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CONTROL THEORY

Criteria for Modal Controllability of Completely Regular Differential-Algebraic Systems with Aftereffect

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Abstract— The problem of control of the coefficients of the characteristic quasipolynomial is studied for linear autonomous completely regular differential-algebraic systems with commensurate delays. Several criteria for modal controllability and weak modal controllability are obtained, schemes for synthesis of the corresponding controllers are proposed, and illustrative examples are given.

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INTRODUCTION

The stabilizability of linear autonomous differential systems, which means the capability of controlling of all unstable natural motions, has been studied sufficiently well for finite-dimensional systems, but for infinite-dimensional ones, where it is a significantly more complicated property [1–5], it still remains poorly investigated. A generalization of the stabilization problem for system with aftereffect is the FSA (finite spectrum assignment) problem [6–10], i.e., the problem of assigning a closed system an arbitrary prescribed finite spectrum which, as a rule, consists of numbers with negative real parts. The problem of reducing a system with delay to a finite (but not arbitrary) spectrum was studied in [11–13]. The set of eigenvalues of a linear system with aftereffect is generally infinite, and hence it is natural to consider the problem of control of all its eigenvalues as the problem of control of the coefficients of the characteristic quasipolynomial; i.e., this is a modal control problem [14–16]. The problem of simultaneous stabilization of plants with delay by a unique controller, which is very important in practical applications, was studied in [17].

Here we present the results of studying the problem of modal controllability of linear autonomous completely regular differential-algebraic systems with commensurate delays in the state and in the control The following two approaches to solving this problem were proposed. In the first of them, it is required to have a state feedback controller that ensures that the closed-loop system has a prescribed characteristic quasipolynomial. The other approach assumes that there is a feedback that permits controlling the quasipolynomial coefficients in the part of the closed-loop system which uniquely determines the dynamics of variation in the variables of the original system. (It is assumed that the closed-loop system contains auxiliary variables.) Several criteria for the solvability of modal control problems under these approaches were obtained, and several methods were developed for the synthesis of appropriate controllers whose implementation is based on standard operations with polynomial matrices and does not require to know the whole spectrum of the system, to calculate the inverse Laplace transform, and to solve an infinite-dimensional interpolation problem. The main idea in this paper is to generalize the methods for using controllers of variable structure [18], which were developed in [13, 15, 16, 19, 20], to solving the problems of controllability and stabilization of linear systems of retarded and neutral types.

1. STATEMENT OF THE PROBLEM

Assume that a linear autonomous completely regular [21, 22] differential-algebraic system with commensurate delays in the state and in the control is given,

$$\frac{d}{dt}(\widetilde{A}_0\widetilde{x}(t)) = \widetilde{A}(\lambda)\widetilde{x}(t) + \widetilde{B}(\lambda)u(t), \qquad t > 0, \tag{1'}$$

where $\widetilde{x} \in \mathbb{R}^n$ is the vector of solution of the original problem, $u \in \mathbb{R}^r$ is the vector of piecewise continuous control, $\widetilde{A}_0 \in \mathbb{R}^{n\times n}$, $\widetilde{A}(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$, $\widetilde{B}(\lambda) \in \mathbb{R}^{n\times r}[\lambda]$ ($\mathbb{R}^{i\times j}[\lambda]$ is the set of polynomial $i \times j$ matrices), and λ is the shift operator ($\lambda f(t) = f(t-h)$ for an arbitrary function f, where $h = \mathrm{const} > 0$ is a constant delay). We write $\operatorname{rank} \widetilde{A}_0 = n_1$. A system is completely regular [21, 22] if the following condition is satisfied: $\deg \widetilde{a}_0(p) = n_1$ for the degree of the polynomial $\widetilde{a}_0(p) = |p\widetilde{A}_0 - \widetilde{A}(0)|$, where $|\cdot|$ is the determinant of a matrix. This condition permits reducing the original system ($1'$) to a form that is simpler for investigations. We choose nonsingular matrices H and H_1 such that $H_1\widetilde{A}_0 H = \operatorname{diag}[I_{n_1}, 0_{n_2\times n_2}]$, where $I_i \in \mathbb{R}^{i\times i}$ is the identity matrix, $0_{i\times j} \in \mathbb{R}^{i\times j}$ is the zero matrix, and $n_2 = n - n_1$. Let

$$H_1\widetilde{A}(\lambda)H = \begin{bmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ \overline{A}_{21}(\lambda) & \overline{A}_{22}(\lambda) \end{bmatrix}, \qquad H_1\widetilde{B}(\lambda) = \begin{bmatrix} B_1(\lambda) \\ \overline{B}_2(\lambda) \end{bmatrix},$$

and $B_1(\lambda) \in \mathbb{R}^{n_1\times r}[\lambda]$, $\overline{B}_2(\lambda) \in \mathbb{R}^{n_2\times r}[\lambda]$. Since the degree satisfies the relation

$$\deg |pH_1\widetilde{A}_0 H - H_1\widetilde{A}(0)H| = \deg \widetilde{a}_0(p),$$

it follows that the determinant $|\overline{A}_{22}(0)|$ is nonzero. We set

$$\begin{aligned} A_{21}(\lambda) &= -(\overline{A}_{22}(0))^{-1}\overline{A}_{21}(\lambda), \qquad A_{22}(\lambda) = -\lambda^{-1}(\overline{A}_{22}(0))^{-1}(\overline{A}_{22}(\lambda) - \overline{A}_{22}(0)), \\ B_2(\lambda) &= -(\overline{A}_{22}(0))^{-1}\overline{B}_2(\lambda), \end{aligned}$$

change variables in system ($1'$) by the formulas $\widetilde{x} = H\operatorname{col}[x, y]$, $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, and obtain the new system

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t) + A_{12}(\lambda)y(t) + B_1(\lambda)u(t), \\ y(t) &= A_{21}(\lambda)x(t) + A_{22}(\lambda)y(t-h) + B_2(\lambda)u(t), \qquad t > 0. \end{aligned} \tag{1}$$

Further, we study system (1) and then restate the results in terms of system ($1'$).

Let

$$W_0(p, e^{-ph}) = \begin{bmatrix} pI_{n_1} - A_{11}(e^{-ph}) & -A_{12}(e^{-ph}) \\ -A_{21}(e^{-ph}) & I_{n_2} - e^{-ph}A_{22}(e^{-ph}) \end{bmatrix}$$

be the characteristic matrix of system (1). The characteristic quasipolynomial of system (1) has the form

$$|W_0(p, e^{-ph})| = \sum_{i=0}^{n_1} p^i \chi_i(e^{-ph}),$$

where the $\chi_i(p)$ are some polynomials and $\chi_{n_1}(0) = 1$.

In this paper, we solve the following problem.

Problem. Given an arbitrary quasipolynomial of the form

$$d(p, e^{-ph}) = \sum_{i=0}^{n_0} p^i d_i(e^{-ph}), \tag{2}$$

where the $d_i(p)$ are certain polynomials, $d_{n_0}(0) = 1$, and $n_0 \geq n_1$ is a number which we call the degree of the quasipolynomial (2), it is required to close system (1) by a linear state feedback $u(t) = u(\{x(\tau), y(\tau - h), \tau \leq t\})$ so that the components of the solution $\operatorname{col}[x, y]$ of the closed-loop system satisfy a linear autonomous completely regular differential-algebraic system with the characteristic quasipolynomial (2).

To solve this problem, we use a controller of the form

$$u(t) = R_{01}(\lambda)X(t) + R_{02}(\lambda)Y(t) + \sum_{i=0}^{\delta} \int_0^h F_{0i}(s)\lambda^i X(t-s)\,ds, \tag{3}$$

$$\dot{x}_1(t) = R_{11}(\lambda)X(t) + R_{12}(\lambda)Y(t) + \sum_{i=0}^{\delta} \int_0^h F_{1i}(s)\lambda^i X(t-s)\,ds, \tag{4}$$

$$y_1(t) = R_{21}(\lambda)X(t) + R_{22}(\lambda)Y(t) + \sum_{i=0}^{\delta} \int_0^h F_{2i}(s)\lambda^i X(t-s)\,ds, \qquad t > 0, \tag{5}$$

where $x_1 \in \mathbb{R}^{\bar{n}_1}$ and $y_1 \in \mathbb{R}^{\bar{n}_2}$ are auxiliary variables, $X = \operatorname{col}[x, x_1]$, $Y = \operatorname{col}[y, y_1]$, $R_{01}(\lambda) \in \mathbb{R}^{r\times(n_1+\bar{n}_1)}[\lambda]$, $R_{02}(\lambda) \in \mathbb{R}^{r\times(n_2+\bar{n}_2)}[\lambda]$, $R_{11}(\lambda) \in \mathbb{R}^{\bar{n}_1\times(n_1+\bar{n}_1)}[\lambda]$, $R_{12}(\lambda) \in \mathbb{R}^{\bar{n}_1\times(n_2+\bar{n}_2)}[\lambda]$, $R_{21}(\lambda) \in \mathbb{R}^{\bar{n}_2\times(n_1+\bar{n}_1)}[\lambda]$, $R_{22}(\lambda) \in \mathbb{R}^{\bar{n}_2\times(n_2+\bar{n}_2)}[\lambda]$, the functions $F_{j,i}(s)$ are defined by

$$F_{j,i}(s) = \sum_{k=0}^{\gamma} e^{\alpha_{k,j,i}s}(\cos(\beta_{k,j,i}s)F_{j,i,1,k}(s) + \sin(\beta_{k,j,i}s)F_{j,i,2,k}(s)), \qquad j = 0,\dots,2, \qquad i = 1,\dots,\delta$$

($\alpha_{k,j,i}, \beta_{k,j,i} \in \mathbb{R}$, $F_{j,i,e,k}(s)$ are polynomial matrices), and $\delta, \gamma \in \mathbb{N} \bigsqcup \{0\}$. To be definite, we assume that $x(t)$, $x_1(t)$, $t < 0$, are arbitrary continuous functions and $y(t)$, $y_1(t)$, $t < 0$, are arbitrary piecewise continuous functions.

Remark 1. If a system contains delays in the control, then the above-stated problem can be solved without an increase in the number of equations of the system only for a very narrow class of plants [16]. Therefore, it is necessary to introduce auxiliary variables in the structure of the controller (3)–(5).

We close system (1) with the controller (3)–(5). Since system (1), (3)–(5) is completely regular, we perform elementary transformations with the rows of its characteristic matrix, order the variables as the vector $\operatorname{col}[x, x_1, y, y_1]$, and obtain a matrix of the form

$$\begin{bmatrix} pI_{n_x} - \Xi_{11}(p, e^{-ph}) & -\Xi_{12}(e^{-ph}) \\ -\Xi_{21}(p, e^{-ph}) & I_{n_y} - e^{-ph}\Xi_{22}(e^{-ph}) \end{bmatrix}, \tag{6}$$

where $n_x = n_1 + \bar{n}_1$ and $n_y = n_2 + \bar{n}_2$, the entries of the matrices $\Xi_{i1}(p, \lambda)$, $i = 1, 2$, are proper fractional-rational functions in the variable p and polynomials in the variable λ, and the entries of the matrices $\Xi_{i2}(\lambda)$, $i = 1, 2$, are polynomials.

A matrix that, up to the similarity transformation, has the form (6), will be called a matrix with CR-structure (CR meaning completely regular). In particular, the characteristic matrix of system (1′) has CR-structure. Thus, a matrix with CR-structure is associated with a linear autonomous completely regular differential-algebraic delay system. Note that the converse is not true. But although the class of matrices of the form (6) does not embrace the whole set of completely regular differential-algebraic delay systems, this class is quite sufficient for the aims of this paper.

By $\mathcal{W}(p, e^{-ph})$ we denote the characteristic matrix of the closed-loop system (1), (3)–(5). When studying the problem stated above, we distinguish the following properties of system (1).

Definition 1. System (1) is said to be *modally controllable* if there exists a number ζ such that, for any prescribed quasipolynomial (2) of degree $n_0 \geq \zeta$, there exists a controller (3)–(5) ensuring that the following conditions are satisfied for the closed-loop system (1), (3)–(5): (i) its characteristic matrix $\mathcal{W}(p, e^{-ph})$ has CR-structure; (ii) its characteristic quasipolynomial $|\mathcal{W}(p, e^{-ph})|$ coincides with the quasipolynomial $d(p, e^{-ph})$.

Definition 2. System (1) is said to be *weakly modally controllable* if there exists a number ζ such that, for any prescribed quasipolynomial (2) of degree $n_0 \geq \zeta$, there exists a controller (3)–(5)

ensuring that the following conditions are satisfied for the closed-loop system (1), (3)–(5) : (i) its characteristic matrix $\mathcal{W}(p, e^{-ph})$ has CR-structure; (ii) there exists a unimodular matrix $\mathcal{P}(\lambda) \in \mathbb{R}^{(n_x+n_y)\times(n_x+n_y)}$ (i.e., $|\mathcal{P}(\lambda)| \equiv \text{const} \neq 0$) such that if the variables of system (1), (3)–(5) are arranged in the order $\operatorname{col}[x, y, x_1, y_1]$, then

$$\mathcal{P}(e^{-ph})\mathcal{W}(p, e^{-ph}) = \begin{bmatrix} \mathcal{W}_{11}(p, e^{-ph}) & 0_{(n_x+n_y-n_*)\times n_*} \\ \mathcal{W}_{21}(p, e^{-ph}) & I_{n_*} - e^{-ph}\mathcal{A}(e^{-ph}) \end{bmatrix}, \tag{7}$$

where the matrix $\mathcal{W}_{11}(p, e^{-ph})$ has CR-structure, the entries of the matrix $\mathcal{W}_{21}(p, \lambda)$ are proper fractional-rational functions in the variable p and polynomials in the variable λ, $n_* \le n_y$, and $\mathcal{A}(\lambda) \in \mathbb{R}^{n_*\times n_*}[\lambda]$; (iii) the identity $|\mathcal{W}_{11}(p, e^{-ph})| = d(p, e^{-ph})$ holds.

Remark 2. Let us explain the property of weak modal controllability. Relation (7) means that there exists a time $\bar{t} > 0$ such that, for $t > \bar{t}$, in system (1) closed with the controller (3)–(5), one can single out a completely regular differential-algebraic subsystem with prescribed characteristic quasipolynomial $|\mathcal{W}_{11}(p, e^{-ph})|$ of the form (2), which uniquely determines the quantity $\operatorname{col}[x, y]$. In this case, the existence of a unimodular matrix $\mathcal{P}(\lambda)$ permits obtaining the above-mentioned subsystem by elementary transformations of the equations in system (1), (3)–(5) that do not contain the differentiation operation. We also note that since the entries of the blocks $\Xi_{i1}(p, e^{-ph})$, $i = 1, 2$, in the matrix (6) are proper fractional-rational functions in the variable p, we can reduce the matrix (6) to the form (7) (i.e., obtain the above-mentioned subsystem) only by "annihilating" its elements located in the first $n_x+n_y-n_*$ rows and in the columns corresponding to some components of the vector y_1. In turn, this can be done only by multiplying the corresponding rows in (5) by certain polynomials and adding them to the other equations in system (1), (3)–(5). It follows that if in the closed-loop system (1), (3)–(5), using elementary transformations (which do not contain the differentiation operation), we separate a completely regular subsystem describing the dynamics of variation in the variables $\operatorname{col}[x, y]$, then the characteristic matrix has the form (7).

Example 1. Consider the system with the characteristic matrix and the control matrix ($h = 1$)

$$W_0(p, e^{-ph}) = \begin{bmatrix} p+1-e^{-p} & 1-e^{-2p} \\ 0 & p+2 \end{bmatrix}, \qquad B_1(e^{-p}) = \begin{bmatrix} e^{-p}-e^{-2p} & 0 \\ e^{-p} & e^{-p} \end{bmatrix}, \tag{8}$$

which is a special case ($n_2 = 0$) of system (1). A simple analysis of the matrices (8) for $p = 0$ shows that it is impossible to use traditional methods [1–10, 14] to eliminate the number 0 from the spectrum. At the same time, using a controller of the form

$$u(t) = \operatorname{col}[1, -1]y_1(t), \qquad y_1(t) = y_1(t-1) + [-1, -\lambda]x(t), \qquad t > 0, \tag{9}$$

we obtain the closed-loop system with the characteristic matrix

$$\mathcal{W}(p, e^{-ph}) = \begin{bmatrix} p+1-e^{-p} & 1-e^{-2p} & -e^{-p}(1-e^{-p}) \\ 0 & p+2 & 0 \\ 1 & e^{-p} & 1-e^{-p} \end{bmatrix}. \tag{10}$$

We multiply the matrix (10) on the left by the matrix

$$\mathcal{P}(e^{-ph}) = \begin{bmatrix} 1 & 0 & e^{-p} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and obtain

$$\mathcal{W}_{11}(p, e^{-p}) = \begin{bmatrix} p+1 & 1 \\ 0 & p+2 \end{bmatrix}, \qquad |\mathcal{W}_{11}(p, e^{-p})| = (p+1)(p+2).$$

Thus, choosing a control of the form (9), we see that the function x is determined by the system

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix} x(t), \qquad t > \bar{t} = 2.$$

Note that the matrix $\mathcal{W}_{11}(p, e^{-p})$ can also be obtained in a different way. We write the second relation in (9) in the form $(1-\lambda)y_1(t) = [-1, -\lambda]x(t)$, $t > 0$. Then

$$B_1(\lambda)u(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda(1-\lambda)y_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} [-1, -\lambda]\lambda x(t), \qquad t > \bar{t},$$

and $\mathcal{W}_{11}(p, e^{-p}) = W_0(p, e^{-p}) - \operatorname{col}[1,\, 0][-\lambda,\, -\lambda^2]$.

Remark 3. For the original system $(1')$, Definitions 1 and 2 are stated in a similar way. It is necessary to take into account the fact that the solutions of systems $(1')$ and (1) are related by the nonsingular transformation $\widetilde{x} = H\operatorname{col}[x, y]$. Moreover, if one of the systems has the property of modal controllability (weak modal controllability), then so does the other system.

2. MODAL CONTROLLABILITY

In this section, we state and prove a criterion for modal controllability of system $(1')$. First, we introduce the required notation. Let Φ and Ψ be fundamental solution matrices of the linear algebraic systems $\Phi\widetilde{A}_0 = 0_{n_2\times n}$ and $\widetilde{A}_0\Psi = 0_{n\times n_2}$. Let us determine the matrices $\Phi_1 = [0_{n_2\times n_1}, I_{n_2}]$ and $\Psi_1 = \operatorname{col}[0_{n_1\times n_2}, I_{n_2}]$. Obviously, we can take

$$\Phi = \Phi_1 H_1, \qquad \Psi = H\Psi_1. \tag{11}$$

By $\mathbb{C}$ we denote the set of complex numbers, and by $B(\lambda)$, the matrix $\operatorname{col}[B_1(\lambda), B_2(\lambda)]$.

Theorem 1. *System* $(1')$ *is modally controllable if and only if the following conditions are satisfied simultaneously*:

$$\operatorname{rank}[p\widetilde{A}_0 - \widetilde{A}(e^{-ph}), \widetilde{B}(e^{-ph})] = n, \qquad p \in \mathbb{C}, \tag{12}$$

$$\operatorname{rank}[\Phi\widetilde{A}(\lambda)\Psi, \Phi\widetilde{B}(\lambda)] = n_2, \qquad \lambda \in \mathbb{C}. \tag{13}$$

To prove Theorem 1, we first obtain a criterion for modal controllability in terms of the transformed system (1) and then prove that it is equivalent to Theorem 1.

Theorem 2. *For system* (1) *to be modally controllable*, *it is necessary and sufficient that the following conditions be satisfied*:

$$\operatorname{rank}[W_0(p, e^{-ph}), B(e^{-ph})] = n, \qquad p \in \mathbb{C}, \tag{14}$$

$$\operatorname{rank}[I_{n_2} - \lambda A_{22}(\lambda), B_2(\lambda)] = n_2, \qquad \lambda \in \mathbb{C}. \tag{15}$$

Proof. Necessity. The proof of necessity of condition (14) does not encounter serious difficulties and hence is omitted.

Let us prove the necessity of condition (15). Assume that, for the closed-loop system (1), (3)–(5), the controller (3)–(5) ensures the characteristic quasipolynomial $d(p, e^{-ph}) = p^{n_0}$, where n_0 is a sufficiently large number. We order the variables in the closed-loop system (1), (3)–(5) as the vector $\operatorname{col}[x, x_1, y, y_1]$ and write the characteristic equation as

$$\begin{vmatrix} pI_{n_0} - A_{11}^0(e^{-ph}) - F_{11}^0(p, e^{-ph}) & -A_{12}^0(e^{-ph}) \\ -A_{21}^0(e^{-ph}) - F_{21}^0(p, e^{-ph}) & I_{n_y} - A_{22}^0(e^{-ph}) \end{vmatrix} = p^{n_0}, \tag{16}$$

where the $A_{ij}^0(\lambda)$, $i = 1, 2$, $j = 1, 2$, are polynomial matrices and the $F_{i1}^0(p, \lambda)$, $i = 1, 2$, are matrices whose entries are proper fractional-rational functions in the variable p and polynomials in the variable λ. Let $|I_{n_y} - A_{22}^0(\lambda)| = d_0(\lambda)$, and let $\Pi_0(\lambda)$ be the matrix associated with the matrix adjoint to the matrix $(I_{n_y} - A_{22}^0(\lambda))$; i.e., $(I_{n_y} - A_{22}^0(\lambda))\Pi_0(\lambda) = d_0(\lambda)I_{n_y}$. Multiplying both sides of Eq. (16) on the left by the respective sides of $|\mathrm{diag}[I_{n_0}, \Pi_0(e^{-ph})]| = (d_0(e^{-ph}))^{n_y-1}$, we obtain the quasipolynomial

$$d^*(p, e^{-ph}) = \begin{vmatrix} pI_{n_0} - A_{11}^0(e^{-ph}) - F_{11}^0(p, e^{-ph}) & -A_{12}^0(e^{-ph})\Pi_0(e^{-ph}) \\ -A_{21}^0(e^{-ph}) - F_{21}^0(p, e^{-ph}) & d_0(e^{-ph})I_{n_y} \end{vmatrix} = p^{n_0}(d_0(e^{-ph}))^{n_y-1}. \quad (17)$$

On the other hand, it follows from the form of the determinant in (17) that the quasipolynomial $d^*(p, e^{-ph})$ can be represented as

$$d^*(p, e^{-ph}) = p^{n_0}(d_0(e^{-ph}))^{n_y} + \tilde{d}(p, e^{-ph}), \quad (18)$$

where $\tilde{d}(p, e^{-ph})$ is a quasipolynomial whose maximum degree in the variable p is less than the number n_0. Comparing the right-hand sides of (17) and (18), we conclude that $d_0(\lambda) \equiv 1$. We write this identity as

$$\begin{vmatrix} I_{n_2} - \lambda A_{22}(\lambda) - B_2(\lambda)R_{02}^1(\lambda) & -B_2(\lambda)R_{02}^2(\lambda) \\ -R_{22}^1(\lambda) & I_{\bar{n}_2} - R_{22}^2(\lambda) \end{vmatrix} \equiv 1, \quad (19)$$

where $R_{i2}^1(\lambda)$, $i = 0, 2$, are the first n_2 columns of the matrix $R_{i2}(\lambda)$, and $R_{i2}^2(\lambda)$ are the remaining $\bar{n}_2$ columns of the matrix $R_{i2}(\lambda)$. If, for some $\lambda = \lambda_0 \in \mathbb{C}$, condition (15) is violated, then, in identity (19), the left-hand side is not equal to the right-hand side for $\lambda = \lambda_0$. The proof of the necessity is complete.

Sufficiency. We show a method for constructing the controller (3)–(5) that equips the closed-loop system (1), (3)–(5) with the prescribed characteristic quasipolynomial (2).

By condition (15), there exist [20] matrices $L_1(\lambda) \in \mathbb{R}^{r\times n_2}[\lambda]$ and $L_2(\lambda) \in \mathbb{R}^{r\times r}[\lambda]$ such that the identity

$$\begin{vmatrix} I_{n_2} - \lambda A_{22}(\lambda) & -B_2(\lambda) \\ -\lambda L_1(\lambda) & I_r - \lambda L_2(\lambda) \end{vmatrix} \equiv 1 \quad (20)$$

holds. We close system (1) with a controller of the form

$$\begin{aligned} u(t) &= y_1(t) + v_1(t), \\ y_1(t) &= L_1(\lambda)y(t-h) + L_2(\lambda)y_1(t-h) + v_2(t), \qquad t > 0, \end{aligned} \quad (21)$$

where y_1 is a new auxiliary variable and $v = \mathrm{col}\,[v_1, v_2] \in \mathbb{R}^{2r}$ is a new control whose choice will be described below. We introduce two matrices,

$$\overline{W}(p, e^{-ph}) = \begin{bmatrix} pI_{n_1} - A_{11}(e^{-ph}) & -A_{12}(e^{-ph}) & -B_1(e^{-ph}) \\ -A_{21}(e^{-ph}) & I_{n_2} - e^{-ph}A_{22}(e^{-ph}) & -B_2(e^{-ph}) \\ 0_{r\times n_1} & -e^{-ph}L_1(e^{-ph}) & I_r - e^{-ph}L_2(e^{-ph}) \end{bmatrix},$$

which is the characteristic matrix of system (1), (21), and $\overline{B}(\lambda) = \mathrm{diag}\,[B(\lambda), I_r]$. By condition (14), we have

$$\mathrm{rank}\,[\overline{W}(p, e^{-ph}), \overline{B}(e^{-ph})] = n + r, \qquad p \in \mathbb{C}. \quad (22)$$

Let

$$\Pi(\lambda) = \begin{bmatrix} \Pi_{11}(\lambda) & \Pi_{12}(\lambda) \\ \Pi_{21}(\lambda) & \Pi_{22}(\lambda) \end{bmatrix}$$

be the inverse of the matrix

$$\begin{bmatrix} I_{n_2} - \lambda A_{22}(\lambda) & -B_2(\lambda) \\ -\lambda L_1(\lambda) & I_r - \lambda L_2(\lambda) \end{bmatrix}.$$

Here the blocks $\Pi_{11}(\lambda)$ and $\Pi_{22}(\lambda)$ are $n_2 \times n_2$ and $r \times r$ matrices, respectively, and the dimensions of the blocks $\Pi_{12}(\lambda)$ and $\Pi_{21}(\lambda)$ are obvious. Since identity (20) holds, it follows that the matrix $\Pi(\lambda)$ is polynomial.

In system (1), (21), we set $\operatorname{col}[y, y_1] = \Pi(\lambda)\operatorname{col}[z_1, z_2]$, where $z_1 \in \mathbb{R}^{n_2}$ and $z_2 \in \mathbb{R}^r$. Then the original system (1), (21) transforms into the system

$$\begin{aligned} &\dot{x}(t) = A_{11}(\lambda)x(t) + C_{12}(\lambda)z_1(t) + C_{13}(\lambda)z_2(t) + B_1(\lambda)v_1(t), \\ &z_1(t) = A_{21}(\lambda)x(t) + B_2(\lambda)v_1(t), \\ &z_2(t) = v_2(t), \qquad t > 0, \end{aligned} \tag{23}$$

where $C_{12}(\lambda) = A_{12}(\lambda)\Pi_{11}(\lambda) + B_1(\lambda)\Pi_{21}(\lambda)$, $C_{13}(\lambda) = A_{12}(\lambda)\Pi_{12}(\lambda) + B_1(\lambda)\Pi_{22}(\lambda)$. Let

$$\widetilde{W}(p, e^{-ph}) = \begin{bmatrix} pI_{n_1} - A_{11}(e^{-ph}) & -C_{12}(e^{-ph}) & -C_{13}(e^{-ph}) \\ -A_{21}(e^{-ph}) & I_{n_2} & 0_{n_2\times r} \\ 0_{r\times n_1} & 0_{r\times n_2} & I_r \end{bmatrix}$$

be the characteristic matrix of system (23). In view of condition (22) and identity (20), we have

$$\operatorname{rank}[\widetilde{W}(p, e^{-ph}), \overline{B}(e^{-ph})] = n + r, \qquad p \in \mathbb{C}. \tag{24}$$

We write $D(\lambda) = A_{11}(\lambda) + C_{12}(\lambda)A_{21}(\lambda)$ and $B^C(\lambda) = [B_1(\lambda) + C_{12}(\lambda)B_2(\lambda), C_{13}(\lambda)]$. Let us prove that

$$\operatorname{rank}[pI_{n_1} - D(e^{-ph}), B^C(e^{-ph})] = n_1, \qquad p \in \mathbb{C}. \tag{25}$$

Indeed, assume that, for a $p_0 \in \mathbb{C}$, Eq. (25) is violated. Then there exists a nonzero vector $g_0 \in \mathbb{R}^{n_1}$ for which $g_0'(p_0 I_{n_1} - D(e^{-p_0 h})) = 0_{1\times n_1}$, $g_0' B^C(e^{-p_0 h}) = 0_{1\times 2r}$ (the prime denotes the operation of transposition). Multiplying the matrix $[\widetilde{W}(p_0, e^{-p_0 h}), \overline{B}(e^{-p_0 h})]$ in (24) on the left by the vector $[g_0', g_0' C_{12}(e^{-p_0 h}), g_0' C_{13}(e^{-p_0 h})]$, we see that condition (24) is violated for $p = p_0$.

We replace the functions z_1 and z_2 in the first equation in system (23) by their expressions determined by the second and third equations and obtain a linear autonomous differential-difference system of retarded type

$$\dot{x}(t) = D(\lambda)x(t) + B^C(\lambda)v(t), \qquad t > 0. \tag{26}$$

First, consider the case of $d_{n_0}(p) \equiv 1$ (see formula (2)). By (25), system (26) is modally controllable [16]; i.e., there exists a controller

$$\begin{aligned} &v(t) = \widetilde{R}_{01}(\lambda)X(t) + \sum_{i=0}^{\delta}\int_0^h \widetilde{F}_{0i}(s)\lambda^i X(t-s)\,ds, \\ &\dot{x}_1(t) = \widetilde{R}_{11}(\lambda)X(t) + \sum_{i=0}^{\delta}\int_0^h \widetilde{F}_{1i}(s)\lambda^i X(t-s)\,ds, \qquad t > 0, \end{aligned} \tag{27}$$

such that the characteristic quasipolynomial of the closed-loop system (26), (27) coincides with any prescribed quasipolynomial $d(p, e^{-ph})$ of the form (2) whose degree is greater than or equal to the number $\zeta = n_1 + 1$. Here $x_1 \in \mathbb{R}^{\bar{n}_1}$ is an auxiliary variable, $\widetilde{R}_{01}(\lambda) \in \mathbb{R}^{2r\times(n_1+\bar{n}_1)}$, $\widetilde{R}_{11}(\lambda) \in \mathbb{R}^{\bar{n}_1\times(n_1+\bar{n}_1)}$, and the functions $\widetilde{F}_{ij}(s)$, $j = 0, 1$, $i = 1, \dots, \delta$, have the same form as the functions $F_{ij}(s)$ in (3)–(5). We close system (23) with the controller (27). Since the matrices of system (26) are obtained by elementary transformations from the rows of the matrix $[\widetilde{W}(p,\lambda), \overline{B}(\lambda)]$, we conclude that the characteristic quasipolynomial of system (23), (27) is equal to $d(p, e^{-ph})$.

We substitute the function v determined by the first equation in (27) into formulas (21), use the obtained controller to close the original system, and supplement the resulting relations with the second equation in (27). As a result, we obtain system (1), (21), (27) which coincides with system (1), (3)–(5) in which, in relations (3)–(5), we set ($\bar{n}_2 = r$),

$$\begin{aligned} &R_{01}(\lambda) = [I_r, 0_{r\times r}]\widetilde{R}_{01}(\lambda), \qquad R_{02}(\lambda) = [0_{r\times n_2}, I_r], \qquad F_{0i}(s) = [I_r, 0_{r\times r}]\widetilde{F}_{0i}(s), \\ &R_{11}(\lambda) = \widetilde{R}_{11}(\lambda), \qquad R_{12}(\lambda) = 0_{\bar{n}_1\times(n_2+\bar{n}_2)}, \qquad F_{1i}(s) = \widetilde{F}_{1i}(s), \\ &R_{21}(\lambda) = [0_{r\times r}, I_r]\widetilde{R}_{01}(\lambda), \qquad R_{22}(\lambda) = [\lambda L_1(\lambda), \lambda L_2(\lambda)], \qquad F_{2i}(s) = [0_{r\times r}, I_r]\widetilde{F}_{0i}(s); \end{aligned} \tag{28}$$

here $i = 1, \ldots, \delta$. It follows from the above that the characteristic quasipolynomial of the closed-loop system (1), (21), (27) coincides with the characteristic quasipolynomial of system (23), (27); i.e., it is equal to $d(p, e^{-ph})$. Thus, the identity $|\mathcal{W}(p, e^{-ph})| = d(p, e^{-ph})$ holds. To verify that system (1), (21), (27) is completely regular, we replace the function $y_1(t)$ in the first relation in (21), with its expression from the second relation.

Now consider the case of $d_{n_0}(p) \not\equiv 1$. Instead of the controller (27), we close system (26) with a controller of the same form as the controller (3)–(5), which we construct by using the scheme (with obvious changes) proposed in [18]. The subsequent proof reproduces the argument in case considered above. The proof of the theorem is complete.

Lemma 1. *For system* (1′) *to satisfy condition* (13), *it is necessary and sufficient that system* (1) *satisfy condition* (15).

Proof. Condition (15) is equivalent to the condition

$$\operatorname{rank}[\overline{A}_{22}(\lambda), \overline{B}_2(\lambda)] = n_2, \qquad \lambda \in \mathbb{C}. \tag{29}$$

Assume that, for some $\lambda_0 \in \mathbb{C}$, relation (13) is violated. Then there exists a nonzero vector $g_0 \in \mathbb{R}^{n_2}$ such that $g_0'\Phi\widetilde{A}(\lambda_0)\Psi = 0_{1\times n_2}$ and $g_0'\Phi\widetilde{B}(\lambda_0) = 0_{1\times r}$. This and (11) imply that $g_0'\overline{A}_{22}(\lambda_0) = 0_{1\times n_2}$ and $g_0'\overline{B}_2(\lambda_0) = 0_{1\times r}$. The resulting relations contradict condition (29). To prove the second part of the lemma, we use the same reasoning but in reverse order. The proof of the lemma is complete.

Proof of Theorem 1. Assume that conditions (12) and (13) are satisfied. Then, by Lemma 1, conditions (14) and (15) are satisfied. Therefore, for the prescribed quasipolynomial (2), the exists a controller (3)–(5) such that the characteristic quasipolynomial of the closed-loop system (1), (3)–(5) coincides with the quasipolynomial (2). We change the variables in the formulas of the controller (3)–(5) as in Remark 3 and obtain a controller ensuring the characteristic quasipolynomial $\widetilde{d}(p, e^{-ph}) = d(p, e^{-ph})|HH_1|$ for the closed-loop system (1′).

If conditions (12), (13) are violated, then system (1) is not modally controllable, and hence system (1′) is not modally controllable either. The proof of Theorem 1 is complete.

3. WEAK MODAL CONTROLLABILITY

Assume that the assumptions of Theorem 1 are violated. Then the following natural question arises: What are the possibilities of controlling the spectrum of system (1′) in this case? In this section, we propose a criterion for weak modal controllability (see Theorem 3), which gives a partial answer to this question.

First, we introduce some auxiliary notions and notation. Let $B(\lambda) = \sum_{i=0}^{m} B^{(i)}\lambda^i$, where $B^{(i)} \in \mathbb{R}^{n\times r}$. As in [22], consider the sequence of vectors q_k, $k = m, m+1, \ldots$, whose terms are defined recursively as the solutions of the difference equation

$$B^{(0)}q_k + \sum_{i=1}^{m} B^{(i)}q_{k-i} = 0, \qquad k = m, m+1, \ldots, \tag{30}$$

with the initial values $q_i = \tilde{q}_i$, $i = 0, \dots, m-1$. The sequence q_k, $k = m, m+1, \dots$, given by Eq. (30) exists if [23] and only if $\tilde{q}_{m-i} = T_i c$ and $i = 1, \dots, m$, where $T_i \in \mathbb{R}^{r \times r_T}$ are certain matrices and $c \in \mathbb{R}^{r_T}$ is an arbitrary constant vector (the same for all matrices T_i). The method for constructing the matrices T_i is given in [22, 23], and hence we do not describe it here. Note that its practical implementation is always possible, because it amounts to solving a finite chain of homogeneous algebraic systems. We denote $T = T_m$ and define a matrix $S \in \mathbb{R}^{r_T \times r_T}$ as the solution of the system of equations

$$B^{(0)} T_1 S + \sum_{i=1}^{m} B^{(i)} T_i = 0_{n \times r_T}, \qquad T_k S = T_{k-1}, \qquad k = 2, \dots, m,$$

whose solvability follows from the definition of the matrices T_i. Note that $\sum_{i=0}^{m} B^{(i)} T S^{m-i} = 0_{n \times r_T}$. Let us introduce the matrices $G^{(0)} = B^{(0)} T$, $G^{(i)} = G^{(i-1)} S + B^{(i)} T$, $i = 1, \dots, m$. In particular, $G^{(m)} = \sum_{i=0}^{m} B^{(i)} T S^{m-i} = 0_{n \times r_T}$. We define the matrix $G(\lambda) = \sum_{i=0}^{m} G^{(i)} \lambda^i$. For a given polynomial matrix $B(\lambda)$, the matrix $G(\lambda)$ constructed by the method described above is called the matrix of additional inputs (this term is justified by Lemma 2 and Theorem 4 below).

Assume that $\widetilde{G}(\lambda)$ is a matrix of additional inputs for the matrix $\widetilde{B}(\lambda)$. Then there exists a matrix H_3, $|H_3| \neq 0$, such that $\widetilde{G}(\lambda) = H_3 G(\lambda)$.

Theorem 3. *System* (1′) *is weakly modally controllable if and only if the following conditions are satisfied simultaneously* :

$$\operatorname{rank}\,[p\widetilde{A}_0 - \widetilde{A}(e^{-ph}), \widetilde{B}(e^{-ph}), \widetilde{G}(e^{-ph})] = n, \qquad p \in \mathbb{C},$$
$$\operatorname{rank}\,[\Phi\widetilde{A}(\lambda)\Psi, \Phi\widetilde{B}(\lambda), \Phi\widetilde{G}(\lambda)] = n_2, \qquad \lambda \in \mathbb{C}.$$

As in the problem of modal controllability, we first prove a criterion for weak modal controllability of the transformed system (1). Let $G_1(\lambda)$ be the matrix consisting of the first n_1 rows of the matrix $G(\lambda)$, and let $G_2(\lambda)$ be the matrix consisting of the last n_2 rows of the matrix $G(\lambda)$; i.e., $G(\lambda) = \operatorname{col}\,[G_1(\lambda), G_2(\lambda)]$.

To system (1), we assign the system

$$\begin{aligned} \dot{x}(t) &= A_{11}(\lambda)x(t) + A_{12}(\lambda)y(t) + B_1(\lambda)v_1^*(t) + G_1(\lambda)v_2^*(t), \\ y(t) &= A_{21}(\lambda)x(t) + A_{22}(\lambda)y(t-h) + B_2(\lambda)v_1^*(t) + G_2(\lambda)v_2^*(t), \qquad t > t_0, \end{aligned} \tag{31}$$

where $v^* = \operatorname{col}\,[v_1^*, v_2^*]$ is a new control and $t_0 > 0$ is a moment of time.

Lemma 2. *Assume that the control* $u(t)$, $t > 0$, *for system* (1) *is defined by the formulas*

$$u(t) = T\psi(t) + v_1^*(t), \qquad \psi(t) = S\psi(t-h) + v_2^*(t), \qquad t > 0, \tag{32}$$

where $v_i^*(t)$, $t > 0$, $i = 1, 2$, *and* $\psi(t)$, $t \le 0$, *are arbitrary piecewise continuous functions. Then for* $t > t_0 = mh$ *system* (1) *has the form of system* (31).

Proof. Consider the chain of relations [19, 20]

$$\begin{aligned} B(\lambda)T\psi(t) &= B^{(0)}T\psi(t) + \sum_{i=1}^{m} B^{(i)}\lambda^i T\psi(t) = G^{(0)}\psi(t) + \sum_{i=1}^{m}(G^{(i)} - G^{(i-1)}S)\lambda^i\psi(t) \\ &= \sum_{i=0}^{m} G^{(i)}\lambda^i\psi(t) - \sum_{i=1}^{m} G^{(i-1)}S\lambda^i\psi(t) = \sum_{i=0}^{m-1} G^{(i)}\lambda^i(I_{r_T} - S\lambda)\psi(t) \\ &= G(\lambda)v_2^*(t), \qquad t > t_0. \end{aligned} \tag{33}$$

We substitute the control u determined by formulas (32) into system (1). Based on the chain of relations (33), we conclude that $B(\lambda)u(t) = B(\lambda)v_1^*(t) + G(\lambda)v_2^*(t)$, $t > t_0$. Therefore, for $t > t_0$ system (1), (32) has the form (31). The proof of the lemma is complete.

Theorem 4. *For system* (1) *to be weakly modally controllable, it is necessary and sufficient that the following conditions be satisfied simultaneously*:

$$\operatorname{rank}[W_0(p,e^{-ph}),\ B(e^{-ph}),\ G(e^{-ph})]=n, \qquad p\in\mathbb{C}, \tag{34}$$

$$\operatorname{rank}[I_{n_2}-\lambda A_{22}(\lambda),B_2(\lambda),G_2(\lambda)]=n_2, \qquad \lambda\in\mathbb{C}. \tag{35}$$

Proof. Necessity. Assume that the controller (3)–(5) ensures that the three conditions in Definition 2 are satisfied. We also assume that $R_{22}(\lambda)=\lambda\widetilde{R}_{22}(\lambda)$, where $\widetilde{R}_{22}(\lambda)\in\mathbb{R}^{\bar{n}_2\times n_y}[\lambda]$. This does not lead to the loss of generality, because the characteristic matrix $\mathcal{W}(p,\lambda)$ of system (1), (3)–(5) has CR-structure (see the transformations taking system (1′) to system (1)). Consider relation (7). We still assume that the variables in system (1), (3)–(5) are ordered as the vector $\operatorname{col}[x,x_1,y,y_1]$. Since the matrix $\mathcal{W}_{11}(p,\lambda)$ has CR-structure and the equations in the original system are reduced to the form (1), it follows that, up to elementary column transformations of the matrix $\mathcal{P}(\lambda)$, this matrix has the form

$$\mathcal{P}(\lambda)=\begin{bmatrix} I_{n_x+n_y-n_*} & \mathcal{P}_1(\lambda)\\ 0_{n_*\times(n_x+n_y-n_*)} & I_{n_*}\end{bmatrix}, \tag{36}$$

where $\mathcal{P}_1(\lambda)\in\mathbb{R}^{(n_x+n_y-n_*)\times n_*}[\lambda]$. Let $Y=\operatorname{col}[y_{11},y_{12}]$, $y_{11}\in\mathbb{R}^{n_y-n_*}$, $y_{12}\in\mathbb{R}^{n_*}$. Then the zero block of the matrix on the right-hand side in (7) corresponds to the components of the vector y_{12}. Our goal is to determine the form of the matrix $\mathcal{P}_{11}(\lambda)$ consisting of the first n rows of the matrix $\mathcal{P}_1(\lambda)$.

We introduce the matrix $R_0(\lambda)$ composed of the last n_* columns of the matrix $R_{02}(\lambda)$ and write (3) as

$$u(t)=R_0(\lambda)y_{12}(t)+V_0 \tag{37}$$

and the last n_* rows of the closed-loop system (1), (3)–(5) as

$$y_{12}(t)=\lambda\mathcal{A}(\lambda)y_{12}(t)+V_1, \tag{38}$$

where the quantities $V_i=V_i(\lambda,X,y_{11})$, $i=0,1$, are independent of the function y_{12} and their form can easily be seen when (37) and (38) are composed. We also note that (38) is the relation between the last n_* components of the vectors in formula (5). It follows from (38) that

$$y_{12}(t)=(\lambda\mathcal{A}(\lambda))^k y_{12}(t)+\sum_{i=0}^{k-1}(\lambda\mathcal{A}(\lambda))^i V_1, \qquad t>(k-1)(\widetilde{k}+1)h, \qquad k=1,2,\dots, \tag{39}$$

where $\widetilde{k}$ is the largest degree of entries of the matrix $\mathcal{A}(\lambda)$. We successively apply formula (39) for $k=m-i$ to the terms $B^{(i)}R_0(\lambda)y_{12}(t-ih)$, $i=0,\dots,m-1$, contained in the sum $B(\lambda)R_0(\lambda)y_{12}(t)$ and write it as

$$B(\lambda)R_0(\lambda)y_{12}(t)=\lambda^m\widetilde{B}(\lambda)y_{12}(t)+\widehat{B}(\lambda)V_1, \qquad t>((m-1)(\widetilde{k}+1)+\widehat{k})h,$$

where the polynomial matrices $\widetilde{B}(\lambda)$ and $\widehat{B}(\lambda)$ are given by the formulas

$$\widetilde{B}(\lambda)=\sum_{i=0}^{m}B_iR_0(\lambda)(\mathcal{A}(\lambda))^{m-i}, \qquad \widehat{B}(\lambda)=\sum_{i=0}^{m-1}\lambda^iB_iR_0(\lambda)\sum_{j=0}^{m-1-i}(\lambda\mathcal{A}(\lambda))^j$$

and the number $\widehat{k}$ is the maximum degree of entries of the matrix $R_0(\lambda)$. We repeatedly apply (39) (we assume that the term with the symbol of the sum is absent for $k=0$) and obtain

$$B(\lambda)R_0(\lambda)y_{12}(t)=\lambda^m\widetilde{B}(\lambda)(\lambda\mathcal{A}(\lambda))^k y_{12}(t)+\left(\lambda^m\widetilde{B}(\lambda)\sum_{i=0}^{k-1}(\lambda\mathcal{A}(\lambda))^i+\widehat{B}(\lambda)\right)V_1,$$
$$k=0,1,\dots, \qquad t>((k+m-1)(\widetilde{k}+1)+\widehat{k})h. \tag{40}$$

Now, based on (37), we write the expression $B(\lambda)u(t)$ as

$$B(\lambda)u(t) = B(\lambda)V_0 + \lambda^m \widetilde{B}(\lambda)(\lambda\mathcal{A}(\lambda))^k y_{12}(t) + \left(\lambda^m \widetilde{B}(\lambda)\sum_{i=0}^{k-1}(\lambda\mathcal{A}(\lambda))^i + \widehat{B}(\lambda)\right)V_1,$$
$$t > ((k+m-1)(\widetilde{k}+1)+\widehat{k})h, \qquad k = 0,1,\ldots \tag{41}$$

It follows from the form of the matrix on the right-hand side in (7) that, for $t > \bar{t}$, where $\bar{t} > 0$ is a sufficiently large number, $B(\lambda)u(t)$ is independent of the function y_{12}. Therefore, for some $k = k^*$ we have (see (41))

$$\lambda^m \widetilde{B}(\lambda)(\lambda\mathcal{A}(\lambda))^{k^*} y_{12}(t) = 0_{n\times 1}, \qquad t > ((k^*+m-1)(\widetilde{k}+1)+\widehat{k})h. \tag{42}$$

By setting $k = k^*$ in (40), by taking into account (42), and by transforming the terms in the sum in an obvious way, we obtain

$$B(\lambda)R_0(\lambda)y_{12}(t) = B(\lambda)R_0(\lambda)\sum_{i=0}^{k^*-1}(\lambda\mathcal{A}(\lambda))^i V_1$$
$$+\sum_{i=1}^{m}\left(\sum_{j=0}^{m-i} B_i R_0(\lambda)(\mathcal{A}(\lambda))^{m-i-j}\right)\lambda^{k^*+m-i}(\mathcal{A}(\lambda))^{k^*}V_1, \tag{43}$$
$$t > ((k^*+m-1)(\widetilde{k}+1)+\widehat{k})h.$$

Consider Eq. (42). Since it holds for any piecewise continuous function $y_{12}(t)$, $t < 0$, we have

$$\widetilde{B}(\lambda)(\mathcal{A}(\lambda))^{k^*} = 0_{n\times n_*}. \tag{44}$$

Let us determine the sequence of matrices $J_k(\lambda) = R_0(\lambda)(\mathcal{A}(\lambda))^{k^*+k}$, $k = 0,1,\ldots$ A straightforward verification shows that, by (44), the elements $J_k(\lambda)$, $k = 0,1,\ldots,$ of this sequence satisfy the equation

$$\sum_{i=0}^{m} B_i J_{k-i}(\lambda) = 0_{n\times n_*}, \qquad k = m, m+1,\ldots$$

Therefore [23], there exists a matrix $J^*(\lambda) \in \mathbb{R}^{r_T\times n_*}[\lambda]$ such that $J_k(\lambda) = TS^k J^*(\lambda)$, $k = 0,\ldots,m-1$, where the matrices T and S are defined in Lemma 2. Based on this representation, we write (43) as

$$B(\lambda)R_0(\lambda)y_{12}(t) = \left(B(\lambda)R_0(\lambda)\sum_{i=0}^{k^*-1}(\lambda\mathcal{A}(\lambda))^i + \lambda^{k^*}G(\lambda)J^*(\lambda)\right)V_1,$$
$$t > ((k^*+m-1)(\widetilde{k}+1)+\widehat{k})h,$$

or, by (38), as

$$B(\lambda)R_0(\lambda)y_{12}(t) = \left(B(\lambda)R_0(\lambda)\sum_{i=0}^{k^*-1}(\lambda\mathcal{A}(\lambda))^i + \lambda^{k^*}G(\lambda)J^*(\lambda)\right)$$
$$\times (I_{n^*} - \lambda\mathcal{A}(\lambda))y_{12}(t), \qquad t > ((k^*+m-1)(\widetilde{k}+1)+\widehat{k})h. \tag{45}$$

We write (41) as

$$B(\lambda)u(t) = B(\lambda)V_0 + \left(B(\lambda)R_0(\lambda)\sum_{i=0}^{k^*-1}(\lambda\mathcal{A}(\lambda))^i + \lambda^{k^*}G(\lambda)J^*(\lambda)\right)V_1, \tag{46}$$
$$t > ((k^*+m-1)(\widetilde{k}+1)+\widehat{k})h.$$

The representations (45) and (46) and the condition $\operatorname{rank}(I_{n_*} - \lambda\mathcal{A}(\lambda)) = n_*$ imply that

$$\mathcal{P}_{11}(\lambda) = B(\lambda)R_0(\lambda)\sum_{i=0}^{k^*-1}(\lambda\mathcal{A}(\lambda))^i + \lambda^{k^*}G(\lambda)J^*(\lambda).$$

Now we determine the matrices

$$\mathcal{B}_0(\lambda) = [B(\lambda), \mathcal{P}_{11}(\lambda)] \quad \text{and} \quad \mathcal{W}_{11}(p,\lambda) = \operatorname{col}[\mathcal{W}_{111}(p,\lambda), \mathcal{W}_{112}(p,\lambda)],$$

where the matrix $\mathcal{W}_{111}(p,\lambda)$ is composed of the first n rows of $\mathcal{W}_{11}(p,\lambda)$ and the matrix $\mathcal{W}_{112}(p,\lambda)$, of the remaining rows of $\mathcal{W}_{11}(p,\lambda)$. There also exist polynomial matrices $\mathcal{R}_i(\lambda)$, $i = 0, 1$, of appropriate dimensions such that $V_i = \mathcal{R}_i(\lambda)\operatorname{col}[X(t), y_{11}(t)]$, $i = 0, 1$. It follows from the form of the matrix $\mathcal{P}_{11}(\lambda)$ that

$$\mathcal{W}_{111}(p,\lambda) = [W_0(p,\lambda), 0_{n\times(\bar{n}_1+\bar{n}_2-n_*)}] - \mathcal{B}_0(\lambda)\operatorname{col}[\mathcal{R}_0(\lambda), \mathcal{R}_1(\lambda)].$$

By Definition 2, the relation $|\mathcal{W}_{11}(p,\lambda)| = d(p,\lambda)$ holds for any prescribed quasipolynomial $d(p,\lambda)$. Therefore, the system determined by the characteristic matrix $W_0(p,\lambda)$ and the matrix $\mathcal{B}_0(\lambda)$ multiplying control is modally controllable. But this is impossible by Theorem 1 if conditions (34) and (35) are violated. The proof of the necessity of conditions (34), (35) is complete.

Sufficiency. Assume that conditions (34) and (35) are satisfied. Then system (31) is modally controllable. Therefore, for any quasipolynomial (2) ($\zeta = n_1 + 1$), there exists a controller

$$v^*(t) = R_{01}^*(\lambda)X^*(t) + R_{02}^*(\lambda)Y^*(t) + \sum_{i=0}^{\delta}\int_0^h F_{0i}^*(s)\lambda^i X^*(t-s)\,ds, \tag{47}$$

$$\dot{x}_1^*(t) = R_{11}^*(\lambda)X^*(t) + R_{12}^*(\lambda)Y^*(t) + \sum_{i=0}^{\delta}\int_0^h F_{1i}^*(s)\lambda^i X^*(t-s)\,ds, \tag{48}$$

$$y_1^*(t) = R_{21}^*(\lambda)X^*(t) + R_{22}^*(\lambda)Y^*(t) + \sum_{i=0}^{\delta}\int_0^h F_{2i}^*(s)\lambda^i X^*(t-s)\,ds, \qquad t > 0, \tag{49}$$

such that the characteristic matrix of the closed-loop system (31), (47)–(49) has CR-structure and the characteristic quasipolynomial coincides with the quasipolynomial (2). Here $x_1^* \in \mathbb{R}^{\bar{n}_1^*}$, $y_1^* \in \mathbb{R}^{\bar{n}_2^*}$ are additional variables, $X^* = \operatorname{col}[x, x_1^*]$, $Y = \operatorname{col}[y, y_1^*]$, and $R_{i,j}^*(\lambda)$, $i = 0, \ldots, 2$, $j = 1, 2$, are polynomial matrices, and the functions $F_{j,i}^*(s)$, $j = 0, \ldots, 2$, $i = 0, \ldots, \delta$, have the same structure as the functions $F_{i,j}(s)$ in (3)–(5).

Let us determine the matrices $E_1 = [I_r, 0_{r\times r_T}]$ and $E_2 = [0_{r_T\times r}, I_{r_T}]$. From the parameters of the controller (47)–(49), we construct the desired controller (3)–(5) for system (1),

$$u(t) = E_1R_{01}^*(\lambda)X(t) + [E_1R_{02}^*(\lambda),\, T]Y(t) + \sum_{i=0}^{\delta}\int_0^h E_1F_{0i}^*(s)\lambda^i X(t-s)\,ds, \tag{50}$$

$$\dot{x}_1(t) = R_{11}^*(\lambda)X(t) + [R_{12}^*(\lambda),\, 0_{\bar{n}_1^*\times r_T}]Y(t) + \sum_{i=0}^{\delta}\int_0^h F_{1i}^*(s)\lambda^i X(t-s)\,ds, \tag{51}$$

$$y_{11}(t) = R_{21}^*(\lambda)X(t) + [R_{22}^*(\lambda),\, 0_{\bar{n}_2^*\times r_T}]Y(t) + \sum_{i=0}^{\delta}\int_0^h F_{2i}^*(s)\lambda^i X(t-s)\,ds, \tag{52}$$

$$y_{12}(t) = E_2R_{01}^*(\lambda)X(t) + [E_2R_{02}^*(\lambda),\, \lambda S]Y(t) + \sum_{i=0}^{\delta}\int_0^h E_2F_{0i}^*(s)\lambda^i X(t-s)\,ds, \quad t > 0, \tag{53}$$

where $X = \operatorname{col}[x, x_1] \in \mathbb{R}^{n_1+\bar{n}_1^*}$, $Y = \operatorname{col}[y, y_{11}, y_{12}] \in \mathbb{R}^{n_2+\bar{n}_2^*+r_T}$ ($y_1 = \operatorname{col}[y_{11}, y_{12}]$).

Let us prove that for system (1) closed with the controller (50)–(53) there exists a matrix $\mathcal{P}(e^{-ph})$ ensuring (7), where the matrix $\mathcal{W}_{11}(p,e^{-ph})$ has CR-structure and $|\mathcal{W}_{11}(p,e^{-ph})| = d(p,e^{-ph})$. To this end, it suffices to show that this controller permits one, by elementary row transformations, to single out an independent subsystem from the closed-loop system (1), (50)–(53), and this subsystem contains the variables x and y as components of the solution and has the characteristic quasipolynomial $|\mathcal{W}_{11}(p,e^{-ph})| = d(p,e^{-ph})$. We write (50) and (53) as

$$u(t) = Ty_{12}(t) + \Gamma_1(X,y,y_{11}), \qquad y_{12}(t) = Sy_{12}(t-h) + \Gamma_2(X,y,y_{11}), \tag{54}$$

where the form of the functions $\Gamma_i(X,y,y_{11})$, $i = 1,2$, can be seen from the expressions (50) and (53). By Lemma 2, system (1) closed with the controller (54) has the form (31), where $v_i^*(t) = \Gamma_i(X,y,y_{11})$, $i = 1,2$. We supplement this system with Eqs. (51), (52) and obtain a system with the same parameters as system (31), (47)–(49). The proof of the theorem is complete.

Remark 4. If the controller for system (1) is determined by formulas (50)–(53), then one can take the matrix $\mathcal{P}(e^{-ph})$ of the form (36) in (7), where $\mathcal{P}_1(\lambda) = \operatorname{col}[G(\lambda),\, 0_{(\bar{n}_1+\bar{n}_2-r_T)\times r_T}]$ and $n_* = r_T$.

Proof of Theorem 3. The proof follows from Lemma 1 and Theorem 4.

Example 1. It follows from the proofs of Theorems 2 and 4 that, in Definitions 1 and 2 one can set $\zeta = n_1+1$ and the number of equations in the closed-loop system (1), (3)–(5) is n_0+n_2+r if one solves a modal controllability problem and $n_0+n_2+r+r_T$ if one solves a weak modal controllability problem. But in several cases, the number of equations in the controller (3)–(5) can be decreased. For example (also see [20]), if one succeeds in constructing polynomial matrices $L_{ij}(\lambda)$, $i = 1,2$, $j = 1,2$, of appropriate dimensions such that the identity

$$\begin{vmatrix} I_n - \lambda A_{22}(\lambda) - \lambda B(\lambda)L_{11}(\lambda) & -\lambda B(\lambda)L_{12}(\lambda) \\ -\lambda L_{21}(\lambda) & I_{r^*} - \lambda L_{22}(\lambda) \end{vmatrix} \equiv 1$$

holds (with a number $r^* \in \mathbb{N} \bigsqcup \{0\}$), then, instead of the controller (21), one can close system (1) with the controller

$$\begin{aligned} u(t) &= L_{11}(\lambda)y(t-h) + L_{12}(\lambda)y_1(t-h) + v_1(t), \\ \dot{y}_1(t) &= L_{21}(\lambda)y(t-h) + L_{22}(\lambda)y_1(t-h) + v_2(t). \end{aligned} \tag{55}$$

The resulting closed-loop system should be treated in the same way as system (1), (21) was treated above.

Example 2. Consider system (1) with the matrices

$$A_{11}(\lambda) = \begin{bmatrix} 0 & 1-\lambda \\ -\lambda & \lambda \end{bmatrix}, \qquad A_{12} = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix}, \qquad A_{21}(\lambda) = [0,-1], \qquad A_{22} = 1,$$

$$B_1(\lambda) = \begin{bmatrix} \lambda-\lambda^2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_2(\lambda) = [\lambda^2,\, \lambda^2], \tag{56}$$

and the delay $h = 1$. A simple verification of the conditions of Theorem 2 shows that the system is not modally controllable. Calculating [23] the matrices

$$T_1 = T_2 = T = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad S = 1, \quad G(\lambda) = \begin{bmatrix} -\lambda \\ 0 \\ 0 \end{bmatrix}, \quad G_1(\lambda) = \begin{bmatrix} -\lambda \\ 0 \end{bmatrix}, \quad G_2(\lambda) = 0,$$

we see that the conditions of Theorem 4 are satisfied; i.e., the given system in weakly modally controllable. For this system, we construct the controller (3)–(5) which ensures a finite stable spectrum for the subsystem of the closed-loop system (1), (3)–(5) which determines the dynamics in the variables x and y. The specific form of the desired characteristic quasipolynomial will be chosen below.

1. Consider system (31) for which we construct the controller (47)–(49). This can be done by using the proof of sufficiency of the conditions of Theorem 1, where the matrix $B(\lambda)$ is replaced with the matrix $[B(\lambda), G(\lambda)]$. First, we determine [20] the polynomial matrices

$$L_1(\lambda) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \qquad L_2(\lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{57}$$

which ensure identity (20). Note that the first and last rows of the matrices $L_i(\lambda)$, $i = 1, 2$, are zero. This permits decreasing their dimensions; i.e., instead of the matrices (57) obtained above, we can set $L_1(\lambda) = -1$, $L_2(\lambda) = -1 - \lambda$, and instead of the controller (21), we can take the controller (55) of the form

$$\begin{gathered} u(t) = \operatorname{col}[v_{11}(t),\, y_1(t) + v_{12}(t),\, v_{13}(t)], \\ y_1(t) = -y(t-1) - y_1(t-1) - y_1(t-2) + v_2(t), \end{gathered} \tag{58}$$

where y_1 is a new auxiliary variable, $v_1 = \operatorname{col}[v_{11}, v_{12}, v_{13}]$, the $v = \operatorname{col}[v_1, v_2]$. We use the controller (58) to close the original system and pass to system (26) with matrices

$$D(\lambda) = \begin{bmatrix} 0 & 1-\lambda \\ -\lambda & \lambda^3 + \lambda^2 + 2\lambda \end{bmatrix}, \quad B^C(\lambda) = \left[\begin{array}{c|c|c|c} \lambda - \lambda^2 & 0 & -\lambda & 0 \\ -\lambda^5 - \lambda^4 - \lambda^3 & -\lambda^5 - \lambda^4 - \lambda^3 & 0 & -\lambda^3 \end{array}\right]. \tag{59}$$

(Since the dimensions of the matrices $L_i(\lambda)$, $i = 1, 2$, have decreased, the calculations insignificantly differ from those in the proof of Theorem 2, and since they are obvious, we do not present them here.) The order of system (26) with matrices (59) is 2, and hence for any quasipolynomial (2) of degree greater than or equal to three (the order of system (26) plus 1), there exists a controller such that quasipolynomial of the closed-loop system coincides with the quasipolynomial (2) [16]. Hence the controller (3)–(5) can take the original system (1) to the same characteristic quasipolynomial. Therefore, we pose the following problem: use feedback to ensure a characteristic quasipolynomial equal, for example, to $(p+1)(p+2)(p+3)$, for a subsystem of the closed-loop system (1) with matrices (56) which determines the components x and y of the original system. We use the results obtained in [16] to close system (26) with matrices (59) by a controller that ensures the characteristic quasipolynomial $(p+1)(p+2)(p+3)$ for the closed-loop system. Since the problem of closing a system of the form (26) was considered in [16], we do not discuss it here in detail but write the final result at once:

$$\begin{gathered} v(t) = \operatorname{col}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \operatorname{col}[0, 0, 1, 0] x_1(t), \\ \dot{x}_1(t) = [\nu_1(p,\lambda), \nu_2(p,\lambda), \nu_3(p,\lambda)] \operatorname{col}[x(t), x_1(t)], \end{gathered}$$

where x_1 is an auxiliary variable,

$$\nu_1(p,\lambda) = \frac{1}{3}(143 + 30\lambda + 12\lambda^2 + 3\lambda^3) - \frac{1}{3}\int_0^1 (-48 + 184\lambda + 95\lambda^2) e^{-ps}\, ds$$
$$+ \frac{1}{3}\int_0^1 (3 + 95\lambda + 114\lambda^2) s e^{-ps}\, ds + \frac{1}{3}\int_0^1 (-9 - 9\lambda - 19\lambda^2) s^2 e^{-ps}\, ds,$$

$$\nu_2(p,\lambda) = \frac{1}{3}(-317 - 137\lambda - 60\lambda^2 - 18\lambda^3 - 3\lambda^4) + \frac{1}{6}\int_0^1 (-294 + 722\lambda + 748\lambda^2 + 190\lambda^3) e^{-ps}\, ds$$
$$- \frac{1}{3}\int_0^1 (24 + 211\lambda + 361\lambda^2 + 114\lambda^3) s e^{-ps}\, ds + \frac{1}{3}\int_0^1 (2+\lambda)(9 + 9\lambda + 19\lambda^2) s^2 e^{-ps}\, ds,$$

$$\nu_3(p,\lambda) = -6 - 2\lambda - \lambda^2 + \frac{1}{3}\int_0^1 (-33 + 59\lambda + 38\lambda^2)e^{-ps}\,ds - \frac{2}{3}\int_0^1 (9 + 9\lambda + 19\lambda^2)se^{-ps}\,ds,$$

and use the notation

$$\int_0^1 \lambda^k s^m e^{-ps}\,ds\,x(t) = \int_0^1 s^m x(t - kh - s)\,ds.$$

Based on formulas (28), we obtain the controller (47)–(49) for system (31) (we set $v^* = v$):

$$\begin{aligned} &v^*(t) = \mathrm{col}\,[0, y_1^*(t), x_1^*(t)],\\ &\dot{x}_1^*(t) = [\nu_1(p,\lambda), \nu_2(p,\lambda), 0, \nu_3(p,\lambda), 0]\mathrm{col}\,[x(t), y(t), x_1^*(t), y_1^*(t)],\\ &y_1^*(t) = [0, 1, -\lambda, 0, -\lambda - \lambda^2]\mathrm{col}\,[x(t), y(t), x_1^*(t), y_1^*(t)]. \end{aligned} \tag{60}$$

2. We substitute the controller (60) into relations (58), write the resulting equations in the form (50)–(53), and finally obtain the controller (3)–(5)

$$\begin{aligned} u(t) &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} Y(t),\\ \dot{x}_1(t) &= [\nu_1(p,\lambda), \nu_2(p,\lambda), \nu_3(p,\lambda)]X(t),\\ \begin{bmatrix} y_{11}(t) \\ y_{12}(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X(t) + \begin{bmatrix} -\lambda & -\lambda - \lambda^2 & 0 \\ 0 & 0 & \lambda \end{bmatrix} Y(t), \end{aligned} \tag{61}$$

where $Y = \mathrm{col}\,[y, y_{11}, y_{12}]$ and $X = \mathrm{col}\,[x, x_1]$. If we close the original system with the controller (61) and multiply the characteristic matrix of the resulting system by the matrix (36) (see Remark 4), then the variable y_{12} is eliminated from the corresponding subsystem of the closed-loop system and, for $t > 2$, the characteristic matrix of the obtained subsystem has the form ($\lambda = e^{-p}$)

$$\mathcal{W}_{11}(p,\lambda) = \begin{bmatrix} p & \lambda - 1 & 0 & \lambda & 0 \\ \lambda & p - \lambda & \lambda & 0 & 0 \\ 0 & 1 & 1 - \lambda & 0 & -\lambda^2 \\ -\nu_1(p,\lambda) & \nu_2(p,\lambda) & 0 & p - \nu_3(p,\lambda) & 0 \\ 0 & -1 & \lambda & 0 & 1 + \lambda + \lambda^2 \end{bmatrix}. \tag{62}$$

(The variables are arranged as $\mathrm{col}\,[x, y, x_1, y_{11}]$.) Straightforward calculations readily show that the determinant of the matrix (62) is

$$|\mathcal{W}_{11}(p,\lambda)| = (p + 1)(p + 2)(p + 3).$$

3. Let us show how the obtained controller (61) can be verified in a different way. Note that (also see the chain of relations (33))

$$B(\lambda)u(t) = \begin{bmatrix} 0 & 0 & -\lambda + \lambda^2 \\ 0 & 0 & 0 \\ 0 & \lambda^2 & 0 \end{bmatrix} Y(t) = \begin{bmatrix} (-\lambda + \lambda^2)y_{12}(t) \\ 0 \\ \lambda^2 y_{11}(t) \end{bmatrix}. \tag{63}$$

From the third relation in (61), we have $y_{12}(t) = x_1(t) + y_{12}(t - h)$, which implies that $-\lambda(1-\lambda)y_{12}(t) = -\lambda x_1(t)$. By this relation, we preserve the first component of the vector (63), then add the remaining relations in the controller (61), and arrange the variables in the required order $\mathrm{col}\,[x, y, x_1, y_{11}]$. As a result, the characteristic matrix $\mathcal{W}_{11}(p,\lambda)$ coincides with the matrix (62).

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