

## V. CONCLUSION

In this note, a generalized Bezout matrix with respect to a polynomials sequence of interpolatory type is discussed. For this new kind of matrix, its operator representation relative to a pair of dual bases, generalized Barnett-type factorization formula and reduction to a block diagonal form by congruence via a generalized Vandermonde matrix are given. Generalized Fujiwara–Hermite and Routh–Hurwitz criteria in terms of this Bezout matrix are obtained.

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## Parametric Eigenstructure Assignment in Second-Order Descriptor Linear Systems

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**Abstract**—This note considers eigenstructure assignment in second-order descriptor linear systems via proportional plus derivative feedback. It is shown that the problem is closely related with a type of so-called second-order Sylvester matrix equations. Through establishing two general parametric solutions to this type of matrix equations, two complete parametric methods for the proposed eigenstructure assignment problem are presented. Both methods give simple complete parametric expressions for the feedback gains and the closed-loop eigenvector matrices. The first one mainly depends on a series of singular value decompositions, and is thus numerically simple and reliable. The second one utilizes the right factorization of the system, and allows the closed-loop eigenvalues to be set undetermined and sought via certain optimization procedures. An example shows the effect of the proposed approaches.

**Index Terms**—Eigenstructure assignment, parametric solutions, proportional-plus-derivative feedback, right factorization, second-order descriptor linear systems, singular value decomposition.

## I. INTRODUCTION

Second-order linear systems capture the dynamic behavior of many natural phenomena, and have found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control and, hence, have attracted much attention [1]–[13]. In this note, we consider the control of the following second-order descriptor dynamical linear system:

$$M\ddot{x} + D\dot{x} + Kx = Bu \quad (1.1)$$

where  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^r$  are the state vector and the control vector, respectively, and  $M, D, K \in \mathbb{R}^{n \times n}$ , and  $B \in \mathbb{R}^{n \times r}$  are the system coefficient matrices. In certain applications, the matrices  $M$ ,  $D$ , and  $K$  being called the mass matrix, the structural damping matrix and the stiffness matrix, respectively. These coefficient matrices satisfy the following assumptions.

*Assumption A1:*  $\text{rank}(M) = n_0$ ,  $0 < n_0 \leq n$ , and  $\text{rank}(B) = r$ .

Concerning the control of the second-order linear system (1.1), most of the results are focused on stabilization (see, for example, [2] and [3]), pole assignment [4]–[6], and partial pole assignment [7], [8]. Furthermore, many theoretical results for second-order systems have been developed via the corresponding extended first-order descriptor state-space model

$$E_e \dot{z} = A_e z + B_e u \quad (1.2)$$

where

$$E_e = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix} \quad A_e = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix} \quad B_e = \begin{bmatrix} 0 \\ B \end{bmatrix}. \quad (1.3)$$

Therefore, these results eventually involve manipulations on  $2n$  dimensional matrices  $E_e$ ,  $A_e$ , and  $B_e$ .

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Eigenstructure assignment is a very important problem in linear control systems design, and has attracted much attention in the last two decades (see [14] and the references therein). However, for eigenstructure assignment in second-order linear systems, there have not been many results [9]–[13]. Reference [9] considers eigenstructure assignment in a special class of second-order linear systems using inverse eigenvalue methods. Reference [10] proposes an algorithm for eigenstructure assignment in the second-order linear system (1.1), with  $M$  and  $K$  being symmetric positive definite, and  $D$  being symmetric positive semidefinite. This algorithm is attractive because it utilizes only the original system data  $M$ ,  $D$ , and  $K$ . In [11], an effective method for partial eigenstructure assignment is proposed for the second-order linear system (1.1) with all the coefficient matrices  $M$ ,  $D$ , and  $K$  are symmetric. In [12], the problem of robust eigenstructure assignment is treated for the second-order linear system (1.1) with the matrix  $M$  nonsingular. The design degree of freedom provided by eigenstructure assignment is utilized to minimize the condition number of the closed-loop system. Eigenstructure assignment using a proportional-plus-derivative feedback in the second-order linear system (1.1), also with  $M$  nonsingular, is considered in [13]. Simple, general, and complete parametric expressions in direct closed forms for both the closed-loop eigenvector matrix and the feedback gains are established. As in [10], the approach utilizes directly the original system data  $M$ ,  $D$  and  $K$ , and involves manipulations on  $n$ -dimensional matrices only. However, the approach has the disadvantage that it requires the controllability of the matrix pair  $(D, B)$ , which is not satisfied in some applications.

This note considers eigenstructure assignment in the second-order linear system (1.1) via proportional-plus-derivative coordinate control. Different from existing results, the only requirement on the system coefficient matrices is that the matrix  $B$  has full-column rank. Based on a series of singular value decompositions and the right factorization of the system, two complete parametric approaches are proposed. Very simple, complete parametric expressions for both the closed-loop eigenvector matrices and the feedback gains are established in terms of a group of parameter vectors and a matrix parameter, which represent the design degrees of freedom. These parameters can be properly further chosen to produce a closed-loop system with some desired system specifications. Particularly, the second approach, which uses the right factorization of the system, is a natural generalization of the parametric approach in [14]–[16] proposed for first-order state-space descriptor linear systems. With this approach, besides the group of parameter vectors, the closed-loop eigenvalues may also be treated as part of the design freedom since they appear directly in the expressions of the eigenvector matrix and the feedback gains, and hence are not necessarily chosen *a priori*.

## II. PROBLEM FORMULATION

For the second-order descriptor dynamical system (1.1), by choosing the following control law:

$$u = F_0 x + F_1 \dot{x} \quad F_0, F_1 \in \mathbb{R}^{r \times n} \quad (2.1)$$

we obtain the closed-loop system as follows:

$$M\ddot{x} + (D - BF_1)\dot{x} + (K - BF_0)x = 0. \quad (2.2)$$

System (2.2) can be written in the first-order state-space form

$$E_{ec}\dot{z} = A_{ec}z \quad (2.3)$$

with

$$E_{ec} = \begin{bmatrix} I_n & 0 \\ 0 & M \end{bmatrix} \quad A_{ec} = \begin{bmatrix} 0 & I \\ -(K - BF_0) & -(D - BF_1) \end{bmatrix}. \quad (2.4)$$

**Definition 1:** The second-order dynamical system (1.1) is called R-controllable (I-controllable) if and only if the corresponding extended first-order state-space representation (1.2)–(1.3) is R-controllable (I-controllable).

Recall the fact that a nondefective matrix possesses eigenvalues which are less sensitive to the parameter perturbations in the matrix, we here require the closed-loop matrix pair  $(E_{ec}, A_{ec})$  to be nondefective, that is, the Jordan form of the matrix pair  $(E_{ec}, A_{ec})$  possesses a diagonal form. Further, following the pole assignment theory for first-order descriptor linear systems, under the R- and I-controllability of system (1.1),  $n + n_0$  finite eigenvalues can be assigned to the closed-loop system (2.3)–(2.4). Therefore, the desired Jordan form of the matrix pair  $(E_{ec}, A_{ec})$  takes the form

$$\Lambda = \text{diag}(s_1, s_2, \dots, s_{n+n_0}) \quad (2.5)$$

where  $s_i, i = 1, 2, \dots, n+n_0$ , are clearly the eigenvalues of the matrix pair  $(E_{ec}, A_{ec})$ .

**Lemma 1:** Let  $E_{ec}, A_{ec}$  be given by (2.4), and  $\Lambda$  by (2.5). Then, the following hold.

- 1) There exist matrices  $V, V' \in \mathbb{C}^{n \times (n+n_0)}$  satisfying

$$A_{ec} \begin{bmatrix} V \\ V' \end{bmatrix} = E_{ec} \begin{bmatrix} V \\ V' \end{bmatrix} \Lambda \quad (2.6)$$

if and only if

$$MV\Lambda^2 + (D - BF_1)V\Lambda + (K - BF_0)V = 0 \quad (2.7)$$

and

$$V' = V\Lambda. \quad (2.8)$$

- 2) There exist matrices  $V_\infty, V'_\infty \in \mathbb{R}^{n \times (n-n_0)}$  satisfying

$$E_{ec} \begin{bmatrix} V'_\infty \\ V_\infty \end{bmatrix} = 0 \quad \text{rank} \left( \begin{bmatrix} V'_\infty \\ V_\infty \end{bmatrix} \right) = n - n_0 \quad (2.9)$$

if and only if  $V'_\infty = 0$  and

$$MV_\infty = 0 \quad \text{rank}(V_\infty) = n - n_0. \quad (2.10)$$

**Proof:** The second conclusion is obvious in view of the structure of the matrix  $E_{ec}$ . Here, we only show the first one.

Since

$$\begin{aligned} A_{ec} \begin{bmatrix} V \\ V' \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -(K - BF_0) & -(D - BF_1) \end{bmatrix} \begin{bmatrix} V \\ V' \end{bmatrix} \\ &= \begin{bmatrix} V' \\ -(K - BF_0)V - (D - BF_1)V' \end{bmatrix} \end{aligned}$$

and

$$E_{ec} \begin{bmatrix} V \\ V' \end{bmatrix} \Lambda = E_{ec} \begin{bmatrix} V\Lambda \\ V'\Lambda \end{bmatrix} = \begin{bmatrix} V\Lambda \\ MV'\Lambda \end{bmatrix}$$

(2.6) is clearly equivalent to (2.8) and

$$-(K - BF_0)V - (D - BF_1)V' = MV'\Lambda. \quad (2.11)$$

Further, substituting (2.8) into (2.11) yields (2.7).  $\square$

The first conclusion of the aforementioned lemma states that the Jordan matrix associated with the matrix pair  $(E_{ec}, A_{ec})$  is  $\Lambda$  if and only if there exists  $V \in \mathbb{C}^{n \times (n+n_0)}$  satisfying (2.7), and in this case the corresponding finite eigenvector matrix of the matrix pair  $(E_{ec}, A_{ec})$  is given by

$$V_{ec}^f = \begin{bmatrix} V \\ V\Lambda \end{bmatrix}. \quad (2.12)$$

The second conclusion of the previous lemma states that the infinite eigenvector matrix of the matrix pair  $(E_{ec}, A_{ec})$ , associated with the infinite eigenvalues, is given by

$$V_{ec}^{\infty} = \begin{bmatrix} 0 \\ V_{\infty} \end{bmatrix} \quad (2.13)$$

where  $V_{\infty}$  satisfies (2.10). Therefore, the entire eigenvector matrix of the matrix pair  $(E_{ec}, A_{ec})$  is

$$V_{ec} = \begin{bmatrix} V & 0 \\ V\Lambda & V_{\infty} \end{bmatrix}. \quad (2.14)$$

With the above understanding, the problem of eigenstructure assignment in the second-order descriptor dynamical system (1.1) via the proportional plus derivative feedback law (2.1) can be stated as follows.

**Problem eigenstructure assignment (ESA):** Given (1.1) satisfying Assumption A1, a matrix  $V_{\infty}$  satisfying (2.10), and the matrix  $\Lambda = \text{diag}(s_1, s_2, \dots, s_{n+n_0})$ , with  $s_i, i = 1, 2, \dots, n+n_0$ , being a group of self-conjugate complex numbers (not necessarily distinct), find a general parametric form for the matrices  $F_0, F_1 \in \mathbb{R}^{r \times n}$  and  $V \in \mathbb{C}^{n \times (n+n_0)}$  such that the matrix equation (2.7) and the condition

$$\det \begin{bmatrix} V & 0 \\ V\Lambda & V_{\infty} \end{bmatrix} \neq 0 \quad (2.15)$$

are satisfied.

Letting

$$W = F_1 V \Lambda + F_0 V = [F_0 \quad F_1] \begin{bmatrix} V \\ V\Lambda \end{bmatrix} \quad (2.16)$$

then (2.7) becomes

$$M V \Lambda^2 + D V \Lambda + K V = B W. \quad (2.17)$$

Clearly, (2.17) becomes the type of generalized Sylvester matrix equation investigated in [15]–[17] when  $M = 0$ . Due to this fact, we call the (2.17) the second-order generalized Sylvester matrix equation.

It follows from the aforementioned deduction that the key step to solve problem ESA is to find a solution to the following problem.

**Problem second-order Sylvester equation (SSE):** Given the matrices  $M, D, K \in \mathbb{R}^{n \times n}$ , and  $B \in \mathbb{R}^{n \times r}$  satisfying Assumption A1, and a diagonal matrix

$$\Lambda = \text{diag}(s_1, s_2, \dots, s_q) \in \mathbb{C}^{q \times q} \quad (2.18)$$

find a parameterization for all the matrices  $V \in \mathbb{C}^{n \times q}$  and  $W \in \mathbb{C}^{r \times q}$  satisfying the second-order Sylvester matrix equation (2.17).

It should be noted that, the number of columns of the matrices  $V$ ,  $W$  and  $\Lambda$  in problem SSE are changed into  $q$  because this makes the problem SSE more general.

### III. SOLUTION TO PROBLEM SSE

Denote

$$V = [v_1 \quad v_2 \quad \cdots \quad v_q] \quad (3.1)$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_q] \quad (3.2)$$

then, in view of (2.18), we can convert the second-order Sylvester matrix equation (2.17) into the following column form:

$$(s_i^2 M + s_i D + K) v_i = B w_i, \quad i = 1, 2, \dots, q. \quad (3.3)$$

#### A. Case of Prescribed $s_i, i = 1, 2, \dots, q$

The equations in (3.3) can be further written in the following form:

$$\Pi_i \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0, \quad i = 1, 2, \dots, q \quad (3.4)$$

where

$$\Pi_i = [s_i^2 M + s_i D + K \quad -B], \quad i = 1, 2, \dots, q. \quad (3.5)$$

This states that

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} \in \ker \Pi_i, \quad i = 1, 2, \dots, q. \quad (3.6)$$

The following algorithm produces two sets of constant matrices  $N_i$  and  $D_i, i = 1, 2, \dots, q$ , to be used in the representation of the solution to the matrix equation (2.17).

#### Algorithm P1 (Solving $N_i$ and $D_i, i = 1, 2, \dots, q$ )

Step 1) Through applying SVD to the matrix  $\Pi_i, i = 1, 2, \dots, q$ , obtain two sets of matrices  $P_i \in \mathbb{C}^{n \times n}$  and  $Q_i \in \mathbb{C}^{(n+r) \times (n+r)}, i = 1, 2, \dots, q$ , satisfying

$$P_i \Pi_i Q_i = \begin{bmatrix} \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{n_i}) & 0 \\ 0 & 0 \end{bmatrix}, \quad i = 1, 2, \dots, q \quad (3.7)$$

where  $\sigma_i > 0, i = 1, 2, \dots, n_i$ , are the singular values of  $\Pi_i$ , and

$$n_i = \text{rank} [s_i^2 M + s_i D + K \quad B], \quad i = 1, 2, \dots, q. \quad (3.8)$$

Step 2) Obtain the matrices  $N_i \in \mathbb{R}^{n \times (n+r-n_i)}$  and  $D_i \in \mathbb{R}^{r \times (n+r-n_i)}, i = 1, 2, \dots, q$ , by partitioning the matrix  $Q_i$  as follows:

$$Q_i = \begin{bmatrix} * & N_i \\ * & D_i \end{bmatrix}, \quad i = 1, 2, \dots, q. \quad (3.9)$$

As a result of (3.7) and (3.9), the matrices  $N_i \in \mathbb{R}^{n \times (n+r-n_i)}$  and  $D_i \in \mathbb{R}^{r \times (n+r-n_i)}, i = 1, 2, \dots, q$ , obtained through Algorithm P1 satisfy

$$\Pi_i \begin{bmatrix} N_i \\ D_i \end{bmatrix} = 0 \quad \text{rank} \begin{bmatrix} N_i \\ D_i \end{bmatrix} = n+r-n_i, \quad i = 1, 2, \dots, q. \quad (3.10)$$

Therefore, the columns of  $\begin{bmatrix} N_i \\ D_i \end{bmatrix}$  form a set of basis for  $\ker \Pi_i$ .

The previous deduction clearly yields the following result.

**Theorem 1:** Let  $n_i, i = 1, 2, \dots, q$ , be defined by (3.8), and  $N_i \in \mathbb{R}^{n \times (n+r-n_i)}$  and  $D_i \in \mathbb{R}^{r \times (n+r-n_i)}, i = 1, 2, \dots, q$ , be obtained via Algorithm P1. Then, all the matrices  $V$  and  $W$  satisfying the second-order Sylvester matrix equation (2.17) can be parameterized by columns as follows:

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N_i \\ D_i \end{bmatrix} f_i, \quad i = 1, 2, \dots, q \quad (3.11)$$

where  $f_i \in \mathbb{C}^{n+r-n_i}, i = 1, 2, \dots, q$ , are a set of arbitrary parameter vectors.

Regarding the R-controllability of (1.1), we have the following basic result which is a general extension of the well-known PHB criterion [18].

**Lemma 2:** The second-order dynamical system (1.1) is R-controllable if and only if

$$\text{rank} [s^2 M + sD + K \quad B] = n \quad \forall s \in \mathbb{C}. \quad (3.12)$$

Based on the aforementioned lemma, the following corollary of Theorem 1 can be immediately derived.

**Corollary 1:** Let (1.1) be R-controllable, and  $\Lambda$  be given by (2.18), then the degrees of freedom existing in the general solution to the second-order Sylvester matrix equation (2.17) is  $qr$ .

*Proof:* Due to the controllability of (1.1), we have from Lemma 2 that  $n_i = n$ ,  $i = 1, 2, \dots, q$ . Thus, the conclusion immediately follows from Theorem 1.  $\square$

### B. Case of Undetermined $s_i$ , $i = 1, 2, \dots, q$

The solution for this case depends on a pair of polynomial matrices  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  satisfying

$$(s^2 M + sD + K)N(s) = BD(s). \quad (3.13)$$

In the case where (1.1) is regular, that is,  $\det(s^2 M + sD + K)$  is not identically zero, the aforementioned equation can be written as

$$(s^2 M + sD + K)^{-1}B = N(s)D^{-1}(s) \quad (3.14)$$

which can be viewed as the right factorization of the following transfer function:

$$G(s) = (s^2 M + sD + K)^{-1}B.$$

For simplicity, we also call (3.13) the right factorization of (1.1).

**Theorem 2:** Let (1.1) be R-controllable, and  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  satisfy the right factorization (3.13). Then, the following hold.

1) The matrices  $V$  and  $W$  given by (3.1)–(3.2), and

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} f_i, \quad i = 1, 2, \dots, q \quad (3.15)$$

satisfy the second-order Sylvester matrix equation (2.17) for all  $f_i \in \mathbb{C}^r$ ,  $i = 1, 2, \dots, q$ .

2) When

$$\text{rank} \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} = r, \quad i = 1, 2, \dots, q \quad (3.16)$$

hold, (3.15) gives all the solutions to Problem SSE.

*Proof:* It follows from (3.13) that

$$(s_i^2 M + s_i D + K) N(s_i) - BD(s_i) = 0, \quad i = 1, 2, \dots, q. \quad (3.17)$$

Using (3.15) and (3.17), yields

$$\begin{aligned} & (s_i^2 M + s_i D + K) v_i - B w_i \\ &= [(s_i^2 M + s_i D + K) N(s_i) - BD(s_i)] f_i \\ &= 0, \quad i = 1, 2, \dots, q. \end{aligned}$$

This states that the equations in (3.3) hold. Therefore, the first conclusion of the theorem is true.

It follows from Corollary 1 that, under the controllability of (1.1), the degrees of freedom existing in the general solution to the matrix equation (2.17), with  $\Lambda$  given by (2.18), is  $qr$ , while in the solution (3.15), the number of free parameters just equal to  $qr$ . Further, it is clear that all these parameters in the solution (3.15) have contributions when condition (3.16) holds. With this, we complete the proof.  $\square$

The right factorization (3.13) performs a fundamental role in the solution (3.15). When (1.1) is regular and  $s_i$ ,  $i = 1, 2, \dots, q$ , are chosen to be different from the zeros of  $\det(s^2 M + sD + K)$ , we can take

$$\begin{cases} N(s) = \text{Adj}(s^2 M + sD + K)B \\ D(s) = \det(s^2 M + sD + K)I_r \end{cases}.$$

For general numerical algorithms solving such right factorizations, one can refer to [19], [20]. The following simple procedure can also be used.

### Algorithm P2 (Right coprime factorization)

Step 1) Under the R-controllability of system (1.1), find a pair of unimodular matrices  $P(s)$  and  $Q(s)$ , of appropriate dimensions, satisfying

$$P(s)[s^2 M + sD + K - B]Q(s) = [I_n \quad 0].$$

Step 2) Obtain the pair of polynomial matrices  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  by partitioning the unimodular matrix  $Q(s)$  as follows:

$$Q(s) = \begin{bmatrix} * & N(s) \\ * & D(s) \end{bmatrix}.$$

It is worth pointing out that the pair of polynomial matrices  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  satisfying the right factorization (3.13) obtained from the Algorithm P2 are right coprime since

$$\text{rank} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = r \quad \forall s \in \mathbb{C}. \quad (3.18)$$

This condition certainly implies (3.16) which ensures the completeness of the solution (3.15).

To finish this section, let us finally give a remark on the extension of the result.

**Remark 1:** The main result in this section can be easily extended into the case that the matrix  $\Lambda$  is a general Jordan form. In fact, when  $\Lambda$  is replaced with the following Jordan matrix:

$$J = \text{Blockdiag}(J_1, J_2, \dots, J_p) \in \mathbb{C}^{q \times q}$$

with

$$J_i = \begin{bmatrix} s_i & 1 & & \\ & s_i & \ddots & \\ & & \ddots & 1 \\ & & & s_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, \quad i = 1, 2, \dots, p.$$

Following the development in [15]–[17], we can show that all the matrices  $V$  and  $W$  satisfying the second-order Sylvester matrix equation (2.17) are given by

$$\begin{cases} V = [V_1 \quad V_2 \quad \cdots \quad V_p] \\ V_i = [v_{i1} \quad v_{i2} \quad \cdots \quad v_{ip_i}] \end{cases}$$

and

$$\begin{cases} W = [W_1 \quad W_2 \quad \cdots \quad W_p] \\ W_i = [w_{i1} \quad w_{i2} \quad \cdots \quad w_{ip_i}] \end{cases}$$

with

$$\begin{aligned} \begin{bmatrix} v_{ik} \\ w_{ik} \end{bmatrix} &= \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} f_k + \begin{bmatrix} N^{(1)}(s_i) \\ D^{(1)}(s_i) \end{bmatrix} f_{k-1} + \cdots + \frac{1}{(k-1)!} \\ &\quad \times \begin{bmatrix} N^{(k-1)}(s_i) \\ D^{(k-1)}(s_i) \end{bmatrix} f_1, \quad k = 1, 2, \dots, p_i, \quad i = 1, 2, \dots, p. \end{aligned} \quad (3.19)$$

where  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  are a pair of polynomial matrices satisfying the right factorization (3.13) and condition (3.16).

#### IV. SOLUTION TO PROBLEM ESA

Following from the results in Section III, we can obtain the following two theorems regarding the solution to Problem ESA.

**Theorem 3:** Let  $n_i, i = 1, 2, \dots, n + n_0$ , be given by (3.8), and  $N_i \in \mathbb{R}^{n \times (n+r-n_i)}$  and  $D_i \in \mathbb{R}^{r \times (n+r-n_i)}, i = 1, 2, \dots, n + n_0$ , be given by Algorithm P1. Then, the following hold.

- 1) Problem ESA has solutions if and only if there exist a group of parameters  $f_i \in \mathbb{C}^{n+r-n_i}, i = 1, 2, \dots, n + n_0$ , satisfying the following constraints.

Constraint C1:  $f_i = \bar{f}_j$  if  $s_i = \bar{s}_j$ .

Constraint C2<sub>a</sub>:  $\det V_{ca} \neq 0$ , with

$$V_{ca} = \begin{bmatrix} N_1 f_1 & N_2 f_2 & \cdots & N_{n+n_0} f_{n+n_0} & 0 \\ s_1 N_1 f_1 & s_2 N_2 f_2 & \cdots & s_{n+n_0} N_{n+n_0} f_{n+n_0} & V_\infty \end{bmatrix}. \quad (4.1)$$

- 2) When this condition is met, all the solutions to problem ESA are given by

$$V = [N_1 f_1 \ N_2 f_2 \ \cdots \ N_{n+n_0} f_{n+n_0}] \quad (4.2)$$

and

$$[F_0 \ F_1] = [D_1 f_1 \ D_2 f_2 \ \cdots \ D_{n+n_0} f_{n+n_0} \ W_\infty] V_{ca}^{-1} \quad (4.3)$$

where  $f_i \in \mathbb{C}^{n+r-n_i}, i = 1, 2, \dots, n + n_0$ , are arbitrary parameter vectors satisfying Constraints C1 and C2<sub>a</sub> and  $W_\infty \in \mathbb{C}^{r \times (n-n_0)}$  is an arbitrary parameter matrix.

**Theorem 4:** Let (1.1) be R-controllable, and  $N(s) \in \mathbb{R}^{n \times r}[s]$  and  $D(s) \in \mathbb{R}^{r \times r}[s]$  be a pair of polynomial matrices satisfying the right factorization (3.13) and condition (3.16). Then

- 1) Problem ESA has solutions if and only if there exist a group of parameters  $f_i \in \mathbb{C}^r, i = 1, 2, \dots, n + n_0$ , satisfying C1 and Constraint C2<sub>b</sub>:  $\det V_{cb} \neq 0$ , with

$$V_{cb} = \begin{bmatrix} N(s_1) f_1 & N(s_2) f_2 & \cdots & N(s_{n+n_0}) f_{n+n_0} & 0 \\ s_1 N(s_1) f_1 & s_2 N(s_2) f_2 & \cdots & s_{n+n_0} N(s_{n+n_0}) f_{n+n_0} & V_\infty \end{bmatrix}. \quad (4.4)$$

- 2) When this condition is met, all the solutions to problem ESA are given by

$$V = [N(s_1) f_1 \ N(s_2) f_2 \ \cdots \ N(s_{n+n_0}) f_{n+n_0}] \quad (4.5)$$

and

$$[F_0 \ F_1] = [D(s_1) f_1 \ D(s_2) f_2 \ \cdots \ D(s_{n+n_0}) f_{n+n_0} \ W_\infty] V_{cb}^{-1} \quad (4.6)$$

where  $f_i \in \mathbb{C}^r, i = 1, 2, \dots, n + n_0$ , are arbitrary parameter vectors satisfying Constraints C1 and C2<sub>b</sub>, and  $W_\infty \in \mathbb{C}^{r \times (n-n_0)}$  is an arbitrary parameter matrix.

The previous two theorems can be proven similarly. Therefore, only a proof of Theorem 4 is given here.

*Proof of Theorem 4:* Introducing the following auxiliary equation:

$$W_\infty = F_1 V_\infty = [F_0 \ F_1] \begin{bmatrix} 0 \\ V_\infty \end{bmatrix}. \quad (4.7)$$

Combining (4.7) with (2.16) gives

$$[W \ W_\infty] = [F_0 \ F_1] V_{cb} \quad (4.8)$$

with

$$V_{cb} = \begin{bmatrix} V & 0 \\ V\Lambda & V_\infty \end{bmatrix} \quad (4.9)$$

where the matrices  $V$  and  $W$  satisfy the second-order Sylvester matrix equation (2.17). Therefore, when (2.15) holds, the feedback gain matrices can be obtained as

$$[F_0 \ F_1] = [W \ W_\infty] V_{cb}^{-1}. \quad (4.10)$$

Substituting the general expressions for the matrices  $V$  and  $W$  given in Theorem 2 into (4.9) and (4.10), respectively, yields (4.4) and (4.6).

As a requirement of the eigenstructure assignment problem, (2.15) needs to be satisfied. In view of Theorem 2, this condition is clearly equivalent to Constraint C2<sub>b</sub>.

Finally, it suffices only to show that Constraint C1 is necessary and sufficient for the matrices  $F_0$  and  $F_1$  given by (4.6) to be real. This is actually a natural extension of the simple fact that the scalar  $k$  satisfying  $\beta = k\alpha$  is real if and only if  $\bar{\beta} = \bar{\alpha}$ .  $\square$

In the rest of this section, let us make some remarks on the previous results.

**Remark 2:** The previous two theorems give complete parametric solutions to the Problem ESA. The free parameter matrix  $W_\infty$  and the free parameter vectors  $f_i, i = 1, 2, \dots, n + n_0$  represent the degrees of freedom in the eigenstructure assignment design, and can be sought to meet certain desired system performances. It should be noted that Constraint C1 is not a restriction at all, it only gives a way of selecting these design parameter vectors.

**Remark 3:** It follows from well-known pole assignment result that Problem ESA has a solution when (1.1) is R- and I-controllable and the closed-loop eigenvalues  $s_i, i = 1, 2, \dots, n + n_0$ , are restricted to be distinct. In this case, there exist parameter vectors  $f_i, i = 1, 2, \dots, n + n_0$ , satisfying Constraint C2<sub>a</sub> or C2<sub>b</sub>. As a matter of fact, it can be reasoned that, in this case, “almost all” parameter vectors  $f_i, i = 1, 2, \dots, n + n_0$ , satisfy Constraint C2<sub>a</sub> or C2<sub>b</sub>. Therefore, in such applications verification of Constraint C2<sub>a</sub> or C2<sub>b</sub> can often be neglected.

**Remark 4:** The solution given in Theorem 3 utilizes only a series of singular value decompositions and, hence, is numerically very simple and reliable. As for the solution given in Theorem 4, it has the advantage that the closed-loop eigenvalues  $s_i, i = 1, 2, \dots, n + n_0$ , can be set undetermined and used as a part of extra design degrees of freedom to be sought with  $f_i, i = 1, 2, \dots, n + n_0$ , by certain optimization procedures. Furthermore, the approaches proposed in this note naturally reduces to those for first-order descriptor linear systems when  $M = 0$ . Especially, it happens that the solution given in Theorem 4 is a natural generalization of the parametric solution in [14]–[16] proposed for eigenstructure assignment in first-order descriptor state-space systems.

**Remark 5:** The eigenstructure assignment results can be easily extended into the defective case, that is, the case that the closed-loop system possesses a general Jordan form (refer to Remark 1). However, from the control systems design point of view, this is not desired since the eigenvalues of defective matrices are more sensitive to parameter perturbations than those of nondefective ones.

**Remark 6:** When  $\text{rank}(M) = n_0 < n$ , both the open-loop system (1.1) and the closed-loop system (2.2) are singular ones. For this case, closed-loop regularity has to be addressed. Note that the control of system (1.1) via the feedback control law (2.1) is equivalent to the state feedback control in the first-order descriptor linear system (1.2)–(1.3).

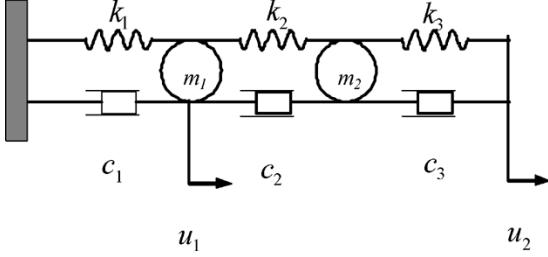


Fig. 1. Three-spring and two-mass system.

Therefore, under the R-controllability condition of system (1.1), the first-order descriptor linear system (1.2)–(1.3) is regularizable via state feedback [21], and hence for “almost all” controllers in the form of (2.1), the corresponding closed-loop system (2.3)–(2.4) is regular [21].

## V. EXAMPLE

Consider a simple linear dynamical system consisting of two lumped masses and three spring dashpots, connected in series and fixed at one end as shown in Fig. 1. The equation of motion can be written in the form of (1.1) with \$M = \text{diag}(m\_1, m\_2, 0)\$, and

$$\begin{aligned} D &= \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \\ K &= \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

When \$m\_1 = m\_2 = 1\$, \$k\_1 = k\_2 = 5\$, \$k\_3 = 20\$, \$c\_1 = c\_3 = 2\$, and \$c\_2 = 0.5\$, we have \$M = \text{diag}(1, 1, 0)\$, and

$$\begin{aligned} D &= \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{bmatrix} \\ K &= \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

It is clear to see that

$$v_\infty = [0 \ 0 \ 1]^T.$$

For simplicity, the closed-loop eigenvalues \$s\_i, i = 1 \sim 5\$, are restricted to be a group of distinct real scalars.

By applying Algorithm P2, we obtain the pair of polynomial matrices satisfying the right coprime factorization (3.13) as follows:

$$\begin{aligned} N(s) &= \begin{bmatrix} 2s+20 & 0 \\ 0 & 2s+20 \\ -0.5s-5 & s^2+2.5s+25 \end{bmatrix} \\ D(s) &= \begin{bmatrix} 2s^3+25s^2+70s+200 & -s^2-20s-100 \\ -s^2-20s-100 & 2s^3+21s^2+20s+100 \end{bmatrix}. \end{aligned}$$

By denoting

$$f_i = [\alpha_i \ \beta_i]^T \in \mathbb{R}^2, \quad i = 1 \sim 5$$

we have

$$v_i = \begin{bmatrix} (2s_i + 20)\alpha_i \\ (2s_i + 20)\beta_i \\ (-0.5s_i - 5)\alpha_i + (s_i^2 + 2.5s_i + 25)\beta_i \end{bmatrix}, \quad i = 1 \sim 5$$

and

$$w_i = \begin{bmatrix} (2s_i^3 + 25s_i^2 + 70s_i + 200)\alpha_i + (-s_i^2 - 20s_i - 100)\beta_i \\ (-s^2 - 20s - 100)\alpha_i + (2s^3 + 21s^2 + 20s + 100)\beta_i \end{bmatrix}, \quad i = 1 \sim 5$$

and the gain matrices are given by

$$[F_0 \ F_1] = [w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_\infty] V_{ec}^{-1}$$

with

$$V_{ec} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & 0 \\ s_1 v_1 & s_2 v_2 & s_3 v_3 & s_4 v_4 & s_5 v_5 & v_\infty \end{bmatrix}.$$

All the design parameters \$\alpha\_i, \beta\_i, s\_i, i = 1 \sim 5\$, and \$w\_\infty\$, can be taken as arbitrary real scalars which make the aforementioned matrix \$V\_{ec}\$ nonsingular (note that \$s\_i, i = 1 \sim 5\$, should be restricted negative due to the closed-loop stability).

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## Optimal Finite-Horizon Production Control in a Defect-Prone Environment

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**Abstract**—In this note, we consider a single-machine, single-part-type production system, operating in a defect-prone environment. It is assumed that there is a random yield proportion of nondefective parts, with known probability distribution. Over each production cycle, it is assumed that there is a single realization of the yield random variable. Furthermore, it is assumed that the system is operated under a periodic-review policy. Thus, the particular realization of the yield proportion cannot be determined prior to the end of the production horizon. The optimal production control, that minimizes a linear combination of expected surplus and shortage costs over the planning horizon is shown to be piecewise constant, and the appropriate production levels and control break-points are determined as functions of the yield rate distribution.

**Index Terms**—Cost minimization, defect-prone, finite-horizon, production control, random yield.

### I. INTRODUCTION

In this note, we provide analytical results for a basic manufacturing system model where the effect of the yield uncertainty is introduced. It is assumed that the probability distribution of the random yield rate is known, but the inventory level is observable only intermittently. The optimal production control, that minimizes a linear combination of expected surplus and shortage costs over the planning horizon, is shown to be piecewise constant, and the appropriate production levels and control break-points are determined as functions of the yield rate distribution. It is interesting to note that, even for the one-machine, one-part-type system, the consideration of random yield leads to a nonintuitive, and nontrivial, optimal production control.

The incorporation of random yield into manufacturing system models has been of interest since as early as 1958 (see [9]). Since then, many authors have considered random yield problems in various

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forms. In 1995, Yano and Lee [13] provided a comprehensive review of the existing literature. Based on the system modeling characteristics, they arranged random yield lot-sizing problems into the following categories: discrete-time models, which include single-stage models (both single and multiple period), multiple stages in tandem, assembly systems, and continuous-time models with constant demand rates or random demand rates.

Subsequently, more generic models have been studied including extensions such as uncertain supply, backlogged demand, imperfect production, and late-stage inspection. Yield variability due to random production capacity is considered in [2] and in [8]. More recently, Grosfeld-Nir and Gerchak [4] examined a model with multiple successive production runs to meet orders. They focused on relatively small volumes of custom-made items and analyzed the yield structure. Liu and Yang [11] considered multiple defect types (reworkable and nonreworkable defects) and determined the optimal lot size. Bollapragada and Morton [1] used myopic heuristics for the random yield problem and obtained promising results. Yao and Zheng [14] studied a two-stage problem in which, in order to coordinate the inspection procedures at the two stages, the optimal policy is characterized by a sequence of thresholds at stage 1 and by a priority structure at stage 2. For an assembly system under random demand and production yield loss, Gurnani *et al.* [5] circumvented the difficulty of solving the original problem by modifying the exact cost function with an approximate one and determined a bound on the difference. Grosfeld-Nir *et al.* [3] included inspection costs as a key part of the problem in a general multiple production run model.

A related problem is the extension of the classical single-period newsboy problem to incorporate yield variability. (See [10] for an excellent survey of the newsboy problem and its extensions.) For example, Henig and Gerchak [7] examine the single-period newsboy problem with random yield. Sipper and Bulfin, Jr. [12] discuss the implications of random yield on material resource planning.

A primary difference between the model considered in this note and the newsboy problem is that the newsboy problem assigns costs, and allows replenishment, at discrete points of time. However, our model considers continuous production and assignment of costs, although the exact inventory position is known only periodically. In effect, the traditional random yield lot-sizing problem is transformed into a continuous-time optimal control problem.

A constructive approach will be used to analyze the problem with the aid of the maximum principle. The maximum principle is a set of optimality conditions, the application of which results in a new objective function, called the Hamiltonian, and a co-state differential equation (see, for example, [6]). A solution (control function) will be constructed that is both feasible and maximizes the Hamiltonian. This ensures optimality.

The remainder of the note is organized as follows. In Section II, the single part-type problem is described, and the original stochastic problem is transformed into an equivalent deterministic problem. Section III details the dual formulation, deriving the co-state equation and describing the form of the optimal control. Then, in Section IV, the optimal control is determined analytically for the piecewise linear instantaneous cost function.

### II. SINGLE-PART-TYPE PROBLEM DESCRIPTION

Consider a single-machine, single-part-type production system, operating in a defect-prone environment. Suppose the production rate of the system,  $u(t)$ , is bounded and controllable, i.e.,

$$0 \leq u(t) \leq U. \quad (1)$$