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The hyperbolicity problem in electrical circuit theory

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The time-domain characterization of qualitative properties of electrical circuits requires the combined use of mathematical concepts and tools coming from digraph theory, applied linear algebra and the theory of differential-algebraic equations. This applies, in particular, to the analysis of the circuit *hyperbolicity*, a key qualitative feature regarding oscillations. A linear circuit is hyperbolic if all of its eigenvalues are away from the imaginary axis. Characterizing the hyperbolicity of a strictly passive circuit family is a two-fold problem, which involves the description of (so-called *topologically non-hyperbolic*) configurations yielding purely imaginary eigenvalues (PIEs) for all circuit parameters and, when this is not the case, the description of the parameter values leading to PIEs. A full characterization of the problem is shown here to be feasible for certain circuit topologies. The analysis is performed in terms of differential-algebraic branch-oriented circuit models, which drive the spectral study to a matrix pencil setting, and makes systematic use of a matrix-based formulation of digraph properties. Several examples illustrate the results. Copyright © 2010 John Wiley & Sons, Ltd.

Keywords: electrical circuit; hyperbolicity; digraph; differential-algebraic equation; matrix pencil

1. Introduction

The study of different features of electrical circuits requires the use of an analytical framework which involves several mathematical disciplines. In particular, the difficulties arising in the computer-based generation of state-space models (namely, models formulated in terms of explicit ordinary differential equations) for the simulation of non-linear circuit dynamics have driven the attention to *semistate* models supported on differential-algebraic equations (DAEs); cf. [1–7]. On the other hand, many circuit properties can be assessed in terms of the network underlying digraph; digraph theory actually plays a prominent role in circuit analysis and design (see e.g. [8–11] and references therein). Both disciplines (DAEs and digraphs) involve in turn a systematic use of applied linear algebra; this is related to the role of matrix pencils in DAE theory and to the matrix-based representation of digraphs, although other topics coming from matrix analysis are also relevant in these fields. Therefore, the time-domain analysis of electrical circuits raises problems which must be tackled in a truly interdisciplinary manner. Certainly, this applies to the analysis of qualitative features of circuit dynamics and, in particular, to the *hyperbolicity* problem introduced below.

It is a well-known fact in electrical circuit theory that LC networks oscillate. This can be expressed in several (equivalent) mathematical forms; in particular, we can say that the eigenvalues of the matrix arising in the state-space equation of a circuit composed of linear, uncoupled, strictly passive inductors and capacitors are purely imaginary. If we consider RLC circuits instead (that is, if linear, strictly passive resistors are allowed) things change. It is not trivial to say whether a given RLC circuit has a purely imaginary eigenvalue (PIE) or not. A linear RLC circuit will be said to be *hyperbolic* if all eigenvalues are away from the imaginary axis, and *non-hyperbolic* if it has at least a null eigenvalue or a (conjugate pair of) purely imaginary eigenvalue(s).

Consider e.g. the circuits displayed in Figure 1.

Assume that all devices are linear, uncoupled and strictly passive; that is, let $K_i > 0$ for $K = R, L, C$ and $i = 1, 2$. As detailed in Section 4, the circuit on the left will exhibit a PIE if and only if the relations $L_1 = L_2$, $C_1 = C_2$ hold. The circuit on the right will have a PIE just if

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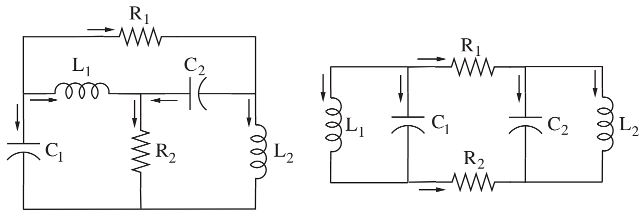

Figure 1. RLC circuits exhibiting PIEs for certain parameter values.

the relation $L_1C_1 = L_2C_2$ is met. The circuits are therefore hyperbolic except for parameter values satisfying these constraints. Note that checking this by inspection is not easy, even in these simple cases, and becomes unfeasible as the circuit complexity increases.

It is known [12–15] that the existence of null eigenvalues in strictly passive RLC circuits depends on the presence of purely inductive loops or purely capacitive cutsets (the 'cutset' notion is explained in Section 2.3). The hyperbolicity problem then relies on the characterization of the existence of (non-vanishing) purely imaginary eigenvalues. This is a two-fold problem; it involves the characterization of (so-called *topologically non-hyperbolic*) circuit configurations yielding PIEs for all circuit parameters and, when this is not the case, the description of the parameter values leading to PIEs.

As in previous research [15], our goal is to discuss these issues in terms of the circuit topology. In particular, we would like to characterize in topological terms the codimension of the manifold defined by the parameter values which yield PIEs; that is, to explain why the number of constraints yielding PIEs in the examples above are two and one, respectively. The fact that these constraints do not depend on the resistance values (as long as resistors are strictly passive) will be proved to hold in general.

The results will be addressed in terms of semistate (differential-algebraic) circuit models, which allow for a direct formulation of the network equations. In this way we avoid the need for any assumptions allowing for a state-space formulation, since such assumptions would be unnecessary from a qualitative point of view. The use of branch-oriented models (cf. Section 2.2; see also [5, 10, 16]) instead of nodal ones [1–3, 6, 7, 17] will turn out to be advantageous. Although the results will also hold for circuits including independent sources (cf. Proposition 1 in Section 3.1), we prefer to frame the analysis in the RLC context since this allows for somewhat simpler statements; the reformulation for VIRLC circuits (namely, circuits including also voltage and current sources) is straightforward.

This paper is structured as follows: The role of qualitative aspects and, in particular, of hyperbolic configurations in circuit theory is discussed in Section 2, which also includes some background material on circuit modeling and digraph theory. In Section 3, we address the main results of this paper. Sections 3.1 and 3.2 include some results of general interest on this topic; in particular (see Proposition 6) we show that strictly passive resistors may in a sense be removed from the analysis, allowing us to tackle the problem in terms of certain properties of the LC-subcircuit. We then present in Sections 3.3 and 3.4 a full solution of the problem for circuits exhibiting just one LC-cutset or one LC-loop, respectively. The study of these configurations is motivated by the fact that the existence of at least one LC-cutset and one LC-loop is known to be a necessary condition for the existence of PIEs [15]. These restricted topologies already pose non-trivial issues, and might help in the analysis of the general problem, which is currently under research. Some examples in Section 4 illustrate the scope of this framework. Concluding remarks are compiled in Section 5.

2. Background

We present in this section a digression on the role of qualitative aspects in the electrical circuit theory, together with some background material on circuit modeling and digraphs.

2.1. Qualitative aspects of circuit dynamics

Qualitative theory plays a prominent role in dynamical systems analysis. It focuses on the characterization of the long-term system behavior, addressing analytical properties of different kinds of invariants and the dependence of these properties on system parameters; see e.g. [18–23]. A qualitative motivation has driven many significant contributions of non-linear circuit analysis to dynamical systems theory since the pioneering research of Van der Pol [24, 25]; it is worth mentioning in this direction the work of Brayton and Moser [26, 27] and Smale [28], as well as the analysis of chaotic effects in Chua's circuit family [29–32]. Qualitative aspects have been emphasized in non-linear circuit theory since the 1970s (cf. [5, 9, 10, 31, 33] and references therein).

Many qualitative features can be tackled via linearization techniques. Different properties of equilibrium points can be characterized in terms of the spectrum (i.e. the set of eigenvalues) of the linearized problem; this spectrum may be that of a Jacobian matrix in state-space descriptions of the dynamics, or may come from a matrix pencil when semistate models are used. In particular, the hyperbolicity and exponential stability of an equilibrium are defined by the conditions $\mathrm{Re}\lambda \neq 0$ and $\mathrm{Re}\lambda < 0$ for all eigenvalues λ in the spectrum. Note that, according to Lyapunov's linearization theorem (see e.g. [34, Theorem 15.6]), the exponential stability of an equilibrium guarantees its asymptotic stability.

The hyperbolicity problem is relevant *per se* in both linear and non-linear circuit contexts. In linear circuits, proper oscillations are precluded in hyperbolic configurations. For the sake of simplicity, we will restrict the attention in this paper to linear problems, although the results apply directly to the linearization of non-linear circuits around an operating point. In particular, our analysis should be of interest in the characterization of Hopf bifurcations (see for instance [19, 20, 22] or, in the circuit literature, [35]), in which the hyperbolicity loss of an equilibrium at the bifurcation value gives birth to a periodic solution.

But hyperbolicity is also important in the asymptotic stability analysis of equilibria in electrical circuits. In the strictly passive working setting defined in Section 2.2 below, all eigenvalues are known to satisfy the condition $\mathrm{Re}\lambda \leqslant 0$, as shown in [15, Proposition 1]. Hence, in this strictly passive context the hyperbolicity of the spectrum (i.e. the condition $\mathrm{Re}\lambda \neq 0$ for all eigenvalues) automatically guarantees the asymptotic stability of the equilibrium point.

The hyperbolicity analysis is therefore important in qualitative studies of electrical circuits. This analysis will be performed in terms of matrix pencils, which naturally arise from the differential-algebraic form of circuit models, as detailed in Section 2.2. In turn, the aim for a graph-theoretic characterization of hyperbolicity involves the use of different properties from digraph theory which are compiled in Section 2.3.

2.2. *Branch-oriented circuit models*

Consider a connected, linear circuit composed of b_c capacitors, b_l inductors and b_r resistors. Splitting the vectors of branch voltages and currents as $v=(v_c, v_l, v_r) \in \mathbb{R}^{b_c} \times \mathbb{R}^{b_l} \times \mathbb{R}^{b_r}$ and $i=(i_c, i_l, i_r) \in \mathbb{R}^{b_c} \times \mathbb{R}^{b_l} \times \mathbb{R}^{b_r}$, the subscripts c, l, r, denoting capacitors, inductors and resistors, the time-domain equations of such a linear RLC circuit read

$$Cv_c' = i_c, \tag{1a}$$

$$Li_l' = v_l, \tag{1b}$$

$$0 = B_c v_c + B_l v_l + B_r v_r, \tag{1c}$$

$$0 = Q_c i_c + Q_l i_l + Q_r i_r, \tag{1d}$$

$$0 = i_r - G v_r. \tag{1e}$$

Here, $C \in \mathbb{R}^{b_c \times b_c}$, $L \in \mathbb{R}^{b_l \times b_l}$ and $G \in \mathbb{R}^{b_r \times b_r}$ are the capacitance, inductance and conductance matrices, respectively. Equations (1c) and (1d) express in matrix form Kirchhoff's voltage and current laws, making use of the reduced loop and cutset matrices $B = (B_c \ \ B_l \ \ B_r)$, $Q = (Q_c \ \ Q_l \ \ Q_r)$. The matrices B/Q describe the relations between branches and loops/cutsets of the circuit (cf. Section 2.3); all of their entries belong to $\{-1, 0, +1\}$ and they have full row rank. Letting $b = b_c + b_l + b_r$ and n stand for the total number of circuit branches and nodes, the matrices B and Q lie on $\mathbb{R}^{(b-n+1)\times b}$ and $\mathbb{R}^{(n-1)\times b}$, and therefore (1) is a system with $2b$ equations and $2b$ unknowns. Notice that the blocks B_c, B_l and B_r belong to $\mathbb{R}^{(b-n+1)\times b_c}$, $\mathbb{R}^{(b-n+1)\times b_l}$ and $\mathbb{R}^{(b-n+1)\times b_r}$, respectively, whereas Q_c, Q_l and Q_r are in $\mathbb{R}^{(n-1)\times b_c}$, $\mathbb{R}^{(n-1)\times b_l}$ and $\mathbb{R}^{(n-1)\times b_r}$.

Working hypothesis. We will assume throughout, without further explicit mention, that the capacitance and inductance matrices C and L are diagonal with positive entries and that the conductance matrix G is positive definite (i.e. that $u^{\mathsf{T}} G u > 0$ for any non-vanishing real vector u; we do not need G to be diagonal, not even symmetric). This means that all devices are linear and strictly passive and that capacitors and inductors are uncoupled.

The eigenvalues of any state-space reduction of (1) can be shown to be defined by the spectrum of the *matrix pencil* [36]

$$\begin{pmatrix} \lambda C & 0 & -I & 0 & 0 & 0 \\ 0 & -I & 0 & \lambda L & 0 & 0 \\ B_c & B_l & 0 & 0 & B_r & 0 \\ 0 & 0 & Q_c & Q_l & 0 & Q_r \\ 0 & 0 & 0 & 0 & G & -I \end{pmatrix}, \tag{2}$$

where the columns have been arranged according to the order of variables $(v_c, v_l, i_c, i_l, v_r, i_r)$. This stems from the work of Bryant [37, 38], and a detailed discussion can be found in [5]. The spectrum of the matrix pencil is just defined by the values of $\lambda \in \mathbb{C}$ which make the $2b \times 2b$ matrix in (2) a singular one. In the time-domain, this approach applies to both linear and non-linear circuits, working locally around an operating point in the latter case. For linear problems and in terms of the Laplace transform, these eigenvalues stand of course for the circuit poles.

In particular, we will be concerned with the search of (conjugate pairs of) purely imaginary eigenvalues $\lambda = \pm\alpha \mathrm{j}$, with $\alpha \in \mathbb{R} - \{0\}$; j stands for $\sqrt{-1}$ in order to avoid confusion with the current vector i. We will often perform the search of such an eigenvalue together with that of an associated right eigenvector. In order to keep the circuit-theoretical meaning of variables, the eigenvalue-eigenvector problem will be written as the search of λ, $(v_c, v_l, i_c, i_l, v_r, i_r)$ yielding non-trivial solutions of the linear system

$$\lambda C v_c = i_c, \tag{3a}$$

$$\lambda L i_l = v_l, \tag{3b}$$

$$B_c v_c + B_l v_l + B_r v_r = 0, \tag{3c}$$

$$Q_c i_c + Q_l i_l + Q_r i_r = 0, \tag{3d}$$

$$i_r = G v_r. \tag{3e}$$

Occasionally, we will also make use of *left* eigenvectors of (2). When unlabeled, the term 'eigenvector' will, as usual, stand for a right eigenvector.

2.3. Digraphs

Throughout this paper, we will characterize different circuit properties in terms of the underlying digraph and, more specifically, using the loop and cutset matrices presented below. The reader is referred to [39–42] for extensive introductions to digraph theory.

Chosen an orientation in every loop, the *loop matrix* $\tilde{B}$ is defined as (b_{ij}), with

$$b_{ij}=\begin{cases} 1 & \text{if branch } j \text{ is in loop } i \text{ with the same orientation} \\ -1 & \text{if branch } j \text{ is in loop } i \text{ with the opposite orientation} \\ 0 & \text{if branch } j \text{ is not in loop } i. \end{cases}$$

This matrix can be shown to have rank $b-n+k$ (see e.g. [40, Theorem 3.20]), where b, n and k stand respectively for the number of branches, nodes and connected components in the digraph. A *reduced loop matrix* B is any $((b-n+k)\times b)$-submatrix of $\tilde{B}$ with full row rank.

A subset K of the set of branches of a digraph is a *cutset,* if the removal of K increases the number of connected components of the digraph and it is minimal with respect to this property, that is, the removal of any proper subset of K does not increase the number of components.

The removal of the branches of a cutset increases the number of connected components by exactly one [40, p. 153]. Furthermore, all the branches of a cutset may be shown to connect the same pair of connected components of the digraph which results from the deletion of the cutset. This makes it possible to define the orientation of a cutset, say from one of these components towards the other. The cutset matrix $\tilde{Q}=(q_{ij})$ is then defined by

$$q_{ij}=\begin{cases} 1 & \text{if branch } j \text{ is in cutset } i \text{ with the same orientation} \\ -1 & \text{if branch } j \text{ is in cutset } i \text{ with the opposite orientation} \\ 0 & \text{if branch } j \text{ is not in cutset } i. \end{cases}$$

The rank of $\tilde{Q}$ can be proved to be $n-k$ [40, Theorem 3.32]; any set of $n-k$ linearly independent rows of $\tilde{Q}$ defines a *reduced cutset matrix* $Q\in\mathbb{R}^{(n-k)\times b}$. We will work with electrical circuits whose underlying digraphs are connected (although certain subcircuits arising in the analysis may not). For a connected digraph, any reduced cutset matrix has order $(n-1)\times b$.

The proof of the following result can be found in [40, Theorem 3.31] or in [42, Section 7.4].

Lemma 1
If the columns of the reduced loop and cutset matrices B, Q of a digraph are arranged according to the same order of branches, then $BQ^{\mathrm{T}}=0$, $QB^{\mathrm{T}}=0$.

Moreover, the relations $\mathrm{im}Q^{\mathrm{T}}=\ker B$ and $\mathrm{im}B^{\mathrm{T}}=\ker Q$ do hold. Hence, the *cut space* $\mathrm{im}Q^{\mathrm{T}}$ spanned by the rows of Q can be described as $\ker B$ and, analogously, the *cycle space* $\mathrm{im}B^{\mathrm{T}}$ spanned by the rows of B equals $\ker Q$ [41]. Note that these spaces are orthogonal to each other since $(\mathrm{im}Q^{\mathrm{T}})^{\perp}=(\ker B)^{\perp}=\mathrm{im}B^{\mathrm{T}}$.

The submatrices B_K and Q_K of B and Q defined by the columns which correspond to a set of branches K partially inherit these properties, as stated in Lemma 2 below; this result is an immediate consequence of the identities $\ker B_K\times\{0\}\subseteq\ker B=\mathrm{im}Q^{\mathrm{T}}$ and $\ker Q_K\times\{0\}\subseteq\ker Q=\mathrm{im}B^{\mathrm{T}}$ (where w.l.o.g. the columns corresponding to K-branches are assumed to be the first ones).

Lemma 2
Let K be a subset of branches of a digraph. Then $\ker B_K$ and $\ker Q_K$ are spanned by maximal sets of independent K-cutsets and independent K-loops.

This means that $\dim\ker B_K$ and $\dim\ker Q_K$ are defined by the number of independent K-cutsets and K-loops, respectively. In particular, K does not contain cutsets (resp. loops) if and only if B_K (resp. Q_K) has full column rank. Here, by a K-cutset (resp. K-loop) we mean a cutset (resp. a loop) formed only by branches belonging to K; similarly, we will use expressions such as I-cutset, LC-loop, etc., to mean circuit cutsets defined only by current sources, loops formed by inductors and capacitors, etc.

The following Lemma will also be useful in Section 3.

Lemma 3
A subset K of branches of a digraph $\mathscr{G}$ does not contain cutsets (resp. loops) if and only if $Q_{\mathscr{G}-K}$ (resp. $B_{\mathscr{G}-K}$) has full row rank.

The claim about cutsets follows from the fact that, by definition, the removal of K increases the number of connected components (making the rank of $Q_{\mathscr{G}-K}$ strictly smaller than that of Q) if and only if K contains at least one cutset. For the proof of the assertion involving loops, we refer the reader to [5, Lemma 5.8].

3. The hyperbolicity problem

3.1. Circuits with independent sources

We begin the analysis of the hyperbolicity problem by showing that the study of circuits with independent voltage and current sources can be driven to the RLC setting.

In the presence of sources, the circuit equations read

$$Cv_c' = i_c, \tag{4a}$$

$$Li_l' = v_l, \tag{4b}$$

$$0 = B_c v_c + B_l v_l + B_r v_r + B_j v_j + B_u v_s(t), \tag{4c}$$

$$0 = Q_c i_c + Q_l i_l + Q_r i_r + Q_j i_s(t) + Q_u i_u, \tag{4d}$$

$$0 = i_r - G v_r. \tag{4e}$$

Here $v_s(t)$ and $i_s(t)$ are the excitations in voltage and current sources. Mind the new variables defined by the current source voltages $v_j \in \mathbb{R}^{b_j}$ and the voltage source currents $i_u \in \mathbb{R}^{b_u}$, where b_j and b_u stand for the numbers of current and voltage sources, respectively. Since the sum of the row dimensions of B and Q always equals the total number of branches, the row dimension of the blocks defined by $B = (B_c\ B_l\ B_r\ B_j\ B_u)$ and $Q = (Q_c\ Q_l\ Q_r\ Q_j\ Q_u)$ now sum up to $b_c + b_l + b_r + b_j + b_u$. System (4) has, therefore, $2(b_c + b_l + b_r) + b_j + b_u$ equations and unknowns. The corresponding matrix pencil reads

$$\begin{pmatrix} \lambda C & 0 & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & \lambda L & 0 & 0 & 0 & 0 \\ B_c & B_l & 0 & 0 & B_r & 0 & B_j & 0 \\ 0 & 0 & Q_c & Q_l & 0 & Q_r & 0 & Q_u \\ 0 & 0 & 0 & 0 & G & -I & 0 & 0 \end{pmatrix}. \tag{5}$$

Proposition 1

Assume that a given VIRLC circuit has neither V-loops nor I-cutsets. The spectrum of the matrix pencil (5) coincides with that of the pencil (2), provided that the latter is defined by the RLC circuit obtained after open-circuiting current sources and short-circuiting voltage sources.

Proof

The key aspect in the proof of this statement is that the loop and cutset matrices can be assumed, without loss of generality, to be *fundamental* ones, that is, to be defined from the choice of a tree. Indeed, every tree branch (twig) defines an unique cutset together with some cotree branches (links) and, similarly, every link defines a unique loop together with some twigs. Additionally, the assumption that the circuit has neither V-loops nor I-cutsets makes it possible to choose the tree in a way such that voltage and current sources correspond to twigs and links, respectively. This gives B and Q the form (see e.g. [5, 8])

$$B = \begin{pmatrix} I_j & 0 & E & F \\ 0 & I_x & H & K \end{pmatrix}, \quad Q = \begin{pmatrix} -E^{\mathrm{T}} & -H^{\mathrm{T}} & I_y & 0 \\ -F^{\mathrm{T}} & -K^{\mathrm{T}} & 0 & I_u \end{pmatrix}.$$

Here the rows corresponding to the identity matrices I_x and I_y are those defined by the cotree and tree elements different from current and voltage sources, respectively. With the notation introduced above we have

$$B_j = \begin{pmatrix} I_j \\ 0 \end{pmatrix}, \quad B_u = \begin{pmatrix} F \\ K \end{pmatrix}, \quad Q_j = \begin{pmatrix} -E^{\mathrm{T}} \\ -F^{\mathrm{T}} \end{pmatrix}, \quad Q_u = \begin{pmatrix} 0 \\ Iu \end{pmatrix}.$$

Note that the submatrices B_u and Q_j do not enter the matrix pencil (5). This fact, together with the structure of B_j and Q_u, implies that the submatrices of B and Q on which the determinant of (5) actually depends are

$$B_x = (I_x\ \ H), \quad Q_y = (-H^{\mathrm{T}}\ \ I_y).$$

On the other hand, it is not difficult to check that these are exactly the loop and circuit matrices of the (so-called reduced) circuit obtained after short-circuiting voltages sources and open-circuiting current sources. This is a consequence of the fact that the fundamental loops of the reduced circuit are those defined by the x-elements (just remarking that voltage sources, which have been short-circuited, disappear from the loops), and the fundamental cutsets are, except for the removed current sources, those defined by the y-elements. This means that the determinant of (5) coincides, maybe up to a '$-$' sign, with that of (2), in the understanding that in the latter the submatrices B_c, B_l, B_r, Q_c, Q_l and Q_r correspond to the reduced circuit. □

We may hence address the results in terms of RLC circuits without loss of generality. We will do so in the remainder of the paper. When working with circuits which include independent sources, and as long as they do not display V-loops or I-cutsets, the results apply just after short-circuiting voltages sources and open-circuiting current sources.

3.2. *General results involving purely imaginary eigenvalues* (*PIEs*)

Proposition 2
The matrix pencil (2) has a zero eigenvalue if and only if at least one of the matrices $(B_c\ B_r)$ and $(Q_l\ Q_r)$ does not have full row rank.

Proof
The pencil has $\lambda=0$ as an eigenvalue if and only if the matrix resulting from the substitution $\lambda=0$ in (2) is singular. The singular nature of (2) when $\lambda=0$ amounts to the singularity of

$$\begin{pmatrix} B_c & 0 & B_r & 0 \\ 0 & Q_l & 0 & Q_r \\ 0 & 0 & G & -I \end{pmatrix}. \tag{6}$$

Let $(x_c, x_l, x_r)\neq 0$ belong to the left-kernel of (6), that is, assume that

$$x_c^{\mathrm{T}} B_c = 0, \tag{7a}$$

$$x_l^{\mathrm{T}} Q_l = 0, \tag{7b}$$

$$x_c^{\mathrm{T}} B_r + x_r^{\mathrm{T}} G = 0, \tag{7c}$$

$$x_l^{\mathrm{T}} Q_r - x_r^{\mathrm{T}} = 0. \tag{7d}$$

Premultiplying the relation $B_c Q_c^{\mathrm{T}} + B_l Q_l^{\mathrm{T}} + B_r Q_r^{\mathrm{T}} = 0$ (which follows from Lemma 1 by x_c^{T} and then multiplying the resulting identity by x_l we get, because of (7a) and (7b),

$$x_c^{\mathrm{T}} B_r Q_r^{\mathrm{T}} x_l = 0,$$

which, in the light of (7c) and (7d), yields $x_r^{\mathrm{T}} G x_r = 0$. In turn, the positive definiteness of G forces $x_r = 0$, and therefore (7) reads $x_c^{\mathrm{T}} B_c = x_c^{\mathrm{T}} B_r = 0$, $x_l^{\mathrm{T}} Q_l = x_l^{\mathrm{T}} Q_r = 0$. The former two identities with $x_c \neq 0$ would indicate $(B_c\ B_r)$ not to have full row rank, whereas the latter two with $x_l \neq 0$ would mean that $(Q_l\ Q_r)$ has not full row rank. Notice that x_c and x_l cannot vanish simultaneously.

Conversely, if at least one of the matrices $(B_c\ B_r)$ and $(Q_l\ Q_r)$ does not have full row rank, then there exist two vectors x_c and x_l not vanishing simultaneously and satisfying $x_c^{\mathrm{T}} B_c = x_c^{\mathrm{T}} B_r = 0$, $x_l^{\mathrm{T}} Q_l = x_l^{\mathrm{T}} Q_r = 0$. The non-vanishing vector $(x_c, x_l, 0)$ then belongs to the left-kernel of (6) and shows that $\lambda=0$ is indeed an eigenvalue. □

Using Lemma 3, we get directly the following reformulation of Proposition 2 in terms of the network topology.

Proposition 3
The matrix pencil (2) has a zero eigenvalue if and only if the circuit has at least one L-loop or one C-cutset.

This property, which in different forms can be traced back at least to [12–14], expresses that L-loops and C-cutsets are particular instances of topologically non-hyperbolic configurations. Note that the result holds for positive-definite G without any assumption on L and C. A different proof, based on the use of nodal circuit models, can be found in [15, Theorem 4.1].

Proposition 4
Necessary conditions for the circuit to exhibit a PIE are the existence of at least one LC-loop and one LC-cutset.

For the proof of this result, we refer the reader to [15, Theorem 4.2], where this statement is proved for circuits including independent sources. Note incidentally that, when open-circuiting current sources and short-circuiting voltage sources (as indicated in Section 3.1), ILC-cutsets are transformed into LC-cutsets and VLC-loops yield LC-loops.

Proposition 4 can be also expressed in terms of matrix criteria, as a direct consequence of Lemma 3.

Proposition 5
If the matrix pencil (2) has a PIE then the matrices B_r and Q_r do not have full row rank.

The following result provides a key simplification in the hyperbolicity analysis.

Proposition 6
Any eigenvector associated with a PIE verifies $v_r = i_r = 0$.

Proof
Note first that any vector $v=(v_c, v_l, v_r)$ verifying $Bv=0$ (according to (3c)) belongs to the cut space and can therefore be written as $v = Q^{\mathrm{T}} u$ for some vector u. Similarly, any $i=(i_c, i_l, i_r)$ satisfying $Qi=0$ (cf. (3d)) reads $i=B^{\mathrm{T}} w$ for a certain w. Letting * stand for the

conjugate transpose, write $v^* = u^* Q$, so that $v^* i = u^* Q B^{\mathrm{T}} w = 0$. Therefore,

$$v_c^* i_c + v_l^* i_l + v_r^* i_r = 0. \tag{8}$$

Using (3a), (3b) and (3e) we transform (8) into

$$\lambda v_c^* C v_c + \overline{\lambda} i_l^* L i_l + v_r^* G v_r = 0. \tag{9}$$

Assume now that λ is a PIE, namely, that $\lambda = \alpha \mathrm{j}$ for some real α. Equation (9) then reads

$$j\alpha (v_c^* C v_c - i_l^* L i_l) + v_r^* G v_r = 0.$$

Splitting $v_r = x_r + \mathrm{j} y_r$ we get

$$j\alpha (v_c^* C v_c - i_l^* L i_l) + x_r^{\mathrm{T}} G x_r + y_r^{\mathrm{T}} G y_r + \mathrm{j}(x_r^{\mathrm{T}} G y_r - y_r^{\mathrm{T}} G x_r) = 0. \tag{10}$$

The fact that C and L are diagonal implies that $v_c^* C v_c$ and $i_l^* L i_l$ are real. Hence, the real part of the left-hand side of (10) amounts to $x_r^{\mathrm{T}} G x_r + y_r^{\mathrm{T}} G y_r$. Since G is positive definite, for this expression to vanish we need $x_r = y_r = 0$. This means that $v_r = 0$ and, in the light of (3e), $i_r = 0$. □

It is worth remarking that the result above remains true if we just assume that C and L are symmetric. Proposition 6 implies that, as long as resistors are strictly passive, the actual resistance values are irrelevant concerning the existence of purely imaginary eigenvalues. In the light of this result, the existence of a PIE will be equivalent to the existence of a non-trivial solution for the system

$$\lambda C v_c = i_c, \tag{11a}$$

$$\lambda L i_l = v_l, \tag{11b}$$

$$B_c v_c + B_l v_l = 0, \tag{11c}$$

$$Q_c i_c + Q_l i_l = 0 \tag{11d}$$

with $\lambda = \alpha \mathrm{j}$.

3.3. A characterization of non-hyperbolic circuits with one LC-cutset

As indicated in the introduction, the full characterization of the existence of purely imaginary eigenvalues involves the description of the (so-called *topologically non-hyperbolic*) configurations which admit PIEs for all positive values of the reactances and, for circuits not displaying these configurations, a characterization of the set of parameter values yielding PIEs. In this generality, this seems to be a difficult problem. However, under certain restrictions these results are feasible, as detailed in this and the next subsection, where we present such a characterization for circuits with just one LC-cutset or just one LC-loop; cf. Theorems 1 and 2. The study of these configurations is motivated by Proposition 4. These partial results are of independent interest (they explain, for instance, why the number of restrictions on the reactive values yielding PIEs in the circuits of Figure 1 are two and one, respectively) and, additionally, should be of help in the analysis of the general problem.

Let us then assume that the RLC circuit has only one LC-cutset. Choose an orientation for this LC-cutset and assume without loss of generality that all of its branches have the same orientation as the cutset itself. According to Lemma 2, the existence of a unique LC-cutset implies that $\dim \ker (B_c \ B_l) = 1$. Actually, this space can be checked to be spanned by the vector $u = (u_c, u_l)$ whose components are defined as $(u_c)_k = 1$ (resp. $(u_l)_k = 1$) if capacitor (resp. inductor) k is in the cutset, and 0 otherwise. This is an easy consequence of the fact that every loop includes an even number of branches from any cutset (see item 3.28 in [40]). Since $\dim \ker (B_c \ B_l) = 1$, any solution (v_c, v_l) to (11c) must be given, up to a non-vanishing multiplicative constant, by the above-defined vector $u = (u_c, u_l)$. This paves the way for the characterization of the existence of PIEs in this setting.

Since the eigenvector is itself defined up to a multiplicative constant, there is no loss of generality in working with $(v_c, v_l) = (u_c, u_l)$ in the eigenvector construction. Set, from (11a) and (11b), $(i_c)_k = \lambda C_k (v_c)_k$ and $(i_l)_k = \lambda^{-1} L_k^{-1} (v_l)_k$. This means that $(i_c)_k = \lambda C_k$ and $(i_l)_k = \lambda^{-1} L_k^{-1}$ for the entries which correspond to branches in the LC-cutset, and $(i_c)_k = 0$, $(i_l)_k = 0$ for the remaining ones.

Restricting the attention to elements in the LC-cutset, let $u_{c\,\mathrm{cut}}$ and $u_{l\,\mathrm{cut}}$ be vectors whose kth entry is given by

$$u_{c\,\mathrm{cut},k} = C_k \quad \text{and} \quad u_{l\,\mathrm{cut},k} = L_k^{-1}, \tag{12}$$

where C_k denotes the kth capacitance in the cutset and L_k stands for the kth inductance in the cutset. Denote by $(B_{c\,\mathrm{cut}} \ B_{l\,\mathrm{cut}})$ and $(Q_{c\,\mathrm{cut}} \ Q_{l\,\mathrm{cut}})$ the submatrices of $(B_c \ B_l)$ and $(Q_c \ Q_l)$, respectively, defined by the columns which correspond to branches in the cutset. A PIE will therefore exist if and only if the relations

$$\lambda Q_{c\,\mathrm{cut}} u_{c\,\mathrm{cut}} + \lambda^{-1} Q_{l\,\mathrm{cut}} u_{l\,\mathrm{cut}} = 0, \tag{13}$$

coming from (11d), hold for some $\lambda = \alpha \mathrm{j}$.

We need to clarify the meaning of these relations. Specifically, we need to check when they do hold for all positive values of the capacitances and inductances in the cutset and, if this is not the case, the nature of the relations that they impose on λ and on the values of these reactances. This is done in the sequel. We denote by $\sum_{\text{LC-cutset}} C$ (resp. $\sum_{\text{LC-cutset}} L^{-1}$) the sum of capacitances (resp. inverse inductances) of the capacitors (resp. inductors) in the LC-cutset.

Lemma 4
Let the circuit have only one LC-cutset and assume that there exists a PIE $\lambda = \pm\alpha \mathrm{j}$, with $\alpha \neq 0$. Then, the LC-cutset is neither an L-cutset nor a C-cutset. Furthermore, α is given by

$$\alpha = \sqrt{\frac{\sum_{\text{LC-cutset}} L^{-1}}{\sum_{\text{LC-cutset}} C}}. \tag{14}$$

Proof
The first row of $(Q_c\ Q_l)$ can be assumed w.l.o.g. to describe the LC-cutset of the circuit. Assume as above that all of its branches have the same orientation as the cutset itself. Consequently, the first row of $(Q_c\ Q_l)$ equals the vector u defined above. With this choice, the first line of (13) reads

$$\lambda \sum_{\text{LC-cutset}} C + \lambda^{-1} \sum_{\text{LC-cutset}} L^{-1} = 0. \tag{15}$$

Since the capacitances and inductances are positive, the existence of a solution $\lambda = \pm\alpha \mathrm{j}$ with $\alpha \neq 0$ means that at least one capacitor and one inductor must be present in the cutset. From (15), the value of α displayed in (14) follows trivially. □

Note, incidentally, that the reasoning above also reflects the fact that a C-cutset yields a zero eigenvalue (cf. Proposition 3), whereas an L-cutset would indicate the existence of an infinite eigenvalue, corresponding to a higher index circuit configuration.

We assume in the sequel that the LC-cutset actually includes at least one capacitor and one inductor.

Lemma 5
The rank of the matrix $(Q_{c\,\text{cut}}\ Q_{l\,\text{cut}})$ equals the number b_{cut} of reactances in the LC-cutset minus the number of independent loops defined just by elements of the LC-cutset, and also the number n_{cut} of nodes incident with the cutset minus the number k_{cut} of connected components that the LC-cutset (considered as a subcircuit) has.

Proof
The first assertion is an immediate consequence of the fact that $\dim \ker Q_K$ equals the number of independent K-loops, as stated in Lemma 2. The second one follows from the fact that the number of independent loops in the subcircuit defined by the LC-cutset reads $b_{\text{cut}} - n_{\text{cut}} + k_{\text{cut}}$. □

Lemma 6
Let the circuit have exactly one LC-cutset. Then, the circuit has a PIE for all positive values of capacitances and inductances if and only if the LC-cutset is a parallel connection.

Proof
There will exist a PIE for all positive values of reactances if and only if (13) imposes no restriction apart from (15), which defines the value of α. This will happen if and only if $\operatorname{rk}(Q_{c\,\text{cut}}\ Q_{l\,\text{cut}}) = 1$.

We know from Lemma 5 that the rank of $(Q_{c\,\text{cut}}\ Q_{l\,\text{cut}})$ is $n_{\text{cut}} - k_{\text{cut}}$. Note that $n_{\text{cut}} \geqslant 2k_{\text{cut}}$, since each connected component of the LC-cutset has at least two nodes. In order to have $n_{\text{cut}} - k_{\text{cut}} = 1$ it must hence be $n_{\text{cut}} = 2$, $k_{\text{cut}} = 1$. This means that all reactances in the LC-cutset connect the same pair of nodes, that is, they define a parallel connection. □

What happens if $n_{\text{cut}} - k_{\text{cut}} \geqslant 2$, that is, if the LC-cutset is not a parallel connection? As before, we may assume w.l.o.g. that the first row of $(Q_{c\,\text{cut}}\ Q_{l\,\text{cut}})$ is defined by the LC-cutset, so that the first row of (13) defines the value for α displayed in (14). Letting $(Q^*_{c\,\text{cut}}\ Q^*_{l\,\text{cut}})$ stand for the submatrix of $(Q_{c\,\text{cut}}\ Q_{l\,\text{cut}})$ defined by the remaining rows, the other equations coming from (13) yield

$$\alpha Q^*_{c\,\text{cut}} u_{c\,\text{cut}} - \alpha^{-1} Q^*_{l\,\text{cut}} u_{l\,\text{cut}} = 0, \tag{16}$$

with α given by (14); recall that $u_{c\,\text{cut}}$ and $u_{l\,\text{cut}}$ denote the vectors defined in (12). Equivalently,

$$\left(\sum_{\text{LC-cutset}} L^{-1}\right) Q^*_{c\,\text{cut}} u_{c\,\text{cut}} = \left(\sum_{\text{LC-cutset}} C\right) Q^*_{l\,\text{cut}} u_{l\,\text{cut}}. \tag{17}$$

Equation (17) says that the existence of a PIE relies on these relations among the reactances in the cutset; note that only $n_{\text{cut}} - k_{\text{cut}} - 1$ of them are independent relations. Keep also in mind that only positive solutions to these equations are guaranteed to yield a PIE, whose frequency α will be given by (14).

These remarks, together with Lemmas 4, 5 and 6, support the following statement.

Theorem 1

Assume that a given RLC circuit has exactly one LC-cutset, and that it includes at least one inductor and one capacitor. Then the circuit has a PIE if and only if

- either the LC-cutset is a parallel connection,
- or, otherwise, the reactances in the LC-cutset verify the algebraic relations (17).

In both cases, the PIE reads $\pm\alpha j$ with α given by (14).

Remark

If the LC-cutset is a parallel connection, it is pretty obvious that every branch in the cutset forms at least one loop together with another branch from the cutset. In a slightly modified form, this property is also true in cases in which the LC-cutset is not a parallel connection. Indeed, equation (13) can be recast as

$$(Q_{c\,\text{cut}}\ \ Q_{l\,\text{cut}})\begin{pmatrix}\alpha u_{c\,\text{cut}}\\ -\alpha^{-1}u_{l\,\text{cut}}\end{pmatrix}=0.$$

The existence of a PIE yields a non-trivial solution $(\alpha u_{c\,\text{cut}}, -\alpha^{-1}u_{l\,\text{cut}})$ to this equation, with α given by (14). Since $(\alpha u_{c\,\text{cut}}, -\alpha^{-1}u_{l\,\text{cut}})\in \ker(Q_{c\,\text{cut}}\ \ Q_{l\,\text{cut}})$, according to Lemma 2 this vector can be described as a linear combination of vectors corresponding to loops defined just by elements of the cutset. Because of the fact that all the components of the vector $(\alpha u_{c\,\text{cut}}, -\alpha^{-1}u_{l\,\text{cut}})$ are non-zero, this means that all the reactances in the cutset define at least one loop together with other reactances coming from the cutset. This is a necessary condition for a PIE to exist which might be of help in the analysis of the general hyperbolicity problem.

3.4. Circuits with one LC-loop

Problems in which the circuit has only one LC-loop can be addressed in a similar way. In this situation we have $\dim\ker(Q_c\ Q_l)=1$. Proceeding as above, assume that all branches in the LC-loop have the same orientation as the loop itself, and set $w=(w_c, w_l)$ with $(w_c)_k=1$ (resp. $(w_l)_k=1$) if capacitor (resp. inductor) k is in the loop, and 0 otherwise. Now this will yield a solution to (11d). Fix $(i_c, i_l)=(w_c, w_l)$ and, from (11a)–(11b), $(v_c)_k=\lambda^{-1}C_k^{-1}(i_c)_k$, $(v_l)_k=\lambda L_k(i_l)_k$. Hence, $(v_c)_k=\lambda^{-1}C_k^{-1}$, $(v_l)_k=\lambda L_k$ for the entries in the loop, the remaining ones vanishing. Denote by $(B_{c\,\text{loop}}\ B_{l\,\text{loop}})$ and $(Q_{c\,\text{loop}}\ Q_{l\,\text{loop}})$ the submatrices of $(B_c\ B_l)$ and $(Q_c\ Q_l)$ defined by branches in the loop.

Now, because of the condition $\dim\ker(Q_c\ Q_l)=1$, a PIE exists if and only if the relations

$$\lambda^{-1}B_{c\,\text{loop}}w_{c\,\text{loop}}+\lambda B_{l\,\text{loop}}w_{l\,\text{loop}}=0, \tag{18}$$

defined by (11c), are met for some $\lambda=\alpha j$. Here $w_{c\,\text{loop}}$ and $w_{l\,\text{loop}}$ are the vectors whose kth entry is given by

$$w_{c\,\text{loop},k}=C_k^{-1}\quad\text{and}\quad w_{l\,\text{loop},k}=L_k, \tag{19}$$

where C_k denotes the kth capacitance in the loop and L_k is the kth inductance in the loop.

Lemma 7

Let the circuit have exactly one LC-loop and assume that there exists a PIE $\lambda=\pm\alpha j$, with $\alpha\neq 0$. Then, the LC-loop is neither an L-loop nor a C-loop. Furthermore, α is given by

$$\alpha=\sqrt{\frac{\sum_{\text{LC-loop}}C^{-1}}{\sum_{\text{LC-loop}}L}}. \tag{20}$$

Proof

The proof parallels that of Lemma 4 just by letting the first row of $(B_c\ B_l)$ be given by the vector w defined above, the first line of (18) now reading

$$\lambda^{-1}\sum_{\text{LC-loop}}C^{-1}+\lambda\sum_{\text{LC-loop}}L=0. \tag{21}$$

$\square$

We will assume in the sequel that the LC-loop includes at least one inductor and one capacitor.

Lemma 8

The rank of the matrix $(B_{c\,\text{loop}}\ B_{l\,\text{loop}})$ equals the number b_{loop} of reactances in the LC-loop minus the number of independent cutsets defined just by elements of the LC-loop, and exceeds by one the quantity $b_{\text{loop}}-k_{\text{loop}}$, where k_{loop} is the number of connected components of the subcircuit which results from the removal of the LC-loop branches.

Proof
The first statement follows again from Lemma 2, since $\dim\ker B_K$ describes the number of independent K-cutsets. The second one is in turn a consequence of the fact that the removal of the branches of c independent cutsets in a connected digraph defines a subgraph with $c+1$ connected components. □

We will say that b branches are connected in series if the current is the same in all of them; that is, we allow additional devices to be connected to the terminal nodes of the branches, as long as they do not define any loop or cutset together with any of these branches. These so-called *blocks*, which are decoupled from the rest of the circuit and are therefore trivial from the electrical point of view, may arise from the removal of current sources or the short-circuit of voltage sources in VIRLC circuits.

Lemma 9
Let the circuit have exactly one LC-loop. Then, the circuit has a PIE for all positive values of capacitances and inductances if and only if the LC-loop is a series connection.

Proof
This will happen if and only if $\mathrm{rk}(B_{c\,\mathrm{loop}}\ B_{l\,\mathrm{loop}})=1$. From the second characterization of the rank of $(B_{c\,\mathrm{loop}}\ B_{l\,\mathrm{loop}})$ in Lemma 8, we need $b_{\mathrm{loop}}=k_{\mathrm{loop}}$. In turn, b_{loop} equals the number of nodes in the LC-loop; for this number not to exceed k_{loop}, it must happen that the remaining devices do not connect any two nodes of the loop. This is equivalent to saying that the additional devices may only define blocks attached to the nodes of the LC-loop, making it a series connection. □

Now, when $b_{\mathrm{loop}}-k_{\mathrm{loop}}\geqslant 1$ (so that $\mathrm{rk}(B_{c\,\mathrm{loop}}\ B_{l\,\mathrm{loop}})\geqslant 2$), certain constraints among the reactances in the loop must be satisfied for a PIE to exist. Assume as above that the first row of $(B_{c\,\mathrm{loop}}\ B_{l\,\mathrm{loop}})$ is defined by the LC-loop. Denoting by $(B^*_{c\,\mathrm{loop}}\ B^*_{l\,\mathrm{loop}})$ the submatrices of $(B_{c\,\mathrm{loop}}\ B_{l\,\mathrm{loop}})$ defined by the remaining rows, and recalling that $w_{c\,\mathrm{loop}}$ and $w_{l\,\mathrm{loop}}$ are the vectors defined in (19), the other equations in (18) read

$$-\alpha^{-1}B^*_{c\,\mathrm{loop}}w_{c\,\mathrm{loop}}+\alpha B^*_{l\,\mathrm{loop}}w_{l\,\mathrm{loop}}=0, \tag{22}$$

which can be equivalently rewritten as

$$\left(\sum_{\text{LC-loop}} L\right)B^*_{c\,\mathrm{loop}}w_{c\,\mathrm{loop}}=\left(\sum_{\text{LC-loop}} C^{-1}\right)B^*_{l\,\mathrm{loop}}w_{l\,\mathrm{loop}}. \tag{23}$$

Equation (23) indicates that $b_{\mathrm{loop}}-k_{\mathrm{loop}}$ independent relations among the reactances in the loop must be satisfied for a PIE to exist.

The following result is then a direct consequence of the discussion above and Lemmas 7, 8 and 9.

Theorem 2
Assume that a given RLC circuit has exactly one LC-loop, and that it includes at least one inductor and one capacitor. Then the circuit has a PIE if and only if

- either the LC-loop is a series connection,
- or, otherwise, the reactances in the LC-loop verify the algebraic relations (23).

In both cases, the PIE reads $\pm\alpha j$ with α given by (20).

Remark
Again as in the one LC-cutset case, if we rewrite (18) as

$$(B_{c\,\mathrm{loop}}\ \ B_{l\,\mathrm{loop}})\begin{pmatrix}-\alpha^{-1}w_{c\,\mathrm{loop}}\\ \alpha w_{l\,\mathrm{loop}}\end{pmatrix}=0$$

it follows as a necessary condition for a PIE to exist that all the reactances in the LC-loop belong to at least one cutset defined only by reactances coming from this loop.

4. Examples

The framework developed in Section 3 makes it easy to assess the hyperbolicity of circuits such as the ones displayed in Figure 1. In both of these circuits, the absence of L-loops and C-cutsets rules out null eigenvalues, according to Proposition 3. Additionally, the two circuits have a single LC-cutset, defined in both cases by the four reactances: the existence of a PIE for them can be then addressed in terms of the results discussed in Section 3.3.

4.1. Example 1

Consider the circuit depicted on the left of Figure 1. A reduced cutset matrix is

$$Q=\begin{pmatrix}1 & 1 & 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1 & 0 & 1\end{pmatrix}, \tag{24}$$

where columns are arranged according to the order $C_1, C_2, L_1, L_2, R_1, R_2$. Note that the first row of (24) corresponds to the LC-cutset.

In the notation of Section 3.3 (cf. Lemma 5) we have $n_{\mathrm{cut}}=4$, $k_{\mathrm{cut}}=1$, and therefore $n_{\mathrm{cut}}-k_{\mathrm{cut}}-1=2$ relations among the values of the reactances are expected to show up as necessary conditions for a PIE to exist.

A loop matrix for this circuit reads

$$B=\begin{pmatrix}-1 & 0 & 1 & 0 & 0 & 1\\ 0 & 1 & 0 & -1 & 0 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\end{pmatrix}. \tag{25}$$

From (24) and (25) we have

$$(Q_{c\,\mathrm{cut}}\ Q_{l\,\mathrm{cut}})=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\end{pmatrix},\quad (B_{c\,\mathrm{cut}}\ B_{l\,\mathrm{cut}})=\begin{pmatrix}-1 & 0 & 1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 1 & -1 & 0\end{pmatrix}.$$

Notice that, consistently with the results of Section 3.3, the submatrix $(Q_{c\,\mathrm{cut}}\ Q_{l\,\mathrm{cut}})$ has rank $n_{\mathrm{cut}}-k_{\mathrm{cut}}=3$. Remark also that the loop matrix is not necessary in the following computations; it is only included for illustrative purposes.

From the first row of $(Q_{c\,\mathrm{cut}}\ Q_{l\,\mathrm{cut}})$ we compute the frequency of an eventual PIE, which in this case reads

$$\alpha=\sqrt{\frac{L_1^{-1}+L_2^{-1}}{C_1+C_2}}. \tag{26}$$

Such a PIE will actually exist if the relations (16) (or (17)) imposed by the two remaining rows of $(Q_{c\,\mathrm{cut}}\ Q_{l\,\mathrm{cut}})$ are satisfied for positive values of the capacitances and reactances. These two relations are easily seen to read

$$\alpha C_1-\alpha^{-1}L_1^{-1}=0$$

$$\alpha C_1-\alpha^{-1}L_2^{-1}=0,$$

which yield $L_1=L_2$ and, together with (26), $C_1=C_2$. This simplifies the value of α to $1/\sqrt{LC}$ with $L=L_1=L_2$, $C=C_1=C_2$.

It is not a trivial task to contrast these results with the actual eigenvalue behavior of the circuit. In this case, we are dealing with a fourth-order circuit and we may check 'by hand' whether these conditions indeed characterize the existence of PIEs; note, however, that the following approach becomes unfeasible as the order of the circuit increases. Since there are no C-loops or L-cutsets reducing the state-space dimension of the problem, the pencil determinant characterizing the spectrum of the circuit will have the form

$$a\lambda^4+b\lambda^3+c\lambda^2+d\lambda+e,$$

and the coefficients can be checked to read

$$a=C_1C_2L_1L_2(R_1+R_2),\quad b=C_1C_2(L_1+L_2)R_1R_2+(C_1+C_2)L_1L_2,$$

$$c=C_1L_1R_2+C_1L_2R_1+C_2L_1R_1+C_2L_2R_2,$$

$$d=(C_1+C_2)R_1R_2+L_1+L_2,\quad e=R_1+R_2.$$

For this polynomial to have a purely imaginary root $\lambda=\alpha j$ the relations

$$a\alpha^4-c\alpha^2+e=0,\quad -b\alpha^3+d\alpha=0$$

must hold, and $\alpha\neq 0$ requires $ad^2-bcd+eb^2=0$. With the coefficients given above, this expression can be shown to admit the factorization

$$-(R_2L_1L_2+R_1R_2^2(C_1L_2+C_2L_1)+R_1^2R_2^3C_1C_2)(C_1L_1-C_2L_2)^2$$

$$-(R_1L_1L_2+R_1^2R_2(C_1L_1+C_2L_2)+R_1^3R_2^2C_1C_2)(C_1L_2-C_2L_1)^2.$$

Since all resistances, capacitances and inductances are assumed to be positive, the vanishing of this expression requires $C_1L_1-C_2L_2=C_1L_2-C_2L_1=0$, a pair of conditions which are easily proved equivalent to $L_1=L_2$, $C_1=C_2$. As expected, these *ad-hoc* computations confirm the result emanating from the analysis of Section 3.3.

4.2. Example 2

Let us now drive the attention to the circuit displayed on the right of Figure 1. Take as the reduced cutset matrix

$$Q=\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \tag{27}$$

where columns are again arranged according to the order C_1, C_2, L_1, L_2, R_1, R_2. As in Example 1, the first row of (27) corresponds to the LC-cutset. Now we have $n_{\text{cut}}=4$, $k_{\text{cut}}=2$, so that $n_{\text{cut}}-k_{\text{cut}}-1=1$ and therefore only one relation among the values of the reactances will be necessary for the circuit to exhibit a PIE.

Now we have

$$(Q_{C\text{cut}}\ Q_{l\text{cut}})=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

with rank $n_{\text{cut}}-k_{\text{cut}}=2$.

The first row of $(Q_{C\text{cut}}\ Q_{l\text{cut}})$ leads to

$$\alpha=\sqrt{\frac{L_1^{-1}+L_2^{-1}}{C_1+C_2}}, \tag{28}$$

whereas for instance the second row yields

$$\alpha C_1-\alpha^{-1}L_1^{-1}=0. \tag{29}$$

Together with (28), Equation (29) leads to the relation $L_1C_1=L_2C_2$. We leave it as an exercise to the reader to check by other means that this is indeed a necessary and sufficient condition for a PIE to exist. Notice that the relation $L_1C_1=L_2C_2$ simplifies the expression given for α to $1/\sqrt{L_1C_1}=1/\sqrt{L_2C_2}$.

5. Concluding remarks

We have discussed in this paper the hyperbolicity problem for electrical circuits including inductors, capacitors, resistors, and voltage and current sources, addressing the existence of purely imaginary eigenvalues (PIEs) in the pencil spectrum characterizing the dynamics. This is an important problem in the qualitative theory of (both linear and non-linear) electrical circuits due to the importance of PIEs regarding oscillatory phenomena, and also in connection to stability aspects. Its study requires the interrelated use of several notions coming from the digraph theory, the theory of differential and differential-algebraic equations and matrix analysis.

Some of our results provide crucial simplifications in the hyperbolicity analysis; specifically, Proposition 1 shows how to tackle the problem in terms of RLC circuits; in turn, Proposition 6 proves that actual resistance values are irrelevant as long as resistors are strictly passive. The necessary existence of an LC-cutset and an LC-loop for a network to exhibit PIEs (cf. Proposition 4) motivates the analysis of circuits with just one LC-cutset or one LC-loop. For these configurations, we have characterized in Theorems 1 and 2 the codimension of the manifold of reactive values yielding PIEs, as well as their oscillation frequency, in terms of the circuit topology; in particular, the codimension-zero cases describe so-called topologically non-hyperbolic configurations.

The description of this codimension for general circuits and the full characterization of topologically non-hyperbolic configurations define the scope of future research. The results here discussed seem to be promising with regard to these problems.

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