J Optim Theory Appl
DOI 10.1007/s10957-012-0212-5

# Controllability Results for Nonlinear Fractional-Order Dynamical Systems

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Received: 7 July 2012 / Accepted: 11 October 2012


**Abstract** This paper establishes a set of sufficient conditions for the controllability of nonlinear fractional dynamical system of order $1 < \alpha < 2$ in finite dimensional spaces. The main tools are the Mittag–Leffler matrix function and the Schaefer's fixed-point theorem. An example is provided to illustrate the theory.

**Keywords** Controllability · Fractional Differential Equations · Mittag–Leffler Matrix Function · Schaefer's Fixed-Point Theorem

## 1 Introduction

Fractional differential equations have gained considerable popularity and importance during the past there decades or so. The class of fractional differential equations of various types play important roles and tools are used from not only mathematics but also physics, control systems, dynamical systems and engineering to create the mathematical modeling of many physical phenomena. The fractional-order integration and differentiation represent a rapidly growing field both in theory and in applications to

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Published online: 19 October 2012

real world problems. The monographs [1–5] are commonly cited for the theory and applications of fractional derivatives and integrals to differential equations of fractional order.

Controllability plays a major role in the development of modern mathematical control theory. Mainly, the problem of controllability of dynamical systems is widely used in analysis and the design of control systems. In this way, fractional-order control systems described by fractional-order differential equations are attracting considerable attention in recent years. The fractional-order models need fractional-order controllers for more effective control of dynamical systems [6]. Clearly, the use of fractional-order derivatives and integrals in control theory leads to better results than integer-order approaches.

In recent years, much attention has been paid to establish sufficient conditions for the controllability of linear fractional dynamical systems of order $0 < \alpha < 1$ by several authors, including a recent monograph [7] and various papers (see, [8–12]). Balachandran et al. [13–16] studied the controllability of nonlinear fractional dynamical systems of order $0 < \alpha < 1$. However, there is no work that reported on the problem of controllability of nonlinear fractional dynamical system of order $1 < \alpha < 2$. Motivated by this fact, in this paper we establish the controllability results for nonlinear fractional dynamical system of order $1 < \alpha < 2$ in finite dimensional spaces by using the Schaefer's fixed-point theorem. An example is provided to illustrate the result.

## 2 Preliminaries

In this section, we introduce definitions and preliminary facts from fractional calculus which are used throughout this paper.

**Definition 2.1** [17] The Caputo fractional derivative of order $\alpha \in \mathbb{C}$ with $n-1 < \alpha < n$, $n \in \mathbb{N}$, for a suitable function $f$ is defined as

$$\left({}^C D_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s)\,\mathrm{d}s$$

where $f^n(s) = \dfrac{d^n f}{ds^n}$. In particular, if $1 < \alpha < 2$ then

$$\left({}^C D_{0+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)\,\mathrm{d}s.$$

For the brevity, the Caputo fractional derivative ${}^C D_{0+}^{\alpha}$ is taken as ${}^C D^{\alpha}$.

**Definition 2.2** [18] The Mittag–Leffler matrix function for an arbitrary square matrix $A$ is

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \beta > 0.$$

$$E_{\alpha,1}(A) = E_{\alpha}(A) \quad \text{with } \beta = 1.$$

Consider the linear fractional dynamical system represented by a fractional differential equation of the form

$$\left.\begin{aligned}&{}^{C}D^{\alpha}x(t)+A^{2}x(t)=Bu(t),\qquad t\in[0,T]:=J\\&x(0)=x_{0},\qquad x'(0)=y_{0},\end{aligned}\right\}\tag{1}$$

with $1<\alpha<2$, $A$ and $B$ the $n\times n$ and $n\times m$ matrices, $x\in\mathbb{R}^{n}$ and $u\in\mathbb{R}^{m}$. The solution of (1) is given by

$$x(t)=\Phi_{0}(t)\,x_{0}+\Phi_{1}(t)\,y_{0}+\int_{0}^{t}\Phi(t-s)Bu(s)\,\mathrm{d}s\tag{2}$$

where

$$\Phi_{0}(t)=E_{\alpha}\bigl(-A^{2}t^{\alpha}\bigr);\qquad \Phi_{1}(t)=tE_{\alpha,2}\bigl(-A^{2}t^{\alpha}\bigr);\qquad \Phi(t)=t^{\alpha-1}E_{\alpha,\alpha}\bigl(-A^{2}t^{\alpha}\bigr)$$

are the Mittag–Leffler matrix functions.

**Definition 2.3** The system (1) is said to be controllable on $J$ iff for each vectors $x_{0},y_{0},x_{1}\in\mathbb{R}^{n}$, there exists a control $u(t)\in L^{2}(J,\mathbb{R}^{m})$ such that the solution of (1) satisfies $x(T)=x_{1}$.

**Lemma 2.1** [19] *Let $f_i$, for $i=1,2,\ldots,n$, be $1\times p$ vector valued continuous functions defined on $[t_1,t_2]$. Let $F$ be an $n\times p$ matrix with $f_i$ as its $i$th row. Then $f_1,f_2,\ldots,f_n$ are linearly independent on $[t_1,t_2]$ if and only if the $n\times n$ constant matrix*

$$M(t_{1},t_{2})=\int_{t_{1}}^{t_{2}}F(t)F^{*}(t)\,\mathrm{d}t$$

*is nonsingular.*

**Theorem 2.1** *The following statements are equivalent.*

(a) *The linear system (1) is controllable on $J$.*
(b) *The rows of $\Phi(t)B$ are linearly independent.*
(c) *The controllability Gramian*

$$W=\int_{0}^{T}\Phi(T-s)BB^{*}\Phi^{*}(T-s)\,\mathrm{d}s\tag{3}$$

*is nonsingular.*

*Proof* First we shall prove that (a) $\Longrightarrow$ (b). Suppose that the system (1) is controllable, but the rows of $\Phi(t)B$ are linearly dependent functions on $J$. Then there exists a nonzero constant $n\times 1$ vector $y$ such that

$$y^{*}\,\Phi(t)\,B=0,\qquad \text{for every } t\in J.\tag{4}$$

Let us choose $x(0)=x_0=0, x'(0)=y_0=0$. Therefore, the solution of (1) becomes

$$x(t)=\int_0^t \Phi(t-s)Bu(s)\,\mathrm{d}s.$$

Since the system (1) is controllable on $J$, taking $x(T)=y$, we get

$$x(T)=y=\int_0^T \Phi(T-s)Bu(s)\,\mathrm{d}s,$$

$$y^*y=\int_0^T y^*\Phi(T-s)Bu(s)\,\mathrm{d}s.$$

From (4), $y^*y=0$ and hence $y=0$. Hence it contradicts our assumption that $y$ is nonzero. Now we prove that (b) $\Longrightarrow$ (a). Suppose that the rows of $\Phi(t)B$ are linearly independent on $J$. Therefore by Lemma 2.1, the $n\times n$ constant matrix

$$W=\int_0^T \Phi(T-s)BB^*\Phi^*(T-s)\,\mathrm{d}s$$

is nonsingular. Now we define the control function as

$$u(t)=B^*\Phi^*(T-t)W^{-1}\big[x_1-\Phi_0(T)x_0-\Phi_1(T)y_0\big]. \tag{5}$$

Substituting (5) in (2), we get $x(T)=x_1$. Thus (1) is controllable. The implications (b) $\Longrightarrow$ (c) and (c) $\Longrightarrow$ (b) follow directly from Lemma 2.1. Hence the desired result. □

In order to prove our main result, we need the following.

**Lemma 2.2** (Schaefer's Theorem [20]) *Let $S$ be a normed space, $T$ a continuous mapping of $S$ into $S$, which is compact on each bounded subset of $X$. Then, either* (i) *the equation $x=\lambda T(x)$ has a solution for $\lambda=1$, or* (ii) *the set of all such solutions $x$ for $0<\lambda<1$ is unbounded.*

Consider the nonlinear fractional dynamical system represented by the fractional differential equation of the form

$$\left.\begin{array}{l} {}^C D^\alpha x(t)+A^2x(t)=Bu(t)+f(t,x(t),{}^C D^\beta x(t)), \quad t\in J \\ x(0)=x_0, \quad x'(0)=y_0, \end{array}\right\} \tag{6}$$

with $1<\alpha<2,\ 0<\beta<1$, $A$ and $B$ defined as above and the nonlinear function $f: J\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ being continuous. The solution of (6) is given by

$$x(t)=\Phi_0(t)x_0+\Phi_1(t)\,y_0+\int_0^t \Phi(t-s)\big[Bu(s)+f\big(s,x(s),{}^C D^\beta x(s)\big)\big]\,\mathrm{d}s.$$

We assume the following hypotheses.

(H1) For each $t \in J$, the function $f(t, \cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the function $f(\cdot, x, y) : J \to \mathbb{R}^n$ is strongly measurable for each $x, y \in \mathbb{R}^n$.

(H2) For every positive constant $k$ there exists $h_k \in L^1(J)$ such that

$$\sup_{\|x\|,\ \|y\| \le k} \left\| f(t, x, y) \right\| \le h_k(t), \quad \text{for almost all } t \in J.$$

(H3) There exists a continuous function $m_1 : J \to [0, \infty[$ such that

$$\left\| f(t, x, y) \right\| \le m_1(t) \Omega\bigl(\|x\| + \|y\|\bigr), \quad t \in J,\ x, y \in \mathbb{R}^n,$$

where $\Omega : ]0, \infty[ \to ]0, \infty[$ is a continuous nondecreasing function and

$$\int_0^T m(s)\,\mathrm{d}s < \int_r^\infty \frac{\mathrm{d}s}{\Omega(s)}.$$

(H4) There exists a constant $M > 0$ and a continuous function $m_2 : J \to [0, \infty[$ such that

$$\frac{k_2 t^{-\beta}}{\Gamma(1-\beta)} + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} m_1(\xi) \Omega\bigl(w(\xi)\bigr)\,\mathrm{d}\xi \\ \le M m_2(t) \Omega\bigl(w(t)\bigr),$$

where

$$n_1 = \sup\bigl\{\left\|\Phi_0(t)\right\|, t \in J\bigr\}; \qquad n_2 = \sup\bigl\{\left\|\Phi_1(t)\right\|, t \in J\bigr\};$$
$$n_3 = \sup\bigl\{\left\|\Phi(t-s)\right\|, t, s \in J\bigr\}; \qquad n_4 = \sup\bigl\{\left\|A^2\Phi(t)\right\|, t \in J\bigr\};$$
$$n_5 = \sup\bigl\{\left\|\Phi_2(t-s)\right\|, t, s \in J\bigr\}; \qquad \bar{x} = x_1 - \Phi_0(T)x_0 - \Phi_1(T)y_0;$$
$$n_6 = n_4\|x_0\| + n_1\|y_0\| + n_3 n_5 T \|B\| \left\|B^*\right\| \left\|W^{-1}\right\| \left[\|\bar{x}\| + n_3 \int_0^T h_q(s)\,\mathrm{d}s\right];$$
$$m(t) = \max\bigl\{n_3 m_1(t), M m_2(t)\bigr\}; \qquad \Phi_2(t) = t^{\alpha-1} E_{\alpha,\alpha-1}\bigl(-A^2 t^\alpha\bigr);$$
$$r = n_1\|x_0\| \\ + n_2\|y_0\| + n_3^2 T \|B\| \left\|B^*\right\| \left\|W^{-1}\right\| \left[\|\bar{x}\| + n_3 \int_0^T m_1(s) \Omega\bigl(w(s)\bigr)\,\mathrm{d}s\right].$$

## 3 Main Result

Now we are able to state and prove our main theorem.

**Theorem 3.1** *Assume that hypotheses* (H1)–(H4) *hold and suppose that the linear system* (1) *is controllable. Then the nonlinear system* (6) *is controllable on* $J$.

*Proof* Consider the space $X=\{x : x' \in C(J,\mathbb{R}^n) \text{ and } {}^C D^\beta x \in C(J,\mathbb{R}^n)\}$ with norm $\|x\|^* = \max\{\|x\|, \|{}^C D^\beta x\|\}$. Using the hypothesis, for an arbitrary function $x(\cdot)$, define the control

$$u(t) = B^* \Phi^*(T-t) W^{-1}\left[\bar{x} - \int_0^T \Phi(T-s) f\big(s, x(s), {}^C D^\beta x(s)\big)\,\mathrm{d}s\right].$$

We shall now show that, when using this control, the nonlinear operator $F : X \to X$ defined by

$$\begin{aligned}(Fx)(t) = {}& \Phi_0(t)\,x_0 + \Phi_1(t)\,y_0 + \int_0^t \Phi(t-s) f\big(s, x(s), {}^C D^\beta x(s)\big)\,\mathrm{d}s \\ &+ \int_0^t \Phi(t-s) B u(s)\,\mathrm{d}s\end{aligned}$$

has a fixed point. This fixed point is then a solution of Eq. (6). Substituting the control $u(t)$ in the above equation we get

$$\begin{aligned}(Fx)(t) = {}& \Phi_0(t)\,x_0 + \Phi_1(t)\,y_0 + \int_0^t \Phi(t-s) f\big(s, x(s), {}^C D^\beta x(s)\big)\,\mathrm{d}s \\ &+ \int_0^t \Phi(t-s) B B^* \Phi^*(T-s) W^{-1} \\ &\times \left[\bar{x} - \int_0^T \Phi(T-\xi) f\big(\xi, x(\xi), {}^C D^\beta x(\xi)\big)\,\mathrm{d}\xi\right] \mathrm{d}s.\end{aligned}$$

Clearly, $(Fx)(T) = x_1$, which means that the control $u$ steers the system from the initial state $x_0$ to $x_1$ in time $T$, provided we obtain a fixed point of the nonlinear operator $F$.

The first step is to obtain a priori bound of the set

$$\zeta(F) = \big\{x \in X : x = \lambda F x \text{ for some } \lambda \in ]0,1[\big\}.$$

Let $x \in \zeta(F)$, then $x = \lambda F x$ for some $0 < \lambda < 1$. Then for each $t \in J$, we have

$$\begin{aligned}x(t) = {}& \lambda \Phi_0(t)\,x_0 + \lambda \Phi_1(t)\,y_0 + \lambda \int_0^t \Phi(t-s) f\big(s, x(s), {}^C D^\beta x(s)\big)\,\mathrm{d}s \\ &+ \lambda \int_0^t \Phi(t-s) B B^* \Phi^*(T-s) W^{-1} \\ &\times \left[\bar{x} - \int_0^T \Phi(T-\xi) f\big(\xi, x(\xi), {}^C D^\beta x(\xi)\big)\,\mathrm{d}\xi\right] \mathrm{d}s;\end{aligned}$$

$$\begin{aligned}\big\|x(t)\big\| \le {}& n_1 \|x_0\| + n_2 \|y_0\| + n_3 \int_0^t m_1(s) \Omega\big(\big\|x(s)\big\| + \big\|{}^C D^\beta x(s)\big\|\big)\,\mathrm{d}s \\ &+ n_3^2 T \|B\| \big\|B^*\big\| \big\|W^{-1}\big\| \left[\|\bar{x}\| + n_3 \int_0^T m_1(s) \Omega\big(\big\|x(s)\big\| + \big\|{}^C D^\beta x(s)\big\|\big)\,\mathrm{d}s\right] \\ \equiv {}& k_1 + n_3 \int_0^t m_1(s) \Omega\big(\big\|x(s)\big\| + \big\|{}^C D^\beta x(s)\big\|\big)\,\mathrm{d}s.\end{aligned}$$

Denoting by $r_1(t)$ the right-hand side of the above inequality, we have $r_1(0) = k_1$, $\|x(t)\| \le r_1(t)$, and

$$r_1'(t) = n_3 m_1(t) \Omega\big(\|x(t)\| + \|{}^C D^\beta x(t)\|\big).$$

Also,

$$\begin{aligned}
x'(t) = {} & -\lambda A^2 \Phi(t)\, x_0 + \lambda \Phi_0(t)\, y_0 + \lambda \int_0^t \Phi_2(t-s) f\big(s, x(s), {}^C D^\beta x(s)\big)\, \mathrm{d}s \\
& + \lambda \int_0^t \Phi_2(t-s) B B^* \Phi^*(T-s) W^{-1} \\
& \times \left[\bar{x} - \int_0^T \Phi(T-\xi) f\big(\xi, x(\xi), {}^C D^\beta x(\xi)\big)\, \mathrm{d}\xi\right] \mathrm{d}s; \\
\|x'(t)\| \le {} & n_4 \|x_0\| + n_1 \|y_0\| + n_5 \int_0^t m_1(s) \Omega\big(\|x(s)\| + \|{}^C D^\beta x(s)\|\big)\, \mathrm{d}s \\
& + n_3 n_5 T \|B\| \|B^*\| \|W^{-1}\| \\
& \times \left[\|\bar{x}\| + n_3 \int_0^T m_1(s) \Omega\big(\|x(s)\| + \|{}^C D^\beta x(s)\|\big)\, \mathrm{d}s\right] \\
\equiv {} & k_2 + n_5 \int_0^t m_1(s) \Omega\big(\|x(s)\| + \|{}^C D^\beta x(s)\|\big)\, \mathrm{d}s.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\|{}^C D^\beta x(t)\| &\le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \|x'(s)\|\, \mathrm{d}s \\
&\le \frac{k_2\, t^{1-\beta}}{\Gamma(2-\beta)} + \frac{n_5}{\Gamma(2-\beta)} \int_0^t (t-\xi)^{1-\beta} m_1(\xi) \Omega\big(\|x(\xi)\| \\
&\quad + \|{}^C D^\beta x(\xi)\|\big)\, \mathrm{d}\xi.
\end{aligned}$$

Denoting by $r_2(t)$ the right-hand side of the above inequality, we have $r_2(0) = 0$, $\|{}^C D^\beta x(t)\| \le r_2(t)$, and

$$r_2'(t) = \frac{k_2\, t^{-\beta}}{\Gamma(1-\beta)} + \frac{n_5}{\Gamma(1-\beta)} \int_0^t (t-\xi)^{-\beta} m_1(\xi) \Omega\big(\|x(\xi)\| + \|{}^C D^\beta x(\xi)\|\big)\, \mathrm{d}\xi.$$

Let $w(t) = r_1(t) + r_2(t),\ t \in J$. Then $w(0) = r_1(0) + r_2(0) = r$ and

$$w'(t) = r_1'(t) + r_2'(t) \le m(t) \Omega\big(w(t)\big).$$

This implies that for each $t \in J$,

$$\int_{w(0)}^{w(t)} \frac{\mathrm{d}s}{\Omega(s)} \le \int_0^T m(s)\, \mathrm{d}s < \int_r^\infty \frac{\mathrm{d}s}{\Omega(s)}.$$

This inequality implies that there exists a constant $K$ such that

$$w(t) = r_1(t) + r_2(t) \leq K, \quad t \in J.$$

Then $\|x(t)\| \leq r_1(t)$ and $\|{}^C D^\beta x(t)\| \leq r_2(t),\ t \in J$, and hence

$$\|x\|^* = \max\{\|x\|, \|{}^C D^\beta x\|\} \leq K$$

and the set $\zeta(F)$ is bounded. Next we shall prove that the operator $F : X \to X$ is completely continuous.

Let $B_q = \{x \in X; \|x\|^* \leq q\}$. We first show that $F$ maps bounded sets $B_q$ into equicontinuous family. Let $x \in B_q$ and $t_1, t_2 \in J$. Then, if $0 < t_1 < t_2 \leq T$,

$$\begin{aligned}
&\|(Fx)(t_2) - (Fx)(t_1)\| \\
&\quad \leq \|\Phi_0(t_2) - \Phi_0(t_1)\| \|x_0\| + \|\Phi_1(t_2) - \Phi_1(t_1)\| \|y_0\| + \int_{t_1}^{t_2} \|\Phi(t_2 - s)\| h_q(s)\, \mathrm{d}s \\
&\qquad + \int_0^{t_1} \|\Phi(t_2 - s) - \Phi(t_1 - s)\| h_q(s)\, \mathrm{d}s + \int_0^{t_1} \|\Phi(t_2 - s) - \Phi(t_1 - s)\| \\
&\qquad \times \|B\| \|B^*\| \|\Phi^*(T - s)\| \|W^{-1}\| \left[\|\bar{x}\| + n_3 \int_0^T h_q(\xi)\, \mathrm{d}\xi\right] \mathrm{d}s \\
&\qquad + \int_{t_1}^{t_2} \|\Phi(t_2 - s)\| \|B\| \|B^*\| \|\Phi^*(T - s)\| \|W^{-1}\| \left[\|\bar{x}\| + n_3 \int_0^T h_q(\xi)\, \mathrm{d}\xi\right] \mathrm{d}s,
\end{aligned} \tag{7}$$

and

$$\|(Fx)'(t)\| \leq n_6 + n_5 \int_0^t h_q(s)\, \mathrm{d}s.$$

Hence, it follows that

$$\begin{aligned}
&\|{}^C D^\beta (Fx)(t_2) - {}^C D^\beta (Fx)(t_1)\| \\
&\quad = \left\| \frac{1}{\Gamma(1-\beta)} \int_0^{t_2} (t_2 - s)^{-\beta} (Fx)'(s)\, \mathrm{d}s - \frac{1}{\Gamma(1-\beta)} \int_0^{t_1} (t_1 - s)^{-\beta} (Fx)'(s)\, \mathrm{d}s \right\| \\
&\quad \leq \frac{n_6}{\Gamma(2-\beta)} \left(t_2^{-\beta} - t_1^{-\beta}\right) + \frac{1}{\Gamma(1-\beta)} \int_{t_1}^{t_2} (t_2 - s)^{-\beta} \left(\int_0^s h_q(\xi)\, \mathrm{d}\xi\right) \mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(2-\beta)} \int_0^{t_1} \left((t_2 - \xi)^{1-\beta} - (t_2 - t_1)^{1-\beta} - (t_1 - \xi)^{1-\beta}\right) h_q(\xi)\, \mathrm{d}\xi.
\end{aligned} \tag{8}$$

The right-hand sides of (7) and (8) tend to zero as $t_2 \to t_1$. Then $F$ maps $B_q$ into an equicontinuous family of functions. It is easy to see that the family $FB_q$ is uniformly bounded.

Next we show that $F$ is a compact operator. It suffices to show that the closure of $FB_q$ is compact. Let $0 \le t \le T$ be fixed and $\epsilon$ a real number satisfying $0 < \epsilon < t$. For $x \in B_q$ we define

$$\begin{aligned}(F_\epsilon x)(t) = \Phi_0(t)x_0 + \Phi_1(t)y_0 + \int_0^{t-\epsilon} \Phi(t-s) f\bigl(s, x(s), {}^C D^\beta x(s)\bigr)\,\mathrm{d}s \\ + \int_0^{t-\epsilon} \Phi(t-s) B B^* \Phi^*(T-s) W^{-1} \\ \times \left[\bar{x} - \int_0^T \Phi(T-\xi) f\bigl(\xi, x(\xi), {}^C D^\beta x(\xi)\bigr)\,\mathrm{d}\xi\right] \mathrm{d}s.\end{aligned}$$

Note that using the same methods in the procedure above we can obtain the bounded and equicontinuous property of $F_\epsilon$. Hence,

$$S_\epsilon(t) = \bigl\{(F_\epsilon x)(t),\ x \in B_q\bigr\}$$

is relatively compact in $X$ for every $0 < \epsilon < t$. Moreover, for every $x \in B_q$,

$$\begin{aligned}\bigl\|(Fx)(t) - (F_\epsilon x)(t)\bigr\| \le \int_{t-\epsilon}^t \bigl\|\Phi(t-s)\bigr\| h_q(s)\,\mathrm{d}s + \int_{t-\epsilon}^t \bigl\|\Phi(t-s)\bigr\| \|B\| \\ \times \bigl\|B^*\bigr\| \bigl\|\Phi^*(T-s)\bigr\| \bigl\|W^{-1}\bigr\| \left[\|\bar{x}\| + n_3 \int_0^T h_q(\xi)\,\mathrm{d}\xi\right] \mathrm{d}s.\end{aligned}$$

Also,

$$\begin{aligned}\bigl\|(Fx)'(t) - (F_\epsilon x)'(t)\bigr\| \le \int_{t-\epsilon}^t \bigl\|\Phi_2(t-s)\bigr\| h_q(s)\,\mathrm{d}s + \int_{t-\epsilon}^t \bigl\|\Phi_2(t-s)\bigr\| \|B\| \\ \times \bigl\|B^*\bigr\| \bigl\|\Phi^*(T-s)\bigr\| \bigl\|W^{-1}\bigr\| \left[\|\bar{x}\| + n_3 \int_0^T h_q(\xi)\,\mathrm{d}\xi\right] \mathrm{d}s.\end{aligned}$$

Since $\|(Fx)(t) - (F_\epsilon x)(t)\| \to 0$ and $\|(Fx)'(t) - (F_\epsilon x)'(t))\| \to 0$, as $\epsilon \to 0$, therefore,

$$\begin{aligned}&\bigl\|{}^C D^\beta (Fx)(t) - {}^C D^\beta (F_\epsilon x)(t)\bigr\| \\ &\quad \le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \bigl\|(Fx)'(t) - (F_\epsilon x)'(t)\bigr\|\,\mathrm{d}s \to 0, \qquad \text{as } \epsilon \to 0.\end{aligned}$$

So, relatively compact sets $S_\epsilon(t) = \{(F_\epsilon x)(t),\ x \in B_q\}$ are arbitrary close to the set $\{(Fx)(t),\ x \in B_q\}$. Hence, $\{(Fx)(t),\ x \in B_q\}$ is compact in $X$ by the Arzela–Ascoli theorem.

It remains to show that $F$ is continuous. Let $\{x_n\}$ be a sequence in $X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Then there is an integer $k$ such that $\|x_n\| \le k$, $\|{}^C D^\beta x_n\| \le k$ for all $n$ and $t \in J$. So, $\|x(t)\| \le k$, $\|{}^C D^\beta x(t)\| \le k$ and $x, {}^C D^\beta x \in X$. By (H1),

$$f\bigl(t, x_n(t), {}^C D^\beta x_n(t)\bigr) \to f\bigl(t, x(t), {}^C D^\beta x(t)\bigr)$$

for each $t \in J$ and since

$$\left\| f\big(t, x_n(t), {}^C D^\beta x_n(t)\big) - f\big(t, x(t), {}^C D^\beta x(t)\big)\right\| \le 2h_k(t),$$

we have by the dominated convergence theorem that

$$\begin{aligned}
&\left\|(Fx_n)(t) - (Fx)(t)\right\| \\
&\quad \le \int_0^T \left\|\Phi(t-s)\big[f\big(s, x_n(s), {}^C D^\beta x_n(s)\big) - f\big(s, x(s), {}^C D^\beta x(s)\big)\big]\right\| \mathrm{d}s \\
&\qquad + \int_0^T \bigg\|\Phi(t-s)BB^*\Phi^*(T-s)W^{-1}\int_0^T \Phi(T-\xi) \\
&\qquad \times \big[f\big(\xi, x_n(\xi), {}^C D^\beta x_n(\xi)\big) - f\big(\xi, x(\xi), {}^C D^\beta x(\xi)\big)\big]\,\mathrm{d}\xi\bigg\| \,\mathrm{d}s \to 0,
\end{aligned}$$

as $n \to \infty$.

Also,

$$\begin{aligned}
&\left\|(Fx_n)'(t) - (Fx)'(t)\right\| \\
&\quad \le \int_0^T \left\|\Phi_2(t-s)\big[f\big(s, x_n(s), {}^C D^\beta x_n(s)\big) - f\big(s, x(s), {}^C D^\beta x(s)\big)\big]\right\| \mathrm{d}s \\
&\qquad + \int_0^T \bigg\|\Phi_2(t-s)BB^*\Phi^*(T-s)W^{-1}\int_0^T \Phi(T-\xi) \\
&\qquad \times \big[f\big(\xi, x_n(\xi), {}^C D^\beta x_n(\xi)\big) - f\big(\xi, x(\xi), {}^C D^\beta x(\xi)\big)\big]\,\mathrm{d}\xi\bigg\| \,\mathrm{d}s \to 0,
\end{aligned}$$

as $n \to \infty$.

This implies that

$$\begin{aligned}
&\left\|{}^C D^\beta (Fx_n)(t) - {}^C D^\beta (Fx)(t)\right\| \\
&\quad \le \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\left\|(Fx_n)'(t) - (Fx)'(t))\right\| \mathrm{d}s \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus $F$ is continuous. Finally, the set $\zeta(F) = \{x \in X;\ x = \lambda Fx,\ \lambda \in ]0,1[$ is bounded as shown in the first step. By Schaefer's theorem, the operator $F$ has a fixed point in $X$. This fixed point is then the solution of (6). Hence the system (6) is controllable on $[0, T]$. □

## 4 An Example

Consider the following fractional dynamical system of order $1 < \alpha < 2$:

$$\left.\begin{aligned}
&{}^C D^\alpha x(t) + A^2 x(t) = Bu(t) + f\big(t, x(t), {}^C D^\beta x(t)\big), \quad 0 < \beta < 1,\ t \in J \\
&x(0) = x_0, \quad x'(0) = y_0,
\end{aligned}\right\} \tag{9}$$

with

$$A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad f\big(t, x, {}^C D^\beta x\big) = \begin{pmatrix} c \\ d \end{pmatrix},$$

where

$$c = \big[\exp(-2t)\big(|x_1| + \big|{}^C D^\beta x_1(t)\big|\big)\big] / \big(1 + \big|x_1(t)\big|\big), \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \text{and}$$

$$d = \big[\exp(-2t)\big(|x_2| + \big|{}^C D^\beta x_2(t)\big|\big)\big] / \big(1 + \big|x_2(t)\big|\big).$$

Using the Mittag–Leffler matrix function for a given matrix $A$, we get

$$\Phi(T - s) = \begin{pmatrix} N(s) & 0 \\ 0 & N(s) \end{pmatrix},$$

where $N(s) = (T - s)^{\alpha-1} E_{\alpha,\alpha}((T - s)^\alpha)$. Then, by simple matrix calculation one can see that the controllability matrix

$$W = \int_0^T \Phi(T - s) B B^* \Phi^*(T - s)\, \mathrm{d}s = \int_0^T \begin{pmatrix} N^2(s) & 0 \\ 0 & N^2(s) \end{pmatrix} \mathrm{d}s$$

is positive definite for any $T > 0$. Further, the nonlinear function $f$ is continuous and satisfies the hypotheses of Theorem 3.1. Observe that the control defined by

$$u(t) = B^* \Phi^*(T - t) W^{-1} \bigg[\bar{x} - \int_0^T \Phi(T - s) f\big(s, x(s), {}^C D^\beta x(s)\big)\, \mathrm{d}s\bigg]$$

steers the system (9) from $x_0$ to $x_1$. Hence the system (9) is controllable on $[0, T]$.

## 5 Conclusion

Controllability is one of the qualitative properties of fractional dynamical system. It means the ability to move a system around entire configuration space using only certain admissible manipulations. In this paper we have proved a set of sufficient conditions for the controllability of nonlinear fractional dynamical system in finite dimensional spaces by using Mittag–Leffler matrix function and Schaefer's fixed-point theorem. Also, we included an example to verify our result.

**Acknowledgements** This work was partially funded by project MTM2010-16499 from the Goverment of Spain.

## References

1. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Oldham, K., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)

3. Petráŝ, I.: Control of fractional order Chua's system. J. Electr. Eng. **53**, 219–222 (2002)
4. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
5. West, B.J., Bologna, M., Grigolini, P.: Physics of Fractal Operators. Springer, Berlin (2003)
6. Podlubny, I.: Fractional-order systems and $PI^{\lambda}D^{\mu}$ controller. IEEE Trans. Auto. Control. **44**, 208–214 (1999)
7. Kaczorek, T.: Selected Problems of Fractional Systems Theory. Springer, Berlin (2011)
8. Adams, J.L., Hartley, T.T.: Finite time controllability of fractional order systems. J. Comput. Nonlinear Dyn. **3**, 0214021 (2008)
9. Bettayeb, M., Djennoune, S.: New results on the controllability and observability of fractional dynamical systems. J. Vib. Control. **14**, 1531–1541 (2008)
10. Chen, Y., Ahn, H.S., Xue, D.: Robust controllability of interval fractional order linear time invariant systems. Signal Process. **86**, 2794–2802 (2006)
11. Matignon, D., d'Andréa-Novel, B.: Some results on controllability and observability of finite dimensional fractional differential systems. In: Proceedings of the IAMCS, IEEE Conference on Systems, Man and Cybernetics, Lille, France, July 9-12, pp. 952–956 (1996)
12. Monje, C.A., Chen, Y.Q., Vinagre, B.M., Xue, X., Feliu, V.: Fractional-Order Systems and Controls: Fundamentals and Applications. Springer, London (2010)
13. Balachandran, K., Kokila, J., Trujillo, J.J.: Relative controllability of fractional dynamical systems with multiple delays in control. Comput. Math. Appl. (in press)
14. Balachandran, K., Park, J.Y., Trujillo, J.J.: Controllability of nonlinear fractional dynamical systems. Nonlinear Anal.: Theory Methods Appl. **75**, 1919–1926 (2012)
15. Balachandran, K., Zhou, Y., Kokila, J.: Relative controllability of fractional dynamical systems with distributed delays in control. Comput. Math. Appl. (in press)
16. Balachandran, K., Kokila, J., Trujillo, J.J.: Relative controllability of fractional dynamical systems with delays in control. Commun. Nonlinear Sci. Numer. Simul. **17**, 3508–3520 (2012)
17. Caputo, M.: Linear model of dissipation whose $Q$ is almost frequency independent. Part II. Geophys. J. R. astr. Soc. **13**, 529–539 (1967)
18. Eidel'man, S.D., Chikrii, A.A.: Dynamic game problems of approach for fractional-order equations. Ukr. Math. J. **52**, 1787–1806 (2000)
19. Chen, C.T.: Linear System Theory and Design. Saunder College Publishing/Harcourt Brace college publishers, New York (1970)
20. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, New York (1974)