Steady-state matching and model reduction for systems of differential-algebraic equations

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Abstract—The problem of model reduction for nonlinear differential-algebraic systems is addressed using the notions of moment and of steady-state response. These notions are formally introduced for this class of systems and families of nonlinear differential-algebraic reduced order models achieving moment matching with additional properties are presented. Stronger results for the special class of linear singular systems are provided. Two simple examples illustrate the proposed technique.

I. INTRODUCTION

SINGULAR *systems*, also known as *descriptor systems* or *differential-algebraic equations*[1] (DAEs) are a class of systems which, combining differential equations with algebraic equations, are used to model constraints and multi-scale behaviors [1]–[3]. In fact, they can be seen as the composition of a fast subsystem (the algebraic part) and a slow subsystem (the differential part). For this reason, systems of differential-algebraic equations are used to model various behaviors and phenomena, *e.g.* multi-body systems, electrical networks, chemical processes, social economic systems [1], and to model large interconnected systems, *e.g.* systems obtained from the application of the Kirchoff's laws or models created by object-oriented modeling languages such as MODELICA [4], [5].

As a result, systems of differential-algebraic equations may have large dimensionality. This has motivated researchers to investigate model reduction techniques. Given a *complex* high dimensional system, the model reduction problem consists in determining a reduced order model, namely a *simple* low dimensional mathematical representation with a behavior similar (in a sense to be defined) to the behavior of the complex system. For linear singular systems several approaches have been presented in the last decades. Some of these techniques are based on balanced truncation, see *e.g.* [6]–[12], on interpolation theory, see *e.g.* [13], and on other variations or *ad hoc* methods, see *e.g.* [14]–[19]. More detail and additional references on model reduction of linear singular systems can be found in the survey paper [5]. Conversely, only a few methods to reduce nonlinear singular systems have been proposed. A method based on balanced truncation has been proposed in [4]. Further results have been obtained using piecewise-linear model reduction, see [20]–[22]. Model reduction of nonlinear singular systems in special forms, *e.g.* bilinear systems, has been investigated in [23], [24]. An approach based on adaptive proper orthogonal

[1]Throughout the technical note we use the expressions "singular system" and "system of differential-algebraic equations" interchangeably.

decomposition of a related class of distributed parameter systems has been proposed in [25], [26].

In this technical note we present, in a unified framework, a model reduction technique for linear and nonlinear singular systems. Exploiting the connection between moment and steady-state response given in [27], we provide a notion of moment for nonlinear DAEs (Section II-A). We propose families of reduced order models for nonlinear singular systems (Section III-A) and we analyze in detail special families for which additional properties, *e.g.* strong stability, can be imposed (Section III-B). In addition we specialize the results to linear singular systems, obtaining stronger properties for this particular class (Sections II-B, III-C, III-D).

Preliminary versions of this technical note have been published in [28] and [29]. The additional contributions of the technical note are as follows: we provide all the proofs of the results (in the Appendix); the presentation has been improved, developing the results for nonlinear systems and then deriving the linear framework as a special case; we study the controllability and observability properties of the linear reduced order models; we illustrate the results with two examples, namely we compute the moment in an academic example and we obtain reduced order models for a nonlinear system describing an electrical circuit.

Notation. We use standard notation. $\mathbb{C}_{<0}$ ($\mathbb{C}_0$) denotes the set of complex numbers with negative (zero) real part. The symbol $\emptyset$ indicates the empty set, I denotes the identity matrix, $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n \times n}$, $\mathrm{rank}(A)$ denotes its rank and $\det(A)$ its determinant. The superscript $\top$ denotes the transposition operator. Given a polynomial p, $\deg(p)$ indicates its degree. Given two functions, $f : Y \to Z$ and $g : X \to Y$, with $f \circ g : X \to Z$ we denote the composite function which maps all $x \in X$ to $f(g(x)) \in Z$ as $(f \circ g)(x) = f(g(x))$.

II. NOTION OF MOMENT FOR CLASSES OF DAEs

In this section we provide the definition of moment for singular systems. We first formulate the notion for nonlinear singular systems. We then conclude the section with an in-depth discussion of the special case of linear singular systems.

A. Moment for nonlinear singular systems

Consider a nonlinear, single-input, single-output, continuous-time, system described by the equations

$$E\dot{x} = f(x, u), \qquad y = h(x), \qquad (1)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $E \in \mathbb{R}^{n \times n}$ and f and h smooth mappings. If E is invertible, then equations (1)

describe a nonlinear system, which we call *normal*. If E is singular, then system (1) is called *singular* system. Without loss of generality we assume that $\mathrm{rank}(E) = r < n$ with $E = \mathrm{diag}(I, 0)$. Throughout the technical note we also assume that the initial condition $x(0)$ is *consistent*, *i.e.* the initial value problem associated to (1) has at least one solution [2]. Consider a signal generator described by the equations

$$\dot{\omega} = s(\omega), \qquad u = l(\omega), \qquad (2)$$

with $\omega(t) \in \mathbb{R}^\nu$, s and l smooth mappings, and the interconnected system

$$\dot{\omega} = s(\omega), \qquad E\dot{x} = f(x, l(\omega)), \qquad y = h(x). \qquad (3)$$

In addition, suppose that $f(0,0) = 0$, $s(0) = 0$, $l(0) = 0$ and $h(0) = 0$. The following assumptions and definitions provide a generalization of the notion of moment.

Definition 1: System (2) is observable if for any pair of initial conditions $\omega_a(0)$ and $\omega_b(0)$, such that $\omega_a(0) \neq \omega_b(0)$, the corresponding output trajectories $l(\omega_a(t))$ and $l(\omega_b(t))$ are such that $l(\omega_a(t)) - l(\omega_b(t)) \not\equiv 0$, *i.e.* the output trajectories are not indistinguishable, see [30, Definition 3.27].

Assumption 1: The signal generator (2) is observable and neutrally stable. Finally, $\omega(0)$ is almost periodic and such that (2) satisfies the excitation rank condition[2] at $\omega(0)$.

Assumption 2: The pair (E, A), with $A = \left. \frac{\partial f(x,0)}{\partial x} \right|_{x=0}$, is strongly stable, *i.e.* for all $\bar{\lambda} \in \mathbb{C}$ such that $\det(\bar{\lambda} E - A) = 0$, $\bar{\lambda} \in \mathbb{C}_{<0}$ and $\deg(\det(\lambda E - A)) = r$.

If the system is normal, Assumption 2 reduces to the hypothesis that the zero equilibrium of $\dot{x} = f(x, 0)$ is asymptotically stable in the first approximation. This is a necessary condition for the existence of a center manifold which we intend to exploit to characterize the notion of moment [31]. However, Assumption 2 is stronger than simple stability because it guarantees that all the trajectories of the system exist and are bounded for all $t \geq 0$ (for sufficiently small initial states), *i.e.* the response of the system is impulse free [34].

Lemma 1: Consider system (1) and the signal generator (2). Suppose Assumptions 1 and 2 hold. Then there is a sufficiently smooth mapping $\pi(\omega)$, with $\pi(0) = 0$, locally defined in a neighborhood of $\omega = 0$ which solves the partial differential-algebraic equation

$$E\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega), l(\omega)). \qquad (4)$$

In addition, for any sufficiently small $x(0)$ and $\omega(0)$, the solution $x(t)$, $\omega(t)$ of (3) exists, is bounded for all $t \geq 0$ and satisfies $\lim_{t\to\infty} x(t) - \pi(\omega(t)) = 0$.

Remark 1: Observability and neutral stability of the generator (2) are the only conditions (from Assumption 1) required to prove Lemma 1. However, we want to exclude some "pathological" situations in which the components of the steady-state may be identically zero for selected initial conditions $\omega(0)$ (*e.g.* we want to exclude the initial condition $\omega(0) \neq 0$). For this reason we require the excitability rank condition to be satisfied at an almost periodic $\omega(0)$. This condition

guarantees persistence of excitation of the signal u [32]. For an in-depth discussion of the relation between this condition and the problem of model reduction see [33].

The result of Lemma 1 implies that the interconnected system (3) possesses an invariant center manifold described by the equation $x = \pi(\omega)$ and that this equation expresses the steady-state response of system (1) driven by (2). In analogy with the definition of moment for nonlinear normal systems given in [27] we define the moment for nonlinear singular systems.

Definition 2: Consider system (1) and the signal generator (2). Suppose Assumption 1 holds. The function $h \circ \pi$, with π solution of equation (4), is the *moment of system (1) at* (s, l).

Remark 2: The results can be extended to the class of uncertain systems $E\dot{x} = f(x, u, \widetilde{\eta})$, $y = h(x, \widetilde{\eta})$, $u = l(\omega, \widetilde{\eta})$, where $\widetilde{\eta} \in \mathbb{R}^{\bar{n}}$ is a vector of unknown parameters of the plant. The results still hold [34, Lemma 8.21] and we can define the *parametric moment of the system at* (s, l) as the mapping $h \circ \pi$, with π solution of the equation $E\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega, \widetilde{\eta}), l(\omega, \widetilde{\eta}), \widetilde{\eta})$. This is illustrated in the next example.

Example 1: Consider the nonlinear uncertain singular system described by the equations

$$
\begin{aligned}
\dot{x}_1 &= 2x_2 + (1 + \widetilde{\eta})x_3, & 0 &= x_2 + x_3, \\
\dot{x}_2 &= -4x_1 - 2x_2 + x_4, & 0 &= -x_1 - \sin(x_2) + x_4 + u, \\
y &= x_4,
\end{aligned}
$$
$$(5)$$

parametrized in $\widetilde{\eta} \in \mathbb{R}$, and the input

$$
\begin{aligned}
\dot{\omega}_1 &= \sqrt{2}\omega_2, & \dot{\omega}_2 &= -\sqrt{2}\omega_1, \\
u &= -\left(3 - \frac{2}{1-\widetilde{\eta}}\right)\omega_1 - \frac{2\sqrt{2}}{1-\widetilde{\eta}}\omega_2 + \sin\left(\frac{\sqrt{2}}{1-\widetilde{\eta}}\omega_2\right).
\end{aligned}
$$
$$(6)$$

The pair (E, A) has two finite[3] eigenvalues $\lambda_{1,2} = -0.5 \pm \sqrt{3\widetilde{\eta} - 2.75}$, both in the left half complex plane if $\widetilde{\eta} < 1$. Simple computation shows that the mapping $\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 \end{bmatrix}^\top$ is given by $\pi_1 = \omega_1$, $\pi_2 = \frac{\sqrt{2}}{1-\widetilde{\eta}}\omega_2$, $\pi_3 = -\frac{\sqrt{2}}{1-\widetilde{\eta}}\omega_2$ and $\pi_4 = (4 - \frac{2}{1-\widetilde{\eta}})\omega_1 + \frac{2\sqrt{2}}{1-\widetilde{\eta}}\omega_2$. For $\widetilde{\eta} < 1$ the pair (E, A) is strongly stable, Assumptions 1 and 2 hold, the steady-state response of system (6) is $\pi(\omega)$ and π_4 is the parametric moment. If $\widetilde{\eta} > 1$, the mapping π is still a solution of the partial differential equation, thus π_4 is the parametric moment, but we cannot establish a relation with the steady-state response of the system.

Remark 3: Remark 2 and Example 1 show a possible approach to deal with constant uncertain parameters of the system to be reduced. If the uncertain parameter obeys a stochastic distribution, then the techniques presented in [35], [36] to deal with stochastic systems could be extended to the present singular framework. Uncertainties can be addressed also formulating the uncertain system as an inclusion, see [37]. This offers another approach that can be extended to singular systems. Finally, if the model is completely unknown, we can extend to differential-algebraic equations the data-driven techniques based on output measurements which have been presented in [38].

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[2]See [31, Chapter 8] for the definition of neutral stability. See [32], [33] for the definition of almost periodicity and excitation rank condition.

[3]The finite eigenvalues of the pair (E, A) are the zeros of $\det(\lambda E - A)$ for finite values of λ [1].

B. Moment for linear singular systems

It is of particular interest to specialize the results we have obtained to the case of linear singular systems. In fact, on one hand, exploiting linearity we can obtain stronger results and provide new insights. On the other hand, since the problem of model reduction for linear singular systems is a classic problem in linear algebra, it is important to relate this framework with the literature.

1) Preliminaries on linear singular systems: in this section we recall some basic results on linear singular systems. Assume system (1) takes the form

$$E\dot{x} = Ax + Bu, \qquad y = Cx, \qquad (7)$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $C \in \mathbb{R}^{1 \times n}$. Assume that $u(t)$ is piecewise continuously differentiable.

Definition 3: [1] Let $\mathcal{E} \subset \mathbb{C}$ be the set of complex numbers $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \neq 0$. For any two matrices E and A, the pencil (E, A) is called *regular* if $\mathcal{E} \neq \emptyset$.

Lemma 2: [1] The pencil (E, A) is regular if and only if there exist two nonsingular matrices Q and P such that $QEP = \text{diag}(I, N)$, $QAP = \text{diag}(A_1, I)$, where $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent with degree[4] d and $A_1 \in \mathbb{R}^{n_1 \times n_1}$, with $n_1 + n_2 = n$.

Assume system (7) is regular, then Q and P can be selected such that system (7) can be written in the so-called *first equivalent form* [1], namely

(*slow subsystem*) $\quad \dot{x}_1 = A_1 x_1 + B_1 u, \quad y_1 = C_1 x_1,$
(*fast subsystem*) $\quad N\dot{x}_2 = x_2 + B_2 u, \quad y_2 = C_2 x_2,$
$$y = y_1 + y_2, \qquad (8)$$

with the coordinate transformation $\begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top = P^{-1}x$, where $x_1 \in \mathbb{R}^{n_1 \times n_1}$, $x_2 \in \mathbb{R}^{n_2 \times n_2}$, $QB = [B_1^\top \ B_2^\top]^\top$ and $CP = [C_1 \ C_2]$. The state response of system (7) is given by

$$x(t) = P \begin{bmatrix} I \\ 0 \end{bmatrix} \left(e^{A_1 t} x_1(0) + \int_0^t e^{A_1(t-\tau)} B_1 u(\tau) d\tau \right)$$
$$- P \begin{bmatrix} 0 \\ I \end{bmatrix} \sum_{i=0}^{d-1} N^i B_2 \frac{d^i}{dt^i} u(t). \qquad (9)$$

For linear singular systems the assumption that the initial condition $x(0) = P \begin{bmatrix} x_1^\top(0) & x_2^\top(0) \end{bmatrix}^\top$ is *consistent* implies that $x(0) = P[I \ 0]^\top x_1(0) - P[0 \ I]^\top \sum_{i=0}^{d-1} N^i B_2 \frac{d^i}{dt^i} u(t)|_{t=0}$, where $x_1(0)$ can be freely selected.

2) Interpolation-based description of moment: in this section we show that the moments of system (7) are uniquely determined by the solution of a generalized Sylvester equation. Let $W(q) = C(qE - A)^{-1}B = C_1(qI - A_1)^{-1}B_1 + C_2(qN - I)^{-1}B_2$ be the transfer function associated to system (7) and assume that the system is minimal, *i.e.*, as for normal systems (see [1, Theorem 2-6.3]), controllable and observable.

Definition 4: [39, Chapter 11]) Let $s_i \in \mathcal{E}$. The 0-*moment of system (7) at* s_i is the complex number $\eta_0(s_i) = C(s_i E - A)^{-1}B$. The k-*moment of system (7) at* s_i is the complex number $\eta_k(s_i) = \frac{(-1)^k}{k!} \left[\frac{d^k}{dq^k} W(q) \right]_{q=s_i}$, with $k \geq 1$ integer.

The next result, which is a direct extension of [27] (see also [40], [41] for analogous results), gives a relation between the moments and the solution of a generalized Sylvester equation.

Lemma 3: Let $s_i \in \mathcal{E}$. Consider system (7), then $[\eta_0(s_i) \ \ldots \ \eta_k(s_i)] = C\widetilde{\Pi}_i \Psi_k$, where $\Psi_k = \text{diag}(1, -1, 1, \ldots, (-1)^k) \in \mathbb{R}^{(k+1) \times (k+1)}$ and $\widetilde{\Pi}_i \in \mathbb{R}^{n \times k+1}$ is the unique solution of the generalized Sylvester equation

$$E\widetilde{\Pi}_i \Sigma_i = A\widetilde{\Pi}_i + BL_i, \qquad (10)$$

with $L_i = [1 \ 0 \ \ldots \ 0] \in \mathbb{R}^{(k+1)}$ and

$$\Sigma_i = \begin{bmatrix} s_i & 1 & 0 & \ldots & 0 \\ 0 & s_i & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & s_i & 1 \\ 0 & \ldots & \ldots & 0 & s_i \end{bmatrix} \in \mathbb{R}^{(k+1) \times (k+1)}.$$

Exploiting this lemma, the following result holds.

Theorem 1: Consider system (7) and a set of numbers $s_i \in \mathcal{E}$, with $i = 1, \ldots, \rho$. Then the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1)$, $\ldots, \eta_0(s_\rho), \ldots, \eta_{k_\rho-1}(s_\rho)$ are uniquely determined by the matrix $C\Pi$, where $\Pi \in \mathbb{R}^{n \times \nu}$ is the unique solution of the generalized Sylvester equation

$$E\Pi S = A\Pi + BL, \qquad (11)$$

with $S \in \mathbb{R}^{\nu \times \nu}$ any non-derogatory[5] matrix with characteristic polynomial

$$c(q) = \prod_{i=1}^{\rho} (q - s_i)^{k_i}, \qquad (12)$$

where $\nu = \sum_{i=1}^{\rho} k_i$, and $L \in \mathbb{R}^{1 \times \nu}$ such that the pair (S, L) is observable.

Remark 4: The Sylvester equation (11) is the linear version of the partial differential-algebraic equation (4).

3) Steady-state-based description of moment: we are now ready to present the "steady-state-based" description of moment for linear singular systems. This result links the nonlinear notion given in Section II-A with the linear framework.

Theorem 2: Let $S \in \mathbb{R}^{\nu \times \nu}$ be any non-derogatory matrix with characteristic polynomial (12), with $k_i = 1$ for all $i = 1, \ldots, \rho$. Consider system (7) and suppose that $\sigma(S) \subset \mathbb{C}_0$ and $\sigma(A) \subset \mathbb{C}_{<0}$. Consider the interconnection of system (7) with the system

$$\dot{\omega} = S\omega, \qquad u = L\omega, \qquad (13)$$

with L and $\omega(0)$ such that the triple $(S, \omega(0), L)$ is minimal. Then the moments $\eta_0(s_1), \eta_0(s_2), \ldots, \eta_0(s_\rho)$ are uniquely determined by the steady-state response of the output of such interconnected system, *i.e.* $C\Pi\omega$.

Remark 5: In the linear case the moments can be defined as in Definition 4 or *equivalently* as the matrix $C\Pi$, as established in Theorem 1, or *equivalently* as steady-state output mapping $C\Pi$, as established in Theorem 2. The definition for nonlinear systems is a nonlinear enhancement of this last characterization of moment for linear systems. While in the linear case the three definitions are equivalent, in the nonlinear case the first one

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[4]A nilpotent matrix is a square matrix N such that $N^k = 0$ for some positive integer k. The smallest such a k is called the degree of N.

[5]A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

does not make sense. For this reason while for linear systems we can define k-moments at s_i, for nonlinear systems we do not define higher order moments.

In the linear framework we can achieve more than the mere restriction of the results of Section II-A. In fact, we now show that the matrix Π is partitioned in "slow" and "fast" parts.

Theorem 3: Let $S \in \mathbb{R}^{\nu \times \nu}$ be any non-derogatory matrix with characteristic polynomial (12). Consider system (8) and suppose that $\sigma(S) \cap \sigma(A_1) = \emptyset$. Then the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_\rho), \ldots, \eta_{k_\rho-1}(s_\rho)$ are uniquely determined by the matrix $C\widehat{\Pi}$, where $\widehat{\Pi} = P^{-1}\Pi = \begin{bmatrix} \Pi_1^\top & \Pi_2^\top \end{bmatrix}^\top$ is the unique solution of the equations

$$A_1\Pi_1 - \Pi_1 S = -B_1 L, \quad \Pi_2 = -\sum_{i=0}^{d-1} N^i B_2 L S^i. \quad (14)$$

Remark 6: The contribution of the fast subsystem to $\widehat{\Pi}$ is separated from the contribution of the slow subsystem. Moreover, the condition for uniqueness of $\widehat{\Pi}$ is simplified and it depends only on the eigenvalues of the slow subsystem. This fact is actually expected since in the first equivalent form the k-th moment of the system can be written as $\eta_k(s_i) = C_1(s_i I - A_1)^{-(k+1)}B_1 + C_2 N^k(s_i N - I)^{-(k+1)}B_2$, in which $s_i N - I$ is full-rank for any s_i. This formulation gives high flexibility in maintaining the properties of the fast and slow subsystems. Depending on the particular application the fast subsystem can be preserved, reduced or eliminated independently of the reduction of the slow subsystem. This possibility is analyzed in detail in the next section.

III. FAMILIES OF REDUCED ORDER MODELS

In this section we provide families of reduced order models for linear and nonlinear singular systems.

A. Nonlinear singular reduced order models

We begin giving the definition of reduced order model in the present framework.

Definition 5: Consider system (1) and the signal generator (2). The system described by the equations

$$\Xi\dot{\xi} = \phi(\xi, u), \quad \psi = \kappa(\xi), \quad (15)$$

with $\xi(t) \in \mathbb{R}^\nu$, $u(t) \in \mathbb{R}$, $\psi(t) \in \mathbb{R}$, $\Xi = \mathrm{diag}(I, 0)$, with $\mathrm{rank}(\Xi) = \bar{r} \leq r$, and ϕ and κ smooth mappings, is a *model of system (1) at* (s, l) if system (15) has the same moment at (s, l) as system (1). System (15) is a *reduced order model of system (1) at* (s, l) if $\nu < n$.

From this definition a rather general result can be presented.

Lemma 4: Consider system (1) and the signal generator (2). Suppose Assumptions 1 and 2 hold. Then system (15) is a model of system (1) at (s, l) if the equation

$$\Xi\frac{\partial p}{\partial \omega}s(\omega) = \phi(p(\omega), l(\omega)) \quad (16)$$

has a unique solution p such that

$$h(\pi(\omega)) = \kappa(p(\omega)), \quad (17)$$

where π solves (4).

This claim follows directly from Definition 5 and the definition of moment.

The family of models identified in Lemma 4 has numerous free design "parameters", *i.e.* Ξ, ϕ and κ, that can be exploited to obtain a convenient representation, as shown in the next section.

B. Nonlinear "identity" family of models

In the case of differential-algebraic systems it is of interest to have the differential part and the algebraic part separated. At the same time we would like to have an easily tunable parameter which allows to span the family of models. To streamline the presentation of the results we indicate with ξ_1, $\bar{s}_1(\omega)$ and $(\frac{\partial p}{\partial \omega})_1$ the first $\bar{r}$ rows of ξ, $s(\omega)$ and $\frac{\partial p}{\partial \omega}$, respectively. For a mapping $\delta : \mathbb{R}^\nu \rightarrow \mathbb{R}^\nu$, δ_1 indicates the first $\bar{r}$ rows, whereas δ_2 denotes the last $\nu - \bar{r}$ rows.

Assumption 3: There exist mappings κ and p such that $\kappa(0) = 0$, $p(0) = 0$, p is locally continuously differentiable, equation (17) holds and p has a local inverse p^{-1}.

Lemma 5: Consider system (1) and the signal generator (2). Suppose Assumptions 1, 2 and 3 hold. Then the system

$$\dot{\xi}_1 = \bar{s}_1(\xi) - \delta_1(\xi)l(\xi) + \delta_1(\xi)u, \quad 0 = -\delta_2(\xi)l(\xi) + \delta_2(\xi)u,$$
$$\psi = h(\pi(\xi)), \quad (18)$$

is a model of system (1) at (s, l) if δ is an arbitrary mapping such that the partial differential-algebraic equations

$$(\frac{\partial p}{\partial \omega})_1 s(\omega) = \bar{s}_1(p(\omega)) - \delta_1(p(\omega))l(p(\omega)) + \delta_1(p(\omega))l(\omega),$$
$$0 = -\delta_2(p(\omega))l(p(\omega)) + \delta_2(p(\omega))l(\omega), \quad (19)$$

have the unique solution $p(\omega) = \omega$.

The nonlinear differential-algebraic model (18) is parametrized in the mappings δ_1, δ_2 which can be used to achieve additional properties. Note that the mapping p may not describe the steady-state response of system (18) unless additional assumptions hold. In fact, from Section II, it follows that we need the additional property that the system be strongly stable.

Lemma 6: Consider system (18) and the signal generator (2). Suppose Assumptions 1 and 2 hold. If δ_1 and δ_2 are chosen such that the pair (Ξ, F), with

$$F = \begin{bmatrix} \frac{\partial}{\partial \xi}\left(\bar{s}_1(\xi) - \delta_1(\xi)l(\xi)\right) \\ -\frac{\partial}{\partial \xi}\left(\delta_2(\xi)l(\xi)\right) \end{bmatrix}_{\xi=0}, \quad (20)$$

is strongly stable, then $p(\omega) = \omega$ solves the partial differential-algebraic equations (19). Moreover, for any sufficiently small $\xi(0)$ and $\omega(0)$, the solution $\xi(t)$, $\omega(t)$ of the interconnection of (18) and (2) exists, is bounded for all $t \geq 0$ and satisfies $\lim_{t\to\infty} \xi(t) - p(\omega(t)) = 0$.

This result is a consequence of Lemma 1 applied to the family of reduced order models (18).

Remark 7: The mapping δ can be selected such that (20) holds. Then the partial differential-algebraic equation (19) is solved and the resulting system (18) is a reduced order model of (1). Moreover, note that condition (20) is a "loose" constraint. In fact, δ can be selected to satisfy the condition and, at

the same time, be used to obtain reduced order models with additional properties, see *e.g.* [27].

Remark 8: To obtain the family of reduced order models (18) we only need to solve equation (4). In general the solution of this partial differential-algebraic equation is difficult to determine analytically. However, the solution can be approximated with numerical methods, see *e.g.* [34], [38].

Remark 9: Nothing prevents to define a normal reduced order model. Normal reduced order models are actually included in the family of models given so far by simply setting $\Xi = I$. In this case the family (18) becomes

$$\dot{\xi} = s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u, \qquad \psi = h(\pi(\xi)), \qquad (21)$$

with δ free, which is the family of reduced order models given in [27].

C. Linear singular reduced order models

Since for linear singular systems stronger results can be obtained, we now specialize the results to this class of systems. The possibility of obtaining purely fast or purely slow singular systems is investigated and a "simple" family of linear singular systems is given. In addition, the problem of maintaining controllability and observability properties is discussed.

Definition 6: Consider system (7) and the signal generator (13). The system described by the equations

$$\begin{aligned} \dot{\xi}_1 &= F_1\xi_1 + G_1 u, & M\dot{\xi}_2 &= \xi_2 + G_2 u, \\ \psi_1 &= H_1\xi_1, & \psi_2 &= H_2\xi_2, \\ \psi &= \psi_1 + \psi_2, \end{aligned} \qquad (22)$$

where $\xi_1(t) \in \mathbb{R}^{\nu_1}$, $\xi_2(t) \in \mathbb{R}^{\nu_2}$, with $\nu_1 + \nu_2 = \nu$, $\psi_1(t) \in \mathbb{R}$, $\psi_2(t) \in \mathbb{R}$, $F_1 \in \mathbb{R}^{\nu_1 \times \nu_1}$, $M \in \mathbb{R}^{\nu_2 \times \nu_2}$ nilpotent with degree $\bar{d} \leq d$, $G_1 \in \mathbb{R}^{\nu_1 \times 1}$, $G_2 \in \mathbb{R}^{\nu_2 \times 1}$, $H_1 \in \mathbb{R}^{1 \times \nu_1}$, and $H_2 \in \mathbb{R}^{1 \times \nu_2}$, is a *model of system (7) at S*, if system (22) has the same moments at S as system (7). System (22) is a *reduced order model of system (7) at S* if $\nu < n$.

Lemma 7: Consider system (7) and let $S \in \mathbb{R}^{\nu \times \nu}$ be any non-derogatory matrix with characteristic polynomial (12). Assume $\sigma(S) \cap \sigma(A_1) = \emptyset$ and let L be such that the pair (S, L) is observable. Then the system (22) is a model of system (7) at S if there exists a unique solution $\begin{bmatrix} \Gamma_1^\top & \Gamma_2^\top \end{bmatrix}^\top$ of the equations

$$F_1\Gamma_1 - \Gamma_1 S = -G_1 L, \quad \Gamma_2 = -\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i, \quad (23)$$

such that

$$C_1\Pi_1 = H_1\Gamma_1, \qquad C_2\Pi_2 = H_2\Gamma_2, \qquad (24)$$

where $\begin{bmatrix} \Pi_1^\top & \Pi_2^\top \end{bmatrix}^\top$ is the unique solution of (14).

This claim follows directly from Definition 6 and the definition of moment.

Similarly to the nonlinear case, there are several parameters that can be exploited to achieve a convenient representation.

D. Linear "identity" family of models

In this section we provide a linear equivalent of the family (18). Since for linear singular systems we can use the special structure of Π highlighted in (14), it seems natural to select $\Gamma_1 = I$ keeping M and G_2 free.

Lemma 8: Consider system (7) and let $S \in \mathbb{R}^{\nu \times \nu}$ be any non-derogatory matrix with characteristic polynomial (12). Assume $\sigma(S) \cap \sigma(A_1) = \emptyset$ and let L be such that the pair (S, L) is observable. Then the system

$$\begin{aligned} \dot{\xi}_1 &= (S - G_1 L)\xi_1 + G_1 u, \\ M\dot{\xi}_2 &= \xi_2 + G_2 u, \\ \psi_1 &= C_1\Pi_1\xi_1, \\ \psi_2 &= C_2 \sum_{i=0}^{\bar{d}-1} N^i B_2 L S^i \left(\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i \right)^{-1} \xi_2, \\ \psi &= \psi_1 + \psi_2, \end{aligned} \qquad (25)$$

where $\begin{bmatrix} \Pi_1^\top & \Pi_2^\top \end{bmatrix}^\top$ is the unique solution of (14), is a model of system (7) at S for any G_1 such that $\sigma(S) \cap \sigma(S - G_1 L) = \emptyset$ and any G_2 and M such that $\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i$ is invertible.

Remark 10: As noted in [13], in the reduction of non-proper transfer functions $W(s)$ the algebraic part should be maintained unchanged to guarantee that the error between the frequency response of the system to be reduced and the frequency response of the reduced order model stays bounded. This can be achieved in the presented framework setting $\nu_2 = n - r$, $\bar{d} = d$. However, in some cases we may not be interested in preserving the rate at which the frequency response grows unbounded, but we may be interested only that it grows unbounded. In other cases, we may be interested only in the output response for low frequencies. The proposed method can deal with these situations leaving to the designer the choice of reducing the fast subsystem, the slow subsystem, or a combination thereof, as shown in the following. Note that a similar discussion can be made for nonlinear singular systems.

Remark 11: Similarly to Remark 9, setting $\nu_1 = \nu$, we can approximate a singular system with a normal system. Note that this was the first approach attempted [6], although inaccurate since the impulsive characteristic of the singular system is lost [42]. However, in particular settings and as a first approximation it can be useful to approximate a singular system with a normal one. In this case the family (25) becomes

$$\dot{\xi} = (S - GL)\xi + Gu, \qquad \psi = C\Pi\xi, \qquad (26)$$

with G free, which is the family of reduced order models given in [27].

Remark 12: Conversely, setting $\nu_2 = \nu$, we can obtain a completely fast reduced order model. In this case the family (25) becomes

$$M\dot{\xi} = \xi + Gu, \qquad \psi = -C\Pi \left(\sum_{i=0}^{\bar{d}-1} M^i GLS^i \right)^{-1} \xi, \qquad (27)$$

with M nilpotent with degree $\bar{d}$ and G free.

Recall now that system (7) is assumed to be controllable and observable. We conclude this section investigating if these structural properties can be maintained with a specific choice of the free parameters of the reduced order models.

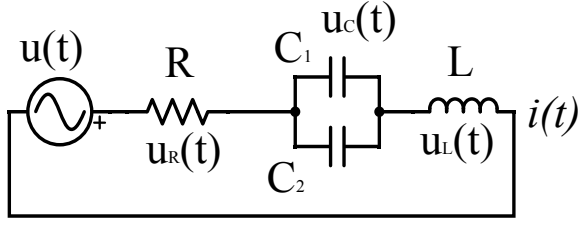*Theorem 4:* System (25) has the following properties.

Fig. 1. The electrical circuit in [4].

(C1) The slow subsystem $\dot{\xi}_1 = (S - G_1 L)\xi_1 + G_1 u$ is controllable. In this case system (25) is said R-controllable (see [1, Definition 2-2.2]).

(C2) The fast subsystem $M\dot{\xi}_2 = \xi_2 + G_2 u$ is controllable.

(C3) System (25) is controllable.

(C4) System (25) is impulse controllable (see [1, Definition 2-2.3]).

Theorem 5: System (25) has the following properties.

(O1) The slow subsystem $\dot{\xi}_1 = (S - G_1 L)\xi_1 + G_1 u$, $\psi_1 = C_1 \Pi_1 \xi_1$ is observable if G_1 is such that this subsystem has relative degree ν_1. In this case system (25) is said R-observable (see [1, Definition 2-3.2]).

(O2) The fast subsystem $M\dot{\xi}_2 = \xi_2 + G_2 u$, $\psi_2 = -C_2 \Pi_2 \left(\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i \right)^{-1} \xi_2$ is observable if M and G_2 are such that

$$\mathrm{rank}\left(\begin{bmatrix} M \\ -C_2 \Pi_2 \left(\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i \right)^{-1} \end{bmatrix} \right) = \nu_2.$$

(O3) System (25) is observable if the conditions in (O1) and (O2) hold simultaneously.

(O4) System (25) is impulse observable (see [1, Definition 2-3.3]) if M and G_2 are such that

$$\mathrm{rank}\left(\begin{bmatrix} M & I \\ 0 & M \\ 0 & -C_2 \Pi_2 \left(\sum_{i=0}^{\bar{d}-1} M^i G_2 L S^i \right)^{-1} \end{bmatrix} \right) = \nu_2 + \mathrm{rank}(M).$$

Example 2: We illustrate the results with a simple example. We consider the model studied in [4], which represents the electrical circuit illustrated in Fig. 1. The model is described by the equations

$$
\begin{aligned}
\dot{u}_{C_1} &= \tfrac{1}{2}\tfrac{i_1}{1+10^{-1}u_{C_1}} + \tfrac{1}{2}\tfrac{i_2}{1+10^{-2}u_{C_1}}, & \dot{\Phi} &= u_L, \\
0 &= u_R - 5i_R - 10i_R^3, & 0 &= \Phi - \arctan(i_R), \\
0 &= \tfrac{i_1}{1+10^{-1}u_{C_1}} - \tfrac{i_2}{1+10^{-2}u_{C_1}}, & 0 &= i - i_1 - i_2, \\
0 &= u - u_R - u_{C_1} - u_L, & y &= u_{C_1},
\end{aligned}
$$
(28)

where u is an ideal voltage source, u_R and i_R are, respectively, the voltage and the current of the nonlinear resistor, i_1, i_2 and u_{C_1} represent, respectively, the currents and the voltage of the capacitors, and Φ and u_L represent, respectively, the saturated flux and the voltage of the inductor. Consider the signal generator (13) with matrices $S = [0, -1; 1, 0]$ and $L = [0.001, -0.0098]$ which generates an input similar to the one used in [4]. We compute an approximation of the moment $h \circ \pi$ (see [38], [43]), namely $h(\pi(\omega)) \approx 10^{-4}(0.0451 + 9.591\omega_1 +$
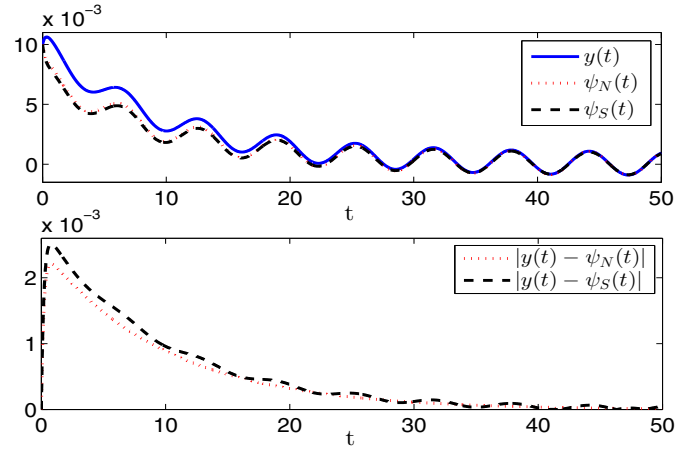

Fig. 2. The top graph shows the time history of the output of system (28) in blue/solid line, of the output of model (29) in red/dotted line and of the output of the model (29) in black/dashed line. The bottom graph shows the absolute errors.

$1.955\omega_2 + 0.0042\omega_1^2 - 0.0005\omega_1\omega_2)$. We determine two reduced order models, a normal model described by the equations

$$
\begin{aligned}
\dot{\xi}_1 &= -0.1021\,\xi_1 + 0.0001\,\xi_2 + 102.1076\,u, \\
\dot{\xi}_2 &= 1.5001\,\xi_1 - 4.8979\,\xi_2 - 500.0748\,u, \\
\psi_N &= h(\pi(\xi)),
\end{aligned}
$$
(29)

and a singular model described by the equations

$$
\begin{aligned}
\dot{\xi}_1 &= -0.1028\,\xi_1 + 0.0073\,\xi_2 + 102.8404\,u, \\
0 &= \xi_1 - 9.7944\,\xi_2 - 1000\,u, \\
\psi_S &= h(\pi(\xi)).
\end{aligned}
$$
(30)

Note that both models are linear systems with nonlinear output map (this is expected since the signal generator is linear). The eigenvalues of the dynamic matrix of the model (29) have been set, using the free parameter G, as $-0.1021, -4.8979$. These eigenvalues are the eigenvalues of the linearization around $u_{C_1} = 0$, $\Phi = 0$ of system (28). Similarly, the only finite eigenvalue of model (30) is set as -0.1021. The resulting two reduced order models are, respectively, asymptotically stable and strongly stable. Fig. 2 (top) shows the time history of the output of system (28) in blue/solid line, of the output of model (29) in red/dotted line and of the output of the model (30) in black/dashed line. The initial conditions of the simulation have been selected as $u_{C_1}(0) = 0.01$, $\Phi(0) = 0.01$, $i_R = 0.01$, $i_1 = 0.005$, $i_2 = 0.005$, $u_L = -0.06$, $u_R = 0.05$, $\omega(0) = [1\ 0]^\top$ and $\xi(0) = [8.6\ 8.8]^\top$. The bottom graph shows the absolute errors $|y - \psi_S|$ (red/dotted line) and $|y - \psi_N|$ (black/dotted line). We see that both the singular model and the normal model are reduced order models of system (28) achieving moment matching in the sense of Definition 5. Hence, from this simulation we are not able to establish which of the two models is preferable.

IV. CONCLUSION

In this technical note we have extended the notion of moment based on the steady-state response to nonlinear singular systems. We have proposed families of reduced order model for nonlinear and linear singular systems. Connections with the moment

matching approach based on the interpolation of the transfer function have been drawn. The results have been illustrated by the reduction of a simple electrical circuit.

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APPENDIX

Proof of Lemma 1: Note that this lemma, which characterizes the steady-state response of system (3) in terms of the mapping π, is a standard result in the literature of output regulation of singular systems. A detailed proof can be found in [34, Lemma 8.9]. As a compromise between completeness and space, we provide a sketch of the proof. Under the stated assumptions the singular system can be rewritten as a normal system. This is proved showing that there exists a change of coordinates such that the algebraic subsystem written in the new coordinates has an obvious solution. The obtained normal system satisfies the assumptions of [31, Proposition 8.1.1]. The proposition establishes that there exists a center manifold which is locally exponentially attractive, is the graph of a mapping solving an equation similar to (4) and can be used to define the steady-state response. $\qquad\square$

Proof of Lemma 3: Let $\widetilde{\Pi}_i = [\widetilde{\Pi}_i^0 \ \widetilde{\Pi}_i^1 \ \ldots \ \widetilde{\Pi}_i^k]$ and note that (10) can be rewritten as

$$
\begin{aligned}
E\widetilde{\Pi}_i^0 s_i &= A\widetilde{\Pi}_i^0 + B, \\
E\widetilde{\Pi}_i^1 s_i + E\widetilde{\Pi}_i^0 &= A\widetilde{\Pi}_i^1, \\
&\vdots \\
E\widetilde{\Pi}_i^k s_i + E\widetilde{\Pi}_i^{k-1} &= A\widetilde{\Pi}_i^k.
\end{aligned}
$$

As a result

$$\widetilde{\Pi}_i^0 = (s_i E - A)^{-1} B$$
$$\widetilde{\Pi}_i^1 = -(s_i E - A)^{-1} E (s_i E - A)^{-1} B = \left[\frac{d}{dq}(qE-A)^{-1}B \right]_{q=s_i}$$
$$\vdots$$
$$\widetilde{\Pi}_i^k = \frac{1}{k!}\left[\frac{d^k}{dq^k}(qE-A)^{-1}B \right]_{q=s_i}$$

which proves the claim. $\qquad\square$

Proof of Theorem 1: Note that (11) has a unique solution if and only if $\prod_{i=1}^{\rho} \det(A - s_i E) \neq 0$. Since the matrix S is non-derogatory, for each distinct eigenvalue s_i there exists only one Jordan block Σ_i of dimension k_i, *i.e.* $\bar{\Sigma} = \bar{T}S\bar{T}^{-1} = \mathrm{diag}(\Sigma_1, \ldots, \Sigma_\rho)$ is in Jordan form and L becomes $\bar{L} = L\bar{T}^{-1} = [\bar{L}_1, \ldots, \bar{L}_\rho]$ where each $\bar{L}_i$ block corresponds to a Σ_i block. We can then consider a pair $(\Sigma_i, \bar{L}_i)$ and note that this is observable. It is easy to see that we can find an invertible matrix $\hat{T}$ such that $\Sigma_i = \hat{T}\Sigma_i\hat{T}^{-1}$ and $L_i = \bar{L}_i\hat{T}^{-1}$, *i.e.* it leaves Σ_i unchanged and transform $\bar{L}_i$ in the canonical form in Lemma 3. Let $T = \hat{T}\bar{T}$. Then equation (11) becomes $A\Pi - E\Pi T^{-1}\,\mathrm{diag}(\Sigma_1, \ldots, \Sigma_\rho)T = -B[L_1, \ldots, L_\rho]T$. The claim follows multiplying on the right by T^{-1}, defining $\Pi = [\widetilde{\Pi}_1, \ldots, \widetilde{\Pi}_\rho]T$ and applying Lemma 3 to each of the resulting ρ blocks. $\qquad\square$

Proof of Theorem 2: Consider the interconnection of system (7) with system (13). By the assumptions on the pencil (E, A) and $\sigma(S)$, the interconnected system has a well-defined invariant manifold given by $\mathcal{M} = \{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi\omega\}$, with Π the unique solution of the generalized Sylvester equation (11). We prove now that $\mathcal{M}$ is attractive. Let $z = x - \Pi\omega$. Consider the equation $E\dot{z} = Ax + BL\omega - E\Pi S\omega$, which substituting (11), yields $E\dot{z} = A(x - \Pi\omega) = Az$. The solution of this last equation is $z(t) = P[I\ 0]^\top e^{A_1 t} z(0)$. Thus, the output response of the interconnected system is $y(t) = C\Pi\omega(t) + CP[I\ 0]^\top e^{A_1 t}(x_1(0) - \Pi\omega(0))$. Observing that $CP[I\ 0]^\top e^{A_1 t}(x_1(0) - \Pi\omega(0))$ describes the transient response (which vanishes exponentially), yields that the steady-state output response of (7) driven by (13) is $C\Pi\omega(t)$. By minimality of $(S, \omega(0), L)$, $C\Pi$ can be uniquely determined by $C\Pi\omega(t)$ computed at ν sample times t_i, see [33], [38]. $\square$

Proof of Theorem 3: Let $\bar{x} = \Pi\omega$ and define $[\bar{x}_1^\top \ \bar{x}_2^\top]^\top = P^{-1}\bar{x} = P^{-1}\Pi\omega = [\Pi_1^\top \ \Pi_2^\top]^\top\omega$. Substituting these relations in (8) yields

$$\Pi_1 S\omega = A_1\Pi_1\omega + B_1 L\omega, \qquad N\Pi_2 S\omega = \Pi_2\omega + B_2 L\omega,$$

from which ω can be eliminated because the two equations hold for any ω. From the first of these equations we readily obtain the first of equations (14). From the second equation we note that Π_2 is unique if and only if $\det(I - s_i N) \neq 0$, for any $s_i \in \sigma(S)$. This holds by definition for any $s_i \in \mathbb{C}$ since N is a nilpotent matrix, *i.e.* $s_i N$ has all zero eigenvalues for any $s_i \in \mathbb{C}$. Then Π is unique if and only if $\sigma(S) \cap \sigma(A_1) = \emptyset$. To obtain the explicit expression of Π_2, we substitute $\bar{x}_2 = \Pi_2\omega$ in the last n_2 equations in (9) obtaining $\Pi_2\omega = -\sum_{i=0}^{d-1} N^i B_2 L \frac{d^i}{dt^i}\omega$. Observing that $\frac{d^i}{dt^i}\omega = S^i\omega$ yields $\Pi_2 = -\sum_{i=0}^{d-1} N^i B_2 L S^i$, since the equation holds for any ω. $\square$

Proof of Lemma 5: Consistently with Lemma 4 and Assumption 3 a family of models that achieves moment matching at (s, l) is described by $\Xi\dot{\xi} = \hat{\phi}(\xi) + \delta(\xi)u$ and $\psi = \kappa(\xi)$, with $\hat{\phi}(\xi) = [\Xi\frac{\partial p(\omega)}{\partial\omega}(s(\omega)) - \delta(p(\omega))l(\omega)]_{\omega=p^{-1}(\xi)}$, where p is the unique solution of (16) and δ is a free mapping. Mappings κ and p that have the required regularity properties are given by the selection $p(\omega) = \omega$ and $\kappa(\omega) = h(\pi(\omega))$. This yields a family of models described by the equations $\Xi\dot{\xi} = \Xi s(\xi) - \delta(\xi)l(\xi) + \delta(\xi)u$ and $\psi = h(\pi(\xi))$, where δ is an arbitrary mapping such that equation (16), namely $\Xi\frac{\partial p}{\partial\omega}s(\omega) = \Xi s(p(\omega)) - \delta(p(\omega))l(p(\omega)) + \delta(p(\omega))l(\omega)$, has the unique solution $p(\omega) = \omega$. The claim is proved rewriting these last equations using the definitions given for Ξ, ξ_1, $\bar{s}_1$, δ_1, δ_2 and $(\frac{\partial p}{\partial\omega})_1$. $\square$

Proof of Lemma 8: Consider model (22). F_1 and H_1 are selected substituting $\Gamma_1 = I$ in (23) and (24), respectively. H_2 is obtained substituting in equation (24) the expressions for Γ_2 and Π_2 given in (23) and (14), respectively. $\square$

Proof of Theorem 4: Claim (C1): Since G_1 is such that $\sigma(S) \cap \sigma(S - G_1 L) = \emptyset$, $\dot{\xi}_1 = (S - G_1 L)\xi_1 + G_1 u$ is controllable, see [44]. Claim (C2): Recall that $\sum_{i=0}^{d-1} M^i G_2 L S^i$ is invertible, *i.e.* $\mathrm{rank}([G_2|MG_2|\ldots|M^{d-1}G_2][L^\top|S^\top L^\top|\ldots|(S^{d-1})^\top|L^\top]^\top) = \nu_2$. Since (S, L) is observable then $\mathrm{rank}([G_2|MG_2|\ldots|M^{d-1}G_2]) = \nu_2$ which is equivalent to the controllability of the fast subsystem $M\dot{\xi}_2 = \xi_2 + G_2 u$ [1, Theorem 2-2.1 (2)(b)]. Claim (C3): (C1) + (C2) imply (C3) [1, Theorem 2-2.1 (3)]. Claim (C4): (C2) implies (C4) [1, p. 37]. $\square$

Proof of Theorem 5: Claim (O1): This is a direct consequence of [27, Theorem 2]. Claim (O2): This is a direct consequence of [1, Theorem 2-3.1 (3)(d)]. Claim (O3): (O1) + (O2) imply (O3) [1, Theorem 2-3.1 (4)]. Claim (O4): The condition is obtained after eliminating full rank blocks from the condition given in [1, Theorem 2-3.4 (vii)]. Note that (O2) implies (O4) but (O4) does not imply (O2). $\square$