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Output reversibility in linear discrete-time dynamical systems ☆

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Abstract

Output reversibility involves dynamical systems where for every initial condition and the corresponding output there exists another initial condition such that the output generated by this initial condition is a time-reversed image of the original output with the time running forward. Through a series of necessary and sufficient conditions, we characterize output reversibility in linear discrete-time dynamical systems in terms of the *geometric symmetry* of its eigenvalue set with respect to the unit circle in the complex plane. Furthermore, we establish that output reversibility of a linear continuous-time system implies output reversibility of its discretization. In addition, we present a control framework that allows to alter the system dynamics in such a way that a discrete-time system, otherwise not output reversible, can be made output reversible. Finally, we present numerical examples involving a discretization of a Hamiltonian system that exhibits output reversibility and an example of a controlled system that is rendered output reversible.

1. Introduction

In continuous-time dynamical systems, Lipschitz continuity of the system dynamics guarantees existence and uniqueness of the system solutions or trajectories. Those trajectories

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evolve as time runs forward and it is due to their uniqueness that the trajectories cannot trace themselves backwards while the time is still running forward. The only possibility for this scenario to occur is to let the time run backwards which is physically not plausible due to the asymmetry of time, also known as the *arrow of time* [1–8]. However, the common sense physics is full of examples when a *part* of the system state or a combination of the system states may be traced backwards while the time is running forward. For example, free motion of a point mass traveling along a straight line with a constant velocity allows for reversibility of its position if an impulse force changes the sign of the point mass velocity while retaining the velocity's magnitude. In this case, with the time running forward, the position of the point mass traces the original position profile precisely backwards, however, the velocity profile is altered. Here, the point mass position or, broadly speaking, part of the system state or a combination of the system states is mathematically regarded as the system output. Other examples include the motion of a harmonic oscillator, orbital motion under the effect of gravity, or a bouncing ball with full restitution. The above examples illustrate the notion of *output reversibility* when for every system output generated by some initial condition, there exists an alternative initial condition such that the corresponding output is the time-reversed image of the original output. We stress here that such time-reversed image is obtained while the time is running forward.

Output reversibility of linear single-output continuous-time dynamical systems was first studied in [9] while the extension to multi-output systems and connections to Poincaré recurrence [10,11] were developed in [12]. Mathematically, output reversibility is characterized by the spectral symmetry of the system state matrix, that is, all system eigenvalues form a set which is symmetric with respect to the origin in the complex plane. Such systems include, as a special case, Hamiltonian systems and rigid body. One of the practical values of the output reversibility is that the system's output can be traced backwards without applying any external input but rather by choosing an appropriate set of initial conditions that generate the time-reversed output; for example, for trajectory tracking in orbital dynamics wherein generalized positions of space station modules need to be tracked backwards to restore their initial configuration while generalized velocities need not be tracked.

In this paper, we study output reversibility of linear discrete-time dynamical systems. Our underlying assumption here is system observability. We explicitly characterize initial conditions that generate discrete outputs which are time-reversed images of the original discrete outputs generated by the original initial conditions. Furthermore, we develop a series of necessary and sufficient conditions for output reversibility which capture the structure of the characteristic polynomial of the system state matrix. Moreover, we present constructive sufficient conditions for output reversibility that involves *geometric symmetry* of the eigenvalue set with respect to the unit circle in the complex plane. In addition, we provide connections between output reversibility of continuous-time and discrete-time systems to establish that output reversibility of a continuous-time system implies output reversibility of its discretization. Furthermore, we provide a framework for designing a dynamic compensator for a linear discrete-time dynamical system such that the closed-loop system is output reversible. This framework allows for reversibility of the *entire* state of the original dynamical system. Finally, we present numerical examples.

2. Mathematical preliminaries

In this section, we introduce notation and definitions needed for developing the results of this paper. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{C}$ denote the set of complex numbers or the complex plane, $j \in \mathbb{C}$ denote the imaginary unit, $\overline{\mathbb{Z}}_+$ denote the set of nonnegative integers, $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, and I or I_n denote the $n \times n$ identity matrix. For $A \in \mathbb{R}^{n \times n}$, the multi-spectrum $\mathrm{mspec}(A) = \{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}}$ is the set of all eigenvalues of A including their multiplicity. We say that the multi-spectrum of A is *geometrically symmetric* with respect to the unit circle in the complex plane if $\{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}} = \{1/\lambda_1, \ldots, 1/\lambda_n\}_{\mathrm{m}}$. We say that, for $A \in \mathbb{R}^{n \times n}$, $\lambda \in \mathrm{mspec}(A)$ is *simple* if the algebraic multiplicity of λ is equal to one. For $A \in \mathbb{R}^{n \times n}$, we write the characteristic polynomial as

$$\chi_A(\lambda) = \det(\lambda I_n - A) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0. \tag{1}$$

Recall that by Cayley–Hamilton theorem [13]

$$\chi_A(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_1 A + \alpha_0 I_n = 0. \tag{2}$$

Consequently, if the inverse of $A \in \mathbb{R}^{n \times n}$ exists or, equivalently, if $\alpha_0 \neq 0$, then A^{-1} can be expressed as

$$A^{-1} = -\frac{1}{\alpha_0}\left(A^{n-1} + \alpha_{n-1}A^{n-2} + \cdots + \alpha_2 A + \alpha_1 I_n\right). \tag{3}$$

Finally, for a pair (A,C), $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{l \times n}$, we define the observability matrix by

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{nl \times n}. \tag{4}$$

Recall that (A,C) is an observable pair if and only if $\operatorname{rank} \mathcal{O} = n$.

We begin by considering the nonlinear discrete-time dynamical system given by

$$x(k+1) = f(x(k)), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{5}$$

with output

$$y(k) = g(x(k)), \tag{6}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^l$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ are continuous. For clarity, we write the solution of Eq. (5) as $x(k, x_0)$ with the output given by $y(k) = y(k, x_0) = g(x(k, x_0))$.

Definition 2.1. The system (5) and (6) is *output reversible* if, for every $x_0 \in \mathbb{R}^n$ and $N \geq n-1$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$y(k, \hat{x}_0) = y(N-k, x_0), \quad k = 0, \ldots, N. \tag{7}$$

In the next section, we consider the special case of linear discrete-time systems for which we determine the necessary and sufficient conditions for output reversibility.

3. Linear output reversible systems

In this section, we consider the linear discrete-time dynamical systems given by

$$x(k+1) = Ax(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{8}$$

$$y(k) = Cx(k), \tag{9}$$

where $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{l \times n}$. For the remainder of the paper, we assume that A is invertible and the pair (A,C) is observable. It follows from Definition 2.1 that Eqs. (8) and (9) are output reversible if and only if, for every $x_0 \in \mathbb{R}^n$ and $N \geq n-1$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$CA^k\hat{x}_0 = CA^{N-k}x_0, \quad k = 0, \ldots, N. \tag{10}$$

Proposition 3.1. *Let $x_0 \in \mathbb{R}^n$ and $N \geq n-1$. Assume that Eqs. (8) and (9) are output reversible and let $\hat{x}_0 \in \mathbb{R}^n$ satisfy Eq. (10). Then $\hat{x}_0$ satisfies*

$$\mathcal{O}\hat{x}_0 = \mathcal{S}(N)x_0, \tag{11}$$

and is given uniquely by

$$\hat{x}_0 = \mathcal{O}^{\dagger}\mathcal{S}(N)x_0, \tag{12}$$

where

$$\mathcal{O} \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{nl\times n}, \quad \mathcal{S}(N) \triangleq \begin{bmatrix} CA^N \\ CA^{N-1} \\ \vdots \\ CA^{N-(n-1)} \end{bmatrix} \in \mathbb{R}^{nl\times n}, \tag{13}$$

and $\mathcal{O}^{\dagger} \in \mathbb{R}^{n\times nl}$ is the Moore–Penrose generalized inverse of $\mathcal{O} \in \mathbb{R}^{nl\times n}$ [13].

Proof. Since Eqs. (8) and (9) are output reversible, it follows from the first n equalities of Eq. (10) ($k = 0, \ldots, n-1$) that Eq. (11) holds. Furthermore, since (A, C) is observable, rank $\mathcal{O} = n$, and hence, the unique solution to Eq. (11) is given by Eq. (12). □

Remark 3.1. In case when Eqs. (8) and (9) are a single output system, $C \in \mathbb{R}^{1\times n}$ and $\mathcal{O} \in \mathbb{R}^{n\times n}$. Hence, since (A, C) is observable, $\mathcal{O}^{\dagger} = \mathcal{O}^{-1}$.

Next, we present a series of necessary and sufficient conditions for output reversibility of linear discrete-time dynamical systems.

Proposition 3.2. *The linear discrete-time dynamical systems (8) and (9) are output reversible if and only if, for any $N \geq n-1$,*

$$CA^k\mathcal{O}^{\dagger}\hat{\mathcal{O}} = CA^{-k}, \quad k = 0, \ldots, N, \tag{14}$$

where

$$\hat{\mathcal{O}} \triangleq \begin{bmatrix} C \\ CA^{-1} \\ \vdots \\ CA^{-(n-1)} \end{bmatrix} \in \mathbb{R}^{nl\times n}. \tag{15}$$

Proof. To show necessity, assume that Eqs. (8) and (9) are output reversible. Let $x_0 \in \mathbb{R}^n$ and $N \geq n-1$. Then it follows from Eqs. (10) and (12) that

$$CA^k\mathcal{O}^{\dagger}\mathcal{S}(N)x_0 = CA^{N-k}x_0, \quad k = 0, \ldots, N, \tag{16}$$

where $\mathcal{S}(N) \in \mathbb{R}^{nl\times n}$ is defined in Eq. (13). Alternatively, Eq. (16) can be rewritten as

$$CA^k\mathcal{O}^{\dagger}\hat{\mathcal{O}}A^Nx_0 = CA^{-k}A^Nx_0, \quad k = 0, \ldots, N. \tag{17}$$

Since A is invertible and x_0 is arbitrary, Eq. (17) implies Eq. (14).

To show sufficiency, assume that Eq. (14) holds. Let $x_0 \in \mathbb{R}^n$, choose $N \geq n-1$, and multiply both sides of Eq. (14) by $A^N x_0$ to obtain

$$CA^k \mathcal{O}^\dagger \hat{\mathcal{O}} A^N x_0 = CA^{N-k} x_0, \quad k=0,\ldots,N. \tag{18}$$

Let $\hat{x}_0 = \mathcal{O}^\dagger \hat{\mathcal{O}} A^N x_0$ and rewrite Eq. (18) as

$$CA^k \hat{x}_0 = CA^{N-k} x_0, \quad k=0,\ldots,N, \tag{19}$$

which implies output reversibility of Eqs. (8) and (9). □

Corollary 3.1. *If $C = I_n$, then the linear discrete-time dynamical systems (8) and (9) are output reversible if and only if $A^2 = I_n$.*

Proof. If $C = I_n$, then it follows from Proposition 3.2 that Eqs. (8) and (9) are output reversible if and only if, for any $N \geq n-1$,

$$A^k \mathcal{O}^\dagger \hat{\mathcal{O}} = A^{-k}, \quad k=0,\ldots,N, \tag{20}$$

where

$$\mathcal{O} = \begin{bmatrix} I_n \\ A \\ \vdots \\ A^{(n-1)} \end{bmatrix} \in \mathbb{R}^{n^2 \times n}, \quad \hat{\mathcal{O}} = \begin{bmatrix} I_n \\ A^{-1} \\ \vdots \\ A^{-(n-1)} \end{bmatrix} \in \mathbb{R}^{n^2 \times n}. \tag{21}$$

Since $\mathcal{O} \in \mathbb{R}^{n^2 \times n}$ is a full rank, it follows from Proposition 6.1.5 of [13] that

$$\mathcal{O}^\dagger = (\mathcal{O}^{\mathrm{T}} \mathcal{O})^{-1} \mathcal{O}^{\mathrm{T}} = (I_n + A^{\mathrm{T}} A + \cdots + (A^{(n-1)})^{\mathrm{T}} A^{(n-1)})^{-1} [I_n \;\; A^{\mathrm{T}} \;\; \cdots \;\; (A^{(n-1)})^{\mathrm{T}}], \tag{22}$$

which implies that

$$\mathcal{O}^\dagger \hat{\mathcal{O}} = (I_n + A^{\mathrm{T}} A + \cdots + (A^{(n-1)})^{\mathrm{T}} A^{(n-1)})^{-1} \cdot (I_n + A^{\mathrm{T}} A^{-1} + \cdots + (A^{(n-1)})^{\mathrm{T}} A^{-(n-1)}). \tag{23}$$

To show sufficiency, assume $A^2 = I_n$. Then it follows from Eq. (23) that $\mathcal{O}^\dagger \hat{\mathcal{O}} = I_n$ and Eq. (20) is satisfied which implies output reversibility of Eqs. (8) and (9) with $C = I_n$. Alternatively, to show necessity, assume that Eqs. (8) and (9) are output reversible with $C = I_n$, which implies that Eq. (20) holds. Thus, for $k=0$, it follows from Eq. (20) that $\mathcal{O}^\dagger \hat{\mathcal{O}} = I_n$ and, for $k=1,\ldots,N$, Eq. (20) implies that $A^2 = I_n$. Note that $A^2 = I_n$ verifies $\mathcal{O}^\dagger \hat{\mathcal{O}} = I_n$ as can be seen from Eq. (23) which proves the result. □

Remark 3.2. Note that if $A^2 = I_n$, then all system trajectories of Eq. (8) are characterized by periodic orbits containing two points. In this case, the system is clearly full state reversible which is also confirmed by Corollary 3.1.

The next two theorems present the main results of the paper.

Theorem 3.1. *Consider the linear discrete-time dynamical systems (8) and (9) and assume that there exists at least one row $c^* \in \mathbb{R}^{1 \times n}$ of $C \in \mathbb{R}^{l \times n}$ such that (A, c^*) is observable. If Eqs. (8) and (9) are output reversible, then*

$$\alpha_0 = \frac{1}{\alpha_0}, \quad \frac{\alpha_{n-i}}{\alpha_0} - \alpha_i = 0, \quad i=1,\ldots,n-1, \tag{24}$$

where α_i, $i=0,\ldots,n-1$, are the coefficients of the characteristic polynomial of $A\in\mathbb{R}^{n\times n}$ given in Eq. (1).

Proof. Let $x_0\in\mathbb{R}^n$. Since Eqs. (8) and (9) are output reversible, it follows from Eq. (10) that, for $N=n-1$, there exists $\hat{x}_0\in\mathbb{R}^n$ such that

$$CA^k\hat{x}_0=CA^{n-1-k}x_0,\quad k=0,\ldots,n-1. \tag{25}$$

Next, for $N=n$, it follows from Eq. (10) that there exists $\tilde{x}_0\in\mathbb{R}^n$ such that

$$CA^k\tilde{x}_0=CA^{n-k}x_0,\quad k=0,\ldots,n. \tag{26}$$

Note that it follows from Eq. (25) that the last n equalities of Eq. (26) ($k=1,\ldots,n$) can be equivalently written as

$$\mathcal{O}A\tilde{x}_0=\mathcal{O}\hat{x}_0, \tag{27}$$

which, since rank $\mathcal{O}=n$, implies $\tilde{x}_0=A^{-1}\hat{x}_0$. Next, substitute $\tilde{x}_0=A^{-1}\hat{x}_0$ into the first equality of Eq. (26) ($k=0$) to obtain

$$CA^{-1}\hat{x}_0=CA^nx_0. \tag{28}$$

Using Eqs. (2) and (3), Eq. (28) can be rewritten as

$$\begin{aligned}&-\frac{1}{\alpha_0}C\left(A^{n-1}+\alpha_{n-1}A^{n-2}+\cdots+\alpha_2A+\alpha_1I_n\right)\hat{x}_0\\&\qquad=-C(\alpha_{n-1}A^{n-1}+\alpha_{n-2}A^{n-2}+\cdots+\alpha_1A+\alpha_0I_n)x_0.\end{aligned} \tag{29}$$

Furthermore, it follows from Eq. (25) that Eq. (29) can be rewritten as

$$\begin{aligned}C\Bigg[&\left(\frac{1}{\alpha_0}-\alpha_0\right)I_n+\left(\frac{\alpha_{n-1}}{\alpha_0}-\alpha_1\right)A\\&+\cdots+\left(\frac{\alpha_2}{\alpha_0}-\alpha_{n-2}\right)A^{n-2}+\left(\frac{\alpha_1}{\alpha_0}-\alpha_{n-1}\right)A^{n-1}\Bigg]x_0=0.\end{aligned} \tag{30}$$

Since $x_0\in\mathbb{R}^n$ is arbitrary, Eq. (30) implies

$$\begin{aligned}&\left(\frac{1}{\alpha_0}-\alpha_0\right)C+\left(\frac{\alpha_{n-1}}{\alpha_0}-\alpha_1\right)CA\\&\quad+\cdots+\left(\frac{\alpha_2}{\alpha_0}-\alpha_{n-2}\right)CA^{n-2}+\left(\frac{\alpha_1}{\alpha_0}-\alpha_{n-1}\right)CA^{n-1}=0.\end{aligned} \tag{31}$$

Since there exists at least one row $c^*\in\mathbb{R}^{1\times n}$ of $C\in\mathbb{R}^{l\times n}$ such that (A,c^*) is observable, it follows from Eq. (31) that Eq. (24) holds which proves the result. □

The following lemma is needed for the next theorem.

Lemma 3.1. *Consider the linear discrete-time dynamical systems* (8) *and* (9). *Let $N^*\in\overline{\mathbb{Z}}_+$ and assume that there exists n linearly independent vectors $x_1,\ldots,x_n\in\mathbb{R}^n$ such that*

$$\operatorname{rank}[\mathcal{O}\ \ \mathcal{S}(N^*)x_i]=n,\quad i=1,\ldots,n. \tag{32}$$

Then, for every $x_0\in\mathbb{R}^n$ and $N\geq n-1$,

$$\operatorname{rank}[\mathcal{O}\ \ \mathcal{S}(N)x_0]=n. \tag{33}$$

Proof. Since (A,C) is observable and rank $\mathcal{O}=n$, it follows from Eq. (32) that each vector $\mathcal{S}(N^*)x_i$, $i=1,\ldots,n$, is a linear combination of the columns of $\mathcal{O}$. Next, let $x_0\in\mathbb{R}^n$, $N\geq n-1$, and note that

$$\mathcal{S}(N) \triangleq \begin{bmatrix} CA^{N} \\ CA^{N-1} \\ \vdots \\ CA^{N-(n-1)} \end{bmatrix} = \begin{bmatrix} CA^{N^*} \\ CA^{N^*-1} \\ \vdots \\ CA^{N^*-(n-1)} \end{bmatrix} A^{N-N^*} = \mathcal{S}(N^*)A^{N-N^*}. \tag{34}$$

Furthermore, since x_i, $i=1,\ldots,n$, are linearly independent, they form a basis in $\mathbb{R}^n$ which implies that there exist $\beta_1,\ldots,\beta_n \in \mathbb{R}$ such that $A^{N-N^*}x_0 = \sum_{i=1}^n \beta_i x_i$. Next, it follows from Eq. (34) that

$$\mathcal{S}(N)x_0 = \mathcal{S}(N^*)A^{N-N^*}x_0 = \sum_{i=1}^{n} \beta_i \mathcal{S}(N^*)x_i, \tag{35}$$

which, since each vector $\mathcal{S}(N^*)x_i$, $i=1,\ldots,n$, is a linear combination of the columns of $\mathcal{O}$, implies that $\mathcal{S}(N)x_0$ is also a linear combination of the columns of $\mathcal{O}$. Hence, rank $[\mathcal{O}\ \mathcal{S}(N)x_0]=n$ which proves the result. □

Theorem 3.2. *Consider the linear discrete-time dynamical systems (8) and (9) and assume that there exist $N^* \in \overline{\mathbb{Z}}_+$ and n linearly independent vectors $x_1,\ldots,x_n \in \mathbb{R}^n$ such that*

$$\mathrm{rank}[\mathcal{O}\ \ \mathcal{S}(N^*)x_i] = n, \quad i=1,\ldots,n. \tag{36}$$

If Eq. (24) holds, then Eqs. (8) and (9) are output reversible.

Proof. Let $x_0 \in \mathbb{R}^n$. We need to show that, for any $N \ge n-1$, there exists $\tilde{x}_0 \in \mathbb{R}^n$ such that

$$CA^k\tilde{x}_0 = CA^{N-k}x_0, \quad k=0,\ldots,N. \tag{37}$$

Let $N=n-1$ and let

$$\hat{x}_0 = \mathcal{O}^{\dagger} \begin{bmatrix} CA^{n-1} \\ CA^{n-2} \\ \vdots \\ CA \\ C \end{bmatrix} x_0 = \mathcal{O}^{\dagger}\mathcal{S}(n-1)x_0. \tag{38}$$

It follows from Lemma 3.1 that rank $[\mathcal{O}\ \ \mathcal{S}(n-1)x_0]=n$ which implies that $\tilde{x}_0=\hat{x}_0$ satisfies Eq. (37) for $N=n-1$.

Next, let $N>n-1$. First, we verify Eq. (37) for $k=N-(n-1),\ldots,N$; that is, we need to show that there exists $\tilde{x}_0 \in \mathbb{R}^n$ such that

$$\begin{aligned} &CA^{N-(n-1)}\tilde{x}_0 = CA^{(n-1)}x_0, \\ &CA^{N-(n-2)}\tilde{x}_0 = CA^{(n-2)}x_0, \\ &\vdots \\ &CA^{N-1}\tilde{x}_0 = CAx_0, \\ &CA^{N}\tilde{x}_0 = Cx_0, \end{aligned}$$

or, equivalently,

$$\mathcal{O}A^{N-(n-1)}\tilde{x}_0 = \begin{bmatrix} CA^{n-1} \\ CA^{n-2} \\ \vdots \\ CA \\ C \end{bmatrix} x_0. \tag{39}$$

It follows from Eq. (38) and Lemma 3.1 that $A^{N-(n-1)}\tilde{x}_0 = \hat{x}_0$ which implies that

$$\tilde{x}_0 = A^{(n-1)-N}\hat{x}_0 \tag{40}$$

satisfies Eq. (37) for $k = N-(n-1), \ldots, N$.

Now, it remains to verify Eq. (37) for $k = 0, \ldots, N-n$. We do this by implementing mathematical induction [14]. First, we show that Eq. (37) is satisfied for $k = N-n$. To see this, note that, using Eqs. (2), (3), (24), (38), and (40), we obtain

$$\begin{aligned} CA^{N-n}\tilde{x}_0 &= CA^{-1}\hat{x}_0 \\ &= -\frac{1}{\alpha_0}C\big(A^{n-1} + \alpha_{n-1}A^{n-2} + \cdots + \alpha_2 A + \alpha_1 I_n\big)\hat{x}_0 \\ &= -\frac{1}{\alpha_0}C\big(I_n + \alpha_{n-1}A + \cdots + \alpha_2 A^{n-2} + \alpha_1 A^{n-1}\big)x_0 \\ &= -C(\alpha_0 I_n + \alpha_1 A + \cdots + \alpha_{n-2}A^{n-2} + \alpha_{n-1}A^{n-1})x_0 \\ &= CA^n x_0, \end{aligned} \tag{41}$$

which verifies Eq. (37) for $k = N-n$. Next, assume that Eq. (37) is satisfied for all $k = p, \ldots, N-n$, $p \in (0, N-n)$. Then, in accordance with the mathematical induction, we need to prove that Eq. (37) is satisfied for $k = p-1$. To see this, note that since Eq. (37) holds for $k = p, \ldots, N$, it follows from Eq. (24) that

$$\begin{aligned} CA^{p-1}\tilde{x}_0 &= CA^{p+n-2-N}\hat{x}_0 \\ &= CA^{-1}A^{p-N+n-1}\hat{x}_0 \\ &= -\frac{1}{\alpha_0}C\big(A^{n-1} + \alpha_{n-1}A^{n-2} + \cdots + \alpha_2 A + \alpha_1 I_n\big)A^{p-N+n-1}\hat{x}_0 \\ &= -\frac{1}{\alpha_0}\big(CA^{p-N+2(n-1)}\hat{x}_0 + \alpha_{n-1}CA^{p-N+2(n-1)-1}\hat{x}_0 \\ &\quad + \cdots + \alpha_2 CA^{p-N+n}\hat{x}_0 + \alpha_1 CA^{p-N+n-1}\hat{x}_0\big) \\ &= -\frac{1}{\alpha_0}\big(CA^{N-p-(n-1)}x_0 + \alpha_{n-1}CA^{N-p-(n-2)}x_0 \\ &\quad + \cdots + \alpha_2 CA^{N-p-1}x_0 + \alpha_1 CA^{N-p}x_0\big) \\ &= -C\left(\frac{1}{\alpha_0}I_n + \frac{\alpha_{n-1}}{\alpha_0}A + \cdots + \frac{\alpha_2}{\alpha_0}A^{n-2} + \frac{\alpha_1}{\alpha_0}A^{n-1}\right)A^{N-p-(n-1)}x_0 \\ &= -C(\alpha_0 I_n + \alpha_1 A + \cdots + \alpha_{n-2}A^{n-2} + \alpha_{n-1}A^{n-1})A^{N-p-(n-1)}x_0 \\ &= CA^{N-(p-1)}x_0, \end{aligned} \tag{42}$$

which verifies Eq. (37) for $k = p-1$. Thus, by mathematical induction, Eq. (37) is satisfied for all $k=0,\ldots,N$ which proves output reversibility of Eqs. (8) and (9). □

Remark 3.3. Note that in case of single output systems, that is when $C \in \mathbb{R}^{1 \times n}$, $\mathcal{O} \in \mathbb{R}^{n \times n}$ is invertible and, hence, condition (36) is satisfied immediately and need *not* be verified.

The next corollary presents a specialization of Theorems 3.1 and 3.2 to the case of single output systems.

Corollary 3.2. *Consider the linear discrete-time dynamical systems* (8) *and* (9) *with* $C \in \mathbb{R}^{1 \times n}$. *Then Eqs.* (8) and (9) *are output reversible if and only if Eq.* (24) *holds.*

Proof. The proof is a direct consequence of Theorems 3.1 and 3.2 by noting that if $C \in \mathbb{R}^{1 \times n}$, then $c^* = C$ and $\mathcal{O} \in \mathbb{R}^{n \times n}$ is invertible. □

Proposition 3.3. *Consider the linear discrete-time dynamical systems* (8) *and* (9). *Assume that there exists at least one row* $c^* \in \mathbb{R}^{1 \times n}$ *of* $C \in \mathbb{R}^{l \times n}$ *such that* (A, c^*) *is observable. If Eqs.* (8) *and* (9) *are output reversible, then the multispectrum of* $A \in \mathbb{R}^{n \times n}$ *is geometrically symmetric with respect to the unit circle in the complex plane, that is,* $\{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}} = \{1/\lambda_1, \ldots, 1/\lambda_n\}_{\mathrm{m}}$.

Proof. Let $\lambda \in \operatorname{mspec}(A)$. In this case,

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda + \alpha_0 = 0, \tag{43}$$

where α_i, $i = 0, \ldots, n-1$, are the coefficients of the characteristic polynomial of A. Since Eqs. (8) and (9) are output reversible, it follows from Theorem 3.1 that Eq. (24) holds. This implies that

$$\begin{aligned} &\frac{1}{\lambda^n} + \alpha_{n-1}\frac{1}{\lambda^{n-1}} + \cdots + \alpha_1 \frac{1}{\lambda} + \alpha_0 \\ &\quad = \frac{\alpha_0}{\lambda^n}\left(\frac{1}{\alpha_0} + \frac{\alpha_{n-1}}{\alpha_0}\lambda + \cdots + \frac{\alpha_1}{\alpha_0}\lambda^{n-1} + \lambda^n\right) \\ &\quad = \frac{\alpha_0}{\lambda^n}\left(\alpha_0 + \alpha_1\lambda + \cdots + \alpha_{n-1}\lambda^{n-1} + \lambda^n\right) \\ &\quad = 0, \end{aligned} \tag{44}$$

which proves the result. □

The next result presents constructive sufficient conditions for output reversibility of linear discrete-time dynamical systems.

Proposition 3.4. *Consider the linear discrete-time dynamical systems* (8) *and* (9). *Assume that there exist* $N^* \in \overline{\mathbb{Z}}_+$ *and* n *linearly independent vectors* $x_1, \ldots, x_n \in \mathbb{R}^n$ *such that*

$$\operatorname{rank}[\mathcal{O} \quad \mathcal{S}(N^*)x_i] = n, \quad i = 1, \ldots, n. \tag{45}$$

If all eigenvalues of $A \in \mathbb{R}^{n \times n}$ *are simple and the multispectrum of* $A \in \mathbb{R}^{n \times n}$ *is geometrically symmetric with respect to the unit circle, then Eqs.* (8) *and* (9) *are output reversible.*

Proof. Let $\lambda_i \in \operatorname{mspec}(A)$, $i = 1,\ldots,n$. Since $\{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}} = \{1/\lambda_1, \ldots, 1/\lambda_n\}_{\mathrm{m}}$, it follows that, for all $i = 1,\ldots,n$,

$$\lambda_i^n + \alpha_{n-1}\lambda_i^{n-1} + \cdots + \alpha_1 \lambda_i + \alpha_0 = 0, \tag{46}$$

and

$$\frac{1}{\lambda_i^n} + \alpha_{n-1}\frac{1}{\lambda_i^{n-1}} + \cdots + \alpha_1 \frac{1}{\lambda_i} + \alpha_0 = 0. \tag{47}$$

Note that Eq. (47) can be equivalently rewritten as

$$\frac{\alpha_0}{\lambda_i^n}\left(\frac{1}{\alpha_0}+\frac{\alpha_{n-1}}{\alpha_0}\lambda_i+\cdots+\frac{\alpha_1}{\alpha_0}\lambda_i^{n-1}+\lambda_i^n\right)=0, \quad i=1,\ldots,n, \tag{48}$$

which, since A is invertible and, hence, $\alpha_0 \neq 0$, implies

$$\lambda_i^n+\frac{\alpha_1}{\alpha_0}\lambda_i^{n-1}+\cdots+\frac{\alpha_{n-1}}{\alpha_0}\lambda_i+\frac{1}{\alpha_0}=0, \quad i=1,\ldots,n. \tag{49}$$

Next, subtract Eq. (49) from Eq. (46) to obtain

$$\left(\alpha_{n-1}-\frac{\alpha_1}{\alpha_0}\right)\lambda_i^{n-1}+\left(\alpha_{n-2}-\frac{\alpha_2}{\alpha_0}\right)\lambda_i^{n-2}+\cdots+\left(\alpha_1-\frac{\alpha_{n-1}}{\alpha_0}\right)\lambda_i+\left(\alpha_0-\frac{1}{\alpha_0}\right)=0,$$
$$i=1,\ldots,n. \tag{50}$$

Note that Eq. (50) can be written in a matrix form, for all $i=1,\ldots,n$, as

$$\Lambda a=0, \tag{51}$$

where

$$\Lambda \triangleq \begin{bmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \cdots & \lambda_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \cdots & \lambda_n & 1 \end{bmatrix}, \quad a \triangleq \begin{bmatrix} \alpha_{n-1}-\dfrac{\alpha_1}{\alpha_0} \\ \alpha_{n-2}-\dfrac{\alpha_2}{\alpha_0} \\ \vdots \\ \alpha_1-\dfrac{\alpha_{n-1}}{\alpha_0} \\ \alpha_0-\dfrac{1}{\alpha_0} \end{bmatrix}. \tag{52}$$

Note that $\Lambda \in \mathbb{R}^{n\times n}$ is a Vandermonde matrix [13] whose determinant is given by

$$\det \Lambda = \prod_{1\le i<j\le n}(\lambda_i-\lambda_j). \tag{53}$$

Since all $\lambda_i \in \text{mspec}(A)$, $i=1,\ldots,n$, are simple, $\det\Lambda \neq 0$. Thus, it follows from Eq. (51) that $a=0$, which is equivalent to Eq. (24). Now, the result is a direct consequence of Theorem 3.2. □

4. Connections between output reversibility of continuous-time and discrete-time linear dynamical systems

The notions of output reversibility of continuous-time and discrete-time dynamical systems are related to each other and, in this section, we draw the connections between these two concepts. Output reversibility of continuous-time dynamical systems was first introduced in [9], with constructive necessary and sufficient conditions for linear systems developed in [12]. For completeness, we briefly present here the main results needed.

Consider the nonlinear dynamical system given by

$$\dot{x}(t)=f(x(t)), \quad x(0)=x_0, \quad t\ge 0, \tag{54}$$

with output

$$y(t)=g(x(t)), \tag{55}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^l$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^l$ are continuous. We assume that solutions of Eq. (54) exist and are unique on all finite intervals $[0, T)$. For clarity, we write the solution of Eq. (54) as $x(t, x_0)$ with the output given by $y(t) = y(t, x_0) = g(x(t, x_0))$.

Definition 4.1 (*Bernstein and Bhat [9]*). The systems (54) and (55) are *output reversible* if, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$y(t, \hat{x}_0) = y(t_1 - t, x_0), \quad t \in [0, t_1]. \tag{56}$$

Next, consider the linear dynamical system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{57}$$

with output

$$y(t) = Cx(t), \tag{58}$$

where $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{l \times n}$. Assume that (A, C) is observable. It follows from Definition 4.1 that Eqs. (57) and (58) are output reversible if and only if, for all $x_0 \in \mathbb{R}^n$ and $t_1 > 0$, there exists $\hat{x}_0 \in \mathbb{R}^n$ such that

$$Ce^{At}\hat{x}_0 = Ce^{A(t_1 - t)}x_0, \quad t \in [0, t_1]. \tag{59}$$

Theorem 4.1 (*Nersesov et al. [12]*). *If the linear dynamical system (57) and (58) are output reversible, then the multispectrum of A is symmetric with respect to the imaginary axis, that is,* $\{\lambda_1, \ldots, \lambda_n\}_{\rm m} = \{-\lambda_1, \ldots, -\lambda_n\}_{\rm m}$.

Next, consider a discretization of the continuous-time dynamical system (57) and (58) given by

$$x(k+1) = A_{\rm d}x(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{60}$$

$$y(k) = Cx(k), \tag{61}$$

where $A_{\rm d} \triangleq e^{At_{\rm s}} \in \mathbb{R}^{n \times n}$ and $t_{\rm s} > 0$ is a constant sample time. Note that if $\operatorname{mspec}(A) = \{\lambda_1, \ldots, \lambda_n\}$, then $\operatorname{mspec}(A_{\rm d}) = \{e^{\lambda_1 t_{\rm s}}, \ldots, e^{\lambda_n t_{\rm s}}\}$. The following theorem is needed for the main result of this section.

Theorem 4.2 (*Kalman et al. [15]*). *If the linear continuous-time dynamical system (57) and (58) are observable and the sampling time $t_{\rm s} > 0$ is such that, for all λ_i, $\lambda_s \in \operatorname{mspec}(A)$,*

$$t_{\rm s}(\lambda_i - \lambda_s) \neq (2\pi j)l, \quad l \in \mathbb{Z}\backslash\{0\}, \tag{62}$$

then the discretization given by Eqs. (60) and (61) is also observable.

Proposition 4.1. *Consider the continuous-time dynamical system (57) and (58). Assume that all eigenvalues of $A \in \mathbb{R}^{n \times n}$ are simple, Eqs. (57) and (58) are output reversible in a sense of Definition 4.1, and Eq. (62) holds. Assume, in addition, that there exist $N^* \in \overline{\mathbb{Z}}_+$ and n linearly independent vectors $x_1, \ldots, x_n \in \mathbb{R}^n$ such that*

$$\operatorname{rank}[\mathcal{O}\ \mathcal{S}(N^*)x_i] = n, \quad i = 1, \ldots, n, \tag{63}$$

where $\mathcal{O} \in \mathbb{R}^{nl \times n}$ and $\mathcal{S}(\cdot) \in \mathbb{R}^{nl \times n}$ are given by Eq. (13) with $A \in \mathbb{R}^{n \times n}$ replaced by $A_{\rm d} = e^{At_{\rm s}} \in \mathbb{R}^{n \times n}$, $t_{\rm s} > 0$. Then the discretization of Eqs. (57) and (58) given by Eqs. (60) and (61) is output reversible in a sense of Definition 2.1.

Proof. Note that $A_{\mathrm{d}} = e^{At_{\mathrm{s}}} \in \mathbb{R}^{n\times n}$ is invertible. Furthermore, if (A,C) is observable and Eq. (62) holds, it follows from Theorem 4.2 that (A_{d}, C) is also observable. Moreover, since Eqs. (57) and (58) are output reversible, it follows from Theorem 4.1 that $\{\lambda_1, \ldots, \lambda_n\}_{\mathrm{m}} = \{-\lambda_1, \ldots, -\lambda_n\}_{\mathrm{m}}$ which further implies that $\{e^{\lambda_1 t_{\mathrm{s}}}, \ldots, e^{\lambda_n t_{\mathrm{s}}}\}_{\mathrm{m}} = \{e^{-\lambda_1 t_{\mathrm{s}}}, \ldots, e^{-\lambda_n t_{\mathrm{s}}}\}_{\mathrm{m}}$. Hence, the multispectrum of A_{d} is geometrically symmetric with respect to the unit circle. Finally, since λ_i, $i=1,\ldots,n$, are simple eigenvalues of A, then $e^{\lambda_i t_{\mathrm{s}}}$, $i=1,\ldots,n$, are also simple eigenvalues of A_{d}. Now, the result is a direct consequence of Proposition 3.4. □

5. Numerical examples

In this section, we consider an example of a continuous-time system that is output reversible and demonstrate the output reversibility of its discretization. Specifically, consider a mechanical system involving two coupled oscillators with masses m_1 and m_2, and spring stiffness coefficients k_1 and k_2 so that the system inertia and stiffness matrices are given by

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}. \tag{64}$$

The state space representation of this system is given by Eqs. (57) and (58) with $A \in \mathbb{R}^{4\times 4}$ given by

$$A \triangleq \begin{bmatrix} 0 & I_2 \\ -M^{-1}K & 0 \end{bmatrix}, \tag{65}$$

where $\mathrm{mspec}(A) = \{\pm j\omega_1, \ \pm j\omega_2\}$, $\omega_1 > 0$, and $\omega_2 > 0$. For the first example, let

$$C = [1 \ \ 0 \ \ 0 \ \ 0], \tag{66}$$

so that the output $y(t)$, $t \geq 0$, in Eq. (58) represents the position of the first mass. If (A,C) is observable, then it follows from Corollary 3.2 of [12] that Eqs. (57) and (58) are output reversible. If, in addition, $\omega_1 \neq \omega_2$ and Eq. (62) holds, then it follows from Proposition 4.1 that the discretization of Eqs. (57) and (58) given by Eqs. (60) and (61) is also output reversible.

For the following simulation, let $m_1 = 3$ kg, $m_2 = 5$ kg, $k_1 = 10$ N/m, and $k_2 = 7$ N/m. In this case, (A,C) is observable and $\mathrm{mspec}(A) = \{\pm j2.515, \ \pm j0.858\}$. Furthermore, let $x_0 = [4, -7, 0, 0]^{\mathrm{T}}$,

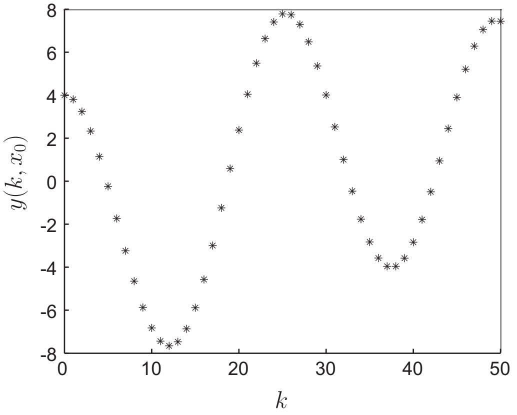

Fig. 1. Original output $y(k, x_0)$.

$t_s = 0.1$ s, and $N=50$. Hence, for the discretized system (60) and (61), it follows from Eq. (12) that $\hat{x}_0 = [7.4419, 0.2725, 2.1258, 3.9957]^T$ generates the output $y(k, \hat{x}_0)$ that traces the original output $y(k, x_0)$ backwards while the index k is running forward. Fig. 1 shows the original output $y(k, x_0)$, while Fig. 2 shows its time-reversed image $y(k, \hat{x}_0)$.

For the next example, let m_1, m_2, k_1, and k_2 be as above and consider reversibility of both mass positions, that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{67}$$

It can be shown using Theorem 3.1 of [12] that the system (57) and (58) with A and C given by Eqs. (65) and (67), respectively, is output reversible. Moreover, with $N^* = 1$ and x_i, $i = 1, \ldots, 4$, being the ith column of I_4, condition (63) is satisfied. Thus, it follows from Proposition 4.1 that the discretization (60) and (61) is output reversible. For the numerical simulation, we use $x_0 = [4, -7, 0, 0]^T$, $t_s = 0.1$ s, and $N=35$. It follows from Eq. (12) that

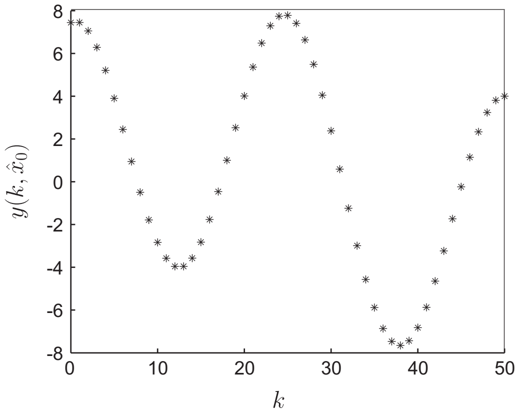

Fig. 2. Time-reversed output $y(k, \hat{x}_0)$.

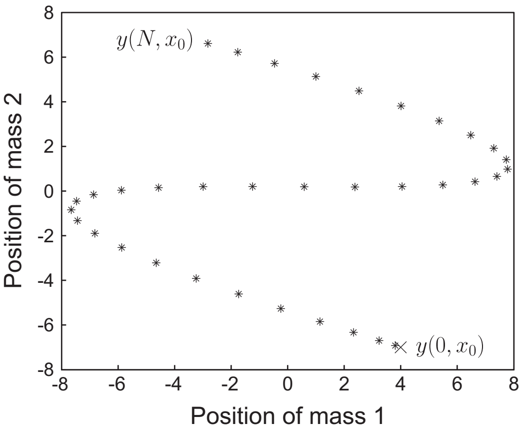

Fig. 3. Original output $y(k, x_0)$.

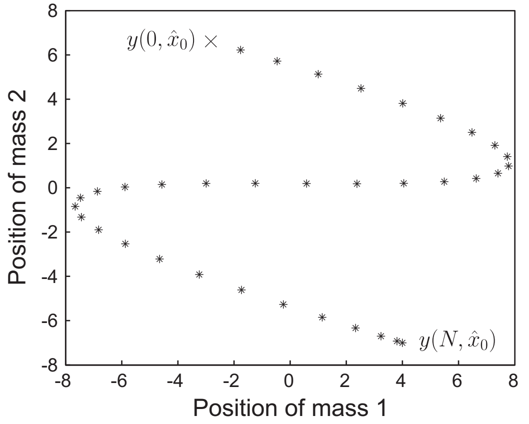

Fig. 4. Time-reversed output $y(k,\hat{x}_0)$.

$\hat{x}_0 = [-2.825, 6.6119, 9.1305, -3.2772]^{\mathrm{T}}$. Fig. 3 shows the original output $y(k,x_0)$, while Fig. 4 shows its time-reversed image $y(k,\hat{x}_0)$.

6. Application to control

In this section, we discuss how a discrete-time system can be altered by means of control in order to make the closed-loop system output reversible. We will show that the reversibility of the *entire* plant state is possible by properly designing a dynamic compensator for the plant. From the practical point of view, the analysis below may be applicable to systems that are required to perform repetitive tasks by tracing a specific trajectory between the two states of the system back and forth. In this case, making the system output reversible has a potential to eliminate hysteresis.

To elucidate our approach, consider a controlled linear discrete-time dynamical system given by

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0, \quad k \in \overline{\mathbb{Z}}_+, \tag{68}$$

$$y(k) = Cx(k), \tag{69}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$. At this point, we do *not* assume that A is invertible neither that (A,C) is observable. Consider the following feedback control law:

$$u(k) = Ky(k) + Gv(k), \tag{70}$$

where $K \in \mathbb{R}^{m\times l}$, $G \in \mathbb{R}^{m\times n}$, and $v \in \mathbb{R}^n$ is the secondary control input. We assume that $A + BKC$ is invertible and consider the following dynamics for the secondary control input v:

$$v(k+1) = (A + BKC)^{-1}v(k), \quad v(0) = v_0, \quad k \in \overline{\mathbb{Z}}_+. \tag{71}$$

Thus, the closed-loop dynamics are given by

$$x(k+1) = (A + BKC)x(k) + BGv(k), \tag{72}$$

$$v(k+1) = (A + BKC)^{-1}v(k), \tag{73}$$

or, equivalently,

$$\tilde{x}(k+1) = \tilde{A}\tilde{x}(k), \tag{74}$$

$$y(k) = \tilde{C}\tilde{x}(k), \tag{75}$$

where $\tilde{x} \triangleq [x^{\mathrm{T}}, v^{\mathrm{T}}]^{\mathrm{T}}$,

$$\tilde{A} = \begin{bmatrix} A+BKC & BG \\ 0_{n\times n} & (A+BKC)^{-1} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \quad \tilde{C} = [C \ \ , 0_{l\times n}] \in \mathbb{R}^{l\times 2n}. \tag{76}$$

Note that the multispectrum of $\tilde{A}$ is geometrically symmetric with respect to the unit circle. Thus, it follows from Proposition 3.4 that if the eigenvalues of $\tilde{A}$ are simple, $(\tilde{A}, \tilde{C})$ is observable, and condition (45) is satisfied, where $\mathcal{O}$ and $\mathcal{S}(N)$ are given in Eq. (13) with A and C replaced by $\tilde{A}$ and $\tilde{C}$, respectively, then the closed-loop system (74) and (75) is output reversible.

For practical implementation of the above technique, one would have to consider the internal dynamics of sensors and actuators as well as noise and system uncertainties which have been omitted here.

Example 6.1. Consider the discrete-time dynamical system (68) and (69) with

$$A = \begin{bmatrix} -2.4 & 5.4 & -0.2 \\ -0.2 & 2.2 & -0.6 \\ -0.6 & 3.6 & -0.8 \end{bmatrix}, \tag{77}$$

$B = I_3$, and $C = I_3$. Note that mspec $(A) = \{-2, 0, 1\}$, and hence, the uncontrolled system (5) and (6) with A given by Eq. (77) cannot be shown to be output reversible since (i) A is not invertible, (ii) the dimensionality of the system is odd, and (iii) $A^2 \neq I_3$ (see Corollary 3.1). Below, we show that, with the control system (70) and (71), the closed-loop system (74) and (75) can be made output reversible. Specifically, let $G = I_3$ and choose $K \in \mathbb{R}^{3\times 3}$ to be

$$K = \begin{bmatrix} 2.4 & -4.4 & 0.2 \\ 0.2 & -2.2 & 1.6 \\ 1.6125 & -6.075 & 3.2 \end{bmatrix} \tag{78}$$

such that mspec $(A+BKC) = \{0.9, 0.75 \pm j0.75\}$. It can be easily verified that $(\tilde{A}, \tilde{C})$ is observable and condition (45) is satisfied with $N^* = 1$ and x_i, $i = 1, \ldots, 6$, being the columns of I_6. For the initial condition $\tilde{x}_0 = [-5, 5, 5, 0, 0, 0]^{\mathrm{T}}$ and N=30, it follows from Eq. (12) that $\hat{\tilde{x}}_0 = [-85.1144, -24.9123, 58.0409, -66.7695, -143.1553, -139.6900]^{\mathrm{T}}$. Fig. 5 shows the original output $y(k, \tilde{x}_0)$ which is the full state of the system (68), while Fig. 6 shows its time-reversed image $y(k, \hat{\tilde{x}}_0)$.

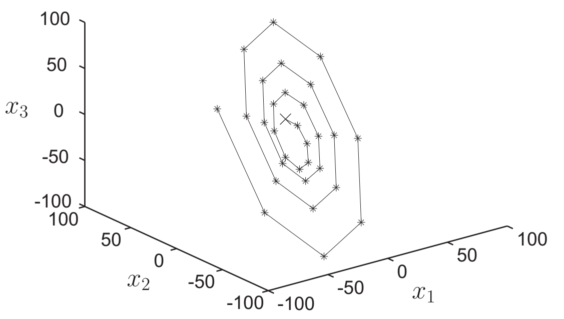

Fig. 5. Original output $y(k, \tilde{x}_0)$ with $y(0, \tilde{x}_0)$ marked by ‘ × ’.

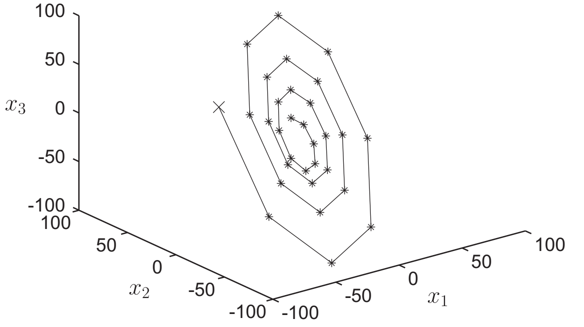

Fig. 6. Time-reversed output $y(k, \tilde{x}_0)$ with $y(0, \tilde{\tilde{x}}_0)$ marked by ' $\times$ '.

7. Conclusion

In this paper, we defined and studied the notion of output reversibility in linear discrete-time dynamical systems. Specifically, we established necessary and sufficient conditions for output reversibility in observable discrete-time systems that involve a special structure of the characteristic polynomial of the system state matrix. Furthermore, we provided constructive sufficient conditions for output reversibility in terms of geometric symmetry of the system eigenvalue set with respect to the unit circle in the complex plane. Moreover, we established that output reversibility of a continuous-time system implies output reversibility of its discretization. In addition, we showed how a discrete-time system can be rendered output reversible by means of control. Finally, we presented numerical examples.

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