Delays. Propagation. Conservation Laws.

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Abstract. Since the very first paper of J. Bernoulli in 1728, a connection exists between initial boundary value problems for hyperbolic Partial Differential Equations (PDE) in the plane (with a single space coordinate accounting for wave propagation) and some associated Functional Equations (FE). The functional equations may be difference equations (in continuous time), delay-differential (mostly of neutral type) or even integral/integro-differential. It is possible to discuss dynamics and control either for PDE or FE since both may be viewed as self contained mathematical objects. A more recent topic is control of systems displaying conservation laws. Conservation laws are described by *nonlinear* hyperbolic PDE belonging to the class "lossless" (conservative). It is not without interest to discuss association of some FE. Lossless implies usually distortionless propagation hence one would expect here also lumped time delays. The paper contains some illustrating applications from various fields: nuclear reactors with circulating fuel, canal flows control, overhead crane, without forgetting the standard classical example of the nonhomogeneous transmission lines for distortionless and lossless propagation. Specific features of the control models are discussed in connection with the control approach wherever it applies.

1 Introduction and Basics

We shall start from two elementary facts. First, any electrical or control engineer has dealt with mathematical models where either a complex domain term like $e^{-\tau s}$ with $\tau > 0$, $s \in \mathbb{C}$, or a time domain term like $u(t - \tau)$, where u was some signal, were present. Such models were called *time delay* or *time lag systems*. A more involved interest to such systems would inevitably have sent to some reference about

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the underlying equations of these models - the *equations with deviating argument*. A still more involved interest would concern origins of these equations: the first differential equation with deviating argument, reported in [16], was published by Johann (Jean) Bernoulli in 1728 [2] and reads as

$$y'(t) = y(t-1) \tag{1}$$

As the title of this paper shows, this equation appears to be associated to a partial differential equation of hyperbolic type - the string equation; it thus sends to the second elementary fact, less known, that propagation is associated to time delay. In order to explain this, we shall discuss a special case of propagation - the *lossless propagation*. By lossless propagation it is understood the phenomenon associated with long (in a definite sense) transmission lines for physical signals. In electrical and electronic engineering there are considered in various applications circuit structures consisting of multipoles connected through LC transmission lines (A long list of references may be provided, starting with [3] and going up to a quite recent book [13]). The lossless propagation occurs also for non-electric signals as water, steam or gas flows and pressures - see e.g. the pioneering papers of [10], [11] on steam pipes for combined heat-electricity generation, the papers dealing with waterhammer and many other. In order to illustrate these assertions, we shall consider one of the early benchmark problems, the nonlinear circuit containing a tunnel diode and a lossless transmission - the so called Nagumo-Shimura circuit described by

$$L\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial \lambda} \, , \, C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial \lambda}, \, 0 \le \lambda \le 1$$

$$E = v(0,t) + R_0 i(0,t) \, , \, -C_0\frac{\mathrm{d}}{\mathrm{d}t}v(1,t) = -i(1,t) + \psi(v(1,t)) \tag{2}$$

Proceeding in "an engineering way" we may apply formally the Laplace transform to compute the solution of the boundary value problem viewed as independent of the differential equation but being nevertheless controlled by it. A similar approach of applying formally the Laplace transform and deducing a characteristic equation accounting for time delays (deviating arguments) was used in the pioneering papers [10, 11, 21] dealing with steam pipes; for water pipes a pioneering paper is [22] where the same approach is applied.

Continuing the investigation of the above benchmark system one may observe that the aggregate

$$v(t) + \sqrt{\frac{L}{C}}i(1,t) \equiv u(1,t) + \sqrt{\frac{L}{C}}i(1,t)$$

represents the so called progressive (forward) wave of the system at the boundary $\lambda = 1$. Both the voltage $u(\lambda,t)$ and the current $i(\lambda,t)$ are linear combinations of the

progressive (forward) and reflected (backward) waves hence it is useful to express (2) in terms of these waves

$$\frac{\partial u_1}{\partial t} + \frac{1}{\sqrt{LC}}\frac{\partial u_1}{\partial \lambda} = 0 \ , \ \frac{\partial u_2}{\partial t} - \frac{1}{\sqrt{LC}}\frac{\partial u_2}{\partial \lambda} = 0$$

$$(1+R_0\sqrt{C/L})u_1(0,t) + (1-R_0\sqrt{C/L})u_2(0,t) = 2E(t) \tag{3}$$

$$u_1(1,t) + u_2(1,t) = 2v(t) \ , \ C_0\frac{dv}{dt} + \psi(v) = \frac{1}{2}\sqrt{\frac{C}{L}}[u_1(\lambda,t) - u_2(\lambda,t)]$$

The propagation (partial differential) equations of the two waves are decoupled; the two waves are exactly the Riemann invariants of the problem. We may consider now the standard version of the d'Alembert method i.e. of integrating along the two families of characteristics; there is a family of increasing characteristics and one of decreasing; as (3) shows, the forward wave should be considered along the increasing characteristics while the backward wave along the decreasing ones. If we perform this integration we shall find

$$u_1(0,t) = u_1(1,t+\sqrt{LC}) \ , \ u_2(1,t) = u_2(0,t+\sqrt{LC}) \tag{4}$$

By denoting $\eta_1(t) = u_1(1,t)$, $\eta_2(t) = u_1(1,t)$ we associate to (2) the following system of equations with delayed argument

$$C_0\frac{dv}{dt} + \psi(v) = \frac{1}{2}\sqrt{\frac{C}{L}}(\eta_1(t) - \eta_2(t-\sqrt{LC})),$$

$$\eta_2(t) = -\rho_0\eta_1(t-\sqrt{LC}) + (1+\rho_0)E(t) \ , \ \eta_1(t) = -\eta_2(t-\sqrt{LC}) + 2v(t) \tag{5}$$

associated in a rigorous way, starting from the solutions of (2); even the initial conditions may be associated in this way. Moreover, the converse association is also possible. Using the representation formulae for the two waves

$$u_1(\lambda,t) = \eta_1(t+(1-\lambda)\sqrt{LC}) \ , \ u_2(\lambda,t) = \eta_2(t+\lambda\sqrt{LC}) \tag{6}$$

we may construct the solutions of (2) starting from the solutions of (5).

To end this introductory discussion we just mention that (4) and (6) define what is usually known as *lossless propagation*. Since the two waves propagate from one boundary to the other in finite time $\tau = \sqrt{LC}$ but without changing their waveform (just with a pure - lumped - time delay) this propagation is also distortionless. In the following we shall discuss both these aspects.

2 Lossless and Distortionless Propagation

A. We shall consider again the Nagumo-Shimura circuit but with a lossy transmission line

$$\frac{\partial u}{\partial \lambda} + L\frac{\partial i}{\partial t} + Ri = 0 \ , \ \frac{\partial i}{\partial \lambda} + C\frac{\partial u}{\partial t} + Gu = 0$$

$$R_0 i(0,t) + u(0,t) = E(t) \ , \ u_1(t) = v(t) \ , \ C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = i(1,t) \tag{7}$$

Introducing the forward and backward waves as previously we find their propagation equations which are no longer decoupled unless the "matching" condition of Heaviside is met i.e. $RC = LG$ which "destroys" the coupling terms. Introducing the new "waves"

$$u_1(\lambda,t) = \mathrm{e}^{-\delta\lambda}w_1(\lambda,t) \ , \ u_2(\lambda,t) = \mathrm{e}^{\delta\lambda}w_2(\lambda,t) \ , \ \delta = R\sqrt{C/L} \tag{8}$$

we obtain a lossless-like system. Moreover, denoting

$$\eta_1(t) = w_1(1,t) \ , \ \eta_2(t) = w_2(0,t) \tag{9}$$

we associate the system

$$C_0\frac{\mathrm{d}v}{\mathrm{d}t} + \psi(v) = \frac{1}{2}\sqrt{\frac{C}{L}}[\mathrm{e}^{-\delta}\eta_1(t) - \mathrm{e}^{\delta}\eta_2(t - \sqrt{LC})]$$

$$\eta_2(t) = -\rho_0\eta_1(t - \sqrt{LC}) + (1 + \rho_0)E(t) \tag{10}$$

$$\eta_1(t) = -\mathrm{e}^{-2\delta}\eta_2(t - \sqrt{LC}) + 2\mathrm{e}^{-\delta}v(t)$$

and an additional damping is introduced in the second difference equation. Adapting (6) to the new case, we have

$$u_1(\lambda,t) = \mathrm{e}^{-\delta\lambda}\eta_1(t + (1 - \lambda)\sqrt{LC}) \ , \ u_2(\lambda,t) = \mathrm{e}^{\delta\lambda}\eta_2(t + \lambda\sqrt{LC}) \tag{11}$$

and it is easily seen that the progressive wave propagates forwards from $\lambda = 0$ to $\lambda = 1$ being retarded and damped along the propagation while the reflected wave propagates backwards from $\lambda = 1$ to $\lambda = 0$ being also retarded and damped. Since the basic waveforms $\eta_i(\cdot)$ are not modified but just retarded during propagation, the propagation is also distortionless.

B. The natural development of the distortionless propagation is to consider the so called inhomogeneous media and transmission lines. The theory of the waveguides is their most straightforward application. The mathematical model of the inhomogeneous transmission line is given by the space varying telegraph equations [4]

$$-\frac{\partial v}{\partial \lambda} = r(\lambda)i(\lambda,t) + l(\lambda)\frac{\partial i}{\partial t} \ , \ -\frac{\partial i}{\partial \lambda} = g(\lambda)v(\lambda,t) + c(\lambda)\frac{\partial v}{\partial t} \tag{12}$$

with the standard notations, the line having length L. Here $l(\lambda) > 0$, $c(\lambda) > 0$ for standard physical reasons. The distortionless definition (*op. cit.*) states that

$$v(\lambda,t) = f(\lambda)\phi(t - \tau(\lambda)) \tag{13}$$

where $f(\cdot)$ is called *attenuation* and $\tau(\cdot)$ is called *propagation delay* while $\phi(\cdot)$ is the waveform. There exist also other cases of interest, for instance the time independent voltage/curent ratio i.e. when the line is resistive. Our approach includes these cases in the general setting of the distortionless propagation. Since (12) are exactly like (7), we introduce the Riemann invariants by

$$u^{\pm}(\lambda,t) = v(\lambda,t) \pm a(\lambda)i(\lambda,t) \tag{14}$$

or by the converse equalities; with the choice $a(\lambda) = \sqrt{l(\lambda)/c(\lambda)}$ which is similar to (8) the cross derivative terms are "destroyed" and the equations for the forward and backward waves are obtained. It appears in these equations that the coupling terms cannot be canceled by a unique choice of the line coefficients. This explains the option in [4] for the distortionless propagation forwards: such choice requires decoupling of the progressive wave $u^{+}(\lambda,t)$. Therefore

$$a'(\lambda) = r(\lambda) - g(\lambda)a^{2}(\lambda) \tag{15}$$

which is a condition on line's parameters. Remark that *this is a Riccati differential equation*. Consequently the equations of the waves become

$$
\begin{aligned}
-\frac{\partial u^{+}}{\partial \lambda} &= \sqrt{l(\lambda)c(\lambda)}\frac{\partial u^{+}}{\partial t} + a(\lambda)g(\lambda)u^{+}(\lambda,t) \\
-\frac{\partial u^{-}}{\partial \lambda} &= -\sqrt{l(\lambda)c(\lambda)}\frac{\partial u^{-}}{\partial t} - \frac{r(\lambda)}{a(\lambda)}u^{-}(\lambda,t) + \\
&\quad + (r(\lambda)/a(\lambda) - g(\lambda)a(\lambda))u^{+}(\lambda,t)
\end{aligned}
\tag{16}
$$

Having in mind (8) we introduce the new "waves" by

$$u^{+}(\lambda,t) = \exp\left(-\int_{0}^{\lambda}g(\sigma)a(\sigma)d\sigma\right)w^{+}(\lambda,t)$$

$$u^{-}(\lambda,t) = \exp\left(-\int_{\lambda}^{1}(r(\sigma)/a(\sigma)d\sigma\right)w^{-}(\lambda,t)$$

to obtain with the corresponding notation

$$-\frac{\partial w^{+}}{\partial \lambda} = \sqrt{l(\lambda)c(\lambda)}\frac{\partial w^{+}}{\partial t} \,,\, \frac{\partial w^{-}}{\partial \lambda} = \sqrt{l(\lambda)c(\lambda)}\frac{\partial w^{-}}{\partial t} + \beta(\lambda)w^{+}(\lambda,t) \tag{17}$$

We perform now integration along the characteristics to find

$$w^{+}(0,t) = w^{+}(L,t+\tau) \tag{18}$$

Denoting $\eta^{+}(t) = w^{+}(L,t)$ the following representation formula is obtained

$$w^+(\lambda,t) = \eta^+\left(t + \int_\lambda^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right) \tag{19}$$

obviously accounting for distortionless propagation of the forward wave. For the backward wave we obtain, by integrating along the decreasing characteristics but taking also into account (19)

$$w^-(L,t) = w^-(0,t+\tau) + \int_0^L \beta(\sigma)\eta^+\left(t + 2\int_\sigma^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right)\mathrm{d}\sigma \tag{20}$$

Denoting $\eta^-(t) = w^-(0,t+\tau)$ the following representation formula is obtained

$$\begin{aligned}
w^-(\lambda,t) &= \eta^-\left(t + \int_0^\lambda \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right) + \\
&+ \int_0^\lambda \beta(\sigma)\eta^+\left(t + \int_\sigma^\lambda \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu + \int_\sigma^L \sqrt{l(\mu)c(\mu)}\mathrm{d}\mu\right)\mathrm{d}\sigma
\end{aligned} \tag{21}$$

and the propagation is clearly associated with the distortions introduced by the integral term. To obtain distortionless of the backward wave, it is necessary to have $\beta(\lambda) = 0$ a.e. This will give finally $a'(\lambda) = 0$ hence the ratio $l(\lambda)/c(\lambda)$ *has to be piecewise constant on* $(0,L)$. *Not only constant coefficients can ensure distortionless propagation for both forward and backward waves!*

3 The Multi-wave Case. Application to the Circulating Fuel Nuclear Reactors

When several transmission lines (channels) are included in the system, several couples of waves are present, leading to the model of e.g. [20]

$$\frac{\partial u}{\partial t} + A(\lambda)\frac{\partial u}{\partial \lambda} = B(\lambda)u \, , \, t > 0 \, , \, 0 \le \lambda \le L \tag{22}$$

where u is a m-dimensional vector and $A(\lambda)$, $B(\lambda)$ are $m \times m$ matrices. Also A is supposed diagonal, having distinct diagonal elements, of which k are strictly positive (corresponding to the forward waves) and $m-k$ are strictly negative (corresponding to the backward waves). If $B(\lambda)$ could be also diagonal then propagation would be distortionless, otherwise it is not.

There exist situations when this diagonal structure is inherent to the basic equations, for instance, the model of the circulating fuel nuclear reactor [7, 8, 9]

$$\begin{cases}
\dfrac{d}{dt}n(t) = \rho n(t) + \displaystyle\sum_{i=1}^m \beta_i(\bar{c}_i(t) - n(t)) \, , \, t \ge t_0 \, , \, 0 \le \eta \le h \\
\bar{c}_i(t) = \displaystyle\int_0^h \phi(\eta)c_i(\eta,t)d\eta \, , \dfrac{\partial c_i}{\partial t} + \dfrac{\partial c_i}{\partial \eta} + \sigma_i c_i = \sigma_i\phi(\eta)n(t) \, , \, i = \overline{1,m} \\
c_i(0,t) = c_i(h,t) \, , \, c_i(\eta,t_0) = q_i^0(\eta) \, , \, n(t_0) = n_0 \, ; \, i = \overline{1,m}.
\end{cases} \tag{23}$$

The boundary conditions are of periodic type. The PDE (partial differential equations) are completely decoupled, the coupling taking place at the level of the differential equation. All eigenvalues of $A(\lambda) = I$ are equal and positive hence there exist m forward waves. Integration along the characteristics and computation of the integral of (23) will give again a system of FDE (functional differential equations) of neutral type [14], where it may be seen that propagation is distortionless.

4   A Control Problem

We shall discuss here the control model of an overhead crane with a flexible cable, given by [6]

$$y_{tt} - (a(s)y_s)_s = 0 , \; t > 0 , \; 0 < s < L ; \; a(s) = g(s + \frac{m}{\rho})$$

$$y_{tt}(0,t) = gy_s(0,t) , \; y(L,t) = X_p(t) \tag{24}$$

$$\ddot{X}_P = K(a(s)y(s,t))(L,t) + u(t) , \; K = \frac{m + \rho L}{ma(L)} = \frac{\rho}{Mg}$$

It will appear in what follows that the nonhomogeneous material properties account for propagation with distortions. But we have to mention first that the basic model of (*op.cit.*) contained the boundary condition $y_s(0,t) = 0$, explained by the physical assumption that the acceleration of the load mass is negligible with respect to the gravitational acceleration g i.e. $y_{tt}(0,t)/g \approx 0$; in fact this is not rigorous and definitely cannot be ascertained for all t; the only valid argument is connected to singular perturbations. For this reason we shall deal with the complete model (24).

If the rated cable length variable $\sigma = s/L$ is introduced, then, with a slight abuse of notation, the following model containing possible small parameters is obtained

$$\frac{L}{g} \cdot \frac{\rho L}{m} y_{tt} - \left(\left(1 + \frac{\rho L}{m} \sigma \right) y_\sigma \right) = 0 , \; 0 \le \sigma \le 1 , \; t > 0$$

$$\frac{L}{g} y_{tt}(0,t) = y_\sigma(0,t) , \; y(1,t) = X_p(t) , \; \frac{L}{g}\ddot{X}_P = \frac{m}{M}\left(1 + \frac{\rho L}{M} \right) y_\sigma(1,t) + \frac{L}{g}u(t) \tag{25}$$

A preliminary comment is useful: supposing we would like to neglect nonuniformity of the cable parameters, this would require the assumption that the cable mass is negligible with respect to the carried mass i.e. $\rho L/m \approx 0$. However, this will destroy the entire distributed dynamics since (25) would become

$$y_{\sigma\sigma} = 0 ; \; \frac{L}{g}y_{tt}(0,t) = y_\sigma(0,t) , \; y(1,t) = X_P$$

$$\frac{L}{g}\ddot{X}_P = \frac{m}{M}y_\sigma(1,t) + \frac{L}{g}u(t) \tag{26}$$

We shall then have $y(\sigma,t) = \phi_1(t)\sigma + \phi_0(t)$ which is substituted in the boundary conditions. Therefore

$$\frac{L}{g}\ddot{\phi}_0 + \phi_0 = X_p \; ; \; \phi_1 = X_p - \phi_0 \; , \; \frac{L}{g}\ddot{X}_p = \frac{m}{M}(X_p - \phi_0) + \frac{L}{g}u(t) \qquad (27)$$

Its uncontrolled dynamics is given by the roots of the characteristic equation

$$\frac{L}{g}s^2\left(\frac{L}{g}s^2 + 1 - \frac{m}{M}\right) = 0 \qquad (28)$$

i.e. by two purely imaginary modes and a double zero mode; this is but well known. Instead of this approach, we start by introducing new functions and by making some other notations

$$v(\sigma,t) := y_t(\sigma,t) \; , \; w(\sigma,t) := (1 + \gamma_0\sigma)y_\sigma(\sigma,t) \; , \; \gamma_0 = \frac{\rho L}{m} \; , \; T^2 = \frac{L}{g} \; , \; \delta_0 = \frac{m}{M}$$
$$(29)$$

Define further the forward and backward waves as below

$$v(\sigma,t) = u^+(\sigma,t) + u^-(\sigma,t) \; , \; w(\sigma,t) = T\sqrt{\gamma_0(1+\gamma_0\sigma)}(u^-(\sigma,t) - u^+(\sigma,t))$$

The following equations are then obtained

$$\frac{\partial u^+}{\partial t} + \frac{1}{T\sqrt{\gamma_0}}\sqrt{(1+\gamma_0\sigma)}\frac{\partial u^+}{\partial t} = \frac{1}{T\sqrt{\gamma_0}} \cdot \frac{\gamma_0}{\sqrt{(1+\gamma_0\sigma)}}(u^- - u^+)$$

$$\frac{\partial u^-}{\partial t} - \frac{1}{T\sqrt{\gamma_0}}\sqrt{(1+\gamma_0\sigma)}\frac{\partial u^-}{\partial t} = \frac{1}{T\sqrt{\gamma_0}} \cdot \frac{\gamma_0}{\sqrt{(1+\gamma_0\sigma)}}(u^- - u^+) \qquad (30)$$

$$T(u_t^- + u_t^+)(0,t) = \sqrt{\gamma_0}(u^-(0,t) - u^+(0,t)) \quad u^-(1,t) + u^+(1,t) = \dot{X}_p$$

$$T\ddot{X}_p = \delta_0\sqrt{\gamma_0(1+\gamma_0)}(u^-(1,t) - u^+(1,t)) + Tu(t)$$

It is clear that *under no conditions can be made this system distortionless*. We may however try to replace this system by an approximation which would be such. To find such an approximation, we turn back to the basic equation of (24) where $a(\cdot)$ is a sufficiently smooth function. With the new variables

$$v(s,t) = y_t(s,t) \; , \; w(s,t) = a(s)y_s(s,t)$$

there are obtained the first order equations of the propagation; the forward and backward waves are defined by

$$v(s,t) = u^-(s,t) + u^+(s,t) \; , \; w(s,t) = \sqrt{a(s)}(u^-(s,t) - u^+(s,t))$$

and satisfy

$$u_t^{\pm} \pm \sqrt{a(s)}u_s^{\pm} = \frac{a'(s)}{4\sqrt{a(s)}}(u^- - u^+) \tag{31}$$

Obviously the distortionless condition is $a'(s) = 0$ a.e., but in our case $a'(s) = g \neq 0$. The piecewise constant approximation is thus the only suitable. This means approximation of $a(s)$ piecewise constantly in order that e.g. the propagation time should remain constant

$$\int_0^L \frac{d\lambda}{\sqrt{a(\lambda)}} = \sum_1^N \frac{l_i}{\sqrt{a_i}} \ , \ \sum_1^N l_i = L \tag{32}$$

We shall not discuss here specific approximation problems such as concatenation conditions and convergence but just take $N = 1$ and write down the associated system. In this simplest case we find

$$T\sqrt{\gamma_0} \int_0^1 \frac{d\sigma}{\sqrt{1 + \gamma_0 \sigma}} = T\sqrt{\gamma_0} \frac{1}{\sqrt{1 + \gamma_1}} \tag{33}$$

If $T_d = T\sqrt{\gamma_0/(1 + \gamma_1)}$ - the propagation time - is introduced and the cyclic variable X_p is eliminated, we obtain a genuine system of neutral type

$$\begin{aligned}
&T_d \frac{d}{dt}(y^+(t) + y^-(t - T_d)) = -\gamma_0(y^+(t) - y^-(t - T_d)) \\
&T_d \frac{d}{dt}(y^-(t) + y^+(t - T_d)) = \gamma_0 \delta_0(y^-(t) - y^+(t - T_d)) + T_d u(t) \\
&\dot{X}_p = y^-(t) + y^+(t - T_d)
\end{aligned} \tag{34}$$

with adequate notations [18]. For $u(t) \equiv 0$ the inherent stability of (34) has been studied [18]. Its characteristic equation is

$$(T_d s + \gamma_0)(T_d s - \gamma_0 \delta_0) - (T_d s - \gamma_0)(T_d s + \gamma_0 \delta_0)e^{-2sT_d} = 0 \tag{35}$$

and obviously has a zero root. The output $\dot{X}_p$ being a cyclic variable, we have here a double zero root of the controlled configuration with $u(t)$ as input and X_p as measurable control output. Since (28) has also a pair of of purely imaginary roots, we may check for purely imaginary roots of (35) and find them to be of the form $\pm \iota x_k/Td$ where x_k are the positive roots of

$$\tan x = \frac{\gamma_0(1 - \delta_0)x}{\gamma_0 \delta_0 + x^2} \tag{36}$$

The equation is well studied [19]: it has real roots of the form $k\pi + \delta_k$ where $\{\delta_k\}_k$ is a positive bounded sequence approaching 0 for $k \to \infty$.

We have thus discovered an infinity of purely imaginary roots; this infinity of oscillating modes is well known in the theory of the elastic rods; mathematically, its presence can be explained by the fact that the difference operator of (34) has its roots on $\iota\mathbb{R}$ - the imaginary axis. Other details may be found in [18].

5 Dynamics and Control for Systems of Conservation Laws

A more contemporary trend in the field of control for systems with distributed parameters consists in applying control theory to general structures that may be considered as benchmark problems. Due to their broad applications, the systems of conservation laws which describe various physical phenomena with a single space parameter distribution are very suitable for such applications [12]. The systems of conservation laws are interesting also for their nonlinear character; when linearized they reduce to the quite well propagation equations - see [17] or the previous sections - and, therefore, a comparison to some known results is also available.

We shall consider in this section a system of two conservation laws on $\mathbb{R}^2$ (one space variable) which reads

$$Y_t + f(Y)_x = 0 \tag{37}$$

where $Y : [0,\infty) \times [0,L] \mapsto \Omega \subseteq \mathbb{R}^2$ is the vector of the two dependent variables and $f : \Omega \subseteq \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the *flux density*. The solution is defined by the initial conditions

$$Y(x,0) = Y_0(x) , \; 0 \le x \le L \tag{38}$$

and by some boundary conditions of Dirichlet type that may contain some control input variables

$$g_0(Y(0,t),u_0(t)) = 0 , \; g_L(Y(L,t),u_L(t)) = 0 , \; t > 0 \tag{39}$$

The standard problem we are approaching reads as follows

For constant control actions $u_i(t) \equiv \overline{u}_i$, $i = 0,L$, a steady state solution is a constant solution $\overline{Y}$ satisfying (37) and (39). Depending on the form of the boundary conditions this steady state solution may be stable or unstable. Accordingly it may be stated the *boundary control problem* - that of defining the control inputs $u_i(t)$ from a feedback structure such that for any smooth enough initial condition in (38) the unique smooth solution should converge to a desired steady solution defined by $\overline{u}_i$ - the controllers' set points.

A. Consider the first application - the control of the flows in open canals [1, 5, 15] By choosing the flow velocity $V(x,t)$ and the cross section $A(x,t)$ (instead of the liquid level) as variables, the standard Saint Venant equations are conservation laws. In the simplest case of the *prismatic level canal* whose geometric parameters are independent of the coordinate x and whose bed is lying at the same constant elevation Y_b they are given by

$$\frac{\partial}{\partial t}\begin{pmatrix} A \\ V \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} AV \\ \frac{1}{2}V^2 + g\psi(A) \end{pmatrix} = 0 \tag{40}$$

where $h = \psi(A)$ is the liquid level and ψ is the inverse of the monotone mapping defining the cross section in a prismatic canal

$$\Phi(h) = \int_0^h \sigma(y)\mathrm{d}y$$

(σ is the canal width corresponding to the liquid elevation y). To these equations we may add the boundary conditions which arise from the canal conditions; for constant flow at $x = 0$ and constant level (area) at $x = L$

$$A(0,t)V(0,t) = Q_0 \ , \ A(L,t) = A_L \tag{41}$$

In [15] the existence of physically significant invariant sets was proved. First

$$-F(A_0) < V(x,t) < F(A_0) \ , \ 0 < F(A(x,t)) < 2F(A_0) \ , \ 0 \le x \le L \ , \ t > 0 \quad (42)$$

This shows both limited flow reversals as well as some limitations of the liquid level. It is not quite clear if these conditions may ensure *the invariance of the Froude number* i.e. $\mathfrak{Fr}(A(x,t),V(x,t)) < 1$ provided $\mathfrak{Fr}(A(x,0),V(x,0)) < 1$ thus ensuring sub-criticality of the flow; actually one can hope that initial conditions that are sufficiently far away from the critical limit will generate sub-critical evolutions.

B. We shall address now to a problem that has been considered much earlier (see [17] but also its references). In the technology of combined heat electricity generation there are steam pipes whose dynamics affect the stability of the control systems for basic operating parameters. The traditional approach of pipe dynamics started from the equations of the hydrodynamic flow

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial l} + \frac{1}{\rho}\frac{\partial p}{\partial l} = 0 \ , \ \frac{\partial \rho}{\partial t} + \rho\frac{\partial w}{\partial l} + w\frac{\partial \rho}{\partial l} = 0 \tag{43}$$

where the flow characteristics (velocity w, mass density ρ and steam pressure p) are also related by the polytropic equation

$$p/p_\infty = (\rho/\rho_\infty)^\kappa \tag{44}$$

with the subscript ∞ accounting for steady state values and $\kappa > 1$ being the polytropic exponent. Instead of the linearization we introduce the rated (per cross section area) mass flow $\phi = \rho w$ and eliminate the pressure p to obtain a system of conservation laws

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \phi\end{pmatrix} + \frac{\partial}{\partial l}\begin{pmatrix}\phi \\ \phi^2/\rho + \gamma_\infty\rho^\kappa\end{pmatrix} = 0 \tag{45}$$

where $\gamma_\infty = p_\infty(\rho_\infty)^{-\kappa}$ is the polytropic steady state constant. The boundary conditions are defined by the controlled admission of the steam into the pipe at $l = 0$ and the steam consumption from the pipe at $l = L$, thus being analogous to those of the previous application

$$\phi(0,t) = \sqrt{2}\,\gamma_\infty\eta\,(\rho(0,t))^{\frac{\kappa+1}{2}} \quad , \quad \phi(L,t) = \sqrt{2}\,\gamma_\infty\frac{f(t)}{F}\,(\rho(L,t))^{\frac{\kappa+1}{2}} \tag{46}$$

Here $f(t)$ - the admission cross section of the steam to the consumer acts as a disturbance; since the steam has to be supplied at constant pressure, this pressure has to be the measured output. Taking into account that the controller acts using the control error, the controller equations might be as follows

$$\psi_m^2 \ddot{\zeta} + \psi_D \dot{\zeta} + \zeta = \gamma_\infty (\rho(0,t))^\kappa - p_\infty \, , \; \dot{\eta} = -\varphi(\zeta + \gamma_0(\eta - \eta_\infty)) \qquad (47)$$

where η_∞ - the steady state of the actuator - may be computed using the steady state equations of (45)-(47). Summarizing there was obtained a boundary value problem for a system of conservation laws.

For this system various problems may be stated, some of them being already mentioned at the previous application: discussion of the hyperbolicity, associated functional equations and basic theory, invariant sets, control synthesis, stability.

In all, the control of the systems of conservation laws is at its beginnings (at least in the nonlinear case). Our point of view is that *the most suitable approach would be to use the energy integral as a Liapunov functional* (possibly for synthesis purposes also: a nonlinear counterpart of the standard results e.g. [17] may be obtained. Unlike [17] here even the partial differential equations are nonlinear. Finding the associated functional equations is still a challenge [15].

6   Conclusions

This survey is an attempt to discuss some dynamical models in automatic control that are connected with distributed parameters in one dimension. These models are described by boundary value problems for hyperbolic partial differential equations. We considered here the functional equations associated to these problems using the integration of the Riemann invariants along the characteristics. Such quite known models correspond to linear partial differential equations with possible nonlinear boundary conditions. The most interesting and significant fact is that these equations arise from the linearization of the equations of the conservation laws. The control of the nonlinear systems of conservation laws is one of the most recent challenges in engineering. It is felt that the energy integral combined with integration along the characteristics could produce new advancement. And, last but not least, the so called *model validation* (basic theory, invariant sets) may turn helpful for better control issues.

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