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Brief paper

Exponential stability of singular systems with multiple time-varying delays ☆

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ABSTRACT

This paper deals with the class of continuous-time singular linear systems with multiple time-varying delays in a range. The global exponential stability problem of this class of systems is addressed. Delay-range-dependent sufficient conditions such that the system is regular, impulse-free and α-stable are developed in the linear matrix inequality (LMI) setting. Moreover, an estimate of the convergence rate of such stable systems is presented. A numerical example is employed to show the usefulness of the proposed results.

1. Introduction

Singular systems with time-delays arise in a variety of practical systems such as chemical processes, lossless transmission lines, etc (Haidar, 2008; Halanay & Rasvan, 1997). Since singular systems with time-delays are matrix delay differential equations coupled with matrix difference equations, the study of such systems is much more complicated than that for standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given singular time-delay system is not always guaranteed and the system can also have undesired impulsive behavior.

Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived using the time domain method, see Fridman (2002), Xu, Van Dooren, Stefan, and Lam (2002), Yue, Lam, and Daniel (2005) and Zhu, Zhang, Cheng, and Feng (2007) and the references therein. However, most of the delay-dependent results in the literature tackled only the case of constant time-delay where two approaches were used to prove the stability of the system. The first approach consists of decomposing the system into fast and slow subsystems and the stability of the slow subsystem is proved using some Lyapunov functional. Then, the fast variables is expressed explicitly by an iterative equation in terms of the slow variables (Xu et al., 2002). The second approach introduced by Fridman (2002) and it consists of constructing a Lyapunov–Krasovskii functional that corresponds directly to the descriptor form of the system. In Yue et al. (2005), where time-varying delays are considered, the response of the fast variables has been bounded by an exponential term using a different approach. Using the approach in Yue et al. (2005), it is not possible to give an estimate of the convergence rate of the states of the system. To the best of the authors' knowledge, the stability problem for singular systems with time-varying delays has not been fully investigated. This is due to the difficulty of guaranteeing the stability of the fast subsystem.

Recently, a free-weighting matrices method was proposed to study the delay-dependent stability for time-delay systems. In 2007, Zhu et al. adopted this technique for singular time-delay systems (Zhu et al., 2007) and show the advantages of their results over the existing results in the literature. Also, delay-range-dependent concept was recently studied for time-delay systems, where the delays are considered to vary in a range and thereby more applicable in practice (He, Wang, Lin, & Wu, 2007).

Formally speaking, these conditions provide only the asymptotic stability of singular systems with time-delays. In Sun (2003), the global delay-independent exponential stability for a class of singular systems with multiple constant time-delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the LMI approach for deriving exponential estimates for solutions of singular time-delay systems.

This paper addresses an important problem that has not been fully investigated. Delay-range-dependent exponential stability

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conditions for singular systems with multiple time-varying delays are established in terms of LMIs. Moreover, an estimate of the convergence rate of such systems is presented. The method used is based on Lyapunov–Krasovskii approach. Some graph theory terminologies have been adopted to model the dependency of the fast variables on past instances which leads to an explicit expression of the fast variables in terms of the initial conditions and the slow variables. This expression is used to prove the stability of the fast subsystem.

The rest of the paper is organized as follows. In Section 2, the problem and some definitions and Lemmas are stated. In Section 3, the main results are given. In Sections 4 and 5, a numerical example and the conclusion are given.

Notation: Throughout this paper, $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximal and minimal eigenvalue of matrix P. $C_\tau = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean norm whereas $\|\phi\|_c = \sup_{-\tau\le t\le 0}\|\phi(t)\|$ stands for the norm of a function $\phi \in C_\tau$. C_τ^v is defined by $C_\tau^v = \{\phi \in C_\tau;\ \|\phi\|_c \langle v, v\rangle 0\}$.

2. Problem statement and definitions

Consider the linear singular time-delay system:

$$\begin{cases} E\dot{x}(t) = Ax(t) + \sum_{k=1}^{p} A_k x(t - d_k(t)) \\ x(t) = \phi(t),\ -\bar{d} \le t \le 0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, the matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we assume that $\mathrm{rank}(E) = r \le n$, A and A_k are known constant matrices, $\phi(t) \in C_\tau^v$ is a compatible vector valued continuous function (see Fridman (2002) for details) and $d_k(t)$, $k = 1, \dots, p$, is the time-delay and that is assumed to satisfy:

$$\begin{cases} 0 < \underline{d}_k \le d_k(t) \le \bar{d}_k \\ \dot{d}_k(t) \le \mu < 1 \end{cases} \tag{2}$$

with $\underline{d}_k$, $\bar{d}_k$ and μ are given scalars. Also, $\bar{d}$ and $\underline{d}$ are positive scalars with $\bar{d} = \max\{\bar{d}_1, \dots, \bar{d}_p\}$ and $\underline{d} = \min\{\underline{d}_1, \dots, \underline{d}_p\}$. Note that this class of systems includes the class of neutral descriptor systems (see Fridman (2002) for details).

The following definitions will be used in what follows.

Definition 1 (*Boukas, 2008*; *Haidar, 2008*).

i. System (1) is said to be regular if the characteristic polynomial, $\det(sE - A)$ is not identically zero.
ii. System (1) is said to be impulse-free if $\deg(\det(sE - A)) = \mathrm{rank}(E)$.
iii. System (1) is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the singular time-delay system satisfies $\|x(t)\| \le \gamma e^{-\sigma t}\|\phi\|_c$.
iv. System (1) is said to be exponentially admissible if it is regular, impulse-free and exponentially stable.

Remark 2. In the rest of this paper, the following terminologies borrowed from graph theory will be used.

- A tree structure is a way of representing the hierarchical nature of a structure in a graphical form (see Fig. 1).
- The topmost node in a tree is called the root node.
- A node is a parent of another node (child) if it is one step higher in the hierarchy and closer to the root node.
- Nodes at the bottommost level of the tree are called leaf nodes.

Lemma 3 (*Haidar, 2008*). *Suppose that system (1) is regular and impulse-free, then the solution to system (1) exists and it is impulse-free and unique on* $(0, \infty)$.

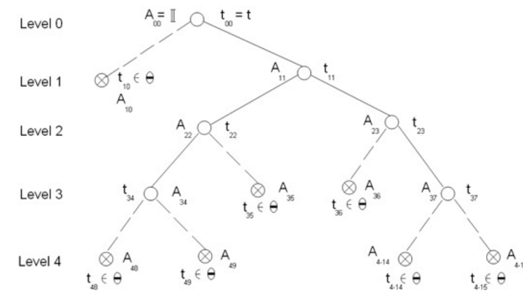

Fig. 1. An example with $p = 2$.

3. Main results

3.1. Delay-Range-dependent exponential stability

Theorem 4. *Given positive scalars* $\underline{d}_k$ *and* $\bar{d}_k$, *with* $\underline{d}_k < \bar{d}_k$, $k = 1, \dots, p$, $0 \le \mu < 1$ *and* $\alpha > 0$. *System (1) is exponentially admissible with* $\sigma = \alpha$ *if there exist a nonsingular matrix* P, *symmetric and positive-definite matrices* Q_{k1}, Q_{k2}, Q_{k3}, Z_{k1} *and* Z_{k2}, $k = 1, \dots, p$, *and matrices* M_{ki}, N_{ki} *and* S_{ki}, $i = 1, 2$, $k = 1, \dots, p$, *such that the following LMI holds*

$$\begin{bmatrix} \Pi & \Upsilon & \tilde{A}U \\ \star & T & 0 \\ \star & \star & -U \end{bmatrix} < 0 \tag{3}$$

with the following constraint

$$E^\top P = P^\top E \ge 0 \tag{4}$$

where

$$T = \mathrm{diag}\left\{ -\frac{2\alpha}{e^{2\alpha\underline{d}_k} - 1} Z_{k1},\ -\frac{2\alpha}{e^{2\alpha\bar{d}_k} - e^{2\alpha\underline{d}_k}}(Z_{k1} + Z_{k2}),\ -\frac{2\alpha}{e^{2\alpha\bar{d}_k} - e^{2\alpha\underline{d}_k}} Z_{k2} \right\},\quad k = 1, \dots, p$$

$$\tilde{A}^\top = \begin{bmatrix} A & \tilde{A}_1 & \dots & \tilde{A}_p \end{bmatrix},\quad \text{with } \tilde{A}_k = \begin{bmatrix} A_k & 0 & 0 \end{bmatrix}$$

$$U = \sum_{k=1}^{p} \left\{ \left(\bar{d}_k Z_{k1} + \bar{\underline{d}}_k Z_{k2} \right) \right\} \quad \text{with } \bar{\underline{d}}_k = \bar{d}_k - \underline{d}_k$$

$$\Pi = \begin{bmatrix} \Pi_1 & F \\ \star & G \end{bmatrix},\qquad \Upsilon = \begin{bmatrix} \tilde{N}_1 & \tilde{S}_1 & \tilde{M}_1 & \dots & \tilde{N}_p & \tilde{S}_p & \tilde{M}_p \end{bmatrix}$$

$$\tilde{N}_k^\top = \begin{bmatrix} N_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & N_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix} \quad k = 1, \dots, p$$

$$\tilde{M}_k^\top = \begin{bmatrix} M_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & M_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix} \quad k = 1, \dots, p$$

$$\tilde{S}_k^\top = \begin{bmatrix} S_{k1}^\top & \mathbf{0}_{n\times 3n(k-1)} & S_{k2}^\top & 0 & 0 & \mathbf{0}_{n\times 3n(p-k)} \end{bmatrix} \quad k = 1, \dots, p$$

$$\Pi_1 = P^\top A + A^\top P + \sum_{k=1}^{p} \left\{ \sum_{i=1}^{3} Q_{ki} + N_{k1}E + (N_{k1}E)^\top \right\} + 2\alpha E^\top P$$

$$F = \begin{bmatrix} W_1 & \dots & W_p \end{bmatrix},\qquad G = \mathrm{diag}\left\{ J_1, \dots, J_p \right\}$$

$$J_k = \begin{bmatrix} \Pi_{k3} & e^{\alpha\underline{d}_k} M_{k2}E & -e^{\alpha\bar{d}_k} S_{k2}E \\ \star & -Q_{k1} & \mathbf{0} \\ \star & \star & -Q_{k2} \end{bmatrix} \quad k = 1, \dots, p$$

$$W_k = \begin{bmatrix} \Pi_{k2} & e^{\alpha\underline{d}_k} M_{k1}E & -e^{\alpha\bar{d}_k} S_{k1}E \end{bmatrix},\quad k = 1, \dots, p$$

$$\Pi_{k2} = P^\top A_k + (N_{k2}E)^\top - N_{k1}E + S_{k1}E - M_{k1}E,\quad k = 1, \dots, p$$

$$\Pi_{k3} = -(1-\mu)e^{-2\alpha\bar{d}_k} Q_{k3} + S_{k2}E + (S_{k2}E)^\top - N_{k2}E - (N_{k2}E)^\top - M_{k2}E - (M_{k2}E)^\top,\quad k = 1, \dots, p.$$

Proof. First, we will show that the system is regular and impulse-free. For this purpose, choose two nonsingular matrices R and L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \tag{5}$$

Now, let $\bar{A}_{kd} = RA_kL$, $\bar{P} = R^{-\top}PL$, $\bar{N}_{ki} = L^\top N_{ki}R^{-1}$, $\bar{Q}_{ki} = L^\top Q_{ki}L$,

$$\bar{A}_{kd} = \begin{bmatrix} A_{kd11} & A_{kd12} \\ A_{kd21} & A_{kd22} \end{bmatrix}, \qquad \bar{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6}$$

$$\bar{N}_{ki} = \begin{bmatrix} N_{ki11} & N_{ki12} \\ N_{ki21} & N_{ki22} \end{bmatrix}, \qquad \bar{Q}_{ki} = \begin{bmatrix} Q_{ki11} & Q_{ki12} \\ Q_{ki21} & Q_{ki22} \end{bmatrix}. \tag{7}$$

From (3)–(7) and some algebraic manipulations, we obtain $A_{22}^\top P_{22} + P_{22}^\top A_{22} < 0$. Therefore A_{22} is nonsingular, which implies in turn that system (1) is regular and impulse-free (see Haidar (2008)). Next, we show the exponential stability of system (1). Since system (1) is regular, there exist two other matrices R and L such that (see Haidar (2008))

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix} \qquad \bar{A} = RAL = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix}. \tag{8}$$

Define $\bar{A}_{kd}$, $\bar{P}$, $\bar{N}_{ki}$, $\bar{Q}_{ki}$ as in (6)–(7), $\bar{M}_{ki}$, $\bar{S}_{ki}$ similar to $\bar{N}_{ki}$, and $\bar{Z}_{ki} = R^{-\top}Z_{ki}R^{-1}$. Using (3) and Schur complement, we get

$$\begin{bmatrix} \Pi_1 & \Xi \\ \star & \Omega \end{bmatrix} < 0 \quad \text{where} \quad \begin{array}{l} \Xi = \left[\Pi_{12}, \ldots \Pi_{p2}\right] \\ \Omega = \operatorname{diag}\left\{\Pi_{13}, \ldots, \Pi_{p3}\right\}. \end{array}$$

Substitute (8) into this inequality, pre- and post-multiply by $\operatorname{diag}\{L^\top, \ldots, L^\top\}$, $\operatorname{diag}\{L, \ldots, L\}$ and using the Schur complement, we have

$$\begin{bmatrix} P_{22}^\top + P_{22} + \sum_{k=1}^{p}\sum_{i=1}^{3} Q_{ki22} & X \\ \star & H \end{bmatrix} < 0 \tag{9}$$

where $X = \begin{bmatrix} P_{22}^\top A_{1d22} & \ldots & P_{22}^\top A_{pd22} \end{bmatrix}$

$H = \operatorname{diag}\left\{-(1-\mu)\mathrm{e}^{-2\alpha\bar{d}_i}Q_{i322}\right\}, \quad i = 1, \ldots, p.$

Pre- and post-multiplying (9) by $\operatorname{diag}\left\{\mathbb{I}, \mathrm{e}^{\alpha\bar{d}_1}\mathbb{I}, \ldots, \mathrm{e}^{\alpha\bar{d}_p}\mathbb{I}\right\}$ and its transpose, respectively, and noting that $\mu \geq 0$, Lemma 2 in Fridman (2002) implies

$$\rho\left(\sum_{k=1}^{p} \mathrm{e}^{\alpha\bar{d}_k}A_{kd22}\right) < 1. \tag{10}$$

From (10), there exist constants $\beta > 1$ and $\gamma \in (0, 1)$ such that

$$\left\|\sum_{k=1}^{p} \mathrm{e}^{\alpha\bar{d}_k}A_{kd22}\right\| \leq \beta\gamma^i, \quad i = 1, 2, \ldots. \tag{11}$$

Let $\zeta(t) = L^{-1}x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (1) becomes equivalent to the following one

$$\dot{\zeta}_1(t) = \widehat{A}\zeta_1(t) + \sum_{k=1}^{p}\{A_{kd11}\zeta_1(t-d_k(t)) + A_{kd12}\zeta_2(t-d_k(t))\} \tag{12}$$

$$0 = \zeta_2(t) + \sum_{k=1}^{p}\{A_{kd21}\zeta_1(t-d_k(t)) + A_{kd22}\zeta_2(t-d_k(t))\}. \tag{13}$$

Now, choose the Lyapunov functional as follows:

$$\begin{aligned} V(\zeta_t) = {} & \zeta^\top(t)\bar{E}^\top\bar{P}\zeta(t) + \sum_{k=1}^{p}\left\{\int_{t-\underline{d}_k}^{t} \zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k1}\zeta(s)\mathrm{d}s\right. \\ & + \int_{t-\bar{d}_k}^{t} \zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k2}\zeta(s)\mathrm{d}s \\ & + \int_{t-d_k(t)}^{t} \zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k3}\zeta(s)\mathrm{d}s \\ & + \int_{-\bar{d}_k}^{0}\int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k1}\bar{E}\dot{\zeta}(s)\mathrm{d}s\mathrm{d}\theta \\ & \left. + \int_{-\bar{d}_k}^{-\underline{d}_k}\int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k2}\bar{E}\dot{\zeta}(s)\mathrm{d}s\mathrm{d}\theta\right\} \end{aligned}$$

where $\zeta_t = \zeta(t-\beta)$, $\beta \in (-\bar{d}, 0]$. Then, the time-derivative of $V(\zeta_t)$ along the solution of (12) and (13) is given by

$$\begin{aligned} \dot{V}(\zeta_t) = {} & 2\zeta^\top(t)\bar{P}^\top\bar{E}\dot{\zeta}(t) + \sum_{k=1}^{p}\left\{\zeta^\top(t)\bar{Q}_{k1}\zeta(t)\right. \\ & - \zeta^\top(t-\underline{d}_k)\mathrm{e}^{-2\alpha\underline{d}_k}\bar{Q}_{k1}\zeta(t-\underline{d}_k) + \zeta^\top(t)\bar{Q}_{k2}\zeta(t) \\ & - \zeta^\top(t-\bar{d}_k)\mathrm{e}^{-2\alpha\bar{d}_k}\bar{Q}_{k2}\zeta(t-\bar{d}_k) + \zeta^\top(t)\bar{Q}_{k3}\zeta(t) \\ & - (1-\dot{d}_k(t))\zeta^\top(t-d_k(t))\mathrm{e}^{-2\alpha d_k(t)}\bar{Q}_{k3}\zeta(t-d_k(t)) \\ & + \bar{d}_k(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_{k1}\bar{E}\dot{\zeta}(t) + (\bar{d}_k-\underline{d}_k)(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_{k2}\bar{E}\dot{\zeta}(t) \\ & - \int_{t-\bar{d}_k}^{t}(\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k1}\bar{E}\dot{\zeta}(s)\mathrm{d}s \\ & - \int_{t-\bar{d}_k}^{t-\underline{d}_k}(\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k2}\bar{E}\dot{\zeta}(s)\mathrm{d}s \\ & -2\alpha\int_{t-\underline{d}_k}^{t}\zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k1}\zeta(s)\mathrm{d}s \\ & -2\alpha\int_{t-\bar{d}_k}^{t}\zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k2}\zeta(s)\mathrm{d}s \\ & -2\alpha\int_{t-d_k(t)}^{t}\zeta^\top(s)\mathrm{e}^{2\alpha(s-t)}\bar{Q}_{k3}\zeta(s)\mathrm{d}s \\ & -2\alpha\int_{-\bar{d}_k}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k1}\bar{E}\dot{\zeta}(s)\mathrm{d}s\mathrm{d}\theta \\ & \left. -2\alpha\int_{-\bar{d}_k}^{-\underline{d}_k}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k2}\bar{E}\dot{\zeta}(s)\mathrm{d}s\mathrm{d}\theta\right\}. \end{aligned} \tag{14}$$

Adding these terms

$$\begin{aligned} & \sum_{k=1}^{p} 2\left[\zeta^\top(t)\bar{N}_{k1} + \zeta^\top(t-d_k(t))\bar{N}_{k2}\right] \\ & \times \left[\bar{E}\zeta(t) - \bar{E}\zeta(t-d_k(t)) - \int_{t-d_k(t)}^{t}\bar{E}\dot{\zeta}(s)\mathrm{d}s\right] \\ & + \sum_{k=1}^{p} 2\left[\zeta^\top(t)\bar{S}_{k1} + \zeta^\top(t-d_k(t))\bar{S}_{k2}\right] \\ & \times \left[\bar{E}\zeta(t-d_k(t)) - \bar{E}\zeta(t-\bar{d}_k) - \int_{t-\bar{d}_k}^{t-d_k(t)}\bar{E}\dot{\zeta}(s)\mathrm{d}s\right] \\ & + \sum_{k=1}^{p} 2\left[\zeta^\top(t)\bar{M}_{k1} + \zeta^\top(t-d_k(t))\bar{M}_{k2}\right] \\ & \times \left[\bar{E}\zeta(t-\underline{d}_k) - \bar{E}\zeta(t-d_k(t)) - \int_{t-d_k(t)}^{t-\underline{d}_k}\bar{E}\dot{\zeta}(s)\mathrm{d}s\right] \\ & + \sum_{k=1}^{p}\left\{\int_{t-\bar{d}_k}^{t}\left[\zeta^\top(t)\bar{N}_{k1} + \zeta^\top(t-d_k(t))\bar{N}_{k2}\right]\bar{Z}_{k1}^{-1}\mathrm{e}^{-2\alpha(s-t)}\right. \end{aligned}$$

$$
\begin{aligned}
&\times \left[\zeta^\top(t)\bar{N}_{k1} + \zeta^\top(t-d_k(t))\bar{N}_{k2}\right]^\top \mathrm{d}s \\
&- \int_{t-d_k(t)}^{t} \left[\zeta^\top(t)\bar{N}_{k1} + \zeta^\top(t-d_k(t))\bar{N}_{k2}\right] \bar{Z}_{k1}^{-1} \mathrm{e}^{-2\alpha(s-t)} \\
&\times \left[\zeta^\top(t)\bar{N}_{k1} + \zeta^\top(t-d_k(t))\bar{N}_{k2}\right]^\top \mathrm{d}s \\
&+ \int_{t-\bar{d}_k}^{t-\underline{d}_k} \left[\zeta^\top(t)\bar{S}_{k1} + \zeta^\top(t-d_k(t))\bar{S}_{k2}\right] \left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1} \\
&\times \mathrm{e}^{-2\alpha(s-t)} \left[\zeta^\top(t)\bar{S}_{k1} + \zeta^\top(t-d_k(t))\bar{S}_{k2}\right]^\top \mathrm{d}s \\
&- \int_{t-\bar{d}_k}^{t-d_k(t)} \left[\zeta^\top(t)\bar{S}_{k1} + \zeta^\top(t-d_k(t))\bar{S}_{k2}\right] \left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1} \\
&\times \mathrm{e}^{-2\alpha(s-t)} \left[\zeta^\top(t)\bar{S}_{k1} + \zeta^\top(t-d_k(t))\bar{S}_{k2}\right]^\top \mathrm{d}s \\
&+ \int_{t-\bar{d}_k}^{t-\underline{d}_k} \left[\zeta^\top(t)\bar{M}_{k1} + \zeta^\top(t-d_k(t))\bar{M}_{k2}\right] \bar{Z}_{k2}^{-1} \mathrm{e}^{-2\alpha(s-t)} \\
&\times \left[\zeta^\top(t)\bar{M}_{k1} + \zeta^\top(t-d_k(t))\bar{M}_{k2}\right]^\top \mathrm{d}s \\
&- \int_{t-d_k(t)}^{t-\underline{d}_k} \left[\zeta^\top(t)\bar{M}_{k1} + \zeta^\top(t-d_k(t))\bar{M}_{k2}\right] \bar{Z}_{k2}^{-1} \mathrm{e}^{-2\alpha(s-t)} \\
&\times \left[\zeta^\top(t)\bar{M}_{k1} + \zeta^\top(t-d_k(t))\bar{M}_{k2}\right]^\top \mathrm{d}s \Bigg\}
\end{aligned}
$$

to (14) gives

$$
\begin{aligned}
\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) &\le \eta^\top(t) \Bigg[\Pi + \sum_{k=1}^{p} \Bigg\{ \widetilde{A}^\top \left(\bar{d}_k \bar{Z}_{k1} + \underline{d}_k \bar{Z}_{k2}\right) \widetilde{A} \\
&+ \frac{\mathrm{e}^{2\alpha\bar{d}_k}-1}{2\alpha} \widetilde{N}_k \bar{Z}_{k1}^{-1} \widetilde{N}_k^\top + \frac{\mathrm{e}^{2\alpha\bar{d}_k}-\mathrm{e}^{2\alpha\underline{d}_k}}{2\alpha} \widetilde{S}_k \left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1} \widetilde{S}_k^\top \\
&+ \frac{\mathrm{e}^{2\alpha\bar{d}_k}-\mathrm{e}^{2\alpha\underline{d}_k}}{2\alpha} \widetilde{M}_k \bar{Z}_{k2}^{-1} \widetilde{M}_k^\top \Bigg\} \Bigg] \eta(t) \\
&- \sum_{k=1}^{p} \Bigg\{ \int_{t-d_k(t)}^{t} \left[\eta^\top(t)\widetilde{N}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k1}\right] \mathrm{e}^{-2\alpha(s-t)} \bar{Z}_{k1}^{-1} \\
&\times \left[\eta^\top(t)\widetilde{N}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k1}\right]^\top \mathrm{d}s \\
&+ \int_{t-\bar{d}_k}^{t-d_k(t)} \left[\eta^\top(t)\widetilde{S}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)\right] \mathrm{e}^{-2\alpha(s-t)} \\
&\times \left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1} \left[\eta^\top(t)\widetilde{S}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)\right]^\top \mathrm{d}s \\
&+ \int_{t-d_k(t)}^{t-\underline{d}_k} \left[\eta^\top(t)\widetilde{M}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k2}\right] \mathrm{e}^{-2\alpha(s-t)} \bar{Z}_{k2}^{-1} \\
&\times \left[\eta^\top(t)\widetilde{M}_k + \bar{E}\dot{\zeta}(s)\mathrm{e}^{2\alpha(s-t)}\bar{Z}_{k2}\right]^\top \mathrm{d}s \Bigg\} \\
&\le \eta^\top(t) \Bigg[\Pi + \sum_{k=1}^{p} \Bigg\{ \widetilde{A}^\top \left(\bar{d}_k \bar{Z}_{k1} + \underline{d}_k \bar{Z}_{k2}\right) \widetilde{A} \\
&+ \frac{\mathrm{e}^{2\alpha\bar{d}_k}-1}{2\alpha} \widetilde{N}_k \bar{Z}_{k1}^{-1} \widetilde{N}_k^\top + \frac{\mathrm{e}^{2\alpha\bar{d}_k}-\mathrm{e}^{2\alpha\underline{d}_k}}{2\alpha} \widetilde{S}_k \left(\bar{Z}_{k1}+\bar{Z}_{k2}\right)^{-1} \widetilde{S}_k^\top \\
&+ \frac{\mathrm{e}^{2\alpha\bar{d}_k}-\mathrm{e}^{2\alpha\underline{d}_k}}{2\alpha} \widetilde{M}_k \bar{Z}_{k2}^{-1} \widetilde{M}_k^\top \Bigg\} \Bigg] \eta(t)
\end{aligned}
$$

where

$$\eta^\top(t) = \begin{bmatrix} \zeta^\top(t) & \omega_1^\top(t) & \dots & \omega_p^\top(t) \end{bmatrix}$$

$$\omega_k(t) = \begin{bmatrix} \zeta^\top(t-d_k(t)) & \zeta^\top(t-\underline{d}_k) & \zeta^\top(t-\bar{d}_k) \end{bmatrix}$$

and $\widetilde{A}$, $\widetilde{N}_k$, $\widetilde{M}_k$, $\widetilde{S}_k$ and Π as defined in Theorem 4 with $\bar{A}$, $\bar{A}_{kd}$, $\bar{N}_{k1}$, $\bar{N}_{k2}$, $\bar{M}_{k1}$, $\bar{M}_{k2}$, $\bar{S}_{k1}$, $\bar{S}_{k2}$ and $\bar{P}$ instead of A, A_k, N_{k1}, N_{k2}, M_{k1}, M_{k2}, S_{k1}, S_{k2} and P. Pre- and post-multiply (3) by $\mathrm{diag}\{L^\top, L^\top, \mathrm{e}^{-\alpha\underline{d}_1}L^\top, \mathrm{e}^{-\alpha\bar{d}_1}L^\top, L^\top, \mathrm{e}^{-\alpha\underline{d}_2}L^\top, \mathrm{e}^{-\alpha\bar{d}_2}L^\top, \dots, L^\top, \mathrm{e}^{-\alpha\underline{d}_p}L^\top, \mathrm{e}^{-\alpha\bar{d}_p}L^\top, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \dots, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}\}$ and its transpose, then using the Schur complement implies

$\dot{V}(\zeta_t) + 2\alpha V(\zeta_t) \le 0$, which leads to, $V(\zeta_t) \le \mathrm{e}^{-2\alpha t} V(\phi(t))$.

Then, the following estimation is obtained

$$\lambda_1 \|\zeta_1(t)\|^2 \le V(\zeta_t) \le \mathrm{e}^{-2\alpha t} V(\phi(t)) \le \lambda_2 \mathrm{e}^{-2\alpha t} \|\phi\|_c^2$$

where $\lambda_1 = \lambda_{\min}(\bar{P}_{11}) > 0$ and $\lambda_2 > 0$ is sufficiently large and can be found since $V(\phi(t))$ is a bounded quadratic functional of $\phi(t)$. This leads to

$$\|\zeta_1(t)\| \le \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c \, \mathrm{e}^{-\alpha t}. \tag{15}$$

In order to prove the exponential stability of the fast subsystem, the relation in (13) should be used. For constant time delay, an explicit equation of $\zeta_2(t)$ can be found easily by an iterative method (Xu et al., 2002). In the case of multiple time-varying or even single delay, such a simple relation cannot be found. Thus, a tree structure will be adopted to model the dependency of $\zeta_2(t)$ on past instances. For this purpose, define

$$t_{00} = t, \qquad t_{ij} = t_{(i-1)\nu_j} - d_{(\kappa_p^j+1)}(t_{(i-1)\nu_j}) \tag{16}$$

$$\Theta = \{t_{ij} \mid t_{ij} \in (-\bar{d}, 0] \quad \text{and} \quad t_{(i-1)\nu_j} \notin (-\bar{d}, 0]\} \tag{17}$$

$$\hat{A}_{00} = \mathbb{I}, \qquad \hat{A}_{ij} = \left(\hat{A}_{(i-1)\nu_j}\right) \times \left(-A_{(\kappa_p^j+1)d22}\right) \tag{18}$$

where ν_j is the greatest integer less than or equal to $\frac{j}{p}$, κ_p^j is the remainder of the integer division $\frac{j}{p}$, and t_{ij} and $\hat{A}_{ij}$ are undefined if $t_{(i-1)\nu_j} \in (-\bar{d}, 0]$. If we let the parents of t_{ij} and $\hat{A}_{ij}$ to be $t_{(i-1)\nu_j}$ and $\hat{A}_{(i-1)\nu_j}$, respectively, t_{ij}'s and $\hat{A}_{ij}$'s will represent two trees with the same structure (see Fig. 1), with roots t and $\mathbb{I}$, respectively. Note that $\zeta_2(t_{ij})$ depends on the value of ζ at all times indicated by the children of t_{ij} in the tree. Note also that the values of the leaf nodes of the t_{ij}'s tree belong to Θ. Noting that $\kappa_p^j = j$ if $j < p$, then from (13), and using the definitions (16)–(18), we get

$$\zeta_2(t) = -\sum_{k=1}^{p} \{A_{kd21}\zeta_1(t-d_k(t)) + A_{kd22}\zeta_2(t-d_k(t))\}$$

$$
\begin{aligned}
\zeta_2(t) &= \sum_{j=0}^{p-1} \left\{ -A_{(j+1)d21}\zeta_1(t_{1j}) + \hat{A}_{1j}\zeta_2(t_{1j}) \right\} \\
&= \sum_{t_{1j}\in\Theta} \left\{ \hat{A}_{1j}\zeta_2(t_{1j}) \right\} + \sum_{j=0}^{p-1} \left\{ -A_{(j+1)d21}\zeta_1(t_{1j}) \right\} \\
&+ \sum_{\substack{j=0 \\ t_{1j}\notin\Theta}}^{p-1} \hat{A}_{1j}\zeta_2(t_{1j})
\end{aligned} \tag{19}
$$

if $t_{1j} \notin \Theta$, from (13) and (16)–(18), we get

$$
\begin{aligned}
\hat{A}_{1j}\zeta_2(t_{1j}) &= \hat{A}_{1j} \sum_{r=jp}^{(j+1)p-1} \left\{ -A_{(\kappa_p^r+1)d21}\zeta_1(t_{2r}) - A_{(\kappa_p^r+1)d22}\zeta_2(t_{2r}) \right\} \\
&= \sum_{r=jp}^{(j+1)p-1} \left\{ -\hat{A}_{1j}A_{(\kappa_p^r+1)d21}\zeta_1(t_{2r}) - \hat{A}_{1j}A_{(\kappa_p^r+1)d22}\zeta_2(t_{2r}) \right\} \\
&= \sum_{r=jp}^{(j+1)p-1} \left\{ -\hat{A}_{1j}A_{(\kappa_p^r+1)d21}\zeta_1(t_{2r}) + \hat{A}_{2r}\zeta_2(t_{2r}) \right\}
\end{aligned}
$$

thus, $\zeta_2(t)$ in (19) can be computed from

$$\zeta_2(t) = \sum_{t_{1j}\in\Theta}\left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} + \sum_{j=0}^{p-1}\left\{-A_{(j+1)d21}\zeta_1(t_{1j})\right\} + \sum_{\substack{j=0\\ t_{1j}\notin\Theta}}^{p-1}\sum_{r=jp}^{(j+1)p-1}\left\{-\hat{A}_{1j}A_{(\kappa_p^r+1)d21}\zeta_1(t_{2r}) + \hat{A}_{2r}\zeta_2(t_{2r})\right\}$$

$$= \sum_{t_{1j}\in\Theta}\left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} + \sum_{j=0}^{p-1}\left\{-A_{(j+1)d21}\zeta_1(t_{1j})\right\} + \sum_{\substack{j=0\\ t_{1\nu_j}\notin\Theta}}^{p^2-1}\left\{-\hat{A}_{1j}A_{(\kappa_p^j+1)d21}\zeta_1(t_{2j}) + \hat{A}_{2j}\zeta_2(t_{2j})\right\}$$

$$= \sum_{t_{1j}\in\Theta}\left\{\hat{A}_{1j}\zeta_2(t_{1j})\right\} - \sum_{i=0}^{1}\sum_{\substack{j=0\\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\kappa_p^j+1)d21} \times \zeta_1(t_{(i+1)j}) + \sum_{\substack{j=0\\ t_{1\nu_j}\notin\Theta}}^{p^2-1}\hat{A}_{2j}\zeta_2(t_{2j})$$

$$= \sum_{i=1}^{2}\sum_{t_{ij}\in\Theta}\left\{\hat{A}_{ij}\zeta_2(t_{ij})\right\} - \sum_{i=0}^{1}\sum_{\substack{j=0\\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\kappa_p^j+1)d21} \times \zeta_1(t_{(i+1)j}) + \sum_{\substack{j=0\\ t_{2j}\notin\Theta}}^{p^2-1}\hat{A}_{2j}\zeta_2(t_{2j}).$$

Continuing in the same manner, if $t_{2j}\notin\Theta$,

$$\hat{A}_{2j}\zeta_2(t_{2j}) = \sum_{r=jp}^{(j+1)p-1}\left\{-\hat{A}_{2j}A_{(\kappa_p^r+1)d21}\zeta_1(t_{3r}) + \hat{A}_{3r}\zeta_2(t_{3r})\right\}$$

we get,

$$\zeta_2(t) = \sum_{i=1}^{3}\sum_{t_{ij}\in\Theta}\left\{\hat{A}_{ij}\zeta_2(t_{ij})\right\} - \sum_{i=0}^{2}\sum_{\substack{j=0\\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\kappa_p^j+1)d21} \times \zeta_1(t_{(i+1)j}) + \sum_{\substack{j=0\\ t_{3j}\notin\Theta}}^{p^3-1}\hat{A}_{3j}\zeta_2(t_{3j}).$$

Note that $t_{ij} = t_{(i-1)\nu_j} - d_{(\kappa_p^j+1)}(t_{(i-1)\nu_j}) \le t_{(i-1)\nu_j} - \underline{d}_{(\kappa_p^j+1)} < t_{(i-1)\nu_j}$

which means that the time of a child is always less than the time of its parent. Therefore, there exists a positive finite integer $k(t)$ such that

$$\zeta_2(t) = \sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\left\{\hat{A}_{ij}\zeta_2(t_{ij})\right\} - \sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\hat{A}_{ij}A_{(\kappa_p^j+1)d21}\zeta_1(t_{(i+1)j}) \tag{20}$$

and $t_{ij}\in[-\bar{d},0]$. Thus, we get

$$\|\zeta_2(t)\| \le \sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\left\{\|\hat{A}_{ij}\|\right\}\|\phi\|_c + \sum_{i=0}^{k(t)-1}\sum_{\substack{j=0\\ t_{i\nu_j}\notin\Theta}}^{p^{i+1}-1}\|\hat{A}_{ij}\|\,\|A_{(\kappa_p^j+1)d21}\|\,\|\zeta_1(t_{(i+1)j})\|. \tag{21}$$

Now, in order to estimate $\|\zeta_2(t)\|$, the two terms in (21) have to be estimated. For the first term, from (18), $\hat{A}_{ij}$ can be written as

$$\hat{A}_{ij} = \left(\hat{A}_{(i-1)\nu_j}\right)\times\left(-A_{(\kappa_p^j+1)d22}\right) = \left(\hat{A}_{(i-1)\nu_j}\right)\times\left(-A_{k_1d22}\right).$$

Iterating on $\hat{A}_{(i-1)\nu_j}$ gives after $(i-1)$ iterations $\hat{A}_{ij} = A_{k_id22}\ldots A_{k_1d22}$, where $k_1,\ldots,k_i$ are integers between 1 and p. Then, we have

$$\hat{A}_{ij} = A_{k_id22}e^{\alpha\bar{d}_{k_i}}\ldots A_{k_1d22}e^{\alpha\bar{d}_{k_1}}e^{-\alpha\hat{d}_{ij}} \tag{22}$$

with $\hat{d}_{ij} = \sum_{e=1}^{i}\bar{d}_{k_e}$. Note also that since $t_{ij}\in\Theta$, $\hat{d}_{ij}$ is greater than or equal to t. Therefore, using (22), the first term in (21) can be bounded by

$$\sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\left\{\|\hat{A}_{ij}\|\right\}\|\phi\|_c \le \sum_{i=1}^{k(t)}\sum_{t_{ij}\in\Theta}\left\{\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\|\right\}\|\phi\|_c\,e^{-\alpha t}. \tag{23}$$

Now, the summation $\sum_{t_{ij}\in\Theta}$ sums over the leaves in level i (see Fig. 1). This summation is bounded by the summation over all the nodes in level i, which has the worst case sum when all the nodes exist in the level (i.e. p^i nodes). Therefore, $\sum_{t_{ij}\in\Theta}\left\{\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\|\right\}$ can be bounded by

$$\sum_{j=0}^{p^i-1}\left\{\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\|\right\} = \sum_{j=0}^{p-1}\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\| + \cdots + \sum_{j=p^{i-1}}^{p^i-1}\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\|.$$

Note that $\hat{A}_{ij}$'s, in each summation, have the same parent. The parent of $\hat{A}_{ij}$'s in the first summation is $\hat{A}_{(i-1)\nu_0}$, in the second summation is $\hat{A}_{(i-1)\nu_1}$ and so on. Therefore, using (16)–(18), $\sum_{j=0}^{p^i-1}\left\{\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\|\right\}$ equals

$$\sum_{j=0}^{p-1}\|\hat{A}_{(i-1)\nu_0}e^{\alpha\hat{d}_{(i-1)\nu_0}}A_{(\kappa_p^j+1)d22}e^{\alpha\bar{d}_{(\kappa_p^j+1)}}\| + \cdots + \sum_{j=p^{i-1}}^{p^i-1}\|\hat{A}_{(i-1)\nu_{(p^{i-1}-1)}}e^{\alpha\hat{d}_{(i-1)\nu_{(p^{i-1}-1)}}}A_{(\kappa_p^j+1)d22}e^{\alpha\bar{d}_{(\kappa_p^j+1)}}\|$$
$$\le \|\hat{A}_{(i-1)\nu_0}e^{\alpha\hat{d}_{(i-1)\nu_0}}\|\sum_{j=0}^{p-1}A_{(\kappa_p^j+1)d22}e^{\alpha\bar{d}_{(\kappa_p^j+1)}} + \cdots + \|\hat{A}_{(i-1)\nu_{(p^{i-1}-1)}}e^{\alpha\hat{d}_{(i-1)\nu_{(p^{i-1}-1)}}}\|\sum_{j=p^{i-1}}^{p^i-1}A_{(\kappa_p^j+1)d22}e^{\alpha\bar{d}_{(\kappa_p^j+1)}}. \tag{24}$$

From the definition of κ_p^j, all the summations in (24) are equal to $\sum_{k=1}^{p}A_{kd22}e^{\alpha\bar{d}_k}$. Therefore,

$$\sum_{j=0}^{p^i-1}\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\| \le \sum_{j=0}^{p^{i-1}-1}\|\hat{A}_{(i-1)j}e^{\alpha\hat{d}_{(i-1)j}}\|\sum_{k=1}^{p}A_{kd22}e^{\alpha\bar{d}_k}.$$

Note that $\sum_{j=0}^{p^{i-1}-1}\|\hat{A}_{(i-1)j}e^{\alpha\hat{d}_{(i-1)j}}\|$ is the summation over all the nodes in the previous level (i.e. level $i-1$). Following the same procedure with level $(i-1)$ and iterating until level 0 gives

$$\sum_{j=0}^{p^i-1}\|\hat{A}_{ij}e^{\alpha\hat{d}_{ij}}\| \le \left[\sum_{k=1}^{p}A_{kd22}e^{\alpha\bar{d}_k}\right]^i. \tag{25}$$

Therefore, using (11) and (23), the first term in (21) can be bounded by

$$\|\phi\|_c \sum_{i=1}^{k(t)} \left[\sum_{k=1}^{p} \left\{ A_{kd22} e^{\alpha \bar{d}_k} \right\} \right]^i e^{-\alpha t} \leq \frac{\beta}{1-\gamma} \|\phi\|_c e^{-\alpha t}. \tag{26}$$

Now, in order to estimate the second term in (21), define

$$\|\check{A}_1\| = \max\{\|A_{1d21}\|, \ldots, \|A_{pd21}\|\}.$$

Then, from (16)–(18), we get

$$\begin{aligned} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})} &\leq \|\hat{A}_{(i-1)\nu_j} A_{(\kappa_p^j+1)d22}\| e^{-\alpha(t_{i\nu_j})} e^{\alpha d_{(\kappa_p^j+1)}(t_{i\nu_j})} \\ &\leq \|\hat{A}_{(i-1)\nu_j} e^{-\alpha(t_{i\nu_j})} A_{(\kappa_p^j+1)d22} e^{\alpha \bar{d}_{(\kappa_p^j+1)}}\| \\ &\leq \|\hat{A}_{(i-1)\nu_j} e^{-\alpha(t_{i\nu_j})} A_{k_1 d22} e^{\alpha \bar{d}_{k_1}}\|. \end{aligned} \tag{27}$$

Iterating on $\hat{A}_{(i-1)\nu_j}$ and $t_{i\nu_j}$ gives after $(i-1)$ iterations,

$$\begin{aligned} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})} &\leq \|\mathbb{I} e^{-\alpha t_{1k_{i+1}}} A_{k_i d22} e^{\alpha \bar{d}_{k_i}} \ldots A_{k_1 d22} e^{\alpha \bar{d}_{k_1}}\| \\ &\leq \|A_{k_i d22} e^{\alpha \bar{d}_{k_i}} \ldots A_{k_1 d22} e^{\alpha \bar{d}_{k_1}}\| e^{-\alpha t} e^{\alpha \bar{d}} \\ &\leq \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| e^{-\alpha t} e^{\alpha \bar{d}}. \end{aligned} \tag{28}$$

Now, using (15) and noting that $\|A_{(\kappa_p^j+1)d21}\| \leq \|\check{A}_1\|$ for any integer j, the second term in (21) can be estimated as

$$\begin{aligned} &\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| \|A_{(\kappa_p^j+1)d21}\| \|\zeta_1(t_{(i+1)j})\| \\ &\leq \left\|\check{A}_1\right\| \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\|_c \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| e^{-\alpha(t_{(i+1)j})}. \end{aligned}$$

Using (28), this term is bounded by

$$\left\|\check{A}_1\right\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha \bar{d}} \sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| \right\} \|\phi\|_c e^{-\alpha t}. \tag{29}$$

Note that for any i, $\sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\check{A}_e\|^i = m \|\check{A}_e\|^i$, where m equals the number of nodes in level $i+1$ (see Fig. 1). It can be seen easily that the worst case is when all the nodes exist in the level (i.e. p^{i+1} nodes), and we get

$$\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| \right\} \leq \sum_{i=0}^{k(t)-1} \sum_{j=0}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| \right\}.$$

Using (11) and (25), we have

$$\sum_{i=0}^{k(t)-1} \sum_{j=0}^{p^{i+1}-1} \left\{ \|\hat{A}_{ij} e^{e^{\alpha \hat{d}_{ij}}}\| \right\} \leq \frac{\beta}{1-\gamma}. \tag{30}$$

Therefore, using (29) and (30), the second term in (21) can be estimated as

$$\begin{aligned} &\sum_{i=0}^{k(t)-1} \sum_{\substack{j=0 \\ t_{i\nu_j} \notin \Theta}}^{p^{i+1}-1} \|\hat{A}_{ij}\| \|A_{(\kappa_p^j+1)d21}\| \|\zeta_1(t_{(i+1)j})\| \\ &\leq \left\|\check{A}_1\right\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha \bar{d}} \frac{\beta}{1-\gamma} \|\phi\|_c e^{-\alpha t}. \end{aligned} \tag{31}$$

Now, from (26) and (31), $\|\zeta_2(t)\|$ in (21) is estimated by

$$\|\zeta_2(t)\| \leq \left[\frac{\beta}{1-\gamma} + \left\|\check{A}_1\right\| \sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha \bar{d}} \frac{\beta}{1-\gamma} \right] \|\phi\|_c e^{-\alpha t}.$$

Thus, the singular time-delay system is exponentially stable with a minimum decaying rate equals α. Finally, as we have shown that this system is also regular and impulse-free, by Definition 1, we conclude that system (1) is exponentially admissible. □

Remark 5. Strict LMI conditions are more desirable than non-strict ones from numerical point of view. Considering this, Eqs. (3) and (4) can be combined into a single strict LMI. Let $\mathscr{P} > 0$ and $S \in R^{n \times (n-r)}$ be any matrix with full column rank that satisfies $E^\top S = 0$. Changing P to $\mathscr{P}E + SQ$ in (3) yields the strict LMI.

As a special case of our class of systems, We present here the result in the case of single time-varying delay.

Corollary 6. *Let $p = 1$, $0 < \underline{d}_1 < \bar{d}_1$, $0 \leq \mu < 1$ and $\alpha > 0$ be given scalars. System (1) is exponentially admissible with $\sigma = \alpha$ if there exist a nonsingular matrix P, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, and matrices M_i, N_i and S_i, $i = 1, 2$ such that the following LMI holds:*

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & e^{\alpha \underline{d}_1} M_1 E & -e^{\alpha \bar{d}_1} S_1 E & c_2 N_1 & c S_1 & c M_1 & \Pi_{18} \\ \star & \Pi_{22} & e^{\alpha \underline{d}_1} M_2 E & -e^{\alpha \bar{d}_1} S_2 E & c_2 N_2 & c S_2 & c M_2 & A_1^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -c_2 Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -c Z_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0$$

with the following constraint: $E^\top P = P^\top E \geq 0$, where

$$\Pi_{11} = P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top + 2\alpha E^\top P$$

$$\Pi_{12} = P^\top A_1 + (N_2 E)^\top - N_1 E + S_1 E - M_1 E$$

$$\begin{aligned} \Pi_{22} &= -(1-\mu) e^{-2\alpha \bar{d}_1} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top \\ &\quad - M_2 E - (M_2 E)^\top, \ \underline{d} = \bar{d}_1 - \underline{d}_1, \ U = \bar{d}_1 Z_1 + \underline{d} Z_2 \end{aligned}$$

$$\Pi_{18} = A^\top U, \qquad c = \frac{e^{2\alpha \bar{d}_1} - e^{2\alpha \underline{d}_1}}{2\alpha} \qquad c_2 = \frac{e^{2\alpha \bar{d}_1} - 1}{2\alpha}.$$

Remark 7. If $E = \mathbb{I}$, the result of Corollary 6 with $P > 0$ as $\alpha \to 0^+$ is equivalent to Theorem 1 in He et al. (2007).

Remark 8. If $\underline{d}_1 = 0$, $\mu = 0$, $Q_1 = \epsilon_1 I$, $Q_2 = \epsilon_2 I$, $Z_2 = \epsilon_3 I$, with $\epsilon_i > 0$, $i = 1, 2, 3$, being sufficiently small scalars, $M_1 = M_2 = S_1 = S_2 = 0$, the result of Corollary 6 as $\alpha \to 0^+$ is equivalent to Theorem 1 in Zhu et al. (2007).

4. Example

Consider the following singular time-delay system:

$$E = \begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}, \qquad A = \begin{bmatrix} -4.7 & 0.4 \\ -4.9 & 0.8 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.7 & -0.95 \\ 1.1 & -1.75 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & -0.8 \\ 1.4 & -1.3 \end{bmatrix}.$$

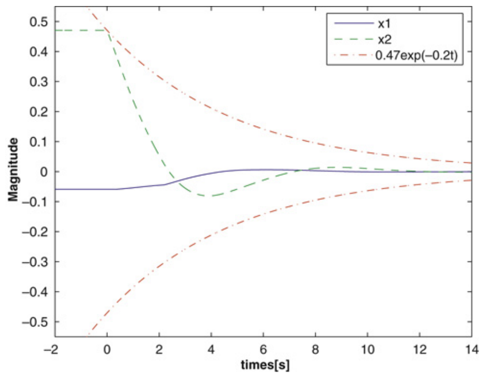

Fig. 2. Solution behavior of x_1 and x_2.

Table 1
Maximum allowable $\overline{d}_2$ for different α.

α	0.01	0.1	0.15	0.2	0.25	0.3	0.35
$\overline{d}_2$	5.73	3.30	2.79	2.43	1.84	1.01	0.32

Simulation results show that this system is unstable for large delays. Now, let $\underline{d}_1 = 0.1$, $\overline{d}_1 = 0.5$, $\underline{d}_2 = 0.2$, and $\mu = 0.3$. For various α, the maximum allowable $\overline{d}_2$, for which the system is exponentially stable are listed in Table 1. Note that as α increases, the maximum allowable $\overline{d}_2$ decreases. Fig. 2 gives the simulation results of x_1 and x_2 when $d_1(t) = 0.3 + 0.1\sin(2t)$, $d_2(t) = 2 + 0.25\sin(t)$ and the initial function is $\phi(t) = [\ -0.059 \quad 0.47\ ]^\top$, $t \in [-2, 0]$. As it is expected from Table 1, $\|x(t)\|$ is bounded by $\gamma e^{-0.2t}$.

5. Conclusion

This paper dealt with the stability of the class of singular systems with multiple delays. A delay-range-dependent exponential stability conditions have been developed for singular systems with time-delays. An estimate of the convergence rate of such stable systems has been presented. The conditions are expressed as feasible LMI conditions.

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