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Observability of Linear Discrete Time Systems of Algebraic and Difference Equations

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The notion of observability for higher order discrete time systems of algebraic and difference equations is studied. Such systems are also known as Polynomial Matrix Descriptions (PMDs). Attention is first given to a special form of descriptor systems with a state lead in the output. This system is transformed into its causal and noncausal subsystems and observability criteria are given in terms of the subsystem's matrices, and the fundamental matrix sequence of the matrix pencil $(\sigma E - A)$. Afterwards, the higher order system is studied. By transforming it into a first order descriptor system of the above form, an observability criterion is provided for the higher order system in terms of the Laurent expansion at infinity of the system's polynomial matrix. In addition, observability is connected with the coprimeness of the polynomial matrices of the higher order system and the coprimeness of the matrix pencils of the descriptor system.

Keywords: discrete-time systems; observability; singular systems; descriptor systems; higher order systems; multivariable systems.

1. Introduction

Let $\mathbb{R}$ be the field of reals, $\mathbb{R}[\sigma]$ the ring of polynomials with coefficients from $\mathbb{R}$ and $\mathbb{R}(\sigma)$ the field of rational functions. By $\mathbb{R}[\sigma]^{p\times m}, \mathbb{R}(\sigma)^{p\times m}, \mathbb{R}_{pr}(\sigma)^{p\times m}$ we denote the sets of $p\times m$ polynomial, rational and proper rational matrices with real coefficients. We are going to study systems of linear algebraic and difference equations, described by the matrix equation

$$A_q\beta(k+q) + \ldots + A_1\beta(k+1) + A_0\beta(k) = B_q u(k+q) + \ldots + B_1 u(k+1) + B_0 u(k) \tag{1a}$$

$$\xi(k) = C_q\beta(k+q) + \ldots + C_1\beta(k+1) + C_0\beta(k) \tag{1b}$$

where $k \in \mathbb{N}$, $A_i \in \mathbb{R}^{r\times r}, B_i \in \mathbb{R}^{r\times l}$, $C_i \in \mathbb{R}^{m\times r}$ and at least one of A_q, C_q is a non zero matrix. The discrete time functions $u(k) : \mathbb{N} \to \mathbb{R}^l$, $\beta(k) : \mathbb{N} \to \mathbb{R}^r$ and $\xi(k) : \mathbb{N} \to \mathbb{R}^m$ define the input, state and output vectors of the system respectively. The maximum number of time shifts q is called the *lag*

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of the system. Using the forward shift operator σ with $\sigma^i \beta(k) = \beta(k+i)$, the system (1) can be rewritten as

$$A(\sigma)\beta(k) = B(\sigma)u(k) \tag{2a}$$
$$\xi(k) = C(\sigma)\beta(k) \tag{2b}$$

with $A(\sigma) = \sum_{i=0}^{q} A_i\sigma^i \in \mathbb{R}[\sigma]^{r\times r}$, $B(\sigma) = \sum_{i=0}^{q} B_i\sigma^i \in \mathbb{R}[\sigma]^{r\times l}$, $C(\sigma) = \sum_{i=0}^{q} C_i\sigma^i \in \mathbb{R}[\sigma]^{m\times r}$ and $\det A(\sigma) \neq 0$. Systems described by (2) are also called *Polynomial Matrix Descriptions (PMDs)*. Systems described by (2a) are also known as *ARMA (Auto-Regressive Moving Average) representations.*

Higher order systems (2) and their continuous time analogues have been a subject of extensive research and often appear in applications, since they can model electrical circuit networks, mechanical, social, biological and economic systems, see for example Campbell (1980); Liu et al. (2008); Moysis et al. (2016) and the references therein. Various methods of approach have been established to compute their solution and study properties like controllability and observability. A *straightforward approach* has been used in Vardulakis (1991); Vardulakis et al. (1999) for the continuous time case and in Karampetakis et al. (2001); Antoniou et al. (1998); Jones et al. (2003); Moysis & Karampetakis (2016); Karampetakis (2004) for the discrete time case, through the use of the *Jordan Pairs* and the *Laurent expansion at infinity* of the polynomial matrix $A(\sigma)$. A different method for continuous time has also been used in Taher & Rachidi (2008). Another approach is to first transform the higher order system into a state-space or descriptor system, as in Karampetakis & Vologiannidis (2009); Antsaklis & Michel (2006); Mahmood et al. (1998); Laub & Arnold (1984). The descriptor system can then be further decomposed using the *Weierstrass* decomposition of the matrix pencil involved, into multiple subsystems, which can then be studied separately, as in Dassios (2012); Kalogeropoulos et al. (2009); Pantelous et al. (2014).

Descriptor systems are a special case of (2) and are described by

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{3}$$

where the matrix E may be singular. If E is nonsingular, then the descriptor systems transforms to a regular *state space system*. Descriptor systems have been studied in Campbell (1980); Brull (2009); Dassios & Kalogeropoulos (2013); Dai (1989); Duan (2010); Mertzios & Lewis (1989); Lewis (1984, 1985); Pantelous et al. (2014); Verghese et al. (1981); Lewis (1986).

The concept of observability was introduced by Kalman in (1959) and has been previously studied by various authors, initially for state space systems in Antsaklis & Michel (2006); Kailath (1980). These results have been extended to descriptor systems in Duan (2010); Koumboulis & Mertzios (1999); Yip & Sincovec (1981); Lewis & Mertzios (1990); Verghese et al. (1981); Ozcaldiran et al. (1992); Bender (1987); Hou & Muller (1999); Berger et al. (2016), for second order descriptor systems in Losse & Mehrmann (2008); Laub & Arnold (1984), for rectangular descriptor systems in Ishihara & Terra (2001) and for positive systems in Kaczorek (2002).

In general, observability refers to the determination of the initial value of the system's state, by knowledge of its input and output values over a finite interval. In contrast to state space systems though where the state and output of the system can be determined by knowledge of $x(0)$, in descriptor systems, knowledge of $x(0)$ may not be necessary, since the state and output are determined by $Ex(0)$. Thus, observability of descriptor systems indicates if different $Ex(0)$ may yield distinct time responses for a fixed input $u(k)$, as commented for the continuous time case in Hou & Muller (1999). So the ability to determine $Ex(0)$ or $x(0)$ are considered two different properties of the system which are defined as *weak* and *complete* observability or simply as *observability*. These terms are used because knowledge of $x(0)$ also gives us knowledge of the vector $Ex(0)$, while the opposite does not hold since E is singular. This is why the ability to determine $x(0)$ is called complete observability. A similar distinction between $x(0)$ and $Ex(0)$ for continuous time systems is made in Ozcaldiran et al. (1992). In the following sections though, we see that for discrete time descriptor systems, which satisfy certain admissibility conditions, the two notions coincide.

This is also the case for higher order systems of the form (2). For such systems, we make the distinction between the initial values $\beta(0), ..., \beta(q-1)$ and the vector

$$\beta_{in} = \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} \tag{4}$$

as it was defined in Pugh et al. (1995).

For continuous time higher order descriptor systems of the form (2), conditions were given in Pugh et al. (1993) for (2b) to be a formal mapping, that is, to uniquely specify an image $\xi(t)$ for each smooth or impulsive solution $\beta(t)$ of (2a). It was concluded that $\delta_M\left(A(\sigma)^T \quad C(\sigma)^T\right)^T = \delta_M\left(A(\sigma)\right)$ must hold, where δ_M denotes the *Mc-Millan degree* of a polynomial matrix (see Vardulakis, 1991, Proposition 1.73). In the discrete time case though, the notion of impulsive solutions does not exist and thus this condition is not required. If we were to study the solution of (2) over a finite time horizon though, then the state $\beta(k)$ would depend not only on past, but also on future values of $\beta(k)$ and $u(k)$, as was shown in Dai (1989) for descriptor systems and in Karampetakis et al. (2001) for higher order systems. Thus, the impulsive behavior in continuous time is exhibited as noncausality in discrete time systems over a finite horizon $[0, N]$, as commented in Lewis (1986).

Contrary to continuous time descriptor systems though, discrete time descriptor systems do not have a solution for every set of initial values, as was commented in Dai (1989) and later in Karampetakis et al. (2001); Karampetakis & Gregoriadou (2014). That is, in order for the descriptor systems (2), (3) to have a solution, the initial values need to satisfy certain admissibility constraints. The concept of admissible initial values also appears in continuous time and gives the conditions under which the solution of the system will be impulse free Yip & Sincovec (1981). So while for discrete time the non satisfaction of the admissibility constraints means that the system has no solution, for continuous time it translates to the system exhibiting impulses.

In the present work we present conditions for the complete observability of (2). This concept of complete observability appears in Ozcaldiran et al. (1992); Duan (2010); Yip & Sincovec (1981) in continuous time (where the determination of $x(0)$ or $x(0-)$ is of interest), and in Coll et al. (2002); Dai (1989) for discrete time systems. To do so, we first transform the system (2) into a descriptor system, using the method proposed in Karampetakis & Vologiannidis (2009). The resulting system is a special form of a descriptor system with output matrix $C(\sigma) = C_0 + C_1\sigma$, that is, the output will depend on $x(k)$ and $x(k+1)$. This is the discrete time analog of continuous time descriptor systems with a derivative in the output, that have been studied in Tan & Zhang (2010). This descriptor system is then further decomposed into two subsystems, the so called *causal* and *noncausal*. In this way, we first derive a criterion for observability in terms of the matrices of the causal and noncausal subsystems. We then extend these results to give criteria that connect observability to the matrices of the descriptor system $(\sigma E - A)$, C, the matrices of the higher order system $A(\sigma)$, $C(\sigma)$, and the Laurent expansion at infinity of $(\sigma E - A)^{-1}$ and $A(\sigma)^{-1}$. Thus, the observability requirements for the resulting descriptor and the original polynomial system are shown to coincide in the analysis that follows.

Possible applications of the results presented in this work can be the problem of observer design, or the problem of pole placement through output feedback of the form $u(k) = K\xi(k)$, which for the continuous time case has been studied in Yu & Duan (2010, 2009); Wu & Duan (2007); Duan & Yu (2006); Zhang & Liu (2012). For the discrete time case, higher order systems of the form (2) can result from the discretization of first or second order systems that appear in constrained mechanics, in structural analysis, spacecraft control and so forth, see for example Bhaya & Desoer

(1985); Moysis et al. (2016); Pantelous et al. (2014); Yu & Duan (2010); Duan & Yu (2006); Kalogeropoulos et al. (2009) and the references therein. In the descritization for such systems, several issues can arise, such as the problem of choosing an appropriate sampling interval and the choice of the discretization method, so that properties like controllability and observability aren't lost. These issues have been studied in Karampetakis (2004); Kalogeropoulos et al. (2009); Mehmood et al. (2016). Another possible application where higher order discrete time systems arise is multichannel signal processing. In MIMO channels, polynomial matrices are used to describe filters that give the convolution of a set of signals that arive at an array of sensors from different paths, having different delays, see for example Foster et al. (2010).

The rest of the paper is organized as follows. In Section 2, some preliminary concepts from the theory of polynomial matrices and matrix pencil theory are presented. Section 3 gives details on the solution and observability of descriptor systems with a state lead in the output. In section 4 higher order systems are presented and transformed into first order descriptor systems of the above special form. Their observability is then studied and the results are showcased through an example. Section 5 concludes the paper.

2. Mathematical Preliminaries

This section introduces some basic background material regarding polynomial matrices that will be used throughout this work.

Consider a polynomial matrix

$$A(\sigma) = A_q\sigma^q + A_{q-1}\sigma^{q-1} + \ldots + A_1\sigma + A_0 \tag{5}$$

with $A_i \in \mathbb{R}^{r\times r}$, $A_q \neq 0$ and $\det(A(\sigma)) \neq 0$.

Definition 1: Vardulakis (1991) A square polynomial matrix $A(\sigma) \in \mathbb{R}[\sigma]^{r\times r}$ is called *unimodular* if $\det A(\sigma) = c \in \mathbb{R}$, $c \neq 0$. A rational matrix $A(\sigma) \in \mathbb{R}_{pr}(\sigma)^{r\times r}$ is called *biproper* if $\lim_{\sigma\to\infty} A(\sigma) = E \in \mathbb{R}^{r\times r}$ with $rankE = r$.

Lemma 1: *Vardulakis (1991) The Laurent expansion of $A(\sigma)^{-1}$ at infinity is given by*

$$A(\sigma)^{-1} = H_{pol}(\sigma) + H_{sp}(\sigma) = H_{\hat{q}_r}\sigma^{\hat{q}_r} + \cdots + H_1\sigma + H_0 + H_{-1}\sigma^{-1} + H_{-2}\sigma^{-2} + \ldots, \tag{6}$$

where $H_{pol}(\sigma)$, $H_{sp}(\sigma)$ denote the polynomial and the strictly proper part of $A(\sigma)^{-1}$, and $\hat{q}_r$ is the maximum order amongst the orders of zeros at infinity of $A(\sigma)$, (see Vardulakis, 1991, Corollary 3.54).

Remark 1: The *fundamental matrix sequence* H_i of $A(\sigma)$ can be effectively computed using the technique proposed in Fragulis et al. (1991).

We shall also need the following results from the theory of rational matrices.

Definition 2: (Vardulakis, 1991, Definitions 1.77, 1.78) Let $T(\sigma) \in \mathbb{R}_{pr}(\sigma)^{m\times r}$. Then a quadruple of matrices (C, J, B, E) with $C \in \mathbb{R}^{m\times p}$, $J \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times r}$, $E \in \mathbb{R}^{m\times r}$, $q \in \mathbb{Z}^+$ such that

$$T(\sigma) = C(\sigma I_p - J)^{-1}B + E \tag{7}$$

is called a realization of $T(\sigma)$. The realization is called minimal if J has the minimal possible order, or equivalently if $p = \delta_M(T)$.

Theorem 1: *(Vardulakis, 1991, Propositions 1.79, 1.88) Let* $T(\sigma) = B(\sigma)A(\sigma)^{-1} = C(\sigma I_r - J)^{-1}B + E \in \mathbb{R}_{pr}(\sigma)^{m\times r}$ *be a proper rational matrix, with* $B(\sigma) \in \mathbb{R}^{m\times r}[\sigma]$, $A(\sigma) \in \mathbb{R}^{r\times r}[\sigma]$ *right coprime and* (C, J, B, E) *a realization of* $T(\sigma)$, *with* $C \in \mathbb{R}^{m\times n}$, $J \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, $E \in \mathbb{R}^{m\times r}$. *The realization* (C, J, B, E) *is a minimal realization of* $T(\sigma)$ *if and only if the following equivalent conditions hold:*

(1) $n = \deg\det A(\sigma)$.
(2) The pairs (J, B) *and* (C, J) *are respectively controllable and observable pairs.*
(3)

$$rank\left(B, JB, \ldots J^{n-1}B\right) = rank\begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{n-1}\end{pmatrix} = n \tag{8}$$

Theorem 2: *(Vardulakis, 1991, Theorem 4.50),(Gohberg et al., 2009, Theorem 7.1) A minimal realization of the strictly proper part of* $A(\sigma)^{-1}$ *is given by* $A(\sigma)^{-1}_{sp} = C(\sigma I_n - J)^{-1}B$, *with* $C \in \mathbb{R}^{r\times n}$, $J \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$, *and the matrix* J *is in Jordan form. For the matrices* C, J, B *it holds that*

$$H_{-i} = CJ^{i-1}B,\ i = 1, 2, \ldots \tag{9}$$

Theorem 3: *(Gohberg et al., 2009, Theorems 7.1, 7.2, 7.7) Let* (X_F, T_F) *and* (X_∞, T_∞) *be the* finite and infinite Jordan Pairs *of* $A(\sigma)$, *with* $X_F \in \mathbb{R}^{r\times n}, T_F \in \mathbb{R}^{n\times n}, X_\infty \in \mathbb{R}^{r\times\mu}, T_\infty \in \mathbb{R}^{\mu\times\mu}$, *and* $\mu = rq - n$. *These pairs satisfy the following properties:*

(1) $\det \sigma^q A(\sigma^{-1})$ *has a zero at* $\lambda = 0$ *with multiplicity* μ.
(2) $\sum_{i=0}^{q} A_i X_F T_F^i = 0$, $\sum_{i=0}^{q} A_{q-i} X_\infty T_\infty^i = 0$
(3)

$$rank\begin{pmatrix} X_F \\ X_F T_F \\ \vdots \\ X_F T_F^{q-1}\end{pmatrix} = n, \quad rank\begin{pmatrix} X_\infty \\ X_\infty T_\infty \\ \vdots \\ X_\infty T_\infty^{q-1}\end{pmatrix} = \mu \tag{10}$$

In addition, a realization of $A(\sigma)^{-1}$ *is given by*

$$A(\sigma)^{-1} = \begin{pmatrix} X_F & X_\infty \end{pmatrix}\begin{pmatrix} \sigma I_n - T_F & 0 \\ 0 & \sigma T_\infty - I_\mu \end{pmatrix}^{-1}\begin{pmatrix} Z_F \\ Z_\infty \end{pmatrix} \tag{11}$$

with $Z_F \in \mathbb{R}^{n\times r}, Z_\infty \in \mathbb{R}^{\mu\times r}$ *and*

$$\begin{pmatrix} Z_F \\ Z_\infty \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & T_\infty^{q-1} \end{pmatrix}\begin{pmatrix} X_F & X_\infty T_\infty^{q-2} \\ \vdots & \vdots \\ X_F T_F^{q-2} & X_\infty \\ A_q X_F T_F^{q-1} & -\sum_{i=0}^{q-1} A_i X_\infty {T_\infty}^{q-1-i} \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \vdots \\ I_r \end{pmatrix} \tag{12}$$

Making use of Theorems 2 and 3, we are going to prove the following result.

Theorem 4: *Given a minimal realization of the strictly proper part of* $A(\sigma)^{-1}$, (C, J, B), *it holds* *that*

$$rank\begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{q-1} \end{pmatrix} = n \tag{13}$$

Proof. We consider two cases. If $n \leq q$, then (13) is straightforward from (8).

Now, let $n > q$. From Theorems 2 and 3, the strictly proper part of $A(\sigma)^{-1}$ is given by

$$A(\sigma)^{-1}_{sp} = X_F(\sigma I_n - T_F)^{-1}Z_F = C(\sigma I_n - J)^{-1}B \tag{14}$$

it holds that

$$CJ^iB = X_F T_F^i Z_F, \qquad i = 0, 1, 2, \ldots \tag{15}$$

In view of (15) it is easy to verify that

$$\underbrace{\begin{pmatrix} C \\ CJ \\ \vdots \\ CJ^{n-1} \end{pmatrix}}_{M_n} \underbrace{\left(B, JB, \ldots J^{n-1}B\right)}_{L_n} = \underbrace{\begin{pmatrix} X_F \\ X_F T_F \\ \vdots \\ X_F T_F^{n-1} \end{pmatrix}}_{\hat{M}_n} \underbrace{\left(Z_F, T_F Z_F, \ldots T_F^{n-1} Z_F\right)}_{\hat{L}_n} \Leftrightarrow \tag{16}$$

$$M_n L_n = \hat{M}_n \hat{L}_n. \tag{17}$$

Now, since from (8), $\text{rank}L_n = n$, it follows that $L_n L_n^T$ is square and invertible. Post-multiplying (17) by L_n^T and in turn by $(L_n L_n^T)^{-1}$, yields

$$M_n = \hat{M}_n W \tag{18}$$

where $W = \hat{L}_n L_n^T (L_n L_n^T)^{-1} \in \mathbb{R}^{n\times n}$. It is easy to check that if $x \in \mathbb{R}^n$ is chosen such that $Wx = 0$, then $M_n x$=0, which implies that $x = 0$, since from (8), $\text{rank}M_n = n$. Thus, W is invertible.

Now, since $n > q$, keeping only the q first block rows of (18) reads

$$\underbrace{\begin{pmatrix} C \\ \vdots \\ CJ^{q-1} \end{pmatrix}}_{M_q} = \underbrace{\begin{pmatrix} X_F \\ \vdots \\ X_F T_F^{q-1} \end{pmatrix}}_{\hat{M}_q} W \tag{19}$$

Taking into account (10) and the invertibility of W it follows that $\text{rank}M_q = \text{rank}\hat{M}_q = n$. □

Now, consider $E, A \in \mathbb{R}^{r\times r}$, with E singular and assume $(\sigma E - A)$ is *regular*, that is $\det(\sigma E - A) \neq 0$. The resolvent matrix of the descriptor system (3) has the Laurent expansion at infinity

$$(\sigma E - A)^{-1} = \sigma^{-1}\sum_{i=-h}^{\infty} \Phi_i \sigma^{-i} = \Phi_{-h}\sigma^{h-1} + \ldots + \Phi_{-1}\sigma^0 + \Phi_0\sigma^{-1} + \Phi_1\sigma^{-2} + \ldots \tag{20}$$

where $h \leq rankE - \deg\det(\sigma E - A) + 1$ is the *index of nilpotency* of the pair (E, A). The sequence of matrices $\Phi_i \in \mathbb{R}^{r\times r}$ is known as the (forward) fundamental matrix sequence of $(\sigma E - A)$.

Remark 2: The fundamental matrix sequence Φ_i of $(\sigma E - A)$ can be computed by the methods presented in Kaczorek (2002); Campbell (1980); Mertzios & Lewis (1989); Schweitzer & Stewart (1993).

The matrices Φ_i satisfy the following properties.

Theorem 5: *Langenhop (1971) Let* $(\sigma E - A)$ *regular and* Φ_i *as in* (20). *The following properties hold*

$$\Phi_i E - \Phi_{i-1} A = I\delta_i \tag{21a}$$

$$E\Phi_i - A\Phi_{i-1} = I\delta_i \tag{21b}$$

$$\Phi_i = \begin{cases} (\Phi_0 A)^i \Phi_0, \; i \geq 0 \\ (-\Phi_{-1} E)^{-i-1} \Phi_{-1}, \; i < 0 \end{cases} \tag{21c}$$

$$\Phi_i E \Phi_j = \Phi_j E \Phi_i \tag{21d}$$

$$\Phi_i E \Phi_j = \begin{cases} -\Phi_{i+j}, \; i < 0, \; j < 0 \\ \Phi_{i+j}, \; i \geq 0, \; j \geq 0 \\ 0, \; otherwise \end{cases} \tag{21e}$$

$$\Phi_i A \Phi_j = \begin{cases} -\Phi_{i+j+1}, \; i < 0, \; j < 0 \\ \Phi_{i+j+1}, \; i \geq 0, \; j \geq 0 \\ 0, \; otherwise \end{cases} \tag{21f}$$

where δ_i *denotes the* Kronecker delta.

3. Descriptor systems with state lead in the output

First, we will study a special case of descriptor systems where there is a state lead in the output. This is a special case of descriptor systems where the output depends not only on $x(k)$ but also on $x(k+1)$ and is the discrete time analog of continuous time descriptor systems with a derivative in the output, that have been studied in Tan & Zhang (2010). Such systems are described by

$$Ex(k+1) = Ax(k) + Bu(k) \tag{22a}$$

$$y(k) = C_1 x(k) + C_2 x(k+1) \tag{22b}$$

with $E, A \in \mathbb{R}^{r\times r}$, $B \in \mathbb{R}^{r\times l}$, $C_1, C_2 \in \mathbb{R}^{m\times r}$. The state response of (22) is the sum of its homogeneous response due to initial value $x(0)$ and its dynamic response due to the input, that is $x(k) = x_{hom}(k) + x_{dynamic}(k)$. The dynamic response $x_{dynamic}(k)$ is always known, since it depends on the input $u(k)$. Thus, since we are interested in determining the intial value of the system (22) we can always define a new system with $\bar{x}(k) = x(k) - x_{dynamic}(k) = x_{hom}(k)$ and study it instead of the original. As a result, we will focus our attention to the homogeneous case of (23), that is

$$Ex(k+1) = Ax(k) \tag{23a}$$

$$y(k) = C_1 x(k) + C_2 x(k+1) \tag{23b}$$

To begin with, it is well established that the state $x(k)$ of (73) is given by (see Dai, 1989; Lewis & Mertzios, 1990)

$$x(k) = \Phi_k E x(0) \tag{24}$$

As mentioned in the Introduction, discrete time descriptor systems do not have solutions for all initial values. Thus, the initial vector $x(0)$ is subject to certain admissibility conditions that are given in the following theorem.

Theorem 6: *Karampetakis et al. (2001) The set of all $x(0)$ that satisfy (23a) for $k = 0$ is called the set of* admissiible initial values *and is given by*

$$H^{ad}_{descriptor} = \{x(0) \in \mathbb{R}^r \,|\, x(0) = \Phi_0 E x(0)\} \tag{25}$$

So for any $x(0)$ that does not satisfy (25), the descriptor system has no solution, in contrast to the continuous time case, where the non satisfaction of the admissible values can lead to impulsive terms in the solution.

For the output of the system, considering (24), we obtain that

$$\begin{aligned} y(k) &= C_1 x(k) + C_2 x(k+1) = && (26)\\ &= C_1 \Phi_k E x(0) + C_2 \Phi_{k+1} E x(0) = && (27)\\ &= (C_1 \Phi_k + C_2 \Phi_{k+1}) E x(0) && (28) \end{aligned}$$

Now that we studied the descriptor system in its original form, we can move on to decompose it into its subsystems. So following Tan & Zhang (2010); Yip & Sincovec (1981); Gantmacher (1959), there exist nonsingular matrices $P, Q \in \mathbb{R}^{r \times r}$ such that

$$PEQ = diag(I_n, N) \tag{29}$$

$$PAQ = diag(A_1, I_\mu) \tag{30}$$

$$C_1 Q = (F_1, F_2) \tag{31}$$

$$C_2 Q = (D_1, D_2) \tag{32}$$

where $A_1 \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{\mu \times \mu}$, where $\mu = r - n$, $n = \deg\det(\sigma E - A)$, $F_1, D_1 \in \mathbb{R}^{m \times n}$, $F_2, D_2 \in \mathbb{R}^{m \times \mu}$ and N is nilpotent with index of nilpotency h.

Applying the transformation

$$x(k) = Q\tilde{x}(k) = Q \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} \tag{33}$$

on (23) and multiplying the first equation from the left by P we get

$$PEQ \begin{pmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{pmatrix} = PAQ \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} \tag{34}$$

$$y(k) = C_1 Q \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} + C_2 Q \begin{pmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{pmatrix} \tag{35}$$

which is equivalent to

$$\begin{pmatrix} I_n & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & I_\mu \end{pmatrix} \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} \tag{36}$$

$$y(k) = (F_1, F_2) \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} + (D_1, D_2) \begin{pmatrix} \tilde{x}_1(k+1) \\ \tilde{x}_2(k+1) \end{pmatrix} = \tag{37}$$

$$= \underbrace{F_1\tilde{x}_1(k) + D_1\tilde{x}_1(k+1)}_{y_1(k)} + \underbrace{F_2\tilde{x}_2(k) + D_2\tilde{x}_2(k+1)}_{y_2(k)} \tag{38}$$

So the system is decomposed as

$$\tilde{x}_1(k+1) = A_1\tilde{x}_1(k) \tag{39a}$$
$$y_1(k) = F_1\tilde{x}_1(k) + D_1\tilde{x}_1(k+1) \overset{(39a)}{=} (F_1 + D_1A_1)\tilde{x}_1(k) \tag{39b}$$

and

$$N\tilde{x}_2(k+1) = \tilde{x}_2(k) \tag{40a}$$
$$y_2(k) = F_2\tilde{x}_2(k) + D_2\tilde{x}_2(k+1) \tag{40b}$$

with output

$$y(k) = y_1(k) + y_2(k) \tag{41}$$

The system (39) is called the *causal* or *forward subsystem* and the system (40) is called the *non-causal* or *backward subsystem*. In the continuous time case, these systems are called the *slow* and *fast* subsystems respectively.

Since the causal subsystem is in state space form, its state is given by $\tilde{x}_1(k) = A_1^k\tilde{x}_1(0)$. For the noncausal subsystem, we have the following result.

Theorem 7: *(Dai, 1989, Section 8.2) The noncausal subsystem* (40) *has only the zero solution* $\tilde{x}_2(k) = 0$

From the above result and the fact that the causal subsystem is in state space form and is thus free of admissibility constraints, we conclude to the following result about the admissible initial values of the transformed system.

Theorem 8: *The set of admissible initial values of the causal-noncausal subsystems* (39)-(40) *are*

$$H^{ad}_{c-nc} = \{\tilde{x}_1(0) \in \mathbb{R}^n, \tilde{x}_2(0) \in \mathbb{R}^\mu \,|\, \tilde{x}_2(0) = 0\} \tag{42}$$

and are equivalent to the set of admissible initial values (25) *of the descriptor system* (23).

Proof. First, we need to consider the Laurent expansion of the matrix $(\sigma E - A)^{-1}$

$$(\sigma E - A)^{-1} = \sigma^{-1}\sum_{i=-h}^{\infty}\Phi_i\sigma^{-i} = \Phi_{-h}\sigma^{h-1} + \ldots + \Phi_{-1}\sigma^0 + \Phi_0\sigma^{-1} + \Phi_1\sigma^{-2} + \ldots \Rightarrow \tag{43}$$
$$Q^{-1}(\sigma E - A)^{-1}P^{-1} = Q^{-1}\sigma^{-1}\sum_{i=-h}^{\infty}\Phi_i\sigma^{-i}P^{-1} \tag{44}$$

For the above matrix we have

$$Q^{-1}(\sigma E - A)^{-1}P^{-1} = (P(\sigma E - A)Q)^{-1} = \tag{45}$$
$$= \begin{pmatrix} \sigma I_n - A_1 & 0 \\ 0 & \sigma N - I_\mu \end{pmatrix}^{-1} = \tag{46}$$

$$= \begin{pmatrix} (\sigma I_n - A_1)^{-1} & 0 \\ 0 & (\sigma N - I_\mu)^{-1} \end{pmatrix} = \tag{47}$$

$$= \begin{pmatrix} \sigma^{-1} I_n + \sigma^{-2} A_1 + \ldots & 0 \\ 0 & -I_\mu - \sigma N - \ldots - \sigma^{h-1} N^{h-1} \end{pmatrix} \tag{48}$$

By equating coefficients among the powers of σ in (44) and (48) we get

$$Q^{-1} \Phi_i P^{-1} = \begin{pmatrix} A_1^i & 0 \\ 0 & 0_\mu \end{pmatrix}, \ i \geq 0 \tag{49}$$

Now, the admissible conditions for the descriptor system are

$$x(0) = \Phi_0 E x(0) \tag{50}$$

Using the transformation (33) the above is equal to

$$Q \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} = \Phi_0 E Q \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} \overset{(29)}{\Rightarrow} \tag{51}$$

$$Q \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} = \Phi_0 P^{-1} \begin{pmatrix} I_n & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} \Rightarrow \tag{52}$$

$$\begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} = Q^{-1} \Phi_0 P^{-1} \begin{pmatrix} \tilde{x}_1(0) \\ N\tilde{x}_2(0) \end{pmatrix} \overset{(49)}{\Rightarrow} \tag{53}$$

$$\begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & 0_\mu \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ N\tilde{x}_2(0) \end{pmatrix} \Rightarrow \tag{54}$$

$$\begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix} = \begin{pmatrix} \tilde{x}_1(0) \\ 0_\mu \end{pmatrix} \tag{55}$$

and thus the admissible initial value sets are equivalent. □

Now we can proceed to study the observability of the descriptor system. First, we give a definition for complete observability.

Definition 3: The first order descriptor system (23) is *completely observable* if the initial value $x(0)$ and consequently the vector $Ex(0)$ can be uniquely determined from knowledge of the output $y(k)$ over a finite time interval.

Since for the discrete time case we only consider systems with consistent initial values, from (25), it can be seen that knowledge of $Ex(0)$ gives us $x(0)$. Thus, for discrete time desciptor and higher order systems, the notions of complete and weak observability coincide.

Since the noncausal subsystem has only the zero solution, from (40b) it is seen that the output of the noncausal subsystem is $y_2(k) = 0$, and the output (41) of the system is equal to the output of the causal subsystem, $y(k) = y_1(k)$. This means that the initial state of the noncausal subsystem is always known and thus the noncausal subsystem is always observable. This leads us to the following result.

Theorem 9: *The system (23) is completely observable if and only if the causal subsystem is observable, or equivalently if*

$$rank \begin{pmatrix} \sigma I_n - A_1 \\ F_1 + D_1 A_1 \end{pmatrix} = n \tag{56}$$

or

$$rank\mathcal{O}_{causal} = rank\begin{pmatrix} F_1 + D_1A_1 \\ (F_1 + D_1A_1)A_1 \\ \vdots \\ (F_1 + D_1A_1)A_1^{n-1} \end{pmatrix} = n \tag{57}$$

where $\mathcal{O}_{causal}$ *is the observability matrix of the causal subsystem.*

We have provided a criterion for the observability connecting it with the causal subsystem's matrices. Now we will connect this to the fundamental matrix sequence of $(\sigma E - A)^{-1}$.

Theorem 10: *The observability criterion proposed in* (57) *is equivalent to*

$$rank\mathcal{O}_{descriptor} = rank\begin{pmatrix} C_1 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_1 & C_2 \end{pmatrix}\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} = n \tag{58}$$

Proof. Multiplying the matrix $\mathcal{O}_{descriptor}$ in (58) from the right by the nonsingular matrix P^{-1} we obtain the following

$$\begin{pmatrix} C_1 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_1 & C_2 \end{pmatrix}\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} P^{-1} \overset{(49)}{=} \begin{pmatrix} C_1 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_1 & C_2 \end{pmatrix}\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}\left(\begin{array}{cc} I_n & 0 \\ 0 & 0_\mu \\ \hline A_1 & 0 \\ 0 & 0_\mu \\ \hline \vdots & \vdots \\ \hline A_1^n & 0 \\ 0 & 0_\mu \end{array}\right) = \tag{59}$$

$$= \begin{pmatrix} C_1Q & C_2Q & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_1Q & C_2Q \end{pmatrix}\left(\begin{array}{cc} I_n & 0 \\ 0 & 0_\mu \\ \hline A_1 & 0 \\ 0 & 0_\mu \\ \hline \vdots & \vdots \\ \hline A_1^n & 0 \\ 0 & 0_\mu \end{array}\right) \overset{(31)}{\underset{(32)}{=}} \tag{60}$$

$$= \begin{pmatrix} F_1 & F_2 & D_1 & D_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & F_1 & F_2 & D_1 & D_2 \end{pmatrix}\left(\begin{array}{cc} I_n & 0 \\ 0 & 0_\mu \\ \hline A_1 & 0 \\ 0 & 0_\mu \\ \hline \vdots & \vdots \\ \hline A_1^n & 0 \\ 0 & 0_\mu \end{array}\right) = \tag{61}$$

$$= \begin{pmatrix} F_1 + D_1A_1 & 0 \\ (F_1 + D_1A_1)A_1 & 0 \\ \vdots & \\ (F_1 + D_1A_1)A_1^{n-1} & 0 \end{pmatrix} \tag{62}$$

and the first column is the observability matrix $\mathcal{O}_{causal}$ of the causal subsystem. □

So far, we have given criteria for the observability of the system. Now we will connect the observability of (23) with the system's compound matrix.

Theorem 11: *The system (23) is observable if and only if the matrices $(\sigma E + A)$ and $(C_1 + C_2\sigma)$ are right coprime, or equivalently*

$$rank \begin{pmatrix} \sigma E + A \\ C_1 + C_2\sigma \end{pmatrix} = r \tag{63}$$

Proof. Since the matrices P, Q are nonsingular, we can multiply the above matrix by them without affecting its rank, so

$$\begin{pmatrix} P & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} \sigma E + A \\ C_1 + C_2\sigma \end{pmatrix} Q = \begin{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & N \end{pmatrix} \sigma - \begin{pmatrix} A_1 & 0 \\ 0 & I_\mu \end{pmatrix} \\ \begin{pmatrix} F_1 & F_2 \end{pmatrix} + \begin{pmatrix} D_1 & D_2 \end{pmatrix} \sigma \end{pmatrix} = \left(\begin{array}{c|c} \sigma I_n - A_1 & 0 \\ 0 & \sigma N - I_\mu \\ F_1 + D_1\sigma & F_2 + D_2\sigma \end{array}\right) \tag{64}$$

The second column block of the above matrix always has full column rank equal to μ. A similar result was showcased in Yip & Sincovec (1981) and is easy to explain. Since N is nilpotent, it is similar to an upper triangular matrix with zeros in its diagonal. Thus, the matrix $\sigma N - I_\mu$ is similar to a matrix with -1 on its diagonal and therefore it has full column rank equal to μ.

For the first column, if we multiply the first row by $-D_1$ from the left and add it to the third column, we get

$$\begin{pmatrix} I_n & 0 \\ -D_1 & I_m \end{pmatrix} \begin{pmatrix} \sigma I_n - A_1 \\ F_1 + D_1\sigma \end{pmatrix} \Rightarrow \begin{pmatrix} \sigma I_n - A_1 \\ F_1 + D_1 A_1 \end{pmatrix} \tag{65}$$

and the above matrix has full column rank iff the causal subsystem is observable. Thus, the matrix (63) has full column rank iff the system is observable. □

4. Transformation of higher order systems to descriptor form

In this section we will show how a higher order descriptor system can be transformed to a first order descriptor system of the form (23). It will be shown that the two systems have the same initial values and thus we can study the observability of the higher order system by studying the first order system.

First, we will give formulas for the state and output of (2). As with the first order case, since our aim is to study the observability of the system, we will focus our attention to the homogeneous case of (2), that is

$$A(\sigma)\beta(k) = 0 \tag{66a}$$
$$\xi(k) = C(\sigma)\beta(k) \tag{66b}$$

Now we can describe the state and output of (66) in terms of the Laurent expansion of $A(\sigma)^{-1}$

Theorem 12: *(Karampetakis et al., 2001) The state $\beta(k)$ of (66) is given by*

$$\beta(k) = \begin{pmatrix} H_{-k-q} & \cdots & H_{-k-1} \end{pmatrix} \beta_{in} \tag{67}$$

where

$$\beta_{in} = \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} \tag{68}$$

As with the first order case, the higher order descriptor system (66) has a solution if the initial values $\beta(0), \ldots, \beta(q-1)$ satisfy certain admissibility constraints, that are given in the following theorem.

Theorem 13: *Karampetakis et al. (2001) The set of* admissible initial values *of* (66a) *is given by*

$$H_{AR}^{ad} = \left\{ \beta(i) \in \mathbb{R}^r,\ i = 0, \ldots, q-1 \,\middle|\, \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} = \begin{pmatrix} H_{-q} & \cdots & H_{-1} \\ \vdots & & \vdots \\ H_{-2q+1} & \cdots & H_{-q} \end{pmatrix} \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} \right\} \tag{69}$$

Theorem 14: *The output $\xi(k)$ of (66) is*

$$\xi(k) = \begin{pmatrix} C_0 & \cdots & C_q \end{pmatrix} \begin{pmatrix} H_{-(k+q)} & \cdots & H_{-(k+1)} \\ \vdots & & \vdots \\ H_{-(k+2q)} & \cdots & H_{-(k+q+1)} \end{pmatrix} \beta_{in} \tag{70}$$

Proof. The above formula can be easily derived by directly substituting (67) into $\xi(k) = C(\sigma)\beta(k)$. Alternatively, the formula for the output can be derived analytically by taking the Z-transform on (66) and making use of (69) □

Now, following the procedure of Karampetakis & Vologiannidis (2009), by defining the new state and output vectors

$$x(k) = \begin{pmatrix} \beta(kq+0) \\ \vdots \\ \beta(kq+q-1) \end{pmatrix}, \quad y(k) = \begin{pmatrix} \xi(kq+0) \\ \vdots \\ \xi(kq+q-1) \end{pmatrix} \tag{71}$$

the system (66) can be rewritten as

$$\underbrace{\begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix}}_{\tilde{E}} x(k+1) = -\underbrace{\begin{pmatrix} A_0 & \cdots & A_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_0 \end{pmatrix}}_{\tilde{A}} x(k) \tag{72a}$$

$$y(k) = \underbrace{\begin{pmatrix} C_0 & \cdots & C_{q-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & C_0 \end{pmatrix}}_{\tilde{C}_1} x(k) + \underbrace{\begin{pmatrix} C_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ C_1 & \cdots & C_q \end{pmatrix}}_{\tilde{C}_2} x(k+1) \tag{72b}$$

Thus, the higher order system (66) has been rewritten as a descriptor system of the form

$$\tilde{E}x(k+1) = -\tilde{A}x(k) \tag{73a}$$

$$y(k) = \tilde{C}_1 x(k) + \tilde{C}_2 x(k+1) \tag{73b}$$

with $\tilde{E}, \tilde{A} \in \mathbb{R}^{rq\times rq}$, $\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}^{mq\times rq}$.

Before we continue, it should be noted that the state vector defined in Karampetakis & Vologiannidis (2009) was $(\beta(kq+q-1)^T, \cdots, \beta(kq+0)^T)^T$, so the resulting system had the block transpose matrices of $\tilde{E}, \tilde{A}$. Thus, it is easily proved that the fundamental matrices Φ_i of the system (73) will be the block transpose of the ones in Karampetakis & Vologiannidis (2009).

For the descriptor system (73) we have the following important results.

Theorem 15: *Karampetakis & Vologiannidis (2009) The fundamental matrix sequences H_i of $A(\sigma)^{-1}$ and Φ_i of $(\sigma\tilde{E}+\tilde{A})^{-1}$ are connected by*

$$\Phi_i = \begin{pmatrix} H_{-q-qi} & \cdots & H_{-qi-1} \\ \vdots & & \vdots \\ H_{-2q-qi+1} & \cdots & H_{-q-qi} \end{pmatrix} \in \mathbb{R}^{rq\times rq} \tag{74}$$

Theorem 16: *Karampetakis & Vologiannidis (2009) The degrees of the determinants of $(\sigma\tilde{E}+\tilde{A})$ and $A(\sigma)$ are equal, that is*

$$\deg\det(\sigma\tilde{E}+\tilde{A}) = \deg\det A(\sigma) = n \tag{75}$$

Theorem 17: *Let $\lambda_i \in \mathbb{C}$, $i = 1,\ldots,\ell$ be the distinct zeros of $A(\sigma)$. Then the zeros of $(\sigma E + A)$ are given by λ_i^q.*

Proof. Let λ_i be a zero of $A(\sigma)$. It thus holds that $\det A(\lambda_i) = 0$. Define the following lower block triangular matrix

$$G = \begin{pmatrix} I_r & 0 & \cdots & 0 \\ \lambda_i I_r & I_r & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ \lambda_i^{q-1} I_r & 0 & \cdots & I_r \end{pmatrix}, \text{ with } \det G = 1 \tag{76}$$

The determinant of $(\sigma E + A)$ for $\sigma = \lambda_i^q$ is equal to

$$\det(\lambda_i^q E + A) = \left|\begin{pmatrix} \lambda_i^q A_q + A_0 & A_1 & \cdots & A_{q-1} \\ \vdots & \ddots & & \\ \lambda_i^q A_1 & \lambda_i^q A_2 & \cdots & \lambda_i^q A_q + A_0 \end{pmatrix}\right| = \tag{77}$$

$$= \left|\begin{pmatrix} \lambda_i^q A_q + A_0 & A_1 & \cdots & A_{q-1} \\ \vdots & \ddots & & \\ \lambda_i^q A_1 & \lambda_i^q A_2 & \cdots & \lambda_i^q A_q + A_0 \end{pmatrix}\begin{pmatrix} I_r & 0 & \cdots & 0 \\ \lambda_i I_r & I_r & \cdots & 0 \\ \vdots & \ddots & \cdots & \vdots \\ \lambda_i^{q-1} I_r & 0 & \cdots & I_r \end{pmatrix}\right| = \tag{78}$$

$$= \left|\begin{pmatrix} A(\lambda_i) & A_1 & \cdots & A_{q-1} \\ \vdots & & \vdots & \\ \lambda_i^{q-1} A(\lambda_i) & \lambda_i^q A_2 & \cdots & \lambda_i^q A_q + A_0 \end{pmatrix}\right| = \tag{79}$$

$$= \left|\begin{pmatrix} I_r & A_1 & \cdots & A_{q-1} \\ \vdots & & \vdots & \\ \lambda_i^{q-1} I_r & \lambda_i^q A_2 & \cdots & \lambda_i^q A_q + A_0 \end{pmatrix}\begin{pmatrix} A(\lambda_i) & & \\ & \ddots & \\ & & I_r \end{pmatrix}\right| = \tag{80}$$

$$= \left|\begin{pmatrix} I_r & A_1 & \cdots & A_{q-1} \\ \vdots & & \vdots & \\ \lambda_i^{q-1} I_r & \lambda_i^q A_2 & \cdots & \lambda_i^q A_q + A_0 \end{pmatrix}\right| |A(\lambda_i)| = 0 \tag{81}$$

and so λ_i^q are zeros of the matrix $(\sigma E + A)$. Since from Theorem 16 the determinants of the two matrices have the same degrees, the complete set of zeros of $(\sigma E + A)$ are given by λ_i^q . □

As in the previous section, there exist nonsingular matrices $P, Q \in \mathbb{R}^{rq\times rq}$ such that the system (73) is decomposed as in (39)-(41), with $A_1 \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{\mu\times\mu}$, where $\mu = rq - n$ is the sum of the orders of the infinite elementary divisors of $A(\sigma)$ as defined in (Vardulakis, 1991, Corollary 4.41), $F_1, D_1 \in \mathbb{R}^{mq\times n}$, $F_2, D_2 \in \mathbb{R}^{mq\times\mu}$ and N is nilpotent with index of nilpotency h.

Now we can proceed with studying the observability of the higher order system (66).

Definition 4: The higher order system (66) is completely observable if the initial values $\beta(0), ..., \beta(q-1)$, and consequently the vector β_{in} can be uniquely determined from knowledge of the output $\xi(k)$ over a finite time interval.

Arranging the initial values $\beta(0), ..., \beta(q-1)$ of (66) in a column it can be easily observed that

$$x(0) = \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix}, \quad \tilde{E}x(0) = \begin{pmatrix} A_q & \cdots & 0 \\ \vdots & \ddots & \vdots \\ A_1 & \cdots & A_q \end{pmatrix} \begin{pmatrix} \beta(0) \\ \vdots \\ \beta(q-1) \end{pmatrix} = \beta_{in} \tag{82}$$

So the two systems (66) and (73) have the same initial conditions. In addition, the admissible initial values of (73) are given by

$$x(0) = \Phi_0 \tilde{E} x(0) \tag{83}$$

and considering (74) and (72a), it is clear that (83) is equivalent to (69). Thus, by also taking into account Theorem 8 we conclude that the higher order system (66), the descriptor system (73) and the causal/noncausal subsystems (39)-(41) have the same set of admissible initial values.

Now, although the higher order system (66) and its corresponding descriptor system (73) may not be connected in a strict system equivalence sense (see Rosenbrock, 1970), there is a 1-1 correspondence between their input vectors $\beta(k)$, $x(k)$ and their output vectors $\xi(k)$, $y(k)$, given by

$$x(k) = \begin{pmatrix} \beta(kq+0) \\ \vdots \\ \beta(kq+q-1) \end{pmatrix} \tag{84}$$

$$\beta(k) = \begin{pmatrix} \delta_{(k \mod q)} I_r & \delta_{(k \mod q)-1} I_r & \cdots & \delta_{(k \mod q)-(q-1)} I_r \end{pmatrix} x\left(\left[\frac{k}{q}\right]\right) \tag{85}$$

$$y(k) = \begin{pmatrix} \xi(kq+0) \\ \vdots \\ \xi(kq+q-1) \end{pmatrix} \tag{86}$$

$$\xi(k) = \begin{pmatrix} \delta_{(k \mod q)} I_m & \delta_{(k \mod q)-1} I_m & \cdots & \delta_{(k \mod q)-(q-1)} I_m \end{pmatrix} y\left(\left[\frac{k}{q}\right]\right) \tag{87}$$

where $k \in \mathbb{N}$, $\delta_i = 0$ for $i \neq 0$ and $\delta_0 = 1$ and $[\cdot]$ denotes the integer part of a number. In addition, since the higher order system (66), the descriptor system (73) and its equivalent causal/noncausal subsystems (39)-(41) all have the same initial values $(\beta(0)^T, ..., \beta(q-1)^T)^T = x(0) = Q(\tilde{x}(0))$, it should be clear that in order to study the observability of the higher order system (66), one can instead study the observability of the causal/noncausal subsystems (39)-(41), since the determination of the initial value $\tilde{x}(0)$ will axiomatically give us the initial values of the higher order system. This leads to the following result.

Remark 3: The higher order system (66) is completely observable if and only if the descriptor system (73) is completely observable.

Corollary 1: *Using the connection between Φ_i and H_i in (74) and the form of $\tilde{C}_1, \tilde{C}_2$ in (72) the observability criterion (58) can be rewritten as*

$$rank\mathcal{O}_{PMD} = rank \begin{pmatrix} C_0 & \cdots & C_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_0 & \cdots & C_q \end{pmatrix} \begin{pmatrix} H_{-q} & \cdots & H_{-1} \\ \vdots & & \vdots \\ H_{-2q-qn+1} & \cdots & H_{-q-qn} \end{pmatrix} = n \tag{88}$$

So overall, the rank conditions in (57), (58) and (88) coincide.

In Theorem 11 we managed to connect the observability of (73) with the coprimeness of the matrices $(\sigma\tilde{E} + \tilde{A})$ and $(\tilde{C}_1 + \tilde{C}_2\sigma)$. The next step is to connect observability with the coprimeness of the original system matrices $A(\sigma)$, $C(\sigma)$. To do so we need the following Theorem.

Theorem 18: *The matrices $(\sigma\tilde{E} + \tilde{A})$ and $(\tilde{C}_1 + \tilde{C}_2\sigma)$ are right coprime, or equivalently*

$$rank \begin{pmatrix} \sigma\tilde{E} + \tilde{A} \\ \tilde{C}_1 + \tilde{C}_2\sigma \end{pmatrix} = rq \tag{89}$$

if and only if the matrices $A(\sigma)$ and $C(\sigma)$ are right coprime.

Proof. Before we proceed to the main part of this proof, we need to consider the division of the matrices $A(\sigma)$, $C(\sigma)$ and $(\sigma\tilde{E} + \tilde{A})$, $(\tilde{C}_1 + \tilde{C}_2\sigma)$ separately.

Considering the division of $A(\sigma)$, $C(\sigma)$, there exist $Q(\sigma), R(\sigma)$ such that

$$C(\sigma) = Q(\sigma)A(\sigma) + R(\sigma) \tag{90}$$

with $R(\sigma)A(\sigma)^{-1}$ strictly proper. From (90), we obtain

$$\begin{pmatrix} A(\sigma) \\ R(\sigma) \end{pmatrix} = \begin{pmatrix} I_r & 0 \\ -Q(\sigma) & I_m \end{pmatrix} \begin{pmatrix} A(\sigma) \\ C(\sigma) \end{pmatrix} \tag{91}$$

From the abve equation, we can easily conclude that $A(\sigma)$, $C(\sigma)$ are right coprime if and only if $A(\sigma)$, $R(\sigma)$ are right coprime. Multiplying (90) from the right by $A(\sigma)^{-1}$ we get

$$C(\sigma)A(\sigma)^{-1} = Q(\sigma) + R(\sigma)A(\sigma)^{-1} \tag{92}$$

Using the Laurent expansion of $A(\sigma)^{-1}$ from (6), $C(\sigma)A(\sigma)^{-1}$ is equal to

$$C(\sigma)A(\sigma)^{-1} = \begin{pmatrix} C_0 & \cdots & C_q \end{pmatrix} \left(\begin{array}{cccc|cc} & & H_{\hat{q}_r} & \cdots & H_0 & H_{-1} & \cdots \\ & \iddots & & & \vdots & \vdots & \cdots \\ H_{\hat{q}_r} & \cdots & & \cdots & H_{-q} & H_{-q-1} & \cdots \end{array}\right) \left(\begin{array}{c} \sigma^{q+\hat{q}_r} I_r \\ \vdots \\ I_r \\ \hline \sigma^{-1} I_r \\ \vdots \end{array}\right) \tag{93}$$

Comparing (92) and (93) we get that the strictly proper matrix $T(\sigma) = R(\sigma)A(\sigma)^{-1}$ is equal to

$$R(\sigma)A(\sigma)^{-1} = \begin{pmatrix} C_0 & \cdots & C_q \end{pmatrix} \begin{pmatrix} H_{-1} & H_{-2} & \cdots \\ \vdots & \vdots & \cdots \\ H_{-q-1} & H_{-q-2} & \cdots \end{pmatrix} \begin{pmatrix} \sigma^{-1} I \\ \sigma^{-2} I_r \\ \vdots \end{pmatrix} \overset{(9)}{=} \tag{94}$$

$$= \begin{pmatrix} C_0 & \cdots & C_q \end{pmatrix} \begin{pmatrix} CB & CJB & \cdots \\ \vdots & \vdots & \cdots \\ CJ^q B & CJ^{q+1} B & \cdots \end{pmatrix} \begin{pmatrix} \sigma^{-1} I_r \\ \sigma^{-2} I_r \\ \vdots \end{pmatrix} = \tag{95}$$

$$= \underbrace{\left(C_0 C + C_1 CJ + \ldots + C_q C J^q \right)}_{K} \underbrace{\begin{pmatrix} I & J & J^2 & \cdots \end{pmatrix} \begin{pmatrix} \sigma^{-1} I_n \\ \sigma^{-2} I_n \\ \vdots \end{pmatrix}}_{(\sigma I_n - J)^{-1}} B = \tag{96}$$

$$= K(\sigma I_n - J)^{-1} B \tag{97}$$

where $n = \deg\det A(\sigma)$. So $T(\sigma) = R(\sigma)A(\sigma)^{-1} = K(\sigma I_n - J)^{-1}B$ with $K \in \mathbb{R}^{m\times n}$, $J \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ and $(K, J, B, 0_{m\times r})$ is a realization of $T(\sigma)$. Now, since $(C, J, B, 0_{m\times r})$ is a minimal realization of the strictly proper part of $A(\sigma)^{-1}$, from Theorem 1 it holds that

$$rank \begin{pmatrix} B & JB & \ldots & J^{n-1}B \end{pmatrix} = n \tag{98}$$

As stated above, if $A(\sigma)$, $C(\sigma)$ are right coprime, then the matrices $A(\sigma)$ and $R(\sigma)$ are coprime as well and $T(\sigma) = R(\sigma)A(\sigma)^{-1}$ is a right coprime matrix fraction description of $T(\sigma)$ and since $J \in \mathbb{R}^{n\times n}$ with $n = \deg\det A(\sigma)$, from Theorem 1 it holds that $(K, J, B, 0_{m\times r})$ is a minimal realization of $T(\sigma)$. So it holds that

$$rank \begin{pmatrix} K \\ KJ \\ \vdots \\ KJ^{n-1} \end{pmatrix} = n \tag{99}$$

In the same fashion, if we consider the division of $(\sigma\tilde{E} + \tilde{A})$ and $(\tilde{C}_1 + \tilde{C}_2\sigma)$ we have

$$(\tilde{C}_1 + \tilde{C}_2\sigma) = \bar{Q}(\sigma)(\sigma\tilde{E} + \tilde{A}) + \bar{R}(\sigma) \tag{100}$$

where $\bar{R}(\sigma)(\sigma\tilde{E} + \tilde{A})^{-1}$ is strictly proper. From (100) we obtain

$$\begin{pmatrix} \sigma\tilde{E} + \tilde{A} \\ \bar{R}(\sigma) \end{pmatrix} = \begin{pmatrix} I_{rq} & 0 \\ -\bar{Q}(\sigma) & I_{mq} \end{pmatrix} \begin{pmatrix} \sigma\tilde{E} + \tilde{A} \\ \tilde{C}_1 + \tilde{C}_2\sigma \end{pmatrix} \tag{101}$$

so $(\sigma\tilde{E}+\tilde{A})$, $(\tilde{C}_1+\tilde{C}_2\sigma)$ are right corime if and only if $(\sigma\tilde{E}+\tilde{A})$, $\bar{R}(\sigma)$ are right coprime. Multiplying (100) from the right by $(\sigma\tilde{E} + \tilde{A})^{-1}$ we get

$$(\tilde{C}_1 + \tilde{C}_2\sigma)(\sigma\tilde{E} + \tilde{A})^{-1} = \bar{Q}(\sigma) + \bar{R}(\sigma)(\sigma\tilde{E} + \tilde{A})^{-1} \tag{102}$$

Using the Laurent expansion of $(\sigma\tilde{E}+\tilde{A})^{-1}$, $(\tilde{C}_1+\tilde{C}_2\sigma)(\sigma\tilde{E}+\tilde{A})^{-1}$ is equal to

$$(\tilde{C}_1+\tilde{C}_2\sigma)(\sigma\tilde{E}+\tilde{A})^{-1}=\begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}\left(\begin{array}{cccc|cc} 0 & \Phi_{-\mu} & \cdots & \Phi_{-1} & \Phi_0 & \cdots \\ \Phi_{-\mu} & \Phi_{-\mu+1} & \cdots & \Phi_0 & \Phi_1 & \cdots \end{array}\right)\left(\begin{array}{c}\sigma^{\mu}I_{rq}\\ \vdots \\ I_{rq} \\ \hline \sigma^{-1}I_{rq} \\ \vdots\end{array}\right) \tag{103}$$

Comparing (102) and (103) we get that the strictly proper matrix $\bar{T}(\sigma)=\bar{R}(\sigma)(\sigma\tilde{E}+\tilde{A})^{-1}$ is equal to

$$\bar{T}(\sigma)=\bar{R}(\sigma)(\sigma\tilde{E}+\tilde{A})^{-1}=\begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}\begin{pmatrix}\Phi_0 & \Phi_1 & \cdots \\ \Phi_1 & \Phi_2 & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}I_{rq}\\ \vdots\end{pmatrix}\overset{(49)}{=} \tag{104}$$

$$=\begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}\begin{pmatrix} Q\begin{pmatrix}I_n & 0\\ 0 & 0_\mu\end{pmatrix}P & Q\begin{pmatrix}A_1 & 0\\ 0 & 0_\mu\end{pmatrix}P & \cdots \\ Q\begin{pmatrix}A_1 & 0\\ 0 & 0_\mu\end{pmatrix}P & Q\begin{pmatrix}A_1^2 & 0\\ 0 & 0_\mu\end{pmatrix}P & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}I_{rq}\\ \vdots\end{pmatrix}= \tag{105}$$

$$=\begin{pmatrix}\tilde{C}_1 & \tilde{C}_2\end{pmatrix}\begin{pmatrix}Q & 0\\ 0 & Q\end{pmatrix}\begin{pmatrix}\begin{pmatrix}I_n & 0\\ 0 & 0_\mu\end{pmatrix} & \begin{pmatrix}A_1 & 0\\ 0 & 0_\mu\end{pmatrix} & \cdots \\ \begin{pmatrix}A_1 & 0\\ 0 & 0_\mu\end{pmatrix} & \begin{pmatrix}A_1^2 & 0\\ 0 & 0_\mu\end{pmatrix} & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}P\\ \vdots\end{pmatrix}\overset{(31)}{\underset{(32)}{=}} \tag{106}$$

$$=\begin{pmatrix}F_1 & F_2 & D_1 & D_2\end{pmatrix}\begin{pmatrix} I_n & 0_{n\times\mu} & A_1 & 0_{n\times\mu} & \cdots \\ 0_{\mu\times n} & 0_\mu & 0_{\mu\times n} & 0_\mu & \cdots \\ A_1 & 0_{n\times\mu} & A_1^2 & 0_{n\times\mu} & \cdots \\ 0_{\mu\times n} & 0_\mu & 0_{\mu\times n} & 0_\mu & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}P\\ \vdots\end{pmatrix}= \tag{107}$$

$$=\begin{pmatrix}F_1 & D_1\end{pmatrix}\begin{pmatrix}I_n & 0_{n\times\mu} & A_1 & 0_{n\times\mu} & \cdots \\ A_1 & 0_{n\times\mu} & A_1^2 & 0_{n\times\mu} & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}P\\ \vdots\end{pmatrix} \tag{108}$$

and by decomposing the matrix P as

$$P=\begin{pmatrix}P_1\\ P_2\end{pmatrix} \tag{109}$$

with $P_1\in\mathbb{R}^{n\times rq}$, $P_2\in\mathbb{R}^{\mu\times rq}$ the above is equal to

$$\bar{T}(\sigma)=\begin{pmatrix}F_1 & D_1\end{pmatrix}\begin{pmatrix}I_n & A_1 & \cdots \\ A_1 & A_1^2 & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}P_1\\ \vdots\end{pmatrix}= \tag{110}$$

$$=(F_1+D_1A_1)\begin{pmatrix}I_n & A_1 & A_1^2 & \cdots\end{pmatrix}\begin{pmatrix}\sigma^{-1}P_1\\ \vdots\end{pmatrix}= \tag{111}$$

$$=(F_1+D_1A_1)(\sigma I_n-A_1)^{-1}P_1 \tag{112}$$

where $n=\deg\det(\sigma\tilde{E}+\tilde{A})=\deg\det A(\sigma)$, from Theorem (16). So $\bar{T}(\sigma)=\bar{R}(\sigma)(\sigma\tilde{E}+\tilde{A})^{-1}=(F_1+D_1A_1)(\sigma I_n-A_1)^{-1}P_1$ and $((F_1+D_1A_1),A_1,P_1,0_{mq\times rq})$ is a realization of $\bar{T}(\sigma)$.

Now we can proceed to the main part of our proof regarding the coprimeness of the matrices $A(\sigma)$, $C(\sigma)$ and $(\tilde{C}_1+\tilde{C}_2\sigma)$, $(\sigma\tilde{E}+\tilde{A})$.

First, assume that $(\tilde{C}_1+\tilde{C}_2\sigma)$ and $(\sigma\tilde{E}+\tilde{A})$ are right coprime. Then the matrices $(\sigma\tilde{E}+\tilde{A})$, $\bar{R}(\sigma)$ are also right coprime and $((F_1+D_1A_1),A_1,P_1,0_{mq\times rq})$ is a minimal realization of $\bar{T}(\sigma)$. It thus holds that

$$rank\begin{pmatrix} F_1+D_1A_1 \\ (F_1+D_1A_1)A_1 \\ \vdots \\ (F_1+D_1A_1)A_1^{n-1} \end{pmatrix} = n \underset{\text{Theorem 10}}{\overset{\text{Proof of}}{\Leftrightarrow}} rank\begin{pmatrix} C_1 & C_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_1 & C_2 \end{pmatrix}\begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} = n \underset{(74)}{\overset{(72)}{\Leftrightarrow}} \tag{113}$$

$$rank\begin{pmatrix} C_0 & \cdots & C_q & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & C_0 & \cdots & C_q \end{pmatrix}\begin{pmatrix} H_{-q} & \cdots & H_{-1} \\ \vdots & & \vdots \\ H_{-2q-qn+1} & \cdots & H_{-q-qn} \end{pmatrix} = n \Leftrightarrow \tag{114}$$

$$rank\begin{pmatrix} KJ^{q-1}B & \cdots & KB \\ \vdots & & \vdots \\ KJ^{q+qn-2}B & \cdots & KJ^{qn-1}B \end{pmatrix} = n \Leftrightarrow \tag{115}$$

$$rank\begin{pmatrix} K \\ KJ \\ \vdots \\ KJ^{qn-1} \end{pmatrix}\begin{pmatrix} J^{q-1}B & \cdots & B \end{pmatrix} = n \tag{116}$$

from the above relation we conclude that

$$rank\begin{pmatrix} K \\ KJ \\ \vdots \\ KJ^{qn-1} \end{pmatrix} = n \tag{117}$$

From the Cayley-Hamilton theorem we know that J, which is in Jordan form, satisfies its characteristic polynomial $p(\lambda) = \det(\lambda I_n - J) = \lambda^n + \sum_{i=0}^{n-1} p_i\lambda^i$. Hence, J satisfies

$$p(J) = J^n + p_{n-1}J^{n-1} + \ldots + p_0 I_n = 0 \tag{118}$$

This means that the matrix $J^n = -p_{n-1}J^{n-1} - \ldots - p_0 I_n$ is linearly dependent to the matrices J^{n-1}, J^{n-2},..., J, I_n. Multiplying the above equation from the right by J and replacing J^n, we get that the same holds for J^{n+1}. Following this procedure, we get that all $J^k, k \geq n$ are linearly dependent to J^{n-1} ,J^{n-2},..., J, I_n. As a result

$$J^i \in \operatorname{Im}\begin{pmatrix} J^{n-1} & \cdots & J & I_n \end{pmatrix} \Rightarrow KJ^i \in \operatorname{Im}\begin{pmatrix} KJ^{n-1} & \cdots & KJ & K \end{pmatrix},\ i \geq n \tag{119}$$

Thus we conclude that the last $mqn - mn = mn(q-1)$ rows are linearly dependent to its first mn rows $K, KJ, \ldots, KJ^{n-1}$. It thus holds that

$$rank\begin{pmatrix} K \\ KJ \\ \vdots \\ KJ^{n-1} \end{pmatrix} = n \tag{120}$$

and $(K, J, B, 0_{m\times r})$ is a minimal realization of $T(\sigma)$ and so $A(\sigma)$, $R(\sigma)$ and $A(\sigma)$, $C(\sigma)$ are right coprime.

For the reverse, assume that $A(\sigma)$ and $C(\sigma)$ are right coprime. The pairs (K, J) and (J, B) are observable and controllable respectively. Considering again the rank of the following matrix

$$rank \begin{pmatrix} F_1 + D_1A_1 \\ (F_1 + D_1A_1)A_1 \\ \vdots \\ (F_1 + D_1A_1)A_1^{n-1} \end{pmatrix} = rank \underbrace{\begin{pmatrix} K \\ KJ \\ \vdots \\ KJ^{qn-1} \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} J^{q-1}B & \cdots & B \end{pmatrix}}_{L} = rank(ML) \tag{121}$$

with $M \in \mathbb{R}^{mqn \times n}$, $L \in \mathbb{R}^{n \times rq}$. Since (K, J) is an observable pair, we can easily conclude that $rankM = n$. For the product ML it holds that (Bernstein, 2009, Proposition 2.5.9):

$$\left. \begin{array}{c} rank(ML) \leq rankM = n \\ rank(ML) \leq rankL \\ rank(ML) \geq rankM + rankL - n = rankL \end{array} \right\} \Rightarrow rank(ML) = rankL \tag{122}$$

Thus, it suffices to show that $rankL = n$. Considering the minimal realization of $A(\sigma)_{sp}^{-1}$ from Theorem 2:

$$A(\sigma)_{sp}^{-1} = C(\sigma I_n - J)^{-1}B \Rightarrow A(\sigma)_{sp}^{T\,-1} = \left(C(\sigma I_n - J)^{-1}B\right)^T = B^T(\sigma I_n - J^T)^{-1}C^T \tag{123}$$

so (B^T, J^T, C^T) is a minimal realization of $A(\sigma)_{sp}^{T\,-1}$, thus from Theorem 4 it holds that

$$rank \begin{pmatrix} B^T \\ \vdots \\ B^T J^{T\,q-1} \end{pmatrix} = n \Rightarrow rankL = n \tag{124}$$

and the pair $(F_1 + D_1A_1, A_1)$ is observable. So

$$rank \begin{pmatrix} \sigma I_n - A_1 \\ F_1 + D_1A_1 \end{pmatrix} = n \underset{\text{Theorem (11)}}{\overset{\text{proof of}}{\Leftrightarrow}} rank \begin{pmatrix} \sigma\tilde{E} + \tilde{A} \\ \tilde{C}_1 + \tilde{C}_2 \end{pmatrix} = rq \tag{125}$$

and the matrices $(\tilde{C}_1 + \tilde{C}_2\sigma)$, $(\sigma\tilde{E} + \tilde{A})$ are right coprime. □

Summarising the results from Theorems 9, 10 11, 18, Remark 3 and Corollary 1 we conclude to the following result.

Theorem 19: *The following statements are equivalent.*

(1) The higher order system (66) is completely observable.
(2) $rank\mathcal{O}_{PMD} = n$.
(3) The matrices $A(\sigma)$ and $C(\sigma)$ are right coprime.
(4) The descriptor system (73) is completely observable.
(5) $rank\mathcal{O}_{descriptor} = n$.
(6) The matrices $(\sigma\tilde{E} + A)$ and $(\tilde{C}_1 + \tilde{C}_2\sigma)$ are right coprime.
(7) The causal subsystem (39) is observable.
(8) $rank\mathcal{O}_{causal} = n$.
(9) The matrices $(\sigma I_n - A_1)$ and $(F_1 + D_1A_1)$ are right coprime.

Remark 4: It is worth noting that using the solution formula (24), $x(k+1)$ can be rewritten as

$$x(k+1) = \Phi_{k+1}Ex(0) \overset{(21e)}{=} \Phi_1 E \Phi_k Ex(0) = \Phi_1 Ex(k) \tag{126}$$

and so the output descriptor system (23) can be written as

$$y(k) = C_1 x(k) + C_2 x(k+1) = (C_1 + C_2 \Phi_1 E) x(k) \tag{127}$$

and can be studied as a standard descriptor system.

It should be noted that the coprimeness of the matrices $A(\sigma)$ and $C(\sigma)$ was taken as an initial assumption in Yu & Duan (2010, 2009); Wu & Duan (2007) where the problem of eigenstructure assignment and observer design via output feedback was considered, for continuous time descriptor systems and higher order systems of algebraic and difference equations.

Example 1: Consider the system

$$\begin{pmatrix} 1+2\sigma+\sigma^2 & 1 \\ 0 & \sigma-1 \end{pmatrix} \beta(k) = 0 \tag{128a}$$

$$\xi(k) = \begin{pmatrix} \sigma^2 & 2+\sigma \end{pmatrix} \beta(k) \tag{128b}$$

For the matrix $A(\sigma)$ we have $\det A(\sigma) = (s+1)^2(s-1)$, so $q = 2$, $r = 2$, $n = 3$ and $n + \mu = rq \Rightarrow \mu = 1$. The triple (C, J, B) is given by

$$C = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & \frac{1}{4} \\ 1 & \frac{1}{2} \end{pmatrix} \tag{129}$$

The corresponding descriptor system is

$$\begin{pmatrix} A_2 & 0 \\ A_1 & A_2 \end{pmatrix} x(k+1) = -\begin{pmatrix} A_0 & A_1 \\ 0 & A_0 \end{pmatrix} x(k) \tag{130a}$$

$$y(k) = \begin{pmatrix} C_0 & C_1 \\ 0 & C_0 \end{pmatrix} x(k) + \begin{pmatrix} C_2 & 0 \\ C_1 & C_2 \end{pmatrix} x(k+1) \tag{130b}$$

or equivalently

$$\underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)}_{\tilde{E}} x(k+1) = -\underbrace{\left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{array}\right)}_{\tilde{A}} x(k) \tag{131a}$$

$$y(k) = \underbrace{\left(\begin{array}{cc|cc} 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right)}_{\tilde{C}_1} x(k) + \underbrace{\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)}_{\tilde{C}_2} x(k+1) \tag{131b}$$

with

$$x(k) = \begin{pmatrix} \beta(2k) \\ \beta(2k+1) \end{pmatrix}, \quad y(k) = \begin{pmatrix} \xi(2k) \\ \xi(2k+1) \end{pmatrix} \tag{132}$$

The matrices that give the decomposition into the causal and noncausal subsystems are

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ -2 & -1 & 1 & 0 \\ -2 & -1 & 2 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; Q = \begin{pmatrix} -1 & -1 & \frac{1}{2} & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix} \tag{133}$$

with

$$P\tilde{E}Q = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \; P(-\tilde{A})Q = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \tag{134}$$

$$\tilde{C}_1 Q = \left(\begin{array}{ccc|c} 6 & 0 & 0 & 1 \\ 4 & 0 & 0 & 2 \end{array}\right), \; \tilde{C}_2 Q = \left(\begin{array}{ccc|c} -1 & -1 & 1/2 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right) \tag{135}$$

and the resulting subsystems are

$$\tilde{x}_1(k+1) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{A_1} \tilde{x}_1(k), \quad y_1(k) = \underbrace{\begin{pmatrix} 5 & -1 & -\frac{1}{2} \\ 6 & 1 & 1 \end{pmatrix}}_{(F_1+D_1A_1)} \tilde{x}_1(k) \tag{136a}$$

and

$$\underbrace{0}_{N} = \tilde{x}_2(k), \quad y_2(k) = \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{F_2} \tilde{x}_2(k) + \underbrace{0_{2\times1}}_{D_2} \tilde{x}_2(k+1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tilde{x}_2(k) = 0_{2\times1} \tag{137a}$$

with $y(k) = y_1(k) + y_2(k)$ and the transformation

$$x(k) = Q \begin{pmatrix} \tilde{x}_1(k) \\ \tilde{x}_2(k) \end{pmatrix} \tag{138}$$

with $\tilde{x}_1(k) \in \mathbb{R}^3$, $\tilde{x}_2(k) \in \mathbb{R}$ and admissibility set $H^{ad}_{c-nc} = \{\tilde{x}_1(0) \in \mathbb{R}^3, \tilde{x}_2(0) \in \mathbb{R} \,|\, \tilde{x}_2(0) = 0\}$. The observability matrix of the causal subsystem is

$$\mathcal{O}_{causal} = \begin{pmatrix} F_1 + D_1 A_1 \\ (F_1 + D_1 A_1) A_1 \\ (F_1 + D_1 A_1) A_1^2 \end{pmatrix} = \left(\begin{array}{ccc} 5 & -1 & -\frac{1}{2} \\ 6 & 1 & 1 \\ \hline 5 & -1 & -\frac{3}{2} \\ 6 & 1 & 2 \\ \hline 5 & -1 & -\frac{5}{2} \\ 6 & 1 & 3 \end{array}\right) \tag{139}$$

and has $rank \mathcal{O}_{causal} = 3$, so the causal subsystem is observable and thus the complete system is observable. The same holds for the matrices $\mathcal{O}_{descriptor}$ and $\mathcal{O}_{PMD}$. In addition, the matrix pairs $A(\sigma)$, $C(\sigma)$ and $(\sigma\tilde{E} + \tilde{A})$, $(\tilde{C}_1 + \tilde{C}_2\sigma)$ are right coprime.

5. Conclusions

The observability of higher order systems of algebraic and difference equations was studied. By first transforming the system into a first order descriptor system with a state lead in the output and then further transforming it into an equivalent causal/noncausal subsystem decomposition, observability criteria have been derived for the higher order system in all its forms. In its original form (66), observability criteria are given in terms of the Laurent expansion of $A(\sigma)^{-1}$ and the coprimeness of $A(\sigma)$, $C(\sigma)$. In its descriptor form (73) observability criteria are given in terms of the Laurent expansion of $(\sigma\tilde{E}+\tilde{A})^{-1}$ and the coprimeness of $(\sigma\tilde{E}+\tilde{A})$, $(\tilde{C}_1+\tilde{C}_2\sigma)$. In its causal/noncausal subsystem form, observability criteria are given in terms of the state space observability matrix of the causal subsystem and the coprimeness of $(\sigma I_n - A_1)$, $(F_1 + D_1 A_1)$. These results can be extended to higher order continuous time systems following the work of Tan & Zhang (2010), or to higher order positive systems. The problems of observer design, or pole placement via output feedback can also be studied.

References

Antoniou, E.N., Vardulakis, A.I.G., & Karampetakis, N.P. (1998). A spectral characterization of the behavior of discrete time AR-representations over a finite time interval.. *Kybernetika*, *34*(5), 555–564.

Antsaklis, Panos J., & Michel, Anthony N. (2006). *Linear systems.*. 2nd corrected printing ed., Boston: Birkhäuser.

Bender, D (1987). Lyapunov-like equations and reachability/observabiliy Gramians for descriptor systems. *IEEE Transactions on Automatic Control*, *32*(4), 343–348.

Berger, Thomas, Reis, Timo, & Trenn, Stephan (2016). Observability of linear differential-algebraic systems - a survey. In *Surveys in differential-algebraic equations IV, Differential-Algebraic Equations Forum* Springer-Verlag, Berlin-Heidelberg.

Bernstein, D. S. (2009). *Matrix mathematics. Theory, facts, and formulas*. Second ed., Princeton, University Press.

Bhaya, Amit, & Desoer, Charles (1985). On the design of large flexible space structures (LFSS). *IEEE Transactions on Automatic Control*, *30*(11), 1118–1120.

Brull, Tobias (2009). Explicit solutions of regular linear discrete-time descriptor systems with constant coefficients.. *ELA. The Electronic Journal of Linear Algebra [electronic only]*, *18*, 317–338.

Campbell, S.L.V. (1980). *Singular systems of differential equations, volume 1*. Research notes in mathematics, Pitman.

Coll, Carmen, Fullana, Màrius J., & Sánchez, Elena (2002). Some invariants of discrete-time descriptor systems.. *Appl. Math. Comput.*, *127*(2-3), 277–287.

Dai, Liyi (1989). *Singular control systems.*. Berlin etc.: Springer-Verlag.

Dassios, Ioannis (2012). On non-homogeneous generalized linear discrete time systems. *Circuits, Systems, and Signal Processing*, *31*(5), 1699–1712.

Dassios, Ioannis, & Kalogeropoulos, Grigoris (2013). On a non-homogeneous singular linear discrete time system with a singular matrix pencil. *Circuits, Systems, and Signal Processing*, *32*(4), 1615–1635.

Duan, Guang-Ren (2010). *Analysis and design of descriptor linear systems.*. Dordrecht: Springer.

Duan, Guang-Ren, & Yu, Hai-Hua (2006). Observer design in high-order descriptor linear systems. In *Sice-icase, 2006. international joint conference*. IEEE, (pp. 870–875).

Foster, Joanne A, McWhirter, John G, Davies, Martin R, & Chambers, Jonathon A (2010). An algorithm for calculating the QR and singular value decompositions of polynomial matrices. *IEEE Transactions on Signal Processing*, *58*(3), 1263–1274.

Fragulis, G., Mertzios, B.G., & Vardulakis, A.I.G. (1991). Computation of the inverse of a polynomial matrix and evaluation of its Laurent expansion. *International Journal of Control*, *53*(2), 431–443.

Gantmacher, F. R. (1959). *The theory of matrices. Vols. 1, 2*. Chelsea Publishing Co., New York.

Gohberg, Israel, Lancaster, Peter, & Rodman, Leiba (2009). *Matrix polynomials*. Reprint of the 1982 original ed., Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM).

Hou, M, & Muller, PC (1999). Causal observability of descriptor systems. *IEEE Transactions on Automatic Control*, *44*(1), 158–163.

Ishihara, Joao Yoshiyuki, & Terra, Marco Henrique (2001). Impulse controllability and observability of rectangular descriptor systems. *Automatic Control, IEEE Transactions on*, *46*(6), 991–994.

Jones, J., Karampetakis, N.P., & Pugh, A.C. (2003). Solution of discrete ARMA-representations via MAPLE.. *Appl. Math. Comput.*, *139*(2-3), 437–489.

Kaczorek, Tadeusz (2002). *Positive 1D and 2D systems.*. London: Springer.

Kailath, Thomas (1980). *Linear systems*. Prentice-Hall Information and System Sciences Series, Prentice-Hall, Inc., Englewood Cliffs, N.J.

Kalman, Rudolf (1959). On the general theory of control systems. *IRE Transactions on Automatic Control*, *4*(3), 110–110.

Kalogeropoulos, GI, Karageorgos, AD, & Pantelous, AA (2009). Discretising effectively a linear singular differential system by choosing an appropriate sampling period. *IET Control Theory & Applications*, *3*(7), 823–833.

Kalogeropoulos, Grigoris I, Karageorgos, Athanasios D, & Pantelous, Athanasios A (2009). Higher-order linear matrix descriptor differential equations of Apostol-Kolodner type. *Electronic Journal of Differential Equations*, *2009*(25), 1–13.

Karampetakis, Nicholas P (2004). On the discretization of singular systems. *IMA Journal of Mathematical Control and Information*, *21*(2), 223–242.

Karampetakis, NP (2004). On the solution space of discrete time AR-representations over a finite time horizon. *Linear algebra and its applications*, *382*, 83–116.

Karampetakis, N.P., & Gregoriadou, A. (2014). Reachability and controllability of discrete-time descriptor systems. *International Journal of Control*, *87*(2), 235–248.

Karampetakis, N.P., Jones, J., & Antoniou, E.N. (2001). Forward, backward, and symmetric solutions of discrete ARMA representations.. *Circuits Syst. Signal Process.*, *20*(1), 89–109.

Karampetakis, N.P., & Vologiannidis, S. (2009). On the fundamental matrix of the inverse of a polynomial matrix and applications to ARMA representations.. *Linear Algebra Appl.*, *431*(11), 2261–2276.

Koumboulis, F.N., & Mertzios, B.G. (1999). On Kalman's controllability and observability criteria for singular systems.. *Circuits Syst. Signal Process.*, *18*(3), 269–290.

Langenhop, CE (1971). The Laurent expansion for a nearly singular matrix. *Linear Algebra and its Applications*, *4*(4), 329–340.

Laub, Alan J., & Arnold, W.F. (1984). Controllability and observability criteria for multivariable linear second-order models.. *IEEE Trans. Autom. Control*, *29*, 163–165.

Lewis, F (1985). Fundamental, reachability, and observability matrices for discrete descriptor systems. *IEEE Transactions on automatic control*, *30*(5), 502–505.

Lewis, F.L., & Mertzios, B.G. (1990). On the analysis of discrete linear time-invariant singular systems.. *IEEE Trans. Autom. Control*, *35*(4), 506–511.

Lewis, Frank (1984). Descriptor systems: Decomposition into forward and backward subsystems. *IEEE Transactions on Automatic Control*, *29*(2), 167–170.

Lewis, Frank L (1986). A survey of linear singular systems. *Circuits, Systems and Signal Processing*, *5*(1), 3–36.

Liu, Peiyong, Zhang, Qingling, Yang, Xiaoguang, & Yang, Li (2008). Passivity and optimal control of descriptor biological complex systems. *IEEE Transactions on Automatic Control*, *53*(Special Issue), 122–125.

Losse, Philip, & Mehrmann, Volker (2008). Controllability and observability of second order descriptor systems.. *SIAM J. Control Optim.*, *47*(3), 1351–1379.

Mahmood, S., Karampetakis, N.P., & Pugh, A.C. (1998). Solvability, reachability, controllability and observability of regular PMDs. *International Journal of Control*, *70*(4), 617–630.

Mehmood, Yasir, Mustafa, Ghulam, Khan, Abdul Qayyum, Abid, Muhammad, & Darouach, Mohamed (2016). Approximate discretization of regular descriptor (singular) systems with impulsive mode. *Automatica*, *73*, 231–236.

Mertzios, BG, & Lewis, FL (1989). Fundamental matrix of discrete singular systems. *Circuits, Systems and Signal Processing*, *8*(3), 341–355.

Moysis, Lazaros, & Karampetakis, Nicholas P (2016). Reachability of discrete time ARMA representations. *IMA Journal of Mathematical Control and Information*.

Moysis, Lazaros, Pantelous, Athanasios A, Antoniou, Efstathios, & Karampetakis, Nicholas P (2016). Closed Form Solution for the Equations of Motion for Constrained Linear Mechanical Systems and Generalisations: An Algebraic Approach. *Journal of the Franklin Institute*, *354*(3), 1421 – 1445.

Ozcaldiran, K, Fountain, DW, & Lewis, FL (1992). Some generalized notions of observability. *IEEE Transactions on Automatic Control*, *37*(6), 856–860.

Pantelous, AA, Karageorgos, AD, & Kalogeropoulos, GI (2014). A new approach for second-order linear matrix descriptor differential equations of Apostol–Kolodner type. *Mathematical Methods in the Applied Sciences*, *37*(2), 257–264.

Pugh, A.C., Karampetakis, N.P., Hayton, G.E., & Vardulakis, A.I. (1993). On a certain McMillan-degree condition appearing in control.. *IMA J. Math. Control Inf.*, *10*(4), 361–373.

Pugh, AC, Karampetakis, NP, Mahmood, S, & Hayton, GE (1995). On the admissible initial conditions of a regular PMD. In *Decision and control, 1995., proceedings of the 34th ieee conference on*. Vol. 1, IEEE, (pp. 307–308).

Rosenbrock, Howard Harry (1970). State-space and multivariable theory. .

Schweitzer, Paul, & Stewart, GW (1993). The Laurent expansion of pencils that are singular at the origin. *Linear Algebra and its Applications*, *183*, 237–254.

Taher, R Ben, & Rachidi, Mustapha (2008). Linear matrix differential equations of higher-order and applications. *Electronic Journal of Differential Equations*, *2008*(95), 1–12.

Tan, Chong, & Zhang, Xian (2010). Observability of linear time-invariant descriptor systems with a derivative in the output. *Journal of Applied Mathematics and Computing*, *33*(1-2), 239–250.

Vardulakis, A.I.G. (1991). *Linear multivariable control. Algebraic analysis and synthesis methods.*. Chichester etc.: John Wiley &— Sons.

Vardulakis, A.I.G., Antoniou, E.N., & Karampetakis, N.P. (1999). On the solution and impulsive behaviour of polynomial matrix descriptions of free linear multivariable systems. *International Journal of Control*, *72*(3), 215–228.

Verghese, George, Lévy, Bernard, & Kailath, Thomas (1981). A generalized state-space for singular systems. *IEEE Transactions on Automatic Control*, *26*(4), 811–831.

Wu, A, & Duan, G (2007). Design of PD observers in descriptor linear systems. *International Journal of Control Automation and Systems*, *5*(1), 93.

Yip, Elizabeth L., & Sincovec, Richard F. (1981). Solvability, controllability, and observability of continuous descriptor systems.. *IEEE Trans. Autom. Control*, *26*, 702–707.

Yu, Hai-Hua, & Duan, Guang-Ren (2009). ESA in high-order linear systems via output feedback. *Asian Journal of Control*, *11*(3), 336–343.

Yu, Hai-Hua, & Duan, Guang-Ren (2010). ESA in high-order descriptor linear systems via output feedback. *International Journal of Control, Automation and Systems*, *8*(2), 408–417.

Zhang, Xuxi, & Liu, Xianping (2012). Output regulation for matrix second order singular systems via measurement output feedback. *Journal of the Franklin Institute*, *349*(6), 2124–2135.