J. Differential Equations 248 (2010) 151–183

Contents lists available at ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Lyapunov functions for trichotomies with growth rates ☆

Luis Barreira *, Claudia Valls

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal

ARTICLE INFO

Article history:
Received 28 March 2009
Revised 30 June 2009
Available online 21 July 2009

MSC:
37D99
93D99

Keywords:
Lyapunov functions
Nonuniform exponential trichotomies

ABSTRACT

We consider linear equations $x' = A(t)x$ that may exhibit stable, unstable and central behaviors in different directions, with respect to arbitrary asymptotic rates $e^{c\rho(t)}$ determined by a function $\rho(t)$. For example, the usual exponential behavior with $\rho(t) = t$ is included as a very special case, and when $\rho(t) = \log t$ we obtain a polynomial behavior. We emphasize that we also consider the general case of nonuniform exponential behavior, which corresponds to the existence of what we call a *ρ-nonuniform exponential trichotomy*. This is known to occur in a large class of nonautonomous linear equations. Our main objective is to give a complete characterization in terms of *strict Lyapunov functions* of the linear equations admitting a ρ-nonuniform exponential trichotomy. This includes criteria for the existence of a ρ-nonuniform exponential trichotomy, as well as inverse theorems providing explicit strict Lyapunov functions for each given exponential trichotomy. In the particular case of *quadratic* Lyapunov functions we show that the existence of strict Lyapunov sequences can be deduced from more algebraic relations between the quadratic forms defining the Lyapunov functions. As an application of the characterization of nonuniform exponential trichotomies in terms of strict Lyapunov functions, we establish the robustness of ρ-nonuniform exponential trichotomies under sufficiently small linear perturbations. We emphasize that in comparison with former works, our proof of the robustness is much simpler even when $\rho(t) = t$.

☆ Partially supported by FCT through CAMGSD, Lisbon.

* Corresponding author.
E-mail addresses: barreira@math.ist.utl.pt (L. Barreira), cvalls@math.ist.utl.pt (C. Valls).
URL: http://www.math.ist.utl.pt/~barreira/ (L. Barreira).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2009.07.001

1. Introduction

1.1. Arbitrary asymptotic rates

We consider a linear equation

$$x' = A(t)x \tag{1}$$

that may exhibit different asymptotic behaviors in different directions, such as stable, unstable and central behaviors. In the present paper, in strong contrast with the usual (exponential) stable, unstable, and central behaviors, we consider asymptotic rates of the form $e^{c\rho(t)}$ determined by an arbitrary function $\rho(t)$. The usual exponential behavior corresponds to the particular case when $\rho(t) = t$. We point out that it is quite easy to give examples of differential equations as in (1) (see Section 3) for which all Lyapunov exponents are infinite (either $+\infty$ or $-\infty$). In this situation, it is sometimes possible to consider more general asymptotic behaviors $e^{c\rho(t)}$ that can replace with success the usual exponential behavior e^{ct}, now for a much larger class of linear equations. These may appear for example as linear variational equations along nonhyperbolic trajectories. In particular, we show in [3] that for ρ in a large class of rate functions, *any* linear equation as in (1) in a finite-dimensional space, with three blocks having asymptotic rates $e^{c\rho(t)}$ respectively with c negative, zero, and positive, admits a ρ-nonuniform exponential trichotomy. To formulate a rigorous result illustrating the ubiquity of the nonuniform exponential behavior without being technical at this point, we consider the simpler case of ρ-nonuniform exponential contractions. Let $T(t,\tau)$ be the linear evolution operator associated to Eq. (1) in the finite-dimensional space $\mathbb{R}^n$. We say that Eq. (1) admits a *ρ-nonuniform exponential contraction* if there exist constants $d, D > 0$ and $\varepsilon \geqslant 0$ such that

$$\big\|T(t,\tau)\big\| \leqslant De^{-d(\rho(t)-\rho(\tau))+\varepsilon|\rho(\tau)|}, \quad t \geqslant \tau > 0. \tag{2}$$

When Eq. (1) admits a ρ-nonuniform exponential contraction with $\varepsilon = 0$ we say that it admits a *ρ-uniform exponential contraction*. The following are two particular cases of the notion of ρ-nonuniform exponential contraction:

1. when $\rho(t) = t$ we recover the usual notion of nonuniform exponential contraction (see [2] for details and references), and (2) reduces to
$$\big\|T(t,\tau)\big\| \leqslant De^{-d(t-\tau)+\varepsilon\tau};$$
2. when $\rho(t) = \log(1+t)$ we recover the notion of *nonuniform polynomial contraction* (introduced in [5]), and (2) reduces to
$$\big\|T(t,\tau)\big\| \leqslant D\left(\frac{1+t}{1+s}\right)^{-d}(1+s)^{\varepsilon}.$$

The following is an example of ρ-nonuniform exponential contraction that is not uniform.

Example 1. Given constants $\omega > \varepsilon > 0$, we consider the equation

$$x' = \big(-3\omega t^2 + 3\varepsilon t^2 \cos t - \varepsilon t^3 \sin t\big)x. \tag{3}$$

It follows from Example 3 below that Eq. (3) admits a ρ-nonuniform exponential contraction with $\rho(t) = t^3$. Moreover, it is shown that the contraction is not uniform.

We emphasize that we do not need to assume that $A(t)$ has bounded coefficients (see Example 1). On the other hand, there are equations with bounded coefficients admitting nonuniform exponential contractions that are not uniform. An example is the following.

Example 2. Given constants $\omega > \varepsilon > 0$, we consider the equation

$$x' = (-\omega + \varepsilon \sin \log t + \varepsilon \cos \log t)x. \tag{4}$$

We have

$$\begin{aligned} T(t,\tau) &= e^{-\omega(t-\tau)+\varepsilon t \sin \log t - \varepsilon \tau \sin \log \tau} \\ &= e^{(-\omega+\varepsilon)(t-\tau)+\varepsilon t(\sin \log t - 1)+\varepsilon \tau(1-\sin \log \tau)} \\ &\leqslant e^{(-\omega+\varepsilon)(t-\tau)+2\varepsilon\tau}, \end{aligned}$$

with equality when $\sin \log t = 1$ and $\sin \log \tau = -1$. Therefore, Eq. (4) admits a ρ-nonuniform exponential contraction with $\rho(t) = t$ that is not uniform.

In addition, one can consider equations with bounded coefficients in some particular class and ask whether they may exhibit genuine nonuniform exponential behavior, that is, whether there are equations in this class that admit nonuniform exponential contractions which are not uniform. First of all, we remark that in the case of constant or periodic coefficients it follows from Floquet's theory that a nonuniform exponential contraction is in fact uniform. One can also ask whether this property of periodic coefficients holds for almost periodic or almost automorphic coefficients. We are not able to give an answer, although we conjecture that it is negative. Incidentally, this question is in fact much older (it appeared in a slightly different context although not entirely rigorously defined). Indeed, in a related direction, Hahn asked in [18] whether for a linear equation $x' = A(t)x$ with almost periodic coefficients the asymptotic stability implies uniform stability (a positive answer to our question would give a partial positive answer to Hahn's question). It turns out that the answer is negative, as shown by Conley and Miller in [12]. There are however some results that give a positive answer to Hahn's question for some classes of linear systems. In particular, it follows from results of Sacker and Sell in [33] that for $A(t)$ almost periodic, if all linear equations $x' = B(t)x$ are asymptotically stable for $B(t)$ in the closure of $\{A_\tau\colon\ \tau \in \mathbb{R}\}$, where $A_\tau(t) = A(t+\tau)$, with the closure taken with respect to the topology of uniform convergence on compact subsets, then $x' = A(t)x$ is uniformly stable. In another direction, it is sometimes possible to reduce (or "almost" reduce, in some precise sense) certain classes of linear equations with quasi-periodic coefficients to equations with constant coefficients, using a KAM-type approach. We refer the reader to [17] for details and references. We note that the observations in this paragraph extend with straightforward changes to dichotomies and trichotomies.

Now we set

$$\lambda = \sup_{x_0 \in \mathbb{R}^n} \limsup_{t \to +\infty} \frac{1}{\rho(t)} \log \|x(t)\|,$$

where $x(t)$ is the solution of Eq. (1) with $x(0) = x_0$ (since the solutions form a linear space, one can easily show that the supremum is attained).

Theorem 1. *If $\lambda < 0$, then for each sufficiently small $\delta > 0$, Eq.* (1) *admits a ρ-nonuniform exponential contraction with $d = \lambda + \delta$.*

Theorem 1 is a simple consequence of much more general results in [3]. We refer to [3] for further explicit examples of ρ-nonuniform exponential behavior, and for sharp bounds for the constant ε in (2). Our study includes as a particular case the notion of exponential dichotomy, going back to

Perron in [27], and which plays a central role for example in the study of stable and unstable invariant manifolds, and in the existence of topological conjugacies between a linear dynamics and its nonlinear perturbations. The theory of exponential dichotomies and its applications are in fact widely developed. In particular, there exist large classes of linear differential equations with exponential dichotomies. We refer to the books [8,14,19,20,35] for details and further references. In a related direction, the notion of exponential trichotomy plays a central role in the study of center manifolds, that can be traced back to the works of Pliss [30] and Kelley [21]. Among the first related references in the literature is the work of Brin and Pesin [6] on partially hyperbolic systems, followed closely by the work of Sacker and Sell [34]. To the best of our knowledge the first explicit use of the term "trichotomy" may be due to Elaydi and Hájek in [16], although the concept was already around for quite some time. A very detailed exposition in the case of autonomous equations is given in [36], adapting results in [38]. See also [24,37] for the case of equations in infinite-dimensional spaces. We refer to [7,10,11,36] for more details and further references.

1.2. Exponential trichotomies and strict Lyapunov functions

Our main objective is to give a complete characterization in terms of *strict Lyapunov functions* of the linear equations admitting a ρ-nonuniform exponential trichotomy. This includes criteria for the existence of a ρ-nonuniform exponential trichotomy, as well as inverse theorems providing explicit strict Lyapunov functions for each given exponential trichotomy. We illustrate our results in the particular case of ρ-nonuniform exponential contractions. We first introduce the notion of strict Lyapunov function (here we consider only a simplification of the notions introduced in Sections 2.1 and 2.2). We assume that there exist $C > 0$ and $\delta \geqslant 0$ such that

$$\bigl|V(t,x)\bigr| \leqslant Ce^{\delta|\rho(t)|}\|x\| \tag{5}$$

for every $t \geqslant 0$ and $x \in \mathbb{R}^p$. Given $\alpha > 0$ and $\gamma \geqslant 0$, we say that a function $V : \mathbb{R}_0^+ \times \mathbb{R}^p \to \mathbb{R}_0^-$ is a *strict Lyapunov function* for Eq. (1) if

$$V\bigl(t, T(t,\tau)x\bigr) \leqslant \alpha^{\rho(t)-\rho(\tau)} V(\tau,x), \quad t \geqslant \tau,$$

and

$$V(\tau,x) \geqslant e^{-\gamma\rho(\tau)}\|x\|/C$$

for every $\tau \geqslant 0$ and $x \in \mathbb{R}^p$. The connection between the notions of ρ-nonuniform exponential contraction and of strict Lyapunov function is given by the following results.

Theorem 2. *If there is a strict Lyapunov function V for Eq.* (1) *and $\alpha e^{\gamma} < 1$, then the equation admits a ρ-nonuniform exponential contraction.*

Theorem 2 is a simple consequence of Theorem 4 below (or of Lemma 1). It has the following converse.

Theorem 3. *If Eq.* (1) *admits a ρ-nonuniform exponential contraction, then it has a strict Lyapunov function.*

Theorem 3 is a simple consequence of Theorem 6 below. An explicit strict Lyapunov function is given by

$$V(t,x) = \sup\bigl\{\bigl\|T(r,t)x\bigr\|e^{d(\rho(r)-\rho(t))} \colon r \geqslant t\bigr\},$$

with the constant d as in (2). The formula is a particular case of (37) below.

1.3. Nonuniform exponential behavior

We emphasize that in (2), in addition to considering arbitrary growth rates given by a function ρ, we also consider the possibility of a nonuniform exponential behavior. This occurs when ε cannot be made equal to zero by taking D sufficiently large. It happens that a uniform exponential behavior substantially restricts the dynamics and it is important to look for more general types of hyperbolic behavior. In this respect we can consider for example the much weaker notion of *nonuniform* exponential trichotomy (see [4] for details and references). It turns out that this notion is much more common than the notion of (uniform) exponential trichotomy, although it still allows the development of a quite rich stability theory, besides having a privileged relation with ergodic theory. For example, *almost all* linear variational equations with nonzero Lyapunov exponents obtained from a measure-preserving flow have a nonuniform exponential dichotomy, in fact with an arbitrarily small nonuniformity, up to an appropriate change of coordinates. We refer to [1] for a detailed exposition of the nonuniform hyperbolicity theory. The theory goes back to the landmark works of Oseledets [26] and particularly Pesin [28,29]. Since then it became an important part of the general theory of dynamical systems and a principal tool in the study of stochastic behavior. Among the most important properties due to nonuniform hyperbolicity are the existence of stable and unstable invariant manifolds, and their absolute continuity property established by Pesin in [28].

1.4. Applications and robustness of exponential trichotomies

It is also important to understand whether the notions of exponential dichotomy and exponential trichotomy are robust under sufficiently small *linear* perturbations. As an application of the characterization of a nonuniform exponential trichotomies in terms of strict Lyapunov functions, we establish the robustness of a large class of ρ-nonuniform exponential trichotomies. We emphasize that in comparison with former works, our proof is much simpler even in the particular case when $\rho(t) = t$. The study of robustness in the case of uniform exponential behavior has a long history. In particular, it was discussed by Massera and Schäffer [22] (building on earlier work of Perron [27]; see also [23]), Coppel [13], and Dalec'kiĭ and Kreĭn [15] in the case of Banach spaces. For more recent works we refer to [9,25,31,32] and the references therein. In particular, Chow and Leiva [9] and Pliss and Sell [31] consider the context of linear skew-product semiflows and give examples of applications in the infinite-dimensional setting, including to parabolic partial differential equations and functional differential equations. We refer to [4] for the study of robustness in the more general setting of nonuniform exponential behavior.

We can also consider *nonlinear* perturbations. While our characterization of nonuniform exponential behavior in terms of Lyapunov functions may be used to decide whether a given linear equation has an exponential behavior, it is also useful to establish the persistence of the (nonuniform) exponential stability or instability under a large class of nonlinear perturbations. Namely, we consider the perturbed equation

$$x' = A(t)x + f(t, x) \tag{6}$$

with $f(t, 0) = 0$ for every t. Assuming that the equation $x' = A(t)x$ admits a nonuniform exponential contraction we can use Lyapunov functions to show that this behavior persists in Eq. (6), in the sense that the corresponding (nonlinear) evolution operator satisfies a similar bound to that in (2). We can also show that the zero solution of Eq. (6) is unstable when equation $x' = A(t)x$ admits a nonuniform exponential dichotomy or a nonuniform exponential trichotomy. The proofs require several new ideas and have a considerable size, due to the additional work required to take care of the nonuniform exponential behavior. For this reason we did not strive to add the material here, and it will appear elsewhere. We emphasize that in comparison with other methods to study the stability under nonlinear perturbations, both in the uniform and in the nonuniform setting, such as fixed point theorems, the use of Lyapunov functions is much more automatic, since it essentially corresponds to a computation.

2. Criterion for exponential behavior

2.1. Lyapunov functions

Let $A:\mathbb{R}\to M_p$ be a continuous function, where M_p is the set of $p\times p$ matrices, and consider Eq. (1). Given a function $V:\mathbb{R}^p\to\mathbb{R}$ we consider the *cones*

$$C^u(V)=\{0\}\cup V^{-1}(0,+\infty) \quad \text{and} \quad C^s(V)=\{0\}\cup V^{-1}(-\infty,0).$$

We say that a continuous function $V:\mathbb{R}\times\mathbb{R}^p\to\mathbb{R}$ is a *Lyapunov function* for Eq. (1) if there exist integers $r_u, r_s\in\mathbb{N}\cup\{0\}$ with $r_u+r_s=p$ such that setting $V_t=V(t,\cdot)$ the following properties hold:

1. r_u and r_s are respectively the maximal dimensions of the linear subspaces inside the cones $C^u(V_t)$ and $C^s(V_t)$, for every $t\in\mathbb{R}$;
2. for every $x\in\mathbb{R}^p$ and $t\geqslant\tau$ we have

$$T(t,\tau)\overline{C^u(V_\tau)}\subset\overline{C^u(V_t)} \quad \text{and} \quad T(\tau,t)\overline{C^s(V_t)}\subset\overline{C^s(V_\tau)}.$$

In view of the compactness of the closed unit ball in $\mathbb{R}^p$, if $(V_t)_{t\in\mathbb{R}}$ is a Lyapunov function for (1), then for each $\tau\in\mathbb{R}$ the sets

$$H^u_\tau=\bigcap_{r\in\mathbb{R}} T(\tau,r)\overline{C^u(V_r)}\subset\overline{C^u(V_\tau)} \tag{7}$$

and

$$H^s_\tau=\bigcap_{r\in\mathbb{R}} T(\tau,r)\overline{C^s(V_r)}\subset\overline{C^s(V_\tau)} \tag{8}$$

contain subspaces respectively of dimensions r_u and r_s. We note that for every $t,\tau\in\mathbb{R}$,

$$T(t,\tau)H^u_\tau=H^u_t \quad \text{and} \quad H(t,\tau)H^s_\tau=H^s_t. \tag{9}$$

2.2. Strictness and exponential behavior

Now we introduce the notion of strict Lyapunov function. Let V be a Lyapunov function and let $\rho:\mathbb{R}\to\mathbb{R}$ be a continuous increasing function with $\rho(0)=0$. We assume that there exist $C>0$ and $\delta\geqslant 0$ such that (5) holds for every $t\in\mathbb{R}$ and $x\in\mathbb{R}^p$. Given $\lambda>\nu>0$ and $\gamma\geqslant 0$, we say that V is a (λ,ν)-*strict Lyapunov function* if for each $\tau\in\mathbb{R}$ and $x\in\mathbb{R}^p$ the following properties hold:

1. if $x\in H^u_\tau$ then

$$V\big(t,T(t,\tau)x\big)\geqslant(\lambda+\nu)^{\rho(t)-\rho(\tau)}V(\tau,x), \quad t\geqslant\tau; \tag{10}$$

2. if $x\in H^s_\tau$ then

$$\big|V\big(t,T(t,\tau)x\big)\big|\leqslant(\lambda-\nu)^{\rho(t)-\rho(\tau)}\big|V(\tau,x)\big|, \quad t\geqslant\tau; \tag{11}$$

3. if $x\in H^u_\tau\cup H^s_\tau$ then

$$\big|V(\tau,x)\big|\geqslant e^{-\gamma|\rho(\tau)|}\|x\|/C. \tag{12}$$

We establish a criterion for the existence of partially hyperbolic behavior in terms of pairs of strict Lyapunov functions. Without loss of generality we consider the same constants δ and γ for the two functions in each pair.

Theorem 4. *Let $\lambda_1 > \lambda_2 > 0$. If there exist a (λ_1, ν_1)-strict Lyapunov function V and a (λ_2, ν_2)-strict Lyapunov function W for Eq.* (1) *with*

$$(\lambda_i + \nu_i)/(\lambda_i - \nu_i) > e^{\delta+\gamma}, \quad i = 1, 2, \tag{13}$$

and

$$\lambda_1 - \lambda_2 > |\nu_1 - \nu_2|, \tag{14}$$

then for each $t, \tau \in \mathbb{R}$:

1. *the sets*

$$F_\tau^s = H_\tau^s(V), \quad F_\tau^u = H_\tau^u(W) \quad \textit{and} \quad F_\tau^c = H_\tau^s(W) \cap H_\tau^u(V)$$

are linear subspaces, with

$$\mathbb{R}^p = F_\tau^u \oplus F_\tau^s \oplus F_\tau^c, \tag{15}$$

and

$$T(t,\tau)F_\tau^u = F_t^u, \qquad T(t,\tau)F_\tau^s = F_t^s, \qquad T(t,\tau)F_\tau^c = F_t^c; \tag{16}$$

2. *for each $t \geqslant \tau$ we have*

$$\begin{aligned} \left\| T(t,\tau)^{-1} \middle| F_t^u \right\| &\leqslant C^2 e^{-(\log(\lambda_2+\nu_2)-\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(t)|}, \\ \left\| T(t,\tau) \middle| F_\tau^s \right\| &\leqslant C^2 e^{(\log(\lambda_1-\nu_1)+\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(\tau)|}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \left\| T(t,\tau) \middle| F_\tau^c \right\| &\leqslant C^2 e^{(\log(\lambda_2-\nu_2)+\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(\tau)|}, \\ \left\| T(t,\tau)^{-1} \middle| F_t^c \right\| &\leqslant C^2 e^{-(\log(\lambda_1+\nu_1)-\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(t)|}. \end{aligned} \tag{18}$$

Proof. We start with an auxiliary result. For each $\tau \in \mathbb{R}$, set

$$\mu_\tau^+(x) = \limsup_{t \to +\infty} \frac{1}{\rho(t)} \log \left\| T(t,\tau)x \right\|$$

and

$$\mu_\tau^-(x) = \limsup_{t \to -\infty} \frac{1}{|\rho(t)|} \log \left\| T(t,\tau)x \right\|.$$

Lemma 1. *If there exists a* (λ, ν)*-strict Lyapunov function* V *for Eq.* (1) *with*

$$(\lambda+\nu)/(\lambda-\nu) > e^{\delta+\gamma}, \tag{19}$$

then for each $t, \tau \in \mathbb{R}$:

1. *the sets* H^u_τ *and* H^s_τ *in* (7) *are linear subspaces respectively of dimensions* r_u *and* r_s, *with*

$$\mathbb{R}^p = H^u_\tau \oplus H^s_\tau,$$

and

$$T(t,\tau)H^s_\tau = H^s_t \quad \text{and} \quad T(t,\tau)H^u_\tau = H^u_t;$$

2. $$\mu^+_\tau(x) \geqslant \log(\lambda+\nu) - \delta \quad \text{for } x \in H^u_\tau, \tag{20}$$

and

$$\mu^+_\tau(x) \leqslant \log(\lambda-\nu) + \gamma \quad \text{for } x \in H^s_\tau; \tag{21}$$

3. $$\mu^-_\tau(x) \leqslant -\log(\lambda+\nu) + \gamma \quad \text{for } x \in H^u_\tau,$$

and

$$\mu^-_\tau(x) \geqslant -\log(\lambda-\nu) - \delta \quad \text{for } x \in H^s_\tau;$$

4. *for each* $t \geqslant \tau$ *we have*

$$\left\| T(t,\tau)^{-1} \middle| H^u_t \right\| \leqslant C^2 e^{-(\log(\lambda+\nu)-\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(t)|},$$

and

$$\left\| T(t,\tau) \middle| H^s_\tau \right\| \leqslant C^2 e^{(\log(\lambda-\nu)+\gamma)(\rho(t)-\rho(\tau))+(\delta+\gamma)|\rho(\tau)|}.$$

Proof. It follows from (12) that the inclusions in (7) and (8) can be replaced by

$$H^u_\tau \subset C^u(V_\tau) \quad \text{and} \quad H^s_\tau \subset C^s(V_\tau). \tag{22}$$

Indeed, if $x \in H^u_\tau \setminus \{0\}$, then by (12) we have $V(\tau,x) > 0$. This establishes the first inclusion in (22). A similar argument establishes the second one. By (22), the function $V(\tau,\cdot)$ is positive in $H^u_\tau \setminus \{0\}$ and negative in $H^s_\tau \setminus \{0\}$. For each $x \in H^s_\tau$, it follows from (11) and (12) that for every $t \geqslant \tau$ we have

$$\begin{aligned} \left\| T(t,\tau)x \right\| &\leqslant C e^{\gamma|\rho(t)|} \left| V\big(t, T(t,\tau)x\big) \right| \\ &\leqslant C e^{\gamma|\rho(t)|} (\lambda-\nu)^{\rho(t)-\rho(\tau)} \left| V(\tau,x) \right|. \end{aligned} \tag{23}$$

Therefore, (21) holds. For each $x \in H^u_\tau$, it follows from (5) and (10) that for every $t \geqslant \tau$ we have

$$\begin{aligned} \left\| T(t,\tau)x \right\| &\geqslant \frac{e^{-\delta|\rho(t)|}}{C} V\big(t, T(t,\tau)x\big) \\ &\geqslant \frac{e^{-\delta|\rho(t)|}}{C} (\lambda+\nu)^{\rho(t)-\rho(\tau)} V(\tau,x). \end{aligned} \tag{24}$$

Therefore, (20) holds. For each $\tau \in \mathbb{R}$, let $D^u_\tau \subset H^u_\tau$ be any r_u-dimensional subspace, and let $D^s_\tau \subset H^s_\tau$ be any r_s-dimensional subspace. By (22), we have $H^u_\tau \cap H^s_\tau = \{0\}$, and hence $D^u_\tau \cap D^s_\tau = \{0\}$. Therefore, $\mathbb{R}^p = D^u_\tau \oplus D^s_\tau$. We want to show that

$$H^s_\tau = D^s_\tau \quad \text{and} \quad H^u_\tau = D^u_\tau.$$

If there exists $x \in H^s_\tau \setminus D^s_\tau$, then we write $x = y + z$ with $y \in D^s_\tau$ and $z \in D^u_\tau \setminus \{0\}$. By (19) we have

$$\log(\lambda + \nu) - \delta > \log(\lambda - \nu) + \gamma.$$

Hence, it follows from (21) and (20) that

$$\mu^+_\tau(x) = \max\{\mu^+_\tau(y), \mu^+_\tau(z)\} = \mu^+_\tau(z) \geqslant \log(\lambda + \nu) + \delta,$$

which contradicts to (21). Therefore, $H^s_\tau = D^s_\tau$ for each $\tau \in \mathbb{R}$. We can show in a similar manner that $H^u_\tau = D^u_\tau$ for each $\tau \in \mathbb{Z}$. By (5) and (23), for every $x \in H^s_\tau$ and $t \geqslant \tau$ we have

$$\begin{aligned}\|T(t,\tau)x\| &\leqslant Ce^{\gamma|\rho(t)|}(\lambda - \nu)^{\rho(t)-\rho(\tau)}|V(\tau,x)| \\ &\leqslant C^2 e^{\gamma|\rho(t)|}(\lambda - \nu)^{\rho(t)-\rho(\tau)} e^{\delta|\rho(\tau)|}\|x\| \\ &= C^2 e^{(\gamma + \log(\lambda-\nu))(\rho(t)-\rho(\tau))} e^{(\delta+\gamma)|\rho(\tau)|}\|x\|.\end{aligned}$$

Moreover, by (12) and (24), for every $x \in H^u_\tau$ and $t \geqslant \tau$ we have

$$\begin{aligned}\|T(t,\tau)x\| &\geqslant \frac{e^{-\delta|\rho(t)|}}{C}(\lambda + \nu)^{\rho(t)-\rho(\tau)} V(\tau,x) \\ &\geqslant \frac{e^{-\delta|\rho(t)|}}{C^2}(\lambda + \nu)^{\rho(t)-\rho(\tau)} e^{-\gamma|\rho(\tau)|}\|x\|.\end{aligned}$$

Hence,

$$\|T(t,\tau)^{-1}x\| \leqslant C^2 e^{(-\log(\lambda+\nu)+\gamma)(\rho(t)-\rho(\tau))} e^{(\delta+\gamma)|\rho(t)|}\|x\|$$

for every $x \in H^u_t$ and $t \geqslant \tau$. This completes the proof of the lemma. □

Lemma 2. *For each $\tau \in \mathbb{R}$ we have*

$$H^s_\tau(V) \subset H^s_\tau(W) \quad \textit{and} \quad H^u_\tau(W) \subset H^u_\tau(V).$$

Proof. If there exists $x \in H^s_\tau(V) \setminus H^s_\tau(W)$, then we write $x = y + z$ with $y \in H^s_\tau(W)$ and $z \in H^u_\tau(W) \setminus \{0\}$. By Lemma 1 and (13) we have

$$\begin{aligned}\log(\lambda_2 - \nu_2) + \gamma \geqslant \mu^+_\tau(x) &= \max\{\mu^+_\tau(y), \mu^+_\tau(z)\} \\ &= \mu^+_\tau(z) \geqslant \log(\lambda_1 + \nu_1) - \delta.\end{aligned}$$

Together with (14) this implies that

$$\frac{\lambda_1 + \nu_1}{\lambda_1 - \nu_1} < \frac{\lambda_1 + \nu_1}{\lambda_2 - \nu_2} \leqslant e^{\delta+\gamma},$$

which contradicts to (13). We obtain $H^s_\tau(V) \subset H^s_\tau(W)$ for every $\tau \in \mathbb{R}$. Similarly, if there exists $x \in H^u_\tau(W) \setminus H^u_\tau(V)$, then we write $x = y + z$ with $y \in H^s_\tau(V)$ and $z \in H^u_\tau(V) \setminus \{0\}$. By Lemma 1 and (13) we have

$$-\log(\lambda_2 + \nu_2) + \gamma \geqslant \mu^-_\tau(x) = \mu^-_\tau(y) \geqslant -\log(\lambda_1 - \nu_1) - \delta.$$

Together with (14) this implies that

$$\frac{\lambda_2 + \nu_2}{\lambda_2 - \nu_2} < \frac{\lambda_1 + \nu_1}{\lambda_2 - \nu_2} \leqslant e^{\delta+\gamma},$$

which again contradicts to (13). Thus, $H^u_\tau(W) \subset H^u_\tau(V)$ for each $\tau \in \mathbb{R}$. □

Lemma 3. *We have*

$$\big(H^s_\tau(W) \cap H^u_\tau(V)\big) \oplus H^s_\tau(V) \oplus H^u_\tau(W) = \mathbb{R}^p. \tag{25}$$

Proof. By Lemma 1 we have

$$\big(H^s_\tau(W) \cap H^u_\tau(V)\big) \cap H^u_\tau(W) = \{0\},$$

and

$$\big(H^s_\tau(W) \cap H^u_\tau(V)\big) \cap H^s_\tau(V) = \{0\}.$$

Moreover, in view of Lemma 2 we have

$$H^s_\tau(V) \cap H^u_\tau(W) \subset H^s_\tau(V) \cap H^u_\tau(V) = \{0\},$$

and thus $H^s_\tau(V) \cap H^u_\tau(W) = \{0\}$. Furthermore,

$$\begin{aligned}\dim\big(H^s_\tau(W) \cap H^u_\tau(V)\big) &= \dim H^s_\tau(W) + \dim H^u_\tau(V) - \dim\big(H^s_\tau(W) + H^u_\tau(V)\big)\\ &= p - \dim H^u_\tau(W) + p - \dim H^s_\tau(V) - \dim\big(H^s_\tau(W) + H^u_\tau(V)\big),\end{aligned}$$

which implies that

$$\begin{aligned}&\dim\big(H^s_\tau(W) \cap H^u_\tau(V)\big) + \dim H^u_\tau(W) + \dim H^s_\tau(V)\\ &\quad = 2p - \dim\big(H^s_\tau(W) + H^u_\tau(V)\big) \geqslant p.\end{aligned}$$

This shows that (25) holds. □

We proceed with the proof of Theorem 4. Property 1 follows easily from Lemma 3 while property 2 follows from Lemma 1. This completes the proof of the theorem. □

2.3. Differentiable Lyapunov functions

Now we consider the particular case of differentiable Lyapunov functions. Set

$$\begin{aligned}\dot V(t,x) &= \frac{d}{dh} V\big(t+h, T(t+h,h)x\big)_{|h=0}\\ &= \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}(t,x)A(t)x.\end{aligned}$$

Proposition 1. *Let V and ρ be C^1 functions.*

1. *For each $x \in H^u_\tau$, property* (10) *is equivalent to*

$$\dot V\big(t, T(t,\tau)x\big) \geqslant V\big(t, T(t,\tau)x\big)\rho'(t)\log(\lambda+\nu), \quad t>\tau. \tag{26}$$

2. *For each $x \in H^s_\tau$, property* (11) *is equivalent to*

$$\dot V\big(t, T(t,\tau)x\big) \geqslant V\big(t, T(t,\tau)x\big)\rho'(t)\log(\lambda-\nu), \quad t>\tau.$$

Proof. Let $x \in H^u_\tau$. By (16) we have $T(t,\tau)x \in H^u_t$ for every $t \in \mathbb{R}$. Now we assume that (10) holds. If $t>\tau$ and $h>0$ then

$$V\big(t+h, T(t+h,\tau)x\big) \geqslant (\lambda+\nu)^{\rho(t+h)-\rho(t)} V\big(t, T(t,\tau)x\big),$$

and

$$\begin{aligned}\lim_{h\to 0^+} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} &\geqslant V\big(t, T(t,\tau)x\big) \lim_{h\to 0^+} \frac{(\lambda+\nu)^{\rho(t+h)-\rho(t)}-1}{h}\\ &= V\big(t, T(t,\tau)x\big)\rho'(t)\log(\lambda+\nu).\end{aligned}$$

Similarly, if $h<0$ is such that $t+h>\tau$, then

$$V\big(t+h, T(t+h,\tau)x\big) \leqslant (\lambda+\nu)^{\rho(t+h)-\rho(t)} V\big(t, T(t,\tau)x\big),$$

and

$$\begin{aligned}\lim_{h\to 0^-} \frac{V(t+h, T(t+h,\tau)x) - V(t, T(t,\tau)x)}{h} &\geqslant V\big(t, T(t,\tau)x\big) \lim_{h\to 0^-} \frac{(\lambda+\nu)^{\rho(t+h)-\rho(t)}-1}{h}\\ &= V\big(t, T(t,\tau)x\big)\rho'(t)\log(\lambda+\nu).\end{aligned}$$

This establishes (26). Now we assume that (26) holds. Given $x \in H^u_\tau \setminus \{0\}$, it follows from (12) that $|V(\tau,x)|>0$, and thus, by (9), $V(t, T(t,\tau)x)>0$ for every $t \in \mathbb{R}$. Therefore, we can rewrite (26) in the form

$$\frac{\dot V(t, T(t,\tau)x)}{V(t, T(t,\tau)x)} \geqslant \log(\lambda+\nu)\rho'(t), \quad t>\tau.$$

This implies that

$$\log V\bigl(t, T(t,\tau)x\bigr) - \log V(\tau, x) = \int_\tau^t \dot V\bigl(v, T(v,\tau)x\bigr)\,dv$$
$$\geqslant \log(\lambda+\nu)\bigl(\rho(t)-\rho(\tau)\bigr)\,dv,$$

and hence (10) holds. The second property is obtained in a similar manner. □

3. Nonuniform exponential trichotomies

We continue to consider a continuous increasing function $\rho:\mathbb{R}\to\mathbb{R}$ with $\rho(0)=0$. We say that Eq. (1) admits a *ρ-nonuniform exponential trichotomy* (in $\mathbb{R}$) if there exist projections $P(t)$, $Q(t)$ and $R(t)$ for $t\in\mathbb{R}$ such that

$$P(t)+Q(t)+R(t)=\mathrm{Id},$$
$$P(t)T(t,\tau)=T(t,\tau)P(\tau), \qquad Q(t)T(t,\tau)=T(t,\tau)Q(\tau),$$
$$R(t)T(t,\tau)=T(t,\tau)R(\tau),$$

for every $t,\tau\in\mathbb{R}$, and there exist constants

$$0\leqslant a<b, \quad 0\leqslant c<d, \quad \varepsilon\geqslant 0 \quad \text{and} \quad D>0 \tag{27}$$

such that for every $t,\tau\in\mathbb{R}$ with $t\geqslant\tau$ we have

$$\begin{aligned}\bigl\|T(t,\tau)P(\tau)\bigr\| &\leqslant De^{-d(\rho(t)-\rho(\tau))+\varepsilon|\rho(\tau)|},\\ \bigl\|T(t,\tau)R(\tau)\bigr\| &\leqslant De^{a(\rho(t)-\rho(\tau))+\varepsilon|\rho(\tau)|},\end{aligned} \tag{28}$$

and

$$\begin{aligned}\bigl\|T(t,\tau)^{-1}Q(t)\bigr\| &\leqslant De^{-b(\rho(t)-\rho(\tau))+\varepsilon|\rho(t)|},\\ \bigl\|T(t,\tau)^{-1}R(t)\bigr\| &\leqslant De^{c(\rho(t)-\rho(\tau))+\varepsilon|\rho(t)|}.\end{aligned} \tag{29}$$

For each $t\in\mathbb{R}$ we define the *stable*, *unstable* and *central subspaces* respectively by

$$E_t^s=P(t)\bigl(\mathbb{R}^p\bigr), \quad E_t^u=Q(t)\bigl(\mathbb{R}^p\bigr) \quad \text{and} \quad E_t^c=R(t)\bigl(\mathbb{R}^p\bigr).$$

We also say that Eq. (1) admits a *ρ-uniform* exponential trichotomy if it admits a ρ-nonuniform exponential trichotomy with $\varepsilon=0$. We present an example of ρ-nonuniform exponential trichotomy.

Example 3. Take constants $\omega>\varepsilon>0$. For each $t\in\mathbb{R}$ let

$$\begin{pmatrix} 0 & 0 & 0\\ 0 & -3\omega t^2+3\varepsilon t^2\cos t-\varepsilon t^3\sin t & 0\\ 0 & 0 & 3\omega t^2-3\varepsilon t^2\cos t+\varepsilon t^3\sin t\end{pmatrix}.$$

Each solution

$$\begin{aligned}v(t)&=\bigl(U(t,s),V_1(t,s),V_2(t,s)\bigr)v(s)\\ &=\bigl(T(t,s)R(s),T(t,s)P(s),T(t,s)Q(s)\bigr)v(s)\end{aligned}$$

of Eq. (1) is given by

$$U(t,s)=1, \qquad V_1(t,s)=e^{-\omega(t^3-s^3)+\varepsilon t^3\cos t-\varepsilon s^3\cos s}, \qquad V_2(t,s)=V_1(s,t). \tag{30}$$

Clearly, for $t\geqslant s\geqslant 0$ we have

$$V_1(t,s)=e^{-(\omega-\varepsilon)(t^3-s^3)+\varepsilon t^3(\cos t-1)-\varepsilon s^3(\cos s-1)}\leqslant e^{-(\omega-\varepsilon)(t^3-s^3)+2\varepsilon s^3}.$$

Moreover, if $t=2\pi+2\pi k$ and $s=\pi+2\pi k$ for some $k\in\mathbb{N}$, then

$$V_1(t,s)=e^{-(\omega-\varepsilon)(t^3-s^3)+2\varepsilon s^3}. \tag{31}$$

Now we assume that $t\geqslant 0$ and $s\leqslant 0$. By (30) we have

$$\begin{aligned}V_1(t,s)&=e^{-(\omega-\varepsilon)(t^3-s^3)+\varepsilon t^3(\cos t-1)+\varepsilon|s|^3(\cos s-1)}\\&\leqslant e^{-(\omega-\varepsilon)(t^3-s^3)}\leqslant e^{-(\omega-\varepsilon)(t^3-s^3)+2\varepsilon s^3}.\end{aligned}$$

Finally, if $t\geqslant s$ and $t,s\leqslant 0$, then again by (30) we have

$$\begin{aligned}V_1(t,s)&=e^{-(\omega-\varepsilon)(t^3-s^3)-\varepsilon|t|^3(\cos t-1)+\varepsilon|s|^3(\cos s-1)}\\&\leqslant e^{-(\omega-\varepsilon)(t^3-s^3)+2\varepsilon|t|^3}\leqslant e^{-(\omega-\varepsilon)(t^3-s^3)+2\varepsilon|s|^3}.\end{aligned}$$

Moreover, for $s\geqslant t\geqslant 0$ we have

$$V_2(t,s)=V_1(s,t)\leqslant e^{-(\omega-\varepsilon)(s^3-t^3)+2\varepsilon t^3}\leqslant e^{-(\omega-\varepsilon)(s^3-t^3)+2\varepsilon s^3}.$$

Similarly, for $s\geqslant 0$ and $t\leqslant 0$ we have

$$V_2(t,s)=V_1(s,t)\leqslant e^{-(\omega-\varepsilon)(s^3-t^3)}\leqslant e^{-(\omega-\varepsilon)(s^3-t^3)+2\varepsilon|s|^3},$$

and for $s\geqslant t$ with $s,t\leqslant 0$ we have

$$V_2(t,s)=V_1(s,t)\leqslant e^{-(\omega-\varepsilon)(s^3-t^3)+2\varepsilon|s|^3}.$$

This shows that Eq. (1) admits a ρ-nonuniform exponential trichotomy in $\mathbb{R}$ with $\rho(t)=t^3$. Moreover, it follows from (31) that the trichotomy is not uniform.

The following is a criterion for the existence of ρ-nonuniform exponential trichotomies.

Theorem 5. *If Eq.* (1) *admits a* (λ_1,ν_1)*-strict Lyapunov function* V *and a* (λ_2,ν_2)*-strict Lyapunov function* W *satisfying* (13) *and* (14)*, and there exist constants* $c_1,c_2>0$ *such that*

$$\angle\left(F_t^s,F_t^u\right),\angle\left(F_t^s,F_t^c\right),\angle\left(F_t^u,F_t^c\right)\geqslant c_1e^{-c_2|\rho(t)|}, \quad t\in\mathbb{R}, \tag{32}$$

then Eq. (1) *admits a* ρ*-nonuniform exponential trichotomy.*

Proof. By (15), for each $t \in \mathbb{R}$ there exist projections

$$U^s(t): \mathbb{R}^p \to F_t^s, \quad U^u(t): \mathbb{R}^p \to F_t^u \quad \text{and} \quad U^c(t): \mathbb{R}^p \to F_t^c,$$

with $U^s(t) + U^u(t) + U^c(t) = \mathrm{Id}$. It follows from (16) that

$$U^s(t)T(t,\tau) = T(t,\tau)U^s(\tau),$$

with similar identities for $U^u(t)$ and $U^c(t)$. To obtain the inequalities in (28) and (29), we first note that for each $t \in \mathbb{R}$,

$$\|U^s(t)\| = \frac{1}{2\sin(\angle(F_t^s, F_t^u \oplus F_t^c)/2)},$$

with similar identities for $U^u(t)$ and $U^c(t)$. Since $2/\pi \leqslant \sin x/x < 1$ for $x \in (0, \pi/2]$, this implies that

$$\|U^s(t)\| \leqslant \frac{\pi}{2\angle(F_t^s, F_t^u \oplus F_t^c)} \leqslant \frac{\pi}{2c_1} e^{c_2|\rho(t)|}, \tag{33}$$

and similarly

$$\begin{aligned} \|U^u(t)\| &\leqslant \frac{\pi}{2\angle(F_t^u, F_t^s \oplus F_t^c)} \leqslant \frac{\pi}{2c_1} e^{c_2|\rho(t)|}, \\ \|U^c(t)\| &\leqslant \frac{\pi}{2\angle(F_t^c, F_t^s \oplus F_t^u)} \leqslant \frac{\pi}{2c_1} e^{c_2|\rho(t)|}. \end{aligned} \tag{34}$$

Together with the inequalities

$$\begin{aligned} \|T(t,\tau)U^s(\tau)\| &\leqslant \|T(t,\tau)|F_\tau^s\| \cdot \|U^s(\tau)\|, \\ \|T(t,\tau)U^c(\tau)\| &\leqslant \|T(t,\tau)|F_\tau^c\| \cdot \|U^c(\tau)\|, \end{aligned}$$

and

$$\begin{aligned} \|T(t,\tau)^{-1}U^u(t)\| &\leqslant \|T(t,\tau)^{-1}|F_t^u\| \cdot \|U^u(t)\|, \\ \|T(t,\tau)^{-1}U^c(t)\| &\leqslant \|T(t,\tau)^{-1}|F_t^c\| \cdot \|U^c(t)\|, \end{aligned}$$

the desired statement follows readily from (33), (34), and statement 2 in Theorem 4. □

Now we obtain a converse statement, by constructing strict Lyapunov functions for each ρ-nonuniform exponential trichotomy.

Theorem 6. *If Eq.* (1) *admits a ρ-nonuniform exponential trichotomy, then it has a (λ_1, ν_1)-strict Lyapunov function V with*

$$\lambda_1 = (e^a + e^b)/2 > 1 \quad \text{and} \quad \nu_1 = (e^b - e^a)/2, \tag{35}$$

and a (λ_2, ν_2)-strict Lyapunov function W with

$$\lambda_2 = (e^{-d} + e^{-c})/2 < 1 \quad \text{and} \quad \nu_2 = (e^{-c} - e^{-d})/2, \tag{36}$$

taking $\delta = 2\varepsilon$ and $\gamma = 0$. Moreover, if ε is sufficiently small, then inequalities (13) *and* (14) *hold.*

Proof. We write $x = y + z$, where $y \in F_t^s$ and $z \in F_t^u \oplus F_t^c$. For each $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$ we set

$$W(t,x) = -W^s(t,y) + W^u(t,z), \tag{37}$$

where

$$W^s(t,x) = \sup\bigl\{\bigl\|T(r,t)y\bigr\|e^{d(\rho(r)-\rho(t))}\colon r \geqslant t\bigr\},$$

and

$$W^u(t,x) = \sup\bigl\{\bigl\|T(r,t)z\bigr\|e^{-c(\rho(t)-\rho(r))}\colon r \leqslant t\bigr\}.$$

Clearly, W satisfies condition 1 in the notion of Lyapunov function, with $r_s = \dim F_t^s$ and $r_u = \dim(F_t^u \oplus F_t^c)$ (which are independent of t). Furthermore, since $c < d$, for every $t \geqslant \tau$ we have

$$\begin{aligned} W^s\bigl(t, T(t,\tau)y\bigr) &= e^{-d(\rho(t)-\rho(\tau))} \sup\bigl\{\bigl\|T(r,t)T(t,\tau)y\bigr\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant t\bigr\} \\ &\leqslant e^{-c(\rho(t)-\rho(\tau))} \sup\bigl\{\bigl\|T(r,\tau)y\bigr\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant \tau\bigr\} \\ &= e^{-c(\rho(t)-\rho(\tau))} W^s(\tau,y), \end{aligned}$$

and

$$\begin{aligned} W^u\bigl(t, T(t,\tau)z\bigr) &= e^{-c(\rho(t)-\rho(\tau))} \sup\bigl\{\bigl\|T(r,t)T(t,\tau)z\bigr\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant t\bigr\} \\ &\geqslant e^{-c(\rho(t)-\rho(\tau))} \sup\bigl\{\bigl\|T(r,\tau)z\bigr\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant \tau\bigr\} \\ &= e^{-c(\rho(t)-\rho(\tau))} W^u(\tau,z). \end{aligned}$$

Since

$$T(t,\tau)F_\tau^s = F_t^s \quad \text{and} \quad T(t,\tau)\bigl(F_\tau^u \oplus F_\tau^c\bigr) = F_t^u \oplus F_t^c,$$

we obtain

$$\begin{aligned} W\bigl(t, T(t,\tau)x\bigr) &= -W^s\bigl(t, T(t,\tau)y\bigr) + W^u\bigl(t, T(t,\tau)z\bigr) \\ &\geqslant e^{-c(\rho(t)-\rho(\tau))}\bigl(-W^s(\tau,y) + W^u(\tau,z)\bigr) \\ &= e^{-c(\rho(t)-\rho(\tau))} W(\tau,x). \end{aligned}$$

This readily implies that

$$T(t,\tau)\overline{C^u(W_\tau)} \subset \overline{C^u(W_t)} \quad \text{and} \quad T(\tau,t)\overline{C^s(W_t)} \subset \overline{C^s(W_\tau)},$$

and W satisfies condition 2 in the notion of Lyapunov function. For every $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$ we have

$$\|y\| \leqslant W^s(t,y) \leqslant De^{\varepsilon|\rho(t)|}\|y\|,$$

and

$$\|z\| \leqslant W^u(t,z) \leqslant De^{\varepsilon|\rho(t)|}\|z\|.$$

Therefore,

$$\bigl|W(t,x)\bigr| \leqslant De^{\varepsilon|\rho(t)|}\bigl(\|y\| + \|z\|\bigr) \leqslant 2D^2e^{2\varepsilon|\rho(t)|}\|x\|,$$

and (5) holds with $\delta = 2\varepsilon$. If $x \in H^s_\tau$, then for $t \geqslant \tau$ we have

$$\begin{aligned}\big|W\big(t, T(t,\tau)x\big)\big| &= W^s\big(t, T(t,\tau)x\big) - W^u\big(t, T(t,\tau)x\big)\\ &= \sup\big\{\big\|T(r,t)T(t,\tau)y\big\|e^{d(\rho(r)-\rho(t))}\colon r \geqslant t\big\}\\ &\quad - \sup\big\{\big\|T(r,t)T(t,\tau)z\big\|e^{-c(\rho(t)-\rho(r))}\colon r \leqslant t\big\}\\ &= e^{-d(\rho(t)-\rho(\tau))}\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant t\big\}\\ &\quad - e^{-c(\rho(t)-\rho(\tau))}\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant t\big\}.\end{aligned}$$

Since $e^{-c} > e^{-d}$ we obtain

$$\begin{aligned}\big|W\big(t, T(t,\tau)x\big)\big| &\leqslant e^{-d(\rho(t)-\rho(\tau))}\big(\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant \tau\big\}\\ &\quad - \sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant \tau\big\}\big)\\ &= e^{-d(\rho(t)-\rho(\tau))}\big|W(\tau,x)\big|,\end{aligned}$$

and (11) holds with $\lambda_2 - \nu_2 = e^{-d}$. Similarly, if $x \in H^u_\tau$, then for $t \geqslant \tau$ we have

$$\begin{aligned}W\big(t, T(t,\tau)x\big) &= -W^s\big(t, T(t,\tau)x\big) + W^u\big(t, T(t,\tau)x\big)\\ &= -e^{-d(\rho(t)-\rho(\tau))}\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant t\big\}\\ &\quad + e^{-c(\rho(t)-\rho(\tau))}\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant t\big\}\\ &\geqslant e^{-c(\rho(t)-\rho(\tau))}\big(-\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant \tau\big\}\\ &\quad + \sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r \leqslant \tau\big\}\big)\\ &= e^{-c(\rho(t)-\rho(\tau))}W(\tau,x),\end{aligned}$$

and (10) holds with $\lambda_2 + \nu_2 = e^{-c}$. Now we show that (12) holds with $\gamma = 0$. If $x \in H^s_\tau$ and $t \geqslant \tau$, then setting $\alpha = \lambda_2^{\rho(t)-\rho(\tau)}$, we have

$$\begin{aligned}\alpha\big|W(\tau,x)\big| &\geqslant \alpha\big|W(\tau,x)\big| - \big|W\big(t, T(t,\tau)x\big)\big|\\ &= \alpha W^s(\tau,y) - W^s\big(t, T(t,\tau)y\big) - \alpha W^u(\tau,z) + W^u\big(t, T(t,\tau)z\big).\end{aligned}\tag{38}$$

Moreover,

$$\begin{aligned}\alpha W^s(\tau,y) - W^s\big(t, T(t,\tau)y\big) &= \alpha\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant \tau\big\}\\ &\quad - e^{d(\rho(\tau)-\rho(t))}\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant t\big\}\\ &\geqslant \big(\alpha - e^{d(\rho(\tau)-\rho(t))}\big)\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r \geqslant \tau\big\}\\ &\geqslant \big(\alpha - e^{-d(\rho(t)-\rho(\tau))}\big)\|y\|\\ &\geqslant \alpha\left(1 - \frac{e^{-d(\rho(t)-\rho(\tau))}}{\lambda_2^{\rho(t)-\rho(\tau)}}\right)\|y\| \geqslant \frac{\alpha}{2}\|y\|,\end{aligned}$$

provided that t is sufficiently large, since $c < d$. Similarly,

$$\begin{aligned}-\alpha W^u(\tau,z)+W^u\big(t,T(t,\tau)z\big)&=e^{-c(\rho(t)-\rho(\tau))}\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant t\big\}\\&\quad-\alpha\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant \tau\big\}\\&\geqslant\big(e^{-c(\rho(t)-\rho(\tau))}-\alpha\big)\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant\tau\big\}\\&\geqslant\big(e^{-c(\rho(t)-\rho(\tau))}-\alpha\big)\|z\|\\&\geqslant\alpha\left(\frac{e^{-c(\rho(t)-\rho(\tau))}}{\lambda_2^{\rho(t)-\rho(\tau)}}-1\right)\|z\|\geqslant\frac{\alpha}{2}\|z\|,\end{aligned}$$

provided that t is sufficiently large, again since $c<d$. By (38) we obtain

$$\big|W(\tau,x)\big|\geqslant\frac{1}{2}\big(\|y\|+\|z\|\big)\geqslant\frac{1}{2}\|x\|.$$

Now we assume that $x\in H_\tau^u$ and $t\leqslant\tau$. Setting

$$\beta=\left(\frac{e^d+e^c}{2}\right)^{\rho(\tau)-\rho(t)},$$

we have

$$\begin{aligned}\beta W(\tau,x)&\geqslant\beta W(\tau,x)-W\big(t,T(t,\tau)x\big)\\&=-\beta W^s(\tau,y)+W^s\big(t,T(t,\tau)y\big)+\beta W^u(\tau,z)-W^u\big(t,T(t,\tau)z\big).\end{aligned}\tag{39}$$

We obtain

$$\begin{aligned}-\beta W^s(\tau,y)+W^s\big(t,T(t,\tau)y\big)&=-\beta\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r\geqslant\tau\big\}\\&\quad+e^{d(\rho(\tau)-\rho(t))}\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r\geqslant t\big\}\\&\geqslant\big(e^{d(\rho(\tau)-\rho(t))}-\beta\big)\sup\big\{\big\|T(r,\tau)y\big\|e^{d(\rho(r)-\rho(\tau))}\colon r\geqslant\tau\big\}\\&\geqslant\big(e^{d(\rho(\tau)-\rho(t))}-\beta\big)\|y\|\\&\geqslant\beta\left(\frac{e^{d(\rho(\tau)-\rho(t))}}{\beta}-1\right)\|y\|\geqslant\frac{\beta}{2}\|y\|,\end{aligned}$$

provided that t is sufficiently small, since $d>c$. Similarly,

$$\begin{aligned}\beta W^u(\tau,z)-W^u\big(t,T(t,\tau)z\big)&=\beta\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant\tau\big\}\\&\quad-e^{c(\rho(\tau)-\rho(t))}\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant t\big\}\\&\geqslant\big(\beta-e^{c(\rho(\tau)-\rho(t))}\big)\sup\big\{\big\|T(r,\tau)z\big\|e^{-c(\rho(\tau)-\rho(r))}\colon r\leqslant\tau\big\}\\&\geqslant\big(\beta-e^{c(\rho(\tau)-\rho(t))}\big)\|z\|\\&\geqslant\beta\left(1-\frac{e^{c(\rho(\tau)-\rho(t))}}{\beta}\right)\|z\|\geqslant\frac{\beta}{2}\|z\|,\end{aligned}$$

provided that t is sufficiently small. By (39) we obtain

$$W(\tau, x) \geqslant \frac{1}{2}\bigl(\|y\| + \|z\|\bigr) \geqslant \frac{1}{2}\|x\|.$$

Therefore, W is a (λ_2, ν_2)-strict Lyapunov function with λ_2 and ν_2 as in (36). Now we write $x = y + z$, where $y \in F_t^s \oplus F_t^c$ and $z \in F_t^u$. For each $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$ we set

$$V(t, x) = -V^s(t, y) + V^u(t, z),$$

where

$$V^s(t, x) = \sup\bigl\{\bigl\|T(r, t)y\bigr\|e^{-a(\rho(r)-\rho(t))}\colon r \geqslant t\bigr\},$$

and

$$V^u(t, x) = \sup\bigl\{\bigl\|T(r, t)z\bigr\|e^{b(\rho(t)-\rho(r))}\colon r \leqslant t\bigr\}.$$

Proceeding in a similar manner to that for W, we find that V is a (λ_1, ν_1)-strict Lyapunov function with λ_1 and ν_1 as in (35). Since $\delta + \gamma = 2\varepsilon$, (13) holds provided that ε is sufficiently small. Moreover, the conditions $\lambda_1 - \lambda_2 > \nu_1 - \nu_2$ and $\lambda_1 - \lambda_2 > \nu_2 - \nu_1$ are equivalent respectively to $a + d > 0$ and $b + c > 0$ (see (35) and (36)), which in view of (27) are always satisfied. Thus, (14) holds. This completes the proof of the theorem. $\square$

4. Quadratic Lyapunov functions

We show in this section that using quadratic Lyapunov functions we can give a complete characterization of ρ-nonuniform exponential trichotomies without the need for condition (32) in Theorem 5. For each $t \in \mathbb{R}$, let $S(t)$ and $T(t)$ be symmetric invertible $p \times p$ matrices. We consider the functions

$$G(t, x) = \bigl\langle S(t)x, x\bigr\rangle, \qquad V(t, x) = -\operatorname{sign} G(t, x)\sqrt{\bigl|G(t, x)\bigr|}, \tag{40}$$

and

$$H(t, x) = \bigl\langle T(t)x, x\bigr\rangle, \qquad W(t, x) = -\operatorname{sign} H(t, x)\sqrt{\bigl|H(t, x)\bigr|}. \tag{41}$$

Any Lyapunov functions V and W as in (40) and (41) are called *quadratic Lyapunov functions*. Notice that when $t \mapsto S(t)$ is differentiable we have

$$\begin{aligned}\dot{V}(t, x) &= \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)A(t)x \\ &= \bigl\langle S'(t)x, x\bigr\rangle + 2\bigl\langle S(t)x, A(t)x\bigr\rangle,\end{aligned}$$

with similar identities for $\dot{W}$. We present in two theorems a characterization of ρ-nonuniform exponential trichotomies in terms of *quadratic* Lyapunov functions that does not require condition (32).

Theorem 7. *Assume that Eq.* (1) *admits a ρ-nonuniform exponential trichotomy. Then there exist symmetric invertible $p \times p$ matrices $S(t)$ and $T(t)$ for $t \in \mathbb{R}$ such that*:

1. *$t \mapsto S(t)$ and $t \mapsto T(t)$ are of class C^1, and*

$$\limsup_{t\to\pm\infty} \frac{1}{|\rho(t)|} \log\|S(t)\| < \infty \tag{42}$$

and

$$\limsup_{t\to\pm\infty} \frac{1}{|\rho(t)|} \log\|T(t)\| < \infty; \tag{43}$$

2. *there exist $K_1 > K_2 > 0$ and $L_1 > L_2 > 0$ such that for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$ we have*

$$\dot{G}(t,x) \leqslant \begin{cases} -K_1\rho'(t)G(t,x) - \frac{1}{2}\|x\|^2\rho'(t) & \text{if } G(t,x) \geqslant 0, \\ -K_2\rho'(t)G(t,x) - \frac{1}{2}\|x\|^2\rho'(t) & \text{if } G(t,x) \leqslant 0, \end{cases} \tag{44}$$

and

$$\dot{H}(t,x) \leqslant \begin{cases} L_2\rho'(t)H(t,x) - \frac{1}{2}\|x\|^2\rho'(t) & \text{if } H(t,x) \geqslant 0, \\ L_1\rho'(t)H(t,x) - \frac{1}{2}\|x\|^2\rho'(t) & \text{if } H(t,x) \leqslant 0. \end{cases} \tag{45}$$

Proof. Take $\varrho_1 \in (0, (d-c)/2)$, and let $\Gamma_1(t) = Q(t) \oplus R(t)$. Consider the matrices

$$\begin{aligned} S(t) = {} & \int_t^\infty T(v,t)^* P(v)^* P(v) T(v,t) e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ & - \int_{-\infty}^t T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv. \end{aligned} \tag{46}$$

Similarly, take $\varrho_2 \in (0, (b-a)/2)$, and let $\Gamma_2(t) = P(t) \oplus R(t)$. Consider also the matrices

$$\begin{aligned} T(t) = {} & \int_t^\infty T(v,t)^*\Gamma_2(v)^*\Gamma_2(v)T(v,t)e^{-2(a+\varrho_2)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ & - \int_{-\infty}^t T(v,t)^* Q(v)^* Q(v) T(v,t) e^{2(b-\varrho_2)(\rho(t)-\rho(v))}\rho'(v)\,dv. \end{aligned} \tag{47}$$

The matrices $S(t)$ and $T(t)$ are symmetric and invertible for each $t \in \mathbb{R}$. We define the functions G and H by (40) and (41). By (28) and (29), since ρ is an increasing function we have

$$\begin{aligned} |G(t,x)| \leqslant {} & \int_t^\infty \|T(v,t)P(t)x\|^2 e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ & + \int_{-\infty}^t \|T(v,t)\Gamma_1(t)x\|^2 e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv \end{aligned}$$

$$\begin{aligned}
&\leqslant D^2\|x\|^2 e^{2\varepsilon|\rho(t)|}\int_t^\infty e^{-2\varrho_1(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\quad + 4D^2\|x\|^2 e^{2\varepsilon|\rho(t)|}\int_{-\infty}^t e^{-2\varrho_1(\rho(t)-\rho(v))}\rho'(v)\,dv\\
&\leqslant \frac{5D^2}{2\varrho_1}\|x\|^2 e^{2\varepsilon|\rho(t)|}.
\end{aligned}$$

Since

$$\frac{\partial}{\partial t}T(\tau,t) = -T(\tau,t)A(t) \quad \text{and} \quad \frac{\partial}{\partial t}T(\tau,t)^* = -A(t)^*T(\tau,t)^*,$$

one can easily verify that $S(t)$ and $T(t)$ are of class C^1 in t. Moreover, since $S(t)$ is symmetric we obtain

$$\big\|S(t)\big\| = \sup_{x\neq 0}\frac{|G(t,x)|}{\|x\|^2} \leqslant \frac{5D^2}{2\varrho_1}e^{2\varepsilon|\rho(t)|}, \tag{48}$$

and (42) holds. Similar arguments apply to $T(t)$ to obtain (43). Furthermore, taking derivatives in (46) we obtain

$$\begin{aligned}
S'(t) &= -P(t)^*P(t)\rho'(t)\\
&\quad - \int_t^\infty A(t)^*T(v,t)^*P(v)^*P(v)T(v,t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\quad - \int_t^\infty T(v,t)^*P(v)^*P(v)T(v,t)A(t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\quad - 2(d-\varrho_1)\rho'(t)\int_t^\infty T(v,t)^*P(v)^*P(v)T(v,t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\quad - \Gamma_1(t)^*\Gamma_1(t)\rho'(t)\\
&\quad + \int_{-\infty}^t A(t)^*T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv\\
&\quad + \int_{-\infty}^t T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)A(t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv\\
&\quad + 2(c+\varrho_1)\rho'(t)\int_{-\infty}^t T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv\\
&= -\big[P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)\big]\rho'(t) - A(t)^*S(t) - S(t)A(t)
\end{aligned}$$

$$-2(d-\varrho_1)\rho'(t)\int_t^\infty T(v,t)^*P(v)^*P(v)T(v,t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv$$
$$+2(c+\varrho_1)\rho'(t)\int_{-\infty}^t T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv.$$

Therefore,

$$\begin{aligned}
&S'(t)+S(t)A(t)+A(t)^*S(t)^*+\big[P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)\big]\rho'(t)\\
&\quad=-2(d-\varrho_1)\rho'(t)\int_t^\infty T(v,t)^*P(v)^*P(v)T(v,t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\qquad+2(c+\varrho_1)\rho'(t)\int_{-\infty}^t T(v,t)^*\Gamma_1(v)^*\Gamma_1(v)T(v,t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv\\
&\quad=-2(d-\varrho_1)\rho'(t)\int_t^\infty \big(T(v,t)P(t)\big)^*T(v,t)P(t)e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&\qquad+2(c+\varrho_1)\rho'(t)\int_{-\infty}^t \big(T(v,t)\Gamma_1(t)\big)^*T(v,t)\Gamma_1(t)e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv.
\end{aligned}\tag{49}$$

Moreover, since

$$2\big\langle\big(P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)\big)x,x\big\rangle\geqslant\big(\|P(t)x\|+\|\Gamma_1(t)x\|\big)^2\geqslant\big\|\big(P(t)+\Gamma_1(t)\big)x\big\|^2=\|x\|^2,\tag{50}$$

we have

$$P(t)^*P(t)+\Gamma_1(t)^*\Gamma_1(t)\geqslant\frac{1}{2}\,\mathrm{Id}\tag{51}$$

(given two $p\times p$ matrices A and B, we say that $A\geqslant B$ if $\langle Ax,x\rangle\geqslant\langle Bx,x\rangle$ for every $x\in\mathbb{R}^p$). Furthermore, if $x(t)$ is a solution of Eq. (1), then

$$\begin{aligned}
\frac{d}{dt}G\big(t,x(t)\big)&=\big\langle S'(t)x(t),x(t)\big\rangle+\big\langle S(t)x'(t),x(t)\big\rangle+\big\langle S(t)x(t),x'(t)\big\rangle\\
&=\big\langle\big(S'(t)+S(t)A(t)+A(t)^*S(t)\big)x(t),x(t)\big\rangle.
\end{aligned}\tag{52}$$

We note that

$$\begin{aligned}
G\big(t,x(t)\big)=&\int_t^\infty \big\|T(v,t)P(t)x(t)\big\|^2e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\\
&-\int_{-\infty}^t \big\|T(v,t)\Gamma_1(t)x(t)\big\|^2e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv.
\end{aligned}$$

If $G(t,x(t)) \geqslant 0$, then by (49), (51), and (52), since $c+\varrho_1 < d-\varrho_1$ and ρ is an increasing function, we obtain

$$\begin{aligned}\frac{d}{dt}G\big(t,x(t)\big) &\leqslant -\frac{1}{2}\big\|x(t)\big\|^2\rho'(t) - 2(d-\varrho_1)\rho'(t)\Bigg(\int_t^\infty \big\|T(v,t)P(t)x(t)\big\|^2 e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ &\quad - \int_{-\infty}^t \big\|T(v,t)\Gamma_1(t)x(t)\big\|^2 e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv\Bigg) \\ &= -\frac{1}{2}\big\|x(t)\big\|^2\rho'(t) - 2(d-\varrho_1)\rho'(t)G\big(t,x(t)\big). \end{aligned} \tag{53}$$

Thus, we can take $K_1 = 2(d-\varrho_1) > 0$. On the other hand, if $G(t,x(t)) \leqslant 0$, then by (49), (51), and (52), again since $c+\varrho_1 < d-\varrho_1$ and ρ is an increasing function, we obtain

$$\begin{aligned}\frac{d}{dt}G\big(t,x(t)\big) &\leqslant -\frac{1}{2}\big\|x(t)\big\|^2\rho'(t) \\ &\quad + 2(c+\varrho_1)\rho'(t)\Bigg(\int_{-\infty}^t \big\|T(v,t)\Gamma_1(t)x(t)\big\|^2 e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv \\ &\quad - \int_t^\infty \big\|T(v,t)P(t)x(t)\big\|^2 e^{2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv\Bigg) \\ &= -\frac{1}{2}\big\|x(t)\big\|^2\rho'(t) - 2(c+\varrho_1)\rho'(t)G\big(t,x(t)\big). \end{aligned}$$

Thus, we can take $K_2 = 2(c+\varrho_1) > 0$. Furthermore,

$$K_1 - K_2 = 2(d-c-2\varrho_1) > 0.$$

Proceeding in a similar manner with $T(t)$ we obtain

$$\begin{aligned}&T'(t) + T(t)A(t) + A(t)^*T(t) + \big[\Gamma_2(t)^*\Gamma_2(t) + Q(t)^*Q(t)\big]\rho'(t) \\ &\quad = 2(a+\varrho_2)\rho'(t)\int_t^\infty \big(T(v,t)\Gamma_2(t)\big)^*\big(T(v,t)\Gamma_2(t)\big)e^{-2(a+\varrho_2)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ &\quad\quad - 2(b-\varrho_2)\rho'(t)\int_{-\infty}^t \big(T(v,t)Q(t)\big)^*\big(T(v,t)Q(t)\big)e^{2(b-\varrho_2)(\rho(t)-\rho(v))}\rho'(v)\,dv, \end{aligned}$$

and we can show that inequalities (45) hold with $L_1 = 2(b-\varrho_2) > 0$ and $L_2 = 2(a+\varrho_2) > 0$. Moreover, by the choice of ϱ_2 we have

$$L_1 - L_2 = 2(b-a-2\varrho_2) > 0.$$

This completes the proof of the theorem. □

The following is a partial converse to Theorem 7.

Theorem 8. *Assume that there exist constants $\gamma, \alpha, \kappa > 0$ such that*

$$\big\| T(t,s) \big\| \leqslant \kappa e^{\alpha|\rho(t)|} \quad \textit{whenever} \quad \big|\rho(t)-\rho(s)\big| \leqslant \gamma. \tag{54}$$

Moreover, assume that ρ is of class C^1, and that there exist symmetric invertible $p \times p$ matrices $S(t)$ and $T(t)$ for $t \in \mathbb{R}$, satisfying conditions 1 and 2 in Theorem 7 with $K_1 - K_2 > 2\alpha$ and $L_1 - L_2 > 2\alpha$. Then Eq. (1) admits a ρ-nonuniform exponential trichotomy with

$$a = L_2/2 + \alpha, \qquad b = L_1/2 - \alpha, \qquad c = K_2/2 + \alpha, \qquad d = K_1/2 - \alpha. \tag{55}$$

Proof. Set

$$I_\tau^s = \{0\} \cup \big\{x \in \mathbb{R}^p\colon\ G\big(t, T(t,\tau)x\big) > 0 \text{ for every } t \geqslant \tau\big\},$$

and

$$I_\tau^u = \{0\} \cup \big\{x \in \mathbb{R}^p\colon\ G\big(t, T(t,\tau)x\big) < 0 \text{ for every } t \geqslant \tau\big\}.$$

Lemma 4. *If $x \in I_\tau^s$, then*

$$G\big(t, T(t,\tau)x\big) \leqslant e^{-K_1(\rho(t)-\rho(\tau))} G(\tau, x), \quad t \geqslant \tau, \tag{56}$$

and if $x \in I_\tau^u$, then

$$\big|G\big(t, T(t,\tau)x\big)\big| \geqslant e^{-K_2(\rho(t)-\rho(\tau))} \big|G(\tau, x)\big|, \quad t \geqslant \tau. \tag{57}$$

Proof. Given $x \in I_\tau^s \setminus \{0\}$, since $G(t, T(t,\tau)x) > 0$ for every $t \geqslant \tau$, it follows from (44) that

$$\frac{\dot{G}(t, T(t,\tau)x)}{G(t, T(t,\tau)x)} \leqslant -K_1 \rho'(t), \quad t > \tau.$$

This implies that

$$\log G\big(t, T(t,\tau)x\big) - \log G(\tau, x) \leqslant -K_1 \int_\tau^t \rho'(v)\,dv = -K_1\big(\rho(t) - \rho(\tau)\big),$$

and hence (56) holds. Similarly, given $x \in I_\tau^u \setminus \{0\}$, since $G(t, T(t,\tau)x) < 0$ for every $t \geqslant \tau$, it follows again from (44) that

$$\frac{\dot{G}(t, T(t,\tau)x)}{G(t, T(t,\tau)x)} \geqslant -K_2 \rho'(t), \quad t > \tau.$$

This implies that

$$\log\big|G\big(t, T(t,\tau)x\big)\big| - \log\big|G(\tau, x)\big| \geqslant -K_2 \int_\tau^t \rho'(v)\,dv = -K_2\big(\rho(t) - \rho(\tau)\big),$$

and hence (57) holds. $\square$

Lemma 5. *If $x \in I_\tau^u \cup I_\tau^s$, then*

$$\left|G(\tau, x)\right| \geqslant \frac{1}{2\kappa^2} \max\left\{\gamma, 1 - e^{-K_2\gamma}\right\} e^{-2\alpha|\rho(\tau)|} \|x\|^2.$$

Proof. Set $x(t) = T(t, \tau)x$. It follows from (44) that if $x \in I_\tau^s$, then

$$\frac{d}{dt} G\big(t, x(t)\big) \leqslant -\frac{1}{2} \left\|x(t)\right\|^2 \rho'(t).$$

Now, given $\tau \in \mathbb{R}$, take $t > \tau$ such that $\rho(t) = \rho(\tau) + \gamma$ (with γ as in (54)). Then

$$\begin{aligned} G\big(t, x(t)\big) - G(\tau, x) &= \int_\tau^t \frac{d}{dr} G\big(r, x(r)\big)\, dr \leqslant -\frac{1}{2} \int_\tau^t \left\|x(r)\right\|^2 \rho'(r)\, dr \\ &= -\frac{1}{2} \int_\tau^t \left\|T(r, \tau)x\right\|^2 \rho'(r)\, dr \leqslant -\frac{1}{2} \|x\|^2 \int_\tau^t \frac{\rho'(r)\, dr}{\|T(\tau, r)\|^2}. \end{aligned}$$

It follows from (54) that

$$\begin{aligned} G\big(t, x(t)\big) - G(\tau, x) &\leqslant -\frac{1}{2} \|x\|^2 \int_\tau^t \frac{\rho'(r)\, dr}{\kappa^2} e^{-2\alpha|\rho(\tau)|} \\ &= -\frac{1}{2\kappa^2} e^{-2\alpha|\rho(\tau)|} \|x\|^2 \big(\rho(t) - \rho(\tau)\big) \\ &= -\frac{\gamma}{2\kappa^2} e^{-2\alpha|\rho(\tau)|} \|x\|^2. \end{aligned}$$

Therefore,

$$G(\tau, x) \geqslant G(\tau, x) - G\big(t, x(t)\big) \geqslant \frac{\gamma}{2\kappa^2} e^{-2\alpha|\rho(\tau)|} \|x\|^2.$$

On the other hand, it follows from (44) that if $x \in I_\tau^u$, then

$$\begin{aligned} \frac{d}{dt}\big(e^{K_2\rho(t)} G\big(t, x(t)\big)\big) &= e^{K_2\rho(t)} \left(\frac{d}{dt} G\big(t, x(t)\big) + K_2 \rho'(t) G\big(t, x(t)\big)\right) \\ &\leqslant -\frac{1}{2} \rho'(t) e^{K_2\rho(t)} \left\|x(t)\right\|^2. \end{aligned}$$

Therefore, given $\tau \in \mathbb{R}$ and $t < \tau$ such that $\rho(\tau) = \rho(t) + \gamma$, using again (54) we obtain

$$\begin{aligned} e^{K_2\rho(\tau)} G(\tau, x) - e^{K_2\rho(t)} G\big(t, x(t)\big) &\leqslant -\frac{1}{2\kappa^2} e^{-2\alpha|\rho(\tau)|} \|x\|^2 \int_t^\tau \rho'(r) e^{K_2\rho(r)}\, dr \\ &= -\frac{1}{2\kappa^2} e^{-2\alpha|\rho(\tau)|} \big[e^{K_2\rho(\tau)} - e^{K_2\rho(t)}\big] \|x\|^2. \end{aligned}$$

Since $G(t,x(t))<0$ and $G(\tau,x)<0$ we have

$$e^{K_2\rho(\tau)}\left|G(\tau,x)\right| \geqslant \frac{1}{2\kappa^2}e^{-2\alpha|\rho(\tau)|}\left[e^{K_2\rho(\tau)}-e^{K_2\rho(t)}\right]\|x\|^2$$

and thus,

$$\begin{aligned}\left|G(\tau,x)\right| &\geqslant \frac{1}{2\kappa^2}e^{-2\alpha|\rho(\tau)|}\left[1-e^{K_2(\rho(t)-\rho(\tau))}\right]\|x\|^2\\ &= \frac{1}{2\kappa^2}e^{-2\alpha|\rho(\tau)|}\left[1-e^{-K_2\gamma}\right]\|x\|^2.\end{aligned}$$

This completes the proof of the lemma. $\square$

Now we set

$$J_\tau^s=\{0\}\cup\left\{x\in\mathbb{R}^p\colon\ H\big(t,T(t,\tau)x\big)>0 \text{ for every } t\geqslant\tau\right\},$$

and

$$J_\tau^u=\{0\}\cup\left\{x\in\mathbb{R}^p\colon\ H\big(t,T(t,\tau)x\big)<0 \text{ for every } t\geqslant\tau\right\}.$$

Proceeding in a similar manner to that in the proofs of Lemmas 4 and 5 we obtain the following statements.

Lemma 6. *If $x\in J_\tau^s$, then*

$$H\big(t,T(t,\tau)x\big)\leqslant e^{L_2(\rho(t)-\rho(\tau))}H(\tau,x),\quad t\geqslant\tau,$$

and if $x\in J_\tau^u$, then

$$\left|H\big(t,T(t,\tau)x\big)\right|\geqslant e^{L_1(\rho(t)-\rho(\tau))}\left|H(\tau,x)\right|,\quad t\geqslant\tau.$$

Lemma 7. *If $x\in J_\tau^u\cup J_\tau^s$, then*

$$\left|H(\tau,x)\right|\geqslant\frac{1}{2\kappa^2}\max\left\{\gamma,1-e^{-L_2\gamma}\right\}e^{-2\alpha|\rho(\tau)|}\|x\|^2.$$

By (42) and (43), for each $\delta>0$ there exists $d>0$ such that

$$\left\|S(t)\right\|\leqslant de^{\delta|\rho(t)|}\quad\text{and}\quad\left\|T(t)\right\|\leqslant de^{\delta|\rho(t)|}$$

for every $t\in\mathbb{R}$. Hence,

$$\left|G(t,x)\right|\leqslant de^{\delta|\rho(t)|}\|x\|^2\quad\text{and}\quad\left|H(t,x)\right|\leqslant de^{\delta|\rho(t)|}\|x\|^2. \tag{58}$$

Lemma 8. *The function V in* (40) *is a (λ_1,ν_1)-strict Lyapunov function for Eq.* (1) *with*

$$\lambda_1=\frac{1}{2}\big(e^{-K_1/2}+e^{-K_2/2}\big)\quad\textit{and}\quad\nu_1=\frac{1}{2}\big(e^{-K_2/2}-e^{-K_1/2}\big).$$

Proof. By (58) and (40) we have that

$$\|V(t,x)\| \leqslant \sqrt{d}e^{\delta|\rho(t)|/2}\|x\|,$$

and (5) holds. Furthermore, by Lemma 5, for $x \in I_\tau^s \cup I_\tau^u = H_\tau^u \cup H_\tau^s$ we have

$$|V(\tau,x)| \geqslant \frac{1}{\sqrt{2}\kappa}\max\{\gamma, 1-e^{-K_2\gamma}\}^{1/2}e^{-\alpha|\rho(\tau)|}\|x\|, \tag{59}$$

and (12) holds. Finally, by Lemma 4, if $x \in I_\tau^s = H_\tau^u$ then

$$|V(t,T(t,\tau)x)| \leqslant e^{-K_1(\rho(t)-\rho(\tau))/2}|V(\tau,x)|, \quad t \geqslant \tau,$$

that is, (11) holds with $\lambda - \nu = e^{-K_1/2}$. Moreover, if $x \in I_\tau^u = H_\tau^u$ then

$$V(t,T(t,\tau)x) \geqslant e^{-K_2(\rho(t)-\rho(\tau))/2}V(\tau,x), \quad t \geqslant \tau,$$

that is, (10) holds with $\lambda + \nu = e^{-K_2/2}$. This concludes the proof of the lemma. □

In an analogous manner we can prove the following result.

Lemma 9. *The function W in (41) is a (λ_2,ν_2)-strict Lyapunov function for Eq. (1) with*

$$\lambda_2 = \frac{1}{2}\left(e^{L_1/2}+e^{L_2/2}\right) \quad \text{and} \quad \nu_2 = \frac{1}{2}\left(e^{L_1/2}-e^{L_2/2}\right).$$

Since V is a strict Lyapunov function, by Lemma 1 there exist subspaces $H_t^u(V)$ and $H_t^s(V)$ such that $\mathbb{R}^p = H_t^u(V) \oplus H_t^s(V)$ for each $t \in \mathbb{R}$. We consider the associated projections

$$P_V(t):\mathbb{R}^p \to H_t^s(V) \quad \text{and} \quad Q_V(t):\mathbb{R}^p \to H_t^u(V).$$

In a similar manner, there exist subspaces $H_t^u(W)$ and $H_t^s(W)$ such that $\mathbb{R}^p = H_t^u(W) \oplus H_t^s(W)$ for each $t \in \mathbb{R}$, and we consider the associated projections

$$P_W(t):\mathbb{R}^p \to H_t^s(W) \quad \text{and} \quad Q_W(t):\mathbb{R}^p \to H_t^u(W).$$

Lemma 10. *There exists $K > 0$ such that for each $t \in \mathbb{R}$ we have*

$$\|P_V(t)\| = \|Q_V(t)\| \leqslant Ke^{2\alpha|\rho(t)|}\|S(t)\|,$$

and

$$\|P_W(t)\| = \|Q_W(t)\| \leqslant Ke^{2\alpha|\rho(t)|}\|T(t)\|.$$

Proof. We only prove the statement for the Lyapunov function V. The proof for W is completely analogous. We note that

$$V(t,P_V(t)x)^2 = \langle S(t)P_V(t)x, P_V(t)x\rangle, \tag{60}$$

and

$$V(t,Q_V(t)x)^2 = -\langle S(t)Q_V(t)x, Q_V(t)x\rangle. \tag{61}$$

Given $x \in \mathbb{R}^p$ we write $x = y + z$ with

$$y = P_V(t)x \in H_t^s(V) \quad \text{and} \quad z = Q_V(t)x \in H_t^u(V).$$

Now take $a(t) > 0$, and set

$$V^s(t, y) = -V(t, y)^2 + a(t)\|y\|^2 = -\langle S(t)y, y\rangle + a(t)\|y\|^2.$$

By (59), there exists $K > 0$ such that

$$V^s(t, y) \leqslant -Ke^{-2\alpha|\rho(t)|}\|y\|^2 + a(t)\|y\|^2 = \big(a(t) - Ke^{-2\alpha|\rho(t)|}\big)\|y\|^2.$$

Similarly, for each $t \in \mathbb{R}$ we set

$$V^u(t, z) = V(t, z)^2 - a(t)\|z\|^2 = -\langle S(t)z, z\rangle - a(t)\|z\|^2.$$

By (59) we have

$$V^u(t, z) \geqslant \big(Ke^{-2\alpha|\rho(t)|} - a(t)\big)\|z\|^2.$$

We conclude that if $a(t) \leqslant Ke^{-2\alpha|\rho(t)|}$, then

$$-V(t, y)^2 + a(t)\|y\|^2 \leqslant 0 \quad \text{and} \quad V(t, z)^2 - a(t)\|z\|^2 \geqslant 0.$$

Thus, it follows from (60) and (61) that

$$-\langle S(t)P_V(t)x, P_V(t)x\rangle + a(t)\|P_V(t)x\|^2 \leqslant 0,$$

and

$$-\langle S(t)Q_V(t)x, Q_V(t)x\rangle - a(t)\|Q_V(t)x\|^2 \geqslant 0.$$

Since $S(t)$ is symmetric, subtracting the two inequalities we obtain

$$\begin{aligned}
0 \geqslant\ & a(t)\|P_V(t)x\|^2 + a(t)\|Q_V(t)x\|^2 \\
& - \langle S(t)P_V(t)x, P_V(t)x\rangle + \langle S(t)Q_V(t)x, Q_V(t)x\rangle \\
=\ & a(t)\|P_V(t)x\|^2 + a(t)\|Q_V(t)x\|^2 + \langle S(t)x, x\rangle - 2\langle S(t)P_V(t)x, x\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& a(t)\left\|P_V(t)x - \frac{1}{2a(t)}S(t)x\right\|^2 + a(t)\left\|Q_V(t)x + \frac{1}{2a(t)}S(t)x\right\|^2 \\
& = a(t)\|P_V(t)x\|^2 + a(t)\|Q_V(t)x\|^2 + \frac{\|S(t)x\|^2}{2a(t)} + \langle S(t)x, x\rangle - 2\langle S(t)P_V(t)x, x\rangle \\
& \leqslant \frac{\|S(t)x\|^2}{2a(t)},
\end{aligned}$$

and

$$\left\| P_V(t)x - \frac{1}{2a(t)} S(t)x \right\|^2 + \left\| Q_V(t)x + \frac{1}{2a(t)} S(t)x \right\|^2 \leqslant \frac{\|S(t)x\|^2}{2a(t)^2}.$$

This implies that

$$\begin{aligned}
\|P_V(t)x\| &= \left\| P_V(t)x - \frac{1}{2a(t)} S(t)x + \frac{1}{2a(t)} S(t)x \right\| \\
&\leqslant \left\| P_V(t)x - \frac{1}{2a(t)} S(t)x \right\| + \frac{1}{2a(t)} \|S(t)x\| \\
&\leqslant \frac{1}{\sqrt{2}a(t)} \|S(t)x\| + \frac{1}{\sqrt{2}a(t)} \|S(t)x\| \leqslant \frac{\sqrt{2}}{a(t)} \|S(t)x\|,
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|Q_V(t)x\| &\leqslant \left\| Q_V(t)x + \frac{1}{2a(t)} S(t)x \right\| + \frac{1}{2a(t)} \|S(t)x\| \\
&\leqslant \frac{1}{\sqrt{2}a(t)} \|S(t)x\| + \frac{1}{2a(t)} \|S(t)x\| \leqslant \frac{\sqrt{2}}{a(t)} \|S(t)x\|.
\end{aligned}$$

Taking $a(t) = Ke^{-2\alpha|\rho(t)|}$ we obtain the desired statement. □

Note that by taking δ sufficiently small we have

$$\frac{\lambda_1 + \nu_1}{\lambda_1 - \nu_1} = e^{(K_1 - K_2)/2} > e^{\alpha + \delta/2}$$

and

$$\frac{\lambda_2 + \nu_2}{\lambda_2 - \nu_2} = e^{(L_1 - L_2)/2} > e^{\alpha + \delta/2}.$$

Moreover, we can easily verify that $\lambda_2 - \lambda_1 > |\nu_2 - \nu_1|$. This allows us to apply Theorem 4 (with V and W interchanged). Therefore, if we set

$$\begin{aligned}
P(\tau) &= P_W(\tau) : \mathbb{R}^p \to F^s_\tau = H^s_\tau(W), \\
Q(\tau) &= Q_V(\tau) : \mathbb{R}^p \to F^u_\tau = H^u_\tau(V),
\end{aligned}$$

and

$$R(\tau) = P_V(\tau) \oplus Q_W(\tau) : \mathbb{R}^p \to F^c_\tau = H^s_\tau(V) \cap H^u_\tau(W),$$

then the subspaces F^s_τ, F^u_τ, and F^c_τ satisfy the properties in Theorem 4. Moreover, for every $t \geqslant \tau$ we have

$$\begin{aligned}
\|T(t,\tau)P(\tau)\| &\leqslant \|T(t,\tau)|F^s_\tau\| \cdot \|P(\tau)\|, \\
\|T(t,\tau)^{-1}Q(t)\| &\leqslant \|T(t,\tau)^{-1}|F^u_t\| \cdot \|Q(t)\|,
\end{aligned}$$

and

$$\|T(t,\tau)R(\tau)\| \leqslant \|T(t,\tau)|F_\tau^c\| \cdot \|R(\tau)\|,$$
$$\|T(t,\tau)^{-1}R(t)\| \leqslant \|T(t,\tau)^{-1}|F_t^c\| \cdot \|R(t)\|.$$

Therefore, by property 2 in Theorem 4 and Lemma 10 there exist constants as in (27) satisfying (28) and (29). In other words, Eq. (1) admits a ρ-nonuniform exponential trichotomy. By (17), (18), and Lemma 10 we can take the constants a, b, c, d in (27) as in (55). This completes the proof of the theorem. $\square$

Now we consider the particular case of uniform exponential trichotomies. The following are simple consequences respectively of Theorems 7 and 8.

Theorem 9. *Assume that Eq.* (1) *admits a ρ-uniform exponential trichotomy. Then there exist symmetric invertible $p \times p$ matrices $S(t)$ and $T(t)$ for $t \in \mathbb{R}$ such that*:

1. *$t \mapsto S(t)$ and $t \mapsto T(t)$ are bounded and of class C^1*;
2. *there exist $K_1 > K_2 > 0$ and $L_1 > L_2 > 0$ such that* (44) *and* (45) *hold for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$.*

Theorem 10. *Assume that there are constants $\gamma, \kappa > 0$ such that*

$$\|T(t,s)\| \leqslant \kappa \quad \text{whenever} \quad |\rho(t) - \rho(s)| \leqslant \gamma.$$

Moreover, assume that ρ is of class C^1, and that there exist symmetric invertible $p \times p$ matrices $S(t)$ and $T(t)$ for $t \in \mathbb{R}$ such that:

1. *$t \mapsto S(t)$ and $t \mapsto T(t)$ are bounded and of class C^1*;
2. *there exist $K_1 > K_2 > 0$ and $L_1 > L_2 > 0$ such that* (44) *and* (45) *hold for every $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$.*

Then Eq. (1) *admits a ρ-uniform exponential trichotomy with*

$$a = L_2/2, \qquad b = L_1/2, \qquad c = K_2/2, \qquad d = K_1/2.$$

5. Application: Robustness of exponential trichotomies

We establish in this section the robustness of ρ-nonuniform exponential trichotomies under perturbations

$$x' = [A(t) + B(t)]x \tag{62}$$

using Lyapunov functions. We continue to denote by M_p the set of $p \times p$ matrices.

Theorem 11. *Let $A, B : \mathbb{R} \to M_p$ be continuous functions. We assume that Eq.* (1) *satisfies* (54) *with $\alpha = \varepsilon$, and that it admits a ρ-nonuniform exponential trichotomy with $b - a > \varepsilon$ and $d - c > \varepsilon$. For any sufficiently small $\delta > 0$, if $\|B(t)\| \leqslant \delta\rho'(t)e^{-2\varepsilon|\rho(t)|}$ for every $t \in \mathbb{R}$, then Eq.* (62) *admits a ρ-nonuniform exponential trichotomy.*

Proof. We want to apply Theorem 8 to Eq. (62). For this we show that under the assumptions of the theorem Eq. (62) also satisfies (54). Indeed, if we denote by $U(t,s)$ the evolution operator associated to (62), then for every $t,s\in\mathbb{R}$ with $|\rho(t)-\rho(s)|\leqslant\gamma$, we have

$$U(t,s)=T(t,s)+\int_s^t T(t,\tau)B(\tau)U(\tau,s)\,d\tau,$$

and hence,

$$\|U(t,s)\|\leqslant\kappa e^{\varepsilon|\rho(t)|}+\int_s^t \kappa e^{\varepsilon|\rho(t)|}\delta\rho'(\tau)e^{-2\varepsilon|\rho(\tau)|}\|U(\tau,s)\|\,d\tau.$$

Setting $\eta(t)=\|U(t,s)\|e^{-\varepsilon|\rho(t)|}$, we obtain

$$\eta(t)\leqslant\kappa+\kappa\delta\int_s^t \rho'(\tau)e^{-\varepsilon|\rho(\tau)|}\eta(\tau)\,d\tau.$$

By Gronwall's lemma, we have

$$\eta(t)\leqslant\kappa e^{\kappa\delta\int_s^t\rho'(\tau)e^{-\varepsilon|\rho(\tau)|}\,d\tau}\leqslant\kappa e^{\kappa\delta\gamma},$$

and hence,

$$\|U(t,s)\|\leqslant\kappa e^{\kappa\delta\gamma}e^{\varepsilon|\rho(t)|}$$

for every $t,s\in\mathbb{R}$ such that $|\rho(t)-\rho(s)|\leqslant\gamma$. This shows that Eq. (62) also satisfies (54) (with $\alpha=\varepsilon$). Now we consider the matrices $S(t)$ and $T(t)$ in (46) and (47), that are associated to Eq. (1), and we show that they can also be used for Eq. (62) (simply with $S(t)$ and $T(t)$ replaced by some constant multiples). The first condition in Theorem 7 follows as in the proof of the theorem (see (48)). Now we show that the second condition also holds when the dynamics of Eq. (1) is replaced by the dynamics of Eq. (62). We first show that

$$S(t)B(t)+B(t)^*S(t)\leqslant\eta\rho'(t)\,\mathrm{Id},$$

and

$$T(t)B(t)+B(t)^*T(t)\leqslant\eta\rho'(t)\,\mathrm{Id}$$

for some constant $\eta<1/2$. By the norm bound for $B(t)$ we have

$$S(t)B(t)+B(t)^*S(t)\leqslant 2\|S(t)\|\cdot\|B(t)\|\,\mathrm{Id}\leqslant\frac{5\delta D^2}{\varrho_1}\rho'(t)\,\mathrm{Id}$$

and

$$T(t)B(t)+B(t)^*T(t)\leqslant 2\|T(t)\|\cdot\|B(t)\|\,\mathrm{Id}\leqslant\frac{5\delta D^2}{\varrho_2}\rho'(t)\,\mathrm{Id}.$$

Thus, we obtain $\eta < 1/2$ by taking δ sufficiently small. Furthermore, if $x(t)$ is a solution of Eq. (62), then

$$\begin{aligned}\frac{d}{dt}G\big(t,x(t)\big) &= \big\langle S'(t)x(t),x(t)\big\rangle \\ &\quad + \big\langle S(t)A(t)x(t),x(t)\big\rangle + \big\langle S(t)B(t)x(t),x(t)\big\rangle \\ &\quad + \big\langle A(t)^*S(t)x(t),x(t)\big\rangle + \big\langle B(t)^*S(t)x(t),x(t)\big\rangle,\end{aligned} \tag{63}$$

and

$$\begin{aligned}\frac{d}{dt}H\big(t,x(t)\big) &= \big\langle T'(t)x(t),x(t)\big\rangle \\ &\quad + \big\langle T(t)A(t)x(t),x(t)\big\rangle + \big\langle T(t)B(t)x(t),x(t)\big\rangle \\ &\quad + \big\langle A(t)^*T(t)x(t),x(t)\big\rangle + \big\langle B(t)^*T(t)x(t),x(t)\big\rangle.\end{aligned}$$

By (49) and (50) we obtain

$$\begin{aligned}&\big\langle S'(t)x(t),x(t)\big\rangle + \big\langle S(t)A(t)x(t),x(t)\big\rangle + \big\langle A(t)^*S(t)x(t),x(t)\big\rangle \\ &\quad \leqslant -\frac{1}{2}\rho'(t)\big\|x(t)\big\|^2 - 2(d-\varrho_1)\rho'(t)\int_t^\infty \big\|T(v,t)P(t)x(t)\big\|^2 e^{-2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ &\qquad + 2(c+\varrho_1)\rho'(t)\int_{-\infty}^t \big\|T(v,t)\Gamma_1(t)x(t)\big\|^2 e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv.\end{aligned} \tag{64}$$

Therefore, it follows from (63) and (64) that if δ is sufficiently small so that $\eta < 1/2$, then

$$\begin{aligned}\frac{d}{dt}G\big(t,x(t)\big) &\leqslant -\left(\frac{1}{2}-\eta\right)\rho'(t)\big\|x(t)\big\|^2 \\ &\quad + 2(d-\varrho_1)\rho'(t)\int_t^\infty \big\|T(v,t)P(t)x(t)\big\|^2 e^{-2(d-\varrho_1)(\rho(v)-\rho(t))}\rho'(v)\,dv \\ &\quad - 2(c+\varrho_1)\rho'(t)\int_{-\infty}^t \big\|T(v,t)\Gamma_1(t)x(t)\big\|^2 e^{-2(c+\varrho_1)(\rho(t)-\rho(v))}\rho'(v)\,dv.\end{aligned}$$

Proceeding in a similar manner to that in (53) (now with $x(t)$ to be a solution of Eq. (62)) we obtain

$$\tilde{G}'(t) \leqslant \begin{cases} -(\frac{1}{2}-\eta)\rho'(t)\|x(t)\|^2 - 2(d-\varrho_1)\rho'(t)\tilde{G}(t) & \text{if } \tilde{G}(t) \geqslant 0, \\ -(\frac{1}{2}-\eta)\rho'(t)\|x(t)\|^2 - 2(c+\varrho_1)\rho'(t)\tilde{G}(t) & \text{if } \tilde{G}(t) \leqslant 0, \end{cases}$$

where $\tilde{G}(t) = G(t,x(t))$. Notice that setting $K_1 = 2(d-\varrho_1)$ and $K_2 = 2(c+\varrho_1)$, we have $K_1 - K_2 = 2(d-c-2\varrho_1) > 2\alpha$ provided that ϱ_1 is sufficiently small. Proceeding in a similar manner we obtain

$$\begin{aligned}
&\big\langle T'(t)x(t), x(t)\big\rangle + \big\langle T(t)A(t)x(t), x(t)\big\rangle + \big\langle A(t)^*T(t)x(t), x(t)\big\rangle \\
&\quad\leqslant -\frac{1}{2}\rho'(t)\big\|x(t)\big\|^2 + 2(a+\varrho_2)\rho'(t)\int_t^\infty \big\|T(v,t)\Gamma_2(t)x(t)\big\|^2 e^{-2(a+\varrho_2)(\rho(v)-\rho(t))}\rho'(v)\,dv \\
&\qquad - 2(b-\varrho_2)\rho'(t)\int_{-\infty}^t \big\|T(v,t)Q(t)x(t)\big\|^2 e^{2(b-\varrho_2)(\rho(t)-\rho(v))}\rho'(v)\,dv,
\end{aligned}$$

and

$$\tilde H(t) \leqslant \begin{cases} -(\frac{1}{2}-\eta)\rho'(t)\|x(t)\|^2 + 2(a+\varrho_2)\rho'(t)\tilde H(t) & \text{if } \tilde H(t) \geqslant 0, \\ -(\frac{1}{2}-\eta)\rho'(t)\|x(t)\|^2 + 2(b-\varrho_2)\rho'(t)\tilde H(t) & \text{if } \tilde H(t) \leqslant 0, \end{cases}$$

where $\tilde H(t) = H(t, x(t))$. Notice that setting $L_1 = 2(b-\varrho_2)$ and $L_2 = 2(a+\varrho_2)$, we have $L_1 - L_2 = 2(b-a-2\varrho_2) > 2\alpha$ provided that ϱ_2 is taken sufficiently small. Hence, the second condition in Theorem 7 holds, with $S(t)$ and $T(t)$ replaced respectively by $S(t)/(1-2\eta)$ and $T(t)/(1-2\eta)$ for each $t \in \mathbb{R}$. Therefore, by Theorem 8 Eq. (62) admits a ρ-nonuniform exponential trichotomy. □

References

[1] L. Barreira, Ya. Pesin, Nonuniform Hyperbolicity, Encyclopedia Math. Appl., vol. 115, Cambridge Univ. Press, 2007.
[2] L. Barreira, C. Valls, Stability of nonautonomous differential equations in Hilbert spaces, J. Differential Equations 217 (2005) 204–248.
[3] L. Barreira, C. Valls, Growth rates and nonuniform hyperbolicity, Discrete Contin. Dyn. Syst. 22 (2008) 509–528.
[4] L. Barreira, C. Valls, Stability of Nonautonomous Differential Equations, Lecture Notes in Math., vol. 1926, Springer, 2008.
[5] L. Barreira, C. Valls, Polynomial growth rates, Nonlinear Anal., in press.
[6] M. Brin, Ja. Pesin, Partially hyperbolic dynamical systems, Math. USSR Izv. 8 (1974) 177–218.
[7] C. Chicone, Yu. Latushkin, Center manifolds for infinite dimensional nonautonomous differential equations, J. Differential Equations 141 (1997) 356–399.
[8] C. Chicone, Yu. Latushkin, Evolution Semigroups in Dynamical Systems and Differential Equations, Math. Surveys Monogr., vol. 70, Amer. Math. Soc., 1999.
[9] S.-N. Chow, H. Leiva, Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces, J. Differential Equations 120 (1995) 429–477.
[10] S.-N. Chow, W. Liu, Y. Yi, Center manifolds for invariant sets, J. Differential Equations 168 (2000) 355–385.
[11] S.-N. Chow, W. Liu, Y. Yi, Center manifolds for smooth invariant manifolds, Trans. Amer. Math. Soc. 352 (2000) 5179–5211.
[12] C. Conley, R. Miller, Asymptotic stability without uniform stability: Almost periodic coefficients, J. Differential Equations 1 (1965) 333–336.
[13] W. Coppel, Dichotomies and reducibility, J. Differential Equations 3 (1967) 500–521.
[14] W. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer, 1978.
[15] Ju. Dalec'kiĭ, M. Kreĭn, Stability of Solutions of Differential Equations in Banach Space, Transl. Math. Monogr., vol. 43, Amer. Math. Soc., 1974.
[16] S. Elaydi, O. Hájek, Exponential trichotomy of differential systems, J. Math. Anal. Appl. 129 (1988) 362–374.
[17] L. Eliasson, Almost reducibility of linear quasi-periodic systems, in: Smooth Ergodic Theory and Its Applications, in: Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., 2001, pp. 679–705.
[18] W. Hahn, The present state of Lyapunov's direct method, in: Nonlinear Problems, University of Wisconsin Press, Madison, 1963, pp. 195–205.
[19] J. Hale, Asymptotic Behavior of Dissipative Systems, Math. Surveys Monogr., vol. 25, Amer. Math. Soc., 1988.
[20] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math., vol. 840, Springer, 1981.
[21] A. Kelley, The stable, center-stable, center, center-unstable, unstable manifolds, J. Differential Equations 3 (1967) 546–570.
[22] J. Massera, J. Schäffer, Linear differential equations and functional analysis, I, Ann. of Math. (2) 67 (1958) 517–573.
[23] J. Massera, J. Schäffer, Linear Differential Equations and Function Spaces, Pure Appl. Math., vol. 21, Academic Press, 1966.
[24] A. Mielke, A reduction principle for nonautonomous systems in infinite-dimensional spaces, J. Differential Equations 65 (1986) 68–88.
[25] R. Naulin, M. Pinto, Admissible perturbations of exponential dichotomy roughness, Nonlinear Anal. 31 (1998) 559–571.
[26] V. Oseledets, A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems, Trans. Moscow Math. Soc. 19 (1968) 197–221.
[27] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703–728.

[28] Ya. Pesin, Families of invariant manifolds corresponding to nonzero characteristic exponents, Math. USSR Izv. 10 (1976) 1261–1305.
[29] Ya. Pesin, Characteristic Ljapunov exponents, and smooth ergodic theory, Russian Math. Surveys 32 (1977) 55–114.
[30] V. Pliss, A reduction principle in the theory of stability of motion, Izv. Akad. Nauk SSSR Ser. Mat. 28 (1964) 1297–1324.
[31] V. Pliss, G. Sell, Robustness of exponential dichotomies in infinite-dimensional dynamical systems, J. Dynam. Differential Equations 11 (1999) 471–513.
[32] L. Popescu, Exponential dichotomy roughness on Banach spaces, J. Math. Anal. Appl. 314 (2006) 436–454.
[33] R. Sacker, G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, I, J. Differential Equations 15 (1974) 429–458.
[34] R. Sacker, G. Sell, Existence of dichotomies and invariant splittings for linear differential systems, III, J. Differential Equations 22 (1976) 497–522.
[35] G. Sell, Y. You, Dynamics of Evolutionary Equations, Appl. Math. Sci., vol. 143, Springer, 2002.
[36] A. Vanderbauwhede, Centre manifolds, normal forms and elementary bifurcations, in: Dynamics Reported 2, Wiley, 1989, pp. 89–169.
[37] A. Vanderbauwhede, G. Iooss, Center manifold theory in infinite dimensions, in: Dynamics Reported (N.S.) 1, Springer, 1992, pp. 125–163.
[38] A. Vanderbauwhede, S. van Gils, Center manifolds and contractions on a scale of Banach spaces, J. Funct. Anal. 72 (1987) 209–224.