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NONNEGATIVITY OF SOLUTIONS OF NONLINEAR FRACTIONAL DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract Nonlinear fractional differential-algebraic equations often arise in simulating integrated circuits with superconductors. How to obtain the nonnegative solutions of the equations is an important scientific problem. As far as we known, the nonnegativity of solutions of the nonlinear fractional differential-algebraic equations is still not studied. In this article, we investigate the nonnegativity of solutions of the equations. Firstly, we discuss the existence of nonnegative solutions of the equations, and then we show that the nonnegative solution can be approached by a monotone waveform relaxation sequence provided the initial iteration is chosen properly. The choice of initial iteration is critical and we give a method of finding it. Finally, we present an example to illustrate the efficiency of our method.

Key words Fractional differential-algebraic equations; nonnegativity of solutions; waveform relaxation; monotone convergence

2010 MR Subject Classification 26A33; 33E12

1 Introduction

Fractional calculus has been used widely to deal with some problems in fluid and continuum mechanics [1, 2], viscoelastic and viscoplastic flow [3], epidemiological models [4, 5], and circuit simulation with superconductor materials [6]. The main advantage of fractional derivatives lies in that they are more suitable for describing memory and hereditary properties of various materials and process in comparison with classical integer-order derivative. In these years, various theory and numerical solutions to fractional differential equations were extensively investigated. For example, collocation methods were applied into solving fractional differential equations ([7, 8]). Gong et al [9] gave an efficient parallel solution for Caputo fractional reaction-diffusion equation with explicit method. The parallel solution is implemented

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with MPI parallel programming model. Stokes et al [10] proposed a method to accelerate the computation of the numerical solution of fractional differential equations. Xu et al [11] applied parareal method into solving time-fractional differential equations. Mohammed Al-Refai and Yuri Luchko gave maximum principle for the multi-term time-fractional diffusion equations with the Riemann-Liouville fractional derivatives [12].

The investigation of positive solutions of different classes of fractional differential equations is a relevant question in real world prolems. For example, some authors in [13, 14] discussed the existence of positive solutions of nonlinear fractional differential equations. Some authors [15, 16] investigated positive solutions of fractional differential equations with integral boundary conditions and multi-point boundary conditions, respectively. Li [17] discussed the nonexistence of positive solution for a semi-linear equation involing the fractional Laplcian in $\mathbb{R}^{N^*}$. Wang et al [18] gave the existence of solutions for nonlinear fractional differential equations using monotone iterative method. In [19], Kaczorek discussed positive linear systems consisting of n subsystems with different orders, where the proposed system can be described by linear fractional differential-algebraic equations. However, as far as we known, nonnegativity of solutions of nonlinear fractional differential-algebraic equations is still not studied.

In this article, we consider nonnegativity of solutions of nonlinear fractional differential-algebraic equations using waveform relaxation (WR) algorithm. It is well known that the WR method is a dynamic iterative method. It was originally proposed to simulate large circuits in [20] and it was widely studied. Until now, the method has been applied into solving ordinary differential equations [21], differential-algebraic equations [22], functional differential equations [23], and fractional differential equations [24]. As usual, the waveform sequence computed by the algorithm is not monotone. In this article, we identify the nonlinear fractional differential-algebraic equations which satisfy certain Lipschitz conditions, such that if the initial iteration waveform is chosen properly, the waveform sequence converges to the nonnegative solution monotonically.

This article is organized as follows. In Section 2, we present some notations, definitions, and assumptions. In Section 3, we firstly examine monotone dependency on initial conditions and inputs. Then, we state the main theorem on nonnegative solution. Finally, we give a method to choose the initial iteration. In Section 4, we present an example with numerical simulations to illustrate the waveform relaxation algorithm.

2 Preliminaries

In this section, we give some basic concepts and notations.

Definition 2.1 ([26]) Let $[a, b]$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville integral $(I^\alpha_{a^+}x)(t)$ and the Riemann-Liouville fractional derivative $(D^\alpha_{a^+}x)(t)$ of order $\alpha > 0$ are defined by

$$(I^\alpha_{a^+}x)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}x(\tau)\mathrm{d}\tau, \quad t > a,$$

and

$$(D^\alpha_{a^+}x)(t) = \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_a^t (t-\tau)^{m-\alpha-1}x(\tau)\mathrm{d}\tau, \quad t > a,$$

respectively, where $m-1<\alpha\leq m$, $m\in\mathbb{N}^+$, and $\Gamma(\cdot)$ denotes the Gamma function.

Of course, one has to impose some conditions on the function x such that the right hand sides are defined for almost all $t\in[a,b]$. For example, the fractional integral $I^\alpha_{a^+}x$ is defined for $x\in L^1(a,b)$.

The Laplace transform of the Riemann-Liouville fractional derivative is given as follows:

$$(\mathcal{L}D^\alpha_{0^+}x)(s)=s^\alpha(\mathcal{L}x)(s)-\sum_{i=0}^{m-1}s^i(D^{\alpha-i-1}_{0^+}x)(0^+),\quad t>0,\ m-1<\alpha\leq m,\ m\in\mathbb{N}^+.$$

However, the practical applicability of the Riemann-Liouville fractional derivative is limited by the absence of the physical interpretation of the limit values of fractional derivatives at the low terminal $t=0$. The mentioned problem does not exist in the Caputo definition of the fractional derivative.

Definition 2.2 ([26]) Let $[a,b]$ be a finite interval on the real axis $\mathbb{R}$, and let $x\in C^m([a,b])$. The Caputo fractional derivative $({}^cD^\alpha_{a^+}x)(t)$ of order $\alpha>0$ is defined by

$$({}^cD^\alpha_{a^+}x)(t)=\frac{1}{\Gamma(m-\alpha)}\int_a^t(t-\tau)^{m-\alpha-1}x^{(m)}(\tau)\mathrm{d}\tau,\quad t>a,m-1<\alpha\leq m,m\in\mathbb{N}^+.$$

The Laplace transform of the Caputo fractional derivative is given by

$$(\mathcal{L}{}^cD^\alpha_{0^+}x)(s)=s^\alpha(\mathcal{L}x)(s)-\sum_{i=0}^{m-1}s^{\alpha-i-1}x^{(i)}(0^+),\quad t>0,\ m-1<\alpha\leq m,\ m\in\mathbb{N}^+.$$

Contrary to the Laplace transform of the Riemann-Liouville fractional derivative, only integer order derivatives of function x appear in the Laplace transform of the Caputo fractional derivative. Thus, it can be useful for solving applied problems leading to linear fractional differential equations with constant coefficients with accompanying initial conditions in traditional form.

Particularly, the Riemann-Liouville fractional derivative and the Caputo fractional derivative are connected with the following relation

$$({}^cD^\alpha_{a^+}x)(t)=D^\alpha_{a^+}\left(x(t)-\sum_{i=0}^{m-1}\frac{x^{(i)}(a)}{i!}(t-a)^i\right),\ t>a,m-1<\alpha\leq m,m\in\mathbb{N}^+.\qquad(2.1)$$

Note that if $x^{(i)}(a)=0$, $i=0,1,\cdots,m-1$, then $({}^cD^\alpha_{a^+}x)(t)$ coincides with $(D^\alpha_{a^+}x)(t)$.

In this article, we consider the following semi-explicit nonlinear fractional differential-algebraic system with two continuous inputs u and e:

$$\begin{cases}M({}^cD^\alpha_{0^+}x)(t)=f(x(t),y(t),u(t),t),\ 0<\alpha<1,\ x(0)=x_0,\\ y(t)=g(x(t),y(t),e(t),t),\ t\in[0,T],\end{cases}\qquad(2.2)$$

where $({}^cD^\alpha_{0^+}x)(t)$ is α-order Caputo fractional derivative of function x, and $f:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^p\times\mathbb{R}^1\to\mathbb{R}^{n_1}$ and $g:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^q\times\mathbb{R}^1\to\mathbb{R}^{n_2}$ are continuous functions. We assume that the initial value x_0 appearing in system is consistent, that is, for a given initial value x_0, the initial value $y(0)$ satisfies $y(0)=g(x_0,y(0),e(0),0)$. We denote $n=n_1+n_2$. We also assume that the matrix M is diagonal and satisfies $M^{-1}\geqslant 0$. Thus, it is obvious that $Mx\geqslant 0$ implies that $x\geqslant 0$, where $x\in\mathbb{R}^{n_1}$. Note that $z(t)=[x(t)^{\mathrm{T}},y(t)^{\mathrm{T}}]^{\mathrm{T}}$ is to be solved.

In the following, we give sufficient conditions for the existence of nonnegative solutions of system (2.2) and construct a monotone waveform relaxation method to approximate the nonnegative solutions. For (2.2), the WR algorithm is described as

$$\begin{cases} M({}^cD_{0^+}^{\alpha}x^{(k+1)})(t) = F(x^{(k+1)}(t), x^{(k)}(t), y^{(k+1)}(t), y^{(k)}(t), u(t), t),\ x^{(k+1)}(0) = x_0, \\ y^{(k+1)}(t) = G(x^{(k+1)}(t), y^{(k+1)}(t), y^{(k)}(t), e(t), t),\ t \in [0,T],\ k = 0,1,2,\cdots, \end{cases} \tag{2.3}$$

where $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{p} \times \mathbb{R}^1 \to \mathbb{R}^{n_1}$ and $G : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_2} \times \mathbb{R}^{q} \times \mathbb{R}^1 \to \mathbb{R}^{n_2}$ are called splitting functions satisfying, for any $x \in \mathbb{R}^{n_1}$, and $y \in \mathbb{R}^{n_2}$,

$$F(x,x,y,y,u,t) = f(x,y,u,t), \quad G(x,y,y,e,t) = g(x,y,e,t), \quad t \in [0,T],$$

and $[(x^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ is a given initial iteration. For brevity, we denote the iterative waveform by $[(x^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ produced by the WR algorithm (2.3) as $z^{(k)}(t)$. Some special and typical splittings for a nonlinear function are the Jacobi splitting and the Gauss-Seidel splitting as in the classical WR algorithm. It is well known that the WR algorithm for an arbitrary initial iteration does not usually produce a monotone convergent sequence. To ensure the monotone convergence of system (2.3), we need to define partial orderings and impose some conditions on the functions F and G.

Definition 2.3 For $x, y \in \mathbb{R}^n$, $x \leqslant y \Longleftrightarrow x_i \leqslant y_i$, for all $i = 1,2,\cdots,n$. For $x(t), y(t) : [0,T] \to \mathbb{R}^n$, $x(t) \leqslant y(t) \Longleftrightarrow x(t) \leqslant y(t)$ for all $t \in [0,T]$.

Definition 2.4 A function $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^l$ is said to be globally Lipschitz continuous with respect to the first argument uniformly over the other arguments if there exists a constant L such that for all $x, y \in \mathbb{R}^n$, $e \in \mathbb{R}^m$, and $t \in [0,T]$, $\|h(x,e,t) - h(y,e,t)\|_{\mathbb{R}^l} \leqslant L\|x-y\|_{\mathbb{R}^n}$, where $\|\cdot\|_{\mathbb{R}^l}$ and $\|\cdot\|_{\mathbb{R}^n}$ are norms in $\mathbb{R}^l$ and $\mathbb{R}^n$, respectively.

Definition 2.5 A function $h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^l$ is said to be monotone increasing with respect to the first argument if for each $e \in \mathbb{R}^m$ and $t \in [0,T]$, $h(x,e,t) \leqslant h(y,e,t)$ when $x \leqslant y$, where $x, y \in \mathbb{R}^n$. A function $k : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^1 \to \mathbb{R}^l$ is said to be quasi-monotone increasing with respect to the first argument if for each $i \in \{1,2,\cdots,n\}$, each $e \in \mathbb{R}^m$, and $t \in [0,T]$, $k_i(x,e,t) \leqslant k_i(y,e,t)$ when $x \leqslant y$ with $x_i = y_i$, where $x, y \in \mathbb{R}^n$.

Clearly, monotone increasing implies that quasi-monotone increasing. In fact, quasi-monotone increasing for functions is a key property used in proving the monotone convergence of iterative waveforms. This concept was carefully stated in [27].

In this article, except Theorem 3.2 of Section 3, we always assume that the functions F and G satisfy the following assumptions 1 and 2.

Assumption 1 For all $t \in [0,T]$, the function $F(\cdot,\cdot,\cdot,\cdot,\cdot,t)$ is globally Lipschitz continuous with respect to each of the first four arguments with Lipschitz constants $L_i (i = 1,2,3,4)$, respectively, uniformly over the other arguments. Likewise, for all $t \in [0,T]$, the function $G(\cdot,\cdot,\cdot,\cdot,t)$ is globally Lipschitz continuous with respect to each of the first three arguments with Lipschitz constants $L_i (i = 5,6,7)$, respectively, uniformly over the other arguments.

Assumption 2 For all $t \in [0,T]$, the function $F(\cdot,\cdot,\cdot,\cdot,\cdot,t)$ is quasi-monotone increasing with respect to the first arguments, and it is monotone increasing with respect to the each of the other four arguments. Likewise, for all $t \in [0,T]$, the function $G(\cdot,\cdot,\cdot,\cdot,t)$ is monotone increasing with respect to the each of the first four arguments.

Finally, we state an existence condition of solutions of system (2.2). This existence condition can be carried out by the approach in [24] with a careful modification on its proof. So, we omit the proof in this article.

Theorem 2.1 Assume that for all $t \in [0,T]$, the functions $f(\cdot,\cdot,\cdot,t)$ and $g(\cdot,\cdot,\cdot,t)$ are globally Lipschitz continuous with respect to each of the first two arguments with Lipschitz constants L_i^f and L_i^g $(i=1,2)$, respectively, uniformly over the other arguments, that is, for any $u \in \mathbb{R}^p$, $e \in \mathbb{R}^q$, $x_i \in \mathbb{R}^{n_1}$, and $y_i \in \mathbb{R}^{n_2}$ $(i=1,2,3,4)$,

$$\|f(x_1,y_1,u,t)-f(x_2,y_2,u,t)\| \leqslant L_1^f\|x_1-x_2\|+L_2^f\|y_1-y_2\|,$$

and

$$\|g(x_3,y_3,e,t)-g(x_4,y_4,e,t)\| \leqslant L_1^g\|x_3-x_4\|+L_2^g\|y_3-y_4\|.$$

If $L_2^g<1$, then system (2.2) has a unique solution $[x(t)^{\mathrm{T}},y(t)^{\mathrm{T}}]^{\mathrm{T}}$ on $[0,T]$.

Similarly, we give a convergence condition for the WR algorithm (2.3).

Theorem 2.2 Assume that the functions F and G satisfy Assumption 1. If $L_7<1$, then the iteration sequence $\{z^{(k)}\}$ produced by the WR algorithm (2.3) converges uniformly to the unique solution $[x(t)^{\mathrm{T}},y(t)^{\mathrm{T}}]^{\mathrm{T}}$ of system (2.2) on $[0,T]$.

3 Monotone Waveform Relaxation Method

3.1 Monotone dependency on initial conditions and inputs

We examine the monotone dependency properties on initial iteration and inputs for system (2.2) for any fixed $k \in \{0,1,2,\cdots.\}$. These properties are useful to show the monotone convergence of the relaxation sequence based on system (2.2). For the sake of clarity, we consider the following nonlinear fractional differential-algebraic system for system (2.2) for some fixed k:

$$\begin{cases} M({}^cD_{0+}^{\alpha}x)(t)=\widetilde{F}(x,y,e_1(\cdot),t),\ x(0)=x_0, \\ y(t)=\widetilde{G}(x,e_2(\cdot),t),\ t\in[0,T], \end{cases} \tag{3.1}$$

where $\widetilde{F}:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{m_1}\times\mathbb{R}^1\to\mathbb{R}^{n_1}$ and $G:\mathbb{R}^{n_1}\times\mathbb{R}^{m_2}\times\mathbb{R}^1\to\mathbb{R}^{n_2}$ are continuous functions. Assume that for any $t\in[0,T]$, the function $\widetilde{F}(\cdot,\cdot,\cdot,t)$ is globally Lipschitz continuous with respect to each of the first two arguments uniformly over the other arguments and the function $\widetilde{G}(\cdot,\cdot,t)$ is globally Lipschitz continuous with respect to the first argument uniformly over the second argument for any $t\in[0,T]$.

For system (3.1), let the inputs $e_1:[0,T]\to\mathbb{R}^{m_1}$ and $e_2:[0,T]\to\mathbb{R}^{m_2}$ be two given continuous functions. Then, we establish the following lemma.

Lemma 3.1 Assume that for each $i\in\{1,2,\cdots,n_1\}$, $\widetilde{F}_i(x,y,e_1(\cdot),t)\geqslant 0$ on $t\in[0,T]$ when $x\geqslant 0$, $x_i=0$, and $y\geqslant 0$. And assume that in system (3.1), $\widetilde{G}(x,e_2(\cdot),t)\geqslant 0$ on $t\in[0,T]$ when $x\geqslant 0$. Then, the solution $z(t)$ of system (3.1) satisfies $z(t)\geqslant 0$ on $[0,T]$ if the initial values subject to $Mx_0\geqslant 0$, and $y(0)\geqslant 0$, where $z(t)=[x(t)^{\mathrm{T}},y(t)^{\mathrm{T}}]^{\mathrm{T}}$.

Proof We apply contradiction to show the statement. Suppose that there exist $t^*>0$ and some subscript l such that $(Mx)_l(t^*)<0$ or $y_l(t^*)<0$. Because $\widetilde{F}(x,G(x,e_2,t),e_1,t)$ is Lipschitz continuous with respect to x, system (3.1) has a unique solution and the solution depends continuously on the initial value and the right-hand continuous disturbance ([26]). Thus, there exists $\delta > 0$ such that the following system

$$\begin{cases} (M({}^cD_{0+}^{\alpha}x))_i(t) = \widetilde{F}_i(x, y, e_1(\cdot), t) + \delta,\ x(0) = x_0,\ i = 1, 2, \cdots, n_1, \\ y_j(t) = \widetilde{G}_j(x, e_2(\cdot), t) + \delta,\ j = 1, 2, \cdots, n_2, \\ y(0) = \widetilde{G}(x(0), e_2(0), 0) + \overrightarrow{\delta} \geqslant 0,\ t \in [0, T], \end{cases} \tag{3.2}$$

has a unique solution $[x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}}]^{\mathrm{T}}$ satisfying $(Mx)_l(t^*) < 0$ or $y_l(t^*) < 0$ for the subscript l, where $\overrightarrow{\delta} = [\delta, \delta, \cdots, \delta]^{\mathrm{T}} \in \mathbb{R}^{n_2}$.

We denote $z(t) = [x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}}]^{\mathrm{T}} = [z_1(t), z_2(t), \cdots, z_{n_1}(t), z_{n_1+1}(t), \cdots, z_n(t)]$, where $n = n_1 + n_2$. Let $K = \{k : z_k(t) < 0 \text{ for some } t > 0\}$ and $t_k = \inf\{t > 0 : z_k(t) < 0\}$ for $k \in K$. By continuity, $z_k(t_k) = 0$ for each $k \in K$. Now, let r be the smallest integer such that $t_r = \min\{t_k\}$. We have $z(t) \geqslant 0$ for $t \leqslant t_r$ in which $z_r(t_r) = 0$.

Let $r \in \{1, 2, \cdots, n_1\}$. When $t \leqslant t_r$, it has $x(t) \geqslant 0$, $x(t_r) = 0$, and $x(t) < 0$ for $t \in (t_r, t_r + \epsilon]$, ϵ is some positive constant. Then, by the Hadamard lemma (see [28], p.17), $x(t)$ leads to the representation $x(t) = (t_r - t)h(t)$, with $h(t) \in C^1([0, t_r])$, $h(t) \geqslant 0$ for $t \in [0, t_r]$, and $h(t) \leqslant 0$ for $t \in [t_r, t_r + \epsilon]$.

We consider the sign of $(D_{0+}^{\alpha}x)(t_r)$. We define a function

$$\Phi(t) = \int_0^t (t - \tau)^{-\alpha} x(\tau) \mathrm{d}\tau.$$

Then, it has

$$\begin{aligned} \Phi(t_r + \Delta t) - \Phi(t_r) &= \int_0^{t_r + \Delta t} (t_r + \Delta t - \tau)^{-\alpha} x(\tau) \mathrm{d}\tau - \int_0^{t_r} (t_r - \tau)^{-\alpha} x(\tau) \mathrm{d}\tau \\ &= \int_0^{t_r} \big((t_r + \Delta t - \tau)^{-\alpha} - (t_r - \tau)^{-\alpha}\big) x(\tau) \mathrm{d}\tau \\ &\quad + \int_{t_r}^{t_r + \Delta t} (t_r + \Delta t - \tau)^{-\alpha} x(\tau) \mathrm{d}\tau \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_0^{t_r} \big((t_r + \Delta t - \tau)^{-\alpha} - (t_r - \tau)^{-\alpha}\big)(t_r - \tau) h(\tau) \mathrm{d}\tau,$$

and

$$I_2 = \int_{t_r}^{t_r + \Delta t} (t_r + \Delta t - \tau)^{-\alpha} (t_r - \tau) h(\tau) \mathrm{d}\tau.$$

As $h(t)$ is continuous on $[0, t_r]$ and the function

$$q(\tau, \Delta t) = \big((t_r + \Delta t - \tau)^{-\alpha} - (t_r - \tau)^{-\alpha}\big)(t_r - \tau)$$

is of one sign and integrable on the interval $[0, t_r]$, the mean value theorem yields the following representation with a ξ, $0 < \xi < t_r$:

$$\begin{aligned} I_1 &= h(\xi) \int_0^{t_r} q(\tau, \Delta t) \mathrm{d}\tau = h(\xi) \int_0^{t_r} \big((t_r + \Delta t - \tau)^{-\alpha} - (t_r - \tau)^{-\alpha}\big)(t_r - \tau) \mathrm{d}\tau \\ &= h(\xi)\bigg(- \frac{\Delta t}{1 - \alpha}\Big((t_r + \Delta t)^{1-\alpha} - (\Delta t)^{1-\alpha}\Big) + \frac{1}{2 - \alpha}\Big((t_r + \Delta t)^{2-\alpha} - (\Delta t)^{2-\alpha} - t_r^{2-\alpha}\Big)\bigg). \end{aligned}$$

It follows from the last representation that

$$
\begin{aligned}
\lim_{\Delta t\to 0}\frac{I_1}{\Delta t} &= \lim_{\Delta t\to 0} h(\xi)\bigg(-\frac{1}{1-\alpha}\Big((t_r+\Delta t)^{1-\alpha}-(\Delta t)^{1-\alpha}\Big)\\
&\quad+\frac{1}{2-\alpha}\bigg(\frac{(t_r+\Delta t)^{2-\alpha}-t_r^{2-\alpha}}{\Delta t}-(\Delta t)^{1-\alpha}\bigg)\bigg)\\
&= \lim_{\Delta t\to 0} h(\xi)\bigg(-\frac{t_r^{1-\alpha}}{1-\alpha}+t_r^{1-\alpha}\bigg)\\
&= -\frac{\alpha}{1-\alpha}t_r^{1-\alpha}\lim_{\Delta t\to 0} h(\xi)\leqslant 0. \qquad (3.3)
\end{aligned}
$$

Because the function h is continuous and non-positive on $[0,t_r]$, the limit $\lim\limits_{\Delta t\to 0} h(\xi)$ exists and is non-negative.

Now, we consider the auxiliary function $p(\tau,\Delta t)=(t_r+\Delta t-\tau)^{-\alpha}(t_r-\tau)$ that is of one sign and integrable on the interval $[t_r,t_r+\Delta t]$. Because the function h is continuous on $[t_r,t_r+\Delta t]$, the mean value theorem applied to the integral I_2 yields the following representation with a ζ, $t_r<\zeta<t_r+\Delta t$:

$$
\begin{aligned}
I_2 &= h(\zeta)\int_{t_r}^{t_r+\Delta t} p(\tau,\Delta t)\mathrm{d}\tau = h(\zeta)\int_{t_r}^{t_r+\Delta t}(t_r+\Delta t-\tau)^{-\alpha}(t_r-\tau)\mathrm{d}\tau\\
&= -h(\zeta)\frac{1}{(1-\alpha)(2-\alpha)}(\Delta t)^{2-\alpha}.
\end{aligned}
$$

Thus, we get the following representation

$$
\lim_{\Delta t\to 0}\frac{I_2}{\Delta t}=\frac{1}{(1-\alpha)(2-\alpha)}\lim_{\Delta t\to 0}h(\zeta)\lim_{\Delta t\to 0}(\Delta t)^{1-\alpha}=0. \qquad (3.4)
$$

On the basis of the relations (3.3) and (3.4), we can obtain $(D_{0^+}^{\alpha}x)(t_r)\leqslant 0$.

Furthermore, by relation (2.1), we can obtain

$$
(D_{0^+}^{\alpha}x)(t_r)=({}^cD_{0^+}^{\alpha}x)(t_r)+\frac{x_0}{\Gamma(1-\alpha)}t_r^{-\alpha}. \qquad (3.5)
$$

This implies that $({}^cD_{0^+}^{\alpha}x)(t_r)\leqslant -\frac{x_0}{\Gamma(1-\alpha)}t_r^{-\alpha}\leqslant 0$, that is, $F_l(x(t_r),y(t_r),e_1(t_r),t_r)+\delta\leqslant 0$. This is impossible because $x(t_r)\geqslant 0$, $x_l(t_r)=0$, and $y(t_r)\geqslant 0$.

Let $r\in\{n_1+1,n_2+2,\cdots,n\}$. Then, $y_l(t_r)=0$, that is, $G_l(x(t_r),e_2(t_r),t_r)+\delta=0$. This is impossible because $x(t_r)\geqslant 0$. This completes the proof of this lemma. □

In the following, we will discuss that for some fixed k, the iteration waveform at each iteration in the WR algorithm is monotonically dependent on the previous iterative waveform and input functions. For this aim, let $z^{(k)}(t)=[(x^{(k)}(t))^{\mathrm{T}},(y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}$, where $x^{(k)}(0)=x_0$ is a solution of system (2.3) with given continuous input functions u, e, and the previous iteration $z^{(k-1)}(t)$. And let $\overline{z}^{(k)}(t)=[(\overline{x}^{(k)}(t))^{\mathrm{T}},(\overline{y}^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}$, where $\overline{x}^{(k)}(0)=\overline{x}_0$ is a solution of system (2.3) with given continuous input functions $\overline{u}$, $\overline{e}$, and the previous iteration $\overline{z}^{(k-1)}(t)$. Likewise, let $z(t)=[x(t)^{\mathrm{T}},y(t)^{\mathrm{T}}]^{\mathrm{T}}$, where $x(0)=x_0$ is a solution of system (2.2) with given continuous input functions u and e. And let $\overline{z}(t)=[\overline{x}(t)^{\mathrm{T}},\overline{y}(t)^{\mathrm{T}}]^{\mathrm{T}}$, where $\overline{x}(0)=\overline{x}_0$ is a solution of system (2.3) with given continuous input functions $\overline{u}$ and $\overline{e}$. Under these conditions, we have the following result.

Lemma 3.2 Assume that the input functions in system (2.3) satisfy $u(t)\geqslant\overline{u}(t)$, $e(t)\geqslant\overline{e}(t)$, and the $(k-1)$st iterations produced by the algorithm (2.3) satisfying $z^{(k-1)}(t)\geqslant\overline{z}^{(k-1)}(t)$, then we have $z^{(k)}(t)\geqslant\overline{z}^{(k)}(t)$ on $[0,T]$ if $Mx_0\geqslant M\overline{x}_0$, and $y^{(k)}(0)\geqslant\overline{y}^{(k)}(0)$.

Likewise, assume that $L_7 < 1$, and the input functions in system (2.2) satisfy $u(t) \geqslant \overline{u}(t)$ and $e(t) \geqslant \overline{e}(t)$, then we have $z(t) \geqslant \overline{z}(t)$ on $[0, T]$ if $Mx_0 \geqslant M\overline{x}_0$, and $y(0) \geqslant \overline{y}(0)$.

Proof The two statements are analogous, so we consider the first. let $\eta_1(t) = x^{(k)}(t) - \overline{x}^{(k)}(t)$, $\eta_2(t) = y^{(k)}(t) - \overline{y}^{(k)}(t)$. Then, $\eta_1(t)$ and $\eta_2(t)$ satisfy the following system:

$$\begin{cases} M(D_{0+}^{\alpha}\eta_1)(t) = F(\overline{x}^{(k)}(t) + \eta_1(t), x^{(k-1)}(t), \overline{y}^{(k)}(t) + \eta_2(t), y^{(k-1)}(t), u(t), t) \\ \qquad\qquad\qquad - F(\overline{x}^{(k)}(t), x^{(k-1)}(t), \overline{y}^{(k)}(t), y^{(k-1)}(t), u(t), t), \\ \eta_2(t) = G(\overline{x}^{(k)}(t) + \eta_1(t), x^{(k-1)}(t), y^{(k-1)}(t), e(t), t) \\ \qquad\quad - G(\overline{x}^{(k)}(t), \overline{x}^{(k-1)}(t), \overline{y}^{(k-1)}(t), \overline{e}(t), t), \\ M\eta_1(0) \geqslant 0, \eta_2(0) \geqslant 0, t \in [0, T]. \end{cases} \tag{3.6}$$

Clearly, the functions of the right-hand sides of system satisfy the conditions of Lemma 3.1, so, we can arrive at $\eta_1(t) \geqslant 0$, and $\eta_2(t) \geqslant 0$ for all $t \in [0, T]$, that is, $[Mx^{(k)}(t), y^{(k)}(t)] \geqslant [M\overline{x}^{(k)}(t), \overline{y}^{(k)}(t)]$ on $[0, T]$.

By Theorem 2.2 and the first statement of this lemma, the second part is obvious. The proof of this lemma is completed. □

3.2 Convergence of monotone waveform relaxation

On the basis of the above statements, we can establish the existence theorem (Theorem 3.3) of nonnegative solutions of system (2.2). And from Theorem 3.1, one can see that the nonnegative solution of system (2.2) can be approximated using the WR algorithm (2.3) if the choice of initial iteration is proper.

Theorem 3.1 Suppose $L_7 < 1$. If the initial iteration $[(x^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ in (2.3) satisfies

$$0 \leqslant [(Mx^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}} \leqslant [(Mx_0)^{\mathrm{T}}, (y(0))^{\mathrm{T}}]^{\mathrm{T}},$$

and

$$[(Mx^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}} \leqslant [(Mx^{(1)}(t))^{\mathrm{T}}, (y^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}},$$

then, for each $k \in \mathbb{N}$, it has

$$[(Mx^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}} \leqslant [(Mx^{(k+1)}(t))^{\mathrm{T}}, (y^{(k+1)}(t))^{\mathrm{T}}]^{\mathrm{T}}, \tag{3.7}$$

and the sequence $\{[(x^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}\}$ is convergent uniformly and monotonically on $[0, T]$, that is, $\lim\limits_{k\to+\infty} x^{(k)}(t) = x(t)$, $\lim\limits_{k\to+\infty} y^{(k)}(t) = y(t)$, and $\{[x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}}]^{\mathrm{T}}\}$ is the nonnegative solution of system (2.2).

Proof From

$$y^{(1)}(0) = G(x_0, x_0, y^{(0)}(0), e(0), 0) \leqslant G(x_0, x_0, y^{(1)}(0), e(0), 0) = y^{(2)}(0),$$

by Lemma 3.2, one can obtain relation (3.7) for each k by induction. On the other hand, as $L_7 < 1$, by Lemma 3.2, the sequence $\{[(x^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}\}$ converges to $[(x(t))^{\mathrm{T}}, (y(t))^{\mathrm{T}}]^{\mathrm{T}}$ as $k \to +\infty$ uniformly and monotonically on [0,T]. This function $[(x(t))^{\mathrm{T}}, (y(t))^{\mathrm{T}}]^{\mathrm{T}}$ satisfies

$$\begin{cases} M({}^{c}D_{0+}^{\alpha}x)(t) = F(x(t), x(t), y(t), y(t), u(t), t), \ x(0) = x_0, \\ y(t) = G(x(t), y(t), y(t), e(t), t), \ t \in [0, T], \end{cases}$$

and $[(Mx^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}} \leqslant [(Mx(t))^{\mathrm{T}}, (y(t))^{\mathrm{T}}]^{\mathrm{T}}$ for each k. Thus, this completes the proof. □

From Theorem 3.1, we know that as long as the initial iteration is chosen properly, namely, $x^{(0)}(t) \leqslant x^{(1)}(t)$ and $y^{(0)}(t) \leqslant y^{(1)}(t)$ on $[0,T]$, then the iterative procedure will monotonically converge to the actual solution of system.

3.3 Initial iterations

The choice of initial iterations is crucial to ensure monotone convergence of the waveforms in the WR algorithm. In the following, we present a choice to deal with this matter. In this subsection, we denote $\|x\|_\infty = \max\{|x_i| : i = 1, 2, \cdots, n\}$ for $x \in \mathbb{R}^n$.

For any given input functions u and e, we assume that

$$\|M^{-1}F(x_1, x_2, y_1, y_2, u, t)\|_\infty \leqslant h_1(\|x_1\|_\infty, \|x_2\|_\infty, \|y_1\|_\infty, \|y_2\|_\infty, t)$$

and

$$\|G(x_1, x_2, y_1, e, t)\|_\infty \leqslant h_2(\|x_1\|_\infty, \|x_2\|_\infty, \|y_1\|_\infty, t)$$

for $x_i \in \mathbb{R}^{n_1}$ and $y_i \in \mathbb{R}^{n_2}$ $(i = 1, 2)$, where $h_1(\cdot, \cdot, \cdot, \cdot, t)$, and $h_2(\cdot, \cdot, \cdot, t)$ are nondecreasing functions for any $t \in [0, T]$.

Now, we need to assume that the following simple two-dimension fractional differential-algebraic system has a positive solution $w(t) = [w_1(t), w_2(t)]^{\mathrm{T}}$:

$$\begin{cases} ({}^cD^\alpha_{0+}w_1)(t) = h_1(w_1(t), w_1(t), w_2(t), w_2(t), t),\ 0 < \alpha < 1,\ w_1(0) = \|x_0\|_\infty, \\ w_2(t) = h_2(w_1(t), w_1(t), w_2(t), t) + 2\|y(0)\|_\infty,\ t \in [0, T], \end{cases} \tag{3.8}$$

where x_0 is the given initial value and $y_0 = g(x_0, y_0, e(0), 0)$ in system (2.2).

We define

$$\beta^1_i(t) = x_i(0) + \|x_0\|_\infty - w_1(t), \quad \beta^2_i(t) = x_i(0) - \|x_0\|_\infty + w_1(t), \quad i = 1, 2, \cdots, n_1,$$

and

$$\gamma^1_i(t) = y_i(0) + \|y(0)\|_\infty - w_2(t), \quad \gamma^2_i(t) = y_i(0) - \|y(0)\|_\infty + w_2(t), \quad i = 1, 2, \cdots, n_2.$$

Let

$$\beta^l(t) = [\beta^l_1(t), \beta^l_2(t), \cdots, \beta^l_{n_1}(t)]^{\mathrm{T}},\ \gamma^l(t) = [\gamma^l_1(t), \gamma^l_2(t), \cdots, \gamma^l_{n_2}(t)]^{\mathrm{T}},$$

and

$$\rho^l(t) = [(\beta^l(t))^{\mathrm{T}}, (\gamma^l(t))^{\mathrm{T}}]^{\mathrm{T}}, \tag{3.9}$$

where $t \in [0, T]$ and $l = 1, 2$. It is obvious that

$$\beta^1(0) \leqslant x_0 \leqslant \beta^2(0), \quad \gamma^1(0) \leqslant y(0) \leqslant \gamma^2(0),$$

and

$$[\rho^1(t), \rho^2(t)] \subseteq [-w(t), w(t)],$$

where $w(t) = [w_1(t)\overrightarrow{e}^{\mathrm{T}}_1, w_2(t)\overrightarrow{e}^{\mathrm{T}}_2]^{\mathrm{T}}$, and $\overrightarrow{e}_1 = [1, 1, \cdots, 1]^{\mathrm{T}} \in \mathbb{R}^{n_l}$, $l = 1, 2$.

Lemma 3.3 If $[(x^{(1)}(t))^{\mathrm{T}}, (y^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ satisfies

$$\begin{cases} M({}^cD^\alpha_{0+}x^{(1)})(t) = F(x^{(1)}(t), x^{(0)}(t), y^{(1)}(t), y^{(0)}(t), u(t), t),\ 0 < \alpha < 1,\ x^{(1)}(0) = x_0, \\ y^{(1)}(t) = G(x^{(1)}(t), x^{(0)}(t), y^{(0)}(t), e(t), t),\ t \in [0, T], \end{cases} \tag{3.10}$$

where $[(x^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}} \in [\rho^1(t), \rho^2(t)]$ on $[0, T]$, then $[(x^{(1)}(t))^{\mathrm{T}}, (y^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}} \in [\rho^1(t), \rho^2(t)]$ on $[0, T]$.

Proof First, we define a sequence $\{\widetilde{z}^{(k)}(t)\} = [(\widetilde{x}^{(k)}(t))^{\mathrm{T}}, (\widetilde{y}^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ such that it satisfies

$$\begin{cases} \widetilde{x}^{(k+1)}(t) = x_0 + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} M^{-1} F(\widetilde{x}^{(k)}(\tau), x^{(0)}(\tau), \widetilde{y}^{(k)}(\tau), y^{(0)}(\tau), u(\tau), \tau)\mathrm{d}\tau, \\ \widetilde{y}^{(k+1)}(t) = G(\widetilde{x}^{(k)}(t), x^{(0)}(t), y^{(0)}(t), e(t), t), \ \ t \in [0, T], \ \ k = 0, 1, \cdots, \end{cases} \tag{3.11}$$

where $[(\widetilde{x}^{(0)}(t))^{\mathrm{T}}, (\widetilde{y}^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}} = [(x^{(0)}(t))^{\mathrm{T}}, (y^{(0)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ on $[0, T]$.

Next, we will prove that $[(\widetilde{x}^{(k)}(t))^{\mathrm{T}}, (\widetilde{y}^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}} \in [\rho^1(t), \rho^2(t)]$ for each k on $[0, T]$. Without loss of generality, we show it for $k = 1$ only. We have

$$\begin{aligned} \|\widetilde{x}^{(1)}(t) - x_0\|_\infty &\leqslant \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|F(\widetilde{x}^{(0)}(\tau), x^{(0)}(\tau), \widetilde{y}^{(0)}(\tau), y^{(0)}(\tau), u(\tau), \tau)\|_\infty \mathrm{d}\tau \\ &\leqslant w_1(t) - \|x_0\|_\infty, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{y}^{(1)}(t) - y(0)\|_\infty &\leqslant \|G(\widetilde{x}^{(0)}(t), x^{(0)}(t), y^{(0)}(t), e(t), t)\|_\infty + \|y(0)\|_\infty \\ &\leqslant w_2(t) - \|y(0)\|_\infty. \end{aligned}$$

This implies that $[(\widetilde{x}^{(1)}(t))^{\mathrm{T}}, (\widetilde{y}^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}} \in [\rho^1(t), \rho^2(t)]$ on $[0, T]$.

Thirdly, we need to prove that the sequence $\{\widetilde{z}^{(k)}(t)\}$ converges to $[(x^{(1)}(t))^{\mathrm{T}}, (y^{(1)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ as $k \to +\infty$ on $[0, T]$. In fact, we have

$$\begin{aligned} \|\widetilde{x}^{(k)}(t) - x^{(1)}(t)\|_\infty \leqslant{}& \frac{1}{\Gamma(\alpha)} \bigg(\int_0^t (t-\tau)^{\alpha-1} \|F(\widetilde{x}^{(k-1)}, x^{(0)}, \widetilde{y}^{(k-1)}, y^{(0)}, u(\tau), \tau) \\ & - F(x^{(1)}, x^{(0)}, \widetilde{y}^{(k-1)}, y^{(0)}, u(\tau), \tau)\|_\infty \mathrm{d}\tau \\ & + \int_0^t (t-\tau)^{\alpha-1} \|F(x^{(1)}, x^{(0)}, \widetilde{y}^{(k-1)}, y^{(0)}, u(\tau), \tau) \\ & - F(x^{(1)}, x^{(0)}, y^{(1)}, y^{(0)}, u(\tau), \tau)\|_\infty \mathrm{d}\tau \bigg) \|M^{-1}\|_\infty \\ \leqslant{}& \bigg(\frac{L_1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\widetilde{x}^{(k-1)} - x^{(1)}\|_\infty \mathrm{d}\tau \\ & + \frac{L_3}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|\widetilde{y}^{(k-1)} - y^{(1)}\|_\infty \mathrm{d}\tau \bigg) \|M^{-1}\|_\infty, \end{aligned}$$

and

$$\begin{aligned} \|\widetilde{y}^{(k)}(t) - y^{(1)}(t)\|_\infty &= \|G(\widetilde{x}^{(k-1)}, x^{(0)}, y^{(0)}, e(t), t) - G(x^{(1)}, x^{(0)}, y^{(0)}, e(t), t)\|_\infty \\ &\leqslant L_5 \|\widetilde{x}^{(k-1)}(t) - x^{(1)}(t)\|_\infty. \end{aligned}$$

On the basis of the above relations, we have the following inequality

$$\begin{aligned} \begin{bmatrix} \|\widetilde{x}^{(k)}(t) - x^{(1)}(t)\|_\infty \\ \|\widetilde{y}^{(k)}(t) - y^{(1)}(t)\|_\infty \end{bmatrix} \leqslant{}& \left(\begin{bmatrix} 0 & 0 \\ L_5 & 0 \end{bmatrix} + \|M^{-1}\|_\infty \begin{bmatrix} L_1 \mathcal{R}_c & L_3 \mathcal{R}_c \\ 0 & 0 \end{bmatrix} \right) \\ & \cdot \begin{bmatrix} \|\widetilde{x}^{(k-1)}(t) - x^{(1)}(t)\|_\infty \\ \|\widetilde{y}^{(k-1)}(t) - y^{(1)}(t)\|_\infty \end{bmatrix}, \end{aligned}$$

where the operator $\mathcal{R}_c$ is defined by

$$(\mathcal{R}_c w)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} w(\tau)\mathrm{d}\tau, \quad w \in C([0, T], \mathbb{R}).$$

By [25], we have $\rho(\mathcal{R}_c) = 0$. Thus, we can derive $\rho(\mathcal{R}) = 0$, where

$$\mathcal{R} = \begin{bmatrix} 0 & 0 \\ L_5 & 0 \end{bmatrix} + \|M^{-1}\|_\infty \begin{bmatrix} L_1\mathcal{R}_c & L_3\mathcal{R}_c \\ 0 & 0 \end{bmatrix}.$$

This completes the proof of this lemma. □

The previous result says that if $x^{(0)}(t)$ is chosen as $\beta^1(t)$ and $y^{(0)}(t)$ is chosen as $\gamma^1(t)$, then $x^{(0)}(t) \leqslant x^{(1)}(t)$, and $y^{(0)}(t) \leqslant y^{(1)}(t)$ on $[0, T]$. Thus, by Lemma 3.3 and Theorem 3.1, we can easily establish the following important result.

Theorem 3.2 Suppose $L_7 < 1$, and let $z(t) = [x(t)^{\mathrm{T}}, y(t)^{\mathrm{T}}]^{\mathrm{T}}$, where $x(0) = x_0$ is a solution of system (2.2) with given continuous input functions u and e. Let the sequence $\{z^{(k)}(t)\}$ be defined by the WR algorithm (2.3) with initial iteration $z^{(0)}(t) = \rho^1(t)$, where $\rho^1(t)$ is defined by (3.9). Then, the sequence $\{z^{(k)}(t)\}$ converges to the unique solution $z(t)$ on $[0, T]$, and the solution z satisfies that $z(t) \geqslant 0$ for all $t \in [0, T]$.

4 Example

In this section, we give a simple example to confirm the monotone convergence properties of the WR algorithm for fractional differential-algebraic equations. In the processing of the numerical computation, the Caputo fractional derivative is computed by the implicit finite difference approximation:

$$({}^cD_{0^+}^\alpha x)(t_k) \approx \sigma_{\alpha,h} \sum_{j=0}^{k} \omega_j^{(\alpha)} (x_{k-j+1} - x_{k-j}),$$

where $x_k = x(t_k)$, and

$$\sigma_{\alpha,h} = \frac{1}{h^\alpha \Gamma(2-\alpha)}, \quad \omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}.$$

The time-step h is adopted as 0.2, and the error is defined by $e^{(k)} = \frac{\|x^{(k)} - x\|}{\|x\|}$, where $x^{(k)}$ is obtained by the WR algorithm, x is the true solution, and $\|\cdot\|$ denotes the 2-norm in $\mathbb{R}^n$.

Example 4.1 Consider the following fractional differential-algebraic system:

$$\begin{cases} ({}^cD_{0^+}^\alpha x_1)(t) = -x_1(t) + \dfrac{\Gamma(3)}{\Gamma(5/2)} t^{3/2} x_2(t) + t^2, \ x_1(0) = 0, \\ ({}^cD_{0^+}^\alpha x_2)(t) = x_1(t) - t^2 x_2(t), \ x_2(0) = 0, \\ y(t) = x_1(t) + x_2(t) + \dfrac{\Gamma(3)}{\Gamma(5/2)} t^{3/2}, \ t \in [0, 10]. \end{cases} \tag{4.1}$$

Its WR algorithm is described as

$$\begin{cases} ({}^cD_{0^+}^\alpha x_1^{(k+1)})(t) = -2x_1^{(k+1)}(t) + x^{(k)}(t) + \dfrac{\Gamma(3)}{\Gamma(5/2)} t^{3/2} x_2^{(k)}(t) + t^2, \ x_1^{(k+1)}(0) = 0, \\ ({}^cD_{0^+}^\alpha x_2^{(k+1)})(t) = x_1^{(k+1)}(t) - t^2 x_2^{(k+1)}(t), \ x_2^{(k+1)}(0) = 0, \\ y^{(k+1)}(t) = x_1^{(k+1)}(t) + x_2^{(k+1)}(t) + \dfrac{\Gamma(3)}{\Gamma(5/2)} t^{3/2}, \ t \in [0, 10], \ k = 0, 1, 2, \cdots, \end{cases} \tag{4.2}$$

where $x_1^{(0)}(t) \equiv 0$ on $t \in [0, 10]$.

One can see that the functions of the right-hand side in (4.2) satisfy Assumptions 1 and 2. Therefore, the sequence $z^{(k)}(t) = [(x^{(k)}(t))^{\mathrm{T}}, (y^{(k)}(t))^{\mathrm{T}}]^{\mathrm{T}}$ obtained by equation (4.2) converges uniformly and monotonically to the solution of equation (4.1). From Figure 1, one can see that the nonnegative solution can be approximated by a monotone waveform sequence. The experiment results agree with the theory analysis.

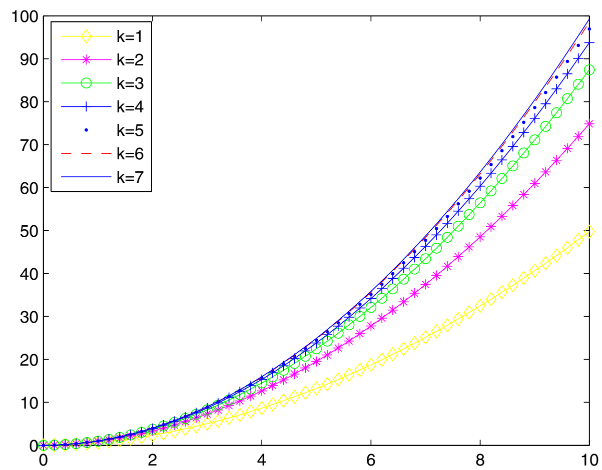

Figure 1 Monotone waveforms of x_1 in system (4.1)

The errors are given in Table 1.

Table 1 The relative errors for the different iterative numbers

k	5	10	15	20	25	30	35
error	0.0289	7.8307e-04	1.9823e-05	4.7014e-07	1.0468e-08	2.1937e-10	4.3384e-12

Appendix

Hadamard's lemma Any smooth function f in a starlike neighborhood of a point z is representable in the form

$$f(x) = f(z) + \sum_{i=1}^{n} (x_i - z_i) g_i(x),$$

where g_i are smooth functions.

In fact, by Hadamard's lemma, any smooth function $f(x)$ is representable in the form $f(x) = f(x_0) + (x - x_0)g(x)$, where $f(x_0) = 0$, and $g(x)$ is a smooth function.

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