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LINEAR ALGEBRA PROPERTIES OF DISSIPATIVE HAMILTONIAN DESCRIPTOR SYSTEMS*

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Abstract. A wide class of matrix pencils connected with dissipative Hamiltonian descriptor systems is investigated. In particular, the following properties are shown: all eigenvalues are in the closed left half plane, the nonzero finite eigenvalues on the imaginary axis are semisimple, the index is at most two, and there are restrictions for the possible left and right minimal indices. For the case that the eigenvalue zero is not semisimple, a structure-preserving method is presented that perturbs the given system into a Lyapunov stable system.

Key words. port-Hamiltonian system, descriptor system, dissipative Hamiltonian system, matrix pencil, singular pencil, Kronecker canonical form, Lyapunov stability

AMS subject classifications. 15A18, 15A21, 15A22

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1. Introduction.

This paper deals with the linear algebra properties of matrix pencils that are associated with linear time-invariant *dissipative Hamiltonian descriptor systems* of the form

$$E\dot{x} = (J - R)\,Qx, \tag{1.1}$$

where $J, R \in \mathbb{F}^{n,n}$, $E, Q \in \mathbb{F}^{n,m}$, $m \leq n$ with $\mathbb{F} \in \{\mathbb{C}, \mathbb{R}\}$. Denoting by $\star$ the complex conjugate transpose for complex matrices and the transpose for real matrices, we require in addition that $J^\star = -J$ and $R^\star = R \geq 0$, where we use the standard notation $W > 0$ ($W \geq 0$) for a real symmetric or complex Hermitian matrix W to be positive (semi-)definite.

These systems arise in energy based modeling of dynamical systems; see, e.g., [2, 3, 16, 21, 27, 30, 32, 31] and are a special case of *port-Hamiltonian (pH) descriptor systems* that were introduced recently in [2]; see also [32].

In the linear time-invariant case pH descriptor systems have the form

$$\begin{aligned} E\dot{x} &= (J - R)\,Qx + (B - P)u, \\ y &= (B + P)^\star Qx + (S + N)u, \end{aligned} \tag{1.2}$$

with $B, P \in \mathbb{F}^{n,p}$, and $S, N \in \mathbb{F}^{p,p}$ with $S = S^\star, N = -N^\star$.

An important quantity (modeling the internal energy of storage elements) associated with the system (1.2) is the *Hamiltonian*, which in the descriptor case takes the form $\mathcal{H}(x) = \frac{1}{2}x^\star E^\star Qx$. To guarantee that $\mathcal{H}(x)$ (as an energy) is real, it is natural

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to assume (see [2] for details) that

$$E^\star Q = Q^\star E. \tag{1.3}$$

In most practical examples the system has the extra property that $Q^\star E$ is positive semidefinite, in which case we speak of a pH descriptor system *with nonnegative Hamiltonian*. The matrix J is the *structure matrix* describing the energy flux among energy storage elements within the system; R is the *dissipation matrix* describing energy dissipation/loss in the system. The matrices $B \pm P \in \mathbb{F}^{n,m}$ are *port* matrices describing the manner in which energy enters and exits the system, and $S+N$ describes the direct *feed-through* from input to output.

To illustrate the broad variety of application areas where pH descriptor systems play a role, we briefly review the following examples from the literature.

Example 1.1. A simple RLC network (see, e.g., [2, 5, 11]) can be modeled by a dissipative Hamiltonian descriptor system of the form

$$\underbrace{\begin{bmatrix} G_c C G_c^\top & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{:=E} \begin{bmatrix} \dot{V} \\ \dot{I}_l \\ \dot{I}_v \end{bmatrix} = \underbrace{\begin{bmatrix} -G_r R_r^{-1} G_r^\top & -G_l & -G_v \\ G_l^\top & 0 & 0 \\ G_v^\top & 0 & 0 \end{bmatrix}}_{:=J-R} \begin{bmatrix} V \\ I_l \\ I_v \end{bmatrix}, \tag{1.4}$$

where $L > 0$, $C > 0$, $R_r > 0$ are real symmetric matrices describing inductances, capacitances, and resistances, respectively. Here, G_v is of full column rank, and the subscripts r, c, l, and v refer to the resistors, capacitors, inductors, and voltage sources, while V denotes the voltage, and I_l, I_v denote the currents through an inductor or voltage source, respectively. Here, J and $-R$ are defined to be the skew-symmetric and symmetric parts, respectively, of the matrix on the right-hand side of (1.4). We see that in this example we have a system of the form (1.1) with the matrix Q being the identity, $E = E^T \geq 0$, $J = -J^T$, and $R \geq 0$.

Example 1.2. Space discretization of the Stokes or Oseen equation in fluid dynamics (see, e.g., [9]) leads to a dissipative Hamiltonian system with pencil

$$\lambda E - (J - R) = \lambda \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ -B^\top & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & B \\ -B^\top & 0 \end{bmatrix}, \quad R = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

where A is a positive semidefinite discretization of the negative Laplace operator, B being a discretized gradient, and M being a positive definite mass matrix, so $R \geq 0$, $Q = I$, and $E = E^\top \geq 0$. Note that the matrix B may be rank deficient, in which case the pencil is singular.

Example 1.3. Space discretization of the Euler equation describing the flow in a gas network [8] leads to a dissipative Hamiltonian descriptor system with

$$E = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -G & 0 \\ G^\top & 0 & K^\top \\ 0 & -K & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q = I,$$

where M_1, M_2, D are positive definite and the matrices N and $\begin{bmatrix} G^\top & K^\top \end{bmatrix}$ have full column rank. Thus, again we have $R \geq 0$ and a nonnegative Hamiltonian, since $Q^\top E = E^\top Q \geq 0$.

Example 1.4. Classical second order representations of linear damped mechanical systems (see, e.g., [37, 38]) have the form

$$M\ddot{x} + D\dot{x} + Kx = f, \tag{1.5}$$

where $M, D, K \in \mathbb{R}^{n,n}$ are symmetric positive semidefinite matrices and represent the mass, damping, and stiffness matrices, respectively, while f is an exiting force. First order formulation leads to a system associated with the matrix pencil

$$\lambda \underbrace{\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}}_{=:E} - \left(\underbrace{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}_{=:J} - \underbrace{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}_{=:R} \right) \underbrace{\begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}}_{=:Q}, \tag{1.6}$$

where we have $E^\top Q = Q^\top E \geq 0$ and $R \geq 0$. As a particular case, freely vibrating strongly damped systems of the form $M\ddot{x} + sD\dot{x} + Kx = 0$ were investigated in [35] for $s \to \infty$. In the limit case, the first order formulation of the equivalent system $\frac{1}{s}M\ddot{x} + D\dot{x} + \frac{1}{s}Kx = 0$ leads to a pencil as in (1.6) with $M = K = 0$. Observe that this pencil is singular if D is singular.

Example 1.5. Consider a mechanical system as in (1.5) with a kinematic constraint $G\dot{x} = 0$. Incorporating it in the dynamical equation using a Lagrange multiplier y, the system can be written as

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} K & G^\top \\ G & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

which in first order formulation gives a system associated with the pencil

$$\lambda \underbrace{\begin{bmatrix} M & & & \\ & 0 & & \\ & & I & \\ & & & I \end{bmatrix}}_{=:E} - \left(\underbrace{\begin{bmatrix} & & -I & \\ & & & -I \\ I & & & \\ & I & & \end{bmatrix}}_{=:J} - \underbrace{\begin{bmatrix} D & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}}_{=:R} \right) \underbrace{\begin{bmatrix} I & & & \\ & I & & \\ & & K & G^\top \\ & & G & 0 \end{bmatrix}}_{=:Q}.$$

Here, both E and Q are symmetric, commuting square matrices and R is positive semidefinite, but, in contrast to the previous examples, $E^\top Q$ is not positive semidefinite in this case. Note also that often the system is modeled with redundant constraints which implies that G is rank deficient and then the pencil is singular.

Examples with singular Q arise also in many other applications; see, e.g., [1, 13, 18]. The singular pencil case then typically arises as a limiting case or when redundant system modeling is used.

To understand the linear algebra properties of pH descriptor systems, we analyze matrix pencils of the form

$$\begin{aligned} &P(\lambda) = \lambda E - LQ \in \mathbb{F}^{n,m}[\lambda] \\ &\quad \text{with } L = J - R, \quad J = -J^\star, \quad R = R^\star \geq 0, \quad E^\star Q = Q^\star E. \end{aligned} \tag{1.7}$$

We will discuss the general case allowing that the pencil is singular. If $Q = I$, and thus $n = m$, and if $R = 0$, then $P(\lambda)$ is an *even matrix pencil*; i.e., the coefficient associated with λ is Hermitian/symmetric and the second coefficient is skew-Hermitian/skew-symmetric. Canonical forms for $P(\lambda)$ in this case are well-known; see, e.g., [36]. If, in addition, E is positive semidefinite, then it easily follows from these forms that all finite eigenvalues are on the imaginary axis and semisimple. Then replacing the condition $R = 0$ by $R \geq 0$ has the effect that now all finite eigenvalues are in the closed left half complex plane, and those on the imaginary axis are still semisimple; see, e.g., [25].

At first sight, it does not seems obvious that the structure of the pencil $P(\lambda)$ in (1.7) preserves any of these restrictions in the spectrum. However, it turns out that the pencil $P(\lambda)$ possesses surprisingly rich linear algebraic properties, in particular under the assumption that $Q^\star E$ is positive semidefinite. To derive these properties is the main purpose of this paper which is organized as follows.

Section 2 reviews the Kronecker canonical form and introduces the concept of regular deflating subspaces of singular matrix pencils. It appears that the properties of the pencil $\lambda E - Q$ are crucial for understanding the pencil $P(\lambda)$. Hence, we first analyze the pencil $\lambda E - Q$ in section 3. In the regular case, a canonical form is derived in Propositions 3.1 and 3.3. In the singular case, we present a condensed form (Theorem 3.7) and show what are the possible left and right minimal indices (Corollary 3.9). Sections 4 and 5 discuss the stability properties and the Kronecker canonical form of the pencil $P(\lambda) = \lambda E - LQ$. In particular, we show in Theorem 4.3 that if the pencil $\lambda E - Q$ does not have left minimal indices larger than zero, then the pencil $P(\lambda)$ has all finite eigenvalues in the closed left half plane, and all eigenvalues on the imaginary axis except zero are semisimple. Furthermore, the index of the pencil (see section 2) is at most two and its right minimal indices (if there are any) are not larger than one. If, in addition, the pencil $\lambda E - Q$ is regular, then the left minimal indices of $P(\lambda)$ are all zero (if there are any). Furthermore, we present many examples that illustrate which properties are lost if some of the assumptions on the pencil $\lambda E - Q$ are weakened. Finally, we discuss in section 6 structure-preserving perturbations of $P(\lambda)$ that make the eigenvalue zero semisimple but keep the Jordan structure of all other eigenvalues invariant.

2. Preliminaries.

For general matrix pencils $\lambda E - A$ the structural properties are characterized via the Kronecker canonical form. Recall the following result for complex matrix pencils [14].

THEOREM 2.1. *Let $E, A \in \mathbb{C}^{n,m}$. Then there exist nonsingular matrices $S \in \mathbb{C}^{n,n}$ and $T \in \mathbb{C}^{m,m}$ such that*

$$S(\lambda E - A)T = \operatorname{diag}(\mathcal{L}_{\epsilon_1}, \dots, \mathcal{L}_{\epsilon_p}, \mathcal{L}^\top_{\eta_1}, \dots, \mathcal{L}^\top_{\eta_q}, \mathcal{J}^{\lambda_1}_{\rho_1}, \dots, \mathcal{J}^{\lambda_r}_{\rho_r}, \mathcal{N}_{\sigma_1}, \dots, \mathcal{N}_{\sigma_s}), \tag{2.1}$$

where the block entries have the following properties:

(i) *Every entry $\mathcal{L}_{\epsilon_j}$ is a bidiagonal block of size $\epsilon_j \times (\epsilon_j + 1)$, $\epsilon_j \in \mathbb{N}_0$, of the form*

$$\lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}.$$

(ii) *Every entry $\mathcal{L}^\top_{\eta_j}$ is a bidiagonal block of size $(\eta_j + 1) \times \eta_j$, $\eta_j \in \mathbb{N}_0$, of the form*

$$\lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}.$$

(iii) *Every entry $\mathcal{J}_{\rho_j}^{\lambda_j}$ is a Jordan block of size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$, of the form*

$$\lambda \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

(iv) *Every entry $\mathcal{N}_{\sigma_j}$ is a nilpotent block of size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$, of the form*

$$\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

The Kronecker canonical form is unique up to permutation of the blocks.

For real matrices there exists a real Kronecker canonical form which is obtained under real transformation matrices S, T. Here, the blocks $\mathcal{J}_{\rho_j}^{\lambda_j}$ with $\lambda_j \in \mathbb{C} \setminus \mathbb{R}$ are in real Jordan canonical form instead, but the other blocks have the same structure as in the complex case.

The sizes η_j and ϵ_i of the rectangular blocks are called the *left and right minimal indices* of $\lambda E - A$, respectively. Furthermore, a value $\lambda_0 \in \mathbb{C}$ is called a (finite) eigenvalue of $\lambda E - A$ if

$$\operatorname{rank}(\lambda_0 E - A) < \operatorname{nrank}(\lambda E - A) := \max_{\alpha \in \mathbb{C}} \operatorname{rank}(\alpha E - A),$$

where $\operatorname{nrank}(\lambda E - A)$ is called the *normal rank* of $\lambda E - A$. Furthermore, $\lambda_0 = \infty$ is said to be an eigenvalue of $\lambda E - A$ if zero is an eigenvalue of $\lambda A - E$. It is obvious from Theorem 2.1 that the blocks $\mathcal{J}_{\rho_j}$ as in (iii) correspond to finite eigenvalues of $\lambda E - A$, whereas blocks $\mathcal{N}_{\sigma_j}$ as in (iv) correspond to the eigenvalue ∞. The sum of all sizes of blocks that are associated with a fixed eigenvalue $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ is called the *algebraic multiplicity* of λ_0. The size of the largest block $\mathcal{N}_{\sigma_j}$ is called the *index* ν of the pencil $\lambda E - A$, where, by convention, $\nu = 0$ if E is invertible.

The matrix pencil $\lambda E - A \in \mathbb{F}^{n,m}[\lambda]$ is called *regular* if $n = m$ and $\det(\lambda_0 E - A) \neq 0$ for some $\lambda_0 \in \mathbb{C}$; otherwise it is called *singular*. A pencil is singular if and only if it has blocks of at least one of the types $\mathcal{L}_{\varepsilon_j}$ or $\mathcal{L}_{\eta_j}^\top$ in the Kronecker canonical form.

While eigenvalues of singular pencils are well-defined, the same is not true for eigenvectors. For example, the pencil

$$\lambda E - A = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda & -1 \end{bmatrix} = \mathcal{J}_1^1 \oplus \mathcal{L}_1 \tag{2.2}$$

has the eigenvalue 1 with algebraic multiplicity 1, but any vector x of the form $x = \begin{bmatrix} \alpha & \beta & \beta \end{bmatrix}^\top$ with $\alpha, \beta \in \mathbb{F}$ satisfies $1 \cdot Ex = Ax$. Hence, the equation $\lambda_0 Ex = Ax$ itself is not suitable (at least for the purpose of the current paper) as a generalization of the notion *eigenvector* in the singular case. We therefore introduce the following concept.

DEFINITION 2.2. Let $E, A \in \mathbb{F}^{n,m}$, and let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $P(\lambda) = \lambda E - A$ with algebraic multiplicity k.

(i) A k-dimensional subspace $\mathcal{X}$ of $\mathbb{F}^m$ is called a *(right) regular deflating subspace of $P(\lambda)$ associated with* λ_0 if there exist nonsingular matrices $X \in \mathbb{C}^{m,m}$ and $Y \in \mathbb{F}^{n,n}$ such that the first k columns of X span $\mathcal{X}$ and such that

$$Y(\lambda E - A)X = \begin{bmatrix} R(\lambda) & 0 \\ 0 & \widetilde{P}(\lambda) \end{bmatrix}, \tag{2.3}$$

where $R(\lambda)$ is a regular $k \times k$ pencil that has only the eigenvalue λ_0 and where $\widetilde{P}(\lambda)$ does not have λ_0 as an eigenvalue.

(ii) A vector $x \in \mathbb{C}^m \setminus \{0\}$ is called a *regular eigenvector* of $P(\lambda)$ associated with λ_0 if $\lambda_0 Ex = Ax$ and if there exists a regular deflating subspace associated with λ_0 that contains x.

Regular deflating subspaces and regular eigenvectors associated with the eigenvalue ∞ are defined in the obvious way by considering the reversed pencil $\lambda A - E$ and the eigenvalue zero.

Clearly, for each eigenvalue of a singular pencil there exists a regular eigenvector and a regular deflating subspace. For the pencil in (2.2) each subspace of the form $\mathcal{X} = \operatorname{span}(\begin{bmatrix} \alpha & \beta & \beta \end{bmatrix}^\top)$ with some $\alpha, \beta \in \mathbb{F}$, $\alpha \neq 0$ is a regular deflating subspace. Indeed, with

$$X = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \beta & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1/\alpha & 0 \\ -\beta/\alpha & 1 \end{bmatrix},$$

one has $Y(\lambda E - A)X = \lambda E - A$. In particular, this shows that a regular deflating subspace associated with an eigenvalue is not unique if the pencil is singular, which is in contrast with the regular case. Furthermore, it is straightforward to show that neither the subspace

$$\operatorname{span}(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top) \quad \text{nor} \quad \operatorname{span}(\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top)$$

is a regular deflating subspace. We have the following key property of regular deflating subspaces.

LEMMA 2.3. *Let $E, A \in \mathbb{F}^{n,m}$, let $\lambda_0 \in \mathbb{C} \cup \{\infty\}$ be an eigenvalue of $\lambda E - A$, and let $\mathcal{X}$ be an associated regular deflating subspace. Then*

$$\mathcal{X} \cap \operatorname{span}\left(\bigcup_{\lambda \in (\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_0\}} \ker(\lambda E - A) \right) = \{0\}$$

with the usual convention $\infty E - A := E$.

Proof. Indeed, let X, Y be as in Definition 2.2 such that (2.3) is satisfied. Furthermore assume that $y \in \ker(\lambda_1 E - A)$ for some $\lambda_1 \in (\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_0\}$, and let $X^{-1}y = \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top$ be split conformably with (2.3). First we show that $y_1 = 0$. Indeed,

$$0 = Y(\lambda_1 E - A)X(X^{-1}y) = \begin{bmatrix} R(\lambda_1)y_1 \\ * \end{bmatrix}.$$

In particular, $y_1 \in \ker R(\lambda_1)$. But the pencil $R(\lambda)$ is regular and has only one eigenvalue $\lambda_0 \neq \lambda_1$. Therefore $y_1 = 0$. In other words, $PX^{-1}y = 0$, where P is the canonical projection onto the first k coordinates. Assume now that $x = \sum_{i=1}^{l} z_i$, with $x \in \mathcal{X}$ and $z_i \in \ker(\lambda_i E - A)$, $\lambda_i \neq \lambda_0$ for $i = 1, \ldots, l$ and some $l > 0$. Then $PX^{-1}x = \sum_{i=1}^{l} Pz_i = 0$. But since x is a vector in $\mathcal{X}$, there exists a vector $v \in \mathbb{F}^k$ such that $x = X \begin{bmatrix} v^\top & 0 \end{bmatrix}^\top$; i.e., $PX^{-1}x = X^{-1}x$. Hence $x = 0$, which finishes the proof. ∎

Let us mention two other concepts associated with eigenvectors and deflating subspaces in the case of singular pencils that are present in the literature. In [7], *reducing subspaces* were introduced to replace the concept of deflating subspaces in the singular case. This concept has the advantage that reducing subspaces associated with an eigenvalue are now uniquely defined. However, they are explicitly allowed to contain vectors *from the singular part*, and thus a result similar to Lemma 2.3 cannot be obtained. For the second concept, that only refers to eigenvectors, we first discuss a different characterization of regular eigenvectors as follows. Let $\mathcal{N}_r(P)$ denote the right nullspace of $P(\lambda)$ interpreted as a matrix over the field of rational functions (thus $\mathcal{N}_r(P)$ is a subspace of the space $\mathbb{C}^m(\lambda)$ of vectors of length m with rational functions as entries), and let

$$\mathcal{N}_r(P)\big|_{\lambda_0} := \left\{ z \in \mathbb{C}^m \,\middle|\, z = v(\lambda_0) \text{ for some } v(\lambda) \in \mathcal{N}_r(P) \right\}$$

denote the *right nullspace of* $P(\lambda)$ *evaluated at* λ_0; see [12]. Then it can be shown that $x \in \mathbb{C}^m$ is a regular eigenvector of $P(\lambda)$ associated with λ_0 if and only if $\lambda_0 Ex = Ax$ and $x \notin \mathcal{N}_r(P)\big|_{\lambda_0}$. This construction allows to define *eigenspaces* uniquely, e.g., as being orthogonal to $\mathcal{N}_r(P)\big|_{\lambda_0}$; cf. [12]. However, this orthogonality property is not preserved under equivalence transformations for pencils.

3. Properties of the pencil $\boldsymbol{\lambda E - Q}$. Many of the properties of the pencil $\lambda E - (J-R)Q$ are direct consequences of the properties of the pencil $\lambda E - Q \in \mathbb{F}^{m,n}[\lambda]$ satisfying the condition (1.3). In this section we therefore study properties of the latter pencil, and we will also frequently assume that, in addition, one (or both) of the conditions

$$EQ^\star = QE^\star \tag{3.1}$$

and

$$E^\star Q \geq 0 \tag{3.2}$$

are satisfied. Although (3.1) does not have a direct physical interpretation, it is satisfied by most classical examples (see the introduction), and together with (1.3) it forms a very strong assumption on the structure of the pencil $\lambda E - Q$. Condition (3.2) has a minor impact on the properties of $\lambda E - Q$, but strongly influences the properties of $P(\lambda)$, as will be seen in subsequent sections. We will not discuss pencils $\lambda E - Q$ for which only (3.1) is assumed, because results for these pencils can easily be obtained from the corresponding results on pencils satisfying (1.3) by considering the pencil $\lambda E^\star - Q^\star$.

Clearly, when E and Q are square Hermitian (or real symmetric matrices) that commute (as is the case in some of our applications), then both $E^\star Q = Q^\star E$ and $EQ^\star = QE^\star$ are satisfied. It is clear that for any unitary $U \in \mathbb{F}^{m,m}$ and invertible $X \in \mathbb{F}^{n,n}$ a transformation of the form $(E, Q) \mapsto (UEX, UQX)$ preserves the conditions (1.3) and (3.2). The following result yields a canonical form for regular pencils under this transformation.

PROPOSITION 3.1. *Let $\lambda E - Q \in \mathbb{F}^{n,n}[\lambda]$ be a regular pencil. Then we have $E^\star Q = Q^\star E$ if and only if there exist a unitary (in the case $\mathbb{F} = \mathbb{R}$ real orthogonal) matrix $U \in \mathbb{F}^{n,n}$ and an invertible $X \in \mathbb{F}^{n,n}$ such that UEX and UQX are diagonal and real and satisfy*

$$(UEX)^2 + (UQX)^2 = I_n.$$

Moreover, U and X can be chosen such that one of the matrices UEX or UQX is nonnegative. If, in addition, we have $E^\star Q \geq 0$, then U and X can be chosen such that UEX and UQX are both nonnegative.

Proof. Suppose that $E^\star Q = Q^\star E$ holds. Then the regularity of the pencil $\lambda E - Q$ implies that the columns of the matrix

$$W := \begin{bmatrix} E \\ Q \end{bmatrix} \tag{3.3}$$

are linearly independent and they span an n-dimensional *Lagrangian subspace*, i.e., a subspace $\mathcal{U} \subset \mathbb{F}^{2n}$ such that

$$u^T \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} v = 0$$

for all $u, v \in \mathcal{U}$. Orthonormalizing the columns of W, i.e., carrying out a factorization, $W = ZT$ with $T \in \mathbb{F}^{n,n}$ invertible, and

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}, \; Z_1, Z_2 \in \mathbb{F}^{n,n}$$

satisfying $Z^\star Z = I_n$, we obtain that the columns of $Z = WT^{-1}$ span the same Lagrangian subspace as the columns of W, and according to the CS decomposition (see, e.g., [28, Theorem 2.1] for the complex case and [26, Lemma 2.1] for the real case), there exists $U, V \in \mathbb{F}^{n,n}$ unitary and such that

$$\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} E \\ Q \end{bmatrix} T^{-1} V = \begin{bmatrix} C \\ S \end{bmatrix}$$

with C and S being diagonal such that $C^2 + S^2 = I_n$. The claim follows now by taking $X := T^{-1}V$. The converse implication is straightforward.

Clearly, by premultiplying U with a diagonal signature matrix that switches the signs of diagonal entries if necessary then yields the desired nonnegativity of one of the matrices UEX or UQX. If, in addition, we have $E^\star Q \geq 0$, then it follows that

$$(UEX)(UQX) = (UEX)^\star (UQX) = X^\star E^\star Q X \geq 0,$$

and thus nonnegativity of one of the diagonal matrices UEX or UQX implies nonnegativity of the other. $\square$

Let us remark that regularity of a square pencil $\lambda E - Q$ satisfying $E^\star Q = Q^\star E$ is in fact equivalent to W in (3.3) having linearly independent columns—this fact will become clear after proving Proposition 3.5. For nonsquare pencils this equivalence is not true; see Example 3.4 below.

A direct corollary of Proposition 3.1 gives a characterization of the index of $\lambda E - Q$.

COROLLARY 3.2. *Let $E, Q \in \mathbb{F}^{n,n}$ be such that $E^\star Q = Q^\star E$ and that the pencil $\lambda E - Q$ is regular. Then it is of index at most one.*

Proof. The proof follows directly from Proposition 3.1, since if both E and Q are transformed to be diagonal, then the regularity implies that the nilpotent part in the Kronecker canonical form is diagonal, and hence the index is zero if E is nonsingular and it is one otherwise. ∎

The next result shows that the assumption of regularity in Proposition 3.1 can be dropped if both conditions (1.3) and (3.1) are satisfied. Note that the rectangular case $m \neq n$ is included. By a diagonal matrix $D = \left[\, d_{ij} \,\right]$ we then mean a matrix with all entries d_{ij} equal to zero unless $i = j$.

PROPOSITION 3.3. *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^\star Q = Q^\star E$ and $EQ^\star = QE^\star$. Then there exist unitary (real orthogonal), matrices $U \in \mathbb{F}^{m,m}, V \in \mathbb{F}^{n,n}$ such that UEV and UQV are diagonal and real. Furthermore, U and V can be chosen such that one of the matrices UEV and UQV is nonnegative, and if in addition $E^\star Q \geq 0$ holds, then U, V can be chosen such that UEV and UQV are both nonnegative.*

Proof. The proof follows from the fact that if both conditions $E^\star Q = Q^\star E$ and $EQ^\star = QE^\star$ hold, then this is equivalent to the fact that the four matrices $E^\star Q$, $Q^\star E$, $Q^\star Q$ and $E^\star E$ are all Hermitian and commuting, as well as the four matrices $EQ^\star$, $QE^\star$, $QQ^\star$ and $EE^\star$. Then the statement follows immediately from [24, Corollary 5] in the complex case and [24, Corollary 9] in the real case. ∎

Surprisingly, the statement of Proposition 3.1 is in general not true for singular pencils $\lambda E - Q$ satisfying only $E^\star Q = Q^\star E$ as the following example shows.

Example 3.4. Let

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = S \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ with } S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $E^\star Q$ is symmetric as

$$E^\star Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

but by construction $\lambda E - Q$ has one singular block with left minimal index two in its Kronecker canonical form and thus cannot be equivalent to a pencil in diagonal form.

Although a direct generalization of Proposition 3.1 to the singular case is not possible, it turns out that there is a restriction in the possible Kronecker structure for pencils satisfying (1.3) and (3.2).

PROPOSITION 3.5. *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^\star Q = Q^\star E$. Then all right minimal indices of $\lambda E - Q$ are zero (if there are any).*

Proof. Let $\lambda E - Q = S(\lambda E_0 - Q_0)T$, where $\lambda E_0 - Q_0$ is in Kronecker canonical form. Clearly, with $E^\star Q$ also $E_0^\star S^\star S Q_0$ is Hermitian. Suppose that $\lambda E - Q$ has a right minimal index which is nonzero. Then $\lambda E_0 - Q_0$ contains a block of the form

$$\lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \tag{3.4}$$

of size $k \times (k+1)$ with $k > 0$. Without loss of generality let this block be the first block in the Kronecker canonical form $\lambda E_0 - Q_0$. Then the kth and $(k+1)$st standard basis vectors $x := e_k$ and $y := e_{k+1}$ satisfy $Q_0 y = E_0 x \neq 0$ and $E_0 y = 0$. This, however, implies that

$$y^\star Q_0^\star S^\star S E_0 x > 0 = y^\star E_0^\star S^\star S Q_0 x,$$

contradicting the fact that $E_0^\star S^\star S Q_0$ is Hermitian. ∎

Clearly, by applying Proposition 3.5 to the pencil $\lambda E^\star - Q^\star$ we get that $EQ^\star = QE^\star$ implies all left minimal indices of $\lambda E - Q$ being zero. Another immediate consequence of Proposition 3.5 is the following corollary.

COROLLARY 3.6. *Let $E, Q \in \mathbb{F}^{n,n}$ be such that $E^\star Q = Q^\star E$. If $\lambda E - Q$ is singular; then E and Q have a common nullspace.*

Proof. The proof follows immediately from Proposition 3.5, the fact that a square singular pencil must have both left and right minimal indices, and the fact that a right minimal index zero corresponds to a common nullspace. ∎

With the help of these intermediate results, we are now able to derive a Kronecker-like condensed form for pencils $\lambda E - Q$ satisfying $E^\star Q = Q^\star E$.

THEOREM 3.7. *Let $E, Q \in \mathbb{F}^{m,n}$ be such that $E^\star Q = Q^\star E$. Then there exist a unitary (real orthogonal) matrix $U \in \mathbb{F}^{m,m}$ and a nonsingular matrix $X \in \mathbb{F}^{n,n}$ such that*

$$UEX = \begin{bmatrix} E_{11} & E_{12} & 0 \\ 0 & E_{22} & 0 \end{bmatrix} \quad \text{and} \quad UQX = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ 0 & Q_{22} & 0 \end{bmatrix}, \tag{3.5}$$

and the following statements hold.

(i) *$E_{11}, Q_{11} \in \mathbb{F}^{n_1,n_1}$ are diagonal, real, and satisfy $E_{11}^2 + Q_{11}^2 = I_{n_1}$. Furthermore, U and X can be chosen such that one of the matrices E_{11} and Q_{11} is nonnegative, and if, in addition, $E^\star Q \geq 0$, then U, X can be chosen such that both E_{11} and Q_{11} are nonnegative.*

(ii) *$Q_{22}, E_{22} \in \mathbb{F}^{m_2,n_2}$, with $m_2 = m - n_1 > n_2$ (or $m_2 = n_2 = 0$), are such that $\lambda E_{22} - Q_{22}$ is singular having left singular blocks (blocks of the form $\mathcal{L}_{\eta_j}^\top$) only.*

(iii) $\operatorname{Ker}(E_{22}) \subseteq \operatorname{Ker}(E_{12})$ *and* $\operatorname{Ker}(Q_{22}) \subseteq \operatorname{Ker}(Q_{12})$.

(iv) *The pencil $\lambda E - Q$ has the same Kronecker canonical form as*

$$\lambda \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & E_{22} & 0 \end{bmatrix} - \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \end{bmatrix}. \tag{3.6}$$

In particular, the regular part of $\lambda E - Q$ is given by $\lambda E_{11} - Q_{11}$, the left minimal indices of $\lambda E - Q$ coincide with those of $\lambda E_{22} - Q_{22}$ and $\lambda E - Q$ has exactly $n - n_1 - n_2$ right minimal indices which are all zero.

Proof. Let S, T be nonsingular matrices such that

$$S(\lambda E - Q)T = \begin{bmatrix} \mathcal{R}(\lambda) & 0 \\ 0 & \mathcal{S}(\lambda) \end{bmatrix} \tag{3.7}$$

is in Kronecker canonical form, where $\mathcal{R}(\lambda) \in \mathbb{F}^{n_1,n_1}$ and $\mathcal{S}(\lambda) \in \mathbb{F}^{m_1,n-n_1}$ are the regular part and singular part, respectively. Without loss of generality, let the singular blocks in $\mathcal{S}(\lambda)$ be ordered in such a way that the left singular blocks (of the form $\mathcal{L}_{\eta_j}^\top$) come first. Then, since by Proposition 3.5 all right minimal indices of $\lambda E - Q$ are zero, we obtain that $\mathcal{S}(\lambda)$ has the form

$$\mathcal{S}(\lambda) = \begin{bmatrix} \lambda E_0 - Q_0 & 0 \end{bmatrix},$$

where $\lambda E_0 - Q_0 \in \mathbb{F}^{m_2,n_2}$ contains all singular blocks $\mathcal{L}_{\eta_j}^\top$. This implies that $m_2 > n_2$ unless there are no blocks $\mathcal{L}_{\eta_j}^\top$; i.e., $m_2 = n_2 = 0$. The zero block is of size $m_2 \times (n - n_1 - n_2)$; i.e., the pencil $\lambda E - Q$ has $n - n_1 - n_2$ right minimal indices all of which are zero. Let

$$S^{-1} = V \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}, \quad \text{with} \quad V^\star V = I$$

be a QR decomposition of S^{-1} partitioned conformably with (3.7). Then, with $\lambda \widetilde{E}_{11} - \widetilde{Q}_{11} := R_{11}\mathcal{R}(\lambda)$ one has

$$V^\star(\lambda E - Q)T = \begin{bmatrix} \lambda \widetilde{E}_{11} - \widetilde{Q}_{11} & R_{12}(\lambda E_0 - Q_0) & 0 \\ 0 & R_{22}(\lambda E_0 - Q_0) & 0 \end{bmatrix}.$$

Furthermore, $E^\star Q = Q^\star E$ implies that $\widetilde{E}_{11}^\star \widetilde{Q}_{11} = \widetilde{Q}_{11}^\star \widetilde{E}_{11}$. Since R, and thus also R_{11} and R_{22} are invertible, it follows that $\lambda \widetilde{E}_{11} - \widetilde{Q}_{11}$ is regular. Thus, by Proposition 3.1 there exists a unitary matrix $U_1 \in \mathbb{F}^{n_1,n_1}$ and a nonsingular matrix $X_1 \in \mathbb{F}^{n_1,n_1}$ such that $E_{11} := U_1 \widetilde{E}_{11} X_1$ and $Q_{11} := U_1 \widetilde{Q}_{11} X_1$ are real diagonal satisfying $E_{11}^2 + Q_{11}^2 = I_{n_1}$, with at least one of the matrices E_1, Q_1 being nonnegative. Setting $U := \operatorname{diag}(U_1, I_{m_2})V^\star$, $X := T \operatorname{diag}(X_1, I_{n-n_1})$, we obtain

$$U(\lambda E - Q)X = \begin{bmatrix} \lambda U_1(\widetilde{E}_{11} - \widetilde{Q}_{11})X_1 & U_1 R_{12}(\lambda E_0 - Q_0) & 0 \\ 0 & R_{22}(\lambda E_0 - Q_0) & 0 \end{bmatrix}, \tag{3.8}$$

which is the desired form (3.5). We also have already shown the first statement of (i) and statement (ii). To finish the proof of (i), note that if $E^\star Q$ is positive semidefinite, then $\widetilde{E}_{11}^\star \widetilde{Q}_{11}$ is positive semidefinite as well, as a principal submatrix of a matrix congruent to $E^\star Q$. Hence, one may apply Proposition 3.1.

Invertibility of R_{22} together with (3.8) implies that statement (iii) holds. To see (iv) note that the equivalence transformation

$$\begin{bmatrix} I & -U_1 R_{12} R_{22}^{-1} \\ 0 & I \end{bmatrix} U(\lambda E - Q)X$$

transforms $\lambda E - Q$ into the pencil in (3.6); see also again (3.8). □

So far we have not seen any restrictions on the left minimal indices of pencils $\lambda E - Q$ satisfying $E^\star Q = Q^\star E$. In fact, the left minimal indices may be arbitrary, even under the additional assumption $E^\star Q \geq 0$ as long as the corresponding singular blocks are of suitable dimensions to be part of the block $\lambda E_{22} - Q_{22}$ in the condensed form of Theorem 3.7. To show this, it would be helpful to obtain a canonical form for pencils $\lambda E - Q$ satisfying $Q^\star E = E^\star Q \geq 0$. The problem in obtaining such a form lies in the fact that left-multiplication with a nonunitary invertible matrix S does not preserve the property of $Q^\star E$ being Hermitian, let alone being Hermitian positive semidefinite. In particular, if $\lambda E_0 - Q_0 = \mathcal{L}_\eta^\top$ is a Kronecker block associated with a left minimal index $\eta > 0$, then it is straightforward to check that $Q_0^\star E_0$ is not a Hermitian matrix. We will therefore investigate the structure of matrices that make the pencil $\lambda E_0 - Q_0$ in Kronecker canonical form Hermitian by left-multiplication.

PROPOSITION 3.8. *Let $S \in \mathbb{F}^{l,k}$, and let $E_1 = SE_0$, $Q_1 = SQ_0$, where*

$$\lambda E_0 - Q_0 = \mathcal{L}_{k-1}^\top = \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} \in \mathbb{C}^{k,k-1}[\lambda]. \tag{3.9}$$

Then the following statements hold.

(i) *We have $E_1^\star Q_1 = Q_1^\star E_1$ if and only if $H = S^\star S$ is a real Hankel matrix, i.e.,*

$$H = \begin{bmatrix} a_0 & a_1 & \dots & a_{k-1} \\ a_1 & \iddots & \iddots & a_k \\ \vdots & \iddots & \iddots & \vdots \\ a_{k-1} & a_k & \dots & a_{2k-1} \end{bmatrix} \in \mathbb{R}^{k,k}. \tag{3.10}$$

(ii) *We have $E_1^\star Q_1 = Q_1^\star E_1 \geq 0$ if and only if $H = S^\star S$ is a real Hankel matrix such that its submatrix $\tilde{H}$ obtained by deleting the first row and last column is positive semidefinite; i.e.,*

$$\tilde{H} = \begin{bmatrix} a_1 & a_2 & \dots & a_{k-1} \\ a_2 & \iddots & \iddots & a_k \\ \vdots & \iddots & \iddots & \vdots \\ a_{k-1} & a_{k+1} & \dots & a_{2k-2} \end{bmatrix} \geq 0. \tag{3.11}$$

Proof. (i) Suppose that $H = S^\star S$ is real and has the form given in (3.10). Then

$$E_1^\star Q_1 = E_0^\star H Q_0 = \tilde{H} = Q_0^\star H E_0 = Q_1^\star E_1, \tag{3.12}$$

where $\tilde{H}$ has the form as in (3.11) (but need not be positive semidefinite). Since this matrix is a real Hankel matrix and thus Hermitian, it follows that $E_1^\star Q_1$ is Hermitian.

Conversely, assume that $E_1^\star Q_1 = Q_1^\star E_1$. Then it follows by comparison of elements in the equation $E_0^\star H Q_0 = Q_0^\star H E_0$ that H is a Hermitian Hankel matrix, or equivalently, a real Hankel matrix.

Statement (ii) of Theorem 3.7 follows immediately from (3.12). ∎

Using these preparations, we now obtain the following result showing that a pencil $\lambda E - Q$ satisfying $Q^\star E = E^\star Q \geq 0$ may have arbitrary left minimal indices as long as they fit the dimension restrictions.

COROLLARY 3.9. *Let $n, m \in \mathbb{N}$, and let $q, \eta_1, \dots, \eta_q$ be arbitrary nonnegative integers satisfying*

$$n - q \leq m, \quad \eta_1 + \cdots + \eta_q \leq n - q.$$

Then there exists a pencil $\lambda E - Q \in \mathbb{F}^{m,n}[\lambda]$ satisfying $Q^\star E = E^\star Q \geq 0$ with precisely q left minimal indices $\eta_1, \dots, \eta_q$.

Proof. First observe that for each $k \geq 1$ there exists a positive definite Hankel matrix H with $\tilde{H} > 0$, where $\tilde{H}$ is defined as in (3.11), when H is as in (3.10). Indeed, let $\xi_1 > \cdots > \xi_k > 0$, and let

$$V = \begin{bmatrix} \xi_1 & \dots & \xi_1^k \\ \vdots & & \vdots \\ \xi_k & \dots & \xi_k^k \end{bmatrix} \in \mathbb{R}^{k,k}$$

be a Vandermonde matrix. Observe that

$$H = V^\top V = \left[\sum_{l=1}^{n} \xi_l^{i+j} \right]_{i,j=1}^{k} \tag{3.13}$$

is a positive definite Hankel matrix. Furthermore, $\tilde{H} > 0$ due to

$$x^\star \tilde{H} x = \sum_{i,j=1}^{k-1} x_i \bar{x}_j \sum_{l=1}^{k} \xi_l^{i+j+1} = \sum_{l=1}^{k} \xi_l \left| \sum_{i=1}^{k-1} x_i \xi_l^i \right|^2 > 0 \quad \text{for } x \in \mathbb{F}^{k-1} \setminus \{0\}.$$

Taking $S = H^{1/2}$ (or S to be the Cholesky factor of H) we get by Proposition 3.8 (ii) a pencil $\lambda E_1 - Q_1$ satisfying $E_1^\star Q_1 \geq 0$ (in fact, we even have $E_1^\star Q_1 > 0$) with left minimal index $k-1$. To obtain a pencil with the desired minimal indices it is enough to take the direct sum of pencils constructed as above for the minimal indices $\eta_1, \ldots, \eta_q$ and if necessary adding to the direct sum an appropriate pencil having no left minimal indices. (It is possible to find such a pencil, because its size is $(n - \eta - q) \times (m - \eta)$ with $\eta = \eta_1 + \cdots + \eta_q$ which means that the number of rows does not exceed the number of columns by the assumption $n - q \leq m$.) ∎

Remark 3.10. Defining an equivalence relation on the set of pencils $\lambda E - Q$ satisfying $E^\star Q = Q^\star E$ via $(\lambda E_1 - Q_1) \sim (\lambda E_2 - Q_2)$ if there exist unitary U and invertible X such that $U(\lambda E_1 - Q_1)X = \lambda E_2 - Q_2$, the question arises what is the canonical form. We have derived this canonical form in the regular case in Proposition 3.1. However, obtaining a canonical form in the singular case turns out to be much more difficult, because even for the case that the pencil only has singular blocks, there seem to be far more invariants under the equivalence relation $\sim$ than those given by the set of minimal indices. Indeed, for any $k > 1$ there is a continuum of nonequivalent (in the sense of the relation $\sim$ defined above) linear pencils $\lambda E - Q \in \mathbb{F}^{k,k-1}[\lambda]$ with $E^\star Q = Q^\star E \geq 0$ having precisely one left minimal index $k-1$. This can be seen in the following way. Suppose that two pencils $\lambda S_1 E_0 - S_1 Q_0$ and $\lambda S_2 E_0 - S_2 Q_0$ as in Proposition 3.8 (ii) are equivalent; i.e., we have $U(\lambda S_1 E_0 - S_1 Q_0)X = \lambda S_2 E_0 - S_2 Q_0$ with unitary U and invertible X. Then by Proposition 3.8 (ii) the matrix $H_i := S_i^\star S_i \in \mathbb{R}^{k,k}$ $(i = 1, 2)$ is a Hankel matrix. The relation $\sim$ implies that

$$X^\star E_0^\star H_1 E_0 X = E_0^\star H_2 E_0, \quad X^\star Q_0^\star H_1 Q_0 X = Q_0^\star H_2 Q_0,$$

and thus the Hermitian pencils $\lambda E_0^\star H_1 E_0 - Q_0^\star H_1 Q_0$, and $\lambda E_0^\star H_2 E_0 - Q_0^\star H_2 Q_0$ are congruent.

Now let $\xi_1 > \cdots > \xi_k > 0$, and let H be the positive definite Hankel matrix defined as in (3.13). Observe that there clearly is a continuum of mutually noncongruent pencils of the form

$$\lambda E_0^\star H E_0 - Q_0^\star H Q_0 = \lambda \left[\sum_{l=1}^{n} \xi_l^{i+j} \right]_{i,j=1}^{k-1} - \left[\sum_{l=1}^{n} \xi_l^{i+j} \right]_{i,j=2}^{k}.$$

One can see this easily, e.g., by investigating the ξ_1 dependence of eigenvalues, keeping $\xi_2, \ldots, \xi_n$ fixed. But this immediately implies that under the equivalence relation $\sim$ there is a continuum of nonequivalent pencils $\lambda E - Q$ with canonical form $\lambda E_0 - Q_0$ that satisfy $E^\star Q = Q^\star E \geq 0$.

In this section we have considered the possible Kronecker canonical forms for possibly singular pencils $\lambda E - Q$ satisfying $E^\star Q = Q^\star E$. In the next section, we will discuss the influence of the Kronecker form of $\lambda E - Q$ on the pencil $P(\lambda)$ from (1.7).

4. Properties of the pencil $P(\lambda) = \lambda E - (J - R)Q$. In this section, we investigate the properties of the pencil $P(\lambda) = \lambda E - (J-R)Q$ from (1.7). For brevity, we will sometimes use the notation $L := J - R$ and thus write our pencil in the form

$$P(\lambda) = \lambda E - LQ, \quad E, Q \in \mathbb{F}^{n,m}, \quad L \in \mathbb{F}^{m,m}, \quad L + L^\star \leq 0, \quad E^\star Q = Q^\star E. \tag{4.1}$$

Note that the decomposition $L = J - R$, with $J = -J^\star$, $R = R^\star$ is unique. Hence, the condition $L + L^\star \leq 0$ is equivalent to the condition $R \geq 0$.

PROPOSITION 4.1. *Suppose that the pencil $P(\lambda)$ in* (4.1) *is regular. Then the pencil $\lambda E - Q$ is regular as well.*

Proof. Suppose that $\lambda E - Q$ is singular. Then by Corollary 3.6 the matrices E and Q have a common right nullspace which implies that E and LQ have a common right nullspace as well, contradicting the hypothesis that $P(\lambda)$ is regular. ◻

In the regular case and with $Q > 0$, it is well-known (see, e.g., [15, 23, 25]) that all finite eigenvalues of the pencil $P(\lambda)$ in (4.1) are in the closed left half complex plane, and those on the imaginary axis are semisimple. We will now investigate if these properties remain valid under the weaker condition $E^\star Q = Q^\star E \geq 0$. Clearly, without this condition, we may have eigenvalues in the right half complex plane as is shown by the scalar example

$$E = \begin{bmatrix} -1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 \end{bmatrix} \quad R = \begin{bmatrix} 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 \end{bmatrix}, \quad L = J - R = \begin{bmatrix} -1 \end{bmatrix},$$

where the corresponding pencil $\lambda E - LQ = 1 - \lambda$ has the eigenvalue $\lambda_0 = 1$. But even if the condition $E^\star Q = Q^\star E \geq 0$ is satisfied we may have eigenvalues in the right half plane if the pencil $\lambda E - Q$ happens to be singular.

Example 4.2. Consider the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

where $a > 0$. Then we have $E^* Q = Q^* E = 0 \geq 0$ and $L + L^\star = 0 \leq 0$, but the pencil

$$P(\lambda) = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$$

has the eigenvalue $a > 0$ in the right half plane.

We highlight that in the previous example, the pencil $\lambda E - Q$ had a left minimal index larger than zero. (We recall that all right minimal indices are automatically zero by Theorem 3.7, but the left minimal indices may be arbitrary up to dimensional restrictions by Corollary 3.9.) In fact, it turns out that under the additional hypothesis that all left minimal indices of $\lambda E - Q$ are zero, the pencil $P(\lambda)$ has all eigenvalues in the closed left half complex plane, and those on the imaginary axis are semisimple with the possible exception of nonsemisimplicity of the eigenvalues zero and infinity. With these preparations we are now able to formulate and prove our main result that does not only give information on the finite eigenvalues, but also restricts the possible index and minimal indices of a pencil as in (4.1).

THEOREM 4.3. *Let $E, Q \in \mathbb{F}^{n,m}$ satisfy $E^\star Q = Q^\star E \geq 0$, and let all left minimal indices of $\lambda E - Q$ be equal to zero (if there are any). Furthermore, let $L \in \mathbb{F}^{m,m}$ be such that we have $R := -\frac{1}{2}(L + L^\star) \geq 0$. Then the following statements hold for the pencil $P(\lambda) = \lambda E - LQ$.*

(i) *If* $\lambda_0 \in \mathbb{C}$ *is an eigenvalue of* $P(\lambda)$, *then* $\mathrm{Re}(\lambda_0) \leq 0$.
(ii) *If* $\omega \in \mathbb{R} \setminus \{0\}$ *and* $\lambda_0 = i\omega$ *is an eigenvalue of* $P(\lambda)$, *then* λ_0 *is semisimple. Moreover, if the columns of* $V \in \mathbb{C}^{m,k}$ *form a basis of a regular deflating subspace of* $P(\lambda)$ *associated with* λ_0, *then* $RQV = 0$.
(iii) *The index of* $P(\lambda)$ *is at most two.*
(iv) *All right minimal indices of* $P(\lambda)$ *are at most one (if there are any).*
(v) *If in addition* $\lambda E - Q$ *is regular, then all left minimal indices of* $P(\lambda)$ *are zero (if there are any).*

Before we give the proof of Theorem 4.3 in the next section, we discuss a few consequences and examples.

Remark 4.4. Part (i) and (ii) of Theorem 4.3 generalize [25, Lemma 3.1] in which these results were proved for the case $E = I$ and $Q > 0$, and they also generalize [15, Theorem 2] where the regular case was covered under the additional assumption that for any eigenvector x associated with a nonzero eigenvalue the condition $Q^\star Ex \neq 0$ is satisfied, but no necessary or sufficient conditions were given when this is the case. Theorem 4.3 now states that this extra assumption is not needed in the regular case (in fact, the proof of Theorem 4.3 shows that this condition is automatically satisfied), and it extends the assertion to the singular case under the additional assumption that the left minimal indices of the pencil $\lambda E - Q$ are all zero.

Example 4.5. We highlight that Theorem 4.3 gives no information about the semisimplicity of the eigenvalue zero of the pencil $P(\lambda)$. Indeed, consider the matrices

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \ D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then with $E = C$ and $Q = D$, the pencil

$$P(\lambda) = \lambda E - (J - R)Q = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies $Q^\star E = D \geq 0$ and has the eigenvalue zero with algebraic multiplicity two but geometric multiplicity one. (We will see in section 6 that sizes of Jordan blocks associated with the eigenvalue zero are also not restricted to be at most two, even under the additional hypothesis that $\lambda E - Q$ is regular.)

Example 4.6. With the matrices from Example 4.5, and $E = D$ and $Q = C$, the pencil

$$P(\lambda) = \lambda E - (J - R)Q = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

satisfies again $Q^\star E = D \geq 0$ and has the eigenvalue ∞ with algebraic multiplicity two but geometric multiplicity one. This shows that the case that the index of the pencil is two may indeed occur even when $\lambda E - Q$ is regular.

Example 4.7. The case that the pencil $P(\lambda)$ as in Theorem 4.3 has a right minimal index equal to one may indeed occur even in the case when $\lambda E - Q$ is regular. For an example, consider the matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \quad LQ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $\lambda E - Q$ is regular, $E^* Q = Q^* E = 0$, and $L + L^* \leq 0$. The pencil $P(\lambda) = \lambda E - LQ$ has one right minimal index equal to one and one left minimal index equal to zero.

Example 4.8. If the pencil $\lambda E - Q$ in Theorem 4.3 is not regular, then not much can be said about the left minimal indices of the pencil $P(\lambda)$. In fact, they can be arbitrarily large as the following example shows. Let

$$E = Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n,n}, \quad R = 0 \in \mathbb{R}^{n,n},$$

and $L = J - R = J$. Then $Q^\star E = E^\star Q = Q^2 \geq 0$, and the pencil $\lambda E - Q$ has one left and right minimal index both being equal to zero. Furthermore, we obtain

$$P(\lambda) = \lambda E - LQ = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ -1 & \lambda & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & \lambda & 0 \\ 0 & \dots & 0 & -1 & 0 \end{bmatrix}.$$

Obviously this pencil has one right minimal index which is zero. Since its normal rank is $n-1$, it has exactly one left minimal index. Observing that $\operatorname{rank} E = \operatorname{rank} P(\lambda_0) = n-1$ for all $\lambda_0 \in \mathbb{C}$, it follows that $P(\lambda)$ does not have finite or infinite eigenvalues, and hence the left minimal index must be $n-1$ which is the largest possible size for a left minimal index of an $n \times n$ singular matrix pencil.

As an application of Theorem 4.3, we obtain the following statement on the Kronecker structure of quadratic matrix polynomials that correspond to damped mechanical systems.

COROLLARY 4.9. *Let $S(\lambda) := \lambda^2 M + \lambda D + K \in \mathbb{F}^{n,n}[\lambda]$ be a quadratic matrix polynomial such that M, D, K are all Hermitian and positive semidefinite matrices. Then the following statements hold.*

(i) *All eigenvalues of $S(\lambda)$ are in the closed left half complex plane and all finite nonzero eigenvalues on the imaginary axis are semisimple.*
(ii) *The possible length of Jordan chains of $S(\lambda)$ associated either with the eigenvalue ∞ or with the eigenvalue zero is at most two.*
(iii) *All left and all right minimal indices of $S(\lambda)$ are zero (if there are any).*

Proof. The proof of (i) and the statement in (ii) on the length of the Jordan chains associated with the eigenvalue ∞ follows immediately from the fact that the companion linearization

$$\begin{aligned} P(\lambda) &= \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \\ &= \lambda \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} - \left(\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} - \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix} \end{aligned}$$

satisfies the hypothesis of Theorem 4.3 with

$$E = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad R = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} I & 0 \\ 0 & K \end{bmatrix}.$$

The remaining statement of (ii) then follows by applying the already proved part of (ii) to the reversal polynomial $\lambda^2 K + \lambda D + M$ of $S(\lambda)$.

To see (iii) observe that by Theorem 4.3 the left minimal indices of $P(\lambda)$ are all zero and the right minimal indices of $P(\lambda)$ are at most one. By [6, Theorem 5.10] the left minimal indices of $S(\lambda)$ coincide with those of $P(\lambda)$, and if $\varepsilon_1, \dots, \varepsilon_k$ are the right minimal indices of $S(\lambda)$, then $\varepsilon_1 + 1, \dots, \varepsilon_k + 1$ are the right minimal indices of $P(\lambda)$. This implies that all minimal indices of $S(\lambda)$ are zero (if there are any). ◻

As another application of Theorem 4.3 we obtain a result for rectangular matrix pencils which in the case $E = I$, $Q > 0$ coincides with the well-known Lyapunov stability condition [22] and can thus be seen as a generalization of this stability criterion. In the following $Q^\dagger$ denotes the Moore–Penrose generalized inverse of a matrix Q; see, e.g., [17].

Corollary 4.10. *Let $A, E \in \mathbb{F}^{n,m}$, and assume that there exists $Q \in \mathbb{F}^{n,m}$ such that all minimal indices of the pencil $\lambda E - Q$ are zero (if there are any) and*

$$E^\star Q \geq 0, \quad AQ^\dagger + (Q^\dagger)^\star A^\star \leq 0, \quad \ker Q \subseteq \ker A. \tag{4.2}$$

Then, the statements (i)–(v) *of Theorem* 4.3 *hold for the pencil $P(\lambda) = \lambda E - A \in \mathbb{F}^{n,m}[\lambda]$.*

Proof. We set $L := AQ^\dagger$. The first and second inequality of (4.2) ensure that the assumptions of Theorem 4.3 are satisfied for this E, L, Q. Furthermore, as $\ker Q \subseteq \ker A$ and $Q^\dagger Q$ is the orthogonal projection onto $\operatorname{im} Q^\star \supset \operatorname{im} A^\star$, we have $LQ{=}AQ^\dagger Q{=}A$. Hence, the statements (i)–(v) of Theorem 4.3 hold for the pencil $P(\lambda)$. ◻

In the case that $n = m$ and Q is invertible, the condition $AQ^\dagger + Q^{\dagger\star} A^\star \leq 0$ can be reformulated as $Q^\star A + A^\star Q \leq 0$, and thus we obtain the following simplified statement.

Corollary 4.11. *Let $A, E \in \mathbb{F}^{n,n}$, and assume that there exist an invertible $Q \in \mathbb{F}^{n,n}$ such that $E^\star Q \geq 0$ and $Q^\star A + A^\star Q \leq 0$. Then, the statements* (i)–(v) *of Theorem* 4.3 *hold for the pencil $P(\lambda) = \lambda E - A \in \mathbb{F}^{n,n}[\lambda]$ with the assumption of regularity of $\lambda E - Q$ in* (v) *being automatically satisfied.*

Other generalizations of Lyapunov's theorem with focus on asymptotic stability have been obtained in [20, 33, 34] for the case of square pencils and in [19] for the rectangular case. Corollaries 4.10 and 4.11 give a generalization to the concept of *stability* in the sense of the conditions (i)–(v) in Theorem 4.3.

We also note that parts of Theorem 4.3 have recently been extended to the linear time varying case in [29].

5. Proof of Theorem 4.3.

In this section we will present a proof of our main result. To show the statements (iii)–(v) we will need the following lemma.

Lemma 5.1. *Let $E, A \in \mathbb{F}^{n,m}$, and $k \geq 2$. Assume that one of the following conditions hold:*

(i) *the pencil $\lambda E - A$ has index k;*
(ii) *the pencil $\lambda E - A$ has a right minimal index $k - 1$.*

Then there exists a vector $x_1 \in \mathbb{F}^m \setminus \{0\}$ orthogonal to the common nullspace of E and A such that the following two statements hold:

(a) *There exist $x_2, \dots, x_k \in \mathbb{F}^m \setminus \{0\}$ such that $Ex_1 = 0$, $Ex_2 = Ax_1$, $\dots$, $Ex_k = Ax_{k-1}$, where in addition we have $Ax_k \neq 0$ in case* (i) *and $Ax_k = 0$ in case* (ii).

(b) *For any choice of vectors* $x_2, \ldots, x_k \in \mathbb{F}^m$ *with* $Ex_1 = 0$, $Ex_2 = Ax_1$, $\ldots$, $Ex_k = Ax_{k-1}$ *we have* $Ax_1, \ldots, Ax_{k-1} \neq 0$.

Proof. First, assume that $\lambda E - A$ is in Kronecker canonical form, where without loss of generality the block $\mathcal{N}_k$ associated with a block of size k of the eigenvalue ∞, or the block associated with the right minimal index $k-1$, respectively, comes first, i.e., we have

$$\lambda E - A = \begin{bmatrix} \lambda E_{11} - A_{11} & 0 \\ 0 & \lambda E_{22} - A_{22} \end{bmatrix},$$

where $\lambda E_{11} - A_{11} = \mathcal{N}_k$, or

$$\lambda E_{11} - A_{11} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \tag{5.1}$$

and the form of the pencil $\lambda E_{22} - A_{22}$ is arbitrary; i.e., it may in particular contain other singular blocks or blocks corresponding to eigenvalue ∞. Also, in (5.1) we did on purpose not use the block $\mathcal{L}_{k-1}$ as the representation of a block with right minimal index $k-1$, but its reversal which is well-known to be equivalent to $\mathcal{L}_{k-1}$. In this way, we can treat the proofs for both cases simultaneously, because the block in (5.1) consists exactly of the first $k-1$ rows of the block $\mathcal{N}_k$.

In the following, let $e_i^{(p)}$ denote the ith standard basis vector of $\mathbb{F}^p$. Then $x_1 := e_1^{(m)}$ is orthogonal to the common nullspace of E and A, and it is straightforward to check that the vectors $x_2 = e_2^{(m)}$, $\ldots$, $x_k = e_k^{(m)}$ satisfy (a). On the other hand, if $x_2, \ldots, x_k \in \mathbb{F}^m$ are chosen such that they satisfy $Ex_1 = 0$, $Ex_2 = Ax_1$, $\ldots$, $Ex_k = Ax_{k-1}$, then a straightforward computation shows that $x_2, \ldots, x_k$ must have the form

$$x_2 = \begin{bmatrix} e_2^{(k)} + \alpha_1 e_1^{(k)} \\ x_{22} \end{bmatrix}, \ldots, \; x_k = \begin{bmatrix} e_k^{(k)} + \alpha_1 e_{k-1}^{(k)} + \cdots + \alpha_{k-1} e_1^{(k)} \\ x_{k,2} \end{bmatrix}$$

for appropriate $\alpha_1, \ldots, \alpha_{k-1} \in \mathbb{F}$ and $x_{22}, \ldots, x_{k,2} \in \mathbb{F}^{m-k}$. Then we obtain

$$Ex_j = Ax_{j-1} = \begin{bmatrix} e_{j-1}^{(p)} + \alpha_1 e_{j-2}^{(p)} + \cdots + \alpha_{j-2} e_1^{(p)} \\ A_{22} x_{j-1,2} \end{bmatrix} \neq 0$$

for $j = 2, \ldots, k$ with $p = k$ in case (i) and $p = k-1$ in case (ii) which shows (b). This also proves (a) and (b) for the general case with the possible exception of the orthogonality condition on x_1. However, note that any vector $y \in \mathbb{F}^m$ satisfying $Ey = 0 = Ay$ can be added to x_1, x_2, $\ldots x_k$ without changing any of the identities in (a) and (b). This shows that x_1 satisfying (a) and (b) can be chosen to be orthogonal to the common nullspace of E and A. □

Proof of Theorem 4.3. Since $Q^\star E \geq 0$, we may assume without loss of generality by Theorem 3.7 that E and Q have the form

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.2}$$

respectively, where E_{11}, Q_{11} are real and diagonal (which implies, in particular, that $Q^\star = Q$) and satisfy $E_{11}^2 + Q_{11}^2 = I_{n_1}$. The fact that E_{22} and Q_{22} (and thus also

E_{12} and Q_{12}) in (3.5) are zero follows from the assumption that $\lambda E - Q$ has only left minimal indices equal to zero (if any). To show the items (i)–(v) for the pencil $P(\lambda) = \lambda E - LQ$ we will again frequently use the decomposition $L = J - R$, where $R := -\frac{1}{2}(L + L^\star) \geq 0$ and $J := \frac{1}{2}(L - L^\star)$.

(i) Let $\lambda_0 \in \mathbb{C}$ be an eigenvalue of $P(\lambda)$, and let $v \neq 0$ be a regular eigenvector associated with λ_0. Then we have $\lambda_0 Ev = LQv = (J - R)Qv$, and thus

$$\lambda_0 v^\star Q^\star Ev = v^\star Q^\star JQv - v^\star Q^\star RQv.$$

Considering the real parts of both sides of this equation, we obtain

$$\mathrm{Re}(\lambda_0) \cdot v^\star Q^\star Ev = -v^\star Q^\star RQv,$$

where we used the fact that $Q^\star E$ and R are Hermitian and J is skew-Hermitian. If $Q^\star Ev = 0$, then the special structure of E and Q implies that $v = y_1 + y_2$, where $Ey_1 = 0$ and $Qy_2 = 0$, and thus v can be expressed as a linear combination of vectors from the kernels of E and LQ, respectively, which by Lemma 2.3 leads to a contradiction. Hence, we have $Q^\star Ev \neq 0$, and since $Q^\star E$ is positive semidefinite, we obtain $v^* Q^\star Ev > 0$ which finally implies

$$\mathrm{Re}(\lambda_0) = -\frac{v^* Q^\star RQv}{v^* Q^\star Ev} \leq 0.$$

For (ii) we first prove the "moreover" part. For this, let the columns of $V \in \mathbb{C}^{m,k}$ form a basis of a regular deflating subspace of $P(\lambda)$ associated with the eigenvalue $\lambda_0 = i\omega$, $\omega \in \mathbb{R} \setminus \{0\}$. Then we have to show $RQV = 0$.

By the definition of regular deflating subspaces and from the Kronecker canonical form it follows that there exists a matrix $W \in \mathbb{C}^{m,k}$ with full column rank such that

$$EV = W \quad \text{and} \quad (J - R)QV = WT, \tag{5.3}$$

where $T \in \mathbb{C}^{k,k}$ only has the eigenvalue $i\omega$. Without loss of generality we may assume that $T = i\omega I_k + N$ is in Jordan canonical form (JCF), where N is strictly upper triangular. Otherwise we choose P such that $P^{-1}TP$ is in JCF and set $\widetilde{W} = WP$ and $\widetilde{V} = VP$.

By the second identity in (5.3) we have $V^\star Q^\star (J - R)QV = V^\star Q^\star WT$, and taking the Hermitian part of both sides we obtain

$$0 \geq -2V^\star Q^\star RQV = V^\star Q^\star WT + T^\star W^\star Q^\star V. \tag{5.4}$$

Since R is positive semidefinite, it remains to show that $V^\star Q^\star RQV = 0$, because then we also have $RQV = 0$. For this, we first note that we have

$$V^\star Q^\star W = W^\star Q^\star V > 0. \tag{5.5}$$

Indeed, it follows from the first identity in (5.3) that $V^\star Q^\star W = V^\star Q^\star EV \geq 0$. If there exists $x \neq 0$ such that $Q^\star EVx = 0$, then with $y = Vx$ one has $QEy = 0$. Due to the specific form of Q and E this implies that $y = y_1 + y_2$ with $Ey_1 = 0$ and $Qy_2 = 0$. Hence, as we assumed that $\lambda_0 \neq 0$, one has

$$y \in \mathrm{span}(\ker E \cup \ker(LQ)) \subseteq \mathrm{span}\left(\bigcup_{\lambda \in (\mathbb{C} \cup \{\infty\}) \setminus \{\lambda_0\}} \ker(\lambda E - A) \right).$$

This by Lemma 2.3 contradicts the fact that the columns of V span a regular deflating subspace associated with $\lambda_0 \neq 0$ and finishes the proof of (5.5).

Now let M be the inverse of the Hermitian positive definite square root of the Hermitian positive definite matrix $V^\star Q^\star W = W^\star Q^\star V$. Then it follows from (5.4) that the matrix

$$M(V^\star Q^\star WT + T^\star V^\star QW)M = M^{-1}TM + MT^\star M^{-1}$$

is Hermitian negative semidefinite, because it is congruent to the right-hand side of (5.4). Moreover, we obtain

$$\begin{aligned}\operatorname{trace}(M^{-1}TM + MT^\star M^{-1}) &= \operatorname{trace}(M^{-1}TM) + \operatorname{trace}(MT^* M^{-1})\\ &= \operatorname{trace}(T + T^\star) = \operatorname{trace}(N + N^\star) = 0,\end{aligned}$$

because N has a zero diagonal. But this implies that $M^{-1}TM + MT^\star M^{-1} = 0$ and hence by (5.4) also $-2V^\star QRQV = 0$, which finishes the proof of the "moreover" part.

Next, we will show that $i\omega \neq 0$ is a semisimple eigenvalue. For this, it remains to show that the matrix $T = i\omega I_k + N$ in (5.3) is diagonal; i.e., $N = 0$. Since purely imaginary eigenvalues of the system correspond to eigenvectors of the nondissipative system $(R = 0)$ that are in the kernel of RQ (see [25]) with $RQV = 0$ the identities in (5.3) simplify to $EV = W$ and $JQV = WT$ which implies that

$$V^\star Q^\star JQV = V^\star Q^\star WT.$$

Using again the inverse M of the Hermitian positive definite square root of $V^\star Q^\star W$, we obtain that

$$M^{-1}TM = MV^\star Q^\star WTM = MV^\star Q^\star JQVM,$$

which implies that T is similar to a matrix which is congruent to J; i.e., T is similar to a skew-Hermitian matrix. This, however, immediately implies that we must have $N = 0$. Thus, $i\omega$ is a semisimple eigenvalue of $\lambda E - (J - R)Q$, and assertion (ii) is proved.

For the remaining part of the proof, we will use a slight variation of (5.2). By scaling the nonzero parts of E and Q in (5.2) if necessary, we may assume without loss of generality that E and Q take the forms

$$(5.6)\qquad E = \begin{bmatrix} I_k & 0 & 0 & 0\\ 0 & I_\ell & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \quad\text{and}\quad Q = \begin{bmatrix} Q_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & I_q & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

where $Q_1 \in \mathbb{F}^{k,k}$ is positive definite and both E and Q are partitioned conformably with the partition (k, ℓ, q, p) of n, where $p, q \geq 0$.

To prove (iii), assume that the index ν of $P(\lambda)$ exceeds two. Then by Lemma 5.1 there exists a vector x_1 orthogonal to the common kernel of E and LQ that satisfies the conditions (a) and (b) of Lemma 5.1. In particular, by (a) there exists vectors $x_2, \dots, x_\nu$ such that

(5.7)

$$Ex_1 = 0,\ Ex_2 = LQx_1 \neq 0,\ Ex_3 = LQx_2 \neq 0,\ \dots,\ Ex_\nu = LQx_{\nu-1} \neq 0,\ LQx_\nu \neq 0.$$

Since x_1 is in the kernel of E, it must be of the form $x_1^\top = \begin{bmatrix} 0 & 0 & x_{13}^\top & 0 \end{bmatrix}^\top$, when partitioned conformably with respect to the partition (k, ℓ, q, p) of n. Then, we can apply a simultaneous similarity transformation to E, L, and Q with a unitary transformation matrix of the form $U = I_k \oplus I_\ell \oplus U_q \oplus I_p$, where $U_q x_{13} = \alpha e_q$ for some $\alpha \in \mathbb{R}$, e.g., we can take U_q to be an appropriate Householder matrix. Note that the simultaneous similarity transformation with U preserves all the properties of E, Q, and L. Finally, x_1 can be appropriately scaled such that $\alpha = 1$.

Next, let us extend the partitioning of E, Q, L, J, and R by splitting the third block row and column in two; i.e., the new partitioning is according to the partition $(k, \ell, q-1, 1, p)$ of n. Then we have

$$E = \operatorname{diag}(I_k, I_\ell, 0, 0, 0), \quad Q = \operatorname{diag}(Q_1, 0, I_{q-1}, 1, 0),$$

and we can express $L = J - R$ as

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} & J_{14} - R_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & J_{24} - R_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & J_{34} - R_{34} & L_{35} \\ -J_{14}^\star - R_{14}^\star & L_{42} & L_{43} & J_{44} - R_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & -J_{45}^\star - R_{45}^\star & L_{55} \end{bmatrix}.$$

Observe that

$$LQx_1 = \begin{bmatrix} J_{14} - R_{14} \\ J_{24} - R_{24} \\ J_{34} - R_{34} \\ J_{44} - R_{44} \\ -J_{45}^\star - R_{45}^\star \end{bmatrix} \in \mathbb{F}^n$$

has to be in the range of E in order to satisfy $Ex_2 = LQx_1$ for some x_2. This implies the identities

$$J_{34} = R_{34}, \quad J_{44} = R_{44}, \quad -J_{45}^\star = R_{45}^\star. \tag{5.8}$$

From this we obtain $J_{44} = R_{44} = 0$, because R_{44} is a real scalar and J_{44} is purely imaginary. But then it follows from the positive semidefiniteness of R that $R_{14} = 0$, $R_{24} = 0$, $R_{34} = 0$, and $R_{45} = 0$ which in turn implies that $J_{34} = 0$ and $J_{45} = 0$ by (5.8). Thus, using $L_{43} = -J_{34}^\star - R_{34}^\star = 0$ we find that LQ and LQx_1 take the simplified form

$$LQ = \begin{bmatrix} L_{11}Q_1 & 0 & L_{13} & J_{14} & 0 \\ L_{21}Q_1 & 0 & L_{23} & J_{24} & 0 \\ L_{31}Q_1 & 0 & L_{33} & 0 & 0 \\ -J_{14}^\star Q_1 & 0 & 0 & 0 & 0 \\ L_{51}Q_1 & 0 & L_{53} & 0 & 0 \end{bmatrix} \quad \text{and} \quad LQx_1 = \begin{bmatrix} J_{14} \\ J_{24} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where we used the fact that the second diagonal block in Q is zero. At this point, we see that x_2 has to be of the form $x_2^\top = \begin{bmatrix} J_{14}^\top & J_{24}^\top & x_{23}^\top & x_{24}^\top & x_{25}^\top \end{bmatrix}^\top$ in order to satisfy

$$\begin{bmatrix} J_{14} \\ J_{24} \\ 0 \\ 0 \\ 0 \end{bmatrix} = LQx_1 = Ex_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_2.$$

Since $Ex_3 = LQx_2$, the vector

$$LQx_2 = \begin{bmatrix} * \\ * \\ * \\ -J_{14}^\star Q_1 J_{14} \\ * \end{bmatrix}$$

has to be in the range of E, which implies that $J_{14}^\star Q_1 J_{14} = 0$, and hence $J_{14} = 0$ as Q_1 is positive definite. But then the vector $\widetilde{x}_2 := \begin{bmatrix} 0 & J_{24}^\top & 0 & 0 & 0 \end{bmatrix}^\top$ satisfies the identities

$$Ex_1 = 0, \quad E\widetilde{x}_2 = LQx_1, \quad LQ\widetilde{x}_2 = 0,$$

which is a contradiction to part (b) of Lemma 5.1.

To prove (iv) assume that $P(\lambda)$ has a right minimal index $\varepsilon \geq 2$. Then, by Lemma 5.1, there exists a vector x_1, orthogonal to the common nullspace of E and LQ, that satisfies the conditions (a) and (b) of Lemma 5.1. In particular, by (a) there exists vectors satisfying

$$Ex_1 = 0,\ Ex_2 = LQx_1 \neq 0,\ Ex_3 = LQx_2 \neq 0,\ \dots,\ Ex_{\varepsilon+1} = LQx_\varepsilon \neq 0,\ LQx_{\varepsilon+1} = 0.$$

Since only x_1, x_2, x_3 have been used in (iii) to obtain a contradiction, we can follow exactly the lines of (iii) to obtain again a contradiction.

(v) Since $\lambda E - Q$ is regular, we have $p = 0$ in the partitioning (5.6). Assume that $P(\lambda)$ has a left minimal index $\eta > 0$. Then by applying Lemma 5.1 to the pencil $P(\lambda)^\star = \lambda E^\star - (LQ)^\star$, there exists a vector x_1, orthogonal to the common nullspace of $E^\star$ and $(LQ)^\star$, such that the conditions (a) and (b) of Lemma 5.1 are satisfied. In particular, (a) implies the existence of vectors $x_2, \dots, x_{\eta+1}$ such that

$$x_1^\star E = 0, \quad x_2^\star E = x_1^\star LQ \neq 0, \quad \cdots, \quad x_{\eta+1}^\star E = x_\eta^\star LQ \neq 0, \quad x_{\eta+1}^\star LQ = 0.$$

By an argument analogous to the one in the proof of (iii), we may assume without loss of generality that $x_1 = e_n$ is the last standard basis vector of $\mathbb{F}^n$. Next, partitioning E, Q, J, and R conformably with respect to the partition $(k, \ell, q-1, 1)$ of n we get

$$E = \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_\ell & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{q-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ -J_{12}^\star & J_{22} & J_{23} & J_{24} \\ -J_{13}^\star & -J_{23}^\star & J_{33} & J_{34} \\ -J_{14}^* & -J_{24}^* & -J_{34}^* & J_{44} \end{bmatrix}, R = \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{12}^\star & R_{22} & R_{23} & R_{24} \\ R_{13}^\star & R_{23}^\star & R_{33} & R_{34} \\ R_{14}^* & R_{24}^* & R_{34}^* & R_{44} \end{bmatrix}.$$

Then, similarly as in the proof of (iii), we obtain the identity

$$x_2^* E = x_1^* LQ = \begin{bmatrix} (-J_{14}^\star - R_{14}^\star)Q_1 & 0 & -J_{34}^\star - R_{34}^\star & J_{44} - R_{44} \end{bmatrix}$$

which implies that $J_{44} = R_{44} = 0$, $J_{34} = R_{34} = 0$, $R_{24} = 0$, and $R_{14} = 0$. Hence, the matrix LQ takes the simplified form

$$LQ = (J - R)Q = \begin{bmatrix} (J_{11} - R_{11})Q_1 & 0 & J_{13} - R_{13} & J_{14} \\ (-J_{12}^\star - R_{12}^\star)Q_1 & 0 & J_{23} - R_{23} & J_{24} \\ (-J_{13}^\star - R_{13}^\star)Q_1 & 0 & J_{33} - R_{33} & 0 \\ -J_{14}^\star Q_1 & 0 & 0 & 0 \end{bmatrix}.$$

This implies that the vector x_2 must be of the form $x_2^\star = \begin{bmatrix} -J_{14}^\star Q_1 & 0 & x_{23}^\star & x_{24}^\star \end{bmatrix}$ for some $x_{23} \in \mathbb{F}^{q-1}, x_{24} \in \mathbb{F}$ in order to satisfy the identity

$$x_2^\star E = x_1^\star LQ = \begin{bmatrix} -J_{14}^\star Q_1 & 0 & 0 & 0 \end{bmatrix}. \tag{5.9}$$

Let us assume first that $\eta > 1$. Then the vector

$$x_2^\star LQ = \begin{bmatrix} * & * & * & -J_{14}^\star Q_1 J_{14} \end{bmatrix}$$

has to satisfy the identity $x_3^\star E = x_2^\star LQ$, which implies $J_{14}^\star Q_1 J_{14} = 0$ and thus $J_{14} = 0$ as Q_1 is positive definite. But this together with (5.9) implies that $x_1^\star LQ = 0$ which contradicts the fact that x_1 is orthogonal to the common left nullspace of E and LQ.

In the second case that $\eta = 1$ we have a chain $x_1^\star E = 0$, $x_2^\star E = x_1^\star LQ \neq 0$ and $x_2^\star LQ = 0$, which again leads to $J_{14} = 0$ and $x_1^\star LQ = 0$, a contradiction. Thus, $P(\lambda)$ cannot have a left minimal index of size larger than zero which finishes the proof. ∎

6. Removing higher order Jordan blocks of critical eigenvalues. In this section we will concentrate on the case that the pencil

$$P(\lambda) = \lambda E - LQ, \quad E, Q, L \in \mathbb{F}^{n,n} \tag{6.1}$$

with $Q^\star E = E^\star Q \geq 0$ and $L + L^\star \leq 0$ is regular.

We have seen in the previous section that in this case the nonzero eigenvalues on the imaginary axis are semisimple while the eigenvalues 0 and ∞ may have Jordan blocks of size larger than one, with the additional restriction that the maximal size of a Jordan block associated with ∞ is two. If the latter case applies, i.e., the system is of index two, then this leads to difficulties for numerical simulation methods, and typically an index reduction is performed. In [2] it was shown that in the case of pH DAEs with variable coefficients it is possible to perform such an index reduction of the given pH system such that the resulting system of index one is again pH. Since the case with constant coefficients is a special case, the index reduction procedure introduced in section 6 of [2] applies so that we can assume that our pencil $P(\lambda)$ is of index at most one.

If the eigenvalue 0 of $P(\lambda)$ is not semisimple, then this would mean that the associated pH system is unstable, having linear growth of the solution. In the following we will therefore provide a method for checking this property, and we also present a method of perturbing L to make 0 a semisimple eigenvalue, i.e. to stabilize the system via perturbation, while keeping the structure assumptions on the pencil and all other eigenvalues and their Kronecker structures intact. The main ingredient of the method is a condensed form that provides particular information on the Kronecker structure of the eigenvalue zero.

THEOREM 6.1. *Assume that $P(\lambda)$ from* (6.1) *with $E^\star Q = Q^\star E \geq 0$ and $L + L^\star \leq 0$ is regular and of index at most one. Then there exist a partition (n_1, n_2, n_3, n_4) of n, a unitary matrix $\widetilde{U}$, and an invertible matrix $\widetilde{X}$ such that partitioning conformably* *with respect to* (n_1,n_2,n_3,n_4) *we have*

$$\widetilde{Q} := \widetilde{U}Q\widetilde{X} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Q_{41} & Q_{42} & 0 & I \end{bmatrix}, \quad \widetilde{E} := \widetilde{U}E\widetilde{X} = \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ E_{21} & E_{22} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{L} := \widetilde{U}L\widetilde{U}^\star = \left[\ L_{ij}\ \right]_{ij=1}^4, \quad \widetilde{A} := \widetilde{U}LQ\widetilde{X} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} & A_{32} & 0 & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \tag{6.2}$$

where E_{11}, E_{22}, A_{11} *are lower-triangular and invertible, and* A_{44}, $\left[\begin{smallmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{smallmatrix}\right]$ *are invertible.*

Proof. Note that the regularity of $P(\lambda)$ implies the regularity of $\lambda E - Q$ by Proposition 4.1. Hence, by Proposition 3.1, there exist a unitary matrix U and invertible matrix X such that

$$\hat{Q} := UQX = \begin{bmatrix} Q_1 & & \\ & 0 & \\ & & I \end{bmatrix}, \quad \hat{E} := UEX = \begin{bmatrix} E_1 & & \\ & I & \\ & & 0 \end{bmatrix}$$

with E_1 and Q_1 being diagonal and positive. The sizes of the block columns and rows are n_1+n_2, n_3, n_4 in both matrices; later on the first block row and column will be split into blocks of sizes n_1, n_2, respectively. Next, let

$$\hat{L} := ULU^* = \left[\hat{L}_{ij}\right]_{ij=1,\dots,3}, \quad ULQX = \begin{bmatrix} \hat{L}_{11}Q_1 & 0 & \hat{L}_{13} \\ \hat{L}_{21}Q_1 & 0 & \hat{L}_{23} \\ \hat{L}_{31}Q_1 & 0 & \hat{L}_{33} \end{bmatrix}$$

be partitioned conformably with $\hat{Q}$ and $\hat{E}$. Since $P(\lambda)$ is of index one, it follows that $\hat{L}_{33}$ is invertible; see, e.g., [4]. Setting

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\hat{L}_{33}^{-1}\hat{L}_{31}Q_1 & 0 & I \end{bmatrix} \tag{6.3}$$

we get that

$$UEXT = \hat{E}, \quad ULQXT = \begin{bmatrix} (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1 & 0 & \hat{L}_{13} \\ (\hat{L}_{21} - \hat{L}_{23}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1 & 0 & \hat{L}_{23} \\ 0 & 0 & \hat{L}_{33} \end{bmatrix}. \tag{6.4}$$

Observe that

$$Q_1E_1^{-1} > 0, \quad (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31}) + (\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})^\star \le 0. \tag{6.5}$$

Indeed, the first inequality results directly from the form of Q_1 and E_1, and the second follows from $\hat{L} + \hat{L}^\star \le 0$ as

$$\hat{L}_{11} - \hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31} = S^\star\hat{L}\hat{S}, \quad S := \begin{bmatrix} I \\ 0 \\ -\hat{L}_{33}^{-1}\hat{L}_{31} \end{bmatrix}.$$

Hence, we can apply [25, Lemma 3.1] to see that zero is a semisimple eigenvalue of the pencil $\lambda I-(\hat{L}_{11}-\hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1E_1^{-1}$ and thus also a semisimple eigenvalue of the pencil $\lambda E_1-(\hat{L}_{11}-\hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1$, which has a generalized Schur form (see, e.g., [17])

$$W_1(\hat{L}_{11}-\hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1W_2=\begin{bmatrix}A_{11}&0\\A_{21}&0\end{bmatrix},\quad W_1E_1W_2=\begin{bmatrix}E_{11}&0\\E_{21}&E_{22}\end{bmatrix}\tag{6.6}$$

with $A_{11},E_{11}\in\mathbb{F}^{n_1,n_1}$, $E_{22}\in\mathbb{F}^{n_2,n_2}$ being lower-triangular and invertible and W_1,W_2 being unitary. Setting

$$\widetilde{U}=\begin{bmatrix}W_1&&\\&I&\\&&I\end{bmatrix}U,\quad \widetilde{X}=XT\begin{bmatrix}W_2&&\\&I&\\&&I\end{bmatrix}$$

and splitting the first column and row conformably with (n_1,n_2), where $n_2=\dim\ker(\hat{L}_{11}-\hat{L}_{13}\hat{L}_{33}^{-1}\hat{L}_{31})Q_1$, finishes the definition of $\tilde{Q},\tilde{X}$ and (n_1,n_2,n_3,n_4).

The particular form of $\tilde{A}$ and $\tilde{E}$ follows now from (6.3), (6.4), and (6.6). Furthermore,

$$\tilde{Q}=\begin{bmatrix}W_1&&\\&I&\\&&I\end{bmatrix}\hat{Q}T\begin{bmatrix}W_2&&\\&I&\\&&I\end{bmatrix},$$

and hence $\begin{bmatrix}Q_{11}&Q_{12}\\Q_{21}&Q_{22}\end{bmatrix}=W_1Q_1W_2$ is invertible and the form of $\tilde{Q}$ follows. ∎

Let us discuss now some properties of the condensed form (6.2). In what follows L_j (respectively, Q_j) denote for $j=1,2,3,4$ the jth block row (block column) of the matrix $\widetilde{L}$ (matrix $\widetilde{Q}$) in (6.2).

COROLLARY 6.2. *With the same notation and assumptions as in Theorem* 6.1*, the following conditions are equivalent:*

(a) *Zero is a semisimple eigenvalue of $P(\lambda)$;*
(b) $A_{32}=0$*;*
(c) $Q_2^\star L_3^\star=0$.

Proof. (a)⇔(b): As A_{44} is invertible, the pencil $\lambda\widetilde{E}-\widetilde{A}$ is equivalent to the pencil

$$\lambda\begin{bmatrix}E_{11}&0&0&0\\E_{12}&E_{22}&0&0\\0&0&I&0\\0&0&0&0\end{bmatrix}-\begin{bmatrix}A_{11}&0&0&0\\A_{21}&0&0&0\\A_{31}&A_{32}&0&0\\0&0&0&A_{44}\end{bmatrix},$$

which has a semisimple eigenvalue at 0 if and only if the matrix

$$C:=\begin{bmatrix}E_{11}&0&0\\E_{12}&E_{22}&0\\0&0&I\end{bmatrix}^{-1}\begin{bmatrix}A_{11}&0&0\\A_{21}&0&0\\A_{31}&A_{32}&0\end{bmatrix}=\begin{bmatrix}E_{11}^{-1}A_{11}&0&0\\ *&0&0\\A_{31}&A_{32}&0\end{bmatrix}$$

has a semisimple eigenvalue at 0. Since $E_{11}^{-1}A_{11}$ is invertible, we immediately see that semisimplicity of the eigenvalue zero is equivalent to $A_{32}=0$.

(b)⇔(c) follows from $A_{32}=L_3Q_2$. ∎

Following the notation of (6.2) we also obtain the following result.

COROLLARY 6.3. *With the same notation and assumptions as in Theorem* 6.1, *any perturbation* $L + M$ *of* L, *where* $\widetilde{M} := \widetilde{U}M\widetilde{U}^\star$ *is of the form*

$$\widetilde{M} = \begin{bmatrix} 0 & 0 & M_{13} & 0 \\ 0 & 0 & M_{23} & 0 \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & 0 & M_{43} & 0 \end{bmatrix},$$

leaves the pencil $P(\lambda)$ *regular and the Kronecker form invariant, except for possible changes in the number and sizes of the blocks corresponding to the eigenvalue* 0.

Proof. The assertion follows directly by computing

$$\widetilde{A} + \widetilde{M}\widetilde{Q} = \begin{bmatrix} A_{11} & 0 & 0 & A_{14} \\ A_{21} & 0 & 0 & A_{24} \\ A_{31} + W_{31} & A_{32} + W_{32} & 0 & A_{34} + W_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \quad W = \left[\ W_{ij}\ \right]_{i,j=1}^{4} := \widetilde{M}\widetilde{Q}.$$

Due to the form of $\widetilde{E}$ in (6.2), the entries W_{31} and W_{34} do not have any influence on the Kronecker form and regularity of $\lambda\widetilde{E} - \widetilde{A}$, (and thus also $\lambda E - A$), and the entry W_{32} may cause changes to the Kronecker structure of the eigenvalue zero only. ∎

Hence, given a pencil $P(\lambda)$ as in (6.1) with an eigenvalue 0 that is not semisimple, it is desirable to construct a stabilizing perturbation of the form above with $W_{32} = -A_{32}$ as this procedure will make the eigenvalue zero semisimple but leaves all other eigenvalues and their Kronecker structure invariant. To express such a perturbation in terms of L and Q we need some preparations. In the following, for a linear subspace $\mathcal{X} \subseteq \mathbb{F}^n$ the symbol $P_{\mathcal{X}} \in \mathbb{F}^{n,n}$ denotes the orthogonal projection onto $\mathcal{X}$.

LEMMA 6.4. *Let* $C \in \mathbb{F}^{m,l}$, $B \in \mathbb{F}^{n,m}$, *and let* $Z = -P_{\operatorname{im} B^\star}C$. *Then* $B(C + Z) = 0$, *and the matrix* Z *has the smallest Frobenius norm among all matrices* Z_0 *satisfying* $B(C + Z_0) = 0$.

Proof. The proof follows directly via a straightforward calculation from the decomposition $\mathbb{F}^m = \ker B \oplus \operatorname{im} B^\star$. ∎

Theorem 1.1 of [10] gives a characterization of positive semidefinite 2×2 block operators. We present here a matrix version for the reader's convenience.

THEOREM 6.5. *Let* $B \in \mathbb{F}^{k,k}$ *and* $F \in \mathbb{F}^{m,m}$ *be Hermitian positive semidefinite, let* $E \in \mathbb{F}^{m,k}$, *and let*

$$T := \begin{bmatrix} B & E^\star \\ E & F \end{bmatrix}.$$

Then $T \geq 0$ *if and only if there exist a linear mapping* $\Gamma : \operatorname{im} F \to \operatorname{im} B$ *with the operator norm* $\|\Gamma\| \leq 1$ *such that* $E^\star = B^{1/2}\Gamma F^{1/2}$. *In particular, if* $T \geq 0$, *then* $\operatorname{im} E^\star \subseteq \operatorname{im} B^{1/2} = \operatorname{im} B$.

The following lemma will serve as a prerequisite for constructing perturbations of R that preserve semidefiniteness.

LEMMA 6.6. *Let* $B \in \mathbb{F}^{k,k}$ *and* $D \in \mathbb{F}^{m,m}$ *be Hermitian, positive semidefinite, and let* $C \in \mathbb{F}^{m,k}$ *be such that*

$$\begin{bmatrix} B & C^\star \\ C & D \end{bmatrix} \geq 0.$$

Then, for any $Y \in \mathbb{F}^{m,k}$ with $\operatorname{im} Y^\star \subseteq \operatorname{im} B$ we have that

$$T(Y,W) := \begin{bmatrix} B & (C+Y)^\star \\ C+Y & D+W \end{bmatrix} \geq 0$$

for $W := (\alpha^2 + 2\alpha)(D + \|Y\|\, I) + \|Y\|\, I$ with $\alpha = \big\|(B|_{\operatorname{im} B})^{-\frac{1}{2}} Y^\star (D + \|Y\|\, I)^{-\frac{1}{2}}\big\|$. (Here $(B|_{\operatorname{im} B})^{-\frac{1}{2}}$ is the inverse of the principal square root of the invertible part of B.)

Proof. Theorem 6.5 adapted to the current situation states that $T(Y,W)$ is positive semidefinite if and only if there exists a linear map $\Gamma : \operatorname{im}(D+W) \to \operatorname{im} B$ with $\|\Gamma\| \leq 1$ such that

$$(C+Y)^\star = B^{1/2}\ \Gamma\ (D+W)^{1/2}. \tag{6.7}$$

Now let W be as in the assertion, and assume that $Y \neq 0$. The case $Y = 0$ is trivial, because then $W = 0$ as well. Observe that $D \geq 0$ implies that $D + W = (\alpha+1)^2(D + \|Y\|\, I) > 0$. Setting

$$\Gamma := (B|_{\operatorname{im} B})^{-\frac{1}{2}}(C+Y)^\star(D+W)^{-\frac{1}{2}} = (1+\alpha)^{-1}(B|_{\operatorname{im} B})^{-\frac{1}{2}}(C+Y)^\star(D+\|Y\|\, I)^{-\frac{1}{2}},$$

by Theorem 6.5 we have that $\operatorname{im} C^\star \subseteq \operatorname{im} B$ and by the assumption that $\operatorname{im} Y^\star \subseteq \operatorname{im} B$ we have that (6.7) is satisfied. Furthermore, by definition of α, we have that

$$\|\Gamma\| \leq (1+\alpha)^{-1}\left(\big\|(B|_{\operatorname{im} B})^{-\frac{1}{2}} C^\star (D + \|Y\|\, I)^{-\frac{1}{2}}\big\| + \alpha\right).$$

Note that by Theorem 6.5, applied to $T(0, \|Y\|\, I) \geq 0$, there exist $\tilde{\Gamma} : \mathbb{F}^m \to \operatorname{im} B$ with $\|\tilde{\Gamma}\| \leq 1$ such that $C^\star = B^{1/2}\ \tilde{\Gamma}\ (D + \|Y\|\, I)^{1/2}$. Hence,

$$\big\|(B|_{\operatorname{im} B})^{-\frac{1}{2}} C^\star (D + \|Y\|\, I)^{-\frac{1}{2}}\big\| \leq \|\tilde{\Gamma}\| \leq 1,$$

which finishes the proof. ∎

We have now prepared the tools for constructing a stabilizing perturbation of a pencil as in (6.1) by perturbing the matrix L in a structure-preserving manner. Below $\|X\|_F$ stands for the Frobenius norm of the matrix X.

THEOREM 6.7. *Assume that $P(\lambda)$ from (6.1), with $E^\star Q = Q^\star E \geq 0$ and $R = -\frac{1}{2}(L + L^\star) \leq 0$, is a regular pencil of index at most one, and let the decomposition (6.2) be given, and let $R = [R_{ij}]$, $i,j = 1,\dots,4$ be partitioned accordingly. Then, for every $Y := \begin{bmatrix} Y_1 & Y_2 & Y_4 \end{bmatrix} \in \mathbb{F}^{n_3, n_1+n_2+n_4}$ with*

$$\operatorname{im} Y^\star \subseteq \operatorname{im} R_0, \quad R_0 := \begin{bmatrix} R_{11} & R_{12} & R_{14} \\ R_{21} & R_{22} & R_{24} \\ R_{41} & R_{42} & R_{44} \end{bmatrix} \in \mathbb{F}^{n_1+n_2+n_4, n_1+n_2+n_4},$$

there exist $\Delta_R = \Delta_R^\star$ such that $R + \Delta_R \geq 0$ and

$$\|\Delta_R\| \leq \|Y\| + \left(\big\|(R_0|_{\operatorname{im} R_0})^{-1}\big\|\, \|Y\| + 2\big\|(R_0|_{\operatorname{im} R_0})^{-\frac{1}{2}}\big\|\, \|Y\|^{\frac{1}{2}}\right)(\|R_{33}\| + \|Y\|), \tag{6.8}$$

and there exist $\Delta_J = -\Delta_J^\star$ with the Frobenius norm bounded by

$$\|\Delta_J\|_F \leq 2\, \|P_{\operatorname{im} Q_2}(G^\star)\|_F\,, \tag{6.9}$$

where

$$G = \begin{bmatrix} L_{31} - Y_1 & L_{32} - Y_2 & 0 & L_{34} - Y_4 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} Q_{21} \\ Q_{22} \\ 0 \\ Q_{24} \end{bmatrix}$$

such that for every $s, t \in \mathbb{R}$ the pencil $\lambda E - (J + s\Delta_J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure for the nonzero eigenvalues (including infinity) as $P(\lambda)$, and zero is a semisimple eigenvalue of $\lambda E - (J + \Delta_J - R - \Delta_R)Q$.

Proof. Without loss of generality we may assume that $\widetilde{U} = I_n$ in the decomposition (6.2). We interchange the last two block rows and last two block columns, so that

$$R = \begin{bmatrix} R_0 & C^* \\ C & R_{33} \end{bmatrix}, \quad C := \begin{bmatrix} R_{31} & R_{32} & R_{34} \end{bmatrix}.$$

Let $Y = \begin{bmatrix} Y_1 & Y_2 & Y_4 \end{bmatrix}$ with $\operatorname{im} Y^\star \subseteq \operatorname{im} R_0$ and let $B := R_0$, $D = R_{33}$. In this setting we apply Lemma 6.6 getting a matrix W such that $R + \Delta_R \geq 0$, where

$$\Delta_R := \begin{bmatrix} 0 & Y^\star \\ Y & W \end{bmatrix}.$$

To prove (6.8) first note that

$$\|\Delta_R\| \leq \left\| \begin{bmatrix} 0 & 0 \\ 0 & W \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & Y^\star \\ Y & 0 \end{bmatrix} \right\| \leq \|W\| + \|Y\|. \tag{6.10}$$

Then, since

$$\left\|(R_{33} + \|Y\|\, I)^{-\frac{1}{2}}\right\| \leq \left\|(\|Y\|\, I)^{-\frac{1}{2}}\right\| = \|Y\|^{-\frac{1}{2}}$$

and $\alpha \leq \left\|(R_0|_{\operatorname{im} R_0})^{-\frac{1}{2}}\right\| \|Y\|^{\frac{1}{2}}$, where α is as in Lemma 6.6, the proof of (6.8) is complete. Furthermore, by Corollary 6.3 for any $t \in \mathbb{C}$ the pencil $\lambda E - (J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure associated with the nonzero eigenvalues (including infinity).

Now let us define Δ_J to make the eigenvalue zero of the perturbed pencil semisimple. We reverse the interchange of the last two block rows and block columns, so that we are in the original setting of (6.2) and

$$\Delta_R = \Delta_R^\star = \begin{bmatrix} 0 & 0 & Y_1^\star & 0 \\ 0 & 0 & Y_2^\star & 0 \\ Y_1 & Y_2 & W & Y_4 \\ 0 & 0 & Y_4^\star & 0 \end{bmatrix}.$$

We also define the minimum Frobenius norm $Z \in \mathbb{F}^{n,n_3}$ as in Lemma 6.4 applied to $B = Q_2^\star$, $C = L_3^\star - \hat{Y}^\star$, where L_3 and Q_2 are defined as before Corollary 6.2 and $\hat{Y} = \begin{bmatrix} Y_1 & Y_2 & W & Y_4 \end{bmatrix}$, getting

$$Q_2^\star(L_3^\star - \hat{Y}^\star + Z) = 0. \tag{6.11}$$

We partition $Z = \begin{bmatrix} Z_1^\top & Z_2^\top & Z_3^\top & Z_4^\top \end{bmatrix}^\top$ accordingly. Due to the minimality of the Frobenius norm of Z in Lemma 6.4 and $Q_{23} = 0$, we have $Z_3 = 0$ and also $Z = -P_{\operatorname{im} Q_2}(G^\star)$. We define

$$\Delta_J := \begin{bmatrix} 0 & 0 & Z_1 & 0 \\ 0 & 0 & Z_2 & 0 \\ -Z_1^\star & -Z_2^\star & 0 & -Z_4^\star \\ 0 & 0 & Z_4 & 0 \end{bmatrix}$$

and note that (6.9) holds. Now we apply Corollary 6.2 to the perturbed matrix $A+(-\Delta_R+\Delta_J)Q$. By (6.11) condition (c) is satisfied, and hence zero is a semisimple eigenvalue. Again by Corollary 6.3 for any $s,t \in \mathbb{R}$ the pencil $\lambda E - (J + s\Delta_J - R - t\Delta_R)Q$ is regular and has the same Kronecker structure at nonzero eigenvalues (including infinity). ∎

Let us conclude this section with some comments.

- Theorem 6.7 can be applied with $Y = 0$. In this way we get a skew-symmetric perturbation of L.
- Note that due to the zeros appearing in the form (6.2) of $\widetilde{A}$ on has that $L_1Q_2 = 0$, $L_2Q_2 = 0$. Hence, condition (b) of Corollary 6.2 is implied by
$$\ker \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \subseteq \ker L_3.$$
This leads to a bound $\|\Delta_J\|_F \le 2\left\|P_{\mathrm{im}[L_1,L_2]}(G^\star)\right\|_F$, which, in general, is weaker than (6.9); however, it does not use the matrix Q.
- By Theorem 6.5, if $\operatorname{im} Y^\star$ is not contained in $\operatorname{im} R_0$, then the matrix $R+\Delta_R$ is not positive semidefinite.
- For sufficiently small Y with $\operatorname{im}(Y+Y^\star) \subseteq \operatorname{im} R$ the perturbation $\widetilde{\Delta}_R := Y + Y^\star$ preserves positive definiteness as well. Note that $\|\widetilde{\Delta}_R\| = \|Y\|$, while $\|\Delta_R\| = \mathcal{O}(\|Y\|^{1/2})$ as $\|Y\|$ tends to zero. However, note that the condition $\operatorname{im}(Y+Y^\star) \subseteq \operatorname{im} R$ is stronger than $\operatorname{im} Y^\star \subseteq \operatorname{im} R_0$ as it involves R_{33} as well. Furthermore, in applications, Y need not have a small norm.
- The matrix Δ_R constructed in the proof of Theorem 6.7 has the same form as the matrix M from Corollary 6.3. Hence, Δ_R is never positive semidefinite (unless it is zero). Therefore, strengthening the statement from $R + \Delta_R$ positive semidefinite to Δ_R positive semidefinite is in general not possible.
- The case $\Delta_J = 0$ can be considered as well. A sufficient condition for the existence of a symmetric perturbation Δ_L (of the form presented in the theorem) is that $P_{\mathrm{im}\,Q_2}L_3^\star \subseteq \operatorname{im} R_0$. In this case the matrix $Y = -L_3P_{\mathrm{im}\,Q_2}$ satisfies $\operatorname{im} Y^\star \subseteq \operatorname{im} R_0$, and for this particular choice of Y we get $\Delta J = 0$.
- The matrices $\widetilde{U}$ and $\widetilde{X}$, as well as the decomposition (6.2) can be computed efficiently, e.g., using the CS decomposition and the generalized Schur decomposition of `MATLAB`.

Conclusions. We have studied matrix pencils associated with dissipative Hamiltonian descriptor systems. We have derived canonical and condensed forms and characterized spectral properties of such pencils. In particular, we have studied the size of Jordan blocks associated with the eigenvalues 0 and ∞ and the restrictions for the left and right minimal indices. When Kronecker blocks of size larger than one arise for the eigenvalue 0 (i.e., the associated system is not Lyapunov stable) then we have constructed minimal norm structure-preserving perturbations that make the system Lyapunov stable.

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