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Stability analysis of arbitrarily high-index, positive delay-descriptor systems

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Abstract This paper deals with the stability analysis of positive delay-descriptor systems with arbitrarily high index. First we discuss the solvability problem, which is followed by the study on characterizations of the (internal) positivity. Finally, we discuss the stability analysis. Numerically verifiable conditions in terms of matrix inequality for the system's coefficients are proposed, and are examined in several examples.

Keywords Positivity · Stability · Delay · Descriptor systems · Singular systems .

Nomenclature

$\mathbb{N}$ ($\mathbb{N}_0$)	the set of natural numbers (including 0)
$\mathbb{R}$ ($\mathbb{R}_+$)	the set of real (non-negative real) numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_-$	the set $\{\lambda \in \mathbb{C} \mid Re\lambda < 0\}$
I (I_n)	the identity matrix (of size $n \times n$)
$x^{(j)}$	the j-th derivative of a function x
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of p-times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$ (for $0 \leqslant p \leqslant \infty$)
$\Vert \cdot \Vert_\infty$	the norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$
$\mathrm{im}_+ W$	the space $\{Ww_1$ for all $w_1 \in \mathbb{R}^n_+\}$
$\mathcal{K}(U, W)$	the matrix $\mathcal{K}(U, W) := \big[W,\ UW,\ \ldots,\ U^{\nu-1}W\big]$.

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1 Introduction

Our focus in the present paper is on the positivity and stability analysis of linear, constant coefficients *delay-descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, $C \in \mathbb{R}^{q,n}$, $x : [t_0-\tau, \infty) \to \mathbb{R}^n$, $f : [t_0, \infty) \to \mathbb{R}^n$, and $\tau > 0$ is a constant delay. Together with (1), we are also concern with the associated *zero-input/free system*

$$E\dot{x}(t) = Ax(t) + A_d x(t-\tau), \quad \text{for all } t \in [t_0, \infty). \tag{2}$$

Systems of the form (1) can be considered as a general combination of two important classes of dynamical systems, namely *differential-algebraic equations (descriptor systems)* (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

where the matrix E is allowed to be singular ($\det E = 0$), and *delay-differential equations* (DDEs)

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t). \tag{4}$$

Delay-descriptor systems of the form (1) have been arisen in various applications, see Ascher and Petzold [1995], Campbell [1980], Hale and Lunel [1993], Shampine and Gahinet [2006], Zhu and Petzold [1997] and the references there in. From the theoretical viewpoint, the study for such systems is much more complicated than that for standard DDEs or DAEs. The dynamics of DDAEs has been strongly enriched, and many interesting properties, which occur neither for DAEs nor for DDEs, have been observed for DDAEs Campbell [1995], Du et al. [2013], Ha [2018]. Due to these reasons, recently more and more attention has been devoted to DDAEs, Campbell and Linh [2009], Fridman [2002], Ha and Mehrmann [2012, 2016], Michiels [2011], Shampine and Gahinet [2006], Tian et al. [2014], Linh and Thuan [2015].

[.... Em nho anh viet bo sung 1 phan gioi thieu ve viec can thiet phai nghien cuu tinh on dinh cua he duong voi chi so cao o day ....]

The short outline of this work is as follows. Firstly, in Section 2, we briefly recall the solvability analysis to system (1) (Theorem 1), followed by a result about solution comparison for the free system (2) (Theorems 3, 4). Based on the explicit solution representation in Section 2, we present a characterization for the positivity of system (1) in Section 3. Numerically verifiable conditions in terms of the matrix coefficients are established there. To follow, in Section 4 we discuss further about the free system (2) under biconditional requirements: stability and positivity (Theorems 6,7). Numerical examples are presented to illustrate the advantages of the proposed methods. Finally, we conclude this research with some discussion and open questions.

2 Preliminaries

In this section we discuss the solvability analysis (i.e., about the existence and uniqueness of a solution), including the solution representation and the comparison principal for the initial value problem (IVP) consisting of (1) with an initial condition

$$x|_{[t_0-\tau,t_0]} = \varphi : [t_0 - \tau, t_0] \to \mathbb{R}^n. \tag{5}$$

Here, φ is a prescribed initial trajectory (preshape function), which is necessary to achieve uniqueness of solutions. Without loss of generality, we assume that $t_0 = 0$.

2.1 Existence, uniqueness and explicit solution formula

It is well-known (e.g. Du et al. [2013]) that we may consider different solution concepts for system (1). The reason is, that $E(0)\dot{x}(0^+)$ which arises from the right hand side in (1) at 0 may not be equal to $E(0)\dot{\varphi}(0^-)$. Moreover, it has been observed in Baker et al. [2002], Campbell [1980], Guglielmi and Hairer [2008] that a discontinuity of $\dot{x}$ at $t = 0$ may propagate with time, and typically $\dot{x}$ is discontinuous at every point $j\tau,\ j \in \mathbb{N}_0$ or x may not even exist on the whole interval $[t_0, \infty)$. To deal with this property of DDAEs, we use the following solution concept.

Definition 1 Let us consider a fixed input function $u(t)$.
i) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *piecewise differentiable solution* of (1), if Ex is piecewise continuously differentiable, x is continuous and satisfies (1) at every $t \in [t_0, \infty) \setminus \bigcup_{j\in\mathbb{N}_0} \{j\tau\}$.
ii) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *classical solution* of (1) if it is at least continuous and satisfies (1) at every $t \in [t_0, \infty)$.

Throughout this paper whenever we speak of a solution, we mean a piecewise differentiable solution. Notice that, like DAEs, DDAEs are not solvable for arbitrary initial conditions, but they have to obey certain consistency conditions.

Definition 2 An initial function φ is called *consistent* with (1) if the associated IVP (1), (5) has at least one solution. System (1) is called *solvable* (resp. *regular*) if for every consistent initial function φ, the IVP (1), (5) has a solution (resp. has a unique solution).

For each $j \in \mathbb{N}$, we introduce sequences of matrix-valued and vector-valued functions f_j, u_j, x_j on the time interval $[0, \tau]$ via

$$\begin{aligned} &f_j(t) = f(t + (j-1)\tau),\ u_j(t) = u(t + (j-1)\tau),\\ &x_j(t) = x(t + (j-1)\tau),\ x_0(t) := \varphi(t - \tau), \end{aligned}$$

we can rewrite the IVP (1)-(5) as a sequence of non-delayed descriptor systems

$$E\dot{x}_j(t) = Ax_j(t) + A_d x_{j-1}(t) + Bu_j(t), \tag{6}$$

for all $t \in (0, \tau)$ and for all $j = 1, 2, ...$ We notice, that for each j, the initial condition $x_j(0)$ is given due to the continuity of the solution $x(t)$ at the point $(j-1)\tau$, i.e.,

$$x_j(0) = x_{j-1}(\tau) \ . \tag{7}$$

In particular, $x_1(0) = \phi(0)$ and the function x_0 is given.

It is well-known (see e.g. Bellman and Cooke [1963], Hale and Lunel [1993]) that in general, time-delayed systems has been classified into three different types (retarded, neutral, advanced). For example, the time-delayed equation

$$a_0\dot{x}(t) + a_1\dot{x}(t-\tau) + b_0 x(t) + b_1 x(t-\tau) = f(t)$$

is retarded if $a_0 \neq 0$ and $a_1 = 0$; is neutral if $a_0 \neq 0$, $a_1 \neq 0$; is advanced if $a_0 = 0$, $a_1 \neq 0$, $b_0 \neq 0$. Obviously, this classification is based on the smoothness comparison between $x_j(t)$ and $x_{j-1}(t)$. In literature, not only the theoretical but also the numerical solution has been studied mainly for retarded and neutral systems, due to their appearance in various applications. For this reason, in Ha [2015], Ha and Mehrmann [2016], Unger [2018] the authors proposed a concept of *non-advancedness* for the free system (2) (see Definition 3 below). We also notice, that even though not clearly proposed, due to the author's knowledge, so far results for delay-descriptor are only obtained for certain classes of non-advanced systems, e.g. Ascher and Petzold [1995], Shampine and Gahinet [2006], Zhu and Petzold [1997, 1998], Michiels [2011], Phat and Sau [2014], Sau et al. [2016], Cui et al. [2018], Ngoc [2018].

Definition 3 A regular delay-descriptor system (1) is called *non-advanced* if for any consistent and continuous initial function φ, there exists a piecewise differentiable solution $x(t)$ to the IVP (1), (5).

Definition 4 Consider the DDAE (1). The matrix triple (E, A, B) is called *regular* if the (two variable) *characteristic polynomial* $\det(\lambda E - A - \omega B)$ is not identically zero. If, in addition, $B = 0$ we say that the matrix pair (E, A) (or the pencil $\lambda E - A$) is regular. The sets $\sigma(E, A, B) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A - e^{-\lambda\tau}B) = 0\}$ and $\rho(E, A, B) = \mathbb{C} \setminus \sigma(E, A, B)$ are called the *spectrum* and the *resolvent set* of (1), respectively.

Provided that the pair (E, A) is regular, we can transform them to the Kronecker-Weierstraß canonical form (see e.g. Dai [1989], Kunkel and Mehrmann [2006]). That is, there exist regular matrices W, $T \in \mathbb{R}^{n,n}$ such that

$$(E, A) = \left(W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, \ W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right) \ , \tag{8}$$

where N is a nilpotent matrix of nilpotency index ν. We also say that the pair (E, A) has a *differentiation index* ν, i.e., $\text{ind}(E, A) = \nu$. Furthermore, the system (1) is called *impulse-free* if in the form (8) $N = 0$.

Remark 1 We notice that the impulse-freeness of system (1) is equivalent to the algebraic condition $\deg(\det(sE - A)) = \operatorname{rank}(E)$. Furthermore, for regular matrix pair (E, A), the impulse-freeness also has other names, such as strangeness-free or index 1 or causal, see Du et al. [2013], Sau et al. [2016], Ngoc [2018].

Remark 2 In general, the two concepts non-advancedness and differentiation index are independent. In details, a non-advanced system can have arbitrarily high index, as can be seen in the following example.

Example 1 Consider the following systems with two parameters ε_1, ε_2.

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E} \dot{x}(t) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} x(t) + \underbrace{\begin{bmatrix} 0 & \varepsilon_1 \\ 0 & \varepsilon_2 \end{bmatrix}}_{A_d} x(t-\tau) \ . \tag{9}$$

In this example $\operatorname{ind}(E, A) = 2$. Furthermore, depending on the value of ε_2, the system will be advanced (if $\varepsilon_2 \neq 0$) and be non-advanced (if $\varepsilon_2 \neq 0$). Analogously, one can construct a non-advanced system which has an arbitrarily high index.

Let E have index $\tilde{\nu}$, i.e., $\operatorname{ind}(E, I_n) = \tilde{\nu}$, the Drazin inverse E^D of E is uniquely defined by the properties

$$E^D E = E E^D, \ E^D E E^D = E^D, \ E^D E^{\tilde{\nu}+1} = E^{\tilde{\nu}}. \tag{10}$$

Lemma 1 *Kunkel and Mehrmann [2006] Let (E,A) be a regular matrix pair. Then for any $\lambda \in \rho(E, A)$, the following matrices commute*

$$\hat{E} := (\lambda E - A)^{-1} E, \ \hat{A} := (\lambda E - A)^{-1} A. \tag{11}$$

Furthermore, the following commutative identities hold true.

$$\hat{E}\hat{A}^D = \hat{A}^D\hat{E}, \ \hat{E}^D\hat{A} = \hat{A}\hat{E}^D, \ \hat{E}^D\hat{A}^D = \hat{A}^D\hat{E}^D \ . \tag{12}$$

We notice that the matrix products $\hat{E}^D\hat{E}$, $\hat{E}^D\hat{A}$, $\hat{E}\hat{A}^D$, $\hat{E}^D\hat{B}$, $\hat{A}^D\hat{B}$ do not depend on the choice of λ (see e.g. Dai [1989]). Furthermore, they can be numerically computed by transforming the pair (E, A) to their Weierstrass canonical form (8) (see e.g. Varga [2019], Virnik [2008]).

For any $\lambda \in \rho(E, A)$, we denote

$$\hat{A}_d := (\lambda E - A)^{-1} A_d, \ \hat{B} := (\lambda E - A)^{-1} B \ . \tag{13}$$

Making use of the Drazin inverse, in the following theorem we present the explicit solution representation of system (1).

Theorem 1 *Consider the delay-descriptor system* (1)*. Assume that* (E, A) *is a regular matrix pair with a differentiation index* $ind(E, A) = \nu$*. Let* $\hat{E}$*,* $\hat{A}$*,* $\hat{A}_d$*,* $\hat{B}$ *be defined as in* (11)*,* (13)*. Furthermore, assume that* u *is sufficiently smooth. Then, every solution* x_j *of the DAE* (6) *has the form*

$$x_j(t) = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left(\hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds$$
$$+ (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(t) + \hat{B} u_j^{(i)}(t) \right), \tag{14}$$

for some vector $v_j \in \mathbb{R}^n$.

Proof. The proof is straightly followed from the explicit solution of DAEs, see [Kunkel and Mehrmann, 2006, Chap. 2]. □

From Theorem 1 and (7), we directly obtain the following corollary.

Corollary 1 *The solution* $x(t)$ *of system* (1) *is continuous at the point* $(j-1)\tau$ *if and only if the following condition holds.*

$$(\hat{E}^D \hat{E} - I)\ x_{j-1}(\tau) = (\hat{E}^D \hat{E} - I) \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(0) + \hat{B} u_j^{(i)}(0) \right) .$$

In particular, for the preshape function $\varphi(t)$*, we must require*

$$(\hat{E}^D \hat{E} - I) \left(\varphi(0) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \left(\hat{A}_d \varphi^{(i)}(-\tau) + \hat{B} u^{(i)}(0) \right) \right) = 0 .$$

Following from (14), we directly obtain a simpler form in case of non-advanced system as follows.

Corollary 2 *Consider system* (1) *and assume that it is regular and non-advanced. Then, we have*

$$x_j(t) = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v_j + \int_0^t e^{\hat{E}^D \hat{A}(t-s)} \hat{E}^D \left(\hat{A}_d x_{j-1}(s) + \hat{B} u_j(s) \right) ds$$
$$+ (\hat{E}^D \hat{E} - I) \left(\hat{A}^D \hat{A}_d x_{j-1}(t) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u_j^{(i)}(t) \right), \tag{15}$$

Furthermore, the consistency condition at $t = 0$ *reads*

$$(\hat{E}^D \hat{E} - I) \left(\varphi(0) + \hat{A}^D \hat{A}_d \varphi(-\tau) + \sum_{i=0}^{\nu-1} (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} u^{(i)}(0) \right) = 0 . \tag{16}$$

2.2 A simple check for the non-advancedness

Assume that the pair (E, A) is regular with index $\operatorname{ind}(E, A) = \nu$. We want to give a simple check whether the free system (2) is non-advanced or not. In analoguous to the case of DAEs, see e.g. Brenan et al. [1996], Kunkel and Mehrmann [2006], we aim to extract the so-called *underlying delay equation* of the form

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{A}_{d0}x(t-h) + \mathbf{A}_{d1}\dot{x}(t-h), \tag{17}$$

from an augmented system consisting of system (2) and its derivatives, which read in details

$$\frac{d^i}{dt^i}\left(E\dot{x}(t) - Ax(t) - A_d x(t-\tau)\right) = 0, \text{ for all } i = 0, 1, \ldots, \nu.$$

We rewrite these equations into the so-called *inflated system*

$$\underbrace{\begin{bmatrix} E & & & \\ -A & E & & \\ & \ddots & \ddots & \\ & & -A & E \end{bmatrix}}_{\mathcal{E}} \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \vdots \\ x^{(\nu+1)} \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(\nu)} \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} A_d & & & \\ & A_d & & \\ & & \ddots & \\ & & & A_d \end{bmatrix}}_{\mathcal{A}_d} \begin{bmatrix} x(t-h) \\ \dot{x}(t-h) \\ \vdots \\ x^{(\nu)}(t-h) \end{bmatrix}. \tag{18}$$

Here the matrix coefficients are $\mathcal{E}, \mathcal{A}, \mathcal{A}_d \in \mathbb{R}^{(\nu+1)n,(\nu+1)n}$. For the reader's convenience, below we will use MATLAB notations. An underlying delay system (17) can be extracted from (18) if and only if there exists a matrix $P = \begin{bmatrix} P_0 & P_1 & \ldots & P_\nu \end{bmatrix}^T$ in $\mathbb{R}^{(\nu+1)n,n}$ such that

$$P^T\mathcal{E} = \begin{bmatrix} I_n & 0_{n,\nu n} \end{bmatrix},$$
$$P^T\mathcal{A}_d = \begin{bmatrix} * & * & 0_{n,(\nu-1)n} \end{bmatrix},$$

where $*$ stands for an arbitrary matrix. Consequently, P is the solution to the following linear systems

$$\begin{bmatrix} \mathcal{E} & \mathcal{A}_d(:, 2n+1 : end) \end{bmatrix}^T P = \begin{bmatrix} I_n & 0_{n,\nu n} & 0_{n,(\nu-1)n} \end{bmatrix}^T.$$

Therefore, making use of Crammer's rule we directly obtain the simple check for the non-advancedness of system (2) in the following theorem.

Theorem 2 *Consider the zero-input descriptor system* (2) *and assume that the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = \nu$. *Then, this system is non-advanced if and only if the following rank condition is satisfied*

$$\operatorname{rank}\begin{bmatrix} \mathcal{E}^T \\ \mathcal{A}_d(:, 2n+1 : end)^T \end{bmatrix} = \operatorname{rank}\left[\begin{array}{c|c} \mathcal{E}^T & I_n \\ \mathcal{A}_d(:, 2n+1 : end)^T & 0_{(2\nu-1)n,n} \end{array}\right] \tag{19}$$

Theorem 2 applied to the index two case straightly gives us the following corollary.

Corollary 3 *Consider the zero-input descriptor system* (2) *and assume that the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = 2$*. Then, system* (2) *is non-advanced if and only if the following identity hold true.*

$$\operatorname{rank} \begin{bmatrix} E^T & -A^T & 0 \\ 0 & E^T & -A^T \\ 0 & 0 & A_d^T \end{bmatrix} = n + \operatorname{rank} \begin{bmatrix} E^T & -A^T \\ 0 & E^T \\ 0 & A_d^T \end{bmatrix} . \tag{20}$$

Example 2 Let us reconsider system (9) in Example 1. Numerical verification of non-advancedness via condition (20) completely agrees with theoretical observation.

2.3 Comparison principal

In this part of Section 2, we will show how to generalize our result to delay-descriptor systems with time-varying delay of the following form

$$E\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)) + Bu(t), \quad \text{for all } t \in [t_0, \infty), \tag{21}$$

where the delay function $\tau(t)$ is preassumed continuous and bounded, i.e. $0 < \underline{\tau} \leqslant \tau(t) \leqslant \bar{\tau}$ for all $t \geq 0$. Here $\underline{\tau}$, $\bar{\tau}$ are two positive constants. Following Ha and Mehrmann [2016], it can be shown that the solution to system (21) exists, unique and totally determined by any consistent initial function φ such that $x(t) = \varphi(t)$ for all $-\bar{\tau} \leqslant t \leqslant 0$. Indeed, also making use of the method of steps, the solution x is constructively built on consecutive interval $[t_{i-1}, t_i]$, $i \in \mathbb{N}$ such that $0 = t_0 < t_1 < t_2 < ....$ and

$$t_i - \tau(t_i) = t_{i-1} \ .$$

As shown in Theorems 3, 4 below, we can directly generalize our result to systems with bounded, time varying delay of the form (21).

Theorem 3 *Consider system* (21) *and assume that the corresponding constant delay system* (1) *is positive and non-advanced. For a fixed input* u*, let* $x(t)$ *(resp.* $\tilde{x}(t)$*) be a state function corresponds to a preshape function* $\varphi(t)$ *(resp.* $\tilde{\varphi}(t)$*). Furthermore, assume that* $\varphi(t) \leqslant \tilde{\varphi}(t)$ *for all* $t \in [-\bar{\tau}, 0]$*. Then, we have* $x(t) \leqslant \tilde{x}(t)$ *for all* $t \geq 0$*.*

Proof. Based on the linearity of system (1), $\tilde{x}(t) - x(t)$ satisfies the free system (2). Furthermore, since this system is non-advanced and positive the non-negativity of $\tilde{\varphi}(t) - \varphi(t)$ implies that $\tilde{x}(t) - x(t) \geq 0$ for all t. □

Theorem 4 *Consider system* (21) *and assume that the corresponding constant delay system* (1) *is positive. Furthermore, assume that*

$$(\hat{E}^D \hat{E} - I) \ (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$$

for all $i = 0, \ldots, \nu - 1$. Let $x(t)$ (resp. $\tilde{x}(t)$) be a state function corresponds to a reference input $u(t)$ (resp. $\tilde{u}(t)$) and a preshape function $\varphi(t)$ (resp. $\tilde{\varphi}(t)$). Then we have $x(t) \leqslant \tilde{x}(t)$ for all $t \geq 0$, provided that the following conditions are fulfilled.
i) $\varphi(t) \leqslant \tilde{\varphi}(t)$ for all $t \in [-\tau, 0]$,
ii) $u^{(i)}(t) \leqslant \tilde{u}^{(i)}(t)$ for all $t \geq 0$ and for all $i \leqslant (\nu - 1)\ \lfloor t/\tau \rfloor$.

Proof. The proof is also straightforward from the solution's representation (14). □

From Theorems 3, 4 above, we see that the time varying delay will not affect our later results on the positivity and the stability of system (1).

3 Characterizations of positive delay-descriptor system

Since most systems occur in application are non-advanced, in this section we focus on the characterization for positivity of non-advanced delay descriptor systems. We, furthermore, notice that the non-advancedness is a necessary condition for the stability (in the Lyapunov sense) of any time-delayed system, see e.g. Hale and Lunel [1993], Du et al. [2013].

Definition 5 Consider the delay-descriptor system (1) and assume that it is non-advanced, and that the pair (E, A) is regular with $\operatorname{ind}(E, A) = \nu$. We call (1) positive if for all $t \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for any input function u and any consistent initial function $\varphi(t)$ that satisfy two following conditions.
i) $\varphi(t) \geq 0$ for all $t \in [-\tau, 0]$,
ii) $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and all $i \leqslant (\nu - 1)\ \lfloor t/\tau \rfloor$.

For nontiaonal convenience, let us denote by

$$\begin{aligned}
&P := \hat{E}^D \hat{E},\ \bar{\mathbf{A}} := \hat{E}^D \hat{A},\ \bar{\mathbf{A}}_d := \hat{E}^D \hat{A}_d,\ \bar{\mathbf{B}} := \hat{E}^D \hat{B}, \\
&\mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D \hat{B}) := \left[\hat{A}^D \hat{B}, \bar{\mathbf{A}}\ \hat{A}^D \hat{B}, \ldots, \bar{\mathbf{A}}^{\nu-1} \hat{A}^D \hat{B}\right]\ .
\end{aligned} \tag{22}$$

Since our systems is linear, time invariant coefficients, it would be sufficient to study the positivity on the first time interval $[0, \tau]$. Making use of (15), and let $j = 1$, we can rewrite the solution $x_1 = x|_{[0,\tau]}$ as follows

$$\begin{aligned}
x_1(t) = &\underbrace{e^{\bar{\mathbf{A}}t} P x_0(\tau) + (P - I)\hat{A}^D \hat{A}_d x_0(t) + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d x_0(s) ds}_{x_{zi}(t)} \\
&+ \underbrace{\int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{B}} u_j(s) ds + (P - I) \sum_{i=0}^{\nu-1} \bar{\mathbf{A}}^i \hat{A}^D \hat{B} u_j^{(i)}(t)}_{x_{zs}(t)}\ .
\end{aligned} \tag{23}$$

In the theory of linear systems, $x_{zi}(t)$ (resp. $x_{zs}(t)$) is often called the *zero input/free* (resp. *zero state*) solution. The characterization for the positivity of the free solution x_{zi} is given in Rami and Napp [2012] as follows.

Proposition 1 *Rami and Napp [2012] The following statements are equivalent.*
i) The non-delayed free system $E\dot{x}(t) = Ax(t)$ is positive.
ii) There exists a Metzler matrix H such that $\bar{\mathbf{A}} = HP$, where P is defined via (22).
iii) There exists a matrix D such that $H := \bar{\mathbf{A}} + D\ (I - P)$ is Metzler.

Lemma 2 *Consider the delay-descriptor system (1) and assume that it is non-advanced. Let the pair (E, A) be regular with index $\mathrm{ind}(E, A) = \nu$. Then, the free system (2) has a non-negative solution $x_{zi}(t) \geq 0$ for all $t \geq 0$ and for all consistent initial function $\varphi(t) \geq 0$ if and only if the following conditions are satisfied.*
i) There exists a Metzler matrix H such that $\bar{\mathbf{A}} = HP$.
ii) $\bar{\mathbf{A}}_d \geq 0$, $(P - I)\hat{A}^D\hat{A}_d \geq 0$.

Proof. "⇒" Consider $x_zi(t)$ in (23). For any fixed $t \in (0, \tau)$, since the integral part $\int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds$ can be arbitrarily small chosen, independent of the two boundary points 0 and t, we see that the sum $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$ must be non-negative for any non-negative vectors $x_0(\tau)$ and $x_0(t)$. The independence of these two vectors leads to the fact that the sum $e^{\bar{\mathbf{A}}t}Px_0(\tau) + (P - I)\hat{A}^D\hat{A}_d x_0(t)$ is non-negative if and only if both terms are non-negative. Thus, due to Proposition 1, the non-negativity of the term $e^{\bar{\mathbf{A}}t}Px_0(\tau)$ is equivalent to the claim i). On the other hand, the non-negativity of the term $(P - I)\hat{A}^D\hat{A}_d x_0(t)$ implies that $(P - I)\hat{A}^D\hat{A}_d \geq 0$.

To prove that $\bar{\mathbf{A}}_d \geq 0$, we assume the contrary, that there exist some indices i, j with $[\bar{\mathbf{A}}_d]_{ij} < 0$. Thus, for the jth unit vector e_j, we have $[\bar{\mathbf{A}}_d e_j]_i < 0$. For a sufficiently small $\varepsilon > 0$, let us choose the initial function x_0 as follows

$$x_0(s) = \begin{cases} \left(1 - \frac{1}{\varepsilon}|t - \varepsilon - s|\right) e_j & \text{for all} \quad |t - \varepsilon - s| \leqslant \varepsilon, \\ 0 \quad \text{otherwise.} \ . \end{cases} \tag{24}$$

The graph of the magnitude of $x_0(s)$ is given in Figure 1. Since $u \equiv 0$, $x_0(0) = x_0(\tau) = 0$, the consistency condition (16) is trivially satisfied. Then, we have that

$$\begin{aligned} x_1(t) &= \int_0^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds = \int_{t-2\varepsilon}^t e^{\bar{\mathbf{A}}(t-s)}\bar{\mathbf{A}}_d x_0(s)ds, \\ &= \int_{t-2\varepsilon}^t \left(I + \bar{\mathbf{A}}(t-s) + \mathcal{O}((t-s)^2)\right)\left(1 - \frac{1}{\varepsilon}|t - \varepsilon - s|\right)\bar{\mathbf{A}}_d e_j ds \ . \end{aligned}$$

Thus, for sufficiently small ε, the coordinate $(x_1(t))_i$ has exactly the same sign as $[\bar{\mathbf{A}}_d e_j]_i$, which is strictly negative. This is contradicted to the non-negativity of the solution $x(t)$, and hence, we conclude that $\bar{\mathbf{A}}_d \geq 0$.
"⇐" It is directly followed from i) and ii) that all three summands of $x_{zi}(t)$ are non-negative. This completes the proof. □

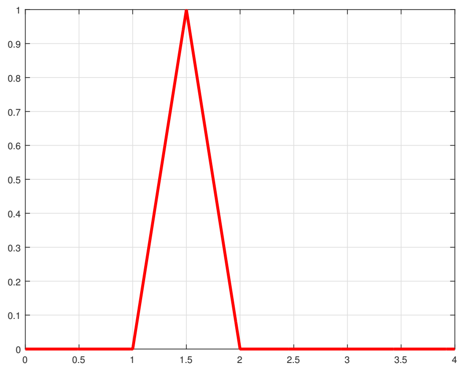

Fig. 1 The function x_0 in (24) with $\tau = 4$, $t = 2$, $\varepsilon = 0.5$.

Theorem 5 *Consider the delay-descriptor system* (1) *and assume that it is non-advanced. Let the pair* (E, A) *be regular with index* $\mathrm{ind}(E, A) = \nu$*. Furthermore, assume that* $(P - I)\ \bar{\mathbf{A}}^i \hat{A}^D \hat{B} \geq 0$ *for all* $i = 0, \ldots, \nu - 1$*. Then, system* (1) *is positive if and only if the following conditions hold.*
i) $\bar{\mathbf{A}} = H\ P$ *for some Metzler matrix* H*.*
ii) $\bar{\mathbf{A}}_d \geq 0$*,* $\bar{\mathbf{B}} \geq 0$*,* $(P - I)\hat{A}^D \hat{A}_d \geq 0$*,*
iii) C *is non-negative on the subspace*

$$\mathcal{X} := \mathrm{im}_+ \left[P,\ (P - I)\hat{A}^D \hat{A}_d,\ (P - I)\ \mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D \hat{B}) \right] . \tag{25}$$

Proof. "$\Rightarrow$" By consecutively choosing $u \equiv 0$ and $\phi \equiv 0$, we see that both the free solution $x_{zi}(t)$ and the zero-state solution $x_{zs}(t)$ are non-negative for all $t \geq 0$. Analogous to the proof of Lemma 2 (the necessity part), the non-negativity of the integral $\int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{B}} u_j(s) ds$ follows that $\bar{\mathbf{B}} \geq 0$. Thus, only the claim iii) needs to be proven. We notice that due to Lemma 1 and the property (10) of the Drazin inverse, we have that P and $\bar{\mathbf{A}}$ commute, and $P\hat{E}^D = \hat{E}^D$, and hence,

$$e^{\bar{\mathbf{A}}} \hat{E}^D = \hat{E}^D e^{\bar{\mathbf{A}}} = \hat{E}^D \hat{E} \hat{E}^D e^{\bar{\mathbf{A}}} = P e^{\bar{\mathbf{A}}} \hat{E}^D .$$

Therefore, we see that

$$\begin{aligned} & e^{\bar{\mathbf{A}} t} P x_0(\tau) + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d x_0(s) ds + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{B}} u_j(s) ds \subseteq \mathrm{im}_+(P), \\ & (P - I)\hat{A}^D \hat{A}_d x_0(t) + (P - I) \sum_{i=0}^{\nu-1} \bar{\mathbf{A}}^i \hat{A}^D \hat{B} u_j^{(i)}(t) \\ & \quad \subseteq \mathrm{im}_+ \left[(P - I)\hat{A}^D \hat{A}_d,\ (P - I)\ \mathcal{K}_\nu(\bar{\mathbf{A}}, \hat{A}^D \hat{B}) \right] . \end{aligned}$$

Thus, the claim iii) is directly followed.
"$\Leftarrow$" It is straightforward that from i) and ii) we obtain the non-negativity of $x(t)$, and due to iii) we obtain the non-negativity of $y(t)$. This completes the proof. $\square$

Theorem 5 applied to the non-delayed case (i.e. $A_d = 0$) gives us the following corollary. We notice that this corollary has slightly improved the result [Virnik, 2008, Thm. 3.4].

Corollary 4 *Consider the descriptor system* (3) *and assume that the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = \nu$. *Furthermore, assume that the inequalities* $(P - I)\ \bar{\mathbf{A}}^i \hat{A}^D \hat{B} \geq 0$ *hold true for* $i = 0, \ldots, \nu - 1$.
Then, system (3) *is positive if and only if the following conditions hold.*
i) $\bar{\mathbf{A}} = H\ P$ *for some Metzler matrix* H.
ii) $\bar{\mathbf{B}} \geq 0$,
iii) C *is non-negative on the subspace* $\mathcal{X}$ *defined in* (25).

4 Stability of positive delay-descriptor system

In this section we focus our attention on systems which is both stable and positive. Firstly, we demonstrate that the non-advancedness is necessary for the stability. Then, we present several sufficient conditions to examining the stability of positive delay-descriptor systems, followed by an illustrate example.

Example 3 Let us recall system (9) with $\varepsilon_2 = -1$, $\varepsilon_1 = 0$. From the second equation we see that $x_2(t) = x_2(t - \tau)$. Inserting this into the first equation we obtain

$$\dot{x}_2(t - \tau) = x_1(t)\ .$$

Therefore, we have $x(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t - \tau) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}(t - \tau)$, which implies that the system is of advanced type. Clearly, the solution formula implies that the system is unstable in the Lyapunov sense.

To study the stability of system (1), we first transform this system to an equivalent impulse-free system, in the sense that the solution of the original system and the transformed system coincide.

Let $y_j(t) := Px_j(t)$ and $z_j(t) := (I - P)x_j(t)$ for all $j \in \mathbb{N}$, $t \geq 0$, then from the solution's representation (14) we obtain

$$x_j(t) = e^{\bar{\mathbf{A}}t} x_j(0) + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d \left(y_{j-1}(s) + z_{j-1}(s) \right) ds + (P - I)\hat{A}^D \hat{A}_d x_{j-1}(t),$$

for all $t \in (0, \tau)$. Premultiply this equation with P and $I - P$, we then obtain the system

$$y_j(t) = e^{\bar{\mathbf{A}}t} y_j(0) + \int_0^t e^{\bar{\mathbf{A}}(t-s)} \bar{\mathbf{A}}_d \left(y_{j-1}(s) + z_{j-1}(s) \right) ds, \tag{26a}$$

$$z_j(t) = (P - I)\hat{A}^D \hat{A}_d \left(y_{j-1}(t) + z_{j-1}(t) \right)\ . \tag{26b}$$

This system can be rewritten as follows.

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_j(t) \\ \dot{z}_j(t) \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_j(t) \\ z_j(t) \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{A}}_d & \bar{\mathbf{A}}_d \\ (P-I)\hat{A}^D\hat{A}_d & (P-I)\hat{A}^D\hat{A}_d \end{bmatrix} \begin{bmatrix} y_{j-1}(t) \\ z_{j-1}(t) \end{bmatrix}. \tag{27}$$

Therefore, we see that this transformed system is impulse-free, and hence we can applied already known results to study the its stability. The following results are directly extended from Cui et al. [2018]

Theorem 6 *Consider the delay-descriptor system* (1). *Assume that the matrix pair (E,A) is regular, and system* (1) *is non-advanced. Then, system* (1) *is positive and asymptotically stable if the following conditions hold true.*

i) $\bar{\mathbf{A}}_d \geq 0, \ (P-I)\hat{A}^D\hat{A}_d \geq 0,$

ii) C *is non-negative on the subspace* $\mathrm{im}_+ \left[P, \ (P-I)\hat{A}^D\hat{A}_d\right]$,

iii) the matrix $\bar{H}$ *is Hurwitz, where*

$$\bar{H} := \begin{bmatrix} \bar{\mathbf{A}}_d + H & \bar{\mathbf{A}}_d \\ (P-I)\hat{A}^D\hat{A} & (P-I)\hat{A}^D\hat{A} - I \end{bmatrix}. \tag{28}$$

Theorem 7 *Consider the delay-descriptor system* (1). *Assume that the matrix pair (E,A) is regular, and system* (1) *is non-advanced. Furthermore, assume that there exists a positive vector* $w \in \mathbb{R}^n_+$ *such that* $(P-I)\hat{A}^D\hat{A}w > 0$. *Then, system* (1) *is positive and asymptotically stable if and only if the following conditions hold true.*

i) $\bar{\mathbf{A}}_d \geq 0, \ (P-I)\hat{A}^D\hat{A}_d \geq 0,$

ii) C *is non-negative on the subspace* $\mathrm{im}_+ \left[P, \ (P-I)\hat{A}^D\hat{A}_d\right]$,

iii) the matrix $\bar{H}$ *is Hurwitz, where* $\bar{H}$ *is defined in* (28).

Remark 3 We stress out that in previous results on positivity of delay-descriptor systems (except Ha [2018]) it is always assumed that the system is impulse-free, which is an unnecessary condition, see for instance Cui et al. [2018], Liu et al. [2009], Phat and Sau [2014], Sau et al. [2016]. In contrast, our result in Theorems 6, 7 provide (necessary and) sufficient conditions for the positivity of (1) without this impulse-free assumption.

In light of Remark 3, we illustrate how Theorem 6 and 7 apply to general situations by presenting an example where system (1) is not impulse-free, but it is positive and also stable. We notice that in this example, the system is of index $\nu(E,A) = 2$, even though arbitrarily high-index system can be constructed in the same fashion.

Example 4 Let us consider system (1) whose the matrix coefficients are

$$E = \begin{bmatrix} -11 & 1 & 0.1521 \\ 0 & 0 & 0.9365 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 0.2 & 0.61 & 0.9236 \\ -1 & 0.6 & 0.4683 \\ 0 & 0 & 0.7722 \end{bmatrix}, \ A_d = \begin{bmatrix} -1 & -0.2 & -1.9298 \\ -0.8 & -0.01 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Direct computation yields that the matrix polynomial $\det(sE - A)$ is

$$\det(sE - A) = -4.32432\ s - 0.563706\ ,$$

and hence the system is not impulse-free, since $\mathrm{rank}(E) = 2$. For $s = 3$ we have $\det(sE - A) \neq 0$, so we obtain

$$\hat{E} = \begin{bmatrix} 0.3765 & -0.034227 & -0.13289 \\ 0.6275 & -0.057045 & -1.7823 \\ 0 & 0 & 0 \end{bmatrix}, \hat{A} = \begin{bmatrix} 0.12949 & -0.10268 & -0.39866 \\ 1.8825 & -1.1711 & -5.3469 \\ 0 & 0 & -1 \end{bmatrix}, \hat{A}_d = \begin{bmatrix} 0.1433 & 0.0082088 & 0.066051 \\ 1.5722 & 0.030348 & 0.11009 \\ 0 & 0 & 0 \end{bmatrix}.$$

We also see that the index of system (1) is $\mathrm{ind}(E, A) = 2$. Corollary 3 applied here implies that the system is non-advanced. Furthermore, we have that

$$\bar{\mathbf{A}} = \begin{bmatrix} -0.0050085 & 0.00045532 & -0.0004569 \\ -0.0083475 & 0.00075887 & -0.0007615 \\ 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{A}}_d = \begin{bmatrix} 4.4908e-05 & 0.00065547 & 0.0067405 \\ 7.4847e-05 & 0.0010925 & 0.011234 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$P = \begin{bmatrix} 0.038421 & -0.0034929 & 0.003505 \\ 0.064036 & -0.0058214 & 0.0058417 \\ 0 & 0 & 0 \end{bmatrix},\ (P - I)\hat{A}^D \hat{A}_d = \begin{bmatrix} 0.14371 & 0.05087 & 0.42647 \\ 1.5494 & 0.10116 & 0.71079 \\ 0 & 0 & 0 \end{bmatrix}.$$

By solving the equality in Theorem 6, we obtain

$$H = \begin{bmatrix} -0.59167 & 0.27679 & 0 \\ 0.27679 & -0.29643 & 0 \\ 0 & 0 & -0.74313 \end{bmatrix}$$

The spectrum of H and $\bar{H}$ are $\sigma(H) = \{-0.7577, -0.7431, -0.1304\}$ and $\sigma(\bar{H}) = \{-1.2832, -0.7577, -0.1221, -0.3001, -0.7431, -1.0000\}$. Therefore, due to Theorem 6 we see that system (1) is both positive and stable.

5 Conclusion

In this paper, we have studied the stability of positive delay-descriptor systems of arbitrarily high index without the impulse-free assumption. Firstly, a necessary and sufficient condition has been proposed to ensure the positivity of delay-descriptor system. Then, stability conditions for positive systems of arbitrarily high index have been established.

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