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Admissibly inertial manifolds for a class of semi-linear evolution equations

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ABSTRACT

Consider the semi-linear evolution equation $\frac{du(t)}{dt} + Au(t) = f(t, u)$. We prove the existence of a new class of inertial manifolds called *admissibly inertial manifolds* for this equation. These manifolds are constituted by trajectories of the solutions belonging to rescaledly admissible function spaces which contain wide classes of function spaces like weighted L_p-spaces, the Lorentz spaces $L_{p,q}$ and many other rescaling function spaces occurring in interpolation theory. The existence of these manifolds is obtained in the case that the partial differential operator A is positive definite and self-adjoint with a discrete spectrum, and the nonlinear forcing term f satisfies the φ-Lipschitz conditions on the domain $D(A^\theta)$, $0 \leqslant \theta < 1$, *i.e.*, $\|f(t,x) - f(t,y)\| \leqslant \varphi(t)\|A^\theta(x-y)\|$ and $\|f(t,x)\| \leqslant \varphi(t)(1 + \|A^\theta x\|)$ for $\varphi(\cdot)$ being a real and positive function which belongs to certain classes of admissible function spaces.

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1. Introduction

Consider the semi-linear evolution equation of the form

$$\frac{dx}{dt} + Ax(t) = f\big(t, x(t)\big), \quad t > s, \; x(s) = x_s, \; s \in \mathbb{R}, \tag{1.1}$$

where A is in general an unbounded linear operator on a Hilbert space X and $f : \mathbb{R} \times X_\theta \to X$ is a nonlinear operator with $X_\theta := D(A^\theta)$ being the domain of the fractional power A^θ for $0 \leqslant \theta < 1$.

In case X is finite-dimensional, A is a matrix, and $\theta = 0$, Hadamard [11], Perron [23,24], Bogoliubov and Mitropolsky [2,3] had found conditions for this equation to have an integral manifold (e.g., a stable, unstable, or center manifold) which was constituted by bounded (or rescaledly bounded) solutions on the positive (or negative) half-line. We refer the reader to Daleckii and Krein [9] for the extension of these results to the case of bounded coefficients acting on Banach spaces, and to Henry [12] for the case of unbounded coefficients (see also [1,5,6,14,21,26] and references therein for recent contributions to the theory and application of integral manifolds).

The methods and results on integral manifolds have been used to derive the notion of inertial manifolds and to obtain their existence and properties (see [7,8,18,26] and references therein). The importance of the discovery of inertial manifolds is that such manifolds are of finite dimensions and exponentially attract all solutions of the evolution equation under consideration. This allows to apply the reduction principles to consider the asymptotic behavior of the partial differential equation by determining the structures of its induced solutions belonging to these inertial manifolds, which turn out to be solutions to some induced ordinary differential equations.

To our best knowledge, the most popular conditions for the existence of inertial manifolds are the spectral gap condition of the linear operator A and the uniform Lipschitz continuity of the nonlinear term $f(t,x)$ (i.e., $\|f(t,x) - f(t,y)\| \leqslant q\|x-y\|$ for a Lipschitz constant q independent of t) (see [7,8,20,26]). However, for equations arising in complicated reaction–diffusion processes, the Lipschitz coefficients may depend on time, and the restricted spectral gap condition may not be fulfilled. Therefore, one tries to extend the conditions on the operator A and the nonlinear term such that they describe more exactly such processes. Moreover, the inertial manifolds considered in the existing literature are mostly constituted by trajectories of solutions bounded (or rescaledly bounded) on the negative half-line. We refer the reader to [7,8,15,18,20,26] and references therein for more information on this matter.

Recently, we have obtained exciting results in [14], where we have used the Lyapunov–Perron method and the characterization of the exponential dichotomy (obtained in [13]) of evolution equations in admissible function spaces to construct the structures of solutions of Eq. (1.1) in a mild form, which belong to some certain classes of admissible spaces on which we could implement some well-known procedures in functional analysis such as: constructing of contraction mappings; using of Implicit Function Theorem, etc. The use of admissible spaces has helped us to define and prove the existence of new classes of invariant manifolds, that are the invariant manifolds of $\mathcal{E}$-class for Eq. (1.1) (see [14, Theorems 3.7, 4.6]). Such manifolds are constituted by trajectories of solutions belonging to the Banach space $\mathcal{E}$ which can be a space of L_p type ($1 \leqslant p \leqslant \infty$) or a Lorent space $L_{p,q}$ or some function spaces occurring in interpolation theory (see [14, Definitions 3.3, 4.2]).

The purpose of the present paper is to establish the existence of a new class of inertial manifolds called admissibly inertial manifolds of $\mathcal{E}$-class (see Definition 3.1 and Remark 3.2 thereafter) under the conditions that the linear operator A is positive definite and self-adjoint with a discrete spectrum, and the nonlinear term $f(t,x)$ is non-uniformly Lipschitz continuous on some interpolation space, i.e., $\|f(t,x) - f(t,y)\| \leqslant \varphi(t)\|A^\theta(x-y)\|$ and $\|f(t,x)\| \leqslant \varphi(t)(1 + \|A^\theta x\|)$ for φ being a real and positive function which belongs to admissible function spaces defined in Definition 2.4 below, and $0 \leqslant \theta < 1$.

Since the admissibly inertial manifold is constituted by trajectories of the solutions belonging to rescaledly general admissible function spaces (see Definition 3.1 and Remark 3.2 thereafter) which are not necessary L_∞-spaces, the techniques and methodology used in the paper [15] cannot be applied here. Instead, we use the arguments of duality to overcome this difficulty. Concretely, we introduce new ingredients which are the space E^θ and the functions h_ν and Θ_ν (see Standing Hypothesis 2.10) representing the association between the spaces of Lipschitz coefficients φ and that of solutions which belong to the admissible inertial manifold. These ingredients are used in the duality arguments together with generalized "Hölder inequalities" (see inequality (2.6)) to obtain necessary estimates corresponding to the spectral gap of A. Then we apply our techniques and results in [14] (see also [13]) of using admissibility of function spaces to construct the solutions of Lyapunov–Perron equation which will be used to derive the existence of admissibly inertial manifolds. Our main results are contained in Lemma 3.5, Theorem 3.6. We also illustrate our results in Example 3.8.

2. Preliminaries

We now recall some notions. Let X be a separable Hilbert space, and suppose that A is a closed linear operator on X satisfying the following standing hypothesis.

Standing Hypothesis 2.1. We suppose that A is a positive definite, self-adjoint operator with a discrete spectrum, say

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \text{each with finite multiplicity and } \lim_{k\to\infty} \lambda_k = \infty,$$

and assume that $\{e_k\}_{k=1}^{\infty}$ is the orthonormal basis in X consisting of the corresponding eigenfunctions of the operator A (i.e., $Ae_k = \lambda_k e_k$). Let now λ_N and λ_{N+1} be two successive and different eigenvalues with $\lambda_N < \lambda_{N+1}$, let further P be the orthogonal projection onto the first N eigenvectors of the operator A.

Denote by $(e^{-tA})_{t\geqslant 0}$ the semigroup generated by $-A$. Since $\operatorname{Im} P$ is of finite dimension, we have that the restriction $(e^{-tA}P)_{t\geqslant 0}$ of the semigroup $(e^{-tA})_{t\geqslant 0}$ to $\operatorname{Im} P$ can be extended to the whole line $\mathbb{R}$.

For $0 \leqslant \theta < 1$ we then recall the following dichotomy estimates (see [7,26]):

$$\begin{aligned}
\left\| e^{-tA}P \right\| &\leqslant M e^{\lambda_N |t|}, \quad t \in \mathbb{R} \text{ for some constant } M \geqslant 1, \\
\left\| A^\theta e^{-tA}P \right\| &\leqslant \lambda_N^\theta M e^{\lambda_N |t|}, \quad t \in \mathbb{R}, \\
\left\| e^{-tA}(I-P) \right\| &\leqslant M e^{-\lambda_{N+1} t}, \quad t \geqslant 0, \\
\left\| A^\theta e^{-tA}(I-P) \right\| &\leqslant M\left[\left(\frac{\theta}{t}\right)^\theta + \lambda_{N+1}^\theta\right] e^{-\lambda_{N+1} t}, \quad t > 0,\ \theta > 0.
\end{aligned} \tag{2.1}$$

We next recall some notions on function spaces and refer to Massera and Schäffer [19], Räbiger and Schnaubelt [25] for concrete applications.

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. The space $L_{1,loc}(\mathbb{R})$ of real-valued locally integrable functions on $\mathbb{R}$ (modulo λ-nullfunctions) becomes a Fréchet space for the seminorms $p_n(f) := \int_{J_n} |f(t)|\,dt$, where $J_n = [n, n+1]$ for each $n \in \mathbb{Z}$ (see [19, Chapt. 2, §20]).

We can now define Banach function spaces as follows.

Definition 2.2. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called *a Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$, λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,

(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and

$$\sup_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E < \infty \quad \text{and} \quad \inf_{t\in\mathbb{R}} \|\chi_{[t,t+1]}\|_E > 0,$$

(3) $E \hookrightarrow L_{1,loc}(\mathbb{R})$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R})$ there exists a number $\beta_{p_n} > 0$ such that $p_n(f) \leqslant \beta_{p_n}\|f\|_E$ for all $f \in E$.

We remark that condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}$ there exists a number $\beta_J \geqslant 0$ such that $\int_J |f(t)|\,dt \leqslant \beta_J\|f\|_E$ for all $f \in E$.

We then define Banach spaces of vector-valued functions corresponding to Banach function spaces as follows.

Definition 2.3. Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}, X) := \bigl\{h : \mathbb{R} \to X \bigm| h \text{ is strongly measurable and } \bigl\|h(\cdot)\bigr\| \in E\bigr\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|h\|_{\mathcal{E}} := \bigl\|\bigl\|h(\cdot)\bigr\|\bigr\|_E.$$

One can easily see that $\mathcal{E}$ is a Banach space. We call it *the Banach space corresponding to the Banach function space E*.

We now introduce the notion of admissibility in the following definition.

Definition 2.4. The Banach function space E is called *admissible* if the following hold:

(i) there is a constant $M \geqslant 1$ such that for every compact interval $[a,b] \subset \mathbb{R}$ we have

$$\int_a^b \bigl|\varphi(t)\bigr|\,dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{2.2}$$

(ii) for $\varphi \in E$ the function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_{t-1}^t \varphi(\tau)\,d\tau$ belongs to E.

(iii) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined, for $\tau \in \mathbb{R}$, by

$$\begin{aligned} T_\tau^+\varphi(t) &:= \varphi(t-\tau) \quad \text{for } t \in \mathbb{R}, \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{2.3}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\| \leqslant N_1$, $\|T_\tau^-\| \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.5. Besides the spaces $L_p(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}(\mathbb{R}) := \left\{ f \in L_{1,loc}(\mathbb{R}) \colon \sup_{t\in\mathbb{R}} \int_{t-1}^t \bigl|f(\tau)\bigr|\,d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\in\mathbb{R}} \int_{t-1}^{t} |f(\tau)|\,d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 < q < \infty$ (see [4, Thm. 3 and p. 284], [27, 1.18.6, 1.19.3]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}, \mathcal{B}, \lambda)$ (see [16, 2.a]) are admissible.

Remark 2.6. If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R})$. Indeed, put $\beta := \inf_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.2(2)). Then, from Definition 2.4(i) we derive

$$\int_{t-1}^{t} \left|\varphi(\tau)\right| d\tau \leqslant \frac{M}{\beta}\|\varphi\|_E \quad \text{for all } t \in \mathbb{R} \text{ and } \varphi \in E. \tag{2.4}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R})$ and $\|\varphi\|_{\mathbf{M}} \leqslant \frac{M}{\beta}\|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R})$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [13, Proposition 2.6] and originally in [19, 23.V.(1)]).

Proposition 2.7. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,loc}(\mathbb{R})$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where Λ_1 is defined as in Definition* 2.4(ii). *For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)\,ds,$$

$$\Lambda''_\sigma\varphi(t) := \int_{t}^{\infty} e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. Moreover, the following estimates hold:

$$\left\|\Lambda'_\sigma\varphi\right\|_E \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E \quad \text{and} \quad \left\|\Lambda''_\sigma\varphi\right\|_E \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_E \tag{2.5}$$

for constants N_1, N_2 defined as in Definition 2.4.

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha|t|}$ for $t \in \mathbb{R}$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) := e^{b|t|}$ for $t \in \mathbb{R}$ and any fixed constant $b > 0$.*

We next define the associate spaces of admissible Banach function spaces on $\mathbb{R}$ as follows.

Definition 2.8. Let E be an admissible Banach function space and denote by $S(E)$ the unit sphere in E. Recall that $L_1 = \{g : \mathbb{R} \to \mathbb{R} \mid g \text{ is measurable and } \int_{\mathbb{R}} |g(t)|\,dt < \infty\}$. Then, we consider the set E' of all measurable real-valued functions ψ on $\mathbb{R}$ such that

$$\varphi\psi \in L_1, \quad \int_{\mathbb{R}} \left|\varphi(t)\psi(t)\right| dt \leqslant k \quad \text{for all } \varphi \in S(E),$$

where k depends only on ψ. Then, E' is a normed space with the norm given by (see [19, Chapt. 2, 22.M]):

$$\|\psi\|_{E'} := \sup\left\{\int_{\mathbb{R}} \left|\varphi(t)\psi(t)\right| dt\colon\ \varphi \in S(E)\right\} \quad \text{for } \psi \in E'.$$

We call E' the *associate space* of E.

Remark 2.9. Let E be an admissible Banach function space and E' be its associate space. Then, from [19, Chapt. 2, 22.M] we also have that the following "Hölder's inequality" holds:

$$\int_{\mathbb{R}} \left|\varphi(t)\psi(t)\right| dt \leqslant \|\varphi\|_E \|\psi\|_{E'} \quad \text{for all } \varphi \in E,\ \psi \in E'. \tag{2.6}$$

In order to study the admissibly inertial manifolds of $\mathcal{E}$-class for reaction–diffusion equations we need some restrictions on the admissible Banach function spaces and assume the following hypothesis.

Standing Hypothesis 2.10. In this paper we will consider the admissible Banach function space E such that its associate space E' is also an admissible Banach function space. Moreover, we suppose that

(1) for $0 \leqslant \theta < 1$ the function space $E^\theta := \{u \in E \mid |u|^{\frac{1+\theta}{1-\theta}} \in E\}$ is also an admissible Banach function space with the norm $\|u\|_\theta := \max\{\|u\|_E, \||u|^{\frac{1+\theta}{1-\theta}}\|_E^{\frac{1-\theta}{1+\theta}}\}$,
(2) E' contains a ν*-exponentially* E*-invariant* function, that is the function $\varphi \geqslant 0$ having the property that, for a fixed $\nu > 0$, the functions h_ν and Θ_ν defined by

$$h_\nu(t) := \left\|e^{-\nu|t-\cdot|}\varphi(\cdot)\right\|_{E'},$$
$$\Theta_\nu(t) := \left\|e^{-\nu\frac{1+\theta}{1-\theta}|t-\cdot|}\varphi^{\frac{1+\theta}{1-\theta}}(\cdot)\right\|_{E'}^{\frac{1-\theta}{1+\theta}} \quad \text{for } t \in \mathbb{R}$$

belong to E.

We also give here some examples of the admissible Banach function spaces and their associate function spaces which satisfy the above standing hypothesis with a ν-exponentially E-invariant function $\varphi(t) = \eta e^{-\beta|t|}$ for $t \in \mathbb{R}$ and any fixed $\eta, \beta, \nu > 0$.

Example 2.11. $L'_p = L_q$ for $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p < \infty$, and, $L'_1 = L_\infty$, $L'_\infty = L_1$.

Besides the above functions of the form $\varphi(t) = \eta e^{-\beta|t|}$, one can see that the functions of the forms $\varphi = c\chi_{[a,b]}$ for any fixed constant $c > 0$ and any finite interval $[a,b] \subset \mathbb{R}$, are also ν-exponentially L_p-invariant functions for any $\nu > 0$.

Remark 2.12. If we replace the whole line $\mathbb{R}$ by a half-infinite interval $(-\infty, t_0]$ for any fixed $t_0 \in \mathbb{R}$, then we have the similar notions of admissible spaces on $t_0 \in \mathbb{R}$ with slight changes as follows:

(1) In Definition 2.4, the translations semigroups T^+_τ and T^-_τ for $\tau \in \mathbb{R}$ should be replaced by T^+_τ and T^-_τ defined for $\tau \leqslant t_0$ and $t \leqslant t_0$ as

$$T_\tau^+\varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \leqslant \tau + t_0, \\ 0 & \text{for } t > t_0 + \tau; \end{cases}$$
$$T_\tau^-\varphi(t) := \begin{cases} \varphi(t+\tau) & \text{for } t \leqslant t_0 - \tau, \\ 0 & \text{for } t > t_0. \end{cases} \tag{2.7}$$

(2) In Proposition 2.7(a), the functions Λ'_σ and Λ''_σ should be replaced by

$$\Lambda'_\sigma\varphi(t) := \int_t^{t_0} e^{-\sigma(s-t)}\varphi(s)\,ds,$$
$$\Lambda''_\sigma\varphi(t) := \int_{-\infty}^{t} e^{-\sigma(t-s)}\varphi(s)\,ds$$

for $t \leqslant t_0$.

(3) In Proposition 2.7(b) and (c) the functions $\psi(t) = e^{-\alpha|t|}$ ($t \in \mathbb{R}$, and fixed $\alpha > 0$) should be replaced by $\psi(t) = e^{\alpha t}$, $t \leqslant t_0$ and fixed $\alpha > 0$; and the functions $f(t) := e^{b|t|}$ ($t \in \mathbb{R}$, and any fixed constant $b > 0$) should be replaced by $f(t) := e^{-bt}$, $t \leqslant t_0$ and fixed $b > 0$.

We denote the admissible function space of the functions defined on $(-\infty, t_0]$ by $E_{(-\infty,t_0]}$. Also, denote $E^\theta_{(-\infty,t_0]} := \{u \in E_{(-\infty,t_0]} \mid |u|^{\frac{1+\theta}{1-\theta}} \in E_{(-\infty,t_0]}\}$ endowed with the norm $\|\cdot\|_\theta$ as in Standing Hypothesis 2.10.

For a function φ defined on the whole line we denote the restriction of φ on $(-\infty, t_0]$ by $\varphi|_{(-\infty,t_0]}$. It is obvious that, if the function $\varphi \in E$, then $\varphi|_{(-\infty,t_0]} \in E_{(-\infty,t_0]}$.

Similarly to Definition 2.3, for a Banach function space $E_{(-\infty,t_0]}$ and a Banach space X endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E}_{(-\infty,t_0]} := \bigl\{h : (-\infty, t_0] \to X \bigm| h \text{ is strongly measurable and } \bigl\|h(\cdot)\bigr\| \in E_{(-\infty,t_0]}\bigr\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|h\|_{\mathcal{E}_{(-\infty,t_0]}} := \bigl\|\bigl\|h(\cdot)\bigr\|\bigr\|_{E_{(-\infty,t_0]}}.$$

Then, $\mathcal{E}_{(-\infty,t_0]}$ is a Banach space called *the Banach space corresponding to the Banach function space* $E_{(-\infty,t_0]}$. Also, denote

$$\mathcal{E}^\theta_{(-\infty,t_0]} := \bigl\{h : (-\infty, t_0] \to X_\theta \bigm| h \text{ is strongly measurable and } \bigl\|h(\cdot)\bigr\| \in E^\theta_{(-\infty,t_0]}\bigr\}$$

endowed with the norm $\|h\|_{\mathcal{E}^\theta_{(-\infty,t_0]}} := \|\|h(\cdot)\|\|_\theta$.

In the case of infinite-dimensional phase spaces, instead of Eq. (1.1), we consider the integral equation

$$u(t) = e^{-(t-s)A}u(s) + \int_s^t e^{-(t-\xi)A} f\bigl(\xi, u(\xi)\bigr)\,d\xi \quad \text{for a.e. } t \geqslant s,\ t, s \in \mathbb{R}. \tag{2.8}$$

By a *solution* of (2.8) we mean a *strongly measurable* function $u(t)$ defined on $\mathbb{R}$ with the values on X_θ that satisfies Eq. (2.8). We note that, the solution u to Eq. (2.8) is called a mild solution of Eq. (1.1).

We refer the reader to Pazy [22] for more detailed treatment on the relations between classical and mild solutions of evolution equations (see also [7,10,17,26]).

To obtain the existence of an inertial manifold for Eq. (2.8), beside the assumptions on the operator A, we also need the φ-Lipschitz property of the nonlinear term f in the following definitions.

Definition 2.13 *(φ-Lipschitz functions).* Let E be an admissible Banach function space on $\mathbb{R}$ and φ be a positive function belonging to E. Put $X_\theta := D(A^\theta)$. Then, a function $f : \mathbb{R} \times X_\theta \to X$ is said to be φ-Lipschitz if f satisfies

(i) $\|f(t,x)\| \leqslant \varphi(t)(1 + \|A^\theta x\|)$ for a.e. $t \in \mathbb{R}$, and all $x \in X_\theta$,
(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|A^\theta(x_1 - x_2)\|$ for a.e. $t \in \mathbb{R}$, and all $x_1, x_2 \in X_\theta$.

3. Inertial manifolds

In this section we will prove the existence of the admissibly inertial manifolds for solutions to Eq. (2.8). We suppose that A satisfies Standing Hypothesis 2.1 and recall that P is the orthogonal projection onto the first N orthonormal eigenvectors of A. We then make precisely the notion of admissibly inertial manifolds of $\mathcal{E}$-class in the following definition.

Definition 3.1. Let E be an admissible function space and $\mathcal{E}$ be a Banach space corresponding to E. An admissibly inertial manifold of $\mathcal{E}$-class for Eq. (2.8) is a collection of Lipschitz surfaces $(\mathbb{M}_t)_{t\in\mathbb{R}}$ in X such that each $\mathbb{M}_t$ is the graph of a Lipschitz function $\Phi_t : \operatorname{Im} P \to (I-P)X_\theta$, i.e.,

$$\mathbb{M}_t = \{x + \Phi_t x \mid x \in \operatorname{Im} P\} \quad \text{for } t \in \mathbb{R},$$

and the following conditions are satisfied:

(i) The Lipschitz constants of Φ_t are independent of t, i.e., there exists a constant C independent of t such that

$$\left\|A^\theta(\Phi_t x_1 - \Phi_t x_2)\right\| \leqslant C\left\|A^\theta(x_1 - x_2)\right\|.$$

(ii) There exists $\gamma > 0$ such that to each $x_0 \in \mathbb{M}_{t_0}$ there corresponds one and only one solution $u(t)$ to Eq. (2.8) on $(-\infty, t_0]$ satisfying that $u(t_0) = x_0$ and the function

$$v(t) = e^{-\gamma(t_0 - t)} A^\theta u(t), \quad t \leqslant t_0, \tag{3.1}$$

belongs to $\mathcal{E}_{(-\infty,t_0]}$ for each $t_0 \in \mathbb{R}$.
(iii) $(\mathbb{M}_t)_{t\in\mathbb{R}}$ is positively invariant under Eq. (2.8), i.e., if a solution $x(t)$, $t \geqslant s$, of Eq. (2.8), satisfies $x(s) \in \mathbb{M}_s$, then we have that $x(t) \in \mathbb{M}_t$ for all $t \geqslant s$.
(iv) $(\mathbb{M}_t)_{t\in\mathbb{R}}$ exponentially attracts all the solutions to Eq. (2.8), i.e., for any solution $u(\cdot)$ of Eq. (2.8) and any fixed $s \in \mathbb{R}$, there is a positive constant H such that

$$dist_{X_\theta}\big(u(t), \mathbb{M}_t\big) \leqslant He^{-\gamma(t-s)} \quad \text{for all } t \geqslant s,$$

where γ is the same constant as the one in item (ii), and $dist_{X_\theta}$ denotes the Hausdorff semi-distance generated by the norm in X_θ.

Remark 3.2. We would like to emphasize some features of this new definition compared with the traditional definition of an inertial manifold:

1. We do not need the existence and uniqueness theorem for Eq. (2.8) since this is explicitly referred in the item (ii) of the above definition of an admissibly inertial manifold.
2. The inertial manifold defined by traditional definition in the present literature, which is constituted of the solutions (rescaledly) bounded in negative half-line, is just a special case of our admissibly inertial manifolds (IM). Indeed, we just take $\mathcal{E} = L_\infty$ to derive the traditional IM from our above definition of admissibly IM.

Let A satisfy the Standing Hypothesis 2.1. Then, we can define the Green function as follows:

$$G(t,\tau) := \begin{cases} e^{-(t-\tau)A}[I-P] & \text{for } t > \tau, \\ -e^{-(t-\tau)A}P & \text{for } t \leqslant \tau. \end{cases} \tag{3.2}$$

Then, one can see that $G(t,s)$ maps X into X_θ. Also, by the dichotomy estimates (2.1) and for $\gamma = (\lambda_N + \lambda_{N+1})/2$ we have

$$\left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \leqslant K(t,\tau) e^{-\alpha|t-\tau|} \quad \text{for all } t \neq \tau, \tag{3.3}$$

where $\alpha = (\lambda_{N+1} - \lambda_N)/2$ and

$$K(t,\tau) = \begin{cases} M\left(\left(\frac{\theta}{t-\tau}\right)^\theta + \lambda_{N+1}^\theta\right) & \text{if } t > \tau, \\ M\lambda_N^\theta & \text{if } t \leqslant \tau. \end{cases}$$

We can now construct the form of the solutions of Eq. (2.8) which belongs to weighted (or rescaling) admissible spaces on the half-line $(-\infty, t_0]$ in the following lemma.

Lemma 3.3. *Let the operator A satisfy Standing Hypothesis 2.1. Let E be an admissible Banach function space, E' be its associate space. Suppose that $\varphi \in E'$ is an α-exponentially E-invariant function defined as in Standing Hypothesis 2.10. Suppose that $f : \mathbb{R} \times X_\theta \to X$ is φ-Lipschitz. For fixed $t_0 \in \mathbb{R}$ let $x(t)$, $t \leqslant t_0$, be a solution of (2.8) such that $x(t) \in X_\theta$ $\forall t \leqslant t_0$, and the function*

$$z(t) = \left\| e^{-\gamma(t_0-t)} A^\theta x(t) \right\| \quad \textit{for } t \leqslant t_0 \textit{ and } \gamma \textit{ being defined as in inequality (3.3)}$$

belongs to $E_{(-\infty,t_0]}$. Then, for $t \leqslant t_0$ this solution $x(t)$ can be rewritten in the form

$$x(t) = e^{-(t-t_0)A} v_1 + \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\, d\tau \quad \textit{for all } t \leqslant t_0, \tag{3.4}$$

where $v_1 \in PX$, and $G(t,\tau)$ is the Green's function defined as in (3.2).

Proof. Put

$$y(t) := \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\, d\tau \quad \text{for } t \leqslant t_0. \tag{3.5}$$

Then $y(t) \in X_\theta$ for all $t \leqslant t_0$.

Since f is φ-Lipschitz, using estimate (3.3) we obtain that

$$\begin{aligned}\left\|A^\theta e^{-\gamma(t_0-t)}y(t)\right\| &\leqslant \int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\|\varphi(\tau)e^{-\gamma(t_0-\tau)}\left(1+\left\|A^\theta x(\tau)\right\|\right)d\tau \\ &= \int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\|\varphi(\tau)\left(e^{-\gamma(t_0-\tau)}+\left\|z(\tau)\right\|\right)d\tau \quad \text{for } t\leqslant t_0. \qquad (3.6)\end{aligned}$$

Putting $w(t) := e^{-\gamma(t_0-t)} + \|z(t)\|$ for $t \leqslant t_0$, we have that the function w belongs to $E_{(-\infty,t_0]}$. Using (2.5), (3.3), and (2.6) we estimate the integral

$$\begin{aligned}&\int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\|\varphi(\tau)w(\tau)\,d\tau \\ &\leqslant \int_{-\infty}^{t} M\left(\left(\frac{\theta}{t-\tau}\right)^\theta+\lambda_{N+1}^\theta\right)e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau + \int_t^{t_0} M\lambda_N^\theta e^{-\alpha(\tau-t)}\varphi(\tau)w(\tau)\,d\tau \\ &\leqslant \int_{-\infty}^{t} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau + M\left(\lambda_{N+1}^\theta+\lambda_N^\theta\right)\left\|e^{-\alpha(t-\cdot)}\varphi(\cdot)\right\|_{E'_{(-\infty,t_0]}}\|w\|_{E_{(-\infty,t_0]}} \\ &\text{where } \alpha = \frac{\lambda_{N+1}-\lambda_N}{2}. \qquad (3.7)\end{aligned}$$

The first integral on the right-hand side is now estimated for $0 \leqslant \theta < 1$ as follows

$$\begin{aligned}&\int_{-\infty}^{t} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau \\ &= \int_{-\infty}^{t-1} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau + \int_{t-1}^{t} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau \\ &\leqslant \int_{-\infty}^{t-1} M\theta^\theta e^{-\alpha(t-\tau)}\varphi(\tau)w(\tau)\,d\tau \\ &\quad + M\theta^\theta\left(\int_{t-1}^{t} \frac{1}{(t-\tau)^{\frac{1+\theta}{2}}}\,d\tau\right)^{\frac{2\theta}{1+\theta}}\left(\int_{t-1}^{t} e^{-\alpha\frac{1+\theta}{1-\theta}(t-\tau)}\left(\varphi(\tau)w(\tau)\right)^{\frac{1+\theta}{1-\theta}}\,d\tau\right)^{\frac{1-\theta}{1+\theta}} \\ &\leqslant M\theta^\theta\left\|e^{-\alpha(t-\cdot)}\varphi(\cdot)\right\|_{E'_{(-\infty,t_0]}}\|w\|_{E_{(-\infty,t_0]}} \\ &\quad + M\theta^\theta\left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}}\left\|e^{-\alpha\frac{1+\theta}{1-\theta}(t-\cdot)}\varphi^{\frac{1+\theta}{1-\theta}}(\cdot)\right\|_{E'_{(-\infty,t_0]}}^{\frac{1-\theta}{1+\theta}}\left\|w^{\frac{1+\theta}{1-\theta}}\right\|_{E_{(-\infty,t_0]}}^{\frac{1-\theta}{1+\theta}}\end{aligned}$$

(here we have used the Hölder inequality for the second term on the right-hand side).

Substituting the above inequality into (3.7) we obtain that

$$\int_{-\infty}^{t_0} \left\| e^{\gamma(t-\tau)} A^\theta G(t,\tau) \right\| \varphi(\tau) w(\tau)\, d\tau \leqslant k(t) \|w\|_\theta \quad \text{for all } t \leqslant t_0 \tag{3.8}$$

where

$$\begin{aligned} k(t) = {} & M\theta^\theta \left(1 + \lambda_{N+1}^\theta + \lambda_N^\theta\right) \\ & \times \left(\left\| e^{-\alpha|t-\cdot|} \varphi(\cdot) \right\|_{E'} + \left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}} \left\| e^{-\alpha \frac{1+\theta}{1-\theta}|t-\cdot|} \varphi^{\frac{1+\theta}{1-\theta}}(\cdot) \right\|_{E'}^{\frac{1-\theta}{1+\theta}} \right). \end{aligned} \tag{3.9}$$

Now, substituting this estimate to (3.6) we have that

$$\left\| A^\theta e^{-\gamma(t_0-t)} y(t) \right\| \leqslant k(t) \|w\|_\theta \quad \text{for all } t \leqslant t_0.$$

Since $k(\cdot)$ belongs to $E_{(-\infty,t_0]}$, using the admissibility of $E_{(-\infty,t_0]}$ we obtain that

$$A^\theta e^{-\gamma(t_0-\cdot)} y(\cdot) \in \mathcal{E}_{(-\infty,t_0]}$$

and

$$\left\| A^\theta e^{-\gamma(t_0-\cdot)} y(\cdot) \right\|_{\mathcal{E}_{(-\infty,t_0]}} \leqslant \left\| k(\cdot) \right\|_{E_{(-\infty,t_0]}} \|w\|_\theta.$$

Next, by computing directly we will verify that $y(\cdot)$ satisfies the integral equation

$$y(t_0) = e^{-(t_0-t)A} y(t) + \int_t^{t_0} e^{-(t_0-\tau)A} f\big(\tau, x(\tau)\big)\, d\tau \quad \text{for } t \leqslant t_0. \tag{3.10}$$

Indeed, substituting y from (3.5) to the right-hand side of (3.10) we obtain

$$\begin{aligned} & e^{-(t_0-t)A} y(t) + \int_t^{t_0} e^{-(t_0-\tau)A} f\big(\tau, x(\tau)\big)\, d\tau \\ & = e^{-(t_0-t)A} \int_{-\infty}^{t_0} G(t,\tau) f\big(\tau, x(\tau)\big)\, d\tau + \int_t^{t_0} e^{-(t_0-\tau)A} f\big(\tau, x(\tau)\big)\, d\tau \\ & = e^{-(t_0-t)A} \int_{-\infty}^{t} e^{-(t-\tau)A} (I-P) f\big(\tau, x(\tau)\big)\, d\tau - e^{-(t_0-t)A} \int_t^{t_0} e^{-(t-\tau)A} P f\big(\tau, x(\tau)\big)\, d\tau \\ & \quad + \int_t^{t_0} e^{-(t_0-\tau)A} f\big(\tau, x(\tau)\big)\, d\tau \end{aligned}$$

$$= \int_{-\infty}^{t} e^{-(t_0-\tau)A}(I-P)f\big(\tau,x(\tau)\big)\,d\tau - \int_{t}^{t_0} e^{-(t_0-t)A}e^{-(t-\tau)A}Pf\big(\tau,x(\tau)\big)\,d\tau$$

$$+ \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau$$

$$= \int_{-\infty}^{t_0} e^{-(t_0-\tau)A}(I-P)f\big(\tau,x(\tau)\big)\,d\tau = \int_{-\infty}^{t_0} G(t_0,\tau)f\big(\tau,x(\tau)\big) = y(t_0),$$

here we use the fact that $e^{-(t_0-t)A}e^{-(t-\tau)A}P = e^{-(t_0-\tau)A}P$ for all $t \leqslant \tau \leqslant t_0$.

Thus, we have

$$y(t_0) = e^{-(t_0-t)A}y(t) + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau.$$

On the other hand,

$$x(t_0) = e^{-(t_0-t)A}x(t) + \int_{t}^{t_0} e^{-(t_0-\tau)A}f\big(\tau,x(\tau)\big)\,d\tau.$$

Then $x(t_0) - y(t_0) = e^{-(t_0-t)A}[x(t) - y(t)]$. We need to prove that $x(t_0) - y(t_0) \in PX$.

Indeed, applying the operator $A^\theta(I-P)$ to the expression $x(t_0) - y(t_0) = e^{-(t_0-t)A}[x(t) - y(t)]$, we have

$$\begin{aligned}\big\|A^\theta(I-P)\big[x(t_0)-y(t_0)\big]\big\| &= \big\|e^{-(t_0-t)A}A^\theta(I-P)\big[x(t)-y(t)\big]\big\| \\ &\leqslant Ne^{-(\lambda_{N+1}-\gamma)(t_0-t)}\|I-P\|\big\|e^{-\gamma(t_0-t)}A^\theta\big(x(t)-y(t)\big)\big\|.\end{aligned}$$

Since $\operatorname{ess\,sup}_{t\leqslant t_0}\|e^{-\gamma(t_0-t)}A^\theta(x(t)-y(t))\| < \infty$, letting $t \to -\infty$ we obtain that

$$\big\|A^\theta(I-P)\big[x(t_0)-y(t_0)\big]\big\| = 0, \quad \text{hence} \quad A^\theta(I-P)\big[x(t_0)-y(t_0)\big] = 0.$$

Since A^θ is injective, it follows that $(I-P)[x(t_0)-y(t_0)] = 0$. Thus, $v_1 := x(t_0) - y(t_0) \in PX$. Using the fact that the restriction of $e^{-(t_0-t)A}$ on PX is invertible with the inverse $e^{-(t-t_0)A}$ we obtain that

$$x(t) = e^{-(t-t_0)A}v_1 + y(t) = e^{-(t-t_0)A}v_1 + \int_{-\infty}^{t_0} G(t,\tau)f\big(\tau,x(\tau)\big)\,d\tau \quad \text{for } t \leqslant t_0,$$

finishing the proof. □

Remark 3.4. Eq. (3.4) is called *Lyapunov–Perron equation* which will be used to determine the admissibly inertial manifold for Eq. (2.8). By computing directly, we can see that the converse of Lemma 3.3 is also true. It means, all solutions of Eq. (3.4) satisfied Eq. (2.8) for $t \leqslant t_0$.

We then have the following lemma which describes the existence and uniqueness of certain solutions belonging to rescaling spaces.

Lemma 3.5. *Let the operator A satisfy the Standing Hypothesis 2.1. Let E and E' be the admissible Banach function space and its associate space, respectively. Suppose that $\varphi \in E'$ is an α-exponentially E-invariant function defined as in the Standing Hypothesis 2.10. For $0 \leqslant \theta < 1$ define the function k by*

$$\begin{aligned} k(t) = {} & M\theta^{\theta}\left(1+\lambda_{N+1}^{\theta}+\lambda_N^{\theta}\right) \\ & \times\left(\left\|e^{-\alpha|t-\cdot|}\varphi(\cdot)\right\|_{E'}+\left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}}\left\|e^{-\alpha\frac{1+\theta}{1-\theta}|t-\cdot|}\varphi^{\frac{1+\theta}{1-\theta}}(\cdot)\right\|_{E'}^{\frac{1-\theta}{1+\theta}}\right) \end{aligned} \tag{3.11}$$

for all $t \in \mathbb{R}$, where $\lambda_N < \lambda_{N+1}$ are two successive eigenvalues of A, and $\alpha = (\lambda_{N+1} - \lambda_N)/2$.

Let $f : \mathbb{R} \times \boldsymbol{X} \to \boldsymbol{X}$ be φ-Lipschitz such that $\|k(\cdot)\|_{\theta} < 1$.

Then, there corresponds to each $v_1 \in PX$ one and only one solution $x(t)$ of Eq. (2.8) on $(-\infty, t_0]$ satisfying the conditions that $Px(t_0) = v_1$ and the function

$$z(t) := \left\|e^{-\gamma(t_0-t)}A^{\theta}x(t)\right\|, \quad t \leqslant t_0,$$

belongs to $E_{(-\infty,t_0]}$ for each $t_0 \in \mathbb{R}$.

Proof. Denote

$$\begin{aligned} \mathcal{E}^{\gamma,t_0,\theta} := \Big\{ & h : (-\infty, t_0] \to X_{\theta} \;\Big|\; h \text{ is strongly measurable and} \\ & \left\|e^{-\gamma(t_0-\cdot)}A^{\theta}h(\cdot)\right\| \text{ belongs to } E^{\theta}_{(-\infty,t_0]}\Big\} \end{aligned}$$

endowed with the norm

$$\|h\|_{\gamma,\theta} := \left\| \left\|e^{-\gamma(t_0-\cdot)}A^{\theta}h(\cdot)\right\| \right\|_{\theta}.$$

For each $t_0 \in \mathbb{R}$ and $v_1 \in PX$ we will prove that the transformation T defined by

$$(Tx)(t) = e^{-(t-t_0)A}v + \int_{-\infty}^{t_0} G(t,\tau) f\left(\tau, x(\tau)\right) d\tau \quad \text{for } t \leqslant t_0$$

acts from $\mathcal{E}^{\gamma,t_0,\theta}$ into itself and is a contraction.

In fact, for $x(\cdot) \in \mathcal{E}^{\gamma,t_0,\theta}$, we have that $\|f(t,x(t))\| \leqslant \varphi(t)(1+\|A^{\theta}x(t)\|)$. Therefore, putting

$$y(t) = e^{-(t-t_0)A}v_1 + \int_{-\infty}^{t_0} G(t,\tau) f\left(\tau, x(\tau)\right) d\tau \quad \text{for } t \leqslant t_0$$

we derive that

$$\begin{aligned} \left\|A^{\theta}e^{-\gamma(t_0-t)}y(t)\right\| \leqslant {} & \lambda_N^{\theta}Me^{-(\gamma-\lambda_N)(t_0-t)}\|v_1\| \\ & + \int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)}A^{\theta}G(t,\tau)\right\|\varphi(\tau)e^{-\gamma(t_0-\tau)}\left(1+\left\|A^{\theta}x(\tau)\right\|\right) d\tau \quad \text{for } t \leqslant t_0. \end{aligned}$$

Using the estimate (3.8) we obtain that

$$\left\| A^\theta e^{-\gamma(t_0-t)} y(t)\right\| \leqslant \lambda_N^\theta M e^{-(\gamma-\lambda_N)(t_0-t)}\|v_1\| + k(t)\|w\|_\theta$$

for all $t \leqslant t_0$, where $w(t) := e^{-\gamma(t_0-t)}(1 + \|A^\theta x(t)\|)$.

Since $e^{-(\gamma-\lambda_N)(t_0-\cdot)}$ and $k(\cdot)$ belong to $E^\theta_{(-\infty,t_0]}$, it follows from the above inequalities that $y(\cdot) \in \mathcal{E}^{\gamma,t_0,\theta}$ and

$$\left\|y(\cdot)\right\|_{\gamma,\theta} \leqslant \lambda_N^\theta M \|v_1\| \left\|e^{-(\gamma-\lambda_N)(t_0-\cdot)}\right\|_\theta + \left\|k(\cdot)\right\|_\theta \|w\|_\theta.$$

Therefore, the transformation T acts from $\mathcal{E}^{\gamma,t_0,\theta}$ to $\mathcal{E}^{\gamma,t_0,\theta}$.

For $x, z \in \mathcal{E}^{\gamma,t_0,\theta}$ we now estimate

$$\begin{aligned}\left\|e^{-\gamma(t_0-t)} A^\theta \big(Tx(t) - Tz(t)\big)\right\| &\leqslant \int_{-\infty}^{t_0} \left\|e^{-\gamma(t_0-t)} A^\theta G(t,\tau)\right\| \left\| f\big(\tau, x(\tau)\big) - f\big(\tau, z(\tau)\big)\right\| d\tau \\ &\leqslant \int_{-\infty}^{t_0} \left\|e^{\gamma(t-\tau)} A^\theta G(t,\tau)\right\| \varphi(\tau) e^{-\gamma(t_0-\tau)} \left\|A^\theta \big(x(\tau)\big) - z(\tau)\right\| d\tau.\end{aligned}$$

Again, using inequality (3.8) we derive

$$\left\|Tx(\cdot) - Tz(\cdot)\right\|_{\gamma,\theta} \leqslant \left\|k(\cdot)\right\|_\theta \left\|x(\cdot) - z(\cdot)\right\|_{\gamma,\theta} \quad \text{where } k \text{ is defined as in (3.11).}$$

Hence, if $\|k(\cdot)\|_\theta < 1$, then we obtain that $T : \mathcal{E}^{\gamma,t_0,\theta} \to \mathcal{E}^{\gamma,t_0,\theta}$ is a contraction. Thus, there exists a unique $u(\cdot) \in \mathcal{E}^{\gamma,t_0,\theta}$ such that $Tu = u$. By definition of T we have that $u(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0,\theta}_\infty$ of Eq. (3.4) for $t \leqslant t_0$. By Lemma 3.3 and Remark 3.4 we have that $u(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0,\theta}$ of Eq. (2.8) for $t \leqslant t_0$. □

We now come to our main result on the existence of an admissibly inertial manifold for Eq. (2.8).

Theorem 3.6. *Let the operator A satisfy the Standing Hypothesis 2.1. Let E be an admissible Banach function space and E' be its associate space. Suppose that $\varphi \in E'$ is an α-exponentially E-invariant function defined as in the Standing Hypothesis 2.10. Denote by e_α the function $e_\alpha(t) := e^{-\alpha|t|}$ for all $t \in \mathbb{R}$ and put*

$$l = M\theta^\theta\big(1 + \lambda_{N+1}^\theta\big)\left(\|\varphi\|_{E'} + \left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}} \left\|\varphi^{\frac{1+\theta}{1-\theta}}\right\|_{E'}^{\frac{1-\theta}{1+\theta}}\right) \quad \text{for } 0 \leqslant \theta < 1$$

and

$$\begin{aligned} m :=& \frac{M^3\lambda_N^{2\theta} N_2 \|e_\alpha\| l + (1 - \|k\|_\theta) M((\theta^\theta + \lambda_{N+1}^\theta)N_1 + \lambda_N^\theta N_2)}{(1 - \|k\|_\theta)(1 - e^{-\alpha})} \|\Lambda_1 \varphi\|_\infty \\ &+ M\theta\left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}} \left\|\Lambda_1 \varphi^{\frac{1+\theta}{1-\theta}}\right\|_\infty^{\frac{1-\theta}{1+\theta}}. \end{aligned} \tag{3.12}$$

Let f be φ-Lipschitz and suppose that

$$\max\big\{\|k\|_\theta;\, m\big\} < 1 \tag{3.13}$$

where the function k is defined by (3.11)*,* $\lambda_N < \lambda_{N+1}$ *are two successive eigenvalues of A, and* $\alpha = (\lambda_{N+1} - \lambda_N)/2$*.*

Then, Eq. (2.8) *has an admissibly inertial manifold of* $\mathcal{E}$*-class.*

Proof. Lemma 3.5 allows us to define a collection of surfaces $(\mathbb{M}_{t_0})_{t_0\in\mathbb{R}}$ by

$$\mathbb{M}_{t_0} := \{y + \Phi_{t_0} y \mid y \in PX\}$$

here $\Phi_{t_0} : PX \to (I-P)X_\theta$ is defined by

$$\Phi_{t_0}(y) = \int_{-\infty}^{t_0} e^{-(t_0-\tau)A}(I-P)f\big(\tau, x(\tau)\big)\,d\tau = (I-P)x(t_0) \tag{3.14}$$

where $x(t)$ is the unique solution in $\mathcal{E}^{\gamma,t_0,\theta}$ of Eq. (2.8) satisfying that $Px(t_0) = y$ (note that the existence and uniqueness of $x(t)$ is proved in Lemma 3.5).

We then prove that Φ_{t_0} is Lipschitz continuous with Lipschitz constant independent of t_0. Indeed, for y_1 and y_2 belonging to PX we have

$$\begin{aligned}
\big\|A^\theta\big(\Phi_{t_0}(y_1) - \Phi_{t_0}(y_2)\big)\big\| &\leqslant \int_{-\infty}^{t_0} \big\|A^\theta e^{-(t_0-s)A}(I-P)\big\|\,\big\|\big(f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big)\big\|\,ds \\
&= \int_{-\infty}^{t_0} e^{\gamma(t_0-s)}\big\|A^\theta G(t_0,s)\big\|\,\big\|e^{-\gamma(t_0-s)}\big(f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big)\big\|\,ds \\
&\leqslant \int_{-\infty}^{t_0} e^{\gamma(t_0-s)}\varphi(s)\big\|A^\theta G(t_0,s)\big\|\,\big\|e^{-\gamma(t_0-s)}A^\theta\big(x_1(s) - x_2(s)\big)\big\|\,ds \\
&\leqslant l\big\|x_1(\cdot) - x_2(\cdot)\big\|_{\gamma,\theta}
\end{aligned} \tag{3.15}$$

(here we use the estimate (3.8) and note that the term λ_N^θ disappears since $t = t_0$).

We now estimate $\|x_1(\cdot) - x_2(\cdot)\|_{\gamma,\theta}$. Since $x_i(\cdot)$ is the unique solution in $\mathcal{E}^{\gamma,t_0,\theta}$ of Eq. (2.8) on $(-\infty, t_0]$ satisfying $Px_i(t_0) = y_i$, $i = 1, 2$, respectively, using the form (3.4) we have that

$$\begin{aligned}
&\big\|e^{-\gamma(t_0-t)}A^\theta\big(x_1(t) - x_2(t)\big)\big\| \\
&= \left\|e^{-\gamma(t_0-t)}A^\theta\left(e^{-(t-t_0)A}(y_1 - y_2) + \int_{-\infty}^{t_0} G(t,\tau)\big(f\big(\tau, x_1(\tau)\big) - f\big(\tau, x_2(\tau)\big)\big)\,d\tau\right)\right\| \\
&\leqslant M\lambda_N^\theta e^{-\alpha(t-t_0)}\big\|A^\theta(y_1 - y_2)\big\| + k(t)\big\|x_1(\cdot) - x_2(\cdot)\big\|_{\gamma,\theta} \\
&\leqslant M\lambda_N^\theta\big(T_{t_0}^+ e_\alpha\big)(t)\big\|A^\theta(y_1 - y_2)\big\| + k(t)\big\|x_1(\cdot) - x_2(\cdot)\big\|_{\gamma,\theta} \quad \text{for all } t \leqslant t_0.
\end{aligned}$$

Hence, we obtain that

$$\big\|x_1(\cdot) - x_2(\cdot)\big\|_{\gamma,\theta} \leqslant M\lambda_N^\theta N_1\|e_\alpha\|\,\big\|A^\theta(y_1 - y_2)\big\| + \|k\|_\theta\,\big\|x_1(\cdot) - x_2(\cdot)\big\|_{\gamma,\theta}.$$

Therefore,

$$\left\|x_1(\cdot)-x_2(\cdot)\right\|_{\gamma,\theta} \leqslant \frac{M\lambda_N^\theta N_1\|e_\alpha\|}{1-\|k\|_\theta}\left\|A^\theta(y_1-y_2)\right\|.$$

Substituting this inequality to (3.15) we obtain that

$$\left\|A^\theta\big(\Phi_{t_0}(y_1)-\Phi_{t_0}(y_2)\big)\right\| \leqslant \frac{M\lambda_N^\theta N_1\|e_\alpha\|l}{1-\|k\|_\theta}\left\|A^\theta(y_1-y_2)\right\|$$

yielding that Φ_{t_0} is Lipschitz continuous with the Lipschitz constant $C=\frac{M\lambda_N^\theta N_1\|e_\alpha\|l}{1-\|k\|_\theta}$ independent of t_0. We thus obtain the property (i) in Definition 3.1 of the admissibly inertial manifold.

The property (ii) follows from Lemmas 3.5, 3.3, and Remark 3.4.

We now prove the property (iii). To do this, let $x(\cdot)$ be a solution to Eq. (2.8) satisfying $x(s)=x_0\in\mathbb{M}_s$, i.e., $x(s)=Px(s)+\Phi_s(Px(s))$. Then, we fix an arbitrary number $t_0\in[s,\infty)$ and define a function $w(t)$ on $[s,\infty)$ by

$$w(t)=\begin{cases} x(t) & \text{if } t\in[s,t_0],\\ u(t) & \text{if } t\in(-\infty,s]\end{cases}$$

where $u(t)$ is the unique solution in $\mathcal{E}^{\gamma,t_0,\theta}$ of Eq. (2.8) satisfying $u(s)=x(s)\in\mathbb{M}_s$. Then, using Eqs. (2.8) and (3.14) we obtain that

$$\begin{aligned}
w(t) &= e^{-(t-s)A}\big(Px(s)+\Phi_s\big(Px(s)\big)\big)+\int_s^t e^{-(t-\tau)A}f\big(\tau,w(\tau)\big)\,d\tau\\
&= e^{-(t-s)A}\big(Px(s)\big)+\int_{-\infty}^t e^{-(t-\tau)A}(I-P)f\big(\tau,w(\tau)\big)\,d\tau\\
&\quad+\int_s^t e^{-(t-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau \quad \text{for } s\leqslant t\leqslant t_0.
\end{aligned}\tag{3.16}$$

Obviously, Eq. (3.16) also remains true for $t\in(-\infty,s]$. Now, in Eq. (3.16) setting $t=t_0$ and applying the projection P we obtain that

$$Pw(t_0)=e^{-(t_0-s)A}\big(Px(s)\big)+\int_s^{t_0} e^{-(t_0-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau \quad \text{for } s\leqslant t_0.$$

Since the restriction of the semigroup $(e^{-tA})_{t\geqslant 0}$ on $\operatorname{Im}P$ can be extended to the group $(e^{-tA}P)_{t\in\mathbb{R}}$ and using the fact that $w(t_0)=x(t_0)$, it follows from the above equation that

$$\begin{aligned}
Px(s) &= e^{(t_0-s)A}\big(Px(t_0)\big)-\int_s^{t_0} e^{(t_0-s)A}e^{-(t_0-\tau)A}Pf\big(\tau,w(\tau)\big)\,d\tau\\
&= e^{-(s-t_0)A}\big(Px(t_0)\big)-\int_s^{t_0} e^{-(s-\tau)A}Pf\big(\tau,w(\tau)\big) \quad \text{for } s\leqslant t_0.
\end{aligned}$$

Substituting this form of $Px(s)$ to Eq. (3.16) we obtain that

$$\begin{aligned} w(t) &= e^{-(t-t_0)A}Px(t_0) + \int_{t_0}^{t} e^{-(t-\tau)A}Pf\big(\tau, w(\tau)\big)\,d\tau + \int_{-\infty}^{t} e^{-(t-\tau)A}(I-P)f\big(\tau, w(\tau)\big)\,d\tau \\ &= e^{-(t-t_0)A}Px(t_0) + \int_{-\infty}^{t_0} G(t,\tau)f\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \leqslant t_0. \end{aligned} \tag{3.17}$$

Therefore, $x(t_0) = w(t_0) = Px(t_0) + \Phi_{t_0}(Px(t_0)) \in \mathbb{M}_{t_0}$ for all $t_0 \geqslant s$.

Lastly, we prove the property (iv) of the admissibly inertial manifold. To do this, we will prove that for any solution $u(\cdot)$ to Eq. (2.8) with $u(s) = u_0$ there is a solution $u^*(\cdot)$ of (2.8) such that $u^*(t) \in \mathbb{M}_t$ for $t \geqslant s$ and

$$\big\|A^\theta\big(u(t) - u^*(t)\big)\big\| \leqslant \frac{M\eta}{1-m}e^{-\gamma(t-s)} \quad \text{for all } t \geqslant s \text{ and some constant } \eta, \tag{3.18}$$

where m is given as in (3.12). Note that this solution $u^*(\cdot)$ is called an *induced trajectory*.

To this purpose, we will find the induced trajectory in the form $u^*(t) = u(t) + w(t)$ such that

$$\|w\|_{s,+} = \operatorname{ess\,sup}_{t\geqslant s}\big\{\big\|e^{\gamma(t-s)}A^\theta\big(w(t)\big)\big\|\big\} < \infty. \tag{3.19}$$

Substituting $u^*(\cdot)$ to Eq. (2.8) we obtain that $u^*(\cdot)$ is a solution to (2.8) for $t \geqslant s$ if and only if $w(\cdot)$ is a solution to the equation

$$w(t) = e^{-(t-s)A}w(s) + \int_s^t e^{-(t-\xi)A}\big[f(u+w,\xi) - f(u,\xi)\big]\,d\xi. \tag{3.20}$$

For the sake of simplicity in the presentation we put $F(w,t) = f(u+w,t) - f(u,t)$ and set

$$L_\infty^{s,+} = \big\{v : [s,\infty) \to X_\theta \;\big|\; v \text{ is strongly measurable and } \operatorname{ess\,sup}_{t\geqslant s}\big\|e^{\gamma(t-s)}A^\theta v(t)\big\| < \infty\big\}$$

endowed with the norm $\|\cdot\|_{s,+}$ defined as in equality (3.19).

Then, by the same way as in Lemma 3.3 and Remark 3.4 we can prove that a function $w(\cdot) \in L_\infty^{s,+}$ is a solution to (3.20) if and only if it satisfies

$$w(t) = e^{-(t-s)A}x_0 + \int_s^\infty G(t,\tau)F\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \geqslant s \text{ and some } x_0 \in (I-P)X_\theta. \tag{3.21}$$

Here the value $x_0 \in (I-P)X_\theta$ is chosen such that $u^*(s) = u(s) + w(s) \in \mathbb{M}_s$, i.e., such that

$$(I-P)\big(u(s) + w(s)\big) = \Phi_s\big(P\big(u(s) + w(s)\big)\big).$$

From (3.21) it follows that

$$w(s) = x_0 - \int_s^\infty e^{-(s-\tau)A}PF\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \geqslant s. \tag{3.22}$$

Hence $P(u(s)+w(s)) = Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF(\tau, w(\tau))\,d\tau$, and therefore

$$x_0 = (I-P)w(s) = -(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, w(\tau)\big)\,d\tau\right). \tag{3.23}$$

Substituting this form of x_0 into (3.21) we obtain that

$$\begin{aligned} w(t) = {} & e^{-(t-s)A}\left[-(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, w(\tau)\big)\,d\tau\right)\right] \\ & + \int_s^\infty G(t,\tau)F\big(\tau, w(\tau)\big)\,d\tau \quad \text{for } t \geqslant s. \end{aligned} \tag{3.24}$$

What we have to do now to prove the existence of u^* satisfying (3.18) is to prove that Eq. (3.24) has a solution $w(\cdot) \in L_\infty^{s,+}$. To do this, we will prove that the transformation T defined by

$$\begin{aligned} (Tx)(t) = {} & e^{-(t-s)A}\left[-(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, x(\tau)\big)\,d\tau\right)\right] \\ & + \int_s^\infty G(t,\tau)F\big(\tau, x(\tau)\big)\,d\tau \quad \text{for } t \geqslant s \end{aligned}$$

acts from $L_\infty^{s,+}$ into itself and is a contraction.

Indeed, for $x(\cdot) \in L_\infty^{s,+}$, we have that $\|F(t,x(t))\| \leqslant \varphi(t)\|A^\theta x(t)\|$, therefore, putting

$$q(x) := -(I-P)u(s) + \Phi_s\left(Pu(s) - \int_s^\infty e^{-(s-\tau)A} PF\big(\tau, x(\tau)\big)\,d\tau\right)$$

we can estimate

$$\begin{aligned} \left\|e^{\gamma(t-s)}A^\theta (Tx)(t)\right\| & \leqslant \left\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\right\| + \int_s^\infty \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\| \varphi(\tau) e^{\gamma(\tau-s)}\left\|A^\theta x(\tau)\right\|\,d\tau \\ & \leqslant \left\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\right\| + \int_s^\infty \left\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\right\| \varphi(\tau)\,d\tau\,\left\|x(\cdot)\right\|_{s,+}. \end{aligned} \tag{3.25}$$

Using Lipschitz property of Φ_s and for $t \geqslant s$ we now estimate the first term in the right-hand side of the above formula. In fact,

$$\begin{aligned} & \left\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A} q(x)\right\| \\ & \quad \leqslant \left\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A}\big(-(I-P)u(s) + \Phi_s\big(Pu(s)\big)\big)\right\| \\ & \qquad + \left\|e^{\gamma(t-s)}A^\theta e^{-(t-s)A}\big(q(x) + (I-P)u(s) - \Phi_s\big(Pu(s)\big)\big)\right\| \end{aligned}$$

$$
\begin{aligned}
&\leqslant Me^{-(\lambda_{N+1}-\gamma)(t-s)}\big(\big\|A^\theta\big(-(I-P)u(s)+\Phi_s\big(Pu(s)\big)\big)\big\| \\
&\quad+\big\|A^\theta\big(q(x)+(I-P)u(s)-\Phi_s\big(Pu(s)\big)\big)\big\|\big) \\
&\leqslant M\eta+M\big\|A^\theta\big(q(x)+(I-P)u(s)-\Phi_s\big(Pu(s)\big)\big)\big\| \\
&\text{here } \eta:=\big\|A^\theta\big(-(I-P)u(s)+\Phi_s\big(Pu(s)\big)\big)\big\| \\
&=M\eta+M\left\|A^\theta\left[\Phi_s\left(Pu(s)-\int_s^\infty e^{-(s-\tau)A}PF\big(\tau,x(\tau)\big)\,d\tau\right)-\Phi_s\big(Pu(s)\big)\right]\right\| \\
&\leqslant M\eta+\frac{M^2\lambda_N^\theta\|e_\alpha\|l}{1-\|k\|_\theta}\left\|\int_s^\infty A^\theta e^{-(s-\tau)A}PF\big(\tau,x(\tau)\big)\,d\tau\right\| \\
&\leqslant M\eta+\frac{M^3\lambda_N^{2\theta}\|e_\alpha\|l}{1-\|k\|_\theta}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\big\|e^{\gamma(\tau-s)}A^\theta x(\tau)\big\|\,d\tau \\
&\leqslant M\eta+\frac{M^3\lambda_N^{2\theta}N_2\|e_\alpha\|l}{(1-\|k\|_\theta)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty\big\|x(\cdot)\big\|_{s,+}
\end{aligned}
\tag{3.26}
$$

where k is defined as in (3.11).

We next estimate the second term of (3.25). In fact,

$$
\begin{aligned}
&\int_s^\infty \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)\,d\tau \\
&\leqslant \int_{-\infty}^\infty \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)\,d\tau \\
&=\int_{-\infty}^t \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)\,d\tau+\int_t^\infty \big\|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\big\|\varphi(\tau)\,d\tau \\
&\leqslant \int_{-\infty}^t M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau+\frac{M(\lambda_{N+1}^\theta N_1+\lambda_N^\theta N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty \\
&\leqslant \int_{-\infty}^{t-1} M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau+\int_{t-1}^t M\left(\frac{\theta}{t-\tau}\right)^\theta e^{-\alpha(t-\tau)}\varphi(\tau)\,d\tau \\
&\quad+\frac{M(\lambda_{N+1}^\theta N_1+\lambda_N^\theta N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty \\
&\leqslant \frac{M(N_1\theta^\theta+\lambda_{N+1}^\theta N_1+\lambda_N^\theta N_2)}{1-e^{-\alpha}}\|\Lambda_1\varphi\|_\infty+M\theta\left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}}\big\|\Lambda_1\varphi^{\frac{1+\theta}{1-\theta}}\big\|_\infty^{\frac{1-\theta}{1+\theta}}.
\end{aligned}
\tag{3.27}
$$

Substituting the estimates (3.26) and (3.27) to (3.25) we obtain

$$\|Tx\|_{s,+} \leqslant M\eta + \frac{M^3\lambda_N^{2\theta}N_2\|e_\alpha\|l + (1-\|k\|_\theta)M((\theta^\theta + \lambda_{N+1}^\theta)N_1 + \lambda_N^\theta N_2)}{(1-\|k\|_\theta)(1-e^{-\alpha})}\|\Lambda_1\varphi\|_\infty \|x(\cdot)\|_{s,+}$$
$$+ M\theta\left(\frac{2}{1-\theta}\right)^{\frac{2\theta}{1+\theta}} \|\Lambda_1\varphi^{\frac{1+\theta}{1-\theta}}\|_\infty^{\frac{1-\theta}{1+\theta}} \|x(\cdot)\|_{s,+} = M\eta + m\|x(\cdot)\|_{s,+}, \tag{3.28}$$

where m is given as in (3.12). Therefore, the transformation T acts from $L_\infty^{s,+}$ to $L_\infty^{s,+}$.

Using the fact that $\|F(w_1,t) - F(w_2,t)\| \leqslant \varphi(t)\|A^\theta(w_1-w_2)\|$ and for $x, z \in L_\infty^{s,+}$ we now estimate

$$\begin{aligned}
\|e^{\gamma(t-s)}A^\theta(Tx(t) - Tz(t))\| &\leqslant \frac{M^2\lambda_N^\theta N_1\|e_\alpha\|l}{1-\|k\|_\theta}\left\|\int_s^\infty A^\theta e^{-\gamma(s-\tau)A}P\big(F(\tau,x(\tau)) - F(\tau,z(\tau))\big)\,d\tau\right\| \\
&\quad + \int_s^\infty \|e^{\gamma(t-s)}A^\theta G(t,\tau)\|\,\|F(\tau,x(\tau)) - F(\tau,z(\tau))\|\,d\tau \\
&\leqslant \frac{M^3\lambda_N^{2\theta}N_1\|e_\alpha\|l}{1-\|k\|_\theta}\int_s^\infty e^{-\alpha(\tau-s)}\varphi(\tau)\|e^{(\tau-s)}A^\theta(x(\tau)-z(\tau))\|\,d\tau \\
&\quad + \int_s^\infty \|e^{\gamma(t-\tau)}A^\theta G(t,\tau)\|\varphi(\tau)e^{\gamma(\tau-s)}\|A^\theta(x(\tau)-z(\tau))\|\,d\tau \\
&\leqslant m\|x(\cdot)-z(\cdot)\|_{s,+} \quad \text{for } t \geqslant s,
\end{aligned}$$

where m is given as above.

Therefore,

$$\|Tx(\cdot) - Tz(\cdot)\|_{s,+} \leqslant m\|x(\cdot)-z(\cdot)\|_{s,+}.$$

Hence, if $m < 1$, then we obtain that $T : L_\infty^{s,+} \to L_\infty^{s,+}$ is a contraction. Thus, there exists a unique $w(\cdot) \in L_\infty^{s,+}$ such that $Tw = w$. By the definition of T we have that $w(\cdot)$ is the unique solution in $L_\infty^{s,+}$ of Eq. (3.24) for $t \geqslant s$. Also, using inequality (3.28) we have the estimate for $\|w(\cdot)\|_{s,+}$ as

$$\|w(\cdot)\|_{s,+} \leqslant \frac{M\eta}{1-m}.$$

Furthermore, by determination of w we obtain the existence of the solution $u^* = u + w$ to Eq. (2.8) such that $u^*(t) \in \mathbb{M}_t$ for $t \geqslant s$, and u^* satisfies the inequality (3.18) yielding that

$$\|A^\theta(u^*(t) - u(t))\| = \|A^\theta w(t)\| \leqslant \frac{M\eta}{1-m}e^{-\gamma(t-s)} \quad \text{for all } t \geqslant s.$$

Putting $H := \frac{M\eta}{1-m}$ it follows from this inequality that

$$dist_{X_\theta}(u(t), \mathbb{M}_t) \leqslant He^{-\gamma(t-s)} \quad \text{for all } t \geqslant s.$$

Therefore, $(\mathbb{M}_t)_{t\in\mathbb{R}}$ exponentially attracts every solution u of (2.8). $\square$

Remark 3.7. By the definition of the function k (see (3.11)) and constant m (see (3.12)) we have that, for $0 \leqslant \theta < 1$, the condition (3.13) is fulfilled if the norms

$$\big\| \big\| e^{-\alpha|t-\cdot|}\varphi(\cdot) \big\|_{E'} \big\|_\theta, \quad \|\Lambda_1 \varphi\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} \int_{t-1}^{t} \varphi(\tau)\, d\tau, \quad \text{and} \quad \big\| \Lambda_1 \varphi^{\frac{1+\theta}{1-\theta}} \big\|_\infty$$

are sufficiently small.

We illustrate our result in the following example.

Example 3.8. Consider the reaction–diffusion problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = g\Big(x, u, \dfrac{\partial u}{\partial x}, t\Big), \quad 0 < x < l,\ t > 0, \\ u|_{x=0} = u|_{x=l} = 0; \quad u|_{t=0} = u_0(x) \end{cases} \tag{3.29}$$

where $g(x, u, \frac{\partial u}{\partial x}, t)$ is a continuous function of its variables such that

$$\big|g(x, u_1, \xi_1, t) - g(x, u_1, \xi_2, t)\big| \leqslant \psi(t)\big(M_1|u_1 - u_2| + M_2|\xi_1 - \xi_2|\big) \quad \text{for all } x \in (0, l),\ t \geqslant 0,$$
$$\big|g(x, 0, 0, t)\big| \leqslant M_3 \psi(t)$$

here M_i, $i = 1, 2, 3$, are positive numbers, and φ belongs to the admissible function space E. We choose the Hilbert space $X = L^2(0, l)$ and consider the operators

$$A = -\frac{d^2}{dx^2} \quad \text{with } D(A) = H_0^1(0, l) \cap H^2(0, l)$$

and

$$f : \mathbb{R} \times D\big(A^{\frac{1}{2}}\big) \to X \quad \text{defined by } f(t, u)(x) = g\Big(x, u, \frac{\partial u}{\partial x}, t\Big).$$

Then, we have that A satisfies the Standing Hypothesis 2.1 with the discrete point spectra being

$$\Big(\frac{\pi}{l}\Big)^2, \Big(\frac{\pi}{l}\Big)^2 4, \ldots, \Big(\frac{\pi}{l}\Big)^2 n^2, \ldots.$$

Obviously,

$$\big\| f(t, u) - f(t, v)\big\| \leqslant \psi(t)\bigg(M_1\|u - v\| + M_2\bigg\|\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\bigg\|\bigg).$$

Using now the Poincaré inequality

$$\bigg\|\frac{\partial u}{\partial x}\bigg\|^2 \geqslant \Big(\frac{\pi}{l}\Big)^2 \|u\|^2 \quad \text{for } u \in H_0^1(0, l),$$

we obtain that

$$\left\| f(t,u) - f(t,v) \right\| \leqslant \psi(t) \frac{(lM_1 + \pi M_2)}{\pi} \left\| A^{\frac{1}{2}}(u-v) \right\| \quad \text{for all } t \in \mathbb{R} \text{ and } u, v \in X_{\frac{1}{2}}.$$

Therefore, f is φ-Lipschitz with $\varphi = \max\{\psi(t), M_3\psi(t), \frac{(lM_1+\pi M_2)}{\pi}\psi(t)\}$. We now choose concretely the space $E = L_p$ and its associate space $E' = L_q$ for $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For fixed constants $c > 1$ and $b > 0$ we take $\psi(t)$ of the form

$$\psi(t) = \begin{cases} |m|^{\frac{1}{3}} & \text{if } t \in [\frac{2m+1}{2} - \frac{1}{e^{cm}}, \ \frac{2m+1}{2} + \frac{1}{e^{cm}}] \text{ for } m = 0, \pm 1, \pm 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

We note that ψ can take any arbitrarily large value but we still have that $\psi \in L_q$ (for $1 \leqslant q < c$) and

$$\|\psi\|_{L_q} \leqslant \frac{4^{\frac{1}{q}}}{(1 - e^{q-c})^{\frac{1}{q}}}.$$

Also, it is straightforward to obtain the estimates $\|\Lambda_1 \psi\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} \int_{t-1}^{t} |\psi(\tau)|\, d\tau \leqslant \frac{1}{2^{c-1}}$,

$$\left\| \Lambda_1 \psi^3 \right\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} \int_{t-1}^{t} \left|\psi(\tau)\right| d\tau \leqslant \frac{3}{2^{c-1}}$$

and

$$\left\| \left\| e^{-\alpha|t-\cdot|}\psi(\cdot) \right\|_{L_q} \right\|_{\frac{1}{2}} \leqslant \frac{3^{\frac{1}{q}} e^{-\frac{\alpha}{2}}}{(p\alpha)^{\frac{1}{2p}} (1 - e^{q-c-q\alpha})^{\frac{1}{2q}}}.$$

Applying Theorem 3.6 we obtain that, for $\theta = 1/2$, if the constant c is sufficiently large, and N is large enough (i.e., the difference $(\frac{\pi}{l})^2(N+1)^2 - (\frac{\pi}{l})^2 N^2$ is large enough), then Eq. (3.29) has an admissibly inertial manifold (of L_p-class).

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References

[1] B. Aulbach, N.V. Minh, Nonlinear semigroups and the existence and stability of semilinear nonautonomous evolution equations, Abstr. Appl. Anal. 1 (1996) 351–380.
[2] N. Bogoliubov, Yu. Mitropolsky, The method of integral manifolds in nonlinear mechanics, Contrib. Differential Equations 2 (1963) 123–196.
[3] N. Bogoliubov, Yu. Mitropolsky, Asymptotic Methods in the Theory of Non-linear Oscillations, Int. Monogr. Adv. Math. Phys., Gordon and Breach Science Publishers, New York, 1961, translated from the second revised Russian edition.
[4] A.P. Calderon, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. 26 (1996) 273–299.
[5] J. Carr, Applications of Centre Manifold Theory, Appl. Math. Sci., vol. 35, Springer-Verlag, New York, Berlin, 1981.
[6] S.N. Chow, K. Lu, Invariant manifolds for flows in Banach spaces, J. Differential Equations 74 (1988) 285–317.
[7] I. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, ACTA Scientific Publishing House, 2002.
[8] P. Constantin, C. Foias, B. Nicolaenko, R. Temam, Integral Manifolds and Inertial Manifolds for Dissipative Partial Differential Equations, Springer, New York, 1989.
[9] Ju.L. Daleckii, M.G. Krein, Stability of Solutions of Differential Equations in Banach Spaces, Transl. Math. Monogr., Amer. Math. Soc., Providence, RI, 1974.

[10] K.J. Engel, R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, Berlin, Heidelberg, 2000.
[11] J. Hadamard, Sur l'interation et les solutions asymptotiques des equations differentielles, Bull. Soc. Math. France 29 (1991) 224–228.
[12] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer, Berlin, Heidelberg, New York, 1981.
[13] Nguyen Thieu Huy, Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line, J. Funct. Anal. 235 (2006) 330–354.
[14] Nguyen Thieu Huy, Invariant manifolds of admissible classes for semi-linear evolution equations, J. Differential Equations 246 (2009) 1820–1844.
[15] Nguyen Thieu Huy, Inertial manifolds for semi-linear parabolic equations in admissible spaces, J. Math. Anal. Appl. 386 (2012) 894–909.
[16] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II: Function Spaces, Springer-Verlag, Berlin, 1979.
[17] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, 1995.
[18] J. Mallet-Paret, G.R. Sell, Inertial manifolds for reaction–diffusion equations in higher space dimensions, J. Amer. Math. Soc. 1 (1988) 805–866.
[19] J.J. Massera, J.J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.
[20] M. Miklavcic, A sharp condition for existence of an inertial manifold, J. Dynam. Differential Equations 3 (1991) 437–456.
[21] N.V. Minh, J. Wu, Invariant manifolds of partial functional differential equations, J. Differential Equations 198 (2004) 381–421.
[22] A. Pazy, Semigroup of Linear Operators and Application to Partial Differential Equations, Springer-Verlag, Berlin, 1983.
[23] O. Perron, Über Stabilität und asymptotisches Verhalten der Integrale von Differential gleichungssystemen, Math. Z. 29 (1) (1929) 129–160 (in German).
[24] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703–728.
[25] F. Räbiger, R. Schaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, Semigroup Forum 48 (1996) 225–239.
[26] G.R. Sell, Y. You, Dynamics of Evolutionary Equations, Appl. Math. Sci., vol. 143, Springer-Verlag, 2002.
[27] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, New York, Oxford, 1978.