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DIFFERENTIAL-ALGEBRAIC EQUATIONS: A TUTORIAL REVIEW

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This article[1] explores some introductory principles of differential-algebraic equations (DAEs) and makes a connection with the theory of dynamical systems. Some results which are new in the field of DAEs are also surveyed.

Most treatments on DAE emphasize the differences that exist when compared with the ODE case. Here we seek to underline the similarities so that readers with a very basic knowledge of nonlinear dynamics can understand some of their consequences in this more general context.

1. Differential-Algebraic Equations

1.1. *Some motivation*

When we study physical, biological, economic, chemical, electrical or other systems, the first part of any mathematical analysis is the modeling process. Sometimes there are underlying physical laws which the model must obey, sometimes the assumptions one makes are of a qualitative or empirical nature.

In many cases, one aims to find assumptions which capture the essence of the real system yet are sufficiently simple so that mathematical techniques can be employed on the resulting equations. However, would nature really be so kind as to always yield models which are linear in their derivatives? The notion that linear systems could completely describe physically realistic systems was rejected some time ago, but it is becoming increasingly clear that even classical nonlinear models are not sufficient to describe all the systems that scientists wish to analyze.

It is not necessary to dig so deep in order to find examples where, perhaps, mathematical simplicity is preferred to more meaningful physical assumptions. Take the humble pendulum as an example.

In elementary courses on applied mathematics one is introduced to the equation

$$\ddot{\theta} + \omega^2\theta = 0 \tag{1}$$

which, one is told, is "the pendulum equation". Students learn how to write down the orbits of this system and how they can be interpreted in a framework of small oscillations.

In the first year of undergraduate studies we are given

$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0 \tag{2}$$

and told that this is the real pendulum equation.

But masses never occupy a single point in space and rods are never infinitely stiff and the motion of the ball at the end of the rod must be seen as constrained to a circle, or else it must move on some manifold to which a circle is an approximation. If the tension T in the rod is included in the model then one arrives at the following system as an improved model of the pendulum,

$$\begin{aligned} \ddot{x} + Tx &= 0 \\ \ddot{y} + Ty + g &= 0 \\ x^2 + y^2 - l^2 &= 0\,. \end{aligned} \tag{3}$$

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We can see that this is a combination of dynamic equations and an algebraic constraint: a DAE. If one truly desires circular motion in the pendulum then the tension in the rod must change to permit the particle on the end of the rod to lie on a circle of radius l. Indeed, substitute into this DAE $x = l\sin\theta$ and $y = -l\cos\theta$ then one recovers the above ODE for the motion of the pendulum.

Of course, this development is just the usual procedure adopted in a modeling and analysis iteration which has led us to a differential-algebraic equation. It is always important to determine how accurate or realistic a model should be and when the modeling process can be terminated. However, in the current age of increasing computational power it is less important to have models which can be dealt with using "by-hand" techniques; many models now are just too large and too complex for analytic techniques to be of use.

For this reason it is important to develop new techniques which can deal with more general models than $\dot{x} = f(x)$, from both analytic and numerical viewpoints. Some are presented in the course of this article, with emphasis on how the new can be viewed as a natural generalization of the old.

1.2. *In the beginning*

Perhaps the most recognisible DAE is the system of equations

$$\dot{x}(t) = f(x(t)), \quad x_0 = x(0)\,, \tag{4}$$

which is a DAE with no algebraic constraints! Although this seems to be a frivolous example, it will be seen later that an ODE system can be viewed as a DAE system of "index zero" where the index is a measure of complexity of the DAE.[2] Suppose that some control is needed within the system modeled by our system of equations (4) and we want the particle to follow a prescribed path, as for the pendulum. One must find a function y such that

$$\dot{x}(t) = f(x(t),\, y(t)), \quad x_0 = x(0)\,, \tag{5}$$

and x follows the path in space. To model this situation, a constraint is introduced into the system of the form

$$g(x(t)) \equiv 0 \tag{6}$$

which describes this path. Examples of systems that may use such an approach are missile guidance systems and shuttle re-entry controllers whose designers may all encounter the coupled differential-constraint system (5) and (6).

Perhaps the type of control one uses depends on the path the particle can take. It is then possible that the control may form part of the constraint and one obtains the system

$$\begin{aligned} \dot{x}(t) &= f(x(t),\, y(t)) \\ 0 &= g(x(t),\, y(t)) \end{aligned} \tag{7}$$

where some initial conditions $(x_0,\, y_0)$ satisfying $g(x_0,\, y_0) = 0$ are given. This system arises in many areas of the applied sciences. In some sense, this is the simplest form of nonlinear DAE and is perhaps the best understood. We shall illustrate some of the phenomena of DAE which do not occur for ODE with this class of system as a prototype. There are many other classes of DAE, both linear and nonlinear.

Suffice to say that one can cast the general DAE problem as that of finding the solution to the implicit equation (the autonomous form of a general DAE)

$$F(x(t),\, \dot{x}(t)) = 0 \tag{8}$$

but some structure is needed to be imposed onto F to ensure that solutions exist. Assume throughout that $F : \mathbb{R}^{2m} \to \mathbb{R}^q$ where $m \le q \le 2m$.

The constrained nature of the equations means that, as is the case for any nonlinear algebraic equation, it is easy to write down inconsistent systems and this must be taken into account in software packages written to solve DAEs. Later on, various strucural assumptions will be made on F which allow progress to be made for realistic problems.

1.3. *Examples*

Here we present some differential systems which come under the umbrella encompassed by the definition of DAE.

1.3.1. *Semi-discretization schemes*

The Navier–Stokes equations model an incompressible, viscous fluid flow and are given as follows.

$$u_t + (u \cdot \nabla)u = -\nabla\rho + \gamma\delta u$$

$$\nabla \cdot u = 0$$

[2] To quote Steve Campbell, there is "a veritable zoo of indexes" of which we shall see but one.

Applying the method of lines (MOL) (see [Brenan *et al.*, 1989]) to this PDE yields the DAE

$$MU_s + (K + N(U))U + CP = f(U, P)$$
$$C^T \cdot U = 0\,.$$

It is seen in [Brenon *et al.*, 1989] that M is usually a nonsingular but sparse matrix. The reason for not multiplying through by M^{-1} is that we would then lose sparsity.

1.3.2. *Chemical engineering*

As can be found in [Kumer & Daoutidis, 1997; Byrne & Ponzi, 1988], it is possible to model many chemical processes by the nonlinear DAE

$$\dot{x} = F(x) + G(x)u + B(x)z$$
$$0 = K(x) + L(x)z\,.$$

Here, x is the differential variable, z is the algebraic variable, u is control and $y = h(x)$ is some output of the system.

In such systems some reactions often occur at a much greater rate than others. The system is then said to operate in a state of quasi-equilibrium as some of the dynamical processes are assumed to occur instantaneously. This introduces constraints into the system which gives the DAE formulation.

1.3.3. *Electrical power systems*

One typical application of DAEs arises with the application of Kirchoff's Law to electrical circuits. An example of such a set of equations is the following which represents an electrical power system [Kwatny *et al.*, 1995] where δ, P, Q and B are parameters and $(\theta, \omega, \phi, v) \in \mathbb{R}^2 \times \mathbb{R}^2$.

$$\begin{aligned}
\dot{\theta} &= \omega \\
\dot{\omega} &= -\omega - V\sin(\theta - \phi) - V\sin(\phi) + \delta \\
0 &= V(\sin(\phi) + \sin(\theta - \phi)) - P \\
0 &= -V(\cos(\phi) + \cos(\theta - \phi)) + (2 - B)V^2 - Q\,.
\end{aligned} \tag{9}$$

Such models of power systems have been analyzed since the 1960's and devices were used to bring the system into a form that could be understood. For instance, it was suggested that the way to analyze the above system, contained in the class of systems (7) would be to modify it and instead consider

$$\dot{x} = f(x, y)$$
$$\varepsilon\dot{y} = g(x, y)$$

and let $\varepsilon > 0$ be small. This gave the engineers the problem of deciding which ε to choose.

1.3.4. *Constrained mechanics*

The equations of rigid body motion are analyzed throughout the literature (e.g. [McPhee, 1996]). Here is a basic formulation to be found in [Simeon, 1997], modeled by the Euler–Lagrange equations,

$$M(p)\ddot{p} = f(p, \dot{p}, t) - G(p)^T\lambda$$
$$0 = g(p)\,,$$

where p denotes the position and orientation of all the bodies, $\lambda(t)$ is the vector of Lagrange multipliers, f defines external forces and M is symmetric and positive definite, with $G(p) = (d/dp)g(p)$. The constraint $g(p) = 0$ is known as a *holonomic constraint* involving a constraint in only one variable.

1.3.5. *Population dynamics*

Assume that a population has individuals which move with biased random walks on a finite domain $\Omega \subset \mathbb{R}^2$ such that each individual seeks to optimize its chances of reproduction, modeled by some function w. One obtains [Grindrod, 1996] the dimensionless system, after taking into account localized smoothing behavior in the populations

$$\begin{aligned}
u_t &= \delta\Delta u - \nabla\cdot(uw) + ruE(u) \\
0 &= d_uE[u](\nabla u) - w + \varepsilon\Delta w\,,
\end{aligned} \tag{10}$$

where $x \in \Omega$, $t \geq 0$ and r, ε, δ are parameters and E is some function. The stability properties of equilibria can be analyzed using comparison principles, but it is easier to view this system as a constrained system on a Banach space. Some results in this direction are presented in [Campbell & Marszalek, 1996]. The MOL applied to this system yeilds a DAE of the form (7).

There are many other applications of DAE which range from mathematical biology [Alscher, 1997] to molecular dynamics to computing space-time curvature.

2. Classifying DAEs

Let us now introduce some notation and terminology used to describe different classes of DAE.

2.1. *Linear DAEs*

Before embarking on a theory of general, nonlinear DAEs it is expedient to first treat the linear cases. So we define the *linear, constant coefficient DAE* to be the following system, assuming that A is a singular, linear mapping.

$$A\dot{x}(t) + Bx(t) = f(t)\,. \tag{11}$$

The *fully-implicit time varying* equivalent is obtained by setting $A = A(t)$ and $B = B(t)$ where $\det A(t) \equiv 0$.

The system

$$A(t, x(t))\dot{x}(t) = f(t, x(t)) \tag{12}$$

is called the *linearly implicit* or *semilinear* DAE. Equation (7) which has been described above is therefore a special case of this class of systems as that equation can be written in the form $M\dot{z} = F(z)$ where $\det(M) = 0$.

2.2. *Semi-explicit*

A system is called *semi-explicit* if it can be written in the form

$$F(\dot{x}(t), x(t), y(t)) = 0$$
$$G(x(t), y(t)) = 0\,.$$

There are many other forms of DAE which are described in great detail in [Brenan *et al.*, 1989]. In this article we shall concentrate on illustrating the theory of DAE using the class of the perhaps simplest (nontrivial) form, namely (7). As shall be seen later, a great deal of structure is present in this form which allows phase space, bifurcation and stability analyses to be performed [Venkatasubramanian, *et al.*, 1992; Beardmore & Song, 1998; Rabier & Rheinoldt, 1994; Chua & Deng, 1989; Chua, 1989; von Sosen, 1994; Hill & Mareels, 1990]. Various authors have studied nonsmooth solutions [Rabier & Rheinoldt, 1995] and even implicit delay equations [Petzold, 1995; Guan *et al.*, 1995; Coroian, 1994] for this class of systems amongst others.

3. Are DAEs Just ODEs?

3.1. *Is linearization possible?*

Considering the following examples we can see that DAEs and ODEs have different characteristics. For instance, suppose that the function F is defined by

$$F(x, y) = x^2 - xy \tag{13}$$

and we seek solutions of the $x = 0$ equilbrium of the DAE $F(x, \dot{x}) = 0$. The linearization procedure taken from ODE theory yields

$$\frac{\partial F}{\partial x}(0, 0) \cdot h + \frac{\partial F}{\partial y}(0, 0) \cdot h_t = 0 \tag{14}$$

which is the equation $0 = 0$ [Reich, 1995]. This tells us that we should not always expect the procedures that we apply to ODEs to be meaningful when we attempt to apply them to DAEs.

3.2. *Smooth forcing does not yield smooth solutions*

Another problem involves the smoothness of the forcing function for the (nonautonomous) DAE

$$F(x, \dot{x}) = f(t)\,. \tag{15}$$

Suppose that we consider the linear, constant-coefficient DAE where $f : \mathbb{R} \to \mathbb{R}^n$ and n is a given integer, define

$$F(x, y) = Ax + By \tag{16}$$

where A and B are $n \times n$ and $\det(A) = 0$. It is easy to construct equations, due to the nature of the constraints induced by the existence of a null space of A, which need "very" smooth forcing functions to allow the existence of smooth solutions.

To see this, consider the *Kronecker Normal Form* of the matices A and B [Brenan *et al.*, 1989]. Suppose that the function $\mu \mapsto \det(\mu A + B)$ is not identically zero, then we may find matrices P and Q such that

$$PAQ = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}$$

where N is a nilpotent matrix of index k, and

$$PBQ = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$$

where each I represents some identity matrix and C is some matrix. The number k is also called the index of the matrix pencil $\mu \mapsto \mu A + B$.

Applying this transformation to the above system, one obtains the equivalent system.

$$\begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}.$$

Which is the same as

$$\dot{u}_1 + Cu_1 = f_1(t)$$

and

$$\left(N \cdot \frac{d}{dt} + C\right) u_2 = f_2(t).$$

The first of these equations represents an ODE and can therefore be solved uniquely for a continuous solution given that f_1 is continuous. Conversely, the second part of the equation can also be solved, since $N^k = 0$ with

$$u_2(t) = \sum_{i=0}^{k-1} \left(-N \cdot \frac{d}{dt}\right)^i f_2(t).$$

Hence, if the forcing function f is not C^k-smooth in the components given by the Kroncker Normal Form computation, then there is no reason to expect smooth solutions to the problem. This contrasts sharply with the ODE case whereby smooth forcing will ensure smooth solutions.

3.3. *Implicit ODEs*

To illustrate a general principle that will be used later on, consider the system (7). Differentiate the constraint with respect to t along a solution (assumed to exist) and one obtains that the trajectory with this initial condition must obey the constraint

$$d_x g[x(t), y(t)](\dot{x}(t)) + d_y g[x(t), y(t)](y_t(t)) \equiv 0.$$

One can therefore write the system in the form

$$\begin{pmatrix} I & 0 \\ d_x g[x, y] & d_y g[x, y] \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ 0 \end{pmatrix}$$

which is an implicit ODE, $M(z)z_t = F(z)$ with $z = (x, y)^T$.

This procedure is perfectly legitimate and tells us that the solution of the DAE must satisfy an implicit ODE. It is possible, with the help of the above equation to show the existence of solutions to (7) provided the resulting $M^{-1}F$ is Lipshitz continuous and if $\det M(x_0, y_0) = \det d_y g[x_0, y_0] \neq 0$.

There is one obvious problem in attempting to show that the existence of a solution of the ODE implies the existence of a solution to the DAE. Namely, (17) has more solutions than (7). This is because if we define f and g to be smooth functions on $X := \mathbb{R}^n$ and $U := \mathbb{R}^m$ respectively, then the phase space of (7) is the closed subset of $Z = X \times Y$ defined by

$$\mathcal{P} := \{z \in Z | g(z) = 0\} = g^{-1}(0). \qquad (17)$$

This means that one is not at liberty to chose arbitrary initial conditions for the DAE whereas the implicit ODE makes sense for any initial conditions. The solution is to consider the ODE when providing initial conditions which do lie on $\mathcal{P}$. In this case, it follows that $(x(t), y(t)) \in \mathcal{P}$ whenever the solution exists: Namely $\mathcal{P}$ is an invariant manifold for the implicit ODE (17) and this is what is needed to rectify the problem.

4. Existence and Uniqueness

We can now prove the following albeit rather trivial theorem on existence. This adapts the usual ODE proof and uses the fact that the DAE gives rise to an ODE whose flow has $\mathcal{P}$ as an invariant subset.

Theorem 1. *Suppose that $f \times g \in C^1(\mathbb{R}^{n+m})$. The DAE*

$$\dot{x}(t) = f(x(t), y(t))$$
$$0 = g(x(t), y(t))$$

where initial conditions $\zeta \in \mathcal{P}$ are given. Suppose that $d_y g[\zeta] \in BL(V)$ is injective, then there is some maximal interval of existence such that $I_\zeta = (\alpha, \omega) \subset \mathbb{R}$ and a unique C^1 function $\phi = \phi^x \times \phi^y : \mathcal{D} \to \mathcal{P}$ where

$$\mathcal{D} := \{(\zeta, t) | t \in I_\zeta, \zeta \in \mathcal{P}\}. \qquad (18)$$

Defining $x(t) = \phi^x(t)$ and $y(t) = \phi^y(t)$ for all $t \in I_\zeta$ gives the unique solution to the DAE.

This follows from Picard's Theorem applied to the system (17) which is obtained after differentiating the DAE. Since the matrix $M(z)$ is an invertible mapping in some neighborhood of the initial condition, the system can be viewed as an ODE restricted to $\mathcal{P}$.

Another, perhaps more instructive way of determining the existence is to see that the constraint in the DAE of the above type can be removed in the same neighborhood using the Implicit Function Theorem (IFT). In this neighborhood we may find a function, y such that $g(x, u) = 0$ if and only if $u = y(x)$. With sufficient smoothness, the DAE has been reduced to the study of

$$\dot{x} = f(x, y(x)) \tag{19}$$

and the existence theorem above follows from Picard's Theorem again.

Proceeding further, the resulting ODE can be used to find stability properties for the original DAE. Notions such as Lyapunov stability, linear stability and bifurcations can all be defined for a large class of DAE.

Note that this theorem is of very limited use in the theory of general DAEs. Using the path control system defined above in (6) we see that the differential of the constraint function $d_y g \equiv 0$, so there is no hope of using the existence theorem for this problem. To make progress in this situation is more complex.

4.1. *A trivial case*

We have defined the most general DAE as the system $F(x, \dot{x}) = 0$ where $(x, y) \mapsto F(x, y)$ is defined and C^1-smooth on some space $Z = X \times Y \simeq \mathbb{R}^{2m}$.

If the map $d_y F[x_0, y_0]^{-1}$ exists then $F(x, y) = 0$ holds in a neighborhood of this point if and only if $y(t) = Y(x(t))$ for some C^1 function Y. The solution of the DAE exists in this case and is given by the solution of $\dot{x}(t) = Y(x(t))$. Even this simple system can differ from ODE in terms of stability and bifurcation properties.

4.2. *What is a solution?*

At this point we define what a solution of a DAE is. We shall avoid certain technicalities that can arise in considering different smoothness properties of various components of a solution using the definition of [Brenan *et al.*, 1989].

Definition 1 (DAE Solvability). Let $\mathcal{I} \subset \mathbb{R}$ be open and $\Omega \subset \mathbb{R}^{2m}$ be an open, connected set and suppose that $F : \Omega \to \mathbb{R}^q$ is differentiable. The DAE (8) is *solvable* on $\mathcal{I}$ in Ω if there is an r-dimensional family of solutions $s(t, c)$ defined on an open, connected set $\mathcal{I} \times \Omega^*$ where $\Omega^* \subset \mathbb{R}^r$, such that

1. $s(t, c)$ is defined on all of $\mathcal{I}$ for each $c \in \Omega^*$,
2. $(s(t, c), s_t(t, c)) \in \Omega$ for $(t, c) \in \mathcal{I} \times \Omega^*$,
3. If one can find any other solution, denoted $\sigma(t)$ with $(\sigma(t), \sigma_t(t)) \in \Omega$ then $\sigma(t) = s(t, c)$ for some $c \in \Omega^*$.
4. The graph of s as a function of (t, c) is an $r + 1$-dimensional manifold.

A practical approach to computing the solution manifold and to solvability in general can be found in [Campbell *et al.*, 1998].

For the DAE (7) it follows from previous analysis that the solution manifold is just some disjoint union of path-connected subsets of $\mathcal{P}$ which are known as *causal regions* for this specific case. Each causal region is separated by a hypersurface called the *singular* or *singularity manifold* which are typically of codimension 1. The singular manifold is merely the subsets of $\mathcal{P}$ where the implicit function theorem cannot be applied to remove the constraint in (7).

This definition of a solution extends some of the properties of the flow-map ϕ given in Theorem 1. In that case it is clear that ϕ can be used to define a dynamical system.

5. The Differential Index

A procedure was used in the previous section to turn a DAE into an ODE, and provided care was taken over the initial conditions, it followed that a vector field and a DAE had the same flow map. This process involved a differentiation and a re-arrangement of the resulting equations, this will be now applied to the general DAE. This is a procedure which is presented in almost all introductions and lectures on DAEs.

Consider the system (8) where for convenience the function $(x, y) \mapsto F(x, y)$ is C^∞ everywhere. Suppose that a solution denoted $x(t)$ exists and differentiate $F(x(t), \dot{x}(t))$ with respect to t. This gives a new system

$$\begin{aligned} d_x F[x(t), y(t)](x_t(t)) + d_y F[x(t), y(t)](x_{tt}(t)) &= 0 \\ F(x, x_t) &= 0\,. \end{aligned}$$

This can be rearranged into a first order system, and after setting $\dot{x} = u$ one obtains

$$\begin{pmatrix} I & 0 \\ d_x F[x, y] & d_y F[x, y] \end{pmatrix} \begin{pmatrix} x_t \\ u_t \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

$$F(x, u) = 0\,.$$

This is another DAE which can be written in the form $F^1(z, z_t) = 0$. If this represents an implicit ODE then denote the index of the system as one. Otherwise the process is continued on the new DAE which is written in the form $F^2(z', z'_t) = 0$ for some function F^2; if the resulting system is an ODE then denote the system as index 2. The index is found by repeating this process until the system $F^k(w, w_t) = 0$ can be put into the form of an ODE

$$w_t = G(w) \tag{20}$$

and the number of times the process is repeated, k, is the differential index of the DAE.

As was mentioned before, every solution of the DAE solves the ODE system (20) but not vice-versa. During the differentiation process constraints are present and if these constraints are imposed onto the solution of the ODE then we do expect to recover only those solutions which also solve the DAE.

The index gives some measure of how difficult the DAE is to solve, lower indexes denoting the easier cases. The notion was first introduced as a tool to determine the numerical properties of the system. The index is a more subtle concept than one might anticipate.

6. DAE Stability, Linearization Properties

6.1. *Equilibria: Can we linearize?*

Equilibria of the general DAE (8) are solutions of the system $F(x, 0) = 0$. Suppose that $x^* \in X$ is an equilibrium and that the DAE is solvable in some neighborhood x^* intersected with the r-dimensional solution manifold in $\mathbb{R}^{2m}$ which comes from the definition of solvability.

In ODE systems, the typical development of the theory of flows near equilibria is to define Lyapunov stability and asymptotic stability. One can then show that for the equation $\dot{x} = f(x) = A(x - x^*) + o(\|x - x^*\|)$, the flow of the system $\bar{\phi}$ is *locally topologically conjugate* to the mapping $t \mapsto \exp(At)$ in some neighborhood of x^* when ever the spectrum of A, $\sigma(A)$ is known to have no eigenvalues of zero imaginary part. This means essentially that the trajectories of the nonlinear system and its linearization that start near the equilibrium have the same character in terms of being attracted to, or repelled away from the equilibrium.

This is the Hartman–Grobman or linear stability theorem for ODEs and it has a counterpart in the theory of PDEs. If this spectrum is contained in the open left-half plane then it follows that the equilibrium is locally, asymptotically stable. Conversely, if the equilibrium is Lyapunov stable, then each eigenvalue in the spectrum must lie in the closed, left-half plane.

The question naturally follows, can linearization methods be used for DAEs by showing that the linearization of a solvable DAE has a flow map which behaves locally like the flow of the DAE itself? Is it possible to define Lyapunov stability and find Lyapunov functions for DAEs? If the DAE is parameterized by a real parameter λ say, can the bifurcation structure of the solutions be obtained?

Not in general. The bifurcation structure of the equilibrium equation is conceptually easy to handle since this would be finding solutions of the parameterized system ((8) with a 1-parameter dependence)

$$F_\lambda(x, 0) = 0\,. \tag{21}$$

To obtain information concerning the bifurcation structure of periodic solutions via the Hopf bifurcation, as well as codimension 2 and higher bifurcations, requires more mathematical machinery. This has been developed by Reich in the paper [Reich, 1995] where information concerning the local structure of the solutions of the DAE can be obtained from that of the linear system

$$d_xF[x^*, 0](h(t)) + d_yF[x^*, 0](h_t(t)) = 0 \tag{22}$$

where $h(t) = x(t) - x^*$.

Reich assumes that (Reich's first regularity assumption) $d_yF[x^*, 0]$ is a singular mapping with constant rank $p < m$ in system (8), in some neighborhood of $(x^*, 0) \in \mathbb{R}^{2m} \cap F^{-1}(0)$ and $dF[x, y]$ has constant rank on the same set.

For simplicity of notation, write this system as the linear DAE

$$Ah_t + Bh = 0\,. \tag{23}$$

In his paper, Reich also uses the following regularity condition which allow the stability of the equilibrium to be analyzed using linear algebra and spectral techniques. He supposes further that $F : \mathbb{R}^{2m} \to \mathbb{R}^m$ and imposes (Reich's second regularity assumption) the condition that $\text{rank}(d_xF[x^*, 0]) = m$.

Hence x_0 is an isolated equilibrium point and it is now possible to deduce some useful properties of the DAE in a neighborhood of x_0. In particular, the DAE has a linearization which is a vector field with a flow equivalent to that of the DAE and whose dimension is equal to that of the dimension of the local solution manifold.

6.1.1. *Stability types*

Continuing the connection with ODE theory we present the basic types of stability which are interpreted in the context of DAEs.

To recap, given the DAE $F(x, \dot{x}) = 0$, where F satisfies the regularity conditions of Reich given above, suppose an equilibrium $x^* \in \mathbb{R}^m$ exists and is therefore isolated. Properties of the trajectories that start "near" this point can be analyzed using the following definitions.

Suppose that ζ is in some small neighborhood of x^*, but is not the equilibrium itself. Suppose that the function $t \mapsto x(t)$ satisfies $x(0) = \zeta$ and for each t in some open interval $\mathcal{I}$ containing 0 we have $F(x(t), \dot{x}(t)) \equiv 0$.

It is a simple matter to define the usual notions of Lyapunov and asymptotic stabilty with some slight modifications which we do not dwell on here. The tools needed to calculate linear stability differ slightly from the ODE case and Reich has dealt with this problem.

6.1.2. *Linear stability*

Definition 2 (Matrix Pencils). A pair of $n \times n$ matrices A and B can be combined to give the matrix pencil $\mu \mapsto \mu A + B$ denoted (A, B).

This notion was used previously to show that nonautonomous linear DAE do not behave as linear ODE, but matrix pencils also arrive in a more general setting so we shall explain the terminology.

If $\det(\mu A + B)$ is *not* identically zero then the matrix pencil (A, B) is said to be regular. This is required to make precise the notion of linear stability.

Suppose that the DAE satisfies Reich's regularity assumptions so that its linearization exists. One can then say that if all the pairs $(\mu, v) \in \mathbb{C}\times\mathbb{C}^m\backslash\{0\}$ which satisfy

$$(\mu A + B)v = 0$$

also have $\Re(\mu) < 0$ then x^* is *linearly stable*. It is well-known for vector fields that linear stability implies local, asymptotic stability and by Reich's work, this is so for regular DAEs.

The related notions of stable, unstable and centre manifolds can be defined for the regular DAE and this is done in [Reich, 1995].

6.1.3. *What linearization does not give*

It is not unfortunately possible to use the linearization of a DAE to find the index of a DAE except in special cases. The following case is known to be such an instance.

Theorem 2. *Suppose that d_yF has constant rank and that $(d_xF[x, y], d_yF[x, y])$ is an index one matrix pencil independently of x and y, then the DAE $F(x, \dot{x}) = 0$ has index one.*

7. The Semi-Explicit, Index One Case

Let us now restrict attention to systems of the form (7). This particular class of DAE is perhaps closest in flavor to an ODE system. We have already seen this where Picard's Theorem was easily adapted to prove existence for this class of system. However, there are phenomena which occur for this DAE which cannot arise in the theory of ODE. We shall use these examples to illustrate the extra difficulties involved when passing from the DAE to ODE case. As one famous paper tells us [Petzold, 1982], "DAEs are not ODEs" but they do have similarities.

7.1. *The simplest case*

We now give an explanation of the stability methods mentioned above where they can be explicitly applied: The semi-explicit index one case. We give the explicit construction of the vector field whose existence is implied by the theory of Reich and use this to construct Lyapunov functions, linear stability criteria and address the bifurcation phenomena in parameterized systems, with one real parameter. In many cases, the theory does not yet exist for many-parameter variations.

This work was originally carried out mainly by electrical engineers where understanding the behavior of the index one DAE is of prime importance and the principle authors in this area have been Venkatasubramanian *et al.* [Venkatasubramanian, 1992; Chua & Deng, 1989; Rabier & Rheinoldt, 1994]. The approach of the engineers differs from that of the mathematical community in several important ways. The main distinction when studying the system, (7) into which we introduce a real parameter $\lambda \in \mathbb{R}$

$$\begin{aligned} \dot{x}(t) &= f(x(t),\, y(t),\, \lambda) \\ 0 &= g(x(t),\, y(t),\, \lambda) \end{aligned} \tag{24}$$

is as follows. The engineer motivated by physical models, assumes that the mapping at the initial condition $d_y g[x_0,\, y_0,\, \lambda_0]$ is invertible, at the state of the system describe by his set of parameters. The mathematical theory, although partly motivated originally during the 1960's and 1970's by electrical models assumes that the same mapping is of constant rank as λ changes. This assumption is not used in the engineering community which leads to interesting behavior not until now used within the mathematics community [Campbell *et al.*, 1998].

7.1.1. *From ODE to DAE*

We shall now take our model DAE, (24) and construct an ODE which has "the same local solution structure". This procedure was followed in [Venkatasubramanian *et al.*, 1992] to analyze this system.

Fix $\lambda \in \mathbb{R}$ and suppose that $g(x_0,\, y_0,\, \lambda) = 0$ and let $z_0 = (x_0,\, y_0)$ be the initial condition for the DAE. Differentiating the constraint in the above system along the solution, which we know exists from previous work, with respect to t, assuming that $d_y g[x_0,\, y_0,\, \lambda]$ is nonsingular and re-arranging one obtains the ODE (omitting the parameter)

$$\begin{aligned} \dot{x} &= f(x,\, y) \\ y_t &= -\frac{\mathrm{adj}(d_y g[x,\, y]) \cdot d_x g[x,\, y](f(x,\, y))}{\det(d_y g[x,\, y])}\,. \end{aligned} \tag{25}$$

With a singular shift in time scale, it is easy to show that the following ODE has a globally defined flow map which is locally conjugate to the above system or to the above system with time reversed, in regions where $d_y g$ is a bijection,

$$\begin{aligned} x_s &= \det(d_y g[x,\, y]) f(x,\, y) \\ y_s &= -\mathrm{adj}(d_y g[x,\, y]) \cdot d_x g[x,\, y](f(x,\, y))\,. \end{aligned} \tag{26}$$

This system can be analyzed and properties inferred for the DAE system and some results in this direction now follow.

7.1.2. *The Lyapunov function*

Let $(x^*,\, y^*) =: z^* \in \mathbb{R}^{n+m}$ be an equilibrium for (7) and denote $F = f \times g$ be the product map so that $F(z^*) = 0$ by definition. If there is a C^1 function V defined in a neighborhood $B(z^*) \subset \mathbb{R}^{n+m}$ such that for all $z = (x,\, y) \in B(z^*)$,

1. $V(z) \geq 0$, $V(z) = 0 \Rightarrow z = z^*$
2. $d_x V(f) - d_y V \cdot d_y g^{-1} \cdot d_x g(f) \leq 0$

then z^* is Lyapunov stable and V is termed a Lyapunov function. If V is strictly, locally positive definite then the equilibrium is locally, asymptotically stable.

7.1.3. *The linearization of a DAE*

The assumption that $d_y g[z_0]$ is a bijection implies that the constraint can be solved locally in (7) for $y = y(x,\, \lambda)$. Assume that the initial condition lies within some neighborhood of the equilibrium z^* where this has been carried out, where $y(x^*,\, \lambda) = y^*$. Therefore the linearization of the function

$$x \mapsto f(x,\, y(x,\, \lambda),\, \lambda)$$

about the equilibrium $(x^*,\, y^*) = z^*$ yields the linearization of the system and the spectrum of the this map yields the linear stability conditions, as follows. Define the *stability mapping*, denoted $\mathcal{J}$ for Jacobian, $\mathcal{J}(\lambda) =$

$$d_x f[z^*,\, \lambda] - d_x f[z^*,\, \lambda] \cdot d_y g[z^*,\, \lambda]^{-1} \cdot d_x g[z^*,\, \lambda]\,. \tag{27}$$

This is also the linearization of the vector field (25) and is given by the partial derivative of $f(x,\, y(x,\, \lambda),\, \lambda)$ with respect to x. The form is that of a Schur Complement which arises in many areas of numerical mathematics.

Bearing in mind that we have considered $\lambda \in \mathbb{R}$ to be a fixed constant. If we drop this assumption then it is clear that as the rank of $d_y g[z^*,\, \lambda]$ (where $z^* = z^*(\lambda)$ is a parameterized equilibrium locus) drops then $\mathcal{J}$ will cease to exist and its spectrum will undergo some dramatic changes. This is the cause of one of the main differences between stability change in ODE and index-one DAE.

7.2. *The classic bifurcations and stability changes*

Adopting the notation used before, define the mapping $F(z,\, \lambda) = (f \times g)(z,\, \lambda)$ and assume that $F(e(\lambda),\, \lambda) \equiv 0$ so that e represents some equilibrium locus as the parameter is varied. It is of great importance to obtain local and global information concerning the solution structure of $F(z, \lambda) = 0$ and one way of viewing this is using a bifurcation analysis.

A bifurcation point $\lambda_0 \in \mathbb{R}$ is some value where to distinct equilibrium locii e_1 and e_2 come together as the parameter λ is varied and passes through λ_0. For more information concerning bifurcation theory, see [Ambrosetti & Prodi, 1992].

7.2.1. *Bifurcating solutions*

Using the above analysis it is possible to use $\mathcal{J}$ to show when oscillating or equilibrium solutions bifurcate from some branch of equilibria.

It is possible to formulate the saddle-node bifurcation (SNB) for index-one DAE and this is stated in [Venkatasubramaniam *et al.*, 1992]. This means that stability loss in one eigenmode of the DAE will result in the coalescence of two equilbrium loci, which are solutions of $F(z, \lambda) = 0$. This says that if one, algebraically simple eigenvalue of $\mathcal{J}(\lambda)$ moves through $0 \in \mathbb{C}$ at λ_0 then λ_0 is a bifurcation point, and generically a saddle-node.

While this is a remarkable fact based around $\mathcal{J}(\lambda)$, it is not necessary to use this theorem to calculate the SNB points. This is because the result applies to *equilibria* of the DAE and the calculations could have equally well arisen from the system $\dot{z} = F(z, \lambda)$. Therefore, it is possible to use the SNB theorem from ODE theory to calculate the SNB points in the index-one DAE (7).

This form of reasoning is not possible in the following case which describes when periodic solutions and an equilirium locus $e(\lambda)$ come together in a Hopf bifurcation as soon in Fig. 1. Here we assume that $e(\lambda_0) = z^*$ is an equilibrium.

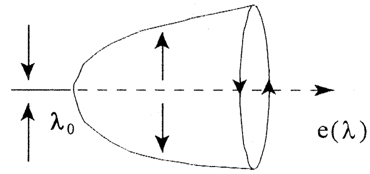

Fig. 1. The trivial equilibrium is enclosed by periodic orbits with increasing amplitude.

Theorem 3 (The Hopf Bifurcation for Index One DAE). *Suppose that $d_y g[z^*, \lambda_0]$ is invertible and $\pm i\omega \in \sigma(\mathcal{J}(\lambda_0))$ for some real, nonzero ω and $\beta(\lambda) \in \sigma(\mathcal{J}(\lambda))$ is an eigenvalue locus (algebraically simple) such that $\beta(\lambda_0) = i\omega$. If $\mathcal{J}(\lambda_0)$ has no other eigenvalue of zero real part, other than $-i\omega$ and $\Re(d/d\lambda)\beta(\lambda_0) \neq 0$ then there is a curve of periodic solutions of (24) "eminating from the equilibrium locus" at $\lambda = \lambda_0$.*

The vagueness of the conclusion to this theorem is necessary since we have not defined what a Hopf bifurcation point is precisely but there are many references and textbooks concerning this bifurcation [Wiggins, 1990; Ambrosetti & Prodi, 1992; Glendinning, 1996] so we shall dwell no further on this.

In this instance there is no possibility of using the usual ODE calculations to obtain Hopf bifurcation points, as the DAE and ODE are different animals from this point of view.

It is in fact possible to develop a Hopf bifurcation theorem for the very general class of DAE (8) as has been done in [Reich, 1995], where the regularity conditions ensure enough structure for the DAE to be linearized, Reich then applies the classical Hopf theorem [Guckenheimer & Holmes, 1986] in this context. This is the same procedure that can be done to justify the Hopf theorem for the index-one case and this small result we have presented is subsumed within the theory of Reich.

7.3. *Smooth systems can be singular in phase space*

The definition of the above DAE involves the function $F = f \times g$ which is smooth everywhere on its domain of definition. We have discussed how, unlike ODE systems, the phase space of the index one DAE can contain codimension-1 (and greater) regions in the set $\mathcal{P}_\lambda = g^{-1}(0)$ where trajectories may terminate in forwards and backwards time, called the *singularity manifold.* Not all trajectories necessarily terminate here but those points on this set where solutions cannot be continued are generally termed *impasse points.*

The singularity manifold is the set of points in $z \in \mathcal{P}_\lambda$, denoted S_λ such that

$$\det(d_y g[z, \lambda]) = 0\,. \tag{28}$$

As has been already stated, one expects that $\mathcal{P}_\lambda$ is separated into path-connected components whose boundaries are characterized by the solution of this equation. The complement of these components in $\mathcal{P}_\lambda$ is the singularity manifold which is the set of points where the implicit function theorem *cannot* be used to remove the constraint and the existence of a solution through this point cannot be guaranteed.

One can say that if a solution approaches the singular manifold (not necessarily in finite time) then it approaches a unique point on that manifold by virtue of the fact that the above ODE has a flow which is identical (modulo a scaling in the time variable) in some small enough neighborhood of any point not on S. A complete analysis of this situation can be found in [Venkatasubramaniam *et al.*, 1992].

7.4. *Singularity-induced instabilities*

There is a generic [Wiggins, 1990] linear stability change, for one-parameter variations, present in (24) that is not present in the theory of continuous ODE bifurcation theory [Venkatasubramaniam, 1994; Beardmore & Song, 1998]. In ODE (and PDE) theory there is famous theorem which says that if the linear stability mapping, in this case this is just the usual linearization, has a single (real and algebraically simple) eigenvalue which moves from one half of the complex plane to the other, then a bifurcation of equilibria results. This is the bifurcation from a simple eigenvalue theorem [Ambrosetti & Prodi, 1992].

In the DAE case it is possible that as the parameter is varied, one eigenvalue of $\mathcal{J}(\lambda)$ moves from one half of the complex plane to the other "by diverging to infinity". More precisely one can say that under certain nondegeneracy assumptions, as λ passes λ_0, there is a real, algebraically simple eigenvalue of the system, which is parameterized by λ and has a simple pole in $\lambda - \lambda_0$.

The statement of this theorem is as follows, assuming for simplicity that $F(0, \lambda) \equiv 0$ for all λ near λ_0, although this assumption can easily be dropped. This theorem was originally named by Venkatasubramanian *et al.* as the singularity-induced bifurcation. We recall that $z = (x, y)^T$ and $F = f \times g$.

Theorem 4. *Suppose that*

1. $N(d_y g[0, \lambda_0]) = \langle k \rangle \neq \{0\}$,
2. $d^2_{\lambda y} g[0, \lambda_0](1, k) \notin R(d_y g[0, \lambda_0])$,
3. $d_x g[0, \lambda_0] \cdot d_y f[0, \lambda_0](k) \notin R(d_y g[0, \lambda_0])$

then $0 \in \mathbb{R}^{n+m}$ *is the unique equilibrium of (24) defined in a neighborhood* $\mathcal{N}$ *of* λ_0 *which is transversal to the singularity manifold at* λ_0. *One eigenvalue of* $\mathcal{J}(\lambda)$ *is simple and real and has a simple pole at* λ_0 *and therefore moves from one open half of the complex plane to the other by diverging to infinity.*

This tells us that the eigenvalue in question $\alpha(\lambda)$, behaves locally ($\lambda \neq \lambda_0$) as

$$\alpha(\lambda) = \frac{\mu}{\lambda - \lambda_0} + \phi(\lambda - \lambda_0) \qquad (29)$$

where $\mu \in \mathbb{R}\backslash\{0\}$ and ϕ is C^{r-2} in this neighborhood of 0.

This is not the original form of the SIB Theorem which was first stated in [Venkatasubramaniam, 1994] but is an essentially equivalent version of it [Beardmore & Song, 1998].

8. Numerical Consequences of the Index

Since the understanding of the system $w_t = G(w)$ is much more advanced in mathematics than that of the system $F(x, \dot{x}) = 0$ it is natural to try and write the latter in terms of the former. This has consequences not only for the analytic and structural properties of the DAE but also affects the numerical methods that may be used to solve the equations.

For instance, suppose that one discretizes time so that $t \in [0, T]$. Set $i = 0, \ldots, N$ and let $h = T/N$. Defining $t_i = hi$, let $x_i \simeq x(t_i)$ along the solution, assumed to exist, and substitute this into the DAE (8). Forcing equality in the resulting equation, one obtains the problem of solving

$$F\left(x_i, \frac{x_{i+1} - x_i}{h}\right) = 0 \qquad (30)$$

for x_{i+1} if x_i is known. This is called the implicit Euler method to which one may apply a classical numerical method such as Newton Iteration. The index is important for this approach since the following result can be found in [Brenan *et al.*, 1989].

Theorem 5. *The condition number of the iteration matrix (in the Newton step) for a system with index* ν *is* $O(h^{-\nu})$.

We can see from this theorem that the greater the index, the greater the difficulty will be in obtaining accurate numerical estimates of the solutions. It has been claimed that indices of greater than 50 have been encountered in industrial-based problems. It is unlikely that anyone has actually calculated the index of such systems and in truth one rarely does since it is not a trivial problem to find the index of an arbitrary problem. It is however, possible to find bounds on the index using methods from graph theory and differential geometry [Le Vey, 1994].

For low (≤ 1) index problems it is possible to write general-purpose software and a BDF (backward differentiation formulae) fortran implementation of numerical methods, called DASSL, to solve such problems can be downloaded. Details on how to obtain this can be found in [Brenan *et al.*, 1989].

9. Conclusions

We have given a very basic survey of some of the problems associated with DAE. There is an ever-growing literature devoted to the subject encompassing techniques from fields as diverse as differential geometry and topology, symbolic computing, numerical analysis, linear algebra, bifurcation theory and many other others.

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