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On the Solutions of Linear Differential Equations with Singular Coefficients*

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Equations of the form $E\dot{x} = Ax + u$, with E and A square matrices and E singular, are considered. The controversy that exists in the literature concerning the solutions of such equations is investigated. Solutions are arrived at through an application of singular perturbation theory.

INTRODUCTION

Recently, in certain areas of engineering and economics, there has been interest in ordinary differential equations of the form

$$E\dot{x}(t) = Ax(t) + u(t) \tag{1}$$

[1–11], where $E, A \in R^{n\times n}$ with E singular. It has been proposed in [3–7] that such equations can be used to describe the behaviour of systems in which a sudden change in structure or parameter values (e.g., as a result of component failure or switching) occurs. The basic rationale is as follows: Assume that switching occurs at $t = 0$ and that for $t > 0$ the physical system is modelled by (1). If $x(t)$ is the response of the system for $t < 0$ (not necessarily described by (1)) and $x(t) \to x_0$ as $t \to 0^-$, then x_0 may be interpreted as an initial condition which, together with (1), determines the system behaviour at the time of switching and for $t > 0$. Clearly, any value of x_0 is possible since nothing has been said about the system structure for $t < 0$.

The main problem with this approach is that, for certain initial conditions, (1) has no solution. For this reason some authors [1–3, 11] have confined themselves to the restricted class of "consistent" initial conditions for which (1) does have a solution. Others [3–7] have proposed certain distributions as "solutions" of (1) due to inconsistent initial conditions.

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To see the essence of the problem more clearly, a canonical decomposition may be employed. We henceforth adopt the standard assumption (as in [1–11]) that $\lambda E - A$ is invertible for some $\lambda \in R$. That is, $sE - A$ is a regular pencil in the sense of Gantmacher [12]. Under this assumption, (1) may be written equivalently as

$$\dot{y}(t) = By(t) + v(t), \tag{2}$$

$$D\dot{z}(t) = z(t) + w(t). \tag{3}$$

If $\det(sE - A)$ has degree $n - p$ then $B \in R^{(n-p)\times(n-p)}$ and $D \in R^{p\times p}$ with D nilpotent. The vector $\left[\begin{smallmatrix} v(t) \\ w(t) \end{smallmatrix}\right]$ is related to $u(t)$ by a nonsingular transformation.

The solutions of (2) are well understood and the forced response of (3) is given by

$$z_f(t) = -\sum_{i=0}^{q-1} D^i w^i(t),$$

where w^i denotes the i-th derivative and q is the index of nilpotency of D. It is the natural response of (3) that is in question so we need only consider the equation

$$D\dot{z} = z. \tag{4}$$

The following result shows that in the conventional sense only one initial condition in (4) corresponds to a solution.

PROPOSITION. *The only distribution that satisfies* (4) *is the trivial distribution.*

Proof. Let a distribution z satisfy the equation. Then

$$D^{q-1}z = D^q\dot{z} = 0.$$

Proceeding inductively, assume that $D^{q-k}z = 0$ for some $1 \leqslant k < q$. Then

$$D^{q-(k+1)}z = D^{q-k}\dot{z} = (D^{q-k}z)^{\boldsymbol{\cdot}} = 0.$$

Hence $D^0 z = 0$. ∎

To account for nonzero initial conditions various arguments have been employed. Doetsch [7] states that we should actually be considering the equation

$$D\dot{z} = z + \delta\, Dz_0, \tag{5}$$

which has solution

$$\Phi(z_0) = -\sum_{i=1}^{q-1} \delta^{i-1} D^i z_0. \tag{6}$$

Note that, in light of the proposition, $\Phi(z_0)$ is not a solution of (4) in the conventional sense unless $Dz_0 = 0$. Also, the value of $\Phi(z_0)$ at $t = 0$ is not defined so it is not clear that calling z_0 an initial condition is justified.

Verghese *et al.* [4–6] agree with (6), basing their arguments on the application of the Laplace transform.

The purpose of this paper is to take a somewhat different approach to the problem. We also agree with (6), but we choose to approach it via singular perturbation theory [13], treating (4) as the limit of a sequence of less ambiguous systems. In this way it will be shown that there is a natural interpretation of $\Phi(z_0)$ in terms of systems approximating (4). Also, z_0 will be seen to arise from the initial conditions of those approximations.

PRELIMINARIES

It is assumed that the reader is familiar with the theory of distributions [14, 15]. In this section we summarize any nonstandard definitions and notation that will be needed later. We denote by K'^p and $K'^{p\times p}$ the spaces of vector and matrix distributions, i.e., the R-vector spaces of continuous linear transformations from the space K of test functions (as defined in [14]) into R^p and $R^{p\times p}$, respectively. Let C_1^{p+} and $C_1^{p\times p+}$ be the continuously differentiable mappings from $[0, \infty)$ into R^p and $R^{p\times p}$, using right-hand differentiation at the origin. C_1^{p+} and $C_1^{p\times p+}$ can be naturally embedded in K'^p and $K'^{p\times p}$.

The Dirac delta of magnitude σ is defined by

$$(\delta\sigma, \phi) = \phi(0)\sigma$$

for all $\phi \in K$ where either $\sigma \in R^p$ or $\sigma \in R^{p\times p}$. The derivative of $f \in C_1^{p+}$ or $f \in C_1^{p\times p+}$ may be defined in two ways: Let $\dot{f}$ denote the derivative of f in the distribution sense and let $f^{(1)}$ be its derivative in the ordinary sense (right hand at the origin). Then

$$\dot{f} = f^{(1)} + \delta f(0).$$

K'^p and $K'^{p\times p}$ become topological vector spaces when the standard topology is defined. One basis of neighbourhoods of the origin is the collection of all sets of the form

$$U_\phi = \{f \mid \|(f, \phi)\| < 1\},$$

where ϕ ranges over K.

For $v \varepsilon R^p$, $M \in R^{p \times p}$, $f \in K'^{p \times p}$, and $g \in K'^p$ define

$$(fv, \phi) = (f, \phi)v,$$
$$(Mf, \phi) = M(f, \phi),$$
$$(Mg, \phi) = M(g, \phi),$$

and let $e(M) \in C_1^{p \times p+}$ be given by

$$e(M)(t) = e^{tM}.$$

We will need the theory of Fourier transforms as developed in [14]. Let C be the complex field and denote by Z'^p and $Z'^{p \times p}$ the C-vector spaces of continuous linear functionals on Z with values in C^p and $C^{p \times p}$. A homeomorphic isomorphism exists between Z'^p and K'^p and is given by

$$(g, \psi) = 2\pi i(f, \phi),^1$$

where $g \in Z'^p$ corresponds to $f \in K'^p$ and $\psi \in Z$ is the transform of $\phi \in K$ (in the conventional sense). Finally, we note that if Λ is a straight line in C parallel to the imaginary axis and $A \in R^{p \times p}$ is nonsingular with eigenvalues $\lambda_1, ..., \lambda_p$ satisfying $\operatorname{Re} \lambda_i < \operatorname{Re} s$ for all $s \in \Lambda$ then $e(A^{-1})$ has transform $g \in Z'^{p \times p}$ given by

$$(g, \psi) = \int_\Lambda \psi(-s)(sA - I)^{-1} A \, ds. \tag{7}$$

If A is nilpotent with index of nilpotency r then $-\sum_{i=1}^{r-1} \delta^{i-1} A^i$ has transform also given by (7).

The Limiting Solution

In attempting to justify (6) we will consider (singular) perturbations of the system (4) and see how the corresponding solutions vary. For example, consider

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{8}$$

[1] Actually, we define $\psi(s) = \int_{-\infty}^{\infty} \phi(t) e^{-st} \, dt$ which corresponds to a 90° rotation of the plane relative to [14]. Consequently, our formulas differ slightly from those of [14].

If we approximate D with nonsingular matrices

$$D_n = \begin{bmatrix} -\frac{1}{n} & 1 \\ 0 & -\frac{1}{n} \end{bmatrix}, \qquad n = 1, 2, 3,\ldots,$$

then the approximating systems

$$D_n z_n^{(1)} = z_n \tag{9}$$

have well-defined solutions

$$z_n(t) = \begin{bmatrix} e^{-nt} & -n^2 te^{-nt} \\ 0 & e^{-nt} \end{bmatrix} z(0).$$

Considering only $t \geqslant 0$, we have $z_n \in C_1^{p+}$ and

$$z_n \to \begin{bmatrix} -\delta z_{02} \\ 0 \end{bmatrix} \tag{10}$$

in the K'^p topology where

$$z(0) = \begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix}.$$

We would like to say that the limit in (10) is the "solution" of (4) when D is given by (8). From a physical viewpoint this is reasonable since the physical system described by (4) is, in reality, probably described more precisely by (9). That is, (4) can be considered an idealized model of a higher-order system.

With these thoughts in mind we make the following definition:

DEFINITION. $z \in K'^p$ is a *limiting solution* of (4) with initial condition $z_0 \in R^p$ if there exist sequences $z_n(0) \to z_0$ and $D_n \to D$ with D_n invertible such that the solutions z_n of (9), subject to initial conditions $z_n(0)$, converge to z in the topology of K'^p.

Several questions come to mind immediately. First, can (4) always be perturbed in a nonsingular manner so that the corresponding sequence of solutions converges? In other words, does a limiting solution always exist?

Second, is the limiting solution unique? In general there are infinitely many ways to perturb (4) so that the solutions converge. It is not clear whether different approximating sequences yield different limits of solutions.

There is also a third important question which we leave until the next section. We now treat existence and uniqueness.

LEMMA 1. *If $D_n \to D$ with D_n invertible and $\alpha_n = \int_0^\infty \|D_n^k e^{tD_n^{-1}}\| \, dt$ is a bounded sequence for some nonnegative integer k then the solution of* (9) *with initial condition z_0 converges to $\Phi(z_0)$ as $n \to \infty$.*

Proof. Let $\phi \in K$. Then there is a real number M such that $|\phi(t)| \leqslant M$ for all $t \in R$ so

$$\|(D_n^{q+k} e(D_n^{-1}), \phi)\| \leqslant \int_0^\infty |\phi(t)| \, \|D_n^{q+k} e^{tD_n^{-1}}\| \, dt$$
$$\leqslant M \, \|D_n^q\| \, \alpha_n.$$

Hence $D_n^{q+k} e(D_n^{-1}) \to 0$. Next, letting $e(D_n^{-1})^{q+k}$ denote the $(q+k)$th derivative of $e(D_n^{-1})$ considered as a distribution, we have

$$e(D_n^{-1})^{q+k} = D_n^{-q-k} e(D_n^{-1}) + \delta \, D_n^{1-q-k} + \cdots + \delta^{q+k-1} I.$$

Solving for $e(D_n^{-1})$,

$$e(D_n^{-1}) = D_n^{q+k} e(D_n^{-1})^{q+k} - \sum_{i=1}^{q+k} \delta^{i-1} D_n^i.$$

Thus

$$e(D_n^{-1}) \to -\sum_{i=1}^{q+k} \delta^{i-1} D^i = -\sum_{i=1}^{q-1} \delta^{i-1} D^i$$

and

$$z_n = e(D_n^{-1}) \, z_0 \to \Phi(z_0). \quad \blacksquare$$

LEMMA 2. *Let $D_n = D - (1/n)I$. Then the sequence $\alpha_n = \int_0^\infty \|D_n^k e^{tD_n^{-1}}\| \, dt$ is bounded for some k.*

Proof. Since

$$\left(D - \frac{1}{n} I\right)^{-1} = -\sum_{i=0}^{q-1} n^{i+1} D^i,$$

we have

$$e^{t(D-(1/n)I)^{-1}} = e^{-nt} \prod_{i=1}^{q-1} e^{-n^{i+1} t D^i}.$$

If $k \geqslant q-1$ then

$$\begin{aligned}\|D_n^k e^{tD_n^{-1}}\| &\leqslant e^{-nt} \left\| \sum_{i=0}^{q-1} \left(-\frac{1}{n}\right)^{k-i} D^i \right\| \prod_{i=1}^{q-1} \left\| \sum_{j=0}^{q-1} \frac{(-tn^{i+1})^j}{j!} D^{ij} \right\| \\ &\leqslant e^{-nt} \sum_{i=0}^{q-1} \frac{1}{n^{k-i}} \|D\|^i \prod_{i=1}^{q-1} \sum_{j=0}^{q-1} \frac{(tn^{i+1})^j}{j!} \|D\|^{ij} \\ &= e^{-nt} \sum_{i=-k}^{M-k} \sum_{j=0}^{N} c_{ij} n^i t^j\end{aligned}$$

for some constants M, N, and c_{ij}. Then

$$\begin{aligned}\alpha_n &\leqslant \sum_{i=-k}^{M-k} \sum_{j=0}^{N} c_{ij} n^i \int_0^\infty t^j e^{-nt}\, dt \\ &= \sum_{i=-k}^{M-k} \sum_{j=0}^{N} j!\, c_{ij} n^{i-j-1} \\ &\leqslant N!\, n^{M-k-1} \max c_{ij}.\end{aligned}$$

Set $k = \max\{M-1, q-1\}$. ∎

Applying the definition we now have

THEOREM 1. *For each $z_0 \in R^p$, $\Phi(z_0)$ is a limiting solution of $D\dot{z} = z$ with initial condition z_0.*

If Eq. (4) were arrived at as part of a singular perturbation problem, it is most likely that the sequence (D_n) considered in the development of Theorem 1 was not the one actually encountered. However, the issue here is whether any sequence (D_n) exists that gives $\Phi(z_0)$ as a limiting solution. Having established an affirmative answer to this question, we may now consider the effect of other approximating sequences. That is, we need to address uniqueness of the limiting solution.

THEOREM 2. *The limiting solution of $D\dot{z} = z$ with initial codition z_0 is unique.*

Proof. Let $D_n \to D$ with D_n invertible. Also let

$$z_n = e(D_n^{-1})\, z_0 \to f \in K'^p.$$

From

$$e(D_n^{-1})^q = D_n^{-q}\, e(D_n^{-1}) + \delta\, D_n^{1-q} + \cdots + \delta^{q-1} I$$

it follows that

$$\begin{aligned} z_n &= D_n^q e(D_n^{-1})^q z_0 - \sum_{i=1}^{q} \delta^{i-1} D_n^i z_0 \\ &= D_n^q z_n^q - \sum_{i=1}^{q} \delta^{i-1} D_n^i z_0 \\ &\to 0 \cdot f + \Phi(z_0) \end{aligned}$$

so

$$f = \Phi(z_0). \quad \blacksquare$$

Unreasonable Perturbations

The third question we must deal with concerns certain pathological approximations to (4). For example, suppose that in (9) we were given

$$D_n = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{bmatrix}. \tag{11}$$

Then the resulting sequence of systems would exhibit instability increasing without bound as $n \to \infty$. It is not resonable to think that this system would yield any sort of limiting solution. However, since the system determined by (11) has solutions that diverge on the entire half line $[0, \infty)$, we would not expect that the limiting system (4) would be a good idealization of (9), (11) to begin with.

To generalize this, note that for all systems of interest, when t is outside a neighbourhood of the origin (i.e., after switching transients have died away), the system behaviour is unambiguously determined by (4). The Proposition and Theorems 1 and 2 show that the solution must be zero. Only in the vicinity of $t = 0$ is the system response questionable. Thus, if an approximating sequence D_n is chosen which does not result in solutions converging to zero on some subinterval of $(0, \infty)$, it can be disregarded as being pathological. Its existence does not constitute an argument against the validity of $\Phi(z_0)$ as the solutions of (4).

We claim that convergence of the solutions of (9) to zero on some subinterval of $(0, \infty)$ is in fact sufficient to guarantee that they also converge on a neighbourhood of the origin. To prove this, a few preliminary results are needed.

LEMMA 3. *If* $s, \lambda \in C$, *and* $r > 0$ *satisfy* $|s-\lambda| \geqslant r$ *then*

$$1 + \frac{1}{r}|s| \geqslant \frac{|\lambda|}{|s-\lambda|}.$$

Proof. The result follows from elementary arguments.

LEMMA 4. *Let* $\psi \in Z$ *and* A *be a* $p \times p$ *matrix, either nilpotent or invertible with eigenvalues* λ_i, $i = 1,..., p$, *satisfying*

$$\operatorname{Re} \lambda_i \leqslant \max \left\{ 0, N \ln \frac{|\operatorname{Im} \lambda_i|}{c} \right\}$$

for some $c > 0$, $N < \infty$. *Let* $\Gamma \subset C$ *be the path parameterized by*

$$\begin{aligned} \gamma(x) &= r + ix && \text{if} \quad x \in [-(c-r)\, e^{r/N}, (c-r)\, e^{r/N}], \\ &= N \ln \frac{|x|}{c-r} + ix && \text{if} \quad x \in R - [-(c-r)\, e^{r/N}, (c-r)\, e^{r/N}], \end{aligned}$$

where $r > 0$. *Finally, let* $\Lambda \subset C$ *be any straight line parallel to the imaginary axis and satisfying* $\operatorname{Re} \lambda_i < \operatorname{Re} s$, $i = 1, 2,..., p$, *for any* $s \in \Lambda$.

Then

$$\int_\Gamma \psi(-s)(sA - I)^{-1} A \, ds$$

exists and equals

$$\int_\Lambda \psi(-s)(sA - I)^{-1} A \, ds$$

Proof. For some $\sigma \in R$,

$$\int_\Lambda \psi(-s)(sA - I)^{-1} A \, ds = \lim_{m \to \infty} i \int_{-m}^{m} \psi(-\sigma - ix)((\sigma + ix)A - I)^{-1} A \, dx.$$

Let

$$\Gamma_m = \{s \in \Gamma \mid |\operatorname{Im} s| \leqslant m\}.$$

If $m > (c-r) \max\{e^{r/N}, e^{\sigma/N}\}$ then $\operatorname{Re} \gamma(m) > \sigma$ and, because of the assumptions on the λ_i, Cauchy's theorem yields

$$
\begin{aligned}
i\int_{-m}^{m} &\psi(-\sigma - ix)((\sigma + ix)A - I)^{-1}A\,dx \\
&= \int_{\sigma}^{\operatorname{Re}\gamma(m)} \psi(-x + im)((x - im)A - I)^{-1}A\,dx \\
&\quad + \int_{\Gamma_m} \psi(-s)(sA - I)^{-1}A\,ds \\
&\quad - \int_{\sigma}^{\operatorname{Re}\gamma(m)} \psi(-x - im)((x + im)A - I)^{-1}A\,dx.
\end{aligned}
$$

Since $\psi \in Z$ there exist $\beta, \alpha_k > 0$, $k = 0, 1, 2, \ldots$ such that

$$|\psi(-s)| \leqslant \alpha_k \frac{e^{\beta|\operatorname{Re} s|}}{|s|^k}$$

for all $s \in C$ (see [14]). Let

$$\Delta_m = \int_{\sigma}^{\operatorname{Re}\gamma(m)} \psi(-x + im)((x - im)A - I)^{-1}A\,dx.$$

Assume A is invertible and let $k > \beta N$. Then

$$
\begin{aligned}
|\Delta_m| &\leqslant \int_{\sigma}^{\operatorname{Re}\gamma(m)} \alpha_k \frac{e^{\beta|x|}}{(x^2 + m^2)^{k/2}} \|((x + im)\,I - A^{-1})^{-1}\|\,dx \\
&\leqslant (\operatorname{Re}\gamma(m) - \sigma)\alpha_k \frac{e^{\beta \operatorname{Re}\gamma(m)}}{m^k} \sup_{x\in R} \|((x + im)\,I - A^{-1})^{-1}\| \\
&= \left(N \ln \frac{m}{c - r} - \sigma\right) \alpha_k \frac{m^{\beta N - k}}{(c - r)^{\beta N}} \sup_{x \in R} \|((x + im)\,I - A^{-1})^{-1}\| \\
&\to 0 \qquad \text{as} \quad m \to \infty.
\end{aligned}
$$

Assume A is nilpotent with index of nilpotency r and set $k > \beta N + r - 2$. Then

$$(sA - I)^{-1}A = -\sum_{i=1}^{r-1} s^{i-1}A^i$$

so

$$
\begin{aligned}
|\Delta_m| &\leqslant \int_{\sigma}^{\operatorname{Re}\gamma(m)} \alpha_k \frac{e^{\beta|x|}}{(x^2 + m^2)^{k/2}} \sum_{i=1}^{r-1} (x^2 + m^2)^{(i-1)/2} \|A\|^i\,dx \\
&\leqslant (\operatorname{Re}\gamma(m) - \sigma)\,\alpha_k\, e^{\beta \operatorname{Re}\gamma(m)} \sum_{i=1}^{r-1} \frac{\|A\|^i}{m^{k-i+1}}
\end{aligned}
$$

$$= \left(N \ln \frac{m}{c-r} - \sigma\right) \alpha_k \frac{m^{\beta N}}{(c-r)^{\beta N}} \sum_{i=1}^{r-1} \frac{\|A\|^i}{m^{k-i+1}}$$

$$\to 0 \qquad \text{as} \quad m \to \infty.$$

Similarly,

$$\int_{\sigma}^{\operatorname{Re} \gamma(m)} \psi(-x - im)((x + im) A - I)^{-1} A \, dx \to 0.$$

Note that

$$\begin{aligned} \gamma'(x) &= i && \text{if} \quad x \in [-(c-r)\, e^{r/N}, (c-r)\, e^{r/N}], \\ &= N/x + i && \text{if} \quad x \in [-(c-r)\, e^{r/N}, (c-r) e^{r/N}], \end{aligned}$$

and let

$$\Omega(x) = \gamma'(x)\, \psi(-\gamma(x))(\gamma(x) A - I)^{-1} A.$$

For A invertible and $k > \beta N$,

$$\begin{aligned} \|\Omega(x)\| &\leqslant \sqrt{1 + \frac{N^2}{x^2}}\, \alpha_k \frac{e^{\beta \operatorname{Re} \gamma(x)}}{|\gamma(x)|^k} \|(\gamma(x)\, I - A^{-1})^{-1}\| \\ &\leqslant \sqrt{1 + \frac{N^2}{x^2}}\, \alpha_k \frac{|x|^{\beta N - k}}{(c-r)^{\beta N}} \|(\gamma(x)\, I - A^{-1})^{-1}\|. \end{aligned}$$

For A nilpotent and $k > \beta N + r - 2$,

$$\begin{aligned} \|\Omega(x)\| &\leqslant \sqrt{1 + \frac{N^2}{x^2}}\, \alpha_k \frac{e^{\beta \operatorname{Re} \gamma(x)}}{|\gamma(x)|^k} \sum_{i=1}^{r-1} |\gamma(x)|^{i-1} \|A\|^i \\ &\leqslant \sqrt{1 + \frac{N^2}{x^2}}\, \alpha_k \frac{|x|^{\beta N}}{(c-r)^{\beta N}} \sum_{i=1}^{r-1} \frac{\|A\|^i}{|x|^{k-i+1}}. \end{aligned}$$

In either case, Ω is integrable on $(-\infty, \infty)$ and

$$\int_{\Gamma} \psi(-s)(sA - I)^{-1} A \, ds = \int_{-\infty}^{\infty} \Omega(x)\, dx.$$

By the dominated convergence theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} \Omega(x)\, dx &= \lim_{m \to \infty} \int_{\Gamma_m} \psi(-s)(sA - I)^{-1} A \, ds \\ &= i \lim_{m \to \infty} \int_{-m}^{m} \psi(-\sigma - ix)((\sigma + ix) A - I)^{-1} A \, dx \\ &= \int_{\Lambda} \psi(-s)(sA - I)^{-1} A \, ds. \qquad \blacksquare \end{aligned}$$

LEMMA 5. *If* $[a, b]$ *is a subinterval of* $(0, \infty)$ *and* $e(D_n^{-1}) \to 0$ *on* $[a, b]$ *in the* L^1 *topology then the eigenvalues* λ_{in}, $i = 1,\ldots, p$ *of* D_n^{-1} *satisfy* $\operatorname{Re} \lambda_{in} \to -\infty$.

Proof. Since $e^{t\lambda_{in}}$ is an eigenvalue of $e^{tD_n^{-1}}$,

$$e^{t \operatorname{Re} \lambda_{in}} \leqslant \| e^{tD_n^{-1}} \|.$$

Hence

$$\int_a^b e^{t \operatorname{Re} \lambda_{in}} \, dt \leqslant \int_a^b \| e^{tD_n^{-1}} \| \, dt \to 0.$$

But

$$\begin{aligned} \int_a^b e^{t \operatorname{Re} \lambda_{in}} \, dt &= \frac{1}{\operatorname{Re} \lambda_{in}} (e^{b \operatorname{Re} \lambda_{in}} - e^{a \operatorname{Re} \lambda_{in}}) \qquad && \text{if} \quad \operatorname{Re} \lambda_{in} \neq 0, \\ &= b - a && \text{if} \quad \operatorname{Re} \lambda_{in} = 0 \end{aligned}$$

so the desired result follows from elementary arguments. ∎

We are now in a position to prove the main result of this section. Together, Lemma 5 and Theorem 3 show that whenever a small perturbation of (4) gives a solution close to the desired zero solution on a subinterval of $(0, \infty)$, the approximating system approximates the idealized system response in the vicinity of $t = 0$ as well. Theorem 3 is also a generalization of a theorem by Francis [16].

THEOREM 3. *If the eigenvalues* λ_{in} *of* D_n^{-1} *satisfy*

$$\operatorname{Re} \lambda_{in} \leqslant \max \left\{ 0, N \ln \frac{|\operatorname{Im} \lambda_{in}|}{c} \right\}$$

for some N, $c > 0$, $i = 1,\ldots, p$, *and* $n = 1, 2, 3,\ldots$, *then*

$$e(D_n^{-1}) \to -\sum_{i=1}^{q-1} \delta^{i-1} D^i.$$

Proof. We will show that the Fourier transforms of $e(D_n^{-1})$ converge to that of $-\sum_{i=1}^{q-1} \delta^{i-1} D^i$ in the topology of Z'^p. From Lemma 4 we need only show that

$$\int_\Gamma \psi(-s)(sD_n - I)^{-1} D_n \, ds \to \int_\Gamma \psi(-s)(sD - I)^{-1} D \, ds$$

for any $\psi \in Z$.

First note that

$$(sD_n - I)^{-1} D_n \to (sD - I)^{-1} D$$

pointwise on Γ. Thus

$$\begin{aligned}\int_\Gamma \psi(-s)(sD_n - I)^{-1} D_n \, ds &= \int_{-\infty}^{\infty} \Omega_n(x)\, dx \\ &\to \int_{-\infty}^{\infty} \gamma'(x)\, \psi(-\gamma(x))(\gamma(x) D - I)^{-1} D \, dx \\ &= \int_\Gamma \psi(-s)(sD - I)^{-1} D \, ds. \quad \blacksquare\end{aligned}$$

The conditions of Theorem 3 are significantly more general than needed to link it with Lemma 5. However, the complexity of the proof is not reduced if the conditions are weakened. Moreover, Theorem 3 may be useful in other branches of singular perturbation theory where more general types of eigenvalue behaviour are present.

Conclusion

We have presented three main results offering further justification of $\Phi(z_0)$ as the solution of (4) due to initial condition z_0. The singular perturbation approach is somewhat more intuitively appealing than the arguments used by other authors to arrive at the same result. This approach allows one to convince oneself of the actual system response by considering a nonidealized approximation to the singular equation. From the results we have presented it is clear that for a given initial condition there is always precisely one meaningful solution to (1). This solution has been designed in such a way that it is justified not only mathematically, but from a physical viewpoint as well.

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