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Stable manifolds for semi-linear evolution equations and admissibility of function spaces on a half-line

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ABSTRACT

Consider an evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ on a half-line $\mathbb{R}_+$ and a semi-linear integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))\,d\xi$. We prove the existence of stable manifolds of solutions to this equation in the case that $(U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy and the nonlinear forcing term $f(t,x)$ satisfies the non-uniform Lipschitz conditions: $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ for φ being a real and positive function which belongs to admissible function spaces which contain wide classes of function spaces like function spaces of L_p type, the Lorentz spaces $L_{p,q}$ and many other function spaces occurring in interpolation theory.

1. Introduction

Consider the semi-linear differential equation

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad t \in [0,+\infty),\ x \in X, \tag{1}$$

where $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed t and $f : \mathbb{R}_+ \times X \to X$ is a nonlinear operator.

One of the center research interests regarding asymptotic behavior of solutions to the above equation is to find conditions for that equation to have integral (stable, unstable or center) manifolds (see, e.g., [1,2,4–7,13,16,21]). To our best knowledge, the most popular conditions for the existence of integral (stable, unstable or center) manifolds are the exponential dichotomy (or trichotomy) of the linear part $\frac{dx}{dt} = A(t)x$ and the uniform Lipschitz continuity (with respect to the second variable x) of the nonlinear part $f(t,x)$ with sufficiently small Lipschitz constants (i.e., $\|f(t,x) - f(t,y)\| \leqslant q\|x-y\|$ for q small enough). Such manifolds are local or global depending on the fact that $f(t,x)$ is locally or globally Lipschitz, respectively. We refer the reader to [1,2,5–7,13,21] and references therein for more information on this matter.

In the present paper, we consider the existence of stable manifolds for Eq. (1) under more general conditions on the nonlinear term $f(t,x)$, that is the non-uniform Lipschitz continuity of f, i.e., $\|f(t,x) - f(t,y)\| \leqslant \varphi(t)\|x-y\|$ for φ being a real and positive function which belongs to admissible function spaces defined in Definition 2.4 below. Under some conditions on φ, we will prove the existence of stable manifolds for Eq. (1) provided that the linear part $\frac{dx}{dt} = A(t)x$ has an exponential dichotomy. In our strategy, we use the characterization of the exponential dichotomy of evolution equations in admissible spaces of functions defined on the half-line $\mathbb{R}_+$. This characterization, which is obtained in [8], allows us to construct the structures of solutions of Eq. (1) in a mild form, which belong to some certain class of admissible spaces on

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which we can implement some well-known procedures in functional analysis such as: constructing of contraction mapping; using of Implicit Function Theorem, etc. The use of admissible spaces helps us to avoid using the smallness of Lipschitz constants in classical sense. Instead, the "smallness" is now understood as the sufficient smallness of $\sup_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)\,d\tau$ (see the conditions in Theorem 3.8 below). Consequently, we obtain the existence of stable manifolds for the mild solutions of Eq. (1) under very general conditions on the nonlinear term $f(t,x)$. Our main results are contained in Theorems 3.7, 3.8, 4.6, 4.7. We also illustrate our results in Example 4.8.

In the case of unbounded $A(t)$, it is more convenient to consider Eq. (1) in a mild form

$$x(t)=U(t,s)x(s)+\int_s^t U(t,\xi)f\big(\xi,x(\xi)\big)\,d\xi \quad \text{for } t\geqslant s\geqslant 0$$

using the evolution family $U(t,s))_{t\geqslant s\geqslant 0}$ arising in well-posed homogeneous Cauchy problems. We now recall the definition of an evolution family.

Definition 1.1. A family of bounded linear operators $\mathcal{U}=(U(t,s))_{t\geqslant s\geqslant 0}$ on a Banach space X is a (strongly continuous, exponential bounded) evolution family on the half-line if

(i) $U(t,t)=\mathrm{Id}$ and $U(t,r)U(r,s)=U(t,s)$ for $t\geqslant r\geqslant s\geqslant 0$,
(ii) the map $(t,s)\mapsto U(t,s)x$ is continuous for every $x\in X$,
(iii) there are constants $K,c\geqslant 0$ such that $\|U(t,s)\|\leqslant Ke^{c(t-s)}$ for $t\geqslant s\geqslant 0$.

This notion of evolution families arises naturally from the theory of Cauchy problems for evolution equations which are well-posed (see, e.g., [17, Chapt. 5], [15,19]). In fact, in the terminology of [17, Chapt. 5] and [15], an evolution family arises from the following well-posed evolution equation

$$\begin{cases} \dfrac{du(t)}{dt}=A(t)u(t), & t\geqslant s\geqslant 0,\\ u(s)=x_s\in X, \end{cases} \tag{2}$$

where $A(t)$ are (in general unbounded) linear operators for $t\geqslant 0$. Roughly speaking, when the Cauchy problem (2) is well-posed, there exists a (strongly continuous, exponential bounded) evolution family $\mathcal{U}=(U(t,s))_{t\geqslant s\geqslant 0}$ solving (2), i.e., the solution of (2) is given by $u(t):=U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [17] (see also Nagel and Nickel [14] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$).

2. Function spaces and admissibility

We recall some notions of function spaces and admissibility. We refer the readers to Massera and Schäffer [11, Chapt. 2] for wide classes of function spaces that play a fundamental role throughout the study of differential equations in the case of bounded coefficients $A(t)$ (see also Räbiger and Schnaubelt [18, §1] for some classes of admissible Banach function spaces of functions defined on the whole line $\mathbb{R}$).

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. As already known, the set of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) that are integrable on every compact subinterval $J\subset\mathbb{R}_+$ becomes, with the topology of convergence in the mean on every such J, a locally convex topological vector space, which we denote by $L_{1,\mathrm{loc}}(\mathbb{R}_+)$. A set of seminorms defining the topology of $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is given by $p_n(f):=\int_{J_n}|f(t)|\,dt$, $n\in\mathbb{N}$, where $\{J_n\}_{n\in\mathbb{N}}=\{[n,n+1]\}_{n\in\mathbb{N}}$ is a countable set of abutting compact intervals whose union is $\mathbb{R}_+$. With this set of seminorms one can see (see [11, Chapt. 2, §20]) that $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a Fréchet space.

Let V be a normed space (with norm $\|\cdot\|_V$) and W be a locally convex Hausdorff topological vector space. Then, we say that V is *stronger than* W if $V\subseteq W$ and the indentity map from V into W is continuous. The latter condition is equivalent to the fact that for each continuous seminorm π of W there exists a number $\beta_\pi>0$ such that $\pi(x)\leqslant\beta_\pi\|x\|_V$ for all $x\in V$. We write $V\hookrightarrow W$ to indicate that V is stronger than W. If, in particular, W is also a normed space (with norm $\|\cdot\|_W$) then the relation $V\hookrightarrow W$ is equivalent to the fact that $V\subseteq W$ and there is a number $\alpha>0$ such that $\|x\|_W\leqslant\alpha\|x\|_V$ for all $x\in V$ (see [11, Chapt. 2] for detailed discussions on this matter).

We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called *a Banach function space* (over $(\mathbb{R}_+,\mathcal{B},\lambda)$) if

(1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E,\|\cdot\|_E)$ is a Banach space, and if $\varphi\in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)|\leqslant|\varphi(\cdot)|\lambda$-a.e., then $\psi\in E$ and $\|\psi\|_E\leqslant\|\varphi\|_E$,

(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geqslant 0}\|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geqslant 0}\|\chi_{[t,t+1]}\|_E > 0$,
(3) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$.

For a Banach function space E we remark that the condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}_+$ there exists a number $\beta_J \geqslant 0$ such that $\int_J |f(t)|\,dt \leqslant \beta_J \|f\|_E$ for all $f \in E$.

We state the following trivial lemma which will be frequently used in our strategy.

Lemma 2.2. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}_+$ such that they coincide with each other outside a compact interval and they are essentially bounded (for instance, continuous) on this compact interval. Then $\varphi \in E$ if and only if $\psi \in E$.*

We then define Banach spaces of vector-valued functions corresponding to Banach function spaces as follows.

Definition 2.3. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \bigl\{ f : \mathbb{R}_+ \to X\colon\ f \text{ is strongly measurable and } \bigl\|f(\cdot)\bigr\| \in E \bigr\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}} := \bigl\| \bigl\|f(\cdot)\bigr\| \bigr\|_E.$$

One can easily see that $\mathcal{E}$ is a Banach space. We call it *the Banach space corresponding to the Banach function space E*.

We now introduce the notion of admissibility in the following definition.

Definition 2.4. The Banach function space E is called admissible if it satisfies

(i) there is a constant $M \geqslant 1$ such that for every compact interval $[a,b] \in \mathbb{R}_+$ we have

$$\int_a^b \bigl|\varphi(t)\bigr|\,dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{3}$$

(ii) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1}\varphi(\tau)\,d\tau$ belongs to E,
(iii) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined, for $\tau \in \mathbb{R}_+$, by

$$T_\tau^+\varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \geqslant \tau \geqslant 0, \\ 0 & \text{for } 0 \leqslant t \leqslant \tau, \end{cases}$$
$$T_\tau^-\varphi(t) := \varphi(t+\tau) \quad \text{for } t \geqslant 0. \tag{4}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\|_E \leqslant N_1$, $\|T_\tau^-\|_E \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.5. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+)\colon\ \sup_{t\geqslant 0}\int_t^{t+1}\bigl|f(\tau)\bigr|\,d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geqslant 0}\int_t^{t+1}|f(\tau)|\,d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 \leqslant q < \infty$ (see [3, Thm. 3 and p. 284], [22, 1.18.6, 1.19.3]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}_+, \mathcal{B}, \lambda)$ (see [9, 2.a]) are admissible.

Remark 2.6. If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$. Indeed, put $\beta := \inf_{t\geqslant 0}\|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.1(2)). Then, from Definition 2.4(i) we derive

$$\int_t^{t+1}\bigl|\varphi(\tau)\bigr|\,d\tau \leqslant \frac{M}{\beta}\|\varphi\|_E \quad \text{for all } t \geqslant 0 \text{ and } \varphi \in E. \tag{5}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\varphi\|_{\mathbf{M}} \leqslant \frac{M}{\beta}\|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [8, Proposition 2.6]).

Proposition 2.7. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is defined as in Definition 2.4(ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int_0^t e^{-\sigma(t-s)}\varphi(s)\,ds,$$

$$\Lambda''_\sigma\varphi(t) := \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)\,d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.6)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for ess sup-norm, we have

$$\left\|\Lambda'_\sigma\varphi\right\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\left\|\Lambda_1 T_1^+\varphi\right\|_\infty \quad \text{and} \quad \left\|\Lambda''_\sigma\varphi\right\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \tag{6}$$

for operator T_1^+ and constants N_1, N_2 defined as in Definition 2.4.

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geqslant 0$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) := e^{bt}$ for $t \geqslant 0$ and any fixed constant $b > 0$.*

Proof. The proof of this proposition is essentially done in [8, Proposition 2.6] and originally in [11, 23.V.(1)]. We present it here for seek of completeness.

(a) We first prove that $\Lambda'_\sigma\varphi$ belongs to E.

Indeed, putting $a_+ := \max\{0, a\}$ for $a \in \mathbb{R}$, we remark that, by the definitions of Λ_1 and T_1^+, the equalities

$$\Lambda_1 T_1^+\varphi(t) = \int_{(t-1)_+}^t \varphi(s)\,ds \quad \text{and}$$

$$T_1^+\Lambda_1\varphi(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant 1, \\ \int_{t-1}^t \varphi(s)\,ds & \text{for } t > 1 \end{cases}$$

hold. Since $T_1^+\Lambda_1\varphi \in E$, by Lemma 2.2, we obtain that $\Lambda_1 T_1^+\varphi$ also belongs to E. We then compute

$$\begin{aligned}\Lambda'_\sigma\varphi(t) &= \sum_{j=0}^\infty \int_{(t-(j+1))_+}^{(t-j)_+} e^{-\sigma(t-s)}\varphi(s)\,ds \leqslant \sum_{j=0}^\infty e^{-j\sigma}\int_{(t-(j+1))_+}^{(t-j)_+}\varphi(s)\,ds \\ &= \sum_{j=0}^\infty e^{-j\sigma}T_j^+\Lambda_1 T_1^+\varphi(t) \quad \text{for all } t \in \mathbb{R}_+.\end{aligned}$$

Moreover, $e^{-j\sigma}T_j^+\Lambda_1T_1^+\varphi \in E$ for all j and

$$\sum_{j=0}^\infty \left\|e^{-j\sigma}T_j^+\Lambda_1 T_1^+\varphi\right\|_E \leqslant \sum_{j=0}^\infty N_1 e^{-j\sigma}\left\|\Lambda_1 T_1^+\varphi\right\|_E = \frac{N_1}{1-e^{-\sigma}}\left\|\Lambda_1 T_1^+\varphi\right\|_E.$$

Since E is a Banach function space, we obtain that $\Lambda'_\sigma\varphi \in E$ and

$$\left\|\Lambda'_\sigma\varphi\right\|_E \leqslant \frac{N_1}{1-e^{-\sigma}}\left\|\Lambda_1 T_1^+\varphi\right\|_E. \tag{7}$$

We now prove that $\Lambda''_\sigma\varphi$ belongs to E. To do that we compute

$$\begin{aligned}\Lambda''_\sigma\varphi(t) &= \sum_{j=0}^\infty \int_{t+j}^{t+j+1} e^{-\sigma(s-t)}\varphi(s)\,ds \leqslant \sum_{j=0}^\infty e^{-j\sigma}\int_{t+j}^{t+j+1}\varphi(s)\,ds \\ &= \sum_{j=0}^\infty e^{-j\sigma}T_j^-\Lambda_1\varphi(t) \quad \text{for all } t \in \mathbb{R}_+.\end{aligned}$$

Furthermore, $e^{-j\sigma}T_j^-\Lambda_1\varphi \in E$ for all j and

$$\sum_{j=0}^{\infty}\left\|e^{-j\sigma}T_j^-\Lambda_1\varphi\right\|_E \leqslant \sum_{j=0}^{\infty}N_2e^{-j\sigma}\|\Lambda_1\varphi\|_E = \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E.$$

Since E is a Banach function space, we obtain that $\Lambda_\sigma''\varphi \in E$ and

$$\left\|\Lambda_\sigma''\varphi\right\|_E \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E. \tag{8}$$

To prove that the condition $\sup_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)\,d\tau < \infty$ implies the boundedness of Λ_σ' and Λ_σ'' we just apply the above result to the admissible Banach function space L_∞. Also, the inequality (6) now follows from (7) and (8).

(b) Since $\chi_{[0,1]}$ belongs to E, using the above assertion (a), for any fixed constant $\alpha > 0$ we have that the function

$$v(t) := \int_0^t e^{-\alpha(t-s)}\chi_{[0,1]}(s)\,ds = \begin{cases} \frac{e^{-\alpha t}(e^{\alpha}-1)}{\alpha} & \text{for } t \geqslant 1, \\ \frac{1-e^{-\alpha t}}{\alpha} & \text{for } 0 \leqslant t < 1 \end{cases}$$

belongs to E. The assertion (b) now follows from Lemma 2.2.

(c) For the purpose of contradiction let the function $f(t) = e^{bt}$ belong to E for some $b > 0$. Then, by the inequality (5) we have that

$$\frac{1}{b}e^{bt}\left(e^b - 1\right) \leqslant \frac{M}{\beta}\|f\|_E \quad \text{for all } t \geqslant 0.$$

This is a contradiction since $\lim_{t\to\infty}\frac{1}{b}e^{bt}(e^b-1) = \infty$. □

We now recall the cone inequality theorem which will be used to compare solutions on the manifolds. Firstly, we introduce the following notion.

A closed subset $\mathcal{K}$ of a Banach space W is called a *cone* if it has the following properties:

(i) $x_0 \in \mathcal{K}$ implies $\lambda x_0 \in \mathcal{K}$ for all $\lambda \geqslant 0$;
(ii) $x_1, x_2 \in \mathcal{K}$ implies $x_1 + x_2 \in \mathcal{K}$;
(iii) $\pm x_0 \in \mathcal{K}$ implies $x_0 = 0$.

Suppose a cone $\mathcal{K}$ is given in a Banach space W. For $x, y \in W$ we will write $x \leqslant y$ if $y - x \in \mathcal{K}$.

If the cone $\mathcal{K}$ is invariant under a linear operator A, then it is easy to see that A preserves the inequality, i.e., $x \leqslant y$ implies $Ax \leqslant Ay$. Also, the following cone inequality theorem is taken from [4, Theorem I.9.3].

Theorem 2.8 *(Cone inequality). Let $\mathcal{K}$ be a cone given in a Banach space W such that $\mathcal{K}$ is invariant under a bounded linear operator $A \in \mathcal{L}(W)$ having spectral radius $r_A < 1$. If a vector $x \in W$ satisfies the inequality*

$$x \leqslant Ax + z \quad \text{for some given } z \in W,$$

then it also satisfies the estimate $x \leqslant y$, where $y \in W$ is the solution of the equation $y = Ay + z$.

3. Local stable manifolds

In this section we consider the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f\big(\xi, u(\xi)\big)\,d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+. \tag{9}$$

We note that, if the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ arises from the well-posed Cauchy problem (2) then the function u, which satisfies (9) for some given function f, is called a mild solution of the inhomogeneous problem

$$\begin{cases} \dfrac{du(t)}{dt} = A(t)u(t) + f\big(t, u(t)\big), & t \geqslant s \geqslant 0, \\ u(s) = x_s \in X \end{cases}$$

(see Pazy [17] for more information on this matter).

We will prove the existence of local stable manifolds for solutions of Eq. (9) under the appropriate conditions imposed on the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ (the linear part) and on the nonlinear term $f(t,x)$. Firstly, for the linear part we need the fact that the evolution family has an exponential dichotomy. We now make precise the notion of exponential dichotomy in the following definition.

Definition 3.1. An evolution family $\mathcal{U} = (U(t,s))_{t \geqslant s \geqslant 0}$ on the Banach space X is said to have an *exponential dichotomy* on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t \geqslant 0$, on X and positive constants N, ν such that

(a) $U(t,s)P(s) = P(t)U(t,s)$, $t \geqslant s \geqslant 0$,
(b) the restriction $U(t,s)_| : \ker P(s) \to \ker P(t)$, $t \geqslant s \geqslant 0$, is an isomorphism (and we denote its inverse by $U(s,t)_| : \ker P(t) \to \ker P(s)$),
(c) $\|U(t,s)x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)X$, $t \geqslant s \geqslant 0$,
(d) $\|U(s,t)_| x\| \leqslant Ne^{-\nu(t-s)}\|x\|$ for $x \in \ker P(t)$, $t \geqslant s \geqslant 0$.

The constants N, ν are called dichotomy constants and the projections $P(t)$, $t \geqslant 0$, are called dichotomy projections. We also denote by $X_0(t) := P(t)X$ and $X_1(t) := (I - P(t))X$.

We remark that properties (a)–(d) of dichotomy projections $P(t)$ already imply that

(i) $H := \sup_{t \geqslant 0} \|P(t)\| < \infty$,
(ii) $t \mapsto P(t)$ is strongly continuous

(see [12, Lemma 4.2]). Furthermore, let E be an admissible Banach function space and $\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X)$ be the Banach space corresponding to E (see Definition 2.3). Then, for each $t_0 \geqslant 0$ the space $X_0(t_0) = P(t_0)X$ can be characterized (see [8]) as:

$$X_0(t_0) = \left\{ x \in X\colon \text{ the function } z(t) := \begin{cases} U(t,t_0)x & \text{for } t \geqslant t_0, \\ 0 & \text{for } t < t_0 \end{cases} \text{ belongs to } \mathcal{E} \right\}.$$

Concretely, taking, e.g., $E = L_\infty$ we have that

$$X_0(t_0) = \left\{ x \in X\colon \sup_{t \geqslant t_0} \big\| U(t,t_0)x \big\| < \infty \right\}.$$

Let $(U(t,s))_{t \geqslant s \geqslant 0}$ have an exponential dichotomy with corresponding dichotomy projections $(P(t))_{t \geqslant 0}$ and dichotomy constants $N, \nu > 0$. Then, we can define the Green's function on a half-line as follows:

$$\mathcal{G}(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t > \tau \geqslant 0, \\ -U(t,\tau)_| (I - P(\tau)) & \text{for } 0 \leqslant t < \tau. \end{cases} \tag{10}$$

Also, $\mathcal{G}(t,\tau)$ satisfies the estimate

$$\big\| \mathcal{G}(t,\tau) \big\| \leqslant (1+H)Ne^{-\nu|t-\tau|} \quad \text{for } t \neq \tau \geqslant 0. \tag{11}$$

For the nonlinear term we have the following definition.

Definition 3.2. Let φ be a positive function belonging to E and B_ρ be the ball with radius ρ in X, i.e., $B_\rho := \{x \in X\colon \|x\| \leqslant \rho\}$. A function $f : [0,\infty) \times B_\rho \to X$ is said to belong to the class (M, φ, ρ) for some positive constants M, ρ if f satisfies

(i) $\|f(t,x)\| \leqslant M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and all $x \in B_\rho$, and
(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ for a.e. $t \in \mathbb{R}_+$ and all $x_1, x_2 \in B_\rho$.

Remark 3.3. If $f(t,0) = 0$ then, the condition (ii) in the above definition already implies that f belongs to class (ρ, φ, ρ).

We then give the definition of local stable manifolds for the solutions to Eq. (9).

Definition 3.4. A set $\mathbf{S} \subset \mathbb{R}_+ \times X$ is said to be a local stable manifold for the solutions of Eq. (9) if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ such that

$$\inf_{t \in \mathbb{R}_+} Sn\big(X_0(t), X_1(t)\big) := \inf_{t \in \mathbb{R}_+} \inf\big\{ \|x_0 + x_1\|\colon\ x_i \in X_i(t),\ \|x_i\| = 1,\ i = 0,1 \big\} > 0,$$

and if there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_0} \cap X_0(t) \to B_{\rho_1} \cap X_1(t), \quad t \in \mathbb{R}_+,$$

with Lipschitz constants independent of t such that

(i) $\mathbf{S} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) \mid t \in \mathbb{R}_+,\ x \in B_{\rho_0} \cap X_0(t)\}$, and we denote by $\mathbf{S}_t := \{x + g_t(x)\colon\ (t, x + g_t(x)) \in \mathbf{S}\}$,
(ii) $\mathbf{S}_t$ is homeomorphic to $B_{\rho_0} \cap X_0(t) := \{x \in X_0(t)\colon\ \|x\| \leqslant \rho_0\}$ for all $t \geqslant 0$,
(iii) to each $x_0 \in \mathbf{S}_{t_0}$ there corresponds one and only one solution $u(t)$ of Eq. (9) on $[t_0, \infty)$ satisfying conditions $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t \geqslant t_0} \|u(t)\| \leqslant \rho$.

The following lemma gives the form of bounded solutions of Eq. (9).

Lemma 3.5. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Suppose that φ is a positive function which belongs to E. Let $f : \mathbb{R}_+ \times B_\rho \to X$ belong to class (M, φ, ρ) for some positive constants M, ρ. Let $u(t)$ be a solution to Eq. (9) such that $\operatorname{ess\,sup}_{t\geqslant t_0} \|u(t)\| \leqslant \rho$ for fixed $t_0 \geqslant 0$. Then, for $t \geqslant t_0$ we have that $u(t)$ can be rewritten in the form*

$$u(t) = U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f\bigl(\tau, u(\tau)\bigr)\, d\tau \quad \text{for some } v_0 \in X_0(t_0) = P(t_0)X, \tag{12}$$

where $\mathcal{G}(t,\tau)$ is the Green's function defined by equality (10).

Proof. Put $y(t) := \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f(\tau, u(\tau))\, d\tau$ for a.e. $t \geqslant t_0$. Since f belongs to class (M, φ, ρ), using estimate (11) we obtain that

$$\bigl\|y(t)\bigr\| \leqslant (1+H)NM \int_0^{\infty} e^{-\nu|t-\tau|} \varphi(\tau)\, d\tau \quad \text{for } t \geqslant t_0.$$

Next, using the estimate (6) we have that

$$\operatorname*{ess\,sup}_{t\geqslant t_0} \bigl\|y(t)\bigr\| \leqslant \frac{(1+H)NM(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\nu}}.$$

Also, it is straightforward to see that $y(\cdot)$ satisfies the equation

$$y(t) = U(t,t_0)y(t_0) + \int_{t_0}^{t} U(t,s) f\bigl(s, u(s)\bigr)\, ds \quad \text{for } t \geqslant t_0.$$

Since $u(t)$ is a solution of Eq. (9) we obtain that $u(t) - y(t) = U(t,t_0)(u(t_0) - y(t_0))$ for $t \geqslant t_0$. Put now $v_0 = u(t_0) - y(t_0)$. The essential boundedness of $u(\cdot)$ and $y(\cdot)$ on $[t_0, \infty)$ implies that $v_0 \in X_0(t_0)$. Finally, since $u(t) = U(t,t_0)v_0 + y(t)$ for $t \geqslant t_0$, the equality (12) follows. □

Remark 3.6. By straightforward computation we can prove that the converse is also true: a solution of Eq. (12) satisfies Eq. (9) for $t \geqslant t_0$.

Using the admissibility, we construct the structure of certain solutions of Eq. (9) in the following theorem.

Theorem 3.7. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Then, for any positive numbers ρ and M we have that, if f belongs to the class (M, φ, ρ) with the positive function $\varphi \in E$ satisfying*

$$\frac{(1+H)N}{1-e^{-\nu}}\bigl(N_1\bigl\|\Lambda_1 T_1^+\varphi\bigr\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\bigr) < \min\left\{1, \frac{\rho}{2M}\right\},$$

then there corresponds to each $v_0 \in B_{\frac{\rho}{2N}} \cap X_0(t_0)$ one and only one solution $u(t)$ of Eq. (9) on $[t_0, \infty)$ satisfying the conditions $P(t_0)u(t_0) = v_0$ and $\operatorname{ess\,sup}_{t\geqslant t_0} \|u(t)\| \leqslant \rho$. Moreover, the following estimate is valid for any two solutions $u_1(t), u_2(t)$ corresponding to different values $v_1, v_2 \in B_{\frac{\rho}{2N}} \cap X_0(t_0)$:

$$\bigl\|u_1(t) - u_2(t)\bigr\| \leqslant C_\mu e^{-\mu(t-t_0)}\|v_1 - v_2\| \quad \text{for } t \geqslant t_0, \tag{13}$$

where μ is a positive number satisfying

$$0 < \mu < \nu + \ln\bigl(1 - (1+H)N\bigl(N_1\bigl\|\Lambda_1 T_1^+\varphi\bigr\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\bigr)\bigr), \quad \text{and}$$
$$C_\mu = \frac{N}{1 - \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}.$$

Proof. We consider in the space $L_\infty(\mathbb{R}_+, X)$ the ball

$$\mathcal{B}_\rho := \Bigl\{x(\cdot) \in L_\infty(\mathbb{R}_+, X)\colon\ \bigl\|x(\cdot)\bigr\|_\infty := \operatorname*{ess\,sup}_{t\geqslant 0}\bigl\|x(t)\bigr\| \leqslant \rho\Bigr\}.$$

For $v_0 \in B_{\frac{\rho}{2N}} \cap X_0(t_0)$ we will prove the transformation T defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f(\tau, x(\tau))\,d\tau & \text{for } t \geqslant t_0, \\ 0 & \text{for } t < t_0 \end{cases}$$

acts from $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$ and is a contraction.

In fact, for $x(\cdot) \in \mathcal{B}_\rho$ we have that $\|f(t,x(t))\| \leqslant M\varphi(t)$, therefore, putting

$$y(t) = \begin{cases} U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f(\tau, x(\tau))\,d\tau & \text{for } t \geqslant t_0, \\ 0 & \text{for } t < t_0 \end{cases}$$

then, $\|y(t)\| \leqslant Ne^{-\nu(t-t_0)}\|v_0\| + (1+H)NM\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\,d\tau$. It follows from the admissibility of L_∞ that, $y(\cdot) \in L_\infty$ and

$$\|y(\cdot)\|_\infty \leqslant N\|v_0\| + \frac{(1+H)NM}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big).$$

Using now the fact that $\|v_0\| \leqslant \frac{\rho}{2N}$ and

$$\frac{(1+H)N}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big) < \frac{\rho}{2M},$$

we obtain that $\|y(\cdot)\|_\infty \leqslant \rho$. Therefore, the transformation T acts from $\mathcal{B}_\rho$ to $\mathcal{B}_\rho$.

We now estimate

$$\begin{aligned} \|Tx(t) - Tz(t)\| &\leqslant \int_0^\infty \|\mathcal{G}(t,\tau)\|\,\big\|f\big(\tau,x(\tau)\big) - f\big(\tau,z(\tau)\big)\big\|\,d\tau \\ &\leqslant (1+H)N\int_0^\infty e^{-\nu|t-\tau|}\varphi(\tau)\,d\tau\,\|x(\cdot)-z(\cdot)\|_\infty. \end{aligned}$$

Therefore,

$$\|Tx(\cdot) - Tz(\cdot)\|_\infty \leqslant \frac{(1+H)N}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big)\|x(\cdot)-z(\cdot)\|_\infty.$$

Hence, if $\frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1$, then we obtain that $T : \mathcal{B}_\rho \to \mathcal{B}_\rho$ is a contraction with the contraction constant $k = \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)$. Thus, there exists a unique $u(\cdot) \in \mathcal{B}_\rho$ such that $Tu = u$. By definition of T we have that $u(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of Eq. (12) for $t \geqslant t_0$. By Lemma 3.5 and Remark 3.6 we have that $u(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of Eq. (9) for $t \geqslant t_0$. The proof of the estimate (13) can be done by the similar way as in [4, Lemma III.2.2]. We present it here for seek of completeness. Let $u_1(t)$ and $u_2(t)$ be two essentially bounded solutions of Eq. (9) corresponding to different values $v_1, v_2 \in B_{\frac{\rho}{2N}} \cap X_0(t_0)$. Then, we have that

$$u_1(t) - u_2(t) = U(t,t_0)(v_1 - v_2) + \int_{t_0}^\infty \mathcal{G}(t,\tau)\big[f\big(\tau,u_1(\tau)\big) - f\big(\tau,u_2(\tau)\big)\big]\,d\tau \quad \text{for } t \geqslant t_0.$$

This follows that

$$\|u_1(t) - u_2(t)\| \leqslant Ne^{-\nu(t-t_0)}\|v_1 - v_2\| + (1+H)N\int_{t_0}^\infty e^{-\nu|t-\tau|}\varphi(\tau)\|u_1(\tau) - u_2(\tau)\|\,d\tau$$

for $t \geqslant t_0$. Put $\phi(t) = \|u_1(t) - u_2(t)\|$. Then $\operatorname{ess\,sup}_{t\geqslant t_0}\phi(t) < \infty$, and

$$\phi(t) \leqslant Ne^{-\nu(t-t_0)}\|v_1 - v_2\| + (1+H)N\int_{t_0}^\infty e^{-\nu|t-\tau|}\varphi(\tau)\phi(\tau)\,d\tau \quad \text{for } t \geqslant t_0. \tag{14}$$

We will use the cone inequality theorem applying to Banach space $W := L_\infty([t_0,\infty))$ which is the space of real-valued functions defined and essentially bounded on $[t_0,\infty)$ (endowed with the sup-norm denoted by $\|\cdot\|_\infty$) with the cone $\mathcal{K}$ being the set of all (a.e.) nonnegative functions. We then consider the linear operator A defined for $u \in L_\infty([t_0,\infty))$ by

$$(Au)(t) = (1+H)N\int_{t_0}^\infty e^{-\nu|t-\tau|}\varphi(\tau)u(\tau)\,d\tau \quad \text{for } t \geqslant t_0.$$

By the inequalities (6) we have that

$$\sup_{t\geqslant t_0}(Au)(t) = \sup_{t\geqslant t_0}(1+H)N\int_{t_0}^{\infty} e^{-\nu|t-\tau|}\varphi(\tau)u(\tau)\,d\tau$$
$$\leqslant \frac{(1+H)N}{1-e^{-\nu}}\left(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\right)\|u\|_\infty.$$

Therefore, $A \in \mathcal{L}(L_\infty([t_0,\infty)))$ and $\|A\| \leqslant \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1$. Obviously, A leaves the cone $\mathcal{K}$ invariant. The inequality (14) can now be rewritten by

$$\phi \leqslant A\phi + z \quad \text{for } z(t) = Ne^{-\nu(t-t_0)}\|v_1 - v_2\|;\ t \geqslant t_0.$$

Hence, by cone inequality Theorem 2.8 we obtain that $\phi \leqslant \psi$, where ψ is a solution in $\mathcal{L}(L_\infty([t_0,\infty)))$ of the equation $\psi = A\psi + z$ which can be rewritten as

$$\psi(t) = Ne^{-\nu(t-t_0)}\|v_1 - v_2\| + (1+H)N\int_{t_0}^{\infty} e^{-\nu|t-\tau|}\varphi(\tau)\psi(\tau)\,d\tau \quad \text{for } t \geqslant t_0. \tag{15}$$

We now estimate ψ. To that purpose, for

$$0 < \mu < \nu + \ln\left(1 - (1+H)N\left(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\right)\right)$$

we set $w(t) = e^{\mu(t-t_0)}\psi(t)$ for $t \geqslant t_0$. Then, by (15) we obtain that

$$w(t) = Ne^{-(\nu-\mu)(t-t_0)}\|v_1 - v_2\| + (1+H)N\int_{t_0}^{\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)w(\tau)\,d\tau \quad \text{for } t \geqslant t_0. \tag{16}$$

We next consider the linear operator D defined for $u \in L_\infty([t_0,\infty))$ by

$$(Du)(t) = (1+H)N\int_{t_0}^{\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)u(\tau)\,d\tau \quad \text{for } t \geqslant t_0.$$

By the inequalities (6) we have that

$$\sup_{t\geqslant t_0}(Du)(t) = \sup_{t\geqslant t_0} N\int_{t_0}^{\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}\varphi(\tau)u(\tau)\,d\tau$$
$$\leqslant \sup_{t\geqslant t_0} N\int_{t_0}^{\infty} e^{-(\nu-\mu)|t-\tau|}\varphi(\tau)u(\tau)\,d\tau$$
$$\leqslant \frac{(1+H)N}{1-e^{-(\nu-\mu)}}\left(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\right)\|u\|_\infty.$$

Therefore, $D \in \mathcal{L}(L_\infty([t_0,\infty)))$ and $\|D\| \leqslant \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)$. Eq. (16) can now be rewritten by

$$w = Dw + z \quad \text{for } z(t) = Ne^{-(\nu-\mu)(t-t_0)}\|v_1 - v_2\|;\ t \geqslant t_0.$$

Since $\mu < \nu + \ln(1 - (1+H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty))$ we obtain that

$$\|D\| \leqslant \frac{(1+H)N}{1-e^{-(\nu-\mu)}}\left(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\right) < 1.$$

Therefore, the equation $w = Dw + z$ is uniquely solvable in $L_\infty([t_0,\infty))$, and its solution is $w = (I-D)^{-1}z$. Hence, we obtain that

$$\|w\|_\infty = \left\|(I-A)^{-1}z\right\|_\infty \leqslant \left\|(I-A)^{-1}\right\|\|z\|_\infty \leqslant \frac{N}{1-\|A\|}\|v_1 - v_2\|$$
$$\leqslant \frac{N}{1 - \frac{(1+H)N}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}\|v_1 - v_2\| := C_\mu\|v_1 - v_2\|.$$

This yields that

$$w(t) \leqslant C_\mu\|v_1 - v_2\| \quad \text{for } t \geqslant t_0.$$

Hence, $\psi(t) = e^{-\mu(t-t_0)} w(t) \leqslant C_\mu e^{-\mu(t-t_0)} \|v_1 - v_2\|$. Since $\|u_1(t) - u_2(t)\| = \phi(t) \leqslant \psi(t)$, we obtain that

$$\|u_1(t) - u_2(t)\| \leqslant C_\mu e^{-\mu(t-t_0)} \|v_1 - v_2\| \quad \text{for } t \geqslant t_0. \qquad \square$$

We now prove our main result of this section.

Theorem 3.8. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Then, for any $\rho > 0$ and $M > 0$ we have that, if f belongs to the class (M, φ, ρ) with the positive function $\varphi \in E$ satisfying $\frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \min\{1, \frac{\rho}{2M}, \frac{1}{N+1}\}$, then there exists a local stable manifold **S** for the solutions of Eq. (9). Moreover, every two solutions $u_1(t)$, $u_2(t)$ on the manifold **S** attract each other exponentially in the sense that, there exist positive constants μ and C_μ independent of $t_0 \geqslant 0$ such that*

$$\|u_1(t) - u_2(t)\| \leqslant C_\mu e^{-\mu(t-t_0)} \|P(t_0)u_1(t_0) - P(t_0)u_2(t_0)\| \quad \textit{for } t \geqslant t_0. \tag{17}$$

Proof. Since the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy, we have that, for each $t \geqslant 0$ the phase space X splits into the direct sum $X = X_0(t) \oplus X_1(t)$, where $X_0(t) = P(t)X$ and $X_1(t) = \ker P(t)$. Furthermore, since $\sup_{t\geqslant 0}\|P(t)\| < \infty$ we obtain that

$$\inf_{t\in\mathbb{R}_+} Sn\big(X_0(t), X_1(t)\big) := \inf_{t\in\mathbb{R}_+} \inf\big\{\|x_0 + x_1\|:\ x_i \in X_i(t),\ \|x_i\| = 1,\ i = 0, 1\big\} > 0.$$

We now construct the family of Lipschitz continuous mapping $(g_t)_{t\geqslant 0}$ satisfying the conditions of Definition 3.4. To do that, for each $t_0 \geqslant 0$ we define a transformation g_{t_0} by

$$g_{t_0}(y) = \int_{t_0}^{\infty} \mathcal{G}(t_0, s) f\big(s, x(s)\big)\, ds, \tag{18}$$

where $y \in B_{\rho/2N} \cap X_0(t_0)$ and $x(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of Eq. 9 on $[t_0, \infty)$ satisfying $P(t_0)x(t_0) = y$ and $x(t) = 0$, $t < t_0$ (note that the existence and uniqueness of $x(\cdot)$ is obtained in Theorem 3.7). It is clear by definition of Green's function that $g_{t_0}(y) \in X_1(t_0)$.

We next estimate $\|g_{t_0}(y)\|$ by

$$\begin{aligned}\|g_{t_0}(y)\| &\leqslant \int_0^\infty \|\mathcal{G}(t_0, s)\| \|f(s, x(s))\|\, ds \leqslant (1+H)NM \int_0^\infty e^{-|t_0 - s|}\varphi(s)\, ds \\ &\leqslant \frac{(1+H)NM}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big) \leqslant \frac{\rho}{2}.\end{aligned}$$

Hence, we obtain that g_{t_0} is a mapping from $B_{\rho/2N} \cap X_0(t_0)$ to $B_{\rho/2} \cap X_1(t_0)$. We then prove that g_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for y_1 and y_2 belonging to $B_{\rho/2N} \cap X_0(t_0)$ we have

$$\begin{aligned}\|g_{t_0}(y_1) - g_{t_0}(y_2)\| &\leqslant \int_0^\infty \|\mathcal{G}(t_0, s)\| \big\|f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big\|\, ds \\ &\leqslant (1+H)N \int_0^\infty e^{-|t_0 - s|}\varphi(s)\|x_1(s) - x_2(s)\|\, ds \\ &\leqslant \frac{(1+H)N}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big)\|x_1(\cdot) - x_2(\cdot)\|_\infty.\end{aligned} \tag{19}$$

We now estimate $\|x_1(\cdot) - x_2(\cdot)\|_\infty$. Since $x_i(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of Eq. (9) on $[t_0, \infty)$ satisfying $P(t_0)x_i(t_0) = y_i$, $i = 1, 2$, respectively, we have that

$$\begin{aligned}\|x_1(t) - x_2(t)\| &= \left\| U(t, t_0)(y_1 - y_2) + \int_{t_0}^\infty \mathcal{G}(t, \tau)\big(f\big(\tau, x_1(\tau)\big) - f\big(\tau, x_2(\tau)\big)\big)\, d\tau \right\| \\ &\leqslant N\|y_1 - y_2\| + \frac{(1+H)N}{1-e^{-\nu}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big)\|x_1(\cdot) - x_2(\cdot)\|_\infty \quad \text{for all } t \geqslant t_0.\end{aligned}$$

Hence, putting $k = \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1$ we obtain that

$$\|x_1(\cdot) - x_2(\cdot)\|_\infty \leqslant N\|y_1 - y_2\| + k\|x_1(\cdot) - x_2(\cdot)\|_\infty.$$

Therefore,

$$\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty \leqslant \frac{N}{1-k}\|y_1 - y_2\|.$$

Substituting this inequality to (19) we obtain that

$$\left\|g_{t_0}(y_1) - g_{t_0}(y_2)\right\| \leqslant \frac{Nk}{1-k}\|y_1 - y_2\|$$

yielding that g_{t_0} is Lipschitz continuous with the Lipschitz constant $\frac{Nk}{1-k}$ independent of t_0. Therefore, putting $\rho_0 := \frac{\rho}{2N}$, $\rho_1 := \frac{\rho}{2}$ we obtain that the above family of mappings $(g_t)_{t\geqslant 0}$ (here $g_t : B_{\rho_0} \cap X_0(t) \to B_{\rho_1} \cap X_1(t)$) are Lipschitz continuous with the Lipschitz constant $\frac{Nk}{1-k}$ independent of t.

Put $\mathbf{S} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) \mid t \in \mathbb{R}_+,\ x \in B_{\rho/2N} \cap X_0(t)\}$. Then, for each $t_0 \geqslant 0$ we prove that $\mathbf{S}_{t_0} := \{x + g_{t_0}(x)\colon\ (t_0, x + g_{t_0}(x)) \in \mathbf{S}\}$ is homeomorphic to $B_{\rho/2N} \cap X_0(t_0)$. In fact, we define a transformation $H : B_{\rho/2N} \cap X_0(t_0) \to \mathbf{S}_{t_0}$ by $Hy := y + g_{t_0}(y)$ for all $y \in B_{\rho/2N} \cap X_0(t_0)$. Then, applying the Implicit Function Theorem for Lipschitz continuous mapping (see [13, Lemma 2.7], [10,16]) we have that, if Lipschitz constant $q = \frac{Nk}{1-k}$ of g_{t_0} satisfies $\frac{Nk}{1-k} < 1$ (or, equivalently, $k = \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \frac{1}{N+1}$), then H is a homeomorphism. Therefore, the condition (ii) in Definition 3.4 follows. The condition (iii) of Definition 3.4 now follows from Theorem 3.7. Finally, the inequality (17) follows from inequality (13) in Theorem 3.7. □

4. Invariant manifolds for bounded solutions

In this section we shall prove the existence of invariant (global) stable manifolds for solutions of Eq. (9). To obtain such an existence we need the following (global) property of the nonlinear term f.

Definition 4.1. Let E be an admissible Banach function space. Let $\varphi \in E$ be a positive function. A function $f : [0,\infty) \times X \to X$ is said to be φ-Lipschitz if f satisfies

(i) $\|f(t,0)\| = 0$ for a.e. $t \in \mathbb{R}_+$, and
(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ for a.e. $t \in \mathbb{R}_+$ and all $x_1, x_2 \in X$.

Remark 4.2. From (i) and (ii) in the above definition, it follows that, if f is φ-Lipschitz, then $\|f(t,x)\| \leqslant \varphi(t)\|x\|$ for a.e. $t \in \mathbb{R}_+$ and all $x \in X$.

We then give the definition of invariant stable manifolds for the solutions to Eq. (9).

Definition 4.3. A set $\mathbf{S} \subset \mathbb{R}_+ \times X$ is said to be an invariant stable manifold for the solutions of Eq. (9) if for every $t \in \mathbb{R}_+$ the phase space X splits into a direct sum $X = X_0(t) \oplus X_1(t)$ such that

$$\inf_{t\in\mathbb{R}_+} Sn\big(X_0(t), X_1(t)\big) := \inf_{t\in\mathbb{R}_+} \inf\big\{\|x_0 + x_1\|\colon\ x_i \in X_i(t),\ \|x_i\| = 1,\ i = 0,1\big\} > 0,$$

and if there exists family of Lipschitz continuous mappings

$$g_t : X_0(t) \to X_1(t), \quad t \in \mathbb{R}_+,$$

with Lipschitz constants independent of t such that

(i) $\mathbf{S} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) \mid t \in \mathbb{R}_+,\ x \in X_0(t)\}$, and we denote by $\mathbf{S}_t := \{x + g_t(x)\colon\ (t, x + g_t(x)) \in \mathbf{S}\}$,
(ii) $\mathbf{S}_t$ is homeomorphic to $X_0(t)$ for all $t \geqslant 0$,
(iii) to each $x_0 \in \mathbf{S}_{t_0}$ there corresponds one and only one solution $u(t)$ of Eq. (9) on $[t_0,\infty)$ satisfying conditions $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t\geqslant t_0}\|u(t)\| < \infty$,
(iv) $\mathbf{S}$ is invariant under Eq. (9) in the sense that, if u is a solution of Eq. (9) satisfying $u(t_0) = u_0 \in \mathbf{S}_{t_0}$ and $\operatorname{ess\,sup}_{t\geqslant t_0}\|u(t)\| < \infty$, then $u(s) \in \mathbf{S}_s$ for all $s \geqslant t_0$.

By the similar way as done in the proof of Lemma 3.5 we can prove the following lemma which gives the form of bounded solutions of Eq. (9).

Lemma 4.4. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Suppose that φ is a positive function which belongs to E. Let $f : \mathbb{R}_+ \times X \to X$ be φ-Lipschitz. Let $u(t)$ be a solution to Eq.* (9) *such that $\operatorname{ess\,sup}_{t\geqslant t_0} \|u(t)\| < \infty$ for fixed $t_0 \geqslant 0$. Then, for $t \geqslant t_0$ we have that $u(t)$ can be rewritten in the form*

$$u(t) = U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f\big(\tau, u(\tau)\big)\, d\tau \quad \text{for some } v_0 \in X_0(t_0) = P(t_0)X, \tag{20}$$

where $\mathcal{G}(t,\tau)$ is the Green's function defined by equality (10)*.*

Remark 4.5. By straightforward computation we can prove that the converse is also true: a solution of Eq. (20) satisfies Eq. (9) for $t \geqslant t_0$.

Similarly to Theorem 3.7, we can now construct the structure of certain solutions of Eq. (9) in the following theorem using the admissibility.

Theorem 4.6. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Suppose that the function f be φ-Lipschitz with the positive function $\varphi \in E$ satisfying $\frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1$. Then, there corresponds to each $v_0 \in X_0(t_0)$ one and only one solution $u(t)$ of Eq.* (9) *on $[t_0,\infty)$ satisfying the conditions $P(t_0)u(t_0) = v_0$ and $\operatorname{ess\,sup}_{t\geqslant t_0}\|u(t)\| < \infty$. Moreover, the following estimate is valid for any two solutions $u_1(t), u_2(t)$ corresponding to different values $v_1, v_2 \in X_0(t_0)$:*

$$\big\|u_1(t) - u_2(t)\big\| \leqslant C_\mu e^{-\mu(t-t_0)}\|v_1 - v_2\| \quad \text{for } t \geqslant t_0, \tag{21}$$

where μ is a positive number satisfying

$$0 < \mu < \nu + \ln\big(1 - (1+H)N\big(N_1\big\|\Lambda_1 T_1^+\varphi\big\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big)\big), \quad \text{and}$$
$$C_\mu = \frac{N}{1 - \frac{N}{1-e^{-(\nu-\mu)}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 3.7. We just have to replace the ball $\mathcal{B}_\rho$ by the space $L_\infty(\mathbb{R}_+, X)$ itself and consider for each $v_0 \in X_0(t_0)$ the transformation $T : L_\infty(\mathbb{R}_+, X) \to L_\infty(\mathbb{R}_+, X)$ defined by

$$(Tx)(t) = \begin{cases} U(t,t_0)v_0 + \int_{t_0}^{\infty} \mathcal{G}(t,\tau) f(\tau, x(\tau))\, d\tau & \text{for } t \geqslant t_0, \\ 0 & \text{for } t < t_0. \end{cases}$$

Then, by the same way as in the proof of Theorem 3.7 we can prove that T is a contraction mapping. Therefore, the assertion of the theorem follows. □

We now prove the existence of the invariant stable manifold.

Theorem 4.7. *Let the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ and dichotomy constants $N, \nu > 0$. Suppose that the function f be φ-Lipschitz with the positive function $\varphi \in E$ satisfying $\frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \min\{1, \frac{1}{N+1}\}$. Then, there exists an invariant stable manifold* **S** *for the solutions of Eq.* (9)*. Moreover, every two solutions $u_1(t)$, $u_2(t)$ on the manifold* **S** *attract each other exponentially in the sense that, there exist positive constants μ and C_μ independent of $t_0 \geqslant 0$ such that*

$$\big\|u_1(t) - u_2(t)\big\| \leqslant C_\mu e^{-\mu(t-t_0)}\big\|P(t_0)u_1(t_0) - P(t_0)u_2(t_0)\big\| \quad \text{for } t \geqslant t_0. \tag{22}$$

Proof. Since the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ have an exponential dichotomy, we have that, for each $t \geqslant 0$ the phase space X splits into the direct sum $X = X_0(t) \oplus X_1(t)$, where $X_0(t) = P(t)X$ and $X_1(t) = \ker P(t)$. Furthermore, since $\sup_{t\geqslant 0}\|P(t)\| < \infty$ we obtain that

$$\inf_{t\in\mathbb{R}_+} Sn\big(X_0(t), X_1(t)\big) := \inf_{t\in\mathbb{R}_+} \inf\big\{\|x_0 + x_1\|\colon\ x_i \in X_i(t),\ \|x_i\| = 1,\ i = 0,1\big\} > 0.$$

We now construct the family of Lipschitz continuous mapping $(g_t)_{t\geqslant 0}$ satisfying the conditions of Definition 3.4. To do that, for each $t_0 \geqslant 0$ we define a transformation g_{t_0} by

$$g_{t_0}(y) = \int_{t_0}^{\infty} \mathcal{G}(t_0,s) f\big(s, x(s)\big)\, ds, \tag{23}$$

where $y \in X_0(t_0)$ and $x(\cdot)$ is the unique solution in $L_\infty(\mathbb{R}_+, X)$ of Eq. (9) on $[t_0, \infty)$ satisfying $P(t_0)x(t_0) = y$ and $x(t) = 0$, $t < t_0$ (note that the existence and uniqueness of $x(\cdot)$ is obtained in Theorem 4.6). It is clear by definition of Green's function that $g_{t_0}(y) \in X_1(t_0)$.

Hence, we obtain that g_{t_0} is a mapping from $X_0(t_0)$ to $X_1(t_0)$. We then prove that g_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for y_1 and y_2 belonging to $X_0(t_0)$ we have

$$\begin{aligned}\left\|g_{t_0}(y_1) - g_{t_0}(y_2)\right\| &\leqslant \int_0^\infty \left\|\mathcal{G}(t_0, s)\right\| \left\|f\bigl(s, x_1(s)\bigr) - f\bigl(s, x_2(s)\bigr)\right\| ds \\ &\leqslant (1+H)N \int_0^\infty e^{-|t_0-s|}\varphi(s)\left\|x_1(s) - x_2(s)\right\| ds \\ &\leqslant \frac{(1+H)N}{1-e^{-\nu}}\bigl(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\bigr)\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty. \end{aligned} \tag{24}$$

We now estimate $\|x_1(\cdot) - x_2(\cdot)\|_\infty$. Since $x_i(\cdot)$ is the unique solution in $L_\infty(\mathbb{R}_+, X)$ of Eq. (9) on $[t_0, \infty)$ satisfying $P(t_0)x_i(t_0) = y_i$, and $x_i(t) = 0$; $t < t_0$, $i = 1, 2$, respectively, we have that

$$\begin{aligned}\left\|x_1(t) - x_2(t)\right\| &= \left\|U(t, t_0)(y_1 - y_2) + \int_{t_0}^\infty \mathcal{G}(t, \tau)\bigl(f\bigl(\tau, x_1(\tau)\bigr) - f\bigl(\tau, x_2(\tau)\bigr)\bigr) d\tau\right\| \\ &\leqslant N\|y_1 - y_2\| + \frac{(1+H)N}{1-e^{-\nu}}\bigl(N_1\left\|\Lambda_1 T_1^+\varphi\right\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\bigr)\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty \quad \text{for all } t \geqslant t_0.\end{aligned}$$

Hence, putting $k = \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < 1$ we obtain that

$$\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty \leqslant N\|y_1 - y_2\| + k\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty.$$

Therefore,

$$\left\|x_1(\cdot) - x_2(\cdot)\right\|_\infty \leqslant \frac{N}{1-k}\|y_1 - y_2\|.$$

Substituting this inequality to (24) we obtain that

$$\left\|g_{t_0}(y_1) - g_{t_0}(y_2)\right\| \leqslant \frac{Nk}{1-k}\|y_1 - y_2\|$$

yielding that g_{t_0} is Lipschitz continuous with the Lipschitz constant $\frac{Nk}{1-k}$ independent of t_0.

Put $\mathbf{S} = \{(t, x + g_t(x)) \in \mathbb{R}_+ \times (X_0(t) \oplus X_1(t)) \mid t \in \mathbb{R}_+,\ x \in X_0(t)\}$. Then, for each $t_0 \geqslant 0$ we prove that $\mathbf{S}_{t_0} := \{x + g_{t_0}(x)$: $(t_0, x + g_{t_0}(x)) \in \mathbf{S}\}$ is homeomorphic to $X_0(t_0)$. In fact, we define a transformation $\mathbf{D} : X_0(t_0) \to \mathbf{S}_{t_0}$ by $\mathbf{D}y := y + g_{t_0}(y)$ for all $y \in X_0(t_0)$. Then, applying the Implicit Function Theorem for Lipschitz continuous mapping (see [13, Lemma 2.7], [10,16]) we have that, if Lipschitz constant $q = \frac{Nk}{1-k}$ of g_{t_0} satisfies $\frac{Nk}{1-k} < 1$ (or, equivalently, $k = \frac{(1+H)N}{1-e^{-\nu}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \frac{1}{N+1}$), then $\mathbf{D}$ is a homeomorphism. Therefore, the condition (ii) in Definition 4.3 follows. The condition (iii) of Definition 4.3 now follows from Theorem 4.6. We now prove that the condition (iv̈) of Definition 4.3 is satisfied. Indeed, let $u(\cdot)$ be solution in $L_\infty(\mathbb{R}_+, X)$ of Eq. (9) such that $u(t_0) = u_0 \in \mathbf{S}_{t_0}$. Then, by Lemma 4.4 we have that, for $s \geqslant t_0$ the solution $u(s)$ can be rewritten in the form

$$u(s) = U(s, t_0)v_0 + \int_{t_0}^\infty \mathcal{G}(s, \tau) f\bigl(\tau, u(\tau)\bigr) d\tau \quad \text{for some } v_0 \in X_0(t_0) = P(t_0)X, \tag{25}$$

where $\mathcal{G}(s, \tau)$ is the Green's function defined by equality (10).

Putting now $w_s := U(s, t_0)v_0 + \int_{t_0}^s \mathcal{G}(s, \tau) f(\tau, u(\tau)) d\tau$ we obtain that $w_s \in P(s)X$ and

$$u(s) = w_s + \int_s^\infty \mathcal{G}(s, \tau) f\bigl(\tau, u(\tau)\bigr) d\tau. \tag{26}$$

Moreover, for $t \geqslant s$, by straightforward computation using the formula (20) we have that

$$u(t) = U(t, s)w_s + \int_s^\infty \mathcal{G}(t, \tau) f\bigl(\tau, u(\tau)\bigr) d\tau. \tag{27}$$

Now, by (26), (27) and the above definition of g_t we obtain that $u(s) = w_s + g_s w_s$ yielding that $u(s) \in \mathbf{S}_s$ for all $s \geqslant t_0$. Finally, the inequality (22) follows from inequality (21) in Theorem 4.6. □

We illustrate our results by the following example from non-autonomous heat equations with diffusion.

Example 4.8. We consider the problem

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \displaystyle\sum_{k,l=1}^{n} D_k a_{kl}(t,x) D_l u(t,x) + \delta u(t,x) + b e^{-\alpha t}\sin\bigl(u(t,x)\bigr) & \text{for } t \geqslant s \geqslant 0,\ x \in \Omega, \\ \displaystyle\sum_{k,l=1}^{n} n_k(x) a_{kl}(t,x) D_l u(t,x) = 0, \quad t \geqslant s \geqslant 0,\ x \in \partial\Omega, \\ u(s,x) = f(x), \quad x \in \Omega. \end{cases} \tag{28}$$

Here $D_k := \frac{\partial}{\partial x_k}$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ oriented by outer unit normal vectors $n(x)$. The coefficients $a_{k,l}(t,x) \in C_b^{\mu}(\mathbb{R}_+, L_\infty(\Omega))$, $\mu > \frac{1}{2}$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^{n} a_{kl}(t,x) v_k v_l \geqslant \eta |v|^2 \quad \text{for a.e. } x \in \Omega \text{ and some constant } \eta > 0.$$

Finally, the real constants $\alpha > 0$ and b are fixed, and the constant δ is defined by

$$\delta := -\frac{1}{2}\eta\lambda,$$

where $\lambda < 0$ denotes the largest eigenvalue of Neumann Laplacian Δ_N on Ω. We now choose the Hilbert space $X = L_2(\Omega)$ and define the operators $C(t)$ via the standard scalar product in X as

$$\bigl(C(t)f, g\bigr) = -\sum_{k,l=1}^{n} \int_{\Omega} a_{kl} D_k f(x)(t,x) \overline{D_l g(t,x)}\, dx$$

with $D(C(t)) = \{f \in W^{2,2}(\Omega):\ \sum_{k,l}^{n} n_k(x) a_{kl}(t,x) D_l f(x) = 0,\ x \in \partial\Omega\}$. We then write the problem (28) as an abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t,\cdot) = A(t)u(t,\cdot) + F\bigl(t, u(t,\cdot)\bigr), \quad t \geqslant s \geqslant 0, \\ u(s,\cdot) = f \in X, \end{cases}$$

where $A(t) := C(t) + \delta$ and $F : \mathbb{R}_+ \times X \to X$ defined by $F(t,f)(x) := b e^{-\alpha t}\sin(f(x))$ for $(t,f) \in \mathbb{R}_+ \times X$.

By Schnaubelt [20, Chapt. 2, Theorem 2.8, Example 2.3], we have that the operators $A(t)$ generate an evolution family having an exponential dichotomy with the dichotomy constants N and ν provided that the Hölder constants of $a_{k,l}$ are sufficiently small. Also, the dichotomy projections $P(t)$, $t \geqslant 0$, satisfy $\sup_{t\geqslant 0}\|P(t)\| \leqslant N$.

We now easily see that F is φ-Lipschitz, where $\varphi(t) := |b|e^{-\alpha t}$ for $t \geqslant 0$. Clearly, $\varphi \in E$. To be concrete, we can take, e.g., $E = L_p(\mathbb{R}_+)$; $1 \leqslant p \leqslant \infty$. In this space, the constants N_1 and N_2 in Definition 2.4 are defined by $N_1 = N_2 = 1$. Also, we have

$$\Lambda_1\varphi(t) = \int_{t}^{t+1} \varphi(\tau)\, d\tau \quad \text{and} \quad \Lambda_1 T_1^{+}\varphi(t) = \int_{(t-1)_+}^{t} \varphi(\tau)\, d\tau,$$

where $(t-1)_+ := \max\{0, t-1\}$. Hence,

$$\bigl\|\Lambda_1 T_1^{+}\varphi\bigr\|_\infty = \|\Lambda_1\varphi\|_\infty = \frac{|b|(1-e^{-\alpha})}{\alpha}.$$

By Theorem 4.7 we then obtain that, if

$$\frac{|b|(1-e^{-\alpha})}{\alpha} < \min\left\{\frac{(1-e^{-\nu})}{2N(N+1)}, \frac{(1-e^{-\nu})}{2N(N+1)^2}\right\},$$

then there is an invariant stable manifold **S** for mild solutions of Eq. (28).

We note that if we replace the function F in the above equation by the other function G which belongs to class (M, φ, ρ) for some positive constants M, ρ, i.e., G is locally φ-Lipschitz (see Definition 3.2), then we obtain the existence of a local stable manifold for mild solutions of Eq. (28) (see Theorem 3.8).

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