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Relative controllability of delay differential systems with impulses and linear parts defined by permutable matrices

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This paper investigates the relative controllability of delay differential systems with linear impulses and linear parts defined by permutable matrices. We use the impulsive delay Grammian matrix to discuss the relatively controllability of impulsive linear delay controlled systems and we use the Krasnoselskii's fixed point theorem to discuss the relatively controllability of impulsive semilinear delay controlled systems. Finally, two examples are presented to illustrate our theoretical results.

KEYWORDS

impulsive delay differential equations, impulsive delay Grammian matrix, relative controllability

1 | INTRODUCTION

Mathematical control theory is an important area of research and classical differential controlled systems in finite and infinite dimensional spaces have been discussed in the literature. However, very little is known for delay differential controlled systems and impulsive delay controlled systems.

Khusainov and Shuklin[1] and Diblík and Khusainov[2] introduced delayed exponential matrix functions and these were used to discuss formulas for solutions to linear delay systems with permutable matrices, stability, inequality, and control problems (see[3-23] and the references therein). Motivated by,[1,2] You and Wang[24,25] extended the single and multiple delayed exponential matrix function in[1] to the impulsive case and used it to discuss the representation and stability of solutions to linear delay systems with linear impulses.

In this paper, we consider the relative controllability of the following impulsive semilinear delay differential systems

$$\begin{cases} v'(t) = Av(t) + Bv(t-\vartheta) + f(t, v(t)) + Du(t),\ t \in \Omega,\ t \notin \mathscr{T},\ \vartheta > 0, \\ \Delta v(t_i) := v(t_i^+) - v(t_i^-) = C_i v(t_i),\ t_i \in \mathscr{T} = \{t_1, t_2, \cdots, t_k\}, \\ v(t) = \psi(t), -\vartheta \le t \le 0, \end{cases} \tag{1}$$

where $A, B, C_i, D \in \mathbb{R}^{n\times n}$, $AB = BA$, $AC_i = C_iA$, $BC_i = C_iB$ for each $i \in \{1, 2, \cdots, k\}$, $v \in C^1([-\vartheta, 0], \mathbb{R}^n) \cup \left[PC(\Omega, \mathbb{R}^n) \cap (\cup_{i=0}^k C^1(t_i, t_{i+1}])\right]$ (here $t_0 = 0$ and $t_{k+1} = \tau_1$), $\psi \in C_\vartheta^1 := C^1([-\vartheta, 0], \mathbb{R}^n)$, $f \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, $\Omega := [0, \tau_1], \tau_1 > 0, 0 < t_1 < t_2 < \cdots < t_k < \tau_1$, and the control function $u(\cdot)$ takes values from $L^2(\Omega, \mathbb{R}^n)$. Set $v(t_i^+) = \lim_{\epsilon\to 0^+} v(t_i + \epsilon)$ and $v(t_i^-) = v(t_i)$ represent respectively the right and left limits of $v(t)$ at $t = t_i$.

First, we investigate the relative controllability of the linear impulsive delay controlled system as follows:

$$\begin{cases} v'(t) = Av(t) + Bv(t-\vartheta) + Du(t),\ t \in \Omega,\ t \notin \mathscr{T},\ \vartheta > 0, \\ \Delta v(t_i) = C_i v(t_i),\ t_i \in \mathscr{T}, \\ v(t) = \psi(t), -\vartheta \le t \le 0, \end{cases} \tag{2}$$

using the impulsive delay Cauchy matrix introduced in.[24] Next, we construct a suitable control function for (1), which means that we give a condition necessary and sufficient for $u \in L^2(\Omega, \mathbb{R}^n)$ to lead the solution of (1) to v_1 at the time τ_1. We apply the Krasnoselskii's fixed point theorem to show that (1) is also relatively controllable under suitable conditions.

The rest of this paper is organized as follows. In Section 2, we give some notations, concepts, and important lemmas. In Section 3, we establish relative controllability results for (1) and (2) respectively. Examples are given to illustrate our main results in the final section.

2 | PRELIMINARIES

Let $\mathbb{R}^n$ be the n-dimensional Euclidian space with the vector norm $\|\cdot\|$ and $\mathbb{R}^{n\times n}$ be the $n \times n$ matrix space with real valued elements. For $v \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$, we introduce the vector infinite-norm $\|v\| = \max_{1\le i\le n} |v_i|$ and the matrix infinite-norm $\|A\| = \max_{1\le i\le n} \sum_{j=1}^n |a_{ij}|$ respectively, where v_i and a_{ij} are the elements of the vector v and matrix A. Denote by $C(\Omega, \mathbb{R}^n)$ the Banach space of vector-valued bounded continuous functions from $\Omega \to \mathbb{R}^n$ and let $C^1(\Omega, \mathbb{R}^n) = \{v \in C(\Omega, \mathbb{R}^n) : v' \in C(\Omega, \mathbb{R}^n)\}$. Let $PC(\Omega, \mathbb{R}^n) := \{v : \Omega \to \mathbb{R}^n : v \in C((t_i, t_{i+1}], \mathbb{R}^n),\ i = 0, 1, \cdots, k,$ and there exists $v(t_i^+), v(t_i^-)$ with $v(t_i^-) = v(t_i),\ i = 1, \cdots, k\}$. Denote by $\mathcal{PC}$, the space $PC(\Omega, \mathbb{R}^n)$ of vector-valued piece-wise continuous functions from $\Omega \to \mathbb{R}^n$ endowed with the norm $\|v\|_{PC} = \sup_{t\in\Omega} \|v(t)\|$ where $\|\cdot\|$ is the norm on $\mathbb{R}^n$. In addition, $\|\psi\|_{PC} = \sup_{t\in[-\vartheta,0]} \|\psi(t)\|$. Let Y_1, Y_2 be two Banach spaces, and let $L_b(Y_1, Y_2)$ denote the space of all bounded linear operators from Y_1 to Y_2. Now $L^p(\Omega, Y_2)$ denotes the Banach space of functions $y : \Omega \to Y_2$ which are Bochner integrable with norm $\|y\|_{L^p(\Omega,Y_2)}$ (here $1 < p < \infty$).

We recall the delayed matrix exponential $e_\vartheta^{\Lambda\cdot} : \mathbb{R} \to \mathbb{R}^{n\times n}$ given by as follows:

$$e_\vartheta^{\Lambda t} = \begin{cases} \Theta, \quad t < -\vartheta,\ \vartheta > 0, \\ E, -\vartheta \le t < 0, \\ E + \Lambda t + \Lambda^2 \frac{(t-\vartheta)^2}{2} + \cdots + \Lambda^z \frac{(t-(z-1)\vartheta)^z}{z!},\ (z-1)\vartheta \le t < z\vartheta,\ z = 1, 2, \cdots, \end{cases} \tag{3}$$

where $\Lambda \in \mathbb{R}^{n\times n}$, and Θ and E are the zero and identity matrices, respectively. From,[1, Lemma 4]

$$\frac{d}{dt} e_\vartheta^{\Lambda t} = \Lambda e_\vartheta^{\Lambda(t-\vartheta)}, \tag{4}$$

for all $t \in \mathbb{R}$.

Let $V(\cdot,\cdot) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ (see[24]) be given by

$$V(t, s) = e^{A(t-s)} X(t, s + \vartheta),\ t > s, \tag{5}$$

and

$$X(t,s) = e_\vartheta^{\check{B}(t-s)} + \sum_{s-\vartheta<t_j\le t} C_j e_\vartheta^{\check{B}(t-\vartheta-t_j)} X(t_j,s),\ \check{B} = e^{-A\vartheta}B. \tag{6}$$

For any solution $v \in C^1([-\vartheta,0],\mathbb{R}^n)\cup\left[PC(\Omega,\mathbb{R}^n)\cap(\cup_{i=0}^k C^1(t_i,t_{i+1}])\right]$ (here $t_0 = 0$ and $t_{k+1} = \tau_1$) of (1), from,[24, Corollary 3.3] we obtain the following:

$$\begin{aligned} v(t) = {} & V(t,-\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^0 V(t,s)[\psi'(s) - A\psi(s)]ds \\ & + \int_0^t V(t,s)[f(s,v(s)) + Du(s)]ds. \end{aligned} \tag{7}$$

The following results are important and we keep the details of the proof for the reader.

Lemma 2.1. *For all $t \ge 0$, we have $\|e_\vartheta^{\Lambda t}\| \le e^{\|\Lambda\|t}$.*

Proof. From (3), without loss of generality, for $(z-1)\vartheta \le t < z\vartheta, z = 1,2,\cdots$, we have the following

$$\begin{aligned} \|e_\vartheta^{\Lambda t}\| &= \left\| \sum_{n=0}^{z} \Lambda^n \frac{(t-(n-1)\vartheta)^n}{n!} \right\| \\ &\le \sum_{n=0}^{z} \|\Lambda\|^n \frac{t^n}{n!} \\ &\le \sum_{n=0}^{\infty} \|\Lambda\|^n \frac{t^n}{n!} = e^{\|\Lambda\|t}. \end{aligned}$$

The result is proved. □

Lemma 2.2. *For any $t > s$, we have the following:*

$$\|X(t,s)\| \le \left(\prod_{s-\vartheta<t_j\le t} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{\|\check{B}\|(t-s)}, \tag{8}$$

and

$$\|V(t,s)\| \le \left(\prod_{s<t_j\le t} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{-\|\check{B}\|\vartheta} e^{a(t-s)}, \tag{9}$$

where $a = \|A\| + \|\check{B}\|$.

Proof. Without loss of generality, we suppose that $t_m \le s - \vartheta < t_{m+1}$ and $t_{m+n} \le t < t_{m+n+1}, m, n = 0,1,2,\cdots$. To complete our proof we use mathematical introduction.

a. For $n = 0$, from Lemma 2.1 via (6),

$$\|X(t,s)\| \le \|e_\vartheta^{\check{B}(t-s)}\| \le e^{\|\check{B}\|(t-s)}.$$

b. For $n = 1$, from Lemmas 2.1 via (6), we have

$$\begin{aligned} \|X(t,s)\| &\le \|e_\vartheta^{\check{B}(t-s)} + C_{m+1} e_\vartheta^{\check{B}(t-\vartheta-t_{m+1})} X(t_{m+1},s)\| \\ &\le e^{\|\check{B}\|(t-s)} + \|C_{m+1}\| e^{\|\check{B}\|(t-\vartheta-t_{m+1})} e^{\|\check{B}\|(t_{m+1}-s)} \\ &= (1 + \|C_{m+1}\| e^{-\|\check{B}\|\vartheta}) e^{\|\check{B}\|(t-s)}. \end{aligned}$$

c. For $n = z$, suppose that

$$\|X(t,s)\| \le \left(\prod_{j=m+1}^{m+z} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{\|\check{B}\|(t-s)} = \left(\prod_{s-\vartheta<t_j\le t} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{\|\check{B}\|(t-s)}.$$

For $n = z + 1$, from Lemmas 2.1 via (6) again,

$$
\begin{aligned}
\|X(t,s)\| \le& \left(\prod_{j=m+1}^{m+z}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{\|\check{B}\|(t-s)} \\
&+\|C_{m+z+1}\|e^{\|\check{B}\|(t-\vartheta-t_{m+z+1})}\left(\prod_{j=m+1}^{m+z}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{\|\check{B}\|(t_{m+z+1}-s)} \\
=& \left(\prod_{j=m+1}^{m+z+1}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{\|\check{B}\|(t-s)} \\
=& \left(\prod_{s-\vartheta<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{\|\check{B}\|(t-s)}.
\end{aligned}
$$

From the mathematical induction principle, we obtain (8).

Finally, using (5) via (8) and $\|e^{At}\| \le e^{\|A\|t}$, one obtains (9) immediately. □

Definition 2.3. (see[26, Definition 4]). System (1) is called relatively controllable, if for an arbitrary initial vector function $\psi \in C^1([-\vartheta,0],\mathbb{R}^n)$, final state of the vector $\nu_1 \in \mathbb{R}^n$ at time τ_1, there exists a control $u \in L^2(\Omega,\mathbb{R}^n)$ such that the system (1) has a solution $\nu \in C^1([-\vartheta,0],\mathbb{R}^n)\cup\left[PC(\Omega,\mathbb{R}^n)\cap(\cup_{i=0}^{k}C^1(t_i,t_{i+1}])\right]$ (here $t_0 = 0$ and $t_{k+1} = \tau_1$) that satisfies $\nu(\tau_1) = \nu_1$.

Lemma 2.4. (Krasnoselskii's fixed point theorem, see[27]). *Let $\mathcal{B}$ be a bounded closed and convex subset of Banach space X and let F_1, F_2 be maps of $\mathcal{B}$ into X such that $F_1x + F_2y \in \mathcal{B}$ for every $x, y \in \mathcal{B}$. If F_1 is a contraction and F_2 is compact and continuous, then the equation $F_1x + F_2x = x$ has a solution on $\mathcal{B}$.*

Theorem 2.5. (PC-type Ascoli–Arzela theorem, see[28, Theorem 2.1]). *Let $\mathcal{Q} \subset PC(\Omega,X)$ where X is a Banach space. Then $\mathcal{Q}$ is a relatively compact subset of $PC(\Omega,X)$ if, (a) $\mathcal{Q}$ is uniformly bounded subset of $PC(\Omega,X)$; (b) $\mathcal{Q}$ is equicontinuous in (t_i,t_{i+1}), $i = 0,1,2,\cdots,k$ (here $t_0 = 0$ and $t_{k+1} = \tau_1$); and (c) $\mathcal{Q}(t) = \{\nu(t)|\nu\in\mathcal{Q}, t\in\Omega\setminus\mathcal{T}\}$, $\mathcal{Q}(t_i^+) = \{\nu(t_i^+)|\nu\in\mathcal{Q}\}$ and $\mathcal{Q}(t_i^-) = \{\nu(t_i^-)|\nu\in\mathcal{Q}\}$ are relatively compact subsets of X.*

3 | RELATIVE CONTROLLABILITY

3.1 | Linear systems

We consider an impulsive delay Grammian matrix, an extension of the classical Grammian matrix for linear differential systems, as follows:

$$W_\vartheta[0,\tau_1] = \int_0^{\tau_1} V(\tau_1,s)DD^TV^T(\tau_1,s)ds. \tag{10}$$

Theorem 3.1. *System (2) is relatively controllable, if and only if $W_\vartheta[0,\tau_1]$ is nonsingular.*

Proof. First, we establish sufficiency. Since $W_\vartheta[0,\tau_1]$ is nonsingular, its inverse $W_\vartheta[0,\tau_1]$ is well-defined. For any final state $\nu_1 \in \mathbb{R}^n$ one can select a control function as follows:

$$u(t) = D^TV^T(\tau_1,t)W_\vartheta^{-1}[0,\tau_1]\eta,$$

where

$$\eta = \nu_1 - V(\tau_1,-\vartheta)\psi(-\vartheta) - \int_{-\vartheta}^{0} V(\tau_1,s)[\psi'(s) - A\psi(s)]ds.$$

Then

$$
\begin{aligned}
\nu(\tau_1) &= V(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} V(\tau_1, s)[\psi'(s) - A\psi(s)]ds \\
&\quad + \int_0^{\tau_1} V(\tau_1, s)Du(s)ds \\
&= V(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} V(\tau_1, s)[\psi'(s) - A\psi(s)]ds \\
&\quad + \int_0^{\tau_1} V(\tau_1, s)DD^T V^T(\tau_1, s)W_\vartheta^{-1}[0, \tau_1]\eta ds \\
&= \nu_1.
\end{aligned}
$$

We argue by contradiction to prove our necessity result. Assume $W_\vartheta[0, \tau_1]$ is singular, ie, there exists at least one nonzero state $\tilde{\nu} \in \mathbb{R}^n$ such that,

$$
\tilde{\nu}^T W_\vartheta[0, \tau_1]\tilde{\nu} = 0.
$$

One obtains the following

$$
\begin{aligned}
0 &= \tilde{\nu}^T W_\vartheta[0, \tau_1]\tilde{\nu} \\
&= \int_0^{\tau_1} \tilde{\nu}^T V(\tau_1, s)DD^T V^T(\tau_1, s)\tilde{\nu} ds \\
&= \int_0^{\tau_1} \|\tilde{\nu}^T V(\tau_1, s)D\|^2 ds,
\end{aligned}
$$

which implies that

$$
\tilde{\nu}^T V(\tau_1, s)D = \underbrace{(0, \cdots, 0)}_{n} := \mathbf{0}^T, \quad \forall s \in \Omega.
$$

Since system (2) is relatively controllable, according to Definition 2.3, there exists a control $u_1(t)$ that drives the initial state to zero at τ_1 ie,

$$
\nu(\tau_1) = V(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} V(\tau_1, s)[\psi'(s) - A\psi(s)]ds + \int_0^{\tau_1} V(\tau_1, s)Du_1(s)ds = \mathbf{0}. \tag{11}
$$

Similarly, there also exists a control $u_2(t)$ that drives the initial state to the (nonzero) state $\tilde{\nu}$ at τ_1, ie,

$$
\begin{aligned}
\nu(\tau_1) &= V(\tau_1, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0} V(\tau_1, s)[\psi'(s) - A\psi(s)]ds \\
&\quad + \int_0^{\tau_1} V(\tau_1, s)Du_2(s)ds \\
&= \tilde{\nu}.
\end{aligned} \tag{12}
$$

Then from (11) and (12), we have

$$
\tilde{\nu} = \int_0^{\tau_1} V(\tau_1, s)D[u_2(s) - u_1(s)]ds. \tag{13}
$$

Multiplying both sides of (13) by $\tilde{\nu}^T$, we obtain

$$
\tilde{\nu}^T \tilde{\nu} = \int_0^{\tau_1} \tilde{\nu}^T V(\tau_1, s)D[u_2(s) - u_1(s)]ds = 0.
$$

Thus $\tilde{\nu} = \mathbf{0}$, which conflicts with $\tilde{\nu}$ being nonzero. Thus, the impulsive delay Grammian matrix $W_\vartheta[0, \tau_1]$ is nonsingular. The proof is finished. □

3.2 | Semilinear systems

We assume the following:

(H_1): The operator $W : L^2(\Omega, \mathbb{R}^n) \to \mathbb{R}^n$ defined by

$$
Wu = \int_0^{\tau_1} V(\tau_1, s)Du(s)ds,
$$

has an inverse operator W^{-1} which takes values in $L^2(\Omega, \mathbb{R}^n)/kerW$. Then we set

$$M = \|W^{-1}\|_{L_b(\mathbb{R}^n, L^2(\Omega,\mathbb{R}^n)/kerW)}.$$

From,[29, Remark 3.3] we know

$$M = \sqrt{\|W_\vartheta[0, \tau_1]^{-1}\|}. \tag{14}$$

(H_2): The function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and there exists a constant $q > 1$ and $L_f(\cdot) \in L^q(\Omega, \mathbb{R}^+)$ such that

$$\|f(t, \nu) - f(t, \mu)\| \le L_f(t)\|\nu - \mu\|, \quad \nu, \mu \in \mathbb{R}^n.$$

Theorem 3.2. *Suppose that* (H_1) *and* (H_2) *are satisfied. Then system (1) is relatively controllable provided that*

$$c\left[1 + \frac{d\|D\|M}{a}(e^{a\tau_1} - 1)\right] < 1, \tag{15}$$

where $d = \left(\prod_{j=1}^k (1 + \|C_j\|e^{-\|\check{B}\|\vartheta})\right) e^{-\|\check{B}\|\vartheta}$ *and* $c = \left(\prod_{j=1}^k (1 + \|C_j\|e^{-\|\check{B}\|\vartheta})\right) e^{-\|\check{B}\|\vartheta} \left[\frac{1}{ap}(e^{ap\tau_1} - 1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega,\mathbb{R}^+)}, \frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1.$

Proof. Let $\tilde{\nu} \in \mathbb{R}^n$ be the final state. Using hypothesis (H_1) for arbitrary $\nu(\cdot) \in \mathcal{PC}$, we define the control function $u_\nu(t)$ by

$$\begin{aligned} u_\nu(t) = W^{-1}(\nu_1 - V(\tau_1, -\vartheta)\psi(-\vartheta) - \int_{-\vartheta}^0 V(\tau_1, s)[\psi'(s) - A\psi(s)]ds \\ - \int_0^{\tau_1} V(\tau_1, s)f(s, \nu(s))ds)(t), \ t \in \Omega. \end{aligned} \tag{16}$$

We show that, using this control, the operator $\mathcal{P} : \mathcal{PC} \to \mathcal{PC}$ defined by

$$\begin{aligned} (\mathcal{F}\nu)(t) = V(t, -\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^0 V(t, s)[\psi'(s) - A\psi(s)]ds \\ + \int_0^t V(t, s)f(s, \nu(s))ds + \int_0^t V(t, s)Du_\nu(s)ds, \end{aligned}$$

has a fixed point ν, which is a mild solution of (1).

We check that $(\mathcal{F}\nu)(\tau_1) = \nu_1$, which means that u_ν steers the system (1) from $(\mathcal{F}\nu)(0)$ to ν_1 in finite time τ_1. This implies system (1) is relatively controllable on Ω.

For each positive number r, let $\mathcal{B}_r = \{\nu \in \mathcal{PC} : \|\nu\|_{PC} \le r\}$ (a bounded, closed and convex set of $\mathcal{PC}$). Set $N = \sup_{t\in\Omega} \|f(t, 0)\|$.

We divide the proof into several steps.

Step 1. We claim that there exists a positive number r such that $\mathcal{F}(\mathcal{B}_r) \subseteq \mathcal{B}_r$.

From (H_2) and Hölder's inequality, we obtain that

$$\begin{aligned} \int_0^t e^{a(t-s)}L_f(s)ds &\le \left(\int_0^t e^{ap(t-s)}ds\right)^{\frac{1}{p}} \left(\int_0^t L_f^q(s)ds\right)^{\frac{1}{q}} \\ &\le \left[\frac{1}{ap}(e^{apt} - 1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega,\mathbb{R}^+)}, \end{aligned}$$

and

$$\int_0^t e^{a(t-s)}\|f(s, 0)\|ds \le N \int_0^t e^{a(t-s)}ds = \frac{N}{a}(e^{at} - 1).$$

From (16), using (H_1) and (H_2), we have the following

$$
\begin{aligned}
\|u_v(t)\| \leq{}& \|W^{-1}\|_{L_b(\mathbb{R}^n, L^2(\Omega,\mathbb{R}^n)/kerW)}(\|v_1\| + \|V(\tau_1,-\vartheta)\|\|\psi(-\vartheta)\| \\
&+ \int_{-\vartheta}^{0} \|V(\tau_1,s)\|\|\psi'(s) - A\psi(s)\|ds \\
&+ \int_{0}^{\tau_1} \|V(\tau_1,s)\|\|f(s,v(s))\|ds) \\
\leq{}& M\left(\|v_1\| + \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(\tau_1+\vartheta)}\|\psi(-\vartheta)\|\right. \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{-\vartheta}^{0} e^{a(\tau_1-s)}\|\psi'(s) - A\psi(s)\|ds \\
&+ \left.\left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{0}^{\tau_1} e^{a(\tau_1-s)}(\|f(s,v(s)) - f(s,0)\| + \|f(s,0)\|)ds\right) \\
\leq{}& M(\|v_1\| + \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(\tau_1+\vartheta)}\|\psi(-\vartheta)\| \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{-\vartheta}^{0} e^{a(\tau_1-s)}\|\psi'(s) - A\psi(s)\|ds \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{0}^{\tau_1} e^{a(\tau_1-s)}(L_f(s)\|v(s)\| + \|f(s,0)\|)ds) \\
\leq{}& M\left(\|v_1\| + (\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(\tau_1+\vartheta)}\|\psi(-\vartheta)\| \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{-\vartheta}^{0} e^{a(\tau_1-s)}\|\psi'(s) - A\psi(s)\|ds \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\left[\frac{1}{ap}(e^{ap\tau_1} - 1)\right]^{\frac{1}{p}}\|L_f\|_{L^q(\Omega,\mathbb{R}^+)}\|v\|_{PC} \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\frac{N}{a}(e^{a\tau_1} - 1)) \\
\leq{}& M\|v_1\| + Mb + Mc\|v\|_{PC},
\end{aligned}
$$

where

$$
\begin{aligned}
b ={}& \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(\tau_1+\vartheta)}\|\psi(-\vartheta)\| \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\int_{-\vartheta}^{0} e^{a(\tau_1-s)}\|\psi'(s) - A\psi(s)\|ds \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\frac{N}{a}(e^{a\tau_1} - 1).
\end{aligned}
$$

From (H_1) and (H_2) we have the following

$$\begin{aligned}
\|(\mathcal{F}v)(t)\| &\le \|V(t,-\vartheta)\|\|\psi(-\vartheta)\| + \int_{-\vartheta}^{0}\|V(t,s)\|\|\psi'(s)-A\psi(s)\|ds \\
&+ \int_0^t \|V(t,s)\|\|f(s,v(s))\|ds + \int_0^t \|V(t,s)\|\|D\|\|u_v(s)\|ds \\
&\le \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t+\vartheta)}\|\psi(-\vartheta)\| \\
&+ \int_{-\vartheta}^{0}\left(\prod_{s<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t-s)}\|\psi'(s)-A\psi(s)\|ds \\
&+ \int_0^t\left(\prod_{s<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t-s)}(L_f(s)\|v(s)\|+\|f(s,0)\|)ds \\
&+ \int_0^t\left(\prod_{s<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t-s)}\|D\|[M\|v_1\|+Mb+Mc\|v\|_{PC}]ds \\
&\le b + c\|v\|_{PC} + \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\|D\| \\
&\times [M\|v_1\|+Mb+Mc\|v\|_{PC}]\int_0^t e^{a(t-s)}ds \\
&\le b\left[1+\left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\frac{\|D\|M}{a}(e^{at}-1)\right] \\
&+ \left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\frac{\|D\|M}{a}(e^{at}-1)\|v_1\| \\
&+ c\left[1+\left(\prod_{j=1}^{k}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}\frac{\|D\|M}{a}(e^{at}-1)\right]\|v\|_{PC} \\
&\le b\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right] + \frac{d\|D\|M}{a}(e^{a\tau_1}-1)\|v_1\| \\
&+ c\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right]r = r,
\end{aligned}$$

where

$$r = \frac{b\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right]+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\|v_1\|}{1-c\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right]}.$$

Hence, we obtain $\mathcal{F}(\mathcal{B}_r) \subseteq \mathcal{B}_r$ for such an r.

Now, we define operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{B}_r$ as follows

$$(\mathcal{F}_1 v)(t) = V(t,-\vartheta)\psi(-\vartheta) + \int_{-\vartheta}^{0}V(t,s)[\psi'(s)-A\psi(s)]ds + \int_0^t V(t,s)Du_v(s)ds,$$

and

$$(\mathcal{F}_2 v)(t) = \int_0^t V(t,s)f(s,v(s))ds,$$

for $t \in \Omega$, respectively.

Step 2. We claim that $\mathcal{F}_1$ is a contraction mapping.

Let $\nu, \gamma \in \mathcal{B}_r$. From (H_1) and (H_2), for each $\in \Omega$, we have the following

$$
\begin{aligned}
&\|u_\nu(t) - u_\gamma(t)\| \leq M \int_0^{\tau_1} \|V(\tau_1, s)\| \|f(s, \nu(s)) - f(s, \gamma(s))\| ds \\
&\leq M \int_0^{\tau_1} \left(\prod_{s<t_j \leq \tau_1} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{-\|\check{B}\|\vartheta} e^{a(\tau_1 - s)} L_f(s)(\|\nu(s) - \gamma(s)\|) ds \\
&\leq M \left(\prod_{j=1}^{k} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{-\|\check{B}\|\vartheta} \left[\frac{1}{ap} (e^{ap\tau_1} - 1) \right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega, \mathbb{R}^+)} \|\nu - \gamma\|_{PC} \\
&\leq Mc\|\nu - \gamma\|_{PC}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|(\mathcal{F}_1\nu)(t) - (\mathcal{F}_1\gamma)(t)\| \\
&\leq \int_0^t \|V(t, s)\| \|D\| \|u_\nu(s) - u_\gamma(s)\| ds \\
&\leq \int_0^t \left(\prod_{s<t_j \leq t} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{-\|\check{B}\|\vartheta} e^{a(t-s)} ds \|D\| Mc \|\nu - \gamma\|_{PC} \\
&\leq \frac{d\|D\|Mc}{a} (e^{a\tau_1} - 1) \|\nu - \gamma\|_{PC},
\end{aligned}
$$

so we obtain

$$
\|\mathcal{F}_1\nu - \mathcal{F}_1\gamma\|_{PC} \leq P\|\nu - \gamma\|_{PC},
$$

where $P = \frac{d\|D\|Mc}{a}(e^{a\tau_1} - 1)$. From (15), we have $P < 1$, so $\mathcal{F}_1$ is a contraction.

Step 3. We claim that $\mathcal{F}_2 : \mathcal{B}_r \to \mathcal{PC}$ is a compact and continuous operator.

Let $\nu_n \in \mathcal{B}_r$ with $\nu_n \to \nu$ in $\mathcal{B}_r$. Using (H_2), we have $f(s, \nu_n(s)) \to f(s, \nu(s))$ in $\mathcal{PC}$ and thus using the Lebesgue dominated convergence theorem we have

$$
\begin{aligned}
&\|(\mathcal{F}_2\nu_n)(t) - (\mathcal{F}_2\nu)(t)\| \\
&\leq \int_0^t \|V(t, s)\| \|f(s, \nu_n(s)) - f(s, \nu(s))\| ds \\
&\leq \left(\prod_{j=1}^{k} (1 + \|C_j\| e^{-\|\check{B}\|\vartheta}) \right) e^{-\|\check{B}\|\vartheta} \int_0^t e^{a(t-s)} \|f(s, \nu_n(s)) - f(s, \nu(s))\| ds \\
&\to 0 \text{ as } n \to 0,
\end{aligned}
$$

which implies that $\mathcal{F}_2$ is continuous on $\mathcal{B}_r$.

To check the compactness of $\mathcal{F}_2 : \mathcal{B}_r \to \mathcal{PC}$, we prove that $\mathcal{F}_2(\mathcal{B}_r)$ is equicontinuous and uniformly bounded. In fact, for any $\nu \in \mathcal{B}_r, t_z < t \leq t + h \leq t_{z+1}, z = 0, 1, \cdots, k$,

$$
\begin{aligned}
&(\mathcal{F}_2\nu)(t + h) - (\mathcal{F}_2\nu)(t) \\
&= \int_0^{t+h} V(t + h, s) f(s, \nu(s)) ds - \int_0^t V(t, s) f(s, \nu(s)) ds \\
&= \int_t^{t+h} V(t + h, s) f(s, \nu(s)) ds \\
&\quad + \int_0^t e^{A(t+h-s)} (X(t + h, s + \vartheta) - X(t, s + \vartheta)) f(s, \nu(s)) ds \\
&\quad + \int_0^t (e^{A(t+h-s)} - e^{A(t-s)}) X(t, s + \vartheta) f(s, \nu(s)) ds.
\end{aligned}
$$

Let

$$S_1 = \int_t^{t+h} V(t+h,s)f(s,v(s))ds,$$
$$S_2 = \int_0^t e^{A(t+h-s)}(X(t+h,s+\vartheta) - X(t,s+\vartheta))f(s,v(s))ds,$$
$$S_3 = \int_0^t (e^{A(t+h-s)} - e^{A(t-s)})X(t,s+\vartheta)f(s,v(s))ds$$
$$= \int_0^t (e^{Ah} - E)V(t,s)f(s,v(s))ds.$$

From above, we see that

$$\|(\mathcal{F}_2 v)(t+h) - (\mathcal{F}_2 v)(t)\| \le \|S_1\| + \|S_2\| + \|S_3\|.$$

Now, we only need to check $\|S_i\| \to 0$ as $h \to 0$, $i = 1,2,3$. Clearly,

$$\begin{aligned}
\|S_1\| &\le \int_t^{t+h} \|V(t+h,s)\|\|f(s,v(s))\|ds \\
&\le \int_t^{t+h} \left(\prod_{s<t_j\le t+h}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right) e^{-\|\check{B}\|\vartheta} e^{a(t+h-s)}(L_f(s)\|v(s)\| + \|f(s,0)\|)ds \\
&\le e^{-\|\check{B}\|\vartheta}\left[\frac{1}{ap}(e^{aph}-1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega,\mathbb{R}^+)}\|v\|_{PC} + e^{-\|\check{B}\|\vartheta}\frac{N}{a}(e^{ah}-1) \\
&\to 0 \text{ as } h \to 0,
\end{aligned}$$

$$\begin{aligned}
\|S_2\| &\le \int_0^t \|e^{A(t+h-s)}\|\|X(t+h,s+\vartheta) - X(t,s+\vartheta)\|\|f(s,v(s))\|ds \\
&\le e^{\|A\|\tau_1}\int_0^t \Bigg\| e_\vartheta^{\check{B}(t+h-\vartheta-s)} - e_\vartheta^{\check{B}(t-\vartheta-s)} + \sum_{s<t_j\le t+h} C_j e_\vartheta^{\check{B}(t+h-\vartheta-t_j)}X(t_j,s+\vartheta) \\
&\quad - \sum_{s<t_j\le t} C_j e_\vartheta^{\check{B}(t-\vartheta-t_j)}X(t_j,s+\vartheta)\Bigg\| (L_f(s)\|v(s)\| + \|f(s,0)\|)ds \\
&\le e^{\|A\|\tau_1}\int_0^t \Bigg\| e_\vartheta^{\check{B}(t+h-\vartheta-s)} - e_\vartheta^{\check{B}(t-\vartheta-s)} \\
&\quad + \sum_{s<t_j\le t} C_j (e_\vartheta^{\check{B}(t+h-\vartheta-t_j)} - e_\vartheta^{\check{B}(t-\vartheta-t_j)})X(t_j,s+\vartheta)\Bigg\| (L_f(s)\|v(s)\| + \|f(s,0)\|)ds \\
&\le e^{\|A\|\tau_1}\int_0^t \|e_\vartheta^{\check{B}(t+h-\vartheta-s)} - e_\vartheta^{\check{B}(t-\vartheta-s)}\|(L_f(s)\|v(s)\| + \|f(s,0)\|)ds \\
&\quad + \sum_{j=1}^k e^{\|A\|\tau_1}\int_0^t \|C_j(e_\vartheta^{\check{B}(t+h-\vartheta-t_j)} - e_\vartheta^{\check{B}(t-\vartheta-t_j)})X(t_j,s+\vartheta)\|L_f(s)\|v(s)\|ds \\
&\quad + \sum_{j=1}^k e^{\|A\|\tau_1}\int_0^t \|C_j(e_\vartheta^{\check{B}(t+h-\vartheta-t_j)} - e_\vartheta^{\check{B}(t-\vartheta-t_j)})X(t_j,s+\vartheta)\|\|f(s,0)\|ds \\
&\le e^{\|A\|\tau_1}\|L_f\|_{L^q(\Omega,\mathbb{R}^+)} r\left(\int_0^t \|e_\vartheta^{\check{B}(t+h-\vartheta-s)} - e_\vartheta^{\check{B}(t-\vartheta-s)}\|^p ds\right)^{\frac{1}{p}} \\
&\quad + e^{\|A\|\tau_1} N \int_0^t \|e_\vartheta^{\check{B}(t+h-\vartheta-s)} - e_\vartheta^{\check{B}(t-\vartheta-s)}\|ds \\
&\quad + \sum_{j=1}^k e^{\|A\|\tau_1}\|L_f\|_{L^q(\Omega,\mathbb{R}^+)} r\left(\int_0^t \|C_j(e_\vartheta^{\check{B}(t+h-\vartheta-t_j)} - e_\vartheta^{\check{B}(t-\vartheta-t_j)})X(t_j,s+\vartheta)\|^p ds\right)^{\frac{1}{p}} \\
&\quad + \sum_{j=1}^k e^{\|A\|\tau_1} N \int_0^t \|C_j(e_\vartheta^{\check{B}(t+h-\vartheta-t_j)} - e_\vartheta^{\check{B}(t-\vartheta-t_j)})X(t_j,s+\vartheta)\|ds \\
&\to 0 \text{ as } h \to 0,
\end{aligned}$$

and

$$\begin{aligned}\|S_3\| &\le \int_0^t \|e^{Ah}-E\|\|V(t,s)\|\|f(s,v(s))\|ds\\ &\le \|e^{Ah}-E\|\int_0^t\left(\prod_{s<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t-s)}(L_f(s)\|v(s)\|+\|f(s,0)\|)ds\\ &\le \|e^{Ah}-E\|d\left(\left[\frac{1}{ap}(e^{apt}-1)\right]^{\frac{1}{p}}\|L_f\|_{L^q(\Omega,\mathbb{R}^+)}r+\frac{N}{a}(e^{at}-1)\right)\\ &\to 0 \text{ as } h\to 0.\end{aligned}$$

As a result, we immediately obtain that

$$\|(\mathcal{F}_2v)(t+h)-(\mathcal{F}_2v)(t)\|\to 0 \text{ as } h\to 0,$$

for all $v\in\mathcal{B}_r$. Therefore, $\mathcal{F}_2(\mathcal{B}_r)$ is equicontinuous in $\mathcal{PC}$.

Next, repeating the above computations, we have the following

$$\begin{aligned}\|(\mathcal{F}_2v)(t)\| &\le \int_0^t \|V(t,s)\|\|f(s,v(s))\|ds\\ &\le \int_0^t\left(\prod_{s<t_j\le t}(1+\|C_j\|e^{-\|\check{B}\|\vartheta})\right)e^{-\|\check{B}\|\vartheta}e^{a(t-s)}(L_f(s)\|v(s)\|+\|f(s,0)\|)ds\\ &\le d\left[\frac{1}{ap}(e^{ap\tau_1}-1)^{\frac{1}{p}}\right]\|L_f\|_{L^q(\Omega,\mathbb{R}^+)}r+\frac{dN}{a}(e^{a\tau_1}-1).\end{aligned}$$

Hence $\mathcal{F}_2(\mathcal{B}_r)$ is uniformly bounded. From Theorem 2.5, $\mathcal{F}_2(\mathcal{B}_r)$ is relatively compact in $\mathcal{PC}$. Thus, $\mathcal{F}_2:\mathcal{B}_r\to\mathcal{PC}$ is a compact and continuous operator.

Hence, using the Krasnoselskii's fixed point theorem, $\mathcal{F}$ has a fixed point v on $\mathcal{B}_r$. Obviously, v is a solution of the system (1) satisfying $v(\tau_1)=v_1$. The boundary condition $v(t)=\psi(t), -\vartheta\le t\le 0$ holds from (7). The proof is complete. □

Remark 3.3. Now, we make a comparison with.[29] Let $C_i=\Theta, i=1,2,\cdots,k$, and then system (1) reduces to.[29, (3)] For relative controllability results for the linear system, one has $V(\cdot,\cdot)=e^{A(\cdot-\cdot)}e_{\vartheta}^{\check{B}(\cdot-\vartheta-\cdot)}$ and the Grammian matrix (10) which is the same as in.[29, (10)] For the nonlinear system, set $C_i=\Theta, a=N,\ c=e^{-\|\check{B}\|\vartheta}M_2,\ d=e^{-\|\check{B}\|\vartheta}$, and then (15) becomes

$$e^{-\|\check{B}\|\vartheta}M_2\left[1+\frac{e^{-\|\check{B}\|\vartheta}\|D\|M}{N}(e^{N\tau_1}-1)\right]<1. \tag{17}$$

Now, (17) with $M_2[1+\frac{M}{N}(e^{Nt_1}-1)\|C\|]<1$,[29, (23)] and one see that the factor (17) improves[29, (23)] since $e^{-\|\check{B}\|\vartheta}$ is in (17).

4 | EXAMPLES

Example 4.1. Set $\tau_1=1$. Consider the following semilinear impulsive delay differential controlled system

$$\begin{cases} v'(t)=Av(t)+Bv(t-0.5)+f(t,v(t))+Du(t),\ t\in\Omega:=[0,1],\ t\notin\mathcal{T},\\ \Delta v(t_i)=C_iv(t_i),\ t_i\in\mathcal{T}:=\{0.5\},\\ \psi(t)=(0.2,0.3)^{\mathrm{T}}, -0.5\le t\le 0,\end{cases} \tag{18}$$

where we set $k=1, \vartheta=0.5, \mathcal{T}=\{0.5\}$ and

$$A=\begin{pmatrix}0.2&0\\0&0.2\end{pmatrix},\quad B=\begin{pmatrix}0.2&0.1\\0&0.3\end{pmatrix},\quad D=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$C_i=\begin{pmatrix}0.3&0.1\\0&0.4\end{pmatrix},\ i=1,\quad f(t,v(t))=\begin{pmatrix}0.2tv_1(t)\\0.3tv_2(t)\end{pmatrix}.$$

Clearly $\|A\| = 0.2$, $\|\check{B}\| = 0.2715$, $\|D\| = 1$ and $\|C_i\| = 0.4 (i = 1)$. Also,

$$AB = \begin{pmatrix} 0.04 & 0.02 \\ 0 & 0.06 \end{pmatrix} = BA, \quad AC_i = \begin{pmatrix} 0.06 & 0.02 \\ 0 & 0.08 \end{pmatrix} = C_iA,$$
$$BC_i = \begin{pmatrix} 0.06 & 0.06 \\ 0 & 0.12 \end{pmatrix} = C_iB, \ i = 1,$$
$$\check{B} = e^{-A\vartheta}B = \begin{pmatrix} 1.8221 & 0.3644 \\ 0 & 1.4577 \end{pmatrix}.$$

Clearly $a = \|A\| + \|\check{B}\| = 0.4715$.

Now, we use (14) to obtain M. For this purpose, we need to obtain $W_\vartheta[0, \tau_1]$ and then $W_\vartheta[0, \tau_1]^{-1}$. The delay Grammian matrix has the following explicit form as follows

$$\begin{aligned} W_\vartheta[0, \tau_1] &= \int_0^{\tau_1} V(\tau_1, s)DD^TV^T(\tau_1, s)ds \\ &= \int_0^1 e^{A(1-s)}X(1, s+0.5)X^T(1, s+0.5)e^{A^T(1-s)}ds \\ &= W_1 + W_2, \end{aligned}$$

where

$$W_1 = \int_0^{0.5} e^{A(1-s)} \left[E + \check{B}(0.5 - s) + C_1\right] \times \left[E + \check{B}^T(0.5 - s) + C_1^T\right] e^{A^T(1-s)}ds,$$

and

$$W_2 = \int_{0.5}^1 e^{A(1-s)}E^2e^{A^T(1-s)}ds.$$

Note

$$W_1 = \begin{pmatrix} 1.2371 & 0.1230 \\ 0.1230 & 1.4622 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.5535 & 0 \\ 0 & 0.5535 \end{pmatrix}.$$

Therefore, we obtain

$$W_\vartheta[0, 1] = \begin{pmatrix} 1.7906 & 0.1230 \\ 0.1230 & 2.0157 \end{pmatrix}, \quad W_\vartheta[0, 1]^{-1} = \begin{pmatrix} 0.5608 & -0.0342 \\ -0.0342 & 0.4982 \end{pmatrix},$$

and

$$M = \sqrt{\|W_\vartheta[0, 1]\|} = \sqrt{0.5266} = 0.7257.$$

For any $\nu, \mu \in \mathbb{R}^n$,

$$\begin{aligned} \|f(t, \nu) - f(t, \mu)\| &= \max\{0.2t\|\nu_1 - \mu_1\|, 0.3t\|\nu_2 - \mu_2\|\} \\ &\le 0.3t \max\{\|\nu_1 - \mu_1\|, \|\nu_2 - \mu_2\|\} \\ &= 0.3t\|\nu - \mu\|. \end{aligned}$$

Now, we set $L_f(t) = 0.3t$, $L_f \in L^q(\Omega, \mathbb{R}^+)$ with $p = q = 2$. Note $\|L_f\|_{L^2(\Omega,\mathbb{R}^+)} = (\int_0^1 (0.3s)^2ds)^{\frac{1}{2}} = 0.1732$,

$$\begin{aligned} d &= \left(\prod_{j=1}^k (1 + \|C_j\|e^{-\|\check{B}\|\vartheta})\right) e^{-\|\check{B}\|\vartheta} \\ &= (1 + \|C_1\|e^{-0.5\|\check{B}\|})^2e^{-0.5\|\check{B}\|} = 1.3681, \\ c &= \left(\prod_{j=1}^k (1 + \|C_j\|e^{-\|\check{B}\|\vartheta})\right) e^{-\|\check{B}\|\vartheta} \left[\frac{1}{ap}(e^{ap\tau_1} - 1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega,\mathbb{R}^+)} \\ &= d\left[\frac{1}{ap}(e^{ap\tau_1} - 1)\right]^{\frac{1}{p}} \|L_f\|_{L^q(\Omega,\mathbb{R}^+)} = 0.3055, \end{aligned}$$

and

$$c\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right] = 0.3055\left[1+\frac{1.3681\times 0.7257}{0.4715}(e^{0.4715}-1)\right] = 0.6930 < 1.$$

Thus all the conditions of Theorem 3.2 are satisfied, so (18) is relatively controllable on $[0, 1]$.

Example 4.2. In Example 4.1, we keep the other parameters unchanged except f and let

$$f(t, v(t)) = \begin{pmatrix} 0.3t^2 v_1(t)\sin t \\ 0.4tv_2(t)\cos 2t \end{pmatrix}. \tag{19}$$

For any $t \in \Omega$ and $v, \mu \in \mathbb{R}^n$,

$$\begin{aligned}\|f(t,v)-f(t,\mu)\| &= \max\{0.3t^2\|(v_1-\mu_1)\sin t\|, 0.4t\|(v_2-\mu_2)\cos 2t\|\}\\ &\le 0.4t\max\{\|v_1-\mu_1\|, \|v_2-\mu_2\|\}\\ &= 0.4t\|v-\mu\|.\end{aligned}$$

Set $L_f(t) = 0.4t$, $L_f \in L^q(\Omega, \mathbb{R}^+)$ with $p = q = 2$. Note

$$\|L_f\|_{L^2(\Omega,\mathbb{R}^+)} = \left(\int_0^1 (0.4s)^2 ds\right)^{\frac{1}{2}} = 0.2309,$$

$$c = d\left[\frac{1}{ap}(e^{ap\tau_1}-1)\right]^{\frac{1}{p}}\|L_f\|_{L^q(\Omega,\mathbb{R}^+)} = 0.4073,$$

and

$$c\left[1+\frac{d\|D\|M}{a}(e^{a\tau_1}-1)\right] = 0.4073\left[1+\frac{1.3681\times 0.7257}{0.4715}(e^{0.4715}-1)\right] = 0.9239 < 1.$$

From Theorem 3.2, we know (18) with (19) is relatively controllable on $[0, 1]$.

Example 4.3. Consider the relative controllability of system (18) (with $f(t, v(t)) \equiv \mathbf{0}$) on Ω, where A, B, C_i, D are defined in Example 4.1.

According to Theorem 3.1, we know that system (18) is relative controllability when $f(t, v(t)) = \mathbf{0}$. Further, one can get

$$\eta = v_1 - V(1,-0.5)\psi(-0.5) - \int_{-0.5}^{0} V(1,s)[\psi'(s) - A\psi(s)]ds = v_1 - \begin{pmatrix} -0.0444 \\ -0.0630 \end{pmatrix}.$$

By using the form of the control $u(t)$, we arrive at

$$\begin{aligned}u(t) &= D^T V^T(\tau_1, t) W_\vartheta^{-1}[0,\tau_1]\eta\\ &= X^T(1, t+0.5)e^{A^T(1-t)}W_\vartheta^{-1}[0,1]\eta\\ &= \begin{cases} \left[E + \check{B}^T(0.5-t) + C_1^T\right] e^{A^T(1-t)}W_\vartheta^{-1}[0,1]\eta, & 0 \le t < 0.5,\\ Ee^{A^T(1-t)}W_\vartheta^{-1}[0,1]\eta, & 0.5 \le t < 1,\end{cases}\end{aligned}$$

where $\check{B}$ and $W_\vartheta^{-1}[0,1]$ are given in Example 4.1.

5 | CONCLUSIONS

The main contribution of this paper is to introduce a relative controllability method for impulsive delay controlled systems with linear parts defined by permutable matrices. We use the impulsive delayed Cauchy matrix to construct the impulsive delay Grammian matrix. We give a sufficient and necessary condition to examine when a linear delay controlled system is relatively controllable. Also, we apply a fixed point method to establish a relative controllability result for impulsive semilinear delay controlled systems. The study on controllability of delay differential systems with impulses has potential for future research to multiple delay problems or to problems with a nonlinear term f depending on delayed arguments.

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