Delay-Dependent Robust Stability Criteria for Two Classes of Uncertain Singular Time-Delay Systems

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Delay-Dependent Robust Stability Criteria for Two Classes of Uncertain Singular Time-Delay Systems

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***Abstract*—In this note, the delay-dependent robust stability criteria for two classes of singular time-delay systems with norm-bounded uncertainties are investigated. First, without using model transformation and bounding technique for cross terms, an improved delay-dependent stability criterion for the nominal singular time-delay system is established in terms of strict linear matrix inequalities (LMIs). Then, based on this criterion, the delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems are proposed, which ensure that the systems are regular, impulse free and asymptotically stable for all admissible uncertainties. Numerical examples are proposed to illustrate the less conservatism of the obtained results.**

***Index Terms*—Delay-dependent stability criteria, linear matrix inequality (LMI), norm-bounded uncertainty, singular time-delay systems.**

I. Introduction

Singular time-delay systems, which are also referred to as implicit time-delay systems, descriptor time-delay systems or generalized differential-difference equations, often appear in various engineering systems, including aircraft attitude control, flexible arm control of robots, large-scale electric network control, chemical engineering systems, lossless transmission lines, etc. (see, e.g., [1] and [2]). Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study for such systems is much more complicated than that for standard state-space time-delay systems or singular systems. First, in order to discuss the uniqueness and stability of solutions, the absence of impulsive solutions and the compatibility of the initial condition should be considered. Moreover, because small delays may change the stability of a singular system [3], the robust stability of singular systems with respect to small delays is required. Recently, more and more attention has been paid to the study of such more general class of delay systems, see [4]–[9].

Since 2001, both delay-independent [4], [5] and delay-dependent [6]–[9] stability conditions for singular time-delay systems have been derived by using the time domain method. Generally speaking, delay-dependent conditions are less conservative than the delay-independent ones, especially when the size of delay is small. Moreover, in engineering practice, information on the delay range is generally available. To obtain delay-dependent conditions, many efforts have been made in the literature, among which the model transformation technique and bounding technique on cross product terms are often used. In [7] and [8], the descriptor system transformation method was adopted, which was an equivalent model transformation. While the applying of inequality on bounding some cross terms still led to relative conservatism. The delay-dependent conditions in [9] was derived without using model transformation and bounding technique, while the authors only proposed the criterion for "E-exponential stability" (i.e., the stability of the "slow" variable x_1), for the purpose of investigating the exponential stability of neutral systems. The stability of the "fast" variable x_2 has not been further discussed. Moreover, the number of variables to be determined in [9] was increased greatly since several slack variables were introduced.

In this note, the problem of delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems with norm-bounded uncertainties is investigated. First, an improved delay-dependent stability criterion for the nominal singular time-delay system is established in terms of LMIs without using any model transformation and bounding technique. Then based on this criterion, the delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems are proposed. Three numerical examples are given to show the effectiveness of the presented results.

Notations

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space over the reals and $\mathbb{R}^{n\times m}$ denotes the set of all $n\times m$ real matrices. For real symmetric matrix X, the notation $X \geq 0 (X > 0)$ means that the matrix X is positive-semidefinite (positive-definite), and $\lambda_{\min}(X)(\lambda_{\max}(X))$ denotes the minimum (maximum) eigenvalue of X. $C_{n,\tau} := C([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$, $x_t := x(t+\theta)$, $\theta\in[-\tau,0]$, $t\geq 0$ denotes the function family defined on $[-\tau,0]$ which is generated by n-dimensional real vector valued continuous function $x(t)$, $t\in[-\tau,+\infty)$. Obviously, $x_t\in C_{n,\tau}$. The following norms will be used, $\|\cdot\|$ refers to the Euclidean vector norm or spectral matrix norm, $\|\phi\|_c := \sup_{-\tau\leq t\leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi\in C_{n,\tau}$. The superscript T represents the transpose. The symbol $*$ will be used in some matrix expressions to induce a symmetric structure, for example, $\begin{bmatrix} X & Y \\ * & Z\end{bmatrix} = \begin{bmatrix} X & Y \\ Y^T & Z\end{bmatrix}$.

II. Problem Formulation and Preliminaries

Consider the uncertain singular time-delay system represented by

$$\begin{cases} E\dot{x}(t) = (A+\triangle A)x(t) + (A_\tau + \triangle A_\tau)x(t-\tau) \\ x(t) = \phi(t), \qquad t\in[-\tau,0]\end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $E\in\mathbb{R}^{n\times n}$ and $0<\operatorname{rank} E = p < n$, A and A_τ are known real constant matrices. τ is an unknown constant delay and satisfies $0<\tau\leq\tau_m$, τ_m is known. $\phi(t)\in C_{n,\tau}$ is a compatible vector valued initial function. $\triangle A$ and $\triangle A_\tau$ are unknown time-invariant matrices representing norm-bounded uncertainties, which are assumed to be of the following two forms:

$$\begin{gathered} A) : [\,\triangle A \quad \triangle A_\tau\,] = DF\,[\,E_1 \quad E_\tau\,] \quad F^TF \leq I_j \\ F\in R^{i\times j}\end{gathered} \tag{2a}$$

where D, E_1, E_τ are known real constant matrices and F is an uncertain real constant matrix

$$B) : \|\triangle A\| \leq \rho_1 \quad \|\triangle A_\tau\| \leq \rho_2 \tag{2b}$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known real constants.

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$\triangle A$ and $\triangle A_\tau$ are said to be admissible if (2a) or (2b) is satisfied.

First of all, we will give some definitions and lemmas about the nominal singular time-delay system of (1)

$$\begin{cases} E\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) & \\ x(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{3}$$

For this purpose, the following notations are needed.

- $S_0 := \{\phi(t) \mid \phi(t) \in C_{n,\tau}, \phi(t)$ is the compatible initial function of system (3)$\}$.
- $S := \{\phi(t) \mid \phi(t) \in S_0$, and there exists a uniquely continuous solution of system (3) on $[0, +\infty)$ for $\phi(t)\}$.
- $B(0,\delta) := \{\phi(t) \mid \phi(t) \in C_{n,\tau}, \| \phi \|_c \leq \delta, \delta > 0\}$.

Definition 1 [10]:

1) The pair (E, A) is said to be regular, if $\det(sE - A) \not\equiv 0$.
2) The pair (E, A) is said to be impulse free, if degree $\{\det(sE - A)\} = \text{rank } E$.

Lemma 1 [4]: If the pair (E, A) is regular and impulse free, then for any compatible initial function $\phi(t) \in C_{n,\tau}$, there exists a uniquely continuous solution of system (3) on $[0, +\infty)$ for $\phi(t)$.

That is, if the pair (E, A) is regular and impulse free, then $S = S_0$.

Definition 2: The singular time-delay system (3) is said to be regular and impulse free, if the pair (E, A) is regular and impulse free.

Definition 3:

1) The zero solution of system (3) is said to be stable, if for any $\varepsilon > 0$, there exists a scalar $\delta(\varepsilon) > 0$ such that for any compatible initial function $\phi \in B(0, \delta(\varepsilon)) \cap S$, the solution $x(t)$ of system (3) satisfies $\| x(t) \| \leq \varepsilon$, $t \geq 0$.
2) The zero solution of system (3) is said to be asymptotically stable, if the zero solution of system (3) is stable, and furthermore, there is a $b_0 > 0$ such that for any compatible initial function $\phi \in B(0, b_0) \cap S$, the solution $x(t) \to 0$ as $t \to \infty$.

Definition 4: System (3) is said to be stable (asymptotically stable), if its zero solution is stable (asymptotically stable).

Without loss of generality, we can assume that the matrices in (1) and (3) have the forms:

$$E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$A_\tau = \begin{bmatrix} A_{\tau 11} & A_{\tau 12} \\ A_{\tau 21} & A_{\tau 22} \end{bmatrix} \tag{4a}$$

$$\begin{aligned}
2x^T(t)P \begin{bmatrix} \dot{x}_1(t) \\ 0 \end{bmatrix} &= 2x^T(t)P\Bigg(\begin{bmatrix} A_{11} + A_{\tau 11} \\ A_{21} + A_{\tau 21} \end{bmatrix} x_1(t) + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x_2(t) \\
&\quad - \begin{bmatrix} A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} \int_{t-\tau}^{t} \dot{x}_1(\alpha) d\alpha + \begin{bmatrix} A_{\tau 12} \\ A_{\tau 22} \end{bmatrix} x_2(t-\tau) \Bigg) \\
&= 2x^T(t)P\left(\begin{bmatrix} A_{11} + A_{\tau 11} \\ A_{21} + A_{\tau 21} \end{bmatrix} x_1(t) + \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x_2(t) \right) \\
&\quad + 2x^T(t)\left(\begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} - P \begin{bmatrix} A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} \right) \int_{t-\tau}^{t} \dot{x}_1(\alpha) d\alpha \\
&\quad + 2x^T(t-\tau) \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} \int_{t-\tau}^{t} \dot{x}_1(\alpha) d\alpha \\
&\quad - \left(2x^T(t) \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix} \int_{t-\tau}^{t} \dot{x}_1(\alpha) d\alpha + 2x^T(t-\tau) \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} \int_{t-\tau}^{t} \dot{x}_1(\alpha) d\alpha \right) \\
&\quad + 2x^T(t)P \begin{bmatrix} A_{\tau 12} \\ A_{\tau 22} \end{bmatrix} x_2(t-\tau) \\
&= \frac{1}{\tau} \int_{t-\tau}^{t} \Bigg\{ 2x^T(t) \left(P \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} x_1(t) + P \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} x_2(t) + Y_1 x_1(t) \right) \\
&\quad + 2x^T(t) \Bigg(P \begin{bmatrix} A_{\tau 11} \\ A_{\tau 21} \end{bmatrix} x_1(t-\tau) + P \begin{bmatrix} A_{\tau 12} \\ A_{\tau 22} \end{bmatrix} x_2(t-\tau) \\
&\quad - Y_1 x_1(t-\tau) \Bigg) + 2x^T(t-\tau) W_1 x_1(t) - 2x^T(t-\tau) W_1 x_1(t-\tau) \\
&\quad - 2x^T(t)\tau Y_1 \dot{x}_1(\alpha) - 2x^T(t-\tau)\tau W_1 \dot{x}_1(\alpha) \Bigg\} d\alpha \\
&= \frac{1}{\tau} \int_{t-\tau}^{t} \Big[2x^T(t)(PA + Y)x(t) + 2x^T(t)(PA_\tau - Y + W^T)x(t-\tau) \\
&\quad - 2x^T(t-\tau)Wx(t-\tau) - 2x^T(t)\tau Y_1 \dot{x}_1(\alpha) \\
&\quad - 2x^T(t-\tau)\tau W_1 \dot{x}_1(\alpha) \Big] d\alpha.
\end{aligned} \tag{13}$$

and denote

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}\quad \phi(t)=\begin{bmatrix}\phi_1(t)\\\phi_2(t)\end{bmatrix} \tag{4b}$$

with $A_{11}\in\mathbb{R}^{p\times p}$, $A_{\tau 11}\in\mathbb{R}^{p\times p}$, $x_1(t)\in\mathbb{R}^p$, and $\phi_1(t)\in\mathbb{R}^p$.

Remark 1: If A_{22} is nonsingular in (4a), then system (3) is regular and impulse free. In this case, the compatible initial condition is

$$0=A_{21}\phi_1(0)+A_{22}\phi_2(0)+A_{\tau 21}\phi_1(-\tau)+A_{\tau 22}\phi_2(-\tau). \tag{5}$$

Define the difference operator $\mathcal{D}: C_{n-p,\tau}\to\mathbb{R}^{n-p}$

$$\mathcal{D}(x_{2t})=x_2(t)+A_{22}^{-1}A_{\tau 22}x_2(t-\tau) \tag{6}$$

then we have the following lemma.

Lemma 2 [7]: If the operator $\mathcal{D}$ is stable and there exist positive numbers α,β,γ and a continuous functional $V: C_{n,\tau}\to\mathbb{R}$ such that

$$\beta\|x_1(t)\|^2\le V(x_t)\le\gamma\|x_t\|_c^2$$
$$\dot V(x_t)\le-\alpha\|x(t)\|^2$$

and the function $\overline{V}(t)=V(x_t)$ is absolutely continuous for x_t satisfying (3), then (3) is asymptotically stable.

III. Main Results

In this section, first of all, we will present a new delay-dependent criterion guaranteeing the nominal system (3) to be regular, impulse free and asymptotically stable, which plays a key role in obtaining the delay-dependent robust stability criteria for the uncertain system (1).

Theorem 1: The singular time-delay system (3) is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0<\tau\le\tau_m$, if there exist matrices

$$P=\begin{bmatrix}P_{11}&P_{12}\\0&P_{22}\end{bmatrix}\quad P_{11}>0\ \ Q>0$$
$$Z=\begin{bmatrix}Z_{11}&Z_{12}\\ *&Z_{22}\end{bmatrix}>0 \tag{7a}$$
$$Y=\begin{bmatrix}Y_{11}&0\\Y_{21}&0\end{bmatrix}\quad W=\begin{bmatrix}W_{11}&0\\W_{21}&0\end{bmatrix} \tag{7b}$$
$$Y_1=\begin{bmatrix}Y_{11}\\Y_{21}\end{bmatrix}\quad W_1=\begin{bmatrix}W_{11}\\W_{21}\end{bmatrix} \tag{7c}$$

with appropriate dimensions and $P_{11}\in\mathbb{R}^{p\times p}$, $Z_{11}\in\mathbb{R}^{p\times p}$, $Y_{11}\in\mathbb{R}^{p\times p}$, $W_{11}\in\mathbb{R}^{p\times p}$ satisfying the following LMI:

$$\begin{bmatrix}\Pi & PA_\tau-Y+W^T+\tau_mA^TZA_\tau & -\tau_mY_1\\ * & -Q-W-W^T+\tau_mA_\tau^TZA_\tau & -\tau_mW_1\\ * & * & -\tau_mZ_{11}\end{bmatrix}<0 \tag{8}$$

where $\Pi=PA+A^TP^T+Y+Y^T+Q+\tau_mA^TZA$.

Proof: Substituting (4a) and (7) into (8) and using Schur complement argument, we have

$$\begin{bmatrix}P_{22}A_{22}+A_{22}^TP_{22}^T+Q_{22} & P_{22}A_{\tau 22}\\ * & -Q_{22}\end{bmatrix}<0 \tag{9}$$

which implies that A_{22} is nonsingular. Therefore, system (3) is regular and impulse free. In addition, from (9) and [7, Lemma 2], we conclude that the difference operator $\mathcal{D}$ is stable.

Construct the Lyapunov–Krasovskii functional for system (3) as

$$V(x_t)=x^T(t)PEx(t)+\int_{t-\tau}^t x^T(s)Qx(s)ds$$
$$+\int_{-\tau}^0\int_{t+\beta}^t \dot x_1^T(\alpha)Z_{11}\dot x_1(\alpha)d\alpha d\beta\quad t\ge\tau \tag{10}$$

then we get

$$\begin{aligned}\lambda_{\min}(P_{11})\|x_1(t)\|^2 &\le V(x_t)\\ &\le\Bigg[\tau^2\Big(\|A^TZA\|+2\|A^TZA_\tau\|\\ &\qquad+\|A_\tau^TZA_\tau\|\Big)+\lambda_{\max}(P_{11})\\ &\qquad+\tau\|Q\|\Bigg]\|x_t\|_c^2\\ t&\ge\tau,\end{aligned} \tag{11}$$

here $\|x_t\|_c=\sup_{\theta\in[-2\tau,0]}\|x(t+\theta)\|$. Taking the time derivative of $V(x_t)$ along with the solution of (3) yields

$$\begin{aligned}\dot V(x_t)|_{(3)} &=2x^T(t)P\begin{bmatrix}\dot x_1(t)\\0\end{bmatrix}\\ &\quad+x^T(t)Qx(t)-x^T(t-\tau)Qx(t-\tau)\\ &\quad+\tau\dot x^T(t)E^TZE\dot x(t)\\ &\quad-\int_{t-\tau}^t\dot x_1^T(\alpha)Z_{11}\dot x_1(\alpha)d\alpha\\ &=2x^T(t)P\begin{bmatrix}\dot x_1(t)\\0\end{bmatrix}\\ &\quad+\frac{1}{\tau}\int_{t-\tau}^t\Big[x^T(t)Qx(t)-x^T(t-\tau)Qx(t-\tau)\\ &\qquad+\tau(Ax(t)+A_\tau x(t-\tau))^T\\ &\qquad\times Z(Ax(t)+A_\tau x(t-\tau))\\ &\qquad-\tau\dot x_1^T(\alpha)Z_{11}\dot x_1(\alpha)\Big]d\alpha.\end{aligned} \tag{12}$$

By Newton–Leibniz formula $x_1(t)-x_1(t-\tau)=\int_{t-\tau}^t\dot x_1(\alpha)d\alpha$, we have (13), as shown at the bottom of the previous page. Combing (13) and (12) we obtain:

$$\dot V(x_t)|_{(3)}=\frac{1}{\tau}\int_{t-\tau}^t\xi^T(t,\alpha)\Lambda(\tau)\xi(t,\alpha)d\alpha\quad t\ge\tau \tag{14}$$

where

$$\xi(t,\alpha)=\begin{bmatrix}x^T(t) & x^T(t-\tau) & \dot x_1^T(\alpha)\end{bmatrix}^T \tag{15a}$$

$$\Lambda(\tau) = \begin{bmatrix} \Lambda_{11}(\tau) & PA_\tau - Y + W^T + \tau A^T Z A_\tau & -\tau Y_1 \\ * & -Q - W - W^T + \tau A_\tau^T Z A_\tau & -\tau W_1 \\ * & * & -\tau Z_{11} \end{bmatrix} \tag{15b}$$

$$\Lambda_{11}(\tau) = PA + A^T P^T + Y + Y^T + Q + \tau A^T Z A. \tag{15c}$$

It follows from (8) and Schur complement argument that for any τ satisfying $0 < \tau \leq \tau_m$

$$\begin{aligned} &\begin{bmatrix} PA + A^T P^T + Y + Y^T + Q & PA_\tau - Y + W^T \\ * & -Q - W - W^T \end{bmatrix} \\ &\quad + \tau \begin{bmatrix} A^T & -Y_1 \\ A_\tau^T & -W_1 \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & Z_{11}^{-1} \end{bmatrix} \begin{bmatrix} A & A_\tau \\ -Y_1^T & -W_1^T \end{bmatrix} \\ &\quad \leq \begin{bmatrix} PA + A^T P^T + Y + Y^T + Q & PA_\tau - Y + W^T \\ * & -Q - W - W^T \end{bmatrix} \\ &\qquad + \tau_m \begin{bmatrix} A^T & -Y_1 \\ A_\tau^T & -W_1 \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & Z_{11}^{-1} \end{bmatrix} \begin{bmatrix} A & A_\tau \\ -Y_1^T & -W_1^T \end{bmatrix} \\ &\quad < 0. \end{aligned} \tag{16}$$

Applying Schur complement argument again, for any $0 < \tau \leq \tau_m$, we have $\Lambda(\tau) < 0$. This, together with (14), means

$$\dot{V}(x_t)|_{(3)} \leq -\lambda_{\min}(-\Lambda(\tau))\|x(t)\|^2. \tag{17}$$

In addition, it is easy to see that Lemma 2 still holds for $\|x_t\|_c = \sup_{\theta\in[-2\tau,0]} \|x(t+\theta)\|$. Thus, from the stability of operator $\mathcal{D}$, (11), (17) and using Lemma 2, it follows that system (3) is asymptotically stable for any $\tau : 0 < \tau \leq \tau_m$. It completes the proof. □

Similar to [11, Cor. 1], the following result can be obtained from Theorem 1.

Corollary 1: The singular time-delay system (3) is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0 < \tau \leq \tau_m$, if there exist matrices P, Q, Z, Y, Y_1, Z_{11} of (7) and X such that the following LMIs hold:

$$\begin{bmatrix} \Phi & PA_\tau - Y + \tau_m A^T Z A_\tau \\ * & -Q + \tau_m A_\tau^T Z A_\tau \end{bmatrix} < 0 \tag{18a}$$

$$\begin{bmatrix} X & Y_1 \\ * & Z_{11} \end{bmatrix} \geq 0 \tag{18b}$$

where $\Phi = PA + A^T P^T + Y + Y^T + Q + \tau_m X + \tau_m A^T Z A$.

Remark 2: As shown by [12], when W is set to be zero, Theorem 1 is equivalent to Corollary 1. However, in this case, the number of the variables to be determined in Theorem 1 is fewer than that in Corollary 1, that is, some variables in Corollary 1 is *redundant*. Thus, from the viewpoint of computational complexity, Theorem 1 is more "powerful." While when W is a matrix to be determined of (7), compared with Corollary 1, the stability condition in Theorem 1 is less conservative, which implies that W is not *redundant*. In fact, all of the above will also be shown by Example 1 in Section IV.

Remark 3: Compared with the delay-dependent stability criterion in [7]–[9], it is worth noticing that there are mainly two advantages in our note. One compared with [7] and [8] is the criterion in Theorem 1 is obtained without using any bounding techniques on the related cross product terms, which deduces some conservatism in some sense. The other is LMI (8) involves fewer variables than those in [7]–[9]. When the system considered is of the form (3) and $x(t) \in \mathbb{R}^n$, $x_1(t) \in \mathbb{R}^p$, the number of the variables to be determined in (8) is $2n^2+p^2/2+np+n+p/2$, while the number of variables is $2n^2+3p^2/2+3np+n+3p/2$ in [7], $3n^2+3p^2+3np+n+p$ in [8] and $8n^2+p^2/2-np+n+p/2$ in [9], respectively. Especially in [9], as the result of six slack matrices introduced, the number of the variables has been greatly increased.

Remark 4: The delay-independent asymptotically stability criterion for the singular time-delay system (3) given in [4] and [5] is stated as: The system (3) is regular, impulse free and asymptotically stable if there exist matrices $Q > 0$ and P of (7) satisfying

$$\begin{bmatrix} PA + A^T P^T + Q & PA_\tau \\ * & -Q \end{bmatrix} < 0. \tag{19}$$

Obviously, if matrices Q, P are solutions of (19), let $Y = 0$, $W = 0$, $Z = \varepsilon I$ ($\varepsilon > 0$ is small enough), then the above P, Q, Y, W, Z satisfy the condition (8) in Theorem 1. It evidently shows that the criterion in Theorem 1 is less conservative than that obtained in [4] and [5].

Remark 5: The criterion in Theorem 1 is given for system (3) with nonzero delay. When $\tau_m = 0$, system (3) becomes

$$E\dot{x}(t) = (A + A_\tau)x(t) \tag{20}$$

which is a singular system without time delay. It is well-known that the necessary and sufficient condition guaranteeing the system (20) to be regular, impulse free and asymptotically stable is: There exists a matrix P of (7) such that

$$P(A + A_\tau) + (A + A_\tau)^T P^T < 0. \tag{21}$$

If matrices P, Q, Y, W of (7) are solutions of (8) with $\tau_m = 0$, that is

$$\begin{bmatrix} PA + A^T P^T + Y + Y^T + Q & PA_\tau - Y + W^T \\ * & -Q - W - W^T \end{bmatrix} < 0 \tag{22}$$

pre- and postmultiplying on both sides of (22) by $\begin{bmatrix} I & I \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$, we have

$$\begin{bmatrix} P(A + A_\tau) + (A + A_\tau)^T P^T & PA_\tau - Y - W - Q \\ * & -Q - W - W^T \end{bmatrix} < 0 \tag{23}$$

which implies that (21) holds. So, when $\tau_m = 0$, the criterion in Theorem 1 is still effective.

Remark 6: We have known from [7] that in order to guarantee robust stability of singular systems with respect to small changes of delay, the delay-dependent condition given in Theorem 1 should be delay-independent with respect to delay in the "fast" variable x_2. In fact, it can follow from the form of the Lyapunov-Krasovskii functional (10) and the proof of Theorem 1. The third term of the functional (10) and the applying of Newton–Leibniz formula $x_1(t) - x_1(t-\tau) = \int_{t-\tau}^{t} \dot{x}_1(\alpha)d\alpha$ lead to delay-dependent conditions, we can see that none of which are associated with the variable x_2.

Remark 7: The result of Theorem 1 can be easily extended to the case of multiple delays. In the interests of economy, the result for the latter case is omitted.

Next, we will present the delay-dependent robust stability criteria for the uncertain singular time-delay system (1) via Theorem 1. The following lemma is needed.

Lemma 3 [13]: Given matrices Ω, Γ and Ξ of appropriate dimensions with Ω symmetrical, then $\Omega + \Gamma F \Xi + (\Gamma F \Xi)^T < 0$ for all F satisfying $FF^T \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that $\Omega + \varepsilon \Gamma\Gamma^T + \varepsilon^{-1}\Xi^T\Xi < 0$.

TABLE I
COMPARISON OF DELAY-DEPENDENT STABILITY CONDITIONS OF EXAMPLE 1

Methods	Maximum τ_m allowed	Number of variables
[7]	1.0000	19
[8]	1.1612	24
[9]	1.1547	33
Corollary 1	0.5773	14
Theorem 1 ($W = 0$)	0.5773	11
Theorem 1	1.2011	13

Theorem 2: The singular time-delay system (1) with respect to uncertainty (2a) is regular, impulse free and robustly asymptotically stable if there exist matrices P, Q, Z, Y, W, Y_1, W_1 of (7) and a scalar $\varepsilon > 0$ satisfying (24), as shown at the bottom of the page, with $\Psi = PA + A^T P^T + Y + Y^T + Q + \varepsilon E_1^T E_1$.

Theorem 3: The singular time-delay system (1) with respect to uncertainty (2b) is regular, impulse free and robustly asymptotically stable if there exist matrices P, Q, Z, Y, W, Y_1, W_1 of (7) and scalars $\varepsilon_i > 0$, $i = 1, 2, 3, 4$, satisfying

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & \Upsilon_{14} & P & P & 0 & 0 \\ * & \Upsilon_{22} & \Upsilon_{23} & \Upsilon_{24} & 0 & 0 & 0 & 0 \\ * & * & \Upsilon_{33} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\tau_m Z & 0 & 0 & \tau_m Z & \tau_m Z \\ * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_2 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_3 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_4 I \end{bmatrix} < 0 \quad (25)$$

with

$$\begin{aligned} \Upsilon_{11} &= PA + A^T P^T + Y + Y^T + Q + \rho_1^2(\varepsilon_1 + \varepsilon_3) I \\ \Upsilon_{12} &= PA_\tau - Y + W^T \quad \Upsilon_{13} = -\tau_m Y_1, \quad \Upsilon_{14} = \tau_m A^T Z \\ \Upsilon_{22} &= -Q - W - W^T + \rho_2^2(\varepsilon_2 + \varepsilon_4) I \\ \Upsilon_{23} &= -\tau_m W_1 \quad \Upsilon_{24} = \tau_m A_\tau^T Z, \quad \Upsilon_{33} = -\tau_m Z_{11}. \end{aligned}$$

By Theorem 1 and Lemma 3 and using the idea of generalized quadratic stability, Theorems 2 and 3 can be easily proved. So the proofs are omitted.

IV. EXAMPLES

1) Example 1: Consider the time-delay system studied in [7, Ex. 3] of the form (3) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix} \quad A_\tau = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have known from [7] that this system is asymptotically stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$, where $\tau^* = \arccos 0.5/\sqrt{3/4} \approx 1.2092$. Table I lists the results compared with [7]–[9] and Corollary 1. It can be seen from Table I that the maximum τ_m allowed by using Theorem 1 is the largest with the fewest variables computed. In addition, we can also see that the maximum τ_m allowed by using Theorem 1 ($W = 0$) is identical to that by using Corollary 1 and smaller than that by Theorem 1, with fewer variables computed, which shows that some variables in Corollary 1 are redundant while W is not redundant. All of the above demonstrates the less conservatism of the delay-dependent stability criterion obtained in Theorem 1.

2) Example 2: Consider the partial element equivalent circuit (PEEC) model in Example 1 in [9] with $\delta = 2$. It can be rewritten as the singular time-delay system (1) with respect to uncertainty (2a) where

$$\begin{aligned} L &= 100 \times \begin{bmatrix} \beta & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix} \\ M &= 100 \times \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix} \\ N &= \frac{1}{72} \times \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix} \quad E = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} \\ A &= \begin{bmatrix} L & I_3 \\ 0 & -I_3 \end{bmatrix} \\ A_\tau &= \begin{bmatrix} 0 & 0 \\ M + NL & N \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ I_3 \end{bmatrix} \\ E_1 &= E_\tau = [\, 2I_3 \quad 0 \,]. \end{aligned}$$

By using Theorem 2, the computational results about maximum of τ_m for various β are obtained identical to those given in [9], which are listed in Table II. However, the LMI in Theorem 2 involves significantly fewer variables than those in [9]. For this example, the number of variables computed in [9] is 282, and ours is 103.

3) Example 3: Consider the singular time-delay system (1) with respect to uncertainty (2b) with

$$\begin{aligned} E &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} -0.5 & 0 \\ 0 & 1 \end{bmatrix} \\ A_\tau &= \begin{bmatrix} -1 & 0.1 \\ 1 & 0 \end{bmatrix} \\ &\| \triangle A \| \leq 0.2 \quad \| \triangle A_\tau \| \leq 0.2. \end{aligned}$$

By Theorem 3, it can be confirmed that system (1) is regular, impulse free and robustly asymptotically stable for any constant delay $\tau \leq 0.7076$. Fig. 1 gives the simulation results of x_1 and x_2 when

$$\begin{bmatrix} \Psi & PA_\tau - Y + W^T + \varepsilon E_1^T E_\tau & -\tau_m Y_1 & \tau_m A^T Z & PD \\ * & -Q - W - W^T + \varepsilon E_\tau^T E_\tau & -\tau_m W_1 & \tau_m A_\tau^T Z & 0 \\ * & * & -\tau_m Z_{11} & 0 & 0 \\ * & * & * & -\tau_m Z & \tau_m Z D \\ * & * & * & * & -\varepsilon I \end{bmatrix} < 0 \quad (24)$$

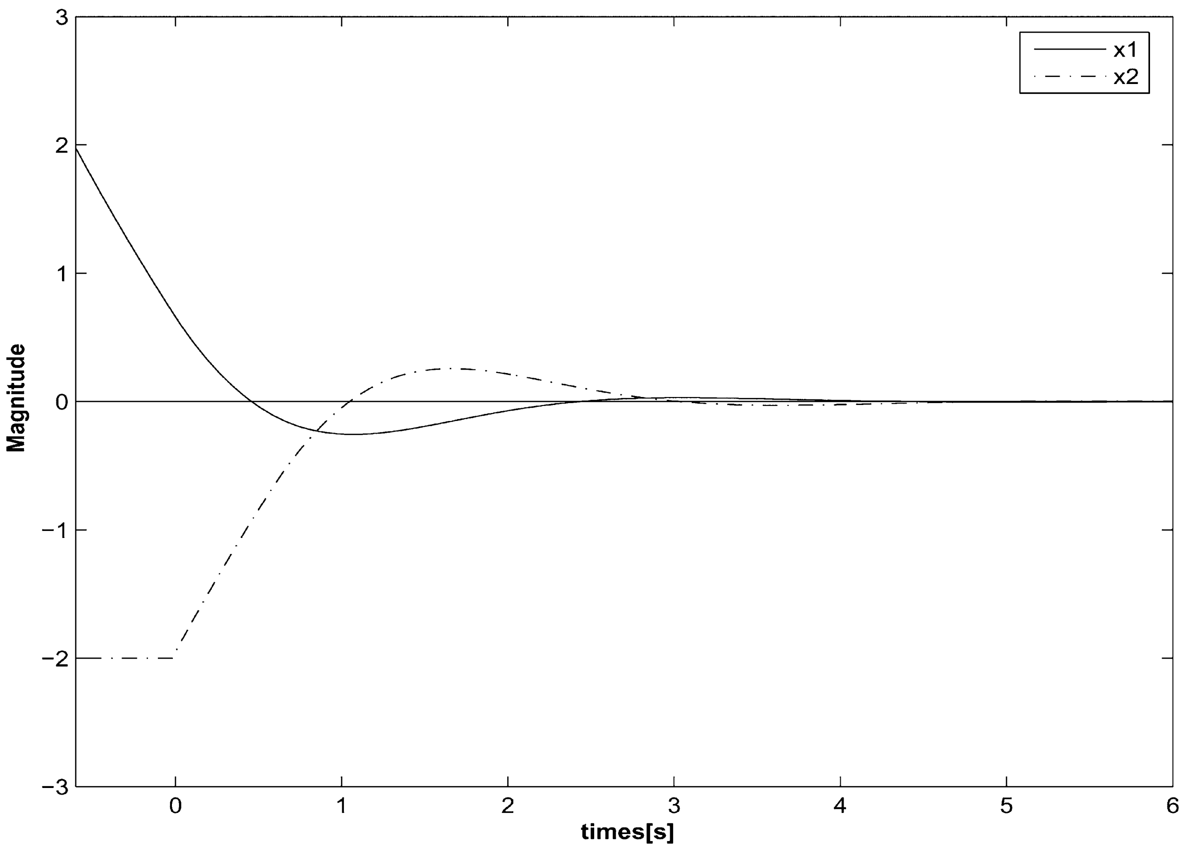

Fig. 1. Simulation results of x_1 and x_2.

TABLE II
MAXIMUM τ_m ALLOWED FOR VARIOUS β OF EXAMPLE 2

β	-2.105	-2.103	-2.1
Maximum τ_m by Theorem 2	0.4064	0.2783	0.2079
Maximum τ_m by [9]	0.4064	0.2783	0.2079

$\triangle A = \triangle A_\tau = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}$, $\tau = 0.6$, and the initial function is $\phi(t) = [2 \quad -2]^T$, $t \in [-\tau, 0]$. From Fig. 1, we can see that the states x_1 and x_2 asymptotically converge to zero.

V. CONCLUSION

In this note, the delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems with norm-bounded uncertainties are investigated. An improved delay-dependent stability criterion is established in terms of Lyapunov technique and strict LMIs, which guarantees the nominal singular time-delay systems to be regular, impulse free and asymptotically stable. The criterion is obtained without using any model transformation and bounding technique. Based on this criterion, the delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems are obtained by taking an idea of generalized quadratic stability. Applying the delay-dependent robust stability criteria proposed here to solve the robust control problem for singular time-delay systems, such as H_∞ control, guaranteed cost control, variable structure control and so on, will be interesting topics for further research.

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REFERENCES

[1] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*. New York: Springer-Verlag, 1993.
[2] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations*. New York: Academic, 1986.
[3] H. Logemann, "Destabilizing effects of small time delays on feedback-controlled descriptor systems," *Linear Alg. Appl.*, vol. 272, pp. 131–153, 1998.
[4] S. Xu, P. Van Dooren, R. Stefan, and J. Lam, "Robust stability and stabilization for singular systems with state delay and parameter uncertainty," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1122–1128, Jul. 2002.
[5] J. Feng, S. Zhu, and Z. Cheng, "Guaranteed cost control of linear uncertain singular time-delay systems," in *Proc. 41st IEEE Conf. Decision Control*, Las Vegas, NV, Dec. 2002, pp. 1802–1807.
[6] E. Fridman, "Effects of small delays on stability of singularly perturbed systems," *Automatica*, vol. 38, pp. 897–902, 2002.
[7] E. Fridman, "Stability of linear descriptor systems with delay: A Lyapunov-based approach," *J. Math. Anal. Appl.*, vol. 273, pp. 24–44, 2002.
[8] E. Fridman and U. Shaked, "H_∞-control of linear state-delay descriptor systems: An LMI approach," *Linear Alg. Appl.*, vol. 351–352, pp. 271–302, 2002.
[9] D. Yue and Q. Han, "A delay-dependent stability criterion of neutral systems and its application to a partial element equivalent circuit model," *IEEE Trans. Circuits Syst. II*, vol. 51, no. 2, pp. 685–689, Dec. 2004.
[10] L. Dai, *Singular Control Systems*. Berlin, Germany: Springer-Verlag, 1989.
[11] S. Xu and J. Lam, "Improved delay-dependent stability criteria for time-delay systems," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 384–387, Mar. 2005.
[12] S. Xu, J. Lam, and Y. Zou, "Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems," *IEE Proc. Control Theory Appl.*, vol. 152, no. 2, pp. 147–151, 2005.
[13] I. R. Peterson, "A stabilization algorithm for a class uncertain linear systems," *Syst. Control Lett.*, vol. 8, pp. 351–357, 1987.