ON EXPONENTIAL DICHOTOMY AND STABLE MANIFOLDS FOR DIFFERENTIAL-ALGEBRAIC EQUATIONS ON THE HALF-LINE

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ABSTRACT. In this work we study linear/semi-linear differential-algebraic equations (DAEs) on the half-line $\mathbb{R}_+$. First we characterize the existence of exponential dichotomy for linear DAEs by invoking the Lyapunov-Perron method. Then we prove the existence of local and global, invariant, stable manifolds for semi-linear DAEs in the case that the corresponding evolution family to an associated linear DAE admits exponential dichotomy and an inhomogeneity function fulfills the non-uniform φ-Lipschitz condition, where the Lipschitz function φ belongs to wide classes of admissible function spaces such as L_p, $1 \leqslant p \leqslant \infty$, $L_{p,q}$, etc.

1. INTRODUCTION AND PRELIMINARIES

Our focus in the present paper is on the existence of (local and global) stable manifolds for semi-linear time varying differential-algebraic equations (DAEs) of the form

$$\begin{array}{l} d \text{ rows} \\ a \text{ rows} \end{array} \underbrace{\begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}}_{E(t)} \dot{x}(t) = \underbrace{\begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix}}_{A(t)} x(t) + \underbrace{\begin{bmatrix} f_1(t, x(t)) \\ f_2(t, x(t)) \end{bmatrix}}_{f(t,x(t))}, \quad t \in \mathbb{R}_+ := [0, +\infty). \tag{1.1}$$

Beside that, we also study the exponential dichotomy of the associated linear system

$$E(t)\dot{x}(t) = A(t)x(t), \quad t \in [0, +\infty). \tag{1.2}$$

Here $E = \begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}$, $A = \begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix}$ are assumed to be matrix-valued functions act on $\mathbb{R}_+$ to $\mathbb{R}^{n,n}$, $x : \mathbb{R}_+ \to \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. Furthermore, we assume that for all t, the matrices $E_1(t)$, $A_2(t)$ have full row rank.

DAE systems of the forms (1.1), (1.2) arise in many applications, include multibody dynamics, electrical circuit, chemical engineering, and many other applications. Due to the rank-deficiency of $E(t)$, the qualitative behavior of DAEs is much richer, in comparison to ordinary differential equations (ODEs). We refer the interested readers to recent monographs [2, 11–13] and the references therein. In particular, even though the stability analysis for DAEs have been intensively discussed (see the survey [11], Chapter 2), there are only a few papers on the spectral theory of DAEs and in particular, the exponential dichotomy for DAEs. We refer to [14] for the concept of exponential dichotomy and its relation to the well conditioning of the associated boundary value problem, to [16] for Lyapunov and other spectra for linear DAEs, to [4, 7] for the robustness of exponential stability and Bohl exponents. Besides that, whenever the exponential dichotomy of the linear, homogeneous system (1.2) is characterized, the next important question in the qualitative theory of DAEs is to study whether integral manifolds (e.g., stable, unstable, center, center-stable, center-unstable) for the semi-linear DAE (1.1) exist, [3, 5]. Unfortunately, till now this question is essentially open for DAEs. In order to shorten these gaps, this paper is devoted to the exponential stability of (1.2) and stable manifolds

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of (1.1). Our method is based on the classical "Lyapunov-Perron method" ([5, 25]) and the admissibility of function spaces ([9, 10]).

The outline of this paper is as follows. In the first section we recall some basis concepts which will be useful later, including an exponential dichotomy and its properties. Then, in Section 2 we present the admissibility of function spaces and their properties. In Section 3 we gives a characterization for the existence of exponential dichotomy for the DAE (1.2). Section 4 contains our main results on the existence and properties of local stable manifold for the semi-linear DAE (1.1). The global version of these results will be presented in Section 5. Finally, we illustrate our results by studying a spatial discretization of Navier-Stokes equations, and we conclude this research by a summary and some open problems.

Now let us recall some basic notions. By $(\mathbb{R}^n,\ \|\cdot\|)$ we denote the n-dimensional real vector space equipped with the Euclidean norm. For any matrix V, by V^T we denote its transpose. For any $p \in \mathbb{N}$, by $C^p([0,\infty),\mathbb{R}^n)$ we denote the space of p-times continuously differentiable functions act on $[0,\infty)$ to $\mathbb{R}^n$. By $C_\infty([0,\infty),\mathbb{R}^n)$ we denote the space of continuous, bounded functions act on $[0,\infty)$ to $\mathbb{R}^n$. This space is a Banach space with the $ess\,$sup-norm $\|f\|_\infty := \sup\{\|f(t)\|,\ t \geqslant 0\}$.

It is well-known (e.g. [3]), that for ordinary differential equations (ODEs), if the Cauchy problem

$$\begin{aligned} \frac{dx(t)}{dt} &= A(t)x(t),\ t \geqslant s \geqslant 0, \\ x(s) &= x_s \in \mathbb{R}^n, \end{aligned} \tag{1.3}$$

is well-posed, then there exists a pointwise nonsingular matrix-valued function $X(t,s) \in \mathbb{R}^{n,n}$ such that the solution of (1.3) is given by $x(t) = X(t,s)x_s$. This fact motivates the existence of an evolution family $(X(t,s))_{t\geqslant s\geqslant 0}$ associated with the matrix function $A(t)$. This family satisfies the condition $X(t,t) = Id$ and the so-called *semi-group property*

$$X(t,r)X(r,s) = X(t,s),\ \text{for all } t \geqslant r \geqslant s \geqslant 0. \tag{1.4}$$

Furthermore, the solution of the corresponding semi-linear ODE

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t,x(t)),\quad \text{for all } t \geqslant s \geqslant 0,$$

is given by the so-called *variational of constant formula*

$$x(t) = X(t,s)x(x) + \int_s^t X(t,\tau)f(\tau,x(\tau))d\tau,\quad \text{for all } t \geqslant s \geqslant 0. \tag{1.5}$$

For more details on the notion and some problems focus on properties and applications of evolution families we refer the readers to Pazy, [21].

Definition 1.1. A given evolution family $\{X(t,s)\}_{t\geqslant s\geqslant 0}$ of the ODE (1.3) is said to have an *exponential dichotomy* on the half-line if there exist a family of projection matrices $\{P(t)\}_{t\geqslant 0}$ and two positive constants N,ν such that the following conditions are satisfied.

i) $P(t)(t)X(t,s) = X(t,s)P(s)$, for all $t \geqslant s \geqslant 0$,

ii) the restriction $X(t,s)_|: \ker P(s) \to \ker P(t)$ is an isomorphism and we denote the inverse of $X(s,t)_|$,

iii) $\|X(t,s)P(s)x\| \leqslant Ne^{-\nu(t-s)}\|P(s)x\|$, for all $t \geqslant s \geqslant 0$, $x \in \mathbb{R}^n$,

iv) $\|X(t,s)_|\ (I-P(s))x\| \leqslant Ne^{\nu(t-s)}\|(I-P(s))x\|$, for all $s \geqslant t \geqslant 0$, $x \in \mathbb{R}^n$.

Here $\{P(t)\}_{t\geqslant 0}$ (reps. N, ν) are called *dichotomy projections* (resp. *dichotomy constants*).

Next we recall some basic concepts and properties for DAEs, starting with *fundamental solution matrix* as below.

Definition 1.2. i) Consider the DAE (1.2). A matrix function $X \in C([0,\infty),\mathbb{R}^{n,k})$, $d \leqslant k \leqslant n$, is called *a fundamental solution matrix* of (1.2) if each of its columns is a solution to (1.2) and $\operatorname{rank} X(t) = d$, for all $t \geqslant 0$.
ii) A fundamental solution matrix is said to be *maximal* if $k = n$ and *minimal* if $k = d$, respectively. A maximal fundamental solution is called *principal* if it satisfies the *projected initial condition*

$$E(0)\,(X(0) - Id) = 0. \tag{1.6}$$

We can easily see that, fundamental solution matrices for DAEs are not necessarily square or of full rank. Furthermore, every fundamental solution matrices has exactly d-linear independent columns, and a minimal fundamental solution matrix can be made maximal by adding $n - d$ zero columns. This is the major difference between ODEs and DAEs. Consequently, we are unable to define the evolution family for a DAE in the classical sense. The modified concept, but still capture the essence of an original one, has been proposed and carefully discussed in [16]. We recall it below, and notice that this concept is equivalent to the one proposed by Lentini and März in [14] within the context of the matrix chains approach and tractability index. Throughout this paper, we will assume the following.

Assumption 1.3. Consider the DAEs (1.1), (1.2). We assume that the function pair (E, A) in these DAEs is *strangeness-free*, i.e.,

$$\operatorname{rank}\begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix} = n,$$

for all $t \geqslant 0$. Furthermore, we assume that $E \in C^1([0,\infty),\mathbb{R}^{n,n})$ and $A \in C^0([0,\infty),\mathbb{R}^{n,n})$.

It should be important to note, that for general linear, homogeneous DAE of the form (1.2), one can transform it to the strangeness-free form without alternating the solution space. For further details, see [12, Chap. 3].

By making use of some smooth factorizations, for example QR or SVD ([6] or [12], Theorem 3.9), we can decouple and then exploit the structure of the DAE (1.2) in the following lemma.

Lemma 1.4. *Consider the DAE* (1.2) *and assume that it satisfies Assumption 1.3. Then, there exists pointwise-orthogonal matrix-valued functions U and $V \in C^1([0,\infty),\mathbb{R}^{n,n})$, such that by changing variable $x(t) = V(t)y(t)$, and scaling* (1.2) *with $U(t)$, we can transform it to the so-called* decoupled system *of the following form*

$$\begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_1(t) & \tilde{A}_2(t) \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \tag{1.7}$$

with pointwise nonsingular matrix functions $\Sigma(t) \in \mathbb{R}^{d,d}$ and $\tilde{A}_4(t) \in \mathbb{R}^{a,a}$.

Proof. Applying an SVD factorization for $E_1(t)$ we can find pointwise-orthogonal matrix functions $U_1(t) \in C^1([0,\infty),\mathbb{R}^{d,d})$ and $V \in C^1([0,\infty),\mathbb{R}^{n,n})$ such that $U_1(t)E_1(t)V(t) = \begin{bmatrix} \Sigma(t) & 0 \end{bmatrix}$, where $\Sigma(t)$ is continuous, pointwise nonsingular, matrix-valued function in $\mathbb{R}^{d,d}$. Changing the variable $x(t) = V(t)y(t)$ and scaling (1.2) with $U(t) := \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}$, we obtain a new system

$$U(t)E(t)V(t)y(t) = U(t)\left(A(t)V(t) - E(t)\dot{V}(t)\right)y(t),$$

which is exactly of the form (1.7). Furthermore, notice that

$$\begin{bmatrix} \Sigma(t) & 0 \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix} = \begin{bmatrix} U_1(t) & 0 \\ 0 & I_a \end{bmatrix}\begin{bmatrix} E_1(t) \\ A_2(t) \end{bmatrix} V,$$

then Assumption 1.3 yields that both Σ and $\tilde{A}_4$ are nonsingular. This completes the proof. $\square$

Let $\hat{A}_3 := -\tilde{A}_4^{-1}(t)\tilde{A}_3(t)$, $\hat{A}_1 := \Sigma^{-1}(t)\tilde{A}_1(t) + \Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t)\tilde{A}_3(t)$, we rewrite the transformed system (1.7) as

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t), \tag{1.8}$$

$$y_2(t) = \hat{A}_3(t)y_1(t). \tag{1.9}$$

Since $V(t)$ is orthogonal for all $t \geqslant 0$, we see that all important qualitative properties of $x(t)$, such as boundedness, exponential stability, contractivity, expansiveness, etc., can be carried out for the function $y(t)$. Clearly, we see that (1.9) gives an *algebraic constraint* that the solution to (1.7) must obey, while (1.8) gives the dynamic of (1.7). For this reason, we call it *an underlying ODE* to (1.7).

Let $\{\hat{Y}_1(t,s)\}_{t\geqslant s\geqslant 0}$ be the evolution family associated with the matrix function $\hat{A}_1(t)$, then we can define the corresponding evolution families for two DAEs (1.7), (1.2) consecutively as follows.

$$\hat{Y}(t,s) := \begin{bmatrix} \hat{Y}_1(t,s) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t,s) & 0 \end{bmatrix},\ \hat{X}(t,s) = V(t)\hat{Y}(t,s)V^T(s),\ \text{for all } t \geqslant s \geqslant 0. \tag{1.10}$$

Nevertheless, since $X(t,s)$ is not invertible, we will define the *reflexive generalized inverse matrix function* as in [16] by

$$\hat{Y}^-(t,s) := \begin{bmatrix} \hat{Y}_1^{-1}(t,s) & 0 \\ \hat{A}_3(s)\hat{Y}_1^{-1}(t,s) & 0 \end{bmatrix},\ \hat{X}^-(t,s) := V(s)\hat{Y}^-(t,s)V^T(t),\ \text{for all } t \geqslant s \geqslant 0. \tag{1.11}$$

Furthermore, we can directly verify the semigroup properties, i.e.

$$\hat{X}(t,r) = \hat{X}(t,s)\hat{X}(s,r),\ \text{for all } t \geqslant s \geqslant r \geqslant 0,$$
$$\hat{X}(t,s) = \hat{X}(t,0)\hat{X}^-(s,0),\ \text{for all } t \geqslant s \geqslant 0.$$

Now we give a solution formula for system (1.1), in comparison to (1.5).

Lemma 1.5. *Consider the DAE* (1.1) *and the evolution family* $(X(t,s))_{t\geqslant s\geqslant 0}$ *defined by* (1.10)*. Then the solution to* (1.1)*, if exists, also satisfies the so-called* mild equation

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \hat{X}(t,s)\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} + \int_s^t \hat{X}(t,\tau)\begin{bmatrix} \hat{f}_1(\tau, x_1(\tau), x_2(\tau)) \\ 0 \end{bmatrix} d\tau \ + \begin{bmatrix} 0 \\ \hat{f}_2(t, x_1(t), x_2(t)) \end{bmatrix},$$

for all $t \geqslant s \geqslant 0$*, where* $\hat{f}_1 := \Sigma^{-1}(t)f_1$ *and* $\hat{f}_2 := -\tilde{A}_4^{-1}(t)f_2$*.*

Proof. The proof can be obtained directly by using Lemma 1.4. Thus, in order to keep the brevity we will omit the details here. □

In the following, for ease of notation, we will use the abbreviation $\hat{X}(t) := \hat{X}(t,0)$, $\hat{X}^-(t) := \hat{X}^-(t,0)$, $\hat{Y}(t) := \hat{Y}(t,0)$ and $\hat{Y}^-(t) := \hat{Y}^-(t,0)$. The concept of exponential dichotomy for the DAE (1.7) is given as below.

Definition 1.6. ([16]) The DAE (1.7) is said to have an *exponential dichotomy* if there exist a family of projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ in $\mathbb{R}^{d,d}$ and positive constants N, ν such that

$$\left\| \hat{Y}(t)\begin{bmatrix} P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}\hat{Y}^-(s) \right\| \leqslant Ne^{-\nu(t-s)},\ \text{for all } t \geqslant s \geqslant 0,$$
$$\left\| \hat{Y}(t)\begin{bmatrix} I_d - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix}\hat{Y}^-(s) \right\| \leqslant Ne^{\nu(t-s)},\ \text{for all } s \geqslant t \geqslant 0, \tag{1.12}$$

Since the Euclidean norm is preserved under orthogonal transformations, due to (1.10)-(1.12) we see that

$$\left\| \hat{X}(t)V^T(0)\begin{bmatrix} P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^{-}(s) \right\| \leqslant Ne^{-\nu(t-s)}, \text{ for all } t \geqslant s \geqslant 0.$$

and

$$\left\| \hat{X}(t)V^T(0)\begin{bmatrix} Id - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)\hat{X}^{-}(s) \right\| \leqslant Ne^{\nu(t-s)}, \text{ for all } s \geqslant t \geqslant 0.$$

In addition, since $V^T(0)\begin{bmatrix} Id - P_{\mathrm{y}}(t) & 0 \\ 0 & 0 \end{bmatrix} V(0)$ is also a projection matrix for any $t \geqslant 0$, we can interpret the exponential dichotomy of (1.2) as the one of (1.7).

2. Function Spaces and Admissibility

In this section we recall some notions of function spaces that play a fundamental role in the study of differential equations and refer to Massera and Schäffer [17, Chap. 2] and Räbiger and Schnaubelt [22, §1] for their concrete applications.

Denote by $\mathcal{B}$ the Borel algebra, by λ the Lebesgue measure on $\mathbb{R}_+$, and by $L_{1,loc}(\mathbb{R}_+)$ the set of real-valued locally integrable functions on $\mathbb{R}_+$ (modulo λ-null functions). With a set of seminorms defining the topology given by $p_n(f) := \int_{J_n} |f(t)|dt,\ n \in \mathbb{N}$, where $\{J_n\}_{n\in\mathbb{N}} = \{[n, n+1]\}_{n\in\mathbb{N}}$, it is well-known (e.g. [17, Chapt. 2])) that $L_{1,loc}(\mathbb{R}_+)$ becomes a Fréchet space. We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo $\lambda-$nullfunctions) is called *a Banach function space* (over $(\mathbb{R}_+, \beta, \lambda)$) if

i) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$, $\lambda-$a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,

ii) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup\limits_{t\geqslant 0}\|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E > 0$,

iii) $E \hookrightarrow L_{1,loc}(\mathbb{R}_+)$, i.e., for each seminorm p_n of $L_{1,loc}(\mathbb{R}_+)$ there exists a positive constant β_n such that $p_n(f) \leqslant \beta_n\|f\|_E$ for all $f \in E$.

We then define the Banach space corresponding to the space E as follows.

Definition 2.2. Consider the Banach space $(\mathbb{R}^n, \|\cdot\|)$ with some arbitrary norm. For a Banach function space E we set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, \mathbb{R}^n) := \{f : \mathbb{R}_+ \to \mathbb{R}^n : f \text{ is strongly measurable and } \|f(\cdot)\| \in E\}$$

(modullo $\lambda-$nullfunctions) endowed with the norm $\|f\|_{\mathcal{E}} := \|\|f(\cdot)\|\|_E$. Thus, one can directly see that $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a Banach space. We call it *the Banach space corresponding to E.*

We now introduce the notion of admissibility in the following definition.

Definition 2.3. The Banach function space E is called *admissible* if for any $\varphi \in E$ the following conditions hold.

i) There exists a constant $M_\varphi \geqslant 1$ such that for every compact interval $[a, b] \in \mathbb{R}_+$ we have

$$\int_a^b |\varphi(t)|dt \leqslant \frac{M_\varphi(b-a)}{\|\chi_{[a,b]}\|_E}\|\varphi\|_E \text{ for all } \varphi \in E. \tag{2.1}$$

ii) The function $\Lambda_1\varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)d\tau$ belongs to E.

iii) For any $\tau \geqslant 0$, the space E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined as

$$\begin{aligned} T_\tau^+\varphi(t) &:= \begin{cases} \varphi(t-\tau) & \text{for } t \geqslant \tau \geqslant 0, \\ 0 & \text{for } 0 \leqslant t \leqslant \tau, \end{cases} \\ T_\tau^-\varphi(t) &:= \varphi(t+\tau) \text{ for } t \geqslant 0. \end{aligned} \tag{2.2}$$

Furthermore, there exist constants N_1, N_2 such that $\|T_\tau^+\|_E \leqslant N_1$, $\|T_\tau^-\|_E \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.4. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}_\alpha(\mathbb{R}_+) := \{h \in L_{1,loc}(\mathbb{R}_+) : \sup_{t\geqslant 0} \int_t^{t+\alpha} |h(\tau)|d\tau < \infty\},$$

(for any fixed $\alpha > 0$), endowed with the norm $\|h\|_{\mathbf{M}_\alpha} := \sup_{t\geqslant 0} \int_t^{t+\alpha} |h(\tau)|d\tau$, many other function spaces occurring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}, 1 < p < \infty, 1 \leqslant q < \infty$ (see [3], [24]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}_+, \beta, \lambda)$ (see [15]) are admissible.

Remark 2.5. Following directly from Definitions 2.1 ii) and 2.3 i) we have that

$$\sup_{t\geqslant 0} \int_t^{t+1} |\varphi(\tau)|d\tau \leqslant \frac{M_\varphi}{\inf_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E} \|\varphi\|_E,$$

and hence, $E \hookrightarrow \mathbf{M}_1(\mathbb{R}_+)$. Furthermore, $C_\infty(\mathbb{R}^+)$ is dense in $\mathbf{M}_1$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [9, Proposition 2.6] and originally in [17, 23.V.(1)]).

Proposition 2.6. Let E be an admissible Banach function space. Then the following assertions hold.
a) Let $\varphi \in L_{1,loc}(\mathbb{R}_+)$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where, Λ_1 is defined as in definition 2.3 (ii). For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by

$$\begin{aligned} \Lambda'_\sigma\varphi(t) &:= \int_0^t e^{-\sigma(t-s)}\varphi(s)ds, \\ \Lambda''_\sigma\varphi(t) &:= \int_t^\infty e^{-\sigma(s-t)}\varphi(s)ds. \end{aligned}$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geqslant 0} \int_t^{t+1} \varphi(\tau)d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see remark 2.5)) then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for $ess\,$sup-norm, we have

$$\|\Lambda'_\sigma\varphi\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty \quad \text{and} \quad \|\Lambda''_\sigma\varphi\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \tag{2.3}$$

for operator T_1^+ and constants N_1, N_2 defined as in Definition 2.3.
b) E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for any constant $\alpha > 0$.
c) E does not contain exponentially growing functions $f(t) := e^{bt}$ for any constant $b > 0$.

3. Exponential Dichotomy for linear DAEs and its properties

In the qualitative analysis of ODEs, one of the central topic is to find sufficient and necessary conditions for the considered system to admit exponential dichotomy. Many researches have been devoted to this topic, and critical results have been achieved for ODEs of both finite and infinite dimensions (e.g. [5, Chap. 4], [25]). For DAEs, the only result that we are aware of is recalled below.

Proposition 3.1. ([16]) The DAE (1.2) has exponential dichotomy if and only if the matrix function $\hat{A}_3(t)$ is bounded, and the corresponding underlying ODE (1.8) also has exponential dichotomy. Moreover, the existence of exponential dichotomy imlies that $\sup\limits_{t\geqslant 0} \|P_{\mathrm{y}}(t)\| < \infty$.

Notice that, even for ODEs, Proposition 3.1 is only valid for finite but not infinite dimensional systems. For this reason, we recall another important result below.

Proposition 3.2. ([5]) Consider the ODE (1.3). Then it has exponential dichotomy if and only if for any continuous, bounded function $f(t)$ on $[0,\infty)$, there exists a continuous, bounded solution $x(t)$.

In view of Proposition 3.2, comparable conditions for the existence of exponential dichotomy have not been considered for DAEs, and hence, this will be our main aim in this section.

Definition 3.3. Consider the matrix functions E, A in system (1.2). Then, any function $f:\mathbb{R}_+ \to \mathbb{R}^n$ satisfies the condition

$$\sup_{t\geqslant 0}\left\{\left\| \begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & \tilde{A}_4^{-1}(t)\end{bmatrix} f(t)\right\|\right\} < +\infty,$$

is called (E,A)-bounded. We denote the set of all continuous, (E,A)-bounded function by $\mathcal{B}^{EA}(\mathbb{R}_+,\mathbb{R}^n)$.

The main result of this section is to prove, that "roughly speaking" the DAE (1.2) admits exponential dichotomy if and only if the mapping $\mathcal{L} := E\dfrac{d}{dt} - A$ is surjective on the space $\mathcal{B}^{EA}(\mathbb{R}_+,\mathbb{R}^n)$. In connection with the solvability of the linear, inhomogeneous DAE

$$\begin{array}{l} d \text{ rows} \\ a \text{ rows}\end{array} \quad \underbrace{\begin{bmatrix} E_1(t) \\ 0 \end{bmatrix}}_{E(t)} \dot{x}(t) = \underbrace{\begin{bmatrix} A_1(t) \\ A_2(t) \end{bmatrix}}_{A(t)} + \underbrace{\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}}_{f(t)},\ t \in [0,+\infty). \tag{3.1}$$

we reform our main result in this section as follows.

Theorem 3.4. *Consider the linear, strangeness-free DAE* (1.2) *and the associated inhomogeneous DAE* (3.1)*. Then the following assertions hold.*
i) If the DAE (1.2) *admits exponential dichotomy then for any continuous,* (E,A)*-bounded function* $f(t)$ *on* $[0,\infty)$*, there exists a continuous, bounded solution* $x(t)$ *to the DAE* (3.1)*.*
ii) If the matrix function $\hat{A}_3(t)$ *is bounded, then the converse of i) is also true.*

Proof. First we notice that, since $\hat{f} = U(t)\begin{bmatrix} \Sigma^{-1}(t) & -\Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t) \\ 0 & -\hat{A}_4^{-1}\end{bmatrix} f(t)$, the (E,A)-boundedness of f is equivalent to the boundedness of $\hat{f}$. Recall that the decoupled system (1.7) reads

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t), \tag{3.2}$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t). \tag{3.3}$$

i) Assuming that the DAE (1.2) admits exponential dichotomy, then (1.7) also has an exponential dichotomy. Proposition 3.1 implies that equation (3.2) has an exponential dichotomy, and the function $\hat{A}_3$ is bounded. Therefore, Proposition 3.2 implies that y_1 is bounded, and consequently, y_2 is also bounded.
ii) Due to Proposition 3.2, it follows that (3.2) has exponential dichotomy. Besides that, due to the boundedness of $\hat{A}_3$, it follows that (1.2) admits exponential dichotomy. □

Remark 3.5. Making use of admissible function spaces, stronger conditions for characterizing the exponential dichotomy of the DAE (1.2) have been obtained in [20], where an inhomogeneity function $f(t)$ belongs to less restricted spaces than $\mathcal{B}^{EA}(\mathbb{R}_+,\mathbb{R}^n)$. There, we also study the robustness of exponential dichotomy under structured-perturbations.

4. Local Stable manifolds for semi-linear DAEs

In this section we study the existence of a local stable manifold for the semi-linear DAE (1.1). Throughout this section we assume that the evolution family $(X(t,s))_{t\geqslant s\geqslant 0}$ associated with the linear, homogeneous DAE (1.2) admits an exponential dichotomy on $\mathbb{R}_+$.

Due to Lemma 1.4, by using orthogonal transformation $x(t) = V(t)y(t)$, where $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} \in \mathbb{R}^{d+a}$ we can transform (1.1) to the coupled system

$$\dot{y}_1(t) = \hat{A}_1(t)y_1(t) + \hat{f}_1(t, y(t)), \tag{4.1}$$

$$y_2(t) = \hat{A}_3(t)y_1(t) + \hat{f}_2(t, y(t)), \tag{4.2}$$

where

$$\hat{f}(t, y(t)) = \begin{bmatrix} \hat{f}_1(t, y(t)) \\ \hat{f}_2(t, y(t)) \end{bmatrix} := \begin{bmatrix} \Sigma^{-1}(t)f_1(t, x(t)) - \Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t)f_2(t, x(t)) \\ -A_4^{-1}(t)f_2(t, x(t)) \end{bmatrix}. \tag{4.3}$$

Notice that, different from the DAEs (1.2) and (3.1), equation (4.2) only gives an implicit algebraic constraint in terms of y_1 and y_2. In order to guarantee the strangeness-free of system (1.1), we need the following assumption.

Assumption 4.1. Assume that for some $\rho > 0$, the function $A_4^{-1}(t)f_2(t, x)$ is a contraction mapping in the ball B_ρ (uniformly in time), i.e.,

$$\|A_4^{-1}(t)f_2(t, x) - A_4^{-1}(t)f_2(t, \tilde{x})\| \leqslant L\|x - \tilde{x}\|,$$

for a.e. $t \in \mathbb{R}_+$, and for all $x, \tilde{x} \in B_\rho$, where the Lipschitz constant L satisfies that $L < 1$.

Lemma 4.2. *Under Assumption 4.1, restricted to the ball $B_\rho \subset \mathbb{R}^n$, y_2 can be uniquely solvable from* (4.2) *in terms of t and y_1.*

Proof. First notice that Assumption 4.1 implies that $\hat{f}_2(t, y)$ is also Lipschitz in y with the same constant L. Then, the desired claim is obtained directly by making use of [18, Lem. 2.7]. ☐

Remark 4.3. Lemma 4.2 leads to one critical fact, that under Assumption 4.1, the coupled system (4.1)-(4.2) is still strangeness-free, as defined in [12, Chap. 4]. Therefore, in analoguous to the linear case, (4.2) is called *an algebraic constraint*, while (4.1) is called *an underlying ODE*.

To obtain the stable manifold we need the following property of the nonlinear part f_1 as be shown in the notion below.

Definition 4.4. Let E be an admissible Banach function space and φ be a positive function belongs to E. A function $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to the class (M, φ, ρ) for some positive constant M, ρ if h satisfies

(i) $\|h(t, x)\| \leqslant M\varphi(t)$ for a.e. $t \in \mathbb{R}_+$ and for all $x \in B_\rho$,
(ii) $\|h(t, x) - h(t, \tilde{x})\| \leqslant \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$, for all $x, \tilde{x} \in B_\rho$.

Assumption 4.5. Assume that the function $\Sigma^{-1}(t)\ f_1(t, x(t)) - \Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t)\ f_2(t, x(t))$ belongs to the class (M, φ, ρ) for some positive constants M, ρ and a positive function $\varphi \in E$.

For notational simplicity, we will study the existence of a local stable manifold for system (4.1)-(4.2). Moreover, we consider the mild/integral-algebraic system which reads

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \hat{Y}(t, s)\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} + \int_s^t \hat{Y}(t, \tau)\begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \tag{4.4}$$

for all $t \geqslant s \geqslant 0$.

Lemma 4.6. *Let Assumptions 4.1 and 4.5 hold true. Then, for all $y, \tilde{y} \in B_\rho$ the following assertions hold.*
i) $\|\hat{f}_1(t,y)\| \leqslant M\varphi(t)$, *for a.e.* $t \in \mathbb{R}_+$,
ii) $\|\hat{f}_1(t,y) - \hat{f}_1(t,\tilde{y})\| \leqslant \varphi(t)\|y-\tilde{y}\|$, *for a.e.* $t \in \mathbb{R}_+$,
iii) $\|\hat{f}_2(t,y) - \hat{f}_2(t,\tilde{y})\| \leqslant L\|y-\tilde{y}\|$, *for a.e.* $t \in \mathbb{R}_+$.

Proof. The proof is trivially followed from Assumptions 4.1 and 4.5 due to the fact that $\|y\| = \|Qy\|$ for any orthogonal matrix V. ☐

Let $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N, \nu > 0$ as in Definition 1.6. Furthermore, due to Proposition 3.1, let us denote by $H_1 := \sup_{t\geqslant 0} \|\hat{A}_3(t)\|$ and $H_2 := \sup_{t\geqslant 0} \|P_{\mathrm{y}}(t)\|$. Then, we can define the Green function on the half-line as follows

$$G(t,\tau) := \begin{cases} \hat{Y}(t,\tau)\begin{bmatrix} P_{\mathrm{y}}(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t,\tau)P_{\mathrm{y}}(\tau) & 0 \\ \hat{A}_3(t)\hat{Y}_1(t,\tau)P_{\mathrm{y}}(\tau) & 0 \end{bmatrix}, & \text{for all } \ t \geqslant \tau \geqslant 0, \\ -\hat{Y}(t,\tau)\begin{bmatrix} I_d - P_{\mathrm{y}}(\tau) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau)) & 0 \\ \hat{A}_3(\tau)\hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau)) & 0 \end{bmatrix}, & \text{for all } \ 0 \leqslant t < \tau. \end{cases} \tag{4.5}$$

Then, we have

$$\|G(t,\tau)\| \leqslant (1+H_1)(1+H_2)\; Ne^{-\nu|t-\tau|} \quad \text{for all } t \neq \tau \geqslant 0. \tag{4.6}$$

In the following lemma, we give an explicit form for bounded solutions to system (4.4).

Lemma 4.7. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system (1.7) has an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumptions 4.1, 4.5 hold true. Let $y(t)$ be any solution to (4.4) such that $\operatorname{ess\,sup}_{t\geqslant t_0} \|y(t)\| \leqslant \rho$ for fixed $t_0 \geqslant 0$ and some $\rho > 0$. Then, for $t \geqslant t_0 \geqslant 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t,t_0)\begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t,y(t)) \end{bmatrix}, \tag{4.7}$$

for some $v_0 \in \mathrm{Im} P_{\mathrm{y}}(t_0)$, where $G(t,\tau)$ is the Green function defined by (4.5).

Proof. Put $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} := \int_{t_0}^{\infty} G(t,\tau)\begin{bmatrix} \hat{f}_1(\tau,y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t,y(t)) \end{bmatrix}$, by direct computation, we can verify that z satisfies the integral equation

$$z(t) = \hat{Y}(t,t_0)\begin{bmatrix} z_1(t_0) \\ z_2(t_0) \end{bmatrix} + \int_{t_0}^{t} \hat{Y}(t,\tau)\begin{bmatrix} \hat{f}_1(\tau,y(\tau)) \\ 0 \end{bmatrix} d\tau \; + \begin{bmatrix} 0 \\ \hat{f}_2(t,y(t)) \end{bmatrix},$$

for all $t \geqslant t_0$. Now let us estimate $\|z(t)\|$. Making use of Lemma 4.6 and (4.6), we see that

$$\|z(t)\| \leqslant \int_{t_0}^{\infty} (1+H_1)(1+H_2)\; Ne^{-\nu|t-\tau|}\; M\varphi(\tau)\; d\tau \; + L\rho,$$

and then, (2.3) follows that

$$\|z(t)\| \leqslant M\;(1+H_1)(1+H_2)\,\frac{N}{1-e^{-\nu}}\;\left(\|\Lambda_1 T_1^+\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right) \; + L\rho\;,$$

for all $t \geqslant t_0$. Thus, $z(t) - y(t)$ is also bounded. Moreover, since

$$z(t) - y(t) = \hat{Y}(t,t_0)\,(z(t_0) - y(t_0)) = \begin{bmatrix} \hat{Y}_1(t,t_0)\;(z_1(t_0) - y_1(t_0)) \\ \hat{A}_3(t)\hat{Y}_1(t,t_0)\;(z_1(t_0) - y_1(t_0)) \end{bmatrix},$$

we see that $v_0 := z_1(t_0) - y_1(t_0) \in \mathrm{Im}P_{\mathrm{y}}(t_0)$. Finally, since $z(t) = y(t) + \hat{Y}(t, t_0)\begin{bmatrix} v_0 \\ 0 \end{bmatrix}$ for all $t \geqslant t_0$, (4.7) follows. $\square$

Remark 4.8. By computing directly, we can see that the converse of Lemma 4.7 is also true. It means, that all solutions to (4.7) also satisfy equation (4.4) for all $t \geqslant t_0$.

Let us denote by

$$H_3 := (1 + H_1)(1 + H_2)\ \frac{N}{1 - e^{-\nu}}\ \left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right) \quad \text{and} \quad \tilde{\rho} := \frac{1 - L}{2N(1 + H_1)}\rho\ . \tag{4.8}$$

Lemma 4.9. *Under the assumptions of Lemma 4.7, let $y(t)$, $\tilde{y}(t)$ be any two functions lie in the ball B_ρ and satisfy (4.7) for v_0, $\tilde{v}_0 \in \mathrm{Im}P_{\mathrm{y}}(t_0)$. If H_3 defined by (4.8) satisfies $H_3 + L < 1$ then we have the following estimation*

$$\|y - \tilde{y}\|_\infty \leqslant \frac{N}{1 - H_3 - L}\ \|v_0 - \tilde{v}_0\|\ . \tag{4.9}$$

Proof. Using the same arguments as in the proof of Lemma 4.6, we see that

$$\begin{aligned}
\|y(t) - \tilde{y}(t)\| &\leqslant N\|v_0 - \tilde{v}_0\| + \int_{t_0}^{\infty} (1 + H_1)(1 + H_2)\ Ne^{-\nu|t-\tau|}\ \varphi(\tau)\|y(\tau) - \tilde{y}(\tau)\|\ d\tau\ + L\|y(t) - \tilde{y}(t)\|, \\
&\leqslant N\|v_0 - \tilde{v}_0\| + (1 + H_1)(1 + H_2)\ \frac{N}{1 - e^{-\nu}}\ \left(\|\Lambda_1 T_1^{+}\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right)\ \|y - \tilde{y}\|_\infty + L\|y(t) - \tilde{y}(t)\|, \\
&\leqslant N\|v_0 - \tilde{v}_0\| + (H_3 + L)\ \|y - \tilde{y}\|_\infty,
\end{aligned}$$

which directly implies (4.9). $\square$

In the following theorem, we exploit the local structure of bounded solutions to (4.4).

Theorem 4.10. *Let the evolution family $(\hat{Y}(t, s))_{t \geqslant s \geqslant 0}$ of system (1.7) have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t \geqslant 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumptions 4.1, 4.5 hold true.*
i) If the condition

$$H_3 < \min\left\{1 - L,\ \frac{(1 - L)\rho}{2M}\right\} \tag{4.10}$$

is fulfilled, then there corresponds to each $v_0 \in B_{\tilde{\rho}} \cap \mathrm{Im}P_{\mathrm{y}}(t_0)$ one and only one solution $y(t)$ to (4.4) on $[t_0, \infty)$ satisfying $P_{\mathrm{y}}(t_0)y_1(t_0) = v_0$ and $\operatorname{ess\,sup}_{t \geqslant t_0}\|y(t)\| \leqslant \rho$.
ii) Moreover, for any two solutions $y(t)$, $\tilde{y}(t)$ corresponding to different v_0, $\tilde{v}_0$ in $B_{\tilde{\rho}} \cap \mathrm{Im}P_{\mathrm{y}}(t_0)$, they are attracted to each other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leqslant H_4\ e^{-\mu(t - t_0)}\ \|v_0 - \tilde{v}_0\|, \quad \textit{for all } t \geqslant t_0, \tag{4.11}$$

for some positive constants H_4, μ.

Proof. i) Consider in the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ the ball $\mathcal{B}_\rho := \{y \in L_\infty(\mathbb{R}_+, \mathbb{R}^n) :\ \|y(\cdot)\|_\infty := \underset{t \geqslant 0}{ess\sup}\|y(t)\| \leqslant \rho\}$. For each fixed $v_0 \in B_{\tilde{\rho}}$ we will prove the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t, t_0)\begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t, \tau)\begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geqslant t_0, \\ 0, & \text{for all } t < t_0, \end{cases} \tag{4.12}$$

is a contraction mapping from $\mathcal{B}_\rho$ to itself. Using the same argument as in the proof of Lemma 4.6, we see that

$$\begin{aligned}\|(Ty)(t)\| &\leqslant (1+H_1)Ne^{-\nu(t-t_0)}\|v_0\| + M\,(1+H_1)(1+H_2)\,\frac{N}{1-e^{-\nu}}\,\left(\|\Lambda_1 T_1^+\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right) + L\rho\,,\\ &\leqslant (1+H_1)N\|v_0\| + MH_3 + L\rho\,,\end{aligned}$$

and due to (4.10) we see that

$$\|(Ty)(t)\| \leqslant (1+H_1)N\tilde{\rho} + \frac{(1-L)\rho}{2} + L\rho = \rho\,.$$

Therefore, T is a mapping from $\mathcal{B}_\rho$ to itself. Now we prove its contraction property. Making use of (4.6), we obtain the following estimation

$$\begin{aligned}\|Ty(t) - T\tilde{y}(t)\| &\leqslant \int_{t_0}^{\infty} \|G(t,\tau)\|\ \|\hat{f}_1(\tau,y(\tau)) - \hat{f}_1(\tau,\tilde{y}(\tau))\|d\tau + \|\hat{f}_2(t,y(t)) - \hat{f}_2(t,\tilde{y}(t))\|,\\ &\leqslant \int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau + L\ \|y(t)-\tilde{y}(t)\|,\\ &\leqslant (H_3+L)\ \|y(t)-\tilde{y}(t)\|_\infty.\end{aligned}$$

Consequently, due to (4.10), we see that T is a contraction mapping with the contraction constant $H_3 + L$. Thus, there exist a unique function $y \in \mathcal{B}_\rho$ such that $y = Ty$, and hence, due to the definition of T, y is the solution to the mild/integral-algebraic system (4.4).

ii) The proof of the estimate (4.11) can be done in a similar way as in [10, Thm 3.7]. We present here for seek of completeness. Let $y(t)$ and $\tilde{y}(t)$ be two essentially bounded solutions of (4.4) corresponding to different values v_0, $\tilde{v}_0 \in B_{\tilde{\rho}} \cap \mathrm{Im}P_{\mathrm{y}}(t_0)$. Then, we have that

$$\begin{aligned}\|y(t)-\tilde{y}(t)\| &\leqslant Y(t,t_0)\|v_0-\tilde{v}_0\| + \int_{t_0}^{\infty}\|G(t,\tau)\|\ \|\hat{f}_1(\tau,y(\tau)) - \hat{f}_1(\tau,\tilde{y}(\tau))\|d\tau + \|\hat{f}_2(t,y(t)) - \hat{f}_2(t,\tilde{y}(t))\|,\\ &\leqslant (1+H_1)Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty}(1+H_1)(1+H_2)\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau + L\ \|y(t)-\tilde{y}(t)\|,\end{aligned}$$

and hence,

$$\|y(t)-\tilde{y}(t)\| \leqslant \frac{1+H_1}{1-L}Ne^{-\nu(t-t_0)} + \int_{t_0}^{\infty}\frac{(1+H_1)(1+H_2)}{1-L}\ Ne^{-\nu|t-\tau|}\varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau\quad.$$

Then, due to the Cone Inequality, [5, Theorem 1.9.3], in analoguous to [19, Theorem 3.7], we obtain the estimation (4.11) with H_4, μ are given by

$$0 < \mu < \nu + \ln\left(1 - \frac{H_3(1-e^{-\nu})}{1-L}\right), \quad H_4 := \frac{(1+H_1)N}{1 - L - \dfrac{H_3(1-e^{-\nu})}{1-e^{\mu-\nu}}}\,.$$

Furthermore, notice that due to (4.10) we see that $\mu < \nu$, which implies the positivity of H_4. This completes the proof. □

Under Assumption 4.1, we define the so-called *constrained manifold*, which all solutions to (4.1)-(4.2) must lie on

$$\mathbb{L}(t,y) := \{(t,y_1,y_2) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^a \mid y_2 = \hat{A}_3(t)y_1 + \hat{f}_2(t,y_1,y_2)\}\,. \qquad (4.13)$$

We further notice that this manifold is of dimension d, which is the degree of freedom to the DAE (4.4). Now we are able to introduce the concept of a local stable manifold for the solutions of the integral-algebraic system (4.4).

Definition 4.11. A subset $\mathbb{M}$ of the constrained manifold $\mathbb{L}(t,y)$ is said to be a *local stable manifold* for solutions to (4.4) if for every $t \in \mathbb{R}_+$ the phase subspace $\mathbb{R}^d$ splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t\in\mathbb{R}_+} Sn(W_1(t),W_2(t)) := \inf_{t\in\mathbb{R}_+} \inf\{\|w_1+w_2\|,\ w_i\in W_i(t),\ \|w_i\|=1,\ i=0,1\} > 0,$$

and if there exist positive constants ρ, ρ_1, ρ_2 and a family of Lipschitz continuous mappings

$$g_t : B_{\rho_1}\cap W_1(t) \to B_{\rho_2}\cap W_2(t), \quad t\in\mathbb{R}_+,$$

with a common Lipschitz constant independent of t such that

(i) $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t)\oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in B_{\rho_1}\cap W_1(t)\}$, and we denote by $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$,

(ii) $\mathbb{M}_t$ is homeomorphic to $B_{\rho_1}\cap W_1(t)$ for all $t \geqslant 0$,

(iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (4.4) satisfying $y_1(t_0) = \tilde{w}$ and $ess\sup_{t\geqslant t_0}\|y(t)\| \leqslant \rho$.

Theorem 4.12. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system (1.7) have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumptions 4.1, 4.5 hold true. If the condition*

$$H_3 < \min\left\{1-L,\ \frac{(1-L)(1+H_1)\rho}{2M},\ \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}\right\}$$

is fulfilled, then there exists a local stable manifold for the solutions of (4.4). Moreover, every two solutions $y(t)$, $\tilde{y}(t)$ on the manifold $\mathbb{M}$ attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0 \geqslant 0$ such that

$$\|y(t)-\tilde{y}(t)\| \leqslant H_4\ e^{-\mu(t-t_0)}\ \|P(t_0)y_1(t_0) - P(t_0)y_2(t_0)\|, \quad \textit{for all } t \geqslant t_0\ . \tag{4.14}$$

Proof. First we notice that the phase subspace $\mathbb{R}^d$ splits into the direct sum $\mathbb{R}^d = \mathrm{Im}P_{\mathrm{y}}(t) \oplus \mathrm{kernel}\,P_{\mathrm{y}}(t)$ for all $t\geqslant 0$. We set $W_1(t) := \mathrm{Im}P_{\mathrm{y}}(t)$ and $W_2(t) := \mathrm{kernel}\,P_{\mathrm{y}}(t)$, then due to Proposition 3.1, we see that $\sup\limits_{t\geqslant 0}\|P_{\mathrm{y}}(t)\| < \infty$, and hence, $\inf_{t\in\mathbb{R}_+} Sn(W_1(t),W_2(t)) > 0$.

For any $\rho>0$ corresponding to Assumptions 4.1, 4.5, let $\rho_1 := \tilde{\rho} = \dfrac{1-L}{2N(1+H_1)}\rho$ and $\rho_2 := \dfrac{(1-L)\rho}{2}$. For each $t \geqslant 0$ we define the mapping g_t acts on $B_{\rho_1}\cap W_1(t)$ as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t,\tau)(I_d - P_{\mathrm{y}}(\tau))f_1(\tau,y(\tau))d\tau,$$

where the function $y(t)$ is uniquely defined via Theorem 4.10 i). Clearly, $g_t(w_1) \in \ker P_{\mathrm{y}}(t) = W_2(t)$.

Now we prove that $\|g_t(w_1)\| \leqslant \rho_2$. Due to Theorem 4.10 i) and Lemma 4.6 i), we see that $\|y(t)\| \leqslant \rho$ and $\|f_1(\tau,y(\tau))\| \leqslant M\varphi(\tau)$ for a.e. $t \geqslant 0$. Therefore,

$$\begin{aligned}\|g_t(w_1)\| &\leqslant \int_t^\infty N\ e^{-\nu(\tau-t)}\ \|f_1(\tau,y(\tau))\|d\tau \ \leqslant\ \int_t^\infty N\ e^{-\nu(\tau-t)}\ M\varphi(\tau)d\tau,\\ &\leqslant M\ (1+H_2)\ \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^+\varphi\|_\infty + \|\Lambda_1\varphi\|_\infty\right) \ =\ \frac{MH_3}{1+H_1} \leqslant \frac{(1-L)\rho}{2}\ ,\end{aligned}$$

and hence, $g_t : B_{\rho_1}\cap W_1(t) \to B_{\rho_2}\cap W_2(t)$.

Notice that both part iii) in Definition 4.11 and estimation (4.14) are followed directly from Theorem 4.10. We now only need to prove that $\mathbb{M}_t$ is homeomorphic to $B_{\rho_1} \cap W_1(t)$. We first prove that g_t is a Lipschitz mapping. This fact can be seen from the following estimation.

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leqslant \int_t^{\infty} N\ e^{-\nu(\tau-t)}\ \|f_1(\tau, y(\tau)) - f_2(\tau, \tilde{y}(\tau))\| d\tau \leqslant \int_t^{\infty} N\ e^{-\nu(\tau-t)}\ \varphi(\tau)\ \|y(\tau) - \tilde{y}(\tau)\| d\tau\ ,$$
$$\leqslant \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^+ \varphi\|_\infty + \|\Lambda_1 \varphi\|_\infty\right)\ \|y - \tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)}\ \|y - \tilde{y}\|_\infty\ ,$$

and hence, (4.9) implies that

$$\|g_t(w_1) - g_t(\tilde{w}_1)\| \leqslant \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)}\ \|w_1 - \tilde{w}_1\|\ .$$

Finally, $H_3 < \dfrac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\dfrac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)} < 1$, and hence, g_t is a contraction mapping for all $t \geqslant 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous mappings ([18, Lem. 2.7]), we see that the mapping $Id + g_t : \mathbb{M}_t \to B_{\rho_1} \cap W_1(t)$ is a homeomorphism. This implies the condition ii) of Definition 4.11, and hence, the proof is finished. □

5. Global stable manifolds for semi-linear DAEs

In this section we study the existence of global stable manifolds for semi-linear DAEs of the form (1.1). We begin with the concept of φ-Lipschitz functions.

Definition 5.1. Let E be an admissible Banach function space and $\varphi \in E$ is a positive function. A function $h : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^d$ is said to be *φ-Lipschitz* if the following conditions hold true.

(i) $\|h(t,0)\| = 0$ for a.e. $t \in \mathbb{R}_+$,

(ii) $\|h(t,x) - h(t,\tilde{x})\| \leqslant \varphi(t)\|x - \tilde{x}\|$ for a.e. $t \in \mathbb{R}_+$ and all $x, \tilde{x} \in \mathbb{R}^n$.

In comparable to Assumptions 4.1, 4.5, we also need some global properties of the nonlinear term f.

Assumption 5.2. Assume that the followings hold true.
i) The function $\Sigma^{-1}(t)\ f_1(t, x(t)) - \Sigma^{-1}(t)\tilde{A}_2(t)\tilde{A}_4^{-1}(t)\ f_2(t, x(t))$ is φ-Lipschitz.
ii) The function $\tilde{A}_4^{-1}(t)\ f_2(t, x(t))$ is a contraction mapping with the Lipschitz constant $L < 1$ for all $(t, x(t))$ lies on the constraint-manifold associated with (1.1) defined by

$$\mathbb{L}(t,x) := \{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid 0 = A_2(t)x + f_2(t,x)\}\ .$$

We can directly verify that orthogonal transformations of the form $x = Vy$ preserves the φ-Lipschitz property, and hence, function $\hat{f}_1$ in (4.1) is also φ-Lipschitz. Besides that, function $\hat{f}_2$ in (4.2) is also a contraction mapping with the Lipschitz constant $L < 1$. For notational simplicity, now we will study the transformed system (1.7) and the integral-algebraic system (4.4).

Definition 5.3. A subset $\mathbb{M}$ of the constrained manifold $\mathbb{L}(t,y)$ is said to be a *global, stable manifold* for solutions to (4.4) if for every $t \in \mathbb{R}_+$ the phase subspace $\mathbb{R}^d$ splits into a direct sum $\mathbb{R}^d = W_1(t) \oplus W_2(t)$ such that

$$\inf_{t \in \mathbb{R}_+} Sn(W_1(t), W_2(t)) := \inf_{t \in \mathbb{R}_+}\ \inf\{\|w_1 + w_2\|,\ w_i \in W_i(t),\ \|w_i\| = 1,\ i = 0,1\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t : W_1(t) \to W_2(t), \quad t \in \mathbb{R}_+,$$

with the Lipschitz constants independent of t such that

(i) $\mathbb{M} = \{(t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{R}_+ \times (W_1(t) \oplus W_2(t)) \times \mathbb{R}^a \mid w_1 \in W_1(t)\}$, and we denote by $\mathbb{M}_t := \{(y_1 = w_1 + g_t(w_1), y_2) \mid (t, y_1 = w_1 + g_t(w_1), y_2) \in \mathbb{M}\}$,
(ii) $\mathbb{M}_t$ is homeomorphic to $W_1(t)$ for all $t \geqslant 0$,
(iii) to each $\tilde{w} \in \mathbb{M}_{t_0}$ there corresponds one and only one solution y to (4.4) satisfying $y_1(t_0) = \tilde{w}$ and $ess\sup_{t \geqslant t_0} \|y(t)\| < \infty$,
(iv) $\mathbb{M}$ is invariant under system (4.4), i.e., if y is a solution to (4.4), and $ess\sup_{t \geqslant t_0} \|y(t)\| < \infty$, then $y(s) \in \mathbb{M}_s$ for all $s \geqslant t_0$.

In analogous to Lemma 4.7, we give the explicit form of bounded solutions to system (4.4) as below.

Lemma 5.4. *Let the evolution family $(\hat{Y}(t,s))_{t \geqslant s \geqslant 0}$ of system* (1.7) *has an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t \geqslant 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumption 5.2 holds true. Let $y(t)$ be any solution to* (4.4) *such that $ess\sup_{t \geqslant t_0} \|y(t)\| < \infty$ for fixed $t_0 \geqslant 0$. Then, for all $t \geqslant t_0 \geqslant 0$, we can rewrite $y(t)$ in the form*

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Y(t,t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, \tag{5.1}$$

for some $v_0 \in \mathrm{Im} P_{\mathrm{y}}(t_0)$, where $G(t,\tau)$ is the Green function defined by (4.5).

Proof. The proof can be achieved by using similar arguments as done in the proof of Lemma 4.2, and we will omit the details here in order to keep the brevity of this research. □

In the following two theorems, we present the global versions of Theorems 4.10 and 4.12, where we construct the structure of bounded solutions to (4.4) and prove the existence of a global, stable manifold, respectively.

Theorem 5.5. *Let the evolution family $(\hat{Y}(t,s))_{t \geqslant s \geqslant 0}$ of system* (1.7) *have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t \geqslant 0}$ and the dichotomy constants $N, \nu > 0$. Furthermore, assume that Assumption 5.2 holds true.*
i) For any fixed $t_0 \geqslant 0$, if the condition

$$H_3 < 1 - L$$

is fulfilled, then there corresponds to each $v_0 \in \mathrm{Im} P_{\mathrm{y}}(t_0)$ one and only one solution $y(t)$ to (4.4) *on $[t_0, \infty)$ satisfying $P_{\mathrm{y}}(t_0) y_1(t_0) = v_0$ and $ess\sup_{t \geqslant t_0} \|y(t)\| < \infty$.*
ii) Moreover, for any two solutions $y(t)$, $\tilde{y}(t)$ corresponding to different v_0, $\tilde{v}_0$ in $\mathrm{Im} P_{\mathrm{y}}(t_0)$, they are attracted to each other exponentially, i.e.,

$$\|y(t) - \tilde{y}(t)\| \leqslant H_4\, e^{-\mu(t-t_0)}\, \|v_0 - \tilde{v}_0\|, \quad \text{for all } t \geqslant t_0,$$

for some positive constants H_4, μ satisfying

$$0 < \mu < \nu + \ln\left(1 - \frac{H_3(1-e^{-\nu})}{1-L}\right), \quad H_4 := \frac{(1+H_1)N}{1 - L - \dfrac{H_3(1-e^{-\nu})}{1-e^{\mu-\nu}}}.$$

Proof. The proof of this theorem is essentially the same as the proof of Theorem 4.10. The only change is, that instead of considering the ball B_ρ we will work with the space $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ itself. Then, we can prove (without any difficulty) that for each fixed $v_0 \in \mathrm{Im} P_{\mathrm{y}}(t_0)$, the transformation T defined by

$$(Ty)(t) = \begin{cases} Y(t,t_0) \begin{bmatrix} v_0 \\ 0 \end{bmatrix} + \int_{t_0}^{\infty} G(t,\tau) \begin{bmatrix} \hat{f}_1(\tau, y(\tau)) \\ 0 \end{bmatrix} d\tau + \begin{bmatrix} 0 \\ \hat{f}_2(t, y(t)) \end{bmatrix}, & \text{for all } t \geqslant t_0, \\ 0, & \text{for all } t < t_0, \end{cases}$$

is a contraction mapping, and therefore, all the assertions of the theorem follows. □

Theorem 5.6. *Let the evolution family $(\hat{Y}(t,s))_{t\geqslant s\geqslant 0}$ of system (1.7) have an exponential dichotomy with the corresponding projection matrices $\{P_{\mathrm{y}}(t)\}_{t\geqslant 0}$ and the dichotomy constants $N,\nu>0$. Furthermore, assume that Assumption 5.2 holds true. If the condition*

$$H_3 < \min\left\{1-L,\ \frac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}\right\}$$

is fulfilled, then there exists a global stable manifold for the solutions of (4.4). Moreover, every two solutions $y(t)$, $\tilde{y}(t)$ on the manifold $\mathbb{M}$ attract each other exponentially in the sense that there exist positive constants H_4 and μ independent of $t_0\geqslant 0$ such that

$$\|y(t)-\tilde{y}(t)\| \leqslant H_4\ e^{-\mu(t-t_0)}\ \|P(t_0)y_1(t_0)-P(t_0)y_2(t_0)\|,\quad \textit{for all } t\geqslant t_0\ .$$

Proof. Analogous to the proof of Theorem 4.12, we consider the decomposition $\mathbb{R}^d=\mathrm{Im}P_{\mathrm{y}}(t)\oplus\mathrm{kernel}\,P_{\mathrm{y}}(t)$ and set $W_1(t):=\mathrm{Im}P_{\mathrm{y}}(t)$ and $W_2(t):=\mathrm{kernel}\,P_{\mathrm{y}}(t)$. Thus, we see that $\inf_{t\in\mathbb{R}_+} Sn(W_1(t),W_2(t))>0$. Now we define the family of mappings $(g_t)_{t\geqslant 0}$ acting on W_1 as

$$g_t(w_1) := \int_t^\infty \hat{Y}_1(t,\tau)(I_d-P_{\mathrm{y}}(\tau))f_1(\tau,y(\tau))d\tau,$$

where the function $y(t)$ is bounded and be uniquely defined via Theorem 5.5 i). Clearly, $g_t(w_1)\in\ker P_{\mathrm{y}}(t)=W_2(t)$. To verify the Lipschitz property of g_t, let us consider two arbitrary elements w_1 and $\tilde{w}_1$ in W_1 and let y and $\tilde{y}$ be the corresponding functions defined via Theorem 5.5 i). Then, we see that

$$\begin{aligned}\|g_t(w_1)-g_t(\tilde{w}_1)\| &\leqslant \int_t^\infty N\ e^{-\nu(\tau-t)}\ \|f_1(\tau,y(\tau))-f_2(\tau,\tilde{y}(\tau))\|d\tau \leqslant \int_t^\infty N\ e^{-\nu(\tau-t)}\ \varphi(\tau)\ \|y(\tau)-\tilde{y}(\tau)\|d\tau\ ,\\ &\leqslant \frac{N}{1-e^{-\nu}}\ \left(\|\Lambda_1 T_1^+\varphi\|_\infty+\|\Lambda_1\varphi\|_\infty\right)\ \|y-\tilde{y}\|_\infty = \frac{H_3}{(1+H_1)(1+H_2)}\ \|y-\tilde{y}\|_\infty\ ,\end{aligned}$$

and hence, (4.9) implies that

$$\|g_t(w_1)-g_t(\tilde{w}_1)\| \leqslant \frac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)}\ \|w_1-\tilde{w}_1\|\ .$$

Finally, $H_3<\dfrac{(1-L)(1+H_1)(1+H_2)}{N+(1+H_1)(1+H_2)}$ yields that $\dfrac{NH_3}{(1+H_1)(1+H_2)(1-H_3-L)}<1$, and hence, g_t is a contraction mapping for all $t\geqslant 0$. Then, applying the Implicit Function Theorem for Lipschitz continuous mapping ([18, Lem. 2.7]), we see that the mapping $Id+g_t:\mathbb{M}_t\to W_1(t)$ is a homeomorphism. This implies the condition ii) of Definition 4.11, and hence, the proof is finished. ☐

Now let us illustrate our results by the following examples.

Example 5.7. The dynamical behavior of a system in fluid mechanics and turbulence modeling is often described by the incompressible Navier-Stokes equation on an open, bounded domain $\Omega\subset\mathbb{R}^k$, $k=2$ or 3, of the form

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nu\Delta u-\nabla p-(u\cdot\nabla)u+f(t,u,p),\\ \nabla\cdot u &= 0,\end{aligned}$$

where $\nu>0$ is the viscosity, $u=u(t,\xi)$ is the velocity field which is a function of the time t and the position ξ, p is the pressure, f is the external force. Then, discretizing the space variable by finite difference, finite volumes, or finite element methods [8], one obtains a differential-algebraic system of the following form.

$$\begin{aligned}&M\dot{U}=(K+N(U))\,U-CP+F(t,U,P),\\ &C^TU=0,\end{aligned}$$

where $U(t)$, $P(t)$ approximate the velocity $u(t,\xi)$ and the pressure $p(t,\xi)$, respectively. Here the leading matrix M is either an identity matrix or a symmetric positive definite matrix depending on the spatial discretization scheme. Furthermore, in many applications, the matrix $C^T M^{-1}\left(C - \dfrac{\partial F}{\partial P}\right)$ is nonsingular. We notice, see e.g. [1], that the differentiation index of this system is two, and hence, it is not strangeness-free, so Assumption 1.3 is violated. Thus, one needs to transform it first in order to obtain a DAE

$$\begin{aligned} M\dot{U} &= -(K + N(U))\, U - CP + F(t,U,P), \\ 0 &= C^T M^{-1} C\, P - C^T M^{-1}\left(F - (K + N(U))\, U\right) . \end{aligned} \tag{5.2}$$

Clearly, we still need to linearize (5.2) to obtain system of the form (1.1). Fortunately, in this case the linearization procedure around a trajectory yields the decoupled form (1.7)

$$\begin{aligned} M\dot{U} &= \tilde{A}_1(t)U + \tilde{A}_2(t)P + g_1(t,U,P), \\ 0 &= C^T M^{-1}\left(C - \frac{\partial F}{\partial P}\right) P - C^T M^{-1}\left(\frac{\partial F}{\partial U} - A(t)\right) U + C^T M^{-1} g_2(t,U,P) . \end{aligned} \tag{5.3}$$

We further notice that since $C^T M^{-1}\left(C - \dfrac{\partial F}{\partial P}\right)$ is nonsingular, from the second equation we can uniquely determine P in term of U, and hence, system (5.2) is indeed strangeness-free. Let

$$\tilde{A}_3(t) := -C^T M^{-1}\left(\frac{\partial F}{\partial U} - A(t)\right), \ \tilde{A}_4(t) := C^T M^{-1}\left(C - \frac{\partial F}{\partial P}\right)$$

Consequently, if the homogenous DAE

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{V} \end{bmatrix} = \begin{bmatrix} \tilde{A}_1(t) & \tilde{A}_2(t) \\ \tilde{A}_3(t) & \tilde{A}_4(t) \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix}$$

admits an exponential dichotomy, and g_1 satisfies the φ-Lipschitz condition, and g_2 is a contraction mapping (uniformly in time), then there exists a stable manifold for the solution to (5.2).

Example 5.8. Consider the nonlinear electrical circuit with the Josephson junction in Figure 1 below. The Josephson junction device on the right hand side, consisting of two super conductors separated by an oxide barrier, is characterized by the sinusoidal relation $i_2 = I_0 \sin(k\phi_2)$, where I_0 and k are positive constants depend on the device itself. Moreover, the resistance R, inductance L and conductance G are positive. Furthermore, i_1 is the current goes through the inductance, v_1 and v_2 are voltage of the inductance and the Josephson junction, respectively. It is important to note that we will consider nonlinear instead of linear

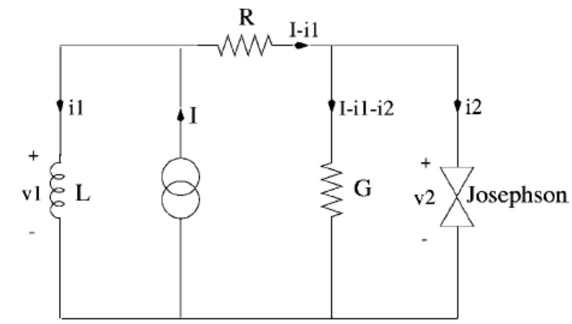

FIGURE 1. Electric circuit with Josephson junction, [23]

resistance, inductance and conductance as in [23], and hence, we see that for the inductance $i_1 = i_L(L, \phi_1)$, for the resistance $v_R = v_R(R, i_1)$, and for the conductance $i_G = i_G(G, v_2)$. Therefore, we obtain the following system, which completely describes the behavior of this circuit.

$$\dot{\phi}_1 = v_1, \tag{5.4a}$$
$$\dot{\phi}_2 = v_2, \tag{5.4b}$$
$$i_1 = i_L(L, \phi_1), \tag{5.4c}$$
$$i_2 = I_0 \sin(k\phi_2), \tag{5.4d}$$
$$0 = v_1 - v_R(R, i_1) + v_2, \tag{5.4e}$$
$$0 = -i_G(G, v_2) + I - i_1 - i_2. \tag{5.4f}$$

From (5.4c)-(5.4f) we obtain an explicit form of v_1 in terms of ϕ_1, i_1 and v_2, so we can compress the system to obtain

$$\dot{\phi}_1 = v_R(R, i_L(L, \phi_1)) + v_2, \tag{5.5a}$$
$$\dot{\phi}_2 = v_2, \tag{5.5b}$$
$$i_1 = i_L(L, \phi_1), \tag{5.5c}$$
$$0 = -i_G(G, v_2) + I - i_L(L, \phi_1) - I_0 \sin(k\phi_2). \tag{5.5d}$$

The linearized version of this system along equilibrium points defined by $v_2 = 0$, $i_1 = I$, $\phi_1 = LI$, $\phi_2 = n\pi/k$, reads

$$\begin{aligned}
\dot{\phi}_1 &= RI - (R/L)\phi_1 + v_2, \\
\dot{\phi}_2 &= v_2, \\
i_1 &= \phi_1/L, \\
0 &= -Gv_2 + I - \phi_1/L - I_0 \sin(k\phi_2),
\end{aligned}$$

will have one positive and one negative eigenvalue (e.g. [23]). Hence, it admits exponential dichotomy for any odd number n. Thus, for φ-Lipschitz function v_R and contraction mapping i_G, we obtain a stable manifold for (5.5).

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