Dichotomies for Linear Evolutionary Equations in Banach Spaces*

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In this paper we present a characterization for the existence of an exponential dichotomy for a linear evolutionary system on a Banach space. The theory we present here applies to general time-varying linear equations in Banach spaces. As a result it gives a description of the behavior of the nonlinear dynamics generated by certain nonlinear evolutionary equations in the vicinity of a compact invariant set. In the case of dissipative systems, our theory applies to the study of the flow in the vicinity of the global attractor. The theory formulated here holds for linear evolutionary systems which are uniformly α-contracting and applies to the study of the linearization of nonlinear equations of the following type: (a) parabolic PDEs, including systems of reaction diffusion equations and the Navier–Stokes equations; (b) hyperbolic PDEs, including the nonlinear wave equation and the nonlinear Schrödinger equation with dissipation; (c) retarded differential equations; and (d) certain neutral differential delay equations.   © 1994 Academic Press, Inc.

1. INTRODUCTION

The theory of exponential dichotomies has played a central role in the study of ordinary differential equations and diffeomorphisms for finite dimensional dynamical systems. This theory, which addresses the issue of

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strong transversality in dynamical systems, originated in the pioneering works of Lyapunov (1892) and Poincaré (1890), who studied the local behavior of a nonlinear dynamical system in the vicinity of a fixed point and a periodic orbit. In particular, the Lyapunov–Perron method, which was introduced by Lyapunov and developed extensively by Perron (1928, 1930), is based on a theory of exponential dichotomies.

In recent years, the theory of exponential dichotomies has proven to be a useful method for studying the stable, unstable, and center manifolds (see Hale, 1969; Henry, 1981, 1985; Kelley, 1967a, 1967b; Pilyugin, 1988; Pliss, 1964, 1977; Sell, 1978; Stokes, 1971; van Gils and Vanderbauwhede, 1987; and Vanderbauwhede, 1989), perturbation theories (see Coppel, 1965, 1967, 1968, 1978; Palmer, 1984, 1987; Pliss, 1977; and Pliss and Sell, 1991), linearization theories (see Sell, 1984, 1985), bifurcation theory (see Braaksma and Broer, 1987; Chow and Hale, 1982; Chow and Mallet-Paret, 1977; Flockerzi, 1984; and Sell, 1979), homoclinic behavior (see Meyer and Sell, 1989; and Palmer, 1984), and the spectral theory of a compact invariant set (see Magalhães, 1987; and Sacker and Sell, 1978, 1980), as well as the theory of inertial manifolds (see Foias, Sell, and Temam, 1988; and Foias, Sell, and Titi, 1989).

During the last few years one finds an ever growing use of exponential dichotomies to study the dynamical structures of various partial differential equations and differential delay equations; see for example Foias, Sell, and Temam (1988), Hale (1988), Henry (1981), and Temam (1988). Because of this one wants to find necessary and sufficient conditions for the occurrence of exponential dichotomies for linear evolutionary systems on an infinite dimensional Banach space. The primary purpose of this paper is to present a theory which addresses these issues.

Our main goal is to be able to analyze the structure of a nonlinear semi-flow in the vicinity of some compact, invariant set $\mathscr{X}$. Because of the invariance, this implies that for every $x \in \mathscr{X}$ there is a global solution $\varphi(t)$, which satisfies $\varphi(t) \in \mathscr{X}$ for all $t \in R$ and $\varphi(0) = x$. Even though the associated linearized evolutionary systems over $\mathscr{X}$ will generate solutions for $t \geqslant 0$, this linear system has time-varying coefficients which are defined for all $t \in R$. In addition to this feature, the linear evolutionary systems studied here will be assumed to have some compactness or asymptotic smoothing property.

The theory of necessary and sufficient conditions for the existence of exponential dichotomies for finite dimensional linear differential systems appears in Sacker and Sell (1974, 1976a). Our goal here can be succinctly stated to be the generalization of the finite dimensional theory to infinite dimensional linear evolutionary systems. As we shall see, this generalization is possible, but it is far from being trivial. The main difficulty one encounters is that *some*, but *not all* solutions of the linear evolutionary system

have extensions defined for $t \leqslant 0$. Also it can happen in some systems that these negative extensions need not be unique.

The starting point for both the finite and infinite dimensional theories is to assume that the given linear evolutionary system and all limits of translates of that system have the property that the only solution ϕ which is bounded for all $t \in R$ is $\phi(t) \equiv 0$. One of our main results is the Alternative Theorem which states that, under the assumption of weak hyperbolicity (which is defined later and which includes the above property), either the linear evolutionary system admits a global exponential dichotomy or the flow on the space of coefficients has a specific gradient-like structure. One step in the proof is to show that one always has an exponential dichotomy over every minimal set. In the Compatibility Theorem we show that if there is an integer $k \geqslant 0$ such that the unstable manifold over every minimal set has dimension k, then the linear evolutionary system admits a global exponential dichotomy.

In the next section we give the definition of the linear evolutionary systems we study in this paper. In Section 3 we present the concept of an exponential dichotomy and we give the precise statement of the theorems to be proved here. Some illustrative examples are presented in Section 4. In Sections 5 and 6 we discuss some of the basic properties of the dynamics of linear evolutionary systems, and the proofs of the theorems are given in Section 7.

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2. Linear Evolutionary Systems

The object of study in this paper is the theory of exponential dichotomies for linear time-varying evolutionary equations on a Banach space. These equations, which typically arise when one linearizes a nonlinear evolutionary equation along solutions lying in a given compact invariant set, give rise to a linear skew-product dynamical system; see Sacker and Sell (1974, 1976a). We call these systems linear evolutionary systems. Let us now be more precise.

Let $\mathscr{E} = X \times \Theta$ be given where X is a fixed Banach space (the state space) and Θ is a compact Hausdorff space. Assume that $\sigma(\theta, t) = \theta \cdot t$ is a (two-sided) flow on Θ; i.e., the mapping $(\theta, t) \to \theta \cdot t$ is continuous, $\theta \cdot 0 = \theta$, and one has $\theta \cdot (s + t) = (\theta \cdot s) \cdot t$ for all $s, t \in R$. A *linear evolutionary system* $\pi = (\Phi, \sigma)$ on $\mathscr{E}$ is a mapping

$$\pi(x, \theta, t) = (\Phi(\theta, t) x, \theta \cdot t), \qquad \text{for} \quad t \geqslant 0$$

with the following four properties:

(1) $\Phi(\theta, 0) = I$, the identity operator, for all $\theta \in \Theta$.

(2) $\lim_{t \to 0^+} \Phi(\theta, t) x = x$, and this limit is uniform in θ. This means that for every $x \in X$ and every $\varepsilon > 0$ there is a $\delta = \delta(x, \varepsilon) > 0$ such that $|\Phi(\theta, t) x - x| \leqslant \varepsilon$, for all $\theta \in \Theta$ whenever $0 \leqslant t \leqslant \delta$.

(3) $\Phi(\theta, t)$ is a bounded linear mapping from X into X that satisfies the cocycle identity:

$$\Phi(\theta, s + t) = \Phi(\theta \cdot t, s) \, \Phi(\theta, t), \qquad \theta \in \Theta, \quad 0 \leqslant s, t. \tag{2.1}$$

(4)   For each $t \geqslant 0$ the mapping of $\mathscr{E}$ into X given by

$$(x, \theta) \to \Phi(\theta, t) x$$

is continuous.

Properties (2) and (3) imply that for each $(x, \theta) \in \mathscr{E}$ the solution operator $t \to \Phi(\theta, t) x$ is right-continuous for $t \geqslant 0$. Indeed, one has

$$|\Phi(\theta, t + h) x - \Phi(\theta, t) x| = |[\Phi(\theta \cdot t, h) - I] \, \Phi(\theta, t) x|,$$

which goes to 0 as $h \to 0^+$.

One can have linear evolutionary systems which are not left-continuous for $t > 0$. Here is an example. Let Φ satisfy $\Phi(\theta, t) = I$ for $0 \leqslant t < 1$ and $\Phi(\theta, 1) = L(\theta)$, where $\theta \to L(\theta)$ is a continuous mapping of Θ into the space of bounded linear operators on X. For $n = 2, 3, \ldots$ we define $\Phi(\theta, n)$ by induction:

$$\Phi(\theta, n) = \Phi(\theta \cdot (n-1), 1) \, \Phi(\theta, n-1)$$
$$= L(\theta \cdot (n-1)) \, L(\theta \cdot (n-2)) \cdots L(\theta).$$

Now set $\Phi(\theta, n + \tau) = \Phi(\theta, n)$ for $0 \leqslant \tau < 1$. Then $\pi = (\Phi, \sigma)$ is a linear evolutionary system on $\mathscr{E}$.

The prototypical example of a linear evolutionary system arises in the finite dimensional setting, wherein $\Phi(\theta, t)$ is the fundamental operator solution of the linear ordinary differential equation

$$x' = A(\theta \cdot t) x \tag{2.2}$$

that satisfies $\Phi(\theta, 0) = I$, where $A(\theta)$ is a continuous $(n \times n)$ matrix-valued function defined on Θ. As we show in Section 4, one can construct similar examples in the infinite dimensional setting.

We use the Kuratowski functional α, which is a measure of noncompactness. Let X be a Banach space and let A be a bounded set in X. We define

$$\alpha(A) \overset{\text{def}}{=} \inf\{d: A \text{ has a finite covering with open sets of diameter } < d\}.$$

The Kuratowski functional satisfies the following properties:

(A) $\alpha(A) = 0$ if and only if Cl A is compact.

(B) $\alpha(A \cup B) = \max[\alpha(A), \alpha(B)]$.

(C) $\alpha(A + B) \leqslant \alpha(A) + \alpha(B)$.

(D) $\alpha(\text{Cl } A) = \alpha(A)$.

(E) $\alpha(A) \leqslant \alpha(B)$ whenever $A \subset B$.

(F) $\alpha(A) \leqslant \text{diam } A$.

(G) Let A_t be a family of nonempty closed bounded sets defined for $t_0 < t < \infty$ with $A_t \subset A_s$ whenever $s \leqslant t$. If $\alpha(A_t) \to 0$ as $t \to \infty$, then $\bigcap_{t > t_0} A_t$ is nonempty and compact.

(H) If $L: X \to X$ is a bounded linear operator, then $\alpha(LA) \leqslant \|L\| \, \alpha(A)$.

See Deimling (1985), Hale (1988), Martin (1976), and Sadovskii (1972).
For any subset $\mathcal{F} \subset \mathcal{E}$ we define the fiber

$$\mathcal{F}(\theta) \stackrel{\text{def}}{=} \{x \in X : (x, \theta) \in \mathcal{F}\}, \qquad \theta \in \Theta.$$

Thus $\mathcal{E}(\theta) = X \times \{\theta\}$ is the fiber of $\mathcal{E}$ over the point $\theta \in \Theta$. If $U \subset \Theta$, then we define

$$\mathcal{F}(U) \stackrel{\text{def}}{=} \bigcup_{\theta \in U} \mathcal{F}(\theta).$$

Also we define $\mathcal{E}_0 = \{(x, \theta) \in \mathcal{E} : x = 0\}$ and $\mathcal{E}_M = \{(x, \theta) \in \mathcal{E} : |x| \leqslant M\}$. If $\mathcal{F} \subset \mathcal{E}_M$ for some M, $0 \leqslant M < \infty$, we define

$$\alpha_\infty(\mathcal{F}) = \sup\{\alpha(\mathcal{F}(\theta)): \theta \in \Theta\}. \tag{2.3}$$

The following lemma, which is a variation of statement (G) above, is easily verified.

2.1. LEMMA. *Let $\mathcal{F}_t$ be a family of nonempty closed sets defined for $t_0 < t < \infty$ that satisfy $\mathcal{F}_t \subset \mathcal{E}_M$ for some M, $0 \leqslant M < \infty$, and $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $s \leqslant t$. If $\alpha_\infty(\mathcal{F}_t) \to 0$ as $t \to \infty$, then $\bigcap_{t > t_0} \mathcal{F}_t$ is a nonempty compact set in $\mathcal{E}_M$.*

A linear evolutionary system $\pi = (\Phi, \sigma)$ is said to be *uniformly α-contracting* if for every bounded set $B = B_M = \{x \in X : |x| \leqslant M\}$ there is a function k with $k(t) \to 0$ as $t \to \infty$ and such that

$$\alpha(\Phi(\theta, t) B) \leqslant k(t) \, \alpha(B), \qquad \theta \in \Theta.$$

If π is uniformly α-contracting, then it follows that $\alpha_\infty(\pi(\mathscr{F}, t)) \leqslant k(t) \alpha_\infty(\mathscr{F})$, where α_∞ is defined by (2.3).

Remarks. 1. Examples of uniformly α-contracting linear evolutionary systems are given in Hale (1988). A typical situation occurs when $\Phi(\theta, t)$ maps bounded sets into compact sets for $t > T$, where $T \geqslant 0$. Another example occurs when $\Phi = \Phi_c + \Phi_s$, where Φ_c maps bounded sets into compact sets for $t > T$ and $|\Phi_s(\theta, t)| \leqslant Ke^{-\beta t}$ for $t \geqslant 0$, where $\beta > 0$ and $K \geqslant 1$ are independent of θ.

2. The theory we present here extends easily to Banach bundles which are locally product spaces. A *Banach bundle* $\mathscr{E}$ with fiber X over a base space Θ with projection p is denoted by $(\mathscr{E}, X, \Theta, p)$, or $\mathscr{E}$ for short, and is defined as follows:

(1) X is a fixed Banach space and Θ is a compact Hausdorff space.

(2)   The mapping $p: \mathscr{E} \to \Theta$ is a continuous mapping.

(3)   For each $\theta \in \Theta$, $p^{-1}(\theta) = \mathscr{E}(\theta)$ is a Banach space, which is referred to as the fiber over θ.

(4)   For each $\theta \in \Theta$, there is an open neighborhood U of θ in Θ and a homeomorphism $\tau: p^{-1}(U) \to X \times U$ such that for each $\eta \in U$, $p^{-1}(\eta)$ is mapped onto $X \times \{\eta\}$ and $\tau: p^{-1}(\eta) \to X \times \{\eta\}$ is a linear isomorphism.

(5)   The norms $|\cdot| = |\cdot|_\theta$ on the fiber $p^{-1}(\theta)$ vary continuously in θ.

One can use the local coordinate notation (x, θ) to denote a typical point in a Banach bundle $\mathscr{E}$. By this we mean that x is an element in the fiber $\mathscr{E}(\theta)$. This is a shortened way to refer to property (4) above.

3. Notation: Statement of Main Theorems

3.1. *Projectors and Subbundles*

Let X denote a given Banach space and let Θ denote a compact Hausdorff space. Set $\mathscr{E} = X \times \Theta$.

A mapping $\mathbf{P}: \mathscr{E} \to \mathscr{E}$ is said to be a *projector* if $\mathbf{P}$ is continuous and has the form $\mathbf{P}(x, \theta) = (P(\theta) x, \theta)$ where $P(\theta)$ is a bounded linear projection on the fiber $\mathscr{E}(\theta)$. For any projector $\mathbf{P}$ we define the *range* and *null space* by

$$\mathscr{R} = \mathscr{R}(\mathbf{P}) = \{(x, \theta) \in \mathscr{E} : P(\theta) x = x\} \qquad \text{and}$$

$$\mathscr{N} = \mathscr{N}(\mathbf{P}) = \{(x, \theta) \in \mathscr{E} : P(\theta)x = 0\}.$$

Since $\mathbf{P}$ is continuous, this means that the fibers $\mathscr{R}(\theta)$ and $\mathscr{N}(\theta)$ vary continuously in θ. This also means that $P(\theta)$ varies continuously in the operator norm. The following result is elementary.

3.1. LEMMA. *Let* **P** *be a projector on* $\mathscr{E}$. *Then* $\mathscr{R}$ *and* $\mathscr{N}$ *are closed subsets in* $\mathscr{E}$, *and one has*

$$\mathscr{R}(\theta) \cap \mathscr{N}(\theta) = \{0\}, \qquad \mathscr{R}(\theta) + \mathscr{N}(\theta) = \mathscr{E}(\theta)$$

for all $\theta \in \Theta$.

A subset $\mathscr{V}$ is said to be a *subbundle* of $\mathscr{E}$ if there is a projector **P** on $\mathscr{E}$ with the property that $\mathscr{R}(\mathbf{P}) = \mathscr{V}$. In this case $\mathscr{W} = \mathscr{N}(\mathbf{P})$ is a *complementary* subbundle, i.e., $\mathscr{E} = \mathscr{V} + \mathscr{W}$ as a Whitney sum of subbundles. The following lemma is a useful for determining whether a set $\mathscr{V}$ is a subbundle.

3.2. LEMMA. *Let* $\mathscr{V} \subset \mathscr{E}$ *have the following properties*:

(1) $\mathscr{V}$ *is closed*,

(2) $\mathscr{V}(\theta)$ *is a linear subspace of* $\mathscr{E}(\theta)$ *for all* $\theta \in \Theta$,

(3)   codim $\mathscr{V}(\theta)$ *is finite for all* $\theta \in \Theta$,

(4)   codim $\mathscr{V}(\theta)$ *is locally constant on* Θ.

Then $\mathscr{V}$ *is a subbundle of* $\mathscr{E}$.

Proof. First note that one can find a finite collection $\theta_1, ..., \theta_n \in \Theta$ and an open covering $U_1, ..., U_n$ of Θ such that

(a) $\theta_i \in U_i$ for $1 \leqslant i \leqslant n$;

(b)   codim $\mathscr{V}(\theta) = \mathscr{V}(\theta_i)$ for $\theta \in \bar{U}_i$, where $\bar{U}_i$ is the closure of U_i;

(c)   there is a complementary space $\mathscr{X}(\theta_i)$ to $\mathscr{V}(\theta_i)$ such that $\mathscr{X}(\theta_i) \cap \mathscr{V}(\theta) = \varnothing$ for all $\theta \in \bar{U}_i$ for $1 \leqslant i \leqslant n$.

It then follows that $\mathscr{X}(\theta_i)$ is a complementary space for $\mathscr{V}(\theta)$ for all $\theta \in \bar{U}_i$. Let $\Re_i = \Re_i(\alpha)$ be the cone in X given by

$$\Re_i = \{x \in X : |v| \leqslant \alpha\, |u|, v \in \mathscr{V}(\theta_i), u \in \mathscr{X}(\theta_i)\}$$

where $\alpha > 0$. Since the sets U_i, $1 \leqslant i \leqslant n$, and α can be made smaller if necessary, there is no loss in generality in assuming that for $1 \leqslant i \leqslant n$ one has

$$\mathscr{V}(\theta) \cap \Re_i = \{0\}, \qquad \text{for all} \quad \theta \in \bar{U}_i. \tag{3.1}$$

A linear projection $P_i(\theta)$ on X is then defined for $\theta \in U_i$ by setting the range $\mathscr{R}(\theta) = \mathscr{V}(\theta)$ and the null space $\mathscr{N}(\theta) = \mathscr{X}(\theta_i)$. Let $Q_i(\theta) = I - P_i(\theta)$ for $\theta \in U_i$.

We claim that Q_i and P_i are continuous on U_i. To prove this, let $\theta_n \in U_i$ and $x_n \in X$ be convergent sequences with $\theta_n \to \theta \in U_i$ and $x_n \to x \in X$. We need to show that $Q_i(\theta_n) x_n \to Q_i(\theta) x$. Let $u_n = Q_i(\theta_n) x_n$ and $v_n = x_n - u_n$. Because of (3.1) we see that $|Q_i(\theta_n)|_{op}$ is uniformly bounded. Consequently the sequence u_n lies in a bounded set in $\mathscr{K}(\theta_i)$. Since $\mathscr{K}(\theta_i)$ is finite dimensional, u_n has a convergent subsequence, say $u_{n_k} \to u \in \mathscr{K}(\theta_i)$. Consequently $v_{n_k} = x_{n_k} - u_{n_k} \to (x - u)$. Since $v_{n_k} \in \mathscr{V}(\theta_{n_k})$ and $\mathscr{V}$ is closed one has $v = x - u \in \mathscr{V}(\theta)$. Since the representation $x = u + v$ is unique, the entire sequence u_n must converge to u; i.e.,

$$Q_i(\theta_n) x_n = u_n \to u = Q_i(\theta) x.$$

Hence Q_i is continuous on U_i, and $P_i = I - Q_i$ is continuous as well.

Next we let $f_1, ..., f_n$ be a partition of unity subordinate to $U_1, ..., U_n$. This means that

  (a)   each f_i is a continuous mapping of Θ into $[0, 1]$;
  (b)   supp $f_i \subset U_i$;
  (c) $\sum_{i=1}^n f_i(\theta) = 1$ for all $\theta \in \Theta$.

Define $Q(\theta)$ for $\theta \in \Theta$ by

$$Q(\theta) = \sum_{i=1}^n f_i(\theta) Q_i(\theta), \qquad \theta \in \Theta,$$

and set $\mathbf{Q}(x, \theta) = (Q(\theta) x, \theta)$. We claim that $\mathbf{Q}$ is a projector on $\mathscr{E}$. The continuity of $\mathbf{Q}$ follows from the continuity of the f_i's and the Q_i's. The fact that $Q(\theta)$ is linear on each fiber $\mathscr{E}(\theta)$ follows from the fact that each Q_i is a projection. The final point is to show that $Q^2(\theta) = Q(\theta)$ for all $\theta \in \Theta$. Let $\theta \in U_i \cap U_j$. Then $Q_i(\theta) Q_j(\theta) = Q_i(\theta)$ since $\mathscr{N}(Q_i(\theta)) = \mathscr{N}(Q_j(\theta)) = \mathscr{V}(\theta)$. One then obtains

$$Q^2 = \left(\sum_i f_i Q_i\right)\left(\sum_j f_j Q_j\right) = \sum_i f_i \left(\sum_j f_j Q_i Q_j\right) = \sum_i f_i \left(\sum_j f_j Q_i\right)$$

$$= \sum_i f_i \left(\sum_j f_j\right) Q_i = \sum_i f_i Q_i = Q,$$

where the θ has been suppressed in the notation.   ∎

3.2. *Linear Evolutionary Equations*

Let $\mathscr{E} = X \times \Theta$ be given where X is a fixed Banach space and Θ is a compact Hausdorff space. Assume that $\sigma(\theta, t) = \theta \cdot t$ is a (two-sided) flow of the reals on Θ, and let $\pi = (\Phi, \sigma)$ be a linear evolutionary system on $\mathscr{E}$. For any set $J \subset R^+$ we will let $\pi(x, \theta, J)$ denote the trajectory arc $\bigcup_{t \in J} \{\pi(x, \theta, t)\}$.

3.3. LEMMA. *Let* $\pi = (\Phi, \sigma)$ *be a linear evolutionary system on* $X \times \Theta$. *Then there exist constants* $M \geqslant 1$, $\omega \in R$ *such that*

$$|\Phi(\theta, t)| \leqslant Me^{\omega t}, \qquad \theta \in \Theta, \quad t \geqslant 0.$$

Proof. First we claim that there is an $h > 0$ such that

$$M \stackrel{\text{def}}{=} \sup\{|\Phi(\theta, t)| : \theta \in \Theta, 0 \leqslant t \leqslant h\} < \infty.$$

If this were not the case, then there are sequences $\theta_n \in \Theta$, $t_n > 0$ such that $t_n \to 0$ and $|\Phi(\theta_n, t_n)| \geqslant n$. The Uniform Boundedness Theorem then implies that there is an $x \in X$ such that $|\Phi(\theta_n, t_n) x|$ is unbounded. But this contradicts the fact that

$$\lim_{t \to 0^+} \Phi(\theta, t) x = x$$

uniformly for $\theta \in \Theta$. Hence $M < \infty$. Since $|\Phi(\theta, 0)| = 1$ it follows that $M \geqslant 1$. Next define $\omega \stackrel{\text{def}}{=} h^{-1} \log M$ and note that any $t \geqslant 0$ can be expressed as $t = nh + \tau$ for some integer $n \geqslant 0$ and $0 \leqslant \tau < h$. The cocycle identity (2.1) then implies that

$$|\Phi(\theta, t)| = |\Phi(\theta \cdot nh, \tau) \Phi(\theta \cdot (n-1) h, h) \cdots \Phi(\theta, h)|$$

$$\leqslant M^{n+1} \leqslant MM^{t/h} \leqslant Me^{\omega t}. \quad \blacksquare$$

3.4. LEMMA. *Let* $\pi = (\Phi, \sigma)$ *be a linear evolutionary system on* $X \times \Theta$. *Assume that* X *is finite dimensional and that* $\Phi(\theta, t)$ *is one-to-one for all* $\theta \in \Theta$ *and* $t \geqslant 0$. *Then* Φ *admits a unique extension for* $t < 0$ *such that the following hold:*

(a) $\Phi^{-1}(\theta, t) = \Phi(\theta \cdot t, -t)$,

(b)   *The cocycle identity*

$$\Phi(\theta, s + t) = \Phi(\theta \cdot t, s) \Phi(\theta, t) \tag{3.2}$$

holds for all $\theta \in \Theta$ *and* $s, t \in R$.

The same conclusion holds when $X \times \Theta$ *is replaced by a finite dimensional vector bundle over* Θ.

Proof. Since X is finite dimensional, the linear mapping Φ is one-to-one if and only if it is invertible. For $t > 0$ define $\Phi(\theta \cdot t, -t) \stackrel{\text{def}}{=} \Phi^{-1}(\theta, t)$. It is now a simple exercise to verify that the cocycle identity (3.2) holds for all $s, t \in R$. $\quad \blacksquare$

3.3. *The Stable and Unstable Sets*: *Weak Hyperbolicity*

A point $(x, \theta) \in \mathscr{E}$ is said to *have a negative continuation* if there exists a continuous function $\phi : (-\infty, 0] \to \mathscr{E}$ satisfying the following properties:

(1) $\phi(t) = (\phi^x(t), \theta \cdot t)$, where $\phi^x : (-\infty, 0] \to X$;

(2) $\phi(0) = (x, \theta)$;

(3) $\phi(s) \in \mathscr{E}(\theta \cdot s)$ for each $s \leqslant 0$;

(4) $\pi(\phi(s), t) = \phi(s + t)$ for all $s \leqslant 0$, and $0 \leqslant t \leqslant -s$.

In this case the function ϕ is said to be a *negative continuation* of the point (x, θ). For any negative continuation ϕ and any $\tau \leqslant 0$, we define $\phi_\tau(t) = \phi(\tau + t)$ for $-\infty < t \leqslant -\tau$. If ϕ is a negative continuation of the point (x, θ), then ϕ_τ is a negative continuation of $\phi(\tau)$. A *partial negative continuation* of the point (x, θ) is a mapping ϕ defined on a finite interval $[\tau, 0]$, where $\tau \leqslant 0$, and such that (1), (2), and (3) hold for s, $(s + t) \in [\tau, 0]$. (Notice that we do not assume uniqueness of a negative continuation.)

Next we define the following sets:

$$\mathscr{M} \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{E} : (x, \theta) \text{ has a negative continuation } \phi\},$$

$$\mathscr{U} \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{M} : \text{some neg. continuation } \phi \text{ of } (x, \theta) \text{ satisfies}$$
$|\phi^x(t)| \to 0 \text{ as } t \to -\infty\}$,

$$\mathscr{B}^- \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{M} : \text{some neg. continuation } \phi \text{ of } (x, \theta) \text{ satisfies}$$
$\sup_{t \leqslant 0} |\phi^x(t)| < \infty\}$,

$$\mathscr{B}_u^- \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{M} : (x, \theta) \text{ has a unique bounded negative continuation } \phi\},$$

$$\mathscr{B}^+ \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{E} : \sup_{t \geqslant 0} |\Phi(\theta, t) x| < \infty\},$$

$$\mathscr{S} \stackrel{\mathrm{def}}{=} \{(x, \theta) \in \mathscr{E} : |\Phi(\theta, t) x| \to 0 \text{ as } t \to \infty\}, \text{ and, finally,}$$

$$\mathscr{B} \stackrel{\mathrm{def}}{=} \mathscr{B}^+ \cap \mathscr{B}^-.$$

The set $\mathscr{U}$ is the *unstable set*, $\mathscr{S}$ is the *stable set*, and $\mathscr{B}$ is the *bounded set*.

Remark. Our theory does allow for the possibility that the linear operator $\Phi(\theta, t)$ need not be one-to-one for some $t \geqslant 0$, i.e., $\Phi(\theta, t)$ may have a nontrivial null space. Because of this, it may be possible for a point $(x, \theta) \in \mathscr{E}$ to have more that one negative continuation. It is easily seen that if $\Phi(\theta, t)$ is one-to-one for all $t > 0$ then every negative continuation, and every partial negative continuation, is unique. Uniqueness of negative continuations is a common feature in the study of partial differential equations; see for example Temam (1988).

For $(x, \theta) \in \mathscr{B}_u^-$ we shall denote the unique bounded negative continuation by $\Phi(\theta, t) x$ for $t \leqslant 0$. This then defines an extension of the mapping Φ. It is clear that for each $\theta \in \Theta$ the fiber $\mathscr{B}_u^-(\theta)$ is a linear subspace of

$\mathscr{E}(\theta)$, and $\Phi(\theta, t)x$ is linear in x for each $t \leqslant 0$; i.e., $\Phi(\theta, t)$ is a linear mapping from $\mathscr{B}_u^-(\theta)$ to $\mathscr{B}_u^-(\theta \cdot t)$ for $t \leqslant 0$. Furthermore, the co-cycle identity

$$\Phi(\theta, s + t)\, x = \Phi(\theta \cdot t, s)\, \Phi(\theta, t)\, x, \qquad s, t \in R$$

is valid for all $(x, \theta) \in \mathscr{B}_u^-$.

The linear evolutionary system $\pi(x, \theta, t) = (\Phi(\theta, t)\, x, \theta \cdot t)$ is said to be *weakly hyperbolic* on $\mathscr{E} = X \times \Theta$ whenever π is uniformly α-contracting and the bounded set $\mathscr{B}$ is trivial, i.e., $\mathscr{B} = \mathscr{E}_0$.

3.4. Exponential Dichotomies: The Shifted Flow

A projector $\mathbf{P}$ on $\mathscr{E}$ is said to be *invariant* if one has

$$P(\theta \cdot t)\, \Phi(\theta, t) = \Phi(\theta, t)\, P(\theta), \qquad t \geqslant 0, \quad \theta \in \Theta. \tag{3.3}$$

We shall say that a linear evolutionary system π on $\mathscr{E}$ has an *exponential dichotomy* over an invariant set $\hat{\Theta}$, where $\hat{\Theta} \subset \Theta$, if there is an invariant projector $\mathbf{P}$ on $\mathscr{E}(\hat{\Theta})$ and constants $K \geqslant 1$, $\beta > 0$ such that dim Range$(I - P(\theta))$ $< \infty$ and Range$(I - P(\theta)) \subset \mathscr{B}_u^-(\theta)$ for $\theta \in \hat{\Theta}$, and the following inequalities hold:

$$
\begin{aligned}
|\Phi(\theta, t)\, P(\theta)|_{\mathrm{op}} &\leqslant K e^{-\beta t}, \qquad t \geqslant 0, \ \theta \in \hat{\Theta}, \\
|\Phi(\theta, t)[I - P(\theta)]|_{\mathrm{op}} &\leqslant K e^{\beta t}, \qquad t \leqslant 0, \ \theta \in \hat{\Theta},
\end{aligned}
\tag{3.4}
$$

where $|L(\theta)|_{\mathrm{op}} = \sup\{|L(\theta)\, x| : x \in \mathscr{E}(\theta) \text{ and } |x| \leqslant 1\}$.

One of the important features of exponential dichotomies is their use in the method of Lyapunov and Perron. This method, as it is applied to the linear evolutionary system $\pi = (\Phi, \sigma)$ generated by (2.2) in the finite dimensional setting, is based on the observation that if the linear evolutionary system π has an exponential dichotomy and if the function $f(t)$ satisfies $f \in L^\infty(R, R^n)$, then the linear inhomogeneous equation

$$x' = A(\theta \cdot t)\, x + f(t)$$

has a unique solution $x \in C^0(R, R^n) \cap L^\infty(R, R^n)$ and x is given by the Lyapunov–Perron formula:

$$
x(t) = \int_{-\infty}^{t} \Phi(\theta \cdot s, t - s)\, P(\theta \cdot s)\, f(s)\, ds
$$

$$
- \int_{t}^{\infty} \Phi(\theta \cdot s, t - s)[I - P(\theta \cdot s)]\, f(s)\, ds.
$$

For the infinite dimensional setting, a similar formula is valid and has been used for such things as the proof of the existence of inertial manifolds; see Foias *et al.* (1988) and Magalhães (1987). See also Lemma 6.2, the Variation of Constants formula.

We observe that if π is a linear evolutionary system on $\mathscr{E} = X \times \Theta$ which admits an exponential dichotomy over Θ, then one has $\mathscr{B} = \mathscr{E}_0$ and π is weakly hyperbolic. Indeed, let $\mathbf{P}$ be the projector on $\mathscr{E}$ that satisfies (3.4). Let $(x, \theta) \in \mathscr{B}$ and set $y = P(\theta) x$ and $z = (I - P(\theta)) x$. One then observes that the three trajectories $\Phi(\theta, t) x$, $\Phi(\theta, t) y$, and $\Phi(\theta, t) z$ admit negative continuations $(\phi^x(t), \theta \cdot t)$, $(\phi^y(t), \theta \cdot t)$, and $(\phi^z(t), \theta \cdot t)$ such that

$$(\phi^x(t), \theta \cdot t) \in \mathscr{B}, \qquad (\phi^y(t), \theta \cdot t) \in \mathscr{B} \cap \mathscr{S}, \qquad (\Phi(\theta, t) z, \theta \cdot t) \in \mathscr{U}$$

for all $t \leqslant 0$. Let $t \leqslant 0$ and set $\tau = -t$ and $\hat{\theta} = \theta \cdot t$. Since $w = \Phi(\theta, \tau) \phi^w(t)$ for $w = x, y, z$, and since the projector $\mathbf{P}$ is invariant, one has

$$y = P(\theta) x = P(\hat{\theta} \cdot \tau) \Phi(\hat{\theta}, \tau) \phi^x(t) = \Phi(\hat{\theta}, \tau) P(\hat{\theta}) \phi^x(t).$$

As a result, $|y| \leqslant K |\phi^x(t)| e^{-\beta |t|}$. Since $|\phi^x(t)|$ is bounded for $t \leqslant 0$, this implies that $y = 0$. Similarly, since $z = [I - P(\theta)] z$ one has

$$z = \Phi(\theta \cdot \tau, t) \Phi(\theta, \tau) z = \Phi(\theta \cdot \tau, t) \Phi(\theta, \tau)[I - P(\theta)] z$$

$$= \Phi(\theta \cdot \tau, t)[I - P(\theta \cdot \tau)] \Phi(\theta, \tau) z.$$

Consequently, $|z| \leqslant Ke^{\beta t} |\Phi(\theta, \tau) z| \leqslant KM |z| e^{\beta t}$ for all $t \leqslant 0$. Therefore, $z = 0$, and consequently $x = y + z = 0$. Next we show that π is uniformly α-contracting. For any bounded set B in X, one has

$$\Phi(\theta, t) B \subset \Phi(\theta, t) P(\theta) B + \Phi(\theta, t)[I - P(\theta)] B,$$

where the last term has compact closure, since $[I - P(\theta)]$ has finite dimensional range. Using the properties of the Kuratowski measure α one has

$$\alpha(\Phi(\theta \cdot t) B) \leqslant \alpha(\Phi(\theta, t) P(\theta) B) \leqslant |\Phi(\theta, t) P(\theta)|_{op} \alpha(B) \leqslant Ke^{-\beta t}\alpha(B).$$

Hence π is weakly hyperbolic.

Remark. In the finite dimensional case, where the linear evolutionary system π is reversible in time t, one has the identity $P(\theta \cdot t) = \Phi(\theta, t) P(\theta) \Phi^{-1}(\theta, t)$. In this case inequalities (3.4) take on the equivalent form

$$|\Phi(\theta, t) P(\theta) \Phi^{-1}(\theta, s)|_{op} \leqslant Ke^{-\beta(t-s)}, \qquad t \geqslant s, \ \theta \in \Theta,$$

$$|\Phi(\theta, t)[I - P(\theta)] \Phi^{-1}(\theta, s)|_{op} \leqslant Ke^{\beta(t-s)}, \qquad t \leqslant s, \ \theta \in \Theta.$$

If $\pi = (\Phi, \sigma)$ is a linear evolutionary system on $\mathscr{E}$ and $\lambda \in R$, we define the *shifted flow* $\pi_\lambda = (\Phi_\lambda, \sigma)$, where $\Phi_\lambda(\theta, t) = e^{-\lambda t}\Phi(\theta, t)$. Note that for all $\lambda \in R$, the shifted flow π_λ is a linear evolutionary system on $\mathscr{E}$. If π is uniformly α-contracting, then π_λ is uniformly α-contracting for each $\lambda > 0$. Following Sacker and Sell (1978) we define the set of all λ for which π_λ admits an exponential dichotomy on $\mathscr{E}$ to be the *resolvent set* of π. The *dynamical spectrum* of π, which we write as $\Sigma(\pi)$, is defined to be the complement, in R, of the resolvent set. (Also see Magalhães, 1987.)

3.5. *The Morse Sets*

Before describing our main results, it is helpful to introduce some additional notation. For any integer $k \geqslant 0$ we define

$$\Theta_k \overset{\text{def}}{=} \{\theta \in \Theta : \dim \mathscr{U}(\theta) = \operatorname{codim} \mathscr{S}(\theta) = k\},$$

$$A_k \overset{\text{def}}{=} \{\theta \in \Theta : \dim \mathscr{U}(\theta) \geqslant k\}, \tag{3.5}$$

$$B_k \overset{\text{def}}{=} \{\theta \in \Theta : \operatorname{codim} \mathscr{S}(\theta) \leqslant k\}.$$

Note that if π is weakly hyperbolic, then $\mathscr{E}(\theta) = \mathscr{S}(\theta) + \mathscr{U}(\theta)$ for all $\theta \in \Theta_k$, and one has $\Theta_k = A_k \cap B_k$. The *Morse set* M_k is defined to be the maximal compact invariant subset of Θ_k. We will show below that each Θ_k is closed and negatively invariant, and consequently M_k is nonempty if and only if Θ_k is nonempty.

3.6. *Main Theorems*

The first set of theorems we give below lead to an alternative theorem which says that if the linear evolutionary system π is weakly hyperbolic, then either π has an exponential dichotomy over Θ or there exists a Morse decomposition on the base space Θ, and π has an exponential dichotomy over each of the Morse sets.

THEOREM A. *Let* $\pi = (\Phi, \sigma)$ *be a weakly hyperbolic linear evolutionary system on* $\mathscr{E}$. *Then the following statements hold:*

(1) $\mathscr{S}$ *and* $\mathscr{U}$ *are closed subsets of* $\mathscr{E}$.

(2) $\mathscr{U} = \mathscr{B}_u^-$.

(3) *There exist constants* $K \geqslant 1$, $\beta > 0$ *such that for all* $(x, \theta) \in \mathscr{S}$ *one has*

$$|\Phi(\theta, t) x| \leqslant K |x| e^{-\beta t}, \qquad t \geqslant 0,$$

and for all $(x, \theta) \in \mathscr{U}$ *one has*

$$|\Phi(\theta, t) x| \leqslant K |x| e^{\beta t}, \qquad t \leqslant 0.$$

(4)   *For all $\theta \in \Theta$ one has $\dim \mathscr{U}(\theta) \leqslant \operatorname{codim} \mathscr{S}(\theta) < \infty$.*

(5)   *The function $\dim \mathscr{U}(\theta)$ is upper semicontinuous in θ, and $\operatorname{codim} \mathscr{S}(\theta)$ is lower semicontinuous in θ.*

THEOREM B.   *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$, and let $\tilde{\Theta}$ denote a closed, invariant subset of Θ with the property that there is an integer k with $\operatorname{codim} \mathscr{S}(\theta) = k$ for all $\theta \in \tilde{\Theta}$. Then the following hold for all $\theta \in \tilde{\Theta}$:*

(1) $\dim \mathscr{U}(\theta) = k$.

(2) $\mathscr{E}(\theta) = \mathscr{S}(\theta) + \mathscr{U}(\theta)$.

(3)   *Let $P(\theta): X \to X$ denote the linear projection on X with range $\mathscr{R}(P(\theta)) = \mathscr{S}(\theta)$ and null space $\mathscr{N}(P(\theta)) = \mathscr{U}(\theta)$. Then $P(\theta)$ varies continuously in the operator norm, and P is invariant.*

(4)   *Let $K \geqslant 1$ and $\beta > 0$ be given by Theorem A. Then*

$$|\Phi(\theta, t) P(\theta)|_{\mathrm{op}} \leqslant K e^{-\beta t}, \qquad t \geqslant 0,$$

$$|\Phi(\theta, t)[I - P(\theta)]|_{\mathrm{op}} \leqslant K e^{\beta t}, \qquad t \leqslant 0.$$

In other words, π admits an exponential dichotomy over $\tilde{\Theta}$.

THEOREM C.   *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$, and let $\tilde{\Theta}$ denote a closed, invariant subset of Θ with the property that there is an integer k with $\dim \mathscr{U}(\theta) = k$ for all $\theta \in \tilde{\Theta}$. Then $\operatorname{codim} \mathscr{S}(\theta) = k$ for all $\theta \in \tilde{\Theta}$ and the other conclusions of Theorem B are valid. In particular, π admits an exponential dichotomy over each nonempty Morse set M_k.*

In the next theorem, we use the concept of chain-recurrence, see Section 7.8 for the definition. What is important to note is that the flow on every α- and every ω-limit set is chain recurrent; see Conley (1978).

THEOREM D.   *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$, and let $\tilde{\Theta}$ denote a closed, invariant subset of Θ with the property that $\tilde{\Theta}$ is chain recurrent. Then $\tilde{\Theta}$ is the union of nonempty Morse sets and π admits an exponential dichotomy over $\tilde{\Theta}$. If in addition, $\tilde{\Theta}$ is connected, then $\tilde{\Theta}$ lies in a single Morse set.*

Since the Morse sets M_k defined above are closed invariant subsets of Θ, Theorem B applies to each of these. We also show that only a finite number of the M_k are nonempty. The Alternative Theorem, which we state next, distinguishes between two cases: (a) there is precisely one nonempty Θ_k and (b) there are at least two nonempty Θ_k. Compare with Sacker and Sell (1974, 1976a).

THEOREM E: THE ALTERNATIVE THEOREM. *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$. Then there are only finitely many nonempty Θ_k, and we have the following*:

(1) *If there is precisely one nonempty Θ_k, then $\Theta = \Theta_k$ and π admits an exponential dichotomy over Θ.*

(2) *If there are at least two nonempty Θ_k, we define*

$$p = \min\{k : \Theta_k \neq \varnothing\}, \qquad q = \max\{k : \Theta_k \neq \varnothing\}.$$

Then for any k, $M_k \neq \varnothing$ if and only if $\Theta_k \neq \varnothing$. Moreover, one has $M_q = \Theta_q$. Also π admits an exponential dichotomy over each M_k. In addition, every motion $\sigma(\theta, t)$ in Θ has its α-limit set in some M_{k_1} and ω-limit set in some M_{k_2} where $k_1 \leqslant k_2$. Moreover, one has $k_1 = k_2 = k$ if and only if $\theta \in M_k$ for some k. Finally, $M_q = \Theta_q$ is a stable attractor and M_p is a stable repeller in the flow σ.

The next result is a useful test for a *global* exponential dichotomy for π. As we shall see, it is a direct consequence of the Alternative Theorem.

THEOREM F: THE COMPATIBILITY THEOREM. *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$. If either* codim $\mathscr{S}(\theta)$ *assumes the same value on all minimal sets in Θ or* dim $\mathscr{U}(\theta)$ *assumes the same value on all minimal sets in Θ, then π admits an exponential dichotomy over Θ.*

Remark. Since Θ is not assumed to be connected, one can apply the last theorem to each component of Θ. In this way, one can construct global dichotomies having unstable manifolds with different dimensions.

4. EXAMPLES OF LINEAR EVOLUTIONARY SYSTEMS

As mentioned above, the linear evolutionary systems oftentimes arise where Φ is the solution operator for a linear system of the form

$$\frac{du}{dt} + Au = B(t)\, u, \tag{4.1}$$

on a Banach space X, where $(-A)$ is the infinitesimal generator of a linear C_0-semigroup on X and $B(t)$ is a suitable linear operator with domain $\mathscr{D}(A) \subset \mathscr{D}(B(t))$. For example, if $\varphi(t)$, $t \in R$, is a globally defined solution of a nonlinear evolutionary equation

$$\frac{du}{dt} + Au = F(u), \tag{4.2}$$

with range in a given compact set $\mathscr{K}$ in X, and F is Frechét differentiable, then $B(t) = DF(\varphi(t))$ is a bounded linear operator for all $t \in R$. In this case (4.1) generates a linear evolutionary system, where the base space Θ is the closure (in the compact-open topology) of the collection of all translates $B_\tau, \tau \in R$, and $B_\tau(t) = B(\tau + t)$; see for example, Henry (1981) and Pazy (1983).

A special case of wide-spread interest occurs when the nonlinear evolutionary equation has an attractor $\mathfrak{A}$, which can be either a local or a global attractor. The linearization of (4.2) over $\mathfrak{A}$ then leads to a linear evolutionary system over Θ, where Θ is the collection of all globally defined solutions φ of (4.2) with $\varphi(t) \in \mathfrak{A}$ for all $t \in R$. The flow on Θ is the translational flow $(\varphi, \tau) \to \varphi_\tau$, where $\varphi_\tau(t) = \varphi(\tau + t)$.

4.1. Navier–Stokes Equations

To be more specific, let us consider the linearized Navier–Stokes equations along a compact, invariant set. Recall that the Navier–Stokes equations on a suitable bounded region Ω in R^n, $n = 2, 3$, can be reduced to an abstract nonlinear evolutionary equation

$$\frac{du}{dt} + \nu Au + B(u, u) = f, \qquad u(0) = u_0, \tag{4.3}$$

on the Hilbert space H, where H is the closure in $L^2(\Omega, R^n)$ of those vector fields $u \in C_0^\infty(\Omega, R^n)$ that satisfy $\nabla \cdot u = 0$, cf. Constantin and Foias (1988), Ladyzhenskaya (1972), and Temam (1977, 1983, 1988). The norm $|\cdot|$ on H is the standard L^2-norm. The operator A in (4.3) is a positive, self-adjoint linear operator on H with compact inverse, and A has domain $V^2 =^{\text{def}} D(A)$ being $H \cap H^2(\Omega, R^n)$. As a result, $(-A)$ generates an analytic semigroup e^{-At}, see Henry (1981) or Pazy (1983). In addition, the domain $V^1 =^{\text{def}} D(A^{1/2})$ of $A^{1/2}$ is contained in $H_0^1(\Omega, R^n)$.

The nonlinear term $B = B(u, v)$ is a bilinear mapping; see the references cited above. In the autonomous case the forcing term f is assumed to lie in H. More generally, in the time-varying case, f is chosen so that its hull $H(f)$ is a compact subset of $C^0(R, H) \cap L^\infty(R, H)$; see Raugel and Sell (1990, 1993abc).

The long-time dynamics of (4.3) in the two-dimensional setting, with f autonomous, is described in references cited above. For the time-varying case, which is of special interest here, the theory of the long-time dynamics, in both the two-dimensional setting and for certain thin three-dimensional domains, is presented by Raugel and Sell (1989, 1993abc). What is important for our purposes is that in these settings, Eq. (4.3) has a attractor $\mathfrak{A}$, and that $\mathfrak{A}$ is a compact subset of $D(A) \times \Omega(f)$, where $\Omega(f)$ is the ω-limit set of the forcing function f. Since $\mathfrak{A}$ is invariant, it follows that for every

$\theta = (u, g)$ in $\mathfrak{A}$ the translate $\theta_\tau = (u_\tau, g_\tau)$ is in $\mathfrak{A}$ for every $\tau \in R$, where $u_\tau(t, x) = u(\tau + t, x)$ and $g_\tau(t, x) = g(\tau + t, x)$. Furthermore, there is a constant K_2 such that for every $(u, g) \in \mathfrak{A}$ one has $|Au| \leqslant K_2$. The attractor $\mathfrak{A}$, or any compact invariant subset of $\mathfrak{A}$, will play the role of Θ in what follows.

The linearized Navier–Stokes equation takes on the form

$$\frac{dv}{dt} + vAv + B(u(t), v) + B(v, u(t)) = 0, \qquad v(0) = v_0, \qquad (4.4)$$

where $\theta = (u, g) \in \Theta$. Equation (4.4) generates a linear evolutionary system π over Θ as follows: For each $\theta = (u, g) \in \Theta$ and $v_0 \in V^1$, one constructs a strong solution $v(t) = \Phi(\theta, t) v_0$ of (4.4). This solution satisfies

$$v(\cdot) \in C^0([0, \infty), V^1) \cap L^\infty_{\text{loc}}((0, \infty), V^2).$$

The following result can be easily derived from the theory presented in Constantin and Foias (1988) or Temam (1977, 1983). It shows that the mapping

$$(v_0, \theta) \to (\Phi(\theta, \tau) v_0, \theta_\tau)$$

generates a linear evolutionary system on $V^1 \times \Theta$, and that for each $\tau > 0$, the linear operator $\Phi(\theta, \tau)$ is compact. In particular, the linear evolutionary system is uniformly α-contracting.

4.1. Lemma. *Let (4.3) represent the Navier–Stokes equations on a suitable bounded region Ω in R^2. Assume that $f \in C^0([0, \infty), H) \cap L^\infty((0, \infty), H)$ is chosen so that its positive hull $H^+(f)$ is a compact subset. Let u_0 be an element in the global attractor $\mathfrak{A}$ for (4.3), and let $u(t) = u(u_0, t)$, $t \geqslant 0$, be the corresponding strong solution of (4.3). Then one has*

$$u(\cdot) \in C^0([0, \infty), V^1) \cap L^\infty((0, \infty), V^1) \cap L^\infty_{\text{loc}}((0, \infty), V^2)$$

and $u_t(\cdot) \in L^2((0, T), H)$ for every $T > 0$. Also for every $v_0 \in V^1$, there is a unique strong solution $v(t) = \Phi(u_0, t) v_0$ of (4.4) satisfying

$$v(\cdot) \in C^0([0, \infty), V^1) \cap L^\infty_{\text{loc}}((0, \infty), V^2)$$

and $v_t(\cdot) \in L^2_{\text{loc}}((0, \infty), H)$. Furthermore, for each $t \geqslant 0$, the mapping

$$(v_0, u_0) \to \Phi(u_0, t) v_0$$

is a continuous mapping of $V^1 \times \mathfrak{A}$ into V^1.

By using the theory presented in Raugel and Sell (1993abc), one can extend the above result to the Navier–Stokes equations on thin 3D domains.

4.2. *Other Partial Differential Equations*

Similar constructions of linear evolutionary systems are possible in the study of other partial differential equations. We will not go into the details here. Instead the reader should consult the following references: For the reaction-diffusion equations, see Henry (1981) and the references contained therein. For hyperbolic partial differential equations, including the nonlinear wave equation and the nonlinear Schrödinger equation with dissipation, see Babin and Vishik (1987), Ghidaglia and Temam (1987, 1988), Hale (1988), Hale and Raugel (1988, 1990), Haraux (1984, 1988), Keller (1983), Massat (1983), and Temam (1988).

4.3. *Differential-Delay Equations*

Linearizations of differential-delay equations, including retarded differential equations and certain neutral differential equations, can be found in Hale (1977, 1988). The issues of the uniform α-contracting property are treated there as well.

5. Dynamics of Linear Evolutionary Equations

In this section we present some of the dynamical properties of uniformly α-contracting linear evolutionary systems which are used below. Throughout this section $\pi = (\Phi, \sigma)$ will denote a uniformly α-contracting linear evolutionary system on $\mathscr{E} = X \times \Theta$, where X is a Banach space and Θ is a compact Hausdorff space. We do not assume here that π is weakly hyperbolic.

A set $\mathscr{F} \subset \mathscr{E}$ is said to be *invariant* if for all $t > 0$ one has $\pi(\mathscr{F}, t) = \mathscr{F}$. Since π maps $\mathscr{F}$ onto $\mathscr{F}$ at each time t, this means that every $(x, \theta) \in \mathscr{F}$ has a negative continuation ϕ with the property that $\phi(s) \in \mathscr{F}$ for all $s \leqslant 0$. In other words, if $\mathscr{F}$ is invariant, then $\mathscr{F} \subset \mathscr{M}$, where $\mathscr{M}$ is defined in Section 3.3. An immediate consequence is the following result.

5.1. Lemma. *Let $\mathscr{F}$ be an invariant set for π. If $\mathscr{F} \subset \mathscr{E}_M$ for some M, $0 \leqslant M < \infty$, then $\mathscr{F} \subset \mathscr{B}$.*

For any set $\mathscr{F} \subset \mathscr{E}$ we define

$$\text{positive trajectory:} \quad \gamma^+(\mathscr{F}) = \pi(\mathscr{F}, R^+),$$

$$\text{positive hull:} \quad H^+(\mathscr{F}) = \operatorname{Cl} \gamma^+(\mathscr{F}),$$

$$\omega\text{-limit set:} \quad \Omega(\mathscr{F}) = \bigcap_{\tau \geqslant 0} H^+(\pi(\mathscr{F}, \tau)).$$

For $(x, \theta) \in \mathcal{M}$ and any negative continuation ϕ of (x, θ), we define

$$\text{negative trajectory:} \quad \gamma^-(x, \theta, \phi) = \phi(R^-),$$

$$\text{negative hull:} \quad H^-(x, \theta, \phi) = \mathrm{Cl}\, \gamma^-(x, \theta, \phi),$$

$$\alpha\text{-limit set:} \quad A(x, \theta, \phi) = \bigcap_{\tau \leqslant 0} H^-(\phi(\tau), \phi_\tau),$$

$$\text{trajectory:} \quad \gamma(x, \theta, \phi) = \gamma^-(x, \theta, \phi) \cup \gamma^+(x, \theta),$$

$$\text{hull:} \quad H(x, \theta, \phi) = \mathrm{Cl}\, \gamma(x, \theta, \phi).$$

For $\theta \in \Theta$ we shall let $\gamma^+(\theta)$, $H^+(\theta)$, etc., denote the same objects in Θ.

There are some elementary properties of hulls and limit sets which we note in the following lemmas. First, for any $M, T \in R^+$ define

$$\mathcal{L}_M \overset{\text{def}}{=} \{(x, \theta) \in \mathcal{E} : |\Phi(\theta, t)\, x| \leqslant M \text{ for } 0 \leqslant t\},$$

$$\mathcal{L}^T_M \overset{\text{def}}{=} \{(x, \theta) \in \mathcal{E} : |\Phi(\theta, t)\, x| \leqslant M \text{ for } 0 \leqslant t \leqslant T\},$$

$$\mathcal{F}^N_M \overset{\text{def}}{=} \bigcap_{T \geqslant 0} \pi(\mathcal{L}^{N+T}_M, T), \qquad \mathcal{F}_M \overset{\text{def}}{=} \bigcap_{N \geqslant 0} \mathcal{F}^N_M.$$

The first four properties listed below are elementary:

   (A) $\mathcal{L}^T_M$ and $\mathcal{L}_M$ are nonempty closed sets in $\mathcal{E}_M$.

   (B) $\pi(\mathcal{L}^{N+T}_M, T)$ and $\pi(\mathcal{L}_M, T)$ are nonempty closed sets in $\mathcal{E}_M$.

   (C) $\pi(\mathcal{L}^{N+T}_M, T) \subset \pi(\mathcal{L}^{N+S}_M, S)$ and $\pi(\mathcal{L}_M, T) \subset \pi(\mathcal{L}_M, S)$ whenever $S \leqslant T$.

   (D) $\mathcal{F}^N_M \subset \mathcal{F}^S_M$ whenever $0 \leqslant S \leqslant N$.

   (E) $\pi(\mathcal{F}^N_M, T) = \mathcal{F}^{N-T}_M$ whenever $0 \leqslant T \leqslant N$.

In order to prove (E) we note that it follows from (C) that

$$\mathcal{F}^N_M = \bigcap_{S \geqslant 0} \pi(\mathcal{L}^{N+S}_M, S) = \bigcap_{S \geqslant T} \pi(\mathcal{L}^{N+S}_M, S)$$

for any $T \geqslant 0$, and

$$\pi\left(\pi\left(\bigcap_{S \geqslant 0} \mathcal{L}^{N+S}_M, S \right), T \right) = \bigcap_{S \geqslant 0} \pi(\pi(\mathcal{L}^{N+S}_M, S), T).$$

Consequently, when $0 \leqslant T \leqslant N$ one has

$$\pi(\mathcal{F}^N_M, T) = \bigcap_{S \geqslant 0} \pi(\pi(\mathcal{L}^{N+S}_M, S), T) = \bigcap_{S \geqslant 0} \pi(\mathcal{L}^{N+S}_M, S+T)$$

$$= \bigcap_{S \geqslant 0} \pi(\mathcal{L}^{N-T+S+T}_M, S+T) = \bigcap_{S \geqslant T} \pi(\mathcal{L}^{N-T+S}_M, S) = \mathcal{F}^{N-T}_M.$$

In the next lemma we show that certain subsets of $\mathscr{E} = X \times \Theta$ necessarily lie in the bounded set $\mathscr{B}$.

5.2. LEMMA. *Let* $0 \leqslant M, N, S, T$, *and let* B *be a nonempty set satisfying* $B \subset \mathscr{L}_M$. *Then the following statements are valid:*

(F) $\mathscr{F}_M^N$ *and* $\mathscr{F}_M$ *are nonempty compact sets in* $\mathscr{E}_M$.

(G) $\Omega(B)$ *and* $\Omega(\mathscr{L}_M)$ *are nonempty compact sets in* $\mathscr{E}_M$ *with* $\Omega(B) \subset \Omega(\mathscr{L}_M)$.

(H)   *For every* $(x, \theta) \in \mathscr{F}_M^N$ *there is a negative continuation* ϕ *of* (x, θ) *with* $\phi(s) \in \mathscr{F}_M^N$ *for all* $s \leqslant 0$.

(I) $\mathscr{F}_M$ *is an invariant set for* π *and* $\mathscr{F}_M \subset \mathscr{B}$.

(J) $\Omega(B)$ *and* $\Omega(\mathscr{L}_M)$ *are invariant sets for* π *and* $\Omega(B) \subset \Omega(\mathscr{L}_M) \subset \mathscr{B}$.

Proof.   The fact that $\mathscr{F}_M^N$ is nonempty and compact follows from (B), (C), and Lemma 2.1 since

$$\alpha_\infty(\pi(\mathscr{L}_M^{N+T}, T)) \leqslant 2Mk(T)$$

for some function k with $k(t) \to 0$ as $t \to \infty$. By (D) $\mathscr{F}_M$ is a nonempty compact set as well. Since $\gamma^+(\pi(\mathscr{L}_M, T)) = \pi(\gamma^+(\mathscr{L}_M), T)$ one has

$$\alpha_\infty(H^+(\pi(B, T))) \leqslant \alpha_\infty(H^+(\pi(\mathscr{L}_M, T))) = \alpha_\infty(\pi(\gamma^+(\mathscr{L}_M), T)) \leqslant 2Mk(T),$$

which implies that $\Omega(B)$ and $\Omega(\mathscr{L}_M)$ are nonempty compact sets. The inclusion of $\Omega(B)$ in $\Omega(\mathscr{L}_M)$ follows directly from the definitions and the fact that $B \subset \mathscr{L}_M$.

Using the fact that $\pi(\mathscr{F}_M^{N+n+1}, 1) = \mathscr{F}_M^{N+n}$ for any integer $n \geqslant 0$, one sees that for any $(x, \theta) \in \mathscr{F}_M^N$ there is a sequence of partial negative continuations $\phi_n : [-n, 0] \to \mathscr{E}$ that satisfy (i) $\phi_n(0) = (x, \theta)$, (ii) $\phi_n(s) \in \mathscr{F}_M^N$ for $-n \leqslant s \leqslant 0$, and (iii) ϕ_{n+1} is an extension of ϕ_n. It then follows that $\phi = \lim_{n \to \infty} \phi_n$ exists and satisfies statement (H).

Statement (I) is proved by noting that from (D) one has

$$\pi\left(\bigcap_{N \geqslant t} \mathscr{F}_M^N, t\right) = \bigcap_{N \geqslant t} \pi(\mathscr{F}_M^N, t) \qquad \text{and} \qquad \mathscr{F}_M = \bigcap_{N \geqslant 0} \mathscr{F}_M^N = \bigcap_{N \geqslant t} \mathscr{F}_M^N.$$

Combining this with (G) one then obtains

$$\pi(\mathscr{F}_M, t) = \pi\left(\bigcap_{N \geqslant t} \mathscr{F}_M^N, t\right) = \bigcap_{N \geqslant t} \pi(\mathscr{F}_M^N, t) = \bigcap_{N \geqslant t} \mathscr{F}_m^{N-t} = \mathscr{F}_M.$$

In order to show that $\Omega(B)$ is an invariant set, we need to verify that $\pi(\Omega(B), t) = \Omega(B)$ for all $t \geqslant 0$. Now $\Omega(B)$ can be characterized as the

collection of all points $(x, \theta) \in \mathscr{E}$ such that there are sequences $(x_n, \theta_n) \in \gamma^+(B)$ and $t_n \to \infty$ such that

$$\bar{\theta}_n \overset{\text{def}}{=} \theta_n \cdot t_n \to \theta \qquad \text{and} \qquad \bar{x}_n \overset{\text{def}}{=} \Phi(\theta_n, t_n) x_n \to x.$$

Let us fix $t \geqslant 0$, and set $\hat{x}_n \overset{\text{def}}{=} \Phi(\theta_n, t_n + t) x_n$ and $\hat{\theta}_n \overset{\text{def}}{=} \theta_n \cdot (t_n + t)$. Since $(\hat{x}_n, \hat{\theta}_n) \in H^+(\pi(B, t_n))$ and since $\alpha_\infty(H^+(B, t_n)) \to 0$ as $n \to \infty$, it follows from Lemma 2.1 that the sequence $(\hat{x}_n, \hat{\theta}_n)$ has a convergent subsequence, which we shall relabel as $(\hat{x}_n, \hat{\theta}_n)$. Hence there is a $(\hat{x}, \hat{\theta}) \in \Omega(B)$ such that $(\hat{x}_n, \hat{\theta}_n) \to (\hat{x}, \hat{\theta})$ as $n \to \infty$. $>$ From the cocycle condition (2.1), one finds that

$$\hat{x} = \lim \hat{x}_n = \lim \Phi(\bar{\theta}_n, t) \bar{x}_n = \Phi(\theta, t) x,$$

where the last step follows from the continuity property (4) in the definition of a linear evolutionary system. Similarly, one finds that $\hat{\theta} = \theta \cdot t$. This shows that $\pi(\Omega(B), t) \subset \Omega(B)$ for $t \geqslant 0$.

Next restrict n so that $t_n - t \geqslant 0$, and set $\tilde{x}_n \overset{\text{def}}{=} \Phi(\theta_n, t_n - t) x_n$ and $\tilde{\theta}_n \overset{\text{def}}{=} \theta_n \cdot (t_n - t)$. Since $(\tilde{x}_n, \tilde{\theta}_n) \in H^+(\pi(B, t_n - t))$ and since $\alpha_\infty(H^+(B, t_n - t)) \to 0$ as $n \to \infty$, it follows from Lemma 2.1 that the sequence $(\tilde{x}_n, \tilde{\theta}_n)$ has a convergent subsequence, which we shall relabel as $(\tilde{x}_n, \tilde{\theta}_n)$. Hence there is a $(\tilde{x}, \tilde{\theta}) \in \Omega(B)$ such that $(\tilde{x}_n, \tilde{\theta}_n) \to (\tilde{x}, \tilde{\theta})$ as $n \to \infty$. From the cocycle condition (2.1), one finds that $\bar{x}_n = \Phi(\tilde{\theta}_n, t) \tilde{x}_n$ and $\bar{\theta}_n = \tilde{\theta}_n \cdot t$. Therefore, as above one has

$$x = \lim \bar{x}_n = \lim \Phi(\tilde{\theta}_n, t) \tilde{x}_n = \Phi(\tilde{\theta}, t) \tilde{x},$$

and $\theta = \tilde{\theta} \cdot t$. Consequently, $\Omega(B) \subset \pi(\Omega(B), t)$ for $t \geqslant 0$, i.e., $\Omega(B)$ is invariant. Similary one shows that $\Omega(\mathscr{L}_M)$ is invariant. The fact that $\Omega(B) \subset \Omega(\mathscr{L}_M) \subset \mathscr{B}$ follows from statement (G) and Lemma 5.1. ∎

5.3. LEMMA. *The following statements are valid:*

(A) *If* $\gamma^+(x, \theta) \subset \mathscr{E}_M$ *for some* $M < \infty$, *then* $\Omega(x, \theta)$ *is a nonempty compact invariant subset of* $\mathscr{E}$ *satisfying* $\Omega(x, \theta) \subset \mathscr{B}$.

(B) *Let* $(x, \theta) \in \mathscr{M}$ *with a negative continuation* ϕ. *If* $\phi(s) \in \mathscr{E}_M$ *for all* $s \leqslant 0$ *and for some* $M < \infty$, *then* $H^-(x, \theta, \phi)$ *and* $A(x, \theta, \phi)$ *are nonempty compact subsets of* $\mathscr{E}$. *Furthermore,* $A(x, \theta, \phi)$ *is invariant and* $A(x, \theta, \phi) \subset \mathscr{B}$.

Proof. Part (A) follows directly from Lemma 5.2 with $B = (x, \theta)$. For part (B) note that for any $n \geqslant 0$ and $t \geqslant 0$ one has

$$\pi(H^-(\phi(-n-t), \phi_{-n-t}), t) = H^-(\phi(-n), \phi_{-n}).$$

By the uniform α-contracting property this implies that $\alpha_\infty(H^-(\phi(-n),\phi_{-n}))$ $= 0$, i.e., $H^-(\phi(-n),\phi_{-n})$ is compact and nonempty for every $n \geqslant 0$. Furthermore, $H^-(\phi(-n),\phi_{-n})$ is monotone (nonincreasing) in n. Hence $A(x,\theta,\phi)$, which is $\bigcap_{n \geqslant 0} H^-(\phi(-n),\phi_{-n})$, is nonempty and compact. By restricting π to the compact set $H^-(\phi(0),\phi)$, it follows from the usual arguments for alpha limit sets that $A(x,\theta,\phi)$ is invariant. Finally $A(x,\theta,\phi) \subset \mathscr{B}$ follows from Lemma 5.1. ∎

In the next lemma we give sufficient conditions for there to exist a point (x,θ) with $x \neq 0$ and such that $(x,\theta) \in \mathscr{B}^-$ or, more specifically, that $(x,\theta) \in \mathscr{B}$.

5.4. LEMMA.   *Assume that there exist sequences x_k, θ_k, s_k, t_k such that $0 \leqslant s_k \leqslant t_k$, $s_k \to \infty$, and for some M, $0 < M < \infty$, and ε, $0 < \varepsilon \leqslant 1$, one has $|\Phi(\theta_k,t)x_k| \leqslant M$ for $0 \leqslant t \leqslant t_k$ and $|\Phi(\theta_k,s_k)x_k| \geqslant \varepsilon M$. Then there is an $(x,\theta) \in \mathscr{E}_M$ with $\varepsilon M \leqslant |x|$, and there is a negative continuation ϕ of (x,θ) satisfying $\phi(t) \in \mathscr{E}_M$ for all $t \leqslant 0$; i.e., $(x,\theta) \in \mathscr{B}^-$.*

If, in addition, one has $t_k - s_k \to \infty$, then $|\Phi(\theta,t)x| \leqslant M$ for all $t \geqslant 0$; in other words, $(x,\theta) \in \mathscr{B}$.

Proof.   The assumption that $|\Phi(\theta_k,s_k)x_k| \geqslant \varepsilon M$ implies that the set

$$A^{N,T} \overset{\text{def}}{=} \pi(\mathscr{L}_M^{N+T},T) \cap \{(x,\theta): \varepsilon M \leqslant |x| \leqslant M\}$$

is nonempty and closed for $0 \leqslant N \leqslant (t_k - s_k)$. As a result of (C), $A^{N,T}$ is monotone (nonincreasing) in T. Since $\alpha_\infty(A^{N,T}) \to 0$ as $T \to \infty$, it follows that $\bigcap_{T \geqslant 0} A^{N,T}$ is a nonempty (compact) subset of $\mathscr{F}_M^0$. From Lemma 5.2(H) it follows that for each $(x,\theta) \in \bigcap_{T \geqslant 0} A^{N,T}$ there is a negative continuation ϕ with $\phi(s) \in \mathscr{E}_M$ for all $s \leqslant 0$. Thus $(x,\theta) \in \mathscr{B}^-$.

If $(t_k - s_k) \to \infty$, then $\bigcap_{N \geqslant 1} \bigcap_{T \geqslant 0} A^{N,T}$ is a nonempty compact set in

$$\mathscr{F}_M \cap \{(x,\theta): \varepsilon M \leqslant |x| \leqslant M\}.$$

By Lemma 5.2(I) each $(x,\theta) \in \bigcap_{N \geqslant 1} \bigcap_{T \geqslant 0} A^{N,T}$ lies in $\mathscr{B}$. ∎

6. INDUCED FLOW ON NORMAL SUBBUNDLE

Let $\pi = (\Phi,\sigma)$ be a linear evolutionary system on $\mathscr{E} = X \times \Theta$. Throughout this section we will not assume π to be uniformly α-contracting. Instead we make the following

SPECIAL HYPOTHESIS.   *$\mathscr{V}$ is a closed positively invariant subset of $\mathscr{E}$ where each fibre $\mathscr{V}(\theta)$ is a linear subspace of $\mathscr{E}(\theta)$ and there exists an integer $k \geqslant 0$ such that*

(1)   codim $\mathscr{V}(\theta) = k$, *for all* $\theta \in \Theta$, *and*

(2)   *whenever* $\Phi(\theta, t) x = 0$ *one has* $(x, \theta) \in \mathscr{V}$.

It then follows from Lemma 3.2 that $\mathscr{V}$ is a subbundle of $\mathscr{E}$ and that there exists a projector $\mathbf{L}$ on $\mathscr{E}$ and a complementary subbundle $\mathscr{W}$ with range $\mathscr{R}(L(\theta)) = \mathscr{W}(\theta)$ and null space $\mathscr{N}(L(\theta)) = \mathscr{V}(\theta)$ for all $\theta \in \Theta$. The complementary subbundle $\mathscr{W}$ is, in general, not invariant for π. Nevertheless, π does induce a flow $\hat{\pi}$ on $\mathscr{W}$ as follows: Define $\hat{\Phi}$ by $\hat{\Phi}(\theta, t) x = L(\theta \cdot t) \Phi(\theta, t) x$, where $(x, \theta) \in \mathscr{W}$. The *induced flow* $\mathscr{W}$ is defined by

$$\hat{\pi}(x, \theta, t) = (\hat{\Phi}(\theta, t) x, \theta \cdot t), \qquad (x, \theta) \in \mathscr{W}, \ t \geqslant 0.$$

The proof that $\hat{\Phi}$ satisfies the co-cycle identity,

$$\hat{\Phi}(\theta, t + s) = \hat{\Phi}(\theta \cdot t, s) \, \hat{\Phi}(\theta, t), \tag{6.1}$$

for $s, t \geqslant 0$, follows from (2.1), the cocycle identity for Φ, and the fact that $\mathscr{V}$ is invariant under π. Indeed, the invariance of $\mathscr{V}$ implies that

$$[I - L(\theta \cdot t)] \, \Phi(\theta, t)[I - L(\theta)] = \Phi(\theta, t)[I - L(\theta)],$$

which in turn yields

$$L(\theta \cdot t) \, \Phi(\theta, t) = L(\theta \cdot t) \, \Phi(\theta, t) \, L(\theta), \qquad \theta \in \Theta, \ t \geqslant 0. \tag{6.2}$$

By using (6.2) with the cocycle identity for Φ, it is easily verified that $\hat{\Phi}$ satisfies (6.1).

Next note that the mapping

$$\hat{\Phi}(\theta, t): \mathscr{W}(\theta) \to \mathscr{W}(\theta \cdot t)$$

is one-to-one for all $t \geqslant 0$ since $\mathscr{N}(\Phi(\theta, t)) \subset \mathscr{V}(\theta)$ and $\mathscr{V}(\theta) \cap \mathscr{W}(\theta) = \varnothing$. Since dim $\mathscr{W}(\theta) = $ dim $\mathscr{W}(\theta \cdot t) = k$ for all $\theta \in \Theta$ and $t \geqslant 0$, it follows as in Lemma 3.4 that $\hat{\Phi}(\theta, t)$ has a continuous inverse, which we denote by $\hat{\Phi}(\theta \cdot t, -t)$. This means that $\hat{\pi} = (\hat{\Phi}, \sigma)$ is a (two-sided) linear skew-product flow on $\mathscr{W}$ and that the cocycle identity (6.1) on $\mathscr{W}$ is valid for *all* $t, s \in R$.

For the induced flow $\hat{\pi}$ on $\mathscr{W}$ we shall let $\hat{\mathscr{B}}^+$, $\hat{\mathscr{B}}$, $\hat{\mathscr{S}}$, and $\hat{\mathscr{U}}$ denote, respectively, the positively bounded, the bounded, the stable, and the unstable sets in $\mathscr{W}$.

Let $(x, \theta) \in \mathscr{E}$ and define $w = L(\theta) x$ and $v = x - L(\theta) x$. Then

$$(x, \theta) = (v + w, \theta)$$

is the unique decomposition of (x, θ) into the sum of two vectors $(v, \theta) \in \mathcal{V}$ and $(w, \theta) \in \mathcal{W}$. Now the trajectories $\Phi(\theta, t) v$ and $\Phi(\theta, t) w$ admit similar decompositions. Since $\mathcal{V}$ is invariant under π one has $(\Phi(\theta, t) v, \theta \cdot t) \in \mathcal{V}$ whenever $(v, \theta) \in \mathcal{V}$. This means that one can write

$$\Phi(\theta, t) v = A(\theta, t) v,$$

where $A(\theta, t)$ is a bounded linear transformation from $\mathcal{V}(\theta)$ to $\mathcal{V}(\theta \cdot t)$. Moreover one has

$$\Phi(\theta, t) w = B(\theta, t) w + D(\theta, t) w,$$

where $B(\theta, t): \mathcal{W}(\theta) \to \mathcal{V}(\theta \cdot t)$ and $D(\theta, t): \mathcal{W}(\theta) \to \mathcal{W}(\theta \cdot t)$ are bounded linear transformations.

Let us now write $x = \binom{v}{w}$. Then the last paragraph can be summarized by using matrix notation and writing

$$\Phi(\theta, t) x = \Phi(\theta, t) \binom{v}{w} = \begin{pmatrix} A(\theta, t) v + B(\theta, t) w \\ D(\theta, t) w \end{pmatrix}$$

$$= \begin{pmatrix} A(\theta, t) & B(\theta, t) \\ 0 & D(\theta, t) \end{pmatrix} \binom{v}{w}.$$

We now obtain the following properties for the linear operators A, B, and D.

6.1. LEMMA. *Assume that the Special Hypothesis is satisfied. Then the linear operators A and D satisfy the cocycle identity*

$$\begin{aligned} A(\theta \cdot t, s) A(\theta, t) &= A(\theta, t + s) \\ D(\theta \cdot t, s) D(\theta, t) &= D(\theta, t + s) \end{aligned} \tag{6.3}$$

for all $\theta \in \Theta$ and $t, s \geq 0$, and $B(\theta, 0) = 0$ for all $\theta \in \Theta$. Furthermore, one has that (i) *the operator $D(\theta, t)$ is invertible,* (ii) *the inverse satisfies $D^{-1}(\theta, t) = D(\theta \cdot t, -t)$, and* (iii) *the D equation in* (6.3) *holds for all $t, s \in R$. Finally, there exist constants $K \geq 1$ and $\omega \in R$ such that*

$$|A(\theta, t)|_{op} \leq Ke^{\omega t}, \qquad |B(\theta, t)|_{op} \leq Ke^{\omega t}, \qquad |D(\theta, t)|_{op} \leq Ke^{\omega t},$$

for all $\theta \in \Theta$ and $t \geq 0$.

Proof. The cocycle identities for A and D follow immediately from (2.1), the cocycle property for Φ. The invertibility properties for D follow as in Lemma 3.4 since $\mathcal{W}(\theta)$ is finite dimensional with $\mathcal{N}(\Phi(\theta, t)) \cap \mathcal{W}(\theta) = \{0\}$. Finally, the exponential estimates for A, B and D follow

from the corresponding estimate for Φ and the facts that the projector **L** is bounded and

$$A(\theta, t) = \Phi(\theta, t)(I - L(\theta)), \qquad B(\theta, t) = (I - L(\theta \cdot t)) \Phi(\theta, t) L(\theta),$$

$$D(\theta, t) = L(\theta \cdot t) \Phi(\theta, t) L(\theta). \quad \blacksquare$$

For $t \geqslant 0$ define v_t and w_t by the formulas

$$\begin{pmatrix} v_t \\ w_t \end{pmatrix} = \Phi(\theta, t) \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} A(\theta, t) v + B(\theta, t) w \\ D(\theta, t) w \end{pmatrix}. \tag{6.4}$$

Thus v_t and w_t denote the time evolution of the v and w coordinates under the flow π. Note that the point $\begin{pmatrix} v_t \\ w_t \end{pmatrix}$ lies in the fiber $\mathscr{E}(\theta \cdot t)$ and that $v_t \in \mathscr{V}(\theta \cdot t)$ and $w_t \in \mathscr{W}(\theta \cdot t)$. The projector **L** can be expressed as $L(\theta)\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ w \end{pmatrix}$ in these coordinates. Consequently, we see that the induced flow $\hat{\pi}$ on $\mathscr{W}$ is given by

$$\hat{\pi}(w, \theta, t) = (D(\theta, t) w, \theta \cdot t),$$

that is, $\hat{\Phi}(\theta, t) = D(\theta, t)$. Finally the mapping

$$(v, \theta, t) \to (A(\theta, t) v, \theta \cdot t)$$

is simply the restriction of the original flow π to the invariant subbundle $\mathscr{V}$.

We need the following variation of constants formula for the triangular system described above.

6.2. VARIATION OF CONSTANTS LEMMA. *Assume that the Special Hypothesis is satisfied. Let $r \in R$, $h > 0$, $0 \leqslant \tau < h$, and let m and n be integers with $m \geqslant 0$ and $n > -m$. Then for any trajectory $\begin{pmatrix} v_s \\ w_s \end{pmatrix}$ which exists for $s \in [r - mh, r + nh + \tau]$ one has*

$$v_{r+nh+\tau} = A(\theta \cdot (r - mh), (n + m) h + \tau) v_{r-mh}$$

$$+ \sum_{j=1-m}^{n} A(\theta \cdot (r + jh), (n - j) h + \tau)$$

$$\times B(\theta \cdot [r + (j-1) h], h) w_{r+(j-1)h} \tag{6.5}$$

$$+ B(\theta \cdot (r + nh), \tau) w_{r+nh},$$

$$w_{r+nh+\tau} = D(\theta \cdot (r - mh), (n + m) h + \tau) w_{r-mh}.$$

Proof. The main issue here is the formula for v_s. If $n = 1 - m$, then (6.5) follows from (6.4). We now use induction to study the case for larger n. First consider the case of $\tau = 0$, and assume that (6.5) is valid for n. (Note

that $B(\theta, 0) = 0$ for all $\theta \in \Theta$.) By using the cocycle identity (6.3) and (6.5) one obtains

$$
A(\theta \cdot (r + nh), h) \, v_{r + nh} = A(\theta \cdot (r - mh), (n + 1 + m) \, h) \, v_{r - mh}
$$
$$
+ \sum_{j = 1 - m}^{n + 1} A(\theta \cdot (r + jh), (n + 1 - j) \, h)
$$
$$
\times B(\theta \cdot [r + (j - 1) \, h], h) \, w_{r + (j - 1) h}.
$$

Consequently from (6.4) one has

$$
v_{r + (n + 1) h} = A(\theta \cdot (r + nh), h) \, v_{r + nh} + B(\theta \cdot (r + nh), h) \, w_{r + nh}
$$
$$
= A(\theta \cdot (r - mh), (n + 1 + m) \, h) \, v_{r - mh}
$$
$$
+ \sum_{j = 1 - m}^{n + 1} A(\theta \cdot (r + jh), (n + 1 - j) \, h)
$$
$$
\times B(\theta \cdot [r + (j - 1) \, h], h) \, w_{r + (j - 1) h}
$$
$$
+ B(\theta \cdot [r + (n + 1) \, h], 0) \, w_{r + (n + 1) h}.
$$

The formula for $v_{r + nh + \tau}$ for $\tau > 0$ is obtained by applying $\Phi(\theta \cdot (r + nh), \tau)$ to $\binom{v_{r + nh}}{w_{r + nh}}$ and using the cocycle identity (6.3) with (6.4). ∎

We next derive some useful identities.

6.3. LEMMA.   *Let $\sigma \in R$, $h > 0$, and let n be an integer such that $\sigma = nh + \tau$, where $0 \leqslant \tau$. Let $s \geqslant 0$. Then one has*

$$
A(\theta \cdot \sigma, s) \, B(\theta \cdot nh, \tau) + B(\theta \cdot \sigma, s) \, D(\theta \cdot nh, \tau) = B(\theta \cdot nh, \tau + s). \qquad (6.6)
$$

Furthermore, if $h \leqslant \tau + s$, then

$$
A(\theta \cdot \sigma, s) \, B(\theta \cdot nh, \tau) + B(\theta \cdot \sigma, s) \, D(\theta \cdot nh, \tau)
$$
$$
= A(\theta \cdot (n + 1) \, h, \lambda) \, B(\theta \cdot nh, h) + B(\theta \cdot (n + 1) \, h, \lambda) \, D(\theta \cdot nh, h), \qquad (6.7)
$$

where $\tau + s = h + \lambda$.

Proof.   We use the cocycle identity for Φ and (6.4) to prove this. For any $v \in \mathscr{V}(\theta \cdot nh)$ and $w \in \mathscr{W}(\theta \cdot nh)$, we define $\binom{v_1}{w_1}$ and $\binom{v_3}{w_3}$ by

$$
\binom{v_1}{w_1} = \Phi(\theta \cdot nh, \tau) \binom{v}{w}, \qquad \binom{v_3}{w_3} = \Phi(\theta \cdot nh, \tau + s) \binom{v}{w}.
$$

By using (6.4) with the fact that

$$
\binom{v_3}{w_3} = \Phi(\theta \cdot \sigma, s) \binom{v_1}{w_1},
$$

one obtains (6.6). In the case that $h \leqslant \tau + s$, we define

$$\begin{pmatrix} v_2 \\ w_2 \end{pmatrix} = \varPhi(\theta \cdot nh, h) \begin{pmatrix} v \\ w \end{pmatrix}.$$

Then (6.7) follows from (6.4) and the fact that

$$\begin{pmatrix} v_3 \\ w_3 \end{pmatrix} = \varPhi(\theta \cdot \sigma, s) \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} = \varPhi(\theta \cdot (n+1) h, \lambda) \begin{pmatrix} v_2 \\ w_2 \end{pmatrix}. \quad \blacksquare$$

Remark. The variation of constants formula can be written in usual integral form provided $B(\theta, t)$ is differentiable in t at $t = 0$. Define

$$b(\theta) \stackrel{\text{def}}{=} \frac{\partial}{\partial t} B(\theta, t)|_{t=0}.$$

Then (6.5) can be written as

$$v_t = A(\theta, t) v + \int_0^t A(\theta \cdot s, t - s) b(\theta \cdot s) D(\theta, s) w \, ds,$$

see Sacker and Sell (1976b) for details.

The induced flow on the normal bundle $\mathscr{W}$ is reversible since its state space is finite dimensional and the Special Hypothesis holds. This means that for every initial condition $(x, \theta) \in \mathscr{W}$ the trajectory $\hat{\varPhi}(\theta, t) x$ has a negative continuation. The next lemma gives a sufficient condition under which this negative continuation in $\mathscr{W}$ gives rise to a related negative continuation for the original flow on $\mathscr{E}$. The next result gives information about solutions bounded on either R or $R^- = (-\infty, 0]$.

6.4. LEMMA. *In addition to the Special Hypothesis assume that there exist constants $K_1 \geqslant 1$ and $\beta > 0$ such that*

$$|A(\theta, t)|_{\text{op}} \leqslant K_1 e^{-\beta t}, \qquad \theta \in \varTheta, \ t \geqslant 0. \tag{6.8}$$

Next assume that there is a trajectory $w_t = D(\theta, t) w_0$ that is bounded for all $t \in R$ (or $t \in R^-$) and set

$$|w|_\infty \stackrel{\text{def}}{=} \sup\{|w_t| : t \in R \ (or \ t \in R^-)\}.$$

Then there exists a constant K_2, which depends only on K, ω, K_1, and β, and there exists a unique $v_0 \in \mathscr{V}(\theta)$ such that the following hold:

(1)   *The point $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix}$ has a negative continuation $\begin{pmatrix} v_t \\ w_t \end{pmatrix}$ with the property that*

$$|v_t| \leqslant K_2 \, |w|_\infty, \qquad \text{for all} \quad t \in R \, (\text{or } t \in R^-), \tag{6.9}$$

i.e., $\begin{pmatrix} v_0 \\ u_0 \end{pmatrix} \in \mathscr{B}$ *(or* $\in \mathscr{B}^-$ *).*

(2)   *The function v_t in Part 1 is uniquely determined and satisfies the identity*

$$v_{nh+\tau} = \sum_{j=-\infty}^{n} A(\theta \cdot jh, (n-j)h + \tau) \, B(\theta \cdot (j-1)h, h) \, w_{(j-1)h}$$
$$+ B(\theta \cdot nh, \tau) \, w_{nh} \tag{6.10}$$

where $t = nh + \tau$, $n \in Z$, $0 \leqslant \tau < h$, and $h > 0$. (In particular, v_t does not depend on h.)

(3) $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *if and only if $w_0 = 0$.*

(4)   *The mapping $T(\theta)$: $w_0 \to \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}$ is a one-to-one bounded linear mapping of $\hat{\mathscr{B}}(\theta)$ into $\mathscr{B}(\theta)$ (or of $\hat{\mathscr{B}}^-(\theta)$ into $\mathscr{B}^-(\theta)$).*

Proof.   Let us first prove the uniqueness of the point $v_0 \in \mathscr{V}(\theta)$ such that statement (1) is valid. If there exist two such points, say v_1 and v_2, then $\begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ would have a bounded negative continuation $\begin{pmatrix} v_t \\ 0 \end{pmatrix}$, where $v_0 = v_1 - v_2$. Then for any $t < 0$ one has $v_0 = A(\theta \cdot t, -t) v_t$, which implies that

$$|v_0| \leqslant K_1 e^{-\beta |t|} \sup_{-\infty < t \leqslant 0} |v_t|.$$

Hence $v_0 = 0$. (This also proves statement 3.)

Next we fix $h > 0$ and define v_t^h by the right-hand side of (6.10), where $t = nh + \tau$, $n \in Z$, and $0 \leqslant \tau < h$. First note that the summation in (6.10) is well-defined because from (6.8) one has

$$\sum_{j=-\infty}^{n} |A(\theta \cdot jh, (n-j)h + \tau)|_{\mathrm{op}} \, |B(\theta \cdot (j-1)h, h)| \, |w_{(j-1)h}|$$

$$\leqslant K_1 \, |B_\infty| \, |w|_\infty \, e^{-\beta h} \sum_{j=-\infty}^{0} e^{\beta j h} = \frac{K_1 \, |B|_\infty}{1 - e^{-\beta h}} \, |w|_\infty,$$

where

$$|B|_\infty = \sup_{\theta \in \Theta, \, 0 \leqslant s \leqslant h} |B(\theta, s)|.$$

In fact, this implies that (6.9) holds for v_t^h, where $K_2 = |B|_\infty + K_1 \, |B|_\infty \, (1 - e^{-\beta h})^{-1}$.

In order to prove that $\binom{v_t^h}{w_t}$ is a negative continuation of $\binom{v_0^h}{w_0}$, we restrict to $t = \sigma < 0$ and we must show that

$$\begin{pmatrix} v_{\sigma+s}^h \\ w_{\sigma+s} \end{pmatrix} = \Phi(\theta \cdot \sigma, s) \begin{pmatrix} v_\sigma^h \\ w_\sigma \end{pmatrix} \tag{6.11}$$

for all $s \geqslant 0$. Because of (2.1), the cocycle identity for Φ, one has

$$\Phi(\theta \cdot \sigma, s) = \Phi(\theta \cdot (\sigma + \ell h), \tau_s) \, \Phi(\theta \cdot (\sigma + (\ell-1)h), h) \cdots \Phi(\theta \cdot \sigma, h),$$

where $s = \ell h + \tau_s$, $0 \leqslant \tau_s < h$. Therefore, it suffices to verify (6.11) for $0 \leqslant s \leqslant h$. Note that $0 \leqslant \tau + s < 2h$.

Since w_σ is a trajectory, it follows that

$$w_{\sigma+s} = D(\theta \cdot \sigma, s) \, w_\sigma.$$

Thus it remains to show that $v_{\sigma+s}^h = y_{\sigma+s}$, where

$$y_{\sigma+s} \overset{\text{def}}{=} A(\theta \cdot \sigma, s) \, v_\sigma^h + B(\theta \cdot \sigma, s) \, w_\sigma.$$

From the definition of v_σ^h one has

$$\begin{aligned}
y_{\sigma+s} = \sum_{j=-\infty}^{n} & A(\theta \cdot \sigma, s) \, A(\theta \cdot jh, (n-j)h+\tau) \\
& \times B(\theta \cdot (j-1)h, h) \, w_{(j-1)h} + \Gamma w_{nh},
\end{aligned}$$

where

$$\Gamma w_{nh} = A(\theta \cdot \sigma, s) \, B(\theta \cdot nh, \tau) \, w_{nh} + B(\theta \cdot \sigma, s) \, D(\theta \cdot nh, \tau) \, w_{nh}.$$

If $\tau + s < h$, we use (6.6) to replace Γw_{nh} with $B(\theta \cdot nh, \tau+s) \, w_{nh}$, and thereby verify that $y_{\sigma+s} = v_{\sigma+s}^h$ in this case. On the other hand, if $h \leqslant \tau + s < 2h$, we use (6.7) to obtain

$$\begin{aligned}
y_{\sigma+s} &= \sum_{j=-\infty}^{n} A(\theta \cdot jh, (n-j)h+\sigma+s) \, B(\theta \cdot (j-1)h, h) \, w_{(j-1)h} \\
&\quad + [A(\theta \cdot (n+1)h, \lambda) \, B(\theta \cdot nh, h) + B(\theta \cdot (n+1)h, \lambda) \, D(\theta \cdot nh, h)] \, w_{nh} \\
&= \sum_{j=-\infty}^{n+1} A(\theta \cdot jh, (n-j)h+\sigma+s) \, B(\theta \cdot (j-1)h, h) \, w_{(j-1)h} \\
&\quad + B(\theta \cdot (n+1)h, \lambda) \, w_{(n+1)h},
\end{aligned}$$

which again agrees with $v_{\sigma+s}^h$. (Note that $v_0 \overset{\text{def}}{=} v_0^h$ is independent of h.)

We have now shown that $\binom{v_t^h}{w_t}$ is a negative continuation for $\binom{v_0}{w_0}$, and that (6.9) is valid for any $h > 0$. In order to show the uniqueness of the

negative continuation, we fix $h > 0$, and let $\binom{v_t}{w_t}$ denote any negative continuation that satisfies (6.9). (For example, v_t may be chosen to satisfy (6.10) for a different value of h.) The Variation of Constants Formula (6.5) (with $r = 0$ and $s = nh + \tau_s$, where $0 \leqslant \tau_s < h$ and $n > -m$) implies that

$$
v_s = A(\theta \cdot (-mh), mh + s)\, v_{-mh} + \sum_{j=1-m}^{n} A(\theta \cdot (jh), (n-j)h + \tau_s)
$$
$$
\times B(\theta \cdot [(j-1)h], h)\, w_{(j-1)h} + B(\theta \cdot (nh), \tau_s)\, w_{nh}.
$$

Now one has

$$
|A(\theta \cdot (-mh), mh + s)\, v_{-mh}| \leqslant K_1 e^{-\beta(s+mh)} \sup_{t \leqslant 0} |v_t|,
$$

which goes to 0 as $m \to \infty$. Hence

$$
v_s = \sum_{j=-\infty}^{n} A(\theta \cdot (jh), (n-j)h + \tau_s)
$$
$$
\times B(\theta \cdot [(j-1)h], h)\, w_{(j-1)h} + B(\theta \cdot (nh), \tau_s)\, w_{nh},
$$

which agrees with v_s^h.

Finally, statement (4) is an immediate consequence of statements (1), (2), and (3). ∎

6.5. LEMMA. *In addition to the Special Hypothesis, assume that there exist constants $K_1 \geqslant 1$ and $\beta > 0$ such that*

$$
|A(\theta, t)|_{\mathrm{op}} \leqslant K_1 e^{-\beta t}, \qquad \theta \in \Theta, \ t \geqslant 0.
$$

Next assume that there is a trajectory $w_t = D(\theta, t)\, w_0$ that is bounded for all $t \geqslant 0$. Define v_t by $\binom{v_t}{w_t} = \Phi(\theta, t)\binom{0}{w_0}$. Then there is a constant K_3 such that $|v_t| \leqslant K_3$ for all $t \geqslant 0$.

Proof. From Lemma 6.2 with $h = 1$ one has

$$
|v_n| \leqslant \sum_{i=1}^{n} K_1 e^{-\beta(n-i)} K e^{\omega} |w|_\infty \leqslant \frac{K K_1 e^{\omega}}{\beta} |w|_\infty,
$$

where $|w|_\infty = \sup\{|w_t| : t \geqslant 0\}$. For $t = n + \tau$, where $0 < \tau < 1$, one has

$$
|v_t| \leqslant \sum_{i=1}^{n} K_1 e^{-\beta(n-i+\tau)} K e^{\omega} |w|_\infty \leqslant \frac{K K_1 e^{\omega}}{\beta} |w|_\infty. \quad ∎
$$

7. Proof of Main Theorems

Throughout this section we will assume that $\pi = (\Phi, \sigma)$ is a weakly hyperbolic linear evolutionary system. In particular π is uniformly α-contracting, and one has $\mathscr{S} \cap \mathscr{U} \subset \mathscr{B} = \mathscr{E}_0$, the zero section.

7.1. *Preliminary Lemmas*

We will first derive a number of preliminary properties of the stable and unstable sets.

7.1. Lemma. *Let $M < \infty$ be given and assume that $(x_k, \theta_k) \in \mathscr{E}_M$ with $(x_k, \theta_k) \to (x, \theta)$. Let $t_k \in R$. Then the following statements hold:*

(A) *If $t_k \to \infty$ and $\pi(x_k, \theta_k, [0, t_k]) \subset \mathscr{E}_M$, then $(x, \theta) \in \mathscr{S}$.*

(B) *If $t_k \to -\infty$ and there exists a partial negative continuation $\phi_k : [t_k, 0] \to \mathscr{E}$ of (x_k, θ_k) with $(\phi_k(s), \theta_k) \in \mathscr{E}_M$ for all k and all $t_k \leqslant s \leqslant 0$, then $(x, \theta) \in \mathscr{U}$.*

(C) *If both (A) and (B) hold, then $x = 0$.*

Proof. (A) It is clear that $\gamma(x, \theta) \subset \mathscr{E}_M$. By Lemma 5.3 the ω-limit set $\Omega(x, \theta)$ is nonempty, compact, and $\Omega(x, \theta) \subset \mathscr{B} \subset \mathscr{E}_0$. Consequently, $|\Phi(\theta, t) x| \to 0$ as $t \to \infty$; i.e., $(x, \theta) \in \mathscr{S}$.

(B) From Lemma 5.4 we conclude that there is a negative continuation ϕ for (x, θ) and ϕ satisfies $\phi(t) \in \mathscr{E}_M$ for $t \leqslant 0$. By Lemma 5.3 the α-limit set is nonempty, compact, and $A(x, \theta, \phi) \subset \mathscr{B} = \mathscr{E}_0$. Consequently $|\phi^x(t)| \to 0$ as $t \to -\infty$; i.e., $(x, \theta) \in \mathscr{U}$.

(C) If both (A) and (B) hold, then $(x, \theta) \in \mathscr{S} \cap \mathscr{U} \subset \mathscr{B} = \mathscr{E}_0$. Hence $x = 0$. ∎

The following corollary is an immediate application of the last result.

7.2. Lemma. *The following statements are valid:*

(A) *One has $\mathscr{B}^+ = \mathscr{S}$ and $\mathscr{B}^- = \mathscr{U}$.*

(B) *For each $(x, \theta) \in \mathscr{U}$, there is a unique bounded negative continuation ϕ. Furthermore, ϕ satisfies $|\phi(s)| \to 0$ as $s \to -\infty$; i.e., one has $\mathscr{B}^- = \mathscr{B}_u^- = \mathscr{U}$.*

Proof. (A) Clearly one has $\mathscr{S} \subset \mathscr{B}^+$ and $\mathscr{U} \subset \mathscr{B}^-$. If $(x, \theta) \in \mathscr{B}^+$, then by Lemma 5.3 the ω-limit set $\Omega(x, \theta)$ is a nonempty, compact invariant subset of $\mathscr{E}$ and $\Omega(x, \theta) \subset \mathscr{B} = \mathscr{E}_0$. Hence one has $|\Phi(\theta, t) x| \to 0$ as $t \to \infty$, i.e., $(x, \theta) \in \mathscr{S}$. The argument that $\mathscr{B}^- = \mathscr{U}$ is similar.

(B) Let ϕ_1 and ϕ_2 denote two bounded negative continuations of (x, θ) and set $x_i = \phi_i^x(s)$, $i = 1, 2$, for some $s < 0$. Since $x_1, x_2 \in \mathscr{U}(\theta \cdot s)$

one has $x_1 - x_2 \in \mathcal{U}(\theta \cdot s)$. However, $x_1 - x_2 \in \mathcal{S}(\theta \cdot s)$ since $\Phi(\theta \cdot s, -s) \times (x_1 - x_2) = 0$. Therefore $x_1 - x_2 = 0$, which implies uniqueness. The fact that $|\phi(s)| \to 0$ as $s \to -\infty$ follows from (A). ∎

Note, therefore, that the extension $\Phi(\theta, t)x$ is defined for all $(x, \theta) \in \mathcal{U}$ and $t \leqslant 0$.

7.3. LEMMA.   *Each of the following two configurations is impossible*:

   (A)   *There exist sequences* $(x_k, \theta_k) \in \mathcal{E}$, $\varepsilon_k \to 0$, $0 \leqslant \tau_k \leqslant t_k$, *such that* $|x_k| \leqslant \varepsilon_k$, $|\Phi(\theta_k, t_k) x_k| \leqslant \varepsilon_k$, *and* $|\Phi(\theta_k, \tau_k) x_k| \geqslant 1$.

   (B)   *There exists* $a > 0$ *and sequences* $(x_k, \theta_k) \in \mathcal{E}$, $B_k \to \infty$, $0 \leqslant \tau_k \leqslant t_k$ *such that* $|x_k| \leqslant a$, $|\Phi(\theta_k, t_k) x_k| \leqslant a$, *and* $|\Phi(\theta_k, \tau_k) x_k| \geqslant B_k$.

Proof. The argument for either statement is the same. Assume the lemma is false. We then replace x_k with $\mu_k x_k$, where μ_k is a positive scalar, and change τ_k to the point $\sigma_k \in [0, t_k]$ where $\Phi(\theta_k, t) x_k$ assumes its maximum value on $[0, t_k]$, so that

   (1) $|\Phi(\theta_k, t) x_k| \leqslant 1$, for $t \in [0, t_k]$,

   (2) $|\Phi(\theta_k, \sigma_k) x_k| = 1$,

   (3) $|\Phi(\theta_k, t) x_k| \leqslant \delta_k$ for $t = 0, t_k$, and $\delta_k \to 0$.

By continuity one has $(t_k - \sigma_k) \to \infty$ and $\sigma_k \to \infty$. It then follows from Lemma 5.4 that there is an $(x, \theta) \in \mathcal{E}$ with $|x| = 1$ and $(x, \theta) \in \mathcal{B}$, a contradiction. ∎

Define the following two sets:

$$\mathcal{A}^+ \overset{\text{def}}{=} \{(x, \theta) \in \mathcal{B}^+ : |\Phi(\theta, t) x| \leqslant 1 \text{ for all } t \geqslant 0\}$$

$$\mathcal{A}^- \overset{\text{def}}{=} \{(x, \theta) \in \mathcal{B}_u^- : |\Phi(\theta, t) x| \leqslant 1 \text{ for all } t \leqslant 0\}.$$

Note that by Lemma 7.2 one has $\mathcal{A}^+ \subset \mathcal{S}$, while $\mathcal{A}^- \subset \mathcal{B}^- = \mathcal{B}_u^- = \mathcal{U}$.

7.4. LEMMA.   *The following statements are valid*:

   (A)   *For* $0 \leqslant \tau \leqslant t$ *one has* $\pi(\mathcal{A}^+, t) \subset \pi(\mathcal{A}^+, \tau) \subset \mathcal{A}^+ \subset \mathcal{S}$.

   (B)   *For each* $t \geqslant 0$ *the set* $\pi(\mathcal{A}^+, t)$ *is closed.*

   (C) $\mathcal{A}^-$ *is compact, and* $\mathcal{A}^- \subset \mathcal{U}$.

Proof. (A) is obvious. (B) follows from the continuity of π. The proof of the compactness of $\mathcal{A}^-$ is similar to the argument used in Lemma 5.3 to show that $H^-(x, \theta, \phi)$ is compact. We omit the details. ∎

7.5. LEMMA. *Let λ be given with $0 < \lambda \leqslant 1$. Then there cannot exist sequences $(x_k, \theta_k) \in \mathscr{A}^+$ and $t_k \to \infty$ such that $|\Phi(\theta_k, t_k) x_k| \geqslant \lambda$.*

Proof. By Lemma 7.4 one has

$$(\hat{x}_k, \hat{\theta}_k) \overset{\text{def}}{=} \pi(x_k, \theta_k, t_k) = (\Phi(\theta_k, t_k) x_k, \theta_k \cdot t_k) \in \pi(\mathscr{A}^+, t_k) \subset \mathscr{A}^+.$$

Since $\alpha_\infty(\pi(\mathscr{A}^+, t)) \leqslant k(t) \alpha_\infty(\mathscr{A}^+) \leqslant 2k(t)$ for some function k with $k(t) \to 0$ as $t \to \infty$, there is a convergent subsequence, say

$$(\hat{x}_k, \hat{\theta}_k) \to (x, \theta) \in \mathscr{A}^+ \subset \mathscr{S}.$$

Furthermore, one has $|x| \geqslant \lambda > 0$. Since $-t_k \to -\infty$, it follows from Lemma 7.1(B) that $(x, \theta) \in \mathscr{U}$. Since $\mathscr{S} \cap \mathscr{U} \subset \mathscr{B} = \mathscr{E}_0$ one has $|x| = 0$, a contradiction. ∎

7.6. LEMMA. *Let $(x, \theta) \in \mathscr{A}^+$ be chosen so that $(\mu x, \theta) \notin \mathscr{A}^+$ whenever $\mu > 1$. Then there exists a $\tau = \tau(x, \theta) \geqslant 0$ such that $|\Phi(\theta, \tau) x| = 1$.*

Proof. First we note that for any $(x, \theta) \in \mathscr{S}$ there is a $\tau \geqslant 0$ such that $|\Phi(\theta, \tau) x| = b$, where $b \overset{\text{def}}{=} \sup\{|\Phi(\theta, t) x|: t \geqslant 0\}$. By linearity one has $\sup\{|\Phi(\theta, t)\mu x|: t \geqslant 0\} = |\mu| b$. Now if (x, θ) satisfies the hypotheses of this lemma, then for $\mu > 1$ one has $\mu b > 1$. Hence $b \geqslant 1$. However, $(x, \theta) \in \mathscr{A}^+$ implies that $b \leqslant 1$. Hence, with τ given as above, one has $|\Phi(\theta, \tau) x| = 1$. ∎

7.7. LEMMA. *Let λ be given where $0 < \lambda \leqslant 1$. Then there cannot exist sequences $(x_k, \theta_k) \in \mathscr{A}^-$, $t_k \to -\infty$, such that $|\Phi(\theta_k, t_k) x_k| \geqslant \lambda$.*

Proof. By Lemma 7.4 $\mathscr{A}^-$ is compact and $\mathscr{A}^- \subset \mathscr{U}$. Since

$$(\hat{x}_k, \hat{\theta}_k) \overset{\text{def}}{=} (\Phi(\theta_k, t_k) x_k, \theta_k \cdot t_k) \in \mathscr{A}^-,$$

we can assume, without loss of generality, that $(\hat{x}_k, \hat{\theta}_k)$ is convergent with limit $(x, \theta) \in \mathscr{A}^- \subset \mathscr{U}$. Also one has $|x| \geqslant \lambda > 0$ and $\pi(\hat{x}_k, \hat{\theta}_k, [0, -t_k]) \subset \mathscr{A}^-$ for all k. Since $-t_k \to \infty$ it follows from Lemma 7.1(A) that $(x, \theta) \in \mathscr{S}$. Since $\mathscr{S} \cap \mathscr{U} \subset \mathscr{B} = \mathscr{E}_0$ one has $|x| = 0$, a contradiction. ∎

7.8. LEMMA. *Let $(x, \theta) \in \mathscr{A}^-$ be chosen so that $(\mu x, \theta) \notin \mathscr{A}^-$ whenever $\mu > 1$. Then there exists a $\tau = \tau(x, \theta) \leqslant 0$ such that $|\Phi(\theta, \tau) x| = 1$.*

Proof. Let $b \overset{\text{def}}{=} \sup\{|\Phi(\theta, s) x|: s \leqslant 0\}$. First note that $b \leqslant 1$ since $(x, \theta) \in \mathscr{A}^-$. Next one cannot have $b < 1$ or else this would violate the condition that $(\mu x, \theta) \notin \mathscr{A}^-$ whenever $\mu > 1$. Since $\mathscr{A}^- \subset \mathscr{U}$ one has $|\Phi(\theta, s) x| \to 0$ as $s \to -\infty$. Consequently, there is a $\tau \leqslant 0$ such that $|\Phi(\theta, \tau) x| = b = 1$. ∎

7.2. *Stable Set: Stability and Exponential Decay*

Our next objective is to show that the decay rate for trajectories beginning in the stable set is exponential. The first step is to derive a form of uniform stability for the stable set.

7.9. LEMMA (UNIFORM STABILITY). *There exists a v, $0 < v \leqslant 1$, such that if $(x, \theta) \in \mathscr{S}$ and $|x| \leqslant v$, then $(x, \theta) \in \mathscr{A}^+$; i.e., $|\Phi(\theta, t) x| \leqslant 1$ for all $t \geqslant 0$.*

Proof. If this were false, then there would exist a sequence $(x_k, \theta_k) \in \mathscr{S}$ such that $|x_k| \to 0$ and $(x_k, \theta_k) \notin \mathscr{A}^+$. Define λ_k by $\lambda_k^{-1} \overset{\text{def}}{=} \sup\{|\Phi(\theta_k, t) x_k| : t \geqslant 0\}$. Then $0 < \lambda_k < 1$ and, by linearity, $(\lambda_k x_k, \theta_k) \in \mathscr{A}^+$. However, $(\mu \lambda_k x_k, \theta_k) \notin \mathscr{A}^+$ whenever $\mu > 1$. By Lemma 7.6, there exists $\tau_k \geqslant 0$ such that $|\Phi(\theta_k, \tau_k) \lambda_k x_k| = 1$. Now $|\lambda_k x_k| \leqslant |x_k| \to 0$. Since $(\lambda_k x_k, \theta_k) \in \mathscr{A}^+ \subset \mathscr{S}$, there exists $t_k \geqslant \tau_k$ such that $|\Phi(\theta_k, t_k) \lambda_k x_k| \leqslant k^{-1}$. However, this violates Lemma 7.3(A). ∎

7.10. LEMMA (EXPONENTIAL DECAY). *There exists a $\tau > 0$ such that for any $(x, \theta) \in \mathscr{S}$ one has $|\Phi(\theta, t) x| \leqslant \frac{1}{2}|x|$ for all $t \geqslant \tau$.*

Proof. If this were false, then there exist sequences $t_k \to \infty$ and $(x_k, \theta_k) \in \mathscr{S}$ such that $|\Phi(\theta_k, t_k) x_k| > \frac{1}{2}|x_k|$. Because of the linearity of Φ and Lemma 7.9, one can assume that $|x_k| = v$, and $(x_k, \theta_k) \in \mathscr{A}^+$. Then $|\Phi(\theta_k, t_k) x_k| > \frac{1}{2}v$, which contradicts Lemma 7.5. ∎

The next result begins to address some of the specific features noted in Theorem A.

7.11. LEMMA. *The following statements are valid:*

   (A) *The stable set $\mathscr{S}$ is closed.*

   (B) *There exist constants $K \geqslant 1$, $\beta > 0$ such that for all $(x, \theta) \in \mathscr{S}$ one has*

$$|\Phi(\theta, t) x| \leqslant K |x| e^{-\beta t}, \qquad t \geqslant 0.$$

Proof. (A) Let (x_k, θ_k) be a sequence in $\mathscr{S}$ with $(x_k, \theta_k) \to (x, \theta)$. If $x = 0$ then $(x, \theta) \in \mathscr{S}$. On the other hand, if $x \neq 0$, then define μ by $2\mu |x| = v$, where v is given by Lemma 7.9. For k sufficiently large, it follows from Lemma 7.9 that $(\mu x_k, \theta_k) \in \mathscr{A}^+$. Since $\mathscr{A}^+$ is closed (Lemma 7.4(B)), one has

$$(\mu x_k, \theta_k) \to (\mu x, \theta) \in \mathscr{A}^+ \subset \mathscr{S}.$$

Consequently, $(x, \theta) \in \mathscr{S}$, since $\mu \neq 0$.

(B)   Let $\tau > 0$ be given by Lemma 7.10. Define β and K by $\beta\tau = \log 2$ and

$$K = e^{\beta\tau} \sup\{ |\Phi(\theta, t)\, x| : (x, \theta) \in \mathscr{S},\ |x| = 1 \text{ and } 0 \leqslant t \leqslant \tau \}.$$

Note that $\beta > 0$ and $K \geqslant 1$. We next present an induction argument to establish

$$|\Phi(\theta, t)\, x| \leqslant K\, |x|\, e^{-\beta(j+1)\tau}, \qquad t \in [\,j\tau, (j+1)\,\tau\,] \tag{7.1}$$

uniformly for $(x, \theta) \in \mathscr{S}$ and $j = 0, 1, 2, \dots$. From the definitions of K and β and the linearity of ϕ, we see that (7.1) is true for $j = 0$. Assuming that (7.1) is true for $j = k$, we verify it for $j = k + 1$. Let $\theta \in \Theta$ and $x \in \mathscr{S}(\theta)$ be chosen arbitrarily. Let $\theta_j \overset{\text{def}}{=} \theta \cdot j\tau$ and $x_j \overset{\text{def}}{=} \Phi(\theta, j\tau)$. One then has $(x_j, \theta_j) \in \mathscr{S}$, and (from Lemma 7.10) $|\Phi(\theta, s)\, x| \leqslant \frac{1}{2}|x|$ for $\tau \leqslant s \leqslant 2\tau$. Now the co-cycle identity implies that $\Phi(\theta_j, s)\, x_j = \Phi(\theta, j\tau + s)\, x$. By setting $t = k\tau + s$, where $(k+1)\,\tau \leqslant t \leqslant (k+2)\,\tau$, i.e., for $\tau \leqslant s \leqslant 2\tau$, we have

$$|\Phi(\theta, t)\, x| \leqslant \tfrac{1}{2}|\Phi(\theta, k\tau)\, x| \leqslant \tfrac{1}{2}K\, |x|\, e^{-\beta(k+1)\tau} = K\, |x|\, e^{-\beta(k+2)\tau},$$

which is precisely (7.1) for $j = k + 1$. Finally, given any $t \geqslant 0$ pick j so that $j\tau \leqslant t < (j+1)\,\tau$. Then from (7.1) one has

$$|\Phi(\theta, t)\, x| \leqslant K\, |x|\, e^{-\beta(j+1)\tau} \leqslant K\, |x|\, e^{-\beta t},$$

which completes the argument.   ∎

7.3. *Unstable Set: Stability and Exponential Decay*

Our next objective is to examine analogous properties for the unstable set $\mathscr{U}$. Since the flow in the infinite dimensional setting is not reversible, we cannot proceed as in the finite dimensional case where one simply replaces t with $-t$ and uses the argument of the last section. Nevertheless there are some inevitable similarities.

Recall that each $(x, \theta) \in \mathscr{U} = \mathscr{B}_u^-$ has a unique bounded negative continuation $\Phi(\theta, t)\, x$, $t \leqslant 0$. Also if $(x, \theta) \in \mathscr{A}^-$ then $|\Phi(\theta, t)\, x| \leqslant 1$, for $t \leqslant 0$.

7.12. **Lemma** (Uniform Stability).   *There exists a v, $0 < v \leqslant 1$ such that if $(x, \theta) \in \mathscr{U}$ and $|x| \leqslant v$, then $|\Phi(\theta, t)\, x| \leqslant 1$ for all $t \leqslant 0$; in particular, one has $(x, \theta) \in \mathscr{A}^-$.*

Proof.   If this were false, then there exists a sequence $(x_k, \theta_k) \in \mathscr{U}$, with $|x_k| \to 0$, and $b_k > 1$ where $b_k \overset{\text{def}}{=} \sup\{ |\Phi(\theta_k, s)\, x_k| : s \leqslant 0 \}$. Since $|x_k| \to 0$ there is a sequence of times $t_k < 0$ that satisfy $|\Phi(\theta_k, t_k)\, x_k| = 1$. Since $|\Phi(\theta_k, s)\, x_k| \to 0$ as $s \to -\infty$, there is a sequence $s_k \leqslant t_k < 0$ such that

$|\Phi(\theta_k, s_k) x_k| \leqslant k^{-1}$. Now set $\hat{x}_k = \Phi(\theta_k, s_k) x_k$, $\hat{\theta}_k = \theta_k \cdot s_k$. One then has $|\hat{x}_k| \leqslant k^{-1}$, $|x_k| = |\Phi(\hat{\theta}_k, -s_k) \hat{x}_k| \to 0$ and $|\Phi(\hat{\theta}_k, t_k - s_k) \hat{x}_k| = |\Phi(\theta_k, t_k) x_k| = 1$, which violates Lemma 7.3(A). ∎

7.13. LEMMA (EXPONENTIAL DECAY). *There is a $\tau < 0$ such that for any* $(x, \theta) \in \mathscr{U}$ *one has* $|\Phi(\theta, s) x| \leqslant \frac{1}{2}|x|$ *for all* $s \leqslant \tau$.

Proof. If false, then there are sequences $s_k \to -\infty$ and $(x_k, \theta_k) \in \mathscr{U}$ with $|\Phi(\theta_k, s) x_k| > \frac{1}{2}|x_k|$. By linearity one can assume that $|x_k| = v$, where v is given by Lemma 7.12. In this case, Lemma 7.12 implies that $(x_k, \theta_k) \in \mathscr{A}^-$ and $|\Phi(\theta_k, s_k) x_k| > \frac{1}{2}v$, which contradicts Lemma 7.7. ∎

Again we return to some of the features cited in Theorem A.

7.14. LEMMA. *The following statements are valid:*

(A)   *The unstable set $\mathscr{U}$ is closed.*

(B)   *There exist constants $K \geqslant 1$, $\beta > 0$ such that for all $(x, \theta) \in \mathscr{U}$ one has*

$$|\Phi(\theta, s) x| \leqslant K |x| e^{\beta s}, \qquad s \leqslant 0. \tag{7.2}$$

(C)   *For any $M > 0$ the ball bundle $\mathscr{U} \cap \mathscr{E}_M$ is compact.*

Proof. (A) Let $(x_k, \theta_k) \in \mathscr{U}$ with $(x_k, \theta_k) \to (x, \theta)$. If $x = 0$ then $(x, \theta) \in \mathscr{U}$. On the other hand, if $x \neq 0$ define μ by $2\mu |x| = v$ where v is given by Lemma 7.12. For k sufficiently large, it then follows from Lemma 7.12 that $(\mu x_k, \theta_k) \in \mathscr{A}^-$. Since $\mathscr{A}^-$ is compact (Lemma 7.4) one has

$$(\mu x_k, \theta_k) \to (\mu x, \theta) \in \mathscr{A}^- \subset \mathscr{U}.$$

Thus $(x, \theta) \in \mathscr{U}$, since $\mu \neq 0$.

(B)   Let $\tau < 0$ be given by Lemma 7.13. Define β by $-\beta\tau = \log 2$ and set

$$K \overset{\text{def}}{=} e^{-\beta\tau} \sup\{|\Phi(\theta, t) x|: (x, \theta) \in \mathscr{U}, |x| \leqslant 1, \tau \leqslant t \leqslant 0\}.$$

One then has $\beta > 0$ and $K \geqslant 1$. Note that if $(x, \theta) \in \mathscr{U}$ then $|\Phi(\theta, t) x| \leqslant K |x| e^{\beta\tau} = \frac{1}{2}K |x|$ for $\tau \leqslant t \leqslant 0$. Let $(x, \theta) \in \mathscr{U}$. We claim that if $t \in [(j+1)\tau, j\tau]$, then one has

$$|\Phi(\theta, t) x| \leqslant K |x| e^{\beta(j+1)\tau} \tag{7.3}$$

uniformly for $(x, \theta) \in \mathscr{U}$. Indeed by the linearity of the flow in x and by the definition of K and β we see that (7.3) is valid for $j = 0$. Now assume that

(7.3) is valid for $j = k$. We then verify it for $j = k + 1$. Let $\theta \in \Theta$ and $x \in \mathcal{U}(\theta)$ be fixed and set $x_j \stackrel{\text{def}}{=} \Phi(\theta, j\tau) x$ and $\theta_j \stackrel{\text{def}}{=} \theta \cdot j\tau$. One then has $(x_j, \theta_j) \in \mathcal{U}$ and (from Lemma 7.13) $|\Phi(\theta_j, s) y| \leqslant \frac{1}{2}|y|$ for $y \in \mathcal{U}(\theta_j)$ and $2\tau \leqslant s \leqslant \tau$. The cocycle identity implies that $\Phi(\theta_j, s) x_j = \Phi(\theta, j\tau + s) x$. Let $t = k\tau + s$, where $2\tau \leqslant s \leqslant \tau$. One then has

$$|\Phi(\theta, t) x| = |\Phi(\theta_k, s) \Phi(\theta, k\tau) x| \leqslant \frac{1}{2}|\Phi(\theta, k\tau) x|$$
$$\leqslant \frac{1}{2}K |x| e^{\beta(k+1)\tau} = K |x| e^{\beta(k+2)\tau}$$

which establishes (7.3) for $j = k + 1$. Finally given any $t \leqslant 0$, pick j so that $(j+1)\tau < t \leqslant j\tau$. From (7.3) one has

$$|\Phi(\theta, t) x| \leqslant K |x| e^{\beta(j+1)\tau} \leqslant K |x| e^{\beta t},$$

which completes the proof of (7.2).

(C) From (7.2) we see that $\pi(\mathcal{U} \cap \mathcal{E}_M, -t) \subset \mathcal{E}_{KM}$ for all $t \geqslant 0$. Hence $\mathcal{U} \cap \mathcal{E}_M \subset \pi(\mathcal{E}_{KM}, t)$ for all $t \geqslant 0$. Since $\alpha_\infty(\mathcal{U} \cap \mathcal{E}_M) \leqslant \alpha_\infty(\pi(\mathcal{E}_{KM}, t)) \leqslant 2KMk(t)$ for some function k with $k(t) \to 0$ as $t \to \infty$, it follows that $\alpha_\infty(\mathcal{U} \cap \mathcal{E}_M) = 0$. Since $\mathcal{U} \cap \mathcal{E}_M$ is closed, it is compact. ∎

7.4. *Semicontinuity of Dimension Functions*

Since $\mathcal{U} \cap \mathcal{E}_1$ is compact and the unit ball in a Banach space is compact if and only if the Banach space has finite dimension, one has the following.

7.15. LEMMA. *For every $\theta \in \Theta$, dim $\mathcal{U}(\theta)$ is finite.*

Throughout this subsection we fix $\theta \in \Theta$ and let $\mathcal{X}(\theta)$ denote an arbitrary closed linear subspace of the fiber $\mathcal{E}(\theta)$ with $\mathcal{X}(\theta) \cap \mathcal{S}(\theta) = \{0\}$. Furthermore, we let

$$\mathcal{X}(\theta) \cdot \tau \stackrel{\text{def}}{=} \Phi(\theta, \tau) \mathcal{X}(\theta), \quad \text{for} \quad \tau \geqslant 0.$$

7.16. LEMMA. *Let $\theta \in \Theta$ and $\mathcal{X}(\theta)$ be a finite dimensional subspace of $\mathcal{E}(\theta)$ that satisfies $\mathcal{S}(\theta) \cap \mathcal{X}(\theta) = \{0\}$. Then there is a $\tau > 0$, which depends on both θ and the subspace $\mathcal{X}(\theta)$, such that for all $x \in \mathcal{X}(\theta)$ one has*

$$|\Phi(\theta, t) x| \geqslant 2 |x|, \quad t \geqslant \tau.$$

Proof. If this is false, then there exist $x_n \in \mathcal{X}(\theta)$ and $\tau_n > 0$ with $|x_n| = 1$, $\tau_n \to \infty$, and such that $|\Phi(\theta, \tau_n) x_n| < 2$. Let $M_n \stackrel{\text{def}}{=} \max\{|\Phi(\theta, t) x_n| : 0 \leqslant t \leqslant \tau_n\}$. From Lemma 7.3(B) one has $\sup_n M_n = M < \infty$. By using a subsequence, if necessary, we may assume that $x_n \to x$ in $\mathcal{X}(\theta)$. Since $\pi(x_n, \theta, [0, \tau_n])$ remains in $\mathcal{E}_M$ for all n and $\tau_n \to \infty$, it follows from Lemma 7.1(A) that $(x, \theta) \in \mathcal{S}$. Finally the facts that $|x| = 1$

and $x \in \mathscr{K}(\theta)$ contradict the disjointness condition $\mathscr{K}(\theta) \cap \mathscr{S}(\theta) = \{0\}$. Since $\Phi(\theta, t) x = x$ at $t = 0$ one must have $\tau > 0$. ∎

7.17. LEMMA. *For all $(x, \theta) \in \mathscr{E}$ one has*

$$\limsup_{t \to \infty} |\Phi(\theta, t) x| = \liminf_{t \to \infty} |\Phi(\theta, t) x|,$$

i.e., $\lim_{t \to \infty} |\Phi(\theta, t) x|$ exists. Moreover, $x \notin \mathscr{S}(\theta)$ if and only if $\lim_{t \to \infty} |\Phi(\theta, t) x| = \infty$.

Proof. Let $(x, \theta) \in \mathscr{E}$ be given and define

$$L = \limsup_{t \to \infty} |\Phi(\theta, t) x|, \qquad \ell = \liminf_{t \to \infty} |\Phi(\theta, t) x|.$$

One then has $0 \leqslant \ell \leqslant L \leqslant \infty$. Furthermore, $L = 0$ if and only if $(x, \theta) \in \mathscr{S}$. Assume then that $L > 0$ or, equivalently, that $x \notin \mathscr{S}(\theta)$. If $L < \infty$, then $(x, \theta) \in \mathscr{B}^+ = \mathscr{S}$ by Lemma 7.2. Therefore we can assume that $L = \infty$. If $\ell < L = \infty$, then there exist sequences s_k, r_k, t_k such that $1 \leqslant s_k < r_k < t_k$ and $|\Phi(\theta, t) x| \leqslant (\ell + 1)$ for $t = s_k$ or t_k while $|\Phi(\theta, t) x| = v_k$ for $t = r_k$ (where v_k is defined by $v_k \overset{\text{def}}{=} \max\{|\Phi(\theta, t) x| : s_k \leqslant t \leqslant t_k\}$) and $v_k \to \infty$ as $k \to \infty$. However this contradicts Lemma 7.3(B). Hence $\ell = L = \infty$. ∎

7.18. LEMMA. *Fix $\theta \in \Theta$ and let ω denote any point in the omega limit set $\Omega(\theta)$. Let $\mathscr{K}(\theta)$ be any finite dimensional linear subspace of $\mathscr{E}(\theta)$ satisfying $\mathscr{S}(\theta) \cap \mathscr{K}(\theta) = \{0\}$. Then one has $\dim \mathscr{K}(\theta) \leqslant \dim \mathscr{U}(\omega)$.*

Proof. Let $k = \dim \mathscr{K}(\theta)$. Next choose a sequence $t_n \to \infty$ so that $t_{n+1} \geqslant t_n \geqslant \tau$, where τ is given by Lemma 7.16, and $\theta_n \overset{\text{def}}{=} \theta \cdot t_n \to \omega$. We claim that there exists a subsequence such that $\mathscr{K}(\theta) \cdot t_n \to \hat{\mathscr{K}}$ where $\hat{\mathscr{K}}$ is a closed linear subspace of $\mathscr{E}(\omega)$ with $\dim \hat{\mathscr{K}} = \dim \mathscr{K}(\theta)$. By the Riesz Theorem there is a basis $\{x_n^1, ..., x_n^k\}$ for $\mathscr{K}(\theta) \cdot t_n$ satisfying $|x_n^i| = 1$ for all i and n and

$$\text{dist}(x_n^{i+1}, \text{Span}\{x_n^1, ..., x_n^i\}) \geqslant \tfrac{1}{2}$$

for $i = 1, ..., k - 1$ and all n. Each (x_n^i, θ_n) admits a negative continuation $\phi_{i,n}(t) = (\phi^{x_n^i}(t), \theta \cdot t)$ for $-t_n \leqslant t \leqslant 0$, $1 \leqslant i \leqslant k$. For each n define

$$N_n \overset{\text{def}}{=} \max_{1 \leqslant i \leqslant k} \sup\{|\phi^{x_n^i}(t)| : -t_n \leqslant t \leqslant 0\}.$$

From Lemma 7.16 one has $|\phi^{x_n^i}(-t_n)| \leqslant \tfrac{1}{2}$. Because of Lemma 7.3(B) one must have $\sup_n N_n = N < \infty$.

Now define $H_n \overset{\text{def}}{=} \{(x_n^1, \theta_n), ..., (x_n^k, \theta_n)\}$, $J_n \overset{\text{def}}{=} \bigcup_{j \geqslant n} H_j$, $C_n \overset{\text{def}}{=} \bigcup_{j=1}^{n-1} H_j$, and $\Gamma_n \overset{\text{def}}{=} \{\phi_{i,j}(-t_n): 1 \leqslant i \leqslant k, j \geqslant n\}$. Then $\pi(\Gamma_n, t_n) = J_n$. Since $t_{n+1} \geqslant t_n$, it follows that $\Gamma_n \subset \mathscr{E}_N$. If we set $\Gamma \overset{\text{def}}{=} \bigcup_{n \geqslant 1} H_n$, we have

$$\Gamma = J_n \cup C_n = \pi(\Gamma_n, t_n) \cup C_n.$$

Since each C_n is compact, one has $\alpha_\infty(C_n) = 0$ for all n, and therefore

$$\alpha_\infty(\Gamma) = \max(\alpha_\infty(\pi(\Gamma_n, t_n)), \alpha_\infty(C_n)) \leqslant k(t_n) \alpha_\infty(\Gamma_n) \leqslant Nk(t_n).$$

Since $k(t_n) \to 0$, one has $\alpha_\infty(\Gamma) = 0$, i.e., Closure($\Gamma$) is compact. Thus one can choose subsequences so that $x_n^i \to y^i$ for $i = 1, ..., k$. Since

$$\text{dist}(y^{i+1}, \text{Span}\{y^1, ..., y^i\}) \geqslant \tfrac{1}{2}, \qquad \text{for} \quad i = 1, ..., k-1,$$

the collection $\{y^1, ..., y^k\}$ is linearly independent and forms a basis for $\hat{\mathscr{X}}$, where $\hat{\mathscr{X}} \overset{\text{def}}{=} \text{Span}\{y^1, ..., y^k\}$. It remains to show that $\hat{\mathscr{X}} \subset \mathscr{U}(\omega)$. In order to prove this it suffices to show that each basis element y^i is in $\mathscr{U}(\omega)$. But this follows from Lemma 7.1(B). ∎

Since codim $\mathscr{S}(\theta)$ is the supremum of the set

$$\{\dim \mathscr{X}(\theta): \mathscr{X}(\theta) \text{ is a finite dimensional subspace of } \mathscr{E}(\theta)$$

$$\text{satisfying } \mathscr{S}(\theta) \cap \mathscr{X}(\theta) = \{0\}\},$$

it follows from the last lemma that codim $\mathscr{S}(\theta) \leqslant \dim \mathscr{U}(\omega)$ for any point $\omega \in \Omega(\theta)$. Since $\mathscr{U}(\theta) \cap \mathscr{S}(\theta) = \{0\}$, one can clearly choose $\mathscr{U}(\theta) \subset \mathscr{X}(\theta)$. As a result of these facts and Lemma 7.15 we obtain the following.

7.19. LEMMA. *For every $\theta \in \Theta$ the stable set $\mathscr{S}(\theta)$ has finite codimension and* codim $\mathscr{S}(\theta) \leqslant \dim \mathscr{U}(\omega)$, *where ω is any point in the omega limit set $\Omega(\theta)$. Furthermore, there exists a closed linear subspace $\mathscr{X}(\theta)$ of finite dimension such that $\mathscr{X}(\theta) \cap \mathscr{S}(\theta) = \varnothing$ and $\mathscr{E}(\theta) = \mathscr{S}(\theta) + \mathscr{X}(\theta)$. Furthermore, $\mathscr{X}(\theta)$ can be chosen so that $\mathscr{U}(\theta) \subset \mathscr{X}(\theta)$. Moreover, for each $\theta \in \Theta$ one has* $\dim \mathscr{U}(\theta) \leqslant$ codim $\mathscr{S}(\theta) < \infty$ *and*

$$\dim \mathscr{U}(\theta) = \text{codim } \mathscr{S}(\theta) \Leftrightarrow \mathscr{E}(\theta) = \mathscr{U}(\theta) + \mathscr{S}(\theta).$$

We are now ready to prove the semi-continuity properties of the functions codim $\mathscr{S}$ and dim $\mathscr{U}$.

7.20. LEMMA. *The function* codim $\mathscr{S}(\theta)$ *is lower semi-continuous on Θ, i.e., one has*

$$\text{codim } \mathscr{S}(\theta) \leqslant \liminf_{\eta \to \theta} \text{codim } \mathscr{S}(\eta), \qquad \theta \in \Theta.$$

Proof. Let $k = \operatorname{codim} \mathscr{S}(\theta)$ and let $x^1, ..., x^k$ be a linearly independent set of vectors with $x^i \notin \mathscr{S}(\theta)$, $i = 1, ..., k$. In order to prove this lemma, it suffices to show that there is a neighborhood $U(\theta)$ such that

$$x^i \notin \mathscr{S}(\eta), \qquad \text{for all } i \text{ and } \eta \in U(\theta). \tag{7.4}$$

Indeed if (7.4) were not true, then there exists a sequence $\theta_n \to \theta$ and i_n, $1 \leqslant i_n \leqslant k$, such that $x^{i_n} \in \mathscr{S}(\theta_n)$. Since the set $\{x^1, ..., x^k\}$ is finite, we can assume, without any loss of generality, that $x^1 \in \mathscr{S}(\theta_n)$ for all n. One then has $(x^1, \theta_n) \in \mathscr{S}$ and $(x^1, \theta_n) \to (x^1, \theta) \in \mathscr{S}$ since $\mathscr{S}$ is closed. This is a contradiction. ∎

7.21. LEMMA. *Let $\mathscr{V}$ be a closed subset of $\mathscr{E}$ where each fibre $\mathscr{V}(\theta)$ is a linear subspace of $\mathscr{E}(\theta)$ and assume that the unit ball bundle $\mathscr{V}_1 \overset{\text{def}}{=} \{(x, \theta) \in \mathscr{V}: |x| = 1\}$ is compact in $\mathscr{E}$. Let θ_n be a sequence in Θ with $\theta_n \to \theta$ and $\dim \mathscr{V}(\theta_n) \geqslant k$ for all n. Then $\dim \mathscr{V}(\theta) \geqslant k$. In other words, $\dim \mathscr{V}(\theta)$ is upper semicontinuous on Θ; i.e., one has*

$$\dim \mathscr{V}(\theta) \geqslant \limsup_{\eta \to \theta} \dim \mathscr{V}(\eta), \qquad \theta \in \Theta.$$

Proof. Let $x_n^1, ..., x_n^k$ be a linearly independent set in $\mathscr{V}(\theta_n)$ and choose $\ell_n^1, ..., \ell_n^k \in X^*$, the dual space to X, such that $|x_n^i| = |\ell_n^i| = 1$ for all i, n and $\ell_n^i(x_n^j) = \delta^{ij}$, the Kronecker delta. By the assumed compactness of $\mathscr{V}_1$ and the weak compactness of the unit ball in X^*, there exist convergent subsequences $x_n^i \to x^i \in \mathscr{V}(\theta)$ and ℓ_n^j converges weakly to m^j; i.e., $\ell_n^j \rightharpoonup m^j$, for all i, j. Let $\varepsilon > 0$ be given. Now fix i, j so that $1 \leqslant i, j \leqslant k$. We claim that for n sufficiently large one has

$$|m^j(x^i) - \delta^{ji}| = |m^j(x^i) - \ell_n^j(x_n^i)| \leqslant |m^j(x^i) - \ell_n^j(x^i)| + |\ell_n^j(x^i) - \ell_n^j(x_n^i)|$$

$$< \varepsilon + |x^i - x_n^i| \leqslant 2\varepsilon.$$

Indeed, the first term in the last inequality follows from the weak convergence of ℓ_n^j while the strong convergence of x_n^i implies the second. Consequently, one has $m^j(x^i) = \delta^{ji}$; i.e., there exist k linearly independent vectors $x^1, ..., x^k \in \mathscr{V}(\theta)$. ∎

The next result concerning the behavior of $\dim \mathscr{U}(\theta)$ follows immediately from Lemmas 7.14(C) and 7.21.

7.22. LEMMA. *The function $\dim \mathscr{U}(\theta)$ is upper semi-continuous on Θ.*

7.23. LEMMA. *For each $\theta \in \Theta$ the function $\operatorname{codim} \mathscr{S}(\theta \cdot t)$ is a non-decreasing function of t. Furthermore, one has $\dim \mathscr{U}(\theta \cdot t)$ is constant for $t \in R$.*

Proof. Let $\mathcal{K}(\theta)$ be a full complement of $\mathcal{S}(\theta)$, i.e., one has $\mathcal{K}(\theta) \cap \mathcal{S}(\theta) = \{0\}$ and $\mathcal{E}(\theta) = \mathcal{K}(\theta) + \mathcal{S}(\theta)$. We claim that

$$\mathcal{K}(\theta) \cdot \tau \cap \mathcal{S}(\theta \cdot \tau) = \{0\}, \qquad \tau \geqslant 0. \tag{7.5}$$

If (7.5) fails for some τ, $0 \leqslant \tau$, then the two sets $\mathcal{K}(\theta) \cdot \tau$ and $\mathcal{S}(\theta \cdot \tau)$ would have a common element $\hat{x} \neq 0$. This means that there is an $x \in \mathcal{K}(\theta)$ with $\hat{x} = \Phi(\theta, \tau) x$. But $\hat{x} \in \mathcal{S}(\theta \cdot \tau)$ implies that $x \in \mathcal{S}(\theta)$. Thus $x = 0$, which contradicts the fact that $\hat{x} \neq 0$.

It follows from (7.5) that

$$\operatorname{codim} \mathcal{S}(\theta \cdot \tau) \geqslant \dim \mathcal{K}(\theta) \cdot \tau = \dim \mathcal{K}(\theta) = \operatorname{codim} \mathcal{S}(\theta)$$

for all $\tau \geqslant 0$. Finally by replacing θ by $\theta \cdot s$ and τ by $(s + t)$ where $s \in R$ and $t \geqslant 0$, one obtains $\operatorname{codim} \mathcal{S}(\theta \cdot (s + t)) \geqslant \mathcal{S}(\theta \cdot s)$.

Note that if $\Phi(\theta, t) x = 0$ for some $t \geqslant 0$, then $x \in \mathcal{S}(\theta)$. Since $\mathcal{S}(\theta) \cap \mathcal{U}(\theta) = \{0\}$, it follows that $\Phi(\theta, t)$ is a one-to-one mapping of $\mathcal{U}(\theta)$ into $\mathcal{U}(\theta \cdot t)$ for $t \geqslant 0$. We claim that $\Phi(\theta, t)$ maps $\mathcal{U}(\theta)$ onto $\mathcal{U}(\theta \cdot t)$. Indeed, let $\phi(\tau) = (\phi^x(\tau), \theta \cdot (\tau + t))$ be a negative continuation of $(x, \theta \cdot t)$ with $x \in \mathcal{U}(\theta \cdot t)$. Since $|\phi^x(\tau)| \to 0$ as $\tau \to -\infty$, one has $\phi^x(-t) \in \mathcal{U}(\theta)$. Thus $x = \Phi(\theta, t) \phi^x(-t) \in \Phi(\theta, t) \mathcal{U}(\theta)$, or $\mathcal{U}(\theta \cdot t) \subset \Phi(\theta, t) \mathcal{U}(\theta)$. Consequently one has $\dim \mathcal{U}(\theta) = \dim \mathcal{U}(\theta \cdot t)$ for all $t \geqslant 0$. By replacing θ by $\theta \cdot s$ for $s \leqslant 0$, we conclude that $\dim \mathcal{U}(\theta \cdot t)$ is constant for $t \in R$. ∎

7.5. *Proof of Theorems* A *and* B

Proof of Theorem A. This now follows directly from Lemmas 7.2, 7.11, 7.14, 7.19, 7.20, and 7.22. ∎

Proof of Theorem B. Let $\tilde{\Theta}$ be a closed invariant subset of Θ and set $\mathcal{E}(\tilde{\Theta}) = X \times \tilde{\Theta}$ and $\mathcal{S}(\tilde{\Theta}) = \{(x, \theta) \in \mathcal{S} : \theta \in \tilde{\Theta}\}$. Assume that $\operatorname{codim} \mathcal{S}(\theta) = k$ for all $\theta \in \tilde{\Theta}$. Since $\mathcal{S}(\tilde{\Theta})$ is closed, it follows from Lemma 3.2 that $\mathcal{S}(\tilde{\Theta})$ is a subbundle of $\mathcal{E}(\tilde{\Theta})$. Consequently there is a projector $\mathbf{L}$ on $\mathcal{E}(\tilde{\Theta})$ with $\mathcal{R}(\mathbf{L}) = \mathcal{S}(\tilde{\Theta})$ and $\mathcal{N}(\mathbf{L}) = \mathcal{K}$, where $\mathcal{K}$ is a complementary subbundle for $\mathcal{S}(\tilde{\Theta})$ in $\mathcal{E}(\tilde{\Theta})$. The Special Hypothesis in Section 6 is satisfied with $\mathcal{V} = \mathcal{S}(\tilde{\Theta})$. Let $\hat{\pi}$ denote the induced flow on $\mathcal{K}$. Note that because of Lemma 7.11 the hypothesis (6.8) of Lemma 6.4 is satisfied. Since $\mathcal{B} = \mathcal{E}_0$ is trivial, it follows from Lemma 6.4 that $\hat{\mathcal{B}}$, the bounded set for the induced flow on $\mathcal{K}$, is also trivial. Since $\mathcal{K}$ is a finite dimensional bundle, it follows from Sacker and Sell (1974, 1976a) and Selgrade (1975) that $\hat{\pi}$ admits an exponential dichotomy over every minimal set $M \subset \tilde{\Theta}$. This implies that $\hat{\mathcal{S}}(\theta) \cap \hat{\mathcal{U}}(\theta) = \{0\}$ and $\mathcal{K}(\theta) = \hat{\mathcal{S}}(\theta) + \hat{\mathcal{U}}(\theta)$ for all $\theta \in M$.

Next we claim that for every minimal set $M \subset \tilde{\Theta}$ one has $\hat{\mathcal{S}}(\theta) = \{0\}$ for all $\theta \in M$. Indeed let $(w, \theta) \in \hat{\mathcal{S}}$ for $\theta \in M$ where M is a minimal set in $\tilde{\Theta}$. Using the (v, w) representation from Section 6, we let $w_t = D(\theta, t) w$ and let

v_t be given by Lemma 6.5. Since $(w, \theta) \in \hat{\mathscr{S}}$ there are constants $K_1 \geqslant 1$ and $\alpha > 0$ such that $|D(\theta, t) w| \leqslant K_1 |w| e^{-\alpha t}$ for $t \geqslant 0$. Since $\mathscr{V} = \mathscr{S}$ there are constants $K_2 \geqslant 1$ and $\beta > 0$ such that $|A(\theta, t-s)| \leqslant K_2 e^{-\beta(t-s)}$ for $0 \leqslant s \leqslant t$. Lemma 6.5 then implies that there is a constant $K_3 > 0$ such that $|v_t| \leqslant K_3$ for all $t \geqslant 0$. In other words, $\binom{v_t}{w_t}$ is bounded for $t \geqslant 0$. Hence $\binom{v_0}{w_0} = \binom{0}{w} \in \mathscr{B}^+(\theta) = \mathscr{S}(\theta)$ by Lemma 7.2. Since $\binom{0}{w} \in \mathscr{K}(\theta)$ one has $w = 0$, which completes the proof that $\hat{\mathscr{S}}(\theta) = \{0\}$ for all $\theta \in M$.

Since $\dim \hat{\mathscr{U}}(\theta) = k$ for θ in any minimal set in $\tilde{\Theta}$, it follows from the finite dimensional Compatibility Theorem in Sacker and Sell (1976a, Theorem 2) that $\hat{\pi}$ has an exponential dichotomy over $\tilde{\Theta}$, which in turn implies that

$$\dim \hat{\mathscr{U}}(\theta) = \dim \mathscr{K}(\theta) = k, \qquad \text{for all} \quad \theta \in \tilde{\Theta}.$$

Since $\hat{\mathscr{U}} = \hat{\mathscr{B}}^- = \mathscr{K}$, the linear mapping $T(\theta): w_0 \to \binom{v_0}{w_0}$ constructed in Lemma 6.4 is defined for all $(w_0, \theta) \in \mathscr{K}$ and the range satisfies $\mathscr{R}(T(\theta)) \subset \mathscr{B}^-(\theta) = \mathscr{U}(\theta)$. Since $T(\theta)$ is one-to-one one has $k \leqslant \dim \mathscr{U}(\theta)$ for $\theta \in \tilde{\Theta}$. From Lemma 7.19 one has $\dim \mathscr{U}(\theta) \leqslant \operatorname{codim} \mathscr{S}(\theta) = k$. Consequently, $T(\theta): \mathscr{K}(\theta) \to \mathscr{U}(\theta)$ is a one-to-one bounded mapping of $\mathscr{K}(\theta)$ onto $\mathscr{U}(\theta)$ for all $\theta \in \tilde{\Theta}$. It then follows that $\mathscr{E}(\theta) = \mathscr{S}(\theta) + \mathscr{U}(\theta)$ for all $\theta \in \tilde{\Theta}$.

For $\theta \in \tilde{\Theta}$ define $M(\theta)$ as the restriction of $(I - L(\theta))$ to $\mathscr{U}(\theta)$, i.e.,

$$M(\theta) \stackrel{\text{def}}{=} (I - L(\theta))|_{\mathscr{U}(\theta)}.$$

Since $\dim \mathscr{K}(\theta) = \dim \mathscr{U}(\theta)$, $M(\theta)$ is a one-to-one mapping of $\mathscr{U}(\theta)$ onto $\mathscr{K}(\theta)$ with inverse $M^{-1}(\theta): \mathscr{K}(\theta) \to \mathscr{U}(\theta)$. Furthermore, one has $M^{-1}(\theta) = T(\theta)$ and

$$(I - L(\theta)) M^{-1}(\theta)(I - L(\theta)) = I - L(\theta). \tag{7.6}$$

Now let $Q(\theta) \stackrel{\text{def}}{=} M^{-1}(\theta)(I - L(\theta))$. Then by (7.6) one has

$$Q^2(\theta) = M^{-1}(\theta)(I - L(\theta)) M^{-1}(\theta)(I - L(\theta)) = M^{-1}(\theta)(I - L(\theta)) = Q(\theta),$$

i.e., $Q(\theta)$ is a projection. It is easily seen that $\mathscr{R}(Q(\theta)) = \mathscr{U}(\theta)$ and $\mathscr{N}(Q(\theta)) = \mathscr{N}(I - L(\theta)) = \mathscr{R}(L(\theta)) = \mathscr{S}(\theta)$. Let $P(\theta) = I - Q(\theta)$. Since $\mathscr{S}$ and $\mathscr{U}$ are closed (Theorem A) and since $L(\theta)$ varies continuously in θ in the operator norm, it follows that $P(\theta)$ is also continuous in $\theta \in \tilde{\Theta}$. This completes the proof of the first three items in Theorem B. Since $\mathscr{S}$ and $\mathscr{U}$ are positively invariant, it is easily verified that the projector P is invariant; i.e., (3.3) holds. The final item, the exponential dichotomy over $\tilde{\Theta}$, is now an immediate consequence of the above properties of $P(\theta)$ and Theorem A. ∎

7.6. *Dichotomies over ω-Limit Sets*

In the next two lemmas we show that the given linear evolutionary system admits an exponential dichotomy over every omega limit set in Θ.

7.24. LEMMA. *Fix $\theta \in \Theta$. Then there exist integers $0 \leqslant k_1 \leqslant k_2$ with the following properties:*

(1) $\lim_{t \to -\infty} \operatorname{codim} \mathscr{S}(\theta \cdot t) = k_1$.

(2) $\lim_{t \to \infty} \operatorname{codim} \mathscr{S}(\theta \cdot t) = k_2$.

(3) *For all t sufficiently large, one has* $\operatorname{codim} \mathscr{S}(\theta \cdot t) = k_2$ *and*

$$\operatorname{codim} \mathscr{S}(\alpha) \leqslant k_1 \leqslant \operatorname{codim} \mathscr{S}(\theta) \leqslant \operatorname{codim} \mathscr{S}(\omega) = k_2, \qquad (7.7)$$

for all $\alpha \in A(\theta)$ and $\omega \in \Omega(\theta)$. In particular, $\operatorname{codim} \mathscr{S}(\cdot)$ is constant on $\Omega(\theta)$, and $\dim \mathscr{U}(\omega) = \operatorname{codim} \mathscr{S}(\omega)$ for all $\omega \in \Omega(\theta)$.

Proof. The limit in (1) exists since $\operatorname{codim} \mathscr{S}(\theta \cdot t)$ is nondecreasing (Lemma 7.23), integer-valued, and bounded below. Let $k_3 \stackrel{\text{def}}{=} \min \{ \dim \mathscr{U}(\omega) : \omega \in \Omega(\theta) \}$. Then by Lemma 7.19, one has $\operatorname{codim} \mathscr{S}(\theta \cdot \tau) \leqslant k_3$ for every $\tau \in R$. Hence $\lim_{t \to \infty} \operatorname{codim} \mathscr{S}(\theta \cdot t) = k_2$ exists and $k_2 \leqslant k_3$. By Lemma 7.23 one has $k_1 \leqslant k_2$. Since $\operatorname{codim} \mathscr{S}(\theta)$ is integer-valued one must have $\operatorname{codim} \mathscr{S}(\theta \cdot t) = k_2$ for all large t.

The first inequality in (7.7) is a result of the lower semicontinuity of $\operatorname{codim} \mathscr{S}(\theta)$. The second follows from the monotonicity of $\operatorname{codim} \mathscr{S}(\theta \cdot t)$. The third inequality follows from Lemma 7.19. The lower semicontinuity of $\operatorname{codim} \mathscr{S}$ implies that for any $\omega \in \Omega(\theta)$ one has $\operatorname{codim} \mathscr{S}(\omega) \leqslant \operatorname{codim} \mathscr{S}(\theta \cdot t)$ for t large. Hence

$$\operatorname{codim} \mathscr{S}(\omega) \leqslant k_2 \leqslant k_3 \leqslant \dim \mathscr{U}(\omega) \leqslant \operatorname{codim} \mathscr{S}(\omega). \quad \blacksquare$$

In the light of Theorem B and Lemma 7.24, the following result is immediate.

7.25. LEMMA. *Fix $\theta \in \Theta$. Then π admits an exponential dichotomy over the omega limit set $\Omega(\theta)$, i.e., there exists an integer $k \geqslant 0$ such that the following properties hold:*

(1) $\operatorname{codim} \mathscr{S}(\omega) = \dim \mathscr{U}(\omega) = k$ *for all $\omega \in \Omega(\theta)$, i.e., $\Omega(\theta)$ lies in the Morse set M_k.*

(2) $\mathscr{E}(\omega) = \mathscr{S}(\omega) + \mathscr{U}(\omega)$ *for all $\omega \in \Omega(\theta)$.*

(3) *Let $P(\omega): X \to X$ denote the linear projection on X with range $\mathscr{S}(\omega)$ and null space $\mathscr{U}(\omega)$. Then $P(\omega)$ varies continuously in the operator norm for all $\omega \in \Omega(\theta)$, and P is invariant.*

(4)   *Let $K \geqslant 1$ and $\beta > 0$ be given by Lemmas 7.11 and 7.15. Then*

$$|\Phi(\omega, t) P(\omega)|_{\text{op}} \leqslant Ke^{-\beta t}, \qquad t \geqslant 0, \ \omega \in \Omega(\theta),$$

$$|\Phi(\omega, t)[I - P(\omega)]|_{\text{op}} \leqslant Ke^{\beta t}, \qquad t \leqslant 0, \ \omega \in \Omega(\theta).$$

Since every minimal set in Θ is an omega limit set we have the following corollary to Lemma 7.25.

7.26. LEMMA.   *For every minimal set $M \subset \Theta$ there is an integer k such that the conclusions of Lemma 7.25 hold over M. In particular, M lies in the Morse set M_k.*

7.27. LEMMA.   *Let $\theta \in \Theta$ and assume that there is an integer $k \geqslant 0$ such that $A(\theta) \cup \Omega(\theta) \subset M_k$. Then $\theta \in H(\theta) \subset M_k$.*

Proof.   From (7.7) one has

$$\text{codim } \mathscr{S}(\alpha) = \text{codim } \mathscr{S}(\theta) = \text{codim } \mathscr{S}(\omega) = k$$

for all $\alpha \in A(\theta)$, and $\omega \in \Omega(\theta)$. In other words, codim $\mathscr{S}(\eta) = k$ for all $\eta \in H(\theta)$. Since the hull $H(\theta)$ is compact and invariant, one has $\theta \in H(\theta) \subset M_k$, by Theorem B.   ∎

7.7. *Dichotomies over the Morse Sets*

We next show that the given linear evolutionary system admits an exponential dichotomy over every nonempty Morse set M_k in Θ. For each integer $k \geqslant 0$, let Θ_k, A_k, B_k be given by (3.4), where $\Theta_k = A_k \cap B_k$, and let M_k denote the Morse set, i.e., M_k is the maximal compact invariant subset of Θ_k. Since dim $\mathscr{U}(\theta)$ is finite everywhere and upper semicontinuous on the compact set Θ, the function dim $\mathscr{U}$ is bounded above. As a result we have the following:

7.28. LEMMA.   *There is a $k < \infty$ such that dim $\mathscr{U}(\theta) \leqslant k$ for all $\theta \in \Theta$.*

7.29. LEMMA.   *The following statements are valid:*

(1)   *Each set Θ_k is a closed set in Θ.*

(2)   *Each set Θ_k is negatively invariant.*

(3)   *The Morse set M_k is nonempty if and only if Θ_k is nonempty.*

(4)   *At least one, and at most finitely many, Θ_k are nonempty.*

(5)   *Let $q = \max\{k : \Theta_k \neq \varnothing\}$. Then q is finite and $M_q = \Theta_q$ is invariant.*

(6)   *codim $\mathscr{S}(\theta) \leqslant q$ for all $\theta \in \Theta$.*

Proof. (1) By Lemmas 7.20 and 7.22 the sets A_k and B_k are closed. Consequently Θ_k is closed.

(2) Let $\theta \in \Theta_k$ and $\tau \leqslant 0$. Since dim $\mathscr{U}(\eta \cdot t)$ is constant in t and codim $\mathscr{S}(\eta \cdot t)$ is nondecreasing in t, Lemma 7.19 implies that

$$\dim \mathscr{U}(\theta) = \dim \mathscr{U}(\theta \cdot \tau) \leqslant \operatorname{codim} \mathscr{S}(\theta \cdot \tau) \leqslant \operatorname{codim} \mathscr{S}(\theta) = \dim \mathscr{U}(\theta) = k.$$

Hence dim $\mathscr{U}(\theta \cdot \tau) = \operatorname{codim} \mathscr{S}(\theta \cdot \tau) = k$, and $\theta \cdot \tau \in \Theta_k$; i.e., Θ_k is negatively invariant.

(3) Since $M_k \subset \Theta_k$, it suffices to show that M_k is nonempty whenever Θ_k is nonempty. Let $\theta \in \Theta_k$. Since Θ_k is closed and negatively invariant, the alpha limit set $A(\theta) \subset \Theta_k$. Since $A(\theta)$ is nonempty and invariant, one has $A(\theta) \subset M_k$; i.e., M_k is nonempty.

(4) The fact that at most finitely many Θ_k are nonempty follows directly from Lemma 7.28. Since the flow on Θ contains a minimal set, it follows from Lemma 7.26 that at least one Θ_k is nonempty.

(5 and 6) From statement (4) q is finite. Let $\theta \in \Theta_q$. Since dim $\mathscr{U}(\theta \cdot t)$ is constant in t and codim $\mathscr{S}(\theta \cdot t)$ is nondecreasing in t, we need only show that codim $\mathscr{S}(\theta \cdot t)$ cannot increase for $t \geqslant 0$. However, if codim $\mathscr{S}(\theta \cdot t) > q$ for some $t \geqslant 0$, then by Lemma 7.24 codim $\mathscr{S}(\omega) > q$ for any $\omega \in \Omega(\theta)$. But then Lemma 7.25 yields $\Omega(\theta) \subset \Theta_k$ for some $k > q$, which is impossible. As a result, Θ_q is invariant. Lastly, since Θ_q is closed one has $M_q = \Theta_q$. ∎

7.30. LEMMA. *Assume that there is precisely one nonempty Θ_k in Θ. Then $\Theta = \Theta_k$ and the conclusions of Theorem B are valid over Θ.*

Proof. If there is precisely one nonempty Θ_k then $\Theta_k = \Theta_q$ where q is defined in Lemma 7.29(5). By Lemma 7.29, Θ_q is compact and invariant. Furthermore, because of Lemmas 7.25 and 7.26, every omega limit set $\Omega(\theta) \subset \Theta_q$ and every minimal set $M \subset \Theta_q$. We claim that codim $\mathscr{S}(\theta) = q$ for all $\theta \in \Theta$. If there is a $\theta_0 \in \Theta$ such that codim $\mathscr{S}(\theta_0) = k < q$ then (Lemma 7.24) for all $\alpha \in A(\theta_0)$ one has codim $\mathscr{S}(\alpha) \leqslant \operatorname{codim} \mathscr{S}(\theta_0) = k < q$. Hence there is a minimal set $M \subset A(\alpha)$ with $M \subset B_k$, i.e., $M \cap \Theta_q = \varnothing$, a contradiction. It then follows from Theorem B that $\Theta = \Theta_q$, and the conclusions of Theorem B are valid over Θ. ∎

7.8. *Proof of Theorems C, D, E, and F*

Proof of Theorem C. Without loss of generality we can assume that $\tilde{\Theta} = \Theta$. (Otherwise, one can restrict the flow to $\tilde{\Theta}$.) Since dim $\mathscr{U}(\theta) = k$ for all $\theta \in \Theta$, it follows from the definition of Θ_k that at most one Θ_k is nonempty. Since there is at least one nonempty Θ_k (Lemma 7.29), Theorem C is now a consequence of Lemma 7.30. ∎

For the remaining theorems we need some properties of chain recurrence which we summarize here. The material from the next few paragraphs is taken from Conley (1978, pp. 32–38).

Let Z be a compact Hausdorff space with a two-sided flow $\sigma(z, t) = z \cdot t$. For any set $U \subset Z$ the α- and ω-limit sets are

$$A(U) = \bigcap_{\tau \leqslant 0} \mathrm{Cl}\{\sigma(U, \tau + t): t \leqslant 0\}, \qquad \Omega(U) = \bigcap_{\tau \geqslant 0} \mathrm{Cl}\{\sigma(U, \tau + t): t \geqslant 0\}.$$

A set $A \subset Z$ is said to be a (local) *attractor* if there is a neighborhood U of A with $\Omega(U) = A$, and A is said to be a (local) *repeller* if $A(U) = A$. Given any attractor $A \subset Z$ one defines A^* to be the set of $z \in Z$ for which $\Omega(z) \cap A = \varnothing$. Then it is shown that A^* is a repeller and is referred to as the *repeller complementary* to A. Furthermore, one has

$$Z \backslash (A \cup A^*) = \{z \in Z : A(z) \subset A^* \text{ and } \Omega(z) \subset A\}. \tag{7.10}$$

Because of (7.10) the set $C(A^*, A) = Z \backslash (A \cup A^*)$ is the collection of all orbits *connecting* A^* to A. In a similar way, given any repeller R one defines the attractor R^* complementary to R to be the set of all $z \in Z$ for which $A(z) \cap R = \varnothing$. Equation (7.10) is also valid in that case with (R^*, R) replacing (A, A^*). Note that if A is an attractor (or R is a repeller), then $A^{**} = A$ (or $R^{**} = R$). The pair (A, A^*) (or (R^*, R)) is called an *attractor–repeller pair*. One of the main properties of attractors and repellers which we need is the following result; see Conley (1978, p. 33). (Also see Sacker (1974).)

7.31. LEMMA.   *Let U be a compact set in Z. Then the following are valid*:

(A)   *If for every $z \in \mathrm{Bdy}\ U$ there is a time $t_z > 0$ such that $z \cdot t_z \in Z \backslash U$, then the maximal invariant set in U is a nonempty repeller.*

(B)   *If, instead, one has $z \cdot (-t_z) \in Z \backslash U$, then the maximal invariant set in U is a nonempty attractor.*

The next concept is that of chain recurrence. We begin with an open covering $\mathcal{U}$ of Z. For example, if Z is a metric space, then $\mathcal{U}$ may be a covering of Z with open balls of diameter ε. Let $L > 0$ be given and let $y, z \in Z$. A $(\mathcal{U}, L)$ *chain* connecting y to z consist of finite sequences $x_1, ..., x_k$ and $t_1, ..., t_{k-1}$ satisfying $x_1 = y$, $x_k = z$, $t_i \geqslant L$, for $1 \leqslant i \leqslant k - 1$, and for each i, $1 \leqslant i \leqslant k - 1$, there is a $U_i \in \mathcal{U}$ such that both $x_i \cdot t_i$ and x_{i+1} lie in U_i. The flow on Z is said to be *chain recurrent* if for every $z \in Z$ and for any covering $\mathcal{U}$ and any $L > 0$ there is an $(\mathcal{U}, L)$ chain connecting z to itself. The key property we need is the following.

7.32. LEMMA.   (A)   *If Z has an attractor-repeller pair (A, A^*) with $C(A^*, A) \neq \varnothing$, then the flow on Z is not chain recurrent.*

(B) *For any $z \in Z$, the flows on the alpha limit set $A(z)$ and on the omega limit set $\Omega(z)$ are chain recurrent.*

The following result is an application of the last two lemmas.

7.33. LEMMA. *Let Z be a compact Hausdorff space with a (two-sided) flow σ. Assume that there are pairwise disjoint sets D, E, and F in Z such that E is nonempty, D and F are compact sets, and $Z = D \cup E \cup F$. Then the following hold.*

(A) *If for all $z \in E$ one has $\Omega(z) \subset F$, then D and F are nonempty and the flow on Z is not chain recurrent. If in addition D is invariant, then D is a repeller.*

(B) *If for all $z \in E$ one has $A(z) \subset D$, then D and F are nonempty and the flow on Z is not chain recurrent. If in addition F is invariant, then F is an attractor.*

Proof. We will prove (A) and note that the proof of (B) is similar. Clearly F must be nonempty since E is nonempty and Z is compact. Let U be a compact set in $D \cup E$ with $D \subset \text{Int } U$. Then $\text{Bdy } U \subset E$. Since for every $z \in \text{Bdy } U$ one has $\Omega(z) \subset F$, there is a time $t_z > 0$ such that $z \cdot t_z \in Z \setminus U$. By Lemma 7.31, the maximal invariant set A^* in U is a repeller. Let A denote the attractor complementary to A^*. Then $A \subset F$. Since $E \subset C(A^*, A)$, it follows from Lemma 7.32 that the flow on Z is not chain recurrent. Since $A^* \subset D$, D is nonempty. If D is invariant, then for all $z \in D$ one has $\Omega(z) \cap A \subset \Omega(z) \cap F = \varnothing$. Consequently, $D \subset A^*$; i.e., $D = A^*$ is a repeller. ∎

Proof of Theorem D. Assume that the flow on Θ is chain recurrent and define q by $q = \max\{\text{codim } \mathscr{S}(\theta) : \theta \in \Theta\}$. Because of Theorem C, we need to show that $\Theta = \bigcup_{i=0}^{q} M_i$. Assume that this is not the case and let k be the smallest integer such that

$$M_{k+1} \cup \cdots \cup M_q = \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) \geqslant k+1\}. \tag{7.11}$$

Note that for $k = q - 1$ one has equality in (7.11) by Lemma 7.29 (5 and 6). Thus $k + 1 \leqslant q$, and by assumption one has $1 \leqslant k + 1$. Set $G \stackrel{\text{def}}{=} \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) \geqslant k\}$. Because of the lower semicontinuity of codim $\mathscr{S}$, the set G is open, and by the definition of k one has

$$F \stackrel{\text{def}}{=} M_k \cup \cdots \cup M_q \subsetneqq G.$$

Note that F is a closed set since the Morse sets M_i are closed. Hence $E \stackrel{\text{def}}{=} G \setminus F$ is nonempty. Furthermore, since $E \subset G$ and since E does not meet any of the Morse sets $M_k, ..., M_q$, one has

$$E = \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) = k, \dim \mathscr{U}(\theta) \leqslant k - 1\}.$$

By Lemmas 7.24 and 7.25 one has $\Omega(\theta) \subset F$ for all $\theta \in E$. Finally,

$$D \overset{\text{def}}{=} B_{k-1} = \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) \leqslant k-1\}$$

is compact, because codim $\mathscr{S}$ is lower semicontinuous. Hence by Lemma 7.33(A) the flow on Θ is not chain recurrent, a contradiction. Finally, since $\Theta = \bigcup_{i=0}^{q} M_i$, each Morse set is both open and closed in Θ. Therefore if Θ is connected, then it lies in a single Morse set. ∎

7.34. LEMMA. *For every $\theta \in \Theta$, the linear evolutionary system $\pi = (\Phi, \sigma)$ admits an exponential dichotomy over the alpha limit set $A(\theta)$, i.e., the conclusions of Theorem* B *hold over $A(\theta)$ for some k. In particular, $A(\theta) \subset M_k$ for this k.*

Proof. We use the fact that every alpha limit set is connected and chain recurrent (Lemma 7.32). One then restricts π to $X \times A(\theta)$ and applies Theorem D. ∎

Proof of Theorem E. Lemma 7.29 assures us that only finitely many Θ_k are nonempty. The conclusion for a single nonempty Θ_k is given in Lemma 7.30. Now assume that there exist at least two nonempty Θ_k and let p and q be respectively the min and max of the set $\{k : \Theta_k \neq \varnothing\}$. As a result of Lemmas 7.24, 7.25, 7.27, 7.29, and 7.34, we need only show that M_q is an attractor and M_p is a repeller in the flow σ on Θ. As we now show, this follows in turn from Lemma 7.33. Indeed, in the case of M_q, we set $D = B_{q-1}$ (where B_{q-1} is defined in Section 3.5), $F = M_q$, and

$$E = \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) = q, \text{ dim } \mathscr{U}(\theta) \leqslant q-1\}.$$

Since codim $\mathscr{S}$ is lower semicontinuous, the set $U \overset{\text{def}}{=} E \cup F$ is an open neighborhood of F. If E is empty, then $\Omega(U) = F$ and therefore F is an attractor. On the other hand, if E is nonempty, then for all $\theta \in E$ one has $\Omega(\theta) \subset F$ and $A(\theta) \subset D$ by Lemmas 7.22, 7.24, and 7.34. Consequently, Lemma 7.33 implies that $F = M_q$ is an attractor. Similarly for M_p, one uses $D = M_p$, $F = A_{p+1}$, and $E = \{\theta \in \Theta : \text{codim } \mathscr{S}(\theta) \geqslant p+1, \text{ dim } \mathscr{U}(\theta) = p\}$. ∎

Proof of Theorem F. Since each nonempty Morse set M_k contains a minimal set, Theorem F is a direct consequence of Lemmas 7.29 and 7.30 and Theorem E. ∎

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