

Robust and LPV control of MIMO systems

Part 1: Tools for analysis and control of dynamical systems

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Some definitions

- Nonlinear dynamical systems

- LTI dynamical systems

Signal and system norms

- Signal norms

- Some topological spaces recalls

- System norms: \mathcal{H}_2 and \mathcal{H}_∞ performances

Stability issues

- Well-posedness

- Internal stability

- Input-Output stability

Introduction to LMIs

- Background in Optimisation

- LMI in control

- To go further: some useful lemmas

Reference books

To be studied during the course

- ▶ S. Skogestad and I. Postlethwaite, Multivariable Feedback Control: analysis and design, John Wiley and Sons, 2005.
www.nt.ntnu.no/users/skoge/book, [chap 1 to 3 available](#)
- ▶ K. Zhou, Essentials of Robust Control, Prentice Hall, New Jersey, 1998.
www.ece.lsu.edu/kemin, [book slides available](#)
- ▶ J.C. Doyle, B.A. Francis, and A.R. Tannenbaum, Feedback control theory, Macmillan Publishing Company, New York, 1992.
<https://sites.google.com/site/brucefranciscontact/Home/publications>, [book available](#)
- ▶ Carsten Scherer's courses
<http://www.dsc.tudelft.nl/~cscherer/>, [Lecture slides available](#) (MSc Course "Robust Control", MSc Course "Linear Matrix Inequalities in Control")
- ▶ + all the MATLAB demo, examples and documentation on the 'Robust Control toolbox' (mathworks.com/products/robust)

Outline

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Nonlinear systems

In control theory, dynamical systems are mostly modeled and analyzed thanks to the use of a set of Ordinary Differential Equations (ODE)¹.

Nonlinear dynamical system modeling is the "most" representative model of a given system. Generally this model is derived thanks to system knowledge, physical equations etc. Nonlinear dynamical systems are described by nonlinear ODEs.

Definition (Nonlinear dynamical system)

For given functions $f : \mathbb{R}^n \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_z}$, a nonlinear dynamical system ($\Sigma_{\mathcal{NL}}$) can be described as:

$$\Sigma_{\mathcal{NL}} : \begin{cases} \dot{x} = f(x(t), w(t)) \\ z = g(x(t), w(t)) \end{cases} \quad (1)$$

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

¹Partial Differential Equations (PDEs) may be used for some applications such as irrigation channels, traffic flow, etc. but methodologies involved are more complex compared to the ones for ODEs.

Comments

- ▶ The main advantage of nonlinear dynamical modeling is that (if it is correctly described) it catches most of the real system phenomena.
- ▶ On the other side, the main drawback is that there is a lack of mathematical and methodological tools; e.g. parameter identification, control and observation synthesis and analysis are complex and non systematic (especially for MIMO systems).
- ▶ In this field, notions of robustness, observability, controllability, closed loop performance etc. are not so obvious. In particular, complex nonlinear problems often need to be reduced in order to become tractable for nonlinear theory, or to apply input-output linearization approaches (e.g. in robot control applications).
- ▶ As nonlinear modeling seems to lead to complex problems, especially for MIMO systems control, the LTI dynamical modeling is often adopted for control and observation purposes. The LTI dynamical modeling consists in describing the system through linear ODEs. According to the previous nonlinear dynamical system definition, LTI modeling leads to a local description of the nonlinear behavior (e.g. it locally describes, around a linearizing point, the real system behavior).
- ▶ From now on only Linear Time-Invariant (LTI) systems are considered.

Definition of LTI systems

Definition (LTI dynamical system)

Given matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$ and $D \in \mathbb{R}^{n_z \times n_w}$, a Linear Time Invariant (LTI) dynamical system (Σ_{LTI}) can be described as:

$$\Sigma_{LTI} : \begin{cases} \dot{x}(t) &= Ax(t) + Bw(t) \\ z(t) &= Cx(t) + Dw(t) \end{cases} \quad (2)$$

where $x(t)$ is the state which takes values in a state space $X \in \mathbb{R}^n$, $w(t)$ is the input taking values in the input space $W \in \mathbb{R}^{n_w}$ and $z(t)$ is the output that belongs to the output space $Z \in \mathbb{R}^{n_z}$.

The LTI system locally describes the real system under consideration and the linearization procedure allows to treat a linear problem instead of a nonlinear one. For this class of problem, many mathematical and control theory tools can be applied like closed loop stability, controllability, observability, performance, robust analysis, etc. for both SISO and MIMO systems. However, the main restriction is that LTI models only describe the system locally, then, compared to nonlinear models, they lack of information and, as a consequence, are incomplete and may not provide global stabilization.

Signal norms

Reader is also invited to refer to the famous book of [Zhou et al., 1996](#), where all the following definitions and additional information are given.

All the following definitions are given assuming signals $x(t) \in \mathbb{C}$, then they will involve the conjugate (denoted as $x^*(t)$). When signals are real (i.e. $x(t) \in \mathbb{R}$), $x^*(t) = x^T(t)$.

Definition (Norm and Normed vector space)

- ▶ Let V be a finite dimension space. Then $\forall p \geq 1$, the application $\|\cdot\|_p$ is a norm, defined as,

$$\|v\|_p = \left(\sum_i |v_i|^p \right)^{1/p} \quad (3)$$

- ▶ Let V be a vector space over \mathbb{C} (or \mathbb{R}) and let $\|\cdot\|$ be a norm defined on V . Then V is a normed space.

\mathcal{L}_\star norms

Definition (\mathcal{L}_1 , \mathcal{L}_2 , \mathcal{L}_∞ norms)

- The 1-Norm of a function $x(t)$ is given by,

$$\|x(t)\|_1 = \int_0^{+\infty} |x(t)| dt \quad (4)$$

- The 2-Norm (that introduces the energy norm) is given by,

$$\begin{aligned} \|x(t)\|_2 &= \sqrt{\int_0^{+\infty} x^*(t)x(t)dt} \\ &= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} X^*(j\omega)X(j\omega)d\omega} \end{aligned} \quad (5)$$

The second equality is obtained by using the Parseval identity.

- The ∞ -Norm is given by,

$$\|x(t)\|_\infty = \sup_t |x(t)| \quad (6)$$

$$\|X\|_\infty = \sup_{\operatorname{Re}(s) \geq 0} \|X(s)\| = \sup_\omega \|X(j\omega)\| \quad (7)$$

if the signals that admit the Laplace transform, analytic in $\operatorname{Re}(s) \geq 0$ (i.e. $\in \mathcal{H}_\infty$).

About vector spaces

Definition (Banach, Hilbert, Hardy and \mathcal{L}_p spaces)

- ▶ A Banach space is a (real or complex) complete (i.e. all Cauchy sequences, of points in K have a limit that is also in K) normed vector space B (with norm $\|\cdot\|_p$).
- ▶ A Hilbert space is a (real or complex) vector space H with an inner product $\langle \cdot, \cdot \rangle$ that is complete under the norm defined by the inner product. The norm of $f \in H$ is then defined by,

$$\|f\| = \sqrt{\langle f, f \rangle} \quad (8)$$

Every Hilbert space is a Banach since a Hilbert space is complete with respect to the norm associated with its inner product.

- ▶ The Hardy spaces (\mathcal{H}_p) are certain spaces of holomorphic functions (functions defined on an open subset of the complex number plane \mathbb{C} with values in \mathbb{C} that are complex-differentiable at every point) on the unit disk or upper half plane.
- ▶ The \mathcal{L}_p space are spaces of p -power integrable functions (function whose integral exists, generally called Lebesgue integral), and corresponding sequence spaces.

For example, \mathbb{R}^n and \mathbb{C}^n with the usual spatial p -norm, $\|\cdot\|_p$ for $1 \leq p < \infty$, are Banach spaces. This means that a Banach space is a vector space B over the real or complex numbers with a norm $\|\cdot\|_p$ such that every Cauchy sequence (with respect to the metric $d(x, y) = \|x - y\|$) in B has a limit in B .

\mathcal{L}_2 and \mathcal{H}_2 spaces

Definition (\mathcal{L}_2 space)

\mathcal{L}_2 is the space of piecewise continuous square integrable functions. It is a Hilbert space of matrix-valued (or scalar-valued) functions on \mathbb{C} and consists of all complex matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, such that,

$$\|f\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[f^*(j\omega)f(j\omega)]d\omega} < \infty \quad (9)$$

The inner product for this Hilbert space is defined as (for $f, g \in \mathcal{L}_2$)

$$\langle f, g \rangle = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[f^*(j\omega)g(j\omega)]d\omega} \quad (10)$$

Definition (\mathcal{H}_2 and \mathcal{RH}_2 spaces)

\mathcal{H}_2 is a subspace (Hardy space) of \mathcal{L}_2 with matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, analytic in $\text{Re}(s) > 0$ (functions that are locally given by a convergent power series and differentiable on each point of its definition set). In particular, the real rational subspace of \mathcal{H}_2 , which consists of all strictly proper and real rational stable transfer matrices, is denoted by \mathcal{RH}_2 .

\mathcal{L}_2 and \mathcal{H}_2 spaces

Example

In control theory

$$\begin{aligned}\frac{s+1}{(s+10)(s+6)} &\in \mathcal{RH}_2 \\ \frac{s+1}{(s-10)(s+6)} &\notin \mathcal{RH}_2 \\ \frac{s+1}{(s+10)} &\notin \mathcal{RH}_2\end{aligned}\tag{11}$$

\mathcal{L}_∞ and \mathcal{H}_∞ spaces

Definition (\mathcal{L}_∞ space)

\mathcal{L}_∞ is the space of piecewise continuous bounded functions. It is a Banach space of matrix-valued (or scalar-valued) functions on \mathbb{C} and consists of all complex bounded matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, such that,

$$\sup_{\omega \in \mathbb{R}} \bar{\sigma}[f(j\omega)] < \infty \quad (12)$$

Definition (\mathcal{H}_∞ and \mathcal{RH}_∞ spaces)

\mathcal{H}_∞ is a (closed) subspace in \mathcal{L}_∞ with matrix functions $f(j\omega)$, $\forall \omega \in \mathbb{R}$, analytic in $\text{Re}(s) > 0$ (open right-half plane). The real rational subspace of \mathcal{H}_∞ which consists of all proper and real rational stable transfer matrices, is denoted by \mathcal{RH}_∞ .

\mathcal{L}_∞ and \mathcal{H}_∞ spaces

Example

In control theory

$$\begin{aligned}\frac{s+1}{(s+10)(s+6)} &\in \mathcal{RH}_\infty \\ \frac{s+1}{(s-10)(s+6)} &\notin \mathcal{RH}_\infty \\ \frac{s+1}{(s+10)} &\in \mathcal{RH}_\infty\end{aligned}\tag{13}$$

\mathcal{H}_2 norm

Definition (\mathcal{H}_2 norm)

The \mathcal{H}_2 norm of a strictly proper LTI system defined as on (2) from input $w(t)$ to output $z(t)$ and which belongs to \mathcal{RH}_2 , is the energy (\mathcal{L}_2 norm) of the impulse response $g(t)$ defined as,

$$\begin{aligned}\|G(j\omega)\|_2 &= \sqrt{\int_{-\infty}^{+\infty} g^*(t)g(t)dt} \\ &= \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}[G^*(j\omega)G(j\omega)]d\omega} \\ &= \sup_{w(s) \in \mathcal{H}_2} \frac{\|z(s)\|_\infty}{\|u(s)\|_2}\end{aligned}\tag{14}$$

The norm \mathcal{H}_2 is finite if and only if $G(s)$ is strictly proper (i.e. $G(s) \in \mathcal{RH}_2$).

Remark

\mathcal{H}_2 physical interpretations and remarks

- ▶ For SISO systems, it represents the area located below the so called Bode diagram.
- ▶ For MIMO systems, the \mathcal{H}_2 norm is the impulse-to-energy gain of $z(t)$ in response to a white noise input $w(t)$ (satisfying $W^*(j\omega)W(j\omega) = I$, i.e. uniform spectral density).
- ▶ The \mathcal{H}_2 norm can be computed analytically (through the use of the controllability and observability Grammians) or numerically (through LMIs).

\mathcal{H}_∞ norm

Definition (\mathcal{H}_∞ norm)

The \mathcal{H}_∞ norm of a proper LTI system defined as on (2) from input $w(t)$ to output $z(t)$ and which belongs to \mathcal{RH}_∞ , is the **induced energy-to-energy gain (induced \mathcal{L}_2 norm)** defined as,

$$\begin{aligned}\|G(j\omega)\|_\infty &= \sup_{\omega \in \mathbb{R}} \bar{\sigma}(G(j\omega)) \\ &= \sup_{w(s) \in \mathcal{H}_2} \frac{\|z(s)\|_2}{\|w(s)\|_2} \\ &= \max_{w(t) \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2}\end{aligned}\tag{15}$$

Remark

\mathcal{H}_∞ physical interpretations

- ▶ This norm represents the maximal gain of the frequency response of the system. It is also called the worst case attenuation level in the sense that it measures the maximum amplification that the system can deliver on the whole frequency set.
- ▶ For SISO (resp. MIMO) systems, it represents the maximal peak value on the Bode magnitude (resp. singular value) plot of $G(j\omega)$, in other words, it is the largest gain if the system is fed by harmonic input signal.
- ▶ Unlike \mathcal{H}_2 , the \mathcal{H}_∞ norm cannot be computed analytically. Only numerical solutions can be obtained (e.g. Bisection algorithm, or LMI resolution).

Recalls on Singular Value Definition

Let $A \in \mathbb{R}^{m \times n}$, There exists unitary matrices:

$$U = [u_1, u_2, \dots, u_m] \text{ and } V = [v_1, v_2, \dots, v_n]$$

such that

$$A = U\Sigma V^T, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_p \end{bmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$, $p = \min\{m, n\}$.

Singular values are good measures of the size of a matrix Singular vectors are good indications of strong/weak input or output directions. Note that:

$Av_i = \sigma_i u_i$ and $A^T u_i = \sigma_i v_i$. Then

$$AA^T u_i = \sigma_i^2 u_i$$

$\bar{\sigma} = \sigma_{\max}(A) = \sigma_1$ = the largest singular value of A .

and

$\underline{\sigma} = \sigma_{\min}(A) = \sigma_p$ = the smallest singular value of A .

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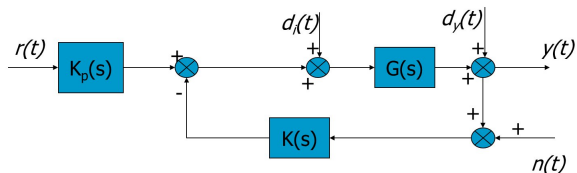
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Well-posedness



Consider

$$G = -\frac{s-1}{s+2}, \quad K = 1$$

Therefore the control input is non proper:

$$u = \frac{s+2}{3}(r - n - d_y) + \frac{s-1}{3}d_i$$

DEF: A closed-loop system is well-posed if all the transfer functions are proper

$$\Leftrightarrow I + K(\infty)G(\infty) \text{ is invertible}$$

In the example $1 + 1 \times (-1) = 0$ Note that if G is strictly proper, this always holds.

Internal stability

DEF: A system is internally stable if all the transfer functions of the closed-loop system are stable

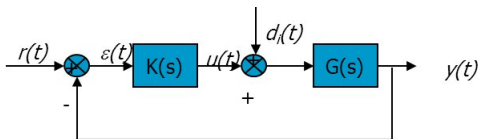


Figure: Control scheme

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} (I + GK)^{-1}GK & (I + GK)^{-1}G \\ K(I + GK)^{-1} & -K(I + GK)^{-1}G \end{pmatrix} \begin{pmatrix} r \\ d_i \end{pmatrix}$$

For instance :

$$G = \frac{1}{s-1}, \quad K = \frac{s-1}{s+1}, \quad \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} & \frac{s+1}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & -\frac{1}{s+2} \end{pmatrix} \begin{pmatrix} r \\ d_i \end{pmatrix}$$

There is one RHP pole (1), which means that this system is not internally stable. This is due here to the pole/zero cancellation (forbidden!!).

Small Gain theorem

Consider the so called $M - \Delta$ loop.

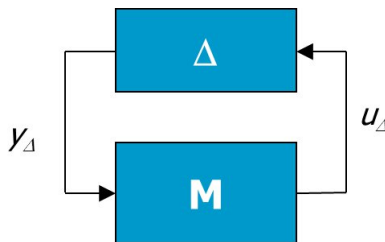


Figure: $M - \Delta$ form

Theorem

Suppose $M(s)$ in RH_∞ and γ a positive scalar. Then the system is well-posed and internally stable for all $\Delta(s)$ in RH_∞ such that $\|\Delta\|_\infty \leq 1/\gamma$ if and only if

$$\|M\|_\infty < \gamma$$

Input-Output Stability

Definition (BIBO stability)

A system G ($\dot{x} = Ax + Bu$; $y = Cx$) is **BIBO stable** if a bounded input $u(\cdot)$ ($\|u\|_\infty < \infty$) maps a bounded output $y(\cdot)$ ($\|y\|_\infty < \infty$).

Now, the quantification (for BIBO stable systems) of the signal amplification (gain) is evaluated as:

$$\gamma_{peak} = \sup_{0 < \|u\|_\infty < \infty} \frac{\|y\|_\infty}{\|u\|_\infty}$$

and is referred to as the **PEAK TO PEAK Gain**.

Definition (\mathcal{L}_2 stability)

A system G ($\dot{x} = Ax + Bu$; $y = Cx$) is **\mathcal{L}_2 stable** if $\|u\|_2 < \infty$ implies $\|y\|_2 < \infty$.

Now, the quantification of the signal amplification (gain) is evaluated as:

$$\gamma_{energy} = \sup_{0 < \|u\|_2 < \infty} \frac{\|y\|_2}{\|u\|_2}$$

and is referred to as the **ENERGY Gain**, and is such that:

$$\gamma_{energy} = \sup \omega \|G(j\omega)\| := \|G\|_\infty$$

For a linear system, these stability definitions are equivalent (but not the quantification criteria).

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Brief on optimisation

Definition (Convex function)

A function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^m$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y) \quad (16)$$

Equivalently, f is convex if and only if its epigraph,

$$\mathbf{epi}(f) = \{(x, \lambda) | f(x) \leq \lambda\} \quad (17)$$

is convex.

Definition ((Strict) LMI constraint)

A Linear Matrix Inequality constraint on a vector $x \in \mathbb{R}^m$ is defined as,

$$F(x) = F_0 + \sum_{i=1}^m F_i x_i \succeq 0 (\succ 0) \quad (18)$$

where $F_0 = F_0^T$ and $F_i = F_i^T \in \mathbb{R}^{n \times n}$ are given, and symbol $F \succeq 0 (\succ 0)$ means that F is symmetric and positive semi-definite ($\succeq 0$) or positive definite ($\succ 0$), i.e. $\{\forall u | u^T F u (>) \geq 0\}$.

Convex to LMIs

Example

Lyapunov equation. A very famous LMI constraint is the Lyapunov inequality of an autonomous system $\dot{x} = Ax$. Then the stability LMI associated is given by,

$$\begin{array}{rcl} x^T P x & > & 0 \\ x^T (A^T P + P A) x & < & 0 \end{array} \quad (19)$$

which is equivalent to,

$$F(P) = \begin{bmatrix} -P & 0 \\ 0 & A^T P + P A \end{bmatrix} \prec 0 \quad (20)$$

where $P = P^T$ is the decision variable. Then, the inequality $F(P) \prec 0$ is linear in P .

LMI constraints $F(x) \succeq 0$ are convex in x , i.e. the set $\{x | F(x) \succeq 0\}$ is convex. Then LMI based optimization falls in the convex optimization. This property is fundamental because it guarantees that the global (or optimal) solution x^* of the the minimization problem under LMI constraints can be found efficiently, in a polynomial time (by optimization algorithms like e.g. Ellipsoid, Interior Point methods).

LMI problem

Two kind of problems can be handled

Feasibility: The question whether or not there exist elements $x \in X$ such that $F(x) < 0$ is called a feasibility problem. The LMI $F(x) < 0$ is called feasible if such x exists, otherwise it is said to be infeasible.

Optimization: Let an objective function $f : S \rightarrow R$ where $S = \{x | F(x) < 0\}$. The problem to determine

$$V_{opt} = \inf_{x \in S} f(x)$$

is called an optimization problem with an LMI constraint. This problem involves the determination of V_{opt} , the calculation of an almost optimal solution x (i.e., for arbitrary $\epsilon > 0$ the calculation of an $x \in S$ such that $V_{opt} \leq f(x) \leq V_{opt} + \epsilon$, or the calculation of a optimal solutions x_{opt} (elements $x_{opt} \in S$ such that $V_{opt} = f(x_{opt})$)).

Examples of LMI problem

Stability analysis is a *feasibility* problem.

LQ control is an optimization problem, formulated as:

LQ control

Consider a controllable system $\dot{x} = Ax + Bu$. Find a state feedback $u(t) = -Kx(t)$ s.t $J = \int_0^\infty (x^T Qx + u^T Ru)dt$ is minimum (given $Q > 0$ and $R > 0$) is an optimisation problem whose solution is obtained solving the Riccati equation:

$$\text{Find } P > 0, \text{ s.t. } A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and then the state feedback is given by:

$$u(t) = -R^{-1}B^T Px(t)$$

which is equivalent to: find $P > 0$ s.t

$$\begin{bmatrix} A^T P + PA + Q & PB \\ B^T P & R \end{bmatrix} > 0$$

Semi-Definite Programming (SDP) Problem

LMI programming is a generalization of the Linear Programming (LP) to cone positive semi-definite matrices, which is defined as the set of all symmetric positive semi-definite matrices of particular dimension.

Definition (SDP problem)

A SDP problem is defined as,

$$\begin{array}{ll} \min & c^T x \\ \text{under constraint} & F(x) \succeq 0 \end{array} \quad (21)$$

where $F(x)$ is an affine symmetric matrix function of $x \in \mathbb{R}^m$ (e.g. LMI) and $c \in \mathbb{R}^m$ is a given real vector, that defines the problem objective.

SDP problems are theoretically tractable and practically:

- ▶ They have a polynomial complexity, i.e. there exists an algorithm able to find the global minimum (for a given a priori fixed precision) in a time polynomial in the size of the problem (given by m , the number of variables and n , the size of the LMI).
- ▶ SDP can be practically and efficiently solved for LMIs of size up to 100×100 and $m \leq 1000$ see [ElGhaoui, 97](#). Note that today, due to extensive developments in this area, it may be even larger.

The state feedback design problem

Stabilisation

Let us consider a controllable system $\dot{x} = Ax + Bu$. The problem is to find a state feedback $u(t) = -Kx(t)$ s.t the closed-loop system is stable.

Using the Lyapunov theorem, this amounts at finding $\mathbf{P} = \mathbf{P}^T > 0$ s.t:

$$\begin{aligned} (A - BK)^T \mathbf{P} + \mathbf{P}(A - BK) &< 0 \\ \Leftrightarrow A^T \mathbf{P} + \mathbf{P}A - K^T B^T P - PBK &< 0 \end{aligned}$$

which is obviously not linear...

Solution; use of change of variables

First, left and right multiplication by \mathbf{P}^{-1} leads to

$$\begin{aligned} \mathbf{P}^{-1}A^T + A\mathbf{P}^{-1} - \mathbf{P}^{-1}K^T B^T - BK\mathbf{P}^{-1} &< 0 \\ \Leftrightarrow \mathbf{Q}A^T + A\mathbf{Q} + \mathbf{Y}^T B^T + B\mathbf{Y} &< 0 \end{aligned}$$

with $\mathbf{Q} = \mathbf{P}^{-1}$ and $\mathbf{Y} = -K\mathbf{P}^{-1}$.

The problem to be solved is therefore formulated as an LMI ! and without any conservatism !

The Bounded Real Lemma

The \mathcal{L}_2 -norm of the output z of a system Σ_{LTI} is uniformly bounded by γ times the \mathcal{L}_2 -norm of the input w (initial condition $x(0) = 0$).

A dynamical system $G = (A, B, C, D)$ is internally stable and with an $\|G\|_\infty < \gamma$ if and only if there exists a positive definite symmetric matrix \mathbf{P} (i.e. $\mathbf{P} = \mathbf{P}^T > 0$ s.t

$$\begin{bmatrix} A^T \mathbf{P} + \mathbf{P} A & \mathbf{P} B & C^T \\ B^T \mathbf{P} & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0, \quad \mathbf{P} > 0. \quad (22)$$

The Bounded Real Lemma (BRL), can also be written as follows (see Scherer)

$$\begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix}^T \left[\begin{array}{cc|cc} 0 & \mathbf{P} & 0 & 0 \\ \mathbf{P} & 0 & 0 & 0 \\ \hline 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & 0 & I \end{array} \right] \begin{bmatrix} I & 0 \\ A & B \\ 0 & I \\ C & D \end{bmatrix} \prec 0 \quad (23)$$

Note that the BRL is an LMI if the only unknown (decision variables) are \mathbf{P} and γ (or γ^2).

Quadratic stability

This concept is very useful for the stability analysis of uncertain systems.
Let us consider an uncertain system

$$\dot{x} = A(\delta)x$$

where δ is an parameter vector that belongs to an uncertainty set Δ .

Definition

The considered system is said to be quadratically stable for all uncertainties $\delta \in \Delta$ if there exists a (single) "Lyapunov function" $\mathbf{P} = \mathbf{P}^T > 0$ s.t

$$A(\delta)^T \mathbf{P} + \mathbf{P} A(\delta) < 0, \text{ for all } \delta \in \Delta \quad (24)$$

This is a sufficient condition for ROBUST Stability which is obtained when $A(\delta)$ is stable for all $\delta \in \Delta$.

Schur lemma

Lemma

Let $Q = Q^T$ and $R = R^T$ be affine matrices of compatible size, then the condition

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succeq 0 \quad (25)$$

is equivalent to

$$\begin{array}{ccc} Q - SR^{-1}S^T & \succeq & 0 \\ & \underline{\succeq} & 0 \end{array} \quad (26)$$

The Schur lemma allows to convert a quadratic constraint (ellipsoidal constraint) into an LMI constraint.

Kalman-Yakubovich-Popov lemma

Lemma

For any triple of matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$,
 $M \in \mathbb{R}^{(n+m) \times (n+m)} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, the following assessments are equivalent:

1. There exists a symmetric $K = K^T \succ 0$ s.t.

$$\begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix} + M < 0$$

2. $M_{22} < 0$ and for all $\omega \in \mathbb{R}$ and complex vectors $\text{col}(x, w) \neq 0$

$$\begin{bmatrix} A - j\omega I & B \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} x \\ w \end{bmatrix}^T M \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

3. If $M = - \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}^T \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A & B \end{bmatrix}$ then the second statement is equivalent to the condition that, for all $\omega \in \mathbb{R}$ with $\det(j\omega I - A) \neq 0$,

$$\begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} I \\ C(j\omega I - A)^{-1}B + D \end{bmatrix} > 0$$

This lemma is used to convert frequency inequalities into Linear Matrix Inequalities.

Projection Lemma

Lemma

For given matrices $W = W^T$, M and N , of appropriate size, there exists a real matrix $K = K^T$ such that,

$$W + MKN^T + NK^TM^T \prec 0 \quad (27)$$

if and only if there exist matrices U and V such that,

$$\begin{aligned} W + MU + U^TM^T &\prec 0 \\ W + NV + V^TN^T &\prec 0 \end{aligned} \quad (28)$$

or, equivalently, if and only if,

$$\begin{aligned} M_{\perp}^T W M_{\perp} &\prec 0 \\ N_{\perp}^T W N_{\perp} &\prec 0 \end{aligned} \quad (29)$$

where M_{\perp} and N_{\perp} are the orthogonal complements of M , N respectively (i.e. $M_{\perp}^T M = 0$).

The projection lemma is also widely used in control theory. It allows to eliminate variable by a change of basis (projection in the kernel basis). It is involved in one of the \mathcal{H}_{∞} solutions ?)see e.g..

Completion Lemma

Lemma

Let $X = X^T, Y = Y^T \in \mathbb{R}^{n \times n}$ such that $X > 0$ and $Y > 0$. The three following statements are equivalent:

1. There exist matrices $X_2, Y_2 \in \mathbb{R}^{n \times r}$ and $X_3, Y_3 \in \mathbb{R}^{r \times r}$ such that,

$$\begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} \succ 0 \text{ and } \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} \quad (30)$$

$$2. \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \text{ and } \text{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n + r$$

$$3. \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succeq 0 \text{ and } \text{rank}[XY - I] \leq r$$

This lemma is useful for solving LMIs. It allow to simplify the number of variables when a matrix and its inverse enter in a LMI.

Finsler's lemma

This Lemma allows the elimination of matrix variables.

Lemma

The following statement are equivalent

- ▶ $x^T A x < 0$ for all $x \neq 0$ s.t: $Bx = 0$
- ▶ $\tilde{B}^T A \tilde{B} < 0$ where $B \tilde{B} = 0$
- ▶ $A + \lambda B^T B < 0$ for some scalar λ
- ▶ $A + X B + B^T X^T < 0$ for some matrix X
- ▶ $B^\perp{}^T A B^\perp < 0$

Interest of LMIs

LMIs allow to formulate complex optimization problems into "Linear" ones, allowing the use of convex optimization tools.

Usually it requires the use of different transformations, changes of variables ... in order to linearize the considered problems: Congruence, Schur complement, projection lemma, Elimination lemma, S-procedure, Finsler's lemma ...

Examples of handled criteria

- ▶ stability
- ▶ H_∞ , H_2 , H_2/H_∞ performances
- ▶ robustness analysis: Small gain theorem, Polytopic uncertainties, LFT representations...
- ▶ Robust control and/or observer design
- ▶ pole placement
- ▶ stability, stabilization with input constraints
- ▶ Passivity constraints
- ▶ Time-delay systems

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