non-linear) capacitors, r_l (possibly coupled and non-linear) inductors, and a given number of DC sources and devices with a non-dynamic $i-v$ characteristic such as resistors, diodes, controlled sources, etc., which define a multiport with exactly $r=r_c+r_l$ ports connected to the above-mentioned reactances.

Let $(q, \phi, i^c, v^c, i^l, v^l)$ be a vector in $\mathbb{R}^{3r}$ describing capacitor charges, inductor fluxes and capacitor/inductor currents/voltages, respectively. Such a circuit is described by the following semiexplicit DAE:

$$q' = i^c \tag{1a}$$

$$\phi' = v^l \tag{1b}$$

$$0 = g^c(q, v^c) \tag{1c}$$

$$0 = g^l(\phi, i^l) \tag{1d}$$

$$0 = g^r(i^c, v^c, i^l, v^l) \tag{1e}$$

where $g^{c} \in C^1(\mathbb{R}^{2r_c}, \mathbb{R}^{r_c})$ (resp. $g^{l} \in C^1(\mathbb{R}^{2r_l}, \mathbb{R}^{r_l})$) describes the constitutive relations of capacitors (resp. inductors), and $g^{r} \in C^1(\mathbb{R}^{2r}, \mathbb{R}^{r})$ stands for the driving-point characteristics of the resistive multiport.

Schemes such as MNA, used in computer simulation programs (e.g. in SPICE), or Tableau Analysis (TA), also formulate the network equations as a differential-algebraic system [3, 10]. The simple form (1) suffices for the stability discussion performed in this work and, from the author's point of view, it helps to avoid confusion with specific details arising while setting up the circuit equations. The matrix pencil tools for the stability analysis discussed in Section 3 seem to have a natural extension to the MNA and TA settings.

Equation (1) has the semiexplicit form

$$x' = f(x, y) \tag{2a}$$

$$0 = g(x, y) \tag{2b}$$

where $x \in \mathbb{R}^r$ stands for the *dynamic variables* ((q, ϕ) in Equation (1)), and $y \in \mathbb{R}^p$ denotes the *algebraic* ones ($(i^{c}, v^{c}, i^{l}, v^{l})$ in Equation (1)), $f \in C^1(\mathbb{R}^{r+p}, \mathbb{R}^r)$, $g \in C^1(\mathbb{R}^{r+p}, \mathbb{R}^p)$. The analytical or numerical study of these equations are usually organized around the *index* concept: in the sequel, we focus for simplicity on index-1 and index-2 DAEs.

2.1. Index-1 problems

Semiexplicit index-1 DAEs: Consider the DAE (2), and assume that both f and g are C^1 functions. Let (x^*, y^*) satisfy $g(x^*, y^*) = 0$. If the derivative $g_y(x^*, y^*)$ defines an invertible matrix, then Equation (2) is said to have differential index one around (x^*, y^*) [11]. This notion is supported on the fact that one differentiation in Equation (2b) suffices to obtain an explicit *underlying* ODE

$$x' = f(x, y) \tag{3a}$$

$$y' = -g_y^{-1}(x, y) g_x(x, y) f(x, y) \tag{3b}$$

for which $g = 0$ is an invariant comprising the solutions of the original DAE.

Under the index-1 assumption, Equation (2b) locally defines a C^1 r-dimensional *solution manifold* $\mathscr{M}_1$ where the solutions of the DAE lie. Also, there exists a local C^1 map ψ such that $g(x, y) = 0 \Leftrightarrow y = \psi(x)$. The dynamical behaviour on $\mathscr{M}_1$ may be described, in the parameterization defined by x, through the *reduced explicit* ODE

$$x' = f(x, \psi(x)) \tag{4}$$

Note that $f(x, \psi(x))$ is a C^1 map.

Index-1 circuit equations: The index-1 condition amounts, in the circuit DAE (1), to the non-singularity of the matrix

$$g_y = \begin{pmatrix} 0 & \dfrac{\partial g^{c}}{\partial v^{c}} & 0 & 0 \\ 0 & 0 & \dfrac{\partial g^{l}}{\partial i^{l}} & 0 \\ \dfrac{\partial g^{r}}{\partial i^{c}} & \dfrac{\partial g^{r}}{\partial v^{c}} & \dfrac{\partial g^{r}}{\partial i^{l}} & \dfrac{\partial g^{r}}{\partial v^{l}} \end{pmatrix} \tag{5}$$

which in turn has the following consequences:

- The matrix $\partial g^{\mathrm{c}}/\partial v^{\mathrm{c}}$ is non-singular. In circuit theoretic terms, this implies that the capacitors admit a local *charge-controlled* description $v^{\mathrm{c}}=\psi^{\mathrm{c}}(q)$;
- The matrix $\partial g^{\mathrm{l}}/\partial i^{\mathrm{l}}$ is also non-singular and, therefore, the inductors have a local *flux-controlled* representation $i^{\mathrm{l}}=\psi^{\mathrm{l}}(\phi)$;
- The matrix $(\partial g^{r}/\partial i^{\mathrm{c}}\ \partial g^{r}/\partial v^{\mathrm{l}})$ is non-singular. This means that the multiport can be locally described through a *hybrid* representation $i^{\mathrm{c}}=\psi_1^{r}(v^{\mathrm{c}},i^{\mathrm{l}}),\ v^{\mathrm{l}}=\psi_2^{r}(v^{\mathrm{c}},i^{\mathrm{l}})$.

Note that these are the standard circuit-theoretic assumptions supporting the existence of a local state–space description in terms of (q,ϕ) [2]:

$$q' = \psi_1^{r}(\psi^{\mathrm{c}}(q),\psi^{\mathrm{l}}(\phi)) \tag{6a}$$

$$\phi' = \psi_2^{r}(\psi^{\mathrm{c}}(q),\psi^{\mathrm{l}}(\phi)) \tag{6b}$$

which represents the circuit analogue of Equation (4). In DAE terms, Equation (6) is simply a description of the reduced flow using the variables (q,ϕ) to parameterize the solution manifold represented by Equations (1c)–(1e). In Section 3, it will be shown that there is no need to perform such a reduction in order to analyse stability properties of equilibria, and that simple statements guaranteeing that a given equilibrium is asymptotically stable or unstable can be directly enunciated in terms of the original equation (1).

It is also worth remarking that the reduced or state–space description need not be unique. Assume that, besides the index-1 hypothesis, the matrices $\partial g^{\mathrm{c}}/\partial q$ and $\partial g^{\mathrm{l}}/\partial \phi$ are non-singular, supporting local *voltage-controlled* and *current-controlled* descriptions $q=\varphi^{\mathrm{c}}(v^{\mathrm{c}}),\ \phi=\varphi^{\mathrm{l}}(i^{\mathrm{l}})$ for the capacitors and inductors, respectively. Differentiation in Equations (1c)–(1d) leads to a quasilinear $v^{\mathrm{c}}-i^{\mathrm{l}}$ description of the dynamics, namely (writing $v\equiv v^{\mathrm{c}},\ i\equiv i^{\mathrm{l}}$ for notational simplicity)

$$\frac{\partial g^{\mathrm{c}}}{\partial v}(\varphi^{\mathrm{c}}(v),v)v' = -\frac{\partial g^{\mathrm{c}}}{\partial q}(\varphi^{\mathrm{c}}(v),v)\psi_1^{r}(v,i)$$

$$\frac{\partial g^{\mathrm{l}}}{\partial i}(\varphi^{\mathrm{l}}(i),i)i' = -\frac{\partial g^{\mathrm{l}}}{\partial \phi}(\varphi^{\mathrm{l}}(i),i)\psi_2^{r}(v,i)$$

or, equivalently,

$$C(v)v' = \psi_1^{r}(v,i)$$

$$L(i)i' = \psi_2^{r}(v,i)$$

where

$$C(v) = -\left(\left(\frac{\partial g^{\mathrm{c}}}{\partial q}\right)^{-1}\frac{\partial g^{\mathrm{c}}}{\partial v}\right)(\varphi^{\mathrm{c}}(v),v) \tag{7a}$$

$$L(i) = -\left(\left(\frac{\partial g^{\mathrm{l}}}{\partial \phi}\right)^{-1}\frac{\partial g^{\mathrm{l}}}{\partial i}\right)(\varphi^{\mathrm{l}}(i),i) \tag{7b}$$

stand for the incremental capacitance and inductance matrices. From the non-singularity of $C(v)$ and $L(i)$ which follows from the index-1 assumption, this description trivially amounts to the explicit one

$$v' = C(v)^{-1}\psi_1^r(v, i) \tag{8a}$$

$$i' = L(i)^{-1}\psi_2^r(v, i) \tag{8b}$$

Equations (6) and (8) can be seen as two descriptions of the same reduced flow using two different parameterizations of the solution manifold (1c)–(1e).

2.2. *Index-2 problems*

Semiexplicit index-2 DAEs: Assume that $g(x^*, y^*) = 0$ but $\mathrm{rk}\, g_y(x^*, y^*) < p$ in the semiexplicit DAE (2), that is, consider a situation in which g_y is a singular matrix. One differentiation in (2b) yields the quasilinear equation

$$x' = f(x, y) \tag{9a}$$

$$g_y(x, y)y' = -g_x(x, y)f(x, y) \tag{9b}$$

which cannot be written in explicit form due to the non-invertibility of g_y. If $\mathrm{rk}\, g_y$ is constant around (x^*, y^*), we are typically led to a higher-index problem. Higher-index DAEs present additional difficulties from both analytical and numerical points of view [11–15]. In the sequel we will define index-2 semiexplicit DAEs and then focus, for the sake of simplicity, on index-2 problems *in Hessenberg form*.

Suppose that f and g in (2) are C^1 and C^2, respectively. Let (x^*, y^*) satisfy not only $g(x^*, y^*) = 0$ but also the *hidden constraint* $(g_x f)(x^*, y^*) \in \mathrm{Im}\, g_y(x^*, y^*)$, and assume that g_y has constant rank $s < p$ around (x^*, y^*). Let $S(x, y)$ be a C^1 non-singular $p \times p$ matrix function such that

$$S(x, y)g_y(x, y) = \begin{pmatrix} S_1(x, y)g_y(x, y) \\ S_2(x, y)g_y(x, y) \end{pmatrix} = \begin{pmatrix} B(x, y) \\ 0 \end{pmatrix}$$

S_1 and S_2 representing $s \times p$ and $(p - s) \times p$ matrices, respectively. In this situation, Equation (2) is said to have differential index-2 around (x^*, y^*) if the matrix

$$\begin{pmatrix} S_1 g_y \\ (S_2 g_x f)_y \end{pmatrix}_{(x^*, y^*)} \tag{10}$$

is non-singular.

Hessenberg form: The meaning of the index-2 notion becomes much clearer if the attention is restricted to semiexplicit DAEs in Hessenberg form of size 2

$$x' = f(x, y) \tag{11a}$$

$$0 = g(x) \tag{11b}$$

where $f \in C^1(\mathbb{R}^{r+p}, \mathbb{R}^r)$, $g \in C^2(\mathbb{R}^r, \mathbb{R}^p)$, and $r > p$. The hidden constraint reads in this case, after differentiation of (11b) and in light of (11a),

$$0 = g_x(x) f(x, y) \tag{12}$$

In this situation, we can take $S = S_2 = I_p$, and the index-2 condition amounts to the non-singularity of the matrix $g_x(x^*) f_y(x^*, y^*)$, that is, to the condition $\mathrm{rk}(g_x(x^*) f_y(x^*, y^*)) = p$. This implies in particular that $\mathrm{rk}\, g_x(x^*) = p$, and then

$$\mathrm{rk} \begin{pmatrix} g_x & 0 \\ (g_x f)_x & g_x f_y \end{pmatrix}_{(x^*, y^*)} = 2p \tag{13}$$

meaning that (11b)–(12) define an $(r - p)$-dimensional C^1 *solution manifold* $\mathscr{M}_2$, where the solutions of the DAE lie.

From the condition $\mathrm{rk}\, g_x(x^*) = p$, it follows that x can be split into $r - p$ variables z (which can be assumed w.l.o.g. to be $x_1, \ldots, x_{r-p}$) and p variables w $(x_{r-p+1}, \ldots, x_r)$ so that, with notational abuse, $\mathrm{rk}\, g_w(x^*) = p$ and, therefore, $g = 0$ amounts to $w = \varphi(z)$ for some local C^2 map φ. Also, from the condition $\mathrm{rk}\,(g_x(x^*) f_y(x^*, y^*)) = p$, we can describe the set $g_x f = 0$ through a relation of the form $y = \eta(z, w)$, η being C^1, and, on the intersection with $g = 0$, $y = \eta(z, \varphi(z)) \equiv \zeta(z)$, where ζ is also C^1. Using these co-ordinates, and splitting $f = (f_1, f_2)$, where f_1 (resp. f_2) denotes the first $r - p$ (resp. last p) components of f, we get from Equation (11a) the following C^1 explicit ODE or state–space equation for the description of reduced dynamics in $\mathscr{M}_2$:

$$z' = h(z) \equiv f_1(z, \varphi(z), \zeta(z)) \tag{14}$$

This is an index-2 analogue of Equation (4).

Index-2 circuit equations in Hessenberg form: In the DAE-modelled circuit (1), a higher index may be displayed if the matrix g_y in Equation (5) is singular, due for example to the existence of certain configurations such as V–C loops or J–L cutsets: see References [3, 9, 10, 36] and the references cited therein. The index-2 condition represented by the non-singularity of Equation (10) is not easy to compute in practical cases, but this is notably simplified in Hessenberg problems. In the following, we illustrate how certain index-2 circuits can be easily modelled through a DAE in Hessenberg form.

Assume that the circuit described by Equation (1) includes a V–C loop. The extension to cases with several V–C loops and/or J–L cutsets is straightforward. Suppose that $k = 1, \ldots, n$ are the subindices of the capacitors in the V–C loop, and let E represent the sum of DC source voltages in the loop. This means that one of the equations in Equation (1e) can be written as

$$\sum_{k=1}^{n} v_k^{\mathrm{c}} = E$$

and, therefore, the hybrid description of the multiport in terms of v^{c} and i^{l} is no longer possible. Suppose that $r_{\mathrm{c}} - 1$ capacitor currents $\hat{i}^{\mathrm{c}}$ (say $\hat{i}^{\mathrm{c}} = (i_2^{\mathrm{c}}, \ldots, i_{r_{\mathrm{c}}}^{\mathrm{c}})$) and all r_{l} inductor voltages can still be obtained from the remaining equations in Equation (1e), that is,

$$\hat{i}^{\mathrm{c}} = \hat{\psi}_1^r(v^{\mathrm{c}}, i^{\mathrm{l}}, i_1^{\mathrm{c}})$$

$$v^{\mathrm{l}} = \hat{\psi}_2^r(v^{\mathrm{c}}, i^{\mathrm{l}}, i_1^{\mathrm{c}})$$

and that the non-singularity assumption on $\partial g^{\mathrm{c}}/\partial v^{\mathrm{c}}$ and $\partial g^{\mathrm{l}}/\partial i^{\mathrm{l}}$, supporting the charge and flux-controlled descriptions

$$v^{\mathrm{c}} = \psi^{\mathrm{c}}(q)$$

$$i^{\mathrm{l}} = \psi^{\mathrm{l}}(\phi)$$

in the capacitors and inductors, is still valid. Performing the corresponding substitutions, we arrive to

$$q' = \begin{pmatrix} i_1^{\mathrm{c}} \\ \hat{\psi}_1^r(\psi^{\mathrm{c}}(q), \psi^{\mathrm{l}}(\phi), i_1^{\mathrm{c}}) \end{pmatrix} \tag{15a}$$

$$\phi' = \hat{\psi}_2^r(\psi^{\mathrm{c}}(q), \psi^{\mathrm{l}}(\phi), i_1^{\mathrm{c}}) \tag{15b}$$

$$0 = \sum_{k=1}^{n} \psi_k^{\mathrm{c}}(q) - E \tag{15c}$$

This is a Hessenberg DAE of size 2. With regard to the general Hessenberg equation of size 2 (11), x would still stand for the dynamic variables (q, ϕ), but now y would denote the unique algebraic variable i_1^{c} which remains in the equation after the above-indicated substitutions. The reduced or state–space equation (14), for the DAE-modelled circuit (15), would describe an explicit ODE in terms of $r-1$ ($r-(p_1+p_2)$ in index-2 cases with p_1 V–C loops and p_2 J–L cutsets) variables z taken from within the set of dynamic variables (q, ϕ).

This Hessenberg formulation is also feasible in terms of capacitor voltages and inductor currents. Assume, additionally, that the matrices $\partial g^{\mathrm{c}}/\partial q$ and $\partial g^{\mathrm{l}}/\partial \phi$ are non-singular, supporting local voltage-controlled and current-controlled descriptions $q = \varphi^{\mathrm{c}}(v^{\mathrm{c}})$, $\phi = \varphi^{\mathrm{l}}(i^{\mathrm{l}})$ in the capacitors and inductors, respectively. Operating as in Section 2.1, and with the notation of Equation (7), it is not difficult to derive the following formulation in terms of $v \equiv v^{\mathrm{c}}$, $i \equiv i^{\mathrm{l}}$, and the algebraic variable i_1^{c}:

$$v' = C(v)^{-1} \begin{pmatrix} i_1^{\mathrm{c}} \\ \hat{\psi}_1^r(v, i, i_1^{\mathrm{c}}) \end{pmatrix} \tag{16a}$$

$$i' = L(i)^{-1} \hat{\psi}_2^r(v, i, i_1^{\mathrm{c}}) \tag{16b}$$

$$0 = \sum_{k=1}^{n} v_k - E \tag{16c}$$

Hessenberg formulations of several instances of index-2 circuits are presented in Section 4.

3. STABILITY OF EQUILIBRIA IN SEMIEXPLICIT DAEs

Equilibrium points of the semiexplicit DAE (2) are defined by the conditions $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$. In the DAE-modelled circuit (1), these conditions amount to $i^{\mathrm{c}} = 0$, $v^{\mathrm{l}} = 0$,

$g^{c}(q, v^{c}) = 0$, $g^{l}(\phi, i^{l}) = 0$, $g^{r}(i^{c}, v^{c}, i^{l}, v^{l}) = 0$. In particular, the conditions $i^{c} = 0$, $v^{l} = 0$ show that equilibria of the DAE exactly correspond to DC operating points of the circuit, which are defined as solutions of the network equation obtained after open-circuiting capacitors and short-circuiting inductors.

As it was indicated in Section 1, stability aspects of equilibria in non-linear circuits have been usually tackled through a state–space reduction (see e.g. References [2, 27–29]). We show in this section that such a reduction is not necessary for the statement of the stability results, which can be enunciated in the original DAE setting through the use of *matrix pencil* theory.

3.1. Matrix pencils

Given two $n \times n$ matrices A, B, the *matrix pencil* $\{A, B\}$ is defined as the one-parameter family $\{\lambda A + B : \lambda \in \mathbb{C}\}$. The importance of this concept stems from its relation with the linearization of a semiexplicit DAE at a given equilibrium, as it will be shown in Sections 3.2 and 3.3.

The *spectrum* $\sigma(A, B)$ of the matrix pencil $\{A, B\}$ is the set $\{\lambda \in \mathbb{C} : \det(\lambda A + B) = 0\}$. Remark that $\det(\lambda A + B)$ is a polynomial in λ with degree $m \leqslant n$: when this polynomial does not vanish identically or, equivalently, when there exists a λ_0 such that $\det(\lambda_0 A + B) \neq 0$, the matrix pencil is called *regular*. In this situation, the matrix pencil may be reduced to the so-called *Kronecker canonical form* [11, 40]. Namely, it may be proved that a regular matrix pencil admits non-singular matrices $E, F \in \mathbb{R}^{n \times n}$ such that

$$EAF = \begin{pmatrix} I_m & 0 \\ 0 & N \end{pmatrix}, \quad EBF = \begin{pmatrix} W & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

where $W \in \mathbb{R}^{m \times m}$ for some $m \leqslant n$ and $N \in \mathbb{R}^{(n-m) \times (n-m)}$ is a nilpotent matrix with index $v \leqslant n - m$, that is, $N^{v} = 0$, $N^{v-1} \neq 0$. In particular, N can be taken in lower-triangular Jordan form. The nilpotency index v is called the *Kronecker index* of the matrix pencil. The pencil is said to have index 0 if $m = n$, which amounts to require that A be a non-singular matrix.

The Kronecker canonical form provides much insight into the spectral behaviour of the matrix pencil. If $\{A, B\}$ is a regular matrix pencil, it is not difficult to check that $\sigma(A, B) = \sigma(-W)$ (see e.g. Reference [34]). This means that the matrix pencil has exactly m eigenvalues or, equivalently, $\det(\lambda A + B)$ is a polynomial in λ with degree m. When the matrix pencil comes from the linearization of a semiexplicit DAE at a given equilibrium point, these eigenvalues will be shown to describe, under mild assumptions, the linearized stability properties of the equilibrium.

3.2. Index-1 DAEs

Consider an equilibrium point (x^*, y^*) of the semiexplicit DAE (2) where $g_y(x^*, y^*)$ is non-singular. This assumption implies that the DAE has index 1 and, therefore, the explicit ODE $x' = f(x, \psi(x))$ displayed in Equation (4), which has an equilibrium at x^*, describes the reduced flow on the solution manifold $\mathscr{M}_1$, as indicated in Section 2.1.

The linear stability features of this equilibrium for the dynamics on $\mathscr{M}_1$ are characterized by the matrix

$$(f_x + f_y \psi_x)(x^*, \psi(x^*)) = (f_x - f_y g_y^{-1} g_x)(x^*, \psi(x^*))$$

but its spectrum may be directly computed from the matrix pencil describing the linearization of the original DAE, as shown below. Although this is a well-known result (see References [33, 34] and the bibliography therein), the proof is included with illustrative purposes.

Lemma 1

Consider the semiexplicit DAE (2) around an equilibrium (x^*, y^*) defined by $f(x^*, y^*) = 0$, $g(x^*, y^*) = 0$. Assume that f and g are C^1, and that the DAE has index 1 around (x^*, y^*), that is, let $g_y(x^*, y^*)$ be non-singular. Denote

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; J_1 = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} (x^*, y^*) \tag{17}$$

Then, $\sigma(f_x - f_y g_y^{-1} g_x)(x^*, y^*) = \sigma(A, -J_1)$.

Proof

1. Let $\lambda \in \sigma(f_x - f_y g_y^{-1} g_x)$, and assume that v is an associated eigenvector, so that

$$(f_x - f_y g_y^{-1} g_x) v = \lambda v \tag{18}$$

Note that all matrices are assumed to be evaluated at (x^*, y^*). Denoting $u_1 = v$, $u_2 = -g_y^{-1} g_x v$, Equation (18) may be rewritten as $f_x u_1 + f_y u_2 = \lambda u_1$ which, together with the fact that $g_x u_1 + g_y u_2 = 0$, yields

$$\lambda \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{19}$$

This implies that $\lambda \in \sigma(A, -J_1)$.

2. Conversely, if $\lambda \in \sigma(A, -J_1)$ and $u = (u_1, u_2)$ is an associated eigenvector, Equation (19) will hold. From this identity we have $u_2 = -g_y^{-1} g_x u_1$ and $\lambda u_1 = (f_x - f_y g_y^{-1} g_x) u_1$. Therefore, Equation (18) is satisfied with $v = u_1$, showing that $\lambda \in \sigma(f_x - f_y g_y^{-1} g_x)$. □

As shown in Appendix A, the non-singularity of $g_y(x^*, y^*)$, which supports the differential index-1 nature of the DAE, is equivalent to the fact that the matrix pencil $\{A, -J_1\}$ has Kronecker index 1. This means that the differential index and the so-called *local index*, that is, the Kronecker index of the linearization, are the same for semiexplicit index-1 DAEs. This simplifies the conditions needed to check the asymptotic stability of a given equilibrium, as shown in Theorem 1. Its proof is a direct consequence of Lemma 1 and the Lyapunov linearization method for the stability analysis of equilibria in explicit ODEs: the reader is referred to [41, Theorem 15.6] for a statement of this result for C^1 explicit ODEs.

Theorem 1

Consider the semiexplicit DAE (2) around an equilibrium (x^*, y^*). Assume that f and g are C^1, and that:

1. The matrix pencil $\{A, -J_1\}$ defined in Equation (17) is regular with Kronecker index 1 or, equivalently, $g_y(x^*, y^*)$ is non-singular;
2. $\sigma(A, -J_1) \subset \mathbb{C}^- \equiv \{z \in \mathbb{C} / \mathrm{Re}(z) < 0\}$.

Then (x^*, y^*) is an asymptotically stable equilibrium for the dynamics of the DAE on the solution manifold $\mathscr{M}_1$ defined by Equation (2b).

3.3. *Index-2 DAEs in Hessenberg form*

The state–space based stability analysis of equilibria in higher-index DAEs becomes more involved than that of index-1 problems. We show below that conditions guaranteeing asymptotic stability of equilibria in index-2 Hessenberg problems can also be stated in matrix pencil terms, avoiding the need for state–space reductions in the local stability analysis of non-linear circuits. These results will be applied in Section 4 to the local stability analysis of circuits containing non-linear devices such as Josephson junctions or MOS transistors. It is worth remarking that more general approaches have been developed for DAEs with either *geometrical* or *tractability* index 2: nevertheless, these techniques require much more elaborated frameworks. The reader is referred to References [38] and [30–32], respectively, for details.

The reduced flow of a Hessenberg index-2 DAE can be described through the C^1 explicit ODE $z' = h(z)$ depicted in Equation (14). Focusing on an equilibrium point (x^*, y^*), which leads to an equilibrium z^* in the reduced equation, linear stability properties are characterized by the matrix $h'(z^*)$. Lemma 2 shows that its spectrum may be directly computed from a certain matrix pencil, avoiding the need for such a state–space reduction in practical cases.

Lemma 2

Consider the Hessenberg DAE (11) around an equilibrium (x^*, y^*) defined by $f(x^*, y^*) = 0$, $g(x^*) = 0$. Assume that f and g are C^1 and C^2, respectively, and that the DAE has index 2 around (x^*, y^*), that is, let $g_x(x^*) f_y(x^*, y^*)$ be non-singular. Denote

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} f_x & f_y \\ g_x & 0 \end{pmatrix} (x^*, y^*) \tag{20}$$

Let z^* stand for the corresponding equilibrium in the reduced ODE $z' = h(z)$ displayed in Equation (14). Then, $\sigma(h'(z^*)) = \sigma(A, -J_2)$.

Proof

0. Recall the notation $x = (z, w)$ and $f = (f_1, f_2)$ presented in Section 2.2, and the description $w = \varphi(z)$, $y = \zeta(z)$ of the solution manifold $\mathscr{M}_2$, from which $h(z)$ was defined as $h(z) \equiv f_1(z, \varphi(z), \zeta(z))$. Hence,

$$h' = f_{1_z} + f_{1_w} \varphi_z + f_{1_y} \zeta_z$$

Since $w = \varphi(z)$ comes from the equation $g = 0$, and w is such that g_w is non-singular, it follows that

$$\varphi_z = -\, g_w^{-1} g_z$$

Similarly, $y = \eta(x) \equiv \eta(z, w)$ arises from the equation $g_x f = 0$ and, therefore,

$$\eta_x = -\, (g_x f_y)^{-1} g_x f_x$$

where we have used that $(g_x f)_x = g_x f_x$ since $f = 0$ at the equilibrium. Also, $y = \zeta(z) = \eta(z, \varphi(z))$ yields

$$\zeta_z = \eta_z + \eta_w \varphi_z = \eta_x \begin{pmatrix} I_{r-p} \\ \varphi_z \end{pmatrix} = -(g_x f_y)^{-1} g_x f_x \begin{pmatrix} I_{r-p} \\ -g_w^{-1} g_z \end{pmatrix}$$

1. Assume first that $\lambda \in \sigma(h'(z^*))$. This means that there exists a non-vanishing, $(r-p)$-dimensional vector v such that

$$\lambda v = f_{1_z} v - f_{1_w} g_w^{-1} g_z v - f_{1_y} (g_x f_y)^{-1} g_x f_x \begin{pmatrix} I_{r-p} \\ -g_w^{-1} g_z \end{pmatrix} v \tag{21}$$

Denote $u_{11} = v$, and define $u_{12} = -g_w^{-1} g_z v$. Thus, $u_1 = (u_{11}, u_{12})$ satisfies by definition $g_z u_{11} + g_w u_{12} = 0$ or, equivalently,

$$g_x u_1 = 0 \tag{22}$$

From the definitions $u_{11} = v$, $u_{12} = -g_w^{-1} g_z v$ and $u_1 = (u_{11}, u_{12})$, Equation (21) can be rewritten as

$$\lambda u_{11} = f_{1_x} u_1 - f_{1_y} (g_x f_y)^{-1} g_x f_x u_1 \tag{23}$$

Define $u_2 = -(g_x f_y)^{-1} g_x f_x u_1$. This transforms (23) into

$$\lambda u_{11} = f_{1_x} u_1 + f_{1_y} u_2 \tag{24}$$

Also, from the definition of u_2, we have $g_x(f_x u_1 + f_y u_2) = 0$, that is,

$$g_z(f_{1_x} u_1 + f_{1_y} u_2) + g_w(f_{2_x} u_1 + f_{2_y} u_2) = 0$$

or

$$f_{2_x} u_1 + f_{2_y} u_2 = -g_w^{-1} g_z (f_{1_x} u_1 + f_{1_y} u_2) = -g_w^{-1} g_z \lambda u_{11} \tag{25}$$

where we have used (24). From the definition of u_{12} and (25) we have

$$\lambda u_{12} = f_{2_x} u_1 + f_{2_y} u_2 \tag{26}$$

Equations (24) and (26) can be written together as

$$\lambda u_1 = f_x u_1 + f_y u_2$$

which, together with Equation (22), yields

$$\lambda \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} f_x & f_y \\ g_x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0 \tag{27}$$

showing that $\lambda \in \sigma(A, -J_2)$.

2. Assume now that $\lambda \in \sigma(A, -J_2)$, and that (27) holds. This equation can be split into

$$\lambda u_1 = f_x u_1 + f_y u_2 \tag{28a}$$

$$0 = g_x u_1 \tag{28b}$$

Denoting $u_1=(u_{11},u_{12})$, where u_{11} (resp. u_{12}) comprises the first $r-p$ (resp. last p) components of u_1, from Equation (28a) we have

$$\lambda u_{11}=f_{1_z}u_{11}+f_{1_w}u_{12}+f_{1_y}u_2 \tag{29}$$

On the other hand, from Equation (28b) it follows that

$$u_{12}=-g_w^{-1}g_z u_{11} \tag{30}$$

Also, multiplying Equation (28b) by λ and using Equation (28a), we have

$$g_x(f_x u_1+f_y u_2)=0$$

and, therefore,

$$u_2=-(g_x f_y)^{-1}g_x f_x u_1=-(g_x f_y)^{-1}g_x f_x\begin{pmatrix} I_{r-p} \\ -g_w^{-1}g_z \end{pmatrix}u_{11} \tag{31}$$

Substituting Equation (30) and (31) into (29), and denoting $v=u_{11}$, it follows that Equation (21) holds, showing that $\lambda\in\sigma(h'(z^*))$. □

Again, it may be shown (see Appendix A) that the matrix pencil $\{A,-J_2\}$ is regular with index 2 if and only if the matrix $g_x(x^*)f_y(x^*,y^*)$ is non-singular. This means that the differential index and the local (Kronecker) index are the same for index-2 DAEs in Hessenberg form. From this fact, together with Lemma 2 and the principle of linearized stability for C^1 explicit ODEs [41, Theorem 15.6], we get the following set of sufficient conditions for the asymptotic stability of equilibria in Hessenberg index-2 DAEs.

Theorem 2
Consider the Hessenberg DAE (11) around an equilibrium (x^*,y^*). Assume that f and g are C^1 and C^2, respectively, and that:

1. The matrix pencil $\{A,-J_2\}$ defined in Equation (20) is regular with Kronecker index 2 or, equivalently, $g_x(x^*)f_y(x^*,y^*)$ is non-singular;
2. $\sigma(A,-J_2)\subset\mathbb{C}^-$.

Then (x^*,y^*) is an asymptotically stable equilibrium for the dynamics of the DAE on the solution manifold $\mathscr{M}_2$ defined by Equations (11b)–(12).

4. LOCAL STABILITY ANALYSIS OF NON-LINEAR CIRCUITS

4.1. Example 1

Consider the circuit displayed in Figure 2. The device on the right of this figure is a *Josephson junction*, consisting of two superconductors separated by an oxide barrier [7] and characterized by the sinusoidal $i-\phi$ relation

$$i_2=I_0\sin k\phi_2$$

Here, $I_0>0$ is a device parameter, whereas $k=4\pi e/h$, e and h standing for the electron charge and Planck's constant, respectively.

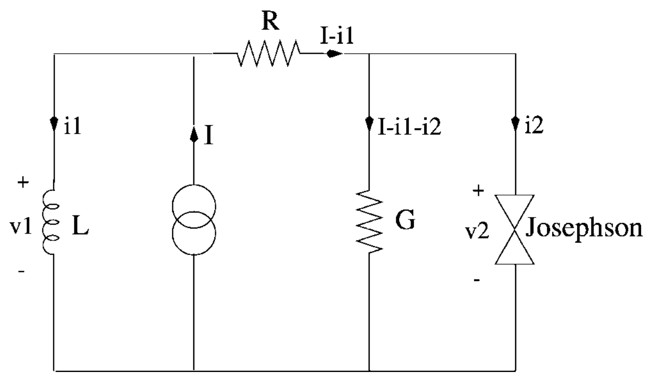

Figure 2. Josephson circuit.

The resistance R and the inductance L are assumed to be positive, whereas for the conductance G we only assume $G \geqslant 0$. The circuit can be easily shown to be modelled by the DAE

$$\phi_1' = v_1 \tag{32a}$$

$$\phi_2' = v_2 \tag{32b}$$

$$0 = Li_1 - \phi_1 \tag{32c}$$

$$0 = i_2 - I_0 \sin k\phi_2 \tag{32d}$$

$$0 = -v_1 + RI - Ri_1 + v_2 \tag{32e}$$

$$0 = -Gv_2 + I - i_1 - i_2 \tag{32f}$$

which has the form depicted in Equation (1). Equilibrium points are defined by the conditions

$$v_1 = v_2 = 0, \quad i_1 = I, \quad \phi_1 = LI, \quad i_2 = 0, \quad \sin k\phi_2 = 0$$

the latter yielding $\phi_2 = n\pi/k$, $n \in \mathbb{Z}$. Our purpose is to analyse the stability of these equilibria in the following two situations:

(a) $G > 0$;
(b) $G = 0$, meaning that the corresponding branch is open-circuited.

The derivation of a state–space equation can be easily performed in the case $G > 0$:

$$\phi_1' = RI - R\phi_1/L + (I - \phi_1/L - I_0 \sin k\phi_2)/G$$
$$\phi_2' = (I - \phi_1/L - I_0 \sin k\phi_2)/G$$

but this reduction is not valid for $G = 0$. Note that a different reduction involving only one dynamic variable ϕ_i would hold in this case. We show below that a unified stability study can be performed in the differential-algebraic setting, without the need for these state–space reductions; the use of matrix pencil tools will additionally allow for a simple analysis of a bifurcation phenomenon occurring in this circuit.

To simplify the discussion, perform the substitutions $i_1 = \phi_1/L$, $i_2 = I_0 \sin k\phi_2$, $v_1 = RI - Ri_1 + v_2$ in Equations (32a), (32b) and (32f), obtaining

$$\phi_1' = RI - (R/L)\phi_1 + v_2 \tag{33a}$$

$$\phi_2' = v_2 \tag{33b}$$

$$0 = -Gv_2 + I - \phi_1/L - I_0 \sin k\phi_2 \tag{33c}$$

This is a flux-oriented analogue of the reductions presented at the end of Section 2.2. The dynamic variables are still (ϕ_1, ϕ_2), and the algebraic one is v_2.

Let us analyse the structure and the index of the network equations (33) in terms of G. If $G > 0$, the derivative of the algebraic constraint (33c) with respect to v_2 is $-G \neq 0$ and, therefore, the DAE has index 1 in case (a). On the other hand, if $G = 0$, the algebraic variable is not present in the constraint and in case (b) we are actually led to a Hessenberg structure of size 2. In circuit-theoretic terms, this is due to the fact that a J–L cutset (see e.g. References [1, 3, 7, 10, 36]) appears in the network after open-circuiting the resistor represented by G. It is easy to check that this Hessenberg DAE will have index 2 if and only if $1/L + I_0 k \cos k\phi_2 \neq 0$.

Assuming for the moment that $1/L + I_0 k \cos k\phi_2 \neq 0$ if $G = 0$, the linear stability of equilibrium points $v_2 = 0$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, with $n \in \mathbb{Z}$, is characterized, in light of Theorems 1 and 2, by the matrix pencil

$$\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -R/L & 0 & 1 \\ 0 & 0 & 1 \\ -1/L & -I_0 k \cos k\phi_2 & -G \end{pmatrix} = \begin{pmatrix} \lambda + R/L & 0 & -1 \\ 0 & \lambda & -1 \\ 1/L & I_0 k \cos k\phi_2 & G \end{pmatrix}$$

The spectrum of this pencil is defined by the roots of the determinant

$$G\lambda^2 + (GR/L + 1/L + I_0 k \cos k\phi_2)\lambda + (R/L) I_0 k \cos k\phi_2 \tag{34}$$

In case (a), defined by $G > 0$, the eigenvalues are

$$\lambda = \frac{-(GR/L + 1/L + I_0 k \cos k\phi_2) \pm \sqrt{(GR/L + 1/L + I_0 k \cos k\phi_2)^2 - 4G(R/L) I_0 k \cos k\phi_2}}{2G}$$

Recall that equilibrium points are defined by the conditions $v_2 = 0$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, with $n \in \mathbb{Z}$. If n is even, that is, if $k\phi_2 = 2m\pi$, then $\cos k\phi_2 = 1$ and, since G, R, L, I_0 and k are positive, both eigenvalues can be easily shown to be real and negative, making those equilibria asymptotically stable. On the contrary, if n is odd, that is, if $k\phi_2 = (2m+1)\pi$, then $\cos k\phi_2 = -1$ and there is one positive eigenvalue. Thus, equilibria with $k\phi_2 = (2m + 1)\pi$ are unstable.

In case (b), defined by the assumption $G = 0$, the network equations yield a Hessenberg DAE of size 2, and the index will be 2 except at points where $1/L + I_0 k \cos k\phi_2 = 0$. Equilibria are still characterized by the conditions $v_2 = 0$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, with $n \in \mathbb{Z}$. Let us analyse their stability in terms of R, L, I_0, k, using the fact that the unique eigenvalue has, from (34), the expression

$$\lambda = \frac{-(R/L)I_0 k \cos k\phi_2}{1/L + I_0 k \cos k\phi_2} = \frac{-RI_0 k \cos k\phi_2}{1 + LI_0 k \cos k\phi_2}$$

If $0 < LI_0 k < 1$, this eigenvalue will be negative if and only if $\cos k\phi_2 > 0$. Hence, equilibrium points with $k\phi_2 = 2m\pi$ will be asymptotically stable, whereas $k\phi_2 = (2m+1)\pi$ define unstable equilibria. On the contrary, it is not difficult to check that, if $LI_0 k > 1$, the eigenvalue is negative for every $\phi_2 = n\pi/k$. This means that all equilibria are asymptotically stable.

It is worth remarking that equilibrium points with $k\phi_2 = (2m+1)\pi$ experience a stability change as the product $LI_0 k$ crosses the critical value $(LI_0 k)^* = 1$. Assume that I_0 and k are fixed, and note that $(I_0 k \cos k\phi_2)^{-1}$ represents the differential inductance of the Josephson junction. When L equals $L^* = (I_0 k)^{-1}$, the inductance L and the incremental inductance in the Josephson junction have the same absolute value with opposite signs, since $\cos k\phi_2 = -1$ at these equilibria. In this situation, the index-2 condition $1/L + I_0 k \cos k\phi_2 \neq 0$ is not satisfied. This means that these operating points are *singular equilibria* when $LI_0 k = 1$; although details are beyond the scope of this paper, one can show that the local index jumps from 2 to 3 as L takes on the value $L^* = (I_0 k)^{-1}$: this minimal index change is responsible for a *singularity-induced bifurcation* (SIB) in the Hessenberg DAE (see References [42, 43] and related results for index-1 singular problems in References [44–48]. Owing to this SIB phenomenon, the eigenvalue goes from $\mathbb{C}^+$ to $\mathbb{C}^-$ by divergence through $\pm\infty$ as L increases through $L^* = (I_0 k)^{-1}$, and this is in turn responsible for the stabilization of equilibria with $k\phi_2 = (2m+1)\pi$ as L crosses this bifurcation value.

4.2. Circuits containing MOS transistors

The existence and stability of DC operating points in MOS circuits have been the focus of considerable recent research [16–18, 22, 27–29]. In particular, the work [29] provides a nice framework for the analysis of asymptotic stability properties of equilibria in state–space models of MOS circuits. Regarding, however, the limitations of the state–space approach pointed out in Section 1, a different framework aimed at semistate models should be of interest within this context. This framework is developed below.

The MOS-transistor model used in Reference [29] includes a loop of capacitors which will lead to an index-2 formulation in the DAE context. The matrix pencil results discussed in Section 3 and, particularly, in Section 3.3 for index-2 problems, will be the fundamental tool in the present approach. The DAE setting will make it possible to relax some assumptions implicitly needed in Reference [29] to derive a state–space equation; in particular, coupling among capacitors, additional V–C loops in the external connection between transistors, and non-monotonic i–v, q–v characteristics in resistors and capacitors, respectively, would be allowed in the present formulation. The results will be applied to an example which is reported to fail in Reference [29].

4.2.1. Transistor model. Figure 3 replicates the n-channel MOS transistor model proposed in Reference [29], based in turn on the Shichman–Hodges model [49]. In this model, the currents i_d and i_{dr} are given by

$$i_d = I(e^{-v_3/v_T} - 1) \tag{35a}$$

$$i_{dr} = \phi(-v_1) - \phi(v_2) \tag{35b}$$

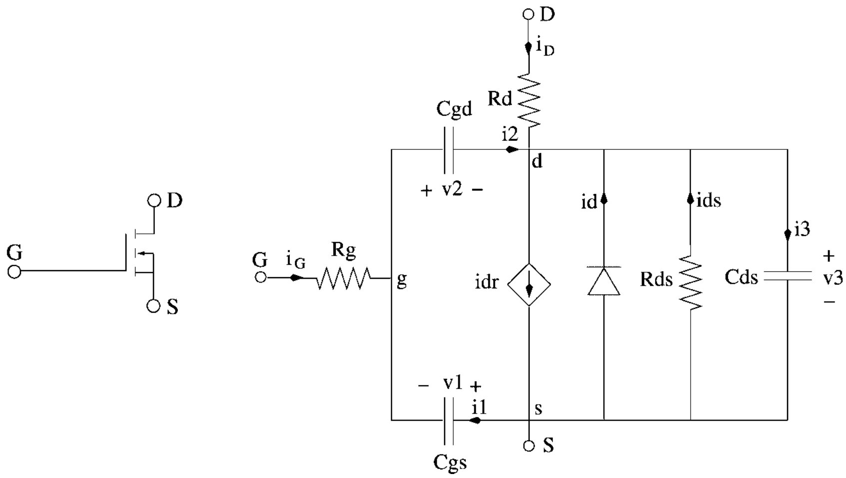

Figure 3. MOS transistor.

with

$$\phi(v) = \begin{cases} k(v - v_{t_0})^2 & \text{if } v \geqslant v_{t_0} \\ 0 & \text{if } v < v_{t_0} \end{cases} \tag{36}$$

v_{t_0} being a threshold voltage, whereas k is a parameter of the transistor depending on several physical constants. This MOS transistor model can be described by the equations

$$q' = i \tag{37a}$$

$$0 = g^{\mathrm{c}}(q, v) \tag{37b}$$

$$0 = v_1 + v_2 + v_3 \tag{37c}$$

$$0 = i_1 - i_2 + i_G \tag{37d}$$

$$0 = -i_3 + i_2 - i_{dr} + i_d - v_3/R_{ds} + i_D \tag{37e}$$

$$0 = -v_G + i_G R_g - v_1 \tag{37f}$$

$$0 = -v_D + i_D R_d + v_3 \tag{37g}$$

where $q = (q_1, q_2, q_3)$ (resp. $i = (i_1, i_2, i_3)$, $v = (v_1, v_2, v_3)$) represents capacitor charges (resp. currents, voltages), i_d and i_{dr} in (37e) are given by Equations (35a) and (35b), respectively, and we have used $i_{ds} = -v_3/R_{ds}$. The function $g^{\mathrm{c}} : \mathbb{R}^6 \to \mathbb{R}^3$ in Equation (37b) represents the constitutive relations of the three capacitors: note that, in this formulation, coupling and non-linear effects in the capacitors are allowed. Finally, v_G, v_D, i_G, i_D represent gate/drain voltages and currents, respectively.

In the kind of applications considered in this paper, MOS transistors will have both the source and the bulk connected to the ground. The gate and drain of all transistors will be

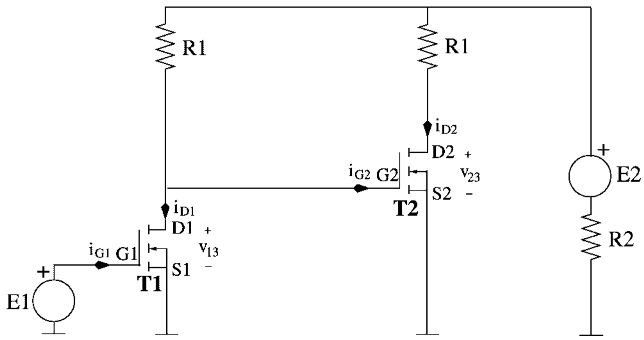

Figure 4. MOS circuit.

connected to an external multiport. These external connections will make it possible to eliminate gate/drain voltages and currents, leading to a semiexplicit DAE in terms of capacitor charges, voltages and currents. The index of this DAE will be at least two, due to the C-loops already present in the transistors, and might be higher depending on the external connections. For the sake of clarity in the discussion below, some variables will be eliminated from Equation (37) to get a simpler, voltage-oriented Hessenberg formulation for circuits containing MOS transistors. The matrix pencil tools presented in Section 3 may then be applied to such Hessenberg systems.

To derive this formulation, use (37d) and (37e) to eliminate the currents i_1 and i_3. Additionally, assume that $\partial g^c/\partial q$ in (37b) is non-singular, indicating that the set of capacitors admits a voltage-controlled description. Using the notation of Equation (7a), we may write

$$C(v)v' = \xi(v, i_2, i_D, i_G) \tag{38a}$$

$$0 = v_1 + v_2 + v_3 \tag{38b}$$

$$0 = -v_G + i_G R_g - v_1 \tag{38c}$$

$$0 = -v_D + i_D R_d + v_3 \tag{38d}$$

where

$$\xi(v, i_2, i_D, i_G) = \begin{pmatrix} i_2 - i_G \\ i_2 \\ i_2 - \phi(-v_1) + \phi(v_2) + I(\mathrm{e}^{-v_3/v_T} - 1) - v_3/R_{ds} + i_D \end{pmatrix} \tag{39}$$

4.2.2. Example 2. Consider the circuit depicted in Figure 4, taken from [29], where the values

$$E_2 = 10\ \mathrm{V}, \quad R_1 = 3\ \mathrm{k\Omega}, \quad R_2 = 10\ \Omega \tag{40}$$

are used. Internal values of MOS transistors are

$$R_d = 0, \ R_g = 1\,\Omega, \quad R_{ds} = 200\ \mathrm{k\Omega}, \quad k = 0.5\ \mathrm{mA/V^2}, \quad v_{t_0} = 2\ \mathrm{V}, \quad I = 10^{-12}\ \mathrm{mA} \tag{41}$$

We will use the notation v_{ij} for the voltage j in transistor i. For instance, the voltage drop in the capacitor C_{ds} in Figure 3 would be denoted as v_{13} for transistor 1. Gate/drain voltages/currents in both transistors can be written in terms of voltages v_{ij} and E_i, using Equations (38c)–(38d) and basic circuit relations in the external connections. Note in particular that, since $R_d = 0$ for this example, v_{13} and v_{23} are the voltage drops between D and S in both transistors (see Figure 4). Some simple computations yield

$$v_{G_1} = E_1 \tag{42a}$$

$$v_{G_2} = v_{13} \tag{42b}$$

$$v_{D_1} = v_{13} \tag{42c}$$

$$v_{D_2} = v_{23} \tag{42d}$$

$$i_{G_1} = E_1 + v_{11} \tag{42e}$$

$$i_{G_2} = v_{13} + v_{21} \tag{42f}$$

$$i_{D_1} = \alpha_1 v_{13} - v_{21} + \alpha_2 v_{23} + \alpha_3 E_2 \tag{42g}$$

$$i_{D_2} = \alpha_2 v_{13} + \alpha_4 v_{23} + \alpha_3 E_2 \tag{42h}$$

with

$$\alpha_1 = -1 - \frac{R_1 + R_2}{R_1^2 + 2R_1R_2}, \quad \alpha_2 = \frac{R_2}{R_1^2 + 2R_1R_2}, \quad \alpha_3 = \frac{1}{R_1 + 2R_2}, \quad \alpha_4 = -\frac{R_1 + R_2}{R_1^2 + 2R_1R_2}$$

Let us assume in the sequel that there is no coupling between capacitors. Denote as $C_{ij}(v_{ij})$ the incremental capacitance in capacitor j of transistor i, and assume that these capacitances do not vanish at equilibria. In light of the expressions (42e)–(42h) for gate and drain currents in both transistors, and using Equations (38a)–(38b), the dynamical behaviour of the whole circuit is given by

$$C_{11}(v_{11})v'_{11} = i_{12} - E_1 - v_{11} \tag{43a}$$

$$C_{12}(v_{12})v'_{12} = i_{12} \tag{43b}$$

$$C_{13}(v_{13})v'_{13} = i_{12} - \phi(-v_{11}) + \phi(v_{12}) + I(\mathrm{e}^{-v_{13}/v_T} - 1) - v_{13}/R_{ds} + \alpha_1 v_{13} - v_{21} + \alpha_2 v_{23} + \alpha_3 E_2 \tag{43c}$$

$$C_{21}(v_{21})v'_{21} = i_{22} - v_{13} - v_{21} \tag{43d}$$

$$C_{22}(v_{22})v'_{22} = i_{22} \tag{43e}$$

$$C_{23}(v_{23})v'_{23} = i_{22} - \phi(-v_{21}) + \phi(v_{22}) + I(\mathrm{e}^{-v_{23}/v_T} - 1) - v_{23}/R_{ds} + \alpha_2 v_{13} + \alpha_4 v_{23} + \alpha_3 E_2 \tag{43f}$$

$$0 = v_{11} + v_{12} + v_{13} \tag{43g}$$

$$0 = v_{21} + v_{22} + v_{23} \tag{43h}$$

Except for the coefficients $C_{ij}(v_{ij})$, which can be trivially moved to the right-hand side of the corresponding equations, system (43) represents a Hessenberg DAE of size 2 such as Equation (11), where the dynamic variables x would represent capacitor voltages v_{ij} $(i=1,2,\ j=1,2,3)$, whereas the algebraic variables y would be $i_{12},\ i_{22}$, which do not appear in the algebraic restrictions (43g)–(43h). In fact, it is easy to check that positive values of C_{ij} guarantee that the problem actually has index 2.

Equilibria: Equilibrium points of Equation (43) are defined by the vanishing of the right-hand side of this system. With the particular values defined by Equations (40) and (41), equilibria will depend on the value of E_1. For $E_1=2.3$ V, computer simulations yield $i_{12}=i_{22}=0$ and

$$\begin{aligned} &v_{11}=-v_{gs1}=-2.300, \quad v_{12}=v_{gd1}=-7.387, \quad v_{13}=v_{ds3}=9.687 \\ &v_{21}=-v_{gs2}=-9.687, \quad v_{22}=v_{gd1}=9.262, \quad v_{23}=v_{ds3}=0.425 \end{aligned} \tag{44}$$

Note that, since $v_{ds}>0$ for both transistors, internal diodes are reversed biased and, therefore, i_d can be approximated by $-I$. For $E_1=3$ V we obtain $i_{12}=i_{22}=0$ and

$$\begin{aligned} &v_{11}=-v_{gs1}=-3.000, \quad v_{12}=v_{gd1}=-5.338, \quad v_{13}=v_{ds3}=8.338 \\ &v_{21}=-v_{gs2}=-8.338, \quad v_{22}=v_{gd1}=7.821, \quad v_{23}=v_{ds3}=0.517 \end{aligned} \tag{45}$$

Both results are coherent with the ones given in Reference [29]. Nevertheless, the method of stability analysis in Reference [29] is reported there to fail in the second case.

Stability: Asymptotic stability of equilibria in the DAE (43) can be checked by studying the spectrum of the matrix pencil which describes the linearization of the problem at equilibria. With regard to the method proposed in Reference [29], the present approach has the drawback of needing actual values for the capacitances in order to study this spectrum. Remark that linearization of (43) at equilibrium would be characterized by the pencil $\{C,-\hat{J}\}$, with

$$C=\text{block\_diag}(K,0_2), \quad K=\text{diag}(C_{11},C_{12},C_{13},C_{21},C_{22},C_{23})$$

whereas $\hat{J}$ would denote the Jacobian of the right-hand side of Equation (43) evaluated at equilibrium. Assuming that C_{ij} do not vanish, it is very easy to check that

$$\sigma(C,-\hat{J})=\sigma(A,-\text{block\_diag}(K^{-1},I_2)\hat{J})$$

where A is defined in Equation (20). Therefore, the result presented in Theorem 2 can be applied here.

Note also that the above-mentioned drawback is attenuated by the fact that only the *relative* values of capacitances do influence the stability of the matrix pencil. Indeed, denoting

$$\tilde{C}=\text{diag}(1,C_{12}/C_{11},C_{13}/C_{11},C_{21}/C_{11},C_{22}/C_{11},C_{23}/C_{11},0,0)$$

it is a simple matter to check that, if $C_{11}>0$, then $\sigma(C,-\hat{J})\subset\mathbb{C}^-\Leftrightarrow\sigma(\tilde{C},-\hat{J})\subset\mathbb{C}^-$, since eigenvalues of the first pencil are transformed into those of the second one through the map $\lambda\rightarrow C_{11}\lambda$. Therefore, in order to apply Theorem 2 we only have to assume certain values for the ratios C_{ij}/C_{11}. Inspired on the values used in the seminal work [49], we get $C_{i1}/C_{11}=C_{i2}/C_{11}=1$, $C_{i3}/C_{11}=1.846$. Using these quantities, the spectrum $\sigma(\tilde{C},-\hat{J})$ for equilibrium (44) (which corresponds to the case $E_1 = 2.3$ V) is given by

$$-1.1215308, \ -0.5184073, \ -0.0029167, \ -0.0000686 \tag{46}$$

Note that this spectrum has four eigenvalues since, in the Hessenberg problem, we have $r = 6$, $p = 2$, and, therefore, the solution manifold $\mathscr{M}_2$ has dimension $r - p = 4$. All eigenvalues are located in $\mathbb{C}^-$ and, therefore, the equilibrium is asymptotically stable, as indicated in Reference [29]. On the contrary, the asymptotic stability of equilibrium (45) (corresponding to the case $E_1 = 3$ V) is not correctly identified by the method presented in Reference [29], as reported there. The spectrum of $\sigma(\tilde{C}, -\hat{J})$ reads in this case

$$-1.1216170, \ -0.5184140, \ -0.0023788, \ -0.0000682 \tag{47}$$

and, therefore, the equilibrium is correctly identified as asymptotically stable.

5. CONCLUDING REMARKS

DAEs provide a general setting for semistate modelling of non-linear circuits. These semistate or differential-algebraic models are nowadays pervasive in circuit simulation programs and, therefore, the development of formal tools for the local stability analysis in this kind of equations seems to be of interest. In the present paper, several results based on matrix pencils have been presented for such a stability analysis in the circuit context. The aim of this approach is to introduce a general mathematical framework in this field: specific circuit-theoretic characterizations for particular families of non-linear networks are the scope of future research.

APPENDIX A

The equivalence between the local (Kronecker) and differential indices around equilibria of semiexplicit index-1 and Hessenberg index-2 DAEs, claimed in the statements of Theorems 1 and 2, can be proved using some linear algebra results coming from the *tractability index* framework [12, 50]. Specifically, we show below that the pencil $\{A, -J_1\}$ defined in Equation (17) is regular with Kronecker index 1 if and only if $g_y(x^*, y^*)$ is non-singular, and that the matrix pencil $\{A, -J_2\}$ defined in Equation (20) is regular with Kronecker index 2 if and only if $g_x(x^*) f_y(x^*, y^*)$ is non-singular.

Both results follow from [50, Theorem 3], where it is shown that the Kronecker index-1 condition on a pencil $\{A, B\}$ with singular A is equivalent to the non-singularity of $A_1 = A + BQ_0$, Q_0 being any projector onto $\operatorname{Ker} A$ (i.e. a linear operator satisfying $Q_0^2 = Q_0$ with $\operatorname{Im} Q_0 = \operatorname{Ker} A$), whereas the pencil is regular with Kronecker index-2 if and only if A_1 is singular but $A_2 = A_1 + B(I - Q_0)Q_1$ is non-singular, for any projector Q_1 onto $\operatorname{Ker} A_1$. These properties do not depend on the specific choices of the projectors Q_0 and Q_1.

A unified approach to prove both cases then follows from the notation

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

with $B_{11} = -f_x(x^*, y^*)$, $B_{12} = -f_y(x^*, y^*)$, whereas $B_{21} = -g_x$ evaluated at (x^*, y^*) or x^*, respectively, and

$$B_{22} = \begin{cases} -g_y(x^*, y^*) & \text{for the pencil } \{A, -J_1\} \\ 0 & \text{for the Hessenberg pencil } \{A, -J_2\} \end{cases}$$

Choose

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix}$$

so that

$$A_1 = A + BQ_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} I_r & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

Note that A_1 is non-singular if and only if so it is B_{22}. This proves that $\{A, -J_1\}$ is regular with Kronecker index 1 if and only if $g_y(x^*, y^*)$ is non-singular, since $B_{22} = -g_y(x^*, y^*)$ in this case.

Regarding the Hessenberg pencil $\{A, -J_2\}$, the condition $B_{22} = 0$ makes A_1 singular. We may choose the projector Q_1 onto $\operatorname{Ker} A_1$ as

$$Q_1 = \begin{pmatrix} 0 & -B_{12} \\ 0 & I_p \end{pmatrix}$$

so that $A_2 = A_1 + B(I - Q_0)Q_1$ reads

$$A_2 = \begin{pmatrix} I_r & B_{12} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B_{11} & 0 \\ B_{21} & 0 \end{pmatrix} \begin{pmatrix} 0 & -B_{12} \\ 0 & I_p \end{pmatrix} = \begin{pmatrix} I_r & B_{12} - B_{11}B_{12} \\ 0 & -B_{21}B_{12} \end{pmatrix}$$

Now, A_2 is non-singular if and only if so it is $B_{21}B_{12}$. This expresses that the Hessenberg pencil $\{A, -J_2\}$ is regular with Kronecker index 2 if and only if $g_x(x^*)f_y(x^*, y^*)$ is non-singular, since $B_{21} = -g_x(x^*)$ and $B_{12} = -f_y(x^*, y^*)$.

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