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Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations[☆]

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ABSTRACT

In this paper, we introduce a concept of delayed two parameters Mittag-Leffler type matrix function, which is an extension of the classical Mittag-Leffler matrix function. With the help of the delayed two parameters Mittag-Leffler type matrix function, we give an explicit formula of solutions to linear nonhomogeneous fractional delay differential equations via the variation of constants method. In addition, we prove the existence and uniqueness of solutions to nonlinear fractional delay differential equations. Thereafter, we present finite time stability results of nonlinear fractional delay differential equations under mild conditions on nonlinear term. Finally, an example is presented to illustrate the validity of the main theorems.

1. Introduction

It is well-known that fractional differential equation is an alternative mathematical model corresponding to integer differential equation, which provides an excellent tool for the description of memory and hereditary properties of various materials and processes [1–6]. Numerous research papers on basic theory analysis for fractional differential equations as well as control problems are investigated by many researchers, see for example, [7–30] and the references cited therein.

Finite time stability (FTS) is concerned with the behavior of systems over a finite intervals, which is not like Lyapunov stability, asymptotic stability, and exponential stability that are concerned with the behavior of systems within an infinite time interval. Recently, FTS of fractional differential equations has recently received a lot of attention and now constitutes a significant branch of automatic control. It is remarkable that Lazarević [31] initially investigate the finite time stability of linear fractional delay differential equations wicontrols. Some sufficient conditions to guarantee the FTS are established. Thereafter, FTS of fractional differential equations has been studied in [32–38] with the help of linear matrix inequality and Gronwall's integral inequality.

Recently, there is a quick development on seeking the explicit formula of solution to delay differential/discrete equations via the concept of continuous/discrete delayed exponential matrix [39,40]. One of the most advantages of continuous/discrete delayed exponential matrix is used to transfer the classical idea to represent the solution of linear ODEs into

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linear delay differential/discrete equations. For more continued contributions, one can refer to existence and stability of solutions to several classes of delay differential/discrete equations [41–50] and some relative controllability problems [51–54].

Very recently, Li and Wang [55] initially study the following linear homogenous fractional delay differential equations:

$$\begin{cases} ({}^cD^{\alpha}_{0+}y)(x) = By(x-\tau),\ \ y(x)\in\mathbb{R}^n,\ \ x\in J:=[0,T], & \tau>0,\\ y(x)=\varphi(x),\ \ -\tau\le x\le 0, & \varphi\in C^1_\tau:=C^1([-\tau,0],\mathbb{R}^n),\end{cases} \tag{1}$$

where ${}^cD^{\alpha}_{0+}y(\cdot)$ is the Caputo derivative of order $\alpha\in(0,1)$ and of lower limit zero, $B\in\mathbb{R}^{n\times n}$ denotes constant matrix, $T=k^*\tau$ for a fixed $k^*\in\Lambda:=\{1,2,\dots\}$, τ is a fixed delay time. We introduced the notation of delayed Mittag-Leffler type matrix $\mathbb{E}^{B,\alpha}_{\tau}$ (see [55, Definition 2.3] or Definition 2.5) for (1), which is an extension of one parameter Mittag-Leffler matrix function $\mathbb{E}_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}$, $\alpha>0, z\in\mathbb{R}^{n\times n}$. Then, the solution $y\in C([-\tau,T],\mathbb{R}^n)$ of (1) can be given by (see [55, Theorem 3.2])

$$y(x)=\mathbb{E}^{Bx^{\alpha}}_{\tau}\varphi(-\tau)+\int_{-\tau}^{0}\mathbb{E}^{B(x-\tau-t)^{\alpha}}_{\tau}\varphi'(t)dt. \tag{2}$$

Moreover, sufficient conditions to guarantee finite time stability of (1) via $\mathbb{E}^{B,\alpha}_{\tau}$ are also presented.

Motivated by Li and Wang [55], the first objective of this paper is to seeking representation of solutions of the following linear nonhomogeneous fractional delay differential equations:

$$\begin{cases} ({}^cD^{\alpha}_{0+}y)(x) = By(x-\tau)+f(x),\ \ x\in J,\ \ \tau>0, & f\in C(J,\mathbb{R}^n),\\ y(x)=\varphi(x), & -\tau\le x\le 0.\end{cases} \tag{3}$$

For this purpose, we introduce the notation of delayed two parameters Mittag-Leffler type matrix $\mathbb{E}^{B,\alpha}_{\tau,\beta}$ (see Definition 2.4), which is an extension of classical two parameters Mittag-Leffler matrix function $\mathbb{E}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\beta)}$, $\alpha>0$, $\beta\in\mathbb{R}, z\in\mathbb{R}^{n\times n}$.

Thereafter, we study existence and FTS of the following nonlinear fractional delay differential equations:

$$\begin{cases} ({}^cD^{\alpha}_{0+}y)(x) = By(x-\tau)+f(x,y(x)), \quad x\in J,\ \ \tau>0,\ \ f\in C(J\times\mathbb{R}^n,\mathbb{R}^n),\\ y(x)=\varphi(x),\ \ -\tau\le x\le 0.\end{cases} \tag{4}$$

Existence and FTS results for solutions of (4) are given by using delayed one parameter and two parameter Mittag-Leffler type matrixes and singular Gronwall inequality.

The rest of the paper is organized as follows. In Section 2, we recall some necessary notations, definitions and give some necessary estimation on delayed one parameter and two parameter Mittag-Leffler type matrixes. In Section 3, we give the explicit formula of solutions to linear nonhomogeneous fractional delay differential equations. In Section 4, sufficient conditions ensuring the existence and uniqueness of solutions is presented by using classical contraction mapping principle. In Section 5, sufficient conditions on FTS of nonlinear delay equations are established. An example is given to illustrate our theoretical results in final section.

2. Preliminaries

Throughout the paper, we denote $\|y\|=\sum_{i=1}^{n}|y_i|$ and $\|A\|=\max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|$, which are the Euclidean vector norm and matrix norm, respectively; y_i and a_{ij} are the elements of the vector y and the matrix A, respectively. Denote by $C(J,\mathbb{R}^n)$ the Banach space of vector-value continuous function from $J\to\mathbb{R}^n$ endowed with the norm $\|x\|_{C(J)}=\max_{t\in J}\|x(t)\|$ for a norm $\|\cdot\|$ on $\mathbb{R}^n$. We introduce a space $C^1(J,\mathbb{R}^n)=\{x\in C(J,\mathbb{R}^n): x'\in C(J,\mathbb{R}^n)\}$. In addition, we note $\|\varphi\|_C=\max_{\theta\in[-\tau,0]}\|\varphi(\theta)\|$.

We recall definitions of Riemann–Liouville and Caputo fractional derivatives and finite time stability.

Definition 2.1. (see [1]) The fractional integral of order $\alpha\in(0,1)$ with the lower limit zero for a function $y:[0,\infty)\to\mathbb{R}$ is defined as $I^{\alpha}y(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{y(t)}{(x-t)^{1-\alpha}}dt$, $x>0$, provided the right side is point-wise defined on $[0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. (see [1]) The Riemann–Liouville derivative of order $\alpha\in(0,1)$ for a function $y:[0,\infty)\to\mathbb{R}$ can be written as $({}^{RL}D^{\alpha}_{0+}y)(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{y(t)}{(x-t)^{\alpha}}dt$, $x>0$.

Definition 2.3. (see [1]) The Caputo derivative of order $\alpha\in(0,1)$ for a function $y:[0,\infty)\to\mathbb{R}$ can be written as $({}^cD^{\alpha}_{0+}y)(x)=({}^{RL}D^{\alpha}_{0+}y)(x)-\frac{y(0)}{\Gamma(1-\alpha)}x^{-\alpha}$, $x>0$.

Definition 2.4. (see [55, Definition 2.3]) Delayed classical Mittag-Leffler type matrix $\mathbb{E}_{\tau}^{B\cdot\alpha}:\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined by

$$\mathbb{E}_{\tau}^{Bx^{\alpha}}=\begin{cases}\Theta, & -\infty<x<-\tau,\\ I, & -\tau\le x\le 0,\\ I+B\dfrac{x^{\alpha}}{\Gamma(\alpha+1)}+B^2\dfrac{(x-\tau)^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+B^k\dfrac{(x-(k-1)\tau)^{k\alpha}}{\Gamma(k\alpha+1)}, & \\ (k-1)\tau<x\le k\tau, & k\in\Lambda,\end{cases}\tag{5}$$

where Θ and I denote the zero and identity matrices, respectively.

Next, we introduce a concept of delayed two parameters Mittag-Leffler type matrix function, which is a generalization of Mittag-Leffler function $\mathbb{E}_{\alpha,\beta}(z)$.

Definition 2.5. Delayed two parameters Mittag-Leffler type matrix $\mathbb{E}_{\tau,\beta}^{B\cdot\alpha}:\mathbb{R}\to\mathbb{R}^{n\times n}$ is defined by

$$\mathbb{E}_{\tau,\beta}^{Bx^{\alpha}}=\begin{cases}\Theta, & -\infty<x\le-\tau,\\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)} & -\tau<x\le 0,\\ I\dfrac{(\tau+x)^{\alpha-1}}{\Gamma(\beta)}+B\dfrac{x^{2\alpha-1}}{\Gamma(\alpha+\beta)}+B^2\dfrac{(x-\tau)^{3\alpha-1}}{\Gamma(2\alpha+\beta)}+\cdots+B^k\dfrac{(x-(k-1)\tau)^{(k+1)\alpha-1}}{\Gamma(k\alpha+\beta)}, & \\ & (k-1)\tau<x\le k\tau,\ k\in\Lambda.\end{cases}\tag{6}$$

Lemma 2.6. *Let $(k-1)\tau<x\le k\tau,\ 0\le s\le t$ and $k\in\Lambda$ is a fixed number, we have*

$$\int_{(k-1)\tau+s}^{x}(x-t)^{-\alpha}(t-(k-1)\tau-s)^{k\alpha-1}dt=(x-(k-1)\tau-s)^{(k-1)\alpha}\mathbb{B}[1-\alpha,k\alpha],$$

where $\mathbb{B}[\xi,\eta]=\int_0^1 s^{\xi-1}(1-s)^{\eta-1}ds$ is a Beta function.

Proof. Integration by parts, one has

$$\begin{aligned}&\int_{(k-1)\tau+s}^{x}(x-t)^{-\alpha}(t-(k-1)\tau-s)^{k\alpha-1}dt\\ \xlongequal{t-(k-1)\tau-s=z}\ &\int_{0}^{x-(k-1)\tau-s}(x-(k-1)\tau-s-z)^{-\alpha}z^{k\alpha-1}dz\\ =\ &\int_{0}^{x-(k-1)\tau-s}(x-(k-1)\tau-s)^{-\alpha}\left(1-\frac{z}{x-(k-1)\tau-s}\right)^{-\alpha}z^{k\alpha-1}dz\\ \xlongequal{y(x-(k-1)\tau-s)=z}\ &\int_0^1(x-(k-1)\tau-s)^{(k-1)\alpha}(1-y)^{-\alpha}y^{k\alpha-1}dy\\ =\ &(x-(k-1)\tau-s)^{(k-1)\alpha}\mathbb{B}[1-\alpha,k\alpha].\end{aligned}$$

The proof is completed. □

Lemma 2.7. *Let $(k-1)\tau<x\le k\tau,\ 0\le s\le t$ and $k\in\Lambda$ is a fixed number, we have*

$$\begin{aligned}\int_s^x(x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dt=&\int_s^x(x-t)^{-\alpha}I\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dt+\int_{\tau+s}^{x}(x-t)^{-\alpha}B\frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}dt\\ &+\cdots+\int_{(k-1)\tau+s}^{x}(x-t)^{-\alpha}B^{k-1}\frac{(t-(k-1)\tau-s)^{k\alpha-1}}{\Gamma(k\alpha)}dt.\end{aligned}\tag{7}$$

Proof. For arbitrary $x\in((k-1)\tau,k\tau],\ 0\le s\le t$ and $k\in\Lambda$ is a fixed number. We adopt mathematical induction to prove our result.

(i) For $k=1,\ 0<x\le\tau$, using $\mathbb{E}_{\tau,\alpha}^{B\cdot\alpha}$ defined in (6), we have

$$\int_s^x(x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dt=\int_s^x(x-t)^{-\alpha}I\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dt.$$

(ii) For $k=2,\ \tau<x\le 2\tau$, using $\mathbb{E}_{\tau,\alpha}^{B\cdot\alpha}$ defined in (6) again, we have

$$\int_s^x(x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dt$$

$$= \int_s^{\tau+s} (x-t)^{-\alpha} \mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} dt + \int_{\tau+s}^x (x-t)^{-\alpha} \mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} dt$$

$$= \int_s^{\tau+s} (x-t)^{-\alpha} I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{\tau+s}^x (x-t)^{-\alpha} \left(I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + B \frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)} \right) dt$$

$$= \int_s^x (x-t)^{-\alpha} I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{\tau+s}^x (x-t)^{-\alpha} B \frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)} dt.$$

(iii) Suppose $k = N$, $(N-1)\tau < x \le N\tau$ and $N \in \Lambda$, the following relation holds:

$$\int_s^x (x-t)^{-\alpha} \mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} dt = \int_s^x (x-t)^{-\alpha} I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{\tau+s}^x (x-t)^{-\alpha} B \frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)} dt$$
$$+ \cdots + \int_{(N-1)\tau+s}^x (x-t)^{-\alpha} B^{N-1} \frac{(t-(N-1)\tau-s)^{N\alpha-1}}{\Gamma(N\alpha)} dt.$$

Next, for $k = N+1$, $N\tau < x \le (N+1)\tau$, we have

$$\int_s^x (x-t)^{-\alpha} \mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} dt$$
$$= \int_s^{\tau+s} (x-t)^{-\alpha} I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{\tau+s}^{2\tau+s} (x-t)^{-\alpha} \left(I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + B \frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)} \right) dt$$
$$+ \cdots + \int_{N\tau+s}^x (x-t)^{-\alpha} \left(I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \cdots + B^N \frac{(t-N\tau-s)^{(N+1)\alpha-1}}{\Gamma((N+1)\alpha)} \right) dt$$
$$= \int_s^x (x-t)^{-\alpha} I \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dt + \int_{\tau+s}^x (x-t)^{-\alpha} B \frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)} dt$$
$$+ \cdots + \int_{N\tau+s}^x (x-t)^{-\alpha} B^N \frac{(t-N\tau-s)^{(N+1)\alpha-1}}{\Gamma((N+1)\alpha)} dt,$$

which implies that (7) holds for any $(k-1)\tau < x \le k\tau$ and $k \in \Lambda$ is a fixed number. The proof is completed. □

Next, we recall the norm estimation of $\mathbb{E}_\tau^{B\cdot^\alpha}$.

Lemma 2.8. *(see [55, Lemma 2.4]) For any $x \in ((k-1)\tau, k\tau]$ and $k \in \Lambda$ is a fixed number, we obtain $\left\| \mathbb{E}_\tau^{Bx^\alpha} \right\| \le \mathbb{E}_\alpha(\|B\| x^\alpha)$.*

Now we establish the norm estimation of $\mathbb{E}_{\tau,\beta}^{B\cdot^\alpha}$, which will be used in the sequel.

Lemma 2.9. *For any $x \in ((k-1)\tau, k\tau]$, $k \in \Lambda$ is a fixed number, and $0 \le t < x$, we have*
(i) For $0 \le t < x-(k-1)\tau$ and $k \in \Lambda$, we have

$$\left\| \mathbb{E}_{\tau,\beta}^{B(x-\tau-t)^\alpha} \right\| \le \sum_{i=1}^k \|B\|^{i-1} \frac{(x-(i-1)\tau-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\beta)}.$$

(ii) For $x-(j-1)\tau \le t < x-(j-2)\tau$ and $j = 2, 3, \ldots k$, we have

$$\left\| \mathbb{E}_{\tau,\beta}^{B(x-\tau-t)^\alpha} \right\| \le \sum_{i=2}^j \|B\|^{i-2} \frac{(x-(i-2)\tau-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\beta)}.$$

Proof. By the formula of (6), one has
(i) For $0 \le t < x-(k-1)\tau$ and $k \in \Lambda$, we have

$$\left\| \mathbb{E}_{\tau,\beta}^{B(x-\tau-t)^\alpha} \right\| = \left\| I \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + B \frac{(x-\tau-t)^{2\alpha-1}}{\Gamma(\alpha+\beta)} + \cdots + B^{k-1} \frac{(x-(k-1)\tau-t)^{k\alpha-1}}{\Gamma((k-1)\alpha+\beta)} \right\|$$
$$\le \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + \|B\| \frac{(x-\tau-t)^{2\alpha-1}}{\Gamma(\alpha+\beta)} + \cdots + \|B\|^{k-1} \frac{(x-(k-1)\tau-t)^{k\alpha-1}}{\Gamma((k-1)\alpha+\beta)}$$
$$\le \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + \sum_{i=2}^k \|B\|^{i-1} \frac{(x-(i-1)\tau-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\beta)} \le \sum_{i=1}^k \|B\|^{i-1} \frac{(x-(i-1)\tau-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\beta)}.$$

(ii) For $x-(j-1)\tau \le t < x-(j-2)\tau$ and $j = 2, 3, \ldots, k$, we have

$$\left\| \mathbb{E}_{\tau,\beta}^{B(x-\tau-t)^\alpha} \right\| \le \left\| I \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + B \frac{(x-\tau-t)^{2\alpha-1}}{\Gamma(\alpha+\beta)} + \cdots + B^{j-2} \frac{(x-(j-2)\tau-t)^{(j-1)\alpha-1}}{\Gamma((j-2)\alpha+\beta)} \right\|$$

$$
\begin{aligned}
&\leq \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + \|B\|\frac{(x-\tau-t)^{2\alpha-1}}{\Gamma(\alpha+\beta)} + \cdots + \|B\|^{j-2}\frac{(x-(j-2)\tau-t)^{(j-1)\alpha-1}}{\Gamma((j-2)\alpha+\beta)} \\
&\leq \frac{(x-t)^{\alpha-1}}{\Gamma(\beta)} + \sum_{i=3}^{j}\|B\|^{i-2}\frac{(x-(i-2)\tau-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\beta)} \\
&\leq \sum_{i=2}^{j}\|B\|^{i-2}\frac{(x-(i-2)\tau-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\beta)}.
\end{aligned}
$$

The proof is completed. □

In what follows, we give the following useful integral inequalities.

Lemma 2.10. *For any $x \in ((k-1)\tau, k\tau]$ and $k \in \Lambda$ is a fixed number, we have*

$$
\left\|\int_{-\tau}^{0}\mathbb{E}_{\tau}^{B(x-\tau-t)^{\alpha}}\varphi'(t)dt\right\| \leq \mathbb{E}_{\alpha}(\|B\|x^{\alpha})\int_{-\tau}^{0}\|\varphi'(t)\|dt.
$$

Proof. According to definition of $\mathbb{E}_{\tau}^{B.\alpha}$, we obtain

$$
\begin{aligned}
&\left\|\int_{-\tau}^{0}\mathbb{E}_{\tau}^{B(x-\tau-t)^{\alpha}}\varphi'(t)dt\right\| \\
&= \left\| \int_{-\tau}^{x-k\tau}\left(I + B\frac{(x-\tau-t)^{\alpha}}{\Gamma(\alpha+1)} + \cdots + B^{k}\frac{(x-k\tau-t)^{k\alpha}}{\Gamma(k\alpha+1)}\right)\varphi'(t)dt \right. \\
&\quad \left. + \int_{x-k\tau}^{0}\left(I + B\frac{(x-\tau-t)^{\alpha}}{\Gamma(\alpha+1)} + \cdots + B^{k-1}\frac{(x-(k-1)\tau-t)^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}\right)\varphi'(t)dt\right\| \\
&\leq \left(\int_{-\tau}^{0}\left(1 + \|B\|\frac{x^{\alpha}}{\Gamma(\alpha+1)} + \cdots + \|B\|^{k-1}\frac{x^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}\right) + \int_{-\tau}^{x-k\tau}\|B\|^{k}\frac{x^{k\alpha}}{\Gamma(k\alpha+1)}\right)\|\varphi'(t)\|dt \\
&\leq \left(I + \frac{\|B\|x^{\alpha}}{\Gamma(\alpha+1)} + \cdots + \frac{\|B\|^{k-1}x^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}\right)\int_{-\tau}^{0}\|\varphi'(t)\|dt + \frac{\|B\|^{k}x^{k\alpha}}{\Gamma(k\alpha+1)}\int_{-\tau}^{x-k\tau}\|\varphi'(t)\|dt \\
&\leq \left(I + \frac{\|B\|x^{\alpha}}{\Gamma(\alpha+1)} + \cdots + \frac{\|B\|^{k-1}x^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)}\right)\int_{-\tau}^{0}\|\varphi'(t)\|dt + \frac{\|B\|^{k}x^{k\alpha}}{\Gamma(k\alpha+1)}\int_{-\tau}^{0}\|\varphi'(t)\|dt \\
&\leq \left(I + \frac{\|B\|x^{\alpha}}{\Gamma(\alpha+1)} + \cdots + \frac{\|B\|^{k-1}x^{(k-1)\alpha}}{\Gamma((k-1)\alpha+1)} + \frac{\|B\|^{k}x^{k\alpha}}{\Gamma(k\alpha+1)}\right)\int_{-\tau}^{0}\|\varphi'(t)\|dt \\
&\leq \sum_{k=0}^{\infty}\frac{(\|B\|x^{\alpha})^{k}}{\Gamma(k\alpha+1)}\int_{-\tau}^{0}\|\varphi'(t)\|dt \\
&\leq \mathbb{E}_{\alpha}(\|B\|x^{\alpha})\int_{-\tau}^{0}\|\varphi'(t)\|dt.
\end{aligned}
$$

The proof is completed. □

Lemma 2.11. *For any $x \in ((k-1)\tau, k\tau]$ $k \in \Lambda$, $j = 2, 3, \ldots, k$ and $\alpha \geq \frac{1}{2}$, we have*

$$
\begin{aligned}
&\int_{0}^{x-(k-1)\tau}\left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}\right\|\|\phi(t)\|dt + \int_{x-(k-1)\tau}^{x}\left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}\right\|\|\phi(t)\|dt \\
&\leq \mathbb{E}_{\alpha,\alpha}(\|B\|x^{\alpha})\int_{0}^{x}(x-t)^{\alpha-1}\|\phi(t)\|dt, \ \phi \in C(J, \mathbb{R}^{n}).
\end{aligned}
$$

Proof. According to definition of $\mathbb{E}_{\tau,\alpha}^{B.\alpha}$, we obtain

$$
\begin{aligned}
&\int_{0}^{x-(k-1)\tau}\left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}\right\|\|\phi(t)\|dt + \int_{x-(k-1)\tau}^{x}\left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}\right\|\|\phi(t)\|dt \\
&= \int_{0}^{x-(k-1)\tau}\left(\sum_{i=1}^{k}\|B\|^{i-1}\frac{(x-(i-1)\tau-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\alpha)}\right)\|\phi(t)\|dt \\
&\quad + \sum_{j=2}^{k}\int_{x-(j-1)\tau}^{x-(j-2)\tau}\left(\sum_{i=2}^{j}\|B\|^{i-2}\frac{(x-(i-2)\tau-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\alpha)}\right)\|\phi(t)\|dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \int_0^{x-(k-1)\tau}\left(\sum_{i=1}^{k}\|B\|^{i-1}\frac{(x-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\alpha)}\right)\|\phi(t)\|dt\\
&\quad+\int_{x-(k-1)\tau}^{x}\left(\sum_{i=2}^{k}\|B\|^{i-2}\frac{(x-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\alpha)}\right)\|\phi(t)\|dt\\
&\leq\left(\int_0^x\sum_{i=2}^{k}\|B\|^{i-2}\frac{(x-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\alpha)}+\int_0^{x-(k-1)\tau}\|B\|^{k-1}\frac{(x-t)^{k\alpha-1}}{\Gamma((k-1)\alpha+\alpha)}\right)\|\phi(t)\|dt\\
&\leq\int_0^x(x-t)^{\alpha-1}\mathbb{E}_{\alpha,\alpha}(\|B\|(x-t)^{\alpha})\|\phi(t)\|dt\\
&\leq\mathbb{E}_{\alpha,\alpha}(\|B\|x^{\alpha})\int_0^x(x-t)^{\alpha-1}\|\phi(t)\|dt,
\end{aligned}
$$

where we use the monotonic property of $f(\cdot)=\cdot^{k\alpha-1}$ for $\alpha\geq\frac{1}{2}$, which implies that $(x-(k-1)\tau-t)^{k\alpha-1}\leq(x-t)^{k\alpha-1}$, $x\in((k-1)\tau,k\tau]$. The proof is completed. □

3. Representation of solutions to (3)

In this section, we seek the explicit formula of solutions to linear nonhomogeneous fractional delay differential equations by adopting the classical ideas to find solution of linear fractional ODEs.

In [55, Theorem 3.2], the formula of solutions to linear homogeneous fractional delay differential equations (1) is given in (2). To achieve our aim, we need to find a special solution to (3). The next theorem will show that how to find this special solution.

Theorem 3.1. *A solution $\bar{y}\in C([-\tau,T],\mathbb{R}^n)$ of (3) satisfying initial conditions $y(x)=0$, $x\in[-\tau,0]$ has a form*

$$\bar{y}(x)=\int_0^x\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}f(t)dt,\qquad x>0.$$

Proof. By using the method of variation of constants, any solution of nonhomogeneous system $\bar{y}(x)$ should be satisfied in the form

$$\bar{y}(x)=\int_0^x\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^{\alpha}}c(t)dt,\tag{8}$$

where $c(t)$, $0\leq t\leq x$ is an unknown vector function and $\bar{y}(0)=0$.

Having Caputo fractional differentiation on both sides of (8), we obtain the following cases:

(i) For $0<x\leq\tau$, according to (3), we have

$$({}^cD_{0+}^{\alpha}\bar{y})(x)=B\bar{y}(x-\tau)+f(x)=B\int_0^{x-\tau}\mathbb{E}_{\tau,\alpha}^{B(x-2\tau-t)^{\alpha}}c(t)dt+f(x)=f(x),$$

where we note that $\mathbb{E}_{\tau,\alpha}^{B(x-2\tau-\cdot)^{\alpha}}=\Theta$ due to (6).

According to Definition 2.3 and Lemma 2.6, we have

$$
\begin{aligned}
({}^cD_{0+}^{\alpha}\bar{y})(x)&=({}^{RL}D_{0+}^{\alpha}\bar{y})(x)\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x(x-t)^{-\alpha}\left(\int_0^t\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}c(s)ds\right)dt\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x c(s)\int_s^x(x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^{\alpha}}dsdt\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x c(s)\left(\int_s^x(x-t)^{-\alpha}I\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dt\right)ds\\
&=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\frac{\mathbb{B}[1-\alpha,\alpha]}{\Gamma(\alpha)}c(s)ds\\
&=c(x).
\end{aligned}
$$

Hence, we obtain $c(x)=f(x)$.

(ii) For $k\tau<x\leq(k+1)\tau$ and $k\in\Lambda$, according to (3), we have

$$({}^cD_{0+}^{\alpha}\bar{y})(x)=B\bar{y}(x-\tau)+f(x)=B\int_0^{x-\tau}\mathbb{E}_{\tau,\alpha}^{B(x-2\tau-t)^{\alpha}}c(t)dt+f(x)$$

$$= B\left(\int_0^{x-\tau} \frac{(x-\tau-t)^{\alpha-1}}{\Gamma(\alpha)} c(t)dt + \int_0^{x-2\tau} B\frac{(x-2\tau-t)^{2\alpha-1}}{\Gamma(2\alpha)} c(t)dt + \cdots + \int_0^{x-k\tau} B^{k-1}\frac{(x-k\tau-t)^{k\alpha-1}}{\Gamma(k\alpha)} c(t)dt\right) + f(x).$$

According to Definition 2.3, we have

$$\begin{aligned}({}^cD^\alpha_{0+}\bar{y})(x) &= ({}^{RL}D^\alpha_{0+}\bar{y})(x)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}\left(\int_0^t \mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} c(s)ds\right)dt\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x\int_0^t (x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha} c(s)dsdt\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x c(s)\left(\int_s^x (x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha}dt\right)ds.\end{aligned}$$

According to Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned}&({}^cD^\alpha_{0+}\bar{y})(x) = ({}^{RL}D^\alpha_{0+}\bar{y})(x)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x c(s)\left(\int_s^x (x-t)^{-\alpha}\mathbb{E}_{\tau,\alpha}^{B(t-\tau-s)^\alpha}dt\right)ds\\ &= \frac{d}{dx}\int_0^x c(s)ds + \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^{x-\tau} c(s)\left(\int_{\tau+s}^x (x-t)^{-\alpha}B\frac{(t-\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}dt\right)ds\\ &\quad+\cdots+\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^{x-k\tau} c(s)\left(\int_{k\tau+s}^x (x-t)^{-\alpha}B^k\frac{(t-k\tau-s)^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)}dt\right)ds\\ &= c(x) + B\int_0^{x-\tau}\frac{(x-\tau-s)^{\alpha-1}}{\Gamma(\alpha)}c(s)ds + B^2\int_0^{x-2\tau}\frac{(x-2\tau-s)^{2\alpha-1}}{\Gamma(2\alpha)}c(s)ds\\ &\quad+\cdots+B^k\int_0^{x-k\tau}\frac{(x-k\tau-s)^{k\alpha-1}}{\Gamma(k\alpha)}c(s)ds\\ &= B\int_0^{x-\tau}\mathbb{E}_{\tau,\alpha}^{B(x-2\tau-t)^\alpha}c(t)dt + f(x).\end{aligned}$$

Hence, we obtain $c(x) = f(x)$. The proof is completed. □

It is obvious that any solution y of (3) can be written as a sum $y(x) = y_0(x) + \bar{y}(x)$, where $y_0(x)$ is a solution of (1) satisfying conditions $y(x) = \varphi(x)$, $-\tau \le x \le 0$ and $\bar{y}(x)$ is a solution of (3) satisfying $y(0) = 0$. The following theorem present the construction of formula of solutions to (3). The proof is straightway, so we omit it here.

Theorem 3.2. *A solution $y \in C([-\tau, T], \mathbb{R}^n)$ of (3) has a form*

$$y(x) = \mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau) + \int_{-\tau}^0 \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt + \int_0^x \mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha} f(t)dt.$$

4. Existence of solutions to (4)

Definition 4.1. A function $y \in C([-\tau, T], \mathbb{R}^n)$ is called the solution of (4) if it satisfies the following integral equation

$$y(x) = \mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau) + \int_{-\tau}^0 \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt + \int_0^x \mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha} f(t, y(t))dt. \tag{9}$$

We introduce the following assumptions:

$[H_1]$ $f \in C(J, \mathbb{R}^n)$ and there exists a $\widetilde{K} > 0$ such that $\|f(x, y) - f(x, z)\| \le \widetilde{K}\|y - z\|$ for any $x \in J$ and all $y, z \in \mathbb{R}^n$.

$[H_2]$ $\rho = \widetilde{K}\left(\sum_{i=1}^k \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(T-(i-1)\tau)^{i\alpha}\right) < 1$, where $k \in \Lambda$ is a fixed number.

Denote $M := \int_{-\tau}^0 \|\varphi'(t)\|dt$. Now we are ready to present the following existence and uniqueness result.

Theorem 4.2. *Assume that $[H_1]$ and $[H_2]$ hold. Then (4) has a unique solution $y \in C([-\tau, T], \mathbb{R}^n)$.*

Proof. Define an operator $\Lambda : C([-\tau, T], \mathbb{R}^n) \to C([-\tau, T], \mathbb{R}^n)$ by

$$(\Lambda y)(x) = \mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau) + \int_{-\tau}^0 \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt + \int_0^x \mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha} f(t, y(t))dt. \tag{10}$$

It is obvious that Λ is well defined due to $[H_1]$. Next, we check that Λ is a contraction mapping.

For any $y, z \in C([-\tau, T], \mathbb{R}^n)$. By Lemma 2.9, we obtain

$$\|(\Lambda y)(x) - (\Lambda z)(x)\| \leq \widetilde{K}\left(\sum_{i=1}^{k} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(x-(i-1)\tau)^{i\alpha}\right)\|y-z\|_{C(J)},$$

which implies that $\|\Lambda y - \Lambda z\|_{C(J)} \leq \rho\|y-z\|_{C(J)}$. By $[H_2]$, one can apply contraction mapping principle to complete the rest proof. □

Next, we give another existence and uniqueness result based on below locally Lipschitz condition.

$[H_1']$ $f \in C([0,\infty), \mathbb{R}^n)$. For every $r>0$ and $x_1>0$ there exists a constant $K := K(x_1, r)$ such that $\|f(x,y) - f(x,z)\| \leq K \|y-z\|$ for all $x \in [0, x_1]$ and $y, z \in B_r = \{y \in \mathbb{R}^n : \|y\| \leq r\}$.

Theorem 4.3. *Assume that $[H_1']$ holds. Then (4) has a unique solution $y \in C([-\tau, x_m), \mathbb{R}^n)$ where $x_m \leq \infty$. Further, if $x_m < \infty$ then $\lim_{x\to x_m}\|y(x)\| = \infty$.*

Proof. Consider $\Lambda : C([-\tau, x_m), \mathbb{R}^n) \to C([-\tau, x_m), \mathbb{R}^n)$ defined in (10) again.

Step 1. A prior estimate.

For arbitrary $x \in [-\tau, 0]$, we have $\|y(x)\| = \|\varphi(x)\|$. Choosing a subinterval $I_1 = [0, x_1]$. Let $\mu(x) = \sum_{i=1}^{k} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(x-(i-1)\tau)^{i\alpha}$, where k comes from set $\max\{k \in \Lambda: k\tau \leq x_1\}$, $r_1 = (\|\varphi\|_C + M)\mathbb{E}_\alpha(\|B\|x_1^\alpha) + 1$, and $N_1 = \sup_{x\in I_1}\|f(x,0)\|$. For any $y \in C([0, x_1], \mathbb{R}^n)$ with $\{\|y(x)\|: x \in [0, x_1]\} \leq r_1$, we have

$$\begin{aligned}
\|(\Lambda y)(x)\| &\leq \|\mathbb{E}_\tau^{Bx^\alpha}\|\|\varphi(-\tau)\| + \left\|\int_{-\tau}^{0} \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt\right\| + \int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|\|f(t,y(t))\|dt \\
&\leq (\|\varphi(-\tau)\| + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|\|f(t,y(t)) - f(t,0)\|dt \\
&\quad + \int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|\|f(t,0)\|dt \\
&\leq (\|\varphi(-\tau)\| + M)\mathbb{E}_\alpha(\|B\|x_1^\alpha) + K\|y\|_{C(I_1)}\int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|dt \\
&\quad + N_1\int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|dt \\
&\leq (\|\varphi\|_C + M)\mathbb{E}_\alpha(\|B\|x_1^\alpha) + (Kr_1 + N_1)\mu(x). \qquad (11)
\end{aligned}$$

Choosing $l_1 = \min\{x_1, x_1^*\}$, where x_1^* comes from set $\left\{x_1^* : \mu(x_1^*) = \frac{1}{Kr_1+N_1}\right\}$. It follows from (11) that $\|(\Lambda y)(x)\| \leq r_1$, for all $x \in [0, l_1]$.

Step 2. Local existence and uniqueness of solution.

For $x \in [0, l_1]$ and $y, z \in B_r$, using Lemma 2.9 to derive

$$\begin{aligned}
\|(\Lambda y)(x) - (\Lambda z)(x)\| &\leq \int_0^x \|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\|\|f(x,y(t)) - f(x,z(t))\|dt \\
&\leq K\left(\int_0^x \|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\|dt\right)\|y-z\|_{C([0,l_1])} \\
&\leq K\left(\sum_{i=1}^{k} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(l_1-(i-1)\tau)^{i\alpha}\right)\|y-z\|_{C([0,l_1])},
\end{aligned}$$

where k comes from set $\max\{k \in \Lambda: k\tau \leq l_1\}$.

Choosing l_1 satisfying $\sum_{i=1}^{k} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(l_1-(i-1)\tau)^{i\alpha} < \frac{1}{2K}$. Therefore, we can obtain

$$\|\Lambda y - \Lambda z\|_{C([0,l_1])} \leq \frac{1}{2}\|y-z\|_{C([0,l_1])}.$$

This implies that Λ has a fixed point on $[0, l_1]$, which reduces a solution.

Step 3. Extension of solution.

Next, we extend the solution for $x \geq l_1$ by solving the fixed point problem, $z = \Lambda z$, where Λ is given by

$$(\Lambda z)(x) = \mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau) + \int_{-\tau}^{0} \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt + \int_0^x \mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha} f(t,z(t))dt, \ x \in [0, l_2],$$

Setting $z(x) = y(x)$ for $x \in [0, l_2]$ where l_2 is defined as follows. Take $x_2 > l_1$ and define $I_2 = [0, x_2]$, $\mu(x) = \sum_{i=1}^{k} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(x-(i-1)\tau)^{i\alpha}$, where k comes from set $\max\{k \in \Lambda: k\tau \leq x_2\}$, $r_2 = (\|\varphi\|_C + M)\mathbb{E}_\alpha(\|B\|x_2^\alpha) + 1$ and $N_2 = \sup_{x\in I_2}\|f(x,0)\|$.

Choosing $l_2 = l_1 + \min\{x_2 - l_1, x_2^*\}$, where x_2^* comes from set $\left\{x_2^* : \mu(x_2^*) = \frac{1}{Kr_2+N_2}\right\}$. For $x \in [l_1, l_2]$, we obtain $\|(\Lambda y)(x)\| \le r_2$. Repeating this procedure, we get the maximal interval of existence of solution. Thus (4) has a unique solution $y \in C([-\tau, x_m), \mathbb{R}^n)$.

Now we verify that $\lim_{x\to x_m} \|y(x_m)\| = \infty$ for x_m is finite. If not, then there exists a sequence $\{l_n\}$ converging to x_m and a finite positive number r such that $\|y(l_n)\| \le r$ for all n. Taking n sufficiently large, so that l_n is infinitesimally close to x_m, one can use the previous arguments to extend the solution beyond x_m, which is a contradiction. The proof is completed. □

5. Finite time stability results for (4)

In this section, we continue to study FTS of (4). We give the basic definition of FTS as follows.

Definition 5.1. (see [32]) Let y be a solution of (4). We say (4) is finite time stable with respect to $\{0, J, \tau, \delta, \eta\}$ if and only if $\|\varphi\|_C < \delta$ implies $\|y(x)\| < \eta, \forall x \in J$, where $\varphi(x),\ -\tau \le x \le 0$ is the initial time of observation, δ, η are real positive numbers and $\delta < \eta$.

We impose the following assumptions:

$[H_3]$ There exists a $\omega(\cdot) \in C(J, \mathbb{R}^+)$ such that $\|f(x, y)\| \le \omega(x)$, for $x \in J$ and $y \in \mathbb{R}^n$.

$[H_4]$ There exists a $\psi(\cdot) \in L^q(J, \mathbb{R}^+)$, $\frac{1}{q} = 1 - \frac{1}{p}$, $p > 1$ such that $\|f(x, y)\| \le \psi(x)$ for $x \in J$, $y \in \mathbb{R}^n$ and $Q(x) := \left(\int_0^x \psi(t)^q dt\right)^{\frac{1}{q}} < \infty$.

$[H_5]$ There exists a $L > 0$ such that $\|f(x, y)\| \le L\|y\|$, $x \in J$ and $y \in \mathbb{R}^n$.

Now we are ready to prove our main results in this section.

Theorem 5.2. *Assume that $[H_1] - [H_3]$ hold. For a fixed $k \in \Lambda$, if*

$$(\delta + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \|\omega\|_{C(J)}\left(\sum_{i=1}^k \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(x - (i-1)\tau)^{i\alpha}\right) < \eta, \ \forall\ x \in J, \tag{12}$$

then (4) is finite time stable with respect to $\{0, J, \tau, \delta, \eta\}$.

Proof. By $[H_1]$ and $[H_2]$ via Theorem 4.2, we know (4) has a unique solution $y \in C([-\tau, T], \mathbb{R}^n)$. By Lemmas 2.8–2.10 and the properties of norm $\|\cdot\|$ via (9), we have

$$\begin{aligned}
\|y(x)\| \le\ & \|\mathbb{E}_\tau^{Bx^\alpha}\varphi(-\tau)\| + \left\|\int_{-\tau}^0 \mathbb{E}_\tau^{B(x-\tau-t)^\alpha}\varphi'(t)dt\right\| + \int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|\omega(t)dt \\
\le\ & (\|\varphi(-\tau)\| + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \|\omega\|_{C(J)}\int_0^{x-(k-1)\tau}\sum_{i=1}^k \|B\|^{i-1}\frac{(x-(i-1)\tau-t)^{i\alpha-1}}{\Gamma((i-1)\alpha+\alpha)}dt \\
& + \|\omega\|_{C(J)}\sum_{j=2}^k \int_{x-(j-1)\tau}^{x-(j-2)\tau}\left(\sum_{i=2}^j \|B\|^{i-2}\frac{(x-(i-2)\tau-t)^{(i-1)\alpha-1}}{\Gamma((i-2)\alpha+\alpha)}\right)dt \\
\le\ & (\delta + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \|\omega\|_{C(J)}\sum_{i=1}^k \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}((x-(i-1)\tau)^{i\alpha} - ((k-i)\tau)^{i\alpha}) \\
& + \|\omega\|_{C(J)}\sum_{j=2}^k\left(\sum_{i=2}^j \frac{\|B\|^{i-2}}{\Gamma((i-1)\alpha+1)}\left[((j-i+1)\tau)^{\alpha(i-1)} - ((j-i)\tau)^{\alpha(i-1)}\right]\right) \\
\le\ & (\delta + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \|\omega\|_{C(J)}\left(\sum_{i=1}^k \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(x-(i-1)\tau)^{i\alpha}\right) < \eta,
\end{aligned}$$

due to (12). The proof is completed. □

Theorem 5.3. *Assume that $[H_1]$, $[H_2]$, $[H_4]$ and $\alpha > 1 - \frac{1}{p}$ $(p > 1)$ hold. For a fixed $k \in \Lambda$, if*

$$(\delta + M)\mathbb{E}_\alpha(\|B\|x^\alpha) + Q(x)\sum_{i=1}^k\left(\frac{\|B\|^{i-1}}{\Gamma(i\alpha)}\cdot\frac{(x-(i-1)\tau)^{i\alpha-1+\frac{1}{p}}}{(pi\alpha-p+1)^{\frac{1}{p}}}\right) < \eta, \ \forall\ x \in J, \tag{13}$$

then (4) is finite time stable with respect to $\{0, J, \tau, \delta, \eta\}$.

Proof. By Lemmas 2.8–2.10 and the properties of norm $\|\cdot\|$ via (9), (13), we have

$$\begin{aligned}
\|y(x)\| &\le \|\varphi(-\tau)\|\|\mathbb{E}_\tau^{Bx^\alpha}\| + \mathbb{E}_\alpha(\|B\|x^\alpha)\int_{-\tau}^{0}\|\varphi'(t)\|dt + \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\psi(t)dt \\
&\quad + \cdots + \int_0^{x-(k-1)\tau}\|B\|^{k-1}\frac{(x-(k-1)\tau-t)^{k\alpha-1}}{\Gamma(k\alpha)}\psi(t)dt \\
&\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \sum_{i=1}^{k}\frac{\|B\|^{i-1}}{\Gamma(i\alpha)}\int_0^{x-(i-1)\tau}(x-(i-1)\tau-t)^{i\alpha-1}\psi(t)dt \\
&\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) \\
&\quad + \sum_{i=1}^{k}\frac{\|B\|^{i-1}}{\Gamma(i\alpha)}\left(\int_0^{x-(i-1)\tau}(x-(i-1)\tau-t)^{p(i\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_0^{x-(i-1)\tau}\psi(t)^q dt\right)^{\frac{1}{q}} \\
&\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) \\
&\quad + \sum_{i=1}^{k}\frac{\|B\|^{i-1}}{\Gamma(i\alpha)}\left(\int_0^{x-(i-1)\tau}(x-(i-1)\tau-t)^{p(i\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_0^{x}\psi(t)^q dt\right)^{\frac{1}{q}} \\
&\le (\delta+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + Q(x)\sum_{i=1}^{k}\left(\frac{\|B\|^{i-1}}{\Gamma(i\alpha)}\cdot\frac{(x-(i-1)\tau)^{i\alpha-1+\frac{1}{p}}}{(pi\alpha-p+1)^{\frac{1}{p}}}\right) < \eta.
\end{aligned}$$

The proof is completed. □

Theorem 5.4. *Assume that* $[H_1]-[H_3]$ *and* $\alpha \ge \frac{1}{2}$ *hold. For a fixed* $k\in\Lambda$, *if*

$$(\delta+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \frac{\|\omega\|_{C(J)}}{\alpha}x^\alpha\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha) < \eta, \quad \forall\, x\in J, \tag{14}$$

then (4) is finite time stable with respect to $\{0, J, \tau, \delta, \eta\}$.

Proof. By Lemmas 2.8–2.11 and the properties of norm $\|\cdot\|$ via (9), (14), we have

$$\begin{aligned}
\|y(x)\| &\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)\int_0^x (x-t)^{\alpha-1}\omega(t)dt \\
&\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \|\omega\|_{C(J)}\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)\int_0^x (x-t)^{\alpha-1}dt \\
&\le (\delta+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + \frac{\|\omega\|_{C(J)}}{\alpha}x^\alpha\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha) < \eta.
\end{aligned}$$

The proof is completed. □

Theorem 5.5. *Assume that* $[H_1]$, $[H_2]$, $[H_5]$ *and* $\alpha \ge \frac{1}{2}$ *hold. For a fixed* $k\in\Lambda$, *if*

$$\mathbb{E}_\alpha(\|B\|x^\alpha)\mathbb{E}_\alpha(L\Gamma(\alpha)\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)x^\alpha) < \frac{\eta}{\delta+M}, \quad \forall\, x\in J, \tag{15}$$

then (4) is finite time stable with respect to $\{0, J, \tau, \delta, \eta\}$.

Proof. For $x\in J$, by Lemmas 2.8–2.11 and the properties of norm $\|\cdot\|$, we have

$$\begin{aligned}
\|y(x)\| &\le \|\varphi(-\tau)\|\|\mathbb{E}_\tau^{Bx^\alpha}\| + \mathbb{E}_\alpha(\|B\|x^\alpha)\int_{-\tau}^{0}\|\varphi'(t)\|dt + \int_0^x \left\|\mathbb{E}_{\tau,\alpha}^{B(x-\tau-t)^\alpha}\right\|\|f(t,y(t))\|dt \\
&\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + L\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)\int_0^x (x-t)^{\alpha-1}\|y(t)\|dt.
\end{aligned}$$

According to [56, Theorem 1] via (15), we have

$$\begin{aligned}
\|y(x)\| &\le (\|\varphi(-\tau)\|+M)\mathbb{E}_\alpha(\|B\|x^\alpha) + L\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)\int_0^x (x-t)^{\alpha-1}\|y(x)\|dt \\
&\le (\delta+M)\mathbb{E}_\alpha(\|B\|x^\alpha)\mathbb{E}_\alpha(L\Gamma(\alpha)\mathbb{E}_{\alpha,\alpha}(\|B\|x^\alpha)x^\alpha) < \eta.
\end{aligned}$$

The proof is completed. □

6. An example

In this section, we give an example to demonstrate the validity of our theoretical results.

Table 1
FTS results of (16) and fixed the time $T = 0.6$.

Theorem	$\|\varphi\|_C$	α	T	τ	δ	$\|y(x)\|$	η	FTS
5.2	0.6	0.6	0.6	0.2	0.61	2.6100	2.62	Yes
5.3	0.6	0.6	0.6	0.2	0.61	2.3235	2.33 (optimal)	Yes
5.4	0.6	0.6	0.6	0.2	0.61	3.0231	3.03	Yes
5.5	0.6	0.6	0.6	0.2	0.61	8.1462	8.15	Yes

Let $\alpha = 0.6,\ \tau = 0.2,\ T = 0.6$ and $k = 3$. Consider

$$\begin{cases} {}^cD^{0.6}_{0^+}y(x) = By(x-0.2) + f(x, y(x)),\ \ x \in [0, 0.6], \\ \varphi(x) = (x, 2x)^\top,\ \ -0.2 \le x \le 0, \end{cases} \tag{16}$$

where

$$B = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.5 \end{pmatrix},\quad f(x, y(x)) = \begin{pmatrix} x^2 \frac{|y_1(x)|}{1+|y_1(x)|} \\ x^2 \frac{|y_2(x)|}{1+|y_2(x)|} \end{pmatrix}.$$

A solution of (16) can be expressed in the following form:

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \mathbb{E}^{Bx^{0.6}}_{0.2}\varphi(-0.2) + \int_{-0.2}^{0} \mathbb{E}^{B(x-0.2-s)^{0.6}}_{0.2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} ds$$
$$+ \int_0^x \mathbb{E}^{B(x-0.2-t)^{0.6}}_{0.2,0.6} \begin{pmatrix} x^2 \frac{|y_1(x)|}{1+|y_1(x)|} \\ x^2 \frac{|y_2(x)|}{1+|y_2(x)|} \end{pmatrix} dt.$$

Obviously, $\|f(x,y) - f(x,z)\| \le 2x^2\|y - z\|$ for $y, z \in \mathbb{R}^n$ and $\|f(x, y)\| < 2x^2, \forall x \in J$. Now put $p = 2, q = 2,\ K = L = 0.72,\ \rho = 0.7078$ and $\omega(x) = \psi(x) = 2x^2$.

By calculation, one has $\|\varphi\|_C = 0.6,\ \ M = \int_{-0.2}^{0}\|\varphi'(t)\|dt = 0.6,\ \ \mathbb{E}_{0.6}(\|B\|T^{0.6}) = 1.572,\ \ \mathbb{E}_{0.6,0.6}(\|B\|T^{0.6}) = 1.2692$, $\mathbb{E}_\alpha(\|B\|T^\alpha)\mathbb{E}_\alpha(L\Gamma(\alpha)\mathbb{E}_{\alpha,\alpha}(\|B\|T^\alpha)T^\alpha) = 4.2827,\ \ \|\omega\|_C = 0.72,\ \ Q(T) = 0.2494$. Next, $\sum_{i=1}^{3} \frac{\|B\|^{i-1}}{\Gamma(i\alpha+1)}(T-(i-1)\tau)^{i\alpha} = 0.9831$ and $\sum_{i=1}^{3} \frac{\|B\|^{i-1}(T-(i-1)\tau)^{i\alpha-1+\frac{1}{p}}}{\Gamma(i\alpha)(pi\alpha-p+1)^{\frac{1}{p}}} = 1.6896$.

Choosing $\delta = 0.61$. We have the following results (see Table 1).

Discussion: Concerning on the definition of FTS (see Definition 5.1), we need to find a certain threshold η, which makes the state $\|y(x)\|$ of (4) does not exceed η on J. By checking the required conditions in Theorems 5.2–5.5 on [0, 0.6], we get a relative optimal threshold $\eta = 2.33$ by comparing the values of η in Table 1.

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