Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: http://www.tandfonline.com/loi/gdea20

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To cite this article: N. H. Du, V. H. Linh & N. T. T. Nga (2016): On stability and Bohl exponent of linear singular systems of difference equations with variable coefficients, Journal of Difference Equations and Applications, DOI: 10.1080/10236198.2016.1198341

To link to this article: http://dx.doi.org/10.1080/10236198.2016.1198341

Published online: 20 Jun 2016.

JOURNAL OF DIFFERENCE EQUATIONS AND APPLICATIONS, 2016
http://dx.doi.org/10.1080/10236198.2016.1198341

On stability and Bohl exponent of linear singular systems of difference equations with variable coefficients

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ABSTRACT

In this paper we investigate the stability of linear singular systems of difference equations with variable coefficients by the projector-based approach. We study the preservation of uniform/exponential stability when the system coefficients are subject to allowable perturbations. A Bohl–Perron type theorem is obtained which provides a necessary and sufficient condition for the boundedness of solutions of nonhomogenous systems. The notion of Bohl exponent is introduced and we characterize the relation between the exponential stability and the Bohl exponent. Finally, robustness of the Bohl exponent with respect to allowable perturbations is investigated.

ARTICLE HISTORY
Received 25 October 2015
Accepted 1 June 2016

KEYWORDS
Singular difference equation; tractability index; canonical projector; Cauchy operator; exponential stability; Bohl exponent

AMS SUBJECT CLASSIFICATIONS
39A06; 39A30

1. Introduction

In this paper, we consider the linear singular difference equations (LSDEs)

$$E_n y(n+1) = A_n y(n) + q_n, \qquad n \in \mathbb{N}(n_0), \tag{1.1}$$

where $E_n, A_n \in \mathbb{K}^{d\times d}$, $q_n \in \mathbb{K}^d$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $\mathbb{N}(n_0)$ denotes the set of integers that are greater than or equal to a given integer n_0. The homogeneous system associated with (1.1) is

$$E_n x(n+1) = A_n x(n), \qquad n \in \mathbb{N}(n_0). \tag{1.2}$$

Throughout the paper, we assume that the leading term E_n is singular for all $n \geqslant n_0$ and rank $E_n \equiv r$, where $1 \leqslant r \leqslant d-1$. For the case of a nonsingular leading term, system (1.1) reduces to ordinary difference equations, whose theory is well known, see [1,2]. Singular difference equations of form (1.1) generalize ordinary difference equations and they arise in many applications, for example, in the system and control theory, population dynamics, economics, numerical analysis, etc. Singular systems of difference equations are also called implicit discrete-time systems or discrete-time descriptor systems, see [3]. They can also be considered as discrete-time analogues of differential–algebraic equations (DAEs) of the form

$$E(t)x'(t) = A(t)x(t) + q(t), \tag{1.3}$$

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which have attracted a lot of researchers' attention due to numerous applications, see [4,5]. If one discretizes the continuous-time system (1.3) by explicit finite difference schemes, then discrete-time systems of form (1.1) are obtained, see [6].

While the theory of DAEs, the continuous-time counterpart of (1.1), has been almost well established, qualitative results in the theory of singular difference equations, particularly those for non-autonomous systems, are very few. Though the first result on LSDEs with variable coefficients was given a long time ago in [3], interesting results on the existence and the stability of solutions have been published only recently, see [7–13]. Most of the stability and robust stability results for singular difference equations are obtained for autonomous systems, e.g. see [14] and the references therein. The main aim of this paper is to extend some well-known stability theorems from ordinary difference equations to singular difference equations. These results complement those in [8,9,13]. They can also be considered as the discrete-time analogues of some recent results for DAEs, see [15–19]. For the rigorous proofs of the main results, we have to overcome the difficulties that are caused simultaneously by both the singularity and the discrete-time nature of the systems. Up to our knowledge, this paper is the first work that uses the Bohl exponent, see [20], to characterize the exponential stability and its robustness for singular discrete-time systems. Furthermore, unlike the problem formulation in [13,17], here we consider a general class of allowable structured pertubations arising in both the coefficients of system (1.2).

The paper is organized as follows. In the next section, we recall some basic results and stability notions in the theory of LSDEs by using the projector approach. In Section 3, we study the preservation of the uniform stability and the exponential stability when the coefficents of system (1.2) are subject to perturbations. In Section 4, we present a Bohl-Perron type theorem that establishes the relation between the exponential stability of the homogeneous system (1.2) and the boundedness of solutions to the nonhomogeneous system (1.1). In Section 5, we give the notion of the Bohl exponent for linear singular systems (1.2) and analyze its properties, including its sensitivity to perturbations occurring in the system coefficients. Finally, we close the paper by conclusions and mentioning some ongoing works in this topic.

2. Preliminaries

In this section we briefly recall the index notion and a decoupling technique for linear singular systems of index 1 by using appropriate projectors, see the review paper [7]. This can be considered as the discrete analogue of the projector approach for DAEs, see [5].

2.1. Solutions of Cauchy problem

Denote $N_n := \ker E_n$ and let Q_n be a projection onto N_n. Put $P_n := I - Q_n$. Let $T_n \in GL(\mathbb{R}^d)$ $(n \geqslant n_0 + 1)$ be such that $T_n|_{N_n}$ is an isomorphism between N_n and N_{n-1}. Associating with Equation (1.1) we introduce matrices and subspaces

$$G_n := E_n - A_n T_n Q_n \quad (n \geqslant n_0 + 1), \quad S_n := \{z \in \mathbb{R}^d : A_n z \in \operatorname{Im} E_n\} \quad (n \geqslant n_0).$$

We have the following lemma (see [7, Lemma 2.3]).

Lemma 2.1: *For $n \geqslant n_0 + 1$, the following conditions are equivalent:*

(i) *The matrix $G_n := E_n - A_n T_n Q_n$ is nonsingular;*
(ii) $N_{n-1} \oplus S_n = \mathbb{R}^d$;
(iii) $N_{n-1} \cap S_n = \{0\}$.

Therefore, by virtue of Lemma 2.1 we can define the so-called LSDEs of tractability index-1 (see [7, Definition 2.2]).

Definition 2.2: The LSDE (1.1) is said to be of tractability index-1 (index-1 for short) if the following two conditions simultaneously hold:

(i) $\operatorname{rank} E_n = r = \text{const}$ for all $n \geqslant n_0$;
(ii) $N_{n-1} \cap S_n = \{0\}$ for all $n \geqslant n_0 + 1$.

In what follows we always assume that $\dim S_{n_0} = r$, and let $E_{n_0-1} \in \mathbb{R}^{d\times d}$ be any fixed matrix satisfying the relation $\mathbb{R}^d = S_{n_0} \oplus \ker E_{n_0-1}$. Thus, the condition (ii) in Definition 2.2 holds for all $n \in \mathbb{N}(n_0)$, and the operators T_n as well as matrices G_n are defined for all $n \in \mathbb{N}(n_0)$.

Lemma 2.3: (See [7]) *Suppose the LSDE (1.1) is of index-1 and Q_n are arbitrary projections onto N_n, $n \geqslant n_0$. Then, the following relations hold:*

$$\text{(i) } P_n = G_n^{-1} E_n, \text{ where } P_n := I - Q_n; \tag{2.1}$$

$$\text{(ii) } P_n G_n^{-1} A_n = P_n G_n^{-1} A_n P_{n-1}; \quad Q_n G_n^{-1} A_n = Q_n G_n^{-1} A_n P_{n-1} - T_n^{-1} Q_{n-1}; \tag{2.2}$$

$$\text{(iii) } \widetilde{Q}_{n-1} := -T_n Q_n G_n^{-1} A_n \text{ is the projector onto } N_{n-1} \text{ along } S_n. \tag{2.3}$$

Due to the claim (iii), the projector $\widetilde{Q}_{n-1}$ defined in Lemma 2.3 is uniquely determined, i.e. it does not depend on the choice of Q_n and T_n. Thus, the corresponding projector $\widetilde{P}_n := I - \widetilde{Q}_n$ is unique, too. The projectors $\widetilde{P}_n$ and $\widetilde{Q}_n$ are said to form the canonical projector pair associated with the LSDE (1.1). We have some properties involving the canonical projector pair as follows.

Lemma 2.4: *The matrices $\widetilde{P}_n G_n^{-1}$ and $T_n Q_n G_n^{-1}$ are independent of the choice of T_n and Q_n.*

Proof: Let Q_n, Q'_n be projectors onto N_n, and set $P_n = I - Q_n, P'_n = I - Q'_n$, respectively. Let T_n, T'_n be operators in $GL(\mathbb{R}^d)$ such that $T_n|_{N_n}, T'_n|_{N_n}$ are isomorphisms between N_n and N_{n-1} and let G'_n be defined similarly to G_n. Then, we have

$$G_n^{-1} G'_n = G_n^{-1}(E_n - A_n T'_n Q'_n) = P_n - G_n^{-1} A_n T_n T_n^{-1} T'_n Q'_n.$$

Note that $\operatorname{im}(T'_n Q'_n) = \ker E_{n-1}$ and $\operatorname{im}(T_n^{-1} T'_n Q'_n) = \ker E_n$ so $T_n^{-1} T'_n Q'_n = Q_n T_n^{-1} T'_n Q'_n$ and $P_n T_n^{-1} T'_n Q'_n = 0$. Hence,

$$\begin{aligned} G_n^{-1} G'_n &= P_n - G_n^{-1} A_n T_n Q_n T_n^{-1} T'_n Q'_n \\ &= P_n + G_n^{-1}(G_n - E_n) T_n^{-1} T'_n Q'_n \\ &= P_n + T_n^{-1} T'_n Q'_n - P_n T_n^{-1} T'_n Q'_n \\ &= P_n + T_n^{-1} T'_n Q'_n \end{aligned}$$

Therefore $\widetilde{P}_n G_n^{-1} = \widetilde{P}_n (P_n + T_n^{-1} T'_n Q'_n) G'^{-1}_n = \widetilde{P}_n G'^{-1}_n$.

To prove that $T_nQ_nG_n^{-1} = T'_nQ'_nG'^{-1}_n$ we use the equality $G_n^{-1} = (P_n+T_n^{-1}T'_nQ'_n)G'^{-1}_n$. It implies that

$$\begin{aligned} T_nQ_nG_n^{-1} &= T_nQ_n(P_n + T_n^{-1}T'_nQ'_n)G'^{-1}_n \\ &= T_nQ_nT_n^{-1}T'_nQ'_nG'^{-1}_n. \end{aligned}$$

On the other hand $P_nT_n^{-1}T'_nQ'_n = 0$. Hence, $T_nQ_nG_n^{-1} = T_n(P_n + Q_n)T_n^{-1}T'_nQ'_nG'^{-1}_n = T'_nQ'_nG'^{-1}_n$. The proof is complete. □

As a consequence of Lemma 2.4, it follows immediately that the matrices $\widetilde{P}_n\widetilde{G}_n^{-1}$ and $T_n\widetilde{Q}_n\widetilde{G}_n^{-1}$ are independent of the choice of T_n. Here, the corresponding scaling matrix $\widetilde{G}_n := E_n - A_nT_n\widetilde{Q}_n$ is set.

We describe shortly the decoupling technique for index-1 LSDEs. By virtue of Lemma 2.1, we see that the matrices G_n are nonsingular for all $n \geqslant n_0$. Hence, multiplying (1.1) by $P_nG_n^{-1}$ and $Q_nG_n^{-1}$, respectively, and applying the formulas (2.1) and (2.2) of Lemma 2.3 we decouple the index-1 LSDE into the system

$$P_ny(n+1) = P_nG_n^{-1}A_nP_{n-1}y(n) + P_nG_n^{-1}q_n, \tag{2.4}$$

$$0 = Q_nG_n^{-1}A_ny(n) + Q_nG_n^{-1}q_n. \tag{2.5}$$

Multiplying both sides of Equation (2.5) by T_n and using the second equality in (2.2), this equation is rewritten as

$$Q_{n-1}y(n) = -\widetilde{Q}_{n-1}P_{n-1}y(n) + T_nQ_nG_n^{-1}q_n. \tag{2.6}$$

Thus, the solution $y(n)$ is decomposed as the sum of two components $P_{n-1}y(n)$ and $Q_{n-1}y(n)$, where the 'dynamic' component $P_{n-1}y(n)$ is governed by the difference Equation (2.4), while the 'algebraic' component is determined by the algebraic Equation (2.6). Inspired by this decoupling procedure, we formulate the correctly stated initial condition for the index-1 LSDE (1.1) as

$$P_{n_0-1}(y(n_0) - y_0) = 0, \quad y_0 \in \mathbb{R}^d \text{ is arbitrarily given.} \tag{2.7}$$

Therefore, the Cauchy problem (1.1)–(2.7) has a unique solution defined on $\mathbb{N}(n_0)$, see [7].

Remark 2.5: In fact the initial condition (2.7) is independent of the choice of P_{n_0-1} since it implies $\widetilde{P}_{n_0-1}(y(n_0) - y_0) = 0$. A given initial vector y_0 is said to be consistent with the LSDE (1.1) if $\widetilde{Q}_{n_0-1}y_0 = T_{n_0}\widetilde{Q}_{n_0}\widetilde{G}_{n_0}^{-1}q_{n_0}$. Then, the Cauchy problem for (1.1) with the consistent initial condition $y(n_0) = y_0$ admits a unique solution.

Next, we consider the homogenous Equation (1.2), where $q_n \equiv 0,\ n \in \mathbb{N}(n_0)$. Let us define $z(n) = P_{n-1}x(n)$. The regular ordinary difference equation

$$z(n+1) = P_nG_n^{-1}A_nz(n),\ n \in \mathbb{N}(n_0) \tag{2.8}$$

is called the *inherent (ordinary) difference equation* of LSDE (1.2) associated with the projector pair $P_n,\ Q_n$.

From now on, without loss of generality and just for sake of simplicity, we use the uniquely defined canonical projector pair $\widetilde{P}_n$ and $\widetilde{Q}_n$ instead of general ones. By Lemma 2.3, equation of (1.1) is decoupled as

$$\begin{aligned} \widetilde{P}_n y(n+1) &= \widetilde{P}_n \widetilde{G}_n^{-1} A_n \widetilde{P}_{n-1} y(n) + \widetilde{P}_n \widetilde{G}_n^{-1} q_n, \ n \in \mathbb{N}(n_0) \\ \widetilde{Q}_{n-1} y(n) &= T_n \widetilde{Q}_n \widetilde{G}_n^{-1} q_n. \end{aligned} \tag{2.9}$$

From (2.9), it is easy to see that (1.2) is equivalent to

$$\begin{aligned} \widetilde{P}_n x(n+1) &= \widetilde{P}_n \widetilde{G}_n^{-1} A_n \widetilde{P}_{n-1} x(n), \ n \in \mathbb{N}(n_0) \\ \widetilde{Q}_{n-1} x(n) &= 0. \end{aligned} \tag{2.10}$$

We now construct the Cauchy operator for the homogeneous Equation (1.2). There exists a unique matrix function denoted by $\{\Phi(n,m)\}_{n \geqslant m}$ satisfying

$$E_n \Phi(n+1,m) = A_n \Phi(n,m), \quad P_{m-1}(\Phi(m,m) - I) = 0.$$

This $\{\Phi(n,m)\}_{n \geqslant m}$ is called the Cauchy operator associated with the LSDE (1.2). By using the decoupled system (2.10) that is constructed with the canonical projector pair $\widetilde{P}_n, \ \widetilde{Q}_n$, we obtain

$$\Phi(n,m) = \prod_{k=n-1}^{m} \widetilde{P}_k \widetilde{G}_k^{-1} A_k; \quad n > m \geqslant n_0, \text{ and } \Phi(m,m) = \widetilde{P}_{m-1}.$$

Due to the first equality of (2.2), the equality $\Phi(n,m) = \Phi(n,m)\Phi(m,m)$ holds for all $n \geqslant m \geqslant n_0$. Clearly, $\{\Phi(n,m)\}_{n \geqslant m}$ satisfies the relation

$$\Phi(n,m) = \Phi(n,k)\Phi(k,m) \quad \text{for any } n \geqslant k \geqslant m.$$

By using (2.9) and using the classical constant-variation formula for inhomogeneous regular difference equations, any solution $y(\cdot)$ of the LSDE (1.1) can be expressed by

$$y(n) = \Phi(n,m)\widetilde{P}_{m-1} y(m) + \sum_{i=m}^{n-1} \Phi(n,i+1)\widetilde{P}_i \widetilde{G}_i^{-1} q_i + T_n \widetilde{Q}_n \widetilde{G}_n^{-1} q_n \tag{2.11}$$

for $n \geqslant m \geqslant n_0$.

Remark 2.6: If we use an arbitrary non-canonical projector pair $P_n, \ Q_n$ for decoupling, then the Cauchy operator $\{\Phi(n,m)\}_{n \geqslant m}$ can be alternatively constructed as follows. First, the Cauchy operator associated with the inherent difference Equation (2.8) is defined by

$$\Phi_0(n,m) = \prod_{k=n-1}^{m} P_k G_k^{-1} A_k, \ n > m \geqslant n_0, \text{ and } \Phi_0(m,m) = I.$$

It is easy to see that the relation $\Phi(n,m) = \widetilde{P}_{n-1}\Phi_0(n,m)$ holds. An alternative formula for the solution can be given as well. Analogously to the construction of 2.11, we obtain

$$y(n) = \Phi(n,m)P_{m-1}y(m) + \sum_{i=m}^{n-1}\Phi(n,i+1)P_iG_i^{-1}q_i + T_nQ_nG_n^{-1}q_n \tag{2.12}$$

for all $n \geqslant m \geqslant n_0$. Due to the fact that $\Phi(n,m) = \Phi(n,m)\widetilde{P}_{m-1}$ and the results of Lemma 2.4, the formulas (2.11) and (2.12) obviously coincide.

2.2. Stability notions for singular difference equations

From now on, we always suppose that the LSDE (1.2) has index-1 and its Cauchy operator $\Phi(n,m)$ is defined as above. The following stability notions generalize those for ordinary difference equations. See also [8,10].

Definition 2.7: The zero solution of Equation (1.2) is said to be *stable* if for any $\varepsilon > 0$ and $n_1 \in \mathbb{N}(n_0)$ there exists a positive constant $\delta = \delta(\varepsilon, n_1)$ such that the inequality $\|\widetilde{P}_{n_1-1}x^1\| < \delta$ implies $\|x(k)\| < \varepsilon$ for all $k \in \mathbb{N}(n_1)$, where $x(\,\cdot\,)$ is the solution of (1.2) satisfying $\widetilde{P}_{n_1-1}(x(n_1) - x^1) = 0$.

The zero solution of Equation (1.2) is said to be *uniformly stable* if it is stable and the above defined δ is independent of n_1.

Definition 2.8: The zero solution of Equation (1.2) is said to be *asymptotically stable* if it is stable and $\lim_{k\to\infty}\|x(k)\| = 0$, where $x(\,\cdot\,)$ is the solution of (1.2) with $\widetilde{P}_{n_1-1}(x(n_1) - x^1) = 0$.

If the zero solution of Equation (1.2) is stable (resp. uniformly stable, asymptotically stable) then we say Equation (1.2) is stable (resp. uniformly stable, asymptotically stable).

Definition 2.9: The Equation (1.2) is said to be *exponentially stable* if there exist constants $K > 0$ and $0 < \omega < 1$ such that $\|x(n)\| \leqslant K\omega^{n-m}\|\widetilde{P}_{m-1}x(m)\| = K\omega^{n-m}\|x(m)\|, n, m \in \mathbb{N}(n_0), n \geqslant m$, for every solution $x(\,\cdot\,)$ of (1.2).

The following characterizations of uniform stability and exponential stability are straightforward generalizations of the well-known results for ordinary difference equations, see [1,2]. Therefore, we omit the details of the proofs.

Theorem 2.10: *Suppose that the projector function $\widetilde{P}_m$ is bounded. Then the LSDE (1.2) is uniformly stable if and only if there exists a constant $C > 0$ such that*

$$\|\Phi(n,m)\| \leqslant C, \quad \textit{for all } n,m \in \mathbb{N}(n_0), n \geqslant m.$$

Theorem 2.11: *Suppose that the projector function $\widetilde{P}_m$ is bounded. Then the LSDE (1.2) is exponentially stable if and only if there exists constants $K > 0$ and $0 < \omega < 1$ such that*

$$\|\Phi(n,m)\| \leqslant K\omega^{n-m}, \quad \textit{for all } n,m \in \mathbb{N}(n_0), n \geqslant m.$$

We note that for the case of unbounded $\widetilde{P}_m$, the sufficiency statements may fail. We give here an example for exponentially stable equations of index-1.

Example 2.12: Consider the following singular difference equation

$$E_n x(n+1) = A_n x(n), \ x(n) \in \mathbb{R}^3, \ n \in \mathbb{N}(2), \tag{2.13}$$

where

$$E_n = \begin{pmatrix} -2n\cos(n+1) & 2n\sin(n+1) & 0 \\ -2n\cos(n+1) & 2n\sin(n+1) & 0 \\ -4n\cos(n+1) & 4n\sin(n+1) & 0 \end{pmatrix},$$
$$A_n = \begin{pmatrix} -(n-1)\cos n + \sin n & (n-1)\sin n + \cos n & 2 \\ -(n-1)\cos n + 2\sin n & (n-1)\sin n + 2\cos n & 1 \\ -(2n-2)\cos n + \sin n & (2n-2)\sin n + \cos n & 1 \end{pmatrix}.$$

Choose the projections as follows

$$Q_n = \begin{pmatrix} \sin^2(n+1) & \sin(n+1)\cos(n+1) & 0 \\ \sin(n+1)\cos(n+1) & \cos^2(n+1) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$P_n = \begin{pmatrix} \cos^2(n+1) & -\sin(n+1)\cos(n+1) & 0 \\ -\sin(n+1)\cos(n+1) & \sin^2(n+1) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that rank $E_n \equiv 1, \ \forall n \in \mathbb{N}(2)$. We choose

$$T_n = \begin{pmatrix} \cos 1 & -\sin 1 & 0 \\ \sin 1 & \cos 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Associated with the projectors Q_n and T_n defined as above, we calculate

$$G_n = \begin{pmatrix} -2n\cos(n+1) - \sin(n+1) & 2n\sin(n+1) - \cos(n+1) & -2 \\ -2n\cos(n+1) - 2\sin(n+1) & 2n\sin(n+1) - 2\cos(n+1) & -1 \\ -4n\cos(n+1) - \sin(n+1) & 4n\sin(n+1) - \cos(n+1) & -1 \end{pmatrix}.$$

It is easy to verify that G_n is invertible for any $n \in \mathbb{N}(2)$.

Thus, Equation (2.13) has index 1. The canonical projectors are

$$\widetilde{Q}_{n-1} = \begin{pmatrix} \sin^2 n & \sin n\cos n & 0 \\ \sin n\cos n & \cos^2 n & 0 \\ 0 & 0 & 1 \end{pmatrix}, \widetilde{P}_{n-1} = \begin{pmatrix} \cos^2 n & -\sin n\cos n & 0 \\ -\sin n\cos n & \sin^2 n & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain

$$\widetilde{P}_n \widetilde{G}_n^{-1} A_n = \frac{n-1}{2n} \begin{pmatrix} \cos n\cos(n+1) & -\sin n\cos(n+1) & 0 \\ -\cos n\sin(n+1) & \sin n\sin(n+1) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the Cauchy operator of (2.13) is

$$\Phi(n,m) = \frac{1}{2^{n-m}}\frac{m-1}{n-1}\begin{pmatrix} \cos m\cos n & -\sin m\cos n & 0 \\ -\cos m\sin n & \sin m\sin n & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\forall n \geqslant m \geqslant 2$. It is not difficult to see that

$$\|\Phi(n,m)\| \leqslant \frac{1}{2^{n-m}}, \ \forall n \geqslant m \geqslant 2.$$

Here the Euclidean norm is used. Thus, any solution $x(n)$ of (2.13) satisfies

$$\|x(n)\| = \|\Phi(n,m)x(m)\| \leqslant \|\Phi(n,m)\|\|x(m)\| \leqslant \frac{1}{2^{n-m}}\|x(m)\|, \ \forall n \geqslant m \geqslant 2.$$

Hence, Equation (2.13) is exponentially stable.

3. Stability of perturbed equations

Due to the remark after Theorems 2.10 and 2.11, from now on we always assume that the canonical projector function $\widetilde{P}_n$ associated with the unperturbed Equation (1.2) is bounded, i.e. there exists a constant $\rho > 0$ such that $\|\widetilde{P}_n\| \leqslant \rho$ for all $n \in \mathbb{N}(n_0)$.

3.1. The case of one-side perturbation

Consider the perturbed equation associated with (1.2)

$$E_n y(n+1) = (A_n + B_n)y(n), \quad n \in \mathbb{N}(n_0), \tag{3.1}$$

where $B_n \in \mathbb{R}^{d\times d}$, $n \in \mathbb{N}(n_0)$, are perturbation matrices. Multiplying both sides of the perturbed Equation (3.1) by $P_n G_n^{-1}$ and $Q_n G_n^{-1}$, respectively, we obtain

$$\begin{aligned} P_n y(n+1) &= P_n G_n^{-1}(A_n + B_n)y(n), \\ 0 &= Q_n G_n^{-1}(A_n + B_n)y(n). \end{aligned} \tag{3.2}$$

Here the auxiliary matrices are the same as those defined in the previous section for the unperturbed Equation (1.2). The term $B_n y(n)$ caused by the perturbation now plays the role of the inhomogeneity. Using the canonical projectors $\widetilde{P}_n$, $\widetilde{Q}_n$, and the same argument as that for the construction of the constant variation formula (2.11), Equation (3.1) is equivalent to the 'integral' equation

$$y(n) = \Phi(n,n_0)\widetilde{P}_{n_0-1}y(n_0) + \sum_{i=n_0}^{n-1}\Phi(n,i+1)\widetilde{P}_i\widetilde{G}_i^{-1}B_i y(i) + T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n y(n), \ n \geqslant n_0.$$

If $I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n$ are invertible for all $n \in \mathbb{N}(n_0)$, then

$$y(n) = (I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^{-1}\left(\Phi(n, n_0)\widetilde{P}_{n_0-1}y(n_0) + \sum_{i=n_0}^{n-1}\Phi(n, i+1)\widetilde{P}_i\widetilde{G}_i^{-1}B_iy(i)\right). \tag{3.3}$$

Thus, the unique solution $y(n)$ of (3.1) can be obtained recursively.

In order to analyze the stability of the perturbed Equation (3.1), we may need some of the following assumptions on the perturbation B_n.

Assumption 1: *The perturbation B_n is such that*

$$\sup_{n\in\mathbb{N}(n_0)} \|T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n\| < 1.$$

The following lemma obviously holds.

Lemma 3.1: *Let Assumption 1 hold. Then, the matrices $I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n$ are invertible and there exists a constant $c_2 > 0$ such that*

$$\|(I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^{-1}\| \leqslant c_2, \ \forall n \in \mathbb{N}(n_0).$$

Assumption 2: *The perturbation B_n is such that*

$$\sum_{n=n_0}^{\infty} \|\widetilde{P}_n\widetilde{G}_n^{-1}B_n\| < \infty.$$

If Assumption 2 is satisfied, then obviously we have

$$\sum_{i=m}^{n-1} \|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\| \leqslant c_4, \forall n > m \geqslant n_0,$$

where $c_4 = \sum_{i=n_0}^{\infty} \|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|$.

Assumption 3: *The perturbation B_n is such that there exist a constant $\delta > 0$ and a number $N \geqslant n_0$ such that*

$$\|\widetilde{P}_n\widetilde{G}_n^{-1}B_n\| \leqslant \delta, \ \forall n \in \mathbb{N}(N).$$

Assumption 4: *The perturbation B_n is such that*

$$\|\widetilde{P}_n\widetilde{G}_n^{-1}B_n\| \to 0 \text{ as } n \to \infty.$$

It is obvious that Assumption 4 implies Assumption 3.

Lemma 3.2: *Suppose that the unperturbed Equation (1.2) has index-1 and the canonical projectors associated with it are bounded. If Assumption 1 holds, then the perturbed Equation (3.1) has index-1 and the canonical projectors associated with it are bounded, too.*

Proof: Due to Lemma 2.1 and Definition 2.2, the perturbed Equation (3.1) has index 1 if and only if the matrix $\hat{G}_n = E_n - (A_n + B_n)T_n\widetilde{Q}_n$ is invertible for all $n \geqslant n_0$. We have $\hat{G}_n = \widetilde{G}_n - B_nT_n\widetilde{Q}_n = \widetilde{G}_n(I - \widetilde{G}_n^{-1}B_nT_n\widetilde{Q}_n)$. If Assumption 1 holds, then $I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n$

is obviously invertible for all $n \in \mathbb{N}(n_0)$. It is not difficult to show that $I - \widetilde{G}_n^{-1}B_nT_n\widetilde{Q}_n$ is invertible, too. Thus, the first statement is proven.

To verify the boundedness of the canonical projector $\hat{Q}_n$ associated with the perturbed Equation (3.1), first we formulate $\hat{Q}_{n-1} = -T_n\widetilde{Q}_n\hat{G}_n^{-1}(A_n + B_n)$. Thus, under Assumption 1, we have

$$\begin{aligned}
\hat{Q}_{n-1} &= -T_n\widetilde{Q}_n(\widetilde{G}_n - B_nT_n\widetilde{Q}_n)^{-1}(A_n + B_n) = -T_n\widetilde{Q}_n(I - \widetilde{G}_n^{-1}B_nT_n\widetilde{Q}_n)^{-1}\widetilde{G}_n^{-1}(A_n + B_n)\\
&= -T_n\widetilde{Q}_n\sum_{i=0}^{\infty}(\widetilde{G}_n^{-1}B_nT_n\widetilde{Q}_n)^i\widetilde{G}_n^{-1}(A_n + B_n)\\
&= -\sum_{i=0}^{\infty}(T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^iT_n\widetilde{Q}_n\widetilde{G}_n^{-1}(A_n + B_n)\\
&= -(I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^{-1}T_n\widetilde{Q}_n\widetilde{G}_n^{-1}(A_n + B_n).
\end{aligned}$$

Finally, the boundedness of $\widetilde{Q}_{n-1} = T_n\widetilde{Q}_n\widetilde{G}_n^{-1}A_n$ and the result of Lemma 3.1 together imply the boundedness of projector function $\hat{Q}_{n-1}$. The proof is complete. □

The following auxiliary lemma (also known as the discrete Gronwall lemma) will be useful in the estimation of the solutions of (3.1).

Lemma 3.3: (See [1], Corollary 4.1.2) *Let p, q be nonnegative real numbers and let $\{u(n)\}$ and $\{f(n)\}$ be nonnegative sequences for all $n \in \mathbb{N}(n_0)$, $n_0 \in \mathbb{N}$ is given. Suppose that*

$$u(n) \leqslant p + q\sum_{l=n_0}^{n-1} f(l)u(l),\ \forall n \in \mathbb{N}(n_0).$$

Then, we have the estimate

$$u(n) \leqslant p\prod_{l=n_0}^{n-1}(1 + qf(l)),\ \forall n \in \mathbb{N}(n_0).$$

Theorem 3.4: *Suppose that the unperturbed Equation (1.2) has index-1 and it is uniformly stable. Let the perturbation B_n satisfy Asssumptions 1 and 2. Then, there exists a constant $C_1 > 0$ such that*

$$\|y(n)\| \leqslant C_1\|y(m)\| \ \text{ for all } \ n > m \geqslant n_0,$$

for every solution $y(\cdot)$ of (3.1). That is, the perturbed Equation (3.1) is uniformly stable, too.

This theorem means that if the perturbation is small enough in the sense of Asssumptions 1 and 2, then the perturbed equation preserves the index-1 property as well as the uniform stability.

Proof: Due to Lemma 3.1 and the formula (3.3), every solution $y(\cdot)$ of (3.1) satisfies the expression

$$y(n) = (I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^{-1}\left(\Phi(n,m)\widetilde{P}_{m-1}y(m) + \sum_{i=m}^{n-1}\Phi(n,i+1)\widetilde{P}_i\widetilde{G}_i^{-1}B_iy(i)\right).$$

Hence, we obtain the estimate

$$\|y(n)\| \leqslant \|(I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}B_n)^{-1}\| \left(\|\Phi(n,m)\widetilde{P}_{m-1}y(m)\| + \sum_{i=m}^{n-1} \|\Phi(n,i+1)\widetilde{P}_i\widetilde{G}_i^{-1}B_i y(i)\| \right).$$

The uniform stability of Equation (1.2) implies that there exists a constant $c_1 > 0$ such that

$$\|\Phi(n,m)\widetilde{P}_{m-1}\| = \|\Phi(n,m)\| \leqslant c_1$$

for all $n > m \geqslant n_0$.

Combining with the estimate in Lemma 3.1, we obtain

$$\|y(n)\| \leqslant c_2c_1\|y(m)\| + c_2c_1 \sum_{i=m}^{n-1} \|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\|y(i)\|.$$

Applying Lemma 3.3, we have

$$\begin{aligned}
\|y(n)\| &\leqslant c_1c_2\|y(m)\| \prod_{i=m}^{n-1} (1 + c_1c_2\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|) \\
&\leqslant c_1c_2\|y(m)\| \prod_{i=m}^{n-1} \exp\left(c_1c_2\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\right) \\
&\leqslant c_1c_2\|y(m)\| \exp\left(c_1c_2 \sum_{i=m}^{n-1} \|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\| \right) \\
&\leqslant c_1c_2 \exp\left(c_1c_2c_4\right)\|y(m)\|.
\end{aligned}$$

Put $C_1 = c_1c_2 \exp(c_1c_2c_4)$, the proof is complete. □

Theorem 3.5: *Suppose that the unperturbed Equation (1.2) has index-1 and it is exponentially stable. Let the perturbation B_n satisfy Assumptions 1 and 3 with a sufficiently small δ. Then, there exist constants $K_1 > 0$ and $0 < \omega_1 < 1$ such that*

$$\|y(n)\| \leqslant K_1\omega_1^{n-m}\|y(m)\|, \text{ for all } \quad n \geqslant m \geqslant n_0,$$

for every solution $y(\cdot)$ of (3.1). That is, the perturbed Equation (3.1) preserves the exponential stability.

This theorem means that the exponential stability of Equation (1.2) is robust under small right-hand side perturbations. A bound for δ will be pointed out in the proof.

Proof: Suppose that $y(n)$ is any solution of (3.1) and that Assumption 3 holds with constants δ and N. We note that for each fixed $N \in \mathbb{N}(n_0)$, there exists a constant $C_N > 0$ such that $\|y(i)\| \leqslant C_N\|y(m)\|,\ n_0 \leqslant m < i \leqslant N$ (where C_N depends only on N).

The exponential stability of Equation (1.2) implies that there exist constants $c_3 > 0$ and $0 < \omega < 1$ such that

$$\|\Phi(n,m)\widetilde{P}_{m-1}\| = \|\Phi(n,m)\| \leqslant c_3\omega^{n-m}$$

for all $n \geqslant m \geqslant n_0$. First, we consider the case $n > N > m$. Using Lemma 3.1 and Assumption 3, we have

$$\begin{aligned}\|y(n)\| &\leqslant c_2c_3\omega^{n-m}\|y(m)\| + c_2\sum_{i=m}^{n-1} c_3\omega^{n-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\|y(i)\|,\\ &\leqslant c_2c_3\omega^{n-m}\|y(m)\| + c_2\sum_{i=m}^{N-1} c_3\omega^{n-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|C_N\|y(m)\|\\ &\quad + c_2\sum_{i=N}^{n-1} c_3\omega^{n-(i+1)}\delta\|y(i)\|.\end{aligned}$$

Multiplying both sides of the above equality with ω^{-n}, we obtain

$$\begin{aligned}\omega^{-n}\|y(n)\| &\leqslant \left(c_2c_3\omega^{-m}\|y(m)\| + c_2c_3\sum_{i=m}^{N-1}\omega^{-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|C_N\|y(m)\|\right)\\ &\quad + \frac{\delta c_2c_3}{\omega}\sum_{i=N}^{n-1}\omega^{-i}\|y(i)\|.\end{aligned}$$

Applying Lemma 3.3, we have

$$\omega^{-n}\|y(n)\| \leqslant \left(c_2c_3\omega^{-m} + C_Nc_2c_3\sum_{i=m}^{N-1}\omega^{-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\right)\|y(m)\|\prod_{i=N}^{n-1}\left(1+\frac{\delta c_2c_3}{\omega}\right).$$

Therefore,

$$\begin{aligned}\|y(n)\| &\leqslant \left(c_2c_3 + C_Nc_2c_3\sum_{i=m}^{N-1}\omega^{m-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\right)\omega^{N-m}(\omega+\delta c_2c_3)^{n-N}\|y(m)\|\\ &\leqslant \left(c_2c_3 + C_Nc_2c_3\sum_{i=m}^{N-1}\omega^{m-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|\right)(\omega+\delta c_2c_3)^{n-m}\|y(m)\|\\ &\leqslant c_5\omega_1^{n-m}\|y(m)\|,\end{aligned}$$

where $c_5 = c_2c_3 + C_Nc_2c_3\sum_{i=n_0}^{N-1}\omega^{n_0-(i+1)}\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\|$, and $\omega_1 = \omega + \delta c_2c_3$. Choose a sufficiently small δ such that $\omega + \delta c_2c_3 < 1$.

The case $n > m \geqslant N$ is treated similarly with even simpler calculations since the estimate $\|\widetilde{P}_i\widetilde{G}_i^{-1}B_i\| \leqslant \delta$ holds for all $i \geqslant m$. Again using Lemma 3.1 and Assumption 3, we have

$$\|y(n)\| \leqslant c_2c_3\omega^{n-m}\|y(m)\| + c_2c_3\sum_{i=m}^{n-1}\omega^{n-(i+1)}\delta\|y(i)\|,$$

which is equivalent to

$$\omega^{-n}\|y(n)\| \leqslant c_2c_3\omega^{-m}\|y(m)\| + \frac{c_2c_3\delta}{\omega}\sum_{i=m}^{n-1}\omega^{-i}\|y(i)\|.$$

Applying Lemma 3.3, we obtain

$$\omega^{-n}\|y(n)\| \leqslant c_2c_3\omega^{-m}\|y(m)\| \prod_{i=m}^{n-1}\left(1+\frac{\delta c_2c_3}{\omega}\right),$$

which implies

$$\|y(n)\| \leqslant c_2c_3(\omega+\delta c_2c_3)^{n-m}\|y(m)\| = c_2c_3\omega_1^{n-m}\|y(m)\|.$$

For the remaining case $n_0 \leqslant m < n \leqslant N$, with $\omega_1 < 1$ defined above we have

$$\|y(n)\| \leqslant \frac{C_N}{\omega_1^N}\omega_1^N\|y(m)\| \leqslant \frac{C_N}{\omega_1^N}\omega_1^{n-m}\|y(m)\|.$$

Thus, by setting $K_1 = \max\{c_5, C_N/\omega_1^N\}$, the proof is complete. □

The following corollary immediately follows.

Corollary 3.6: *Suppose that the unperturbed Equation (1.2) has index 1 and it is exponentially stable. Let the perturbation B_n satisfy Assumptions 1 and 4. Then, the perturbed Equation (3.1) preserves the exponential stability.*

3.2. *The case of two-side perturbation*

Supposing again that Equation (1.2) has index 1, now we consider Equation (1.2) subject to two-side perturbations

$$(E_n+F_n)y(n+1) = (A_n+B_n)y(n), \quad n \in \mathbb{N}(n_0), \tag{3.4}$$

where $F_n, B_n \in \mathbb{K}^{d\times d}$ are perturbation matrices. Here, we always assume that F_n is an allowable structured perturbation, i.e. $\ker(E_n+F_n) = \ker E_n$, for all $n \in \mathbb{N}(n_0)$, see [16,21].

The following example shows that if F_n does not belong to the family of allowable structured perturbations, then the asymptotic behaviour of solutions of the perturbed LSDEs (3.4) and the asymptotic behaviour of solutions of the unperturbed one may be quite different, even if the pertubation F is arbitrarily small or $\|F_n\| \to 0$ as $n \to \infty$ or $\sum_{n=n_0}^{\infty}\|F_n\| < \infty$.

Example 3.7: Consider the index-1 LSDE

$$\begin{pmatrix}3 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1(n+1)\\ x_2(n+1)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x_1(n)\\ x_2(n)\end{pmatrix}, \quad n \in \mathbb{N}.$$

It is easy to obtain the solution $x_1(n) = \frac{1}{3^n}x_1(0)$ and $x_2(n) = 0$ for all $n \in \mathbb{N}$. After that, we consider the following perturbed LSDE

$$\begin{pmatrix}3 & 0\\ 0 & \dfrac{1}{(n+1)^2}\end{pmatrix}\begin{pmatrix}y_1(n+1)\\ y_2(n+1)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}y_1(n)\\ y_2(n)\end{pmatrix}, \quad n \in \mathbb{N}. \tag{3.5}$$

From the first equation of (3.5), it follows that $y_1(n) = x_1(n) = \frac{1}{3^n}x_1(0)$. However, the second component is $y_2(n) = (n!)^2 u_2(0)$, which tends to ∞ as $n \to \infty$ for a nonzero initial value $y_2(0)$. That is, an arbitrarily small perturbation arising in the leading coefficient can completely change the behaviour of solutions.

Let G_n be the auxiliary matrix associated with the unperturbed Equation (1.2), which is nonsingular for all $n \in \mathbb{N}(n_0)$. We will use the following assumption on F_n.

Assumption 5: *The perturbation F_n is of allowable structure and satisfies*

$$\sup_{n\in\mathbb{N}(n_0)} \|F_n G_n^{-1}\| < 1.$$

Let us define $\overline{G}_n = E_n + F_n - A_n T_n Q_n = G_n + F_n$. It is easy to see that Assumption 5 implies the invertibility of $\overline{G}_n$ for all $n \in \mathbb{N}(n_0)$. Furthermore, we have

$$\overline{G}_n^{-1} = G_n^{-1} - G_n^{-1} F_n (G_n + F_n)^{-1}. \tag{3.6}$$

Multiplying both sides of (3.4) by $P_n\overline{G}_n^{-1}$ and $Q_n\overline{G}_n^{-1}$, respectively, and applying the formula (2.1) of Lemma 2.3 and (3.6), we decouple Equation (3.4) into the system

$$\begin{aligned} P_n y(n+1) &= P_n[G_n^{-1} - G_n^{-1}F_n(G_n+F_n)^{-1}](A_n+B_n)y(n),\\ 0 &= Q_n[G_n^{-1} - G_n^{-1}F_n(G_n+F_n)^{-1}](A_n+B_n)y(n). \end{aligned}$$

Let us define

$$\overline{B}_n = B_n - F_n(G_n+F_n)^{-1}(A_n+B_n), \tag{3.7}$$

we rewrite the above system as

$$\begin{aligned} P_n y(n+1) &= P_n G_n^{-1}(A_n+\overline{B}_n)y(n),\\ 0 &= Q_n G_n^{-1}(A_n+\overline{B}_n)y(n). \end{aligned} \tag{3.8}$$

Remark 3.8: Since $F_nG_n^{-1} = F_n(\widetilde{P}_n + \widetilde{Q}_n)G_n^{-1} = F_n\widetilde{P}_nG_n^{-1}$ and the property that $\widetilde{P}_nG_n^{-1}$ does not depend on the choice of T_n and Q_n, we conclude that $F_nG_n^{-1}$ does not depend on the choice of T_n and Q_n, too. Hence, it is not difficult to see that $F_n(G_n+F_n)^{-1}$ as well as $\overline{B}_n$ is independent of the choice of T_n and Q_n.

If we use again the canonical projectors $\widetilde{P}_n$ and $\widetilde{Q}_n$, then the system (3.8) looks similar to the system (3.2) with the only difference that $\overline{B}_n$ replaces B_n. If the perturbations F_n and $\overline{B}_n$ satisfy Asssumptions 5 and 1, then due to Lemma 3.2, the perturbed Equation (3.4) preserves the index-1 property. Analogously to the case of one-side perturbation, we also have the following stability results for the perturbed Equation (3.4).

Theorem 3.9: *Consider the perturbed Equation (3.4). Suppose that the unperturbed Equation (1.2) has index 1 and it is uniformly stable. Let the perturbations F_n and $\overline{B}_n$ satisfy Asssumptions 5, 1, and 2. Then, there exists a constant $C_2 > 0$ such that*

$$\|y(n)\| \leqslant C_2\|y(m)\| \quad \textit{for all } n \geqslant m \geqslant n_0$$

for every solution $y(\cdot)$ of (3.4). That is, the perturbed Equation (3.4) is uniformly stable, too.

Proof: Repeating the proof of Theorem 3.4 applied to Equation (3.8), the proof is complete. □

Theorem 3.10: *Consider the perturbed Equation (3.4). Suppose that the unperturbed Equation (1.2) has index 1 and it is exponentially stable. Let the perturbations F_n and $\overline{B}_n$ satisfy Assumptions 5, 1, and 3 with a sufficiently small δ. Then, there exist constants $K_2 > 0$ and $0 < \omega_2 < 1$ such that*

$$\|y(n)\| \leqslant K_2\omega_2^{n-m}\|y(m)\|, \quad \text{for all } n \geqslant m \geqslant n_0$$

for every solution $y(\cdot)$ of (3.4). That is, the perturbed Equation (3.4) preserves the exponential stability.

Proof: Repeating the proof of Theorem 3.5 applied to Equation (3.8), the proof is complete. □

Corollary 3.11: *Consider the perturbed Equation (3.4). Suppose that the unperturbed Equation (1.2) has index 1 and it is exponentially stable. Let the perturbations F_n and $\overline{B}_n$ satisfy Assumptions 5, 1, and 4. Then, the perturbed Equation (3.4) preserves the exponential stability.*

4. Boundedness of solutions of nonhomogeneous equations

Next, we investigate the relation between the exponential stability of LSDE (1.2) and the boundedness of solutions of nonhomogenous Equation (1.1). This relation is known in the theory of ordinary differential equations and difference equations as Bohl-Perron theorems, see [22–24].

Consider the Cauchy problem (denoted by $CP(n_0)$) associated with $q = \{q_n\}_{n\geqslant n_0}$

$$\begin{aligned} E_n y(n+1) &= A_n y(n) + q_n, \quad n \in \mathbb{N}(n_0), \\ \widetilde{P}_{n_0-1}y(n_0) &= 0. \end{aligned}$$

and another Cauchy problem (denoted by $\overline{CP}(n_0)$) associated with $\overline{q} = \{\overline{q}_n\}_{n\geqslant n_0}$

$$\begin{aligned} z(n+1) &= \widetilde{P}_n\widetilde{G}_n^{-1}A_n z(n) + \overline{q}_n, \quad n \in \mathbb{N}(n_0), \\ z(n_0) &= 0. \end{aligned}$$

Set

$$\begin{aligned} LP(n_0) &= \Big\{\{q_n\}_{n\geqslant n_0} : q_n \in \operatorname{im}\widetilde{P}_n \text{ and } \{q_n\}_{n\geqslant n_0} \text{ is bounded}\Big\}, \\ L(n_0) &= \Big\{\{q_n\}_{n\geqslant n_0} : \sup_{n\in\mathbb{N}(n_0)} \|T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n\| < \infty \\ &\qquad \text{and} \sup_{n\in\mathbb{N}(n_0)} \|\widetilde{P}_n\widetilde{G}_n^{-1}q_n\| < \infty\Big\}. \end{aligned}$$

By Lemma 2.4, $LP(n_0)$ and $L(n_0)$ do not depend on the choice of T_n and Q_n. It is not difficult to verify that $LP(n_0)$ endowed with the norm $\|q\| = \sup_{n\in\mathbb{N}(n_0)}\|q_n\|$ is a Banach space.

Lemma 4.1: *Let $n_1 \in \mathbb{N}(n_0)$ be given. If for every sequence $\overline{q} = \{\overline{q}_n\}_{n\geqslant n_0}$ in $LP(n_0)$, the solution of the Cauchy problem $\overline{CP}(n_0)$ is bounded then the problem $\overline{CP}(n_1)$ has the* *bounded solution for every sequence* $\widetilde{q} = \{\widetilde{q}_n\}_{n \geqslant n_1}$ *in* $LP(n_1)$*. Moreover there exists a constant* k *(independent of* n_1*) such that* $\|\widetilde{z}(n)\| \leqslant k\|\widetilde{q}\|$ *for all* $n \geqslant n_1$*, where* $\widetilde{z}(\cdot)$ *is the solution of* $\overline{CP}(n_1)$ *associated with* $\{\widetilde{q}_n\}_{n \geqslant n_1}$ *in* $LP(n_1)$*.*

Proof: We define a family of operators $\{V_n\}_{n \geqslant n_0}$ as follows

$$\begin{aligned} V_n : \quad LP(n_0) &\longrightarrow \mathbb{R}^d \\ \overline{q} &\longmapsto V_n\overline{q} := z(n), \end{aligned}$$

where $z(\cdot)$ is the solution of $\overline{CP}(n_0)$ associated with $\overline{q} = \{\overline{q}_n\}_{n \geqslant n_0}$. From the assumption that the solution of the Cauchy problem $\overline{CP}(n_0)$ associated with $\overline{q} = \{\overline{q}_n\}_{n \geqslant n_0}$ in $LP(n_0)$ is bounded, it implies

$$\sup_{n \in \mathbb{N}(n_0)} \|V_n\overline{q}\| = \sup_{n \in \mathbb{N}(n_0)} \|z(n)\| < \infty.$$

Moreover, $LP(n_0)$ and $\mathbb{R}^d$ are Banach spaces. Using the Uniform Boundedness Principle, see [25], there exists a constant $k > 1$ such that

$$\|z(n)\| = \|V_n\overline{q}\| \leqslant k\|\overline{q}\|, \ \forall n \in \mathbb{N}(n_0). \tag{4.1}$$

With an arbitrary n_1 in $\mathbb{N}(n_0)$, corresponding to every sequence $\widetilde{q} = \{\widetilde{q}_n\}_{n \geqslant n_1}$ in $LP(n_1)$, we define $\overline{q} = \{\overline{q}_n\}_{n \geqslant n_0}$ in $LP(n_0)$ as follows: if $i < n_1$ then $\overline{q}_i = 0$, else $\overline{q}_i = \widetilde{q}_i$. It is easy to see that

$$\begin{cases} z(i) = 0, & \text{if } n_0 \leqslant i \leqslant n_1, \\ z(i) = \widetilde{z}(i), & \text{otherwise,} \end{cases}$$

where $z(\cdot)$ (resp. $\widetilde{z}(\cdot)$) is the solution of $\overline{CP}(n_0)$ (resp. $\overline{CP}(n_1)$) associated with $\overline{q}$ (res. $\widetilde{q}$). Hence, we have $\|\overline{q}\| = \|\widetilde{q}\|$ and $\|z(n)\| = \|\widetilde{z}(n)\|$ for all n in $\mathbb{N}(n_1)$. From (4.1), we have

$$\|\widetilde{z}(n)\| = \|z(n)\| \leqslant k\|\overline{q}\| = k\|\widetilde{q}\|, \ \forall n \in \mathbb{N}(n_1).$$

The proof is complete. □

Theorem 4.2: *The solutions of* $CP(n_0)$ *associated with every* $q = \{q_n\}_{n \geqslant n_0}$ *in* $L(n_0)$ *are bounded if and only if the index-1 LSDE (1.2) is exponentially stable.*

Proof: The proof contains two parts.

Necessity First, we prove that if all the solutions of $CP(n_0)$ associated with $q = \{q_n\}_{n \geqslant n_0}$ in $L(n_0)$ are bounded then the LSDE (1.2) is exponentially stable.

Let n_1 be an arbitrary number in $\mathbb{N}(n_0)$. For each $\overline{q} = \{\overline{q}_n\}_{n \geqslant n_0}$ in $LP(n_0)$, put $q = \widetilde{G}\overline{q} = \{\widetilde{G}_n\overline{q}_n\}_{n \geqslant n_0}$. Since $\overline{q} \in LP(n_0)$, it implies that $\overline{q}_n = \widetilde{P}_n\overline{q}_n$, i.e. $\widetilde{Q}_n\overline{q}_n = 0$ for all $n \in \mathbb{N}(n_0)$. Thus, we have $T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n = T_n\widetilde{Q}_n\widetilde{G}_n^{-1}\widetilde{G}_n\overline{q}_n = 0$ and $\widetilde{P}_n\widetilde{G}_n^{-1}q_n = \widetilde{P}_n\widetilde{G}_n^{-1}\widetilde{G}_n\overline{q}_n = \overline{q}_n$. Hence, $q = \widetilde{G}\overline{q} \in L(n_0)$. Therefore, the solution of $CP(n_0)$ corresponding to q is bounded. Moreover, the solution of $CP(n_0)$ associated with q is exactly the same as the solution of $\overline{CP}(n_0)$ associated with $\overline{q}$. It implies that for every sequence $\overline{q} = \{\overline{q}_n\}_{n \geqslant n_0}$ in $LP(n_0)$, the solution of the Cauchy problem $\overline{CP}(n_0)$ is bounded. Applying Lemma 4.1, there exists a constant $k > 1$ which is independent of n_1 such that

$$\|y(n)\| = \|z(n)\| \leqslant k\|\overline{q}\|, \ \forall n \in \mathbb{N}(n_1), \tag{4.2}$$

where $y(\cdot)$ is the solution of $CP(n_1)$ corresponding to $q = \widetilde{G}\overline{q}$ and $z(\cdot)$ is the solution of $\overline{CP}(n_1)$ corresponding to $\overline{q} = \{\overline{q}_n\}_{n\geqslant n_1}$ in $LP(n_1)$.

For any $v \in \mathbb{R}^d$, set $\chi(i) = \|\Phi(i+1, n_1)\|, \quad i \in \mathbb{N}(n_1)$, where $\Phi(\cdot,\cdot)$ is the Cauchy operator of the corresponding homogeneous equation, we consider the sequence $\overline{q}_i = \dfrac{\Phi(i+1,n_1)}{\chi(i)}v, \quad i \in \mathbb{N}(n_1)$. It is obvious that

$$\|\overline{q}_i\| \leqslant \frac{\|\Phi(i+1,n_1)\|}{\chi(i)}\|v\| = \|v\|, \tag{4.3}$$

and $\overline{q}_i \in \operatorname{im}\widetilde{P}_i, \quad \forall i \in \mathbb{N}(n_1)$. Hence $\overline{q} = \{\overline{q}_i\}_{i\geqslant n_1} \in LP(n_1)$ and $q = \widetilde{G}\overline{q} = \{\widetilde{G}_i\overline{q}_i\}_{i\geqslant n_1} \in L(n_1)$. Let $y(\cdot)$ be the solution of $CP(n_1)$ corresponding to q and let $z(\cdot)$ be the solution of $\overline{CP}(n_1)$ corresponding to $\overline{q}$. By the constant-variation formula, for all $n \geqslant n_1$, we obtain

$$\begin{aligned} y(n) &= \Phi(n,n_1)\widetilde{P}_{n_1-1}y(n_1) + \sum_{i=n_1}^{n-1}\Phi(n,i+1)\widetilde{P}_i\widetilde{G}_i^{-1}q_i + T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n \\ &= \sum_{i=n_1}^{n-1}\Phi(n,i+1)\widetilde{P}_i\frac{\Phi(i+1,n_1)}{\chi(i)}v \\ &= \sum_{i=n_1}^{n-1}\frac{\Phi(n,n_1)}{\chi(i)}v. \end{aligned}$$

Put $\Psi(n) = \sum_{i=n_1}^{n-1}\dfrac{1}{\chi(i)}$, we have

$$y(n) = \Phi(n,n_1)\Psi(n)v. \tag{4.4}$$

From (4.2)–(4.4), we obtain

$$\|y(n)\| = \|\Phi(n,n_1)\Psi(n)v\| = \|\Phi(n,n_1)v\|\Psi(n) \leqslant k\|\overline{q}\| \leqslant k\|v\|.$$

Therefore, we have $\|\Phi(n,n_1)\| \leqslant \dfrac{k}{\Psi(n)}$, which implies that

$$\frac{1}{\Psi(n)-\Psi(n-1)} \leqslant \frac{k}{\Psi(n)} \text{ or } \Psi(n)-\Psi(n-1) \geqslant \frac{\Psi(n)}{k}.$$

It follows that

$$\|\Phi(n,n_1)\| \leqslant \frac{k}{\Psi(n)} \leqslant \cdots \leqslant k\left(\frac{k-1}{k}\right)^{n-n_1-1}\frac{1}{\Psi(n_1+1)}.$$

Hence, we get

$$\|\Phi(n,n_1)\| \leqslant \frac{k^2}{k-1}\omega^{n-n_1}\|\Phi(n_1+1,n_1)\|, \tag{4.5}$$

where $0 < \omega = \dfrac{k-1}{k} < 1$.

On the other hand, for any $w \in \mathbb{R}^d$ choose $\overline{q} = \{\overline{q}_i\}_{i \geqslant n_0} \in LP(n_0)$ defined by $\overline{q}_{n_1-1} = \widetilde{P}_{n_1-1}w$ and $\overline{q}_i = 0$ for $i \in \mathbb{N}(n_0), i \neq n_1 - 1$, then for $n = n_1 + 1$, we have

$$\|z(n_1+1)\| = \|\sum_{i=n_0}^{n_1} \Phi(n_1+1, i+1)\overline{q}_i\| = \|\Phi(n_1+1, n_1)\widetilde{P}_{n_1-1}w\| \leqslant k\|\overline{q}\|,$$

which yields

$$\|\Phi(n_1+1, n_1)w\| \leqslant k\|\widetilde{P}_{n_1-1}w\| \leqslant k\rho\|w\|.$$

Thus, we obtain

$$\|\Phi(n_1+1, n_1)\| \leqslant k\rho. \tag{4.6}$$

From (4.5) and (4.6), the estimate

$$\|\Phi(n, n_1)\| \leqslant \frac{k^3}{k-1}\rho\omega^{n-n_1} \leqslant K\omega^{n-n_1},$$

where $K = \dfrac{k^3}{k-1}\rho$, holds for all $n \geqslant n_1 \geqslant n_0$. This clearly shows that (1.2) is exponentially stable.

Sufficiency To complete the proof, we will show that if (1.2) is exponentially stable then all the solutions of $CP(n_0)$ associated with $q = \{q_n\}_{n \geqslant n_0}$ in $L(n_0)$ are bounded.

Indeed, since (1.2) is exponentially stable, there exist constants $K > 0$ and $0 < \omega < 1$ such that

$$\|\Phi(n, m)\| \leqslant K\omega^{n-m}, \ \forall n \geqslant m \geqslant n_0.$$

Let $y(\cdot)$ be the solution of $CP(n_0)$ associated with some $q = \{q_n\}_{n \geqslant n_0}$ in $L(n_0)$. According to the constant-variation formula (2.11), for any $n \in \mathbb{N}(n_0)$ we have

$$y(n) = \sum_{i=n_0}^{n-1} \Phi(n, i+1)\widetilde{P}_i\widetilde{G}_i^{-1}q_i + T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n.$$

Hence, it follows that

$$\begin{aligned}
\|y(n)\| &\leqslant \sum_{i=n_0}^{n-1} \|\Phi(n, i+1)\|\|\widetilde{P}_i\widetilde{G}_i^{-1}q_i\| + \|T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n\| \\
&\leqslant CK\sum_{i=n_0}^{n-1} \omega^{n-i-1} + C \\
&\leqslant \frac{CK}{1-\omega} + C,
\end{aligned}$$

where $C = \max\{\sup_{n \in \mathbb{N}(n_0)} \|T_n\widetilde{Q}_n\widetilde{G}_n^{-1}q_n\|, \sup_{n \in \mathbb{N}(n_0)} \|\widetilde{P}_n\widetilde{G}_n^{-1}q_n\|\}$. That is, the solution $y(\cdot)$ is bounded.

The proof is complete. □

Remark 4.3: In [15,17], similar Bohl-Perron type theorems are obtained for continuous-time linear singular systems, i.e. linear DAEs of the form (1.3). We would like to comment on a comparison of Theorem 4.2 with Theorem 5.5 in [15] and Theorem 4.12 in [17]. In [17], the projector-based approach is used for the treatment of index-1 DAEs, too. However, the assumptions given there on the auxiliary matrices are more restrictive than those in this paper. In [15], the author proves a Bohl-Perron theorem for general linear DAEs, but the existence of the Cauchy operator and of the projector associated with the solution space is assumed in advance. Moreover, the family of inhomogeneity that is considered in [15] is supposed to be of highly structured. In this paper, thanks to the index-1 property, we are able to give a detailed construction of the Cauchy operator as well as to consider inhomogeneity of very general form.

5. Bohl exponents and exponential stability

5.1. Bohl exponents and their basic properties

The aim of this section is to extend the notion of Bohl exponent introduced by Bohl, see [20,24,26], to the case of linear implicit difference equations of index-1. Bohl exponents for DAEs have been investigated recently in [15,17,18].

Definition 5.1: Let $x = \{x(n)\}, n \in \mathbb{N}(n_0)$, be a nonvanishing sequence in $\mathbb{K}^d$. The (upper) Bohl exponent of x, denoted by $\kappa_B[x]$, is defined by

$$\kappa_B[x] = \inf\{\omega \in \mathbb{R}; \exists M_\omega : \|x(n)\| \leqslant M_\omega \omega^{n-m} \|x(m)\|, \ \forall n \geqslant m \geqslant n_0\}.$$

Thus, the Bohl exponent characterizes the uniform exponential growth rate of x.

Definition 5.2: Let the LSDE (1.2) have index 1, $\Phi(n, m)$ be the Cauchy operator of (1.2). Then, the (upper) Bohl exponent (also called the upper general exponent) of LSDE (1.2) is given by

$$\kappa_B(E, A) = \inf\{\omega \in \mathbb{R}; \exists M_\omega : \|\Phi(n, m)\| \leqslant M_\omega \omega^{n-m}, \forall n \geqslant m \geqslant n_0\}.$$

Proposition 5.3: *Let the LSDE (1.2) have index-1, $\Phi(n, m)$ be the Cauchy operator of (1.2). If $\kappa_B(E, A)$ is finite then*

$$\kappa_B(E, A) = \overline{\lim_{\substack{m\to\infty \\ n-m\to\infty}}} \|\Phi(n, m)\|^{\frac{1}{n-m}}.$$

Proof: Since $\kappa_B(E, A)$ is finite, let us denote $\kappa_1 = \kappa_B(E, A)$ and

$$\kappa_2 = \overline{\lim_{\substack{m\to\infty \\ n-m\to\infty}}} \|\Phi(n, m)\|^{\frac{1}{n-m}}.$$

We will prove that $\kappa_1 \leqslant \kappa_2$ and $\kappa_2 \leqslant \kappa_1$.

First, we will prove that $\kappa_2 \leqslant \kappa_1$. By the definition of infimum, for all $\varepsilon > 0$ there exists $M > 1$ such that

$$\|\Phi(n, m)\| \leqslant M(\kappa_1 + \varepsilon)^{n-m}, \ \forall m, n \in \mathbb{N}(n_0), n \geqslant m.$$

Hence, we have

$$\overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\Phi(n,m)\|^{\frac{1}{n-m}} \leqslant \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} M^{\frac{1}{n-m}}(\kappa_1+\varepsilon) = \kappa_1+\varepsilon,$$

which implies that

$$\kappa_2 = \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\Phi(n,m)\|^{\frac{1}{n-m}} \leqslant \kappa_1.$$

Now we prove the converse inequality $\kappa_1 \leqslant \kappa_2$. Using the definition of limit superior, for any $\varepsilon > 0$ there exist integers N, T large enough such that

$$\|\Phi(n,m)\|^{\frac{1}{n-m}} \leqslant \kappa_2+\varepsilon,\ \forall m,n \in \mathbb{N}(n_0), m \geqslant N \text{ and } n-m \geqslant T,$$

equivalently

$$\|\Phi(n,m)\| \leqslant (\kappa_2+\varepsilon)^{n-m},\ \forall m,n \in \mathbb{N}(n_0), m \geqslant N \text{ and } n-m \geqslant T. \tag{5.1}$$

We need to prove that there exists $M > 0$ such that

$$\|\Phi(n,m)\| \leqslant M(\kappa_2+\varepsilon)^{n-m},\ \forall m,n \in \mathbb{N}(n_0), n \geqslant m.$$

Note that κ_1 is finite, so there exist positive numbers ω and M_ω such that

$$\|\Phi(n,m)\| \leqslant M_\omega \omega^{n-m},\ \forall m,n \in \mathbb{N}(n_0), n \geqslant m. \tag{5.2}$$

We consider the following cases:

(i) Case 1: Suppose that $n-m \geqslant T$ and $m \geqslant N$. Due to (5.1), we have

$$\|\Phi(n,m)\| \leqslant (\kappa_2+\varepsilon)^{n-m}.$$

(ii) Case 2: Suppose that $n-m < T$. It implies from (5.2) that

$$\begin{aligned}\|\Phi(n,m)\| \leqslant M_\omega\omega^{n-m} &= \frac{M_\omega\omega^{n-m}}{(\kappa_2+\varepsilon)^{n-m}}(\kappa_2+\varepsilon)^{n-m}\\ &\leqslant M_1(\kappa_2+\varepsilon)^{n-m},\end{aligned}$$

where $M_1 = \max\left\{M_\omega, M_\omega\left(\frac{\omega}{\kappa_2+\varepsilon}\right)^T\right\}$.

(iii) Case 3: Suppose that $n-m \geqslant T$, $m < N$ and $n < N+T$. From (5.2), we obtain

$$\begin{aligned}\|\Phi(n,m)\| \leqslant M_\omega\omega^{n-m} &= \frac{M_\omega\omega^{n-m}}{(\kappa_2+\varepsilon)^{n-m}}(\kappa_2+\varepsilon)^{n-m}\\ &\leqslant M_2(\kappa_2+\varepsilon)^{n-m},\end{aligned}$$

where $M_2 = \max\left\{M_\omega, M_\omega\left(\frac{\omega}{\kappa_2+\varepsilon}\right)^{N+T}\right\}$.

(iv) Case 4: Suppose that $n-m \geqslant T$, $m < N$ and $n \geqslant N+T$. Since $n-N \geqslant T$, applying (5.1), we get

$$\|\Phi(n,N)\| \leqslant (\kappa_2+\varepsilon)^{n-N}.$$

Applying (5.2) with $N \geqslant m$, it follows that

$$\|\Phi(N,m)\| \leqslant M_\omega \omega^{N-m}.$$

Therefore,

$$\begin{aligned}\|\Phi(n,m)\| \leqslant \|\Phi(n,N)\|\|\Phi(N,m)\| &\leqslant (\kappa_2+\varepsilon)^{n-N} M_\omega \omega^{N-m}\\ &= (\kappa_2+\varepsilon)^{n-m} M_\omega \left(\frac{\omega}{\kappa_2+\varepsilon}\right)^{N-m}.\end{aligned}$$

Hence, we obtain

$$\|\Phi(n,m)\| \leqslant M_3(\kappa_2+\varepsilon)^{n-m},$$

where $M_3 = \max\left\{M_\omega, M_\omega\left(\frac{\omega}{\kappa_2+\varepsilon}\right)^N\right\}$.

Now, choose $M = \max\{1, M_1, M_2, M_3, \}$, we have

$$\|\Phi(n,m)\| \leqslant M(\kappa_2+\varepsilon)^{n-m}, \ \forall m,n \in \mathbb{N}(n_0), n \geqslant m.$$

It implies that $\kappa_1 \leqslant \kappa_2+\varepsilon, \ \forall \varepsilon > 0$. So $\kappa_1 \leqslant \kappa_2$. The proof is complete. □

Remark 5.4: If the function $\widetilde{P}_n \widetilde{G}_n^{-1} A_n$ is bounded, then the Bohl exponent of LSDE (1.2) is finite. Moreover, since we suppose that $\|\widetilde{P}_n\|$ is bounded, we have

$$\begin{aligned}\kappa_B(E,A) = \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\Phi(n,m)\|^{\frac{1}{n-m}} &= \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\widetilde{P}_{n-1}\Phi_0(n,m)\|^{\frac{1}{n-m}}\\ &\leqslant \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\Phi_0(n,m)\|^{\frac{1}{n-m}} \rho^{\frac{1}{n-m}}\\ &= \overline{\lim_{\substack{m\to\infty\\ n-m\to\infty}}} \|\Phi_0(n,m)\|^{\frac{1}{n-m}}.\end{aligned}$$

Definition 5.5: The Bohl exponent of LSDE (1.2) is said to be strict if and only if it is finite and

$$\kappa_B(E,A) = \lim_{\substack{m\to\infty\\ n-m\to\infty}} \|\Phi(n,m)\|^{\frac{1}{n-m}}.$$

Remark 5.6: Let μ be a nonzero constant. Suppose that $\kappa_B(E,A)$ is finite, then $\kappa_B(E,\mu A)$ is finite and $\kappa_B(E,\mu A) = |\mu|\kappa_B(E,A)$.

Proof: Consider the new system with E and μA. We define $\widetilde{G}'_n = E_n - \mu A_n T_n \widetilde{Q}_n = E_n - A_n T'_n \widetilde{Q}_n$, where $T'_n = \mu T_n$. Thus, it is easy to see that $\widetilde{G}'_n$ is nonsingular and $\widetilde{P}_n \widetilde{G}_n'^{-1} = \widetilde{P}_n \widetilde{G}_n^{-1}$. Hence, we have $\kappa_B(E,\mu A) = \overline{\lim}_{\substack{m\to\infty\\ n-m\to\infty}} \|\mu^{n-m}\Phi(n,m)\|^{\frac{1}{n-m}} = |\mu|\kappa_B(E,A)$. □

Returning to LSDE (1.2), by introducing the variable change $x(n) = U_n z(n)$ and scaling (1.2) by W_n, where $U_n, W_n \in \mathbb{R}^{d\times d}$ are nonsingular for all $n \in \mathbb{N}(n_0)$, we arrive at a new equation

$$\hat{E}_n z(n+1) = \hat{A}_n z(n), \tag{5.3}$$

where $\hat{E}_n = W_n E_n U_{n+1}, \quad \hat{A}_n = W_n A_n U_n$. It is not difficult to prove that

$$\hat{Q}_n = U_{n+1}^{-1} Q_n U_{n+1}, \quad \hat{P}_n = U_{n+1}^{-1} P_n U_{n+1}, \quad \hat{T}_n = U_n^{-1} T_n U_{n+1}$$
$$\hat{G}_n = W_n G_n U_{n+1}, \quad \widetilde{\hat{P}}_n = U_{n+1}^{-1} \widetilde{P}_n U_{n+1}.$$

Therefore, the Cauchy operator of (5.3) satisfies

$$\hat{\Phi}(n,m) = U_n^{-1} \Phi(n,m) U_m, \ n \geqslant m \geqslant n_0.$$

Lemma 5.7: *If ν is a real number such that there exists M_ν and the inequality $\|U_n^{-1}\|\|U_m\| \leqslant M_\nu \nu^{|n-m|}, \forall n,m \in \mathbb{N}(n_0)$ holds, then $\nu \geqslant 1$.*

Proof: Suppose the contrary statement that $\nu < 1$. We have

$$\|U_n^{-1}\|\|U_{n_0}\| \leqslant M_\nu \nu^{|n-n_0|} \text{ and } \|U_{n_0}^{-1}\|\|U_n\| \leqslant M_\nu \nu^{|n-n_0|}, \forall n \geqslant n_0.$$

Hence,

$$\|U_n^{-1}\| \leqslant \frac{M_\nu}{\|U_{n_0}\|} \nu^{|n-n_0|} \text{ and } \|U_n\| \leqslant \frac{M_\nu}{\|U_{n_0}^{-1}\|} \nu^{|n-n_0|}, \forall n \geqslant n_0.$$

Letting $n \to \infty$, we have $\|U_n^{-1}\| \to 0$ and $\|U_n\| \to 0$. But $\|U_n^{-1}\|\|U_n\| \geqslant \|U_n^{-1} U_n\| = 1$. This is a contradiction. Therefore, the value of ν cannot be less than 1, i.e. we have proved that $\nu \geqslant 1$. □

Definition 5.8: The transformation with a matrix function $U : \mathbb{N}(n_0) \longrightarrow \mathbb{R}^{d\times d}$ is said to be a Bohl transformation if

$$\inf\{\nu \in \mathbb{R}; \exists M_\nu : \|U_n^{-1}\|\|U_m\| \leqslant M_\nu \nu^{|n-m|}, \forall n,m \in \mathbb{N}(n_0)\} = 1.$$

Trivially, if both U and its inverse U^{-1} are bounded, then the transformation is a Bohl transformation. Clearly, the exponential stability of (1.2) is preserved under a Bohl transformation.

Proposition 5.9:

(i) *The set of Bohl transformations forms a group with respect to pointwise multiplication.*
(ii) *The Bohl exponent is invariant with respect to Bohl transformations.*

Proof:

(i) Firstly, we prove that the set of Bohl transformations is closed with pointwise multiplication. Indeed, suppose that $U, V : \mathbb{N}(n_0) \longrightarrow \mathbb{R}^{d\times d}$ are Bohl transformation. We define $(U \circ V)_n = U_n V_n, \ n \in \mathbb{N}(n_0)$. According to Bohl transformation properties, we have

$$\forall \varepsilon' > 0 \quad \exists M_1, M_2 : \|U_n^{-1}\|\|U_m\| \leqslant M_1(1+\varepsilon')^{|n-m|}$$

$$\text{and } \|V_n^{-1}\|\|V_m\| \leqslant M_2(1+\varepsilon')^{|n-m|}, \ \forall m,n \in \mathbb{N}(n_0).$$

Hence, $\forall m,n \in \mathbb{N}(n_0)$ we obtain

$$\begin{aligned}\|(U_nV_n)^{-1}\|\|U_mV_m\| &\leqslant (\|U_n^{-1}\|\|U_m\|)(\|V_n^{-1}\|\|V_m\|)\\ &\leqslant M_1M_2[(1+\varepsilon')^2]^{|n-m|}.\end{aligned}$$

For every $\varepsilon > 0$, choose ε' small enough such that $(1+\varepsilon')^2 < 1+\varepsilon$. Choose $M = M_1M_2$

$$\|(U_nV_n)^{-1}\|\|U_mV_m\| \leqslant M(1+\varepsilon)^{|n-m|}.$$

Thus, we get

$$\inf\{\alpha \in \mathbb{R};\, \exists M_\alpha : \|(U_nV_n)^{-1}\|\|(U_mV_m)\| \leqslant M_\alpha \alpha^{|n-m|},\ \forall n,m \in \mathbb{N}(n_0)\} = 1.$$

Therefore $U \circ V$ is a Bohl transformation, too.

It is easy to see that the identity element is I.

Let $U : \mathbb{N}(n_0) \longrightarrow \mathbb{R}^{d\times d}$ is a Bohl transformation, the inversion of U is $\overline{U}$ defined by $\overline{U}_n = U_n^{-1}, \forall n \in \mathbb{N}(n_0)$. It is clear that $\overline{U}$ is a Bohl transformation, too.

We have proved that the set of Bohl transformations forms a group.

(ii) Let U be a Bohl transformation, the relation between the Cauchy operators of the new equation and the original equation is

$$\hat{\Phi}(n,m) = U_n^{-1}\Phi(n,m)U_m.$$

From the definition of Bohl transformation, it is not difficult to show that $\kappa_B(E,A) = \kappa_B(\hat{E},\hat{A})$. □

5.2. Robustness of Bohl exponents

The following proposition show the relation between exponential stability and Bohl exponent of LSDE (1.2).

Proposition 5.10: *The LSDE (1.2) is exponentially stable if and only if*

$$\kappa_B(E,A) < 1.$$

Proof: The proof is divided into two parts.

Necessity Firstly, we prove that if LSDE (1.2) is exponentially stable then $\kappa_B(E,A) < 1$. Since LSDE (1.2) is exponentially stable, there exist positive constants $\omega < 1$ and $M > 0$ such that

$$\|\Phi(n,m)\| \leqslant M\omega^{n-m},\ \forall n \geqslant m, n,m \in \mathbb{N}(n_0).$$

It implies that $\kappa_B(E,A) \leqslant \omega < 1$.

Sufficiency To finish the proof, we need to prove that if $\kappa_B(E,A) < 1$ then LSDE (1.2) is exponentially stable. From $\kappa_B(E,A) < 1$, there exist $\varepsilon > 0$ and $M > 0$ such that $\omega = \kappa_B(E,A) + \varepsilon < 1$ and $\|\Phi(n,m)\| \leqslant M\omega^{n-m}$, $\forall n \geqslant m, n,m \in \mathbb{N}(n_0)$. Therefore, LSDE (1.2) is exponentially stable. □

Now, we will investigate how sensitive the Bohl exponent of a system is to perturbations in the coefficients. This is the problem of the robustness (or the continuity) of Bohl exponents. The following auxiliary lemma will be useful for the later estimation.

Lemma 5.11: *Let α_i be nonnegative numbers, $i = \overline{1,n}$ and $\beta \geqslant 0$. Suppose that*

$$\sqrt[n]{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} \leqslant 1+\gamma$$

for some positive γ. If $\beta \geqslant 1$, then the inequality

$$\sqrt[n]{(1+\beta\alpha_1)(1+\beta\alpha_2)\cdots(1+\beta\alpha_n)} \leqslant 1+\beta\gamma \tag{5.4}$$

holds. Otherwise, we have

$$\sqrt[n]{(1+\beta\alpha_1)(1+\beta\alpha_2)\cdots(1+\beta\alpha_n)} \leqslant 1+\gamma. \tag{5.5}$$

Proof: The inequality (5.5) is trivial in the case of $\beta < 1$. For the case of $\beta \geqslant 1$, we only need to prove that if

$$\sqrt[n]{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} = 1+\gamma \tag{5.6}$$

then (5.4) holds.

From (5.6), we have $\gamma = \sqrt[n]{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} - 1$. Substituting this formula into (5.4), we need to prove that

$$\sqrt[n]{(1+\beta\alpha_1)(1+\beta\alpha_2)\cdots(1+\beta\alpha_n)} \leqslant 1+\beta(\sqrt[n]{(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_n)} - 1),$$

which is rewritten as

$$\sqrt[n]{(1+\beta\alpha_1)\cdots(1+\beta\alpha_n)} + \sqrt[n]{(\beta-1)\cdots(\beta-1)} \leqslant \sqrt[n]{(\beta+\beta\alpha_1)\cdots(\beta+\beta\alpha_n)}.$$

The latter is equivalent to

$$\sqrt[n]{\frac{1+\beta\alpha_1}{\beta+\beta\alpha_1}\cdots\frac{1+\beta\alpha_n}{\beta+\beta\alpha_n}} + \sqrt[n]{\frac{\beta-1}{\beta+\beta\alpha_1}\cdots\frac{\beta-1}{\beta+\beta\alpha_n}} \leqslant 1. \tag{5.7}$$

Applying the AM-GM inequality to n nonnegative numbers $\frac{1+\beta\alpha_1}{\beta+\beta\alpha_1}, \cdots, \frac{1+\beta\alpha_n}{\beta+\beta\alpha_n}$, and to n nonnegative numbers $\frac{\beta-1}{\beta+\beta\alpha_1}, \cdots, \frac{\beta-1}{\beta+\beta\alpha_n}$, we obtain

$$\sqrt[n]{\frac{1+\beta\alpha_1}{\beta+\beta\alpha_1}\cdots\frac{1+\beta\alpha_n}{\beta+\beta\alpha_n}} \leqslant \frac{1}{n}\sum_{i=1}^{n}\frac{1+\beta\alpha_i}{\beta+\beta\alpha_i}$$

and

$$\sqrt[n]{\frac{\beta-1}{\beta+\beta\alpha_1}\cdots\frac{\beta-1}{\beta+\beta\alpha_n}} \leqslant \frac{1}{n}\sum_{i=1}^{n}\frac{\beta-1}{\beta+\beta\alpha_i}.$$

Hence, we have

$$\sqrt[n]{\frac{1+\beta\alpha_1}{\beta+\beta\alpha_1}\cdots\frac{1+\beta\alpha_n}{\beta+\beta\alpha_n}} + \sqrt[n]{\frac{\beta-1}{\beta+\beta\alpha_1}\cdots\frac{\beta-1}{\beta+\beta\alpha_n}} \leqslant \sum_{i=1}^{n}(\frac{1+\beta\alpha_i}{\beta+\beta\alpha_i} + \frac{\beta-1}{\beta+\beta\alpha_i}) = 1.$$

Thus (5.7) is proved. Therefore, the inequality (5.4) holds true. □

Theorem 5.12: *Consider the perturbed Equation (3.4). Suppose that the unperturbed Equation (1.2) has a finite Bohl exponent $\kappa_B(E,A)$. Then for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that for all perturbations F_n and $\overline{B}_n$ which satisfy Asssumptions 5 and 1 and the following condition*

$$\varlimsup_{\substack{m\to\infty\\ n-m\to\infty}} \frac{1}{n-m}\sum_{i=m}^{n-1} \|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\| \leqslant \delta, \tag{5.8}$$

the estimate $\kappa_B(E+F, A+B) \leqslant \kappa_B(E,A) + \varepsilon$ holds.

Proof: Let $y(\cdot)$ be any solution of LSDE (3.4). Due to Assumptions 5 and 1, analogously to (3.3), we have

$$y(n) = (I - T_n\widetilde{Q}_n\widetilde{G}_n^{-1}\overline{B}_n)^{-1}\left(\Phi(n,m)\widetilde{P}_{m-1}y(m) + \sum_{i=m}^{n-1}\Phi(n,i+1)\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i y(i)\right),$$

$\forall n \geqslant m, n, m \in \mathbb{N}(n_0)$. By virtue of Definition 5.2, there exists constant $\overline{M}$ such that

$$\|\Phi(n,m)\widetilde{P}_{m-1}\| = \|\Phi(n,m)\| \leqslant \overline{M}\alpha^{n-m},\ \forall n \geqslant m, n, m \in \mathbb{N}(n_0),$$

where $\alpha = \kappa_B(E,A) + \dfrac{\varepsilon}{2}$. It follows that

$$\|y(n)\| \leqslant c_2\left(\overline{M}\alpha^{n-m}\|y(m)\| + \sum_{i=m}^{n-1}\overline{M}\alpha^{n-i-1}\|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\|\|y(i)\|\right). \tag{5.9}$$

Multiplying both sides of (5.9) by α^{-n}, we have

$$\alpha^{-n}\|y(n)\| \leqslant c_2\overline{M}\alpha^{-m}\|y(m)\| + \frac{c_2\overline{M}}{\alpha}\sum_{i=m}^{n-1}\alpha^{-i}\|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\|\|y(i)\|.$$

Applying Lemma 3.3, we obtain

$$\alpha^{-n}\|y(n)\| \leqslant c_2\overline{M}\alpha^{-m}\|y(m)\|\prod_{i=m}^{n-1}\left(1 + \frac{c_2\overline{M}}{\alpha}\|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\|\right).$$

Hence, we get

$$\|y(n)\| \leqslant c_2\overline{M}\alpha^{n-m}\|y(m)\|\prod_{i=m}^{n-1}\left(1 + \frac{c_2\overline{M}}{\alpha}\|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\|\right). \tag{5.10}$$

From (5.8) and Cauchy's inequality, there exist constants N and T such that

$$\left[\prod_{i=m}^{n-1}(1 + \|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\|)\right]^{\frac{1}{n-m}} \leqslant 1 + \delta,\ \forall m \geqslant N, n-m \geqslant T, m, n \in \mathbb{N}(n_0). \tag{5.11}$$

Two cases occur as follows.

(i) If $m \geqslant N$ and $n - m \geqslant T$ then from (5.10), (5.11) and applying Lemma 5.11, we obtain the estimate

$$\|y(n)\| \leqslant c_2\overline{M}\alpha^{n-m}\|y(m)\|(1+\beta\delta)^{n-m}.$$

That is, $\|y(n)\| \leqslant c_2\overline{M}(\alpha+\alpha\beta\delta)^{n-m}\|y(m)\|$, where $\beta = \max\{1, \frac{c_2\overline{M}}{\alpha}\}$.

(ii) If $m \geqslant N$ and $n - m < T$, then

$$\|y(n)\| \leqslant c_2\overline{M}\alpha^{n-m}\|y(m)\| \prod_{i=m}^{m+T-1} (1 + \frac{c_2\overline{M}}{\alpha}\|\widetilde{P}_i G_i^{-1}\overline{B}_i\|).$$

Applying (5.11) and Lemma 5.11, we have

$$\|y(n)\| \leqslant c_2\overline{M}\alpha^{n-m}\|y(m)\|(1+\beta\delta)^T \leqslant c_2\overline{M}(1+\beta\delta)^T(\alpha+\alpha\beta\delta)^{n-m}\|y(m)\|.$$

Therefore, $\forall\ n \geqslant m \geqslant N$, the estimate

$$\|y(n)\| \leqslant c_2\overline{M}(1+\beta\delta)^T(\alpha+\alpha\beta\delta)^{n-m}\|y(m)\|$$

holds. Since the estimates hold for any solution of LSDE (3.4), they imply

$$\kappa_B(E+F, A+B) \leqslant \alpha + \alpha\beta\delta = \kappa_B(E, A) + \frac{\varepsilon}{2} + \alpha\beta\delta.$$

Finally, it remains to choose $\delta = \frac{\varepsilon}{2\alpha\beta}$. The proof is complete. □

We remark that now Theorem 3.10 can also be obtained as a direct consequence of Proposition 5.10 and Theorem 5.12. We have a further consequence on the characterization of perturbation such that the Bohl exponents of the unperturbed and perturbed systems are equal.

Corollary 5.13: *Consider the perturbed Equation (3.4). Suppose that the unperturbed Equation (1.2) has index 1 and that its Bohl exponent $\kappa_B(E, A)$ is finite. Let the perturbations F_n and $\overline{B}_n$ satisfy Asssumptions 5 and 1.*

(i) *If*

$$\overline{\lim_{\substack{m\to\infty \\ n-m\to\infty}}} \frac{1}{n-m} \sum_{i=m}^{n-1} \|\widetilde{P}_i\widetilde{G}_i^{-1}\overline{B}_i\| = 0,$$

then $\kappa_B(E+F, A+B) = \kappa_B(E, A)$.

(ii) *In particular, if $\lim_{n\to\infty} \|\widetilde{P}_n\widetilde{G}_n^{-1}\overline{B}_n\| = 0$, then $\kappa_B(E+F, A+B) = \kappa_B(E, A)$.*

6. Conclusion

In this paper we have investigated the stability of linear singular systems of difference equations. To treat the singularity, we use the projector approach and decouple the system into difference and algebraic subsystems. We have characterized the stability of the system under perturbations. We have also established the relation between the exponential stability of the homogeneous system and the boundedness of solutions of nonhomogeneous systems. The notion of Bohl exponent, which is known to be useful in the robust stability analysis of linear nonautonomous systems, has been extended to linear singular systems and its properties have been presented. In this current work, we restrict the analysis to the case of bounded canonical projectors. The case of unbounded canonical projectors will be discussed in a separate work since many arguments used in this work must be modified. Furthermore, extensions of other variants of Bohl-Perron type theorems to LSDEs may also be interesting topics for future investigation.

Acknowledgements

The authors would like to thank the referees for their useful comments that led to the improvement of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The research of V.H. Linh and N.T.T. Nga was supported by *Vietnam National Foundation for Science and Technology Development (NAFOSTED) Project [101.02-2014.05]*.

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