Geometric invariants for singular delay systems by systems over a ring

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Disturbance Decoupling Problem for Singular Time-Delay Systems

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Abstract

The Disturbance Decoupling Problem is discussed for linear singular systems with a finite number of commensurable point delays using as models systems with coefficients in a ring. A geometric notion of controlled invariant submodule is introduced for this class of systems over a ring and a design procedure is presented for constructing a dynamic feedback law that achieves the decoupling.

1. Introduction

Singular systems are dynamical systems whose behaviors are governed by both differential equations and algebraic equations. Such systems arise in electrical networks, power systems, large-scale systems, etc. Over the years, the problem of time-delay systems has been explored because delay is commonly encountered various engineering systems, such as chemical processes or in long transmission lines or electric networks. The existence of time delays may cause undesirable system transient response, or even instability. In general, the introduction of time delay factors makes the analysis much more complicated.

In the past two decades, a considerable amount of research concerning singular systems (also referred to as descriptor, generalized, degenerate, differential-algebraic or semi-state systems) has been reported because of their extensive applications. and many efforts have been devoted to investigating structural properties of singular systems.
Up to now not many of these results have been extended to singular systems with delays, In literature we have several results concerning stability and state observation (see for instance, [1], [2], [3]).

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In this paper, by means of mathematical models with coefficients in a suitable ring, or systems over a ring, we will extend to quite general class of singular systems with delays, that contains also neutral delay systems, several fundamental notions of the geometric approach and we will discuss their relation.

2. Singular time delay systems

Assume that Σ_d is the linear, time invariant system with a finite number of commensurable point delays, described by the equations

$$
\begin{array}{rcl}
E\dot{x}(t) & = & \sum_{i=0}^{a} A_i x(t-ih) + \sum_{i=0}^{b} B_i u(t-ih) \\
& & \qquad \sum_{i=0}^{h} D_i q(t-ih) \\
y(t) & = & \sum_{i=0}^{c} C_i x(t-ih) \\
& & \qquad x(t) = \varphi(t),\ t \in [-\alpha\, h, 0] \quad \alpha > 0
\end{array}
\tag{1}
$$

where, denoting by $\mathbb{R}$ the field of real numbers, the state $x(t) \in \mathbb{R}^n$, the control input $u(t) \in \mathbb{R}^m$, the disturbance $q(t) \in \mathbb{R}^h$, the output $y(t) \in \mathbb{R}^p$, $h \in \mathbb{R}^+$ is the delay, $\alpha = max(a,\ b,\ c)$, $\varphi(t)$ is a consistent initial condition, E, A_i, B_i, C_i and D_i are matrices of suitable dimensions with entries in $\mathbb{R}$.

Remark that the class of singular systems we consider is more general than that usually considered in the literature (see, for instance [4]) and also contains delay systems of neutral type.

Definition 2.1 *(see [4]) A singular system Σ (1) is regular if the matrix pencil $\lambda E - A_0$ is regular, namely* $\det(\lambda E - A_0)$ *is not identically equal to zero.*
The degree of the regular system Σ or degree of the matrix pencil $\lambda E - A_0$ is the degree of the polynomial $\det(\lambda E - A_0)$.

Regularity of a singular system, ensures that, given any fixed initial condition $x(0^-)$, the solution $x(t)$ of $E\dot{x}(t) = Ax(t)$ is unique, and given any fixed consistent initial condition $x(0^-)$ and sufficiently smooth input function $u(t)$, the solution $x(t)$ of $E\dot{x}(t) = Ax(t) + Bu(t)$ exists and it is unique.

Definition 2.2 *(see [4]) A singular system Σ (1) is said impulse free if* $\deg(\det(sE - A_0)) = \text{rank}E$

The regularity and the absence of impulses of the pair (E, A_0) ensure the existence and uniqueness of an impulse free solution to this system,(see [4, Lemma 1]).

3. Singular systems over rings

By introducing the delay operator δ defined, for any time function $f(t)$, by $\delta f(t) := f(t-h)$, the system Σ_d can be written in the following form

$$\left\{ \begin{array}{lcl} E\dot{x}(t) & = & \sum_{i=0}^{a} A_i\delta^i x(t) + \sum_{i=0}^{b} B_i\delta^i u(t) \\ & & \quad + \sum_{i=0}^{h} D_i\delta^i q(t) \\ y(t) & = & \sum_{i=0}^{c} C_i\delta^i x(t). \end{array} \right.$$

By formally replacing the delay operator δ with the algebraic indeterminate Δ and defining

$$\begin{array}{ll} A = \sum_{i=0}^{a} A_i\Delta^i, & B = \sum_{i=0}^{b} B_i\Delta^i, \\ C = \sum_{i=0}^{c} C_i\Delta^i, & D = \sum_{i=0}^{h} D_i\Delta^i \end{array}$$

we can associate to Σ_d (1) the system $\Sigma = (E,\ A,\ B,\ C,\ D)$ over the ring $\mathscr{R} = \mathbb{R}[\Delta]$ of real polynomials in one indeterminate defined buy the following equations.

$$\Sigma = \left\{ \begin{array}{rcl} Ex(t+1) & = & Ax(t) + Bu(t) + Dq(t) \\ y(t) & = & Cx(t). \end{array} \right. \tag{2}$$

where the state $x(t)$, the control input $u(t)$, the disturbance $q(t)$ and the output $y(t)$ belong, respectively, to the free modules $\mathscr{X} = \mathscr{R}^n$, $\mathscr{U} = \mathscr{R}^m$, $\mathscr{Q} = \mathscr{R}^h$, and $\mathscr{Y} = \mathscr{R}^p$. E, A, B, C and D are matrices of suitable dimension with entries in $\mathscr{R}$ and E is singular, i.e. $\det(E) = 0$.
In the following we'll assume B monic and C epic.
The two systems Σ_d and Σ are different objects from a dynamical point of view, but they share the structural properties that depend only on the defining matrices, so that many control problems concerning the input/output behavior of Σ_d, can be naturally formulated in terms of the input/output behavior of Σ and solved in the framework of systems over rings. Substituting back the delay operator δ to the algebraic indeterminate Δ, the solutions can be interpreted in the original delay-differential framework (see for instance [5], [6], [7], [8]).

Following [9], $\mathscr{X}$ will be denoted $\mathscr{X}_d$ when considered as domain of the maps E, A, C (descriptor space and $\mathscr{X}_c$ when considered as codomain of the maps E, A, B (equation space, following. With these notations we have

$$\begin{array}{c} E, A : \mathscr{X}_d \to \mathscr{X}_c, \quad B : \mathscr{U} \to \mathscr{X}_c \\ D : \mathscr{Q} \to \mathscr{X}_c \quad C : \mathscr{X}_d \to \mathscr{Y} \end{array}$$

4. Canonical form of singular systems

It is well known, since [10], that a regular matrix pencil $sE - A$, admits a *canonical decomposition* also called *Weierstrass decomposition*, namely there exist nonsingular real $n \times n$ matrices P, Q, such that

$$P(sE - A)Q = s\begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} - \begin{bmatrix} A_s & 0 \\ 0 & I \end{bmatrix} \tag{3}$$

where J is strictly upper triangular with E, A identically partitioned. Then the system (2) may be decomposed, by a suitable change of basis, into two subsystems:

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_s x_s(t) + B_s u(t) \\ y_s(t) & = & C_s x_s(t) \end{array} \right. \tag{4}$$

$$\left\{ \begin{array}{rcl} J\dot{x}(t) & = & x_f(t) + B_f u(t) \\ y_f(t) & = & C_f x_f(t) \end{array} \right. \tag{5}$$

$$y(t) = y_s(t) + y_f(t)$$

with $x_s \in \mathbb{R}^{n_s}$, $x_f \in \mathbb{R}^{n_f}$, $n_s = deg(\det(sE - A))$ and $n_s + n_f = n$, $A_s : \mathbb{R}^{n_s} \to \mathbb{R}^{n_s}$, $B_s : \mathbb{R}^m \to \mathbb{R}^{n_s}$, $C_s : \mathbb{R}^{n_s} \to \mathbb{R}^p$, $E_f : \mathbb{R}^{n_f} \to \mathbb{R}^{n_f}$, J nilpotent matrix with index of nilpotence μ, i.e. $J^\mu = 0$, $B_f : \mathbb{R}^m \to \mathbb{R}^{n_f}$, $C_f : \mathbb{R}^{n_f} \to \mathbb{R}^p$.

The subscripts $_f$ and $_s$ refer to the fact that often in the literature (e.g. [11]) subsystem (4) is called *slow subsystem*, and subsystem (5) is called *fast subsystem*. Other authors (e.g.[12]) call subsystem (4) the *finite subsystem* because its dynamics matrix A_s represents the finite structure of $(sE - A)$, and subsystem (5) the *infinite subsystem*, referring to the poles at infinity (the poles of $(E - sA)$ at $s = 0$), corresponding to infinite-frequency behavior.

4.1. Canonical form over the ring

For systems over a ring (2), we can consider the action on the pair (E, A) of the group G of all couples of unimodular matrices (P, Q)

$$(E, A) \cdot (P, Q) = (PEQ, PAQ), \tag{6}$$

whose orbits form a partition of $\mathscr{R}^{n \times n} \times \mathscr{R}^{n \times n}$.

Definition 4.1 *[13] The pair (E,A) is algebraically solvable if its orbit under (6) contains a member in standard canonical form, i.e. satisfying* **??**.

Algebraic solvability may be difficult to establish, and does not necessarily holds for all regular singular system. Easier, necessary conditions have been established in [13] by means of the following less restrictive property.

Proposition 4.2 *[13] The pair of matrices (E,A) is said pre-solvable if all the following conditions hold:*

$$\begin{array}{ll} PS1) & \mathrm{Im}E + A\ker E = \mathscr{R}^n \\ PS2) & \mathrm{Im}E \cap A\ker E \neq \{0\} \\ PS3) & \ker E \cap \ker A \neq \{0\} \end{array} \tag{7}$$

Pre-solvability is a necessary condition for algebraic solvability.

The following example describe a singular system over the ring $\mathbb{R}[\Delta]$ whose defining pair E, A is not pre-solvable, then not algebraically solvable.

Example 4.3 *Consider the system (2), where*

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 1 & \Delta & -\Delta & 1 \\ 1 & 1 & \Delta & 0 \\ 1 & -1 & 1 & \Delta \end{pmatrix}$$
$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In this case, none of the conditions (7) holds, since we have

$$\mathrm{Im}E + A\ker E = \mathrm{span}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \neq \mathscr{R}^n$$
$$\mathrm{Im}E \cap A\ker E = \{0\}, \quad \ker E \cap \ker A = \{0\}.$$

The pair (E,A) is not pre-solvable and therefore it cannot be algebraically solvable, i.e. the system Σ does not have a standard canonical form, i.e. the dynamical part and the anticipative part cannot be decoupled.

5. Invariants in generalized systems over rings and duality

Let us now introduce the fundamental notions of invariance for singular systems over a ring, comparing them with the notions introduced in the field case by [14] and [15] and also with other approaches to singular systems.

5.1. (E,A,B)-controlled invariants

Definition 5.1 *A submodule $\mathscr{V}$ of $\mathscr{X}_d$ is (E,A,B)- invariant or controlled invariant if and only if $A\mathscr{V} \subseteq E\mathscr{V} + \mathrm{Im}B$,*
$(E,A+BF)$- invariant or feedback invariant if and only if there exists a map $F : \mathscr{X}_d \to \mathscr{U}$ such that $(A+BF)\mathscr{V} \subseteq E\mathscr{V}$.

The above definition generalizes that given in the field case in [14] and if $E = I$ coincides with the notion of controlled invariance for systems over a ring defined in [16].

Given a singular systems algebraically solvable we can assume the form (4), (5) standard canonical form, any controlled invariant submodule $\mathscr{V}$ can be written as $\mathscr{V} = \mathscr{V}_s \oplus \mathscr{V}_f$, where $\mathscr{V}_s \subseteq \mathscr{R}^{n_s}$ and $\mathscr{V}_f \subseteq \mathscr{R}^{n_f}$. This reduced two the following conditions on the decoupled subspaces :

1. $A_s\mathscr{V}_s \subseteq \mathscr{V}_s + \mathrm{Im}B_s$ in the slow subsystem;
2. $\mathscr{V}_f \subseteq J\mathscr{V}_f + \mathrm{Im}B_f$ in the fast subsystem.

In particular, the second condition implies that the $\mathscr{V}_f$ is a subset of the reachable subspace $\langle J|B_f\rangle$, and it is null if the system is autonomous.

Definition 5.2 *Given a regular system (2), the maximum (A,E,B)-invariant submodule contained in a given submodule $\mathscr{K} \subseteq \mathscr{X}_d$ is denoted $\mathscr{V}^*(\mathscr{K})$.*

In the important case $\mathscr{K} = Ker\ C$ we shall simply denote it by $\mathscr{V}^*$.

Algorithm 5.3 *$\mathscr{V}^*$ can be computed as the limit as $k \to \infty$ of the following non increasing sequence:*

$$\begin{cases} \mathscr{V}_0 := \mathscr{X} \quad (Malabre) \\ \mathscr{V}_0 := \ker C \quad (io) \\ \mathscr{V}_{k+1} := A^{-1}\left(E\mathscr{V}_k + \mathrm{Im}B\right) \cap \ker C \end{cases} \tag{8}$$

The proof is analogous to the one presented in [14] for the field case. What may happen in the ring case is that the algorithm does not converge in a finite number of steps.

5.2. (C,E,A)-conditioned invariants for singular systems

Definition 5.4 *[**?**] A submodule $\mathscr{S}$ of $\mathscr{X}_c$ is (C,E,A)-invariant or conditioned invariant if and only if $A\left(\ker C \cap E^{-1}(\mathscr{S})\right) \subseteq \mathscr{S}$.*
$(E,A+GC)$-invariant or injection invariant if and only if there exists a map $G : \mathscr{Y} \to \mathscr{X}_c$ such that $(A+GC)E^{-1}(\mathscr{S}) \subseteq \mathscr{S}$.

Algorithm 5.5 *The set of all (C,A,E)-conditioned invariant subspace containing* ImB *has a minimum element denoted by $\mathscr{S}^*$ that can be computed by the following algorithm.*

$$\begin{cases} \underline{\mathscr{S}}_0 := \mathrm{Im}B \quad (io) \\ \\ \underline{\mathscr{S}}_{k+1} := A\left(E^{-1}(\underline{\mathscr{S}}_k) \cap \ker C\right) + \mathrm{Im}B \end{cases} \tag{9}$$

This algorithm generates an ascending sequence of submodules of $\mathscr{X}_c$, so it converges in a finite number of steps for systems over a Noetherian ring and, in particular, over $\mathscr{R}$.

The definitions 5.1 and 5.4 are independent from the algebraic solvability of the system as the following example shows.

Example 5.6 *The singular system of Example 4.3, is not algebraicaqlly solvable, but the maximal (E,A,B)-invariant contained in* ker C *is* span $\begin{pmatrix} 0 & \Delta \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and the minimal (C,E,A)-conditioned invariant containing* ImB *is* span $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

The notion of conditioned invariance is a fundamental step for the solution of the unknown input observation problem for singular delay systems presented in [**?**].

5.3. Duality does not hold for singular systems

There exists a duality between controlled and conditioned invariant submodules in the field case, also for singular systems. This does not happens for systems with coefficient in a ring, not even in the case $E = I$, see [17]. What happens for singular systems is described hereafter. Let us briefly recall a few definitions and results that will be used in the following.

Definition 5.7 *Let $\mathscr{R}$ be a P.I.D. and $\mathscr{S} \subseteq \mathscr{T}$ be submodules of $\mathscr{X} = \mathscr{R}^n$.*
i)*The closure of $\mathscr{S}$ in $\mathscr{T}$ is the submodule $\bar{\mathscr{S}}_{\mathscr{T}} = \{x \in \mathscr{T} \,|$there exists $a \in \mathscr{R}$, $ax \in \mathscr{S}$, $a \neq 0\}$. The submodule $\mathscr{S}$ is closed if $\mathscr{S} = \bar{\mathscr{S}}_{\mathscr{T}}$.*
ii) *The* orthogonal *of $\mathscr{S}$ is the submodule*
$\mathscr{S}^\perp := \{x \in \mathscr{X} \mid x^\top s = 0 \,\forall\, s \in \mathscr{S}\}$.
iii)*A full column rank matrix whose columns generate $\mathscr{S}$ is called a basis matrix for $\mathscr{S}$.*
The closure of $\mathscr{S}$ in $\mathscr{X}$ is $\bar{\mathscr{S}} := (\mathscr{S}^\perp)^\perp$, see [17].

Proposition 5.8 *[6] Let $\mathscr{R}$ be a P.I.D., then a submodule $\mathscr{S} \subseteq \mathscr{X} = \mathscr{R}^n$ is closed if and only if $\mathscr{S}$ is a direct summand of $\mathscr{X}$, i.e. there exists a closed submodule $\mathscr{W}$ such that $\mathscr{X} = \mathscr{S} \oplus \mathscr{W}$.*

Proposition 5.9 *[6] $\bar{\mathscr{S}}$ is the smallest closed submodule containing $\mathscr{S}$, and* $\dim \bar{\mathscr{S}} = \dim \mathscr{S}$. *If $\mathscr{S} \subseteq \mathscr{R}^n$ is conditioned invariant, then $\bar{\mathscr{S}}$ is injection invariant.*

Proposition 5.10 *Let $\mathscr{S}$ be a closed and let S be a basis matrix for $\mathscr{S}$ and W is a basis matrix for $\mathscr{S}^\perp$. Then,*

$$\mathscr{S}^\perp = \ker S^\top \quad \text{and} \quad \mathscr{S} = (\mathscr{S}^\perp)^\perp = \ker W^\top. \tag{10}$$

Theorem 5.11 *Let $\mathscr{V} \subset \mathscr{X}_d$ be a closed (E,A,B)-controlled invariant.*
If $\mathscr{V}$ is $(E,A+BF)$- invariant, then $\mathscr{V}^\perp$ is $(E^t, A^t + F^tB^t)$ - invariant.

Proof $\mathscr{V}$ is $(E,A+BF)$- invariant, then there exists a map $F : \mathscr{X}_d \to \mathscr{U}$ such that $(A+BF)\mathscr{V} \subseteq E\mathscr{V}$; $E\mathscr{V}$ might be not closed, nevertheless by virtue of Remark **??**,

$$(A+BF)^t (E\mathscr{V})^\perp \subseteq \mathscr{V}^\perp. \tag{11}$$

Let us remember that being $\mathscr{V}$ closed, Proposition **??** applies, and in particular $(E\mathscr{V})^\perp = (E^t)^{-1}\mathscr{V}^\perp$. Then, equation (11) can be rewritten

$$(A^t + F^tB^t)\,(E^t)^{-1}\mathscr{V}^\perp \subseteq \mathscr{V}^\perp \tag{12}$$

which implies $(E^t, A^t + F^tB^t)$-invariance of $\mathscr{V}^\perp$. ■

The converse is not true, as the following example shows.

Example 5.12

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t+1) = \begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & \Delta & 1 & -1 \\ 0 & -\Delta & \Delta & 1 \\ 2 & 1 & 0 & \Delta \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ -1 & 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) \tag{13}$$

We have shown in example 4.3 that the pair (E,A) is not algebraically solvable, i.e. the system Σ cannot be put in standard canonical form, therefore the dynamical part and the anticipative part cannot be decoupled. However it is possible to compute the minimal (C,A,E)-invariant submodule containing ImB*, and we will denote it $\mathscr{S}*$.*

Computing $\mathscr{S}*$ by means of algorithm 5.5, we obtain

$$\mathscr{S}^* = \text{span}\begin{pmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}$$

which is also $(E, A+GC)$-injection invariant submodule with $G = \begin{pmatrix}-1 & -1\\ g_1 & g_2\\ -\Delta & -1\\ g_3 & g_4\end{pmatrix}$ where g_1, g_2, g_3, g_4 are arbitrary parameters.
The submodule of $\mathscr{X}_c$ orthogonal to $\mathscr{S}^*$ is

$$(\mathscr{S}^*)^\perp = \text{span}\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

but it is not feedback invariant, since $(A^t + C^t * G^t)(\mathscr{S}^*)^\perp = \text{span}\begin{pmatrix}-1 & 0\\ 1 & -\Delta\\ 0 & 0\\ 0 & 0\end{pmatrix}$ which is not contained in $E^t(\mathscr{S}^*)^\perp = \text{span}\begin{pmatrix}1 & 0\\ 0 & \Delta\\ 0 & 0\\ 0 & 0\end{pmatrix}$ $(\mathscr{S}^*)^\perp$ is not even (E^t, A^t, C^t)-invariant because $A^t(\mathscr{S}^*)^\perp = \text{span}\begin{pmatrix}-1 & 0\\ 1 & -\Delta\\ 1 & \Delta\\ 1 & 1\end{pmatrix} \not\subseteq$

$$\text{span}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & x & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = E^t(\mathscr{S}^*) + \text{Im}C^t$$

6. The Disturbance Decoupling Problem

Given a system Σ of the form (2), find, if possible, integers n_a and m_i and a feedback law of the form

$$\begin{cases} x_a(t+1) = A_1x(t) + A_2x_a(t) + \sum_{i=1}^k G_{ai}v_i(t)\\ u(t) = Fx(t) + Hx_a(t) + \sum_{i=1}^k G_iv_i(t)\end{cases}, \tag{14}$$

where $x_a \in \mathscr{X}_a = R^{n_a}$; $v_i \in R^{m_i}$, $i = 1, 2, \ldots, k$; A_1, A_2, F, H, G_i and G_{ai} are matrices of suitable dimensions with entries in the ring $\mathscr{R}$, such that in the compensated system $\Sigma_{F,G}$, given by the equations

$$\begin{cases} Ex(t+1) &= (A+BF)x(t) + BH_ax_a(t) + Dq(t) +\\ & \quad + \sum_{i=1}^k BG_iv_i(t)\\ x_a(t+1) &= A_1x(t) + A_2x_a(t) + \sum_{i=1}^k G_{ai}v_i(t)\\ y_(t) &= Cx(t)\end{cases} \tag{15}$$

the disturbance $q(t)$ is completely decoupled from the output $y_(t)$. We will speak of *Restricted Decoupling Problem* when the feedback (14) is constrained to be static, namely $n_a = 0$, and of *Extended Decoupling Problem* otherwise.

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Appendix

Proof[of the Corollary **??**]
The reasoning is analogue to the proof of Theorem **??**. The map $E : \mathscr{X}_d \to \mathscr{X}_c$ can be easily restrict to $\mathscr{S}_C \subseteq \mathscr{X}$ without changes, so we can have $E_{|\mathscr{S}_C} : \mathscr{S}_C \to \mathscr{S}$; it is well-defined because

$$E(\mathscr{S}_C) \subseteq E(E^{-1}\mathscr{S}) \subseteq \mathscr{S}.$$

The diagram

$$\begin{array}{ccc} \mathscr{S}_C & \longrightarrow & \mathscr{X}_d \\ \downarrow E & & \downarrow E \\ \mathscr{S} & \longrightarrow & \mathscr{X}_c \end{array}$$

commutes because if $x \in E^{-1}\mathscr{S} \cap \ker C$, we have

$$\begin{array}{ccc} x \mapsto^i & x \mapsto^E & Ex \in X \\ x \mapsto^E & Ex \mapsto^i & Ex \in X. \end{array}$$

We would try to build a map $\tilde{E} : X/\mathscr{S}_C \to X/\mathscr{S}$ such that the diagram commutes

$$\begin{array}{ccc} \mathscr{X}_d & \xrightarrow{\pi_d} & \mathscr{X}_d/\mathscr{S}_C \\ \downarrow E & & \downarrow \tilde{E} \\ \mathscr{X}_c & \xrightarrow{\pi_c} & \mathscr{X}_c/\mathscr{S} \end{array}$$

i.e. $\tilde{E} \circ \pi_d = \pi_c \circ E$.

Let $x \in \mathscr{X}_d$. If $\mathscr{S}_C$ is closed, we can write $x = x_0 + \underline{x}$ with $\underline{x} \in \mathscr{S}_C$.

$$x \xmapsto{\pi_d} [x]_{\mathscr{S}_C} = [x_0]_{\mathscr{S}_C} \xmapsto{\tilde{E}} \tilde{E}[x_0]_{\mathscr{S}_C}$$

$$x \xmapsto{E} Ex \xmapsto{\pi_c} [Ex]_{\mathscr{S}} = [Ex_0]_{\mathscr{S}}$$

because $[Ex]_{\mathscr{S}} = [E(x_0 + \underline{x})]_{\mathscr{S}} = [Ex_0 + E\underline{x}]_{\mathscr{S}} = [Ex_0]_{\mathscr{S}}$, being $E\underline{x} \in \mathscr{S}$.

So we define $\tilde{E}[x_0]_{\mathscr{S}_C} := [Ex_0]_{\mathscr{S}}$.

$\tilde{E}$ is well defined because:

- $\tilde{E}[x + x']_{\mathscr{S}_C} = [E(x + x')]_{\mathscr{S}} = [Ex + Ex']_{\mathscr{S}} = [Ex]_{\mathscr{S}} + [Ex']_{\mathscr{S}} = \tilde{E}[x]_{\mathscr{S}_C} + \tilde{E}[x']_{\mathscr{S}_C}$;
- $\tilde{E}[0]_{\mathscr{S}_C C} = [E(0)]_{\mathscr{S}} = [0]_{\mathscr{S}}$.
  If $x \in \mathscr{S}_C, x \neq 0$, $Ex \in \mathscr{S}$ so $\tilde{E}[x]_{\mathscr{S}_C} = [Ex]_{\mathscr{S}} = [0]_{\mathscr{S}}$;
- $\tilde{E}[\lambda x]_{\mathscr{S}_C} = [E\lambda x]_{\mathscr{S}} = [\lambda Ex]_{\mathscr{S}} = \lambda[Ex]_{\mathscr{S}} = \lambda\tilde{E}[x]_{\mathscr{S}_C}$ $\forall\, \lambda \in \mathscr{R}, \forall\, x \in \mathscr{X}_d$;
- if $[x]_{\mathscr{S}_C} = [x']_{\mathscr{S}_C}$, then $x - x' \in \mathscr{S}_C$, so
  $[x]_{\mathscr{S}_C} - [x']_{\mathscr{S}_C} = [x - x']_{\mathscr{S}_C} = [0]_{\mathscr{S}_C}$.
  Thus $\tilde{E}([x]_{\mathscr{S}_C} - [x']_{\mathscr{S}_C}) = [0]_{\mathscr{S}}$ and $\tilde{E}[x]_{\mathscr{S}_C} = \tilde{E}[x']_{\mathscr{S}_C}$.

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