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Mean-square stability of analytic solution and Euler–Maruyama method for impulsive stochastic differential equations

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ABSTRACT

From the view of algebra, the mean-square stability of analytic solutions and numerical solutions for impulsive stochastic differential equations are considered. By the logarithmic norm, the conditions under which the analytic and numerical solutions for a linear impulsive stochastic differential equation are mean-square stable (MS-stable) respectively are obtained. The conditions are simple and easy to use. Some numerical experiments are given to illustrate the results.

1. Introduction

Impulsive effects exist widely in many evolution processes in which states are changed abruptly at certain moments of time, involving such fields as medicine, biology, economics, mechanics, electronics and telecommunications, etc. (see [1,8]). In recent years, there have been intensive studies on the impulsive differential equations (see [9–11]). However, besides impulsive effects, stochastic effects likewise exist widely in real systems (see [7]). It has received considerable attention. For example, the existence and uniqueness of the analytic solution for stochastic differential equations (SDEs) has been studied by Gard [5] and Mao [13], the stability of Euler–Maruyama (EM) method for SDEs has been studied by Cao et al. [3] and Liu et al. [14], the stability and convergence of composite Milstein method for SDEs has been studied by Omar et al. [15], the convergence of EM method for SDEs has been studied by Buckwar [2], etc.

Therefore, it is necessary to consider the stability of solutions of impulsive stochastic differential equations. It has attracted interests of many researchers. Wu and Sun [17], Yang et al. [19,20] and Zhao et al. [21] have studied the pth moment stability of impulsive stochastic differential equations (ISDEs). Wu and Ding [18] has studied the convergence and stability of Euler method for ISDEs and Zhao et al. [22] has studied pth moment stability of EM method for ISDEs. Liu at al. [12] has studied the mean-square stability of semi-implicit method for one-dimensional ISDEs.

Like many other equations, explicit solutions can rarely be obtained for the impulsive stochastic differential equations. Thus, it is necessary to develop numerical methods and to study the properties of these methods.

Like as ordinary differential equations, we just discuss the test equations. Unfortunately, the diffusion and drift coefficients can not be diagonalized at the same time.

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Hence, in this paper, we consider the stability of both analytic and EM solution of the following equation

$$\begin{cases} dX(t) = AX(t)dt + BX(t)dW(t), & t \neq \tau_k, \quad t > 0, \\ X(\tau_k^+) = H_k X(\tau_k), & t = \tau_k, \\ X(0^+) = X_0, \end{cases} \tag{1.1}$$

where $A = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^{n\times n}$, $B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, $H_k = (H_{kij})_{n\times n} \in \mathbb{R}^{n\times n} (k = 1, 2, 3, \ldots)$ are constant matrices, $W(t)$ is standard Brownian motion, $X(t) = (X^1(t), \ldots, X^n(t))^T, X(t^+)$ is the right limit of $X(t), 0 < \tau_1 < \tau_2 < \tau_3 < \ldots$, $\lim_{k\to\infty}\tau_k = \infty$. Denote $\tau_0 = 0$, $\mathbb{N} = \{1, 2, \ldots\}$.

By Gard [5] and Mao [13], the Eq. (1.1) has on $(0, \infty)$ unique solution

$$\begin{cases} X(t) = \exp\left\{\left(A - \frac{1}{2}b^2\right)t + BW(t)\right\}X_0, \quad t \in (0, \tau_1], \\ X(t) = \exp\left\{\left(A - \frac{1}{2}b^2\right)(t - \tau_k) + B(W(t) - W(\tau_k))\right\} \quad t \in (\tau_k, \tau_{k+1}], \quad k \in \mathbb{N} \\ \qquad \cdot\prod_{i=1}^{k}\left[H_i \exp\left\{\left(A - \frac{1}{2}b^2\right)(\tau_i - \tau_{i-1}) + B(W(\tau_i) - W(\tau_{i-1}))\right\}\right]X_0. \end{cases}$$

In this paper, we always assume that there exist two constants θ_1 and θ_2 such that

$$0 < \theta_1 \leqslant \tau_k - \tau_{k-1} \leqslant \theta_2 < \infty, \text{ for all } k \in \mathbb{N}.$$

In Section 2, MS-stability of Eq. (1.1) is analyzed. The results obtained in Section 2 are coincide with existing results. Such as, Corollary 2.7 is Theorem 1 in [16], Corollary 2.8 is Theorem 2.1 in [12]. In Section 3, conditions of MS-stability of the Euler–Maruyama scheme corresponding to Section 2 are obtained. The results in Section 3 are also coincide with existing results. Corollary 3.8 is coincide with Theorem 2 in [16]. The main result with $\theta = 0$ in Section 3 of [12] is coincide with Corollary 3.9. In Section 4, some numerical experiments confirming our stability analysis in Section 3 are given.

2. The stability of the analytic solution

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with a filtration $(\mathscr{F}_t)_{t\geqslant 0}$, which satisfies the usual conditions. Let $W(t), t \geqslant 0$ in Eq. (1.1) be $\mathscr{F}_t$-adapted and independent of $\mathscr{F}_0$. We assume X_0 to be $\mathscr{F}_0$-measurable.

In this section we will give the sufficient conditions under which the analytic solution of (1.1) is MS-stable. First, we will give the definition of MS-stability.

Definition 2.1 [16]. The zero solution of the system (1.1) is MS-stable if

$$\mathbb{E}(|X(t)|^2) \to 0 \quad \text{as} \quad t \to \infty,$$

where $|\cdot|$ stands for the Euclidean norm, $\mathbb{E}(\cdot)$ denotes the expectation with respect to $\mathbb{P}$.

Let $F(t) = \mathbb{E}(X(t)X(t)^T)$ be the $n \times n$ matrix-valued second moment of the solution of (1.1). By the Itô formula, $F(t)$ obeys the initial value problem of the following impulsive ordinary differential equation

$$\begin{cases} dF(t) = (AF(t) + F(t)A^T + BF(t)B^T)dt, & t \neq \tau_k, \quad t > 0, \\ F(\tau_k^+) = H_k F(\tau_k) H_k^T, & t = \tau_k, \\ F(0^+) = \mathbb{E}(X_0 X_0^T). \end{cases} \tag{2.1}$$

By virtue of the symmetry of the matrix $F(t)$, we have its governing ordinary differential equation of $\frac{n(n+1)}{2}$-dimension.

$$\begin{cases} \dot{Y}(t) = MY(t), & t \neq \tau_k, \quad t > 0, \\ Y(\tau_k^+) = D_k Y(\tau_k), & t = \tau_k, \\ Y(0^+) = Y_0, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} &Y(t) = (\mathbb{E}(X^1(t))^2, \mathbb{E}(X^1(t)X^2(t)), \ldots, \mathbb{E}(X^1(t)X^n(t)), \ldots, \mathbb{E}(X^i(t))^2, \mathbb{E}(X^i(t)X^{i+1}(t)), \ldots, \mathbb{E}(X^i(t)X^n(t)), \ldots, \mathbb{E}(X^n(t))^2)^T, \\ &M = M1 + diag(2\lambda_1, \ldots, 2\lambda_i, \lambda_i + \lambda_{i+1}, \ldots, \lambda_i\lambda_n, \ldots, 2\lambda_n), \end{aligned} \tag{2.3}$$

$$M1 = \begin{bmatrix} b_{11}^2 & 2b_{11}b_{12} & \cdots & 2b_{11}b_{1n} & \cdots & b_{1p}^2 & 2b_{1p}b_{1(p+1)} \\ b_{11}b_{21} & b_{11}b_{22}+b_{12}b_{21} & \cdots & b_{11}b_{2n}+b_{1n}b_{21} & \cdots & b_{1p}b_{2p} & b_{1p}b_{2(p+1)}+b_{2p}b_{1(p+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{11}b_{n1} & b_{11}b_{n2}+b_{12}b_{n1} & \cdots & b_{11}b_{nn}+b_{1n}b_{n1} & \cdots & b_{1p}b_{np} & b_{1p}b_{n(p+1)}+b_{np}b_{1(p+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{i1}b_{j1} & b_{i1}b_{j2}+b_{i2}b_{j1} & \cdots & b_{i1}b_{jn}+b_{in}b_{j1} & \cdots & b_{ip}b_{jp} & b_{ip}b_{j(p+1)}+b_{jp}b_{i(p+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n1}^2 & 2b_{n1}b_{n2} & \cdots & 2b_{n1}b_{nn} & \cdots & b_{np}^2 & 2b_{np}b_{n(p+1)} \end{bmatrix.}$$

$$\begin{bmatrix.} 2b_{1p}b_{1(p+2)} & \cdots & 2b_{1p}b_{1n} & \cdots & b_{1n}^2 \\ b_{1p}b_{2(p+2)}+b_{2p}b_{1(p+2)} & \cdots & b_{1p}b_{2n}+b_{2p}b_{1n} & \cdots & b_{1n}b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{ip}b_{j(p+2)}+b_{jp}b_{i(p+2)} & \cdots & b_{ip}b_{jn}+b_{jp}b_{in} & \cdots & b_{in}b_{jn} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2b_{np}b_{n(p+2)} & \cdots & 2b_{np}b_{nn} & \cdots & b_{nn}^2 \end{bmatrix},$$

replacing the element b_{ij} in $M1$ by $H_{k_{ij}}$, we can get the matrices D_k $(k = 1, 2, \ldots)$. Denote $Y_0 = Y(0^+)$.

Note that the system (1.1) is MS-stable, if and only if $Y(t) \to 0$ as $t \to \infty$.

Definition 2.2 [1]. The function $Y : (0, +\infty) \to \mathbb{R}^{n(n+1)/2}$ is said to be a solution of the system (2.2), if the following conditions are satisfied:

(1) $\lim_{t\to 0^+} Y(t) = Y_0$;
(2) for $t \in (0, +\infty), t \neq \tau_k, k \in \mathbb{N}$, the function $Y(t)$ is differentiable and $\dot{Y}(t) = MY(t)$;
(3) the function $Y(t)$ is left continuous in $(0, +\infty)$ and if $t \in (0, +\infty), t = \tau_k$, then $Y(t^+) = D_k Y(t)$.

Denote $U_j = D_j e^{M(\tau_j - \tau_{j-1})}, j \in \mathbb{N}$. System (2.2) has on $(0, +\infty)$ a unique solution

$$\begin{cases} Y(t) = e^{Mt} Y_0, & t \in (0, \tau_1], \\ Y(t) = e^{M(t-\tau_k)} U_k U_{k-1} \cdots U_1 Y_0, & t \in (\tau_k, \tau_{k+1}], \quad k \in \mathbb{N}. \end{cases}$$

In the following, $\|\cdot\|$ denotes any arbitrary norm on $\mathbb{R}^{\bar{n}}$ $(\bar{n} \in \mathbb{N})$. The same symbol will be used for the matrix norm subordinate to the given vector norm. Let $\bar{A} = (\bar{a}_{i,j})_{\bar{n}\times\bar{n}}$. The logarithmic norm $\mu[\bar{A}]$ of $\bar{A}$ is (see [4])

$$\mu[\bar{A}] = \lim_{\delta\to 0^+} \frac{\|I + \delta\bar{A}\| - 1}{\delta} \quad (I \text{ is a unite matrix}, \delta \in \mathbb{R}).$$

For convenience, we introduce some well known identities for our results.

$$\|\bar{A}\|_\infty = \max_{1\leqslant i\leqslant n} \left\{ \sum_{j=1}^n |\bar{a}_{ij}| \right\}, \quad \mu_\infty[\bar{A}] = \max_{1\leqslant i\leqslant n} \left\{ \bar{a}_{ii} + \sum_{j\neq i} |\bar{a}_{ij}| \right\}.$$

Theorem 2.3. *The zero solution of the system (1.1) is MS-stable, if*

$$\limsup_{k\to\infty} \sum_{j=1}^k \frac{1}{\tau_k} \log(\|D_j\|) < -\mu[M]. \tag{2.4}$$

Proof. For $t \in (\tau_k, \tau_{k+1}]$, we have

$$\frac{\log\|Y(t)\|}{\tau_k} = \frac{\log\left(\|e^{M(t-\tau_k)}\|\|\prod_{j=1}^k \|D_j e^{M(\tau_j-\tau_{j-1})}\|\|Y_0\|\right)}{\tau_k} \leqslant \frac{\mu[M](t-\tau_k)}{\tau_k} + \frac{\log(\|Y_0\|)}{\tau_k} + \frac{\log\left(\prod_{j=1}^k \|D_j\|\right)}{\tau_k} + \mu[M].$$

Since $0 < \theta_1 \leqslant \tau_j - \tau_{j-1} \leqslant \theta_2$, $\lim_{k\to\infty}\tau_k = \infty$,

$$\limsup_{k\to\infty}\frac{\log\|Y(t)\|}{\tau_k} \leqslant \mu[M] + \limsup_{k\to\infty}\frac{1}{\tau_k}\log\left(\prod_{j=1}^{k}\|D_j\|\right) = \mu[M] + \limsup_{k\to\infty}\sum_{j=1}^{k}\frac{1}{\tau_k}\log(\|D_j\|).$$

From (2.4), we have

$$\limsup_{k\to\infty}\frac{\log\|Y(t)\|}{\tau_k} < 0.$$

Hence $\log\|Y(t)\| \to -\infty$, as $t\to\infty$. That is $\|Y(t)\| \to 0$ as $t\to\infty$. □

Corollary 2.4. *The zero solution of the system (1.1) is MS-stable if there exists a constant $q \in [0,1)$, such that*

$$\|D_j\|e^{\mu[M](\tau_j-\tau_{j-1})} \leqslant q < 1, \quad \forall j \in \mathbb{N}. \tag{2.5}$$

Proof. Since

$$\frac{1}{\tau_k}\log\left(\prod_{j=1}^{k}\|D_j\|\right) + \mu[M] = \frac{\log\left(\prod_{j=1}^{k}\|D_j\|e^{\mu[M](\tau_j-\tau_{j-1})}\right)}{\tau_k}.$$

From (2.5), we obtain

$$\limsup_{k\to\infty}\frac{1}{\tau_k}\log\left(\prod_{j=1}^{k}\|D_j\|\right) + \mu[M] \leqslant \limsup_{k\to\infty}\frac{k}{k\theta_1}\log q < 0.$$

From Theorem 2.3, we obtain the result. □

We will study the MS-stability w.r.t. $\|\cdot\|_\infty$ for the system (1.1).

Theorem 2.5. *The system (1.1) is MS-stable, if*

$$\limsup_{p\to\infty}\frac{1}{\tau_p}\log\left(\prod_{k=1}^{p}d_k\right) < -L \tag{2.6}$$

where

$$L = \max\left\{\left(\sum_{p=1}^{n}|b_{ip}|\right)^2 + 2\lambda_i : i = 1,\ldots,n\right\}, \quad d_k = \max\left\{\left(\sum_{p=1}^{n}|H_{k_{ip}}|\right)^2 : i = 1,\ldots,n\right\}.$$

Proof. Let $Q_{ii} = \sum_{p=1}^{n}|b_{ip}|^2 + \sum_{p=1,l>p}^{n}2|b_{ip}b_{il}| + 2\lambda_i$,

$$Q_{ij} = \sum_{p=1}^{n}|b_{ip}b_{jp}| + \sum_{p=1,l>p,p\neq i,l\neq j}^{n}(|b_{ip}b_{jl} + b_{il}b_{jp}|) + b_{ii}b_{jj} + b_{ij}b_{ji} + \lambda_i + \lambda_j, \quad i \neq j.$$

Eq. (2.3) implies that

$$\mu_\infty[M] = \max\{Q_{ij} : 1 \leqslant i \leqslant j \leqslant n\}.$$

$$Q_{ii} = \sum_{p=1}^{n}|b_{ip}|^2 + \sum_{p=1,l>p}^{n}2|b_{ip}b_{il}| + 2\lambda_i = \left(\sum_{p=1}^{n}|b_{ip}|\right)^2 + 2\lambda_i,$$

$$\begin{aligned}
Q_{ij} &= \sum_{p=1}^{n}|b_{ip}b_{jp}| + \sum_{p=1,l>p,p\neq i,l\neq j}^{n}(|b_{ip}b_{jl} + b_{il}b_{jp}|) + b_{ii}b_{jj} + b_{ij}b_{ji} + \lambda_i + \lambda_j \\
&\leqslant \sum_{p=1}^{n}|b_{ip}b_{jp}| + \sum_{p=1,l>p}^{n}(|b_{ip}b_{jl} + b_{il}b_{jp}|) + \lambda_i + \lambda_j \\
&\leqslant \left(\left(\sum_{p=1}^{n}|b_{ip}|\right)^2 + 2\lambda_i\right)\Big/2 + \left(\left(\sum_{p=1}^{n}|b_{jp}|\right)^2 + 2\lambda_j\right)\Big/2, \quad i \neq j.
\end{aligned}$$

Clearly, $\mu_\infty[M] = L$. Let

$$G_{k_{ii}} = \sum_{p=1}^{n} |H_{k_{ip}}|^2 + \sum_{p=1,l>p}^{n} 2|H_{k_{ip}} H_{k_{il}}|,$$

$$G_{k_{ij}} = \sum_{p=1}^{n} |H_{k_{ip}} H_{k_{jp}}| + \sum_{p=1,l>p}^{n} (|H_{k_{ip}} H_{k_{jl}} + H_{k_{il}} H_{k_{jp}}|), \quad i \neq j.$$

Eq. (2.3) implies that

$$\|D_k\|_\infty = \max\{G_{k_{ij}} : 1 \leqslant i \leqslant j \leqslant n\}.$$

$$G_{k_{ii}} = \sum_{p=1}^{n} H_{k_{ip}}^2 + \sum_{p=1,l>p}^{n} 2|H_{k_{ip}} H_{k_{il}}| = \left(\sum_{p=1}^{n} |H_{k_{ip}}|\right)^2,$$

$$G_{k_{ij}} = \sum_{p=1}^{n} |H_{k_{ip}} H_{k_{jp}}| + \sum_{p=1,l>p}^{n} (|H_{k_{ip}} H_{k_{jl}} + H_{k_{il}} H_{k_{jp}}|) \leqslant \left(\sum_{p=1}^{n} |H_{k_{ip}}|\right)^2 / 2 + \left(\sum_{p=1}^{n} |H_{k_{jp}}|\right)^2 / 2, \quad i \neq j.$$

It shows that $d_k = \|D_k\|_\infty$. Thus, we get the result by Theorem 2.3. $\square$

In the remainder, we will show that results obtained in this section are coincide with some existing results.

Corollary 2.6. *The system (1.1) is MS-stable if there exists a constant $q \in [0, 1)$ such that*

$$d_k e^{L(\tau_k - \tau_{k-1})} \leqslant q < 1, \quad \forall k \in \mathbb{N}, \tag{2.7}$$

where L and d_k have been defined in Theorem 2.5.

By Corollary 2.6, we can get Theorem 1 in [16] as following.

Corollary 2.7. *Assume that $n = 2, H_k = I$ for $k = 1, 2, \ldots$. The system (1.1) is MS-stable, if the following estimation*

$$\max\{2\lambda_1 + (|b_{11}| + |b_{12}|)^2, 2\lambda_2 + (|b_{21}| + |b_{22}|)^2\} < 0$$

holds.

By Corollary 2.6, we can get the following result which is coincide with Theorem 2.1 in [12].

Corollary 2.8. *Let $A = a \in \mathbb{R}, B = b \in \mathbb{R}, H_k = 1 + \beta_k \in \mathbb{R}, \tau_k = k$. The system (1.1) is MS-stable, if the following estimation*

$$|1 + \beta_k|^2 exp\{2a + b^2\} \leqslant q < 1, \quad \forall k \in \mathbb{N}$$

holds. That is

$$\sup\{|1 + \beta_k| : k \in \mathbb{N}\} exp\left\{a + \frac{1}{2} b^2\right\} < 1.$$

3. MS-stability of Euler–Maruyama scheme

In this section, we shall investigate the MS-stability of EM scheme with variable step-size for system (1.1). The adaption of EM scheme to Eq. (1.1) leads to a numerical solution of the following type

$$\begin{cases} X_{km,l} = (I + Ah_k + B\Delta W_{km,l-1}) X_{km,l-1}, & k = 0, 1, \ldots, \ l = 1, \ldots, m, \\ X_{km,0} = H_k X_{(k-1)m,m}, \\ X_{0,0} = X_0, \end{cases} \tag{3.1}$$

where $h_k = \frac{\tau_{k+1} - \tau_k}{m}$ is a step-size for a positive integer m, I is a unit matrix, the discrete point $t_{km,l} = \tau_k + lh_k, 0 \leqslant l \leqslant m, \Delta W_{km,l} = W(t_{km,l+1}) - W(t_{km,l})$ are independent $N(0, h_k)$-distributed Gaussian random variables, $X_{km,l} \approx X(t_{km,l}), 1 \leqslant l \leqslant m$ and $X_{km,0} \approx X(\tau_k^+) = \lim_{t \to \tau_k^+} X(t)$. Furthermore $X_{km,l}$ is $\mathscr{F}_{t_{km,l}}$-measurable.

Theorem 3.1. *From Theorem 10.22 in [6], we can obtain that the method (3.1) converges strongly.*

Let $\overline{F}_{km,l} = \mathbb{E}(X_{km,l}X_{km,l}^T)$ be the $n \times n$ matrix-valued second moment of $X_{km,l}$. $\overline{F}_{km,l}$ obeys the following difference equation

$$\begin{cases} \overline{F}_{km,l} = (A\overline{F}_{km,l-1} + \overline{F}_{km,l-1}A^T + B\overline{F}_{km,l-1}B^T)h_k \\ \qquad\qquad + \overline{F}_{km,l-1} + A\overline{F}_{km,l-1}A^T h_k^2, \quad l = 1, \dots, m, \\ \overline{F}_{km,0} = D_k \overline{F}_{(k-1)m,m}, \\ \overline{F}_{0,0} = \mathbb{E}(X_0X_0^T). \end{cases} \tag{3.2}$$

By the symmetry of $\overline{F}_{km,l}$, we obtain a one-step difference equation of the form

$$\begin{cases} \overline{Y}_{km,l} = \overline{M}_k \overline{Y}_{km,l-1}, \quad l = 1, \dots, m, \\ \overline{Y}_{km,0} = D_k \overline{Y}_{(k-1)m,m}, \\ \overline{Y}_{0,0} = Y_0, \end{cases} \tag{3.3}$$

where $\overline{M}_k = Mh_k + I + Nh_k^2, M$ and D_k are the values occurring in (2.3), and

$$N = diag(\lambda_1^2, \dots, \lambda_1\lambda_n, \dots, \lambda_i^2, \dots, \lambda_i\lambda_n, \dots, \lambda_n^2),$$
$$\overline{Y}_{km,l} = \left(\mathbb{E}(X^1_{km,l}X^1_{km,l}), \mathbb{E}(X^1_{km,l}X^2_{km,l}), \dots, \mathbb{E}(X^1_{km,l}X^n_{km,l}), \dots, \mathbb{E}(X^i_{km,l}X^i_{km,l}), \mathbb{E}(X^i_{km,l}X^{i+1}_{km,l}), \dots, \mathbb{E}(X^i_{km,l}X^n_{km,l}), \dots, \mathbb{E}(X^n_{km,l}X^n_{km,l})\right)^T.$$

Denote $\overline{U}_j = D_j\overline{M}_{j-1}^m$ $j \in \mathbb{N}$. From Eq. (3.3) we have

$$\begin{cases} \overline{Y}_{0,l} = \overline{M_0}^l Y_0, & 0 \leqslant l \leqslant m, \\ \overline{Y}_{km,l} = \overline{M}_k^l \overline{U}_k \overline{U}_{k-1} \cdots \overline{U}_1 Y_0, & 0 \leqslant l \leqslant m, \quad k \in \mathbb{N}. \end{cases} \tag{3.4}$$

Let $n = km + l, X_n = X_{km+l}$, where $k \in \mathbb{N}, 1 \leqslant l \leqslant m$.

Definition 3.2 [16]. The numerical method (3.1) is said to be MS-stable if

$\mathbb{E}(|X_n|^2) \to 0$ as $n \to \infty$,

Note that, $\mathbb{E}(|X_n|^2) \to 0$ as $n \to \infty$, if and only if $\|\overline{Y}_{km,l}\| \to 0$, as $k \to \infty$ for all $1 \leqslant l \leqslant m$.

Theorem 3.3. *The numerical solution (3.1) for (1.1) is MS-stable, if*

$$(1) \qquad \limsup_{k\to\infty} \sum_{j=1}^{k} \left[\frac{\log(\|D_j\|)}{\tau_k} + \frac{\log\|\overline{M}_{j-1}\|}{\sum_{j=1}^k h_{j-1}} \right] < 0, \tag{3.5}$$

or

(2) *the condition (2.4) holds and*

$$\limsup_{k\to\infty} \sum_{j=1}^{k} (\log(\|\overline{M}_{j-1}\|) - \mu[M]h_{j-1}) \leq 0. \tag{3.6}$$

Proof. From (3.4), for $K \in \mathbb{N}$, we have

$$\frac{\log\|\overline{Y}_{km,l}\|}{\tau_k} \leqslant \frac{l\log\|\overline{M}_k\|}{\tau_k} + \frac{\log\|Y_0\|}{\tau_k} + \frac{\log\left(\prod_{j=1}^k\|D_j\|\right)}{\tau_k} + \frac{\sum_{j=1}^k m\log\|\overline{M}_{j-1}\|}{\tau_k}$$
$$= \frac{l\log\|\overline{M}_k\|}{\tau_k} + \frac{\log\|Y_0\|}{\tau_k} + \frac{\log\left(\prod_{j=1}^k\|D_j\|\right)}{\tau_k} + \frac{\sum_{j=1}^k \log\|\overline{M}_{j-1}\|}{\sum_{j=1}^k h_{j-1}}.$$

Hence

$$\limsup_{k\to\infty} \frac{\log\|\overline{Y}_{km,l}\|}{\tau_k} \leqslant \limsup_{k\to\infty} \left[\frac{\log\left(\prod_{j=1}^k\|D_j\|\right)}{\tau_k} + \frac{\sum_{j=1}^k \log\|\overline{M}_{j-1}\|}{\sum_{j=1}^k h_{j-1}} \right]$$
$$= \limsup_{k\to\infty} \sum_{j=1}^{k} \left[\frac{\log(\|D_j\|)}{\tau_k} + \frac{\log\|\overline{M}_{j-1}\|}{\sum_{j=1}^k h_{j-1}} \right].$$

From (1) or (2), we can obtain

$$\limsup_{k\to\infty} \frac{\log \|\overline{Y}_{km,l}\|}{\tau_k} < 0.$$

That is, $\|\overline{Y}_{km,l}\| \to 0$, as $k \to \infty$. □

Remark 3.4. The condition (1) in Theorem 3.3 contains the condition (2). Under condition (2), the numerical solution and analytical solution are both stable. But, under condition (1), the numerical solution is stable, the analytical solution may be unstable.

Corollary 3.5. *The numerical solution (3.1) for (1.1) is MS-stable, if*

(1) *there exists a constant $q \in [0,1)$ such that*

$$\|D_j\|\|\overline{M}_{j-1}\|^m \leqslant q < 1, \quad \forall j \in \mathbb{N}, \tag{3.7}$$

or

(2) *the condition (2.4) holds and*

$$\|\overline{M}_{j-1}\|e^{-\mu[M]h_{j-1}} < 1, \quad \forall j \in \mathbb{N}, \tag{3.8}$$

Proof. From Theorem 3.3, we can easily get the case (2). Next, we will prove the case (1). From (3.5), we have

$$\sum_{j=1}^{k}\left[\frac{\log(\|D_j\|)}{\tau_k} + \frac{\log\|\overline{M}_{j-1}\|}{\sum_{j=1}^{k} h_{j-1}}\right] = \sum_{j=1}^{k}\left[\frac{\log(\|D_j\|\|\overline{M}_{j-1}\|^m)}{\tau_k}\right] \leqslant \frac{\log q}{\theta_1} < 0.$$

Hence, we obtain the case (1). □

We will study the MS-stability of the EM method w.r.t $\|\cdot\|_\infty$.

Theorem 3.6. *The numerical solution (3.1) for (1.1) is MS-stable w.r.t. $\|\cdot\|_\infty$, if*

$$(1) \quad \limsup_{p\to\infty}\sum_{k=1}^{p}\left[\frac{\log(d_k)}{\tau_p} + \frac{\log\|\sigma_{k-1}\|}{\sum_{k=1}^{p} h_{k-1}}\right] < 0, \tag{3.9}$$

or

(2) *the condition (2.6) holds and*

$$\limsup_{p\to\infty}\sum_{k=1}^{p}(\log(\sigma_{k-1}) - Lh_{k-1}) \le 0, \tag{3.10}$$

where d_k and L have been defined in Theorem 2.5 and

$$\sigma_k = \max\left\{h_k\left(\sum_{p=1}^{n}|b_{ip}|\right)^2 + (1+\lambda_i h_k)^2 : i = 1,2,\ldots,n\right\}, \quad k = 0,1,\ldots.$$

Proof. From (2.3) and (3.3), we know that

$$\|\overline{M}_k\|_\infty = \max\left\{h_k\left(\sum_{p=1,p\neq i}^{n}|b_{ip}|^2 + \sum_{p=1,l>p}^{n}2|b_{ip}b_{il}|\right) + |b_{ii}^2 h_k + 2\lambda_i h_k + 1 + \lambda_i^2 h_k^2|, h_k\left(\sum_{p=1}^{n}|b_{ip}b_{jp}| + \sum_{p=1,l>p,p\neq i,l\neq j}^{n}(|b_{ip}b_{jl} + b_{il}b_{jp}|)\right)\right.$$
$$\left. + \left|(b_{ii}b_{jj} + b_{ij}b_{ji})h_k + (\lambda_i+\lambda_j)h_k + 1 + \lambda_i\lambda_j h_k^2\right| : 1 \leqslant i < j \leqslant n\right\}.$$

By the similar calculation in Theorem 2.5, we can easily get that $\sigma_k = \|\overline{M}_k\|_\infty$. Hence, by Theorem 3.3, we obtain the result. □

Corollary 3.7. *The numerical solution (3.1) for (1.1) is MS-stable, if*

(1) *there exists a constant $q \in [0,1)$ such that*

$$d_k\sigma_{k-1}^m \leqslant q < 1, \quad \forall k \in \mathbb{N}, \tag{3.11}$$

or

(2) *the condition* (2.6) *holds and*

$$\sigma_{k-1}e^{-Lh_{k-1}} < 1, \quad \forall k \in \mathbb{N}, \tag{3.12}$$

where d_k *and* L *have been defined in* Theorem 2.5, *and* σ_k *has defined in* Theorem 3.6.

In the remainder, we will show that results obtained in this section are coincide with some existing results.

In particular, If $H_k = I$. Set $\tau_k = k, h_k = h$, $\overline{M}_k = M$ for all $k = 0, 1, \ldots$. Thus the EM solution (3.1) is stable if and only if $\|\overline{M}\| < 1$.

Hence we can get Theorem 2 in [16] as following.

Corollary 3.8. *Assumed that* $n = 2, H_k = I$. *The numerical method* (3.1) *is MS-stable w.r.t.* $\|\cdot\|_\infty$ *if and only if the following inequality*

$$\max\{h(|b_{11}| + |b_{12}|)^2 + (1 + \lambda_1 h)^2, h(|b_{21}| + |b_{22}|)^2 + (1 + \lambda_2 h)^2\} < 1$$

holds.

By Theorem 3.3, we can get the following result which is coincide with the main result with $\theta = 0$ in section 3 of [12].

Corollary 3.9. *Let* $A = a \in \mathbb{R}, B = b \in \mathbb{R}, H_k = 1 + \beta_k \in \mathbb{R}, \tau_k = k$. *The numerical method* (3.1) *is MS-stable, if*

$$(1 + ah)^2 + b^2 h \leqslant e^{(2a+b^2)h}.$$

4. Numerical experiments

Example 4.1

$$\begin{cases} dX(t) = 2X(t)dt + X(t)dW(t), & t \neq k, \\ X(k^+) = H_k X(k), & t = k, \\ X(0^+) = 1, \end{cases} \tag{4.1}$$

where $H_k \in \mathbb{R}, k = 1, 2, \ldots$.

We will confirm the main result in this paper through numerical experiments. First, we choose (4.1) as test equations to illustrate the convergence.

From [5,13], we can obtain the explicit solution of (4.1).

$$X(t) = \begin{cases} \exp\{1.5t + W(t)\}X(0^+), & \text{for} \quad t \in (0, 1], \\ \exp\{1.5(t-k) + (W(t) - W(k))\}\cdot & \\ \prod_{i=1}^{k} H_i \exp\{1.5 + (W(i) - W(i-1))\}X(0^+) & \text{for} \quad t \in (k, k+1]. \end{cases}$$

In Fig. 1, we plot the explicit solution of (4.1) together with the EM method for step-size $h = 1/128$. Owing to the convergence of the EM method, the figures illustrate that the numerical solution has the same stability property with its analytical solution. The system (4.1) without impulsive effects that means that $H_k = 1, k = 0, 1, \ldots$.

In the following tests, we show that the influence of impulsive effects and the step-size on the MS-stability of the EM method. The data used in the following figures are obtained by the mean square of data by 10,000 trajectories, that is, $w_i : 1 \leqslant i \leq 10,000$, if $X_n = (X_n^1, X_n^2)$, then

$$Y^1 = 1/10,000\Sigma_{i=1}^{10,000}(X_n^1(w_i))^2, \quad Y^2 = 1/10,000\Sigma_{i=1}^{10,000}X_n^1(w_i)X_n^2(w_i),$$

$$Y^3 = 1/10,000\Sigma_{i=1}^{10,000}(X_n^2(w_i))^2,$$

if $X_n = (X_n^1, X_n^2, X_n^3)$, then

$$Y^1 = 1/10,000\Sigma_{i=1}^{10,000}(X_n^1(w_i))^2, \quad Y^2 = 1/10,000\Sigma_{i=1}^{10,000}X_n^1(w_i)X_n^2(w_i),$$

$$Y^3 = 1/10,000\Sigma_{i=1}^{10,000}X_n^1(w_i)X_n^3(w_i), \quad Y^4 = 1/10,000\Sigma_{i=1}^{10,000}(X_n^2(w_i))^2,$$

$$Y^5 = 1/10,000\Sigma_{i=1}^{10,000}X_n^2(w_i)X_n^3(w_i), \quad Y^6 = 1/10,000\Sigma_{i=1}^{10,000}(X_n^3(w_i))^2.$$

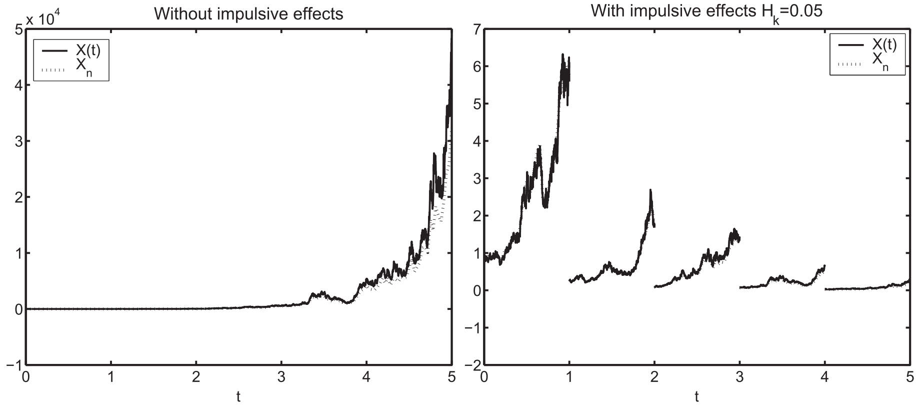

Fig. 1. Convergence of EM method for (4.1).

Example 4.2

$$\begin{cases} dX(t) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} X(t)dt + \begin{bmatrix} -0.1 & 0.5 & 0.4 \\ 0.5 & 0.1 & -0.4 \\ -0.1 & -0.4 & 0.5 \end{bmatrix} X(t)dW(t), & t \neq \tau_k, \\ X(\tau_k{}^+) = H_k X(\tau_k), & t = \tau_k, \\ X(0^+) = (2,2,2)^T, \end{cases} \tag{4.2}$$

where $\tau_k = 3 + \frac{3(k-2)}{2}$ if k is even; $\tau_k = 1 + \frac{3(k-1)}{2}$ if k is odd.

In Figs. 2 and 3, we choose $m = 10$ $\left(h_k = \frac{\tau_{k+1}-\tau_k}{m}\right)$.

For $H_k = I, d_k e^{L(\tau_k - \tau_{k-1})} \geqslant e^2 > 1$, Fig. 2 shows that the EM solution of (4.2) is unstable.

For $H_k = 10^{-k-1}\begin{bmatrix} 5 & 2 & 1 \\ -2 & -1 & 5 \\ 1 & -5 & 2 \end{bmatrix}$, the conditions in Theorem 3.6 are satisfied. Fig. 3 shows that the EM solution of (4.2) is MS-stable. Figs. 2 and 3 show that the impulsive effects drive unstable system to stable system.

Example 4.3

$$\begin{cases} dX(t) = \begin{bmatrix} -0.2 & 0 \\ 0 & -3.5 \end{bmatrix} X(t)dt + \begin{bmatrix} 0.2 & 0.4 \\ -0.5 & 1.5 \end{bmatrix} X(t)dW(t), & t \neq \tau_k, \\ X(k^+) = H_k X(k), & t = \tau_k, \\ X(0^+) = (1,-1)^T, \end{cases} \tag{4.3}$$

where $\tau_k = \frac{k}{2}$. In Fig. 4, we choose $m = 10$ $(h_k = \frac{\tau_{k+1}-\tau_k}{m})$.

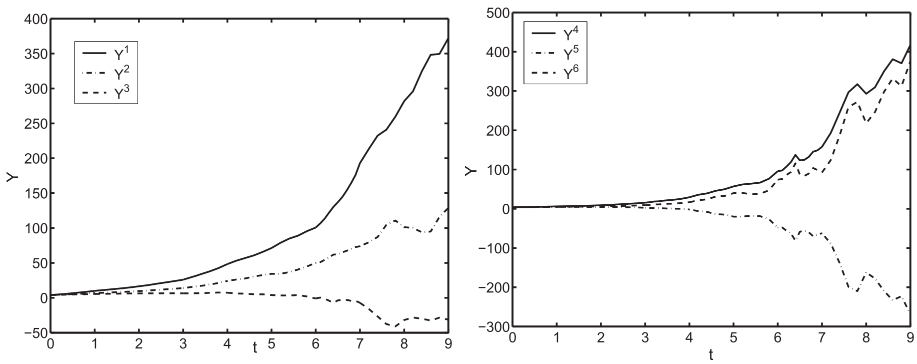

Fig. 2. Simulation for (4.2) without impulsive effects.

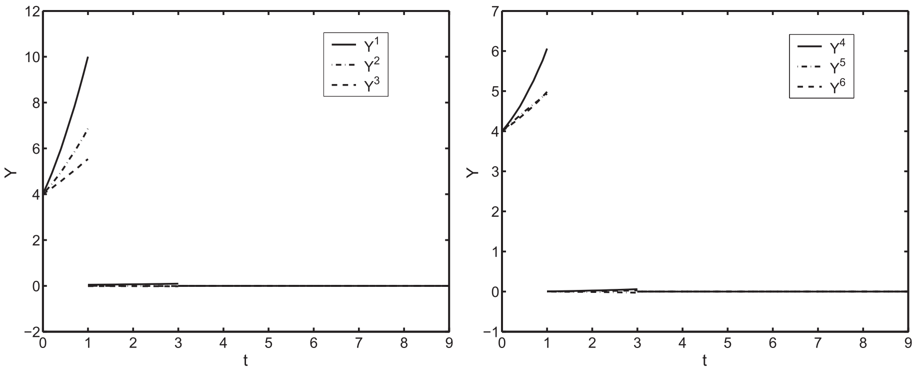

Fig. 3. Simulation for (4.2) with impulsive effects.

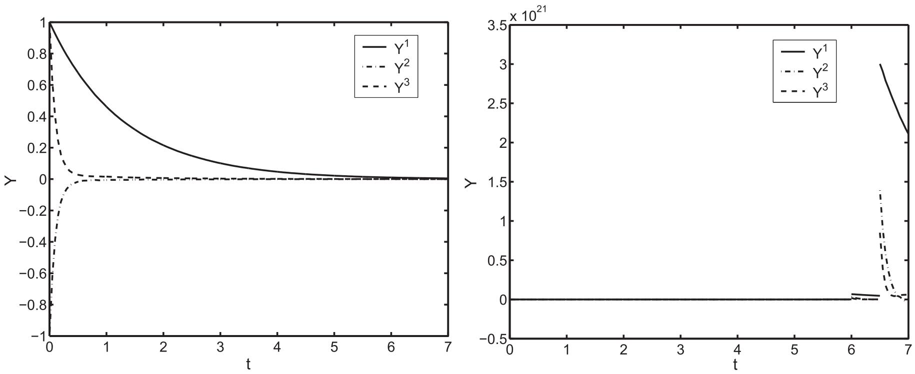

Fig. 4. Simulation for (4.3).

For $H_k = I$, by Theorem 3.6, the EM solution is stable. The left of Figure gives a hard evidence. For $H_k = \begin{bmatrix} 8 & 5\times(-1)^k \\ 4 & 10 \end{bmatrix}$, $d_k e^{L(\tau_k - \tau_{k-1})} = 196e^{-0.02} > 1$. The right of Fig. 4 shows that the EM solutio of (4.3) is unstable. It shows that the impulsive effects can change the stable system into unstable.

Example 4.4

$$\begin{cases} dX(t) = \begin{bmatrix} -8 & 0 \\ 0 & -6 \end{bmatrix} X(t)dt + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} X(t)dW(t), & t \neq k, \\ X(k^+) = \begin{bmatrix} 3 & 1.4 \\ 1.4 & 3 \end{bmatrix} X(k), & t = k, \\ X(0^+) = (1, -1)^T. \end{cases} \tag{4.4}$$

Fig. 5 show that the EM method for (4.4) is stable when $h = h_k = 1/100$, unstable when $h = h_k = 1/5$. It shows that the stability of EM method depends on the choice of step-size.

Example 4.5

$$\begin{cases} dX(t) = \begin{bmatrix} -1.5 & 0 \\ 0 & -1.5 \end{bmatrix} X(t)dt + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X(t)dW(t), & t \neq k, \\ X(k^+) = \begin{bmatrix} \frac{7}{6}e + (-1)^k \frac{5}{6}e & 0 \\ 0 & 0 \end{bmatrix} X(k), & t = k, \\ X(0^+) = (1, -1)^T. \end{cases} \tag{4.5}$$

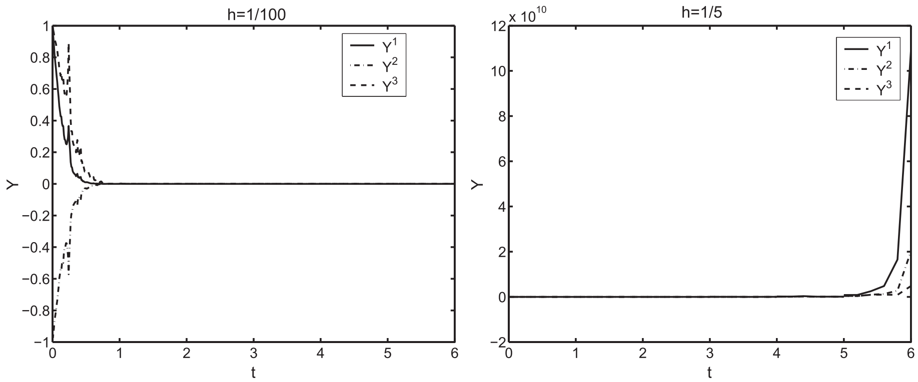

Fig. 5. The simulation for (4.4).

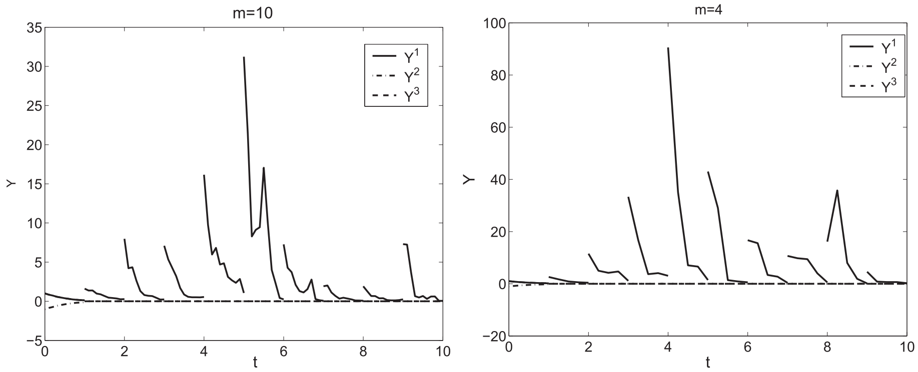

Fig. 6. The simulation for (4.5).

By calculating, $\mu_\infty[M] = -2, \|D_j\|_\infty = \frac{1}{9}e^2$ when j is odd, $\|D_j\|_\infty = 4e^2$ when j is even. Clearly the condition (2.7) does not hold for Eq. (4.5), but the condition (2.6) holds. It shows that Corollary 2.6 is more restrictive than Theorem 2.5. By calculating, we know that the condition (3.9) holds for $m = 10$ and $m = 4$. Fig. 6 gives a hard evidence. At the same, by calculating, we know that the condition (3.10) and condition (3.11) do not hold for $m = 4$. It shows that the case (2) in Theorem 3.6 is more restrictive than the case (1), the Corollary 3.7 is more restrictive than Theorem 3.6.

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