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BIFURCATION WITHOUT PARAMETERS IN CIRCUITS WITH MEMRISTORS: A DAE APPROACH

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Abstract. Bifurcations without parameters describe qualitative changes in the local dynamics of nonlinear ODEs when normal hyperbolicity of a manifold of equilibria fails. Non-isolated equilibrium points are systematically exhibited by nonlinear circuits with *memristors*; a memristor is a nonlinear device recently introduced in circuit theory and which is expected to play a key role in electronics in the near future. In this communication we provide a graph-theoretic analysis of the transcritical bifurcation without parameters in memristive circuits, owing to the presence of a locally active memristor. The results are crucially based on the use of differential-algebraic circuit models.

1. **Introduction.** From Van der Pol's oscillator to Chua's circuit, much research on ordinary differential equations (ODEs) has been motivated by applications in nonlinear circuit theory. The results here reported stem from the discovery, in 2008, of a two-terminal nonlinear device behaving as a memristor [26] which has had a great impact in circuit theory and, more generally, in the electronic engineering community, in particular at the nanometer scale. This device poses challenging analytical and numerical problems, and in this communication we focus on the fact that it systematically yields manifolds of non-isolated equilibrium points; in particular, we perform an analysis of transcritical bifurcations without parameters in memristive circuits, resulting from the loss of normal hyperbolicity along a line of equilibria when a memristor becomes locally active. Our work is framed in the theory of Fiedler et al. [11, 12, 13] and, more specifically, we extend the results presented in [25].

Memristors are circuit elements characterized by a nonlinear relation between the charge and the flux, and their existence was predicted by Leon Chua in 1971 [7] for symmetry reasons. For a sample of recent research on memristive circuits, see [3, 4, 5, 9, 10, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24]. In a charge-controlled setting, this device is governed by a nonlinear, differentiable flux-charge relation

$$\varphi = \phi(q). \tag{1}$$

Time-derivation yields, via the relations $\varphi' = v$, $q' = i$, the voltage-current characteristic

$$v = M(q)i, \tag{2}$$

where $M(q) = \phi'(q)$ is the so-called *memristance*. Note that (2) is reminiscent of Ohm's law, but the "resistance" $M(q)$ now depends on the charge q, which is the time-integral of the current i; for this reason the device' characteristic keeps track of its own history. The name *memristor*, which is an abbreviation of *memory-resistor*, comes from this remark [7]. A memristor is said to be strictly locally passive at a given value of q if $M(q) > 0$.

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The dual device is a flux-controlled memristor, which is defined by a nonlinear relation $q = \sigma(\varphi)$; in this case $W(\varphi) = \sigma'(\varphi)$ is the *memductance*; the current-voltage characteristic is now $i = W(\varphi)v$. We will focus on the charge-controlled form (2), but the results also hold in the flux-controlled setting.

For reasons detailed later, circuits with memristors systematically exhibit non-isolated equilibria. Our goal in this communication is to present a systematic analysis of transcritical bifurcations without parameters (TBWP) in circuits with one memristor, which exhibit *lines* of equilibria; the study is restricted to circuits including, in addition, resistors, capacitors and inductors, all of which may be nonlinear. Specifically, the TBWP will result from the vanishing of $M(q)$ at a given value of q. The phenomenon here addressed can be easily introduced by means of the simple circuit example presented below.

An introductory example. Consider the circuit depicted in Figure 1, defined by a linear inductor with inductance L, and a memristor.

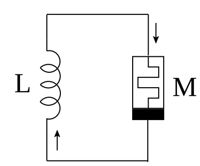

FIGURE 1. A simple memristive circuit

Elementary circuit theory yields the model

$$\begin{aligned} q' &= i \\ Li' &= -M(q)i, \end{aligned}$$

where q is the memristor charge and i is the current flowing in the loop. Note that equilibria are not isolated, but define the line $i = 0$. It is worth remarking that, with $L = 1$, $M(q) = -q$, $q \equiv x$, $i \equiv y$, we get the normal form for a transcritical bifurcation without parameters [11]:

$$\begin{aligned} x' &= y \\ y' &= xy. \end{aligned}$$

Here $y = 0$ is a line of equilibria, which are stable if $x < 0$ and unstable if $x > 0$; the stability change along the equilibrium line occurs at $x = 0$. In circuit-theoretic terms, the incremental memristance $M(q) = -q$ becomes negative for positive values of q, meaning that the memristor becomes strictly locally active, and this results in a stability loss in the circuit for positive values of q.

To a large extent, this example captures the circuit-theoretic essence of a transcritical bifurcation without parameters, as detailed in what follows.

2. **Background.**

2.1. **Circuit model.**

A key problem in time-domain circuit analysis is how to describe the circuit dynamics in terms of an ordinary differential equation (ODE). In most cases, the first description of a nonlinear circuit takes the form of an implicit ODE or, more precisely, of a *differential-algebraic equation* (DAE). Whenever possible, it is of interest to address qualitative properties directly in the DAE setting, since this avoids the need for assumptions yielding an explicit ODE model: these assumptions may be unnecessary in a qualitative analysis. The reader is referred to [23, 27] for detailed introductions to DAE circuit modelling.

We will consider circuits including a unique memristor, and possibly nonlinear capacitors, inductors and resistors. The dynamics of such a circuit can be described in terms of the model

$$\begin{aligned} q_m' &= i_m && \text{(3a)}\\ C(v_c)v_c' &= i_c && \text{(3b)}\\ L(i_l)i_l' &= v_l && \text{(3c)}\\ 0 &= B_m M(q_m) i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) && \text{(3d)}\\ 0 &= Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r, && \text{(3e)} \end{aligned}$$

where the subscripts m, c, l, r correspond to memristors, capacitors, inductors and resistors; q, v and i stand for charge, voltage and current, respectively. Equations (3a), (3b) and (3c) are nothing but the standard differential relations for memristors, capacitors and inductors; $C(v_c)$ and $L(i_l)$ are the incremental capacitance and inductance matrices, which depend on v_c and i_l when the corresponding devices are nonlinear. In turn, (3d) and (3e) are Kirchhoff laws, which are written in the form $Bv = 0$, $Qi = 0$ in terms of the so-called reduced loop and cutset matrices B and Q, whose columns are split according to the nature of the different devices to yield $B = (B_m \;\; B_c \;\; B_l \;\; B_r)$, $Q = (Q_m \;\; Q_c \;\; Q_l \;\; Q_r)$ (find details in [8, 23]). In (3d), $M(q_m)$ is the memristance (cf. (2)), whereas $\gamma(i_r)$ is the characteristic of resistors, which may in general be nonlinear. We assume throughout that ϕ in (1) and γ are differentiable; additional smoothness requirements are explicitly indicated when needed. For simplicity we assume that the characteristic of resistors verifies $\gamma(0) = 0$ and, for later use, we write the incremental resistance matrix $\gamma'(i_r)$ as $R(i_r)$; in linear cases, we would just have $\gamma(i_r) = Ri_r$, and the resistance matrix would not depend on i_r. All variables in (3) are vector-valued, except for q_m and i_m, since the circuit is assumed to include a single memristor.

Provided that $C(v_c)$ and $L(i_l)$ are non-singular (invertible) they can be trivially driven to the right-hand side of (3b) and (3c) to get

$$\begin{aligned} q_m' &= i_m && \text{(4a)}\\ v_c' &= (C(v_c))^{-1} i_c && \text{(4b)}\\ i_l' &= (L(i_l))^{-1} v_l && \text{(4c)}\\ 0 &= B_m M(q_m) i_m + B_c v_c + B_l v_l + B_r \gamma(i_r) && \text{(4d)}\\ 0 &= Q_m i_m + Q_c i_c + Q_l i_l + Q_r i_r, && \text{(4e)} \end{aligned}$$

and the circuit model has a semiexplicit differential-algebraic form:

$$\begin{aligned} y' &= h(y, z) && \text{(5a)}\\ 0 &= g(y, z), && \text{(5b)} \end{aligned}$$

with $y = (q_m, v_c, i_l)$ and $z = (i_m, i_c, v_l, i_r)$.

2.2. **Lines of equilibria.** Equilibria are defined by the vanishing of the right-hand side of (5), that is, by the conditions $h(y, z) = 0$, $g(y, z) = 0$. In light of (3) or (4), these conditions are easily translated into

$$i_m = i_c = v_l = 0, \;\; B_c v_c + B_r \gamma(i_r) = 0, \;\; Q_l i_l + Q_r i_r = 0.$$

Note that q_m does not enter these equations, meaning that equilibrium points are not isolated. Because of $\gamma(0) = 0$, we may focus on the line of equilibria defined by

$$i_m = i_c = v_l = 0, \;\; v_c = i_r = i_l = 0, \tag{6}$$

whereas q_m can be freely assigned; this variable parameterizes the equilibrium line.

2.3. **Some notions and results involving digraphs.** The reader is referred e.g. to [1, 2, 6] for detailed introductions to graph and digraph theory, and to [8, 23, 27] for systematic applications of this theory to nonlinear circuit analysis. A *loop* or *cycle* in a directed graph (or digraph) is the set of branches in a closed path without self-intersections. A *cutset* K is a set of branches whose removal increases the number of connected components of the digraph, and which is minimal with respect to this property, that is, the removal of any proper subset of K does not increase the number of components.

The existence or absence of certain loops/cutsets plays a relevant role in many aspects of circuit theory. For instance, C-loops are loops defined only by capacitors; L-cutsets are cutsets defined only by inductors; both configurations play a key role in several analytical properties of electrical circuits (find detailed discussions in this regard in [23, 27]). Regarding the terminology to be used, LC-loops are loops defined only by inductors and/or capacitors, and they include L-loops and C-loops as particular cases. Other types of loops and cutsets arising in our analysis are defined analogously.

The absence of loops or cutsets of a given type can be characterized in terms of the reduced loop and cycle matrices B, Q. As detailed in the aforementioned references, a set K of branches does not include cutsets if and only if B_K has full column rank (i.e. $\ker B_K = \{0\}$) and, analogously, it does not include loops if and only if Q_K has full column rank. Here B_K (resp. Q_K) is the submatrix of B (resp. of Q) defined by the columns which correspond to K-branches.

The following result will be used at several points in our analysis; its proof is easy and can be found in [25]. Split the branches of a given digraph in four pairwise disjoint sets K_1, K_2, K_3, K_4, and denote by B_i, Q_i the submatrices of B and Q defined by the columns which correspond to branches in K_i. If P is a positive definite matrix, and denoting

$$M = \begin{pmatrix} B_1 & 0 & B_3P \\ 0 & Q_2 & Q_3 \end{pmatrix}, \tag{7}$$

then the identity

$$\ker M = \ker B_1 \times \ker Q_2 \times \{0\} \tag{8}$$

holds. In particular, $\ker M = \{0\}$ if and only if the digraph has neither K_1-cutsets nor K_2-loops.

2.4. **Previous results.** As detailed in [25], if a given circuit (with sources) does not display VC-loops, VL-loops, IL-cutsets or IC-cutsets, and provided that the memristance and resistance matrices M, R are positive definite at a given equilibrium, then the linearization of the differential equations modelling the circuit has a zero eigenvalue whose geometric and algebraic multiplicities equal the number of memristors. If, additionally, the capacitance and inductance matrices C, L are symmetric and non-singular, either the absence of ICL-cutsets or that of VCL-loops implies that the linearization has no purely imaginary eigenvalues $i\omega$ with $\omega \neq 0$.

Altogether, these conditions imply the normal hyperbolicity of the manifold of equilibria in memristive circuits. The absence of the, say, degenerate configurations compiled above guarantees that the positive definiteness assumptions rules out purely imaginary eigenvalues or additional vanishing ones. Our goal below is to address what happens in a circuit with a unique memristor (yielding a *line* of equilibria) when the positive definiteness assumption on the memristance fails at a given point, that is, when the memristor becomes locally active. The key aspect is again to perform the analysis in circuit-theoretic terms, that is, to unravel the circuit configurations which yield the bifurcation.

3. Transcritical bifurcation without parameters in memristive circuits.

Theorem 3.1. *Consider an electrical circuit including a unique memristor, with a twice differentiable characteristic ϕ and for which $M(0) = 0$, $M'(0) \neq 0$, together with some* *(possibly nonlinear) resistors, with $\gamma(0) = 0$ and $R(0)$ positive definite, and some (possibly nonlinear) capacitors and inductors, with symmetric and non-singular capacitance and inductance matrices $C(0)$, $L(0)$.*

Assume that the circuit has a unique ML-loop, including the memristor and at least one inductor, and that it does not display LC-cutsets, MC-loops or MC-cutsets.

Then the circuit undergoes a transcritical bifurcation without parameters at the origin as the memristor charge q crosses 0; specifically, if $C(0)$ and $L(0)$ are positive definite, there is a loss of stability along the equilibrium line defined by (6) as q becomes positive (resp. negative) provided that $M'(0) < 0$ (resp. $M'(0) > 0$).

Here M' is the second derivative of the function ϕ defining the memristor characteristic (1) (cf. (2)). Before proceeding with the proof, note that the absence of LC-cutsets precludes in particular L- and C-cutsets; analogously, C-loops and L-loops are ruled out by the other conditions. Note also that the bifurcation is located at the origin because of the assumption $\gamma(0) = 0$ (cf. (6)).

Proof. The first step of the proof shows that, via the implicit function theorem, the variables i_m, i_c and v_l can be written, locally around the origin, in terms of q_m, v_c, i_l using (3d) and (3e). In terms of (5), the z-variables ($z = (i_m, i_c, v_l, i_r)$) are written in terms of $y = (q_m, v_c, i_l)$ by means of a local map $z = \psi(y)$, using (5b), to recast (5a) as an explicit ODE, namely

$$y' = h(y, \psi(y)). \tag{9}$$

This relies on the non-singularity of the matrix of partial derivatives of g with respect to z. In light of (3d) and (3e), this matrix is easily checked to read, at a generic point,

$$\begin{pmatrix} B_m M(q_m) & 0 & B_l & B_r R(i_r) \\ Q_m & Q_c & 0 & Q_r \end{pmatrix} \tag{10}$$

and, at the origin,

$$\begin{pmatrix} 0 & 0 & B_l & B_r R(0) \\ Q_m & Q_c & 0 & Q_r \end{pmatrix}, \tag{11}$$

because of the assumption $M(0) = 0$. After an obvious column reordering, (11) has the form depicted in (7), and therefore the absence of L-cutsets and MC-loops, together with the positive definiteness of $R(0)$, makes (11) non-singular. This means that a reduction of the form (9) is locally feasible in terms of $y = (q_m, v_c, i_l)$.

As detailed in the sequel, the linearization of this reduction at the origin has an index two, double zero eigenvalue, the other eigenvalues being away from the imaginary axis. Together with the additional second order condition detailed later, this will complete the requirements which, as proved by Fiedler et al. in [11], guarantee that, along the line of equilibria, the system undergoes a transcritical bifurcation without parameters at the origin.

The characteristic polynomial of (3) at a generic equilibrium (cf. (6)) is defined by the determinant of

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda C(0) & 0 & 0 & -I_c & 0 & 0 \\ 0 & 0 & \lambda L(0) & 0 & 0 & -I_l & 0 \\ 0 & B_c & 0 & B_m M(q_m) & 0 & B_l & B_r R(0) \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}, \tag{12}$$

which at the origin amounts to that of

$$\begin{pmatrix} \lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \lambda C(0) & 0 & 0 & -I_c & 0 & 0 \\ 0 & 0 & \lambda L(0) & 0 & 0 & -I_l & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(0) \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}. \tag{13}$$

The first column already shows that $\lambda = 0$ is an eigenvalue, consistently with the existence of a line of equilibria. The determinant of (13) reads as $\lambda \det D(\lambda)$, with

$$D(\lambda) = \begin{pmatrix} \lambda C(0) & 0 & 0 & -I_c & 0 & 0 \\ 0 & \lambda L(0) & 0 & 0 & -I_l & 0 \\ B_c & 0 & 0 & 0 & B_l & B_r R(0) \\ 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}. \tag{14}$$

Note that $\lambda = 0$ is also a root of $\det D(\lambda)$, as an easy consequence of the singularity of the matrix

$$\begin{pmatrix} B_c & 0 & 0 & B_r R(0) \\ 0 & Q_l & Q_m & Q_r \end{pmatrix}, \tag{15}$$

obtained after the determinantal expansion across the rows defined by the blocks $-I_c$ and $-I_l$ in (14) for $\lambda = 0$. The singularity of (15) owes to the assumed existence of an ML-loop, which makes the columns of $(Q_l \;\; Q_m)$ linearly dependent.

This means that $\lambda = 0$ is indeed a multiple eigenvalue when $q = 0$. The fact that it is a *double* eigenvalue follows from the fact that

$$\frac{d}{d\lambda} \det D(\lambda)\bigg|_{\lambda=0} \neq 0,$$

a relation which is equivalent to

$$D'(0)u \notin \operatorname{im} D(0) \tag{16}$$

for $u \in \ker D(0) - \{0\}$. Because C-cutsets are precluded, in light of (7), (8) and (14) vectors in $\ker D(0)$ have the form

$$u = \begin{pmatrix} 0 \\ u_l \\ u_m \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{17}$$

with

$$\begin{pmatrix} u_l \\ u_m \end{pmatrix} \in \ker(Q_l \;\; Q_m). \tag{18}$$

For u not to vanish it must happen that neither u_l nor u_m vanish, since (18) indicates the existence of an ML-loop and the circuit is assumed to have only one, which include both the memristor and some inductors. The condition (16) is then

$$\begin{pmatrix} 0 \\ L(0)u_l \\ 0 \\ 0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} 0 & 0 & 0 & -I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_l & 0 \\ B_c & 0 & 0 & 0 & B_l & B_r R(0) \\ 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}, \tag{19}$$

but the assumed absence of LC-cutsets implies $\ker(B_c \;\; B_l) = \{0\}$, and this guarantees (cf. again (7) and (8)) that (19) holds because of the condition $u_l \neq 0$ and the non-singularity of $L(0)$. Hence, the zero eigenvalue is a double one, as claimed.

Additionally, the null eigenvalue is index two; to check this it is enough to note that, when $\lambda = 0$, the matrix (12) has corank one. This follows from the fact that the columns of

$$\begin{pmatrix} B_c & 0 & B_r R(0) \\ 0 & Q_l & Q_r \end{pmatrix} \tag{20}$$

are linearly independent, because of the absence of both C-cutsets and L-loops. Together with the properties discussed above, we may conclude that the null eigenvalue is an index two, double one.

The absence of other purely imaginary eigenvalues when $q = 0$ follows from the results reported in subsection 2.4; specifically, the condition $M(0) = 0$ makes the circuit topology at the origin equivalent to the one obtained after short-circuiting the memristor or, equivalently, after replacing it by a voltage source with null voltage. Because of the assumption that $C(0)$ and $L(0)$ are symmetric and non-singular, the resulting circuit falls in the framework considered in [25] and, as indicated in subsection 2.4, the absence of ILC-cutsets (which amount in our setting to LC-cutsets) is enough to rule out other purely imaginary eigenvalues.

In order to apply the results of Fiedler et al. we finally need to prove the second order condition detailed in [11], which in our setting reads as

$$f''(0,0)\overline{p}\overline{q} \notin \operatorname{im} f'(0,0). \tag{21}$$

Here f groups together the maps arising in the right-hand side of (4) and, in terms of (5), we may write $f(y,z) = (h(y,z), g(y,z))$. Additionally, the vector $\overline{p}$ verifies

$$\overline{p} \in \ker f'(0,0) - \{0\},$$

whereas $\overline{q}$ is defined by the condition $Pw = f'(0,0)\overline{q}$, given that $w \in \ker f'(0,0)$ and $Pw \in \operatorname{im} f'(0,0) - \{0\}$, P denoting the projection $(y,z) \to (y,0)$. Note that $\overline{p}$ and $\overline{q}$ yield respectively an eigenvector and a generalized eigenvector of the null eigenvalue in the linearization of (9) at the origin; the existence of w (and hence of $\overline{q}$) follows from the index-two nature of the double zero eigenvalue proved above.

The matrix $f'(0,0)$ is, in light of (4),

$$f'(0,0) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_l & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(0) \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}, \tag{22}$$

and therefore vectors in $\ker f'(0,0) - \{0\}$ have the first scalar component as the unique non-vanishing one; this is of course consistent with the fact that the condition $f = 0$ describes the line of equilibria, which is parameterized by the first scalar variable (namely, the memristor charge q). In turn, it is not difficult to check that this implies that the vector $\overline{q}$ must have the form

$$\overline{q} = \begin{pmatrix} u_1 \\ 0 \\ u_l \\ u_m \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{23}$$

with the vector (u_l, u_m) satisfying (18). With these expressions for $\overline{p}$ and $\overline{q}$, the second order condition (21) may be checked to read

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ B_m M'(0) \\ 0 \end{pmatrix} \notin \operatorname{im} \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_l & 0 \\ 0 & B_c & 0 & 0 & 0 & B_l & B_r R(0) \\ 0 & 0 & Q_l & Q_m & Q_c & 0 & Q_r \end{pmatrix}. \tag{24}$$

It is worth emphasizing that the left-hand side of (24) does not vanish; B_m has a single column, which has at least one non-vanishing entry (because the memristor enters at least one loop), and $M'(0) \neq 0$ by hypothesis.

Assume that (24) does not hold; this would mean that there exists a nontrivial solution for the system

$$\begin{pmatrix} B_m M'(0) \\ 0 \end{pmatrix} = \begin{pmatrix} B_c & 0 & B_r R(0) \\ 0 & Q_l & Q_r \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{25}$$

or, equivalently, a non-vanishing vector in the kernel of

$$\begin{pmatrix} B_m & B_c & 0 & B_r R(0) \\ 0 & 0 & Q_l & Q_r \end{pmatrix}, \tag{26}$$

but, again as a consequence of (7)-(8)), this would be in contradiction with the absence of MC-cutsets assumed in the statement of our result.

This shows that the results of Fiedler et al. in [11] apply in our context, and therefore one real eigenvalue changes sign as q crosses the origin along the line of equilibria. The additional claim in Theorem 3.1, involving the positive definite assumption on $C(0)$ and $L(0)$, follows directly from the fact that $R(0)$ is positive definite as well; therefore, when $M(q)$ is also positive the circuit is passive and, according to the results in [25], no eigenvalue may have a positive real part. This means that the eigenvalue transition is from the negative to the positive real semiaxis (hence the stability loss) as q increases if $M'(0) < 0$, and vice-versa. This completes the proof. □

4. **Concluding remarks.** Memristive circuits are likely to play a relevant role in electronics in the near future; these circuits pose challenging analytical problems, some of which must be framed in a dynamical systems context. In this communication we have addressed certain qualitative phenomena arising from the systematic presence of non-isolated equilibria in circuits with memristors; specifically, we have performed a graph-theoretic analysis of the transcritical bifurcation without parameters, stemming from the transition of a memristor to a locally active region. Other related bifurcations, such as the Hopf bifurcation without parameters, are in the scope of future research in this field.

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