Brief paper

Robust stabilization of uncertain descriptor fractional-order systems☆

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ABSTRACT

This paper presents sufficient conditions for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional order α satisfying $0 < \alpha < 2$. The results are obtained in terms of linear matrix inequalities. The parameter uncertainties are assumed to be time-invariant and norm-bounded appearing in the state matrix. A necessary and sufficient condition for the normalization of uncertain descriptor fractional-order systems is given via linear matrix inequality (LMI) formulation. The state feedback control to robustly stabilize such uncertain descriptor fractional-order systems with the fractional order α belonging to $0 < \alpha < 2$ is derived. Two numerical examples are given to demonstrate the applicability of the proposed approach.

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1. Introduction

Recently, fractional-order systems have been studied by many authors in engineering science from an application point of view (see Podlubny, 1999, Hilfer, 2001, Kilbas, Srivastava, & Trujillo, 2006 and references therein). Many systems can be described with the help of fractional derivatives: electromagnetic systems (Engheta, 1996), dielectric polarization (Sun, Abdelwahad, & Onaral, 1984), viscoelastic systems (Bagley & Calico, 1991; Rossikhin & Shitikova, 1997).

Moreover, descriptor systems have been of great interest in the specialized literature because they have many applications (Dai, 1989) in electrical circuit networks, robotics and economics. Intuitively, a descriptor state space description of linear systems is more general than a conventional state space description.

Descriptor systems are described by a mixture of differential equations and algebraic equations. The method of transforming these systems into normal ones is called a normalization or regularization (Dai, 1988). The question of stability is very important in control theory. For descriptor fractional-order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded-input bounded-output stability, internal stability, etc. So far, very few works exist for the stability and stabilization issue for descriptor fractional-order systems. In N'Doye, Zasadzinski, Darouach, and Radhy (2010), for the stabilization of such descriptor fractional-order systems with fractional commensurate orders α, $1 < \alpha < 2$, an LMI formulation is used for, and only for the nominal stabilization case.

In this paper, we consider the problem of the robust asymptotical stabilization for uncertain descriptor fractional-order systems. The structure of the uncertainty has been extensively used in many papers when dealing with the problem of robust stabilization for uncertain standard state-space systems in both continuous and discrete time frameworks (Xie & de Souza, 1990; Xu, Yang, Niu, & Lam, 2001). However, concerning robust stability and stabilization for uncertain descriptor fractional-order systems, such a structure of the uncertainty has not yet been introduced. The descriptor fractional-order system under consideration is subject to unstructured time-invariant parameter uncertainties in the state matrix.

A necessary and sufficient condition for the normalization of descriptor fractional-order systems in an LMI formulation is given. A novel form of normalizing stabilizing controllers that involve the action of derivative and a state feedbacks is introduced. The use of a proportional and derivative feedback has a strong engineering motivation and so far there have been many research papers published to address the importance and the engineering motivation of proportional and derivative state feedback and proportional and derivative output feedback (Bunse-Gerstner, Byers, Mehrmann, & Nichols, 1999; Lin, Wang, & Lee, 2005). With the above motivation, the important issues of stability check and stabilization of uncertain descriptor fractional-order systems are further investigated in this paper.

First, we normalize the descriptor fractional-order systems by applying a derivative controller and second, a state feedback controller is given to achieve the robust asymptotical stabilization of the obtained normalized fractional-order systems. The problem of asymptotical stability and stabilization problems for descriptor fractional-order are still open problems to our best knowledge.

This paper is organized as follows. In Section 2, we provide some preliminaries results on the fractional derivative, the stability conditions of the fractional-order systems, the linear algebra and the matrix theory. In Section 3, the problem of normalization by full state derivative controller is firstly studied. The condition of existence of this controller is formulated in terms of linear matrix inequalities (LMIs). Then sufficient robust stabilization of uncertain descriptor fractional-order systems conditions are derived in terms of linear matrix inequalities. Finally, two illustrative examples are presented to illustrate of our proposed results.

Notations. M^T is the transpose of M, Sym$\{X\}$ is used to denote $X^T + X$, $\otimes$ stands for the Kronecker product and D^α represents initialized αth order differintegration. Let $A \in \mathbb{C}^{n \times n}$ be a complex matrix, then $A^* = \overline{A}^T$ and $\overline{A}$ denotes the complex conjugate of A. The null space of matrix A is defined by $\mathcal{N}(A) = \{x \in \mathcal{F}^n$ such that $Ax = 0\}$, where $\mathcal{F}$ represents either $\mathbb{R}$ or $\mathbb{C}$.

2. Preliminary results

In this section, we present some preliminaries results on the fractional derivative systems, the linear algebra and matrix theory which will be used in the sequel of this paper. Formulations of noninteger-order derivatives fall into two main classes: the Riemann–Liouville derivative and the Grûnward–Letnikov derivative, on one hand, defined as (Podlubny, 1999)

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,$$
$$n - 1 \leqslant \alpha < n \tag{1}$$

or the Caputo derivative on the other, defined as (Podlubny, 2002),

$$D^\alpha f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{\frac{d^n f(\tau)}{dt^n}}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n - 1 \leqslant \alpha < n \tag{2}$$

with $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^+$, where $\Gamma(.)$ is the Gamma function and is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

The physical interpretation of the fractional derivatives and the solution of fractional differential equations are given in Podlubny (2002). Here and throughout the paper, only the Caputo definition is used since this Laplace transform allows using initial values of classical integer-order derivatives with clear physical interpretations. In the rest of this paper, D^α is used to denote the Caputo fractional derivative of order α.

Now, consider the following linear fractional-order systems

$$\begin{cases} D^\alpha x(t) = Ax(t) \\ x(0) = x_0 \end{cases} \quad 0 < \alpha < 2 \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and matrix $A \in \mathbb{R}^{n \times n}$.

It has been shown that system (3) is stable if the following condition is satisfied (Matignon, 1996, 1998) for $0 < \alpha \leqslant 1$, (Sabatier, Moze, & Farges, 2008) for $1 < \alpha < 2$

$$|\arg(\mathrm{spec}(A))| > \alpha \frac{\pi}{2} \tag{4}$$

where spec(A) represents the eigenvalues of matrix A.

The necessary and sufficient LMIs conditions to satisfy condition (4) when the fractional-order α belonging to $0 < \alpha < 2$ are given in the two following lemmas.

Lemma 1 (*Chilali, Gahinet, & Apkarian, 1999; Sabatier et al., 2008; Sabatier, Moze, & Farges, 2010*). *Let $A \in \mathbb{R}^{n \times n}$, then $|\arg(\mathrm{spec}(A))| > \alpha \frac{\pi}{2}$, where $1 \leqslant \alpha < 2$, if and only if there exits a matrix $P_0 = P_0^T > 0$ such that*

$$\begin{bmatrix} (AP_0 + P_0 A^T) \sin\theta & (AP_0 - P_0 A^T) \cos\theta \\ (P_0 A^T - AP_0) \cos\theta & (AP_0 + P_0 A^T) \sin\theta \end{bmatrix} < 0$$

where $\theta = \pi - \alpha \frac{\pi}{2}$. $\square$

Lemma 2 (*Lu & Chen, 2010*). *Let $A \in \mathbb{R}^{n \times n}$ and $0 < \alpha < 1$. The fractional-order system $D^\alpha x(t) = Ax(t)$ is asymptotically stable (i.e. $|\arg(\mathrm{spec}(A))| > \alpha \frac{\pi}{2}$) if and only if there exist two real symmetric matrices $P_{k1} \in \mathbb{R}^{n \times n}$, $k = 1, 2$, and two skew-symmetric matrices $P_{k2} \in \mathbb{R}^{n \times n}$, $k = 1, 2$, such that*

$$\sum_{i=1}^2 \sum_{j=1}^2 \mathrm{Sym}\{\Gamma_{ij} \otimes (AP_{ij})\} < 0 \tag{5}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ -P_{12} & P_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} P_{21} & P_{22} \\ -P_{22} & P_{21} \end{bmatrix} > 0, \tag{6}$$

where

$$\Gamma_{11} = \begin{bmatrix} \sin\left(\alpha\frac{\pi}{2}\right) & -\cos\left(\alpha\frac{\pi}{2}\right) \\ \cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \end{bmatrix},$$

$$\Gamma_{12} = \begin{bmatrix} \cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ -\sin\left(\alpha\frac{\pi}{2}\right) & \cos\left(\alpha\frac{\pi}{2}\right) \end{bmatrix},$$

$$\Gamma_{21} = \begin{bmatrix} \sin\left(\alpha\frac{\pi}{2}\right) & \cos\left(\alpha\frac{\pi}{2}\right) \\ -\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \end{bmatrix},$$

$$\Gamma_{22} = \begin{bmatrix} -\cos\left(\alpha\frac{\pi}{2}\right) & \sin\left(\alpha\frac{\pi}{2}\right) \\ -\sin\left(\alpha\frac{\pi}{2}\right) & -\cos\left(\alpha\frac{\pi}{2}\right) \end{bmatrix}. \quad \square \tag{7}$$

Notice that the conditions given in Lemma 2 are equivalent to those given in Sabatier et al. (2010) and Farges, Moze, and Sabatier (2010).

To prove the main results in the next section, we need the following result.

Lemma 3. *Let $A \in \mathbb{C}^{n \times n}$ be a complex matrix. Then A is nonsingular if and only if there exists a nonsingular matrix $X \in \mathbb{C}^{n \times n}$ such that*

$$AX + X^* A^* < 0. \quad \square \tag{8}$$

Proof. Assume that there exists a nonsingular matrix $X \in \mathbb{C}^{n \times n}$ such that (8) is satisfied and A be singular. Then, for any $x \in \mathcal{N}(A^*)$ we have

$$x^*(AX + X^*A^*)x = 0,$$

which is in contradiction with (8). This concludes that (8) is a sufficient condition for matrix A to be nonsingular.

Now, to prove the necessity, assume that $\det(A) \neq 0$, then from Skelton, Iwasaki, and Grigoriadis (1998), there exists a matrix $X \in \mathbb{C}^{n \times n}$ such that (8) is satisfied. To prove that X is nonsingular, it is sufficient to use the same reasoning as in the sufficiency. □

The following lemma proved in Khargonakar, Petersen, and Zhou (1990), generally used for the study of uncertain systems, will be used in the next section.

Lemma 4 (*Khargonakar et al., 1990*)**.** *For any matrices X and Y with appropriate dimensions, we have*

$$X^T Y + Y^T X \leqslant \delta X^T X + \delta^{-1} Y^T Y \tag{9}$$

for any $\delta > 0$. □

3. Robust stabilization of uncertain descriptor fractional-order systems

Consider the following uncertain descriptor fractional-order systems

$$\begin{cases} E D^{\alpha} x(t) = (A + \Delta_A)x(t) + Bu(t) \\ x(0) = x_0 \end{cases} \quad 0 < \alpha < 2 \tag{10}$$

where $x(t) \in \mathbb{R}^n$ is the semi-state vector, $u(t) \in \mathbb{R}^m$ is the control input. Matrix $E \in \mathbb{R}^{n \times n}$ is a singular square matrix and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ are constant matrices. Δ_A is time-invariant matrix representing a norm-bounded parameter uncertainty, and is assumed to be of the following form

$$\Delta_A = M_A \Delta N_A \tag{11}$$

where M_A and N_A are known real constant matrices of appropriate dimensions, and the uncertain matrix Δ satisfies

$$\Delta \Delta^T \leqslant I. \tag{12}$$

We consider the feedback control for system (10) in the following form

$$u(t) = -L D^{\alpha} x(t) + K x(t), \tag{13}$$

where $L \in \mathbb{R}^{m \times n}$ and $K \in \mathbb{R}^{m \times n}$ are gain matrices which must be determined such that system (10) is normalized and the obtained normalized system is asymptotically stable.

In this case, the closed-loop system becomes

$$(E + BL)D^{\alpha} x(t) = (A + \Delta_A + BK)x(t). \tag{14}$$

Now, we recall from Dai (1989) the definition of a normalizable system.

The following section will be devoted to the determination of the gain matrix L such that system (10) is normalized.

3.1. Normalization of uncertain descriptor fractional-order systems

Let us consider the closed-loop system (14). As in Dai (1989), we make the following lemma which generalizes well known results

of "integer-order" singular systems to fractional-order singular systems.

Lemma 5. *System (10) is normalizable if and only if*

$$\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n. \quad \square$$

The following theorem gives the conditions for the existence, in the LMI formulation of the gain matrix L such that system (10) is normalizable.

Theorem 1. *System (10) is normalizable if and only if there exist a nonsingular matrix P and a matrix Y such that the following LMI*

$$EP + BY + P^T E^T + Y^T B^T < 0 \tag{15}$$

is satisfied. In this case, the gain matrix L is given by

$$L = YP^{-1}. \quad \square$$

Proof. System (10) is normalizable if and only if there exists a gain matrix L such that the real matrix $(E + BL)$ is nonsingular. By applying the result of Lemma 3 to $(E + BL)$ we obtain, the following condition, there exists a gain matrix L if and only if there exists a nonsingular matrix P such that

$$[(E + BL)P + P^T (E + BL)^T] < 0.$$

The above inequality is equivalent to the LMI (15) by taking $Y = LP$. □

In the sequel of this paper, we assume that system (10) is normalizable, *i.e.* rank $\begin{bmatrix} E & B \end{bmatrix} = n$, in this case the gain matrix L can be determined from the result of Theorem 1.

Remark 1. The normalization of system (10) uses the derivative $D^{\alpha} x(t)$, see (13). In the standard systems it corresponds to $\dot{x}(t)$ and can be accessible to the control feedback, it can correspond to the velocity of the system for example. In the fractional systems this derivative can be deduced from the available semi-state $x(t)$ by using a continuous filter or an existing approximation function, presented in Matlab N-integer Toolbox (see Petráš, Podlubny, O'Leary, Dorcák, & Vinagre, 2002; Valério, 2005; Vinagre, Podlubny, Hernández, & Feliu, 2000). □

The following section will be devoted to the design of the gain matrix K such that the obtained normalized system is asymptotically stable.

3.2. Robust stabilization of the normalized fractional-order systems

In this section, the design of the gain matrix K such that the feedback (13) asymptotically stabilizes the uncertain descriptor fractional-order system (10) is investigated. Based on the result of Lemma 5 we obtain the following normalized system

$$D^{\alpha} x(t) = (A_1 + B_1 K + E_1 \Delta_A)x(t) \tag{16}$$

where $E_1 = (E + BL)^{-1}$, $A_1 = E_1 A$ and $B_1 = E_1 B$ stand for the new resulting nominal matrices. The design of the gain matrix K which robustly stabilize the descriptor fractional-order system (10) for the fractional order α belonging to $1 \leqslant \alpha < 2$ and $0 < \alpha < 1$ are derived.

3.2.1. Case one: $1 \leqslant \alpha < 2$

In this case, the robust asymptotical stabilization of the uncertain descriptor fractional-order system (10) is based on the Lemma 1. It is derived from the robust $\mathcal{D}$-stability in a given LMI

region (see Chilali & Gahinet, 1996; Chilali et al., 1999; Peaucelle, Arzelier, Bachelier, & Bernussou, 2000).

The following theorem gives a sufficient condition to design the gain matrix K.

Theorem 2. *Assume that (10) is normalizable, then there exists a gain matrix K such that the uncertain descriptor fractional-order system (10) with fractional-order $1 \leqslant \alpha < 2$ controlled by the control (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m \times n}$, $P_0 = P_0^T > 0 \in \mathbb{R}^{n \times n}$ and a real scalar $\delta > 0$, such that*

$$\begin{bmatrix} \Omega_{11} & \bullet & \bullet & \bullet \\ \Omega_{21} & \Omega_{22} & \bullet & \bullet \\ N_A P_0 & 0 & -\delta I & \bullet \\ 0 & N_A P_0 & 0 & -\delta I \end{bmatrix} < 0, \qquad (17)$$

where

$$\begin{aligned} \Omega_{11} &= \Omega_{22} \\ &= (A_1 P_0 + P_0 A_1^T + B_1 X + X^T B_1^T) \sin\theta + \delta E_1 M_A (E_1 M_A)^T, \end{aligned}$$

$$\Omega_{21} = (P_0 A_1^T - A_1 P_0 + X^T B_1^T - B_1 X) \cos\theta,$$

with $\theta = \pi - \alpha \frac{\pi}{2}$ and matrices P and Y are given by LMI (15).
Moreover, the gain matrix K is given by

$$K = X P_0^{-1}. \quad \Box \qquad (18)$$

3.2.2. Case two: $0 < \alpha < 1$

In this case, a sufficient condition is derived for robust asymptotical stabilization of the uncertain descriptor fractional-order systems (10) in terms of linear matrix inequalities (LMIs).

Theorem 3. *Assume that (10) is normalizable, then there exists a gain matrix K such that the uncertain descriptor fractional-order system (10) with fractional-order $0 < \alpha < 1$ controlled by the control (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m \times n}$, $P_0 = P_0^T > 0 \in \mathbb{R}^{n \times n}$ and two real scalars $\delta_i > 0$ $(i = 1, 2)$, such that*

$$\begin{bmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{bmatrix} < 0 \qquad (19)$$

where

$$\begin{aligned} W_{11} &= \sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (A_1 P_0 + B_1 X)\} \\ &\quad + \sum_{i=1}^{2} \delta_i \{I_2 \otimes (E_1 M_A)(E_1 M_A)^T\} \end{aligned}$$

$$W_{12} = \begin{bmatrix} I_2 \otimes (N_A P_0)^T & I_2 \otimes (N_A P_0)^T \end{bmatrix}$$
$$W_{22} = -\text{diag}(\delta_1, \delta_2) \otimes I_{2n}$$

Γ_{i1} $(i = 1, 2)$ satisfy (7) and matrices P and Y are given by LMI (15).
Moreover, the robustly asymptotically stabilizing state-feedback gain matrix is given by

$$K = X P_0^{-1}. \quad \Box \qquad (20)$$

Proof. Under the assumption that system (10) is normalizable from Theorem 1, there exits a gain matrix L such that system (10) can be written in the form (16), in this case the matrix K can be determined from the stability of system (16).

It follows from Lemma 2 that $|\arg(\text{spec}(\bar{A}))| > \alpha \frac{\pi}{2}$ is equivalent to

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \text{Sym}\{\Gamma_{ij} \otimes (\bar{A} P_{ij})\} < 0 \qquad (21)$$

where $\bar{A} = A_1 + B_1 K + E_1 \Delta_A$ and Γ_{ij} $(i, j = 1, 2)$ satisfy (7).

By setting $P_{11} = P_{21} = P_0$ et $P_{12} = P_{22} = 0$ in (21), one can conclude that

$$\text{Sym}\{\Gamma_{11} \otimes (\bar{A} P_0)\} + \text{Sym}\{\Gamma_{21} \otimes (\bar{A} P_0)\} < 0. \qquad (22)$$

Suppose that there exist matrices $X \in \mathbb{R}^{m \times n}$ and $P_0 = P_0^T > 0 \in \mathbb{R}^{n \times n}$ such that

$$\sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (\bar{A} P_0)\} < 0. \qquad (23)$$

Substituting $\bar{A} = A_1 + B_1 K + E_1 \Delta_A$ in (23) with $K = X P_0^{-1}$, we obtain

$$\begin{aligned} &\sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (A_1 P_0 + B_1 X)\} \\ &\quad + \sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (E_1 \Delta_A P_0)\} < 0. \end{aligned} \qquad (24)$$

Note that $\Delta \Delta^T \leqslant I$, this leads to

$$(I_2 \otimes \Delta)(I_2 \otimes \Delta)^T = (I_2 \otimes \Delta)(I_2 \otimes \Delta^T) = (I_2 \otimes \Delta \Delta^T) < I. \qquad (25)$$

Note that $\Gamma_{ij} \Gamma_{ij}^T (i = 1, 2) = I_2$, it follows from (25) and Lemma 4 that for any real scalar $\delta > 0$

$$\begin{aligned} &\sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (E_1 \Delta_A P_0)\} \\ &= \sum_{i=1}^{2} \text{Sym}\{(\Gamma_{i1} \otimes (E_1 M_A))(I_2 \otimes \Delta)(I_2 \otimes N_A P_0)\} \\ &\leqslant \sum_{i=1}^{2} \delta_i (\Gamma_{i1} \otimes (E_1 M_A))(I_2 \otimes \Delta)(I_2 \otimes \Delta)^T (\Gamma_{i1} \otimes (E_1 M_A))^T \\ &\quad + \sum_{i=1}^{2} \delta_i^{-1} (I_2 \otimes N_A P_0)^T (I_2 \otimes N_A P_0). \end{aligned} \qquad (26)$$

Using (25), we obtain

$$\begin{aligned} &\sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (E_1 \Delta_A P_0)\} \leqslant \sum_{i=1}^{2} \delta_i \{I_2 \otimes (E_1 M_A)(E_1 M_A)^T\} \\ &\quad + \sum_{i=1}^{2} \delta_i^{-1} (I_2 \otimes N_A P_0)^T (I_2 \otimes N_A P_0). \end{aligned} \qquad (27)$$

Substituting (27) into (24), we have

$$\begin{aligned} &\sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (\bar{A} P_0)\} \leqslant \sum_{i=1}^{2} \text{Sym}\{\Gamma_{i1} \otimes (A_1 P_0 + B_1 X)\} \\ &\quad + \sum_{i=1}^{2} \delta_i \{I_2 \otimes (E_1 M_A)(E_1 M_A)^T\} \\ &\quad + \sum_{i=1}^{2} \delta_i^{-1} (I_2 \otimes N_A P_0)^T (I_2 \otimes N_A P_0). \end{aligned} \qquad (28)$$

Inequality (28) is equivalent to (19) by the well-known Schur complement (Boyd, El Ghaoui, Féron, & Balakrishnan, 1994). This ends the proof. $\quad \Box$

Remark 2. The robust asymptotical stability is a particular case of the robust asymptotical stabilization. In fact, to obtain the stability of the uncertain open-loop system

$$D^\alpha x(t) = (A_1 + \Delta_A) x(t), \qquad (29)$$

for the fractional order α belonging to $0 < \alpha < 2$, it is sufficient to take $K = 0$ or $X = 0$ and $E_1 = I$ in Theorems 2 and 3. $\quad \Box$

4. Numerical examples

In this section, we provide two numerical examples to illustrate the applicability of the proposed method.

4.1. Example 1

Consider the robust stabilization of the following uncertain descriptor fractional-order system described in (10) with parameters as follows

$$E = \begin{bmatrix} 1 & 0 & 0.5 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 2.4 & 0.2 & 1.2 \\ 4 & 1.5 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 4 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},$$

$$M_A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0.3 & 0.4 \\ 0 & 0.2 & 0 \end{bmatrix}, \qquad N_A = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0.4 \\ 0 & 0.1 & 0 \end{bmatrix},$$

where $\alpha = 1.23$.

It is easy to see that the nominal descriptor fractional-order system is not regular and unstable. The purpose of this example is the design of a state feedback control law such that the resulting normalized fractional-order system is asymptotically stable.

The solution of the LMI (15) with respect to P and Y gives

$$P = 10^8 \times \begin{bmatrix} 0.2613 & -1.5587 & 1.0635 \\ -1.5587 & -1.2010 & -0.7272 \\ 1.0635 & -0.7272 & 0.4664 \end{bmatrix},$$

$$Y = 10^8 \times \begin{bmatrix} -1.4992 & 0.7485 & 0.3345 \\ 0.6535 & -0.2534 & -2.4425 \end{bmatrix},$$

and the gain matrix L is given by

$$L = YP^{-1} = \begin{bmatrix} 1.0224 & -0.5004 & -2.3943 \\ -2.8015 & 1.6202 & 3.6775 \end{bmatrix}.$$

This means that the system can be normalized by the full state controller. Using the full state derivative controller, we obtain the new matrices

$$A_1 = \begin{bmatrix} 0.9963 & 0.4454 & 0.4981 \\ 1.9559 & 0.5388 & 0.9780 \\ -0.1605 & 0.1136 & -0.0802 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.3541 & -0.3010 \\ 1.5889 & 0.9091 \\ -1.0030 & -0.3770 \end{bmatrix}.$$

A feasible solution of LMI (17) is as follows

$$P_0 = \begin{bmatrix} 27.6967 & -6.7669 & -14.8319 \\ -6.7669 & 9.7767 & 1.0669 \\ -14.8319 & 1.0669 & 11.3889 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.0118 & 0.0004 & -0.0023 \\ -37.6733 & -2.1008 & 18.7534 \end{bmatrix}, \quad \delta = 1.784.$$

Finally, the asymptotically stabilizing state-feedback gain is obtained as

$$K = XP_0^{-1} = \begin{bmatrix} 0.0017 & 0.0010 & 0.0019 \\ -2.8547 & -1.9850 & -1.8851 \end{bmatrix}.$$

The state responses of the selected system when

$$\Delta = \begin{bmatrix} \cos(0.8) & 0 & 0 \\ 0 & e^{-0.8} & 0 \\ 0 & 0 & \sin(0.1) \end{bmatrix}$$
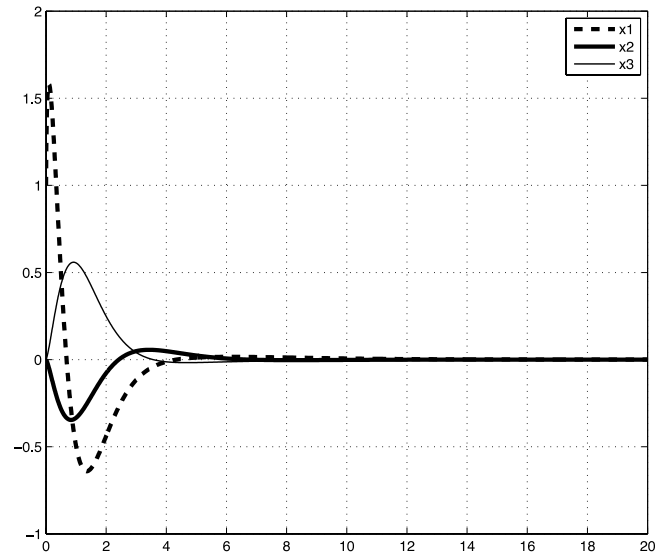

Fig. 1. State responses of the selected system in Example-1 with fractional order $\alpha = 1.23$.

is shown in Fig. 1, which shows that it is asymptotically stable and its states converge to zero.

4.2. Example 2

Consider the robust stabilization of the following uncertain fractional-order system described in (10) for the fractional-order α belonging to $0 < \alpha < 1$ and with parameters as follows

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad M_A = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.32 \end{bmatrix},$$

$$N_A = \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.21 \end{bmatrix} \quad \text{where } \alpha = 0.8.$$

It is easy to show that the pair (E, A) is unstable. The purpose of this example is the design of a state feedback control law such that the resulting normalized fractional-order system is asymptotically stable.

The solution of the LMI (15) with respect to P and Y gives

$$P = 10^8 \times \begin{bmatrix} -1.2834 & 1.9964 \\ 1.9964 & -3.2084 \end{bmatrix},$$

$$Y = 10^8 \times \begin{bmatrix} -1.5686 & -1.2121 \end{bmatrix},$$

and the gain matrix L is given by

$$L = YP^{-1} = \begin{bmatrix} -56.4382 & 35.4960 \end{bmatrix}.$$

This means that the system can be normalized by the full state controller. Using the full state derivative controller, we obtain the new matrices

$$A_1 = \begin{bmatrix} 0.0134 & 0.2443 \\ 0.0067 & -0.3778 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0.0134 \\ 0.0067 \end{bmatrix}.$$

A feasible solution of LMI (17) is as follows

$$P_0 = \begin{bmatrix} 0.5347 & -0.3264 \\ -0.3264 & 0.7377 \end{bmatrix}, \qquad X = \begin{bmatrix} -5.3942 & -6.1060 \end{bmatrix},$$

$\delta_1 = 1.0467$ and $\delta_2 = 1.2085$.

From Theorem 2, the asymptotically stabilizing state-feedback gain is obtained as

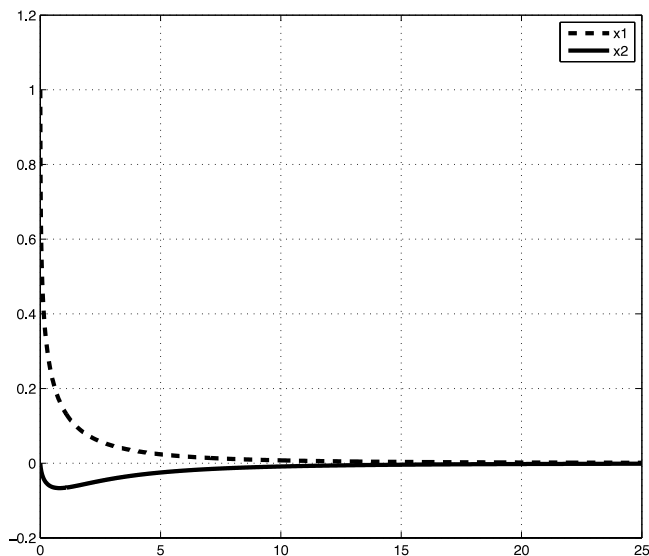$$K = XP_0^{-1} = \begin{bmatrix} -60.2017 & -40.3963 \end{bmatrix}.$$

Fig. 2. State responses of the selected system in Example-2 with fractional order $\alpha = 0.8$.

The state responses of the selected system when

$$\Delta = \begin{bmatrix} e^{-0.8} & 0 \\ 0 & \sin(0.1) \end{bmatrix}$$

is shown in Fig. 2, which shows that it is asymptotically stable and its states converge to zero.

5. Conclusion

In this paper, the robust stabilization of uncertain descriptor fractional-order systems for the fractional-order α belonging to $0 < \alpha < 2$ with parameter uncertainties in the state matrix have been studied. The problem of normalization of descriptor fractional-order systems by derivative controller has been proposed. A necessary and sufficient condition for the existence of such normalizing feedback is given in an LMI formulation. Based on this, the robust asymptotical stabilization of uncertain descriptor fractional-order systems has been solved by the feedback control design. Two illustrative examples have shown the effectiveness of our results. Our results are based on the fractional derivative of the state. An extension of these results to the robust stabilization of descriptor fractional-order systems without using the concept of the normalization is under study.

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