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Research Article

Stability Analysis of Fractional Differential Systems with Order Lying in $(1, 2)$

Fengrong Zhang[1, 2] and Changpin Li[1]

[1] *Department of Mathematics, Shanghai University, Shanghai 200444, China*
[2] *School of Mathematics and Computational Science, China University of Petroleum (East China), Dongying 257061, China*

Correspondence should be addressed to Changpin Li, lcp@shu.edu.cn

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The stability of n-dimensional linear fractional differential systems with commensurate order $1 < \alpha < 2$ and the corresponding perturbed systems is investigated. By using the Laplace transform, the asymptotic expansion of the Mittag-Leffler function, and the Gronwall inequality, some conditions on stability and asymptotic stability are given.

1. Introduction

Fractional calculus has a long history with more than three hundred years [1–3]. Up to now, it has been proved that fractional calculus is very useful. Many mathematical models of real problems arising in various fields of science and engineering were established with the help of fractional calculus, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, and electromagnetic waves [4–7].

Recently, the stability theory of fractional differential equations (FDEs) is of main interest in physical systems. Moreover, some stability results have been found [8–17]. These stability results are almost about the linear fractional differential systems with commensurate order (i.e., the fractional derivative order has to be an integer multiple of minimal fractional order [18]). For example, a necessary and sufficient condition on asymptotic stability of linear fractional differential system with order $0 < \alpha \leq 1$ was first given in [9]. Then, some literatures on the stability of linear fractional differential systems with order $0 < \alpha < 1$ have been appeared [11–15]. However, not all the fractional differential systems have fractional orders in $(0, 1)$. There exist fractional models which have fractional orders lying in $(1, 2)$, for example, super-diffusion [19]. Hence, the stability of linear fractional differential systems with order

$1 < \alpha < 2$ has also been considered by using the conversion methods and transfer function [8, 10]. Almost all of the above literatures dealt with the fractional differential systems with Caputo derivative. Recently, Qian et al. [16] have investigated the stability of fractional differential systems with Riemann-Liouville derivative whose order α lies in $(0, 1)$ in details. It is worth mentioning that not all of the stability conditions are parallel to the corresponding classical integer-order differential equations because of nonlocality and weak singularities of fractional calculus. For example, the solution to an autonomous fractional differential equation cannot define a dynamical system in the sense of semigroup [20]. Of course, some of the mathematical tools for the integer-order differential equation can be applied to fractional kinetics. In [20], the authors first define the Lyapunov exponents for fractional differential system then determine their bounds, where the basic ideas and techniques are borrowed from [21, 22].

In this paper, we study the stability of autonomous linear fractional differential systems, nonautonomous linear fractional differential systems, and the corresponding perturbed systems with order $1 < \alpha < 2$ by using the properties of Mittag-Leffler functions and the Gronwall inequality.

The paper is organized as follows. In Section 2, we first recall some definitions and lemmas used throughout the paper. In Section 3, the stability analysis is presented for autonomous linear fractional differential systems with order $1 < \alpha < 2$. The stability of nonautonomous linear fractional differential systems and the corresponding perturbed systems are studied in Sections 4 and 5, respectively. Conclusions and comments are included in Section 6.

2. Preliminaries

Let us denote by $\mathbb{R}$ the set of real numbers, denote by $\mathbb{R}_+$ the set of positive real numbers, denote by $\mathbb{Z}_+$ the set of positive integer numbers, and denote by $\mathbb{C}$ the set of complex numbers.

In this section, we recall the most commonly used definitions and properties of fractional derivatives, Mittag-Leffler functions, and their asymptotic expansions.

Definition 2.1. The Riemann-Liouville derivative with order α of function $x(t)$ is defined as follows:

$$ {}_{\mathrm{RL}}D^{\alpha}_{t_0,t}x(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x(\tau)d\tau, \tag{2.1}$$

where $m - 1 \leq \alpha < m \in \mathbb{Z}_+$, $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. The Caputo derivative with order α of function $x(t)$ is defined as follows:

$$ {}_{C}D^{\alpha}_{t_0,t}x(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)d\tau, \tag{2.2}$$

where $m - 1 < \alpha < m \in \mathbb{Z}_+$.

Their Laplace transforms for $t_0 = 0$ are given as follows [23]:

$$\mathcal{L}\left\{ {}_{\mathrm{RL}}D_{0,t}^{\alpha}\, x(t); s \right\} = s^{\alpha}\mathcal{L}\{x(t)\} - \sum_{k=0}^{m-1} s^{k}\left[{}_{\mathrm{RL}}D_{0,t}^{\alpha-k-1}\, x(t) \right]_{t=0} \quad (m-1 \le \alpha < m), \tag{2.3}$$

$$\mathcal{L}\left\{ {}_{C}D_{0,t}^{\alpha}\, x(t); s \right\} = s^{\alpha}\mathcal{L}\{x(t)\} - \sum_{k=0}^{m-1} s^{\alpha-k-1} x^{(k)}(0) \quad (m-1 < \alpha \le m). \tag{2.4}$$

Definition 2.3. The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(k\alpha+1)}, \tag{2.5}$$

where the real part of α, that is, $\Re\alpha > 0$, $z \in \mathbb{C}$. The two-parameter Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(k\alpha+\beta)}, \tag{2.6}$$

where $\Re\alpha > 0$ and $\beta \in \mathbb{C}$, $z \in \mathbb{C}$.

One can see $E_{\alpha}(z) = E_{\alpha,1}(z)$ from the above equations. By analogy with (2.6), for $A \in \mathbb{C}^{n\times n}$, we introduce a matrix Mittag-Leffler function defined by [24]

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^{k}}{\Gamma(k\alpha+\beta)}. \tag{2.7}$$

The following definitions of stability are introduced.

Definition 2.4. The constant x_{eq} is an equilibrium of fractional differential system $\mathfrak{D}_{t_0,t}^{\alpha}x(t) = f(t,x)$ if and only if $f(t,x_{\mathrm{eq}}) = \mathfrak{D}_{t_0,t}^{\alpha}x(t)|_{x(t)=x_{\mathrm{eq}}}$ for all $t > t_0$, where the operator $\mathfrak{D}_{t_0,t}$ denotes either ${}_{\mathrm{RL}}D_{t_0,t}^{\alpha}$ or ${}_{C}D_{t_0,t}^{\alpha}$.

Without loss of generality, let the equilibrium be $x_{\mathrm{eq}} = 0$, we introduce the following definition.

Definition 2.5. The zero solution of $\mathfrak{D}_{t_0,t}^{\alpha}x(t) = f(t,x(t))$ with order $1 < \alpha < 2$ is said to be stable if, for any initial values x_k $(k = 0,1)$, there exists $\varepsilon > 0$ such that $\|x(t)\| \le \varepsilon$ for all $t > t_0$. The zero solution is said to be asymptotically stable if, in addition to being stable, $\|x(t)\| \to 0$ as $t \to +\infty$.

It is useful to recall the following asymptotic formulas for our developments in the sequel.

Lemma 2.6. *If $0 < \alpha < 2$, β is an arbitrary complex number and μ is an arbitrary real number such that*

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}, \tag{2.8}$$

then for an arbitrary integer $p \geq 1$, the following expansions hold:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-p-1}\right), \tag{2.9}$$

with $|z| \rightarrow \infty, |\arg(z)| \leq \mu$ and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - \alpha k)} + O\left(|z|^{-p-1}\right), \tag{2.10}$$

with $|z| \rightarrow \infty$, $\mu \leq |\arg(z)| \leq \pi$.

Proof. These results were proved in [23]. □

Especially, taking into account the Lemma 2.6 and derivatives of the Mittag-Leffler function, we obtain

$$t^{\alpha j+\beta-1} E^{(j)}_{\alpha,\beta}(\lambda t^{\alpha}) \sim \left(\frac{\partial}{\partial \lambda}\right)^{j} \left[\frac{1}{\alpha} \lambda^{(1-\beta)/\alpha} \exp\left(\lambda^{1/\alpha} t\right)\right], \tag{2.11}$$

with $t \rightarrow +\infty$, $|\arg(\lambda)| \leq \mu$ and

$$t^{\alpha j+\beta-1} E^{(j)}_{\alpha,\beta}(\lambda t^{\alpha}) \sim (-1)^{j+1} \left[\frac{j!\lambda^{-j-1}}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1} + \frac{(j+1)!\lambda^{-j-2}}{\Gamma(\beta-2\alpha)} t^{\beta-2\alpha-1}\right], \tag{2.12}$$

with $t \rightarrow +\infty$, $\mu \leq |\arg(z)| \leq \pi$, $j = 0, 1, 2, \ldots$.

Lemma 2.7 (see [25]). *If $A \in \mathbb{C}^{n\times n}$ and $0 < \alpha < 2$, β is an arbitrary real number, μ satisfies $\pi\alpha/2 < \mu < \min\{\pi, \pi\alpha\}$, and $C > 0$ is a real constant, then*

$$\|E_{\alpha,\beta}(A)\| \leq \frac{C}{1 + \|A\|}, \tag{2.13}$$

where $\mu \leq |\arg(\operatorname{spec}(A))| \leq \pi$, $\operatorname{spec}(A)$ denotes the eigenvalues of matrix A and $\|\cdot\|$ denotes the l_2-norm.

Lemma 2.8 (Jordan Decomposition [26]). *Let A be a square complex matrix, then there exists an invertible matrix P such that*

$$P^{-1}AP = J_1 \oplus \cdots \oplus J_s, \tag{2.14}$$

where the J_l are the Jordan blocks of A with the eigenvalues of A on the diagonal. The Jordan blocks are uniquely determined by A.

Lemma 2.9 (see [27]). *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \quad t \in [t_0, T), \tag{2.15}$$

where all the functions involved are continuous on $[t_0, T)$, $T \leq +\infty$, and $k(t) \geq 0$, then $x(t)$ satisfies

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)h(s) \exp\left[\int_{s}^{t} k(u)du\right] ds, \quad t \in [t_0, T). \tag{2.16}$$

If, in addition, $h(t)$ is nondecreasing, then

$$x(t) \leq h(t) \exp\left(\int_{t_0}^{t} k(s)ds\right), \quad t \in [t_0, T). \tag{2.17}$$

3. Stability of Autonomous Linear Fractional Differential Systems

3.1. The Riemann-Liouville Derivative Case

In this subsection, we consider the following system of fractional differential equations:

$$_{\mathrm{RL}}D^{\alpha}_{t_0,t}\, x(t) = Ax(t), \quad t > t_0, \tag{3.1}$$

with the initial conditions

$$_{\mathrm{RL}}D^{\alpha-k}_{t_0,t}\, x(t)|_{t=t_0} = x_{k-1} \quad (k = 1, 2), \tag{3.2}$$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, and $1 < \alpha < 2$. Then, by analyzing the solutions of the above initial value problem (3.1)-(3.2), one can find the following result.

Theorem 3.1. *The autonomous fractional differential system* (3.1) *with Riemann-Liouville derivative and the initial conditions* (3.2) *is asymptotically stable iff $|\arg(\operatorname{spec}(A))| > \alpha\pi/2$. In this case, the components of the state decay towards 0 like $t^{-\alpha-1}$. Moreover, the system* (3.1) *is stable iff either it is asymptotically stable, or those critical eigenvalues which satisfy $|\arg(\operatorname{spec}(A))| = \alpha\pi/2$ have the same algebraic and geometric multiplicities.*

Proof. Applying the Laplace transform, we can get the solution of (3.1)-(3.2),

$$\begin{aligned} x(t) &= (t-t_0)^{\alpha-1} E_{\alpha,\alpha}\left(A(t-t_0)^{\alpha}\right)x_0 + (t-t_0)^{\alpha-2} E_{\alpha,\alpha-1}\left(A(t-t_0)^{\alpha}\right)x_1 \\ &\triangleq \sum_{k=0}^{1} (t-t_0)^{\alpha-k-1} E_{\alpha,\alpha-k}\left(A(t-t_0)^{\alpha}\right)x_k. \end{aligned} \tag{3.3}$$

Firstly, we study the properties of the elements of matrixes $(t-t_0)^{\alpha-k-1} \cdot E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})$, $k = 0, 1$. With regard to matrix A, there exists an invertible matrix P, such that

$$A = PJP^{-1} = P\,\mathrm{diag}(J_1, J_2, \dots, J_s)P^{-1}, \tag{3.4}$$

from Lemma 2.8, where the Jordan block

$$J_l = \begin{pmatrix} \lambda_l & 1 & & & \\ & \lambda_l & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_l & 1 \\ & & & & \lambda_l \end{pmatrix}_{n_l \times n_l}, \tag{3.5}$$

$l = 1, 2, \dots, s$, $\lambda_l \in \mathbb{C}$ is the eigenvalue of matrix A and $\sum_{l=1}^{s} n_l = n$. Substituting (3.4) into $(t-t_0)^{\alpha-k-1} E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})$, we yield

$$\begin{aligned} &(t-t_0)^{\alpha-k-1} E_{\alpha,\alpha-k}\left(A(t-t_0)^{\alpha}\right) \\ &\quad = (t-t_0)^{\alpha-k-1} P \sum_{m=0}^{\infty} \frac{\mathrm{diag}\left(J_1^m, J_2^m, \dots, J_s^m\right)(t-t_0)^{\alpha m}}{\Gamma(\alpha m + \alpha - k)} P^{-1} \\ &\quad = (t-t_0)^{\alpha-k-1} P \begin{pmatrix} E_{\alpha,\alpha-k}\left(J_1(t-t_0)^{\alpha}\right) & & \\ & \ddots & \\ & & E_{\alpha,\alpha-k}\left(J_s(t-t_0)^{\alpha}\right) \end{pmatrix} P^{-1}, \end{aligned} \tag{3.6}$$

where $k = 0, 1$. The matrix $(t-t_0)^{\alpha-k-1} E_{\alpha,\alpha-k}(J_l(t-t_0)^{\alpha})$ can be written as follows by computing

$$T\left(E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right)\big|_{\lambda=\lambda_l}, \tag{3.7}$$

where the operator T is given as follows:

$$T = (t-t_0)^{\alpha-k-1} \begin{pmatrix} 1 & \dfrac{\partial}{\partial\lambda} & \dfrac{1}{2!}\left(\dfrac{\partial}{\partial\lambda}\right)^2 & \cdots & \dfrac{1}{(n_l-1)!}\left(\dfrac{\partial}{\partial\lambda}\right)^{n_l-1} \\ & 1 & \dfrac{\partial}{\partial\lambda} & \cdots & \dfrac{1}{(n_l-2)!}\left(\dfrac{\partial}{\partial\lambda}\right)^{n_l-2} \\ & & \ddots & \ddots & \vdots \\ & & & 1 & \dfrac{\partial}{\partial\lambda} \\ & & & & 1 \end{pmatrix}. \tag{3.8}$$

The nonzero elements of $(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(J_l(t-t_0)^{\alpha})$ can be described uniformly as

$$(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\Bigg|_{\lambda=\lambda_l}\quad(j=1,2,\ldots,n_l). \tag{3.9}$$

(i) If $\lambda_l=0$, then

$$(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\Bigg|_{\lambda=\lambda_l=0}=\frac{(t-t_0)^{j\alpha-k-1}}{\Gamma(j\alpha-k)}. \tag{3.10}$$

It is obvious that $(t-t_0)^{j\alpha-k-1}/\Gamma(j\alpha-k)\to\infty$ $(t\to+\infty)$ for $k=0$ and $j\geq 1$. Thus, $\|x(t)\|\to\infty$ $(t\to+\infty)$.

(ii) If $\lambda_l\neq 0$, three cases will be considered separately.

Case 1 $(|\arg(\operatorname{spec}(A))|=|\arg(\lambda_l)|>\alpha\pi/2)$. If $|\arg(\lambda_l)|>\alpha\pi/2$ and $t\to+\infty$, then

$$\begin{aligned}&(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\Bigg|_{\lambda=\lambda_l}\\&\quad\sim(-1)^j\left[\frac{\lambda_l^{-j}(t-t_0)^{-k-1}}{\Gamma(-k)}+\frac{j\lambda_l^{-j-1}(t-t_0)^{-k-\alpha-1}}{\Gamma(-k-\alpha)}\right].\end{aligned} \tag{3.11}$$

That is to say, $(t-t_0)^{\alpha-k-1}(1/(j-1)!)\{(\partial/\partial\lambda)^{j-1}E_{\alpha,\alpha-k}(\lambda(t-t_0)^{\alpha})\}|_{\lambda=\lambda_l}\to 0$ $(t\to+\infty)$ from the asymptotic expansion (2.12) and $\|x(t)\|\to 0$ $(t\to+\infty)$. Moreover, the components of the state decay towards 0 like $t^{-\alpha-1}$. Taking into account the entire function $E_{\alpha,\alpha-k}(\lambda(t-t_0)^{\alpha})$, we also get the boundedness of $((t-t_0)^{\alpha-k-1}/(j-1)!)\{(\partial/\partial\lambda)^{j-1}E_{\alpha,\alpha-k}(\lambda(t-t_0)^{\alpha})\}|_{\lambda=\lambda_l}$ $(j=1,2,\ldots,n_l;\, l=1,2,\ldots,s)$.

Case 2 $(|\arg(\operatorname{spec}(A))|=|\arg(\lambda_l)|<\alpha\pi/2)$. If $|\arg(\operatorname{spec}(A))|=|\arg(\lambda_l)|<\alpha\pi/2$ and $t\to+\infty$, from the asymptotic expansion (2.11), we have

$$\begin{aligned}&(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\Bigg|_{\lambda=\lambda_l}\\&\quad=\frac{(t-t_0)^{j\alpha-k-1}}{(j-1)!}E^{(j-1)}_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\Big|_{\lambda=\lambda_l}\\&\quad\sim\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}\left[\frac{1}{\alpha}\lambda^{(1-\alpha+k)/\alpha}e^{\lambda^{1/\alpha}(t-t_0)}\right]\right\}\Bigg|_{\lambda=\lambda_l}\end{aligned}$$

$$= \frac{1}{(j-1)!}\Bigg\{\frac{(1+k-\alpha)(1+k-2\alpha)\cdots(1+k-(j-1)\alpha)}{\alpha^j}\lambda_l^{1+k-j\alpha/\alpha}+\cdots$$
$$+\frac{j(j-1)/2-(j-1)(\alpha-k)-(j-1)(j-2)/2\alpha}{\alpha^j}$$
$$\times\lambda_l^{(j(1-\alpha)+k-1)/\alpha}(t-t_0)^{j-2}+\frac{1}{\alpha^j}\lambda_l^{(j(1-\alpha)+k)/\alpha}(t-t_0)^{j-1}\Bigg\}\exp\left(\lambda_l^{1/\alpha}(t-t_0)\right), \tag{3.12}$$

then

$$\left|(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\Big|_{\lambda=\lambda_l}\right|$$
$$\sim\frac{1}{(j-1)!}\Bigg|\frac{(1+k-\alpha)(1+k-2\alpha)\cdots(1+k-(j-1)\alpha)}{\alpha^j}\lambda_l^{(1+k-j\alpha)/\alpha}+\cdots$$
$$+\frac{j(j-1)/2-(j-1)(\alpha-k)-((j-1)(j-2)/2)\alpha}{\alpha^j}\lambda_l^{(j(1-\alpha)+k-1)/\alpha}(t-t_0)^{j-2}$$
$$+\frac{1}{\alpha^j}\lambda_l^{(j(1-\alpha)+k)/\alpha}(t-t_0)^{j-1}\Bigg|\exp\left\{|\lambda_l|^{1/\alpha}\cos\left(\frac{\arg(\lambda_l)}{\alpha}\right)(t-t_0)\right\}$$
$$\longrightarrow+\infty\quad\text{as } t\longrightarrow+\infty,\ j=1,2,\ldots,n_l, \tag{3.13}$$

because of $|(\arg(\lambda_l))/\alpha|<\pi/2$, that is, $\cos((\arg(\lambda\_l))/\alpha)>0$.

So, $\|x(t)\|=\|\sum_{k=0}^{1}(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})x_k\|\ \to\ +\infty\ (t\ \to\ +\infty)$.

Case 3 ($|\arg(\mathrm{spec}(A))|=|\arg(\lambda_l)|=\alpha\pi/2$). Let $\lambda_l=r(\cos(\alpha\pi/2)+i\,\sin(\alpha\pi/2))$, where r is the modulus of λ_l and $i^2=-1$.

Firstly, suppose that the critical eigenvalue λ_l has the same algebraic and geometric multiplicities, that is, the matrix J_l is a diagonal matrix, then, according to (3.7), we have

$$(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}\left(J_l(t-t_0)^{\alpha}\right)=(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}\left(\lambda_l(t-t_0)^{\alpha}\right)\mathrm{diag}(1,\ldots,1). \tag{3.14}$$

If $|\arg(\lambda_l)|=\alpha\pi/2$, we have the diagonal elements of matrix (3.14) $|(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(\lambda_l(t-t_0)^{\alpha})|(\sim 1/\alpha)r^{(1+k-\alpha)/\alpha}$ $(t\ \to\ +\infty)$ from the asymptotic expansion (2.11). So, the solution of (3.1) is stable in this case.

Next, suppose that the algebraic multiplicity of critical eigenvalue λ_l is not equal to the geometric multiplicity, that is, the matrix J_l is a Jordan block matrix, and matrix

$(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(J_l(t-t_0)^{\alpha})$ is the same as (3.7), then the nondiagonal elements of $(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(J_l(t-t_0)^{\alpha})$ can be evaluated from (3.12) as follows:

$$\begin{aligned}
&(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\bigg|_{\lambda=\lambda_l}\\
&\quad\sim\frac{1}{(j-1)!}\left\{\frac{(1+k-\alpha)(1+k-2\alpha)\cdots(1+k-(j-1)\alpha)}{\alpha^j}\lambda_l^{(1+k-j\alpha)/\alpha}+\cdots\right.\\
&\qquad+\frac{\dfrac{j(j-1)}{2}-(j-1)(\alpha-k)-((j-1)(j-2)/2)\alpha}{\alpha^j}\lambda_l^{(j(1-\alpha)+k-1)/\alpha}(t-t_0)^{j-2}\\
&\qquad\left.+\frac{1}{\alpha^j}\lambda_l^{(j(1-\alpha)+k)/\alpha}(t-t_0)^{j-1}\right\}\exp\left\{ir^{1/\alpha}(t-t_0)\right\},\quad j=2,\ldots,n_l.
\end{aligned}\tag{3.15}$$

So,

$$\begin{aligned}
&\left|(t-t_0)^{\alpha-k-1}\frac{1}{(j-1)!}\left\{\left(\frac{\partial}{\partial\lambda}\right)^{j-1}E_{\alpha,\alpha-k}\left(\lambda(t-t_0)^{\alpha}\right)\right\}\bigg|_{\lambda=\lambda_l}\right|\\
&\quad\sim\frac{1}{(j-1)!}\left|\frac{(1+k-\alpha)(1+k-2\alpha)\cdots(1+k-(j-1)\alpha)}{\alpha^j}\lambda_l^{(1+k-j\alpha)/\alpha}+\cdots\right.\\
&\qquad+\frac{j(j-1)/2-(j-1)(\alpha-k)-((j-1)(j-2)/2)\alpha}{\alpha^j}\lambda_l^{(j(1-\alpha)+k-1)/\alpha}(t-t_0)^{j-2}\\
&\qquad\left.+\frac{1}{\alpha^j}\lambda_l^{(j(1-\alpha)+k)/\alpha}(t-t_0)^{j-1}\right|\\
&\quad\longrightarrow+\infty\quad\text{as } t\longrightarrow+\infty,\ j=2,\ldots,n_l,
\end{aligned}\tag{3.16}$$

that is, $\|x(t)\|\to+\infty$ as $t\to+\infty$.

According to the above discussions, the proof is completed. □

Remark 3.2. (1) If $|\arg(\operatorname{spec}(A))|<\alpha\pi/2$, then system (3.1) is not stable.

(2) If A has zero eigenvalue, system (3.1) is not stable.

(3) If A has critical eigenvalue(s) λ_c, that is, $|\arg(\lambda_c)|=\alpha\pi/2$, and the arguments of the rest eigenvalues in absolute values are greater than $\alpha\pi/2$, then system (3.1) is not stable provided that λ_c has different geometric and algebraic multiplicities.

3.2. The Caputo Derivative Case

Now, we consider the fractional differential system with Caputo derivative

$$ {}_{C}D^{\alpha}_{t_0,t}\, x(t) = Ax(t), \quad t > t_0, \tag{3.17}$$

under the initial conditions

$$ x^{(k)}(t_0) = x_k \quad (k = 0, 1), \tag{3.18}$$

where x, A, and α are as in Section 3.1. Then, one can get the following theorem.

Theorem 3.3. *The autonomous fractional differential system* (3.17) *with Caputo derivative and initial conditions* (3.18) *is asymptotically stable iff* $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$*. In this case, the components of the state decay towards 0 like* $t^{-\alpha+1}$*. Moreover, the system* (3.17) *is stable iff either it is asymptotically stable, or those critical eigenvalues which satisfy* $|\arg(\mathrm{spec}(A))| = \alpha\pi/2$ *have the same algebraic and geometric multiplicities.*

Proof. This theorem can be proved in the same manner as that in the proof of Theorem 3.1, so it is omitted here. □

4. Stability of Nonautonomous Linear Fractional Differential Systems

4.1. The Riemann-Liouville Derivative Case

We will consider a nonautonomous fractional differential system with Riemann-Liouville derivative

$$ {}_{\mathrm{RL}}D^{\alpha}_{t_0,t} x(t) = Ax(t) + B(t)x(t), \quad t > t_0, \tag{4.1}$$

under the initial conditions

$$ {}_{\mathrm{RL}}D^{\alpha-k}_{t_0,t}\, x(t)\Big|_{t=t_0} = x_{k-1} \quad (k = 1, 2), \tag{4.2}$$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, $B(t) : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is a continuous matrix, and $1 < \alpha < 2$. The main results of this subsection are derived as follows.

Theorem 4.1. *If the matrix* A *such that* $|\mathrm{spec}(A)| \neq 0, |\arg(\mathrm{spec}(A))| \geq \alpha\pi/2$*, the critical eigenvalues which satisfy* $|\arg(\mathrm{spec}(A))| = \alpha\pi/2$ *have the same algebraic and geometric multiplicities, and* $\int_{t_0}^{\infty} \|B(t)\| dt$ *is bounded, then the zero solution of* (4.1) *is stable.*

Proof. Applying the Laplace transform, we can get the solution of (4.1)-(4.2),

$$\begin{aligned} x(t) &= (t-t_0)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-t_0)^{\alpha}\right)x_0 + (t-t_0)^{\alpha-2}E_{\alpha,\alpha-1}\left(A(t-t_0)^{\alpha}\right)x_1 \\ &\quad + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)B(\tau)x(\tau)d\tau \\ &\triangleq \sum_{k=0}^{1}(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}\left(A(t-t_0)^{\alpha}\right)x_k + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)B(\tau)x(\tau)d\tau. \end{aligned} \tag{4.3}$$

From the proof of Theorem 3.1, the matrix $(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})$ is bounded for $k=0,1$. Therefore, there exist positive numbers M_k, such that $\|(t-t_0)^{\alpha-k-1}E_{\alpha,\alpha-k}(A(t-t_0)^{\alpha})\| \le M_k$ $(k=0,1)$. Now, we can get the estimate of solution $x(t)$

$$\|x(t)\| \le M_0\|x_0\| + M_1\|x_1\| + \int_{t_0}^{t} M_0\|B(\tau)\|\cdot\|x(\tau)\|d\tau. \tag{4.4}$$

Applying the Gronwall inequality (2.17) leads to

$$\|x(t)\| \le (M_0\|x_0\| + M_1\|x_1\|)\exp\left(M_0\int_{t_0}^{t}\|B(\tau)\|d\tau\right). \tag{4.5}$$

Thus, we derive that $\|x(t)\|$ is bounded according to the condition $\int_{t_0}^{\infty}\|B(t)\|dt < \infty$, that is, the zero solution of (4.1) is stable. The proof is completed. □

Similarly, we can derive the following conclusion.

Theorem 4.2. *If the matrix A such that $|\mathrm{spec}(A)| \neq 0$, $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$, and $\|B(t)\| = O(t-t_0)^{\gamma}$ $(-1<\gamma<1-\alpha,\ t_0>0)$ for $t \ge t_0$, then the zero solution of* (4.1) *is asymptotically stable.*

Proof. From the proof of Theorem 3.1, the following expression is valid:

$$\|x(t)\| \le (t-t_0)^{\alpha-2}L_1\|x_0\| + (t-t_0)^{\alpha-2}L_2\|x_1\| + L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\cdot\|x(\tau)\|d\tau, \tag{4.6}$$

where $L_1, L_2 > 0$ such that $\|(t-t_0)E_{\alpha,\alpha}(A(t-t_0)^{\alpha})\| < L_1$ and $\|E_{\alpha,\alpha-1}(A(t-t_0)^{\alpha})\| < L_2$. Moreover, from (4.4) and (2.17), one has

$$\begin{aligned} \|x(t)\| &\le M_0\|x_0\| + M_1\|x_1\| + L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\cdot\|x(\tau)\|d\tau \\ &\le (M_0\|x_0\| + M_1\|x_1\|)\exp\left(L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|d\tau\right). \end{aligned} \tag{4.7}$$

Substituting (4.7) into (4.6), we have

$$\|x(t)\| \leq (t-t_0)^{\alpha-2}(L_1\|x_0\| + L_2\|x_1\|) + M_{01}\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\| e^{L_1\int_{t_0}^{\tau}(\tau-\eta)^{\alpha-2}\|B(\eta)\|d\eta}d\tau, \quad (4.8)$$

where $M_{01} = L_1(M_0\|x_0\| + M_1\|x_1\|)$. It follows from the condition $\|B(t)\| = O(t-t_0)^{\gamma}$ $(-1 < \gamma < 1-\alpha,\ t_0 > 0)$ for $t \geq t_0$ that there exists a constant $M > 0$, such that $\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|d\tau < M$ and

$$\begin{aligned}\|x(t)\| &\leq (t-t_0)^{\alpha-2}(L_1\|x_0\| + L_2\|x_1\|) + M_{01}e^{L_1 M}\int_{t_0}^{t}(t-\tau)^{\alpha-2}O(\tau-t_0)^{\gamma}d\tau \\ &= (t-t_0)^{\alpha-2}(L_1\|x_0\| + L_2\|x_1\|) + M_{01}e^{L_1 M}\frac{\Gamma(\alpha-1)\Gamma(1+\gamma)}{\Gamma(\alpha+\gamma)}O(t-t_0)^{\gamma+\alpha-1}.\end{aligned} \quad (4.9)$$

So, the zero solution of (4.1) is asymptotically stable. □

4.2. The Caputo Derivative Case

In this subsection, we consider a nonautonomous fractional differential system with Caputo derivative

$$_{C}D_{t_0,t}^{\alpha}x(t) = Ax(t) + B(t)x(t), \quad t > t_0, \quad (4.10)$$

under the initial conditions

$$x^{(k)}(t_0) = x_k \quad (k = 0, 1), \quad (4.11)$$

where x, A, and α are as in Section 4.1, $B(t) : [t_0, \infty) \to \mathbb{R}^{n\times n}$ is a continuously differentiable matrix. We can get the solution of (4.10)-(4.11) by using the Laplace transform and Laplace inverse transform

$$\begin{aligned}x(t) &= E_{\alpha}\left(A(t-t_0)^{\alpha}\right)x_0 + (t-t_0)E_{\alpha,2}\left(A(t-t_0)^{\alpha}\right)x_1 \\ &\quad + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(A(t-\tau)^{\alpha}\right)B(\tau)x(\tau)d\tau.\end{aligned} \quad (4.12)$$

The main stability results of this subsection are derived as follows.

Theorem 4.3. *If the matrix A such that $|\mathrm{spec}(A)| \neq 0$, $|\arg(\mathrm{spec}(A))| \geq \alpha\pi/2$, the critical eigenvalues which satisfy $|\arg(\mathrm{spec}(A))| = \alpha\pi/2$ have the same algebraic and geometric multiplicities, and $\int_{t_0}^{\infty}\|B(t)\|dt$ is bounded, then the zero solution of* (4.10) *is stable.*

Proof. The proof line is similar to that of Theorem 4.1. □

Theorem 4.4. *If the matrix A such that $|\mathrm{spec}(A)| \neq 0$, $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$, and $\|B(t)\| = O(t-t_0)^{\gamma}$ $(-1 < \gamma < 1-\alpha,\ t_0 > 0)$ for $t \geq t_0$, then the zero solution of* (4.10) *is asymptotically stable.*

Proof. From the solution (4.12) and Lemma 2.7, we can directly get

$$\|x(t)\| \leq \frac{C_0\|x_0\|}{1+\|A\|(t-t_0)^{\alpha}} + \frac{C_1(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^{\alpha}} + L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\cdot\|x(\tau)\|d\tau, \tag{4.13}$$

where $C_0, C_1 > 0$ and $L_1 > 0$, such that $\|(t-t_0)E_{\alpha,\alpha}(A(t-t_0)^{\alpha})\| < L_1$. Furthermore, there exists a constant $M_0 > 0$ such that

$$\frac{C_0\|x_0\|}{1+\|A\|(t-t_0)^{\alpha}} + \frac{C_1(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^{\alpha}} \leq M_0, \tag{4.14}$$

that is,

$$\begin{aligned}\|x(t)\| &\leq M_0 + L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|\cdot\|x(\tau)\|d\tau \\ &\leq M_0\exp\left(L_1\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|d\tau\right).\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.13) gives

$$\begin{aligned}\|x(t)\| \leq{}& \frac{C_0\|x_0\|}{1+\|A\|(t-t_0)^{\alpha}} + \frac{C_1(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^{\alpha}} \\ &+ L_1M_0\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|e^{L_1\int_{t_0}^{\tau}(\tau-\eta)^{\alpha-2}\|B(\eta)\|d\eta}d\tau.\end{aligned} \tag{4.16}$$

It follows from the condition $\|B(t)\| = O(t-t_0)^{\gamma}$ $(-1 < \gamma < 1-\alpha,\ t_0 > 0)$ for $t \geq t_0$ that there exists a constant $M > 0$, such that $\int_{t_0}^{t}(t-\tau)^{\alpha-2}\|B(\tau)\|d\tau < M$ and

$$\begin{aligned}\|x(t)\| &\leq \frac{C_0\|x_0\|}{1+\|A\|(t-t_0)^{\alpha}} + \frac{C_1(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^{\alpha}} + L_1M_0e^{L_1M}\int_{t_0}^{t}(t-\tau)^{\alpha-2}O(\tau-t_0)^{\gamma}d\tau \\ &= \frac{C_0\|x_0\|}{1+\|A\|(t-t_0)^{\alpha}} + \frac{C_1(t-t_0)\|x_1\|}{1+\|A\|(t-t_0)^{\alpha}} + L_1M_0e^{L_1M}\frac{\Gamma(\alpha-1)\Gamma(1+\gamma)}{\Gamma(\alpha+\gamma)}O(t-t_0)^{\gamma+\alpha-1}.\end{aligned} \tag{4.17}$$

So, the zero solution of (4.10) is asymptotically stable. □

5. Stability of the Perturbed Systems

In this section, we only study the perturbed system of a linear fractional differential system with Riemann-Liouville derivative

$$ {}_{\mathrm{RL}}D^{\alpha}_{t_0,t}x(t) = Ax(t) + f(t, x(t)), \quad t > t_0, \tag{5.1} $$

under the initial conditions

$$ {}_{\mathrm{RL}}D^{\alpha-k}_{t_0,t}x(t)\Big|_{t=t_0} = x_{k-1} \quad (k = 1, 2), \tag{5.2} $$

where $x \in \mathbb{R}^n$, matrix $A \in \mathbb{R}^{n\times n}$, and $1 < \alpha < 2$. $f(t, x) : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function in which $f(t, 0) = 0$; moreover, $f(t, x)$ fulfils the Lipschitz condition with respect to x. Then, the unique solution of (5.1)-(5.2) can be written as

$$ \begin{aligned} x(t) &= (t-t_0)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-t_0)^{\alpha}\big)x_0 + (t-t_0)^{\alpha-2}E_{\alpha,\alpha-1}\big(A(t-t_0)^{\alpha}\big)x_1 \\ &\quad + \int_{t_0}^{t}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-\tau)^{\alpha}\big)f(\tau, x(\tau))d\tau. \end{aligned} \tag{5.3} $$

The following theorem can be proved by the same argument used in the proof of Theorem 4.1.

Theorem 5.1. *If the matrix A such that $|\mathrm{spec}(A)| \neq 0$, $|\arg(\mathrm{spec}(A))| \geq \alpha\pi/2$, the critical eigenvalues which satisfy $|\arg(\mathrm{spec}(A))| = \alpha\pi/2$ have the same algebraic and geometric multiplicities. Moreover, suppose that there exists a positive function $\gamma(t)$ which satisfies the following conditions:*

(i) $\int_{t_0}^{\infty}\gamma(t)dt$ *is bounded,*

(ii) $\|f(t, x)\| \leq \gamma(t)\|x(t)\|$,

then the zero solution of (5.1) *is stable.*

Theorem 5.2. *If the matrix A such that $|\mathrm{spec}(A)| \neq 0$, $|\arg(\mathrm{spec}(A))| > \alpha\pi/2$, $\alpha + 1/\|A\| < 2$, and suppose that the function $f(t, x)$ satisfies uniformly*

$$ \lim_{x\to 0}\frac{\|f(t, x)\|}{\|x\|} = 0, \quad t \in [t_0, \infty), \tag{5.4} $$

then the zero solution of (5.1) *is asymptotically stable.*

Proof. According to the proof of Theorem 3.1 and Lemma 2.7, we have

$$ \|x(t)\| \leq (t-t_0)^{\alpha-2}L_1\|x_0\| + (t-t_0)^{\alpha-2}L_2\|x_1\| + \int_{t_0}^{t}\frac{C_1(t-\tau)^{\alpha-1}}{1+\|A\|(t-\tau)^{\alpha}}\|f(\tau, x(\tau))\|d\tau, \tag{5.5} $$

where $C_1, L_1, L_2 > 0$ such that $\|(t-t_0)E_{\alpha,\alpha}(A(t-t_0)^{\alpha})\| < L_1$ and $\|E_{\alpha,\alpha-1}(A(t-t_0)^{\alpha})\| < L_2$. Taking into account the condition (5.4), there exists a constant $\delta > 0$, such that

$$\|f(t,x(t))\| < \frac{1}{C_1}\|x(t)\| \quad \text{as } \|x(t)\| < \delta. \tag{5.6}$$

Then,

$$\|x(t)\| \leq M(t-t_0)^{\alpha-2} + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{1+\|A\|(t-\tau)^{\alpha}}\|x(\tau)\|d\tau, \tag{5.7}$$

where $M = L_1\|x_0\| + L_2\|x_1\|$. Applying the Gronwall inequality (2.16) to (5.7) yields

$$\begin{aligned}
\|x(t)\| &\leq M(t-t_0)^{\alpha-2} + M\int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha-2}}{1+\|A\|(t-\tau)^{\alpha}} e^{\int_{\tau}^{t}((t-s)^{\alpha-1}/(1+\|A\|(t-s)^{\alpha}))ds}d\tau \\
&= M(t-t_0)^{\alpha-2} + M\int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}(\tau-t_0)^{\alpha-2}}{\left(1+\|A\|(t-\tau)^{\alpha}\right)^{1-1/(\alpha\|A\|)}}d\tau \\
&\leq M(t-t_0)^{\alpha-2} + M\|A\|^{1/\alpha\|A\|-1}\int_{t_0}^{t} (t-\tau)^{1/\|A\|-1}(\tau-t_0)^{\alpha-2}d\tau \\
&= M(t-t_0)^{\alpha-2} + \frac{M\|A\|^{1/\alpha\|A\|-1}\Gamma(1/\|A\|)\Gamma(\alpha-1)}{\Gamma(\alpha+1/\|A\|-1)}(t-t_0)^{1/\|A\|+\alpha-2}.
\end{aligned} \tag{5.8}$$

So, the zero solution of (5.1) is asymptotically stable due to $\alpha + 1/\|A\| - 2 < 0$. The proof is thus finished. □

6. Conclusion

It is well know that many physical phenomena having memory and genetic characteristics can be described by using the fractional differential systems. Especially, the fractional differential systems with order $1 < \alpha < 2$ have recently gained an increasing attention [23, 28–31]. It should be noted that [29, 30] are earlier and interesting work on fractional interval systems. Motivated by the above research activities, in this paper, we have studied the stability of linear fractional differential systems and the corresponding perturbed systems with Rimann-Liouville derivative and Caputo derivative for the commensurate order $1 < \alpha < 2$. The main analytic tools used in this paper are the Mittag-Leffler function and the Gronwall inequality. For the autonomous linear fractional differential systems with order $1 < \alpha < 2$, the necessary and sufficient conditions on stability and asymptotic stability are given, which are almost the same as those with the fractional derivative order $\alpha \in (0, 1)$. But the components of the state decay towards 0 like $t^{-\alpha+1}$, which is different from the case with Caputo derivative order $\alpha \in (0,1)$. For the nonautonomous linear fractional differential systems, we have derived some sufficient conditions on stability and asymptotic stability. We have further given the asymptotic stability results of the perturbed systems with order $1 < \alpha < 2$.

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