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Stability analysis and a numerical scheme for fractional Klein-Gordon equations

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Fractional order nonlinear Klein-Gordon equations (KGEs) have been widely studied in the fields like; nonlinear optics, solid state physics, and quantum field theory. In this article, with help of the Sumudu decomposition method (SDM), a numerical scheme is developed for the solution of fractional order nonlinear KGEs involving the Caputo's fractional derivative. The coupled method provides us very efficient numerical scheme in terms of convergent series. The iterative scheme is applied to illustrative examples for the demonstration and applications.

KEYWORDS
Adomian decomposition, Caputo's fractional derivative, Klein-Gordon equation, Sumudu transform

1 | INTRODUCTION

In last few decades, analysis, computation, error estimation, and stabilities of fractional differential equations (FDEs) have gained a considerable attention of researchers in different fields. Several physical and biological problems were mathematically modeled via FDEs, which gave high quality accuracy than the models by integer order differential equations. One can see in the references of the paper some useful applications of FDEs in viscoelasticity, bioengineering, damping laws, rheology, thermodynamics, synchronization, dynamical system, electrical circuits, signal processing, and fluid mechanics, see.[1-13] The fractional order Klein-Gordon equation (KGE) is derived from the KGE of the integer order by switching time derivative by noninteger order ($\theta \in [0,1]$) derivative. The fractional order KGE can be illustrated as below:

$$\mathcal{D}_t^{\theta}\psi(x,t) = \mathcal{D}_x^{2}\psi(x,t) + \varrho\psi(x,t) + \sigma\mathcal{G}(\psi(x,t)) = \mathfrak{F}(x,t),\ \ t \geq 0, \tag{1}$$

with conditions

$$\frac{\partial}{\partial t}\psi(x,0) = g_1(x), \quad \psi(x,0) = g_0(x),$$

where $\mathfrak{F}(x,t), g_0(x), g_1(x)$ are analytical functions, ϱ, σ are constants, $\mathcal{G}(\psi)$ is a nonlinear function and ψ is a function of x and t to be determined. The nonlinear KGEs are arising from quantum mechanics and classical relativistic, such type of equations have a lot of applications in different areas like, physics, relativity, advance quantum mechanics, and the quantum theory of field.

Recently, many mathematical methods were established for the solutions of FDEs including the modified Adomian decomposition method, Kudryashov method, homotopy decomposition method, fractional variational iteration method (FVIM), generalized Kudryashov method, and many others.[14-17] These analytical methods are very much useful and efficient in the approximation of FDEs. In 1993, Watugala established Sumudu transform (ST) for the numerical solutions of many research problems in engineering.[18,19] Recently, Gomez-Aguilar[20] considered alcoholism model for analytical solutions via Laplace and STs and presented the usefulness of their work via testing examples. The inverse of the transform was suggested by Weerakoon.[21] The applications of ST in the approximation of integer order differential equation were studied by Belgacem's in 2006. Atangana and Kiliman[22] studied nonlinear fractional heat like equations by ST. Khan et al.[23] studied stability and existence theorem for a fractional order nonlinear differential equation.

In the mathematical description of physical scientific problems, there is an important role of linear and nonlinear fractional partial differential equations (FPDEs). Among the nonlinear FPDEs, fractional order KGEs were widely studied in the fields like; nonlinear optics, solid state physics ,and quantum field theory.[24] Fractional order KGEs have have been considered by a large number of scientists for its new and better approximation via different numerical methods. For instance, Merdan[17] considered fractional order KGE for the approximate solution via FVIM and studied graphically different features. Vong and Wang[25] studied two dimensional KGE via finite different scheme and studied stability and convergence for their numerical scheme. Hosseini et al.[26] considered nonlinear time-fractional KGE in the sense of conformable fractional derivative with the help of modified Kudryashov technique. Onate et al.[27] presented analytical solutions for KGE with a combined potential.

Motivated from the aforementioned techniques and their applications to real world problems, we use Sumudu decomposition method (SDM) to get analytical solutions for fractional order KGEs. For this purpose, we produce an iterative scheme and apply the scheme to some examples for the the illustration of the work. We get very good results.

2 | BASIC LITERATURE

This section has reserved for some related literature including definitions of Riemann-Liouville fractional integral operator (FIO), Caputo's fractional derivative, and definition of the ST.

Definition 1. Riemann-Liouville FIO of $\theta > 0$ order of a function $\mathcal{F}(t) \in C_{\mu}$, $\mu \geq -1$ is given by the following:

$$\mathcal{J}^{\theta}\mathfrak{F}(t) = \frac{1}{\Gamma\theta}\int_{0}^{t}(t-\lambda)^{\theta-1}\mathfrak{F}d\lambda, \tag{2}$$

$$\mathcal{J}^{0}\mathfrak{F}(t) = \mathfrak{F}(t). \tag{3}$$

For Riemann-Liouville FIO, we have

$$\mathcal{J}^{\theta}t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta+\theta+1)}t^{(\theta+\eta)}. \tag{4}$$

Definition 2. The fractional derivative of function $\mathcal{F}(t)$, in Caputo sense is defined as follows:

$$\mathcal{D}^{\theta}\mathfrak{F}(t) = \mathcal{J}^{n-\theta}\mathcal{D}^{\theta}\mathfrak{F}(t) = \frac{1}{\Gamma(n-\theta)}\int_{0}^{t}(t-\mu)^{n-\theta-1}\mathfrak{F}^{n}d\lambda, \tag{5}$$

where $n \in \mathcal{N}, n-1 < \theta \leq n, t > 0$.

Definition 3. The ST defined over the set of function

$$\mathbb{A} = \{\mathfrak{F}(t) | \exists\ \mathbb{M}, \lambda_1, \lambda_2 > 0, |\mathfrak{F}(t)| < \mathbb{M}e^{|t|/\lambda_j}, \text{if } t \in (-1)^j \times [0,\infty)\}, \tag{6}$$

is defined by the following formula

$$\mathfrak{F}\bar{(}u) = \mathcal{S}[\mathfrak{F}(t)] = \mathcal{J}^{n-\theta}\mathcal{D}^{\theta}\mathfrak{F}(t) = \int_0^{\infty} \mathfrak{F}(ut)e^{-t}dt, \ u \in (-\mu_1, \mu_2). \tag{7}$$

Definition 4. The ST of a function with Caputo fractional derivative is defined given by

$$\mathcal{S}[\mathcal{D}^{\theta}\mathfrak{F}(t)] = u^{-\theta}\mathcal{S}[\mathfrak{F}(t)] - \sum_{k=0}^{m} u^{-\theta+k}\mathfrak{F}^{(k)}(0_+), \ m-1 < \theta \leq m. \tag{8}$$

The Mittag-Leffler function is also applicable our work and is defined by

$$\mathrm{E}_{\theta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\theta n + 1)} \ (z \in \mathrm{C}, Re(\theta) > 0). \tag{9}$$

3 | DESCRIPTION OF SDM

Here to illustrate the basic idea for the mention method, we consider the following general fractional nonlinear nonhomogeneous partial differential equation with initial conditions:

$$\mathcal{D}_t^{\theta}\psi(x,t) + \mathcal{R}\psi(x,t) + \mathcal{N}\psi(x,t) = g(x,t), \ 0 < \theta \leq 1, \tag{10}$$

$$\psi(x,t) = h(x), \tag{11}$$

where $\mathcal{D}_t^{\theta}$ denotes Caputo derivative of the function $\psi(x,t)$, $\mathcal{R}$ represents linear differential operator, $\mathcal{N}$ denotes nonlinear operator, and $g(x,t)$ is a source term.

Applying the ST to both side of the (10), we proceed

$$\mathcal{S}[\mathcal{D}_t^{\theta}\psi(x,t)] + \mathcal{S}[\mathcal{R}\psi(x,t)] + \mathcal{S}[\mathcal{N}\psi(x,t)] = \mathcal{S}[g(x,t)]. \tag{12}$$

By using differentiation property of ST with initial conditions, we have

$$\mathcal{S}[\psi(x,t)] = h(x) - u^{\theta}\mathcal{S}[\mathcal{R}\psi(x,t) + \mathcal{N}\psi(x,t)] + u^{\theta}\mathcal{S}[g(x,t)]. \tag{13}$$

Now applying inverse ST to both sides of (13), we have

$$\psi(x,t) = \mathcal{G}(x,t) - \mathcal{S}^{-1}[u^{\theta}\mathcal{S}[\mathcal{R}\psi(x,t) + \mathcal{N}\psi(x,t)]], \tag{14}$$

where $\mathcal{G}(x,t)$ denotes the calculation of source term and initial condition. For linear term, we have to use

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t), \tag{15}$$

for the nonlinear term we have

$$\mathcal{N}\psi(x,t) = \sum_{n=0}^{\infty}\mathcal{Q}_n(x,t). \tag{16}$$

The Adomian polynomials $\mathcal{Q}_n(x,t)$ is given by

$$\mathcal{Q}_n(x,t) = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[\mathcal{N}\left(\sum_{k=0}^{n}\mu^k\psi_n(x,t)\right)\right]_{\mu=0}, \ n = 0,1,2,3,\cdots, \tag{17}$$

putting (15) and (16) in (14), we get

$$\sum_{n=0}^{\infty}\psi_n(x,t) = \mathcal{G}(x,t) - \mathcal{S}^{-1}\left[u^{\theta}\mathcal{S}\left[\mathcal{R}\sum_{n=0}^{\infty}\psi_n(x,t) + \sum_{n=0}^{\infty}\mathcal{Q}_n(x,t)\right]\right]. \tag{18}$$

Comparing the coefficients we evaluate the following approximations:

$$
\begin{aligned}
&\psi_0(x,t) = \mathcal{G}(x,t),\\
&\psi_1(x,t) = -\mathcal{S}^{-1}[u^\theta \mathcal{S}[\mathcal{R}\psi_0(x,t) + \mathcal{Q}_0(x,t)]],\\
&\psi_2(x,t) = -\mathcal{S}^{-1}[u^\theta \mathcal{S}[\mathcal{R}\psi_1(x,t) + \mathcal{Q}_1(x,t)]],\\
&\psi_3(x,t) = -\mathcal{S}^{-1}[u^\theta \mathcal{S}[\mathcal{R}\psi_2(x,t) + \mathcal{Q}_2(x,t)]],\\
&.\\
&.\\
&.\\
&\psi_{n+1}(x,t) = -\mathcal{S}^{-1}[u^\theta \mathcal{S}[\mathcal{R}\psi_n(x,t) + \mathcal{Q}_n(x,t)]].
\end{aligned}
$$

By similarly process, the elements of $\psi_n(x,t)$ are obtained.

4 | STABILITY ANALYSIS

In this section, we are going to produce stability of our numerical scheme based on the SDM. For this, we consider a Banach space $(\mathcal{Y}, \|.\|)$ and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$. Let $\psi_{n+1} = f(\mathcal{T}, \psi_n)$ be a recursive technique. Let $\mathcal{F}(\mathcal{T})$ be the set of fixed points of $\mathcal{T}$ at least containing one point say $p \in \mathcal{F}(\mathcal{T})$. Assume that $\psi_n \subset \mathcal{Y}$ and define $err_n = ||\psi_{n+1} - f(\mathcal{T}, \psi_n)||$. If $\lim_{n\to\infty}\psi^n = p$ then $\psi_{n+1} = f(\mathcal{T}, \psi_n)$ is said to be H-stable. We will use the following theorem of Atangana[28]:

Theorem 1. *Let $(\mathcal{Y}, \|.\|)$ be a Banach space and $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$, satisfying*

$$
\|\mathcal{T}_x - \mathcal{T}_y\| \le C\|x - \mathcal{T}_x\| + c\|x - y\|, \tag{19}
$$

for all $x, y \in \mathcal{Y}$, $C \ge 0, 0 \le c \le 1$, then $\mathcal{T}$ is Picard $\mathcal{T}$-Stable.

Theorem 2. *Let $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$ be a defined by*

$$
\mathcal{T}(\psi_n(x,t)) = \psi_{n+1}(x,t) = -\mathcal{S}^{-1}[u^\theta \mathcal{S}[\mathcal{R}\psi_n(x,t) + \mathcal{Q}_n(x,t)]], \tag{20}
$$

where $\mathcal{R} = \mathcal{D}_x^2 + \vartheta\mathcal{I}$, $\vartheta \in [0,1]$ and $\mathcal{Q}_n(x,t) = \psi_n^k(x,t)$, for $k = 2, 3$. Then $\mathcal{T}$ is Picard $\mathcal{T}$-Stable provided that for some $\lambda_i > 0 (i = 1, 2, 3)$, the following contractions are satisfied:

- *(A)$\|\mathcal{D}_x^2\psi_n(x,t) - \mathcal{D}_x^2\psi_m(x,t)\| \le \lambda_1\|\psi_n(x,t) - \psi_m(x,t)\|$,*
- *(B)$\|\psi_n^k(x,t) - \psi_m^k(x,t)\| \le \lambda_k\|\psi_n(x,t) - \psi_m(x,t)\|$, for $k = 2, 3$,*

and $(\lambda_1 + \vartheta + \lambda_k) < 1$.

Proof. In order to show the existence of a fixed point of the operator $\mathcal{T}$, for this, we consider for $n, m \in \mathbb{N}$, and

$$
\begin{aligned}
\|\mathcal{T}(\psi_n(x,t)) - \mathcal{T}(\psi_m(x,t))\| = \|&\mathcal{S}^{-1}[u^\theta \mathcal{S}[(\mathcal{D}_x^2 + \vartheta\mathcal{I})\psi_n(x,t) + \psi_n^k(x,t)]]\\
&- \mathcal{S}^{-1}[u^\theta \mathcal{S}[(\mathcal{D}_x^2 + \vartheta\mathcal{I})\psi_m(x,t) + \psi_m^k(x,t)]]\|.
\end{aligned} \tag{21}
$$

Using the properties of integral transform $\mathcal{S}$ in (21), we proceed

$$
\begin{aligned}
\|\mathcal{T}(\psi_n(x,t)) - \mathcal{T}(\psi_m(x,t))\| = \mathcal{S}^{-1}[u^\theta \mathcal{S}[&\|(\mathcal{D}_x^2 + \vartheta\mathcal{I})\psi_n(x,t) - (\mathcal{D}_x^2 + \vartheta\mathcal{I})\psi_m(x,t)\|\\
&+ \|\psi_n^k(x,t) - \psi_m^k(x,t)\|]].
\end{aligned} \tag{22}
$$

With the help of (A), (B), and (22), we have

$$
\begin{aligned}
\|\mathcal{T}(\psi_n(x,t)) - \mathcal{T}(\psi_m(x,t))\| &\le (\lambda_1 + \vartheta)\|\psi_n(x,t) - \psi_m(x,t)\| + \lambda_k\|\psi_n(x,t) - \psi_m(x,t)\|]\\
&= (\lambda_1 + \vartheta + \lambda_k)\|\psi_n(x,t) - \psi_m(x,t)\|.
\end{aligned} \tag{23}
$$

Consequently, with the help of Theorem 1, the operator $\mathcal{T}$ is a Picard $\mathcal{T}$-Stable. □

5 | NUMERICAL EXAMPLES

In this section, we apply the ST with Adomian polynomial to fractional KGEs to check the applicability and simplicity of the proposed method. The simulation results shows quality accuracy of the aforementioned technique.

Example 3. Let us consider the following one dimensional nonlinear fractional KGE

$$\mathcal{D}_t^{\theta}\psi(x,t) - \mathcal{D}_x^2\psi(x,t) + \psi^2(x,t) = 0,\ 0 < \theta \leq 1,\ \ x,t \geq 0, \tag{24}$$

with initial condition

$$\psi(x,0) = 1 + \sin(x). \tag{25}$$

Applying ST to (24) with initial condition, we get

$$\mathcal{S}[\psi(x,t)] = \sin(x) + 1 + u^{\theta}[\mathcal{S}[\mathcal{D}_x^2\psi(x,t) - \psi^2(x,t)]], \tag{26}$$

applying inverse of ST to (26), we obtain the following

$$\psi(x,t) = \sin(x) + 1 + \mathcal{S}^{-1}[u^{\theta}[\mathcal{S}[\mathcal{D}_x^2\psi(x,t) - \psi^2(x,t)]]]. \tag{27}$$

By using ST with Adomian polynomial, we get

$$\sum_{n=0}^{\infty}\psi_n(x,t) = \mathcal{S}^{-1}\left[u^{\theta}\left[\mathcal{S}\left[\mathcal{D}_x^2\sum_{n=0}^{\infty}\psi_n(x,t) - \sum_{n=0}^{\infty}\mathcal{Q}_n(x,t)\right]\right]\right] + 1 + \sin(x), \tag{28}$$

where Adomian polynomials $\mathcal{Q}_n(x,t)$ is given by

$$\mathcal{Q}_n(x,t) = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[\sum_{k=0}^{n}(\mu^k\psi_n(x,t))^2\right]_{\mu=0},\quad n = 0,1,2,3,\cdots. \tag{29}$$

Comparing the coefficients, we have

$$\begin{aligned}
\psi_0(x,t) &= \sin(x) + 1,\\
\psi_1(x,t) &= -(1 + 3\sin(x) + \sin^2(x))\frac{t^{\theta}}{\Gamma(\theta+1)},\\
\psi_2(x,t) &= (2 + 5\sin(x) + 2\cos(2x) + 8\sin^2(x) + 2\sin^3(x))\frac{t^{2\theta}}{\Gamma(2\theta+1)},\\
\psi_3(x,t) &= (6 - 16\cos(2x) - 10\sin^4(x) - 70\sin^3(x) - 106\sin^2(x) - 33\sin(x),\\
&- 8\sin(x)\cos(2x))\frac{t^{3\theta}}{\Gamma(3\theta+1)},
\end{aligned}$$

and so on. We take few terms of the approximate solutions as given below

$$\begin{aligned}
\psi(x,t) &= \sin(x) + 1 - (1 + 3\sin(x) + \sin^2(x))\frac{t^{\theta}}{\Gamma(\theta+1)} + (2 + 5\sin(x) + 2\cos(2x)\\
&+ 8\sin^2(x) + 2\sin^3(x))\frac{t^{2\theta}}{\Gamma(2\theta+1)} + (6 - 16\cos(2x) - 10\sin^4(x) - 70\sin^3(x)\\
&- 106\sin^2(x) - 33\sin(x) - 8\sin(x)\cos(2x))\frac{t^{3\theta}}{\Gamma(3\theta+1)} + \cdots
\end{aligned}$$

In Figure 1, we have shown our approximate solution for the problem given in 3 keeping $x = 2$. Figure 2 presents approximate solution of the the model given in example 3 for fixed value $t = 1.5$. In Figure 3, we have approximate solution of the problem given in the example 3 for $\theta = 0.6$ and various values of x and t.

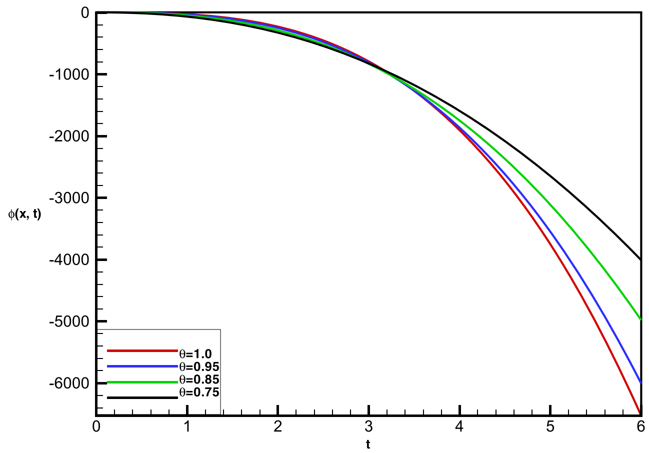

FIGURE 1 Plot of approximate solution at $x = 2$ of example 3 at different fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

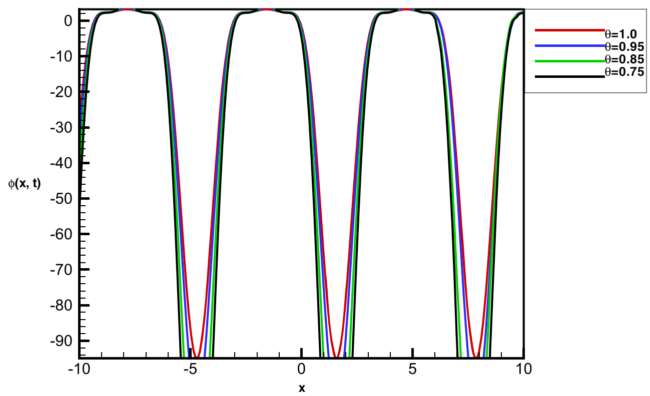

FIGURE 2 Plot of approximate solution at $t = 1.5$ of example 3 at different fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

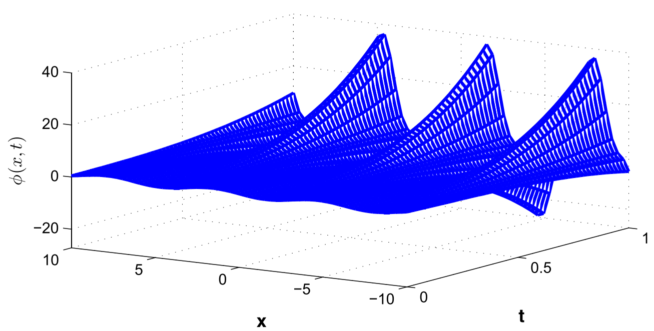

FIGURE 3 Plot of approximate solution at $\theta = 0.6$ of example 3 [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4. Consider the following nonlinear fractional KGE

$$\mathcal{D}_t^{\theta}\psi(x,t) - \mathcal{D}_x^{2}\psi(x,t) + \psi(x,t) - \psi^{3}(x,t) = 0, \ 0 < \theta \leq 1, \ \ x, t \geq 0, \tag{30}$$

with initial condition

$$\psi(x,0) = -sech(x). \tag{31}$$

Applying ST to (24) with initial condition, we get

$$\mathcal{S}[\psi(x,t)] = -sech(x) + u^{\theta}\left[\mathcal{S}\left[\mathcal{D}_x^{2}\psi(x,t) - \psi(x,t) + \psi^{3}(x,t)\right]\right], \tag{32}$$

applying inverse of ST to (30), we have

$$\psi(x,t) = -sech(x) + \mathcal{S}^{-1}\left[u^{\theta}\left[\mathcal{S}\left[\mathcal{D}_x^2\psi(x,t) - \psi(x,t) + \psi^3(x,t)\right]\right]\right]. \tag{33}$$

By using Sumudu-Adomain decomposition method, we get

$$\sum_{n=0}^{\infty}\psi_n(x,t) = -sech(x) + \mathcal{S}^{-1}\left[u^{\theta}\left[\mathcal{S}\left[\mathcal{D}_x^2\sum_{n=0}^{\infty}\psi_n(x,t) - \sum_{n=0}^{\infty}\psi_n(x,t) + \sum_{n=0}^{\infty}\mathcal{Q}_n(x,t)\right]\right]\right]. \tag{34}$$

Where Adomian polynomials $\mathcal{Q}_n(x,t)$ is given by

$$\mathcal{Q}_n(x,t) = \frac{1}{n!}\frac{d^n}{d\mu^n}\left[\sum_{k=0}^{n}(\mu^k\psi_n(x,t))^3\right]_{\mu=0}, \quad n = 0,1,2,3,\cdots. \tag{35}$$

Comparing the coefficients, we proceed

$$\begin{aligned}
\psi_0(x,t) &= -\sec h(x),\\
\psi_1(x,t) &= sech^3(x)\frac{t^{\theta}}{\Gamma(\theta+1)},\\
\psi_2(x,t) &= (8\sec h^3(x) - 9\sec h^5(x))\frac{t^{2\theta}}{\Gamma(2\theta+1)},\\
\psi_3(x,t) &= (64sech^3(x) - 288sech^5(x) + 240sech^7(x))\frac{t^{3\theta}}{\Gamma(3\theta+1)},
\end{aligned}$$

and so on. Therefore, with the help of the proposed method, we get series solutions of the given nonlinear problem. Approximate solutions up to three terms is given.

$$\begin{aligned}
\psi(x,t) = {}& -sech(x) + sech^3(x)\frac{t^{\theta}}{\Gamma(\theta+1)} + (8sech^3(x) - 9sech^5(x))\frac{t^{2\theta}}{\Gamma(2\theta+1)}\\
& + (64sech^3(x) - 288sech^5(x) + 240sech^7(x))\frac{t^{3\theta}}{\Gamma(3\theta+1)}.
\end{aligned} \tag{36}$$

From Table 1, we see that the absolute error is decreasing on increasing number of terms. This phenomenon shows the convergence of the procedure to the exact solution of the problem. Figure 4 represents approximate solutions at different fractional orders and various values of t by taking $x = 1.5$. Similarly in Figure 5, we have provided the images at various fractional orders and different values of x by taking $t = 1.5$. In Figure 6, we have provided the plot of approximate solution up to three terms at various values of x, t, and taking $\theta = 0.5$

TABLE 1 Absolute error for three terms solution and four terms solutions in Example 4 at different values of variable x, t

Error analysis of example 4 at $\theta = 0.9$				
(x,t)	**Three terms solution**	**Absolute error**	**Four terms solution**	**Absolute error**
(0, 0)	−1.00000	0.00000	−1.00000	0.00000
(0.2, 0.1)	−0.00382341681	7.00345×10^{-4}	−0.007823976	1.05411×10^{-7}
(0.3, 0.4)	−0.00456741681	6.2345670×10^{-4}	−0.00910785704	2.012287×10^{-7}
(0.4, 0.6)	−0.00561233456	5.67589×10^{-4}	−0.001078570462	7.20636×10^{-7}
(0.5, 0.5)	−0.00432341345	4.09730×10^{-4}	−0.00107851232	2.4567×10^{-7}
(0.6, 0.5)	−0.00523856371	6.456789×10^{-4}	−0.00107857123	3.123450×10^{-7}
(0.8, 0.3)	−0.00914228023	5.81730×10^{-4}	−0.007800999	1.00464×10^{-7}
(0.3, 0.5)	−0.00741442678	6.78965×10^{-4}	−0.00966785704	2.16899×10^{-7}
(0.5, 0.6)	−0.00567285971	5.09876×10^{-4}	−0.007654321	2.12385×10^{-7}
(0.9, 0.9)	−0.008305000007	7.12345×10^{-4}	−0.0010785700	2.70230×10^{-7}
(1.0, 1.0)	−0.–07654321875	2.81730×10^{-4}	−0.00107857046	3.12128×10^{-7}

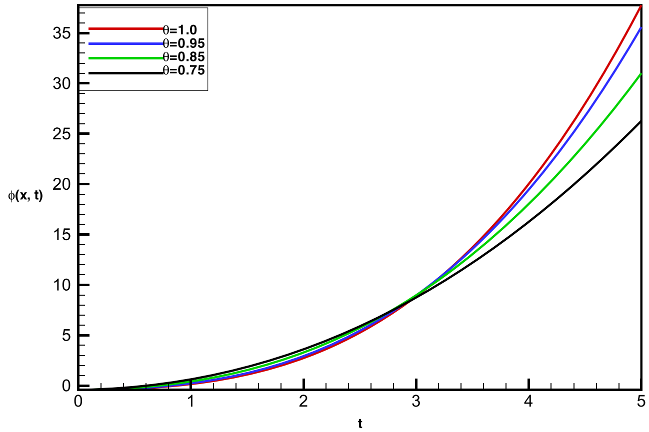

FIGURE 4 Plot of approximate solution at $x = 1.5$ of example 4 at different fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

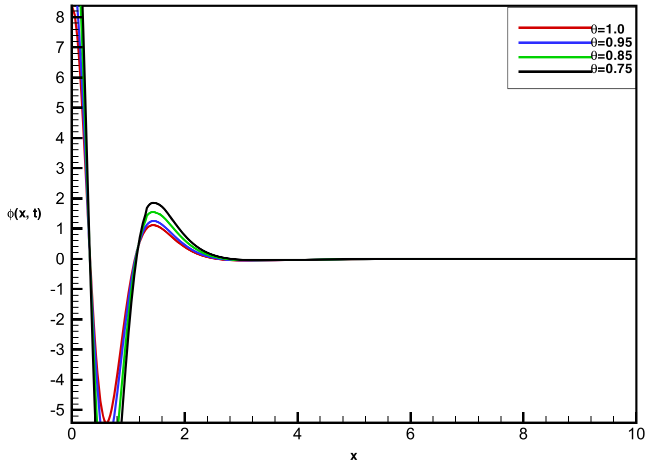

FIGURE 5 Plot of approximate solution at $t = 1.5$ of example 4 at different fractional order [Colour figure can be viewed at wileyonlinelibrary.com]

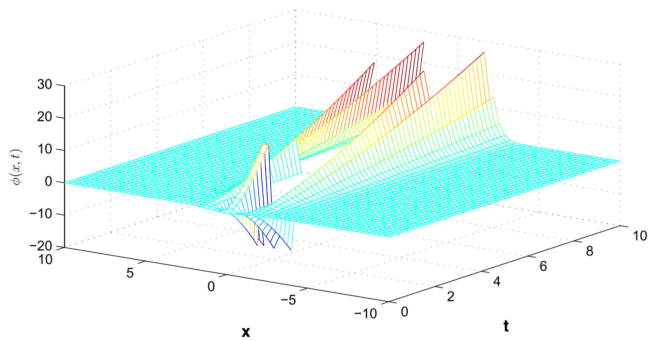

FIGURE 6 Plot of approximate solution at $\theta = 0.5$ of example 4 [Colour figure can be viewed at wileyonlinelibrary.com]

6 | CONCLUSIONS

In this paper, two aspects have been considered for the fractional order nonlinear KGEs. They are, numerical scheme for the approximate solution of the equations and Picard $\mathcal{T}$-stability. The numerical scheme is based on the SDM which is a coupling of Sumudu integral transform and Adomian decomposition method. For the efficiency of the scheme, two nonlinear illustrative examples are included. In order to continue study about the fractional order KGEs, we suggest the readers to consider the study of variable order fractional KGEs for one, two, and three dimensions. One may also consider the suggested equation for the Atangana-Baleanu fractional derivative and many others in place of Caputo.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

FINANCIAL DISCLOSURE

none reported.

AUTHOR CONTRIBUTIONS

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