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Positivity and Stability of Standard and Fractional Descriptor Continuous-Time Linear and Nonlinear Systems

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Abstract: The positivity and stability of standard and fractional descriptor continuous-time linear and nonlinear systems are addressed. Necessary and sufficient conditions for the positivity of descriptor linear and sufficient conditions for nonlinear systems are established. Using an extension of Lyapunov method sufficient conditions for the stability of positive nonlinear systems are given. The considerations are extended to fractional nonlinear systems.

Keywords: descriptor, linear, nonlinear, fractional, system, positivity, stability

1 Introduction

A dynamical system is called fractional if it is described by fractional order differential equation [1–6]. The fundamentals of fractional differential equations and systems have been given in [3–6]. The stability of linear and nonlinear systems has been analyzed in Refs [1, 7–15].

Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology, medicine, etc. An overview of state of the art in positive systems theory is given in monographs [16–18].

Positive linear systems with different fractional orders have been addressed in Refs [19–21]. Descriptor (singular) linear systems have been analyzed in Refs [22–27]. The stability of a class of nonlinear fractional-order systems has been analyzed in Refs [10, 15, 28]. Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems has been presented in Ref. [29]. The stability and stabilization of nonlinear fractional systems have been analyzed in Refs [30,31, 32].

In this paper the positivity and stability of standard and fractional descriptor continuous-time linear and nonlinear systems will be investigated.

The paper is organized as follows. In Section 2 some definitions and theorems concerning positivity and stability of standard and fractional linear systems are recalled. Positivity of descriptor linear systems is analyzed in Section 3 and the positivity and stability of descriptor nonlinear systems in Section 4. The positivity and stability of fractional descriptor nonlinear systems is addressed in Section 5. Concluding remarks are given in Section 6.

The following notations will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2 Preliminaries

Consider the continuous-time linear system

$$\dot{x}=Ax+Bu, \tag{1}$$

where $x=x(t)\in\Re^n$, $u=u(t)\in\Re^m$ are the state and input vectors and $A=[a_{ij}]\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

Definition 2.1 [1, 17]. The system (1) is called (internally) positive if $x(t)\in\Re_+^n$, $t\geq 0$ for all $x(0)\in\Re_+^n$ and every $u(t)\in\Re_+^m$, $t\geq 0$.

Theorem 2.1 [1, 17]. The system (1) is positive if and only if

$$A\in M_n,\ B\in\Re_+^{n\times m}. \tag{2}$$

Definition 2.2 [1, 17]. The positive system (1) for $u(t)=0$ is called asymptotically stable if

$$\lim_{t\to\infty}x(t)=0,\ \text{forall}\, x(0)\in\Re_+^n. \tag{3}$$

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Theorem 2.2 [1, 17]. The positive system (1) for $u(t)=0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0 \tag{4}$$

are positive, i. e. $a_i > 0$ for $i = 0, 1, \dots, n-1$.

2) All principal minors $\bar{M}_i$, $i=1,\dots,n$ of the matrix $-A$ are positive, i.e.

$$\bar{M}_1 = |-a_{11}| > 0,\ \bar{M}_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0,\ \dots,$$
$$\bar{M}_n = \det[-A] > 0. \tag{5}$$

3) There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k=1,\dots,n$ such that

$$A\lambda < 0. \tag{6}$$

If $\det A \neq 0$ then we may choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is any strictly positive vector.

Theorem 2.3 [1]. If $A \in M_n$ is asymptotically stable then

$$-A^{-1} \in \Re_+^{n\times n}. \tag{7}$$

Caputo fractional derivative of α order is defined as follows:

$${}_0D_t^\alpha f(t) = \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{f}(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau,\ 0<\alpha<1, \tag{8}$$

where $\dot{f}(\tau) = \frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$, $\mathrm{Re}(x) > 0$ is the Euler gamma function.

Consider the fractional continuous-time linear system

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t),\ 0<\alpha<1, \tag{9}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Theorem 2.4 [1]. The solution to the eq. (9) has the form

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)\mathrm{d}\tau,\ x(0)=x_0, \tag{10}$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{11}$$

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{12}$$

are the Mittag-Leffler functions.

Definition 2.3 [1]. The fractional system (9) is called (internally) positive if $x(t) \in \Re_+^n$, $t \geq 0$ for all initial conditions $x(0) \in \Re_+^n$ and every $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 2.5 [1]. The fractional system (9) is positive if and only if

$$A \in M_n,\ B \in \Re_+^{n\times m}. \tag{13}$$

If $A \in M_n$ then

$$\Phi_0(t) \in \Re_+^{n\times n} \text{ and } \Phi(t) \in \Re_+^{n\times n} \quad \text{for} \quad t \geq 0. \tag{14}$$

3 Positivity of descriptor linear systems

Consider the descriptor continuous-time linear system

$$E\dot{x} = Ax + Bu, \tag{15}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

It is assumed that:

(A1) The matrix E has $n_1 < n$ linearly independent columns (the remaining columns are zero).

(A2) The pencil of eq. (15) is regular, i.e.

$$\det[Es - A] \neq 0 \text{ for some } s \in \mathbf{C} \text{ (thefieldof complex numbers)} \tag{16}$$

Defining the new state vector

$$\bar{x} = \bar{x}(t) = P^{-1}x(t) = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix},\ \bar{x}_1 \in \Re^{n_1},\ \bar{x}_2 \in \Re^{n_2},\ n_2 = n - n_1 \tag{17}$$

and premultiplying the eq. (15) by the matrix $Q \in \Re^{n\times n}$ we obtain

$$QEPP^{-1}\dot{x} = QAPP^{-1}x + QBu \tag{18}$$

and

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u, \tag{19}$$

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + \bar{B}_2 u, \tag{20}$$

where

$$QEP=\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}, \bar{A}=QAP=\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}, \bar{A}_{11}\in\Re^{n_1\times n_1},$$
$$\bar{A}_{22}\in\Re^{n_2\times n_2}, \bar{B}=QB=\begin{bmatrix} \bar{B}_1\\ \bar{B}_2\end{bmatrix}, \bar{B}_1\in\Re^{n_1\times m}, \bar{B}_2\in\Re^{n_2\times m}. \tag{21}$$

The matrices Q and P can be obtained by the use of the following elementary row and column operations [18]:

(1) Multiplication of the i-th row (column) by a real number c. This operation will be denoted by $L[i\times c](R[i\times c])$.
(2) Addition to the i-th row (column) of the j-th row (column) multiplied by a real number c. This operation will be denoted by $L[i+j\times c](R[i+j\times c])$.
(3) Interchange of the i-th and j-th rows (columns). These operations will be denoted by $L[i,j](R[i,j])$.

From assumption (A1) it follows that the matrix P is a permutation matrix and $P^{-1}=P^T\in\Re_+^{n\times n}$ (T denote transpose). Therefore, if $x(t)\in\Re_+^n$, $t\geq 0$ then $\bar{x}(t)=P^{-1}x(t)\in\Re_+^n$, $t\geq 0$. The matrix $Q\in\Re^{n\times n}$ can be obtained by performing the elementary row operations on the identity matrix I_n (see Example 3.1).

Theorem 3.1. The descriptor linear system (15) satisfying the assumptions (A1) and (A2) is positive and asymptotically stable if and only if the matrix $\bar{A}\in M_n$ is asymptotically stable and $\bar{B}\in\Re_+^{n\times m}$.

Proof. If $\bar{A}\in M_n$ is asymptotically stable then $\bar{A}_{22}\in M_{n_2}$ and by Theorem 2.3 $-\bar{A}_{22}^{-1}\in\Re_+^{n_2\times n_2}$. From eq. (20) we have

$$\bar{x}_2=-\bar{A}_{22}^{-1}\bar{A}_{21}x_1-\bar{A}_{22}^{-1}\bar{B}_2u. \tag{22}$$

Note that $\bar{x}_2(t)\in\Re_+^{n_2}$ for $\bar{x}_1(t)\in\Re_+^{n_1}$ and $u(t)\in\Re_+^m$, $t\geq 0$ since $-\bar{A}_{22}^{-1}\bar{A}_{21}\in\Re_+^{n_2\times n_1}$ and $-\bar{A}_{22}^{-1}\bar{B}_2\in\Re_+^{n_2\times m}$.

Substitution of eq. (22) into eq. (19) yields

$$\dot{\bar{x}}_1=\bar{A}_1x_1+\tilde{B}_1u, \tag{23}$$

where

$$\bar{A}_1=\bar{A}_{11}-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21},\ \tilde{B}_1=\bar{B}_1-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2. \tag{24}$$

From the assumption $\bar{A}\in M_n$, $\bar{B}\in\Re_+^{n\times m}$ and $-\bar{A}_{22}^{-1}\in\Re_+^{n_2\times n_2}$ it follows that $\bar{A}_1\in M_{n_1}$, $\tilde{B}_1\in\Re_+^{n_1\times m}$ since $\bar{A}_{11}\in M_{n_1}$ and $-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\in\Re_+^{n_1\times n_1}$, $-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2\in\Re_+^{n_1\times m}$.

Therefore, $\bar{x}_1(t)\in\Re_+^{n_1}$ and $\bar{x}_2(t)\in\Re_+^{n_2}$, $t\geq 0$ and the descriptor system (15) is positive.

Premultiplying the equation

$$\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}\frac{d\bar{x}}{dt}=\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}\bar{x} \tag{25}$$

by the matrix

$$\begin{bmatrix} I_{n_1} & -\bar{A}_{12}\bar{A}_{22}^{-1}\\ 0 & I_{n_2}\end{bmatrix} \tag{26}$$

we obtain

$$\begin{bmatrix} I_{n_1} & 0\\ 0 & 0\end{bmatrix}\frac{d\bar{x}}{dt}=\begin{bmatrix} \bar{A}_1 & 0\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}\bar{x}. \tag{27}$$

From eq. (27) it follows that the matrix $\bar{A}_1$ is asymptotically stable if and only if the matrix $\bar{A}\in M_n$ is asymptotically stable.

Example 3.1. Consider the descriptor system (15) with the matrices

$$E=\begin{bmatrix} 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 2 & 0 & 0 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}, A=\begin{bmatrix} 1 & 0 & 2 & -4\\ -3 & 1 & 0 & 1\\ -8 & 3 & -6 & 11\\ 2 & -2 & 2 & -4\end{bmatrix},$$
$$B=\begin{bmatrix} 0 & 2\\ 1 & 0\\ 2 & -3\\ 1 & 2\end{bmatrix}. \tag{28}$$

The system satisfies the assumptions (A1) and (A2) since the matrix E has only $n_1=2$ nonzero columns and

$$\det[Es-A]=\begin{vmatrix} -1 & 0 & -2 & s+4\\ s+3 & -1 & 0 & -1\\ 2s+8 & -3 & 6 & -2s-11\\ -2 & 2 & -2 & s+4\end{vmatrix}=4s^2+22s+24. \tag{29}$$

The permutation matrix P has the form

$$P=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} \tag{30}$$

and it can be obtained from I_4 by performing the column operation $R[2,4]$.

The matrix

$$Q=\begin{bmatrix} 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 2 & -2 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix} \tag{31}$$

can be obtained from I_4 by performing the row operations $L[1,2]$, $L[3+1\times(-2)]$, $L[3+2\times 2]$ and $L[4+2\times(-1)]$.

Using eqs. (21), (25), (30) and (31) we obtain

$$QEP = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A} = QAP = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & -4 \\ -3 & 1 & 0 & 1 \\ -8 & 3 & -4 & 11 \\ 2 & -2 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 1 \\ 1 & -4 & 2 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}, \bar{B} = QB = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 2 & -3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix}. \tag{32}$$

From eq. (32) it follows that $\bar{A} \in M_4$, $\bar{B} \in \Re_+^{4\times 2}$ and the matrix $\bar{A}$ is asymptotically stable since all its principal minors are positive:

$$\bar{M}_1 = |-a_{11}| = 3 > 0, \bar{M}_2 = \begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} = 11 > 0,$$
$$\bar{M}_3 = \begin{vmatrix} 3 & -1 & 0 \\ 1 & 4 & -2 \\ 0 & 1 & 2 \end{vmatrix} = 16 > 0, \tag{33}$$
$$\bar{M}_4 = \det[-\bar{A}] = \begin{vmatrix} 3 & -1 & 0 & -1 \\ -1 & 4 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 2 \end{vmatrix} = 24 > 0.$$

Therefore, by Theorem 2.2 the descriptor system with eq. (28) is positive. Using eqs. (23), (24) and (32) we obtain

$$\bar{A}_1 = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21} = \begin{bmatrix} -3 & 1 \\ 1 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2.5 & 1 \\ 0.5 & -3 \end{bmatrix},$$
$$\tilde{B}_1 = \bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1.5 & 0 \\ 0.5 & 3 \end{bmatrix} \tag{34}$$

and the positive system eqs. (23), (24) has the form

$$\dot{\bar{x}}_1 = \begin{bmatrix} -2.5 & 1 \\ 0.5 & -3 \end{bmatrix} \bar{x}_1 + \begin{bmatrix} 1.5 & 0 \\ 0.5 & 3 \end{bmatrix} u \tag{35}$$

and its solution is given by

$$\bar{x}_1(t) = e^{\bar{A}_1 t}\bar{x}_1(0) + \int_0^t e^{\bar{A}_1(t-\tau)}\tilde{B}_1 u(\tau)\mathrm{d}\tau, \tag{36}$$

where

$$e^{\bar{A}_1 t} = \begin{bmatrix} \frac{2}{3}e^{-2t} + \frac{1}{3}e^{-\frac{7}{2}t} & \frac{2}{3}e^{-2t} - \frac{2}{3}e^{-\frac{7}{2}t} \\ \frac{1}{3}e^{-2t} - \frac{1}{3}e^{-\frac{7}{2}t} & \frac{1}{3}e^{-2t} + \frac{2}{3}e^{-\frac{7}{2}t} \end{bmatrix}. \tag{37}$$

Knowing the solution $\bar{x}_1(t)$ of eq. (35) and using eq. (22) we can find

$$\begin{aligned} \bar{x}_2(t) &= -\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1(t) - \bar{A}_{22}^{-1}\bar{B}_2 u(t) \\ &= -\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{x}_1(t) \\ &\quad - \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ &= \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0 \end{bmatrix} \bar{x}_1(t) + \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0 \end{bmatrix} u(t). \end{aligned} \tag{38}$$

4 Positivity and stability of descriptor nonlinear systems

Consider the descriptor continuous-time nonlinear system

$$E\dot{x} = Ax + f(x, u), \tag{39}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors, $E, A \in \Re^{n\times n}$ and $f(x, u) \in \Re^n$ is the continuous vector function of x and u.

It is assumed that the matrices E and A satisfy the assumptions (A1) and (A2) of Section 3. Defining the new state vector by eq. (17) and premultiplying the eq. (39) by the matrix $Q \in \Re^{n\times n}$ we obtain

$$QEPP^{-1}\dot{x} = QAPP^{-1}x + Qf(x,u), \tag{40}$$

and

$$\dot{\bar{x}}_1 = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + f_1(\bar{x},u), \tag{41}$$

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + f_2(\bar{x},u), \tag{42}$$

where $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$ is defined by eq. (21) and

$$Qf(x,u) = Qf(P\bar{x},u) = \begin{bmatrix} f_1(\bar{x},u) \\ f_2(\bar{x},u) \end{bmatrix}, f_1(\bar{x},u) \in \Re^{n_1}, f_2(\bar{x},u) \in \Re^{n_2}. \tag{43}$$

The permutation matrix $P \in \Re^{n\times n}$ and the elementary row operations matrix $Q \in \Re^{n\times n}$ are defined in the same way as in Section 3 for system (15).

Theorem 4.1. The descriptor nonlinear system (39) satisfying the assumptions (A1), (A2) is positive if the matrix $\bar{A} \in M_n$ is asymptotically stable and

$$\begin{bmatrix} f_1(\bar{x},u) \\ f_2(\bar{x},u) \end{bmatrix} \in \Re_+^n \text{ for } \bar{x}(t) \in \Re_+^n \text{ and } u(t) \in \Re_+^m,\ t \ge 0. \tag{44}$$

Proof. If $\bar{A} \in M_n$ is asymptotically stable then $\bar{A}_{22} \in M_{n_2}$ and by Theorem 2.3 we have $-\bar{A}_{22}^{-1} \in \Re_+^{n_2\times n_2}$. From eq. (42) we have

$$\bar{x}_2 = -\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1 - \bar{A}_{22}^{-1}f_2(\bar{x},u) \in \Re_+^{n_2},\ t \ge 0 \tag{45}$$

since by eq. (44) $f_2(\bar{x},u) \in \Re_+^{n_2}$, $t \ge 0$.

Substituting eq. (45) into eq. (41) we obtain

$$\dot{\bar{x}}_1 = \bar{A}_1\bar{x}_1 + \bar{f}_1(\bar{x},u), \tag{46}$$

where $\bar{A}_1$ is defined by eq. (24) and

$$\bar{f}_1(\bar{x},u) = f_1(\bar{x},u) - \bar{A}_{12}\bar{A}_{22}^{-1}f_2(\bar{x},u) \in \Re_+^{n_1},\ t \ge 0 \tag{47}$$

since eq. (44) holds.

Therefore, $\bar{x}_1 \in \Re_+^{n_1}$ and $\bar{x}_2 \in \Re_+^{n_2}$ for $t \ge 0$ and the descriptor nonlinear system (39) is positive.

Example 4.1. (Continuation of Example 3.1) Consider the descriptor nonlinear system with the matrices E and A given by eq. (28) and

$$f(x,u) = \begin{bmatrix} e^{-t}(1-\sin t) \\ \bar{x}_{21}^2 + \bar{x}_{22} \\ \bar{x}_{11}\bar{x}_{22} \\ 1-e^{-t} \end{bmatrix} \in \Re_+^4,\ t \ge 0,\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \tag{48}$$

Using the matrices eqs. (30)–(32) obtained in Example 3.1 we have

$$\dot{\bar{x}}_1 = \begin{bmatrix} \dot{\bar{x}}_{11} \\ \dot{\bar{x}}_{12} \end{bmatrix} = \begin{bmatrix} -2.5 & 1 \\ 0.5 & -3 \end{bmatrix}\begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} + \begin{bmatrix} e^{-t}(1-\sin t) \\ 1-e^{-t} \end{bmatrix} \tag{49}$$

and

$$\bar{x}_2 = \begin{bmatrix} \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} + \begin{bmatrix} 0.5 & 0.25 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} \bar{x}_{11}\bar{x}_{12} \\ e^{-t} + \bar{x}_{11}^2 \end{bmatrix}. \tag{50}$$

Solving the positive linear system (49) for given $\bar{x}_1(0) \in \Re_+^2$ we obtain $\bar{x}_1(t) \in \Re_+^2$, $t \ge 0$ and next from eq. (50) $\bar{x}_2(t) \in \Re_+^2$, $t \ge 0$.

Therefore, the descriptor nonlinear system is positive.

Definition 4.1. The positive descriptor nonlinear system (39) for $u = 0$ is called asymptotically stable in the region $D \in \Re_+^n$ if $x(t) \in \Re_+^n$, $t \ge 0$ and

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x(0) \in D. \tag{51}$$

To test the asymptotic stability of the positive nonlinear system (39) the following extension of the Lyapunov method will be used.

As a candidate of Lyapunov function we choose

$$V(x) = c^T x > 0 \text{ for } x = x(t) \in D \in \Re_+^n,\ t \ge 0 \tag{52}$$

where $c \in \Re_+^n$ is a vector with strictly positive components, $c_k > 0$ for $k = 1, \ldots, n$.

Note that the positive descriptor nonlinear system (39) is asymptotically stable in the region $D \in \Re_+^n$ if the nonlinear subsystem (46) for $u = 0$ is asymptotically stable in D and

$$\lim_{t\to\infty} f_1(\bar{x},0) = 0 \text{ for } x(t) \in D. \tag{53}$$

Using eq. (52) for $x = \bar{x}_1$ and eq. (46) we obtain

$$\frac{\mathrm{d}V(\bar{x}_1)}{\mathrm{d}t} = c_1^T\dot{\bar{x}}_1 = c_1^T[\bar{A}_1\bar{x}_1 + \bar{f}_1(\bar{x},0)] < 0 \tag{54}$$

for

$$\bar{A}_1\bar{x}_1 + \bar{f}_1(\bar{x},0) < 0 \text{ for } x(t) \in D \tag{55}$$

since $c_1 \in \Re_+^{n_1}$ is strictly positive.

Therefore, the following theorem has been proved.

Theorem 4.2. The positive descriptor nonlinear system (39) for $u = 0$ is asymptotically stable in the region $D \in \Re_+^n$ if the conditions (53) and (55) are satisfied.

Example 4.2. (Continuation of Examples 3.1 and 4.1) The positive subsystem (49) for $u = 0$ is linear and asymptotically stable for all $\bar{x}_1(0) \in \Re_+^2$ since the characteristic polynomial

$$\det[I_2 s - \bar{A}_1] = \begin{vmatrix} s+2.5 & -1 \\ -0.5 & s+3 \end{vmatrix} = s^2 + 5.5s + 7 \tag{56}$$

has all positive coefficients (Theorem 2.2).

Therefore, the asymptotic stability region $D \in \Re_+^n$ of the nonlinear system (28), (48) is determined by the equality

$$\begin{bmatrix} \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix} = \begin{bmatrix} 0.25 & 0.5 \\ 0.5 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} + \begin{bmatrix} 0.5 & 0.25 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} \bar{x}_{11}\bar{x}_{12} \\ \bar{x}_{11}^2 \end{bmatrix} \quad \text{for } \bar{x}_1 = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{12} \end{bmatrix} \in \Re_+^2 \tag{57}$$

or

$$\bar{x}_{21} = 0.25(\bar{x}_{11} + \bar{x}_{11}^2) + 0.5(\bar{x}_{12} + \bar{x}_{11}\bar{x}_{12}),\ \bar{x}_{22} = 0.5(\bar{x}_{11} + \bar{x}_{11}^2) \quad \text{for } \bar{x}_{11}, \bar{x}_{12} \in \Re_+. \tag{58}$$

5 Positivity and stability of fractional descriptor nonlinear systems

Consider the fractional descriptor continuous-time linear system

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + Bu,\ 0 < \alpha < 1 \tag{59}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that the system (59) satisfies the conditions (A1) and (A2) of the Section 3.

For the linear system we introduce the new state vector eq. (17) and choose the matrix $Q \in \Re^{n\times n}$ so that eq. (59) takes the form

$$\frac{d^\alpha \bar{x}_1}{dt^\alpha} = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + \bar{B}_1 u,\ \bar{x}_1 \in \Re^{n_1}, \tag{60}$$

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + \bar{B}_2 u,\ \bar{x}_2 \in \Re^{n_2}, \tag{61}$$

where $\bar{A}_{11}$, $\bar{A}_{12}$, $\bar{A}_{21}$, $\bar{A}_{22}$ and $\bar{B}_1$, $\bar{B}_2$ are defined by eq. (21).

The matrices P and Q can be obtained by the use of elementary row and column operators in a similar way as for the linear system (15).

Theorem 5.1. The fractional descriptor linear system (59) satisfying the assumptions (A1) and (A2) is positive if the matrix $\bar{A} \in M_n$ is asymptotically stable and $\bar{B} \in \Re_+^{n\times m}$.

Proof. The main idea of the proof is similar to the proof of Theorem 3.1. In this case instead of the first derivative of the state vector the fractional derivative of order α should be used.

Now let us consider the fractional descriptor continuous-time nonlinear system

$$E\frac{d^\alpha x}{dt^\alpha} = Ax + f(x,u),\ 0 < \alpha < 1, \tag{62}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors $E, A \in \Re^{n\times n}$ and $f(x,u) \in \Re^n$ is the continuous vector function of x and u.

It is assumed that E, A satisfy the assumptions (A1) and (A2).

In a similar way as for the system (39) we define the new state vector eq. (17) and premultiplying the eq. (59) by the matrix $Q \in \Re^{n\times n}$ we obtain

$$QEPP^{-1}\frac{d^\alpha x}{dt^\alpha} = QAPP^{-1}x + Qf(x,u), \tag{63}$$

and

$$\frac{d^\alpha \bar{x}_1}{dt^\alpha} = \bar{A}_{11}\bar{x}_1 + \bar{A}_{12}\bar{x}_2 + f_1(\bar{x},u), \tag{64}$$

$$0 = \bar{A}_{21}\bar{x}_1 + \bar{A}_{22}\bar{x}_2 + f_2(\bar{x},u), \tag{65}$$

where $\bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}$ is defined by eq. (21) and $\begin{bmatrix} f_1(\bar{x},u) \\ f_2(\bar{x},u) \end{bmatrix}$ by eq. (43).

Theorem 5.2. The fractional descriptor nonlinear system (59) satisfying the assumptions (A1), (A2) is positive if the matrix $\bar{A} \in M_n$ is asymptotically stable and eq. (44) holds.

Proof. The main idea of the proof is similar to the proof of Theorem 4.1. In this case instead of the first derivative of the state vector the fractional derivative of order α should be used.

Example 5.1. Consider the fractional nonlinear system (62) with

$$E = \begin{bmatrix} 0 & -3 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 0 \end{bmatrix},\ A = \begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix},\ f(x,u) = \begin{bmatrix} -(3+2\bar{x}_{11}^2)e^{-t} \\ (2+\bar{x}_{11}^2)e^{-t} \\ (1-e^{-t})\bar{x}_{11}e^{-t} \end{bmatrix},\ \bar{x} = \begin{bmatrix} \bar{x}_{11} \\ \bar{x}_{21} \\ \bar{x}_{22} \end{bmatrix},\ u = e^{-t}. \tag{66}$$

It is easy to check that the fractional nonlinear system satisfies the assumptions (A1), (A2) and $n_1 = 1$, $n_2 = 2$.

In this case the matrices Q and P have the forms

$$Q=\begin{bmatrix}1 & 2 & 0\\ -2 & -3 & 0\\ 1 & 2 & 1\end{bmatrix},\ P=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{67}$$

and can be obtained by performing elementary row and column operations on I_3.

Using eqs. (66) and (67) we obtain

$$QEP=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ QAP=\begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 1\\ 2 & 0 & -1\end{bmatrix}=\begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix},$$
$$\bar{x}=P^{-1}x=\begin{bmatrix}\bar{x}_{11}\\ \bar{x}_{21}\\ \bar{x}_{22}\end{bmatrix},\ Qf(\bar{x},u)=\begin{bmatrix}e^{-t}\\ \bar{x}_{11}^2 e^{-t}\\ \bar{x}_{11}(1-e^{-t})\end{bmatrix} \tag{68}$$

and

$$\frac{d^\alpha \bar{x}_{11}}{dt^\alpha}=e^{-t}, \tag{69}$$

$$\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 2\end{bmatrix}\bar{x}_{11}+\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}\bar{x}_{21}\\ \bar{x}_{22}\end{bmatrix}+\begin{bmatrix}\bar{x}_{11}^2 e^{-t}\\ \bar{x}_{11}(1-e^{-t})\end{bmatrix}. \tag{70}$$

The solution of the fractional eq. (69) has the form

$$\begin{aligned}\bar{x}_{11}(t)&=\Phi_0(t)x_0+\int_0^t \Phi(\tau)u(t-\tau)d\tau\\ &=x_0+\frac{e^{-t}}{\Gamma(\alpha)}\int_0^t \tau^{\alpha-1}e^{\tau}d\tau\geq 0,\ t\geq 0.\end{aligned} \tag{71}$$

From eq. (70) we have

$$\begin{aligned}\begin{bmatrix}\bar{x}_{21}\\ \bar{x}_{22}\end{bmatrix}&=-\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}^{1}\left\{\begin{bmatrix}1\\ 2\end{bmatrix}\bar{x}_{11}+\begin{bmatrix}\bar{x}_{11}^2 e^{-t}\\ \bar{x}_{11}(1-e^{-t})\end{bmatrix}\right\}\\ &=\begin{bmatrix}3\\ 2\end{bmatrix}\bar{x}_{11}+\begin{bmatrix}(\bar{x}_{11}^2-\bar{x}_{11})e^{-t}+\bar{x}_{11}\\ \bar{x}_{11}(1-e^{-t})\end{bmatrix}\in\Re_+^2,\ t\geq 0.\end{aligned} \tag{72}$$

From eqs. (71) and (72) it follows that the fractional nonlinear system is positive but not asymptotically stable.

If $\bar{A}\in M_n$ is asymptotically stable then $\bar{A}_{22}\in M_{n_2}$ and by Theorem 2.3 we have $-\bar{A}_{22}^{-1}\in\Re_+^{n_2\times n_2}$ and from eq. (65) we obtain

$$\bar{x}_2=-\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1-\bar{A}_{22}^{-1}f_2(\bar{x},u). \tag{73}$$

Substituting eq. (73) into eq. (64) we obtain

$$\frac{d^\alpha \bar{x}_1}{dt^\alpha}=\bar{A}_1\bar{x}_1+\bar{f}_1(\bar{x},u), \tag{74}$$

where

$$\bar{A}_1=\bar{A}_{11}-\bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21},\ \bar{f}_1(\bar{x},u)=f_1(\bar{x},u)-\bar{A}_{12}\bar{A}_{22}^{-1}f_2(\bar{x},u). \tag{75}$$

Definition 5.1. The fractional positive nonlinear system (59) for $u=0$ is called asymptotically stable in the region $D\in\Re_+^n$ if $x(t)\in\Re_+^n$, $t\geq 0$ and

$$\lim_{t\to\infty}x(t)=0 \text{ for all } x(0)\in D. \tag{76}$$

To test the asymptotic stability of eq. (59) the following extension of the Lyapunov method will be used.

As a candidate of the Lyapunov function we choose eq. (52). Note that it is the linear function of the state vector x and $c\in\Re_+^n$ is strictly positive.

For the fractional positive nonlinear system (59) the Lyapunov stability theorem can be stated as follows: The fractional positive nonlinear system (59) is asymptotically stable in $D\in\Re_+^n$ if the nonlinear subsystem (64) is asymptotically stable in the region $D\in\Re_+^n$ and the condition (54) is satisfied.

Using eq. (52) for $x=\bar{x}_1$ and eq. (74) we obtain

$$\frac{d^\alpha V(\bar{x}_1)}{dt^\alpha}=c_1^T\frac{d^\alpha \bar{x}_1}{dt^\alpha}=c_1^T[\bar{A}_1\bar{x}_1+\bar{f}_1(\bar{x},0)]<0 \tag{77}$$

for

$$\bar{A}_1\bar{x}_1+\bar{f}_1(\bar{x},0)<0 \text{ for } x(t)\in D\in\Re_+^n \tag{78}$$

since $c_1\in\Re_+^{n_1}$ is strictly positive.

Therefore, the following theorem has been proved.

Theorem 5.3. The fractional descriptor nonlinear system (59) satisfying the assumptions (A1), (A2) for $u=0$ is asymptotically stable in the region $D\in\Re_+^n$ if the conditions (76) and (78) are satisfied.

Example 5.2. (Continuation of Example 5.1) From eq. (71) it follows that $\bar{x}_{11}(t)$ does not decrease to zero for $t\to\infty$ since $x_0=\bar{x}_{11}(0)\neq 0$. Substituting in eq. (72) $u=e^{-t}=0$ we obtain

$$\begin{bmatrix}\bar{x}_{21}\\ \bar{x}_{22}\end{bmatrix}=\begin{bmatrix}4\\ 2\end{bmatrix}\bar{x}_{11}. \tag{79}$$

Therefore, the fractional nonlinear system (62) with eq. (66) is not asymptotically stable.

Corollary 5.1. The sufficient conditions for the asymptotic stability of positive descriptor nonlinear systems are independent of the order of α of the differential equation describing the system (are the same for $\alpha=1$ and $0<\alpha<1$).

Note that for positive nonlinear systems, the choice of the Lyapunov function as a linear form of the state vector x

is independent of the order α. Therefore, the Lyapunov function for $\alpha = 1$ and for $0 < \alpha < 1$ has the same form.

In particular case for positive linear systems the fractional system ($0 < \alpha < 1$) is asymptotically stable if and only if the standard system ($\alpha = 1$) is asymptotically stable.

6 Concluding remarks

The positivity and stability of standard and fractional descriptor continuous-time linear and nonlinear systems have been addressed. Necessary and sufficient conditions for positivity of descriptor linear systems have been established (Theorem 3.1). Sufficient conditions for positivity of descriptor nonlinear systems have been proposed (Theorem 4.1). Using an extension of Lyapunov method sufficient conditions for the stability of positive nonlinear systems have been also given. Next the considerations have been extended to fractional descriptor nonlinear systems (Theorems 5.1–5.3). The considerations have been illustrated by numerical examples.

The considerations can be extended to standard and fractional descriptor linear and nonlinear discrete-time systems.

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