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SHORT COMMUNICATIONS

A Remark on the Controllability Problem for Differential-Algebraic Dynamical Systems

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Consider the control system

$$A_0\dot{x} = Ax + Bu, \qquad t \ge 0, \tag{1}$$

with the initial condition $x(0) = x_0$, where $x \in R^n$, $u \in R^r$, A_0, A, and B are constant matrices of appropriate sizes, $x_0 \in R^n$, and $\det A_0 = 0$.

Let system (1) satisfy the consistency condition [1]; then its solution can be represented in the form

$$x(t) = e^{A_0^d A t} A_0 A_0^d q + \int\limits_0^t e^{A_0^d A(t-s)} A_0^d B u(s)ds + \left(E_n - A_0 A_0^d\right) \sum_{i=0}^{k-1} (-1)^i \left(A_0 A^d\right)^i A^d B u^{(i)}(t), \tag{2}$$

$$x(0) = x_0 = A_0 A_0^d q + \left(E_n - A_0 A_0^d\right) \sum_{i=0}^{k-1} (-1)^i \left(A_0 A^d\right)^i A^d B u^{(i)}(0), \tag{3}$$

where A_0^d and A^d are the Drazin inverses of A_0 and A, respectively, the number k is the index of the matrix A_0, $q \in R^n$, and $u^{(i)}(0) \in R^r$, $i = 0, \ldots, k-1$. Here we assume that the control $u(t)$, $t \ge 0$, is a sufficiently smooth r-dimensional vector function.

In this case, one can show that the solution (2) of system (1) is the output of the system

$$\dot{Y} = \hat{A}Y + \hat{B}v, \qquad x = CY \tag{4}$$

with the initial condition $Y(0) = Y_0 = \left(q, u^{(i)}(0),\ i = 1, \ldots, k\right)$, where $Y = \left(y, u^1, \ldots, u^k\right)$, $v = u^{(k)}$, and $\hat{A} = \left(\hat{A}_{pq}\right)$ and $\hat{B} = \left(\hat{B}_{p1}\right)$, $p = 1, \ldots, k+1$, $q = 1, \ldots, k+1$, are block matrices; moreover, $\hat{A}_{11} = A_0^d A$, $\hat{A}_{12} = A_0^d B$, $\hat{A}_{23} = \hat{A}_{34} = \cdots = \hat{A}_{k,k+1} = E_r$, $\hat{A}_{ij} = 0$ for all remaining indices p and q, $\hat{B}_{k+1,1} = E_r$, $\hat{B}_i = 0$, $i = 1, \ldots, k$, and

$$C = \left[A_0 A_0^d, \left(E_n - A_0 A_0^d\right) A^d B, \ldots, (-1)^{k-1} \left(E_n - A_0 A_0^d\right) \left(A_0 A^d\right)^{k-1} A^d B\right].$$

We set

$$\Omega_0 = \Big\{ z \in R^n \mid z = A_0 A_0^d q + \left(E_n - A_0 A_0^d\right) \sum_{i=0}^{k-1} (-1)^i \left(A_0 A^d\right)^i A^d B u^{(i)}(0),$$
$$q \in R^n,\ u^{(i)}(0) \in R^r,\ i = 0, \ldots, k-1 \Big\}.$$

Let H be a constant $n \times n$ matrix.

Definition 1. System (1) is said to be *H-controllable* if for each $x_0 \in \Omega_0$, there exists a time $t_1 < +\infty$ and a smooth control $u(t)$, $t \in [0, t_1]$, such that $x(0) = x_0$ and $Hx(t_1) = 0$.

Definition 2. System (1) is said to be *completely H-controllable* if for each $x_0 \in \Omega_0$, there exists a time $t_1 < +\infty$ and a smooth control $u(t)$, $t \geq 0$, such that the solution $x(t)$, $t \geq 0$, of system (1) satisfies the conditions $x(0) = x_0$ and $Hx(t) \equiv 0$, $t \geq t_1$.

The following theorems hold.

Theorem 1. *System* (1) *is H-controllable if and only if*

$$\begin{aligned}
&\operatorname{rank}\Big(HA_0A_0^d, (-1)^j H\left(E_n - A_0A_0^d\right)\left(A_0A^d\right)^j A^dB,\ j = 0,\dots,k-1;\\
&\qquad H\left(A_0^dA\right)^i A_0^dB,\ i = 0,\dots,n-1\Big)\\
&\quad = \operatorname{rank}\Big((-1)^j H\left(E_n - A_0A_0^d\right)\left(A_0A^d\right)^j A^dB,\ j = 0,\dots,k-1;\\
&\qquad H\left(A_0^dA\right)^i A_0^dB,\ i = 0,\dots,n-1\Big).
\end{aligned}$$

Theorem 2. *System* (1) *is completely H-controllable if and only if*

$$\operatorname{rank}\left(L, \bar{H}\left(A_0^dA\right)^i A_0^dB,\ i = 0,\dots,k-1\right) = \operatorname{rank}\left(L, \bar{H}\right),$$

where

$$L = \begin{bmatrix} H_1 & \dots & H_k \\ HA_0^dB & \dots & H_{k-1} \\ H\left(A_0^dA\right)B & \dots & H_{k-2} \\ \dots & \dots & \dots \\ H\left(A_0^dA\right)^{n+k-2}A_0^dB & \dots & H\left(A_0^dA\right)^{n-1}A_0^dB \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} HA_0^dA_0 \\ HA_0^dA \\ H\left(A_0^dA\right)^2 \\ \dots \\ H\left(A_0^dA\right)^{n+k-1} \end{bmatrix},$$

$$H_{j+1} = (-1)^j H\left(E_n - A_0A_0^d\right)\left(A_0A^d\right)^j A^dB, \qquad j = 0,\dots,k-1.$$

The proofs of Theorems 1 and 2 readily follow from the representation of system (1) in the form (4) with the use of the results in [2].

Consider systems (1) and (4). We introduce the correspondences

$$x(t) \to X_t; \qquad u(t) \to U_t; \qquad y(t) \to Y_t; \qquad u^i(t) \to U_t^i; \qquad p \to \Delta. \tag{5}$$

Here X_t, Y_t, U_t, and U_t^i are matrices of sizes $n \times r$ and $r \times r$; $p \equiv d/dt$ is the differentiation operator, and Δ is the shift operator ($\Delta^i X_t = X_{t+i}$; $\Delta^i U_t = U_{t+i}$; $\Delta^i U_t^j = U_{t+i}^j$).

By using Eq. (4) and the correspondences (5), we pass to the recursion relations

$$\begin{aligned}
&Y_{t+1} = A_0^dAY_t + A_0^dBU_t^1, \qquad U_{t+1}^i = U_t^{i+1}, \qquad i = 1,\dots,k-1,\\
&U_{t+1}^k = U_{t+k}, \qquad X_t = C\left[Y_t, U_t^1, \dots, U_t^k\right], \quad t \geq 0,
\end{aligned} \tag{6}$$

under the condition that $Y_t \equiv 0$, $t = 0,\dots,k-1$, $U_t \equiv 0$, $t \neq k$, and $U_k = E_r$. Relations (6) are referred to as *determining equations* for the control system (1). By X_t^* we denote the solution of the determining equations (6) with $Y_1 = E_n$ and $U_t \equiv 0$, $t \geq 0$. Then Theorems 1 and 2 can be stated in terms of solutions of the determining equations.

Theorem 1′. *System* (1) *is H-controllable if and only if*

$$\operatorname{rank}\left(HX_1^*,\ HX_i,\ i = 1,\dots,n+k\right) = \operatorname{rank}\left(HX_i,\ i = 1,\dots,n+k\right).$$

Theorem 2′. *System* (1) *is completely* H*-controllable if and only if*

$$\operatorname{rank}\left(L,\ \bar{H}X_i,\ i = k+1, \ldots, n+k\right) = \operatorname{rank}\left(L, \bar{H}\right),$$

where

$$L = \begin{bmatrix} HX_k & \ldots & HX_1 \\ \ldots & \ldots & \ldots \\ HX_{n+2k-1} & \ldots & HX_{n+k} \end{bmatrix}, \qquad \bar{H} = \begin{bmatrix} HX_1^* \\ \ldots \\ HX_{n+k}^* \end{bmatrix}.$$

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