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# Existence and uniqueness of solutions of linear variable coefficient discrete-time descriptor systems ☆,☆☆

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ABSTRACT

We consider linear discrete-time descriptor systems, i.e., systems of linear equations of the form $E_k x^{k+1} = A_k x^k + f^k$ for $k \in \mathbb{Z}$, where all $E_k$ and $A_k$ are matrices, $f_k$ are vectors and $x_k$ are the vectors of the solution we are looking for. We study the existence and uniqueness of solutions. A strangeness index is defined for such systems. Compared to the continuous-time case, see [P. Kunkel, V. Mehrmann, Differential-Algebraic Equations – Analysis and Numerical Solution, European Mathematical Society, Zürich, 2006], in the discrete-time case we have to account for the fact that it makes a difference, if one has an initial condition and one wants a solution in the future or if one has an initial condition and one wants a solution into the past and the future at the same time.

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## 1. Introduction

Let $\mathbb{I} \subset \mathbb{R}$ be an interval and let $\mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$ denote the space of all continuous functions mapping $\mathbb{I}$ into the space of all complex valued $m$-by-$n$ matrices. Also, let $\mathcal{C}(\mathbb{I}, \mathbb{C}^{n})$ denote the space $\mathcal{C}(\mathbb{I}, \mathbb{C}^{n,1})$. Consider the *linear time-varying continuous-time descriptor system*

$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0, \quad t \in \mathbb{I}, \tag{1}$$

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where $E, A \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m,n})$, $x \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{n})$ is the state vector, $f \in \mathcal{C}(\mathbb{I}, \mathbb{C}^{m})$ is the inhomogeneity and $x_0 \in \mathbb{R}^n$ is an initial condition given at the point $t_0 \in \mathbb{I}$. Using some constant rank assumptions, a canonical form for systems of the form (1) is developed in [7, Chapter 3] and there the notion of the strangeness index is introduced. Note, that for continuous-time systems (which satisfy some regularity conditions) it does not matter if the initial condition is given at a point $t_0$ which belongs to the interior of $\mathbb{I}$ or at a point $t_0$ which belongs to the boundary of $\mathbb{I}$.

The purpose of this paper is to obtain corresponding results for the discrete-time case. Therefore, let us first define two discrete intervals in the following way.

$$\mathbb{K} := \{k \in \mathbb{Z} : k_b \leqslant k \leqslant k_f\}, \quad k_b \in \mathbb{Z} \cup \{-\infty\}, \quad k_f \in \mathbb{Z} \cup \{\infty\},$$
$$\mathbb{K}^+ := \begin{cases} \mathbb{K} & \text{if } k_f = \infty, \\ \mathbb{K} \cup \{k_f + 1\} & \text{if } k_f < \infty. \end{cases}$$

With this definition we call

$$E_k x^{k+1} = A_k x^k + f^k, \quad x^{k_0} = x_0, \quad k \in \mathbb{K}, \tag{2}$$

a *linear time-varying discrete-time descriptor system*, where $E_k, A_k \in \mathbb{C}^{m,n}$ for $k \in \mathbb{K}$, $x^k \in \mathbb{C}^n$ for $k \in \mathbb{K}^+$ are the state vectors, $f^k \in \mathbb{C}^m$ for $k \in \mathbb{K}$ are the inhomogeneities and $x_0 \in \mathbb{R}^n$ is an initial condition given at the point $k_0 \in \mathbb{K}^+$. Such equations arise naturally from equations of type (1) by approximating $\dot{x}(t)$ via an explicit finite difference. Other applications of Eq. (2) include Singular Leontief Systems [4,11] and the Backward Leslie Model [5]. Also the discretization of partial differential equations in both the time- and the space-domain at the same time leads to systems of the form (2).

Systems of the form (2) have also received some theoretical attention, e.g., solvability of systems of the form (2) has already been studied in [9]. Periodic systems have been investigated in [14]. Some work regarding the associated control problem has been done, see [8,12,13,15].

In [9] only finite sequences of solutions are considered, i.e., system (2) with $\mathbb{K}$ being a finite subset of $\mathbb{Z}$. The problem of this approach is that one has to introduce two initial conditions (one at the beginning and one at the end) in order to fix a unique solution. We will almost only consider systems where at least one end is open, i.e., either $k_f = \infty$ or $k_b = -\infty$, because in this case we only need one initial condition to fix a unique solution. Also one can verify in a way similar to [3] that by moving between finite and infinite intervals only the last few elements of the finite solution are changed, with the exact number of changed elements depending on the strangeness index (see Definition 8) of the system.

In contrast to the continuous-time case, it makes a difference if the initial condition is fixed in the interior and we are looking for a solution on all of $\mathbb{Z}$ (also called the *two-way case*) or at the beginning of the interval $k_b$ and we are only looking for a solution for all $k \geqslant k_b$ (also called the *forward case*). Curiously enough, the forward case is more closely related to the continuous-time case than the two-way case. For instance, we first define a strangeness index for the forward case, analogously to [7], where continuous-time systems are considered and then have some additional work to do in order to transfer the results to the two-way case.

Throughout the paper we will use the notation $\{a_k\}_{k\in\mathbb{K}}$ or $\{b^k\}_{k\in\mathbb{K}}$ to denote a sequence and $a_k$ or $b^k$ to denote the $k$th element of the sequence. Also, if $a_k$ are matrices, we use the notation $a_k^{(i,j)}$ to denote the $(i,j)$th (block-)element of the matrix $a_k$ and the notation $b_i^k$ to denote the $i$th (block-)row of the vector $b^k$.

Considering a discrete-time descriptor system of the form (2) with variable coefficients and an initial condition given at $k_0 = 0$ we see that the original system

$$E_k x^{k+1} = A_k x^k + f^k, \quad x^0 = \hat{x}$$

is equivalent to the transformed system

$$P_k E_k Q_{k+1} Q_{k+1}^{-1} x^{k+1} = P_k A_k Q_k Q_k^{-1} x^k + P_k f^k, \quad Q_0^{-1} x^0 = Q_0^{-1} \hat{x}, \tag{3}$$

as long as all $P_k$ and $Q_k$ are invertible matrices. Defining $\widetilde{E}_k := P_k E_k Q_{k+1}$, $\widetilde{A}_k := P_k A_k Q_k$, $\tilde{x}^k := Q_k^{-1} x^k$, and $\tilde{f}_k := P_k f^k$ for all $k$, we can also write system (3) as

$$\widetilde{E}_k\tilde{x}^{k+1} = \widetilde{A}_k\tilde{x}^k + \tilde{f}^k, \quad \tilde{x}^0 = Q_0^{-1}\hat{x}.$$

This observation leads to the definition of the following equivalence relation.

**Definition 1.** Let $E_k, A_k, \widetilde{E}_k, \widetilde{A}_k \in \mathbb{C}^{m,n}$ for all $k \in \mathbb{K}$. Then two sequences of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{K}}$ and $\{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{K}}$ are called *globally equivalent* (on $\mathbb{K}$) if there exist two pointwise nonsingular matrix sequences $\{P_k\}_{k\in\mathbb{K}}$ with $P_k \in \mathbb{C}^{m,m}$ and $\{Q_k\}_{k\in\mathbb{K}^+}$ with $Q_k \in \mathbb{C}^{n,n}$ such that $P_kE_kQ_{k+1} = \widetilde{E}_k$ and $P_kA_kQ_k = \widetilde{A}_k$ for all $k \in \mathbb{K}$. We denote this equivalence by $\{(E_k, A_k)\}_{k\in\mathbb{K}} \sim \{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{K}}$.

Following the approach in [7] we first concentrate on one particular matrix pair in the sequence of matrix pairs to find out which transformations can be applied to this single matrix pair.

**Definition 2.** Two pairs of matrices $(E, A), (\widetilde{E}, \widetilde{A}) \in \mathbb{C}^{m,n}$ are called *locally equivalent* if there exist matrices $P \in \mathbb{C}^{m,m}$ and $Q, R \in \mathbb{C}^{n,n}$ that are all nonsingular, such that

$$\widetilde{E} = PEQ \quad \text{and} \quad \widetilde{A} = PAR.$$

Again, we denote this equivalence by $(E, A) \sim (\widetilde{E}, \widetilde{A})$.

Once we have seen that global equivalence is an equivalence relation it is easy to see that local equivalence is an equivalence relation, since we only have to consider the special case that $\mathbb{K} = \{1\}$.

## 2. Local invariants

In this section, we take a closer look at local equivalence and try to identify characteristic values, i.e., values that are invariant under local equivalence, since these characteristic values also have to be invariant under global equivalence. In other words, we adjust the result [7, Theorem 3.7] to the discrete-time case.

For convenience, we say in the following that a matrix is a basis of a vector space if this is valid for its columns. For matrix pairs of block matrices we also use the convention that corresponding blocks (i.e., blocks in the same block row and block column) have the same number of rows and columns.

**Theorem 3.** *Let $E, A \in \mathbb{C}^{m,n}$. Let the matrix $Z$ be a basis of* $\operatorname{corange}(E) = \operatorname{kernel}\left(E^H\right)$ *and let the matrix $Y$ be a basis of* $\operatorname{corange}(A) = \operatorname{kernel}\left(A^H\right)$. *Then, the quantities*

$$\begin{aligned}
r_f &= \operatorname{rank}(E) && (\textit{corresponds to forward direction})\\
r_b &= \operatorname{rank}(A) && (\textit{corresponds to backward direction})\\
h_f &= \operatorname{rank}\left(Z^HA\right) && (\textit{rank of } Z^HA;\ \textit{forward})\\
h_b &= \operatorname{rank}\left(Y^HE\right) && (\textit{rank of } Y^HE;\ \textit{backward})\\
&= r_f + h_f - r_b, && \\
c &= r_b - h_f && (\textit{common part})\\
a &= \min(h_f, n - r_f) && (\textit{algebraic part})\\
s &= h_f - a && (\textit{strangeness})\\
d &= r_f - c - s && (\textit{difference part})\\
u &= n - r_f - a && (\textit{undetermined variables})\\
v &= m - r_f - h_f && (\textit{vanishing equations})
\end{aligned}$$

*are invariant under local equivalence, and $(E, A)$ is locally equivalent to the canonical form*

$$(\widetilde{E}, \widetilde{A}) = \left( \begin{bmatrix} I_s & 0 & 0 & 0 & 0\\ 0 & I_d & 0 & 0 & 0\\ 0 & 0 & I_c & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I_c & 0 & 0\\ 0 & 0 & 0 & I_a & 0\\ I_s & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right), \tag{4}$$

*where the last block column has $u$ columns and the last block row has $v$ rows. We have that either $s = 0$, $u = 0$ or $s = u = 0$. The quantities defined above are called local characteristics or local invariants of the matrix pair $(E, A)$.*

**Proof.** One could first reduce $(E, A)$ to Kronecker canonical form and then apply further allowed local equivalence transformations. For a proof not employing the Kronecker canonical form see the extended version of the present paper [2, Theorem 3]. □

Comparing this result to the analogous continuous-time result [7, Theorem 3.7] one notices the additional "common" part. This part cannot be eliminated, since local equivalence does not allow changes of the matrix $A$ by means of the matrix $E$.

## 3. Forward global canonical form

Similar to the results in [3], where the time-invariant case is studied, we first concentrate on the case where one starts at some time point (here this time point is always $k = 0$) and calculates into the future, i.e., one tries to get a solution for $k \geqslant 0$. In order to derive a global canonical form, some constant rank assumptions are introduced. Milder assumptions are necessary in this case, than in the case where one wants to get a solution for all $k \in \mathbb{Z}$. Despite the issue that we only want to get a solution for $k \geqslant 0$ we consider linear descriptor systems with equations for all $k \in \mathbb{Z}$ (i.e., systems of the type (2) with $\mathbb{K} = \mathbb{Z}$), since this simplifies moving to the case where we want to get a solution for all $k \in \mathbb{Z}$. This is no restriction, since every linear descriptor system of the form $\{(E_k, A_k)\}_{k \in \mathbb{N}_0}$ can be extended to one of the form $\{(E_k, A_k)\}_{k \in \mathbb{Z}}$ by choosing $E_k = E_0$, $A_k = A_0$, and $f_k = f_0$ for all $k < 0$. This extension could also be done by choosing $E_k = 0$, $A_k = 0$, and $f_k = 0$ for all $k < 0$, but this sequence of matrix pairs would in general not have a well-defined strangeness index (see Definition 8) any more.

Note that we use here the term canonical form in a way that differs from the terminology of abstract algebra, where a canonical form is required to be the most canonical representative of an equivalence class. The form (6) merely displays all the information that we are interested in although further reductions could be applied which, however, would unnecessarily complicate things.

**Lemma 4.** *Consider system (2) and introduce the matrix sequence $\{Z_k\}_{k \in \mathbb{K}}$, where $Z_k$ is a basis of* corange $(E_k) = \text{kernel}\left(E_k^H\right)$ *for all $k \in \mathbb{K}$. Let*

$$r_f^k = \text{rank}\,(E_k)\,, \quad k \in \mathbb{K},$$
$$r_b^k = \text{rank}\,(A_k)\,, \quad k \in \mathbb{K},$$
$$h_f^k = \text{rank}\left({Z_k}^H A_k\right), \quad k \in \mathbb{K}$$

*be the local characteristics of each matrix pencil $(E_k, A_k)$ with $k \in \mathbb{K}$. Then, these characteristic sequences are invariant under global equivalence.*

*Assume further that the two local characteristic sequences*

$$r_f \equiv r_f^k \quad \text{and} \quad h_f \equiv h_f^k \tag{5}$$

*are constant for all $k \in \mathbb{K}$. Then the sequence of matrix pairs $\{(E_k, A_k)\}_{k \in \mathbb{K}}$ is globally equivalent to the sequence*

$$\left\{\left(\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1)} & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k \in \mathbb{K}}, \tag{6}$$

*where all matrices $\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \end{bmatrix}$ have full row rank, i.e., they all are of rank $r_f$.*

**Proof.** The invariance of the local characteristics follows directly from Theorem 3. Let $Z_k'$ be a basis of range $(E_k)$ for all $k \in \mathbb{K}$. Then $[Z_k' \quad Z_k]$ is invertible for all $k \in \mathbb{K}$ and ${Z_k'}^H E_k$ has full row rank $r_f$. Transforming with $[Z_k' \quad Z_k]^H$ from the left yields the assertion since

$$\begin{aligned}\{(E_k, A_k)\}_{k\in\mathbb{K}} &\sim \left\{\left(\begin{bmatrix}{Z_k'}^H E_k\\ 0\end{bmatrix}, \begin{bmatrix}{Z_k'}^H A_k\\ Z_k^H A_k\end{bmatrix}\right)\right\}_{k\in\mathbb{K}}\\ &\sim \left\{\left(\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\\ 0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}A_k^{(1)} & A_k^{(2)}\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{K}}\\ &\sim \left\{\left(\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\\ 0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}A_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{K}}. \qquad \square\end{aligned}$$

Writing down the equations from (2) connected with the form (6) one obtains the system

$$\begin{aligned}&E_k^{(1)} x_1^{k+1} + E_k^{(2)} x_2^{k+1} = A_k^{(1)} x_1^k + f_1^k,\\ &0 = x_2^k + f_2^k,\\ &0 = f_3^k\end{aligned}$$

for $k \in \mathbb{K}$. Assuming that $\mathbb{K} = \mathbb{Z}$ (or $\mathbb{K} = \mathbb{N}_0$), this system is equivalent to the system given by

$$\begin{aligned}&E_k^{(1)} x_1^{k+1} = A_k^{(1)} x_1^k + \tilde{f}_1^k,\\ &0 = x_2^k + f_2^k,\\ &0 = f_3^k\end{aligned}$$

(where $\tilde{f}_1^k = f_1^k + E_k^{(2)} f_2^{k+1}$) which is connected with the sequence of matrix pairs

$$\left\{\left(\begin{bmatrix}E_k^{(1)} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}A_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{K}}. \tag{7}$$

**Remark 5.** Since the reduction from (6) to (7) is reversible the set of solution sequences is not altered by the reduction. One may also notice that the new inhomogeneity $\tilde{f}^k$ can depend on the inhomogeneity of the former successive inhomogeneity $f^{k+1}$. Hence, looking at the results in [3] one can interpret this step as an index reduction.

Although the reduction step from (6) to (7) does not alter the set of solutions it is not a global equivalence transformation in the sense of Definition 1. One could therefore introduce the reduction from (6) to (7) in a more formal definition by extending the notion of equivalence to triples of the form $\{(E_k, A_k, f_k)\}_{k\in\mathbb{Z}}$, i.e., one could include the inhomogeneity $f^k$ in the equivalence transformations. Further analysis is necessary to develop a proper definition and a suitable notation. Here we settle for only using an informal definition for the sake of shortness of the presentation.

Analogous to [7, Theorem 3.14], the following Theorem 6 states, that reduced sequences of matrix pairs of the form (7) are still globally equivalent, if the original sequences (6) have been.

**Theorem 6.** *Assume that the sequences of matrix pairs*

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{\left(\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\\ 0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}A_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

*and*

$$\{(\widetilde{E}_k,\widetilde{A}_k)\}_{k\in\mathbb{Z}} = \left\{\left(\begin{bmatrix}\widetilde{E}_k^{(1)} & \widetilde{E}_k^{(2)}\\ 0 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}\widetilde{A}_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

*are globally equivalent on $\mathbb{Z}$ and in the form (6). In particular, suppose that (5) holds and that all $\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\end{bmatrix}$ and all $\begin{bmatrix}\widetilde{E}_k^{(1)} & \widetilde{E}_k^{(2)}\end{bmatrix}$ have full row rank $r_f$. Then we also have the following global equivalence relation:*

$$\left\{\left(\begin{bmatrix}E_k^{(1)} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}A_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix}\widetilde{E}_k^{(1)} & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}\widetilde{A}_k^{(1)} & 0\\ 0 & I_{h_f}\\ 0 & 0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

**Proof.** By assumption, there exist two pointwise nonsingular matrix sequences $\{P_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^{m,m}$ and $\{Q_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}^{n,n}$, such that

$$P_kE_k = \widetilde{E}_kQ_{k+1} \quad \text{and} \quad P_kA_k = \widetilde{A}_kQ_k \tag{8}$$

for all $k \in \mathbb{Z}$. By partitioning the transforming matrices $P_k$ and $Q_k$ according to the block structure of $E_k$ and $A_k$ as

$$P_k = \begin{bmatrix}P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)}\\ P_k^{(2,1)} & P_k^{(2,2)} & P_k^{(2,3)}\\ P_k^{(3,1)} & P_k^{(3,2)} & P_k^{(3,3)}\end{bmatrix} \quad \text{and} \quad Q_k = \begin{bmatrix}Q_k^{(1,1)} & Q_k^{(1,2)}\\ Q_k^{(2,1)} & Q_k^{(2,2)}\end{bmatrix},$$

we obtain from (8) that $P_k^{(3,2)} = 0$ for all $k \in \mathbb{Z}$. From (8) we also obtain that $P_k^{(2,1)} = 0$, $P_k^{(3,1)} = 0$ for all $k \in \mathbb{Z}$, since we assumed that all matrices $\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\end{bmatrix}$ have full row rank. This shows that the left transforming matrices take the form

$$P_k = \begin{bmatrix}P_k^{(1,1)} & P_k^{(1,2)} & P_k^{(1,3)}\\ 0 & P_k^{(2,2)} & P_k^{(2,3)}\\ 0 & 0 & P_k^{(3,3)}\end{bmatrix}.$$

Hence, the diagonal matrices $P_k^{(1,1)}, P_k^{(2,2)}, P_k^{(3,3)}$ have to be nonsingular. Since from (8) we also get that $Q_k^{(2,1)} = P_k^{(2,1)}A_k^{(1)} = 0$, it follows that all matrices $Q_k^{(1,1)}, Q_k^{(2,2)}$ are also invertible, which proves the claim. For a more detailed proof, see the extended version of the present paper [2, Corollary 6]. □

In this section we have obtained the canonical forms (6) and (7). A more advanced canonical form that requires the notion of the strangeness index will be introduced in the next chapter.

## 4. The strangeness index

The preceding results allow for an inductive procedure closely related to the corresponding procedure for continuous-time systems [7]. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}} =: \{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}}$ we define a sequence (of sequences of matrix pairs) $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ by Lemma 4 to the form (6) assuming that its local invariants are constant and we set

$$r_f =: r_{f,i} \quad \text{and} \quad h_f =: h_{f,i}. \tag{9}$$

Then we reduce the so obtained sequence of matrix pairs to the form (7) which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}}$. This whole iterative process (although derived from [7]) is very similar to Luenberger's shuffle algorithm, which is described in [10] for discrete-time descriptor systems with constant coefficients.

Observe that we have to have the constant rank assumptions of the form (5) for every step of the iterative procedure, i.e., for every sequence $\{E_{k,i}, A_{k,i}\}_{k\in\mathbb{Z}}$ with $i \in \mathbb{N}_0$. Due to Theorem 6 the so obtained sequence of global invariants $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs. Several properties of this sequence are summed up in the following Lemma 7.

**Lemma 7.** *Let the sequences $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ and $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ be defined as in (9). In particular, let the constant rank assumptions (5) hold for every $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$. Defining the quantities*

$$h_{f,-1} := 0, \tag{10a}$$
$$s_i := r_{f,i} - r_{f,i+1}, \tag{10b}$$
$$v_i := m - r_{f,i} - h_{f,i} \tag{10c}$$

*for all $i \in \mathbb{N}_0$ we have that*

$$r_{f,i} \geqslant r_{f,i+1}, \tag{11a}$$
$$h_{f,i} \leqslant h_{f,i+1}, \tag{11b}$$
$$v_{i+1} \geqslant v_i, \tag{11c}$$
$$s_i \geqslant s_{i+1}, \tag{11d}$$
$$s_i, v_i \geqslant 0 \tag{11e}$$

*for all $i \in \mathbb{N}_0$. Further there exists a $\mu \in \mathbb{N}_0$ so that*

$$s_{i+\mu} = 0 \tag{11f}$$

*for all $i \in \mathbb{N}_0$.*

**Proof.** Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Lemma 4 that

$$\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

Thus from the Definition of the iterative process at (9) we have

$$\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 \\ 0 & I_{h_{f,i}} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{12}$$

which shows that

$$r_{f,i} = \operatorname{rank}\left(\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix}\right) \geqslant \operatorname{rank}\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 \end{bmatrix}\right) = r_{f,i+1}$$

and thus (11a) follows. This means that the sequence $\{r_{f,i}\}$ is non-increasing and it is also bounded from below by zero. Thus it becomes stationary at some point $\mu$, which implies (11f). Since $\begin{bmatrix} E_{k,i}^{(1)} & E_{k,i}^{(2)} \end{bmatrix}$ has full row rank $r_{f,i}$, we get

$$\dim\left(\operatorname{range}\left(E_{k,i}^{(1)}\right)\right) + \dim\left(\operatorname{corange}\left(E_{k,i}^{(1)}\right)\right) = r_{f,i}$$

and independent of this

$$\dim\left(\operatorname{range}\left(E_{k,i}^{(1)}\right)\right) = r_{f,i+1},$$

since the constant rank assumptions (5) also holds for $\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}}$. For $k \in \mathbb{Z}$ let $Z_k$ be a basis of $\operatorname{corange}\left(E_{k,i}^{(1)}\right)$. Then we know that

$$h_{f,i+1} = h_{f,i} + \operatorname{rank}\left(Z_k^H A_{k,i}^{(1)}\right) \leqslant h_{f,i} + \dim\left(\operatorname{corange}\left(E_{k,i}^{(1)}\right)\right),$$

which implies (11b) and also that

$$\begin{aligned} r_{f,i} + h_{f,i} &= \dim\left(\text{range}\left(E_{k,i}^{(1)}\right)\right) + \dim\left(\text{corange}\left(E_{k,i}^{(1)}\right)\right) + h_{f,i} \\ &\geqslant r_{f,i+1} + h_{f,i+1} \end{aligned}$$

from which we see (11c). The remainder of the proof involves a rather sophisticated inductive argument and can be found in the extended version of the present paper [2, Lemma 7]. □

Lemma 7 leads to the following Definition 8, according to [7, Definition 3.15].

**Definition 8.** Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs and let the corresponding sequence of characteristic values $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ as in (9) be well defined. In particular, let (5) hold for every entry $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ of the sequence (of sequences of matrix pairs). Then, with definition (10b) we call

$$\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\}, \tag{13}$$

the *strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and of the associated descriptor system (2). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and (2) *strangeness-free*.

From (12) we see that (under some constant rank assumptions) after $\mu + 1$ reduction steps from (6) to (7) and $\mu + 1$ equivalence transformations every sequence of matrix pairs $\{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}}$ can be transformed to a sequence of the form

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_{k,\mu}^{(1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 \\ 0 & I_{h_{f,\mu}} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

with all $E_{k,\mu}^{(1)}$ having full row rank $r_{f,\mu+1} = r_{f,\mu}$ since $s_\mu = 0$. In a last step, one can further reduce all those matrices $E_{k,\mu}^{(1)}$ to the echelon form $[I_{r_{f,\mu}} \;\; 0]$ by global equivalence achieving (with adapted indexing)

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} I_{r_{f,\mu}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & A_{k,\mu}^{(2)} \\ 0 & I_{h_{f,\mu}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{14}$$

which can be regarded as a canonical form. One notices that in general not only $\mu$ but $\mu + 1$ reduction steps are necessary to get to the canonical form, although after $\mu$ reduction steps a strangeness-free sequence has already been reached. This situation can be avoided by introducing a further constant rank assumption in every step of the reduction process described at (9) (see [6]).

**Remark 9.** It is also possible to obtain a canonical form for sequences of matrix pairs with well-defined strangeness index without performing the reduction from (6) to (7), see [2, Theorem 9]. This canonical form is more complicated and similar to the results of Theorem 11 it can also be used to make (more complicated) statements about the existence and uniqueness of solutions.

**Remark 10.** For the discrete-time case studied here one can also obtain results corresponding to [7, Section 3.2], i.e., one can determine the characteristic values $\{(r_{f,i}, h_{f,i})\}_{i\in\mathbb{N}_0}$ from the local characteristic values (compare Theorem 3) of the so called inflated descriptor systems. The according theorems and proofs can be found in [1, Section 3.2.1].

*4.1. Existence and uniqueness of solutions*

Concerning existence and uniqueness of sequences of matrix pairs with well-defined strangeness index we get similar results as in [7].

**Theorem 11.** *Let the strangeness index $\mu$ of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ as in (13) be well defined. Then the discrete-time descriptor system (2) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete-time descriptor system of the form*

$$\begin{array}{rll} x_1^{k+1} & = A_k^{(1)} x_1^k + A_k^{(3)} x_3^k + f_1^k, & r_{f,\mu} \\ 0 & = x_2^k + f_2^k, & h_{f,\mu} \\ 0 & = f_3^k, & v_\mu \end{array}$$

*where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_3^k \in \mathbb{C}^{u_\mu}$ and each inhomogeneity $f_1^k, f_2^k, f_3^k$ is determined by the original inhomogeneities $f^k, \ldots, f^{k+\mu+1}$ as in (2) for all $k \in \mathbb{Z}$. For the associated forward problem*

$$E_k x^{k+1} = A_k x^k + f^k \quad \text{for all } k \in \mathbb{N}_0, \tag{15}$$

*we obtain the following existence and uniqueness results. System (15) is solvable if and only if the $v_\mu$ consistency conditions $f_3^k = 0$ are fulfilled for all $k \in \mathbb{N}_0$. An initial condition $x^0 = \hat{x}$ is consistent with system (15) if and only if in addition the $h_{f,\mu}$ conditions $x_2^0 = \hat{x}_2 = -f_2^0$ are satisfied. The corresponding initial value problem is uniquely solvable if and only if in addition $u_\mu = 0$ holds.*

**Proof.** Under the assumptions of Theorem 11 we see that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ can be transformed to the form (14) by $\mu + 1$ reduction steps from (6) to (7) and proper global equivalence transformations. Both of these operations generate a one-to-one correspondence of solutions. Similar to Remark 5 we see that the strangeness index $\mu$ tells us how far the system looks into the future, i.e., that for all $k \in \mathbb{Z}$ each inhomogeneity $f_1^k, f_2^k, f_3^k$ is really only determined by the original inhomogeneities $f^k, \ldots, f^{k+\mu+1}$ from (15). $\square$

## 5. Backward global canonical form

In the previous section we have constructed a canonical form which allows for statements about the solvability of descriptor systems where one starts at some point $k_0$ and computes a solution into the future. Let us now have a short look at the case where one starts at a point in time $k_0$, and calculates into the past, i.e., one calculates a solution $\{x^k\}_{k\leqslant k_0}$. This case is closely related to the first case. To see this, suppose that a descriptor system of the form

$$E_k x^{k+1} = A_k x^k + f^k, \quad k \leqslant k_0 - 1,$$

together with the initial condition $x^{k_0} = \hat{x}$ is given and we are looking for a solution. Substituting $k$ by $-k$ then yields

$$E_{-k} x^{-k+1} = A_{-k} x^{-k} + f^{-k}, \quad k \geqslant 1 - k_0.$$

Defining $y^k := x^{-k+1}$ and $g^k := -f^{-k}$, this system is equivalent to

$$A_{-k} y^{k+1} = E_{-k} y^k + g^k, \quad k \geqslant -k_0 + 1.$$

By calculating the solution of the very last system into the future with the initial condition $y^{-k_0+1} = \hat{x}$, i.e., by calculating $\{y^k\}_{k\geqslant -k_0+1}$, we see through resubstitution that we have obtained a solution

$$\begin{aligned} \{y^k\}_{k\geqslant -k_0+1} &= \{x^{-k+1}\}_{k\geqslant -k_0+1} = \{x^{k+1}\}_{-k\geqslant -k_0+1} \\ &= \{x^{k+1}\}_{k\leqslant k_0-1} = \{x^k\}_{k-1\leqslant k_0-1} = \{x^k\}_{k\leqslant k_0}, \end{aligned}$$

i.e., a solution of the backward problem with "initial" condition. Thus, we do not have to consider the backward case separately. Instead we make the following definition.

**Definition 12.** Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Then

$$\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}} \tag{16}$$

is called the *reversed sequence of matrix pairs*. Analogously, the descriptor system corresponding to (16) is called the *reversed descriptor system*. Also, the strangeness index of (16) is call *reversed strangeness* *index* and is denoted by $\mu_b$ (for backwards). In contrast to this, the strangeness index of the original sequence is also called *forward strangeness index* and denoted by $\mu_f$.

The following Lemma 13 is obvious.

**Lemma 13.** *Let* $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ *and* $\{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{Z}}$ *be two globally equivalent sequences of matrix pairs. Then the reversed sequences are also globally equivalent.*

## 6. A two-way global canonical form

Finally, we consider the case where we want to obtain a solution for all $k \in \mathbb{Z}$. This case is somehow different from the forward and backward case. To see this consider the form (14). The problem is that in this (forward strangeness-free) form (14) the $A^{(1)}_{k,\mu}$ are allowed to be arbitrary. Consider a descriptor system which only consists of the (1,1) block in (14). For such a system one can easily compute the unique value of $x^{k_0+1}$ once the value of $x^{k_0}$ is given. In contrast, if the value for $x^{k_0}$ is given there may be many choices of appropriate $x^{k_0-1}$ values (e.g., for the system given by the equations $x^{k_0} = 0x^{k_0-1}$, $x^{k_0-1} = x^{k_0-2}$, $x^{k_0-2} = x^{k_0-3}$, ...) or even no possible choice of an appropriate $x^{k_0-1}$ value (e.g., for the system given by the equations $x^{k_0} = x^{k_0-1}$, $x^{k_0-1} = 0x^{k_0-2}$, provided that $x^{k_0} \neq 0$), depending on the sequence of the $A^{(1)}_{k,\mu}$ matrices. Also, the solvability may vary from iterate to iterate. It seems that additional rank assumptions are appropriate to obtain a canonical form which allows for statements about the solvability in the two-way case.

One approach that suggests itself is to not only demand the system itself to have well-defined strangeness index but to also demand the reversed system to have well-defined strangeness index, i.e., to demand the system to have a well-defined reversed strangeness index. To study such systems, the following Lemma 14 is very helpful.

**Lemma 14.** *For* $k \in \mathbb{Z}$ *let* $E_k, A_k \in \mathbb{C}^{m,n}$ *be such matrices, that the strangeness index* $\mu_f$ *and the reversed strangeness index* $\mu_b$ *of* $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ *are both well defined. Perform one step of index reduction from (6) to (7) on the reversed sequence* $\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}}$ *and denote the so obtained sequence by* $\{(\widetilde{A}_{-k}, \widetilde{E}_{-k})\}_{k\in\mathbb{Z}}$. *Then, not only the reversed strangeness index* $\tilde{\mu}_b$ *(i.e., the strangeness index of* $\{(\widetilde{A}_{-k}, \widetilde{E}_{-k})\}_{k\in\mathbb{Z}}$*) but also the strangeness index* $\tilde{\mu}_f$ *of* $\{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{Z}}$ *is well defined. We have* $\tilde{\mu}_f \leqslant \mu_f$ *and* $\tilde{\mu}_b \leqslant \mu_b$.

**Proof.** The very complicated proof can be found in the extended version of the present paper [2, Lemma 14]. It involves the derivation of a canonical form for descriptor systems with well-defined strangeness index which only uses global equivalence transformations (but no reductions from (6) to (7), compare Remark 9). Reversing this canonical form, performing the index reduction from (6) to (7), and reversing the reduced system back we can show that the final system again is in the canonical form mentioned in Remark 9, which implies that it has a well-defined strangeness index. □

The index reduction performed in Lemma 14 will be used frequently, which is why we introduce the following Definition.

**Definition 15.** Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Then performing one step of index reduction from form (6) to (7) on the reversed sequence $\{(A_{-k}, E_{-k})\}_{k\in\mathbb{Z}}$ and re-reversing the so obtained reduced sequence is called one step of *reversed index reduction*. In contrast to this, the index reduction from (6) to (7) on the original sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is also called *forward index reduction*.

Lemma 14 shows that under the assumption that both the strangeness index and the reversed strangeness index are well defined one can perform forward and reversed index reduction steps at will.

One may conjecture, that one obtains globally equivalent sequences of matrix pairs, as long as one performs the same number of forward and reversed index reduction steps, but as we will see in the following Example 16 this is false. Example 16 also shows that one step of reversed index reduction can alter the forward strangeness index.

**Example 16.** Consider the constant sequence of matrix pairs

$$\left\{\left(\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{17}$$

First performing one forward step of index reduction on (17) yields the sequence

$$\left\{\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{18}$$

Performing a step of reversed index reduction on (18) does not alter the sequence any more. First performing one step of reversed index reduction on (17), however, yields the sequence

$$\left\{\left(\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{19}$$

which again is not altered anymore by a further step of forward index reduction. Comparing (18) with (19) and using Lemma 4 clearly shows that these two sequences are not globally equivalent, since those sequences of matrix pairs do not have the same characteristic values.

Let us first derive a canonical form under the assumption that both the strangeness index and the reversed strangeness index are well defined.

**Theorem 17.** *For $k\in\mathbb{Z}$ let $E_k, A_k\in\mathbb{C}^{m,n}$ be matrices, such that the strangeness index and the reversed strangeness index of $\{(E_k,A_k)\}_{k\in\mathbb{Z}}$ are both well defined. Introduce the matrix sequences $\{Z_k\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$ so that $Z_k$ is a basis of* corange $(E_k)$ *and $Y_k$ is a basis of* corange $(A_k)$ *for $k\in\mathbb{Z}$. Then, there exist $h_f, h_b, q\in\mathbb{N}_0$ such that for all $k\in\mathbb{Z}$ we have*

$$\begin{aligned}
h_f &= \operatorname{rank}\left(Z_k^H A_k\right) \qquad \textit{(forward direction)}\\
h_b &= \operatorname{rank}\left(Y_k^H E_k\right) \qquad \textit{(backward direction)}\\
q &= h_f + h_b - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k\\ Z_{k+1}^H A_{k+1}\end{bmatrix}\right).
\end{aligned} \tag{20}$$

*These quantities in (20) are invariant under global equivalence and we have*

$$\{(E_k,A_k)\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix} E_k^{(1)} & 0 & 0 & E_k^{(2)}\\ 0 & I_{h_b-q} & 0 & 0\\ 0 & 0 & I_q & 0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\end{bmatrix},\begin{bmatrix} A_k^{(1)} & A_k^{(2)} & 0 & 0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&I_q&0\\ 0&0&0&I_{h_f-q}\\ 0&0&0&0\end{bmatrix}\right)\right\}, \tag{21}$$

*where for all $k\in\mathbb{Z}$ the matrices $\begin{bmatrix}E_k^{(1)} & E_k^{(2)}\end{bmatrix}$ and $\begin{bmatrix}A_k^{(1)} & A_k^{(2)}\end{bmatrix}$ have full row rank.*

**Proof.** That rank $\left(Z_k^H A_k\right)$ and rank $\left(Y_k^H E_k\right)$ are invariant under global equivalence and constant for all $k\in\mathbb{Z}$ follows from Lemma 3 and the assumption that the strangeness index and the reversed strangeness index are both well defined.

To show that $q$ is independent of the choice of the bases and thus well defined for a given sequence of matrix pairs, let $Y_k$ and $\tilde{Y}_k$ be bases of corange $(A_k)$ and let $Z_k$ and $\tilde{Z}_k$ be bases of corange $(E_k)$ for all $k \in \mathbb{Z}$. Then for all $k \in \mathbb{Z}$ there exists invertible matrices $M_{Y_k}$ and $M_{Z_k}$ such that $Y_k = \tilde{Y}_k M_{Y_k}$ and $Z_k = \tilde{Z}_k M_{Z_k}$. This shows that

$$\operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} M_{Y_k}^{-H} Y_k^H E_k \\ M_{Z_{k+1}}^{-H} Z_{k+1}^H A_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \tilde{Y}_k^H E_k \\ \tilde{Z}_{k+1}^H A_{k+1} \end{bmatrix}\right)$$

and thus, that $q$ is independent of the choice of the bases. To show the invariance under global equivalence, let $\{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{Z}}$ be globally equivalent to $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$, i.e., let $Q_k$ and $P_k$ be invertible matrices, such that for all $k \in \mathbb{Z}$ we have $E_k = P_k \widetilde{E}_k Q_{k+1}$ and $A_k = P_k \widetilde{A}_k Q_k$. Since

$$0 = Y_k^H A_k = Y_k^H P_k \widetilde{A}_k Q_k = \left(P_k^H Y_k\right)^H \widetilde{A}_k Q_k,$$
$$0 = Z_k^H E_k = Z_k^H P_k \widetilde{E}_k Q_{k+1} = \left(P_k^H Z_k\right)^H \widetilde{E}_k Q_{k+1},$$

it is clear that $\widehat{Y}_k := P_k^H Y_k$ is a basis of corange $(\widetilde{A}_k)$ and that $\hat{Z}_k := P_k^H Z_k$ is a basis of corange $(\widetilde{E}_k)$. With this we see that

$$\operatorname{rank}\left(\begin{bmatrix} \widehat{Y}_k^H \widetilde{E}_k \\ \hat{Z}_{k+1}^H \widetilde{A}_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \widehat{Y}_k^H \widetilde{E}_k Q_{k+1} \\ \hat{Z}_{k+1}^H \widetilde{A}_{k+1} Q_{k+1} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right),$$

which means that $q$ does only depend on the equivalence class.

Since the strangeness index of $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is well defined, it is clear that the sequence can be transformed to the form (6). Since the reversed strangeness index is also well defined, we also know that all $A_k$ have constant rank. Thus, in (6) all $A_k^{(1)}$ matrices also have to have constant rank. Thus, by transforming the first block row of (6) from the left we have that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is equivalent to a sequence of matrix pairs of the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \tag{22}$$

with all $A_k^{(1,1)}$ having full (constant) row rank. Performing one (ordinary) reduction step from (6) to (7) on (22) yields the sequence

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 \\ E_k^{(2,1)} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{23}$$

Then it follows from Lemma 14 that (23) still has well-defined reversed strangeness index. Let $\tilde{Y}_k$ be bases of the second matrices in (23) for all $k \in \mathbb{Z}$. Then clearly $\tilde{Y}_k = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}^T$, since all $A_k^{(1,1)}$ have full row rank. Thus, since the reversed strangeness index of (23) is well defined, we know that for every $k \in \mathbb{Z}$ the matrix $E_k^{(2,1)}$ has to have constant rank, which will be called $\hat{g}$ in the following. By reducing all $E_k^{(2,1)}$ in (22) to echelon form and adapting the indexing we then see that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of matrix pairs of the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} & E_k^{(1,3)} \\ 0 & I_{\hat{g}} & E_k^{(2,3)} \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

$$\sim \left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & \widetilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_f} \\ 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

where all $\begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} \end{bmatrix}$ have full row rank. Also, all $\begin{bmatrix} E_k^{(1,1)} & 0 & \widetilde{E}_k^{(1,3)} \\ 0 & I_{\hat{g}} & 0 \\ 0 & 0 & E_k^{(3,3)} \end{bmatrix}$ have full row rank, since those matrices are equivalent to the matrices $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ as in (22), which have full row rank. So all $E_k^{(3,3)}$ also have full row rank. Reducing all $E_k^{(3,3)}$ to echelon form then finally shows that $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of matrix pairs of the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & E_k^{(1,3)} & E_k^{(1,4)} \\ 0 & I_{\hat{g}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f-\hat{q}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

$$\sim \left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & E_k^{(1,4)} \\ 0 & I_{h_b-\hat{q}} & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_k^{(1,1)} & A_k^{(1,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{\hat{q}} & 0 \\ 0 & 0 & 0 & I_{h_f-\hat{q}} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}, \tag{24}$$

where $h_b := \hat{g} + \hat{q}$ has been used. Finally, we have $q = \hat{q}$, since (as shown above) the quantity defined in (20) is invariant under global equivalence and $q$ can directly be calculated from (24). $\square$

From the form (21) one may conjecture that it is also possible to show Theorem 17 by defining

$$q = h_f + h_b - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_k^H A_k \end{bmatrix}\right) \tag{25}$$

instead of (20). This is not the case. If one would do so, $q$ would not be invariant under global equivalence any more as shown by the following Example 18.

**Example 18.** Define the (constant) sequence of matrix pairs

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} := \left\{\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}},$$

which has $h_f = 1, h_b = 1$ and with both (25) or (20) $q = 1$. Transforming this sequence from the right by the sequence $\{Q_k\}_{k\in\mathbb{Z}}$ defined through

$$Q_{2k} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad Q_{2k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for all } k \in \mathbb{Z}$$

will yield a sequence $\{(\widetilde{E}_k, \widetilde{A}_k)\}_{k\in\mathbb{Z}} = \{(E_k Q_{k+1}, A_k Q_k)\}_{k\in\mathbb{Z}}$ which satisfies

$$(\widetilde{E}_{2k}, \widetilde{A}_{2k}) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right) \quad \text{and}$$

$$(\widetilde{E}_{2k+1},\widetilde{A}_{2k+1}) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) \quad \text{for all } k \in \mathbb{Z}.$$

This sequence would have $q = 0$ if one would apply definition (25).

The same result as in Theorem 17 can be obtained under a weaker assumption.

**Corollary 19.** *Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence and introduce the matrix sequences $\{Z_k\}_{k\in\mathbb{Z}}$ and $\{Y_k\}_{k\in\mathbb{Z}}$ so that $Z_k$ is a basis of* corange $(E_k)$ *and $Y_k$ is a basis of* corange $(A_k)$ *for $k \in \mathbb{Z}$. Assume that the quantities*

$$r_f = r_{f,k} \equiv \operatorname{rank}(E_k), \tag{26a}$$

$$h_f = h_{f,k} \equiv \operatorname{rank}\left(Z_k^H A_k\right), \tag{26b}$$

$$h_b = h_{b,k} \equiv \operatorname{rank}\left(Y_k^H E_k\right), \tag{26c}$$

$$q = q_k \equiv h_{f,k} + h_{b,k} - \operatorname{rank}\left(\begin{bmatrix} Y_k^H E_k \\ Z_{k+1}^H A_{k+1} \end{bmatrix}\right) \tag{26d}$$

(*which are invariant under global equivalence as shown in Theorem* 17) *are constant for all $k \in \mathbb{Z}$. Then we also have the relation* (21), *where for all $k \in \mathbb{Z}$ the matrices $\begin{bmatrix} E_k^{(1)} & E_k^{(2)} \end{bmatrix}$ and $\begin{bmatrix} A_k^{(1)} & A_k^{(2)} \end{bmatrix}$ have full row rank*.

**Proof.** First we note that under the given assumptions the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is globally equivalent to a sequence of the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} A_k^{(1,1)} & 0 \\ 0 & 0 \\ 0 & I_{h_f} \\ 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}$$

with all $\begin{bmatrix} E_k^{(1,1)} & E_k^{(1,2)} \\ E_k^{(2,1)} & E_k^{(2,2)} \end{bmatrix}$ and all $A_k^{(1,1)}$ having full row rank. Since $q$ is invariant under global equivalence, it is clear that $\operatorname{rank}\left(\begin{bmatrix} E_k^{(2,1)} & E_k^{(2,2)} \\ 0 & I_{h_f} \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} E_k^{(2,1)} & 0 \\ 0 & I_{h_f} \end{bmatrix}\right)$ has to be constant for all $k \in \mathbb{Z}$. Thus, also all $E_k^{(2,1)}$ have to have constant rank. The remainder of the proof can then be carried out analogously to the proof of Theorem 17. $\square$

Applying one step of forward and one step of reversed index reduction to the form (21) yields the form

$$\left\{\left(\begin{bmatrix} E_k^{(1,1)} & 0 & 0 & 0 \\ 0 & I_{h_b-q} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} A_k^{(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_q & 0 \\ 0 & 0 & 0 & I_{h_f-q} \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}. \tag{27}$$

It is clear that first applying one step of reversed and then one step of forward index reduction will yield another form (i.e., the $I_q$ block then stays in the left matrices and is therefore missing in the right matrices), compare Example 16. Using the preceding results we can adapt the process at (9) to systems that fulfill even harder constant rank assumptions. For an original sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}} =: \{(E_{k,0}, A_{k,0})\}_{k\in\mathbb{Z}}$ we define a sequence (of sequences of matrix pairs) $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ by the following procedure. First we reduce $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ by Corollary 19 to the form (21) assuming that the local invariants

$$r_f \;=:\; r_{f,i}, \quad h_f \;=:\; h_{f,i}, \quad h_b \;=:\; h_{b,i}, \quad \text{and} \quad q \;=:\; q_i \tag{28}$$

are constant for all matrix pairs on the whole interval $\mathbb{Z}$. Then we reduce the so obtained sequence of matrix pairs first by one step of forward and then by one step of reversed index reduction to the form (27), which yields the next sequence of matrix pairs $\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}}$. Due to Theorem 6, Corollary 19 and Lemma 13 the so obtained sequence of quadruples $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ is characteristic for a given equivalence class of sequences of matrix pairs. Note that we have to have the constant rank assumptions of the form (27) for every step of the iterative procedure.

**Remark 20.** Under the assumption that the strangeness index and the reversed strangeness index of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ are both well defined, all constant rank assumptions which are required in the process at (28) are satisfied, because of Lemma 14 and Theorem 17.

To define a strangeness index under the assumptions of the process at (28) we need Lemma 21 which is an adaption of Lemma 7 to the two-way case.

**Lemma 21.** *Let the sequence $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ and the sequence (of sequences of matrix pairs) $\{\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}\}_{i\in\mathbb{N}_0}$ be defined as in the process at (28). In particular, let the constant rank assumptions (26) hold for every step of the reduction process at (28). Defining the quantities*

$$\begin{aligned}
r_{b,i} &:= r_{f,i} - h_{b,i} + h_{f,i},\\
s_{E,i} &:= r_{f,i} - r_{f,i+1},\\
s_{A,i} &:= r_{b,i} - r_{b,i+1},\\
s_i &:= s_{E,i} + s_{A,i}
\end{aligned} \tag{29}$$

*for all $i \in \mathbb{N}_0$, there exists a $\mu \in \mathbb{N}_0$ so that*

$$r_{b,i} = \operatorname{rank}(A_{k,i}), \tag{30a}$$
$$r_{f,i+1} \leqslant r_{f,i}, \tag{30b}$$
$$r_{b,i+1} \leqslant r_{b,i}, \tag{30c}$$
$$s_{E,\mu+i} = s_{A,\mu+i} = s_{\mu+i} = 0 \tag{30d}$$

*for all $i \in \mathbb{N}_0$ and $k \in \mathbb{Z}$.*

**Proof.** (30a) follows directly from the identity $h_b = r_f + h_f - r_b$ in Theorem 3. Let $i \in \mathbb{N}_0$ be any non-negative integer. Then we know from Corollary 19 that we have the global equivalence relation

$$\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}} \quad \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & E_{k,i}^{(2)}\\ 0 & I_{h_{b,i}-q_i} & 0 & 0\\ 0 & 0 & I_{q_i} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & A_{k,i}^{(2)} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & I_{q_i} & 0\\ 0 & 0 & 0 & I_{h_{f,i}-q_i}\\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

which implies

$$\{(E_{k,i+1}, A_{k,i+1})\}_{k\in\mathbb{Z}} \quad \sim \left\{\left(\begin{bmatrix} E_{k,i}^{(1)} & 0 & 0 & 0\\ 0 & I_{h_{b,i}-q_i} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,i}^{(1)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & I_{q_i} & 0\\ 0 & 0 & 0 & I_{h_{f,i}-q_i}\\ 0 & 0 & 0 & 0 \end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

This clearly shows that we have rank $(E_{k,i+1}) \leqslant$ rank $(E_{k,i})$ and rank $(A_{k,i+1}) \leqslant$ rank $(A_{k,i})$, from which (30b) and (30c) follows. Since we know that both of the sequences $\{r_{f,i}\}_{i\in\mathbb{N}_0}$ and $\{r_{b,i}\}_{i\in\mathbb{N}_0}$ are non-increasing and bounded by zero, they have to become stationary a some point $\mu$, which shows (30d). $\square$

Lemma 21 leads to the following Definition.

**Definition 22.** Let $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ be a sequence of matrix pairs. Let the sequence $\{(r_{f,i}, h_{f,i}, h_{b,i}, q_i)\}_{i\in\mathbb{N}_0}$ from (28) be well defined. In particular, let (26) hold for every entry $\{(E_{k,i}, A_{k,i})\}_{k\in\mathbb{Z}}$ of the sequence (of sequences of matrix pairs) defined at (28). Then, with the definitions (29) we call

$$\mu = \min\{i \in \mathbb{N}_0 \mid s_i = 0\}, \tag{31}$$

the *two-way strangeness index* of the sequence of matrix pairs $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and of the associated descriptor system (2). In the case that $\mu = 0$ we call $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ and (2) *two-way strangeness-free*.

Connections between the two-way strangeness index and the forward and backward strangeness index are investigated in the following. Since one step of the iterative procedure described in the process at (28) involves one step of forward and one step of reversed index reduction it may happen that the two-way strangeness index is smaller than the forward strangeness index.

**Example 23.** Consider the sequence of (constant) matrix pairs

$$\{(E_k, A_k)\}_{k\in\mathbb{Z}} = \left\{\left(\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

With Definition 8 we get the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, s_0) &= (2, 1, 1),\\
(r_{f,1}, h_{f,1}, s_1) &= (1, 2, 1),\\
(r_{f,2}, h_{f,2}, s_2) &= (0, 3, 0),\\
(r_{f,3}, h_{f,3}, s_3) &= (0, 3, 0),\\
(r_{f,4}, h_{f,4}, s_4) &= (0, 3, 0),\\
&\vdots
\end{aligned}$$

and thus a forward strangeness index of 2. With Definition 22, however, we face the reduction process

$$\begin{aligned}
\left\{\left(\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} &\sim \left\{\left(\begin{bmatrix}0 & 1\\1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}\\
&\overset{\text{reduction}}{\sim} \left\{\left(\begin{bmatrix}0 & 0\\1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\\0 & 1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}
\end{aligned}$$

and thus the sequences

$$\begin{aligned}
(r_{f,0}, h_{f,0}, h_{b,0}, q_0, r_{b,0}, s_{E,0}, s_{A,0}, s_0) &= (2, 1, 1, 0, 2, 1, 1, 2),\\
(r_{f,1}, h_{f,1}, h_{b,1}, q_1, r_{b,1}, s_{E,1}, s_{A,1}, s_1) &= (1, 1, 1, 0, 1, 0, 0, 0),\\
(r_{f,2}, h_{f,2}, h_{b,2}, q_2, r_{b,2}, s_{E,2}, s_{A,2}, s_2) &= (1, 1, 1, 0, 1, 0, 0, 0),\\
&\vdots
\end{aligned}$$

which shows that the two-way strangeness index is 1.

In Example 23 we see that the forward strangeness index can be twice as big as the two-way strangeness index. From Example 16 we see that one step of forward index reduction can alter the backward strangeness index by one. Thus, if one first applies one step of forward index reduction this will definitely change the forward strangeness index (unless the system is forward strangeness-free). The subsequent step of backward index reduction can then again decrease the forward strangeness index by one. In other words, one step of two-way index reduction can decrease the forward strangeness index (and the backward strangeness index analogously) by two. This is why one could suppose that the forward and backward strangeness index of a system are always less than or equal to the two-way strangeness index multiplied by two. Also, we observe that one step of two-way index reduction involves one step of forward index reduction, which is why we can guess that the two-way strangeness index is always less than or equal to the forward strangeness index.

Of course the forward strangeness index and the two-way strangeness index can also coincide as one can see by applying Definitions 8 and 22 to a (constant) sequence of matrix pairs of the form

$$\left\{\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}}.$$

In fact, any constant sequence of regular pencils will do, as long as the pencil does not have the eigenvalue 0. Based on these observations one can presume that if a sequence of matrix pairs $\{(E_k,A_k)\}_{k\in\mathbb{Z}}$ has well-defined two-way strangeness index $\mu$ it also has to have a well-defined forward strangeness index $\mu_f$ and a well-defined reversed strangeness index $\mu_b$ which are related by the inequalities $2\mu \geqslant \max(\mu_f,\mu_b) \geqslant \mu$. A proof might turn out to be very complicated and is not known to the author. At least for two-way strangeness-free sequences $\{(E_k,A_k)\}_{k\in\mathbb{Z}}$ the presumption is clear, since we know due to Theorem 17 that in this case we have

$$\{(E_k,A_k)\}_{k\in\mathbb{Z}} \sim \left\{\left(\begin{bmatrix}E_k^{(1)}&0&E_k^{(2)}\\0&I_{h_b}&0\\0&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}A_k^{(1)}&A_k^{(2)}&0\\0&0&0\\0&0&I_{h_f}\\0&0&0\end{bmatrix}\right)\right\}_{k\in\mathbb{Z}} \tag{32}$$

with all $E_k^{(1)}$ and all $A_k^{(1)}$ having full row rank and we can easily perform one step of forward/backward index reduction on (32).

### 6.1. Existence and uniqueness of solutions

With the notation of the process at (28) and Corollary 19 we know that for the two-way strangeness index $\mu$ we have

$$\{(E_{k,\mu},A_{k,\mu})\} \sim \left\{\left(\begin{bmatrix}E_{k,\mu}^{(1)}&0&0&E_{k,\mu}^{(2)}\\0&I_{h_{b,\mu}-q_\mu}&0&0\\0&0&I_{q_\mu}&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}A_{k,\mu}^{(1)}&A_{k,\mu}^{(2)}&0&0\\0&0&0&0\\0&0&0&0\\0&0&I_{q_\mu}&0\\0&0&0&I_{h_{f,\mu}-q_\mu}\\0&0&0&0\end{bmatrix}\right)\right\}$$

and thus

$$\{(E_{k,\mu+1},A_{k,\mu+1})\} \sim \left\{\left(\begin{bmatrix}E_{k,\mu}^{(1)}&0&0&0\\0&I_{h_{b,\mu}-q_\mu}&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\begin{bmatrix}A_{k,\mu}^{(1)}&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&I_{q_\mu}&0\\0&0&0&I_{h_{f,\mu}-q_\mu}\\0&0&0&0\end{bmatrix}\right)\right\}.$$

But we also know from the definitions (29) that $\operatorname{rank}\left(E_{k,\mu}\right) = \operatorname{rank}\left(E_{k,\mu+1}\right)$, from which we see that $q_\mu = 0$ and that $E_{k,\mu}^{(1)}$ is a matrix with full row rank for all $k \in \mathbb{Z}$, since from Corollary 19 we know that all $\begin{bmatrix} E_{k,\mu}^{(1)} & E_{k,\mu}^{(2)} \end{bmatrix}$ had full row rank. From rank $(A_{k,\mu})$ = rank $(A_{k,\mu+1})$, we analogously see that all $A_{k,\mu}^{(1)}$ already have full row rank. Thus, every sequence with well-defined two-way strangeness index can be transformed by $\mu + 1$ index reduction steps from (21)–(27) and appropriate global equivalence transformations to a two-way strangeness-free sequence of the form

$$\{(E_{k,\mu+1}, A_{k,\mu+1})\} \sim \left\{ \left( \begin{bmatrix} E_{k,\mu}^{(1)} & 0 & 0 \\ 0 & I_{h_{b,\mu}} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{k,\mu}^{(1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{h_{f,\mu}} \\ 0 & 0 & 0 \end{bmatrix} \right) \right\}_{k \in \mathbb{Z}}, \tag{33}$$

where all $A_{k,\mu}^{(1)}$ and all $E_{k,\mu}^{(1)}$ have full row rank. By transformations of the (1,1)-block in (33) one can achieve through global equivalence transformations that $E_{k,\mu}^{(1)} = \begin{bmatrix} I & 0 \end{bmatrix}$ for all $k \in \mathbb{N}_0$ and $A_{k,\mu}^{(1)} = \begin{bmatrix} I & 0 \end{bmatrix}$ for all $k \leqslant -1$. Thus, from (33) one can derive a statement similar to Theorem 11 for the case where one wants to get a solution for all $k \in \mathbb{Z}$.

**Theorem 24.** *Assume that the two-way strangeness index $\mu$ of the sequence $\{(E_k, A_k)\}_{k\in\mathbb{Z}}$ is well defined as in (31). Then the discrete-time descriptor system (2) is equivalent (in the sense that there is a one-to-one correspondence between the solution/sequence spaces) to a discrete-time descriptor system of the form*

$$\begin{array}{ll}
x_1^{k+1} = A_k^{(1)} x_1^k + A_k^{(2)} x_4^k + f_1^k, \quad k \geqslant 0, & r_{f,\mu} - h_{b,\mu} \\
x_1^{k-1} = E_{k-1}^{(1)} x_1^k + E_{k-1}^{(2)} x_4^k - f_1^{k-1}, \quad k \leqslant 0, & r_{b,\mu} - h_{f,\mu} \\
x_2^{k+1} = f_2^k, & h_{b,\mu} \\
0 = x_3^k + f_3^k, & h_{f,\mu} \\
0 = f_4^k, & m - r_{f,\mu} - h_{f,\mu}
\end{array}$$

*where with $u_\mu := n - r_{f,\mu} - h_{f,\mu}$ we have $x_4^k \in \mathbb{C}^{u_\mu}$ and each of the inhomogeneities $f_1^k, f_2^k, f_3^k, f_4^k$ is determined by the original inhomogeneities $f^{k-\mu-1}, \ldots, f^k, \ldots, f^{k+\mu+1}$ as in (2) for all $k \in \mathbb{Z}$. For the problem*

$$E_k x^{k+1} = A_k x^k + f^k, \quad k \in \mathbb{Z}, \tag{34}$$

*we obtain the following existence and uniqueness results. System (34) is solvable if and only if the $v_\mu := m - r_{f,\mu} - h_{f,\mu}$ consistency conditions $f_4^k = 0$ are fulfilled for all $k \in \mathbb{Z}$. An initial condition $x^0 = \hat{x}$ is consistent with (34) if and only if in addition the $h_{f,\mu} + h_{b,\mu}$ conditions $x_2^0 = \hat{x}_2 = f_2^{-1}$ and $x_3^0 = \hat{x}_3 = -f_3^0$ are satisfied. The corresponding initial value problem is uniquely solvable if and only if in addition $u_\mu = 0$ holds.*

**Proof.** The proof can be carried out as the proof of Theorem 11 by using (33). □

## 7. Conclusion

Analogously to [7], in this paper we have first derived a canonical form that allows for statements about the existence and uniqueness of solutions for forward discrete-time descriptor systems. To obtain this canonical form one has to make constant rank assumptions about the involved sequences of matrix pairs. An index was defined for descriptor systems that fulfill these constant rank assumptions. In every step of the reduction procedure only two characteristics (9) have to be constant for the sequence of matrix pairs, whereas in the continuous-time case there have to be three characteristics constant for the pair of matrix valued functions. However, it seems less natural to make constant rank assumptions in the discrete-time case than in the continuous-time case because in the discrete-time case one can never be sure if there is an interval in which the constant rank assumptions hold as one can be in the continuous-time case with smooth matrix valued functions (see [7, Theorem 3.25]).

After the analysis of the forward case we took a short look at the backward case and then continued to understand the two-way case. A different canonical form has been obtained for two-way descriptor systems and the two-way strangeness index has been defined. More constant rank assumptions than in the forward case were necessary to obtain the results.

The processes that lead to the forward and the two-way canonical form both imply a way to efficiently compute solutions of linear time-varying descriptor systems that satisfy the corresponding constant rank assumptions. A first attempt to do so has been done in [1]. The same algorithm (with a different implementation) could also be used to efficiently compute the solution of time-invariant descriptor systems. Time-invariant descriptor systems always satisfy any constant rank assumption. Anyway, it is possible and desirable to generalize the time-variant algorithm to linear descriptor systems, that do not fulfill the constant rank assumptions, although this might be a quite complicated undertaking.

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