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Approximate Solutions of Fractional Order Differential-Algebraic Equations

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Mathematics

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بسم الله الرحمن الرحيم

(نَرْفَعُ دَرَجَٰتٍ مَّن نَّشَاءُ وَفَوْقَ كُلِّ ذِي عِلْمٍ عَلِيمٌ)

صدق الله العظيم

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SUPERVISOR'S CERTIFICATION

We certify that this thesis, entitled "Approximate solutions of Fractional Order Differential-Algebraic Equations", was prepared under our supervision at the Department of Mathematics and Computer Applications, College of Science, Al-Nahrain University as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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We certify that we have read this thesis entitled "*Approximate Solutions of Fractional Order Differential-Algebraic Equations*" and as examining committee examined the student *(Zahraa Shakir Shayee Alobaidi)* in its contents and in what it connected with, and that, in our opinion, it meets the standards of a thesis for the degree of Master of Science in Mathematics.

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ABSTRACT

The main theme of this thesis is oriented about two objectives the first one is to study the fundamental concepts of the differential-algebraic equations and the fractional calculus which are needed for finding the approximate solution of the differential-algebraic equations of fractional order.

The second objective is to approximate the solution of the differential-algebraic equations of fractional order using two approximate methods which are so called the differential transform method and the Adomian decomposition method both of these methods expressed the solution of the differential-algebraic equations of fractional order as an infinite series in which its coefficients can be evaluated in a very simple way.

CONTENTS

INTRODUCTION

Differential-Algebraic Equations (DAEs) are system of differential equations where the unknown functions satisfy additional algebraic equations. They arise in many areas of science and engineering such as robotics (via Lagrange's equations with independent coordinates), biomechanics, control theory, electrical engineering (via Kirchoff's laws), and fluid dynamics (via the Navier-Stokes equations for incompressible flows). DAEs present both numerical and analytical difficulties as compared with systems of ordinary differential equations (ODEs), [Rang and Angermann, 2005].

This has made them an active area of research for engineers, applied mathematicians, and numerical analytic ever since the 1960's when the advantages of working with DAEs, directly rather than attempting to convert them to ODEs was first recognized [Bernan and et al., 1989], [Schwerin, 1999].

In particular, since the mid of 1980's there has been a flurry of research regarding the numerical solution of DAEs. The sequential regularization method and related predicted sequential regularization method are recent numerical methods designed to deal with certain classes of DAEs, specifically index-2 problems with and without singularities and index-3 problems that arise from so-called multibody systems.

The notion of the so-called index of a DAE plays a fundamental role in both theoretical and numerical investigations of such problems. It has turned out to give insight into the solution properties, as well as into the numerical difficulties to be expected when solving these problems, i.e., how to obtain consistent initial data if there are hidden constraints. To a certain extent the DAE index is a measure of the singularity of the DAE, [Rang and Angermann, 2005].

There are various types of indices known, for example, the differentiation index and the perturbation index to mention the best known indices.

Griepentrog and März [Griepentrog and März, 1986] described the theoretical background and define some indices, whereas Brenan, Campbell and Petzold in [Brenan and et al., 1996] presented a theoretical overview and numerical aspects.

The second main subject which deals with our work is the so called fractional calculus. Fractional calculus of mathematics which grows out of the traditional definitions of the calculus integral operators in which the same by fractional exponents in an out growth of exponents with integral value. Consider the physical meaning of the exponent, according to our primary school teacher, exponents provide a short notation for what is essentially a repeated multiplication of numerical value. This concept in itself is easy to grasp and straight forward. However, the physical definitions can clearly become confused when considering exponent of non integer value, [Loverro, 2004].

Most authors on this topic will cite a particular date as the birthday of so called 'Fractional Calculus'. In a letter dated September 30th, 1695 L'Hopital wrote to Leibniz asking him about a particular notation, he had been used in his publication for the nth-derivative of the linear function $f(x) = \frac{D^n}{Dx^n}$. L'Hopital's posed the question to Leibniz, what would the result be if n = ½. Leibniz's response "An apparent paradox, from which one day useful consequences will be drawn". In these words fractional calculus was born.

Following L'Hoptial's and Leibniz's first inquisition, fractional calculus was primary a study reserved for the best minds in mathematics. Fourier, Euler, Laplace are among the many that dabbled with fractional calculus and the mathematical consequences [Nishimoto, 1991].

Many found, using their own notation and methodology, definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions that have been popularized in the world of fractional calculus are the Riemann-Liouville and Grunwald-Letnikov definition. Most of the mathematical theory applicable to the study of fractional calculus was developed prior to the turn of the 20th century. However, it is in the past 100 years that the most intriguing leaps in engineering and scientific application have been found.

The mathematics has in some cases to change to meet the requirements of physical reality. Caputo reformulated the more 'classic' definition of the Riemann-Liouville fractional derivative in order to use integer order initial conditions to solve his fractional order differential equations [Podlubny, 1999]. However, during the last ten years fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various; especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [He, 1998], and the fluid-dynamic traffic model with fractional derivatives [He, 1999] can eliminate the deficiency arising from the assumption of continuum traffic flow.

Recently, many important mathematical models can be expressed in terms of systems of DAEs of fractional order. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential-algebraic equations. Only approximate solutions can be derived using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of systems of fractional ODEs and DAEs, [Zugrigat and et al., 2010].

In this thesis the approximate solution of DAEs of fractional order will be presented using the differential transform method and Adomain decomposition method. This work consists of three chapters as well as this introduction. In chapter one, the fundamental concepts of DAEs and fractional calculus are given. While in chapter two the approximate solution of DAEs of fractional order using DTM is presented. Finally the approximate solution of DAEs of fractional order by using ADM will be given in chapter three.

It is important to notice that, the calculations in chapter two and three are simplified using MATHCAD 2001i computer software. The results are presented by graph and in a tabulated form.

CHAPTER ONE

BASIC CONCEPTS

1.1 INTRODUCTION

In this chapter, we shall presents some general concepts related to this work, including the DAEs and fractional calculus.

This chapter consists of two sections as well as this introduction. In section two a historical background and basic concepts of DAEs are given including its definitions, the index of the DAEs and some real life problems in terms of DAEs. Finally, analytical solution of DAEs using Laplace transformation method.

While in section three primitive concepts with definitions related to fractional calculus are given, including gamma function, beta function, Riemann-Liouville formula of fractional integral, Weyle fractional integral and Abel Riemann (A-R) fractional integral as well as the Riemann-Liouville formula of fractional derivative, the A-R fractional derivative, the Caputo fractional derivative and the Grünwald fractional derivative, finally analytical solution of fractional differential equations.

1.2 BASIC CONCEPTS OF DIFFERENTIAL-ALGEBRAIC EQUATIONS

1.2.1 Historical Background of DAEs:

Many authors and researchers studied systems of DAEs say, [Gear, 1970] discussed a unified method for handling the mixed differential and algebraic equation of the type that commonly occur in the transient analysis of large networks or in continuous system simulation, [Starner, 1976] described a numerical algorithm for solution of implicit algebraic differential systems of equations, [Petzold, 1982] outlines number of difficulties which can arise when numerical methods are used to solve systems of DAEs some of the differential-algebraic system may be solved using numerical methods which are commonly used for solving stiff systems of ordinary differential equations, other problems can be solved using codes based on the stiff method, but only after extensive modifications to the error estimates and other strategies in the code, [Hank, 1987] solved the DAE of index-2 by using regularization method, [Lubicn and Hairer, 1988] used the extrapolation polynomial method to solve system of DAEs, [Hariharan and Harris, 1993] discussed the connection between systems of non-linear DAEs and singularly perturbed control systems in nonstandard form.

Pasic [Pasic, 1997] presented on algorithm for a numerical solution of system of DAEs, [Feng and Ji, 1997] studied, the differential evolution algorithm is applied for determining the optimal control solution for problems described by DAEs, [Wenjie and Petzold, 1998] proved that for Hessenberg delay DAEs of related type, the direct linearization along the stationary solution is valid. This validity is obtained by showing the equivalence between the direct linearization and the linearization of the state space form of the original problem, which is assured to be legitimate, [Dan and at el, 1999] used an implicit Runge-Kutta Method for integration the DAEs.

Shengtai and Petzold [Shengtai and Petzold, 2000] outlined some algorithms for sensitivity analysis of large-scale system DAE of DAEs and present algorithms and software for sensitivity analysis of large-scale of index up to two, [Danielle and Harry, 2002] described solution was using tools from geometric control theory, higher index differential-algebraic systems are shown to be inherently unstable about their solution manifold, [Yang and at el, 2003] presented adjoint sensitivity method for parameter-dependent DAE systems and discuses numerical stability is maintained for the adjoint system for the augmented adjoint system, [Kunkel and Mehrmann, 2005] described characterization of classes of singular linear DAE [Azizi and et al., 2006] gave a new method for distributed simulation of DAE systems was developed based on purely decentralized sliding mode control. Due to the large amount of computation and communication associated with large scale matrix inversion problems in the existing centralized approaches [Laurent and et al., 2007] presented second order extensions of the Hilber Hughes-Taylor method for systems of over determined DAEs arising, [Nguyen, 2007] discussed the stability radii of DAEs with structured perturbations, [Kunkel and Mehrmann, 2007] studied different stability concepts for differential-algebraic equations as well as stabilization techniques for numerical methods and spin-stabilized discretization, [Johan, 2008] presented optimal control and model reduction of nonlinear DAEs models. [Campbell and Linh, 2009] concerned with the asymptotic stability of DAEs with multiple delays and their numerical solutions.

1.2.2 Differential-Algebraic Equations, [Asher and Petzold, 1998]:

Consider an implicit ODE:

$$F(t, y(t), y'(t)) = 0$$

where y, y' and F are n-dimensional vectors and F is assumed to be smooth and if $\frac{\partial F}{\partial y'}$ then the system can, in principle, be written in the explicit ODE form :

$$y' = f(t, y(t))$$

Consider next on extension of the explicit ODE, that of an ODE with constraints:

$$x' = f(t, x(t), z(t)) \quad \text{(1.1a)}$$

$$g(t, x(t), z(t)) = 0 \quad \text{(1.1b)}$$

Here the ODE (1.1a) for x(t) depends on additional algebraic variables z(t), and the solution is forced in addition to satisfy the algebraic constraints (1.1b). The system (1.1) is a semi-explicit system of differential-algebraic equations (DAEs). Obviously, we can cast (1.1) in the form of an implicit ODE

$$F(t, y(t), y'(t)) = 0$$

where $\frac{\partial F}{\partial y'}$ may be singular.

Definition (1.1), [Brennan and et al., 1989]:

The general form of system of the first order linear systems of differential-algebraic equations with non-constant coefficients is:

$$A(t)\, y'(t) + B(t)\, y(t) = f(t) \qquad \text{(1.2)}$$

where A and B are $n\times n$ known function matrices, f is $n\times 1$ known vector function and y is $n\times 1$ unknown vector function, such that A is singular and B is nonsingular.

Example (1.1), [Khaleel, 2008]:

Consider the system of the first order nonlinear ordinary differential equations with constant coefficients:

$$y_1'(t) = 5y_1(t) + 7\sin y_2(t) + 7t^2$$

together with the nonlinear algebraic equation

$$e^{y_1(t)y_2(t)} - \sin y_2(t) + 5t^2 = 0$$

Example (1.2), [Khaleel, 2008]:

Consider system of the first order linear systems of differential-algebraic equations with constants coefficients:

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 7 & 1 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} t^2 \\ 5t \end{bmatrix}$$

Example (1.3), [Khaleel, 2008]:

Consider the initial value problem that consists of the system of first order linear systems of differential-algebraic equations with non constant coefficients

$$\begin{bmatrix} 2t & t^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}+\begin{bmatrix} t & 3t \\ t^2 & 0 \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} 4 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ t^2 \end{bmatrix}, \ t \geq 0$$

together with the initial conditions:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Definition (1.2), [Macdonald, 2001]:

The system of the first order differential-algebraic equations given by eq. (1.1) is in Hessenberg form of size r if it can be written in the form:

$$\begin{aligned} y_1'(t) &= F_1(t, y_1(t), y_2(t), \ldots, y_r(t)), \\ y_2'(t) &= F_2(t, y_1(t), y_2(t), \ldots, y_{r-1}(t)), \\ &\vdots \\ y_i'(t) &= F_i(t, y_{i-1}(t), y_i(t), \ldots, y_{r-1}(t)) \ , \ i = 3, 4, \ldots, r-1, \\ 0 &= F_r(t, y_{r-1}(t)), \end{aligned}$$

Therefore, the system of the first order nonlinear systems of differential-algebraic equations that takes the form:

$$\begin{aligned} y_1'(t) &= F_1(t, y_1(t), y_2(t), y_3(t)), \\ y_2'(t) &= F_2(t, y_1(t), y_2(t)) \\ 0 &= F_3(t, y_2(t)) \end{aligned}$$

is in Hessenberg form of size three.

Example (1.4), [Khaleel, 2008]:

Consider the Hessenberg form of size two:

$$\begin{aligned} y_1'(t) &= 2y_1(t) + 5y_2^2(t) + 8t^2 \\ 0 &= y_1^2(t) - e^{y_1(t)} + t^2 \end{aligned}$$

Example (1.5), [Khaleel, 2008]:

Consider the Hessenberg form of size three:

$$\begin{aligned} y_1'(t) &= 2y_1(t)\, y_2(t) + 5y_3^2(t) \\ y_2'(t) &= 8y_1(t) + 17y_2^2(t) \\ 0 &= 5y_2^2(t) + \cosh y_2(t) \end{aligned}$$

1.2.3 Index of DAE, [Asher and Petzold, 1998]:

A DAE involves a mixture of differentiations and integrations, one may hope that applying analytical differentiations to a given system and eliminating as needed, repeatedly if necessary, will yield an explicit ODE system for all the unknowns. This turns out to be true unless the problem is singular. The number of differentiations needed for this transformation is called the index of the DAE. Thus, ODEs have index 0. We will redefine this definition later, but first let us consider some simple examples.

Example (1.6), [Asher and Petzold, 1998]:

Let q(t) be a given smooth function, and consider the following problems for y(t).

The scalar equation:

$$y = q(t) \quad \text{.......} (1.3)$$

is a (trivial) index-1 DAE, because it takes one differentiation to obtain an ODE for y.

For the system

$$\left.\begin{aligned} y_1 &= q(t) \\ y_2 &= y_1' \end{aligned}\right\} \quad \text{.......} (1.4)$$

we differentiate the first equation to get:

$$y_2 = y_1' = q'(t)$$

and then:

$$y_2' = y_1'' = q''(t)$$

The index is 2 because two differentiations of q(t) were needed.

A similar treatment for the system:

$$\left.\begin{aligned} u &= q(t) \\ y_3 &= u'' \end{aligned}\right\} \quad \text{.......} (1.5)$$

necessitates three differentiations to obtain an ODE for y_3, hence the index is 3.

<u>Example (1.7), [Asher and Petzold, 1998]:</u>

Consider the DAE system for

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$y_1' = y_3,$$

$$0 = y_2(1 - y_2),$$

$$0 = y_1y_2 + y_3(1 - y_2) - t.$$

The third equation has two solutions $y_2 = 0$ and $y_2 = 1$, and it is given that $y_2(t)$ does not switch arbitrary between these two value (e.g., another equation involving y_2' and y_4' is prescribed with $y_4(0)$ given, implying continuity of $y_2(t)$).

1. Setting $y_2 = 0$, we get from the fourth equation $y_3 = t$. then from the first equation, $y_1 = y_1(0) + t^2/2$. The system has index-1 and the solution is

$$y(t) = \begin{pmatrix} y_1(0) + t^2/2 \\ 0 \\ t \end{pmatrix}.$$

   Note that this is an index-1 system in semi-explicit form.

2. Setting $y_2 = 1$, the fourth equation reads $y_1 = t$. Then, upon differentiating the first equation, $y_3 = 1$. The system has index-2 and the solution is

$$y(t) = \begin{pmatrix} t \\ 1 \\ 1 \end{pmatrix}.$$

   Note that, unlike in the index-1 case, no initial value is required.

If we replace the algebraic equation involving y_2 by its derivative and simplify, we obtain the DAE.

$y_1' = y_3$

$y_2' = 0$

$0 = y_1 y_2 + y_3(1 - y_2) - t$

Now the index depends on the initial conditions. If $y_2(0) = 0$ the index is 1, and if $y_2(0) = 1$ the index equals 2. We are ready to define the index of a DAE.

For general DAE system:

$F(t, y, y') = 0$

The index along a solution y(t) is the minimum number of differentiations of the system which would be required to solve for y' uniquely in terms of y and t (i.e., to define an ODE for y). Thus, the index is defined in terms of the over determined system:

$$\left.\begin{aligned} & F(t, y, y') = 0 \\ & \frac{\partial F}{\partial t}(t, y, y', y'') = 0 \\ & \qquad \vdots \\ & \frac{\partial^p F}{\partial t^p}(t, y, y', \ldots, y^{(p+1)}) = 0 \end{aligned}\right\} \qquad (1.6)$$

to be the smallest integer p so that y' in eq.(1.6) can be solved for in terms of y and t.

1.2.4 Some Real Life Applications of the System of Differential-Algebraic Equations, [Kunkel and Mehrmann, 2006]:

In this section, we give some real life applications for the system of differential-algebraic equations, namely electrical network and chemical reactor.

Example (1.8) (Electrical Network):

To obtain a mathematical model for the charging of capacitor via a resister, we associate a potential $x_i(t)$, i = 1, 2, 3 with each node of the circuit, the voltage source increases the potential $x_3(t)$ to $x_1(t)$ by U, i.e., $x_1(t) - x_3(t) - U = 0$. By Kirchoff's first law, the sum of the currents vanishes in each node. Hence, assuming ideal electronic units for the second node we obtain that:

$$\frac{C[x_3'(t) - x_2'(t)] + [x_1(t) - x_2(t)]}{R} = 0$$

where R is the resister and C is the capacity of the capacitor, when $x_3(t) = 0$, this take the form:

$$x_1(t) - x_3(t) - U = 0$$

$$\frac{C[x_3'(t) - x_2'(t)] + [x_1(t) - x_2(t)]}{R} = 0$$

$$x_3(t) = 0$$

it is clear that this system can be solved for $x_3(t)$ and $x_1(t)$ to obtain an ordinary systems of differential equation for $x_2(t)$ only, combined with algebraic equations for $x_1(t)$, $x_3(t)$. This system has index one.

<u>Example (1.9) (Chemical Reactor):</u>

Consider the model of chemical reactor in which a first order isomerization reaction takes place and which is externally cooled, this model take the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C'(t) \\ T'(t) \\ R'(t) \end{bmatrix} = \begin{bmatrix} k_1(C_0 - C_1(t)) - R(t) \\ k_1(T_0 - T(t)) + k_2 R(t) - k_3(T(t) - T_c) \\ R(t) - k_3 \exp\left(-\dfrac{k_4}{T(t)}\right) C(t) \end{bmatrix}$$

where, C_0 is the given feed reactant concentration, T_0 the initial temperature, $C_1(t)$ and $T(t)$ the concentration and temperature at time t, and by R the reaction rate per unit volume, T_c is the cooling temperature (which can be used as control input) and k_1, k_2, k_3, k_4, are constants. If T_c is given, this system has index one.

1.2.5 Analytical Solution for Differential-Algebraic Equations:

In this section we shall give the solution of the initial value problems for system of the linear (DAE) via Laplace Transform.

First recall that the Laplace transform of a function f specified at $t > 0$, denoted by $L\{f(t)\}$ (or F(s)), is defined by:

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{...(1.7)}$$

where s is a complex number. Laplace transform of f exists if the above integral converges for some values of s, [Murry, 1965]. Moreover, it is easy to check that:

$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$.

In this section, we use Laplace transform method to solve the initial value problem that consists of system of the first order linear ordinary differential equations with constant coefficients:

$$y'(t) = Ax(t) + By(t) + f(t),\ t > 0 \quad \text{(1.7a)}$$

together with the system of linear algebraic equations:

$$Cx(t) + Dy(t) + g(t) = 0,\ t \geq 0 \quad \text{(1.7b)}$$

and the initial condition

$$y(0) = \alpha \quad \text{(1.7c)}$$

where A, B, C and D are $n\times n$ constant matrices, f and g are $n\times 1$ vector functions, α is $n\times 1$ known constant vector and x, y are the $n\times 1$ vector functions that must be determined.

To do this we consider the following two cases:

Case (1):

If x is a differentiable $n\times 1$ vector function then by differentiating eq. (1.7b) with respect to t one can get:

$$Cx'(t) + Dy'(t) + g'(t) = 0 \quad \text{(1.8)}$$

By substituting eq. (1.7a) into eq. (1.8), one can have:

$$Cx'(t) + DAx(t) + DBy(t) + Df(t) + g'(t) = 0.$$

Now, if C is a nonsingular matrix then the above equation can be rewritten as:

$$x'(t) + C^{-1}DAx(t) + C^{-1}DBy(t) + C^{-1}Df(t) + C^{-1}g'(t) = 0 \quad \text{(1.9a)}$$

But

$$y'(t) = Ax(t) + By(t) + f(t) \quad \text{...(1.9b)}$$

In this case the original system of differential-algebraic given by eq's. (1.7a)-(1.7b) reduced to the system of the differential equations given by eq's. (1.9a)-(1.9b). Moreover, by substituting t = 0 in eq. (1.7b) one can get:

$$Cx(0) + Dy(0) + g(0) = 0$$

but, C^{-1} exist and

$$y(0) = \alpha \quad \text{..(1.9c)}$$

thus:

$$x(0) = C^{-1}[-g(x) - D\alpha] \quad \text{...(1.9d)}$$

Therefore the initial value problem given by eq's. (1.7) reduces to the initial value problem given by eq's. (1.9). This initial value problem can be solved by any suitable method, say Laplace transform method, and numerical methods namely Euler method.

On the other hand, if C is singular matrix, then by taking the Laplace transform of both sides of eq's. (1.8) and eq. (1.7a) one can get:

$$C[s\mathcal{L}\{x(t)\} - x(0)] + D[s\mathcal{L}\{y(t)\} - y(0)] + s\mathcal{L}\{g(t)\} - g(0) = 0$$

and

$$s\mathcal{L}\{y(t)\} - y(0) = A\mathcal{L}\{x(t)\} + B\mathcal{L}\{y(t)\} + \mathcal{L}\{f(t)\}$$

respectively.

But $y(0) = \alpha$ and $Cx(0) = -g(0) - Dy(0) = -g(0) - D\alpha$, therefore the above equations becomes:

$$sC\mathcal{L}\{x(t)\} + g(0) + D\alpha + sD\mathcal{L}\{y(t)\} - D\alpha + s\mathcal{L}\{g(t)\} - g(0) = 0$$

and

$$s\mathcal{L}\{y(t)\} - \alpha = A\mathcal{L}\{x(t)\} + B\mathcal{L}\{y(t)\} + \mathcal{L}\{f(t)\}.$$

Hence, the above two equations can be rewritten in the matrix form:

$$\begin{bmatrix} sC & sD \\ A & B - sI \end{bmatrix} \begin{bmatrix} \mathcal{L}\{x(t)\} \\ \mathcal{L}\{y(t)\} \end{bmatrix} = \begin{bmatrix} -s\mathcal{L}\{g(t)\} \\ -\alpha - \mathcal{L}\{f(t)\} \end{bmatrix}$$

where I is $n \times n$ identity matrix. If the above matrix is nonsingular for some values of s then

$$\begin{bmatrix} \mathcal{L}\{x(t)\} \\ \mathcal{L}\{y(t)\} \end{bmatrix} = \begin{bmatrix} sC & sD \\ A & B - sI \end{bmatrix}^{-1} \begin{bmatrix} -s\mathcal{L}\{g(t)\} \\ -\alpha - \mathcal{L}\{f(t)\} \end{bmatrix}$$

and by taking the inverse Laplace transform of $\mathcal{L}\{x(t)\}$ and $\mathcal{L}\{y(t)\}$ one can get x(t) and y(t) that satisfy eqs. (1.7).

Case 2:

If x is a nondifferentiable $n \times 1$ vector function, then the algebraic equation given by eq.(1.7a) can not be transformed into an ordinary differential equations. In this case if C is a nonsingular matrix then:

$$x(t) = -C^{-1}Dy(t) - C^{-1}g(t) \quad \text{.......................................(1.10)}$$

By substituting the above equation in eq. (1.7a), one can get:

$$y'(t) = (-AC^{-1}D + B)y(t) + f(t) - AC^{-1}g(t)$$

this equation can be solved together with the initial condition given by eq. (1.7c) by using Laplace transform method, to get:

$$(sI + AC^{-1}D - B)L\{y(t)\} = L\{f(t)\} - AC^{-1}L\{g(t)\} + \alpha$$

Therefore

$$L\{y(t)\} = (sI + AC^{-1}D - B)^{-1}[L\{f(t)\} - AC^{-1}L\{g(t)\} + \alpha]$$

provided that $sI + AC^{-1}D - B$ is a nonsingular matrix for some values of s. By taking the inverse Laplace transform of both sides of the above equation one can get y(t) which can be substituted in eq. (1.10) to get x(t). On the other hand, if C is a singular matrix then by taking the Laplace transform of both sides of eq. (1.7a) and eq. (1.7b), one can get:

$$AL\{x(t)\} + (B - sI)L\{y(t)\} + L\{f(t)\} + \alpha = 0$$

and

$$CL\{x(t)\} + DL\{y(t)\} + L\{g(t)\} = 0$$

The above system of algebraic equations may be rewritten in the matrix form:

$$\begin{bmatrix} A & B - sI \\ C & D \end{bmatrix} \begin{bmatrix} L\{x(t)\} \\ L\{y(t)\} \end{bmatrix} = \begin{bmatrix} -\alpha - L\{f(t)\} \\ -L\{g(t)\} \end{bmatrix}$$

which has the solution:

$$\begin{bmatrix} L\{x(t)\} \\ L\{y(t)\} \end{bmatrix} = \begin{bmatrix} A & B - sI \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} -\alpha - L\{f(t)\} \\ -L\{g(t)\} \end{bmatrix}$$

provided that $\begin{bmatrix} A & B - sI \\ C & D \end{bmatrix}$ is a nonsingular matrix for some values of s. Therefore by taking the Laplace transform of both sides of the above equations one can obtain x(t) and y(t).

To illustrate this method we consider the following example:

<u>Example (1.10), [Khaleel, 2008]:</u>

Consider the initial value problem that consists of the system of the first order linear differential equation with constant coefficients:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 12 & 7 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -11+7t^3 \\ t-2t^2-t^3 \end{bmatrix} \text{.... (1.11a)}$$

together with the system of the algebraic equations:

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -1-3t^3+4t^2 \\ -t-t^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{... (1.11b)}$$

and the initial conditions:

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we solve this example by using Laplace transform method. To do this, let $C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$, then $C^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$. Therefore (1.11b) becomes:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -17/6 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} -1+4t^2+3t \\ -t-t^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{......... (1.12)}$$

by substituting the above equations in eq. (1.11a) one can obtain:

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 29 & -48 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 1+48t^2-29t \\ 2t-2t^2 \end{bmatrix}$$

Then by taking the Laplace transform of the both system, one can have:

$$\begin{bmatrix} s-29 & 48 \\ 0 & s-2 \end{bmatrix}\begin{bmatrix} L\{y_1(t)\} \\ L\{y_2(t)\} \end{bmatrix}=\begin{bmatrix} \frac{1}{s}+\frac{96}{s^3}-\frac{29}{s^2} \\ \frac{2}{s^2}-\frac{4}{s^3} \end{bmatrix}$$

Therefore:

$$\begin{bmatrix} L\{y_1(t)\} \\ L\{y_2(t)\} \end{bmatrix}=\begin{bmatrix} s-29 & 48 \\ 0 & s-2 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{s}+\frac{96}{s^3}-\frac{29}{s^2} \\ \frac{2}{s^2}-\frac{4}{s^3} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{(s-29)}\left(\frac{1}{s}+\frac{96}{s^3}-\frac{29}{s^2}\right)-\frac{48}{(s-29)(s-2)}\left(\frac{2}{s^2}-\frac{4}{s^3}\right) \\ \frac{1}{(s-2)}\left(\frac{2}{s^2}-\frac{4}{s^3}\right) \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{s^2} \\ \frac{2}{s^3} \end{bmatrix}, \; s \neq 0,\, 2,\, 29.$$

Then by taking the Laplace transformation of the both system one can get:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} t \\ t^2 \end{bmatrix}$$

By substituting the $y_1(t)$ and $y_2(t)$ in eq. (1.12) one can have:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} 1-\frac{1}{6}t \\ t^3 \end{bmatrix}$$

1.3 BASIC CONCEPTS OF FRACTIONAL CALCULUS

In this section, we introduce some of the basic and fundamental concepts and definitions related to the subject of fractional calculus for completeness purpose

1.3.1 The Gamma and Beta Functions, [Oldham, 1974]:

Gamma and Beta fractions are two of the most important notations in fractional calculus, since they play an important role in fractional differentiation and integration.

First, the gamma function $\Gamma(x)$ of a positive real x, is defined by:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \ x > 0 \quad \text{...} (1.13)$$

Following are some of the most important properties of the gamma function:

1. $\Gamma(1) = 1.$
2. $\Gamma(x + 1) = x\Gamma(x).$
3. $\Gamma(x + 1) = x!.$
4. $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n n!\sqrt{\pi}}{(2n)!}.$
5. $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!\sqrt{\pi}}{4^n n!}.$
6. $\Gamma(-x) = \frac{-\pi\csc(\pi x)}{\Gamma(x+1)}.$
7. $\Gamma(nx) = \sqrt{\frac{2\pi}{n}} \left[\frac{n^x}{\sqrt{2\pi}}\right] \prod_{k=0}^{n-1} \left(n + \frac{k}{n}\right).$

The second function is the beta function with positive parameters p and q is defined by:

$$\beta(p,q)=\int_0^1 y^{p-1}(1-y)^{q-1}\,dy \quad \text{...}(1.14)$$

If either p or q is non-positive, the integral diverges.

The incomplete beta function can be defined in terms of the gamma function by the following relationship:

$$\beta(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \text{ for all p and q}$$

The beta function of argument t is defined by the integral:

$$\beta_t(p,q)=\int_0^t y^{p-1}(1-y)^{q-1}\,dy \quad \text{...}(1.15)$$

1.3.2 Fractional Integral:

There are many literatures introduces different definitions of fractional integrations, such as:

1. Riemann-Liouville integral, [Oldham, 1974]:

The generalization to non-integer q of Riemann-Liouville integral can be written for suitable function f(x) ($x \in R$) as:

$$\frac{d^q}{dx^q}f(x)=\frac{1}{\Gamma(-q)}\int_0^x (x-y)^{-q-1}f(y)\,dy,\ q<0 \quad \text{.......................}(1.16)$$

2. Wely fractional integral, [Oldham, 1974]:

The left hand fractional order integral of order q > 0 of a given function f is defined as:

$$ {}_{-\infty}D_x^q f(x) = \frac{1}{\Gamma(q)} \int_{-\infty}^{x} \frac{f(y)}{(x-y)^{1-q}} dy, \ x > -\infty \ \ldots\ldots\ldots\ldots(1.17) $$

And the right hand fractional order integral of order $q > 0$ of a given function f is defined as:

$$ {}_{\infty}D_x^q f(x) = \frac{1}{\Gamma(q)} \int_{x}^{\infty} \frac{f(y)}{(y-x)^{1-q}} dy, x < \infty $$

3. Abel-Riemann fractional integral, [Mittal, 2008]:

The Abel-Riemann (A-R) fractional integral of any order $\alpha > 0$ for a function x(t) with $t \in \mathbb{R}^+$ is defined as

$$ J^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau, \ t > 0 \ \alpha > 0 \ \ldots\ldots\ldots\ldots(1.18) $$

$J^0 = I$ (Identity operator).

The A-R integral posses the semigroup property:

$J^{\alpha}J^{\beta} = J^{\alpha+\beta}$, for all $\alpha, \beta \geq 0$ ....(1.19)

1.3.3 Fractional Derivatives:

Many literatures discussed and presented fractional derivatives of certain functions; therefore in this subsection some definitions of fractional derivatives are presented:

1. Riemann-Liouville formula of fractional differentiation, [Oldham, 1974], [Nishimoto, 1983]:

Among the most important formula used in fractional calculus is the Riemann-Liouville formula. For a given function f(x), $\forall\ x \in [a, b]$, the left and right hand Riemann-Liouville fractional derivatives of order $q > 0$ and m is a natural number, are given by:

$$ {}_{x}D_{a+}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{q-m+1}} dt \quad \text{.........................(1.20)} $$

$$ {}_{x}D_{b-}^{q} f(x) = \frac{(-1)^m}{\Gamma(m-q)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(x-t)^{q-m+1}} dt \quad \text{.........................(1.21)} $$

where $m - 1 < q \leq m$, $m \in \mathbb{N}$. These equations are usually named as the Riemann-Liouville fractional derivatives.

2. The A-R fractional derivative, [Mittal, 2008]:

The A-R fractional derivative (of order $\alpha > 0$) is defined as the left inverse of corresponding A-R fractional integral, i.e.,

$$ D^{\alpha} J^{\alpha} = I \quad \text{..(1.22)} $$

For positive integer m such that $m-1 < \alpha \leq m$,

$$ (D^m J^{m-\alpha}) J^{\alpha} = D^m (J^{m-\alpha} J^{\alpha}) = D^m J^m = I, \text{ i.e.,} $$

$$ D^{\alpha} x(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \dfrac{d^m}{dt^m} \displaystyle\int_0^t \dfrac{x(t)}{(t-\tau)^{\alpha+1-m}} d\tau \ , & m-1 < \alpha < m \\ \dfrac{d^m}{dt^m} x(t) & \alpha = m \end{cases} \quad \text{...(1.23)} $$

Properties of the operator J^{α} and D^{α} can be found in (Podulbny 1999), we mention the following

$$ J^{\alpha} t^{y} = \frac{\Gamma(y+1)}{\Gamma(y+1+\alpha)} t^{y+\alpha} $$

$$ D^{\alpha} t^{y} = \frac{\Gamma(y+1)}{\Gamma(y+1-\alpha)} t^{y-\alpha} $$

for $t > 0$, $\alpha \geq 0$, $y \geq -1$

3. Caputo fractional derivative, [Caputo, 1967], [Mirandi, 1997]:

In the late sixties an alternative definition of fractional derivatives was introduced by Caputo. Caputo and Mirandi used this definition in their work on the theory of viscoelasticity. According to Caputo's definition

$$D_*^{\alpha} = J^{m-\alpha} D^m \text{, for } m-1 < \alpha \le m$$

which means that:

$$D_*^{\alpha} x(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t \dfrac{x^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \text{ , } m-1 < \alpha < m \\ \dfrac{d^m}{dt^m} x(t), \qquad \alpha = m \end{cases}$$

The basic properties of the Caputo fractional derivative are:

1. Caputo introduced an alternative, definition, which has the advantage of defining integer order initial conditions for fractional order differential equations.

2. $J^{\alpha} D_*^{\alpha} x(t) = x(t) - \sum_{k=0}^{m-1} x^{(k)}(0^+) \dfrac{t^k}{k!}$.

3. Caputo's fractional differentiation is a linear operation, similar to integer order differentiation

$$D_*^{\alpha}[\lambda f(t) + M g(t)] = \lambda D_*^{\alpha} f(t) + M D_*^{\alpha} g(t) .$$

where λ and M are constants.

4. Gruünwald fractional derivatives, [Oldham, 1974]:

The Gruünwald derivatives of any integer order to any fraction order derivative which takes the form:

$$\frac{d^q}{dx^q} f(x) = \lim_{N\to\infty} \left\{ \frac{\left(\frac{x}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x}{N}\right)\right) \right\} \text{............}(1.24)$$

1.3.4 Analytic Methods for Solving Fractional Order Differential Equations, [Oldham, 1974]:

In this present subsection, some analytical methods are proposed for solving fractional order differential equations, and among such methods:

1.3.4.1 The inverse operator method:

Let f be an unknown function and let q be an arbitrary real number, F is known function, then we can construct the simplest of all fractional order differential equations by:

$$\frac{d^q f}{dx^q} = F \text{ ....}(1.25)$$

hence upon taking the inverse operator $\frac{d^{-q}}{dx^{-q}}$, gives:

$$f = \frac{d^{-q} F}{dx^{-q}}$$

where it is clear that it is not always the case that they are equal, but this is not the most general solution, [Oldham, 1974]:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q}f = 0 \quad \text{...(1.26)}$$

additional terms must be added to equation (1.26), which are:

c_1x^{q-1}, c_2x^{q-2}, …, c_mx^{q-m}

and hence:

$$f - \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q}f = c_1x^{q-1} + c_2x^{q-2} + ... + c_mx^{q-m}$$

where c_1, c_2, …, c_m are an arbitrary constants to be determined from the initial conditions and $m-1 < q \le m$. Thus:

$$f - c_1x^{q-1} - c_2x^{q-2} - ... - c_mx^{q-m} = \frac{d^{-q}}{dx^{-q}}\frac{d^q}{dx^q}f$$

$$= \frac{d^{-q}}{dx^{-q}}F$$

Hence, the most general solution of eq. (1.25) is given by:

$$f = \frac{d^{-q}}{dx^{-q}}F = c_1x^{q-1} + c_2x^{q-2} + ... + c_mx^{q-m}$$

where $m-1 < q \le m$.

As an illustration, we consider the following example:

Example (1.11):

Consider the fractional order differential equation:

$$\frac{d^{3/2}f(x)}{d^{3/2}} = x^5 \quad \text{...(1.27)}$$

with initial condition:

$$\frac{d^{1/2} f(0)}{dx^{1/2}} = k_0 \, , \; \frac{d^{-1/2} f(0)}{dx^{-1/2}} = k_1$$

Applying $\frac{d^{-3/2}}{dx^{-3/2}}$ to the both sides of eq. (1.27), we get:

$$f(x) = \frac{d^{-3/2} x^5}{dx^{-3/2}} + c_1 x^{1/2} + c_2 x^{-1/2}$$

and from the initial condition, we have $c_1 = \frac{k_0}{\Gamma(3/2)}, c_2 = \frac{k_1}{\Gamma(1/2)}$

therefore:

$$f(x) = \frac{\Gamma(6)}{\Gamma(15/2)} x^{13/2} + \frac{k_0 x^{1/2}}{\Gamma(3/2)} + \frac{k_1 x^{-1/2}}{\Gamma(1/2)}$$

1.3.4.2 Laplace transform method:

In this subsection, we seek a Laplace transform of $d^q f / dx^q$ for all q and differintegrable function f, i.e., we wish to relate:

$$L\left\{\frac{d^q f}{dx^q}\right\} = \int_0^\infty \exp(-sx) \, \frac{d^q f}{dx^q} dx$$

to the Laplace transform $L\{f\}$ of the differintegrable function. Let us first recall the well-known transforms on integer-order derivatives:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{q-1} s^{q-1-k} \frac{d^k f(0)}{dx^k} \;\;, \;\; q = 1, 2, \ldots$$

and multiple integrals:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q \, L\{f\} \, , q = 0, -1, \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.28)$$

and note that both formulas are embraced by:

$$L\left\{\frac{d^q f}{dx^q}\right\}=s^q L\{f\}-\sum_{k=0}^{q-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}} \ , \ q=0,\mp 1,\ldots \ \ldots\ldots\ldots\ldots(1.29)$$

Also, formula (1.29), can be generalized to include non integer $q \in \mathbb{R}$, as:

$$L\left\{\frac{d^q f}{dx^q}\right\}=s^q L\{f\}-\sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}} \ , \ \text{for al } q \ \ldots\ldots\ldots\ldots\ldots\ldots(1.30)$$

where n is integer such that $n-1 < q \leq n$. The sum vanishes when $q \leq 0$. In proving (1.30), we first consider $q < 0$, so that the Riemann-Liouville definition:

$$\frac{d^q f}{dx^q}=\frac{1}{\Gamma(-q)}\int_0^x \frac{f(y)}{[x-y]^{q+1}}dy \ , q<0$$

may be adopted and upon direct application of the convolution theorem [Churchill, 1948]:

$$L\left\{\int_0^x f_1(x-y) f_2(y)\, dy\right\}=L\{f_1\} L\{f_2\}$$

Then gives:

$$L\left\{\frac{d^q f}{dx^q}\right\}=\frac{1}{\Gamma(-q)} L\{x^{-1-q}\} L\{f\}=s^q L\{f\}, \ q<0 \ \ldots\ldots\ldots\ldots\ldots\ldots(1.31)$$

For positive non integer q, we use the result, [Oldham, 1974]:

$$\left[\frac{d^q f}{dx^q}\right]=\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right] \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.32)$$

$$\frac{d^q f}{dx^q} = \frac{d^n}{dx^n}\frac{d^{q-n} f}{dx^{q-n}}$$

where n is an integer number such that $n-1 < q \le n$. Now, on application of the formula (1.29), we find that:

$$L\left\{\frac{d^q f}{dx^q}\right\} = L\left\{\frac{d^n}{dx^n}\left[\frac{d^{q-n} f}{dx^{q-n}}\right]\right\}$$

$$= s^n L\left\{\frac{d^{q-n} f}{dx^{q-n}}\right\} - \sum_{k=0}^{n-1} s^k \frac{d^{n-1-k}}{dx^{n-1-k}}\left[\frac{d^{q-n} f(0)}{dx^{q-n}}\right]$$

The difference q–n being negative, the first right-hand term may be applied to the terms within the summation. The result:

$$L\left\{\frac{d^q f}{dx^q}\right\} = s^q L\{f\} - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f(0)}{dx^{q-1-k}},\ 0 < q \ne 1, 2, \ldots$$

Follows from these two operations and is seen to be incorporated (1.30). The transformation (1.30) is a very simple generalization of the classical formula for the Laplace transform of the derivative or integral of f. No similar generalization exists, however, for the classical formulas, [Oldham, 1974]:

$$L\left\{\frac{-f}{x}\right\} = \frac{d^{-1} L\{f\}}{ds^{-1}}(s) - \frac{d^{-1} L\{f\}}{ds^{-1}}(\infty)$$

$$L\{-xf\} = \frac{dL\{f\}}{ds}$$

$$L\left\{[-x]^n f\right\} = \frac{d^n L\{f\}}{ds^n},\ n = 1, 2, \ldots \quad \text{......................................(1.33)}$$

As a final result of this section we shall establish the useful formula:

$$L\left\{\exp(-kx)\frac{d^q}{dx^q}[fe^{kx}]\right\}=[s+k]^q L\{f\},\ q\geq 0 \ ....(1.34)$$

in which equation (1.31), may be regarded as a special case, when k = 0 in equation (1.34).

As an illustration, we consider the following example:

Example (1.12), [Abdulkhalik, 2008]:

Consider the semi differential equation:

$$\frac{d^{1/2}f(x)}{dx^{1/2}}+\frac{d^{-1/2}f(x)}{dx^{-1/2}}+2f(x)=\frac{2}{\sqrt{\pi x}}+6\sqrt{\frac{x}{\pi}}+\frac{4x^{3/2}}{3\sqrt{\pi}}+2x+4 \ ...(1.35)$$

and in order to solve this equation using Laplace transformation method, first we take the Laplace transformation to the both sides of equations (1.35):

$$L\left\{\frac{d^{1/2}f(x)}{dx^{1/2}}\right\}+L\left\{\frac{d^{-1/2}f(x)}{dx^{-1/2}}\right\}+2L\{f(x)\}=\frac{2}{\sqrt{\pi}}L\left\{\frac{1}{\sqrt{x}}\right\}+$$

$$\frac{6}{\sqrt{\pi}}L\left\{\sqrt{x}\right\}+\frac{4}{3\sqrt{\pi}}L\left\{x^{3/2}\right\}+2L\{x\}+L\{4\}$$

use eq. (1.31), leads to:

$$L(f)=\frac{2s^2+3s+1+2\sqrt{s}+4s\sqrt{s}}{s^2\left(s+1+2\sqrt{s}\right)}$$

$$=\frac{(2s+1)+\left(s+1+2\sqrt{s}\right)}{s^2\left(s+1+2\sqrt{s}\right)}$$

$$=\frac{2}{s}+\frac{1}{s^2}$$

Then upon using the inverse Laplace transform, we have:

$$f(x) = 2 + x$$

as the solution of the fractional order differential equation.

The work of this thesis is concerned with the following type of the problem.

Let us consider the following system of fractional differential-algebraic equation with variable coefficients:

$$A(t)[D_*^q x(t)] + B(t)\, x(t) = f(t),\ 0 < q \le 1$$

with initial value:

$$x(t_0) = x_0$$

where D_*^q represent the Caputo fractional derivative of order q and with A(t) is $n \times n$ singular matrix, B(t) is $n \times n$ nonsingular matrix and f is $n \times 1$ vector function, x_0 is $n \times 1$ known constant vector and x(t) is the $n \times 1$ vector function that must be determined.

The above problem will be treated in chapter two and three respectively.

CHAPTER TWO

DIFFERENTIAL TRANSFORM METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

2.1 INTRODUCTION

In this chapter, we describe the application of fractional differential transform method (DTM) for solving Fractional Differential-Algebraic Equations. This chapter consists of five sections in section (2.2) the literature review of the DTM is given. In section (2.3) we introduce the analysis of the DTM and its related theorems. In section (2.4) the fractional DTM is presented. Finally, the implementation of the fractional DTM to the fractional DAEs is given in section (2.5) in addition to the approximate results of an illustrative examples.

2.2 LITERATURE REVIEW

The (DTM) is a numerical method for solving differential equations. The concept of the DTM was first proposed by [Zhou, 1986]. Using one-dimensional differential transform, [Ho and Chen, 1998] proposed a method to solve eigenvalue problems. The method has been applied to the partial differential equations [Chen and Ho, 1999], [Jang and et al., 2001].

Hassan applied the DTM to solve eigenvalues and normalized eigenfunctions for a Sturm-Liouville eigenvalue problem [Hassan, 2002]. Application of two dimensional DTM was studied by [Ayaz, 2003] for the solution of the partial differential equations.

Chen and Ju are used the DTM to predict the advective-dispersive transport problems [Chen and Ju, 2004]. The method is also used for the solution of DAEs of index-1 by [Ayaz, 2004]. Liu and Song [Liu and Song, 2007] analyzed higher index DAEs using this technique where he showed that the method is effective in case of index-2 DAEs but not suitable for DAEs of index-3. Fourth order boundary value problem was studied by [Erturk and Momani, 2007].

Comparison of this method with ADM was done by [Hassan, 2008] to solve PDEs. The solution of systems of fractional differential equations using DTM was studies by [Erturk and Momani, 2008]. DTM for solving Volterra integral equation with separable kernels is given by [Odibat, 2008].

Tari,Rahimi,Shahmorad and Talati [Tari and et al., 2009] solve a class of 2-dimensional linear and nonlinear Voltera integral equations by the DTM, [Siraj-Ul Islam and et al., 2009] study the numerical solution of special 12th-order boundary value problems using DTM, [Arikoglu and Ozkol, 2009] solve the fractional integro-differential equations by using fractional DTM, [Nazari and Shahmorad, 2010] applied the fractional DTM to fractional order-integro-differential equations with non local boundary conditions.

2.3 ANALYSIS OF THE DIFFERENTIAL TRANSFORM METHOD[Odibat, 2008]:

The differential transform of the k^{th} derivative of a function f(x) is defined as follows:

$$F(k)=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \quad \text{...}(2.1)$$

where f(x) is the original function and F(k) is the transformed function. The inverse differential transform of F(k) is defined as:

$$f(x)=\sum_{k=0}^{\infty} F(k)(x-x_0)^k \quad \text{...}(2.2)$$

From eq. (2.1) and (2.2), we get

$$f(x)=\sum_{k=0}^{\infty}\frac{(x-x_0)^k}{k!}\left.\frac{d^k f(x)}{dx^k}\right|_{x=x_0} \quad \text{...}(2.3)$$

which implies that the concept of differential transform is derived from Taylor series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivatives are calculated by an iterative way which are described by the transformed equations of the original function. In real applications, the function f(x) is expressed by a finite series and eq.(2.2) can be written as:

$$f(x)=\sum_{k=0}^{n} F(k)(x-x_0)^k \quad \text{...}(2.4)$$

Here n decided by the convergence of series coefficients.

The fundamental operations preformed by differential transform can readily be obtained are listed in Table (1).

Table (1)

Operation of differential transformation.

Original function	***Transformed function***
$f(x) = u(x)$ m $v(x)$	$F(k) = U(k)$ m $V(k)$
$f(x) = \alpha\, u(x)$	$F(k) = \alpha U(k)$
$f(x) = \dfrac{du(x)}{dx}$	$F(k) = (k + 1)U(k + 1)$
$f(x) = \dfrac{d^m u(x)}{dx^m}, m \in$ ¢$^+$	$F(k) = (k+1)(k+2)\ldots(k+m)\, U(k+m)$
$f(x) = \int_{x_0}^{x} u(t)dt$	$F(k) = \dfrac{U(k+1)}{k}, k \ge 1,\ F(0) = 0$
$f(x) = x^m \quad m \in$ ¢$^+$	$F(k) = \delta(k-m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$
$f(x) = \exp(\lambda x)$	$F(k) = \dfrac{\lambda^k}{k!}$
$f(x) = \sin(wx + \alpha)\ \alpha \in$ ¢$^+$	$F(k) = \dfrac{w^k}{k!}\sin\left(\pi k/2 + \alpha\right)$
$f(x) = \cos(wx + \alpha)\ \alpha \in$ ¢$^+$	$F(k) = \dfrac{w^k}{k!}\cos\left(\pi k/2 + \alpha\right)$

2.4 FRACTIONAL DIFFERENTIAL TRANSFORM METHOD [ARIKOGLU AND OZKOL, 2009]

DTM has been developed as follows:

The fractional differentiation in Riemann-Liouville sense was defined by

$$D_{x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m}\left[\int_{x_0}^{x} \frac{f(t)}{(x-t)^{1+q-m}}\,dt\right] \text{.........................} (2.5)$$

for $m-1 < q \le m$, $m \in ¢^{+}$, $x > x_0$. Let us expand the analytical and continuous function f(x) in terms of fractional power series as follows:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\alpha} \text{..} (2.6)$$

where α is the order of fraction and F(k) is the fractional differential transform of f(x).

In order to avoid fractional initial and boundary conditions we define the fractional derivatives in the Caputo sense. The relationship between the Riemann-Liouville operator and Caputo operator is given by:

$$D_{*x_0}^{q} f(x) = D_{x_0}^{q}\left[f(x) - \sum_{k=0}^{m-1} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)\right] \text{..................} (2.7)$$

And using eq.(2.5), we get:

$$D_{*x_0}^{q} f(x) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m}\left\{\int_{x_0}^{x}\left[\frac{f(t) - \sum_{k=0}^{m-1}(1/k)!\,(t-x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}}\right]dt\right.$$

$$\text{................................} (2.8)$$

Since the initial conditions are implemented to the integer order derivatives, the transformations of the initial conditions are defined as follows:

$$F(k)=\begin{cases} k/\alpha \in \mathbb{Z}^+, \ \dfrac{1}{(k/\alpha)!}\left[\dfrac{d^{k/\alpha}f(x)}{dx^{k/\alpha}}\right]_{x=x_0} & \text{, for } k=0,1,\dots,q\alpha-1 \\ k/\alpha \notin \mathbb{Z}^+, \ 0 & \end{cases}$$

................................(2.9)

where q is the order of fractional derivative. The following theorems that can be deduced from eq.(2.5) and (2.6) are given below:

Theorem (2.1):

If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$.

Theorem (2.2):

If $f(x) = g(x)\, h(x)$, then $F(k) = \sum_{r=0}^{k} G(r)H(k-r)$

Theorem (2.3):

If $f(x) = g_1(x)\, g_2(x) \dots g_{n-1}(x)\, g_n(x)$, then:

$$F(k)=\sum_{k_{n-1}=0}^{k}\ \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3}\ \sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\dots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$$

Theorem (2.4):

If $f(x) = (x - x_0)^p$, then $F(k) = \delta(k - \alpha p)$, where:

$$\delta(k) = \begin{cases} 1 & \text{if } \; k = 0 \\ 0 & \text{if } \; k \neq 0 \end{cases}$$

Theorem (2.5):

If $f(x) = D_{x_0}^{q}\left[g(x)\right]$, then:

$$F(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k+\alpha q)$$

Proof:

Utilizing eqs.(2.6), (2.8) and (2.9), we get:

$$D_{x_0}^{q}[g(x)] = \frac{1}{\Gamma(m-q)} \frac{d^m}{dx^m}\left\{ \int_{x_0}^{x} \left[\frac{\sum_{k=0}^{\infty} G(k)(t-x_0)^{k/\alpha} - \sum_{k=0}^{q\alpha-1} G(k)(t-x_0)^{k/\alpha}}{(x-t)^{1+q-m}} \right] dt \right\}$$

$$= \frac{1}{\Gamma(m-q)} \sum_{k=\alpha q}^{\infty} G(k) \frac{d^m}{dx^m}\left[\int_{x_0}^{x} \frac{(t-x_0)^{k/\alpha}}{(x-t)^{1+q-m}} dt \right]$$

$$= \sum_{k=\alpha q}^{\infty} \frac{\Gamma(1+k/\alpha)}{\Gamma(1-q+k/\alpha)} G(k)(x-x_0)^{k/\alpha - q}$$

Starting the index of this series from k = 0, we have:

$$f(x) = \sum_{k=0}^{\infty} \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k+\alpha q)(x-x_0)^{k/\alpha}$$

From eq.(2.6) the following expression is obtained:

$$F(k) = \frac{\Gamma(q+1+k/\alpha)}{\Gamma(1+k/\alpha)} G(k+\alpha q). <$$

<u>Theorem (2.6):</u>

For the production of fractional derivatives in the most general form

$$f(x)=\frac{d^{q_1}}{dx^{q_1}}[g_1(x)]\,\frac{d^{q_2}}{dx^{q_2}}[g_2(x)]\cdots\frac{d^{q_{n-1}}}{dx^{q_{n-1}}}[g_{n-1}(x)]\frac{d^{q_n}}{dx^{q_n}}[g_n(x)]$$

Then:

$$\begin{aligned}F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}&\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}\\ &\frac{\Gamma(q_2+1+(k_2-k_1)/\alpha)}{\Gamma(1+(k_2-k_1)/\alpha)}\cdots\frac{\Gamma(q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha)}{\Gamma(1+(k_{n-1}-k_{n-2})/\alpha)}\\ &\frac{\Gamma(q_n+1+(k-k_{n-1})/\alpha)}{\Gamma(1+(k-k_{n-1})/\alpha)}G_1(k_1+\alpha q_1)\times G_2(k_2-k_1+\alpha q_2)\cdots\\ &G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})\times G_n(k-k_{n-1}+\alpha q_n)\end{aligned}$$

Where $\alpha q_i \in \mathbb{R}^{+}$, for i = 1, 2, …, n.

Proof:

Let the differential transform of $\frac{d^{q_i}}{dx^{q_i}}[g_i(x)]$ be $C_i(k)$ at $x = x_0$ for i = 1, 2, …, n.

Then by using Theorem (2.3), we have the fractional differential transform of f(x) as

$$\begin{aligned}F(k)=\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}&C_1(k_1)C_2(k_2-k_1)\cdots\\ &C_{n-1}(k_{n-1}-k_{n-2})C_n(k-k_{n-1})\end{aligned}$$

and using Theorem (2.5) one can deduce that

$$C_1(k_1)=\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}G_1(k_1+\alpha q_1)$$

$$C_2(k_2-k_1)=\frac{\Gamma\left[q_2+1+(k_2-k_1)/\alpha\right]}{\Gamma\left[1+(k_2-k_1)/\alpha\right]}G_2(k_2-k_1+\alpha q_2),\ldots,$$

$$C_{n-1}(k_{n-1}-k_{n-2})=\frac{\Gamma\left[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha\right]}{\Gamma\left[1+(k_{n-1}-k_{n-2})/\alpha\right]}G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})$$

$$C_n(k-k_{n-1})=\frac{\Gamma\left[q_n+1+(k-k_{n-1})/\alpha\right]}{\Gamma\left[1+(k-k_{n-1})/\alpha\right]}G_n(k-k_{n-1}+\alpha q_n)$$

By utilizing these values, we have:

$$\begin{aligned}F(k)=&\sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}\frac{\Gamma(q_1+1+k_1/\alpha)}{\Gamma(1+k_1/\alpha)}\\
&\frac{\Gamma\left[q_2+1+(k_2-k_1)/\alpha\right]}{\Gamma\left[1+(k_2-k_1)/\alpha\right]}\cdots\frac{\Gamma\left[q_{n-1}+1+(k_{n-1}-k_{n-2})/\alpha\right]}{\Gamma\left[1+(k_{n-1}-k_{n-2})/\alpha\right]}\\
&\frac{\Gamma\left[q_n+1+(k-k_{n-1})/\alpha\right]}{\Gamma\left[1+(k-k_{n-1})/\alpha\right]}G_1(k_1+\alpha q_1)\,G_2(k_2-k_1+\alpha q_2)\ldots\\
&G_{n-1}(k_{n-1}-k_{n-2}+\alpha q_{n-1})\;G_n(k-k_{n-1}+\alpha q_n)\end{aligned}$$

Where $\alpha q_i \in$ ¢ $^+$, for $i = 1, 2, \ldots, n$. <

<u>Theorem (2.7):</u>

If $f(x)=\int_{x_0}^{x} g(t)\,dt$, then $F(k)=\alpha\frac{G(k-\alpha)}{k}$, where $k \ge \alpha$.

Proof:

Using the fractional power series expansion, we have:

$$f(x)=\int_{x_0}^{x}\sum_{k=0}^{\infty}G(k)(t-x_0)^{k/\alpha}dt$$

$$= \sum_{k=0}^{\infty} \int_{x_0}^{x} G(k)(t-x_0)^{k/\alpha} dt$$

$$= \sum_{k=0}^{\infty} \left[\alpha \frac{G(k)}{k+\alpha} (t-x_0)^{k/\alpha+1} \Big|_{x_0}^{x} \right]$$

$$= \sum_{k=0}^{\infty} \alpha \frac{G(k)}{k+\alpha} (x-x_0)^{k/\alpha+1}$$

Starting the index of series from $k = \alpha$

$$f(x) = \sum_{k=\alpha}^{\infty} \alpha \frac{G(k-\alpha)}{k} (x-x_0)^{k/\alpha}$$

From the definition of transform in

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\alpha}$$

we get:

$$F(k) = \alpha \frac{G(k-\alpha)}{k}, \text{ where } k \geq \alpha. \quad <$$

<u>Theorem (2.8):</u>

If $f(x) = g(x) \int_{x_0}^{x} h(t)dt$, then:

$$F(k) = \alpha \sum_{k_1=\alpha}^{k} \frac{H(k_1-\alpha)}{k_1} G(k-k_1), \text{ where } k \geq \alpha.$$

Proof:

Let the differential transform of $\int_{x_0}^{x} h(t)dt$ be $G(k)$

Then, by using Theorem (2.2), we get:

$$F(k)=\sum_{k_1=0}^{k} C(k_1)G(k-k_1)$$

where Theorem (2.7) states that

$$C(k)=\alpha\frac{H(k-\alpha)}{k},\ k\geq\alpha$$

Which implies to:

$$C(k_1)=\alpha\frac{H(k_1-\alpha)}{k_1},\ k_1\geq\alpha$$

Using this value, we evaluate:

$$F(k)=\alpha\sum_{k_1=\alpha}^{k}\frac{H(k_1-\alpha)}{k_1}G(k-k_1)\,. \qquad <$$

<u>Theorem (2.9):</u>

$$\text{If } f(x)=\int_{x_0}^{x} h_1(t)h_2(t)\ldots h_{n-1}(t)h_n(t)dt,\ \text{then:}$$

$$F(k)=\frac{\alpha}{k}\sum_{k_{n-1}=0}^{k-\alpha}\ \sum_{k_{n-2}=0}^{k_{n-1}}\ \mathbf{L}\sum_{k_2=0}^{k_3}\ \sum_{k_1=0}^{k_2} H_1(k_1)H_2(k_2-k_1)\,\mathbf{L}$$
$$H_{n-1}(k_{n-1}-k_{n-2})H_n(k-k_{n-1}-\alpha)$$

Proof:

Let $C(t) = h_1(t)\ h_2(t)\ \ldots\ h_{n-1}(t)\ h_n(t)$ and $C(k)$ be the transform of $C(t)$. then by using Theorem (2.7), we have:

$$F(k)=\alpha\frac{C(k-\alpha)}{k},\ k\geq\alpha$$

And using Theorem (2.3), one can deduce

Utilizing these values we get

$$F(k)=\frac{\alpha}{k}\sum_{k_{n-1}=0}^{k-\alpha}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}H_1(k_1)H_2(k_2-k_1)\cdots$$
$$H_{n-1}(k_{n-1}-k_{n-2})H_n(k-k_{n-1}-\alpha),\ k\geq\alpha. \quad <$$

Theorem (2.10):

If $f(x)=[g_1(x)g_2(x)\ldots g_{m-1}(x)g_m(x)]\int_{x_0}^{x}h_1(t)h_2(t)\ldots h_{n-1}(t)h_n(t)dt$

$$F(k)=\alpha\sum_{k_1=\alpha}^{k}\frac{1}{k_1}\sum_{j_{n-1}=0}^{k_1-\alpha}\sum_{j_{n-2}=0}^{j_{n-1}}\cdots\sum_{j_2=0}^{j_3}\sum_{j_1=0}^{j_2}\sum_{i_{m-1}=0}^{k-k_1}\sum_{i_{m-2}=0}^{i_{m-1}}\cdots$$
$$\sum_{i_2=0}^{i_3}\sum_{i_1=0}^{i_2}G_1(i_1)G_2(i_2-i_1)\cdots G_{m-1}(i_{m-1}-i_{m-2})$$
$$G_m(k-i_{m-1}-k_1)\,H_1(j_1)H_2(j_2-j_1)\cdots$$
$$H_{n-1}(j_{n-1}-j_{n-2})H_n(k_1-j_{n-1}-\alpha)$$

where $k\geq\alpha$. Let:

$$C_1(x)=g_1(x)g_2(x)\ldots g_{m-1}(x)g_m(x)$$

and

$$C_2(t)=h_1(t)h_2(t)\ldots h_{n-1}(t)h_n(t)$$

Then by using Theorem (2.8), we get:

$$F(k)=\alpha\sum_{k_1=\alpha}^{k}\frac{C_2(k_1-\alpha)}{k_1}C_1(k-k_1),\ \text{where } k\geq\alpha$$

and Theorem (2.3) states that:

$$C_1(k) = \sum_{i_{m-1}=0}^{k} \sum_{i_{m-2}=0}^{i_{m-1}} \ldots \sum_{i_2=0}^{i_3} \sum_{i_1=0}^{i_2} G_1(i_1)G_2(i_2-i_1)\ldots$$

$$G_{m-1}(i_{m-1}-i_{m-2})G_m(k-i_{m-1})$$

$$C_2(k) = \sum_{j_{n-1}=0}^{k} \sum_{j_{n-2}=0}^{j_{n-1}} \ldots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} H_1(j_1)H_2(j_2-j_1)\ldots$$

$$H_{n-1}(j_{n-1}-j_{n-2})H_n(k-j_{n-1})$$

Utilizing these values, we get:

$$G_1(i_1)G_2(i_2-i_1)\ \ldots\ G_{m-1}(i_{m-1}-i_{m-2})\ G_m(k-i_{m-1}-k_1)$$

$$H_1(j_1)H_2(j_2-j_1)\ \ldots\ H_{n-1}(j_{n-1}-j_{n-2})\ H_n(k_1-j_{n-1}-\alpha)$$

where $k \geq \alpha$.

2.5 APPLICATION OF THE FRACTIONAL DIFFERENTIAL TRANSFORM METHOD FOR SOLVING FRACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

In this section, we employ the fractional DTM to solve the fractional DAEs and for this purpose let us consider the following system of fractional DAE with variable coefficients:

$$A(t)[D_*^q x(t)] + B(t)\, x(t) = f(t),\ 0 < q \leq 1 \qquad (2.10)$$

with initial value:

$$x(t_0) = x_0 \qquad (2.11)$$

where D_*^q represent the Caputo fractional derivative of order q and with A(t) is $n\times n$ singular matrix, B(t) is $n\times n$ nonsingular matrix and f is $n\times 1$ vector function, x_0 is $n\times 1$ known constant vector and x(t) is the $n\times 1$ vector function that must be determined.

The method of solution is begin by rewriting equation (2.10) as a system of equations and then by using the basic properties of the fractional DTM given in section (2.4), one can get an explicit iterative form for the vector X(k). Finally, the vector function x(t) will be expressed by a finite series which can be written as

$$x(t)=\sum_{k=0}^{m} X(k)(t-t_0)^{k/\alpha} \quad \text{...}(2.12)$$

which represent the approximate solution of eq. (2.10), (2.11) and to demonstrate this steps of solution let us consider the following examples:

<u>Example (2.1):</u>

Consider the following fractional differential-algebraic equation:

$$\begin{pmatrix} 1 & -x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} D_*^q v_1(x) \\ D_*^q v_2(x) \end{pmatrix}+\begin{pmatrix} 1 & -(1+x) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}=\begin{pmatrix} 0 \\ \sin x \end{pmatrix} \quad \text{........}(2.13)$$

With the following initial conditions

$$\begin{pmatrix} v_1(0) \\ v_2(0) \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{...}(2.14)$$

The exact solution of the above example as given in [Celik and et al., 2006] for is q=1 is:

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} e^{-x} + x\sin x \\ \sin x \end{pmatrix}$$

Equivalently eq. (2.13) can be written as

$$D_*^q v_1(x) - x[D_*^q v_2(x)] + v_1(x) - v_2(x) - xv_2(x) = 0 \quad \text{...............(2.15)}$$

$$v_2(x) = \sin x \quad \text{...(2.16)}$$

or

$$D_*^q v_1(x) = x[D_*^q v_2(x)] - v_1(x) + v_2(x) + xv_2(x)$$

$$v_2(x) = \sin x$$

By using the basic properties of fractional differential transform method Theorems (2.1), (2.2), (2.4), (2.5) and applying the transformation on (2.15) and (2.16) one can obtain that

$$\begin{aligned} V_1(k+\alpha q) = \frac{\Gamma(1+k/\alpha)}{\Gamma(q+1+k/\alpha)} \Bigg\{ & \sum_{r=0}^{k} \frac{\Gamma(q+1+(k-r)/\alpha)}{\Gamma(1+(k-r)/\alpha)} \delta(r-\alpha) \\ & V_2(k-r+\alpha q) - V_1(k) + V_2(k) + \sum_{r=0}^{k} \delta(r-\alpha) V_2(k-r) \Bigg\} \end{aligned}$$

$$\text{...............................(2.17)}$$

$$V_2(k) = \begin{cases} 0 & , \text{if } k/\alpha \notin \text{¢}^{+} \\ \dfrac{\sin\left(\dfrac{k\pi}{2\alpha}\right)}{(k/\alpha)!} & , \text{if } k/\alpha \in \text{¢}^{+} \end{cases} \quad \text{..(2.18)}$$

Case 1:

From eq. (2.17), (2.18) and for k = 0, 1, … ; the differential transforms $V_1(k)$ and $V_2(k)$ can be obtained for $q = \alpha = 1$ as:

$$V_1(k+1) = \frac{\Gamma(1+k)}{\Gamma(2+k)} \left\{ \sum_{r=0}^{k} \frac{\Gamma(2+(k-r))}{\Gamma(1+k-r)} \delta(r-1)\, V_2(k-r+1) - \right.$$
$$\left. V_1(k) + V_2(k) + \sum_{r=0}^{k} \delta(r-1)\, V_2(k-r) \right\}$$

$$V_2(k) = \frac{\sin\left(\frac{k\pi}{2}\right)}{k!}, \quad k \in \mathbb{Z}^{+}$$

And hence

$$V_1(1) = -1,\ V_1(2) = \frac{3}{2!},\ V_1(3) = \frac{-1}{3!},\ V_1(4) = \frac{-3}{4!},\ V_1(5) = \frac{-1}{5!},$$

$$V_1(6) = \frac{7}{6!},\ V_1(7) = \frac{-1}{7!},\ V_1(8) = \frac{-7}{8!},\ V_1(9) = \frac{-1}{9!},\ V_1(10) = \frac{11}{10!},$$

$$V_1(11) = \frac{-1}{11!},\ V_1(12) = \frac{-11}{12!},\ V_1(13) = \frac{-1}{13!},\ V_1(14) = \frac{15}{14!},$$

$$V_1(15) = \frac{-1}{15!},\qquad V_1(16) = \frac{-15}{16!},\ V_1(17) = \frac{-1}{17!},\ V_1(18) = \frac{19}{18!},$$

$$V_1(19) = \frac{-1}{19!},\ V_1(20) = \frac{-19}{20!},\ \ldots$$

and in general $V_1(k)$ may be formulated as follows:

$$V_1(k) = \begin{cases} \dfrac{-1}{k!}, & k \text{ is odd} \\[2ex] \dfrac{k+1}{k!}, & k \text{ is even and } \dfrac{k+2}{2} \text{ is even} \\[2ex] \dfrac{-(k-1)}{k!}, & k \text{ is even and } \dfrac{k+2}{2} \text{ is odd} \end{cases}$$

Consequently, by substituting the values of $V_1(k)$ into eq. (2.12) up to 20 terms it yields to

$$v_1(x) = 1 - x + \frac{3}{2}x^2 - \frac{1}{3!}x^3 - \frac{3}{4!}x^4 - \frac{1}{5!}x^5 + \frac{7}{6!}x^6 - \frac{1}{7!}x^7 - \frac{7}{8!}x^8 - \frac{1}{9!}x^9 + \frac{11}{10!}x^{10}$$
$$\frac{-1}{11!}x^{11} - \frac{11}{12!}x^{12} - \frac{1}{13!}x^{13} + \frac{15}{14!}x^{14} - \frac{1}{15!}x^{15} - \frac{15}{16!}x^{16} - \frac{1}{17!}x^{17} + \frac{19}{18!}x^{18} - \frac{1}{19!}x^{19} - \frac{19}{20!}x^{20}$$

..............................(2.19)

Following Table (2), represent a comparison of the approximate solution of $v_1(x)$ with the exact solution

Table (2)

Comparison of the approximate solution of $v_1(x)$ with the exact solution.

x	**Exact solution of $v_1(x)$**	**Approximate solution of $v_1(x)$**	**Absolute error**
0.1	0.91482076	0.91482076	0.0
0.2	0.858464619	0.858464619	0.0
0.3	0.829474283	0.829474283	0.0
0.4	0.826087383	0.826087383	0.0
0.5	0.846243429	0.846243429	0.0
0.6	0.88759712	0.88759712	0.0
0.7	0.947537684	0.947537685	0.1×10^{-8}
0.8	1.023213837	1.023213837	0.0
0.9	1.111563878	1.111563878	0.0
1	1.209350426	1.209350426	0.0

Case 2:

From equation (2.17), (2.18) and for k= 0, 1, … ; the differential transforms $V_1(k)$ and $V_2(k)$ with $q = \frac{1}{2}$, $\alpha = 2$, is given as follows:

$$V_1(k+1)=\frac{\Gamma(1+k/2)}{\Gamma(1/2+1+k/2)}\left\{\sum_{r=0}^{k}\frac{\Gamma(1/2+1+(k-r)/2)}{\Gamma(1+(k-r)/2)}\delta(r-2)\right.$$

$$\left. V_2(k-r+1)-V_1(k)+V_2(k)+\sum_{r=0}^{k}\delta(r-2)\,V_2(k-r)\right\}$$

and

$$V_2(k)=\begin{cases}0, & k/2\notin \text{¢}^{+}\\[2mm] \dfrac{\sin\left(\dfrac{k\pi}{4}\right)}{(k/2)!}, & k/2\in \text{¢}^{+}\end{cases}$$

Thus:

$V_1(1)=-1.128$, $V_1(2)=1$, $V_1(3)=0$, $V_1(4)=0.75$, $V_1(5)=0.15$,

$V_1(6)=-0.083$, $V_1(7)=-0.043$, $V_1(8)=-0.125$, $V_1(9)=0.057$,

$V_1(10)=-0.025$, $V_1(11)=-0.014$, $V_1(12)=2.041\times10^{-3}$,

$V_1(13)=2.421\times10^{-3}$, $V_1(14)=-8.989\times10^{-4}$,

$V_1(15)=2.516\times10^{-4}$, $V_1(16)=-2.736\times10^{-4}$,

$V_1(17)=2.541\times10^{-5}$, $V_1(18)=-8.353\times10^{-6}$,

$V_1(19)=3.557\times10^{-6}$, $V_1(20)=1.507\times10^{-6}$, … .

Then substituting these values of $V_1(k)$ into eq. (2.12) up to 20 terms, we get:

$$\begin{aligned} v_1(x)=\;&1-1.128x^{1/2}+x+0.75x^{2}+0.15x^{5/2}-0.083x^{3}-0.043x^{7/2}-\\ &0.125x^{4}+0.057x^{9/2}-0.025x^{5}-0.014x^{11/2}+2.041\times10^{-3}x^{6}+\\ &2.421\times10^{-3}x^{13/2}-8.989\times10^{4}x^{7}+2.516\times10^{-4}x^{15/2}-2.736\times10^{-4}x^{8}+\\ &2.541\times10^{-5}x^{17/2}-8.353\times10^{-6}x^{9}+3.557\times10^{-6}x^{19/2}+1.507\times10^{-6}x^{10}\end{aligned}$$

Following Table (3) represent the approximate solution of $v_1(x)$ when $q = \frac{1}{2}$.

Table (3)

The approximate solution of $v_1(x)$ for $q = \frac{1}{2}$.

x	**Approximate solution of $v_1(x)$**
0.1	0.751161923
0.2	0.727243483
0.3	0.753386842
0.4	0.812287372
0.5	0.896511031
0.6	1.00154078
0.7	1.124044788
0.8	1.261275747
0.9	1.410832026
1	1.570563221

Case 3:

From equation (2.17), (2.18) and for k= 0, 1, … ; the differential transforms $V_1(k)$ and $V_2(k)$ with $q = \frac{1}{4}$, $\alpha = 4$, is given as follows:

$$V_1(k+1) = \frac{\Gamma(1+k/4)}{\Gamma(1/4+1+k/4)}\left\{\sum_{r=0}^{k}\frac{\Gamma(1/4+1+(k-r)/4)}{\Gamma(1+(k-r)/4)}\delta(r-4)\right.$$
$$\left. V_2(k-r+1) - V_1(k) + V_2(k) + \sum_{r=0}^{k}\delta(r-4)\,V_2(k-r)\right\}$$

and

$$V_2(k)=\begin{cases}0, & k/4 \notin ¢^+\\ \dfrac{\sin\left(\dfrac{k\pi}{8}\right)}{(k/4)!}, & k/4 \in ¢^+\end{cases}$$

Hence:

$V_1(1) = -1.103$, $V_1(2) = 1.128$, $V_1(3) = -1.088$, $V_1(4) = 1$,

$V_1(5) = 0$, $V_1(6) = 0$, $V_1(7) = 0$, $V_1(8) = 0.875$,

$V_1(9) = 0.098$,$V_1(10) = -0.075$, $V_1(11) = 0.056$, $V_1(12) = -0.041$,

$V_1(13) = -0.091$, $V_1(14) = 0.065$, $V_1(15) = -0.046$,

$V_1(16) = -0.124$, $V_1(17) = -0.029$, $V_1(18) = 0.02$,

$V_1(19) = -0.013$, $V_1(20) = 8.535\times 10^{-3}$, …

Then substituting the values of $V_1(k)$ into eq. (2.12) up to 20 terms we get:

$$\begin{aligned} v_1(x) = {} & 1-1.103x^{1/4}+1.128x^{1/2}-1.088x^{3/4}+x+0.875x^2+0.098x^{9/4}\\ & -0.075x^{10/4}+0.056x^{11/4}-0.041x^3-0.091x^{13/4}+0.065x^{14/4}-\\ & 0.046x^{15/4}-0.124x^4-0.029x^{17/4}+0.02x^{18/4}-0.013x^{19/4}+\\ & 8.353\times 10^{-3}x^5. \end{aligned}$$

Following Table (4) represent the approximate solution of $v_1(x)$ when $q = \dfrac{1}{4}$.

Table (4)

The approximate solution of $v_1(x)$ for $q = \frac{1}{4}$.

x	Approximate solution of $v_1(x)$
0.1	0.652035803
0.2	0.677487264
0.3	0.740532352
0.4	0.82875184
0.5	0.93662663
0.6	1.060228555
0.7	1.196030569
0.8	1.340509631
0.9	1.489983519
1	1.640535

Example (2.2):

Consider the fractional differential-algebraic equation:

$$\begin{pmatrix} 1 & -x & x^2 \\ 0 & 1 & -x \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} D_*^q v_1(x) \\ D_*^q v_2(x) \\ D_*^q v_3(x) \end{pmatrix} + \begin{pmatrix} 1 & -(x+1) & x^2+2x \\ 0 & 1 & -(1+x) \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sin x \end{pmatrix}$$

................................(2.20)

with the following initial condition:

$$\begin{pmatrix} v_1(0) \\ v_2(0) \\ v_3(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{...(2.21)}$$

where the exact solution as given in [Celik and et al., 2006] when q = 1, is:

$$\begin{pmatrix} v_1(x) \\ v_2(x) \\ v_3(x) \end{pmatrix} = \begin{pmatrix} e^{-x} + xe^{-x} \\ e^{x} + x\sin x \\ \sin x \end{pmatrix}$$

System of eq. (2.20) can be written equivalently as:

$$D_*^q v_1(x) - x[D_*^q v_2(x)] + x^2[D_*^q v_3(x)] + v_1(x) - xv_2(x) - v_2(x) + x^2 v_3(x) + 2xv_3(x) = 0$$

$$D_*^q v_2(x) - x[D_*^q v_3(x)] - v_2(x) + xv_3(x) - v_3(x) = 0$$

$$v_3(x) = \sin x$$

or

$$D_*^q v_1(x) = 2xv_2(x) - 2x^2 v_3(x) - xv_3(x) - v_1(x) + v_2(x)$$

$$D_*^q v_2(x) = x[D_*^q v_3(x)] + v_2(x) - xv_3(x) + v_3(x)$$

$$v_3(x) = \sin x$$

By taking the differential transform of the above equations and by using the fundamental operation of fractional DTM which are given in section four we have :

$$V_1(k+\alpha q) = \frac{\Gamma(1+k/\alpha)}{\Gamma(q+1+k/\alpha)}\left\{2\sum_{r=0}^{k}\delta(r-\alpha)\,V_2(k-r) - 2\sum_{r=0}^{k}\delta(r-2\alpha)\right.$$
$$\left. V_3(k-r) - \sum_{r=0}^{k}\delta(r-\alpha)V_3(k-r) - V_1(k) + V_2(k)\right\} \ldots(2.22)$$

$$V_2(k+\alpha q)=\frac{\Gamma(1+k/\alpha)}{\Gamma(q+1+k/\alpha)}\left\{\sum_{r=0}^{k}\frac{\Gamma(q+1+(k-r)/\alpha)}{\Gamma(1+(k-r)/\alpha)}\delta(r-\alpha)\right.$$

$$\left.V_3(k-r+\alpha q)+V_2(k)-\sum_{r=0}^{k}\delta(r-\alpha)V_3(k-r)+V_3(k)\right\} \quad ..(2.23)$$

$$V_3(k)=\begin{cases}0, & k/\alpha\notin \text{¢}^+\\ \dfrac{\sin\left(\dfrac{k\pi}{2\alpha}\right)}{(k/\alpha)!}, & k/\alpha\in \text{¢}^+\end{cases} \quad ....(2.24)$$

Case 1:

For k = 0, 1, … ; $V_1(k)$, $V_2(k)$ and $V_3(k)$ can be calculated from eqs.(2.22), (2.23) and (2.24)when $q=\alpha=1$ as follows:

$$V_1(k+1)=\frac{\Gamma(1+k)}{\Gamma(2+k)}\left\{2\sum_{r=0}^{k}\delta(r-1)V_2(k-r)\ -2\sum_{r=0}^{k}\delta(r-1)V_3(k-r)-\right.$$

$$\left.\sum_{r=0}^{k}\delta(r-1)V_3(k-r)-V_1(k)+V_2(k)\right\}.$$

$$V_2(k+1)=\frac{\Gamma(1+k)}{\Gamma(2+k)}\left\{\sum_{r=0}^{k}\frac{\Gamma(2+(k-r))}{\Gamma(1+k-r)}\delta(r-1)V_3(k-r+1)+V_2(k)-\right.$$

$$\left.\sum_{r=0}^{k}\delta(r-1)V_3(k-r)+V_3(k)\right\}.$$

$$V_3(k)=\frac{\sin\left(\dfrac{k\pi}{2}\right)}{k!},\quad k\in \text{¢}^+$$

$$V_1(1)=0,\ V_1(2)=\frac{3}{2!},\ V_1(3)=\frac{2}{3!},\ V_1(4)=\frac{5}{4!},\ V_1(5)=\frac{4}{5!},$$

$$V_1(6)=\frac{7}{6!},\ V_1(7)=\frac{6}{7!},\ V_1(8)=\frac{9}{8!},\ V_1(9)=\frac{8}{9!},\ V_1(10)=\frac{11}{10!},$$

$$V_1(11)=\frac{10}{11!},\ V_1(12)=\frac{13}{12!},\ V_1(13)=\frac{12}{13!},\ V_1(14)=\frac{15}{14!},$$

$$V_1(15)=\frac{14}{15!},\ V_1(16)=\frac{17}{16!},\ V_1(17)=\frac{16}{17!},\ V_1(18)=\frac{19}{18!},$$

$$V_1(19)=\frac{18}{19!},\ V_1(20)=\frac{21}{20!},\ldots$$

$$V_2(1)=1,\ V_2(2)=\frac{3}{2!},\ V_2(3)=\frac{1}{3!},\ V_2(4)=\frac{-3}{4!},\ V_2(5)=\frac{1}{5!},$$

$$V_2(6)=\frac{7}{6!},\ V_2(7)=\frac{1}{7!},\ V_2(8)=\frac{-7}{8!},\ V_2(9)=\frac{1}{9!},\ V_2(10)=\frac{11}{10!},$$

$$V_2(11)=\frac{1}{11!},\ V_2(12)=\frac{-11}{12!},\ V_2(13)=\frac{1}{13!},\ V_2(14)=\frac{15}{14!},$$

$$V_2(15)=\frac{1}{15!},V_2(16)=\frac{-15}{16!},\ V_2(17)=\frac{1}{17!},\ V_2(18)=\frac{19}{18!},$$

$$V_2(19)=\frac{1}{19!},\ V_2(20)=\frac{-19}{20!},\ \ldots$$

As a result, these coefficients can be generalized by the following formula:

$$V_1(k)=\begin{cases}\dfrac{k+1}{k!}, & \text{k is even}\\[2ex] \dfrac{k-1}{k!}, & \text{k is odd}\end{cases}$$

$$V_2(k)=\begin{cases}\dfrac{1}{k!}, & \text{k is odd}\\ \dfrac{k+1}{k!}, & \text{k is even and } \dfrac{k+2}{2} \text{ is even}\\ \dfrac{(1-k)}{k!}, & \text{k is even and } \dfrac{k+2}{2} \text{ is odd}\end{cases}$$

Consequently, by substituting these coefficients in the finite series of solutions given by eq. (2.12) to the $v_1(x)$ and $v_2(x)$ up to 20 terms, thus we have:

$$v_1(x)=1+\frac{3}{2!}x^2+\frac{2}{3!}x^3+\frac{5}{4!}x^4+\frac{4}{5!}x^5+\frac{7}{6!}x^6+\frac{6}{7!}x^7+\frac{9}{8!}x^8+\frac{8}{9!}x^9+\frac{11}{10!}x^{10}+\frac{10}{11!}x^{11}+\frac{13}{12!}x^{12}+\frac{12}{13!}x^{13}+\frac{15}{14!}x^{14}+\frac{14}{15!}x^{15}+\frac{17}{16!}x^{16}+\frac{16}{17!}x^{17}+\frac{19}{18!}x^{18}+\frac{18}{19!}x^{19}+\frac{21}{20!}x^{20}.$$

$$v_2(x)=1+x+\frac{3}{2!}x^2+\frac{1}{3!}x^3-\frac{3}{4!}x^4+\frac{1}{5!}x^5+\frac{7}{6!}x^6+\frac{1}{7!}x^7-\frac{7}{8!}x^8+\frac{1}{9!}x^9+\frac{11}{10!}x^{10}+\frac{1}{11!}x^{11}-\frac{11}{12!}x^{12}+\frac{1}{13!}x^{13}+\frac{15}{14!}x^{14}+\frac{1}{15!}x^{15}-\frac{15}{16!}x^{16}+\frac{1}{17!}x^{17}+\frac{19}{18!}x^{18}+\frac{1}{19!}x^{19}-\frac{19}{20!}x^{20}.$$

Following tables (5) and (6) represent a comparison between the approximate result and the exact solution of $v_1(x)$ and $v_2(x)$ respectively when $q=\alpha=1$

Table (5)

Comparison of the approximate solution of $v_1(x)$ with the exact solution.

x	**Exact solution of $v_1(x)$**	**Approximate solution of $v_1(x)$**	**Absolute error**
0.1	1.01535451	1.01535451	0.0
0.2	1.063011305	1.063011305	0.0
0.3	1.145775863	1.145775863	0.0
0.4	1.267049925	1.267049925	0.0
0.5	1.430891295	1.430891295	0.0
0.6	1.642082916	1.642082916	0.0
0.7	1.906212199	1.906212199	0.0
0.8	2.229761706	2.229761707	0.1×10^{-8}
0.9	2.62021246	2.62021246	0.0
1	3.08616126	3.08616127	0.1×10^{-8}

Table (6)

Comparison of the approximate solution of $v_2(x)$ with the exact solution.

x	**Exact solution of $v_2(x)$**	**Approximate solution of $v_2(x)$**	**Absolute error**
0.1	1.11515426	1.11515426	0.0
0.2	1.261136624	1.261136625	0.1×10^{-8}
0.3	1.43851487	1.43851487	0.0
0.4	1.647592035	1.647592035	0.0
0.5	1.88843404	1.88843404	0.0
0.6	2.160904284	2.160904284	0.0
0.7	2.464705088	2.464705089	0.1×10^{-8}
0.8	2.799423801	2.799425801	0.0
0.9	3.16459733	3.16459733	0.0
1	3.559752813	3.559752813	0.0

Case 2:

When $q = \frac{1}{2}$, $\alpha = 2$, substituting these values into eq. (2.22) eq. (2.23) and (2.24), thus we have:

$$V_1(k+1) = \frac{\Gamma(1+k/2)}{\Gamma(1/2+1+k/2)}\left\{2\sum_{r=0}^{k}\delta(r-2)\,V_2(k-r) - 2\sum_{r=0}^{k}\delta(r-4)\right.$$
$$\left. V_3(k-r) - \sum_{r=0}^{k}\delta(r-2)\,V_3(k-r) - V_1(k) + V_2(k)\right\}$$

$$V_2(k+1) = \frac{\Gamma(1+k/2)}{\Gamma(1/2+1+k/2)}\left\{\sum_{r=0}^{k}\frac{\Gamma(1/2+1+(k-r)/2)}{\Gamma(1+(k-r)/2)}\delta(r-2)\right.$$
$$\left. V_3(k-r+1) + V_2(k) - \sum_{r=0}^{k}\delta(r-2)\,V_3(k-r) + V_3(k)\right\}$$

$$V_3(k) = \begin{cases} 0, & k/2 \notin ¢^{+} \\ \dfrac{\sin\left(\dfrac{k\pi}{4}\right)}{(k/2)!}, & k/2 \in ¢^{+} \end{cases}$$

$V_1(1) = 0$, $V_1(2) = 1$, $V_1(3) = 1.1505$, $V_1(4) = 1.499$,

$V_1(5) = 0.753$, $V_1(6) = 1.5$, $V_1(7) = 0.129$, $V_1(8) = 0.395$,

$V_1(9) = 0.067$, $V_1(10) = -0.05$, $V_1(11) = 0.031$, $V_1(12) = -0.128$,

$V_1(13) = 1.119\times10^{-3}$, $V_1(14) = -0.017$, $V_1(15) = -8.411\times10^{-4}$,

$V_1(16) = -2.32\times10^{-3}$, $V_1(17) = -1.557\times10^{-4}$,

$V_1(18) = -3.218\times10^{-4}$, $V_1(19) = -2.692\times10^{-4}$,

$V_1(20) = 4.55\times10^{-5}$, …

$V_2(1) = 1.128$, $V_2(2) = 1$, $V_2(3) = 1.505$, $V_2(4) = 1.75$,

$V_2(5) = 0.451$, $V_2(6) = 0.25$, $V_2(7) = 0.043$, $V_2(8) = -0.125$,

$V_2(9) = -0.134$, $V_2(10) = -0.058$, $V_2(11) = -0.021$,

$V_2(12) = -7.578\times10^{-4}$, $V_2(13) = -3.498\times10^{-3}$,

$V_2(14) = -1.299\times10^{-3}$, $V_2(15) = -5.377\times10^{-4}$,

$V_2(16) = -3.732\times10^{-4}$, $V_2(17) = -5.908\times10^{-5}$,

$V_2(18) = -1.942\times10^{-5}$, $V_2(19) = -5.337\times10^{-6}$,

$V_2(20) = 9.515\times10^{-7}, \ldots$

Then by substituting these coefficients into eq. (2.12) the series of the solutions to the $v_1(x)$ and $v_2(x)$ up to 20 terms, are:

$$\begin{aligned} v_1(x) = {} & 1 + x^1 + 1.505x^{3/2} + 1.499x^2 + 0.753x^{5/2} + 1.5x^3 + 0.129x^{7/2} + \\ & 0.395x^4 + 0.067x^{9/2} - 0.05x^5 + 0.031x^{11/2} - 0.128x^6 + \\ & 1.119\times10^{-3}x^{13/2} - 0.017x^7 - 8.911\times10^{-4}x^{15/2} - \\ & 2.312\times10^{-3}x^8 - 1.557\times10^{-4}x^{17/2} - 3.218\times10^{-4}x^9 - \\ & 2.692\times10^{-4}x^{19/2} + 4.55\times10^{-5}x^{10} \end{aligned}$$

$$\begin{aligned} v_2(x) = {} & 1 + 1.128x^{1/2} + x + 1.505x^{3/2} + 1.75x^2 + 0.451x^{5/2} + 0.25x^3 + \\ & 0.043x^{7/2} - 0.125x^4 - 0.134x^{9/2} - 0.058x^5 - 0.021x^{11/2} - \\ & 7.578\times10^{-4}x^6 - 3.498\times10^{-3}x^{13/2} - 1.299\times10^{-3}x^7 - \\ & 5.377\times10^{-4}x^{15/2} - 3.732\times10^{-4}x^8 - 5.908\times10^{-5}x^{17/2} - \\ & 1.942\times10^{-5}x^9 - 5.336\times10^{-6}x^{19/2} + 9.515\times10^{-7}x^{10} \end{aligned}$$

The approximate results of $v_1(x)$ and $v_2(x)$ are tabulated in Table (7) and (8) respectively

Table (7)

The numerical solution of $v_1(x)$ for $q = \frac{1}{2}$.

x	Approximate solution of $v_1(x)$
0.1	1.166545355
0.2	1.421162879
0.3	1.765053263
0.4	2.208331711
0.5	2.763498032
0.6	3.444086339
0.7	4.263835794
0.8	5.23590118
0.9	6.371968528
1	7.6812147

Table (8)

The approximate solution of $v_2(x)$ for $q = \frac{1}{2}$.

x	Approximate solution of $v_2(x)$
0.1	1.523469598
0.2	1.918972185
0.3	2.350467531
0.4	2.831411954
0.5	3.36591496
0.6	3.95451982
0.7	4.595422724
0.8	5.284744524
0.9	6.016501991
1	6.782451415

Case 3:

When $q = \frac{1}{4}$, $\alpha = 4$, substituting these values into eqs. (2.22) eq. (2.23) and (2.24), thus we have:

$$V_1(k+1) = \frac{\Gamma(1+k/4)}{\Gamma(1/2+1+k/2)} \left\{ 2\sum_{r=0}^{k} \delta(r-4)\, V_2(k-r) - 2\sum_{r=0}^{k} \delta(r-8) V_3(k-r) - \sum_{r=0}^{k} \delta(r-4)\, V_3(k-r) - V_1(k) + V_2(k) \right\}$$

$$V_2(k+1) = \frac{\Gamma(1+k/4)}{\Gamma(1/2+1+k/2)} \left\{ \sum_{r=0}^{k} \frac{\Gamma(1/4+1+(k-r)/4)}{\Gamma(1+(k-r)/2)} \delta(r-4) V_3(k-r+1) + V_2(k) - \sum_{r=0}^{k} \delta(r-4)\, V_3(k-r) + V_3(k) \right\}$$

$$V_3(k) = \begin{cases} 0, & k/4 \notin ¢^{+} \\ \dfrac{\sin\left(\frac{k\pi}{8}\right)}{(k/4)!}, & k/4 \in ¢^{+} \end{cases}$$

Hence:

$V_1(1) = 0$, $V_1(2) = 1.128$, $V_1(3) = 0$, $V_1(4) = 1$, $V_1(5) = 1.765$,

$V_1(6) = 1.88$, $V_1(7) = 1.554$, $V_1(8) = 1.5$, $V_1(9) = 1.079$,

$V_1(10) = 2.406$, $V_1(11) = 0.848$, $V_1(12) = 1.499$, $V_1(13) = 1.841$,

$V_1(14) = -0.27$, $V_1(15) = 0.972$, $V_1(16) = -0.95$, $V_1(17) = 0.49$,

$V_1(18) = -0.19$, $V_1(19) = 0.224$, $V_1(20) = -0.079$, …

$V_2(1) = 1.103$, $V_2(2) = 1.128$, $V_2(3) = 1.088$, $V_2(4) = 1$,

$V_2(5) = 1.765$, $V_2(6) = 1.504$, $V_2(7) = 1.243$, $V_2(8) = 1.875$,

$V_2(9) = 0.686$, $V_2(10) = 0.526$, $V_2(11) = 0.395$, $V_2(12) = 0.291$,

$V_2(13) = 0.09$, $V_2(14) = 0.064$, $V_2(15) = 0.045$, $V_2(16) = -0.125$,

$V_2(17) = 0.028$, $V_2(18) = 0.019$, $V_2(19) = 0.013$,

$V_2(20) = 8.535\times10^{-3}$, …

Then substituting these coefficients into eq. (2.12) the series of the solution to the $v_1(x)$ and $v_2(x)$ up to 20 terms, are:

$$\begin{aligned} v_1(x) = {} & 1.128x^{1/2} + x + 1.765x^{5/4} + 1.88x^{6/4} + 1.554x^{7/4} + 1.5x^2 + \\ & 1.079x^{9/4} + 2.406x^{10/4} + 0.848x^{11/4} + 1.499x^3 + 1.841x^{13/4} - \\ & 0.27x^{14/4} + 0.972x^{15/4} - 0.095x^4 + 0.49x^{17/4} - 0.19x^{18/4} + \\ & 0.224x^{19/4} - 0.079x^5 \end{aligned}$$

$$\begin{aligned} v_2(x) = {} & 1.103x^{1/4} + 1.128x^{1/2} + 1.088x^{3/4} + x + 1.765x^{5/4} + 1.504x^{6/4} + \\ & 1.243x^{7/4} + 1.875x^2 + 0.686x^{9/4} + 0.526x^{10/4} + 0.395x^{11/4} + \\ & 0.291x^3 + 0.09x^{13/4} + 0.064x^{14/4} + 0.045x^{15/4} - 0.125x^4 + \\ & 0.028x^{17/4} + 0.019x^{18/4} + 0.013x^{19/4} + 8.535\times10^{-3}x^5. \end{aligned}$$

The approximate results of $v_1(x)$ and $v_2(x)$ are tabled in table (9) and (10) respectively.

Table (9)

The approximate solution of $v_1(x)$ when $q = \frac{1}{4}$.

x	Approximate solution of $v_1(x)$
0.1	1.675864507
0.2	2.367191608
0.3	3.249783755
0.4	4.368675847
0.5	5.766146891
0.6	7.485964262
0.7	9.574273096
0.8	12.07987224
0.9	15.05432287
1	18.552

Table (10)

The approximate solution of $v_2(x)$ when $q = \frac{1}{4}$.

x	Approximate solution of $v_2(x)$
0.1	2.464695398
0.2	3.322884854
0.3	4.230519029
0.4	5.22733384
0.5	6.330185917
0.6	7.54968756
0.7	8.894080152
0.8	10.370564003
0.9	11.985868032
1	13.746535

CHAPTER THREE

ADOMIAN DECOMPOSITION METHOD FOR SOLVING FRACTIONAL-DIFFERENTIAL ALGEBRAIC EQUATIONS

3.1 INTRODUCTION

In this chapter, we present the application of Adomian decomposition method (ADM) for solving fractional DAEs. This chapter consists of four sections, where in section (3.2) the literature review of (ADM) is given. In section (3.3) we introduce the ADM. In section (3.4) we apply the (ADM) for solving fractional DAEs with the approximate results of an illustrative examples.

3.2 LITERATURE REVIEW

Most of the phenomena that arise in real world are described by nonlinear differential and integral equations. However, most of the methods developed in mathematics are usually used in solving linear differential and integral equations.

The recently developed decomposition method proposed by American mathematician, Georg Adomian has been receiving much attention in recent years in applied mathematics. The ADM emerged as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, integro-differential equations, and partial differential equations. Thus yields rapidly convergent series solutions for both linear and nonlinear deterministic and stochastic equations; it has many advantages over the classical techniques, namely, it avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculations and avoidance of physically unrealistic assumptions, the theoretical treatment of convergence of the decomposition method has been considered in [Seng and et al., 1996] and the obtained results about the speed of convergence of this method. The solution of the fractional differential equation has been obtained through the Adomian decomposition by [Ray and Bera, 2004].

However, El-Sayed and Kaya proposed ADM to approximate the numerical and analytical solution of system two-dimensional Burger's equations with initial conditions in [El-Sayed and Kaya, 2004], and the advantages of this work is that the decomposition method reduces the computational work and improves with regards to its accuracy and rapid convergence. The convergence of decomposition method is proved as [Inc, 2005], in [Celik and et al., 2006] applied ADM to obtain the approximate solution for the DAEs system and the result obtained by this method indicate a high degree of accuracy through the comparison with the analytic solutions. In [Hosseini, 2006 a], [Hosseini, 2006 b] standard and modified ADMs are applied to solve non-linear DAEs. While, the error analysis of Adomian series solution to a class of nonlinear differential equation, where as numerical experiments show that Adomian solution using this formula converges faster is discussed in [El-Kala, 2007]. Also, a new discrete ADM to approximate the theoretical solution of discrete nonlinear Schrodinger equations is presented in [Bratsos and Ehrhardt, 2007] where this examined for plane waves and single solution waves in case of continuous, semi discrete and fully discrete Schrodinger equations.

Momani and Jafari, [Momani and Jafari, 2008] presented numerical study of system of fractional differential equation by ADM.

3.3 THE ADOMIAN DECOMPOSITION METHOD (ADM)

To introduce the basic idea of the ADM, we consider the operator equation Fu = G, where F represents a general nonlinear ordinary differential operator and G is a given function. The linear part of F can be decomposed as:

$$Lu + Ru + Nu = G \quad \text{......(3.1)}$$

where, N is a nonlinear operator, L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order less than L and G is the nonhomogeneous term.

The method is based by applying the operator L^{-1} formally to the expression

$$Lu = G - Ru - Nu \quad \text{......(3.2)}$$

so by using the given conditions, we obtain:

$$u = h + L^{-1}G - L^{-1}Ru - L^{-1}Nu \quad \text{......(3.3)}$$

where, h is the solution of the homogeneous equation Lu = 0, with the initial-boundary conditions. The problem now is the decomposition of the nonlinear term Nu. To do this, Adomian developed a very elegant technique as follows:

Define the decomposition parameter λ as:

$$u = \sum_{n=0}^{\infty} \lambda^n u_n$$

then N(u) will be a function of λ, u_0, u_1, … . Next expanding N(u) in Maclurian series with respect to λ we obtain $N(u) = \sum_{n=0}^{\infty} \lambda^n A_n$, where:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n} \lambda^k u_k\right)\right]_{\lambda=0} \qquad \text{...(3.4)}$$

where, the components of A_n are the so called Adomian polynomials, they are generated for each nonlinearity, for example, for N(u) = f(u), the Adomian polynomials, are given as:

$$A_0 = f(u_0)$$

$$A_1 = u_1 f'(u_0)$$

$$A_2 = u_2 f'(u_0) + \frac{u_1^2}{2} f''(u_0)$$

$$A_3 = u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0)$$

$$\vdots$$

Now, we parameterize eq.(3.3) in the form:

$$u = h + L^{-1}G - \lambda L^{-1}Ru - \lambda L^{-1}Nu \quad \text{.......................................} (3.5)$$

where, λ is just an identifier for collecting terms in a suitable way such that u_n depends on $u_0, u_1, \ldots, u_n$ and we will later set $\lambda = 1$

$$\sum_{n=0}^{\infty} \lambda^n u_n = h + L^{-1}G - \lambda L^{-1}R \sum_{n=0}^{\infty} \lambda^n u_n - \lambda L^{-1} \sum_{n=0}^{\infty} \lambda^n A_n \quad \text{.......} (3.6)$$

Equating the coefficients of equal powers of λ, we obtain:

$$\left.\begin{aligned} u_0 &= h + L^{-1}G \\ u_1 &= -L^{-1}(Ru_0) - L^{-1}(A_0) \\ u_2 &= -L^{-1}(Ru_1) - L^{-1}(A_1) \\ &\vdots \end{aligned}\right\} \quad \text{...} (3.7)$$

and in general

$$u_n = -L^{-1}(Ru_{n-1}) - L^{-1}(A_{n-1}),\ n \geq 1$$

Finally, an N-terms that approximate the solution is given by:

$$\phi_N(x) = \sum_{n=0}^{N-1} u_n(x),\ N \geq 1$$

and the exact solution is $u(x) = \lim_{N\to\infty} \phi_N$.

3.4 ADM FOR SOLVING FACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS

In this section ADM will be applied to find the solution of the fractional DAEs and to do this, let as consider as in chapter two the following system of fractional DAEs with variable coefficients

$$A(t)\left[D_*^q x(t)\right] + B(t)\,x(t) = f(t),\ 0 < q \le 1 \quad\text{.............................}(3.8)$$

with initial value:

$$x(t_0) = x_0 \quad\text{...}(3.9)$$

where D_*^q is the Caputo fractional derivatives of order q, A(t) is n×n singular matrix, B(t) is n×n nonsingular matrix and f is n×1 vector function, x_0 is n×1 known constant vector and x(t) is the n×1 vector function that must be determined.

The procedure of solution is begins by writing system (3.8), equivalently as a system of fractional DAEs, then by substituting the algebraic variable in the previous equations, we get a system of ordinary differential equations, which can be considered as:

$$\left.\begin{aligned} D_*^q x_1 &= f_1(t, x_1, x_2, \ldots, x_{n-1}) \\ D_*^q x_2 &= f_2(t, x_1, x_2, \ldots, x_{n-1}) \\ &\vdots \\ D_*^q x_{n-1} &= f_{n-1}(t, x_1, x_2, \ldots, x_{n-1}) \end{aligned}\right\} \quad\text{...}(3.10)$$

where each equation represents the fractional derivative of order q of one of the unknown functions as a mapping depending on the independent variable t and n – 1 unknown functions $x_1, x_2, \ldots, x_{n-1}$.

We can present the system (3.10), by using the i^{th} equation as:

$$Lx_i = f_i(t, x_1, x_2, \ldots, x_{n-1}),\ i = 1, 2, \ldots, n-1 \quad\text{..........................}(3.11)$$

where $L = D_*^q$ is the Caputo fractional derivative of order q with the inverse J^q which is the Riemman-Louville fractional integration of order

q. Applying the inverse operator J^q on (3.11) we shall get the following form:

$$x_i = \sum_{k=0}^{n-1} x^k(0^+)\frac{t^k}{k!} + J^q f_i(t, x_1, x_2, ..., x_{n-1}) \quad \text{..............................}(3.12)$$

As usual in ADM the solution of equation (3.8) and (3.9) is considered to be the infinite of a series

$$x_i = \sum_{j=0}^{\infty} x_{ij}(x),\ i = 1, 2, \ldots, n-1 \quad \text{..}(3.13)$$

Where

$$x_{i,0} = x_i(0)$$
$$x_{i,n+1} = J^q f_i,\ n = 0,1,2...$$

And if the integrand f_i in eq. (3.12) is nonlinear, we define:

$$\left.\begin{aligned} x_{i,0} &= x_i(0) \\ x_{i,n+1} &= J^q A_{i,n},\ n = 0,1,2... \end{aligned}\right\} \text{..}(3.14)$$

Where $A_{i,n}$ are the Adomian polynomial which is illustrated in section three.

Now, we shall consider the following examples in order to illustrate the above procedure:

<u>Example (3.1):</u>

Consider the following system of fractional differential-algebraic equations (DAE's):

$$D_*^q u(x) - x\left[D_*^q v(x)\right] + u(x) - (1+x)\,v(x) = 0 \text{.......................}(3.15)$$

$$v(x) = \sin x \qquad (3.16)$$

with initial condition:

$$u(0) = 1,\ v(0) = 0 \qquad (3.17)$$

with the exact solution as given in [Celik and et al., 2006] for $q = 1$

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} e^{-x} + x\sin x \\ \sin x \end{pmatrix}$$

Equivalently eq. (3.15) and (3.16) can be written as:

$$D_*^q u(x) = x\left[D_*^q v(x)\right] - u(x) + (1+x)\,v(x) \qquad (3.18)$$

$$v(x) = \sin x \qquad (3.19)$$

And in order to avoid the difficulty of the fractional differentiation and integration respectively we shall substitute the Maclurain series of v(x) up to certain terms in eq.(3.18) and upon applying the inverse operator of the fractional differential operator J^q to the both sides of eq.(3.18) we shall get:

$$u(x) = \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{x^k}{k!} + J^q\left\{xD_*^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\} - J^q[u(x)] + J^q\left\{(1+x)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right]\right\}$$

Hence:

$$\left.\begin{aligned}
u_0(x) &= \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{x^k}{k!} + J^q\left[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q} - \frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q} + \frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q} - \right.\\
&\quad \left.\frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q} + \frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}\right] + J^q\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right] + \\
&\quad J^q\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right]\\
u_{k+1}(x) &= -J^q[u_k(x)]
\end{aligned}\right\} \qquad \text{.............................(3.20)}$$

Case 1:

From eq.(3.20) the functions $u_k(x)$ for k = 0, 1, …; can be obtained for q = 1, as:

$$u_0(x) = 1 + J\left[\frac{\Gamma(2)}{\Gamma(1)}x - \frac{\Gamma(4)}{3!\Gamma(3)}x^3 + \frac{\Gamma(6)}{5!\Gamma(5)}x^5 - \frac{\Gamma(8)}{7!\Gamma(7)}x^7 + \frac{\Gamma(10)}{9!\Gamma(9)}x^9\right] +$$
$$J\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right] + J\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right].$$

$$u_{k+1}(x) = -J[u_k(x)]$$

and thus, by taking the series (3.13) up to 10 terms we have:

$$u(x) = u_0(x) + u_1(x) + \ldots + u_{10}(x)$$

which represent the approximate solution of u(x) when q = 1.

Following Table (1), gives a comparison between the approximate solution of u(x) with the exact solution.

Table (1)

Comparison of the approximate solution of u(x)with the exact solution.

x	Exact solution of u(x)	Approximate solution of u(x)	Absolute error
0.1	0.91482076	0.91482076	0.0
0.2	0.858464619	0.858464619	0.0
0.3	0.829474283	0.829474283	0.0
0.4	0.826087383	0.826087383	0.0
0.5	0.846243429	0.846243429	0.0
0.6	0.88759712	0.887597126	0.6×10^{-8}
0.7	0.947537685	0.94753776	0.75×10^{-7}
0.8	1.023213837	1.023214508	0.671×10^{-6}
0.9	1.111563878	1.111568391	0.4513×10^{-5}
1	1.209350426	1.209374996	0.2457×10^{-4}

Case 2:

From eq. (3.20) the functions $u_k(x)$ for $k = 0, 1, \ldots$;can be obtained for $q = \frac{1}{2}$, as:

$$u_0(x) = 1 + J^{\frac{1}{2}}\left[\frac{\Gamma(2)}{\Gamma(3/2)}x^{3/2} - \frac{\Gamma(4)}{3!\Gamma(7/2)}x^{7/2} + \frac{\Gamma(6)}{5!\Gamma(11/2)}x^{11/2} - \frac{\Gamma(8)}{7!\Gamma(15/2)}x^{15/2} + \frac{\Gamma(10)}{9!\Gamma(19/2)}x^{19/2}\right] + J^{\frac{1}{2}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right] + J^{\frac{1}{2}}\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right].$$

$$u_{k+1}(x) = -J^{\frac{1}{2}}[u_k(x)]$$

Substituting the above functions up to 10 terms into eq.(3.13).One can get the approximate solution of u(x) when $q = \frac{1}{2}$.

Following Table (2), represent the approximate result of u(x) when $q = \frac{1}{2}$.

Table (2)

The approximate solution of u(x), when q = $\frac{1}{2}$.

x	**Approximate solution u(x)**
0.1	0.75104093
0.2	0.727031608
0.3	0.75290395
0.4	0.811047383
0.5	0.893511908
0.6	0.995100324
0.7	1.111543574
0.8	1.238915952
0.9	1.373370585
1	1.511027198

Case 3:

From eq. (3.20) the functions $u_k(x)$ for k = 0, 1, …; can be obtained for $q = \frac{1}{4}$, as:

$$u_0(x)=1+J^{\frac{1}{4}}\left[\frac{\Gamma(2)}{\Gamma(7/4)}x^{7/4}-\frac{\Gamma(4)}{3!\Gamma(15/4)}x^{15/4}+\frac{\Gamma(6)}{5!\Gamma(23/4)}x^{23/4}-\frac{\Gamma(8)}{7!\Gamma(31/4)}x^{31/4}+\frac{\Gamma(10)}{9!\Gamma(39/4)}x^{39/4}\right]+J^{\frac{1}{4}}\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}\right]+J^{\frac{1}{4}}\left[x^2-\frac{x^4}{3!}+\frac{x^6}{5!}-\frac{x^8}{7!}+\frac{x^{10}}{9!}\right].$$

$$u_{k+1}(x)=-J^{\frac{1}{4}}[u_k(x)]$$

After substituting the functions $u_0(x)$, $u_1(x)$, …; up to 10 terms into eq.(3.13) one can get the approximate solution of u(x) when $q=\frac{1}{4}$.

Following table (3), represent the approximate solution of u(x) when $q=\frac{1}{4}$.

Table (3)

The approximate solution of u(x) when q = $\frac{1}{4}$.

x	**Approximate solution of u(x)**
0.1	0.651319941
0.2	0.673763976
0.3	0.730424793
0.4	0.807957669
0.5	0.899892734
0.6	1.001297664
0.7	1.107591215
0.8	1.214166383
0.9	1.316262741
1	1.408929446

Example (3.2):

Consider the following fractional differential-algebraic equations:

$$D_*^q u(x) - x\left[D_*^q v(x)\right] + x^2\left[D_*^q z(x)\right] + u(x) - (x+1)\,v(x) + (x^2+2x)z(x) = 0 \quad \ldots\ldots (3.21)$$

$$D_*^q v(x) - x\left[D_*^q z(x)\right] - v(x) + (x-1)\,z(x) = 0 \quad \ldots\ldots (3.22)$$

$$z(x) = \sin x \quad \ldots\ldots (3.23)$$

with initial condition:

$$u(0) = 1,\ v(0) = 1,\ z(0) = 0$$

with the exact solution as given in [Celik and et al., 2006] for $q = 1$

$$u(x) = e^{-x} + xe^{x}$$

$$v(x) = e^{x} + x\sin x$$

$$z(x) = \sin x$$

Equivalently eqs.(3.21), (3.22) and (3.23) can be written as:

$$D_*^q u(x) = x\left[D_*^q v(x)\right] - x^2\left[D_*^q z(x)\right] - u(x) + (x+1)\,v(x) - (x^2+2x)z(x) \quad \ldots\ldots (3.24)$$

$$D_*^q v(x) = x\left[D_*^q z(x)\right] + v(x) - (x-1)\,z(x) = 0 \quad \ldots\ldots (3.25)$$

$$z(x) = \sin x \quad \ldots\ldots (3.26)$$

As we do in example (3.1) we apply the inverse fractional differential operator J^q to the both sides of equations (3.24) and (3.25) after substituting the Maclurain series of z(x) into these equations in order to avoid the difficulty of fractional differentiation and integration respectively we get:

$$D_*^q u(x) = -2\left[x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \frac{x^{11}}{9!}\right] - \left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right] - u(x) + 2xv(x) + v(x)$$

$$D_*^q v(x) = x\left[D_*^q\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right)\right] + v(x) - (x-1)\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}\right]$$

By taking the fractional integration to the both sides of the above equations we get:

$$u(x) = \sum_{k=0}^{m-1} u^{(k)}(0^+)\frac{x^k}{k!} + J^q\left[2xv(x)\right] + J^q\left[v(x)\right] - J^q\left\{2\left[x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \frac{x^{11}}{9!}\right]\right\} - J^q\left[x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \frac{x^{10}}{9!}\right] - J^q\left[u(x)\right]$$

$$v(x) = \sum_{k=0}^{m-1} v^k(0^+)\frac{x^k}{k!} + J^q\left[\frac{\Gamma(2)}{\Gamma(2-q)}x^{2-q} - \frac{\Gamma(4)}{3!\Gamma(4-q)}x^{4-q} + \frac{\Gamma(6)}{5!\Gamma(6-q)}x^{6-q} - \frac{\Gamma(8)}{7!\Gamma(8-q)}x^{8-q} + \frac{\Gamma(10)}{9!\Gamma(10-q)}x^{10-q}\right] + J^q\left[v(x) - x^2 + \frac{x^4}{3!} - \frac{x^6}{5!} + \frac{x^8}{7!} - \frac{x^{10}}{9!} + \frac{x^3}{3!} + \frac{x^5}{5!} - x - \frac{x^7}{7!} + \frac{x^9}{9!}\right]$$

Then:

$$u_0(x)=1-\frac{2\Gamma(4)}{\Gamma(4+q)}x^{3+q}+\frac{2\Gamma(6)}{3!\Gamma(6+q)}x^{5+q}-\frac{2\Gamma(8)}{5!\Gamma(8+q)}x^{7+q}+\frac{2\Gamma(10)}{7!\Gamma(10+q)}x^{9+q}-\frac{2\Gamma(12)}{9!\Gamma(12+q)}x^{11+q}-\frac{\Gamma(3)}{\Gamma(3+q)}x^{2+q}+\frac{\Gamma(5)}{3!\Gamma(5+q)}x^{4+q}-\frac{\Gamma(7)}{5!\Gamma(7+q)}x^{6+q}+\frac{\Gamma(9)}{7!\Gamma(9+q)}x^{8+q}-\frac{\Gamma(11)}{9!\Gamma(11+q)}x^{10+q} \quad \ldots\ldots (3.27)$$

$$u_{k+1}(x)=-J^q\left[u_k(x)\right]+J^q\left[v_k(x)\right]+J^q\left[2xv_k(x)\right] \quad \ldots\ldots (3.28)$$

$$v_0(x)=\sum_{k=0}^{m-1}v^k(0^+)\frac{x^k}{k!}+\frac{\Gamma(3-q)}{\Gamma(2-q)\Gamma(3)}x^2-\frac{\Gamma(5-q)}{\Gamma(4-q)\Gamma(5)}x^4+\frac{\Gamma(7-q)}{\Gamma(6-q)\Gamma(7)}x^6-\frac{\Gamma(9-q)}{\Gamma(8-q)\Gamma(9)}x^8+\frac{\Gamma(11-q)}{\Gamma(10-q)\Gamma(11)}x^{10}-\frac{\Gamma(3)}{\Gamma(3+q)}x^{2+q}+\frac{\Gamma(5)}{3!\Gamma(5+q)}x^{4+q}-\frac{\Gamma(7)}{5!\Gamma(7+q)}x^{6+q}+\frac{\Gamma(9)}{7!\Gamma(9+q)}x^{8+q}-\frac{\Gamma(11)}{9!\Gamma(11+q)}x^{10+q}+\frac{\Gamma(2)}{\Gamma(2+q)}x^{1+q}-\frac{\Gamma(4)}{3!\Gamma(4+q)}x^{3+q}+\frac{\Gamma(6)}{5!\Gamma(6+q)}x^{5+q}-\frac{\Gamma(8)}{7!\Gamma(8+q)}x^{7+q}+\frac{\Gamma(10)}{9!\Gamma(10+q)}x^{9+q} \quad \ldots\ldots (3.29)$$

$$v_{k+1}(x)=J^q\left[v_k(x)\right] \quad \ldots\ldots (3.30)$$

Case 1:

From eq.(3.28), (3.29) and (3.30) the functions $u_k(x)$ and $v_k(x)$ for $k = 0, 1, \ldots$; can be obtained for $q = 1$ as:

$$u_0(x)=1-\frac{2\Gamma(4)}{\Gamma(5)}x^4+\frac{2\Gamma(6)}{3!\Gamma(7)}x^6-\frac{2\Gamma(8)}{5!\Gamma(9)}x^8+\frac{2\Gamma(10)}{7!\Gamma(11)}x^{10}-\frac{2\Gamma(12)}{9!\Gamma(13)}x^{12}-$$
$$\frac{\Gamma(3)}{\Gamma(4)}x^3+\frac{\Gamma(5)}{3!\Gamma(6)}x^5-\frac{\Gamma(7)}{5!\Gamma(8)}x^7+\frac{\Gamma(9)}{7!\Gamma(10)}x^9-\frac{\Gamma(11)}{9!\Gamma(12)}x^{11}$$

$$u_{k+1}(x)=-J[u_k(x)]+J[v_k(x)]+J[2xv_k(x)]$$

$$v_0(x)=1+\frac{\Gamma(2)}{\Gamma(3)}x^2-\frac{\Gamma(4)}{\Gamma(3)\Gamma(5)}x^4+\frac{\Gamma(6)}{\Gamma(5)\Gamma(7)}x^6-\frac{\Gamma(8)}{\Gamma(7)\Gamma(9)}x^8+$$
$$\frac{\Gamma(10)}{\Gamma(9)\Gamma(11)}x^{10}-\frac{\Gamma(3)}{\Gamma(4)}x^3+\frac{\Gamma(5)}{3!\Gamma(6)}x^5-\frac{\Gamma(7)}{5!\Gamma(8)}x^7+$$
$$\frac{\Gamma(9)}{7!\Gamma(10)}x^9-\frac{\Gamma(11)}{9!\Gamma(12)}x^{11}+\frac{\Gamma(2)}{\Gamma(3)}x^2-\frac{\Gamma(4)}{3!\Gamma(5)}x^4+\frac{\Gamma(6)}{5!\Gamma(7)}x^6-$$
$$\frac{\Gamma(8)}{7!\Gamma(9)}x^8+\frac{\Gamma(10)}{9!\Gamma(11)}x^{10}$$

$$v_{k+1}(x)=J[v_k(x)]$$

After determined the components $u_k(x)$ and $v_k(x)$, for k = 0, 1, …; and substituting them into eq. (3.13) up to 10 terms one can get the approximate solution for u(x) and v(x) respectively for q = 1.

Following Tables (4) and (5) gives a comparison between approximate solution of u(x) and v(x) with the exact solution.

Table (4)

Comparison of the approximate solution of u(x) with the exact solution.

x	**Exact solution of u(x)**	**Approximate solution of u(x)**	**Absolute error**
0.1	1.01535451	1.02035898	0.5×10^{-2}
0.2	1.063011305	1.05086707	0.012
0.3	1.145775863	1.191172025	0.45×10^{-1}
0.4	1.267049925	1.348330569	0.813×10^{-1}
0.5	1.430891295	1.559045046	0.128
0.6	1.642082916	1.828593336	0.187
0.7	1.906212199	2.163095124	0.257
0.8	2.229761707	2.569713698	0.34
0.9	2.62021246	3.057488401	0.407
1	3.08616127	3.64018646	0.554

Table (5)

Comparison of the approximate solution of v(x) with the exact solution.

x	**Exact solution of v(x)**	**Approximate solution of v(x)**	**Absolute error**
0.1	1.11515426	1.11515426	0.0
0.2	1.261136624	1.261136624	0.0
0.3	1.43851487	1.43851487	0.0
0.4	1.647592035	1.647592035	0.0
0.5	1.88843404	1.88843803	0.399×10^{-5}
0.6	2.160904284	2.16091408	0.9796×10^{-5}
0.7	2.464705089	2.464789866	0.8478×10^{-4}
0.8	2.799425801	2.799975556	0.5498×10^{-3}
0.9	3.16459733	3.167456927	0.286×10^{-2}
1	3.559752813	3.57225281	0.125×10^{-1}

Case 2:

From eq.(3.27), (3.28), (3.29) and (3.30) the functions $u_k(x)$ and $v_k(x)$ for $k = 0, 1, \ldots$; can be obtained for $q = \frac{1}{2}$ as:

$$u_0(x) = 1 - \frac{2\Gamma(4)}{\Gamma(9/2)}x^{7/2} + \frac{2\Gamma(6)}{3!\Gamma(13/2)}x^{11/2} - \frac{2\Gamma(8)}{5!\Gamma(17/2)}x^{15/2} + \frac{2\Gamma(10)}{7!\Gamma(21/2)}x^{19/2} - \frac{2\Gamma(12)}{9!\Gamma(25/2)}x^{23/2} - \frac{\Gamma(3)}{\Gamma(7/2)}x^{5/2} + \frac{\Gamma(5)}{3!\Gamma(11/2)}x^{9/2} - \frac{\Gamma(7)}{5!\Gamma(15/2)}x^{13/2} + \frac{\Gamma(9)}{7!\Gamma(19/2)}x^{17/2} - \frac{\Gamma(11)}{9!\Gamma(23/2)}x^{21/2}$$

$$u_{k+1}(x) = -J^{1/2}\left[u_k(x)\right] + J^{1/2}\left[v_k(x)\right] + J^{1/2}\left[2xv_k(x)\right]$$

$$v_0(x) = 1 + \frac{\Gamma(5/2)}{\Gamma(3/2)\Gamma(3)}x^2 - \frac{\Gamma(9/2)}{\Gamma(7/2)\Gamma(5)}x^4 + \frac{\Gamma(13/2)}{\Gamma(11/2)\Gamma(7)}x^6 - \frac{\Gamma(17/2)}{\Gamma(15/2)\Gamma(9)}x^8 + \frac{\Gamma(21/2)}{\Gamma(19/2)\Gamma(11)}x^{10} - \frac{\Gamma(3)}{\Gamma(7/2)}x^{5/2} + \frac{\Gamma(5)}{3!\Gamma(11/2)}x^{9/2} - \frac{\Gamma(7)}{5!\Gamma(15/2)}x^{13/2} + \frac{\Gamma(9)}{7!\Gamma(19/2)}x^{17/2} - \frac{\Gamma(11)}{9!\Gamma(23/2)}x^{21/2} + \frac{\Gamma(2)}{\Gamma(5/2)}x^{3/2} - \frac{\Gamma(4)}{3!\Gamma(9/2)}x^{7/2} + \frac{\Gamma(6)}{5!\Gamma(13/2)}x^{11/2} - \frac{\Gamma(8)}{7!\Gamma(17/2)}x^{15/2} + \frac{\Gamma(10)}{9!\Gamma(21/2)}x^{19/2}$$

$$v_{k+1}(x) = J^{1/2}\left[v_k(x)\right]$$

After determined the components $u_k(x)$ and $v_k(x)$ for k = 0, 1, 2, …; up to 10 terms and substituting them into eq. (3.13) one can get the approximate solution of u(x) and v(x) for $q=\frac{1}{2}$.

Following Tables (6) and (7) represent the approximate solution of u(x) and v(x) when $q=\frac{1}{2}$.

Table (6)

The approximate solution of u(x), when q = $\frac{1}{2}$.

x	**Approximate solution of u(x)**
0.1	1.277006028
0.2	1.665134105
0.3	2.169140178
0.4	2.8031011221
0.5	3.583897244
0.6	4.530008671
0.7	5.660975446
0.8	6.99705369
0.9	8.558960728
1	10.367683549

Table (7)

The approximate solution of v(x), when q = $\frac{1}{2}$.

x	Approximate solution of v(x)
0.1	1.523580567
0.2	1.919238804
0.3	2.351493063
0.4	2.834930619
0.5	3.375655793
0.6	3.977318952
0.7	4.642596074
0.8	5.373741033
0.9	6.172866973
1	7.04214453062

Case 3:

From eq. (3.27), (3.28), (3.29) and (3.30) the functions $u_k(x)$ and $v_k(x)$ for k=0,1,2,… ; can be obtained for $q=\frac{1}{4}$ as:

$$\begin{aligned}u_0(x) = 1 &- \frac{2\Gamma(4)}{\Gamma(17/4)}x^{13/4} + \frac{2\Gamma(6)}{3!\Gamma(25/4)}x^{21/4} - \frac{2\Gamma(8)}{5!\Gamma(33/4)}x^{29/4} + \\ &\frac{2\Gamma(10)}{7!\Gamma(41/4)}x^{37/4} - \frac{2\Gamma(12)}{9!\Gamma(49/4)}x^{45/4} - \frac{\Gamma(3)}{\Gamma(13/4)}x^{9/4} + \\ &\frac{\Gamma(5)}{3!\Gamma(21/4)}x^{17/4} - \frac{\Gamma(7)}{5!\Gamma(29/4)}x^{25/4} + \frac{\Gamma(9)}{7!\Gamma(37/4)}x^{33/4} - \\ &\frac{\Gamma(11)}{9!\Gamma(45/4)}x^{41/4}\end{aligned}$$

$$u_{k+1}(x) = -J^{1/4}\left[u_k(x)\right] + J^{1/4}\left[v_k(x)\right] + J^{1/4}\left[2xv_k(x)\right]$$

$$v_0(x) = 1 + \frac{\Gamma(1/4)}{\Gamma(7/4)\Gamma(3)}x^2 - \frac{\Gamma(19/4)}{\Gamma(15/4)\Gamma(5)}x^4 + \frac{\Gamma(27/4)}{\Gamma(23/4)\Gamma(7)}x^6 -$$
$$\frac{\Gamma(35/4)}{\Gamma(31/4)\Gamma(9)}x^8 + \frac{\Gamma(43/4)}{\Gamma(39/4)\Gamma(11)}x^{10} - \frac{\Gamma(3)}{\Gamma(13/4)}x^{9/4} +$$
$$\frac{\Gamma(5)}{3!\Gamma(21/4)}x^{17/4} - \frac{\Gamma(7)}{5!\Gamma(29/4)}x^{25/4} + \frac{\Gamma(9)}{7!\Gamma(37/4)}x^{33/4} -$$
$$\frac{\Gamma(11)}{9!\Gamma(45/4)}x^{41/4} + \frac{\Gamma(2)}{\Gamma(9/4)}x^{5/4} - \frac{\Gamma(4)}{3!\Gamma(17/4)}x^{13/4} +$$
$$\frac{\Gamma(6)}{5!\Gamma(25/4)}x^{21/4} - \frac{\Gamma(8)}{7!\Gamma(33/4)}x^{29/4} + \frac{\Gamma(10)}{9!\Gamma(41/4)}x^{37/4}$$

$$v_{k+1}(x) = J^{1/4}\left[v_k(x)\right]$$

After determining the components $u_k(x)$ and $v_k(x)$ for $k = 0, 1, \ldots$; up to 10 terms and substituting them into eq. (3.13) one can get the approximate solutions of u(x) and v(x) for $q = \frac{1}{4}$.

Following Tables (8) and (9) represent the approximate solution of u(x) and v(x) when $q = \frac{1}{4}$.

Table (8)

The approximate solution of u(x), when q = $\frac{1}{4}$.

x	Approximate solution of u(x)
0.1	2.211092121
0.2	3.315848939
0.3	4.646807546
0.4	6.34597496
0.5	8.174991978
0.6	10.438230868
0.7	13.070609363
0.8	16.094584143
0.9	19.528479512
1	23.386085215

Table (9)

The approximate solution of v(x), when q = $\frac{1}{4}$.

x	Approximate solution of v(x)
0.1	2.464330552
0.2	3.318012629
0.3	4.21280831
0.4	5.183356657
0.5	6.240572066
0.6	7.388237238
0.7	8.626851519
0.8	9.954951053
0.9	11.369683314
1	12.8671116036

CONCLUSIONS AND FUTURE WORK

From the present work, we can conclude the following:

1. The (DTM) and (ADM) reduces the computational difficulties of many numerical methods such as Homotopy analysis method if it's used to solve fractional differential-algebraic equations and all the calculations can be made very simple.

2. As we see in chapters two and three the differential transform method is more accurate than the (ADM) because in some times we need to evaluate fractional order differentiation or integration of some special functions such as sin(x) in the (ADM) and this is difficult to do, and as a technique to avoid this difficulty we approximate these functions by taking its Maclurian series, while in the (DTM) we do not need to do this process.

3. The (DTM) and (ADM) can be applied to other types of non-linear problems in the field of fractional calculus.

Also, we can recommend the following problems as a future work:

1. Studying the numerical solution of stochastic differential-algebraic equations of fractional order.

2. Studying the numerical solution of partial fractional differential-algebraic equations.

3. Studying the numerical solution of fractional delay differential-algebraic equations.

REFERENCES

[1]Abdulkhalik F., (2008), "Solution of Fractional Order Delay Differential Equations", M.Sc. Thesis, Department of Mathematics and Computer Application, Al-Nahrain University, Baghdad, Iraq.

[2]Arikoglu A. and Ozkol I., (2009), "Solution of fractional integro-differential equations by using fractional differential transform method", Chaos, Solution and Fractals 40 pp. (521-529).

[3]Asher M. and Petzold L., (1998), "Computer methods for ordinary differential equations and differential-equations", by the society for industrial and Applied Mathematics.

[4]Ayaz F., (2003), "On the two-dimensional transform method", Applied Mathematics and Computation", 143, pp.(361-374).

[5]Ayaz F., (2004), "Application of differential transform method to differential-algebraic equation", Applied Mathematics and Comput., 152, pp.(649-57).

[6]Azizi S., Gordon B. and Ramachandran V., (2006), "A Decentralized Sliding Control Approach for Distribution Simulation of Differential-Algebraic Equation Systems", IEEE J. Vol. 26, No.7, pp.1166-1187.

[7]Bratsos A., and Ehrhardt M., (2007), "A discrete Adomian decomposition method for the discrete nonlinear Schrödinger equation", Appl. Math. And Comput.

[8]Brenan K. E., Cambell S. L. and Petzold L. R., (1996), "Numerical Solution of Initial-Value Problems in DAEs", Classics in Applied Mathematics, Vol. 14, SIAM, Philadelphia PA.

[9] Brenan K., Campbell S. and Petzold L. R., (1989), "Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations", New York, North Holland.

[10] Campbell S. and Linh V., (2009), "Stability Criteria for differential-algebraic equations with multiple delays and their numerical solutions", Applied Math. and Computation 208 pp.(397-415).

[11] Caputo M., (1967), "Linear models of dissipation whose Q is almost frequency independent II", Geophys. J. Roy. Astronom. Soc. 13, pp. (529-539).

[12] Celik E., Bayram M. and Yeloglu T., (2006), "Solution of differential-algebraic equations by Adomian Decomposition Method", Journal Pure and Appl. Math. ISSN 0972-9828 Vol.3 No.1, pp. (93-100).

[13] Chen C. K. and Ho S. H., (1999), "Solving partial differential equations by two dimensional differential transform method", Applied Mathematics and Computation, 106, pp. (171-179).

[14] Chen C. K. and Ju S. P., (2004), "Application of differential transformation to transient a dvective-dispersive transport equation", Applied Mathematics and computation, 155, pp. (25-38).

[15] Churchill R. V., (1948), "Operational Mathematics", McGraw-Hill, New York.

[16] Dan N., Edward J. and Horatiu C., (1999), "An Implicit Runge-Kutta Method for Integration of Differential-Algebraic Equations of Multibody Dynamics", Technical Report, the University of Iowa.

[17] Danielle C. and Harry H., (2002), "On Nature and Stability of Differential-Algebraic Systems", Proceeding of the American Control Conference Anchorage.

[18] El-Kala I. L., (2007), "Error analysis of Adomian series solution to a class of non-linear differential equations", ISSN, pp.(1607-2510).

[19] El-Sayed S. M. and Kaya D., (2004), "On the numerical solution of the system of two-dimensional Burgers equations by decomposition method", Appl. Math. and Comput. Vol. 158, pp. (101-109).

[20] Ertürk V. S., and Momani S., (2008), "Solving system of fractional differential equations using differential transform method", Journal of Computational and Applied Mathematics 215, pp. (142-151).

[21] Ertürk V. S., and Momani S., (2007), "Comparing numerical methods for solving fourth-order boundary value problems", Appl. Math. Comput.,188(2), pp. (1963-8).

[22] Feng S. and Ji P., (1997), "Differential Evolution for Dynamic Optimization of Differential-Algebraic Systems", IEEE, pp. (7803-3949).

[23] Gear C., (1970), "The Simultaneous Numerical Solution of Differential-algebraic equations", SLAC-PUB-723 (Revised).

[24] Griepentrog E. and März R., (1986), "Differential-Algebraic Equations and Their Numerical Treatment", Teubner-Texte zur Mathematics, Vol. 88, Teubner, Leipzing.

[25] Hank M., (1987), "On the Regularization of Index-2 Differential-Algebraic Equations", Section Mathematics, Berlin.

[26] Hariharan K. and Harris M., (1983), "On the Connection between Nonlinear Differential-Algebraic Equation and Singularly Perturbed Control Systems In Nonstandard form", Proceeding of the 32nd Conference on Decision and Control San Antanio, Texas December.

[27] Hassan IHA., (2002), "On solving some eigenvalue problems by using a differential transformation", Applied Mathematics and Computation, 127, pp. (1-22).

[28] Hassan IHA., (2008), "Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems", Chaos Solutions Fract: 36(1); pp. (53-65).

[29] He J. H., (1998), "Nonlinear oscillation with fractional derivative and its applications", International Conference on Vibrating Engineering'98, Dalian ,China, pp. (288-291).

[30] He J. H., (1999), "Some applications of nonlinear fractional differential equations and their approximation", Bull, Sci. Technol., 15(2), pp. (86-90).

[31] Ho S. H., and Chen C. K., (1998), "Analysis of general elastically end restrained non-uniform beams using differential transform", Appl. Math. Mod., 22, pp. (219-234).

[32] Hosseini M. M., (2006) a, "Adomian decomposition method for solution of differential-algebraic equations", J. Comput. Appl. Math. in press. 197 pp. (495-501).

[33] Hosseini M. M., (2006) b, "Adomian decomposition method for solution of nonlinear differential-algebraic equations", Applied Math. and Computation 181pp.(1737-1744).

[34] Inc M., (2005), "On the numerical solutions of one-dimensional nonlinear Burgers equations and convergence of the decomposition method", Appl. Math. Comput. Vol.170, pp. (76-85).

[35] Jang M. J., Chen C. L. and Lity Y. C., (2001), "Two-dimensional differential transform for partial differential equations", Applied Mathematics and Computation, 121, pp. (261-270).

[36] Johan S., (2008), "Optimal Control and Model Reduction of Nonlinear Differential-Algebraic Equation Models", Linköping studies in science and technology Dissertations No.1166, Linköping University.

[37] Khaleel A., (2008), "Some Methods for Solving the Differential-Algebraic Equations", M.Sc. Thesis, Department of Mathematics and Computer Application, Al-Nahrain University, Baghdad, Iraq.

[38] Kowankar K. M. and Gangal A. D., (1996), "Fractional Differentiability of nowhere differentiable functions and dimensions", CHAOS V.6, No. 4, American Institute of Physics.

[39] Kunkel P. and Mehrmann V, (2006), "Analytical and Numerical Treatment of Initial Value problems for Differential-Algebraic Equations", European Mathematical Society.

[40] Kunkel P. and Mehrmann V., (2007), "Stability Properties of Differential-Algebraic Equation tability properties of differential-

algebraic equation and spin-stabilized discretization", Electronic J. Vol.26, pp. (385-420).

[41] Laurent O. Jay and Dan N., (2007), "A second order extension of the generalized-α method for constrained systems in mechanics", Milano, Italy pp.(25-28), June.

[42] Liu H., and Song Y., (2007), "Differential transform method applied to high index differential-algebraic equations", Appl. Math. Comput., 184: pp.(748-53).

[43] Loverro A., (2004), Fractional Calculus: History, "Definition and Application for Engineer", Department of Aerospace and Mechanical Engineering University of Notre Dame, IN 46556, USA, May 8,.

[44] Lubich C. and Hairer E., (1988), "Extrapolation at Differential-Algebraic equations", Numer-Math. Vol.52, pp.(377-400).

[45] Macdonald C., (2001), "The Predicted Sequential Regularization Method for Differential-Algebraic Equations", M.Sc. Thesis, Ince Direct Vol.129, No.11, pp. (1201-1298), Acadia University (2001).

[46] Minardi F., (1997), "Fractional Calculus: Some basic problems in continuum and statistical mechanics", A carpinteri and F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, pp.(291-348) New York: Springer-Verlag.

[47] Mittal R. C. and Nigam R., (2008),"Solution of fractional integro-differential equations by Adomian decomposition method", Int. J. Appl. Math. and Mech. 4(2): pp.(87-94).

[48] Momani S. and Jafari H., (2008) ,"Numerical study of System of Fractional Differential Equations by Using the Decomposition Method", Southeast Asian Bulletin of Mathematics 32: pp.(721-730).

[49] Murry R., (1965), "Laplace Transform Theory and Problems", New York.

[50] Nazari D. and Shahmorad S., (2010), "Application of the Fractional Differential Transform Method to fractional Order Integro-Differential Equations with Non local Boundary Conditions", journal of computational and applied Math. 234, pp. (883-891).

[51] Nguyen H., (2007), "Stability Radii of Differential-Algebraic Equations with structured Perturbations", International Journal Pure and Applied Mathematical Science, Vol.5, No.1, pp.(77-85).

[52] Nishimoto K., (1983), Fractional Calculus: "Integrations and Differentiations of Arbitrary Order", Descartes Press Co. Koriyama Japan.

[53] Nishimoto K., (1991), "An essence of Nishimoto’s Fractional Calculus", Descartes Press Co.

[54] Odibat Z. M., (2008), "Differential transform method for solving Volterra integral equation with separable kernels", Mathematics and Computer Modeling 48, pp.(1144-1149).

[55] Oldham K. B. and Spanir J., (1974),"The Fractional Calculus, Academic Press", New York and London.

[56] Pasic H., (1997), "An Algorithm for a Numerical Solution of Differential-Algebraic Equations", Proceedings of the 36th

Conference on Decision & Control San Diego, California USA December.

[57] Peter K. and Volker M., (2005), "Characterizations of classes of singular linear differential-algebraic equations", Electronic Journal of Linear Algebra ISSN 1081-3810 A publication of the International Linear Algebra Society Volume 13, pp.(359-386), November.

[58] Petzold L., (1982), "Differential-Algebraic equations are not ordinary differential equation", SIAM J. Sc. Stat. comp. Vol.3, No.3, September.

[59] Podulbny I., (1999), "Fractional Differential Equations, "Mathematics in science and Engineering V198", Academic Press.

[60] Rang J. and Angermann L., (2005), "Perturbation index of linear partial-algebraic equations", Applied Numerical Mathematics 53, pp(437-456).

[61] Ray S. S. and Bera R. K., (2004), "Solution of an extraordinary differential equation by Adomian decomposition method". J. Appl. Math. Vol.4, pp. (331-338).

[62] Schwerin R., (1999), Multibody System Simulation: "Numerical Methods", Algorithms, and Software, Lecture notes in computational science and engineering, 7, Springer-Verlag, Berlin.

[63] Seng V., Abbaoui K. and Cherruantly Y., (1996), "Adomian polynomials for non-linear operators", Math. Comput. Model. Vol.24, No.1, pp. (59-65).

[64] Shengtai Li. and Petzold L., (2000), "Software and algorithms for sensitivity analysis of large-scale differential-algebraic systems", Journal of Computational and Applied Math. Volume 125, Issues 1-2, 15 December pp.(131-146).

[65] Siraj-Ul Islam, Sirajul Haq and Javid Ali, (2009), "Numerical solution of special-12th-order boundary value problems using differential transform method", Commun nonlinear Sci. Numer. Simulat. 14, pp.(1132-1138).

[66] Starner J., (1976), "A Numerical Algorithm for the solution of implicit Differential-Algebraic Systems of equations", Ph.D. thesis, Dept. of Mathematics and Statistics, University of New Mexico.

[67] Tari A., Rahimi M. Y., Shah Morad S. and Talati F., (2009), "Solving a class of two-dimensional linear and nonlinear Volterra integral equations by the differential transform method", journal of Computational and Applied Math. 228, pp.(70-76).

[68] Wenjie Z. and Petzold L., (1998), "Asymptotic Stability of Hessenberg Delay Differential-Algebraic Equations of Retarded or Neutral Type, Applied Numerical Mathematics", Vol.27, pp.(309-325).

[69] Yang C., Shengtai L., Petzold L. and Radu S., (2003), "Adjoint Sensitivity Analysis for Differential-Algebraic Equations: The Adjoint DAE System and its Numerical Solution", SIAM J. Sci. Computer., Vol.24, No.3, pp.(1076-1089) .

[70] Zhou J. K., (1986), "Differential Transformation and its Applications for Electrical Circuits", Huazhong University Press, Wuhan, China, (in Chinese).

[71] Zuigat M., Momani S. and Alawneh A., (2010), "Analytical approximate solutions of systems of fractional algebraic-differential equations by homotopy analysis method", Computes and Mathematics with applications 59, pp.(1227-1235).

المستخلص

الغرض الرئيسي لهذه الرسالة يدور حول هدفين:

الهدف الاول هو دراسة المبادئ الاساسـية للمعـادلات التفاضـلية الجبريـة (Differential-Algebraic Equations) وحسبان التفاضل والتكامـل ذات الرتـب الكسرية (Fractional Calculus) والتي سوف نحتاج اليها عند ايجاد الحل التقريبـي للمعادلات التفاضلية الجبرية ذات الرتب الكسرية.

الهدف الثاني هو تقريب حل العادلات التفاضلية-الجبرية ذات الرتب الكسرية باسـتخدام طريقتـان تقريبيتـان همـا طريقـة التحويـل التفاضـلي (Differential Transform Method) وطريقـة ادومـين للتجزئـة (Adomian Decomposition Method) كلتا الطريقتين تمثل الحل علـى شـكل متسلسلة لانهائية والتي يمكن حساب معاملاتها بشكل سهل جدا.

جمهـورية العـراق
وزارة التعليم العالي والبحث العلمي
جــامعة النهـــريـن
كليــة العلوم
قسم الريـاضيات وتطبيقات الحاسوب

حلول تقريبية للمعادلات التفاضلية الجبرية ذات الرتب الكسرية

رسالة
مقدمة إلى كلية العلوم - جامعة النهرين
وهي جزء من متطلبات نيل درجة ماجستير علوم
في الرياضيات

من قبل
زهراء شاكر شايع العبيدي
(بكالوريوس رياضيات / كلية العلوم /، جامعة النهرين، ٢٠٠٨)

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ربيع الاول   شباط
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