Numerical Simulation of Stochastic Differential Algebraic Equations for Power System Transient Stability with Random Loads

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***Abstract*—This paper summarizes numerical methods for Stochastic Differential Algebraic Equations (SDAEs) with which power system are modeled. The loads are modeled as random variables which appear in algebraic equations. The properties of numerical methods for Differential Algebraic Equations (DAE) and Stochastic Differential Equations (SDE) are reviewed and the first-order backward euler method is proposed for SDAE in power system transient stability simulation. Illustration examples are given on a single-machine-infinite-bus (SMIB) system.**

***Index Terms*—Stochastic Differential Algebraic Equations, Backward Euler Integration, Power System Transient Stability Analysis, Monte Carlo Simulation**

I. Introduction

Transient stability analysis in power systems plays an important role in dynamic security assessment to validate the system's ability to remain in synchronism following a major disturbance such as a short circuit. Usually a large set of nonlinear differential-algebraic equations (DAEs) is used to describe system behavior, which can be called semi-explicit index-1 DAEs of the form

$$\dot{x} = f(x, y) \tag{1a}$$
$$0 = g(x, y) \tag{1b}$$

where x and y are the vectors of differential variables and algebraic variables respectively and f and g are sufficiently differentiable vector functions of the same dimensions as x and y, respectively.

Directly determining the dynamic behavior of the system described by (1) is a very difficult task and is frequently accomplished by direct methods such as the energy function method or through time-domain simulation. Direct Methods are fast but suffer from generating conservative results and difficulties of employing complex models [1]. An alternative and more practical approach is time-domain simulation, which is utilized in transient stability analysis by both industrial engineers and academic researchers. A considerable amount of effort has been devoted to the numerical solution of DAEs [2]-[3] and the application to power systems [4]-[15]. Considering the large scale of a real power system, the effort to improve the computational efficiency is continually on-going. The parallel implementation of the transient stability simulation includes waveform relaxation [4]-[6] and other methods [7]-[9]. Moreover, even in a serial environment, many accelerating techniques are proposed, such as variable-step methods [10]-[11], mutlirate methods[12]-[14], and flexible reconfiguration of explicit or implicit integration methods [11], [15].

These traditional approaches have all been developed based on deterministic DAEs and cannot deal with the stochastic phenomena. However, many types of stochastic processes are encountered in power systems which cannot be ignored. Power system loads may be modeled as random variables. Renewable energy sources, such as wind and PV may also be modeled as random variables. Therefore, in this paper we propose to replace the DAEs as stochastic DAEs (SDAEs) in the transient stability analysis. SDAEs are the generalization of both deterministic differential-algebraic equations (DAEs) and stochastic differential equations (SDEs) [16] .

Much research has been devoted to the numerical solution of SDEs [16]-[17]. Recently, there have been several incipient works on stochastic differential algebraic equations (SDAEs) [18]-[20]. To model a random external perturbation, an additional term is added to the differential-algebraic equations, in the form of an stochastic process such as Gaussian white noise. The solution will then be a stochastic process instead of a deterministic function.

In [18]-[20], the focus is primarily on linear SDAEs and where the stochastic terms only appear in differential equations. In this paper we investigate the solution of semi-explicit index-1 SDAEs with constant coefficients and additive white noise which appear in the algebraic equations:

$$\dot{x} = f(x, y) \tag{2a}$$
$$0 = g(x, y) + b\xi \tag{2b}$$

where ξ is a white noise and b is a constant vector of the same dimension as y.

The numerical methods for SDAE (2) , same with SDE, require to run Monte Carlo Simulation to simulate sufficient

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trajectories to obtain the statistic accuracy. Considering the significant computational effort caused by Monte Carlo Simulation, the Backward Euler Method (BEM) might be a good choice [21]. Due to the fact that BEM has larger absolute stability regions than Euler Method and is more efficient than the Trapezoidal Method, BEM is a trade-off to balance the requirements of solution accuracy and completeness, on one hand, and computational efficiency, on the other hand.

This paper is organized as follows: The properties of numerical methods for for Differential Algebraic Equations (DAE) and Stochastic Differential Equations (SDE) are reviewed in Section II and Section III respectively. A SMIB system is used to illustrate the power system model for SDAE in Section IV. The backward euler method is proposed in Section V. The simulation results are shown and discussed in Section VI. Section VII is the conclusion.

II. Differential Algebraic Equations

A. Singular Perturbation Problems and Index 1 Problems

DAEs are not ordinary differential equations (ODEs). DAEs often arise from singular perturbation problems (SPP) in which they are a special class of ODEs containing a parameter ϵ, given by

$$\dot{x} = f(x, y) \tag{3a}$$
$$\epsilon\dot{y} = g(x, y) \tag{3b}$$

where ϵ is a parameters that determines the system "stiffness." When ϵ is small, the corresponding ODE (3) is stiff. When ϵ tends to zero, the ODE (3) becomes a DAE (1). Considerable insight can be obtained by studying the numerical solution for $\epsilon \to 0$.

Considering the index-1 DAE (1), the general assumptions in this paper will be

- A-i initial values are *consistent*: $0 = g(x_0, y_0)$;
- A-ii the Jacobian matrix $g_y(x, y) = \partial g/\partial y$ is nonsingular in a neighborhood of the solution of (1);
- A-iii the eigenvalues of $g_y(x, y)$ all have negative real parts.

Assumption (ii) implies that equation (1b) possesses a locally unique solution $y = \tilde{g}(x)$ which inserted into (1a) gives

$$\dot{x} = f(x, \tilde{g}(x)) \tag{4}$$

the so-called "state space form" which is an ordinary differential system.

We now want to study the behavior of the numerical solution of equation (3) for $\epsilon \to 0$ and to find the connection between the solution of DAE (1) and SPP (3). Let us illustrate this approach for one-step *theta*-methods which is the family of methods shown in Table I. The Table I gives the parameter θ and the corresponding algorithms.

B. Convergence and Error Analysis

There are two primary methods to solve the index 1 DAE: the *state space form* method and the *ϵ-embedding* method. If the former approach is used, the one-step *theta*-method with a stepsize h applied to the index 1 DAE system gives

TABLE I
ONE-STEP METHOD

Method	θ	type	order
Euler	0	explicit	1
backward Euler	1	implicit	1
trapezoidal rule	$\frac{1}{2}$	implicit	2

$$x_{n+1} = x_n + h[\theta f_{n+1} + (1-\theta)f_n] \tag{5a}$$
$$0 = g_{n+1} \tag{5b}$$

where $f_i = f(x_i, y_i)$ and $g_i = g(x_i, y_i)$, $i = 0, \ldots, N-1$, and $h = T/N$.

However, the *ϵ-embedding* approach replaces equation (5b) with the singularly perturbed ODE and yields

$$x_{n+1} = x_n + h[\theta f_{n+1} + (1-\theta)f_n] \tag{6a}$$
$$\epsilon y_{n+1} = \epsilon y_n + h[\theta g_{n+1} + (1-\theta)g_n] \tag{6b}$$

By setting $\epsilon = 0$ we obtain

$$x_{n+1} = x_n + h[\theta f_{n+1} + (1-\theta)f_n] \tag{7a}$$
$$0 = h[\theta g_{n+1} + (1-\theta)g_n] \tag{7b}$$

Remark: the *ϵ-embedding* approach will usually not guarantee that the numerical solution (x_{n+1}, y_{n+1}) will lie on the algebraic manifold $g(x, y) = 0$.

We cannot use the ODE approach directly to study the convergence of one-step *theta*-methods when applied to singular perturbation problems. The reason is the Lipschitz constant of SPP (3) is of size $O(\epsilon^{-1})$ but we are interested in estimates that hold for $\epsilon \to 0$. There are two Theorems presented in Appendix, which establish the rigorous estimates of error bound for the *theta*-method applied to SPP. The Theorems used in this paper can be seen as the special case of the theorems for multistep methods by Hairer. Interested readers can refer to [3].

III. Stochastic Differential Equations[16]

A. Wiener Processes and White Noise

We define a *standard Wiener process* $W = \{W(t), t \geq 0\}$ to be a stochastic process that satisfies the following conditions.

1) $W(t)$is continuous and $W(0) = 0$ (with probability 1);
2) $\forall t \geq 0, W(t) \sim \sqrt{t}\mathcal{N}(0, 1)$
3) Independent increments: $W(t+\delta t) - W(t) \sim \sqrt{\delta t}\mathcal{N}(0, 1)$ and is independent of the history of the process up to time t

where $\mathcal{N}(0, 1)$ denotes a normally distributed random variable with zero mean and unit variance.

We can approximate a standard Wiener process in distribution over $t = [0\ T]$ with N subintervals. We thus set the step size $h = T/N$ and let W_i denotes $W(t_i)$ with $t_i = ih$. Condition (3) indicates that

$$W_i = W_{i-1} + dW_i, \quad i = 1, 2, \ldots, N \tag{8}$$

where each dW_i is an independent random variable of the form $\sqrt{h}\mathcal{N}(0,1)$. The discretized Wiener process is shown in Fig. 1. The random variable dW_i is generated by Monte Carlo trials and the sample paths are 1000 steps. Note that the mean of $W(t)$ is close to zero but that some paths have increasing variance with time t, since $E(W(t)) = 0$ and $var(W(t)) = t$.

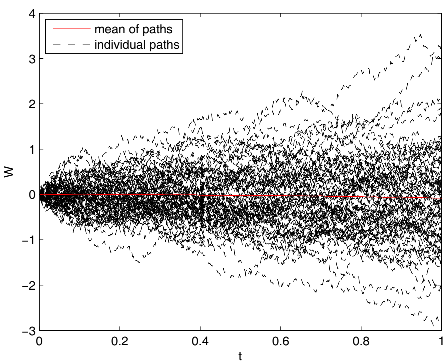

Fig. 1. Wiener Process

The sample paths of a Wiener process are not differentiable anywhere. However, Gaussian white noise process is defined as the derivative of a Wiener process is denoted as dW_t. It is a zero-mean wide-sense stationary process with constant nonzero spectral density. Gaussian white noise cannot be a stochastic process in the usual sense, but must be in interpreted in the sense of generalized functions.

B. Stochastic Differential Equations

Consider an ordinary differential equation,

$$\dot{x}(t) \quad = \quad f(x(t), t); \; x(0) = x_0 \tag{9}$$

The solution of (9) is

$$x(t) \quad = \quad x_0 + \int_0^t f(x(s), s) ds \tag{10}$$

Consider an stochastic differential equation,

$$dX(t) = f(X(t), t)dt + \alpha(X(t), t)dW; \; X(0) = X_0 \tag{11}$$

where $W(t)$ is a Wiener process. The formulations of differential equations are different between (9) and (11) because the Wiener process is (with probability one) nowhere differentiable. Therefore, (11) defines $dW = \xi dt$ and ξ are the Gaussian random variables for each t. The solution of (11) for each sample path is

$$X(t) = X_0 + \int_0^t f(X(s), s) ds + \int_0^t \alpha(X(s), s) dW(s) \tag{12}$$

where $f(X(t), t)$ is the drift vector and $\alpha(X(t), t)$ is the diffusion matrix.

C. Ito Stochastic Integrals

The first integral in (12) is an ordinary Riemann integral, defined as

$$\int_0^T f(t)dt = \lim_{n\to\infty} \sum_{j=0}^{N-1} f(t_j)(t_{j+1} - t_j) \tag{13}$$

for $t_j = jT/N$. However, the second integral in (12) cannot be interpreted as Riemann-Stieltjes integral even for each sample path, because the continuous sample paths of a Wiener process are not of bounded variation on any bounded time interval. The *Ito Stochastic Integral*, or *Ito Integral*, is introduced to fix this problem, defined as

$$\int_0^T \alpha(t)dW(t) = m.s. \lim_{n\to\infty} \sum_{j=0}^{N-1} \alpha(t_j)(W_{j+1} - W_j) \tag{14}$$

where $m.s.$ is “mean square.” The Ito integral has the peculiar property amongst Riemann integrals that

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T \tag{15}$$

with probability 1, in contrast to

$$\int_0^t w(t)dw(t) = \frac{1}{2}w(t)^2 \tag{16}$$

from classical calculus for a differentiable function $w(t)$ with $w(0) = 0$. The equality (15) follows from the algebraic rearrangement

$$\sum_{j=0}^{N-1} W_j(W_{j+1} - W_j) = \frac{1}{2}W(T)^2 - \frac{1}{2}\sum_{j=0}^{N-1}(W_{j+1} - W_j)^2 \tag{17}$$

since the mean-square limit of the sum of squares on the right is equal to T. The Ito integral (15), compared with the integral (16) , has a additional correction term $-\frac{1}{2}T$ to guarantee $E[\int_0^T W(t)dW(t)] = 0$.

D. Numerical Discretization Techniques

Considering the SDE (11), by integrating both sides of the equation from t_n to t_{n+1} yields

$$X_{n+1} = X_n + \int_{t_n}^{t_{n+1}} f(X, s)ds + \int_{t_n}^{t_{n+1}} \alpha(X, s)dW(s) \tag{18}$$

where $X_{n+1} = X(t_{n+1})$ and $X_n = X(t_n)$. In the next sections we use different methods to approximate the deterministic and stochastic integrals on the right hand side (for details see [16]).

1) Strong and weak convergence: A discretization of a stochastic differential equation governing the behavior of the random variable $X(t)$ is said to have *strong order of convergence* n if we can define a constant C such that

$$E|X_n - X(nh)| \leq Ch^n \tag{19}$$

where X_n is the discrete solution and $X(nh)$ is the exact solution. Strong convergence therefore depends on the expected value of the error of the solution at some point in time. A discretization of a stochastic differential equation is said to have *weak order of convergence* n if we can define a constant C such that

$$|E(X_n) - E(X(nh))| \leq Ch^n \tag{20}$$

Weak convergence is not as strict as strong convergence because it is a function of the error of the mean rather than the mean of the error.

2) The Euler-Maruyama method: In the Euler-Maruyama (EM) method, the deterministic integral in equation (18) is approximated with the rectangular rule and the Ito rule is used to compute the stochastic integral to yield

$$X_{n+1} = X_n + hf_n + \alpha_n \Delta W_n \tag{21}$$

where $f_n = f(X_n, t_n)$, $\alpha_n = \alpha(X_n, t_n)$ and $\Delta W_n = W(t_{n+1} - W(t_n)) \sim \mathcal{N}(0, h)$. We can see that this method reduces to the forward Euler method for deterministic ODEs if $\alpha = 0$. The EM method has a strong order of convergence of $n = 1/2$ and a weak order of convergence of $n = 1$.

3) Milstein's higher order method: The EM method can be made to converge strongly to first order by keeping higher order terms in the stochastic integral in equation (18). Using this method, the discretized stochastic differential equation becomes

$$X_{n+1} = X_n + hf_n + \alpha_n \Delta W_n + \frac{1}{2}\alpha_n \alpha_n'((\Delta W_n)^2 - h) \tag{22}$$

where $\alpha_n' = d\alpha/dx$. f_n, α_n and ΔW_n are defined the same as in (21).

4) The theta method: Similarly with the EM method, the *theta*-method applied to the generalized one-step method for ODEs yields

$$X_{n+1} = X_n + h(\theta f_{n+1} + (1 - \theta)f_n) + \alpha_n \Delta W_n \tag{23}$$

where $\theta \in \{0, \frac{1}{2}, 1\}$. For $\theta = 0$, (23) reduces to Euler-Maruyama. The cases $\theta = 1$ and $\theta = \frac{1}{2}$ are addressed as backward Euler and trapezoidal rule, respectively.

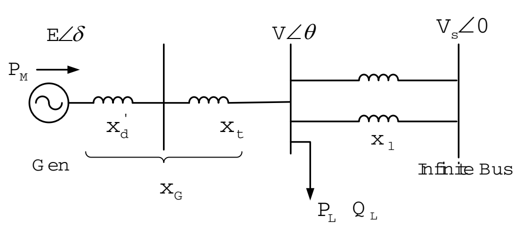

Fig. 2. Single Machine Infinite Bus System

IV. Power System Model

In order to focus on the SDAE itself without loss of generality, the simple single-machine-infinite-bus (SMIB) system is used to illustrate the numerical solution of SDAE for power systems. This system has one generator modeled with classical model and one infinite bus with fixed bus voltage magnitude and angle, as shown in Fig. 2. There are three nodes in this system, one is a PQ node, one is a PV node and the other is a slack node for power flow context. The DAE model for SMIB system is written as

$$\begin{aligned}
\dot{\delta} &= \omega_s(\omega - 1) && \text{(24a)}\\
\dot{\omega} &= \frac{1}{M}\left(P_m - \frac{EV}{X_G}\sin(\delta - \theta) - D(\omega - 1)\right) && \text{(24b)}\\
0 &= P_L + \frac{EV}{X_G}\sin(\theta - \delta) + \frac{V_s V}{X_L}\sin(\theta) && \text{(24c)}\\
0 &= Q_L + \frac{V^2}{X_G} - \frac{VE}{X_G}\cos(\theta - \delta) \\
&\quad + \frac{V^2}{X_L} - \frac{VV_s}{X_L}\cos(\theta) && \text{(24d)}
\end{aligned}$$

Assume the loads are the stochastic processes

$$\begin{aligned}
P_L &= (1 + \lambda\xi)P_{L0}\\
Q_L &= (1 + \lambda\xi)Q_{L0}
\end{aligned}$$

where P_{L0} and Q_{L0} are the initial active load and reactive load respectively and λ is the magnitude of the random variation .

Then, the DAE (24) becomes the SDAE as following,

$$\begin{aligned}
\dot{x} &= f(x, y) && \text{(25a)}\\
0 &= g(x, y) + b\xi && \text{(25b)}
\end{aligned}$$

where

$$\begin{aligned}
x &= [x_1 \;\; x_2]^T = [\delta \;\; \omega]^T\\
y &= [y_1 \;\; y_2]^T = [\theta \;\; V]^T
\end{aligned}$$

$$\begin{aligned}
f &= \begin{bmatrix} \omega_s x_2 \\ \frac{P_m}{M} - \frac{B_G E}{M} y_2 \sin(x_1 - y_1) - \frac{D}{M}x_2 \end{bmatrix}\\
g &= \begin{bmatrix} P_{L0} + B_G E y_2 \sin(y_1 - x_1) + B_L V_s y_2 \sin(y_1) \\ Q_{L0} + (B_G + B_L)y_2^2 - B_G E y_2 \cos(y_1 - x_1) \\ -B_L V_s y_2 \cos(y_1) \end{bmatrix}\\
b &= \begin{bmatrix} \lambda P_{L0} \\ \lambda Q_{L0} \end{bmatrix}
\end{aligned}$$

where $B_G = 1/X_G$ and $B_L = 1/X_L$.

V. Backward Euler Method

The SDAE (25) can be transferred into SDAE-SPP through incoorpating a small parameter ϵ and writing as stochastic differential

$$\begin{aligned}
dX &= f(X, Y)dt && \text{(26a)}\\
\epsilon dY &= g(X, Y)dt + bdW && \text{(26b)}
\end{aligned}$$

where W is Wiener process and $\xi = dW$ is the derivative of W. It is usual to rewrite (26) as integral equations on $t \in [0\; T]$,

$$X_t = X_{t0} + \int_{t_0}^{t} f(X_s, Y_s)ds \tag{27a}$$

$$Y_t = Y_{t0} + \int_{t_0}^{t} \frac{g(X_s, Y_s)}{\epsilon}ds + \int_{t_0}^{t} \frac{b}{\epsilon}dW_s \tag{27b}$$

where the second integral of (27b) is an *Ito* integral.

Usually implicit algorithms, such as Adams-Moulton, Gear (BDF), Implicit Runge-Kutta (IRK), are applied to solve the stiff ODE and DAE problems considering numerical stability [3]. The backward Euler method is the simplest algorithm in the entire implicit family, which can be classified as 1st-order Adams-Moulton or 1st-order Gear's algorithm. We will apply backward Euler method to (27):

$$X_{n+1} = X_n + hf(X_{n+1}, Y_{n+1}) \tag{28a}$$

$$Y_{n+1} = Y_n + \frac{h}{\epsilon}g(X_{n+1}, Y_{n+1}) + \frac{b}{\epsilon}\Delta W_n \tag{28b}$$

where $h = T/N$ is step size and $t_n = nh$, $n = 0, \ldots, N-1$. $\Delta W_n = W(t_{n+1}) - W(t_n) \sim \mathcal{N}(0, h)$. The Newton-Raphson iteration is ulitilized because equation (28) are nonlinear. The flowchart of the numerical integration is shown in Fig. 3.

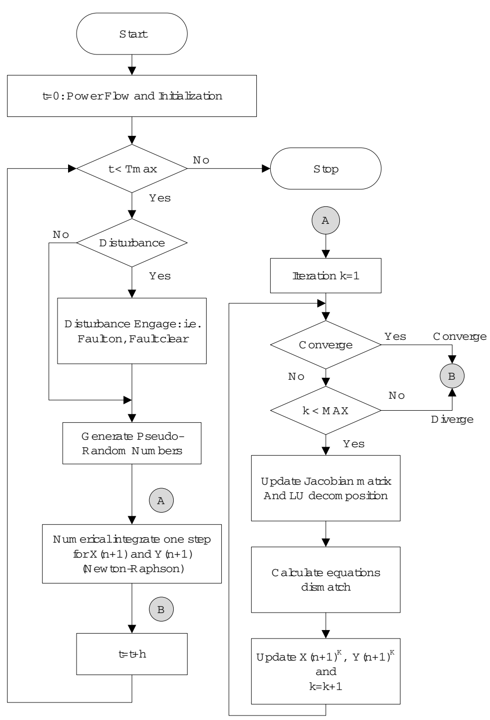

Fig. 3. Flowchart of SDAE integration

VI. Illustrative Example

A. Case I

In this case, we are integrating only with respect to the load stochatic process. There are no other perturbations applied on the system. The related parameters are given as $\epsilon = 10^{-5}$, $h = 10^{-3}$ and $\lambda = 5e-4$. There are 100 paths integrated on $t \in [0\ 5]$. The results are shown in Fig. 4-Fig. 7. The bold lines show the averaged values over 100 paths. The dashed lines show the individual paths. The following observations can be made from these results:

- The algebraic variables, which are the magnitude and angle of the bus voltages, behave as very fast variables in the singular perturbation problem. The deviations caused by random load propagate to the algebraic variables in the network almost instantaneously. Therefore, the stochastic processes of algebraic variables show the behaviors similar to white noise.
- The differential variables, which are the angle and frequency of the generators, behave as relatively slow variables in the singular perturbation problem. The stochastic processes of differential variables show the behaviors more similar to Wiener process.

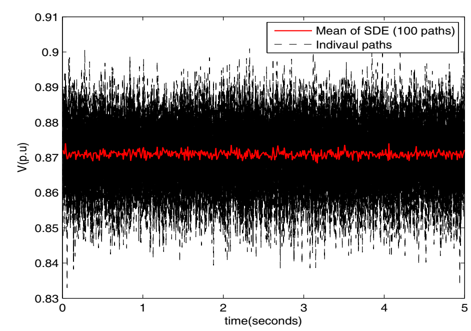

Fig. 4. Case I: Bus Voltage Magnitude

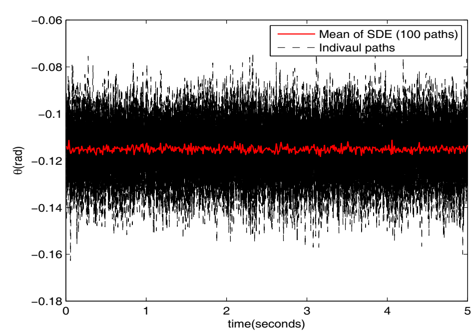

Fig. 5. Case I: Bus Voltage Angle

B. Case II

In this case, a line between the load bus and the infinite bus is tripped at t=1.0s and reclosed at t=1.1s. The dynamic responses are shown as Fig. 8-Fig. 11. The bold lines show the deterministic solution of the DAE and the dotted lines show the averaged value over 100 paths of the SDAE. The observations from these results can be made as follows:

- The algebraic variables, bus voltage magnitude and angle, show more apparent noise in the results of the mean of 100 paths of SDAE, compared with deterministic solution

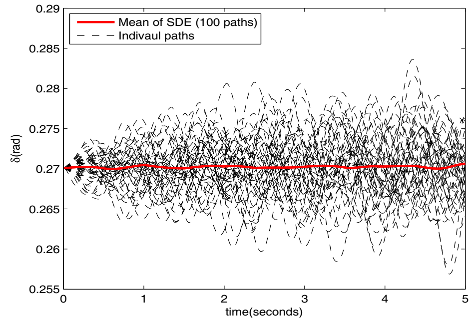

Fig. 6. Case I: Generator Angle

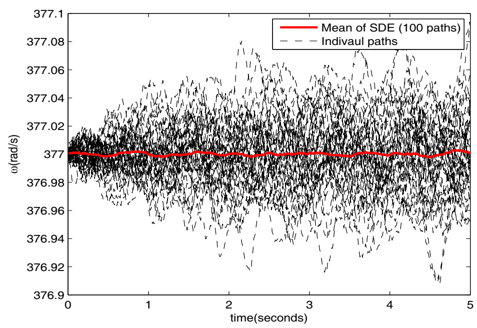

Fig. 7. Case I: Generator frequency

DAE. A reasonable interpretation is the fact that the noise sources are random loads which are propagated to algebraic variables through power flow equations directly.

- The differential variables, generator frequency and angle, show more close response in the comparison between the mean of 100 paths of SDAE and deterministic solution DAE because the differential variables are only coupled with noise source through the network.

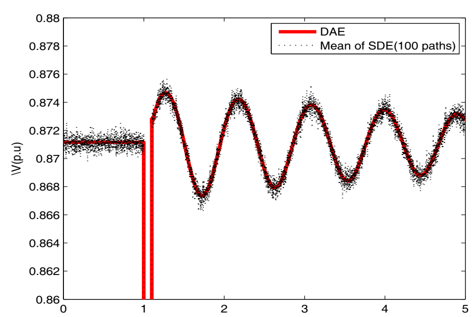

Fig. 8. Case II: Bus Voltage Magnitude

VII. Conclusion

This paper introduces the numerical method for Stochastic Differential Algebraic Equations (SDAEs) for power system applications. This algorithm can provide the insight into many

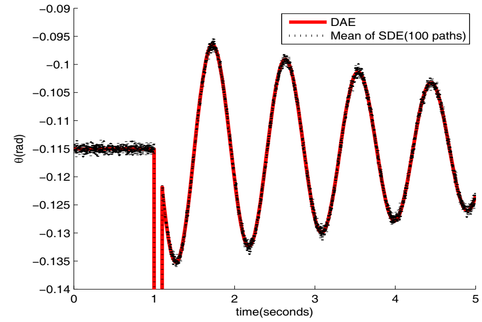

Fig. 9. Case II: Bus Voltage Angle

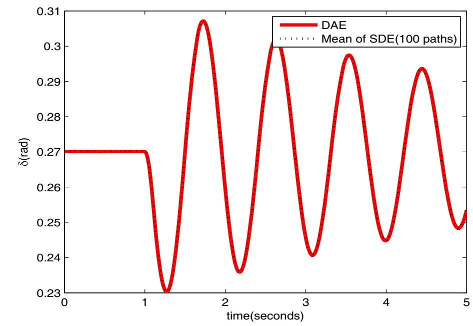

Fig. 10. Case II: Generator Angle

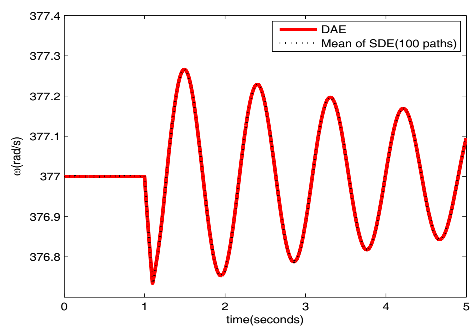

Fig. 11. Case II: Generator frequency

stochastic phenomena encountered power system. The Backward Euler Method is employed as integration method. The example is illustrating the effectiveness of this approach on a SMIB system. However, there are many interesting theoretical questions that have not been covered in present paper. Our long-term purpose is to achieve the compressive analysis of SDAE, including stability, consistency and convergency, as far as possible a theory similar to that of SDE and ODEs.

Appendix A
Two Theorems for DAE-SPP

1) Error Analysis: [3] and [2] both introduced the convergence of multistep method for singular perturbation problems. Since the one-step *theta*-method can be considered as multistep with step $k = 1$, the following statement follows from the results by Hairer (Theorem VI.2.1 [3]).

Theorem 1: (global error of theta-method for index 1 problem) Suppose that the system (1) satisfies assumption (ii). Consider a *theta*-method of order p and suppose that the error of the starting values x_i, y_i for $i = 0, \ldots, n-1$ is $O(h^p)$. Then the global error of (5) (for the *state space form* method) or (7) (for ϵ-*embedding* approach) satisfies

$$x_n - x(nh) = O(h^p), y_n - y(nh) = O(h^p)$$

for $x_n - x_0 = nh \leq C$ where C is a constant, where x_n denotes the numerical solution and $x(t)$ the exact solution. If the ϵ-*embedding* approach is adopted, then an additional condition has to be satisfied:

$$\infty \in S$$

which means infinity lies in the stability region of the integration method.

2) Convergence: We will study the convergence of the *theta*-method for SPP, based on the results by Lubich (1991) (Theorem VI.2.2 [3]). The Lubich theorem gives the error bounds for arbitrary multistep methods, which includes the one-step *theta*-method.

The Jacobian of the system (3) is of the form

$$\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \epsilon^{-1}\frac{\partial g}{\partial x} & \epsilon^{-1}\frac{\partial g}{\partial y} \end{pmatrix}$$

and its dominant eigenvalues are noted to be close to $\epsilon^{-1}\lambda$ where λ represents the eigenvalues of $\partial g/\partial y$. The assumption for stability is that λ of $\partial g/\partial y$ have negative real part. More precisely, we assume that

$$\lambda \text{ of } \partial g/\partial y \text{ lies in } \mid \arg \lambda - \pi \mid < \alpha \qquad (29)$$

for some (sufficiently small) $\alpha > 0$

Theorem 2: (convergence of *theta*-method for index 1 problem) Suppose that the one-step θ-method is of order p, $A(\alpha)$-stable and strictly stable at infinity. If the problem (3) satisfies (29), then the error is bounded for $h \gg \epsilon > 0$ by

$$\parallel x_n - x(nh) \parallel + \parallel y_n - y(nh) \parallel \leq CB(x, y)$$

for independent constant $C > 0$ and bounded function $B(x, y)$, which is defined as

$$\begin{aligned} B(x, y) = \max_{0\leq i\leq n} \parallel x_i - x(ih) \parallel + h^p \int_0^{nh} \parallel x^{p+1}(t) \parallel dt \\ + (h + \varrho^n) \max_{0\leq i\leq n} \parallel y_i - y(ih) \parallel \\ + \epsilon h^p \max_{0\leq t\leq nh} \parallel y^{p+1}(t) \parallel \end{aligned}$$

with independent constant $0 < \varrho < 1$.

Remark 1: We cannot use the ODE approach directly to study the convergence of one-step *theta*-methods when applied to singular perturbation problems. The reason is the Lipschitz constant of SPP (3) is of size $O(\epsilon^{-1})$ but we are interested in estimates that hold for $\epsilon \to 0$. Theorem 1 and 2 establish the rigorous estimates of error bound for the *theta*-method applied to SPP. The Theorems in this paper can be seen as the special case of the theorems for multistep methods. Interested readers can refer to [3].

APPENDIX B
SIMULATION PARAMETERS

$D = 2, M = 7, \omega_s = 377, E = 1.1, P_m = 0.8, X_G = 0.45, V_s = 1.0, X_L = 0.5, P_L = 1.0, Q_L = 0.5$.

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