

1                   **EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR**  
 2                   **DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE**

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ABSTRACT. We study linear and semi-linear differential-algebraic equations (DAEs) on the half-line  $\mathbb{R}_+$ . Firstly, we characterize the existence of exponential dichotomy for linear DAEs based on the Lyapunov-Perron method. Then, we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the evolution family corresponding to linear DAE admits an exponential dichotomy and the nonlinear forcing function fulfills the non-uniform  $\varphi$ -Lipschitz condition, in which the Lipschitz function  $\varphi$  belongs to wide classes of admissible function spaces such as  $L_p$ ,  $1 \leq p \leq \infty$ ,  $L_{p,q}$ , etc.

4                   1. INTRODUCTION AND PRELIMINARIES

5         The present paper focuses on the existence of invariant (local and global) stable manifolds for semi-linear  
 6         non-autonomous differential-algebraic equations (DAEs) of the form

$$\begin{array}{ll} d \text{ rows} & \left[ \begin{array}{c} E_1(t) \\ 0 \end{array} \right] \dot{x}(t) = \underbrace{\left[ \begin{array}{c} A_1(t) \\ A_2(t) \end{array} \right]}_{A(t)} x(t) + \underbrace{\left[ \begin{array}{c} f_1(t, x(t)) \\ f_2(t, x(t)) \end{array} \right]}_{f(t, x(t))}, \\ a \text{ rows} & \end{array} \quad t \in \mathbb{R}_+ := [0, +\infty). \quad (\text{eq1})$$

7         To do that, we start by investigating the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \quad (\text{eq2})$$

8         Here  $E = \begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}$ ,  $A = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix}$  are assumed to be matrix-valued functions acting on  $\mathbb{R}_+$  to  $\mathbb{R}^{n,n}$ ,  $x : \mathbb{R}_+ \rightarrow$   
 9          $\mathbb{R}^n$ ,  $f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Furthermore, we assume that for all  $t$ , the matrices  $E_1(t)$ ,  $A_2(t)$  have full row rank.

10         DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuits, chemical engineering, and many other applications. Due to the rank-deficiency of  $E(t)$ , the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the reader to recent monographs [2, 12–14] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [12, Chapter 2]), there are only few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [15] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [17] for Lyapunov and other spectra for linear DAEs, to [5, 8] for the robustness of exponential stability and Bohl exponents.

19         On the other hand, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is  
 20         characterized, the next important question in the qualitative theory of DAEs is to study the existence of  
 21         integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1)  
 22         [4, 6]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps,  
 23         this paper is devoted to investigation of the exponential dichotomy of (1.2) and stable manifolds of (1.1).

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*Key words and phrases.* Exponential dichotomy, semilinear, differential-algebraic equation, admissibility of function spaces, stable manifold.

24 Our method is based on the classical "Lyapunov-Perron method" ([6, 25]) and the admissibility of function  
 25 spaces ([10, 11]).

26 The outline of this paper is as follows. In the rest of this first section we recall some basis concepts for  
 27 later use, including the notion of the exponential dichotomy and its properties, as well as some important  
 28 features of admissible function spaces. In Section 2 we give a characterization for the existence of expo-  
 29 nential dichotomy for the DAE (1.2). Section 3 contains our main results on the existence and properties of  
 30 local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in  
 31 Section 4. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations,  
 32 and we conclude this research by a summary and some open problems.  
 33

34 **1.1. Evolution Families and Exponential Dichotomies.** Let us now recall some basic notions. By  $(\mathbb{R}^n,$   
 35  $\|\cdot\|)$  we denote the  $n$ -dimensional real vector space equipped with the Euclidean norm. For any matrix  
 36  $V$ , by  $V^T$  we denote its transpose. For any  $p \in \mathbb{N}$ , by  $C^p([0, \infty), \mathbb{R}^n)$  we denote the space of  $p$ -times  
 37 continuously differentiable functions acting on  $[0, \infty)$  with values in  $\mathbb{R}^n$ . By  $C_b([0, \infty), \mathbb{R}^n)$  we denote the  
 38 space of continuous and bounded functions mapping from  $[0, \infty)$  into  $\mathbb{R}^n$ . This space is a Banach space with  
 39 the *ess sup*-norm  $\|f\|_\infty := \sup\{\|f(t)\|, t \geq 0\}$ .

It is well-known (e.g. [4]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t), \quad t \geq s \geq 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

40 is well-posed, then there exists a pointwise nonsingular matrix-valued function  $(t, s) \mapsto X(t, s) \in \mathbb{R}^{n,n}$  such  
 41 that the solution of (1.3) is given by  $x(t) = X(t, s)x_s$ . This fact motivates the existence of an evolution family  
 42  $(X(t, s))_{t \geq s \geq 0}$  associated with the matrix function  $A(t)$ . This family satisfies the condition  $X(t, t) = Id$  and  
 43 the so-called *semi-group property*

$$X(t, r)X(r, s) = X(t, s), \quad \text{for all } t \geq r \geq s \geq 0. \tag{1.4}$$

44 Furthermore, every solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x(t)), \quad \text{for all } t \geq s \geq 0,$$

45 also satisfies the so-called *variation-of-constant formula*

$$x(t) = X(t, s)x(s) + \int_s^t X(t, \tau)f(\tau, x(\tau))d\tau, \quad \text{for all } t \geq s \geq 0. \tag{1.5}$$

46 For more details on the notion and discussion on properties and applications of evolution families we refer  
 47 the readers to Pazy [21].

48 **Definition 1.1.** A given evolution family  $\{X(t, s)\}_{t \geq s \geq 0}$  of the ODE (1.3) is said to have an *exponential*  
 49 *dichotomy* on the half-line if there exist a family of projection matrices  $\{P(t)\}_{t \geq 0}$  and two positive constants  
 50  $N, \nu$  such that the following conditions are satisfied.

- 51 i)  $P(t)X(t, s) = X(t, s)P(s)$  for all  $t \geq s \geq 0$ ,
  - 52 ii) for all  $t \geq s \geq 0$ , the restriction  $X(t, s)| : \ker P(s) \rightarrow \ker P(t)$  is an isomorphism, and we denote its  
 53 inverse by  $X(s, t)|$ ,
  - 54 iii)  $\|X(t, s)P(s)x\| \leq Ne^{-\nu(t-s)}\|P(s)x\|$ , for all  $t \geq s \geq 0$ ,  $x \in \mathbb{R}^n$ ,
  - 55 iv)  $\|X(t, s)|(I - P(s))x\| \leq Ne^{\nu(t-s)}\|(I - P(s))x\|$ , for all  $s \geq t \geq 0$ ,  $x \in \mathbb{R}^n$ .
- 56 Here  $\{P(t)\}_{t \geq 0}$  (reps.  $N, \nu$ ) are called *dichotomy projections* (resp. *dichotomy constants*).

57 The concept exponential dichotomy means that the state space  $\mathbb{R}^n$  has been splitted into the stable  
 58 subspace ( $\text{Im}(P(t))$ ) and the unstable subspace ( $\ker(P(t))$ ).

59 **1.2. A short review of DAE solvability theory.**

60 **Definition 1.2.** (i) Consider the DAE (1.2). A matrix function  $X \in C([0, \infty), \mathbb{R}^{n,k})$ ,  $d \leq k \leq n$ , is called a  
 61 *fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and  $\text{rank } X(t) = d$ , for all  
 62  $t \geq 0$ .

63 (ii) A fundamental solution matrix is said to be *maximal* if  $k = n$  and *minimal* if  $k = d$ , respectively. A  
 64 maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)(X(0) - Id) = 0. \quad \text{projected intial conditon} \quad (1.6)$$

65 We can easily see that, the fundamental solution matrices for DAEs are not necessarily square or of  
 66 full rank. Furthermore, each fundamental solution matrix has exactly  $d$ -linear independent columns, and  
 67 a minimal fundamental solution matrix can be made maximal by adding  $n - d$  zero columns. This is the  
 68 major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a  
 69 DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been  
 70 proposed and carefully discussed in [17]. We recall it below, and notice that this concept is equivalent to the  
 71 one proposed by Lentini and März in [15] within the context of the matrix chains approach and tractability  
 72 index. Throughout this paper, we will assume the following.

73 **Assumption 1.3.** Assume that the function pair  $(E, A)$  in the DAEs (1.1) and (1.2) is *strangeness-free*,  
 74 i.e.,

$$\text{rank} \begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix} = n,$$

75 for all  $t \geq 0$ . Furthermore, we assume that  $E \in C^1([0, \infty), \mathbb{R}^{n,n})$  and  $A \in C^0([0, \infty), \mathbb{R}^{n,n})$ .

76 *Remark 1.4.* In general, not all systems of the type or (1.2) is strangeness-free. Nevertheless, under some  
 77 constant rank assumption, one can always transform it, without alternating the solution space, to the  
 78 strangeness-free form. For further details, see [3, 13, 14].

79 By making use of some smooth factorizations, for example QR or SVD ([7] or [13], Theorem 3.9), we can  
 80 decouple and then exploit the structure of the DAE (1.2) in the following lemma.

81 **Lemma 1.5.** Consider the DAE (1.2) and assume that it satisfies Assumption 1.3. Then, there exist  
 82 pointwise-orthogonal matrix-valued functions  $U \in C^0([0, \infty), \mathbb{R}^{d,d})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$ , such that after  
 83 changing variable  $x(t) = V(t)y(t)$ , and scaling (1.2) with  $U(t)$ , we can transform it to the so-called semi-  
 84 explicit system of the following form

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_1(t) & \tilde{A}_2(t) \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \quad \text{semi-explicit system} \quad (1.7)$$

85 with pointwise nonsingular matrix-valued functions  $\Sigma(t) \in \mathbb{R}^{d,d}$  and  $\tilde{A}_4(t) \in \mathbb{R}^{a,a}$ .

86 *Proof.* Applying an SVD factorization for  $E_1(t)$  we can find pointwise-orthogonal matrix functions  $U_1(t) \in$   
 87  $C^1([0, \infty), \mathbb{R}^{d,d})$  and  $V \in C^1([0, \infty), \mathbb{R}^{n,n})$  such that  $U_1(t)E_1(t)V(t) = [\Sigma(t) \ 0]$ , where  $\Sigma(t)$  is a continuous,  
 88 pointwise nonsingular function with values in  $\mathbb{R}^{d,d}$ . Changing the variable  $x(t) = V(t)y(t)$  and scaling (1.2)  
 89 with  $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$ , we obtain a new system

$$U(t)E(t)V(t)y(t) = U(t) \left( A(t)V(t) - E(t)\dot{V}(t) \right) y(t),$$

which is exactly of the form (1.7). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix} \begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix} V,$$

then Assumption 1.3 yields that both  $\Sigma$  and  $\tilde{A}_4$  are nonsingular. This completes the proof.  $\square$

Let  $\hat{A}_3 := -\tilde{A}_4^{-1}\tilde{A}_3$ ,  $\hat{A}_s := \tilde{A}_1 + \tilde{A}_2\tilde{A}_4^{-1}\tilde{A}_3$  is the Schur complement, we rewrite system (1.7) as

$$\begin{aligned} \dot{y}_1(t) &= \Sigma^{-1}(t)\hat{A}_s(t)y_1(t), \\ y_2(t) &= \hat{A}_3(t)y_1(t). \end{aligned} \quad \begin{array}{l} \text{eq5.1} \\ \text{eq5.2} \end{array}$$

Since  $V(t)$  is orthogonal for all  $t \geq 0$ , we see that all important qualitative properties of  $x(t)$ , such as boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function  $y(t)$ . Clearly, we see that (1.9) gives an *algebraic constraint* that the solution to (1.7) must obey, while (1.8) gives the dynamic of (1.7). For this reason, we call it *an underlying ODE* to (1.7).

Let  $\{\hat{Y}_1(t, s)\}_{t \geq s \geq 0}$  be the evolution family associated with the matrix function  $\hat{A}_1(t)$ , then we can define the corresponding evolution families for two DAEs (1.7), (1.2) consecutively as follows.

$$\hat{Y}(t, s) := \begin{bmatrix} \hat{Y}_1(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1(t, s) & 0 \end{bmatrix}, \quad \hat{X}(t, s) := V(t)\hat{Y}(t, s)V^T(s), \quad \text{for all } t \geq s \geq 0. \quad (1.10)$$

Nevertheless, since  $X(t, s)$  is not invertible, we will define the *reflexive generalized inverse matrix function* as in [17] by

$$\hat{Y}^-(t, s) := \begin{bmatrix} \hat{Y}_1^{-1}(t, s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t, s) & 0 \end{bmatrix}, \quad \hat{X}^-(t, s) := V(s)\hat{Y}^-(t, s)V^T(t), \quad \text{for all } t \geq s \geq 0. \quad (1.11)$$

Then, we can directly verify the semigroup properties, i.e.

$$\begin{aligned} \hat{X}(t, r) &= \hat{X}(t, s)\hat{X}(s, r), \quad \text{for all } t \geq s \geq r \geq 0, \\ \hat{X}(t, s) &= \hat{X}(t, 0)\hat{X}^-(s, 0), \quad \text{for all } t \geq s \geq 0. \end{aligned}$$

Furthermore, Lemma 1.6 below shows that the family  $\{\hat{X}(t, s)\}$  play the same role as the evolution family  $(X(t, s))_{t \geq s \geq 0}$ , in comparison to (1.5).

**Lemma 1.6.** *Consider the DAE (1.1) and the evolution family  $(X(t, s))_{t \geq s \geq 0}$  defined by (1.10). Furthermore, we also consider the pointwise-orthogonal matrix-valued functions  $U, V$  defined in Lemma 1.6. Then, the solution to (1.1), if exists, also satisfies the so-called mild equation*

$$\begin{aligned} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \hat{X}(t, s) \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t, \tau) \begin{bmatrix} \Sigma^{-1}(\tau) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & 0 \end{bmatrix} U(\tau)f(\tau, x(\tau))d\tau \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & -\tilde{A}_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t)), \end{aligned}$$

for all  $t \geq s \geq 0$ .

*Proof.* The proof can be obtained directly by using Lemma 1.5. Thus, in order to keep the brevity we will omit the details here.  $\square$

In the following, for ease of notation, we will use the abbreviation  $\hat{X}(t) := \hat{X}(t, 0)$ ,  $\hat{X}^-(t) := \hat{X}^-(t, 0)$ ,  $\hat{Y}(t) := \hat{Y}(t, 0)$  and  $\hat{Y}^-(t) := \hat{Y}^-(t, 0)$ . The concept of exponential dichotomy for the DAE (1.7) is given as below.

**Definition 1.7.** ([17]) The DAE (1.7) is said to have an *exponential dichotomy* if there exist a family of projection matrices  $\{P_y(t)\}_{t \geq 0}$  in  $\mathbb{R}^{d,d}$  and positive constants  $N, \nu$  such that

$$\begin{aligned} \left\| \hat{Y}(t) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0, \\ \left\| \hat{Y}(t) \begin{bmatrix} I_d - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} \hat{Y}^-(s) \right\| &\leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0, \end{aligned} \quad (1.12)$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.10)-(1.12) we see that

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0) \hat{X}^-(s) \right\| \leq N e^{-\nu(t-s)}, \text{ for all } t \geq s \geq 0.$$

and

$$\left\| \hat{X}(t)V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0) \hat{X}^-(s) \right\| \leq N e^{\nu(t-s)}, \text{ for all } s \geq t \geq 0.$$

In addition, since  $V^T(0) \begin{bmatrix} Id - P_y(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$  is also a projection matrix for any  $t \geq 0$ , we can interpret the exponential dichotomy of (1.2) as the one of (1.7).

1.3. **Function Spaces and Admissibility.** In this subsection we recall some notions of function spaces that play a fundamental role in the study of differential equations and refer to Nguyen [10], Massera and Schäffer [18, Chap. 2] and Räbiger and Schnaubelt [22, §1] for various applications.

Let  $E$  (endowed with the norm  $\|\cdot\|_E$ ) be Banach function space of real-valued functions defined as in [10]. We then recall the Banach space corresponding to the space  $E$  as follows.

**Definition 1.8** ([10]). Consider the Banach space  $(\mathbb{R}^n, \|\cdot\|)$ . For a Banach function space  $E$  we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \rightarrow \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

endowed with the norm  $\|f\|_{\mathcal{E}} := \|f(\cdot)\|_E$ . Thus, one can directly see that  $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$  is a Banach space. We call it *the Banach space corresponding to  $E$* .

We now introduce the notion of admissibility in the following definition.

**Definition 1.9** ([10]). The Banach function space  $E$  is called *admissible* if for any  $\varphi \in E$  the following conditions hold.

(i) There exists a constant  $M \geq 1$  such that for every compact interval  $[a, b] \subset \mathbb{R}_+$  we have

$$\int_a^b |\varphi(t)| dt \leq \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \text{ for all } \varphi \in E, \quad (1.13)$$

where  $\chi_{[a,b]}$  is the indicator function of  $[a, b]$ .

(ii) The function  $\Lambda_1 \varphi$  defined by  $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$  belongs to  $E$ .

(iii) For any  $\tau \geq 0$ , the space  $E$  is  $T_\tau^+$ -invariant and  $T_\tau^-$ -invariant, where  $T_\tau^+$  and  $T_\tau^-$  are defined as

$$\begin{aligned} T_\tau^+ \varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geq \tau \geq 0, \\ 0 & \text{for } 0 \leq t \leq \tau, \end{cases} \\ T_\tau^- \varphi(t) &:= \varphi(t+\tau) \text{ for } t \geq 0. \end{aligned} \quad (1.14)$$

Furthermore, there exist constants  $N_1, N_2$  such that  $\|T_\tau^+\|_E \leq N_1$ ,  $\|T_\tau^-\|_E \leq N_2$  for all  $\tau \in \mathbb{R}_+$ .

<sup>127</sup> **Example 1.10.** Besides the spaces  $L_p(\mathbb{R}_+)$ ,  $1 \leq p \leq \infty$ , and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau < \infty\},$$

<sup>128</sup> (for any fixed  $\alpha > 0$ ), endowed with the norm  $\|h\|_{\mathbf{M}_\alpha} := \sup_{t \geq 0} \int_t^{t+\alpha} |h(\tau)| d\tau$ , many other function spaces  
<sup>129</sup> occurring in interpolation theory, e.g. the Lorentz spaces  $L_{p,q}$ ,  $1 < p < \infty$ ,  $1 \leq q < \infty$  (see [4], [24]) and,  
<sup>130</sup> more general, the class of rearrangement invariant function spaces (see [16]) are admissible.

<sup>131</sup> *Remark 1.11.* Following directly from Definition 1.9 we have that

$$\sup_{t \geq 0} \int_t^{t+1} |\varphi(\tau)| d\tau \leq \frac{M}{\inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

<sup>132</sup> and hence,  $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$ . Furthermore,  $C_b(\mathbb{R}^+)$  is dense in  $\mathbf{M}_1$ .

<sup>133</sup> We present here some important features of admissible spaces in the following proposition (see [10, Propo-  
<sup>134</sup> sition 2.6] and originally in [18, 23.V.(1)]).

**Proposition 1.12** ([10]). Let  $E$  be an admissible Banach function space. Then the following assertions hold.

a) Let  $\varphi \in L_{1,loc}(\mathbb{R}_+)$  such that  $\varphi \geq 0$  and  $\Lambda_1 \varphi \in E$ , where,  $\Lambda_1$  is defined as in definition 1.9 (ii). For  $\sigma > 0$  we define functions  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  by

$$\begin{aligned} \Lambda'_\sigma \varphi(t) &:= \int_0^t e^{-\sigma(t-s)} \varphi(s) ds, \\ \Lambda''_\sigma \varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)} \varphi(s) ds. \end{aligned}$$

<sup>135</sup> Then,  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  belong to  $E$ . In particular, if  $\sup_{t \geq 0} \int_t^{t+1} \varphi(\tau) d\tau < \infty$  (this will be satisfied if  $\varphi \in E$  (see  
<sup>136</sup> remark 1.11)) then  $\Lambda'_\sigma \varphi$  and  $\Lambda''_\sigma \varphi$  are bounded. Moreover, denoted by  $\|\cdot\|_\infty$  for *ess sup*-norm, we have

$$\|\Lambda'_\sigma \varphi\|_\infty \leq \frac{N_1}{1 - e^{-\sigma}} \|\Lambda_1 T_1^+ \varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma \varphi\|_\infty \leq \frac{N_2}{1 - e^{-\sigma}} \|\Lambda_1 \varphi\|_\infty \quad (\text{eq2.1})$$

<sup>137</sup> for operator  $T_1^+$  and constants  $N_1, N_2$  defined as in Definition 1.9.

<sup>138</sup> b)  $E$  contains exponentially decaying functions  $\psi(t) = e^{-\alpha t}$  for any constant  $\alpha > 0$ .

<sup>139</sup> c)  $E$  does not contain exponentially growing functions  $f(t) := e^{bt}$  for any constant  $b > 0$ .

## 140 2. EXPONENTIAL DICHOTOMY FOR LINEAR DAEs

<sup>141</sup> In the qualitative analysis of ODEs, one of the central topic is to find sufficient and necessary conditions  
<sup>142</sup> for the considered systems to admit exponential dichotomy. Many researches have been devoted to this  
<sup>143</sup> topic, and critical results have been achieved for ODEs in finite and infinite dimensional phase spaces (e.g.  
<sup>144</sup> [6, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

<sup>145</sup> **Proposition 2.1.** ([17]) The DAE (1.7) has an exponential dichotomy if and only if the corresponding  
<sup>146</sup> underlying ODE (1.8) also has exponential dichotomy, and the matrix function  $\hat{A}_3(t)$  is bounded. Moreover,  
<sup>147</sup> the existence of exponential dichotomy implies that  $\sup_{t \geq 0} \|P_y(t)\| < \infty$ .

<sup>148</sup> Together with (1.2), let us consider the following system

$$E(t) \dot{x}(t) = A(t)x(t) + g(t) . \quad (\text{eq3.1})$$

149 Notice that, even for ODEs, Proposition 2.1 is only valid for finite-dimensional systems but not for infinite  
 150 dimensional systems, [6]. For this reason, we recall a classical result by Perron below.

151 **Proposition 2.2.** ([6]) The ODE (1.3) has an exponential dichotomy if and only if for any continuous,  
 152 bounded function  $g(t)$  on  $[0, \infty)$ , there exists a continuous, bounded solution  $x(t)$  to the system

$$\dot{x}(t) = A(t)x(t) + g(t). \quad \text{eq31} \quad (2.2)$$

153 In view of Proposition 2.2, comparable result has not been achieved for DAEs, and hence, this will be our  
 154 main aim in this section.

155 **Definition 2.3.** Consider the matrix functions  $E, A$  in system (1.2). Then, a function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}^n$   
 156 satisfying the condition

$$\sup_{t \geq 0} \left\{ \left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & \tilde{A}_4^{-1}(t) \end{bmatrix} g(t) \right\| \right\} < +\infty,$$

157 is called  $(E, A)$ -bounded. We denote the set of all continuous and  $(E, A)$ -bounded functions by  $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$ .

158 The main result of this section is to prove a characterization of the exponential dichotomy for DAEs.  
 159 Roughly speaking, the DAE (1.2) admits exponential dichotomy if and only if the mapping  $\mathcal{L} := E \frac{d}{dt} - A$   
 160 is surjective on the space  $\mathcal{B}^{EA}(\mathbb{R}_+, \mathbb{R}^n)$ . We formulate our main result in this section as follows.

161 **Theorem 2.4.** Consider the linear, strangeness-free DAE (1.2) and the associated inhomogeneous DAE  
 162 (2.1). Then the following assertions hold.

- 163 (i) If the DAE (1.2) admits exponential dichotomy then for any continuous,  $(E, A)$ -bounded function  $g(t)$   
 164 on  $[0, \infty)$ , there exists a continuous, bounded solution  $x(t)$  to the DAE (2.1).
- 165 (ii) If the matrix function  $\hat{A}_3(t)$  is bounded, then the converse of assertion (i) holds true.

*Proof.* Firstly, we notice that, since  $\hat{g} = \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1} \end{bmatrix} U(t)g(t)$ , the  $(E, A)$ -boundedness of  $f$  is equivalent to the boundedness of  $\hat{f}$ . Recall that the semi-explicit system (1.7) reads

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t) + \hat{g}_1(t), & \text{eq3(2.3)} \\ y_2(t) &= \hat{A}_3(t)y_1(t) + \hat{g}_2(t). & \text{eq3(2.4)} \end{aligned}$$

166 (i) Assuming that the DAE (1.2) admits exponential dichotomy, then (1.7) also has an exponential di-  
 167 chotomy. Proposition 2.1 implies that equation (2.3) has an exponential dichotomy, and the function  $\hat{A}_3$  is  
 168 bounded. Therefore, Proposition 2.2 implies that  $y_1$  is bounded, and consequently,  $y_2$  is also bounded.

169 (ii) From Proposition 2.2, it follows that (2.3) has exponential dichotomy **May be wrong, since the**  
 170 **domains of g and hg are not the same.** On the other hand, the boundedness of  $\hat{A}_3$  implies that (1.2)  
 171 admits exponential dichotomy.  $\square$

### 172 3. LOCAL STABLE MANIFOLDS FOR SEMI-LINEAR DAEs

173 In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout  
 174 this section we assume that the evolution family  $(X(t, s))_{t \geq s \geq 0}$  associated with the linear, homogeneous DAE  
 175 (1.2) admits an exponential dichotomy on  $\mathbb{R}_+$ .

From Lemma 1.5, by using orthogonal transformation  $x(t) = V(t)y(t)$ , where  $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$  we can transform (1.1) to the coupled system

$$\begin{aligned} \dot{y}_1(t) &= \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), & \text{eq4, 1a} \\ y_2(t) &= \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), & \text{eq4, 1b} \end{aligned} \quad (3.2)$$

<sup>176</sup> where

$$\begin{aligned} \hat{f}(t, y(t)) &= \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & -\tilde{A}_4^{-1} \end{bmatrix} U(t) \begin{bmatrix} f_1(t, y(t)) \\ f_2(t, y(t)) \end{bmatrix}. & \text{eq4, 2} \\ \hat{f} &= \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t) & 0 \\ 0 & -\tilde{A}_4^{-1}(t) \end{bmatrix} U(t)f(t, x(t)) \end{aligned}$$

<sup>177</sup> Notice that, unlike the DAEs (1.2) and (2.1), equation (3.2) only gives an implicit algebraic constraint  
<sup>178</sup> in terms of  $y_1$  and  $y_2$ . In order to guarantee the strangeness-free of system (1.1), we need the following  
<sup>179</sup> assumption.

<sup>180</sup> **Assumption 3.1.** Assume that for some  $\rho > 0$ , the function  $\tilde{A}_4^{-1}(t)f_2(t, x)$  is a contraction mapping in the  
<sup>181</sup> ball  $B_\rho := \{x \in \mathbb{R}^n \mid \|x\| \leq \rho\}$  (uniformly in time), i.e.,

$$\|\tilde{A}_4^{-1}(t)(f_2(t, x) - f_2(t, \tilde{x}))\| \leq L\|x - \tilde{x}\|,$$

<sup>182</sup> for a.e.  $t \in \mathbb{R}_+$ , and for all  $x, \tilde{x} \in B_\rho$  where the Lipschitz constant  $L$  satisfies that  $L < 1$ .

<sup>183</sup> **Lemma 3.2.** Under Assumption 3.1 and given  $y_1 \in B_\rho$ , there exists a unique function  $y_2 \in \mathcal{B}_\rho$  satisfying  
<sup>184</sup> (3.2).

<sup>185</sup> *Proof.* Firstly, notice that Assumption 3.1 implies that  $\hat{f}_2(t, y)$  is also Lipschitz in  $y$  with the same constant  
<sup>186</sup>  $L$ . Then, the desired claim is obtained directly by making use of [19, Lem. 2.7].  $\square$

<sup>187</sup> *Remark 3.3.* Lemma 3.2 leads to one important fact, that under Assumption 3.1, the coupled system (3.1)-  
<sup>188</sup> (3.2) is still strangeness-free, as defined in [13, Chap. 4]. Therefore, in analogue to the linear case, (3.2) is  
<sup>189</sup> called *an algebraic constraint*, whereas (3.1) is called *an underlying ODE*.

<sup>190</sup> To obtain the existence of a stable manifold we need the following property of the nonlinear part  $f_1$   
<sup>191</sup> defined as follows.

<sup>192</sup> **Definition 3.4.** Let  $\varphi$  be a positive function belonging to an admissible Banach function space  $E$ . A  
<sup>193</sup> function  $h : \mathbb{R}_+ \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is said to belong to the class  $(M, \varphi, \rho)$  for some positive constant  $M, \rho$  if  $h$   
<sup>194</sup> satisfies

- <sup>195</sup> (i)  $\|h(t, x)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$  and for all  $x \in B_\rho$ ,
- <sup>196</sup> (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$ , for all  $x, \tilde{x} \in B_\rho$ .

<sup>197</sup> **Assumption 3.5.** Assume that the function  $t \mapsto \Sigma^{-1}(t)[I_d \ -\tilde{A}_2(t)\tilde{A}_4^{-1}(t)]f(t, x(t))$  belongs to class  
<sup>198</sup>  $(M, \varphi, \rho)$  for some positive constants  $M, \rho$  and a positive function  $\varphi \in E$ .

<sup>199</sup> The following proposition gives one sufficient condition for examining Assumptions 3.1, 3.5.

<sup>200</sup> **Proposition 3.6.** Consider the semi-linear DAE (1.1). Furthermore, assume that all three functions  $\Sigma^{-1}$ ,  
<sup>201</sup>  $\tilde{A}_4^{-1}, \Sigma^{-1}\tilde{A}_2\tilde{A}_4^{-1}$  are bounded. If the function  $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$  belongs to the class  $(M, \varphi, \rho)$  then the following  
<sup>202</sup> claims hold true.

- <sup>203</sup> i)  $\hat{f}_1$  belongs to the class  $(M, \varphi, \rho)$ , and
- <sup>204</sup> ii)  $f_2$  is Lipschitz with the Lipschitz constant  $\varphi \sup_{t \geq 0} \|\tilde{A}_4^{-1}\|$ .

205 We notice that a sufficient condition for Assumption (3.1) is that

$$\|f_2(t, x) - f_2(t, \tilde{x})\| \leq \frac{L}{\|\tilde{A}_4^{-1}(t)\|} \|x - \tilde{x}\|. \quad \text{Lipschitz (3.4)}$$

206 For the simplicity of presentation, we will study the existence of a local stable manifold for system (3.1)-  
207 (3.2). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s) \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad \text{mild equation (3.5)}$$

208 for all  $t \geq s \geq 0$ .

209 **Lemma 3.7.** *Let Assumptions 3.1 and 3.5 hold true. Then, for all  $y, \tilde{y} \in B_\rho$  the following assertions hold.*

210 (i)  $\|\hat{f}_1(t, y)\| \leq M\varphi(t)$  for a.e.  $t \in \mathbb{R}_+$ ,

211 (ii)  $\|\hat{f}_1(t, y) - \hat{f}_1(t, \tilde{y})\| \leq \varphi(t)\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ ,

212 (iii)  $\|\hat{f}_2(t, y) - \hat{f}_2(t, \tilde{y})\| \leq L\|y - \tilde{y}\|$  for a.e.  $t \in \mathbb{R}_+$ .

213 *Proof.* The proof is trivially followed from Assumptions 3.1 and 3.5 due to the fact that  $\|y\| = \|Qy\|$  for any  
214 orthogonal matrix  $V$ .  $\square$

215 Let  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  has an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$   
216 and the dichotomy constants  $N, \nu > 0$  as in Definition 1.7. Furthermore, as in Proposition 2.1, let us denote  
217 by  $H_1 := \sup_{t \geq 0} \|\hat{A}_3(t)\|$  and  $H_2 := \sup_{t \geq 0} \|P_y(t)\|$ . Then, we can define the Green function on the half-line as  
218 follows

$$G(t, \tau) := \begin{cases} \hat{Y}(t, \tau) \begin{bmatrix} P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)P_y(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t, \tau)P_y(\tau) & 0 \end{bmatrix}, & \text{for all } t \geq \tau \geq 0, \\ -\hat{Y}(t, \tau) \begin{bmatrix} I_d - P_y(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t, \tau)(I_d - P_y(\tau)) & 0 \end{bmatrix}, & \text{for all } 0 \leq t < \tau. \end{cases} \quad \text{(3.6)}$$

219 Then, we have

$$\|G(t, \tau)\| \leq (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geq 0. \quad \text{(3.7)}$$

220 In the following lemma, we give an explicit form for bounded solutions to system (3.5).

221 **Lemma 3.8.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the  
222 corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume  
223 that Assumptions 3.1, 3.5 hold true. Let  $y(t)$  be any solution to (3.5) such that  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$  for  
224 fixed  $t_0 \geq 0$  and some  $\rho > 0$ . Then, for  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad \text{(3.8)}$$

225 for some  $v_0 \in \text{Im}P_y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.6).

*Proof.* Put

$$z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}.$$

226 By direct computation, we can verify that  $z$  satisfies the integral equation

$$z(t) = \hat{Y}(t, t_0) \begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^t \hat{Y}(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix},$$

227 for all  $t \geq t_0$ . Now let us estimate  $\|z(t)\|$ . Making use of Lemma 3.7 and (3.7), we see that

$$\|z(t)\| \leq \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} M\varphi(\tau) d\tau + L\rho,$$

228 and then, from (1.15) it follows that

$$\|z(t)\| \leq M (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho ,$$

229 for all  $t \geq t_0$ . Thus,  $z(t) - y(t)$  is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t, t_0) (z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t) \hat{Y}_1(t, t_0) (z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

230 we see that  $v_0 := z_1(t_0) - y_1(t_0) \in \text{Im}P_y(t_0)$ . Finally, since  $z(t) = y(t) + \hat{Y}(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix}$  for all  $t \geq t_0$ , equality  
231 (3.8) follows.  $\square$

232 *Remark 3.9.* By computing directly, we can see that the converse of Lemma 3.8 is also true. It means, that  
233 all solutions to (3.8) also satisfy equation (3.5) for all  $t \geq t_0$ .

234 Let us denote by

$$H_3 := (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)} \rho . \quad (3.9)^8$$

235 **Lemma 3.10.** *Under the assumptions of Lemma 3.8, let  $y(t)$ ,  $\tilde{y}(t)$  be any two functions lying in the ball  
236  $B_\rho$  and satisfy (3.8) for  $v_0, \tilde{v}_0 \in \text{Im}P_y(t_0)$ . If  $H_3$  defined as in (3.9) satisfies  $H_3 + L < 1$  then the following  
237 estimate holds true:*

$$\|y - \tilde{y}\|_\infty \leq \frac{N}{1 - H_3 - L} \|v_0 - \tilde{v}_0\|. \quad (3.10)^9$$

*Proof.* Using the same arguments as in the proof of Lemma 3.7, we see that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq N\|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2) Ne^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty + L\|y(t) - \tilde{y}(t)\|, \\ &\leq N\|v_0 - \tilde{v}_0\| + (H_3 + L) \|y - \tilde{y}\|_\infty, \end{aligned}$$

238 which directly implies (3.10).  $\square$

239 In the following theorem, we exploit the local structure of bounded solutions to (3.5).

240 **Theorem 3.11.** *Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy  
241 with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore,  
242 assume that Assumptions 3.1, 3.5 hold true, and constant  $H_3$  defined as in (3.9). Then, the following  
243 assertions hold true.*

244 (i) *If*

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)\rho}{2M} \right\}, \quad (3.11)^7$$

245 *then there corresponds to each  $v_0 \in B_{\tilde{\rho}} \cap \text{Im}P_y(t_0)$  one and only one solution  $y(t)$  to (3.5) on  $[t_0, \infty)$  satisfying  
246  $P_y(t_0)y_1(t_0) = v_0$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ .*

<sup>247</sup> (ii) Moreover, any two solutions  $y(t), \tilde{y}(t)$  corresponding to different  $v_0, \tilde{v}_0$  in  $B_{\bar{\rho}} \cap \text{Im}P_y(t_0)$  attract each  
<sup>248</sup> other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0, \quad \text{eq4.8} \quad (3.12)$$

<sup>249</sup> for some positive constants  $H_4, \mu$ .

<sup>250</sup> Proof. (i) Consider in the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  the ball  $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) : \|y(\cdot)\|_\infty := \underset{t \geq 0}{\text{esssup}} \|y(t)\| \leq \rho\}$ .

<sup>251</sup> For each fixed  $v_0 \in B_{\bar{\rho}}$  we will prove the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix} & \text{for all } t \geq t_0, \\ 0 & \text{for all } t < t_0, \end{cases} \quad (3.13)$$

<sup>252</sup> is a contraction mapping from  $\mathcal{B}_\rho$  to itself. Using the same argument as in the proof of Lemma 3.7, we see  
<sup>253</sup> that

$$\begin{aligned} \|(Ty)(t)\| &\leq (1 + H_1)N e^{-\nu(t-t_0)} \|v_0\| + M(1 + H_1)(1 + H_2) \frac{N}{1 - e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) + L\rho, \\ &\leq (1 + H_1)N \|v_0\| + MH_3 + L\rho \quad \text{for all } t \geq 0, \end{aligned}$$

<sup>254</sup> and by (3.11) we see that

$$\|(Ty)(t)\| \leq (1 + H_1)N \tilde{\rho} + \frac{(1 - L)\rho}{2} + L\rho = \rho \quad \text{for all } t \geq 0.$$

Therefore,  $T$  is a mapping from  $\mathcal{B}_\rho$  to itself. Now we prove its contraction property. Indeed, making use of (3.7), we obtain the following estimate:

$$\begin{aligned} \|Ty(t) - T\tilde{y}(t)\| &\leq \int_{t_0}^\infty \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq \int_{t_0}^\infty (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \\ &\leq (H_3 + L) \|y(\cdot) - \tilde{y}(\cdot)\|_\infty \quad \text{for all } t \geq 0. \end{aligned}$$

<sup>255</sup> Consequently, due to (3.11), we see that  $T$  is a contraction mapping with the contraction constant  $H_3 + L$ .  
<sup>256</sup> Thus, there exist a unique function  $y \in \mathcal{B}_\rho$  such that  $y = Ty$ , and hence, due to the definition of  $T$ ,  $y$  is the  
<sup>257</sup> solution to the mild/integral-algebraic system (3.5).

(ii) The proof of the estimate (3.12) can be done in a similar way as in [11, Thm 3.7]. We present here  
 for seek of completeness. Let  $y(t)$  and  $\tilde{y}(t)$  be two essentially bounded solutions of (3.5) corresponding to  
 different values  $v_0, \tilde{v}_0 \in B_{\bar{\rho}} \cap \text{Im}P_y(t_0)$ . Then, we have that

$$\begin{aligned} \|y(t) - \tilde{y}(t)\| &\leq Y(t, t_0) \|v_0 - \tilde{v}_0\| + \int_{t_0}^\infty \|G(t, \tau)\| \|\hat{f}_1(\tau, y(\tau)) - \hat{f}_1(\tau, \tilde{y}(\tau))\| d\tau + \|\hat{f}_2(t, y(t)) - \hat{f}_2(t, \tilde{y}(t))\|, \\ &\leq (1 + H_1)N e^{-\nu(t-t_0)} + \int_{t_0}^\infty (1 + H_1)(1 + H_2) N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau + L \|y(t) - \tilde{y}(t)\|, \end{aligned}$$

<sup>258</sup> and hence,

$$\|y(t) - \tilde{y}(t)\| \leq \frac{1 + H_1}{1 - L} N e^{-\nu(t-t_0)} + \int_{t_0}^\infty \frac{(1 + H_1)(1 + H_2)}{1 - L} N e^{-\nu|t-\tau|} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau.$$

Then, due to the Cone Inequality, [6, Theorem 1.9.3], in analogue to [20, Theorem 3.7], we obtain the estimation (3.12) with  $H_4, \mu$  are given by

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1 - e^{-\nu})}{1 - L} \right), \quad H_4 := \frac{(1 + H_1)N}{1 - L - \frac{H_3(1 - e^{-\nu})}{1 - e^{\mu - \nu}}}. \quad (3.12)$$

Furthermore, notice that from (3.11) it follows that  $\mu < \nu$  implying the positivity of  $H_4$ . This completes the proof.  $\square$

Under Assumption 3.1, we then define the so-called *constrained manifold*, which all solutions to (3.1)-(3.2) must belong to

$$\mathbb{L}(t, y) := \{(t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t, y_1, y_2)\} \text{ . constraint manifold} \quad (3.14)$$

We further notice that this manifold is of dimension  $d$ , which is the degree of freedom to the DAE (3.5).

Now, we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic system (3.5).

**Definition 3.12.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *local stable manifold* for solutions to (3.5) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf \{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exist positive constants  $\rho, \rho_1, \rho_2$  and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t), \quad t \in \mathbb{R}_+,$$

with a common Lipschitz constant independent of  $t$  such that

- (i)  $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1} \cap W_1(t)\}$ , and we denote by  $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$ ,
- (ii)  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$  for all  $t \geq 0$ ,
- (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.5) satisfying  $y_1(t_0) = \tilde{w}$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| \leq \rho$ .

We now state and prove our main result on the existence of a local stable manifold for DAEs.

**Theorem 3.13.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumptions 3.1, 3.5 hold true. If

$$H_3 < \min \left\{ 1 - L, \frac{(1 - L)(1 + H_1)\rho}{2M}, \frac{(1 - L)(1 + H_1)(1 + H_2)}{N + (1 + H_1)(1 + H_2)} \right\},$$

then there exists a local stable manifold for the solutions of (3.5). Moreover, every two solutions  $y(t), \tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\|, \quad \text{for all } t \geq t_0. \quad \text{eq4.12} \quad (3.15)$$

*Proof.* First we notice that the phase subspace  $\mathbb{R}^d$  splits into the direct sum  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{kernel}P_y(t)$  for all  $t \geq 0$ . We set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{kernel}P_y(t)$ , then due to Proposition 2.1, we see that  $\sup_{t \geq 0} \|P_y(t)\| < \infty$ , and hence,  $\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$ .

280 For any  $\rho > 0$  defined as in Assumptions 3.1, 3.5, let  $\rho_1 := \tilde{\rho} = \frac{1-L}{2N(1+H_1)}\rho$  and  $\rho_2 := \frac{(1-L)\rho}{2}$ . For  
 281 each  $t \geq 0$  we define the mapping  $g_t$  acting on  $B_{\rho_1} \cap W_1(t)$  as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

282 where the function  $y(t)$  is uniquely defined via Theorem 3.11 i). Clearly,  $g_t(w_1) \in \ker P_y(t) = W_2(t)$ .

283 Now, we prove that  $\|g_t(w_1)\| \leq \rho_2$ . Due to Theorem 3.11 (i) and Lemma 3.7 (i), we have that  $\|y(t)\| \leq \rho$   
 284 and  $\|f_1(\tau, y(\tau))\| \leq M\varphi(\tau)$  for a.e.  $t \geq 0$ . Therefore,

$$\begin{aligned} \|g_t(w_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} M\varphi(\tau) d\tau, \\ &\leq M (1+H_2) \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) = \frac{MH_3}{1+H_1} \leq \frac{(1-L)\rho}{2}, \end{aligned}$$

286 and hence,  $g_t : B_{\rho_1} \cap W_1(t) \rightarrow B_{\rho_2} \cap W_2(t)$ .

287

Notice that both part (iii) in Definition 3.12 and estimation (3.15) are followed directly from Theorem 3.11. We now only need to prove that  $\mathbb{M}_t$  is homeomorphic to  $B_{\rho_1} \cap W_1(t)$ . We first prove that  $g_t$  is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

288 and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

289 Finally,  $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$  yields that  $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$ , and hence,  $g_t$  is a  
 290 contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous  
 291 mappings ([19, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow B_{\rho_1} \cap W_1(t)$  is a homeomorphism.  
 292 This implies the condition (ii) of Definition 3.12 finishing the proof.  $\square$

#### 4. GLOBAL INVARIANT STABLE MANIFOLDS FOR SEMI-LINEAR DAES

294 In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1).  
 295 We begin with the concept of  $\varphi$ -Lipschitz functions.

296 **Definition 4.1.** Let  $E$  be an admissible Banach function space and  $\varphi \in E$  be a positive function. A function  
 297  $h : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^d$  is said to be  $\varphi$ -Lipschitz if the following conditions hold true.

- 298 (i)  $\|h(t, 0)\| = 0$  for a.e.  $t \in \mathbb{R}_+$ ,
- 299 (ii)  $\|h(t, x) - h(t, \tilde{x})\| \leq \varphi(t)\|x - \tilde{x}\|$  for a.e.  $t \in \mathbb{R}_+$  and all  $x, \tilde{x} \in \mathbb{R}^n$ .

300 In comparability to Assumptions 3.1, 3.5, we also need some global properties of the nonlinear term  $f$ .

301 **Assumption 4.2.** Assume that the following hypotheses hold true.

- 302 (i) The function  $\Sigma^{-1}(t) f_1(t, x(t)) - \Sigma^{-1}(t) \tilde{A}_2(t) \tilde{A}_4^{-1}(t) f_2(t, x(t))$  is  $\varphi$ -Lipschitz.

- (ii) The function  $\tilde{A}_4^{-1}(t)f_2(t, x(t))$  is a contraction mapping with the Lipschitz constant  $L < 1$  for all  $(t, x(t))$  lying on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t, x) := \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid A_2(t)x + f_2(t, x) = 0\}.$$

We can directly verify that orthogonal transformations of the form  $x = Vy$  preserves the  $\varphi$ -Lipschitz property, and hence, function  $\hat{f}_1$  in (3.1) is also  $\varphi$ -Lipschitz. Besides that, function  $\hat{f}_2$  in (3.2) is also a contraction mapping with the Lipschitz constant  $L < 1$ . For notational simplicity, now we will study the transformed system (1.7) and the integral-algebraic system (3.5).

**Definition 4.3.** A subset  $\mathbb{M}$  of the constrained manifold  $\mathbb{L}(t, y)$  is said to be a *global, invariant stable manifold* for solutions to (3.5) if for every  $t \in \mathbb{R}_+$  the phase subspace  $\mathbb{R}^d$  splits into a direct sum  $\mathbb{R}^d = W_1(t) \oplus W_2(t)$  such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+} \inf\{\|w_1 + w_2\|, w_i \in W_i(t), \|w_i\| = 1, i = 0, 1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \rightarrow W_2(t), \quad t \in \mathbb{R}_+,$$

with the Lipschitz constants independent of  $t$  such that

- (i)  $\mathbb{M} = \{(t, w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$ , and we denote by  $\mathbb{M}_t := \{(y_1, y_2) \mid (t, y_1, y_2) \in \mathbb{M}\}$ ,
- (ii)  $\mathbb{M}_t$  is homeomorphic to  $W_1(t)$  for all  $t \geq 0$ ,
- (iii) to each  $\tilde{w} \in \mathbb{M}_{t_0}$  there corresponds one and only one solution  $y$  to (3.5) satisfying  $y_1(t_0) = \tilde{w}$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ ,
- (iv)  $\mathbb{M}$  is invariant under system (3.5), i.e., if  $y$  is a solution to (3.5), and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ , then  $y(s) \in \mathbb{M}_s$  for all  $s \geq t_0$ .

Analogously to Lemma 3.8, we give the explicit form of bounded solutions to system (3.5) as below.

**Lemma 4.4.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumption 4.2 holds true. Let  $y(t)$  be any solution to (3.5) such that  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$  for a fixed  $t_0 \geq 0$ . Then, for all  $t \geq t_0 \geq 0$ , we can rewrite  $y(t)$  in the form

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \quad (4.1)^4$$

for some  $v_0 \in \text{Im}P_y(t_0)$ , where  $G(t, \tau)$  is the Green function defined by (3.6).

*Proof.* The proof can be done by using similar arguments as in the proof of Lemma 3.2.  $\square$

In the following two theorems, we present the global versions of Theorems 3.11 and 3.13, where we construct the structure of bounded solutions to (3.5) and prove the existence of a global, stable manifold, respectively.

**Theorem 4.5.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumption 4.2 holds true.

- (i) For any fixed  $t_0 \geq 0$ , if  $H_3 < 1 - L$  then there corresponds to each  $v_0 \in \text{Im}P_y(t_0)$  one and only one solution  $y(t)$  to (3.5) on  $[t_0, \infty)$  satisfying  $P_y(t_0)y_1(t_0) = v_0$  and  $\text{ess sup}_{t \geq t_0} \|y(t)\| < \infty$ .

(ii) Any two solutions  $y(t)$ ,  $\tilde{y}(t)$  corresponding to different initial conditions  $v_0$ ,  $\tilde{v}_0$  in  $\text{Im}P_y(t_0)$ , are exponentially attracted to each other, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|v_0 - \tilde{v}_0\| \quad \text{for all } t \geq t_0,$$

with some positive constants  $H_4$ ,  $\mu$  satisfying

$$0 < \mu < \nu + \ln \left( 1 - \frac{H_3(1-e^{-\nu})}{1-L} \right), \quad H_4 := \frac{(1+H_1)N}{1-L - \frac{H_3(1-e^{-\nu})}{1-e^{\mu-\nu}}},$$

*Proof.* The proof of this theorem is essentially the same as the proof of Theorem 3.11. The only change is, that instead of considering the ball  $B_\rho$  we will work with the space  $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$  itself. Then, we can prove (without any difficulty) that for each fixed  $v_0 \in \text{Im}P_y(t_0)$ , the transformation  $T$  defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^\infty G(t, \tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geq t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

is a contraction mapping, and therefore, all the assertions of the theorem follows.  $\square$

**Theorem 4.6.** Let the evolution family  $(\hat{Y}(t, s))_{t \geq s \geq 0}$  of system (1.7) have an exponential dichotomy with the corresponding projection matrices  $\{P_y(t)\}_{t \geq 0}$  and the dichotomy constants  $N, \nu > 0$ . Furthermore, assume that Assumption 4.2 holds true. If

$$H_3 < \min \left\{ 1 - L, \frac{(1-L)(1+H_1)(1+H_2)}{N + (1+H_1)(1+H_2)} \right\},$$

then there exists a global invariant stable manifold for the solutions of (3.5). Moreover, every two solutions  $y(t)$ ,  $\tilde{y}(t)$  on the manifold  $\mathbb{M}$  attract each other exponentially in the sense that there exist positive constants  $H_4$  and  $\mu$  independent of  $t_0 \geq 0$  such that

$$\|y(t) - \tilde{y}(t)\| \leq H_4 e^{-\mu(t-t_0)} \|P(t_0)y_1(t_0) - P(t_0)\tilde{y}_1(t_0)\| \quad \text{for all } t \geq t_0.$$

*Proof.* Analogous to the proof of Theorem 3.13, we consider the decomposition  $\mathbb{R}^d = \text{Im}P_y(t) \oplus \text{ker}P_y(t)$  and set  $W_1(t) := \text{Im}P_y(t)$  and  $W_2(t) := \text{ker}P_y(t)$ . Thus, we see that  $\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) > 0$ .

Now we define the family of mappings  $(g_t)_{t \geq 0}$  acting on  $W_1$  as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t, \tau)(I_d - P_y(\tau))f_1(\tau, y(\tau))d\tau,$$

where the function  $y(t)$  is bounded and be uniquely defined via Theorem 4.5 i). Clearly,  $g_t(w_1) \in \text{ker}P_y(t) = W_2(t)$ . To verify the Lipschitz property of  $g_t$ , let us consider two arbitrary elements  $w_1$  and  $\tilde{w}_1$  in  $W_1$  and let  $y$  and  $\tilde{y}$  be the corresponding functions defined via Theorem 4.5 i). Then, we see that

$$\begin{aligned} \|g_t(w_1) - g_t(\tilde{w}_1)\| &\leq \int_t^\infty N e^{-\nu(\tau-t)} \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leq \int_t^\infty N e^{-\nu(\tau-t)} \varphi(\tau) \|y(\tau) - \tilde{y}(\tau)\| d\tau, \\ &\leq \frac{N}{1-e^{-\nu}} (\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty) \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)} \|y - \tilde{y}\|_\infty, \end{aligned}$$

and hence, (3.10) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leq \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} \|w_1 - \tilde{w}_1\|.$$

Finally,  $H_3 < \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$  yields that  $\frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$ , and hence,  $g_t$  is a contraction mapping for all  $t \geq 0$ . Then, applying the Implicit Function Theorem for Lipschitz continuous mapping ([19, Lem. 2.7]), we see that the mapping  $Id + g_t : \mathbb{M}_t \rightarrow W_1(t)$  is a homeomorphism. This implies the condition ii) of Definition 3.12, and hence, the proof is finished.  $\square$

Now let us illustrate our results by the following examples.

**Example 4.7.** The dynamical behavior of a system in fluid mechanics and turbulence modeling is often described by the incompressible Navier-Stokes equation on an open, bounded domain  $\Omega \subset \mathbb{R}^k$ ,  $k = 2$  or  $3$ , of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nu \Delta u - \nabla p - (u \cdot \nabla) u + f(t, u, p), \\ \nabla \cdot u &= 0, \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0,\end{aligned}$$

where  $\nu > 0$  is the viscosity,  $u = u(t, \xi)$  is the velocity field which is a function of the time  $t$  and the position  $\xi$ ,  $p$  is the pressure,  $f$  is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [9], one obtains a differential-algebraic system of the following form.

$$\begin{aligned}\dot{M}U &= (K + N(U)) U - CP + F(t, U, P), \\ C^T U &= 0,\end{aligned}$$

where  $U(t)$ ,  $P(t)$  approximate the velocity  $u(t, \xi)$  and the pressure  $p(t, \xi)$ , respectively. Here the leading matrix  $M$  is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix  $C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$  is nonsingular. We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free, so Assumption 1.3 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned}\dot{M}U &= -(K + N(U)) U - CP + F(t, U, P), \\ 0 &= C^T M^{-1} C P - C^T M^{-1} (F - (K + N(U)) U).\end{aligned}\tag{4.2}^{eq5,3}$$

Clearly, we still need to linearize (4.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.7)

$$\begin{aligned}\dot{M}U &= \tilde{A}_1(t)U + \tilde{A}_2(t)P + g_1(t, U, P), \\ 0 &= C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right) P - C^T M^{-1} \left( \frac{\partial F}{\partial U} - A(t) \right) U + C^T M^{-1} g_2(t, U, P).\end{aligned}\tag{4.3}^{eq5,4}$$

We further notice that since  $C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$  is nonsingular, from the second equation we can uniquely determine  $P$  in term of  $U$ , and hence, system (4.2) is indeed strangeness-free. Let

$$\tilde{A}_3(t) := -C^T M^{-1} \left( \frac{\partial F}{\partial U} - A(t) \right), \quad \tilde{A}_4(t) := C^T M^{-1} \left( C - \frac{\partial F}{\partial P} \right)$$

Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} = \begin{bmatrix} \tilde{A}_1(t) & \tilde{A}_2(t) \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

<sup>346</sup> admits an exponential dichotomy, and  $g_1$  satisfies the  $\varphi$ -Lipschitz condition, and  $g_2$  is a contraction mapping  
<sup>347</sup> (uniformly in time), then there exists a stable manifold for the solution to (4.2).

**Example 4.8.** Consider the nonlinear electrical circuit with Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation  $i_2 = I_0 \sin(k\phi_2)$ , where  $I_0$  and  $k$  are positive constants depend on the device itself. Moreover, the resistance  $R$ , inductance  $L$  and conductance  $G$  are positive. Furthermore,  $i_1$  is the current going through the inductance,  $v_1$  and  $v_2$  are voltage drops across the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead

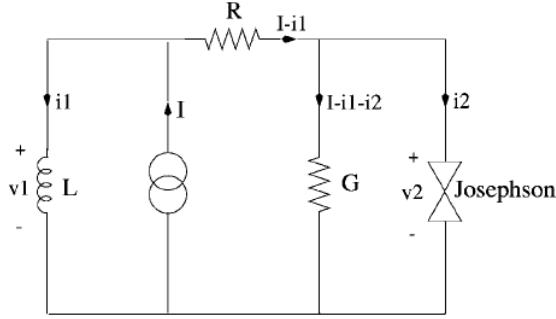


FIGURE 1. Electric circuit with Josephson junction, [23]

of linear resistance, inductance and conductance as in [23], and hence, we see that for the inductance  $i_1 = i_L(L, \phi_1)$ , for the resistance  $v_R = v_R(R, i_1)$ , and for the conductance  $i_G = i_G(G, v_2)$ . Therefore, we obtain the following system, which completely describes the behavior of this circuit.

$$\begin{aligned} \dot{\phi}_1 &= v_1, & (4.4a) \\ \dot{\phi}_2 &= v_2, & (4.4b) \\ i_1 &= i_L(L, \phi_1), & (4.4c) \\ i_2 &= I_0 \sin(k\phi_2), & (4.4d) \\ 0 &= v_1 - v_R(R, i_1) + v_2, & (4.4e) \\ 0 &= -i_G(G, v_2) + I - i_1 - i_2. & (4.4f) \end{aligned}$$

From (4.4c)-(4.4f) we obtain an explicit form of  $v_1$  in terms of  $\phi_1$ ,  $i_1$  and  $v_2$ , so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \quad (4.5a)$$

$$\dot{\phi}_2 = v_2, \quad (4.5b)$$

$$i_1 = i_L(L, \phi_1), \quad (4.5c)$$

$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \quad (4.5d)$$

The linearized version of this system along equilibrium points defined by  $v_2 = 0$ ,  $i_1 = I$ ,  $\phi_1 = LI$ ,  $\phi_2 = n\pi/k$ , reads

$$\begin{aligned}\dot{\phi}_1 &= RI - (R/L)\phi_1 + v_2, \\ \dot{\phi}_2 &= v_2, \\ \dot{i}_1 &= \phi_1/L, \\ 0 &= -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2),\end{aligned}$$

will have a positive eigenvalue and a negative one (e.g. [23]). Hence, it admits exponential dichotomy for any odd number  $n$ . Thus, for  $\varphi$ -Lipschitz function  $v_R$  and contraction mapping  $i_G$ , we obtain a stable manifold for (4.5).<sup>6</sup>

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