Characterization and Stability of Autonomous Positive Descriptor Systems

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***Abstract*—This note addresses two problems: the characterization of positivity of autonomous descriptor systems and the problem of finding conditions for their stability. For these fundamental issues, we provide necessary and sufficient conditions complemented with a numerically reliable computational approach. The given conditions can be checked by means of linear programming (LP). Also, we derive a necessary and sufficient condition for stability that is connected to the underlying eigenstructure of the system.**

I. Introduction

In recent years, descriptor systems (also referred to as singular systems, semistate systems, implicit systems, differential-algebraic systems, or generalized state-space systems) have attracted significant attention of many researchers. This is mainly due to the fact that in many real world applications the underlying systems are represented by differential equations for which the derivative is not given explicitly in function of the state. These systems emerge naturally in many practical areas such as robotics, compartmental systems, economics systems, etc. . . ., see [1]–[5]. Descriptor systems for which the states represent quantities that are intrinsically nonnegative, for instance, data packets in a network, electric charge, populations, concentrations, volumes of liquids, number of molecules; are called *positive descriptor systems*. General linear descriptor systems (without positivity constraint on the state) have been extensively studied and they are, to a large extend, well understood, see for instance [2]–[4]. Although standard positive systems have been an active field of research during recent years [6]–[8], little is done about positive descriptor systems. Most of the advances on fundamental properties of positive descriptor systems are concerned with the discrete-time case [9]–[13]. The continuous time case has been recently treated in [14]. We stress out that the existing results on the positivity of descriptor systems were obtained by assuming a non necessary condition, that is, the matrix that represents the projector on the set of admissible initial conditions, is nonnegative. Under this assumption, the stability issue has been addressed in [14] by using a generalized Perron-Frobenius type condition [15], and in particular, a Lyapunov-type stability condition was derived.

In this note, we confine ourself to unforced descriptor systems and we provide a complete characterization of the positivity of these systems without any *a priori* condition. This characterization can be checked easily by using linear programming (LP). Also, by using structural properties of descriptor systems, the stability problem is studied based on a direct analysis. Note that the characterization of positivity and stability of a linear time-invariant system (even if it is not a descriptor system) is not, in general, an easy task for a specified set of initial conditions that is not the whole space. In this note, we deal with stability for a specific conic set. The derived conditions for which a descriptor system is positive and stable are necessary and sufficient. In addition, equivalent stability conditions that can be expressed in terms of LP are provided. This, in turn, complements our theoretical results and represents a simple computational method to infer whether or not a given autonomous descriptor system is positive and stable.

The note is structured as follows: Section II gives some preliminary results on descriptor systems. In Section III we analyze the positivity of the system and provide a characterization of it. In Section IV some necessary and sufficient conditions for stability are given. Section V is devoted to illustrate the proposed computational approach.

Notation: $\mathbb{R}^n_+$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. A real matrix (or a vector) M is called nonnegative, denoted by $M \geq 0$ (respectively, $M > 0$), if all its components are nonnegative (respectively, strictly positive). M^+ denotes the Moore-Penrose pseudo-inverse of the matrix M, and $im M$ represents its image. $Re(w)$ the real part of a complex number $w \in \mathbb{C}$.

II. Preliminaries

This time-invariant homogeneous descriptor system is considered

$$E\dot{x}(t) = Ax(t) \tag{1}$$

with $E, A \in \mathbb{R}^{n\times n}$.

Notice that unlike standard linear systems, (1) may not admit a solution for arbitrary initial conditions. It is obvious to see that when the matrix E is nonsingular, (1) reduces to the standard linear system $\dot{x}(t) = E^{-1}Ax(t)$, and of course it has always a solution for any given initial condition. In the sequel, we shall characterize the set of admissible initial conditions for which there exist solutions to (1).

In the general case (i.e. when E is not necessarily invertible) Campbell derived in [16] (see also [4]) a result on the admissible set of initial conditions and the characterization of their associated solutions for the descriptor system (1). This result makes use of the Drazin inverse, such inverse possesses the following properties [17], [18].

For any matrix $M \in \mathbb{R}^{n\times n}$ there always exists a unique matrix M^D which is called the Drazin inverse of M, such that $M^D M = MM^D$, $M^D MM^D = M^D$, and $M^D M^{\nu+1} = M^\nu$ where ν is the smallest nonnegative integer such that $\mathrm{rank} M^\nu = \mathrm{rank} M^{\nu+1}$. Moreover, the Drazin inverse can be computed as follows [4], see also [19]. By using the Jordan canonical form, any matrix M can be decomposed as

$$M = T\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix} T^{-1} \tag{2}$$

where C is invertible and N is a nilpotent matrix. Then, its Drazin inverse is given by

$$M^D = T\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{3}$$

Definition 2.1: The set of initial conditions for which (1) has at least one solution is called the *set of admissible initial conditions* and it is denoted by $\mathcal{X}_0$.

Next, we present the following well-known result due to Campbell, which combines the results of [16, Th. 3.1.1 and Th. 3.1.3, , pp. 34–37].

Theorem 2.2: The descriptor (1) admits a unique solution for each admissible initial condition if and only if (E, A) is regular (i.e., $(\lambda E - A)$ is invertible for some $\lambda \in \mathbb{C}$). In this case, the set of all admissible initial conditions is given by $\mathcal{X}_0 = \mathrm{im}\widehat{E}^D\widehat{E}$ and the solutions of (1) have the following form:

$$x(t) = e^{\widehat{E}^D\widehat{A}t}\widehat{E}^D\widehat{E}v \tag{4}$$

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where v is an arbitrary vector in $\mathbb{R}^n$, the matrices $\hat{A}$ and $\hat{E}$ are given by

$$\hat{E} = (\hat{\lambda}E - A)^{-1}E, \ \hat{A} = (\hat{\lambda}E - A)^{-1}A$$

where $\hat{\lambda}$ is a complex number such that $(\hat{\lambda}E - A)^{-1}$ exists, and $\widehat{E}^D$ is the Drazin inverse of $\widehat{E}$.

Remark 2.3: Note that the trajectory (4) is solution of the difference equation $\dot{x}(t) = \widehat{E}^D\widehat{A}x(t)$ with $x(0) = \widehat{E}^D\widehat{E}v \in \operatorname{im} \widehat{E}^D\widehat{E}$.

We assume throughout the rest of the note that (E, A) is regular as a consequence of Theorem 2.2. Also, it is worth mentioning that (4) does not depend on the value of λ used to define $\hat{E}$ and $\hat{A}$, see [4] and [16] for more details on this fact.

III. Positivity Analysis

This section is concerned with the positivity notion according to the following definition.

Definition 3.1: We say that (1) is *positive* if for any nonnegative admissible initial condition $x(0) \geq 0$ we have that $x(t) \geq 0$ for all $t \geq 0$.

It is clear that when the matrix E is nonsingular, (1) reduces to the standard linear system $\dot{x}(t) = E^{-1}Ax(t)$. In this case, it can be shown that (1) is positive if and only if $E^{-1}A$ is a Metzler matrix (i.e., all its off-diagonal elements are nonnegative) see [20]. In the following, we investigate conditions under which (1) is positive when E is not necessarily invertible. For this purpose, we seize the previous result of Theorem 2.2 and Remark 2.3 and instead of (1) we study the following equivalent system (in the sense that they admit the same solutions):

$$\begin{aligned} \dot{x}(t) &= \overline{A}x(t) \\ x(0) &\in \operatorname{im} P \end{aligned} \tag{5}$$

where, for simplicity of notation, we denote $\overline{A} = \widehat{E}^D\widehat{A}$ and $P = \widehat{E}^D\widehat{E}$.

Our main result on the positivity of (5) hinges on some preliminary results. In particular, we will make extensive use of the following lemma.

Lemma 3.2: Let $\overline{A} = \widehat{E}^D\widehat{A}$ and $P = \widehat{E}^D\widehat{E}$. The following holds true.

(i) P is idempotent or projector (i.e., $P^2 = P$).

(ii) $P\overline{A} = \overline{A}P = \overline{A}$.

(iii) For any solution $x(t)$ to (1) [or equivalently to (5)] $Px(t) = x(t)$.

Proof: The first statement follows directly from the properties of the Drazin inverse.

To prove the second statement, we need to use in addition that $\widehat{E}$ and $\widehat{A}$, and $\widehat{E}^D$ and $\widehat{A}$ commute, see for instance, [4, Lemma 2.31 and Lemma 2.21] for a proof of these facts. Then, we have that $P\overline{A} = \widehat{E}^D\widehat{E}\widehat{E}^D\widehat{A} = \widehat{E}^D\widehat{E}\widehat{A}\widehat{E}^D = \widehat{E}^D\widehat{A}\widehat{E}\widehat{E}^D = \widehat{E}^D\widehat{A}\widehat{E}^D\widehat{E} = \overline{A}P$.

Finally, let $x(t)$ be any solution of (1) with $x(0) = Pv_0$ for some $v_0 \in \mathbb{R}^n$. Using the previous statements (i) and (ii) of this lemma we obtain that $Px(t) = Pe^{\overline{A}t}x(0) = Pe^{\overline{A}t}Pv_0 = (P+tP\overline{A}+(t^2/2)P\overline{A}^2+\ldots)Pv_0 = e^{\overline{A}t}PPv_0 = e^{\overline{A}t}Pv_0 = x(t)$. ■

The following result is connected to positive invariance and it can be inferred easily from the results of [21].

Lemma 3.3: Let $F \in \mathbb{R}^{p\times n}, M \in \mathbb{R}^{n\times n}$ and consider the linear system $\dot{z}(t) = Mz(t)$. Then, the following implication holds true:

$$[Fz(0) \geq 0] \Rightarrow [Fz(t) \geq 0 \forall t \geq 0]$$

if and only if there exists a Metzler matrix H such that

$$FM = HF.$$

In order to enhance readability, we include the following well-known lemma due to Penrose, see for instance [22].

Lemma 3.4: The matrix system $SM = N$ has a solution in the variable S if and only if

$$N(I - M^+M) = 0.$$

Moreover, all the solutions are given by

$$S = NM^+ + D(I - MM^+)$$

where D is an arbitrary matrix.

Now we are in position to state our main result.

Theorem 3.5: The following statements are equivalent.

1) System (5) is positive.

2) There exists a Metzler matrix H such that

$$\overline{A} = HP.$$

3) There exists a matrix D such that

$$H := \overline{A} + D(I - P) \text{ is Metzler}.$$

Proof: $(1 \Leftrightarrow 2)$: Let $x(0) = Pv_0$ for $v_0 \in \mathbb{R}^n$. Then by statement (ii) of Lemma 3.2, $P\overline{A} = \overline{A}P$, and therefore we can write any solution of (5) as

$$\begin{aligned} x(t) &= e^{\overline{A}t}x(0) = e^{\overline{A}t}Pv_0 = \left(P + t\overline{A}P + \frac{t^2}{2}\overline{A}^2P + \ldots\right)v_0 \\ &= Pe^{\overline{A}t}v_0. \end{aligned}$$

Hence $x(t)$ can be viewed as $x(t) = Pz(t)$, where $z(t) = e^{\overline{A}t}v_0$ is a solution to

$$\dot{z}(t) = \overline{A}z(t), z(0) = v_0 \in \mathbb{R}^n.$$

Thus, any (arbitrary) initial condition $z(0) = v_0$ for (6) gives rise to an admissible initial condition $x(0) = Pv_0$ for (5) and therefore the positivity of (5) is equivalent to the fact that

$$[x(0) = Pv_0 = Pz(0) \geq 0] \Rightarrow [x(t) = Pz(t) \geq 0]. \tag{6}$$

Now applying Lemma 3.3 for $F = P$ and $M = \overline{A}$ one obtains that the implication (6) amounts to saying that there exists a Metzler matrix H such that $P\overline{A} = HP$. Since by statement (ii) of Lemma 3.2, $P\overline{A} = \overline{A}$, we obtain that $\overline{A} = HP$.

$(2 \Rightarrow 3)$: We first note that the MoorePenrose pseudoinverse of any idempotent matrix $P = P^2$ equals P, i.e., $P^+ = P$. Applying Lemma 3.4 for $N = \overline{A}$ $S = H$ and $M = P$ we have that

$$H = \overline{A}P - D(I - PP^+),$$

for some matrix D. Since $\overline{A}P = \overline{A}$ and $PP^+ = P^2 = P$, we readily obtain the desired equivalence.

$(3 \Rightarrow 2)$: It follows easily after multiplying H by P from the right and using the statements (i) and (ii) of Lemma 3.2.

Remark 3.6: We stress out that in previous results on positivity of autonomous descriptor systems (the case when $u(t) = 0$) it is assumed that $P \geq 0$, which is an unnecessary condition, see for instance [11], [14]. In contrast, our result in Theorem 3.5 provides necessary and sufficient conditions for the positivity of (1) without any *a priori* assumptions on the projector P.

Remark 3.7: An explicit solution to $E\dot{x}(t) = Ax(t) + Bu(t)$ is given by the closed form formula

$$x(t) = e^{\overline{A}t}Pv + \int_0^t e^{\overline{A}(t-\tau)}\widehat{E}^D\widehat{B}u(\tau)d\tau - (I-P)\sum_{i=0}^{\nu-1}(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}u^{(i)}(t) \qquad (7)$$

for some $v \in \mathbb{R}^n$ where ν is the smallest integer such that $\mathrm{rank}\widehat{E}^\nu = \mathrm{rank}\widehat{E}^{\nu+1}$, see [16]. It seems that the use of the solution's formula given in (7) is not appropriate for the analysis of the input-output positivity of descriptor systems because even if the input is nonnegative the components of its derivatives can have any sign. This fact complicates much more the input-output positivity analysis based on the (7). Note that the only work on input-output positivity of continuous descriptor systems is treated in [14] where the input $u(t)$ and its derivatives $u^1(t), \ldots, u^\nu(t)$ are required to be nonnegative. However, this is a quite restrictive condition, for instance, the state-feedback inputs may fail to satisfy such condition.

In light of Remark 3.6, we illustrate how Theorem 3.5 applies to general situations by presenting an example where P is not positive and $\overline{A}$ is not Metzler, but the system is nevertheless positive.

Example 3.8: Let (1) be given by

$$E = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. By (3) it is easy to see that $E = E^D = \widehat{E} = \widehat{E}^D$ and $A = \widehat{A}$ with $\widehat{\lambda} = 0$, which consequently furnishes

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \overline{A}.$$

Obviously one can take, for instance, $H = I$ in statement 2 of Theorem 3.5 which is a Metzler matrix and therefore we conclude that the system is positive.

The result of Theorem 3.5 in the statement 3 involves a Metzler matrix H that is a function of an arbitrary matrix D. Similar statement was established in [14, Th. 3.4] (under the assumption that $P \geq 0$) with a special matrix D equals to a scaled identity $D = \alpha I$. Let us consider [14, Example 4.6] and show that different structures of the matrix D are possible.

Example 3.9: Let $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Using (3) one readily sees that $E = E^D = \widehat{E} = \widehat{E}^D$ and $A = A^D = \widehat{A} = \widehat{A}^D$ with $\widehat{\lambda} = 0$. Consequently, $\overline{A} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, thus

$$H = \overline{A} + D(I-P) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

where $D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then, it is easy to check that H can be constructed to be Metzler by choosing, for instance, $D = I$, $D = 0$ or *any* matrix D with only $\beta \geq 0$.

IV. Stability Analysis

In this section, we study the stability of descriptor system (1) under the positivity constraint. We shall consider the following definitions.

Definition 4.1: We say that (1) is *stable* if $x(t) \to 0$ as $t \to \infty$ for all $x(0) \in \mathrm{im}P \cap \mathbb{R}^n_+$.

Remark 4.2: We stress out that the characterization of positivity and stability of a linear time-invariant systems is not, in general, an easy task for a specified set of initial conditions that is not the whole space, like for the conic set $\mathrm{im}P \cap \mathbb{R}^n_+$.

Definition 4.3: Let $M \in \mathbb{R}^{n\times n}$, $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ be such that $Mv = \lambda v$. If $Re(\lambda) < 0$, then λ is said to be a *stable eigenvalue* and v a *stable eigenvector* (associated with λ).

In the sequel, we need the following result.

Lemma 4.4: Let A be a Metzler matrix. Then the following are equivalent.

1) A is Hurwitz.
2) There exists $\gamma \in \mathbb{R}^n$ such that
$$\gamma > 0 \text{ and } A\gamma < 0.$$
3) There exists $\lambda \in \mathbb{R}^n$ such that
$$\lambda > 0 \text{ and } \lambda^T A < 0.$$

Proof: The equivalence between statements 1) and 2) is well known [7], [23] (see a simple proof in [24]). The remaining equivalence results from the fact that $\sigma(A) = \sigma(A^T)$ and A^T is Metzler if A is Metzler. Also, it is evident that statement 2) for A^T implies statement 3). ■

Next, we present a characterization of the stability of (1), or equivalently, of (5).

Theorem 4.5: Assume there exists $v_0 \in \mathbb{R}^n$ such that $Pv_0 > 0$ (i.e., $\mathrm{im}P \cap \dot{\mathbb{R}}^n_+ \neq \emptyset$). Then the following statements are equivalent.

1) System (5) is positive and stable for all initial conditions such that $x(0) \in \mathrm{im}P \cap \mathbb{R}^n_+$.
2) There exists a matrix D such that
$$H := \overline{A} + D(I-P) \text{ is Metzler and Hurwitz.} \qquad (8)$$

Proof: ($1 \Rightarrow 2$): Assume that $x(t) \to 0$ as $t \to \infty$. Then, by integrating (5) we obtain that

$$x(T) - x(0) = \overline{A}\int_0^T x(t)dt.$$

If we consider $x(0) = Pv_0 > 0$ we must have that $\gamma := \int_0^T x(t)dt > 0$ since $x(0)$ is strictly positive and $x(t)$ is continuous. Moreover, it easy to see that for a sufficiently large T we get that $\overline{A}\int_0^T x(t)dt = \overline{A}\gamma = x(T) - x(0) < 0$. Further, note that $P\gamma = \gamma$ because by statement (ii) of Lemma 3.2 we have that $Px(t) = x(t)$. Now, since the system is positive then by Theorem 3.5 there exists a Metzler matrix of the form $H = \overline{A} + D(I-P)$. Thus, since $(I-P)\gamma = 0$ we have that $H\gamma = \overline{A}\gamma < 0$, which implies, by Lemma 4.4, that H is Hurwitz.

($2 \Rightarrow 1$): By using statement (ii) in Lemma 3.2, that is $Px(t) = x(t)$, we obtain that $\dot{x}(t) = \overline{A}x(t) = Hx(t)$ and therefore (5) is positive and stable as H is Metzler and Hurwitz. ■

Remark 4.6: It is worth mentioning that from the proof of Theorem 4.5 it follows that if (1) is positive and stable then *any* matrix of the form $H = \overline{A} + D(I-P)$ that is Metzler it is necessarily Hurwitz.

Remark 4.7: Note that the assumption $\mathrm{im}P \cap \dot{\mathbb{R}}^n_+ \neq \emptyset$ simply means that we require that (5) admits trajectories that are not just in the boundary of $\mathbb{R}^n_+$ and therefore it can be considered as a plausible assumption.

In the following, we show how we can check stability based on LP.

Theorem 4.8: Assume there exists $v_0 \in \mathbb{R}^n$ such that $Pv_0 > 0$ (i.e., $\mathrm{im}P \cap \dot{\mathbb{R}}^n_+ \neq \emptyset$) and denote $\bar{1} = (1\,1\cdots 1) \in \mathbb{R}^{1\times n}$. Then the following statements are equivalent.

1) System (5) is positive and stable for all initial conditions such that $x(0) \in \mathrm{im}P \cap \mathbb{R}^n_+$.
2) There exists $\lambda \in \mathbb{R}^n, Z \in \mathbb{R}^{n\times n}, \alpha \in \mathbb{R}$ such that

$$\lambda^T\overline{A} + \bar{1}Z(I-P) < 0 \tag{9}$$
$$\lambda > 0 \tag{10}$$
$$diag(\lambda)\overline{A} + Z(I-P) + \alpha I \geq 0. \tag{11}$$

Proof: It is enough to show the equivalence between (2) and (2) of Theorem 4.5, that is there exists a matrix of the form $\overline{A} - D(I-P)$ which is Metzler and Hurwitz.

By premultiplication of (11) by $diag(\lambda)^{-1}$ from the left we obtain that

$$\overline{A} + diag(\lambda)^{-1}Z(I-P) + diag(\lambda)^{-1}\alpha I \geq 0. \tag{12}$$

From this it is easy to see that the existence of λ, Z and α such that (12) holds is equivalent to the existence of λ and Z such that

$$\overline{A} + diag(\lambda)^{-1}Z(I-P) \text{ is Metzler} \tag{13}$$

which is to say that $\overline{A} + D(I-P)$ is Metzler with $D = diag(\lambda)^{-1}Z$. Finally, using the fact that $\lambda^T = \bar{1}diag(\lambda)$ we have that

$$\begin{aligned}\lambda^T\overline{A} + \bar{1}Z(I-P) &= \lambda^T\overline{A} + \bar{1}\,diag(\lambda)diag(\lambda)^{-1}Z(I-P)\\ &= \lambda^T\overline{A} + \lambda^T D(I-P)\\ &= \lambda^T(\overline{A} - D(I-P)) < 0.\end{aligned}$$

This together with $\lambda > 0$ is equivalent, by Lemma 4.4, to $\overline{A} - D(I-P)$ being a Hurwitz matrix. The proof of the reverse implication follows similar arguments. This concludes the proof. ■

Now, we provide a stability result involving the eigenstructure of the matrix $\overline{A}$.

Theorem 4.9: Assume there exists $v_0 \in \mathbb{R}^n$ such that $Pv_0 > 0$ (i.e., $\mathrm{im}P \cap \dot{\mathbb{R}}^n_+ \neq \emptyset$). Then the following statements are equivalent.

1) System (1) [or (5)] is positive and stable for all $x(0) \in \mathrm{im}P \cap \mathbb{R}^n_+$.
2) Any unstable eigenvector of $\overline{A}$ is in the kernel of P, i.e.

$$[\overline{A}w = \alpha w \text{ with } Re(\alpha) \geq 0] \Rightarrow [Pw = 0].$$

Proof: $(1 \Rightarrow 2)$: Assume that the system is stable. Then by Theorem 4.5 there exists a matrix D such that $H = \overline{A} + D(I-P)$ is Hurwitz and Metzler. Assume that there exists an unstable eigenvector of $\overline{A}$ that is not in the kernel of P, i.e. there exists $w \in \mathbb{R}^n$ with $Pw \neq 0$ such that $\overline{A}w = \alpha w$ and $Re(\alpha) \geq 0$. Then, by using the identity $\overline{A}P = P\overline{A}$ given by statement (iii) in Lemma 3.2 we obtain $\overline{A}Pw = P\overline{A}w = \alpha Pw$. This together with the fact that $P^2 = P$ [statement (i) of Lemma 3.2] and $\overline{A}Pw = (HP)Pw = HP^2w = HPw$, yields $HPw = \alpha Pw$, i.e., Pw is an unstable eigenvalue of H, which is a contradiction with the fact the H is Hurwitz.

$(2 \Rightarrow 1)$: This part of the proof will be achieved by using the following well-known fact of generalized eigenvectors associated to the Jordan form. If $v_1, \ldots, v_k$ are generalized eigenvectors of $\overline{A}$ associated to an eigenvalue λ of multiplicity k, then $\overline{A}v_1 = \lambda v_1$, $\overline{A}v_2 = v_1 + \lambda v_2, \ldots, \overline{A}v_k = v_{k-1} + \lambda v_k$, and it also holds that $(\overline{A} - \lambda I)^i v_i = 0$, for $i = 1, \ldots, k$.

Now if $Re(\lambda) \geq 0$, then by assumption $Pv_1 = 0$. Further, as $Pv_1 = 0$, $P\overline{A}v_2 = Pv_1 + \lambda Pv_2$ and since $P\overline{A} = \overline{A}P$ (see statement (ii) of Lemma 3.2) then we have $\overline{A}Pv_2 = \lambda Pv_2$ and therefore necessarily $Pv_2 = 0$ since otherwise Pv_2 would be an unstable eigenvector.

Next, let $x(0) = Pw_0$, $w_0 \in \mathbb{R}^n$ and $w_0 = \sum_{i=1}^n \alpha_i v_i$ where $\{v_1, \ldots, v_\ell, v_{\ell+1}\ldots, v_n\}$ is a basis of $\mathbb{R}^n$ of generalized eigenvectors of $\overline{A}$ and $\{v_1, \ldots v_\ell\}$ and $\{v_{\ell+1}, \ldots, v_n\}$ are associated with stable and unstable eigenvalues respectively. Since $Pv_i = 0$ for $i = \ell + 1, \ldots, n$, so that any solution of (5) has the form

$$x(t) = e^{t\overline{A}}Px(0) = \sum_{i=1}^{\ell}\alpha_i e^{t\overline{A}}Pv_i.$$

Thus, it is enough to show that $e^{t\overline{A}}Pv_i \to 0$ as $t \to \infty$, for $i = 1, \ldots, \ell$.

Let v_i be associated to λ_i with multiplicity k. As $\overline{A}$ and $\lambda_i I$ commute we get that $e^{t(\overline{A}-\lambda_i I)} = e^{t\overline{A}}e^{-t\lambda_i}$. In the other hand $e^{t(\overline{A}-\lambda_i I)}Pv_i = (I + t(\overline{A} - \lambda_i I) + \cdots + (t^k/k!)(\overline{A} - \lambda_i I)^k)Pv_i =: M(t)$, since $(\overline{A} - \lambda_i I)^N Pv_i = 0$ for $N > k$. Hence, we get that $e^{t\overline{A}}Pv_i = e^{t\lambda_i}M(t) \to 0$ as $t \to \infty$.

V. Numerical Illustration

In this section, we illustrate the proposed method by means of two numerical examples. First, we shall show how one can use standard LP to check the proposed positivity condition. For this, one can use the well-known Kronecker product $\otimes$ and the $\mathbf{vec}$ operation (which consists of taking the columns of a given matrix from left to right and stack them one above the other). It is well known that for any matrices M, N and X with appropriate sizes, we have that $\mathbf{vec}(MXN) = [N^T \otimes M]\mathbf{vec}(X)$.

Thus, by using the above property of $\mathbf{vec}$ and Kronecker product, the positivity criterium given in Theorem 3.5 can be reformulated as standard LP problem. Note that our positivity condition: there exists a matrix $D \in R^{n\times n}$ satisfying

$$H := \overline{A} + D(I-P) \text{ is Metzler}$$

is equivalent to the following matrix inequality condition:

$$\overline{A} + D(I-P) + \alpha I \geq 0,$$

in the unknown variables $D \in R^{n\times n}$ and $\alpha \in \mathbb{R}$. Hence, such matrix inequality can be reexpressed as the following standard linear inequality:

$$\begin{bmatrix}(P^T - I)\otimes I & -\mathbf{vec}(I)\end{bmatrix}\begin{bmatrix}x\\ \alpha\end{bmatrix} \leq \mathbf{vec}(\overline{A}) \tag{14}$$

where $x = \mathbf{vec}(D)$.

In order to solve this standard linear problem (14) we can use, for instance, the well-known linprog in Matlab, Cplex, or Sedumi. In particular, we make use of linprog Matlab function to treat the following two illustrative examples.

Example 5.1: Let (1) be given by the matrices

$$E = \begin{pmatrix}1.0000 & 1.0000 & 1.0000\\ 1.0000 & 0 & 1.0000\\ 1.5000 & 1.0000 & 1.5000\end{pmatrix}$$

and

$$A = \begin{pmatrix}1.0000 & 0.2700 & 0.0398\\ 0.0500 & 0.2700 & 0.0502\\ 0.1000 & 0.1000 & 0.4000\end{pmatrix}.$$

For this example we select $\widehat{E} = (E-A)^{-1}E$ and $\widehat{A} = (E-A)^{-1}A$. Then, by using Jordan decomposition and formula (3) we compute the Drazin inverse of $\widehat{E}$ and obtain

$$\widehat{E}^D = \begin{pmatrix} 0.3119 & -0.3701 & 0.3119 \\ -0.2450 & 1.2325 & -0.2450 \\ 0.6380 & 0.1001 & 0.6380 \end{pmatrix}.$$

Thus, the given system can be rewritten as in the form (5) where its dynamic matrix and its projector are given by

$$\overline{A} = \widehat{E}^D\overline{A} = \begin{pmatrix} -0.0460 & 0.1281 & -0.0460 \\ 0.2450 & -0.2325 & 0.2450 \\ 0.0961 & 0.1420 & 0.0961 \end{pmatrix}$$
$$P = \widehat{E}^D\widehat{E} = \begin{pmatrix} 0.2659 & -0.2420 & 0.2659 \\ 0.0000 & 1.0000 & 0.0000 \\ 0.7341 & 0.2420 & 0.7341 \end{pmatrix}$$

respectively. By solving the standard LP (14) we obtain

$$H = \begin{pmatrix} -0.1740 & 0.0859 & 0.0004 \\ 0.6509 & -0.0986 & 0.0979 \\ 119.0394 & 39.3611 & -42.9935 \end{pmatrix}.$$

Thus, the matrix H is a Metzler matrix and therefore, by Theorem 3.5, the system is positive. For stability analysis, we first note that there exists a v_0 satisfying $Pv_0 > 0$, for instance, by using linprog function we have found $v_0 = (72.1541\ 57.1705\ 72.1541)^T$. Thus, we can apply Theorem 4.5 and Remark 4.6 to show that the system is not stable. Effectively, the spectrum of H is given by $\sigma(H) = \{-43.0838, -0.3846, .2023\}$, and then it is not Hurwitz, so that with the given matrices E and A, (1) is not stable.

Note that one can alternatively apply Theorem 4.9 to conclude that the system is not stable. The spectrum of $\overline{A}$ is $\sigma(\overline{A}) = \{0.2023, 0, -0.3846\}$ and it can be checked that the unstable eigenvector of $\overline{A}$ associated to 0.2023, $v_2 = (0.1488\ 0.6474\ 1.0000)^T$, is not in the kernel of P.

Example 5.2: Let (1) be given by the matrices

$$E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -4 & 1 \\ -1 & 2 & -3 \end{pmatrix}.$$

As in the previous example we consider $\widehat{E} = (E-A)^{-1}E$ and $\widehat{A} = (E-A)^{-1}A$ and derive the Drazin inverse of $\widehat{E}$ from which we obtain the following associated matrices to (5)

$$\overline{A} = \begin{pmatrix} 6.6667 & -12.7778 & 6.6667 \\ 4.0000 & -7.6667 & 4.0000 \\ -9.6667 & 18.1111 & -9.6667 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 0.0000 & 1.6667 & 0.0000 \\ 0.0000 & 1.0000 & 0.0000 \\ 1.0000 & -1.6667 & 1.0000 \end{pmatrix}.$$

By solving the LP problem (14) we get

$$H = \begin{pmatrix} -53.7867 & 87.9778 & 6.6667 \\ 47.1065 & -79.5108 & 4.0000 \\ 0.0444 & 1.9260 & -9.6667 \end{pmatrix}.$$

Since its spectrum is given by

$$\sigma(H) = \{-132.2975\ -0.1586\ -10.5081\}$$

we can see that H is Metzler and Hurwitz. By taking into account the fact that $Pv_0 > 0$ for $v_0 = (83.2864\ 54.4789\ 83.2864)^T$, we can apply Theorem 4.5 to conclude that the system is positive and stable.

As previously, we can perform direct stability analysis by looking to the spectrum of $\overline{A}$

$$\sigma(\overline{A}) = \{-10.5081, -0.1586, 0\}.$$

The only unstable eigenvalue is zero, which is associate to $v = (-1\ 0\ 1)$. We can see that $Pv = 0$ and then it suffices to apply the result of Theorem 4.9 to conclude that this system is stable.

VI. Conclusion

A complete characterization of positivity of descriptor systems was given. The provided condition under which the descriptor system is positive is necessary and sufficient and can be numerically checked by using LP. The stability issue was also addressed and two approaches were presented. The first one makes use of an LP necessary and sufficient condition and the second one is connected to the underlying eigenstructure of the system.

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Constrained Nonlinear Polynomial Time-Delay Systems: A Sum-of-Squares Approach to Estimate the Domain of Attraction

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***Abstract*—A sum-of-squares (SOS) approach is proposed to estimate the domain of attraction (DoA) of regulated nonlinear polynomial time-delay systems subject to input constraints. It will be shown that the joint use of semialgebraic geometric arguments and semidefinite programming methods provides an efficient tool for Lyapunov-based control system analysis and design for this class of nonlinear delayed systems. Specifically, a two-step procedure is presented which consists of deriving first a Lyapunov-Krasovskii functional/controller pair complying with the prescribed constraints and computing then an estimate of the DoA.**

***Index Terms*—Domain of attraction (DoA), sum-of-squares (SOS).**

I. Introduction

Significant achievements have been recently obtained in the development of techniques and methods to check the stability and to design control schemes for nonlinear time-delay systems (NTDS) via time- (Lyapunov) and frequency-domain methods. In the literature, the time-delay paradigm is usually employed to model physical phenomena involving transport and propagation of either materials or data, see [16] and references therein for more details.

Time-delay systems are described via functional differential equations (FDE), which differ from ordinary differential equations (ODE) because they do not admit in general a finite dimensional state representation. Performance analysis and control design for such systems suffer therefore from unavoidable structural complications. However, recent advancements on sum-of-squares (SOS) methodologies and semidefinite programming algorithms [19] have significantly improved the research scenario and now efficient computational techniques are available to deal adequately with these analysis and control synthesis challenging problems.

Nonetheless, contributions on convex optimization techniques for nonlinear time-delay control systems are few in the technical literature and mainly confined to the computation of the largest time-delay value still guaranteeing asymptotic stability, see, e.g. [14], [13], [18], [20], and references therein. Specifically, in [14] asymptotic stability is guaranteed via Lyapunov functionals having a piecewise-linear kernels structure, which can be obtained by solving a set of LMIs. In [18], an explicit parametrization of positive definite operators, SOS decomposition techniques and semidefinite programming formulations have been proposed as numerical tools for stability analysis and synthesis. Relevant improvements in fact have been achieved in constructing suitable Lyapunov-Krasovskii (L-K) functionals to assess the asymptotic stability in both delay-independent and delay-dependent cases for nonlinear polynomial time-delay systems (NPTDS). Of interest here are two papers: [17] and [11]. In the first, a SOS-based scheme has been presented for the synthesis of L-K robust stabilizing controllers, which act on the basis of actual and delayed states. In the second, a linearization-based robust model predictive control strategy for input-constrained nonlinear polynomial time-delay systems has been proposed and studied. Other contributions and more general purpose application papers can be found in [16]. It must be noted that, when designing regulation strategies for nonlinear systems whose equilibria are locally stable, it is important to dispose of reliable methods for possibly enlarging the domain of attraction (DoA) estimate, see, e.g. [5]. In this respect, a noticeable contribution is given by [4], where Takagi-Sugeno (TS) fuzzy models are exploited to estimate the DoA of time-delay systems. Unfortunately, the DoA structure is complicated in the present context and analytical representations are impossible to be derived except for trivial cases (see, e.g., [7] and references therein). Because of this, the DoA cannot never be computed exactly but only approximated by sets having simple shapes (ellipsoids, polyhedra, and so on). A significant contribution to reduce the DoA estimation conservativeness is provided in [3], where saturation-dependent Lyapunov functionals are considered. However, such schemes are obviously more awkward from a computational point of view because the constraints and the time-delay need to be explicitly taken into account.

Moving from the above considerations, the purpose of this note is to study the problem of improving, via semialgebraic methods, DoA inner approximations of input-constrained nonlinear polynomial time-delay systems, regulated by feedback control laws acting on the basis of both actual and delayed states. The contribution is twofold: first, delay-dependent (DD) closed-loop stability is guaranteed for input-constrained time-delay systems. In particular, we provide explicit SOS conditions in the form of an algorithmic procedure which is able to overcome all technical obstacles. Then, the DoA estimate is computed by solving a SDP optimization which is based on a series of convex approximations of the original problem obtained by means of semi-algebraic sets theory and set-invariance concepts.

A SOS-based scheme for the computation of DoA inner estimates is here proposed whose tightness increases with the degrees of the polynomials involved in the L-K functional and controller descriptions. The procedure consists in a two-step algorithm: first, a L-K functional/controller pair complying with the prescribed constraints is derived. Then, the maximal level set *contained* into the *true* DoA determined. The effectiveness of the proposed SOS procedure is finally tested on a numerical example taken from the literature by contrasting the size of DoA

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