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Positivity and stability analysis for linear implicit difference delay equations

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ABSTRACT

This paper deals with positivity and stability of linear implicit difference delay equations. Being different from the Lyapunov function approach commonly used in stability analysis, the method employed in this paper gives a way to solve the exponential stability of linear implicit difference equations with time-varying delay. By decomposition state-space and mathematical induction method, new necessary and sufficient conditions for positivity and stability of such systems are derived. Numerical examples are given to illustrate the proposed results.

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1. Introduction

During the past few decades, a considerable amount of research has been done in the field of stability of singular systems with delays [1–6]. Compared with the regular systems, the stability problem of singular systems, in particular, of positive singular systems, is much more complicated since regularity and absence of impulses are necessary to be considered simultaneously. Positive systems, whose state variables naturally take nonnegative values, have received considerable attention because of their engineering insight in communication systems, formation flying, and systems theory [7,8]. In positive models, the variables represent concentrations, population numbers of bacteria or cells or, in general, measures that are nonnegative. Solutions of singular positive systems cannot be solved via the well-established methods for general linear systems. The main reason is that the states of singular positive systems are defined on cones rather than in the whole space. Although many fundamental issues have been well investigated for a class of positive linear systems [9–13], they have not been sufficiently investigated for positive linear implicit difference delay equations. The stability analysis of positive linear implicit difference systems have been considered in [14–18], however, for the system either without delays or with constant delays. In these references, the stability results were obtained by using Lyapunov functional method and the conditions are presented in terms of solutions of some linear matrix inequalities (LMIs).

In this paper, we present some generalizations of exponential stability for the case of positive linear implicit difference systems with time-varying delay. The main contribution of this paper lies in two aspects. First, a novel lemma is proved which not only identifies some specific properties of the decomposed systems, but also enables us to derive new criteria for positivity and stability of the system with interval time-varying delay without using Lyapunov function method. Second, based on the singular value decomposition and mathematical induction approach, delay-dependent necessary and sufficient conditions for the positivity and exponential stability of the systems are derived. Also, numerical examples are presented to illustrate the effectiveness of our results.

The paper is organized as follows. In Section 2, necessary preliminaries are presented and some lemmas are provided. Section 3 proposes necessary and sufficient conditions for the positivity of linear singular difference positive systems with time-varying delay. In Section 4, the corresponding problem is treated for such systems with necessary and sufficient conditions for the exponential stability. Numerical examples are given in these sections to show the effectiveness of the proposed results.

2. Preliminaries

The following notation will be used in this paper. R^n denotes the Euclidean n-dimensional space with the vector norm $\|x\|$; $R^n_{0,+}$ denotes the space of all nonnegative vectors in R^n; $R^{m\times n}$ denotes the set of all real $(m \times n)$ matrices; $\mathbb{N}$ denotes the set of nonnegative integers. If $x = (x_1, x_2, ..., x_n)^T \in R^n$ and $B = (b_{ij}) \in R^{l\times q}$, we define

$|x| = (|x_1|, |x_2|, ..., |x_n|)^T$ and $|B| = (|b_{ij}|)_{l\times q}$, respectively. A vector $x \in R^n$ is called nonnegative if all entries are nonnegative. A vector $x \in R^n$ is called positive ($x \succ 0$) if $x_i > 0, i = 1, 2, ..., n$. I_n is the n-dimensional identity matrix, 0_n is the n-dimensional zero matrix. A matrix is called a monomial matrix if its every row and its every column contains only one positive entry and the remaining entries are zero. A matrix $B \in R^{m\times n}$ is called nonnegative ($B \succeq 0$) if all its entries are nonnegative. A matrix B is called positive ($B \succ 0$) if all its entries are positive. The notation $A \succeq B\,(A \succ B)$ means that $A - B \succeq 0\,(A - B \succ 0)$.

Consider the following linear implicit difference delay system:

$$\begin{cases} Ex(k+1) = A_0x(k) + A_1x(k-h(k)), \quad k \in \mathbb{N}, \\ x(k) = \varphi(k), \quad k \in \{-\tau,\, -(\tau-1),\, ...,\, 0\}, \end{cases} \tag{2.1}$$

where $x(k) \in R^n, k \in \mathbb{N}$ is the state, $A_0, A_1 \in R^{n\times n}$, the matrix $E \in R^{n\times n}$ is singular and assume that rank $E = r < n$; $h(k) \in \mathbb{N}$ is the delay function satisfying $0 < h(k) \leq \tau$; $k,\, \tau \in \mathbb{N}$; $\varphi(\cdot) : \{-\tau, \cdots, 0\} \to \mathbb{R}^n$ is the vector-valued initial function with the norm $\|\varphi\| = \max\limits_{k\in\{-\tau,-(\tau-1),...,0\}} \|\varphi(k)\|$.

Definition 2.1. ([7]). System (2.1) is said to be positive if for any initial condition $\varphi(\cdot) \succeq 0$ the corresponding solution $x(k; \varphi) \succeq 0$ for all $k \in \mathbb{N}$.

Definition 2.2. ([2]). (i) System (2.1) is said to be regular if the characteristic polynomial $det(zE - A_0)$ is not identically zero. (ii) System (2.1) is said to be causal if $deg(det(zE - A_0)) = rank(E) = r$.

Remark 2.1. Note that the definition of causality adopted from [2] imposes a stronger requirement than causality in the usual system theoretic meaning. As we shall see, the assumptions of causality in the sense of [2] and regularity of (E, A_0) imply that the index is one.

Definition 2.3. The implicit difference system (2.1) is said to be exponentially stable if there exist numbers $M > 0, \alpha \in (0, 1)$ such that for any initial condition $\varphi(\cdot)$ the solution $x(k;\, \varphi)$ satisfies

$$\|x(k; \varphi)\| \leq M\|\varphi\|\alpha^k, \quad \forall k \in \mathbb{N}.$$

We introduce the following technical propositions, which will be used in the proof of the main results.

Lemma 2.1. ([19]). *Let M, N be matrices with appropriate dimensions. The following statements are equivalent:*

(i) $Mx \succeq 0$ implies that $Nx \succeq 0$.

(ii) There exists $H \succeq 0$ such that $N = HM$.

As shown in [2], the regularity and causality of the system (2.1) imply that there exist two invertible matrices P, Q such that

$$PEQ = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}, \quad PA_0Q = \begin{pmatrix} A_{01} & 0 \\ 0 & I_{n-r} \end{pmatrix}, \quad PA_1Q = \begin{pmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{pmatrix}.$$

By simple computation, the solution of systems (2.1) is given by

$$\begin{cases} x(k) = \overline{A}_{01}^k P_1 x(0) + \sum\limits_{i=0}^{k-1} \overline{A}_{01}^{k-1-i} \bar{A}_1 x(i-h(i)) + \bar{A}_2 x(k-h(k)), \\ x(k) = \varphi(k), \quad k \in \{-\tau, -(\tau-1), ..., 0\}, \end{cases} \tag{2.2}$$

where

$$\overline{A}_{01} = Q \begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}, \, P_1 = Q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} Q^{-1},$$

$$\bar{A}_1 = Q \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} Q^{-1}, \quad \bar{A}_2 = Q \begin{bmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{bmatrix} Q^{-1}.$$

Lemma 2.2. *Assume that the system (2.1) is regular and causal. Let $x(k;\varphi)$ be a solution of system (2.1). We have the following properties*

(i) $P_1 x(k;\varphi) = x(k;\varphi) - \bar{A}_2 x(k-h(k);\varphi)$, $\forall k \in \mathbb{N}$.

(ii) $x(k+1;\varphi) = \overline{A}_{01} x(k;\varphi) + \bar{A}_1 x(k-h(k);\varphi) + \bar{A}_2 x(k+1-h(k+1);\varphi)$, $k \in \mathbb{N}$.

(iii) $x(k;\alpha\varphi) = \alpha x(k;\varphi)$, $\forall \alpha > 0, k \in \mathbb{N}$.

Proof. (i) We shall prove (i) by mathematical induction. For the sake of simplicity, we denote $x(k) := x(k;\varphi)$. For $k = 0$, we prove that $P_1 x(0) = x(0) - \bar{A}_2 x(-h(0))$. By using the transformation $y(k) = Q^{-1}x(k) = [y_1(k), y_2(k)]$ where $y_1(k) \in R^r$, $y_2(k) \in R^{n-r}$, the system (2.1) is reduced to two subsystems:

$$\begin{cases} y_1(k+1) = A_{01} y_1(k) + A_{11} y_1(k-h(k)) + A_{12} y_2(k-h(k)), \\ y_2(k) = -A_{13} y_1(k-h(k)) - A_{14} y_2(k-h(k)), \end{cases} \tag{2.3}$$

so, the solution of (2.3) is described as

$$\begin{cases} y_1(k) = A_{01}^k \, y_1(0) + \sum\limits_{i=0}^{k-1} A_{01}^{k-1-i} \big[A_{11} \, y_1(i-h(i)) + A_{12} \, y_2(i-h(i))\big], \\ y_2(k) = -A_{13} y_1(k-h(k)) - A_{14} y_2(k-h(k)). \end{cases} \tag{2.4}$$

Therefore, we have

$$y(0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} y(0) + \begin{pmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{pmatrix} y(-h(0)). \tag{2.5}$$

From $y(k) = Q^{-1}x(k)$ and (2.5) it follows that

$$Q^{-1}x(0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}x(0) + \begin{pmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{pmatrix} Q^{-1}x(-h(0)).$$

Hence,

$$\begin{aligned} x(0) &= Q \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}x(0) + Q \begin{pmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{pmatrix} Q^{-1}x(-h(0)) \\ &= P_1 x(0) + \bar{A}_2 x(-h(0)). \end{aligned}$$

Assume that (i) holds for all $l \leq k$:

$$P_1 x(l) = x(l) - \bar{A}_2 x(l - h(l)), \quad \forall l \leq k, \tag{2.6}$$

we shall prove that it holds for $k+1$: $P_1 x(k+1) = x(k+1) - \bar{A}_2 x(k+1-h(k+1))$. Indeed, using (2.2) gives

$$\begin{aligned} x(k+1) &= \overline{A}_{01}^{k+1} P_1 x(0) + \sum_{i=0}^{k} \overline{A}_{01}^{k-i} \bar{A}_1 x(i - h(i)) + \bar{A}_2 x(k+1-h(k+1)) \\ &= \overline{A}_{01}^{k+1} P_1 x(0) + \sum_{i=0}^{k-1} \overline{A}_{01}^{k-i} \bar{A}_1 x(i - h(i)) \\ &\quad + \bar{A}_1 x(k - h(k)) + \bar{A}_2 x(k+1-h(k+1)) \\ &= \overline{A}_{01} \Big(\overline{A}_{01}^{k} P_1 x(0) + \sum_{i=0}^{k-1} \overline{A}_{01}^{k-1-i} \bar{A}_1 x(i - h(i)) \Big) \\ &\quad + \bar{A}_1 x(k - h(k)) + \bar{A}_2 x(k+1-h(k+1)). \end{aligned} \tag{2.7}$$

Since

$$\overline{A}_{01}^{k} P_1 x(0) + \sum_{i=0}^{k-1} \overline{A}_{01}^{k-1-i} \bar{A}_1 x(i - h(i)) = x(k) - \bar{A}_2 x(k - h(k)), \tag{2.8}$$

it follows from (2.7) and (2.8) that

$$x(k+1) = \overline{A}_{01}\big(x(k) - \bar{A}_2 x(k - h(k))\big) + \bar{A}_1 x(k - h(k)) + \bar{A}_2 x(k+1-h(k+1)). \tag{2.9}$$

Using the inductive assumption and (2.9) gives

$$x(k+1) = \overline{A}_{01} P_1 x(k) + \bar{A}_1 x(k - h(k)) + \bar{A}_2 x(k+1-h(k+1)), \tag{2.10}$$

and hence

$$P_1x(k+1) = P_1\overline{A}_{01}P_1x(k) + P_1\bar{A}_1x(k-h(k)) + P_1\bar{A}_2x(k+1-h(k+1)). \tag{2.11}$$

On the other hand, note that

$$\begin{aligned} P_1\overline{A}_{01} &= Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}Q\begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1} = \overline{A}_{01}, \\ P_1\bar{A}_1 &= Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}Q\begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}Q^{-1} = \bar{A}_1, \\ P_1\bar{A}_2 &= Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}Q\begin{bmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{bmatrix}Q^{-1} = 0. \end{aligned}$$

Hence, from (2.11) we have

$$\begin{aligned} P_1x(k+1) =& \overline{A}_{01}P_1x(k) + \bar{A}_1x(k-h(k)) \\ &= \overline{A}_{01}P_1x(k) + \bar{A}_1x(k-h(k)) + \bar{A}_2x(k+1-h(k+1)) \\ &- \bar{A}_2x(k+1-h(k+1)). \end{aligned} \tag{2.12}$$

Therefore, taking (2.10) and (2.12) into account we get

$$\begin{aligned} P_1x(k+1) &= \Big(\overline{A}_{01}P_1x(k) + \bar{A}_1x(k-h(k)) + \bar{A}_2x(k+1-h(k+1))\Big) \\ &\quad - \bar{A}_2x(k+1-h(k+1)) \\ &= x(k+1) - \bar{A}_2x(k+1-h(k+1)), \end{aligned}$$

which proves the condition (i) for $k+1$.
(ii) By using (2.2), we have

$$\begin{aligned} x(k+1;\varphi) &= \overline{A}_{01}^{k+1}P_1x(0;\varphi) + \sum_{i=0}^{k}\overline{A}_{01}^{k-i}\bar{A}_1x(i-h(i);\varphi) + \bar{A}_2x(k+1-h(k+1);\varphi) \\ &= \overline{A}_{01}^{k+1}P_1x(0;\varphi) + \sum_{i=0}^{k-1}\overline{A}_{01}^{k-i}\bar{A}_1x(i-h(i);\varphi) + \bar{A}_1x(k-h(k);\varphi) \\ &\quad + \bar{A}_2x(k+1-h(k+1);\varphi) \\ &= \overline{A}_{01}\Big(\overline{A}_{01}^{k}P_1x(0;\varphi) + \sum_{i=0}^{k-1}\overline{A}_{01}^{k-1-i}\bar{A}_1x(i-h(i);\varphi)\Big) + \bar{A}_1x(k-h(k);\varphi) \\ &\quad + \bar{A}_2x(k+1-h(k+1);\varphi). \end{aligned}$$

Moreover, note that $\overline{A}_{01}\bar{A}_2 = Q\begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}Q\begin{bmatrix} 0 & 0 \\ -A_{13} & -A_{14} \end{bmatrix} Q^{-1} = 0$, from which it follows that

$$\begin{aligned} x(k+1;\varphi) =& \overline{A}_{01}\Big(\overline{A}_{01}^{k} P_1 x(0;\varphi) + \sum_{i=0}^{k-1} \overline{A}_{01}^{k-1-i} \bar{A}_1 x(i-h(i);\varphi) + \bar{A}_2 x(k-h(k);\varphi)\Big) \\ &+ \bar{A}_1 x(k-h(k);\varphi) + \bar{A}_2 x(k+1-h(k+1);\varphi) \\ =& \overline{A}_{01} x(k;\varphi) + \bar{A}_1 x(k-h(k);\varphi) + \bar{A}_2 x(k+1-h(k+1);\varphi). \end{aligned}$$

(iii) We shall prove (iii) by mathematical induction. For $k = 1$, from (2.2) we have

$$\begin{aligned} x(1;\alpha\varphi) &= \overline{A}_{01} P_1 x(0;\alpha\varphi) + \bar{A}_1 x(-h(0);\alpha\varphi) + \bar{A}_2 x(1-h(1);\alpha\varphi) \\ &= \overline{A}_{01} P_1 \alpha\varphi(0) + \bar{A}_1 \alpha\varphi(-h(0)) + \bar{A}_2 \alpha\varphi(1-h(1)) \\ &= \alpha\Big(\overline{A}_{01} P_1 \varphi(0) + \bar{A}_1 \varphi(-h(0)) + \bar{A}_2 \varphi(1-h(1))\Big) \\ &= \alpha x(1;\varphi). \end{aligned}$$

Assume that (iii) holds for all $l \le k$:

$$x(l;\alpha\varphi) = \alpha x(l;\varphi), \ \forall l \le k, \tag{2.13}$$

we shall prove (iii) holds for $k + 1$. Indeed, from (ii) and (2.13) it follows that

$$\begin{aligned} x(k+1;\alpha\varphi) &= \overline{A}_{01} x(k;\alpha\varphi) + \bar{A}_1 x(k-h(k);\alpha\varphi) + \bar{A}_2 x(k+1-h(k+1);\alpha\varphi) \\ &= \alpha\Big(\overline{A}_{01} x(k;\varphi) + \bar{A}_1 x(k-h(k);\varphi) + \bar{A}_2 x(k+1-h(k+1);\varphi)\Big) \\ &= \alpha x(k+1;\varphi), \end{aligned}$$

which proves (iii) for $k + 1$. □

3. Positivity

In this section, we present necessary and sufficient conditions for the positivity of system (2.1).

Theorem 3.1. *Assume that the system (2.1) is regular and causal. The following statements are equivalent.*

(i) System (2.1) is positive.

(ii) $\bar{A}_2 \succeq 0$, *there exist matrices* $H_1 \succeq 0$, $H_2 \succeq 0$ *such that:*

$$\overline{A}_{01} = H_1 P_1; \quad \bar{A}_1 = H_1 \bar{A}_2 + H_2.$$

Proof. *Necessity.* Suppose that the system (2.1) is positive. Using (i) of Lemma 2.2 for the solution at $k = 0, 1$, we have

$$\begin{aligned} x(0) &= P_1 x(0) + \bar{A}_2 x(-h(0)) = P_1\varphi(0) + \bar{A}_2\varphi(-h(0)) \succeq 0, \\ x(1) &= \overline{A}_{01} x(0) + \bar{A}_1 x(-h(0)) + \bar{A}_2 x(1 - h(1)) \\ &= \overline{A}_{01}\varphi(0) + \bar{A}_1\varphi(-h(0)) + \bar{A}_2\varphi(1 - h(1)) \succeq 0, \end{aligned}$$

and hence

$$\begin{bmatrix} P_1 & \bar{A}_2 & 0_n \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \varphi(-h(0)) \\ \varphi(1-h(1)) \end{bmatrix} \succeq 0, \;\; k \in \{-\tau, -(\tau-1), ..., 0\},$$
$$\begin{bmatrix} \overline{A}_{01} & \bar{A}_1 & \bar{A}_2 \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \varphi(-h(0)) \\ \varphi(1-h(1)) \end{bmatrix} \succeq 0, \;\; k \in \{-\tau, -(\tau-1), ..., 0\}.$$

Applying the property (i) of Lemma 2.1 with

$$M = \begin{bmatrix} P_1 & \bar{A}_2 & 0_n \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix}, \quad N = \begin{bmatrix} \overline{A}_{01} & \bar{A}_1 & \bar{A}_2 \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix}, \quad x = \begin{bmatrix} \varphi(0) \\ \varphi(-h(0)) \\ \varphi(1-h(1)) \end{bmatrix},$$

there is a matrix $\mathcal{H}$:

$$\mathcal{H} = \begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \\ H_7 & H_8 & H_9 \end{bmatrix} \succeq 0,$$

such that

$$\begin{bmatrix} \overline{A}_{01} & \bar{A}_1 & \bar{A}_2 \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix} = \begin{bmatrix} H_1 & H_2 & H_3 \\ H_4 & H_5 & H_6 \\ H_7 & H_8 & H_9 \end{bmatrix} \begin{bmatrix} P_1 & \bar{A}_2 & 0_n \\ 0_n & I_n & 0_n \\ 0_n & 0_n & I_n \end{bmatrix}.$$

Therefore, the non-negativity of $\mathcal{H}$ implies $\bar{A}_2 = H_3 \succeq 0, H_1 \succeq 0, H_2 \succeq 0$ and

$$\overline{A}_{01} = H_1 P_1; \quad \bar{A}_1 = H_1\bar{A}_2 + H_2.$$

Sufficiency. Assume that $\bar{A}_2 \succeq 0$, and there exist matrices $H_1 \succeq 0, H_2 \succeq 0$ satisfying $\overline{A}_{01} = H_1 P_1, \bar{A}_1 = H_1\bar{A}_2 + H_2$. Using mathematical induction method, we show that the system (2.1) is positive, i.e. for any initial condition $\varphi(k) \succeq 0$, $k \in \{-\tau, -(\tau-1), ..., 0\}$ we have: $x(k) \succeq 0$, for all $k \in \mathbb{N}$. Indeed, for $k = 1$, we have

$$
\begin{aligned}
x(1) &= \overline{A}_{01}x(0) + \bar{A}_1 x(-h(0)) + \bar{A}_2 x(1-h(1)) \\
&= H_1 P_1 x(0) + (H_1\bar{A}_2 + H_2)x(-h(0)) + \bar{A}_2 x(1-h(1)) \\
&= H_1[\,P_1 x(0) + \bar{A}_2 x(-h(0))\,] + H_2 x(-h(0)) + \bar{A}_2 x(1-h(1)).
\end{aligned}
$$

Using condition (i) of Lemma 2.2 gives

$$x(1) = H_1 x(0) + H_2\varphi(-h(0)) + \bar{A}_2\varphi(1-h(1)) \succeq 0.$$

Assuming that $x(l) \succeq 0$, $\forall l \le k$, we show $x(k+1) \succeq 0$. From (ii) of Lemma 2.2 it follows that

$$
\begin{aligned}
x(k+1) &= \overline{A}_{01}x(k) + \bar{A}_1 x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1)) \\
&= H_1 P_1 x(k) + (H_1\bar{A}_2 + H_2)x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1)).
\end{aligned}
$$

By (i) of Lemma 2.2, we have

$$
\begin{aligned}
x(k+1) = H_1 &\Big(x(k) - \bar{A}_2 x(k-h(k))\Big) + (H_1\bar{A}_2 + H_2)x(k-h(k)) \\
&+ \bar{A}_2 x(k+1-h(k+1)) \\
&= H_1 x(k) + H_2 x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1)).
\end{aligned}
$$

Using the inductive hypothesis: $x(k) \succeq 0$, $x(k-h(k)) \succeq 0$, $x(k+1-h(k+1)) \succeq 0$ and $\bar{A}_2 \succeq 0$, $H_1 \succeq 0$, $H_2 \succeq 0$, we obtain

$$x(k+1) = H_1 x(k) + H_2 x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1)) \succeq 0,$$

which completes the proof. □

Remark 3.1. It is worth noting that Theorem 3.1 extends the result of [17] for the case of system without delays. Indeed, the linear implicit difference system $Ex(k+1) = A_0 x(k)$ is, as shown in [17, Theorem 3.3], positive if and only if there exists a matrix H such that $H \succeq 0$, $\hat{E}^D\hat{A}_0 = H\hat{E}^D\hat{E}$, where the matrices $\hat{A}_0$ and $\hat{E}$ are given by $\hat{E} = (\lambda E - A_0)^{-1} E$, $\hat{A}_0 = (\lambda E - A_0)^{-1} A_0$, λ is any complex number: $(\lambda E - A_0)^{-1}$ and $\hat{E}^D$ is the Drazin inverse of $\hat{E}$. In the case, (E, A_0) is regular and causal, we have:

$$\hat{E} = Q\begin{bmatrix} (\lambda I_r - A_{01})^{-1} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}, \quad \hat{A}_0 = Q\begin{bmatrix} (\lambda I_r - A_{01})^{-1}A_{01} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}.$$

The Drazin inverse of $\hat{E}$ is $\hat{E}^D = Q\begin{bmatrix} \lambda I_r - A_{01} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$. Therefore, as similarly shown in [18] we have $\hat{E}^D\hat{E} = Q\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$, $\hat{E}^D\hat{A}_0 = Q\begin{bmatrix} A_{01} & 0 \\ 0 & 0 \end{bmatrix} Q^{-1}$. Thus, the condition $H \succeq 0$, $\hat{E}^D\hat{A}_0 = H\hat{E}^D\hat{E}$ is equivalent the conditions (ii) of Theorem 3.1 with $A_1 = 0$.

Remark 3.2. If Q is monomial matrix then the condition (ii) of Theorem 3.1 is equivalent to the condition: $A_{01} \succeq 0$, $A_{11} \succeq 0$, $A_{12} \succeq 0$, $A_{13} \preceq 0$, $A_{14} \preceq 0$. Therefore, for the case when the system is undelayed ($A_1 = 0$,) the result of [20, Proposition 2] is derived from our Theorem 3.1.

Example 3.1. Consider system (2.1) with the following parameters

$$E = \begin{bmatrix} -0.25 & 0.25 \\ 0.25 & -0.25 \end{bmatrix}, A_0 = \begin{bmatrix} -0.5 & 0.5 \\ 1.75 & -0.75 \end{bmatrix}, A_1 = \begin{bmatrix} 2 & 6 \\ -3 & -6 \end{bmatrix}.$$

Direct calculation shows that $\det(sE-A_0) = \frac{1}{4}s-\frac{1}{2} \neq 0$ for some $s \in R$ and $\deg(\det(sE-A_0)) = rank(E) = 1$. Then the system is regular and causal. Moreover, there are two nonsingular matrices

$$P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 5 & 1 \end{bmatrix},$$

such that E, A_0, A_1 are partitioned accordingly as

$$PEQ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, PA_0Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, PA_1Q = \begin{bmatrix} 32 & 8 \\ -1 & -1 \end{bmatrix}.$$

Thus we have

$$\overline{A}_{01} = \begin{bmatrix} -0.5 & 0.5 \\ -2.5 & 2.5 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 2 & 6 \\ 10 & 30 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} -0.25 & 0.25 \\ -1.25 & 1.25 \end{bmatrix},$$

and the matrices H_1, H_2 of Theorem 3.1 are defined by

$$H_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, H_2 = \begin{bmatrix} 0 & 6 \\ 8 & 30 \end{bmatrix}.$$

Therefore, by Theorem 3.1 the system is positive.

4. Exponential stability

This section proposes necessary and sufficient conditions for exponential stability of system (2.1).

Theorem 4.1. *Assume that (E, A_0) is regular and causal. Let $\overline{A}_{01}$, $\bar{A}_1$, $\bar{A}_2$, P_1 be defined in (2.2). Then, the following statements are equivalent.*

(i) The system (2.1) is positive and exponentially stable.

(ii) $\bar{A}_2 \succeq 0$, *and there exist matrices* $H_1 \succeq 0$, $H_2 \succeq 0$, *a vector* $p \succ 0$ *and a number* $\delta \in (0, 1)$ *such that*

$$\overline{A}_{01} = H_1 P_1, \quad \bar{A}_1 = H_1 \bar{A}_2 + H_2. \tag{4.1}$$

$$\big(H_1 + H_2 + \bar{A}_2\big) p \preceq \delta\, p. \tag{4.2}$$

Proof. $(i) \Rightarrow (ii)$: Assume that the system (2.1) is positive and exponentially stable. Using Theorem 3.1, we have $\bar{A}_2 \succeq 0$ and there exist matrices $H_1 \succeq 0$, $H_2 \succeq 0$ satisfying the condition (4.1). Since the system (2.1) is exponentially stable for the delay $0 < h(k) \leq \tau$, then as shown in [21], the system

$$\begin{cases} Ex(k+1) = A_0 x(k) + A_1 x(k-\tau), \quad k \in \mathbb{N}, \\ x(k) = \varphi(k), \quad k \in \{-\tau, -(\tau-1), ..., 0\}. \end{cases} \tag{4.3}$$

is also exponentially stable. From (ii) of Lemma 2.2 it follows that

$$\begin{aligned} x(k+1) =& \overline{A}_{01} x(k) + \bar{A}_1 x(k-\tau) + \bar{A}_2 x(k-(\tau-1)),\ k \in \mathbb{N}, \\ x(k) =& \varphi(k), \quad k \in \{-\tau, -(\tau-1), ..., 0\}, \end{aligned}$$

and hence

$$x(k+1) = H_1 P_1 x(k) + \big(H_1 \bar{A}_2 + H_2\big) x(k-\tau) + \bar{A}_2 x(k-(\tau-1)).$$

Using the property (i) of Lemma 2.2 gives

$$\begin{aligned} x(k+1) &= H_1 P_1 x(k) + \big(H_1 \bar{A}_2 + H_2\big) x(k-\tau) + \bar{A}_2 x(k-(\tau-1)) \\ &= H_1 \big(x(k) - \bar{A}_2 x(k-\tau)\big) + \big(H_1 \bar{A}_2 + H_2\big) x(k-\tau) + \bar{A}_2 x(k-(\tau-1)) \\ &= H_1 x(k) + H_2 x(k-\tau) + \bar{A}_2 x(k-(\tau-1)). \end{aligned}$$

Then

$$\begin{aligned} x(k) - x(0) &= (x(k) - x(k-1)) + (x(k-1) - x(k-2)) + ... + (x(1) - x(0)) \\ &= (H_1 - I_n) x(k-1) + H_2 x(k-1-\tau) + \bar{A}_2 x(k-1-(\tau-1)) + ... \\ &\quad ... + (H_1 - I_n) x(0) + H_2 x(-\tau) + \bar{A}_2 x(-(\tau-1)) \\ &= (H_1 - I_n) \sum_{i=0}^{k-1} x(i) + H_2 \sum_{i=0}^{k-1} x(i-\tau) + \bar{A}_2 \sum_{i=0}^{k-1} x(i-(\tau-1)). \end{aligned} \tag{4.4}$$

Applying the identity $x(i-\tau) = x(i) + x(i-\tau) - x(i)$ and $x(i-(\tau-1)) = x(i) + x(i-(\tau-1)) - x(i)$ into (4.4) implies

$$\begin{aligned} x(k)-x(0)=&(H_1-I_n)\sum_{i=0}^{k-1}x(i)+H_2\sum_{i=0}^{k-1}\big(x(i)+x(i-\tau)-x(i)\big)\\ &+\bar{A}_2\sum_{i=0}^{k-1}\big(x(i)+x(i-(\tau-1))-x(i)\big)\\ =&\Big(H_1+H_2+\bar{A}_2-I_n\Big)\sum_{i=0}^{k-1}x(i)+H_2\sum_{i=0}^{k-1}\big(x(i-\tau)-x(i)\big)\\ &+\bar{A}_2\sum_{i=0}^{k-1}\big(x(i-(\tau-1))-x(i)\big). \end{aligned}$$

As the system (4.3) is exponentially stable, we have $\|x(k)\| \to 0$ when $k \to \infty$, and since $x(k) \succeq 0$, $\forall k \in \mathbb{N}$, we have $x(k) \to 0$ for $k \to \infty$. On the other hand, due to $\sum_{i=0}^{\infty}\|x(i)\| \leq \sum_{i=0}^{\infty} N\alpha^k < \infty$, $\alpha \in (0,1)$, we have $\sum_{i=0}^{\infty} x(i) \prec \infty$. Hence

$$\begin{aligned} -x(0)=&\Big(H_1+H_2+\bar{A}_2-I_n\Big)\sum_{i=0}^{\infty}x(i)+H_2\sum_{i=0}^{\infty}\big(x(i-\tau)-x(i)\big)\\ &+\bar{A}_2\sum_{i=0}^{\infty}\big(x(i-(\tau-1))-x(i)\big)\\ =&\Big(H_1+H_2+\bar{A}_2-I_n\Big)\sum_{i=0}^{\infty}x(i)+H_2\sum_{i=-\tau}^{-1}x(i)+\bar{A}_2\sum_{i=1-\tau}^{-1}x(i), \end{aligned}$$

which gives

$$\Big(H_1+H_2+\bar{A}_2-I_n\Big)\sum_{i=0}^{\infty}x(i)=-x(0)-H_2\sum_{i=-\tau}^{-1}x(i)-\bar{A}_2\sum_{i=1-\tau}^{-1}x(i)\prec 0. \tag{4.5}$$

Taking the initial condition $x(k) = \varphi(k) \succ 0$, $\forall k \in \{-\tau, -(\tau-1), ..., 0\}$. Since system (2.1) is positive, we have $x(k) \succeq 0$, $\forall k \in \mathbb{N}$. On the other hand, we have $x(0) = \varphi(0) \succ 0$, $\sum_{i=-\tau}^{-1} x(i) = \sum_{i=-\tau}^{-1} \varphi(i) \succ 0$, $\sum_{i=1-\tau}^{-1} x(i) = \sum_{i=1-\tau}^{-1} \varphi(i) \succ 0$. We shall prove that $\sum_{i=0}^{\infty} x(i) \succ 0$. Indeed, suppose that there exists an index j, $j \in \{1, 2, ..., n\}$ such that $\sum_{i=0}^{\infty} x_j(i) = 0$. As $x(k) \succeq 0$, $\forall k \in \mathbb{N}$, we have $\sum_{i=0}^{\infty} x_j(i) = 0$ if and only if $x_j(i) = 0$, $\forall i \in \mathbb{N}$, hence $x_j(0) = 0$ which contradicts the fact that $x(0) \succ 0$. Therefore, we can define $p := \sum_{i=0}^{\infty} x(i) = (p_1, p_2, ..., p_n) \succ 0$ and hence $q := (H_1 + H_2 + \bar{A}_2)p = (q_1, q_2, ..., q_n) \succeq 0$. If $q_i = 0, \forall i = 1, 2, ..., n$, then (4.2) holds for any $\delta \in (0, 1)$. If $q \neq 0$, then we set $\delta := \max_{i=1,2,...,n} \left\{\frac{q_i}{p_i}\right\}$ so that the inequality (4.5) implies $q \prec p$ (i.e. $q_i < p_i$, $\forall i = 1, 2, ..., n$,) hence $\delta := \max_{i=1,2,\ldots,n} \left\{\frac{q_i}{p_i}\right\} \in (0,\, 1)$. Moreover, since $q_i \preceq \delta p_i$, for all $i = 1, 2, ..., n$, we obtain $q \preceq \delta p$, which gives the condition (4.2).

$(ii) \Rightarrow (i)$: Let the conditions (4.1), (4.2) be satisfied. By Theorem 3.1, the system (2.1) is positive. Consider solution $x(k;\varphi)$ of system (2.1) with the initial condition $\varphi \in S_1 := \{\varphi : \|\varphi\| = 1\}$. We first show that

$$\|x(k,\varphi)\| \leq M\alpha^k,\ \forall k \in \mathbb{N},\ \forall \varphi \in S_1. \tag{4.6}$$

Since $p \succ 0$, there is a positive number $K > 1$ such that

$$Kp \succ |\varphi(j)|,\ \forall j \in \{-\tau, -(\tau-1), ..., 0\},\ \forall \varphi \in S_1.$$

Define $u(k) := K\alpha^k p$, $k \in \mathbb{Z}$, where $\alpha := \sqrt[\tau+1]{\delta}$. In order to derive (4.6), where $M = K\|p\|$, it suffices to prove that

$$u(k) \succ |x(k)|, \quad k \in \mathbb{Z}. \tag{4.7}$$

Since $x(k) = \varphi(k)$, $k \in \{-\tau, -(\tau-1), ..., 0\}$ it is obvious that

$$u(k) \succ |x(k)|,\ k \in \{-\tau, -(\tau-1), ..., 0\}.$$

The system (2.1) is positive and hence by Theorem 3.1, there exist matrices $H_1 \succeq 0$, $H_2 \succeq 0$ satisfying $\overline{A}_{01} = H_1 P_1$, $\bar{A}_1 = H_1\bar{A}_2 + H_2$. For $k = 1$, we have

$$\begin{aligned}
|x(1)| &= \left|\overline{A}_{01}x(0) + \bar{A}_1 x(-h(0)) + \bar{A}_2 x(1-h(1))\right| \\
&= \left|H_1P_1x(0) + (H_1\bar{A}_2 + H_2)x(-h(0)) + \bar{A}_2x(1-h(1))\right| .
\end{aligned}$$

Taking the property (i) of Lemma 2.2 into account, we have

$$\begin{aligned}
|x(1)| &= \left|H_1\big(x(0) - \bar{A}_2x(-h(0))\big) + (H_1\bar{A}_2 + H_2)x(-h(0)) + \bar{A}_2x(1-h(1))\right| \\
&= \left|H_1x(0) - H_1\bar{A}_2x(-h(0)) + (H_1\bar{A}_2 + H_2)x(-h(0)) + \bar{A}_2x(1-h(1))\right| \\
&= \left|H_1x(0) + H_2x(-h(0)) + \bar{A}_2x(1-h(1))\right| \\
&\preceq H_1\,|x(0)| + H_2\,|x(-h(0))| + \bar{A}_2\,|x(1-h(1))| \\
&\preceq H_1Kp + H_2Kp\,\alpha^{-h(0)} + \bar{A}_2Kp\,\alpha^{1-h(1)} \preceq H_1Kp + H_2Kp\,\alpha^{-\tau} + \bar{A}_2Kp\,\alpha^{-\tau} \\
&\preceq \Big(H_1 + H_2 + \bar{A}_2\Big)Kp\,\alpha^{-\tau} \preceq Kp\,\delta\,\alpha^{-\tau} \preceq K\alpha\,p = u(1),
\end{aligned}$$

which shows the condition (4.7) for $k = 1$. Assume that (4.7) holds for all $l \leq k$:

$$u(l) \succeq |x(l)|,\ \forall l \leq k.$$

From $(i), (ii)$ of Lemma 2.2 it follows

$$\begin{aligned}
|x(k+1)| &= \left|\overline{A}_{01}x(k) + \bar{A}_1 x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1))\right| \\
&= \left|H_1 P_1 x(k) + (H_1\bar{A}_2 + H_2)x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1))\right| \\
&= \big|H_1\big(x(k) - \bar{A}_2 x(k-h(k))\big) + (H_1\bar{A}_2 + H_2)x(k-h(k)) \\
&\quad + \bar{A}_2 x(k+1-h(k+1))\big| \\
&= \left|H_1 x(k) + H_2 x(k-h(k)) + \bar{A}_2 x(k+1-h(k+1))\right| \\
&\preceq H_1 |x(k)| + H_2 |x(k-h(k))| + \bar{A}_2 |x(k+1-h(k+1))| .
\end{aligned}$$

By the inductive hypothesis, we have

$$\begin{aligned}
|x(k+1)| &\preceq H_1 u(k) + H_2 u(k-h(k)) + \bar{A}_2 u(k+1-h(k+1)) \\
&= H_1 K\alpha^k p + H_2 K\alpha^{k-h(k)} p + \bar{A}_2 K\alpha^{k+1-h(k+1)} p \\
&\preceq K\alpha^k \Big(H_1 + H_2\alpha^{-\tau} + \bar{A}_2\alpha^{-\tau}\Big)p \preceq K\alpha^{-\tau}\,\alpha^k \Big(H_1 + H_2 + \bar{A}_2\Big)p \\
&\preceq K\alpha^{-\tau}\alpha^k\,\delta\, p = K\alpha^{k+1}p = u(k+1),
\end{aligned}$$

which shows (4.7) for $k+1$. Therefore, by induction we conclude that

$$\|x(k;\varphi)\| \leq K\|p\|\alpha^k = M\alpha^k, \quad \forall k \in \mathbb{N}, \quad \forall \varphi \in S_1.$$

We are now in position to prove exponential stability of system (2.1) as follows. Let $x(k,\varphi)$ be any solution of system (2.1) with any initial condition $\varphi(\cdot)$. From (iii) of Lemma 2.2 it follows that

$$\frac{1}{\|\varphi\|}\|x(k;\varphi)\| = \left\| x\left(k; \frac{\varphi}{\|\varphi\|}\right)\right\| \leq M\alpha^k, \, \forall k \in \mathbb{N}.$$

Therefore

$$\|x(k;\varphi)\| \leq M\|\varphi\|\alpha^k, \, \forall k \in \mathbb{N},$$

which completes the proof of the theorem. $\square$

Remark 4.1. Note that the condition (4.2) can be reduced to the condition $\big(H_1 + H_2 + \bar{A}_2\big)p \preceq \alpha^{\tau+1}p$, provided $\alpha \in (0,1)$. It is easy to see that if this condition is satisfied for one delay value τ, it is satisfied for every other delay value τ (when the delay changes accordingly by another $\alpha \in (0,1)$). Therefore, Theorem 4.1 gives the delay-independent conditions for the positivity and exponential stability of system (2.1).

Remark 4.2. The condition (4.2) in Theorem 4.1 involves bilinear matrix inequalities (BMI) with respect to H_i, p, δ, which can be solved by the branch and bound methods proposed in [22] or the homotopy-based algorithm in [23]. However, the conditions (4.1), (4.2) can be separately solved as a standard LP problem: since the condition (4.1) is independent on p, we first find the solutions H_i from linear matrix equation (4.1), and then matrix inequality (4.2) w.r.t. p, δ can be solved by LP problem by giving a fixed δ.

Example 4.1. Consider system (2.1) with the following parameters

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_0 = \begin{bmatrix} 0.25 & 0 & 0 \\ 0.2 & 0.55 & 0 \\ 0 & 0 & -1 \end{bmatrix}, A_1 = \begin{bmatrix} 0.3 & 0.25 & 0 \\ 0 & 0.1 & 0 \\ 0.2 & 0 & 0.2 \end{bmatrix}.$$

Direct calculation shows that $\det(sE - A_0) = s^2 - 0.8s - 0.1375 \neq 0$ for some $s \in R$ and $\deg(\det(sE - A_0)) = rank(E) = 2$. Then the system is regular and causal. Moreover, there are two nonsingular matrices

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

such that E, A_0, A_1 are partitioned accordingly as

$$PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, PA_0Q = \begin{bmatrix} 0.25 & 0 & 0 \\ 0.2 & 0.55 & 0 \\ 0 & 0 & 1 \end{bmatrix}, PA_1Q = \begin{bmatrix} 0.3 & 0.25 & 0 \\ 0 & 0.1 & 0 \\ 0.2 & 0 & -0.2 \end{bmatrix},$$

and hence

$$\overline{A}_{01} = \begin{bmatrix} 0.25 & 0 & 0 \\ 0.2 & 0.55 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0.3 & 0.25 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.2 & 0 & 0.2 \end{bmatrix}, P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The unknown variables p, δ, H_1, H_2 are defined by $p = (\frac{1}{10}, \frac{1}{10}, \frac{1}{10}) \succ 0$, $\delta = 0.9 < 1$ and

$$H_1 = \begin{bmatrix} 0.25 & 0 & 0 \\ 0.2 & 0.55 & 0 \\ 0 & 0 & 0 \end{bmatrix}, H_2 = \begin{bmatrix} 0.3 & 0.25 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Theorem 4.1 the system is positive and exponentially stable.

5. Conclusions

In this paper, we have investigated the problem of positivity and exponential stability of linear implicit difference equations with time-varying delay without using Lyapunov function method. Based on the singular value decomposition and mathematical induction approach, we have established new necessary and sufficient conditions for the positivity and exponential stability of such systems. Numerical examples are given to illustrate the proposed results.

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