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On the Decay Rate of Discrete-time Linear Delay Systems with Cone Invariance

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***Abstract*—This paper is concerned with the decay rate constrained stability analysis for linear delay systems that possess cone-invariant property. In order to capture the decay rate of such systems, we introduce a nondecreasing positive function whose reciprocal represents the decay rate. Under mild assumptions on the growth rate of this function, an explicit condition is given to ensure that a cone-preserving linear system with unbounded time-varying delays is asymptotically stable with a given decay rate. As typical cases, necessary and sufficient conditions are given to characterize the decay rate when the delay is restricted by a linear, sublinear or logarithmic growth rate. Finally, some numerical examples are given to illustrate the effectiveness of the theoretical results.**

***Index Terms*—cone invariance, decay rate, positive systems, time-delay systems.**

I. Introduction

Invariant set is an important concept for dynamic systems that are subjected to various state constraints. This paper focuses on a particular class of systems with a proper cone being an invariant set. Such systems, also referred to as monotone systems [1], have broad applications in rendezvous of multiple agents [2], including cooperative control of unmanned air vehicle and satellite formation flying. As a typical example of stochastic systems, it was pointed out in [3] that the second moment dynamics of the geometric Brownian motion can be transformed into a matrix differential system invariant with respect to the the cone of positive semidefinite matrices. To handle linear systems with cone invariance one may resort to mathematical tools involving partial ordering over proper cones and cone-invariant operators [4]. On the other hand, positive systems are often used to capture physical systems whose state variables have intrinsically constant sign. The state and output of this particular type of systems are always confined to be nonnegative given that the input and the initial conditions are nonnegative. Positive systems can be viewed as a special class of cone-invariant systems with the first orthant being the invariant cone. Such systems are often encountered in a variety of disciplines, ranging from pharmacokinetics [5], passing through systems biology [6], to ecology [7]. Positive systems naturally arise in the modeling of compartmental systems [8], air traffic flow control [9] and vehicle platoons. In the past decade, many fundamental results concerning behavioral analysis and synthesis for positive systems have been reported in the literature, one may refer to [10]–[14] and the references therein.

The analysis of the effects of time delays in positive systems is a fundamental and hot topic owing to the fact that time delays are ubiquitous in various control systems. Recent developments on positive delay systems have shown that, in general, asymptotic stability of a positive system is insensitive to the magnitude of delays while the decay rate is dependent on the size of the delays. In the following, we give a brief review on the stability analysis of positive delay systems. In [15], a theorem of Perron-Frobenius type was established for delayed cooperative systems with an additional irreducibility assumption. By virtue of linear copositive Lyapunov-Krasovskii functionals, the asymptotic stability for positive systems with constant delays was considered in [16] and it turns out that the delay system is asymptotically stable as long as the corresponding delay-free system is asymptotically stable. Following this result, a necessary and sufficient condition was given in [17] to characterize the decay rate of a positive linear system with constant delays. For cone-preserving systems with constant delays, it was also proved in [3] that the magnitude of the delays does not affect its asymptotic stability using the fact that the spectral radius of the cone-preserving transfer function along the imaginary axis attains its maximum at frequency zero. The delay insensitivity for positive linear systems with bounded time-varying delays was proved in [18], [19], through covering the state trajectory of the system with a family of nonnegative vector-valued functions. More recently, a simple alternative proof based on reductio ad absurdum was given in [20]. To summarize, the linear copositive Lyapunov-Krasovskii functional is related to the L_1-norm of the state trajectory, which is often handicapped when time-varying delays are encountered unless some additional assumptions on the delay derivative are imposed. On the other hand, the L_∞-norm based method [21] is closely related to the partial ordering with respect to the first orthant and is very convenient for the analysis of positive systems with time-varying delays, see for example [22], [23]. Based on a max-type copositive Lyapunov function [24], the decay rate for homogeneous positive systems with bounded and unbounded delays was explicitly characterized in [25], [26] and [27], respectively. However, the effect of delays in cone-preserving systems has received relatively little attention. By resorting to the monotonicity of the state trajectory with respect to a given proper cone, it was recently shown in [28] that the asymptotic stability of a cone-preserving system with bounded time-

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varying delays is insensitive to the magnitude of the delays. These techniques, however, do not seem to be applicable for the decay rate analysis of delay systems with cone invariance, which motivates the current work.

In this paper, we aim at developing a general framework to capture the decay rate of cone-preserving systems with unbounded time-varying delays. Inspired by the μ-stability notion introduced in [29], [30], we employ a nondecreasing positive function whose reciprocal characterizes how the system decays. By imposing some mild growth conditions on the time delays and resorting to the partial ordering induced by the given cone, we give an explicit condition to ensure that a linear delay system with cone invariance is asymptotically stable with guaranteed decay rate. As several typical cases, some necessary and sufficient characterizations on the decay rate are also given, respectively, for the cases when the delay is constrained by a linear, sublinear or logarithmic growth rate. This paper reveals that many results in positive delay systems can be understood as a consequence of the cone invariance rather than positivity.

II. Notations and Preliminaries

In this section, we will introduce some elementary notations and lemmas concerning cones and matrices leaving a cone invariant. One can refer to the book [4, Chapter 1] for further details. $\mathbb{R}$, $\mathbb{R}_+$, $\bar{\mathbb{R}}_+$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_+$ denote the set of real numbers, positive real numbers, nonnegative real numbers, integers, nonnegative integers and positive integers, respectively. x_i denotes the ith entry of a column vector $x \in \mathbb{R}^n$. For a sequence $\{c_k\}_{k=1}^{\infty}$, $\overline{\lim}_{k\to\infty} c_k$ and $\underline{\lim}_{k\to\infty} c_k$ denote, respectively, its limit superior and limit inferior. We will denote the interior of a set S by $\text{int}(S)$. Given a set S, S^G denotes the set generated by S, which consists of all finite nonnegative linear combinations of elements of S. A set is defined to be a cone if $K = K^G$. A convex cone is pointed if $K \cap (-K) = \{0\}$ and solid if $\text{int}(K) \neq \emptyset$. A closed, pointed, solid convex cone K is called a proper cone. Throughout this paper, we always assume that the given cone is proper. A proper cone K induces a partial ordering in $\mathbb{R}^n$ via $y \preceq_K x$ if and only if $x - y \in K$. In addition, $y \prec_K x$ means that $x - y \in \text{int}(K)$.

Given a proper cone $K \subset \mathbb{R}^n$, a matrix A is called K-nonnegative if $Ax \in K$ for all vectors $x \in K$. For the special case when $K = \bar{\mathbb{R}}^n_+$, a matrix A is $\bar{\mathbb{R}}^n_+$-nonnegative (or nonnegative) if and only if all its entries are nonnegative.

The following definition on cone-induced vector norm will be needed in the sequel.

Definition 1: [4, pp. 5–6] Let $K \subset \mathbb{R}^n$ be a proper cone and let $v \in \text{int}(K)$, then one can define an order interval

$$B_v = \{x \in \mathbb{R}^n : -v \preceq_K x \preceq_K v\}.$$

The set B_v induces a norm on $\mathbb{R}^n$:

$$\|x\|_v = \inf\{t \geq 0 : x \in tB_v\}.$$

With a little abuse of notation, for a sequence $\phi : [a,b] \cap \mathbb{Z} \to \mathbb{R}^n$, we define its cone-induced norm as $\|\phi\|_v \triangleq \max_{k\in[a,b]\cap\mathbb{Z}} \|\phi(k)\|_v$. For the particular case when $K = \bar{\mathbb{R}}^n_+$ and $v = [1,1,\ldots,1]^T$, the cone-induced norm becomes the ∞-norm, which is defined as $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$.

III. Main Results

In this subsection, we consider discrete-time linear systems described by the following difference equation:

$$\begin{cases} x(k+1) = Ax(k) + A_\tau x(k - \tau(k)), & k \in [k_0, \infty) \cap \mathbb{Z}, \\ \quad x(s) = \phi(s), & s \in [k_0 - \tau, k_0] \cap \mathbb{Z}, \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ represents the system state. We assume that $\tau(k)$ satisfies $k - \tau(k) \to +\infty$ as $k \to +\infty$, and thus one can find a sufficiently large $T_0 \in \mathbb{N}$ (and without loss of generality, $T_0 > k_0$), such that for any $k > T_0$, it holds that $k - \tau(k) > k_0$. Therefore, the initial condition only needs be specified on $[k_0 - \tau, k_0]$, where $k_0 - \tau = \min_{k\in[k_0,T_0]\cap\mathbb{N}}\{k - \tau(k)\}$. The delay functions $\tau(\cdot)$ are assumed to be within a set Ω, where the set Ω consists of all possible time-varying delay functions under investigation. For instance, $\Omega = \{\tau : \tau(k) \in \{1,2,3\},\ \forall k \in \mathbb{N}\}$ captures a typical class of interval time-varying delays. The trajectory of system (1) with initial time instant k_0 and initial condition $\phi(\cdot)$ is denoted by $x(k; k_0, \phi)$. The characterization of the cone invariance of system (1) with an external input is straightforward by mathematical induction, which is stated in the following lemma.

Lemma 1: Given a proper cone $K \subset \mathbb{R}^n$ and suppose that A, A_τ and B are all K-nonnegative matrices, then for any initial condition $\phi(s) \in K$ ($\forall s \in [k_0 - \tau, k_0] \cap \mathbb{Z}$) and any input $u(k) \in K$ ($\forall k \geq k_0$), the state trajectory of the discrete-time system $x(k+1) = Ax(k) + A_\tau x(k - \tau(k)) + Bu(k)$ satisfies that $x(k) \in K$ for all $k \geq k_0$.

Throughout this paper, we always make the following assumptions on system (1).

Assumption 1: Given a proper cone $K \subset \mathbb{R}^n$, it is assumed that system (1) fulfills the following conditions:

(i) A and A_τ are all K-nonnegative matrices.
(ii) Each time delay $\tau \in \Omega$ satisfies that $k - \tau(k) \to +\infty$ as $k \to \infty$.

The decay rate of system (1) is defined as follows.

Definition 2: [30] Given a nondecreasing strictly positive function $p(k) : [-\tau, +\infty] \cap \mathbb{Z} \to \mathbb{R}_+$ satisfying that $p(k) \to +\infty$ as $k \to \infty$, system (1) (with some delay $\tau \in \Omega$) is said to be asymptotically stable with decay rate $O(1/p(k))$, if for any $k_0 \in \mathbb{N}$, there exists a scalar $M(k_0) > 0$, such that the trajectory of system (1) satisfies $\|x(k; k_0, \phi)\| \leq M(k_0)\|\phi\|/p(k)$ for all $\phi : [k_0 - \tau, k_0] \cap \mathbb{Z} \to \mathbb{R}^n$ and $k \geq k_0$.

Remark 1: $\|\cdot\|$ can denote any vector norm defined on $\mathbb{R}^n$ since all norms in finite dimensional space are equivalent.

By the linearity and cone invariance of system (1), the following lemma directly follows.

Lemma 2: Suppose that A and A_τ are K-nonnegative matrices, then $\phi_1(s) \preceq_K \phi_2(s)$ ($s \in [k_0 - \tau, k_0] \cap \mathbb{Z}$) implies that $x(k; k_0, \phi_1) \preceq_K x(k; k_0, \phi_2)$ for all $k \geq k_0$.

The asymptotic stability condition of cone-invariant system (1) is given in the following proposition.

Proposition 1: Suppose that system (1) satisfies Assumption 1, then system (1) is asymptotically stable if there exists $\lambda \succ_K 0$, such that $(A + A_\tau)\lambda \prec_K \lambda$.

Proof: It suffices to prove that $\lim_{k\to\infty} x(k;k_0,v)=0$ for some $v\succ_K 0$. Suppose that there exists $\lambda\succ_K 0$, such that $(A+A_\tau)\lambda\prec_K\lambda$, which implies the existence of a scalar $0<\theta<1$, such that $(A+A_\tau)\lambda\prec_K\theta\lambda$. We intend to prove by induction that a sequence $\{k_m\}_{m=0}^{\infty}$ can be found, such that $x(k;k_0,\lambda)\preceq_K \theta^m\lambda$ for $k\geq k_m$. It can be readily shown by induction that $x(k;k_0,\lambda)\preceq_K\lambda$ for all $k\geq k_0$. Now suppose that $x(k;k_0,\lambda)\preceq_K \theta^m\lambda$ for $k\geq k_m$. Since $k-\tau(k)\to\infty$ as $k\to\infty$, one can find a sufficiently large $k_{m+1}>k_m$, such that $k-\tau(k)\geq k_m$ for all $k\geq k_{m+1}$. For this index k_{m+1}, we can conclude that $x(k;k_0,\lambda)\preceq_K (A+A_\tau)\theta^m\lambda\preceq_K\theta^{m+1}\lambda$ for all $k\geq k_{m+1}$. This completes the proof. ■

Our goal is to give a characterization on the decay rate for system (1) with cone invariance. To this end, we introduce a nondecreasing strictly positive function $p(k):[-\tau,+\infty]\cap\mathbb{Z}\to\mathbb{R}_+$ and make some assumptions on the growth rate of this function. Then, a sufficient condition can be given, under which system (1) is asymptotically stable with a given decay rate being the reciprocal of $p(k)$.

Theorem 1: Suppose that system (1) satisfies Assumption 1 and that $p(k):[-\tau,+\infty]\cap\mathbb{Z}\to\mathbb{R}_+$ is a nondecreasing function satisfying $p(k)\to+\infty$ as $k\to\infty$ and $\overline{\lim}_{k\to\infty}\frac{p(k+1)}{p(k)}\leq\alpha$ ($\alpha\geq 1$). Further assume that each delay $\tau\in\Omega$ satisfies that $\overline{\lim}_{k\to\infty}\frac{p(k+1)}{p(k-\tau(k))}\leq\beta$ ($\beta\geq 1$). If there exists $\lambda\in\mathrm{int}(K)$, such that the following inequality holds:

$$(\alpha A+\beta A_\tau-I)\lambda\prec_K 0, \tag{2}$$

then system (1) is asymptotically stable with decay rate $O(1/p(k))$ for all delays $\tau\in\Omega$.

Proof: Without loss of generality, we assume that the initial condition $\phi(\cdot)$ satisfies that $\|\phi\|_\lambda=1$. By Lemma 2, it can be concluded that for any initial time instant $k_0\in\mathbb{N}$ and any initial condition $\phi(\cdot)$ with $\|\phi\|_\lambda=1$, $x(k;k_0,-\lambda)\preceq_K x(k;k_0,\phi)\preceq_K x(k;k_0,\lambda)$ holds for all $k\geq k_0$. Therefore, it suffices to focus on one particular constant initial condition $\phi(s)\equiv\lambda$ ($s\in[k_0-\tau,k_0]\cap\mathbb{Z}$). By inequality (2), we can find a sufficiently small $\varepsilon>0$, such that $(\alpha+\varepsilon)A\lambda+(\beta+\varepsilon)A_\tau\lambda\prec_K\lambda$. For this ε and any given $k_0\in\mathbb{N}$, one can find a sufficiently large integer $T>k_0$, such that $\frac{p(k+1)}{p(k)}<\alpha+\varepsilon$ and $\frac{p(k+1)}{p(k-\tau(k))}<\beta+\varepsilon$ hold for $k\geq T$. Define $V(k)\triangleq p(k)x(k;k_0,\lambda)$, one can always find a sufficiently large scalar $M>0$ (dependent on k_0), such that $V(k)=p(k)x(k;k_0,\lambda)\preceq_K M\lambda$ for $k=k_0-\tau,k_0-\tau+1,\ldots,T$. We aim to prove that $V(k)\preceq_K M\lambda$ holds for all $k>T$. We will prove it by mathematical induction. Suppose that $V(k)\preceq_K M\lambda$ holds for $k=T,T+1,\ldots,m$, then for $k=m+1$, we have that

$$\begin{aligned}
&V(m+1)\\
&=p(m+1)x(m+1;k_0,\lambda)\\
&=p(m+1)Ax(m;k_0,\lambda)+p(m+1)A_\tau x(m-\tau(m);k_0,\lambda)\\
&=\frac{p(m+1)}{p(m)}AV(m)+\frac{p(m+1)}{p(m-\tau(m))}A_\tau V(m-\tau(m))\\
&\preceq_K\frac{p(m+1)}{p(m)}AM\lambda+\frac{p(m+1)}{p(m-\tau(m))}A_\tau M\lambda\\
&\preceq_K M((\alpha+\varepsilon)A\lambda+(\beta+\varepsilon)A_\tau\lambda)\prec_K M\lambda.
\end{aligned}$$

By mathematical induction, it can be concluded that $V(k)\preceq_K M\lambda$ holds for all $k>T$. Therefore, it immediately follows that $x(k;k_0,\lambda)\preceq_K M\lambda/p(k)$ for all $k\geq k_0$, which completes the proof. ■

Remark 2: Note that in Theorem 1, the initial conditions of system (1) are not restricted in the cone K. For the case when the initial conditions do not belong to the cone K, the decay rate constrained stability can still be ensured although the state trajectory may not stay in the cone K.

In order to give a necessary condition under which certain decay rate of system (1) can be guaranteed, the following lemma is needed.

Lemma 3: Assume that system (1) satisfies Assumption 1. If for some $k_0\in\mathbb{N}$, system (1) satisfies $\lim_{k\to\infty}x(k;k_0,\phi)=0$ for any $\phi:[k_0-\tau,k_0]\cap\mathbb{Z}\to\mathbb{R}^n$, then there exists $\lambda\in\mathrm{int}(K)$, such that $(A+A_\tau-I)\lambda\prec_K 0$.

Proof: We first prove that $A+A_\tau-I$ is invertible. Suppose on the contrary that $A+A_\tau-I$ is a singular matrix, then one can find a nonzero vector ξ, such that $(A+A_\tau)\xi=\xi$. This implies that the linear space $\{c\xi\mid c\in\mathbb{R}\}$ is an equilibrium manifold of system (1) and therefore the equilibrium zero is not globally asymptotically stable, which leads to a contradiction. Therefore, $A+A_\tau-I$ is invertible and thus we can define $y(k)=x(k;k_0,(A+A_\tau-I)^{-1}v)-(A+A_\tau-I)^{-1}v$ for some given $v\in\mathrm{int}(K)$. Then $y(k)$ satisfies the following system equation:

$$\begin{cases} y(k+1)=Ay(k)+A_\tau y(k-\tau(k))+v,\\ \quad y(s)=0,\quad s\in[k_0-\tau,k_0]\cap\mathbb{Z}.\end{cases}\tag{3}$$

Note that $\lim_{k\to\infty}x(k;k_0,\phi)=0$ implies that $\lim_{k\to\infty}y(k)$ exists. Note that system (3) is a cone-preserving system with zero initial conditions and an input $v\in\mathrm{int}(K)$, hence $y(k)\in K$ for all $k\geq k_0$, which further implies that $\tilde{\lambda}\triangleq\lim_{k\to\infty}y(k)\in K$ since the cone K is closed. Letting $k\to\infty$ on both sides of equation (3), it can be deduced that $(A+A_\tau-I)\tilde{\lambda}+v=0$. Define $\lambda\triangleq\tilde{\lambda}+\varepsilon v$, for sufficiently small $\varepsilon>0$, we have $\lambda\in\mathrm{int}(K)$ and $(A+A_\tau-I)\lambda=-v+\varepsilon(A+A_\tau-I)v\in-\mathrm{int}(K)$. This completes the proof. ■

In the following, a necessary condition will be given to ensure that cone-preserving system (1) is asymptotically stable with certain decay rate.

Theorem 2: Suppose that system (1) satisfies Assumption 1 and that $p(k):[-\tau,+\infty]\cap\mathbb{Z}\to\mathbb{R}_+$ is a nondecreasing function satisfying that $p(k)\to+\infty$ as $k\to\infty$ and $\underline{\lim}_{k\to\infty}\frac{p(k+1)}{p(k)}\geq\alpha\geq 1$. Further assume that each delay $\tau\in\Omega$ satisfies that $\underline{\lim}_{k\to\infty}\frac{p(k+1)}{p(k-\tau(k))}\geq\beta\geq 1$. If there exists a delay $\tau\in\Omega$, such that system (1) is asymptotically stable with decay rate $O(1/p(k))$, then there exists a nonzero vector $\lambda\in K$, such that $(\alpha A+\beta A_\tau-I)\lambda\preceq_K 0$.

Proof: For system (1) with decay rate $O(1/p(k))$, we define $z(k;T,\phi)\triangleq x(k;T,\phi)p(k)^{1-\varepsilon}$, then it follows that $\|z(k;T,\phi)\|=\|x(k;T,\phi)\|p(k)^{1-\varepsilon}\leq M(T)\|\phi\|p(k)^{-\varepsilon}$ for all initial condition ϕ and $k\geq T$, which implies that $\lim_{k\to\infty}z(k;T,\phi)=0$. Since $\underline{\lim}_{k\to\infty}\frac{p(k+1)}{p(k)}\geq\alpha\geq 1$ and $\underline{\lim}_{k\to\infty}\frac{p(k)}{p(k-\tau(k))}\geq\beta\geq 1$, for any positive scalar $\varepsilon<1$, one can find a sufficiently large $T\in\mathbb{N}$, such that $\frac{p(k+1)}{p(k)}>\alpha-\varepsilon$

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and $\frac{p(k)}{p(k-\tau(k))} > \beta - \varepsilon$ hold for all $k \geq T$. Note that for any initial condition $\phi(s) \in K$ ($s \in [T-\tau, T] \cap \mathbb{Z}$), $z(k;T,\phi) \in K$ holds for $k \geq T$ since $x(k;T,\phi) \in K$. By simple manipulations, one can check that $z(k;T,\phi)$ satisfies the following equation:

$$\begin{aligned} & z(k+1) \\ &= \left(\frac{p(k+1)}{p(k)}\right)^{1-\varepsilon} Az(k) + \left(\frac{p(k+1)}{p(k-\tau(k))}\right)^{1-\varepsilon} A_\tau z(k-\tau(k)) \\ &= (\alpha-\varepsilon)^{1-\varepsilon} Az(k) + (\beta-\varepsilon)^{1-\varepsilon} A_\tau z(k-\tau(k)) + u(k), \end{aligned} \tag{4}$$

where

$$\begin{aligned} u(k) = &\left(\left(\frac{p(k+1)}{p(k)}\right)^{1-\varepsilon} - (\alpha-\varepsilon)^{1-\varepsilon}\right) Az(k) \\ &+ \left(\left(\frac{p(k+1)}{p(k-\tau(k))}\right)^{1-\varepsilon} - (\beta-\varepsilon)^{1-\varepsilon}\right) A_\tau z(k-\tau(k)). \end{aligned}$$

It is clear that $u(k) \in K$ for $k \geq T$ since $z(k;T,\phi) \in K$ for $k \geq T$. Consider an auxiliary system given by

$$\underline{z}(k+1) = (\alpha-\varepsilon)^{1-\varepsilon} A\underline{z}(k) + (\beta-\varepsilon)^{1-\varepsilon} A_\tau \underline{z}(k-\tau(k)). \tag{5}$$

It is easy to see that $z(k;T;\phi) \succeq_K \underline{z}(k;T,\phi) \succeq_K 0$ for $k \geq T$ and all $\phi(s) \in K$ ($s \in [T-\tau, T] \cap \mathbb{Z}$). Therefore, it follows that for any initial condition $\phi(s) \in K$ ($s \in [T-\tau, T] \cap \mathbb{Z}$), we have that $\lim_{k\to\infty} \underline{z}(k;T,\phi) = 0$. The same conclusion holds for any initial condition $\phi : [T-\tau, T] \cap \mathbb{Z} \to \mathbb{R}^n$, since any proper cone is reproducing [4, Chapter 1]. By Lemma 3, it can be concluded that for any $0 < \varepsilon < 1$, there exists $\lambda_\varepsilon \in \mathrm{int}(K)$, such that $((\alpha-\varepsilon)^{1-\varepsilon} A + (\beta-\varepsilon)^{1-\varepsilon} A_\tau - I)\lambda_\varepsilon \prec_K 0$ (by suitable scaling, one can assume that $\|\lambda_\varepsilon\| = 1$). Note that a sequence $\varepsilon_j \to 0$ ($j \to \infty$) can always be found, such that $\lambda_{\varepsilon_j} \to \lambda$ as $j \to \infty$, since λ_ε is within a compact set. Now we can conclude that there exists $\lambda \in K$, $\lambda \neq 0$, such that $(\alpha A + \beta A_\tau - I)\lambda \preceq_K 0$. This completes the proof. ■

Remark 3: We remark that while Theorem 1 provides the "worst case" decay rate of the system, Theorem 2 gives fastest decay rate for system (1) with delays $\tau \in \Omega$. Also note that the condition given in [26] requires, *a priori*, the existence of the limit $\lim_{k\to\infty} \frac{p(k+1)}{p(k-\tau(k))}$. This limit does not, in general, exist when we consider all possible delays within the set Ω, therefore the result in [26] seems not easily applicable even for the bounded interval time-varying delay case. Hence, the results in [26] cannot predict the exact decay rate of a positive system even if a particular delay range is given.

Remark 4: Almost all the results on the decay rate of general linear systems (see, for example, [31]) have suffered from certain conservatism owing to the use of Lyapunov-Krasovskii functional. However, combining Theorem 1 and Theorem 2, we can characterize the exact decay rate of an interval time-varying delay system using the cone-invariance property.

IV. Explicit Characterizations for Several Typical Cases

In the light of the two theorems in the last section, we point out that our results encompass several typical cases. When the delay $\tau(k)$ is bounded, system (1) is exponentially stable. When $\tau(k)$ is unbounded but constrained by a linear growth rate, system (1) possesses a power law decay rate. An exact decay rate is also given for the case when $\tau(k)$ is restricted by a sublinear growth rate and a logarithmic one.

Corollary 1: Given scalars $\alpha > 1$, $\tau \in \mathbb{N}_+$, the following two statements hold:

(i) If there exists a vector $\lambda \in \mathrm{int}(K)$, such that the following inequality holds:
$$(\alpha A + \alpha^{(\tau+1)} A_\tau - I)\lambda \prec_K 0, \tag{6}$$
then for any delay satisfying $0 \leq \tau(k) \leq \tau$, system (1) is asymptotically stable with decay rate $O(\alpha^{-k})$.

(ii) Conversely, if for the constant delay $\tau(k) \equiv \tau$, system (1) is asymptotically stable with decay rate $O(\alpha^{-k})$, then there exists a nonzero vector $\lambda \in K$, such that $(\alpha A + \alpha^{(\tau+1)} A_\tau - I)\lambda \preceq_K 0$.

Proof: By letting $p(k) = \alpha^k$ in Theorem 1, it is easy to check that $\lim_{k\to\infty} \frac{p(k+1)}{p(k)} = \alpha$ and $\overline{\lim}_{k\to\infty} \frac{p(k+1)}{p(k-\tau(k))} \leq \alpha^{\tau+1}$ ($\lim_{k\to\infty} \frac{p(k+1)}{p(k-\tau(k))} = \alpha^{\tau+1}$ when $\tau(k) = \tau$). Then, statement (i) directly follows from Theorem 1 while statement (ii) can be proved by applying Theorem 2 to $\Omega = \{\tau : \tau(k) = \tau,\ \forall k \in \mathbb{N}\}$. ■

Remark 5: The exponential stability condition for positive systems with bounded time-varying delays reported in [32] can be regarded as a special case of Corollary 1, when the cone is specified as the nonnegative orthant.

In the following corollaries, the appropriate converse statements also hold due to Theorem 2, but are omitted for simplicity.

Corollary 2: Given scalars $0 < \beta < 1$, $\gamma > 0$ and $\delta \in \mathbb{N}$, if there exists a vector $\lambda \in \mathrm{int}(K)$, such that $(A + (1-\beta)^{-\gamma} A_\tau - I)\lambda \prec_K 0$, then for any delay satisfying $0 \leq \tau(k) \leq \beta k$ ($k \geq \delta$), there exists $a > 0$, such that system (1) is asymptotically stable with decay rate $O((k+a)^{-\gamma})$.

Proof: Let $p(k) = (k+a)^\gamma$, where a is chosen sufficiently large such that $-a < \min_{k\in[0,\delta]}\{k - \tau(k)\}$, then it can be readily obtained that $\lim_{k\to\infty} \frac{p(k+1)}{p(k)} = 1$ and $\overline{\lim}_{k\to\infty} \frac{p(k)}{p(k-\tau(k))} \leq (1-\beta)^{-\gamma}$. By Theorem 1, the conclusion immediately follows. ■

Corollary 3: Given scalars $\alpha > 1$, $\beta > 0$, $0 < \gamma < 1$ and $\delta \in \mathbb{N}$, if there exists $\lambda \in \mathrm{int}(K)$, such that $(A + \alpha^{\beta(1-\gamma)} A_\tau - I)\lambda \prec_K 0$, then for all delays satisfying $0 \leq \tau(k) \leq \beta k^\gamma$ ($k \geq \delta$), system (1) is asymptotically stable with decay rate $O(\alpha^{-k^{1-\gamma}})$.

Proof: Define $p(k) = \alpha^{k^{1-\gamma}}$ ($k \geq 0$), then we have that $\lim_{k\to\infty} \frac{p(k+1)}{p(k)} = 1$. Taking into account that for any $h > 0$, $\lim_{x\to 0} \frac{(1+x)^h - 1}{hx} = 1$, it follows that $\overline{\lim}_{k\to\infty} \log_\alpha \frac{p(k)}{p(k-\tau(k))} \leq \lim_{k\to\infty} \left(k^{1-\gamma} - (k - \beta k^\gamma)^{1-\gamma}\right) = \lim_{k\to\infty} \frac{1 - (1-\beta k^{\gamma-1})^{1-\gamma}}{k^{\gamma-1}} = \beta(1-\gamma)$. Therefore, $\overline{\lim}_{k\to\infty} \frac{p(k)}{p(k-\tau(k))} \leq \alpha^{\beta(1-\gamma)}$. The statement follows from Theorem 1. ■

Corollary 4: Given scalars $\alpha > 1$, $\beta > 0$, and $\delta \in \mathbb{N}$, if there exists $\lambda \in \mathrm{int}(K)$, such that $(A + \alpha^\beta A_\tau - I)\lambda \prec_K 0$, then for all delays satisfying $0 \leq \tau(k) \leq \beta \ln(k+1)$ ($k \geq \delta$), there exists $a > 0$, such that system (1) is asymptotically stable with decay rate $O(\alpha^{-k/\ln(k+a)})$.

Proof: Let $p(k) = \alpha^{k/\ln(k+a)}$ for some sufficiently large $a > 0$, one can see that $\lim_{k\to\infty} \frac{p(k+1)}{p(k)} = 1$.

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Simple manipulation yields that $\overline{\lim}_{k\to\infty} \log_\alpha \frac{p(k)}{p(k-\tau(k))} \leq \lim_{k\to\infty}\left(\frac{k}{\ln k} - \frac{k}{\ln(k-\beta\ln k)}\right) + \beta \lim_{k\to\infty} \frac{\ln k}{\ln(k-\beta\ln k)}$. By taking into account that $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, we can obtain that $\lim_{k\to\infty}\left(\frac{k}{\ln k} - \frac{k}{\ln(k-\beta\ln k)}\right) = 0$ and that $\lim_{k\to\infty} \frac{\ln k}{\ln(k-\beta\ln k)} = 1$, and thus $\overline{\lim}_{k\to\infty} \frac{p(k)}{p(k-\tau(k))} \leq \alpha^\beta$. The conclusion then directly follows from Theorem 1. ■

V. Illustrative Example

Consider discrete-time linear system (1) with the following system matrices:

$$A = \begin{bmatrix} 0.25 & 0.04 & -0.05 \\ 0.11 & 0.12 & 0.1 \\ 0.1 & -0.03 & 0.15 \end{bmatrix},$$

$$A_\tau = \begin{bmatrix} 0.15 & 0.05 & -0.1 \\ 0.25 & 0.1 & 0.15 \\ -0.05 & 0 & 0 \end{bmatrix}.$$

We consider a polyhedral cone K generated through non-negative linear combinations of four vectors $[\ 1 \quad 0 \quad 0\ ]^T$, $[\ 0 \quad 1 \quad 0\ ]^T$, $[\ 1 \quad 1 \quad 1\ ]^T$ and $[\ 1 \quad 1 \quad -1\ ]^T$. Note that the polyhedral cone K can alternatively be described by the set $K = \{x : Fx \in \bar{\mathbb{R}}^4_+\}$ where matrix $F = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. According to [12, Lemma 3.2], it can be easily checked that A is K-nonnegative since one can find a nonnegative matrix

$$H = \begin{bmatrix} 0.1400 & 0.0700 & 0 & 0.0100 \\ 0.0100 & 0.0950 & 0.0550 & 0 \\ 0.0250 & 0 & 0.0900 & 0.1850 \\ 0.1300 & 0 & 0.0100 & 0.2200 \end{bmatrix},$$

such that $FA = HF$. Similarly, one can check that A_τ is K-nonnegative. Therefore, the polyhedral cone K is an invariant set of system (1). Consider the constant delay case $\tau_1(k) = 5$ $(k \in \mathbb{N})$. By Corollary 1, the decay rate can be obtained by solving the following linear program together with a one dimensional search on α:

$$F\lambda \in \mathbb{R}^4_+, \quad -F(\alpha A + \alpha^{\tau+1} A_\tau - I)\lambda \in \mathbb{R}^4_+.$$

It can then be readily checked that the maximum α that can be achieved is $\alpha = 1.186$. According to Corollary 1, it can be concluded that the trajectory of system (1) satisfies $\|x(k;0,\phi)\|_\infty \leq M\|\phi\|_\infty \alpha^{-k}$ $(k \in \mathbb{N})$. Under initial condition $\phi(s) = [5-\sin(0.1s), 1+\cos(0.1s), -4]^T$ $(s \in [-5,0])$, the state trajectory of system (1) is plotted in Fig. 1, which confirms this conclusion.

Now consider the case when $\tau_2(k) = \lfloor 0.6k \rfloor$, where $\lfloor \cdot \rfloor$ denotes the standard floor function. By Corollary 2, it can be concluded that the decay rate is $O((k+1)^{-1.195})$. Under the same initial conditions, the state trajectory of system (1) is depicted in Fig. 2. For the case when $\tau_3(k) = \lfloor 3\sqrt{k} \rfloor$, due to Corollary 3, the decay rate of system (1) can be given by $O(2.076^{-\sqrt{k+3}})$. From Fig. 3, it can be observed that the state trajectory of system (1) satisfies that $\log_{2.076} \|x(k)\|_\infty \leq M - \sqrt{k+3}$. Finally, we consider the case when the delay $\tau_4(k) = \ln(k+1)$. By Corollary 4, the decay rate of system (1) is given by $O(3^{-k/\ln(k+2)})$. From Fig. 4, one can see that the state trajectory of system (1) satisfies that $\log_3 \|x(k)\|_\infty \leq M - k/\ln(k+2)$.

In the following, let us consider system (1) with interval delays $\tau : \mathbb{N} \to \mathbb{N}$ belonging to a set $\Omega \triangleq \{\tau : 2 \leq \tau(k) \leq \frac{1}{2}k,\ \forall k \in \mathbb{N}\}$. From Theorem 2, we can conclude that the trajectories of system (1) with delays $\tau \in \Omega$ cannot decay faster than an exponential decay rate $O(1/1.373^k)$. From Theorem 1 and Corollary 2, it can be concluded that system (1) possesses a power law decay rate $O(k^{-1.58})$ for all delays $\tau \in \Omega$. This example shows that the best and worst case decay rate may be of different types.

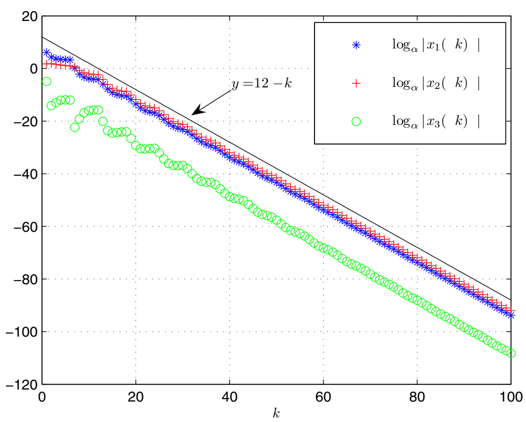

Fig. 1. Evolution of $\log_\alpha |x_i(k)|$ $(\alpha = 1.186)$ versus k for system (1) with delay $\tau_1(k)$.

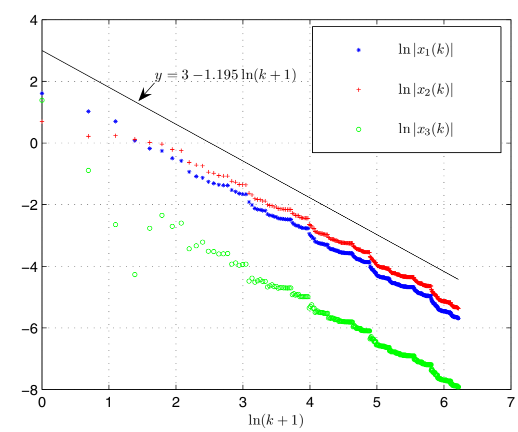

Fig. 2. Evolution of $\ln|x_i(k)|$ versus $\ln(k+1)$ for system (1) with delay $\tau_2(k)$.

VI. Conclusions

In this paper, explicit conditions have been given to ensure that a cone-preserving system with unbounded delays is asymptotically stable with certain decay rate. Relationships have been established between the decay rate and the type of delays. Through introducing a nondecreasing function whose

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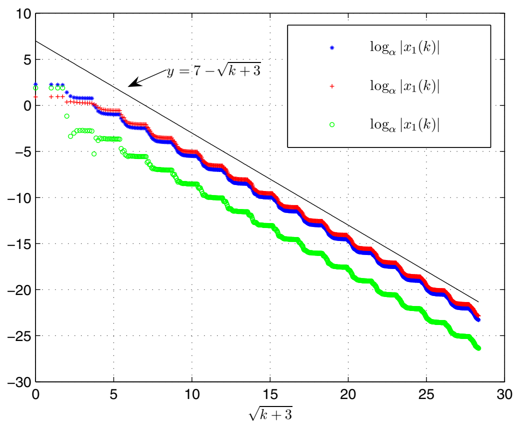

Fig. 3. Evolution of $\log_\alpha |x_i(k)|$ ($\alpha = 2.076$) versus $\sqrt{k+3}$ for system (1) with delay $\tau_3(k)$.

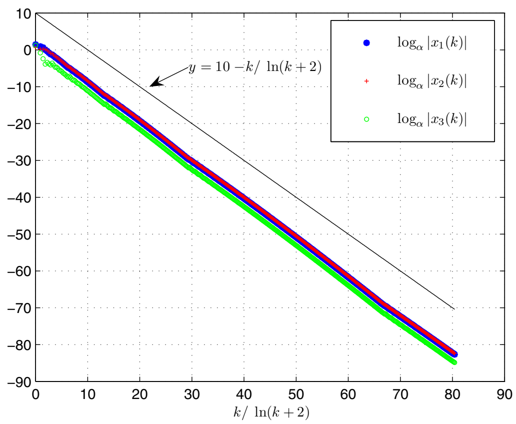

Fig. 4. Evolution of $\log_\alpha |x_i(k)|$ ($\alpha = 3$) versus $k/\ln(k+2)$ for system (1) with delay $\tau_4(k)$.

reciprocal represents the decay rate, an explicit condition has been provided to characterize the decay rate of a linear delay system with cone invariance. Some typical cases involving bounded delays, unbounded delays that are restricted by a linear, sublinear or logarithmic growth rate have been carefully analyzed. Finally, a numerical example is presented on a linear delay system which is invariant on a polyhedral cone.

REFERENCES

[1] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[2] R. Bhattacharya, A. Tiwari, J. Fung, and R. Murray, "Cone invariance and rendezvous of multiple agents," *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, vol. 223, no. 6, pp. 779–789, 2009.

[3] T. Tanaka, C. Langbort, and V. Ugrinovskii, "DC-dominant property of cone-preserving transfer functions," *Systems & Control Letters*, vol. 62, no. 8, pp. 699–707, 2013.

[4] A. Berman and R. J. Plemmons, *Nonnegative Matrices*. SIAM: Philadephia, PA, 1994.

[5] J. A. Jacquez, *Compartmental Analysis in Biology and Medicine*. Ann Arbor, MI: Univ. Michigan Press, 1985.

[6] H. de Jong, "Modeling and simulation of genetic regulatory systems: a literature review," *Journal of Computational Biology*, vol. 9, no. 1, pp. 67–103, 2002.

[7] H. Caswell, *Matrix Population Models: Construction, Analysis and Interpretation*. Sunderland, MA: Sinauer Assoc., 2001.

[8] J. A. Jacquez and C. P. Simon, "Qualitative theory of compartmental systems," *SIAM Review*, vol. 35, no. 1, pp. 43–79, 1993.

[9] P. Li and J. Lam, "Decentralized control of compartmental networks with H_∞ tracking performance," *IEEE Transactions on Industrial Electronics*, vol. 60, no. 2, pp. 546–553, 2013.

[10] T. Kaczorek, *Positive 1D and 2D Systems*. London: Springer Verlag, 2002.

[11] ——, "Positive linear systems consisting of n subsystems with different fractional orders," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 58, no. 6, pp. 1203–1210, 2011.

[12] M. Ait Rami and D. Napp, "Positivity of discrete singular systems and their stability: An LP-based approach," *Automatica*, vol. 50, no. 1, pp. 84–91, 2014.

[13] ——, "Characterization and stability of autonomous positive descriptor systems," *IEEE Transations on Automatic Control*, vol. 57, no. 10, pp. 2668–2673, 2012.

[14] X. Zhao, L. Zhang, P. Shi, and M. Liu, "Stability of switched positive linear systems with average dwell time switching," *Automatica*, vol. 48, no. 6, pp. 1132–1137, 2012.

[15] H. L. Smith, *Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems*. American Mathematical Society, 1995.

[16] W. M. Haddad and V. S. Chellaboina, "Stability theory for nonnegative and compartmental dynamical systems with time delay," *Systems & Control Letters*, vol. 51, no. 5, pp. 355–361, 2004.

[17] S. Zhu, Z. Li, and C. Zhang, "Exponential stability analysis for positive systems with delays," *IET Control Theory & Applications*, vol. 6, no. 6, pp. 761–767, 2012.

[18] X. Liu, W. Yu, and L. Wang, "Stability analysis of positive systems with bounded time-varying delays," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 56, no. 7, pp. 600–604, 2009.

[19] ——, "Stability analysis for continuous-time positive systems with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 4, pp. 1024–1028, 2010.

[20] P. H. A. Ngoc, "Stability of positive differential systems with delay," *IEEE Transactions on Automatic Control*, vol. 58, no. 1, pp. 203–209, 2013.

[21] C. Briat, "Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization," *International Journal of Robust and Nonlinear Control*, vol. 23, no. 17, pp. 1932–1954, 2013.

[22] J. Shen and J. Lam, "L_∞-gain analysis for positive systems with distributed delays," *Automatica*, vol. 50, no. 1, pp. 175–179, 2014.

[23] ——, "ℓ_∞ / l_∞-gain analysis for positive linear systems with unbounded time-varying delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 857–862, 2015.

[24] O. Pastravanu and M. H. Matcovschi, "Max-type copositive lyapunov functions for switching positive linear systems," *Automatica*, vol. 50, no. 12, pp. 3323–3327, 2014.

[25] H. Feyzmahdavian, T. Charalambous, and M. Johansson, "Exponential stability of homogeneous positive systems of degree one with time-varying delays," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1594–1599, 2014.

[26] ——, "Asymptotic stability and decay rates of homogeneous positive systems with bounded and unbounded delays," *SIAM Journal on Control and Optimization*, vol. 52, no. 4, pp. 2623–2650, 2014.

[27] J. Dong, "On the decay rates of homogenous positive systems of any degree with time-varying delays," *IEEE Transactions on Automatic Control*, 2015, doi: 10.1109/TAC.2015.2414793.

[28] J. Shen and W. X. Zheng, "Stability analysis of linear delay systems with cone invariance," *Automatica*, vol. 53, pp. 30–36, 2015.

[29] X. Liu, W. Lu, and T. Chen, "Consensus of multi-agent systems with unbounded time-varying delays," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2396–2401, 2010.

[30] T. Chen and L. Wang, "Global μ-stability of delayed neural networks with unbounded time-varying delays," *IEEE Transactions on Neural Networks*, vol. 18, no. 6, pp. 1836–1840, 2007.

[31] S. Mondié and V. L. Kharitonov, "Exponential estimates for retarded time-delay systems: an LMI approach," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 268–273, 2005.

[32] S. Zhu, M. Meng, and C. Zhang, "Exponential stability for positive systems with bounded time-varying delays and static output feedback stabilization," *Journal of the Franklin Institute*, vol. 350, no. 3, pp. 617–636, 2013.