On the stability analysis of sampled-data systems with delays

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Chapter 1
On the stability analysis of sampled-data systems with delays

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Abstract Controlling a system through a network amounts to solve certain difficulties such as, among others, the consideration of aperiodic sampling schemes and (time-varying) delays. In most of the existing works, delays have been involved in the input channel through which the system is controlled, thereby delaying in a continuous way the control input computed by the controller. We consider here a different setup where the delay acts in a way that the current control input depends on past state samples, possibly including the current one, which is equivalent to considering a discrete-time delay, at the sample level, in the feedback loop. An approach based on the combination of a discrete-time Lyapunov-Krasovskii functional and a looped-functional is proposed and used to obtain tailored stability conditions that explicitly consider the presence of delays and the aperiodic nature of the sampling events. The stability conditions are expressed in terms of linear matrix inequalities and the efficiency of the approach is illustrated on an academic example.

1.1 Introduction

Sampled-data systems are an important class of systems that have been extensively studied in the literature [9] as they arise, for instance, in digital control [23] and networked control systems [16, 38]. The aperiodic nature of the sampling schemes creates additional difficulties in the analysis and the control of such systems as those schemes are much less understood than their periodic counterpart. Several ap-

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proaches have been proposed in order to characterize the behavior of such systems. Those based on the discretization of the sampled-data system have been discussed, for instance, in [10,17,27,35]. In these works, the sampling-period-dependent matrices of the discrete-time system are embedded in a convex polytope and the analysis is carried out using standard robust analysis techniques. This approach leads to efficient and tractable conditions that can be easily used for control design. A limitation of the approach, however, is that it only applies to unperturbed linear time-invariant systems. A second approach is based on the so-called "input-delay approach" which consists of reformulating the original sampled-data system into a time-delay system subject to a sawtooth input-delay [11, 12, 20, 29]. This framework allows for the application of well-known analysis techniques developed for time-delay systems, such as those based on the Lyapunov-Krasovskii theorem. Its main advantage is its applicability to uncertain, time-varying and even nonlinear systems. A limitation, however, is the difficulty of designing controllers with such an approach. Robust analysis techniques based, for instance, on small-gain results [24], Integral Quadratic Constraints [14, 18, 19] or well-posedness theory [1] have also been successfully applied. Approaches based on impulsive systems using Lyapunov functionals [25] or clock-dependent Lyapunov functions [3] also exist. Notably, the latter approach is able to characterize the stability of periodic and aperiodic sampled-data systems subject to both time-invariant and time-varying uncertainties. Even more interestingly, convex robust stabilization conditions for sampled-data systems can also be easily obtained using this approach. In this regard, this framework combines the advantages of discrete-time and functional-based approaches. Finally, approaches based on looped-functionals have been proposed in [7, 8, 31] in order to obtain stability conditions for sampled-data and impulsive systems. This particular type of functional has the interesting property of relaxing the positivity requirement which is necessary in Lyapunov-based approaches. Instead of that, one demands the fulfillment of a "looping condition", a certain boundary condition that can be made structurally satisfied while constructing the functional. In this regard, this class of functionals is therefore more general than Lyapunov(-Krasovsksii) functionals as the looping-condition turns out to be a weaker condition than the positive definiteness condition; see e.g. [7, 8, 31].

We propose to derive here stability conditions for (uncertain) aperiodic sampled-data systems with discrete-time input delay. While the delayed sampled-data systems considered in [22, 30] are subject to a continuous-time delay (the delay is expressed in seconds), the systems we are interested in here involve a discrete-time delay (the delay is expressed in a number of samples). A solution to this problem, based on state augmentation, has been proposed in [32] for the constant delay case. This approach yields quite accurate results at the expense of a rather high computational cost, restricting then its application to small delay values. In order to remove this limitation, an alternative approach relying on the direct use of a blend of Lyapunov-Krasovskii and looped-functionals has been proposed in [33] in the case of constant time-delays. The objective of the current chapter is to extend these conditions to the case of time-varying delays. These conditions are expressed in terms of LMIs and illustrated on a simple example.

The chapter is organized as follows. Section 2 states the considered problem while Section 3 presents several preliminary results on looped-functionals. The main results of the chapter are proved in Section 4 where the cases of constant and time-varying discrete delays are considered. An illustrative example and a discussion of the results are finally treated in Section 5.

Notations: Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $|\cdot|$ and $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The sets $\mathbb{S}_n$ and $\mathbb{S}_n^+$ represent the set of symmetric and symmetric positive definite matrices of dimension n, respectively. Moreover, for two matrices $A, B \in \mathbb{S}_n$, the inequality $A \prec B$ means that $A - B$ is negative definite. In symmetric matrices, the $*$'s are a shorthand for symmetric terms. For any square matrix $A \in \mathbb{R}^{n\times n}$, we also define $\mathrm{Sym}(A) := A + A^T$. Finally, I represents the identity matrix of appropriate dimension while 0 stands for the zero-matrix.

1.2 Problem formulation

1.2.1 System definition

Let us consider here linear continuous-time systems of the form

$$\begin{array}{l}\dot{x}(t) = Ax(t) + Bu(t),\ t \geq 0,\\ x(0) = x_0,\end{array} \tag{1.1}$$

where $x, x_0 \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state of the system, the initial condition and the control input, respectively. Above, the matrices A and B are not necessarily perfectly known but may be uncertain and/or time-varying. The control input u is assumed to be given by the following equation

$$u(t) = Kx(t_{k-h}),\ t \in [t_k, t_{k+1}),\ k \in \mathbb{N}, \tag{1.2}$$

where $K \in \mathbb{R}^{m\times n}$ is a controller gain and the sequence $\{t_k\}_{k\in\mathbb{N}}$ is the sequence of sampling instants. It is assumed that this sequence is strictly increasing and does not admit any accumulation point, that is, we have that $t_k \to \infty$ as $k \to \infty$. We also make the additional assumption that the difference $T_k := t_{k+1} - t_k$ belongs, for all $k \in \mathbb{N}$, to the interval $[T_{min}, T_{max}]$ where $T_{min} \leq T_{max}$. The delay h will either be considered to be constant or bounded and time-varying. In the latter case, the delay will be denoted by h_k to emphasize its time-varying nature.

The closed-loop system obtained from the interconnection of (1.1) and (1.2) is given, for all k in $\mathbb{N}$, by

$$\begin{array}{l}\dot{x}(t) = Ax(t) + BKx(t_{k-h}),\ t \in [t_k, t_{k+1}),\\ x(\theta) = x_0,\ \theta \leq 0.\end{array} \tag{1.3}$$

The discretized version of the previous system is given by

$$x(t_{k+1}) = A_d(T_k)x(t_k) + B_d(T_k)x(t_{k-h}),\ k \geq 0, \tag{1.4}$$

where $A_d(T_k) = e^{AT_k}$ and $B_d(T_k) = \int_0^{T_k} e^{A\tau} BK \mathrm{d}\tau$.

When the sampling period is fixed and known, the stability of the system (1.4) can be established by either augmenting the model with past state values and using then a quadratic discrete-time Lyapunov function, or by using a discrete-time Lyapunov-Krasovskii functional directly on the delayed system. [13, 36]. When the sampling is aperiodic, however, discrete-time methods can still be used by embedding the uncertain matrices $A_d(T_k)$ and $B_d(T_k)$ into a polytope [10, 15, 17]. Unfortunately, this approach is only applicable when the matrices (A, B) of the system are constant and perfectly known. To overcome this limitation, several methods can be applied. The first one is the so-called input-delay approach [11, 12] and is based on the reformulation of the sampled state into a delayed state with sawtooth delay. The analysis is then carried out using, for instance, Lyapunov-Krasovskii functionals. The second one is based on the reformulation of a sampled-data system into an impulsive system. The stability of the underlying impulsive system can then be established out using Lyapunov functionals [25], looped-functionals [8, 31] or clock-dependent Lyapunov functions [3, 4, 6]. In this chapter, we will opt for an approach based on a combination of a looped-functional and a discrete-time Lyapunov-Krasovskii functional, and demonstrate its applicability. Note that an approach based on a looped-functional combined with a Lyapunov function has been considered in [32] together with a state-augmentation approach for the system. A major drawback is that the dimension of the augmented system is hn and, therefore, LMI-based methods will not scale very well with the delay size. The consideration of a Lyapunov-Krasovskii functional in the current paper aims at overcoming such a difficulty by working directly on the original system.

Remark 1. It is worth mentioning that the class of systems considered in this paper differs from the class of systems described by

$$\begin{array}{l} \dot{x}(t) = Ax(t) + BKx(t_k - \tau),\ t \in [t_k, t_{k+1}), \\ x(\theta) = \phi(\theta),\ \theta \in [-\tau,\ 0]. \end{array} \tag{1.5}$$

where τ in a positive scalar. Such systems have been extensively studied in the literature; see e.g. [22, 26, 30]. Note, however, that when the sampling-period T is constant and the delay τ satisfies $\tau = hT$, then the two classes of systems coincide with each other. In this regard, none of these classes is included in the other meaning, therefore, that distinct methods need to be developed for each class.

1.3 Preliminaries

1.3.1 An appropriate modeling using lifting

The looped-functional approach relies on the characterization of the trajectories of system (1.3) in a lifted domain [8,37]. Therefore, we view the entire state-trajectory as a sequence of functions $\{x(t_k+\tau),\ \tau \in (0, T_k]\}_{k\in\mathbb{N}}$ with elements having a unique continuous extension to $[0, T_k]$ defined as

$$\chi_k(\tau) := x(t_k+\tau) \text{ with } \chi_k(0) = \lim_{s\downarrow t_k} x(s). \tag{1.6}$$

Finally we define $\mathbb{K}_{[T_{min}, T_{max}]}$ as the set defined by

$$\mathbb{K}_{[T_{min}, T_{max}]} := \bigcup_{T\in[T_{min},\ T_{max}]} \mathcal{C}([0,\ T], \mathbb{R}^n)$$

where $\mathcal{C}([0,\ T], \mathbb{R}^n)$ denotes the set of continuous functions mapping $[0, T]$ to $\mathbb{R}^n$. Using this notation, system (1.3) can be rewritten as

$$\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_{k-h}(0),\ \tau \in [0,\ T_k), \quad \forall k \in \mathbb{N}. \tag{1.7}$$

Looped-functionals consider this state definition for assessing stability in an efficient and flexible manner. Notably, the positivity requirement of the functional can be shown to be relaxed and the resulting stability condition can be generally written as a convex expression of the system data, see e.g. [7,8], allowing then for an easy application of these results to time-varying systems.

Up to now, looped-functionals have not been used to obtain stability conditions for sampled-data systems with discrete time-delay h. We, therefore, propose to extend the results initially proposed in [7,8,31] to this case. In what follows, we will denote by χ_k^h the function collecting the sampled and delayed values of the state, i.e.

$$\forall \theta = -h, -h+1, \ldots, 0, \quad \chi_k^h(\theta) = \chi_{k+\theta}(0) = x(t_{k+\theta}). \tag{1.8}$$

We, finally, define the set $\mathcal{D}_h$ as

$$\mathcal{D}_h = \{X\ :\ \{-h, \ldots, 0\} \to \mathbb{R}^n\}$$

which contains all possible sequences from $\{-h, \ldots, 0\}$ to $\mathbb{R}^n$.

1.3.2 Functional-based results

The following technical definition is necessary before stating the main general result about looped-functionals. result.

Definition 1 ([8]). Let $0 < T_{min} \leq T_{max} < +\infty$. A functional

$$f : [0, T_{max}] \times \mathbb{K}_{[T_{min}, T_{max}]} \times [T_{min}, T_{max}] \to \mathbb{R}$$

is said to be a **looped-functional** if the following conditions hold

1. the equality
$$f(0, z, T) = f(T, z, T) \tag{1.9}$$
holds for all functions $z \in C([0, T], \mathbb{R}^n) \subset \mathbb{K}_{[T_{min}, T_{max}]}$ and all $T \in [T_{min}, T_{max}]$, and
2. it is differentiable with respect to the first variable with the standard definition of the derivative.

The set of all such functionals is denoted by $\mathfrak{LF}([T_{min}, T_{max}])$.

The idea for proving stability of (1.3) is to look at a positive definite quadratic form $V(x)$ such that the sequence $\{V(\chi_k(T_k))\}_{k\in\mathbb{N}}$ is monotonically decreasing. This is formalized below through a functional existence result:

Theorem 1. *Let $T_{min} \leq T_{max}$ be two finite positive scalars and $V : \mathbb{R}^n \times \mathcal{D}_{h_{max}} \to \mathbb{R}_+$ be a form verifying*

$$X \in \mathcal{D}_{h_{max}}, \quad \mu_1 ||X||^2_{h_{max}} \leq V(X(0), X) \leq \mu_2 ||X||^2_{h_{max}}, \tag{1.10}$$

for some scalars $0 < \mu_1 \leq \mu_2$. Assume that one of the following equivalent statements hold:

(i) The sequence $\{V(\chi_k(T_k), \chi_k^h)\}_{k\in\mathbb{N}}$ is decreasing

(ii) There exists a looped-functional $\mathcal{V} \in \mathfrak{LF}([T_{min}, T_{max}])$ such that the functional $\mathcal{W}_k$ as

$$\mathcal{W}_k(\tau, \chi_k, \chi_k^h) := \frac{\tau}{T_k} \Lambda_k + V(\chi_k(\tau), \chi_k^h) + \mathcal{V}(\tau, \chi_k, T_k), \tag{1.11}$$

where $\Lambda_k = V(\chi_k(T_k), \chi_{k+1}^h) - V(\chi_k(T_k), \chi_k^h)$, has a derivative along the trajectories of system $\dot{\chi}_k(\tau) = A\chi_k(\tau) + BK\chi_{k-h(k)}(0)$, $\tau \in [0, T_k]$

$$\frac{d}{d\tau}\mathcal{W}_k(\tau, \chi_k, \chi_k^h) := \frac{1}{T_k}\Lambda_k + \frac{d}{d\tau}V(\chi_k(\tau), \chi_k^h) + \frac{d}{d\tau}\mathcal{V}(\tau, \chi_k, T_k), \tag{1.12}$$

which is negative definite for all $\tau \in (0, T_k)$, $T_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

Then, the solutions of system (1.3) are asymptotically stable for any sequence $\{t_k\}_{k\in\mathbb{N}}$ satisfying $t_{k+1} - t_k \in [T_{min}, T_{max}]$, $k \in \mathbb{N}$.

Proof. The proof is omitted but is similar to the proof provided in [8, 31].

In the remainder of the chapter, we will propose three stability conditions addressing the cases of constant and time-varying delay h for both certain and uncertain aperiodic sampled-data systems.

1.4 Main results

1.4.1 Stability analysis for constant delay h

This section provides a stability result for aperiodic sampled-data systems with a constant delay h. We have the following result:

Theorem 2. *The sampled-data system (1.7) with the delay h and $T_k := t_{k+1} - t_k \in [T_{min},\ T_{max}]$, $k \in \mathbb{N}$, is asymptotically stable if there exist matrices $R, Q, Z \in \mathbb{S}^n_+$, $P, X \in \mathbb{S}^{2n}$, $S \in \mathbb{S}^n$, $U \in \mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{4n\times n}$ such that the LMIs*

$$\begin{aligned}
\Phi_0 \quad &:= \begin{bmatrix} I \\ I \end{bmatrix} P \begin{bmatrix} I \\ I \end{bmatrix} \quad \succ 0, \\
\Phi_1(\theta) &:= \begin{bmatrix} F_0(\theta) & \theta Y \\ \star & -\theta Z \end{bmatrix} \prec 0, \\
\Phi_2(\theta) &:= F_0(\theta) + \theta F_1 \quad \prec 0,
\end{aligned} \tag{1.13}$$

hold for all $\theta \in \{T_{min}, T_{max}\}$ where

$$\begin{aligned}
F_0(\theta) &= F_{00} + F_{01} + \theta\,\mathrm{Sym}\left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^{\intercal} P \begin{bmatrix} M_0 \\ 0 \end{bmatrix}\right), \\
F_{00} &= M_\Delta^{\intercal} S M_\Delta + \theta M_T^{\intercal} X M_T + \mathrm{Sym}(M_\Delta^{\intercal} U M_T + M_\Delta Y), \\
F_{01} &= M_2^{\intercal} \Phi_0 M_2 - M_T^{\intercal} P M_T + M_3^{\intercal} Q M_3 - M_4^{\intercal} Q M_4 + h^2 M_\delta^{\intercal} R M_\delta - M_h^{\intercal} R M_h, \\
F_1 &= M_0^{\intercal} Z M_0 + \mathrm{Sym}(M_0^{\intercal}(S M_\Delta + U M_T)) - 2 M_T^{\intercal} X M_T,
\end{aligned} \tag{1.14}$$

with

$$\begin{array}{lll}
M_0 = \begin{bmatrix} A\ 0\ 0\ BK \end{bmatrix}, & M_1 = \begin{bmatrix} I\ 0\ 0\ 0 \end{bmatrix}, & M_2 = \begin{bmatrix} 0\ I\ \ 0\ \ 0 \end{bmatrix}, \\
M_3 = \begin{bmatrix} 0\ 0\ I\ 0 \end{bmatrix}, & M_4 = \begin{bmatrix} 0\ 0\ \ 0\ \ I \end{bmatrix}, & M_\Delta = \begin{bmatrix} I\ -I\ 0\ 0 \end{bmatrix}, \\
M_\delta = \begin{bmatrix} 0\ 0\ I\ -I \end{bmatrix}, & M_h = \begin{bmatrix} 0\ 0\ I\ -I \end{bmatrix}, & M_T = \begin{bmatrix} M_2^{\intercal}\ M_3^{\intercal} \end{bmatrix}^{\intercal}.
\end{array}$$

Proof. Consider a Lyapunov function for the discrete-time system (1.4) given by

$$\begin{aligned}
V(\chi_k(\tau), \chi_k^h) = \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^{\intercal} P \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix} + \sum_{i=k-h}^{k-1} \chi_i^{\intercal}(0) Q \chi_i(0) \\
+ h \sum_{i=-h}^{-1} \sum_{j=k+i}^{k-1} \delta_i^{\intercal}(0) R \delta_i(0),
\end{aligned} \tag{1.15}$$

where $\delta_i(0) = \chi_{i+1}(0) - \chi_i(0)$. On the other hand, we define the functional $\mathcal{V}$ as follows

$$
\begin{aligned}
T_k\mathcal{V}(\tau,\chi_k,T_k) = & \tau(T_k-\tau)\begin{bmatrix}\chi_k(T_k)\\ \chi_k(0)\end{bmatrix}^{\intercal} X \begin{bmatrix}\chi_k(T_k)\\ \chi_k(0)\end{bmatrix} \\
& +\tau(\chi_k(\tau)-\chi_k(T_k))^{\intercal} S(\chi_k(\tau)-\chi_k(T_k)) \\
& +2\tau(\chi_k(\tau)-\chi_k(T_k))^{\intercal} U \begin{bmatrix}\chi_k(T_k)\\ \chi_k(0)\end{bmatrix} \\
& -\tau\int_{\tau}^{T_k}\dot{\chi}_k^{\intercal}(s)Z\dot{\chi}_k(s)ds,
\end{aligned}
\tag{1.16}
$$

where the matrices above are such that $Z \in \mathbb{S}_+^n$, $S \in \mathbb{S}^n$, $X \in \mathbb{S}^{2n}$ and $U \in \mathbb{R}^{n\times 2n}$. This functional has been build in order to satisfy the looped condition. Indeed, one can easily verify that

$$
\mathcal{V}(0,\chi_k,T_k) = \mathcal{V}(T_k,\chi_k,T_k) = 0,
$$

for all $T_k \in [T_{min},\, T_{max}]$. As already highlighted in [31] and [8], the consideration of looped-functionals allows to enlarge the set of acceptable functionals in comparison to Lyapunov-Krasovskii functionals. Firstly, the matrices S, X and U are sign-indefinite in the current setting while they would have been required to be positive definite in usual Lyapunov approaches such as the one in [11, 25]. Secondly, the proposed functional includes more components than it is usually proposed in the literature (see for instance [11, 25, 31]). Indeed, looped-functionals allow one to include terms like $\chi_k(T_k)$ which would have been difficult to consider in the Lyapunov-Krasovskii framework. Following Theorem 1, let us consider

$$
\dot{\mathcal{W}}_k(\tau,\chi_k,\chi_k^h) = \frac{1}{T_k}\left(\Lambda_k + T_k\dot{V}(\chi_k(\tau),\chi_k^h) + T_k\dot{\mathcal{V}}(\tau,\chi_k,T_k)\right), \tag{1.17}
$$

where Λ_k is defined in Theorem 1. By virtue of the same theorem, the asymptotic stability of the system (1.7) is then proved if $V(\chi_k(0),\chi_k^h)$ is positive definite and $\dot{\mathcal{W}}_k$ is negative definite. Note that the necessity is lost by choosing specific forms for the functionals (1.15)-(1.16). Regarding the first condition, we have that

$$
\begin{aligned}
T_k V(\chi_k(0),\chi_k^h) = & \chi_k^{\intercal}(0)\Phi_0\chi_k(0) + \sum_{i=k-h}^{k-1}\chi_i^{\intercal}(0)Q\chi_i(0) \\
& +h\sum_{i=-h}^{-1}\sum_{j=k+i}^{k-1}\delta_i^{\intercal}(0)R\delta_i(0)
\end{aligned}
$$

which is positive definite provided that the matrices Φ_0, Q and R are positive definite as well. Let us focus now on the condition on $\mathcal{W}_k$ for which we will provide an upper-bound expressed in terms of the augmented vector

$$
\xi_k(\tau) := \mathrm{col}(\chi_k(\tau),\chi_k(T_k),\chi_k(0),\chi_{k-h}(0)).
$$

By virtue of the above definition, Λ_k can be rewritten as

$$
\begin{aligned}
\Lambda_k = & \begin{bmatrix} \chi_{k+1}(0) \\ \chi_{k+1}(0) \end{bmatrix}^{\intercal} P \begin{bmatrix} \chi_{k+1}(0) \\ \chi_{k+1}(0) \end{bmatrix} - \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^{\intercal} P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\
& + \sum_{i=k-h+1}^{k} \chi_i^{\intercal}(0) Q \chi_i(0) - \sum_{i=k-h}^{k-1} \chi_i^{\intercal}(0) Q \chi_i(0) \\
& + h \sum_{i=-h}^{-1} \left(\sum_{j=k+i+1}^{k} \delta_i^{\intercal}(0) R \delta_i(0) - \sum_{j=k+i}^{k-1} \delta_i^{\intercal}(0) R \delta_i(0) \right).
\end{aligned}
$$

Since $\chi_{k+1}(0) = \chi_k(T_k)$, the previous expression can be easily reexpressed in terms of the augmented vector $\xi_k(\tau)$. Applying then Jensen's inequality yields

$$
\begin{aligned}
\Lambda_k \leq & \, \xi_k^{\intercal}(\tau) \left(M_2^{\intercal} \Phi_0 M_2 - \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}^{\intercal} P \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} + M_3^{\intercal} Q M_3 - M_4^{\intercal} Q M_4 \right. \\
& \left. + h^2 M_\delta^{\intercal} R M_\delta - M_h^{\intercal} R M_h \right) \xi_k(\tau) \\
= & \, \xi_k^{\intercal}(\tau) F_{01} \xi_k(\tau).
\end{aligned} \tag{1.18}
$$

Let us focus now on the second term of $\dot{\mathcal{W}}$, as defined in (1.17), given by

$$
\begin{aligned}
T_k \dot{V}(\chi_k(\tau), X) &= 2T_k \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^{\intercal} P \begin{bmatrix} \dot{\chi}_k(\tau) \\ 0 \end{bmatrix} \\
&= T_k \xi_k^{\intercal}(\tau) \, \mathrm{Sym} \left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^{\intercal} P \begin{bmatrix} M_0 \\ 0 \end{bmatrix} \right) \xi_k(\tau).
\end{aligned} \tag{1.19}
$$

Finally, the last term of $\dot{\mathcal{W}}$, as defined in (1.17), is given by

$$
\begin{aligned}
T_k \dot{\mathcal{V}}(\tau, \chi_k, T_k) = & (T_k - 2\tau) \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^{\intercal} X \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\
& + (\chi_k(\tau) - \chi_k(T_k))^{\intercal} S (\chi_k(\tau) - \chi_k(T_k)) + 2\tau \dot{\chi}_k^{\intercal}(\tau) S (\chi_k(\tau) - \chi_k(T_k)) \\
& + 2(\chi_k(\tau) - \chi_k(T_k))^{\intercal} U \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} + 2\tau \dot{\chi}_k^{\intercal}(\tau) U \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix} \\
& + \tau \dot{\chi}_k^{\intercal}(\tau) Z \dot{\chi}_k(\tau) - \int_{\tau}^{T_k} \dot{\chi}_k^{\intercal}(s) Z \dot{\chi}_k(s) ds.
\end{aligned} \tag{1.20}
$$

We can rewrite the above expression in terms of the matrices F_{00}, F_1, F_2 defined in Theorem 2 to get

$$
T_k \dot{\mathcal{V}}(\chi_k, \tau) = \xi_k^{\intercal}(\tau) \left[F_{00} + \tau F_1 - \mathrm{Sym}(Y M_\Delta) \right] \xi_k(\tau) - \int_{\tau}^{T_k} \dot{\chi}_k^{\intercal}(s) Z \dot{\chi}_k(s) ds. \tag{1.21}
$$

In order to find a convenient upper-bound on the last integral term, we propose to consider the affine version of Jensen's inequality, discussed in [2], to get that

$$
-\int_{\tau}^{T_k} \dot{\chi}_k^{\intercal}(s) Z \dot{\chi}_k(s) ds \leq \xi_k^{\intercal}(\tau) \left[\mathrm{Sym}(Y M_\Delta) + (T_k - \tau) Y Z^{-1} Y^{\intercal} \right] \xi_k(\tau),
$$

where $Y \in \mathbb{R}^{4n \times n}$ is a free matrix. The benefit of the affine version of Jensen's inequality is, in essence, only of computational nature. It has indeed been discussed in [2] that when the interval of integration is uncertain or time-varying, it is preferable to use the affine version to limit the increase of conservatism. The price to pay is a moderate increase of the computational complexity through the presence of the additional matrix Y. Substituting then this inequality back into (1.21), leads to

$$T_k \dot{\mathcal{V}}(\chi_k, \tau) \leq \xi_k^\intercal(\tau) \left[F_{00} + \tau F_1 + (T_k - \tau) Y Z^{-1} Y^\intercal\right] \xi_k(\tau), \tag{1.22}$$

where F_{00} and F_1 are given in Theorem 2. Summing then (1.18)-(1.19)-(1.22) all together, we get that $\dot{\mathcal{W}}$ is negative definite if

$$F_0(T_k) + \tau F_1 + (T_k - \tau) Y Z^{-1} Y^\intercal \tag{1.23}$$

is negative definite for all $(\tau, T_k) \in \mathcal{S}$ where

$$\mathcal{S} := \{(\tau, T) \in \mathbb{R}_+^2 : \tau \in [0, T],\ T \in [T_{min}, T_{max}]\},$$

and where $F_0(T_k)$ is defined in Theorem 2. Exploiting the fact that the matrix (1.23) is affine in τ and T_k, hence convex in these variables, allows us to easily conclude that the matrix (1.23) is negative definite for all $\tau \in [0, T_k]$ and all $T_k \in [T_{min}, T_{max}]$ if and only if it is negative definite at the vertices of the set $\mathcal{S}$ or, equivalently, negative definite on the set $\{(T_{min}, T_{min}), (T_{max}, T_{max}), (0, T_{min}), (0, T_{max})\}$. Each one of these points leads to one of the following LMI conditions:

$$\begin{array}{rcll}
\Phi_1(T_{min}) & = & F_{00} + T_{min} F_1 & \prec 0, \\
\Phi_1(T_{max}) & = & F_{00} + T_{max} F_1 & \prec 0, \\
\tilde{\Phi}_2(T_{min}) & := & F_{00} + T_{min} Y Z^{-1} Y^\intercal & \prec 0, \\
\tilde{\Phi}_2(T_{max}) & := & F_{00} + T_{max} Y Z^{-1} Y^\intercal & \prec 0.
\end{array}$$

Applying finally the Schur complement with respect to the last term in $\tilde{\Phi}_2(\cdot)$ yields $\Phi_2(\cdot)$. The proof is complete.

A similar approach is considered In [33] with the difference that another looped-functional $\mathcal{V}$ is used. Another notable difference is the use of the reciprocally convex combination lemma of [28] yielding less conservative conditions without the introduction of the slack variable Y.

1.4.2 Stability analysis for time-varying delay h_k

Interestingly, Theorem 2 can be easily extended to cope with time-varying delays. In this respect, we now consider that the delay is time-varying and belongs to $\{0, \dots, \bar{h}\}$, $\bar{h} \in \mathbb{N}$. This leads to the following result:

Corollary 1. *The sampled-data system (1.7) with time-varying delay $h_k \in \{0, \dots, \bar{h}\}$ and $T_k := t_{k+1} - t_k \in [T_{min},\ T_{max}]$, $k \in \mathbb{N}$, is asymptotically stable if there exist matrices $R, Z \in \mathbb{S}^n_+$, $P, X \in \mathbb{S}^{2n}$, $S \in \mathbb{S}^n$, $U \in \mathbb{R}^{n\times n}$ and a matrix $Y \in \mathbb{R}^{4n\times n}$ such that the LMIs*

$$\begin{aligned}
\Psi_0 \quad &:= \begin{bmatrix} I \\ I \end{bmatrix} P \begin{bmatrix} I \\ I \end{bmatrix} \quad \succ 0,\\
\Psi_1(\theta) &:= \begin{bmatrix} G_0(\theta) & \theta Y \\ \star & -\theta Z \end{bmatrix} \prec 0,\\
\Psi_2(\theta) &:= G_0(\theta) + \theta G_1 \quad \prec 0,
\end{aligned} \tag{1.24}$$

hold for all $\theta \in \{T_{min}, T_{max}\}$ where $G_{00} = F_{00}$, $G_1 = F_1$ and

$$\begin{aligned}
G_0(\theta) &= G_{00} + G_{01} + \theta \operatorname{Sym}\left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^\intercal P \begin{bmatrix} M_0 \\ 0 \end{bmatrix} \right),\\
G_{01} &= M_2^\intercal \Phi_0 M_2 - M_T^\intercal P M_T + \bar{h}^2 M_\delta^\intercal R M_\delta - M_h^\intercal R M_h.
\end{aligned} \tag{1.25}$$

Proof. As in the proof of Theorem 2, we consider the looped-functional $\mathcal{V}$ given in (1.15). However, we shall consider here the Lyapunov-Krasovskii functional V given by

$$V(\chi_k(\tau), \chi_k^h) \;=\; \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix}^\intercal P \begin{bmatrix} \chi_k(\tau) \\ \chi_k(0) \end{bmatrix} + \bar{h} \sum_{i=-\bar{h}}^{-1} \sum_{j=k+i}^{k-1} \delta_i^\intercal(0) R \delta_i(0), \tag{1.26}$$

where $\delta_i(0) = \chi_{i+1}(0) - \chi_i(0)$. This functional is nothing else but the one we considered for establishing Theorem 2 in which the matrix Q has been set to zero. The proof is now very similar to the one of Theorem 2 and, therefore, only the part pertaining on Λ_k is detailed because of its dissimilarity. Simple calculations show that

$$\begin{aligned}
\Lambda_k = &\begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix}^\intercal P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix} - \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^\intercal P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}\\
&+ \bar{h}^2 (\chi_{k+1}(0) - \chi_k(0))^\intercal R (\chi_{k+1}(0) - \chi_k(0)) - \bar{h} \sum_{i=k-\bar{h}}^{k-1} \delta_i^\intercal(0) R \delta_i(0).
\end{aligned}$$

Since $h_k \leq \bar{h}$ it holds that

$$\begin{aligned}
\Lambda_k \leq &\begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix}^\intercal P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(T_k) \end{bmatrix} - \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}^\intercal P \begin{bmatrix} \chi_k(T_k) \\ \chi_k(0) \end{bmatrix}\\
&+ \bar{h}^2 (\chi_{k+1}(0) - \chi_k(0))^\intercal R (\chi_{k+1}(0) - \chi_k(0)) - h_k \sum_{i=k-h_k}^{k-1} \delta_i^\intercal(0) R \delta_i(0).
\end{aligned}$$

Applying Jensen's inequality to the last summation term, and using the definition of the matrices M_2, M_3, M_δ yields

$$\begin{aligned}\Lambda_k &= \xi_k^{\intercal}(\tau)\left(M_2^{\intercal}\Phi_0 M_2 - \begin{bmatrix} M_2 \\ M_3 \end{bmatrix}^{\intercal} P \begin{bmatrix} M_2 \\ M_3 \end{bmatrix} + \bar{h}^2 M_\delta^{\intercal} R M_\delta - M_h^{\intercal} R M_h\right)\xi_k(\tau) \\ &= \xi_k^{\intercal}(\tau) G_{01}\xi_k(\tau).\end{aligned} \tag{1.27}$$

The rest of the proof is identical to the proof of Theorem 2.

1.4.3 Robust stability analysis

One of the main advantages of the proposed method based lies in the possibility of extending the stability conditions to the case of uncertain systems. Assume now that the matrices of the system are time-varying/uncertain and can be written as

$$\begin{bmatrix} A(t) & B(t) \end{bmatrix} = \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} A_i & B_i \end{bmatrix}, \tag{1.28}$$

where N is a positive integer, A_i and B_i, $i = 1, \dots, N$, are some matrices of appropriate dimensions and the vector $\lambda(t)$ evolves in the N unit simplex defined as

$$\mathcal{U} := \left\{ \lambda \in \mathbb{R}_+^N : \sum_{i=1}^{N} \lambda_i = 1 \right\}. \tag{1.29}$$

This leads us to the following result:

Corollary 2. *The sampled-data system (1.7)-*(1.28) *with constant delay h and $T_k := t_{k+1} - t_k \in [T_{min},\ T_{max}]$, $k \in \mathbb{N}$, is asymptotically stable if there exist matrices $R, Q, Z \in \mathbb{S}_+^n$, $P, X \in \mathbb{S}^{2n}$, $S \in \mathbb{S}^n$, $U \in \mathbb{R}^{n\times n}$ and some matrices $Y_i \in \mathbb{R}^{4n\times n}$ such that the LMIs*

$$\begin{aligned}\Phi_0 \quad &:= \begin{bmatrix} I \\ I \end{bmatrix} P \begin{bmatrix} I \\ I \end{bmatrix} \succ 0, \\ \Phi_1^i(\theta) &:= \begin{bmatrix} F_0^i(\theta) & \theta Y_i \\ \star & -\theta Z \end{bmatrix} \prec 0, \\ \Phi_2^i(\theta) &:= F_0^i(\theta) + \theta F_1^i \prec 0,\end{aligned} \tag{1.30}$$

hold for all $\theta \in \{T_{min}, T_{max}\}$ where

$$\begin{aligned}F_0^i(\theta) &= F_{00}^i + F_{01} + \theta\,\mathrm{Sym}\left(\begin{bmatrix} M_1 \\ M_3 \end{bmatrix}^{\intercal} P \begin{bmatrix} M_0^i \\ 0 \end{bmatrix}\right), \\ F_{00}^i &= M_\Delta^{\intercal} S M_\Delta + \theta M_T^{\intercal} X M_T + \mathrm{Sym}(M_\Delta^{\intercal} U M_T + M_\Delta Y_i) \\ F_{01}^i &= M_2^{\intercal}\Phi_0 M_2 - M_T^{\intercal} P M_T + M_3^{\intercal} Q M_3 - M_4^{\intercal} Q M_4 + h^2 M_\delta^{\intercal} R M_\delta - M_h^{\intercal} R M_h, \\ F_1 &= M_0^{i\intercal} Z M_0^i + \mathrm{Sym}(M_0^{i\intercal}(S M_\Delta + U M_T)) - 2 M_T^{\intercal} X M_T,\end{aligned} \tag{1.31}$$

with $M_0^i = \begin{bmatrix} A_i & 0 & 0 & B_i K \end{bmatrix}$.

Proof. The proof is straightforward by noting that the LMI conditions in Theorem 2 are convex in the matrix M_0. Remarking also that

$$M_0 = \begin{bmatrix} A(t) & 0 & 0 & B(t)K \end{bmatrix} = \sum_{i=1}^{N} \lambda_i(t) \begin{bmatrix} A_i & 0 & 0 & B_i K \end{bmatrix}$$

implies that the LMI conditions are convex in the matrices of the system $A(t)$ and $B(t)$. By virtue of standard results on systems with polytopic uncertainties (see e.g. [5]), it is enough to check the feasibility of the LMI at the vertices of the set $\mathcal{U}$ and the result directly follows.

Remark 2. The above result can be easily extended to the time-varying delay case by setting the matrix Q to 0. This is not presented for brevity.

1.5 Example

Example 1 ([39]). Let us consider the sampled-data system (1.3) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 3.75 & 11.5 \end{bmatrix}.$$

Using an eigenvalue-based analysis, theoretical stability-preserving upper-bounds for the constant sampling period can be determined for any fixed delay h. These values can be understood as a theoretical limit for the upper-bounds obtained in the aperiodic case. These theoretical upper-bounds and the results computed by solving the conditions of Theorem 2 are given in Tables 1.1 and 1.2. More particularly, Table 1.1 compares the maximal allowable sampling period $T = T_{max} = T_{min}$ obtained using Theorem 2 and previous results of the literature. We can immediately see that the numerical values obtained using Theorem 2 are slightly more conservative than those previously obtained by the same authors. Yet, the obtained numerical values are close to the theoretical value.

On the other hand, Table 1.2 compares the obtained results with those obtained with the methods developed in [32, 33]. It can be seen again that the results of [32] are less conservative for small values of the delay h. The main reason for this is that the current paper uses Jensen's inequality, which is more conservative than the integral inequality considered in [32] for large values of the delay. However, the computational burden of the approach of [32] increases exponentially with the delay h, making it inapplicable for systems with large delays.

Finally, Table 1.3 shows the results obtained using Corollary 1, which addresses the case of time-varying delay h_k. We can observe a notable decrease of the maximal allowable sampling period.

Table 1.1 Maximal allowable sampling period $T_{max} = T_{min}$ for Example 1 with periodic samplings for several values of h (-* means "untested because of a too high computational complexity").

h	0	1	2	5	10
Theoretical bounds	1.729	0.763	0.463	0.216	0.112
[26] (with $\tau = hT$)	1.278	0.499	0.333	0.166	0.090
[21] (with $\tau = hT$)	1.638	0.573	0.371	0.179	0.096
[30] (with $\tau = hT$)	1.721	0.701	0.431	0.197	0.103
[33]	1.728	0.761	0.448	0.199	0.103
[32]	1.729	0.763	0.463	-*	-*
Theorem 2	1.720	0.536	0.318	0.146	0.077

Table 1.2 Maximal allowable sampling period T_{max} for Example 1 with $T_{min} = 10^{-2}$ and for several values of h (-* means "untested because of a too high computational complexity").

h	0	1	2	5	10
[33]	1.708	0.618	0.377	0.176	0.094
[32]	1.729	0.763	0.463	-*	-*
Theorem 2	1.245	0.460	0.283	0.132	0.071

Table 1.3 Maximal allowable sampling period T_{max} for Example 1 with $T_{min} = 10^{-2}$ and for several values of the upper bound $\bar{h}$ of the time-varying delay h.

h	0	1	2	5	10
Corollary 1, $T_{min} = T_{max}$	-	0.465	0.264	0.115	0.059
Corollary 1, $T_{min} < T_{max}$	-	0.402	0.240	0.109	0.057

1.6 Conclusions

In this chapter, a way for analyzing stability of periodic and aperiodic uncertain sampled-data systems with discrete-time delays is presented. Instead of using a discrete-time criterion that would prevent the generalization of the approach to uncertain systems with time-varying uncertainties, an alternative approach based on looped-functionals has been preferred. The main novelty of the method relies on the stability analysis, which merges the continuous-time and discrete-time criteria at the same time. This is combination of discrete- and continuous-time approach has been possible by the introduction of a lifted version of the state vector. Further extensions aims at reducing the conseratism of the stability conditions by employing recent and more efficient inequalities such as the reciprocally convex combination lemma [28] and Wirtinger-based integral inequality [34].

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