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An algorithm to check the nonnegativity of singular systems

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Abstract

In the literature, an important class of generalized inverse matrices corresponds to the group inverse, that is, matrices of index 1. Recently, the nonnegativity of a singular system has been applied to different fields. In this paper, an algorithm to check the nonnegativity of a singular linear control system of index 1 is presented. To this purpose, the nonnegativity of this kind of systems is characterized using a block partition of the original matrices. In this way, we can work with matrices having smaller sizes and keeping the original information. Finally, numerical examples illustrating the obtained results are shown.

Keywords: Nonnegative singular systems; Drazin inverse; Group inverse; Block matrices; Algorithm; Numerical examples

1. Introduction and background

A very important part of engineering work concerns the establishment of mathematical models. Due to its capacity to describe the dynamic and algebraic relationships between state variables simultaneously, a singular system permits to give a mathematical description for many practical dynamic systems, which do not admit a standard state–space model representation [1].

A dynamical system is said to be nonnegative if it leaves the first orthant of $\mathbb{R}^n$ invariant for future times when initiated in this orthant. Over the past two decades, these systems have gained much attention appearing in a wide variety of applied areas such as biology, chemistry, and sociology [2–4].

The nonnegativity of the matrices plays an important role in control theory. In particular, the nonnegativity of the coefficient matrices characterizes the nonnegative discrete standard systems. However, it is not the case for the singular systems because this characterization is made by means of products of the Drazin inverses of the state matrices. In the literature, singular systems are also called descriptor systems, implicit systems or generalized state-space systems. Some properties of them can be found in [1,5]. In some situations these systems have the symmetry properties and then its solution involves the group inverse of certain matrices [6]. Singular systems appear when modelling some physical phenomena and interconnected systems such as electrical,

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mechanical, and chemical processes [7,8]. Concretely, they arise when solving computational problems in the analysis and design of standard linear systems.

For a given matrix $F \in \mathbb{R}^{n\times n}$, a matrix $G \in \mathbb{R}^{n\times n}$ is called its Drazin inverse if the properties $GFG = G, FG = GF$ and $F^{k+1}G = F^k$ hold, where $k = \mathrm{ind}(F)$ is the index of F, that is, the smallest nonnegative integer such that $\mathrm{rank}(F^{k+1}) = \mathrm{rank}(F^k)$. This matrix will be denoted by $\mathrm{F^D}$ and it is unique. If $k = 1$, this matrix is called group inverse and is denoted by $F^{\#}$ [9]. The group inverse has been widely studied in the literature and applied to solve real problems. For instance, it is applied in model electric networks, Markov chains, symmetric singular control systems, etc. [6,7,9,10].

Moreover, we will stand $A \geqslant O$ for a matrix A with nonnegative elements.

The following results will be useful in this paper.

Theorem 1.1 (Theorem 1, [11]). *Let $E \in \mathbb{R}^{n\times n}$ be a nonnegative matrix of rank r. Then the following conditions are equivalent*:

(i) $EE^{\#} \geqslant O$.
(ii) *There exists a permutation matrix P such that*

$$PEP^t = \begin{bmatrix} XTY & XTYM & O & O \\ O & O & O & O \\ NXTY & NXTYM & O & O \\ O & O & O & O \end{bmatrix}, \tag{1}$$

where the diagonal blocks are square, $T \in \mathbb{R}^{r\times r}$ is a nonnegative nonsingular matrix, $X = \mathrm{diag}(x_1, x_2, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^t, y_2^t, \ldots, y_r^t)$, x_i and $y_{i)}$ are positive unit vectors such that $y_i^t x_i = 1$, and M, N are nonnegative matrices of appropriate size. In this case,

$$PE^{\#}P^t = \begin{bmatrix} XT^{-1}Y & XT^{-1}YM & O & O \\ O & O & O & O \\ NXT^{-1}Y & NXT^{-1}YM & O & O \\ O & O & O & O \end{bmatrix}. \tag{2}$$

Theorem 1.2 (Theorem 7.7.1, [10]). *Let*

$$M = \begin{bmatrix} A & B \\ O & C \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where A and C are square matrices, and $k = \mathrm{ind}(A)$, $l = \mathrm{ind}(C)$. Then

$$M^D = \begin{bmatrix} A^D & X \\ O & C^D \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where

$$X = (A^D)^2 \left[\sum_{i=0}^{l-1} (A^D)^i BC^i\right](I - CC^D) + (I - AA^D)\left[\sum_{i=0}^{k-1} A^i B (C^D)^i\right](C^D)^2 - A^D BC^D \tag{3}$$

being $O^0 = I$ by convention.

Theorem 1.3 (Corollary 7.7.1, [10]). *Let*

$$L = \begin{bmatrix} C & O \\ B & A \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where A and C are square matrices, and $k = \text{ind}(A)$, $l = \text{ind}(C)$. *Then*

$$L^D = \begin{bmatrix} C^D & O \\ X & A^D \end{bmatrix} \in \mathbb{R}^{n\times n},$$

where X is the matrix given in (3).

In this paper we consider singular discrete-time control systems

$$\begin{cases} Ex(k+1) = Ax(k) + Bu(k), \\ y(k) = Cx(k), \end{cases} \tag{4}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(k) \in \mathbb{R}^{n\times 1}$, $u(k) \in \mathbb{R}^{m\times 1}$, and $y(k) \in \mathbb{R}^{p\times 1}$. In general, this system is denoted by (E, A, B, C) and when $C = I$, it is denoted by (E, A, B). We will suppose that this system satisfy the regularity condition, that is, there exists a scalar α such that $\det(\alpha E + A) \neq 0$. In this case, the system (4) can be transformed into the equivalent system

$$\begin{cases} \widehat{E}x(k+1) = \widehat{A}x(k) + \widehat{B}u(k), \\ y(k) = \widehat{C}x(k), \end{cases} \tag{5}$$

where $\widehat{E} = (\alpha E + A)^{-1}E$, $\widehat{A} = (\alpha E + A)^{-1}A$, $\widehat{B} = (\alpha E + A)^{-1}B$, and $\widehat{C} = C$. This new system (5) satisfies the conditions

(i) $\widehat{E}\widehat{A} = \widehat{A}\widehat{E}$,
(ii) $\text{Ker}(\widehat{E}) \cap \text{Ker}(\widehat{A}) = \{0\}$, where $\text{Ker}(\cdot)$ denotes the null space of $(\cdot)$,
(iii) $\widehat{A} = I - \alpha\widehat{E}$.

The regularity condition assures that the system (5) has solution and it is given by $y(k) = \widehat{C}x(k)$ where

$$x(k) = (\widehat{E}^D\widehat{A})^k\widehat{E}^D\widehat{E}x(0) + \sum_{i=0}^{k-1}\widehat{E}^D(\widehat{E}^D\widehat{A})^{k-i-1}\widehat{B}u(i) - (I - \widehat{E}^D\widehat{E})\sum_{i=0}^{q-1}(\widehat{E}\widehat{A}^D)^i\widehat{A}^D\widehat{B}u(k+i), \tag{6}$$

with $q = \text{ind}(\widehat{E})$ and $x(0)$ an initial admissible condition [5]. The set of admissible initial conditions is given by $\text{Im}[\,\widehat{E}^D\widehat{E} \quad H_0 \quad \ldots \quad H_{q-1}\,]$, where $H_i = (I - \widehat{E}^D\widehat{E})(\widehat{E}\widehat{A}^D)^i\widehat{A}^D$, $i = 0, \ldots, q-1$ and $\text{Im}[\cdot]$ denotes the range of $[\cdot]$.

It is well-known that a system (E, A, B, C) is called nonnegative if, for every admissible initial state $x(0) \geqslant 0$ and for every nonnegative control sequence $u(i)$, $i = 0, 1, \ldots, k-1+\text{ind}(E)$, the states $x(k)$ are nonnegative and the outputs y(k) are nonnegative, $\forall k \in \mathbb{Z}^+$. The following result assures the nonnegativity of the system (E, A, B).

Theorem 1.4 (Proposition 1, [12]). *Let* (E, A, B) *be a discrete-time singular system such that* $E^DE \geqslant O$, $EA = AE$, *and* $\text{Ker}(E) \cap \text{Ker}(A) = \{0\}$. *The system* (E, A, B) *is nonnegative if and only if* $E^DA \geqslant O$, $E^DB \geqslant O$, *and* $-(I - EE^D)(EA^D)^iA^DB \geqslant O$, $i = 0, 1, \ldots, \text{ind}(E) - 1$.

The main aim of this paper is to construct an algorithm which allows to decide if the nonnegativity property holds for a singular control system. In order to check the nonnegativity we give conditions over the singular system using information about blocks of the matrices E, A, B, and C instead of conditions on the proper matrices. This fact may reduce considerably the size of the matrices to be studied for guarantyng the nonnegativity of the system (E, A, B, C).

2. Characterization of the nonnegativity

We start this section with a result that gives necessary and sufficient conditions on the matrices E, A, B, and C such that the system (E, A, B, C) is nonnegative.

Theorem 2.1. *Let (E, A, B, C) be a discrete-time singular system such that $E^D E \geqslant O$, $EA = AE$, and $\mathrm{Ker}(E) \cap \mathrm{Ker}(A) = \{0\}$. The system (E, A, B, C) is nonnegative if and only if for each $i = 0, 1, \ldots, \mathrm{ind}(E) - 1$, the following conditions hold*:

(a) $E^D A \geqslant O$,
(b) $E^D B \geqslant O$,
(c) $CE^D E \geqslant O$,
(d) $-(I - EE^D)(EA^D)^i A^D B \geqslant O$, *and*
(e) $-C(I - EE^D)(EA^D)^i A^D B \geqslant O$.

Proof. From Theorem 1.4, it is clear that the nonnegativity of the system (E, A, B) implies (a), (b), and (d).

Since $E^D E \geqslant 0$ and for each $j = 1, \ldots, n$, the vectors $x(0) = E^D E e_j \geqslant 0$ are admissible initial conditions (where e_j is the canonical vector with 1 in the jth component and 0's otherwise). Then, $y(0) = CE^D E e_j \geqslant 0$, for each $j = 1, \ldots, n$ and for every nonnegative control sequence $u(i)$. Hence

$$CE^D E = CE^D E[e_1 \quad e_2 \quad \ldots \quad e_n] = \begin{bmatrix} CE^D E e_1 & CE^D E e_2 & \ldots & CE^D E e_n \end{bmatrix} \geqslant O$$

and (c) holds.

For each $j = 1, \ldots, n$, if we choose $x(0) = 0$, $u(k + h) = e_j$, and $u(i) = 0$, $i \in \{0, 1, \ldots, k - 1 + \mathrm{ind}(E)\} - \{k + h\}$, we have that

$$y(k) = Cx(k) = -C(I - EE^D)(EA^D)^h A^D B e_i \geqslant 0,$$

then $-C(I - EE^D)(EA^D)^h A^D B \geqslant O$ for all $h = 0, 1, \ldots \mathrm{ind}(E) - 1$ and (e) holds.

Conversely, from $E^D A = AE^D$ and $E^D E E^D = E^D$, it is easy to check that (a), (b), (c), (d), and (e) imply that $x(k) \geqslant 0$ and $y(k) \geqslant 0$ for every $u(i) \geqslant 0$, $i = 1, \ldots, k - 1 + \mathrm{ind}(E)$ and every $x(0)$ nonnegative admissible initial condition. $\square$

Next, we obtain a relationship on the index of the involved matrices E and $\widehat{E}$. In general, neither $\mathrm{ind}(E) = 1$ imply $\mathrm{ind}(\widehat{E}) = 1$ nor $\mathrm{ind}(\widehat{E}) = 1$ imply $\mathrm{ind}(E) = 1$. For instance, given the matrices

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and setting $\alpha = 0$, one has that

$$\widehat{E} = (\alpha E + A)^{-1} E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix},$$

with $\mathrm{ind}(\widehat{E}) = 2$ while $\mathrm{ind}(E) = 1 \neq 2$. In the same way, setting

$$\widehat{E} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and $\alpha = 0$, one has that $E = A\widehat{E}$. So, $\mathrm{ind}(\widehat{E}) = 1$ and $\mathrm{ind}(E) = 2 \neq 1$.

However, we can prove the following result using the Corollary 7.2.2 in [10].

Lemma 2.2. *Let $E \in \mathbb{R}^{n \times n}$ be a matrix with $\mathrm{ind}(E) = 1$, that is,*

$$E = Q \begin{bmatrix} C & O \\ O & O \end{bmatrix} Q^{-1},$$

where C is a nonsingular matrix. Let $M \in \mathbb{R}^{n \times n}$ a nonsingular matrix such that $Q^{-1} M Q$ is a block upper-triangular matrix. Then $\mathrm{ind}(ME) = 1$.

Proof. Since $Q^{-1}MQ$ is a block upper-triangular matrix,

$$Q^{-1}MQ = \begin{bmatrix} R_1 & R_2 \\ O & R_3 \end{bmatrix}.$$

Then,

$$ME = Q\begin{bmatrix} R_1C & O \\ O & O \end{bmatrix}Q^{-1}.$$

The nonsingularity of M implies that $Q^{-1}MQ$ is nonsingular. So, R_1 is nonsingular and, by Corollary 7.2.2 in [10], $\operatorname{ind}(ME) = 1$ because C is nonsingular. $\square$

From now on, we will consider the singular control system (5) in which $\widehat{E}$ is a nonnegative matrix with index equals 1 and $\widehat{E}^{\#}\widehat{E} \geqslant O$. From Theorem 1.1, the matrix $\widehat{E}$ has the form (1). The orthogonal matrix P appearing in (1) allows to transform the system (5) into the following equivalent system:

$$\begin{cases} \widetilde{E}z(k+1) = \widetilde{A}z(k) + \widetilde{B}u(k), \\ y(k) = \widetilde{C}z(k), \end{cases} \tag{7}$$

where $z(k) = Px(k)$,

$$\widetilde{E} = P\widehat{E}P^t = \begin{bmatrix} XTY & XTYM & O & O \\ O & O & O & O \\ NXTY & NXTYM & O & O \\ O & O & O & O \end{bmatrix},$$

$$\widetilde{A} = P\widehat{A}P^t = I - \alpha\widetilde{E} = \begin{bmatrix} I - \alpha XTY & -\alpha XTYM & O & O \\ O & I & O & O \\ -\alpha NXTY & -\alpha NXTYM & I & O \\ O & O & O & I \end{bmatrix},$$

$$\widetilde{B} = P\widehat{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}, \tag{8}$$

and

$$\widetilde{C} = \widehat{C}P^t = [\,C_1 \quad C_2 \quad C_3 \quad C_4\,]. \tag{9}$$

An easy computation allows to show that $\operatorname{ind}(\widehat{E}) = 1$ implies $\operatorname{ind}(\widetilde{E}) = 1$ (actually, the equivalence between $\operatorname{ind}(\widehat{E}) = 1$ and $\operatorname{ind}(\widetilde{E}) = 1$ is valid). Moreover, the conditions $\widehat{E}^{\#}\widehat{E} \geqslant O$, $(\widehat{E}^{\#}\widehat{A})^k \geqslant O$, $(\widehat{E}^{\#}\widehat{A})^k\widehat{E}^{\#}\widehat{E} \geqslant O$, $\widehat{E}^{\#}\widehat{B} \geqslant O$, $-(I - \widehat{E}\widehat{E}^{\#})\widehat{A}^D\widehat{B} \geqslant O$, $\widehat{C}\widehat{E}^{\#}\widehat{E} \geqslant O$, and $-\widehat{C}(I - \widehat{E}\widehat{E}^{\#})\widehat{A}^D\widehat{B} \geqslant O$ are equivalent to $\widetilde{E}^{\#}\widetilde{E} \geqslant O$, $(\widetilde{E}^{\#}\widetilde{A})^k \geqslant O$, $(\widetilde{E}^{\#}\widetilde{A})^k\widetilde{E}^{\#}\widetilde{E} \geqslant O$, $\widetilde{E}^{\#}\widetilde{B} \geqslant O$, $-(I - \widetilde{E}\widetilde{E}^{\#})\widetilde{A}^D\widetilde{B} \geqslant O$, $\widetilde{C}\widetilde{E}^{\#}\widetilde{E} \geqslant O$, and $-\widetilde{C}(I - \widetilde{E}\widetilde{E}^{\#})\widetilde{A}^D\widetilde{B} \geqslant O$, because $(P\widehat{A}P^{-1})^D = P\widehat{A}^DP^{-1}$.

So, we will use the system (7) where, for simplicity, the matrices will be denoted by E, A, B, and C instead of $\widetilde{E}$, $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$. We are interested on finding conditions on the matrices E, A, B, and C such that $E^{\#}E \geqslant O$, $E^{\#}A \geqslant O$, $(E^{\#}A)^kE^{\#}E \geqslant O$, $E^{\#}B \geqslant O$, $-(I - EE^{\#})A^DB \geqslant O$, $CE^{\#}E \geqslant O$, and $-C(I - EE^{\#})A^DB \geqslant O$, under the circumstances $E \geqslant O$, $\operatorname{ind}(E) = 1$, and $A = I - \alpha E$.

A series of lemmas will lead to the main result of this section

Lemma 2.3. *Let $E, A \in \mathbb{R}^{n\times n}$ such that $\operatorname{ind}(E) = 1$, $E \geqslant O$, $EE^{\#} \geqslant O$, and $A = I - \alpha E$, for some scalar α. Then $(E^{\#}A)^k \geqslant O$ if and only if $T^{-1} - \alpha I \geqslant O$, where X, Y, and T are the matrices defined in (1).*

Proof. Since E and A are the coefficient matrices of the system (7), we get

$$E^{\#}A = E^{\#} - \alpha E^{\#}E = \begin{bmatrix} X(T^{-1}-\alpha I)Y & X(T^{-1}-\alpha I)YM & O & O \\ O & O & O & O \\ NX(T^{-1}-\alpha I)Y & NX(T^{-1}-\alpha I)YM & O & O \\ O & O & O & O \end{bmatrix}.$$

By induction one has

$$(E^{\#}A)^k = \begin{bmatrix} (X(T^{-1}-\alpha I)Y)^k & (X(T^{-1}-\alpha I)Y)^k M & O & O \\ O & O & O & O \\ N(X(T^{-1}-\alpha I)Y)^k & N(X(T^{-1}-\alpha I)Y)^k M & O & O \\ O & O & O & O \end{bmatrix} \tag{10}$$

for every positive integer k. Given that $YX = I$, it follows $(X(T^{-1}-\alpha I)Y)^k = X(T^{-1}-\alpha I)^k Y$ and then the nonnegativity of $(E^{\#}A)^k$ holds if and only if the following conditions are fulfilled:

(i) $X(T^{-1}-\alpha I)^k Y \geqslant O$,
(ii) $X(T^{-1}-\alpha I)^k YM \geqslant O$,
(iii) $NX(T^{-1}-\alpha I)^k Y \geqslant O$, and
(iv) $NX(T^{-1}-\alpha I)^k YM \geqslant O$,

which can be reduced to the only condition $X(T^{-1}-\alpha I)^k Y \geqslant O$, because M and N are nonnegative matrices. Now, since $X \geqslant O$, $Y \geqslant O$, and $YX = I$, one has that $X(T^{-1}-\alpha I)^k Y \geqslant O$ is equivalent to $(T^{-1}-\alpha I)^k \geqslant O$, for all positive integer k, and in particular for $k = 1$. So, the lemma has been proved. □

Until now, conditions on each one of the factors of $(E^{\#}A)^k(E^{\#}E)$ have been given to obtain its nonnegativity. In the following result the whole product is analyzed.

Corollary 2.4. *Let $E, A \in \mathbb{R}^{n\times n}$ such that* $\operatorname{ind}(E) = 1$, *$E \geqslant O$, $EE^{\#} \geqslant O$, and $A = I - \alpha E$ for some scalar α. Then the following conditions are equivalent*:

(a) $(E^{\#}A)^k \geqslant O$,
(b) $T^{-1} - \alpha I \geqslant O$,
(c) $(E^{\#}A)^k E^{\#}E \geqslant O$,

for all nonnegative integer k.

Proof. From Lemma 2.3, it is evident that conditions (a) and (b) are equivalent. From expression (1), (2), and (10) we get

$$(E^{\#}A)^k E^{\#}E = P^t \begin{bmatrix} X(T^{-1}-\alpha I)^k Y & X(T^{-1}-\alpha I)^k YM & O & O \\ O & O & O & O \\ NX(T^{-1}-\alpha I)^k Y & NX(T^{-1}-\alpha I)^k YM & O & O \\ O & O & O & O \end{bmatrix} P.$$

So, the equivalence between (b) and (c) can be easily seen, because M, N, X, and Y are nonnegative matrices and $YX = I$. □

Lemma 2.5. *Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ such that* $\operatorname{ind}(E) = 1$, *$E \geqslant O$, $EE^{\#} \geqslant O$, and $A = I - \alpha E$, for some scalar α, and B partitioned as in (8). Then $E^{\#}B \geqslant O$ if and only if $T^{-1}Y(B_1 + MB_2) \geqslant O$, where T, Y, and M are the matrices defined in (1).*

Proof. From (2) we have

$$E^{\#}B = \begin{bmatrix} XT^{-1}YB_1 + XT^{-1}YMB_2 \\ O \\ NXT^{-1}YB_1 + NXT^{-1}YMB_2 \\ O \end{bmatrix}.$$

Hence, $E^{\#}B \geqslant O$ if and only if $XT^{-1}Y(B_1 + MB_2) \geqslant O$ (since $N \geqslant O$). Premultipliying this last inequality by Y and using that $YX = I$ and that Y is a nonnegative matrix, we get that $E^{\#}B \geqslant O$ if and only if $T^{-1}Y(B_1 + MB_2) \geqslant O$. $\square$

It is clear that a necessary condition to obtain $E^{\#}B \geqslant O$ is $Y(B_1 + MB_2) \geqslant O$ since $TT^{-1} = I$ and $T \geqslant O$.
Now, in order to study the product $(I - EE^{\#})A^DB$, we proceed to compute A^D. In fact, since

$$A = \left[\begin{array}{cc|cc} I - \alpha XTY & -\alpha XTYM & O & O \\ O & I & O & O \\ \hline -\alpha NXTY & -\alpha NXTYM & I & O \\ O & O & O & I \end{array}\right] =: \left[\begin{array}{c|c} F & O \\ \hline G & I \end{array}\right], \tag{11}$$

where F and G have the adequate sizes, applying the Theorem 1.3 we obtain that

$$A^D = \begin{bmatrix} F^D & O \\ \widetilde{G} & I \end{bmatrix},$$

where

$$\widetilde{G} = \left[\sum_{i=0}^{l-1} GF^i\right](I - FF^D) - GF^D \tag{12}$$

being l the index of F.

Note that A, F and $H := I - \alpha XTY$ have the same index. In fact, from (11) it is easy to see that $\operatorname{rank}(A^k) = \operatorname{rank}(F^k) + \operatorname{rank}(I)$ and $\operatorname{rank}(A^{k-1}) = \operatorname{rank}(F^{k-1}) + \operatorname{rank}(I)$. So, $\operatorname{rank}(A^k) - \operatorname{rank}(A^{k-1}) = \operatorname{rank}(F^k) - \operatorname{rank}(F^{k-1})$, which implies that $\operatorname{ind}(A) = \operatorname{ind}(F)$. Similarly, one gets that $\operatorname{rank}(F^k) = \operatorname{rank}(H^k) + \operatorname{rank}(I)$, for all nonnegative integer k, and thus $\operatorname{ind}(F) = \operatorname{ind}(H)$. This shows that the relevant information of A is in its (1,1) block.

From Theorem 1.2 one has that the Drazin inverse of F is

$$F^D = \begin{bmatrix} H^D & S \\ O & I \end{bmatrix},$$

being

$$S = (I - HH^D)\left[\sum_{i=0}^{l-1} H^iJ\right] - H^DJ \tag{13}$$

and $J := -\alpha XTYM$. Relative to the matrix $\widetilde{G}$ appearing in (12), we have that

$$F^i = \begin{bmatrix} H^i & \sum_{j=0}^{i-1} H^jJ \\ O & I \end{bmatrix},$$

and then

$$\sum_{i=0}^{l-1} F^i = I + \begin{bmatrix} \sum_{j=1}^{l-1} H^j & \sum_{p=1}^{l-1} pH^{l-p-1}J \\ O & (l-1)I \end{bmatrix}.$$

So,

$$\widetilde{G} = \begin{bmatrix} U & V \\ O & O \end{bmatrix},$$

where

$$U := -\alpha NXTY\left(\sum_{j=0}^{l-1} H^j(I - HH^D) - H^D\right) \tag{14}$$

and

$$V := \alpha NXTY\left(\sum_{j=0}^{l-1} H^j(HS + J) + S\right). \tag{15}$$

Summarizing the previous computations, the Drazin inverse of A is

$$A^D = \begin{bmatrix} H^D & S & O & O \\ O & I & O & O \\ U & V & I & O \\ O & O & O & I \end{bmatrix}, \tag{16}$$

with $H = I - \alpha XTY$ and S, U, and V defined in (13)–(15), respectively. Observe that the matrices A and A^{D} have the same block structure.

A similar reasoning as before allows to give the following results.

Lemma 2.6. *Let* $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ *such that* $\mathrm{ind}(E) = 1$, $E \geqslant O$, $EE^{\#} \geqslant O$, $A = I - \alpha E$, *for some scalar* α, *and* B *partitioned as in* (8). *Then* $-(I - E^{\#}E)A^D B \geqslant O$ *if and only if*

(a) $-B_2 \geqslant O$,
(b) $-B_4 \geqslant O$,
(c) $(XY - I)(H^D B_1 + SB_2) + XYMB_2 \geqslant O$,
(d) $NXY(H^D B_1 + SB_2 + MB_2) - UB_1 - VB_2 - B_3 \geqslant O$.

where X, Y, M, N, *and* T *are the matrices defined in* (1), $H = I - \alpha XTY$, *and* S, U, *and* V *given in* (13)–(15), *respectively.*

Proof. From (1), (2), (16) and (8) an easy calculation leads to

$$(I - EE^{\#})A^D B = \begin{bmatrix} (I - XY)(H^D B_1 + SB_2) - XYMB_2 \\ B_2 \\ -NXY(H^D B_1 + SB_2) - NXYMB_2 + UB_1 + VB_2 + B_3 \\ B_4 \end{bmatrix}, \tag{17}$$

and then the result follows directly.

Lemma 2.7. *Let* $E, A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ *such that* $\mathrm{ind}(E) = 1$, $E \geqslant O$, $EE^{\#} \geqslant O$, $A = I - \alpha E$, *for some scalar* α, *and* C *partitioned as in* (9). *Then* $CE^{\#}E \geqslant O$ *if and only if* $(C_1 + C_3 N)X \geqslant O$, *where* X *and* N *are the matrices defined in* (1).

Proof. From (9),(1) and (2) a direct computation leads to

$$CE^{\#}E = [\,C_1 \quad C_2 \quad C_3 \quad C_4\,]\begin{bmatrix} XY & XYM & O & O \\ O & O & O & O \\ NXY & NXYM & O & O \\ O & O & O & O \end{bmatrix} = [\,(C_1 + C_3N)XY \quad (C_1 + C_3N)XYM \quad O \quad O\,].$$

Then, the equivalence between $CE^{\#}E \geqslant O$ and the two conditions $(C_1 + C_3N)XY \geqslant O$ and $(C_1 + C_3N)XYM \geqslant O$ is clear. Since $M \geqslant O$, $Y \geqslant O$, and $YX = I$, both conditions can be reduced to $(C_1 + C_3N)X \geqslant O$. $\square$

Lemma 2.8. *Let $E, A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$ such that $\operatorname{ind}(E) = 1$, $E \geqslant O$, $EE^{\#} \geqslant O$, $A = I - \alpha E$, for some scalar α, and B and C partitioned as in (8) and (9), respectively. Then $-C(I - E^{\#}E)A^DB \geqslant O$ if and only if $C_1[(XY - I)(H^DB_1 + SB_2) + XYMB_2] - C_2B_2 + C_3[NXY(H^DB_1 + SB_2) + NXYMB_2 - UB_1 - VB_2 - B_3] - C_4B_4 \geqslant O$, where X, Y, M, N, and T are the matrices defined in (1), $H = I - \alpha XTY$, and S, U, and V are given in (13)–(15), respectively.*

Proof. The analysis of the nonnegativity of the product of the matrices C and $(I - E^{\#}E)A^DB$, given by the expressions (9) and (17) respectively, permits to show the lemma. $\square$

We close this section with the a result which summarize all the previous information.

Theorem 2.9. *Let (E,A,B,C) be a singular system where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ are matrices such that $\operatorname{ind}(E) = 1$, $E \geqslant O$, $EE^{\#} \geqslant O$, $A = I - \alpha E$, for some scalar α, and B and C are partitioned as in (8) and (9), respectively. Then (E,A,B,C) is nonnegative if and only if the following conditions hold*

(a) $T^{-1} - \alpha I \geqslant O$,
(b) $T^{-1}Y(B_1 + MB_2) \geqslant O$,
(c) $-B_2 \geqslant O$,
(d) $-B_4 \geqslant O$,
(e) $(XY - I)(H^DB_1 + SB_2) + XYMB_2 \geqslant O$,
(f) $NXY(H^DB_1 + SB_2 + MB_2) - UB_1 - VB_2 - B_3 \geqslant O$,
(g) $(C_1 + C_3N)X \geqslant O$,
(h) $C_1[(XY - I)(H^DB_1 + SB_2) + XYMB_2] - C_2B_2 + C_3[NXY(H^DB_1 + SB_2 + MB_2) - UB_1 - VB_2 - B_3] - C_4B_4 \geqslant O$.

where X, Y, M, N, and T are the matrices defined in (1), $H = I - \alpha XTY$, and S, U, and V are given in (13)–(15), respectively.

Proof. It is an immediate application of Corollary 2.4 and Lemmas 2.5–2.8. $\square$

3. Algorithm and examples

The algorithm presented below permits to decide when a singular control linear system (withind$(\widehat{E}) = 1$) is nonnegative.

ALGORITHM *Inputs*: Singular system (E, A, B, C).

Step 1 Find α such that $\det(\alpha E + A) \neq 0$.
Step 2 Compute $\widehat{E}$, $\widehat{B}$ and $\widehat{C}$ as in (5) and set $\widehat{A} = I - \alpha\widehat{E}$.
Step 3 Compute $\operatorname{ind}(\widehat{E})$ and $\widehat{E}^{\#}$.
Step 4 If $\operatorname{ind}(\widehat{E}) \neq 1$ or $\widehat{E} \ngeqslant O$ or $\widehat{E}\widehat{E}^{\#} \ngeqslant O$ then go to End.
Step 5 Find a permutation P such that $\widehat{E}$ is in the form (1).
Step 6 Set $E = P\widehat{E}P^t$ and $A = P\widehat{A}P^t$, and partition $B = P\widehat{B}$ and $C = \widehat{C}P^t$ as in (8) and (9), respectively.
Step 7 Compute $H = I - \alpha XTY$, $J = -\alpha XTYM$, S, U and V as in (13)–(15), respectively.
Step 8 If all the conditions (a)–(h) in Theorem 2.9 are satisfied the '*The original system is nonnegative*' else '*The original system is not nonnegative*'.

End

Next, we illustrate the obtained results with an example depending on parameters.

Example 3.1. Let (E, A, B, C) be the singular system given by the matrices

$$E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad A = I - \alpha E = \begin{bmatrix} 1 - \frac{\alpha}{2} & -\frac{\alpha}{2} & 0 \\ 0 & 1 & 0 \\ -\frac{\alpha}{2} & -\frac{\alpha}{2} & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix}, \quad \text{and} \quad C = [\, C_1 \quad C_2 \quad C_3 \,],$$

with $\alpha < 2$, the scalars b_1 and b_3 satisfying that $b_1 \geqslant 0$, $b_3 \leqslant b_1$, and the vectors C_1 and C_3 satisfying $C_3 \geqslant O$ and $C_1 + C_3 \geqslant O$.

Then, Theorem 2.9 assures that the system is nonnegative. In fact, from expression (6) the state vector is given by

$$\begin{aligned} x(k) &= (E^{\#}A)^k E^{\#} E x(0) + \sum_{i=0}^{k-1} E^{\#}(E^{\#}A)^{k-i-1} B u(i) - (I - E^{\#}E) A^D B u(k) \\ &= (2-\alpha)^k \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} x(0) + 2b_1 \sum_{i=0}^{k-1} (2-\alpha)^{k-i-1} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u(i) - \begin{bmatrix} 0 \\ 0 \\ b_3 - b_1 \end{bmatrix} u(k) \end{aligned}$$

and the output vector is

$$y(k) = Cx(k) = (2-\alpha)^k [\, C_1 + C_3 \quad C_1 + C_3 \quad O \,] x(0) + 2b_1 \sum_{i=0}^{k-1} (2-\alpha)^{k-i-1} (C_1 + C_3) u(i) - C_3 (b_3 - b_1) u(k).$$

We have identified $P = I$, $T = \frac{1}{2}$, and $X = Y = M = N = 1$ as in Theorem 1.1. The Fig. 1 shows the nonnegativity of the output sequence $y(k)$ for different values of the parameter α and different initial conditions $x(0)$. In both cases, we have chosen $B = [\, 2 \quad 0 \quad 1 \,]^t$, $C = [\, 1 \quad 0 \quad 1 \,]$ and the control sequence $u(i) = i$ for $i = 1, \ldots, k$.

Note that in this example we have considered $\alpha < 2$ obtaining a nonsingular matrix A. Then, the nonnegativity of the system has been checked using the Theorem 2.9. In a similar way, it is possible to check the nonnegativity of this system using the Theorem 2.1 with a similar effort. However, if we consider the example with $\alpha = 2$ then it is easier to apply the Theorem 2.9 than the Theorem 2.1, ought to in the last one we have to compute a Drazin inverse because the matrix A is singular.

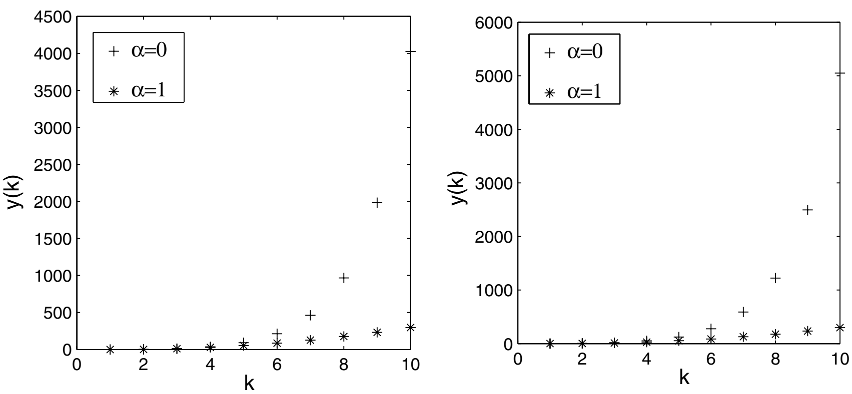

Fig. 1. The left figure represents the outputs y(k) for $x(0) = [0, 0, 0]^t$ and the right one plots the outputs y(k) for $x(0) = [1, 0, 1]^t$.

4. Conclusions

In order to decide if a singular control system (E,A,B,C) in nonnegative we could firstly apply the Theorem 2.1. The inconvenient of this theorem is due to the fact that the Drazin inverse not always can be computed in an easy way. So, we have designed an algorithm by means of a technique that involves matrices of smaller sizes partitioning in blocks the original matrices in a suitable way. In this sense, our algorithm improves the result given in Theorem 2.1, where the whole matrices are used to check the nonnegativity of the system.

Moreover, the step to obtain the mentioned partition only uses permutation matrices, that is, only changes the distribution of the information in the original matrices without making any computation.

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