A Generalized State-Space for Singular Systems

GEORGE C. VERGHESE, MEMBER, IEEE, BERNARD C. LÉVY, MEMBER, IEEE, AND THOMAS KAILATH, FELLOW, IEEE

Abstract—Systems of the form $E\dot{x}=Ax+Bu$, $y=Cx$, with E singular, are studied. Of particular interest are the impulsive modes that may appear in the free-response of such systems when arbitrary initial conditions are permitted, modes that are associated with natural system frequencies at infinity. A generalized definition of system order that incorporates these impulsive degrees of freedom is proposed, and concepts of controllability and observability are defined for the impulsive modes. Allowable equivalence transformations of such singular systems are specified. The present framework is shown to overcome several difficulties inherent in other treatments of singular systems, and to extend, in a natural and satisfying way, many results previously known only for regular state-space systems.

I. Introduction

A. Basic Facts

We consider the time-invariant system of r first-order coupled linear differential equations

$$E\dot{x}(t)=Ax(t)+Bu(t) \tag{1a}$$

$$y(t)=Cx(t), \qquad t\geqslant 0 \tag{1b}$$

where x is an r-vector of internal variables, u is an m-vector of control inputs or forcing functions, and y is a p-vector of outputs.

If $Ex(0-)$ is known and $u(t)$ specified for $t\geqslant 0$, then the solution of (1) may be written in the Laplace transform or frequency domain (see, for example, Doetsch [1]) as

$$X(s)=(sE-A)^{-1}[Ex(0-)+BU(s)] \tag{2a}$$

$$Y(s)=CX(s). \tag{2b}$$

Here X, U, Y denote the transforms of x, u, y. We assume invertibility of $(sE-A)$ so that unique solutions of (1) are obtained for all $Ex(0-)$ and $U(s)$. The output and input under zero initial conditions (i.e., $Ex(0-)=0$) are related by the transfer function $G(s)$, as follows:

$$Y(s)=G(s)U(s)$$

where

$$G(s)=C(sE-A)^{-1}B. \tag{3}$$

The case where E is nonsingular has been quite thoroughly studied and is now rather well understood. In this case (1) is normally rewritten in the form

$$\begin{aligned}\dot{x}(t)&=(E^{-1}A)x(t)+(E^{-1}B)u(t)\\&=\bar{A}x(t)+\bar{B}u(t)\end{aligned} \tag{4a}$$

$$y(t)=Cx(t) \tag{4b}$$

and is said to define a *regular state-space system*, with state (vector) x. Certain features of this case that are of special interest may be listed (see, for example, [2], [45]), and will serve as points of contrast with the case of singular E.

i) It is easily seen from (2a) that knowledge of $Ex(0-)$ is necessary and sufficient to completely determine $x(t)$ for $t\geqslant 0$, given $u(t)$ for $t\geqslant 0$, and for this case of nonsingular E it is evident that the r-vector $Ex(0-)$ can take r independent values. The system (1a) thus has r degrees of freedom and this is termed the (regular) *order* of the (regular) state-space system.

ii) The transfer function $G(s)$ is *strictly proper*, i.e., $G(s)\to 0$ as $s\to\infty$.

iii) The free-response of the system, i.e., $x(t)$ for $t\geqslant 0$ when $u(\cdot)\equiv 0$, consists of combinations of exponential motions or modes at those so-called natural or characteristic frequencies $s=\lambda$ for which $(sE-A)$ is singular, namely the r finite (possibly nondistinct) roots of $|sE-A|$. [In fact, the detailed connection of these modes with the Smith zero structure of $(sE-A)$ has been known for some time (see [3] for example).]

When E is singular in (1), resulting in what we shall term a *generalized state-space system* or a *singular system*, this behavior is considerably modified. In contrast to i)–iii), we find the following.

ia) The number of degrees of freedom of the system, i.e., the number of independent values that $Ex(0-)$ can take, is now evidently reduced to

$$f\triangleq \operatorname{rank} E<r. \tag{5}$$

We propose the term *generalized order* for f.

Manuscript received February 8, 1978; revised February 23, 1981 and March 16, 1981. Paper recommended by M. K. Sain, Past Chairman of the Linear Systems Committee. This work was supported by the National Science Foundation under Contract ENG-78-1003, by the Air Force Office of Scientific Research, Air Force Systems Command, under Contract AF49-620-79-C-0058, and by the Joint Services Electronics Program under Contract DAAG-29-C-0047.

G. C. Verghese is with the Electric Power Systems Engineering Laboratory, Massachusetts Institute of Technology, Cambridge, MA 02139.

B. C. Lévy is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139.

T. Kailath is with the Information Systems Laboratory, Stanford University, Stanford, CA 94305.

iia) The transfer function $G(s)$ may no longer be strictly proper, as we shall soon show, in which case it may be written as the sum of a strictly proper part $\overline{G}(s)$ and a polynomial part $D(s)$.

iiia) For this case of singular E,

$$\text{degree of} |sE-A| \triangleq n \leqslant f < r. \tag{6}$$

The free-response of the system in this case exhibits exponential motions, as before, at the n finite frequencies $s=\lambda$ (possibly nondistinct) where $(sE-A)$ is singular. In addition, however, it contains *f-n impulsive* (i.e., *distributional*) motions, or "infinite-frequency" modes (corresponding essentially to $(sE-A)$ losing rank at $s=\infty$). The detailed demonstration of this fact will be presented later in this paper. [We note that, paralleling the already mentioned connection between the Smith zero structure of $(sE-A)$ and the exponential modes, we have the recently established connection between the *infinite-zero* structure of $(sE-A)$, defined via its Smith–McMillan form, and the impulsive modes referred to above. (For details, we refer the reader to [4], [5].) While the results in these latter references, obtained after the first version of this paper was completed, significantly complement our results here and extend them to systems more general than the form (1), we shall give here a different and quite self-contained presentation. We nevertheless urge the interested reader to examine [4] and also [5].]

The following example is the simplest one that illustrates ia)–iiia).

Example 1.1: Consider

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad t \geqslant 0, \tag{7a}$$

$$y(t) = [1 \quad 0][x_1(t) \quad x_2(t)]'. \tag{7b}$$

ia) The system has generalized order

$$f = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 1. \tag{8}$$

iia) The system transfer function is

$$G(s) = -s. \tag{9}$$

iiia) $n=$ degree of

$$\begin{vmatrix} -1 & s \\ 0 & -1 \end{vmatrix} = 0.$$

The system's free-response therefore exhibits *no* exponential motions, but does display $f-n=1$ impulsive motion. To see this, note from (7a) with $u(t) \equiv 0$ that

$$x_1(t) = \dot{x}_2(t) \tag{10a}$$

$$x_2(t) = 0, \qquad t \geqslant 0. \tag{10b}$$

This pair of equations has the impulsive solution

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -x_2(0-)\delta(t) \\ 0 \end{bmatrix}, \qquad t \geqslant 0 \tag{11}$$

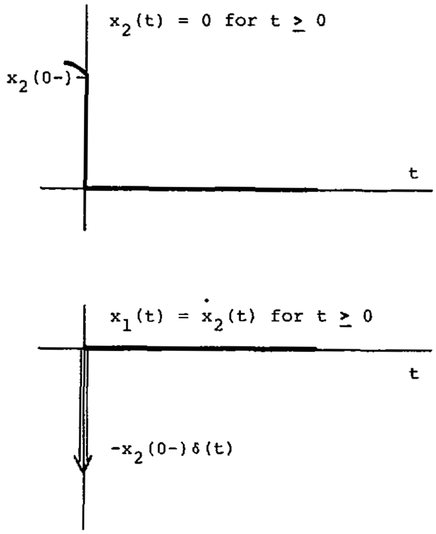


Fig. 1. Free-response of the system in Example 1.1.

because the step that results from $x_2(t)$ falling from its value $x_2(0-)$ to the value 0 required by (10b) for $t \geqslant 0$ is differentiated in accordance with (10a) and appears as an impulse in $x_1(t)$; see Fig. 1. The same answer could of course have been obtained by using the Laplace transformed version of (7a), as in (2a). $\triangle\triangle$

It is the purpose of this paper to explore these and other consequences of having a singular E in the system (1).

B. Background and Previous Work

Descriptions of dynamical systems in the form (1) arise especially naturally when these systems are formed from interconnected subsystems (see, for example, Rosenbrock and Pugh [6]). In fact, any system can generally be viewed as an interconnection of subsystems, and when the natural differential equations and algebraic constraints describing the system are first written down, they involve the internal variables of the subsystems in a description of the form (1), usually with a singular coefficient matrix E. This happens, for example, when describing electrical networks using the element currents and voltages as internal variables (see, for instance, Dziurla and Newcomb [7], who term equations of the form (1) "semistate equations"). Even in cases where E is nonsingular, it may be of interest to relate the behavior of the system (1) to that of a simplified or idealized model of it in which E is singular; this is characteristic of studies of singularly perturbed systems (see, Kokotovic *et al.* [8].) We shall return later in this paper to briefly consider both electrical networks and singularly perturbed systems. Discrete-time versions of (1) also arise quite commonly (see Luenberger [9], [10], who calls these "descriptor form" systems).

Quite naturally, therefore, systems of the form (1) have been the subject of much attention. The use of the "L_--transform," i.e., the Laplace transform with initial conditions specified at $t=0-$, to obtain solutions of such systems, including the impulsive parts of such solutions, is explained in texts of fifteen years ago (see Doetsch [1], Erdelyi [11], Liverman [12]). However, while the detailed structure of the finite frequency or exponential modes of (1) is well known, [3], the corresponding structure of the *impulsive* modes does not appear to have been pursued prior to [4], [5]. The following subsections make the reasons for this apparent.

1) The Regular Theory: With the exception of a paper of Rosenbrock [13] (and a few more recent ones [50]–[53] that have, most of them independently, arrived at similar frameworks to [13]), essentially all of the work that we are aware of on equations of the form (1) does not address the sorts of structural and dynamical issues concerning (1) that are treated in [13], [50]–[53], and the present paper; instead, the usual procedure for dealing with (1) is to explicitly or implicitly reduce it to an "equivalent" regular state-space system, of order $n=\text{degree }|sE-A|$, and of the form

$$\dot{\bar{x}}=\bar{A}\bar{x}+\bar{B}u \tag{12a}$$

$$y=\bar{C}\bar{x}+D(p)u. \tag{12b}$$

Here $p=d/dt$ and $D(s)$ is the polynomial part of the transfer function $G(s)$ of (1), cf. iia) in the preceding. Established methods and results are then used to solve and analyze (12). The reduction procedure is in essence contained in the discussion by Gantmacher [14] of the Kronecker form of $(sE-A)$. It may also be found on specializing the algorithm of Polak [15]. In the circuit theory literature Fettweis [16] and Dervisoglu and Desoer [17], for example, present procedures that yield an $\bar{x}$ that is an untransformed subvector of x, which may be desirable in some situations (but which may require one to have an input matrix $\bar{B}(p)$, instead of a constant $\bar{B}$, in (12) above). Luenberger [10] presents a related procedure, his so-called "shuffle algorithm;" though stated for discrete-time systems, it applies to the reduction of (1) also. In the numerical analysis literature, Wilkinson [18], for example, presents a reduction procedure. A rather different approach that is implicitly a reduction procedure is described by Campbell, Meyer and Rose [19], using the language of Drazin inverses, see also [7]. Gohberg and Rodman [20] have yet another procedure that may be construed as an implicit reduction procedure. (The reason for terming some of these "implicit" reductions is that, while in appearance they may not result in (12), in effect they do.)

All of this work employs essentially the same definition of "equivalence," though it is only fairly recently that the concept (and with it the class of allowable transformation and reduction procedures) has been formalized and extended to a very general setting, through results of Rosenbrock [21], Wolovich [22], Pernebo [23], Furhmann [24], and Lévy *et al.* [25]; Rosenbrock terms the concept "strict system equivalence" (s.s.e.). In essence, an equivalent system in the context of the *regular theory* [as we shall from now on refer to the body of work that explicitly or implicitly deals with (12)] is one in which the structure of the *finite*-frequency modes and of their coupling to the system inputs and outputs is preserved; the only part of the *infinite*-frequency modal structure that is preserved is the polynomial part $D(s)$ of the transfer function. As a result, none of this previous work has identified the structure of the impulsive motions that the free-response of (1) can display. The papers [13], [51]–[53], also miss the impulsive free-response but as our discussion of [13] will show, the treatments in these papers (and in Cobb [50], which we shall discuss briefly in a later section) are distinguished by the fact that they consider in more detail the part of the system that generates $D(s)$.

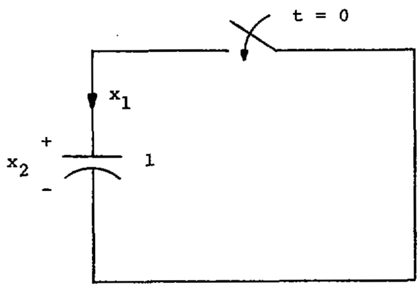


Fig. 2. Circuit realization of the system in Example 1.1.

2) The Generalized Theory: The reason the loss of information about the impulsive modes that results from transforming (1) to (12) has not been of concern in system theory is that it is almost invariably assumed, usually only tacitly (an exception being Rosenbrock [21, pp. 47–49]), that (1) holds for $t<0$ also (with $u(t)\equiv 0$ for $t<0$, though this is not critical). In this case the initial condition vector $Ex(0-)$ is itself *constrained to satisfy* (1), and this constraint turns out to guarantee that (1) *never has an impulsive free-response*. The system (1) in this case effectively has only n (rather than f) degrees of freedom, and the equivalence of (12) to (1) is as complete as one could wish.

When, however, equation (1) describes a system formed *at* $t=0$, as a result perhaps of *switching* or *component failure* in some other system, the regular assumptions no longer hold, and it becomes important to consider the possibility of impulsive solutions in response to unconstrained values of $Ex(0-)$ (or perhaps values that satisfy constraints other than (1)); such initial conditions have been termed "inconsistent" (and even "inadmissible"). Doetsch [1] and Blomberg [26] are among the few who have mentioned the physical significance of such initial conditions. (A physical example corresponding to Example 1.1 is shown in Fig. 2.) Sincovec *et al.* [27] consider algorithms for the numerical solution of (1) with inconsistent initial conditions (their paradigm also being a switched system) but seem to miss the fact that there may be impulsive solutions in this case.

There are other situations where it is important to consider inconsistent initial conditions. For example, the matrix E in (1) may be singular because of idealizations or approximations made in some other "singularly perturbed" system that *could* have initial conditions unconstrained by (1); we shall restate this problem more explicitly later. For another example, we note that even in unswitched composite systems it is often important to know how subsystem properties are reflected into those of the composite, and for this it is necessary to have a theory that can identify the consequences of interconnection; some of our results in this paper will illustrate this point.

It is our belief that our results represent a useful step towards developing a more comprehensive framework for systems of the form (1) and for treating problems of the type mentioned above. We shall refer to results in this vein as elements of the *generalized theory* to distinguish them from results of the regular theory. It will become apparent, however, that several new insights and results on the regular theory have emerged from the generalized theory.

C. Outline of the Paper

In [13] Rosenbrock took the first formal step away from the regular theory by introducing what he termed "restricted system equivalence" (r.s.e.), in an attempt to define allowable manipulations of (1) that would preserve "important properties" related to system behavior "at infinity"; he also defined "decoupling zeros at infinity." With these elements, he was able to prove certain theorems that mirrored results from the regular theory. In Section II, we discuss some of the difficulties with his definitions (difficulties that Rosenbrock himself was first to point out) and show that they result from unnecessary restrictions on parts of the system that may reasonably be assumed to have no dynamical significance.

A modified equivalence definition that effectively ignores these nondynamic parts and that we term "strong equivalence" (str.eq.) is presented in Section III. The definition of decoupling zeros at infinity is also altered, in Section IV, to include only the dynamical part of the system. With these new definitions, the aforementioned difficulties are overcome while the desired features remain.

In Section V, we present several results that follow from the framework constructed in the earlier sections, and make reference to other related results that we have published elsewhere. We obtain in this framework (as did Rosenbrock in his [13]) extensions to singular systems of Kalman's decomposition [28] and equivalence [29] theorems for regular state-space systems. Our structure, in contrast to Rosenbrock's, is shown to mesh smoothly with McMillan degree theory. We provide satisfying extensions of several other results, previously known only for strictly proper rational matrices (and regular realizations of them), to the case of arbitrary rational matrices (and generalized realizations of them). These include results on composite systems: we present necessary and sufficient conditions for the McMillan degree of the (possibly improper) transfer function of a composite to equal the sum of the McMillan degrees of its subsystem transfer functions, conditions that have apparently eluded previous workers (see [6] for example); the concept of "complexity" defined in [6] is discarded here, in favor of our definition of generalized order. We also present some illustrative examples involving electrical networks, and point out problems of potential interest in the area of singular perturbations. The Conclusion suggests directions for future work.

II. Discussion of Restricted System Equivalence

We may rewrite (2) in the form

$$\left[\begin{array}{c|c} sE-A & -B \\ \hline C & 0 \end{array}\right]\left[\begin{array}{c} X(s) \\ \hline U(s) \end{array}\right]=\left[\begin{array}{c} Ex(0-) \\ \hline Y(s) \end{array}\right] \tag{13}$$

where the coefficient matrix on the left is (following [21]) termed the "system matrix" corresponding to (1) or (2). The word "system," when used without qualification, will be understood to signify a (generalized state-space or) singular system in what follows.

Definition 2.1 (Rosenbrock [13]): Two systems S and S_1 are termed *restricted system equivalent*(r.s.e.) if their associated system matrices are related by

$$\begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}\left[\begin{array}{c|c} sE-A & -B \\ \hline C & 0 \end{array}\right]\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}=\left[\begin{array}{c|c} sE_1-A_1 & -B_1 \\ \hline C_1 & 0 \end{array}\right] \tag{14}$$

where M and N are nonsingular matrices. (Subscript 1 refers, of course, to parameters of S_1.) $\triangle\triangle$

The operations of r.s.e. correspond to constant nonsingular transformations of (1a) itself and of the basis in the space of internal variables x. The behavior of x in the original system may thus be simply recovered from the behavior of any system r.s.e. to it. These operations therefore constitute an eminently safe set of transformations that are unlikely to destroy any "important properties" of the system. Furthermore, as we shall see in a moment, such operations suffice to display the detailed structure of the original system (via transformation to the Kronecker form). These, then, are Rosenbrock's reasons for introducing r.s.e.

A more detailed analysis (outlined in the rest of this section) shows, however, that there are directions in the space of the x-variable where the behavior of x is *nondynamic*, in the following sense: the initial conditions in these directions are immaterial to the system behavior, and the evolution of the system in these directions is determined by simply multiplying the input by a constant matrix, a nondynamic operation. By treating these directions in the same manner as the dynamically interesting directions, r.s.e. and the other elements of Rosenbrock's framework for systems of the form (1) become unnecessarily restricted.

To proceed, it is convenient to work with a *standard form under r.s.e.* (following [13]). First recall from the theory of "regular pencils," Gantmacher [14], that there exist nonsingular M and N such that

$$M(sE-A)N=\begin{bmatrix} sI_n-\bar{A} & 0 \\ 0 & I_{r-n}-s\tilde{E} \end{bmatrix} \tag{15}$$

where $\tilde{E}$ is *nilpotent* (i.e., has all eigenvalues$=0$). In particular, $\tilde{E}$ may be chosen to have Jordan canonical form $\tilde{J}$, with all entries zero except perhaps for entries of one in certain positions on the first superdiagonal. The matrix (15) is termed the Kronecker form of $(sE-A)$. In (15), $n=$degree $|sE-A|$, as before. It now follows that (13) is r.s.e. to the following system:

$$\left[\begin{array}{cc|c} sI_n-\bar{A} & 0 & -\bar{B} \\ 0 & I_{r-n}-s\tilde{E} & -\tilde{B} \\ \hline \bar{C} & \tilde{C} & 0 \end{array}\right]\left[\begin{array}{c} \bar{X}(s) \\ \tilde{X}(s) \\ \hline U(s) \end{array}\right]=\left[\begin{array}{c} \bar{x}(0-) \\ -\tilde{E}\tilde{x}(0-) \\ \hline Y(s) \end{array}\right] \tag{16}$$

(where, of course, $\bar{B}$ and $\tilde{B}$ denote appropriate subblocks of MB, and so on). We may, therefore, work with this standard form without loss of generality.

The transfer function of the system in (16), and hence that of (13) (since r.s.e. preserves the transfer function), is now seen to be

$$G(s)=\bar{G}(s)+D(s) \tag{17a}$$

where

$$\bar{G}(s)=\bar{C}(sI-\bar{A})^{-1}\bar{B}, \qquad \text{strictly proper} \tag{17b}$$

and

$$\begin{aligned} D(s)&=\tilde{C}(I-s\tilde{E})^{-1}\tilde{B} \\ &=\tilde{C}(I+s\tilde{E}+\cdots+s^v\tilde{E}^v)\tilde{B}, \qquad \text{polynomial.} \end{aligned} \tag{17c}$$

Here v is less than the size of $\tilde{E}$, since $\tilde{E}$ is nilpotent. (The so-called "index of nilpotency" of $\tilde{E}$ is $v+1$, and equals the size of the largest Jordan block in the Jordan form $\tilde{J}$ of $\tilde{E}$; thus $\tilde{E}^{v+1}=0$.) Note that (17) verifies claim iia) of the Introduction. The standard form (16) serves to separate the portions of the system that generate the strictly proper part $\bar{G}(s)$ and the polynomial part $D(s)$ of the transfer function $G(s)$. In fact, we have the following result to show that for systems that are r.s.e. these two portions are *separately* r.s.e., and conversely.

Lemma 2.1: Two systems S and S_1 are r.s.e. if and only if standard forms for them are related as follows:

$$\left[\begin{array}{cc|c} sI-\bar{A}_1 & & -\bar{B}_1 \\ & I-s\tilde{E}_1 & -\tilde{B}_1 \\ \hline \bar{C}_1 & \tilde{C}_1 & 0 \end{array}\right]=\left[\begin{array}{cc|c} \bar{M} & 0 & 0 \\ 0 & \tilde{M} & \\ \hline & 0 & I \end{array}\right] \cdot\left[\begin{array}{cc|c} sI-\bar{A} & & -\bar{B} \\ & I-s\tilde{E} & -\tilde{B} \\ \hline \bar{C} & \tilde{C} & 0 \end{array}\right]\left[\begin{array}{cc|c} \bar{M}^{-1} & 0 & 0 \\ 0 & \tilde{M}^{-1} & \\ \hline & 0 & I \end{array}\right]. \tag{18}$$

Proof: Sufficiency is immediate from the definition of r.s.e. For necessity, assume the systems are r.s.e., then for some nonsingular M, L (partitioned as below) we have

$$\begin{bmatrix} \bar{M} & M_a \\ M_b & \tilde{M} \end{bmatrix}\begin{bmatrix} sI-\bar{A} & 0 \\ 0 & I-s\tilde{E} \end{bmatrix} =\begin{bmatrix} sI-\bar{A}_1 & 0 \\ 0 & I-s\tilde{E}_1 \end{bmatrix}\begin{bmatrix} \bar{L} & L_a \\ L_b & \tilde{L} \end{bmatrix}.$$

Now: i) $\bar{M}(sI-\bar{A})=(sI-\bar{A}_1)\bar{L}$, and comparing coefficients of s gives $\bar{M}=\bar{L}$; ii) $(sI-\bar{A}_1)^{-1}M_a=L_a(I-s\tilde{E})^{-1}$, but the left side is strictly proper and the right side is polynomial, so both are zero and $M_a=0=L_a$; similarly, iii) $\tilde{M}=\tilde{L}$, and iv) $M_b=0=L_b$. The desired relation (18) follows easily.

The variables $\bar{x}$ in (16) are governed by regular equations, cf. (12a), and their behavior is well understood. We focus here on the behavior of the variables $\tilde{x}$, and consider without loss of generality the system

$$\left[\begin{array}{c|c} I-s\tilde{J} & -\tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]\left[\begin{array}{c} \tilde{X}(s) \\ \hline U(s) \end{array}\right]=\left[\begin{array}{c} -\tilde{J}\tilde{x}(0-) \\ \hline \tilde{Y}(s) \end{array}\right] \tag{20}$$

where $\tilde{J}$ is nilpotent and in Jordan form, i.e., is a block diagonal matrix whose diagonal blocks have the form

$$J_i=\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}, \qquad \text{size } k_i\times k_i \tag{21}$$

and, for simplicity in later notation, are ordered by size, with $k_i \geqslant k_{i+1}$. [We have written $\tilde{Y}$ in (20) in order to distinguish the output of this portion of the system, associated with $\tilde{X}$, from the output Y of the whole system in (16).]

The block diagonal structure implies that (20) is actually composed of equations of the form

$$\left[\begin{array}{cccc|c} 1 & -s & & 0 & -\tilde{b}_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & -s & \\ 0 & & & 1 & -\tilde{b}_{k_i} \\ \hline \tilde{c}_1 & \cdots & & \tilde{c}_{k_i} & 0 \end{array}\right]\left[\begin{array}{c} \tilde{X}_1(s) \\ \vdots \\ \\ \tilde{X}_{k_i}(s) \\ \hline U(s) \end{array}\right]=\left[\begin{array}{c} -\tilde{x}_2(0-) \\ \vdots \\ -\tilde{x}_{k_i}(0-) \\ 0 \\ \hline \tilde{Y}_i(s) \end{array}\right]. \tag{22}$$

(We have not flagged $\tilde{c}_1$, etc., with an extra subscript that would show it was associated with the ith Jordan block, as there should be no confusion here, but we *shall* use such subscripts in the sequel, when necessary, and write $\tilde{c}_{i1}$, etc.) It will be useful at this point for the reader to reexamine Example 1.1 and recognize it as essentially a special case of the above, with $k_i=2$. Just as in the Example 1.1, the free-response of the system (22) comprises impulsive motions, in this case k_i-1 independent ones. This is evident on solving (22) with $U\equiv 0$ to get

$$\begin{bmatrix} \tilde{X}_1(s) \\ \\ \vdots \\ \tilde{X}_{k_i}(s) \end{bmatrix}=\begin{bmatrix} 1 & s & \cdots & s^{k_i-1} \\ & \ddots & \ddots & \vdots \\ & & \ddots & s \\ 0 & & & 1 \end{bmatrix}\begin{bmatrix} -\tilde{x}_2(0-) \\ \vdots \\ -\tilde{x}_{k_i}(0-) \\ 0 \end{bmatrix} \tag{23}$$

or, in the time domain,

$$\begin{aligned} \tilde{x}_1(t)&=-\tilde{x}_2(0-)\delta(t)-\cdots-\tilde{x}_{k_i}(0-)\delta^{(k_i-2)}(t) \\ &\vdots \\ \tilde{x}_{k_i-1}(t)&=-\tilde{x}_{k_i}(0-)\delta(t) \\ \tilde{x}_{k_i}(t)&=0. \end{aligned} \tag{24}$$

In particular, note the following:

a) the relevant initial conditions are those on all but the *first*-position variables in each Jordan block, and thus number

$$k_i - 1 = \operatorname{rank} J_i \tag{25}$$

for the ith block;

b) the free-response consists of $k_i - 1$ independent impulsive motions for the ith block, and these can occur on all but the *last*-position variables in each Jordan block.

Thus, in its initial condition information or in its impulsive response to these initial conditions, each $k \times k$ block exhibits $k-1$ degrees of freedom. [This is directly related to the fact that $I - sJ_i$, for J_i given by (21), has a Smith–McMillan zero of order $k_i - 1$ at $s = \infty$ (see [4], [5] for detailed analysis from this point of view).] It follows from this and (25) that the system (20) displays

$$\tilde{f} \triangleq \operatorname{rank} \tilde{J} \tag{26}$$

degrees of freedom, all associated with impulsive behavior, and that the system (16) displays

$$n + \tilde{f} = \operatorname{rank} \begin{bmatrix} I_n & 0 \\ 0 & -\tilde{E} \end{bmatrix} \qquad (\tilde{f} = \operatorname{rank} \tilde{E}) \tag{27}$$

degrees of freedom, n of them associated with the exponential behavior of the regular portion of (16) and the remaining associated with impulsive behavior. Since (16) and (13) are r.s.e., the claims in iiia) of the Introduction, for systems of the form (1), (2) or (13), are thereby verified.

It also follows from a), b) of the preceding that the 1×1 Jordan blocks have *no* degrees of freedom. The initial conditions on the variables associated with these blocks affect none of the future response of the system, and the behavior of these variables is instantly determined by the input alone via simple multiplication by the appropriate rows of the input matrix. The only interesting feature of these blocks, it would seem, is their contribution to a constant feedthrough term in the transfer function: every block of the form (22) with $k_i = 1$ contributes a term

$$\tilde{Y}_i(s) = \tilde{c}_{i1} \tilde{b}_{i1} U(s) \tag{28}$$

to the output, i.e., a term $\tilde{c}_{i1}\tilde{b}_{i1}$ to $D(s)$ in (17c). We shall from now on refer to the variables associated with the (trivial) 1×1 Jordan blocks as *nondynamic* variables, and to the other variables as *dynamic* variables.

The main difficulty with r.s.e. may now be identified: r.s.e. treats nondynamic and dynamic variables alike, and by insisting on preserving the integrity of the nondynamic variables it becomes unnecessarily restricted. This is made evident when, for example, we trivially augment (1) with equations of the form

$$z_j(t) \equiv 0, \qquad j = 1 \text{ to } q \tag{29}$$

so that the system matrix in (13) is modified to

$$\left[\begin{array}{cc|c} sE-A & 0 & -B \\ 0 & I_q & 0 \\ \hline C & 0 & 0 \end{array}\right]. \tag{30}$$

The new system matrix, although only trivially different from the old one in its dynamical properties (since all we have actually done is add q nondynamic variables), is *not* r.s.e. to the old one, and is considered by Rosenbrock's definitions to have q more "infinite input–output decoupling zeros" than the old one. One consequence of such difficulties is that a connection between Rosenbrock's decoupling zeros and the McMillan degree of the system transfer function is "awkward" to establish, as noted by Rosenbrock [13].

Modifying Rosenbrock's definitions in accordance with the previous discussion, mainly by allowing dynamical considerations to dictate the definitions, we shall in the following sections present our definitions of strong equivalence and of decoupling zeros at infinity (i.e., unobservable and/or uncontrollable impulsive modes), and shall describe some of the features and consequences of these definitions.

To conclude this section, we note that in view of our discussions here it would appear reasonable to think of $Ex(t-)$ as *the state* of the system (1) at time t. It in effect contains that information on the outputs of "integrators" and inputs of "differentiators" in the system that is necessary for solving the system.

III. Strong Equivalence

In view of the discussion in the last section we decide to allow, beyond the operations permitted by r.s.e., any operations that add or eliminate *nondynamic* variables, provided that these operations do not modify the constant term in the system transfer function [cf. discussion of (28)]. For an illustration, we first partition the $\tilde{J}$ in (20) as

$$\tilde{J} = \begin{bmatrix} \hat{J} & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

where $\hat{J}$ has *no* 1×1 Jordan blocks (i.e., all the 1×1 blocks of $\tilde{J}$ are absorbed in the matrix 0 on the diagonal in (31)), so that (20) becomes

$$\left[\begin{array}{cc|c} I - s\hat{J} & & -\hat{B} \\ & I & -B^* \\ \hline \hat{C} & C^* & 0 \end{array}\right] \left[\begin{array}{c} \hat{X}(s) \\ X^*(s) \\ \hline U(s) \end{array}\right] = \left[\begin{array}{c} -\hat{J}\hat{x}(0-) \\ 0 \\ \hline \tilde{Y}(s) \end{array}\right] \tag{32}$$

(where the corresponding partitioning of $\tilde{B}$ into $\hat{B}$ and B^*, etc., should be clear). Now our new definition of equivalence, which we are going to term *strong equivalence* (str. eq.), should allow, for example, that (20), (32) are equivalent to

$$\left[\begin{array}{c|c} I - s\hat{J} & -\hat{B} \\ \hline \hat{C} & C^* B^* \end{array}\right] \left[\begin{array}{c} \hat{X}(s) \\ \hline U(s) \end{array}\right] = \left[\begin{array}{c} -\hat{J}\hat{x}(0-) \\ \hline \tilde{Y}(s) \end{array}\right] \tag{33}$$

where we have eliminated $X^*(s)$ and retained only its contribution, via the term C^*B^*, to the transfer function.

The system matrix in (33) is of special interest because its associated system contains *no* nondynamic internal variables. We shall label such a system matrix as being in *standard form under strong equivalence*. The phrase "standard form" will, unless otherwise specified, be taken from now on to mean that under strong equivalence, and not the standard form under r.s.e. (16). The system matrix in (33) is then termed a standard form for the one in (20) and (32). More generally, the system matrix in (13) is said to be in standard form if it is impossible to find nonsingular M and N for which

$$M(sE-A)N=\begin{bmatrix} s\hat{E}-\hat{A} & 0 \\ 0 & I \end{bmatrix}. \tag{34}$$

If (13) is not in standard form, then clearly it can be brought to standard form by an operation of r.s.e. involving the M and N of (34), followed by elimination of nondynamic variables. A simple algorithm for reduction to standard form is described in [4].

The following observation now suggests that a more general relationship than r.s.e. should, together with addition or elimination of nondynamic variables, form the foundation for our set of allowable equivalence transformations.

Observation 3.1: Given two r.s.e. systems S and S_1 of the form (20), we find that the standard forms obtained from them on elimination of nondynamic variables are no longer necessarily r.s.e., but are related as follows:

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\left[\begin{array}{c|c} I-s\hat{J} & -\hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]\begin{bmatrix} N & R \\ 0 & I \end{bmatrix}=\left[\begin{array}{c|c} I-s\hat{J}_1 & -\hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_1 \end{array}\right] \tag{35a}$$

where

$$M,N \text{ are nonsingular} \tag{35b}$$

and

$$Q\hat{J}=0=\hat{J}R. \tag{35c}$$

The straightforward verification of this is carried out in [4] and not reproduced here, since we use the observation only to motivate what follows. △△

We certainly want to consider the system matrices in (35a) to be strongly equivalent, since they are obtained from parent systems S and S_1 that are r.s.e. by simply eliminating nondynamic variables. The problem, however, is that if we are only given the relationship in (35a), we have no method for determining that the system matrices there do in fact come from parent systems that are r.s.e. The situation is satisfactorily and simply resolved by noting that transformations of the form (35), while more general than r.s.e., are nevertheless as "safe" as we require for our purposes, and can, therefore, be made the basis for our definition of strong equivalence. The rest of this section is devoted to the development of the latter statement.

We begin with the following definitions.

Definition 3.1: Given a system S specified by the system matrix

$$\left[\begin{array}{c|c} sE-A & -B \\ \hline C & D \end{array}\right] \tag{36}$$

the operation on it that is embodied in a transformation of the form

$$\begin{bmatrix} M & 0 \\ Q & I \end{bmatrix}\left[\begin{array}{c|c} sE-A & -B \\ \hline C & D \end{array}\right]\begin{bmatrix} N & R \\ 0 & I \end{bmatrix}=\left[\begin{array}{c|c} sE_1-A_1 & -B_1 \\ \hline C_1 & D_1 \end{array}\right], \tag{37a}$$

$$M,N \text{ nonsingular} \tag{37b}$$

$$QE=0=ER \tag{37c}$$

(yielding the system matrix of a transformed system S_1) will be termed an *operation of strong equivalence* (*str. eq. operation*). △△

(We note in passing that we have included a constant feedthrough term D in (36), for adequate generality; even if S did not contain it, S_1 in general would, after the transformation (37).)

Definition 3.2: The modification of the system matrix in (36) to the form

$$\left[\begin{array}{cc|c} sE-A & 0 & -B \\ 0 & I & 0 \\ \hline C & 0 & D \end{array}\right] \tag{38}$$

corresponding to the definition of trivial additional variables as in (29), will be termed a *trivial augmentation* of (36) or of the associated system S. The reverse process, namely going from (38) to (36), will be termed a *trivial deflation*, and corresponds to the deletion of trivial variables. △△

The transformations of Definitions 3.1 and 3.2 will be termed *allowed transformations*. Note that they preserve the generalized order of the original system, and its transfer function.

Definition 3.3 (Strong Equivalence): Two systems are termed *strongly equivalent* if one can be obtained from the other by some sequence of allowed transformations (i.e., operations of strong equivalence, and/or trivial augmentations, and/or trivial deflations); the number of transformations of each type and the order in which they are applied are *not* of consequence. △△

The remarks that follow present the facts that render the above definitions acceptable and useful.

Remark 3.1: It is easily verified that Definition 3.3 does indeed correspond to an equivalence relation (reflexivity and transitivity are immediate, while symmetry follows from the fact that the allowed transformations in Definitions 3.1 and 3.2 have inverses that are themselves allowed transformations). △△

Remark 3.2: It is the restriction to *constant* M, N, Q, R that makes algebraic operations on the system matrix, as in (37a), meaningful. We would otherwise be forced (because we allow "inconsistent" initial conditions) to perform a separate accounting of the effects of initial conditions. This issue, which does not arise in the regular theory, is briefly explored in [4, example 4.6]. Furthermore, the constraint (37c) ensures that the allowed transformations of a generalized state-space system yield a new system of the same form. △△

Some elaboration of the following two remarks is contained in the Appendix. The results are modeled on those of Pernebo [23] for regular systems.

Remark 3.3: Given two strongly equivalent systems S and S_1, it can be seen (on examination of the effects of the allowed transformations) that there is a *bijective mapping* of the form

$$x_1(t)=T_1x(t)+V_1u(t), \qquad t\geqslant 0 \tag{39}$$

between the solutions of the systems S and S_1 for any given input $u(t)$, such that the outputs $y(t)$ of the two systems are identical under the mapping, and where the corresponding initial conditions are related by

$$E_1x_1(0-)=P_1Ex(0-). \tag{40a}$$

Here T_1, V_1, P_1 are constant matrices that are independent of the input or initial conditions. One also finds (again by examining the effects of the allowed transformations) that

$$E_1T_1=P_1E \quad \text{and} \quad E_1V_1=0 \tag{41}$$

which, in conjunction with (39), yields the following extension of (40a):

$$E_1x_1(t)=P_1Ex(t), \qquad t\geqslant 0. \tag{40b}$$

The inverse mapping to (39) is of the form

$$x(t)=Tx_1(t)+Vu(t), \qquad t\geqslant 0 \tag{42}$$

with

$$Ex(0-)=PE_1x_1(0-). \tag{43a}$$

Furthermore,

$$ET=PE_1 \quad \text{and} \quad EV=0, \tag{44}$$

which in conjunction with (42) yields

$$Ex(t)=PE_1x_1(t), \qquad t\geqslant 0. \tag{43b}$$

We therefore have a natural isomorphism between the solutions of two strongly equivalent systems. △△

Remark 3.4: We also have the converse of the previous result: if two systems have isomorphic solutions in the above sense, then they are strongly equivalent. The demonstration of this fact (see Appendix) is carried out by first showing that for two systems in standard form the isomorphism implies that their system matrices must be related as in (37), i.e., by just a str. eq. operation. A corollary to these results is that two systems S and S_1 are strongly equivalent if and only if standard forms for them are related by a str. eq. operation, as in (37). △△

IV. Observability and Controllability of the Impulsive Modes

A. Modal Observability

A free-response mode of the system (1) that gives rise to a zero output, $y(t)\equiv 0$ for $t\geqslant 0$, is said to be an *unobservable mode*. In the regular theory, a system with no unobservable *exponential* modes is simply termed *observable*. It is well known, [21], [45], that an observable system is characterized by a "modal observability matrix"

$$\mathcal{O}(s)\triangleq\begin{bmatrix} sE-A \\ C\end{bmatrix} \tag{45}$$

that has full column rank for all finite s or, equivalently, has no Smith zeros. [Unobservable exponential modes are associated with Smith zeros of $\mathcal{O}(s)$, which Rosenbrock [21] terms the (finite) output-decoupling zeros.] It is also well known that $x(t)$ for $t>0$ can be uniquely determined from $u(t)$ and $y(t)$ for $t\geqslant 0$ if and only if the system is observable.

We are interested here in the unobservable *impulsive* modes of (1). It is straightforward to see from (2) that there exists an unobservable impulsive mode if and only if for some constant vectors v and $x(0-)$ we have

$$\begin{bmatrix} sE-A \\ C\end{bmatrix}v=\begin{bmatrix} Ex(0-) \\ 0\end{bmatrix}, \qquad v\neq 0 \tag{46}$$

corresponding to an impulsive free-response mode $x(t)=v\,\delta(t)$ for which $y(t)\equiv 0$, $t\geqslant 0$. [Such modes are, in the framework of [4], [5], associated with Smith-McMillan zeros of $\mathcal{O}(s)$ at infinity, which we may thus term infinite output-decoupling zeros. For an interpretation of v in (46) as an eigenvector corresponding to an infinite zero of $(sE-A)$, see [30].] Just as in the regular theory, one may now directly claim that any impulsive behavior in $x(t)$ at $t=0$ may be uniquely determined from $u(t)$ and $y(t)$ for $t\geqslant 0$ if and only if the system (1) has no unobservable impulsive modes. We shall refer to a system with no unobservable impulsive modes as being "observable at infinity;" if in addition it is observable in the sense of the regular theory, we shall term it *strongly observable* (an appellation that is perhaps natural in the present context but one that we must note has been used from time to time for other concepts quite distinct from ours).

Several other characterizations of observability at infinity may be obtained from that in (46). Note first of all by examination of (46) that the observability or otherwise of the impulsive modes is unaffected by the allowed transformations of Definitions 3.1, 3.2; the observability of the exponential modes is known, from the regular theory, to be similarly unaffected by these transformations. Thus, two strongly equivalent systems have the same observability properties (which is not surprising in view of Remarks 3.3, 3.4). This permits us to transform a given system into a strongly equivalent system whose observability properties are more clearly displayed. The following tests for observability at infinity illustrate this point.

Test 4.1: Given a system of the form (1), (13), or (36), use an allowed transformation to bring its modal observability matrix (45) to the form

$$\left[\begin{array}{cc} sE_1-A_1 & A_2 \\ \hline C_1 & C_2 \end{array}\right] \text{ with } E_1 \text{ of full column rank.} \tag{47}$$

Then the system is observable at infinity if and only if

$$\begin{bmatrix} E_1 & A_2 \\ 0 & C_2 \end{bmatrix} \text{ has full column rank.} \tag{48}$$

$\triangle\triangle$

Test 4.2: Use an allowed transformation to bring the modal observability matrix of the given system to the form

$$\left[\begin{array}{c} sE_1-A_1 \\ A_2 \\ \hline C \end{array}\right] \text{ with } E_1 \text{ of full row rank.} \tag{49}$$

Then the system is observable at infinity if and only if

$$\begin{bmatrix} E_1 \\ A_2 \\ C \end{bmatrix} \text{ has full column rank.} \tag{50}$$

$\triangle\triangle$

The validity of the above tests follows directly from the characterization (46), but see [4] or [31] for another viewpoint.

Note from the above that a regular state-space system is always observable at infinity. It is interesting to see what form the above tests take for a system in the special form (20)–(22). Application of either of the tests quickly shows that the system is observable at infinity if and only if those columns of $\tilde{C}$ corresponding to the *first position* of each *nontrivial* (i.e., non-1×1) Jordan block are independent. In the notation of (21), (22), these are the columns $\tilde{c}_{i1}$ corresponding to the ith nontrivial block ($k_i>1$). (We may refer to these columns as the "significant first-position columns" of $\tilde{C}$.)

It is shown in [4] that a system matrix of the form (36) can always be brought by allowed transformations to the form

$$\left[\begin{array}{cc|c} sE_{\bar{o}}-A_{\bar{o}} & K & -B_{\bar{o}} \\ 0 & sE_o-A_o & -B_o \\ \hline 0 & C_o & D \end{array}\right] \tag{51}$$

in which the subsystem

$$\left[\begin{array}{c|c} sE_o-A_o & -B_o \\ \hline C_o & D \end{array}\right] \tag{52}$$

is strongly observable. The special structure of (51) shows that this subsystem has the same transfer function as the original system. It also shows that the unobservable impulsive modes of the original system are the impulsive free-response modes associated with $(sE_{\bar{o}}-A_{\bar{o}})$. We refer to (52) as the part of the original system that is strongly observable, and note that its generalized order ($=\operatorname{rank} E_o$) differs from that of the original system ($=\operatorname{rank} E_{\bar{o}}+\operatorname{rank} E_o$) by precisely the number of unobservable exponential and impulsive modes ($=\operatorname{rank} E_{\bar{o}}$).

We shall not derive the decomposition (51) here, but instead give a canonical example that contains the essence of a general derivation. We refer the reader to [4] for a simple algorithm for extracting from a given system the part (52) that is strongly observable; the algorithm is typical of those presented by Van Dooren [32], [33] as being particularly suited to problems involving generalized state-space or singular systems.

Example 4.1: We start with a system whose modal observability matrix is

$$\left[\begin{array}{c} sE-A \\ \hline C \end{array}\right] = \left[\begin{array}{ccc|cccc} 1 & -s & & & & & \\ & 1 & -s & & & 0 & \\ & & 1 & & & & \\ \hline & & & 1 & -s & & \\ & 0 & & & 1 & -s & \\ & & & & & 1 & -s \\ & & & & & & 1 \\ \hline c & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{53}$$

Since the first-position columns $[c \quad 0]'$ and $[1 \quad 0]'$ of C are dependent (indicating at least one unobservable impulsive mode), we may force the first-position column of the smaller block to 0 by an allowed transformation of the associated system, obtaining

$$\left[\begin{array}{ccc|cccc} 1 & -s & & & & & \\ & 1 & -s & & & & \\ & & 1 & & & & \\ \hline -c & & & 1 & -s & & \\ & & & & 1 & -s & \\ & & & & & 1 & -s \\ & & & & & & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{54}$$

Now a sequence of allowed row and column transformations can be found that will eliminate the term $-c$ in (54): add $c\times$row 1 to row 4; add $-c\times$column 5 to column 2; add $c\times$row 2 to row 5; add $-c\times$column 6 to column 3; add $c\times$row 3 to row 6. The resulting modal observability matrix has the form

$$\left[\begin{array}{c:c} 1 & sL \\ \hdashline 0 & I-sJ_o \\ \hline 0 & C_o \end{array}\right]=\left[\begin{array}{c:cc:cccc} 1 & -s & & & & & \\ \hdashline & 1 & -s & & & & \\ & & 1 & & & & \\ \hdashline & & & 1 & -s & & \\ 0 & & & & 1 & -s & \\ & & & & & 1 & -s \\ & & & & & & 1 \\ \hline & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & & & & & & \\ & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{55}$$

where

$$\begin{bmatrix} I-sJ_o \\ C_o \end{bmatrix}$$

is the modal observability matrix of that part of the original system that is observable at infinity (since the first-position columns $[0 \quad 1]'$ and $[1 \quad 0]'$ of C_o are independent). No further transformations are required if one is only interested in displaying this part of the original system. If, however, one wishes to display the part generating the unobservable impulsive modes, then a trivial augmentation of the system associated with (55), followed by further allowed transformations, will bring its modal observability matrix to the form (cf. (51))

$$\left[\begin{array}{cc:c} 1 & -s & \\ & 1 & -1 \\ \hdashline & & I-sJ_o \\ \hline & 0 & C_o \end{array}\right] \tag{56}$$

demonstrating the existence of one (and only one) unobservable impulsive mode, associated with

$$\begin{bmatrix} 1 & -s \\ & 1 \end{bmatrix}. \qquad \triangle\triangle$$

In concluding this subsection we note that Rosenbrock's definition in [13] of infinite output-decoupling zeros differs from ours in that he in effect includes the trivial nondynamic variables in his definition. For example, a system in the form (20)–(22) is said to have no infinite output-decoupling zeros, in his framework, if those columns of $\bar{C}$ corresponding to the first position of *all* Jordan blocks are independent. More generally, he requires that $[E' \quad C']'$ have full column rank for the system (1) to be considered to have no infinite output-decoupling zeros. This definition leads him to consider (30) as having q more decoupling zeros than (1). (The definitions in [50], [52], [53] are effectively the same as Rosenbrock's.) Our definition is less stringent: any system with no decoupling zeros in Rosenbrock's sense has none in our sense, but not conversely.

B. Modal Controllability

A dual development to that of the previous subsection may in principle be carried out for the concept of controllability. A free-response mode that can be alternatively excited from *zero* initial conditions by means of an input that contains no component at the modal frequency is termed a *controllable* mode. In the regular theory, a system whose exponential modes are all controllable is simply termed a controllable system, and is characterized by a "modal controllability matrix"

$$\mathcal{C}(s) \triangleq [sE-A \quad B] \tag{57}$$

that has full row rank for all finite s or, equivalently, has no Smith zeros. [Uncontrollable exponential modes are associated with Smith zeros of $\mathcal{C}(s)$, which Rosenbrock [21] terms the (finite) input-decoupling zeros.]

Controllable *impulsive* modes are those that can be excited from zero initial conditions using nonimpulsive inputs. Uncontrollable impulsive modes cannot be thus excited. [They correspond to Smith–McMillan zeros of $\mathcal{C}(s)$ at infinity, which we may term infinite input-decoupling zeros.] These characterizations lead us to the duals of the characterization (46), Tests 4.1, 4.2, and the decomposition (51), (52). We omit elaboration here.

If the impulsive modes of a system are controllable, we shall term the system "controllable at infinity." If in addition the system is controllable in the regular sense, we shall term it *strongly controllable.*

C. Decomposition and Irreducibility

The Kalman decomposition [28] of a regular state-space system into four parts is well known: a similarity transformation serves to display i) the controllable and observable (or irreducible) part, ii) the controllable but unobservable part, iii) the observable but uncontrollable part, and iv) the uncontrollable and unobservable part. In the same way, one can bring a generalized state-space system by allowed transformations into a form that displays these four parts for both the exponential and the impulsive modes. For example, a preliminary transformation can be made to the form (16). The regular Kalman decomposition is then carried out on the regular part of the system. The remaining part of the system, which is of the form (20), is unaffected by this; it contains only impulsive modes, and may be decomposed into the four parts by appropriately applying the transformation to the form (51) (which separates the observable impulsive modes from the unobservable ones) and the dual transformation referred to in Section IV-B (which separates the controllable impulsive modes from the uncontrollable ones). The general form of the resulting system is then

$$\left[\begin{array}{cccc|c} sE_{\bar{o}c}-A_{\bar{o}c} & * & * & * & -B_{\bar{o}c} \\ & sE_{\bar{o}\bar{c}}-A_{\bar{o}\bar{c}} & 0 & * & 0 \\ 0 & & sE_{oc}-A_{oc} & * & -B_{oc} \\ & & & sE_{o\bar{c}}-A_{o\bar{c}} & 0 \\ \hline 0 & 0 & C_{oc} & C_{o\bar{c}} & D \end{array}\right] \tag{58}$$

where the $*$ denote constant matrices, and the subscripts o, $\bar{o}$, c, $\bar{c}$ denote strongly observable, unobservable, strongly controllable, and uncontrollable, respectively. In particular, the subsystem

$$\left[\begin{array}{c|c} sE_{oc}-A_{oc} & -B_{oc} \\ \hline C_{oc} & D \end{array}\right] \tag{59}$$

is strongly observable *and* strongly controllable. We shall term it *strongly irreducible*. The structure of (58) indicates that (59) has the same transfer function as the original system. Furthermore, denoting the generalized order of the original system by f ($=\text{rank}\, E_{\bar{o}c}+\text{rank}\, E_{\bar{o}\bar{c}}+\text{rank}\, E_{oc}+\text{rank}\, E_{o\bar{c}}$) and that of the strongly irreducible part by f_{oc} ($=\text{rank}\, E_{oc}$), we see that

$$\begin{aligned} f &= f_{oc} + \# \quad \text{uncontrollable and/or unobservable exponential and impulsive modes} \\ &= f_{oc} + \# \quad \text{"decoupling zeros" (finite and infinite).} \end{aligned} \tag{60}$$

This relationship (which is the natural extension of a standard result of the regular theory) will be useful in the next section, where we shall also establish several special properties of strongly irreducible systems.

[*Notes added in proof:* The recent paper [60] presents yet another development of structural properties of generalized state-space systems; its approach appears to lie between Rosenbrock's and ours, but detailed connections remain to be worked out. The thesis [61] discusses the Kalman decomposition for regular systems in great detail; a similar scrutiny of the decomposition in (58) is to be desired.]

V. Some Consequences of Our Definitions

A. Results Related to Strong Irreducibility

The results of this subsection are intended to illustrate that strong irreducibility is the natural generalization of the concept of irreducibility (or minimality) in the regular theory. We first show that a strongly irreducible singular system is determined, to within strong equivalence, by its transfer function; the corresponding result of Kalman [29] for regular systems is well known. We need the following lemma for our proof.

Lemma 5.1: Given a strongly irreducible system of the form

$$\left[\begin{array}{c|c} I-sJ & -B \\ \hline C & 0 \end{array}\right] \tag{61}$$

where J is a nilpotent Jordan matrix as before, the system

$$\left[\begin{array}{c|c} sI-J & -B \\ \hline C & 0 \end{array}\right] \tag{62}$$

will not in general be irreducible in the regular sense (for it may possess decoupling zeros at $s=0$). However, (61) may be brought by allowed transformations to a system of the same form, but one for which (61) *is* irreducible.

Proof: Write the given system (61) in the form (32), namely,

$$\left[\begin{array}{c|c} I-sJ & -B \\ \hline C & 0 \end{array}\right]=\left[\begin{array}{cc|c} I-s\hat{J} & & -\hat{B} \\ & I & -B^* \\ \hline \hat{C} & C^* & 0 \end{array}\right] \tag{63}$$

where $\hat{J}$ has no 1×1 Jordan blocks. The strong irreducibility of the above is equivalent to the following two statements (see Sections IV-A–IV-C):

i) The first-position columns of $\hat{C}$ are mutually independent.

ii) The last-position rows of $\hat{B}$ are mutually independent.

(Here "first-position" and "last-position" denote positions relative to the associated Jordan blocks of $\hat{J}$.)

Now (62) is irreducible in the regular sense if and only if (see [2]):

ia) The first-position columns of $\hat{C}$ *and* the columns of C^* are mutually independent.

iia) The last-position rows of $\hat{B}$ *and* the rows of B^* are mutually independent.

(The latter conditions are equivalent to requiring that (61) have no infinite decoupling zeros in Rosenbrock's sense, [13].) What we wish to show is that by allowed transformations we can bring (61), (63) to a system of the same form but one for which the conditions ia), iia) are satisfied.

Assume that some column c_k^* of C^* in (63) is dependent on the first-position columns $\{\hat{c}_{j1}\}$ of $\hat{C}$. It is easy to see that a str. eq. operation may be used on (63) to add an appropriate weighted combination of the $\{\hat{c}_{j1}\}$ to c_k^* and convert it to 0, without altering anything else in (63) except the *first*-position rows of $\hat{B}$. The dual operation can be used to convert to 0 any row b_m^* of B^* that is dependent on last-position rows of $\hat{B}$, without altering anything but the *last*-position columns of $\hat{C}$. After trivial deflation of the resulting system, we obtain a system that is strongly equivalent to the old one but that satisfies ia), iia) of the preceeding. △△

We may now prove the following.

Theorem 5.1: Two strongly irreducible systems S_1 and S_2 are strongly equivalent if and only if they have the same transfer function.

Proof: Necessity is immediate from the fact that str. eq. preserves the transfer function.

For sufficiency, bring each system to the form (16), i.e.,

$$\left[\begin{array}{cc|c} sI-\bar{A}_i & & -\bar{B}_i \\ & I-s\tilde{J}_i & -\tilde{B}_i \\ \hline \bar{C}_i & \tilde{C}_i & 0 \end{array}\right], \qquad i=1,2 \tag{64}$$

by allowed transformations. (This can be done even if the systems originally contained constant direct-feedthrough terms, as can be easily verified.) The resulting systems are of course still strongly irreducible and have the same transfer function. Thus,

$$\left[\begin{array}{c|c} sI-\bar{A}_i & -\bar{B}_i \\ \hline \bar{C}_i & 0 \end{array}\right], \qquad i=1,2 \tag{65}$$

are irreducible and generate the same strictly proper transfer function $\bar{G}(s)$. They are, therefore, related by a similarity transformation, by the result of Kalman [29] (see also [21], [34]). Also,

$$\left[\begin{array}{c|c} I-s\tilde{J}_i & -\tilde{B}_i \\ \hline \tilde{C}_i & 0 \end{array}\right], \qquad i=1,2 \tag{66}$$

are strongly irreducible and generate the same polynomial transfer function $D(s)$. By Lemma 5.1 we can assume that the systems in (66) are such that

$$\left[\begin{array}{c|c} sI-\tilde{J}_i & -\tilde{B}_i \\ \hline \tilde{C}_i & 0 \end{array}\right], \qquad i=1,2 \tag{67}$$

are irreducible. The latter are easily seen to generate the same transfer function $s^{-1}D(s^{-1})$, and are thus related by a similarity transformation. It follows that the systems (66) are r.s.e.

Putting together the similarity of the systems in (65), and the restricted system equivalence of those in (66), we see that the systems in (64) are r.s.e., and hence S_1 and S_2 are strongly equivalent. △△

The above theorem, though proved under our less restricted definitions of equivalence and infinite decoupling zeros, has the same essential structure as that obtained by Rosenbrock [13] in attempting to extend Kalman's result [29] on regular systems. It would have been disappointing indeed if our definitions, however else they may have been motivated or justified, were incapable of leading us to such a result; the theorem provides some reassurance that our definitions are meaningful and compatible.

The theorem establishes the transfer function as a "complete invariant" [35] for strongly irreducible singular systems under the allowed transformations of strong equivalence. (It is also an "independent invariant," in the sense that any rational matrix may be obtained as the transfer function of such a system; the demonstration of this is easy, see [4] or [31].) It follows that those structural properties of strongly irreducible systems that remain invariant under strong equivalence must be obtainable from examination of their transfer functions; the results of [31], [36], relating pole, zero and null-space structures of system matrices to those of transfer functions, substantiate this, and are of interest even in the regular theory.

One of the difficulties with Rosenbrock's framework for singular systems [13] was that the connection with McMillan degree theory could not be established. Our framework, on the other hand, meshes cleanly and naturally with McMillan degree concepts. We recall one definition of McMillan degree here (see [37], [21], [38] for further discussion).

Definition 5.1: The McMillan degree of a rational matrix $G(s)$ is denoted by $\delta[G(s)]$, and given by

$$\delta[G(s)]=\nu\left[\bar{G}(s)\right]+\nu\left[D(s^{-1})\right] \tag{68}$$

where $\bar{G}(s)$, $D(s)$ are the strictly proper and polynomial parts of $G(s)$, respectively, and $\nu[\ \ ]$ denotes the regular order of an irreducible (regular) state-space realization of the associated strictly proper matrix. △△

With this definition, we have the next result.

Theorem 5.2: For the system (1)–(3), we have the following relation between its generalized order $f(=\operatorname{rank} E)$ and the McMillan degree of its transfer function:

$$f=\delta[G(s)]+\#\text{finite and infinite decoupling zeros.} \tag{69}$$

Proof: We have noted that the system (1)–(3) may be brought to the form (58) by allowed operations where the strongly irreducible part (59) has the same transfer function $G(s)$ as the original system, and where (60) holds. All that remains to be shown is that for the strongly irreducible system (59) we have

$$f_{oc}=\delta[G(s)]. \tag{70}$$

Bring (59) to the form

$$\left[\begin{array}{ccc|c} sI_{\bar{n}}-\bar{A} & & & -\bar{B} \\ & I-s\hat{J} & & -\hat{B} \\ & & I & -B^* \\ \hline \bar{C} & \hat{C} & \overset{*}{C} & 0 \end{array}\right], \qquad \hat{J}\text{ nilpotent Jordan form with no } 1\times 1 \text{ blocks,} \tag{71}$$

by allowed operations. (71) is of course strongly irreducible. Now

$$f_{oc}=\bar{n}+\operatorname{rank}[\hat{J}] \tag{72}$$

and

$$\bar{n}=\nu\left[\bar{G}(s)\right]. \tag{73}$$

We, therefore, only need show that

$$\hat{f}\triangleq\operatorname{rank}[\hat{J}]=\nu\left[D(s^{-1})\right]. \tag{74}$$

For this, note from (71) that (cf. (17c))

$$\begin{aligned} D(s)&=C^*B^*+\hat{C}(I-s\hat{J})^{-1}\hat{B} \\ &=C^*B^*+\hat{C}\hat{B}+s\hat{C}\hat{J}\hat{B}+\cdots+s^j\hat{C}\hat{J}^j\hat{B} \\ &=C^*B^*+\hat{C}\hat{B}+s\underline{C}\underline{B}+s^2\underline{C}\underline{J}\underline{B}+\cdots+s^j\underline{C}\underline{J}^{j-1}\underline{B} \end{aligned} \tag{75}$$

where a) $\underline{C}$, $\underline{B}$ are obtained from $\hat{C}$, $\hat{B}$ by *dropping* all last-position columns, first-position rows, respectively; and b) $\underline{J}$ is obtained from $\hat{J}$ by contracting each $k\times k$ Jordan block to a $(k-1)\times(k-1)$ Jordan block; in particular, note that $\underline{J}$ is a matrix of *size* $\hat{f}\times\hat{f}$.

Now

$$\left[\begin{array}{c|c} sI_{\hat{f}}-\underline{J} & -\underline{B} \\ \hline \underline{C} & C^*B^*+\hat{C}\hat{B} \end{array}\right] \tag{76}$$

constitutes an irreducible regular state-space realization of $D(s^{-1})$. This proves (74). △△

Remark 5.1: It follows that $f=\delta[G(s)]$ if and only if the system is strongly irreducible, and that $\delta[G(s)]$ is precisely the minimal generalized order of all realizations of $G(s)$.

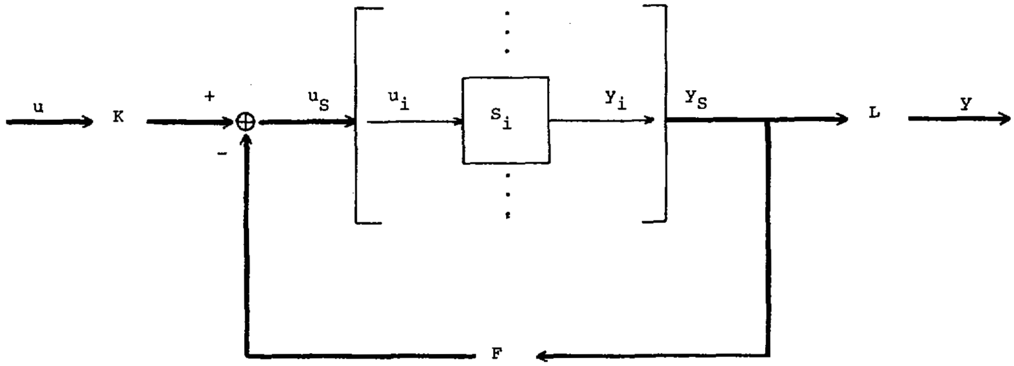


Fig. 3. Composite system.

Remark 5.2: The 'Rosenbrock degree' defined in [39] is the sum of $\nu[D(s^{-1})]$ and the regular order n=degree $|sE-A|$ of the system. Hence,

$$f=d+\#\text{ infinite decoupling zeros.} \tag{77}$$

Remark 5.3: While the regular order of the system (1) is the determinantal degree of the 'denominator' matrix $(sE-A)$, the generalized order is actually the McMillan degree of $(sE-A)$.

B. Results on Composite Systems

The results in this subsection are stimulated by the work of Rosenbrock and Pugh [6] on a "hierarchical theory of systems." Consider a system that is an interconnection of subsystems $S_1, S_2, \cdots, S_N$. Let the inputs $\{u_i\}$ and outputs $\{y_i\}$ of the subsystems $\{S_i\}$ be assembled into the vectors

$$\begin{aligned} u_S &= [u_1' \ u_2' \cdots u_N']' \\ y_S &= [y_1' \ y_2' \cdots y_N']' \end{aligned} \tag{78}$$

and let the input and output of the composite system be denoted by u and y, respectively. Then a general interconnection scheme that covers a large class of systems of practical interest (except for electrical networks, though our results hold generally in that case too—see later) is determined by the following interconnection rule (see Fig. 3):

$$\begin{aligned} u_S &= -Fy_S + Ku \\ y &= Ly_S. \end{aligned} \tag{79}$$

When the subsystems $\{S_i\}$ have strictly proper transfer functions, Rosenbrock and Pugh [6] showed that the (regular) order n of the composite system is the sum of the (regular) orders n_i of the subsystems: $n=\Sigma_1^N n_i$. They state that, while this result may seem "intuitively obvious", no corresponding theorem is true if any of the subsystem transfer functions is not strictly proper. They are then led to search for other properties of the subsystems that have the additive property under the interconnection scheme (79), and settle on the *complexity* of the subsystems as the quantity of interest (see Rosenbrock [39] for detailed definitions). There are, however, several facts that discourage one from according complexity a central role in a general theory (although its possible significance in some problems cannot be ruled out).

The complexity of a system of differential equations is essentially defined as the minimum number of initial conditions on the variables *as written* that are needed to solve the system (or equivalently, that are needed to Laplace transform the system). For example, if

$$\begin{bmatrix} p & p \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0, \qquad p=d/dt \tag{80}$$

then one needs $x_1(0-)$ and $x_2(0-)$ to solve (80), so the complexity of (80) is 2. However, a simple constant (invertible) change of basis in (80) leads to the equations

$$\begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1+x_2 \\ x_2 \end{bmatrix}=0 \tag{81}$$

whose complexity is only 1, corresponding to the fact that really only $x_1(0-)+x_2(0-)$ is needed to solve (81) *or* (80). The same result on the true number of degrees of freedom is evidently obtained if we recognize (80) as a special case of (1), with

$$E=\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

of rank 1, cf. (5). This simple example suggests that while complexity may give us some information about the variables we have chosen to use in writing our equations, it has no dynamical significance for the system of equations.

The following theorem now shows that when the subsystems are in the form

$$\left[\begin{array}{c|c} sE_i-A_i & -B_i \\ \hline C_i & D_i \end{array}\right], \qquad i=1 \text{ to } N \tag{82}$$

(with corresponding internal variables x_i), and thus have possibly *nonstrictly* proper transfer functions, the *generalized order has the additive property.* Since the generalized order has a basic dynamical significance and is preserved under a large class of transformations, namely the allowed transformations of strong equivalence on the subsystems or on the composite, this result is felt to be more interesting than the analogous result for complexity.

Theorem 5.3: Consider a composite system comprising subsystems as in (82), interconnected according to the prescription (79). The generalized order of the composite is the sum of the generalized orders of the subsystems.

Proof: The proof is quite trivial, as we now show. Define

$$\begin{aligned} &E_S=\operatorname{diag}\{E_i\} && x_S'=[x_1',\cdots,x_N'] \\ &A_S=\operatorname{diag}\{A_i\} && B_S=\operatorname{diag}\{B_i\} \\ &C_S=\operatorname{diag}\{C_i\} && D_S=\operatorname{diag}\{D_i\}. \end{aligned} \tag{83}$$

Then the composite system is represented by the following system matrix (with associated variables indicated):

$$\left[\begin{array}{ccc|c} pE_S-A_S & -B_S & & 0 \\ C_S & D_S & -I & 0 \\ & I & F & -K \\ \hline 0 & 0 & L & 0 \end{array}\right]\begin{bmatrix} x_S \\ u_S \\ y_S \\ \overline{u} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \overline{y} \end{bmatrix}, \quad p=d/dt. \tag{84}$$

We assume that the denominator $(pE-A)$, i.e., the (1, 1) submatrix of the system matrix in (84), is nonsingular (when considered as a polynomial matrix in the symbol p) so that the composite is well defined. Now, since $x_S(0-)$ can be arbitrarily chosen at $t=0-$ if the $x_i(0-)$ can, we see that the generalized order of the composite system in (84) is simply

$$f=\operatorname{rank}\begin{bmatrix} E_S & 0 \\ 0 & 0 \end{bmatrix}=\sum_1^N \operatorname{rank} E_i=\sum_1^N f_i. \tag{85}$$

$\triangle\triangle$

The above result can actually be obtained from a result of Pugh's [40], which states that the McMillan degree of the system matrix is additive under the composition we are considering here; cf. Remark 5.3. Pugh's interpretations of this McMillan degree are limited, however, whereas the generalized framework we have built up allows us to go considerably further, as we illustrate now with reference to some other results of Pugh's.

In [41], Pugh showed that under certain sufficient but *not* necessary conditions the Rosenbrock degrees (see Remark 5.2) of systems as well as the McMillan degrees of their transfer functions have the previous additive property under compositions of the form (79). His results are of limited scope, however, since his sufficient conditions are severe (effectively requiring that the composite system input feed directly into every subsystem input independently, and that the composite system output directly contain independent measurements of every subsystem output). In contrast we find here that we can, with little effort, obtain easily stated and intuitively appealing necessary and sufficient conditions for additivity of the Rosenbrock and McMillan degrees.

First note that the block diagonal structure of the matrices in (83), (84) shows that any decoupling zeros (finite or infinite) of the subsystems remain decoupling zeros of the composite, while *additional* ones may be created as a result of the interconnection. Denoting the transfer functions and Rosenbrock degrees of the subsystems and composite by $\{G_i(s)\}$, $\{d_i\}$ and $G(s)$, d, respectively, and with other notation as earlier, we then obtain the following result.

Theorem 5.4: i) $d=\sum d_i$ if and only if no infinite decoupling zeros are created by the interconnection.

ii) $\delta[G(s)]=\sum \delta[G_i(s)]$ if and only if no decoupling zeros, finite or infinite, are created by the interconnection.

Proof: i) follows immediately from substituting (77), written for both the composite and the subsystems in (85). ii) follows immediately from substituting (69), written for both the composite and the subsystems in (85). $\triangle\triangle$

The results presented in this subsection have been simply stated and proved in our framework, and in this framework they are "intuitively obvious" perhaps. Nevertheless, they are believed to be of some importance and interest (particularly part ii) of Theorem 5.4, especially since they seem to have eluded previous formulations (except when restricted to very special cases, e.g. interconnections of proper 'unimodules' as in Singh and Liu [42]). Note especially that these structural results are applicable to systems of the regular theory, i.e., to those that have been in existence prior to $t=0$. The moral seems to be that in order to determine how subsystem properties are reflected into those of a composite system, one needs a framework that can admit the (perhaps hypothetical) act of interconnection at some instant $t=0$.

C. Electrical Networks—Some Examples

Electrical networks are natural candidates for analysis as composite systems, as they are formed by interconnecting in various configurations a few standard components whose detailed behavior is known. They are, however, governed by interconnection rules that are not as simple as those of the previous subsection, (79), the reason being that the component subsystems are no longer "directed." We do not present any general results on electrical networks here, but merely illustrate with respect to *LCR* networks (i.e., those composed of inductors, capacitors, and resistors only) how our present framework helps to clarify the description and analysis. More general results can, we believe, be obtained along the same lines, but these are left to future work.

Once again, the stimulus for our work comes from a paper of Rosenbrock's [43]. From [43] or [6] it may be seen that an *LCR* network may be described by a system matrix similar to (84), namely,

$$\left[\begin{array}{cc|c} pE_S-A_S & -I & 0 \\ F_1 & F_2 & -K \\ \hline L_1 & L_2 & 0 \end{array}\right] \tag{86}$$

with

$$\begin{aligned} f=\operatorname{rank} E_S=\#&\text{reactive elements} \\ &\text{(i.e., inductors and capacitors).} \end{aligned} \tag{87}$$

The first block row in (86) defines the individual components, the second block row describes their interconnection constraints.

The analysis made by Rosenbrock and Pugh in [6], [39], [43] involves the use of the "complexity" of the equations (86) (see previous subsection) because, with the particular way they have chosen to set up their initial description

(86), it turns out that the complexity equals the number of reactive elements. In view of our objections in the previous subsection to their concept of complexity, it seems preferable to us to simply substitute "number of reactive elements" for the term "complexity" in reading their results. Moreover, the term "order of complexity" is in rather standard use in the circuit theory literature, where it denotes the regular order of a circuit, i.e., the number of exponential modes (see, for example, Milic [44]); this number can be less than the complexity defined in [6], [39], for it is well known, see [44] and references therein, that for an LCR network the number of reactive elements may exceed the regular order, and that the excess is caused by the presence of capacitor loops and/or inductor cutsets. In our framework, and using (87), the excess indicates the presence of impulsive modes that may be obtained by assembling the circuit at $t=0$ with arbitrary initial conditions (leading to impulsive currents in capacitor loops and impulsive voltages across inductor cutsets). Thus,

$$\#\text{reactive elements} - \text{regular order} = \#\text{impulsive modes}. \tag{88}$$

The discussion in [43] is for circuits defined with port current inputs and associated port voltage outputs, and with corresponding transfer function or impedance matrix $G(s)$. Necessary and sufficient conditions for a matrix $G(s)$ to be realizable as the impedance of an $LCRT$ network, i.e., an LCR network *with transformers*, have been known since the work of McMillan [37], who also showed that in this case no more than $\delta[G(s)]$ reactive elements are required and no fewer can be used. In contrast, for an LCR network such necessary and sufficient conditions on a $G(s)$ are not known, and furthermore all that can be said in general of the number of reactive elements required to obtain realizable impedance matrices is that

$$\#\text{reactive elements} - \delta[G(s)] \geqslant 0. \tag{89}$$

Following [43], we may term the quantity in (89) the "redundancy" of the LCR realization of $G(s)$. It is tempting to conjecture (based on intuition developed from the regular theory) that this redundancy is entirely accounted for by the presence of finite decoupling zeros. Rosenbrock [43] presents examples, however, to show that this is untrue, i.e., to show that for some circuits

$$\text{redundancy} > \#\text{finite decoupling zeros}. \tag{90}$$

The remaining redundancy, i.e., that not accounted for by the finite decoupling zeros, is however *not* accounted for in [43] in any satisfactory way, and is merely labeled (as being "due to" the realization not having "least complexity"). Our framework, on the other hand, clarifies the situation considerably. Using (69) and (87) we see immediately that

$$\text{redundancy} = \#\text{finite } \textit{and} \text{ infinite decoupling zeros}. \tag{91}$$

[This is easily shown to be consistent with (88).] Since we have at this point a fair understanding of the interpretation and properties of the infinite decoupling zeros, our result is of some interest. We point out again that structural results such as (88), (91) show that concepts of the generalized theory are useful even for systems that have existed prior to $t=0$; what is important for the structural results is whether and how impulsive behavior *would* occur *if* the circuit was put together at $t=0$.

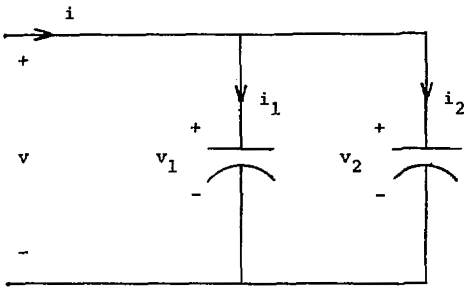


Fig. 4. Circuit of Example 5.1.

Rosenbrock has also shown [43] that (in the LCR realization of an impedance $G(s)$) for every unobservable mode at a finite frequency there is an uncontrollable mode at the same frequency, and vice versa, with some of these being both unobservable *and* uncontrollable. We conjecture that the same is true of the unobservable/uncontrollable impulsive modes.

Example 5.1: The system of Fig. 4 has often been used, see [39], [6], [26]. It may be described by the following equations:

$$\left[\begin{array}{cccc|c} p & 0 & -1 & 0 & 0 \\ 0 & p & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1 \\ v_2 \\ i_1 \\ i_2 \\ \hline i \end{array}\right] = \left[\begin{array}{c} \\ \\ 0 \\ \\ \hline v \end{array}\right]. \tag{92}$$

The preceding equations are in the standard form (86). Note that the first two equations of (92) define the components, while the next two equations describe their interconnection constraints.

By operations of strong equivalence we may bring (92) to the form

$$\left[\begin{array}{c:cc:c|c} p & & & & -1 \\ \hdashline & 1 & -p & & 0 \\ & & 1 & & 0 \\ \hdashline & & & 1 & -1 \\ \hline 1/2 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_1+v_2 \\ i_1-i_2 \\ v_1-v_2 \\ i_1+i_2 \\ \hline i \end{array}\right] = \left[\begin{array}{c} \\ 0 \\ \\ \\ \hline v \end{array}\right] \tag{93}$$

which immediately tells us (as (92) could have told us, but not this transparently) that the generalized order of the system is 2, and that the system has one exponential mode (at frequency zero) that is observable and controllable, and one impulsive mode that is neither observable nor controllable.

The above analysis makes no use of the concept of complexity, and provides a sufficiently detailed and satisfying explanation of the impulsive behavior of the system. It may be compared with the analysis of Rosenbrock in [39], who merely states that the system has (regular) order 1 and complexity 2; it is not shown that there is a precise and

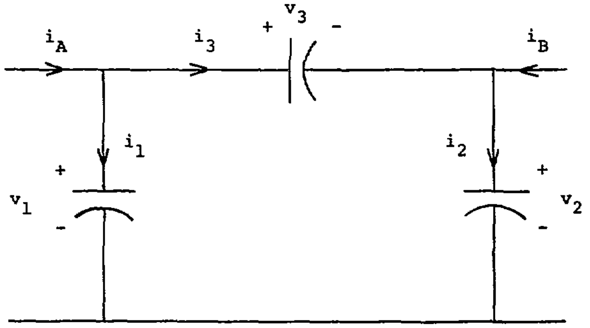


Fig. 5. Circuit of Example 5.2.

useful sense in which the system order is 2, and in which the system is uncontrollable and unobservable. The application of his r.s.e. framework of [13] to this example also leads to very unsatisfactory results (for we find 3 infinite decoupling zeros). △△

While the situation of Fig. 4 is somewhat pathological, the next example illustrates the more significant situation, where an unavoidable redundancy is present, in the sense that no *LCR* realization of the given $G(s)$ can have zero redundancy [where the latter is the quantity defined by (89)].

Example 5.2: The circuit of Fig. 5 is analyzed in [6], [43] as an *LCR* realization of

$$G(s)=\begin{bmatrix} 2/3s & 1/3s \\ 1/3s & 2/3s \end{bmatrix} \tag{94}$$

that uses the smallest possible number of reactive elements. Describing the circuit in the standard form (86), i.e., keeping separate the equations that define the components and those that describe the interconnection constraints, and subsequently transforming by operations of strong equivalence, we obtain the description

$$\left[\begin{array}{cc:cc|cc} p & & & & -1 & 0 \\ & p & & & 0 & -1 \\ \hdashline & & 1 & -p & 0 & 0 \\ & & & 1 & 0 & 0 \\ \hline 2/3 & 1/3 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 1/3 & 2/3 & 0 & 0 & & \end{array}\right]\left[\begin{array}{c} v_1+v_3 \\ v_2-v_3 \\ i_1-i_2-i_3 \\ v_1-v_2-v_3 \\ \hline i_A \\ i_B \end{array}\right]=\left[\begin{array}{c} \\ \\ 0 \\ \\ \hline v_1 \\ v_2 \end{array}\right] \tag{95}$$

The generalized order of the system is 3, with two modes at the origin and one at infinity, the latter being unobservable and uncontrollable. △△

D. Singularly Perturbed Systems

Coupled equations of the form

$$\begin{aligned} \dot{x}&=A_{11}x+A_{12}z+B_1u \\ \epsilon\dot{z}&=A_{21}x+A_{22}z+B_2u, \qquad \epsilon>0 \end{aligned} \tag{96}$$

where ϵ is a "small" parameter, arise naturally in describing systems that are obtained by, for example, coupling subsystems with widely separated natural frequencies (as occur when parasitic elements in a system are to be modeled, or when a relatively sluggish mechanical system such as an electrical generator is connected to an electrical transmission network). For our purposes, a recent overview by Kokotovic *et al.* [8] suffices as background (see also [46]).

When $\epsilon=0$ in (96), the regular order of the system drops, which leads to the description "singularly perturbed" for such systems. The aim of singular-perturbation analysis is to approximate the behavior of (96) via perturbations of the behavior of the system

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{z}} \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix}+\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u. \tag{97}$$

From (97) we find, if A_{22} is invertible, that

$$\bar{z}=-A_{22}^{-1}(A_{21}\bar{x}+B_2u) \tag{98}$$

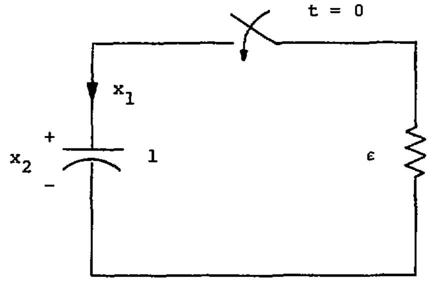
so

$$\dot{\bar{x}}=(A_{11}-A_{12}A_{22}^{-1}A_{21})\bar{x}+(B_1-A_{12}A_{22}^{-1}B_2)u. \tag{99}$$

Now it can be shown that if A_{22} has all its eigenvalues in the open left-half-plane (which in turn assures its invertibility) then the solution $\bar{x}$ of the reduced system in (99), with the initial condition $x(0-)$ specified for the original problem (96), approximates to order ϵ the solution for x in the original problem, while the solution $\bar{z}$ of (98) approximates to order ϵ the solution for z in the original problem, except for a "boundary layer" correction near $t=0$ that enables the transition from $z(0-)$ specified in the original problem to the value $\bar{z}(0+)$ required by (98).

Now (97) may be viewed in our language as a singular system, and we may consider imposing the obvious *unconstrained* initial conditions $\bar{x}(0-)=x(0-)$ and $\bar{z}(0-)=z(0-)$. The assumption, however, that A_{22} is nonsingular, apparently a standard assumption in the literature, guarantees that no impulsive motions occur in response to arbitrary initial conditions in (97), since under this assumption (97) reduces to the regular system (99) and the nondynamic equation (98). Conversely, one can show that if A_{22} is singular the system (97), if solvable, exhibits impulsive free-response motions (for certain initial conditions).

The above observation suggests that relaxing the assumption of nonsingular A_{22} will lead to the development of a perturbation theory for problems involving systems that exhibit "nearly impulsive" behavior. Such problems may, for example, be expected to arise when subsystems are switched together through "imperfect" switches. [For example, let the situation of Fig. 2 be replaced by that of Fig. 6, where a charged capacitor is shorted through an ϵ resistor; we now obtain a spike of current that, in the limit of $\epsilon\to 0$, tends precisely (in a distributional sense) to the impulsive current calculated in Example 1.1 for this same system but with $\epsilon=0$. The describing equations for this system are not quite in the form (96), but (on simply interchanging the variables in the equations in Fig. 6) may be written in the suggestive form



$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varepsilon & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad t \geq 0$$

$$x_1(t) = -\frac{1}{\varepsilon} e^{-t/\varepsilon} x_2(0-), \quad t \geq 0$$

Fig. 6. Perturbed version of Fig. 2, displaying nearly impulsive behavior.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & \epsilon \end{bmatrix}\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \qquad (100)$$

which for $\epsilon=0$ is of the form (97) but with A_{22} singular.] Another situation of interest concerns regulation of systems using cheap control (see, for example, [47]).

There is some literature on the case of singular A_{22}; the system (96) is then referred to as a "singular singularly perturbed" system, see for example [48], [49]. For our purposes, however, the most interesting results are those of Francis [59] and Cobb [50]. Francis shows that if

$$\text{determinant}\begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{22} & \epsilon sI-A_{22} \end{bmatrix}$$

has roots in the left half-plane for $0<\epsilon<\epsilon_0$ and if it is nonzero for $\epsilon=0$—the latter condition guarantees solvability of the singular system (97)—then, with no further conditions on A_{22}, the free-response of (96) converges in a distributional sense to that of the singular system, including the impulsive solutions. He also makes application to the problem of cheap control.

Cobb [50] provides answers to several questions suggested by our framework, and makes substantial headway on many others. He considers the case of systems of the form

$$E(\epsilon)\dot{x}=A(\epsilon)x+B(\epsilon)u \qquad (101)$$

where ϵ is now allowed to be a vector of parameters. Suppose $E(\epsilon)$ is singular for $\epsilon=0$, and not otherwise. Again, the most important question here is under what conditions the solutions of (101) converge (as $\epsilon \to 0$) to those—including the impulsive solutions— of the singular system that results for $\epsilon=0$. If a given system (101) satisfies such conditions then there is justification for setting $\epsilon=0$ and working with the (possibly simpler) singular system, at least for a first pass at analysis (with perhaps subsequent iterative refinement). It is easy to construct examples where there is no convergence to the solutions (exponential and/or impulsive) of the singular system. Cobb shows, however, that if the solutions converge at all, they converge (in the sense of distributions) to those solutions of the singular system that we obtained directly by other methods (e.g., use of Laplace transforms). He also gives certain sufficient conditions for convergence for $t>0$, but these are far from simple to verify for a given system (101). [It would be desirable to at least have simply verifiable conditions for special classes of systems of the form (101). We can show, for example, that systems of the form $(E-\epsilon A)\dot{x}=Ax$, for scalar $\epsilon>0$ and singular E, always have convergent solutions for $t\geqslant 0$, i.e., including the impulsive part.] It would appear that Francis' approach and his results can be extended to systems of the form (101), but this is left to future work.

Cobb also obtains certain results for singular systems. His definition of controllability turns out to be the same as that of [13], [52], [53]. He discusses pole placement by feedback, and in particular the question of when impulsive modes can be eliminated by feedback. (It turns out, interestingly enough, that such elimination of impulsive modes is possible if and only if the singular system is controllable in *our* sense. Several other results of his also suggest that our definition of controllability is perhaps more useful and natural for singular systems.) Additional results of Cobb concern, for example, the controllability of (101) in the neighborhood of $\epsilon=0$.

Before concluding this subsection, we may mention recent work of Hazewinkel [54] and important results of Willems [55], [56] that may be expected to tie in with our work here. Much remains to be done in terms of perturbation results, in order to go beyond any esthetic value our framework may have and into the realm of genuine practical usefulness, but the indications are certainly encouraging [63].

VI. Conclusion

The main contribution of this paper has in our opinion been the elaboration of the existence and structure of impulsive modes in singular systems and of their coupling to system inputs and outputs. The outcome is a satisfyingly consistent and useful framework for dealing with such systems, and one that opens the door to interesting generalizations of results previously restricted to regular state-space systems with strictly proper transfer functions. Moreover, this framework has formed the starting point for the extensions in [4], [5] to systems of higher order equations, again beginning with the analysis of impulsive modes in such systems (but now using a generalized Smith–McMillan form rather than the Kronecker form in order to display system structure).

There are several directions in which this work requires development. Note that while we have used the Kronecker form of $(sE-A)$ fairly heavily for our discussions and proofs, in actual computation one would want to use less special forms. The algorithms in [32], [33] are indicative of the sorts of procedures one ought to be looking for. Also, results such as the fourfold decomposition of (58) have been derived via detailed manipulation of a canonical form; a more general derivation would be preferable. The geometric language of [57], adapted to singular systems in [50], could be enlightening here if appropriately modulated by concrete matrix interpretations. The study of invariant subspaces, especially those corresponding to the impulsive modes, needs to be built up on our understanding of system dynamics, and [30] contains the basis for this. Work remains to be done in connection with perturbation and asymptotic properties, feedback control, observer design, and other such problems that may be of practical interest in the context of the generalized theory for singular systems. The ability to represent arbitrary rational matrices in the form $C(sE-A)^{-1}B$ will undoubtedly turn out to be useful, in view of results such as those in [31]; one would hope, in this connection, to make contact with the work in [20] and [58]. We conclude, rather arbitrarily, with a different sort of question: if (1) is obtained by linearization of an implicit nonlinear system of the form $h(\dot{x}, x, u)=0$, what (if anything) can be inferred about the nonlinear system by studying its linearized version? Reference [62] may contain some leads on this.

Appendix
Elaboration of Remarks 3.3, 3.4

1) The isomorphism embodied in (39)–(44) may be quite easily verified for the case of S and S_1 related by a *single* allowed transformation, namely left or right transformation of the system matrix as in (37), or trivial augmentation or deflation. The general validity of (39)–(44) then follows from the fact (again simply verified) that if S_2 and S_1 are related in a way similar to S_1 and S, i.e., by equations of the form (39)–(44), then S_2 and S are also related in a similar way.

2) We wish to show that the isomorphism between solutions expressed by (39)–(44) implies, for systems in standard form, that they are related as in (37), i.e., by just a str. eq. operation. From this will follow the fact that two systems are str. eq. if and only if standard forms for them are related as in (37). We also obtain from this the result of Remark 3.4, since if two systems have isomorphic solutions then standard forms for them will also have isomorphic solutions and therefore be related as in (37), implying strong equivalence of the original systems.

We restrict our demonstration to systems

$$S_i:\left[\begin{array}{c|c} I-sJ_i & -B_i \\ \hline C_i & D_i \end{array}\right], \qquad i=1,2 \tag{A.1}$$

that are in standard form (i.e., the nilpotent Jordan matrices J_i have no 1×1 Jordan blocks). Extension to systems of the type (1) is then straightforward, on incorporating the results of Pernebo [23] on regular systems.

Assume, therefore, that there is a bijective mapping of the form

$$x_2(t)=T_2x_1(t)+V_2u(t), \qquad t\geqslant 0 \tag{A.2}$$

between the solutions of S_1 and S_2 for a given $u(t)$, with

$$J_2x_2(0-)=P_2J_1x_1(0-) \tag{A.3}$$

and

$$J_2T_2=P_2J_1, \qquad J_2V_2=0. \tag{A.4}$$

Also assume that the output $y(t)$ is preserved under the above mapping. Now

$$(I-sJ_i)X_i(s)=B_iU(s)-J_ix_i(0-), \qquad i=1,2 \tag{A.5}$$

or, using (A.2), (A.3),

$$(I-sJ_2)\left[T_2(I-sJ_1)^{-1}(B_1U(s)-J_1x_1(0-))+V_2U(s)\right] =B_2U(s)-P_2J_1x_1(0-). \tag{A.6}$$

Setting $U(s)=0$ and letting $x_1(0-)$ be arbitrary, we find from (A.6) that

$$T_2J_1=P_2J_1 \;(=J_2T_2 \text{ from (A.4)}). \tag{A.7}$$

Setting $x_1(0-)=0$ and letting $U(s)$ be arbitrary, we find that

$$T_2B_1+V_2=B_2. \tag{A.8}$$

From the fact that the system outputs are identical under the mapping (A.2), we also find that

$$C_2(T_2x_1+V_2u)+D_2u=C_1x_1+D_1u$$

or

$$(C_2T_2-C_1)\left[(I-sJ_1)^{-1}(B_1U(s)-J_1x_1(0-))\right] =(D_1-C_2V_2-D_2)U(s). \tag{A.9}$$

Setting $U(s)=0$ and letting $x_1(0-)$ be arbitrary shows that

$$(C_2T_2-C_1)J_1=0 \tag{A.10}$$

while setting $x_1(0-)=0$ and letting $U(s)$ be arbitrary shows that

$$(C_2T_2-C_1)B_1=D_1-C_2V_2-D_2. \tag{A.11}$$

Now, using (A.4), (A.7), (A.8), (A.10), (A.11), we discover that

$$\begin{bmatrix} T_2 & 0 \\ W_2 & I \end{bmatrix}\left[\begin{array}{c|c} I-sJ_1 & -B_1 \\ \hline C_1 & D_1 \end{array}\right]=\left[\begin{array}{c|c} I-sJ_2 & -B_2 \\ \hline C_2 & D_2 \end{array}\right]\begin{bmatrix} T_2 & V_2 \\ 0 & I \end{bmatrix} \tag{A.12}$$

where

$$W_2=C_2T_2-C_1. \tag{A.13}$$

Note from (A.4), (A.10) that

$$W_2J_1=0=J_2V_2. \tag{A.14}$$

We have not thus far used the bijectivity of (A.2), or the fact that the systems in (A.1) are in standard form. These enter into showing that T_2 in (A.12) is actually nonsingular, and that the systems (A.1) are therefore related by a str. eq. operation as in (37). For this, suppose T_2 did not have full *column* rank, i.e., assume

$$T_2 v=0, \qquad \text{for some vector } v\neq 0. \tag{A.15}$$

Then, from (A.7),

$$T_2 J_1 v=0. \tag{A.16}$$

We must either have $J_1 v=0$ or $J_1 v\neq 0$. It is easily verified that in the former case we can write

$$v=J_1\bar{v}, \qquad \bar{v}\neq 0 \tag{A.17}$$

as a consequence of the fact that J_1 has no 1×1 blocks. This would mean, however, that if we picked

$$u\equiv 0, \qquad x_1(0-)=\bar{v} \tag{A.18}$$

we would obtain, from (A.5),

$$X_1(s)=(I-sJ_1)^{-1}J_1\bar{v}\neq 0 \tag{A.19}$$

and, under the mapping (A.2),

$$\begin{aligned} X_2(s)&=T_2X_1(s)\\ &=T_2(I+sJ_1+\cdots)\,v, \qquad \text{from (A.19)},\\ &=0, \qquad \text{using (A.15), (A.7)}. \end{aligned} \tag{A.20}$$

This would contradict the injectivity of the mapping (A.2), since injectivity implies that a nonzero $X_1(s)$ cannot be mapped into a zero $X_2(s)$.

If, on the other hand, $J_1 v\neq 0$, we could choose

$$u\equiv 0, \quad x_1(0-)=v \tag{A.21}$$

to obtain

$$X_1(s)=(I-sJ_1)^{-1}J_1 v\neq 0 \tag{A.22}$$

and

$$\begin{aligned} X_2(s)&=T_2X_1(s)\\ &=T_2(I+sJ_1+\cdots)J_1 v\\ &=0, \qquad \text{using (A.16), (A.7)} \end{aligned} \tag{A.23}$$

which leads us to the same contradiction as before. Thus, T_2 must have full column rank.

A dual argument, using the surjectivity of (A.2), shows that T_2 must have full row rank. For suppose we had

$$v'T_2=0, \qquad v'\neq 0 \tag{A.24}$$

then

$$v'J_2T_2=0, \qquad \text{from (A.7)}. \tag{A.25}$$

If $v'J_2=0$, then

$$v'=\bar{v}'J_2, \qquad \bar{v}\neq 0 \tag{A.26}$$

since J_2 has no 1×1 Jordan blocks. Hence,

$$\begin{aligned} \bar{v}'X_2(s)&=\bar{v}'(I-sJ_2)^{-1}J_2x_2(0-)\\ &=\bar{v}'(I+sJ_2+\cdots)J_2x_2(0-)\\ &\neq 0, \qquad \text{for some } x_2(0-), \end{aligned} \tag{A.27}$$

but

$$\begin{aligned} \bar{v}'T_2X_1(s)&=\bar{v}'T_2(I-sJ_1)^{-1}J_1x_1(0-)\\ &=\bar{v}'J_2T_2(I-sJ_1)^{-1}x_1(0-), \qquad \text{by (A.7)},\\ &=0, \qquad \text{by (A.24), (A.26)}. \end{aligned} \tag{A.28}$$

This implies that the $X_2(s)$ of (A.27) cannot be obtained by the mapping (A.2), which contradicts the surjectivity of this mapping.

If $v'J_2\neq 0$, one can similarly show that

$$v'X_2(s)\neq 0, \qquad \text{for some } x_2(0-) \tag{A.29}$$

but that

$$v'T_2X_1(s)=0 \tag{A.30}$$

which leads to the same contradiction. Thus, T_2 has full row rank also, and is therefore nonsingular. The desired results then follow easily from (A.12)–(A.14).

We note in conclusion that although (A.3), (A.4) were presented as part of the characterization of the isomorphism between solutions of S_1 and S_2, and were used in the preceding proof, they may be replaced by the perhaps more pleasing statement that (A.2) holds at $t=0-$ as well, in a limited sense, namely,

$$J_2x_2(0-)=J_2T_2x_1(0-)+J_2V_2u(0-). \tag{A.28}$$

Substitution of this in (A.6) then yields the equalities required for the rest of the proof, namely $T_2J_2=J_2T_2$ and $J_2V_2=0$.

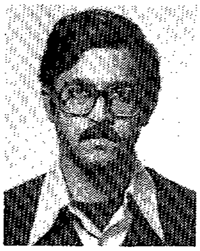
Acknowledgment

The authors would like to thank: Prof. Rosenbrock for discussions that stimulated their interest in singular systems; Dr. P. Van Dooren and Prof. P. Dewilde for much patient instruction, especially on the structure at infinity of rational matrices; Dr. L. Pernebo, for important insights that have filtered into this paper; and the reviewers, for comments that we believe helped improve the paper considerably. The first author would also like to thank his colleagues and friends in the M.I.T Electric Power Systems Engineering Lab., for their uncomplaining support of work such as this, and their unstinting interest, encouragement and help since he joined the lab.

Note Added in Proof: References [27], [33], and [56] have appeared in the February 1981 issue of this Transactions, and [38] has appeared in the April 1981 issue; [36] has appeared in *Int. J. Contr.*, vol. 31, p. 1007, 1980; [51] has appeared in *J. Optimiz. Theory Appl.*, vol. 30, April 1980; Cobb presents further discussion of issues relevant to Section V-D in "On the solutions of linear differential equations with singular coefficients," preprint, Dep. Electrical Engineering, Univ. of Toronto, 1981.

References

[1] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation*. New York: Springer-Verlag, 1974.
[2] C. T. Chen, *Introduction to Linear System Theory*. New York: Rinehart and Winston, 1970.
[3] S. Perlis, *Theory of Matrices*. Cambridge, MA: Addison-Wesley, 1952.
[4] G. Verghese, "Infinite-frequency behavior in generalized dynamical systems," Ph.D. dissertation, Dep. Electrical Engineering, Stanford Univ., Dec. 1978.
[5] G. Verghese and T. Kailath, "Impulsive behavior in dynamical systems: Structure and significance," in *Proc. 4th Int. Symp. Mathematical Theory of Networks and Systems*, Delft, The Netherlands, July 1979.
[6] H. H. Rosenbrock and A. C. Pugh, "Contributions to a hierarchical theory of systems," *Int. J. Contr.*, vol. 19, no. 5, pp. 845–867, 1974.
[7] B. Dziurla and R. Newcomb, "The Drazin inverse and semi-state equations," in *Proc. 4th Int. Symp. Mathematical Theory of Networks and Systems*, Delft, The Netherlands, pp. 283–289, July 1979; R. Newcomb, "The semistate description of nonlinear time-variable circuits," *IEEE Trans. Circuits Syst.*, vol. CAS-28, Jan. 1981.
[8] P. V. Kokotovic, R. E. O'Malley, Jr., and P. Sannuti, "Singular perturbations and order reduction in control theory—An overview," *Automatica*, vol. 12, pp. 123–132, Mar. 1976.
[9] D. G. Luenberger, "Dynamic equations in descriptor form," *IEEE Trans. Automat. Contr.*, vol. AC-22, pp. 312–321, Mar. 1977.
[10] —, "Time-invariant descriptor systems," in *Proc. JACC*, pp. 725–730, San Francisco, CA, 1977.
[11] A. Erdelyi, *Operational Calculus and Generalized Functions*. New York: Holt, Rinehart and Winston, 1962.
[12] T. P. G. Liverman, *Generalized Functions and Direct Operational Methods*. Englewood Cliffs, NJ: Prentice-Hall, 1964.
[13] H. H. Rosenbrock, "Structural properties of linear dynamical systems," *Int. J. Contr.*, vol. 20, no. 2, pp. 191–202, 1974.
[14] F. R. Gantmacher, *Theory of Matrices* (2 volumes). New York: Chelsea, 1959.
[15] E. Polak, "An algorithm for reducing a linear, time-invariant differential system to state form," *IEEE Trans. Automat. Contr.*, vol. AC-11, pp. 577–579, July 1966.
[16] A. Fettweis, "On the algebraic derivation of the state equations," *IEEE Trans. Circuit Theory*, vol. CT-16, pp. 171–175, May 1969.
[17] A. Dervisoglu and C. A. Desoer, "Degenerate networks and minimal differential equations," *IEEE Trans. Circuits and Syst.*, vol. CAS-22, pp. 769–775, Oct. 1975.
[18] J. H. Wilkinson, "The differential system $B\dot{x}=Ax$ and the generalized eigenvalue problem $Au=\lambda Bu$," Nat. Phys. Lab., Britain, Rep. NAC73, Jan. 1977.
[19] S. L. Campbell, C. D. Meyer, and N. J. Rose, "Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients," *SIAM J. Appl. Math.*, vol. 31, no. 3, pp. 411–425, Nov. 1976.
[20] I. Gohberg and L. Rodman, "On spectral analysis of non-monic matrix polynomials, I. Reduction to monic polynomials," *Israel J. Math.*, vol. 30, nos. 1–2, pp. 133–151, 1978.
[21] H. H. Rosenbrock, *State-space and Multivariable Theory*. New York: Wiley, 1970.
[22] W. A. Wolovich, *Linear Multivariable Systems*. New York: Springer-Verlag, 1974.
[23] L. Pernebo, "Notes on strict system equivalence," *Int. J. Contr.*, vol. 25, no. 1, pp. 21–38, 1977.
[24] P. A. Fuhrmann, "On strict system equivalence and similarity," *Int. J. Contr.*, vol. 25, no. 1, pp. 5–10, 1977.
[25] B. Lèvy, S-Y. Kung, M. Morf, and T. Kailath, "A unification of system equivalence definitions," in *Proc. IEEE Conf. Decision and Control* (New Orleans, LA), pp. 795–801, 1977.
[26] H. Blomberg, "Systems and interconnections," Helsinki Univ. Tech., Helsinski, Finland, Systems Theory Lab., B24, 1975.
[27] R. F. Sincovec, *et al.*, "Numerical analysis of descriptor systems," in *Systems Engineering for Power: Organizational Forms for Large Scale Systems, DOE Conf. Proc.*, vol. I, Davos, Switzerland, Sept. 1979.
[28] R. E. Kalman, *Proc. Nat. Acad. Sci.*, Washington, DC, vol. 48, p. 596, 1962.
[29] —, *SIAM J. Contr.*, A, vol. 1, p. 152, 1963.
[30] G. Verghese and T. Kailath, "Eigenvector chains for finite and infinite zeros of rational matrices," in *Proc. 18th IEEE Conf. Decision and Control*, Fort Lauderdale, FL, Dec. 1979.
[31] G. Verghese, P. Van Dooren, and T. Kailath, "Properties of the system matrix of a generalized state-space system," *Int. J. Contr.*, vol. 30, 1979.
[32] P. Van Dooren, "The generalized eigenstructure problem—Applications in linear system theory," Ph.D. dissertation, Katholicke Univ. Leuven, Leuven, Belgium, 1979.
[33] —, "The generalized eigenstructure problem in linear system theory," Information Systems Lab., Stanford Univ., Stanford, CA, to be published.
[34] B. L. Ho and R. E. Kalman, "Effective construction of linear state-variable models from input/output functions," *Regelungstech.*, vol. 14, pp. 545–548, 1966.
[35] S. MacLane and G. Birkhoff, *Algebra*, New York: Mcmillan, 1967.
[36] G. Verghese, "Comments on 'Properties of the system matrix of a generalized state-space system,'" *Int. J. Contr.*, to be published.
[37] B. McMillan, "Introduction to formal realizability theory," *Bell Syst. Tech. J.*, vol. 31, no. 2, pp. 217–279, Mar. 1952; vol. 31, no. 4, pp. 541–600, May 1952.
[38] G. Verghese and T. Kailath, "Rational matrix structure," *IEEE Trans. Automat. Contr.*, vol. AC-26, pp. 434–439, Apr. 1981; *Proc. 18th IEEE Conf. Decision and Control*, Fort Lauderdale, FL, Dec. 1979.
[39] H. H. Rosenbrock, "Order, degree and complexity," *Int. J. Contr.*, vol. 19, no. 2, pp. 323–331, 1974.
[40] A. C. Pugh, "The McMillan degree of a polynomial system matrix," *Int. J. Contr.*, vol. 24, no. 1, pp. 129–135, 1976.
[41] —, "On composite systems that preserve the degrees of their subsystems," *Int. J. Control*, vol. 21, no. 3, pp. 465–473, 1975.
[42] R. S. Singh and R-W. Liu, "Irreducibility of large-scale dynamical systems of uni-modules," *IEEE Trans. Circuits Syst.*, vol. CAS-23, pp. 122–124, Feb. 1976.
[43] H. H. Rosenbrock, "Non-minimal *LCR* multiports," *Int. J. Contr.*, vol. 20, no. 1, pp. 1–16, 1974.
[44] M. M. Milic, "General passive networks—Solvability, degeneracies, and order of complexity," *IEEE Trans. Circuits Syst.*, vol. CAS-21, pp. 177–183, Mar. 1974.
[45] T. Kailath, *Linear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1980.
[46] P. V. Kokotovic, J. J. Allemong, J. R. Winkelman, and J. H. Chow, "Singular perturbation and iterative separation of time scales," *Automatica*, vol. 16, no. 1, pp. 23–34, Jan. 1980.
[47] R. E. O'Malley and A. Jameson, "Singular perturbations and singular arcs, I," *IEEE Trans. Automat. Contr.*, vol. AC-20, pp. 218–226, 1975.
[48] R. E. O'Malley, "A singular singularly-perturbed linear boundary value problem," *SIAM J. Math. Anal.*, vol. 10, pp. 695–708, 1979.
[49] S. L. Campbell, *Singular Systems of Differential Equations*. Pitman Publ., 1980.
[50] D. J. Cobb, "Descriptor variable and generalized singularly perturbed systems: A geometric approach," Ph.D. dissertation, Dep. Electrical Engineering, Univ. Illinois, Urbana, 1980.
[51] L. Pandolfi, "Controllability and stabilization for linear systems of algebraic and differential equations," Istituto Matematico, Univ. Degli Studi Di Firenze, Italy, 1978.
[52] E. L. Yip and R. F. Sincovec, "Solvability, controllability and observability of continuous descriptor systems," Boeing Computer Services, Seattle, WA, to be published.
[53] C. E. Langenhop, "Controllability and stabilizability of regular singular linear systems with constant coefficients," Dep. Mathematics, Southern Illinois Univ., to be published.
[54] M. Hazewinkel, "On families of linear systems: Degeneration phenomena," in *Proc. AMS Summer Seminar on Algebraic and Geometric Methods in Linear System Theory*, 1980.
[55] J. C. Willems, "Almost A(mod-B)-invariant subspaces," *Asterisque*, 1980.
[56] —, "Almost invariant subspaces: An approach to high gain feedback design," Mathematics Inst., Groningen, The Netherlands, to be published.
[57] W. M. Wonham, *Linear Multivariable Control—A Geometric Approach*. New York: Springer-Verlag, 1974.
[58] H. Bart, I. Gohberg, and M. A. Kaashoek, *Minimal Factorization of Matrix and Operator Functions*. Basel, Switzerland: Birkhauser Verlag, 1979.
[59] B. Francis, "Convergence in the boundary layer for singularly perturbed equations," *Automatica*, to be published.
[60] A. J. J. Van Der Weiden and O. H. Bosgra, "The determination of structural properties of a linear multivariable system by operations of system similarity—2. Non-proper systems in generalized state-space form," *Int. J. Contr.*, vol. 32, no. 3, pp. 489–537, 1980.
[61] D. Boley, "Computing the controllability-observability decomposition of a linear time-invariant dynamic system—A numerical approach," Ph.D. dissertation, Dep. Computer Science, Stanford Univ., Stanford, CA, 1981.
[62] S. S. Sastry, C. A. Desoer, and P. P. Varaiya, "Jump behavior of circuits and systems," Electronics Research Lab, Univ. California, Berkeley, Memo No. UCB/ERL M80/44, Oct. 1980.



George C. Verghese (S'74–M'78) was born in Addis Ababa, Ethiopia, on August 15, 1953. He received the B. Tech. degree in electronics from the Indian Institute of Technology, Madras, India, in 1974, the M.S. degree in electrical science from the State University of New York, Stony Brook, in 1975, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA, in 1978.

He held various appointments as teaching assistant, teaching fellow, and research assistant during his graduate studies. Since January 1979 he has been an Assistant Professor of Electrical Engineering at Massachusetts Institute of Technology, Cambridge, attached to the Electric Power Systems Engineering Laboratory. His research interests are in the areas of systems, control and estimation, and their applications to power systems and power electronics.

Thomas Kailath (S'57–M'62–F'70) was born in Poona, India, on June 7, 1935. He received the B.E. degree in telecommunications engineering from the University of Poona, Poona, India, in 1956, and the S.M. and Sc.D. degrees in electrical engineering from the Massachusetts Institute of Technology, Cambridge, in 1959 and 1961, respectively.

During 1961–1962 he worked at the Jet Propulsion Laboratories, Pasadena, CA, where he also taught part-time at the California Institute of Technology. From 1963 to 1968, he was Associate Professor and from 1968, Professor of Electrical Engineering at Stanford University, Stanford, CA, where he is now Director of the Information Systems Laboratory. During January–June 1963, he was a Visiting Scholar at the University of California, Berkeley. He held a Guggenheim Fellowship in 1969–1970, and in 1977 he was elected a Fellow of Churchill College, Cambridge University, Cambridge, England. His research interests are generally in statistical signal processing, especially in communications and control. He is Editor of a Prentice-Hall series of books in Information and Systems Sciences and a member of the IEEE Press Board and the IEEE Publications Board.

Dr. Kailath is a Fellow of the Institute of Mathematical Statistics and a member of SIAM, AMS, and several other scientific societies. Since 1971, he has been a member of the Administrative Committee of the IEEE Professional Group on Information Theory and of the IEEE Control Systems Society, and during 1975 he served as President of the Information Theory Group.

Bernard C. Lévy (S'74–M'78) was born in Princeton, NJ, on July 31, 1951. He received the engineer's degree from the Ecole National Supérieure des Mines, Paris, France, and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA.

While at Stanford University, he held an IRIA Fellowship and worked as a Research Assistant. Since Fall 1978 he has been associated with the Laboratory for Information and Decision Systems at Massachusetts Institute of Technology, Cambridge, first as a Research Associate and then, since Fall 1979, as Assistant Professor of Electrical Engineering. His research interests are in the areas of multidimensional systems, estimation, and control.