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# On convergence of continuous half-explicit Runge-Kutta methods for a class of delay differential-algebraic equations

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## Abstract

In this paper, we propose and investigate continuous Runge-Kutta methods for solving a class of nonlinear differential-algebraic equations (DAEs) with constant delay. Real-life processes that involve simultaneously time-delay effect and constraints are usually described by delay DAEs. Solving delay DAEs is more complicated than solving non-delay ones since we should focus on both the time-delay and DAE aspects. Recently, we have revisited linear multistep methods and Runge-Kutta methods for a class of nonlinear DAEs (without delay) and shown the advantages of appropriately modified discretizations. In this work, we extend the use of half-explicit Runge-Kutta methods to a similar class of structured strangeness-free DAEs with constant delay. Approximation of solutions at delayed time is obtained by continuous extensions of discrete solution, i.e., continuous output formulas. Convergence analysis for continuous Runge-Kutta methods is presented. It is shown that order reduction that may happen with DAEs is avoided if we discretize an appropriately reformulated delay DAE (DDAE) instead of the original one. Difficulties arising in the implementation are discussed as well. Finally, numerical experiments are given for illustration.

**Keywords** Delay differential-algebraic equations · Strangeness-free form · Runge-Kutta methods · Convergence · Continuous extension · Superconvergence

**Mathematics Subject Classification (2010)** 65L80 · 65L03 · 65L05 · 65L06 · 65L20

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## 1 Introduction

In many areas of applications such as electrical circuit design, real-time simulation of mechanical systems, chemical engineering, power systems, control and optimal control, etc, differential equations with time-delay effect and constraints often arise in the course of modeling (see [2, 4, 14, 21]). Such systems are rather complicated mixture of delay differential equations, difference equations, and algebraic equations. They are called delay differential-algebraic equations (DDAEs). The theory and numerical analysis of delay (ordinary) differential equations (DDEs) as well as those of DAEs (without delay) have been well established (see [3–6, 11, 13], respectively). For delay DAEs, many problems in the theory and numerical analysis are still open. Though some early studies of DDAEs were done a long time ago (see [7, 8]), even the solvability of general linear delay DAEs has been investigated only very recently in [14, 15, 22]. Very few papers have been devoted to the convergence analysis of numerical methods for delay DAEs, and most of them are restricted to the consideration of delay DAEs in semi-explicit form. Usually implicit numerical schemes are suggested which may be very costly and have complicated implementation (see [9]). In [1], the convergence of backward differentiation formula (BDF) and collocation Runge-Kutta methods for semi-explicit DDAEs of retarded and neutral type with single delay was analyzed. Later, the use of collocation methods was extended to retarded DDAEs of index two with state-dependent delay (see [12]). The convergence of linear multistep and one-leg methods for semi-explicit index 2 DDAEs with variable delay was given in [20]. Difficulties that arise in solving DDAEs were discussed in [2, 4, 21]. Generally speaking, DDAEs are neither DAEs nor DDEs. However, under certain conditions, a DDAE may be reduced to a DDE of retarded or neutral type [1].

Recently, Ha et al. in [14, 15] investigated general linear variable coefficient DDAEs with constant delay

$$E(t)x'(t) = A(t)x(t) + B(t)x(t - \tau) + \gamma(t), \quad (1)$$

where  $E$ ,  $A$ , and  $B$  are sufficiently smooth matrix functions of size  $m \times m$ ,  $m \geq 2$ , and  $\gamma$  is a sufficiently smooth vector function. Under certain assumptions, they proposed an algorithm that reduces (1) to the form

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \end{bmatrix} x'(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \end{bmatrix} x(t) + \begin{bmatrix} \hat{B}_1(t) \\ \hat{B}_2(t) \end{bmatrix} x(t - \tau) + \hat{\gamma}(t), \quad (2)$$

where  $\begin{bmatrix} \hat{E}_1(t) \\ \hat{A}_2(t) \end{bmatrix}$  is pointwise invertible. Then, it was also shown that  $s$ -stage collocation methods are convergent of order at least  $s$  for initial value problems (IVPs) for (2). We note that the  $s$ -stage collocation methods are in fact some certain implicit Runge-Kutta methods (see [11, 13]).

In this work, we consider a class of nonlinear structured DDAEs of the form

$$\begin{aligned} f(t, x(t), x(t - \tau), E(t)x'(t)) &= 0, \\ g(t, x(t), x(t - \tau)) &= 0, \end{aligned} \quad (3)$$

for all  $t \in \mathbb{I} = [0, T]$ ,  $\tau > 0$  is a constant delay. Here, we assume that  $E(t) : \mathbb{I} \rightarrow \mathbb{R}^{m_1 \times m}$  is a sufficiently smooth matrix function and  $f(t, u, v, w) : \mathbb{I} \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m_1} \rightarrow \mathbb{R}^{m_1}$ ,  $g(t, u, v) : \mathbb{I} \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^{m_2}$ ,  $m_1 + m_2 = m$  are sufficiently smooth functions. Given an initial condition

$$x(t) = \phi(t) \quad \text{for } t \in [-\tau, 0], \quad (4)$$

where  $\phi \in C([-\tau, 0], \mathbb{R}^m)$ , we suppose that the IVP (3)–(4) has a unique solution  $x(t)$ . Here,  $x$  is said to be a solution if it is continuous and piecewise continuously differentiable on  $\mathbb{I}$ , if it satisfies the DDAE (3) for  $t \in \mathbb{I}$  pointwise except for a finite number of discontinuity points, and it satisfies the initial condition (4) [14, 15]. In this paper, we assume that partial derivatives of functions  $f$  and  $g$  are bounded and that the Jacobian

$$\begin{bmatrix} f_w E(t) \\ g_u \end{bmatrix} \quad \text{is invertible} \quad (5)$$

in a neighborhood of the reference solution  $x(t)$  for all  $t \in \mathbb{I}$ . In addition, the initial function  $\phi$  is assumed to be consistent, that is  $g(0, \phi(0), \phi(-\tau)) = 0$ . Then, following the strangeness index theory that was developed for DAEs in [13] and recently extended to linear DDAEs in [14, 15], the nonlinear DDAE (3) is said to be strangeness free (in a sufficiently small neighborhood of  $x$ ). Semi-explicit DDAEs of (differentiation) index one that were considered in [1, 2, 21] and linear DDAEs (2) are obviously only special cases of (3).

For solving DAEs in general, implicit methods such as BDF and Radau methods are the most popular [6, 11, 13]. Half-explicit methods which combine implicit and explicit methods can provide an alternative and cheaper approach for semi-explicit DAEs of low index [11]. Recently, half-explicit Runge-Kutta (HERK) methods were revisited for strangeness-free DAEs and their advantages were demonstrated in [16, 17]. The main aim of this paper is to extend the use of the HERK methods to DDAEs (3). Runge-Kutta discretizations applied directly to DAEs as in [16] certainly lead to order reduction (see numerical comparisons in [17]). However, the special structure of (3) can be well exploited. Similar to the approach recently used in [17, 18] and instead of direct discretization, numerical methods are applied to the reformulated form

$$\begin{aligned} f(t, x(t), x(t - \tau), (Ex)'(t) - E'(t)x(t)) &= 0, \\ g(t, x(t), x(t - \tau)) &= 0. \end{aligned} \quad (6)$$

Very recently in [19], linear multistep methods equipped with additional interpolation for solving IVPs for (6) is investigated and their convergence is established. However, there are certain limitations in the practical implementation of linear multistep methods. For example, it is difficult to change stepsize; the method must be restarted at the discontinuity points; and the proper use of interpolation is rather complicated. Here we exploit the advantages of Runge-Kutta methods with continuous extension which have been proven to be efficient methods for delay ODEs [3, 4]. It will be shown that the reformulated form (6) does not lead to the order reduction observed in the form (3). The reason is that, as it will be pointed out in Section 2, in the reformulated form (6) the term  $Ex$  plays essentially the role of a separate differential component, while in the original form (3) differential and algebraic variables are totally mixed. These

half-explicit Runge-Kutta methods provide not only an alternative approach in addition to the existing collocation implicit methods, but also yield cheaper numerical solutions in the case of large-sized and nonstiff problems.

The organization of the paper is as follows. In Section 2, we present some preliminaries including the method of steps, an analysis of DDAEs (3) by using transformation and reduction, and some auxiliary results on Runge-Kutta methods with continuous extension. Construction and convergence analysis of half-explicit Runge-Kutta (HERK) methods combined with continuous extension are given in Section 3. In Section 4, some numerical experiments are given to confirm the theoretical results. We close the paper by some conclusions in Section 5.

## 2 Preliminary

### 2.1 Method of steps

We will exploit the use of the method of steps which is well known for DDEs with constant delay (see [3]). The idea is that the IVP (6), (4) is replaced by a sequence of the IVPs on the time intervals  $[l\tau, (l+1)\tau]$  for a nonnegative integer  $l$  provided that  $x$  is known on the interval  $[(l-1)\tau, l\tau]$ . Namely, we solve a sequence of “local” IVPs for non-delay strangeness-free DAEs of the form

$$\begin{aligned} f(t, x_{l+1}(t), x_l(t-\tau), (Ex_{l+1})'(t) - E'(t)x_{l+1}(t)) &= 0, \\ g(t, x_{l+1}(t), x_l(t-\tau)) &= 0, \end{aligned} \quad t \in [l\tau, (l+1)\tau] \quad (7)$$

together with the initial conditions

$$x_{l+1}(l\tau) = x_l(l\tau), \quad l = 0, 1, \dots \quad (8)$$

For  $l = 0$ , we define  $x_0(t) = \phi(t - \tau)$ ,  $0 \leq t \leq \tau$ . If the IVPs (7), (8) for all  $l = 0, 1, \dots$  are all uniquely solvable, then the original IVP (6), (4) has the unique solution defined by

$$x(t) = x_{l+1}(t) \quad \text{if } t \in [l\tau, (l+1)\tau], \quad l = 0, 1, \dots$$

Clearly, the “global” solution  $x(t)$  is continuous. At the connecting points  $l\tau$ , discontinuity in the first or higher derivatives of  $x$  is typical. For both DDEs and DDAEs with a single constant delay  $\tau$ , the discontinuity happens at points  $l\tau$ ,  $l = 0, 1, \dots$  (see [3, 4, 15, 22]). Throughout this paper, we assume that the IVP (3)–(4) has the unique solution  $x$ , which is continuous on  $[0, T]$  and sufficiently smooth on each sub-interval  $[l\tau, (l+1)\tau]$ .

### 2.2 Reformulation and conditioning

In general, a problem is said to be well conditioned if it has a unique solution and its solution is not too sensitive to small changes in the problem data. Conditioning analysis of IVPs for semi-explicit DDAEs of retarded and neutral type with single delay was considered in [1]. It was shown that the IVP for a semi-explicit DDAE of index 1

is well conditioned if that for the essential underlying delay ODE (EUDODE) associated with the DDAE is well conditioned. For implicit DDAEs like (3) or even more general ones, the difficulty relies in the definition of EUDODEs and the classification of the problem because differential and algebraic variables are not separated as in the case of semi-explicit DDAEs.

Recently in [19], by the same approach as in [17, 18], by rewriting the DDAEs (3) into the new form (6), a classification and a conditioning analyses are given. We briefly repeat the results here. By the assumption (5), there exists a pointwise invertible and sufficiently smooth matrix function  $Q(t) = [Q^{(1)}(t) \ Q^{(2)}(t)]$ , where  $Q^{(1)}(t) : \mathbb{I} \rightarrow \mathbb{R}^{m,m_1}$ ,  $Q^{(2)}(t) : \mathbb{I} \rightarrow \mathbb{R}^{m,m_2}$ , such that  $E(t)Q(t) = [I \ 0]$ ,  $t \in \mathbb{I}$ . Furthermore, there exist sufficiently smooth functions  $\tilde{f}$ ,  $\tilde{g}$  such that by the change of variable  $x = Qy$ , the DDAE (6) leads to the following semi-explicit system

$$\begin{aligned} y_1'(t) &= \tilde{f}(t, y_1(t), y_2(t), y_1(t - \tau), y_2(t - \tau)), \\ 0 &= \tilde{g}(t, y_1(t), y_2(t), y_1(t - \tau), y_2(t - \tau)). \end{aligned} \quad (9)$$

Here  $y = [y_1^T \ y_2^T]^T$  is decomposed appropriately such that  $Ex = EQy = y_1$ . Under the assumption (5), it is also true that  $\frac{\partial \tilde{g}}{\partial y_2}$  is nonsingular. Therefore, the system (9) is a semi-explicit index-1 DDAE of retarded or neutral type [1]. All the discussions on the conditioning of the IVPs for (9) given in [1] remain true. By the implicit function theorem, there exists a sufficiently smooth function  $\bar{g}$  such that the algebraic variable  $y_2$  of (9) is expressed in the form

$$y_2(t) = \bar{g}(t, y_1(t), y_1(t - \tau), y_2(t - \tau)). \quad (10)$$

We consider the following two cases.

- a) If  $y_2(t - \tau)$  does not appear in the second equation of (9), then the (10) becomes

$$y_2(t) = \bar{g}(t, y_1(t), y_1(t - \tau)). \quad (11)$$

Inserting (11) and the corresponding expression for  $y_2(t - \tau)$  into the first equation of (9), we obtain a DODE of retarded type

$$y_1'(t) = \tilde{f}(t, y_1(t), \bar{g}(t, y_1(t), y_1(t - \tau)), y_1(t - \tau), \bar{g}(t - \tau, y_1(t - \tau), y_1(t - 2\tau))), \quad (12)$$

which can be also written by notation as

$$y_1'(t) = \hat{f}(t, y_1(t), y_1(t - \tau), y_1(t - 2\tau)), \quad t \geq 2\tau. \quad (13)$$

The IVP for DDAE (3) is well conditioned if the IVP for DODE (13) is well conditioned and the transformation  $Q$  is well conditioned.

- b) In the general case, we have to propagate the recursion in (10) back from  $t$  to  $t - l\tau$ , where  $-\tau \leq t - l\tau \leq 0$ , i.e.  $t \in [(l - 1)\tau, l\tau]$  is assumed. For the sake of simplicity, we suppose that linearization is applied, the (10) is written in the form

$$y_2(t) = \hat{g}(t, y_1(t), y_1(t - \tau)) + R(t)y_2(t - \tau) \quad (14)$$

where  $R(t) = -(g_u Q^{(2)}(t))^{-1} g_v Q^{(2)}(t - \tau)$ . This gives

$$y_2(t) = \left[ \prod_{j=0}^{l-1} R(t - j\tau) \right] y_2(t - l\tau) + \sum_{i=0}^{l-1} \left[ \prod_{j=0}^{i-1} R(t - j\tau) \right] \hat{g}(t - i\tau, y_1(t - i\tau), y_1(t - i\tau - \tau)). \quad (15)$$

Substituting (15) and the corresponding expression for  $y_2(t - \tau)$  into the first equation of (9), we get a DODE with  $l$  lags

$$y_1'(t) = \mathbf{F}(t, y_1(t), y_1(t - \tau), y_1(t - 2\tau), \dots, y_1(t - l\tau), y_2(t - l\tau)), \\ t \in [(l - 1)\tau, l\tau], \quad (16)$$

where  $\mathbf{F}$  is determined by  $\tilde{f}$ ,  $\hat{g}$ , and  $R$ . The latter equation is actually a neutral DODE since the number of lags is increased by one as  $t$  moves to the next sub-interval. Therefore, we say that the DDAE (16) is of neutral type. The well conditioning of the IVP for DDAEs (9) depends not only on this DODE but also on the factor  $R$ . If  $\sup_{t \geq 0} \|R(t)\| < 1$ ,  $Q$  is well conditioned, and the IVP for DODE (16) is well conditioned, then the IVP for DDAE (9) is well conditioned, too.

The above analysis is demonstrated in the following example.

**Example 1** (See also [19]) We consider a strangeness-free DDAE with constant delay  $\tau > 0$  of the form

$$\begin{bmatrix} 1 & -\omega t \\ 0 & 0 \end{bmatrix} x'(t) = \begin{bmatrix} \lambda & \omega(1 - \lambda t) \\ -1 & (1 + \omega t) \end{bmatrix} x(t) + \begin{bmatrix} 0 & a \\ b & c - b\omega(t - \tau) \end{bmatrix} x(t - \tau) - \begin{bmatrix} ae^{\lambda(t-\tau)} \\ (b + c)e^{\lambda(t-\tau)} \end{bmatrix} \quad (17)$$

on an interval  $[0, T]$  with real parameters  $a, b, c, \omega$ , and  $\lambda$ . The system (17) possesses an analytical solution

$$x = \begin{bmatrix} e^{\lambda t}(1 + \omega t) \\ e^{\lambda t} \end{bmatrix}$$

provided that the initial function is given equal to the exact solution on  $[-\tau, 0]$ .

Let us take  $Q(t) = \begin{bmatrix} 1 & \omega t \\ 0 & 1 \end{bmatrix}$ . Using the change of variables  $x(t) = Q(t)y(t)$  and the decomposition  $y(t) = [y_1(t) \ y_2(t)]^T$ , we get the following semi-explicit DDAE

$$\begin{aligned} y_1'(t) &= \lambda y_1(t) + a y_2(t - \tau) - a e^{\lambda(t-\tau)}, \\ y_2(t) &= -c y_2(t - \tau) + y_1(t) - b y_1(t - \tau) + (b + c) e^{\lambda(t-\tau)}. \end{aligned} \quad (18)$$

If  $c = 0$ , we obtain an algebraic constraint

$$y_2(t) = y_1(t) - b y_1(t - \tau) + b e^{\lambda(t-\tau)},$$

and a DODE of retarded type with two lags

$$y_1'(t) = \lambda y_1(t) + a y_1(t - \tau) - a b y_1(t - 2\tau) + a(b - 1)e^{\lambda(t-\tau)}. \quad (19)$$

If the IVP for DODE (19) is well conditioned and  $\omega T$  is of moderate size, then the IVP for DDAE (17) is well conditioned, too. This property depends on the choice of parameters  $\lambda$ ,  $a$ , and  $b$ .

If  $c \neq 0$ , the second equation of (18) leads to

$$y_2(t) = c^l y_2(t - l\tau) + \mathbf{H}(t, y_1(t), y_1(t - \tau), \dots, y_1(t - l\tau)), \quad (20)$$

where the function  $\mathbf{H}$  is determined from (15) and  $-\tau < t - l\tau \leq 0$ . Then, we obtain a neutral DODE

$$y_1'(t) = \lambda y_1(t) + \bar{\mathbf{H}}(t, y_1(t - \tau), y_1(t - 2\tau), \dots, y_1(t - l\tau)) + a c^{l-1} y_2(t - l\tau). \quad (21)$$

If the IVP for DODE (21) is well conditioned, the time interval is not too large and the constant  $c$  satisfying  $|c| \leq 1$  or  $|c| > 1$ , but of moderate size, then the IVP for (17) is well conditioned.

Obviously, conditioning of the problem affects numerical solutions. For illustration, we present the exact solution (Exact Sol), the numerical solution (Num. Sol), and the actual errors of a well-conditioned problems (17) defined by the parameter sets  $\lambda = -1.5$ ,  $\omega = 10$ ,  $a = 0.5$ ,  $b = 1$ ,  $c = 0.8$ ,  $\tau = 1$ , and  $T = 20$  in Fig. 1.

The results of an ill-conditioned problem (17) with the parameter sets  $\lambda = -1.5$ ,  $\omega = 1$ ,  $a = -2$ ,  $b = 1.5$ ,  $c = 2$ ,  $\tau = 0.2$ , and  $T = 5$  are plotted in Fig. 2.

The essential difference between the numerical solutions of the well-conditioned problem and the ill-conditioned one is clearly seen. For the above illustrations, the numerical solutions are computed by the half-explicit midpoint method with a natural continuous extension, which is described in the next section.

In addition to the conditioning analysis, the reduction to the semi-explicit DDAE (9) together with the obtained underlying DODEs (13) and (16) plays a key role in the convergence analysis of the numerical methods proposed in this paper.

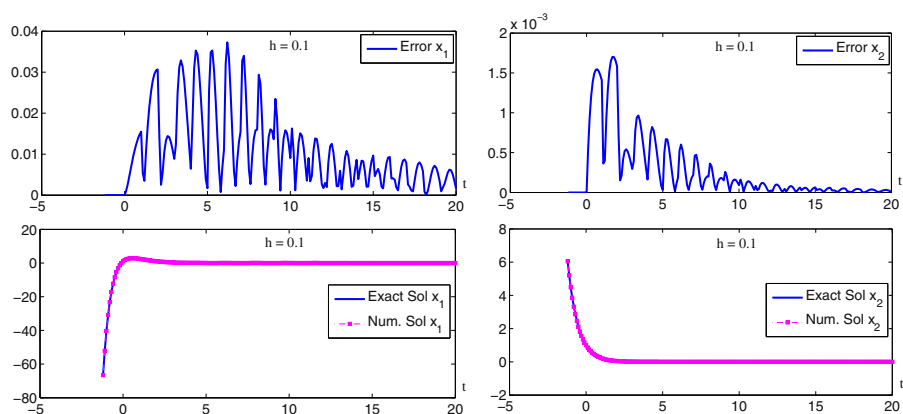
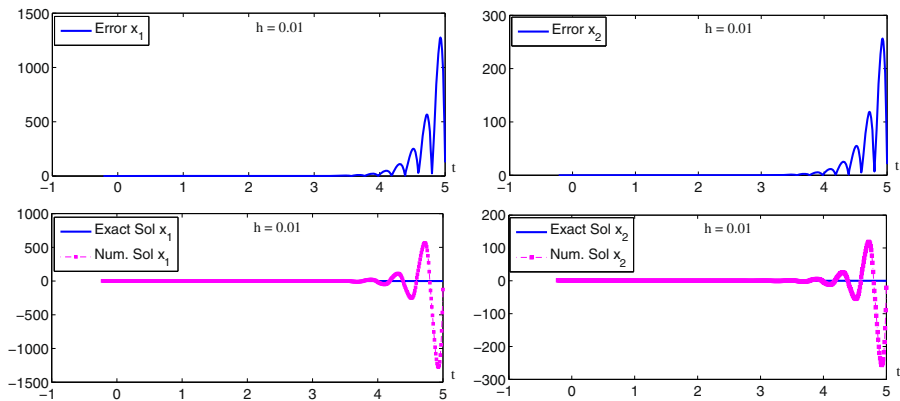


Fig. 1 Exact/numerical solution and errors for a well-conditioned problem (17)



**Fig. 2** Exact/numerical solution and errors for an ill-conditioned problem (17)

### 2.3 Continuous extension of Runge-Kutta methods

In many applications, we want to approximate the solution of an IVP at points which do not belong to the mesh. Continuous extensions of ODE methods have been studied since 1980s (see [10, Chap. II.6]). One-step (Runge-Kutta) methods with continuous extension have great advantage in solving delay ODEs. In the following, we review preliminary results on RK methods with continuous extension given in [3].

For the sake of simplicity, we consider only interpolations of the first class, i.e., the interpolations are constructed without using extra stages at each step. Consider an  $s$ -stage RK method whose coefficients are given by  $A = [a_{ij}]_{s \times s}$ ,  $b = [b_1, b_2, \dots, b_s]^T$ , and  $c = [c_1, c_2, \dots, c_s]^T$ . Given a mesh  $\pi = \{t_0 < t_1 < t_2 < \dots < t_N = T\}$ , a continuous RK method applied to the IVP for ODEs

$$\begin{aligned} y'(t) &= \chi(t, y), \quad t_0 \leq t \leq T, \\ y(t_0) &= y_0, \end{aligned} \quad (22)$$

has the standard form

$$Y_{n,i} = y_n + h_n \sum_{j=1}^s a_{ij} \chi(T_{n,j}, Y_{n,j}) \quad i = 1, 2, \dots, s, \quad (23a)$$

$$y_{n+1} = y_n + h_n \sum_{i=1}^s b_i \chi(T_{n,i}, Y_{n,i}), \quad (23b)$$

where  $h_n = t_{n+1} - t_n$ ,  $T_{n,i} = t_n + c_i h_n$ ,  $y_n \approx y(t_n)$ , and  $Y_{n,i} \approx y(T_{n,i})$ . A continuous extension  $\eta(t)$  of the RK method (23a) is defined by a one-step formula, i.e., a continuous quadrature rule of the form

$$\eta(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^s b_i(\theta) \chi(T_{n,i}, Y_{n,i}), \quad 0 < \theta < 1, \quad (24)$$

or in the  $K$  notation

$$\eta(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^s b_i(\theta) K_{n,i}, \quad 0 < \theta < 1, \quad (25)$$

where  $K_{n,i} = \chi(T_{n,i}, Y_{n,i}) \approx y'(T_{n,i})$  and the coefficients  $b_i(\theta)$  are polynomials of suitable degree less or equal to  $s$  satisfying  $b_i(0) = 0$  and  $b_i(1) = b_i$  at least. We will denote by  $(A, b(\theta))$  the continuous RK (CRK) method (23a), (24) and say that  $(A, b)$  is the underlying discrete RK method.

**Definition 1** We say that the RK method (23a) is consistent of (discrete) order  $k_d$  if  $k_d \geq 1$  is the largest integer such that, for all sufficiently smooth right-hand-side function  $\chi(t, y)$  and for all mesh points, we have that

$$\|z_{n+1}(t_{n+1}) - y_{n+1}\| = \mathcal{O}\left(h_n^{k_d+1}\right)$$

holds uniformly with respect to  $y_n^*$  in any bounded subset of  $\mathbb{R}^m$  and to  $n = 0, 1, \dots, N-1$ , where  $z_{n+1}(t)$  is the unique solution to the local problem

$$\begin{aligned} z'_{n+1}(t) &= \chi(t, z_{n+1}(t)), \quad t_n \leq t \leq t_{n+1}, \\ z_{n+1}(t_n) &= y_n^*. \end{aligned} \quad (26)$$

We say that the interpolant (24) is consistent of uniform order  $k_u$  if  $k_u \geq 1$  is the largest integer such that, for all sufficiently smooth right-hand-side function  $\chi(t, y)$  and for all mesh points, we have that

$$\max_{t_n \leq t \leq t_{n+1}} \|z_{n+1}(t) - \eta(t)\| = \mathcal{O}\left(h_n^{k_u+1}\right).$$

When applying a CRK method to delay ODEs, the convergence order of the numerical solution (even at the meshpoints) is usually reduced to the minimum of the discrete order and the uniform order. A special class of continuous extensions called natural continuous extension (NCE) allows the superconvergence as they are applied to DDEs on a particular mesh. For constant-delay problems, the simplest particular mesh is a uniform mesh  $\pi^*$  with a constant stepsize  $h = \frac{\tau}{\nu}$  for some integer  $\nu \geq 1$ . For variable-delay problems, a so-called constrained mesh can be used (see [3, Definition 6.3.2]).

**Definition 2** We say that the interpolant  $\eta(t)$  defined by (24) is a natural continuous extension (NCE) of the RK method (23a) of order  $k_d$  if the polynomials  $b_i(\theta), i = 1, 2, \dots, s$  are such that  $\eta(t)$  satisfies the additional asymptotic orthogonality condition

$$\left\| \int_{t_n}^{t_{n+1}} G(t) [z'_{n+1}(t) - \eta'(t)] dt \right\| = \mathcal{O}\left(h_n^{k_d+1}\right) \quad (27)$$

for every sufficiently smooth matrix-valued function  $G$ , uniformly with respect to  $n = 0, 1, \dots, N-1$ , where  $z_{n+1}(t)$  is the solution of the local problem (26).

Various Runge-Kutta methods with (natural) continuous extension are presented in [3, pages 125–127]. Here, we introduce three popular ERK methods with NCE which we will use later for numerical experiments.

**Example 2 1.** Explicit midpoint method with NCE (MID-NCE2): The explicit midpoint method given by Butcher tableau

$$\begin{array}{c|cc} 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

has discrete order  $k_d = 2$ . An NCE defined by  $b_1(\theta) = -\theta^2 + \theta$  and  $b_2(\theta) = \theta^2$  has uniform order  $k_u = 2$ .

2. Explicit four-stage Runge-Kutta with NCE of order 2 (ERK4-NCE2): The classical ERK method given by the Butcher tableau

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

has discrete order  $k_d = 4$ . An NCE defined by  $b_1(\theta) = \left(-\frac{1}{2}\theta + \frac{2}{3}\right)\theta$ ,  $b_2(\theta) = \frac{1}{3}\theta = b_3(\theta)$ , and  $b_4(\theta) = \left(\frac{1}{2}\theta - \frac{1}{3}\right)\theta$  has uniform order  $k_u = 2$ .

3. Explicit four-stage Runge-Kutta with NCE of order 3 (ERK4-NCE3): For the same classical ERK method, an NCE defined by  $b_1(\theta) = \left(\frac{2}{3}\theta^2 - \frac{3}{2}\theta + 1\right)\theta$ ,  $b_2(\theta) = \left(-\frac{2}{3}\theta + 1\right)\theta^2 = b_3(\theta)$ , and  $b_4(\theta) = \left(\frac{2}{3}\theta - \frac{1}{2}\right)\theta^2$  has uniform order  $k_u = 3$ .

In the following, we recall some convergence results of the CRK methods applied to delay differential equations. Consider the IVP for a DDE with constant delay

$$\begin{aligned} y'(t) &= \varphi(t, y(t), y(t - \tau)), \quad t_0 \leq t \leq T, \\ y(t) &= \phi(t), \quad t \in [t_0 - \tau, t_0], \end{aligned} \quad (28)$$

where the right-hand side  $\varphi(t, y, x)$  and the initial function  $\phi(t)$  are sufficiently smooth such that the unique solution  $y$  exists and it is piecewise  $p$ -times continuously differentiable. We consider a mesh  $\pi = \{t_0, t_1, \dots, t_N = T\}$  that includes all the possible discontinuity points  $\xi_l = l\tau$ ,  $l = 1, 2, \dots$ . Applying a CRK method with the coefficients  $(A, b(\theta))$  to the DDE (28), we write

$$Y_{n,i} = y_n + h_n \sum_{j=1}^s a_{ij} \varphi(T_{n,j}, Y_{n,j}, \eta(T_{n,j} - \tau)), \quad i = 1, 2, \dots, s, \quad (29a)$$

$$\eta(t_n + \theta h_n) = y_n + h_n \sum_{i=1}^s b_i(\theta) \varphi(T_{n,i}, Y_{n,i}, \eta(T_{n,i} - \tau)), \quad 0 \leq \theta \leq 1 \quad (29b)$$

where  $h_n = t_{n+1} - t_n$ ,  $T_{n,i} = t_n + c_i h_n$ .

The following convergence results of the CRK methods (29a) are given in [3].

**Theorem 1** [3, Theorem 6.3.1] *If the underlying CRK method  $(A, b(\theta))$  has discrete order  $k_d$  and uniform order  $k_u$ , then the CRK method (29a) is convergent and it has discrete global order and uniform global order  $q = \min\{p, k_d, k_u + 1\}$ , i.e.,*

$$\max_{0 \leq n \leq N} \|y(t_n) - y_n\| = \mathcal{O}(h^q), \quad \max_{t_0 \leq t \leq T} \|y(t) - \eta(t)\| = \mathcal{O}(h^q),$$

where  $h = \max_{1 \leq n \leq N} h_n$ .

**Theorem 2** [3, Theorem 6.3.3] *Consider the CRK method (29a) with a uniform mesh  $\pi^*$  with stepsize  $h = \frac{\tau}{\nu}$ , for some positive integer  $\nu$  (or a constrained mesh). If the underlying CRK method  $(A, b(\theta))$  has discrete order  $k_d$  and the interpolant  $\eta(t)$  is an NCE of order  $k_u$ , then superconvergence happens at the meshpoints, i.e., the continuous numerical solution  $\eta(t)$  satisfies*

$$\max_{t_0 \leq t \leq T} \|y(t) - \eta(t)\| = \mathcal{O}(h^q), \quad \max_{0 \leq n \leq N} \|y(t_n) - \eta(t_n)\| = \mathcal{O}(h^{q'}),$$

where  $q = \min\{p, k_d, k_u + 1\}$ ,  $q' = \min\{p, k_d\}$ .

For the neutral delay case, similar convergence results hold true (see [3, Theorem 6.3.5]). In order to extend the convergence of CRK methods to DDAEs, we need the following auxiliary result. In general, approximation by continuous extension and variable transformation do not commute. However, the NCE property of a continuous extension formula is invariant under equivalence transformations.

**Lemma 1** *Let an interpolant  $\eta(t)$  of uniform order  $k_u$  be an NCE of the RK method (23a) which has discrete order  $k_d$ . Given an invertible matrix function  $Q(t)$ , the interpolant  $\gamma(t) = Q(t)\eta(t)$  is also an NCE of the RK method (23a) of uniform order  $k_u$  for the correspondingly transformed differential equation.*

*Proof* Consider the local problem (26), we introduce the change of variable  $z_{n+1}(t) = P(t)x_{n+1}(t)$  and  $x_n^* = Q(t_n)y_n^*$ , where  $P(t) = Q^{-1}(t)$ . The problem (26) becomes

$$\begin{aligned} P(t)x'_{n+1}(t) &= \chi(t, P(t)x_{n+1}(t)) - P'(t)x_{n+1}(t), \quad t_n \leq t \leq t_{n+1}, \\ x_{n+1}(t_n) &= x_n^*. \end{aligned} \quad (30)$$

It is equivalent to

$$\begin{aligned} x'_{n+1}(t) &= \tilde{\chi}(t, x_{n+1}(t)), \quad t_n \leq t \leq t_{n+1}, \\ x_{n+1}(t_n) &= x_n^*, \end{aligned} \quad (31)$$

where  $\tilde{\chi}(t, x_{n+1}(t)) = Q(t)\chi(t, P(t)x_{n+1}(t)) - Q(t)P'(t)x_{n+1}(t)$ . From the condition (27), by using partial integration, it is easy to see that

$$\left\| \int_{t_n}^{t_{n+1}} H(t)[z_{n+1}(t) - \eta(t)]dt \right\| = \mathcal{O}(h_n^{k_d+1}) \quad (32)$$

for any sufficiently smooth matrix-valued function  $H$ . We also have

$$\begin{aligned} \left\| \int_{t_n}^{t_{n+1}} G(t)[x'_{n+1}(t) - \gamma'(t)]dt \right\| &\leq \left\| \int_{t_n}^{t_{n+1}} G(t)Q(t)[z'_{n+1}(t) - \eta'(t)]dt \right\| \\ &\quad + \left\| \int_{t_n}^{t_{n+1}} G(t)Q'(t)[z_{n+1}(t) - \eta(t)]dt \right\|. \end{aligned} \quad (33)$$

where  $x_{n+1}$  is the solution of the local problem (31). Combining (27), (32) with (33), we obtain

$$\left\| \int_{t_n}^{t_{n+1}} G(t)[x'_{n+1}(t) - \gamma'(t)]dt \right\| = \mathcal{O}(h_n^{k_d+1}) \quad (34)$$

for every sufficiently smooth matrix-valued function  $G$ . Furthermore, we have

$$\max_{t_n \leq t \leq t_{n+1}} \|x_{n+1}(t) - \gamma(t)\| = \max_{t_n \leq t \leq t_{n+1}} \|Q(t)z_{n+1}(t) - Q(t)\eta(t)\| \quad (35)$$

$$\begin{aligned} &\leq \max_{t_n \leq t \leq t_{n+1}} \|Q(t)\| \max_{t_n \leq t \leq t_{n+1}} \|z_{n+1}(t) - \eta(t)\| \\ &= \mathcal{O}(h_n^{k_u+1}). \end{aligned} \quad (36)$$

Therefore, we conclude that  $\gamma(t)$  is an NCE of the RK method of uniform order  $k_u$  for the problem (31).  $\square$

### 3 Half-explicit Runge-Kutta methods with continuous extension

Let us consider an  $s$ -stage explicit Runge-Kutta (ERK) method together with a continuous extension  $\eta(t)$ . We suppose that the IVP (3)–(4) is well conditioned and the initial function (4) is consistent and sufficiently smooth such that its solution  $x(t)$  is piecewise  $p$ -times continuously differentiable and the  $(p + 1)$ th derivative exists and bounded except for a finite number of points. We choose a mesh  $\pi = \{0 = t_0 < t_1 < t_2 < \dots < t_N = T\}$  that includes all the possible discontinuity points  $l\tau, l = 0, 1, \dots$ . On an interval  $[t_n, t_{n+1}]$  satisfying  $l\tau \leq t_n < t_{n+1} \leq (l+1)\tau$  for a nonnegative integer  $l$ , we assume that the approximations  $x_i \approx x(t_i)$  with  $i \leq n$  are given. The retarded values  $x(T_{n,i} - \tau)$ ,  $x(t_{n+1} - \tau)$  of  $x$  at the stage and mesh points are approximated by  $\eta(T_{n,i} - \tau)$  and  $\eta(t_{n+1} - \tau)$ , respectively. Let us denote  $h_n = t_{n+1} - t_n$ ,  $T_{n,i} = t_n + c_i h$ ,  $X_{n,i} \approx x(T_{n,i})$ , and  $W_{n,i} \approx (Ex)'(T_{n,i})$ . By extending the discretization scheme by HERK methods for non-delay DAEs in [11, 16,

[17], the  $s$ -stage HERK scheme with the continuous extension  $\eta(t)$  (CHERK method) applied to the reformulated DDAEs (6) is given as follows

$$X_{n,1} = x_n, \quad (36a)$$

$$E_{n,i} X_{n,i} = E(t_n)x_n + h_n \sum_{j=1}^{i-1} a_{ij} W_{n,j}, \quad i = 2, 3, \dots, s, \quad (36b)$$

$$0 = f(T_{n,i-1}, X_{n,i-1}, \eta(T_{n,i-1}-\tau), W_{n,i-1} - E'_{n,i-1} X_{n,i-1}), \quad (36c)$$

$$0 = g(T_{n,i}, X_{n,i}, \eta(T_{n,i}-\tau)), \quad (36d)$$

$$E(t_{n+1})x_{n+1} = E(t_n)x_n + h_n \sum_{i=1}^s b_i W_{n,i}, \quad (36e)$$

$$0 = f(T_{n,s}, X_{n,s}, \eta(T_{n,s}-\tau), W_{n,s} - E'_{n,s} X_{n,s}), \quad (36f)$$

$$0 = g(t_{n+1}, x_{n+1}, \eta(t_{n+1}-\tau)), \quad (36g)$$

where  $E_{n,i} = E(T_{n,i})$ ,  $E'_{n,i} = E'(T_{n,i})$ , ( $i = 1, 2, \dots, s$ ). The continuous extension  $\eta(t)$  is defined by the system

$$\begin{aligned} E(t_n + \theta h_n)\eta(t_n + \theta h_n) &= E(t_n)x_n + h_n \sum_{i=1}^s b_i(\theta) W_{n,i}, \\ 0 &= g(t_n + \theta h_n, \eta(t_n + \theta h_n), \eta(t_n + \theta h_n - \tau)), \end{aligned} \quad (37)$$

for  $0 < \theta < 1$ .

For the implementation, we calculate step by step on consecutive intervals  $[l\tau, (l+1)\tau]$ ,  $l \geq 0$ . On the first sub-interval  $[0, \tau]$ , we consider an interval  $[t_n, t_{n+1}]$ , suppose that the approximate  $x_n$  is given and the retarded values are taken by the initial function  $\eta(T_{n,i}-\tau) = \phi(T_{n,i}-\tau)$ , ( $i = 1, 2, \dots, s$ ), and  $\eta(t_{n+1}-\tau) = \phi(t_{n+1}-\tau)$ . Then, the implementation of the method is done analogously to that of the HERK method in the non-delay case (see [17] for more details). We obtain approximations  $X_{n,i}$ ,  $W_{n,i}$ , ( $i = 1, 2, \dots, s$ ) and  $x_{n+1}$  by solving nonlinear systems (36b–36d) and (36e–36g), respectively.

If we assume in addition that  $a_{i,i-1} \neq 0$  for  $i = 2, \dots, s$  and  $b_s \neq 0$ , then the size of nonlinear systems (36b–36d) and (36e–36g) is reduced by eliminating  $W_{n,i}$ ,  $i = 1, 2, \dots, s$ , as follows. The (36b) and (36e) yield

$$W_{n,1} = \frac{E_{n,2}X_{n,2} - E(t_n)X_{n,1}}{h_n a_{21}},$$

$$W_{n,i-1} = \left( \frac{E_{n,i}X_{n,i} - E(t_n)X_{n,1}}{h_n} - \sum_{j=1}^{i-2} a_{i,j} W_{n,j} \right) \frac{1}{a_{i,i-1}}, \quad i = 3, \dots, s,$$

and

$$W_{n,s} = \left( \frac{E(t_{n+1})x_{n+1} - E(t_n)X_{n,1}}{h_n} - \sum_{i=1}^{s-1} b_i W_{n,i} \right) \frac{1}{b_s}.$$

At the  $i$ th stage ( $i = 2, \dots, s$ ), the approximation  $X_{n,i}$  can be determined from a nonlinear system  $\mathcal{F}_{n,i}(X_{n,i}) = 0$  defined by

$$\begin{aligned} 0 &= h_n f \left( T_{n,i-1}, X_{n,i-1}, \eta(T_{n,i-1} - \tau), \left( \frac{E_{n,i} X_{n,i} - E(t_n) X_{n,1}}{h_n} - \sum_{j=1}^{i-2} a_{i,j} W_{n,j} \right) \frac{1}{a_{i,i-1}} \right. \\ &\quad \left. - E'_{n,i-1} X_{n,i-1} \right), \\ 0 &= g(T_{n,i}, X_{n,i}, \eta(T_{n,i} - \tau)). \end{aligned} \quad (38)$$

Here, we suppose that the approximations  $X_{n,j}$ ,  $j = 1, \dots, i-1$ , and  $W_{n,k}$ ,  $k = 1, \dots, i-2$ , are given sufficiently close to the exact values. The Jacobian matrix of  $\mathcal{F}_{n,i}$  with respect to  $X_{n,i}$  is

$$\frac{\partial \mathcal{F}_{n,i}}{\partial X_{n,i}} = \begin{bmatrix} \frac{1}{a_{i,i-1}} f_v(T_{n,i-1}, X_{n,i-1}, \eta(T_{n,i-1} - \tau), W_{n,i-1} - E'_{n,i-1} X_{n,i-1}) E_{n,i} \\ g_u(T_{n,i}, X_{n,i}, \eta(T_{n,i} - \tau)) \end{bmatrix}. \quad (39)$$

For sufficiently small  $h_n$ , the  $m$ -dimensional nonlinear system (38) has a locally unique solution  $X_{n,i}^*$ , which can be approximated by Newton's iterative method. Then, the approximation  $W_{n,i}$  immediately follows.

Finally, a unique solution  $x_{n+1}$  at time step  $t = t_{n+1}$  is determined by the system  $\mathcal{G}_n(x_{n+1}) = 0$  which is defined by

$$\begin{aligned} 0 &= h_n f \left( T_{n,s}, X_{n,s}, \eta(T_{n,s} - \tau), \left( \frac{E(t_{n+1}) x_{n+1} - E(t_n) x_n}{h_n} - \sum_{i=1}^{s-1} b_i W_{n,i} \right) \frac{1}{b_s} \right. \\ &\quad \left. - E'_{n,s} X_{n,s} \right), \\ 0 &= g(t_{n+1}, x_{n+1}, \eta(t_{n+1} - \tau)), \end{aligned} \quad (40)$$

where  $X_{n,1}, X_{n,2}, \dots, X_{n,s}, K_1, K_2, \dots, K_{s-1}$  have already been obtained. Here the Jacobian

$$\frac{\partial \mathcal{G}_n}{\partial x_{n+1}} = \begin{bmatrix} \frac{1}{b_s} f_v(T_{n,s}, X_{n,s}, \eta(T_{n,s} - \tau), W_{n,s} - E'_{n,s} U_{n,s}) E(t_{n+1}) \\ g_u(t_{n+1}, x_{n+1}, \eta(t_{n+1} - \tau)) \end{bmatrix} \quad (41)$$

is boundedly invertible for sufficiently small  $h_n$ . The locally unique solution  $x_{n+1}^*$  can be approximated by Newton's iterative method, as well.

**Remark 1** We note that the first equations of (38) and (40) are scaled by  $h_n$ . If we do not apply the scaling, then the first block rows of the Jacobians in (39) and (41) are multiplied by  $1/h_n$ , which could increase the condition numbers of the Jacobians, in particular when the stepsize  $h_n$  is very small.

This procedure is repeated in each sub-interval  $[l\tau, (l+1)\tau]$ . Consider the sub-interval  $[l\tau, (l+1)\tau]$ ,  $l \geq 1$ , and a meshpoint  $t = t_n$  such that  $[t_n, t_{n+1}]$  is included in  $[l\tau, (l+1)\tau]$ . Suppose that the approximate solution  $x_n$  is given. By solving (36a–

36g), we compute the numerical solution  $x_{n+1}$ . The continuous extension  $\eta(t)$  in  $[t_n, t_{n+1}]$  is determined as the solution of the system (37) which is written as

$$\mathcal{G}(t_n, \eta(t_n + \theta h_n), \eta(t_n + \theta h_n - \tau), W_{n,1}, \dots, W_{n,s}, h_n, \theta) = 0.$$

Here, the Jacobian matrix of  $\mathcal{G}$  with respect to  $\eta(t_n + \theta h_n)$  is

$$\frac{\partial \mathcal{G}}{\partial \eta(t_n + \theta h_n)} = \begin{bmatrix} E(t_n + \theta h_n) \\ g_u(t_n + \theta h_n, \eta(t_n + \theta h_n), \eta(t_n + \theta h_n - \tau)) \end{bmatrix}$$

which is nonsingular. Thus,  $\eta(t_n + \theta h_n)$  for  $0 < \theta < 1$  is uniquely determined. Therefore, the numerical solution and the continuous extension  $\eta(t)$  are obtained in the  $l$ th sub-interval  $[l\tau, (l+1)\tau]$ . Continuing in this way, we obtain the numerical solution of the DDAEs (3) on the whole interval  $\mathbb{I}$ .

The convergence of the CHERK scheme (36a) is obtained as follows.

**Theorem 3** *We consider a well-conditioned IVP (3)–(4) and suppose that the solution  $x(t)$  is piecewise  $(p+1)$ -times differentiable with bounded derivatives. Given an  $s$ -stage explicit Runge-Kutta method with a continuous extension  $\eta(t)$  which has discrete order  $k_d$  and uniform order  $k_u$ , the  $s$ -stage CHERK scheme (36a), (37) has discrete global order and uniform global order  $q = \min(p, k_d, k_u + 1)$ , i.e.,*

$$\max_{0 \leq n \leq N} \|x(t_n) - x_n\| = \mathcal{O}(h^q), \quad \max_{0 \leq t \leq T} \|x(t) - \eta(t)\| = \mathcal{O}(h^q), \quad (42)$$

where  $h = \max_{0 \leq n \leq N-1} h_n$ .

*Proof* The framework of the proof is as follows. First, we transform the discretization scheme into a form that can be in essence reduced to the semi-explicit form. Then, we solve the discretized algebraic equation for the algebraic component and insert it into the discretized differential part to obtain a CRK scheme for the underlying DODE. Then, using the well-known convergence result for the CRK methods for delay ODEs, we obtain an error estimate for the differential component. Afterward, an error estimate for the algebraic component is obtained.

We now consider the CHERK methods (36a) and (37). Using the change of variables  $\eta(t) = Q(t)\gamma(t) = Q(t)[\gamma_1^T(t) \ \gamma_2^T(t)]^T$  and

$$\begin{aligned} x_n &= Q(t_n) \begin{bmatrix} y_{1,n}^T & y_{2,n}^T \end{bmatrix}^T = Q^{(1)}(t_n) y_{1,n} + Q^{(2)}(t_n) y_{2,n}, \quad n = 0, 1, \dots, N, \\ X_{n,i} &= Q_{n,i} \begin{bmatrix} U_{n,i}^T & V_{n,i}^T \end{bmatrix}^T = Q_{n,i}^{(1)} U_{n,i} + Q_{n,i}^{(2)} V_{n,i}, \quad i = 1, 2, \dots, s, \end{aligned}$$

where  $Q_{n,i} = [Q_{n,i}^{(1)} \ Q_{n,i}^{(2)}] = Q(T_{n,i})$ . Here  $Q(t)$  is the pointwise invertible matrix function that is defined in Section 2. The system (37) is rewritten as

$$\begin{aligned} \gamma_1(t_n + \theta h_n) &= y_{1,n} + h \sum_{i=1}^s b_i(\theta) W_{n,i}, \quad g(t_n + \theta h_n, Q(t_n + \theta h_n) \gamma(t_n + \theta h_n), \\ Q(t_n + \theta h_n - \tau) \gamma(t_n + \theta h_n - \tau)) &= 0. \end{aligned} \quad (43)$$

By the definitions of the functions  $\tilde{g}$  and  $\bar{g}$  in (9) and (10), the second equation of (42) can be rewritten as follows:

$$\gamma_2(t_n + \theta h_n) = \bar{g}(t_n + \theta h_n, \gamma_1(t_n + \theta h_n), \gamma_1(t_n + \theta h_n - \tau), \gamma_2(t_n + \theta h_n - \tau)). \quad (44)$$

Moreover, the system (36a) becomes

$$U_{n,1} = y_{1,n}, \quad V_{n,1} = y_{2,n}, \quad (45a)$$

$$U_{n,i} = y_{1,n} + h \sum_{j=1}^{i-1} a_{ij} W_{n,j}, \quad i = 2, 3, \dots, s, \quad (45b)$$

$$0 = f \left( T_{n,i-1}, Q_{n,i-1} \begin{bmatrix} U_{n,i-1}^T & V_{n,i-1}^T \end{bmatrix}^T, Q(T_{n,i-1} - \tau) \gamma(T_{n,i-1} - \tau), W_{n,i-1} \right. \\ \left. - E'_{n,i-1} Q_{n,i-1} \begin{bmatrix} U_{n,i-1}^T & V_{n,i-1}^T \end{bmatrix}^T \right), \quad (45c)$$

$$0 = g \left( T_{n,i}, Q_{n,i} \begin{bmatrix} U_{n,i}^T & V_{n,i}^T \end{bmatrix}^T, Q(T_{n,i} - \tau) \gamma(T_{n,i} - \tau) \right), \quad (45d)$$

$$y_{1,n+1} = y_{1,n} + h \sum_{i=1}^s b_i W_{n,i}, \quad (45e)$$

$$0 = f \left( T_{n,s}, Q_{n,s} \begin{bmatrix} U_{n,s}^T & V_{n,s}^T \end{bmatrix}^T, Q(T_{n,s} - \tau) \gamma(T_{n,s} - \tau), W_{n,s} \right. \\ \left. - E'_{n,s} Q_{n,s} \begin{bmatrix} U_{n,s}^T & V_{n,s}^T \end{bmatrix}^T \right), \quad (45f)$$

$$-E'_{n,s} Q_{n,s} \begin{bmatrix} U_{n,s}^T & V_{n,s}^T \end{bmatrix}^T \right), \quad (45g)$$

$$0 = g \left( t_{n+1}, Q(t_{n+1}) [y_{1,n+1}^T \ y_{2,n+1}^T]^T, Q(t_{n+1} - \tau) \gamma(t_{n+1} - \tau) \right). \quad (45h)$$

In the same spirit, (45d) and (45h) are rewritten as

$$V_{n,i} = \bar{g}(T_{n,i}, U_{n,i}, \gamma_1(T_{n,i} - \tau), \gamma_2(T_{n,i} - \tau)), \quad i = 2, 3, \dots, s, \\ y_{2,n+1} = \bar{g}(t_{n+1}, y_{1,n+1}, \gamma_1(t_{n+1} - \tau), \gamma_2(t_{n+1} - \tau)). \quad (46)$$

Since the matrix  $f_w$  is nonsingular, by the definition of the function  $\tilde{f}$  in (9), (45c) and (45g) become

$$W_{n,i} = \tilde{f}(T_{n,i}, U_{n,i}, V_{n,i}, \gamma_1(T_{n,i} - \tau), \gamma_2(T_{n,i} - \tau)), \quad i = 1, 2, \dots, s. \quad (47)$$

Next, we consider two cases as classified in Section 2.

- a) If the term  $y_2(t - \tau)$  does not appear in the second equation of (9), then (44) and (46) derive

$$\gamma_2(t_n + \theta h_n) = \bar{g}(t_n + \theta h_n, \gamma_1(t_n + \theta h_n), \gamma_1(t_n + \theta h_n - \tau)), \\ V_{n,i} = \bar{g}(T_{n,i}, U_{n,i}, \gamma_1(T_{n,i} - \tau)), \quad i = 2, 3, \dots, s, \\ y_{2,n+1} = \bar{g}(t_{n+1}, y_{1,n+1}, \gamma_1(t_{n+1} - \tau)). \quad (48)$$

From the first equation of (48), we deduce that

$$\gamma_2(T_{n,i} - \tau) = \bar{g}(T_{n,i} - \tau, \gamma_1(T_{n,i} - \tau), \gamma_1(T_{n,i} - 2\tau)). \quad (49)$$

Inserting (48) and (49) into (47) yields

$$W_{n,i} = \tilde{f}(T_{n,i}, U_{n,i}, \bar{g}(T_{n,i}, U_{n,i}, \gamma_1(T_{n,i} - \tau)), \gamma_1(T_{n,i} - \tau), \bar{g}(T_{n,i} - \tau, \gamma_1(T_{n,i} - \tau), \gamma_1(T_{n,i} - 2\tau))), \quad (50)$$

which is rewritten as

$$W_{n,i} = \hat{f}(T_{n,i}, U_{n,i}, \gamma_1(T_{n,i} - \tau), \gamma_1(T_{n,i} - 2\tau)), \quad i = 1, 2, \dots, s. \quad (51)$$

From (45a), (45b), (45e), (51), and the first equation of (43), we obtain

$$U_{n,1} = y_{1,n}, \quad (52a)$$

$$U_{n,i} = y_{1,n} + h_n \sum_{j=1}^{i-1} a_{ij} W_{n,j}, \quad i = 2, 3, \dots, s, \quad (52b)$$

$$W_{n,i} = \hat{f}(T_{n,i}, U_{n,i}, \gamma_1(T_{n,i} - \tau), \gamma_1(T_{n,i} - 2\tau)) \quad (52c)$$

$$y_{1,n+1} = y_{1,n} + h_n \sum_{i=1}^s b_i W_{n,i}, \quad (52d)$$

and

$$\gamma_1(t_n + \theta h_n) = y_{1,n} + h_n \sum_{i=1}^s b_i(\theta) W_{n,i}. \quad (53)$$

This is exactly the discretization scheme by the corresponding continuous ERK method applied to the underlying DODE (13). Since problem (13) is well conditioned and by Theorem 1, it follows that

$$\max_{0 \leq n \leq N} \|y_1(t_n) - y_{1,n}\| = \mathcal{O}(h^q), \quad \max_{0 \leq t \leq T} \|y_1(t) - \gamma_1(t)\| = \mathcal{O}(h^q), \quad (54)$$

where  $q = \min(p, k_d, k_u + 1)$ .

Moreover, from (11) we have

$$y_2(t_n) = \bar{g}(t_n, y_1(t_n), y_1(t_n - \tau)). \quad (55)$$

Combining (54) and (55) with the third equation of (48), we obtain that

$$\begin{aligned} \max_{0 \leq n \leq N} \|y_2(t_n) - y_{2,n}\| &= \max_{0 \leq n \leq N} \|\bar{g}(t_n, y_1(t_n), y_1(t_n - \tau)) - \bar{g}(t_n, y_{1,n}, \gamma_1(t_n - \tau))\| \\ &\leq \mathcal{C}_1 \max_{0 \leq n \leq N} \|y_1(t_n) - y_{1,n}\| + \mathcal{C}_2 \max_{0 \leq n \leq N} \|y_1(t_n - \tau) - \gamma_1(t_n - \tau)\| = \mathcal{O}(h^q), \end{aligned} \quad (56)$$

where the constants  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are independent of  $h$ . Similarly, we get that

$$\max_{0 \leq t \leq T} \|y_2(t) - \gamma_2(t)\| = \mathcal{O}(h^q).$$

Due to the change of variables, we have

$$\begin{aligned} \max_{0 \leq n \leq N} \|x(t_n) - x_n\| &\leq \|Q\|_\infty \left( \max_{0 \leq n \leq N} \|y_1(t_n) - y_{1,n}\| + \max_{0 \leq n \leq N} \|y_2(t_n) - y_{2,n}\| \right), \\ \max_{0 \leq t \leq T} \|x(t) - \eta(t)\| &\leq \|Q\|_\infty \left( \max_{0 \leq t \leq T} \|y_1(t) - \gamma_1(t)\| + \max_{0 \leq t \leq T} \|y_2(t) - \gamma_2(t)\| \right), \end{aligned} \quad (57)$$

where  $\|Q\|_\infty := \max_{0 \leq t \leq T} \|Q(t)\|$ . By using estimations (54) and (56) and the first equation of (57), it is easy to see that the first estimate of (42) holds true. The second estimate of (42) also follows.

- b) In the general case, by propagating the argument of  $\gamma_1$  in (44) back to  $t_n + \theta h_n - l\tau \in [-\tau, 0]$ , it follows that

$$\begin{aligned} \gamma_2(t_n + \theta h_n) &= \bar{g}(t_n + \theta h_n, \gamma_1(t_n + \theta h_n), \gamma_1(t_n + \theta h_n - \tau), \\ &\quad \bar{g}(t_n + \theta h_n - \tau, \gamma_1(t_n + \theta h_n - \tau), \gamma_1(t_n + \theta h_n - 2\tau), \bar{g}(t_n + \theta h_n - 2\tau, \dots \\ &\quad \bar{g}(t_n + \theta h_n - l\tau, \gamma_1(t_n + \theta h_n - l\tau), \gamma_1(t_n + \theta h_n - (l+1)\tau), \\ &\quad \gamma_2(t_n + \theta h_n - (l+1)\tau) \dots). \end{aligned} \quad (58)$$

Repeating the argument as of the retarded case, we also obtain an ERK method with continuous extension applied to the underlying DODE (16). Since the DODE (16) is well conditioned, we deduce that the estimate (54) holds true. Then, we can assert that

$$\max_{0 \leq n \leq N} \|y_2(t_n) - y_{2,n}\| = \mathcal{O}(h^q), \quad \max_{0 \leq t \leq T} \|y_2(t) - \gamma_2(t)\| = \mathcal{O}(h^q). \quad (59)$$

Finally, using again (57), the estimates (42) are obtained.  $\square$

**Theorem 4** *Let the assumptions of Theorem 3 hold true. If  $\eta(t)$  is an NCE and a uniform mesh  $\pi^*$  is used, then the CHERK scheme (36a), (37) is convergent and superconvergence is attainable at meshpoints. Namely, it has discrete order  $q' = \min(p, k_d)$  and uniform order  $q = \min(p, k_d, k_u + 1)$ , i.e.,*

$$\max_{0 \leq n \leq N} \|x(t_n) - x_n\| = \mathcal{O}(h^{q'}), \quad \max_{0 \leq t \leq T} \|x(t) - \eta(t)\| = \mathcal{O}(h^q). \quad (60)$$

*Proof* For simplicity, we give a proof for the retarded case in which the term  $y_2(t - \tau)$  does not appear in the second equation of (9). The proof for the general case is carried out analogously, but with some more complicated formulations.

By Lemma 1 and the invertibility of  $Q(t)$ , we conclude that if  $\eta(t)$  is an NCE then the interpolant  $\gamma(t)$  is an NCE as well. It follows that the scheme (52a), (53) is an ERK method with natural continuous extension applied to the underlying DODE (16) on the uniform mesh  $\pi^*$ . According to Theorem 2, we obtain that

$$\max_{0 \leq n \leq N} \|y_1(t_n) - y_{1,n}\| = \mathcal{O}(h^{q'}), \quad \max_{0 \leq t \leq T} \|y_1(t) - \gamma_1(t)\| = \mathcal{O}(h^q), \quad (61)$$

where  $q' = \min(p, k_d)$ . Moreover, since  $\pi^*$  is a uniform mesh with a constant step-size  $h = \frac{\tau}{\nu}$  for some integer  $\nu \geq 1$ , we have  $t_n - \tau = t_{n-\nu}$ . Then, we obtain that

$$\max_{0 \leq n \leq N} \|y_1(t_n - \tau) - \gamma_1(t_n - \tau)\| = \max_{0 \leq n \leq N} \|y_1(t_{n-\nu}) - \gamma_1(t_{n-\nu})\| = \mathcal{O}(h^{q'}),$$

and

$$\max_{0 \leq t \leq T} \|y_1(t - \tau) - \gamma_1(t - \tau)\| = \mathcal{O}(h^q).$$

Using (56) and a similar estimate for the continuous solution, we have

$$\max_{0 \leq n \leq N} \|y_2(t_n) - y_{2,n}\| = \mathcal{O}(h^{q'}), \quad \max_{0 \leq t \leq T} \|y_2(t) - \gamma_2(t)\| = \mathcal{O}(h^q). \quad (62)$$

Combining (57), (61), and (62), we obtain immediately the estimation (60).  $\square$

**Remark 2** In practical implementation, in addition to interpolation errors associated with the continuous output formulas, other computational errors do arise, e.g., errors associated with numerical differentiation for  $E'$ , rounding errors, and errors due to finite-time termination of Newton iterations. Thanks to the zero-stability of continuous Runge-Kutta methods for delay ODEs [3], an error bound similar to that in [17, Theorem 3.8] can be obtained.

**Remark 3** The computation of  $\eta$  is a computational challenge since the evaluation of  $\eta$  in the past time is needed which may require another call of nonlinear system solver. To avoid this, we can proceed in one of the following ways. First, if a uniform mesh with  $h = \tau/\nu$  for some positive integer  $\nu$  is chosen, then the situation becomes very simple because the delayed times  $t_n - \tau$  and  $T_{n,i} - \tau$  are exactly a meshpoint and stage-points belonging to a past sub-interval. Thus, the approximations to the solution at meshpoints and stage-points are computed and then stored for a later use. Note that we do not need to store all the approximations but only those at time from  $t_n - \tau$  forward. In the case of non-uniform mesh, instead of solving additional nonlinear systems which may be costly, we suggest to use a continuous output formula which is obtained by quadrature formulas similar to the ODE case. Suppose that we need to compute  $\eta$  at the time  $t_k + \theta h_k$  for  $\theta \in (0, 1)$  and some  $k$ . First, we approximate  $D_{k,i} \approx x'(T_{k,i})$ ,  $i = 1, \dots, s$  by the formulas of the Runge-Kutta methods

$$\begin{aligned} D_{k,1} &= \frac{X_{k,2} - X_{k,1}}{a_{2,1}h_k}, \\ D_{k,i} &= \frac{X_{k,i+1} - X_{k,i}}{a_{i+1,i}h_k} - \sum_{j=1}^{i-1} \frac{a_{i+1,j}}{a_{i+1,i}} D_{k,j}, \quad i = 2, 3, \dots, s-1, \\ D_{k,s} &= \frac{x_{k+1} - X_{k,1}}{b_s h_k} - \sum_{j=1}^{s-1} \frac{b_j}{b_s} D_{k,j}. \end{aligned} \quad (63)$$

Then, we use the continuous extension

$$\eta(t_k + \theta h_k) = x_k + h_k \sum_{i=1}^s b_i(\theta) D_{k,i}, \quad \theta \in (0, 1),$$

for the approximation of the solution in the past time.

## 4 Numerical experiments

In this section, we implement some CHERK methods and carry out numerical experiments in MATLAB. We consider again Example 1 with time-delay  $\tau = 1$ . The system (17) is reformulated as follows:

$$\begin{aligned} (x_1(t) - \omega t x_2(t))' &= \lambda x_1(t) - \lambda \omega t x_2(t) + a x_2(t-1) - a e^{\lambda(t-1)} \\ x_1(t) - (1 + \omega t) x_2(t) &= b x_1(t-1) + (c + b\omega - b\omega t) x_2(t-1) - (b+c) e^{\lambda(t-1)}. \end{aligned} \quad (64)$$

We implement the classical 4-stage Runge-Kutta method with two different natural continuous extensions given in Example 2 and check the global discrete order  $q' = \min(p, k_d)$  and the global uniform order  $q = \min(p, k_d, k_u + 1)$  which have been stated in Theorem 4. We have solved the well-conditioned IVP (17) on interval  $t \in [0, 50]$  with  $\lambda = -1.5$ ,  $\omega = 10$ ,  $a = 0.5$ ,  $b = 1$ , and  $c = 0.8$  by the HERK4-NCE2 method. Actual errors of  $x_i$  at the mesh points are computed by  $e_i(h) = \max |x_i(t_n) - x_{i,n}|$ ,  $i = 1, 2$  for various step-sizes. Then, we estimate the discrete convergence rate in  $x_i$ . In order to illustrate the uniform order, we compute the actual errors of the interpolant  $\eta(t)$  at  $t = t_n + \theta h$  for some concrete values of  $\theta \in (0, 1)$ . These errors are defined by  $\max |x_i(t_n + \theta h) - \eta_i(t_n + \theta h)|$ ,  $i = 1, 2$ . Next, the convergence rate of the interpolant  $\eta(t)$  is estimated, as well. Numerical results in Table 1 illustrate clearly that the discrete order of the HERK4-NCE2 method is  $q' = 4$  and the HERK4-NCE2 method has a sole inner superconvergence point  $\theta = 0.5$ , which is mentioned in [3, Example 5.2.7]. By Table 2, we see that the uniform convergence order of the HERK4-NCE2 method is third which is due to the NCE order  $k_u = 2$ . The

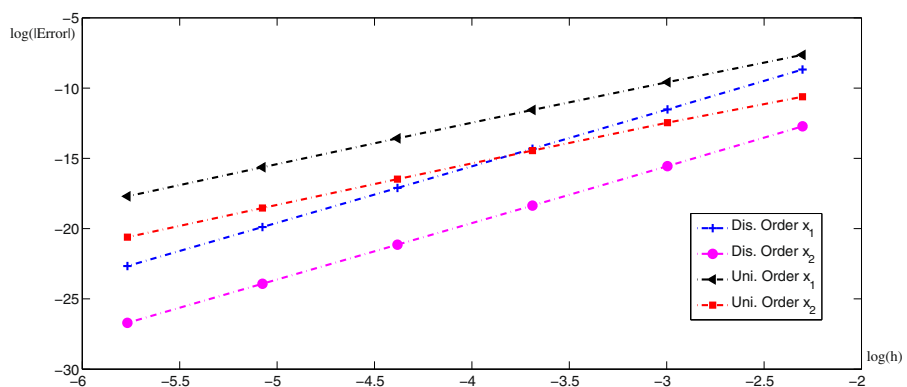
**Table 1** Numerical results for IVP (17) by the HERK4-NCE2 method

$h = 0.1$	Error in $x_1$	Rate in $x_1$	Error in $x_2$	Rate in $x_2$
Errors and convergence rate of $x$ at meshpoints				
$h$	1.6964e-04	–	2.9837e-06	–
$h/2$	9.9611e-06	4.0900	1.7564e-07	4.0864
$h/4$	6.0478e-07	4.0418	1.0655e-08	4.0431
$h/8$	3.7249e-08	4.0211	6.5587e-10	4.0219
$h/16$	2.3107e-09	4.0108	4.0680e-11	4.0110
$h/32$	1.4377e-10	4.0065	2.5227e-12	4.0113
Errors and convergence rate of $\eta(t)$ for $\theta = 0.5$				
$h$	2.3339e-04	–	8.9014e-06	–
$h/2$	1.5471e-05	3.9151	5.6622e-07	3.9746
$h/4$	9.9502e-07	3.9587	3.5704e-08	3.9872
$h/8$	6.3073e-08	3.9796	2.2414e-09	3.9936
$h/16$	3.9700e-09	3.9898	1.4040e-10	3.9968
$h/32$	2.4908e-10	3.9944	8.7648e-12	4.0016

**Table 2** Numerical results for IVP (17) by the HERK4-NCE2 method

$h = 0.1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_2$	Rate in $\eta_2$
Errors and convergence rate of $\eta(t)$ for $\theta = 0.3$				
$h$	4.8248e-04	—	2.4548e-05	—
$h/2$	6.9003e-05	2.8058	3.8641e-06	2.6674
$h/4$	9.6147e-06	2.8433	5.3260e-07	2.8590
$h/8$	1.2657e-06	2.9253	6.9669e-08	2.9345
$h/16$	1.6227e-07	2.9635	8.9019e-09	2.9683
$h/32$	2.0539e-08	2.9820	1.1248e-09	2.9844
Errors and convergence rate of $\eta(t)$ for $\theta = 0.6$				
$h$	5.8113e-04	—	2.2012e-05	—
$h/2$	6.1705e-05	3.2354	2.3079e-06	3.2537
$h/4$	6.9251e-06	3.1555	2.5946e-07	3.1530
$h/8$	8.1301e-07	3.0905	3.0576e-08	3.0850
$h/16$	9.8231e-08	3.0490	3.7047e-09	3.0450
$h/32$	1.2063e-08	3.0256	4.5568e-10	3.0233

difference between the discrete order and the uniform order of the HERK4-NCE2 method is illustrated by Fig. 3, where the log plots of actual errors versus stepsizes are displayed. For the illustration of the uniform order, the numerical approximations at points  $t = t_n + \theta h$  with  $\theta = 0.3$  are used. For the NCE of order  $k_u = 3$ , the numerical results in Table 3 show that the HERK4-NCE3 method has the same discrete global order and uniform global order  $q = 4$ .



**Fig. 3** Errors versus stepsizes for IVP (17) by the HERK4-NCE2 method

**Table 3** Numerical results for IVP (17) by the HERK4-NCE3 method

Meshpoints			$\theta = 0.3$		$\theta = 0.5$	
$h = 0.1$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	1.6964e-04	–	1.6608e-04	–	2.3339e-04	–
$h/2$	9.9611e-06	4.0900	1.0675e-05	3.9596	1.5471e-05	3.9151
$h/4$	6.0478e-07	4.0418	6.7657e-07	3.9799	9.9502e-07	3.9587
$h/8$	3.7249e-08	4.0211	4.2580e-08	3.9900	6.3073e-08	3.9796
$h/16$	2.3107e-09	4.0108	2.6706e-09	3.9949	3.9700e-09	3.9898
$h/32$	1.4377e-10	4.0065	1.6715e-10	3.9979	2.4908e-10	3.9944
$h = 0.1$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	2.9837e-06	–	5.9374e-06	–	8.9014e-06	–
$h/2$	1.7564e-07	4.0864	3.7148e-07	3.9985	5.6622e-07	3.9746
$h/4$	1.0655e-08	4.0431	2.3234e-08	3.9989	3.5704e-08	3.9872
$h/8$	6.5587e-10	4.0219	1.4527e-09	3.9994	2.2414e-09	3.9936
$h/16$	4.0680e-11	4.0110	9.0811e-11	3.9998	1.4040e-10	3.9968
$h/32$	2.5227e-12	4.0113	5.6516e-12	4.0061	8.7648e-12	4.0016

**Example 3** The second experiment is performed on a following nonlinear system

$$\begin{aligned}
 x_1(t) \left( x_1'(t) + (t^2 + 2 \sin t) x_2'(t) \right) &= x_1(t) x_2(t) e^{-t} + x_1(t) \sin(2t) + e^{-2t} x_2(t - \pi) \\
 &\quad + t^2 e^{-t} \cos t - e^{-2t}, \\
 0 &= e^t x_1(t) - x_2(t) - x_2(t - \pi) - 1,
 \end{aligned} \tag{65}$$

for  $t \in [0, 10\pi]$ . This problem has a smooth solution  $x_1 = e^{-t}$ ,  $x_2 = \sin t$  if the initial data for  $t \leq 0$  is given by the exact solution.

We solve the IVP for the reformulated DDAE again by the HERK4-NCE2 and HERK4-NCE3 methods. Here the tolerances used for the stopping criterion for Newton's iteration are taken of size  $h^{p+1}$  if the method is of order  $p$ . The numerical results are displayed in Tables 4 and 5, which again confirm the statements in Theorem 4.

**Example 4** We consider the following linear DDAE

$$\begin{aligned}
 x_1'(t) - \omega t x_2'(t) &= \omega x_1(t) + x_2(t - 1), \\
 0 &= -x_1(t) + (1 + \omega t) x_2(t) + x_2(t - 1),
 \end{aligned} \tag{66}$$

**Table 4** Numerical results for IVP (65) by the HERK4-NCE2 method

Meshpoints			$\theta = 0.3$		$\theta = 0.5$	
$h = \pi/10$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	4.8790e-03	–	4.9359e-03	–	5.4483e-03	–
$h/2$	4.5527e-04	3.4218	4.7560e-04	3.3755	5.1454e-04	3.4045
$h/4$	3.4495e-05	3.7223	4.3577e-05	3.4481	3.9162e-05	3.7157
$h/8$	2.3693e-06	3.8638	3.9706e-06	3.4562	2.6930e-06	3.8622
$h/16$	1.5507e-07	3.9335	3.8674e-07	3.3599	1.7642e-07	3.9321
$h/32$	9.9166e-09	3.9669	4.0924e-08	3.2403	1.1286e-08	3.9664
$h = \pi/10$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	1.2276e-01	–	1.2230e-01	–	1.2065e-01	–
$h/2$	1.0304e-02	3.5746	1.0328e-02	3.5658	1.0265e-02	3.5550
$h/4$	7.7280e-04	3.7369	7.7728e-04	3.7320	7.7209e-04	3.7328
$h/8$	5.2951e-05	3.8674	5.3539e-05	3.8598	5.2940e-05	3.8664
$h/16$	3.4633e-06	3.9344	3.5373e-06	3.9199	3.4632e-06	3.9342
$h/32$	2.2139e-07	3.9675	2.3067e-07	3.9388	2.2139e-07	3.9674

for  $t \geq 0$ . This system is actually a modified version of an example in [2]. For the initial condition  $x_1(t) = \frac{1}{2}\omega t + 1$ ,  $x_2(t) = \frac{1}{2}$ ,  $t \leq 0$ , the IVP has the unique continuous and piecewise smooth solution

$$x_1(t) = \begin{cases} \frac{1}{2}\omega t^2 + \left(\frac{1}{2}\omega + \frac{1}{2}\right)t + 1, & \forall t \in (0, 1], \\ \frac{1}{4}\omega t^3 + \left(\frac{1}{4} - \frac{1}{2}\omega\right)t^2 + \frac{5}{4}\omega t + \frac{5}{4}, & \forall t \in (1, 2], \\ \frac{1}{12}\omega t^4 + \left(\frac{1}{12} - \frac{3}{4}\omega\right)t^3 + \left(3\omega - \frac{1}{2}\right)t^2 + \left(2 - \frac{29}{12}\omega\right)t - \frac{5}{12}, & \forall t \in (2, 3], \\ \frac{\omega}{48}t^5 + \left(\frac{1}{48} - \frac{5}{12}\omega\right)t^4 + \left(\frac{27}{8}\omega - \frac{1}{3}\right)t^3 + \left(\frac{19}{8} - 11\omega\right)t^2 \\ + \left(\frac{685}{48}\omega - \frac{25}{4}\right)t + \frac{385}{48}, & \forall t \in (3, 4], \\ \frac{\omega}{240}t^6 + \left(\frac{1}{240} - \frac{7}{48}\omega\right)t^5 + \left(\frac{25}{12}\omega - \frac{1}{8}\right)t^4 + \left(\frac{19}{12} - \frac{343}{24}\omega\right)t^3 + \\ \left(\frac{289}{6}\omega - \frac{229}{24}\right)t^2 + \left(\frac{349}{12} - \frac{14719}{240}\omega\right)t - \frac{7739}{240}, & \forall t \in (4, 5], \\ \vdots & \end{cases} \quad (67)$$

and

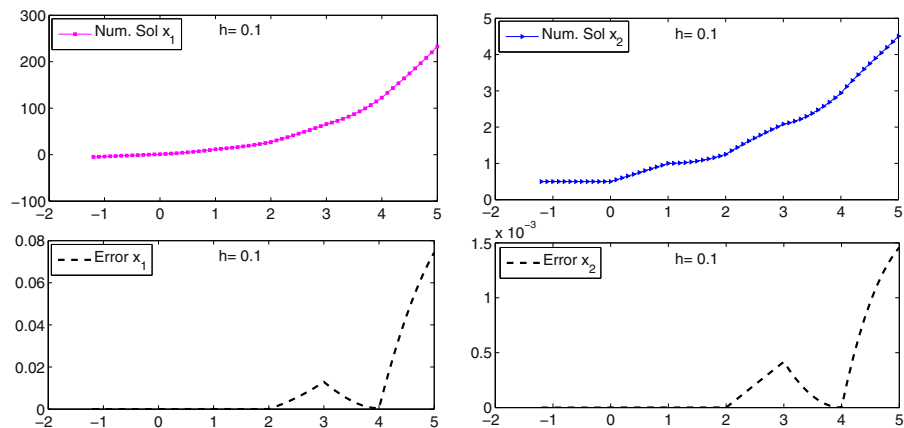
$$x_2(t) = \begin{cases} \frac{1}{2}t + \frac{1}{2}, & \forall t \in (0, 1], \\ \frac{1}{4}t^2 - \frac{1}{2}t + \frac{5}{4}, & \forall t \in (1, 2], \\ \frac{1}{12}t^3 - \frac{3}{4}t^2 + 3t - \frac{29}{12}, & \forall t \in (2, 3], \\ \frac{1}{48}t^4 - \frac{5}{12}t^3 + \frac{27}{8}t^2 - 11t + \frac{685}{48}, & \forall t \in (3, 4], \\ \frac{1}{240}t^5 - \frac{7}{48}t^4 + \frac{25}{12}t^3 - \frac{343}{24}t^2 + \frac{289}{6}t - \frac{14719}{240}, & \forall t \in (4, 5], \\ \vdots & \end{cases} \quad (68)$$

**Table 5** Numerical results for IVP (65) by the HERK4-NCE3 method

	Meshpoints		$\theta = 0.3$		$\theta = 0.5$	
$h = \pi/10$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	4.8790e-03	–	4.8728e-03	–	5.4483e-03	–
$h/2$	4.5527e-04	3.4218	4.7120e-04	3.3703	5.1454e-04	3.4045
$h/4$	3.4495e-05	3.7223	3.5840e-05	3.7167	3.9162e-05	3.7157
$h/8$	2.3693e-06	3.8638	2.4697e-06	3.8591	2.6930e-06	3.8622
$h/16$	1.5507e-07	3.9335	1.6178e-07	3.9323	1.7642e-07	3.9321
$h/32$	9.9166e-09	3.9669	1.0349e-08	3.9664	1.1286e-08	3.9664
$h = \pi/10$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	1.2276e-01	–	1.2199e-01	–	1.2065e-01	–
$h/2$	1.0304e-02	3.5746	1.0290e-02	3.5674	1.0265e-02	3.5550
$h/4$	7.7280e-04	3.7369	7.7256e-04	3.7355	7.7209e-04	3.7328
$h/8$	5.2951e-05	3.8674	5.2948e-05	3.8670	5.2940e-05	3.8664
$h/16$	3.4633e-06	3.9344	3.4633e-06	3.9344	3.4632e-06	3.9342
$h/32$	2.2139e-07	3.9675	2.2139e-07	3.9675	2.2139e-07	3.9674

The numerical solution and actual errors for the above IVP with  $\omega = 10$  on interval  $[0, 5]$  by the half-explicit midpoint method with natural continuous extension (HEMID-NCE2) are shown in Fig. 4.

We also compute actual errors and numerical convergence rate of the CHERK methods for this problem. The numerical results for the IVP (66) with  $\omega = 10$  on interval  $[0, 5]$  by the HERK4-NCE2 and HERK4-NCE3 methods are presented in Tables 6, 7, and 8. All of these numerical results again confirm the statements in Theorem 4.



**Fig. 4** Numerical results for IVP (66) by the HEMID-NCE2 method

**Table 6** Numerical results for IVP (66) by the HEMID-NCE2 method

Meshpoints			$\theta = 0.3$		$\theta = 0.5$	
$h = 0.1$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	7.4407e-02	–	7.0537e-02	–	7.1978e-02	–
$h/2$	1.8596e-02	2.0005	1.8123e-02	1.9606	1.8299e-02	1.9758
$h/4$	4.6486e-03	2.0001	4.5902e-03	1.9812	4.6119e-03	1.9883
$h/8$	1.1621e-03	2.0000	1.1549e-03	1.9908	1.1576e-03	1.9943
$h/16$	2.9053e-04	2.0000	2.8962e-04	1.9955	2.8996e-04	1.9972
$h/32$	7.2632e-05	2.0000	7.2519e-05	1.9977	7.2561e-05	1.9986
$h = 0.1$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	1.4590e-03	–	1.4022e-03	–	1.4253e-03	–
$h/2$	3.6462e-04	2.0005	3.5780e-04	1.9705	3.6057e-04	1.9829
$h/4$	9.1148e-05	2.0001	9.0312e-05	1.9862	9.0651e-05	1.9919
$h/8$	2.2787e-05	2.0000	2.2683e-05	1.9933	2.2725e-05	1.9960
$h/16$	5.6966e-06	2.0000	5.6837e-06	1.9967	5.6890e-06	1.9980
$h/32$	1.4242e-06	2.0000	1.4225e-06	1.9984	1.4232e-06	1.9990

We have also carried out numerical experiments for a semi-explicit DDAE of index-1 given in [1] by these HERK methods with continuous extension. For semi-explicit DDAEs, the implementation of the HERK methods is significantly simplified. The obtained numerical results again confirm the theoretical findings.

**Table 7** Numerical results for IVP (66) by the HERK4-NCE3 method

Meshpoints			$\theta = 0.3$		$\theta = 0.5$	
$h = 0.2$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	5.6667e-05	–	1.7436e-04	–	2.4700e-04	–
$h/2$	3.5417e-06	4.0000	1.1055e-05	3.9793	1.5666e-05	3.9788
$h/4$	2.2135e-07	4.0000	6.9577e-07	3.9899	9.8607e-07	3.9898
$h/8$	1.3843e-08	3.9991	4.3631e-08	3.9952	6.1839e-08	3.9951
$h/16$	8.6010e-10	4.0085	2.7287e-09	3.9991	3.8712e-09	3.9977
$h/32$	5.4001e-11	3.9934	1.6639e-10	4.0355	2.3894e-10	4.0181
$h = 0.2$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	1.1111e-06	–	4.2268e-06	–	5.9306e-06	–
$h/2$	6.9444e-08	4.0000	2.6990e-07	3.9691	3.8064e-07	3.9617
$h/4$	4.3402e-09	4.0000	1.7048e-08	3.9848	2.4102e-08	3.9812
$h/8$	2.7143e-10	3.9991	1.0710e-09	3.9926	1.5161e-09	3.9908
$h/16$	1.6864e-11	4.0086	6.7037e-11	3.9978	9.5048e-11	3.9955
$h/32$	1.0640e-12	3.9863	4.0914e-12	4.0343	5.8646e-12	4.0185

**Table 8** Numerical results for IVP (66) by the HERK4-NCE2 method

Meshpoints			$\theta = 0.3$		$\theta = 0.5$	
$h = 0.2$	Error in $x_1$	Rate in $x_1$	Error in $\eta_1$	Rate in $\eta_1$	Error in $\eta_1$	Rate in $\eta_1$
$h$	5.6667e-05	–	5.2548e-03	–	2.4700e-04	–
$h/2$	3.5417e-06	4.0000	6.8343e-04	2.9428	1.5666e-05	3.9788
$h/4$	2.2135e-07	4.0000	8.7113e-05	2.9718	9.8607e-07	3.9898
$h/8$	1.3843e-08	3.9991	1.0995e-05	2.9860	6.1839e-08	3.9951
$h/16$	8.6010e-10	4.0085	1.3810e-06	2.9930	3.8712e-09	3.9977
$h/32$	5.4001e-11	3.9934	1.7305e-07	2.9965	2.3894e-10	4.0181
$h = 0.2$	Error in $x_2$	Rate in $x_2$	Error in $\eta_2$	Rate in $\eta_2$	Error in $\eta_2$	Rate in $\eta_2$
$h$	1.1111e-06	–	1.2756e-04	–	5.9306e-06	–
$h/2$	6.9444e-08	4.0000	1.6711e-05	2.9323	3.8064e-07	3.9617
$h/4$	4.3402e-09	4.0000	2.1378e-06	2.9666	2.4102e-08	3.9812
$h/8$	2.7143e-10	3.9991	2.7032e-07	2.9834	1.5161e-09	3.9908
$h/16$	1.6864e-11	4.0086	3.3985e-08	2.9917	9.5048e-11	3.9955
$h/32$	1.0640e-12	3.9863	4.2604e-09	2.9958	5.8646e-12	4.0185

## 5 Conclusion

In this work, we have proposed half-explicit Runge-Kutta methods with continuous extension for a class of structured strangeness-free DDAEs with constant delay (3). Instead of direct discretization, it has been shown that it is more preferable to discretize the appropriately reformulated DDAEs. Concretely, discrete and uniform convergence orders of the underlying continuous ERK methods are preserved in this modified discretization. The numerical experiments have been given to confirm the theoretical results.

There are numerous difficulties that arise in solving general DDAEs. From the analysis and results of this paper, it seems that there are still many interesting works in the future. First, a correct definition of solutions as well as a careful analysis for the solvability of general DDAEs under mild conditions should be addressed, which would be of interest from the view point of control theory. Second, we should also design a software package for solving the DDAEs (3) by using the CHERK methods with error control and automatic stepsize selection. As an application, the problem of approximating spectral intervals for linear DDAEs should be investigated. Finally, we could extend the analysis of CRK methods for general strangeness-free DDAEs with non-constant (time-varying, state-dependent) delay, for which more difficulties are expected.

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