

<sup>1</sup> **Stability analysis of arbitrarily high-index, positive  
<sup>2</sup> delay-descriptor systems**

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<sup>6</sup> **Abstract** This paper deals with the stability analysis of positive delay-descrip-  
<sup>7</sup> tor systems with arbitrarily high index. First we discuss the solvability prob-  
<sup>8</sup> lem, which is followed by the study on characterizations of the (internal) pos-  
<sup>9</sup> itivity. Finally, we discuss the stability analysis. Numerically verifiable condi-  
<sup>10</sup> tions in terms of matrix inequality for the system's coefficients are proposed,  
<sup>11</sup> and are examined in several examples.

<sup>12</sup> **Keywords** Positivity · Stability · Delay · Descriptor systems · Singular  
<sup>13</sup> systems .

<sup>14</sup> **Nomenclature**

$\mathbb{N}$ ( $\mathbb{N}_0$ )	the set of natural numbers (including 0)
$\mathbb{R}$ ( $\mathbb{R}_+$ )	the set of real (non-negative real) numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_-$	the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$
$I$ ( $I_n$ )	the identity matrix (of size $n \times n$ )
$x^{(j)}$	the $j$ -th derivative of a function $x$
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of $p$ -times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$ (for $0 \leq p \leq \infty$ )
$\ \cdot\ _\infty$	the norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$
$\operatorname{im}_+ W$	the space $\{Ww_1 \text{ for all } w_1 \in \mathbb{R}_+^n\}$
$\mathcal{K}(U, W)$	the matrix $\mathcal{K}(U, W) := [W, UW, \dots, U^{\nu-1}W]$ .

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**16 1 Introduction**

Our focus in the present paper is on the positivity and stability analysis of linear, constant coefficients *delay-descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_dx(t - \tau) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \\ y(t) &= Cx(t), \end{aligned} \quad (1)$$

17 where  $E, A \in \mathbb{R}^{n,n}$ ,  $B \in \mathbb{R}^{n,p}$ ,  $C \in \mathbb{R}^{q,n}$ ,  $x : [t_0 - \tau, \infty) \rightarrow \mathbb{R}^n$ ,  $f : [t_0, \infty) \rightarrow \mathbb{R}^n$ ,  
 18 and  $\tau > 0$  is a constant delay. Together with (1), we are also concern with  
 19 the associated *zero-input/free system*

$$E\dot{x}(t) = Ax(t) + A_dx(t - \tau), \quad \text{for all } t \in [t_0, \infty). \quad (2)$$

20 Systems of the form (1) can be considered as a general combination of two  
 21 important classes of dynamical systems, namely *differential-algebraic equations*  
 22 (*descriptor systems*) (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad (3)$$

23 where the matrix  $E$  is allowed to be singular ( $\det E = 0$ ), and *delay-differential*  
 24 *equations* (DDEs)

$$\dot{x}(t) = Ax(t) + A_dx(t - \tau) + Bu(t). \quad (4)$$

25 Delay-descriptor systems of the form (1) have been arisen in various applica-  
 26 tions, see Ascher and Petzold [1995], Campbell [1980], Hale and Lunel [1993],  
 27 Shampine and Gahinet [2006], Zhu and Petzold [1997] and the references there  
 28 in. From the theoretical viewpoint, the study for such systems is much more  
 29 complicated than that for standard DDEs or DAEs. The dynamics of DDAEs  
 30 has been strongly enriched, and many interesting properties, which occur nei-  
 31 ther for DAEs nor for DDEs, have been observed for DDAEs Campbell [1995],  
 32 Du et al. [2013], Ha [2018]. Due to these reasons, recently more and more  
 33 attention has been devoted to DDAEs, Campbell and Linh [2009], Fridman  
 34 [2002], Ha and Mehrmann [2012, 2016], Michiels [2011], Shampine and Gahinet  
 35 [2006], Tian et al. [2014], Linh and Thuan [2015].

36  
 37 [.... Em nho anh viet bo sung 1 phan gioi thieu ve viec can thiet phai nghien  
 38 cuu tinh on dinh cua he duong voi chi so cao o day .... ]

39  
 40 The short outline of this work is as follows. Firstly, in Section 2, we briefly  
 41 recall the solvability analysis to system (1) (Theorem 1), followed by a result  
 42 about solution comparison for the free system (2) (Theorems 3, 4). Based on  
 43 the explicit solution representation in Section 2, we present a characterization  
 44 for the positivity of system (1) in Section 3. Numerically verifiable conditions  
 45 in terms of the matrix coefficients are established there. To follow, in Section 4  
 46 we discuss further about the free system (2) under biconditional requirements:  
 47 stability and positivity (Theorems 6,7). Numerical examples are presented to  
 48 illustrate the advantages of the proposed methods. Finally, we conclude this  
 49 research with some discussion and open questions.

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**50 2 Preliminaries**

51 In this section we discuss the solvability analysis (i.e., about the existence  
 52 and uniqueness of a solution), including the solution representation and the  
 53 comparison principal for the initial value problem (IVP) consisting of (1) with  
 54 an initial condition

$$x|_{[t_0-\tau, t_0]} = \varphi : [t_0 - \tau, t_0] \rightarrow \mathbb{R}^n. \quad (5)$$

55 Here,  $\varphi$  is a prescribed initial trajectory (preshape function), which is necessary  
 56 to achieve uniqueness of solutions. Without loss of generality, we assume that  
 57  $t_0 = 0$ .

58 **2.1 Existence, uniqueness and explicit solution formula**

59 It is well-known (e.g. Du et al. [2013]) that we may consider different solution  
 60 concepts for system (1). The reason is, that  $E(0)\dot{x}(0^+)$  which arises from  
 61 the right hand side in (1) at 0 may not be equal to  $E(0)\dot{\varphi}(0^-)$ . Moreover,  
 62 it has been observed in Baker et al. [2002], Campbell [1980], Guglielmi and  
 63 Hairer [2008] that a discontinuity of  $\dot{x}$  at  $t = 0$  may propagate with time, and  
 64 typically  $\dot{x}$  is discontinuous at every point  $j\tau$ ,  $j \in \mathbb{N}_0$  or  $x$  may not even exist  
 65 on the whole interval  $[t_0, \infty)$ . To deal with this property of DDAEs, we use  
 66 the following solution concept.

67 **Definition 1** Let us consider a fixed input function  $u(t)$ .

- 68 i) A function  $x : [-\tau, \infty) \rightarrow \mathbb{R}^n$  is called a *piecewise differentiable solution* of  
 69 (1), if  $Ex$  is piecewise continuously differentiable,  $x$  is continuous and satisfies  
 70 (1) at every  $t \in [t_0, \infty) \setminus \bigcup_{j \in \mathbb{N}_0} \{j\tau\}$ .  
 71 ii) A function  $x : [-\tau, \infty) \rightarrow \mathbb{R}^n$  is called a *classical solution* of (1) if it is at  
 72 least continuous and satisfies (1) at every  $t \in [t_0, \infty)$ .

73 Throughout this paper whenever we speak of a solution, we mean a piece-  
 74 wise differentiable solution. Notice that, like DAEs, DDAEs are not solvable  
 75 for arbitrary initial conditions, but they have to obey certain consistency con-  
 76 ditions.

77 **Definition 2** An initial function  $\varphi$  is called *consistent* with (1) if the associ-  
 78 ated IVP (1), (5) has at least one solution. System (1) is called *solvable* (resp.  
 79 *regular*) if for every consistent initial function  $\varphi$ , the IVP (1), (5) has a solution  
 80 (resp. has a unique solution).

For each  $j \in \mathbb{N}$ , we introduce sequences of matrix-valued and vector-valued functions  $f_j, u_j, x_j$  on the time interval  $[0, \tau]$  via

$$\begin{aligned} f_j(t) &= f(t + (j-1)\tau), \quad u_j(t) = u(t + (j-1)\tau), \\ x_j(t) &= x(t + (j-1)\tau), \quad x_0(t) := \varphi(t - \tau), \end{aligned}$$

<sup>81</sup> we can rewrite the IVP (1)-(5) as a sequence of non-delayed descriptor systems

$$E\dot{x}_j(t) = Ax_j(t) + A_dx_{j-1}(t) + Bu_j(t), \quad (6)$$

<sup>82</sup> for all  $t \in (0, \tau)$  and for all  $j = 1, 2, \dots$ . We notice, that for each  $j$ , the initial  
<sup>83</sup> condition  $x_j(0)$  is given due to the continuity of the solution  $x(t)$  at the point  
<sup>84</sup>  $(j-1)\tau$ , i.e.,

$$x_j(0) = x_{j-1}(\tau). \quad (7)$$

<sup>85</sup> In particular,  $x_1(0) = \phi(0)$  and the function  $x_0$  is given.

<sup>86</sup>

<sup>87</sup> It is well-known (see e.g. Bellman and Cooke [1963], Hale and Lunel [1993])  
<sup>88</sup> that in general, time-delayed systems has been classified into three different  
<sup>89</sup> types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0\dot{x}(t) + a_1\dot{x}(t - \tau) + b_0x(t) + b_1x(t - \tau) = f(t)$$

<sup>90</sup> is retarded if  $a_0 \neq 0$  and  $a_1 = 0$ ; is neutral if  $a_0 \neq 0$ ,  $a_1 \neq 0$ ; is advanced  
<sup>91</sup> if  $a_0 = 0$ ,  $a_1 \neq 0$ ,  $b_0 \neq 0$ . Obviously, this classification is based on the  
<sup>92</sup> smoothness comparison between  $x_j(t)$  and  $x_{j-1}(t)$ . In literature, not only the  
<sup>93</sup> theoretical but also the numerical solution has been studied mainly for re-  
<sup>94</sup> retarded and neutral systems, due to their appearance in various applications.  
<sup>95</sup> For this reason, in Ha [2015], Ha and Mehrmann [2016], Unger [2018] the  
<sup>96</sup> authors proposed a concept of *non-advancedness* for the free system (2) (see  
<sup>97</sup> Definition 3 below). We also notice, that even though not clearly proposed,  
<sup>98</sup> due to the author's knowledge, so far results for delay-descriptor are only ob-  
<sup>99</sup> tained for certain classes of non-advanced systems, e.g. Ascher and Petzold  
<sup>100</sup> [1995], Shampine and Gahinet [2006], Zhu and Petzold [1997, 1998], Michiels  
<sup>101</sup> [2011], Phat and Sau [2014], Sau et al. [2016], Cui et al. [2018], Ngoc [2018].

<sup>102</sup> **Definition 3** A regular delay-descriptor system (1) is called *non-advanced* if  
<sup>103</sup> for any consistent and continuous initial function  $\varphi$ , there exists a piecewise  
<sup>104</sup> differentiable solution  $x(t)$  to the IVP (1), (5).

<sup>105</sup> **Definition 4** Consider the DDAE (1). The matrix triple  $(E, A, B)$  is called  
<sup>106</sup> *regular* if the (two variable) *characteristic polynomial*  $\det(\lambda E - A - \omega B)$  is  
<sup>107</sup> not identically zero. If, in addition,  $B = 0$  we say that the matrix pair  $(E, A)$   
<sup>108</sup> (or the pencil  $\lambda E - A$ ) is regular. The sets  $\sigma(E, A, B) := \{\lambda \in \mathbb{C} \mid \det(\lambda E -$   
<sup>109</sup>  $A - e^{-\lambda\tau}B) = 0\}$  and  $\rho(E, A, B) = \mathbb{C} \setminus \sigma(E, A, B)$  are called the *spectrum* and  
<sup>110</sup> the *resolvent set* of (1), respectively.

<sup>111</sup> Provided that the pair  $(E, A)$  is regular, we can transform them to the  
<sup>112</sup> Kronecker-Weierstraß canonical form (see e.g. Dai [1989], Kunkel and Mehrmann  
<sup>113</sup> [2006]). That is, there exist regular matrices  $W, T \in \mathbb{R}^{n,n}$  such that

$$(E, A) = \left( W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right), \quad (8)$$

<sup>114</sup> where  $N$  is a nilpotent matrix of nilpotency index  $\nu$ . We also say that the  
<sup>115</sup> pair  $(E, A)$  has a *differentiation index*  $\nu$ , i.e.,  $\text{ind}(E, A) = \nu$ . Furthermore, the  
<sup>116</sup> system (1) is called *impulse-free* if in the form (8)  $N = 0$ .

117 *Remark 1* We notice that the impulse-freeness of system (1) is equivalent to  
118 the algebraic condition  $\deg(\det(sE - A)) = \text{rank}(E)$ . Furthermore, for reg-  
119 ular matrix pair  $(E, A)$ , the impulse-freeness also has other names, such as  
120 strangeness-free or index 1 or causal, see Du et al. [2013], Sau et al. [2016],  
121 Ngoc [2018].

122 *Remark 2* In general, the two concepts non-advancedness and differentiation  
123 index are independent. In details, a non-advanced system can have arbitrarily  
124 high index, as can be seen in the following example.

125 *Example 1* Consider the following systems with two parameters  $\varepsilon_1, \varepsilon_2$ .

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_E \dot{x}(t) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_A x(t) + \underbrace{\begin{bmatrix} 0 & \varepsilon_1 \\ 0 & \varepsilon_2 \end{bmatrix}}_{A_d} x(t - \tau). \quad (9)$$

126 In this example  $\text{ind}(E, A) = 2$ . Furthermore, depending on the value of  $\varepsilon_2$ ,  
127 the system will be advanced (if  $\varepsilon_2 \neq 0$ ) and be non-advanced (if  $\varepsilon_2 = 0$ ).  
128 Analogously, one can construct a non-advanced system which has an arbitrarily  
129 high index.

130 **Definition 5** The null solution  $x = 0$  of the free system (2) is called *expo-*  
131 *nentially stable* if there exist positive constants  $\delta$  and  $\gamma$  such that for any  
132 consistent initial function  $\varphi \in C([-\tau, 0], \mathbb{R}^n)$ , the solution  $x = x(t, \varphi)$  of the  
133 corresponding IVP to (2) satisfies

$$\|x(t)\| \leq \delta e^{-\gamma t} \|\varphi\|_\infty, \quad \text{for every } t \geq 0.$$

134 Let  $E$  have index  $\tilde{\nu}$ , i.e.,  $\text{ind}(E, I_n) = \tilde{\nu}$ , the Drazin inverse  $E^D$  of  $E$  is  
135 uniquely defined by the properties

$$E^D E = E E^D, \quad E^D E E^D = E^D, \quad E^D E^{\tilde{\nu}+1} = E^{\tilde{\nu}}. \quad (10)$$

136

137 **Lemma 1** Kunkel and Mehrmann [2006] Let  $(E, A)$  be a regular matrix pair.  
138 Then for any  $\lambda \in \rho(E, A)$ , the following matrices commute

$$\hat{E} := (\lambda E - A)^{-1} E, \quad \hat{A} := (\lambda E - A)^{-1} A. \quad (11)$$

139 Furthermore, the following commutative identities hold true.

$$\hat{E} \hat{A}^D = \hat{A}^D \hat{E}, \quad \hat{E}^D \hat{A} = \hat{A} \hat{E}^D, \quad \hat{E}^D \hat{A}^D = \hat{A}^D \hat{E}^D. \quad (12)$$

140 We notice that the matrix products  $\hat{E}^D \hat{E}$ ,  $\hat{E}^D \hat{A}$ ,  $\hat{E} \hat{A}^D$ ,  $\hat{E}^D \hat{B}$ ,  $\hat{A}^D \hat{B}$  do  
141 not depend on the choice of  $\lambda$  (see e.g. Dai [1989]). Furthermore, they can  
142 be numerically computed by transforming the pair  $(E, A)$  to their Weierstrass  
143 canonical form (8) (see e.g. Varga [2019], Virnik [2008]).

144 For any  $\lambda \in \rho(E, A)$ , we denote

$$\hat{A}_d := (\lambda E - A)^{-1} A_d, \quad \hat{B} := (\lambda E - A)^{-1} B. \quad (13)$$

145 Making use of the Drazin inverse, in the following theorem we present the  
146 explicit solution representation of system (1).

**Theorem 1** Consider the delay-descriptor system (1). Assume that  $(E, A)$  is a regular matrix pair with a differentiation index  $\text{ind}(E, A) = \nu$ . Let  $\hat{E}$ ,  $\hat{A}$ ,  $\hat{A}_d$ ,  $\hat{B}$  be defined as in (11), (13). Furthermore, assume that  $u$  is sufficiently smooth. Then, every solution  $x_j$  of the DAE (6) has the form

$$\begin{aligned} x_j(t) &= e^{\hat{E}^D \hat{A}t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left( \hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds \\ &+ (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left( \hat{A}_d x_{j-1}^{(i)}(t) + \hat{B} u_j^{(i)}(t) \right), \end{aligned} \quad (14)$$

for some vector  $v_j \in \mathbb{R}^n$ .

*Proof.* The proof is straightly followed from the explicit solution of DAEs, see [Kunkel and Mehrmann, 2006, Chap. 2].  $\square$

From Theorem 1 and (7), we directly obtain the following corollary.

**Corollary 1** The solution  $x(t)$  of system (1) is continuous at the point  $(j-1)\tau$  if and only if the following condition holds.

$$(\hat{E}^D \hat{E} - I) x_{j-1}(\tau) = (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left( \hat{A}_d x_{j-1}^{(i)}(0) + \hat{B} u_j^{(i)}(0) \right).$$

In particular, for the preshape function  $\varphi(t)$ , we must require

$$(\hat{E}^D \hat{E} - I) \left( \varphi(0) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left( \hat{A}_d \varphi^{(i)}(-\tau) + \hat{B} u^{(i)}(0) \right) \right) = 0.$$

Following from (14), we directly obtain a simpler form in case of non-advanced system as follows.

**Corollary 2** Consider system (1) and assume that it is regular and non-advanced. Then, we have

$$\begin{aligned} x_j(t) &= e^{\hat{E}^D \hat{A}t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left( \hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds \\ &+ (\hat{E}^D \hat{E} - I) \left( \hat{A}^D \hat{A}_d x_{j-1}(t) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u_j^{(i)}(t) \right), \end{aligned} \quad (15)$$

Furthermore, the consistency condition at  $t = 0$  reads

$$(\hat{E}^D \hat{E} - I) \left( \varphi(0) + \hat{A}^D \hat{A}_d \varphi(-\tau) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u^{(i)}(0) \right) = 0. \quad (16)$$

157 2.2 A simple check for the non-advancedness

158 Assume that the pair  $(E, A)$  is regular with index  $\text{ind}(E, A) = \nu$ . We want  
 159 to give a simple check whether the free system (2) is non-advanced or not.  
 160 In analogous to the case of DAEs, see e.g. Brenan et al. [1996], Kunkel and  
 161 Mehrmann [2006], we aim to extract the so-called *underlying delay equation*  
 162 of the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{A}_{d0}x(t-h) + \mathbf{A}_{d1}\dot{x}(t-h), \quad (17)$$

163 from an augmented system consisting of system (2) and its derivatives, which  
 164 read in details

$$\frac{d^i}{dt^i} (Ex(t) - Ax(t) - A_dx(t-\tau)) = 0, \text{ for all } i = 0, 1, \dots, \nu.$$

We rewrite these equations into the so-called *inflated system*

$$\begin{aligned} & \underbrace{\begin{bmatrix} E \\ -A & E \\ & \ddots & \ddots \\ & & -A & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix} \\ & + \underbrace{\begin{bmatrix} A_d & & & \\ & A_d & & \\ & & \ddots & \\ & & & A_d \end{bmatrix}}_{\mathcal{A}_d} \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \\ \vdots \\ x^{(\nu)}(t-h) \end{bmatrix}. \end{aligned} \quad (18)$$

Here the matrix coefficients are  $\mathcal{E}, \mathcal{A}, \mathcal{A}_d \in \mathbb{R}^{(\nu+1)n, (\nu+1)n}$ . For the reader's convenience, below we will use MATLAB notations. An underlying delay system (17) can be extracted from (18) if and only if there exists a matrix  $P = [P_0 \ P_1 \ \dots \ P_\nu]^T$  in  $\mathbb{R}^{(\nu+1)n, n}$  such that

$$\begin{aligned} P^T \mathcal{E} &= [I_n \ 0_{n, \nu n}], \\ P^T \mathcal{A}_d &= [* \ * \ 0_{n, (\nu-1)n}], \end{aligned}$$

165 where  $*$  stands for an arbitrary matrix. Consequently,  $P$  is the solution to the  
 166 following linear systems

$$[\mathcal{E} \ \mathcal{A}_d(:, 2n+1 : end)]^T P = [I_n \ 0_{n, \nu n} \ 0_{n, (\nu-1)n}]^T.$$

167 Therefore, making use of Crammer's rule we directly obtain the simple check  
 168 for the non-advancedness of system (2) in the following theorem.

169 **Theorem 2** Consider the zero-input descriptor system (2) and assume that  
 170 the pair  $(E, A)$  is regular with index  $\text{ind}(E, A) = \nu$ . Then, this system is non-  
 171 advanced if and only if the following rank condition is satisfied

$$\text{rank} \begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_d(:, 2n+1 : end)^T \end{bmatrix} = \text{rank} \begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_d(:, 2n+1 : end)^T \mid I_n \\ 0_{(\nu-1)n, n} \end{bmatrix} \quad (19)$$

172 Theorem 2 applied to the index two case straightly gives us the following  
173 corollary.

174 **Corollary 3** Consider the zero-input descriptor system (2) and assume that  
175 the pair  $(E, A)$  is regular with index  $\text{ind}(E, A) = 2$ . Then, system (2) is non-  
176 advanced if and only if the following identity hold true.

$$\text{rank} \begin{bmatrix} E^T & -A^T & 0 \\ 0 & E^T & -A^T \\ 0 & 0 & E^T \\ 0 & 0 & A_d^T \end{bmatrix} = n + \text{rank} \begin{bmatrix} E^T & -A^T \\ 0 & E^T \\ 0 & A_d^T \end{bmatrix}. \quad (20)$$

177 *Example 2* Let us reconsider system (9) in Example 1. Numerical verification  
178 of non-advancedness via condition (20) completely agrees with theoretical ob-  
179 servation.

### 180 2.3 Comparison principal

181 In this part of Section 2, we will show how to generalize our result to delay-  
182 descriptor systems with time-varying delay of the following form

$$E\dot{x}(t) = Ax(t) + A_dx(t - \tau(t)) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \quad (21)$$

183 where the delay function  $\tau(t)$  is preassumed continuous and bounded, i.e.  
184  $0 < \underline{\tau} \leq \tau(t) \leq \bar{\tau}$  for all  $t \geq 0$ . Here  $\underline{\tau}, \bar{\tau}$  are two positive constants. Following  
185 Ha and Mehrmann [2016], it can be shown that the solution to system (21)  
186 exists, unique and totally determined by any consistent initial function  $\varphi$  such  
187 that  $x(t) = \varphi(t)$  for all  $-\bar{\tau} \leq t \leq 0$ . Indeed, also making use of the method  
188 of steps, the solution  $x$  is constructively built on consecutive interval  $[t_{i-1}, t_i]$ ,  
189  $i \in \mathbb{N}$  such that  $0 = t_0 < t_1 < t_2 < \dots$  and

$$t_i - \tau(t_i) = t_{i-1}.$$

190 As shown in Theorems 3, 4 below, we can directly generalize our result to  
191 systems with bounded, time varying delay of the form (21).

192 **Theorem 3** Consider system (21) and assume that the corresponding con-  
193 stant delay system (1) is positive and non-advanced. For a fixed input  $u$ , let  
194  $x(t)$  (resp.  $\tilde{x}(t)$ ) be a state function corresponds to a preshape function  $\varphi(t)$   
195 (resp.  $\tilde{\varphi}(t)$ ). Furthermore, assume that  $\varphi(t) \leq \tilde{\varphi}(t)$  for all  $t \in [-\bar{\tau}, 0]$ . Then,  
196 we have  $x(t) \leq \tilde{x}(t)$  for all  $t \geq 0$ .

197 *Proof.* Based on the linearity of system (1),  $\tilde{x}(t) - x(t)$  satisfies the free system  
198 (2). Furthermore, since this system is non-advanced and positive the non-  
199 negativity of  $\tilde{\varphi}(t) - \varphi(t)$  implies that  $\tilde{x}(t) - x(t) \geq 0$  for all  $t$ .  $\square$

200 **Theorem 4** Consider system (21) and assume that the corresponding con-  
201 stant delay system (1) is positive. Furthermore, assume that

$$(\hat{E}^D \hat{E} - I) (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$$

202 for all  $i = 0, \dots, \nu - 1$ . Let  $x(t)$  (resp.  $\tilde{x}(t)$ ) be a state function corresponds to  
203 a reference input  $u(t)$  (resp.  $\tilde{u}(t)$ ) and a preshape function  $\varphi(t)$  (resp.  $\tilde{\varphi}(t)$ ).  
204 Then we have  $x(t) \leq \tilde{x}(t)$  for all  $t \geq 0$ , provided that the following conditions  
205 are fulfilled.  
206 i)  $\varphi(t) \leq \tilde{\varphi}(t)$  for all  $t \in [-\tau, 0]$ ,  
207 ii)  $u^{(i)}(t) \leq \tilde{u}^{(i)}(t)$  for all  $t \geq 0$  and for all  $i \leq (\nu - 1) \lfloor t/\tau \rfloor$ .

208

209 *Proof.* The proof is also straightforward from the solution's representation  
210 (14).  $\square$

211 From Theorems 3, 4 above, we see that the time varying delay will not  
212 affect our later results on the positivity and the stability of system (1).

### 213 3 Characterizations of positive delay-descriptor system

214 Since most systems occur in application are non-advanced, in this section we  
215 focus on the characterization for positivity of non-advanced delay descriptor  
216 systems. We, furthermore, notice that the non-advancedness is a necessary  
217 condition for the stability (in the Lyapunov sense) of any time-delayed system,  
218 see e.g. Hale and Lunel [1993], Du et al. [2013].

219 **Definition 6** Consider the delay-descriptor system (1) and assume that it is  
220 non-advanced, and that the pair  $(E, A)$  is regular with  $\text{ind}(E, A) = \nu$ . We call  
221 (1) positive if for all  $t \geq 0$  we have  $x(t) \geq 0$  and  $y(t) \geq 0$  for any input function  
222  $u$  and any consistent initial function  $\varphi(t)$  that satisfy two following conditions.  
223 i)  $\varphi(t) \geq 0$  for all  $t \in [-\tau, 0]$ ,  
224 ii) For any  $t \geq 0$ ,  $u^{(i)}(t) \geq 0$  for all  $i = 0, 1, \dots, (\nu - 1) \lfloor t/\tau \rfloor$ .

225 For nontiaonal convenience, let us denote by

$$\begin{aligned} P &:= \hat{E}^D \hat{E}, \quad \bar{A} := \hat{E}^D \hat{A}, \quad \bar{A}_d := \hat{E}^D \hat{A}_d, \quad \bar{B} := \hat{E}^D \hat{B}, \\ \mathcal{K}_\nu(\bar{A}, \hat{A}^D \hat{B}) &:= [\hat{A}^D \hat{B}, \bar{A} \hat{A}^D \hat{B}, \dots, \bar{A}^{\nu-1} \hat{A}^D \hat{B}] . \end{aligned} \quad (22)$$

Since our systems is linear, time invariant coefficients, it would be sufficient to study the positivity on the first time interval  $[0, \tau]$ . Making use of (15), and

let  $j = 1$ , we can rewrite the solution  $x_1 = x|_{[0,\tau]}$  as follows

$$\begin{aligned} x_1(t) &= \underbrace{e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)}_{x_{zi}(t)} + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds \\ &\quad + \underbrace{\int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{B}}u_j(s)ds + (P - I)\sum_{i=0}^{\nu-1}\bar{\mathbf{A}}^i\hat{A}^D\hat{B}u_j^{(i)}(t)}_{x_{zs}(t)}. \end{aligned} \quad (23)$$

In the theory of linear systems,  $x_{zi}(t)$  (resp.  $x_{zs}(t)$ ) is often called the *zero input/free* (resp. *zero state*) solution. The characterization for the positivity of the free solution  $x_{zi}$  is given in Rami and Napp [2012] as follows.

**Proposition 1** Rami and Napp [2012] *The following statements are equivalent.*

- i) *The non-delayed free system  $E\dot{x}(t) = Ax(t)$  is positive.*
- ii) *There exists a Metzler matrix  $H$  such that  $\bar{\mathbf{A}} = HP$ , where  $P$  is defined via (22).*
- iii) *There exists a matrix  $D$  such that  $H := \bar{\mathbf{A}} + D(I - P)$  is Metzler.*

**Lemma 2** Consider the delay-descriptor system (1) and assume that it is non-advanced. Let the pair  $(E, A)$  be regular with index  $\text{ind}(E, A) = \nu$ . Then, the free system (2) has a non-negative solution  $x_{zi}(t) \geq 0$  for all  $t \geq 0$  and for all consistent initial function  $\varphi(t) \geq 0$  if and only if the following conditions are satisfied.

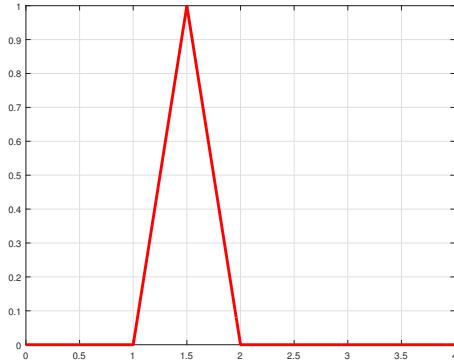
- i) *There exists a Metzler matrix  $H$  such that  $\bar{\mathbf{A}} = HP$ .*
- ii)  *$\bar{\mathbf{A}}_d \geq 0$ ,  $(P - I)\hat{A}^D\hat{A}_d \geq 0$ .*

*Proof.* “ $\Rightarrow$ ” Consider  $x_{zi}(t)$  in (23). For any fixed  $t \in (0, \tau)$ , since the integral part  $\int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds$  can be arbitrarily small chosen, independent of the two boundary points 0 and  $t$ , we see that the sum  $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$  must be non-negative for any non-negative vectors  $x_0(\tau)$  and  $x_0(t)$ . The independence of these two vectors leads to the fact that the sum  $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$  is non-negative if and only if both terms are non-negative. Thus, due to Proposition 1, the non-negativity of the term  $e^{\bar{\mathbf{A}}t}Px_0(\tau)$  is equivalent to the claim i). On the other hand, the non-negativity of the term  $(P - I)\hat{A}^D\hat{A}_d x_0(t)$  implies that  $(P - I)\hat{A}^D\hat{A}_d \geq 0$ .

To prove that  $\bar{\mathbf{A}}_d \geq 0$ , we assume the contrary, that there exist some indices  $i, j$  with  $[\bar{\mathbf{A}}_d]_{ij} < 0$ . Thus, for the  $j$ th unit vector  $e_j$ , we have  $[\bar{\mathbf{A}}_d e_j]_i < 0$ . For a sufficiently small  $\varepsilon > 0$ , let us choose the initial function  $x_0$  as follows

$$x_0(s) = \begin{cases} \left(1 - \frac{1}{\varepsilon}|t - \varepsilon - s|\right)e_j & \text{for all } |t - \varepsilon - s| \leq \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \quad (24)$$

The graph of the magnitude of  $x_0(s)$  is given in Figure 1. Since  $u \equiv 0$ ,



**Fig. 1** The function  $x_0$  in (24) with  $\tau = 4$ ,  $t = 2$ ,  $\varepsilon = 0.5$ .

$x_0(0) = x_0(\tau) = 0$ , the consistency condition (16) is trivially satisfied. Then, we have that

$$\begin{aligned} x_1(t) &= \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d x_0(s) ds = \int_{t-2\varepsilon}^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d x_0(s) ds, \\ &= \int_{t-2\varepsilon}^t (I + \bar{\mathbf{A}}(t-s) + \mathcal{O}((t-s)^2)) \left(1 - \frac{1}{\varepsilon}|t-\varepsilon-s|\right) \bar{\mathbf{A}}_d e_j ds. \end{aligned}$$

254 Thus, for sufficiently small  $\varepsilon$ , the coordinate  $(x_1(t))_i$  has exactly the same sign  
255 as  $[\bar{\mathbf{A}}_d e_j]_i$ , which is strictly negative. This is contradicted to the non-negativity  
256 of the solution  $x(t)$ , and hence, we conclude that  $\bar{\mathbf{A}}_d \geq 0$ .

257 “ $\Leftarrow$ ” It is directly followed from i) and ii) that all three summands of  $x_{zi}(t)$   
258 are non-negative. This completes the proof.  $\square$

259 **Theorem 5** Consider the delay-descriptor system (1) and assume that it is  
260 non-advanced. Let the pair  $(E, A)$  be regular with index  $\text{ind}(E, A) = \nu$ . Fur-  
261 thermore, assume that  $(P - I)\hat{A}^i \hat{A}^D \hat{B} \geq 0$  for all  $i = 0, \dots, \nu - 1$ . Then,  
262 system (1) is positive if and only if the following conditions hold.

- 263 i)  $\bar{\mathbf{A}} = H P$  for some Metzler matrix  $H$ .  
264 ii)  $\bar{\mathbf{A}}_d \geq 0$ ,  $\bar{\mathbf{B}} \geq 0$ ,  $(P - I)\hat{A}^D \hat{A}_d \geq 0$ ,  
265 iii)  $C$  is non-negative on the subspace

$$\mathcal{X} := \text{im}_+ \left[ P, (P - I)\hat{A}^D \hat{A}_d, (P - I) \mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D \hat{B}) \right]. \quad (25)$$

266 *Proof.* “ $\Rightarrow$ ” By consecutively choosing  $u \equiv 0$  and  $\phi \equiv 0$ , we see that both  
267 the free solution  $x_{zi}(t)$  and the zero-state solution  $x_{zs}(t)$  are non-negative for  
268 all  $t \geq 0$ . Analogous to the proof of Lemma 2 (the necessity part), the non-  
269 negativity of the integral  $\int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{B}} u_j(s) ds$  follows that  $\bar{\mathbf{B}} \geq 0$ . Thus, only  
270 the claim iii) needs to be proven. We notice that due to Lemma 1 and the  
271 property (10) of the Drazin inverse, we have that  $P$  and  $\bar{\mathbf{A}}$  commute, and  
272  $P \hat{E}^D = \hat{E}^D$ , and hence,

$$e^{\bar{\mathbf{A}}} \hat{E}^D = \hat{E}^D e^{\bar{\mathbf{A}}} = \hat{E}^D \hat{E} \hat{E}^D e^{\bar{\mathbf{A}}} = P e^{\bar{\mathbf{A}}} \hat{E}^D.$$

Therefore, we see that

$$\begin{aligned} e^{\bar{\mathbf{A}}t}Px_0(\tau) + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_dx_0(s)ds + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{B}}u_j(s)ds &\subseteq \text{im}_+(P), \\ (P - I)\hat{A}^D\hat{A}_dx_0(t) + (P - I)\sum_{i=0}^{\nu-1} \bar{\mathbf{A}}^i\hat{A}^D\hat{B}u_j^{(i)}(t) \\ &\subseteq \text{im}_+ \left[ (P - I)\hat{A}^D\hat{A}_d, (P - I)\mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D\hat{B}) \right]. \end{aligned}$$

273 Thus, the claim iii) is directly followed.

274 “ $\Leftarrow$ ” It is straightforward that from i) and ii) we obtain the non-negativity of  
275  $x(t)$ , and due to iii) we obtain the non-negativity of  $y(t)$ . This completes the  
276 proof.  $\square$

277 Theorem 5 applied to the non-delayed case (i.e.  $A_d = 0$ ) gives us the  
278 following corollary. We notice that this corollary has slightly improved the  
279 result [Virnik, 2008, Thm. 3.4].

280 **Corollary 4** Consider the descriptor system (3) and assume that the pair  
281  $(E, A)$  is regular with index  $\text{ind}(E, A) = \nu$ . Furthermore, assume that the  
282 inequalities  $(P - I)\bar{\mathbf{A}}^i\hat{A}^D\hat{B} \geq 0$  hold true for  $i = 0, \dots, \nu - 1$ .

283 Then, system (3) is positive if and only if the following conditions hold.

284 i)  $\bar{\mathbf{A}} = H P$  for some Metzler matrix  $H$ .

285 ii)  $\bar{\mathbf{B}} \geq 0$ ,

286 iii)  $C$  is non-negative on the subspace  $\mathcal{X}$  defined in (25).

#### 287 4 Stability of positive delay-descriptor system

288 In this section we focus our attention on systems which is both stable and  
289 positive. Firstly, we demonstrate that the non-advancedness is necessary for  
290 the stability. Then, we present several sufficient conditions to examining the  
291 stability of positive delay-descriptor systems, followed by an illustrate example.

292 *Example 3* Let us recall system (9) with  $\varepsilon_2 = -1$ ,  $\varepsilon_1 = 0$ . From the second  
293 equation we see that  $x_2(t) = x_2(t - \tau)$ . Inserting this into the first equation  
294 we obtain

$$\dot{x}_2(t - \tau) = x_1(t).$$

295 Therefore, we have  $x(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}x(t - \tau) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\dot{x}(t - \tau)$ , which implies that  
296 the system is of advanced type. Clearly, the solution formula implies that the  
297 system is unstable in the Lyapunov sense.

298 To study the stability of system (1), we first transform this system to an  
299 equivalent impulse-free system, in the sense that the solution of the original  
300 system and the transformed system coincide.

Let  $y_j(t) := Px_j(t)$  and  $z_j(t) := (I - P)x_j(t)$  for all  $j \in \mathbb{N}$ ,  $t \geq 0$ , then from the solution's representation (14) we obtain

$$x_j(t) = e^{\bar{\mathbf{A}}t}x_j(0) + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d(y_{j-1}(s) + z_{j-1}(s))ds + (P-I)\hat{A}^D\hat{A}_d x_{j-1}(t),$$

for all  $t \in (0, \tau)$ . Premultiply this equation with  $P$  and  $I - P$ , we then obtain the system

$$y_j(t) = e^{\bar{\mathbf{A}}t}y_j(0) + \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d(y_{j-1}(s) + z_{j-1}(s))ds, \quad (26a)$$

$$z_j(t) = (P - I)\hat{A}^D\hat{A}_d(y_{j-1}(t) + z_{j-1}(t)). \quad (26b)$$

301 This system can be rewritten as follows.

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_j(t) \\ \dot{z}_j(t) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_j(t) \\ z_j(t) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{A}}_d & \bar{\mathbf{A}}_d \\ (P - I)\hat{A}^D\hat{A}_d & (P - I)\hat{A}^D\hat{A}_d \end{bmatrix} \begin{bmatrix} y_{j-1}(t) \\ z_{j-1}(t) \end{bmatrix}. \quad (27)$$

302 Therefore, we see that this transformed system is impulse-free, and hence we  
303 can applied already known results to study the its stability. The following  
304 results are directly extended from Cui et al. [2018]

305 **Theorem 6** Consider the delay-descriptor system (1). Assume that the ma-  
306 trix pair  $(E, A)$  is regular, and system (1) is non-advanced. Then, system (1)  
307 is positive and asymptotically stable if the following conditions hold true.

308 i)  $\bar{\mathbf{A}}_d \geq 0$ ,  $(P - I)\hat{A}^D\hat{A}_d \geq 0$ ,

309 ii)  $C$  is non-negative on the subspace  $\text{im}_+ [P, (P - I)\hat{A}^D\hat{A}_d]$ ,

310 iii) the matrix  $\bar{H}$  is Hurwitz, where

$$\bar{H} := \begin{bmatrix} \bar{\mathbf{A}}_d + H & \bar{\mathbf{A}}_d \\ (P - I)\hat{A}^D\hat{A} & (P - I)\hat{A}^D\hat{A} - I \end{bmatrix}. \quad (28)$$

311 **Theorem 7** Consider the delay-descriptor system (1). Assume that the ma-  
312 trix pair  $(E, A)$  is regular, and system (1) is non-advanced. Furthermore, as-  
313 sume that there exists a positive vector  $w \in \mathbb{R}_+^n$  such that  $(P - I)\hat{A}^D\hat{A}w > 0$ .  
314 Then, system (1) is positive and asymptotically stable if and only if the fol-  
315 lowing conditions hold true.

316 i)  $\bar{\mathbf{A}}_d \geq 0$ ,  $(P - I)\hat{A}^D\hat{A}_d \geq 0$ ,

317 ii)  $C$  is non-negative on the subspace  $\text{im}_+ [P, (P - I)\hat{A}^D\hat{A}_d]$ ,

318 iii) the matrix  $\bar{H}$  is Hurwitz, where  $\bar{H}$  is defined in (28).

319 **Remark 3** We stress out that in previous results on positivity of delay-descriptor  
320 systems (except Ha [2018]) it is always assumed that the system is impulse-  
321 free, which is an unnecessary condition, see for instance Cui et al. [2018], Liu  
322 et al. [2009], Phat and Sau [2014], Sau et al. [2016]. In contrast, our result in  
323 Theorems 6, 7 provide (necessary and) sufficient conditions for the positivity  
324 of (1) without this impulse-free assumption.

In light of Remark 3, we illustrate how Theorem 6 and 7 apply to general situations by presenting an example where system (1) is not impulse-free, but it is positive and also stable. We notice that in this example, the system is of index  $\nu(E, A) = 2$ , even though arbitrarily high-index system can be constructed in the same fashion.

*Example 4* Let us consider system (1) whose the matrix coefficients are

$$E = \begin{bmatrix} -11 & 1 & 0.1521 \\ 0 & 0 & 0.9365 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 0.61 & 0.9236 \\ -1 & 0.6 & 0.4683 \\ 0 & 0 & 0.7722 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & -0.2 & -1.9298 \\ -0.8 & -0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Direct computation yields that the matrix polynomial  $\det(sE - A)$  is

$$\det(sE - A) = -4.32432 s - 0.563706,$$

and hence the system is not impulse-free, since  $\text{rank}(E) = 2$ . For  $s = 3$  we have  $\det(sE - A) \neq 0$ , so we obtain

$$\hat{E} = \begin{bmatrix} 0.3765 & -0.034227 & -0.13289 \\ 0.6275 & -0.057045 & -1.7823 \\ 0 & 0 & 0 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0.12949 & -0.10268 & -0.39866 \\ 1.8825 & -1.1711 & -5.3469 \\ 0 & 0 & -1 \end{bmatrix}, \quad \hat{A}_d = \begin{bmatrix} 0.1433 & 0.0082088 & 0.066051 \\ 1.5722 & 0.030348 & 0.11009 \\ 0 & 0 & 0 \end{bmatrix}.$$

We also see that the index of system (1) is  $\text{ind}(E, A) = 2$ . Corollary 3 applied here implies that the system is non-advanced. Furthermore, we have that

$$\bar{A} = \begin{bmatrix} -0.0050085 & 0.00045532 & -0.0004569 \\ -0.0083475 & 0.00075887 & -0.0007615 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{A}_d = \begin{bmatrix} 4.4908e-05 & 0.00065547 & 0.0067405 \\ 7.4847e-05 & 0.0010925 & 0.011234 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0.038421 & -0.0034929 & 0.003505 \\ 0.064036 & -0.0058214 & 0.0058417 \\ 0 & 0 & 0 \end{bmatrix}, \quad (P - I)\hat{A}^D\hat{A}_d = \begin{bmatrix} 0.14371 & 0.05087 & 0.42647 \\ 1.5494 & 0.10116 & 0.71079 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving the equality in Theorem 6, we obtain

$$H = \begin{bmatrix} -0.59167 & 0.27679 & 0 \\ 0.27679 & -0.29643 & 0 \\ 0 & 0 & -0.74313 \end{bmatrix}$$

The spectrum of  $H$  and  $\bar{H}$  are  $\sigma(H) = \{-0.7577, -0.7431, -0.1304\}$  and  $\sigma(\bar{H}) = \{-1.2832, -0.7577, -0.1221, -0.3001, -0.7431, -1.0000\}$ . Therefore, due to Theorem 6 we see that system (1) is both positive and stable.

## 5 Conclusion

In this paper, we have studied the stability of positive delay-descriptor systems of arbitrarily high index without the impulse-free assumption. Firstly, a necessary and sufficient condition has been proposed to ensure the positivity of delay-descriptor system. Then, stability conditions for positive systems of arbitrarily high index have been established.

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