Positivity and Stability of a Class of Fractional Descriptor Discrete-Time Nonlinear Systems

Tadeusz Kaczorek

Abstract A method of analysis of the fractional descriptor nonlinear discrete-time systems with regular pencils of linear part is proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencils. Necessary and sufficient conditions for the positivity of the nonlinear systems are established. A procedure for computing the solution to the equations describing the nonlinear systems are proposed. Using an extension of the Lyapunov method to positive nonlinear systems, sufficient conditions for the asymptotic stability are derived.

Keywords Fractional · Descriptor · Nonlinear · System · Weierstrass-Kronecker decomposition · Positivity · Lyapunov method · Stability

1 Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–17]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [4, 15, 18] and the minimum energy control of descriptor linear systems in [19–21]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [16]. The positive linear systems with different fractional orders have been addressed in [20]. Selected problems in theory of fractional linear systems has been given in monograph [13].

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [22]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

T. Kaczorek (✉)
Faculty of Electrical Engineering, Bialystok University of Technology, Bialystok, Poland
e-mail: kaczorek@isep.pw.edu.pl

S. Domek and P. Dworak (eds.), *Theoretical Developments and Applications of Non-Integer Order Systems*, Lecture Notes in Electrical Engineering 357,
DOI 10.1007/978-3-319-23039-9_12

Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [1–4, 13, 14, 23]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [24]. The stability of positive descriptor systems has been investigated in [17]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [11]. A new class of descriptor fractional linear discrete-time systems has been introduced in [12]. The standard and positive descriptor discrete-time nonlinear systems have been addressed in [10].

In this paper a method of analysis of the fractional descriptor standard and positive nonlinear discrete-time systems with regular pencils will be proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencil of the linear part of the equation describing the nonlinear system and on an extension of the Lyapunov method to positive nonlinear systems.

The paper is organized as follows. In Sect. 2 the Weierstrass-Kronecker decomposition is applied to analysis of the descriptor nonlinear systems. Necessary and sufficient conditions for the positivity of the nonlinear systems are established in Sect. 3. In Sect. 4 the stability of positive nonlinear systems by the use of extended Lyapunov method is analyzed. Concluding remarks are given in Sect. 5.

The following notation will be used: $\mathfrak{R}$—the set of real numbers, $\mathfrak{R}^{n\times m}$—the set of $n\times m$—real matrices, Z_+—the set of nonnegative integers, $\mathfrak{R}_+^{n\times m}$—the set of $n\times m$ matrices with nonnegative entries and $\mathfrak{R}_+^n = \mathfrak{R}_+^{n\times 1}$, I_n—the $n\times n$ identity matrix.

2 Fractional Descriptor Discrete-Time Nonlinear Systems and Their Solution

Consider the fractional descriptor discrete-time nonlinear system

$$E\Delta^\alpha x_{i+1} = Ax_i + f(x_i, u_i), i\in Z_+ = \{0, 1, \ldots\}, 0<\alpha<1 \tag{2.1a}$$

$$y_i = g(x_i, u_i), \tag{2.1b}$$

where $x_i\in\mathfrak{R}^n$, $u_i\in\mathfrak{R}^m$, $y_i\in\mathfrak{R}^p$, $i\in Z_+$ are the state, input and output vectors, $f(x_i,u_i)\in\mathfrak{R}^n$, $g(x_i,u_i)\in\mathfrak{R}^p$ are continuous and bounded vector functions of x_i and u_i satisfying the conditions $f(0,0)=0$, $g(0,0)=0$ and $E, A\in\mathfrak{R}^{n\times n}$ and

$$\Delta^\alpha x_i = \sum_{j=0}^{i} (-1)^j \binom{\alpha}{j} x_{i-j} \tag{2.1c}$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j=0 \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,\ldots \end{cases}. \tag{2.1d}$$

is the fractional $\alpha\in\mathfrak{R}$ order difference of x_i.

It is assumed that $\det E = 0$ and the

$$\det[Ez - A] \neq 0 \text{ for some } z \in \mathrm{C}(\text{the field of complex numbers}). \tag{2.2}$$

Substituting (2.1c) into (2.1a) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} c_j E x_{i-j+1} + f(x_i, u_i) \tag{2.3a}$$

where

$$A_\alpha = A + E\alpha, \quad c_j = (-1)^{j+1}\binom{\alpha}{j}. \tag{2.3b}$$

It is well-known [14] that if (2.2) holds then there exist nonsingular matrices $P, Q \in \Re^{n \times n}$ such that

$$P[Ez - A_\alpha]Q = \begin{bmatrix} I_{n_1} z - A_{1\alpha} & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}, A_{1\alpha} \in \Re^{n_1 \times n_1}, N \in \Re^{n_2 \times n_2} \tag{2.4}$$

where $n_1 = {}^\circ\{\det[Ez - A_\alpha]\}$, $n_2 = n - n_1$ and N is the nilpotent matrix with the index μ, i.e. $N^{\mu-1} \neq 0$, $N^\mu = 0$.

The matrices P and Q can be computed using procedures given in [14, 16].

Premultiplying (2.3a) by the matrix P and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} = Q^{-1}x_i, \bar{x}_{1,i} \in \Re^{n_1}, \bar{x}_{2,i} \in \Re^{n_2}. \tag{2.5}$$

From (2.3a) and (2.5) we obtain

$$PEQQ^{-1}x_{i+1} = PA_\alpha QQ^{-1}x_i + \sum_{j=2}^{i+1} c_j PEQQ^{-1} x_{i-j+1} + Pf(Q\bar{x}_i, u_i), \tag{2.6}$$

and

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_i, u_i), \tag{2.7a}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1} - \bar{f}_2(\bar{x}_i, u_i), \tag{2.7b}$$

where

$$\begin{bmatrix} \bar{f}_1(\bar{x}_i, u_i) \\ -\bar{f}_2(\bar{x}_i, u_i) \end{bmatrix} = Pf(Q\bar{x}_i, u_i). \tag{2.7c}$$

Note that if $0 < \alpha < 1$ then

$$c_j = (-1)^{j+1} \binom{\alpha}{j} > 0 \text{ for } j = 1, 2, \ldots, i+1. \tag{2.8}$$

To simplify the notation it is assumed that the nilpotent matrix contains only one block, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \Re^{n_2 \times n_2}. \tag{2.9}$$

In this case the solution to the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) for given initial conditions $x_0 \in \Re^n$ and input $u_i \in \Re^m$ for i = 0,1,... can be computed iteratively as follows.

From (2.7b) and (2.9) for i = 0 we have

$$\begin{aligned} \bar{x}_{22,1} &= \bar{x}_{21,0} - f_{21}(\bar{x}_0, u_0) \\ \bar{x}_{23,1} &= \bar{x}_{22,0} - f_{22}(\bar{x}_0, u_0) \\ &\vdots \\ \bar{x}_{2n_2,1} &= \bar{x}_{2n_2-1,0} - f_{2n_2-1}(\bar{x}_0, u_0) \end{aligned} \tag{2.10a}$$

$$\bar{x}_{2n_2,0} = f_{2n_2}(\bar{x}_0, u_0) \tag{2.10b}$$

where

$$\bar{x}_{2,i} = [\bar{x}_{21,i} \quad \bar{x}_{22,i} \quad \ldots \quad \bar{x}_{2n_2,i}]^T \tag{2.10c}$$

$$\bar{f}_2(\bar{x}_0, u_0) = [f_{21}(\bar{x}_0, u_0) \quad f_{22}(\bar{x}_0, u_0) \quad \ldots \quad f_{2n_2}(\bar{x}_0, u_0)]^T. \tag{2.10d}$$

From (2.10a) and (2.10c) it follows that $\bar{x}_{21,1}$ can be chosen arbitrary and $\bar{x}_{2n_2,0}$ should satisfy the condition (2.10b).

Next using (2.7a) for i = 0 we have

$$\bar{x}_{1,1} = \begin{bmatrix} \bar{x}_{11,1} \\ \bar{x}_{12,1} \\ \vdots \\ \bar{x}_{1n_1,1} \end{bmatrix} = A_{1\alpha}\bar{x}_{1,0} + \bar{f}_1(\bar{x}_0, u_0).$$

Knowing $\bar{x}_1$ we can compute from (2.7b) for $i = 1$

$$\begin{aligned}
&\bar{x}_{22,2} = \bar{x}_{21,1} + c_2\bar{x}_{22,0} - f_{21}(\bar{x}_1, u_1) \\
&\bar{x}_{23,2} = \bar{x}_{22,1} + c_2\bar{x}_{23,0} - f_{22}(\bar{x}_1, u_1) \\
&\vdots \\
&\bar{x}_{2n_2,2} = \bar{x}_{2n_2-1,1} + c_2\bar{x}_{2n_2,0} - f_{2n_2-1}(\bar{x}_1, u_1)
\end{aligned} \tag{2.12a}$$

$$\bar{x}_{2n_2,1} = f_{2n_2}(\bar{x}_1, u_1) \tag{2.12b}$$

and next from (2.7a)

$$\bar{x}_{1,2} = \begin{bmatrix} \bar{x}_{11,1} \\ \bar{x}_{12,1} \\ \vdots \\ \bar{x}_{1n_1,1} \end{bmatrix} = A_{1\alpha}\bar{x}_{1,1} + c_2\bar{x}_{1,0} + f_1(\bar{x}_1, u_1) \tag{2.14}$$

where $c_2 = \frac{\alpha(1-\alpha)}{2}$.

Repeating the procedure we may compute the state vector $\bar{x}_i$ for i = 1,2,… and next from the equality

$$x_i = Q\bar{x}_i \tag{2.12}$$

the desired solution x_i of the Eq. (2.1a).

3 Positive Fractional Descriptor Nonlinear Systems

Consider the descriptor discrete-time nonlinear system (2.1a), (2.1b), (2.1c), (2.1d).

Definition 3.1 The fractional descriptor discrete-time nonlinear system (2.1a), (2.1b), (2.1c), (2.1d) is called positive if $x_i \in \Re^n_+$, $y_i \in \Re^p_+$, $i \in Z_+$ for any consistent initial conditions $x_0 \in X_0 \in \Re^n_+$ and all admissible inputs $u_i \in U_a \in \Re^m_+$.

Note that for positive systems (2.1a), (2.1b), (2.1c), (2.1d) $\bar{x}_i = Q^{-1}x_i \in \Re^n_+$ if and only if the matrix $Q \in \Re^{n\times n}_+$ is monomial. In this case $Q^{-1} \in \Re^{n\times n}_+$.

Note that for fractional positive systems (2.7a) $\bar{x}_i = Q^{-1}x_i \in \Re^n_+$ for $i \in Z_+$ if and only if

$$A_{1\alpha} \in \Re^{n_1\times n_1}_+ \text{ and } \bar{f}_1(\bar{x}_i, u_i) \in \Re^{n_1}_+ \text{ for all } \bar{x}_i \in \Re^n_+ \text{ and, } u_i \in \Re^m_+ \; i \in Z_+. \tag{3.1}$$

From the structure of the matrix (2.9) and the Eq. (2.7b) it follows that $\bar{x}_{2,i} \in \Re^{n_2}_+$, $i \in Z_+$ if and only if

$$\bar{f}_2(\bar{x}_i, u_i) \in \mathfrak{R}_+^{n_2} \text{ for all } \bar{x}_i \in \mathfrak{R}_+^n \text{ and } u_i \in \mathfrak{R}_+^m,\ i \in Z_+. \tag{3.2}$$

The solution of the Eqs. (2.7a), (2.7b), (2.7c) $\bar{x}_i \in \mathfrak{R}_+^n$ if and only if the conditions (3.1) and (3.2) are satisfied.

Therefore, the following theorem of the positivity of the system (2.1a), (2.1b), (2.1c), (2.1d) has been proved.

Theorem 3.1 *The fractional descriptor nonlinear system* (2.1a), (2.1b), (2.1c), (2.1d) *is positive if and only if the conditions* (3.1) *and* (3.2) *are satisfied, the matrix* $Q \in \mathfrak{R}_+^{n \times n}$ *is monomial and*

$$g(x_i, u_i) \in \mathfrak{R}_+^p \text{ for } x_i \in \mathfrak{R}_+^n \text{ and } u_i \in \mathfrak{R}_+^m,\ i \in Z_+. \tag{3.3}$$

Remark 3.1 If the nilpotent matrix N consist of q block then the condition (2.10b) should be substituted by suitable q conditions of each for the blocks.

Remark 3.2 If the nilpotent matrix N consists of q blocks then for each of the blocks one state variable can be chosen arbitrarily.

Example 3.1 Consider the fractional descriptor nonlinear system (2.1a), (2.1b), (2.1c), (2.1d) with $\alpha = 0.5$ and

$$E = \begin{bmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -0.5 & -0.25 & 0.25 \\ 0.6 & 0 & 0.4 & -0.2 \\ 0.5 & 0.5 & -0.25 & -0.25 \\ 0.3 & 0 & 0.2 & 0.4 \end{bmatrix},$$

$$f(x_i, u_i) = \begin{bmatrix} 0.5x_{3,i}^2 - x_{2,i}^2 + e^{-i} - 0.5 \\ 0.2x_{1,i}^2 + 0.2e^{-i} + 0.4(1+i^2) \\ x_{2,i}^2 + 0.5x_{3,i}^2 + 0.5 \\ 0.2(1+i^2) - 0.4x_{1,i}^2 - 0.4e^{-i} \end{bmatrix}, \tag{3.4a}$$

with the initial conditions

$$x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \\ x_{4,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}. \tag{3.4b}$$

The assumption (2.2) is satisfied since

$$\det E = \begin{vmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{vmatrix} = 0 \tag{3.5}$$

and

$$\det[Ez - A_\alpha] = \begin{vmatrix} -0.5 & 0.5 & 0.5z & -0.5z \\ 0.4z - 0.8 & 0 & -0.4 & 0.2 \\ -0.5 & -0.5 & 0.5z & 0.5z \\ 0.2z - 0.4 & 0 & -0.2 & -0.4 \end{vmatrix} = 0.1z^2 - 0.2z - 0.1 \neq 0. \tag{3.6}$$

In this case

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

Using (2.4), (2.7a), (2.7b), (2.7c) and (3.7) we obtain

$$P[Ez - A_\alpha]Q = \begin{bmatrix} I_{n_1}z - A_{1\alpha} & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}, A_{1\alpha} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, n_1 = n_2 = 2 \tag{3.8}$$

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} = Q^{-1}x_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \end{bmatrix} = \begin{bmatrix} x_{3,i} \\ x_{1,i} \\ x_{2,i} \\ x_{4,i} \end{bmatrix}, \tag{3.9}$$

$$Pf(\bar{x}_i, u_i) = \begin{bmatrix} \bar{f}_1(\bar{x}_i, u_i) \\ -\bar{f}_2(\bar{x}_i, u_i) \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \\ 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ \bar{x}_{2,i}^2 + e^{-i} \end{bmatrix} \tag{3.10}$$

and

$$\begin{bmatrix} \bar{x}_{1,i+1} \\ \bar{x}_{2,i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} \bar{x}_{1,i-j+1} \\ \bar{x}_{2,i-j+1} \end{bmatrix} + \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \end{bmatrix}, \tag{3.11}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{3,i+1} \\ \bar{x}_{4,i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_{3,i-j+1} \\ \bar{x}_{4,i-j+1} \end{bmatrix} + \begin{bmatrix} 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ -\bar{x}_{2,i}^2 - e^{-i} \end{bmatrix} \tag{3.12}$$

with the initial conditions

$$\bar{x}_0 = Q^{-1} x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}. \tag{3.13}$$

The fractional descriptor system (2.1a), (2.1b), (2.1c), (2.1d) with (3.4a), (3.4b) is positive since the conditions (3.1) and (3.2) are satisfied and the matrix Q defined by (3.7) is monomial.

Using the procedure presented in Sect. 3 we obtain the following:

From (3.12) for $i = 0$ we have

$$\bar{x}_{4,1} = \bar{x}_{3,0} + 2\bar{x}_{3,0}^2 - e^0 + 1 = 0, \tag{3.14a}$$

and the condition (3.2) is satisfied since

$$\bar{x}_{4,0} = \bar{x}_{2,0}^2 + 1 = 2. \tag{3.14b}$$

Using (3.11) for $i = 0$ and (3.13) we obtain

$$\begin{aligned} \bar{x}_{1,1} &= \bar{x}_{2,0} + \bar{x}_{1,0}^2 + e^0 = 3, \\ \bar{x}_{2,1} &= \bar{x}_{1,0} + 2\bar{x}_{2,0} + 1 = 4 \end{aligned} \tag{3.15}$$

and from (3.12) for $i = 1$

$$\begin{aligned} \bar{x}_{4,2} &= \bar{x}_{3,1} + 0.125\bar{x}_{4,1} + 2\bar{x}_{3,1}^2 - e^{-1} + 1, \\ \bar{x}_{4,1} &= \bar{x}_{2,1}^2 + e^{-1} \end{aligned} \tag{3.16}$$

for arbitrary $\bar{x}_{3,1} \geq 0$.

From (3.11) for $i = 1$ we have

$$\begin{aligned} \bar{x}_{1,2} &= \bar{x}_{2,1} + 0.125\bar{x}_{1,0} + \bar{x}_{1,1}^2 + e^{-1}, \\ \bar{x}_{2,2} &= \bar{x}_{1,1} + 2\bar{x}_{2,1} + 0.125\bar{x}_{2,0} + 2 \end{aligned} \tag{3.17}$$

and from (3.12) for $i = 2$

$$\begin{aligned} \bar{x}_{4,3} &= \bar{x}_{3,2} + 0.125\bar{x}_{4,1} + 0.0625\bar{x}_{4,0} + 2\bar{x}_{3,2}^2 - e^{-2} + 1, \\ \bar{x}_{4,2} &= \bar{x}_{2,2}^2 + e^{-2} \end{aligned} \tag{3.18}$$

for arbitrary $\bar{x}_{3,2} \geq 0$.

Continuing the procedure we may compute the solution $\bar{x}_i$ of the Eqs. (3.11) and (3.12) and next the solution

$$x_i = Q\bar{x}_i = \begin{bmatrix} \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{1,i} \\ \bar{x}_{4,i} \end{bmatrix} \tag{3.19}$$

of the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with (3.4a), (3.4b).

4 Stability of Positive Nonlinear Systems

Consider the fractional descriptor nonlinear system (2.1a) decomposed into two nonlinear subsystems (2.7a) and (2.7b) of the form

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i}), \tag{4.1}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1} - \bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}), \tag{4.2}$$

where $\bar{f}_1(\bar{x}_{1,i}) = \bar{f}_1(\bar{x}_{1,i}, 0)$ and $\bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}) = \bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}, 0)$.

Definition 4.1 The positive nonlinear system (2.1a) is called asymptotically stable in the region D if $x_i \in \Re_+^n$, $i \in Z_+$ satisfies the condition

$$\lim_{i \to \infty} x_i = 0 \text{ for any finite } x_0 \in \Re_+^n. \tag{4.3}$$

Note that the positive nonlinear system (2.1a) is asymptotically stable if and only if the positive nonlinear subsystem (4.1) and (4.2) are asymptotically stable, since from (2.5) for the monomial matrix Q we have

$$\lim_{i \to \infty} x_i = 0 \text{ if and only if } \lim_{i \to \infty} \bar{x}_i = 0. \tag{4.4}$$

Remark 4.1 The positive nonlinear system (2.1a) is asymptotically stable only if the linear part of the system (4.1), (4.2) (for $\bar{f}_1(\bar{x}_{1,i}) = 0$, $\bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}) = 0$) is asymptotically stable. The asymptotic stability of this positive linear system

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1}, \tag{4.5a}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1}, \tag{4.5b}$$

can be verified using the tests presented in [13].

The subsystem (4.5b) is asymptotically stable since its solution $\bar{x}_{2,i} = 0$ for $i = 1, 2, \ldots$.

To investigate the asymptotic stability of the positive nonlinear subsystem (4.1) we will apply the Lyapunov method. As a candidate of the Lyapunov function for the subsystem (4.5a) we choose

$$V(\bar{x}_{1,i}) = d\bar{x}_{1,i} > 0 \text{ for } \bar{x}_{1,i} \in \mathfrak{R}_+^{n_1}, i \in Z_+, \tag{4.6}$$

where $d = [d_1 \quad \ldots \quad d_{n_1}]^T$ is a strictly positive vector with $d_k > 0$ for $k = 1, 2, \ldots, n_1$.

Using (4.6) and (4.1) we obtain

$$\Delta V(\bar{x}_{1,i}) = V(\bar{x}_{1,i+1}) - V(\bar{x}_{1,i}) = d\left[(A_{1\alpha} - I_{n_1})\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i})\right]. \tag{4.7}$$

From (4.7) it follows that $\Delta V(\bar{x}_{1,i}) < 0$ if

$$(A_{1\alpha} - I_{n_1})\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i}) < 0 \text{ for } \bar{x}_{1,i} \in D \in \mathfrak{R}_+^{n_1} \text{ and } i \in Z_+ \tag{4.8}$$

since d is a strictly positive vector.

Therefore, the following theorem has been proved.

Theorem 4.1 The positive nonlinear system (4.1) is asymptotically stable in the region D if the condition (4.8) is satisfied.

Theorem 4.2 *The positive nonlinear system* (4.2) *is asymptotically stable if the positive nonlinear system* (4.1) *is asymptotically stable and*

$$\lim_{i \to \infty} \bar{f}_1(\bar{x}_{1,i}) = 0. \tag{4.9}$$

Proof If the positive nonlinear subsystem (4.1) is asymptotically stable then $\lim_{i \to \infty} \bar{x}_{1,i} = 0$ and from (4.2) it follows that $\lim_{i \to \infty} \bar{x}_{2,i} = 0$ if the condition (4.9) is satisfied. □

Example 4.1 Consider the fractional nonlinear subsystems for $\alpha = 0.5$

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} x_{1,i-j+1} \\ x_{2,i-j+1} \end{bmatrix} + \begin{bmatrix} x_{1,i}^2 \\ x_{2,i}x_{2,i} \end{bmatrix}, \tag{4.10a}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{3,i+1} \\ x_{4,i+1} \end{bmatrix} = \begin{bmatrix} x_{3,i} \\ x_{4,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{3,i-j+1} \\ x_{4,i-j+1} \end{bmatrix} - \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix}. \tag{4.10b}$$

From comparison of (4.10a), (4.10b) and (4.1) it follows that

$$\bar{x}_{1,i} = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \quad A_{1\alpha} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \quad \bar{f}_1(\bar{x}_i) = \begin{bmatrix} x_{1,i}^2 \\ x_{2,i}x_{2,i} \end{bmatrix}, \tag{4.11a}$$

and

$$\bar{x}_{2,i} = \begin{bmatrix} x_{3,i} \\ x_{4,i} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{f}_2(\bar{x}_i) = \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix}. \tag{4.11b}$$

The nonlinear system (4.10a), (4.10b) is positive since $A_{1\alpha} \in \mathfrak{R}_+^{2\times 2}, \bar{f}_1(\bar{x}_i) \in \mathfrak{R}_+^2$, $\bar{f}_2(\bar{x}_i) \in \mathfrak{R}_+^2$ for $\bar{x}_i \in \mathfrak{R}_+^2$, $u_i \in \mathfrak{R}_+$, $i \in Z_+$ and the conditions of Theorem 3.1 are satisfied.

Note that the linear part of the nonlinear system (4.11a) is asymptotically stable since the eigenvalues of the matrix $A_{1\alpha}$ are $z_1 = -0.1$, $z_2 = -0.2$.

The nonlinear subsystem (4.11b) is also asymptotically stable since the condition (4.9) is satisfied i.e.

$$\lim_{i\to\infty} \bar{f}_2(\bar{x}_i) = \lim_{i\to\infty} \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.12}$$

5 Concluding Remarks

A method of analysis of the fractional descriptor nonlinear discrete-time systems described by the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with regular pencils (2.2) based on the Weierstrass-Kronecker decomposition of the pencil has been proposed. Necessary and sufficient conditions for the positivity of the nonlinear systems have been established (Theorem 3.1). A procedure for computing the solution to the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with given initial conditions and input sequences has been proposed and illustrated by numerical example. Using an extension of the Lyapunov method to positive nonlinear systems sufficient conditions for the asymptotic stability have been derived (Theorems 4.1, 4.2). The proposed method can be applied for example to analysis of descriptor nonlinear discrete-time electrical circuits. The considerations can be extended to fractional descriptor nonlinear discrete-time systems.

Acknowledgment This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

References

1. Bru, R., Coll, C., Romero-Vivo, S., Sanchez, E.: Some problems about structural properties of positive descriptor systems, positive systems (Rome, 2003). Lecture Notes in Control and Information Science, vol. 294, pp. 233–240. Springer, Berlin (2003)
2. Bru, R., Coll, C., Sanchez, E.: About positively discrete-time singular systems, System and Control: theory and applications, pp. 44–48. Electr. Comput. Eng. Ser., World Sci. Eng. Soc. Press, Athens (2000)
3. Bru, R., Coll, C., Sanchez, E.: Structural properties of positive linear time-invariant difference-algebraic equations. Linear Algebra Appl. **349**, 1–10 (2002)
4. Campbell, S.L., Meyer, C.D., Rose, N.J.: Applications of the Drazin inverse to linear systems of differential equations with singular constructions. SIAMJ Appl. Math. **31**(3), 411–425 (1976)
5. Dai, L.: Singular control systems. Lectures Notes in Control and Information Sciences. Springer, Berlin (1989)
6. Dodig, M., Stosic, M.: Singular systems state feedbacks problems. Linear Algebra Appl. **431** (8), 1267–1292 (2009)
7. Fahmy, M.M., O'Reill, J.: Matrix pencil of closed-loop descriptor systems: infinite-eigenvalues assignment. Int. J. Control **49**(4), 1421–1431 (1989)
8. Guang-Ren, D.: Analysis and Design of Descriptor Linear Systems. Springer, New York (2010)
9. Kaczorek, T.: Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils. Int. J. Appl. Math. Comput. Sci. **23**(1), 29–34 (2013)
10. Kaczorek, T.: Descriptor standard and positive discrete-time nonlinear systems, Automatyzacja Procesów Dyskretnych, Teoria i Zastosowania, Tom I, Gliwice, 113–126 (2014)
11. Kaczorek, T.: Reduction and decomposition of singular fractional discrete-time linear systems". Acta Mechanica et Automatica **5**(4), 62–66 (2011)
12. Kaczorek, T.: Singular fractional discrete-time linear systems. Control Cybern. **40**(3), 753–761 (2011)
13. Kaczorek, T.: Selected Problems of Fractional Systems Theory. Springer, Berlin (2011)
14. Kaczorek, T.: Linear Control Systems, vol. 1. Research Studies Press J. Wiley, New York (1992)
15. Kucera, V., Zagalak, P.: Fundamental theorem of state feedback for singular systems. Automatica **24**(5), 653–658 (1988)
16. Van Dooren, P.: The computation of Kronecker's canonical form of a singular pencil. Linear Algebra Appl. **27**, 103–140 (1979)
17. Virnik, E.: Stability analysis of positive descriptor systems. Linear Algebra Appl. **429**, 2640–2659 (2008)
18. Kaczorek, T.: Infinite eigenvalue assignment by output-feedbacks for singular systems. Int. J. Appl. Math. Comput. Sci. **14**(1), 19–23 (2004)

19. Kaczorek, T.: Minimum energy control of descriptor positive discrete-time linear systems. COMPEL: Int. J. Comput. Math. Electr. Electr. Eng. **33**(3), 976–988 (2014)
20. Kaczorek, T.: Positive linear systems with different fractional orders. Bull. Pol. Ac. Sci. Techn. **58**(3), 453–458 (2010)
21. Kaczorek, T.: Minimum energy control of positive fractional descriptor continuous-time linear systems. Control Theory Appl. IET **8**(4), 215–225 (2014)
22. Kaczorek, T.: Positive 1D and 2D Systems. Springer, London (2002)
23. Kaczorek, T.: Drazin inverse matrix method for fractional descriptor continuous-time linear systems. Bull. Pol. Acad. Tech. **62**, 2 (2014)
24. Kaczorek, T.: Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm. Arch. Control Sci. **21**(3), 287–298 (2011)