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New Class of Discrete-Time Models for Nonlinear Systems through Discretization of Integration Gains

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Abstract— A new approach is proposed for obtaining discrete-time models of a nonlinear autonomous continuous-time system based on classification of, what is called in this study, a discrete-time integration gain. The models are expressed as a product of this gain and the system function that has the same structure as that of the continuous-time system. Sufficient conditions on this gain to make the model exact, in the sense defined in the paper, are presented. A new discrete-time model is proposed for nonlinear systems, which is approximate in general, but exact for linear systems. The method is applicable to any system that has a Jacobian matrix. As examples, van der Pol and Lorenz oscillators are examined and simulated to show that the proposed discrete-time models performs better than other discrete-time models that are known to the authors to be on-line computable, and tends to retain such key features as limit cycles and chaos, even for relatively large sampling periods.

Keywords – Discretization, nonlinear system, discrete-time model, integration gain, van der Pol oscillator, Lorenz oscillator.

1 INTRODUCTION

Digital computations involving a continuous-time system requires conversion into a discrete-time format somewhere in the process, and a number of researchers have worked on this topic for nonlinear systems in a variety of fields, including engineering systems and control [1]-[7]. A linearized discrete-time model is often used as an approximation of the given nonlinear system, first by linearizing the continuous-time system around an operating point, and then discretizing the resulting linear system, for which a number of discretization methods are available. Recent approaches are to obtain nonlinear discrete-time models for nonlinear continuous-time systems, and try to capture such key nonlinear phenomena as limit cycles and chaos. While accurate discretization methods are available for off-line simulations, those that lead to on-line computable algorithms are still relatively rare. Of those, the simplest discretization method is the forward-difference model, whose form and parameter values are chosen to be the same as those of the continuous-time system and only the differentiation is replaced with its Euler discrete-time equivalent. Due to its simplicity and applicability, this model is widely used. However its accuracy is usually poor even for linear cases unless a high sampling frequency is used. This is true also for nonlinear digital control systems that are designed based on the forward-difference model, which can complicate a subsequent digital controller design in an effort to take the discretization error into account [5]. A discretization method was proposed in [6] based on bi-linearization. Although this technique seems to be applicable to some important classes of nonlinear systems, it usually does not lead to an exact discrete-time model, which gives state responses that match those of the continuous-time system exactly at any discrete-time instants for any sampling period. The so-called non-standard models have been proposed in [7], which uses non-local discretization grids with constant gains based on the linear portion of the nonlinear equation, and is applicable to a wide range of nonlinear systems. It has been shown through simulations that the non-standard method is superior to the forward-difference model in terms of computational accuracies. However, care has to be exercised in determining the order in which the state equations are updated on-line. This formulation also makes the relationship between discrete-time and continuous-time systems less clear.

An approach that is based on the exact linearization of nonlinear systems has been presented for systems governed by a differential Riccati equation [8], where the gain called the discrete-time integration gain played an important role. This role is more visible using delta-operator form [9] than the conventional shift form. The integration gain for linear systems is a function of continuous-time parameters and a sampling interval [10], while that for nonlinear systems, it is also a function of system states [8]. Extensions of the exact gain to non-exact cases have been attempted for a nonlinear oscillator [11], where the integration gain is chosen such that the system looks linear in form. Based on insight gained from these attempts, the present study develops an equation showing a condition for the discrete-time integration gain so that a discrete-time system is a valid model. When this equation is solved exactly, exact discrete-time models are obtained, while when it is solved approximately, approximate models are derived.

The paper is organized as follows: In Section 2, some definitions are given to clarify the meaning of discretization and discrete-time models used throughout the paper. Discrete-time signals are not always exact but only "similar" to continuous-time signals. A way to accommodate this sense of similarity, a flexible definition proposed in [12], which is based on the point-wise convergence, is used. A condition on the model to be an exact discrete-time model is presented using the integration gain. Section 3 presents a new discrete-time model for nonlinear systems, based on an approximation, in general, of the integration gain. Section 4 examines the model as applied to van der Pol and Lorentz nonlinear oscillators and presents simulation results to illustrate the good performance of the model. Conclusions are given in Section 5.

2 DISCRETE-TIME MODELS OF CONTINUOUS-TIME SYSTEMS

2.1 Exact Discretization

Consider a continuous-time system given by the following state space equation:

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \mathbf{f}\left(\bar{\mathbf{x}}(t)\right), \quad \bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0, \tag{1}$$

where $\bar{\mathbf{x}} \in R^n$ is a function of continuous-time variable t and called the continuous-time state. The function $\mathbf{f}: R^n \to R^n$ depends, in general, on this state and is assumed to be expandable into Taylor series. Thus, it satisfies the Lipschitz condition and has a unique solution for a given initial condition. For a given continuous-time system, there are a number of associated discrete-time systems. In the present study they are expressed in delta form [4] with the discrete-time period of T, as

$$\delta \mathbf{x}_k = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{T} = \mathbf{\Gamma}\left(\mathbf{x}_k, T\right)\mathbf{f}\left(\mathbf{x}_k\right), \quad \mathbf{x}_{k_0} = \bar{\mathbf{x}}_0, \tag{2}$$

where δ is defined as $\delta = (q-1)/T$ with q being the shift-left operator such that $q\mathbf{x}_k = \mathbf{x}_{k+1}$. $\mathbf{x}_k \in R^n$ is a function of discrete-time variable kT and called the discrete-time state. It should be emphasized that the function $\mathbf{f}$ in eq. (2) is the same function as that in eq. (1). The initial time t_0 is arbitrary and may be changed arbitrarily, as long as it is synchronized between (1) and (2) such that

$$t_0 = k_0 T \ . \tag{3}$$

$\mathbf{\Gamma}(\mathbf{x}_k, T) \in R^{n\times n}$, which is called the discrete-time integration gain in the present study, is a bounded function of both, in general, T and $\mathbf{x}_k$, and is assumed to be differentiable with respect to T. This gain plays a key role in the present paper in the development of discrete-time models. The discrete-time system has a unique solution given an initial condition.

Remark 1: The integration gain can be used to classify various discrete-time models. For linear systems given by $\mathbf{f}(\overline{\mathbf{x}}) = \mathbf{A}\overline{\mathbf{x}}$, the so called step-invariant model is one given by $\mathbf{\Gamma}(\mathbf{x}_k, T) = \left(\int_0^T e^{\mathbf{A}\tau} d\tau\right)/T$, which is the average of the state-transition matrix over a discrete-time period, and is known to give state responses that match those of the continuous-time system under zero-order hold exactly at all discrete-time instants for any period T. For linear and nonlinear systems, the forward difference model is one with the integration gain set to identify matrix, or $\mathbf{\Gamma}(\mathbf{x}_k, T) = \mathbf{I}$, and can have good accuracy if T is sufficiently small.

The following definition is one that seems to be accepted widely as proper discretization of continuous-time signals, often termed as “sampled” signals, but is called “exact discretization” in this paper.

***Definition 1* [7].** A discrete-time state $\mathbf{x}_k$ of system (2) is said to be an exact discretization of a continuous-time state $\overline{\mathbf{x}}(t)$ of system (1) if the following relationship holds for any T and k:

$$\mathbf{x}_k = \overline{\mathbf{x}}(kT) . \tag{4}$$

In this case, a discrete-time system, whose state is $\mathbf{x}_k$, is said to be an exact discrete-time model of a continuous-time system, whose state is $\overline{\mathbf{x}}(t)$.

The existence of an exact discrete-time model is guaranteed under the standard assumption of existence of a solution to eq. (1) [7]. The following theorem states, in essence, the same fact, but looked at from the other side; given a discrete-time system with a mild condition, there is a continuous-time system for which the discrete-time system is exact to. This fact will be used later to derive not only exact but approximate models.

Theorem 1. Consider a discrete-time system (2), where $\boldsymbol{\Gamma}(\mathbf{x}_k,T)$ satisfies the following conditions:

(i) $\boldsymbol{\Gamma}(\mathbf{x}_k,T)$, $\dfrac{\partial\boldsymbol{\Gamma}(\mathbf{x}_k,T)}{\partial\mathbf{x}_k}$, $\dfrac{\partial}{\partial\mathbf{x}_k}\left(\dfrac{\partial\boldsymbol{\Gamma}(\mathbf{x}_k,T)}{\partial T}\right)$ are continuous functions of $\mathbf{x}_k$.

(ii) $\dfrac{T\partial\left[\boldsymbol{\Gamma}(\mathbf{x}_k,T)\mathbf{f}(\mathbf{x}_k)\right]}{\partial\mathbf{x}_k} \neq -1$.

There exists a continuous-time system with state $\overline{\mathbf{x}}^*(t)$ for which system (2) is an exact discrete-time model, if $\overline{\mathbf{x}}^*(t)$ satisfies the following continuous-time relationship for $t > t_0$:

$$\frac{\overline{\mathbf{x}}^*(t)-\overline{\mathbf{x}}_0}{t-t_0} = \boldsymbol{\Gamma}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0), \quad \overline{\mathbf{x}}^*(t_0)=\overline{\mathbf{x}}_0 . \tag{5}$$

Proof. Relationship given by eq. (5) yields

$$\overline{\mathbf{x}}^*(t) = \overline{\mathbf{x}}_0 + \mathbf{H}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0) , \tag{6}$$

where the continuous-time function $\mathbf{H}(\overline{\mathbf{x}}_0,t-t_0)$ is defined as

$$\mathbf{H}(\overline{\mathbf{x}}_0,t-t_0) = (t-t_0)\boldsymbol{\Gamma}(\overline{\mathbf{x}}_0,t-t_0) . \tag{7}$$

Differentiation of eq. (6) with respect to time gives

$$\dot{\overline{\mathbf{x}}}^{*}(t)=\dot{\mathbf{H}}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0) \tag{8}$$

If the state $\overline{\mathbf{x}}^{*}(t)$ is to be a solution of a differential equation given by

$$\dot{\overline{\mathbf{x}}}^{*}(t)=\mathbf{F}\left(\overline{\mathbf{x}}^{*},t\right), \tag{9}$$

then, eqs. (6), (8), and (9) lead to

$$\frac{\partial \mathbf{F}\left(\overline{\mathbf{x}}^{*},t\right)}{\partial \overline{\mathbf{x}}^{*}}=\frac{\dfrac{\partial\left[\dot{\mathbf{H}}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0)\right]}{\partial \overline{\mathbf{x}}_0}}{\dfrac{\partial\left[\overline{\mathbf{x}}_0+\mathbf{H}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0)\right]}{\partial \overline{\mathbf{x}}_0}} =\frac{\dfrac{\partial\left[\dot{\mathbf{H}}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0)\right]}{\partial \overline{\mathbf{x}}_0}}{1+\dfrac{\partial\left[\mathbf{H}(\overline{\mathbf{x}}_0,t-t_0)\mathbf{f}(\overline{\mathbf{x}}_0)\right]}{\partial \overline{\mathbf{x}}_0}}. \tag{10}$$

When conditions (i) and (ii) are satisfied, then $\dfrac{\partial \mathbf{F}\left(\overline{\mathbf{x}}^{*},t\right)}{\partial \overline{\mathbf{x}}^{*}}$ is a continuous function of $\overline{\mathbf{x}}^{*}$ and, thus, $\mathbf{F}(\overline{\mathbf{x}}^{*},t)$ satisfies Lipchitz condition. This implies that $\overline{\mathbf{x}}^{*}(t)$ is the unique solution of eq. (9) and, thus, $\overline{\mathbf{x}}^{*}(t)$ satisfies the group property [13]. Therefore, eq. (5) yields

$$\frac{\overline{\mathbf{x}}^{*}(t+T)-\overline{\mathbf{x}}^{*}(t)}{T}=\mathbf{\Gamma}\left(\overline{\mathbf{x}}^{*}(t),T\right)\mathbf{f}\left(\overline{\mathbf{x}}^{*}(t)\right) \tag{11}$$

for any positive t and T. Let us consider such time instants as $t=kT$ in eq. (11). For $k=0$, $\mathbf{x}_0=\overline{\mathbf{x}}^{*}(t_0)$ by assumption. If $\mathbf{x}_k=\overline{\mathbf{x}}^{*}(kT)$ holds, then eq. (2) and eq. (11) give $\mathbf{x}_{k+1}=\overline{\mathbf{x}}^{*}\left((k+1)T\right)$. Therefore,

$$\mathbf{x}_k=\overline{\mathbf{x}}^{*}(kT). \tag{12}$$

for any positive k and T.

Remark 2: It should be noted that, since $\overline{\mathbf{x}}^*(t)$ satisfies the group property, eq. (5) can be written, for arbitrary values of t_1, as

$$\frac{\overline{\mathbf{x}}^*(t)-\overline{\mathbf{x}}^*(t_1)}{t-t_1}=\mathbf{\Gamma}\left(\overline{\mathbf{x}}^*(t_1),t-t_1\right)\mathbf{f}\left(\overline{\mathbf{x}}^*(t_1)\right), \tag{13}$$

where $\overline{\mathbf{x}}^*(t_1)$ satisfies

$$\frac{\overline{\mathbf{x}}^*(t_1)-\overline{\mathbf{x}}_0}{t_1-t_0}=\mathbf{\Gamma}\left(\overline{\mathbf{x}}_0,t_1-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right). \tag{14}$$

In other word, the form of eq. (5) is for any value of pair $(\mathbf{x}_0,t_0)$. Eq. (11) is a special case of eq. (5), which suffices here. However, the more general eq. (5) will be needed later. It should also be noted that condition (5) is a continuous-time relationship imposed on the discrete-time state of system (2) to be an exact discretization.

2.2 Discretization

Discrete-time signals and systems are not always exact in the sense of Definition 1, but only "similar." A way to accommodate this sense of similarity, a more flexible definition is proposed in [12] for linear systems, and is used in the present study for nonlinear systems.

Definition 2 [12]. The discrete-time state $\mathbf{x}_k$ is said to be a discretization of the continuous-time state $\overline{\mathbf{x}}(t)$ if, for any fixed instant τ, the following relationship holds:

$$\lim_{\substack{T\to 0\\ kT\le\tau<(k+1)T}} \mathbf{x}_k=\overline{\mathbf{x}}(\tau) \tag{15}$$

.

Such a discrete-time system is said to be a discrete-time model of the original continuous-time system.

It should be noted that the above definition is based on a point-wise convergence. Moreover, the convergence of

$$\lim_{\substack{T\to 0\\ kT\le\tau<(k+1)T}} kT=\tau \tag{16}$$

may not be monotonic.

Insights obtained from [10] and [12] for linear systems have hinted the following theorem:

Theorem 2. A discrete-time state $\mathbf{x}_k$ of system (2) is a discretization of the continuous-time state $\overline{\mathbf{x}}(t)$ of system (1), if the integration gain in eq. (2) satisfies conditions (i), (ii) in Theorem 1, and

$$\lim_{T\to 0}\mathbf{\Gamma}(\mathbf{x}_k,T)=\mathbf{I}, \tag{17}$$

where $\mathbf{I}$ is an identity matrix. When this condition is satisfied, system (2) is a discrete-time model of system (1).

Proof. From Theorem 1, the discrete-time state $\mathbf{x}_k$ of system (2) is an exact discretization of a continuous-time state $\overline{\mathbf{x}}^*(t)$ that satisfies eq. (5) for any t_0. Therefore, letting $t_0 = kT$ and $t=\tau$ in eq. (5), one has

$$\frac{\overline{\mathbf{x}}^*(\tau)-\overline{\mathbf{x}}^*(kT)}{\tau-kT}=\mathbf{\Gamma}\left(\overline{\mathbf{x}}^*(kT),\tau-kT\right)\mathbf{f}\left(\overline{\mathbf{x}}^*(kT)\right), \tag{18}$$

for any fixed τ. On the other hand, since

$$\lim_{\substack{T\to 0\\ kT\le\tau<(k+1)T}} kT=\tau \tag{19}$$

and the limit of eq. (17) is satisfied for arbitrary values of k, the continuous-time gain satisfies

$$\lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \mathbf{\Gamma}\left(\overline{\mathbf{x}}^*(kT),\tau-kT\right)=\lim_{(\tau-kT)\to 0}\mathbf{\Gamma}\left(\overline{\mathbf{x}}^*(kT),\tau-kT\right)=\mathbf{I}. \tag{20}$$

Noting that

$$\lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \overline{\mathbf{x}}^*(kT)=\overline{\mathbf{x}}^*(\tau), \tag{21}$$

the limit of eq. (18) yields

$$\lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \frac{\overline{\mathbf{x}}^*(\tau)-\overline{\mathbf{x}}^*(kT)}{\tau-kT} = \lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \mathbf{\Gamma}\left(\overline{\mathbf{x}}^*(kT),\tau-kT\right)\mathbf{f}\left(\overline{\mathbf{x}}^*(kT)\right) = \lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \mathbf{f}\left(\overline{\mathbf{x}}^*(kT)\right)$$
$$= \mathbf{f}\left(\lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \overline{\mathbf{x}}^*(kT)\right) = \mathbf{f}\left(\overline{\mathbf{x}}^*(\tau)\right). \tag{22}$$

As T approaches zero, so that condition (17) holds, the above gives

$$\frac{d\overline{\mathbf{x}}^*(\tau)}{dt} = \mathbf{f}\left(\overline{\mathbf{x}}^*(\tau)\right) \tag{23}$$

for all τ in the limit. Since there is only one solution of eq. (1) at any time instant, it follows that

$$\overline{\mathbf{x}}^*(\tau) = \overline{\mathbf{x}}(\tau), \tag{24}$$

while the exactness of discretization of $\mathbf{x}_k$ to $\overline{\mathbf{x}}^*(t)$ gives

$$\mathbf{x}_k = \overline{\mathbf{x}}^*(kT). \tag{25}$$

Finally, for eq. (25), take the limit of T approaching zero and choosing k such that $kT \le \tau < (k+1)T$, and use eq. (21) and eq. (24), to obtain

$$\lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \mathbf{x}_k = \lim_{\substack{T\to 0\\ kT\le T<(k+1)T}} \overline{\mathbf{x}}^*(kT) = \overline{\mathbf{x}}^*(\tau) = \overline{\mathbf{x}}(\tau) \tag{26}$$

for any τ. Therefore, system (2) is a discrete-time model of the continuous-time system (1) in the sense of Definition 2.

3 DISCRETE-TIME MODELS FROM THE INTEGRATION-GAIN POINT OF VIEW

3.1 Exact Discretization

The following theorem states that any discrete-time system (2) can be an exact model of a continuous-time system (1) if the integration gain $\mathbf{\Gamma}$ is chosen in a certain manner.

Theorem 3. A discrete-time system (2), is an exact model of a continuous-time system (1), if the discrete-time integration gain $\mathbf{\Gamma}$ in eq. (2) is chosen to satisfy the following relationship exactly:

$$\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)=\sum_{|\ell|=0}^{N-1}\frac{\partial^\ell\mathbf{f}\left(\overline{\mathbf{x}}_0\right)}{\ell!}\left(\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)^\ell+\mathbf{R}^N\left(\overline{\mathbf{x}}_0,\,\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right), \tag{27}$$

where the continuous-time function $\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)$ is defined as

$$\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)=\left(t-t_0\right)\mathbf{\Gamma}\left(\overline{\mathbf{x}}_0,t-t_0\right) \tag{28}$$

and $\mathbf{R}^N$ is the remainder after N terms, $\ell=\left(\ell_1,...,\ell_n\right)$ is an *n*-dimensional multi-index such that

$$\begin{aligned}
&|\ell|\equiv\ell_1+\ell_2+\cdots+\ell_n,\ \ell_i:\ \text{positive integer}\\
&\partial^\ell\equiv\partial_1^{\ell_1}\partial_2^{\ell_2}\cdots\partial_n^{\ell_n},\ \partial_i^{\ell_i}\equiv\frac{\partial^{\ell_i}}{\partial x_i^{\ell_i}}\\
&\ell!\equiv\left(\ell_1!\right)\left(\ell_2!\right)\cdots\left(\ell_n!\right)\\
&\nu^\ell\equiv\nu_1^{\ell_1}\nu_2^{\ell_2}...\nu_n^{\ell_n},\ \text{where}\ \nu=\left(\nu_1,\nu_2,...,\nu_n\right)^T,\ \nu_i:\ \text{scalar}
\end{aligned} \tag{29}$$

Proof. From Theorem 1, the discrete-time state $\mathbf{x}_k$ of system (2) is an exact discretization of a continuous-time state $\overline{\mathbf{x}}^*(t)$ that satisfies eq. (5), which can be rewritten as

$$\overline{\mathbf{x}}^*(t)=\overline{\mathbf{x}}_0+\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right). \tag{30}$$

Differentiation of eq. (30) gives

$$\dot{\overline{\mathbf{x}}}^*(t)=\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right). \tag{31}$$

If the above state $\overline{\mathbf{x}}^*(t)$ is to be a solution of differential equation (1), the function $\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)$ must be chosen such that

$$\dot{\overline{\mathbf{x}}}^*(t)=\mathbf{f}\left(\overline{\mathbf{x}}^*(t)\right). \tag{32}$$

Using eq. (30) and eq. (31), eq. (32) can be rewritten as

$$\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)=\mathbf{f}\left(\overline{\mathbf{x}}_0+\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right). \tag{33}$$

Taylor series expansion of $\mathbf{f}\left(\overline{\mathbf{x}}_0+\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)$, which exists by assumption, about an arbitrary $\overline{\mathbf{x}}_0$ can be written as

$$\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)=\sum_{\ell=0}^{N-1}\frac{\partial^{\ell}\mathbf{f}\left(\overline{\mathbf{x}}_0\right)}{\ell!}\left(\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)^{\ell}+\mathbf{R}^{N}\left(\overline{\mathbf{x}}_0,\,\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right) \tag{34}$$

For the case of $N=3$, eq. (27) can be written as

$$\begin{aligned}\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)&=\mathbf{f}\left(\overline{\mathbf{x}}_0\right)+D\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\left(\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)\\&+\frac{1}{2!}\left(\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)^{T}\left\{D^2\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right\}\left(\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right)\\&+\mathbf{R}^3\left(\overline{\mathbf{x}}_0,\,\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right)\right),\end{aligned} \tag{35}$$

where $D\mathbf{f}\left(\overline{\mathbf{x}}\right)$ is Jacobian matrix of $\mathbf{f}\left(\overline{\mathbf{x}}\right)$ at $\overline{\mathbf{x}}$, and $D^2\mathbf{f}\left(\overline{\mathbf{x}}\right)=\left(D^2f_1\left(\overline{\mathbf{x}}\right),...,D^2f_n\left(\overline{\mathbf{x}}\right)\right)$ with $D^2f_i\left(\overline{\mathbf{x}}\right)$ being Hessian matrix of $f_i\left(\overline{\mathbf{x}}\right)$.

Remark 3: When the expansion of eq. (27) is exact, its solution $\mathbf{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)$ leads to the exact integration gain $\mathbf{\Gamma}\left(\overline{\mathbf{x}}_0,t-t_0\right)$ using eq. (28). The resulting discrete-time system (2) is the exact discrete-time model of a continuous-time system in the sense of Definition 1. For example, for a logistic system given by

$$\dot{\overline{x}}=\overline{x}\left(1-\varepsilon\overline{x}\right)=\overline{x}-\varepsilon\overline{x}^2, \tag{36}$$

where ε is a positive parameter, the expansion of eq. (34) terminates after three terms, as

$$\dot{\mathrm{H}}\left(\overline{\mathbf{x}}_0,t-t_0\right)=1+\left(1-2\varepsilon\overline{x}_0\right)\mathrm{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)-\varepsilon\overline{x}_0\left(1-\varepsilon\overline{x}_0\right)\left[\mathrm{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)\right]^2. \tag{37}$$

This can be solved exactly, using $\mathrm{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)=0$ at $t=t_0$, as

$$\mathrm{H}\left(\overline{\mathbf{x}}_0,t-t_0\right)=\frac{e^{t-t_0}-1}{1+\varepsilon\overline{x}_0\left(e^{t-t_0}-1\right)}. \tag{38}$$

Therefore, one obtains

$$\Gamma\left(\overline{\mathbf{x}}_0, t-t_0\right)=\frac{1}{t-t_0}\frac{e^{t-t_0}-1}{1+\varepsilon\overline{x}_0\left(e^{t-t_0}-1\right)}, \tag{39}$$

which leads to the exact discrete-time integration gain given by

$$\Gamma\left(x_k, T\right)=\frac{1}{T}\frac{e^{T}-1}{1+\varepsilon x_k\left(e^{T}-1\right)} \quad \to I\left(T \to 0\right). \tag{40}$$

Exact discrete-time models of logistic systems can be obtained via exact linearization using variable transformation [8]. However, the present method is easier.

3.2 Proposed Discretization Method

The following model is widely applicable to nonlinear systems given in the form of eq. (1) and will be called the proposed discrete-time model.

Theorem 4 [The Proposed Model]. A nonlinear discrete-time system given by eq. (2) with the discrete-time integrator gain given by

$$\mathbf{\Gamma}\left(\mathbf{x}_k, T\right)=\frac{1}{T}\int_o^T e^{\left[D\mathbf{f}\left(\mathbf{x}_k\right)\right]\tau}d\tau, \tag{41}$$

where $D\mathbf{f}\left(\mathbf{x}_k\right)$ is a Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}_k$, is a discrete-time model of the continuous-time model given by eq. (1). When Jacobian $D\mathbf{f}\left(\overline{\mathbf{x}}_k\right)$ is non-singular, the gain can also be written as

$$\mathbf{\Gamma}\left(\mathbf{x}_k, T\right)=\frac{e^{\left[D\mathbf{f}\left(\mathbf{x}_k\right)\right]T}-\mathbf{I}}{T}\left[D\mathbf{f}\left(\mathbf{x}_k\right)\right]^{-1}. \tag{42}$$

Proof. Using l'Hospital's Rule, one obtains

$$\lim_{T\to 0}\mathbf{\Gamma}\left(\mathbf{x}_k, T\right)=\lim_{T\to 0}\frac{1}{T}\int_0^T e^{\left[D\mathbf{f}\left(\mathbf{x}_k\right)\right]\tau}d\tau=\mathbf{I}, \tag{43}$$

so that a discrete-time system (2) satisfies Theorem 2 and is a discrete-time model of continuous-time system in the sense of Definition 2.

The derivation for eq. (41) is as follows: When the Taylor series expansion in eq. (27) is truncated with the first two terms, one requires, for arbitrary $\bar{\mathbf{x}}_0$,

$$\dot{\mathbf{H}}\left(\bar{\mathbf{x}}_0, t-t_0\right) = \mathbf{I} + \left[D\mathbf{f}\left(\bar{\mathbf{x}}_0\right)\right]\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right) \tag{44}$$

Noting that $\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right)$ is, in general, a function of $\bar{\mathbf{x}}_0$ and $\left(t-t_0\right)$, and that $\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right)=0$ when $t=t_0$, a solution to the above linear differential equation is found to be

$$\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right) = \int_0^{t-t_0} e^{\left[D\mathbf{f}\left(\bar{\mathbf{x}}_0\right)\right]\tau} d\tau \tag{45}$$

Substituting the above into eq. (28) gives the following continuous-time gain function:

$$\mathbf{\Gamma}(\bar{\mathbf{x}}_0, t-t_0) = \frac{1}{t-t_0}\int_0^{t-t_0} e^{\left[D\mathbf{f}\left(\bar{\mathbf{x}}_0\right)\right]\tau} d\tau \tag{46}$$

Using the same form of function $\mathbf{\Gamma}$, the discrete-time integration gain is obtained as

$$\mathbf{\Gamma}(\mathbf{x}_k, T) = \frac{1}{T}\int_0^{T} e^{\left[D\mathbf{f}\left(\bar{\mathbf{x}}_k\right)\right]\tau} d\tau \tag{47}$$

The proposed discrete-time model can be found for any nonlinear system (1) as long as its Jacobian matrix exists. The fact that this model gives good results for many nonlinear systems and retain important features for van der Pol and Lorentz oscillators will be shown by simulations shortly.

Remark 4: For a linear system, the proposed method gives the exact discrete-time model; for $\mathbf{f}$ in eq. (1) given by

$$\mathbf{f}\left(\mathbf{x}\right) = \mathbf{A}\mathbf{x} + \mathbf{b} \tag{48}$$

where $\mathbf{A}$ and $\mathbf{b}$ are system parameters of compatible dimensions, eq. (27) can be written exactly as a linear equation $\dot{\mathbf{H}}\left(\bar{\mathbf{x}}_0, t-t_0\right) = \mathbf{I} + \mathbf{A}\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right)$, which is independent of $\bar{\mathbf{x}}_0$ and whose solution is $\mathbf{H}\left(\bar{\mathbf{x}}_0, t-t_0\right) = \int_0^{t-t_0} e^{\mathbf{A}\tau} d\tau$. This leads to the integration gain of the exact discrete-time model [10] as

$$\mathbf{\Gamma}\left(\mathbf{x}_k, T\right) = \frac{1}{T}\int_{o}^{T} e^{\mathbf{A}\tau} d\tau \quad (49)$$
.

When $\mathbf{A}$ is invertible, this gain can also be written as

$$\mathbf{\Gamma}\left(\mathbf{x}_k, T\right) = \frac{e^{\mathbf{A}T} - \mathbf{I}}{T}\mathbf{A}^{-1}. \quad (50)$$

Remark 5: Choosing $N=1$ and ignoring the remainder term, eq. (27) yields $\dot{\mathbf{H}}\left(\overline{\mathbf{x}}_0, t-t_0\right)\mathbf{f}\left(\overline{\mathbf{x}}_0\right) = \mathbf{f}\left(\overline{\mathbf{x}}_0\right)$, so that $\mathbf{H}\left(\overline{\mathbf{x}}_0, t-t_0\right) = \left(t-t_0\right)\mathbf{I}$ for arbitrary $\overline{\mathbf{x}}_0$. This leads to $\mathbf{\Gamma}(\mathbf{x}_k, T) = \mathbf{I}$, which satisfies Theorem 2 and is a discrete-time model. This known as the forward difference model.

4 Examples of Discrete-time Models

4.1 Discrete-Time Models of a van der Pol Oscillator

Consider the van der Pol oscillator modeled by

$$\ddot{x} = -x + \varepsilon\left(1 - x^2\right)\dot{x}, \quad (51)$$

where ε is a positive parameter that characterizes a degree of nonlinearity. This can be rewritten in a form of

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + \varepsilon\left(1 - x^2\right)y \end{cases} \quad (52)$$
,

whose Jacobian matrix is non-singular and given by

$$D\mathbf{f}(\overline{\mathbf{x}}) = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon xy & \varepsilon\left(1 - x^2\right) \end{bmatrix} \quad (53)$$
.

Using the discrete-time integration gain given by eq. (41), the proposed discrete-time model is obtained as

$$\begin{bmatrix} \delta x_k \\ \delta y_k \end{bmatrix} = \frac{e^{\left[D\mathbf{f}(\mathbf{x}_k)\right]T} - \mathbf{I}}{T}\left[D\mathbf{f}\left(\mathbf{x}_k\right)\right]^{-1}\begin{bmatrix} y_k \\ -x_k + \varepsilon\left(1 - x_k^2\right)y_k \end{bmatrix} \quad (54)$$
,

where

$$D\mathbf{f}(\mathbf{x}_k)=\begin{bmatrix} 0 & 1 \\ -1-2\varepsilon x_k y_k & \varepsilon\left(1-x_k{}^2\right) \end{bmatrix}. \tag{55}$$

Mickens' model for the van der Pol oscillator [7] is given by

$$\begin{bmatrix} \delta x_k \\ \delta y_k \end{bmatrix}=\phi\begin{bmatrix} y_k \\ -x_k+\varepsilon\left(1-x_{k+1}^2\right)y_k \end{bmatrix}+\varphi\begin{bmatrix} x_k \\ y_k \end{bmatrix}, \tag{56}$$

where ϕ and φ are given by

$$\varphi=\frac{e^{T\frac{\varepsilon}{2}}\left\{\cos\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}T-\frac{\varepsilon\sin\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}T}{2\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}}\right\}-1}{T}. \tag{57}$$

$$\phi=e^{\frac{\varepsilon}{2}T}\frac{\sin\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}T}{\sqrt{1-\left(\frac{\varepsilon}{2}\right)^2}T} \tag{58}$$

In this discretization scheme, the eigenvalues λ_1 and λ_2 of a linear part of the nonlinear function must be distinct, and the parameter be such that $\varepsilon<2$. It should be noted that the right-hand-side of the above equation for δy_k contains the term x_{k+1}, which makes this method nonlocal and nonstandard [7]. Therefore, variable x_k must be updated before y_k in computations.

The forward difference model is obtained simply as

$$\begin{bmatrix} \delta x_k \\ \delta y_k \end{bmatrix}=\begin{bmatrix} y_k \\ -x_k+\varepsilon\left(1-x_k^2\right)y_k \end{bmatrix}. \tag{59}$$

Extensive simulations have been carried out with the van der Pol oscillator (52) and some typical results of phase-planes and time responses are shown in Figs. 1 and 2. They compare the original continuous-time oscillator computed using the Runge-Kutta method, the proposed model, Micken's model, and the forward-difference models.

In Fig. 1, the phase plane plots are traced from 0 to 1,000 seconds and the time response for the first 10 seconds, for $\varepsilon = 1.5$, $T = 0.1$ s, and the initial condition of $x_0 = -1.0$ and $y_0 = -1.5$, which is inside the limit cycle. At this sampling interval, all discrete-time models give responses that are more or less results similar to those of the continuous-time model, although the proposed model is closest. Fig. 2 shows the results under the same conditions except for the sampling interval, which is increased to 0.3 seconds. At this sampling interval, neither Mickens' model nor the forward-difference model could give steady results, while the proposed model is still very close to the continuous-time model. Although not shown here, in all simulation tests for larger nonlinear parameter values of ε and ranges of T and the initial conditions, it was found that the proposed discrete-time model consistently gave results closer to the continuous-time van der Pol model than the forward-difference and Mickens models.

4.2 Discrete-Time Models of a Lorenz Oscillator

Consider the Lorenz oscillator given by the following equation:

$$\begin{cases} \dot{x} = \sigma\left(y - x\right) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}, \tag{60}$$

where all parameters are positive; i.e., $\sigma, r, b > 0$. In particular, σ is the Prandtl number and r the Rayleigh number. Jacobian matrix of this system is

$$D\mathbf{f}(\overline{\mathbf{x}}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{bmatrix}. \tag{61}$$

The proposed discrete-time model can be obtained as

$$\begin{bmatrix} \delta x_k \\ \delta y_k \\ \delta z_k \end{bmatrix} = \frac{e^{\left[D\mathbf{f}(\mathbf{x}_k, T)\right]T} - \mathbf{I}}{T}\left[D\mathbf{f}\left(\mathbf{x}_k, T\right)\right]^{-1} \begin{bmatrix} \sigma\left(y_k - x_k\right) \\ rx_k - y_k - x_k z_k \\ -bz_k + x_k y_k \end{bmatrix}, \tag{62}$$

where

$$
D\mathbf{f}(\mathbf{x}_k, T) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - z_k & -1 & -x_k \\ y_k & x_k & -b \end{bmatrix}. \tag{63}
$$

Mickens' discrete-time model [14] is given by

$$
\begin{bmatrix} \delta x_k \\ \delta y_k \\ \delta z_k \end{bmatrix} = \begin{bmatrix} \dfrac{1-e^{-\sigma T}}{\sigma T} & 0 & 0 \\ 0 & 1-e^{-T} & 0 \\ 0 & 0 & \dfrac{1-e^{-bT}}{bT} \end{bmatrix} \begin{bmatrix} -\sigma x_k + \sigma y_k \\ r x_{k+1} - y_k - x_{k+1} z_k \\ -b z_k + x_{k+1} y_{k+1} \end{bmatrix}. \tag{64}
$$

It can be seen that there are such nonlinear terms as $x_{k+1} z_k$ and $x_{k+1} y_{k+1}$ in the above. The forward difference model is obtained as

$$
\begin{bmatrix} \delta x_k \\ \delta y_k \\ \delta z_k \end{bmatrix} = \begin{bmatrix} \sigma(y_k - x_k) \\ r x_k - y_k - x_k z_k \\ -b z_k + x_k y_k \end{bmatrix}. \tag{65}
$$

Simulations have been carried also for the Lorenz system (60), where $\sigma = 10$ and $b = 8/3$. For a Rayleigh number r smaller than 24.06, the system state approaches one of two fixed-point attractors. Otherwise the system is chaotic. Fig. 3 shows the $x - z$ plane of the continuous model, the proposed model, Mickens' model, and the forward-difference model, for $r = 28$, $T = 0.0075 s$, and the initial condition of $x_0 = 1, y_0 = 2, z_0 = 3$, for the first 1,000 seconds.

The forward-difference model does not produce reasonable results, while both the Mickens' and the proposed models retain the chaotic behavior of the continuous-time model. The proposed model is closer, however, to the continuous-time model than the Micken's model. Fig. 4 is for the same system under the same conditions except that $r = 17$ and $T = 0.05 s$. Although not shown, Mickens' model becomes divergent for a sampling interval greater than $0.072 s$, while the proposed model still retains the general shape of the continuous-time trajectory.

5 CONCLUSIONS

Nonlinear discrete-time models have been looked at from the view-point of discrete-time integration gains, which led to new models. The form of models is fixed to be the product of this gain and the system function that has the same structure as that of the continuous-time original. Sufficient conditions for this gain to make the discrete-time model exact are presented. The model is exact when the gain is exact, while it is approximate when the gain is approximate. The main contribution of the present paper is the development of a systematic procedure to obtain discrete-time models based on approximated gains, even when the exact gain is unknown. As long as a Jacobian matrix exists for the nonlinear state equation, the gain can be found approximately and the proposed model derived. Simulations show that the proposed models give performances that are superior to on-line computable methods such as the forward-difference and Mickens models as applied to van der Pol and Lorentz nonlinear oscillators.

Although the proposed model is developed based on the Taylor expansion of the nonlinear function, it is used to determine the integration gain and the form of the model is nonlinear. This is different from the linearization of a system itself based on the Taylor expansion. The proposed model corresponds to one with the first two terms in the series expansion in a differential equation concerning this gain, and can always be found as long as the expansion exists, whereas the well-known forward difference model corresponds to one with the first term only.

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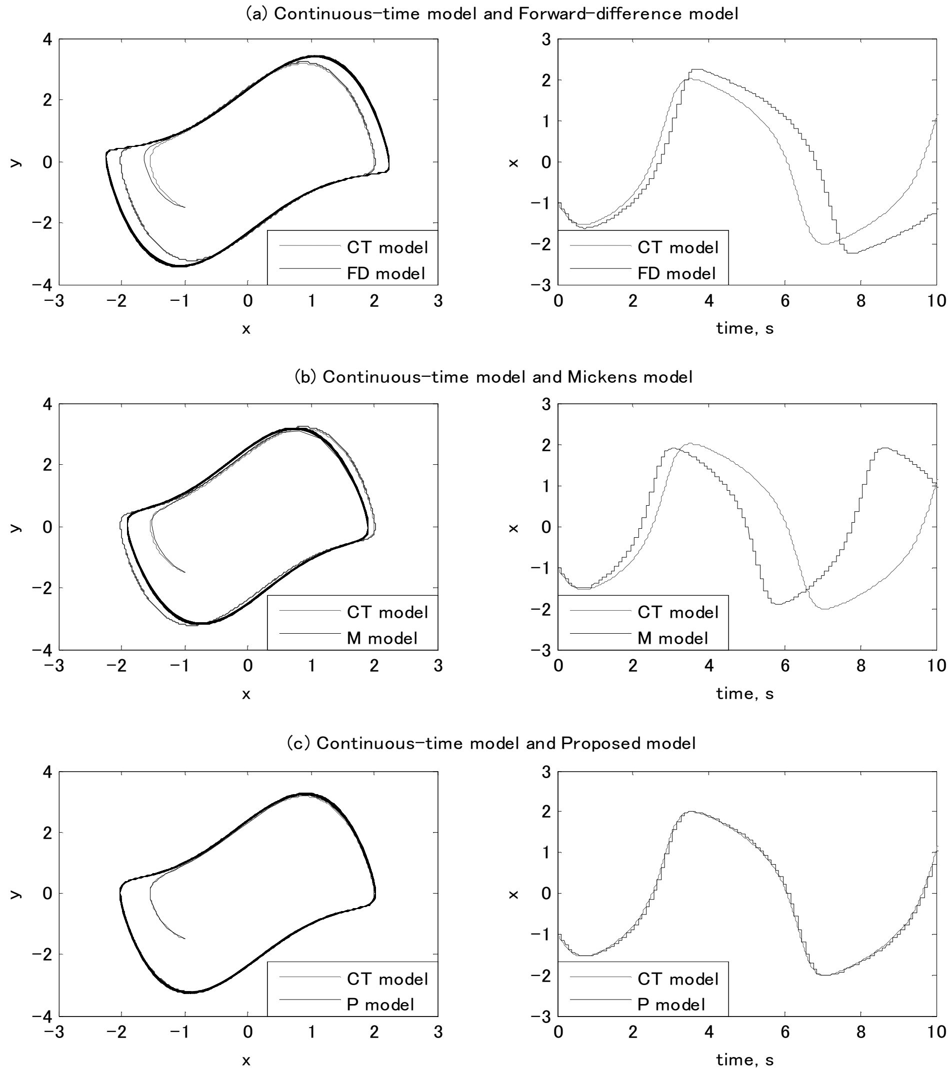

Fig. 1: Phase plane and time response of the continuous-time, the forward-difference, Mickens', and the proposed models, for $\varepsilon = 1.5$, $T = 0.1\,\text{s}$, and the initial condition of $x_0 = -1.0$, and $y_0 = -1.5$.

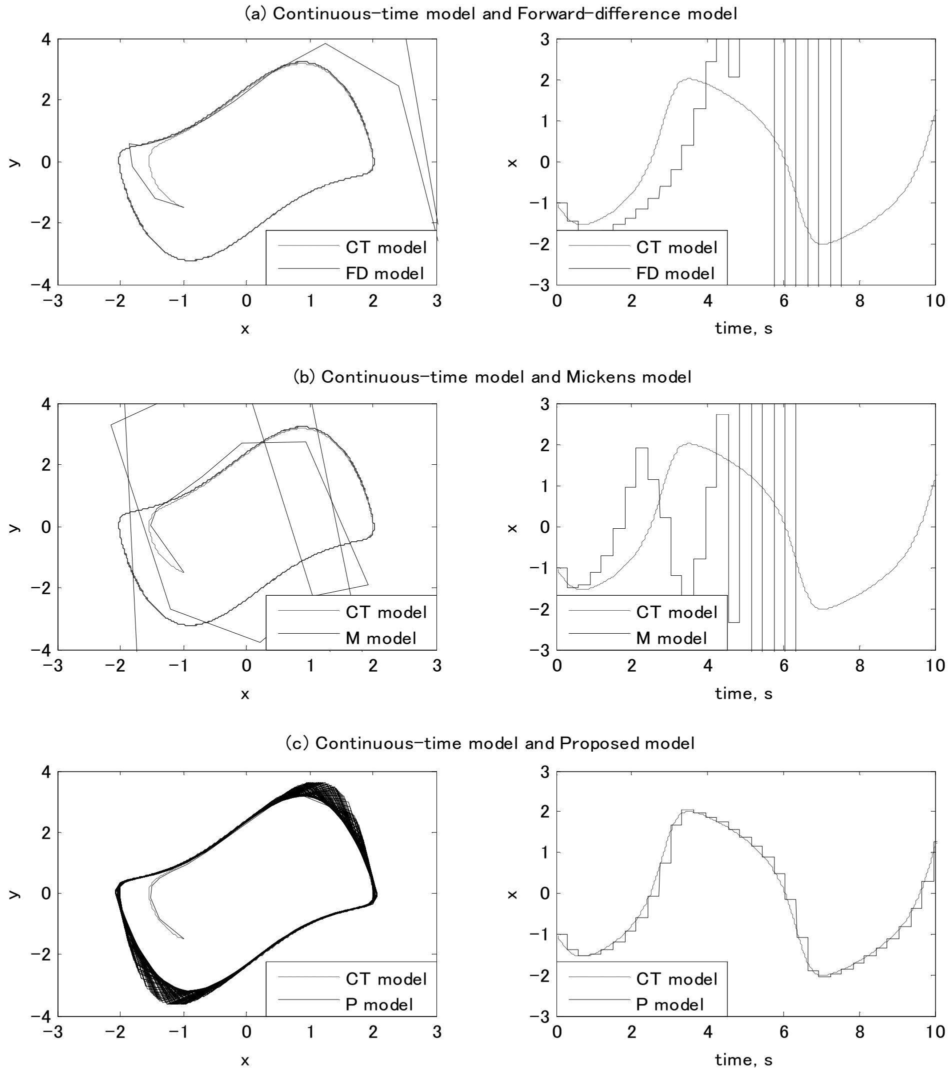

Fig. 2: Phase plane and time response of the continuous-time, the forward-difference, Mickens', and the proposed models, for $\varepsilon = 1.5$, $T = 0.3$ s, and the initial condition of $x_0 = -1.0$, and $y_0 = -1.5$.

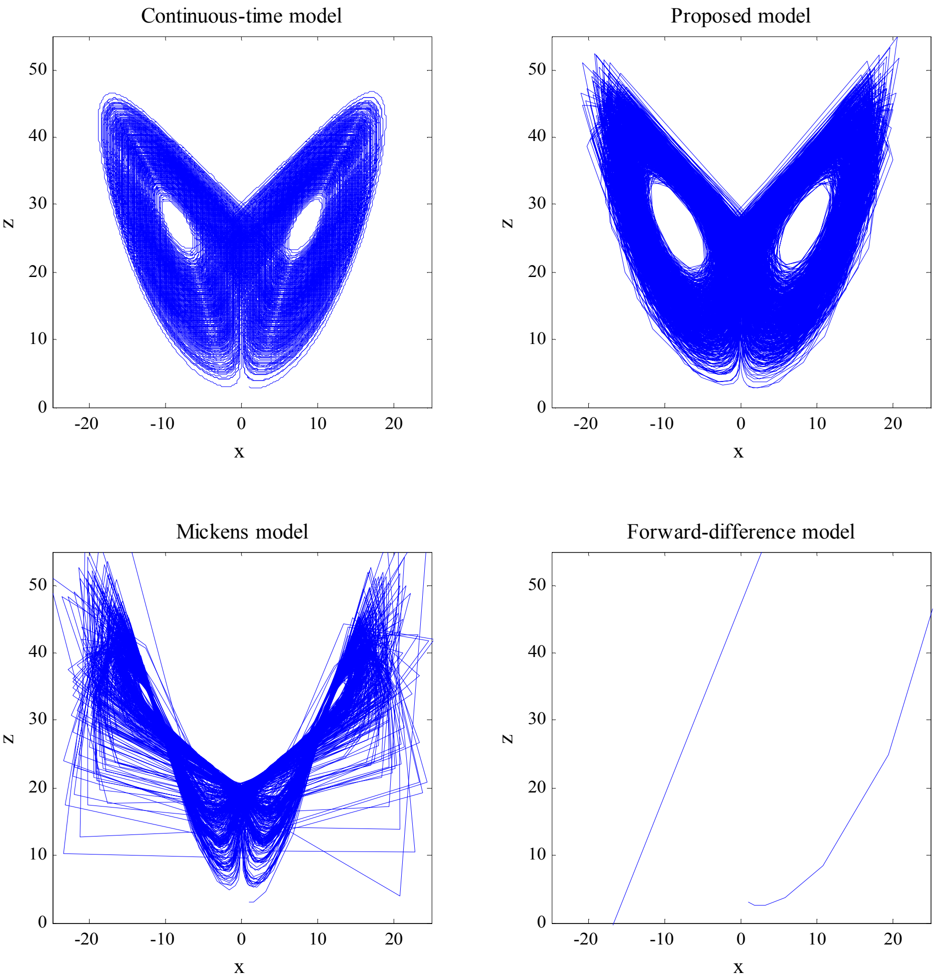

Fig. 3: $x-z$ plane of the continuous-time, the forward-difference, Mickens', and the proposed models for $\sigma=10, b=8/3, r=28$, $T=0.075$ s, and the initial condition of $x_0=1$, $y_0=2$, and $z_0=3$.

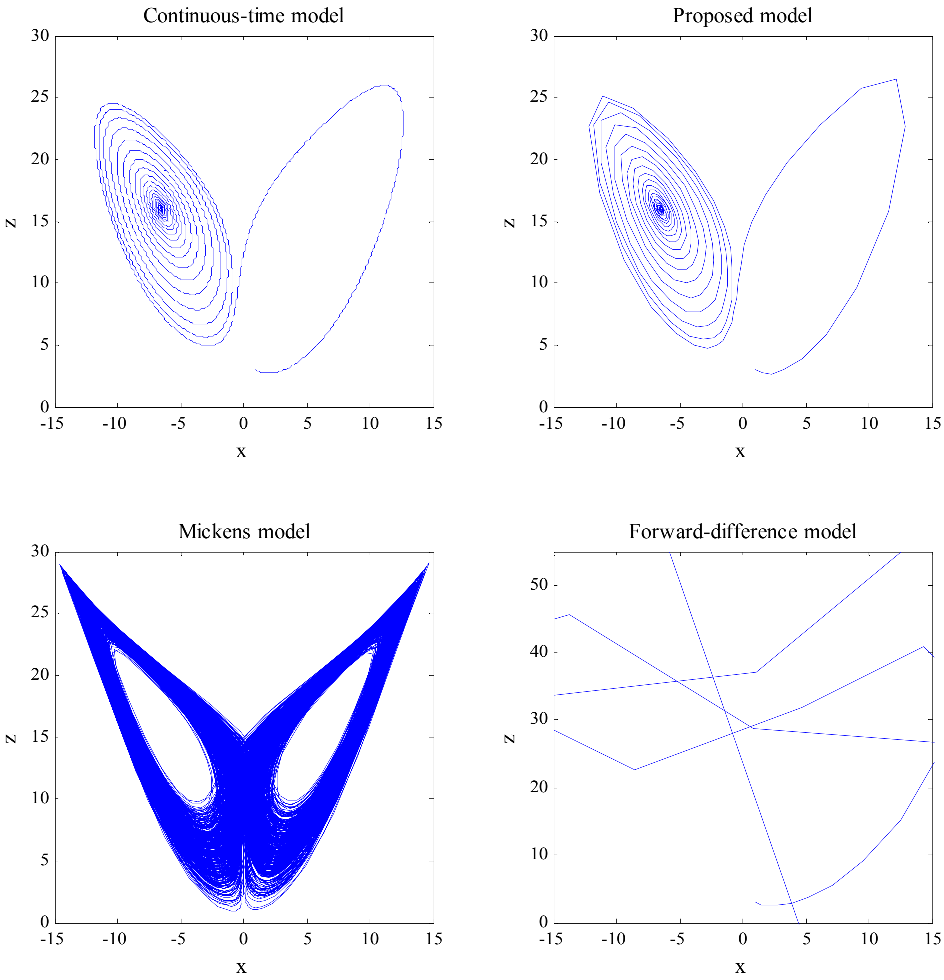

Fig. 4: $x-z$ plane of the continuous-time, the forward-difference, Mickens', and the proposed models for $\sigma=10, b=8/3, r=17$, $T=0.05$ s, and the initial condition of $x_0=1$, $y_0=2$, and $z_0=3$.