Chapter 14
Stability Analysis of Discrete-Time Periodic Positive Systems with Delays

Tiantong Pu and Qinzhen Huang

Abstract Addressed in this paper is the stability issue of discrete-time periodic positive systems with constant delay. The positivity condition of the periodic system with constant delay is given, and three sufficient and necessary conditions for the asymptotic stability of the considered system are established. Note that the positivity condition in this paper is different from that in the Ref. (Bougatef et al, On the stabilization of a class of periodic positive discrete time systems, 2010. [3]), which gives the sufficient condition of the system without delay. The sufficient and necessary condition of the positivity of the system with delay is produced in this paper. Finally, a numerical example is given to demonstrate the effectiveness.

Keywords Positive system · Periodic system · Delay · Stability

14.1 Introduction

A positive system is a system whose state remains the positive orthant forever provided that the initial state is nonnegative and can model a large number of systems in real life, such as ecosystems, chemical reaction systems, physical systems and economic systems [2, 6, 8, 16]. The study on the dynamic properties of the positive system can be traced back to the last century.

Many experts and scholars have explored the positive system and obtained various results. Among these studies, stability analysis and controller design are very crucial and fundamental properties and tasks. Discussed in the Ref. [7] is the property of the positive system from the perspective of non-negative matrix theory. The equivalence relation between the asymptotic stability of the positive system and the diagonal quadratic stability is discussed in the Ref. [4], which can help us to analyze and

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design the complex positive system. In 2007, Rami et al. first established the stability criterion of positive system by using linear programming method. Thereafter, In the Refs. [9, 12–14], the "covering method" was adopted to successfully solve the stability problem of time-varying delay positive system. In the Ref. [10], switched linear copositive Lyapunov function method was adopted to solve the problem of stability of switching linear positive system.

The periodic system has a very important position in engineering and theory, which has attracted the attention of scholars [1, 5, 15, 17]. A periodic system is a type of hybrid dynamical system that combines discrete states and continuous states. Informally, it consists of a number of dynamical subsystems and all the subsystems are acted under the same switching order. Because of the periodicity of the periodic system, it generates interesting dynamic properties and can model many systems in our life.

There is a special kind of systems in periodic systems called periodic positive systems with all the subsystems of them are positive. In 2010, Bougatef et al. proposed the concept of periodic positive system [3], and carried out a preliminary study. The original intention of the study for the class of systems is that the positive system has many elegant attributes, and people intend to promote these interesting attributes of the positive system to the periodic positive system. However, Because the research started late and the study of the periodic positive system was much more difficult than the positive system, Now the results about the periodic positive system are not perfect, and people need to continue to explore. The study of the stability analysis of discrete-time periodic positive system without delay has been successfully solved [11]. In this paper, author deals with the discrete-time periodic positive systems with constant delay, and the stability problem of the system will be verified.

The structure of this paper is as follows: Sect. 14.2 introduces the basic knowledge of the periodic positive system, and Sect. 14.3 is the main content of this paper. In this section, the stability criterion of discrete-time periodic positive systems with constant delay is given by the augment method. With the criterion, the other two sufficient and necessary conditions for the asymptotic stability of the considered system are established. Section 14.4 gives a concrete numerical example. Finally, Sect. 14.5 summarizes the main work of this paper, and points out that some further research is needed in this field.

14.2 Problem Statements and Preliminaries

Notations:
$A \succeq 0 (\preceq 0)$: All elements of matrix A are non-negative (non-positive).
$A \succ 0 (\prec 0)$: All elements of matrix A are positive (negative).
$A > 0 (< 0)$: The matrix A is positive definite (negative definite).
A^{T} : Transpose of matrix A.
$\mathbb{R}(\mathbb{R}_{0,+}, \mathbb{R}_{+})$: The set of real numbers (nonnegative real numbers, positive real numbers).

$\mathbb{R}^n(\mathbb{R}^n_{0,+}, \mathbb{R}^n_+)$: The set of n dimensional real vectors (nonnegative vectors, positive vectors).
$\mathbb{R}^{n\times m}(\mathbb{R}^{n\times m}_{0,+}, \mathbb{R}^{n\times m}_+)$: The set of $n \times m$ dimensional real matrices (nonnegative matrices, positive matrices).
$\mathbb{N}$: $\{1, 2, 3, \ldots\}$.
$\mathbb{N}_0$: $\{0\} \bigcup \mathbb{N}$.
$[a, b]$: The least common multiple of a and b.
$0_{n\times m}$: $n \times m$ dimensional zero matrix.
0: $n \times n$ dimensional zero matrix.
I: $n \times n$ dimensional unit matrix.
A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is called class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for every fixed $t > 0$ and $\beta(r, t)$ is approaching to zero as $t \to \infty$ for every fixed $r \geq 0$.

Consider the following system

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k), \quad k \in \mathbb{N}_0 \tag{14.1}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$ is the state variable, and $A(k) \in \mathbb{R}^{n\times n}$ is the system matrix.

Definition 14.1 ([3]) If there exists $p \in \mathbb{N}$ such that $A(k+p) = A(k)$ holds for $\forall k \in \mathbb{N}_0$, system (14.1) is called periodic; if p is the smallest positive integer satisfying $A(k+p) = A(k)$, the system (14.1) is called p-periodic.

Definition 14.2 ([3]) If $\boldsymbol{x}(0) \succeq 0$, $\boldsymbol{x}(k) \succeq 0$ holds for all $k \in \mathbb{N}$, the system (14.1) is called positive.

Lemma 14.1 ([11]) *the p-periodic system 14.1 is positive if and only if*

$$\prod_{i=0}^{l} A(i) = A(l)A(l-1)\ldots A(0) \succeq 0, \quad \forall l \in \{0, 1, 2, \ldots, p-1\} \tag{14.2}$$

holds.

The special case of system (14.1) is:

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k), \quad k \in \mathbb{N}_0, \tag{14.3}$$

for which the following Lemmas hold.

Lemma 14.2 ([4]) *The system (14.3) is positive if and only if $A \succeq 0$.*

Lemma 14.3 ([4]) *The positive system (14.3) is asymptotically stable if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ such that $(A - I)\boldsymbol{\lambda} \prec 0$.*

Lemma 14.4 ([4]) *The positive system (14.3) is asymptotically stable if and only if there exists a diagonal positive definite matrix P satisfying $A^{\mathrm{T}}PA - P < 0$.*

Lemma 14.5 *The positive system (14.3) is asymptotically stable if and only if all eigenvalues of the matrix A are in the unit circle.*

14.3 Main Results

Consider the following discrete-time system with constant delays:

$$\begin{aligned} \boldsymbol{x}(k+1) &= A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{x}(k-\tau), \quad k \in \mathbb{N}_0, \tau \in \mathbb{N} \\ \boldsymbol{x}(k) &= \varphi(k), \quad k = -\tau, \dots, 0 \end{aligned} \tag{14.4}$$

The system (14.4) that we study here need to satisfy the following two conditions:

1. A(k) is q_1-periodic.
2. B(k) is q_2-periodic.

Theorem 14.1 *Let $q = [q_1, q_2]$. The following statements hold:*

1. System (13.4) is q-periodic and positive if and only if

$$\prod_{i=0}^{l} C(i) \succeq 0 \quad \forall l \in \{0, 1, 2, \dots, q-1\}, \textit{where}$$
$$C(k) = \begin{bmatrix} A(k) & 0_{n\times(\tau-1)n} & B(k) \\ I & & 0_{n\times\tau n} \\ \vdots & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & I & 0 \end{bmatrix} \forall k \in \mathbb{N}_0. \tag{14.5}$$

2. Suppose that the above condition holds, system (14.4) is asymptotically stable if and only if there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^n_+$ satisfying the following formula:

$$\left(\prod_{i=0}^{q-1} C(i) - I\right)\boldsymbol{\lambda} \prec 0. \tag{14.6}$$

Proof The system (14.4) can be rewritten as the following augmented system:

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \boldsymbol{x}(k) \\ \vdots \\ \boldsymbol{x}(k-\tau+1) \end{bmatrix} = \begin{bmatrix} A(k) & 0_{n\times(\tau-1)n} & B(k) \\ I & & 0_{n\times\tau n} \\ \vdots & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & I & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \boldsymbol{x}(k-1) \\ \vdots \\ \boldsymbol{x}(k-\tau) \end{bmatrix} \tag{14.7}$$

Set $\boldsymbol{y}(k+1) = [\boldsymbol{x}^{\mathrm{T}}(k+1), \boldsymbol{x}^{\mathrm{T}}(k), \dots, \boldsymbol{x}^{\mathrm{T}}(k-\tau+1)]^{\mathrm{T}}$, $\boldsymbol{y}(k) = [\boldsymbol{x}^{\mathrm{T}}(k), \boldsymbol{x}^{\mathrm{T}}(k-1), \dots, \boldsymbol{x}^{\mathrm{T}}(k-\tau)]^{\mathrm{T}}$. The system (13.7) can be abbreviated as the following form:

$$\boldsymbol{y}(k+1) = C(k)\boldsymbol{y}(k) \qquad \forall k \in \mathbb{N}_0 \tag{14.8}$$

If $q = [q_1, q_2]$, it is obtained from the condition 1. that $A(k+q) = A(k), \forall k \in N_0$ holds.

If $q = [q_1, q_2]$, it is obtained from the condition 2. that $B(k + q) = B(k), \forall k \in N_0$ holds.

Therefore, it is obtained that

$$
\begin{aligned}
C(k+q) &= \begin{bmatrix} A(k+q) & 0_{n\times(\tau-1)n} & & B(k+q) \\ I & & & 0_{n\times\tau n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & & I & 0 \end{bmatrix} \\
&= \begin{bmatrix} A(k) & 0_{n\times(\tau-1)n} & & B(k) \\ I & & & 0_{n\times\tau n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & & I & 0 \end{bmatrix}, \\
&= C(k)
\end{aligned}
\tag{14.9}
$$

$\forall k \in N_0$ holds.

Now prove that the system (14.9) is q-periodic system by contradiction.

Assume that there exists a $\bar{q} \in \mathbb{N} < q \in \mathbb{N}$ such that $C(k + \bar{q}) = C(k)$ $\forall k \in N_0$ holds. That is to say that

$$
\begin{aligned}
&\begin{bmatrix} A(k+\bar{q}) & 0_{n\times(\tau-1)n} & & B(k+\bar{q}) \\ I & & & 0_{n\times\tau n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & & I & 0 \end{bmatrix} \\
&= \begin{bmatrix} A(k) & 0_{n\times(\tau-1)n} & & B(k) \\ I & & & 0_{n\times\tau n} \\ \vdots & & \ddots & \vdots \\ 0_{n\times(\tau-1)n} & & I & 0 \end{bmatrix}
\end{aligned}
\tag{14.10}
$$

It is yielded from the above equation that $A(k + \bar{q}) = A(k)$ and $B(k + \bar{q}) = B(k)$ $\forall k \in N_0$ holds. Therefore, $\bar{q}$ is the common multiple of q1 and q2, which is in contradiction to q = [q1, q2]. Therefore, it can be obtained from Definition 14.2 that (14.9) is the q-periodic system. It can be obtained from Lemma 14.1 and (14.5) that (14.9) is the q-periodic positive system.

Sufficiency. Assume that the condition (14.7) holds. Because the system (14.9) is the q-periodic positive system, it is obtained from the Lemma that

$$
\prod_{i=0}^{q-1} C(i) \succeq 0. \tag{14.11}
$$

According to the Lemma 14.1, it is easy to obtain that the following system

$$\boldsymbol{z}(k+1) = \bar{A}\boldsymbol{z}(k), \quad \forall k \in \mathbb{N}_0, \tag{14.12}$$

is asymptotically stable, where $\bar{A} = \prod_{i=0}^{q-1} C(i)$. In other words, There exists a class $\mathcal{KL}$ function

$$\beta : [0, a) \times [0, \infty) \mapsto [0, \infty). \tag{14.13}$$

such that $\|\boldsymbol{z}(k)\| \leq \beta(\|\boldsymbol{z}(0)\|, k), \forall k \in \mathbb{N}_0\ holds$. Set

$$M = \max_{l \in \{0,1,2,\ldots,q-1\}} \| \prod_{i=0}^{l} C(i) \| . \tag{14.14}$$

Apparently, M is bounded. For the system (14.8) and the system (14.12), if we use the same initial condition, $\boldsymbol{z}(0) = \boldsymbol{y}(0)$, it holds that

$$\boldsymbol{y}(qk) = \boldsymbol{z}(k), \forall k \in \mathbb{N}. \tag{14.15}$$

Therefore,

$$\|\boldsymbol{y}(qk)\| \leq \beta(\|\boldsymbol{y}(0)\|, k), \forall k \in \mathbb{N}_0. \tag{14.16}$$

On the other hand, for $\forall l \in \{1, 2, \ldots, q\}$, it holds that

$$\begin{aligned} \boldsymbol{y}(qk+l) &= C(qk+l-1)\boldsymbol{y}(qk+l-1) \\ &= C(qk+l-1)C(qk+l-2)\ldots C(qk)\boldsymbol{y}(qk) \\ &= C(l-1)C(l-2)\ldots C(0)\boldsymbol{y}(qk) \end{aligned} \tag{14.17}$$

Therefore,

$$\begin{aligned} \|\boldsymbol{y}(qk+l)\| &= \|C(l-1)C(l-2)\ldots C(0)\boldsymbol{y}(qk)\| \\ &\leq \|C(l-1)C(l-2)\ldots C(0)\|\|\boldsymbol{y}(qk)\| \\ &\leq M\|\boldsymbol{y}(qk)\|, \forall k \in \mathbb{N}_0 \end{aligned} \tag{14.18}$$

Let's set $\bar{M} = max\{1, M\}$ and let $\bar{\beta}(s,t) = \bar{M}\beta(s,t)$. It is easy to see that $\bar{\beta}(s,t)$ is class $\mathcal{KL}$ function. It can obtained from the formula (14.16) and (14.18) that

$$\|\boldsymbol{y}(qk+l)\| \leq \bar{\beta}(\|\boldsymbol{y}(0)\|, k), \forall k \in \mathbb{N}_0. \tag{14.19}$$

If we set that $\hat{\beta}(\|y(0)\|, k) = \bar{\beta}(\|y(0)\|, i)$, $\forall k \in \{iq+1, iq+2, \ldots, (i+1)q\}$, $\forall i \in \mathbb{N}_0$, then (14.19) can be rewritten as the following form:

$$\|\boldsymbol{y}(k)\| \leq \hat{\beta}(\|\boldsymbol{y}(0)\|, k), \forall k \in \mathbb{N}. \tag{14.20}$$

Because $\hat{\beta}(\|\boldsymbol{y}(0)\|, k)$ here is class $\mathcal{KL}$ function, the formula above shows that the system (14.8) is asymptotically stable.

Necessity. Assume that the condition (14.6) doesn't hold. By the Lemma 14.2, the system (14.12) isn't asymptotically stable. In other words, if $k \mapsto +\infty$, the sequence $\|\boldsymbol{z}(k)\|$ doesn't converge to zero. Setting $\boldsymbol{z}(0) = \boldsymbol{y}(0)$, we can obtain from (14.19) that there exists a subsequence $\{\boldsymbol{y}(qk) : k \in \mathbb{N}_0\}$ from the solution of the system (14.12) $\boldsymbol{y}(k)$ and the sequence $\{\boldsymbol{y}(qk) : k \in \mathbb{N}_0\}$ doesn't converge to zero when $k \mapsto +\infty$. Therefore the system (14.8) can not be asymptotically stable, which is contradiction to the condition. □

Theorem 14.2 *Suppose that 1 holds, system (14.4) is asymptotically stable if and only if there exists a diagonal positive definite matrix P satisfying the following formula:*

$$\left(\prod_{i=0}^{q-1} C(i)\right)^{\mathrm{T}} P \left(\prod_{i=0}^{q-1} C(i)\right) - P < 0. \tag{14.21}$$

Proof By Theorem 14.1 and Lemma 14.4, the conclusion can be obtained directly. □

Theorem 14.3 *Suppose that 1 holds, system (14.4) is asymptotically stable if and only if all eigenvalues of the matrix $\prod_{i=0}^{q-1} C(i)$ are in the unit circle.*

Proof By Theorem 14.1 and Lemma 14.5, the conclusion can be yielded directly. □

14.4 Example

Consider the following system:

$$\boldsymbol{x}(k+1) = A(k)\boldsymbol{x}(k) + B(k)\boldsymbol{x}(k-\tau) \tag{14.22}$$

where $\boldsymbol{x}(k) = [\boldsymbol{x}_1(k), \boldsymbol{x}_2(k)]^{\mathrm{T}} \in \mathbb{R}^2$, $\tau = 3$, $A(k)$ is 2-periodic, with

$$A(0) = \begin{bmatrix} 0.01 & 0.2 \\ 0.5 & 0.06 \end{bmatrix}, \quad A(1) = \begin{bmatrix} 0.04 & -0.0001 \\ 0.6 & 0.02 \end{bmatrix}, \tag{14.23}$$

and $B(k)$ is 3-periodic, with

$$\begin{aligned} B(0) &= \begin{bmatrix} 0.04 & 0.5 \\ 0.8 & 0.04 \end{bmatrix}, \quad B(1) = \begin{bmatrix} 0.01 & 0.8 \\ 0.6 & 0.01 \end{bmatrix}, \\ B(2) &= \begin{bmatrix} 0.03 & 0.2 \\ 0.9 & 0.02 \end{bmatrix}. \end{aligned} \tag{14.24}$$

It is easily verified that $\prod_{i=0}^{l} C(i) = C(l)C(l-1)\ldots C(0) \succeq 0, \ \forall l \in \{0, 1, 2, \ldots, 5\}$. Using the routine linprog.m in MATLAB, we can easily get that there exists a vector$\lambda = [27.02, 105.03, 81.53, 111.64, 92.28, 61.84, 59.18, 125.80]^{\mathrm{T}} \in$

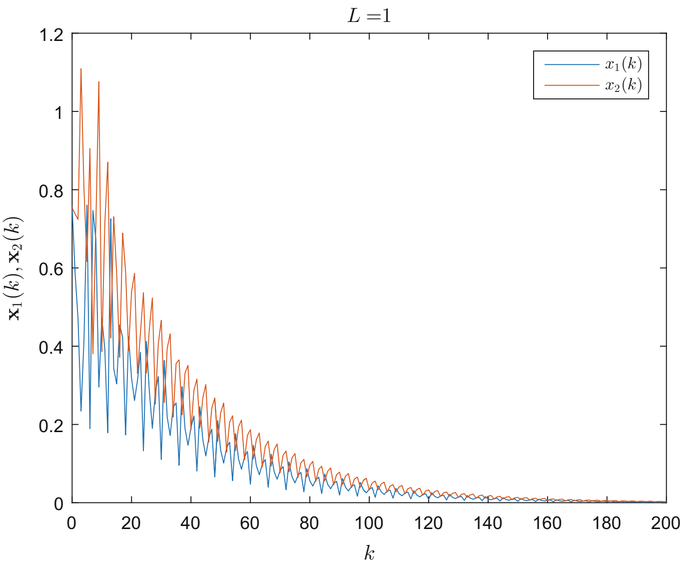

Fig. 14.1 Evolution of the system (14.22), $\varphi(0) = [0.4218, 0.9157]^{\mathrm{T}}$

$\mathbb{R}^8_+$ satisfying the $(\prod_{i=0}^{5} C(i) - I)\lambda \prec 0$. Therefore, the Theorem 14.1 is verified. Simultaneously, It can be calculated that the eigenvalues of $\prod_{i=0}^{5} C(i)$ are in the unit circle. The simulation diagram is shown (Fig. 14.1).

14.5 Conclusions

In this paper, we first try to study the problem of stability of discrete-time periodic positive systems with constant delay, and give the sufficient and necessary conditions of its asymptotic stability. The validity of the results is verified by the examples in the end. periodic positive system is a branch of a positive system discipline that has been emerging in recent years. The research is still in its infancy. Continuous time periodic positive system has not been studied. When the system contains time-varying delays, the problem becomes more complex and more attractive.

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