Applied Mathematics and Computation 298 (2017) 336–350

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Fractional differential equations with a constant delay: Stability and asymptotics of solutions

Jan Čermák[a,*], Zuzana Došlá[b], Tomáš Kisela[a]

[a] Institute of Mathematics, Brno University of Technology, Technická 2, 616 69 Brno, Czech Republic
[b] Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic

ARTICLE INFO

*MSC:*
34K37
34K20
34K25

*Keywords:*
Delay differential equation
Fractional-order derivative
Stability
Asymptotic behaviour

ABSTRACT

The paper discusses stability and asymptotic properties of a fractional-order differential equation involving both delayed as well as non-delayed terms. As the main results, explicit necessary and sufficient conditions guaranteeing asymptotic stability of the zero solution are presented, including asymptotic formulae for all solutions. The studied equation represents a basic test equation for numerical analysis of delay differential equations of fractional type. Therefore, the knowledge of optimal stability conditions is crucial, among others, for numerical stability investigations of such equations. Theoretical conclusions are supported by comments and comparisons distinguishing behaviour of a fractional-order delay equation from its integer-order pattern.



## 1. Introduction

We investigate stability and asymptotic properties of the fractional delay differential equation

$$\mathrm{D}^{\alpha}y(t) = ay(t) + by(t-\tau), \qquad t > 0 \tag{1}$$

with real coefficients $a, b$, a positive real lag $\tau$ and the fractional Caputo derivative operator $\mathrm{D}^{\alpha}$ ($0 < \alpha < 1$ is assumed to be a real number).

Letting $\alpha \to 1$ from the left, $\mathrm{D}^{\alpha}y(t)$ becomes $y'(t)$ and (1) is reduced to the classical delay differential equation

$$y'(t) = ay(t) + by(t-\tau), \qquad t > 0\,, \tag{2}$$

studied frequently due to its theoretical as well as practical importance (see, e.g. [12]). This equation serves, among others, as the basic test equation for stability analysis of various numerical discretizations of delay differential equations (see, e.g. [1,9]). In this connection, stability conditions for (2) are traditionally required in the optimal form, i.e. as the necessary and sufficient ones. There are known two types of such conditions for asymptotic stability of the zero solution of (2) that we recall in the following two assertions (see [12]). As it is customary, by asymptotic stability of the zero solution of (2) we understand the property that any solution $y$ of (2) is eventually tending to the zero solution.

* Corresponding author.
*E-mail addresses:* cermak.j@fme.vutbr.cz (J. Čermák), dosla@math.muni.cz (Z. Došlá), kisela@fme.vutbr.cz (T. Kisela).

http://dx.doi.org/10.1016/j.amc.2016.11.016

**Theorem 1.** *Let $a, b$ and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if the couple $(a, b)$ is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve*

$$a = \varphi \cot(\tau\varphi), \quad b = -\frac{\varphi}{\sin(\tau\varphi)}, \qquad \varphi \in \left(0, \frac{\pi}{\tau}\right)$$

*from below.*

**Theorem 2.** *Let $a, b$ and $\tau > 0$ be real numbers. The zero solution of (2) is asymptotically stable if and only if either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary}, \tag{3}$$

*or*

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \frac{\arccos(-a/b)}{(b^2 - a^2)^{1/2}}. \tag{4}$$

While Theorem 1 describes the stability boundary for (2) in the $(a, b)$-plane, the conditions of Theorem 2 seem to be more explicit. In particular, $(4)_2$ presents the value of the stability switch when (2) loses its stability property as the delay $\tau$ is monotonically increasing.

It is also well-known that asymptotic stability of the zero solution of (2), described via the conditions of Theorems 1 and 2, is of exponential type, i.e. there exists $\delta > 0$ such that $y(t) = \mathcal{O}(\exp[-\delta t])$ as $t \to \infty$ for any solution $y$ of (2). The value $\delta$ depends on location of roots of the characteristic equation

$$s - a - b\exp[-s\tau] = 0 \tag{5}$$

with respect to the imaginary axis (its estimates are discussed, e.g. in [8]).

The involvement of fractional-order derivatives into delay differential equations represents a new type combining advantages of both delayed and non-integer derivative terms, especially hereditary properties, more degrees of freedom and other advantages of fractional modelling. Since application areas of fractional delay differential equations are especially control theory and robotics, the question of their stability (and asymptotics) is again of main interest. In general, stability and asymptotic analysis of fractional delay differential equations is just at the beginning. As it is evident from the literature (see, e.g. [6,16,17,21]), almost all the existing stability results on autonomous equations of this type are based on the root locus of appropriate characteristic equations, and they do not provide universally acceptable effective criteria for testing stability of a given fractional delay equation.

Therefore, the main goal of this paper is to extend the above stated properties of (2) to (1). Since (1) may serve as a basic prototype of fractional delay differential equations, formulation of non-improvable stability conditions and related asymptotic formulae is of a great importance in qualitative as well as numerical analysis of fractional delay differential equations.

Following the classical case, we say that the zero solution of (1) is asymptotically stable if any solution $y$ of (1) is eventually tending to the zero solution. Similarly, the zero solution of (1) is called stable if any its solution is eventually bounded. To describe asymptotics of solutions of (1), we shall use the following asymptotic symbols:

$$f(t) \sim g(t) \qquad \text{if} \qquad \lim_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0,$$

$$f(t) \sim_{sup} g(t) \qquad \text{if} \qquad \limsup_{t\to\infty} \frac{|f(t)|}{|g(t)|} = K > 0.$$

We note that (1) has been studied in the purely delayed case (when $a = 0$) in [5,15]. While the first paper presents stability criterion based on a transcendent inequality involving the fractional Lambert function, the latter paper already contains explicit conditions (which are, as we have already mentioned, very rare for fractional delay differential equations). We recall here the explicit relevant results (reformulated to our notation) characterizing stability and asymptotics of (1) with $a = 0$ (see [15, Theorem 5.1]).

**Theorem 3.** *Let $0 < \alpha < 1$, $b \neq 0$ and $\tau > 0$ be real numbers.*

(i) *The zero solution of*

$$\mathrm{D}^\alpha y(t) = by(t - \tau), \qquad t > 0 \tag{6}$$

*is asymptotically stable if and only if*

$$-\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^\alpha < b < 0.$$

*In this case, $y(t) \sim t^{-\alpha}$ as $t \to \infty$ for any solution $y$ of (6).*

(ii) *The zero solution of (6) is stable, but not asymptotically stable, if and only if*

$$b = -\left(\frac{\pi - \alpha\pi/2}{\tau}\right)^\alpha.$$

A vector extension of Theorem 3 has been recently derived in [4]. As we have declared above, the goal of this paper is to provide its another extension, namely to (1) involving both the delayed as well as non-delayed term (for some preliminary stability results on (1) we refer to [2,11]). More precisely, we are going to derive direct fractional extensions of Theorems 1 and 2, including asymptotic descriptions of solutions of (1).

The paper is organized as follows. Section 2 recalls some basic necessary notions and properties of fractional calculus, especially those related to the Laplace transform. In Section 3, we discuss the characteristic equation associated with (1) and describe some of its root properties. Section 4 presents main results on stability and asymptotics of (1). In particular, we formulate here fractional analogues of Theorems 1 and 2 and present asymptotic formulae for solutions of (1). Some comments and comparisons related to these results are mentioned as well. Section 5 is devoted to the proof of a key auxiliary result describing asymptotics of (1) in terms of roots location of the associate characteristic equation. The final section summarizes the results and outlines perspectives of a future research.

## 2. Preliminaries

Throughout this paper, we use the following standard definitions of the fractional integral of a real function $f$

$$\mathrm{D}^{-\nu} f(t) = \int_0^t \frac{(t-\xi)^{\nu-1}}{\Gamma(\nu)} f(\xi)\mathrm{d}\xi, \qquad \nu > 0, \quad t > 0$$

and the Caputo fractional derivative of $f$

$$\mathrm{D}^{\alpha} f(t) = \mathrm{D}^{-(1-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t} f(t)\right), \qquad 0 < \alpha < 1, \quad t > 0$$

where we put $\mathrm{D}^0 f(t) = f(t)$ (for more details on basics of fractional calculus theory we refer, e.g. to [14,20,25]).

The main analytical technique used in stability investigations of linear fractional differential equations is based on the well-known Laplace transform. For a given real function $f$, it is introduced via

$$\mathcal{L}(f(t))(s) = \int_0^{\infty} f(t)\exp[-st]\mathrm{d}t \ \big(\equiv F(s)\big), \qquad s \in \mathfrak{D}$$

where $\mathfrak{D} \subset \mathbb{C}$ contains all complex $s$ such that the integral converges. The inverse Laplace transform of $F$ can be expressed via the contour integral

$$\mathcal{L}^{-1}(F(s))(t) = \int_{\mathcal{C}} F(s)\exp[st]\mathrm{d}s$$

where the contour $\mathcal{C} \subset \mathfrak{D}$ is usually considered as a line $\Re(s) = c$ such that all singularities of $F$ lie to the left of this line. However, in view of the fundamental theory of complex integration, this line can be changed into arbitrary contour $\mathcal{C}$ such that its orientation with respect to singularities of $F$ remains preserved (we recall that contour is an oriented piecewise smooth curve). In our next analysis, we utilize the contour

$$\gamma(\mu, \vartheta) = \gamma_1 + \gamma_2 + \gamma_3 \tag{7}$$

such that its three oriented segments are given via

$$\begin{aligned}
\gamma_1 &= \{s \in \mathbb{C} : s = -u\exp[-\mathrm{i}\vartheta], u \in (-\infty, -\mu)\},\\
\gamma_2 &= \{s \in \mathbb{C} : s = \mu\exp[\mathrm{i}u], u \in [-\vartheta, \vartheta]\},\\
\gamma_3 &= \{s \in \mathbb{C} : s = u\exp[\mathrm{i}\vartheta], u \in (\mu, \infty)\}
\end{aligned}$$

where $\mu > 0$ and $\vartheta \in (-\pi, \pi]$. This contour is depicted on Fig. 1.

A key computational property, namely the Laplace transform of fractional derivative of $f$, is given by the formula

$$\mathcal{L}(D^{\alpha} f(t))(s) = s^{\alpha}\mathcal{L}(f(t))(s) - s^{\alpha-1} f(0), \qquad 0 < \alpha < 1 \tag{8}$$

extending the classical first-order case (see, e.g. [20]). To avoid a possible misunderstanding, we emphasize that the principal branch of the power functions occurring in (8) has to be considered. It is a consequence of the following property (for more details see, e.g. [7, pp. 8–10]).

**Proposition 1.** *Let $q > -1$. Then $\mathcal{L}(t^q)(s)$ is a single-valued function given by*

$$\mathcal{L}(t^q)(s) = s^{-q-1}\Gamma(q+1)$$

*where the principal branch of the complex power function is considered.*

As the last step of this section, we introduce the family of functions

$$\mathcal{R}_{\alpha,\beta}^{a,b,\tau}(t) = \mathcal{L}^{-1}\left(\frac{s^{\alpha-\beta}}{s^{\alpha} - a - b\exp[-s\tau]}\right)(t) \tag{9}$$

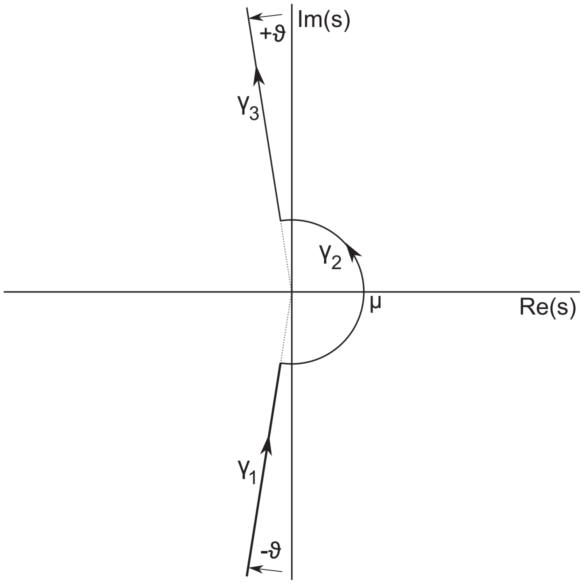


**Fig. 1.** The contour $\gamma(\mu, \vartheta) = \gamma_1 + \gamma_2 + \gamma_3$ in the complex plane.

where $\alpha$, $\beta$, $\tau > 0$ and $a, b \in \mathbb{R}$. This family involves certain special functions appearing in fractional calculus such as

$$\mathcal{R}^{0,0,\tau}_{\alpha,\beta}(t) = t^{\beta-1}/\Gamma(\beta)\,,$$
$$\mathcal{R}^{a,0,\tau}_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})\,,$$
$$\mathcal{R}^{0,b,\tau}_{\alpha,\beta}(t) = G^{b,\tau,0}_{\alpha,\beta}(t)\,,$$
$$\mathcal{R}^{a,b,0}_{\alpha,\beta}(t) = t^{\beta-1}E_{\alpha,\beta}((a+b)t^{\alpha})\,.$$

Here, $E_{\alpha,\,\beta}$ is the two-parameter Mittag–Leffler function (see, e.g. [20]) and $G^{b,\tau,0}_{\alpha,\beta}$ is the generalized delay exponential function of Mittag–Leffler type (see [4]). We can see that for $\alpha = \beta = 1$ the functions coincide with the standard exponential functions or delayed exponential functions known from the theory of delay differential equations (see, e.g. [3]).

The functions $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ generalize the notion of fundamental solution for classical delay differential equations (see, e.g. [10]); in particular, the representation formula

$$y(t) = \phi(0)\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) + b\int_{-\tau}^{0} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)h(t-\tau-u)\phi(u)\mathrm{d}u \tag{10}$$

holds for all $t > 0$. Here, $y$ is a solution of (1), $h$ is the Heaviside step function and $\phi$ is the associated initial function (which is supposed to be piecewise continuous on $[-\tau, 0]$), i.e. it holds

$$y(t) = \phi(t)\,, \qquad -\tau < t \le 0\,. \tag{11}$$

The verification of (10) is analogous to that employed in [4, Theorem 3] and therefore it is omitted. Based on (9) and (10),

$$Q(s) \equiv s^{\alpha} - a - b\exp[-s\tau] = 0 \tag{12}$$

plays the role of a characteristic equation associated with (1). A more detailed explanation of this role and its connection to stability properties of (1) is discussed in the next sections.

## 3. Distribution of roots of the characteristic equation

Eq. (12) admits infinitely many complex roots. In this section, we state some of their basic properties and analyse their location in the complex plane.

**Proposition 2.** *Let $0 < \alpha < 1$, $a, b$ and $\tau > 0$ be real numbers.*

(i) *If $a + b \ge 0$ then (12) has a non-negative real root.*
(ii) *If $s$ is a root of (12) then its complex conjugate $\bar{s}$ is also a root of (12).*
(iii) *Let $0 < \omega < \pi$ be arbitrary. Then (12) has no more than a finite number of roots $s$ such that $|\arg(s)| \le \omega$.*
(iv) *If $Q(s) = Q'(s) = 0$ for some complex $s$, then $s$ is a positive real number.*

**Proof.** If $a+b \geq 0$ then $Q(0) = -a-b \leq 0$. At the same time, $Q(s) \to \infty$ as $s \to \infty$, $s$ being real. This implies the property (i) (more precisely, (12) has the zero root if $a+b=0$ and a positive real root if $a+b>0$).

Now let

$$s = \varrho \exp(\mathrm{i}\psi)\,, \tag{13}$$

$\varrho = |s|$ and $\psi = \arg(s) \in (-\pi, \pi]$, be a root of (12). Then

$$\varrho^\alpha \exp(\mathrm{i}\alpha\psi) - a - b\exp[-\varrho\tau\exp(\mathrm{i}\psi)] = 0\,,$$

i.e. $\varrho$ and $\psi$ satisfy

$$\varrho^\alpha \cos(\alpha\psi) - a - b\exp[-\varrho\tau\cos(\psi)]\cos[\varrho\tau\sin(\psi)] = 0\,, \tag{14}$$

$$\varrho^\alpha \sin(\alpha\psi) + b\exp[-\varrho\tau\cos(\psi)]\sin[\varrho\tau\sin(\psi)] = 0\,. \tag{15}$$

This immediately yields the property (ii). To prove (iii), we first show the existence of a real $\varrho_1 > 0$ large enough such that (12) has no roots $s$ with $|s| > \varrho_1$ and $|\arg(s)| \leq \omega$. Squaring and adding (14) and (15) we get the relation

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 = b^2\exp[-2\varrho\tau\cos(\psi)] \tag{16}$$

as the necessary condition for $\varrho$, $\psi$ to satisfy (14) and (15). We distinguish two cases.

Let $\omega \leq \pi/2$. Then the property $|\arg(s)| \leq \omega$ implies $|\psi| \leq \pi/2$, i.e.

$$b^2\exp[-2\varrho\tau\cos(\psi)] \leq b^2\,.$$

On the other hand,

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 > (\varrho^\alpha - |a|)^2\,.$$

This particularly implies that (16) cannot be satisfied for any $\varrho \geq \varrho_1'$, where

$$\varrho_1' = (|a| + |b|)^{1/\alpha}\,,$$

hence (12) has no roots with $|s| \geq \varrho_1'$ and $|\arg(s)| \leq \pi/2$.

Now let $\omega > \pi/2$. Put $\kappa = -\tau\cos(\omega) > 0$. Since

$$\varrho^{2\alpha} - 2a\varrho^\alpha\cos(\alpha\psi) + a^2 < (\varrho^\alpha + |a|)^2\,,$$

it is enough to find $\varrho_1'' > 0$ such that

$$\varrho^\alpha + |a| < |b|\exp[\kappa\varrho]$$

for all $\varrho \geq \varrho_1''$. The existence of such a value is obvious and, using appropriate elementary calculations, we can give various specifications of $\varrho_1''$. To summarize it, for any given $0 < \omega < \pi$, we can find $\rho_1 > 0$ such that (12) has no roots $s$ with $|s| > \varrho_1$ and $|\arg(s)| \leq \omega$.

Further, we can easily observe the existence of a real $\varrho_2 > 0$ sufficiently small such that (12) has no nonzero roots lying inside the circle $|s| < \varrho_2$. Indeed, the Taylor expansion enables to write (12) as

$$s^\alpha = a + b(1 - s\tau + s^2\tau^2/2! - \cdots)\,. \tag{17}$$

If $a+b \neq 0$ then (17) has no roots $s$ with $|s| < \varrho_2$, $\varrho_2 > 0$ being sufficiently small. If $a+b=0$, then $s=0$ is a root of (17) and, assuming $s \neq 0$, (17) yields

$$s^{\alpha-1} = -b(\tau - s\tau^2/2! + \cdots)\,,$$

hence (17) has no nonzero roots $s$ with $|s| < \varrho_2$, $\varrho_2 > 0$ being sufficiently small.

Finally, let $\Omega$ be the compact set of complex $s$ such that $\varrho_1 \leq |s| \leq \varrho_2$ and $|\arg(s)| \leq \omega$. Then $Q$ is analytic on $\Omega$, hence it cannot have infinitely many roots in $\Omega$. This completes the proof of (iii).

To prove the property (iv), we assume that the relations

$$s^\alpha - a - b\exp[-s\tau] = 0 \qquad \text{and} \qquad \alpha s^{\alpha-1} + \tau b\exp[-s\tau] = 0 \tag{18}$$

hold for a suitable nonzero complex $s$ (obviously, $s=0$ cannot satisfy (18)). Elimination of the exponential term yields

$$a = s^{\alpha-1}\Big(s + \frac{\alpha}{\tau}\Big)\,. \tag{19}$$

If we substitute (13) into (19) and compare imaginary parts, we obtain

$$\tau\varrho\sin(\alpha\psi) = \alpha\sin[(1-\alpha)\psi]\,. \tag{20}$$

Similarly, (18) implies

$$b = -\frac{\alpha}{\tau}s^{\alpha-1}\exp[s\tau] \tag{21}$$

and an analogous argumentation as used above leads to

$$\tau\varrho\sin(\psi)+(\alpha-1)\psi=k\pi \tag{22}$$

for a suitable integer $k$.

Now assume that $\psi\neq 0$. Then we can express $\tau\varrho$ from (20) and (22) to compare their corresponding right-hand sides. This yields

$$g_1(\psi)\equiv\frac{\alpha\sin(\psi)}{\sin(\alpha\psi)}=\frac{(1-\alpha)\psi+k\pi}{\sin[(1-\alpha)\psi]}\equiv g_2(\psi)\,. \tag{23}$$

We show that

$$g_1(\psi)<g_2(\psi)\qquad\text{for all } -\pi<\psi\le\pi,\ \psi\neq 0 \text{ and all integers } k\,. \tag{24}$$

Obviously, it is enough to check (24) for all $0<\psi\le\pi$ and $k=0$. In such a case, $g_1(\psi)=g_2(\psi)=1$ as $\psi\to 0$ and $g_1'(\psi)<0$, $g_2'(\psi)>0$ for all $0<\psi<\pi$ (verification of these sing derivative properties requires some routine and straightforward calculations which are omitted). This confirms (24), hence (23) cannot occur for any $-\pi<\psi\le\pi$, $\psi\neq 0$ and any integer $k$. In other words, (18) cannot be satisfied if $s$ is a complex number with a nonzero argument $\psi$.

If $\psi=0$ then $s=\varrho$ and

$$a=\varrho^{\alpha-1}\left(\varrho+\frac{\alpha}{\tau}\right),\qquad b=-\frac{\alpha}{\tau}\varrho^{\alpha-1}\exp[\varrho\tau] \tag{25}$$

by use of (19) and (21). For all $\varrho>0$, (25) defines the set of all couples $(a, b)$ such that there exists a positive real $s$ satisfying (18). □

**Remark 1.**

(a) We can easily check that the condition $Q(s)=Q'(s)=Q''(s)=0$ does not hold for any complex $s$. In other words, if $s$ is a multiple root of $Q$, then it is a (positive real) double root.

(b) The property Proposition 2 (iii) generalizes the well-known fact, namely that (5) has only finitely many roots with positive real parts (indeed, if we put $\omega=\pi/2$, we get such a conclusion also for (12)).

In classical analysis of (2), asymptotic stability of the zero solution occurs if and only if all roots of (5) have negative real parts (effective conditions ensuring this property are involved in Theorems 1 and 2). In the next section, we deduce an analogous requirement on distribution of roots of (12). Therefore, the next aim of this section is to derive effective conditions on parameters of (12) ensuring that all its roots have negative real parts.

In the case of standard (integer-order) delay differential equations and their associated characteristic equations, this matter is solved via the D-partition method (see, e.g. [12]). This method is applicable also in the fractional-order case and we perform its basic steps (routine calculations will be omitted).

Let $BL_\alpha^\tau(a,b)$ be the boundary locus for (12), i.e. the set of all real couples $(a, b)$ such that (12) admits a purely imaginary root $s=\mathrm{i}\varphi=\varphi\exp[\mathrm{i}\pi/2]$ (note that it is enough to consider only the case $\varphi\ge 0$ due to the property Proposition 2 (ii)). By (14) and (15), such couples $(a, b)$ satisfy

$$\varphi^\alpha\cos(\alpha\pi/2)-a-b\cos(\varphi\tau)=0\,, \tag{26}$$

$$\varphi^\alpha\sin(\alpha\pi/2)+b\sin(\varphi\tau)=0\,. \tag{27}$$

Now we distinguish two cases. First let $\varphi\tau=m\pi$ for a suitable integer $m$. Then (27) implies $\varphi=0$, i.e. $m=0$, and the relevant part of $BL_\alpha^\tau(a,b)$ corresponding to this case consists of the line $a+b=0$.

Now let $\varphi\tau\neq m\pi$ for all integers $m$. Then (26) and (27) yield the solution

$$a=\frac{\varphi^\alpha\sin(\varphi\tau+\alpha\pi/2)}{\sin(\varphi\tau)},\qquad b=-\frac{\varphi^\alpha\sin(\alpha\pi/2)}{\sin(\varphi\tau)}\,.$$

Hence, $BL_\alpha^\tau(a,b)$ corresponding to this case is formed by the system of curves

$$a_m(\varphi)=\frac{\varphi^\alpha\sin(\varphi\tau+\alpha\pi/2)}{\sin(\varphi\tau)},\qquad b_m(\varphi)=-\frac{\varphi^\alpha\sin(\alpha\pi/2)}{\sin(\varphi\tau)}\,, \tag{28}$$

$m\pi/\tau<\varphi<(m+1)\pi/\tau$, $m=0,1,\ldots$. The set $BL_\alpha^\tau(a,b)$ is depicted in the $(a, b)$-plane on Fig. 2. We note that the construction of this set for (12) involving multi-valued function $s^\alpha$ has been performed in [11]. As we have emphasized in the part preceding Proposition 1, it is enough to consider appropriate single-valued function in this problem.

Our next analysis of the area of couples $(a, b)$ such that (12) has all roots with negative real parts originates from continuous dependance of roots of (12) on the coefficients $a, b$. This property particularly implies that the number of roots of (12) with positive real parts can be changed only when its coefficients $a, b$ are crossing an element of $BL_\alpha^\tau(a,b)$. In other words, the number of roots of (12) having a positive real part remains unchanged in all open sets whose boundaries are formed by the line $a+b=0$ or by some curves (28) (or their parts). Then it is enough to choose representatives of these open sets to specify the number of roots of (12) with positive real parts within these sets.

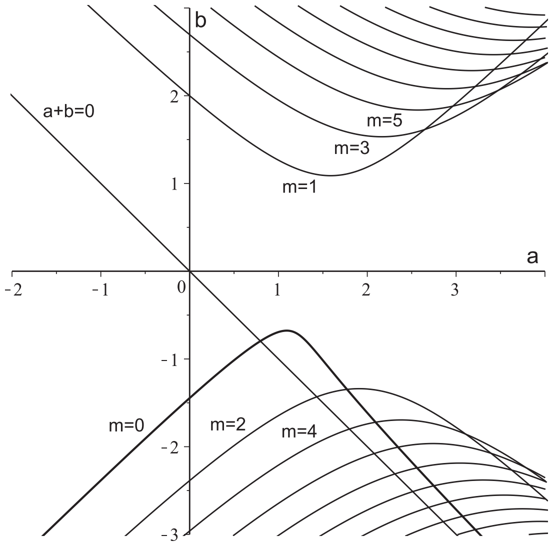


**Fig. 2.** The set $BL_\alpha^\tau(a,b)$ for $\alpha=0.4,\ \tau=1$.

Since we are looking for the couples $(a, b)$ such that (12) has all roots with negative real parts, Proposition 2 (i) implies that it is enough to restrict on the half-plane $a+b<0$. It is easy to check that the curves (28) with even $m$ intersect the line $a+b=0$ when $\varphi=(m+1-\alpha)\pi/\tau$, hence it remains to investigate their parts

$$(a_m(\varphi),\ b_m(\varphi)),\qquad (m+1-\alpha)\pi/\tau<\varphi<(m+1)\pi/\tau,\quad m=0,2,4,\dots$$

lying to the left of this line. A more detailed computational analysis shows that these curves cross the $b$-axis at the points $b_m^*=-[(2m+2-\alpha)\pi/(2\tau)]^\alpha$, tend to the asymptotes $b=a-[(m+1)\pi/\tau]^\alpha\cos(\alpha\pi/2)$ as $\varphi\to(m+1)\pi/\tau$ from the left and, moreover, these curves do not intersect each other (see also Fig. 2). Then, choosing appropriate points on the $b$-axis as suitable representatives, previous considerations along with Theorem 3 yield that (12) has all roots with negative real parts if and only if the couple $(a, b)$ is an interior point of the area bounded by the line $a+b=0$ from above and by the parametric curve

$$a=\frac{\varphi^\alpha\sin(\tau\varphi+\alpha\pi/2)}{\sin(\tau\varphi)},\quad b=-\frac{\varphi^\alpha\sin(\alpha\pi/2)}{\sin(\tau\varphi)},\qquad \varphi\in\left(\frac{(1-\alpha)\pi}{\tau},\frac{\pi}{\tau}\right)\tag{29}$$

from below.

Note also that an alternative way how to deduce such an area from a given boundary locus is based on differentiation of (12) with respect to some of its parameters. Following this way, it can be shown that $d(\Re(s))/da>0$ at $s=\mathrm{i}\varphi$. This implies that all roots of (12) cross the imaginary axis at $s=\mathrm{i}\varphi$ from the left to the right as $a$ increases, and thus we arrive at the same conclusion as above.

We can summarize the previous considerations in the following

**Proposition 3.** *Let $0<\alpha<1$, $a, b$ and $\tau>0$ be real numbers. Then all roots of (12) have negative real parts if and only if the couple $(a, b)$ is an interior point of the area bounded by the line $a+b=0$ from above and by the parametric curve (29) from below.*

Now we rewrite the parametric form (29) into the explicit one. Doing this, (29) implies that

$$\frac{a}{-b}=\frac{\sin(\varphi\tau+\alpha\pi/2)}{\sin(\alpha\pi/2)}\,.$$

Taking into account the restriction $\varphi\in((1-\alpha)\pi/\tau,\pi/\tau)$, we can express $\varphi$ and eliminate it via its substitution into $(29)_2$. This yields

$$b=-\frac{\big((1-\alpha)\pi/2+\arccos[(-a/b)\sin(\alpha\pi/2)]\big)^\alpha\sin(\alpha\pi/2)}{\tau^\alpha\sin\big((1-\alpha)\pi/2+\arccos[(-a/b)\sin(\alpha\pi/2)]\big)}\,.$$

Obviously, some additional calculations enable to determine the delay $\tau$ from this relation explicitly. Consequently, this argumentation implies that (29) can be written as $\tau=\tau^*$ where

$$\tau^*=\frac{(1-\alpha)\pi/2+\arccos[(-a/b)\sin(\alpha\pi/2)]}{[a\cos(\alpha\pi/2)+(b^2-a^2\sin^2(\alpha\pi/2))^{1/2}]^{1/\alpha}}\tag{30}$$

and $|a|+b<0$. Using this notation we have

**Proposition 4.** *Let $0 < \alpha < 1$, $a, b$ and $\tau > 0$ be real numbers. Then all roots of (12) have negative real parts if and only if it holds either*

$$a \le b < -a \qquad \text{and} \qquad \tau \text{ is arbitrary}, \tag{31}$$

*or*

$$|a| + b < 0 \qquad \text{and} \qquad \tau < \tau^* . \tag{32}$$

**Remark 2.** A similar issue has been discussed in [2] where (12) is analysed under the restrictive assumption $a < 0$ (in our notation). The computational technique used in [2] is different from ours and leads to a formally slightly more complicated evaluation of $\tau^*$.

## 4. Stability and asymptotics of solutions

This section presents our main results on (1). First, we formulate the auxiliary assertion providing a key tool in stability and asymptotic analysis of (1).

**Lemma 1.** *Let $0 < \alpha < 1$, $a, b$ and $\tau > 0$ be real numbers and let $y$ be a solution of (1).*

(i) *Let all roots of (12) have negative real parts. Then*

$$y(t) \sim t^{-\alpha} \qquad \text{or} \qquad y(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty . \tag{33}$$

(ii) *Let all roots of (12) have non-positive real parts and let there exist a root with the zero real part. If (12) has no purely imaginary roots, then*

$$y(t) \sim 1 \qquad \text{or} \qquad y(t) = \mathcal{O}(t^{\alpha-1}) \qquad \text{as } t \to \infty . \tag{34}$$

*If (12) has purely imaginary roots, then*

$$y(t) \sim_{sup} 1 \qquad \text{or} \qquad y(t) = \mathcal{O}(1) \qquad \text{as } t \to \infty . \tag{35}$$

(iii) *Let there exist a root of (12) with a positive real part. Then $y(t) = \mathcal{O}(t \exp[Mt])$ as $t \to \infty$, where $M = \max_{s_i}(\Re(s_i))$, $s_i$ being roots of (12). Moreover, there exists a solution $y$ of (1) such that*

$$y(t) \sim_{sup} t \exp[Mt] \qquad \text{or} \qquad y(t) \sim_{sup} \exp[Mt] \qquad \text{as } t \to \infty . \tag{36}$$

**Remark 3.** The asymptotic property $(33)_1$ or $(33)_2$ occurs if $\phi(0) \neq 0$ or $\phi(0) = 0$, respectively, where $\phi$ is the initial function implying a particular solution of (1) via (11). The same comment holds also for (34) and (35) (a more detailed analysis is involved in Section 5). In particular, under the assumption of the part (ii), there always exist solutions of (1) not tending to the zero solution. Regarding (36), the type of asymptotic behaviour depends on multiplicity of the root of (12) with the maximal positive real part (see also Proposition 2 (iv) and Remark 1).

Now we are in a position to formulate our main stability and asymptotic criteria on (1). If we put through Propositions 3, 4 and Lemma 1, we arrive at the following two results.

**Theorem 4.** *Let $0 < \alpha < 1$, $a, b$ and $\tau > 0$ be real numbers.*

(i) *The zero solution of (1) is asymptotically stable if and only if the couple $(a, b)$ is an interior point of the area bounded by the line $a + b = 0$ from above and by the parametric curve (29) from below. In this case, (33) holds for any solution $y$ of (1).*

(ii) *The zero solution of (1) is stable, but not asymptotically stable, if and only if either*

$$a + b = 0, \qquad a \le \frac{[\pi(1-\alpha)]^{\alpha}}{2\tau^{\alpha}\cos(\alpha\pi/2)} , \tag{37}$$

*or $a, b$ satisfy (29) for an admissible value $\varphi$. In this case, (34) or (35) holds for any solution $y$ of (1).*

**Theorem 5.** *Let $0 < \alpha < 1$, $a, b$ and $\tau > 0$ be real numbers.*

(i) *The zero solution of (1) is asymptotically stable if and only if (31) or (32) is satisfied. In this case, (33) holds for any solution $y$ of (1).*

(ii) *The zero solution of (1) is stable, but not asymptotically stable, if and only if either (37), or*

$$|a| + b < 0, \qquad \tau = \tau^*, \qquad \tau^* \text{ being given by (30)}$$

*is satisfied. In this case, (34) or (35) holds for any solution $y$ of (1).*

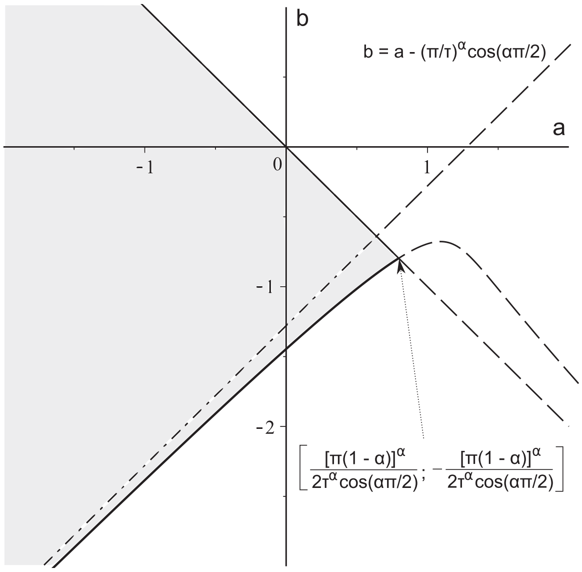


**Fig. 3.** The stability region for (1) with $\alpha = 0.4$ and $\tau = 1$.

The stability area for (1), described via the conditions of Theorems 4 and 5, is depicted in the $(a, b)$-plane on Fig. 3.

**Remark 4.** The assertions of Theorems 4 and 5 provide a direct extension of Theorems 1 and 2, respectively. Indeed, letting $\alpha \to 1$ from the left, one can observe a full agreement between appropriate formulae for the fractional and classical case. In particular, the value (30) of the fractional stability switch, when asymptotic stability of the zero solution of (1) is turning into instability as the delay $\tau$ is monotonically increasing, becomes the appropriate value from (4). Notice also, that if we put $a = 0$ in Theorem 5, we obtain just the assertion of Theorem 3 (where the condition $\phi(0) \neq 0$ is considered).

Now we make some comments on two qualitative dissimilarities between the fractional and classical delayed case. First, the decay rate of solutions in the asymptotically stable case is exponential when $\alpha = 1$, but only algebraic when $0 < \alpha < 1$. Second, the situation on the stability boundary differs at the cusp point

$$P_\alpha = \left[\frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}; -\frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}\right]$$

where the line $a + b = 0$ intersects the transcendental curve (29). The cusp point $P_\alpha$ corresponds to the zero (simple) root of (12) and Theorem 4 (ii) (as well as Theorem 5 (ii)) immediately yields

**Corollary 1.** *Let $0 < \alpha < 1$ and $\tau > 0$ be real numbers. The zero solution of*

$$\mathrm{D}^\alpha y(t) = \frac{[\pi(1-\alpha)]^\alpha}{2\tau^\alpha \cos(\alpha\pi/2)}[y(t) - y(t-\tau)], \qquad t > 0 \tag{38}$$

*is stable, but not asymptotically stable.*

Letting $\alpha \to 1$ from the left, (38) turns into

$$y'(t) = \frac{1}{\tau}[y(t) - y(t-\tau)], \qquad t > 0 \tag{39}$$

and the cusp point $P_\alpha$ becomes $P = [1/\tau; -1/\tau]$. A direct procedure shows that $P$ corresponds to the double zero root of (5) with $a = -b = 1/\tau$, hence the zero solution of (39) cannot be stable. Indeed, we can easily check that (39) admits an unbounded solution, namely $y(t) = t$. Consequently, the stability property of (38) is not transferable to its limit case (39). Thus, this example confirms a positive impact of derivatives of real orders between 0 and 1 on stability properties of studied delay equation.

## 5. Proof of Lemma 1

First, we present several technical assertions where we use the notation

$$\sigma = \gamma(\varepsilon/t, \pi/2 + \delta), \qquad t > 0, \tag{40}$$

$$\sigma' = \gamma(\varepsilon^\alpha, \alpha\pi/2 + \alpha\delta) \tag{41}$$

with real constants $\varepsilon$, $\delta > 0$, $0 < \alpha < 1$ and the contour $\gamma$ given by (7).

**Proposition 5.** *For given $0 < \alpha < 1$ and $\tau > 0$, we denote*

$$\omega_{x,n}(t) = \frac{1}{2\pi\alpha \mathrm{i}} \int_{\sigma'} \frac{u^x \exp[u^{1/\alpha}(1+n\tau/t)]}{(u-at^\alpha)\exp[u^{1/\alpha}\tau/t]-bt^\alpha} \mathrm{d}u\,, \quad x > -1,\ n = 2, 3, \dots .$$

*(i) If there exists $\eta_0 > 0$ such that*

$$|(u-at^\alpha)\exp[u^{1/\alpha}\tau/t]-bt^\alpha| \ge \eta_0 |\exp[u^{1/\alpha}\tau/t]| \tag{42}$$

*for all $u \in \sigma'$, then*

$$\omega_{x,n}(t) = \mathcal{O}(1) \qquad \text{as } t \to \infty\,. \tag{43}$$

*(ii) If there exists $\eta_1 > 0$ such that*

$$|(u-at^\alpha)\exp[u^{1/\alpha}\tau/t]-bt^\alpha| \ge \eta_1 t^\alpha |\exp[u^{1/\alpha}\tau/t]| \tag{44}$$

*for all $u \in \sigma'$, then*

$$\omega_{x,n}(t) = \mathcal{O}(t^{-\alpha}) \qquad \text{as } t \to \infty\,, \tag{45}$$

$$\omega_{x,n+m}(t) = \omega_{x,n}(t) + \mathcal{O}(t^{-1-\alpha}) \qquad \text{as } t \to \infty \tag{46}$$

*where $m \in \mathbb{Z}^+$ is arbitrary.*

**Proof.** First, we prove (43) and (45) simultaneously. Let $\kappa \in \{-\alpha, 0\}$. Utilizing (42) and (44), we obtain

$$\begin{aligned}
|\omega_{x,n}(t)| &\le \frac{t^\kappa}{2\pi\alpha\eta_0} \int_{\sigma'} |u^x||\exp[u^{1/\alpha}(1+(n-1)\tau/t)]||\mathrm{d}u| \\
&\le \frac{t^\kappa}{2\pi\alpha\eta_0} \Bigg( \int_{-\alpha\pi/2-\alpha\delta}^{\alpha\pi/2+\alpha\delta} \varepsilon^{\alpha(x+1)} \exp[\varepsilon(1+(n-1)\tau/t)\cos(\varphi/\alpha)]\mathrm{d}\varphi \\
&\quad + 2\int_{\varepsilon^\alpha}^{\infty} r^x \exp[r^{1/\alpha}(1+(n-1)\tau/t)\cos(\pi/2+\delta)]\mathrm{d}r \Bigg) \\
&\le \frac{t^\kappa}{2\pi\alpha\eta_0} \left( \alpha(\pi+2\delta) + \frac{\alpha\Gamma(\alpha(x+1))}{(\cos(\pi/2-\delta))^{\alpha(x+1)}} \right) = \mathcal{O}(t^\kappa) \qquad \text{as } t \to \infty
\end{aligned}$$

where we have employed the integral definition of the Gamma function (computational details are omitted). This proves (43) and (45).

The formula (46) can be derived via a direct calculation:

$$\begin{aligned}
\omega_{x,n+m}(t) &= \frac{1}{2\pi\alpha \mathrm{i}} \int_{\sigma'} \frac{u^x \exp[u^{1/\alpha}(1+n\tau/t)]}{(u-at^\alpha)\exp[u^{1/\alpha}\tau/t]-bt^\alpha} \sum_{j=0}^{\infty} \frac{(m\tau)^j}{j!t^j} u^{j/\alpha} \mathrm{d}u \\
&= \omega_{x,n}(t) + \frac{1}{2\pi\alpha \mathrm{i}} \sum_{j=1}^{\infty} \frac{(m\tau)^j}{j!t^j} \int_{\sigma'} \frac{u^{x+j/\alpha} \exp[u^{1/\alpha}(1+n\tau/t)]}{(u-at^\alpha)\exp[u^{1/\alpha}\tau/t]-bt^\alpha} \mathrm{d}u \\
&= \omega_{x,n}(t) + \sum_{j=1}^{\infty} \frac{(m\tau)^j}{j!t^j} \omega_{j/\alpha+x,n}(t) = \omega_{x,n}(t) + \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.
\end{aligned}$$

□

**Proposition 6.** *For given $0 < \alpha < 1$ and $\tau > 0$, we denote*

$$\theta_{x,n}(t) = \frac{1}{2\pi\alpha \mathrm{i}} \int_{\sigma'} u^x \exp[u^{1/\alpha}(1+n\tau/t)]\mathrm{d}u\,, \quad x > -1,\ n = 2, 3, \dots .$$

*Then*

$$\theta_{x,n}(t) = \frac{t^{\alpha x+\alpha}(t+n\tau)^{-\alpha x-\alpha}}{\Gamma(1-\alpha x-\alpha)}\,.$$

**Proof.** We set $v = u(1+n\tau/t)^\alpha$ to get

$$\theta_{x,n}(t) = \frac{1}{2\pi\alpha \mathrm{i}} \left(\frac{t}{t+n\tau}\right)^{\alpha x+\alpha} \int_{\gamma((1+n\tau/t)^\alpha\varepsilon^\alpha,\alpha\pi/2+\alpha\delta)} v^x \exp[v^{1/\alpha}]\mathrm{d}v\,.$$

Applying the formula

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\alpha \mathrm{i}} \int_{\gamma(r,\psi)} \exp[\xi^{1/\alpha}]\xi^{1/\alpha-z/\alpha-1}\mathrm{d}\xi\,, \quad r > 0,\ \pi\alpha/2 < \psi < \alpha\pi$$

(see, e.g. [20]), we obtain the result. □

Now we can formulate the key auxilliary assertion of this section dealing with asymptotic properties of the class of $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}$ functions.

**Lemma 2.** *Let $0<\alpha<1$, $0<\beta\le 1$, $a$, $b$ and $\tau>0$ be real numbers and let $s_i$ be roots of (12).*
*(i) If $\Re(s_i)<0$ for all $s_i$ then*

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim t^{\beta-\alpha-1}\ \text{for}\ \alpha\neq\beta \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t)=\mathcal{O}(t^{-\alpha-1}) \qquad \text{as}\ t\to\infty\,.$$

*(ii) If there exists the zero root of (12) and $\Re(s_i)<0$ otherwise, then*

$$\mathcal{R}^{a,b,\tau}_{\alpha,1}(t)\sim 1 \quad \text{and} \quad \mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)=\mathcal{O}(t^{\beta-1})\ \text{for}\ \beta<1 \qquad \text{as}\ t\to\infty\,.$$

*(iii) If $\Re(s_i)\le 0$ for all $s_i$ and some of the roots are purely imaginary, then*

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim_{sup} 1 \qquad \text{as}\ t\to\infty\,.$$

*(iv) If $\Re(s_i)>0$ for some $s_i$ then*

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)\sim_{sup}(Bt+C)\exp[Mt] \qquad \text{as}\ t\to\infty$$

*where $M=\max_{s_i}(\Re(s_i))$ and reals $B$, $C\ge 0$ are such that $B+C>0$.*

**Proof.** It is enough to consider the case $b\neq 0$ (for the case $b=0$ we refer to [18]). Proposition 2 (iii) implies that there exists $\delta>0$ such that all nonzero roots $s_i$ of (12) satisfy $|\arg(s_i)|\neq\pi/2+\delta$ and, moreover, there are only finitely many of them satisfying $|\arg(s_i)|<\pi/2+\delta$. Consequently, for every $t>1$, we can choose $R>\varepsilon>0$ such that all $s_i$ lie to the left of $\gamma(R,\pi/2+\delta)$ and those satisfying $|\arg(s_i)|<\pi/2+\delta$ are located to the right of $\sigma$ given by (40). Thus, we can split the inverse Laplace transform formula into

$$\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma(R,\frac{\pi}{2}+\delta)}\frac{s^{\alpha-\beta}\exp[ts]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=I_1(t)+I_2(t)\,,$$

where $I_1$ and $I_2$ denote integrals over $\gamma(R,\pi/2+\delta)-\sigma$ and $\sigma$, respectively.

I. First, we analyse $I_1$. Clearly, the contour $\gamma(R,\frac{\pi}{2}+\delta)-\sigma$ is a simple positively oriented closed curve. Proposition 2 (iii) implies that the integrand has only finitely many poles $s_i$ $(i=1,\dots,N)$ lying in the interior of $\gamma(R,\frac{\pi}{2}+\delta)-\sigma$. The residue theorem yields

$$I_1(t)=\frac{1}{2\pi\mathrm{i}}\int_{\gamma(R,\frac{\pi}{2}+\delta)-\sigma}\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=\sum_{i=1}^{N}\mathrm{Res}_{s=s_i}\left(\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\right).$$

Proposition 2 (iv) shows that all poles $s_i$ with nonzero imaginary parts are simple, while real poles (more precisely, positive real poles) admit double multiplicity. Thus, for every pole $s_i$, we can utilize the Laurent expansions

$$\frac{s^{\alpha-\beta}}{s^\alpha-a-b\exp[-\tau s]}=a^i_{-2}(s-s_i)^{-2}+a^i_{-1}(s-s_i)^{-1}+a^i_0+a^i_1(s-s_i)+a^i_2(s-s_i)^2+\cdots,$$
$$\exp[st]=\exp[s_it]\left(1+t(s-s_i)+\frac{t^2}{2!}(s-s_i)^2+\frac{t^3}{3!}(s-s_i)^3+\cdots\right)$$

where $a_j$ $(j=-2,-1,0\dots)$ are complex constants independent of $t$. Hence, a multiplication of both expansions enables us to write the sum of residues (i.e. the coefficients at $(s-s_i)^{-1}$) as

$$I_1(t)=\sum_{i=1}^{N}(a^i_{-1}+a^i_{-2}t)\exp[s_it]\,,$$

where $a^i_{-1},a^i_{-2}$ $(i=1,\dots,N)$ are complex constants such that for any $i$ at least one of the terms is nonzero.

II. Now, we analyse the term $I_2$. Employing the change of variables $s=u^{1/\alpha}/t$, which transforms the contour $\sigma$ into $\sigma'$ (given by (41)), we get

$$I_2(t)=\frac{1}{2\pi\mathrm{i}}\int_\sigma\frac{s^{\alpha-\beta}\exp[st]}{s^\alpha-a-b\exp[-\tau s]}\mathrm{d}s=t^{\beta-1}\omega_{(1-\beta)/\alpha,1}(t) \tag{47}$$

where $\omega_{(1-\beta)/\alpha,1}$ is introduced in Proposition 5.

II-1. Let (12) have the zero root, i.e. $a+b=0$. Although $\sigma$ is chosen so that it does not contain any roots of (12), it approaches the zero root as $t\to\infty$. The closest points $s$ of $\sigma$ with respect to the root $s=0$ satisfy $s=\varepsilon/t\exp[\mathrm{i}\varphi]$, i.e. we can write

$$|s^\alpha-a-b\exp[-\tau s]|\ge|\varepsilon^\alpha t^{-\alpha}\exp[\mathrm{i}\alpha\varphi]-a-b\exp[-\varepsilon\tau/t]|\ge\eta_0t^{-\alpha}$$

for a suitable $\eta_0 > 0$ and sufficiently large $t$. Applying the change of variable $s = u^{1/\alpha}/t$, we can directly employ (43) to obtain the asymptotic estimate

$$I_2(t) = \mathcal{O}(t^{\beta-1}) \qquad \text{as } t \to \infty .$$

II-2. Let (12) have only nonzero roots, i.e. $a + b \neq 0$. Since the contour $\sigma$ does not contain any root $s_i$ of (12), there exists $\eta_1 > 0$ such that

$$|s^\alpha - a - b\exp[-\tau s]| \geq \eta_1$$

for all $s \in \sigma$. Similarly to II-1, as a direct consequence of (45) we get

$$I_2(t) = \mathcal{O}(t^{\beta-\alpha-1}) \qquad \text{as } t \to \infty .$$

In the sequel, we precise this estimate.

II-2-a. Let $a \neq b$. Substituting the above suggested change of variable, we get (along with some simple calculations) that (44) is satisfied. Now, we use the identity

$$\frac{1}{\xi - z} = -\frac{1}{z} - \frac{\xi}{z^2} + \frac{\xi^2}{z^2(\xi - z)} \tag{48}$$

(with $z = bt^\alpha$ and $\xi = (u - at^\alpha)\exp[u^{1/\alpha}\tau/t]$) and (45), (46) to get

$$I_2(t) = -\frac{t^{\beta-\alpha-1}}{b}\theta_{(1-\beta)/\alpha,1}(t) - \frac{t^{\beta-2\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha+1,2}(t) + \frac{at^{\beta-\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha,2}(t)$$
$$+ \frac{t^{\beta-2\alpha-1}}{b^2}\int_{\sigma'} \frac{u^{(1-\beta)/\alpha}(u - at^\alpha)^2\exp[u^{1/\alpha}(1 + 3\tau/t)]}{(u - at^\alpha)\exp[u^{1/\alpha}\tau/t] - bt^\alpha}\mathrm{d}u .$$

Further decomposition of the last term yields

$$\begin{aligned}
I_2(t) = &-\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}} \\
&+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \frac{a^2\omega_{(1-\beta)/\alpha,3}(t)}{b^2t^{-\beta+1}} \\
= &-\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}} \\
&+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \frac{a^2}{b^2}I_2(t) + \mathcal{O}(t^{\beta-\alpha-2})
\end{aligned}$$

as $t \to \infty$. Rearranging the expression, we arrive at

$$\begin{aligned}
I_2(t) = \frac{b^2}{b^2 - a^2}\Bigg(&-\frac{\theta_{(1-\beta)/\alpha,1}(t)}{bt^{-\beta+\alpha+1}} - \frac{\theta_{(1-\beta)/\alpha+1,2}(t)}{b^2t^{-\beta+2\alpha+1}} + \frac{a\theta_{(1-\beta)/\alpha,2}(t)}{b^2t^{-\beta+\alpha+1}} \\
&+ \frac{\omega_{(1-\beta)/\alpha+2,3}(t)}{b^2t^{-\beta+2\alpha+1}} - \frac{2a\omega_{(1-\beta)/\alpha+1,3}(t)}{b^2t^{-\beta+\alpha+1}} + \mathcal{O}(t^{\beta-\alpha-2})\Bigg)
\end{aligned}$$

as $t \to \infty$. An analogous argumentation yields

$$\begin{aligned}
\omega_{(1-\beta)/\alpha+1,3}(t) = \frac{b^2}{b^2 - a^2}\Bigg(&-\frac{\theta_{(1-\beta)/\alpha+1,3}(t)}{bt^\alpha} - \frac{\theta_{(1-\beta)/\alpha+2,4}(t)}{b^2t^{2\alpha}} \\
&+ \frac{a\theta_{(1-\beta)/\alpha+1,4}(t)}{b^2t^\alpha} + \frac{\omega_{(1-\beta)/\alpha+3,5}(t)}{b^2t^{2\alpha}} - \frac{2a\omega_{(1-\beta)/\alpha+2,5}(t)}{b^2t^\alpha} + \mathcal{O}(t^{-\alpha-1})\Bigg)
\end{aligned}$$

as $t \to \infty$. Combining these expressions with Propositions 5 (ii) and 6, we get

$$\begin{aligned}
I_2(t) = \frac{b^2}{b^2 - a^2}\Bigg(&-\frac{(t+\tau)^{\beta-\alpha-1}}{b\Gamma(\beta-\alpha)} - \frac{(t+2\tau)^{\beta-2\alpha-1}}{b^2\Gamma(\beta-2\alpha)} + \frac{a(t+2\tau)^{\beta-\alpha-1}}{b^2\Gamma(\beta-\alpha)} \\
&+ \mathcal{O}(t^{\beta-3\alpha-1}) + \frac{2a(t+3\tau)^{\beta-2\alpha-1}}{b(b^2-a^2)\Gamma(\beta-2\alpha)} + \frac{2a(t+4\tau)^{\beta-3\alpha-1}}{b^2(b^2-a^2)\Gamma(\beta-3\alpha)} \\
&- \frac{2a^2(t+4\tau)^{\beta-2\alpha-1}}{b^2(b^2-a^2)\Gamma(\beta-2\alpha)} + \mathcal{O}(t^{\beta-4\alpha-1}) + \mathcal{O}(t^{\beta-3\alpha-1}) + \mathcal{O}(t^{\beta-2\alpha-2}) + \mathcal{O}(t^{\beta-\alpha-2})\Bigg) \\
= &\frac{b(t+\tau)^{\beta-\alpha-1} - a(t+2\tau)^{\beta-\alpha-1}}{-(b^2-a^2)\Gamma(\beta-\alpha)} \\
&+ \frac{(b^2-a^2)(t+2\tau)^{\beta-2\alpha-1} - 2ab(t+3\tau)^{\beta-2\alpha-1} + 2a^2(t+4\tau)^{\beta-2\alpha-1}}{-(b^2-a^2)^2\Gamma(\beta-2\alpha)} + \mathcal{O}(t^{\beta-\alpha-1-\min(2\alpha,1)})
\end{aligned}$$

as $t \to \infty$. Taking into account the property $1/\Gamma(0) = 0$, we can derive

$$\lim_{t\to\infty} \frac{|I_2(t)|}{t^{\beta-\alpha-1}} = \frac{-(a+b)^{-1}}{\Gamma(\beta-\alpha)} \quad \text{and} \quad \lim_{t\to\infty} \frac{|I_2(t)|}{t^{\beta-2\alpha-1}} = \frac{-(a+b)^{-2}}{\Gamma(\beta-2\alpha)} \tag{49}$$

for $\alpha \neq \beta$ and $\alpha = \beta$, respectively. Thus, the asymptotic behaviour of $I_2$ is described for $a \neq b$.

II-2-b. Now, let $a = b \neq 0$ and $\alpha \neq \beta$. We show that $(49)_1$ is transferable to this case through $a = b + \zeta$ as $\zeta \to 0$. Based on (47) and $(49)_1$, we denote

$$I_2^{\zeta}(t) = \frac{t^{\beta-1}}{2\pi\alpha\mathrm{i}} \int_{\sigma'} \frac{u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+\tau/t)]}{(u-(b+\zeta)t^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha}} \mathrm{d}u\,,$$

$$L(\zeta) = \frac{-1}{(2b+\zeta)\Gamma(\beta-\alpha)}\,.$$

Some straightforward (but tedious) calculations enable us to write

$$\begin{aligned} I_2^{\zeta}(t) - I_2^{0}(t) = {} & \frac{\zeta t^{\beta-\alpha-1}}{b^2}\theta_{(1-\beta)/\alpha,2}(t) + \frac{t^{\beta-2\alpha-1}}{b^2} \\ & \times \int_{\sigma'} \left( \frac{\zeta u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+4\tau/t)](-u^2 t^{\alpha} + u(2b+\zeta)t^{2\alpha} - b(b+\zeta)t^{3\alpha})}{[(u-(b+\zeta)t^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha}][(u-bt^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha}]} \right. \\ & \left. + \frac{\zeta u^{(1-\beta)/\alpha} \exp[u^{1/\alpha}(1+3\tau/t)](2ubt^{2\alpha} - b(2b+\zeta)t^{3\alpha})}{[(u-(b+\zeta)t^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha}][(u-bt^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha}]} \right) \mathrm{d}u\,. \end{aligned}$$

Utilizing similar estimates as in the proof of Proposition 5, we arrive at

$$|I_2^{\zeta}(t) - I_2^{0}(t)| \le |\zeta| K t^{\beta-\alpha-1}$$

where $K > 0$ is a suitable real constant (independent of $t$ and $\zeta$). Further, continuity of $L$ at $\zeta = 0$ yields that for every $\varepsilon > 0$ there exists $\chi(\varepsilon) > 0$ such that if $|\zeta| < \chi(\varepsilon)$ then $|L(\zeta) - L(0)| < \varepsilon/3$. Similarly, (49) implies that for every $\varepsilon > 0$ and every $\zeta$ there exists $N(\varepsilon, \zeta) > 0$ such that if $t > N(\varepsilon, \zeta)$ then $|t^{1+\alpha-\beta}|I_2^{\zeta}(t)| - L(\zeta)| < \varepsilon/3$.

Employing the above stated properties, we can conclude that $(49)_1$ holds also for $a = b \neq 0$ and $\alpha \neq \beta$, i.e. for every $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ such that if $t > N(\varepsilon)$ then $|t^{1+\alpha-\beta}|I_2^{0}(t)| - L(0)| < \varepsilon$. Indeed, it holds

$$\begin{aligned} |t^{1+\alpha-\beta}|I_2^{0}(t)| - L(0)| \le {} & |t^{1+\alpha-\beta}|I_2^{\zeta}(t)| - L(\zeta)| \\ & + t^{1+\alpha-\beta}|I_2^{\zeta}(t) - I_2^{0}(t)| + |L(\zeta) - L(0)| \le \varepsilon/3 + \zeta K + \varepsilon/3 < \varepsilon \end{aligned}$$

where $\zeta$ is chosen such that $|\zeta|K < \varepsilon/3$.

II-2-c. Finally, we discuss the case $a = b \neq 0$ and $\alpha = \beta$. Applying (48) with $z = bt^{\alpha}(\exp[u^{1/\alpha}\tau/t] + 1)$ and $\xi = u\exp[u^{1/\alpha}\tau/t]$ to (47), we obtain

$$\begin{aligned} I_2(t) = {} & \frac{t^{\alpha-1}}{2\pi\alpha\mathrm{i}} \int_{\sigma'} \left( \frac{-u^{1/\alpha-1}\exp[u^{1/\alpha}(1+\tau/t)]}{bt^{\alpha}(\exp[u^{1/\alpha}\tau/t]+1)} - \frac{u^{1/\alpha}\exp[u^{1/\alpha}(1+2\tau/t)]}{b^2 t^{2\alpha}(\exp[u^{1/\alpha}\tau/t]+1)^2} \right. \\ & \left. + \frac{u^{1/\alpha+1}\exp[u^{1/\alpha}(1+3\tau/t)]}{b^2 t^{2\alpha}(\exp[u^{1/\alpha}\tau/t]+1)^2((u-bt^{\alpha})\exp[u^{1/\alpha}\tau/t] - bt^{\alpha})} \right) \mathrm{d}u\,. \end{aligned}$$

Estimates employed in the proof of Proposition 5 yield

$$I_2(t) = \frac{-t^{-1}}{2\pi\alpha b\mathrm{i}} G(t) + \mathcal{O}(t^{-\alpha-1}) + \mathcal{O}(t^{-2\alpha-1}) \qquad \text{as } t \to \infty$$

where $G$, utilizing the change of variable $u^{1/\alpha} = p$, is given by

$$G(t) = \int_{\sigma'} \frac{u^{1/\alpha-1}\exp[u^{1/\alpha}(1+\tau/t)]}{\exp[u^{1/\alpha}\tau/t]+1} \mathrm{d}u = \alpha \int_{\gamma(\varepsilon,\pi/2+\delta)} \frac{\exp[p(1+\tau/t)]}{\exp[p\tau/t]+1} \mathrm{d}p\,.$$

We can see that the integrand of the second expression is a function analytical in the complex plane, hence integral of this function taken over any closed contour is zero. Utilizing this property, we can write

$$0 = \lim_{R\to\infty} \oint_{\Sigma(\varepsilon,\delta,R)} \frac{\exp[p(1+\tau/t)]}{\exp[p\tau/t]+1} \mathrm{d}p = \frac{G(t)}{\alpha} + \lim_{R\to\infty} G_R(t)$$

with the contour $\Sigma(\varepsilon, \delta, R)$ depicted on Fig. 4. $G_R$ can be estimated as

$$|G_R(t)| = \left| \int_{\pi/2+\delta}^{3\pi/2-\delta} \frac{\exp[R(1+\tau/t)(\cos(\varphi) + \mathrm{i}\sin(\varphi))]}{\exp[R\tau/t(\cos(\varphi) + \mathrm{i}\sin(\varphi))] + 1} R\mathrm{i}\exp[\mathrm{i}\varphi]\mathrm{d}\varphi \right|$$

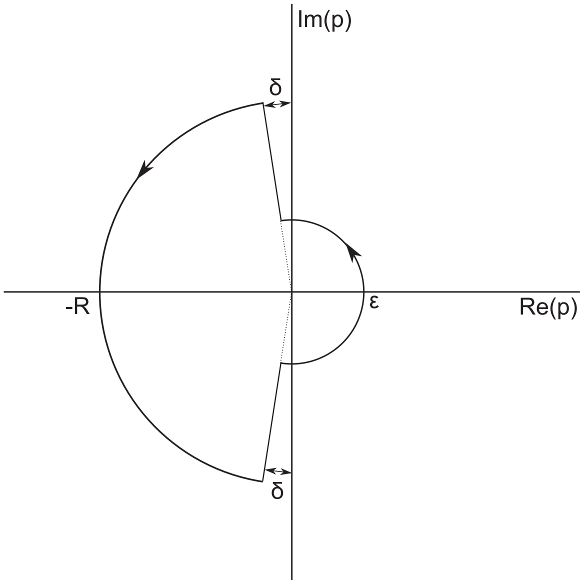


**Fig. 4.** The oriented closed contour $\Sigma(\varepsilon, \delta, R)$ in the complex plane.

$$\leq \frac{R}{2}\exp[-R(1+\tau/t)\sin(\delta)](\pi - 2\delta)\,.$$

Since $G_R(t) \to 0$ as $R \to \infty$ for any $t > 0$, we arrive at $G(t) \equiv 0$. Thus, it holds

$$I_2(t) = \mathcal{O}(t^{-\alpha-1}) \qquad \text{as } t \to \infty\,.$$

III. Combining all the obtained results, we get Lemma 2 (i), (iii) and (iv). In particular, if all the roots of (12) have negative real parts, then algebraic decay becomes dominant over the exponential with negative argument.

If (12) has the zero root, we have $\mathcal{R}^{a,b,\tau}_{\alpha,\beta}(t) = \mathcal{O}(t^{\beta-1})$ as $t \to \infty$ (note that $I_1$ equals zero) which corresponds to Lemma 2 (ii) for $\beta < 1$. In the case $\beta = 1$, we can rewrite (9) as

$$\begin{aligned}\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) &= \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{bs^{-1}(1-\exp[-\tau s])}{s^\alpha + b(1-\exp[-\tau s])}\right)(t) = 1 - b\int_{t-\tau}^{t} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(\xi)\mathrm{d}\xi \\ &= 1 - b\tau\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-c(t)) = 1 + \mathcal{O}(t^{\alpha-1}) \sim 1 \qquad \text{as } t \to \infty\end{aligned}$$

$(0 \leq c(t) \leq \tau)$ which concludes the proof of Lemma 2 (ii). □

**Remark 5.** We note that our technique enables even a stronger formulation of Lemma 2 (i). In particular, if $a \neq b$ then $\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) \sim t^{-\alpha-1}$ as $t \to \infty$.

To complete the proof of Lemma 1, we consider (10) implying that every solution $y$ of (1) consists of terms

$$\phi(0)\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \qquad \text{and} \qquad b\int_{-\tau}^{0} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)h(t-\tau-u)\phi(u)\mathrm{d}u\,.$$

Let all the roots of (12) have negative real parts. By Lemma 2 (i), we have $\mathcal{R}^{a,b,\tau}_{\alpha,1}(t) \sim t^{-\alpha}$ and $\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t) = \mathcal{O}(t^{-\alpha-1})$ as $t \to \infty$ as $t \to \infty$. For $t > \tau$, we get

$$\Big|\int_{-\tau}^{0} \mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(t-\tau-u)\phi(u)\mathrm{d}u\Big| \leq \sup_{t-\tau\leq u\leq t} |\mathcal{R}^{a,b,\tau}_{\alpha,\alpha}(u)| \int_{-\tau}^{0} |\phi(u)|\mathrm{d}u = \mathcal{O}(t^{-\alpha-1})$$

as $t \to \infty$. Thus, we arrive at $(33)_1$ for $\phi(0) \neq 0$ and $(33)_2$ for $\phi(0) = 0$.

Analogously, Lemma 2 (ii), (iii) and (iv) implies (34), (35) and (36), respectively. Moreover, the existence of a constant solution of (1) in the case $a+b=0$ (when the zero root of (12) occurs) can be checked via a direct substitution into (1). Also, the property $(36)_1$ or $(36)_2$ occurs if the root of (12) with the maximal real part is double or single, respectively.

## 6. Concluding remarks

In this paper, we have discussed stability and asymptotic behaviour of a linear prototype of fractional delay differential equations. The main results (Theorems 4 and 5) have formulated two types of necessary and sufficient conditions for stability and asymptotic stability of the zero solution of (1), including asymptotic formulae for its solutions. The derived stability conditions confirm expectations on positive influence of fractional-order derivative $0 < \alpha < 1$ on stability properties of (1) compared with the classical case (2) when $\alpha = 1$. The stability region for (1) is wider than that for (2) and, moreover, the zero solution of (1) is stable for all couples $(a, b)$ lying on the stability boundary (this property does not hold in the case of (2) when the zero solution is not stable at the cusp point of the stability boundary). Another distinction consists in the form of decay rate of solutions of (1) in the asymptotically stable case, which is not exponential, but algebraic.

Following the topic of this paper, we outline some next possible research directions. The most preferable direction is an extension of our results to the vector case when (1) becomes

$$\mathrm{D}^{\alpha}y(t) = Ay(t) + By(t-\tau), \qquad t > 0 \tag{50}$$

with $d \times d$ real matrices $A$, $B$. The knowledge od stability conditions for (50) is important from a theoretical as well as practical view ((50) describes, among others, the fractional-order time delay state space model of $PD^{\alpha}$ control of Newcastle robot, see [16]). It should be noted that the problem of necessary and sufficient conditions for (asymptotic) stability of the zero solution of (50) seems to be a very complicated matter which is not answered neither in the integer-order case $\alpha = 1$ (for a particular result on this problem we refer to [13]). Also, these stability investigations can be extended to other types of fractional equations involving more general types of delays (see, e.g. [19,22,23]), to fractional evolution equations (see, e.g. [24,26]), or to basic numerical discretizations of (1). From this viewpoint, we hope that results of this paper might be a starting point for numerical stability investigations of fractional delay differential equations.

## Acknowledgements

The research was supported by the project FSI–S–14–2290 of Brno University of Technology.

The authors are grateful to a referee for reading this paper and his/her recommendations which helped to improve its content.

## References

[1] A. Bellen, M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, Oxford, 2003.
[2] S. Bhalekar, Stability analysis of a class of fractional delay differential equations, Pramana-J. Phys. 81 (2) (2013) 215–224.
[3] A. Boichuk, J. Diblík, D. Khusainov, M. Růžičková, Fredholm's boundary-value problems for differential systems with a single delay, Nonlinear Anal. 72 (2010) 2251–2258.
[4] J. Čermák, J. Horníček, T. Kisela, Stability regions for fractional differential systems with a time delay, Commun. Nonlinear. Sci. Numer. Simul. 31 (2016) 108–123.
[5] Y. Chen, K.L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems, Nonlinear Dyn. 29 (2002) 191–200.
[6] W. Deng, C. Li, J. Lü, Stability analysis of linear fractional differential system with multiple time delays, Nonlinear Dyn. 48 (2007) 409–416.
[7] G. Doetsch, Introduction to the Theory and Application of the Laplace Transformation, Springer-Verlag, 1974.
[8] M.V.S. Frasson, On the dominance of roots of characteristic equations for neutral functional differential equations, Appl. Math. Comput. 214 (1) (2009) 66–72.
[9] N. Guglielmi, Delay dependent stability regions of $\theta$-methods for delay differential equations, IMA J. Numer. Anal. 18 (1998) 399–418.
[10] J.K. Hale, S.M.V. Lunel, Introduction to Functional Differential Equations, Springer-Verlag, 1993.
[11] E. Kaslik, S. Sivasundaram, Analytical and numerical methods for the stability analysis of linear fractional delay differential equations, J. Comput. Appl. Math. 236 (2012) 4027–4041.
[12] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications for Functional Differential Equations, Kluwer Academic Publishers, 1999.
[13] T. Khokhlova, M. Kipnis, V. Malygina, The stability cone for a delay differential matrix equation, Appl. Math. Lett. 24 (2011) 742–745.
[14] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, 2006.
[15] K. Krol, Asymptotic properties of fractional delay differential equations, Appl. Math. Comput. 218 (2011) 1515–1532.
[16] M. Lazarević, Stability and stabilization of fractional order time delay systems, Sci. Tech. Rev. 61 (2011) 31–44.
[17] C.P. Li, F.R. Zhang, A survey on the stability of fractional differential equations, Eur. Phys. J. Special Topics 193 (2011) 27–47.
[18] D. Matignon, Stability results on fractional differential equations with applications to control processing, in: Proceedings of IMACS-SMC, Lille, France, 1996, pp. 963–968.
[19] M. Medved', M. Pospíšil, L. Škripková, On exponential stability of nonlinear fractional multidelay integrodifferential equations defined by pairwise permutable matrices, Appl. Math. Comput. 227 (2014) 456–468.
[20] I. Podlubný, Fractional Differential Equations, Academic Press, 1999.
[21] D. Qian, C. Li, R.P. Agarwal, P.J.Y. Wong, Stability analysis of fractional differential system with riemann-liouville derivative, Math. Comput. Model. 52 (5–6) (2010) 862–874.
[22] J.R. Wang, Y. Zhou, M. Medved', On the solvability and optimal controls of fractional integrodifferential evolution systems with infinite delay, J. Optim. Theory Appl. 152 (2012) 31–50.
[23] Y. Zhou, F. Jiao, J. Pecaric, Abstract cauchy problem for fractional functional differential equations in banach spaces, Topol. Methods Nonlinear Anal. 42 (2013a) 119–136.
[24] Y. Zhou, L. Zhang, X.H. Shen, Existence of mild solutions for fractional evolution equations, J. Integral Equations Appl. 25 (2013b) 557–586.
[25] Y. Zhou, Basic Theory of Fractional Differential Equations, World Scientific, 2014.
[26] Y. Zhou, V. Vijayakumar, R. Murugesu, Controllability for fractional evolution inclusions without compactness, Evol. Equ. Control Theory 4 (2015) 507–524.