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Error bounds for asymptotic solutions of second-order linear difference equations

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Abstract

Error bounds for asymptotic solutions of second-order linear difference equation

$$y(n+2)+a(n)y(n+1)+b(n)y(n)=0,$$

where $a(n), b(n)$ have asymptotic expansions

$$a(n)\sim\sum_{s=0}^{\infty}\frac{a_s}{n^s},\qquad b(n)\sim\sum_{s=0}^{\infty}\frac{b_s}{n^s},\qquad n\to+\infty,$$

$(b_0\neq 0)$ are discussed in detail in this paper.

Keywords: Error bound for asymptotic expansion; Asymptotic expansion; Linear difference equation

AMS(MOS) Subject Classification: 39 A10, 41 A60

1. Introduction

Asymptotic solutions of second-order linear difference equation

$$y(n+2)+a(n)y(n+1)+b(n)y(n)=0,\tag{1.1}$$

where

$$a(n)\sim\sum_{s=0}^{\infty}\frac{a_s}{n^s},\qquad b(n)\sim\sum_{s=0}^{\infty}\frac{b_s}{n^s},\qquad n\to\infty,\tag{1.2}$$

$(b_o\neq 0)$ were first studied in [3, 4, 1], but as pointed out in [6], these papers are very long and complicated, and they are not easily understood even by most of the specialists in asymptotics.

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Recently, Wong and Li [6] used the method similar to that employed by Olver to treat asymptotic solutions of second-order linear differential equations, systematically discussed asymptotic solutions of Eq. (1.1) under different cases of $a(n), b(n)$. But the method used in [6] would not lead to the estimation of error bounds of these asymptotic solutions. The purpose of this paper is to estimate error bounds for asymptotic solutions of difference equation (1.1).

From [6] we know that according to different cases of the characteristic equation $\rho^2+a_0\rho+b_0=0$ and the auxiliary equation $a_1\rho+b_1=0$, forms of asymptotic expansions of solutions are also unlike.

1.1. Normal case

Let ρ_1, ρ_2 be two roots of the characteristic equation

$$\rho^2+a_0\rho+b_0=0, \tag{1.3}$$

i.e.,

$$\rho_1,\rho_2=-\tfrac{1}{2}a_0\pm\left(\tfrac{1}{4}a_0^2-b_0\right)^{1/2}. \tag{1.4}$$

If $\rho_1\neq\rho_2$, i.e., $a_0^2\neq 4b_0$, then Eq. (1.1) has two linearly independent solutions, $y_1(n), y_2(n)$,

$$y_i(n)\sim\rho_i^n n^{\alpha_i}\sum_{s=0}^{\infty}\frac{c_s^{(i)}}{n^s},\qquad n\to+\infty,\quad i=1,2, \tag{1.5}$$

where

$$\alpha_i=-\frac{a_1\rho_i+b_1}{2\rho_i^2+\rho_i a_0}=\frac{a_1\rho_i+b_1}{a_0\rho_i+2b_0},\qquad i=1,2, \tag{1.6}$$

and constants $c_s^{(i)}$ $(s=0,1,2,\ldots,\ i=1,2)$ are determined by a_j,b_j $(j=0,1,2,\ldots,s)$.

1.2. Subnormal case

If $\rho_1=\rho_2$, but their common value $\rho=-\frac{1}{2}a_0$ is not a root of the auxiliary equation

$$a_1\rho+b_1=0, \tag{1.7}$$

i.e., $2b_1\neq a_0a_1$, then there are two linearly independent solutions of (1.1), $y_1(n), y_2(n)$,

$$y_i(n)\sim\rho^n n^{\alpha}\mathrm{e}^{\gamma_i\sqrt{n}}\sum_{s=0}^{\infty}\frac{c_s^{(i)}}{n^{s/2}},\qquad n\to+\infty,\quad i=1,2, \tag{1.8}$$

where

$$\gamma_i=(-1)^{i+1}2\sqrt{\frac{a_0a_1-2b_1}{2b_0}},\qquad i=1,2, \tag{1.9}$$

$$\alpha=\frac{1}{4}+\frac{b_1}{2b_0}, \tag{1.10}$$

and constants $c_s^{(i)}$ $(s=0,1,2,\ldots,i=1,2)$ are determined by a_j,b_j $(j=0,1,2,\ldots,[\frac{1}{2}(s+3)])$.

1.3. Exceptional cases

If the characteristic equation (1.3) has a common root, and the common root is also a root of the auxiliary equation (1.7), then let $\alpha_1, \alpha_2\,(\mathrm{Re}\,\alpha_2 \geqslant \mathrm{Re}\,\alpha_1)$ be two zeros of the indicial polynomial

$$q(\alpha) = \alpha(\alpha - 1)\rho^2 + (a_1\alpha + a_2)\rho + b_2. \tag{1.11}$$

According to different cases of α_1, α_2,

$$\begin{aligned} &\text{(i)} \quad \alpha_2 - \alpha_1 \neq 0, 1, 2, \ldots; \\ &\text{(ii)} \quad \alpha_2 - \alpha_1 = 1, 2, \ldots; \\ &\text{(iii)} \quad \alpha_2 - \alpha_1 = 0; \end{aligned} \tag{1.12}$$

Eq. (1.1) has two linearly independent solutions $y_1(n), y_2(n)$,

$$\text{(i)} \qquad y_i(n) \sim \rho^n n^{\alpha_i} \sum_{s=0}^{\infty} \frac{c_s^{(i)}}{n^s}, \qquad n \to +\infty, \quad i = 1, 2, \tag{1.13}$$

where constants $c_s^{(i)}$ $(s = 0, 1, 2, \ldots,\ i = 1, 2,)$ are determined by a_j, b_j $(j = 0, 1, 2, \ldots, s + 2)$.

$$\text{(ii)} \quad y_1(n) \sim \rho^n n^{\alpha_1} \sum_{s=0}^{\infty} \frac{c_s^{(1)}}{n^s}, \qquad n \to +\infty, \tag{1.14}$$

$$y_2(n) = z(n) + c(\log n) y_1(n), \tag{1.15}$$

$$z(n) \sim \rho^n n^{\alpha_2} \sum_{s=0}^{\infty} \frac{d_s}{n^s}, \qquad n \to +\infty, \tag{1.16}$$

where c_s, d_s are determined by a_j, b_j $(j = 0, 1, 2, \ldots, s+2)$, $s = 0, 1, 2, \ldots,$ and c is a constant which may happen to be zero.

$$\text{(iii)} \quad y_1(n) \sim \rho^n n^{\alpha_1} \sum_{s=0}^{\infty} \frac{c_s}{n^s}, \qquad n \to +\infty, \tag{1.17}$$

$$y_2(n) = z(n) + c(\log n) y_1(n), \tag{1.18}$$

$$z(n) \sim \rho^n n^{\alpha_1 - Q + 2} \sum_{s=0}^{\infty} \frac{d_s}{n^s}, \qquad n \to +\infty, \tag{1.19}$$

where $Q \geqslant 3$ is a positive integer which is determined in [6], and c_s, d_s are determined by a_j, b_j $(j = 0, 1, 2, \ldots, s + Q - 1)$, $s = 0, 1, 2, \ldots,$ and c is a nonzero constant.

In the following sections, we will discuss in detail error bounds of the proceeding asymptotic solutions of Eq. (1.1).

2. Error bounds in normal case

In this section, without loss of generality, we suppose that $|\rho_2| \geqslant |\rho_1|$ (if $|\rho_2| = |\rho_1|$, let $\mathrm{Re}\,\alpha_2 \geqslant \mathrm{Re}\,\alpha_1$). Let

$$y_i(n) = L_N^{(i)}(n) + \varepsilon_N^{(i)}(n), \qquad i = 1, 2, \tag{2.1}$$

where from (1.5), $L_N^{(i)}(n) = \rho_i^n n^{\alpha_i} \sum_{s=0}^{N-1} c_s^{(i)} n^{-s}$, and $\varepsilon_N^{(i)}(n)$ are error terms. From [6] we know that $\varepsilon_N^{(i)}(n)$ satisfy second-order linear difference equations

$$\varepsilon_N^{(i)}(n+2) + a(n)\varepsilon_N^{(i)}(n+1) + b(n)\varepsilon_N^{(i)}(n) = -\rho_i^n n^{\alpha_i} R_N^{(i)}(n), \quad i = 1,2, \tag{2.2}$$

where $R_N^{(i)}(n) = \rho_i^{-n} n^{-\alpha_i}[L_N^{(i)}(n+2) + a(n)L_N^{(i)}(n+1) + b(n)L_N^{(i)}(n)]$, $i = 1,2$, and $r_N^{(i)} = \sup_n\{n^{N+1}\,|R_N^{(i)}(n)|\}$ are constants. In the following, we estimate $\varepsilon_N^{(1)}(n), \varepsilon_N^{(2)}(n)$, respectively.

2.1. The error bound for the asymptotic expansion of $y_1(n)$

First of all, we consider the bound of the "sum equation"

$$h(n) = \sum_{j=n}^{\infty} K(n;j)\{\phi(j) + \psi(j)h(j+1) + \eta(j)h(j)\}. \tag{2.3}$$

Lemma 1. *Let $K(n;j)$, $\phi(j)$, $\psi(j)$, $\eta(j)$ be real or complex functions of integer variables n,j, if there exists an integer n_0, for $j \geqslant n \geqslant n_0$,*

$$|K(n;j)| \leqslant M\frac{P(n)}{P(j)} j^{\theta},$$

$$|\phi(j)| \leqslant \phi_0 P(j) j^{-\xi(N+\beta)},$$

$$|\psi(j)| \leqslant \psi_0 j^{-\tau},$$

$$|\eta(j)| \leqslant \eta_0 j^{-\tau},$$

where $M, \phi_0, \psi_0, \eta_0, \beta, \tau, \theta, \xi$ are nonnegative constants which satisfy

$$\xi(N+\beta) - \theta > 1, \quad \tau - \theta > 1,$$

N is a positive integer, $P(n)$ is a positive function of integer variable n, and $p_0 = \sup_n[P(n+1)/P(n)]$ is a constant; then Eq. (2.3) has a solution, $h(n)$, which satisfies

$$|h(n)| \leqslant \frac{2M\phi_0}{\xi(N+\beta) - \theta - 1 - 2M(\psi_0 p_0 + \eta_0)} P(n) n^{-\xi(N+\beta)+\theta+1}.$$

Proof. Set $h_0(n) = 0$,

$$h_{s+1}(n) = \sum_{j=n}^{\infty} K(n;j)\{\phi(j) + \psi(j)h_s(j+1) + \eta(j)h_s(j)\}, \quad s = 0,1,2,\ldots;$$

then for $n \geqslant n_0 > \xi(N+\beta) - \theta - 1$,

$$\begin{aligned} |h_1(n)| &\leqslant \sum_{j=n}^{\infty} |K(n;j)| \cdot |\phi(j)| \\ &\leqslant M\phi_0 P(n) \sum_{j=n}^{\infty} j^{-\xi(N+\beta)+\theta} \\ &\leqslant \frac{2M\phi_0}{\xi(N+\beta) - \theta - 1} P(n) n^{-\xi(N+\beta)+\theta+1}. \end{aligned}$$

The inequality

$$\sum_{j=n}^{\infty} j^{-p} \leqslant \frac{2}{p-1} n^{-(p-1)}, \qquad n \geqslant p-1 > 0$$

is used here. Assuming

$$|h_s(n) - h_{s-1}(n)| \leqslant \frac{2M\phi_0}{\xi(N+\beta)-\theta-1} \lambda^{s-1} P(n) n^{-\xi(N+\beta)+\theta+1},$$

where $\lambda = 2M(\psi_0 p_0 + \eta_0)/(\xi(N+\beta)-\theta-1)$, we have

$$\begin{aligned}
|h_{s+1}(n) - h_s(n)| &\leqslant \sum_{j=n}^{\infty} |K(n;j)| \cdot \{|\psi(j)| \cdot |h_s(j+1) - h_{s-1}(j+1)| \\
&\quad + |\eta(j)| \cdot |h_s(j) - h_{s-1}(j)|\} \\
&\leqslant M\phi_0 \lambda^s P(n) \sum_{j=n}^{\infty} j^{-\xi(N+\beta)+(\theta-\tau+1)+\theta} \\
&\leqslant M\phi_0 \lambda^s p(n) \sum_{j=n}^{\infty} j^{-\xi(N+\beta)+\theta} \\
&\leqslant \frac{2M\phi_0}{\xi(N+\beta)-\theta-1} \cdot \lambda^s P(n) n^{-\xi(N+\beta)+\theta+1}.
\end{aligned}$$

By induction, the inequality holds for any integer s. It is now evident that series $\sum_{s=0}^{\infty}\{h_{s+1}(n) - h_s(n)\}$ is uniformly convergent in n, and the sum of the series is $h(n)$, a solution of Eq. (2.3). Furthermore, $h(n)$ satisfies

$$\begin{aligned}
|h(n)| &\leqslant \sum_{s=0}^{\infty} |h_{s+1}(n) - h_s(n)| \\
&\leqslant \frac{2M\phi_0}{\xi(N+\beta)-\theta-1} P(n) n^{-\xi(N+\beta)+\theta+1} \sum_{s=0}^{\infty} \lambda^s \\
&\leqslant \frac{2M\phi_0}{\xi(N+\beta)-\theta-1-2M(\psi_0 p_0 + \eta_0)} P(n) n^{-\xi(N+\beta)+\theta+1}
\end{aligned}$$

for all $n \geqslant n_0$ and sufficient large N such that $0 < \lambda < 1$. □

In the following, the discussion of the error bound for asymptotic expansion of $y_1(n)$ is divided into two parts according to whether $a_0 = 0$ or not.

(i) If $a_0 \neq 0$, then let

$$\begin{aligned}
x(n) &= -\frac{\rho_1^2(1+(2/(n-1)))^{\alpha_1} - \rho_2^2(1+(2/(n-1)))^{\alpha_2}}{(a_0 + (a_1/(n-1)))[\rho_1(1+(1/(n-1)))^{\alpha_1} - \rho_2(1+(1/(n-1)))^{\alpha_2}]}, \qquad n \geqslant 2, \\
l(n) &= -\frac{\rho_1^2(1+(2/n))^{\alpha_1} + (a_0 + (a_1/n))\rho_1(1+(1/n))^{\alpha_1} x(n+1)}{x(n)x(n+1)} - b_0 - \frac{b_1}{n},
\end{aligned} \tag{2.4}$$

it can be easily verified that

$$z_i(n) = \rho_i^n n^{\alpha_i} \prod_{k=n}^{\infty} x(k), \qquad i = 1,2 \tag{2.5}$$

are two linearly independent solutions of the difference equation

$$z(n+2)+\left(a_0+\frac{a_1}{n}\right)z(n+1)+\left[b_0+\frac{b_1}{n}+l(n)\right]z(n)=0. \tag{2.6}$$

The two-term approximation of $x(n)$ is

$$x(n)=1-\left(\frac{a_1}{a_0}+\frac{\rho_1\alpha_1-\rho_2\alpha_2}{\rho_1-\rho_2}-2\frac{\rho_1^2\alpha_1-\rho_2^2\alpha_2}{\rho_1^2-\rho_2^2}\right)\frac{1}{n}+w(n), \tag{2.7}$$

where $w(n)$ is the remainder and $w_0=\sup_n\{n^2|w(n)|\}$ is a constant. By expressions of α_i, ρ_i, (1.4), (1.6), we know

$$\frac{a_1}{a_0}+\frac{\rho_1\alpha_1-\rho_2\alpha_2}{\rho_1-\rho_2}-2\frac{\rho_1^2\alpha_1-\rho_2^2\alpha_2}{\rho_1^2-\rho_2^2}=0,$$

and $l_0=\sup_n\{n^2|l(n)|\}$ is a constant. For bounds of $z_i(n)$, we have

Lemma 2. *If $x(k)=1+w(k)$, where $w(k)$ is a real or complex function, and $w_0=\sup_k\{k^2|w(k)|\}$ is a constant, then for $n\geqslant\sqrt{2w_0}$,*

$$\exp\left(-\frac{\pi^2}{3}w_0\right)\leqslant\left|\prod_{k=n}^{\infty}x(k)\right|\leqslant\exp\left(\frac{\pi^2}{3}w_0\right).$$

Proof. $\ln|\prod_{k=n}^{\infty}x(k)|=\sum_{k=n}^{\infty}\ln|x(k)|\leqslant\sum_{k=n}^{\infty}\ln(1+w_0/k^2)\leqslant\sum_{k=n}^{\infty}w_0/k^2\leqslant\frac{1}{6}\pi^2w_0$, so $|\prod_{k=n}^{\infty}x(k)|\leqslant\exp(\frac{1}{6}\pi^2w_0)\leqslant\exp(\frac{1}{3}\pi^2w_0)$. For the other part of this inequality, we have

$$\ln\left|\prod_{k=n}^{\infty}x(k)\right|\geqslant\sum_{k=n}^{\infty}\ln\left(1-\frac{w_0}{k^2}\right)\geqslant\sum_{k=n}^{\infty}\frac{-\frac{w_0}{k^2}}{1-\frac{w_0}{k^2}}=-\sum_{k=n}^{\infty}\frac{w_0}{k^2-w_0}. \tag{2.8}$$

Since $n\geqslant\sqrt{2w_0}$,

$$-\sum_{k=n}^{\infty}\frac{w_0}{k^2-w_0}\geqslant-\sum_{k=n}^{\infty}\frac{2w_0}{k^2}\geqslant-\frac{\pi^2}{3}w_0,$$

so $|\prod_{k=n}^{\infty}x(k)|\geqslant\exp(-\frac{1}{3}\pi^2w_0)$. □

From [2], any solution of the "sum equation"

$$\varepsilon_N^{(1)}(n)=\sum_{j=n}^{\infty}K(n;j)\Big\{-\rho_1^j j^{\alpha_1}R_N^{(1)}(j)$$
$$-\left[a(j)-a_0-\frac{a_1}{j}\right]\varepsilon_N^{(1)}(j+1)-\left[b(j)-b_0-\frac{b_1}{j}-l(j)\right]\varepsilon_N^{(1)}(j)\Big\} \tag{2.9}$$

is a solution of Eq. (2.2), where

$$K(n;j)=\frac{z_1(n)z_2(j+1)-z_1(j+1)z_2(n)}{z_1(j+1)z_2(j+2)-z_1(j+2)z_2(j+1)}$$

$$=\frac{z_1(n)}{z_1(j+1)}\cdot\frac{x(j+1)}{\rho_2}\left(\frac{j+1}{j+2}\right)^{\alpha_2}\frac{1-(\rho_1/\rho_2)^{j-n+1}(n/(j+1))^{\alpha_2-\alpha_1}}{1-(\rho_1/\rho_2)((j+1)/(j+2))^{\alpha_2-\alpha_1}}. \tag{2.10}$$

For the fact $\rho_1\neq\rho_2$, there exists a positive integer m which is large enough such that $m>\sqrt{2w_0}$ when $j\geqslant n\geqslant m$,

$$\left|1-\frac{\rho_1}{\rho_2}\left(\frac{j+1}{j+2}\right)^{\alpha_2-\alpha_1}\right|=\left|\left(1-\frac{\rho_1}{\rho_2}\right)+\frac{\rho_1}{\rho_2}\left[1-\left(\frac{j+1}{j+2}\right)^{\alpha_2-\alpha_1}\right]\right|$$
$$\geqslant\frac{1}{2}\left|1-\frac{\rho_1}{\rho_2}\right|.$$

From the assumption, $|\rho_2|\geqslant|\rho_1|$ (if $|\rho_2|=|\rho_1|$, then $\operatorname{Re}\alpha_2\geqslant\operatorname{Re}\alpha_1$),

$$q_0=\sup_{j\geqslant n\geqslant m}\left\{\left|1-\left(\frac{\rho_1}{\rho_2}\right)^{j-n+1}\left(\frac{n}{j+1}\right)^{\alpha_2-\alpha_1}\right|\right\}$$

is a constant. Note

$$M=2q_0\frac{1}{|\rho_1|}\left|1-\frac{\rho_1}{\rho_2}\right|^{-1}\exp\left(\frac{2\pi^2}{3}w_0\right)\sup_{j\geqslant n\geqslant m}\left\{\left|\frac{x(j+1)}{\rho_2}\left(\frac{j+1}{j+2}\right)^{\alpha_2}\left(\frac{j}{j+1}\right)^{\alpha_1}\right|\right\};$$

then from (2.10), $|K(n;j)|\leqslant MP(n)/P(j)$, where $P(n)=|\rho_1|^n n^{\operatorname{Re}\alpha_1}$. Let $\phi_0=r_N^{(1)}$, $\psi_0=\sup_{n\geqslant m}\{n^2|a(n)-a_0-a_1/n|\}$, $\eta_0=\sup_{n\geqslant m}\{n^2|b(n)-b_0-(b_1/n)-l(n)|\}$, $\theta=0$, $\tau=2$, $\xi=1$, $\beta=1$, $p_0=|\rho_1|\sup_{n\geqslant m}((n+1)/n)^{\operatorname{Re}\alpha_1}$; then from Lemma 1,

$$|\varepsilon_N^{(1)}(n)|\leqslant\frac{2M\phi_0}{N-2M(p_0\psi_0+\eta_0)}|\rho_1|^n n^{\operatorname{Re}\alpha_1-N},\qquad n\geqslant m. \tag{2.11}$$

(ii) If $a_0=0$, then let $\tilde{y}_1(n)=n^{-(b_1-2b_0)/2b_0}y_1(n)$; obviously, $\tilde{y}_1(n)$ satisfies

$$\tilde{y}_1(n+2)+\tilde{a}(n)\tilde{y}_1(n+1)+\tilde{b}(n)\tilde{y}_1(n)=0,$$

where

$$\tilde{a}(n)\sim\sum_{s=0}^{\infty}\frac{\tilde{a}_s}{n^s},\quad \tilde{b}(n)\sim\sum_{s=0}^{\infty}\frac{\tilde{b}_s}{n^s},\qquad n\to\infty,$$

and $\tilde{a}_0=0$, $\tilde{a}_1=a_1$, $\tilde{b}_0=b_0$, $\tilde{b}_1=2b_0$. Let $\tilde{\rho}_i$, $\tilde{\alpha}_i$, $\tilde{x}(n)$, $\tilde{l}(n)$, $\tilde{z}_i(n)$ be the same as those expressed by (1.4), (1.6), (2.4), (2.5) except that a_0, a_1, b_0, b_1, are replaced by $\tilde{a}_0,\tilde{a}_1,\tilde{b}_0,\tilde{b}_1$, respectively. For the fact $\tilde{b}_1=2\tilde{b}_0$, we have that $\tilde{x}(n)=1+\tilde{w}(n)$, where $\tilde{w}_0=\sup_n\{n^2|w(n)|\}$ is a constant, and $\tilde{l}_0=$

$\sup_n\{n^2|\tilde{l}(n)|\}$ is also a constant. Let $\tilde{y}_1(n) = \tilde{L}_N^{(1)}(n) + \tilde{\varepsilon}_N^{(1)}(n)$, where $\tilde{L}_N^{(1)}(n) = n^{-(b_1-2b_0)/(2b_0)}L_N^{(1)}(n)$; then similar to (2.11), we have the estimation of $\tilde{\varepsilon}_N^{(1)}(n)$:

$$|\tilde{\varepsilon}_N^{(1)}(n)| \leqslant \frac{2\tilde{M}\tilde{\phi}_0}{N - 2\tilde{M}(\tilde{p}_0\tilde{\psi}_0 + \tilde{\eta}_0)}|\tilde{\rho}_1|^n n^{\mathrm{Re}\,\tilde{\alpha}_1 - N}, \qquad n \geqslant \tilde{m},$$

where $\tilde{M}$, $\tilde{\phi}_0$, $\tilde{\psi}_0$, $\tilde{\eta}_0$, $\tilde{p}_0$, $\tilde{m}$ are the same as those in the previous subsection excepting $a(n)$, $b(n)$ are replaced by $\tilde{a}(n)$, $\tilde{b}(n)$, respectively. The estimation of $\varepsilon_N^{(1)}(n)$ is

$$|\varepsilon_N^{(1)}(n)| \leqslant \frac{2\tilde{M}\tilde{\phi}_0}{N - 2\tilde{M}(\tilde{p}_0\tilde{\psi}_0 + \tilde{\eta}_0)}|\rho_1|^n n^{\mathrm{Re}\,\alpha_1 - N}, \qquad n \geqslant \tilde{m}. \tag{2.11a}$$

2.2. The error bound for the asymptotic expansion of $y_2(n)$

Let

$$\varepsilon_N^{(2)}(n) = y_1(n)\delta_N(n); \tag{2.12}$$

then from Eq. (2.2) we have

$$y_1(n+2)\delta_N(n+2) + a(n)y_1(n+1)\delta_N(n+1) + b(n)y_1(n)\delta_N(n) = -\rho_2^n n^{\alpha_2} R_N^{(2)}(n). \tag{2.13}$$

For $y_1(n)$ being a solution of Eq. (1.1), let

$$\Delta_N(n) = \delta_N(n+1) - \delta_N(n); \tag{2.14}$$

then $\Delta_N(n)$ satisfies the first-order linear difference equation

$$y_1(n+2)\Delta_N(n+1) - b(n)y_1(n)\Delta_N(n) = -\rho_2^n n^{\alpha_2} R_N^{(2)}(n). \tag{2.15}$$

The solution of Eq. (2.15) is

$$\Delta_N(n) = -\sum_{j=n}^{\infty} \frac{X(n)}{X(j+1)} \frac{\rho_2^j j^{\alpha_2} R_N^{(2)}(j)}{y_1(j+2)}, \tag{2.16}$$

where $X(n) = X(m)\prod_{k=m}^{n-1} b(k)y_1(k)/y_1(k+2)$, $X(m)$ is an arbitrary constant, m is an integer which is large enough such that when $j \geqslant n \geqslant m$, $|y_1(j)| = |\rho_1|^n n^{\mathrm{Re}\,\alpha_1}\;|1+\varepsilon_1^{(1)}(j)| \geqslant \frac{1}{2}|\rho_1|^n n^{\mathrm{Re}\,\alpha}$. Now, let us consider the bound of $X(n)$. The two-term approximation of the function $b(k)y_1(k)/y_1(k+2)$ is

$$\begin{aligned}\frac{b(k)y_1(k)}{y_1(k+2)} &= \frac{b_0}{\rho_1^2}\left[1 + \frac{b_1/b_0 - 2\alpha_1}{k} + \sigma(k)\right] \\ &= \frac{\rho_2}{\rho_1}\left[1 + \frac{\alpha_2 - \alpha_1}{k} + \sigma(k)\right],\end{aligned}$$

where $\sigma(k)$ is the remainder, and $\sigma_0 = \sup_k\{k^2|\sigma(k)|\}$ is a constant. For the bound of $X(n)$, we introduce the following lemma.

Lemma 3. *If* $Y(n) = \prod_{k=m}^{n-1}[1 + \alpha/k + \sigma(k)]$, *where* α *is a constant, real or complex,* $\mathrm{Re}\,\alpha > 0$, $\sigma_0 = \sup_k\{k^2|\sigma(k)|\}$ *is a constant, then*

$$\mathrm{e}^{-k_1} n^{\mathrm{Re}\,\alpha} \leqslant |Y(n)| \leqslant \mathrm{e}^{k_1} n^{\mathrm{Re}\,\alpha},$$

where

$$k_1 = |\alpha| \left(\frac{1}{m} + \frac{1}{6m^2} + \frac{1}{60m^4} + \ln\, m \right) + \frac{\pi^2}{6}\tilde{\sigma}_0,$$

and

$$\tilde{\sigma}_0 = \sup_k \left\{ k^2 \left| \ln \left[1 + \frac{\alpha}{k} + \sigma(k) \right] - \frac{\alpha}{k} \right| \right\}.$$

Proof. For $Y(n) = \prod_{k=m}^{n-1}[1 + \alpha/k + \sigma(k)]$,

$$\ln Y(n) + 2l\pi\,\mathrm{i} = \sum_{k=m}^{n-1} \ln \left[1 + \frac{\alpha}{k} + \sigma(k) \right],$$

where l is an integer, $\ln Y$ is said to be the principal value of $\ln Y$,

$$\ln Y(n) + 2l\pi\,\mathrm{i} = \sum_{k=m}^{n-1} \left[\frac{\alpha}{k} + \tilde{\sigma}(k) \right],$$

where $\tilde{\sigma}(k)$ is the remainder of one-term approximation of $\ln[1+\alpha/k+\sigma(k)]$, and $\tilde{\sigma}_0 = \sup_k\{k^2|\tilde{\sigma}(k)|\}$ is a constant. From [5, p. 292],

$$\ln Y(n) + 2l\pi\,\mathrm{i} = \alpha \left(\ln\, n - \ln\, m - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120}\frac{\theta_{2,n}}{n^4} + \frac{1}{2m} + \frac{1}{12m^2} - \frac{1}{120}\frac{\theta_{2,m}}{m^4} \right) + \sum_{k=m}^{n-1} \tilde{\sigma}(k),$$

where $\theta_{m,n} \in (0,1)$. For $\sum_{k=1}^{\infty} 1/k^2 = \frac{1}{6}\pi^2$,

$$\mathrm{e}^{-k_1} n^{\mathrm{Re}\,\alpha} \leqslant |Y(n)| \leqslant \mathrm{e}^{k_1} n^{\mathrm{Re}\,\alpha}. \qquad \square$$

Set $\alpha = \alpha_2 - \alpha_1$ in Lemma 3; then we have the bound of $X(n)$,

$$|X(m)|\ \mathrm{e}^{-k_1} \left| \frac{\rho_2}{\rho_1} \right|^{n-m} n^{\mathrm{Re}(\alpha_2-\alpha_1)} \leqslant |X(n)| \leqslant |X(m)| \mathrm{e}^{k_1} \left| \frac{\rho_2}{\rho_1} \right|^{n-m} n^{\mathrm{Re}(\alpha_2-\alpha_1)}. \tag{2.17}$$

Substituting (2.17) into (2.16), we have

$$\begin{aligned} |\Delta_N(n)| &\leqslant \frac{2r_N^{(2)}\mathrm{e}^{2k_1}}{|\rho_1\rho_2|} \sup_{j\geqslant m} \left(\frac{j}{j+2} \right)^{\mathrm{Re}\,\alpha_1} \left| \frac{\rho_2}{\rho_1} \right|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)} \sum_{j=n}^{\infty} j^{-N-1} \\ &\leqslant \frac{4r_N^{(2)}\,\mathrm{e}^{2k_1}}{N|\rho_1\rho_2|} \sup_{j\geqslant m} \left(\frac{j}{j+2} \right)^{\mathrm{Re}\,\alpha_1} \left| \frac{\rho_2}{\rho_1} \right|^n n^{\mathrm{Re}(\alpha_2-\alpha_1)-N}. \end{aligned} \tag{2.18}$$

From (2.14),

$$\delta_N(n) = \delta_N(m) + \sum_{k=m}^{n-1} \Delta_N(k),$$

where $\delta_N(m)$ is an arbitrary constant,

$$|\delta_N(n)| \leqslant |\delta_N(m)| + \frac{4r_N^{(2)}\mathrm{e}^{2k_1}}{N|\rho_1\rho_2|}\sup_{j\geqslant m}\left(\frac{j}{j+2}\right)^{\operatorname{Re}\alpha_1}\sum_{k=m}^{n-1}\left|\frac{\rho_2}{\rho_1}\right|^k k^{\operatorname{Re}(\alpha_2-\alpha_1)-N}, \quad n\geqslant m. \tag{2.19}$$

Next, our discussion will be divided into three parts, according to different cases of ρ_i, α_i.

(i) When $|\rho_1| < |\rho_2|$, first of all, we estimate the bound of the series $\sum_{k=m}^{n} f(k)$, where $f(x) = \rho^x x^\alpha$.

Lemma 4. *Let $f(x) = \rho^x x^\alpha$. If $\rho > 1$, α is an arbitrary real constant, m is a positive integer which is large enough such that $f(x)/x$ is increasing when $x \geqslant m$, then we have*

$$\sum_{k=m}^{n} f(k) \leqslant \left[\frac{1+|\alpha|}{\ln\rho} + \frac{1}{8}\left(\ln\rho + \frac{|\alpha|}{n}\right) + \frac{1}{2}\frac{f(m)}{f(n)} + \frac{1}{2}\right] f(n).$$

Proof. By Euler–Maclaurin formula [5],

$$\sum_{k=m}^{n} f(k) = \int_m^n f(x)\,\mathrm{d}x + \tfrac{1}{2}f(m) + \tfrac{1}{2}f(n) + R_1(n),$$

where $|R_1(n)| \leqslant \frac{1}{8}|f'(n) - f'(m)|$. Integrating by parts, we can get

$$I = \int_m^n f(x)\,\mathrm{d}x = \frac{1}{\ln\rho}[f(n) - f(m)] - \frac{\alpha}{\ln\rho}\int_m^n \frac{f(x)}{x}\,\mathrm{d}x,$$

$$|I| \leqslant \frac{1+|\alpha|}{\ln\rho} f(n).$$

Obviously, $|R_1(n)| \leqslant \frac{1}{8}(\ln\rho + |\alpha|/n) f(n)$, so we complete the proof. □

Set $\rho = |\rho_2/\rho_1| > 1$, $\alpha = \operatorname{Re}(\alpha_2 - \alpha_1) - N$, then from (2.19) and Lemma 4,

$$|\delta_N(n)| \leqslant \mu \left|\frac{\rho_2}{\rho_1}\right|^n n^{\operatorname{Re}(\alpha_2-\alpha_1)-N},$$

where

$$\mu = |\delta_N(m)| \sup_{n\geqslant m}\left\{\left|\frac{\rho_1}{\rho_2}\right|^n n^{\operatorname{Re}(\alpha_1-\alpha_2)+N}\right\} + \frac{4r_N^{(2)}\mathrm{e}^{2k_1}}{N|\rho_1\rho_2|}\sup_{j\geqslant m}\left(\frac{j}{j+2}\right)^{\operatorname{Re}\alpha_1}\left[\frac{1+|\alpha|}{\ln\rho} + \frac{1}{8}\left(\ln\rho + \frac{|\alpha|}{m}\right) + 1\right].$$

From (2.1), (2.11) and (2.12),

$$|\varepsilon_N^{(2)}(n)| \leqslant \mu \sup_{n\geqslant m}\{|\rho_1|^{-n} n^{-\operatorname{Re}\alpha_1}|y_1(n)|\}|\rho_2|^n n^{\operatorname{Re}\alpha_2 - N}, \qquad n\geqslant m. \tag{2.20}$$

(ii) When $|\rho_1| = |\rho_2|$, $\operatorname{Re}(\alpha_2 - \alpha_1) \geqslant N$, then directly from (2.19),

$$|\delta_N(n)| \leqslant \mu n^{\operatorname{Re}(\alpha_2-\alpha_1)-N+1},$$

where $\mu = |\delta_N(m)|n^{\mathrm{Re}(\alpha_1-\alpha_2)+N-1} + (8r_N^{(2)}e^{2k_1}/N|\rho_1|^2)\sup_{j\geqslant n}(j/(j+1))^{\mathrm{Re}\,\alpha_1}$. Similar to (2.20),

$$|\varepsilon_N^{(2)}(n)| \leqslant \mu \sup_{n\geqslant m}\{|\rho_1|^{-n}n^{-\mathrm{Re}\,\alpha_1}|y_1(n)|\}|\rho_2|^n n^{\mathrm{Re}\,\alpha_2-N+1}, \qquad n\geqslant m. \tag{2.21}$$

(iii) When $|\rho_1| = |\rho_2|$, $\mathrm{Re}(\alpha_2-\alpha_1) < N$, then we know, similar to (2.9), any solution of the equation

$$\begin{aligned}\varepsilon_N^{(2)}(n) = \sum_{j=n}^{\infty}K(n;j)\Big\{&-\rho_2^j j^{\alpha_2}R_N^{(2)}(j)\\ &-\left[a(j)-a_0-\frac{a_1}{j}\right]\varepsilon_N^{(2)}(j+1)-\left[b(j)-b_0-\frac{b_1}{j}-l(j)\right]\varepsilon_N^{(2)}(j)\Big\}\end{aligned} \tag{2.22}$$

is a solution of Eq. (2.2) for $i = 2$, where

$$\begin{aligned}K(n;j) &= \frac{z_2(n)z_1(j+1)-z_1(n)z_2(j+1)}{z_2(j+1)z_1(j+2)-z_1(j+1)z_2(j+2)}\\ &= \frac{z_2(n)}{z_2(j+1)}\left(\frac{n}{j}\right)^{\alpha_1-\alpha_2}\frac{x(j+1)}{\rho_1}\left(\frac{j+1}{j+2}\right)^{\alpha_1}\\ &\quad\times\frac{(j/n)^{\alpha_1-\alpha_2}-(\rho_2/\rho_1)^{j-n+1}(j/(j+1))^{\alpha_1-\alpha_2}}{1-(\rho_2/\rho_1)((j+1)/(j+2))^{\alpha_1-\alpha_2}}.\end{aligned}$$

For $\rho_1 \neq \rho_2$, let m be large enough such that when $j\geqslant m\geqslant\sqrt{2w_0}$,

$$\left|1-\frac{\rho_2}{\rho_1}\left(\frac{j+1}{j+2}\right)^{\alpha_1-\alpha_2}\right| \geqslant \frac{1}{2}\left|1-\frac{\rho_2}{\rho_1}\right|.$$

Under conditions $|\rho_1| = |\rho_2|$, $0\leqslant \mathrm{Re}(\alpha_2-\alpha_1) < N$, we have

$$k_0 = \sup_{j\geqslant m}\left\{\left|\frac{x(j+1)}{\rho_1\rho_2}\left(\frac{j+1}{j+2}\right)^{\alpha_1}\left(\frac{j}{j+1}\right)^{\alpha_2}\frac{(j/n)^{\alpha_1-\alpha_2}-(\rho_2/\rho_1)^{j-n+1}(j/(j+1))^{\alpha_1-\alpha_2}}{1-(\rho_2/\rho_1)((j+1)/(j+2))^{\alpha_1-\alpha_2}}\right|\right\}$$

is a constant, and $|K(n;j)|\leqslant M|\rho_2|^n n^{\mathrm{Re}\,\alpha_1}/|\rho_2|^j j^{\mathrm{Re}\,\alpha_1}$, where from (2.8), $M = k_0\exp(\frac{2}{3}\pi^2 w_0)$. Similar to Lemma 1, we have

$$|\varepsilon_N^{(2)}(n)| \leqslant \frac{2Mr_N^{(2)}}{N-\mathrm{Re}(\alpha_2-\alpha_1)-2M(p_0\psi_0+\eta_0)}|\rho_2|^n n^{\mathrm{Re}\,\alpha_2-N}, \qquad n\geqslant m,$$

where ψ_0, η_0, p_0 are listed in Lemma 1.

3. Error bounds in subnormal case

In this section, without loss of generality, we suppose that $\mathrm{Re}\,\gamma_1\leqslant\mathrm{Re}\,\gamma_2$, and $a_0 = -2$, $b_0 = 1$. Thus $\rho = 1$, $\alpha = \frac{1}{4}+\frac{1}{2}b_1$. Let

$$y(n) = L_N^{(i)}(n) + \varepsilon_N^{(i)}(n), \qquad i = 1,2, \tag{3.1}$$

where from (1.8), $L_N^{(i)}(n) = n^\alpha e^{\gamma_i\sqrt{n}}\sum_{s=0}^{N-1} c_s^{(i)}/n^{s/2}$, and $\varepsilon_N^{(i)}(n)$ are error terms of asymptotic expansions of $y_i(n)$ $(i = 1,2)$. From [6] we know that $\varepsilon_N^{(i)}(n)$ satisfy second-order linear difference equations

$$\varepsilon_N^{(i)}(n+2) + a(n)\varepsilon_N^{(i)}(n+1) + b(n)\varepsilon_N^{(i)}(n) = -n^\alpha e^{\gamma_i\sqrt{n}} R_N^{(i)}(n), \tag{3.2}$$

where $R_N^{(i)}(n) = n^{-\alpha}e^{-\gamma_i\sqrt{n}}[L_N^{(i)}(n+2)+a(n)L_N^{(i)}(n+1)+b(n)L_N^{(i)}(n)]$, and $r_N^{(i)} = \sup_n \left\{n^{(N+3)/2}|R_N^{(i)}(n)|\right\}$ are constants, $i = 1,2$.

3.1. The error bound for the asymptotic expansion of $y_1(n)$

Let

$$x(n) = -\frac{(1+(1/n))^\alpha}{a_0 + (a_1/(n-1))}\,\frac{e^{\gamma_2(\sqrt{n+1}-\sqrt{n-1})} - e^{\gamma_1(\sqrt{n+1}-\sqrt{n-1})}}{e^{\gamma_2(\sqrt{n}-\sqrt{n-1})} - e^{\gamma_1(\sqrt{n}-\sqrt{n-1})}},$$

$$l(n) = -\frac{e^{\gamma_1(\sqrt{n+2}-\sqrt{n})}(1+(2/n))^\alpha + (a_0 + (a_1/n))e^{\gamma_1(\sqrt{n+1}-\sqrt{n})}(1+(1/n))^\alpha x(n+1)}{x(n+1)x(n+2)} - b_0 - \frac{b_1}{n},$$

then it can be easily verified from expressions of α and γ_i (1.9), (1.10) that $w_0 = \sup_n\{n^2|x(n)-1|\}$, $l_0 = \sup_n\{n^2|l(n)|\}$ are constants and

$$z_i(n) = e^{\gamma_i\sqrt{n}} n^\alpha \prod_{k=n}^{\infty} x(k), \qquad i = 1,2 \tag{3.3}$$

are two linearly independent solutions of the auxiliary difference equation

$$z(n+2) + \left(a_0 + \frac{a_1}{n}\right) z(n+1) + \left[b_0 + \frac{b_0}{n} + l(n)\right] z(n) = 0. \tag{3.4}$$

By Lemma 2, we have when $n \geqslant \sqrt{2w_0}$,

$$\exp\left(-\tfrac{1}{3}\pi w_0\right) e^{\mathrm{Re}\,\gamma_i\sqrt{n}} n^{\mathrm{Re}\,\alpha} \leqslant |z_i(n)| \leqslant \exp\left(\tfrac{1}{3}\pi w_0\right) e^{\mathrm{Re}\,\gamma_i\sqrt{n}} n^{\mathrm{Re}\,\alpha}. \tag{3.5}$$

We rewrite Eq. (3.2) as

$$\begin{aligned}
\varepsilon_N^{(1)}(n+2) &+ \left(a_0 + \frac{a_1}{n}\right)\varepsilon_N^{(1)}(n+1) + \left[b_0 + \frac{b_1}{n} + l(n)\right]\varepsilon_N^{(1)}(n) \\
&= -n^\alpha e^{\gamma_1\sqrt{n}} R_N^{(1)}(n) - \left[a(n) - a_0 - \frac{a_1}{n}\right]\varepsilon_N^{(1)}(n+1) \\
&\quad - \left[b(n) - b_0 - \frac{b_1}{n} - l(n)\right]\varepsilon_N^{(1)}(n).
\end{aligned}$$

From [2], any solution of the equation

$$\varepsilon_N^{(1)}(n) = \sum_{j=n}^{\infty} K(n;j)\left\{-\mathrm{e}^{\gamma_1\sqrt{j}} j^{\alpha} R_N^{(1)}(j) - \left[a(j) - a_0 - \frac{a_1}{j}\right]\varepsilon_N^{(1)}(j+1) \right.$$
$$\left. - \left[b(j) - b_0 - \frac{b_1}{j} - l(j)\right]\varepsilon_N^{(1)}(j)\right\} \tag{3.6}$$

is a solution of Eq. (3.2), where

$$\begin{aligned} K(n;j) &= -\frac{z_1(j+1)z_2(n) - z_2(j+1)z_1(n)}{z_1(j+1)z_2(j+2) - z_2(j+1)z_1(j+2)} \\ &= \frac{z_1(n)}{z_1(j)} x(j)x(j+1)\left(\frac{j}{j+2}\right)^{\alpha} \exp[\gamma_1(\sqrt{j} - \sqrt{j+1}) + \gamma_2(\sqrt{j+1} - \sqrt{j+2})] \\ &\quad \times \frac{1 - e^{(\gamma_1-\gamma_2)(\sqrt{j+1}-\sqrt{n})}}{1 - e^{(\gamma_1-\gamma_2)(\sqrt{j+2}-\sqrt{j+1})}}. \end{aligned}$$

For the fact $\operatorname{Re}\gamma_1 \leqslant \operatorname{Re}\gamma_2$, we have $\operatorname{Re}\gamma_1 \leqslant 0$, $\operatorname{Re}\gamma_2 \geqslant 0$, and

$$|1 - \mathrm{e}^{(\gamma_1-\gamma_2)(\sqrt{j+1}-\sqrt{n})}| \leqslant 2 \qquad (j \geqslant n);$$
$$|1 - \mathrm{e}^{(\gamma_1-\gamma_2)(\sqrt{j+2}-\sqrt{j+1})}| \geqslant \frac{1}{2}\frac{|\gamma_1 - \gamma_2|}{\sqrt{j}}.$$

Thus,

$$k_0 = \sup_{j \geqslant n}\left\{\left|x(j)x(j+1)\left(\frac{j}{j+2}\right)^{\alpha} \exp[\gamma_1(\sqrt{j} - \sqrt{j+1}) + \gamma_2(\sqrt{j+1} - \sqrt{j+2})]\right|\right\}$$

is a constant, and

$$|K(n;j)| \leqslant \frac{4k_0}{|\gamma_1 - \gamma_2|}\sqrt{j}\left|\frac{z_1(n)}{z_2(j)}\right|.$$

Let $M = (4k_0/|\gamma_1 - \gamma_2|)\exp(\frac{2}{3}\pi^2 w_0)$, $\phi_0 = r_N^{(1)}$, $\psi_0 = \sup_n\{n^2|a(n) - a_0 - a_1/n|\}$, $\eta_0 = \sup_n\{n^2\ |b(n) - b_0 - b_1/n - l(n)|\}$, $P(n) = \mathrm{e}^{\operatorname{Re}\gamma_1\sqrt{n}} n^{\operatorname{Re}\alpha}$, $p_0 = \sup_n[P(n+1)/p(n)]$, $\theta = \frac{1}{2}$, $\tau = 2$, $\xi = \frac{1}{2}, \beta = 3$; then we have from Lemma 1,

$$|\varepsilon_N^{(1)}(n)| \leqslant \frac{4Mr_N^{(1)}}{N - 4M(p_0\psi_0 + \eta_0)}\mathrm{e}^{\operatorname{Re}\gamma_1\sqrt{n}} n^{\operatorname{Re}\alpha - (N/2)}, \qquad n \geqslant \sqrt{2w_0}. \tag{3.7}$$

3.2. The error bound of the asymptotic expansion of $y_2(n)$

Let

$$\varepsilon_N^{(2)}(n) = \delta_N(n) y_1(n), \tag{3.8}$$

then by (3.2), $\delta_N(n)$ satisfies the difference equation

$$y_1(n+2)\delta_N(n+2) + a(n)y_1(n+1)\delta_N(n+1) + b(n)y_1(n)\delta_N(n) = -e^{\gamma_2\sqrt{n}}n^{\alpha}R_N^{(2)}(n). \tag{3.9}$$

Set

$$\Delta_N(n) = \delta_N(n+1) - \delta_N(n), \tag{3.10}$$

then for $y_1(n)$ being a solution of Eq. (1.1),

$$y_1(n+2)\Delta_N(n+1) - b(n)y_1(n)\Delta_N(n) = -e^{\gamma_2\sqrt{n}}n^{\alpha}R_N^{(2)}(n). \tag{3.11}$$

Obviously, the solution of the first-order linear difference equation (3.11) is

$$\Delta_N(n) = -\sum_{j=n}^{\infty}\frac{X(n)}{X(j+1)}\,\frac{e^{\gamma_2\sqrt{j}}j^{\alpha}R_N^{(2)}(j)}{y_1(j+2)}, \qquad n\geqslant m, \tag{3.12}$$

where m is an integer which is large enough such that when $j\geqslant m$,

$$\begin{aligned}|y_1(j+2)| &= e^{\operatorname{Re}\gamma_1\sqrt{j}}j^{\operatorname{Re}\alpha}\left|1+\frac{\gamma_1+c_1^{(1)}}{\sqrt{j}}+\frac{c_2^{(1)}+\gamma_1c_1^{(1)}+\frac{1}{2}\gamma_1^2+2\alpha}{j}+\sigma(j)\right|\\ &\geqslant \tfrac{1}{2}e^{\operatorname{Re}\gamma_1\sqrt{j}}j^{\operatorname{Re}\alpha},\end{aligned}$$

$\sigma(j)$ is the remainder of the three-term approximation of the function $y_1(j+2)$, $\sigma_0 = \sup_j\{j^{3/2}|\sigma(j)|\}$ is a constant, and

$$X(n) = X(m)\prod_{k=m}^{n-1}\frac{b(k)y_1(k)}{y_1(k+2)},$$

$X(m)$ is an arbitrary constant. The three-term approximation of $b(k)y_1(k)/y_1(k+2)$ is

$$\frac{b(k)y_1(k)}{y_1(k+2)} = 1 - \frac{\gamma_1}{\sqrt{k}} + \frac{\frac{1}{2}\gamma_1^2-2\alpha+b_1}{k} + \tilde{\sigma}(k),$$

where $\tilde{\sigma}(k)$ is the remainder and $\tilde{\sigma}_0 = \sup_k\{k^{3/2}|\tilde{\sigma}(k)|\}$ is a constant. The two-term approximation of $\ln[b(k)y_1(k)/y_1(k+2)]$ (here we take the principal value of the logarithm function) is

$$\ln\left[\frac{b(k)y_1(k)}{y_1(k+1)}\right] = -\frac{\gamma_1}{\sqrt{k}} + \frac{b_1-2\alpha}{k} + u(k),$$

where $u(k)$ is the remainder, $u_0 = \sup_n\{k^{3/2}|u(k)|\}$ is a constant. Form [5], we know that

$$\sum_{k=1}^{n-1}\frac{1}{\sqrt{k}} = \zeta\left(\frac{1}{2}\right) + 2\sqrt{n} + v(n),$$

where $\zeta(x)$ is zeta function, the remainder $v(n)$ satisfies

$$|v(n)| \leqslant \frac{1}{2\sqrt{n}} + \frac{1}{12n^{3/2}}.$$

Similar to Lemma 3, we have for $b_1 - 2\alpha = -\frac{1}{2}$,

$$|X(m)|e^{-k_2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}}n^{-1/2} \leqslant |X(n)| \leqslant |X(m)|\, e^{k_2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}}n^{-1/2}, \tag{3.13}$$

where

$$k_2 = |\gamma_1|\left(2\sqrt{m} + \frac{1}{\sqrt{m}} + \frac{1}{6m^{3/2}}\right) + \frac{1}{2}\ln\ m + \frac{4u_0}{\sqrt{m}} + \frac{1}{2m} + \frac{1}{2m^2} + \frac{1}{120m^4}.$$

Substituting (3.13) into (3.12), for $\gamma_1 + \gamma_2 = 0$ and $\mathrm{Re}\,r_1 \leqslant 0$,

$$\begin{aligned} |\Delta_N(n)| &\leqslant 4e^{2k_2}r_N^{(2)}n^{-1/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}}\sum_{j=n}^{\infty}j^{-(N+2)/2} \\ &\leqslant \frac{16}{N}r_N^{(2)}e^{2k_2}n^{-(N+1)/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}}, \qquad n \geqslant m. \end{aligned} \tag{3.14}$$

In the following, our discussion will be divided into two parts according to whether $\mathrm{Re}\,\gamma_1 = 0$.

(i) If $\mathrm{Re}\,\gamma_1 = 0$, then from (3.10), (3.14), when $N \geqslant 2$,

$$\begin{aligned} |\delta_N(n)| &= \left|\sum_{k=n}^{\infty}\Delta_N(k)\right| \\ &\leqslant \frac{16}{N}r_N^{(2)}e^{2k_2}\sum_{k=n}^{\infty}k^{-(N+1)/2} \\ &\leqslant \frac{64}{N-1}r_N^{(2)}e^{2k_1}n^{-(N-1)/2}, \end{aligned}$$

from (3.8),

$$|\varepsilon_N^{(2)}(n)| \leqslant \frac{64}{N-1}r_N^{(2)}e^{2k_1}\ \sup_{n\geqslant m}\{n^{-\mathrm{Re}\,\alpha}|y_1(n)|\}n^{\mathrm{Re}\,\alpha-(N-1)/2}.$$

(ii) If $\mathrm{Re}\,\gamma_1 \neq 0$, then $\mathrm{Re}\,\gamma_1 < 0$, from (3.10), (3.14),

$$\begin{aligned} |\delta_N(n)| &= \left|\delta_N(m) + \sum_{k=m}^{n-1}\Delta_N(k)\right| \\ &\leqslant |\delta_N(m)| \ + \frac{16}{N}r_N^{(2)}e^{2k_2}\sum_{k=m}^{n-1}k^{-(N+1)/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{k}}, \end{aligned}$$

where $\delta_N(m)$ is an arbitrary constant. Similar to Lemma 4, we have

$$\sum_{k=m}^{n-1}k^{-(N+1)/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{k}} \leqslant \left[-\frac{N+1}{\mathrm{Re}\,\gamma_1} + \frac{1}{\sqrt{n}} - \frac{\mathrm{Re}\,\gamma_1}{8n} + \frac{N+1}{16n^{3/2}}\right]n^{-N/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}},$$

so

$$|\delta_N(n)| \leqslant \mu n^{-N/2}e^{-2\mathrm{Re}\,\gamma_1\sqrt{n}},$$

where

$$\mu = |\delta_N(m)| \sup_{n\geqslant m}\{n^{N/2}e^{2\mathrm{Re}\,\gamma_1\sqrt{n}}\} + \frac{16}{N}r_N^{(2)}e^{2k_2}\left[-\frac{N+1}{\mathrm{Re}\,\gamma_1}+\frac{1}{\sqrt{n}}-\frac{\mathrm{Re}\,\gamma_1}{8n}+\frac{N+1}{16n^{3/2}}\right],$$

$$|\varepsilon_N^{(2)}(n)| \leqslant \mu \sup_{n\geqslant m}\{n^{-\mathrm{Re}\,\alpha}e^{-\mathrm{Re}\,\gamma_1\sqrt{n}}\ |y_1(n)|\ \}n^{\mathrm{Re}\,\alpha-N/2}e^{\mathrm{Re}\,\gamma_2\sqrt{n}}, \qquad n\geqslant m.$$

4. Error bounds in exceptional cases

In this section, without loss of generality, we suppose that $a_0=-2,\ b_0=1$.

4.1. Error bounds in the first exceptional case

In this subsection, without loss of generality, we suppose that $\mathrm{Re}\,\alpha_1\leqslant\mathrm{Re}\,\alpha_2$. Let

$$y_i(n)=L_N^{(i)}(n)+\varepsilon_N^{(i)}(n), \qquad i=1,2, \tag{4.1}$$

where from (1.13), $L_N^{(i)}(n)=n^{\alpha_i}\sum_{s=0}^{N-1}c_s^{(i)}/n^s$, $i=1,2$, and $\varepsilon_N^{(i)}(n)$ are error terms. From [6] we know that

$$\varepsilon_N^{(i)}(n+2)+a(n)\varepsilon_N^{(i)}(n+1)+b(n)\varepsilon_N^{(i)}(n)=-n^{\alpha_i}R_N^{(i)}(n), \qquad i=1,2, \tag{4.2}$$

where $R_N^{(i)}(n)=n^{-\alpha_i}[L_N^{(i)}(n+2)+a(n)L_N^{(i)}(n+1)+b(n)L_N^{(i)}(n)]$, and $r_N^{(i)}=\sup_n\{n^{N+2}\ |R_N^{(i)}(n)|\}$ are constants, $i=1,2$.

4.1.1 The error bound of the asymptotic expansion of $y_1(n)$

Let

$$x(n) = -\frac{(1+(2/(n-1)))^{\alpha_1}-(1+(2/(n-1)))^{\alpha_2}}{[a_0+(a_1/(n-1))+(a_2/(n-1)^2)][(1+(1/(n-1)))^{\alpha_1}-(1+(1/(n-1)))^{\alpha_2}]}, \qquad n\geqslant 2,$$

$$l(n) = -\frac{(1+(2/n))^{\alpha_1}+(a_0+(a_1/n)+(a_2/n^2))(1+(1/n))^{\alpha_1}x(n+1)}{x(n+1)x(n)}-b_0-\frac{b_1}{n}-\frac{b_2}{n^2};$$

then it can be easily verified that

$$z_i(n)=n^{\alpha_i}\prod_{k=n}^{\infty}x(k), \qquad i=1,2$$

are two linearly independent solutions of the difference equation

$$z(n+2)+\left(a_0+\frac{a_1}{n}+\frac{a_2}{n^2}\right)z(n+1)+\left[b_0+\frac{b_1}{n}+\frac{b_2}{n^2}+l(n)\right]z(n)=0. \tag{4.3}$$

By expressions of α_i, we also know that $x(n)$ has two-term approximation $x(n)=1+(\frac{1}{2}a_1+\frac{1}{4}a_1^2-\frac{1}{2}b_2)/n^2+w(n)$, where $w(n)$ is the remainder, $w_0=\sup_n\{n^3|w(n)|\}$, $l_0=\sup_n\{n^3|l(n)|\}$ are constants. So from Lemma 2,

$$\exp\left(-\tfrac{1}{3}\pi^2\tilde{w}_0\right)n^{\mathrm{Re}\,\alpha_i}\leqslant|z_i(n)|\leqslant\exp\left(\tfrac{1}{3}\pi^2\tilde{w}_0\right)n^{\mathrm{Re}\,\alpha_i}, \qquad n\geqslant\sqrt{2\tilde{w}_0}, \tag{4.4}$$

where $\tilde{w}_0 = \sup_n[|\frac{1}{2}a_1 + \frac{1}{4}a_1^2 - \frac{1}{2}b_2| + n^2|w(n)|]$. Rewriting Eq. (4.2),

$$\varepsilon_N^{(1)}(n+2) + \left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2}\right)\varepsilon_N^{(1)}(n+1) + \left[b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + l(n)\right]\varepsilon_N^{(1)}(n)$$
$$= -n^{\alpha_1}R_N^{(1)}(n) - \left[a(n) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2}\right]\varepsilon_N^{(1)}(n+1)$$
$$- \left[b(n) - b_0 - \frac{b_1}{n} - \frac{b_2}{n^2} - l(n)\right]\varepsilon_N^{(1)}(n), \tag{4.5}$$

we can easily verify that any solution of the equation

$$\varepsilon_N^{(1)}(n) = \sum_{j=n}^{\infty} K(n;j)\left\{-j^{\alpha_1}R_N^{(1)}(j) - \left[a(j) - a_0 - \frac{a_1}{j} - \frac{a_2}{j^2}\right]\varepsilon_N^{(1)}(j+1)\right.$$
$$\left. - \left[b(j) - b_0 - \frac{b_1}{j} - \frac{b_2}{j^2} - l(j)\right]\varepsilon_N^{(1)}(j)\right\} \tag{4.6}$$

is a solution of Eq. (4.5), where

$$K(n;j) = \frac{z_1(j+1)z_2(n) - z_2(j+1)z_1(n)}{z_1(j+2)z_2(j+1) - z_2(j+2)z_1(j+1)}$$
$$= \frac{z_1(n)}{z_1(j+1)} \frac{(n/(j+1))^{\alpha_2-\alpha_1} - 1}{(1+(1/(j+1)))^{\alpha_1} - (1+(1/(j+1)))^{\alpha_2}} x(j+1).$$

For $\operatorname{Re}\alpha_1 \leqslant \operatorname{Re}\alpha_2$, $|(n/(j+1))^{\alpha_2-\alpha_1} - 1| \leqslant 2$, $j \geqslant n$. Obviously,

$$\left(1 + \frac{1}{j+1}\right)^{\alpha_1} - \left(1 + \frac{1}{j+1}\right)^{\alpha_2} = \frac{\alpha_1 - \alpha_2}{j} + \sigma(j),$$

where $\sigma(j)$ is the remainder of the one-term approximation of the function, and $\sigma_0 = \sup_j\{j^2|\sigma(j)|\}$ is a constant. Let m be an integer which is large enough such that $m \geqslant \sqrt{2\tilde{w}_0}$ and when $j \geqslant m$, $j|\sigma(j)| \leqslant \frac{1}{2}|\alpha_1 - \alpha_2|$ and $|x(j+1)| \leqslant 2$. Then when $j \geqslant n \geqslant m$,

$$|K(n;j)| \leqslant \frac{8}{|\alpha_1 - \alpha_2|}\, j \left|\frac{z_1(n)}{z_1(j+1)}\right| \leqslant M\frac{n^{\operatorname{Re}\alpha_1}}{j^{\operatorname{Re}\alpha_1 - 1}},$$

where

$$M = \frac{8\exp(2\pi^2\tilde{w}_0/3)}{|\alpha_1 - \alpha_2|} \sup_n \left(\frac{n}{n+1}\right)^{\operatorname{Re}\alpha_1}.$$

Let $\phi_0 = r_N^{(1)}$, $\psi_0 = \sup_{n\geqslant m}\{n^3|a(n) - a_0 - (a_1/n) - (a_2/n^2)|\}$, $\eta_0 = \sup_{n\geqslant m}\{n^3|b(n) - b_0 - (b_1/n) - (b_2/n^2) - l(n)|\}$, $P(n) = n^{\operatorname{Re}\alpha_1}$, $p_0 = \sup_{n\geqslant m}\{P(n+1)/P(n)\}$, $\theta = 1$, $\xi = 1$, $\beta = 2$, $\tau = 3$; then from Lemma 1

$$|\varepsilon_N^{(1)}(n)| \leqslant \frac{2M\phi_0}{N - 2M(\psi_0 p_0 + \eta_0)} n^{\operatorname{Re}\alpha_1 - N}, \qquad n \geqslant \sqrt{2\tilde{w}_0}. \tag{4.7}$$

4.1.2. The error bound of the asymptotic expansion of $y_2(n)$

Let

$$\varepsilon_N^{(2)}(n) = \delta_N(n)y_1(n), \qquad \Delta_N(n) = \delta_N(n+1) - \delta_N(n), \tag{4.8}$$

then from (4.2),

$$y_1(n+2)\Delta_N(n+1) - b(n)y_1(n)\Delta_N(n) = -n^{\alpha_2}R_N^{(2)}(n).$$

The solution of the first-order linear difference equation is

$$\Delta_N(n) = -\sum_{j=n}^{\infty} \frac{X(n)}{X(j+1)} \frac{j^{\alpha_2}R_N^{(2)}(j)}{y_1(j+2)}, \qquad n \geqslant m,$$

where $X(n) = X(m)\prod_{k=m}^{n-1}(b(k)y_1(k))/(y_1(k+2))$, $X(m)$ is an arbitrary constant, and m is an integer which is large enough such that when $n \geqslant m$,

$$|n^{-\alpha_1}y_1(n)| = |1 + \varepsilon_1^{(1)}(n)n^{-\mathrm{Re}\,\alpha_1}| \geqslant \tfrac{1}{2}.$$

The function $b(k)y_1(k)/y_1(k+2)$ has a two-term approximation

$$\frac{b(k)y_1(k)}{y_1(k+2)} = 1 + \frac{b_1 - 2\alpha_1}{k} + u(k),$$

where $u(k)$ is the remainder, and $u_0 = \sup_k\{k^2|u(k)|\}$ is a constant. For $b_1 - 2\alpha_1 = \alpha_2 - \alpha_1 - 1$, then from Lemma 3,

$$|X(m)|\mathrm{e}^{-k_1}n^{\mathrm{Re}(\alpha_2-\alpha_1)-1} \leqslant |X(n)| \leqslant |X(m)|\mathrm{e}^{k_1}n^{\mathrm{Re}(\alpha_2-\alpha_1)-1},$$

where k_1 is the same as in Lemma 3. So

$$|\Delta_N(n)| \leqslant 2\mathrm{e}^{2k_1}r_N^{(2)}n^{\mathrm{Re}(\alpha_2-\alpha_1)-1}\sum_{j=n}^{\infty} j^{-N-1} \leqslant \frac{4}{N}\mathrm{e}^{2k_1}r_N^{(2)}n^{\mathrm{Re}(\alpha_2-\alpha_1)-N-1}. \tag{4.9}$$

In the following, we discuss the bound of $\varepsilon_N^{(2)}(n)$ in three cases.

(i) If $\mathrm{Re}(\alpha_2 - \alpha_1) < N$, then from (4.8)

$$\begin{aligned}
\delta_N(n) &= -\sum_{j=n}^{\infty}\Delta_N(j),\\
|\delta_N(n)| &\leqslant \sum_{j=n}^{\infty}|\Delta_N(j)| \leqslant \frac{4}{N}\mathrm{e}^{2k_1}r_N^{(2)}\sum_{j=n}^{\infty} j^{\mathrm{Re}(\alpha_2-\alpha_1)-N-1}\\
&\leqslant \frac{8r_N^{(2)}\mathrm{e}^{2k_1}}{N[N-\mathrm{Re}(\alpha_2-\alpha_1)]}n^{\mathrm{Re}(\alpha_2-\alpha_1)-N},
\end{aligned}$$

where $n \geqslant m_0$. Also from (4.8),

$$|\varepsilon_N^{(2)}(n)| \leqslant \frac{8r_N^{(2)}\mathrm{e}^{2k_1}}{N[N-\mathrm{Re}(\alpha_2-\alpha_1)]}\sup_{n\geqslant m}\{n^{-\mathrm{Re}\,\alpha_1}|y_1(n)|\}n^{\mathrm{Re}\,\alpha_2-N}, \qquad n \geqslant m.$$

(ii) If $\mathrm{Re}(\alpha_2 - \alpha_1) > N$, then from (4.8)

$$\delta_N(n) = \delta_N(m) + \sum_{j=m}^{n-1} \Delta_N(j), \tag{4.10}$$

where $\delta_N(m)$ is an arbitrary constant,

$$|\delta_N(n)| \leqslant |\delta_N(m)| + \frac{4}{N} e^{2k_1} r_N^{(2)} \sum_{j=m}^{n-1} j^{\mathrm{Re}(\alpha_2-\alpha_1)-N-1}.$$

By Abel–Plana formula [5, p.292], we know that

$$\sum_{j=1}^{n-1} j^{\mathrm{Re}(\alpha_2-\alpha_1)-N-1} = \zeta\,[\mathrm{Re}(\alpha_1-\alpha_2)+N+1] + \frac{n^{\mathrm{Re}(\alpha_2-\alpha_1)-N}}{\mathrm{Re}(\alpha_2-\alpha_1)-N} + \sigma(n),$$

where $\zeta(x)$ is zeta function, and

$$|\sigma(n)| \leqslant n^{\mathrm{Re}(\alpha_2-\alpha_1)-N} \left[\frac{1}{2} + \frac{\mathrm{Re}(\alpha_2-\alpha_1)-N-1}{6n}\right].$$

So

$$|\delta_N(n)| \leqslant \mu n^{\mathrm{Re}(\alpha_2-\alpha_1)-N},$$

where

$$\begin{aligned}\mu = {} & |\delta_N(m)| n^{-\mathrm{Re}(\alpha_2-\alpha_1)+N} + \frac{4}{N} e^{2k_1} r_N^{(2)} \\ & \times \left[\frac{2}{\mathrm{Re}(\alpha_2-\alpha_1)-N} + \frac{\mathrm{Re}(\alpha_2-\alpha_1)-N-1}{6}\left(\frac{1}{n}+\frac{1}{m}\right)+1\right],\end{aligned}$$

$$|\varepsilon_N^{(2)}(n)| \leqslant \mu \sup_{n\geqslant m}\{n^{\mathrm{Re}\,\alpha_1}|y_1(n)|\} n^{\mathrm{Re}\,\alpha_2-N}, \quad n \geqslant m.$$

(iii) If $\mathrm{Re}(\alpha_2 - \alpha_1) = N$, then from (4.10)

$$\begin{aligned}|\delta_N(n)| & \leqslant |\delta_N(m)| + \sum_{j=m}^{n-1} |\Delta_N(j)| \leqslant |\delta_N(m)| + \frac{4}{N} e^{2k_1} r_N^{(2)} \sum_{j=m}^{n-1} \frac{1}{j} \\ & \leqslant |\delta_N(m)| + \frac{4}{N} e^{2k_1} r_N^{(2)} (\ln n + \gamma),\end{aligned}$$

where γ is Euler constant. And when $n \geqslant m$

$$|\varepsilon_N^{(2)}(n)| \leqslant \left[|\delta_N(m)| + \frac{4}{N} e^{2k_1} r_N^{(2)} (\ln n + \gamma)\right] \sup\{n^{-\mathrm{Re}\,\alpha_1}|y_1(n)|\} n^{\mathrm{Re}\,\alpha_2 - N}.$$

4.2. Error bounds in the second exceptional case

Let $y_1(n) = L_N^{(1)}(n) + \varepsilon_N^{(1)}(n)$, $y_2(n) = L_N^{(2)}(n) + \varepsilon_N^{(2)}(n) + c(\ln n) y_1(n)$, where from (1.14), (1.16), $L_N^{(1)}(n) = n^{\alpha_1}\sum_{s=0}^{N-1} c_s/n^s$, $L_N^{(2)} = n^{\alpha_2}\sum_{s=0}^{N-1} d_s/n^s$; then $\varepsilon_N^{(1)}(n)$, $\varepsilon_N^{(2)}(n)$ satisfy difference equations [6],

$$\varepsilon_N^{(i)}(n+2) + a(n)\varepsilon_N^{(i)}(n+1) + b(n)\varepsilon_N^{(i)}(n) = -n^{\alpha_i} R_N^{(i)}(n), \qquad i = 1,2, \tag{4.11}$$

where

$$\begin{aligned}R_N^{(1)}(n) &= n^{-\alpha_1}[L_N^{(1)}(n+2) + a(n)L_N^{(1)}(n+1) + b(n)L_N^{(1)}(n)],\\ R_N^{(2)}(n) &= n^{-\alpha_2}\{L_N^{(2)}(n+2) + c[\ln(n+2)]y_1(n+2)\\ &\qquad + a(n)[L_N^{(2)}(n+1) + c\ln(n+1)y_1(n+1)] + b(n)[L_N^{(2)}(n) + c\ln n\cdot y_1(n)]\},\end{aligned}$$

$r_N^{(i)} = \sup_n\{n^{N+2}|R_N^{(i)}(n)|\}$ are constants, $i=1,2$. Eq. (4.11) is the same as Eq. (4.2). So bounds of $\varepsilon_N^{(1)}(n)$, $\varepsilon_N^{(2)}(n)$ are given in the previous subsection.

4.3. Error bounds in the last exceptional case

First of all, we consider solutions of the auxiliary difference equation

$$z(n+2) + \left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2}\right)z(n+1) + \left[b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + l(n)\right]z(n) = 0. \tag{4.12}$$

Obviously, $z_1(n) = n^\alpha$ is a solution of Eq. (4.12), when

$$l(n) = -\frac{(n+2)^\alpha + (a_0 + (a_1/n) + (a_2/n^2))(n+1)^\alpha}{n^\alpha} - b_0 - \frac{b_1}{n} - \frac{b_2}{n^2}.$$

The three-term approximation of $l(n)$ is

$$\begin{aligned}l(n) &= (-1 - a_0 - b_0) - \frac{(a_1+b_1)}{n} - \left[\frac{\alpha(\alpha-1) + a_1\alpha + a_2 + b_2}{n^2}\right] + \sigma(n)\\ &= \sigma(n),\end{aligned}$$

where $\sigma(n)$ is the remainder, and $\sigma_0 = \sup\{n^3|\sigma(n)|\}$ is a constant. For looking for $z_2(n)$, a solution of (4.12) which is linearly independent of $z_1(n)$, let

$$\begin{aligned}z_2(n) &= z_1(n)v(n),\\ \Delta(n) &= v(n+1) - v(n),\end{aligned}$$

then we have

$$(n+2)^\alpha\Delta(n+1) = \left[b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + l(n)\right]n^\alpha\Delta(n),$$

$$\frac{\Delta(n+1)}{\Delta(n)} = \frac{1 + \dfrac{b_1}{n} + \dfrac{b_2}{n^2} + l(n)}{(1+(2/n))^\alpha} = 1 + \frac{b_1 - 2\alpha}{n} + w(n),$$

where $w(n)$ is the remainder of the two-term approximation of $\Delta(n+1)/\Delta(n)$, and $w_0 = \sup\{n^2|w(n)|\}$ is a constant. For the fact $\alpha = (1-a_1)/2$, $a_1 + b_1 = 0$, we have $b_1 - 2\alpha = -1$, and

$$\frac{\Delta(n+1)}{\Delta(n)} = 1 - \frac{1}{n} + w(n). \tag{4.13}$$

4.3.1. The error bound of the asymptotic expansion of $y_1(n)$

Let

$$y_1(n) = L_N^{(1)}(n) + \varepsilon_N^{(1)}(n), \tag{4.14}$$

where $L_N^{(1)}(n) = n^\alpha \sum_{s=0}^{N-1} C_s/n^s$; then from [6],

$$\varepsilon_N^{(1)}(n+2) + a(n)\varepsilon_N^{(1)}(n+1) + b(n)\varepsilon_N^{(1)}(n) = -n^\alpha R_N^{(1)}(n), \tag{4.15}$$

where $r_N^{(1)} = \sup_n\{n^{N+2}|R_N^{(1)}(n)|\}$ is a constant. Rewriting (4.15) as

$$\begin{aligned}\varepsilon_N^{(1)}(n+2) &+ \left(a_0 + \frac{a_1}{n} + \frac{a_2}{n^2}\right)\varepsilon_N^{(1)}(n+1) + \left[b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + l(n)\right]\varepsilon_N^{(1)}(n)\\ &= -n^\alpha R_N^{(1)}(n) - \left[a(n) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2}\right]\varepsilon_N^{(1)}(n+1)\\ &\quad - \left[b(n) - b_0 - \frac{b_1}{n} - \frac{b_2}{n^2} - l(n)\right]\varepsilon_N^{(1)}(n),\end{aligned}$$

obviously, any solution of the equation

$$\begin{aligned}\varepsilon_N^{(1)}(n) = \sum_{j=n}^{\infty} K(n;j)\Bigg\{&-j^\alpha R_N^{(1)}(j) - \left[a(j) - a_0 - \frac{a_1}{j} - \frac{a_2}{j^2}\right]\varepsilon_N^{(1)}(j+1)\\ &- \left[b(j) - b_0 - \frac{b_1}{j} - \frac{b_2}{j^2} - l(j)\right]\varepsilon_N^{(1)}(j)\Bigg\}\end{aligned} \tag{4.16}$$

is a solution of (4.15), where

$$\begin{aligned}K(n;j) &= \frac{z_1(j+1)z_2(n) - z_2(j+1)z_1(n)}{z_1(j+2)z_2(j+1) - z_2(j+2)z_1(j+1)}\\ &= \frac{z_1(n)}{z_1(j+2)}\frac{v(n) - v(j+1)}{v(j+1) - v(j+2)}\\ &= \frac{z_1(n)}{z_1(j+2)}\sum_{i=n}^{j}\frac{\Delta(i)}{\Delta(j+1)}\\ &= \frac{z_1(n)}{z_1(j+2)}\sum_{i=n}^{j}\left\{\prod_{k=i}^{j}\frac{1}{1-(1/k)+w(k)}\right\}\\ &= \frac{z_1(n)}{z_1(j+2)}\sum_{i=n}^{j}\left\{\prod_{k=i}^{j}\left[1 + \frac{1}{k} + \tilde{w}(k)\right]\right\},\end{aligned}$$

where $\tilde{w}(k)$ is the remainder of two-term approximation of the function $[1-(1/k)+w(k)]^{-1}$. So

$$\begin{aligned}|K(n;j)| &\leqslant \frac{n^{\operatorname{Re}\alpha}}{j^{\operatorname{Re}\alpha}}\left(\frac{j}{j+2}\right)^{\operatorname{Re}\alpha}\sum_{i=n}^{j} c_0\frac{j}{i}\\ &\leqslant \tilde{c}_0\frac{n^{\operatorname{Re}\alpha}}{j^{\operatorname{Re}\alpha}}\frac{j^2}{n},\end{aligned}$$

where

$$c_0 = \exp\left(\frac{1}{2n} + \frac{1}{12n^2} + \frac{\pi^2}{6}\tilde{w}_0\right), \qquad \tilde{w}_0 = \sup_k \left\{k^2 \left|\ln\left[1 + \frac{1}{k} + \tilde{w}(k)\right] - \frac{1}{k}\right|\right\},$$

$\tilde{c}_0 = \sup_j c_0(j/(j+2))^{\operatorname{Re}\alpha}$.

Let

$$M = \tilde{c}_0, \quad \psi_0 = \sup_n \left\{n^3 \left|a(n) - a_0 - \frac{a_1}{n} - \frac{a_2}{n^2}\right|\right\}, \quad \eta_0 = \sup_n \left\{n^3 \left|b(n) - b_0 - \frac{b_1}{n} - \frac{b_2}{n^2} - l(n)\right|\right\},$$

$$\phi_0 = r_N^{(1)}, \quad P(n) = n^{\operatorname{Re}\alpha - 1}, \quad p_0 = \sup_n\{P(n+1)/P(n)\}, \tag{4.17}$$

$\xi = 1,\ \beta = 1,\ \theta = 1,\ \tau = 3$; then we have from Lemma 1,

$$|\varepsilon_N^{(1)}(n)| \leqslant \frac{2M\phi_0}{N - 1 - 2M(\psi_0 p_0 + \eta_0)} n^{\operatorname{Re}\alpha - N}. \tag{4.18}$$

4.3.2. The error bound of the asymptotic expansion of $y_2(n)$

Let

$$y_2(n) = L_N^{(2)}(n) + \varepsilon_N^{(2)}(n) + c(\log n) y_1(n), \tag{4.19}$$

where from (1.18), (1.19), $L_N^{(2)}(n) = n^{\alpha - Q + 2}\sum_{s=0}^{N-1} d_s/n^s$, and $\varepsilon_N^{(2)}(n)$ is the error term. From [6], $\varepsilon_N^{(2)}(n)$ satisfies

$$\varepsilon_N^{(2)}(n+2) + a(n)\varepsilon_N^{(2)}(n+1) + b(n)\varepsilon_N^{(2)}(n) = -n^{\alpha - Q + 2} R_N^{(2)}(n), \tag{4.20}$$

where $r_N^{(2)} = \sup_n\{n^{N+2}|R_N^{(2)}(n)|\}$ is a constant. Eq. (4.20) is similar to Eq. (4.15). Let $\beta = Q - 1$, and other constants are same as in (4.17); then we have

$$|\varepsilon_N^{(2)}(n)| \leqslant \frac{2M\phi_0}{N + Q - 3 - 2M(\psi_0 p_0 + \eta_0)} n^{\operatorname{Re}\alpha - Q + 2 - N}.$$

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