Discretization of Nonautonomous Nonlinear Systems Based on Continualization of an Exact Discrete-Time Model

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An innovative approach is proposed for generating discrete-time models of a class of continuous-time, nonautonomous, and nonlinear systems. By continualizing a given discrete-time system, sufficient conditions are presented for it to be an exact model of a continuous-time system for any sampling periods. This condition can be solved exactly for linear and certain nonlinear systems, in which case exact discrete-time models can be found. A new model is proposed by approximately solving this condition, which can always be found as long as a Jacobian matrix of the nonlinear system exists. As an example of the proposed method, a van der Pol oscillator driven by a forcing sinusoidal function is discretized and simulated under various conditions, which show that the proposed model tends to retain such key features as limit cycles and space-filling oscillations even for large sampling periods, and out-performs the forward difference model, which is a well-known, widely-used, and on-line computable model. [DOI: 10.1115/1.4025711]

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1 Introduction

A large number of results have been reported on the study of nonlinear oscillations under forcing inputs, which exhibit such interesting phenomena as bifurcation, period doubling, strange attractors, and chaos [1,2]. Since analytical solutions to general nonautonomous nonlinear systems are not available, numerical investigations are important avenues to pursue. This calls for discretization of continuous-time plants themselves and of control systems designed for them [3–9]. While accurate discretization methods are available for off-line simulations, those that lead to on-line computable algorithms, such as those used for digital control, are still relatively rare. Of those, the simplest and widely-applicable discretization method is the forward-difference model, whose form and parameter values are the same as those of the continuous-time system and only the operation of differentiation is replaced with its Euler discrete-time equivalent [10]. Although very simple and widely applicable to most systems, this model has a performance that is usually poor even for linear cases unless a sufficiently small sampling interval is used. This is true also for nonlinear digital control systems that are designed based on the forward-difference model, which can complicate a subsequent digital controller design in an effort to take the discretization error into account [7]. A bilinearization technique is shown in Ref. [8] to be useful for discretizing some important classes of nonlinear autonomous systems, although it does not necessarily yields an exact discrete-time model. A nonstandard discretization method is proposed in Ref. [9], which seems to be applicable only to autonomous cases for nonlinear systems. In all these methods, a discrete-time model is obtained by discretizing a given continuous-time system.

The present study proposes a new discretization method that uses a relationship between continuous-time systems and their discrete-time models. This relationship is obtained by starting from a discrete-time system and continualizing it to have a continuous-time system to which the discrete-time system is exact at discrete-time instants. When the discrete-time model is determined to satisfy a condition for this system to approach the original continuous-time system at the limit of sampling interval approaching zero, an exact discrete-time model can be obtained. When the condition is satisfied approximately, an approximate discrete-time model may be obtained. Either way, the method is applicable to nonautonomous nonlinear systems, including autonomous systems as a special case.

The paper is organized as follows: In Sec. 2, some definitions are presented to clarify the meanings of discretization and continualization of signals and systems used throughout the paper. Section 3 develops an expression of a continuous-time system based on information of a given discrete-time system, to which the discrete-time model is exact. A new discrete-time model is proposed, then, by solving this expression approximately. Section 4 examines the proposed method as applied to a van der Pol oscillator under excitation by a sinusoidal input and presents simulation results to illustrate its performance in terms of phase-plane traces and time responses. Conclusions are given in Sec. 5.

2 Preliminaries on Discretization and Continualization

Let a continuous-time model of a forced nonlinear oscillator be given by the following state space equation:

$$\frac{d\bar{\mathbf{x}}(t)}{dt} = \bar{\boldsymbol{\Gamma}}(\bar{\mathbf{x}}(t), t) = \bar{\mathbf{f}}(\bar{\mathbf{x}}(t)) + \bar{\mathbf{g}}(t),\ \bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0 \tag{1}$$

where $\bar{\mathbf{x}} \in R^n$ is a state vector of continuous time variable t, $\bar{\mathbf{f}} : R^n \to R^n$ is a function of the state vector, and $\bar{\mathbf{g}} \in R^n$ is a known forcing function. Function $\bar{\mathbf{f}}$ is assumed to be expandable into Taylor series, so that $\bar{\boldsymbol{\Gamma}}$ satisfies the Lipschitz condition and Eq. (1) has a unique solution for a given initial condition.

For the continuous-time system (1), a number of discrete-time systems can be associated. In the present study, they are expressed in delta form [6] with a uniform discrete-time period of T, as

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$$\delta \mathbf{x}_k = \frac{\mathbf{x}_{k+1} - \mathbf{x}_k}{T} = \mathbf{\Gamma}(\mathbf{x}_k, T), \quad \mathbf{x}_{k_0} = \bar{\mathbf{x}}_0 \tag{2}$$

where $\mathbf{x}_k = \mathbf{x}(kT) \in R^n$ is the discrete-time state vector and δ is the delta operator defined as $\delta = (q-1)/T$ with q being the shift-left operator such that $q\mathbf{x}_k = \mathbf{x}_{k+1}$. Since the present paper will be paying a close attention to relationships that exist between the continuous-time system and its discrete-time models, the delta operator form will be more convenient than the more conventional shift form [10]. It is assumed that the initial time k_0 is synchronized between the continuous-time and the discrete-time systems, such that

$$t_0 = k_0 T \tag{3}$$

Given an appropriate initial condition, the discrete-time Eq. (2) has a unique solution as long as $\mathbf{\Gamma}$ is defined for each of its arguments [11], which is a rather mild condition compared with the condition assumed on the continuous-time system.

Since discretization and continualization are the main topics of the present paper, some definitions related to them are presented for clarification.

Definition 1. *[Exact Discretization]. A discrete-time state $\mathbf{x}_k$ of system (2) is said to be an exact discretization of a continuous-time state $\bar{\mathbf{x}}(t)$ of system (1) if the following relationship holds for any k* and T [9]:

$$\mathbf{x}_k = \bar{\mathbf{x}}(kT) \tag{4}$$

In this case, a discrete-time system, whose state is $\mathbf{x}_k$, is said to be an exact discrete-time model of a continuous-time system, whose state is $\bar{\mathbf{x}}(t)$. □

The existence of an exact discrete-time model is guaranteed under the standard assumption of existence of a solution to Eq. (1) [9]. The state of an exact discrete-time model satisfies Eq. (4) for any T. When T is changed, $\mathbf{x}_k$ will represent a new discrete-time sequence.

The above definition is widely accepted as a proper discretization of a continuous-time signal and is sometimes called a sampled-data signal. However, discrete-time signals and systems are not always exact in the sense of Definition 1, but only "similar." This sense of similarity is accommodated in a more general definition given below:

Definition 2. *[Discretization]: The discrete-time state $\mathbf{x}_k$ is said to be a discretization of the continuous-time state $\bar{\mathbf{x}}(t)$ if the following relationship holds for any fixed instant τ [12]*:

$$\lim_{\substack{T \to 0 \\ kT \le \tau < (k+1)T}} \mathbf{x}_k = \bar{\mathbf{x}}(\tau) \tag{5}$$

Such a discrete-time system is said to be a discrete-time model of the original continuous-time system (1). □

It should be noted that time instant τ is fixed and kT is varied as T is changed, so the above definition uses a point-wise convergence. In this definition, τ can be anywhere between the two successive sampling instants and T approaches zero continuously. This is more general than the fixed-station-convergence used in Ref. [3], where T is limited to be such that τ/T is integer as it approaches zero.

A process that can be considered as a sort of inverse operation of discretization is continualization, which is the role of hold devices, such as a zero-order-hold (ZOH) used in digital control. The definition proposed below is a generalization of this concept and will play a key role in the development of new discretization methods. It is more general in the sense that this does not have to be on-line computable, but is used to clarify conditions at the limit of T approaching zero.

Definition 3. *[Signal Continualization]: Given the discrete-time state $\mathbf{x}_k$ of Eq. (2), the following continuous-time signal $\bar{\mathbf{x}}^*(t)$ is said to be a continualization of $\mathbf{x}_k$: In each interval $kT \le t < (k+1)T$*

$$\bar{\mathbf{x}}^*(t) = \bar{\mathbf{x}}^*(kT) + (t - kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t - kT) \tag{6}$$

where $\bar{\mathbf{x}}^*(kT) = \mathbf{x}_k$. □

Remark 1. The discrete-time state of system (2) is an exact discretization of the continualized signal (6), since $\bar{\mathbf{x}}^*(kT) = \mathbf{x}_k$ for any k and T. However, there are a number of continuous-time signals that can pass through the same discrete-time sequence for a finite T. It should be noted that this continuous-time signal is a function of both t and T.

The concept of a continualized signal can be extended to a system, as follows:

Definition 4. *[System Continualization]: Using $\mathbf{\Gamma}$ of a given discrete-time system (2), the continuous-time system given by*

$$\dot{\bar{\mathbf{x}}}^*(t) = \bar{\mathbf{\Gamma}}^*(\bar{\mathbf{x}}^*(t), t) = \frac{d}{dt}((t - kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t - kT)) \tag{7}$$

where $\bar{\mathbf{x}}^*(t)$ is generated by Eq. (6) in each $kT \le t < (k+1)T$, is said to be the continualized system of discrete-time system (2). □

Remark 2. Since $\mathbf{x}_k = \bar{\mathbf{x}}^*(kT)$ for all positive k and T, which satisfies Definition 1, the discrete-time system given by Eq. (2) is an exact discrete-time model of the continualized system (7), but not necessarily of system (1).

3 New Discretization Techniques

Theorem 1. *[Exact Discretization]: A discrete-time system given by Eq. (2) is an exact discrete-time model of system (1) if function $\mathbf{\Gamma}(\mathbf{x}_k, T)$ satisfies the following three conditions*:

(i) $\partial\mathbf{\Gamma}((\mathbf{x}_k, s)/\partial\mathbf{x}_k)$, $\partial/\partial\mathbf{x}_k(\partial(\mathbf{\Gamma}(\mathbf{x}_k, s))/\partial s)$ are continuous functions of $\mathbf{x}_k$, where $s \in \mathbf{R}^+$

(ii) $s(\partial\mathbf{\Gamma}(\mathbf{x}_k, s)/\partial\mathbf{x}_k) \ne -1$

(iii) $$\frac{d}{dt}((t-kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) = \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT) + (t-kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) + \bar{\mathbf{g}}(t) \tag{8}$$

for each interval $kT \le t < (k+1)T$.

The proof of the above theorem will be shown using the following lemma:

Lemma 1. *When the conditions (i) and (ii) are met, the state $\bar{\mathbf{x}}^*(t)$ that satisfies Eq. (6) is a unique solution of Eq. (7) in each interval $kT \le t < (k+1)T$.*

Proof. A sufficient condition for Eq. (7) to have a unique solution in each interval is that $\bar{\mathbf{\Gamma}}^*(\bar{\mathbf{x}}^*(t), t)$ is a continuous function of $\bar{\mathbf{x}}^*$. Since

$$\frac{\partial \bar{\mathbf{\Gamma}}^*(\bar{\mathbf{x}}^*(t), t)}{\partial \bar{\mathbf{x}}^*(t)} = \frac{\dfrac{\partial}{\partial \bar{\mathbf{x}}^*(kT)} \dfrac{\partial}{\partial t}((t-kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT))}{\dfrac{\partial[\bar{\mathbf{x}}^*(kT) + (t-kT)\mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)]}{\partial \bar{\mathbf{x}}^*(kT)}}$$
$$= \frac{\dfrac{\partial \mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)}{\partial \bar{\mathbf{x}}^*(kT)} + (t-kT)\dfrac{\partial}{\partial \bar{\mathbf{x}}^*(kT)} \dfrac{\partial \mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)}{\partial t}}{1 + (t-kT)\dfrac{\partial \mathbf{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)}{\partial \bar{\mathbf{x}}^*(kT)}} \tag{9}$$

$\bar{\mathbf{\Gamma}}^*(\bar{\mathbf{x}}^*(t), t)$ is a continuous function of $\bar{\mathbf{x}}^*$ if conditions (i) and (ii) are met. □

Proof of Theorem 1. From Remark 2 and Lemma 1, discrete-time system (2) is an exact discretization of continualized system (7). Condition (iii) implies that, for each $kT \le t < (k+1)T$, the state $\bar{\mathbf{x}}^*(t)$ of the continualized system (7) is a unique solution of the original continuous-time system (1). □

Remark 3. Condition (iii) is a requirement that function $\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)$ be such that Eq. (7) equals Eq. (1); that is, using Eq. (6), in each interval

$$\begin{aligned}\frac{d}{dt}((t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) &= \dot{\bar{\mathbf{x}}}^*(t) \\ &= \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(t)) + \bar{\mathbf{g}}(t) \\ &= \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT) + (t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) + \bar{\mathbf{g}}(t)\end{aligned} \tag{10}$$

When Eq. (8) can be solved for $\boldsymbol{\Gamma}$, the exact discrete-time model is found. However, Eq. (8) is a nonlinear differential equation, which is generally unsolvable analytically. By solving it approximately, an approximate discrete-time model may be obtained. One such model is proposed below, which is applicable to a class of non-autonomous nonlinear and linear systems, as long as they have a Jacobian matrix.

THEOREM 2. *[The Proposed Model]: A discrete-time system given by Eq. (2) where function $\boldsymbol{\Gamma}$ is chosen as*

$$\begin{aligned}\boldsymbol{\Gamma}(\mathbf{x}_k, T) &= \frac{1}{T}\left(\int_0^T e^{\left[D\bar{\mathbf{f}}(\mathbf{x}_k)\right](T-\lambda)} d\lambda\right)\bar{\mathbf{f}}(\mathbf{x}_k) \\ &\quad + \frac{1}{T}\int_0^T e^{\left[D\bar{\mathbf{f}}(\mathbf{x}_k)\right](T-\lambda)}\bar{\mathbf{g}}(\lambda+kT)d\lambda\end{aligned} \tag{11}$$

with $D\bar{\mathbf{f}}(\mathbf{x}_k)$ being a Jacobian matrix of $\bar{\mathbf{f}}$ at $\mathbf{x}_k$, is a discrete-time model, in the sense of Definition 2, of the continuous-time system given by Eq. (1).

Proof. Equation (6) with $\boldsymbol{\Gamma}$ given by Eq. (11) holds for a fixed time $t=\tau$ in each sampling interval such that

$$\begin{aligned}\frac{\bar{\mathbf{x}}^*(\tau)-\bar{\mathbf{x}}^*(kT)}{\tau-kT} &= \frac{1}{\tau-kT}\left(\int_0^{\tau-kT} e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\tau-kT-\lambda)}d\lambda\right)\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) \\ &\quad + \frac{1}{\tau-kT}\int_0^{\tau-kT} e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\tau-kT-\lambda)}\bar{\mathbf{g}}(\lambda+kT)d\lambda\end{aligned} \tag{12}$$

and this also holds at the limit of T approaching zero while k is chosen such that $kT \le \tau < (k+1)T$. Thus, noting that, for a fixed τ

$$\lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} kT = \tau \tag{13}$$

and $\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(\tau))$ and $\bar{\mathbf{g}}(\tau)$ are finite, the use of l'Hospital's Rule on the right-hand-side of Eq. (12) yields

$$\begin{aligned}&\lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}}\left[\frac{1}{\tau-kT}\left(\int_0^{\tau-kT} e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\tau-kT-\lambda)}d\lambda\right)\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right] \\ &+ \lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}}\left[+\frac{1}{\tau-kT}\int_0^{\tau-kT} e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\tau-kT-\lambda)}\bar{\mathbf{g}}(\lambda+kT)d\lambda\right] \\ &\quad = \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(\tau)) + \bar{\mathbf{g}}(\tau)\end{aligned} \tag{14}$$

The left-hand-side, on the other hand, gives

$$\lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} \frac{\bar{\mathbf{x}}^*(\tau)-\bar{\mathbf{x}}^*(kT)}{\tau-kT} = \frac{d\bar{\mathbf{x}}^*(\tau)}{d\tau} \tag{15}$$

Therefore, at the limit of T approaching zero and for all τ such that $kT \le \tau < (k+1)T$, Eq. (12) gives

$$\frac{d\bar{\mathbf{x}}^*(\tau)}{d\tau} = \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(\tau)) + \bar{\mathbf{g}}(\tau) \tag{16}$$

Since Eq. (1) has a unique solution given an initial condition, it follows that

$$\lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} \bar{\mathbf{x}}^*(\tau) = \bar{\mathbf{x}}(\tau) \tag{17}$$

Therefore, $\mathbf{x}_k = \bar{\mathbf{x}}^*(kT)$, Eqs. (13) and (17) lead, for any τ with $kT \le \tau < (k+1)T$, to

$$\lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} \mathbf{x}_k = \lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} \bar{\mathbf{x}}^*(kT) = \lim_{\substack{T\to 0 \\ kT\le\tau<(k+1)T}} \bar{\mathbf{x}}^*(\tau) = \bar{\mathbf{x}}(\tau) \tag{18}$$

In view of Definition 2, system (2) is a discrete-time model of continuous-time system (1). □

Since the proposed model has been shown above to be a valid model in the sense of Definition 2, a more constructive procedure is shown below. When function $\bar{\mathbf{f}}$ in Eq. (8) is expanded into the Taylor series and truncated with the first two terms as

$$\begin{aligned}&\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT) + (t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) \\ &\quad = \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) + \left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)\end{aligned} \tag{19}$$

Equation (8) can always be solved and an approximate discrete-time model obtained. That is, for arbitrary $\bar{\mathbf{x}}^*(kT)$, Eq. (8) be expressed as

$$\begin{aligned}\frac{d}{dt}((t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT)) &= \left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right] \\ &\quad \times (t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), t-kT) + \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) + \bar{\mathbf{g}}(t)\end{aligned} \tag{20}$$

Defining ζ as $\zeta = t-kT$, Eq. (20) can be written as

$$\begin{aligned}\frac{d}{d\zeta}(\zeta\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), \zeta)) &= \left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right]\zeta\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), \zeta) + \bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) \\ &\quad + \bar{\mathbf{g}}(\zeta+kT)\end{aligned} \tag{21}$$

where $0 \le \zeta < T$. Noting that $\zeta\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), \zeta) = 0$ at $\zeta = 0$, a solution to the above linear differential equation in $\zeta\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), \zeta)$ gives the following continuous-time function [13]:

$$\begin{aligned}\zeta\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT), \zeta) &= \int_0^\zeta e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\zeta-\lambda)}\left(\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) + \bar{\mathbf{g}}(kT+\lambda)\right)d\lambda \\ &= \int_0^\zeta e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\zeta-\lambda)}d\lambda\cdot\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)) \\ &\quad + \int_0^\zeta e^{\left[D\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))\right](\zeta-\lambda)}\bar{\mathbf{g}}(kT+\lambda)d\lambda\end{aligned} \tag{22}$$

Adopting this form of function, the discrete-time function is obtained as

$$\begin{aligned}\boldsymbol{\Gamma}(\mathbf{x}_k,T)=&\frac{1}{T}\int_o^T e^{[D\mathbf{f}(\mathbf{x}_k)](T-\lambda)}d\lambda\cdot\mathbf{f}(x_k)\\&+\frac{1}{T}\int_o^T e^{[D\mathbf{f}(\mathbf{x}_k)](T-\lambda)}\bar{\mathbf{g}}(kT+\lambda)d\lambda\end{aligned} \tag{23}$$

which is Eq. (11).

Remark 4. When Eq. (8) can be solved exactly, an exact discrete-time model can be found. For instance for a linear system, the proposed method gives the exact discrete-time model; i.e., for $\bar{\boldsymbol{\Gamma}}$ in Eq. (1) given by

$$\bar{\boldsymbol{\Gamma}}(\bar{\mathbf{x}}(t),t)=\mathbf{A}\mathbf{x}+\bar{\mathbf{g}}(t) \tag{24}$$

where $\mathbf{A}$ is system matrix of compatible dimension, Eq. (8) can be written exactly as a linear differential equation as

$$\begin{aligned}&\frac{d}{dt}((t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT))-\mathbf{A}(t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT)\\&\quad-\mathbf{A}\mathbf{x}^*(kT)-\bar{\mathbf{g}}(t)=0\end{aligned} \tag{25}$$

whose solution is [14]

$$\begin{aligned}(t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT)=&\int_0^{t-kT}e^{\mathbf{A}(t-kT-\tau)}d\tau\cdot\mathbf{A}\mathbf{x}^*(kT)\\&+\int_0^{t-kT}e^{\mathbf{A}(t-kT-\tau)}\bar{\mathbf{g}}(kT+\tau)d\tau\end{aligned}$$

This leads to the exact discrete-time model [14] as

$$\boldsymbol{\Gamma}(\mathbf{x}_k,T)=\frac{1}{T}\int_0^T e^{\mathbf{A}(T-\tau)}d\tau\cdot\mathbf{A}\mathbf{x}_k+\frac{1}{T}\int_0^T e^{\mathbf{A}(T-\tau)}\bar{\mathbf{g}}(\tau+kT)d\tau \tag{26}$$

Remark 5. When the Taylor series expansion of $\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT)+(t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT))$ is truncated after the first term and $\mathbf{g}(t)$ is constant in the interval $[kT,(k+1)T)$, Eq. (8) yields $d((t-kT)\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT))/dt=\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))+\bar{\mathbf{g}}(kT)$, so that

$$\boldsymbol{\Gamma}(\bar{\mathbf{x}}^*(kT),t-kT)=\bar{\mathbf{f}}(\bar{\mathbf{x}}^*(kT))+\bar{\mathbf{g}}(kT) \tag{27}$$

This is known as the forward difference model.

Remark 6. When $\bar{\mathbf{g}}(t)$ is a stair-case function given by

$$\bar{\mathbf{g}}(t)=\alpha_k,\quad kT\le t<(k+1)T \tag{28}$$

integration of Eq. (23) gives

$$\boldsymbol{\Gamma}(\mathbf{x}_k,T)=\frac{1}{T}\int_0^T e^{[D\mathbf{f}(\mathbf{x}_k)](T-\tau)}d\tau\{\mathbf{f}(\mathbf{x}_k)+\alpha_k\} \tag{29}$$

Furthermore, when Jacobian matrix $D\mathbf{f}(\mathbf{x}_k)$ is nonsingular, Eq. (29) can be written as

$$\boldsymbol{\Gamma}(\mathbf{x}_k,T)=\frac{e^{[D\mathbf{f}(\mathbf{x}_k)]T}-\mathbf{I}}{T}[D\mathbf{f}(\mathbf{x}_k)]^{-1}\{\mathbf{f}(\mathbf{x}_k)+\alpha_k\} \tag{30}$$

Remark 7. When the Jacobian matrix is nonsingular and a specific form is given for $\bar{\mathbf{g}}(t)$, the model (23) can be written in a more specific form. For instance, for a sinusoidal function given by

$$\bar{\mathbf{g}}(t)=\mathbf{A}\cos(\omega t) \tag{31}$$

the second term in Eq. (23) can be integrated by parts. Noting $d^2\bar{\mathbf{g}}(t)/dt^2=-\omega^2\bar{\mathbf{g}}(t)$, it yields

$$\begin{aligned}\boldsymbol{\Gamma}(\mathbf{x}_k,T)=&\frac{e^{[D\mathbf{f}(\mathbf{x}_k)]T}-\mathbf{I}}{T}[D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{f}(x_k)\\&-\left(\omega^2\mathbf{I}+[D\mathbf{f}(\mathbf{x}_k)]^2\right)^{-1}\left\{D\mathbf{f}(\mathbf{x}_k)\left(\bar{\mathbf{g}}_{k+1}-e^{[D\mathbf{f}(\mathbf{x}_k)]T}\bar{\mathbf{g}}_k\right)\right.\\&\left.+\left(\dot{\bar{\mathbf{g}}}_{k+1}-e^{[D\mathbf{f}(\mathbf{x}_k)]T}\dot{\bar{\mathbf{g}}}_k\right)\right\}\end{aligned} \tag{32}$$

where $\bar{\mathbf{g}}_k$ and $\dot{\bar{\mathbf{g}}}_k$ are defined as

$$\bar{\mathbf{g}}_k=\bar{\mathbf{g}}(kT)=\mathbf{A}\cos(\omega kT),\quad \dot{\bar{\mathbf{g}}}_k=\dot{\bar{\mathbf{g}}}(kT)=-\mathbf{A}\omega\sin(\omega kT) \tag{33}$$

4 Discrete-Time Models of a Forced van der Pol Oscillator

Consider the forced van der Pol oscillator modeled by

$$\ddot{x}+x-\varepsilon(1-x^2)\dot{x}-A\cos(\omega t)=0 \tag{34}$$

where ε is a positive parameter, A the amplitude of the forcing function, and ω its angular velocity. This can be rewritten in the form of state-space equation given by

$$\frac{d\bar{\mathbf{x}}(t)}{dt}=\bar{\mathbf{f}}(\bar{\mathbf{x}}(t))+\bar{\mathbf{g}}(t) \tag{35}$$

where

$$\bar{\mathbf{x}}=\begin{bmatrix}x\\y\end{bmatrix} \tag{36}$$

$$\bar{\mathbf{f}}(\bar{\mathbf{x}}(t))=\begin{bmatrix}y\\-x+\varepsilon(1-x^2)y\end{bmatrix} \tag{37}$$

$$\bar{\mathbf{g}}(t)=\begin{bmatrix}0\\A\cos(\omega t)\end{bmatrix} \tag{38}$$

Table 1 Conditions used for the simulations

Conditions	Initial state	ε	A	ω	T
C1(a) Self-sustained oscillation	$x_0=-1$, $y_0=-1.5$	$\varepsilon=1.5$	$A=1$	$\omega=2$	$T=0.1$
C1(b) Quasi-periodic oscillation			$A=3$		
C1(c) Fundamental oscillation				$\omega=3$	
C1(d) Harmonic oscillation			$A=8$		
C2(a)		$\varepsilon=3$	$A=1$	$\omega=2$	
C2(b)		$\varepsilon=4$			
C2(c)		$\varepsilon=5$			
C3(a)		$\varepsilon=1.5$			$T=0.2$
C3(b)					$T=0.3$
C3(c)					$T=0.5$

This system reduces to the relaxation oscillator when $A = 0$, which yields a stable limit cycle. When the system is excited with $A \neq 0$, it exhibits various types of behaviors such as chaos and stable orbit with more than one closed cycle [15]. Jacobian matrix of $\bar{\mathbf{f}}$ for this system is non-singular and given by

$$D\bar{\mathbf{f}}(\bar{\mathbf{x}}) = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon xy & \varepsilon(1 - x^2) \end{bmatrix} \tag{39}$$

The proposed model: Using the discrete-time function given by Eq. (32), the proposed discrete-time model is obtained as

$$\begin{aligned} \delta\mathbf{x}_k = {} & \frac{e^{[D\mathbf{f}(\mathbf{x}_k)]T} - \mathbf{I}}{T}[D\mathbf{f}(\mathbf{x}_k)]^{-1}\mathbf{f}(x_k) \\ & - \left(\omega^2\mathbf{I} + [D\mathbf{f}(\mathbf{x}_k)]^2\right)^{-1}\Big\{D\mathbf{f}(\mathbf{x}_k)\left(\mathbf{g}_{k+1} - e^{[D\mathbf{f}(\mathbf{x}_k)]T}\mathbf{g}_k\right) \\ & + \left(\dot{\mathbf{g}}_{k+1} - e^{[D\mathbf{f}(\mathbf{x}_k)]T}\dot{\mathbf{g}}_k\right)\Big\} \end{aligned} \tag{40}$$

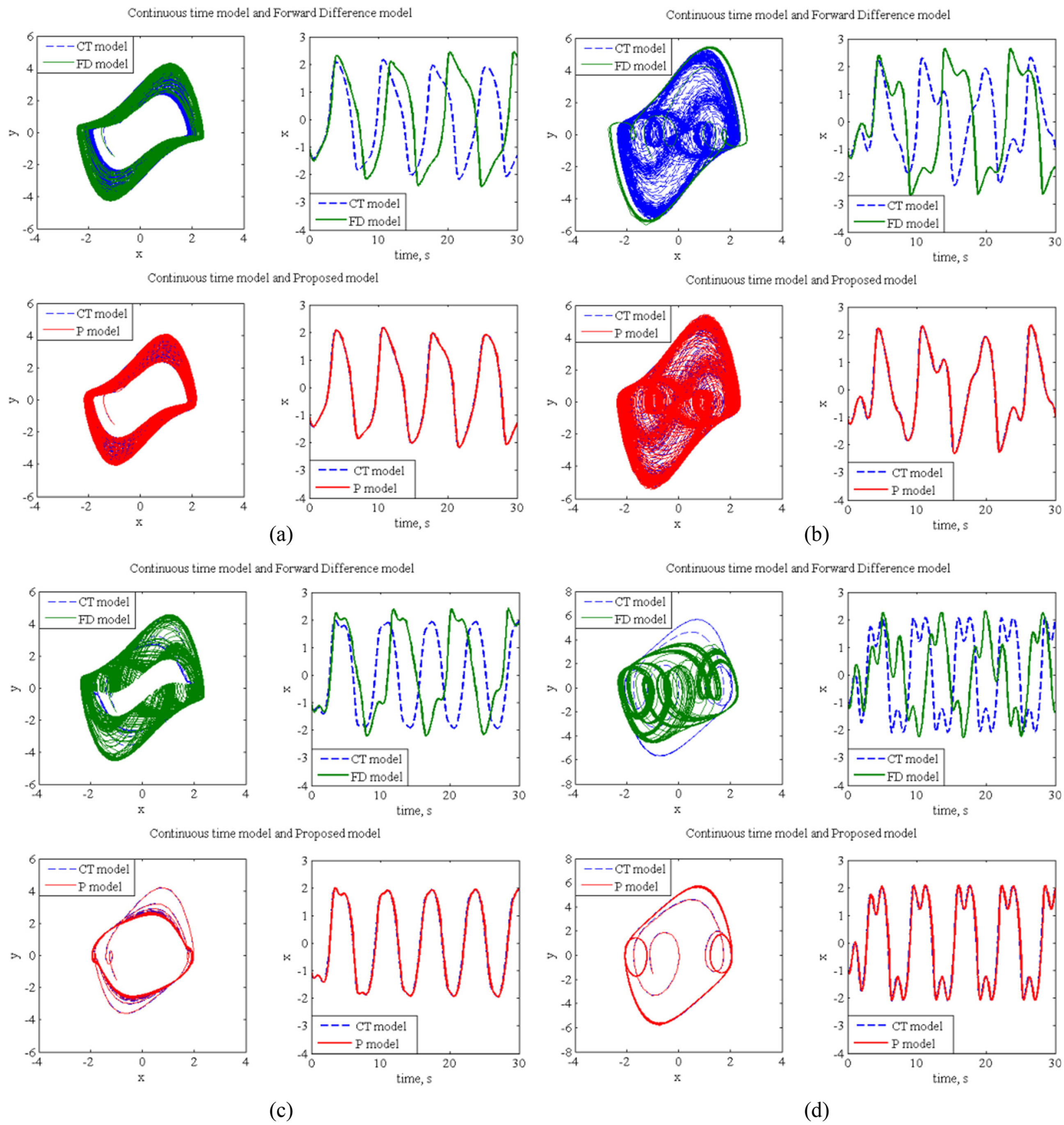

Fig. 1 (*a*) **Self-sustained oscillation (C1(s)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, $T = 0.1$. (*b*) Quasi-periodic oscillation (C(1(b)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 3$, $\omega = 2$, $T = 0.1$. (*c*) Fundamental oscillation (C1(c)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 3$, $\omega = 3$, $T = 0.1$. (*d*) Harmonic oscillation (C1(d)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 8$, $\omega = 3$, $T = 0.1$.**

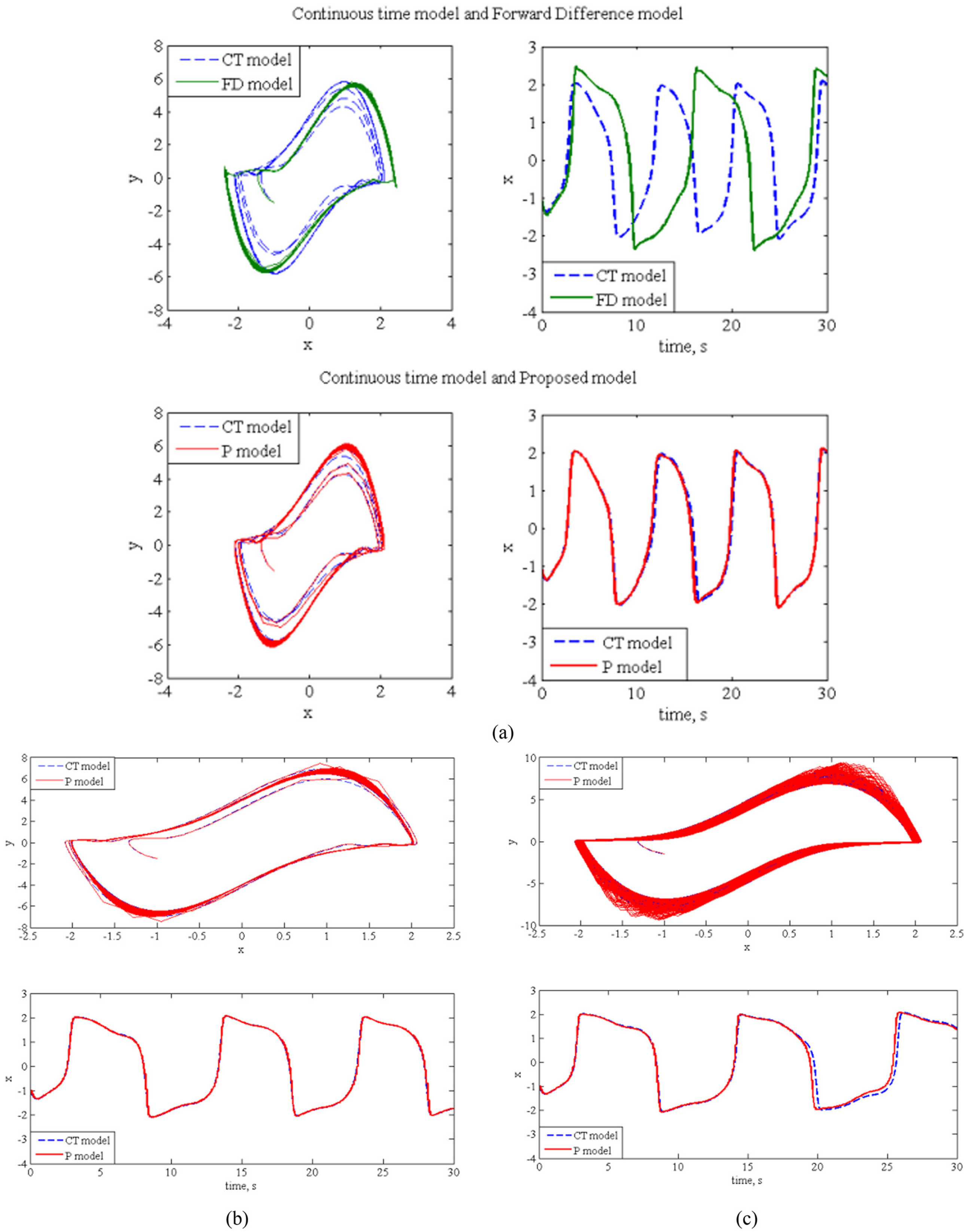

Fig. 2 (*a*) (C2(a)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 3$, $A = 1$, $\omega = 2$, $T = 0.1$. (*b*) (C2(b)): Phase plane and time response of the continuous-time and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 4$, $A = 1$, $\omega = 2$, $T = 0.1$. (*c*) (C2(c)): Phase plane and time response of the continuous-time, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 5$, $A = 1$, $\omega = 2$, $T = 0.1$.

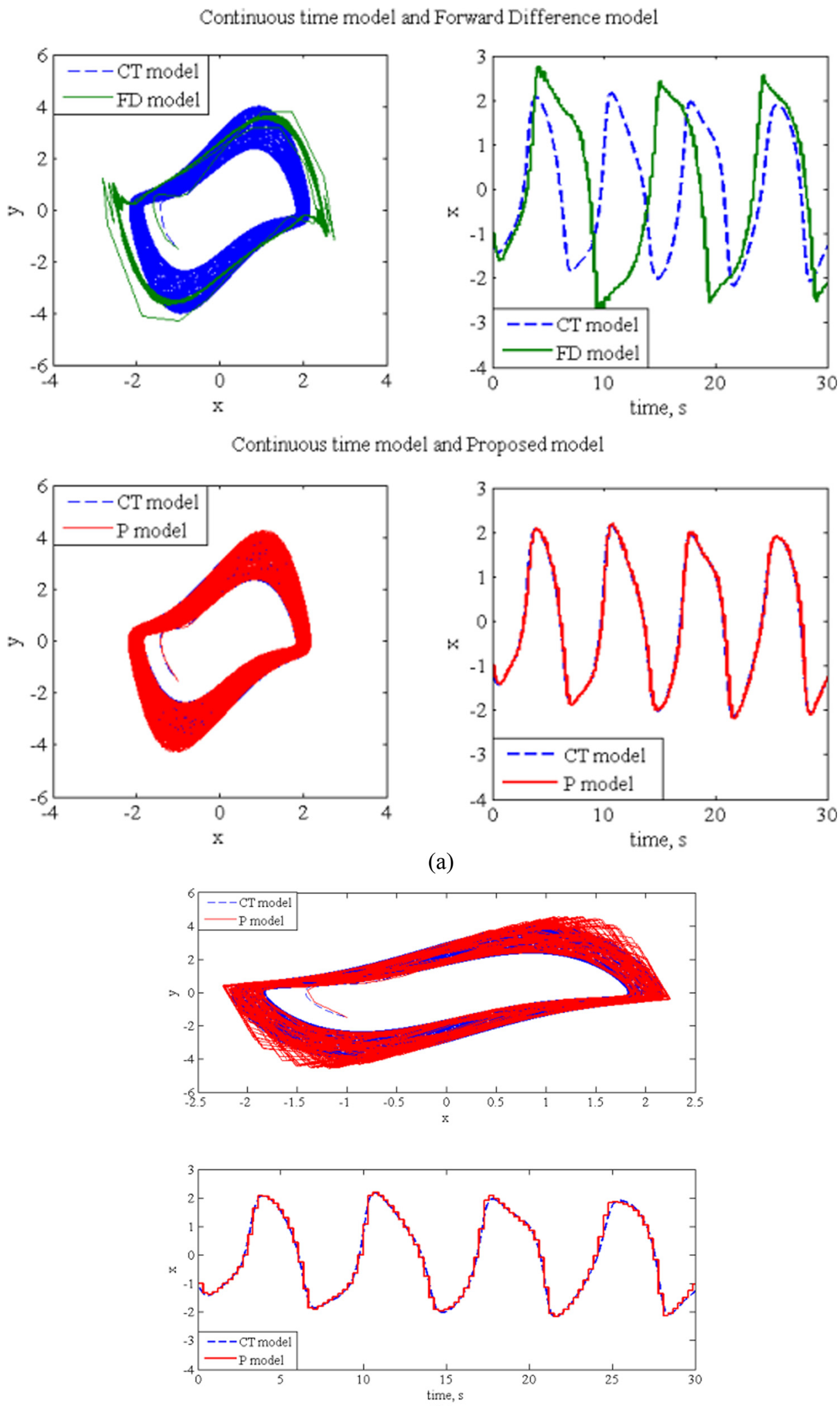

Fig. 3 (*a*) (C3(a)): Phase plane and time response of the continuous-time, the forward-difference, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, $T = 0.2$. (*b*) (C3(a)): Phase plane and time response of the continuous-time, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, $T = 0.3$.

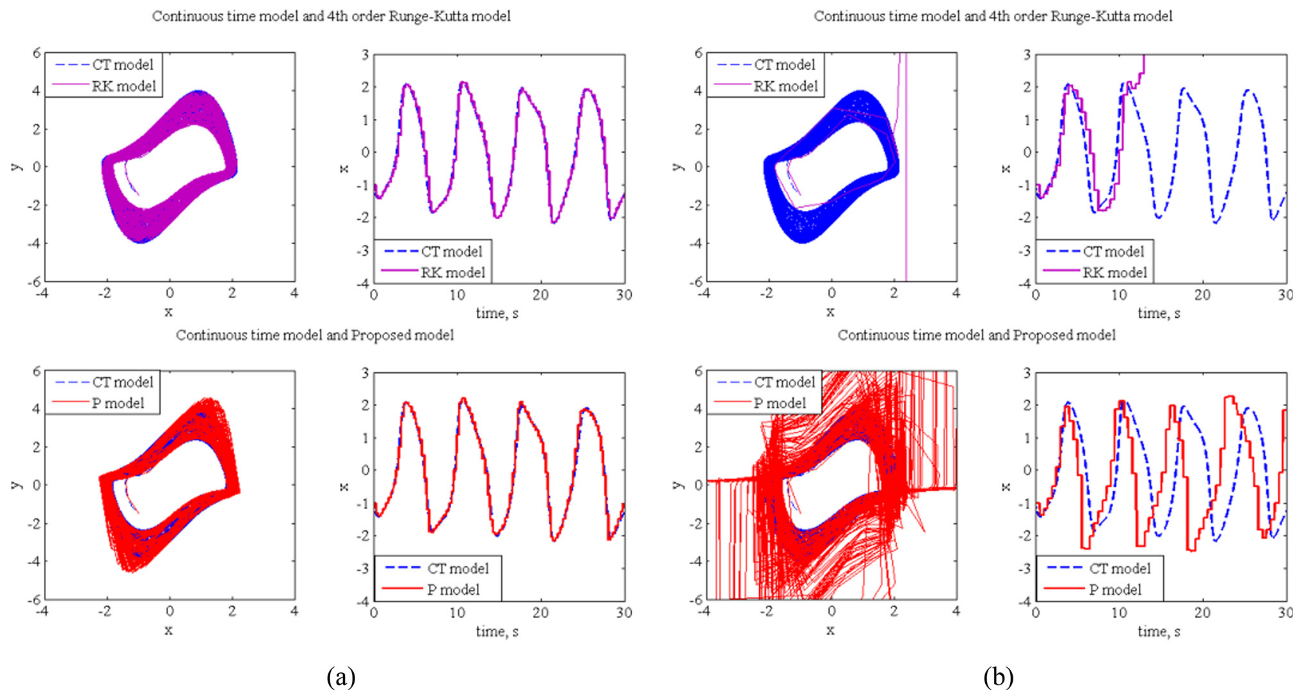

Fig. 4 **(*a*) (C3(b)): Phase plane and time response of the continuous-time, 4th order Runge-Kutta, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, $T = 0.3$. (*b*) (C3(c)): Phase plane and time response of the continuous-time, 4th order Runge-Kutta, and the proposed models, for $x_0 = -1$, $y_0 = -1.5$, $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, $T = 0.5$.**

where

$$\mathbf{g}_k = \bar{\mathbf{g}}(t)|_{t=kT} = \begin{bmatrix} 0 \\ A\cos(\omega kT) \end{bmatrix} \tag{41}$$

and

$$\dot{\mathbf{g}}_k = \dot{\bar{\mathbf{g}}}(t)|_{t=kT} = \begin{bmatrix} 0 \\ -A\omega\sin(\omega kT) \end{bmatrix} \tag{42}$$

The forward-difference model: This model is obtained as

$$\begin{bmatrix} \delta x_k \\ \delta x_k \end{bmatrix} = \begin{bmatrix} y_k \\ -x_k + \varepsilon(1 - x_k^2)y_k \end{bmatrix} + \begin{bmatrix} 0 \\ A\cos(\omega kT) \end{bmatrix} \tag{43}$$

Extensive simulations have been carried out for the forced van der Pol oscillator (34). While the unforced oscillator has a stable limit cycle, the forced oscillator exhibits several interesting phenomena, such as periodic and space-filling responses [15]. The conditions used for simulations are summarized in Table 1, where Cases C1(a)–C(d) cover basic four conditions; namely, self-sustained, quasi-periodic, fundamental, and harmonic oscillations.

All figures given below show typical phase-planes (for the first 1000 s) and time responses (for the first 30 s) starting from the initial condition of $x_0 = -1.0$ and $y_0 = -1.5$. Figures 1(*a*)–1(*d*) are for the case of $\varepsilon = 1.5$ and $T = 0.1$ s, covering four types of oscillations caused under different combinations of input amplitudes ($A = 1$, 3, 8) and frequencies ($\omega = 2$, 3). The proposed model and the forward-difference model are compared with those of the original continuous-time oscillator that is computed using ode45 (Dormand-Prince). The input is assumed to be applied through a ZOH in all discrete-time models.

To see if the proposed method gives consistently better results than others, simulations under different conditions have been carried out. For instance, Figs. 1(*a*) and 2(*a*)–2(*c*) are for the case of $A = 1$, $\omega = 2$, $T = 0.1$, covering different values of nonlinearity parameter $\varepsilon = 1.5$, 3, 4, 5. Figures 1(*a*), 3(*a*), and 3(*b*) are for the case of $\varepsilon = 1.5$, $A = 1$, $\omega = 2$, covering different values of $T = 0.1$, 0.2, 0.3. It can be seen that the forward-difference model is not capable of yielding satisfactory results, and becomes unstable for cases 2(b), 2(c), and 3(b). In contrast, the proposed model gives responses that are very close to those of the continuous-time model in all cases.

Comparisons have also been made between the proposed method and 4th-order Runge-Kutta (R-K), as shown in Figs. 4(*a*) and 4(*b*). It was found that the R-K method has a better performance than the proposed method up to about $T = 0.4$ s. However, when the sampling interval increases to 0.5 s, 4th order Runge-Kutta method suddenly becomes numerically unstable and the computations stops at about 13.7 s. However, the proposed method remains stable at this sampling interval. For the forced van der Pol system under consideration, both the bilinear method of Hirota [8] and the nonstandard method of Mickens [9] do not seem to be applicable.

5 Conclusions

Nonlinear nonautonomous discrete-time models and their relationships with continuous-time systems have been investigated such that a certain differential equation is obtained as a sufficient condition for the model to be exact. When the solution of this equation can be obtained exactly, such as linear and certain nonlinear systems, the discrete-time model will be exact. When an exact model is not known, the proposed model can always be obtained as an approximate model as long as a Jacobian matrix exists for the given continuous-time system. However, the model is not a linear approximation based on this Jacobian matrix, but a nonlinear approximation. The method is applied to a nonautonomous van der Pol oscillator driven by a sinusoidal input. Simulations show that the proposed model gives performances that are superior to the popular on-line computable method known as the forward-difference model.

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