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Stability analysis of impulsive fractional differential systems with delay

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ABSTRACT

In this paper, a class of impulsive fractional differential systems with finite delay is considered. Some sufficient conditions for the finite-time stability of above systems are obtained by using generalized Bellman–Gronwall's inequality, which extend some known results.

1. Introduction

Recently, finite-time stability analysis of fractional differential systems with delay is presented and reported [1,2]. The main approach is generalized Bellman–Gronwall's inequality [3]. For some advances in the control theory of fractional dynamical systems for stability, we refer the reader to [4–12]. For more details of impulsive fractional differential equations with its applications, we refer the reader to [13–23].

Motivated by [1,2], we will investigate the finite-time stability of the following nonautonomous impulsive fractional differential systems

$$\begin{cases} D^{\alpha}x(t) = A_0x(t) + A_1x(t-\tau) + B_0u(t), & t \in J' := J \setminus \{t_1, t_2, \dots, t_m\},\ J = [0, T], \\ \Delta x(t_k) = C_k x(t_k^-), & k = 1, 2, \dots, m, \end{cases} \tag{1.1}$$

with it's autonomous type

$$\begin{cases} D^{\alpha}x(t) = A_0x(t) + A_1x(t-\tau), & t \in J' := J \setminus \{t_1, t_2, \dots, t_m\},\ J = [0, T], \\ \Delta x(t_k) = C_k x(t_k^-), & k = 1, 2, \dots, m, \end{cases} \tag{1.2}$$

with initial condition $x(t) = \psi_x(t), t \in [-\tau, 0]$, where D^{α} denotes the Caputo fractional derivative of order $\alpha \in (0, 1)$, $x(t) \in R^n$ is a state vector, $u(t) \in R^m$ is a control vector, A_0, A_1, B_0, C_k are constant matrices of appropriate dimensions, and $\tau > 0$ is a constant time delay. t_k satisfy $t_0 < t_1 < \cdots t_m = T$, $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$ and $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_k$. Dynamical behavior of system (1.1), (1.2), with a given initial function is defined over time interval $J = [t_0, t_0 + T]$, $J \subset R$, where quantity T may be either a positive real number or a symbol $+\infty$, so finite-time stability and practical stability can be treated simultaneously. Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded. Let index "ε" stands for the set of all allowable states of the system and index "δ" for the set of all initial states of the system, such that the set $S_\delta \subseteq S_\varepsilon$. Let $S_\rho = \{x : \|x\|_Q^2 < \rho\}$, $\rho \in [\delta, \varepsilon]$,

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where Q will be assumed to be a symmetric, positive definite, real matrix. S_{α_u} denotes the set of all allowable control actions. Let $|x|_{(\cdot)}$ be any vector norm (e.g., $\cdot = 1, 2, \infty$) and $\|x\|_{(\cdot)}$ the matrix norm induced by this vector. The initial function can be written as: $x(t_0+\theta) = \psi_x(\theta), \theta \in [-\tau, 0], \psi_x(\theta) \in C([-\tau, 0], R)$, where t_0 is the initial time of observation of the system (1.1), (1.2) and $C([-\tau, 0], R)$ is a Banach space of continuous functions over a time interval of length τ, mapping the interval $[t-\tau, t]$ into R^n with the norm defined in the following manner: $\|\psi\|_C = \max_{\theta\in[-\tau,0]}\{\|\psi(\theta)\|\}$. It is assumed that the usual smoothness condition is present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to initial data.

Definition 1.1. System given by (1.2) satisfying initial condition $x(t) = \psi_x(t), t \in [-\tau, 0]$ is finite stable w.r.t. $\{t_0, J, \delta, \epsilon, \tau\}, \delta < \epsilon$ if and only if $\|\psi\|_C < \delta$ implies $|x| < \epsilon, \forall t \in J$, where t_0 denotes the initial time of observation of the system and J denotes time interval $J = [t_0, t_0 + T], J \subset R$.

Definition 1.2. The system given by (1.2) satisfying initial condition $x(t) = \psi_x(t), t \in [-\tau, 0]$ is finite stable w.r.t. $\{t_0, J, \delta, \epsilon, \tau\}, \delta < \epsilon$ if and only if $\|\psi\|_C < \delta$ and $\|u(t)\| < \alpha_u$ imply $|x| < \epsilon, \forall t \in J$, where t_0 denotes the initial time of observation of the system and J denotes time interval $J = [t_0, t_0 + T], J \subset R$.

The organization of this paper is as follows. In Section 2, we introduce some preliminaries. In Section 3, by using the generalized Gronwall inequality, we obtain some results of stability for system (1.1) and (1.2).

2. Preliminaries

Let us recall some definitions of fractional calculus. For more details see [24].

Definition 2.1. The fractional integral of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$$I_t^\gamma f(t) = \frac{1}{\Gamma(\gamma)}\int_0^t \frac{f(s)}{(t-s)^{1-\gamma}}ds, \quad t > 0,\ n-1 < \gamma < n,$$

provided the right side is point-wise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2. The Riemann–Liouville derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$${}^L D_t^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dt^n}\int_0^t \frac{f(s)}{(t-s)^{\gamma+1-n}}ds, \quad t > 0,\ n-1 < \gamma < n.$$

Definition 2.3. The Caputo derivative of order γ with the lower limit zero for a function $f : [0, \infty) \to R$ is defined as

$${}^c D_t^\gamma f(t) = {}^L D_t^\gamma\Big[f(t) - \sum_{k=0}^{n-1}\frac{t^k}{k!}f^{(k)}(0)\Big], \quad t > 0,\ n-1 < \gamma < n.$$

Lemma 2.1 (*[3] Generalized Gronwall Inequality*)**.** *Suppose $x(t)$, $a(t)$ are nonnegative and local integrable on $0 \le t < T$ some $T \le +\infty$, and $g(t)$ is a nonnegative, nondecreasing continuous function defined on $0 \le t < T$, $g(t) \le M =$ constance , $\alpha > 0$ with*

$$x(t) \le a(t) + g(t)\int_0^t (t-s)^{\alpha-1}x(s)ds$$

on this interval. Then

$$x(t) \le a(t) + g(t)\int_0^t \Big[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\Big]ds, \quad 0 \le t < T.$$

Corollary 2.1 (*[3] Theorem 3.2.*)**.** *Under the hypothesis of Theorem 3.2, let $a(t)$ be a nondecreasing function on $[0, T)$. Then we have*

$$x(t) \le a(t)E_\alpha(g(t)\Gamma(\alpha)t^\alpha),$$

where $E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+1)}$, $z \in \mathbf{C}$, $Re(\beta) > 0$.

Similar to the proof of Lemma 2.8 of [25], we have

Lemma 2.2. *Let $u \in PC(J, R)$ satisfy the following inequality*

$$\|u(t)\| \le c_1(t) + c_2\int_0^t (t-s)^{q-1}\|u(s)\|ds + \sum_{0<t_k<t}\theta_k\|u(t_k)\|,$$

where $c_1(t)$ is nonnegative continuous and nondecreasing on J, and c_2, $\theta_k \geq 0$ are constants. Then

$$\|u(t)\| \leq c_1(t)(1+\theta E_\beta(c_2\Gamma(\beta)t^\beta))^k E_\beta(c_2\Gamma(\beta)t^\beta), \quad \text{for } t \in (t_k, t_{k+1}],$$

where $\theta = \max\{\theta_k : k = 1, 2, \ldots, m\}$.

3. Main results

Theorem 3.1. *Assume that (H_1) holds. If the following equality is satisfied*

$$\delta\Big(1+\frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\Big)E_\alpha(\sigma_{\max 01}t^\alpha) + \sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\| + \frac{\alpha_u\sigma_{\max}(B_0)t^\alpha}{\delta\Gamma(\alpha+1)} \leq \epsilon, \quad \forall\, t \in [0, T), \tag{$*$}$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1)$. Then system (1.1) satisfying initial condition $x(t) = \psi_x(t)$, $t \in [-\tau, 0]$ is finite-time stable.

Proof. In accordance with the property of the fractional order $0 < \alpha < 1$, one can obtain a solution in the form of the equivalent Volterra integral equation:

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(A_0x(s) + A_1x(s-\tau) + B_0u(s))ds + \sum_{0<t_k<t} C_kx(t_k). \tag{3.1}$$

Applying the norm $\|(\cdot)\|$ on Eq. (3.1) and using appropriate property of the norm, it follows that:

$$\|x(t)\| \leq \|x(0)\| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|A_0x(s) + A_1x(s-\tau) + B_0u(s)\|ds + \sum_{0<t_k<t}\|C_k\|\|x(t_k)\|. \tag{3.2}$$

Also, applying the norm $\|(\cdot)\|$ on Eq. (1.1), one can obtain:

$$\begin{aligned}|D^\alpha x(t)| &\leq \|A_0\|\|x(t)\| + \|A_1\|\|x(t-\tau)\| + \|B_0\|\|u(t)\| \\ &\leq \sigma_{\max}(A_0)\|x(t)\| + \sigma_{\max}(A_1)\|x(t-\tau)\| + \sigma_{\max(B_0)}\|u(t)\|,\end{aligned} \tag{3.3}$$

where $\|A\|$ denotes the induced norm of a matrix A, as well as,

$$\|x(t-\tau)\|_C \leq \sup_{\bar{t}\in[t-\tau,t]}\|x(\bar{t})\|. \tag{3.4}$$

Applying this inequality, from Eq. (3.3), we have

$$\begin{aligned}|D^\alpha x(t)| &\leq \sigma_{\max}(A_0)\|x(t)\| + \sigma_{\max}(A_1)\sup_{\bar{t}\in[t-\tau,t]}\|x(\bar{t})\| + \sigma_{\max}(B_0)\|u(t)\| \\ &\leq \sigma_{\max 01}\sup_{\bar{t}\in[t-\tau,t]}\|x(\bar{t})\| + \sigma_{\max}(B_0)\|u(t)\|, \quad t > t_0+\tau,\end{aligned} \tag{3.5}$$

or

$$\|A_0x(t) + A_1x(t-\tau) + B_0u(t)\| \leq \sigma_{\max 01}\Big(\sup_{\bar{t}\in[t-\tau,t]}\|x(\bar{t})\| + \|\psi_x\|_C\Big) + \sigma_{\max}(B_0)\|u\|, \quad t > t_0+\tau. \tag{3.6}$$

Taking into account (3.6) and (3.2), we have

$$\begin{aligned}\|x(t)\| &\leq \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\{\sup_{\bar{t}\in[s-\tau,s]}\|x(\bar{t})\| + \sigma_{\max}(B_0)|u(s)|\Big\}ds \\ &\quad + \sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\| \\ &= \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\Big) + \frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{\bar{t}\in[s-\tau,s]}\|x(\bar{t})\|ds \\ &\quad + \frac{\alpha_u\sigma_{\max}(B_0)t^\alpha}{\Gamma(\alpha+1)} + \sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\|.\end{aligned} \tag{3.7}$$

Let $a(t) = \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^\alpha}{\Gamma(\alpha+1)}\Big) + \sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\| + \frac{\alpha_u\sigma_{\max}(B_0)t^\alpha}{\Gamma(\alpha+1)}$, then $a(t)$ is nondecreasing function. By Corollary 2.1, we have

$$\|x(t)\| \leq \sup_{\bar{t}\in[t-\tau,t]}\|x(\bar{t})\| \leq a(t)E_\alpha\Big(\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\Gamma(\alpha)t^\alpha\Big) = a(t)E_\alpha\Big(\sigma_{\max 01}t^\alpha\Big), \tag{3.8}$$

then

$$\|x(t)\| \le \delta\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)E_{\alpha}\Big(\sigma_{\max 01}t^{\alpha}\Big)+\sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\|+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.9}$$

Hence by the condition $(*)$, we have

$$|x(t)| \le \epsilon, \quad \forall\, t \in J_0=[0,T).$$

Similar to the proof of Theorem 3.1, we have the following Theorems.

Theorem 3.2. *Assume that $\sum_{0<t_k<t}\sigma_{\max}(C_k) < 1$ holds. If the following equality is satisfied*

$$\frac{\delta\big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}E_{\alpha}\Big(\frac{\sigma_{\max 01}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}t^{\alpha}\Big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)} < \epsilon, \quad \forall t\in[0,T), \tag{$**$}$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$. Then system (1.1) satisfying initial condition $x(t)=\psi_x(t)$, $t\in[-\tau,0]$ is finite-time stable.

Proof. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\|x(t)\| \le\; & \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\{\sup_{\bar t\in[s-\tau,s]}\|x(\bar t)\|+\sigma_{\max}(B_0)\|u(s)\|\Big\}ds\\ & +\sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\|.\end{aligned}$$

By the condition that $\sum_{0<t_k<t}\sigma_{\max}(C_k) < 1$, we obtain

$$\begin{aligned}&\Bigg(1-\sum_{0<t_k<t}\sigma_{\max}(C_k)\Bigg)\|x(t)\|\\ &\quad\le \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big\{\sup_{\bar t\in[s-\tau,s]}\|x(\bar t)\|+\sigma_{\max}(B_0)\|u(s)\|\Big\}ds\\ &\quad= \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sup_{\bar t\in[s-\tau,s]}\|x(\bar t)\|ds+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned}$$

Then

$$\|x(t)\| \le \frac{\|\psi_x\|_C\big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}+\frac{\sigma_{\max 01}}{\Gamma(\alpha)\Bigg(1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)\Bigg)}\int_0^t (t-s)^{\alpha-1}\sup_{\bar t\in[s-\tau,s]}\|x(\bar t)\|ds.$$

Let $a(t)=\frac{\|\psi_x\|_C\left(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\right)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}}{1-\sum_{0<t_k<t}\sigma_{\max}(C_k)}$, then $a(t)$ is nondecreasing function. By Corollary 2.1, we have

$$\begin{aligned}\|x(t)\| &\le \sup_{\bar t\in[t-\tau,t]}\|x(\bar t)\| \le a(t)E_{\alpha}\Bigg(\frac{\sigma_{\max 01}}{\Gamma(\alpha)\Bigg(1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)\Bigg)}\Gamma(\alpha)t^{\alpha}\Bigg)\\ &= a(t)E_{\alpha}\Bigg(\frac{\sigma_{\max 01}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}t^{\alpha}\Bigg),\end{aligned} \tag{3.10}$$

then

$$\|x(t)\| \le \frac{\delta\big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}E_{\alpha}\Big(\frac{\sigma_{\max 01}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}t^{\alpha}\Big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}. \tag{3.11}$$

Hence by the condition $(**)$, we have

$$\|x(t)\| \le \epsilon, \quad \forall\, t\in J_0=[0,T).$$

Theorem 3.3. *Assume that (H_1) holds. If the following equality is satisfied*

$$\left\{\delta\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}\right\}(1+CE_\beta(c_2\Gamma(\beta)t^\beta))^kE_\beta(c_2\Gamma(\beta)t^\beta)<\epsilon, \tag{***}$$

for $\forall\, t\in(t_k,t_{k+1}]\subset J_0=[0,T)$, $k=1,2,\dots,m$, where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$. Then system (1.1) satisfying initial condition $x(t)=\psi_x(t)$, $t\in[-\tau,0]$ is finite-time stable.

Proof. Similar to the proof of Theorem 3.1, we have

$$\begin{aligned}\|x(t)\| \le \|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\sigma_{\max 01}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sup_{\bar t\in[s-\tau,s]}\|x(\bar t)\|ds\\ +\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}+\sum_{0<t_k<t}\sigma_{\max}(C_k)\|x(t_k)\|.\end{aligned} \tag{3.12}$$

Let $c_1(t)=\|\psi_x\|_C\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}$, $c_2=\frac{\sigma_{\max 01}}{\Gamma(\alpha)}$, $C=\max\{\sigma_{\max}(C_k),k=1,2,\dots,m\}$, then $c_1(t)$ is nondecreasing function and $c_2, C\ge 0$. By Lemma 2.1, we have

$$\|x(t)\|\le\sup_{\bar t\in[t-\tau,t]}\|x(\bar t)\|\le c_1(t)(1+CE_\beta(c_2\Gamma(\beta)t^\beta))^kE_\beta(c_2\Gamma(\beta)t^\beta),\quad \text{for } t\in(t_k,t_{k+1}], \tag{3.13}$$

then

$$\|x(t)\|\le\left\{\delta\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)+\frac{\alpha_u\sigma_{\max}(B_0)t^{\alpha}}{\Gamma(\alpha+1)}\right\}(1+CE_\beta(c_2\Gamma(\beta)t^\beta))^kE_\beta(c_2\Gamma(\beta)t^\beta),\quad \text{for } t\in(t_k,t_{k+1}]. \tag{3.14}$$

Hence by the condition $(*)$, we have

$$|x(t)|\le\epsilon,\quad \forall\, t\in(t_k,t_{k+1}]\subset J_0=[0,T),\ k=1,2,\dots,m.$$

Similar to the proof of Theorems 3.1–3.3, respectively, we have the following Theorems.

Theorem 3.4. *Assume that (H_1) holds. If the following equality is satisfied*

$$\delta\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)E_\alpha(\sigma_{\max 01}t^\alpha)+\sum_{0<t_k<t}\sigma_{\max}(C_k)\|xt_k\|<\epsilon,\quad \forall t\in[0,T), \tag{*'}$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$. Then system (1.2) satisfying initial condition $x(t)=\psi_x(t)$, $t\in[-\tau,0]$ is finite-time stable.

Theorem 3.5. *Assume that (H_1) and $\sum_{0<t_k<t}\sigma_{\max}(C_k)<1$ hold. If the following equality is satisfied*

$$\frac{\delta\big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\big)}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}E_\alpha\Big(\frac{\sigma_{\max 01}}{1-\sum\limits_{0<t_k<t}\sigma_{\max}(C_k)}t^\alpha\Big)<\epsilon,\quad \forall t\in[0,T), \tag{**'}$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$. Then system (1.2) satisfying initial condition $x(t)=\psi_x(t)$, $t\in[-\tau,0]$ is finite-time stable w.r.t.

Theorem 3.6. *Assume that (H_1) and $\sum_{0<t_k<t}\sigma_{\max}(C_k)<1$ hold. If the following equality is satisfied*

$$\delta\Big(1+\frac{\sigma_{\max 01}t^{\alpha}}{\Gamma(\alpha+1)}\Big)(1+CE_\beta(c_2\Gamma(\beta)t^\beta))^kE_\beta(c_2\Gamma(\beta)t^\beta)<\epsilon, \tag{***'}$$

for $\forall\, t\in(t_k,t_{k+1}]\subset J_0=[0,T)$, $k=1,2,\dots,m$, where $\sigma_{\max}(\cdot)$ being the largest singular value of matrix $(\cdot)$, with $\sigma_{\max 01}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)$. Then system (1.2) satisfying initial condition $x(t)=\psi_x(t)$, $t\in[-\tau,0]$ is finite-time stable.

Remark 3.1. By comparing with [2], we extend the main results of [2].

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