

Stability of positive delay systems with delayed impulses

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Abstract: This study investigates the global exponential stability of positive delay systems with delayed impulses. By using linear copositive Lyapunov functions together with Razumikhin techniques, a number of criteria for global exponential stability of positive delay systems with delayed impulses are provided. It should be noted that it is the first time that the Razumikhin type exponential stability results for positive delay systems with delayed impulses are given. Numerical examples are provided to demonstrate the effectiveness of the derived results.

1 Introduction

Many physical systems in the real-world involve variables that have non-negative sign, just like population values, electric current, concentration of substances etc. Such systems are said to be positive, which means that their states and outputs are non-negative whenever the initial conditions and inputs are non-negative, see [1] and the references therein. Positive systems play a key role in many areas such as engineering, physics, chemistry, biology, economics etc. Therefore in recent years, positive systems have drawn considerable research interest in the control community, see [2–12] and the references therein.

On the one hand, because the feedback is delayed and the delayed measurements, time delay is the inherent feature of many physical processes, time delay is also an important source of instability and poor performance, even makes systems out of control. Recently, there are many researchers considering the problem related to the time-delayed positive systems and some valuable results have been established, see [13–19] and the references therein.

On the other hand, the state of many real-world systems is subject to instantaneous changes, and will experience abrupt changes at certain moments of time, these kinds of systems can be modelled as impulsive systems. Impulsive systems have applications in the problems arising in economics, mechanics, chemistry, biological phenomena, population dynamics etc. In recent years, the theory of impulsive systems has become an important area of investigation, see [20–44] and the references therein. Among them, recently, the theory of impulsive positive systems has attracted the interest of many researchers [37–44] etc, some researchers have proved that impulsive positive systems can be used to represent certain classes of population models [42], epidemiology [43], ecosystems [44] etc, which having deterministic jumps in their dynamics. By using linear co-positive Lyapunov functions, Zhang *et al.* [37] established some criteria of stability for impulsive positive systems. By using a co-positive Lyapunov–Krasovskii functional and the average impulsive interval method, Wang *et al.* [38] further considered the exponential stability of impulsive positive systems with time delay. On the basis of the mode-dependent average dwell-time approach, Liu *et al.* [39] analysed the asynchronously finite-time control of discrete impulsive switched positive time-delay systems and obtained the existence of a family of asynchronously switched controllers. Briat [40] investigated the dwell-time stability and stabilisation for linear positive impulsive and switched systems under arbitrary, constant, minimum, maximum and range dwell time. Hu *et al.* [41] investigated the stability and stabilisation for

impulsive positive linear systems under minimum, maximum dwell time. We can easily see that in those previous works, the authors always suppose that the state variables on the impulses are only related to the nearest state variables, but in most cases it is more applicable that the state variables on the impulses are also related to the time delays, see [28, 35] and the references therein. To the best of the authors' knowledge, however, up to now, stability of positive delay systems with delayed impulses has not been addressed.

Motivated by the above discussion, the focus of this paper is on the stability problem for a class of positive delay systems with delayed impulses. By using linear co-positive Lyapunov functions together with Razumikhin techniques, some criteria for global exponential stability of positive delay systems with delayed impulses are provided. The main contribution of this paper is three-fold: (i) the paper represents the first of few attempts to deal with the positivity problem for a class of positive delay systems with delayed impulses; (ii) some sufficient conditions for global exponential stability with respect to destabilising delayed impulses and stabilising delayed impulses are established; and (iii) some sufficient conditions for global exponential stability of non-linear positive delay systems with delayed impulses are provided.

The remainder of this paper is organised as follows. In Section 2, notations and necessary preliminaries are presented. The main results are presented in Section 3, where some conditions are proposed for guaranteeing positivity of the system and a number of stability criteria for positive delay systems with delayed impulses are given. Some examples are presented in Section 4 to illustrate the effectiveness of the proposed theoretical results. The conclusions are finally drawn in Section 5.

2 Notations and preliminaries

In this section, we introduce some notations, definitions and preliminary facts which are used throughout this paper.

Let \mathbf{R} denote the set of real numbers, \mathbf{R}_+ denote the set of non-negative real numbers. Let \mathbf{Z}^+ denote the set of positive integers, i.e. $\mathbf{Z}^+ = \{1, 2, \dots\}$. $\mathbf{R}^n(\mathbf{R}_+^n)$ is the n -dimensional real (non-negative) space and $\mathbf{R}^{n \times m}$ is the set of all real matrices of $(n \times m)$ -dimension. $\mathbf{R}_+^{n \times m}$ is the set of all real matrices of $(n \times m)$ -dimension, whose elements are non-negative. The superscript T stands for matrix transposition, the notation $\|\cdot\|$ refers to the vector 1-norm. For a matrix $A \in \mathbf{R}^{n \times n}$, $A > 0$ (≥ 0) means that all elements of the matrix A are positive (non-negative). For a vector $v \in \mathbf{R}^n$, $v > 0$ (≥ 0 , < 0 , ≤ 0) means that all elements of the vector v are positive (non-negative, negative and non-positive),

$\max \{v\}$ and $\min \{v\}$ denote the maximum and the minimum element of the vector v , respectively. Let \mathbf{M} denote the set of Metzler matrices whose off-diagonal entries are non-negative. Let I denote an identity matrix with appropriate dimension. For $\tau > 0$, let $PC([- \tau, 0], \mathbf{R}_+^n)$ denote the set of piecewise right continuous function with the norm defined by $\|\varphi\|_\tau = \sup_{-\tau \leq s \leq 0} \|\varphi(s)\|$, where $\varphi \in PC([- \tau, 0], \mathbf{R}_+^n)$. Matrices, if not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

Consider the following positive linear systems with time-varying delay and delayed impulses:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau_1(t)) & t \neq t_k \\ x(t_k) = C_k x(t_k^-) + D_k x(t_k^- - \tau_2(t_k^-)) & k \in \mathbf{Z}^+ \\ x(t_0 + s) = \varphi(s) & s \in [-\tau, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbf{R}_+^n$ is the state variable, $\lim_{h \rightarrow 0^-} x(t+h) = x(t^-)$, $\lim_{h \rightarrow 0^+} x(t+h) = x(t^+)$, $\tau_1(t)$ and $\tau_2(t)$ denote the time-varying state delay and impulse input delays, respectively. $\tau_1(t), \tau_2(t) \in [\mathbf{R}_+, \mathbf{R}_+]$ are bounded, $r = \sup_{t \in \mathbf{R}_+} \tau_1(t)$, $d = \sup_{t \in \mathbf{R}_+} \tau_2(t)$, $\tau = \max \{r, d\}$, $\varphi(\cdot) \in PC([- \tau, 0], \mathbf{R}_+^n)$ is a vector-valued initial continuous function. The impulse sequence $\{t_k\}_{k=1}^\infty$ satisfies $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots, t_k \rightarrow \infty$ for $k \rightarrow \infty$. Without loss of generality, we assume $x(t_k^+) = x(t_k)$, which implies that the solution of system (1) is right continuous at t_k . Moreover, we make the following assumption on system (1):

$$(A_1) A, B, C_k, D_k \in \mathbf{R}^{n \times n}, k \in \mathbf{Z}^+.$$

It was shown in [21] that under assumption (A_1) , for any $\varphi(\cdot) \in PC([- \tau, 0], \mathbf{R}^n)$, system (1) admits a unique solution $x(t, t_0, \varphi)$ exists in a maximal interval $[t_0 - \tau, \infty)$. Set $x(t) = x(t, t_0, \varphi)$.

Remark 1: Compared to the delayed linear systems with delayed impulses, delayed positive linear systems with delayed impulses (1) must possess positivity, that is, $x(t) \in \mathbf{R}_+^n$ for all $t \geq t_0$, which will be more challenging for stability analysis of such system.

Before proceeding, we need to introduce some definitions to develop our theories and results in what follows.

Definition 1: System (1) is said to be a delayed positive linear system with delayed impulses if for any initial condition $\varphi(\cdot) \in PC([- \tau, 0], \mathbf{R}_+^n)$, the corresponding trajectory $x(t) \geq 0$ holds for all $t \geq t_0$.

Definition 2: System (1) is said to be globally exponentially stable if there exist some constants $\zeta > 0$ and $\lambda > 0$ such that for any initial condition $\varphi(\cdot) \in PC([- \tau, 0], \mathbf{R}_+^n)$

$$\|x(t)\| \leq \zeta e^{-\lambda(t-t_0)} \|\varphi\|_\tau, \quad t \geq t_0.$$

Definition 3 [21]: The function $V: \mathbf{R}_+ \times \mathbf{R}^n \rightarrow \mathbf{R}_+$ belongs to class ν_0 if

- i. The function V is continuous in each of the sets $[t_{k-1}, t_k] \times \mathbf{R}^n$ and for each $x, y \in \mathbf{R}^n, t \in [t_{k-1}, t_k], k \in \mathbf{Z}^+$, $\lim_{(t,y) \rightarrow (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists.
- ii. $V(t, x(t))$ is locally Lipschitzian in all $x \in \mathbf{R}^n$, and for all $t \geq t_0, V(t, 0) = 0$.

Definition 4 [21]: Given a function $V \in \nu_0$, the upper right-hand derivative of V along the solution $x(t)$ of system (1) is defined by

$$D^+V(t, x(t)) = \lim_{\varepsilon \rightarrow 0^+} \sup \frac{1}{\varepsilon} [V(t + \varepsilon, x(t + \varepsilon)) - V(t, x(t))].$$

Lemma 1 [45]: Let $A \in \mathbf{M}$, then the following holds true:

- i. $A \in \mathbf{M} \Leftrightarrow e^{At} \geq 0$ for $t \geq 0$.
- ii. If a vector $v \geq 0$, then $e^{At}v \geq 0$ for $t \geq 0$.

3 Main results

3.1 Global exponential stability of delayed positive linear systems with delayed impulses

We derive here several stability results for delayed positive linear systems with delayed impulses. The positivity of system (1) is considered in Section 3.1.1, which is followed by some sufficient stability criteria for system (1) when impulses are stabilising in Section 3.1.2 and one sufficient stability criterion for system (1) when impulses potentially destroy the stability of the positive systems in Section 3.1.3.

3.1.1 Positivity of the system : First, let us consider the positivity of system (1). We have the following result.

Proposition 1: Consider system (1) satisfying assumption (A_1) . System (1) is a delayed positive linear systems with delayed impulses if $A \in \mathbf{M}, B \geq 0, C_k \geq 0$ and $D_k \geq 0, \forall k \in \mathbf{Z}^+$.

Proof: See Section 8 of Appendix 1. \square

3.1.2 Impulsive stabilisation: Let us consider the stability under impulsive stabilisation, which means the original system without impulses may be unstable. We have the following results.

Theorem 1: Consider system (1) satisfying assumption (A_1) , $A \in \mathbf{M}, B \geq 0, C_k \geq 0$ and $D_k \geq 0, k \in \mathbf{Z}^+$, if there exist constants $\lambda_1 > 0, \lambda_2 > 0, \lambda_{3(k-1)} > 0, \lambda_{4(k-1)} > 0, k \in \mathbf{Z}^+, \sigma > \eta > 0$ and vector $v > 0$ such that $0 < \lambda_{3(k-1)} + \lambda_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$ and the following conditions hold for all $k \in \mathbf{Z}^+$:

$$(H_0) [A^T - \lambda_1 I]v \leq 0,$$

$$(H_1) [B^T - \lambda_2 I]v \leq 0,$$

$$(H_2) [C_k^T - \lambda_{3k}]v \leq 0,$$

$$(H_3) [D_k^T - \lambda_{4k}]v \leq 0,$$

$$(H_4) \lambda_1 + \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}} - (\sigma - \eta) \leq 0 \text{ and}$$

$$(H_5) \tau < t_k - t_{k-1} \leq -\frac{\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)})}{\sigma + \eta}, \text{ then system (1) is globally exponentially stable with convergence rate } \eta.$$

Proof: See Section 9 of Appendix 2. \square

Remark 2: It should be noted that in view of inequality (15): $D^+V(t) \leq \lambda_1 x^T(t)v + \lambda_2 x^T(t - \tau_1(t))v$, we can find that the original system without impulses may be unstable. Theorem 1 shows that impulses can exponentially stabilise an unstable delayed positive linear systems. When impulses are used to exponentially stabilise an unstable system, condition (H_5) of Theorem 1 means that the impulses should act frequently.

Note that σ and η exist in conditions (H_4) and (H_5) of Theorem 1, which may make conditions (H_4) and (H_5) of Theorem 1 difficult to be verified, we then give the following result which may be more easier to be applied.

Theorem 2: Consider system (1) satisfying assumption (A_1) , $A \in \mathbf{M}, B \geq 0, C_k \geq 0$ and $D_k \geq 0, k \in \mathbf{Z}^+$, if there exist constants $\lambda_1 > 0, \lambda_2 > 0, \lambda_{3(k-1)} > 0, \lambda_{4(k-1)} > 0, k \in \mathbf{Z}^+$ and vector $v > 0$ such that $0 < \lambda_{3(k-1)} + \lambda_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$, conditions $(H_0) - (H_3)$

of Theorem 1 are satisfied, conditions (H_4) and (H_5) of Theorem 1 are replaced by

$$(H_4^*) \quad \tau < t_k - t_{k-1} < -\left(\lambda_1 + \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right)^{-1} \ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) k \in \mathbf{Z}^+,$$

then system (1) is globally exponentially stable.

Proof: In view of condition (H_4^*) , we have

$$-\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) - \left(\lambda_1 + \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right) \times (t_k - t_{k-1}) > 0.$$

Then we can choose sufficiently small $\eta > 0$ such that

$$-\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) + \left(-2\eta - \lambda_1 - \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right) \times (t_k - t_{k-1}) \geq 0.$$

Thus, for $\forall \sigma > 0$, we have

$$\begin{aligned} & \left(\sigma - \eta - \lambda_1 - \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right)(t_k - t_{k-1}) \\ & -\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) - (\sigma + \eta)(t_k - t_{k-1}) \geq 0, \end{aligned} \quad (2)$$

let $\sigma = \eta + \lambda_1 + (\lambda_2 / (\lambda_{3(k-1)} + \lambda_{4(k-1)}))$, we have

$$\sigma - \eta - \lambda_1 - \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}} = 0,$$

condition (H_4) of Theorem 1 is satisfied.

In view of inequality (2), we have

$$-\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) - (\sigma + \eta)(t_k - t_{k-1}) \geq 0,$$

which together with condition (H_4^*) gives

$$\tau < t_k - t_{k-1} \leq -\frac{\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)})}{\sigma + \eta},$$

condition (H_5) of Theorem 1 is satisfied. Thus by Theorem 1, the trivial solution of system (1) is globally exponentially stable. The proof of Theorem 2 is thus completed. \square

Remark 3: It should be noted that compared to conditions (H_4) and (H_5) of Theorem 1, condition (H_4^*) of Theorem 2 is much easier to be verified and more efficient to be applied in a practical problem.

Now, we consider the impulsive positive system that has been studied in [37]. Compared to Theorem 10 in [37], the following result can be seen as an improved and complemented result.

Corollary 1: Consider system (1) satisfying assumption (A_1) , $A \in \mathbf{M}$, $B = 0$, $C_k \geq 0$ and $D_k = 0$, $k \in \mathbf{Z}^+$, if there exist constants $\lambda_1 > 0$, $\lambda_{3(k-1)} > 0$, $k \in \mathbf{Z}^+$ and vector $v > 0$ such that $0 < \lambda_{3(k-1)} < 1$ for all $k \in \mathbf{Z}^+$, conditions (H_0) , (H_2) of Theorem 1 are satisfied, conditions (H_4^*) of Theorem 2 is replaced by

$$(H_4^{**}) \quad t_k - t_{k-1} < -\frac{\ln \lambda_{3(k-1)}}{\lambda_1}, \quad k \in \mathbf{Z}^+,$$

then system (1) is globally exponentially stable.

Proof: In view of condition (H_4^{**}) , we have

$$\ln \lambda_{3(k-1)} + \lambda_1(t_k - t_{k-1}) < 0 \quad k \in \mathbf{Z}^+.$$

In view of the above inequality and $0 < \lambda_{3(k-1)} < 1$, we can choose sufficiently small $\lambda_2 > 0$ and $\lambda_{4(k-1)} > 0$, $k \in \mathbf{Z}^+$ such that

$$\begin{aligned} & 0 < \lambda_{3(k-1)} + \lambda_{4(k-1)} < 1 \quad k \in \mathbf{Z}^+, \\ & \ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) + \left(\lambda_1 + \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right) \\ & \times (t_k - t_{k-1}) < 0 \quad k \in \mathbf{Z}^+, \end{aligned}$$

when $B = 0$, $D_k = 0$, $k \in \mathbf{Z}^+$, there are no state delay and impulse input delays in system (1), we have $\tau = 0$, which together with the above inequality gives

$$\begin{aligned} & 0 = \tau < t_k - t_{k-1} < -\left(\lambda_1 + \frac{\lambda_2}{\lambda_{3(k-1)} + \lambda_{4(k-1)}}\right)^{-1} \\ & \times \ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) \quad k \in \mathbf{Z}^+. \end{aligned}$$

Thus, condition (H_4^*) of Theorem 2 is satisfied.

In view of $B = 0$, $D_k = 0$, $\lambda_2 > 0$, $\lambda_{4(k-1)} > 0$, $k \in \mathbf{Z}^+$, we have

$$\begin{aligned} & [B^T - \lambda_2 I]v < 0 \\ & [D_k^T - \lambda_{4k} I]v < 0 \quad k \in \mathbf{Z}^+, \end{aligned}$$

conditions (H_1) and (H_3) of Theorem 1 are satisfied. Thus by Theorem 2, the trivial solution of system (1) is globally exponentially stable. The proof of Corollary 1 is thus completed. \square

Remark 4: When $B = 0$, $D_k = 0$, $k \in \mathbf{Z}^+$, i.e. there are no state delay and impulse input delays in system (1), the stability and global asymptotic stability of impulsive positive system (1) with $B = 0$, $D_k = 0$ ($k \in \mathbf{Z}^+$) was studied by means of Lyapunov function methods in the part (3) of Theorem 10 in [37]. The exponential stability are not considered in [37]. Hence, the obtained result in this paper complements and improves those results in [37].

3.1.3 Stability under impulsive perturbations.: In what follows, one sufficient criterion for exponential stability of system (1) under impulsive perturbations is presented. Here, impulsive perturbations mean that the original system without impulses is stable, the impulses potentially destroy the stability property of the original system. We have the following result.

Theorem 3: Consider system (1) satisfying assumption (A_1) , $A \in \mathbf{M}$, $B \geq 0$, $C_k \geq 0$ and $D_k \geq 0$, $k \in \mathbf{Z}^+$, if there exist constants $\mu > 0$, $d > 0$, $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_{3(k-1)} > 0$, $0 < \lambda_{4(k-1)} < 1$, $k \in \mathbf{Z}^+$ and vector $v > 0$ such that $\lambda_1 + \lambda_2 < 0$, $\lambda_{3(k-1)} + \lambda_{4(k-1)} > 1$, $k \in \mathbf{Z}^+$, conditions (H_0) – (H_3) of Theorem 1 hold and the following conditions hold for all $k \in \mathbf{Z}^+$:

$$(H_4^*) \quad (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)}) (t_k - t_{k-1}) + \mu d \leq 0 \text{ and}$$

$$(H_5^*) \quad \lambda_{3k} e^{-\mu d} + \lambda_{4k} e^{\mu d} < 1,$$

then system (1) is globally exponentially stable with convergence rate μ .

Proof: See Section 10 of Appendix 3. \square

Remark 5: It should be noted that in view of condition (H_4^*) : $(\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)}) (t_k - t_{k-1}) + \mu d \leq 0$. By direct computation, we have $0 \leq (\mu + \lambda_2 e^{\mu(r+d)}) (t_k - t_{k-1}) + \mu d \leq -\lambda_1 (t_k - t_{k-1})$. If r or d is getting bigger, then μ will be smaller. Which implies that state delay and impulse input delays will affect the speed and process of global exponential stability.

3.2 Global exponential stability of quasi-linear positive delay systems with delayed impulses

The results in Theorems 1–3 can be further extended to deal with the stability problems for the following quasi-linear positive delay systems with delayed impulses:

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t), x(t - \tau_1(t))) & t \neq t_k \\ x(t_k^-) = g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \\ \quad + C_k x(t_k^-) & k \in \mathbf{Z}^+ \\ x(t_0 + s) = \varphi(s), s \in [-\tau, 0], \end{cases} \quad (3)$$

where $A, C_k, \tau_1(t), \tau_2(t), r, d, \tau, t_k, x(t_k), x(t^+), x(t^-), \varphi(s)$ are defined as those in system (1). In this section, we assume that functions $f(t, x(t), x(t - \tau_1(t))), g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))), k \in \mathbf{Z}^+$ satisfy all necessary conditions for the global existence and uniqueness of solutions $x(t, t_0, \varphi)$ for all $t \geq t_0 - \tau$ [46]. Assumption $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$ for all $t \in \mathbf{R}_+$ enables that system (3) admits a trivial solution $x(t) \equiv 0$.

For system (3), the definitions of positivity and global exponential stability are similar to system (1), so we omit these definitions for system (3). The positivity of system (3) is considered in Section 3.2.1, which is followed by two sufficient stability criteria for system (3) when impulses are stabilising in Section 3.2.2 and one sufficient stability criterion for system (3) when impulses potentially destroy the stability of the positive systems in Section 3.2.3.

3.2.1 Positivity of the system: Let us consider the positivity of system (3). Compared to linear system (1), quasi-linear system (3) are more general. We have the following result.

Proposition 2: System (3) is a quasi-linear positive delay systems with delayed impulses if $A \in \mathbf{M}$, $C_k \geq 0$, $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) \geq 0$, $k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) \geq 0$.

Proof: See Section 11 of Appendix 4. \square

3.2.2 Impulsive stabilisation.: In what follows, two sufficient criteria for impulsive stabilisation of system (3) are presented.

Theorem 4: Consider system (3) satisfying $A \in \mathbf{M}$, $C_k \geq 0$, $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) \geq 0$, $k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) \geq 0$, if there exist constants $p_1 \geq 0$, $p_2 > 0$, $p_{3k} \geq 0$, $p_{4k} > 0$, $\lambda_1 > 0$, $\lambda_{3(k-1)} > 0$, $k \in \mathbf{Z}^+$, $\sigma > \eta > 0$ and vector $v > 0$ such that $0 < \lambda_{3(k-1)} + p_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$ are satisfied and the following conditions hold for all $k \in \mathbf{Z}^+$:

$$(H_0) f^T(t, x(t), x(t - \tau_1(t)))v \leq p_1 x^T(t)v + p_2 x^T(t - \tau_1(t))v,$$

$$(H_1) g^T(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-)))v \leq p_{3k} x^T(t_k^-)v + p_{4k} x^T(t_k^- - \tau_2(t_k^-))v$$

$$(H_2) [A^T + p_1 I - \lambda_1 I]v \leq 0,$$

$$(H_3) [C_k^T + p_{3k} I - \lambda_{3k} I]v \leq 0,$$

$$(H_4) \lambda_1 + \frac{p_2}{\lambda_{3(k-1)} + p_{4(k-1)}} - (\sigma - \eta) \leq 0 \text{ and}$$

$$(H_5) \tau < t_k - t_{k-1} \leq -\frac{\ln(\lambda_{3(k-1)} + p_{4(k-1)})}{\sigma + \eta},$$

then system (3) is globally exponentially stable with convergence rate η .

Proof: Let $x(t, t_0, \varphi)$ be any solution of system (3) through (t_0, φ) . From Proposition 2, we have

$$x^T(t) \geq 0 \quad t \geq t_0. \quad (4)$$

Choose a linear co-positive Lyapunov function in the form of $V(t, x(t)) = x^T(t)v$. Set $V(t) = V(t, x(t))$. Let $\alpha = \min \{v\}$, $\beta = \max \{v\}$, we have

$$\alpha \|x(t)\| \leq V(t) \leq \beta \|x(t)\|.$$

When $t \neq t_k, k \in \mathbf{Z}^+$, in view of conditions $(H_0), (H_2)$ and inequality (4), we have

$$\begin{aligned} D^+V(t) &= \dot{x}^T(t)v \\ &= (Ax(t) + f(t, x(t), x(t - \tau_1(t))))^T v \\ &= x^T(t)A^T v + f^T(t, x(t), x(t - \tau_1(t)))v \\ &\leq x^T(t)A^T v + p_1 x^T(t)v + p_2 x^T(t - \tau_1(t))v \\ &\leq \lambda_1 x^T(t)v + p_2 x^T(t - \tau_1(t))v. \end{aligned} \quad (5)$$

When $t = t_k, k \in \mathbf{Z}^+$, in view of conditions $(H_1), (H_3)$ and inequality (4), we have

$$\begin{aligned} V(t_k) &= x^T(t_k)v \\ &= (C_k x(t_k^-) + g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))))^T v \\ &= x^T(t_k^-)C_k^T v + g^T(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-)))v \\ &\leq x^T(t_k^-)C_k^T v + p_{3k} x^T(t_k^-)v + p_{4k} x^T(t_k^- - \tau_2(t_k^-))v \\ &\leq \lambda_{3k} x^T(t_k^-)v + p_{4k} x^T(t_k^- - \tau_2(t_k^-))v. \end{aligned} \quad (6)$$

Then, the rest of the proof is similar to that of Theorem 1 of system (1). Thus, the detail is omitted. \square

Theorem 5: Consider system (3) satisfying $A \in \mathbf{M}$, $C_k \geq 0$, $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) \geq 0$, $k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) \geq 0$, if there exist constants $p_1 \geq 0$, $p_2 > 0$, $p_{3k} \geq 0$, $p_{4k} > 0$, $\lambda_1 > 0$, $\lambda_{3(k-1)} > 0$, $k \in \mathbf{Z}^+$ and vector $v > 0$ such that $0 < \lambda_{3(k-1)} + p_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$, conditions $(H_0) - (H_3)$ of Theorem 4 hold, conditions (H_4) and (H_5) of Theorem 4 are replaced by

$$(H_4^*) \tau < t_k - t_{k-1} < -\left(\lambda_1 + \frac{p_2}{\lambda_{3(k-1)} + p_{4(k-1)}}\right)^{-1} \ln(\lambda_{3(k-1)} + p_{4(k-1)})k \in \mathbf{Z}^+,$$

then system (3) is globally exponentially stable.

The proof is similar to that of Theorem 2 of system (1). Thus, the detail is omitted.

3.2.3 Stability under impulsive perturbations.: In what follows, one sufficient criterion for exponential stability of system (3) under impulsive perturbations is presented.

Theorem 6: Consider system (3) satisfying $A \in \mathbf{M}$, $C_k \geq 0$, $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) \geq 0$, $k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) \geq 0$, if there exist constants $\mu > 0$, $d > 0$, $p_1 \geq 0$, $p_2 > 0$, $p_{3k} \geq 0$, $\lambda_1 < 0$, $\lambda_{3(k-1)} > 0$, $0 < p_{4(k-1)} < 1$, $k \in \mathbf{Z}^+$ and vector $v > 0$ such that $\lambda_1 + p_2 < 0$, $\lambda_{3(k-1)} + p_{4(k-1)} > 1$, $k \in \mathbf{Z}^+$, conditions $(H_0) - (H_3)$ of Theorem 4 hold and the following conditions hold for all $k \in \mathbf{Z}^+$:

$$(H_4^*) (\mu + \lambda_1 + p_2 e^{\mu(r+d)})(t_k - t_{k-1}) + \mu d \leq 0 \text{ and}$$

$$(H_5^*) \lambda_{3k} e^{-\mu d} + p_{4k} e^{\mu d} < 1,$$

then system (3) is globally exponentially stable with convergence rate μ .

Proof: Let $x(t, t_0, \varphi)$ be any solution of system (3) through (t_0, φ) . From Proposition 2, we have $x^T(t) \geq 0, t \geq t_0$. Choose a linear co-positive Lyapunov function in the form of $V(t, x(t)) = x^T(t)v$. Define $W(t, x(t)) = e^{\mu(t-t_0-d)}V(t, x(t))$. Set $V(t) = V(t, x(t))$, $W(t) = W(t, x(t))$.

When $t \neq t_k, k \in \mathbf{Z}^+$, in view of inequality (5), we have

$$\begin{aligned}
D^+W(t) &= \mu e^{\mu(t-t_0-d)}V(t) + e^{\mu(t-t_0-d)}D^+V(t) \\
&\leq \mu W(t) + e^{\mu(t-t_0-d)}(\lambda_1 x^T(t)v + p_2 x^T(t - \tau_1(t))v) \\
&= (\mu + \lambda_1)W(t) + p_2 e^{\mu\tau_1(t)}W(t - \tau_1(t)) \\
&\leq (\mu + \lambda_1)W(t) + p_2 e^{\mu r}W(t - \tau_1(t)).
\end{aligned}$$

When $t = t_k, k \in \mathbf{Z}^+$, in view of inequality (6), we have

$$\begin{aligned}
W(t_k) &= e^{\mu(t_k-t_0-d)}V(t_k) \\
&\leq e^{\mu(t_k-t_0-d)}(\lambda_{3k}x^T(t_k^-)v + p_{4k}x^T(t_k^- - \tau_2(t_k^-))v) \\
&= \lambda_{3k}W(t_k^-) + p_{4k}e^{\mu\tau_2(t_k^-)}W(t_k^- - \tau_2(t_k^-)) \\
&\leq \lambda_{3k}W(t_k^-) + p_{4k}e^{\mu d}W(t_k^- - \tau_2(t_k^-)).
\end{aligned}$$

Then, the rest of the proof is similar to that of Theorem 3 of system (1). Thus, the detail is omitted. \square

3.3 Global exponential stability of non-linear positive delay systems with delayed impulses

In this section, stability of the following non-linear positive delay systems with delayed impulses will be investigated:

$$\begin{cases} \dot{x}(t) = f(t, x(t), x(t - \tau_1(t))) & t \neq t_k \\ x(t_k) = g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) & k \in \mathbf{Z}^+ \\ x(t_0 + s) = \varphi(s), s \in [-\tau, 0], \end{cases} \quad (7)$$

where $\tau_1(t), \tau_2(t), r, d, \tau, t_k, x(t_k), x(t^+), x(t^-), \varphi(s)$ are defined as those in system (1). In this section, we assume that functions $f(t, x(t), x(t - \tau_1(t))), g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))), k \in \mathbf{Z}^+$ satisfy all necessary conditions for the global existence and uniqueness of solutions $x(t, t_0, \varphi)$ for all $t \geq t_0 - \tau$ [46]. Assumption $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$ for all $t \in \mathbf{R}_+$ enables that system (7) admits a trivial solution $x(t) \equiv 0$.

For system (7), the definitions of positivity and global exponential stability are similar to system (1), so we omit these definitions for system (7). The positivity of system (7) is considered in Section 3.3.1, then two sufficient criteria for impulsive stabilisation of system (7) are given in Section 3.3.2.

3.3.1 Positivity of the system .. Let us consider the positivity of system (7). We have the following result.

Proposition 3: System (7) is a non-linear positive delay systems with delayed impulses if $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0, x(t_k^- - \tau_2(t_k^-)) \geq 0, k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0, x(t - \tau_1(t)) \geq 0$.

Proof: When $A = 0, C_k = 0, k \in \mathbf{Z}^+$, since $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0, x(t_k^- - \tau_2(t_k^-)) \geq 0, k \in \mathbf{Z}^+$ and $f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0, x(t - \tau_1(t)) \geq 0$, it is easy to see that all conditions of Proposition 2 are satisfied, then we have $x(t) \geq 0$ for all $t \geq t_0$. The proof of Proposition 3 is thus completed. \square

3.3.2 Impulsive stabilisation.: In what follows, two sufficient criteria for impulsive stabilisation of system (7) are presented.

Theorem 7: Consider system (7) satisfying $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0, x(t_k^- - \tau_2(t_k^-)) \geq 0, k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0, x(t - \tau_1(t)) \geq 0$, if there exist constants $p_1 > 0, p_2 > 0, p_{3(k-1)} > 0, p_{4(k-1)} > 0, k \in \mathbf{Z}^+, \sigma > \eta > 0$ and vector $v > 0$ such that $0 < p_{3(k-1)} + p_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$, conditions $(H_0) - (H_1)$ of Theorem 4 hold and the following conditions hold for all $k \in \mathbf{Z}^+$:

$$(H_2) \quad p_1 + \frac{p_2}{p_{3(k-1)} + p_{4(k-1)}} - (\sigma - \eta) \leq 0 \text{ and}$$

$$(H_3) \quad \tau < t_k - t_{k-1} \leq -\frac{\ln(p_{3(k-1)} + p_{4(k-1)})}{\sigma + \eta},$$

then system (7) is globally exponentially stable with convergence rate η .

Proof: Let $x(t, t_0, \varphi)$ be any solution of system (7) through (t_0, φ) . From Proposition 3, we have

$$x^T(t) \geq 0 \quad t \geq t_0. \quad (8)$$

Choose a linear co-positive Lyapunov function in the form of $V(t, x(t)) = x^T(t)v$. Let $\alpha = \min \{v\}, \beta = \max \{v\}$, we have

$$\alpha \|x(t)\| \leq V(t, x(t)) \leq \beta \|x(t)\|.$$

When $t \neq t_k, k \in \mathbf{Z}^+$, in view of condition (H_0) and inequality (8), we have

$$\begin{aligned}
D^+V(t, x(t)) &= \dot{x}^T(t)v \\
&= f^T(t, x(t), x(t - \tau_1(t)))v \\
&\leq p_1 x^T(t)v + p_2 x^T(t - \tau_1(t))v,
\end{aligned}$$

When $t = t_k, k \in \mathbf{Z}^+$, in view of condition (H_1) and inequality (8), we have

$$\begin{aligned}
V(t_k, x(t_k)) &= x^T(t_k)v \\
&= (g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))))^T v \\
&\leq p_{3k}x^T(t_k^-)v + p_{4k}x^T(t_k^- - \tau_2(t_k^-))v.
\end{aligned}$$

Then, the rest of the proof is similar to that of Theorem 1 of system (1). Thus, the detail is omitted. \square

Theorem 8: Consider system (3) satisfying $g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \geq 0$ for all $x(t_k^-) \geq 0, x(t_k^- - \tau_2(t_k^-)) \geq 0, k \in \mathbf{Z}^+, f(t, x(t), x(t - \tau_1(t))) \geq 0$ for all $x(t) \geq 0, x(t - \tau_1(t)) \geq 0$, if there exist constants $p_1 > 0, p_2 > 0, p_{3(k-1)} > 0, p_{4(k-1)} > 0, k \in \mathbf{Z}^+$ and vector $v > 0$ such that $0 < p_{3(k-1)} + p_{4(k-1)} < 1$ for all $k \in \mathbf{Z}^+$, conditions $(H_0) - (H_1)$ of Theorem 4 hold and the following condition holds for all $k \in \mathbf{Z}^+$:

$$(H_2^*) \quad \tau < t_k - t_{k-1} < -\left(p_1 + \frac{p_2}{p_{3(k-1)} + p_{4(k-1)}}\right)^{-1} \ln(p_{3(k-1)}) + p_{4(k-1)}, \quad k \in \mathbf{Z}^+$$

then system (7) is globally exponentially stable.

The proof is similar to that of Theorem 2 of system (1). Thus the detail is omitted.

4 Numerical examples

In this section, several numerical examples will be presented to demonstrate the applicability and validity of our theoretical results.

Example 1: Consider a delayed impulsive positive systems (1) with the following system data:

$$\begin{aligned}
A &= \begin{pmatrix} 0.5 & 0.1 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0.1 & 0.4 \end{pmatrix} & B &= \begin{pmatrix} 0.1 & 0.02 & 0.01 \\ 0.08 & 0.1 & 0.02 \\ 0.04 & 0.03 & 0.05 \end{pmatrix} \\
C_k &= \begin{pmatrix} 0.15 & 0.04 & 0.02 \\ 0.07 & 0.1 & 0.03 \\ 0.01 & 0.07 & 0.08 \end{pmatrix} & D_k &= \begin{pmatrix} 0.12 & 0.05 & 0.02 \\ 0.09 & 0.08 & 0.03 \\ 0.06 & 0.04 & 0.06 \end{pmatrix} \quad k \in \mathbf{Z}^+ \\
\tau_1(t) &= 0.1(1 - \sin t) & \tau_2(t) &= 0.25(1 - \cos t).
\end{aligned}$$

Obviously, $A \in \mathbf{M}, B \geq 0, C_k \geq 0$ and $D_k \geq 0, k \in \mathbf{Z}^+$. Choosing $\lambda_1 = 1, \lambda_2 = 0.2, \lambda_{3(k-1)} = 0.2, \lambda_{4(k-1)} = 0.2$ for all $k \in \mathbf{Z}^+$ and

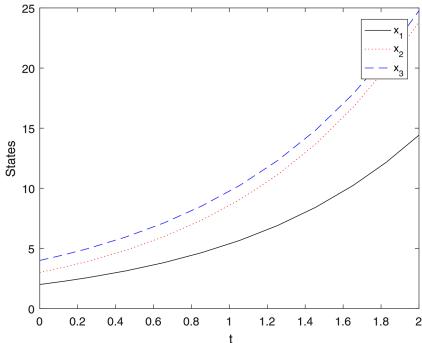


Fig. 1 State trajectories of the system in Example 1 without impulses

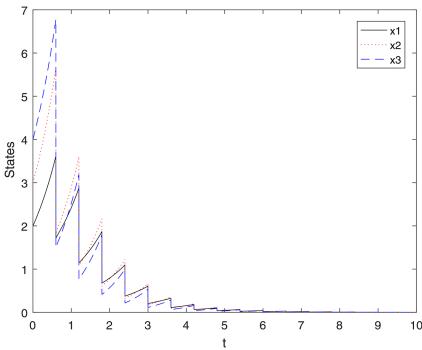


Fig. 2 State trajectories of the system in Example 1

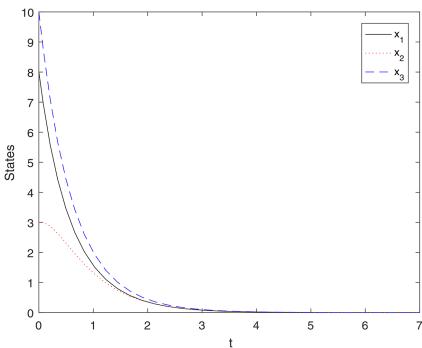


Fig. 3 State trajectories of the system in Example 2 without impulses

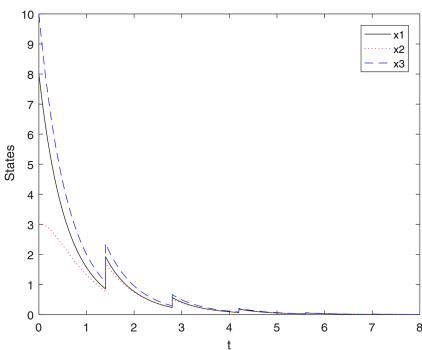


Fig. 4 State trajectories of the system in Example 2

vectors $v = [0.3 \ 0.2 \ 0.1]^T$. By direct computation, we have $\alpha = 0.1, \beta = 0.3, r = 0.2, d = 0.5, \tau = 0.5, [A^T - \lambda_1 I]v = (-0.01 - 0.06 - 0.01)^T, [B^T - \lambda_2 I]v = (-0.01 - 0.011 - 0.008)^T, [C_k^T - \lambda_{3k} I]v = (0 - 0.001 \ 0)^T, [D_k^T - \lambda_{4k} I]v = (0 - 0.005 - 0.002)^T, 0 < \lambda_{3(k-1)} + \lambda_{4(k-1)} = 0.4 < 1, -(\lambda_1 + (\lambda_2 / (\lambda_{3(k-1)} + \lambda_{4(k-1)})))^{-1} \ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) \simeq 0.61086$ for all $k \in \mathbb{Z}^+$. Thus, if $0.5 = \tau < t_k - t_{k-1} < -(\lambda_1 + (\lambda_2 / (\lambda_{3(k-1)} + \lambda_{4(k-1)})))^{-1} \ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) \simeq 0.61086$, it is easy to see that assumption (A₁), conditions (H₀) – (H₃) of Theorem 1 and condition (H₄^{*}) of

Theorem 2 are satisfied, then the zero solution of this system is globally exponentially stable.

Figs. 1 and 2 give the simulations for the state trajectories $x_1(t)$ and $x_2(t)$ of the system in Example 1 without impulses and the system in Example 1 with impulses (when $t_k - t_{k-1} = 0.6$), respectively. Here, the initial condition is given as $\varphi = (2 \ 3 \ 4)^T$.

Remark 6: From Fig. 1, we can find that the impulse-free system in Example 1 is not stable. It is shown in Fig. 2 that impulses can stabilise an unstable system.

Example 2: Consider a delayed impulsive positive system (1) with the following system data:

$$A = \begin{pmatrix} -3 & 0.4 & 0.8 \\ 0.3 & -2.5 & 0.5 \\ 0.2 & 0.3 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0.02 & 0.04 & 0.03 \\ 0.01 & 0.06 & 0.04 \\ 0.01 & 0.01 & 0.07 \end{pmatrix}$$

$$C_k = \begin{pmatrix} 1.2 & 0.5 & 0.4 \\ 0.2 & 1.5 & 0.1 \\ 0.1 & 0.1 & 1.8 \end{pmatrix}, \quad D_k = \begin{pmatrix} 0.01 & 0.03 & 0.04 \\ 0.01 & 0.02 & 0.01 \\ 0.005 & 0.01 & 0.04 \end{pmatrix}, \quad k \in \mathbb{Z}^+,$$

$$\tau_1(t) = 0.1(1 - \cos t), \quad \tau_2(t) = 0.9(1 - \sin t).$$

Obviously, $A \in M$, $B \geq 0$, $C_k \geq 0$ and $D_k \geq 0$, $k \in \mathbb{Z}^+$. Choosing $\mu = 0.5$, $\lambda_1 = -1.5$, $\lambda_2 = 0.1$, $\lambda_{3(k-1)} = 2$, $\lambda_{4(k-1)} = 0.06$ for all $k \in \mathbb{Z}^+$ and vectors $v = [1 \ 2 \ 4]^T$. By direct computation, we have $r = 0.2$, $d = 1.8$, $[A^T - \lambda_1 I]v = (-0.1 - 0.4 - 0.2)^T$, $[B^T - \lambda_2 I]v = (-0.02 \ 0 - 0.01)^T$, $[C_k^T - \lambda_{3k} I]v = (0 - 0.1 - 0.2)^T$, $[D_k^T - \lambda_{4k} I]v = (-0.01 - 0.01 - 0.02)^T$, $\lambda_1 + \lambda_2 = -1.4 < 0$, $\lambda_{3(k-1)} + \lambda_{4(k-1)} = 2.06 > 1$, $\lambda_{3k}e^{-\mu d} + \lambda_{4k}e^{\mu d} \simeq 0.9607 < 1$, $-(\mu d / (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)})) \simeq 1.236$ for all $k \in \mathbb{Z}^+$. Thus, if $t_k - t_{k-1} \geq -(\mu d / (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)})) \simeq 1.236$, $k \in \mathbb{Z}^+$, it is easy to see that assumption (A₁), conditions (H₀) – (H₃) of Theorem 1 and conditions (H₄^{*}) – (H₅^{*}) of Theorem 3 are satisfied, then the zero solution of this system is globally exponentially stable.

Figs. 3 and 4 give the simulations for the state trajectories $x_1(t)$ and $x_2(t)$ of the system in Example 2 without impulses and the system in Example 2 with impulses (when $t_k - t_{k-1} = 1.4$), respectively. Here, the initial condition is given as $\varphi = (8 \ 3 \ 10)^T$.

Remark 7: From Fig. 3, we can find that the impulse-free system in Example 2 is stable. It is shown in Fig. 4 that the impulses potentially destroy the stability property of the original system.

Example 3: Consider a non-linear delayed impulsive positive systems (7) with the following system data:

$$f(t, x(t), x(t - \tau_1(t))) = 0.25(1 - \cos t)x(t - 0.2e^{-t}) + 0.5(1 - \sin t)x(t),$$

$$g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) = 0.15(1 - \cos(t_k^-))x(t_k^-) + 0.05(1 - \sin(t_k^-))x(t_k^- - 0.3e^{-t_k^-}) \quad k \in \mathbb{Z}^+.$$

Obviously, $0 \leq g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \leq 0.3x(t_k^-) + 0.1x(t_k^- - 0.3e^{-t_k^-})$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) = x(t_k^- - 0.3e^{-t_k^-}) \geq 0$, $k \in \mathbb{Z}^+$, $0 \leq f(t, x(t), x(t - \tau_1(t))) \leq x(t) + 0.5 \times x(t - 0.2e^{-t})$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) = x(t - 0.2e^{-t}) \geq 0$. Choosing $p_1 = 1$, $p_2 = 0.5$, $p_{3(k-1)} = 0.3$, $p_{4(k-1)} = 0.1$ for all $k \in \mathbb{Z}^+$ and vectors $v = [1 \ 3 \ 5]^T$. By direct computation, we have $r = 0.2$, $d = 0.3$, $\tau = 0.3$, $0 < p_{3(k-1)} + p_{4(k-1)} = 0.4 < 1$, $-\ln(p_{3(k-1)} + p_{4(k-1)}) / (p_1 + (p_2 / (p_{3(k-1)} + p_{4(k-1)})))^{-1} \simeq 0.40724$ for all $k \in \mathbb{Z}^+$. Thus, if $0.3 = \tau < t_k - t_{k-1} < -\ln(p_{3(k-1)} + p_{4(k-1)}) / ((p_2 / (p_{3(k-1)} + p_{4(k-1)})) + p_1)^{-1} \simeq 0.40724$, $k \in \mathbb{Z}^+$, it is easy to see that conditions (H₀) – (H₁) of Theorem 4 and conditions (H₂^{*}) of Theorem 8 are

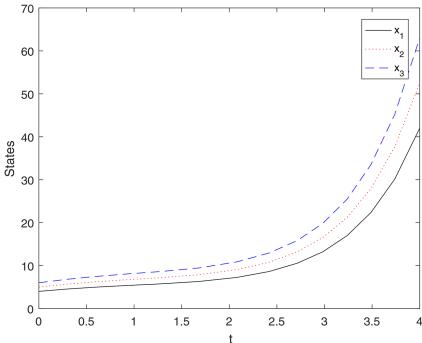


Fig. 5 State trajectories of the system in Example 3 without impulses

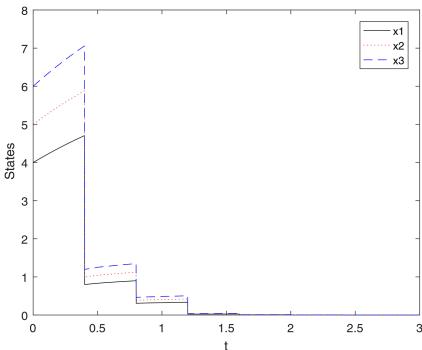


Fig. 6 State trajectories of the system in Example 3

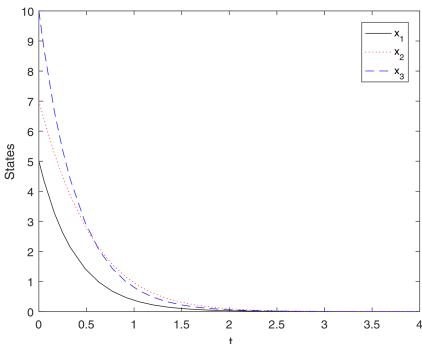


Fig. 7 State trajectories of the system in Example 4 without impulses

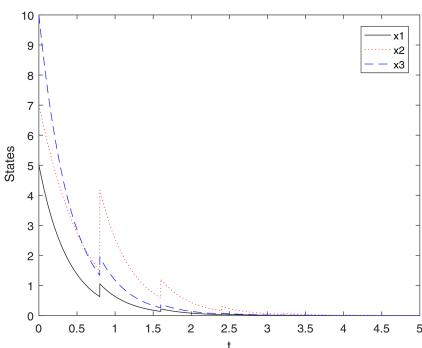


Fig. 8 State trajectories of the system in Example 4

satisfied, then the zero solution of this system is globally exponentially stable.

Figs. 5 and 6 give the simulations for the state trajectories $x_1(t)$ and $x_2(t)$ of the system in Example 3 without impulses and the system in Example 3 with impulses (when $t_k - t_{k-1} = 0.4$), respectively. Here, the initial condition is given as $\varphi = (4\ 5\ 6)^T$.

Remark 8: From Fig. 5, we can find that the impulse-free system in Example 3 is not stable. This example shows that impulses can stabilise a non-linear unstable positive system.

Example 4: Consider a non-linear delayed impulsive positive system (3) with the following system data:

$$A = \begin{pmatrix} -3 & 0.06 & 0.1 \\ 0.6 & -2.8 & 0.4 \\ 0.1 & 0.05 & -2.7 \end{pmatrix}, \quad C_k = \begin{pmatrix} 1.3 & 0.02 & 0.1 \\ 0.2 & 1.4 & 1.3 \\ 0.1 & 0.05 & 1.1 \end{pmatrix}$$

$$g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) = 0.05(1 - \cos(t_k^-))$$

$$\times x(t_k^- - e^{-3t_k^-}) \quad k \in \mathbb{Z}^+$$

$$f(t, x(t), x(t - \tau_1(t))) = 0.1(1 - \sin t)x(t - 0.3e^{-2t}).$$

Obviously, $A \in \mathbf{M}$, $C_k \geq 0$, $0 \leq g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \leq 0.1$, $x(t - e^{-3t_k^-}) \geq 0$ for all $x(t_k^-) \geq 0$, $x(t_k^- - \tau_2(t_k^-)) = x(t_k^- - e^{-3t_k^-}) \geq 0$, $k \in \mathbb{Z}^+$, $0 \leq f(t, x(t), x(t - \tau_1(t))) \leq 0.2x(t - 0.3e^{-2t})$ for all $x(t) \geq 0$, $x(t - \tau_1(t)) = x(t - 0.3e^{-2t}) \geq 0$. Choosing $\mu = 0.8$, $p_1 = 0$, $p_2 = 0.2$, $p_{4(k-1)} = 0.1$, $p_{3k} = 0$, $\lambda_1 = -2.5$, $\lambda_{3(k-1)} = 1.6$ for all $k \in \mathbb{Z}^+$ and vectors $v = [2\ 1\ 3]^T$. By direct computation, we have $d = 1$, $r = 0.3$, $\lambda_1 + p_2 = -2.3 < 0$, $\lambda_{3(k-1)} + p_{4(k-1)} = 1.7 > 1$, $[A^T + p_1 I - \lambda_1 I]v = (-0.1 - 0.03\ 0)^T$, $[C_k^T + p_{3k} I - \lambda_{3k} I]v = (-0.1 - 0.01\ 0)^T$, $(-\mu d / (\mu + \lambda_1 + p_2 e^{\mu(r+d)})) \approx 0.7054$, $\lambda_{3k} e^{-\mu d} + p_{4k} e^{\mu d} \approx 0.9415 < 1$ for all $k \in \mathbb{Z}^+$. Thus, if $t_k - t_{k-1} \geq -(\mu d / (\mu + \lambda_1 + p_2 e^{\mu(r+d)})) \approx 0.7054$, $k \in \mathbb{Z}^+$, it is easy to see that conditions $(H_0) - (H_3)$ of Theorem 4 and conditions $(H_4^*) - (H_5^*)$ of Theorem 6 are satisfied, then the zero solution of this system is globally exponentially stable.

Figs. 7 and 8 give the simulations for the state trajectories $x_1(t)$ and $x_2(t)$ of the system in Example 4 without impulses and the system in Example 4 with impulses (when $t_k - t_{k-1} = 0.8$), respectively. Here, the initial condition is given as $\varphi = (5\ 7\ 10)^T$.

Remark 9: It is worth pointing out that stability of Examples 1–4 cannot be studied by [37, 38, 40, 41], since delayed impulses exist in the systems of these examples.

5 Conclusion

In this paper, by using linear co-positive Lyapunov functions together with Razumikhin techniques, a number of stability criteria for positive delay systems with delayed impulses have been provided. The obtained results show that to stabilise the original unstable system, the impulses must act frequently; to keep the stability of the original system, the impulses should act occasionally. The obtained results improve and complement some recent works. Some examples have also been given to illustrate the effectiveness and the advantage of the obtained results. Inspired by some enlightening works [30, 40], in the future, it is worthwhile to investigate the stability under range dwell time for positive delay systems with delayed impulses.

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7 References

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8 Appendix 1. Proof of Proposition 1

Proof: Let $e = (1, 1, \dots, 1)^T$, we construct an auxiliary-delayed positive linear systems with delayed impulses

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t - \tau_1(t)) + ee & t \neq t_k \\ x(t_k) = C_k x(t_k^-) + D_k x(t_k^- - \tau_2(t_k^-)) & k \in \mathbb{Z}^+ \\ x(t_0 + s) = \varphi(s) & s \in [-\tau, 0], \end{cases} \quad (9)$$

where $\varepsilon > 0$ is a arbitrarily small constant. Obviously, system (9) admits a unique solution $\tilde{x}(t, t_0, \varphi)$ for all $t \geq t_0 - \tau$. We have

$$\tilde{x}(t, t_0, \varphi) \rightarrow x(t, t_0, \varphi) \quad \varepsilon \rightarrow 0, \quad (10)$$

for $t \geq t_0$ by [47]. Let $\tilde{x}(t) = \tilde{x}(t, t_0, \varphi)$.

For system (1) and (9) with any initial condition $\varphi(\cdot) \geq 0$, respectively, we have

$$x(t) \geq 0 \quad \tilde{x}(t) \geq 0 \quad t \in [t_0 - \tau, t_0]. \quad (11)$$

Assume that $\tilde{x}(t) \geq 0$ for all $t \in (t_0, t_1)$ is false, then there exist $t_1^k \in (t_0, t_1)$ such that $\tilde{x}_k(t_1^k) < 0$ for some $k \in \{1, 2, \dots, n\}$, let $J = \{k \in \{1, 2, \dots, n\} : \tilde{x}_k(t_1^k) < 0, t_1^k \in (t_0, t_1)\}$. Set $t^* = \inf\{t \in (t_0, t_1) : \tilde{x}_l(t) < 0 \text{ for some } l \in J\}$, then $t^* \in (t_0, t_1)$, $\tilde{x}(t) \geq 0$ for $t \in (t_0, t^*]$ and there exists a $i \in \{1, 2, \dots, n\}$ such that $\tilde{x}_i(t^*) = 0$. It would follow that $\tilde{x}_i(t^*) \leq 0$, $\tilde{x}(t) \geq 0$ for $t \in (t_0, t^*]$, which together with inequality (11) gives $\tilde{x}(t) \geq 0$ for $t \in [t_0 - \tau, t^*]$. In view of $t - \tau_1(t) \in [t_0 - \tau, t^*]$ for $t \in (t_0, t^*]$, we have $\tilde{x}(t - \tau_1(t)) \geq 0$ for $t \in (t_0, t^*]$. From system (9), $A \in \mathbf{M}$ and $B \geq 0$, we have

$$\begin{aligned} \dot{\tilde{x}}_i(t^*) &= \sum_{j=1, i \neq j}^n a_{ij} \tilde{x}_j(t^*) + a_{ii} \tilde{x}_i(t^*) + \sum_{j=1}^n b_{ij} \tilde{x}_j(t^* - \tau_1(t^*)) + \varepsilon \\ &> 0, \end{aligned}$$

where a_{ij} is the (i, j) th component of A and b_{ij} is the (i, j) th component of B . This contradicts with $\dot{\tilde{x}}_i(t^*) \leq 0$. Thus, $\tilde{x}(t) \geq 0$ for all $t \in (t_0, t_1)$. From (10), we have $x(t) \geq 0$ for all $t \in (t_0, t_1)$.

When $t = t_1$, $x(t_1) = C_1 x(t_1^-) + D_1 x(t_1^- - \tau_2(t_1^-))$, $\tilde{x}(t_1) = C_1 \tilde{x}(t_1^-) + D_1 \tilde{x}(t_1^- - \tau_2(t_1^-))$. In view of $t_1 - \tau_2(t_1) \in [t_0 - \tau, t_1]$, $x(t) \geq 0$, $\tilde{x}(t) \geq 0$ for all $t \in [t_0 - \tau, t_1]$, we have $x(t_1^-) \geq 0$, $x(t_1^- - \tau_2(t_1^-)) \geq 0$, $\tilde{x}(t_1^-) \geq 0$, $\tilde{x}(t_1^- - \tau_2(t_1^-)) \geq 0$. Since $C_1 \geq 0$, $D_1 \geq 0$, it is straightforward to obtain $x(t_1) = C_1 x(t_1^-) + D_1 x(t_1^- - \tau_2(t_1^-)) \geq 0$, $\tilde{x}(t_1) = D_1 \tilde{x}(t_1^- - \tau_2(t_1^-)) + C_1 \tilde{x}(t_1^-) \geq 0$.

From what has been discussed above, we have

$$x(t) \geq 0 \quad \tilde{x}(t) \geq 0 \quad t \in [t_0 - \tau, t_1]. \quad (12)$$

Now, similar to the above processes, assume that $\tilde{x}(t) \geq 0$ for all $t \in (t_1, t_2)$ is false, then there exist $t_2^k \in (t_1, t_2)$ such that $\tilde{x}_k(t_2^k) < 0$ for some $k \in \{1, 2, \dots, n\}$, let $J^* = \{k \in \{1, 2, \dots, n\} : \tilde{x}_k(t_2^k) < 0, t_2^k \in (t_1, t_2)\}$. Set $t_* = \inf\{t \in (t_1, t_2) : \tilde{x}_l(t) < 0 \text{ for some } l \in J^*\}$, then $t_* \in (t_1, t_2)$, $\tilde{x}(t) \geq 0$ for $t \in (t_1, t_*]$ and there exists a $i \in \{1, 2, \dots, n\}$ such that $\tilde{x}_i(t_*) = 0$. It would follow that $\tilde{x}_i(t_*) \leq 0$, $\tilde{x}(t) \geq 0$ for $t \in (t_1, t_*]$, which together with inequality (12) gives $\tilde{x}(t) \geq 0$ for $t \in [t_0 - \tau, t_*]$. In view of $t - \tau_1(t) \in [t_0 - \tau, t_*]$ for

$t \in (t_1, t_*]$, we have $\tilde{x}(t - \tau_1(t)) \geq 0$ for $t \in (t_1, t_*]$. From system (9), $A \in \mathbf{M}$ and $B \geq 0$, we have $\dot{\tilde{x}}_i(t_*) = \sum_{j=1, i \neq j}^n a_{ij} \tilde{x}_j(t_*) + a_{ii} \tilde{x}_i(t_*) + \sum_{j=1}^n b_{ij} \tilde{x}_j(t_* - \tau_1(t_*)) + \varepsilon > 0$. This contradicts with $\dot{\tilde{x}}_i(t_*) \leq 0$. Thus, $\tilde{x}(t) \geq 0$ for all $t \in (t_1, t_2)$. From (10), we have $x(t) \geq 0$ for all $t \in (t_1, t_2)$.

When $t = t_2$, $x(t_2) = C_2 x(t_2^-) + D_2 x(t_2^- - \tau_2(t_2^-))$, $\tilde{x}(t_2) = C_2 \tilde{x}(t_2^-) + D_2 \tilde{x}(t_2^- - \tau_2(t_2^-))$. In view of $t_2^- - \tau_2(t_2^-) \in [t_0 - \tau, t_2]$, $x(t) \geq 0$, $\tilde{x}(t) \geq 0$ for all $t \in [t_0 - \tau, t_2]$, we have $x(t_2^-) \geq 0$, $x(t_2^- - \tau_2(t_2^-)) \geq 0$, $\tilde{x}(t_2^-) \geq 0$, $\tilde{x}(t_2^- - \tau_2(t_2^-)) \geq 0$. Since $C_2 \geq 0$, $D_2 \geq 0$, it is straightforward to obtain $x(t_2) = C_2 x(t_2^-) + D_2 x(t_2^- - \tau_2(t_2^-)) \geq 0$, $\tilde{x}(t_2) = D_2 \tilde{x}(t_2^- - \tau_2(t_2^-)) + C_2 \tilde{x}(t_2^-) \geq 0$.

By repeating the above procedure, we can easily obtain that $x(t) \geq 0$ for all $t \geq t_0$. Then, system (1) is a delayed positive linear systems with delayed impulses. The proof of Proposition 1 is thus completed. \square

9 Appendix 2. Proof of Theorem 1

Let $x(t, t_0, \varphi)$ be any solution of system (1) through (t_0, φ) . From Proposition 1, we have

$$x^T(t) \geq 0 \quad t \geq t_0. \quad (13)$$

Choose a linear co-positive Lyapunov function in the form of $V(t, x(t)) = x^T(t)v$. Set $V(t) = V(t, x(t))$. Let $\alpha = \min \{v\}$, $\beta = \max \{v\}$, we have

$$\alpha \|x(t)\| \leq V(t) \leq \beta \|x(t)\|. \quad (14)$$

When $t \neq t_k, k \in \mathbf{Z}^+$, in view of inequality (13), conditions (H_0) and (H_1) , we have

$$\begin{aligned} D^+V(t) &= \dot{x}^T(t)v \\ &= (Ax(t) + Bx(t - \tau_1(t)))^T v \\ &= x^T(t)A^T v + x^T(t - \tau_1(t))B^T v \\ &\leq \lambda_1 x^T(t)v + \lambda_2 x^T(t - \tau_1(t))v. \end{aligned} \quad (15)$$

When $t = t_k, k \in \mathbf{Z}^+$, in view of inequality (13), conditions (H_2) and (H_3) , we have

$$\begin{aligned} V(t_k) &= x^T(t_k)v \\ &= (C_k x(t_k^-) + D_k x(t_k^- - \tau_2(t_k^-)))^T v \\ &= x^T(t_k^-)C_k^T v + x^T(t_k^- - \tau_2(t_k^-))D_k^T v \\ &\leq \lambda_{3k} x^T(t_k^-)v + \lambda_{4k} x^T(t_k^- - \tau_2(t_k^-))v. \end{aligned} \quad (16)$$

The following inequality can be obtained from condition (H_5) :

$$-\ln(\lambda_{3(k-1)} + \lambda_{4(k-1)}) - (\sigma + \eta)(t_k - t_{k-1}) \geq 0 \quad k \in \mathbf{Z}^+,$$

then we can choose $M > 1$ such that

$$\begin{aligned} 1 &< e^{(\sigma + \eta)(t_1 - t_0)} \leq M \\ &\leq e^{-\ln(\lambda_{30} + \lambda_{40}) - (\sigma + \eta)(t_1 - t_0)} e^{(\sigma + \eta)(t_1 - t_0)} \\ &= \frac{1}{\lambda_{30} + \lambda_{40}}, \end{aligned} \quad (17)$$

$$\begin{aligned} 1 &< e^{(\sigma + \eta)(t_{k+1} - t_k)} \\ &\leq e^{-\ln(\lambda_{3k} + \lambda_{4k}) - (\sigma + \eta)(t_{k+1} - t_k)} e^{(\sigma + \eta)(t_{k+1} - t_k)} \\ &= \frac{1}{\lambda_{3k} + \lambda_{4k}} \quad k \in \mathbf{Z}^+, \end{aligned} \quad (18)$$

and thus

$$1 < e^{\sigma(t_1 - t_0)} \leq M e^{-\eta(t_1 - t_0)}, \quad (19)$$

$$(\lambda_{3k} + \lambda_{4k}) e^{(\sigma + \eta)(t_{k+1} - t_k)} \leq 1 \quad k \in \mathbf{Z}^+. \quad (20)$$

We will prove that

$$V(t) \leq \beta M e^{-\eta(t_k - t_0)} \|\varphi\|_\tau \quad t_{k-1} \leq t < t_k \quad k \in \mathbf{Z}^+. \quad (21)$$

First, we will prove that

$$V(t) \leq \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau \quad t \in [t_0 - \tau, t_1]. \quad (22)$$

In view of inequalities (14) and (19), we have

$$V(t) \leq \beta \|\varphi\|_\tau < \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau \quad t \in [t_0 - \tau, t_0], \quad (23)$$

thus, inequality (22) holds for $t \in [t_0 - \tau, t_0]$, so we only need to prove that

$$V(t) \leq \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau \quad t_0 < t < t_1. \quad (24)$$

If inequality (24) is not true, then there exists a $\bar{t} \in (t_0, t_1)$ such that

$$\begin{aligned} V(\bar{t}) &> \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau \stackrel{(19)}{\geq} \beta e^{\sigma(t_1 - t_0)} \|\varphi\|_\tau \\ &> \beta \|\varphi\|_\tau \geq V(t_0). \end{aligned}$$

Set $t^* = \inf \{t \in (t_0, t_1) : V(t) \geq \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau\}$, then we have $t^* \in (t_0, t_1)$, $V(t^*) = \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau$ and $V(t) \leq \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau$ for $t \leq t^*$. Set $t^{**} = \sup \{t \in [t_0, t^*] : V(t) \leq \beta \|\varphi\|_\tau\}$, then $t^{**} \in [t_0, t^*]$ and $V(t^{**}) = \beta \|\varphi\|_\tau \leq V(t) \leq \beta M \|\varphi\|_\tau e^{-\eta(t_1 - t_0)}$, $t \in [t^{**}, t^*]$. For $t \in [t^{**}, t^*]$ and $s \in [-\tau, 0]$, in view of inequalities (17) and (23), we have

$$\begin{aligned} V(t+s) &\leq \beta M e^{-\eta(t_1 - t_0)} \|\varphi\|_\tau < \beta M \|\varphi\|_\tau \\ &\leq \frac{1}{\lambda_{30} + \lambda_{40}} \beta \|\varphi\|_\tau \leq \frac{1}{\lambda_{30} + \lambda_{40}} V(t). \end{aligned} \quad (25)$$

From condition (H_4) , inequalities (15) and (25), we have

$$\begin{aligned} D^+V(t) &\leq \lambda_1 x^T(t)v + \lambda_2 x^T(t - \tau_1(t))v \\ &< (\lambda_1 + \frac{\lambda_2}{\lambda_{30} + \lambda_{40}}) x^T(t)v \\ &\leq (\sigma - \eta) V(t) \quad t \in [t^{**}, t^*]. \end{aligned} \quad (26)$$

Now, integrating inequality (26) from t^{**} to t^* , we obtain

$$\begin{aligned} V(t^*) &< V(t^{**}) e^{(\sigma - \eta)(t^* - t^{**})} \\ &= \beta \|\varphi\|_\tau e^{(\sigma - \eta)(t^* - t^{**})} \\ &< \beta \|\varphi\|_\tau e^{\sigma(t_1 - t_0)} \\ &\stackrel{(19)}{\leq} \beta M \|\varphi\|_\tau e^{-\eta(t_1 - t_0)} \\ &= V(t_0), \end{aligned}$$

contradiction is obtained and inequality (24) is proved, so inequality (22) holds.

Now assume that inequality (21) holds for $k = 1, 2, \dots, m$, i.e.

$$V(t, x(t)) \leq \beta M e^{-\eta(t_k - t_0)} \|\varphi\|_\tau, \quad t \in [t_{k-1}, t_k] \quad k = 1, 2, \dots, m. \quad (27)$$

We will prove that inequality (21) holds for $k = m + 1$, i.e.

$$V(t) \leq \beta M e^{-\eta(t_{m+1} - t_0)} \|\varphi\|_\tau \quad t \in [t_m, t_{m+1}]. \quad (28)$$

When $t = t_m$, in view of condition (H_5) , inequalities (16), (20) and (27), we have

$$\begin{aligned} V(t_m) &\stackrel{(16)}{\leq} \lambda_{3m}x^T(t_m^-)v + \lambda_{4m}x^T(t_m^- - \tau_2(t_m^-))v \\ &\stackrel{(27)}{\leq} (\lambda_{3m} + \lambda_{4m})\beta Me^{-\eta(t_m - t_0)} \|\varphi\|_\tau \\ &\stackrel{(20)}{\leq} e^{(-\eta - \sigma)(t_m + 1 - t_m)}\beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \\ &= e^{-\sigma(t_m + 1 - t_m)}\beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \\ &< \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau. \end{aligned} \quad (29)$$

Thus, inequality (28) holds for $t = t_m$, so we only need to prove that

$$V(t) \leq \beta Me^{-\eta(t_{m+1} - t_0)} \|\varphi\|_\tau \quad t_m < t < t_{m+1}, \quad (30)$$

If inequality (30) is not true, then there exists a $\tilde{t} \in (t_m, t_{m+1})$ such that

$$\begin{aligned} V(\tilde{t}) &> \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \\ &> (\lambda_{3m} + \lambda_{4m})\beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \stackrel{(29)}{\geq} V(t_m). \end{aligned}$$

Set $t_* = \inf \{t \in (t_m, t_{m+1}): V(t) \geq \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau\}$, then we have $t_* \in (t_m, t_{m+1})$, $V(t_*) = \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau$ and $V(t) \leq \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau$ for $t_m \leq t \leq t_*$. Set $t_{**} = \sup \{t \in [t_m, t_*]: V(t) \leq (\lambda_{3m} + \lambda_{4m})\beta Me^{-\eta(t_m - t_0)} \|\varphi\|_\tau\}$, then we have $t_{**} \in [t_m, t_*]$ and $V(t_{**}) = (\lambda_{3m} + \lambda_{4m})\beta M \|\varphi\|_\tau \times e^{-\eta(t_m - t_0)}$. Since $V(t) \leq \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau$, $t \in [t_{**}, t_*]$. For $t \in [t_{**}, t_*]$ and $s \in [-\tau, 0]$, we have

$$\begin{aligned} V(t+s) &\leq \beta Me^{-\eta(t_m - t_0)} \|\varphi\|_\tau \\ &= \frac{1}{\lambda_{3m} + \lambda_{4m}} V(t_{**}) \\ &\leq \frac{1}{\lambda_{3m} + \lambda_{4m}} V(t), t \in [t_{**}, t_*]. \end{aligned} \quad (31)$$

From condition (H_4) , inequalities (15) and (31), we have

$$\begin{aligned} D^+V(t) &\leq \lambda_1 x^T(t)v + \lambda_2 x^T(t - \tau_1(t))v \\ &\leq (\lambda_1 + \frac{\lambda_2}{\lambda_{3m} + \lambda_{4m}})x^T(t)v \\ &\leq (\sigma - \eta)V(t), t \in [t_{**}, t_*]. \end{aligned} \quad (32)$$

Now, integrating inequality (32) from t_{**} to t_* , we have

$$\begin{aligned} V(t_*) &\leq V(t_{**})e^{(\sigma - \eta)(t_* - t_{**})} \\ &= (\lambda_{3m} + \lambda_{4m})\beta Me^{-\eta(t_m - t_0)}e^{(\sigma - \eta)(t_* - t_{**})} \|\varphi\|_\tau \\ &\leq (\lambda_{3m} + \lambda_{4m})\beta Me^{-\eta(t_m - t_0)}e^{(\sigma - \eta)(t_{m+1} + 1 - t_m)} \|\varphi\|_\tau \\ &= (\lambda_{3m} + \lambda_{4m})e^{\sigma(t_m + 1 - t_m)}\beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \\ &< \beta Me^{-\eta(t_m + 1 - t_0)} \|\varphi\|_\tau \stackrel{(20)}{=} V(t_*), \end{aligned}$$

contradiction is obtained and inequality (30) is proved, so inequality (28) holds.

By the principle of mathematical induction, we have proved that (21) holds for any $k \in \mathbb{Z}^+$, therefore

$$V(t) \leq \beta Me^{-\eta(t_k - t_0)} \|\varphi\|_\tau < \beta Me^{-\eta(t - t_0)} \|\varphi\|_\tau, \quad t \in [t_{k-1}, t_k]$$

which together with inequality (14) gives

$$\|x(t)\| \leq \frac{\beta}{\alpha} M e^{-\eta(t-t_0)} \|\varphi\|_\tau \quad t \geq t_0,$$

which implies that the trivial solution of system (1) is globally exponentially stable with convergence rate η . The proof of Theorem 1 is thus completed.

10 Appendix 3. Proof of Theorem 3

Let $x(t, t_0, \varphi)$ be any solution of system (1) through (t_0, φ) . From Proposition 1, we have

$$x^T(t) \geq 0 \quad t \geq t_0. \quad (33)$$

Choose a linear co-positive Lyapunov function in the form of $V(t, x(t)) = x^T(t)v$. Define $W(t, x(t)) = e^{\mu(t-t_0-d)}V(t, x(t))$. Set $V(t) = V(t, x(t))$, $W(t) = W(t, x(t))$. Let $\alpha = \min \{v\}$, $\beta = \max \{v\}$, we have

$$\alpha \|x(t)\| \leq V(t) \leq \beta \|x(t)\|. \quad (34)$$

When $t \neq t_k, k \in \mathbb{Z}^+$, in view of inequalities (15) and (33), we have

$$\begin{aligned} D^+W(t) &= \mu e^{\mu(t-t_0-d)}V(t) + e^{\mu(t-t_0-d)}D^+V(t) \\ &\leq \mu W(t) + e^{\mu(t-t_0-d)}\lambda_1 V(t) \\ &\quad + e^{\mu(t-t_0-d)}\lambda_2 x^T(t - \tau_1(t))v \\ &= (\mu + \lambda_1)W(t) + \lambda_2 e^{\mu\tau_1(t)}W(t - \tau_1(t)) \\ &\leq (\mu + \lambda_1)W(t) + \lambda_2 e^{\mu t}W(t - \tau_1(t)). \end{aligned} \quad (35)$$

When $t = t_k, k \in \mathbb{Z}^+$, in view of inequalities (16) and (33), we have

$$\begin{aligned} W(t_k) &= e^{\mu(t_k-t_0-d)}V(t_k) \\ &\leq e^{\mu(t_k-t_0-d)}(\lambda_{3k}x^T(t_k^-)v + \lambda_{4k}x^T(t_k^- - \tau_2(t_k^-))v) \\ &= \lambda_{3k}W(t_k^-) + \lambda_{4k}e^{\mu\tau_2(t_k^-)}W(t_k^- - \tau_2(t_k^-)) \\ &\leq \lambda_{3k}W(t_k^-) + \lambda_{4k}e^{\mu d}W(t_k^- - \tau_2(t_k^-)). \end{aligned} \quad (36)$$

We will prove that

$$W(t) \leq \beta \|\varphi\|_\tau \quad t_{k-1} \leq t < t_k \quad k \in \mathbb{Z}^+. \quad (37)$$

First, we will prove that

$$W(t) \leq \beta \|\varphi\|_\tau \quad t \in [t_0 - \tau, t_1]. \quad (38)$$

In view of inequality (34), we have

$$W(t) = e^{\mu(t-t_0-d)}V(t) \leq e^{-\mu d}\beta \|\varphi\|_\tau < \beta \|\varphi\|_\tau \quad t \in [t_0 - \tau, t_1]$$

thus, inequality (38) holds for $t \in [t_0 - \tau, t_0]$, so we only need to prove that

$$W(t) \leq \beta \|\varphi\|_\tau \quad t_0 < t < t_1. \quad (39)$$

If inequality (39) is not true, then there exists a $\bar{t} \in (t_0, t_1)$ such that

$$W(\bar{t}) > \beta \|\varphi\|_\tau > e^{-\mu d}\beta \|\varphi\|_\tau \geq e^{-\mu d}V(t_0) = W(t_0).$$

Set $t^* = \inf \{t \in (t_0, t_1): W(t) \geq \beta \|\varphi\|_\tau\}$, then we have $t^* \in (t_0, t_1)$, $W(t^*) = \beta \|\varphi\|_\tau$ and $W(t) \leq \beta \|\varphi\|_\tau$ for $t \leq t^*$. Set $t^{**} = \sup \{t \in [t_0, t^*]: W(t) \leq e^{-\mu d}\beta \|\varphi\|_\tau\}$, then $t^{**} \in [t_0, t^*]$

and $W(t^{**}) = e^{-\mu d} \beta \| \varphi \|_{\tau} \leq W(t) \leq \beta \| \varphi \|_{\tau}$, $t \in [t^{**}, t^*]$. For $t \in [t^{**}, t^*]$ and $s \in [-\tau, 0]$, we have

$$e^{\mu d} W(t) \geq \beta \| \varphi \|_{\tau} \geq W(t+s), \quad (40)$$

from condition (H_4^*) , inequalities (35) and (40), we have

$$\begin{aligned} D^+ W(t) &\leq (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)}) W(t) \\ &\leq \frac{-\mu d}{t_k - t_{k-1}} W(t) \leq 0 \quad t \in [t^{**}, t^*], \end{aligned} \quad (41)$$

which yields

$$\beta \| \varphi \|_{\tau} = W(t^*) \leq W(t^{**}) = e^{-\mu d} \beta \| \varphi \|_{\tau},$$

contradiction is obtained and inequality (39) is proved, so inequality (38) holds.

Now assume that inequality (37) holds for $k = 1, 2, \dots, m$, i.e.

$$W(t) \leq \beta \| \varphi \|_{\tau} \quad t \in [t_{k-1}, t_k] \quad k = 1, 2, \dots, m. \quad (42)$$

We will prove that inequality (37) holds for $k = m + 1$, i.e.

$$W(t) \leq \beta \| \varphi \|_{\tau} \quad t \in [t_m, t_{m+1}). \quad (43)$$

Towards this end, we first use the method by contradiction to show

$$W(t_m^-) \leq e^{-\mu d} \beta \| \varphi \|_{\tau}. \quad (44)$$

Assume that $W(t_m^-) > e^{-\mu d} \beta \| \varphi \|_{\tau}$. We examine two possible cases.

Case (a): $W(t) > e^{-\mu d} \beta \| \varphi \|_{\tau}$, $\forall t \in [t_{m-1}, t_m]$. In this case, from inequality (42), for $t \in [t_{m-1}, t_m]$, we have $e^{\mu d} W(t) > \beta \| \varphi \|_{\tau} \geq W(t+s)$ for $s \in [-\tau, 0]$. In view of inequality (35), we have

$$W(t) \leq (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)}) W(t) \quad t \in [t_{m-1}, t_m] \quad (45)$$

Now, integrating inequality (45) from t_{m-1} to t_m^- , in view of condition (H_4^*) , we have

$$\begin{aligned} W(t_m^-) &\leq e^{(\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)})(t_m - t_{m-1})} W(t_{m-1}) \\ &\leq e^{(\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)})(t_m - t_{m-1})} \beta \| \varphi \|_{\tau} \\ &\leq e^{-\mu d} \beta \| \varphi \|_{\tau}. \end{aligned}$$

This contradicts with the assumption $W(t_m^-) > e^{-\mu d} \beta \| \varphi \|_{\tau}$.

Case (b): There exist some $t \in [t_{m-1}, t_m]$ such that $W(t) \leq e^{-\mu d} \beta \| \varphi \|_{\tau}$. In this case, set $\bar{t} = \sup \{t \in [t_{m-1}, t_m] : W(t) \leq e^{-\mu d} \beta \| \varphi \|_{\tau}\}$, then $W(\bar{t}) = e^{-\mu d} \beta \| \varphi \|_{\tau}$ and $W(t) \geq e^{-\mu d} \beta \| \varphi \|_{\tau}$ for $t \in [\bar{t}, t_m]$. Thus from inequality (42), when $t \in [\bar{t}, t_m]$, $e^{\mu d} W(t) \geq \beta \| \varphi \|_{\tau} \geq W(t+s)$ for $s \in [-\tau, 0]$. In view of condition (H_4^*) and inequality (35), we have

$$\begin{aligned} &+ \lambda_1 + \lambda_2 e^{\mu(r+d)} W(t) \leq \bar{t} \\ &t \in [\bar{t}, t_m], \end{aligned} \quad (46)$$

which yields

$$t_m^- \leq W(\bar{t}) = e^{-\mu d} \beta \| \varphi \|_{\tau}$$

This contradicts with the assumption $W(t_m^-) > e^{-\mu d} \beta \| \varphi \|_{\tau}$. Thus in either case, we have a contradiction. Thus, inequality (44) holds.

When $t = t_m$, in view of condition (H_5^*) , inequalities (36), (42) and (44), we have

$$\begin{aligned} W(t_m) &\leq \lambda_{3m} W(t_m^-) + \lambda_{4m} e^{\mu d} W(t_m^- - \tau_2(t_m^-)) \\ &\leq \lambda_{3m} e^{-\mu d} \beta \| \varphi \|_{\tau} + \lambda_{4m} e^{\mu d} \beta \| \varphi \|_{\tau} < \beta \| \varphi \|_{\tau}. \end{aligned} \quad (47)$$

Thus, inequality (37) holds for $t = t_m$, so we only need to prove that

$$W(t) \leq \beta \| \varphi \|_{\tau} \quad t_m < t < t_{m+1}, \quad (48)$$

If inequality (48) is not true, then there exists a $\tilde{t} \in (t_m, t_{m+1})$ such that $V(\tilde{t}, x(\tilde{t})) > \beta \| \varphi \|_{\tau}$.

Set $t_* = \inf \{t \in (t_m, t_{m+1}) : W(t) \geq \beta \| \varphi \|_{\tau}\}$, then we have $t_* \in (t_m, t_{m+1})$, $W(t_*) = \beta \| \varphi \|_{\tau}$ and $W(t) \leq \beta \| \varphi \|_{\tau}$ for $t \leq t_*$. If $W(t) > e^{-\mu d} \beta \| \varphi \|_{\tau}$ for all $t \in [t_m, t_*]$, set $t_{**} = t_m$, then $W(t_{**}) = W(t_m) > e^{-\mu d} \beta \| \varphi \|_{\tau}$; otherwise, set $t_{**} = \sup \{t \in [t_m, t_*] : W(t) \leq e^{-\mu d} \beta \| \varphi \|_{\tau}\}$, then $t_{**} \in [t_m, t_*]$ and $W(t_{**}) = e^{-\mu d} \beta \| \varphi \|_{\tau}$. Hence, for $t \in [t_{**}, t_*]$, $e^{\mu d} W(t) \geq \beta \| \varphi \|_{\tau} \geq W(t+s)$, $s \in [-\tau, 0]$. In view of condition (H_4^*) and inequality (35), we have

$$\begin{aligned} D^+ W(t) &\leq (\mu + \lambda_1 + \lambda_2 e^{\mu(r+d)}) W(t) \\ &\leq \frac{-\mu d}{t_k - t_{k-1}} W(t) \leq 0, \quad t \in [t_{**}, t_*], \end{aligned}$$

which yields

$$\beta \| \varphi \|_{\tau} = W(t_*) \leq W(t_{**}) = W(t_m) < \beta \| \varphi \|_{\tau} \quad (47)$$

or

$$\beta \| \varphi \|_{\tau} = W(t_*) \leq W(t_{**}) = e^{-\mu d} \beta \| \varphi \|_{\tau}.$$

contradiction is obtained and inequality (48) is proved, so inequality (43) holds.

By the principle of mathematical induction, we have proved that (37) holds for any $k \in \mathbb{Z}^+$; therefore

$$e^{\mu(t-t_0-d)} V(t) = W(t) \leq \beta \| \varphi \|_{\tau} \quad t_{k-1} \leq t < t_k \quad k \in \mathbb{Z}^+,$$

then

$$V(t) \leq e^{\mu(t-t_0-d)} \beta \| \varphi \|_{\tau} \quad t_{k-1} \leq t < t_k \quad k \in \mathbb{Z}^+.$$

Which together with inequality (34), gives

$$\|x(t)\| \leq \frac{\beta}{\alpha} e^{-\mu(t-t_0-d)} \| \varphi \|_{\tau} \quad t_{k-1} \leq t < t_k \quad k \in \mathbb{Z}^+,$$

which implies that the trivial solution of system (1) is globally exponentially stable with convergence rate μ . The proof of Theorem 3 is thus completed.

11 Appendix 4. Proof of Proposition 2

Let $e = (1, 1, \dots, 1)^T$, $\tilde{f}(t, x(t), x(t-\tau_1(t))) = f(t, x(t), x(t-\tau_1(t))) + \varepsilon e$ for arbitrarily small $\varepsilon > 0$. We construct an auxiliary quasi-linear positive delay systems with delayed impulses

$$\begin{cases} \dot{x}(t) = Ax(t) + \tilde{f}(t, x(t), x(t - \tau_1(t))) & t \neq t_k \\ x(t_k^-) = g(t_k^-, x(t_k^-), x(t_k^- - \tau_2(t_k^-))) \\ \quad + C_k x(t_k^-) \\ x(t_0 + s) = \varphi(s), s \in [-\tau, 0], \end{cases} \quad k \in \mathbb{Z}^+ \quad (49)$$

Obviously, system (49) admits a unique solution $\tilde{x}(t, t_0, \varphi)$ for all $t \geq t_0 - \tau$. We have

$$\tilde{x}(t, t_0, \varphi) \rightarrow x(t, t_0, \varphi), \varepsilon \rightarrow 0, \quad (50)$$

for $t \geq t_0$ by [47]. Let $\tilde{x}(t) = \tilde{x}(t, t_0, \varphi)$.

For system (3) and (49) with any initial condition $\varphi(\cdot) \geq 0$, we have

$$x(t) \geq 0 \quad \tilde{x}(t) \geq 0 \quad t \in [t_0 - \tau, t_0]. \quad (51)$$

Assume that $\tilde{x}(t) \geq 0$ for all $t \in (t_0, t_1)$ is false, then there exist $t_1^k \in (t_0, t_1)$ such that $\tilde{x}_k(t_1^k) < 0$ for some $k \in \{1, 2, \dots, n\}$, let $J = \{k \in \{1, 2, \dots, n\} : \tilde{x}_k(t_1^k) < 0, t_1^k \in (t_0, t_1)\}$. Set $t^* = \inf \{t \in (t_0, t_1) : \tilde{x}_l(t) < 0 \text{ for some } l \in J\}$, then $t^* \in (t_0, t_1)$, $\tilde{x}(t) \geq 0$ for $t \in (t_0, t^*]$ and there exists a $i \in \{1, 2, \dots, n\}$ such that $\tilde{x}_i(t^*) = 0$. It would follow that $\dot{\tilde{x}}_i(t^*) \leq 0$, $\tilde{x}(t) \geq 0$ for $t \in (t_0, t^*]$, which together with inequality (51) gives $\tilde{x}(t) \geq 0$ for $t \in [t_0 - \tau, t^*]$. In view of $t - \tau_1(t) \in [t_0 - \tau, t^*]$ for $t \in (t_0, t^*]$, we have $\tilde{x}(t - \tau_1(t)) \geq 0$ for $t \in (t_0, t^*]$, by conditions of Proposition 2, we have $f(t, \tilde{x}(t), \tilde{x}(t - \tau_1(t))) \geq 0$ for $t \in (t_0, t^*]$, thus $\tilde{f}(t, \tilde{x}(t), \tilde{x}(t - \tau_1(t))) > 0$ for $t \in (t_0, t^*]$. From system (49) and $A \in \mathbf{M}$, we have $\dot{\tilde{x}}_i(t^*) = \sum_{j=1, i \neq j}^n a_{ij} \tilde{x}_j(t^*) + a_{ii} \tilde{x}_i(t^*) + \tilde{f}_i(t^*, \tilde{x}(t^*), \tilde{x}(t^* - \tau_1(t^*))) > 0$, where a_{ij} is the (i, j) th component of A . This contradicts with $\dot{\tilde{x}}_i(t^*) \leq 0$. Thus, $\tilde{x}(t) \geq 0$ for all $t \in (t_0, t_1)$. From (50), we have $x(t) \geq 0$ for all $t \in (t_0, t_1)$.

When $t = t_1$, $x(t_1) = C_1 x(t_1^-) + g(t_1^-, x(t_1^-), x(t_1^- - \tau_2(t_1^-)))$, $\tilde{x}(t_1) = C_1 \tilde{x}(t_1^-) + g(t_1^-, \tilde{x}(t_1^-), \tilde{x}(t_1^- - \tau_2(t_1^-)))$. In view of $t_1^- - \tau_2(t_1^-) \in [t_0 - \tau, t_1]$, $x(t) \geq 0, \tilde{x}(t) \geq 0$ for all $t \in [t_0 - \tau, t_1]$, we have $x(t_1^-) \geq 0, x(t_1^- - \tau_2(t_1^-)) \geq 0, \tilde{x}(t_1^-) \geq 0, \tilde{x}(t_1^- - \tau_2(t_1^-)) \geq 0$, by

conditions of Proposition 2, we have $g(t_1^-, x(t_1^-), x(t_1^- - \tau_2(t_1^-))) \geq 0$, $g(t_1^-, \tilde{x}(t_1^-), \tilde{x}(t_1^- - \tau_2(t_1^-))) \geq 0$. Since $C_1 \geq 0$, it is straightforward to obtain $x(t_1) = C_1 x(t_1^-) + g(t_1^-, x(t_1^-), x(t_1^- - \tau_2(t_1^-))) \geq 0$, $\tilde{x}(t_1) = C_1 \tilde{x}(t_1^-) + g(t_1^-, \tilde{x}(t_1^-), \tilde{x}(t_1^- - \tau_2(t_1^-))) \geq 0$.

From what has been discussed above, we have

$$x(t) \geq 0 \quad \tilde{x}(t) \geq 0 \quad t \in [t_0 - \tau, t_1]. \quad (52)$$

Now, similar to the above processes, assume that $\tilde{x}(t) \geq 0$ for all $t \in (t_1, t_2)$ is false, there exist $t_2^k \in (t_1, t_2)$ such that $\tilde{x}_k(t_2^k) < 0$ for some $k \in \{1, 2, \dots, n\}$, let $J_* = \{k \in \{1, 2, \dots, n\} : \tilde{x}_k(t_2^k) < 0, t_2^k \in (t_1, t_2)\}$. Set $t_* = \inf \{t \in (t_1, t_2) : \tilde{x}_l(t) < 0 \text{ for some } l \in J_*\}$, then $t_* \in (t_1, t_2)$, $\tilde{x}(t) \geq 0$ for $t \in (t_1, t_*]$ and there exists an $i \in \{1, 2, \dots, n\}$ such that $\tilde{x}_i(t_*) = 0$. It would follow that $\dot{\tilde{x}}_i(t_*) \leq 0$, $\tilde{x}(t) \geq 0$ for $t \in (t_1, t_*)$, which together with inequality (52) gives $\tilde{x}(t) \geq 0$ for $t \in [t_0 - \tau, t_*]$. In view of $t - \tau_1(t) \in [t_0 - \tau, t_*]$ for $t \in (t_1, t_*)$, we have $\tilde{x}(t - \tau_1(t)) \geq 0$ for $t \in (t_1, t_*]$, by conditions of Proposition 2, we have $f(t, \tilde{x}(t), \tilde{x}(t - \tau_1(t))) \geq 0$ for $t \in (t_1, t_*]$, thus, $\tilde{f}(t, \tilde{x}(t), \tilde{x}(t - \tau_1(t))) > 0$ for $t \in (t_1, t_*)$. From system (49) and $A \in \mathbf{M}$, we have $\dot{\tilde{x}}_i(t_*) = \sum_{j=1, i \neq j}^n a_{ij} \tilde{x}_j(t_*) + a_{ii} \tilde{x}_i(t_*) + \tilde{f}_i(t_*, \tilde{x}(t_*), \tilde{x}(t_* - \tau_1(t_*))) > 0$. This contradicts with $\dot{\tilde{x}}_i(t_*) \leq 0$. Thus, $\tilde{x}(t) \geq 0$ for all $t \in (t_1, t_2)$. From (50), we have $x(t) \geq 0$ for all $t \in (t_1, t_2)$.

When $t = t_2$, $x(t_2) = C_2 x(t_2^-) + g(t_2^-, x(t_2^-), x(t_2^- - \tau_2(t_2^-)))$, $\tilde{x}(t_2) = C_2 \tilde{x}(t_2^-) + g(t_2^-, \tilde{x}(t_2^-), \tilde{x}(t_2^- - \tau_2(t_2^-)))$. In view of $t_2^- - \tau_2(t_2^-) \in [t_0 - \tau, t_2]$, $x(t) \geq 0, \tilde{x}(t) \geq 0$ for all $t \in [t_0 - \tau, t_2]$, we have $x(t_2^-) \geq 0, x(t_2^- - \tau_2(t_2^-)) \geq 0, \tilde{x}(t_2^-) \geq 0, \tilde{x}(t_2^- - \tau_2(t_2^-)) \geq 0$, by conditions of Proposition 2, we have $g(t_2^-, x(t_2^-), x(t_2^- - \tau_2(t_2^-))) \geq 0$, $g(t_2^-, \tilde{x}(t_2^-), \tilde{x}(t_2^- - \tau_2(t_2^-))) \geq 0$. Since $C_2 \geq 0$, it is straightforward to obtain $x(t_2) = C_2 x(t_2^-) + g(t_2^-, x(t_2^-), x(t_2^- - \tau_2(t_2^-))) \geq 0$, $\tilde{x}(t_2) = C_2 \tilde{x}(t_2^-) + g(t_2^-, \tilde{x}(t_2^-), \tilde{x}(t_2^- - \tau_2(t_2^-))) \geq 0$.

By repeating the above procedure, we can easily obtain that $x(t) \geq 0$ for all $t \geq t_0$. Then, system (3) is a quasi-linear positive delay systems with delayed impulses. The proof of Proposition 2 is thus completed.