International Journal of Pure and Applied Mathematics
Volume 27 No. 4 2006, 465-476

# SECOND ORDER GENERALIZED LINEAR SYSTEMS. STRUCTURAL INVARIANTS AND CONTROLLABILITY

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**Abstract:** Let $(E, A_1, A_2, B)$ be a quadruple of matrices representing a two-order generalized time-invariant linear system, $E\ddot{x} = A_1\dot{x} + A_2x + Bu$.

Structural invariants under second order derivative feedback are obtained and they are applied to obtain conditions for controllability of the systems. In particular, a necessary and sufficient condition for controllability related with golden mean is given.

**AMS Subject Classification:** 15A21, 93B52
**Key Words:** two-order generalized linear systems, feedback, controllability

## 1. Introduction

The study of generalized linear systems have a great interest in recent years. First order are applied in engineering for example they are used in modelling a three-link planar manipulator by M. Hou [8]. Second order generalized systems are applied to power systems by Campbell and Rose [1].

A second order generalized linear system is described by the following state space equation

$$E\ddot{x} = A_1\dot{x} + A_2x + Bu, \tag{1}$$

where $A_i$ are $n$-square complex matrices and $B$ a $n \times m$-rectangular complex matrix in adequate size. We denote this kind of systems by quadruples of matrices $(E, A_1, A_2, B)$, and the space of all quadruples by $\mathcal{M}_{n,m}$:

Received: March 10, 2006 

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$$\mathcal{M}_{n,m} = \{(E, A_1, A_2, B) \mid E, A_1, A_2 \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})\}\,.$$

Controllability is a qualitative property of second order linear dynamical systems largely studied (see [9], [7] for example). In this paper we will go to study the controllability property for second order generalized linear systems.

We recall that a second order generalized linear system is called controllable if, for any $t_1 > 0$, $x(0), \dot{x}(0) \in \mathbb{C}^n$ and $w, w_1 \in \mathbb{C}^n$, there exists a control $u(t)$ such that $x(t_1) = w$, $\dot{x}(t_1) = w_1$. This definition is a natural generalization of controllability concept in the first order linear systems. The controllability can be characterized in terms of the rank of suitable matrices related to the pair $(E, B)$ and the transfer function in the following manner.

**Proposition 2.** (see [2]) *A second order generalized linear system* $(E, A_1, A_2, B)$, *is controllable if and only if*

*i)* rank $\begin{pmatrix} E & B \end{pmatrix} = n$.

*ii)* rank $\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = n\,, \quad \forall s \in \mathbb{C}$.

**Remark 1.** Condition i) ensures that there exists a second order derivative feedback $F$ such that $E + BF$ is regular and premultiplying the system by $(E + BF)^{-1}$ the new system is standard. We will call standardizable this kind of systems.

In order to simplify the study of controllability, in the space of second order generalized linear systems, we can consider the following equivalence relation corresponding to the following standard transformations: basis changes for the state and input spaces: $x(t) = Px_1(t)$, $u(t) = Qu_1(t)$, state feedback: $u(t) = F_2x(t) + w(t)$, derivative feedback: $u(t) = F_1\dot{x}(t) + w(t)$ and second order derivative feedback: $u(t) = -F_3\ddot{x}(t) + w(t)$.

**Definition 1.** Two quadruples $(E, A_1, A_2, B)$, $(E', A'_1, A'_2, B') \in \mathcal{M}_{n,m}$, are equivalent if and only if there exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$ and $F_1, F_2, F_3 \in M_{m\times n}(\mathbb{C})$ such that these equalities $E' = P^{-1}EP + P^{-1}BF_3$, $A'_1 = P^{-1}A_1P + P^{-1}BF_1, A'_2 = P^{-1}A_2P + P^{-1}BF_2, B' = P^{-1}BQ$ hold.

This condition may also be written in a matrix form:

$$\begin{pmatrix} E' & A'_1 & A'_2 & B' \end{pmatrix} = P^{-1} \begin{pmatrix} E & A_1 & A_2 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ F_3 & F_1 & F_2 & Q \end{pmatrix}. \tag{2}$$

and we remark that the controllability condition is invariant under this equivalence relation.

Disposing of an equivalence relation preserving the properties that we want to study, we can change a given quadruple by other in a simpler form simplifying the study. One no dispose in general, of a canonical reduced form, for some special cases canonical forms are presented in this paper.

In Section 2, a collection of invariants that permit us to obtain reduced forms for the cases of one input systems with $(E, B)$ controllable, are presented. Notice that, the set of systems with $(E, B)$ controllable is open and dense. For one input two dimensional standardizable generalized second order linear systems as complete systems of invariants are presented.

In Section 3, the reduced forms are used to obtain necessary and sufficient conditions for controllability. In particular the controllability condition is given in terms of the golden mean.

## 2. Structural Invariants and Reduced Forms

As in the case of first order generalized linear systems we can associate polynomial matrices in terms of the matrices defining the system, providing a collection set of invariants under equivalence relation considered.

The following two-parameter polynomial matrices $\begin{pmatrix} A_1 - \lambda_1 I - \lambda_2 E & B \end{pmatrix}$, $\begin{pmatrix} A_2 - \mu_1 I - \mu_2 E & B \end{pmatrix}$ and $\begin{pmatrix} \lambda E + \mu A_1 - A_2 & B \end{pmatrix}$ might be associated to quadruples of matrices $(E, A_1, A_2, B)$ representing second order generalized linear systems considering the equivalence relation defined.

We recall that given a polynomial matrix $P(\alpha_1, \alpha_2)$ one define the rank of the polynomial as $\operatorname{rank} P(\alpha_1, \alpha_2) = \max\limits_{\alpha_1, \alpha_2 \in \mathbb{C} \times \mathbb{C}} \operatorname{rank} P(\alpha_1, \alpha_2)$ (see [6] for more details).

So, we define the following numbers

**Definition 2.** 1. $r_1 = \operatorname{rank} \begin{pmatrix} A_1 - \lambda_1 I - \lambda_2 E & B \end{pmatrix}$,

2. $r_2 = \operatorname{rank} \begin{pmatrix} A_2 - \mu_1 I - \mu_2 E & B \end{pmatrix}$,

3. $r_3 = \operatorname{rank} \begin{pmatrix} \lambda E + \mu A_1 - A_2 & B \end{pmatrix}$.

**Proposition 2.** *The numbers $r_1$, $r_2$ and $r_3$ are invariant under equivalence relation considered.*

*Proof.* Let $(E', A'_1, A'_2, B')$ equivalent to $(E, A_1, A_2, B)$ so there exist matrices $P \in Gl(n; \mathbb{C})$, $Q \in Gl(m; \mathbb{C})$ and $F_1, F_2, F_3 \in M_{m \times n}(\mathbb{C})$ such that

$$\begin{pmatrix} E' & A'_1 & A'_2 & B' \end{pmatrix} = P^{-1} \begin{pmatrix} E & A_1 & A_2 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ F_3 & F_1 & F_2 & Q \end{pmatrix}.$$

So,

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix} A_1' - \lambda_1 I - \lambda_2 E' & B'\end{pmatrix}\\ &= \operatorname{rank} P^{-1}\begin{pmatrix} A_1 - \lambda_1 I - \lambda_2 E & B\end{pmatrix}\begin{pmatrix} P & 0\\ F_1 - \lambda_2 F_3 & Q\end{pmatrix}\\ &= \operatorname{rank}\begin{pmatrix} A_1 - \lambda_1 I - \lambda_2 E & B\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix} A_2' - \mu_1 I - \mu_2 E' & B'\end{pmatrix}\\ &= \operatorname{rank} P^{-1}\begin{pmatrix} A_2 - \mu_1 I - \mu_2 E & B\end{pmatrix}\begin{pmatrix} P & 0\\ F_2 - \lambda_2 F_3 & Q\end{pmatrix}\\ &= \operatorname{rank}\begin{pmatrix} A_2 - \mu_1 I - \mu_2 E & B\end{pmatrix},\end{aligned}$$

$$\begin{aligned}\operatorname{rank}&\begin{pmatrix} \lambda E' + \mu A_1' - A_2' & B'\end{pmatrix}\\ &= \operatorname{rank} P^{-1}\begin{pmatrix} \lambda E + \mu A_1 - A_2 & B\end{pmatrix}\begin{pmatrix} P & 0\\ \lambda F_3 + \mu F_1 - F_2 & Q\end{pmatrix}\\ &= \operatorname{rank}\begin{pmatrix} \lambda E + \mu A_1 - A_2 & B\end{pmatrix}.\end{aligned}$$

After definition of rank of a polynomial matrix, we have that the values $r_1$, $r_2$ and $r_3$ are the size of the greatest minor non identically zero in the respective matrices.

**Corollary 1.** *Let $(\lambda_1, \lambda_2), (\mu_1, \mu_2), (\lambda, \mu) \in \mathbb{C}^2$ be the pairs of values reducing the rank of the polynomial matrices $\big(A_1 - \lambda_1 I - \lambda_2 E\, B\big)$, $\big(A_2 - \mu_1 I - \mu_2 E\, B\big)$ and $\big(\lambda E + \mu A_1 - A_2\, B\big)$ respectively. Then, these pairs of complex numbers are invariant under equivalence relation considered.*

Recall that, if $(E, A_1, A_2, B)$ and $(E', A_1', A_2', B')$ are equivalent quadruples then the pairs $(E, B)$, $(A_1, B)$ and $(A_2, B)$ are equivalent under feedback equivalence to the pairs $(E', B')$, $(A_1', B')$ and $(A_2', B')$ respectively. As a consequence we have that the following proposition.

**Proposition 3.** *The collection of invariants of the pairs $(E, B)$, $(A_1, B)$ and $(A_2, B)$ under feedback equivalence, are also invariants for the quadruple $(E, A_1, A_2, B)$ under the equivalence considered.*

**Corollary 2.** *Let $(E, A_1, A_2, B)$ be a quadruple of matrices and $(E_c, B_c)$ a canonical reduced form of the pair $(E, B)$ under feedback equivalence. Then, there exist square matrices $A_1', A_2' \in M_n(\mathbb{C})$ such that $(E, A_1, A_2, B)$ is equivalent to $(E_c, A_1', A_2', B_c)$.*

**Remark 2.** A similar result we can obtain considering canonical reduced forms for the pairs $(A_1, B)$ or $(A_2, B)$.

So, using the reduction to the Kronecker reduced form $(\mathbf{E},\mathbf{B})$ for the pair $(E,B)$ under feedback equivalence we have

**Theorem 1.** *Let $(E,A_1,A_2,B)\in\mathcal{M}_{n,1}$ be a quadruple of matrices representing one input second order generalized linear system. Suppose that the pair $(E,B)$ is controllable, then there exist matrices $P\in Gl(n;\mathbb{C})$, $Q\in Gl(m;\mathbb{C})$ and $F_1,F_2,F_3\in M_{m\times n}(\mathbb{C})$ such that $(E,A_1,A_2,B)$ is equivalent to $(\mathbf{E},\mathbf{A}_1,\mathbf{A}_2,\mathbf{B})$ with*

$$\mathbf{E}=\begin{pmatrix}0 & I_{n-1}\\ 0 & 0\end{pmatrix},\ \mathbf{A}_1=(a^1_{i,j}),\ \mathbf{A}_2=(a^2_{i,j}),\ \mathbf{B}=\begin{pmatrix}0_{n-1}\\ 1\end{pmatrix},$$

*with $a^1_{n,j}=a^2_{n,j}=0$.*

*Proof.* Let $P\in Gl(n;\mathbb{C})$, $Q\in\mathbb{C}$, $F_3\in M_{1\times n}(\mathbb{C})$ such that $(E,B)$ is feedback equivalent to $(\mathbf{E},\mathbf{B})$. So,

$$\left(\mathbf{E},A'_1,A'_2,\mathbf{B}\right)=P^{-1}\left(E\quad A_1,A_2,B\right)\begin{pmatrix}P & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ F_3 & 0 & 0 & Q\end{pmatrix}.$$

It is straightforward that there exist matrices $F_1,F_2\in M_{1\times n}(\mathbb{C})$ such that $\left(\mathbf{E},A'_1,A'_2,\mathbf{B}\right)$ is equivalent to $\left(\mathbf{E},\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}\right)$. $\square$

**Proposition 3.** *Numbers $a^1_{ij}$ in matrix $\mathbf{A}_1$ and $a^2_{ij}$ in matrix $\mathbf{A}_2$ characterize the equivalence class of quadruples of matrices.*

That is to say, if the quadruple $(E,A_1,A_2,B)$ is equivalent to $(\mathbf{E},\mathbf{A}_1,\mathbf{A}_2,\mathbf{B})$ as in the above proposition, and equivalent to $(\mathbf{E},\mathbf{A}'_1,\mathbf{A}'_2,\mathbf{B})$, $\mathbf{A}'_1=(a'^1_{i,j})$, $\mathbf{A}'_2=(a'^2_{i,j})$, with $a'^1_{n,j}=a'^2_{n,j}=0$. Then $a^1_{ij}=a'^1_{ij}$ and $a^2_{ij}=a'^2_{ij}$.

**Proposition 4.** *For $n=2$, $a^k_{ij}$ are the values for $\lambda_1,\lambda_2,\mu_1,\mu_2$ such that reduce the rank of the matrices $\left(A_1-\lambda_1 I-\lambda_2 E\quad B\right)$ and $\left(A_2-\mu_1 I-\mu_2 E\quad B\right)$. Concretely: $\lambda_1=a^1_{11}$, $\lambda_2=a^1_{12}$, $\mu_1=a^2_{11}$, $\mu_2=a^2_{12}$.*

*Proof.* Taking into account the invariance under equivalence relation considered of the pairs of values reducing the ranks $r_1$ and $r_2$, it suffices to compute this numbers in the equivalent reduced form. $\square$

Now and for $n=2$, $m=1$, we are going to analyze quadruples such that $B$ is a non-zero matrix but $(E,B)$ is a non-controllable pair.

**Proposition 5.** *Let $(E,A_1,A_2,B)$ be a quadruple of matrices with $B\neq 0$, and $(E,B)$ non-controllable. Then:*

*i) If $(A_1,B)$ is a controllable pair of matrices, the quadruple is equivalent to*

$$\left(\begin{pmatrix}0 & 0\\ 0 & a\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ x & y\end{pmatrix},\begin{pmatrix}1\\ 0\end{pmatrix}\right).$$

*1. If $(E, A_1, A_2, B)$ is standardizable, the eigenvalue of $(E, B)$ is $a \neq 0$, $\lambda = \dfrac{y}{a}$ and $\mu = x$ are the values such that rank $\big(\lambda E + \mu A_1 - A_2\, B\big) = 1$.*

*2. $a$ is the eigenvalue of the pair $(E, B)$. If $(A_2, B)$ is controllable $x \neq 0$, otherwise $x = 0$ and $y$ is the eigenvalue of the pair $(A_2, B)$.*

*ii) If $(A_1, B)$ is a non-controllable pair of matrices but $(A_2, B)$ is controllable, the quadruple is equivalent to*

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

*and $a$, $x$ are the eigenvalues of the pairs $(E, B)$ and $(A_1, B)$ respectively.*

*iii) If $(A_i, B)$ $i = 1, 2$ are non-controllable pairs of matrices, the quadruple is equivalent to*

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

*and $a$, $x$, $y$ are the eigenvalues of the pairs $(E, B)$, $(A_1, B)$ and $(A_2, B)$ respectively.*

*Proof.* Let $P$, $F_3$ and $Q$ matrices reducing $(E, B)$ to the Kronecker form. So there exist $F_1, F_2$ such that $(E, A_1, A_2, B)$ can be reduced to

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right).$$

i) First of all, we observe that the controllability of the pair $(A_1, B)$ implies $x \neq 0$. Then

$$\begin{pmatrix} 1 & \frac{y}{x} \\ 0 & \frac{1}{x} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & x & y & z & t & 0 \end{pmatrix} \begin{pmatrix} 1 & -y & 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -y & 0 & 0 & 0 \\ 0 & 0 & 0 & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -y & 0 \\ 0 & 0 & 0 & 0 & 0 & x & 0 \\ 0 & -ay & -y & 0 & -\frac{yz}{x} & \frac{y^2 z}{x} - yt & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 1 & 0 & x' & y' & 0 \end{pmatrix}.$$

Now, it suffices to prove that the quadruple $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z & t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is equivalent to $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z' & t' \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ if and only if $z = z'$ and $t = t'$

ii) The no controllability of the pair $(A_1, B)$ and the controllability of the pair $(A_2, B)$ implies $x = 0$ and $z \neq 0$. So

$$\begin{pmatrix} z & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & y & z & t & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{z} & -\frac{t}{z} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z} & -\frac{t}{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{z} & -\frac{t}{z} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\frac{at}{z} & 0 & -\frac{yt}{z} & -\frac{t}{z} & 0 & \frac{1}{z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & y & 1 & 0 & 0 \end{pmatrix}.$$

Now it suffices to prove that the quadruple $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is equivalent to $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ if and only if $y = y'$

iii) The no controllability of the pairs $(A_1, B)$ and $(A_2, B)$ implies $x = z = 0$. Then, it suffices to prove that the quadruple $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ is equivalent to $\left(\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & y' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & t' \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ if and only if $y = y'$ and $t = t'$

Now we are going to analyze an special case for $n$-dimensional one input systems.

**Theorem 2.** *Let $(E, A_1, A_2, B) \in \mathcal{M}_{n,1}$. Suppose that the pairs $(E, B)$, $(A_1, B)$, $(A_2, B)$ are controllable and*

$$\begin{aligned}
&\operatorname{rank} \begin{pmatrix} B & (E - A_1)B \end{pmatrix} = 1, \\
&\operatorname{rank} \begin{pmatrix} B & (E - A_2)B \end{pmatrix} = 1, \\
&\operatorname{rank} \begin{pmatrix} B & EB & (E^2 - A_1^2)B \end{pmatrix} = 2, \\
&\operatorname{rank} \begin{pmatrix} B & EB & (E^2 - A_2^2)B \end{pmatrix} = 2, \\
&\vdots \\
&\operatorname{rank} \begin{pmatrix} B & EB & \dots & E^{n-2}B & (E^{n-1} - A_1^{n-1})B \end{pmatrix} = n - 1, \\
&\operatorname{rank} \begin{pmatrix} B & EB & \dots & E^{n-2}B & (E^{n-1} - A_2^{n-1})B \end{pmatrix} = n - 1.
\end{aligned}$$

*Then the quadruple is equivalent to* $(E', A'_1, A'_2, B')$ *with*

$$E' = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, A'_1 = \begin{pmatrix} a^1_{11} & 1 & \dots & 0 & 0 \\ a^1_{21} & a^1_{22} & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ a^1_{n-11} & a^1_{n-12} & \dots & a^1_{n-1n-1} & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$A'_2 = \begin{pmatrix} a^2_{11} & 1 & \dots & 0 & 0 \\ a^2_{21} & a^2_{22} & \dots & 0 & 0 \\ & \ddots & \ddots & & \\ a^1_{n-11} & a^2_{n-12} & \dots & a^2_{n-1n-1} & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \ B' = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

*Proof.* It suffices to apply Proposition 4.2 in [4]. □

## 3. Controllability

In this section, using the reduced forms obtained in the last section, we are going to obtain conditions for controllability of quadruples of matrices representing a second order generalized system that can be reduced in the forms presented in the Section 2. In particular we prove that for $n$-dimensional one input systems verifying Theorem 2, the controllability condition is given in terms of golden mean.

First of all we observe that if $(E, B)$ is a controllable pair, all quadruples $(E, A_1, A_2, B)$ verify the first condition for controllability: rank $\begin{pmatrix} E & B \end{pmatrix} = n$.

**Theorem 3.** *Let* $(E, A_1, A_2, B) \in \mathcal{M}_{2,1}$ *be a quadruple with* $(E, B)$ *a controllable pair of matrices and suppose that there exists a matrix* $R$ *such that* $A_2 = A_1 + BR$*. A necessary and sufficient condition for controllability is that* $a^1_{11} = a^2_{11} \neq 0$*.*

*Proof.* We can consider the quadruple in its equivalent reduced form. We observe that the condition $A_2 = A_1 + BR$ ensures $a^1_{11} = a^2_{11}$ and $a^1_{12} = a^2_{12}$. So, the quadruple is reduced to $(E', A'_1, A'_2, B')$ with

$$E' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ A'_1 = A'_2 = \begin{pmatrix} a^1_{11} & a^1_{12} \\ 0 & 0 \end{pmatrix}, \ B' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Computing the rank $\begin{pmatrix} s^2E' - sA'_1 - A'_1 & B' \end{pmatrix}$ we have.

$$\operatorname{rank}\begin{pmatrix} s^2E' - sA_1' - A_1' & B' \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -sa_{11}^1 - a_{11}^1 & s^2 - sa_{12}^1 - a_{12}^1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2, \quad \forall s \in \mathbb{C}\,,$$

because $-sa_{11}^1 - a_{11}^1 = 0$ if and only if $s = -1$ but $(-1)^2 - (-1)a_{12}^1 - a_{12}^1 = 1$. □

**Corollary 3.** *In the same conditions as in Theorem 3, and if the pair $(A_1, B)$ is non-controllable, the quadruple $(E, A_1, A_2, B)$ is controllable if and only if the eigenvalue of the pair $(A_1, B)$ is non zero.*

**Theorem 4.** *Let $(E, A_1, A_2, B) \in \mathcal{M}_{2,1}$ be a quadruple of matrices with $(E, B)$ a controllable pair. A necessary and sufficient condition for controllability of the quadruples is*

$$(a_{11}^2)^2 + a_{11}^2 a_{12}^1 a_{11}^1 - (a_{11}^1)^2 a_{12}^2 \neq 0$$

*being $a_{1j}^k$ the numbers obtained in (4).*

*Proof.* The matrices of the quadruple have the form

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So,

$$\operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} -sa_{11}^1 - a_{11}^2 & s^2 - sa_{12}^1 - a_{12}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = r\,.$$

Suppose that the system is controllable, then $r = 2$ for all $s \in \mathbb{C}$, consequently $a_{11}^1$ and $a_{11}^2$ being not zero simultaneously.

If $a_{11}^1 = 0$ then $a_{11}^2 \neq 0$ and

$$(a_{11}^2)^2 + a_{11}^2 a_{12}^1 a_{11}^1 - (a_{11}^1)^2 a_{12}^2 \neq 0\,.$$

Suppose now $a_{11}^1 \neq 0$ then $-sa_{11}^1 - a_{11}^2 = 0$ if and only if $s = \frac{-a_{11}^2}{a_{11}^1}$ then $(\frac{-a_{11}^2}{a_{11}^1})^2 - (\frac{-a_{11}^2}{a_{11}^1})a_{12}^1 - a_{12}^2 \neq 0$ if and only if $(a_{11}^2)^2 + a_{11}^2 a_{12}^1 a_{11}^1 - (a_{11}^1)^2 a_{12}^2 \neq 0$.

Conversely. Suppose that $(a_{11}^2)^2 + a_{11}^2 a_{12}^1 a_{11}^1 - (a_{11}^1)^2 a_{12}^2 \neq 0$, then $a_{11}^1$ and $a_{11}^2$ being not zero simultaneously, and now it suffices to compute the rank $r$. □

**Corollary 4.** *In the same conditions as in theorem before, and if the pairs $(A_i, B)$ $i = 1, 2$ are non-controllable. A sufficient condition for controllability of the quadruple $(E, A_1, A_2, B)$ is that the eigenvalue of the pair $(A_2, B)$ is non zero.*

Now we present some results for the case where the pair $(E,B)$ is non controllable but $B \neq 0$.

**Theorem 5.** *Let $(E,A_1,A_2,B) \in \mathcal{M}_{2,1}$ be a quadruple of matrices with $B \neq 0$, $(E,B)$ non-controllable with eigenvalue $a$.*

*i) Suppose $(A_1,B)$ is a controllable pair of matrices. Then the quadruple $(E,A_1,A_2,B)$ is controllable if and only if $y \neq ax^2$, $a \neq 0$, where $a,x,y$ are the continuous invariants obtained in Proposition 5.*

*In particular, if $(A_2,B)$ is not controllable, the quadruple $(E,A_1,A_2,B)$ is controllable if and only if the eigenvalue of $(A_2,B)$ is non-zero.*

*ii) Suppose $(A_1,B)$ is a non-controllable pair of matrices but $(A_2,B)$ is controllable. Then, if $a \neq 0$, the quadruple $(E,A_1,A_2,B)$ is controllable.*

*iii) Suppose $(A_i,B)$ $i=1,2$ are non-controllable pairs of matrices. Then the quadruple $(E,A_1,A_2,B)$ is not controllable.*

*Proof.* If $a=0$,

$$\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

and the quadruple $(E,A_1,A_2,B)$ is not controllable. So, from now on, we suppose that the eigenvalue of the pair $(E,B)$ is non-zero $a \neq 0$. Then

$$\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ 0 & a & 0 \end{pmatrix} = 2$$

and the first condition for controllability is verified.

i) $\operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ -s-x & s^2a-y & 0 \end{pmatrix} = r = 2$, $\forall s \in \mathbb{C}$. Observe that $r<2$ for some $s \in \mathbb{C}$ if and only if $s$ is a solution of the system

$$\begin{cases} -s-x=0\,, \\ s^2a-y=0\,, \end{cases}$$

and this is only possible if and only if $y = ax^2$.

ii) $\operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ -1 & s^2a-sx & 0 \end{pmatrix} = 2\,,\ \forall s \in \mathbb{C}\,,$ and the quadruple $(E,A_1,A_2,B)$ is controllable.

iii) $\operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix}$

$= \operatorname{rank}\left(\begin{pmatrix} 0 & 0 \\ 0 & s^2a \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & sx \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$= \operatorname{rank}\begin{pmatrix} 0 & 0 & 1 \\ 0 & s^2a-sx-y & 0 \end{pmatrix}$. Taking into account that $a \neq 0$, there exists

$s \in \mathbb{C}$ such that $s^2 a - sx - y = 0$. Hence the quadruple $(E, A_1, A_2, B)$ is not controllable. $\square$

**Theorem 5.** *Let $(E, A_1, A_2, B) \in \mathcal{M}_{n,1}$ as in proposition (2). Then the system is controllable if and only if*

$$\frac{1 \pm \sqrt{5}}{2} a_{ii}^1 + a_{ii}^2 \neq 0, \quad \forall i = 1, \dots, n-1.$$

*Proof.* $\operatorname{rank}\begin{pmatrix} s^2 E - sA_1 - A_2 & B \end{pmatrix}$

$$= \operatorname{rank}\begin{pmatrix} -sa_{11}^1 - a_{11}^2 & s^2 - s - 1 & 0 & \dots & 0 & 0 \\ -sa_{21}^1 - a_{21}^2 & -sa_{22}^1 - a_{22}^2 & s^2 - s - 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -sa_{n-11}^1 - a_{n-11}^2 & -sa_{n-12}^1 - a_{n-12}^2 & -sa_{n-13}^1 - a_{n-13}^2 & \dots & s^2 - s - 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

the roots of $s^2 - s - 1$ are $\frac{1 \pm \sqrt{5}}{2}$. $\square$

**Corollary 4.** *In the same conditions a sufficient condition for controllability is that*

$$\frac{1 \pm \sqrt{5}}{2} \lambda_i + \mu_j \neq 0, \quad \forall i, j = 1, \dots, n-1,$$

*where $\lambda_i$ and $\mu_j$ are the eigenvalues of the pairs $\begin{pmatrix} A_1 - E & B \end{pmatrix}$ and $\begin{pmatrix} A_2 - E & B \end{pmatrix}$ respectively.*

*Proof.* It suffice to proof that $a_{ii}^1$ and $a_{jj}^2$ are the eigenvalues of the pairs $\begin{pmatrix} A_1 - E & B \end{pmatrix}$ and $\begin{pmatrix} A_2 - E & B \end{pmatrix}$ respectively, and that this matrices are invariant under our equivalence relation. $\square$

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