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On Multipoint Boundary Value Problems for Discrete Equations

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1. INTRODUCTION

In recent years there has been considerable interest in studying the discrete analogues of several results known for the continuous case (see [1, 9] and references therein). Since in the discrete case there is no analogue of basic Rolle's lemma, which is required almost everywhere in the continuous case, this study is interesting and requires new proofs generally based on algebra. The discrete problem and its relationship to the continuous problem is in fact one of the main sources of the constructive study. In the case of boundary value problems for ordinary differential equations besides this relationship, the basic question about the existence, uniqueness, etc. of discrete problems has been discussed [6, 7, 10–12]. Here, the reason can be explained by simple examples: The continuous problem $y'' + (\pi^2/n^2)\,y = 0$; $y(0) = y(n) = 0$ has an infinite number of solutions $y(t) = k\sin(\pi/n)\,t$ (k is arbitrary) whereas its discrete analogue $\Delta\nabla y(t) + (\pi^2/n^2)\,y(t) = 0$; $y(0) = y(n) = 0$ has only one solution $y(t) \equiv 0$. The problem $y'' + (\pi^2/4n^2)\,y = 0$; $y(0) = 0$, $y(n) = 1$ has only one solution $y(t) = \sin(\pi/2n)\,t$, and its discrete analogue $\Delta\nabla(t) + (\pi^2/4n^2)\,y(t) = 0$; $y(0) = 0$, $y(n) = 1$ also has one solution. The continuous problem $y'' + 4\sin^2(\pi/2n)\,y = 0$; $y(0) = 0$, $y(n) = \varepsilon(\neq 0)$ has only one solution $y(t) = \varepsilon\sin[(2\sin\pi/2n)\,t]/\sin[(2\sin\pi/2n)\,n]$, whereas its discrete analogue $\Delta\nabla y(t) + 4\sin^2(\pi/2n)\,y(t) = 0$; $y(0) = 0$, $y(n) = \varepsilon$ $(\neq 0)$ has no solution. Thus, the nature of the solution changes when a continuous boundary value problem is discretised. Moreover, two-point boundary value problems involving derivatives lead to multipoint problems in the discrete case.

In this paper we shall provide several results for the existence and uniqueness of the solutions of discrete systems together with multipoint boundary conditions. All the results proved here except Theorem 3.1 are

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constructive. Several other theoretical and practical questions concerning the relationship with the continuous problems (the work of Urabe [14, 15]), boundary data falling between two grid points, stability, and other efficient methods for discrete problems have motivated our work with continuous problems [2–4] and the methods for discrete problems proposed in [5, 8]; such problems shall be considered elsewhere.

2. Linear Systems

We shall consider the set of n linear difference equations with variable coefficients

$$y(t+1) = A(t)\,y(t) + f(t), \qquad t \in I_\infty, \tag{2.1}$$

together with the boundary conditions

$$\sum_{i=0}^{N} L_i y(t_i) = l, \tag{2.2}$$

where $A(t)$ is an $n \times n$ matrix with elements $a_{ij}(t)$, $1 \leqslant i, j \leqslant n$; $y(t)$ is an $n \times 1$ vector with components $y_i(t)$, $1 \leqslant i \leqslant n$; $f(t)$ is an $n \times 1$ vector with components $f_i(t)$, $1 \leqslant i \leqslant n$; I_∞ is the set of discrete points $\{0, 1, 2, \ldots\}$; L_i, $0 \leqslant i \leqslant N$ are given $n \times n$ matrices with elements c^i_{pq}, $1 \leqslant p, q \leqslant n$; l is a given $n \times 1$ vector with components l_i, $1 \leqslant i \leqslant n$; the t_i are ordered so that $t_i < t_{i+1}$, with $t_0 = 0$.

In what follows we shall assume that $A(t)$ and $f(t)$ are defined for all $t \in I_\infty$. Further, we shall denote $Y(t)$ as the fundamental matrix solution of the homogeneous difference system

$$y(t+1) = A(t)\,y(t), \qquad t \in I_\infty, \tag{2.3}$$

such that $Y(0) = E$(unit matrix). The matrix G is defined as

$$G = \sum_{i=0}^{N} L_i Y(t_i). \tag{2.4}$$

Theorem 2.1. *Let the matrix G be nonsingular. Then* (2.1) *and* (2.2) *have a unique solution for an arbitrary vector l and it can be represented as*

$$y(t) = Y(t)\,G^{-1} l + \sum_{s=0}^{t_N - 1} H(t, s)\,f(s), \tag{2.5}$$

where $H(t,s)$ is the Green's matrix such that for $t_{k-1} \leqslant s \leqslant t_k - 1$, $1 \leqslant k \leqslant N$,

$$\begin{aligned} H(t,s) &= Y(t)\left[E - G^{-1}\sum_{i=k}^{N} L_i Y(t_i)\right] Y(s+1)^{-1}, \qquad t_{k-1} \leqslant s < t \leqslant t_k - 1, \\ &= -Y(t)\, G^{-1} \sum_{i=k}^{N} L_i Y(t_i)\, Y(s+1)^{-1}, \qquad t_{k-1} \leqslant t \leqslant s \leqslant t_k - 1. \end{aligned} \tag{2.6}$$

Proof. Any solution of (2.1) can be expressed as

$$y(t) = Y(t)\, y_0 + Y(t) \sum_{s=0}^{t-1} Y(s+1)^{-1} f(s), \tag{2.7}$$

where y_0 is a constant vector. The solution (2.7) satisfies the boundary conditions (2.2) if and only if

$$G y_0 = l - \sum_{i=0}^{N} L_i Y(t_i) \sum_{s=0}^{t_i - 1} Y(s+1)^{-1} f(s), \tag{2.8}$$

since det $G \neq 0$, we have on rearranging the terms

$$y_0 = G^{-1} l - G^{-1} \sum_{k=1}^{N} \sum_{s=t_{k-1}}^{t_k - 1} \sum_{i=k}^{N} L_i Y(t_i)\, Y(s+1)^{-1} f(s),$$

and hence the solution of (2.1), (2.2) is

$$\begin{aligned} y(t) = {} & Y(t)\, G^{-1} l - Y(t)\, G^{-1} \sum_{k=1}^{N} \sum_{s=t_{k-1}}^{t_k - 1} \sum_{i=k}^{N} L_i Y(t_i)\, Y(s+1)^{-1} f(s) \\ & + Y(t) \sum_{s=0}^{t-1} Y(s+1)^{-1} f(s), \end{aligned}$$

which from the definition of $H(t,s)$ is the same as (2.5).

Remark 1. The fundamental matrix $Y(t)$ has the representation $Y(t) = \prod_{s=t-1}^{0} A(s)$, where $\prod_{s=-1}^{0} A(s) = E$, hence the computation of the solution $y(t)$ requires only operations with the known matrices.

Remark 2. Theorem 2.1 covers Theorems 3 and 4 of Sugiyama [13].

Next, we shall consider the case when the matrix G has rank $n-m$ $(1 \leqslant m \leqslant n)$. We shall need the following:

LEMMA 2.2 [15]. *Given a system of linear algebraic equations*

$$Ax = b, \tag{2.9}$$

where A is an $n \times n$ matrix and x and b are both n-dimensional vectors, suppose that the rank of A is $n - m$ $(1 \leqslant m \leqslant n)$.

The linear algebraic system (2.9) *possesses a solution if and only if*

$$\Delta b = 0, \tag{2.10}$$

where Δ is an $m \times n$ matrix whose row vectors are linearly independent vectors d_α, $1 \leqslant \alpha \leqslant m$, satisfying

$$d_\alpha A = 0. \tag{2.11}$$

In case (2.10) *holds, any solution of* (2.9) *can be given by*

$$x = \sum_{\alpha=1}^{m} k_\alpha c_\alpha + Sb, \tag{2.12}$$

where k_α, $1 \leqslant \alpha \leqslant m$, are arbitrary constants; c_α, $1 \leqslant \alpha \leqslant m$, are m linearly independent column vectors satisfying

$$Ac_\alpha = 0, \tag{2.13}$$

and S is an $n \times n$ matrix independent of b such that

$$ASp = p \tag{2.14}$$

for any column vector p satisfying

$$\Delta p = 0. \tag{2.15}$$

THEOREM 2.3. *Let the rank of the matrix G be $n - m$ $(1 \leqslant m \leqslant n)$. Then,* (2.1), (2.2) *possesses a solution if and only if*

$$\Delta l - \Delta \sum_{i=0}^{N} L_i Y(t_i) \sum_{s=0}^{t_i - 1} Y(s+1)^{-1} f(s) = 0, \tag{2.16}$$

where Δ is an $m \times n$ matrix whose row vectors are linearly independent vectors d_α, $1 \leqslant \alpha \leqslant m$, satisfying

$$d_\alpha G = 0. \tag{2.17}$$

In case (2.16) *is valid for given l and $f(t)$ any solution of* (2.1), (2.2) *can be given by*

$$y(t) = \sum_{\alpha=1}^{m} k_\alpha u_\alpha(t) + Y(t)\, Sl + \sum_{s=0}^{t_N - 1} H(t, s) f(s), \tag{2.18}$$

where k_α, $1 \leqslant \alpha \leqslant m$, *are arbitrary constants*; $u_\alpha(t)$, $1 \leqslant \alpha \leqslant m$, *are* m *linearly independent solutions of* (2.3) *satisfying the boundary conditions*

$$\sum_{i=0}^{N} L_i u_\alpha(t_i) = 0, \tag{2.19}$$

S *is a matrix independent of* $f(t)$ *and* l *such that*

$$GSp = p \tag{2.20}$$

for any n-dimensional vector p *satisfying*

$$\Delta p = 0, \tag{2.21}$$

and $H(t, s)$ *is the Green's matrix such that, for* $t_{k-1} \leqslant s \leqslant t_k - 1$, $1 \leqslant k \leqslant N$,

$$\begin{aligned} H(t, s) &= Y(t)\left[E - S \sum_{i=k}^{N} L_i Y(t_i)\right] Y(s+1)^{-1}, \quad && t_{k-1} \leqslant s < t \leqslant t_k - 1, \\ &= -Y(t)\, S \sum_{i=k}^{N} L_i Y(t_i)\, Y(s+1)^{-1}, && t_{k-1} \leqslant t \leqslant s \leqslant t_k - 1. \end{aligned} \tag{2.22}$$

Proof. Any solution of (2.1) can be expressed as (2.7); it satisfies condition (2.2) if and only if (2.8) holds. Since G has rank $n - m$, from Lemma 2.2 there exits a solution of (2.1), (2.2) if and only if (2.16) is true. When (2.16) holds, by Lemma 2.2 the constant vector y_0 satisfying (2.8) can be given by

$$y_0 = \sum_{\alpha=1}^{m} k_\alpha c_\alpha + S\left[l - \sum_{i=0}^{N} L_i Y(t_i) \sum_{s=0}^{t_i - 1} Y(s+1)^{-1} f(s)\right], \tag{2.23}$$

where k_α, $1 \leqslant \alpha \leqslant m$, are arbitrary constants; c_α, $1 \leqslant \alpha \leqslant m$, are m linearly independent column vectors satisfying

$$Gc_\alpha = 0, \tag{2.24}$$

and S is an $n \times n$ matrix independent of the right member of (2.8) such that (2.20) holds for any vector p satisfying (2.21). Let

$$Y(t)\, c_\alpha = u_\alpha(t); \tag{2.25}$$

then $u_\alpha(t)$, $1 \leqslant \alpha \leqslant m$, are linearly independent and satisfy (2.3). Moreover, from (2.24) we find

$$\sum_{i=0}^{N} L_i Y(t_i)\, c_\alpha = \sum_{i=0}^{N} L_i u_\alpha(t_i) = Gc_\alpha = 0,$$

and hence $u_\alpha(t)$ satisfy the boundary condition (2.19).

Now substituting (2.23) into (2.7) and making use of (2.25), we find

$$y(t) = \sum_{\alpha=1}^{m} k_\alpha u_\alpha(t) + Y(t)\,Sl - Y(t)\,S \sum_{i=0}^{N} L_i Y(t_i) \sum_{s=0}^{t_i-1} Y(s+1)^{-1} f(s)$$
$$+ Y(t) \sum_{s=0}^{t-1} Y(s+1)^{-1} f(s),$$

which is from (2.22) is same as (2.18).

3. Nonlinear Systems

Here, we shall consider the set of n nonlinear difference equations

$$y(t+1) = f(t, y(t)), \qquad t \in I_\infty, \tag{3.1}$$

together with the boundary conditions (2.2). In (3.1), $f(t, y(t))$ is an $n \times 1$ vector with components $f_i(t, y(t))$, $1 \leqslant i \leqslant n$, which will be assumed continuous in its arguments $(t, y) \in [t_0, t_N] \times R^n$.

In what follows, we shall denote by B the set of n-vector valued functions defined for all $t \in I_\infty$ with the norm

$$\| y \| = \max_{1 \leqslant i \leqslant n} \max_{\substack{t \in I_\infty \\ 0 \leqslant t \leqslant t_N}} \{ | y_i(t) | \}.$$

THEOREM 3.1. *Let* (i) $A(t)$ *be some* $n \times n$ *matrix such that the fundamental solution* $Y(t)$ *of* (2.3) *exist and the matrix* G *defined in* (2.4) *is nonsingular and* (ii) *let* K *be a given positive constant and* $\| f(t, y) - A(t)\, y \| \leqslant Q$, *over the compact set*

$$\{(t, y) \colon t \in I_\infty, 0 \leqslant t \leqslant t_N;\ \| y \| \leqslant 2K \}.$$

Then, there exists a solution of (3; 1), (2, 2) *provided*

$$\| Y(t)\, G^{-1}\, l \| \leqslant K, \tag{3.2}$$

$$Q \, \| H \| \leqslant K, \tag{3.3}$$

where

$$\| H \| = \max_{1 \leqslant i \leqslant n} \max_{\substack{t \in I_\infty \\ 0 \leqslant t \leqslant t_N}} \sum_{j=1}^{n} \sum_{s=0}^{t_N-1} | H_{ij}(t, s) |.$$

Proof. Problem (3.1), (2.2) is equivalent to finding the solution of

$$y(t) \doteq Y(t)\, G^{-1} l + \sum_{s=0}^{t_N - 1} H(t, s)[f(s, y(s)) - A(s)\, y(s)].$$

The mapping $T\colon B \to B$ defined by

$$Ty(t) = Y(t)\, G^{-1} l + \sum_{s=0}^{t_N - 1} H(t, s)[f(s, y(s)) - A(s)\, y(s)] \tag{3.4}$$

is completely continuous.

The subset $S \subset B$, $S = \{ y(t) \in B\colon \| y \| \leqslant 2K \}$ is a closed convex subset of the Banach space B. For $y(t) \in S$, it follows that

$$\begin{aligned} \| Ty \| &\leqslant \| Y(t)\, G^{-1} l \| + \max_{1 \leqslant i \leqslant n} \max_{\substack{t \in I_\infty \\ 0 \leqslant t \leqslant t_N}} \sum_{j=1}^{n} \sum_{s=0}^{t_N - 1} |H_{ij}(t, s)| \\ &\quad \times \| f(s, y(s)) - A(s)\, y(s) \| \\ &\leqslant 2K. \end{aligned}$$

Thus T maps K into itself, and it follows from the Schauder's fixed point theorem that T has a fixed point in S. The fixed point is a solution of (3.1), (2.2).

DEFINITION. The function f is said to be of Lipschitz class, if for all $t \in I_\infty$, $0 \leqslant t \leqslant t_N$; $u(t), v(t) \in B$ the following is satisfied:

$$\| f(t, u(t)) - f(t, v(t)) \| \leqslant L \, \| u - v \|. \tag{3.5}$$

THEOREM 3.2. *Let $A(t)$ be as in Theorem 3.1 and f be of Lipschitz class. Then, there exists a unique solution of (3.1), (2.2) provided*

$$\theta = \max_{1 \leqslant i \leqslant n} \max_{\substack{t \in I_\infty \\ 0 \leqslant t \leqslant t_N}} \sum_{j=1}^{n} \sum_{s=0}^{t_N - 1} P_{ij}(t, s) < 1, \tag{3.6}$$

where

$$P(t, s) = |H(t, s)| \, [L + |A(s)|].$$

Proof. We shall show that the mapping T defined by (3.4) on B is contracting. For any $y_1(t), y_2(t) \in B$ it follows that

$$
\begin{aligned}
|Ty_1(t) - Ty_2(t)| &\leqslant \sum_{s=0}^{t_N-1} |H(t,s)|\, |f(s, y_1(s)) - A(s)\, y_1(s) - f(s, y_2(s)) \\
&\quad + A(s)\, y_2(s)| \\
&\leqslant \sum_{s=0}^{t_N-1} |H(t,s)|\, [L + |A(s)|]\, \| y_1 - y_2 \|,
\end{aligned}
$$

and, hence

$$
\| Ty_1 - Ty_2 \| \leqslant \theta \, \| y_1 - y_2 \|.
$$

Since, $\theta < 1$, the mapping T on B has one fixed point and this is equivalent to the existence and uniqueness of the solution of (3.1), (2.2).

4. Approximations of the Iterates

The fact that T is a contraction mapping on B means among other things that under the conditions of Theorem 3.2 the iterates

$$
\begin{gathered}
y_{n+1}(t) = A(t)\, y_{n+1}(t) + f(t, y_n(t)) - A(t)\, y_n(t), \\
y_0(t) = Y(t)\, G^{-1} l, \qquad n = 0, 1, \ldots,
\end{gathered}
\tag{4.1}
$$

converge to the unique solution of the problem (3.1), (2.2). However, in practical evaluation the theoretical sequence $\{y_n(t)\}$ generated by (4.1) is approximated by the computed sequence, say $\{y_n^*(t)\}$. The function f is approximated by say f^*. Therefore, the computed sequence $\{y_n^*(t)\}$ satisfies the recurrence relation

$$
\begin{aligned}
y_{n+1}^*(t) &= A(t)\, y_{n+1}^*(t) + f^*(t, y_n^*(t)) - A(t)\, y_n^*(t), \\
y_0^*(t) &= Y(t)\, G^{-1} l, \qquad n = 0, 1, \ldots.
\end{aligned}
\tag{4.2}
$$

Let the calculations are performed exactly, neglecting rounding error and

$$
\| f(t, y_n^*(t)) - f^*(t, y_n^*(t)) \| \leqslant \varepsilon
$$

for all n. Then, it follows from (4.1) and (4.2) that

$$
\begin{aligned}
|y_{n+1}(t) - y_{n+1}^*(t)| &\leqslant \sum_{s=0}^{t_N-1} |H(t,s)|\, |f(s, y_n(s)) - f(s, y_n^*(s)) \\
&\quad + f(s, y_n^*(s)) - f^*(s, y_n^*(s)) \\
&\quad - A(s)\, y_n(s) + A(s)\, y_n^*(s)| \\
&\leqslant \sum_{s=0}^{t_N-1} |H(t,s)|\, \{[L + |A(s)|]\, \| y_n - y_n^* \| + \varepsilon\},
\end{aligned}
$$

and hence

$$\| y_{n+1} - y_{n+1}^* \| \leqslant \theta \| y_n - y_n^* \| + \varepsilon \|H\|. \tag{4.3}$$

From (4.3), it follows that

$$\| y_n - y_n^* \| \leqslant \frac{\varepsilon \|H\|}{1-\theta}$$

for all n. Thus, if ε is small the accumulated error obtained by computing the sequence $\{y_n^*(t)\}$ is small.

5. Isolated Solution

Here, we shall assume that the function $f(t, y)$ is continuously differentiable in y for all $t \in [t_0, t_N]$ and $y \in R^n$.

THEOREM 5.1. *Let $y(t)$ be a solution of* (3.1), (2.2) *and let the matrix*

$$G = \sum_{i=0}^{N} L_i X(t_i) \tag{5.1}$$

be nonsingular, where $X(t)$ is the fundamental matrix solution of the first variational equation

$$x(t+1) = J(t, y(t))\, x(t), \qquad t \in I_\infty, \tag{5.2}$$

such that $X(0) = E$. Then, besides the solution $y(t)$, there is no other solution of (3.1), (2.2) *in a small neighborhood of $y(t)$, i.e., the solution is isolated.*

Proof. Let ε be a positive number so that $\varepsilon \|H\| < 1$. For such ε, by the continuity of $J(t, y)$ there is a positive constant δ such that $D_\delta = \{y : \| y - y(t)\| \leqslant \delta\} \subset R^n$, for all $t \in [t_0, t_N]$ and $\|J(t, y(t) + z(t)) - J(t, y(t))\| \leqslant \varepsilon$ on $[t_0, t_N]$ whenever $\|z(t)\| \leqslant \delta$. We shall show that in D_δ, $y(t)$ is the only solution.

For, if not, let $y_1(t)$ be another solution and define $x(t) = y_1(t) - y(t)$, then it follows that $\|x(t)\| \leqslant \delta$ and

$$x(t+1) = f(t, y_1(t)) - f(t, y(t)) = \int_0^1 J(t, y(t) + \theta x(t))\, x(t)\, d\theta.$$

Thus, we find

$$x(t+1) = J(t, y(t))\, x(t) + \int_0^1 [J(t, y(t) + \theta x(t)) - J(t, y(t))]\, x(t)\, d\theta,$$

or

$$x(t) = \sum_{s=0}^{t_N - 1} H(t, s) \left\{ \int_0^1 [J(s, y(s) + \theta x(s)) - J(s, y(s))] \, x(s) \, d\theta \right\},$$

which provides

$$\|x(t)\| \leqslant \varepsilon \, \|H\| \, \|x(t)\|.$$

Since $\varepsilon \|H\| < 1$, we get $x(t) \equiv 0$ or $y_1(t) \equiv y(t)$ on $[t_0, t_N]$.

Remark 3. In Theorem 5.1, $H(t, s)$ is the Green's matrix corresponding to the Jacobian matrix $J(t, y(t))$.

DEFINITION. A function $\bar{y}(t)$ defined on $[t_0, t_N]$ will be called an approximate solution of (3.1), (2.2) if there exist r and ε nonnegative constants such that

$$\|\bar{y}(t+1) - f(t, \bar{y}(t))\| \leqslant r, \tag{5.3}$$

and

$$\left\| \sum_{i=0}^{N} L_i \, \bar{y}(t_i) - 1 \right\| \leqslant \varepsilon. \tag{5.4}$$

From the definition it follows that there exist a function $\eta(t)$ and a constant vector l' such that $\bar{y}(t)$ satisfies

$$\bar{y}(t+1) = f(t, \bar{y}(t)) + \eta(t) \tag{5.5}$$

and

$$\sum_{i=0}^{N} L_i \bar{y}(t_i) = l'; \tag{5.6}$$

also,

$$\|\eta(t)\| \leqslant r \tag{5.7}$$

and

$$\|l - l'\| \leqslant \varepsilon. \tag{5.8}$$

LEMMA 5.2 [6]. *Let T map a ball $D_\delta = \{w : \|w - y_0\| \leqslant \delta\}$ of a complete normed linear space S into itself. If there is an $k \in (0, 1)$ such that for $u, v \in D_\delta$*

$$\|Tu - Tv\| \leqslant k \, \|u - v\| \tag{5.9}$$

and if

$$\| Ty_0 - y_0 \| \leqslant \delta(1-k) \tag{5.10}$$

then T has a unique fixed point y in D_δ.

THEOREM 5.3. *Let there exist an approximate solution $\bar{y}(t)$ of* (3.1), (2.2). *For this $\bar{y}(t)$ assume that there exist some $n \times n$ matrix $A(t)$ such that the fundamental solution $Y(t)$ of* (2.3) *exists and the matrix G defined in* (2.4) *is nonsingular, also there exists a positive constant δ and a constant $k \in (0, 1)$ such that*

$$D_\delta = \{ y \colon \| y - \bar{y}(t) \| \leqslant \delta \} \subset R^n, \qquad \text{for all } t \in [t_0, t_N] \tag{5.11}$$

$$\| J(t, y(t)) - A(t) \| \leqslant \frac{k}{M_1}, \qquad \text{for any } y \in D_\delta \text{ and } t \in [t_0, t_N], \tag{5.12}$$

$$rM_1 + \varepsilon M_2 \leqslant \delta(1-k), \tag{5.13}$$

where $J(t, y(t))$ is the Jacobian matrix of $f(t, y(t))$ with respect to $y(t)$; M_1 and M_2 are positive constants such that

$$\| H \| \leqslant M_1, \qquad \| Y(t)\, G^{-1} \| \leqslant M_2 \tag{5.14}$$

and $H(t, s)$ is the Green's matrix corresponding to $A(t)$.

Then, problem (3.1), (2.2) *has a unique solution $y(t)$ in D_δ, and this is an isolated solution. Further, it holds that*

$$\| y(t) - \bar{y}(t) \| \leqslant \frac{rM_1 + \varepsilon M_2}{1-k}. \tag{5.15}$$

Proof. The approximate solution $\bar{y}(t)$ can be expressed as

$$\begin{aligned} \bar{y}(t) = Y(t)\, G^{-1} l' + \sum_{s=0}^{t_N - 1} H(t, s)[f(t, \bar{y}(t)) \\ - A(s)\, \bar{y}(s) + \eta(s)]. \end{aligned} \tag{5.16}$$

We shall consider in Lemma 5.2 the space S as the space B and the ball D_δ as defined in (5.11) with $y_0(t) = \bar{y}(t)$, and show that the mapping T defined in (3.4) satisfy (5.9) and (5.10). Let $y_1(t)$ and $y_2(t) \in D_\delta$, then from (3.4), we find

$$\begin{aligned} Ty_1(t) - Ty_2(t) &= \sum_{s=0}^{t_N-1} H(t,s)[f(s,y_1(s)) - A(s)\,y_1(s) \\ &\quad - f(s,y_2(s)) + A(s)\,y_2(s)] \\ &= \sum_{s=0}^{t_N-1} H(t,s)\left[\int_0^1 \{J(s, y_1(s) + \theta(y_2(s) - y_1(s))) \right. \\ &\quad \left. - A(s)\}(y_1(s) - y_2(s))\,d\theta\right], \end{aligned}$$

and hence, from (5.12), (5.14) and the fact that $y_1(t) + \theta X (y_2(t) - y_1(t)) \in D_\delta$, we find

$$\|Ty_1(t) - Ty_2(t)\| \leqslant M_1 \times \frac{k}{M_1} \times \|y_1 - y_2\|,$$

thus, (5.9) is satisfied.

Next, from (5.14), and (5.13) it follows that

$$\begin{aligned} T\bar{y}(t) - \bar{y}(t) &= Ty_0(t) - y_0(t) \\ &= Y(t)\,G^{-1}(l - l') + \sum_{s=0}^{t_N-1} H(t,s)(-\eta(s)), \end{aligned}$$

$$\begin{aligned} \|Ty_0(t) - y_0(t)\| &\leqslant M_2\varepsilon + M_1 r \\ &\leqslant \delta(1-k). \end{aligned}$$

Thus, there exist a unique solution $y(t)$ of (3.1), (2.2) in D_δ. This solution can be constructed from the iterative process:

$$y_{n+1}(t) = Y(t)\,G^{-1}l + \sum_{s=0}^{t_N-1} H(t,s)[f(s,y_n(s)) - A(s)\,y_n(s)], \qquad n = 0, 1, \ldots,$$

with $y_0(t) = \bar{y}(t)$.

To prove, (5.15) we note that

$$\begin{aligned} y(t) - \bar{y}(t) &= Y(t)\,G^{-1}(l - l') + \sum_{s=0}^{t_N-1} H(t,s) \\ &\quad \times [f(s,y(s)) - A(s)\,y(s) - f(s,\bar{y}(s)) \\ &\quad + A(s)\,\bar{y}(s) - \eta(s)] \\ &= Y(t)\,G^{-1}(l - l') + \sum_{s=0}^{t_N-1} H(t,s) \\ &\quad \times \left[\int_0^1 \{J(s, y(s) + \theta(\bar{y}(s) - y(s))) - A(s)\}\right. \\ &\quad \left. \times (y(s) - \bar{y}(s))\,d\theta - \eta(s)\right], \end{aligned}$$

and from this,

$$\| y(t) - \bar{y}(t) \| \leqslant M_2 \varepsilon + M_1 \left[\frac{k}{M_1} \| y(t) - \bar{y}(t) \| + r \right],$$

and hence,

$$\| y(t) - \bar{y}(t) \| \leqslant \frac{M_2 \varepsilon + M_1 r}{1 - k}.$$

Finally, we shall show that $y(t)$ is an isolated solution. Let $X(t)$ be the fundamental matrix solution of $x(t+1) = J(t, y(t)) \times x(t)$, $t \in I_\infty$, such that $X(0) = E$. We define $\bar{G} = \sum_{i=0}^{N} L_i X(t_i)$; if $y(t)$ is not isolated then $\bar{G}$ is singular, consequently there is a nontrivial vector c such that $\bar{G}c = 0$. Put $x(t) = X(t)\, c$, then $x(t)$ is a solution of $x(t+1) = J(t, y(t))\, x(t)$ satisfying $\sum_{i=0}^{N} L_i x(t_i) = 0$. But,

$$x(t) = \sum_{s=0}^{t_N - 1} H(t, s)[J(s, y(s))\, x(s) - A(s)\, x(s)],$$

and hence,

$$\|x(t)\| \leqslant M_1 X \frac{k}{M_1} \times \|x(t)\|,$$

which implies that $x(t) \equiv 0$, on $[t_0, t_N]$. Since $X(t)$ is nonsingular we find $c = 0$ which is a contradiction and hence $y(t)$ is isolated.

6. An Example

For the two point boundary value problem

$$\begin{vmatrix} y_1(t+1) \\ y_2(t+1) \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix} \begin{vmatrix} y_1(t) \\ y_2(t) \end{vmatrix} + \begin{vmatrix} 0 \\ \dfrac{\lambda}{N^2} e^{\alpha y_2(t)} \end{vmatrix}, \qquad 0 \leqslant t \leqslant N-1, \tag{6.1}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} y_1(0) \\ y_2(0) \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2(N) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}, \qquad N \geqslant 2, \tag{6.2}$$

where $|\lambda| \leqslant 1$ and $|\alpha| \leqslant 1$, we consider the approximate solution $\bar{y}(t) \equiv 0$ and find $\varepsilon = 0$, $r = 1/N^2$. For this approximate solution, we take

$$A(t) = \begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix},$$

then the following are easy to compute,

$$Y(t) = \begin{vmatrix} 1-t & t \\ -t & 1+t \end{vmatrix},$$

$$G = \begin{vmatrix} 1 & 0 \\ 1-N & N \end{vmatrix},$$

$$Y(t)\, G^{-1} = \frac{1}{N} \begin{vmatrix} N-t & t \\ N-t-1 & 1+t \end{vmatrix},$$

$$H(t, s) = \frac{1}{N} \begin{vmatrix} (N-t)(2+s) & -(s+1)(N-t) \\ (N-t-1)(2+s) & -(s+1)(N-t-1) \end{vmatrix},$$

$$0 \leqslant s < t \leqslant N-1,$$

$$H(t, s) = -\frac{1}{N} \begin{vmatrix} t(2+s-N) & t(N-s-1) \\ (1+t)(2+s-N) & (1+t)(N-s-1) \end{vmatrix},$$

$$0 \leqslant t \leqslant s \leqslant N-1.$$

Thus, it follows that

$$\| Y(t)\, G^{-1} \| = 1 = M_2,$$

$$\sum_{s=0}^{N-1} |H(t, s)| = \frac{1}{2N} \begin{vmatrix} t(4+N^2-tN) & N(N-t)t \\ (N-t-1)(Nt+N-2)+2+2t & N(n-t-1)(1+t) \end{vmatrix},$$

and

$$\sum_{j=1}^{2} \sum_{s=0}^{N-1} |H_{ij}(t, s)| = \frac{1}{2N} \begin{vmatrix} t(4+2N^2-2tN) \\ 2(N-t-1)(N+Nt-1)+2+2t \end{vmatrix};$$

also,

$$\|H\| = \frac{1}{2N}(N^3 + 3N - 4) \leqslant \frac{5}{8} N^2 = M_1.$$

Thus, from Theorem 5.3 there exists a unique solution of (6.1), (6.2) provided the following two inequalities are satisfied:

$$\|J(t, y(t)) - A(t)\| \leqslant \frac{1}{N^2} e^{\delta} \leqslant \frac{8}{5N^2} k \tag{6.3}$$

and

$$\frac{1}{N^2} \times \frac{5}{8} \times N^2 \leqslant \delta(1-k); \tag{6.4}$$

also $0 < k < 1$.

From inequality (6.3), it follows that $0 < k < 1$ as long as $\delta < \log 1.6 \cong 0.47$ and for this δ, inequality (6.4) is also satisfied. Thus, problem (6.1), (6.2) has a unique solution in $D_\delta = \{y : \|y\| \leqslant 0.47\}$.

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