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DAEs Arising from Traveling Wave Solutions of PDEs II

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Abstract—The study of traveling wave solutions of PDEs sometimes leads to systems of Differential Algebraic Equations (DAEs). This paper examines a family of DAEs that arise in this manner from the Magnetohydrodynamics (MHD) equations. These DAEs are of interest in their own right, as a source of test problems for DAE numerical integrators, and because of their relationship to the MHD equations.

Keywords—Differential algebraic equation, Magnetohydrodynamics, Dynamical systems.

1. INTRODUCTION

DAEs are implicitly defined systems of differential equations

$$F(y', y, t) = 0 \tag{1}$$

with $\frac{\partial F}{\partial y'}$ identically singular. DAEs arise in many different areas and a variety of numerical and analytical tools for working with DAEs have been developed over the last decade [1–4]. More general methods are under development [5–9]. The increased use of computer generated models for complex systems means that DAE integrators of the future will have to deal with systems which may not be in the usual forms such as arise in constrained mechanics. There is a need for good test problems which exhibit a rich variety of known behavior. Ideally these test problems should be representative of the difficulties that arise in various applications.

Examining traveling waves for PDEs can naturally lead to the consideration of DAEs. Reference [10] provided a general treatment. In this paper, we focus on a particular system of nonlinear PDEs; the dissipative magnetohydrodynamic (MHD) equations with resistivity, viscosity, and thermal conductivity from [11] which were briefly considered in [10]. Here we consider them in much more detail and exhibit several additional types of behavior. The MHD DAEs that

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we examine are for the special one-dimensional case. This special case is important in its own right as an interesting system. It also is important in understanding the full systems behavior when certain parameters are small but nonzero. However, our purpose in this paper is to develop a better understanding of the MHD DAEs as a source of test problems for DAE integrators and DAE theoretical algorithms. We shall see that the equations exhibit a rich variety of structure. Most importantly, we shall see that this structure changes with the values of various parameters. Thus, these systems can be used to test the behavior of integrators close to singular and nearly singular points, near various kinds of equilibria, and on disconnected solution manifolds but with nearby components. Of course, this system can provide good test problems only if the behavior of the solutions is understood.

As a side benefit, it is also shown that by utilizing a DAE perspective, we can more easily analyze and explain some properties of the MHD DAE than if we tried to reduce it to an Ordinary Differential Equation (ODE).

The starting point of our analysis is the Riemann problem for systems of nondissipative MHD equations. It is known that under mild conditions the existence of shock solutions is equivalent to the existence of traveling waves in the respective system of dissipative MHD equations [12]. A specific type of traveling wave that connects equilibria is of importance here. In order for this specific traveling wave to exist certain conditions need to be satisfied. They are discussed in Section 3. We are interested in equilibria, their stability, and the geometry of the solution manifold since these are properties that might be incorrectly determined by some numerical methods.

In order to make this paper reasonably self-contained, Section 2 reviews some basic DAE terminology and needed information from [10]. The MHD equations are given in Section 3 where several needed properties are developed. Section 4 discusses dissipative mechanisms and lays the foundation for the next sections. Specific cases illustrating different behavior are in Section 5. Finally, Section 6 examines singularity induced bifurcations.

2. SOME DAE BACKGROUND

In this section, we briefly review some facts about DAEs. Classically, when we are looking for a traveling wave that connects two equilibria we would have to reduce the problem to an ODE. Such a reduction might be extremely difficult or even impossible without making simplifying assumptions. Also different reductions might be necessary for different parameter values. By considering a DAE instead, no simplification may be needed. The same DAE model can serve for a variety of parameter values.

DAEs differ from ODEs in several ways. The DAE is called solvable if there is a well-defined constant-dimensional manifold of solutions and the solutions are uniquely determined by consistent initial conditions. Solvability sometimes also includes the idea of solutions existing for a class of forcing functions. The solution manifold may be thought of as being defined by a family of constraints. In some applications these constraints are given explicitly but in other problems some or all of the constraints may be defined implicitly. Determining to what extent a DAE integrator preserves constraints under different situations is important. We will see that the MHD DAE provides good test problems.

The index is one measure of how singular a DAE is. There are several definitions of the index of a DAE like (1). For our purposes the most important is the following. If (1) is differentiated k times with respect to t, we get the $(k+1)n$ *derivative array equations* [1]

$$\begin{bmatrix} F(y', y, t) \\ F_t(y', y, t) + F_y(y', y, t)y' + F_{y'}(y', y, t)y'' \\ \vdots \\ \dfrac{d^k}{dt^k} F(y', y, t) \end{bmatrix} = G(y', y, t, w) = 0, \tag{2}$$

where $w = [y^{(2)}, \dots, y^{(k+1)}]$. In particular applications, different equations in $F = 0$ are often differentiated a different number of times. This has no effect on our discussion.

For our purposes, it suffices to say that the DAE (1) is index k if (2) uniquely determines y' given consistent (t, y) and no smaller value of k has this property. A more careful discussion of the index is in [13].

If we are fortunate enough that our DAE takes the form of

$$\phi_1' = f(\phi_1, \phi_2), \tag{3a}$$

$$0 = g(\phi_1, \phi_2), \tag{3b}$$

and $\frac{\partial g}{\partial \phi_2}$ is invertible, the DAE is called semiexplicit index one and the solution manifold is given by (3b). If $\frac{\partial g}{\partial \phi_2}$ is singular for some, but not all, values of ϕ_1, ϕ_2, then we say that $\det[\frac{\partial g}{\partial \phi_2}] = 0$ defines the singularity manifold of the DAE (3).

The situation is more complex if the DAE is not semiexplicit or it is not index one. Let k be as in (2) and define

$$\overline{J}_k = [\, G_{y'} \quad G_w \,], \qquad J_k = [\, G_{y'} \quad G_w \quad G_y \,].$$

The following assumptions on G, for sufficiently large k, permit a robust numerical least squares solution of the derivative array equations. They are also the basis of a general theory for DAEs.

(A1) Sufficient smoothness of G.
(A2) Consistency of $G = 0$ as an algebraic equation.
(A3) $\overline{J}_k = [\, G_{y'} \quad G_w \,]$ is 1-full with respect to y' and has constant rank independent of (t, y, y', w).
(A4) $J_k = [\, G_{y'} \quad G_w \quad G_y \,]$ has full row rank independent of (t, y, y', w).

Here the matrix C of the equation $Cx = b$ is said to be *1-full* with respect to x_1 if there is a nonsingular matrix Q such that

$$QA = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Conditions (A1)–(A4) are numerically verifiable using a combination of symbolic and numeric software [14]. General procedures in terms of rank and continuity conditions on $\overline{J}_k, J_k$ exist for determining the index, the dimension of the solution manifold, and getting a local characterization of the solution manifold [5,14].

One has to be very careful with linearizing DAEs [15–18]. However, the situation is more straightforward around equilibria [15,19]. We use the following result to determine the stability behavior of equilibria directly from the DAE.

THEOREM 1. *Suppose that $\overline{y}$ is an equilibrium of $F(y', y) = 0$. Suppose that the DAE satisfies (A1)–(A4) in a neighborhood of $(0, \overline{y}, 0)$. Let $A = F_{y'}(0, \overline{y})$, $B = F_y(0, \overline{y})$. Suppose that B is nonsingular. Let y be n dimensional and r be the difference in rank of $[G_{y'}, G_w]$ and $[G_{y'}, G_y, G_w]$ for this system at $(0, \overline{y}, 0)$. Then*

1. *the local linearization $A\widetilde{y}' + B\widetilde{y} = B\overline{y}$ and the original DAE $F(y', y) = 0$ have the same dimensional solution manifold in a neighborhood of $\overline{y}$,*
2. *the dimension of the solution manifold is $n - r$,*
3. *if the pencil $\lambda A + B$ has $n - r$ finite eigenvalues with nonzero real part, then they will determine the stability properties of $\overline{y}$ on the solution manifold of $F(y', y) = 0$.*

3. BASIC PROPERTIES OF THE TRAVELING WAVE MHD DAES

We are interested in traveling wave solutions which connect equilibria of the dissipative MHD system. In this paper, we restrict ourselves to equations in two variables x, t defined in a region Ω

in the form

$$[h(u)]_t + [p(u)]_x = \mu[D(u)u_x]_x. \tag{4}$$

It is assumed that $p(u)$ in (4) is such that the Jacobian $\frac{dp(u)}{du}$ has n real and distinct eigenvalues in the domain of interest. This yields a complete set of eigenvectors. Such a system is strictly hyperbolic [12]. There is then an equivalence between the shock solution of the hyperbolic system (i.e., (4) with $D(u) \equiv 0$) and traveling wave solutions in a parabolic system (i.e., (4) with $D(u) \neq 0$).

DEFINITION 1. *We say that (4) admits a traveling wave solution u (with constant wave speed s), if there exists $s \in \mathbf{R}$ and $u^r, u^l \in \mathbf{R}^n$, and function $\phi(z)$ such that $u(x,t) = \phi(x - st)$ in Ω and $\lim_{\psi \to -\infty} \phi(\psi) = u^l$, $\lim_{\psi \to +\infty} \phi(\psi) = u^r$, and $\lim_{\psi \to \pm\infty} \phi'(\psi) = 0$, where $\psi \equiv x - st$ and u satisfies (4).*

Clearly Ω must contain the line $\{x - st : t \in \mathbf{R}\}$. To simplify our notation we again denote ϕ by u. The traveling wave solution for (4) satisfies the equation

$$-sh'(u)u' + [p(u)]' = \mu[D(u)u']'. \tag{5}$$

We want our solution to go from a left equilibrium u^l to a right equilibrium u^r. Integrating (5) from $-\infty$ to t using u^l as the left endpoint, and using the fact that u' has zero limit as $t \to -\infty$, gives the linearly implicit DAE

$$-s\left(h(u) - h\left(u^l\right)\right) + p(u) - p\left(u^l\right) = \mu D(u)u'. \tag{6}$$

An equilibrium for (6) must satisfy

$$-s\left(h(u) - h\left(u^l\right)\right) + p(u) - p\left(u^l\right) = 0. \tag{7}$$

At a given equilibrium $\overline{u}$, the Jacobian for (7) is $R(\overline{u}) = -sh'(\overline{u}) + p'(\overline{u})$. If there is a traveling wave solution connecting the equilibria, then the left and right equilibria determine the wave speed via (7). A type of converse holds.

THEOREM 2. *(See [10].) Fix u^l. Suppose that there exists a solution $\widehat{u}$ of the DAE (6) which connects the equilibrium u^l with the equilibrium u^r with wave speed $\widehat{s}$. Suppose that the assumptions (A1)–(A4) hold for (6) in a neighborhood of $\widehat{u}$ and u^l, u^r and for s near $\widehat{s}$.*

1. *If $-\widehat{s}h'(u^r) + p'(u^r)$ is nonsingular, then for s near $\widehat{s}$ there will be a right equilibrium $u^r(s)$ and a solution of (6) connecting u^l to $u^r(s)$.*
2. *If $[-\widehat{s}h'(u^r) + p'(u^r), h(u^r) - h(u^l)]$ is invertible when its i^{th} column is deleted, then we may vary the i^{th} component of u^r and there will exist a new right equilibrium $\widetilde{u}^r$ with this same i^{th} component and a solution connecting u^l and $\widetilde{u}^r$ for a wave speed near $\widehat{s}$.*

3.1. Magnetohydrodynamics Equations

The dissipative MHD equations with resistivity, viscosity, and thermal conductivity, are [11]

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \tag{8a}$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} = -\nabla \cdot \left(\rho \mathbf{v}\mathbf{v} + \mathbf{I}\left(p + \frac{B^2}{2}\right) - \mathbf{B}\mathbf{B}\right) + \nu \nabla^2 \mathbf{v} + \left(\mu + \frac{4}{3}\nu\right)\nabla(\nabla \cdot \mathbf{v}), \tag{8b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{8c}$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot \left(\left(\frac{\rho v^2}{2} + \frac{p}{\gamma - 1} + p\right)\mathbf{v} + \mathbf{E} \times \mathbf{B}\right) + \nabla \cdot \sigma \cdot \mathbf{v} + \kappa \nabla^2\left(\frac{p}{\rho}\right), \tag{8d}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{8e}$$

The specific meaning of these variables is discussed in [11]. If we consider one-dimensional flow in the x direction only, then the x component B^x of the magnetic induction is constant from (8e) and the y and z components, B^y, B^z can be taken as functions of x, t. For notational convenience let μ replace $\mu + (4/3)\nu$. Letting $\rho = u_1$, the three components of $\mathbf{v}$ be u_2, u_3, u_4, and $[B^y, B^z, E] = [u_5, u_6, u_7]$, we get the DAE (5) is

$$-su_1' + (u_1u_2)' = 0, \tag{9a}$$

$$-s(u_1u_2)' + \left(u_1u_2^2 + P^*\right)' = \mu u_2'', \tag{9b}$$

$$-s(u_1u_3)' + (u_1u_2u_3 - B^x u_5)' = \nu u_3'', \tag{9c}$$

$$-s(u_1u_4)' + (u_1u_2u_4 - B^x u_6)' = \nu u_4'', \tag{9d}$$

$$-su_5' + (u_2u_5 - B^x u_3)' = \eta u_5'', \tag{9e}$$

$$-su_6' + (u_2u_6 - B^x u_4)' = \eta u_6'', \tag{9f}$$

$$\begin{aligned} -su_7' + \left[(u_7 + P^*)\, u_2 - B^x \left(B^x u_2 + u_3u_5 + u_4u_6\right)\right]' &= \frac{\mu}{2}\left(u_2^2\right)'' + \frac{\nu}{2}\left(u_3^2 + u_4^2\right)'' \\ &\quad + \frac{\eta}{2}\left(u_5^2 + u_6^2\right)'' + \kappa\left(\frac{p}{u_1}\right)''. \end{aligned} \tag{9g}$$

The parameters η, κ, μ, ν are resistivity, thermal conductivity, and two viscosity coefficients. P^* and p are given by

$$P^* = p + \frac{1}{2}\left(\left(B^x\right)^2 + u_5^2 + u_6^2\right), \tag{10a}$$

$$p = \frac{\gamma - 1}{2}\left[2u_7 - u_1\left(u_2^2 + u_3^2 + u_4^2\right) - \left(u_5^2 + u_6^2 + \left(B^x\right)^2\right)\right], \tag{10b}$$

where B^x and γ are constants.

If we integrate (9) between u^l and u and take into account that u^l is an equilibrium, then the traveling wave DAE (6) is

$$-su_1 + u_1u_2 - c_1 = 0, \tag{11a}$$

$$-su_1u_2 + u_1u_2^2 + P^* - c_2 = \mu u_2', \tag{11b}$$

$$-su_1u_3 + u_1u_2u_3 - B^x u_5 - c_3 = \nu u_3', \tag{11c}$$

$$-su_1u_4 + u_1u_2u_4 - B^x u_6 - c_4 = \nu u_4', \tag{11d}$$

$$-su_5 + u_2u_5 - B^x u_3 - c_5 = \eta u_5', \tag{11e}$$

$$-su_6 + u_2u_6 - B^x u_4 - c_6 = \eta u_6', \tag{11f}$$

$$\begin{aligned} -su_7 + (u_7 + P^*)u_2 - B^x \left(B^x u_2 + u_3u_5 + u_4u_6\right) - c_7 &= \mu u_2u_2' + \nu\left(u_3u_3' + u_4u_4'\right) \\ &\quad + \eta\left(u_5u_5' + u_6u_6'\right) + \kappa\left(\frac{p}{u_1}\right)', \end{aligned} \tag{11g}$$

where c_i is the i^{th} entry of the vector $sh(u^l) - p(u^l)$ in (6).

The remainder of this paper concerns the analysis of (11) which we shall refer to as the MHD DAE. As noted earlier, (11) is linearly implicit. However, if $\kappa \neq 0$, then the nullspace of the coefficient of u' will not be constant as required by some integrators. If $\kappa = 0$ but at least one of η, μ, ν are nonzero, then the nullspace will be constant but the range will not be. The DAE can be made semiexplicit by introducing another variable but then the DAE will be index two.

The DAE (11) provides a rich family of test problems. The equations are simple to program and are of modest size. Yet, as we shall show, the system properties can undergo dramatic changes as the parameters are varied. Some nonzero parameter values are more physically realistic. However, all values are of interest for developing test problems. Also, the behavior of a system with some parameters nearly zero is often best reflected by considering the zero parameter case.

3.2. Properties of the MHD DAE

We first analyze the existence and number of possible equilibria in the MHD DAE (11). Note that by construction, the left state is an equilibrium.

THEOREM 3. *The real roots of the polynomial*

$$p(u_6) = \left(au_6^3 + bu_6^2 + cu_6 + d\right)\left(u_6 - u_6^l\right) \tag{12}$$

with

$$a = (B^x)^2 \left[\left(u_5^l\right)^2 + \left(u_6^l\right)^2\right], \tag{13a}$$

$$b = u_6^l \left[(u_6)^2 + \left(u_5^l\right)^2\right]\left[u_1^l(2-\gamma)\left(s-u_2^l\right)^2 + (B^x)^2(\gamma-1)\right], \tag{13b}$$

$$c = -\left[(B^x)^2 - \left(u_2^l - s\right)^2 u_1^l\right] c^*, \tag{13c}$$

$$\begin{aligned} c^* = {} & \left[\left(u_5^l\right)^2 + \left(u_6^l\right)^2\right](2-\gamma)\gamma + u_1^l(1-\gamma)\left(\gamma\left[\left(u_2^l\right)^2 + \left(u_3^l\right)^2 + \left(u_4^l\right)^2\right] - \left(u_2^l - s\right)^2\right) \\ & - (B^x)^2\left(\gamma^2+1\right) - 2\gamma u_7^l(1-\gamma), \end{aligned} \tag{13d}$$

$$d = -\left(u_6^l\right)^3(\gamma+1)\left[\left(u_2^l - s\right)^2 u_1^l - (B^x)^2\right]^2 \tag{13e}$$

are the values of u_6 at the equilibria of (11). Other components of u are functions of u_6.

Thus, if the wave speed and all components of the left state are given, then the traveling wave MHD DAE can admit at most four equilibria (including the left state).

PROOF. One can show using symbolic calculations that when the right-hand side of (11) is zero, then by elimination of six out of seven variables, we obtain a 4^{th} degree polynomial equation (12) in the remaining component. This polynomial equation has at most four real solutions. The other variables can then be computed as functions of the variable in that polynomial equation. The Appendix presents a MAPLE code for finding the polynomial equation in u_6. ∎

The various components of the left state and parameters γ, B^x, and s influence coefficients a, b, c, and d, as well as the number of equilibria. There is always one equilibrium, u^l. The number of other distinct equilibria can range from 0 to 3 as the following examples show.

If

$$u_2^l - s = \pm\sqrt{\frac{(B^x)^2}{u_1^l}}, \tag{14}$$

then $c = d = 0$ and $p(u_6) = (u_6^l)^2(au_6 + b)(u_6 - u_6^l)$. We have three roots u_6^l, $-u_6^l(= -b/a)$, and 0 (double root). It is well known that the right-hand side of (14) is the value of the Alfvén wave [11].

If $u_6^l = 0$, then $b = d = 0$ and the roots are $\{0, 0, \pm\sqrt{-c/a}\}$, provided that c/a is negative. On the other hand, if $u_5^l = u_6^l = 0$ (two components of the magnetic field are zero), then $a = b = d = 0$ (but $c \neq 0$ in general) and the only roots are a double root of 0.

4. DISSIPATIVE MECHANISM AND VARIOUS DAES

The dissipative mechanism represented by coefficients η, μ, ν, κ gives different structure to the resulting DAEs, depending on which of those coefficients are zero and which are nonzero. For a particular problem it is usually possible to eliminate some variables. This is not necessary for a general analysis [10]. Also, the simplification can be complex. We could easily perform the following analysis without eliminating any variables. However, we shall do so in order to obtain graphical representations. In particular, note that (11a) is linear in u_1 so that if $\mu = 0$ we can eliminate u_1 from the equations. Similarly, (11c) and (11d) can be used to eliminate u_3, u_4 if $\eta = 0$ and (11e),(11f) can be used to eliminate u_5, u_6 if $\eta = 0$. The result will sometimes be a

DAE or an ODE depending on the problem. The most interesting from the DAE point of view, are those cases where after reduction the system is still a DAE since then the geometry of the solution manifold can be nontrivial. We shall consider the following cases which exhibit a variety of interesting behavior. Specific examples will be examined more carefully in Sections 5 and 6.

4.1. η only

If the dissipative mechanism is due to a nonzero η only, then after eliminating variables u_1, u_3, u_4, and u_7 we obtain a semiexplicit DAE

$$u_5' = f_1(u_5, u_2, p), \tag{15a}$$
$$u_6' = f_2(u_6, u_2, p), \tag{15b}$$
$$0 = g(u_5, u_6, u_2, p), \tag{15c}$$

where g is quadratic in all three variables and p is a vector of parameters.

If $g_{u_2} \neq 0$, g defines the constraint manifold. In the MHD literature this manifold is usually divided into two separate branches: *supersonic* and *subsonic*. These branches have common points known as the *sonic* points, where the wave speed equals the *sound* speed. One can show for this example that the singular points (common points of $g = 0$ and $g_{u_2} = 0$) are the sonic points of the MHD system [20]. While of interest, we omit the proof since it is not important for our discussion here.

4.2. η and κ only

If η and κ are the only nonzero dissipative coefficients, then we obtain a DAE (or implicit ODE)

$$\kappa h(u_2, u_5, u_6) u_2' = g(u_2, u_5, u_6), \tag{16a}$$
$$\eta u_5' = f_1(u_2, u_5, u_6), \tag{16b}$$
$$\eta u_6' = f_2(u_2, u_5, u_6), \tag{16c}$$

$h = 0$ defines the singularity manifold. If $\kappa = 0$, then with slightly different f_1, f_2, g, (16) is (15).

5. NUMERICAL EXAMPLES

We now turn to examining some of the kinds of DAEs that can arise. In this section, we give three numerical examples. In Section 6, we look at the singularity induced bifurcation.

5.1. Example with κ and η only

Suppose κ and η only are nonzero. In the planar case one assumes that $u_4 = u_6 = 0$ in (11). Thus, we have a DAE in u_2 and u_5 only. Let

$$u^l = [8, 0.1, 2, -0.1, 10], \qquad B^x = 2, \quad \gamma = 1.4, \quad s = 1, \quad \kappa = 1, \quad \eta = 0.1. \tag{17}$$

Then we obtain four equilibria. Three of them (two saddles and a node) are in the *subsonic* region and one saddle is in the *supersonic* region. The regions are separated by the singularity manifold: $h(u_2, u_5) = \kappa\eta(1.5463 - 2u_2 + 0.06944u_5^2) = 0$. The phase portrait is shown in Figure 1. In Figures 1, 5, and 6, the $h = 0$ curve is indicated by the curve of large dots. The nearby trajectories move away from the singularity surface so that equilibria in the two regions cannot be connected.

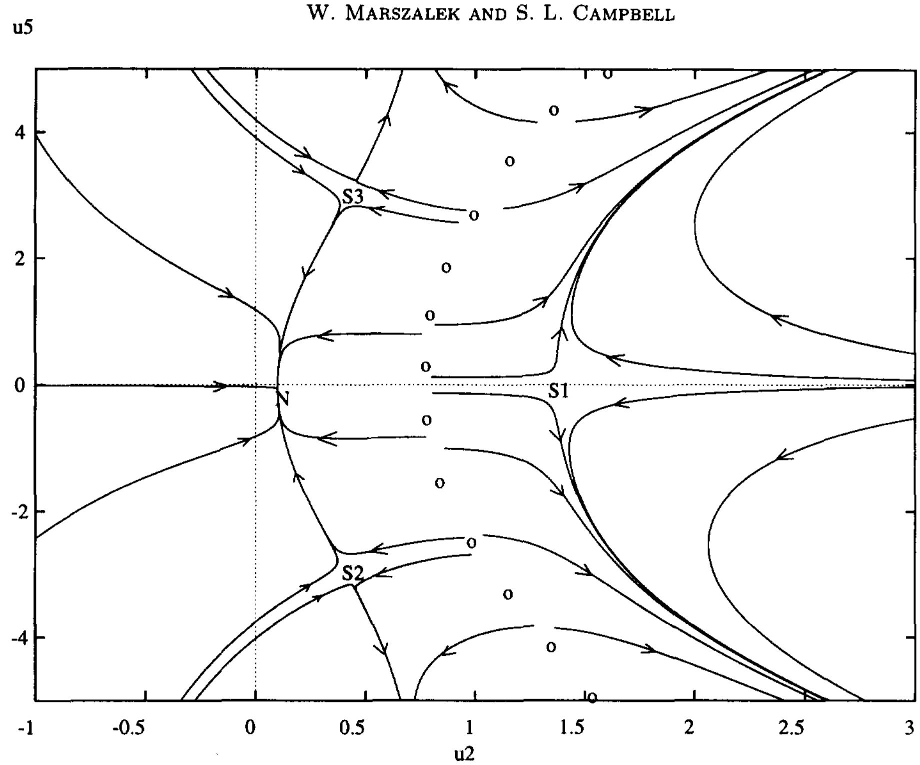

Figure 1. The phase portrait for Example 5.1.

5.2. Example with η Only: Equilibria on Different Components

One interesting feature of the MHD DAEs is that the solution manifold need not be connected. The one example in [10] had all equilibria on the same component. However, it is possible for the solution manifold to not be connected and the equilibria to be on different components.

Consider nonzero η only and the planar case with $u^l = [1, 0.5, 0.2, -1, 10]$, $B^x = 2$, $\gamma = 1.4$, $s = 1$. The constraint manifold consists of two components as shown in Figure 2. There are two equilibria u^l and $[0.122, -3.093, 3.878, -1.920, 4.853]$ which lie on different components so that no smooth solution exists between them.

5.3. Example with Varying Connectivity

The next example is very interesting in that it shows how the topology of the solution manifold can vary with changes in the parameters. As Figures 3–7 show, varying the correct parameter causes the "egg" shaped component to distort, then touch the other component, and then form one component. By choosing parameter values close to the touching value the behavior of integrators can be carefully examined. We have observed, for example in the almost touching state, numerical methods jump from one component onto the other and start tracking a different solution. Correct interpretation of this numerical solution is only possible with an understanding of the topology of the solution manifold.

We begin with the parameter values of $u^l = [15, 0.5, 0.2, 0.2, -1, -1.10]$, $B^x = 2$, $\gamma = 1.4$, and $s = 1$. This produces the solution manifold given in Figure 3.

We now vary the parameter u_1^l. Figures 3–7 show the solution manifold change connectivity.

6. SINGULARITY INDUCED BIFURCATION

As shown in the previous section, the existence of traveling waves in MHD DAEs is often restricted by the presence of singularities which do not allow for smooth connections of equilibria on different sides of the singularity. An interesting case is when an equilibrium is at the singularity. If a DAE depends on a parameter, such as one of the components of the left equilibrium, then it

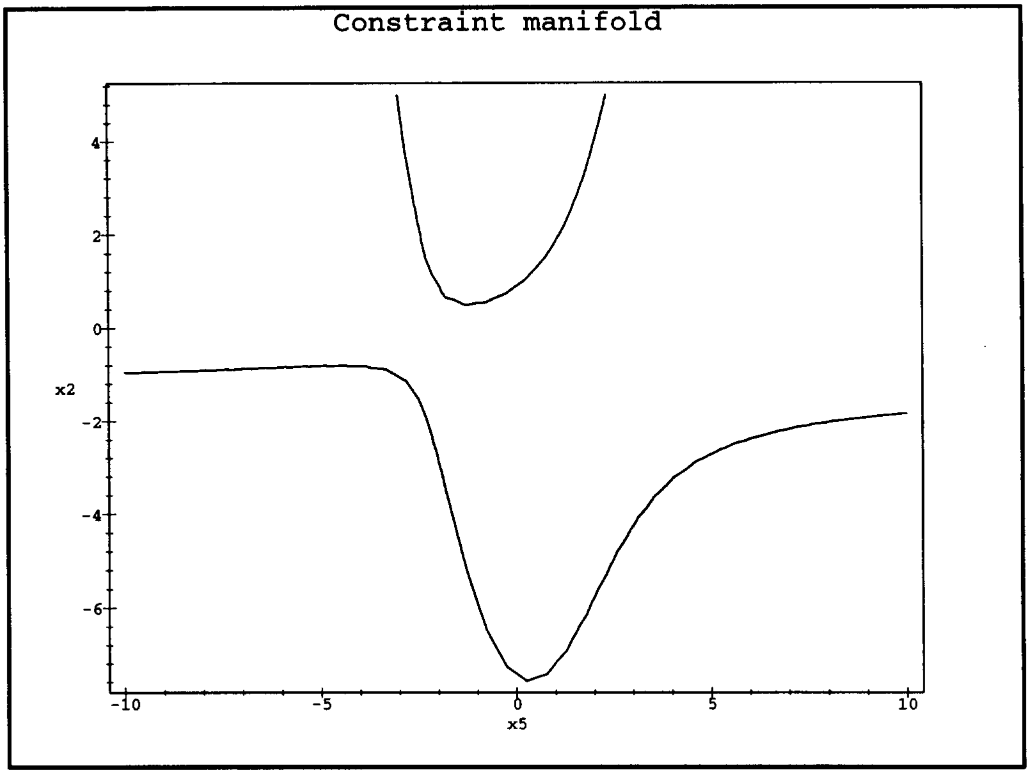

Figure 2. Equilibria on disjoint components.

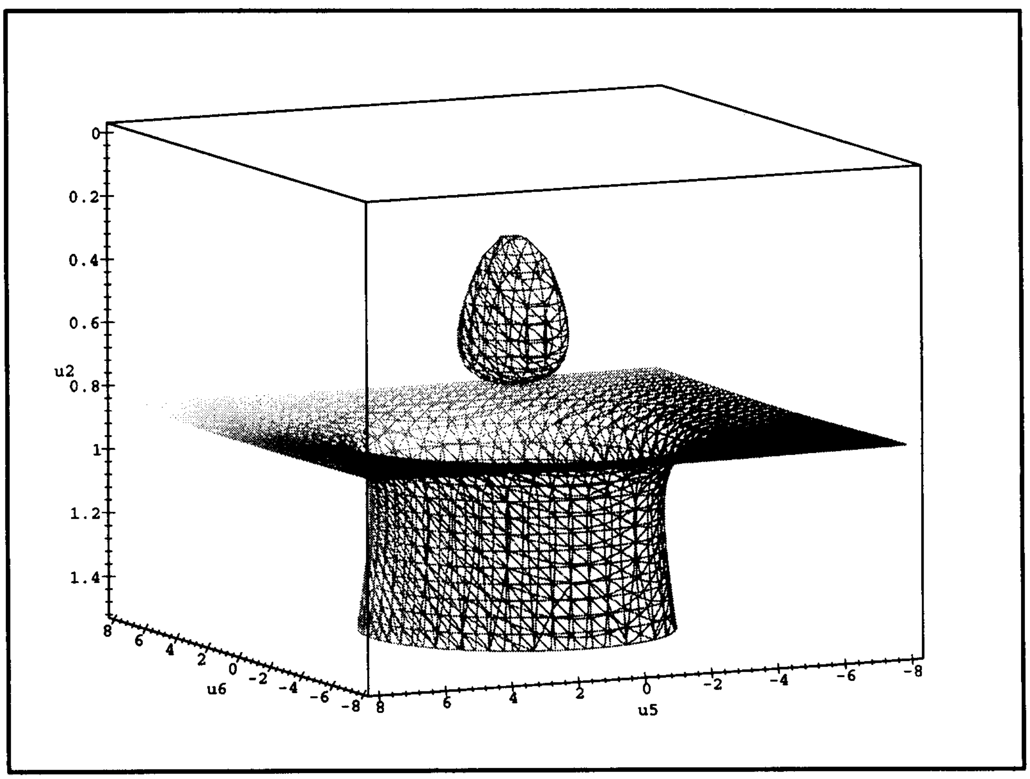

Figure 3. Example 5.3 with $u_1^l = 15$.

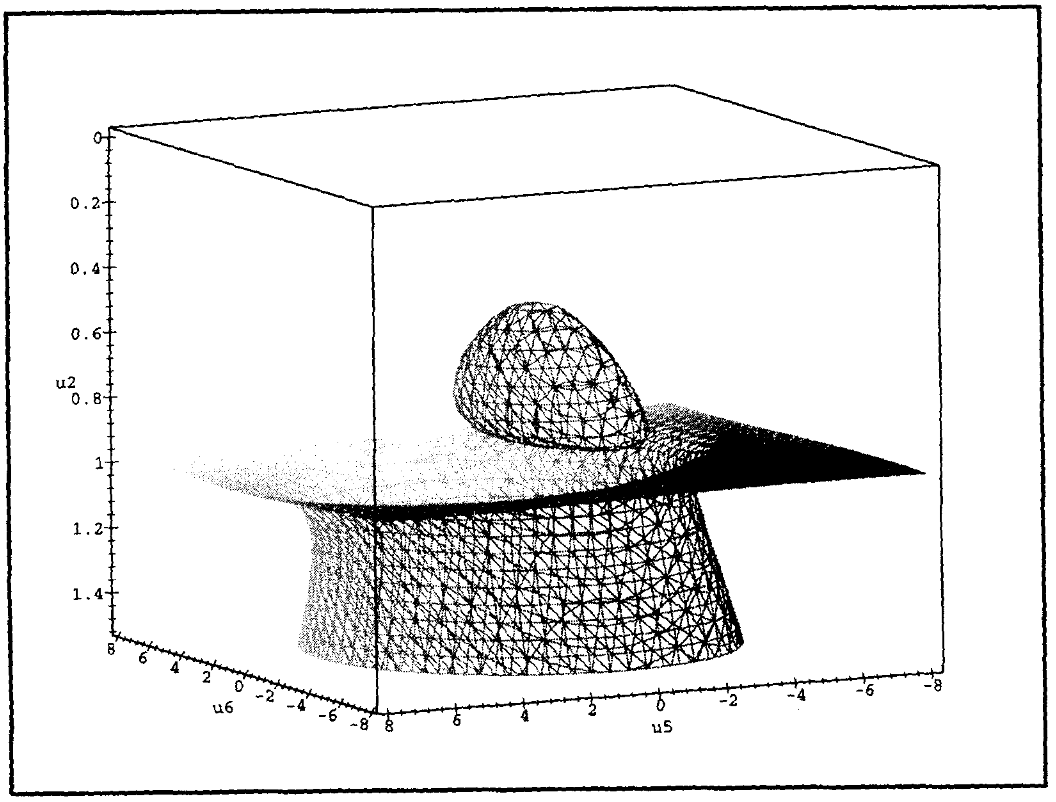

Figure 4. Example 5.3 with $u_1^l = 35$.

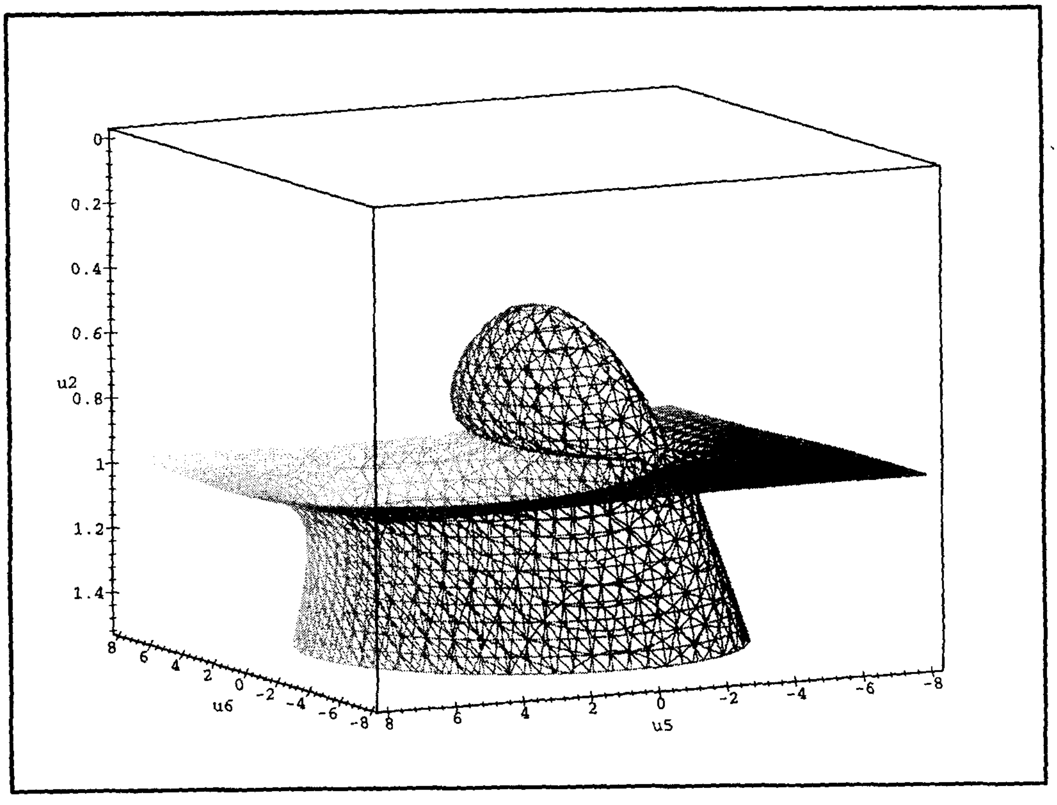

Figure 5. Example 5.3 with $u_1^l = 37.5$.

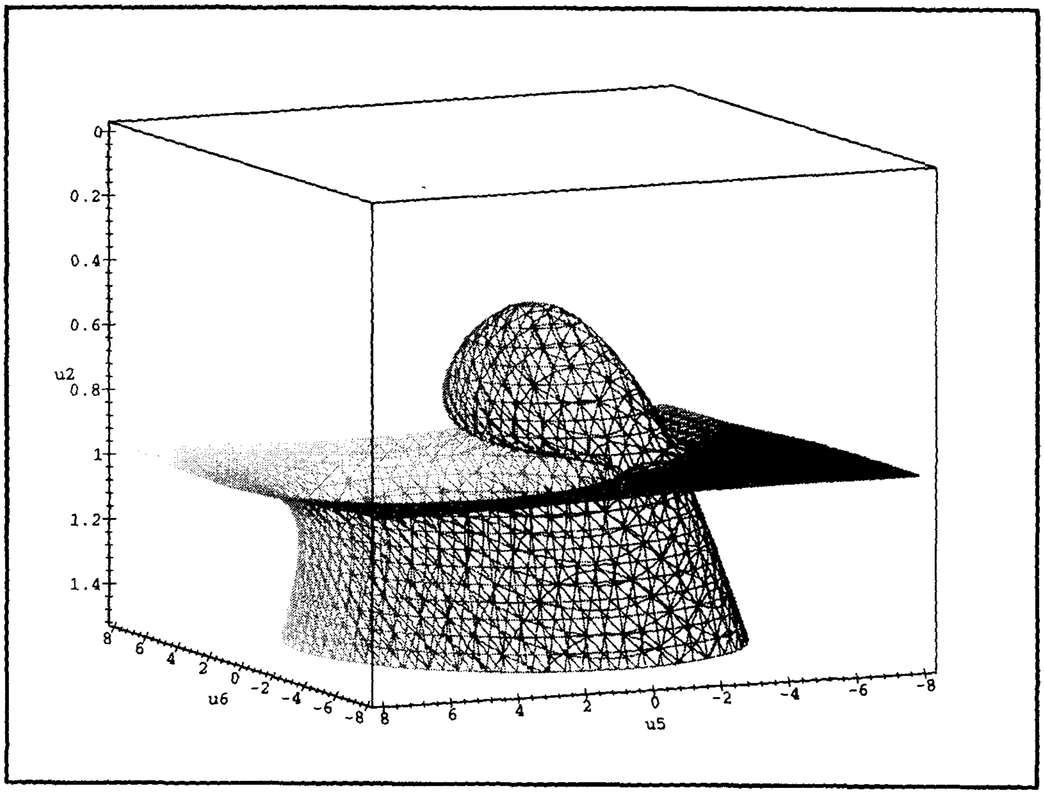

Figure 6. Example 5.3 with $u_1^l = 40$.

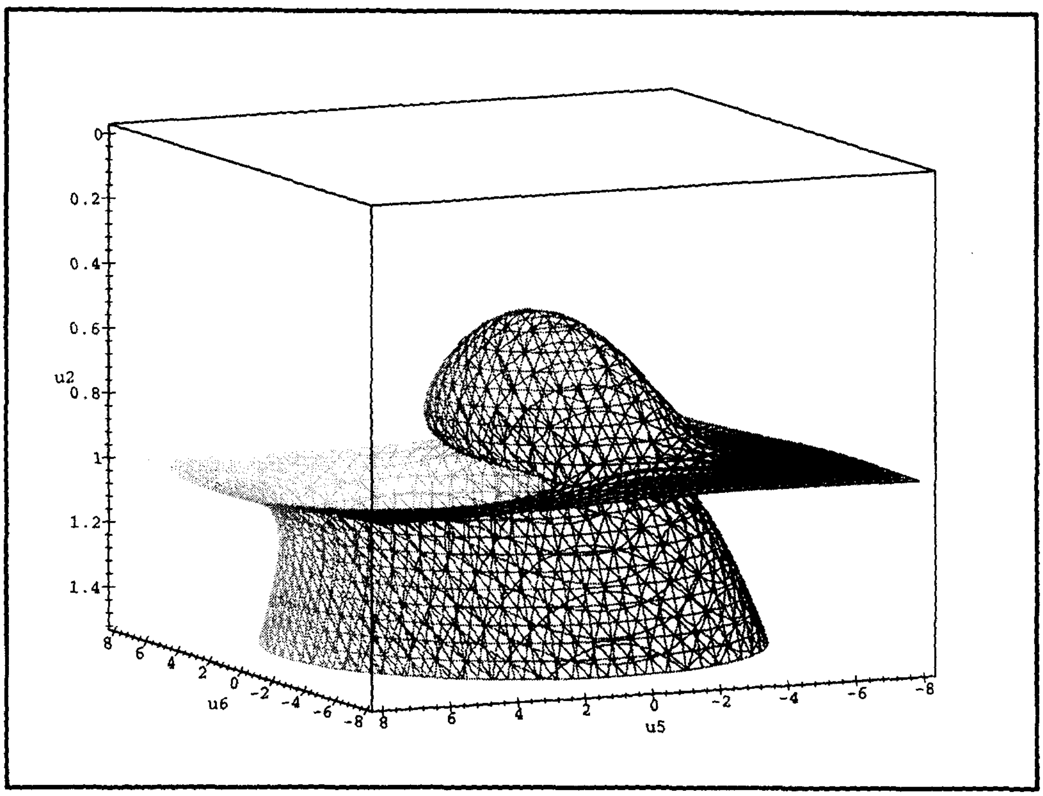

Figure 7. Example 5.3 with $u_1^l = 50$.

may happen that by changing this parameter we are able to shift an equilibrium to the singularity. This problem is known in the DAE literature as a singularity induced bifurcation and has been applied in the last few years to the analysis of electric power systems [21,22]. Knowing which values of the parameters causes a singularity induced bifurcation is very useful in developing test problems, since for parameter values near the critical one, trajectories can be hard to determine numerically as is stability. The following theorem from [21] is the basis of our analysis.

THEOREM 4. *(See [21].) Consider a parameter dependent DAE*

$$u' = f(u, v, p), \tag{18a}$$

$$0 = g(u, v, p), \tag{18b}$$

with $f : \mathbf{R}^{n+m+q} \to \mathbf{R}^n$, $g : \mathbf{R}^{n+m+q} \to \mathbf{R}^m$, $u \in U \subset \mathbf{R}^n$, $v \in V \subset \mathbf{R}^m$, $p \in P \subset \mathbf{R}$. *If* $\Delta(u, v, p) \equiv \det[\frac{\partial g(u,v,p)}{\partial v}]$, *and*

1. $f(0, 0, p_0) = 0$, $g(0, 0, p_0) = 0$, $\frac{\partial g}{\partial v}$ *has a simple zero and* trace $[\frac{\partial f}{\partial v} \operatorname{adj}(\frac{\partial g}{\partial v}) \frac{\partial g}{\partial u}] \neq 0$,
2.
$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}$$
is nonsingular,
3.
$$\begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial p} \\ \frac{\partial \Delta}{\partial u} & \frac{\partial \Delta}{\partial v} & \frac{\partial \Delta}{\partial p} \end{bmatrix}$$
is nonsingular,

then there exists a smooth curve of equilibria in $\mathbf{R}^{n+m+1}$ *which passes through* $(0, 0, p_0)$ *and is transversal to the singular surface at* $(0, 0, p_0)$. *When* p *increases through* p_0, *one eigenvalue of the system, (i.e., an eigenvalue of*

$$J = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} \left(\frac{\partial g}{\partial v}\right)^{-1} \frac{\partial g}{\partial u} \tag{19}$$

evaluated along the equilibrium locus) moves from $\mathbf{C}^-$ *to* $\mathbf{C}^+$ *if* $b/c > 0$ *(respectively, from* $\mathbf{C}^+$ *to* $\mathbf{C}^-$ *if* $b/c < 0$*) along the real axis by diverging through* ∞. *The other* $n-1$ *eigenvalues remain bounded and stay away from the origin. The constants* b *and* c *can be computed by evaluating*

$$b = -\operatorname{trace}\left[\frac{\partial f}{\partial v} \operatorname{adj}\left(\frac{\partial g}{\partial v}\right) \frac{\partial g}{\partial u}\right], \tag{20a}$$

$$c = \frac{\partial \Delta}{\partial p} - \left[\frac{\partial \Delta}{\partial u}, \frac{\partial \Delta}{\partial v}\right] \begin{bmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial p} \end{bmatrix}. \tag{20b}$$

Note that when an equilibrium is placed at the singularity, then one of its eigenvalues changes from either $-\infty$ to $+\infty$ (or from $+\infty$ to $-\infty$). Intuitively, this means that a solution approaches and leaves the equilibrium with an infinite speed. As a consequence of Theorem 4 and the analysis in [21], we have the following corollary.

COLLARARY 1. *If a DAE satisfies the conditions given in Theorem 4, then there always exists a trajectory to and from the equilibrium placed at the singularity.*

PROOF. Suppose p is such that the equilibrium is at the singularity. Solutions of the DAE (18) are included in those of

$$u' = f(u, v, p), \tag{21a}$$
$$0 = g_u u' + g_v v', \tag{21b}$$

or equivalently

$$u' = f(u, v, p), \tag{22a}$$
$$v' = -(g_v)^{-1} g_u f. \tag{22b}$$

Let $\Delta = \det(g_v)$. The trajectories of (22) away from $\Delta = 0$ are also trajectories of

$$u' = \Delta f(u, v, p), \tag{23a}$$
$$v' = -\Delta (g_v)^{-1} g_u f \tag{23b}$$

away from $\Delta = 0$. The assumptions give us a nontrivial trajectory of (23) with $\Delta \neq 0$ along the trajectory which goes to and from the equilibrium on the singularity surface $\Delta = 0$. This in turn gives us a trajectory of (22) which goes toward the singularity. However, $g = c$ is an invariant of (22). Since $g = 0$ by construction at the equilibrium we must have $g = 0$ along the trajectory. But then this trajectory is a solution of (21). ∎

The theory of [21] does not say that the trajectories of Corollary 1 exit or arrive in finite time. However, this turns out to be true for special cases of the MHD equations. This is due to a square root type of local behavior of the vector field along the trajectory and will be proven for the special case that follows.

In the two examples below, we will show how to apply Theorem 4 in the case of $n = 2$, $m = p = 1$, and $n = m = p = 1$, respectively. Theorem 4 can easily be applied directly to the DAE (11). We will consider the special cases of Section 5 since that permits us to get a nice graphical representation.

6.1. Singularity Induced Bifurcation with η Only

Consider the traveling wave MHD DAE with nonzero η only. Such a system can be reduced to form (15). By solving the system of four nonlinear equations,

$$f_1(u_5, u_2, p) = 0, \tag{24a}$$
$$f_2(u_6, u_2, p) = 0, \tag{24b}$$
$$g(u_5, u_6, u_2, p) = 0, \tag{24c}$$
$$g_{u_2}(u_5, u_6, u_2, p) = 0, \tag{24d}$$

one is able to place equilibria at the singularity.

Suppose that the singularity induced bifurcation theorem holds at this equilibrium point which, as we shall shortly see, is often the case. By Corollary 1 there are trajectories going in and out of the equilibrium. We wish to see whether they do so in finite time. Using equation (11a), we get that (11c)–(11f) become

$$c_1 u_3 - B^x u_5 - c_3 = 0, \tag{25a}$$
$$c_1 u_4 - B^x u_6 - c_4 = 0, \tag{25b}$$
$$-s u_5 + u_2 u_5 - B^x u_3 - c_5 = \eta u_5', \tag{25c}$$
$$-s u_6 + u_2 u_6 - B^x u_4 - c_6 = \eta u_6'. \tag{25d}$$

Suppose that $c_1 \neq 0$. Then we can solve (25a) and (25b) for u_3, u_4 to get

$$-su_5 + u_2u_5 - B^x c_1^{-1}\left[B^x u_5 + c_3\right] - c_5 = \eta u_5', \tag{26a}$$
$$-su_6 + u_2u_6 - B^x c_1^{-1}\left[B^x u_6 + c_4\right] - c_6 = \eta u_6'. \tag{26b}$$

This leads to the following result.

LEMMA 1. *Suppose that $c_1 \neq 0$. Then there exists constants α_1, α_2, not both zero, such that $\alpha_1 u_5 + \alpha_2 u_6 = 0$ is invariant under the solutions of (11).*

PROOF. Let $z = \alpha_1 u_5 + \alpha_2 u_6$. Then from (26), we have that $z' = [-s + u_2 - c_1^{-1}(B^x)^2]z + \phi$ where $\phi = -\alpha_1(c_1^{-1}c_3B^x + c_5) - \alpha_2(c_1^{-1}c_4B^x + c_6)$. Take α_1, α_2 so that ϕ is zero. Then z satisfies a linear homogeneous differential equation

$$z' = \left[-s + u_2 - c_1^{-1}(B^x)^2\right] z. \tag{27}$$

If $z(t)$ is zero for some t_0, then it is zero for all t. ∎

Now suppose that at the equilibrium the value of u_2 is such that $[-s + u_2 - c_1^{-1}(B^x)^2] \neq 0$. Suppose that the trajectories given by the singularity induced bifurcation theorem do not reach or leave the origin in finite time. Then we can conclude by integrating either to the singularity, or backward in time to it depending on the sign of $[-s + u_2 - c_1^{-1}(B^x)^2]$, that the singularity lies on $z = 0$. We focus now on the trajectories on $z = 0$. On this curve, we can solve for one of u_5, u_6 in terms of the other. We assume it is u_5. Then we get that (15) reduces to a system in just u_5, u_2. We translate the equilibrium to the origin. Keeping the same name for our new variables and using the fact that (15c) is quadratic in u_5 and u_6 with no products of these terms, we have

$$u_5' = \alpha u_5 + \beta u_2 + u_5 u_2, \tag{28a}$$
$$0 = u_2^2 + \left(a + bu_5 + cu_5^2\right) u_2 + du_5 + eu_5^2. \tag{28b}$$

The additional requirement that the origin is a singularity gives $a = 0$ so that

$$u_5' = \alpha u_5 + \beta u_2 + u_5 u_2, \tag{29a}$$
$$0 = u_2^2 + \left(bu_5 + cu_5^2\right) u_2 + du_5 + eu_5^2. \tag{29b}$$

We are considering the case where the origin lies on a manifold defined by (29b). Thus, (29b) must have real solutions for u_2 for u_5 near zero. Thus,

$$\left(bu_5 + cu_5^2\right)^2 - 4\left(du_5 + eu_5^2\right) \geq 0$$

for u_5 near zero. Assume $d \neq 0$. The largest term is $-4du_5$. Thus, we must have $-4du_5 > 0$. Then u_5 does not change sign near the origin. Let $u_5 = \kappa v^2$ where κ is either 1 or -1 depending on the sign of u_5. Then we have

$$2\kappa v v' = \kappa \alpha v^2 + \beta u_2 + \kappa v^2 u_2, \tag{30a}$$
$$0 = u_2^2 + (b\kappa v^2 + cv^4)u_2 - |d|v^2 + ev^4. \tag{30b}$$

From (30b), we then get that $u_2 = 2\sqrt{|d|}v + o(v)$ and (30a) becomes $2v' \approx \alpha v + \kappa\beta\sqrt{|d|} + vu_2$. But $v \to 0$ and u_2 is bounded near the equilibrium. Thus along the trajectory, near the equilibrium (origin), we have that v' is bounded away from zero. Thus, the v trajectories leave and arrive in finite time. Hence, the same holds for u_5. To summarize, we have the following.

THEOREM 5. *Suppose that only $\eta \neq 0$. Suppose that the equilibrium u^e is placed at the singularity by choosing u_2^l. Let u^l be such that $c_1 \neq 0$ in (11) . Suppose that u_5^e and u_6^e are nonzero and $d \neq 0$ in (28). Then there exist solutions of (11) which reach and leave u^e in finite time.*

6.1.1. A specific example

We will illustrate the above analysis by considering a particular example. Let the left state be

$$u^l = [p, 0.5, 0.2, 0.2, 1, 1, 10], \qquad B^x = 2, \quad \gamma = 1.4, \; s = 1, \tag{31}$$

where p is a parameter (it represents density at the left equilibrium). For the particular choice of u^l in (31), the traveling wave MHD DAE is

$$u_5' = -u_5 + 0.5 + u_5 u_2 + \frac{8(u_5 - 1)}{p}, \tag{32a}$$

$$u_6' = -u_6 + 0.5 + u_6 u_2 + \frac{8(u_6 - 1)}{p}, \tag{32b}$$

$$\begin{aligned} 0 = {} & 1.5u_2^2 p - 2.356 u_2 p + 13.3 u_2 - 1.75 u_5^2 u_2 - 1.75 u_6^2 u_2 + 1.75 u_5^2, \\ & + 1.75 u_6^2 - 0.5 u_5 - 0.5 u_6 + 0.803 p - 7.4 + \frac{(8u_5 + 8u_6 - 4u_5^2 - 4u_6^2 - 8)}{p}. \end{aligned} \tag{32c}$$

One can check that all conditions in Theorem 4 are satisfied.

Solving (24), we obtain the bifurcation parameter $p_0 = 17.97829718$ together with the following equilibrium at the singularity: $[u_2, u_5, u_6] = [0.6082, -1.0346, -1.0346]$. Note that in general the solution of (24) is not unique. The other possible solutions for p_0 are: $\{3.941980273, 11.44859813, 183.7727649\}$. Each of these p_0's corresponds to different equilibrium being moved to the singularity. Table 1 illustrates the location of one of the equilibria and values of its eigenvalues when p changes between 17.50 and 18.50 during the bifurcation process. The DAE (32) has three other equilibria. The constraint manifold for $p = p_0$ is similar to that of Figure 3. Both the constraint and singularity manifolds for $p = p_0$ are shown in Figure 8.

Note that the left state does not move with the change of p. For $p < p_0$, three of the equilibria lie on the supersonic branch of the constraint manifold, whereas for $p > p_0$, we have two equilibria on each (subsonic and supersonic) branch.

Table 1. Equilibrium and its eigenvalues during bifurcation.

Parameter p	Equilibrium: (u_2, u_5, u_6)			Eigenvalues	
17.85	0.6018	−1.0360	−1.0360	7.6068	0.0500
17.95	0.6068	−1.0350	−1.0350	33.5211	0.0525
17.97	0.6079	−1.0347	−1.0347	113.6500	0.0530
17.97829718	0.6082	−1.0346	−1.0346	$\pm\infty$	0.0533
18.00	0.6092	−1.0343	−1.0343	−43.0570	0.0537
18.20	0.6191	−1.0306	−1.0306	−3.9483	0.0586

In Lemma 1, we can take $\alpha_1 = 1$, $\alpha_2 = -1$. For this problem, $u_2 \to 0.6082$. Thus, $z = u_5 - u_6$ satisfies (27) which is $z' = (7 - u_2)z$. All of the conditions of Theorem 5 are met and there are trajectories reaching and leaving the equilibrium in finite time.

6.2. Singularity Induced Bifurcation with κ and η Only

Consider now the case when κ and η are nonzero (Section 5.1). The MHD DAE has the following structure in the planar case:

$$\kappa\eta h(u_2, u_5, p) u_2' = f(u_2, u_5, p), \tag{33a}$$

$$\eta u_5' = g(u_2, u_5, p), \tag{33b}$$

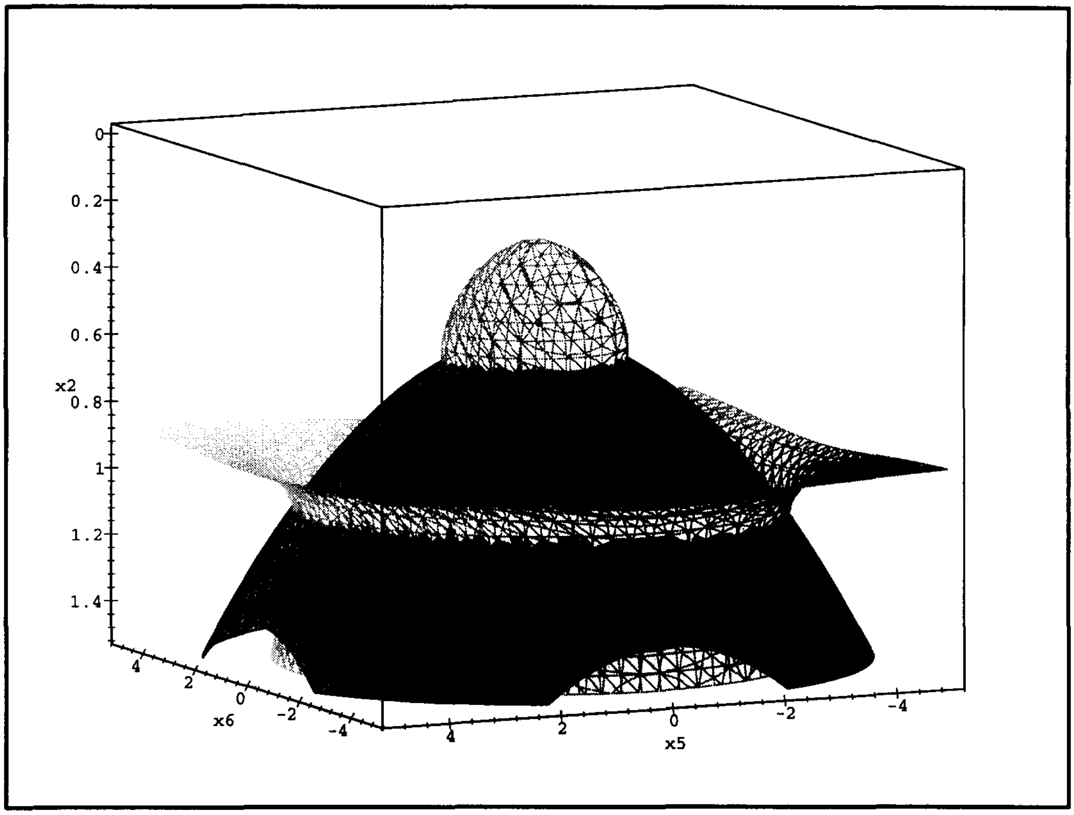

Figure 8. The constraint and singularity manifolds for $p = p_0$.

where $h(u_2, u_5, p) = 0$ defines the singularity manifold and h, f, and g are polynomial functions. We can place an equilibrium at the singularity by solving the system

$$h(u_2, u_5, p) = 0, \tag{34a}$$

$$f(u_2, u_5, p) = 0, \tag{34b}$$

$$g(u_2, u_5, p) = 0. \tag{34c}$$

Let the planar MHD system have the following left state and parameter values:

$$u^l = [p, 0.5, 0.2, 1, 10], \qquad B^x = 2, \quad \gamma = 1.4, \quad s = 1,$$

and $\kappa = 1$, $\eta = 1$. Then (33) is

$$\begin{aligned}\left(250pu_2 - 202p - 125u_5^2 + 875\right) u_2' = {} & -0.125 \left(-20000u_5^2u_2^2p + 1500u_2^2p^3 - 160000u_5^2u_2\right.\\ & +40000u_5^2u_2p - 2328u_2p^3 + 160000u_2u_5 - 1750u_5^2u_2p^2\\ & +12250u_2p^2 - 10000u_2u_5p + 160000u_5^2 + 18000u_5p - 24000u_5^2p\\ & \left. - 4000p + 1750u_5^2p^2 - 160000u_5 + 789p^3 - 6500p^2 - 500u_5p^2\right),\end{aligned} \tag{36a}$$

$$u_5' = -10u_5 + 10u_2u_5 + 5 + \frac{80(u_5 - 1)}{p}. \tag{36b}$$

One of the possible bifurcation values of p is $p_0 = 18.72228883$ for which a saddle is placed at the singularity. The phase portrait shown in Figure 9 (and enlarged in Figure 10) indicates a possible connection between a saddle S_1 and a node N lying in separate regions divided by the

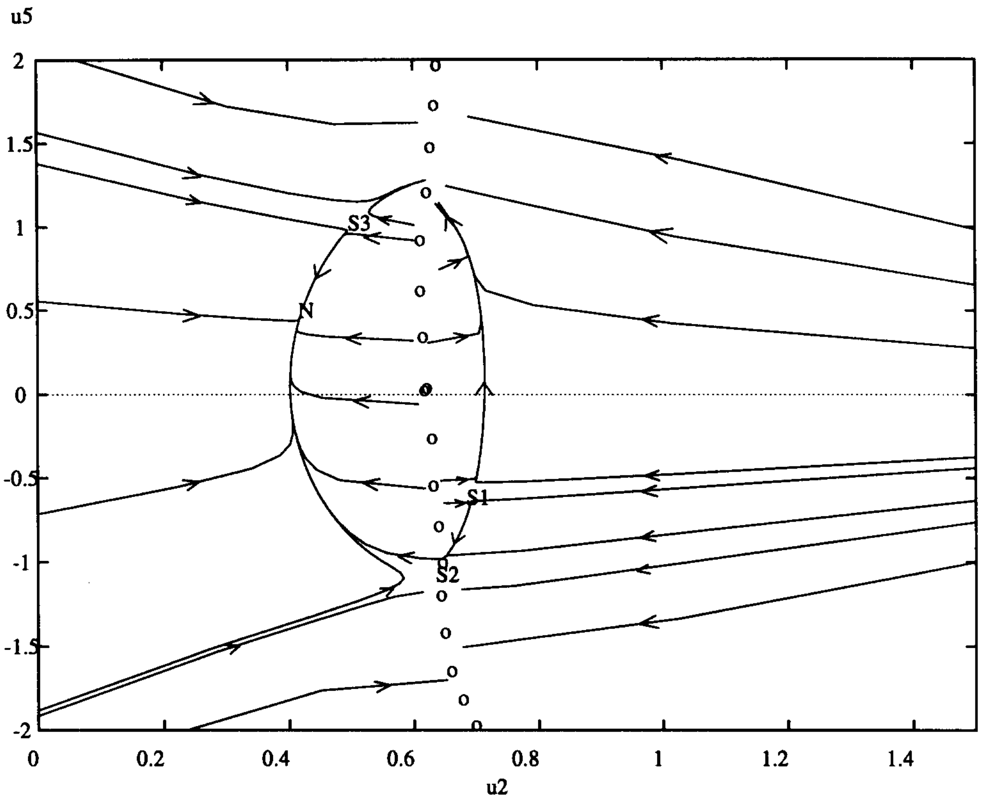

Figure 9. The phase portrait of Example 6.2 with saddle S_2.

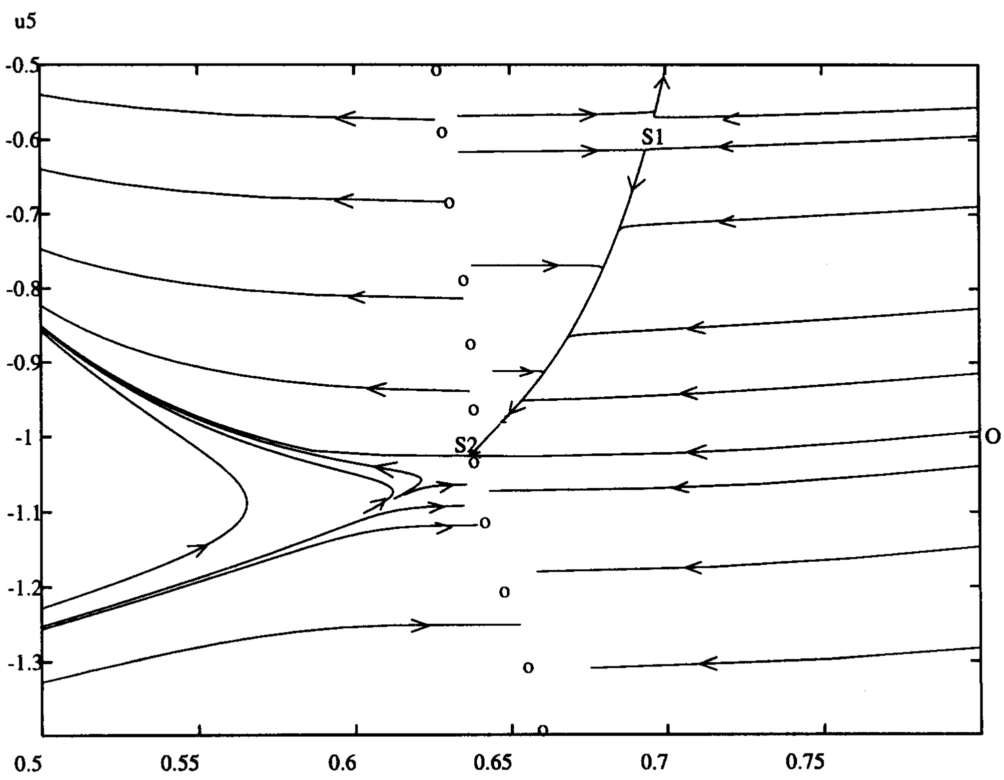

Figure 10. The phase portrait zoomed in around saddle S_2.

singularity curve $h = 0$. The connection is via an intermediate equilibrium (saddle S_2) placed at the singularity. The stable manifold of that saddle is not changed during the bifurcation process, but the unstable manifold is changed so that divergence of the corresponding eigenvalue through $\pm\infty$ occurs. In this particular example, there exist two traveling wave solutions: one going through the singularity from saddle S_1 to node N and the other from saddle S_3 to node N without crossing the singularity.

It is also interesting to note that placing a node with two stable eigenvalues at the singularity changes one of the eigenvalues so that it is $-\infty$ on one side of the singularity and $+\infty$ on the other one. Therefore, both a node and a saddle have the same eigenvalue structure when placed at the singularity. Another interesting property of this system is an invariant manifold $O - S_2 - N$ which crosses the singularity manifold at the saddle S_2.

7. CONCLUSIONS

The traveling wave solution of a system of conservation laws naturally leads to DAEs. A family of DAEs that arise in this way from the MHD equations have been examined. These DAEs exhibit a variety of behavior that makes them good test problems for DAE integrators. They usually have singularities which prevent traveling waves from connecting some of the equilibria. Solution manifolds can change topology as parameters vary. Components can be made close together. However, by placing an equilibrium at the singularity and satisfying rather mild conditions (Theorem 4) one is sometimes able to create solutions through the singularity and connect equilibria on both sides of singularity. In some cases, the equilibrium at the singularity is reached and left in finite time.

APPENDIX

In the proof below of Theorem 3, only the most important responses from MAPLE are given, that is, formulas for $u1$, $u2$, $u3$, $u4$, $u5$, and $u7$ as functions of $u6$. Some other intermediate formulas are very long. Other choices than u_6 are possible.

In what follows, we do all calculations symbolically for the seven arbitrary numbers $u^l = \{ul1, \dots, ul7\}$, wave speed s, and constant parameters gam (γ), Bx (B^x). We compute first the 'left' value of Pstar ($= P^*$), where Pstar is given by (10). Note that p (static pressure) is eliminated from the two equations given below using (10)

$$P\text{star}\, l := (\text{gam} - 1) * \left(ul7 - \frac{(Bx^2 + ul5^2 + ul6^2)}{2} - ul1 * \frac{(ul2^2 + ul3^2 + ul4^2)}{2} \right) + \frac{(Bx^2 + ul5^2 + ul6^2)}{2}.$$

The general expression for Pstar in terms of u is

$$P\text{star} := (\text{gam} - 1) * \left(u7 - \frac{(Bx^2 + u5^2 + u6^2)}{2} - u1 * \frac{(u2^2 + u3^2 + u4^2)}{2} \right) + \frac{(Bx * Bx + u5 * u5 + u6 * u6)}{2}.$$

Solve the first equation in (11) for $u1$: $u1 := \text{solve}\,(-s * u1 + s * ul1 + u1 * u2 - ul1 * ul2 = 0, u1)$.

Substitute $u1$ into the third equation (11c) with the right-hand side (RHS) set equal to 0 : $u3 := \text{solve}\,(\text{simplify}\,(-s*u1*u3+s*ul1*ul3+u1*u2*u3-Bx*u5-ul1*ul2*ul3+Bx*ul5) = 0, u3)$;

$$u3 := -\frac{s\, ul1\, ul3 - Bx\, u5 - ul1\, ul2\, ul3 + Bx\, ul5}{-s\, ul1 + ul1\, ul2}.$$

Substitute ul into the fourth equation (11d) with the RHS $= 0$: $u4 :=$ solve (simplify $(-s * ul * u4 + s * ul1 * ul4 + ul * u2 * u4 - Bx * u6 - ul1 * ul2 * ul4 + Bx * ul6) = 0, u4)$;

$$u4 := -\frac{s\, ul1\, ul4 - Bx\, u6 - ul1\, ul2\, ul4 + Bx\, ul6}{-s\, ul1 + ul1\, ul2}.$$

Substitute ul, $u3$, and $u4$ into the above formula for Pstar

$$P\text{star} := \text{simplify}\left((\text{gam} - 1) * \left(u7 - \frac{(Bx^2 + u5^2 + u6^2)}{2} - ul * \frac{(u2^2 + u3^2 + u4^2)}{2}\right) + \frac{(Bx * Bx + u5 * u5 + u6 * u6)}{2}\right).$$

Use this value of Pstar and solve the second equation in (11) (with the RHS $= 0$) for $u7$.
$u7 :=$ solve $(-s * ul * u2 + s * ul1 * ul2 + ul * u2 * u2 + P\text{star} - ul1 * ul2 * ul2 - P\text{star}\, l = 0, u7)$.
Use $u3$ computed above and solve the fifth equation in (11) (with the RHS $= 0$) for $u2$.
$u2 :=$ simplify (solve $((-s * (u5 - ul5) + u5 * u2 - Bx * u3 - (ul2 * ul5 - Bx * ul3)) = 0, u2)$).
Use the $u2$ computed above and solve the sixth equation in (11f) (with the RHS $= 0$) for $u5$.
$u5 :=$ simplify (solve $((-s * (u6 - ul6) + u6 * u2 - Bx * u4 - (ul2 * ul6 - Bx * ul4)) = 0, u5)$);

$$u5 := \frac{u6\, ul5}{ul6}.$$

Use $u5$ to simplify $u2$
$u2 :=$ simplify $(u2)$;

$$u2 := \left(-u6\, s^2\, ul1 + u6\, s\, ul1\, ul2 + s^2\, ul1\, ul6 - 2\, s\, ul1\, ul2\, ul6 + u6\, Bx^2 - Bx^2\, ul6 + ul2^2\, ul6\, ul1\right) \\ (ul1\,(-s + ul2)\, u6).$$

Simplify $u3$ so that now $u3$ is a function of $u6$: $u3 :=$ simplify $(u3)$;

$$u3 := \frac{-s\, ul1\, ul3\, ul6 + Bx\, u6\, ul5 + ul1\, ul2\, ul3\, ul6 - Bx\, ul5\, ul6}{(-s + ul2)\, ul6\, ul1}.$$

Simplify ul so that now ul is a function of $u6$: $ul :=$ simplify (ul);

$$ul := \frac{ul1^2\,(-s + ul2)^2\, u6}{s^2\, ul1\, ul6 - 2\, s\, ul1\, ul2\, ul6 + u6\, Bx^2 - Bx^2\, ul6 + ul2^2\, ul6\, ul1}.$$

Simplify Pstar so that now Pstar is a function of $u6$: Pstar := simplify (Pstar).

Simplify $u7$ so that now $u7$ is a function of $u6$: $u7 :=$ simplify $(u7)$.

Since ul, $u2$, $u3$, $u4$, $u5$, and $u7$ are given above (all are functions of $u6$), we can use the seventh equation in (11) (with the RHS $= 0$) and simplify it so that its LHS will be a polynomial in $u6$ only, say poly $(u6)$. Then solving the equation poly $(u6) = 0$ one gets several equilibria for $u6$. The corresponding values for ul, $u2$, $u3$, $u4$, $u5$, and $u7$ can be easily found by using the formulas derived above. The poly $(u6)$ is of fourth degree as the following calculation shows: $f(u6) :=$ simplify $(-s * u7 + s * ul7 + (u7 + P\text{star}) * u2 - Bx * (Bx * u2 + u3 * u5 + u4 * u6) - (ul7 + P\text{star}\, l) * ul2 + Bx * (Bx * ul2 + ul3 * ul5 + ul4 * ul6))$.

Finally, poly $(u6)$ =numerator of $f(u)$: poly $(u6) :=$ simplify $(f(u6) * (ul2 - s) * ul1 * ul6^2 * u6^2 * (\text{gam} - 1) * (-2))$.

Since deg [poly $(u6)$] $= 4$, we can have at most four equilibria of the traveling wave DAEs (11). Note that one of these equilibria is the assumed 'left' equilibrium, $ul6$. If the 'left' equilibrium is hyperbolic, then one can assure the existence of at least one more equilibrium.

Additional analysis shows that if $ul6 \neq 0$, then $ul6$ is a single root. Close examination of the 4^{th} degree polynomial poly $(u6)$ shows that its leading coefficient is nonnegative and equal to $Bx^2(ul6^2 + ul5^2)$. Also the constant coefficient of this polynomial can be written in the closed form $ul6^4(\text{gam}+1)[(s-ul2)^2 ul1 - Bx^2]^2$, that is, it is nonnegative. The above facts are equivalent to the fact that there exists at least one more root besides $ul6$ of poly $(u6)$. In addition, this additional root has the same sign as $ul6$ or equals zero since the constant and leading coefficients are both nonnegative. Therefore, the MHD traveling wave DAE has at most four and at least two equilibria.

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