On Linear Ordinary Differential Equations With Functionally Commutative Coefficient Matrices

J. Zhu*
Department of Electrical and Computer Engineering
University of Alabama in Huntsville
Huntsville, Alabama 35899

and

C. H. Morales
Department of Mathematical Sciences
University of Alabama in Huntsville
Huntsville, Alabama 35899

Submitted by Richard A. Brualdi

ABSTRACT

A system of linear ordinary differential equations $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, $t \in I$, is called *semiproper* if $\mathbf{A}(t)\mathbf{A}(\tau) = \mathbf{A}(\tau)\mathbf{A}(t)$, $t, \tau \in I$ [also known as *functional commutativity* of $\mathbf{A}(t)$]. It is known that a semiproper system has a closed-form fundamental solution matrix $\mathbf{X}_{\mathbf{A}}(t) = \exp\int^{t}\mathbf{A}(\tau)\,d\tau$, where the matrix exponential is defined by the power series $\exp(\cdot) = \sum_{k=1}^{\infty}(\cdot)^k/k!$. Therefore the problem of solving a semiproper system amounts to that of finding a finite-form expression for the matrix exponential. Based on some recent results obtained by the authors for decomposing semiproper matrix functions, a systematic approach is developed for finding a finite-form analytical solution for the entire family of semiproper systems. This solution is then used to derive a number of important and practical stability criteria for semiproper systems. Applications of the new results are also discussed.

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1. INTRODUCTION

*Currently with the Department of Electrical and Computer Engineering, Louisiana State University, Baton Rouge, LA 70803.

655 Avenue of the Americas, New York, NY 10010 0024-3795/92/$5.00

Let $\mathbb{R}$ be the field of real numbers and I be an interval in $\mathbb{R}$. Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^n$ be the n-dimensional Cartesian space over $\mathbb{C}$. Let $\mathbb{C}^{n\times n}$ be the space of matrices $\mathbf{G} = [g_{ij}]$, $g_{ij} \in \mathbb{C}$, $i, j = 1, 2, \dots, n$. Following [24], a matrix function $\mathbf{F}: I \to \mathbb{C}^{n\times n}$ is called *semiproper* on I if $\mathbf{F}(t)\mathbf{F}(\tau) = \mathbf{F}(\tau)\mathbf{F}(t)$, $t, \tau \in I$ (also known as functional commutativity on I), and $\mathbf{F}$ is called *proper*[1] on I if $\mathbf{F}(t) = f(t, \mathbf{G})$ for all $t \in I$, where $f: I \times D \to \mathbb{C}$ is a scalar function and $\mathbf{G} \in \mathbb{C}^{n\times n}$. Note that proper matrix functions are special cases of semiproper ones (cf. Theorem 3 of [24]).

In this paper, we consider the system of linear ordinary differential equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) \tag{1a}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0, \qquad t_0, t \in I, \tag{1b}$$

where $\mathbf{x}: I \to \mathbb{R}^n$ is an unknown vector-valued function, and $\mathbf{A}: I \to \mathbb{R}^{n\times n}$ is locally Riemann integrable. In the sequel, the space of locally integrable matrix functions $\mathbf{A}(t) = [a_{ij}(t)]$ on an interval $I \subseteq \mathbb{R}$ will be denoted by $\mathbb{K}^{n\times n}$.

The class of linear differential equations (1) plays an important role in diverse engineering and scientific fields. It is well known that the (unique) solution satisfying both the equation (1a) and the initial condition (1b) can be written as

$$\mathbf{x}(t) = \mathbf{X}_{\mathbf{A}}(t)\mathbf{X}_{\mathbf{A}}^{-1}(t_0)\mathbf{x}_0, \tag{2}$$

where $\mathbf{X}_{\mathbf{A}}(t)$ is *any* continuous nonsingular matrix function which is differentiable almost everywhere such that

$$\dot{\mathbf{X}}_{\mathbf{A}}(t) = \mathbf{A}(t)\mathbf{X}_{\mathbf{A}}(t). \tag{3}$$

The matrix $\mathbf{X}_{\mathbf{A}}(t)$ is known as a *fundamental solution matrix* of (1). Therefore, instead of solving for a (unique) solution to the initial-value problem (1), we may, without loss of generality, solve for an arbitrary solution $\mathbf{X}_{\mathbf{A}}(t)$ to the matrix equation (3). The problem of obtaining a closed-form analytical solution $\mathbf{X}_{\mathbf{A}}(t)$ for (3), in general, has been extensively studied with limited

[1]Formal definition of proper matrix functions is given in [24]; it is also included in Section 2 of this paper.

success. However, if $\mathbf{A}(t)$ is semiproper, then it is well known that Equation (3) possesses a *closed-form* solution given by

$$\mathbf{X}_{\mathbf{A}}(t) = \mathrm{e}^{\int^t \mathbf{A}(\tau)\,d\tau}, \tag{4a}$$

where the matrix exponential function is defined by the power series

$$\mathrm{e}^{(\cdot)} = \sum_{k=0}^{\infty} \frac{(\cdot)^k}{k!}. \tag{4b}$$

Hereafter a system (1) or (3) will be called *semiproper* (*proper*) if $\mathbf{A}(t)$ is semiproper (proper).

Although (4) gives a concise analytical expression for the solution to semiproper systems (3), the infinite power series (4b) hampers its applicability. In order to gain theoretical insight (e.g. into stability) and practical feasibility for semiproper systems, it is desirable to replace the power series on the right-hand side of (4b) with a finite sum of finite powers of matrix functions, not necessarily in the original form of $\mathbf{A}(t)$. If this can be done, (4) is said to be of *finite form*. It is well known that when $\mathbf{A}(t)$ is a scalar function ($n = 1$) or a constant matrix, the system (3) is semiproper and $\mathbf{X}_{\mathbf{A}}(t)$ can be expressed in finite form. Moreover, in this case, explicit stability criteria for (1) can also be obtained from the finite-form solutions. The purpose of the present paper is to extend those known results to the general class of semiproper systems.

The family of semiproper systems has attracted the attention of many researchers, and a bibliography on the topic is included in our list of references. One of the most remarkable results in this area was obtained by Martin [16] (1967). It appears that Theorem 2 of [16] is the earliest complete characterization of the entire family of semiproper matrix functions as commutative algebras generated by a basis of pairwise commutative constant matrices.

It is remarked that the sufficient condition in Theorem 2 of [16] was known prior to Martin's work, and can be found in Erugin [5]. Erugin also recognized the special form of second-order semiproper matrix functions.

Another remarkable result concerning semiproper systems can be found in Wiburg [22, p. 118] (1971), where it is pointed out, through informally, that if a matrix function $\mathbf{A}(t)$ is semiproper on I and simple[2] for all $t \in I$,

[2]A matrix $\mathbf{A} \in \mathbb{C}^{r \times n}$ is said to be simple if and only if it is diagonalizable by a similarity transformation over $\mathbb{C}$.

then the semiproper system (1) can be solved analytically to obtain a finite-form solution. This result is based on a classical theorem which states that any number of mutually commutative simple matrices are simultaneously diagonalizable by a constant similarity transformation (see, for instance, [8, p. 224]).

One of the most recent papers on this topic is Kotin and Epstein [11] (1982). In that paper a special family of semiproper matrix functions which can be expressed as a polynomial in one constant matrix is identified. The members of this family are subjected to a number of constraints, so that even matrices of the form $\mathbf{A}(t) = \alpha(t)\mathbf{I}$, where $\alpha(t)$ is a scalar function and $\mathbf{I}$ is the identity matrix, are not included.

It seems appropriate here to make the following comments on a paper by Wu and Sherif [23] (1976). Theorem 1(a) in [23] seems to be more general than Martin's Theorem 1 of [16]. However, the proof of the "only if" part, which is the most difficult part, is not addressed in [23]. Moreover, that "only if" statement is, in fact, false according to Martin's Theorem 1 of [16]. Also, Theorem 1(c) in [23] was a well-known result at that time and can be found, for example, in [22], which was published five years earlier. In addition, in Theorem 1(b) of [23] and the accompanying remarks, it is claimed that a closed-form analytical solution to any time-varying systems of the *commutative class* (which is a wider class than the class of semiproper systems) can be easily obtained. However, this assertion is not explicitly justified either. In fact, in the examples of [23] all the matrices but a trivial one are of simple structure and this can be solved by the already known result of Theorem 1(c) of [23].

A more rigorous and more complete account on the commutative-class linear systems (1) is found in a recently published book [14] (1982). In particular, in Remark 7.3.2, p. 142 of [14], an analytical solution procedure is suggested for solving the commutative class of linear systems (1) in finite form. However, that procedure relies on the variable eigenvalues of the coefficient matrix $\mathbf{A}(t)$, which, in general, are not easily obtainable for $n \geqslant 4$. Moreover, that procedure is symbolic in nature and therefore does not reveal much information about linear systems of the commutative class. In addition to the analytical solution technique, some sufficient stability criteria for the commutative class of linear systems (1) are also given in [14, pp. 160–162, 175–179].

From the above brief historical review, it appears that to date the problem of how to obtain finite-form solutions and their stability in general for semiproper systems (1) has not been completely resolved. Using some recent results on decomposition of semiproper matrix functions obtained in [24], we can now develop the following results: (i) a general systematic procedure for obtaining a finite-form solution for the entire family of semiproper systems (1); (ii) a necessary and sufficient stability criterion for the entire family of proper systems (1); and (iii) a sufficient stability criterion for semiproper systems (1) (which has been extended to a necessary and sufficient stability criterion in a separate paper).

Our new results constitute a natural generalization of the well-known solution techniques and stability criteria for systems (1) with constant or scalar variable coefficients. They also serve to unify many known results on semiproper systems [2–7, 9, 11, 17–29]. Moreover, these new results will promote several advances in control theory and the theory of linear ordinary differential equations, as indicated in Section 4 of this paper.

2. PRELIMINARIES

To facilitate the development of the new results of this paper, in this section we recall some of the main results established in [24], along with some notation and basic definitions that will be used in the sequel.

DEFINITION 1. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$. The ordered set of all distinct eigenvalues of $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$, denoted by $\Lambda_{\mathbf{A}}$. By a multiset we mean a collection of objects that need not be distinct. The ordered multiset of all roots of the minimal polynomial $\psi_{\mathbf{A}}(\lambda)$ of $\mathbf{A}$, counting multiplicity, will be called the *extended spectrum* of $\mathbf{A}$, denoted by $\Gamma_{\mathbf{A}}$. Moreover, $\mathbf{A}$ is said to be *simple* if all elements of $\Gamma_{\mathbf{A}}$ are of unit multiplicity.

EXAMPLE 1. Suppose that the minimal polynomial of a matrix $\mathbf{A}$ of order 10 is given by

$$\psi_{\mathbf{A}}(\lambda) = \sum_{k=1}^{7} \alpha_k \lambda^{k-1} = (\lambda - \lambda_1)^3 (\lambda - \lambda_2)^2 (\lambda - \lambda_3)(\lambda - \lambda_4),$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are distinct complex numbers. We represent $\Gamma_{\mathbf{A}}$ with the following notations:

$$\begin{aligned}\Gamma_{\mathbf{A}} &= \{\lambda_1, \lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_4\} \\ &= \{3\cdot\lambda_1, 2\cdot\lambda_2, 1\cdot\lambda_3, 1\cdot\lambda_4\}.\end{aligned}$$

The numbers 3, 2, 1, 1 in the second expression are called the *repetition numbers* of the elements.

For every matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$, there is a canonical matrix, known as the *generalized Vandermonde matrix*, associated with its minimal polynomial $\psi_{\mathbf{A}}(\lambda)$, as defined below.

DEFINITION 2. Let $\psi_{\mathbf{A}}(\lambda)$ be the minimal polynomial of $\mathbf{A} \in \mathbb{C}^{n \times n}$ such that

$$\psi_{\mathbf{A}}(\lambda) = (\lambda - \lambda_1)^{d_1}(\lambda - \lambda_2)^{d_2} \cdots (\lambda - \lambda_r)^{d_r}$$

where $d_1 + d_2 + \cdots + d_r = m = \deg \psi_{\mathbf{A}}(\lambda)$. Let

$$\mathrm{v}(\lambda) = [1, \lambda, \lambda^2, \ldots, \lambda^{m-1}]^{\mathrm{T}},$$

and

$$\mathrm{v}^{(p)}(\lambda) = \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p}\mathrm{v}(\lambda).$$

Let $\mathbf{W}_j \in \mathbb{C}^{m \times d_j}$ be the rectangular block matrix with columns $\mathrm{v}^{(p)}(\lambda_j)/p!$, $p = 0, 1, \ldots, d_j - 1$. Then the *generalized Vandermonde matrix* associated with $\psi_{\mathbf{A}}(\lambda)$ is defined by

$$\mathbf{V} = [\mathbf{W}_1 \mid \mathbf{W}_2 \mid \cdots \mid \mathbf{W}_r].$$

An important feature of the generalized Vandermonde matrix $\mathbf{V}(\lambda)$ is that

$$\det \mathbf{V}(\lambda) = \prod_{1 \leqslant i < j \leqslant n} (\lambda_j - \lambda_i)^{d_i d_j},$$

where λ_i is the ith root with multiplicity d_i of the associated polynomial. Therefore the generalized Vandermonde matrix $\mathbf{V}$ associated with any given polynomial is always nonsingular [3].

The special class of (upper) triangular matrices given by the following definition plays an important role in the theory of functions of a matrix and the theory of commutative matrices.

DEFINITION 3

(a) A matrix $\mathbf{A} = [a_{ij}] \in \mathbb{C}^{n \times n}$ is called *regular upper-triangular* if $a_{i+1, j+1} = a_{ij}$ for all $i, j = 1, 2, \ldots, n-1$, and $a_{ij} = 0$ for all $j < i$. Such a matrix $\mathbf{A}$ will be denoted by

$$\mathbf{A} = \mathrm{rut}[a_{11}, a_{12}, \ldots, a_{1n}].$$

The first row vector $\mathbf{A}_{1*} = [a_{11}, a_{12}, \ldots, a_{1n}]$ of $\mathbf{A}$ is called the *representative vector* of $\mathbf{A}$, and its entries a_{ij} are called the *representative entries* of $\mathbf{A}$.

(b) A rectangular matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ is called a *regular upper-triangular matrix* if

(i) for $m < n$, $\mathbf{A} = [\mathbf{0} \,|\, \mathbf{B}]$, where $\mathbf{B} \in \mathbb{C}^{m\times m}$ is a regular upper-triangular block matrix as defined in (a);
(ii) for $m > n$,

$$\mathbf{A} = \left[\begin{array}{c}\mathbf{B}\\ \hline \mathbf{0}\end{array}\right],$$

where $\mathbf{B} \in \mathbb{C}^{n\times n}$ is a regular upper-triangular block matrix as defined in (a).
In either case, $\mathbf{A}$ will be assumed to have the same representative vector and representative entries of $\mathbf{B}$, and $\mathbf{A}$ will be denoted by

$$\mathbf{A} = \mathrm{rut}_{mn}\left[b_{11}, b_{12}, \ldots, b_{1p}\right],$$

where $p = \min\{m, n\}$.

(c) Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be such that $\mathbf{A} = \mathrm{diag}[\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r]$, where $\mathbf{A}_i = \mathrm{rut}[a_{i1}, a_{i2}, \ldots, a_{id_i}]$, $\Sigma_{i=1}^{r} d_i = n$. Then $\mathbf{A}$ is called a *regular upper-triangular block-diagonal matrix*, and the n-vector

$$\mathbf{a} = [a_{11}, a_{12}, \ldots, a_{ik_i}, \ldots, a_{rd_r}],$$

$k_i = 1, 2, \ldots, d_i$, $i = 1, 2, \ldots, r$, is called the *companion vector* of $\mathbf{A}$.

We now formally define what we call *proper matrix functions* and recall from [24] some useful results for such functions. To begin with, let $\mathbb{L} = \mathbb{L}(I, \mathbb{C})$ be any linear space of scalar functions $f: I \to \mathbb{C}$ over the field $\mathbb{C}$. We shall define $\mathbf{F} \in \mathbb{L}^{n\times n}$ to mean $\mathbf{F}: I \to \mathbb{C}^{n\times n}$ such that $\mathbf{F} = [f_{ij}]$ and $f_{ij} \in \mathbb{L}$.

DEFINITION 4.

(a) Let $D \subseteq \mathbb{C}$, and let $f: I \times D \to \mathbb{C}$ be a given function such that $f(\cdot, \lambda) \in \mathbb{L}$ for all $\lambda \in D$. Let $\mathbf{G} \in \mathbb{C}^{n\times n}$ with extended spectrum $\{d_i \cdot \lambda_i\}_{i=1}^{r} =$

$\Gamma_{\mathbf{G}} \subseteq D$. Then f is said to be *defined* on $\Gamma_{\mathbf{G}}$ if the set $f(\cdot, \Gamma_{\mathbf{G}}) \subseteq \mathbb{L}$, where

$$f(\cdot, \Gamma_{\mathbf{G}}) = \left\{ \frac{f_\lambda^{(k_i)}(t, \lambda_i)}{k_i!} \right\}, \tag{5}$$

where $k_i = 0, 1, \ldots, d_i - 1$, $i = 1, 2, \ldots, r$, and

$$f_\lambda^{(k)}(t, \lambda_i) = \left. \frac{\partial^k f(t, \lambda)}{\partial \lambda^k} \right|_{\lambda = \lambda_i}.$$

Now let

$$g(t, \lambda) = \sum_{k=1}^{p} \alpha_k(t) \lambda^{k-1}, \qquad \alpha_k \in \mathbb{L},$$

for some integer p such that $f(t, \Gamma_{\mathbf{G}}) = g(t, \Gamma_{\mathbf{G}})$, for all $t \in I$. Then the matrix value of $f(t, \mathbf{G})$ at each $t \in I$ is given by

$$f(t, \mathbf{G}) = \sum_{k=1}^{p} \alpha_k(t) \mathbf{G}^{k-1}.$$

(b) Let $\mathbf{F} \in \mathbb{L}^{n \times n}$ be a matrix function defined by $\mathbf{F}(t) = f(t, \mathbf{G})$. The function f is called a *primitive function* for $\mathbf{F}$, and the matrix $\mathbf{G}$ is called a *generating matrix* of $\mathbf{F}$ with respect to f. We shall also use $[f(\cdot, \Gamma_{\mathbf{G}})]$ to denote the row vector consisting of the scalar functions in the set $f(\cdot, \Gamma_{\mathbf{G}})$ given by (5).

(c) A matrix function $\mathbf{F} \in \mathbb{L}^{n \times n}$ is said to be *proper* if $\mathbf{F}(t) = f(t, \mathbf{G})$ for some primitive function f and generating matrix $\mathbf{G}$. Otherwise, $\mathbf{F}$ is said to be *improper*.

REMARKS.

(1) This definition is a generalization of the classical theory of functions of a constant matrix $f(\mathbf{G})$, where the primitive functions f are independent of t. Therefore, a constant matrix $\mathbf{G}$ and the functions $f(\mathbf{G})$ are also proper.

(2) In general, the set $D \subseteq C$ in Definition 4 needs to be open, due to the partial-differentiability requirement (5) for the primitive function $f(\cdot, \cdot)$ with respect to the second variable. However, when the generating matrix $\mathbf{G}$ is simple, then D does not need to be open.

We now state the following two theorems obtained in [24] which characterize the class of proper matrix functions and will be used in the development of our new results.

THEOREM 1. *A matrix function* $\mathbf{F} \in \mathbb{L}^{n \times n}$ *is proper on* I *if and only if there exists a nonsingular matrix* $\mathbf{L} \in \mathbb{C}^{n \times n}$ *such that for all* $t \in I$,

$$\mathbf{P}(t) = \mathbf{L}^{-1}\mathbf{F}(t)\mathbf{L} = \mathrm{diag}[\mathbf{P}_1(t), \mathbf{P}_2(t), \ldots, \mathbf{P}_r(t)],$$

where $\mathbf{P}_i(t) = rut[\gamma_{i1}(t), \gamma_{i2}(t), \ldots, \gamma_{id_i}(t)]$, $\gamma_{ij} \in \mathbb{L}$, $j = 1, 2, \ldots, d_i$, $i = 1, 2, \ldots, r$.

THEOREM 2. *Let* $\mathbf{F} \in \mathbb{L}^{n \times n}$ *be proper and* $\mathbf{F}(t) = f(t, \mathbf{G})$. *Let* $\psi_{\mathbf{G}}(\lambda)$ *be the minimal polynomial of* $\mathbf{G}$ *with deg* $\psi_{\mathbf{G}}(\lambda) = m$, *and* $\Gamma_{\mathbf{G}} = \{d_i \cdot \lambda_i\}_{i=1}^{r}$ *be the extended spectrum of* $\mathbf{G}$. *Then*:

(a) *There exists a unique polynomial representation of degree* $m-1$ *for* $\mathbf{F}$ *such that* $\mathbf{F}(t) = \sum_{k=1}^{m} \alpha_k(t)\mathbf{G}^{k-1}$, *where the scalar functions* $\alpha_k \in \mathbb{L}$ *can be found from the equation* $\boldsymbol{\alpha}(t) = \boldsymbol{\gamma}(t)\mathbf{V}^{-1}$, *where* $\boldsymbol{\alpha}(t) = [\alpha_1(t), \alpha_2(t), \ldots, \alpha_m(t)]$, $\mathbf{V}$ *is the generalized Vandermonde matrix associated with* $\Gamma_{\mathbf{G}}$, *and* $\boldsymbol{\gamma}(t) = [\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t)] = [f(t, \Gamma_{\mathbf{G}})]$ *is the companion vector of the regular upper-triangular block-diagonal matrix* $\mathbf{P}(t) = \mathbf{L}^{-1}\mathbf{F}(t)\mathbf{L}$, *where* $\mathbf{L} \in \mathbb{C}^{n \times n}$ *is a constant, nonsingular matrix consisting of eigenvectors and generalized eigenvectors of* $\mathbf{G}$.

(b) *Let* $\{\mathbf{Z}_{ik}\}_{k=0, i=1}^{d_i, r}$ *be the set of components*[3] *of* $\mathbf{G}$. *Then* $\mathbf{F}$ *has a component expansion on* I:

$$\mathbf{F}(t) = \sum_{i=1}^{r} \sum_{k=0}^{d_i - 1} \frac{f_\lambda^{(k)}(t, \lambda_i)}{k!} \mathbf{Z}_{ik}.$$

(c) *For any scalar function* $\alpha \in \mathbb{L}$, $\mathbf{H} = \alpha\mathbf{F}$ *is proper and* $\mathbf{H}(t) = h(t, \mathbf{G})$, *where* $h(t, \lambda) = \alpha(t) f(t, \lambda)$, *for all* $t \in I$.

(d) *Let* $\mathbf{F}_1, \mathbf{F}_2 \in \mathbb{L}^{n \times n}$ *be proper and* $\mathbf{F}_i(t) = f_i(t, \mathbf{G})$, $i = 1, 2$. *Then* $\mathbf{H} = \mathbf{F}_1 + \mathbf{F}_2$, $\mathbf{P} = \mathbf{F}_1 \cdot \mathbf{F}_2$ (*where the* "$\cdot$" *denotes pointwise multiplication*) *are proper, and* $\mathbf{H}(t) = h(t, \mathbf{G})$, $\mathbf{P}(t) = p(t, \mathbf{G})$, *where* $h(t, \lambda) = f_1(t, \lambda) + f_2(t, \lambda)$, $p(t, \lambda) = f_1(t, \lambda) \cdot f_2(t, \lambda)$, *for all* $t \in I$.

(e) *If* $\mathbf{F}$ *is differentiable on* I *and* $\mathbf{P}(t) = \dot{\mathbf{F}}(t)$, *then* $\mathbf{P}$ *is proper and* $\mathbf{P}(t) = p(t, \mathbf{G})$, *where* $p(t, \lambda) = \partial f(t, \lambda)/\partial t$. *If* $\mathbf{F}$ *is Riemann* (*or Lebesgue*)

[3]The components $\mathbf{Z}_{ik}$ used here and in the sequel differ from those defined in Gantmacher [8, Vol. I, p. 104] by a constant coefficient $1/k!$.

integrable on I and $\mathbf{P}(t) = \int^t \mathbf{F}(\tau)\,d\tau$, *then* $\mathbf{P}$ *is proper and* $\mathbf{P}(t) = p(t, \mathbf{G})$, *where* $p(t, \lambda) = \int^t f(\tau, \lambda)\,d\tau$.

(f) *For any scalar function* $h : \mathbb{C} \to \mathbb{C}$ *such that the set* $h \circ f(\cdot, \Gamma_G) \subseteq \mathbb{L}$, *the matrix* $\mathbf{P} = h(\mathbf{F})$ *is proper and* $\mathbf{P}(t) = g(t, \mathbf{G})$, *where* $g(t, \lambda) = h(f(t, \lambda))$.

(g) *Let* $\Lambda_{\mathbf{F}(t)}$ *and* $\Lambda_{\mathbf{G}}$ *be the spectra of* $\mathbf{F}(t)$ *and* $\mathbf{G}$, *respectively. Then* $\Lambda_{\mathbf{F}(t)} = \Lambda_{f(t, \mathbf{G})} = f(t, \Lambda_{\mathbf{G}})$.

REMARKS.

(1) Theorem 1 not only establishes a characterization for the class of proper matrix functions, but also facilitates a systematic procedure developed in [24] for finding a primitive function and a generating matrix for a given proper matrix function.

(2) The polynomial representation of a proper matrix function given in Theorem 2(a) is an extension of the classical Lagrange-Sylvester interpolating polynomial representation for functions of a constant matrix (cf. [8, Vol. I, p. 101]).

(3) As a consequence of Theorem 2(a), if a proper matrix function $\mathbf{F}(t) = f(t, \mathbf{G})$, $t \in I$, is integrable on I, then $f(t, \lambda)$ is integrable on I for any $\lambda \in \Gamma_{\mathbf{G}}$. This property will be used in the sequel without an explicit mention.

The next result, also obtained in [24], gives a general characterization of the class of semiproper matrix functions in terms of proper ones.

THEOREM 3. *Let* $\mathbf{F} \in \mathbb{L}^{n \times n}$. *Then* $\mathbf{F}$ *is semiproper on I if and only if* $\mathbf{F}$ *can be decomposed into*

$$\mathbf{F}(t) = \sum_{i=1}^{N} \mathbf{F}_i(t) = \sum_{i=1}^{N} f_i(t, \mathbf{G}_i), \tag{6}$$

where $\mathbf{G}_i \mathbf{G}_j = \mathbf{G}_j \mathbf{G}_i$, $i, j \leqslant N$.

REMARKS. Theorem 3 decomposes a semiproper matrix function $\mathbf{F}(t)$ into a finite number of mutually commutative proper matrix functions $\mathbf{F}_i(t) = f_i(t, \mathbf{G}_i)$ by successive projections of $\mathbf{F}(t)$ onto the subspaces spanned by the generating matrices $\{\mathbf{G}_i^k\}$. This decomposition has been called *spatial decomposition* in [24], and has been used there to develop a systematic procedure for finding the primitive functions f_i and the mutually commutative generating matrices $\mathbf{G}_i$. Theorem 3, together with the spatial-decomposition procedure, will be used to develop the results of this paper.

In the sequel, we will focus on the class of locally integrable matrix functions $\mathbf{A} \in \mathbb{K}^{n \times n}$, where, by convention, $\mathbf{A}(t)$ is used instead of $\mathbf{F}(t)$ for the variable coefficient matrices of (1) and (3).

3. FINITE-FORM SOLUTIONS FOR SEMIPROPER SYSTEMS

In this section, we derive our first main result of this paper, which gives a finite-form analytical solution for the entire family of semiproper systems (3).

THEOREM 4. *If $\mathbf{A} \in \mathbb{K}^{n \times n}$ is semiproper on I having a spatial decomposition*

$$\mathbf{A}(t) = \sum_{i=1}^{m} \mathbf{A}_i(t) = \sum_{i=1}^{m} f_i(t, \mathbf{G}_i),$$

then the semiproper system (3) *has a finite-form solution given by*

$$\mathbf{X}_{\mathbf{A}}(t) = \prod_{i=1}^{m} g_i(t, \mathbf{G}_i), \tag{7}$$

where

$$g_i(t, \lambda) = \mathrm{e}^{\int^t f_i(\tau, \lambda)\, d\tau}.$$

In order to prove Theorem 4, we need to establish first the following important lemma.

LEMMA 1. *If $\mathbf{A} \in \mathbb{K}^{n \times n}$ is proper on I such that $\mathbf{A}(t) = f(t, \mathbf{G})$, then the system* (3) *has a fundamental solution matrix*

$$\mathbf{X}_{\mathbf{A}}(t) = g(t, \mathbf{G}), \tag{8}$$

where

$$g(t, \lambda) = \mathrm{e}^{\int^t f(\tau, \lambda)\, d\tau}.$$

Lemma 1 follows directly from Theorem 2(e), so the proof is omitted. The following well-known lemma is also needed in the proof of Theorem 4.

LEMMA 2 [1, p. 167]. *Let* $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{C}^{n\times n}$. *If* $\mathbf{G}_1\mathbf{G}_2 = \mathbf{G}_2\mathbf{G}_1$, *then* $\exp(\mathbf{G}_1 + \mathbf{G}_2) = \exp(\mathbf{G}_1)\exp(\mathbf{G}_2)$.

Proof of Theorem 4. Suppose that $\mathbf{A} \in \mathbb{K}^{n\times n}$ is semiproper on I and

$$\mathbf{A}(t) = \sum_{i=1}^{m} \mathbf{A}_i(t) = \sum_{i=1}^{m} f_i(t, \mathbf{G}_i),$$

where $\mathbf{G}_i$ are pairwise commutative generating matrices for $\mathbf{A}_i(t)$, and $f_i(t,\lambda)$ are the corresponding primitive functions, which, by Theorem 2(a), can be chosen such that

$$\mathbf{A}_i(t) = \sum_{k=1}^{m_i} \alpha_{ik}(t)\mathbf{G}_i^{k-1}.$$

Now let $\mathbf{F}_i(t) = \int^t \mathbf{A}_i(\tau)\,d\tau$. Since $\mathbf{G}_i\mathbf{G}_j = \mathbf{G}_j\mathbf{G}_i$, it is easily verified that $\mathbf{F}_i(t)\mathbf{F}_j(t) = \mathbf{F}_j(t)\mathbf{F}_i(t)$, $i, j \leqslant m$. It then follows from Lemmas 1, 2 that

$$\begin{aligned}\mathbf{X}_\mathbf{A}(t) &= e^{\int^t \mathbf{A}(\tau)\,d\tau}\\ &= e^{\int^t \sum_{i=1}^{m} \mathbf{A}_i(\tau)\,d\tau}\\ &= \prod_{i=1}^{m} e^{\int^t \mathbf{A}_i(\tau)\,d\tau}\\ &= \prod_{i=1}^{m} g_i(t, \mathbf{G}_i),\end{aligned}$$

where

$$g_i(\tau, \lambda) = e^{\int^t f_i(\tau,\lambda)\,d\tau}. \qquad \blacksquare$$

REMARKS.

(1) In essence, Lemma 1 and Theorem 4 state that the fundamental matrix $\mathbf{X}_\mathbf{A}(t)$ of a proper (semiproper) system (3) is also proper (semiproper). Since a proper matrix function can be written as a *polynomial* of a constant matrix, the analytical solution expressions (7) and (8) for $\mathbf{X}_\mathbf{A}(t)$ have effectively reduced the infinite-series expression (4) to a finite form.

(2) Lemma 1 is an important result in itself, because it not only gives a finite-form solution for the entire class of proper systems (3), but also sheds

some new light on the controversial issue of using variable ("time-varying") eigenvalues of $\mathbf{A}(t)$ to predict stability of linear systems (1) [27].

(3) A systematic procedure has been developed in [24] for obtaining a spatial decomposition (6) for any given semiproper matrix function. That procedure makes Theorem 4 a practical and potentially useful result.

(4) Lemma 1 and Theorem 4 constitute a generalization and unification of the well-known analytical solution technique for linear systems (1) with constant and scalar variable coefficients.

The following two examples are given to illustrate the applications of Lemma 1 and Theorem 4.

EXAMPLE 2. In this example, we apply Lemma 1 to find a finite-form solution for the proper system (3) with $\mathbf{A}(t)=f(t,\mathbf{G})=\sin \mathbf{G}t$, $t\in\mathbb{R}$, and

$$\mathbf{G}=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},$$

with extended spectrum $\Gamma_{\mathbf{G}}=\{1,2\}$. By Theorem 2(b), for any function $f(t,\lambda)$ defined on $\Gamma_{\mathscr{G}}$, we have $f(t,\mathbf{G})=f(t,1)\mathbf{Z}_1+f(t,2)\mathbf{Z}_2$, where $\mathbf{Z}_1,\mathbf{Z}_2$ are the components of $\mathbf{G}$ and are uniquely found to be

$$\mathbf{Z}_1=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix},\qquad \mathbf{Z}_2=\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}.$$

Now for $f(t,\lambda)=\sin\lambda t$,

$$\begin{aligned}\mathbf{A}(t)&=f(t,\mathbf{G})\\ &=(\sin t)\mathbf{Z}_1+(\sin 2t)\mathbf{Z}_2\\ &=\begin{bmatrix}\sin t & \sin 2t-\sin t\\ 0 & \sin 2t\end{bmatrix}.\end{aligned}$$

Since $\mathbf{A}(t)$ is proper, by Theorem 4,

$$\begin{aligned}g(t,\lambda)&=e^{\int^t f(\tau,\lambda)\,d\tau}\\ &=e^{\int^t \sin\lambda\tau\,d\tau}\\ &=Ce^{-(1/\lambda)\cos\lambda\tau},\end{aligned}$$

where $C \in \mathbb{C}$ is an arbitrary constant. Thus a fundamental solution matrix is found to be

$$\mathbf{X_A}(t) = g(t,1)\mathbf{Z}_1 + g(t,2)\mathbf{Z}_2$$

$$= \begin{bmatrix} C_1 e^{-\cos t} & C_2 e^{-\frac{1}{2}\cos 2t} - C_1 e^{-\cos t} \\ 0 & C_2 e^{-\frac{1}{2}\cos 2t} \end{bmatrix},$$

where $C_1, C_2 \in \mathbb{C}$ are arbitrary constants.

The next example illustrates the application of Theorem 4 for a strictly semiproper (i.e. semiproper and improper) system (3).

EXAMPLE 3. Consider the strictly semiproper matrix function $\mathbf{A}(t)$ in Example 6 of [24]:

$$\mathbf{A}(t) = \begin{bmatrix} t & 1 & t^2 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} = \mathbf{A}_1(t) + \mathbf{A}_2(t), \qquad t \geqslant 0,$$

where[4]

$$\mathbf{A}_1(t) = f_1(t, \mathbf{G}_1) = \begin{bmatrix} t & 1 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}, \qquad t \geqslant 0,$$

with $f_1(t,\lambda) = t + \lambda$ and $\mathbf{G}_1 = \mathbf{A}(0)$, and

$$\mathbf{A}_2(t) = f_2(t, \mathbf{G}_2) = \begin{bmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad t \geqslant 0,$$

with $f_2(t,\lambda) = t^2\lambda$ and $\mathbf{G}_2 = \mathbf{A}_2(1)$. Now let $g_i(t,\lambda) = \exp \int^t f_i(\tau,\lambda)\, d\tau$. By

[4]The proper matrix functions $\mathbf{A}_1(t), \mathbf{A}_2(t)$ are obtained using the spatial decomposition procedure developed in [24].

applying Lemma 1 we obtain

$$g_1(t,\mathbf{G}_1)=\begin{bmatrix} e^{\frac{1}{2}t^2} & te^{\frac{1}{2}t^2} & 0 \\ 0 & e^{\frac{1}{2}t^2} & 0 \\ 0 & 0 & e^{\frac{1}{2}t^2} \end{bmatrix} C_1, \qquad C_1 \in \mathbb{C},$$

and

$$g_2(t,\mathbf{G}_2)=\begin{bmatrix} 1 & 0 & \dfrac{t^3}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} C_2, \qquad C_2 \in \mathbb{C}.$$

By Theorem 4,

$$\begin{aligned} \mathbf{X}_{\mathbf{A}}(t) &= e^{\int^t \mathbf{A}(\tau)\,d\tau} \\ &= \prod_{i=1}^{2} g_i(t,\mathbf{G}_i) \\ &= \begin{bmatrix} e^{\frac{1}{2}t^2} & te^{\frac{1}{2}t^2} & \dfrac{t^3}{3}e^{\frac{1}{2}t^2} \\ 0 & e^{\frac{1}{2}t^2} & 0 \\ 0 & 0 & e^{\frac{1}{2}t^2} \end{bmatrix} C, \qquad C = C_1C_2 \in \mathbb{C}. \end{aligned}$$

4. STABILITY OF PROPER AND SEMIPROPER SYSTEMS

In this section, we shall establish our new results on stability (in the sense of Lyapunov) of proper and semiproper linear systems (1). For this purpose, we first define the norms for vector-valued functions and matrix-valued functions which, unless otherwise stated, will be used in the sequel.

DEFINITION 5. Let $\mathbf{v} \in \mathbb{C}^n$. The *Euclidean norm* of $\mathbf{v}$ is defined by

$$\|\mathbf{v}\| = \left[\sum_{i=1}^{n} |v_i|^2\right]^{1/2}.$$

Let $\mathbf{x}: I \to \mathbb{C}^n$, $I \subseteq \mathbb{R}$, such that $\sup\|\mathbf{x}(t)\| < \infty$ for all $t \in I$. Then the *norm* of $\mathbf{x}$ is defined by

$$\|\mathbf{x}\| = \sup_{t \in I} \| \mathbf{x}(t) \|.$$

Let $\mathbf{G} \in \mathbb{C}^{n \times n}$. The *Euclidean norm* of $\mathbf{G}$ is defined by

$$\|\mathbf{G}\| = \left[\sum_{i=1}^{n} \sum_{j=1}^{n} |g_{ij}|^2 \right]^{1/2}.$$

Let $\mathbf{F}: I \to \mathbb{C}^{n \times n}$ such that $\sup\|\mathbf{F}(t)\| < \infty$ for all $t \in I$. Then the *norm* of $\mathbf{F}$ is defined by

$$\|\mathbf{F}\| = \sup_{t \in I} \| \mathbf{F}(t) \|.$$

For convenience, the notion of stability, in the sense of Lyapunov, is restated below (cf., for instance, [10, p. 26]).

DEFINITION 6. The zero solution of the linear system (1) is said to be

(a) *stable* if, given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that $\|\mathbf{x}(t)\| < \epsilon$ for all $t \geqslant t_0$ whenever $\|\mathbf{x}_0\| < \delta$;

(b) *uniformly stable* on (T, ∞) if in addition to being stable for any $t_0 \geqslant T$, the positive number δ in (a) is independent of t_0, i.e., $\delta = \delta(\epsilon)$ for all $t_0 \geqslant T$;

(c) *asymptotically stable* if in addition to being stable, $\|\mathbf{x}(t)\| \to 0$ as $t \to \infty$;

(d) *uniformly asymptotically stable* on (T, ∞) if both (b) and (c) hold.

Some notation and terminology that will be used in the sequel are introduced in the following two definitions.

DEFINITION 7. Let $\mathbf{A} \in \mathbb{K}^{n \times n}$ be a proper matrix function on $I = [T_0, \infty)$ such that $\mathbf{A}(t) = f(t, \mathbf{G})$ having a generating matrix $\mathbf{G} \in \mathbb{C}^{n \times n}$ with extended spectrum $\Gamma_{\mathbf{G}}$ and a primitive function $f(t, \lambda)$ defined on $\Gamma_{\mathbf{G}}$. Let

$$g(t, \lambda) = \exp^{\int^t f(\tau, \lambda) d\tau}$$

and

$$\phi(t,t_0,\lambda) = \exp^{\int_{t_0}^{t} f(\sigma,\lambda)\,d\sigma}, \qquad t \geqslant t_0.$$

Then,

(a) the function $g(t,\lambda)$ is said to be *bounded on* Γ_G if for every $t_0 \in I$ there exists an $M = M(t_0) > 0$ such that $\|[g(\cdot,\Gamma_G)]\| \leqslant M$ on $[t_0,\infty)$;

(b) the function $g(t,\lambda)$ is said to *converge to zero on* Γ_G *as* $t \to \infty$ if $g(t,\lambda)$ is bounded on Γ_G and $\|[g(t,\Gamma_G)]\| \to 0$ as $t \to \infty$;

(c) the function $\phi(t,t_0,\lambda)$ is said to be *uniformly bounded on* Γ_G for all $t_0 \in I$ if there exists an $M = M(T_0) > 0$, independent of t_0, such that $\|[\phi(t,\tau,\Gamma_G)]\| \leqslant M$ for all t,τ satisfying $T_0 \leqslant t_0 \leqslant \tau \leqslant t < \infty$;

(d) the function $\phi(t,t_0,\lambda)$ is said to *converge to zero exponentially on* Γ_G *as* $t \to \infty$ if for all $\tau \in I$, $\|[\phi(t,\tau,\Gamma_G)]\| \to 0$ exponentially as $t \to \infty$, i.e., there exist $M = M(T_0) > 0$ and $C = C(T_0) > 0$, both independent of t_0, such that $\|[\phi(t,\tau,\Gamma_G)]\| < Me^{-C(t-\tau)}$ for all t,τ such that $T_0 \leqslant t_0 \leqslant \tau \leqslant t < \infty$.

Using this notation and terminology, we now present the following basic theorem regarding stability of proper systems (1) with coefficient matrices $\mathbf{A} \in \mathbb{K}^{n\times n}$.

THEOREM 5. *Let* $\mathbf{A} \in \mathbb{K}^{n\times n}$ *be a proper matrix function on* $I = [T_0,\infty)$ *such that* $\mathbf{A}(t) = f(t,\mathbf{G})$ *with the extended spectrum* Γ_G *of* $\mathbf{G}$. *Let*

$$g(t,\lambda) = e^{\int^{t} f(\tau,\lambda)\,d\tau},$$

and

$$\phi(t,\tau,\lambda) = e^{\int_{\tau}^{t} f(\sigma,\lambda)\,d\sigma}.$$

Then the proper system (1) *is*

(a) *stable on* I *if and only if* $g(t,\lambda)$ *is bounded on* Γ_G;

(b) *asymptotically stable on* I *if and only if* $g(t,\lambda)$ *is bounded on* Γ_G *and converges to zero on* Γ_G *as* $t \to \infty$.

(c) *uniformly stable on* I *if and only if* $\phi(t,t_0,\lambda)$ *is uniformly bounded on* Γ_G *for all* $t_0 \geqslant T_0$;

(d) *uniformly asymptotically stable on* I *if and only if* $\phi(t,t_0,\lambda)$ *is uniformly bounded on* Γ_G *and converges to zero exponentially on* Γ_G *as* $t \to \infty$ *for all* $t_0 \geqslant T_0$.

The following two lemmas are needed in order to prove Theorem 5. The first one is a well-known result (cf. [10, p. 84]).

LEMMA 3. *Let $\mathbf{X}_A(t)$ be a fundamental solution matrix of the system* (1) *on $I=[T_0,\infty]$. Then the linear system* (1) *is:*

(a) *stable on I if and only if for any $t_0 \in I$ there exists an $M = M(t_0) > 0$ such that $\|\mathbf{X}_A\| \leqslant M$ on $[t_0,\infty)$;*

(b) *asymptotically stable on I if and only if for any $t_0 \in I$, $\|\mathbf{X}_A(t)\| \to 0$ as $t \to \infty$;*

(c) *uniformly stable on I if and only if for all $t_0 \geqslant T_0$ there exists an $M = M(T_0) > 0$, independent of t_0, such that $\|\mathbf{X}_A(t)\mathbf{X}_A^{-1}(\tau)\| \leqslant M$ for all t and τ such that $T_0 \leqslant t_0 \leqslant \tau \leqslant t < \infty$;*

(d) *uniformly asymptotically stable on I for all $t_0 \geqslant T_0$ if and only if there exist $M = M(T_0) > 0$ and $C = C(T_0) > 0$, both independent of t_0, such that $\|\mathbf{X}_A(t)\mathbf{X}_A^{-1}(\tau)\| \leqslant Me^{-C(t-\tau)}$ for all t and τ such that $T_0 \leqslant t_0 \leqslant \tau \leqslant t < \infty$.*

The next lemma is a variation of a standard result in functional analysis (cf. [12, p. 72]).

LEMMA 4. *Let $\alpha_i : I \to \mathbb{C}$, $i = 1,2,\dots,m$, be functions defined on $I = [T_0,\infty)$. Let $\{z_i\}_{i=1}^m$ be a linearly independent set of vectors in an n-dimensional space $\mathbb{V}$ over $\mathbb{C}$, and let $z(t) = \sum_{i=1}^m \alpha_i(t) z_i$. Then*

(a) *$\|z\| < \infty$ if and only if $\|\alpha_i\| < \infty$ for all $i \leqslant m$;*
(b) *$\|z(t)\| \to 0$ as $t \to \infty$ if and only if $\alpha_i(t) \to 0$ as $t \to \infty$ for all $i \leqslant m$,*

where $\|z(t)\|$ is the norm for $z(t)$ induced by the standard inner product defined on $\mathbb{V}$ (i.e. the Euclidian norm for the coordinate vectors of $z(t)$ with respect to some basis for $\mathbb{V}$), and $\|z\| = \sup_{t \in I} \|z(t)\|$.

Proof. Let $\mathbf{Z} = [\boldsymbol{\zeta}_1 | \boldsymbol{\zeta}_2 | \cdots | \boldsymbol{\zeta}_m] \in \mathbb{C}^{n \times m}$, where $\boldsymbol{\zeta}_i \in \mathbb{C}^n$ are the coordinate vectors for $z_i \in \mathbb{V}$ with respect to some basis for $\mathbb{V}$. Since $\{\boldsymbol{\zeta}_i\}_{i=1}^m$ is linearly independent, $\mathbf{Z}^*\mathbf{Z} = \mathbf{H} \in \mathbb{C}^{m \times m}$ is a positive definite Hermitian matrix. Let $\boldsymbol{\alpha}(t) = [\alpha_1(t), \alpha_2(t), \dots, \alpha_m(t)]^{\mathrm{T}}$, and use the second canonical Hermitian form [13, p. 204]; we have $\|z(t)\|^2 = z^*(t)z(t) = \langle \boldsymbol{\alpha}(t), \mathbf{H}\boldsymbol{\alpha}(t) \rangle = \sum_{i=1}^m \lambda_i |\alpha_i(t)|^2$, where $\langle \cdot,\cdot \rangle$ denotes the standard inner product on $\mathbb{C}^m$ defined by $\langle \mathbf{x},\mathbf{y} \rangle = \mathbf{x}^*\mathbf{y}$, $\mathbf{x},\mathbf{y} \in \mathbb{C}^m$, and $\lambda_i > 0$ are the eigenvalues of $\mathbf{H}$. Consequently, $\mu\|\boldsymbol{\alpha}\| \leqslant \|z\| \leqslant \nu\|\boldsymbol{\alpha}\|$, where $\mu = \min_{1 \leqslant i \leqslant m}\{\lambda_i\}$ and $\nu = \max_{1 \leqslant i \leqslant m}\{\lambda_i\}$. Lemma 4 then follows immediately. ∎

Proof of Theorem 5. By Lemma 1 and Theorem 2(b), we may write

$$\mathbf{X}_{\mathbf{A}}(t)=g(t,\mathbf{G})=\sum_{i=1}^{r}\sum_{k=1}^{d_i}\frac{g_\lambda^{(k-1)}(t,\lambda_i)}{(k-1)!}\mathbf{Z}_{ik},$$

where $\mathbf{Z}_{ik}$, $k=1,2,\dots,d_i$, $i=1,2,\dots,r$, are the components of $\mathbf{G}$. Since the components of $\mathbf{G}$ are linearly independent and $g_\lambda^{(k)}(\cdot,\lambda_i)$ are defined on I, (a) and (b) follow, respectively, from Lemmas 3(a), 4(a) and Lemmas 3(b), 4(b). To prove parts (c) and (d), note that, by Theorem 2(b), (d), (e), the normalized fundamental matrix (state-transition matrix) $\mathbf{\Phi}_{\mathbf{A}}(t,\tau)$ is given by

$$\mathbf{\Phi}_{\mathbf{A}}(t,\tau)=\mathbf{X}_{\mathbf{A}}(t)\mathbf{X}_{\mathbf{A}}^{-1}(\tau)=\phi(t,\tau,\mathbf{G})=\sum_{i=1}^{r}\sum_{k=1}^{d_i}\frac{\phi_\lambda^{(k-1)}(t,\tau,\lambda_i)}{(k-1)!}\mathbf{Z}_{ik}.$$

Then (c) follows easily from Lemmas 3(c), 4(a), and (d) follows from Lemmas 3(d), 4(b). ∎

The following examples show that, in Theorem 5, the boundedness and convergence requirements for the derivatives $g_\lambda^{(k)}(t,\lambda_i)$ are necessary in order to determine stability of proper systems (1).

EXAMPLE 4

(a) Let

$$g(t,\lambda)=e^{(1-\lambda^2)t}.$$

Then

$$g'_\lambda(t,\lambda)=-2\lambda t e^{(1-\lambda^2)t}.$$

Consider the following generating matrix:

$$\mathbf{G}=\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}$$

with the components $\mathbf{Z}_{11}$, $\mathbf{Z}_{12}$ of $\mathbf{G}$ given by

$$\mathbf{Z}_{11}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\qquad \mathbf{Z}_{12}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

and the extended spectrum of $\mathbf{G}$ given by $\Gamma_{\mathbf{G}}=\{-1,-1\}$. Thus, $g(t,-1)=1$

is bounded, but $g'_\lambda(t,-1) = 2t$ is unbounded. Consequently,

$$\|g(t,\mathbf{G})\| = \|g(t,-1)\mathbf{Z}_{11} + g'_\lambda(t,-1)\mathbf{Z}_{12}\|$$

$$= \left\| \begin{bmatrix} 1 & 2t \\ 0 & 1 \end{bmatrix} \right\| \to \infty \quad \text{as} \quad t \to \infty.$$

(b) Let

$$g(t,\lambda) = e^{(1-\lambda^2)e^{2t}-t}.$$

Then

$$g'_\lambda(t,\lambda) = -2\lambda e^{2t} e^{(1-\lambda^2)e^{2t}-t}.$$

Consider the same generating matrix $\mathbf{G}$ given above, We see that

$$g(t,-1) = e^{-t} \to 0 \quad \text{as} \quad t \to \infty,$$

but

$$g'_\lambda(t,-1) = 2e^t \to \infty \quad \text{as} \quad t \to \infty.$$

Consequently,

$$\|g(t,\mathbf{G})\| = \|g(t,-1)\mathbf{Z}_{11} + g'_\lambda(t,-1)\mathbf{Z}_{12}\|$$

$$= \left\| \begin{bmatrix} e^{-t} & 2e^t \\ 0 & e^{-t} \end{bmatrix} \right\| \to \infty \quad \text{as} \quad t \to \infty.$$

Despite Example 4, under certain circumstances, the restrictions imposed on the derivatives $g_\lambda^{(k)}(t,\lambda_i)$ can be lifted. For instance, note that systems (1) having a constant coefficient matrix $\mathbf{A}(t) \equiv \mathbf{A}$ are proper. In this case, if $\mathbf{A}$ has *no* multiple eigenvalues on the imaginary axis (including the origin), the conditions $\|[g(\cdot,\Gamma_{\mathbf{G}})]\| < \infty$ and $\|[g(\cdot,\Gamma_{\mathbf{G}})]\| \to 0$ as $t \to \infty$ in Theorem 5 for stability and asymptotic stability of proper systems can be relaxed, respectively, to $\|[g(\cdot,\Lambda_{\mathbf{G}})]\| < \infty$ and $\|[g(\cdot,\Lambda_{\mathbf{G}})]\| \to 0$ as $t \to \infty$, where $\Lambda_{\mathbf{G}} = \{\lambda_i\}_{i=1}^r$ is the spectrum of $\mathbf{G}$ consisting of *distinct* eigenvalues of $\mathbf{G}$, and $[g(t,\Lambda_{\mathbf{G}})]$ denotes the row vector consisting of the scalar functions in the ordered set

$g(t, \Lambda_G) = \{g(t, \lambda_i), \lambda_i \in \Lambda_G\}_{i=1}^r$. In other words, under the given circumstances, the derivatives $g_\lambda^{(k)}(t, \lambda_i)$ for repeated eigenvalues λ_i of the generating matrix $\mathbf{G}$ [$\mathbf{G} = \mathbf{A}$ when $\mathbf{A}(t) \equiv \mathbf{A}$] need not be considered for determining stability of systems (1). Therefore, it is natural to ask to what further extent this relaxation is valid for *variable-coefficient* (*time-varying*) proper systems. The following corollary of Theorem 5 gives a sufficient condition for the extended validity, using the fact that $\Gamma_G = \Lambda_G$ when $\mathbf{G}$ is *simple* [8, p. 73].[5]

COROLLARY 1. *Let* $\mathbf{A} \in \mathbb{K}^{n \times n}$ *be proper such that* $\mathbf{A}(t) = f(t, \mathbf{G})$ *with a simple generating matrix* $\mathbf{G}$. *Let*

$$g(t, \lambda) = e^{\int^t f(\tau, \lambda) d\tau},$$

and

$$\phi(t, \tau, \lambda) = e^{\int_\tau^t f(\sigma, \lambda) d\sigma}.$$

Then the system (1) *is*

(a) *stable on I if and only if $g(t, \lambda)$ is bounded on Λ_G;*

(b) *asymptotically stable on I if and only if $g(t, \lambda)$ is bounded on Λ_G and converges to zero on Λ_G as $t \to \infty$;*

(c) *uniformly stable on I if and only if $\phi(t, t_0, \lambda)$ is uniformly bounded on Λ_G for all $t_0 \in I$;*

(d) *uniformly asymptotically stable on I if and only if $\phi(t, t_0, \lambda)$ is uniformly bounded on Λ_G and converges to zero exponentially on Λ_G as $t \to \infty$, for all $t_0 \in I$.*

Now, for semiproper systems (1), Theorem 3 states that every semiproper matrix function $\mathbf{A}(t)$ can be decomposed as $\mathbf{A}(t) = \sum_{k=1}^m \mathbf{A}_k(t)$, where $\mathbf{A}_k(t)$ are proper matrix functions with pairwise commutative generating matrices. We shall call the proper systems defined by $\dot{\mathbf{x}} = \mathbf{A}_k(t)\mathbf{x}$ *proper subsystems* of the underlying semiproper system (1). With this terminology we can now establish the following sufficient criterion for semiproper systems (1) in terms of the stability of their proper subsystems.

[5] Let $\Lambda_G = \{\lambda_i\}_{i=1}^r$, $\Gamma_G = \{d_i \cdot \lambda_i\}_{i=1}^r$, where $r \leqslant n$ is the number of distinct eigenvalues of $\mathbf{G}$. When $\mathbf{G}$ is *simple*, $\Gamma_G = \{1 \cdot \lambda_i\}_{i=1}^r$. Thus, we may, by abuse of notation, write $\Gamma_G = \{\lambda_i\}_{i=1}^r = \Lambda_G$.

THEOREM 6. *Let* $\mathbf{A} \in \mathbb{K}^{n \times n}$ *be semiproper on* $I = [T_0, \infty)$. *Then the semiproper system* (1) *is* (*uniformly*) *stable if all of its proper subsystems are* (*uniformly*) *stable, and is* (*uniformly*) *asymptotically stable if, in addition to being* (*uniformly*) *stable, one of the proper subsystems is* (*uniformly*) *asymptotically stable.*

Proof. We first prove the statements for stability and asymptotic stability. Let $\mathbf{A} \in \mathbb{K}^{n \times n}$ such that $\mathbf{A}(t) = \sum_{k=1}^{m} \mathbf{A}_k(t)$, where the $\mathbf{A}_k(t)$ are proper matrix functions with pairwise commutative generating matrices. By Theorem 4, a fundamental solution to the semiproper system (1) can be written as

$$\mathbf{X}_{\mathbf{A}}(t) = \prod_{k=1}^{m} \mathbf{X}_{\mathbf{A}_k}(t),$$

where $\mathbf{X}_{\mathbf{A}_k}(t)$ is a fundamental solution of the proper subsystem defined by $\mathbf{A}_k(t)$. Suppose that all the proper subsystems are stable. By Lemma 2(a), there exists an $M > 0$ such that $\|\mathbf{X}_{\mathbf{A}_k}\| < M$ for all $k = 1, 2, \ldots, m$. Then we have

$$\|\mathbf{X}_{\mathbf{A}}\| \leqslant \prod_{k=1}^{m} \|\mathbf{X}_{\mathbf{A}_k}\| \leqslant M^m = N.$$

By Lemma 2(a), the system is stable. Now in addition assume, without loss of generality, that $\|\mathbf{X}_{\mathbf{A}_1}(t)\| \to 0$ as $t \to \infty$. Then

$$\|\mathbf{X}_{\mathbf{A}}(t)\| \leqslant \prod_{k=1}^{m} \|\mathbf{X}_{\mathbf{A}_k}(t)\| \leqslant \|\mathbf{X}_{\mathbf{A}_1}(t)\| M^{m-1} \to 0 \qquad \text{as} \quad t \to \infty.$$

Therefore $\|\mathbf{X}_{\mathbf{A}}(t)\| \to 0$ as $t \to \infty$. By Lemma 2c, the system (1) is asymptotically stable. The statements for uniform stable and uniform asymptotic stability can be proved similarly. ∎

5. SUMMARY AND CONCLUSIONS

Using the results obtained in [24], we have developed, in this paper, systematic procedures for obtaining finite-form solutions for proper (Lemma 1) and for semiproper (Theorem 4) linear systems (1). In addition, we have derived a necessary and sufficient stability criterion (Theorem 5) for proper linear systems and a sufficient stability criterion (Theorem 6) for semiproper linear systems.

The results obtained in this paper have important theoretical and practical applications. For instance, Lemma 1 and Theorem 5 have been used in [27] to establish a new notion called *coeigenvalues* for proper matrices $\mathbf{A}(t)$ and proper linear systems (1). This new notion, together with the conventional (time-varying) eigenvalues of $\mathbf{A}(t)$, has been successfully used to construct *finite-form* analytical solutions and a *necessary and sufficient* stability criterion for variable-coefficient (time-varying) proper linear systems in a manner like that for constant-coefficient (time-invariant) linear systems (1). Moreover, this latter result has been used to demonstrate why the (well-known false) *time-varying eigenvalue conjecture* that states "variable-coefficient (time-varying) systems (1) are asymptotically stable if and only if the (time-varying) eigenvalues $\lambda_i(t)$ of $\mathbf{A}(t)$ stay in the open left half plane for all $t \geqslant t_0$" fails both in necessity and in sufficiency.

Under some reasonable restrictions, Theorem 6 has been extended in [25] to a necessary and sufficient stability criterion for (time-varying) *semiproper* linear systems.

The results obtained here, together with the results obtained in [27] and [25], constitute a foundation for further investigations of more general variable-coefficient (time-varying) linear systems (1) using a recently developed mathematical tool called *D-similarity transformations* [26] (see also [28–32]).

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REFERENCES

1 R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.
2 Y. S. Bogdanov, Matrices that commute with their derivatives, *Izv. Vyssh. Uchebn. Zaved. Mat.* No. 4 (11), 1959.
3 H. D'angelo, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn and Bacon, Boston, 1970.
4 I. J., Epstein, Conditions for a matrix to commute with its integral, *Proc. Amer. Math. Soc.* 14:266–270 (1963).
5 N. P. Erugin, *Linear Systems of Ordinary Differential Equations with Periodic and Quasi-periodic Coefficients*, Academic, New York, 1966.
6 H. I. Freedman and J. D. Lawson, Solution for ordinary differential equations, *Linear Algebra Appl.* 8:369–374 (1974).

7 H. I. Freedman, Functionally commutative matrices and matrices with constant eigenvectors, *Linear and Multilinear Algebra* 4:107–113 (1976).
8 F. R. Gantmacher, *The Theory of Matrices*, Chelsea, New York, 1959.
9 S. Goff, Hermitian function matrices which commute with their derivatives, *Linear Algebra Appl.* 36:33–40 1981.
10 J. K. Hale, *Ordinary Differential Equations*, Wiley, New York, 1969.
11 L. Kotin and I. J. Epstein, On matrices which commute with their derivatives, *Linear and Multilinear Algebra* 12:57–72 (1982/1983).
12 E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York, 1978.
13 P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed., Academic, New York, 1985.
14 D. L. Lukes, *Differential Equations: Classical to Controlled*, Academic, 1982.
15 W. Magnus, On the exponential solution of differential equations for a linear operator, *Comm. Pure Appl. Math.* 7:649–673 (1954).
16 J. F. P. Martin, Some results on matrices that commute with their derivatives, *SIAM J. Appl. Math.* 14:1171–1183 (1967).
17 V. V. Morozov, On commutative matrices, *Uchen, Zap. Karel Ped. Inst. Ser. Fiz.-Mat. Nauk* 9 (1952).
18 I. V. Parnev, Some classes of matrices that commute with their derivatives, *Trudy Rjazan. Radiotekhn. Inst. Vyp. 42 Differencial'nye Uravnenija*, 1972, pp. 142–154.
19 I. V. Parnev, A theorem on functionally commutative matrices, *Trudy Rjazan. Radiotekhn. Inst. Vyp. 69 Differencial'nye Uravnenija*, 1975, pp. 78–81.
20 N. J. Rose, On the eigenvalues of a matrix which commutes with its derivative, *Proc. Amer. Math. Soc.* 16:752–754 (1965).
21 E. I. Troickii, A study of the solution of three linear homogeneous differential equations whose matrix commutes with its integral, *Trudy Rjazan. Radiotekhn. Inst. Vyp. 53 Differencial'nye Uravnenija*, 1974, pp. 188–189.
22 D. M. Wiburg, *State Space and Linear Systems*, McGraw-Hill, New York, 1971.
23 M. Y. Wu and A. Sherif, On the commutative class of linear time-varying systems, *Internat. J. Control* 23(3):433–444 (1976).
24 J. Zhu and C. H. Morales, Spatial decomposition of functionally commutative matrices, *Linear Algebra Appl.* 131:71–92 (Apr. 1990).
25 J. Zhu and C. D. Johnson, Necessary and sufficient stability criteria for semiproper linear time-varying dynamical systems, in *Proceedings, 1988 Southeastern Symposium on system Theory*, Mar. 1988, pp. 178–182.
26 J. Zhu and C. D. Johnson, New results in the reduction of linear time-varying dynamical systems, *SIAM J. Control Optim.* 27(3):476–494 (May 1989).
27 J. Zhu and C. D. Johnson, New results for the stability analysis of time-varying linear systems; part I: The case of reduced systems, in *Proceedings, The 1988 American Control Conference*, June 1988, pp. 1494–1499; Part II, in preparation.
28 J. Zhu and C. D. Johnson, Unified canonical forms for linear time-varying dynamical systems under D-similarity transformations; part I and II, in *Proceedings, IEEE 21st Southeastern Symposium on System Theory*, Mar. 1989,

pp. 57–63, 74–81.
29 J. Zhu and C. D. Johnson, Unified canonical forms for matrices over a differential ring, *Linear Algebra Appl.* 147:201–248 (Mar. 1991).
30 J. Zhu and C. D. Johnson, Invariants of linear time-varying dynamical systems under *D*-similarity transformations, in *Proceedings, The 1989 American Control Conference*, June 1989, pp. 827–833.
31 J. Zhu and C. D. Johnson, Stability analysis of periodic linear dynamical systems using a new time-varying eigenvalue concept; part I: Theoretical aspects, in *Proceedings, IEEE Southeastscon*, Apr. 1989, pp. 437–443.
32 J. Zhu and C. D. Johnson, Stability analysis of periodic linear dynamical systems using a new time-varying eigenvalue concept; part II: Computational aspects, in *Proceedings, Conference on Information Sciences and Systems*, Mar. 1989, pp. 390–396.

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