Applied Mathematics and Computation 215 (2010) 4008–4021

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Convergence and stability of numerical solutions to a class of index 1 stochastic differential algebraic equations with time delay

Xiaomei Qu [a,*], Chengming Huang [a], Chao Liu [b]

[a] School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, PR China
[b] College of Computer Science and Technology, Huanggang Normal University, Huangzhou 438000, PR China

ARTICLE INFO

Keywords:
Stochastic delay differential algebraic equation
Convergence
Mean-square
Stability
Stochastic theta method

ABSTRACT

In this paper, we study the convergence and stability of the stochastic theta method (STM) for a class of index 1 stochastic delay differential algebraic equations. First, in the case of constrained mesh, i.e., the stepsize is a submultiple of the delay, it is proved that the method is strongly consistent and convergent with order 1/2 in the mean-square sense. Then, the result is further extended to the case of non-constrained mesh where we employ linear interpolation to approximate the delay argument. Later, under a sufficient condition for mean-square stability of the analytical solution, it is proved that, when the stepsizes are sufficiently small, the STM approximations reproduce the stability of the analytical solution. Finally, some numerical experiments are presented to illustrate the theoretical findings.

1. Introduction

In recent years, because stochastic differential equations (SDEs) play an important role in modeling of problems in many branches of science and industry, their numerical analysis has received considerable attention; see, e.g., [9–12,14,15]. During the same period of time, much work has also been done in the field of numerical solution of differential algebraic equations (DAEs) (see [1,7,16,18]). These systems can be found in a wide variety of scientific and engineering applications, including circuit analysis, computer-aided design and real-time simulation of mechanical systems, and optimal control. While delay differential equations (DDEs) serve as models of physical processes whose time evolution depends on their past history, which arise from, for example, circuit simulation and power systems. The same as SDEs and DAEs, DDEs are also well known and intensively studied (cf. [2,5,19]).

A generalization of SDEs and DAEs is stochastic differential algebraic equations (SDAEs). Not much work has been done on numerical methods for SDAEs. In [20], Winkler gives a introduction to index 1 SDAEs and discusses the convergence and stability of their numerical methods.

Delay differential algebraic equations (DDAEs), which can be seen as an extension of DAEs and DDEs, appear frequently in many problems [22]. Until now, there is little work on numerical analysis of DDAEs, and most of papers are concerned with linear problems. Here, we refer to [6] for a recent discussion of numerical stability for linear DDAEs.

Stochastic delay differential algebraic equations (SDDAEs), which have both delay and algebraic constraints, possess stochastic nature as well. Relatively little is known about the numerical analysis of SDDAEs. In this paper, we consider a class of index 1 SDDAEs. Their stochastic theta method is established, and the convergence and mean-square stability are analyzed.

This paper is organized as follows.

* Corresponding author.
E-mail addresses: maggiequ111@163.com (X. Qu), chengming_huang@hotmail.com (C. Huang), leochal@163.com (C. Liu).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.12.007

In Section 2, a class of index 1 SDDAEs is given and the initial value problems are formulated. We introduce some necessary notations and assumptions, then we present the conditions of existence and uniqueness of strong solutions, and establish the stochastic theta method for index 1 SDDAEs.

In Section 3, we provide a proof of mean-square consistency and convergence for the stochastic theta method applied to index 1 SDDAEs. This generalizes some results of strong convergence for index 1 SDAEs in [20] to index 1 SDAEs with time delay. We introduce an attempt to treat the delay argument by linear interpolation, and also prove that the stochastic theta method for SDDAEs with linear interpolation procedure is consistent and convergent.

In Section 4, we first discuss the stability of analytical solution for a class of index 1 SDDAEs. Then we prove that the STM approximate solutions can preserve MS-stability of the analytical solution, when the stepsizes are sufficiently small.

Finally, in Section 5 some numerical experiments are presented to illustrate the theoretical findings.

2. Index 1 SDDAEs and the stochastic theta method

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant t_0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant t_0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $W = (W_1, W_2, \ldots, W_s)^T$ be a s-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ be the Euclidean norm in R^d. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by

$$|A| = \sqrt{trace(A^TA)}.$$

Let $\tau > 0, J = [t_0, T]$ and $R_+ = [0,\infty)$. Denote by $C([-\tau, 0], R^d)$ the family of continuous functions from $[-\tau, 0]$ to R^d with the norm $\|\varphi\| = \sup_{-\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$. Let $p > 0$, and denote by $L^p_{\mathcal{F}_{t_0}}(J, R^d)$ the family of $\mathcal{F}_{t_0}-$ measurable $C(J, R^d)$-valued random variables $\xi = \{\xi(\theta) : \theta \in J\}$ such that $E\|\xi\|^p < \infty$. The norm and the inner product on $L^2(\Omega, \mathcal{F}, P)$ are defined by

$$\|x\|_{L^2} = (E|x|^2)^{\frac{1}{2}}, \langle x, y\rangle = E(x^Ty).$$

2.1. Index 1 SDDAEs

In the following, we consider a class of index 1 stochastic delay differential algebraic equations (SDDAEs) in Itô sense

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), y(t))dt + g(x(t), x(t-\tau), y(t))dW(t), & t \in J, \\ 0 = u(x(t), x(t-\tau), y(t)), & t \in J, \end{cases} \tag{2.1}$$

with initial data $x(t) = \varphi(t), y(t) = \psi(t), t_0 - \tau \leqslant t \leqslant t_0 \cdot x(t)$ and $y(t)$ are R^d and R^l-valued processes, respectively, $f : R^d \times R^d \times R^l \to R^d$, $u : R^d \times R^d \times R^l \to R^l$, $g : R^d \times R^d \times R^l \to R^{d\times s}$ are given continuous functions, $\varphi(t)$ and $\psi(t)$ are $\mathcal{F}_{t_0}$-measurable and

$$E\|\varphi\|^2 < \infty, \quad E\|\psi\|^2 < \infty\left(\|\varphi\| = \sup_{t_0-\tau\leqslant t\leqslant t_0}|\varphi(t)|\right).$$

For $u(x, v, y) = 0$, the Jacobian matrix $\partial u/\partial y$ is supposed to have a uniformly bounded inverse. Without loss of generality, we suppose

$$|(\partial u(x, v, y)/\partial y)^{-1}| \leqslant L_1, |\partial u(x, v, y)/\partial x| \leqslant L_2, |\partial u(x, v, y)/\partial v| \leqslant L_3. \tag{2.2}$$

Integrating on both sides of (2.1) we can obtain a stochastic integral problem

$$\begin{cases} x(t) = \varphi(t_0) + \int_{t_0}^t f(x(s), x(s-\tau), y(s))ds + \int_{t_0}^t g(x(s), x(s-\tau), y(s))dW(s), & t \in J, \\ 0 = u(x(t), x(t-\tau), y(t)), \\ x(t) = \varphi(t), y(t) = \psi(t), t_0 - \tau \leqslant t \leqslant t_0, \end{cases} \tag{2.3}$$

where the second integral in the first equation of (2.3) is an Itô integral.

2.2. Existence and uniqueness of the solution

In order to ensure the existence and uniqueness of the solution, we further assume that f and g are smooth enough and satisfy

$$(i) \quad |f(x, v, y) - f(\bar{x}, \bar{v}, \bar{y})| \vee |g(x, v, y) - g(\bar{x}, \bar{v}, \bar{y})| \leqslant \beta_1(|x - \bar{x}| + |v - \bar{v}| + |y - \bar{y}|), \tag{2.4}$$

$$(ii) \quad |f(x, v, y)| \vee |g(x, v, y)| \leqslant \beta_2(1 + |x| + |v| + |y|), \tag{2.5}$$

$$(iii) \quad u(\varphi(t), \varphi(t-\tau), \psi(t)) = 0, \ t_0 - \tau \leqslant t \leqslant t_0, \tag{2.6}$$

for all $x, v, \bar{x}, \bar{v}, \in R^d, y, \bar{y} \in R^l$, where $\beta_1, \beta_2 > 0$ are constants (Generally, (i) is called global Lipschitz condition (GLC), (ii) is known as linear growth condition (LGC)). Because the Jacobian matrix $\partial u/\partial y$ is supposed to have a uniformly bounded inverse, so using the existence theorem of implicit function, then for $u(x, v, y) = 0$ there exists a unique solution

$$y = U(x, v),$$

which implies

$$y(t) = U(x(t), x(t-\tau)). \tag{2.7}$$

Substituting (2.7) into (2.1), we can get

$$dx(t) = f(x(t), x(t-\tau), U(x(t), x(t-\tau)))dt + g(x(t), x(t-\tau), U(x(t), x(t-\tau)))dW(t), \quad t \in J. \tag{2.8}$$

Thus, the index 1 SDDAE (2.1) can be transformed to a stochastic delay differential equation theoretically. Nevertheless, it is normally difficult to find the explicit expression of $U(x, v)$ so that we have to solve the original SDDAE.

Lemma 2.1 [12]. *For stochastic delay differential equations*

$$\begin{cases} dx(t) = \tilde{f}(t, x(t), x(t-\tau))dt + \tilde{g}(t, x(t), x(t-\tau))dW(t), & t > t_0, \\ x(t) = \xi, \quad t \leqslant t_0, \end{cases} \tag{2.9}$$

suppose f and g are globally Lipschitz-continuous with respect to x, continuous with respect to t, satisfy the linear growth condition (LGC), and that initial data $E\|\xi\|^2 < \infty$*, which is independent of the Brownian motion W and with finite second moments. Then there exists a solution process* $x(\cdot)$ *of SDDEs (2.9) that is pathwise unique. Moreover, the solution process* $x(\cdot)$ *is square-integrable and fulfills*

$$E\left(\sup_{t_0-\tau \leqslant s \leqslant t} |x(s)|^2\right) \leqslant \check{c}_1 E(1 + \|\xi\|^2) e^{\check{c}_2(t-t_0)},$$

with constants $\check{c}_1, \check{c}_2$.

The regular SDDE (2.9) together with the assembling of the solution (2.7) is equivalent to the problem (2.8). Based on this fact we are now able to give our theorem on the existence and uniqueness of solution to index 1 SDDAEs.

Theorem 2.1. *Suppose f and g satisfy (i)–(iii), and that* $\varphi(t) \in L^2_{\mathscr{F}_{t_0}}(J, R^d), \psi(t) \in L^2_{\mathscr{F}_{t_0}}(J, R^l)$ *are independent of the Brownian motion W and with finite second moments. Then there exists a solution process* $x(\cdot)$ *of index 1 SDDAEs (2.1) that is pathwise unique. Moreover, the solution process* $x(t)$ *is square-integrable.*

The proof of this theorem is analogue to the theorem on existence and uniqueness of solution to neutral stochastic functional differential equations with infinite delay in [17]. Hence, throughout this paper, we assume that (2.1) has a unique global solution $(x(t), y(t))$.

2.3. Stochastic theta method for index 1 SDDAEs

Apply stochastic theta method to (2.1), then we have

$$\begin{cases} x_{n+1} = x_n + h[\theta f(x_{n+1}, \bar{x}_{n+1}, y_{n+1}) + (1-\theta) f(x_n, \bar{x}_n, y_n)] + g(x_n, \bar{x}_n, y_n)\triangle W_n, & n > 0, \\ 0 = u(x_{n+1}, \bar{x}_{n+1}, y_{n+1}), \\ x_n = \varphi(t_n), y_n = \psi(t_n), \quad n \leqslant 0, \end{cases} \tag{2.10}$$

where the stepsize $h > 0$, the parameter $\theta \in [0, 1], t_n = nh, n = 0, 1, \ldots N$, $\triangle W_n = W(t_{n+1}) - W(t_n)$, each x_n is an approximation to $x(t_n)$. For $\theta = 0$, (2.10) reduces to Euler–Maruyama (EM). The $\theta = 1$ case is addressed as backward Euler (BE).

The argument $\bar{x}_n$ denotes an approximation to $x(t_n - \tau)$, that is obtained by an interpolation procedure at the point $t = t_n - \tau$ using values $\{x_k\}_{k \leqslant n}$.

Let $\tau = (m - \delta)h$ with m a positive integer and $\delta \in [0, 1)$. We can define

$$\bar{x}_k = \delta x_{k-m+1} + (1-\delta) x_{k-m}, \tag{2.11}$$

here when $k \leqslant 0, x_k = \varphi(t_0 + kh), y_k = \psi(t_0 + kh)$.

Integrating from t_n to t_{n+1} on (2.1), we have

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} f(x(t), x(t-\tau), y(t))dt + \int_{t_n}^{t_{n+1}} g(x(t), x(t-\tau), y(t))dW(t), \quad t \in [t_n, t_{n+1}]. \tag{2.12}$$

3. Convergence analysis

Throughout this section, we always assume that conditions (i)–(iii) are satisfied. For the convenience of discussion, we denote

$$\begin{cases} r_n := x(t_{n+1}) - x(t_n) - h[\theta f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1})) \\ \qquad +(1-\theta) f(x(t_n), x(t_n-\tau), y(t_n))] - g(x(t_n), x(t_n-\tau), y(t_n))\triangle W_n, \end{cases} \tag{3.1}$$

and $E_t(x) = E(x|\mathscr{F}_t)$. In accordance with [4], we present the following definitions.

Defintion 3.1. The numerical method (2.10) is strongly consistent with order p in the mean-square sense if the following estimates hold

$$\max_{0\leqslant n\leqslant N} \|E_{t_n}(r_n)\|_{L^2} \leqslant Ch^{p+1}, \quad \max_{0\leqslant n\leqslant N} \|r_n\|_{L^2} \leqslant Ch^{p+1/2},$$

where C is a positive constant independent of h.

Defintion 3.2. The numerical method (2.10) is strongly convergent with order p in the mean-square sense if the following estimate holds

$$\max_{0\leqslant n\leqslant N}(\|x_n - x(t_n)\|_{L^2} + \|y_n - y(t_n)\|_{L^2}) \leqslant Ch^p,$$

where C is a positive constant independent of h.

In order to prove our main results, we need to introduce a lemma, whose proof can be found in [12].

Lemma 3.1. *If $(x(t), y(t))$ is the analytical solution of (2.1), and conditions (i)–(iii) hold, then*

$$E\left(\sup_{t_0-\tau\leqslant t\leqslant T} |x(t)|^2\right) \leqslant c_1, \tag{3.2}$$

and for all $t_0 - \tau \leqslant s < t \leqslant T$, the estimate holds

$$E|x(t) - x(s)|^2 \leqslant c_2(t-s), \tag{3.3}$$

where c_1, c_2 are constants independent of s and t.

Theorem 3.1. *The stochastic theta method (2.10) applied to index 1 SDDAEs (2.1) is strongly consistent with order $\frac{1}{2}$.*

Proof. By the properties of Itô integral (see [12]), we have

$$E_{t_n}\left(\int_{t_n}^{t_{n+1}} [g(x(t), x(t-\tau), y(t)) - g(x(t_n), x(t_n-\tau), y(t_n))]dW(t)\right) = 0.$$

Substituting (2.12) into (3.1), we have

$$\begin{aligned} r_n = &\int_{t_n}^{t_{n+1}} f(x(t), x(t-\tau), y(t))dt - \int_{t_n}^{t_{n+1}} [\theta f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1})) + (1-\theta) f(x(t_n), x(t_n-\tau), y(t_n))]dt \\ &+ \int_{t_n}^{t_{n+1}} [g(x(t), x(t-\tau), y(t)) - g(x(t_n), x(t_n-\tau), y(t_n))]dW(t). \end{aligned} \tag{3.4}$$

By the inequality $(EX)^2 \leqslant EX^2$, we arrive at

$$\begin{aligned} \|E_{t_n}(r_n)\|_{L^2}^2 &= E(|E_{t_n}(r_n)|^2) \\ &= E\left(\left|E_{t_n}[\int_{t_n}^{t_{n+1}} (f(x(t), x(t-\tau), y(t)) - \theta f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1})) - (1-\theta) f(x(t_n), x(t_n-\tau), y(t_n)))dt]\right|^2\right) \\ &\leqslant E\left(E_{t_n}\left|\theta \int_{t_n}^{t_{n+1}} [f(x(t), x(t-\tau), y(t)) - f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1}))]dt \right.\right. \\ &\quad \left.\left. + (1-\theta)\int_{t_n}^{t_{n+1}} [f(x(t), x(t-\tau), y(t)) - f(x(t_n), x(t_n-\tau), y(t_n))]dt\right|^2\right). \end{aligned} \tag{3.5}$$

By the inequality $(\theta a + (1-\theta)b)^2 \leqslant \theta a^2 + (1-\theta)b^2, \theta \in [0,1]$ and Hölder inequality $\int_a^b 1\cdot|f|dt \leqslant \sqrt{|b-a|}\left(\int_a^b |f|^2 dt\right)^{\frac{1}{2}}$, we can see from (3.5) that

$$
\begin{aligned}
\|E_{t_n}(r_n)\|_{L^2}^2 &\leqslant E\left[\theta E_{t_n}\left(\int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_{n+1}),x(t_{n+1}-\tau),y(t_{n+1}))|dt\right)^2\right.\\
&\quad \left. + (1-\theta)E_{t_n}\left(\int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_n),x(t_n-\tau),y(t_n))|dt\right)^2\right]\\
&\leqslant hE\left[\theta E_{t_n}\int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_{n+1}),x(t_{n+1}-\tau),y(t_{n+1}))|^2 dt\right.\\
&\quad \left. +(1-\theta)E_{t_n}\int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_n),x(t_n-\tau),y(t_n))|^2 dt\right].
\end{aligned}
\tag{3.6}
$$

We set $R_i = \int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_i),x(t_i-\tau),y(t_i))|^2 dt$, for $i = n, n+1$. By GLC (2.4), we derive that

$$
R_i \leqslant \beta_1^2 \int_{t_n}^{t_{n+1}} [|x(t)-x(t_i)| + |x(t-\tau)-x(t_i-\tau)| + |y(t)-y(t_i)|]^2 dt. \tag{3.7}
$$

Because the Jacobian matrix $\partial u/\partial y$ is supposed to have a globally bounded inverse, so using the existence theorem of implicit function and (2.2), one has

$$
|U_x(x,v)| = \left|-\left(\frac{\partial u(x,v,y)}{\partial y}\right)^{-1}\frac{\partial u(x,v,y)}{\partial x}\right| \leqslant L_1L_2, \tag{3.8}
$$

$$
|U_v(x,v)| = \left|-\left(\frac{\partial u(x,v,y)}{\partial y}\right)^{-1}\frac{\partial u(x,v,y)}{\partial v}\right| \leqslant L_1L_3. \tag{3.9}
$$

By (3.8) and (3.9), we may ensure that

$$
|y(t)-y(t_n)| \leqslant L_1L_2|x(t)-x(t_n)| + L_1L_3|x(t-\tau)-x(t_n-\tau)|. \tag{3.10}
$$

Taking conditional expectation on both sides of inequality (3.7) and using (3.10) and Lemma 3.1, we have

$$
\begin{aligned}
E_{t_n}|R_i| &\leqslant \beta_1^2 E\int_{t_n}^{t_{n+1}} [(1+L_1L_2)|x(t)-x(t_i)| + (1+L_1L_3)|x(t-\tau)-x(t_i-\tau)|]^2 dt\\
&\leqslant 2\beta_1^2\left[(1+L_1L_2)^2\int_{t_n}^{t_{n+1}} E|x(t)-x(t_i)|^2 dt + (1+L_1L_3)^2\int_{t_n}^{t_{n+1}} E|x(t-\tau)-x(t_i-\tau)|^2 dt\right]\\
&\leqslant c_2\beta_1^2h^2[(1+L_1L_2)^2 + (1+L_1L_3)^2],
\end{aligned}
\tag{3.11}
$$

for $i = n, n+1$. Substituting (3.11) into (3.6) gives

$$
\|E_{t_n}(r_n)\|_{L^2}^2 \leqslant c_2\beta_1^2[(1+L_1L_2)^2 + (1+L_1L_3)^2]h^3 \leqslant c_2\beta_1^2(2+L_1L_2+L_1L_3)^2h^3, \tag{3.12}
$$

which implies

$$
\|E_{t_n}(r_n)\|_{L^2} \leqslant \sqrt{c_2}\beta_1(2+L_1L_2+L_1L_3)h^{\frac{3}{2}}. \tag{3.13}
$$

Moreover, because $g(x(t),x(t-\tau),y(t)) \in L^2(J,R^{d\times s})$, we have (see [12])

$$
\begin{aligned}
&E\left|\int_{t_n}^{t_{n+1}} [g(x(t),x(t-\tau),y(t)) - g(x(t_n),x(t_n-\tau),y(t_n))]dw(t)\right|^2\\
&\quad = E\left(\int_{t_n}^{t_{n+1}} |g(x(t),x(t-\tau),y(t)) - g(x(t_n),x(t_n-\tau),y(t_n))|^2 dt\right).
\end{aligned}
$$

By $(a+b)^2 \leqslant 2a^2 + 2b^2$, we can compute

$$
\begin{aligned}
\|r_n\|_{L^2}^2 &= E|r_n|^2\\
&\leqslant 2hE\left[\int_{t_n}^{t_{n+1}} \theta|f(x(t),x(t-\tau),y(t)) - f(x(t_{n+1}),x(t_{n+1}-\tau),y(t_{n+1}))|^2 dt\right.\\
&\quad \left. + (1-\theta)\int_{t_n}^{t_{n+1}} |f(x(t),x(t-\tau),y(t)) - f(x(t_n),x(t_n-\tau),y(t_n))|^2 dt\right]\\
&\quad + 2\int_{t_n}^{t_{n+1}} E|g(x(t),x(t-\tau),y(t)) - g(x(t_n),x(t_n-\tau),y(t_n))|^2 dt\\
&\leqslant 2h[\theta ER_{n+1} + (1-\theta)ER_n] + 4\beta_1^2\int_{t_n}^{t_{n+1}}\left[(1+L_1L_2)^2E|x(t)-x(t_n)|^2 + (1+L_1L_3)^2E|x(t-\tau)-x(t_n-\tau)|^2\right]dt\\
&\leqslant 4\beta_1^2c_2(2+L_1L_2+L_1L_3)^2h^2,
\end{aligned}
\tag{3.14}
$$

hence

$$\|r_n\|_{L^2} \leqslant 2\sqrt{c_2}\beta_1(2+L_1L_2+L_1L_3)h. \tag{3.15}$$

This completes the proof. □

Theorem 3.2. *Let $\tau = mh$ with m a positive integer, then the method (2.10) for index 1 SDDAEs (2.1) is strongly convergent with order 1/2.*

Proof. By (2.10) and (3.1), we can obtain

$$\begin{aligned} x_{n+1} - x(t_{n+1}) = x_n - x(t_n) &+ \theta h[f(x_{n+1}, x_{n-m+1}, y_{n+1}) - f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1}))] + (1-\theta)h[f(x_n, x_{n-m}, y_n) \\ &- f(x(t_n), x(t_n-\tau), y(t_n))] + [g(x_n, x_{n-m}, y_n) - g(x(t_n), x(t_n-\tau), y(t_n))]\triangle W_n - r_n. \end{aligned} \tag{3.16}$$

Here $\tau = mh$ and $t_n - \tau = t_{n-m}$. Denote

$$\begin{aligned} M_n &= x_n - x(t_n), \\ \phi f_n &= f(x_n, x_{n-m}, y_n) - f(x(t_n), x(t_n-\tau), y(t_n)), \\ \phi g_n &= g(x_n, x_{n-m}, y_n) - g(x(t_n), x(t_n-\tau), y(t_n)), \\ \bar{\phi} f_n &= \theta\phi f_{n+1} + (1-\theta)\phi f_n. \end{aligned} \tag{3.17}$$

Therefore,

$$M_{n+1} = M_n + h\bar{\phi}f_n + \phi g_n \triangle W_n - r_n, \tag{3.18}$$

which implies

$$\begin{aligned} \|M_{n+1}\|_{L^2}^2 &= \langle M_n + h\bar{\phi}f_n + \phi g_n\triangle W_n - r_n, M_n + h\bar{\phi}f_n + \phi g_n\triangle W_n - r_n\rangle \\ &= E\Big[|M_n|^2 + |\bar{\phi}f_n|^2h^2 + |r_n|^2 + 2hM_n^T\bar{\phi}f_n + 2M_n^T\phi g_n\triangle W_n - 2M_n^Tr_n \\ &\quad + 2h(\bar{\phi}f_n)^T\phi g_n\triangle W_n - 2(\phi g_n\triangle W_n)^Tr_n - 2h(\bar{\phi}f_n)^Tr_n\Big] + \langle \phi g_n\triangle W_n, \phi g_n\triangle W_n\rangle. \end{aligned} \tag{3.19}$$

Since $W(t)$ is a standard s-dimensional Brownian motion, we have $E[M_n^T\phi g_n\triangle W_n|\mathscr{F}_{t_n}] = 0$, $E[(\bar{\phi}f_n)^T\phi g_n\triangle W_n|\mathscr{F}_{t_n}] = 0$, $E[(\phi g_n\triangle W_n)^Tr_n|\mathscr{F}_{t_n}] = 0$, and

$$\|M_{n+1}\|_{L^2}^2 = E|M_n|^2 + h^2E|\bar{\phi}f_n|^2 + hE|\phi g_n|^2 + E|r_n|^2 + 2hE\Big(M_n^T\bar{\phi}f_n\Big) - 2E\Big(M_n^Tr_n\Big) - 2hE(\bar{\phi}f_n)^Tr_n. \tag{3.20}$$

Similar to (3.10), one has

$$|y_n - y(t_n)| \leqslant L_1L_2|x_n - x(t_n)| + L_1L_3|x_{n-m} - x(t_{n-m})|. \tag{3.21}$$

By (3.21) and GLC (2.4), it follows that

$$|\phi f_n| \leqslant \beta_1[(1+L_1L_2)|M_n| + (1+L_1L_3)|M_{n-m}|]. \tag{3.22}$$

Denote $K_1 := \beta_1(1+L_1L_2)$ and $K_2 := \beta_1(1+L_1L_3)$. Considering $(\theta x + (1-\theta)y)^2 \leqslant \theta x^2 + (1-\theta)y^2$, we have

$$|\bar{\phi}f_n|^2 \leqslant \theta|\phi f_{n+1}|^2 + (1-\theta)|\phi f_n|^2 \leqslant 2\theta\Big(K_1^2|M_{n+1}|^2 + K_2^2|M_{n+1-m}|^2\Big) + 2(1-\theta)\Big(K_1^2|M_n|^2 + K_2^2|M_{n-m}|^2\Big).$$

It is known that $x_n, x(t_n), x_n - x(t_n)$ are $\mathscr{F}_{t_n}$-measurable, thus by the fundamental inequality $2xy \leqslant hx^2 + h^{-1}y^2, 0 < h < 1$ and (3.12), we have

$$\Big|E\Big(M_n^Tr_n\Big)\Big| = \Big|E\Big(E_{t_n}\Big(M_n^Tr_n\Big)\Big)\Big| = \Big|E\Big(M_n^TE_{t_n}(r_n)\Big)\Big| \leqslant \frac{1}{2}[hE|M_n|^2 + h^{-1}E(|(E_{t_n}(r_n))|^2)] = \frac{1}{2}\Big[hE|M_n|^2 + K_3^2h^2\Big], \tag{3.23}$$

where $K_3 = \sqrt{c_2}\beta_1(2+L_1L_2+L_1L_3)$. Additionally,

$$\begin{aligned} 2|E\Big(M_n^T\bar{\phi}f_n\Big)| &= 2\theta|E\Big(M_n^T\phi f_{n+1}\Big)| + 2(1-\theta)|E\Big(M_n^T\phi f_n\Big)| \\ &\leqslant K_1\theta E|M_{n+1}|^2 + (K_1(2-\theta)+K_2)E|M_n|^2 + K_2\theta E|M_{n-m+1}|^2 + K_2(1-\theta)E|M_{n-m}|^2, \end{aligned} \tag{3.24}$$

$$\begin{aligned} |\phi g_n|^2 &= |g(x_n, x_{n-m}, y_n) - g(x(t_n), x(t_n-\tau), y(t_n))|^2 \\ &\leqslant \beta_1^2[|M_n| + |M_{n-m}| + |y_n - y(t_n)|]^2 \\ &\leqslant 2\beta_1^2[(1+L_1L_2)^2|M_n|^2 + (1+L_1L_3)^2|M_{n-m}|^2] \\ &= 2K_1^2|M_n|^2 + 2K_2^2|M_{n-m}|^2, \end{aligned} \tag{3.25}$$

$$2h|E((\bar{\phi}f_n)^Tr_n)| \leqslant hE(|\bar{\phi}f_n|^2 + |r_n|^2). \tag{3.26}$$

Substituting (3.23)–(3.26) into (3.20), we can show that

$$\begin{aligned}\|M_{n+1}\|_{L^2}^2 \leqslant{} & E|M_n|^2 + 2h\left(K_1^2E|M_n|^2 + K_2^2E|M_{n-m}|^2\right) + 4K_3^2(1+h)h^2 + (h^2+h)\Big[2\theta\left(K_1^2E|M_{n+1}|^2 + K_2^2E|M_{n+1-m}|^2\right) \\ & + 2(1-\theta)\left(K_1^2E|M_n|^2 + K_2^2E|M_{n-m}|^2\right)\Big] + K_1\theta hE|M_{n+1}|^2 + (K_1(2-\theta)+K_2)hE|M_n|^2 \\ & + K_2\theta hE|M_{n-m+1}|^2 + K_2(1-\theta)hE|M_{n-m}|^2 + hE|M_n|^2 + K_3^2h^2 \\ ={} & \left(2K_1^2\theta(h^2+h) + K_1\theta h\right)E|M_{n+1}|^2 + \Big[1 + 2K_1^2(1-\theta)(h^2+h) + 2K_1^2h + (K_1(2-\theta)+K_2)h + h\Big]E|M_n|^2 \\ & + \Big[2K_2^2\theta(h^2+h) + K_2\theta h\Big]E|M_{n-m+1}|^2 + (5+4h)K_3^2h^2 + \Big[2K_2^2(1-\theta)(h^2+h) + 2K_2^2h + K_2(1-\theta)h\Big]E|M_{n-m}|^2.\end{aligned}$$

Set $h_0 = \min\left\{1, \frac{1}{2(1+4K_1)K_1}\right\}$. When $h < h_0$, we have $0 < 1 - 2K_1^2\theta(h^2+h) - K_1\theta h < 1$. Noting that $h \leqslant 1$ and $0 \leqslant \theta, 1-\theta \leqslant 1$, there must exist a sufficiently large constant K such that $K\left[1 - 2K_1^2\theta\left(h_0^2 + h_0\right) - K_1\theta h_0\right] \geqslant 1$, then

$$\begin{aligned}\left(1 - 2K_1^2\theta(h^2+h) - K_1\theta h\right)E|M_{n+1}|^2 \leqslant{} & \Big[1 + \left(2K_1^2(1-\theta) + 2K_2^2\right)(h^2+h) \\ & + \left(K_1(2-\theta) + 2K_2 + 1 + 2K_1^2 + 2K_2^2\right)h\Big] \max_{n-m\leqslant i\leqslant n} E|M_i|^2 + 9K_3^2h^2,\end{aligned} \tag{3.27}$$

$$\begin{aligned}E|M_{n+1}|^2 \leqslant{} & \frac{1}{1 - 2K_1^2\theta(h^2+h) - K_1\theta h}\Big(\Big[1 + \left(2K_1^2(1-\theta) + 2K_2^2\right)(h^2+h) \\ & + \left(K_1(2-\theta) + 2K_2 + 1 + 2K_1^2 + 2K_2^2\right)h\Big] \max_{n-m\leqslant i\leqslant n} EM_i^2 + 9K_3^2h^2\Big) \\ ={} & \left[1 + \frac{2\left(K_1^2+K_2^2\right)(h^2+h) + \left(2K_1 + 2K_2 + 1 + 2K_1^2 + 2K_2^2\right)h}{1 - 2K_1^2\theta(h^2+h) - K_1\theta h}\right] \max_{n-m\leqslant i\leqslant n} EM_i^2 \\ & + \frac{9K_3^2h^2}{1 - 2K_1^2\theta(h^2+h) - K_1\theta h} \\ \leqslant{} & \left(1 + K\left[6\left(K_1^2+K_2^2\right) + 2K_1 + 2K_2 + 1\right]h\right) \max_{n-m\leqslant i\leqslant n} E|M_i|^2 + 9KK_3^2h^2.\end{aligned} \tag{3.28}$$

Let $\alpha = K\left(6K_1^2 + 6K_2^2 + 2K_1 + 2K_2 + 1\right), \beta = 9KK_3^2$. Clearly, α, β are constants independent of h. Then (3.28) turns to be

$$E|M_{n+1}|^2 \leqslant (1+\alpha h) \max_{n-m\leqslant i\leqslant n} E|M_i|^2 + \beta h^2,$$

which implies

$$E|M_{n+1}|^2 \leqslant (1+\alpha h)^n \max_{-m\leqslant i\leqslant 0} E|M_i|^2 + (e^{\alpha T} - 1)\beta\alpha^{-1}h.$$

Therefore, we have $\|x_{n+1} - x(t_{n+1})\|_{L^2} \leqslant C_1h^{\frac{1}{2}}$, where $C_1 = \sqrt{(e^{\alpha T}-1)\beta\alpha^{-1}}$. Furthermore, by (3.12), there exists a non-negative constant C_2 such that

$$\|y_{n+1} - y(t_{n+1})\|_{L^2} \leqslant L_1L_2\|M_{n+1}\|_{L^2} + L_1L_3\|M_{n+1-m}\|_{L^2} \leqslant C_2h^{\frac{1}{2}}.$$

From here, we can conclude that there exists a non-negative constant C such that

$$\max_{0\leqslant n\leqslant N}(\|x_n - x(t_n)\|_{L^2} + \|y_n - y(t_n)\|_{L^2}) \leqslant Ch^{\frac{1}{2}}, \tag{3.29}$$

where C_1, C_2, C are independent of h, which proves the theorem. □

Remark 3.1. Applying Theorems 3.1 and 3.1 to the case of systems without delay, we can also obtain that the stochastic theta method is strongly consistent and convergent with order 1/2, which corresponds to the results in [20]. Therefore, our conclusions of convergence can be seen as an extension of the corresponding result in [20].

Theorem 3.3. *Let $\tau = (m-\delta)h$ with m a positive integer and $\delta \in [0,1)$, then stochastic theta method (2.10) with linear interpolation (2.11) is strongly convergent with order $\frac{1}{2}$.*

Proof. By (3.1) and (2.12), it is easy to see that

$$\begin{aligned}x_{n+1} - x(t_{n+1}) ={} & x_n - x(t_n) + \theta h[f(x_{n+1}, \bar{x}_{n+1}, y_{n+1}) - f(x(t_{n+1}), x(t_{n+1}-\tau), y(t_{n+1}))] + (1-\theta)h[f(x_n, \bar{x}_n, y_n) \\ & - f(x(t_n), x(t_n-\tau), y(t_n))] + [g(x_n, \bar{x}_n, y_n) - g(x(t_n), x(t_n-\tau), y(t_n))]\triangle W_n - r_n.\end{aligned}$$

Denote

$$
\begin{aligned}
M_n &= x_n - x(t_n),\\
\tilde{\phi} f_n &= f(x_n, \bar{x}_n, y_n) - f(x(t_n), x(t_n - \tau), y(t_n)),\\
\tilde{\phi} g_n &= g(x_n, \bar{x}_n, y_n) - g(x(t_n), x(t_n - \tau), y(t_n)),\\
\hat{\phi} f_n &= \theta \tilde{\phi} f_{n+1} + (1-\theta)\tilde{\phi} f_n,
\end{aligned}
$$

then

$$M_{n+1} = M_n + h\hat{\phi} f_n + \tilde{\phi} g_n \triangle W_n - r_n. \tag{3.30}$$

Similar to (3.20), we have

$$\|M_{n+1}\|_{L^2}^2 = E|M_n|^2 + h^2 E|\hat{\phi} f_n|^2 + hE|\tilde{\phi} g_n|^2 + E|r_n|^2 + 2hE\left(M_n^T \hat{\phi} f_n\right) - 2E\left(M_n^T r_n\right) - 2hE((\hat{\phi} f_n)^T r_n). \tag{3.31}$$

By the Hölder and Cauchy inequalities, we can show that

$$
\begin{aligned}
&2h|E\left(M_n^T \hat{\phi} f_n\right)| \leqslant 2h\sqrt{E|M_n|^2}\sqrt{E|\hat{\phi} f_n|^2} \leqslant h\left(E|M_n|^2 + E|\hat{\phi} f_n|^2\right),\\
&2h|E((\hat{\phi} f_n)^T r_n)| \leqslant 2h\sqrt{E|r_n|^2}\sqrt{E|\hat{\phi} f_n|^2} \leqslant h\left(E|r_n|^2 + E|\hat{\phi} f_n|^2\right).
\end{aligned}
$$

Therefore, a combination of (3.23) and (3.31) implies

$$E|M_{n+1}|^2 \leqslant (1+2h)E|M_n|^2 + (h^2+2h)E|\hat{\phi} f_n|^2 + hE|\tilde{\phi} g_n|^2 + (1+h)E|r_n|^2 + K_3^2 h^2. \tag{3.32}$$

Using (2.11), we have

$$
\begin{aligned}
|\bar{x}_n - x(t_n - \tau)| &= |\delta x_{n+1-m} + (1-\delta)x_{n-m} - x(t_n - \tau)|\\
&\leqslant \delta|M_{n-m+1}| + (1-\delta)|M_{n-m}| + \delta|x(t_{n+1-m}) - x(t_n - \tau)| + (1-\delta)|x(t_{n-m}) - x(t_n - \tau)|,
\end{aligned} \tag{3.33}
$$

then

$$
\begin{aligned}
|\tilde{\phi} f_n| &\leqslant \beta_1[(1+L_1L_2)|M_n| + (1+L_1L_3)[\delta|M_{n-m+1}| + (1-\delta)|M_{n-m}| + \delta|x(t_{n+1-m}) - x(t_n-\tau)| + (1-\delta)|x(t_{n-m}) - x(t_n-\tau)|]]\\
&= K_1|M_n| + K_2[\delta|M_{n-m+1}| + (1-\delta)|M_{n-m}| + \delta|x(t_{n+1-m}) - x(t_n-\tau)| + (1-\delta)|x(t_{n-m}) - x(t_n-\tau)|].
\end{aligned}
$$

By Lemma 3.1 and the above inequality, we have

$$
\begin{aligned}
E|\tilde{\phi} f_n|^2 &\leqslant E[K_1|M_n| + K_2[\delta|M_{n-m+1}| + (1-\delta)|M_{n-m}| + \delta|x(t_{n+1-m}) - x(t_n-\tau)| + (1-\delta)|x(t_{n-m}) - x(t_n-\tau)|]]^2\\
&\leqslant 2K_1^2 E|M_n|^2 + 8K_2^2\delta^2 E|M_{n+1-m}|^2 + 8K_2^2(1-\delta)^2 E|M_{n-m}|^2 + 8K_2^2\delta^2 E|x(t_{n+1-m}) - x(t_n-\tau)|^2\\
&\quad + 8K_2^2(1-\delta)^2 E|x(t_{n-m}) - x(t_n-\tau)|^2\\
&\leqslant 2K_1^2 E|M_n|^2 + 8K_2^2\delta^2 E|M_{n+1-m}|^2 + 8K_2^2(1-\delta)^2 E|M_{n-m}|^2 + 8K_2^2 c_2[\delta^2(1-\delta) + \delta(1-\delta)^2]h,
\end{aligned}
$$

hence

$$
\begin{aligned}
E|\hat{\phi} f_n|^2 &\leqslant \theta E|\tilde{\phi} f_{n+1}|^2 + (1-\theta)E|\tilde{\phi} f_n|^2\\
&\leqslant \left[2K_1^2 E|M_{n+1}|^2 + 8K_2^2\delta^2 E|M_{n+2-m}|^2 + 8K_2^2(1-\delta)^2 E|M_{n+1-m}|^2\right]\theta\\
&\quad + \left[2K_1^2 E|M_n|^2 + 8K_2^2\delta^2 E|M_{n+1-m}|^2 + 8K_2^2(1-\delta)^2 E|M_{n-m}|^2\right](1-\theta) + 8K_2^2 c_2\delta(1-\delta)h.
\end{aligned} \tag{3.34}
$$

Noting that (3.21), we have

$$
\begin{aligned}
E|\tilde{\phi} g_n|^2 &= E|g(x_n, \bar{x}_n, y_n) - g(x(t_n), x(t_n-\tau), y(t_n))|^2 \leqslant 2\beta_1^2(1+L_1L_2)^2 E|M_n|^2 + 2\beta_1^2(1+L_1L_3)^2 E|\bar{x}_n - x(t_n-\tau)|^2\\
&\leqslant 2K_1^2 E|M_n|^2 + 8K_2^2\delta^2 E|M_{n+1-m}|^2 + 8K_2^2(1-\delta)^2 E|M_{n-m}|^2 + 8K_2^2 c_2\delta(1-\delta)h.
\end{aligned} \tag{3.35}
$$

It therefore follows that

$$
\begin{aligned}
E|M_{n+1}|^2 &\leqslant 2K_1^2\theta(h^2+2h)E|M_{n+1}|^2 + 8K_2^2\theta\delta^2(h^2+2h)E|M_{n+2-m}|^2 + [1 + 2h + 2K_1^2 h + 2K_1^2(1-\theta)(h^2+2h)]EM_n^2\\
&\quad + (5+4h)K_3^2 h^2 + \left[8K_2^2[(1-\delta)^2\theta + \delta^2(1-\theta)](h^2+2h) + 8K_2^2\delta^2 h\right]E|M_{n+1-m}|^2 + 8K_2^2(1-\delta)^2 h[1 + (1\\
&\quad - \theta)(2+h)]E|M_{n-m}|^2 + 8K_2^2 c_2\delta(1-\delta)(3+h)h^2.
\end{aligned} \tag{3.36}
$$

Next we consider the two cases $m = 1$ and $m \geqslant 2$ separately.

1. In the case of $m = 1$, (3.36) turns to be

$$E|M_{n+1}|^2 \leqslant \left(2K_1^2 + 8K_2^2\delta^2\right)\theta(h^2+2h)E|M_{n+1}|^2 + (5+4h)K_3^2h^2 + 8K_2^2c_2\delta(1-\delta)(3+h)h^2 + \left[1 + \left(2 + 2K_1^2 + 8K_2^2(\delta^2 + (1-\delta)^2)\right)h + \left[8K_2^2(\delta^2(1-\theta) + (1-\delta)^2) + 2K_1^2(1-\theta)\right](h^2+2h)\right] \max_{n-m\leqslant i\leqslant n} E|M_i|^2,$$

then

$$\left(1 - \left(2K_1^2 + 8K_2^2\delta^2\right)\theta(h^2+2h)\right)E|M_{n+1}|^2 \leqslant \left[1 + \left(2 + 2K_1^2 + 8K_2^2\right)h + \left(8K_2^2 + 2K_1^2(1-\theta)\right)(h^2+2h)\right] \max_{n-m\leqslant i\leqslant n} E|M_i|^2 + (5+4h)K_3^2h^2 + 8K_2^2c_2\delta(1-\delta)(3+h)h^2. \tag{3.37}$$

2. In the case of $m \geqslant 2$, we can obtain from (3.36)

$$\left(1 - 2K_1^2\theta(h^2+2h)\right)E|M_{n+1}|^2 \leqslant \left[1 + \left(2 + 2K_1^2 + 8K_2^2\right)h + \left(2K_1^2(1-\theta) + 8K_2^2\right)(h^2+2h)\right] \max_{n-m\leqslant i\leqslant n} E|M_i|^2 + (5+4h)K_3^2h^2 + 8K_2^2c_2\delta(1-\delta)(3+h)h^2. \tag{3.38}$$

Therefore, either (3.37) or (3.38) is just different from the inequality (3.27) in the coefficients. Using the same way as in the proof of Theorem 3.2, we can show the assertion (3.29). □

Remark 3.2. Our work of this section is motivated by a recent meeting report [21], which discusses the convergence of Euler–Maruyama method for a class of stochastic differential algebraic system of index 1 with delay.

Remark 3.3. Applying Theorems 3.1–3.3 to stochastic delay differential equations, we also obtain that the stochastic theta method is convergent for SDDEs. Therefore our results generalize the conclusion of Theorem 7 in [3].

4. Stability analysis

In this section, we aim at the stability properties of analytical and numerical solutions for (2.1).

Various stability properties of numerical methods for SDDEs have been studied (see [4] for example), while not much work for that of SDAEs can be found (see [20]). For the purpose of stability analysis, we further assume that $u(0,0,0) = 0$.

Let us state a theorem which provides us a criterion on the asymptotically stability in the mean-square sense (abbreviated as MS-stability) of the exact solution for Eq. (2.1).

Theorem 4.1. *If f,g satisfy*

$$\begin{cases} x^T f(x,0,0) \leqslant -\bar{a}|x|^2, \\ |f(x,v,y_1) - f(x_1,v_1,y)| \leqslant \alpha_0|x - x_1| + \alpha_1|v - v_1| + \alpha_2|y - y_1|, \\ |g(x,v,y)|^2 \leqslant \bar{\beta}_0|x|^2 + \bar{\beta}_1|v|^2 + \bar{\beta}_2|y|^2, \\ 2\bar{a} > 2\alpha_1 + \bar{\beta}_0 + \bar{\beta}_1 + 2\alpha_2L_1(L_2+L_3) + 2\bar{\beta}_2L_1^2\left(L_2^2 + L_3^2\right), \end{cases} \tag{4.1}$$

for $\forall x, x_1, v, v_1 \in R^d, y, y_1 \in R^l$, where $\bar{a}, \alpha_0, \alpha_1, \alpha_2, \bar{\beta}_0, \bar{\beta}_1, \bar{\beta}_2$ are non-negative constants, then the solution of index 1 SDDAEs (2.1) is mean-square stable.

We will prove this conclusion by considering the equivalent form (2.8) of (2.1). To this end, we first state a stability result concerning SDDEs.

Lemma 4.1. [12]. *For SDDEs (2.9), if f,g satisfy*

$$\begin{cases} 2x^T\tilde{f}(t,x,v) + |\tilde{g}(t,x,v)|^2 \leqslant -\lambda_0|x|^2 + \lambda_1|v|^2 \\ \lambda_0 > \lambda_1 \end{cases}$$

where $t \geqslant 0, x, v \in R^d, \lambda_0, \lambda_1 > 0$ are constants, then the zero solution of SDDEs (2.9) is mean-square stable.

Proof of Theorem 4.1.

For the equivalent form (2.8) of (2.1), considering inequality $|U(x,v)| \leqslant L_1L_2|x| + L_1L_3|v|$, we have

$$\begin{aligned} 2x^Tf(t,x,v,U(x,v)) + |g(t,x,v,U(x,v))|^2 &\leqslant 2x^T|f(t,x,v,U(x,v)) - f(t,x,0,0) + f(t,x,0,0)| + |g(t,x,v,U(x,v))|^2 \\ &\leqslant 2x^T(\alpha_1v + \alpha_2U(x,v)) - 2\bar{a}|x|^2 + \bar{\beta}_0|x|^2 + \bar{\beta}_1|v|^2 + \bar{\beta}_2|U(x,v)|^2 \\ &\leqslant (\alpha_1 + 2\alpha_2L_1L_2 + \alpha_2L_1L_3)|x|^2 + (\alpha_1 + \alpha_2L_1L_3)|v|^2 \end{aligned}$$

$$
\begin{aligned}
&-2\bar{a}|x|^2+\bar{\beta}_0|x|^2+\bar{\beta}_1|v|^2+\bar{\beta}_2|U(x,v)|^2\\
&\leqslant(\alpha_1+2\alpha_2L_1L_2+\alpha_2L_1L_3+\bar{\beta}_0-2\bar{a}+2\bar{\beta}_2L_1^2L_2^2)|x|^2\\
&\quad+(\alpha_1+\bar{\beta}_1+\alpha_2L_1L_3+2\bar{\beta}_2L_1^2L_3^2)|v|^2=-\lambda_0|x|^2+\lambda_1|v|^2,
\end{aligned}
$$

where $\lambda_0=2\bar{a}-\alpha_1-\bar{\beta}_0-2\alpha_2L_1L_2-\alpha_2L_1L_3-2\bar{\beta}_2L_1^2L_2^2>0$, $\lambda_1=\alpha_1+\bar{\beta}_1+\alpha_2L_1L_3+2\bar{\beta}_2L_1^2L_3^2$. Using (4.1), we can easily verify $\lambda_0>\lambda_1$. Because $|y(t)|\leqslant L_1L_2|x(t)|+L_1L_3|x(t-\tau)|$, so $(x(t),y(t))$ is mean-square stable. This completes the proof of Theorem 4.1.

Remark 4.1. Apply Theorem 4.1 to SDDEs, which implies $\alpha_2=\bar{\beta}_2=0$. Then the last condition in (4.1) turns to be $2\bar{a}>2\alpha_1+\bar{\beta}_0+\bar{\beta}_1$. This corresponds to results in [12,13].

Next we discuss MS-stability of the stochastic theta method for (2.1). For brevity, we assume the Brownian motion is 1-D.

Defintion 4.1. The numerical method (2.10) is said to be MS-stable if there exists a positive constant h_0 such that for all $h\in(0,h_0)$, we have

$$\lim_{n\to\infty}E|x_n|^2=0.$$

Theorem 4.2. *Assume that all the conditions of Theorem 4.1 hold, and $\tau=mh$ with m a positive integer, then the stochastic theta method (2.10) for index 1 SDDAEs is MS-stable.*

Proof. Noting that $\tau=mh$, we have

$$x_{n+1}-\theta f(x_{n+1},x_{n+1-m},y_{n+1})h=x_n+(1-\theta)f(x_n,x_{n-m},y_n)h+g(x_n,x_{n-m},y_n)\triangle W_n,$$

then

$$
\begin{aligned}
|x_{n+1}|^2&=|x_n|^2+2h\theta x_{n+1}^Tf(x_{n+1},x_{n-m+1},y_{n+1})-h^2\theta^2|f(x_{n+1},x_{n-m+1},y_{n+1})|^2+h^2(1-\theta)^2|f(x_n,x_{n-m},y_n)|^2\\
&\quad+|g(x_n,x_{n-m},y_n)\triangle W_n|^2+2h(1-\theta)x_n^Tf(x_n,x_{n-m},y_n)+2x_n^Tg(x_n,x_{n-m},y_n)\triangle W_n\\
&\quad+2h(1-\theta)f(x_n,x_{n-m},y_n)^Tg(x_n,x_{n-m},y_n)\triangle W_n.
\end{aligned}\tag{4.2}
$$

Using (3.8) and (3.9), we can obtain

$$|y_n|\leqslant L_1L_2|x_n|+L_1L_3|x_{n-m}|,\tag{4.3}$$

therefore

$$
\begin{aligned}
2x_n^Tf(x_n,x_{n-m},y_n)h&\leqslant 2hx_n^T[f(x_n,x_{n-m},y_n)-f(x_n,0,0)+f(x_n,0,0)]\leqslant 2hx_n^T[\alpha_1|x_{n-m}|+\alpha_2|y_n|]-2\bar{a}h|x_n|^2\\
&\leqslant 2hx_n^T[(\alpha_1+\alpha_2L_1L_3)|x_{n-m}|+\alpha_2L_1L_2|x_n|]-2\bar{a}h|x_n|^2\\
&\leqslant(\alpha_1+\alpha_2L_1L_3-2\bar{a}+2\alpha_2L_1L_2)h|x_n|^2+(\alpha_1+\alpha_2L_1L_3)h|x_{n-m}|^2.
\end{aligned}\tag{4.4}
$$

By (4.1), we have

$$
\begin{aligned}
&|g(x_n,x_{n-m},y_n)|^2\leqslant\bar{\beta}_0|x_n|^2+\bar{\beta}_1|x_{n-m}|^2+\bar{\beta}_2|y_n|^2\leqslant\left(\bar{\beta}_0+2\bar{\beta}_2L_1^2L_2^2\right)|x_n|^2+\left(\bar{\beta}_1+2\bar{\beta}_2L_1^2L_3^2\right)|x_{n-m}|^2,\\
&|f(x_n,x_{n-m},y_n)|^2\leqslant(\alpha_0|x_n|^2+\alpha_1|x_{n-m}|^2+\alpha_2|y_n|^2)^2\leqslant 2\widetilde{K}_1^2|x_n|^2+2\widetilde{K}_2^2|x_{n-m}|^2,
\end{aligned}\tag{4.5}
$$

where $\widetilde{K}_1=\alpha_0+\alpha_2L_1L_2$, $\widetilde{K}_2=\alpha_1+\alpha_2L_1L_3$. Because $E(\triangle W_n|\mathscr{F}_{t_n})=0$, $E[(\triangle W_n)^2|\mathscr{F}_{t_n}]=h$, and x_n,x_{n-m} are $\mathscr{F}_{t_n}$-measurable, we can compute that

$$
\begin{aligned}
&E\big[x_n^Tg(x_n,x_{n-m},y_n)\triangle W_n|\mathscr{F}_{t_n}\big]=0,\\
&E\Big[f(x_n,x_{n-m},y_n)^Tg(x_n,x_{n-m},y_n)\triangle W_n|\mathscr{F}_{t_n}\Big]=0.
\end{aligned}
$$

Substituting (4.4), (4.5) and the above inequalities into (4.2), then taking expectation on both sides, we have

$$
\begin{aligned}
E|x_{n+1}|^2&\leqslant E|x_{n+1}|^2+h^2\theta^2E|f(x_{n+1},x_{n-m+1},y_{n+1})|^2\\
&\leqslant E|x_n|^2+h\theta\Big[(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})E|x_{n+1}|^2+(\alpha_1+\alpha_2L_1L_3)E|x_{n+1-m}|^2\Big]\\
&\quad+2h^2(1-\theta)^2\Big[\widetilde{K}_1^2E|x_n|^2+\widetilde{K}_2^2E|x_{n-m}|^2\Big]+h\Big(\Big(\bar{\beta}_0+2\bar{\beta}_2L_1^2L_2^2\Big)E|x_n|^2+\Big(\bar{\beta}_1+2-\mathrm{ar}\beta_2L_1^2L_3^2\Big)E|x_{n-m}|^2\Big)\\
&\quad+h(1-\theta)\Big[(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})E|x_n|^2+(\alpha_1+\alpha_2L_1L_3)E|x_{n-m}|^2\Big],
\end{aligned}
$$

which implies

$$\begin{aligned}(1-h\theta(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a}))E|x_{n+1}|^2 \leqslant{}& \Big[1+h(1-\theta)(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})+h\Big(\bar{\beta}_0+2\bar{\beta}_2L_1^2L_2^2\Big)\\ &+2h^2(1-\theta)^2\widetilde{K}_1^2\Big]E|x_n|^2+(\alpha_1+\alpha_2L_1L_3)\theta hE|x_{n+1-m}|^2\\ &+\Big[2\widetilde{K}_2^2h^2(1-\theta)^2+\Big(\bar{\beta}_1+2\bar{\beta}_2L_1^2L_3^2\Big)h+(\alpha_1+\alpha_2L_1L_3)(1-\theta)h\Big]E|x_{n-m}|^2.\end{aligned} \tag{4.6}$$

Denote

$$\begin{aligned}P&=1+(1-\theta)(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})h+\Big(\bar{\beta}_0+2\bar{\beta}_2L_1^2L_2^2\Big)h+2(1-\theta)^2\widetilde{K}_1^2h^2,\\ Q&=(\alpha_1+\alpha_2L_1L_3)\theta h,\\ R&=2\widetilde{K}_2^2h^2(1-\theta)^2+\Big(\bar{\beta}_1+2\bar{\beta}_2L_1^2L_3^2\Big)h+(\alpha_1+\alpha_2L_1L_3)(1-\theta)h,\end{aligned}$$

then it obeys that

$$(1-\theta(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})h)E|x_{n+1}|^2\leqslant PE|x_n|^2+QE|x_{n+1-m}|^2+RE|x_{n-m}|^2. \tag{4.7}$$

Since (4.1) holds, we have $\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a}<0$ and $1-h\theta(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})>0$. Denote $A=2(1-\theta)^2\widetilde{K}_1^2$, $B=\bar{\beta}_0+2\bar{\beta}_2L_1^2L_2^2+(1-\theta)(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})$. If$B^2-4A<0$, then $P=1+Bh+Ah^2>0$. Otherwise, when $h<\frac{|B|-\sqrt{B^2-4A}}{2A}$, $P>0$ holds. Clearly, $Q,R>0$. Hence (4.7) can be turned to

$$E|x_{n+1}|^2\leqslant\frac{P+Q+R}{1-S}\max_{n-m\leqslant i\leqslant n}E|x_i|^2,$$

where $S=h\theta(\alpha_1+\alpha_2L_1L_3+2\alpha_2L_1L_2-2\bar{a})<0$. Once $P+Q+R<1-S$, we have $\lim_{n\to\infty}E|x_n|^2=0$. Consequently, we discuss the inequality $P+Q+R<1-S$, which is equivalent to

$$2(1-\theta)^2\Big(\widetilde{K}_1^2+\widetilde{K}_2^2\Big)h+\Big(2\alpha_1+\bar{\beta}_0+\bar{\beta}_1+2\alpha_2L_1(L_2+L_3)+2\bar{\beta}_2L_1^2\Big(L_2^2+L_3^2\Big)-2\bar{a}\Big)<0.$$

When $h<\frac{2\bar{a}-2\alpha_1-\bar{\beta}_0-\bar{\beta}_1-2\alpha_2L_1(L_2+L_3)-2\bar{\beta}_2L_1^2(L_2^2+L_3^2)}{2(1-\theta)^2(\widetilde{K}_1^2+\widetilde{K}_2^2)}$, the above inequality is valid.

To sum up, let all the conditions of Theorem 4.1 hold, and choose

$$h_0=\begin{cases}\frac{C}{2(1-\theta)^2\left(\widetilde{K}_1^2+\widetilde{K}_2^2\right)}, & B>0 \text{ or } B^2-4A<0,\\ \min\left(\frac{C}{2(1-\theta)^2(\widetilde{K}_1^2+\widetilde{K}_2^2)},\frac{|B|-\sqrt{B^2-4A}}{2A}\right), & \text{otherwise},\end{cases}$$

where $C=2\bar{a}-2\alpha_1-\bar{\beta}_0-\bar{\beta}_1-2\alpha_2L_1(L_2+L_3)-2\bar{\beta}_2L_1^2\Big(L_2^2+L_3^2\Big)$, then for all $h<h_0$, the stochastic theta method for index 1 SDDAEs is MS-stable. □

Remark 4.2. If we remove the restriction $\tau=mh$, then we can similarly prove that the stochastic theta method with linear interpolation (2.11) is MS-stable.

5. Numerical examples

We use the equation

$$\begin{cases}dx(t)=[ax(t)+bx(t-1)+y(t)]dt+[cx(t)+dx(t-1)]dW(t), & t\geqslant 0,\\ 0=l\sin x(t-1)+qx(t)-y(t), & t\geqslant 0,\\ x(t)=t+1, y(t)=l\sin t+q(t+1), & t\in[-1,0],\end{cases} \tag{5.1}$$

as a test equation, where a,b,c,d,l,q are constants. This equation can be called the semi-explicit index-1 SDDAEs. The explicit solution on $t\in[0,1]$ is given by

$$\begin{aligned}x(t)&=\phi_{t,0}\left(1+\int_0^t\phi_{s,0}^{-1}(bs+l\sin s-cds)ds+\int_0^t ds\phi_{s,0}^{-1}dW(s)\right),\\ \phi_{t,0}&=\exp\left[\left(a+q-\frac{c^2}{2}\right)t+cW(t)\right].\end{aligned}$$

We obtain the explicit solution on the second interval [1,2] by using the above solution as the new initial function. In fact, on every interval the computing method is the same as the method for stochastic differential equation. We compute the integral in the solution by the linear interpolation, and we choose the stepsize as 1/1024.

First, let us consider the influence of parameters for (5.1). In all figures t_n denotes the mesh-point. We have used the set of coefficients I: $a = -1, b = 0, c = 1, d = 0, l = 0, q = 0$, II: $a = -1, b = 1, c = 1, d = 1, l = 0, q = 0$, III: $a = -1, b = 1, c = 1, d = 1, l = 0.1, q = -10$. We all use $\theta = 0.2, h = 2^{-5}$ from Figs. 1–3. The data used in these figures are obtained by the mean of data by 1000 trajectories. One may consider (5.1) with coefficients I to be a linear stochastic ordinary differential equation. When $a < -\frac{c^2}{2}$, the solution of this equation is mean-square stable [12]. Case II for (5.1) can be seen as a linear stochastic delay differential equation. When $a < -|b| - \frac{(|c|+|d|)^2}{2}$, the solution of this equation is mean-square stable [8]. While case III for (5.1) can be considered as SDDAEs. From Figs. 1–3, we observe a change from stable to unstable then stable, varying from the coefficients.

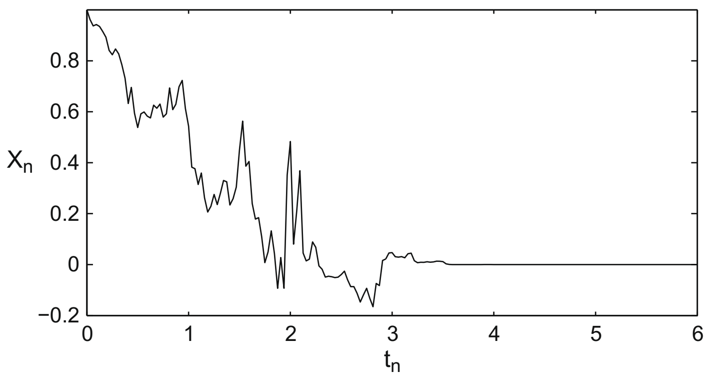

Fig. 1. Simulations with coefficients I for (5.1).

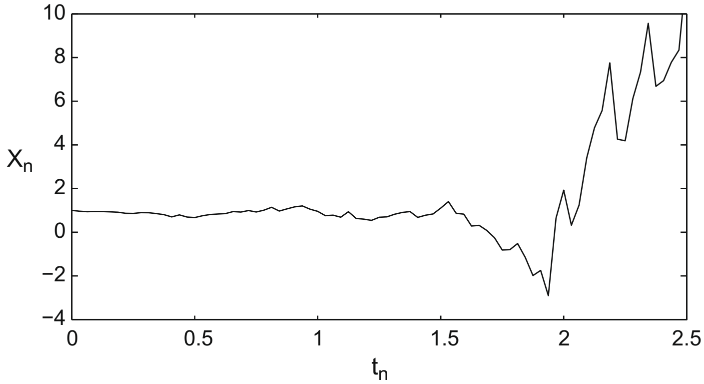

Fig. 2. Simulations with coefficients II for (5.1).

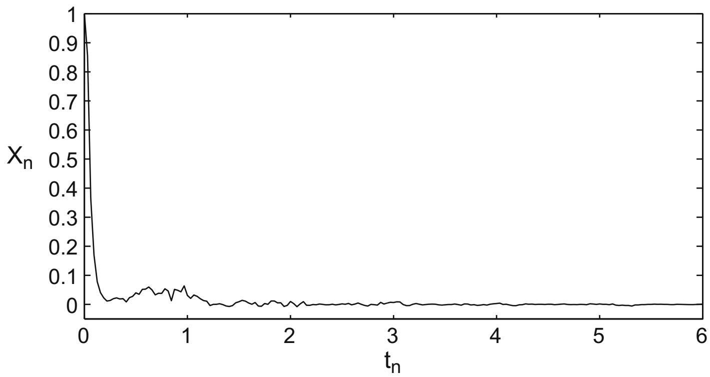

Fig. 3. Simulations with coefficients III for (5.1).

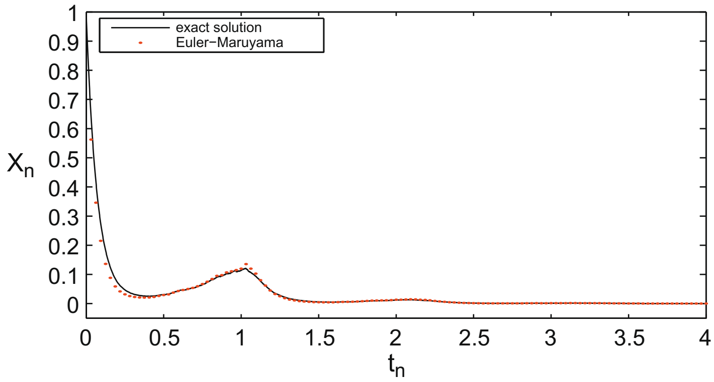

Fig. 4. The stable analytical solution and numerical solution with coefficients IV for (5.1).

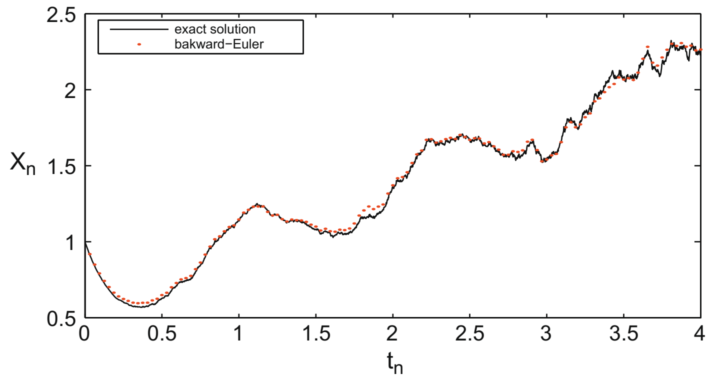

Fig. 5. The unstable analytical solution and numerical solution with coefficients V for (5.1).

Next, we consider the theoretical order of strong convergence. We have used the coefficients IV: $a = -8, b = 2, c = 0, d = 1, l = 1, q = 1$, and V: $a = -4, b = 1, c = 0, d = 3, l = 1, q = 2$. We choose the stepsize $h = 2^{-5}$ and $\theta = 0$ for Fig. 4, $\theta = 1$ for Fig. 5 separately. The mean-square error $E|x(T) - X_N|^2$ at the final time $T = 4$ is estimated in the following way. A set of 20 blocks each containing 100 outcomes $(W_{ij} : 1 \leqslant i \leqslant 20, 1 \leqslant j \leqslant 100)$ are simulated and for each block the estimator

$$\varepsilon_i = \frac{1}{100}\sum_{j=1}^{100}|x(T, W_{ij}) - X_N(W_{ij})|^2$$

is formed. In Table 1, ε denotes the mean of this estimator, which is itself estimated in the usual way:

$$\frac{1}{20}\sum_{i=1}^{20}\varepsilon_i.$$

The numerical results and the explicit solution are compared in Figs. 4 and 5, the results show that numerical solutions converge to explicit solution.

Finally, we check whether the stability conditions (4.1) is valid. We use the coefficients VI: $a = -4, b = 1, c = 0, d = 1, l = 0.2, q = 1$. For the SDDAEs, we can choose $L_1 = 1, L_2 = q = 1, L_3 = l = 0.2, \bar{a} = -a = 4, \bar{\beta}_0 = \bar{\beta}_2 = 0, \bar{\beta}_1 = 1$. By validating conditions (4.1), we ensure that the solution of this equation is mean-square stable. The data is obtained by the mean-square of 1000 trajectories, that is, $W_i : 1 \leqslant i \leqslant 1000, X_n = \frac{1}{1000}\sum_{i=1}^{1000}|X_n(W_i)|^2$. Choosing $\theta = 0.5, h = 2^{-5}$, the simulation results are shown in Fig. 6. Clearly, the STM reveals the stability of the solution.

Table 1
The convergence of BE method applied to (5.1) with IV.

Stepsize	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
ε	2.9806-04	1.5971e−04	7.2109e−05	4.045e−05	2.3559e−05

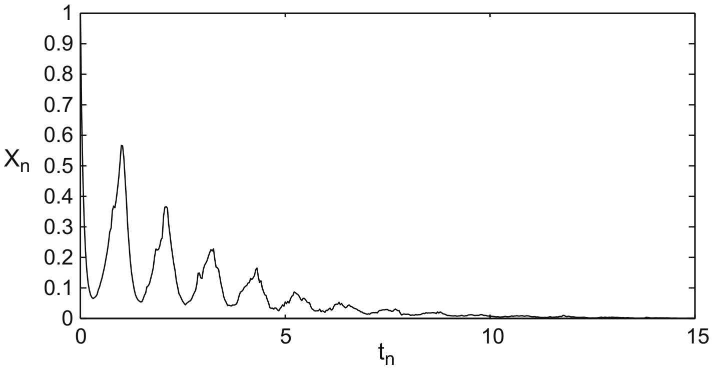

Fig. 6. Simulations with coefficients VI for (5.1).

Acknowledgement

The authors wish to thank the anonymous referees for their kind and valued comments. This work was supported by NSF of China (No. 10971077) and by Program for NCET, the State Education Ministry of China.

References

[1] M. Arnold, Stability of numerical methods for differential-algebraic equations of high index, Appl. Numer. Math. 13 (1993) 5–14.
[2] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, UK, 2003.
[3] E. Buckwar, Introduction to the numerical analysis of stochastic delay differential equations, J. Comput. Appl. Math. 125 (2000) 297–307.
[4] E. Buckwar, One-step approximations for stochastic functional differential equations, Appl. Numer. Math. 56 (2006) 667–681.
[5] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge University Press, UK, 2004.
[6] S.L. Campbell, V.H. Linh, Stability criteria for differential-algebraic equations with multiple delays and their numerical solutions, Appl. Math. Comput. 208 (2009) 397–415.
[7] S.L. Campbell, Nonregular 2D descriptor delay systems, IMA J. Math. Cont. Inf. 12 (1995) 57–67.
[8] W. Cao, M. Liu, Z. Fan, MS-stability of the Euler–Maruyama method for stochastic differential delay equations, Appl. Math. Comput. 159 (2004) 127–135.
[9] D.J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Rev. 43 (2001) 525–546.
[10] A. Jenten, P.E. Kloeden, A. Neuenkirch, Pathwise approximation of stochastic differential equations on domains: higher order convergence rates without global Lipschitz coefficients, Numer. Math. 112 (2009) 41–64.
[11] P.E. Kloden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[12] X. Mao, Stochastic Differential Equations and Applications, Horwood, England, 1997.
[13] X. Mao, S. Sabanis, Numerical solutions of stochastic differential delay equations under local Lipschitiz condition, J. Comput. Appl. Math. 151 (2003) 215–227.
[14] X. Mao, Exponential Stability of Stochastic Differential Equations, Marcel Dekker, New York, 1994.
[15] E. Platen, An introduction to numerical methods for stochastic differential equations, Acta Numer. 8 (1999) 197–246.
[16] G. Reißig, W.S. Martinson, P. Barton, Differential-algebraic equations of index 1 may have an arbitrarily high structural index, SIAM J. Sci. Comput. 21 (2000) 1987–1990.
[17] Y. Ren, N. Xia, A note on the existence and uniqueness of the solution to neutral stochastic functional differential equations with infinite delay, Appl. Math. Comput. 214 (2009) 457–461.
[18] O. Schein, G. Denk, Numerical solution of stochastic differential-algebraic equations with applications to transient noise simulation of microelectronic circuits, J. Comput. Appl. Math. 100 (1998) 77–92.
[19] L. Torelli, A sufficient condition for GRN-stability for delay differential equations, Numer. Math. 59 (1991) 311–320.
[20] R. Winkler, Stochastic differential algebraic equations of index 1 and applications in circuit simulation, J. Comput. Appl. Math. 157 (2003) 477–505.
[21] F. Xiao, C. Zhang, Euler–Maruyama method for a class of stochastic differential algebraic system with delay, in: The 11th Conference on Numerical Methods of Differential Equations, Qingdao, China, 2008.
[22] W. Zhu, L. Petzold, Asymptotic stability of linear delay differential-algebraic equations and numerical methods, Appl. Numer. Math. 24 (1997) 247–264.