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Almost Periodic Solutions for Delay-Differential Equations with Infinite Delays

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1. Introduction

One of the most widely methods for establishing the existence of almost periodic (a.p. for short) solutions of systems or ordinary differential equations is based on a general result due to Amerio [1], a generalization of an earlier result for linear a.p. systems due to Favard. In essence, this basic method consists of showing that if a solution is in a compact set in the state-space for all t and has a certain separation property with respect to other such solutions, then if this same property holds for all systems in the so-called hull of the given system, this solution will be a.p. This result in fact relates to a necessary as well as sufficient stability condition for a fairly general dynamical system to have an a.p. solution; cf. [2, Chap. 5]. In fact Miller [3] used the idea of embedding the a.p. system in a more general dynamical system and was thus able to obtain a local stability condition for the existence of a.p. solutions without using the separation conditions due to Favard and Amerio. Most stability conditions for the existence of a.p. solutions known prior to Miller's result were in terms of global stability conditions. For a fairly complete and comprehensive discussion of a.p. solutions for a.p. systems, the books by Fink [4] and Yoshizawa [5] are recommended. For a more general discussion of the idea of embedding a.p. systems in dynamical systems, cf. Sell [6].

For ordinary differential equations with fixed finite time delays, much of the method based on Amerio's separation result can and has been adapted to obtain similar stability conditions for the existence of a.p. solutions; cf. [7], for example. Miller in [3] shows that local stability conditions can also be used for such delay-differential equations. Results for delay equations with infinite time delays based on Amerio's separation condition have also been obtained; cf., for example, Hino [8]. However, for such systems which do not involve substantially fading memory, not much seems to have been done; one of the main difficulties in such infinite delay systems arises from the fact that bounded solutions may no longer remain in compact subsets of the state

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space, now consisting of a set of functions on an infinite interval with certain smoothness properties. A very general theory for such infinite delay state spaces has been proposed by Hale and Kato [9]; however, in the opinion of the author, the conditions required on such systems even for local existence of solutions are still essentially of fading memory type.

In [10], Kartsatos uses a somewhat obscure but quite useful result due to Medvedev [11] to obtain conditions for the existence of a.p. solutions of certain a.p. systems of ordinary differential equations. Medvedev's condition is basically a stability condition, but in a form which is in a sense usually easier to check than are the well-known stability conditions involving, for example, Liapunov functions, linear approximations, etc.

In this paper we use Medvedev's method for certain a.p. systems with infinite time delays, and following Kartsatos, obtain an existence theorem for a.p. solutions in a very direct way. As in the case of Miller's local stability conditions, we do not obtain uniqueness for such a.p. solutions in closed bounded sets of the state space. Global stability conditions usually yield such uniqueness.

In the last section of this paper we apply our general result to a first order scalar equation with infinite delay which may arise in models for population sizer variations of a single species interacting with itself; cf. Cushing [12, p. 123].

2. Notation, Definitions, and Main Results

Let R^n denote the set of real n-vectors, and $|x|$ any convenient norm for $x \in R^n$; also let $R = R^1$.

By CB we denote the set of R^n-valued functions continuous and bounded on $(-\infty, 0]$; for each $\phi \in CB$ we define $\|\phi = \sup\{|\phi(s)|: s \leqslant 0\}$. Thus $\{CB, \|\ \|\}$ is a real Banach space.

If $x(t)$ is an R^n-valued function on $(-\infty, b)$, $b \leqslant \infty$, we define for each $t \in (-\infty, b)\, x_t(s) = x(t+s)$, $s \leqslant \infty$. Clearly if $x(t)$ is continuous and bounded on each interval $(-\infty, b_1]$, $b_1 < b$, then $x_t \in CB$ for $t \in (-\infty, b)$.

The R^n-valued function $F(t, x, \phi)$ on $R \times R^n \times CB$ is said to satisfy condition:

(H1) *if it is a.p. in t uniformly for (x, ϕ) in closed bounded subsets of $R^n \times CB$: i.e., if $S \subset R^n \times CB$ is closed and bounded, then $\{F(t, x, \phi)$: $(x, \phi) \in S\}$ is a uniformly a.p. family in the sense of Fink* [4, p. 17];

(H2) *if for each $r > 0$ there exists a $M(r) > 0$ such that*

$$|F(t, 0, \phi)| \leqslant M(r) \qquad \textit{for } t \in R \quad \textit{and} \quad \|\phi\| \leqslant r;$$

(H3) *if for $x(t)$ uniformly continuous and bounded on R, $F(t, x(t), x_t)$ is uniformly continuous on R, and $F(t, y, x_t)$ is continuous in (t, y) on $R \times R^n$;*

(H4) *if there exist positive numbers p, H, and r such that $pH < 1$, $p \geqslant M(r)/r$ where $M(r)$ is as in* (H2), *such that*

(i) $|x - y + h(F(t, x, \phi) - F(t, y, \phi))| \leqslant (1 - ph)|x - y|$ *for* $0 < h < H$, $t \in R$, $|x| \leqslant 2r$, $|y| \leqslant 2r$, $\phi \in CB$, $\|\phi\| \leqslant r$, *and*

(ii) $|x(t) - y(t) + h(F(t, x(t), x_t) - F(t, y(t), y_t))| \leqslant (1 - ph)\|x_t - y_t\|$ *for* $t \in R$, $0 < h < H$, *and any functions* $x(t)$, $y(t)$ *uniformly continuous and such that* $|x(t)| \leqslant r$, $|y(t)| \leqslant r$ *on* R;

(H5) *if for $x^k(t)$ and $y^k(t)$ continuous and such that $|x^k(t)| \leqslant r$, $|y^k(t)| \leqslant r$ for all $t \in R$ and $k = 1, 2, \ldots$ and $x^k(t) \to x(t)$, $y^k(t) \to y(t)$ as $k \to \infty$ uniformly on compact subsets of R, we have*

$$F(t, x^k(t), y_t^k) \to F(t, x(t), y_t) \qquad \text{as} \quad k \to \infty$$

uniformly on compact subsets of R.

Remark 1. It follows easily that if F is uniformly continuous on $R \times R^n \times CB$, it satisfies (H3). In this case, however, the functions $F(t, x(t), x_t)$ and $F(t, y, x_t)$ are not necessarily continuous in t for $x(t)$ only continuous and bounded on R. This follows because x_t need not be continuous in t for such $x(t)$.

Remark 2. From (i) in (H4) and (H2) it follows easily that there exists $M_1(r) > 0$ such that

$$|F(t, x, \phi)| \leqslant M_1(r) \qquad \text{for } |x| \leqslant r, \quad \|\phi\| \leqslant r, \quad t \in R.$$

The following lemma is a result essentially due to Medvedev [11]; some obvious modifications in its proof are required for our non-global version.

LEMMA 1. *Let $A(t, x)$ and $g(t)$ be R^n-valued functions continuous on $R \times R^n$, and $A(t, 0) = 0$. Suppose there exist positive constants p, H, M such that $pH < 1$, $g(t) \leqslant M$ for all $t \in R$, and*

$$|x - y + h(A(t, x) - A(t, y))| \leqslant (1 - ph)|x - y| \tag{1.1}$$

for all $t \in R$, $|x| \leqslant 2M/p$, $|y| \leqslant 2M/p$ and $0 < h < H$. Then the equation

$$x' = A(t, x) + g(t) \tag{1.2}$$

has a unique solution $\bar{x}(t)$ bounded on R and such that $|\bar{x}(t)| \leqslant M/p$ for $t \in R$.

We now consider the delay-differential equation

$$x'(t) = F(t, x(t), x_t). \tag{2.1}$$

An equivalent form is

$$x'(t) = A(t, x(t), x_t) + B(t, x_t) \tag{2.11}$$

where $A(t, x, \phi) = F(t, x, \phi) - F(t, 0, \phi)$ and $B(t, \phi) = F(t, 0, \phi)$.

THEOREM 1. *Let F have properties* (H2), (H3), (i) *of* (H4) *and* (H5). *Then there exists a solution $\bar{x}(t)$ of* (2.1) *such that $|\bar{x}(t)| \leqslant r$ for $t \in R$; here r is as in* (H4).

Proof. With r as in (H4), let $M_1(r)$ be as in Remark 2. Define S_r to be the set of all R^n-valued functions f such that $|f(t)| \leqslant r$ and $|f(t) - f(t')| \leqslant M_1(r)\,|t - t'|$ for all t, $t' \in R$. With the topology of uniform convergence on compact subsets of R, the set S_r is clearly a compact convex subset of the set $C(R)$ of all R^n-valued functions continuous on R. Note that $C(R)$ is a locally convex linear topological space under this topology.

We define a mapping $T\colon S_r \to S_r$ as follows: $x(t) = Tf(t)$ is the unique bounded solution of

$$x'(t) = A(t, x(t), f_t) + B(t, f_t), \tag{2.2}$$

where $f \in S_r$. Such a solution exists by Lemma 1; if we put $A(t, x, f_t) = A(t, x)$, then by (H3), $A(t, x)$ is continuous in (t, x), and $B(t, f_t)$ is continuous on R. Also $|B(t, f_t)| \leqslant M(r)$ for $t \in R$ by (H2), and since also (i) of (H4) holds, (1.1) holds, and by the lemma, the solution $x(t)$ exists as asserted and satisfies $|x(t)| \leqslant M(r)/p \leqslant r$ for $t \in R$.

We next show that T is continuous on S_r in the topology of convergence on compact subsets of R. Let $f^k \to f$ as $k \to \infty$ in this topology where f^k and f are in S_r. Put $x^k = Tf^k$ and $x = Tf$. We show that each subsequence of $\{x^k\}$ contains a subsequence which converges to x; this will show that T is continuous at $f \in S_r$. To simplify notation, we denote this arbitrary subsequence again by $\{x^k\}$. Since $x^k \in S_r$, $\{x^k(t)\}$ is equicontinuous and uniformly bounded on R. By Ascoli's lemma, there exists a subsequence $\{k_j\}$ of the sequence of integers and a $u \in S_r$ such that $x^{k_j}(t) \to u(t)$ as $j \to \infty$ uniformly on compact subsets of R. It follows that $u(t)$ is a solution of

$$x'(t) = F(t, x(t), f_t). \tag{2.3}$$

To see this, fix a positive integer n. For $t \in [-n, n]$,

$$\left| u(t) - u(-n) - \int_{-n}^{t} F(s, u(s), f_s)\, ds \right|$$

$$= \left| u(t) - x^{k_j}(t) + x^{k_j}(-n) + \int_{-n}^{t} F(s, x^{k_j}(s), f^{k_j}(s))\, ds \right.$$

$$\left. + \int_{-n}^{t} F(s, u(s), f_s)\, ds - u(-n) \right| \leqslant |u(t) - x^{k_j}(t)| + |x^{k_j}(-n) - u(-n)|$$

$$+ \int_{-n}^{t} |F(s, x^{k_j}(s), f_s^{k_j}) - F(s, u(s) f_s)|\, ds.$$

Using (H5) we see that as $j \to \infty$ this last expression approaches zero; since n is arbitrary, $u(t)$ solves (2.3) for all $t \in R$, and $|u(t)| \leqslant r$, $t \in R$. Since $x(t)$ also solves (2.3) with $|x(t)| \leqslant r$, $t \in R$, and since by Lemma 1 such solutions are unique, we must have $u(t) = x(t)$ on R. Thus we conclude that T is continuous at each $f \in S_r$.

By the Tychonov fixed point theorem (cf., for example, [13, p. 11]) there exists a $\bar{x} \in S_r$ such that $T\bar{x} = \bar{x}$; clearly $\bar{x}(t)$ is the desired solution of (2.11); i.e., of (2.1). This proves the theorem.

Remark 3. If (H1)–(H5) hold for F, it follows that they also hold for any function G in the hull of F; cf. [4] for a definition of hull. Thus if $\bar{x} \in S_r$ were unique for each equation

$$x'(t) = G(t, x(t), x_t)$$

it would follow from well-known theory on the existence of a.p. solutions of such equations that $\bar{x}(t)$ is a.p. However, $\bar{x}$ is not necessarily unique, and so the following theorem does not trivially follow from Theorem 1.

THEOREM 2. *Let F satisfy conditions* (H1)–(H5). *Then* (2.1) *has an a.p. solution $\bar{x}(t)$ such that $|\bar{x}(t)| \leqslant r$ for $t \in R$.*

Proof. Let $\bar{x}(t)$ be the solution of (2.1) with $|\bar{x}(t)| \leqslant r$ for $t \in R$ which exists by Theorem 1. Note that $\bar{x}(t)$ is also uniformly continuous on R.

For $h \neq 0$ define

$$\begin{aligned}\alpha(t, h) &= h^{-1}(\bar{x}(t+h) - \bar{x}(t)) - F(t, \bar{x}(t), \bar{x}_t)\\ &= h^{-1} \int_{t}^{t+h} (F(s, \bar{x}(s), \bar{x}_s) - F(t, \bar{x}(t), \bar{x}_t))\, ds.\end{aligned}$$

Given $\varepsilon \geqslant 0$, there exists h, $0 < h < H$, such that

$$|F(t, \bar{x}(t), \bar{x}_t) - F(s, \bar{x}(s), \bar{x}_s)| < \varepsilon$$

for $|t-s|<h$; here H is as in (H4). This follows from (H3). So for such h, $|\alpha(t,h)|<\varepsilon$ for $t\in R$. Also for this h and any $t\in R$ and $\tau\in R$, we have

$$\begin{aligned}(\bar{x}(t+\tau+h)-\bar{x}(t+h))&-(\bar{x}(t+\tau)-\bar{x}(t))\\ &=(\alpha(t+\tau,h)-\alpha(t,h))\,h\\ &\quad+h(F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t),\bar{x}_t));\end{aligned}$$

hence

$$\begin{aligned}&\bar{x}(t+\tau+h)-\bar{x}(t+h)\\ &\quad=\bar{x}(t+\tau)-\bar{x}(t)+h\{(F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})\\ &\qquad-F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau}))+(F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t),\bar{x}_t))\}\\ &\qquad+h(\alpha(t+\tau,h)-\alpha(t,h)).\end{aligned}$$

From this we get

$$\begin{aligned}&|\bar{x}(t+\tau+h)-\bar{x}(t+h)|\\ &\quad\leqslant|\bar{x}(t+\tau)-\bar{x}(t)+h(F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t),\bar{x}_t))|\\ &\qquad+h\,|F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau})|+2h\varepsilon.\end{aligned}$$

Using (ii) in (H4) with $x(t)=\bar{x}(t+\tau)$, $u(t)=\bar{x}(t)$, we get

$$\begin{aligned}&|\bar{x}(t+\tau+h)-\bar{x}(t+h)|\\ &\quad\leqslant(1-ph)\,\|x_{t+\tau}-x_t\|\\ &\qquad+h\,|F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau})|+2h\varepsilon.\end{aligned}\tag{2.4}$$

By (H1), there exists a $L(\varepsilon)>0$ such that every interval of R of length $L(\varepsilon)$ contains a $\tau=\tau(\varepsilon)$ such that

$$|F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})-F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau})|<\varepsilon,$$

for all $t\in R$. Thus for such a τ, it follows from (2.4) that

$$|\bar{x}(t+\tau+h)-\bar{x}(t+h)|\leqslant(1-ph)\sup_{s\leqslant 0}|\bar{x}(t+\tau+s)-\bar{x}(t+s)|+3h\varepsilon$$

and from this we easily obtain

$$\sup_{t\in R}|\bar{x}(t+\tau)-\bar{x}(t)|\leqslant 3h\varepsilon/ph=3\varepsilon/p.$$

Thus τ is an $3\varepsilon/p$-translation number for $\bar{x}(\mathrm{t})$, and since $\varepsilon>0$ is arbitrary, $\bar{x}(t)$ is a.p. This proves the theorem.

Under the condition (H1) and the boundedness condition in Remark 2, it can be shown that corresponding to any closed and bounded subset S of $R^n \times CB$, there exists a sequence of Fourier exponents for $F(t, x, \phi)$ provided $(x, \phi) \in S$; a proof of this follows using Theorem 2.2 and 4.5 in [4] and the fact that the set of Fourier exponents of a single a.p. function is countable. It follows by a simple argument that there exists a sequence $\{\lambda_j : j = 1, 2,...\}$ of Fourier exponents for $F(t, x, \phi)$ independent of $(x, \phi) \in R^n \times CB$. The set of real numbers of the form $n_1\lambda_1 + \cdots + n_j\lambda_j$, where n_j are integers, is called the module of F. From the proof of Theorem 2 it can be shown that the module of the a.p. solution $\bar{x}(t)$, similarly defined, is contained in the module of F; cf. Theorem 4.5 on p. 61 in [4].

Hence the following

COROLLARY. *If F is periodic in t with period T independent of (x, ϕ), then if F also satisfies* (H2)–(H5), *there exists a periodic solution of* (2.1) *of period T/m for some integer $m \geqslant 1$.*

3. An Application

The following equation can arise in a study of te dynamics of a single-species population model; cf. Cushing [12, p. 123]:

$$N'(t) = N(t)\left(1 - k(t)\,N(t) - l(t)\int_{-\infty}^{0} N(t+s)\,d\eta(s)\right). \tag{3.0}$$

Since in this equation $N(t)$ represents population density we are only concerned with positive solutions, and can make the change of variables $x = \log N$ to get the equation

$$x'(t) = 1 - k(t)\exp x(t) - l(t)\int_{-\infty}^{0} (\exp x(t+s))\,d\eta(s). \tag{3.1}$$

This is in the form of the scalar case of (2.1) with

$$F(t, x, \phi) = 1 - k(t)\exp x(t) - l(t)\int_{-\infty}^{0} (\exp \phi(s))\,d\eta(s).$$

We assume that

(i) $\eta(s)$ is nondecreasing on $(-\infty, 0]$ with

$$\int_{-\infty}^{0} d\eta(s) = B < \infty.$$

(ii) $k(t)$ and $l(t)$ are a.p. with

$$\inf\{k(t) : t \in R = k > 0\}, \qquad \text{and} \qquad l(t) \geqslant 0 \quad \text{for } t \in R.$$

Define $p(t) = k(t) - 1$, $P = \sup\{|p(t)|: t \in R\}$, and $L = \sup\{l(t): t \in R\}$. Thus in this case

$$B(t, \phi) = -p(t) - l(t) \int_{-\infty}^{0} (\exp \phi(s))\, d\eta(s),$$

and hence $|B(t, \phi)| \leqslant P + e^r BL$ for $\|\phi\| \leqslant r$, $t \in R$. So $M(r) = P + e^r BL$, and hence (H2) holds. Since we need $M(r) \leqslant pr$, we need

$$P + e^r BL \leqslant pr. \tag{3.2}$$

Condition (i) of (H3) obviously holds provided

$$|x - y - hk(t)(e^x - e^y)| \leqslant (1 - ph)\,|x - y| \tag{3.3}$$

for $0 < h < H$, $pH < 1$, $|x| \leqslant 2r$, $|y| \leqslant 2r$. Using the Mean Value theorem, (3.3) is essentially equivalent to $|1 - hk(t)\, e^{\bar{x}}| \leqslant 1 - ph$ where $|\bar{x}| \leqslant 2r$. By fixing $H > 0$ such that $e^{2r}HK < 1$, this becomes

$$1 - hk(t)\, e^{\bar{x}} \leqslant 1 - ph,$$

or just $k(t)\, e^{\bar{x}} \geqslant p$. This last condition clearly holds if

$$ke^{-2r} = p. \tag{3.4}$$

To satisfy (ii) of (H4), we observe, as above, that this is implied by

$$\left| 1 - hk(t)\, e^{\bar{x}(t)} - hl(t) \int_{-\infty}^{0} e^{\bar{x}(t+s)}\, d\eta(s) \right| \leqslant (1 - ph)\, \|x_t - y_t\| / |x(t) - y(t)| \tag{3.5}$$

for all $t \in R$, $0 < h < H$, and $|x(t)| \leqslant r$, $|y(t)| \leqslant r$, $x(t) \neq y(t)$; here $\bar{x}(t) = y(t) + \theta(t)(x(t) - y(t))$ for some function $\theta(t)$, $0 < \theta(t) < 1$. We now assume $H > 0$ is such that

$$H(K + LB)\, e^{2r} < 1 \tag{3.6}$$

which clearly implies the previous condition on H, and also enables (3.5) to hold with the absolute value signs on the left removed. The resulting inequality then will clearly hold if (3.4) holds; note that $\|x_t - y_t\| / |x(t) - y(t)| \geqslant 1$. So finally, (H4) is implied by (3.2) if H is satisfies (3.6) and p is fixed by (3.4). Thus the condition we impose is

$$P + e^r BL \leqslant rke^{-2r}. \tag{3.7}$$

We conclude that for any P, B, L, k for which an $r > 0$ exists for which (3.7) holds, we have conditions (H2) and (H4) for our case (3.1). Clearly (3.7) will hold for some $r > 0$ provided P and B or L are sufficiently small relative to k; note that P small implies that k must be near 1.

The remaining conditions (H1), (H3), and (H5) easily seen to hold for our F in (3.1). Thus we conclude

THEOREM 3. *If* (i) *and* (ii) *hold,* $P = \sup\{|k(t) - 1|: t \in R\}$ *is sufficiently small, and either* $L = \sup\{|l(t)|: t \in R\}$ *or* $B = \int_{-\infty}^{0} d\eta(s)\, ds$ *is sufficiently small, then* (3.1), *and hence also* (3.0), *has an a.p. solution whose module is contained in the module of* $\{l(t), k(t)\}$. *Moreover, this a.p. solution* $\bar{x}(t)$ *of* (3.1) *satisfies* $|\bar{x}(t)| \leqslant r$ *for* $t \in R$ *where* r *is the smallest positive number satisfying* (3.7) *where* $k = \inf\{|k(t)|: t \in R\}$.

4. SOME CONCLUDING REMARKS

It may be of interest to observe that (*i*) of (H4) cannot hold for functions F independent of x. In fact, in the application considered in the previous section, this is quite substantially reflected in that the coefficient $k(t)$ of e^x in the F considered there (Eq. (3.1)) must be positive and near 1 for all values of t.

It is clear that results of a much sharper quantitative nature than Theorem 3 can be obtained for Eq. (3.1) at the expense of more complicated conditions on the F in this equation. In fact, elementary calculus methods can be used to analyze the positive roots of the equation $a + be^r = cre^{-2r}$ as function of a, b, c; here of course $a = p$, $b = LB$, and $c = k$, as in (3.7).

The following open questions, obviously of some interest, suggests themselves for further study. Is the a.p. solution of Theorem 2 unique with respect to the set of solutions of (2.1) in S_r? What can be said about the stability properties of this a.p. solution?

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