

MITTAG-LEFFLER STABILITY OF IMPULSIVE DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

BY

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Abstract. In this paper we consider a nonlinear system of impulsive differential equations of fractional order. Applying the definition of Mittag-Leffler stability introduced by Podlubny and his co-authors and the fractional Lyapunov method, we give sufficient conditions for Mittag-Leffler stability and uniform asymptotic stability of the zero solution of the system under consideration.

1. Introduction. Impulsive differential equations represent useful mathematical machinery in modeling many real processes and phenomena. Important impulsive mathematical models have been recently introduced in population dynamics, medicine, population ecology, mechanics, radio engineering, communication security, neural networks, economics, etc.; see [2], [7], [11] and the references therein.

On the other hand, fractional-order models are found to be more adequate than integer-order models in some real world problems. In fact, many systems exhibit the fractional phenomena, such as motions in complex media/environments, random walk of bacteria in fractal substance, the chemotaxi behavior, food seeking of microbes, etc. For examples and details, see [4, 6, 10, 12].

Although the tools of impulsive fractional differential equations are applicable to various fields of study, the investigation of the theory of such equations has only been started quite recently [1, 3, 5]. However, most of the existing studies offered fundamental theory results. The work on the stability theory of such equations is relatively sparse; see [13].

For extending the application of fractional calculus in nonlinear systems, Podlubny and his co-authors propose in [9] the Mittag-Leffler stability and the fractional Lyapunov direct method with a view to enrich the knowledge of both system theory and fractional calculus.

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In this paper, motivated by the above considerations, we study a nonlinear system of impulsive differential equations of fractional order. We apply the Mittag-Leffler stability concept to the system under consideration, and we investigate the effect of the impulses on the stability behavior of the zero solution. The application of continuous Lyapunov functions [8,9] to the investigation of impulsive systems restricts the possibilities of the Lyapunov method. In our paper we use some analogies of the Lyapunov functions which have discontinuities of the first kind [7]. With this research we extend the fractional Lyapunov method to the impulsive fractional-order case. An example is discussed to illustrate the theory.

2. Preliminaries. Let \mathbb{R}^n be the n -dimensional Euclidean space with norm $\|.\|$, Ω be an open set in \mathbb{R}^n containing the origin, and let $\mathbb{R}_+ = [0, \infty)$. Following the notation in [9], we define the fractional integral of order α on the interval $[a, t]$ by

$${}_a\mathcal{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau,$$

where $0 < \alpha < 1$, f is an arbitrary integrable function and $\Gamma(\cdot)$ is the Gamma function. According to [9], for an arbitrary real number p we denote the Riemann-Liouville and Caputo fractional derivatives of order p , respectively, as

$${}_a D_t^p f(t) = \frac{d^{[p]+1}}{dt^{[p]+1}} \left[{}_a \mathcal{D}_t^{-([p]-p+1)} f(t) \right]$$

and

$${}_a^C D_t^p f(t) = {}_a \mathcal{D}_t^{-([p]-p+1)} \left[\frac{d^{[p]+1}}{dt^{[p]+1}} f(t) \right],$$

where $[p]$ stands for the integer part of p ; D and ${}^C D$ denote the *Riemann-Liouville* and *Caputo* fractional derivatives, respectively.

Let $t_0 \in \mathbb{R}_+$. Consider the following system of impulsive fractional differential equations

$$\begin{cases} {}_{t_0} \mathcal{D}_t^\alpha x(t) = f(t, x(t)), & t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \dots, \end{cases} \quad (2.1)$$

where $f : [t_0, \infty) \times \Omega \rightarrow \mathbb{R}^n$, ${}_{t_0} \mathcal{D}_t^\alpha$ denotes either the Caputo or Riemann-Liouville fractional derivative of order α , $0 < \alpha < 1$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $I_k : \Omega \rightarrow \mathbb{R}^n$, $k = 1, 2, \dots$, $t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$.

Let $x_0 \in \Omega$. Denote by $x(t) = x(t; t_0, x_0)$ the solution of system (2.1) with ${}_{t_0} \mathcal{D}_t^\alpha = {}_{t_0}^C D_t^\alpha$, satisfying the initial condition

$$x(t_0^+; t_0, x_0) = x_0. \quad (2.2)$$

In the case when the Riemann-Liouville operator is used, i.e. when ${}_{t_0} \mathcal{D}_t^\alpha = {}_{t_0} D_t^\alpha$, the initial condition (2.2) is given by ${}_{t_0} D_t^{\alpha-1} x(t_0^+; t_0, x_0) = x_0$. We suppose that the functions f and I_k , $k = 1, 2, \dots$, are smooth enough on $[t_0, \infty) \times \Omega$ and Ω , respectively, to guarantee existence, uniqueness and continuability of the solution $x(t) = x(t; t_0, x_0)$ of the initial value problem (IVP) (2.1), (2.2) on the interval $[t_0, \infty)$ for each $x_0 \in \Omega$ and $t \geq t_0$. The solutions $x(t; t_0, x_0)$ are, in general, piecewise continuous functions with

points of discontinuity of the first type at which they are left continuous; that is, at the moments t_k , $k = 1, 2, \dots$, the following relations are satisfied [13]:

$$x(t_k^-) = x(t_k) \quad \text{and} \quad x(t_k^+) = x(t_k) + I_k(x(t_k)).$$

In the next section, we shall use the Mittag-Leffler function which plays an important role in the solution of noninteger order differential equations. The standard Mittag-Leffler function (see [9]) is given as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)},$$

where $\alpha > 0$. It is also common to represent the Mittag-Leffler function in two parameters, α and β , such that

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha > 0$ and $\beta > 0$. For $\beta = 1$, we have $E_\alpha(z) = E_{\alpha,1}(z)$. Also, $E_{1,1}(z) = e^z$. Moreover, the Laplace transform of the Mittag-Leffler function in two parameters is

$$\mathcal{L}[t^{\beta-1} E_{\alpha,\beta}(-\gamma t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha + \gamma}, \quad (\Re(s) > |\gamma|^{\frac{1}{\alpha}}),$$

where t and s are, respectively, the variables in the time domain and Laplace domain, $\Re(s)$ denotes the real part of s , $\gamma \in \mathbb{R}$, and \mathcal{L} stands for the Laplace transform.

We shall introduce definitions of Mittag-Leffler stability of the zero solution of system (2.1) which are analogous to the definitions given in [9].

DEFINITION 2.1. The zero solution $x(t) \equiv 0$ of system (2.1) is said to be:

(a) *Mittag-Leffler stable* if

$$\|x(t; t_0, x_0)\| \leq \{m[x(t_0)]E_\alpha(-\mu(t - t_0)^\alpha)\}^b,$$

where $\alpha \in (0, 1)$, $\mu > 0$, $b > 0$, $m(0) = 0$, $m(x) \geq 0$, and $m(x)$ is locally Lipschitz with respect to $x \in \Omega$ with Lipschitz constant m_0 ;

(b) *globally Mittag-Leffler stable* if (a) holds for $\Omega \equiv \mathbb{R}^n$.

We shall also use the following stability definition.

DEFINITION 2.2. The zero solution $x(t) \equiv 0$ of system (2.1) is said to be:

(a) *stable* if

$$(\forall t_0 \in \mathbb{R}_+)(\forall \varepsilon > 0)(\exists \delta = \delta(t_0, \varepsilon) > 0)$$

$$(\forall x_0 \in \Omega : \|x_0\| < \delta)(\forall t \geq t_0 : \|x(t; t_0, x_0)\| < \varepsilon);$$

(b) *uniformly stable* if the number δ in (a) is independent of $t_0 \in \mathbb{R}_+$;

(c) *attractive* if

$$(\forall t_0 \in \mathbb{R}_+)(\exists \lambda = \lambda(t_0) > 0)(\forall x_0 \in \Omega : \|x_0\| < \lambda)$$

$$\lim_{t \rightarrow \infty} x(t; t_0, x_0) = 0;$$

(d) *equi-attractive* if

$$(\forall t_0 \in \mathbb{R}_+)(\exists \lambda = \lambda(t_0) > 0)(\forall \varepsilon > 0)(\exists T = T(t_0, \varepsilon) > 0)$$

$$(\forall x_0 \in \Omega : \|x_0\| < \lambda)(\forall t \geq t_0 + T : \|x(t; t_0, x_0)\| < \varepsilon);$$

- (e) *uniformly attractive* if the numbers λ and T in (d) are independent of $t_0 \in \mathbb{R}_+$;
- (f) *asymptotically stable* if it is stable and attractive;
- (g) *uniformly asymptotically stable* if it is uniformly stable and uniformly attractive.

REMARK 2.1. Mittag-Leffler stability implies asymptotic stability.

Let $G_k = (t_{k-1}, t_k) \times \Omega$, $k = 1, 2, \dots$ and $G = \bigcup_{k=1}^{\infty} G_k$. In further considerations, we shall use piecewise continuous auxiliary functions [9], which belong to the class $V_0 = \{V : R_+ \times \Omega \rightarrow \mathbb{R}_+ : V \in C[G, \mathbb{R}_+]$, V is locally Lipschitz continuous with respect to its second argument on each of the sets G_k , $V(t_k^-, x) = V(t_k, x)$ and $V(t_k^+, x) = \lim_{\substack{t \rightarrow t_k \\ t > t_k}} V(t, x)$ exists}.

Let the function $V \in V_0$ and $(t, x) \in G$. We define the derivative

$$\dot{V}_{(2.1)}(t, x) = \frac{dV(t, x)}{dt}.$$

In the proof of the main results we shall use the following lemmas:

LEMMA 2.1 ([9]). Let $0 < \beta < 1$, and $M(0) \geq 0$. Then

$${}_0^C D_t^\beta M(t) \leq {}_0 D_t^\beta M(t), \quad t \geq 0.$$

LEMMA 2.2 ([9]). Let $\alpha \geq 0$, and $f(t, x)$ be continuous on $\mathbb{R}_+ \times \Omega$.

Then

$$\|{}_{t_0} \mathcal{D}_t^{-\alpha} f(t, x)\| \leq {}_{t_0} \mathcal{D}_t^{-\alpha} \|f(t, x)\|, \quad (t, x) \in \mathbb{R}_+ \times \Omega.$$

We shall also use the following class of functions:

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(r) \text{ is strictly increasing and } a(0) = 0\}.$$

3. Fractional comparison principle. First we will prove a generalization of the comparison principle in [9].

LEMMA 3.1. Let $x, y \in C[[0, \infty), \Omega]$ and

$${}_0^C D_t^\beta x(t) \leq {}_0^C D_t^\beta y(t), \quad t \geq 0,$$

where $\beta \in (0, 1)$. Then, if $x(0) \leq y(0)$, we have

$$x(t) \leq y(t), \quad t \in [0, \infty).$$

Proof. It follows from ${}_0^C D_t^\beta x(t) \leq {}_0^C D_t^\beta y(t)$ that there exists a nonnegative function $W(t)$ such that

$${}_0^C D_t^\beta x(t) + W(t) = {}_0^C D_t^\beta y(t). \quad (3.1)$$

Taking the Laplace transform of equation (3.1) yields

$$s^\beta X(s) - s^{\beta-1} x(0) + W(s) = s^\beta Y(s) - s^{\beta-1} y(0), \quad (3.2)$$

where $W(s) = \mathcal{L}[W(t)]$. Since $x(0) \leq y(0)$, there exists a constant $W_0 \geq 0$ such that

$$x(0) + W_0 = y(0). \quad (3.3)$$

From (3.2) and (3.3) we have

$$X(s) + s^{-\beta}W(s) + s^{-1}W_0 = Y(s).$$

Applying the inverse Laplace transform to the above equation gives

$$x(t) + {}_0\mathcal{D}_t^{-\beta}W(t) + W_0 = y(t).$$

It follows from $W(t) \geq 0$, $t \geq 0$ and $W_0 \geq 0$ that $x(t) \leq y(t)$ for $t \geq 0$. \square

REMARK 3.1. The fractional comparison principle was proved in [9]. However, the results in [9] hold only for $x(0) = y(0)$.

DEFINITION 3.1. The function $\psi : \Omega \rightarrow \mathbb{R}^n$ is said to be:

- (a) *non-decreasing* in Ω if $\psi(u) \geq \psi(v)$ for $u \geq v$, $u, v \in \Omega$;
- (b) *monotone increasing* in Ω if $\psi(u) > \psi(v)$ for $u > v$ and $\psi(u) \geq \psi(v)$ for $u \geq v$, $u, v \in \Omega$.

The following comparison principle is an impulsive extension of the comparison principle in [8] and [9]. Let $t_0 = 0$, $x(0) = x_0 \in \Omega$, and $y(0) = y_0 \in \Omega$.

THEOREM 3.1. Assume that:

1. The function $f : [0, \infty) \times \Omega \rightarrow \mathbb{R}^n$ is continuous in each of the sets $(t_{k-1}, t_k] \times \Omega$, $k = 1, 2, \dots$ and ${}_0\mathcal{D}_t^\alpha = {}_0^C\mathcal{D}_t^\alpha$.
2. The functions $I_k \in C[\Omega, \mathbb{R}^n]$, $k = 1, 2, \dots$, are nondecreasing in Ω and $(E + I_k) : \Omega \rightarrow \Omega$, $k = 1, 2, \dots$, where E is the identity in Ω .
3. The inequality

$${}_0^C\mathcal{D}_t^\alpha x(t) \leq {}_0^C\mathcal{D}_t^\alpha y(t)$$

is valid for $t \geq 0$ and $t \neq t_k$, $k = 1, 2, \dots$

Then $x_0 \leq y_0$ implies

$$x(t; 0, x_0) \leq y(t; 0, y_0), \quad t \in [0, \infty). \quad (3.4)$$

Proof. Let $x(t) = x(t; 0, x_0)$ be the solution of IVP (2.1), (2.2) with ${}_0\mathcal{D}_t^\alpha = {}_0^C\mathcal{D}_t^\alpha$ existing on $[0, \infty)$. Straightforward calculations show that if we set $u_k(t; t_k, x_k^+)$ for the solution of the equation without impulses ${}_0^C\mathcal{D}_t^\alpha x = f(t, x)$ in the interval $(t_k, t_{k+1}]$, $k = 0, 1, 2, \dots$, for which $x_k^+ = x(t_k^+) = x(t_k) + I_k(x(t_k))$, $k = 1, 2, \dots$ and $x_0^+ = x_0$, then we have that

$$x(t) = \begin{cases} u_0(t; t_0, x_0^+), & t_0 < t \leq t_1, \\ u_1(t; t_1, x_1^+), & t_1 < t \leq t_2, \\ \dots \\ u_k(t; t_k, x_k^+), & t_k < t \leq t_{k+1}, \\ \dots \end{cases}$$

Analogously, the solution $y(t) = y(t; 0, y_0)$ of system (2.1) is defined.

Let $t \in (0, t_1]$. Then, from Lemma 3.1, it follows that

$$x(t; 0, x_0) \leq y(t; 0, y_0);$$

i.e. the inequality (3.4) is valid for $t \in (0, t_1]$.

Suppose that (3.4) is satisfied for $t \in (t_{k-1}, t_k]$, $k > 1$. Then, using the fact that the function I_k is nondecreasing, we obtain

$$\begin{aligned} x(t_k^+; 0, x_0) &= x(t_k; 0, x_0) + I_k(x(t_k; 0, x_0)) \\ &\leq y(t_k; 0, y_0) + I_k(y(t_k; 0, y_0)) = y_k(t_k^+; 0, y_0). \end{aligned}$$

We again apply the comparison Lemma 3.1 in the interval $(t_k, t_{k+1}]$ and obtain

$$x(t; 0, x_0) \leq y(t; 0, y_0);$$

i.e. the inequality (3.4) is valid for $t \in (t_k, t_{k+1}]$.

The proof is completed by induction. \square

In the case when $x(t_0) = x_0 \in \Omega$, and $y(t; t_0, y_0) \equiv y_0$ is a solution of (2.1) for $t \in [t_0, \infty)$, we deduce the following corollary from Theorem 3.1.

COROLLARY 3.1. Assume that conditions 1 and 2 of Theorem 3.1 are fulfilled, and the inequality

$${}_{t_0}^C D_t^\alpha x(t) \leq {}_{t_0}^C D_t^\alpha y_0$$

is valid for $t \geq t_0$ and $t \neq t_k$, $k = 1, 2, \dots$

Then

$$x(t; t_0, x_0) \leq y_0, \quad t \in [t_0, \infty).$$

4. Mittag-Leffler stability. Let $t_0 = 0$. We shall investigate the Mittag-Leffler stability of the zero solution of system (2.1). That is why the following conditions will be used:

- A4.1. $f(t, 0) = 0$, $t \geq 0$.
- A4.2. $I_k(0) = 0$, $k = 1, 2, \dots$

In the proofs of our main theorems in this section we shall use piecewise continuous Lyapunov functions $V : \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}_+$, $V \in V_0$ for which the following condition is true:

- A4.3. $V(t, 0) = 0$, $t \geq 0$.

THEOREM 4.1. Assume that:

- 1. Conditions A4.1 and A4.2 hold.
- 2. Conditions 1 and 2 of Theorem 3.1 hold.
- 3. There exists a function $V \in V_0$ such that A4.3 holds, and

$$\alpha_1 \|x\|^a \leq V(t, x) \leq \alpha_2 \|x\|^{ab}, \quad (t, x) \in [0, \infty) \times \Omega, \quad (4.1)$$

$$V(t^+, x + I_k(x)) \leq V(t, x), \quad x \in \Omega, \quad t = t_k, \quad k = 1, 2, \dots, \quad (4.2)$$

$${}_{t_k}^C D_t^\beta V(t, x) \leq -\alpha_3 \|x\|^{ab}, \quad (t, x) \in G_k, \quad k = 1, 2, \dots, \quad (4.3)$$

where $t \in [0, \infty)$, $\beta \in (0, 1)$, α_1 , α_2 , α_3 , a and b are arbitrary positive constants.

Then the zero solution of system (2.1) is Mittag-Leffler stable. If the assumptions hold globally on \mathbb{R}^n , then the zero solution of system (2.1) is globally Mittag-Leffler stable.

Proof. It follows from (4.1) and (4.3) that

$${}_{t_k}^C D_t^\beta V(t, x(t)) \leq -\frac{\alpha_3}{\alpha_2} V(t, x(t)), \quad t \neq t_k, \quad k = 1, 2, \dots$$

There exists a nonnegative function $W(t)$ satisfying

$${}_{t_k}^C D_t^\beta V(t, x(t)) + W(t) = -\alpha_3 \alpha_2^{-1} V(t, x(t)), \quad t \neq t_k, \quad k = 1, 2, \dots \quad (4.4)$$

Taking the Laplace transform of (4.4) for $t \neq t_k$, $k = 1, 2, \dots$ gives

$$s^\beta V(s) - s^{\beta-1}V(0) + W(s) = -\alpha_3 \alpha_2^{-1} V(s),$$

where $V(0) = V(0, x(0))$ and $V(s) = \mathcal{L}[V(t, x(t))]$. From the last equality we obtain

$$V(s) = \frac{V(0)s^{\beta-1} - W(s)}{s^\beta + \frac{\alpha_3}{\alpha_2}}.$$

It follows from the properties of the function V and from the fractional uniqueness and existence theorem for the continuous case [10] that the unique solution of (4.4) is

$$\begin{aligned} V(t, x(t)) &= V(0, x(0))E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right) \\ &- W(t) * \left[t^{\beta-1}E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)\right], \quad t \neq t_k, \quad k = 1, 2, \dots, \end{aligned}$$

where $*$ denotes the convolution operator. Since both $t^{\beta-1}$ and $E_{\beta,\beta}\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)$ [9] are nonnegative for $t \in (t_{k-1}, t_k)$, $k = 1, 2, \dots$, it follows that for any closed interval contained in $t \in (t_{k-1}, t_k]$,

$$V(t, x(t)) \leq V(0, x(0))E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right).$$

Set $R = V(0, x(0))E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right)$. From condition (4.2) it follows that if $V(t_k, x(t_k)) < R$, then

$$V(t_k^+, x(t_k^+)) = V(t_k, x(t_k) + I_k(x(t_k))) \leq V(t_k, x(t_k)) < R, \quad k = 1, 2, \dots;$$

i.e. $x(t)$ cannot exceed R by jump. Therefore,

$$V(t, x(t)) \leq V(0, x(0))E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right), \quad t \geq 0.$$

From the last inequality and (4.1) we have

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right) \right]^{\frac{1}{\alpha}}, \quad t \geq 0.$$

Let $m = \frac{V(0)}{\alpha_1} = \frac{V(0, x(0))}{\alpha_1} \geq 0$. Then we have

$$\|x(t)\| \leq \left[m E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right) \right]^{\frac{1}{\alpha}}, \quad t \geq 0,$$

where $m = 0$ holds only if $x(0) = 0$. From the properties of the function V it follows that m is Lipschitz with respect to $x(0)$ and $m(0) = 0$, which implies the Mittag-Leffler stability of the zero solution of system (2.1). \square

THEOREM 4.2. If in condition (4.3) of Theorem 4.1 the Caputo fractional derivative ${}_0^C D_t^\beta$ is replaced by the Riemann-Liouville fractional derivative ${}_0 D_t^\beta$, then the zero solution of system (2.1) is Mittag-Leffler stable.

Proof. It follows from Lemma 2.1 and $V(t, x(t)) \geq 0$ that

$${}_0^C D_t^\beta V(t, x(t)) \leq {}_0 D_t^\beta V(t, x(t)), \quad t \neq t_k, \quad k = 1, 2, \dots,$$

which implies (4.3). Following the same proof as for Theorem 4.1 yields

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} E_\beta\left(-\frac{\alpha_3}{\alpha_2}t^\beta\right) \right]^{\frac{1}{\alpha}}, \quad t \geq 0;$$

i.e. the zero solution of system (2.1) is Mittag-Leffler stable. \square

THEOREM 4.3. Assume that for system (2.1) with ${}_0\mathcal{D}_t^\alpha = {}_0D_t^\alpha$:

1. Conditions 1 and 2 of Theorem 4.1 hold.

2. The function $f(t, x)$ is Lipschitz continuous with respect to $x \in \Omega$ with Lipschitz constant $l > 0$.

3. There exists a function $V \in V_0$ such that A4.3 and (4.2) hold, and

$$\alpha_1 \|x\|^a \leq V(t, x) \leq \alpha_2 \|x\|, \quad (t, x) \in [0, \infty) \times \Omega, \quad (4.5)$$

$$\dot{V}_{(2.1)}(t, x) \leq -\alpha_3 \|x\|, \quad (t, x) \in G_k, \quad k = 1, 2, \dots, \quad (4.6)$$

where $t \in [0, \infty)$, a , α_1 , α_2 , α_3 are arbitrary positive constants.

Then the zero solution of system (2.1) is Mittag-Leffler stable.

Proof. It follows from (4.5), (4.6) and Lemma 2.2 that for $t \neq t_k$, $k = 1, 2, \dots$, we have

$$\begin{aligned} {}_0^C D_t^{1-\alpha} V(t, x(t)) &= {}_0 D_t^{-\alpha} \dot{V}_{(2.1)}(t, x(t)) \leq -\alpha_3 ({}_0 D_t^{-\alpha} \|x(t)\|) \\ &\leq -\alpha_3 l^{-1} {}_0 D_t^{-\alpha} \|f(t, x(t))\| \leq -\alpha_3 l^{-1} \|{}_0 D_t^{-\alpha} f(t, x(t))\| = -\alpha_3 l^{-1} \|x(t)\|, \end{aligned}$$

where ${}_0 D_t^{\alpha-1} x(0) = 0$. Following the same proof as for Theorem 4.1 yields

$$\|x(t)\| \leq \left[\frac{V(0)}{\alpha_1} E_{1-\alpha} \left(-\frac{\alpha_3}{\alpha_2} t^{1-\alpha} \right) \right]^{\frac{1}{a}}, \quad t \geq 0;$$

i.e. the zero solution of system (2.1) is Mittag-Leffler stable. \square

5. Uniform asymptotic stability.

THEOREM 5.1. Assume that:

1. Conditions 1 and 2 of Theorem 4.1 hold.

2. There exists a function $V \in V_0$ such that A4.3 and (4.2) hold,

$$\alpha_1 (\|x\|) \leq V(t, x), \quad \alpha_1 \in K, \quad (t, x) \in [t_0, \infty) \times \Omega, \quad (5.1)$$

$${}_0^C D_t^\beta V(t, x) \leq 0, \quad (t, x) \in G_k, \quad k = 1, 2, \dots, \quad (5.2)$$

where $t \in [t_0, \infty)$, $\beta \in (0, 1)$.

Then the zero solution of system (2.1) is stable.

Proof. Let $\varepsilon > 0$. From the properties of the function V , it follows that there exists a constant $\delta = \delta(t_0, \varepsilon) > 0$ such that if $\|x\| \leq \delta$, then

$$\sup_{\|x\| \leq \delta} V(t_0^+, x) < \alpha_1(\varepsilon). \quad (5.3)$$

Let $x_0 \in \Omega$: $\|x_0\| < \delta$, and let $x(t) = x(t; t_0, x_0)$ be the solution of IVP (2.1), (2.2). We shall prove that $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geq t_0$.

Suppose that this is not true. Then, there exist a solution $x(t; t_0, x_0)$ of (2.1) for which $\|x_0\| < \delta$ and $t^* > t_0$, $t_k < t^* \leq t_{k+1}$, for some fixed integer k , such that

$$\|x(t^*)\| \geq \varepsilon \quad \text{and} \quad \|x(t; t_0, x_0)\| < \varepsilon, \quad t \in [t_0, t_k].$$

Then, due to (4.2) and condition 2 of Theorem 3.1, we can find t^0 , $t_k < t^0 \leq t^*$, such that

$$\|x(t^0)\| > \varepsilon \text{ and } x(t^0; t_0, x_0) \in \Omega. \quad (5.4)$$

For $t \in [t_0, t^0]$ it follows from Corollary 3.1 that

$$V(t, x(t; t_0, x_0)) \leq V(t_0^+, x_0), \quad t \in [t_0, t^0]. \quad (5.5)$$

From (5.4), (5.1), (5.5) and (5.3) follow the inequalities

$$\alpha_1(\varepsilon) < \alpha_1(\|x(t^0; t_0, x_0)\|) \leq V(t^0, x(t^0; t_0, x_0)) \leq V(t_0^+, x_0) < \alpha_1(\varepsilon).$$

The contradiction obtained shows that

$$\|x(t; t_0, x_0)\| < \varepsilon$$

for $\|x_0\| < \delta$ and $t \geq t_0$. This implies that the zero solution of system (2.1) is stable. \square

THEOREM 5.2. Let the conditions of Theorem 5.1 hold, and let a function $\alpha_2 \in K$ exist such that

$$V(t, x) \leq \alpha_2(\|x\|), \quad (t, x) \in (t_0, \infty) \times \Omega. \quad (5.6)$$

Then the zero solution of system (2.1) is uniformly stable.

Proof. Let $\varepsilon > 0$ be chosen. Choose $\delta = \delta(\varepsilon) > 0$ so that $\alpha_2(\delta) < \alpha_1(\varepsilon)$.

Let $x_0 \in \Omega$: $\|x_0\| < \delta$ and $x(t) = x(t; t_0, x_0)$ be the solution of problem (2.1), (2.2).

It follows from Corollary 3.1 that

$$\alpha_1(\|x(t; t_0, x_0)\|) \leq V(t, x(t; t_0, x_0)) \leq V(t_0^+, x_0), \quad t \geq t_0.$$

From the above inequalities and (5.6), we get the inequalities

$$\alpha_1(\|x(t; t_0, x_0)\|) \leq V(t_0^+, x_0) \leq \alpha_2(\|x_0\|) < \alpha_2(\delta) < \alpha_1(\varepsilon),$$

from which it follows that $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geq t_0$. This proves the uniform stability of the zero solution of system (2.1). \square

THEOREM 5.3. Assume that:

1. Conditions 1 and 2 of Theorem 4.1 hold.
2. There exists a function $V \in V_0$ such that A4.3 and (4.2) hold, and

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|), \quad \alpha_1, \alpha_2 \in K, \quad (t, x) \in [t_0, \infty) \times \Omega, \quad (5.7)$$

$${}_{t_0}^C D_t^\beta V(t, x) \leq -\alpha_3(\|x\|), \quad \alpha_3 \in K, \quad (t, x) \in G_k, \quad k = 1, 2, \dots, \quad (5.8)$$

where $t \in [t_0, \infty)$, $\beta \in (0, 1)$.

Then the zero solution of system (2.1) is uniformly asymptotically stable.

Proof. 1. Let $H = \text{const} > 0$: $\{x \in \mathbb{R}^n : \|x\| \leq H\} \subset \Omega$.

For any $t \in [t_0, \infty)$ denote

$$V_{t,H}^{-1} = \{x \in \Omega : V(t^+, x) \leq \alpha_1(H)\}.$$

From (5.7), we deduce

$$V_{t,H}^{-1} \subset \{x \in \mathbb{R}^n : \|x\| \leq H\} \subset \Omega.$$

From condition 2 of Theorem 5.3, it follows that for any $t_0 \in \mathbb{R}_+$ and any $x_0 \in V_{t_0, H}^{-1}$ we have $x(t; t_0, x_0) \in V_{t, H}^{-1}$, $t \geq t_0$.

Let $\varepsilon > 0$ be chosen. Choose $\eta = \eta(\varepsilon)$ so that $\alpha_2(\eta) < \alpha_1(\varepsilon)$, and let

$$T > \left[\frac{\alpha_2(H)\Gamma(1+\beta)}{\alpha_3(\eta)} \right]^{1/\beta}.$$

If we assume that for each $t \in [t_0, t_0 + T]$ the inequality $\|x(t; t_0, x_0)\| \geq \eta$ is valid, then from (4.2), (5.8), we get

$$\begin{aligned} V(t, x(t; t_0, x_0)) &\leq V(t_0^+, x_0) \\ -\frac{1}{\Gamma(\beta)} \int_{t_0}^t \alpha_3(\|x(\tau; t_0, x_0)\|)(t-\tau)^{\beta-1} d\tau &\leq \alpha_2(H) - \frac{\alpha_3(\eta)T^\beta}{\Gamma(\beta+1)} < 0, \end{aligned}$$

contradicting (5.7). The contradiction obtained shows that there exists $t^* \in [t_0, t_0 + T]$ such that $\|x(t^*; t_0, x_0)\| < \eta$.

Then from (4.2), (5.7) and (5.8) it follows that for $t \geq t^*$ (hence for any $t \geq t_0 + T$) the following inequalities hold:

$$\begin{aligned} \alpha_1(\|x(t; t_0, x_0)\|) &\leq V(t; x(t; t_0, x_0)) \leq V(t^*, x(t^*; t_0, x_0)) \\ &\leq \alpha_2(\|x(t^*; t_0, x_0)\|) < \alpha_2(\eta) < \alpha_1(\varepsilon). \end{aligned}$$

Therefore, $\|x(t; t_0, x_0)\| < \varepsilon$ for $t \geq t_0 + T$.

2. Let $\lambda = \text{const} > 0$ be such that $\alpha_2(\lambda) < \alpha_1(H)$. Then, if $x_0 \in \Omega : \|x_0\| < \lambda$, (5.7) implies

$$V(t_0^+, x_0) \leq \alpha_2(\|x_0\|) < \alpha_2(\lambda) < \alpha_1(H),$$

which shows that $x_0 \in V_{t_0, H}^{-1}$. From what we proved in item 1, it follows that the zero solution of system (2.1) is uniformly attractive, and since Theorem 5.2 implies that it is uniformly stable, then the solution $x \equiv 0$ is uniformly asymptotically stable. \square

THEOREM 5.4. Let the conditions of Theorem 5.3 hold, except replacing ${}_0^C D_t^\beta$ by ${}_{t_0} D_t^\beta$. Then the zero solution of system (2.1) is uniformly asymptotically stable.

Proof. The proof of Theorem 5.4 is analogous to the proof of Theorem 5.3. It uses the fact that

$${}_0^C D_t^\beta V(t, x(t)) \leq {}_{t_0} D_t^\beta V(t, x(t)), \quad t \neq t_k, \quad k = 1, 2, \dots,$$

which implies (5.8). \square

6. An example.

Consider the impulsive fractional system

$$\begin{cases} {}_0 D_t^\alpha \|x(t)\| = -q\|x(t)\|, & t > 0, t \neq t_k, \\ \Delta x(t_k) = c_k x(t_k), & k = 1, 2, \dots, \end{cases} \quad (6.1)$$

where $x \in \Omega$, $q > 0$, $0 < \alpha < 1$, $-1 < c_k \leq 0$, $k = 1, 2, \dots$, $0 < t_1 < t_2 < \dots < t_k < t_{k+1} < \dots$ and $\lim_{k \rightarrow \infty} t_k = \infty$.

Let $V(t, x) = \|x\|$. Then, for $t \geq 0$ and $t \neq t_k$ from Lemma 2.1 we have

$${}_0^C D_t^\alpha V(t, x(t)) = {}_0^C D_t^\alpha \|x(t)\| \leq {}_0 D_t^\alpha \|x(t)\| = {}_0 D_t^\alpha V(t, x(t)) = -q\|x(t)\|.$$

Also, for $k = 1, 2, \dots$,

$$V(t_k^+, x + c_k x) = \|1 + c_k\| \|x\| \leq V(t_k, x).$$

Thus by Theorem 4.1 for $a = b = \alpha_1 = \alpha_2 = 1$ and $\alpha_3 = q$, the trivial solution of model (6.1) is Mittag-Leffler stable.

REMARK 6.1. Since the zero solution of (6.1) is Mittag-Leffler stable, then it is asymptotically stable.

7. Conclusions. By using the fractional Lyapunov method [8,9], sufficient conditions for Mittag-Leffler stability and uniform asymptotic stability of the zero solution of a nonlinear system of impulsive differential equations of fractional order are obtained. Some known results are improved and generalized. The technique can be applied to studying the effect of impulsive perturbations on the stability behavior of different types of impulsive fractional order models, such as neural networks, biological networks, delay systems, impulsive control systems, etc.

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