Modeling of time delays in numerical solution of an uncertain large-scale system

CHAPTER · DECEMBER 2013

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Abstract: An uncertain continuous-time system composed of N-coupled subsystems is derived including time delays. The problem definition uncovers some new features of the investigated biomechanical thorax-impactor system. The problem of time-varying position- and velocity-dependent uncertainties that are the particular mechanisms of modeling of the rheological connections of physical systems with time delays have been solved numerically. Rheological properties observed in connections between directly coupled rigid bodies (a point focused masses) of the model of mechanical idealization are introduced in two places: 1) If a relative displacement between front and back sides of the thorax ranges over d, then a parameter of stiffness doubles its value. Stiffness of the interior of the thorax increases at this condition about two times to approximate more adequately the real thorax's compression. 2) If a relative velocity measured between front and back sides of the thorax becomes negative, then a parameter of viscous damping doubles its value as well. Stiffness of the rheological coupling increases discontinuously with regard to a greater than d compression of the thorax. Damping ability of the coupling varies in time as the thorax is subject to a suitable compression or depression. Such discontinuity in the system's stiffness and damping vectors produce the position- and velocity-dependent time-varying uncertainties, in general. Therefore, a special attention during the following estimation has been paid. Numerical solutions have been presented on a few time history plots presenting the influence of the time delays.

1. Introduction

Numerical methods are today popular in biodynamics and, in particular, in dynamical analysis of communication accidents [6]. Generally, frontal impacts are considered to be the most common vehicle collision and causing many injuries [8]. When a human body is exposed to an impact load, soft tissues of the internal organs can sustain large stress and strain rate. To investigate the mechanical responses of the internal organs, sometimes complex modeling of the organs is required. Homogeneous and linear elasticity material properties are assigned to each part of the model, whereas the human cartilages and bones may have different material properties.

In this paper the problem of modeling of a multi-body biomechanical system of thorax is considered. On the basis of the theory of large-scale continuous-time systems an uncertain model of thorax has been written in the representation allowing to define its parametric uncertainties and complex interactions between its subsystems. The parametric uncertainties make the discontinuous system more difficult to solve, but it is compensated by more accurate numerical solution and a possibility of inclusion of time delays in the impulse responses.

The fact is that, if we consider an impulse response that takes in the analyzed thoratic response about 0.05 [s], then the time delays superposed on each of the system's body play an important role.

A large-scale dynamical system can be characterized by a large number of state variables, system parametric uncertainties, and a complex interaction between subsystems [7, 9]. In view of reliability and practical implementation, time delays have to be incorporated into the numerical modeling of the large-scale physical systems due to the real transport of mass, propagation of vibrations and computation times.

2. Biomechanical model of the idealized thorax supported from behind

An idealized biomechanical model of a thorax supported from behind has been depicted in Fig. 1. The concept bases on Lobdell model that was developed by General Motors to study the response of the human thorax in automobile crashes [4]. An application of the model has been presented in [2], where the model consisted of a configuration of springs and dashpot elements. Injury which results from the pressure wave released by an explosion is referred to as primary blast injury [10]. Primary blast injury most significantly affects the air-containing organs of the body [3]. In this contribution it has been idealized by a first body of mass m_1 (the impactor) springily attached to the second body of mass m_2 (front chest wall of the thorax) and applying a non-zero initial condition superposed on its velocity (see Fig. 1).

The Lobdell model was developed through measuring the thoracic response of a human subject to an impact load. Use of the Lobdell model has been extended by researchers to the field of blast protection, to predict the thoracic response to a blast wave [1].

In this paper the Lobdell model and associated investigations mentioned above have been extended by adding a support of the thorax from behind and in approximation of the real pressure wave by an elastically impacting solid body.

2.1. Formulation of the large-scale problem

Let us take into consideration a class of uncertain continuous-time system composed of N-coupled subsystems as follows

$$\frac{d\bar{x}_i(t)}{dt} = \big(\mathbf{A}_i + \Delta\mathbf{A}_{ii}(t)\big)\bar{x}_i(t) + \sum\nolimits_{j\neq i}^{N}\big(\mathbf{A}_{ij} + \Delta\mathbf{A}_{ij}(t)\big)\bar{x}_j(t-\tau_j)$$
$$+ \big(\mathbf{B}_i + \Delta\mathbf{B}_i(t)\big)\bar{u}_i(t)\ , \tag{1a}$$
$$\bar{y}_i(t) = \mathbf{C}_i\bar{x}_i(t) + \mathbf{D}_i\bar{u}_i(t), \quad i = 1,\ldots,N\ , \tag{1b}$$

where $\bar{x}_i(t) \in R^{n_i}$, $\bar{u}_i(t) \in R^{m_i}$, and $y_i(t) \in R^{l_i}$ denote, respectively, vectors of system states, control inputs, and system outputs. The dynamical system (1) of i coupled subsystems is described by the internal behavior time-independent state matrices $\mathbf{A}_i^{n_i\times n_i}$, while the control inputs matrix $\mathbf{B}_i^{n_i\times m_i}$, the system output matrix $\mathbf{C}_i^{l_i\times n_i}$ and the control inputs transition matrix $\mathbf{D}^{l_i\times m_i}$ are time-independent as well but represent connections between the external world and the system, τ_j is the time delay of j-th coupled system. It will be assumed in the investigations that control inputs do not directly influence the system outputs, therefore the matrix $\mathbf{D}_i^{l_i\times m_i}$ is zero. For the purpose of solution of the proposed problem there are introduced in Eq. (1) time-dependent matrices $\Delta\mathbf{A}_{ii}^{n_i\times n_i}(t)$, $\Delta\mathbf{A}_{ji}^{n_i\times n_i}(t)$ and $\Delta\mathbf{B}_i^{n_i\times m_i}$ that define, respectively, the system's state and control input uncertainties. It allows for a more or less precise inclusion of the system's parameters disturbances given in a form of known time-dependent function or in a quite different form of a dynamically dependent function on the internal system state (time- or state-varying properties are allowed to be modeled as well). Matrices $\Delta\mathbf{A}_{ji}^{n_i\times n_i}(t)$ represent all types of connections between interconnected subsystems of the entire system.

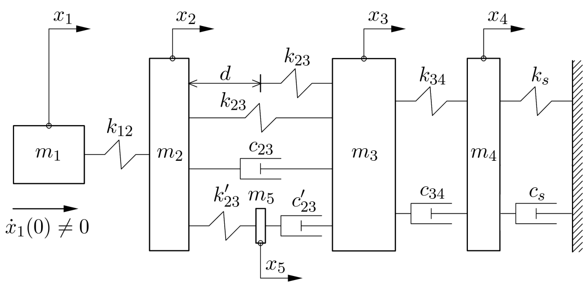


Figure 1. The biomechanical model of a thorax subject to an elastic impact

The biomechanical model of a thorax subject to an elastic impact has been shown in Fig. 1. From the left side: impactor's body of mass m_1, front body of mass m_2 in posterior surface of the chest wall, rear body of mass m_3 in anterior surface of the chest wall, m_4 –

mass of the support, m_5 – mass of the internal organs and tissues creating the rheological connections.

Equations (1) can be expanded to the following form

$$\begin{bmatrix}\dot{x}_{11}(t)\\ \dot{x}_{12}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{-k_{12}}{m_1} & 0\end{bmatrix}\begin{bmatrix}x_{11}(t)\\ x_{12}(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{k_{12}}{m_1} & 0\end{bmatrix}\begin{bmatrix}x_{21}(t-\tau_2)\\ x_{22}(t-\tau_2)\end{bmatrix}, \tag{2a}$$

$$\begin{aligned}\begin{bmatrix}\dot{x}_{21}(t)\\ \dot{x}_{22}(t)\end{bmatrix} = &\left(\begin{bmatrix}0 & 1\\ \frac{-k_{12}-\hat{k}_{23}-k'_{23}}{m_2} & \frac{-\hat{c}_{23}}{m_2}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{-k_{23}(t)}{m_2} & \frac{-c_{23}(t)}{m_2}\end{bmatrix}\right)\begin{bmatrix}x_{21}(t)\\ x_{22}(t)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ \frac{\hat{k}_{12}}{m_2} & 0\end{bmatrix}\begin{bmatrix}x_{11}(t-\tau_1)\\ x_{12}(t-\tau_1)\end{bmatrix}\\ &+ \left(\begin{bmatrix}0 & 0\\ \frac{\hat{k}_{23}}{m_2} & \frac{\hat{c}_{23}}{m_2}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{k_{23}(t)}{m_2} & \frac{c_{23}(t)}{m_2}\end{bmatrix}\right)\begin{bmatrix}x_{31}(t-\tau_3)\\ x_{32}(t-\tau_3)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ \frac{k'_{23}}{m_2} & 0\end{bmatrix}\begin{bmatrix}x_{51}(t-\tau_5)\\ x_{52}(t-\tau_5)\end{bmatrix},\end{aligned} \tag{2b}$$

$$\begin{aligned}\begin{bmatrix}\dot{x}_{31}(t)\\ \dot{x}_{32}(t)\end{bmatrix} = &\left(\begin{bmatrix}0 & 1\\ \frac{-\hat{k}_{23}-k_{34}}{m_3} & \frac{-\hat{c}_{23}-c'_{23}-c_{34}}{m_3}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{-k_{23}(t)}{m_3} & \frac{-c_{23}(t)}{m_3}\end{bmatrix}\right)\begin{bmatrix}x_{31}(t)\\ x_{32}(t)\end{bmatrix}\\ &+ \left(\begin{bmatrix}0 & 0\\ \frac{\hat{k}_{23}}{m_3} & \frac{\hat{c}_{23}}{m_3}\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{k_{23}(t)}{m_3} & \frac{c_{23}(t)}{m_3}\end{bmatrix}\right)\begin{bmatrix}x_{21}(t-\tau_2)\\ x_{22}(t-\tau_2)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ \frac{k_{34}}{m_3} & \frac{c_{34}}{m_3}\end{bmatrix}\begin{bmatrix}x_{41}(t-\tau_4)\\ x_{42}(t-\tau_4)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & \frac{c'_{23}}{m_3}\end{bmatrix}\begin{bmatrix}x_{51}(t-\tau_5)\\ x_{52}(t-\tau_5)\end{bmatrix},\end{aligned} \tag{2c}$$

$$\begin{bmatrix}\dot{x}_{41}(t)\\ \dot{x}_{42}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ \frac{-k_{34}-k_s}{m_4} & \frac{-c_{34}-c_s}{m_4}\end{bmatrix}\begin{bmatrix}x_{41}(t)\\ x_{42}(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{k_{34}}{m_4} & \frac{c_{34}}{m_4}\end{bmatrix}\begin{bmatrix}x_{31}(t-\tau_3)\\ x_{32}(t-\tau_3)\end{bmatrix}, \tag{2d}$$

$$\begin{aligned}\begin{bmatrix}\dot{x}_{51}(t)\\ \dot{x}_{52}(t)\end{bmatrix} = &\begin{bmatrix}0 & 1\\ \frac{-k'_{23}}{m_5} & \frac{-c'_{23}}{m_5}\end{bmatrix}\begin{bmatrix}x_{51}(t)\\ x_{52}(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ \frac{k'_{23}}{m_5} & 0\end{bmatrix}\begin{bmatrix}x_{21}(t-\tau_2)\\ x_{22}(t-\tau_2)\end{bmatrix}\\ &+ \begin{bmatrix}0 & 0\\ 0 & \frac{c'_{23}}{m_5}\end{bmatrix}\begin{bmatrix}x_{31}(t-\tau_3)\\ x_{32}(t-\tau_3)\end{bmatrix},\end{aligned} \tag{2e}$$

$$\begin{bmatrix}y_1(t)\\ \vdots\\ y_N(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}^T\begin{bmatrix}x_{11}(t) & \dots & x_{N1}(t)\\ x_{12}(t) & \dots & x_{N2}(t)\end{bmatrix}, \tag{2f}$$

where $x_{21}(t) - x_{31}(t)$ will be the controlled system output (the difference in displacements of bodies denoted by m_2 and m_3). In Eq. (2a) all the uncertainty matrices $\Delta A_{1j} = 0$ for $j = 1 \ldots N$, coupling matrices $A_{13} = A_{14} = A_{15} = 0$ (it means that there is no physical connection between bodies m_1 and m_j, for $j = 3 \ldots N$). By a similar description in (2b): $\Delta A_{21} = \Delta A_{24} = \Delta A_{25} = 0$, $A_{24} = 0$; in (2c): $\Delta A_{31} = \Delta A_{34} = \Delta A_{35} = 0$, $A_{31} = 0$; in (2d): $\Delta A_{3j} = 0$ for $j = 1 \ldots N$, $A_{41} = A_{42} = A_{45} = 0$, $B_4 \neq 0$, but $B_i = 0$ for $i = 1, 2, 3, 5$ (it means that any control force could be attached to the body m_4); in (2e): $\Delta A_{5j} = 0$ for $j = 1 \ldots N$, $A_{51} = A_{54} = 0$. The rest of terms take a non-zero values.

2.2. Uncertainties and the switching matrices

The problem definition uncovers some new features of the investigated biomechanical chest-impactor system. The problem of time-varying position- and velocity-dependent uncertainties that are the particular mechanisms of modeling of the rheological connections of physical systems with time delays have been solved numerically. Rheological properties observed in connections between directly coupled rigid bodies (a point focused masses) of the model of mechanical idealization are introduced in two places: 1) If a relative displacement $x_r(t) = x_{21}(t) - x_{31}(t)$ between front and back walls of the chest (represented by bodies m_2 and m_3, respectively) ranges over $d = 3.8$ [cm] then the stiffness parameter k_{23} doubles its value. Stiffness of interior of the thorax increases at this condition about two times to approximate more adequately the real chest's compression. 2) If a relative velocity $v_r(t) = x_{22}(t) - x_{32}(t)$ measured between front and back walls of the chest becomes negative then the parameter of viscous damping c_{23} doubles its value as well. It obviously means, that stiffness of the rheological coupling increases discontinuously with regard to a greater than d compression of the thorax x_r, and that damping ability of the coupling varies in time as the thorax is subject to a suitable compression or depression. Such discontinuity in the system's stiffness and damping vectors produce the position- and velocity-dependent time-varying uncertainties, in general. Therefore, a special attention during the following estimation has to be paid.

One can encounter in Eqs. (2b) and (2c) four $\Delta A_{ij}(t)$ parameter uncertainties of the analyzed dynamical system:

$$\Delta A_{ij}(t) = \begin{bmatrix} 0 & 0 \\ \dfrac{\sigma(i,j) k_{23}(t)}{m_i} & \dfrac{\sigma(i,j) c_{23}(t)}{m_i} \end{bmatrix} = D_i F(t) E_{ij} \ , \tag{3a}$$

$$\sigma(i,j) = -\mathrm{sgn}\big((-1)^{i+j}\big) \ , \tag{3b}$$

where $i, j = 2, 3$.

Distribution of entries in $\Delta A_{ij}(t)$ does not change over their all possibilities, therefore the unknown time-varying real matrix $F(t)$ is assumed to be defined in the same way, but $F^T(t)F(t) \leq I$ must hold for $t \in R^+$, I is the identity matrix. The introduced decomposition DFE includes some known constant real-valued matrices D_i and E_{ij} of appropriate dimensions that need to be estimated to use them, for instance, in a solution of LMI problems [5,7].

Dependently on $x_r(t)$ and $v_r(t)$ the following forms of the uncertainty matrix $\Delta A_{22}(t)$ in Eq. (2b) are possible:

$$\Delta A_{22}(t) = \Delta A_{22}^{(k)} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } s_1 \text{ then } k = 1 \ , \\ \begin{bmatrix} 0 & 0 \\ \frac{-\bar{k}_{23}}{m_2} & 0 \end{bmatrix} & \text{if } s_2 \text{ then } k = 2 \ , \\ \begin{bmatrix} 0 & 0 \\ 0 & \frac{-\bar{c}_{23}}{m_2} \end{bmatrix} & \text{if } s_3 \text{ then } k = 3 \ , \\ \begin{bmatrix} 0 & 0 \\ \frac{-\bar{k}_{23}}{m_2} & \frac{-\bar{c}_{23}}{m_2} \end{bmatrix} & \text{if } s_4 \text{ then } k = 4 \ , \end{cases} \tag{4}$$

where the remaining $\Delta A_{ij}(t)$, for $i, j = 2, 3$ are to be defined in a similar way, $s_k = \{x_r(t), v_r(t) : x_r(t) < d \wedge v_r(t) > 0; \, x_r(t) \geq d \wedge v_r(t) > 0; \, x_r(t) < d \wedge v_r(t) \leq 0; \, x_r(t) \geq d \wedge v_r(t) \leq 0\}$ defines rheological properties of the biomechanical system. Switching conditions s_i will select only one of k possibilities $\Delta A_{ij}^{(k)}$ for $k = 1 \dots 4$ dependently on values of pairs $(x_r(t), v_r(t))$ creating the discontinuous time history of uncertainties $\Delta A_{ij}(t)$ of the state matrices of coupled two degrees-of-freedom neighboring subsystems.

To find better description of the existing switching nature it is now required to choice D_i and E_{ij} matrices of a decomposition. For instance, an exemplary decomposition of $\Delta A_{22}^{(k)}$, for $k = 3$ could be made accordingly to the following scheme.

Matrix $F(t)$ will depend on k cases that have been delivered in the case statement (4). Therefore, with regard to $F^T(t)F(t) \leq I$ let us assume $F^{(3)T}F^{(3)} = \gamma I$ that holds for $k = 3$ and $\gamma \leq 1$, then

$$F^{(3)T}F^{(3)} = \begin{bmatrix} f_1 & f_3 \\ f_2 & f_4 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix} ,$$

$$\begin{cases} f_1^2 + f_3^2 = \gamma, \\ f_2^2 + f_4^2 = \gamma, \\ f_1 f_2 + f_3 f_4 = 0 \ . \end{cases} \tag{5}$$

In next step, expansion of Eq. (3) holds

$$\Delta A_{22}^{(3)} = \begin{bmatrix} d_1 & d_2 \\ d_3 & d_4 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} \begin{bmatrix} e_1 & e_2 \\ e_3 & e_4 \end{bmatrix} = \begin{bmatrix} p_1e_1 + p_2e_3 & p_1e_2 + p_2e_4 \\ p_3e_1 + p_4e_3 & p_3e_2 + p_4e_4 \end{bmatrix} , \tag{6}$$

where: $p_1 = d_1f_1 + d_2f_3$, $p_2 = d_1f_2 + d_2f_4$, $p_3 = d_3f_1 + d_4f_3$, $p_4 = d_3f_2 + d_4f_4$. Comparison of Eq. (6) with $\Delta A_{22}^{(3)}$ in Eq. (4) yields

$$p_1e_1 + p_2e_3 = 0 , \tag{7a}$$

$$p_1e_2 + p_2e_4 = 0 , \tag{7b}$$

$$p_3e_1 + p_4e_3 = 0 , \tag{7c}$$

$$p_3e_2 + p_4e_4 = \beta_4 , \tag{7d}$$

where β_4 represents a non-zero entry of the decomposed matrix $\Delta A_{22}^{(3)}$, while other entries of the matrix are equal to zero. Let us reduce number of equations (7). Putting e_1 from (7a) to (7c) and e_2 from (7b) to (7d) one finds $e_3\pi = 0$, and it will be satisfied if $e_3 = 0$ or $\pi = 0$, but with regard to $e_4\pi = \beta_4$, (where $\beta_4 \neq 0$ and $\pi = p_4 - p_3p_2/p_1$) π and e_4 cannot be zero, so $e_3 = 0$ must be set. After that assumption one gets $e_1 = 0$, and in a consequence, Eq. (7a) and (7c) vanish. Two equations remaining in (7) can be rewritten

$$d_1\phi_1 + d_2\phi_2 = 0 , \tag{8a}$$

$$d_3\phi_1 + d_4\phi_2 = \beta_4 , \tag{8b}$$

where:

$$\phi_1 = f_1e_2 + f_2e_4 , \tag{9a}$$

$$\phi_2 = f_3e_2 + f_4e_4 . \tag{9b}$$

The two cases can be distinguished: if $\phi_1 = 0$, then from (8b) $\phi_2 \neq 0$, so $d_2 = 0$ to satisfy Eq. (8a); if $\phi_2 = 0$, then from (8b) $\phi_1 \neq 0$, so $d_1 = 0$ to satisfy (8a). Let us choice the first case, then $d_4 = \beta_4/\phi_2$ for $\phi_2 \neq 0$.

Accordingly to Eqs. (8), $\phi_2 = \beta_4d_1/(d_1d_4 - d_2d_3)$. Selecting $d_3 = 0$, $\phi_2 = \beta_4/d_4$ is confirmed while $d_4 \neq 0$, and $\phi_1 = d_2 = 0$ in (8a), so let $d_1 = 0$ be arbitrarily set.

Having the above derived let $f_3 = 0$ in Eq. (5), then $f_1 = f_4 = \pm\gamma$ and $f_2 = 0$. Now, one writes from Eq. (8b) that $d_4f_4e_4 = \beta_4$ at $\phi_1 = 0$. Putting $d_4 = \delta_4 \neq 0$, one gets $e_4 = \pm\beta_4/\gamma$. Finally, $e_2 = 0$ to satisfy Eq. (9a), and the decomposition (6) finds the following continuation

$$\Delta A_{22}^{(3)} = \begin{bmatrix} 0 & 0 \\ 0 & \delta_4 \end{bmatrix} \begin{bmatrix} \pm\gamma & 0 \\ 0 & \pm\gamma \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \pm\dfrac{\beta_4}{\gamma} \end{bmatrix} . \tag{10}$$

It is possible to obtain for $k = 2$ and $k = 4$ the remaining cases of the uncertainty matrix $\Delta A_{22}^{(k)}$ (see Eq. (4)) in a similar way getting as below

$$\Delta A_{22}^{(2)} = \begin{bmatrix} 0 & 0 \\ 0 & \delta_4 \end{bmatrix} \begin{bmatrix} 0 & \pm\gamma \\ \pm\gamma & 0 \end{bmatrix} \begin{bmatrix} \pm\dfrac{\beta_1}{\gamma} & 0 \\ 0 & 0 \end{bmatrix} , \tag{11a}$$

$$\Delta A_{22}^{(4)} = \begin{bmatrix} 0 & 0 \\ 0 & \delta_4 \end{bmatrix} \begin{bmatrix} 0 & \pm\gamma \\ \pm\gamma & \pm\gamma \end{bmatrix} \begin{bmatrix} \pm\dfrac{\beta_1}{\gamma} & 0 \\ 0 & \pm\dfrac{\beta_4}{\gamma} \end{bmatrix} . \tag{11b}$$

Using Eq. (3) and generalizing Eqs. (10) and (11) on four uncertainty matrices the following formula reads

$$\Delta A_{i,j}^{(k)} = D_i F^{(k)} E_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & \delta_{4i} \end{bmatrix} F^{(k)} \begin{bmatrix} \dfrac{\sigma(i,j)\beta_1}{\gamma} & 0 \\ 0 & \dfrac{\sigma(i,j)\beta_4}{\gamma} \end{bmatrix}, \quad k = 1 \ldots 4 , \tag{12}$$

where $i, j = 2, 3$, $\sigma(i, j)$ is given in Eq. (3b), and $F(t)$ will switch between

$$F^{(k)} = \begin{cases} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } s_1 \text{ then } k = 1 , \\ \begin{bmatrix} 0 & \pm\gamma \\ \pm\gamma & 0 \end{bmatrix} & \text{if } s_2 \text{ then } k = 2 , \\ \begin{bmatrix} \pm\gamma & 0 \\ 0 & \pm\gamma \end{bmatrix} & \text{if } s_3 \text{ then } k = 3 , \\ \begin{bmatrix} 0 & \pm\gamma \\ \pm\gamma & \pm\gamma \end{bmatrix} & \text{if } s_4 \text{ then } k = 4 . \end{cases} \tag{13}$$

The case statement (13) captures switching properties of $\Delta A_{ij}(t)$ described in comments to (4). As it was expected, $F(t)$ is defined in the same way for four uncertainties of the model, matrices D_i and E_{ij} are constant and their entries will depend on the subsystem that they are related to.

To check correctness of the previously made estimations let the following parameters be assigned for all uncertainties: $\delta_{42} = 1/m_2$, $\delta_{43} = 1/m_3$, $\beta_1 = k_{23}$, $\beta_4 = c_{23}$, $\gamma \neq 0$. For example, $\Delta A_{2,2}^{(4)} = D_2 \bar{F}^{(4)} E_{22} = [[0, 0], [0, 1/m_2]] \cdot [[0, \gamma], [\gamma, \gamma]] \cdot [[-k_{23}/\gamma, 0], [0, -c_{23}/\gamma]] = [[0, 0], [-k_{23}/m_2, -c_{23}/m_2]]$ what is in agreement with the fourth case of Eq. (4).

The last condition to secure reads: $F^{(k)T} F^{(k)} \leq I \iff \xi^{(k)} = F^{(k)T} F^{(k)} - I \leq 0$ for $k = 1 \ldots 4$. The $n \times n$ real symmetric matrix $\xi^{(k)}$ is negative semi-definite if $z^T \xi^{(k)} z \leq 0$, for all non-zero vectors $z \in R^n$, where z^T denotes the transpose of z. One could check that $\xi^{(k)}$ is symmetric and in particular: $\xi^{(1)} = 0$, $\xi^{(2)} = \xi^{(3)} = (\gamma^2 - 1)(z_1^2 + z_2^2)$, $\xi^{(4)} = \gamma^2(z_1^2 + 2z_1 z_2 + 2z_2^2) - z_1^2 - z_2^2$, and then, for $\xi^{(k)} \leq 0$ the following bounds are determined $\{\gamma : -1 \leq \gamma \leq 1, |\gamma| \leq (z_1^2 + z_2^2)^{1/2}(z_1^2 + 2z_1 z_2 + 2z_2^2)^{-1/2}\}$. It is seen that only the fourth condition on $\xi^{(4)}$ is sensitive on selection of γ parameter. For the case of presence of two entries in the bottom row of E_{ij} one would require to accomplish the task: maximize γ subject to $\xi^{(4)}(z_1, z_2) \leq 0$, $z_1 \neq 0$, $z_2 \neq 0$. The maximum value of parameter $\gamma^* = 0.618033$ was estimated numerically and the corresponding $\xi^{(4)}(z_1, z_2)$ surface plot shown in Fig. 2c.

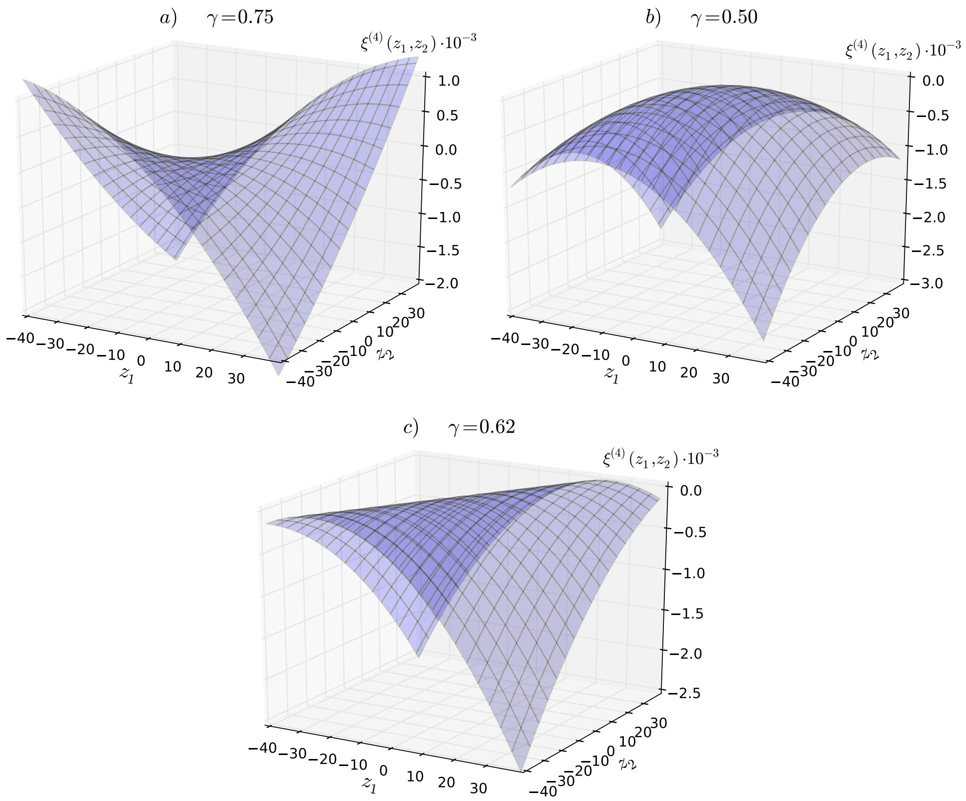
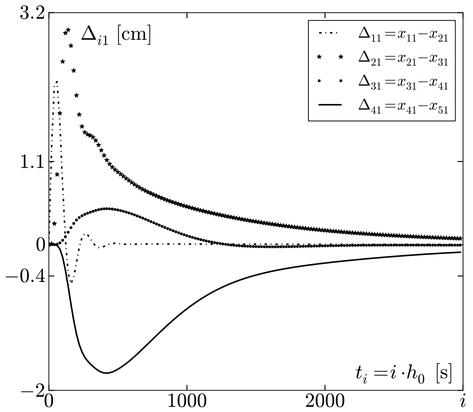


Figure 2. γ-parameter dependent $\xi^{(4)}(z_1, z_2)$ plots.

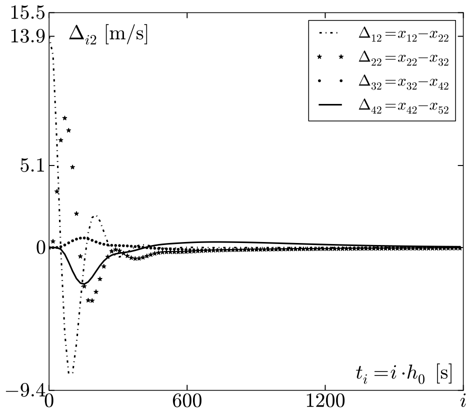
The γ estimate belongs to $-\gamma^* \leq \gamma \leq \gamma^*$, but to achieve the similar decomposition the presented derivation could follow another way. To sum up, the results are useful in numerical integration of a class of discontinuous state large-scale dynamical systems. The switching nature of the system parameters defined by a set of matrices $F^{(k)}$ must be assumed if an exact numerical modeling of the investigated multi-body system has to be achieved.

3. Numerical solution of the investigated discontinuous dynamical system

The time-delayed and not time-delayed impulse responses of the multi-body mechanical system have been presented in Fig. 3 and 4. There are shown trajectories of the relative differences Δ_i of the corresponding state variables. There is shown in Fig. 4 an influence of time delay of each state variable on the relative differences in displacement (a) and velocities (b) of bodies denoted by m_2 and m_3 being the representatives of the posterior and anterior walls of the chest model.



(a)



(b)

Figure 3. Trajectories of the relative differences Δ_i of the corresponding state variables of the investigated dynamical system. Model parameters: $m_1 = 1.6$, $m_2 = 0.45$, $m_3 = 27.21$, $m_4 = 10$, $m_5 = 0.1$ [kg], $d = 0.0381$ [m], $\bar{k} = [281, 26.3, 13.2, 50, 10] \cdot 10^3$ [N/m], $\bar{c} = [0.52, 0.18, 0.11, 1] \cdot 10^3$ [Ns/m]. Zero initial conditions are assumed except the impactor's velocity $x_{12} = 13.9$ [m/s]. Index i denotes number of iterations of the numerical integration procedure

It is clearly visible in Fig. 4, that at the time delays $\bar{\tau}_k = \{0, 10, 20, 30, 40\} \cdot h_0$ (for $k = 1 \dots 5$, $h_0 = 0.00005$) assumed for each body with regard to the zero time delay of the impactor, the time trajectories of solutions existing in the system without (dotted line) and with the time delay (solid line) diverge from each other quite rapidly. This observation points to the need to take into account some time delays characterizing most multidimensional systems.

It is also reasonable in practice, because if one would imagine a real impact, when the impactor at a time of contact with the posterior surface (front chest wall) hits the thorax then in the anterior surface the rear chest wall experiences the shift with a time delay dependent on biophysical parameters of the system and the speed of propagation of the blast wave.

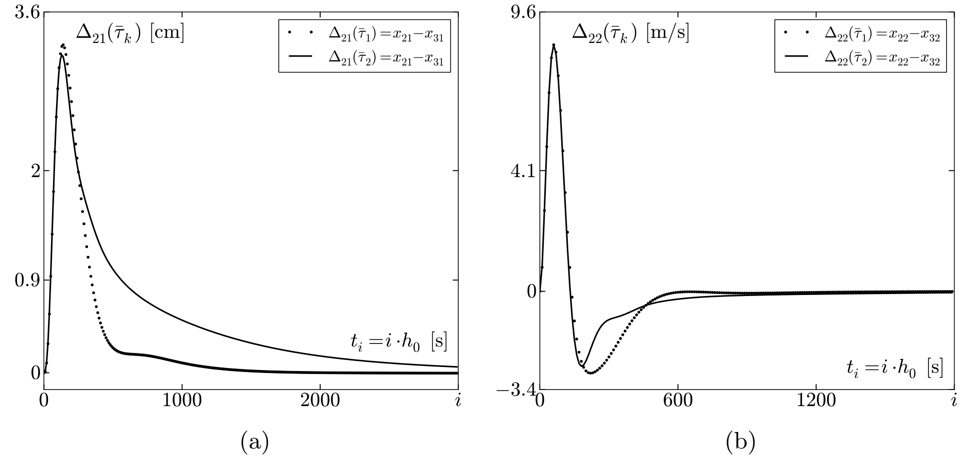


Figure 4. Influence of the time delay $\bar{\tau}$ of each state variable on the relative differences in displacements (a) and velocities (b) between bodies m_2 and m_3

4. Conclusions

Numerical solution of differential equations with delays shown in the extended representation of a linear system of five interconnected subsystems with time-varying parameters significantly increases the accuracy of the simulation of complex physical phenomena. For example, in a numerical procedure not taking into account the time lag between connected subsystems, states x_{21} and x_{22} describing the dynamics of the second sub-system are updated already in the first iteration. As from the second sub-system (the front body of the chest wall) receiving the impact, the subsystem representing the support will take into account the excitation of force as early as the fourth iteration, i.e., in the step where the state variables x_{41} and x_{42} of the rear body's subsystem adopt non-zero values. Therefore, one can conclude that the explicit omission of time delays in the mathematical description and in the numerical procedure, consequently, causes a prior assumption of the shortest possible time delay of the variables describing the state of dynamic inertial objects. After inclusion of time delays in motion of each body being a part of the biomechanical system, the dynamic thoratic response took about 0.05 [s], and the time delays have influenced the solution dramatically.

Acknowledgments

The authors have been supported by the National Center of Science under the grant MAESTRO 2, No. 2012/04/A/ ST8/00738 for years 2012-2015 (Poland).

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