Stability criteria for nonlinear fully implicit differential-algebraic systems

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Kurzzusammenfassung

Die vorliegende Arbeit leistet einen Beitrag zur qualitativen Theorie differential-algebraischer Systeme (engl. differential-algebraic equations, DAEs), indem neue Stabilitätskriterien für eine Klasse nichtlinearer, voll-impliziter DAEs mit proper formuliertem Hauptterm und Traktabilitätsindex 1 und 2 hergeleitet werden.

Das Theorem von Andronov-Witt wird auf asymptotische orbitale Stabilität von periodischen Lösungen voll-impliziter autonomer DAEs verallgemeinert. Zu diesem Zweck wird eine Zustandsraumdarstellung von differential-algebraischen Systemen um eine Referenzlösung herausgearbeitet, welche u.a. die autonome Struktur einer DAE korrekt wiederspiegelt. Sie basiert auf einer Verfeinerung der vollständigen Entkopplung im Rahmen des Traktabilitätsindex und setzt daher nur moderate Differenzierbarkeit der Systemgleichungen voraus. Die Transformation auf diese Zustandsraumdarstellung kommutiert mit Linearisierung entlang gleicher Lösung, folglich können charakteristische Multiplikatoren des Variationssystems der inhärenten Dynamik einer DAE in Termen des Ausgangssystems formuliert werden.

Ein weiterer Schwerpunkt der Dissertation ist die Verallgemeinerung der direkten Methode von Lyapunov auf differential-algebraische Systeme. Es werden neue Definitionen von Lyapunov-Funktionen für differenzierbare Lösungskomponenten einer proper formulierten DAE aufgestellt, bei denen die Monotonie entlang von Lösungen in Termen des Ausgangssystems ausgedrückt wird. Es stellt sich heraus, dass ein zylindrischer Definitionsbereich der inhärenten Dynamik neben der Existenz einer solchen Lyapunov Funktion entscheidend ist, um die Lösbarkeit auf unbeschränkten Intervallen zu garantieren. Dabei werden praktische Stabilitätskriterien für beschränkte Lösungen von autonomen DAEs und für allgemeine Lösungen von DAEs mit beschränkten partiellen Ableitungen der Systemgleichungen bewiesen. Der Zugang erlaubt auch eine Interpretation bekannter Kontraktivitätsbegriffe für differential-algebraische Systeme.

Schlagwörter: nichtlineare differential-algebraische Systeme, Traktabilitätsindex, asymptotische Stabilität, orbitale Stabilität, Andronov-Witt Theorem, Lyapunov Funktion, Kontraktivität, charakteristische Multiplikatoren, Zustandsraumdarstellung, Dissipativität.

Abstract

This thesis contributes to the qualitative theory of differential-algebraic equations (DAEs) by providing new stability criteria for solutions of a class of nonlinear, fully implicit DAEs with a properly stated derivative term and tractability index one and two.

A generalization of the Andronov-Witt Theorem addressing orbital stability is proved. To this purpose, a state space representation of differential-algebraic systems based on the tractability index is developed which has advantageous properties, e.g. moderate smoothness requirements, commutativity with linearization and an autonomous structure in case of autonomous DAEs. It allows a suitable definition of characteristic multipliers referring to the inherent dynamics, but given in terms of the DAE.

Furthermore, the fundamentals of Lyapunov's direct method with respect to differential-algebraic systems are worked out. Novel definitions of Lyapunov functions for differentiable solution components of a DAE are stated, where the monotonically decreasing total time derivative of a Lyapunov function along DAE solutions is expressed in terms of the original system. The topology of the domain of the inherent dynamics turns out to be decisive for nonlocal existence of solutions given a Lyapunov function. As a result, practical stability criteria for bounded solutions of autonomous DAEs and for general solutions of DAEs with bounded partial derivatives of the constitutive function arise. Known contractivity definitions for DAEs can be interpreted in the context of this approach.

Key words: nonlinear differential-algebraic systems, tractability index, asymptotic stability, orbital stability, Andronov-Witt theorem, Lyapunov function, contractivity, characteristic multipliers, state space form, dissipativity.

Contents

Introduction

Differential-algebraic equations (DAEs) emerge from several areas of mathematics, including important applications like modelling of electronic circuits, constrained mechanical systems or solution of continuous optimization problems using Pontryagin's maximum principle. A thorough and systematical research on DAEs correlates with advancing development of computer technology in the last four decades. Considerable progress has been achieved in this period of time resulting e.g. in different concepts of an index of a DAE as a measure of its structural complexity. In particular, stability of solutions of differential-algebraic equations appears on the agenda due to its practical importance. Although elaborate results are available for ordinary differential equations, a comprehensive stability analysis of differential-algebraic systems is an issue of ongoing research. The goal of the present work is to contribute to the qualitative theory of differential-algebraic systems by providing new asymptotic stability criteria for a class of nonlinear, fully implicit DAEs with tractability index two. Here, stability refers to the propagation of perturbations in initial values of exact DAEs on $[t_0, \infty)$. *We aim at practical stability criteria under acceptable requirements using an integrative framework.*

Differential-algebraic equations are dynamical systems expressed in redundant coordinates. Most commonly, DAEs are regarded as coupled systems of ordinary differential equations and algebraic constraints or as vector fields on manifolds. Both approaches are closely related, the latter (geometric) approach being the coordinate-free formulation of the first one. That is, the mentioned vector field can be locally represented by an ordinary differential equation using a parametrization of the the tangent bundle of the constraint manifold. Thus the concept of a *local state space form* (SSF) of a DAE, i.e. the formulation as a differential equation in minimal coordinates is introduced. Anyway, one has to ascertain intrinsic properties having access only to an implicit representation (like $f(x'(t), x(t), t) = 0$) of the desired vector field or SSF of the DAE. In this regard we prefer the state space form for the stability investigations because it results in quite usable assumptions on the given DAE. In order to obtain the state space form, we take up and refine the decoupling approach in the context of the tractability index. This index is applied because of its algorithmic definition and favourable properties. Under certain conditions, we prove that a differentiable manifold is generated by solutions of differential-algebraic index-2 equations in a neighbourhood of a reference trajectory. The number of effective degrees of freedom equals the dimension of this manifold which is less than dimension of the embedding vector space. Decoupling a differential-algebraic equation means to reduce the dynamics to a lower dimensional state space, that is to express the dynamics in minimal coordinates. Thereby, an explicit representation of algebraic constraints acts as a parametrization of a section of the solution set. The presented construction of the local state space form necessitates less smoothness than comparable approaches using derivative arrays.

A distinguishing mark of the present work is that stability of the state space form is ensured using criteria in terms of the original DAE only. It is understood that neither an explicit access to the state space form nor a parametrisation of the tangent bundle of the constraint manifold are necessary in order to formulate our stability criteria.

First, we present is a generalization of the Andronov-Witt theorem to fully implicit differential algebraic equations of index $k = 1, 2$. Our result in Theorem 5.8 covers *self-oscillating* systems, i.e. autonomous DAEs with a periodic solution, hence being a reasonable endorsement of the available stability theory with respect to *forced oscillations*, i.e. periodic solutions of periodic DAEs. It allows a nice geometric interpretation via the state space form. Besides, the idea of orbital stability is barely considered in the context of the tractability index up to the present. Given sufficient smoothness, necessary structural conditions are prevalent in a class of DAEs resulting from the charge oriented modified nodal analysis in circuit simulation. One of them, i.e. constancy a systemic subspace along the reference solution is already known to be essential for the adequate qualitative behaviour of Runge-Kutta and BDF-discretizations. In other words, Theorem 5.8 could be of interest in practice.

We set up partial stability criteria using presumably new definitions of a Lyapunov function for differential-algebraic systems. An important feature of such a Lyapunov function referring to Dx resp. DP_1-components of the solution vector is the dissipation inequality (6.7) resp. (6.13) expressing the decrease of the Lyapunov function along integral curves of the DAE implicitly, but given in terms of the original DAE. Our investigations reveal that nonlocal existence of solutions is the crucial issue in case of differential-algebraic systems and this part is linked to an appropriate topology of the domain of the inherent dynamics which has to contain a cylindrical region around the reference solution. Related stability criteria covering a class of differential-algebraic systems omitted in previous publications are set up. Among others, practical stability criteria addressing the entire solution vector are stated for bounded solutions of autonomous DAEs with index $k = 1, 2$ and for general index-1 systems with bounded derivatives. Our definitions fit well into the established theory, e.g. offering an interpretation of P_0-contractivity.

In the process, a manifold enclosing the solution manifold of the DAE is constructed replacing some constraints by formal differentiation of those equations along suitable functions. The procedure is proved to decrease the tractability index of the resulting differential-algebraic system. If we transform the index reduced system to its state space form, we get an implicit parametrisation of the enclosing manifold which can be restricted to the constraint manifold of the original system. Strictly speaking, our definition of a Lyapunov function of an index-2 DAE applies to the inherent dynamics of the index reduced system. Due to a higher dimensional state space form, it is possible to prove stability of a superset of dynamical components.[1] As a matter of principle, stability of the state space form does not allow to deduce the asymptotic behaviour of

[1] We call a component of the solution vector of a DAE *dynamical,* if it is possible to assign an unrestricted initial value to that component.

the whole solution vector. It is conceivable that the expansion of the solution manifold in time is much faster than the convergence rate of DAE solutions with perturbed initial values towards a reference trajectory. Obviously, this cannot happen in case of autonomous or periodic differential-algebraic systems, in this case stronger stability results are achieved. The same applies to DAEs having a parametrisation of the constraint manifold subject to a uniform Lipschitz condition. Unfortunately, feasible sufficient conditions guaranteeing this property are very restrictive. Introducing the notion of *M-component-stability* as an adequate modification of partial stability in the sense of Lyapunov seems to be a legitimate trade-off between the above considerations and workable assumptions on the DAE.

To sum up, the state space analysis of differential-algebraic systems substantiates the aspiration that it is possible to adapt intricate results of the qualitative theory of differential equations to differential-algebraic systems. For example, the present dissertation enriches the existing arsenal of mathematical tools for DAEs relating to both direct and indirect method of Lyapunov.

This thesis is structured in two parts. The first one deals with the state space analysis of differential-algebraic equations adapting the concepts behind the tractability index to our stability investigations. For that purpose we introduce the tractability index together with a proper formulation of the derivative term of a DAE in the first chapter. We do not confine ourselves to technical details, but also indicate the origin of the tractability index of nonlinear DAEs and ponder a geometric interpretation of some systemic subspaces of the matrix chain. Moreover, the inherent dynamics and fundamental matrices of linear DAEs are considered. In the second chapter, we construct a state space representation of the inherent dynamics for a class of nonlinear index-2 DAEs. Structural conditions on autonomous DAEs which imply the existence of an autonomous state space form are stated. We prove commutativity of our transformation to the state space form and linearization as foundation of Lyapunov's indirect method for DAEs. The third chapter is dealing with the reduction of the tractability index two via differentiation of suitable constraints. From the geometric point of view, a manifold enclosing the constraint manifold of the given DAE is constructed. It turns out to be an easy way of verifying local constancy of the tractability index. This approach is particularly useful for defining Lyapunov functions.

In the second part of this thesis, the state space analysis is applied to obtain asymptotic stability criteria for solutions of nonlinear DAEs. In Chapter 4, stability concepts for differential-algebraic systems, particularly orbital stability of autonomous DAEs and the notion of M-component stability are introduced. We present a definition of characteristic multipliers of periodic solutions in case of autonomous or periodic DAEs exhibiting a properly stated derivative term in Chapter 5. Our approach is targeted on the state space representation, but is given strictly in terms of the differential-algebraic system. Therefore, we are able to make the Andronov-Witt Theorem and a further stability result accessible to fully implicit index-2 DAEs. Chapter 6 is dealing with Lyapunov's direct method for differential-algebraic equations, mainly with suitable Lyapunov functions in terms of the initial system for systems of index one and two. We specify the requirements needed to use the existence of such Lyapunov function as a criterion for (asymptotic) stability. The presented approach seems to be an interesting alternative in comparison to existing definitions of Lyapunov functions for singular differential equations and DAEs. To begin with, our modification of the stability in the sense of Lyapunov enables us to do without requiring a bounded solution set of the differential-algebraic system under consideration. The analytic approach considered in Chapter 6 proves to be the right framework to interpret known contractivity definitions for index-2 systems. Apparently, a new formulation of a least upper bound logarithmic Lipschitz constraint for DAEs directly linked to the notion of D-component contractivity is also possible. This is pointed out in Chapter 7. Moreover, it is shown that the reduction of the tractability index indicates a generalization of a regularization approach for Hessenberg-2 DAEs to fully implicit systems with certain structural assumptions. Chapter 7 also comprises an outline of how to apply presented methods to properly formulated DAEs with tractability index 3. Some hints at an algorithmic approach to compute a Lyapunov function for differential-algebraic systems are presented. Finally, we mention some open problems encountered during the research.

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Notation

$\exists$	there exists
$\forall$	for all
$x \in M$	x is an element of the set M
$M \subseteq N$	M is a subset of N
$N \cup M$	union of sets N, M
$N \cap M$	intersection of sets N, M
$\mathbb{R}$	set of real numbers
$\mathbb{R}^n$	n-times direct product with canonical structure of a n-dimensional vector space over $\mathbb{R}$
$\dim V$	algebraic dimension of a vector space V
$\ker A$	the kernel of a matrix A
$\operatorname{tr} A$	$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}$ the trace of a matrix $A \in \mathbb{R}^{n\times n}$
$\operatorname{im} A$	the range of a matrix A
$\operatorname{rk} A$	the rank of a matrix A
$\operatorname{cork} A$	$\operatorname{cork} A = \dim(\ker A)$ the nullity or co-rank of a matrix A
$f(M)$	$f(M) := \{f(x) \mid x \in M\}$ for a mapping $f:\ M_1 \supseteq M \to M_2$
LM	$LM := L(M)$ for a linear mapping L
$\mathrm{GL}_n(\mathbb{R})$	general linear group, group of invertible matrices in $\mathbb{R}^{n\times n}$
$0_{n\times m}$	null matrix in $\mathbb{R}^{n\times m}$
0_n	$0_n := 0_{n\times n}$, the additive neutral element in $GL_n(\mathbb{R})$
I_n	the identity matrix $(I_n)_{ij} = \delta_{ij} := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ in $\mathbb{R}^{n\times n}$
$A_{n\times m}$	symbolizes that A is an $n \times m$-matrix
$\mathbf{1}_n$	the vector $(1, \ldots, 1)^T \in \mathbb{R}^n$
$e_i \in \mathbb{R}^n$	the i-th column vector in I_n
$\overline{M}$	the closure of a set $M \subseteq \mathbb{R}^m$
M^0	the open interior of a set $M \subseteq \mathbb{R}^m$
∂M	the boundary $\partial M := \overline{M} \backslash M^0$ of $M \subseteq \mathbb{R}^m$
$\operatorname{dist}(N, M)$	$\operatorname{dist}(N, M) := \inf\{\Vert x - y\Vert \mid x \in N, y \in M\}$ the distance between two subsets N, M of a normed linear space V
$\mathrm{B}_r(x)$	$\mathrm{B}_r(x) := \{z \in V \mid \Vert z - x\Vert \leq r\}$ open spherical region around x with radius r in a normed space V
$C^k(U, \mathbb{R}^n)$	linear space of k-times ($k \geq 0$) continuously differentiable functions $f : U \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^m$ is an open set

$C(U,\mathbb{R}^n)$, $C^0(U,\mathbb{R}^n)$	$C(U,\mathbb{R}^n) = C^0(U,\mathbb{R}^n)$ linear space of continuous functions $f: U \to \mathbb{R}^n$ where $U \subseteq \mathbb{R}^m$ is an open set
$L_b(X,Y)$	the set of linear and bounded operators $X \to Y$ between normed spaces X, Y
$\frac{\partial}{\partial v} f(x_0)$	$\frac{\partial}{\partial v} f(x_0) = lim_{s\to 0} \frac{1}{s}[f(x_0+sv) - f(x_0)]$ directional derivative of f in the direction v at the point x_0
$f_y(y,x,t)$	$f_y(y,x,t) := \frac{\partial}{\partial y} f(y,x,t)$ the partial derivative of f with respect to y at (y,x,t)
$\mathrm{D}f(x_0)$	Jacobian of a differentiable function f at the point x_0
$x'(t)$, $\dot{x}(t)$	$x'(t) = \dot{x}(t) = \frac{\partial x}{\partial t}$ for a differentiable function $x = x(t)$
$\mathrm{T}_x\mathcal{M}$	tangent space to the manifold $\mathcal{M}$ at the point x
$\mathrm{T}\mathcal{M}$	tangent bundle to the manifold $\mathcal{M}$

DAE	Differential Algebraic Equation
ODE	Ordinary Differential Equation (meant to be explicit/regular/in the canonical form $x'(t) = f(x(t), t)$)
IVP	Initial Value Problem
BVP	Boundary Value Problem
IRODE	Inherent Regular Ordinary Differential Equation
IR-DAE	Index Reduced Differential-Algebraic Equation
MNA	Modified Nodal Analysis

Let $I \subseteq \mathbb{R}$, $D(t) \in \mathbb{R}^{n\times m}$ and $x_* \in C^1_D(I,\mathbb{R}^m)$:

integral curve of x_*	$\{(x_*(t), t) \in \mathbb{R}^m \times I \,\|\, t \in I\}$
extended integral curve of x_*	$\left\{\left((Dx_*)'(t), x_*(t), t\right) \in \mathbb{R}^n \times \mathbb{R}^m \times I \,\|\, t \in I\right\}$
trajectory of x_*	$\{x_*(t) \in \mathbb{R}^m \,\|\, t \in I\}$
extended trajectory of x_*	$\left\{\left((Dx_*)'(t), x_*(t)\right) \in \mathbb{R}^n \times \mathbb{R}^m \,\|\, t \in I\right\}$

Part I

State space analysis of differential-algebraic equations

1 Properly formulated DAEs with tractability index 2

Introduction

> Only for you, children of doctrine and learning, have we written this work. Examine this book, ponder the meaning we have dispersed in various places and gathered again; what we have concealed in one place we have disclosed in another, that it may be understood by your wisdom.
>
> *(Heinrich Cornelius Agrippa von Nettesheim, De occulta philosophia, 3)*

Undoubtedly, differential equations are the most important tool of mathematical modelling. Since the early days of calculus, research on ordinary differential equations has been leading to qualitative considerations. The French Academy of Sciences formulated the goal to obtain stability criteria for ODEs about the year 1820, which was a considerable milestone. Consecutively, stability criteria for linear systems with constant coefficients were developed by famous mathematicians like Hermite (1856), Routh (1877) and Hurwitz (1895), predominantly by algebraic approaches. Important contributions to nonlinear systems were made by Henri Poincare. The most common concept is stability in the sense of Lyapunov in honor of the Russian mathematician A. M. Lyapunov. His outstanding treatise "The general problem of the stability of motion" (1892) has a lasting influence on this area of research. In the middle of the last century, Lyapunov's direct method generalizing mechanical potentials (by means of so called Lyapunov functions) was strengthened by the achievements of LaSalle, Zubov, Yoshizawa, Malkin and many others. Besides, Lyapunov's indirect method based upon the linearization principle makes interesting stability criteria possible.

Fundamental research on differential-algebraic systems is far less advanced compared to explicit ODEs. That is why we are going to clarify the mathematical background of a class of nonlinear fully implicit systems with tractability index up to two in the first part of this thesis. In doing so, we do not aim at an all-embracing introduction neither to differential-algebraic equations nor to stability of ODEs. Instead, we focus on a self-contained and goal-oriented presentation. We try to clarify and motivate the analytical approaches to DAEs used in this thesis and to present our stability results in an understandable way. The publications listed in the necessarily incomplete bibliography might be helpful in order to delve into the subject and to classify the presented results. Additional references can be found in the cited papers.

The first chapter contains the necessary fundamentals associated with differential-algebraic equations and the tractability index. They are used to construct a state space representation (SSF) of DAEs with properly stated derivative term nearby a reference solution. Details on the construction of the SSF and its properties are dealt with in Chapter 2. These results are essential for the indirect method of Lyapunov applied to DAEs as presented in Chapter 6. Lyapunov functions for differential-algebraic systems are based on reduction of the tractability index via differentiation of constraints presented in Chapter 3.

Definition 1.1. Let $\mathcal{G} \subseteq \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ be a region and $f = f(y, x, t) \in C(\mathcal{G}, \mathbb{R}^m)$ continuously partial differentiable with respect to y. If $\frac{\partial}{\partial y} f(y, x, t)$ is not of full rank on $\mathcal{G}$ then the implicit ordinary differential equation

$$f(x'(t), x(t), t) = 0 \tag{1.1}$$

is called *differential-algebraic equation*, abbreviated *DAE.*

Accordingly, DAEs are implicit ordinary differential equations which cannot be solved for the derivative using the implicit function theorem on its entire domain. Differential-algebraic systems are also known as *semi-state* or *semistate systems*, *singular systems* or *descriptor systems*, confer [Ria08, § 1.1] and the abundant (historic) references therein. The structural complexity of ODEs pales in comparison to differential-algebraic equations. Hence unique solvability cannot be proved using the same functional analytic means and one has to demand additional structural characteristics. As a general rule, it is recommendable to investigate systems possessing a certain upper bound on structural complexity, in other words a fixed index.

Varying rank of the partial derivative f_y might impact seriously on considered differential-algebraic systems. For example, a transition from $\mathrm{rk}\, f_y = k > 0$ to $\mathrm{rk}\, f_y = 0$ implies that derivatives of solution components cease to exist in the equations. Following the lines of [GM86, §1.2.2], we restrict our attention to DAEs satisfying the constant rank condition $\mathrm{rk}\, f_y\,(y, x, t) = \text{const.}$ on the entire domain in order to avoid singularities. Such DAEs are called *normal DAEs* in [GM86, p. 30]. In addition, we use a modified representation of a DAE which has proved to be more suitable both for theoretical analysis and numerical integration schemes.

1.1 Properly stated derivative term

It is common practice to refer to an idempotent ($P^2 = P$) linear mapping P as projector. P projects onto the subspace V_1 if $\mathrm{im}\, P = V_1$ and P projects along V_2 if $\ker P = V_2$. Some basic properties of projectors are given in Lemma 8.1.

Definition 1.2. Let $\mathcal{G} \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ be a region containing the connecting segment for any $(y, x, t), (z, x, t) \in \mathcal{G}$. Let $I = \{t \in \mathbb{R} \mid \ (y, x, t) \in \mathcal{G}\}$ and

$$D \in C(I, \mathbb{R}^{n\times m}),\ f = f(y, x, t) \in C(\mathcal{G}, \mathbb{R}^m) \quad \text{with} \quad f_y \in C\left(\mathcal{G}, \mathbb{R}^{m\times n}\right).$$

The implicit system

$$f\left((D(t)x(t))', x(t), t\right) = 0 \tag{1.2}$$

is called a *differential-algebraic system with a properly stated derivative term* or *properly formulated DAE*, if

$$\ker f_y(y,x,t) \oplus \operatorname{im} D(t) = \mathbb{R}^n \qquad \text{for all } (y,x,t) \in \mathcal{G} \tag{1.3}$$

and there is a projector $R \in C^1(I, \mathbb{R}^{n\times n})$ onto $\operatorname{im} D(t)$ along $\ker f_y(y,x,t)$.

We tacitly assume $\operatorname{rk} f_y < m$ so the properly formulated system is a differential-algebraic one according to Definition (1.1). In order to avoid singularities, normal DAEs only are considered, i.e.

$$\operatorname{rk} D(t) \equiv r \ \text{ and } \ \operatorname{rk} f_y(y,x,t) \equiv n-r$$

Lemma 1.3. *Condition* (1.3) *is equivalent to the following one:*

$$\left.\begin{array}{l} \ker f_y\,(y,x,t) \cap \operatorname{im} D(t) = \{0\} \\ \operatorname{im} f_y\,(y,x,t)\,D(t) = \operatorname{im} f_y\,(y,x,t) \\ \ker f_y\,(y,x,t)\,D(t) = \ker D(t) \end{array}\right\} \text{ for all } (y,x,t) \in \mathcal{G}$$

Proof. See [MHT03a, Lemma 29]. □

Essentially, the proper formulation of the derivative term is a suitable restriction on $\ker f_y\,(y,x,t)$. In particular, $\ker f_y(y,x,t) = \ker R(t)$ is independent of y and x because of (1.3), $\operatorname{im} D(t)$ and $\ker f_y\,(y,x,t)$ have continuously differentiable basis functions due to projector $R \in C^1$ and a constant dimension. It follows that a properly formulated DAE is equivalent to

$$f\left(R(t)\,(Dx)'\,(t), x(t), t\right) = 0$$

due to the mean value theorem

$$f\,(y,x,t) - f\,(R(t)y,x,t) = \int_0^1 f_y\,(R(t)y + s\,(I - R\,(t))\,y,x,t)\,(I - R\,(t))\,y\mathrm{d}s = 0$$

We recognize that differentiability of solution components in $\ker f_y = \ker D(t)$ is completely unnecessary, therefore it is possible to weaken the classical notion of solution of a differential equation.

Definition 1.4. Consider the DAE (1.2) with properly stated derivative term. A function

$$x \in C^1_D\,(I_0, \mathbb{R}^m) := \{x \in C\,(I_0, \mathbb{R}^m) \mid Dx \in C^1\,(I_0, \mathbb{R}^n)\}$$

satisfying the DAE on the t-interval $I_0 \subseteq I$ is called a *solution* of the DAE on I_0.

$C^1_D\,(I_0, \mathbb{R}^m)$ is a vector space over $\mathbb{R}$ using pointwise addition and scalar multiplication. In this context, it is natural to equip the space with the modified C^1-norm

$$\|x\|_{C^1_D} := \|x\|_\infty + \|(Dx)'\|_\infty$$

Then $(C^1_D, \|.\|_{C^1_D})$ is a Banach space (compare [GM86, theorem 9]). We are going to denote such function spaces with restricted differentiability by $C^1_M = \{x \in C^0 \mid Mx \in C^1\}$, $M \in C\,(I, \mathbb{R}^{s\times m})$.

Definition 1.5. We call the set

$$\mathcal{M}_0(t) := \{x \in \mathbb{R}^m \mid \exists y \in \mathbb{R}^n : (y, x, t) \in \mathcal{G} \text{ and } f(y, x, t) = 0\}$$

obvious or *first-level constraint* of the DAE (1.2) in $t \in I$.

For any solution $x \in C_D^1(I_0, \mathbb{R}^m)$ of a differential-algebraic equation $x(t) \in \mathcal{M}_0(t)$ holds because of $\left((Dx)'(t), x(t), t\right) \in \mathcal{G}$ and $f\left((Dx)'(t), x(t), t\right) = 0$. In Lemma 1.19, we show that for properly formulated DAEs

$$\forall x \in \mathcal{M}_0(t)\, \exists! y = R(t)\, y : f(y, x, t) = 0$$

is valid, i.e. the components $R(t)(Dx)'(t)$ proceeding in im $D(t)$ are uniquely defined. Notice that in general $R(Dx)' = (Dx)' + R'Dx \neq (Dx)'$.

Definition 1.6. The *solution set* or *configuration space* of a differential-algebraic system (evaluated at time $t \in I$) is a subset $\mathcal{M}(t) \subseteq \mathcal{M}_0(t) \subseteq \mathbb{R}^m$ possessing the property that $\forall t \in I$, $x_0 \in \mathcal{M}(t)$ there exists a solution $x \in C_D^1$ of the DAE fulfilling $x(t) = x_0$. The elements $x_0 \in \mathcal{M}(t)$ are called *consistent initial values* of the given system.

The notion of the configuration space of DAEs seems to be abstract. Later on we prove that $\mathcal{M}_0(t)$ is the solution set of an index-1 DAE. Thus we denote the configuration space of an index-k system by $\mathcal{M}_{k-1}(t)$.or

Definition 1.7. Consider a matrix $A \in \mathbb{R}^{m\times n}$ and projectors $R \in \mathbb{R}^{n\times n}$ along $\ker A$ and $S \in \mathbb{R}^{m\times m}$ onto im A. The unique *reflexive generalized inverse* $A^- \in \mathbb{R}^{n\times m}$ (also called *pseudoinverse)* with

$$\begin{array}{lcll} A^-AA^- & = & A^- & \text{(generalized inverse)} \\ AA^-A & = & A & \text{(reflexive)} \end{array}$$

is well-defined by

$$A^-A = R, \quad AA^- = S$$

Let us mention that

$$\text{im } A^- = \text{im } R, \quad \ker A^- = \ker S$$

is true due to

$$\begin{array}{ll} \text{im } A^- = \text{im } A^-AA^- \subseteq \text{im } A^-A = \text{im } R, & \text{im } R = \text{im } A^-A \subseteq \text{im } A^- \\ \ker A^- = \ker A^-AA^- \supseteq \ker AA^- = \ker S, & \ker A^- \subseteq \ker AA^- = \ker S \end{array}$$

In accordance with Lemma 8.1, projector R is fixed by the subspace im A^- complementary to $\ker A$. The same principle is used to specify S via the complementary space $\ker A^-$ to im A. Using respective orthogonal complementary spaces, the *Moore-Penrose inverse* A^+ is obtained. [Zie79] provides a classification of generalized inverses including construction of the reflexive pseudoinverse emanating from a singular value decomposition of the given matrix A.

1.2 Tractability index

Consider the nonlinear DAE (1.2) with properly stated derivative term. Additionally, assume f to be continuously partially differentiable with respect to x. Following the lines of [MH04], [Mä03] and [Mä01], the tractability index is defined using a chain of suitable matrices, projectors and subspaces. Denote

$$\begin{aligned} G_0(y,x,t) &= f_y(y,x,t)D(t) \\ N_0(y,x,t) &= \ker G_0(y,x,t) \\ S_0(y,x,t) &= \{z \in \mathbb{R}^m \mid f_x(y,x,t)z \in \text{im } G_0(y,x,t)\} \end{aligned}$$

By Lemma 1.3 it follows that $N_0 = \ker D(t)$ so we are going to choose a continuous projector $Q_0(t)$ onto $N_0(t)$ and its complementary projector $P_0(t) = I - Q_0(t)$ t-dependent only. Next, define

$$\begin{aligned} G_1(y,x,t) &= G_0(y,x,t) + f_x(y,x,t)Q_0(t) \\ N_1(y,x,t) &= \ker G_1(y,x,t) \\ S_1(y,x,t) &= \{z \in \mathbb{R}^m \mid f_x(y,x,t)P_0(t)z \in \text{im } G_1(y,x,t)\} \\ G_2(y,x,t) &= G_1(y,x,t) + f_x(y,x,t)P_0(t)Q_1(y,x,t) \end{aligned}$$

where $Q_1(y,x,t)$ is a continuous projector onto $N_1(y,x,t)$ satisfying

$$Q_1(y,x,t)Q_0(t) \equiv 0$$

Such a projector Q_1 is called *admissible*.

Definition 1.8. The nonlinear DAE (1.2) with properly stated derivative term is referred to as

1. a system having *tractability index 1* in a point $(y,x,t) \in \mathcal{G}$ of its domain, if

   $$\text{rk } G_0(y,x,t) = r_0 > 0, \quad N_0(t) \cap S_0(y,x,t) = \{0\}$$

   We speak about tractability index 1 on $\tilde{\mathcal{G}}$ if both properties hold on the entire open subset $\tilde{\mathcal{G}} \subseteq \mathcal{G}$.
2. a DAE possessing *tractability index 2* on $\tilde{\mathcal{G}}$ if

   $$\dim N_0(t) \cap S_0(y,x,t) = r_1 > 0, \quad N_1(y,x,t) \cap S_1(y,x,t) = \{0\}$$

   is true for all $(y,x,t) \in \tilde{\mathcal{G}}$.

Remark 1.9. Theorem A.13 in [GM86] implies that even in case of non-regular matrix pencils the following three conditions are equivalent

$$N_0(t) \cap S_0(y,x,t) = \{0\},\ N_0(t) \oplus S_0(y,x,t) = \mathbb{R}^m,\ G_1(y,x,t) \in \text{GL}_m(\mathbb{R})$$

in $(y,x,t) \in \mathcal{G}$. Similarly, the conditions

$$N_1(y,x,t) \cap S_1(y,x,t) = \{0\},\ N_1(y,x,t) \cap S_1(y,x,t) = \mathbb{R}^m,\ G_2(y,x,t) \in \text{GL}_m(\mathbb{R})$$

are equivalent. We are going to make use of these characterizations of the second condition in Definition 1.8 later on.

The matrix valued functions G_i are continuous due to definition, same holds for im G_i. Therefore, we are allowed to define continuous projectors

$$\begin{array}{lll} W_0(y,x,t) & \text{along} & \text{im } G_0\,(y,x,t) = \text{im } f_y(y,x,t) \\ W_1(y,x,t) & \text{along} & \text{im } G_1\,(y,x,t) \end{array}$$

in order to describe the systemic important subspaces S_i by

$$\begin{aligned} S_0(y,x,t) &= \ker W_0(y,x,t) f_x(y,x,t) \\ S_1(y,x,t) &= \ker W_1(y,x,t) f_x(y,x,t) P_0(t) \end{aligned}$$

Additionally, we take $N_0\,(t) \cap S_0\,(y,x,t)$ into consideration and denote a continuous projector onto this subspace by $T\,(y,x,t)$ and $U\,(y,x,t) := I_m - T\,(y,x,t)$.

In this thesis, the reflexive generalized inverse $D^-\,(t)$ related to the properly formulated leading derivative and the tractability index via

$$D^-\,(t)\,D\,(t) = P_0\,(t) \;\text{ and }\; D\,(t)\,D^-\,(t) = R\,(t)$$

is used exclusively.

Lemma 1.10. *Considering the DAE* (1.2)*, the following properties of the matrix chain of the tractability index are valid on entire domain:*

1. $N_0(t) \subseteq S_1\,(y,x,t)$
2. $\text{cork } G_1\,(y,x,t) = \dim N_1\,(y,x,t) = \dim N_0(t) \cap S_0\,(y,x,t)$
3. $\text{im } Q_0(t) Q_1\,(y,x,t) = N_0(t) \cap S_0\,(y,x,t)$

Proof. Ad 1) Choose an arbitrary $(y,x,t) \in \mathcal{G}$. Obviously, for $x \in N\,(t)$

$$x = Q_0(t)x \in \ker W_1\,(y,x,t)\,f_x\,(y,x,t)\,P_0(t) = S_1\,(y,x,t)$$

and consequently $N_0(t) \subseteq S_1\,(y,x,t)$ hold.

Ad 2) Using the representation

$$G_1 = G_0 + f_x Q_0 = G_0 + W_0 f_x Q_0 + (I - W_0) f_x Q_0 = G_0 + W_0 f_x Q_0 + G_0 G_0^- f_x Q_0$$

together with the reflexive pseudoinverse $G_0^-\,(y,x,t)$ which is defined pointwise by

$$G_0^-\,(y,x,t)\,G_0\,(y,x,t) = P_0(t), \quad G_0\,(y,x,t)\,G_0^-\,(y,x,t) = I - W_0\,(y,x,t)$$

we obtain $Q_0 G_0^- = (I - P_0) G_0^- = 0$ and

$$G_1 = (G_0 + W_0 f_x Q_0)(I + G_0^- f_x Q_0) =: HF$$

Let us point out that $(I + MN)^{-1} = I - MN$ if $NM = 0$. Therefore, $F^{-1} = I - G_0^- f_x Q_0$ due to $Q_0 G_0^- = 0$. Now,

$$\begin{aligned} \ker H\,(y,x,t) &= \ker G_0\,(y,x,t) \cap \ker\,(W_0 f_x Q_0)\,(y,x,t) \\ &= N_0(t) \cap S_0\,(y,x,t) \end{aligned}$$

because W_0 is a projector. Finally,

$$\begin{aligned} N_1(y,x,t) &= \ker H(y,x,t)\,F(y,x,t) = F^{-1}(y,x,t)\ker H(y,x,t) \\ &= F^{-1}(y,x,t)\,(N_0(t)\cap S_0(y,x,t)) \end{aligned}$$

In particular, $\dim N_0(t)\cap S_0(y,x,t) = \text{const.}$ is proved to be equivalent to a constant rank assumption on $G_1(y,x,t)$.

Ad 3) $\operatorname{im} Q_0Q_1 \subseteq N_0\cap S_0$ is valid due to $Q_0Q_1x\in N_0$ and

$$W_0f_xQ_0Q_1 = W_0(AD+f_xQ_0)\,Q_1 = W_0G_1Q_1 = 0$$

that is $Q_0Q_1x\in\ker W_0f_x = S_0$. Conversely, for $x\in N_0\cap S_0$ per Definition $x=Q_0x$ and $f_xx = ADz$ for a $z\in\mathbb{R}^m$. With $u = x-P_0z$ and

$$G_1u = G_1Q_0x - G_1P_0z = f_xx - ADz = 0$$

it follows $u=Q_1u$ and $x = Q_0u = Q_0Q_1u\in\operatorname{im} Q_0Q_1$. □

In case of index-2 systems $\mathbb{R}^m = S_1\oplus N_1$ is valid, therefore Q_1 can be chosen as the unique projector $Q_{1,c}$ onto N_1 along S_1. This choice is admissible because of $\operatorname{im} Q_0 = N_0\subseteq S_1 = \ker Q_{1,c}$ and $Q_{1,c}$ is called *canonical* index-2 projector.

Lemma 1.11. *The canonical projector $Q_{1,c}$ admits the representation*

$$Q_{1,c}(y,x,t) = Q_1(y,x,t)\,G_2^{-1}(y,x,t)\,f_x(y,x,t)\,P_0(t) \tag{1.4}$$

Thereby, G_2 is constructed using an arbitrary projector Q_0 and an admissible Q_1.

Proof. A projector is defined by $Q_* := Q_1G_2^{-1}f_xP_0$ due to $f_xP_0Q_1 = G_2Q_1$ and

$$Q_1G_2^{-1}f_xP_0Q_1G_2^{-1}f_xP_0 = Q_1G_2^{-1}G_2Q_1G_2^{-1}f_xP_0 = Q_1G_2^{-1}f_xP_0$$

Further, for $x\in N_1$ follows that $Q_*x = \left(Q_1G_2^{-1}f_xP_0\right)Q_1x = Q_1x = x$ and $\operatorname{im} Q_* = N_1$. $Q_1G_2^{-1}G_1 = Q_1G_2^{-1}G_2P_1 = 0$ implies $Q_*x = Q_1G_2^{-1}W_1f_xP_0x$ and subsequently $S_1\subseteq \ker Q_*$, due to $S_1 = \ker W_1f_xP_0$. By reason of dimension, Q_* projects along S_1 implying (1.4) to be a representation of the canonical projector. □

Lemma 1.12. *If $Q_1(y,x,t)$ projects onto a subspace $V_1(y,x,t)$ along $V_2(y,x,t)$ and $N_0(t)\subseteq V_2(y,x,t)$ then DQ_1D^- and DP_1D^- are projectors as well,*

DQ_1D^- projects onto DV_1 along $DV_2\times\ker A$, DP_1D^- onto DV_2 along $DV_1\times\ker A$

Proof. Due to assumptions we made, $Q_1(y,x,t)\,Q_0(t) = 0$ is valid and consequently,

$$\begin{aligned} (DQ_1D^-)(DQ_1D^-) &= DQ_1(I-Q_0)Q_1D^- = DQ_1D^- \\ (DP_1D^-)(DP_1D^-) &= D(I-Q_1)P_0(I-Q_1)D^- = D(I-Q_1)D^- = DP_1D^- \end{aligned}$$

In addition,

$$\begin{aligned}
DV_1 &= D(\mathrm{im}\ Q_1) = \mathrm{im}\ DQ_1 \supseteq \mathrm{im}\ DQ_1D^- \supseteq \mathrm{im}\ DQ_1D^-D = \mathrm{im}\ D(Q_1P_0) = \mathrm{im}\ DQ_1 \\
DV_2 &= \mathrm{im}\ DP_1 \supseteq \mathrm{im}\ DP_1D^- \supseteq \mathrm{im}\ DP_1P_0 = \mathrm{im}\ D(I-Q_1)(I-Q_0) = \mathrm{im}\ DP_1
\end{aligned}$$

imply $\mathrm{im}\ DQ_1D^- = DV_1$ and $\mathrm{im}\ DP_1D^- = DV_2$. Moreover,

$$\begin{aligned}
(DP_1D^-)(DQ_1D^-) = 0 = (DQ_1D^-)(DP_1D^-),\\
(I-R)(DP_1D^-) = 0 = (I-R)(DQ_1D^-),
\end{aligned}$$

Q_1 being a projector implies $V_1 \oplus V_2 = \mathbb{R}^m$ and the proper formulation leads to $DV_1 \oplus DV_2 \oplus \ker A = \mathbb{R}^n$ thereby proving the assertion on range and kernel of DQ_1D^-, DP_1D^-. □

There are special types of differential-algebraic systems which occur frequently in applications:

Definition 1.13. Let $I \subseteq \mathbb{R}$ be an interval and $\hat{\mathcal{G}} \subseteq \mathbb{R}^m$ a region. Furthermore,

$$A \in C(\hat{\mathcal{G}} \times I, \mathbb{R}^{m\times n}), \quad D \in C(I, \mathbb{R}^{n\times m}), \quad b \in C(\hat{\mathcal{G}} \times I, \mathbb{R}^m)$$

A differential-algebraic equation in the shape of

$$A(x(t),t)(D(t)x(t))' + b(x(t),t) = 0 \tag{1.5}$$

is called *linear implicit*.

Modelling mechanical systems results in semi-explicit differential-algebraic equations of a special structure. Following the lines of [AP97, p. 238ff.], we define

Definition 1.14. Let $\mathcal{G} \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}$ be a region and $h \in C(\mathcal{G}, \mathbb{R}^n)$, $g \in C(\mathcal{G}, \mathbb{R}^{m-n})$. The differential-algebraic system

$$\begin{aligned}
x_1'(t) &= h(x_1(t), x_2(t), t) \\
0 &= g(x_1(t), t)
\end{aligned} \tag{1.6}$$

is called a *Hessenberg system* with index 2, if the partial derivatives h_{x_2}, g_{x_1} are continuous and the matrix $g_{x_1}(x_1,t)\, h_{x_2}(x_1,x_2,t)$ is invertible on $\mathcal{G}$.

General Assumptions

In this thesis, index of a DAE will mean tractability index, otherwise the type of index is denoted explicitly. With no modifiers, differential-algebraic equations are assumed to have a properly formulated derivative term. For convenience we only consider square systems. Domains of functions are meant to be regions, that is open and connected. We omit the arguments of a function in some places to keep track of formulae, if it is possible without confusion. For example, dependency on t could be omitted in the matrix chain of linear DAEs. Functions of multiple arguments are meant to be evaluated along the (extended) integral curve of a reference function x_*, if they are noted t-dependent only, e.g. $f = f(y,x,t)$ and $f(t) = f\left((Dx_*)'(t), x_*(t), t\right)$.

Structural conditions in a nutshell

Figure 1.1.2 illustrates the fundamental structural conditions used in this thesis. We arrange the structural conditions for properly formulated DAEs regarding the important classes of Hessenberg-2 DAEs and linear implicit DAEs resulting from the Modified Nodal Analysis of electrical circuits (MNA-equations, cf. § 5.2.2.1). In detail, we investigate sufficiently smooth index-2 systems exhibiting

- *Fully implicit DAE - tractability index reduction*: $f\left((Dx)'(t), x(t), t\right) = 0$ with $\ker D = \text{const.}$ and $\operatorname{im} G_1(y,x,t)$ dependent on $P_0 x$, t
- *Fully implicit DAEs - $T = T(t)$*: $f\left((Dx)'(t), x(t), t\right) = 0$ and $N_0(t) \cap S_0(y,x,t)$ independent of y, x in a neighbourhood of the extended integral curve of a reference solution x_*.
- *Fully implicit DAEs - complete decoupling*: assume $\operatorname{im} \begin{pmatrix} T \\ -f_y^- f_x T \end{pmatrix}(y,x,t)$ t-dependent only around the extended integral curve of x_*. Additionally, we require constant subspaces $\operatorname{im} DP_1$ and $\operatorname{im} DQ_1$ along x_* in order to obtain an autonomous state space form for autonomous DAEs.
- *Linear implicit DAEs - complete decoupling*: $A(t)(Dx)'(t) + b(x(t), t) = 0$ and $N_0(t) \cap S_0(y,x,t)$ independent of y, x in a neighbourhood of a reference solution x_*. In fact, we have to impose $\operatorname{im} DP_1$ and $\operatorname{im} DQ_1$ constant along x_* thus making the regarded class of DAEs smaller.

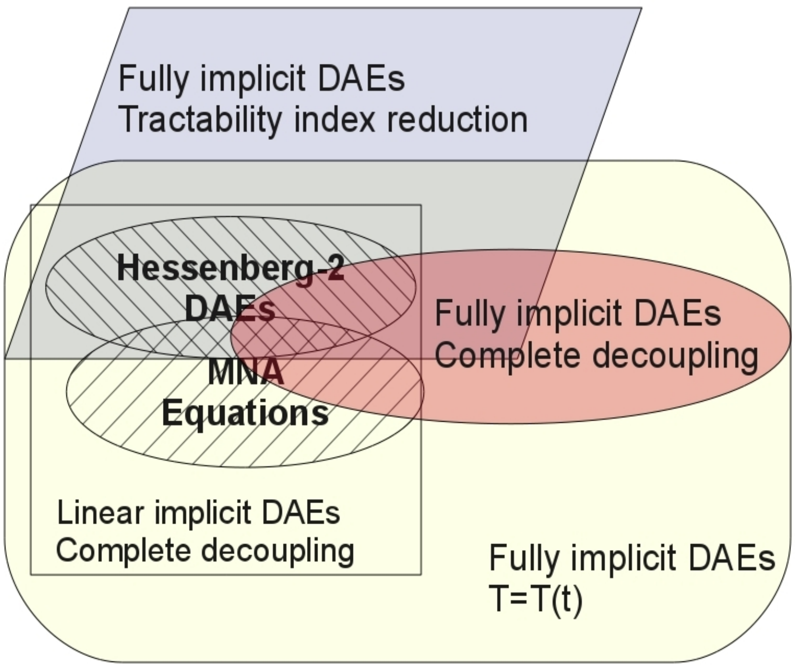

Figure 1.1: Set diagram of important structural conditions in use

1.2.1 Excursus: geometric index

Formally, the tractability index is based on linearization and a generalization of the Kronecker index to time dependent linear DAEs. The first elements of the matrix chain have a geometric interpretation enabling us to interpret the index one condition in Definition 1.2. Denote the projection onto the second component by $\mathrm{pr}_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $\mathrm{pr}_2 (y, x) := x$. It follows that the obvious constraint of the properly stated DAE (1.2) is $\mathcal{M}_0(t) = \mathrm{pr}_2 \mathcal{N}(t)$ referring to

$$\mathcal{N}(t) := \left\{(y, x) \in \mathbb{R}^{n+m} \mid (y, x, t) \in \mathcal{G},\ f(y, x, t) = 0\right\}$$

Thereby $\mathcal{N}(t)$ represents locally a differentiable manifold, assuming rk $\begin{pmatrix} f_y & f_x \end{pmatrix}$ be locally constant. Then, the tangent space at $(y_0, x_0) \in \mathcal{N}(t)$ is

$$\mathrm{T}_{(y_0,x_0)}\mathcal{N}(t) \;=\; \ker\begin{pmatrix} f_y (y_0, x_0, t) & f_x (y_0, x_0, t) \end{pmatrix}$$

and

$$\begin{aligned} \mathrm{pr}_2\mathrm{T}_{(y_0,x_0)}\mathcal{N}(t) &= \{x \in \mathbb{R}^m \mid \exists y \in \mathbb{R}^n :\ f_y (y_0, x_0, t)\, y + f_x (y_0, x_0, t)\, x = 0\} \\ &= S_0 (y_0, x_0, t) \end{aligned}$$

The projection $\mathcal{M}_0 (t)$ of $\mathcal{N} (t)$ does not necessarily inherit the structure of a manifold. In general, if $\mathcal{M}_0 (t)$ is a manifold then $S_0(y, x, t) \subseteq \mathrm{T}_x\mathcal{M}_0(t)$ is true. Just consider a $z \in S_0 (y, x, t)$, there exists an $\epsilon > 0$ and functions v, w such that $\forall s \in (-\epsilon, \epsilon)$: $(v (s) , w (s)) \in \mathcal{N} (t)$, $(v (0) , w (0)) = (y, x)$, $(v, w) (s)$ is differentiable in $s = 0$ and $z = w' (0)$. Obviously, $\forall s \in (-\epsilon, \epsilon)$: $w (s) \in \mathcal{M}_0 (t) = \mathrm{pr}_2\mathcal{N} (t)$ and $w (0) = x$. Therefore, $z = w' (0) \in \mathrm{T}_x\mathcal{M}_0 (t)$.

Additionally, in case of properly formulated index-1 DAEs, there exists a function $y = y (x)$ such that $f (y, x, t) = 0 \Leftrightarrow f (y (x) , x, t) = 0$ due to Lemma 1.19. Accordingly,

$$\dim S_0 (y, x, t) = \dim \mathcal{N} (t) = \dim \mathcal{M}_0 (t) = \dim \mathrm{T}_x\mathcal{M}_0 (t)$$

resulting in $S_0 (y, x, t) = \mathrm{T}_x\mathcal{M}_0 (t)$.

The connection between autonomous linear implicit DAEs

$$A (x(t))\, x'(t) + b (x(t)) = 0$$

and vector fields on manifolds is enlightened in [Rei89]. Hereby, a set $\mathcal{N}$ corresponding to the given DAE is defined likewise. One becomes aware that in case of a classical solution $x \in C^1 (I, \mathbb{R}^m)$ it holds

1. $(x'(t), x(t)) \in \mathcal{N}$, therefore $x(t) \in \mathcal{M}_0 = \mathrm{pr}_2\mathcal{N}$
2. $x'(t)$ is a tangent vector to $\mathcal{M}_0$, that is $(x'(t), x(t)) \in \mathrm{T}\mathcal{M}_0$ is an element of the tangent bundle to $\mathcal{M}_0$

if we suppose $\mathcal{M}_0$ to be a manifold. Necessarily, a C^1-solution of the given system has the property

$$\forall \in I :\ (x'(t), x(t)) \in \mathcal{N} \cap \mathrm{T}\mathcal{M}_0$$

Therefore, the solutions evaluated in t belong to the set $\mathcal{M}_1(t) := \mathrm{pr}_2\,(\mathcal{N} \cap \mathrm{T}\mathcal{M}_0)$. If $\mathcal{M}_1(t)$ is a manifold and $\mathcal{M}_1(t) \neq \{0\}$ then we are able to apply the reasoning to this constraint set. A sequence $\mathcal{M}_i(t)$ of *constraint manifolds* for classical solutions of the DAE is constructed iteratively using this approach. The maximal index i with $\mathcal{M}_i \neq \mathcal{M}_{i-1}$ and $\mathcal{M}_{i+1} = \mathcal{M}_i$ is called the degree of the given system. A DAE of degree s possessing the property that $\mathcal{N} \cap \mathrm{T}\mathcal{M}_s$ is a continuous manifold and $|\mathcal{N} \cap \mathrm{T}_x\mathcal{M}_s| = 1$ can be realized as a vector field on $\mathcal{M}_s$. Such differential-algebraic system is called *regular* and $\mathcal{M}_s$ is the configuration space of the DAE, compare [Rei89, S.36] resp. [Rei91] regarding non autonomous DAEs. This approach is picked up and formulated more precisely in [RR02, Ch. IV], it is leading to a *geometric* notion of index for DAEs. Assuming differentiability and certain constant rank conditions, the tractability index one condition $N_0\,(t) \oplus S_0\,(y, x, t) = \mathbb{R}^m$ implies

$$\mathcal{M}_1\,(t) = \mathrm{pr}_2\,(\mathrm{T}\mathcal{M}_0 \cap \mathcal{N}) = \mathrm{pr}_2\mathcal{N} = \mathcal{M}_0\,(t)$$

that is the geometric index is one. For higher index DAEs ($k > 1$) the connection between both index definitions is barely investigated.

The geometric index enables to use qualitative theory of vector fields on manifolds in order to derive new results for differential-algebraic equations like it is done in [Rei95]. On the other hand, the geometric index is proved to be a geometric interpretation of the differentiation index in reasonable settings. Sufficiently smooth functions are required in order to perform the operations. Moreover, we have to provide criteria for $\mathcal{M}_i$ being manifolds. The given geometric interpretation does not support the suitable development of the standard form (1.1) to properly formulated systems (1.2). These restrictions on f and the methods of proof presented in this thesis are good reasons to stick to the state space representation of a DAE instead of its realization as a vector field on the configuration space.

1.2.2 Excursus: alternative formulation of the matrix chain

The matrix chain of the tractability index can be defined in an alternative fashion. For example, such matrix chain is constructed in [Mä02b] for every possible index $k \in \mathbb{N}$ aiming at linear systems. An extension to linear implicit DAEs takes place in [Mä05] defining the tractability index $k \in \mathbb{N}$ in a way that index k of every linearization along functions $x_* \in C^1_D$, which map to a region containing $(y, x, t) \in G$ implies the same tractability index of the given system. The projector $DP_0P_1D^- = DP_1D^-$ is assumed to be independent of y and continuously differentiable in order to define tractability index 2 [Mä05]. In addition, the element $\tilde{G}_2$ of the new matrix chain reads

$$\begin{aligned}\tilde{G}_2(z, y, x, t) \;=\;& G_1(y, x, t) + f_x(y, x, t)P_0Q_1(y, x, t) - G_1(y, x, t)D^-(t)\cdot \\ & \cdot\left(\left(DP_1D^-\right)_x(x, t)\,z + \left(DP_1D^-\right)_t(x, t)\right) D(t)P_0(t)Q_1(y, x, t)\end{aligned}$$

with[1] $\left(\left(DP_1D^-\right)_x(x, t)\,z\right)_{ij} := \sum_k \frac{\partial}{\partial x_k}\left(DP_1D^-\right)_{ij}(x, t)\,z_k$. Requiring these properties, the differential-algebraic system (1.5) is said to be an index-2 DAE, if G_1 is singular

[1] Just consider a function $\xi \in C^1\left((t-\epsilon, t+\epsilon), \mathbb{R}^m\right)$ with $\xi\,(t) = x$ and $\xi'\,(t) = z$ and compute $\frac{\mathrm{d}}{\mathrm{d}s}\left(DP_1D^-\right)(\xi\,(s), s)$ evaluated in t via chain rule in order to obtain the expression $\left(DP_1D^-\right)_x(x, t)\,z + \left(DP_1D^-\right)_t(x, t)$.

with a constant rank and $\tilde{G}_2$ is nonsingular using an admissible projector Q_1. The presented matrix chain is applied e.g. in [Voi06].

The mentioned definition of index $k = 1, 2$ based on the alternative matrix chain is equivalent to Definition (1.8) due to

$$\tilde{G}_2 = (G_1 + f_x P_0 Q_1)\left(I - P_1 D^- \left(\left(DP_1D^-\right)_x (x,t)\, z + \left(DP_1D^-\right)_t (x,t)\right) DP_0Q_1\right)$$

We notice that

$$\begin{gathered}\left(I - P_1 D^- \left(\left(DP_1D^-\right)_x (x,t)\, z + \left(DP_1D^-\right)_t (x,t)\right) DP_0Q_1\right)^{-1} = \\ I + P_1 D^- \left(\left(DP_1D^-\right)_x (x,t)\, z + \left(DP_1D^-\right)_t (x,t)\right) DP_0Q_1\end{gathered}$$

so $I - P_1 D^- \left(\left(DP_1D^-\right)_x (x,t)\, z + \left(DP_1D^-\right)_t (x,t)\right) DP_0Q_1 \in GL_m(\mathbb{R})$ for all (z, y, x, t) in the respective domain. Therefore $\tilde{G}_2\,(z, y, x, t)$ is nonsingular if and only if $G_2\,(y, x, t)$ defined in Section 1.2 (page 7) has the same property and this is equivalent to

$$N_1\,(y, x, t) \cap S_1\,(y, x, t) = \{0\}$$

The linear subspaces N_1 and S_1 are identical in both matrix chains implying equivalence of the described approaches to the tractability index. For the purpose of our stability investigations, the simpler matrix chain turns out to be sophisticated enough in order to formulate new stability criteria.

1.2.3 Excursus: Hessenberg systems

In order to exemplify the tractability index we construct the matrix chain for general Hessenberg-2 DAEs. Later we prove that these systems conform to requirements of the transformation into the state space representation and to those of the index reduction via differentiation.

Lemma 1.15. *If h_{x_1} is continuous then* (1.6) *possesses the tractability index two.*

Proof. Write (1.6) as $f\left(x_1'(t), (x_1(t), x_2(t)), t\right) = 0$ with $f(y, x, t) = \begin{pmatrix} y - h(x_1, x_2, t) \\ g(x_1, t) \end{pmatrix}$ and $x = (x_1, x_2)^T \in \mathbb{R}^n \times \mathbb{R}^{m-n}$. It holds

$$f_y(y,x,t) = \begin{pmatrix} I_n \\ 0 \end{pmatrix},\; f_x(y,x,t) = \begin{pmatrix} -h_{x_1}(x_1,x_2,t) & -h_{x_2}(x_1,x_2,t) \\ g_{x_1}(x_1,t) & 0 \end{pmatrix},\; D^- = \begin{pmatrix} I_n \\ 0 \end{pmatrix}$$

as well as $D = \begin{pmatrix} I_n & 0 \end{pmatrix}$, $R = I_n$. The first projector $Q_0 = \begin{pmatrix} 0 & \\ & I_{m-n} \end{pmatrix}$ in the matrix chain is fixed by $N_0 = \ker G_0 = \{0\} \times \mathbb{R}^{m-n}$. Obviously, $\ker Q_0 = \operatorname{im} G_0$ and we can choose $W_0 = Q_0$. Furthermore,

$$S_0(x,t) = \ker W_0 f_x(x,t) = \ker \begin{pmatrix} 0 & 0 \\ g_{x_1}(x_1,t) & 0 \end{pmatrix} = \ker g_1(x_1,t) \times \mathbb{R}^{m-n}$$

implies $N_0 \cap S_0(x,t) = N_0$ to be constant. Due to $G_1 = G_0 + f_x Q_0 = \begin{pmatrix} I_n & -h_{x2} \\ 0 & 0 \end{pmatrix}$ and im $G_1 =$ im G_0 we may choose $W_1 = W_0$ leading to

$$\begin{aligned} N_1(x,t) &= \ker G_1(x,t) = \left\{(z_1,z_2)^T \in R^m \mid z_1 = h_{x_2}(x_1,x_2,t)z_2\right\} \\ S_1(x,t) &= \ker W_0 f_x(x,t) P_0 = \ker \begin{pmatrix} 0 & 0 \\ g_{x_1}(x_1,t) & 0 \end{pmatrix} = \ker g_{x_1}(x_1,t) \times \mathbb{R}^{m-n} \\ N_1(x,t) \cap S_1(x,t) &= \left\{(z_1,z_2)^T \in R^m \mid z_1 = h_{x_2}(x_1,x_2,t)z_2,\ g_{x_1}(x_1,t)z_1 = 0\right\} \end{aligned}$$

The nonsingularity of $g_{x_1} h_{x_2}$ implies $z_2 = 0$ so that $z_1 = 0$ and $(N_1 \cap S_1)(x,t) = \{0\}$. For this reason both conditions of the tractability index two are fulfilled on $\mathcal{G}$. □

There is little freedom in choosing admissible projectors Q_1 because of the Hessenberg structure. Condition $Q_1 Q_0 = 0$ implies $Q_1 = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix}$ with $A^2 = A$, $BA = B$ due to idempotence of Q_1. From $\forall x \in N_1 : Q_1 x = x$ we conclude $A h_{x_2} = h_{x_2}$ (that is im $h_{x_2} \subseteq$ im A) and $B h_{x_2} = I_{m-n}$. Left multiplication of the last condition by h_{x_2} and right multiplication by B yield $B h_{x_2} B = B$, $h_{x_2} B h_{x_2} = h_{x_2}$ that is $B = h_{x_2}^-$. Due to regularity of $g_{x_1} h_{x_2}$ it follows $n \geq m-n$ where h_{x_2} is necessarily injective and g_{x_1} surjective. In particular I_n is the only possible projector along $\ker h_{x_2}$ such that $h_{x_2}^- h_{x_2} = I_n$ fixes the reflexive generalized inverse partly. By $h_{x_2} h_{x_2}^- = A$ the unique pseudoinverse $h_{x_2}^- = B$ is fixed completely because $h_{x_2}^- A = h_{x_2}^-$ means that A projects onto im h_{x_2}. All in all, an admissible projector Q_1 onto N_1 has necessarily the form

$$Q_1(x,t) = \begin{pmatrix} h_{x_2}(x_1,x_2,t) h_{x_2}^-(x_1,x_2,t) & 0 \\ h_{x_2}^-(x_1,x_2,t) & 0 \end{pmatrix} \tag{1.7}$$

It is easy to prove that (1.7) defines an admissible projector onto $N_1(x,t)$. Due to $\ker h_{x_2} h_{x_2}^- \subseteq \ker h_{x_2}^- h_{x_2} h_{x_2}^- \subseteq \ker h_{x_2}^- \subseteq \ker h_{x_2} h_{x_2}^-$ we get

$$\ker Q_1 = \ker h_{x_2} h_{x_2}^- \times \mathbb{R}^{m-n}$$

Using the regularity of $g_{x_1} h_{x_2}$ we arrive at im $h_{x_2} \cap \ker g_{x_1} = \{0\}$ so $h_{x_2} h_{x_2}^-$ can be chosen as the projector onto im h_{x_2} along $\ker g_{x_1}$ exhibiting the representation $h_{x_2} \left(g_{x_1} h_{x_2}\right)^{-1} g_{x_1}$. This choice of A has the canonical projector Q_1 as a consequence because $\ker Q_1 = \ker h_{x_2} h_{x_2}^- \times \mathbb{R}^{m-n} = S_1 = \ker g_{x_1} \times \mathbb{R}^{m-n} = S_1$. Particularly, the canonical projector of a Hessenberg system has the nice representation

$$Q_1 = \begin{pmatrix} h_{x_2} \left(g_{x_1} h_{x_2}\right)^{-1} g_{x_1} & 0 \\ \left(g_{x_1} h_{x_2}\right)^{-1} g_{x_1} & 0 \end{pmatrix}$$

and $DN_1(x_*(t),t)) =$ im $h_{x_2}(x_*(t),t))$, $DS_1(x_*(t),t)) = \ker g_{x_1}(x_*(t),t))$ is valid.

1.3 Linearization of DAEs

Let us take a look at the development of the tractability index in order to motivate this index concept for nonlinear DAEs. Due to a statement of Roswitha März expressed in [Mä95] concerning the tractability index:

> "Our notion of index-2 tractability is a straightforward generalization of the corresponding definition for the linear case, which, in turn, represents a generalization of the Kronecker index."

We follow the lines of [Mä95] and correspond the index of nonlinear systems to linearization in the sense of functional analysis and the implicit function theorem on Banach spaces. Consider the unique solvability of initial value problems

$$(DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)(x(t_0) - x_0) = 0$$

of the DAE (1.2) in a neighbourhood of a reference solution x_*. To this end we formulate the DAE as an operator equation

$$F(x) = 0$$

with

$$\begin{aligned}
&F: \ \mathrm{B}_\epsilon(x_*) \subseteq C_D^1(I, \mathbb{R}^m) \to C(I, \mathbb{R}^m) \times L \\
&F(x)(t) := \left(f\left((D(t)x(t))', x(t), t\right), (DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)(x(t_0) - x_0)\right)
\end{aligned}$$

and $L := \mathrm{im}\ (DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)$. It is common practice in applied mathematics to use more restrictive assumptions in order to prove a stronger result. Therefore, we consider the operator equation

$$H(x, q) = 0 \tag{1.8}$$

representing a sufficiently smooth perturbation of $F(x) = 0$, namely

$$\begin{aligned}
&H: \ \mathrm{B}_\epsilon(x_*) \times C(I, \mathbb{R}^m) \longrightarrow C(I, \mathbb{R}^m) \times L \\
&H(x,q)(t) := \left(f\left((D(t)x(t))', x(t), t\right) - q(t),\ (DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)(x(t_0) - x_0)\right)
\end{aligned}$$

Obviously, $(x_*, 0) \in \mathrm{B}_\epsilon(x_*) \times C(I, \mathbb{R}^m)$ solves Equation (1.8). Due to the implicit function theorem (e.g. [ea73, Theorem 1.7]) there exists locally a unique function

$$w: \mathrm{B}_{\hat{\epsilon}}(0) \subseteq C(I, \mathbb{R}^m) \longrightarrow \mathrm{B}_\epsilon(x_*) \subseteq C_D^1(I, \mathbb{R}^m)$$

satisfying

$$H(w(q), q) = 0, \qquad w(0) = x_*$$

if H is continuous and the Fréchet differential has a bounded inverse, i.e.

$$H_x(x_*, 0) = F_x(x_*) \in L_b\left(C_D^1(I, \mathbb{R}^m), C(I, \mathbb{R}^m) \times L\right)$$

Consequently, these assumptions imply that the above IVP aiming at (1.2) is *well posed*. Later, we show

$$F_x(x_*)h = \begin{pmatrix} f_y\left((Dx_*)'(t), x_*(t), t\right)(Dh)'(t) + f_x\left((Dx_*)'(t), x_*(t), t\right)h(t), \\ (DP_1)\left((Dx_*)'(t), x_*(t), t\right)(h(t_0) - h_0) \end{pmatrix}$$

Therefore $F_x(x_*)h = q$ corresponds with the IVP $(DP_1)(t_0)(h(t_0) - h_0) = 0$ of the linearization

$$f_y\left((Dx_*)'(t), x_*(t), t\right)(Dh)'(t) + f_x\left((Dx_*)'(t), x_*(t), t\right)h(t) = q(t) \tag{1.9}$$

of the given nonlinear DAE. Equation (1.9) has index two and complies with requirements from [Mä04] respectively [BM00], [BM04] per constructionem. Accordingly, the IVP is uniquely solvable for all $q \in C^1_{DQ_1G_2^{-1}}(I, \mathbb{R}^m)$, $h_0 \in \mathbb{R}^m$ and the solution depends continuously on q and $(DP_1)(t_0)\, h_0$. In other words,

$$F_x(x_*):\ C^1_D(I, \mathbb{R}^m) \longrightarrow C^1_{DQ_1G_2^{-1}}(I, \mathbb{R}^m) \times \operatorname{im}\ (DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)$$

has a bounded inverse.

The above approach works, if we are able to guarantee

$$F\left(\mathrm{B}_\epsilon(x_*)\right) \subseteq C^1_{DQ_1G_2^{-1}}(I, \mathbb{R}^m) \times \operatorname{im}\ (DP_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right) \tag{1.10}$$

The smoothness property (1.10) reads

$$\forall x \in \mathrm{B}_\epsilon(x_*):\ \left(DQ_1G_2^{-1}\right)(t)\, f\left((Dx)'(t), x(t), t\right) \in C^1(I, \mathbb{R}^m)$$

so appropriate structural conditions have to be imposed in addition to sufficient smoothness of f and x_*. For example, if im $G_1(y, x, t)$ is t-dependent only, we get $W_1 f = (W_1 f)(P_0 x, t)$ and $DQ_1G_2^{-1}G_1 = 0$ due to $G_1 = G_2P_1$. Therefore,

$$\left(DQ_1G_2^{-1}\right)(t) f\left((Dx)'(t), x(t), t\right) = \left(DQ_1G_2^{-1}\right)(t)\,(W_1 f)\,(P_0x(t), t)$$

and the derivative

$$\frac{d}{dt}\left(DQ_1G_2^{-1}\right)(t)\, f\left((Dx)'(t), x(t), t\right) = \begin{array}{l} \left(DQ_1G_2^{-1}\right)'(t)\,(W_1 f)\,((P_0x)(t), t) \\ + \left(DQ_1G_2^{-1}\right)(t)\left((W_1 f)_x\,(P_0x)'(t) + (W_1 f)_t\right) \end{array}$$

exists for all $x \in C^1_D(I, \mathbb{R}^m)$, if $\left(DQ_1G_2^{-1}\right)(t), W_1 f \in C^1$.

Lemma 1.16. *Consider the properly stated DAE* (1.2) *with f continuously differentiable with respect to x. If $x_* \in C^1_D(I, \mathbb{R}^m)$ satisfies $\forall t \in I \subseteq I_0:\ \left((Dx_*)'(t), x_*(t), t\right) \in \mathcal{G}$ and $\|x_*\|_{C^1_D}$ is bounded then there exists an $\epsilon > 0$ so that the operator equation $\mathcal{F}(x) = 0$ on $B_\epsilon(x_*) \subseteq C^1_D(I, \mathbb{R}^m)$,*

$$\mathcal{F}: B_\epsilon(x_*) \to C(I, \mathbb{R}^m), \quad \mathcal{F}(x)(t) := f((D(t)x(t))', x(t), t)$$

is Fréchet differentiable *at x_*. The corresponding* Fréchet differential *reads*

$$\mathcal{F}_x(x_*)h = f_y\left((Dx_*)'(t), x_*(t), t\right)(Dh)'(t) + f_x\left((Dx_*)'(t), x_*(t), t\right) h(t) \tag{1.11}$$

Proof. The *Gâteaux derivative* of $\mathcal{F}$ at the point x_* in the direction $h \in C^1_D(I, \mathbb{R}^m)$ is

$$\begin{aligned} D\mathcal{F}(x_*, h) &:= \lim_{s \to 0} \frac{1}{s}\left(\mathcal{F}(x_* + sh) - \mathcal{F}(x_*)\right) \\ &= \lim_{s \to 0} \frac{1}{s} \begin{pmatrix} f\left((Dx_*)'(t) + s\,(Dh)'(t), x_*(t) + sh(t), t\right) \\ -f\left((Dx_*)'(t), x_*(t), t\right) \end{pmatrix} \end{aligned}$$

Adding the zero-sum

$$f\left((Dx_*)'(t) + s\,(Dh)'(t), x_*(t), t\right) - f\left((Dx_*)'(t) + s\,(Dh)'(t), x_*(t), t\right)$$

allows for rewriting $DF(x_*, h)$ as a sum of two limits

$$\lim_{s\to 0} \frac{1}{s} \left[\begin{array}{l} f\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t) + sh(t), t\right) \\ -f\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t), t\right) \end{array} \right] \tag{1.12}$$

and

$$\lim_{s\to 0} \frac{1}{s} \left[f\left((Dx_*)'(t) + s\,(Dh)'(t), x_*(t), t\right) - f\left((Dx_*)'(t), x_*(t), t\right)\right] \tag{1.13}$$

Due to the smoothness assumptions, (1.13) equals

$$f_y\left((Dx_*)'(t) + s\,(Dh)'(t), x_*(t), t\right)(Dh)'(t)$$

Applying the mean value theorem to (1.12) reveals

$$\begin{array}{l} \frac{1}{s}\left(\begin{array}{l} f\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t) + sh(t), t\right) \\ -f\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t), t\right) \end{array} \right) = \\ \int_0^1 f_x\left((D(t)\,x_*(t))' + s\,(Dh)'(t), x_*(t) + \mu s h(t), t\right) \mathrm{d}\mu\, h(t) \end{array}$$

Now $f_x\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t) + \mu sh(t), t\right)$ is a continuous function so integration and the limit process $s \to 0$ commute according to [For05, p.frm[o]–14, Theorem 1]. Expression (1.12) can be simplified to

$$\begin{array}{l} \lim_{s\to 0} \int_0^1 f_x\left((D(t)x_*(t))' + s\,(Dh)'(t), x_*(t) + \mu sh(t), t\right) \mathrm{d}\mu\, h(t) \\ = f_x\left((D(t)x_*(t))', x_*(t), t\right) h(t) \end{array}$$

This derivative is linear in h. Because of $\|x_*\|_{C_D^1} < C$ there exists a compact set $K \subseteq G$ with $\forall_{t\in I}\left((Dx_*)'(t), x_*(t), t\right) \in K$ including a neighbourhood of the extended integral curve of x_*. The continuous partial derivatives f_y and f_x are bounded on K implying (1.11) to be continuous at h. We have proved the representation (1.11) of the Gâteaux derivative.

Theorem 2 in [SL68, p. 310] or [Wer, p. 113 ff. and Theorem III 5.4 c)] state that $\mathcal{F}$ is Fréchet differentiable, if the Gâteaux derivative $D\mathcal{F}(x,h)$ exists for $x \in B_\epsilon(x_*)$, $\epsilon > 0$ sufficiently small and $\mathrm{D}\mathcal{F}(x,h)$ is continuous at x_*. In this case $\mathcal{F}_x(x_*)h = D\mathcal{F}(x_*, h)$ is true. Using the triangle inequality and (1.11)

$$\|D\mathcal{F}(x_*,h) - D\mathcal{F}(x,h)\| \le \left\{ \begin{array}{l} \left\|f_y\left((Dx_*)'(t), x_*(t), t\right) - f_y\left((Dx)'(t), x(t), t\right)\right\|_\infty \left\|(Dh)'\right\|_\infty \\ + \left\|f_x\left((Dx_*)'(t), x_*(t), t\right) - f_x\left((Dx)'(t), x(t), t\right)\right\|_\infty \|h\|_\infty \end{array} \right.$$

Continuity of f_y, f_x implies that $\left\|f_y\left((Dx_*)'(t), x_*(t), t\right) - f_y\left((Dx)'(t), x(t), t\right)\right\|_\infty$ and $\left\|f_x\left((Dx_*)'(t), x_*(t), t\right) - f_x\left((Dx)'(t), x(t), t\right)\right\|_\infty$ are bounded, if $\|x - x_*\|_{C_D^1}$ is sufficiently small. Consequently, $\|D\mathcal{F}(x_*,h) - D\mathcal{F}(x,h)\|_\infty \le C\,\|h\|_{C_D^1}$ holds uniformly in h for $x \in \mathrm{B}_{\hat{\epsilon}}(x_*) \subseteq C_D^1(I, \mathbb{R}^m)$ so the Gâteaux differential (1.11) is continuous at x_*. Therefore, $\mathrm{D}\mathcal{F}(x,h)$ equals the Fréchet differential $\mathcal{F}_x(x_*)$. □

The above lemma establishes the common definition of the linearization of a differential-algebraic equation as a linearization of f.

Definition 1.17. Consider the DAE (1.2) with $f_x \in C^0$ and $x_* \in C^1_D(I, \mathbb{R}^m)$ satisfying $\forall t \in I : \ \left((Dx_*)'(t), x_*(t), t\right) \in G$. The linear system

$$f_y\left((Dx_*)'(t), x_*(t), t\right)(Dx)'(t) + f_x\left((Dx_*)'(t), x_*(t), t\right)x(t) = 0 \tag{1.14}$$

is called the *linearization* of the given DAE (1.2) around x_*.

Remark. The linearization of an ODE around a solution x_* is also called the corresponding *system of variational equations* around x_*. This system describes the linear approximation of the propagation of the difference between an arbitrary solution of the ODE and x_*, i.e. the propagation of the variation in the initial values. The fundamental matrix $X(t)$ of the system of variational equations having $X(t_0) = I$ corresponds to the derivative of the flow with respect to the initial values, i.e. $X(t) = \frac{\partial}{\partial x_0} x(t; t_0, x_0)$.

1.4 Transformation of fully implicit DAEs to a linear implicit form

A properly stated derivative term allows to transform a general system (1.2) into a properly formulated linear implicit DAE (1.5) with $A = A(t)$. Hereby, additional structural assumptions for the complete decoupling arise, but they can be formulated in terms of the original DAE.

Augment a properly stated DAE

$$f\left((Dx)'(t), x(t), t\right) = 0 \Leftrightarrow f\left(R(t)(Dx)'(t), x(t), t\right) = 0$$

by introducing the new variable $z(t) := R(t)(Dx)'(t)$, i.e.

$$\begin{pmatrix} R(t) \\ 0_{m\times n} \end{pmatrix}\left[\begin{pmatrix} D(t) & 0_n \end{pmatrix}\begin{pmatrix} x(t) \\ z(t) \end{pmatrix}\right]' + \begin{pmatrix} -z(t) \\ f(z(t), x(t), t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.15}$$

Denote the augmented system by

$$\tilde{A}(t)\left(\tilde{D}\tilde{x}\right)'(t) + \tilde{b}(\tilde{x}(t), t) = 0$$

with

$$\tilde{x} = \begin{pmatrix} x \\ z \end{pmatrix},\ \tilde{A}(t) = \begin{pmatrix} R(t) \\ 0_{m\times n} \end{pmatrix},\ \tilde{D}(t) = \begin{pmatrix} D(t) & 0_n \end{pmatrix},\ \tilde{b}(\tilde{x}, t) = \begin{pmatrix} -z \\ f(z, x, t) \end{pmatrix}$$

Lemma 1.18. *Given a properly stated DAE* (1.2) *with tractability index* $k = 1, 2$*, the augmented system* (1.15) *inherits the properly stated derivative term and the tractability index* k *as well. Furthermore, in case of index-2 DAEs it holds*

$$\left(\tilde{D}\tilde{N}_1\right)(z, x, t) = (DN_1)(z, x, t),\ \left(\tilde{D}\tilde{S}_1\right)(z, x, t) = (DS_1)(z, x, t)$$

and $\tilde{N}_0(t) \cap \tilde{S}_0(z, x, t) = \left(\tilde{N}_0 \cap \tilde{S}_0\right)(t)$ *is equivalent to*

$$\operatorname{im}\begin{pmatrix} T(z, x, t) \\ -\left(f_y^- f_x T\right)(z, x, t) \end{pmatrix} \text{ independent of } z, x \tag{1.16}$$

Proof. We construct a matrix chain for the augmented DAE (1.15) (denoted by tilde) based on a given matrix chain for (1.2). Obviously, $\ker \tilde{A}(t) = \ker R(t)$ and $\operatorname{im} \tilde{D}(t) = \operatorname{im} D(t) = \operatorname{im} R(t)$, i.e. the derivative term is properly stated. Moreover,

$$\tilde{N}_0(t) = \ker \tilde{D}(t) = N_0(t) \times \mathbb{R}^n,\ \tilde{Q}_0(t) = \begin{pmatrix} Q_0(t) & \\ & I_n \end{pmatrix},\ \tilde{P}_0(t) = \begin{pmatrix} P_0(t) & \\ & 0_n \end{pmatrix}$$

and $\tilde{D}^-(t) = \begin{pmatrix} D^-(t) \\ 0_n \end{pmatrix}$ is fixed by $\tilde{R}(t) = R(t)$ and $\tilde{P}_0(t)$. $\operatorname{im} \tilde{A}(t) = \operatorname{im} D(t) \times \{0\}$ implies

$$\tilde{W}_0(t) = \begin{pmatrix} I_n - R(t) & \\ & I_m \end{pmatrix}$$

to be a projector along $\operatorname{im} \tilde{A}$. It holds

$$\tilde{b}_{\tilde{x}}(\tilde{x}, t) = \begin{pmatrix} \tilde{b}_x & \tilde{b}_z \end{pmatrix} = \begin{pmatrix} 0_{n\times m} & -I_n \\ f_x(z,x,t) & f_y(z,x,t) \end{pmatrix},$$

$$\tilde{S}_0(\tilde{x}, t) = \ker \tilde{W}_0(t)\tilde{b}_{\tilde{x}}(\tilde{x}, t) = \ker \begin{pmatrix} 0_{n\times m} & -(I_n - R(t)) \\ f_x(z,x,t) & f_y(z,x,t) \end{pmatrix}$$

$(\xi, \mu)^T \in \tilde{S}_0(z,x,t)$ are characterized by $\mu = R(t)\mu$ and $f_x(z,x,t)\,\xi + f_y(z,x,t)\,\mu = 0$. Using the reflexive generalized inverse $f_y^-(z,x,t)$ defined pointwise by

$$f_y^-(z,x,t)\, f_y(z,x,t) = R(t), \quad f_y(z,x,t)\, f_y^-(z,x,t) = (I - W_0(z,x,t))$$

we conclude

$$\tilde{S}_0(\tilde{x}, t) = \left\{ \begin{pmatrix} \xi \\ -f_y^-(z,x,t)\, f_x(z,x,t)\,\xi \end{pmatrix} \in \mathbb{R}^{m+n} \mid \xi \in S_0(z,x,t) \right\}$$

for $\mu = -f_y^-(z,x,t)\, f_x(z,x,t)\,\xi$ satisfies $R(t)\mu = \mu$ and

$$f_y(z,x,t)\,\mu = -(I - W_0(z,x,t))\, f_x(z,x,t)\,\xi = -f_x(z,x,t)\,\xi$$

due to $\xi \in S_0(z,x,t) = \ker W_0(z,x,t)\, f_x(z,x,t)$. $\mu = R(t)\mu$ is unique because $\mu_1,\ \mu_2 \in \operatorname{im} D(t)$ with

$$f_x\xi + f_y\mu_1 = 0 = f_x\xi + f_y\mu_2$$

imply $\mu_1 - \mu_2 \in \ker f_y \cap \operatorname{im} D(t) = \{0\}$, that is $\mu_1 = \mu_2$. Consequently,

$$\tilde{N}_0(t) \cap \tilde{S}_0(\tilde{x}, t) = \left\{ \begin{pmatrix} \xi \\ -f_y^-(z,x,t)\, f_x(z,x,t)\,\xi \end{pmatrix} \in \mathbb{R}^{m+n} \mid \xi \in N_0(t) \cap S_0(z,x,t) \right\}$$

Obviously, $\tilde{N}_0(t) \cap \tilde{S}_0(z,x,t) = \left(\tilde{N}_0 \cap \tilde{S}_0\right)(t)$ is equivalent to

$$\begin{pmatrix} I_m \\ -f_y^-(z,x,t)\, f_x(z,x,t) \end{pmatrix} (N_0(t) \cap S_0(z,x,t)) = \operatorname{im} \begin{pmatrix} T(z,x,t) \\ -\left(f_y^- f_x T\right)(z,x,t) \end{pmatrix}$$

being independent of z, x and this property is captured by (1.16).

Now choose

$$\tilde{T}(\tilde{x},t) = \begin{pmatrix} T(z,x,t) & 0_{m\times n} \\ -f_y^-(z,x,t)\, f_x(z,x,t)\, T(z,x,t) & 0_n \end{pmatrix}$$

as a projector onto $\tilde{N}_0(t) \cap \tilde{S}_0(\tilde{x},t)$, the complementary projector is

$$\tilde{U}(\tilde{x},t) = I_{n+m} - \tilde{U}(\tilde{x},t) = \begin{pmatrix} U(z,x,t) & 0_{m\times n} \\ f_y^-(z,x,t)\, f_x(z,x,t)\, T(z,x,t) & I_n \end{pmatrix}$$

We obtain the representations

$$\tilde{G}_1(\tilde{x},t) = \begin{pmatrix} D(t) & -I_n \\ f_x(z,x,t)\, Q_0(t) & f_y(z,x,t) \end{pmatrix}$$

$$\tilde{N}_1(\tilde{x},t) = \ker \tilde{G}_1(\tilde{x},t) = \left\{ \begin{pmatrix} \xi \\ D(t)\xi \end{pmatrix} \mid \xi \in N_1(z,x,t) \right\}$$

That is why

$$\tilde{Q}_1(\tilde{x},t) = \begin{pmatrix} Q_1(z,x,t) & 0_{m\times n} \\ D(t)Q_1(z,x,t) & 0_n \end{pmatrix}$$

is an admissible projector onto $\tilde{N}_1(\tilde{x},t)$ if $Q_1(z,x,t)$ is admissible.

$$\tilde{P}_1(\tilde{x},t) = \begin{pmatrix} P_1(z,x,t) & 0_{m\times n} \\ -D(t)Q_1(z,x,t) & I_n \end{pmatrix}, \; \tilde{D}(t)\tilde{Q}_1(z,x,t) = \begin{pmatrix} D(t)Q_1(z,x,t) & 0_n \end{pmatrix},$$
$$\tilde{D}(t)\tilde{P}_1(z,x,t) = \begin{pmatrix} D(t)P_1(z,x,t) & 0_n \end{pmatrix}$$

Per definitionem,

$$\begin{aligned} \tilde{S}_1(\tilde{x},t) &= \left\{ \begin{pmatrix} \xi \\ \mu \end{pmatrix} \in \mathbb{R}^{n+m} \mid \tilde{b}_{\tilde{x}}(\tilde{x},t)\, \tilde{P}_0(t) \begin{pmatrix} \xi \\ \mu \end{pmatrix} \in \operatorname{im} \tilde{G}_1(\tilde{x},t) \right\} \\ &= \left\{ \begin{pmatrix} \xi \\ \mu \end{pmatrix} \in \mathbb{R}^{n+m} \mid \begin{array}{l} \exists \tilde{\xi} \in \mathbb{R}^m, \tilde{\mu} \in \mathbb{R}^n : \; D(t)\tilde{\xi} = \tilde{\mu}, \\ f_x(z,x,t)\, P_0(t)\xi = f_x(z,x,t)\, Q_0(t)\tilde{\xi} + f_y(z,x,t)\, \tilde{\mu} \end{array} \right\} \\ &= S_1(z,x,t) \times \mathbb{R}^n \end{aligned}$$

Finally,

$$\tilde{N}_1(\tilde{x},t) \cap \tilde{S}_1(\tilde{x},t) = \left\{ \begin{pmatrix} \xi \\ D(t)\xi \end{pmatrix} \mid \xi \in N_1(z,x,t) \cap S_1(z,x,t) \right\}$$

Case 1. Let $\operatorname{rk} \tilde{G}_0(t) = \operatorname{rk} D(t) = r > 0$ be constant. If (1.2) has the tractability index 1 then $N_0(t) \cap S_0(z,x,t) = \{0\}$ implies $\tilde{N}_0(t) \cap \tilde{S}_0(\tilde{x},t) = \{0\}$ immediately. This is equivalent to index 1 of the augmented system (1.15).

Case 2. If (1.2) has index two on the whole domain $\mathcal{G}$ then

$$\forall (z,x,t) \in \mathcal{G} : \; \dim S_0(y,x,t) \cap N_0(t) = \text{const.} \quad N_1(z,x,t) \cap S_1(z,x,t) = \{0\}$$

The second condition implies $\tilde{N}_1(z,x,t) \cap \tilde{S}_1(z,x,t) = \{0\}$. Due to Lemma 1.10 $\dim N_1 = \dim N_0 \cap S_0$ is valid, so $\dim N_1$ is constant. Therefore,

$$\dim \tilde{N}_0(t) \cap \tilde{S}_0(z,x,t) = \dim \tilde{N}_1(\tilde{x},t) = \dim \begin{pmatrix} I_m \\ D(t) \end{pmatrix} (N_1(z,x,t))$$

has a constant value because the matrix valued function $\begin{pmatrix} I_m \\ D(t) \end{pmatrix} \in \mathbb{R}^{(m+n)\times m}$ is injective for any t. We recognize (1.15) to possess the tractability index two.

In the index-2 case we get $\tilde{D}\tilde{Q}_1\tilde{D}^- = DQ_1D^-$ and $\tilde{D}\tilde{P}_1\tilde{D}^- = DP_1D^-$,

$$\left(\tilde{D}\tilde{N}_1\right)(z,x,t) = (DN_1)(z,x,t), \ \left(\tilde{D}\tilde{S}_1\right)(z,x,t) = (DS_1)(z,x,t)$$

□

If we consider $\mu = f_y^- f_x T\xi$ then

$$G_2^{-1} f_y \mu = G_2^{-1} f_x Q_0 T\xi = P_1 T\xi = T\xi$$

and $G_2^{-1} f_y = G_2^{-1}(f_y D) D^- = P_1 D^-$ so it is worth noting that

$$\operatorname{im} \begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix} (z,x,t) = \begin{pmatrix} P_1(z,x,t)\, D^-(t) \\ I_n \end{pmatrix} \left(\operatorname{im} \left(f_y^- f_x T\right)(z,x,t)\right)$$

The last element of the matrix chain belonging to the augmented DAE is

$$\tilde{G}_2(\tilde{x},t) = \begin{pmatrix} D(t) & -I_n \\ f_x(z,x,t)\left(Q_0(t) + P_0(t)Q_1(z,x,t)\right) & f_y(z,x,t) \end{pmatrix}$$

Using a block matrix structure, it is possible to formulate equations designating $\tilde{G}_2\tilde{G}_2^{-1} = I_{m+n}$ and $\tilde{G}\tilde{G}_2 = I_{m+n}$. Computations reveal the representation

$$\tilde{G}_2^{-1}(\tilde{x},t) = \begin{pmatrix} G_2^{-1}(z,x,t)\, f_y(z,x,t) & G_2^{-1}(z,x,t) \\ D(t)G_2^{-1}(z,x,t)\, f_y(z,x,t) - I_n & D(t)G_2^{-1}(z,x,t) \end{pmatrix} \tag{1.17}$$

The augmented differential-algebraic system is useful to prove the following valuable lemma:

Lemma 1.19. *Consider the DAE* (1.2) *with a properly stated derivative term. Then,*

$$\forall x \in \mathcal{M}_0(t)\, \exists! y = R(t)\, y : \ f(y,x,t) = 0$$

is true.

Proof. Consider a fixed $t \in I$ and $x \in \mathcal{M}_0(t)$. Choose $y_1, y_2 \in \mathbb{R}^n$ satisfying

$$f(y_1, x, t) = 0 = f(y_2, x, t)$$

The proper formulation implies $f_y(y, x, t) = f_y(y, x, t)R(t)$, therefore the mean value theorem results in

$$0 = f\left(y_2, x, t\right) - f\left(y_1, x, t\right) = \int_0^1 f_y\left(y_1 + s(y_2 - y_1), x, t\right) \mathrm{d}s\, R(t)(y_2 - y_1)$$

In case of linear implicit DAEs (1.5) we obtain

$$0 = \int_0^1 A\left(x, t\right) \mathrm{d}s\, R\left(t\right)\left(y_2 - y_1\right) = A\left(x, t\right)\left(y_2 - y_1\right)$$

According to (1.3) $R\left(t\right)\left(y_2 - y_1\right) = 0$ is valid.

The general case can be traced back to this fact using the augmented DAE (1.15). Denote the first-level constraint of the augmented system $\tilde{f}\left(\tilde{y}, \tilde{x}, t\right) = 0$ by $\tilde{\mathcal{M}}_0\left(t\right)$, i.e.

$$\tilde{\mathcal{M}}_0\left(t\right) = \left\{\left(\left(x, z\right), t\right) \in \mathbb{R}^{m+n+1} \mid \left(z, x, t\right) \in \mathcal{G} \text{ and } \exists \tilde{y} \in \mathbb{R}^n :\ f\left(z, x, t\right) = 0,\ R\left(t\right)\tilde{y} = z\right\}$$

Notice that $R\left(t\right)$ also realizes the decomposition of the properly stated derivative term of (1.15). We have already shown that in case of linear-implicit augmented DAE $\tilde{f}\left(\left(\tilde{D}\tilde{x}\right)'\left(t\right), \tilde{x}\left(t\right), t\right) = 0$ it follows

$$\forall \tilde{x} \in \tilde{\mathcal{M}}_0\left(t\right) \exists! \tilde{y} = R\left(t\right)\tilde{y} :\ \tilde{f}\left(\tilde{y}, \tilde{x}, t\right) = 0$$

In detail, for all $\tilde{x} = \left(x, z\right)^T$ such that there exists a $\tilde{y} \in \mathbb{R}^n$ satisfying $R\left(t\right)\tilde{y} = z$ and $f\left(z, x, t\right) = 0$, the z-components are uniquely determined in $\operatorname{im} D\left(t\right)$. Consequently, for $x \in \mathcal{M}_0\left(t\right)$ and $z \in \mathbb{R}^n$ such that $f\left(z, x, t\right) = f\left(R\left(t\right)z, x, t\right) = 0$ it holds $\left(x, z\right) \in \tilde{\mathcal{M}}_0\left(t\right)$ and the associated vector $R\left(t\right)z$ is unique. $\square$

Per constructionem, the augmentation of fully implicit DAEs to (1.15) commutes with the linearization along a solution x_*. In other words, the linearized system corresponding to (1.15) is equivalent to the augmented system

$$\begin{aligned} R(t)\left(Dx\right)'(t) - z(t) &= 0 \\ f_y(t)z(t) + f_x(t)x(t) &= 0 \end{aligned}$$

with $f_y\left(t\right)\left(Dx\right)'\left(t\right) + f_x(t)x(t) = 0$ being the linearization of the given DAE, $f_x\left(t\right) = f_x\left(\left(Dx_*\right)'\left(t\right), x_*\left(t\right), t\right)$. Given a solution $x \in C^1_D$ of the DAE (1.2), $\begin{pmatrix} x(\cdot) \\ R(\cdot)\left(Dx\right)'(\cdot) \end{pmatrix}$ solves the augmented system (1.15) and every solution of (1.15) is of this type.

Nonlinear derivative term $d\left(x(t), t\right)'$

Generally, differential-algebraic equations exhibiting the structure

$$f\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(d\left(x\left(t\right), t\right)\right), x(t), t\right) = 0 \tag{1.18}$$

have to be analysed. We follow the lines of [Mä01] and call the leading terms of these DAEs properly stated, if

$$\forall y, x, t : \; f_y(y, x, t) \oplus \operatorname{im}\, d_x(x, t) = \mathbb{R}^n$$

and there exists a projector $R \in C^1(I_0, \mathbb{R}^{n\times n})$ realizing the above decomposition of $\mathbb{R}^n$. The tractability index two is defined by the matrix chain in Definition 1.8 and conditions $N_1 \cap S_1 = \{0\}$ plus $N_0 \cap S_0 = \text{const}$. The only formal difference is that we use $D = D(x,t) = d_x(x,t)$ instead of $D(t)$. Considering (1.18), we are interested in continuous solutions having $d(x(t),t) \in C^1$, which satisfy the DAE pointwise. Unfortunately, this function class does not constitute a vector space in general. A certain simplification can be obtained by augmenting the given DAE (1.18),

$$\begin{aligned} f\left((R(t)y(t))', x(t), t\right) &= 0 \\ y(t) - d(x(t), t) &= 0 \end{aligned} \tag{1.19}$$

and the resulting system is of the type (1.2). If x solves (1.18) then $(x(t), d(x(t), t))$ is a solution of the augmented DAE (1.19). In addition, every solution of (1.19) can be represented that way. Consequently, the solution sets of (1.18) and of the corresponding augmented system can be identified. In [Mä01] it is proved that the proper formulation and the tractability index $k = 1, 2$ holds simultaneously for both DAEs. In this spirit (1.18) is equivalent to (1.19) and one can restrict oneself to linear implicit DAEs with $A = A\,(t)$ due to Lemma 1.18.

Remark 1.20. Differential-algebraic equations stemming from the charge oriented Modified Nodal Analysis in circuit simulation are given by

$$A \frac{\mathrm{d}}{\mathrm{d}t}\left(d\left(x\left(t\right), t\right)\right) + b(x(t), t) = 0$$

with A, b like in case of linear implicit DAEs (1.5) and $d \in C(D_0 \times I_0)$, $d_x \in C^0$, cf. [Mä03], [EST00], [ESFM+03] or [Voi06, § 1]. Such DAEs can be cast into linear implicit form with a t-dependent matrix pair $A(t)$, $D(t)$ as a derivative term. Due to higher dimension, it is not advisable to apply numerical methods to the linear-implicit system with a simple derivative term $\hat{A}\,(t) \left(\hat{D}\hat{x}\right)'(t)$ resulting after two augmentation steps. If the discretization scheme commutes with (1.19) and (1.15), that is the augmented system having the simplified derivative term $\hat{A}\,(t) \left(\hat{D}\hat{x}\right)'(t)$ is integrated by the numerical method, then it is sufficient to prove convergence of the numerical method applied to linear implicit DAEs in order to obtain convergence for the general form of differential-algebraic systems.

1.5 Analysis of linear systems

Linear differential-algebraic systems with t-dependent coefficients

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t) \tag{1.20}$$

are of vital importance. On the one hand, a deeper insight can be achieved using the linearity, e.g. most of the index concepts were first aiming at linear systems. In addition, linear DAEs arise from simplified models in many applications. From the viewpoint of stability criteria, the crucial reason to deal with linear systems is that the functional analytical linearization of general DAEs (1.2), that is the Fréchet derivative of the corresponding operator, is of the form (1.20). Accordingly, linear systems form the basis of qualitative investigations of nonlinear dynamics using the linearization principle. For example, the theorems of Perron and Andronov-Witt for ODEs work this way. Anticipating, certain aspects of the local qualitative behaviour of DAE solutions can be traced back to properties of the system of variational equations around a reference solution.

On the contrary, linear systems have the disadvantage to exhibit global stability properties only so they are not adequate in the context of nonlinear phenomena. Linearization might turn out to be a too coarse approximation even in simple setups and that causes the indirect method of Lyapunov to fail. In such cases, nonlinear approaches have to be used. Lyapunov's second or direct method is based on the notion of Lyapunov functions as a criterion for asymptotic stability of nonlinear ODEs. With regard to Definition 6.2, Lyapunov functions can be interpreted as a suitable abstraction of the idea behind a potential. The one sided Lipschitz condition (6.24) is a related dissipation inequality used for characterizing contractivity of ODEs. In this thesis, both nonlinear stability criteria for solutions of differential-algebraic equations with index $k = 1, 2$ and some based upon linearization are developed. The latter require a thorough investigation of the relevant properties of linear DAEs. Although a large number of publications dealing with linear DAEs is available, we have to clarify several aspects in order to proceed. Some technical difficulties are due to our goal to ensure a constant invariant subspace of the inherent regular ODE. For that purpose, the space DN_1 or DS_1 plus a corresponding complementary space is required to be constant. In contrast to [BM00], [BM04] and [MHT03b] we are forced to decouple without the canonical projector Q_1 so that suitable representations of the inherent dynamics and that of fundamental matrices of homogeneous DAEs have to be derived.

Remark 1.21. Later on, it is proved that the invariant subspaces of the inherent dynamics of an autonomous nonlinear DAE and of its linearization around the same solution coincide. In principal, the nonlinear decoupling works for time-dependent state spaces as well, but using our methodology it is not possible to conclude that the state space representation is *autonomous* in that case. Unfortunately, we cannot prove the analogon of the Andronov-Witt Theorem in case of autonomous index-2 DAEs without this crucial property.

1.5.1 Representation of the inherent dynamics on $\operatorname{im} DP_1$

An intermediate step towards extraction of the inherent dynamics of nonlinear index-2 systems via complete decoupling necessitates the decoupling of the linearization of a DAE. In order to analyze linear differential-algebraic equations (1.20) we are going to use the matrix chain with an admissible projector $Q_1(t)$ onto $N_1(t)$ along $K(t)$. The complementary space K has to satisfy $N_0(t) \subseteq K(t)$ in order to ensure $(Q_1Q_0)(t) = 0$.

Assumption. *The subspaces* $(DN_1)(t)$ *and* $(DK)(t)$ *are constant.*

Additionally we introduce a projector $T(t)$ onto $N_0(t) \cap S_0(t)$ and $U(t) := I - T(t)$ implying the identities $Q_0T = T$, $P_0U = P_0$ and

$$I = P_0P_1 + (UQ_0 + Q_1)(P_0Q_1 + UQ_0) + TQ_0P_1$$

Notice that $UQ_0 + P_0Q_1$, TQ_0, P_0P_1 and TQ_0P_1 are projectors due to

$$\begin{aligned}
(UQ_0 + P_0Q_1)^2 &= UQ_0(I-T)Q_0 + P_0Q_1P_0Q_1 = UQ_0 + P_0Q_1 \\
(TQ_0)^2 &= TQ_0Q_0 = TQ_0 \\
(TQ_0P_1)^2 &= TQ_0(I-Q_1)TQ_0P_1 = TQ_0P_1
\end{aligned}$$

In the next step, (1.20) is multiplied by $G_2^{-1}(t)$,

$$G_2^{-1}(t)\,A(t)(D(t)x(t))' + G_2^{-1}(t)\,B(t)x(t) = G_2^{-1}(t)\,q(t)$$

and then splitted into three parts via multiplication by DP_1, $P_0Q_1 + UQ_0$ and TQ_0P_1, where the resulting system is still equivalent to (1.20).

1. Multiplication by DP_1 results in

$$DP_1D^-(Dx)' + DP_1G_2^{-1}BP_0P_1x = DP_1G_2^{-1}q$$

   due to

$$G_2^{-1}A = G_2^{-1}AR = G_2^{-1}(G_2P_1P_0)D^- = P_1D^-$$

   and the decomposition $I = P_0P_1 + P_0Q_1 + Q_0$ in

$$B = BP_0P_1 + BP_0Q_1 + BQ_0 = BP_0P_1 + G_2Q_1 + G_1Q_0 = BP_0P_1 + G_2Q_1 + (G_2P_1)Q_0$$

2. Multiplication by TQ_0P_1 reveals

$$-Q_0Q_1D^-(Dx)' + TQ_0P_1G_2^{-1}BD^-(DP_1x) + TQ_0x = TQ_0P_1G_2^{-1}q$$

   because the computations above and $TQ_0P_1Q_0 = TQ_0$, $\operatorname{im} Q_0Q_1 = \operatorname{im} T$ imply

$$\begin{aligned}
TQ_0P_1D^-(Dx)' &= TQ_0(D^-D)D^-(Dx)' - TQ_0Q_1D^-(Dx)' \\
&= -Q_0Q_1D^-(Dx)'
\end{aligned}$$

3. Multiplication by $P_0Q_1 + UQ_0$ leads to

$$(UQ_0 + P_0Q_1)\, G_2^{-1} BD^-(DP_1x) + (UQ_0 + P_0Q_1)\, x = (UQ_0 + P_0Q_1) G_2^{-1} q$$

with

$$UQ_0P_1D^-(Dx)' = UQ_0D^-(Dx)' = UQ_0(D^-D)D^-(Dx)' = 0$$

$(UQ_0 + P_0Q_1)Q_1 = P_0Q_1$ and $(UQ_0 + P_0Q_1)Q_0 = UQ_0$.

DP_1D^- and DQ_1D^- project onto DK along $DN_1 \oplus \ker A$ respectively onto DN_1 along $DK \oplus \ker A$ due to Lemma 1.12. Taking the representation (8.1) and the C^1-bases of DN_1, DK and $\ker A$ into consideration (cf. Lemma 2.15), we obtain $DP_1D^-, DQ_1D^- \in C^1(I, \mathbb{R}^{n\times n})$. Application of the chain rule results in (1.20) being equivalent to

$$\begin{aligned} (DP_1x)' - \left(DP_1D^-\right)' Dx + DP_1G_2^{-1}BD^-(DP_1x) &= DP_1G_2^{-1}q \qquad (1.21)\\ (UQ_0 + P_0Q_1)\, G_2^{-1}BD^-(DP_1x) + (PQ_1 + UQ_0)\, x &= (PQ_1 + UQ_0)\, G_2^{-1}q \\ -Q_0Q_1D^-(Dx)' + TQ_0P_1G_2^{-1}BD^-(DP_1x) + TQ_0x &= TQ_0P_1G_2^{-1}q \end{aligned}$$

Using a constant auxiliary projector $\tilde{P} \in \mathbb{R}^{n\times n}$ onto $DK = \text{im } DP_1D^-$,

$$\left(DP_1D^-\right)' DP_1 = \left(DP_1D^-\right)' \tilde{P}DP_1 = \left(DP_1D^-\tilde{P}\right)' DP_1 = \tilde{P}'DP_1 = 0.$$

A constant projector $\tilde{Q} \in \mathbb{R}^{n\times n}$ onto $\text{im } DQ_1D^- = DN_1$ implies

$$\left(DP_1D^-\right)' DQ_1 = \left(DP_1D^-\right)' \tilde{Q}DQ_1 = \left(DP_1D^-\tilde{Q}\right)' DQ_1 = 0$$

due to $\text{im } \tilde{Q} \subseteq \ker DP_1D^-$. Altogether with $\text{im } DP_1D^- \subseteq \ker Q_0Q_1D^-$

$$Q_0Q_1D^-(DP_1x)' = Q_0Q_1D^-(\tilde{P}DP_1x)' = Q_0Q_1D^-\tilde{P}(DP_1x)' = 0$$

Therefore, the given linear DAE is equivalent to

$$(DP_1x)' + DP_1G_2^{-1}BD^-(DP_1x) = DP_1G_2^{-1}q \qquad (1.22)$$

$$\begin{aligned}(UQ_0 + P_0Q_1)\, G_2^{-1}BD^-(DP_1x)\\ + (PQ_1 + UQ_0)\, x\end{aligned} = (UQ_0 + P_0Q_1)G_2^{-1}q \qquad (1.23)$$

$$\begin{aligned}TQ_0P_1G_2^{-1}BD^-(DP_1x) + TQ_0x\\ -Q_0Q_1D^-(D(P_0Q_1 + UQ_0)x)'\end{aligned} = TQ_0P_1G_2^{-1}q \qquad (1.24)$$

Hereby (1.22) constitutes an explicit differential equation governing the behaviour of the solution components DP_1x. Denote $u = DP_1x$ in order to obtain

$$u'(t) = -\left(DP_1G_2^{-1}BD^-\right)(t)u(t) - \left(DP_1G_2^{-1}\right)(t)q(t) \qquad (1.25)$$

This differential equation is called the *inherent regular ODE (IRODE)* of (1.20) within the context of the tractability index. Besides, $DK = \text{im } DP_1D^-$ is an invariant subspace of (1.25) because $v(t) := (I - DP_1D^-)(t)u(t)$ satisfies the homogeneous ODE

$$v'(t) = -\left(DP_1D^-\right)'(t)v(t)$$

which can be proved multiplying (1.25) by $I - DP_1D^-$. Denote

$$\begin{array}{l} u(t) = (DP_1)(t)x(t),\ w(t) = (TQ_0)(t)x(t),\ y(t) = (UQ_0 + P_0Q_1)(t)x(t), \\ u'(t) = (DP_1x)'(t),\ (Dy)'(t) = (DQ_1x)'(t) \end{array} \tag{1.26}$$

Then,

$$\begin{array}{lcl} y & = & -(UQ_0 + P_0Q_1)G_2^{-1}BD^-u + (UQ_0 + P_0Q_1)G_2^{-1}q \\ (Dy)' & = & -\left(DQ_1G_2^{-1}BD^-u\right)' + \left(DQ_1G_2^{-1}q\right)' \\ & = & \left\{ \begin{array}{l} -\left(DQ_1G_2^{-1}BD^-\right)'u + DQ_1G_2^{-1}BP_0P_1G_2^{-1}BD^-u \\ -DQ_1G_2^{-1}BP_0P_1G_2^{-1}q + \left(DQ_1G_2^{-1}q\right)' \end{array} \right. \\ w & = & -TQ_0P_1G_2^{-1}BD^-u + TQ_0P_1G_2^{-1}q + Q_0Q_1D^-(Dy)' \\ & = & \left\{ \begin{array}{l} \left(TQ_0P_1G_2^{-1} - Q_0Q_1G_2^{-1}BP_0P_1G_2^{-1}\right)q + Q_0Q_1D^-\left(DQ_1G_2^{-1}q\right)' \\ \left(-TQ_0P_1G_2^{-1}BD^- - Q_0Q_1D^-\left(DQ_1G_2^{-1}BD^-\right)'\right)u \\ +Q_0Q_1G_2^{-1}BP_0P_1G_2^{-1}BD^-u \end{array} \right. \end{array}$$

The solution vector is

$$x = D^-u + y + w = \mathcal{K}u + Mq + Q_0Q_1D^-\left(DQ_1G_2^{-1}q\right)'$$

with

$$\mathcal{K} := \begin{array}{l} D^- - (UQ_0 + P_0Q_1)G_2^{-1}BD^- - TQ_0P_1G_2^{-1}BD^- \\ -Q_0Q_1D^-\left(DQ_1G_2^{-1}BD^-\right)' + Q_0Q_1G_2^{-1}BP_0P_1G_2^{-1}BD^- \end{array}$$

resulting in the solution representation $x(t) = \mathcal{K}(t)u(t)$ for homogeneous linear DAEs. Let $\mathcal{U}(t), \mathcal{U}(t_0) = I$ be the normalized fundamental matrix of the homogeneous IRODE

$$u' = -DP_1G_2^{-1}BD^-u.$$

Solutions of the IRODE on DK are given by $u(t) = U(t)(DP_1)(t_0)u_0$. Particularly,

$$X(t) = \mathcal{K}(t)\mathcal{U}(t)(DP_1)(t_0) \tag{1.27}$$

is a representation of the quadratic fundamental system of the homogeneous DAE (1.20) with $(DP_1)(t_0)(X(t_0) - I_m) = 0$.

For the sake of stability analysis of τ-periodic solutions of τ-periodic DAEs we are going to be in need of characteristic multipliers of the state space representation. They are defined to coincide with the non-zero eigenvalues of $\mathcal{U}(\tau)(DP_1D^-)(0)$.

$$D\mathcal{K} = R - DQ_1G_2^{-1}BP_0D^- = R - DQ_{1,c}D^- = DP_{1,c}D^-$$

whereby $P_{1,c}$ denotes the canonical projector. Because DP_1D^- and $DP_{1,c}D^-$ both project along $\ker A \oplus DN_1$, $DP_1 = DP_1P_0$ and $DK = \operatorname{im} DP_1D^-$ is an invariant subspace of the IRODE,

$$\begin{array}{rcl} (DP_1)(\tau)X(\tau)D^-(0) & = & (DP_1D^-)(\tau)\mathcal{U}(\tau)(DP_1D^-)(0) \\ & = & \mathcal{U}(\tau)(DP_1D^-)(0). \end{array} \tag{1.28}$$

Remark 1.22. Choosing $Q_1 = Q_{1,c}$ in our computations results in DS_1 being the invariant subspace of the IRODE. Then the simplified representation

$$D(\tau)X(\tau)D^-(0) = \mathcal{U}(\tau)\left(DP_1D^-\right)(0)$$

is valid. Mimicking the approach in [MHT03b], this choice of Q_1 for the complete decoupling requires both subspaces DS_1 and DN_1 to be constant. In this special case one can fall back to the representation of the fundamental matrix published in [BM00].

1.5.2 Decoupling using a projector $\tilde{P}_1$ onto S_1

Let $\tilde{P}_1(t)$ be a projector onto $S_1(t)$ along $\tilde{K}(t)$ and require the structural conditions

Assumption. *The subspaces DS_1 and $D\tilde{K}$ are constant.*

We aim at a solution representation using the identity

$$I = P_0\tilde{P}_1 + TQ_0 + \left(UQ_0 + P_0\tilde{Q}_1\right).$$

To this end, we consider the above calculations with $Q_1 = Q_{1,c}$ but we do not require DN_1 to be constant. Due to the property (1.4) we achieve a simplification in (1.21), namely

$$\begin{aligned} \left(DP_1D^-\right)(Dx)' + DP_1G_2^{-1}BD^-(DP_1x) &= DP_1G_2^{-1}q \\ UQ_0G_2^{-1}BD^-(DP_1x) + (PQ_1 + UQ_0)\,x &= (PQ_1 + UQ_0)\,G_2^{-1}q \\ -Q_0Q_1D^-(Dx)' + TQ_0P_1G_2^{-1}BD^-(DP_1x) + TQ_0x &= TQ_0P_1G_2^{-1}q. \end{aligned}$$

The auxiliary projectors $\hat{P} \in \mathbb{R}^{n\times n}$ onto $\operatorname{im} D\tilde{P}_1D^- = DS_1$ and $\hat{Q} \in \mathbb{R}^{n\times n}$ onto $\operatorname{im} D\tilde{K}$ imply

$$Q_0Q_1D^-(Dx)' = Q_0Q_1D^-\left(\hat{P}\left(D\tilde{P}_1x\right)' + \left(D\tilde{Q}_1x\right)'\right) = Q_0Q_1D^-(D\tilde{Q}_1x)'$$

due to $DS_1 \subseteq \ker Q_0Q_1D^-$. Moreover,

$$\begin{aligned} D\tilde{P}_1D^-(Dx)' &= \begin{array}{l} \left(D\tilde{P}_1x\right)' - \left(D\tilde{P}_1D^-\hat{P}\right)' D\tilde{P}_1x \\ + \left(D\tilde{P}_1D^-\right)\hat{Q}\left(D\tilde{Q}_1x\right)' \end{array} &= \left(D\tilde{P}_1x\right)' \\ D\tilde{Q}_1D^-(Dx)' &= \begin{array}{l} \left(D\tilde{Q}_1D^-\right)\hat{P}\left(D\tilde{P}_1x\right)' + \left(D\tilde{Q}_1x\right)' \\ - \left(D\tilde{Q}_1D^-\hat{Q}\right)' D\tilde{Q}_1x. \end{array} &= \left(D\tilde{Q}_1x\right)' \end{aligned}$$

We notice that $N_0 \subseteq S_1$ implies $\tilde{P}_1Q_0 = Q_0$, i.e. $\tilde{Q}_1Q_0 = 0$. Since DP_1D^- and $D\tilde{P}_1D^-$ are projecting onto DS_1,

$$DP_1D^- = DP_1\tilde{P}_1D^- + DP_1\tilde{Q}_1D^- = D\tilde{P}_1D^- + \left(DP_1D^-\right)\left(D\tilde{Q}_1D^-\right),$$

$$DP_1D^-(Dx)' = \left(D\tilde{P}_1x\right)' + (DP_1D^-)\left(D\tilde{Q}_1D^-\right)\left(D\tilde{Q}_1x\right)'$$

Taking advantage of (1.4) and of $Q_1\tilde{P}_1 = 0$,

$$P_0Q_1 = P_0\left(\tilde{Q}_1 + \tilde{P}_1\right)Q_1 = P_0\tilde{Q}_1 + P_0\tilde{P}_1Q_1\left(\tilde{P}_1 + \tilde{Q}_1\right) = P_0\tilde{Q}_1 + P_0\tilde{P}_1Q_1G_2^{-1}BP_0\tilde{Q}_1$$

resulting in the following representation of (1.21):

$$\begin{array}{rcl} u' + \left(DP_1\tilde{Q}_1D^-\right)(Dy)' & & \\ +DP_1G_2^{-1}BD^-u + DP_1G_2^{-1}BP_0P_1\tilde{Q}_1y & = & DP_1G_2^{-1}q \end{array} \tag{1.29}$$

$$\begin{array}{rcl} \left(I + \left(UQ_0 + P_0\tilde{P}_1\right)G_2^{-1}BP_0P_1\tilde{Q}_1\right)y & & \\ +UQ_0G_2^{-1}BD^-u & = & (PQ_1 + UQ_0)\,G_2^{-1}q \end{array} \tag{1.30}$$

$$\begin{array}{rcl} TQ_0P_1G_2^{-1}BD^-u - Q_0Q_1D^-(Dy)' & & \\ +TQ_0P_1G_2^{-1}BP_0P_1\tilde{Q}_1y + w & = & TQ_0P_1G_2^{-1}q \end{array} \tag{1.31}$$

whereby

$$\begin{array}{l} u(t) = \left(D\tilde{P}_1\right)(t)x(t),\ w(t) = (TQ_0)\,(t)x(t),\ y(t) = \left(UQ_0 + P_0\tilde{Q}_1\right)(t)x(t), \\ u'(t) = \left(D\tilde{P}_1x\right)'(t),\ (Dy)'(t) = \left(D\tilde{Q}_1x\right)'(t) \end{array} \tag{1.32}$$

Now, the property $\tilde{Q}_1\left(UQ_0 + P_0\tilde{P}_1\right) = 0$ implies

$$\left(I + \left(UQ_0 + P_0\tilde{P}_1\right)G_2^{-1}BP_0P_1\tilde{Q}_1\right)^{-1} = \left(I - \left(UQ_0 + P_0\tilde{P}_1\right)G_2^{-1}BP_0P_1\tilde{Q}_1\right)$$

and (1.30) reveals

$$Dy = \left(DQ_1G_2^{-1} - D\tilde{P}_1G_2^{-1}BP_0P_1\tilde{Q}_1G_2^{-1}\right)q$$

Considering homogeneous DAEs, (1.29)-(1.31) result in

$$y = -UQ_0G_2^{-1}BD^-u \qquad w = -TQ_0P_1G_2^{-1}BD^-u$$

and there exists the solution representation $x = D^-u + y + w = \tilde{\mathcal{K}}u$ with

$$\tilde{\mathcal{K}} := D^- - UQ_0G_2^{-1}BD^- - TQ_0P_1G_2^{-1}BD^-$$

Due to $\tilde{Q}_1y = \tilde{Q}_1D^-(Dy) = 0$ the IRODE reduces to $u' = -DP_1G_2^{-1}BD^-u$ on the invariant subspace DS_1. Let $\mathcal{U}(t)$ be the normalized ($\mathcal{U}(t_0) = I$) fundamental matrix of the inherent regular ODE. Then,

$$X(t) = \tilde{\mathcal{K}}(t)\mathcal{U}(t)\left(D\tilde{P}_1\right)(t_0) \tag{1.33}$$

is a quadratic fundamental matrix of the homogeneous DAE (1.20). As a result of the invariance of DS_1,

$$\begin{aligned} D(t)X(t)D^-(t_0) &= R(t)\mathcal{U}(t)\left(D\tilde{P}_1D^-\right)(t_0) \qquad (1.34) \\ &= \left(D\tilde{P}_1D^-\right)(t)\mathcal{U}(t)\left(D\tilde{P}_1D^-\right)(t_0) \\ &= \mathcal{U}(t)\left(D\tilde{P}_1D^-\right)(t_0) \end{aligned}$$

Remark 1.23. It is possible to construct a nonlinear decoupling based on a projector $\tilde{P}_1$ onto the invariant subspace DS_1 but the term $DP_1\tilde{Q}_1D^-\,(Dy)'$ in (1.29) complicates the analysis seriously. We present such a decoupling procedure in the appendix. As a result of the technical difficulties, the autonomous structure of the SSF (of autonomous index-2 DAEs) and commutativity between decoupling and linearization are waived. The successful analysis based on the structural condition (1.5.1) is per se a weakening of the requirements in [MHT03b], where both DN_1 and DS_1 are required constant.

Remark 1.24. The regularity of $\tilde{G}_2 := G_1 + BP_0\tilde{Q}_1$ can be proved by means of a suitable factorization. Unfortunately, multiplication of the DAE by $\tilde{G}_2^{-1}$ does not yield a convenient decoupling because

$$\begin{aligned} D\tilde{P}_1\tilde{G}_2^{-1}A(Dx)' &= \left(D\tilde{P}_1x\right)' + D\tilde{P}_1\tilde{G}_2^{-1}G_1\tilde{Q}_1D^-\,(Dx)' \\ &= \left(D\tilde{P}_1x\right)' + \left(D\tilde{P}_1\tilde{G}_2^{-1}G_1\tilde{Q}_1x\right)' - \left(D\tilde{P}_1\tilde{G}_2^{-1}G_1\right)'\tilde{Q}_1x \end{aligned}$$

Here again, $\left(\tilde{Q}_1x\right)'$ appears in the equation together with the derivative of the dynamical components $\left(D\tilde{P}_1x\right)'$ and this fact results in serious problems if nonlinear DAEs are considered.

2 The state space form

It's not what you know, it's what you can prove.

(Alonzo Harris, Training Day (2001))

The goal of this chapter is to set up a *state space form* (SSF) of the inherent dynamics for a class of fully implicit nonlinear index-2 DAEs. Subsequently, two important properties of the transformation to the state space representation are proved. Given an autonomous DAE with constant subspaces DN_1 and DK along x_*, the SSF inherits the autonomous structure. In addition, the presented transformation to the state space form commutes with linearization around the same reference solution x_*, i.e. the diagram

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{DAE}\\ f\left((Dx)'(t), x(t), t\right) = 0\end{array}} \arrow[r, "\text{linearization}"] \arrow[d, "\begin{array}{l}\text{transformation}\\ \text{to the SSF}\end{array}"'] \arrow[dr, phantom, "\circlearrowright"] & \boxed{\begin{array}{c}\text{Linearized DAE}\\ f_y(t)\,(Dx)'(t) + f_x(t)x(t) = 0\end{array}} \arrow[d, "\begin{array}{l}\text{transformation}\\ \text{to the SSF}\end{array}"] \\
\boxed{\begin{array}{c}\text{State Space Form of the DAE}\\ \xi'(t) = g\left(\xi(t), t\right)\end{array}} \arrow[r, "\text{linearization}"] & \boxed{\begin{array}{c}\text{SSF of the linearized DAE}\\ z'(t) = g_\xi\left(\xi_*(t), t\right) z(t)\end{array}}
\end{tikzcd} \tag{2.1}$$

where ξ_* is the respective transformation of x_*, is commutative. This allows to formulate the direct method of Lyapunov for index-2 DAEs referring to the linearized DAE only.

The state space form is a convenient tool to analyse the stability behaviour of solutions nearby x_*. Its construction relies on the matrix chain, a suitable parametrization of systemic subspaces and the implicit function theorem. Generally, one cannot expect to compute the transformation numerically in reasonable time.

A preliminary note on the decoupling of nonlinear DAEs

The most laborious intermediate step of our transformation is a complete decoupling of the properly stated differential-algebraic system into an ODE on an invariant subspace together with some appropriate constraints. It is meant to generalize the solution representation for linear DAEs in § 1.5 that is to state that the solution vector $x(t)$ is already determined by its components

$$u(t) = (DP_1)\left((Dx_*)'(t), x_*(t), t\right) x(t)$$

where u solves an explicit ODE on an invariant subspace. Fortunately, we do not need to start from the scratch due to the fundamental results on decoupling of nonlinear differential-algebraic systems in the context of the tractability index achieved by the DAE research group at the Humboldt university of Berlin. The decoupling of nonlinear index-1 DAEs (1.1) dates back not later than to [GM86, § 1.2]. Although it is an elegant way to reveal the obvious constraints in the index-1 case by a single application of the implicit function theorem, this approach is not adaptable to higher index systems. An alternative approach to decouple index-1 DAEs (1.1) relying on a modified Taylor expansion was formulated by Caren Tischendorf in [Tis94, Theorem 2.1]. The basic idea there is to separate the linear part of the system and the remaining nonlinear term and to add a suitable zero-sum in order to enforce certain properties of the remainder term. Subsequently the DAE is splitted into an equivalent system applying the linear decoupling method of the linearized DAE to the entire system. The critical part is to ensure that the implicit function theorem can be applied to the resulting constraints thus leading to an ODE representation of the inherent dynamics. This task was mastered in the thesis [Tis96] for linear implicit index-2 DAEs resulting in structural assumptions on the DAE which allow the astonishingly simple formulation (2.15) using the matrix chain and are shown to be of great practical interest in the circuit simulation. Another advantage of this approach is its versatility which is underlined by an outline of the application to index-3 DAEs in the summary of this thesis. It is not surprising that there are several adaptations of the method. For example, [San00, § 3.1.2] improves the complete decoupling in [Tis96, Theorem 3.12] by replacing the canonical projector $Q_{1,\mathrm{c}}$ with an admissible projector Q_1 onto N_1. Alternatively, a projector onto S_1 is taken into consideration aiming at commutativity between the complete decoupling and the BDF and stiffly accurate Runge-Kutta methods for DAEs, cf. [San00, § 3.2]. Meanwhile, a strong emphasis on a properly stated derivative term of differential-algebraic systems in a series of publications [BM00, Mä01, Mä02a, Mä05] is leading to new findings with regard to the benefits of considering such DAEs which arise naturally in some applications, e.g. the Modified Nodal Analysis (MNA) equations ([Mä03]). An intermediate result of the thesis [Voi06] is to prove the existence of a solution of properly formulated index-2 DAEs $A(t)(D(t)x(t))' + b(x(t), t) = 0$ assuming the mentioned structural condition $N_0(t) \cap S_0(y, x, t)$ independent of y, x. There, an incomplete decoupling of the DAE is used which actually follows the modified Taylor expansion approach similar to [Tis96]. The intention of the present chapter is threefold: first, we point out crucial technical details of the complete decoupling originating from [Tis96, Theorem 3.12] and adjust them to properly stated derivative term. In doing so we hope that the structural conditions on index-2 DAEs are going to appear more or less canonical to the reader. The second goal is to enhance the decoupling approach by passing over to a state space representation and to reveal important properties of this transformation, more precisely the commutativity between decoupling and inheritance of the autonomous structure of the dynamics. In our opinion, the already mentioned publications on DAEs with a properly stated derivative term together with [MHT03a, MHT03b] tend to focus mainly on the numerical benefits of the Formulation (1.2). Therefore, we felt intrigued to investigate the influence a proper formulation or some constant subspaces might have on the DAE from a qualitative point of view, regarding exact solutions.

2.1 Decoupling nonlinear DAEs

In order to decouple nonlinear differential-algebraic systems, we represent them adding up the linearization terms around the reference solution $x_* \in C_D^1$, the nonlinear remainder term plus a suitable zero-sum.

We have already proved the linearization of a given DAE (1.2) around x_* to be

$$f_y^*(t)\,(Dx)'(t) + f_x^*(t)x(t) = 0$$

denoting $f_y^*(t) := f_y\left((Dx_*)'(t), x_*(t), t\right)$ and $f_x^*(t) := f_x\left((Dx_*)'(t), x_*(t), t\right)$. Taking the remainder term into consideration leads to the equivalent representation

$$f_y^*(t)\,(Dx)'(t) + f_x^*(t)x(t) + \hat{h}\left((Dx)'(t), x(t), t\right) = 0$$

of (1.2) whereas

$$\hat{h}\left((Dx)'(t), x(t), t\right) := f\left((Dx)'(t), x(t), t\right) - \left(f_y^*(t)\,(Dx)'(t) + f_x^*(t)x(t)\right)$$

The remainder term does not fulfill $\hat{h}\left((Dx_*)'(t), x_*(t), t\right) \equiv 0$ so we use a fitting

$$r(t) := \left(f_y^*(t)\,(Dx_*)'(t) + f_x^*(t)x_*(t)\right) - f\left((Dx_*)'(t), x_*(t), t\right)$$

such that

$$\tilde{h}(y,x,t) := \hat{h}(y,x,t) + r(t) = \begin{array}{l} f(y,x,t) - f\left((Dx_*)'(t), x_*(t), t\right) \\ +f_y^*(t)\left((Dx_*)'(t) - y\right) + f_x^*(t)\left(x_*(t) - x\right) \end{array}$$

satisfies the following properties

$$\tilde{h}\left((Dx_*)'(t), x_*(t), t\right) \equiv 0,\ \tilde{h}_x\left((Dx_*)'(t), x_*(t), t\right) \equiv 0,\ \tilde{h}_y\left((Dx_*)'(t), x_*(t), t\right) \equiv 0 \tag{2.2}$$

which turn out to be favourable because they simplify calculations and play a decisive role in proving commutativity of (2.21). Due to the t-dependent zero-sum $r(t)$ the original DAE (1.2) is equivalent to the corresponding *modified Taylor expansion*

$$f_y^*(t)\,(Dx)'(t) + f_x^*(t)x(t) + \tilde{h}\left((Dx)'(t), x(t), t\right) - r(t) = 0 \tag{2.3}$$

Motivation of a nonlinear decoupling

We split the modified Taylor expansion (2.3) into an equivalent system of three equations multiplying by the matrix valued functions used to decouple the linearization of (1.2). Keep in mind the initial agreement to denote

$$Q_1(t) = Q_1\left((Dx_*)'(t), x_*(t), t)\right)$$

and so on. We require $Q_1(y,x,t)$ to be an admissible projector on $N_1(y,x,t)$ along $K(y,x,t)$. Moreover, all projectors in the matrix chain should be at least continuous, i.e. $Q_1(t)$, $G_2(t)$ etc. are continuous as well. Using the notation (1.26),

$$I = P_0P_1 + TQ_0 + (UQ_0 + P_0Q_1)$$

and the auxiliary function

$$h\left(u', (Dy)', u, w, y, t\right) := \tilde{h}\begin{pmatrix} u' + (Dy)', (P_0P_1D^-)(t)u \\ +(TQ_0)(t)w + (UQ_0 + P_0Q_1)(t)y, t \end{pmatrix}$$

the splitted DAE exhibits the representation

$$\begin{aligned}
&u'(t) + \left(DP_1G_2^{-1}f_x^*D^-\right)(t)u(t) \\
&+ \left(DP_1G_2^{-1}\right)(t)\left(h\left(u'(t), (Dy)'(t), u(t), w(t), y(t), t\right) - r(t)\right) &= 0 \\
&(UQ_0 + P_0Q_1)(t)\left(G_2^{-1}f_x^*D^-\right)(t)u(t) + y(t) \\
&+ (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\left(h\left(u'(t), (Dy)'(t), u(t), w(t), y(t), t\right) - r(t)\right) &= 0 \\
&-(Q_0Q_1D^-)(t)(Dy)'(t) + w(t) + \left(TQ_0P_1G_2^{-1}f_x^*D^-\right)(t)u(t) \\
&+ \left(TQ_0P_1G_2^{-1}\right)(t)\left(h\left(u'(t), (Dy)'(t), u(t), w(t), y(t), t\right) - r(t)\right) &= 0
\end{aligned} \tag{2.4}$$

Per constructionem, h, $h_{u'}$, $h_{(Dy)'}$, h_u, h_w, h_y vanish along the extended integral curve of x_*, where $u_*(t) := (DP_1)(t)x_*(t)$ etc.

Regarding (2.4) as an algebraic system in formal variables $(Dy)'$, u', y, w, u, it is possible to eliminate y and w locally and to solve the first equation in (2.4) for u' applying the implicit function theorem three times. However, we have to incorporate that $(Dy)'$ in the equivalent representation of a DAE is determined by differentiation of the Dy solution components. If $(Dy)'(t) = v(u(t), t)$ holds then the reformulated first equation in (2.4) will represent an ordinary differential equation for the solution components $u(t) = (DP_1)(t)x(t)$. In the following we have to prove such a representation of $(Dy)'$. Differentiating the algebraic relation $D(t)y(t) = D(t)\tilde{v}\left((Dy)'(t), u(t), t\right)$ we get

$$(Dy)'(t) = \frac{\partial}{\partial t}\left(D(t)\tilde{v}\left((Dy)'(t), u(t), t\right)\right)$$

which is equivalent to an algebraic relation $\tilde{V}\left((Dy)'(t), u(t), t\right) = 0$, if the coefficient of the second derivative $\frac{\partial^2}{(\partial t)^2}(Dy)(t)$ vanish. The latter turns out to be the main structural condition we have to impose on (1.2). Unfortunately, this approach results in sophisticated assumptions if formulated in terms of the given DAE. Even the formulation of $y = \tilde{v}\left((Dy)'(t), u(t), t\right)$ necessitates two implicitly defined functions for u' and w whose partial derivatives enter the above structural assumption. Nevertheless, the approach turns out to be practicable for linear implicit DAEs with $A(x,t) = A(t)$ where $\tilde{h} = \tilde{h}(x,t)$ resp. $h = h(u, w, y, t)$.

2.1.1 Linear implicit DAEs

Consider a linear implicit index-2 DAE

$$A(t)(Dx)'(t) + b(x(t), t) = 0 \tag{2.5}$$

having a solution x_*. The differential-algebraic system is equivalent to its modified Taylor expansion (2.3) with

$$\begin{aligned}\tilde{h}(x,t) &= b(x,t) - b(x_*,t) + b_x^*(t)\,(x_*(t) - x)\\ r(t) &= b_x^*(t)\,x_*(t) - b\,(x_*(t),t)\end{aligned}$$

and $b_x^*(t) := b_x(x_*(t),t)$. In addition, consider a subspace $K(x,t) \subseteq \mathbb{R}^m$ having a basis of continuous functions satisfying $N_1(x,t) \oplus K(x,t) = \mathbb{R}^m$ and $K(x,t) \supseteq N_0(t)$. Then we choose the matrix chain of the tractability index such that the admissible projector $Q_1(x,t)$ maps $N_1(x,t)$ along $K(x,t)$. We need

Assumption 2.1. *The subspaces $(DN_1)(x_*(t),t)$ and $(DK)(x_*(t),t)$ are constant.*

The equivalent representation (2.4) of (2.5) reads

$$\begin{array}{l} u'(t) + \left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) \\ +\left(DP_1G_2^{-1}\right)(t)\left(h\left(u(t),w(t),y(t),t\right) - r(t)\right)\end{array} = 0 \qquad (2.6)$$

$$\begin{array}{l} (UQ_0 + P_0Q_1)(t)\left(G_2^{-1}b_x^*D^-\right)(t)u(t) + y(t) \\ +(UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\left(h\left(u(t),w(t),y(t),t\right) - r(t)\right)\end{array} = 0 \qquad (2.7)$$

$$\begin{array}{l} w(t) - (Q_0Q_1D^-)(t)\,(Dy)'(t) + \left(TQ_0P_1G_2^{-1}b_x^*D^-\right)(t)u(t) \\ +\left(TQ_0P_1G_2^{-1}\right)(t)\left(h\left(u(t),w(t),y(t),t\right) - r(t)\right)\end{array} = 0 \qquad (2.8)$$

using the notation (1.26) for the particular solution components and their derivatives and the subscript-* notation $u_*(t) = (DP_1)(t)x_*(t)$ for the reference solution x_*. The auxiliary function h is

$$h(u,w,y,t) = \tilde{h}\left(\left(P_0P_1D^-\right)(t)u + (TQ_0)(t)w + (UQ_0 + P_0Q_1)(t)y, t\right).$$

Solving for $(UQ_0 + P_0Q_1)(t)\,x$

Equation (2.7) can be written as

$$M\left(u(t),w(t),(UQ_0 + P_0Q_1)(t)y(t),t\right) = 0 \qquad (2.9)$$

by means of the function

$$M(u,w,y,t) := \begin{array}{l}\left((UQ_0 + P_0Q_1)\,G_2^{-1}b_x^*D^-\right)(t)\,u + y \\ +\left((UQ_0 + P_0Q_1)\,G_2^{-1}\right)(t)\left(h(u,w,y,t) - r(t)\right)\end{array}$$

M is continuous and continuously partial differentiable with respect to u, w, y. Set

$$Z(t) := (UQ_0 + P_0Q_1)(t), \quad z := Z(t)\,y$$

to obtain

$$\frac{\partial}{\partial z}M(u,w,z,t) = M_y(u,w,Z(t)\,y,t) = I + (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)h_y(u,w,Z(t)y,t)$$

According to (2.7), $M\left(u_*(t), w_*(t), Z(t)y_*(t), t\right) \equiv 0$ holds for $y_*\left(t\right) = Z\left(t\right) y_*\left(t\right)$ and $M_z\left(u_*(t), w_*(t), Z(t)y_*(t), t\right) \equiv I$ due to the properties of the remaining term h. Let $t_0 \in I$ be arbitrary, but fixed.

$$0 = M\left(u, w, \left(UQ_0 + P_0Q_1\right)(t)y, t\right) = M\left(u, w, z, t\right)$$

can be solved for z in a neighbourhood of $(u_*(t_0), w_*(t_0), y_*(t_0), t_0)$, i.e there exists a unique implicitly defined function $m = m\left(u, w, t\right)$ on

$$\left\{(u, w, t) \in \mathbb{R}^{n+m+1} \quad | \quad \left\|(u, w, t) - \left(u_*\left(t_0\right), w_*\left(t_0\right), t_0\right)\right\| < c_1\right\}$$

satisfying

$$\begin{aligned} m\left(u_*(t_0), w_*(t_0), t_0\right) &= y_*(t_0) \\ M\left(u, w, Z(t)m\left(u, w, t\right), t\right) &= 0 \\ m\left(u, w, t\right) &= \left(UQ_0 + P_0Q_1\right)(t)m\left(u, w, t\right) \end{aligned}$$

The last property is due to $Z = Z^2$ for both $m\left(u, w, t\right)$ and $Z(t)m\left(u, w, t\right)$ satisfy the constraint $M(u, w, Z(t)y, t) = 0$. Because of the uniqueness, $m\left(u, w, t\right) = Z(t)m\left(u, w, t\right)$ is valid.

Solving for the index-1 components $\left(TQ_0\right)(t)\, x$

Preliminary considerations about the third equation in (2.4) corresponding to (2.8) motivate the structural assumption

Assumption 2.2. *$Q_1(t)m_w\left(u, w, t\right) \equiv 0$ in a neighbourhood of $(u_*(t), w_*(t), t)$.*

The above assumption implies $Q_0Q_1m = \left(Q_0Q_1m\right)(u, t)$ locally around the integral curve of x_*. Let us assume $Q_0Q_1D^- \in C^1$ then the chain rule ensures

$$\begin{aligned} &\left(Q_0Q_1D^-\right)(t)\tfrac{\partial}{\partial t}\left(Dm\left(u(t), w(t), t\right)\right) = \\ &= \tfrac{\partial}{\partial t}\left(Q_0Q_1m\right)\left(u(t), t\right) - \left(Q_0Q_1D^-\right)'(t)\left(Dm\right)\left(u(t), w(t), t\right) \\ &= \begin{array}{l} \left(Q_0Q_1m\right)_u\left(u(t), t\right)u'(t) + \left(\left(Q_0Q_1D^-\right)\left(Dm\right)\right)_t\left(u(t), t\right) \\ -\left(Q_0Q_1D^-\right)'(t)\left(Dm\right)\left(u(t), w(t), t\right) \end{array} \\ &= \left(Q_0Q_1\right)(t)m_u\left(u(t), w(t), t\right)u'(t) + \left(Q_0Q_1D^-\right)(t)\left(Dm\right)_t\left(u(t), w(t), t\right) \end{aligned}$$

Using (2.6) to replace $u'(t)$ results in

$$\begin{aligned} &\left(Q_0Q_1D^-\right)(t)\tfrac{\partial}{\partial t}\left(D(t)y(t)\right) = \\ &\left\{\begin{array}{l} \left(Q_0Q_1D^-\right)(t)\left(Dm\right)_t\left(u(t), w(t), t\right) \\ -\left(Q_0Q_1\right)(t)m_u\left(u(t), w(t), t\right)\left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) \\ -\left(Q_0Q_1\right)(t)m_u\left(u(t), w(t), t\right)\left(DP_1G_2^{-1}\right)(t)\left(h\left(u(t), w(t), y(t), t\right) - r(t)\right) \end{array}\right. \end{aligned}$$

Inserting this expression into (2.8) leads to

$$\mathcal{K}\left(u(t), \left(TQ_0\right)(t)w(t), t\right) = 0 \tag{2.10}$$

with

$$\mathcal{K}(u,w,t) := \begin{cases} -\left(Q_0Q_1D^-\right)(t)\left(Dm\right)_t(u,w,t) + \left(Q_0Q_1m\right)_u(u,t)\left(DP_1G_2^{-1}b_x^*D^-\right)(t)u \\ +\left(Q_0Q_1m\right)_u(u,t)\left(DP_1G_2^{-1}\right)(t)\left(h\left(u,w,m\left(u,w,t\right),t\right)-r(t)\right) \\ +\left(TQ_0P_1G_2^{-1}b_x^*\right)(t)D^-(t)u + w \\ +\left(TQ_0P_1G_2^{-1}\right)(t)\left(h\left(u,w,m\left(u,w,t\right),t\right)-r(t)\right) \end{cases}$$

Remark 2.3. The constraint (2.8) is equivalent to (2.10) under the Assumption 2.2. In other words, the analytic relation between the formal variable $(Dy)'$ and its meaning as the derivative of Dy in the DAE is depicted correctly.

Now the algebraic equation (2.10) has to be solved for $\xi := (TQ_0)(t)\,w$.

$$\frac{\partial}{\partial\xi}\mathcal{K}(u,\xi,t) = \begin{array}{l} I - \left(Q_0Q_1D^-\right)(t)\left(Dm\right)_{tw}(u,\xi,t) \\ +\left(TQ_0P_1G_2^{-1}\right)(t)\left(h_w + h_y m_w\right)(u,\xi,t) \\ +\left(Q_0Q_1m\right)_u(u,t)\left(DP_1G_2^{-1}\right)(t)\left(h_w + h_y m_w\right)(u,\xi,t) \end{array}$$

The partial derivatives $h_w,\ h_y$ vanish in $(x_*(t),t)$. For the (i,j)-th entry of $(Dm)_{tw}$ holds:

$$\begin{aligned} \left[(Dm)_{tw}\right]_{ij} &= \frac{\partial\left[(Dm)_t\right]_i}{\partial w_j} = \frac{\partial}{\partial w_j}\left((Dm)_i\right)_t = \left((Dm)_i\right)_{tw_j} \\ &= \left((Dm)_i\right)_{w_jt} = \frac{\partial}{\partial t}\left[(Dm)_w\right]_{ij} = \left[(Dm)_{wt}\right]_{ij} \end{aligned}$$

if the partial derivatives $(Dw)_{tw}$, $(Dw)_{wt}$ exist and commute. The outcome of this is

$$\begin{aligned} \left(Q_0Q_1D^-\right)(t)\left(Dm\right)_{tw}(u,\xi,t) &= \left(Q_0Q_1D^-\right)(t)\left(Dm\right)_{wt}(u,\xi,t) \\ &= \left(\frac{\partial}{\partial t}\left(Q_0Q_1m\right)_w - \left(Q_0Q_1D^-\right)'(t)\left(Dm\right)_w\right)(u,\xi,t) \\ &= -\left(Q_0Q_1D^-\right)'(t)\left(DQ_1D^-\right)(t)\left(Dm\right)_w(u,\xi,t) = 0 \end{aligned}$$

in a neighbourhood of $(u_*(t),(TQ_0)(t)w_*(t),t)$[1] due to Assumptions 2.2 and 2.1. Therefore,

$$\frac{\partial}{\partial\xi}\mathcal{K}\left(u_*(t),\xi_*(t),t\right) \equiv I$$

The components u_*, ξ_* solve the equation $\mathcal{K}\left(u_*(t),\xi_*(t),t\right) = 0$ as a consequence of x_* being a solution of the given DAE. According to the implicit theorem function, there exists a unique resolution of $\mathcal{K}(u,\xi,t) = 0$ for ξ on $B_{c_2}\left(u_*(t_0),t_0\right)$ exhibiting

$$\begin{aligned} k\left(u_*(t_0),t_0\right) &= \xi_*(t_0) \\ \mathcal{K}\left(u,(TQ_0)(t)\,k\left(u,t\right),t\right) &= 0 \\ k\left(u,t\right) &= (TQ_0)(t)k\left(u,t\right) \end{aligned}$$

The last property is due to $TQ_0 = (TQ_0)^2$ and the fact that both $k\left(u_*(t),t\right)$ and $(TQ_0)(t)\,k\left(u_*(t),t\right)$ satisfy (2.10) so $k\left(u,t\right) = (TQ_0)(t)\,k\left(u,t\right)$ due to uniqueness of the implicit function k.

[1] where $\xi_*(t) = (TQ_0)(t)\,x_*(t) = (TQ_0)(t)\,w_*(t)$

Remark 2.4. At this point it is crucial to ask for a reference solution x_* of the DAE (2.5). Starting with an *inconsistent* initial value $x_0 \in \mathcal{M}_0(t_0)$, the representation (2.3) in (y_0, x_0, t_0) is defined and one can even solve the algebraic constraint (2.9) for $(UQ_0 + P_0Q_1)(x_0, t_0)\, x_0$. The partial differential

$$\frac{\partial}{\partial \xi_0}\mathcal{K}\left((DP_1)(x_0, t_0)\, x_0, \xi_0, t_0\right)$$

$\xi_0 = (TQ_0)(x_0, t_0)\, x_0$ is invertible as well. Since there is no DAE solution passing through x_0, the hidden constraint

$$\mathcal{K}\left((DP_1)(x_0, t_0)\, x_0, (TQ_0)(x_0, t_0)x_0, t_0\right) = 0$$

cannot be satisfied. Consequently, the implicit function theorem cannot ensure the resolution for $w = (TQ_0)(x_0, t_0)\, x$.

We have proved the following representations for two solution components of the linear implicit DAE (2.5) in a neighbourhood of $(x_*(t_0), t_0)$:

$$\begin{aligned}(TQ_0)(t)x(t) &= k\left((DP_1)(t)x(t), t\right), \\ (UQ_0 + P_0Q_1)(t)x(t) &= m\left((DP_1)(t)x(t), k\left((DP_1)(t)x(t), t\right), t\right)\end{aligned} \tag{2.11}$$

Inserting these representations into (2.6) demonstrates that the DP_1-components of a solution x of (2.5) nearby $(x_*(t_0), t_0)$ satisfy the ordinary differential equation

$$u'(t) = \begin{array}{l} -\left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) + \left(DP_1G_2^{-1}\right)(t)r(t) \\ -\left(DP_1G_2^{-1}\right)(t)h\left(u(t), k\left(u(t), t\right), m\left(u(t), k\left(u(t), t\right), t\right), t\right). \end{array} \tag{2.12}$$

Definition 2.5. The ODE (2.12) is called the *inherent regular ODE*, abbr. *IRODE* of the given DAE (2.5) locally around x_*.

It is convenient to talk about *the* IRODE although this differential equation is not unique at all.

Lemma 2.6. *The IRODE* (2.12) *has the invariant subspace* $DK = \operatorname{im} DP_1D^-$ *if Assumption 2.1 is valid.*

Proof. Multiplying (2.12) by $(I - DP_1D^-)$ results in $0 = \left(I - (DP_1D^-)(t)\right)u'(t)$. Moreover,

$$\begin{aligned}\left(I - DP_1D^-\right)' &= -\left(I - DP_1D^-\right)\left(DP_1D^-\right)' - \left(DP_1D^-\right)'\left(I - DP_1D^-\right) \\ &= -\left(DP_1D^-\right)'\left(I - DP_1D^-\right)\end{aligned}$$

because $(DQ_1D^-)(DP_1D^-)' = 0$ is true due to Assumption 2.1. Define

$$v(t) := \left(I - DP_1D^-\right)(t)u(t)$$

Then,

$$\begin{aligned} v'(t) &= \left(I - DP_1D^-\right)(t)u'(t) + \left(I - DP_1D^-\right)'(t)u(t) \\ &= \left(I - DP_1D^-\right)'(t)u(t) \\ &= -\left(DP_1D^-\right)(t)v(t) \end{aligned}$$

If $u(t_0) \in (DK)(t_0) \Longleftrightarrow v(t_0) = 0$ holds true then solutions v of the above homogeneous ODE feature $v(t) \equiv 0$. $\square$

Summarizing the assumptions of the complete decoupling, we obtain

Theorem 2.7. *[Complete decoupling of linear implicit DAEs]*

Let $x_* \in C^1_D(I, \mathbb{R}^m)$ be a solution of the properly formulated index-2 DAE

$$A(t)(Dx)'(t) + b(x(t), t) = 0$$

on a compact interval $I \subseteq \mathbb{R}$. Let the smoothness assumptions

- $(Q_0Q_1D^-)(x_*(t), t)$ *continuously differentiable*
- $\left((UQ_0 + P_0Q_1)G_2^{-1}b_x^*P_0P_1\right)(t)$ *and*
- $\left((UQ_0 + P_0Q_1)G_2^{-1}\right)(t)\left(b(x, t) - b_x^*(t)x\right)$ *twice continuously differentiable*

be satisfied. Assume $Q_1(x, t)$ to be an admissible projector on $N_1(x, t)$ along $K(x, t)$ with

$$(DN_1)(x_*(t), t) = const., \qquad (DK)(x_*(t), t) = const.$$

Furthermore, let the structural assumption

$$\begin{aligned} &Q_1(t)\left(I + (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\left(b_x(x, t) - b_x(x_*(t), t)\right)(UQ_0 + P_0Q_1)(t)\right)^{-1} \cdot \\ &\cdot (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\left(b_x(x, t) - b_x(x_*(t), t)\right)(TQ_0)(t) \end{aligned} = 0 \tag{2.13}$$

be valid in a neighbourhood of the integral curve of x_*. Then, the inherent dynamics of the DAE locally around the trajectory of x_* referring to the components $u(t) = (DP_1)(x_*(t), t)$ is represented by the inherent regular ODE

$$u'(t) = \begin{array}{l} -\left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) + \left(DP_1G_2^{-1}\right)(t)r(t) \\ -\left(DP_1G_2^{-1}\right)(t)h\left(u(t), k(u(t), t), m\left(u(t), k(u(t), t), t\right), t\right) \end{array}$$

on the invariant subspace DK.

Proof. Let $t_0 \in I$ be arbitrary, but fixed.

The constancy of DN_1 and DK along the integral curve of x_* corresponds to Assumption 2.1. The given smoothness implies the function M in (2.9) to be twice continuously differentiable. According to the relation

$$M_z\left(u_*(t_0), w_*(t_0), Z(t_0)y_*(t_0), t_0\right) = I_m$$

the constraint (2.7) can be solved for $z = (UQ_0 + P_0Q_1)(t)\, y$ locally around $(x_*(t_0), t_0)$. Then, the derivative of the resolution satisfies

$$m_w(u, w, t) = -\left(M_z^{-1} M_w\right)(u, w, m(u, w, t), t)$$

with

$$\begin{aligned} M_z(u, w, m(u, w, t), t) &= I + (UQ_0 + P_0Q_1)(t) G_2^{-1}(t) h_y(u, w, m(u, w, t), t) \\ M_w(u, w, m(u, w, t), t) &= (UQ_0 + P_0Q_1)(t) G_2^{-1}(t) h_w(u, w, m(u, w, t), t) \end{aligned}$$

Hence, (2.13) expresses Assumption 2.2 in terms of the given DAE. In this case (2.8) is known to be equivalent to the constraint (2.10). Application of the implicit function theorem to (2.10) results in the local resolution function $k(u, t) = (TQ_0)(t)\, k(u, t)$ due to $\mathcal{K}_\xi(u_*(t), w_*(t), t) \equiv I_m$. Inserting the algebraic relations k and m in (2.6), we arrive at the desired representation of the inherent regular ODE. The initial values are $u_0 = (DP_1x)(t_0) \in DK$ so according to Lemma 2.6 $\forall t \in I : u(t) \in DK$ is valid, i.e. the inherent dynamics take place in the invariant subspace DK. The expression "complete decoupling of the DAE" is used interchangeably to these implicitly defined functions $m(u, w, t)$, $k(u, t)$ and the right hand side of the IRODE.

It follows that an open cover $\bigcup_{i\in\mathcal{J}} \mathrm{B}_{\epsilon_i}((x_*(t_i), t_i))$ of the integral curve C belonging to x_* exists such that $\forall i \in \mathcal{J} : t_i \in I$ and the complete decoupling of the initial system 2.5 is feasible on $\mathrm{B}_{\epsilon_i}((x_*(t_i), t_i)) \subseteq \mathbb{R}^{m+1}$. The implicit functions m,k defined there coincide on

$$\mathrm{B}_{\epsilon_i}((x_*(t_i), t_i)) \cap \mathrm{B}_{\epsilon_j}((x_*(t_j), t_j))$$

due to local uniqueness. From now, m and k are meant to denote the maximal extension of the local functions to the entire region $\bigcup_{i\in\mathcal{J}} \mathrm{B}_{\epsilon_i}((x_*(t_i), t_i))$. The integral curve of a continuous function on a compact interval is compact as well, so a finite subcover $C \subseteq \bigcup_{i=1,\dots,s} \mathrm{B}_{\epsilon_i}((x_*(t_i), t_i))$ exists. Accordingly, there is an $\epsilon > 0$ such that the complete decoupling of the initial system (2.5) is feasible on the closed ϵ-tube $\mathcal{T}_{\epsilon,C} \subset \bigcup_{i=1,\dots,s} \mathrm{B}_{\epsilon_i}((x_*(t_i), t_i))$ around C.

Per constructionem, a function

$$x \in \mathrm{B}_\epsilon(x_*) \subseteq C_D^1(I, \mathbb{R}^m)$$

solves the DAE (2.5) if and only if the solution components

$$u(t) = (DP_1)(t)\, x(t),\ y(t) = (UQ_0 + P_0Q_1)(t)\, x(t),\ w(t) = (TQ_0)(t)\, x(t)$$

satisfy the equivalent system (2.6)-(2.8). Due to the above implicit functions, the solution components $(TQ_0)(t)\, x(t)$ and $(UQ_0 + P_0Q_1)(t)\, x(t)$ of the mentioned system fulfill Representation (2.11). Inserting this representation into (2.6) proves that DP_1x solves necessarily the inherent regular ODE (2.12). Due to $(DP_1)(t_0)x(t_0) \in (DK)(t_0)$ and Lemma 2.6, this solution of the IRODE belongs the constant invariant subspace DK for all $t \in I$.

In order to construct a solution of the DAE from a solution of the associated IRODE on DK, consider the function

$$x(t; u) := D^-(t)u + k(u, t) + m(u, k(u, t), t) \tag{2.14}$$

Due to its continuity, there exists a $\delta > 0$ such that $x(t,u) \in \mathrm{B}_{\epsilon}(x_*(t_0)) \subseteq \mathbb{R}^m$ for all $(t,u) \in \mathrm{B}_{\delta}(t_0, (DP_1x_*)(t_0)) \subseteq \mathbb{R}^{1+n}$. From the theory of ordinary differential equations we also know that there exists a $0 < \hat{\delta} \leq \delta$ such that solutions $u(t;t_0,u_0)$ of initial value problems $u(t_0;t_0,u_0) = u_0$ of the inherent regular ODE with $u_0 \in \mathrm{B}_{\hat{\delta}}((DP_1x_*)(t_0))$ exist on $I_{\hat{\delta}} := \left(t_0 - \hat{\delta}, t_0 + \hat{\delta}\right) \subseteq I$ and satisfy $u(t;t_0,u_0) \in \mathrm{B}_{\delta}((DP_1x_*)(t_0))$ for all $t \in I_{\hat{\delta}}$. It follows that

1. $\forall t \in I_{\hat{\delta}}, u_0 \in \mathrm{B}_{\hat{\delta}}((DP_1x_*)(t_0)) \cap DK : \; x(t;u(t;t_0,u_0)) \in \mathrm{B}_{\epsilon}(x_*(t_0))$
2. $x(\cdot;u(\cdot;t_0,u_0)) \in C_D^1(I_{\hat{\delta}}, \mathbb{R}^m)$ for
$$D(t)\,x(t;u(t;t_0,u_0)) = R(t)\,u(t;t_0,u_0) + D(t)\,Q_1(t)\,m(u(t;t_0,u_0),t)$$
due to Assumption 2.2 and $R(\cdot)\,u(\cdot;t_0,u_0) \in C^1(I_{\hat{\delta}}, \mathbb{R}^n)$, $Q_1m_u \in C^0$.
3. $x(\cdot;u(\cdot;t_0,u_0))$ solves (2.6)-(2.8) due to the properties of the implicit functions k and m.

Consequently, $x(\cdot;u(\cdot;t_0,u_0))$ is a solution of the given linear implicit DAE on $I_{\hat{\delta}} \subseteq I$ proceeding in the closed ϵ-tube $\mathcal{T}_{\epsilon,C} \subset \bigcup_{i=1,\ldots,s} \mathrm{B}_{\epsilon_i}((x_*(t_i),t_i))$ around the integral curve of x_*. $\square$

Remark 2.8. Representation (2.14) together with already known continuous dependence of the IRODE solution $u(t;t_0,u_0)$ on initial values (t_0,u_0) shows that this property also holds for linear implicit index-2 DAEs under consideration.

The structural assumption (2.13), i.e. $Q_1(t)\,m_w(u,w,t) \equiv 0$ appears quite impractical. If we use a sufficient precondition implying $m_w = 0$ then less smoothness is required, e.g. $M \in C^2$ turns out to be evitable.

Lemma 2.9. *[Alternative structural assumption]*

Given a linear implicit DAE (2.5) *satisfying*

$$\boxed{N_0(t) \cap S_0(x,t) \;\; \textit{independent of } x} \tag{2.15}$$

in a neighbourhood of the integral curve of x_* and an admissible projector $Q_1(x,t)$ on $N_1(x,t)$ along $K(x,t)$ with DN_1, DK constant along x_*. Then the structural Assumption 2.13 holds and $Q_0Q_1D^- \in C^1$ is not necessary any longer in Theorem 2.7. If

$$\left((UQ_0 + P_0Q_1)\,G_2^{-1}b_x^*P_0P_1\right),\; \left((UQ_0 + P_0Q_1)\,G_2^{-1}\right)(t)\,(b(x,t) - b_x^*(t)x) \in C^1$$

then the right hand side of the IRODE (2.12) *is continuous. It is continuously differentiable in the case of $D \in C^1$ and*

$$\left((UQ_0 + P_0Q_1)\,G_2^{-1}b_x^*P_0P_1\right),\; \left((UQ_0 + P_0Q_1)\,G_2^{-1}\right)(t)\,(b(x,t) - b_x^*(t)x) \in C^2.$$

Proof. Denote

$$H(x,t) := (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\,(b_x(x,t) - b_x(x_*(t),t))$$

Now Assumption 2.13 reads

$$Q_1(t)\left(I - H(x,t)\left(UQ_0 + P_0Q_1\right)(t)\right)^{-1} H(x,t)\left(TQ_0\right)(t) = 0$$

for (x,t) in a neighbourhood of the integral curve of x_*. We are going to show that (2.15) is a sufficient condition for $H(x,t)(TQ_0)(t) = 0$.

$(TQ_0)(t)$ projects on a subspace of $N_0(t) \cap S_0(x_*(t),t)$ where $S_0(x,t) = \ker W_0(t) b_x(x,t)$. The identity

$$W_0(t) b_x(x,t)(TQ_0)(t) = 0$$

is valid because $N_0 \cap S_0$ is independent of x nearby the integral curve of x_*. Additionally, $G_0 = G_2 P_1 P_0$, i.e.

$$(UQ_0 + P_0Q_1) G_2^{-1} G_0 = (UQ_0 + P_0Q_1)(I - Q_1) P_0 = 0$$

respectively

$$(UQ_0 + P_0Q_1)(t) G_2^{-1}(t) = (UQ_0 + P_0Q_1)(t) G_2^{-1}(t) W_0(t)$$

We obtain

$$H(x,t)(TQ_0)(t) \;=\; \begin{array}{l}(UQ_0 + P_0Q_1)(t) G_2^{-1}(t) W_0(t) \cdot \\ \cdot \left(b_x(x,t) - b_x(x_*(t),t)\right)(TQ_0)(t)\end{array} = 0$$

in a sufficiently small neighbourhood of the integral curve of x_*. In particular, $m_w = 0$ and $Q_0 Q_1 D^- \frac{\partial}{\partial t}(Dm)(u(t),t)$ independent of w' hold without assuming differentiability neither of $Q_0 Q_1 D^-$ nor of $(Dm)_{tw} = (Dm)_{wt}$.

In principle the decoupling already works if $M \in C^1$ and the implicit function theorem ensures $m \in C^1$. In this case it is possible that the partial derivative $(Dm)_{tw}$ does not exist, so a priori the right hand side of the IRODE is continuous only. Requiring $M \in C^2$, $D \in C^1$ results in $(Dm)_{tw} \in C^0$ and therefore a continuously differentiable right hand side of the inherent regular ODE. □

Above considerations give a simple sufficient condition for the complete decoupling of linear implicit DAEs (2.5). Moreover, the underlying smoothness properties to ensure the feasibility of the decoupling procedure are quite low, that is $A, D \in C^0$, $b \in C^1$ and continuously differentiable projectors evaluated in the reference solution. However, we are in need of a differentiable right hand side of the IRODE for qualitative investigations so $(Dm)_{tw} \in C^0$ or, loosely speaking, C^2-functions of the DAE have to be required. We notice that the simplified structural Assumption 2.15 is very important because it is automatically fulfilled in many systems originating from circuit simulation and Hessenberg-2 DAEs.

Enhancements of the decoupling approach

The preferential analytical goal of decoupling approaches in the context of the tractability index was to derive local existence results for linear implicit DAEs. For example, the existence of a solution is proved via a partial decoupling in [Voi06, Th. 6.7]. That is to say, the inherent dynamics are represented by a fully implicit DAE with differentiation index one in order to avoid the problem of inconsistent initial values in Remark 2.4. Local solvability of IVPs in differential-algebraic equations of tractability index 2 with t-dependent perturbations plus the verification of the perturbation index two are aspired in [MT94] and [Tis96]. In order to do so, the given system is decoupled completely in a neighbourhood of a reference solution. In the final analysis, all mentioned decouplings seem to originate from the same principle with certain adaptations to the desired field of application. One of our intentions was to clarify the essence of this method yielding the structural conditions in Assumption 2.13 in a canonical way. From this point of view, the results from [Tis96, Th. 3.12] are attested to linear implicit systems exhibiting properly stated derivative term as well. Similarly, we show that $\operatorname{im}\begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix}(y,x,t)$ independent of y, x and constancy of the systemic subspaces DK, DN_1 along the extended integral curve of x_* allow to decouple fully implicit DAEs.

Only few applications of mentioned decoupling approaches to the topic of stability are known to us. At this, the generalization of *Perron's Theorem* to index-2 DAEs with constant coefficients of the linearization and a small nonlinearity in [Mä98, Th. 3.3] is definitely the most important stability result. Due to its practicability and simplicity, it is the only criterion which entered recent textbooks on DAEs slightly modified as Th. 3.5 plus Th. 6.5 (2) [Ria08] for MNA equations and as [RR02, Th. 59.2] for linear implicit DAEs $A(x(t))\,x'(t) = G(x(t))$ with a geometric index $\nu \geq 1$. Unfortunately, the topic of asymptotic stability is not explicitly dealt with in [KM06] and monographs focused on numerical methods for DAEs like [HLR89, HW01, ESF98, AP97]. Stability of periodic solutions of (non-autonomous) periodic DAEs can be reduced to above case, cf. the proof of [LMW03, Th. 4.2]. We have observed that sometimes it is cumbersome to handle the inherent dynamics represented by the IRODE on an invariant subspace. Instead, we propose to enhance the method considering a transformation to a state space with minimal dimension. In other words, the DAE is locally equivalent to an ODE defined on a region (called the associated *state space form*) plus a parametrization of the solution set. This beneficial formulation allows to interpret some quantities like characteristic multipliers of DAEs in terms of the inherent dynamics quite easily. As a matter of course we take fully implicit nonlinear differential-algebraic systems into consideration. One of the objectives of this thesis is to enable stability analysis of *self-oscillating systems*, i.e. autonomous DAEs exhibiting periodic solutions. To this purpose we have to prove suitable properties of the decoupling first. At first glance, the IRODE (2.12) appears to be non-autonomous, but we are able to prove an *autonomous* representation of the IRODE on DK leading to an autonomous state space form on $\mathbb{R}^l$, $l = \dim DK$. Later on we show that the characteristic multipliers of the variational system of the SSF coincide with those of the linearization of the given DAE along x_*. This proposition follows from commutativity between transformation to the SSF and linearization as outlined in Diagram (2.21). Asymptotic stability of periodic solutions of periodic DAEs can be traced back to a theorem for ordinary differential equations so an alternative proof for [LMW03, Theorem 4.2] addressing a class of fully implicit DAEs is given. It demonstrates that our approach provides more insight into the dynamics of the DAE and avoids confusing technical estimates. Actually, the latter case is easier to handle because the autonomous structure of the SSF is waived.

2.1.2 Fully implicit DAEs

The transition to fully implicit DAEs (1.2) is unproblematic due to the preparatory work in Lemma 1.18 and Lemma 2.9. The only drawback is that we have to require the additional structural condition (1.16). As already mentioned, sufficient differentiability is required in order to obtain a differentiable state space representation of the inherent dynamics.

Theorem 2.10. *[Decoupling of fully implicit DAEs]*
Let $x_* \in C_D^1(I, \mathbb{R}^m)$ be a solution of the properly stated index-2 DAE

$$f\left((Dx)'(t), x(t), t\right) = 0$$

on a compact interval I and $Q_1(y, x, t)$ be a projector onto $N_1(y, x, t)$ along $K(y, x, t)$ with

$$(DN_1)\left((Dx_*)'(t), x_*(t), t\right) = const. \qquad and \qquad (DK)\left((Dx_*)'(t), x_*(t), t\right) = const. \tag{2.16}$$

Moreover, let

$$\operatorname{im}\begin{pmatrix} T(y,x,t) \\ -\left(f_y^- f_x T\right)(y,x,t) \end{pmatrix} \text{ independent of } y, x \tag{2.17}$$

be valid in a neighbourhood of the extended integral curve of x_*. Additionally, let $D \in C^1(I, \mathbb{R}^{n\times m})$ and the functions

$$\begin{array}{l} (UQ_0 + P_0Q_1)(t)\, G_2^{-1}(t)\, f_x^*(t)\, (P_0P_1)(t)\,, \\ \left(f_y^- f_x T Q_0 + D\right)(t)\, G_2^{-1}(t)\, f_x^*(t)\, (P_0P_1)(t)\,, \\ \left(f_y^- f_x T Q_0 + D\right)(t)\, G_2^{-1}(t) \left(f(y,x,t) - f_x^*(t)\, x - f_y^*(t)\, y\right), \\ (UQ_0 + P_0Q_1)(t)\, G_2^{-1}(t) \left(f(y,x,t) - f_x^*(t)\, x - f_y^*(t)\, y\right) \end{array} \tag{2.18}$$

be twice continuously differentiable. Then, the inherent dynamics of the DAE around x_* for $u(t) = (DP_1)(t)\, x(t)$ are determined by the IRODE

$$\begin{aligned} u'(t) &= -\left(DP_1G_2^{-1}\right)(t)\left(f_x^*(t) D^-(t)\, u(t) - r(t) + \tilde{h}\left(s\left(u(t), t\right), t\right)\right), \qquad (2.19) \\ s(u,t) &:= \left(\tilde{k}_2(u,t) + \tilde{m}_2\left(u, \tilde{k}(u,t), t\right), D^-(t)\, u + \tilde{k}_1(u,t) + \tilde{m}_1\left(u, \tilde{k}(u,t), t\right)\right) \end{aligned}$$

on the constant invariant subspace $(DK)\left((Dx_*)'(t), x_*(t), t\right)$.

Proof. Augment the DAE introducing $y = R(Dx)'$ and consider from now on the related linear implicit system (1.15) with a t-dependent derivative term. According to Lemma 1.18, the augmented system (1.15) inherits the properly stated derivative term and the tractability index two from the original DAE. In addition, the subspaces $\tilde{D}\tilde{N}_1$, $\tilde{D}\tilde{K}$ coincide with DN_1, DK are therefore constant along $\tilde{x}_*(t) :=$

$\left(x_*\left(t\right), R\left(t\right)\left(Dx_*\right)'\left(t\right)\right)$. We have also shown that (2.17) is equivalent to $\tilde{N}_0 \cap \tilde{S}_0$ being t-dependent only. It remains to check sufficient differentiability in order to apply Lemma 2.9 to the augmented DAE (1.15). In doing so we benefit from the explicit construction of the matrix chain belonging to (1.15) in the proof of Lemma 1.18.

Notice that $\tilde{x} = (x, z)^T$, $\tilde{D}\tilde{P}_1 = \begin{pmatrix} DP_1 & 0 \end{pmatrix}$ i.e. $\tilde{u} = u$ and

$$\tilde{D}\tilde{P}_1\tilde{G}_2^{-1} = \begin{pmatrix} DP_1G_2^{-1}f_y & DP_1G_2^{-1} \end{pmatrix}$$

Moreover,

$$\left(\left(\tilde{U}\tilde{Q}_0 + \tilde{P}_0\tilde{Q}_1\right)\tilde{G}_2\right)(t) = \begin{pmatrix} \left(UQ_0 + P_0Q_1\right)G_2^{-1}f_y & \left(UQ_0 + P_0Q_1\right)G_2^{-1} \\ f_y^- f_x TQ_0G_2^{-1}f_y + DG_2^{-1}f_y - I & f_y^- f_x TQ_0G_2^{-1} + DG_2^{-1} \end{pmatrix}(t)$$

and

$$\tilde{b}_x^*\tilde{P}_0\tilde{P}_1 = \begin{pmatrix} 0 & 0 \\ f_x^* P_0P_1 & 0 \end{pmatrix}, \; \tilde{b}\left(\tilde{x}, t\right) - \tilde{b}_x^*(t)\tilde{x} = \begin{pmatrix} 0 \\ f\left(y, x, t\right) - f_x^* x - f_y^* y \end{pmatrix}$$

are valid with $f_y^-\left(y, x, t\right)$ fixed by

$$f_y^-\left(y, x, t\right) f_y\left(y, x, t\right) = R\left(t\right), \quad f_y\left(y, x, t\right) f_y^-\left(y, x, t\right) = I - W_0\left(y, x, t\right)$$

The differentiability requirements of Theorem 2.10 ensure the smoothness of (1.15) necessary to apply Lemma 2.9.

Denote the terms of the modified Taylor expansion (2.3) of the augmented DAE (1.15) by $\tilde{\tilde{h}}$,$\tilde{r}$ and the corresponding terms of the initial system (1.2) as usual by $\tilde{h}$, r. Obviously,

$$\tilde{\tilde{h}}\left(\tilde{x}, t\right) = \begin{pmatrix} 0 \\ \tilde{h}\left(z, x, t\right) \end{pmatrix}, \; \tilde{r}(t) = \begin{pmatrix} 0 \\ r(t) \end{pmatrix}.$$

Furthermore, denote the constraints (2.11) of the augmented index-2 DAE (1.15) by $\tilde{k} = \left(\tilde{k}_1, \tilde{k}_1\right)^T$ and $\tilde{m} = \left(\tilde{m}_1, \tilde{m}_2\right)^T$. Now, the inherent regular ODE of (1.15) possesses the representation

$$\tilde{u}'(t) = \begin{array}{l} \left(\tilde{D}\tilde{P}_1\tilde{G}_2^{-1}\right)(t)\left(\tilde{r}\left(t\right) - \left(\tilde{b}_x^*\tilde{D}^-\right)(t)\tilde{u}(t)\right) \\ -\left(\tilde{D}\tilde{P}_1\tilde{G}_2^{-1}\right)(t)\tilde{\tilde{h}}\left(\tilde{D}^-\left(t\right)\tilde{u}(t) + \tilde{k}\left(\tilde{u}(t), t\right) + \tilde{m}\left(\tilde{u}(t), \tilde{k}\left(\tilde{u}(t), t\right), t\right), t\right) \end{array}$$

Exploiting the mentioned structure of the augmented system[2] and setting

$$s\left(u, t\right) := \left(\tilde{k}_2\left(u, t\right) + \tilde{m}_2\left(u, \tilde{k}\left(u, t\right), t\right), D^-\left(t\right)u + \tilde{k}_1\left(u, t\right) + \tilde{m}_1\left(u, \tilde{k}\left(u, t\right), t\right)\right)$$

we obtain the target representation of the inherent dynamics of the given DAE (1.2) for DP_1x components, namely

$$u'(t) = -\left(DP_1G_2^{-1}\right)(t)\left(f_x^*(t)D^-\left(t\right)u(t) - r(t) + \tilde{h}\left(s\left(u(t), t\right), t\right)\right)$$

□

[2]Pay attention to $\tilde{u} = u$ and $\tilde{x}\left(t\right) = \left(x\left(t\right), \left(Dx\right)'\left(t\right)\right)^T$ in the definition of $s\left(u, t\right)$.

At the first glance, it is questionable whether the structural condition (1.16) could be met at all. To this purpose, consider a linear implicit DAE satisfying Assumption (2.15) with $T = T(t)$ exhibiting the following structure:

$$A(U(t)x(t))(Dx)'(t) + b(U(t)x(t), t) + C(t)T(t)x(t) = 0 \tag{2.20}$$

Then, $f_x(z,x,t)T(t) = C(t)T(t)$ and (2.17) reduces to

$$\operatorname{im}\begin{pmatrix} T(t) \\ -A^-(U(t)x,t)C(t)T(t) \end{pmatrix} \text{ independent of } U(t)x$$

The structure (2.20) with $A = A(t)$ is a well-known case of special MNA equations so a decoupling approach can be found in [Voi06, § 5.1]. In the present thesis $A^-(U(t)x,t)C(t)T(t)$ need not to be independent of $U(t)x$ in order to proceed. For example, this matrix-valued function is x_3-dependent in the self-oscillating autonomous index-2 DAE (5.9), nevertheless (2.17) is satisfied. Besides, the above decoupling is applicable to fully implicit index-2 systems of the type (1.2).

Actually, Condition (2.16) can be weakened, especially for autonomous DAEs.

Lemma 2.11. *Consider the properly formulated index-2 DAE* (1.2) *possessing a solution x_*. If* $\operatorname{im} D(t)$ *and* $\ker f_y(y,x,t)$ *are constant then*

$$(DN_1)\big((Dx_*)'(t), x_*(t), t\big) = \mathit{const.}$$

ensures the existence of a subspace $K(y,x,t) \subseteq \mathbb{R}^m$ complementary to $N_1(y,x,t)$ satisfying

$$D(t)K\big((Dx_*)'(t), x_*(t), t\big) = \mathit{constant}$$

such that the projector $Q_1(y,x,t)$ onto $N_1(y,x,t)$ along K is admissible.

Proof. Choose a constant projector R onto $\operatorname{im} D(t)$ along $\ker f_y(y,x,t)$ and a reflexive generalized inverse $D^-(t)$ such that $D^-(t)D(t) = P_0(t)$ and $D(t)D^-(t) = R$. Therefore, $\operatorname{im} D^-(t) = \operatorname{im} P_0(t)$ is complementary to $N_0(t)$. It is already known that the tractability index 2 implies $N_1(y,x,t) \oplus S_1(y,x,t) = \mathbb{R}^m$, so consider the following subspace

$$K(y,x,t) := Q_0(t)S_1(y,x,t) \oplus D^-(t)(DN_1)^c(y,x,t)$$

where $(DN_1)^c(y,x,t)$ denotes a constant subspace complementary to $(DN_1)(y,x,t)$ such that $(DN_1)^c$ along the extended integral curve of x_* is a constant complementary space to $(DN_1)\big((Dx_*)'(t), x_*(t), t\big)$. Then,

$$\begin{aligned} N_1(y,x,t) + K(y,x,t) &= (P_0N_1 \oplus Q_0N_1 + P_0K \oplus Q_0K)(y,x,t) \\ &= Q_0(t)(N_1 \oplus S_1)(y,x,t) + D^-(t)(DN_1 \oplus (DN_1)^c)(y,x,t) \\ &= \operatorname{im} Q_0(t) + \operatorname{im} D^-(t) = \mathbb{R}^m \end{aligned}$$

and $N_1(y,x,t) \cap K(y,x,t) = \{0\}$ hold, the latter because of

$$\begin{aligned} N_1(y,x,t) \cap Q_0(t)K(y,x,t) &= Q_0(t)N_1(y,x,t) \cap Q_0(t)K(y,x,t) \\ &= Q_0(t)(N_1 \cap S_1)(y,x,t) = \{0\}, \\ N_1(y,x,t) \cap P_0(t)K(y,x,t) &= P_0(t)N_1(y,x,t) \cap P_0(t)K(y,x,t) \\ &= D_0^-(t)(DN_1 \cap (DN_1)^c)(y,x,t) = \{0\}. \end{aligned}$$

Per constructionem

$$D\,(t)\,K\left((Dx_*)'\,(t)\,,x_*\,(t)\,,t\right) = D\,(t)\,D^-\,(t)\,(DN_1)^c = R\,(DN_1)^c$$

is a constant subspace and $N_0\,(t) \subseteq S_1\,(y,x,t)$ implies

$$N_0\,(t) \subseteq Q_0\,(t)\,S_1\,(y,x,t) \subseteq K\,(y,x,t)$$

□

Notice $s_2\,(u,t) := D^-\,(t)\,u + \tilde{k}_1\,(u,t) + \tilde{m}_1\left(u,\tilde{k}\,(u,t)\,,t\right)$ represents a parametrization of the solution set $\mathcal{M}_1\,(t)$ where $u \in (DK)\,(t)$. It can be convenient to parametrize the invariant subspace $(DK)\,(t)$ in order to obtain a parametrization of the configuration space of the given index-2 DAE in minimal coordinates, i.e. depending on $\xi \in \mathbb{R}^l$, $l = \dim DK$. This idea leads to a local state space form of the DAE.

2.2 Commutativity between decoupling and linearization

In the section on functional analytical linearization of differential-algebraic systems the linearization is recognized to be fundamental in order to analyze nonlinear systems, e.g. to define the tractability index. A complete decoupling is another possibility to reveal the the inherent dynamics of a DAE raising the question of compatibility between the two operations to reduce complexity of nonlinear DAEs. This aspect was neglected in publications concerning the tractability index so far.

Lemma 2.12. *Consider an index-2 DAE* (1.2) *with a solution $x_* \in C^1_D(I,\mathbb{R}^m)$ on a compact interval I. If all preconditions of Theorem 2.10 are satisfied then the following diagram commutes.*

$$\begin{tikzcd} DAE \arrow[r, "decoupling"] \arrow[d, "linearization"'] \arrow[dr, phantom, "\circlearrowright"] & IRODE \arrow[d, "linearization"] \\ LinDAE \arrow[r, "decoupling"] & LinIRODE \end{tikzcd} \tag{2.21}$$

i.e. the IRODE of the linearization $f_y^*(t)\,(Dx)'\,(t) + f_x^*(t)x(t) = 0$ of (1.2) *around x_* coincides with the linearization of the IRODE* (2.19) *of* (1.2) *around $(DP_1)\,(t)\,x_*\,(t)$.*

Proof. We obtain a complete decoupling of the given DAE in a neighbourhood of the reference trajectory x_* using Theorem 2.10. The DP_1-components of solutions nearby x_* satisfy the IRODE (2.19)

$$\begin{aligned} u'(t) &= -\left(DP_1G_2^{-1}\right)(t)\left(f_x^*(t)D^-\,(t)\,u(t) - r(t) + \tilde{h}\,(s\,(u(t),t)\,,t)\right), \\ s\,(u,t) &:= \left(\tilde{k}_2\,(u,t) + \tilde{m}_2\left(u,\tilde{k}\,(u,t)\,,t\right), D^-\,(t)\,u + \tilde{k}_1\,(u,t) + \tilde{m}_1\left(u,\tilde{k}\,(u,t)\,,t\right)\right) \end{aligned}$$

Using notation from Lemma 1.18 it follows that $s\left(u_*(t),t\right) = \begin{pmatrix} x_*(t) \\ \left(Dx_*\right)'(t) \end{pmatrix} = \tilde{x}_*(t)$ is valid. The system of variational equations belonging to the IRODE (2.19) around $u_*(t) = \left(DP_1\right)(t)x_*(t)$ is

$$z'(t) \quad = \quad \begin{array}{l} -\left(DP_1G_2^{-1}f_x^*D^-\right)(t)z(t) \\ -\left(DP_1G_2^{-1}\right)(t)\mathrm{D}_{(y,x)}\tilde{h}\left(s\left(u_*(t),t\right),t\right)\mathrm{D}_u s\left(u_*(t),t\right)z(t) \end{array}$$

Recognize that $\mathrm{D}_{(y,x)}\tilde{h}$ vanishes identically on the integral curve of x_* due to the crucial properties $\tilde{h}_y\left(\left(Dx_*\right)'(t),x_*(t),t\right) = 0$, $\tilde{h}_x\left(\left(Dx_*\right)'(t),x_*(t),t\right) = 0$ of the modified Taylor expansion of (1.2). Consequently, this variational system reads

$$z'(t) \quad = \quad -\left(DP_1G_2^{-1}f_x^*D^-\right)(t)z(t)$$

The linearization of the DAE (1.2) is $f_y^*(t)\left(Dx\right)'(t) + f_x^*(t)x(t) = 0$. Applying the induced matrix chain, i.e. the one used for (1.2) evaluated in the extended integral curve of x_*, we arrive at an inherent regular ODE

$$u'(t) = -\left(DP_1G_2^{-1}f_x^*D^-\right)(t)u(t)$$

representing the inherent dynamics of the linearization. Therewith, commutativity of diagram (2.21) is proved in case of fully implicit DAEs. □

Corollary 2.13. *Given a linear implicit DAE $A(t)\left(Dx\right)'(t)+b\left(x(t),t\right) = 0$ satisfying preconditions of Lemma 2.9, linearization along x_* and complete decoupling around x_* commute.*

Proof. Straightforward adaptation of the above argument using Lemma 2.9 and the complete decoupling in Theorem 2.7. □

Remark 2.14. An interesting proposition about the linearization of DAEs is discussed in [Rei95, Th. 2]. There, autonomous DAEs $A\left(x(t)\right)x'(t) + b\left(x(t)\right) = 0$ with geometric index s nearby a stationary point x_0 are considered as vector fields v on the respective constraint manifold $\mathcal{M}_s$. Geometrically, the linearization $f_y\left(x_0\right)x'(t) + b_x\left(x_0\right)x(t) = 0$ in x_0 corresponds to the linearized vector field $\mathrm{D}v\left(x_0\right)$ on $T_{x_0}\mathcal{M}_s$. In this spirit, the geometric transformation of a DAE into a vector field on a constraint manifold commutes with linearization. This fact is used to derive criteria on the spectrum of the linearized vector field $\mathrm{D}v\left(x_0\right)$ by means of the spectrum of the matrix pencil $\left(f_y\left(0,x_0\right),f_x\left(0,x_0\right)\right)$, i.e. in the original problem setting. From this point of view, Lemma 2.12 serves a similar purpose. It traces properties of the linearization of the underlying dynamics back to the corresponding properties of the linearization of the initial DAE. Even more, we are able to deal with arbitrary periodic solutions whereas [Rei95] is confined to the stationary ones.

2.3 State space representation of the IRODE

We are interested in solutions of the DAE on the invariant subspace DK because the dynamics of the DAE refer to $\left(DP_1\right)(t)\,x(t)$. It turns out to be analytically convenient to introduce an additional linear coordinate transformation in order to represent the inherent dynamics as an ODE on a region of $\mathbb{R}^l$, $l = \dim DK$. This representation of the dynamics in minimal coordinates is meant if spoken about the state space form (SSF).

Lemma 2.15. *In case of properly formulated index-2 DAEs* (1.2) *satisfying the structural condition* (2.16) *there exists a continuously differentiable matrix valued function* $M \in C^1(I, \mathbb{R}^{n\times n})$ *such that* $\forall t \in I : M(t) \in GL_n(\mathbb{R})$ *and*

$$\begin{aligned}\left(DP_1D^-\right)(t) &= M(t)\begin{pmatrix} I_l & & \\ & 0_{r-l} & \\ & & 0_{n-r}\end{pmatrix}M^{-1}(t) \\ \left(DQ_1D^-\right)(t) &= M(t)\begin{pmatrix} 0 & & \\ & I_{r-l} & \\ & & 0_{n-r}\end{pmatrix}M^{-1}(t)\end{aligned} \tag{2.22}$$

Proof. These representations of result from suitable bases of systemic subspaces. Because of the tractability index 2, $N_1 \oplus S_1 = \mathbb{R}^m$ holds in addition to the properly stated leading term, i.e. $\operatorname{im} D \oplus \ker A = \mathbb{R}^n$ implying $DS_1 \oplus DN_1 \oplus \ker A = \mathbb{R}^n$ with $r \equiv \operatorname{rk} D(t)$, $\operatorname{cork} A(t) \equiv n - r$. Furthermore, $l \equiv \dim DK$ and $\dim DN_1 \equiv r - l$ are valid because of (2.16). Denote

$\{Ds_1, \ldots, Ds_l\}$	a constant basis of DK
$\{Dn_1, \ldots, Dn_{r-l}\}$	a constant basis of DN_1
$\{\alpha_1(t), \ldots, \alpha_{n-r}(t)\}$	a t-dependent C^1-basis of $\ker A(t)$

Create a matrix valued function

$$M(t) := (Ds_1, \ldots, Ds_l, Dn_1, \ldots, Dn_{r-l}, \alpha_1(t), \ldots, \alpha_{n-r}(t))$$

using these basis vectors of $\mathbb{R}^n$ as columns. The projectors DP_1D^- and DQ_1D^- possess the following representations

$$\left[DP_1D^-\right]^{\{Ds_i,Dn_i,\alpha_i\}}_{\{Ds_i,Dn_i,\alpha_i\}} = \begin{pmatrix} I_l & & \\ & 0_{r-l} & \\ & & 0_{n-r}\end{pmatrix}, \left[DQ_1D^-\right]^{\{Ds_i,Dn_i,\alpha_i\}}_{\{Ds_i,Dn_i,\alpha_i\}} = \begin{pmatrix} 0_l & & \\ & I_{r-l} & \\ & & 0_{n-r}\end{pmatrix}$$

whereas $[L]^{\{a_i\}}_{\{b_i\}}$ denotes the matrix representation of a linear mapping L according to the bases $\{a_i\}$ of input vectors and $\{b_i\}$ of output vectors. Due to the change of the base,

$$\begin{aligned}\left(DP_1D^-\right)(t) &= \left[DP_1D^-\right]^{\{e_i\}}_{\{e_i\}} \\ &= [I]^{\{Ds_i,Dn_i,\alpha_i\}}_{\{e_i\}}\left[DP_1D^-\right]^{\{Ds_i,Dn_i,\alpha_i\}}_{\{Ds_i,Dn_i,\alpha_i\}}[I]^{\{e_i\}}_{\{Ds_i,Dn_i,\alpha_i\}} \\ &= M(t)\begin{pmatrix} I_l & & \\ & 0_{r-l} & \\ & & 0_{n-r}\end{pmatrix}M^{-1}(t)\end{aligned}$$

because $M = [I]^{\{Ds_i,Dn_i,\alpha_i\}}_{\{e_i\}}$ is the matrix representation of the mapping of the basis $\{Ds_i, Dn_i, \alpha_i\}$ onto the canonical basis $\{e_i\}$ of $\mathbb{R}^n$. We obtain the intended representations

$$\left(DP_1D^-\right)(t) = M(t)\begin{pmatrix} I_l & & \\ & 0_{r-l} & \\ & & 0_{n-r}\end{pmatrix} M^{-1}(t)$$

and

$$\left(DQ_1D^-\right)(t) = M(t)\begin{pmatrix} 0_l & & \\ & I_{r-l} & \\ & & 0_{n-r}\end{pmatrix} M^{-1}(t)$$

□

Lemma 2.16. *[State space form of fully implicit index-2 DAEs]*

If the assumptions of Theorem 2.10 are satisfied then the inherent regular ODE of the index-2 DAE $f\left((Dx)'(t), x(t), t\right) = 0$ features a local state space representation $\xi_1'(t) = \tilde{g}\left(\xi_1(t), t\right)$ on an open subset of $\mathbb{R}^l$. Furthermore, Diagram (2.21) is commutative.

Proof. According to Theorem 2.10 there is a representation of the system dynamics as the IRODE (2.19) on DK referring to $u(t) = \left(DP_1D^-\right)(t)x(t)$. Precisely,

$$\begin{aligned} u'(t) &= \left(DP_1D^-\right)(t) g\left(u(t), t\right) \\ g\left(u,t\right) &:= -\left(DP_1G_2^{-1}\right)(t)\left(f_x^*(t)D^-(t)\,u - r(t) + \tilde{h}\left(s\left(u,t\right),t\right)\right) \\ s\left(u,t\right) &:= \left(\tilde{k}_2\left(u,t\right) + \tilde{m}_2\left(u, \tilde{k}\left(u,t\right),t\right), D^-(t)\,u + \tilde{k}_1\left(u,t\right) + \tilde{m}_1\left(u,\tilde{k}\left(u,t\right),t\right)\right) \end{aligned} \tag{2.23}$$

Using the transformation

$$\xi(t) = \left(\xi_1, \xi_2, \xi_3\right)^T(t) := M^{-1}(t)u(t)$$

together with projector representation (2.22) and DK being an invariant subspace of the IRODE, we obtain the identity

$$\begin{aligned} u(t) &= \left(DP_1D^-\right)(t)u(t) = M(t)\begin{pmatrix} I_l & & \\ & 0_{r-l} & \\ & & 0_{n-r}\end{pmatrix} M^{-1}(t)u(t) \\ &= M(t)\begin{pmatrix} I_{l\times l} \\ 0_{(r-l)\times l} \\ 0_{(n-r)\times l}\end{pmatrix}\xi_1(t) \end{aligned}$$

DK is assumed to be constant thus $u'(t) = \left(DP_1D^-\right)(t)u'(t)$ and

$$\begin{aligned} M^{-1}(t)u'(t) &= \begin{pmatrix} I_l & & \\ & 0 & \\ & & 0\end{pmatrix} M^{-1}(t)\left(M(t)\begin{pmatrix} I_l \\ 0 \\ 0\end{pmatrix}\xi_1(t)\right)' \\ &= \begin{pmatrix} I_l & & \\ & 0 & \\ & & 0\end{pmatrix} M^{-1}(t)M'(t)\begin{pmatrix} I_l \\ 0 \\ 0\end{pmatrix}\xi_1(t) + \begin{pmatrix} I_l \\ 0 \\ 0\end{pmatrix}\xi_1'(t) \end{aligned}$$

Multiplication by $M^{-1}(t)$ reveals that the IRODE (2.23) is equivalent to

$$\begin{pmatrix} \xi_1'(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} I_l & & \\ & 0 & \\ & & 0 \end{pmatrix} M^{-1}(t) \left(g\left(M(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1(t), t \right) - M'(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1(t) \right)$$

respectively to this ordinary differential equation for $\xi_1 \in \mathbb{R}^l$:

$$\begin{aligned} \xi_1'(t) &= \tilde{g}(\xi_1(t), t) \qquad (2.24) \\ \tilde{g}(\xi_1, t) &:= \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t) \left(g\left(M(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1(t), t \right) - M'(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1(t) \right) \end{aligned}$$

Summing up, a state space form of the IRODE is constructed. The system of variational equations of the SSF (2.24) around

$$\xi_1^*(t) := \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t) D(t) x_*(t)$$

reads

$$\begin{aligned} z'(t) &= \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t) \left(g_u\left(M(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1^*(t), t \right) M(t) - M'(t) \right) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} z(t) \\ &= -\begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t) \left(\left(DP_1 G_2^{-1} f_x^* D^- \right)(t) M(t) + M'(t) \right) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} z(t) \qquad (2.25) \end{aligned}$$

because

$$M(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1^*(t) = \left(DP_1 D^- \right)(t) u_*(t) = u_*(t)$$

and $g_u(u_*(t), t) = -\left(DP_1 G_2^{-1} f_x^* D^- \right)(t)$ according to Lemma 2.12.

On the other hand we already know the representation

$$u'(t) = -\left(DP_1 G_2^{-1} f_x^* D^- \right)(t) u(t)$$

of the IRODE of the linearization $f_y^*(t)(Dx)'(t) + f_x^*(t)x(t) = 0$ of (1.2) around x_*. The linear transformation of the IRODE on its invariant subspace DK via

$$u(t) = M(t) \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi_1(t)$$

and the equivalent term replacing $M^{-1}(t)u'(t)$ lead to the SSF of the linearization which exhibits the representation (2.25). □

Definition 2.17. The ODE (2.24) on $\mathbb{R}^l$ is called a *state space form* (abbrev. *SSF*) of the differential-algebraic index-2 system (1.2) nearby x_*.

The above representation of the state space form under consideration depends on the choice of DK. However, uniqueness of a SSF is dispensable with regard to aspired stability analysis.

Presented transformation of a fully implicit DAE to its state space representation reveals that the system dynamics are determined by the ODE (2.24) in a neighbourhood of the trajectory of ξ_1^* in $\mathbb{R}^l$. The entire solution vector is obtained evaluating the local parametrization

$$p(\xi_1, t) := s_2\left(M(t)\begin{pmatrix} I_l & 0 & 0 \end{pmatrix}\xi_1, t\right)$$

of the solution manifold $M_1(t)$ nearby $x_*(t)$ with s_2 as defined in Theorem 2.10.

All in all, the transformations can be visualized in Figure 2.1 with

$$p_{\text{lin}}(\xi_1, t) := \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t)\,\xi_1$$

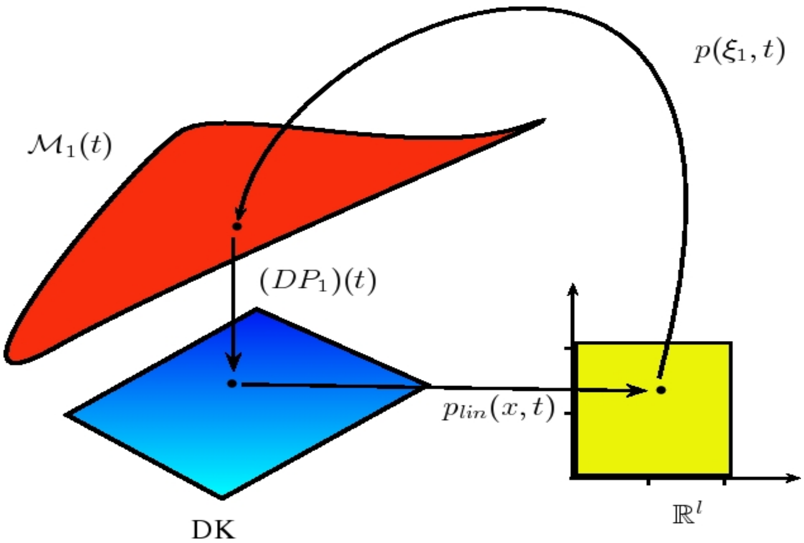

Figure 2.1: construction of the state space representation

State space representation of index-1 DAEs

A state space form of index-1 DAEs on $\mathbb{R}^r$ can be constructed in a way such that the compatibility between transformation to the SSF and linearization is valid as well. Moreover, no structural assumptions except the tractability index one are necessary. It is possible to decouple by splitting a modified Taylor expansion and applying the implicit function theorem twice but the decoupling is clearly arranged if the simple structure of index-1 DAEs is utilized directly, like it is done in [MH04].

Lemma 2.18. *[SSF of index-1 systems]*

If $x_* \in C_D^1(I, \mathbb{R}^m)$ solves the properly formulated DAE (1.2) *exhibiting tractability index one then Diagram* (2.21) *commutes with respect to the index-1 decoupling approach.*

Proof. The given DAE (1.2) is equivalent to

$$f\left(D(t)D^{-}(t)\left(Dx\right)'(t), D^{-}(t)\left(Dx\right)(t) + Q_0(t)x(t), t\right) = 0$$

Denote

$$w(t) := D^{-}(t)(Dx)'(t) + Q_0(t)x(t),\ u(t) := D(t)x(t)$$

then $R(t)\left(Dx\right)'(t) = D(t)w(t)$, $Q_0(t)x(t) = Q_0(t)w(t)$ and (1.2) can be written as

$$f\left(D(t)w(t), D^{-}(t)u(t) + Q_0(t)w(t), t\right) = 0$$

Consider the algebraic constraint

$$F(w,u,t) = f\left(D(t)w, D^{-}(t)u + Q_0(t)w, t\right)$$

The differentiability assumptions of the tractability index one ensure f, f_x and f_y to be continuous. The condition $N_0 \cap S_0 = \{0\}$ is equivalent to

$$F_w = f_y D + f_x Q_0 = G_1\left(D(t)w, D^{-}(t)u + Q_0(t)w, t\right)$$

being invertible on the respective domain. Therefore, $F\left(w,u,t\right) = 0$ is locally solvable for w i.e. there exists a unique implicitly defined function $w = w(u,t)$ satisfying

$$F\left(w\left(u,t\right), u, t\right) = 0, \quad w\left(\left(Dx_*\right)(t), t\right) = D^{-}(t)(Dx_*)'(t) + Q_0(t)x_*(t)$$

Now it becomes clear that D-components of a solution of (1.2) fulfill the ODE

$$R(t)\left(Dx\right)'(t) = D(t)w\left(\left(Dx\right)(t), t\right)$$

Due to $(Dx)' = \left(R\left(Dx\right)\right)' = R'\left(Dx\right) + R\left(Dx\right)'$ the components $u(t) = D(t)x(t)$ solve the associated IRODE

$$u'(t) = -R'(t)u(t) + D(t)w\left(u(t), t\right) \tag{2.26}$$

Multiplying the IRODE by $(I - R\left(t\right))$ it follows that $v\left(t\right) := \left(I - R\left(t\right)\right)u\left(t\right)$ satisfies the ODE $v'\left(t\right) = R'\left(t\right)v\left(t\right)$, i.e im $D\left(t\right)$ is an invariant subspace of the inherent regular ODE (2.26). It holds

$$\begin{aligned} w_u(u,t) &= -F_w^{-1}\left(w(u,t), u, t\right)\right) F_u\left(w(u,t), u, t\right)\right) = \\ &= -G_1^{-1}\left(D(t)w, D^{-}(t)u + Q_0(t)w, t\right) f_x\left(D(t)w, D^{-}(t)u + Q_0(t)w, t\right) D^{-}(t) \end{aligned}$$

The linearization of (2.26) around $u_*(t) = \left(Dx_*\right)(t)$ reads

$$\begin{aligned} z'(t) &= R'(t)z(t) + D(t)w_u\left(u_*(t), t\right)z(t) \\ &= R'(t)z(t) - G_1^{-1}\left(\left(Dx_*\right)(t), x_*(t), t\right) f_x^*(t)D^{-}(t)z(t) \end{aligned}$$

because the implicit function ensures

$$\left(D(t)w\left(u_*(t), t\right), D^{-}(t)u_*(t) + Q_0(t)w\left(u_*(t), t\right), t\right) = \left(\left(Dx_*\right)'(t), x_*(t), t\right)$$

Decoupling the linearization $f_y^*(t)(Dx)'(t) + f_x^*(t)x(t) = 0$ of (1.2) around x_* results in the linear IRODE

$$u'(t) = R'(t)u(t) - D(t)\,G_1^{-1}(t)f_x^*(t)D^-(t)u(t)$$

Choosing the induced matrix chain for the linearization,

$$G_1^{-1}(t) = G_1^{-1}\left((Dx_*)(t), x_*(t), t\right)$$

and the IRODE of the linearization of (1.2) around x_* corresponds to the linearization of the IRODE around Dx_*.

Now we have to go one step further and consider a state space representation of the IRODE on its invariant subspace im $D(t)$. The properly formulated derivative term ensures the existence of a C^1-basis $\{\alpha_1(t), \ldots, \alpha_{n-r}(t)\}$ of $\ker f_y(y, x, t)$ and a C^1-basis $\{d_1(t), \ldots, d_r(t)\}$ of im $D(t)$ with $r \equiv \operatorname{rk} D(t)$. Using the matrix-valued function

$$V(t) := (d_1(t), \ldots, d_r(t), \alpha_1(t), \ldots, \alpha_{n-r}(t))$$

we obtain the representation $R(t) = V(t)\begin{pmatrix} I_r & \\ & 0_{n-r}\end{pmatrix}V^{-1}(t)$ similarly to (2.22). The invariant subspace im $D(t)$ suggests to set

$$\xi_1(t) := \begin{pmatrix} I_r & 0_{r\times n-r}\end{pmatrix}V^{-1}(t)\,u(t)$$

because of

$$\begin{aligned} u(t) &= R(t)\,u(t) = V(t)\begin{pmatrix} I_r & \\ & 0_{n-r}\end{pmatrix}V^{-1}(t) \\ &= V(t)\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\left(\begin{pmatrix} I_r & 0_{r\times n-r}\end{pmatrix}V^{-1}u\right)(t) \end{aligned}$$

The relevant components $\xi_1(t)$ are uniquely determined by the following ODE

$$\xi_1'(t) = \begin{array}{l} \begin{pmatrix} I_r & 0_{r\times n-r}\end{pmatrix}V^{-1}(t)\,D(t)\,w\left(V(t)\begin{pmatrix} I_r \\ 0\end{pmatrix}\xi_1(t), t\right) \\ -\begin{pmatrix} I_r & 0_{r\times n-r}\end{pmatrix}V^{-1}(t)\left(R'(t)\,V(t) + V'(t)\right)\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\xi_1(t) \end{array} \tag{2.27}$$

defined on an open set of $\mathbb{R}^r$ which is called the state space form of the original index-1 DAE. Linearization of (2.27) around $\xi_1^*(t) := \begin{pmatrix} I_r & 0_{r\times n-r}\end{pmatrix}V^{-1}(t)\,D(t)\,x_*(t)$ coincides with the SSF derived likewise from the linearization of (1.2) around x_*. □

2.3.1 Autonomous state space form

Solutions of autonomous DAEs are invariant under translations i.e. if $x \in C_D^1$ solves

$$f\left((Dx(t))', x(t)\right) = 0 \tag{2.28}$$

on the interval $I \subseteq \mathbb{R}$ then $\forall c \in \mathbb{R}: \; x(t+c)$ is also a solution of the given system on $I-c$. An intuitive conjecture is that autonomous DAEs could be governed by autonomous inherent dynamics. In fact, this is true for general index-1 systems due to $w = w(u)$ in Lemma 2.18. Now we are going to prove the conjecture in case of periodic solutions x_* of autonomous linear implicit resp. fully implicit DAEs (2.28) exhibiting the tractability index 2.

Lemma 2.19. *Let x_* be a periodic solution of the autonomous index-2 DAE* (2.28) *satisfying the preconditions of Theorem 2.10. Then the IRODE* (2.19) *is autonomous on the invariant subspace DK as well.*

Proof. We are able to chose constant projectors R and Q_0 due to the autonomous structure of the given DAE. Consequently, the reflexive pseudoinverse D^- is constant, too. Here, $\ker f_y(y,x) = \text{const.}$ and Assumption (2.16) are fulfilled, so it is possible to choose constant basis functions in Lemma 2.15 resulting in a constant representation (2.22) of DP_1D^-, DQ_1D^-, P_0P_1 and P_0Q_1. Due to the simplifying assumption (2.15) we choose a constant projector T onto $N_0 \cap S_0$ locally around the extended trajectory of x_*.

Using these constant projectors we obtain the representation (2.19) of the inherent regular ODE, i.e.

$$\begin{aligned} u'(t) &= DP_1 g\left(u(t),t\right), \qquad (2.29)\\ g\left(u,t\right) &:= -G_2^{-1}(t)\left(f_x^*(t)D^-u - r(t) + \tilde{h}\left(s\left(u,t\right),t\right)\right),\\ s\left(u,t\right) &:= \left(\tilde{k}_2\left(u,t\right) + \tilde{m}_2\left(u,\tilde{k}\left(u,t\right),t\right), D^-\left(t\right)u + \tilde{k}_1\left(u,t\right) + \tilde{m}_1\left(u,\tilde{k}\left(u,t\right),t\right)\right) \end{aligned}$$

As a result of the differentiability requirements (2.18) in Theorem 2.10, the right hand side of the IRODE (2.29) fulfills $g, g_u \in C^0$. Consider the continuation of the periodic solution x_* on entire $\mathbb{R}$, hence $g(u,t)$ is defined for all $t \in \mathbb{R}$ and is periodic in t.

Let us suppose the existence of $u_0 \in DK$ and $t_0, t_1 \in \mathbb{R}$ with $DP_1 g(u_0,t_0) \neq DP_1 g(u_0,t_1)$. The theorem of Picard-Lindelöf ensures the existence of a unique solution

$$u \in C^1\left(\left(t_0-\epsilon, t_0+\epsilon\right), \mathbb{R}^m\right)$$

of the IVP $(DP_1)(x(t_0) - D^-u_0) = 0$ of the IRODE (2.29). We are able to construct a unique solution $x \in C_D^1\left(\left(t_0-\epsilon,t_0+\epsilon\right),\mathbb{R}^m\right)$ of the IVP $(DP_1)(x(t_0) - D^-u_0) = 0$ for the original DAE (2.28) like in Theorem 2.10. Thereby the last-mentioned initial value problem denotes a consistent initial value $x(t_0)$ with fixed DP_1-components. The solution components $u(t) = DP_1x(t)$ satisfy the IVP $u(t_0) = u_0$ of the IRODE $u'(t) = DP_1 g(u(t),t)$ on the invariant subspace DK.

Set $c := t_1 - t_0$. The invariance of solutions of the given autonomous DAE (2.28) under translations signifies that

$$\tilde{x}(t) = x\left(t-c\right)$$

is also a solution of (2.28) on $(t_1-\epsilon, t_1+\epsilon)$ having the consistent initial value $\tilde{x}(t_1) = x(t_0)$ with $(DP_1)(\tilde{x}(t_1) - D^-u_0) = 0$. Now $\tilde{x}(t)$ belongs to a neighbourhood of the extended trajectory of x_* where the decoupling procedure applies, at least for values of t nearby t_1. Consequently, $\tilde{u}(t) = DP_1\tilde{x}(t) = u\,(t)$ solves the inherent regular ODE $u'(t) = DP_1 g\,(u(t), t)$ on DK due to Theorem 2.10. Per constructionem,

$$u(t_0) = u_0 = \tilde{u}(t_1) \text{ and } u'(t_0) = \tilde{u}'(t_1)$$

is valid. This contradicts to the assumption

$$\tilde{u}'(t_1) = DP_1 g\,(\tilde{u}(t_1), t_1) = DP_1 g\,(u_0, t_1) \neq DP_1 g\,(u_0, t_0) = u'(t_0)$$

Therefore, $DP_1 g_t\,(u, t) \equiv 0$ with $u \in DK$, $t \in \mathbb{R}$ is proved, i.e. the existence of the autonomous representation

$$u'(t) = DP_1 g\,(u(t), 0)$$

of the IRODE of the given DAE (2.28) on DK. □

Remark 2.20. Lemma 2.19 fails if the state space DK of the above representation of the inherent dynamics is not constant. Then,

$$\tilde{u}_0 = (DP_1)\,(t_1)\tilde{x}(t_1) = DP_1(t_1)x(t_0) \neq DP_1(t_0)x(t_0) = u_0$$

would be valid in general because $\tilde{u}_0$ and u_0 belong to different subspaces $DK(t_1)$ resp. $DK(t_0)$ of $\mathbb{R}^n$. We emphasize that the autonomous representation of the IRODE is proved on $DK \times \mathbb{R}$ only. Outside of this region, the inherent regular ODE (2.29) is a mere formal construct and it is questionable whether there is any link to the given DAE. Consequently, we cannot state any property of the IRODE in terms of the given DAE except on $DK \times \mathbb{R}$.

Remark 2.21. Observe that in case of τ-periodic solutions of autonomous or τ-periodic DAEs, the constraints $s\,(u, t)$ in (2.29) are also τ-periodic with respect to t. For that purpose the initial system is transformed into (1.15) thereby the augmented DAE remains autonomous resp. τ-periodic. Due to τ-periodicity of the projectors in use, $\tilde{M}\,(u, w, y, t + \tau) = \tilde{M}\,(u, w, y, t)$ is valid for $\tilde{M}\,(u, w, y, t)$. Hence,

$$\tilde{M}\,(u, w, \tilde{m}\,(u, w, t)\,, t) = 0 = \tilde{M}\,(u, w, \tilde{m}\,(u, w, t)\,, t + \tau)$$

on the domain of the corresponding resolution function $\tilde{m}$. Due to local uniqueness, $\tilde{m}\,(u, w, t + \tau) = \tilde{m}\,(u, w, t)$. The identity $\tilde{k}\,(u, t + \tau) = \tilde{k}\,(u, t)$ is proved analogously.

Summarizing, the decoupling approach in the context of tractability index has the serious drawback that autonomous structure is destroyed in an artificial way by the modified Taylor expansion (2.3). A priori, a non-autonomous representation (2.29) of the IRODE on DK is obtained. This disadvantage can be partly avoided by Lemma 2.19 i.e. an autonomous representation of the SSF is revealed. The developed parametrization of the constant configuration set $\mathcal{M}_1$ is still t-dependent but τ-periodic and therefore does not disturb the aspired stability analysis.[3]

The matrix M in the representation (2.24) of the state space form of the DAE can be assumed constant resulting in

[3] This is not perfectly correct. If we had an autonomous parametrization of $\mathcal{M}_1$ in addition to the autonomous SSF then asymptotic D- or DP_1-component stability (defined in § 4.2) of a periodic solution would imply that the solution is stationary. At the moment, this situation is known to be true only for the asymptotic stability concept of R. März.

Theorem 2.22. *The autonomous index-2 DAE* $f\left((Dx(t))', x(t)\right) = 0$ *exhibits an autonomous state space form*

$$\xi_1'(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1} g \left(M \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \xi_1(t), 0 \right) \tag{2.30}$$

on $\mathbb{R}^l$ *under the assumptions of Lemma 2.19. Moreover, linearization and the transformation into* (2.30) *locally around* x_* *commute.*

Proof. Constant systemic subspaces enable the choice $M(t) \equiv M$ in Lemma 2.15. Therefore R, DP_1 and DQ_1 are assumed to be constant. Lemma 2.19 guarantees the autonomous representation $u'(t) = DP_1 g(u(t), 0)$ of the IRODE $f\left((Dx(t))', x(t)\right) = 0$. Thus, (2.24) implies (2.30) to be a possible autonomous representation of the SSF of the given differential-algebraic system. Diagram 2.21 is commutable due to Lemma 2.16. □

3 Index reduction via differentiation

Up to now, a representation of the inherent dynamics around a reference solution x_* for fully implicit DAEs $f\left((Dx)'(t), x(t), t\right) = 0$ of tractability index one and two is proved. The mentioned representation in minimal coordinates is called the state space form of the DAE. In doing so, the structural assumptions

- $N_0(t) \cap S_0(y,x,t) = (N_0 \cap S_0)(t)$ respectively $\operatorname{im} \begin{pmatrix} T \\ -f_y^- f_x T \end{pmatrix}(y,x,t)$ independent of y, x in a neighbourhood of the extended integral curve of x_*
- subspaces DN_1 and DK constant along x_*

are required. The configuration space $\mathcal{M}_1(t)$ of the given DAE nearby x_* exhibits the local parametrization

$$s\left(M(t)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\xi, t\right), \quad \xi \in \mathbb{R}^l$$

as defined in Lemma 2.16. Consequently, $\mathcal{M}_1(t)$ turns out to be a C^1-manifold locally around the trajectory of x_*. In contrast to index-1 DAEs where the constraints $\mathcal{M}_0(t)$ are purely algebraic, the above parametrization of $\mathcal{M}_1(t)$ features partial differentials of some equations of f in case of the tractability index 2.

The index reduction via differentiation is another important analytic tool in context of differential-algebraic systems of higher index $k \geq 2$. The index reduction is based on replacement of constraints by the respective differentiated form whose representation corresponds to formal differentiation of constraints along C_D^1-functions under certain structural assumptions. For example, given a solution $(x_1, x_2)^T$ of a Hessenberg DAE (1.6), the algebraic constraint $0 = g(x_1(t), t)$ is replaced by

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(x_1(t), t) = g_{x_1}(x_1(t), t)\, x_1'(t) = g_{x_1}(x_1(t), t)\, f(x_1(t), x_2(t), t)$$

The resulting semi-explicit system

$$\begin{aligned} x_1'(t) &= f(x_1(t), x_2(t), t) \\ 0 &= g_{x_1}(x_1(t), t)\, f(x_1(t), x_2(t), t) \end{aligned}$$

has the tractability index 1 because the algebraic constraint is uniquely solvable for x_2 due to regularity of $g_{x_1} f_{x_2}$. The main problem dealing with fully implicit DAEs is to identify the right constraints for differentiation. Iterative formal differentiation of the entire system $f\left((Dx)'(t), x(t), t\right) = 0$ with respect to t yields a derivative array. The approach can be modified to eliminate the second order derivatives in every differentiation step resulting in a condensed derivative array, cf. [Rei95]. For the present work it is important to ensure that the tractability index is decremented by differentiation of the constraints, otherwise we could not use the index reduction for a possible definition of a Lyapunov function for differential-algebraic systems with tractability index 2 later on. This objective is achieved by determining the constraints to differentiate using a suitable projector from the matrix chain under certain structural assumptions.

The index reduction via differentiation has a geometric interpretation in case of autonomous DAEs: algebraic constraints $g(x) = 0$ define a constraint manifold $\mathcal{M}$. Formal differentiation of these equations implies $\mathrm{D}g(x(t))\,x'(t) = 0$, i.e. the condition $x'(t) \in T_{x(t)}\mathcal{M}$. If the occurring derivative is replaced by algebraic relations like it is done for Hessenberg DAEs then the heuristics behind the differentiation step is the transition to the tangent bundle of the constraint manifold advancing iteratively towards the configuration space of a regular DAE in the sense of § 1.2.1. Our interpretation of the index reduction is more pragmatic: the differentiation of constraints is considered as a means of constructing a manifold enclosing the configuration space, i.e. $\tilde{\mathcal{M}}_0(t) \supseteq \mathcal{M}_1(t)$. For this purpose it is fundamental that $\tilde{\mathcal{M}}_0(t)$ represents the obvious constraint of an index-1 DAE, the so called *index reduced DAE* (abbr. IR-DAE) and that $\mathcal{M}_1(t)$ is an invariant set with respect to solutions of the index reduced system. Per constructionem a parametrization of $\tilde{\mathcal{M}}_0(t)$ therewith of $\mathcal{M}_1(t) = \tilde{\mathcal{M}}_0(t) \cap \mathcal{M}_1(t)$ is obtained with $\dim \tilde{\mathcal{M}}_0(t) > \dim \mathcal{M}_1(t)$ degrees of freedom. That is the reason why our definition of Lyapunov functions or some already known contractivity definitions for index-2 DAEs ensure stability of the superset Dx of the dynamic solution components DP_1x.

3.1 Extraction of the constraints for differentiation

Consider fully implicit DAEs (1.2), that is

$$f\left((Dx)'(t), x(t), t\right) = 0$$

with a properly formulated derivative term and tractability index 2. The above DAE is equivalent to the splitted system

$$\begin{cases} \left(I - W_1\left((Dx)'(t), x(t), t\right)\right) f\left((Dx)'(t), x(t), t\right) & = & 0 \\ W_1\left((Dx)'(t), x(t), t\right) f\left((Dx)'(t), x(t), t\right) & = & 0 \end{cases}$$

The essential structural condition of this chapter is that the equations $W_1(y, x, t)\, f(y, x, t)$ of the differential-algebraic system are independent of y and of $Q_0(t)\,x$.

Assumption 3.1. $(W_1 f)(y, x, t) = (W_1 f)(P_0(t)x, t)$ *is valid on the domain* $\mathcal{G}$ *of the given DAE.*

In addition, assuming $D^- \in C^1(I, \mathbb{R}^{m\times n})$ and sufficiently smooth functions W_1 and f, the relation

$$W_1(P_0(t)\,x(t), t)\, f\left((Dx)'(t), x(t), t\right) = \text{const.}$$

for C_D^1-functions x can be replaced with

$$W_1\left(P_0\left(t\right)x(t),t\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(W_1 f\right)\left(P_0(t)x(t),t\right)=0$$

which is equivalent to the following formulation using formal differentiation of constraints:

$$\begin{array}{l} W_1\left(P_0(t)x(t),t\right)\left(W_1 f\right)_x\left(P_0(t)x(t),t\right)\left(\left(D^-\right)'(t)\left(Dx\right)(t)+D^-(t)\left(Dx\right)'(t)\right)\\ +W_1\left(P_0(t)x(t),t\right)\left(W_1 f\right)_t\left(P_0(t)x(t),t\right)=0. \end{array} \tag{3.1}$$

All in all, the above approach results in the following differential-algebraic system:

$$\begin{array}{rcl} \left(I-W_1\left(\left(Dx\right)'(t),x(t),t\right)\right)f\left(\left(Dx\right)'(t),x(t),t\right) & = & 0\\ W_1\left(\left(Dx\right)'(t),x(t),t\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(W_1 f\right)\left(P_0(t)x(t),t\right) & = & 0 \end{array}$$

which abbreviates

$$\begin{array}{l} \left(I-W_1\left(\left(Dx\right)'(t),x(t),t\right)\right)f\left(\left(Dx\right)'(t),x(t),t\right)\\ +W_1\left(\left(Dx\right)'(t),x(t),t\right)\Bigg(\left(W_1 f\right)_t\left(P_0(t)x(t),t\right)\\ +\left(W_1 f\right)_x\left(P_0(t)x(t),t\right)\Big(\left(D^-\right)'(t)\left(Dx\right)(t)+D^-(t)\left(Dx\right)'(t)\Big)\Bigg)=0 \end{array} \tag{3.2}$$

due to the chain rule. Obviously, if the initial values x_0 satisfy $\left(W_1 f\right)\left(P_0\left(t\right)x_0,t\right)=0$ then solutions of (3.2) fulfill the original DAE (1.2). In the following, the properly stated leading terms and the tractability index one of (3.2) are proved in order to justify the notion of an index reduced DAE.

We suppress the evident arguments of involved functions for the sake of a better readability. The indices i,j,k,l represent the coordinates of matrix entries but the indices x,v etc. denote partial derivatives with respect to the given variable.

Notation. We agreed to denote the partial derivative of a vector-valued function $h\left(x_1,x_2\right)$ with respect to x_i at $\left(\xi_1,\xi_2\right)^{\mathrm{T}}$ by $h_{x_i}\left(\xi_1,\xi_2\right)$. This term represents a linear map whose argument z is usually written omitting the brackets because the evaluation $h_{x_i}\left(\xi_1,\xi_2\right)(z)=h_{x_i}\left(\xi_1,\xi_2\right)z$ is a matrix multiplication. Another notation to be used is $h_{x_i}\left(\xi_1,\xi_2\right)z=\left[h\left(\xi_1,\xi_2\right)\right]_{x_i}(z)$. Dealing with *matrix-valued* functions $M\left(x\right)$, the derivative with respect to x at x_* evaluated in (z,v) is denoted as the bilinear form $\left[M\left(x_*\right)\right]_x(z,v)$.

We are in need of the following auxiliary result:

Lemma 3.2. *Consider an open set $\tilde{\mathcal{G}}\subseteq\mathbb{R}^m$ and a matrix valued function $M\in C^1(\tilde{\mathcal{G}},\mathbb{R}^{k\times s})$ together with $b\in C^1(\tilde{\mathcal{G}},\mathbb{R}^s)$. The Jacobian $\left[M\left(x_*\right)b\left(x_*\right)\right]_x$ has the following representation*

$$\left[M\left(x_*\right)b\left(x_*\right)\right]_x z=M\left(x_*\right)b_x\left(x_*\right)z+\left[M\left(x_*\right)\right]_x\left(z,b\left(x_*\right)\right)$$

with $z \in \mathbb{R}^m$ and the bilinear form $[M(x_*)]_x$ defined by

$$[M(x_*)]_x(z,v) := \sum_{j=1}^{m} z_j \left[\frac{\partial}{\partial x_j} M(x_*)\right] v \tag{3.3}$$

Moreover, the i-th component of $[M(x_*)]_x$ features the representation

$$([M(x_*)]_x(z,v))_i = \sum_{l=1}^{s} v_l \left(\mathrm{D}M_{il}(x_*)\, z\right) = \sum_{l=1}^{s} v_l \frac{\partial}{\partial z} M_{il}(x_*) \tag{3.4}$$

Proof. see Lemma 8.3 in Appendix. □

There exists a simple sufficient condition in terms of the matrix chain implying Assumption 3.1.

Lemma 3.3. *Consider a properly formulated DAE* (1.2) *exhibiting the tractability index 2. Let $W_1 f \in C^1(\mathcal{G}, \mathbb{R}^m)$ and the domain $\mathcal{G}$ be convex with respect to x, i.e. the line segment between $(y,x,t), (y,\tilde{x},t) \in \mathcal{G}$ also belongs to $\mathcal{G}$. Then, the structural assumption*

$$\text{im } G_1(y,x,t) \quad \textit{dependent of} \quad (P_0(t)x,t) \;\; \textit{only} \tag{3.5}$$

implies $W_1 f = (W_1 f)(P_0(t)x,t)$ on entire $\mathcal{G}$.

Proof. Choose a projector $W_1 = W_1(P_0(t)x,t)$ along im $G_1(y,x,t)$ due to (3.5). Then,

$$(W_1 f)_y = W_1 f_y R = W_1 (f_y D) D^- = W_1 G_1 P_0 D^- = 0$$

By definition of the properly formulated DAE, $\mathcal{G}$ is convex with respect to y and $W_1 f$ is assumed to be sufficiently smooth to apply the mean value theorem (e.g. [For05, p. 70]). It follows that $(W_1 f)(y,x,t) = (W_1 f)(x,t)$.

Similarly, due to convexity of the domain of f w.r.t. x and $W_1 f \in C^1$ we are able to apply the mean value theorem to $W_1 f$ resulting in

$$(W_1 f)(x,t) - (W_1 f)(P_0(t)x,t) = \int_0^1 (W_1 f)_x (P_0(t)x + sQ_0(t)x, t)\, Q_0(t) x \mathrm{d}s$$

According to Lemma 3.2,

$$(W_1 f)_x (y,x,t)\, Q_0(t)x = \begin{array}{l} W_1(P_0(t)x,t)\, f_x(y,x,t)\, Q_0(t)x \\ +[W_1(P_0(t)x,t)]_x (Q_0(t)x, f(y,x,t)). \end{array}$$

The right hand side is zero because of (8.2) and $W_1 = W_1(P_0(t)x,t)$. Precisely, $z = Q_0(t)z$ implies

$$\begin{aligned} ((W_1(P_0(t)x,t))_x(z,v))_i &= \sum_{l=1}^{m} v_l \frac{\partial}{\partial z} (W_1)_{il}(P_0(t)x,t) \\ &= \sum_{l=1}^{m} v_l \frac{\partial}{\partial z} (W_1)_{il}(P_0(t)x,t)\, Q_0(t) = 0 \end{aligned}$$

for all components i. □

3.2 Properties of the index reduced system

Denote the above differential-algebraic system (3.2) resulting from differentiation of constraints by

$$\begin{aligned}&\tilde f\left((Dx)'(t), x(t), t\right) = 0,\\ &\tilde f(y,x,t) := (I - W_1)f + W_1(W_1f)_x D^- y + W_1(W_1f)_x(D^-)'Dx + W_1(W_1f)_t\end{aligned}$$

Observe that $W_1 = W_1(P_0(t)x,t)$ and $(W_1f)(P_0(t)x,t)$ imply $(W_1f)_y = W_1f_y$ and

$$\tilde f_y = [(I-W_1)f]_y + W_1(W_1f)_x D^- = f_y + W_1(W_1f)_x D^-$$

W_1 is a projector, so we get

$$\ker \tilde f_y = \ker f_y \cap \ker W_1(W_1f)_x D^- = \ker f_y$$

due to the property $\ker D^-(t) = \ker R(t) = \ker f_y(y,x,t)$ of the reflexive generalized inverse D^-. Consequently, the leading derivative of (3.2) is properly stated and the projector $R(t)$ realizes the decomposition $\ker \tilde f_y(y,x,t) \oplus \mathrm{im}\ D(t) = \mathbb{R}^n$.

Theorem 3.4. *[Index reduction via differentiation]*
Let the DAE (1.2) *be defined on a region $\mathcal{G}$ which is convex with respect to x. Assume $W_1 \in C^1$ and $W_1f \in C^2$. Furthermore, let* $\ker D(t) = \text{const.}$, *the precondition* (3.5) *and*

$$\mathrm{rk}\ f_y(y,x,t)\,D(t)\,,\ \mathrm{rk}\ W_1(P_0x,t)\,(W_1f)_x\,(P_0x,t)\ \textit{ locally constant} \tag{3.6}$$

be valid. Then the associated differential-algebraic system (3.2) *possesses the tractability index one in a neighbourhood of each $(y,x,t) \in \mathcal{G}$, $f(y,x,t) = 0$ where the given DAE* (1.2) *has index two.*

Proof. Lemma 3.3 ensures the Assumption 3.1 so we are able to replace

$$W_1(P_0(t)x,t)\,f(y,x,t) = 0$$

in the original DAE by its differentiated form (3.1) in order to obtain the associated DAE (3.2). We have already proved that the leading derivative of (3.2) is properly stated. Mark the elements of the matrix chain of (3.2) by tilde, i.e $\tilde G_i$ and so on. As the leading terms of both (1.2) and (3.2) are formed with the same matrix D and the index definition does not depend on the admissible matrix chain, we are able to chose $Q_0 = \tilde Q_0$.

W_1 projects along $\mathrm{im}\ G_1 \supseteq \mathrm{im}\ f_y$ so $\mathrm{im}\ f_y \cap \mathrm{im}\ W_1 = \{0\}$ and it is sufficient to demand constant rank of both addends in $\tilde f_y D$ to obtain the same property for $\tilde G_0$. Therefore the rank condition(3.6) has a locally constant $\mathrm{rk}\ \tilde G_0(y,x,t)$ as a consequence. Note that $\mathrm{rk}\ f_yD$ is already locally constant if the given DAE 1.2 exhibits the index 2 locally around the point $(y,x,t) \in \mathcal{G}$ under consideration.

It suffices to prove

$$\forall\,(y,x,t)\ \text{ with } f(y,x,t) = 0,\ (N_1 \cap S_1)(y,x,t) = \{0\}:\ \tilde G_1(y,x,t)\,z = 0 \Rightarrow z = 0$$

because this implies $\det \tilde{G}_1(y,x,t) \neq 0$ hence $\det \tilde{G}_1 \neq 0$ locally around (y,x,t). Per definitionem, $\tilde{G}_1 z = \tilde{f}_y Dz + \tilde{f}_x Q_0 z$ with $\tilde{f}_y Dz = f_y Dz + W_1(W_1 f)_x z$ and

$$\tilde{f}_x Q_0 z = \begin{array}{l} f_x Q_0 z - (W_1 f)_x Q_0 z + \left[W_1 (W_1 f)_x D^- y\right]_x Q_0 z \\ + \left[W_1 (W_1 f)_x (D^-)' Dx\right]_x Q_0 z + \left[W_1 (W_1 f)_t\right]_x Q_0 z \end{array}$$

It holds

1. $(W_1 f)_x Q_0(t) = 0$ due to $W_1 f = (W_1 f)(P_0(t)x,t)$
2. $$\begin{array}{l} \left[W_1 (P_0(t)x,t)(W_1 f)_x (P_0(t)x,t) D^-(t) y\right]_x Q_0(t) z = \\ = \left[W_1 (W_1 f)_x (P_0(t)x,t)\right]_x (Q_0(t)z, D^-(t)y) = 0 \end{array}$$
   as a result of (8.3) applied to the i-th component
   $$\left(\left[W_1(W_1 f)_x\right]_x \left(Q_0(t)z, D^-(t)y\right)\right)_i = \sum_l \left(D^-(t)y\right)_l \frac{\partial}{\partial z}\left(W_1 (W_1 f)_x\right)_{il} = 0$$
   for $(W_1 f)_x (y,x,t) = (W_1 f)(P_0(t)x,t)$ and $z = Q_0(t)z$.
3. $$\begin{array}{l} \left[W_1 (W_1 f)_x (D^-)' Dx\right]_x Q_0(t) z = \\ \left[W_1(W_1 f)_x\right]_x \left(Q_0(t)z, (D^-)'(t) D(t) x\right) + W_1 (W_1 f)_x (D^-)' D Q_0 z = 0 \end{array}$$
   again due to $W_1(W_1 f)_x$ being independent of $Q_0(t)x$.
4. $$\begin{array}{rcl} \left[W_1(W_1 f)_t\right]_x Q_0 z & = & \begin{array}{l}\left[W_1 (P_0(t)x,t)\right]_x (Q_0(t)z, (W_1 f)_t (P_0(t)x,t)) \\ + W_1 (W_1 f)_{tx} Q_0 z\end{array} \\ & = & W_1 (P_0(t)x,t) \left[(W_1 f)(P_0(t)x,t)\right]_{tx} Q_0(t) z \end{array}$$

Combining these results,

$$\begin{array}{rcl} \tilde{G}_1(y,x,t)z & = & \begin{array}{l} f_y(y,x,t)D(t)z + W_1(P_0(t)x,t)(W_1 f)_x(P_0(t)x,t)z \\ + f_x(y,x,t)Q_0(t)z + W_1 (P_0(t)x,t) \left[(W_1 f)(P_0(t)x,t)\right]_{tx} Q_0(t) z \end{array} \\ & = & \begin{array}{l} G_1(y,x,t)z + W_1(P_0(t)x,t)(W_1 f)_x(P_0(t)x,t)z \\ + W_1 (P_0(t)x,t) \left[(W_1 f)(P_0(t)x,t)\right]_{tx} Q_0(t) z \end{array} \end{array}$$

is obtained. Now $\tilde{G}_1(y,x,t)z = 0$ is equivalent to

$$\begin{array}{rcl} G_1(y,x,t)z & = & 0 \\ W_1(P_0(t)x,t)\Big((W_1 f)_x (P_0(t)x,t) z + \left[(W_1 f)(P_0(t)x,t)\right]_{tx} Q_0(t) z\Big) & = & 0 \end{array} \tag{3.7}$$

because W_1 is a projector. The first equation reveals that $z \in N_1(y,x,t)$, in other words $z = Q_1(y,x,t)z$ is a necessary condition for $\tilde{G}(y,x,t)z = 0$. The canonical projector Q_1 implies

$$z = Q_1(y,x,t)z = Q_1(y,x,t)G_2^{-1}(y,x,t)f_x(y,x,t)P_0(t)z$$

Along with $Q_1 G_2^{-1} G_1 = Q_1 G_2^{-1} G_2 P_1 = 0$ we get

$$z = Q_1\left(y,x,t\right) G_2^{-1}\left(y,x,t\right) W_1\left(y,x,t\right) f_x\left(y,x,t\right) P_0(t)z$$

Finally,

$$\begin{aligned}
&W_1\left(P_0\left(t\right)x,t\right)\left(W_1 f\right)_x\left(P_0\left(t\right)x,t\right)P_0\left(t\right)z = \\
&= W_1\left(P_0\left(t\right)x,t\right)\Big(\left[W_1\left(P_0\left(t\right)x,t\right)\right]_x\left(P_0\left(t\right)z,\underbrace{f\left(y,x,t\right)}_{=0}\right) + f_x\left(y,x,t\right)P_0\left(t\right)z\Big)
\end{aligned}$$

and

$$\begin{aligned}
z &= Q_1(y,x,t)G_2^{-1}\left(y,x,t\right)W_1(P_0(t)x,t)f_x(y,x,t)P_0(t)z \\
&= Q_1(y,x,t)G_2^{-1}\left(y,x,t\right)W_1(P_0(t)x,t)f_x\left(y,x,t\right)z
\end{aligned}$$

The second equation of (3.7) results in

$$W_1\left(P_0\left(t\right)x,t\right)\left(W_1 f\right)_x\left(P_0\left(t\right)x,t\right)z = -W_1\left(P_0\left(t\right)x,t\right)\left[\left(W_1 f\right)\left(P_0\left(t\right)x,t\right)\right]_{tx} Q_0\left(t\right)z$$

Due to symmetry of second derivatives,

$$\begin{aligned}
W_1\left[\left(W_1 f\right)\left(P_0\left(t\right)x,t\right)\right]_{tx} Q_0(t)z &= W_1\left[\left(W_1 f\right)\left(P_0\left(t\right)x,t\right)\right]_{xt} Q_0(t)z \\
&= W_1\left((W_1 f)_x Q_0(t)\right)_t z - W_1(W_1 f)_x Q_0'(t)z \\
&= -W_1(W_1 f)_x Q_0'(t)z
\end{aligned}$$

If $N_0(t) = \mathrm{im}\ Q_0(t)$ is constant then $Q_0'(t) = Q_0(t)Q_0'(t)$ and the above term vanishes. Then,

$$\tilde{G}_1(y,x,t)z = 0 \Rightarrow z = -Q_1 G_2^{-1} W_1 (W_1 f)_{tx} Q_0 z = Q_1 G_2^{-1} W_1\left((W_1 f)_x Q_0\right)_t Q_0' z = 0$$

The matrix-valued function $\tilde{G}_1$ is continuous and therefore nonsingular in a sufficiently small neighbourhood of (y,x,t). □

The above approach decreases the tractability index via differentiation of the constraints $\left(W_1 f\right)\left(P_0 x,t\right) = 0$. In case of Hessenberg-2 systems (1.6), this procedure requires $f \in C^1$, $g \in C^2$ and rk g_1 locally constant.

Definition 3.5. Under the assumptions of Theorem 3.4, the differential-algebraic system (3.2) is called the *index reduced DAE (IR-DAE)* associated to (1.2).

The index reduced system is a powerful tool to analyze index-2 DAEs. An essential question for the original index-2 DAE is unique solvability of initial value problems which is linked to a parametrization of the solution set $\mathcal{M}_1(t)$.

3.2.1 Description of the solution set $\mathcal{M}_1(t)$

We confine ourselves to fully implicit differential-algebraic systems (1.2) with index 2 satisfying the requirements of Theorem 3.4. Solutions $x_* \in C_D^1$ of (1.2) comply with the constraint $(W_1 f)(P_0(t)x(t), t) \equiv 0$ and its corresponding differentiated form

$$0 = W_1 \frac{\mathrm{d}}{\mathrm{d}t}(W_1 f)(P_0(t)x(t), t) = W_1(W_1 f)_x D^-(Dx)' + W_1(W_1 f)_x(D^-)'Dx + W_1(W_1 f)_t$$

For convenience, choose a constant projector Q_0. As a result of the projector W_1, the IR-DAE (3.2) is equivalent to

$$(I - W_1)(P_0 x(t), t)(P_0 x(t), t) f\left((Dx)'(t), x(t), t\right) = 0 \quad (3.8)$$

$$\begin{aligned}&(W_1 (W_1 f)_x)(P_0 x(t), t)\left(D^-(t)(Dx)'(t) + (D^-)'(t)(Dx)(t)\right)\\ &+ (W_1 (W_1 f)_t)(P_0 x(t), t)\end{aligned} = 0 \quad (3.9)$$

In particular, solutions of the original DAE solve the IR-DAE so

$$\mathcal{M}_1(t) \subseteq \mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t)$$

whereas $\mathcal{M}_0(t)$ and $\tilde{\mathcal{M}}_0(t)$ denote the first-level constraints of the given DAE (1.2) or its index reduced system $\tilde{f}((Dx)'(t), x(t), t) = 0$, respectively.

Theorem 3.6. *Let the index-2 DAE* (1.2) *comply with requirements of Theorem 3.4 and let* (3.6) *be valid on entire domain $\mathcal{G}$. Then the configuration space of the original DAE has the representation*

$$\begin{aligned}\mathcal{M}_1(t) &= \mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t)\\ &= \begin{array}{l}\{x \in \mathbb{R}^m \mid \exists y \in \mathbb{R}^n : \ f(y, x, t) = 0\} \cap\\ \{x \in \mathbb{R}^m \mid \exists y \in \mathbb{R}^n : \ W_1((W_1 f)_x D^- y + (W_1 f)_t + (W_1 f)_x (D^-)'Dx) = 0\}\end{array}\end{aligned}$$

and $\mathcal{M}_1(t)$ is covered by solutions of (1.2), *e.g.*

$$\forall t,\ x_0 \in \mathcal{M}_1(t)\ \exists!\ \textit{solution}\ x \in C_D^1(I, \mathbb{R}^m) : \ x(t) = x_0$$

Proof. It remains to prove $\mathcal{M}_1(t) \supseteq \mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t)$.

Due to the tractability index one of the IR-DAE, the index-1 decoupling from Lemma 2.18 ensures the unique solvability of initial value problems $x(t_0) \in \tilde{\mathcal{M}}_0(t_0)$ of this system. Thus, a unique solution of (3.2) passes through each $x_0 \in \tilde{\mathcal{M}}_0(t)$. In addition, $\mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t)$ is an invariant set of the index reduced system and every solution of the IR-DAE on $\mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t)$ complies with the original DAE (1.2) providing $\mathcal{M}_0(t) \cap \tilde{\mathcal{M}}_0(t) \subseteq \mathcal{M}_1(t)$.

To that purpose let $x \in C_D^1(I, \mathbb{R}^m)$ be a solution of the IR-DAE. Observe that $s(t) := (W_1 f)((P_0 x)(t), t)$ satisfies

$$\begin{aligned}s'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(W_1(P_0 x(t), t)\, s(t))\\ &= = (W_1)_t (P_0 x(t), t)\, s(t) + W_1(P_0 x(t), t)\, s'(t)\\ &= (W_1)_t (P_0(t), t)\, s(t)\end{aligned}$$

because (3.9) ensures $W_1((P_0x)(t),t)s'(t) = 0$. If $x(t_0) \in \tilde{\mathcal{M}}_0(t_0) \cap \mathcal{M}_0(t)$ then

$$0 = (W_1 f)((P_0x)(t_0), t_0) = s(t_0)$$

and the above homogeneous ODE results in $s\,(t) \equiv 0$, that is (3.9) is satisfied. Moreover, x solves (3.8) and therefore is a solution of the original DAE yielding $x(t) \in \mathcal{M}_1(t)$ for all $t \geq t_0$. □

The equations representing the configuration space $\mathcal{M}_1(t)$ of linear implicit DAEs $A(x(t),t)(D(t)x(t))' + b(x(t),t) = 0$ can be formulated quite easily under the assumptions of Theorem 3.6. The obvious constraints are

$$\{x \in \mathbb{R}^m \mid W_0\,(P_0x,t)\,b\,(x,t) = 0\}$$

but they are insufficient to determine the configuration space of higher index DAEs. Further constraints, the so called *hidden constraints* occur. They are given by taking the W_1-components of the differentiation of

$$0 = (W_1 f)\,(P_0x,t) = (W_1 b)\,(P_0x,t)$$

along $x \in C^1_D$, i.e.

$$W_1\,(P_0x,t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,(W_1b)\,(P_0x,t) = W_1\,(W_1b)_x\left(D^-\,(Dx)' + \left(D^-\right)'Dx\right) + W_1(W_1b)_t = 0$$

Fix the reflexive pseudoinverse A^- of A by $AA^- = W_0$ and $A^-A = R$ in order to extract

$$(D(t)x(t))' = -A^-(x(t),t)b(x(t),t)$$

from the DAE. Hence, the following representation of the configuration space is obtained:

$$\mathcal{M}_1(t) = \left\{ x \in \mathbb{R}^m \;\middle|\; \begin{array}{l} (W_0b)\,(P_0x,t) = 0, \\ (W_0\,(W_1b)_t)\,(P_0x,t) + (W_1(W_1b)_x)\,(P_0x,t)\,(D^-)'\,(t)\,D\,(t)\,x \\ -\,(W_1(W_1b)_x)\,(P_0x,t)\,D^-(t)A^-\,(x,t)\,b\,(x,t) \end{array} = 0 \right\} \tag{3.10}$$

3.2.2 Locally constant tractability index 2

The tractability index one of a differential-algebraic system is locally constant if the rank of $G_0\,(y,x,t)$ exhibits this property. In this case, continuity of G_1 and $\det G_1\,(y_0,x_0,t_0) \neq 0$ imply regularity of G_1 in a neighbourhood of the point under consideration. The reason is the following representation

$$\det G_1\,(y,x,t) = \sum_{\pi \in S_n} sign\,(\pi)\,(G_1)_{1,\pi(1)}\,(y,x,t) \cdots (G_1)_{m,\pi(m)}\,(y,x,t)$$

(cf. [KM03, p. 114]) of $\det G_1\,(y,x,t)$ as a continuous function. Then, of course, Definition 1.8 of the tractability index one is met in a certain neighbourhood of $(y_0,x_0,t_0) \in \tilde{\mathcal{G}}$. This property is not true for general systems exhibiting the tractability index 2, as indicated in [Mä95]. One has to demand further structural assumptions in order to ensure the index 2 in a neighbourhood of $(y_0,x_0,t_0) \in \mathcal{G}$ (cf. Lemma 4.1, ibid.) and the unique solvability of the given system.

Lemma 3.7. *Consider a properly formulated DAE 1.2 featuring*

$$\dim\left(N_0 \cap S_0(y,x,t)\right) \text{ locally constant}$$

Additionally, let the preconditions of Theorem 3.4 apply to the system. If the DAE (1.2) *has index two at* $(y_0, x_0, t_0) \in \tilde{\mathcal{G}}$ *with* $f(y_0, x_0, t_0) = 0$ *then* $\exists \epsilon > 0$ *such that this property is valid in*

$$\mathcal{U} = \mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \{(y,x,t) \in \mathcal{G} \,|\, f\left(y,x,t\right) = 0\}$$

Proof. Denote the matrix chain of (1.2) as usual and that of the IR-DAE (3.2) by $\tilde{P}_i$, $\tilde{G}_i$. Without loss of generality, use $\tilde{Q}_0 = Q_0$. The requirements of Theorem 3.4 are satisfied so the IR-DAE (3.2) has the tractability index 1 in (y_0, x_0, t_0), i.e. $\tilde{G}_1(y_0, x_0, t_0)$ is invertible. All participating matrix-valued functions are at least continuous, therefore $\tilde{G}_1$ is invertible on $\mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \mathcal{G}$ for a sufficiently small $\epsilon > 0$. Adapt ϵ in a way to ensure

$$\forall\,(y,x,t) \in \mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \mathcal{G}:\; \dim N_0 \cap S_0(y,x,t) \equiv r_0$$

Then, we need to prove

$$\forall (y,x,t) \in \{(y,x,t) \in \mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \mathcal{G} \mid f\left(y,x,t\right) = 0\}:\; (N_1 \cap S_1)(y,x,t) = \{0\}$$

Per definitionem, $S_1(y,x,t) = \ker W_1(x,t) f_x(y,x,t) P_0$. Due to bilinearity of $(W_1)_x$ and $f\left(y,x,t\right) = 0$ we get

$$W_1 (W_1 f)_x z = W_1 (W_1 f)_x P_0 z = W_1 f_x P_0 z + W_1 (W_1)_x (P_0 z, f(y,x,t)) = W_1 f_x P_0 z$$

Assumption 3.1 results in $0 = \left(W_1 f_x Q_0\right)_t = \left(W_1 f\right)_{xt} Q_0$, hence

$$\forall z \in S_1\left(y,x,t\right):\; 0 = W_1 f_x P_0 z = W_1 (W_1 f)_x z = W_1 (W_1 f)_x z + W_1 (W_1 f)_{tx} Q_0 z$$

For $z \in (N_1 \cap S_1)(y,x,t)$ this condition is valid together with $z = Q_1\left(y,x,t\right) z$, which is proved to be equivalent to $\tilde{G}_1(y,x,t) z = 0$ in Theorem 3.4. Now the set

$$\mathcal{U} = \{(y,x,t) \in \mathcal{G} \cap \mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \;|\; f\left(y,x,t\right) = 0\}$$

is chosen in a way to guarantee the regularity of $\tilde{G}_1(y,x,t)$. That is why $\tilde{G}_1(y,x,t) z = 0$ implies $z = 0$ and consequently

$$\forall\,(y,x,t) \in \mathcal{U}:\; \left(N_1 \cap S_1\right)\left(y,x,t\right) = \{0\}$$

The tractability index 2 at the point (y_0, x_0, t_0) implies $r_0 = \dim N_0 \cap S_0(y_0, x_0, t_0) > 0$. Assumption (3.6) causes

$$\dim N_0 \cap S_0(y,x,t) \equiv r_0 > 0 \text{ on } \mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \{(y,x,t) \in \mathcal{G} \,|\, f\left(y,x,t\right) = 0\}$$

Combined with $(N_1 \cap S_1)(y,x,t) = \{0\}$, we obtain the tractability index 2 on entire $\mathrm{B}_\epsilon\left(y_0, x_0, t_0\right) \cap \{(y,x,t) \in \mathcal{G} \,|\, f\left(y,x,t\right) = 0\}$. □

Further appearance of the index reduction in the context of the tractability index

The index reduction via differentiation of certain constraints is a standard technique to analyze differential-algebraic systems leading to derivative array approaches and the primarily tool with regard to systems of Hessenberg type. In context of the tractability index, a series of publications on consistent initialization of DAEs - for example [ES99], [ESL01], [ES01], [BTS10] - are using this method. All but the last mentioned consider DAEs in the standard formulation (1.1).

For linear implicit DAEs (1.5) having a properly stated derivative term, Lemma 3.3 is already proved in [Mä01] and Representation (3.10) of the configuration space is given with reservations. R. März declares in [Mä01, p. 17]:

> "It can be supposed that there are no further hidden constraints for $\mu = 2$ and that $\mathcal{M}_1(t) = \mathcal{M}_0(t) \cap \mathcal{H}_1(t)$, $t \in \mathcal{I}$ represents the sharp geometrical location of the solution of the DAE (6.1) that is of index $\mu = 2$.".

This conjecture possibly refers to the index reduction described for a special type of properly formulated linear implicit DAEs in [San00, § 3.3].

Enhancement and application of the index reduction in this thesis

In our opinion, Theorem 3.3.6 in [San00] is stated imprecisely because the tractability index 1 has to be given on an open subset instead of solitary points (y_0, x_0, t_0) in order to achieve local solvability of the DAE. To this end a constant dimension of $N_0\,(t) \cap \tilde{S}_0\,(y, x, t)$ is required and this property is not ensured until the rank assumption (3.6) is given. Moreover, Lemma 3.2 is favourable to proceed with the computations.

The proof of local constancy of the tractability index via the associated index reduced system is an interesting new quality of the index reduction procedure aiming at fully implicit DAEs with a properly stated leading derivative. Besides, we noticed that the index reduction is linked to some known regularization techniques, cf § 7.2.

The most important aspect of index reduction in connection with the present thesis is certainly the innovative use of the methodology in the context of dissipation inequalities, contractivity and Lyapunov functions for index-2 DAEs in Part II.

Part II

Stability criteria for differential-algebraic systems

4 Stability definitions for DAEs

Usually, the term *stability* of a reference solution $x_* \in C^1(I, \mathbb{R}^m)$, $I = [t_s, \infty)$ of an ODE characterizes the property that slightly perturbed IVPs of that ODE are uniquely solvable on I and the effect of the perturbation in the initial values is bounded. Precisely, the classic definition of stability in the sense of Lyapunov (cf. [LWY07, pp. 21 ff.]) reads

$$\forall \epsilon > 0, t_0 \in I \exists \delta > 0 : \|x_0 - x_*(t_0)\| < \delta \Rightarrow \begin{array}{l} \exists x(t; t_0, x_0) \text{ solution on } [t_0, \infty) : \\ \forall t \geq t_0 : \|x(t; t_0, x_0) - x_*(t)\| < \epsilon \end{array}$$

At this, $x(t; t_0, x_0)$ denotes the solution of the initial value problem $x(t_0; t_0, x_0) = x_0$ of the ODE under consideration. If solutions of slightly perturbed IVPs are stable and converge toward the stable reference solution, i.e.

$$\exists_{\delta_0 > 0} \forall x_0 \in B_{\delta_0}(x_*(t_0)) : \lim_{t \to \infty} \|x(t; t_0, x_0) - x_*(t)\| = 0$$

then x_* is called *asymptotically stable* in the sense of Lyapunov. Finally, the reference solution is called *unstable* in case of not being stable.

We are talking about *partial stability* or *stability with respect to the partial variable* y if $x = (y, z)^T$ and

$$\forall \epsilon > 0, t_0 \in I \exists \delta > 0 : \|x_0 - x_*(t_0)\| < \delta \Rightarrow \begin{array}{l} \exists x(t; t_0, x_0) \text{ solution on } [t_0, \infty) : \\ \forall t \geq t_0 : \|y(t; t_0, x_0) - y_*(t)\| < \epsilon \end{array}$$

that is a small deviation in all components of the initial values is required in order to get an estimation of certain y-coordinates of the solution vector of an ODE. Generalizations of asymptotic stability or instability with respect to the partial variable y are straightforward.

Remark 4.1. The stability definitions do not depend on $t_0 \geq t_s$ because specifications of initial values at time t_s correspond to perturbed initial values at time t_0. The qualitative behaviour does not change because solutions of ODEs are known to depend continuously on initial values. This continuity with respect to consistent initial values is true for index-2 DAEs as well, assumed the preconditions of Theorem 2.19 hold. That is why the simplification $t_s = t_0$ is tacitly assumed in the following.

Peculiarities of differential-algebraic systems

For differential-algebraic systems one should distinguish between two notions of stability. For that purpose, we quote [TA02, p. 3550]:

> "Finally, it is to be cautioned that the issue of stability of the solution trajectory about an equilibrium point or trajectory on manifold S, which is indicative of the stability of the underlying physical system, is totally distinct from the issue of stability of the DAE representation about its solution manifold. While all higher index DAEs are unstable about S, the unperturbed solution trajectory starting on S can end up at an equilibrium point or equilibrium trajectory of the system on S."

This statement is understood in context of certain nonlinear semi-explicit DAEs with control terms. Applying a suitable coordinate transformation results in an inherent ODE for such systems. In [TA02, Proposition 4] it is shown that solutions of that inherent ODE starting arbitrarily close to the solution manifold S of the DAE could move infinitely away from S in case of differentiation index $\nu \geq 2$. This result is one reason to consider stability of DAE solutions relating to *consistent* perturbations of initial values only. Another reason is explained in Remark 2.20 - we are not able to access the qualitative behaviour of the associated IRODE outside of its invariant subspace DK in terms of the given differential-algebraic system. The last and most obvious reason is that a differential-algebraic system is simply not solvable starting with inconsistent perturbations of the given initial values.

4.1 M-component stability

Let $x_* \in C_D^1([t_0,\infty),\mathbb{R}^m)$ be the reference solution of a given differential-algebraic system (1.2) exhibiting properly stated derivative term and the tractability index $k = 1, 2$. We are going to define a variation of an established stability definition in the sense of Lyapunov for these DAEs.

Definition 4.2. Let $M \in C([t_0,\infty),\mathbb{R}^{s\times m})$. The solution $x_* : [t_0,\infty) \to \mathbb{R}^m$ of a properly formulated DAE (1.2) having the index $k = 1, 2$ is called

- *M-component stable* (in the sense of Lyapunov) if

$$\begin{aligned}&\forall\epsilon > 0\exists\delta > 0\forall x_0 \in \mathcal{M}_{k-1}(t_0),\ \|M(t_0)(x_0 - x_*(t_0))\| < \delta :\\ &\exists x(t;t_0,x_0)\ \text{ solution on } [t_0,\infty)\,\forall t \geq t_0 :\ \|M(t)(x(t;t_0,x_0) - x_*(t))\| < \epsilon\end{aligned}$$

- *asymptotically M-component stable* if x_* is M-component stable and there exists a $\delta_0 > 0$ such that for all $x_0 \in \mathcal{M}_{k-1}(t_0)$

$$\|M(t_0)(x_0 - x_*(t_0))\| < \delta_0 \implies \lim_{t\to\infty}\|M(t)(x(t;t_0,x_0) - x_*(t))\| = 0$$

We use the matrix valued function $M(t)$ to specify solution components whose stability has to be analyzed. In doing so we have to ensure that dynamical components[1] of the solution vector are taken into consideration. Solution components P_0x, Dx or DP_1x are

[1] Dynamical components are solution components admitting to set initial values without restriction. In general, there are different feasible choices of such components.

used predominably in scientific publications applying the tractability index. For example, the notion of contractivity of nonlinear index-1 DAEs in [MHT03a] relates to Dx and the contractivity of linear index-2 DAEs in [MHT03b] to DP_1-components, respectively. Moreover there exists a notion of P-contractivity for nonlinear index-2 systems (cf. [San00]) which implies an exponential estimate for $\|P_0(t)(x(t;x_0,t_0) - x_*(t))\|$.

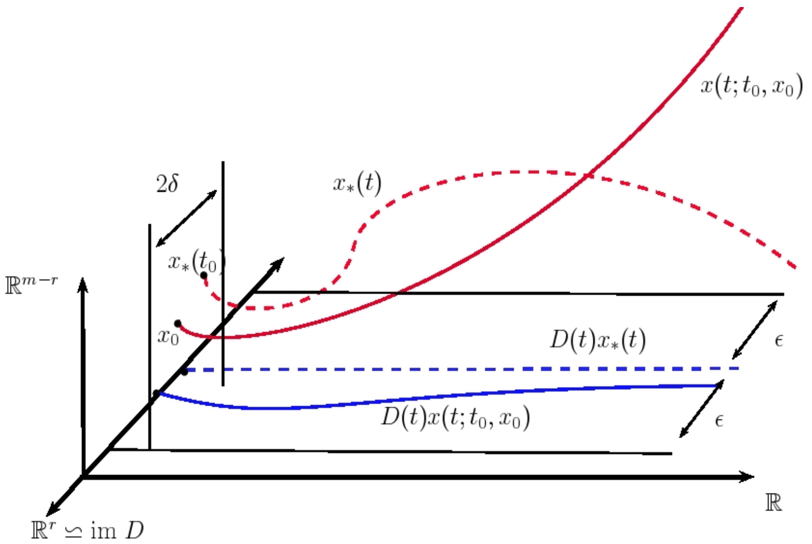

Figure 4.1: D-component stability

So what is the nature of M-component stability? Considering $M(t) \equiv I$ and the condition $\|\Pi_k(x_0 - x_*(t_0))\| < \delta$ we obtain the Lyapunov stability in compliance with [GM86, p. 74] resp. [Tis94, p. 147] which represents a generalization of the classic definition for differential-algebraic equations on their own. Let us formulate their definition for DAEs having a properly stated derivative term:

Definition 4.3. Let (1.2) be properly formulated and index-k ($k = 1, 2$) tractable. A solution $x_* \in C_D^1([t_0,\infty), \mathbb{R}^m)$ is called *stable in Lyapunov's sense* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that all IVPs[2] $\Pi_k x(t_0) = \Pi_k x_0$ where $\|\Pi_k(x(t_0) - x_0)\| \leq \delta$ have a unique C_D^1-solution which are defined on $[t_0, \infty)$ and

$$\|\Pi_k(x(t_0) - x_*(t_0))\| \leq \delta \Longrightarrow \forall t \geq t_0 : \ \|x(t) - x_*(t)\| < \epsilon$$

Thereby $\Pi_k \in \mathbb{R}^{m\times m}$ denotes a projector extracting the dynamical solution components of the index-k DAE at time t_0. We have already proved that $\Pi_1 = P_0(t_0)$ and $\Pi_2 = (P_0P_1)((Dx_*)'(t_0), x_*(t_0), t_0)$ are an admissible choice under certain structural assumptions.

There are two main differences between Definition 4.3 of Lyapunov stability and the one applied in case of ordinary differential equations. As already explained, in the context of an index-k DAE it is a good idea to assume the perturbation x_0 of the initial

[2]Here, $x_0 \in \mathbb{R}^m$ and consistent initial values $x(t_0) \in \mathcal{M}_{k-1}(t_0)$ with prescribed components $\Pi_k x(t_0) = \Pi_k x_0$ are considered.

values $x_*(t_0) \in \mathcal{M}_{k-1}(t_0)$ to be consistent, i.e. $x_0 \in \mathcal{M}_{k-1}(t_0)$. The second difference is to require only the estimation $\|\Pi_k(x_0 - x_*(t_0))\| < \delta$ in order to ensure a bounded propagation of the error, i.e. $\forall t \geq t_0 : \|x_*(t) - x(t; t_0, x_0)\| < \epsilon$. This is a consequence of the DAE being a description of the inherent dynamics in redundant coordinates. In other words, if there exists a decoupling of the given system then $x_*(t_0)$ and $x(t_0)$ can be expressed as evaluation of at least continuous functions of the dynamical components $\Pi_1 x_0$ resp $\Pi_1 x_*(t_0)$ so $\|\Pi_k(x_0 - x_*(t_0))\| < \delta$ implies $\|x_0 - x_*(t_0)\| < c\delta$ for a $c > 0$. From this point of view, the *M-component stability* is the (classical) DAE-stability in the sense of E. Griepentrog, R. März and C. Tischendorf with respect to the $M(t)$-components of the DAE solution where the condition on the initial values $\|\Pi_k(x_0 - x_*(t_0))\| < \delta$ is stated in a redundant way[3] via $\|M(t_0)(x_0 - x_*(t_0))\| < \delta$. In Chapter 6 the reader is going to recognize that this simple concept of a partial stability definition for DAEs is suitable for several nice stability criteria.

4.2 Orbital stability

The notion of Lyapunov stability is inappropriate for periodic solutions of autonomous differential equations. Periodic solutions x_* of autonomous systems exhibiting asymptotic stability in the sense of Lyapunov as specified in [Tis94, p. 147] turn out to be stationary. Because of translation invariance we may choose initial values of the solution $\tilde{x}(t) = x_*(t + c)$ with $0 < c \ll 1$ arbitrarily close to $x_*(t_0)$. Then $\lim_{t\to\infty} \|\tilde{x}(t) - x_*(t)\| = 0$ implies $x_*(t)$ to be constant due to its periodicity. This problem was already noticed in the beginning of the qualitative theory of ODEs, it can be avoided abolishing the parametrization of the reference solution in t. Instead, the orbits, i.e. solution trajectories in the phase space are considered together with their respective distances to the reference trajectory C. Let us formulate this having the peculiarities of differential-algebraic systems in mind:

Definition 4.4. The solution $x_* \in C^1_D([t_0, \infty), \mathbb{R}^m)$ of a properly formulated DAE (1.2) with tractability index $k = 1, 2$ is called

- *orbitally stable*, if

$$\begin{aligned}&\forall \epsilon > 0 \exists \delta > 0 \forall x_0 \in \mathcal{M}_{k-1}(t_0),\ \operatorname{dist}(x_0, C) < \delta :\\ &\exists \text{ solution } x(t; t_0, x_0) \text{ of the IVP on } [t_0, \infty)\, \forall t \geq t_0 : \operatorname{dist}(x(t; t_0, x_0), C) < \epsilon\end{aligned}$$

  where C denotes the trajectory of x_* in $\mathbb{R}^m$ and dist $(x, C) = \inf_{y \in C} \|x - y\|$.

- *orbitally asymptotically stable*, if additionally

$$\exists_{\delta_0 > 0} \forall x_0 \in \mathcal{M}_{k-1}(t_0),\ \operatorname{dist}(x_0, C) < \delta_0 : \lim_{t\to\infty} \operatorname{dist}(x(t; t_0, x_0), C) = 0$$

In [Leo07] the terminology *Poincaré-stability* or *stability in the sense of Poincaré* is used as a synonym for orbital stability.

[3] this is true if im $M(t)$ encompasses the dynamical components of the DAE solution

Remark. In publications dealing with explicit ordinary differential equations the stability notion of Poincaré applies to autonomous systems only. We decided to state the definition of orbital stability for solutions of (possibly) non-autonomous DAEs.

Following [Far94, Def. 5.1.3] resp. [Dem67, p. 306] we define

Definition 4.5. The solution $x_* \in C^1_D([t_0,\infty),\mathbb{R}^m)$ of a properly formulated index-k ($k = 1, 2$) DAE (1.2) is called orbitally asymptotically stable *with asymptotic phase property* if x_* is orbitally asymptotically stable and

$$\forall x_0 \in \mathcal{M}_{k-1}(t_0),\ \operatorname{dist}(x_0, C) < \delta\, \exists \alpha_0 \in \mathbb{R} : \lim_{t\to\infty} \|x(t+\alpha_0; t_0, x_0) - x_*(t)\| = 0$$

Remark 4.6. This restriction on orbital asymptotic stability does not coincide with asymptotic stability in the sense of Zhukovsky (cf. Definition 4.7) because in general $\tau(t) := t + \alpha_0 \notin \operatorname{Hom}(I)$.

It is astonishing that orbital stability takes only an insignificant part in the theory of differential-algebraic systems, especially with regard to the Andronov-Witt Theorem and its generalizations by Demidovich and Leonov. So far, the only criterion for orbital stability of nonlinear index-2 DAEs known to us is an Andronov-Witt-like theorem for Hessenberg systems in [Fra98], which does not apply to *fully implicit* autonomous index-2 DAEs (2.28).

4.3 Excursus: Stability in the sense of Zhukovsky

Another way to allow asymptotically stable periodic solutions of autonomous differential equations is to extend the concept of Lyapunov stability by the means of certain reparametrizations. According to [Leo07], the notion of stability *in the sense of Zhukovsky* was published in [Zhu82] prior to Lyapunov's famous thesis [Lja66]. In today's terminology it corresponds to Lyapunov stability after reparametrization of perturbed solutions. We can also interpret Zhukovsky stability as orbital stability exhibiting a certain kind of asymptotic phase property.

An adaptation of this concept to differential-algebraic equations is as follows:

Definition 4.7. The solution $x_* : [t_0,\infty) \to \mathbb{R}^m$ of a properly stated DAE (1.2) with index $k = 1, 2$ is called

· *stable in the sense of Zhukovsky*, if

$$\begin{aligned}&\forall \epsilon > 0 \exists \delta > 0 \forall x_0 \in \mathcal{M}_{k-1}(t_0),\ \|\Pi_k(x_0 - x_*(t_0))\| < \delta\\ &\exists x(t; t_0, x_0) \text{ solution on } [t_0,\infty),\ \tau \in \operatorname{Hom}(I)\, \forall t \geq t_0 :\\ &\|x(\tau(t); t_0, x_0) - x_*(t)\| < \epsilon\end{aligned}$$

with

$$\operatorname{Hom}(I) := \left\{ \tau \in C([t_0,\infty),\mathbb{R}) \text{ homeomorphism} \;\middle|\; \begin{array}{l}\tau([t_0,\infty)) \subseteq [t_0,\infty),\\ \tau(t_0) = t_0\end{array} \right\}$$

- *asymptotically stable in the sense of Zhukovsky* if, additionally, for all $x_0 \in \mathcal{M}_{k-1}(t_0)$ satisfying $\|\Pi_k(x_0 - x_*(t_0))\| < \delta_0$ and the associated $\tau \in \operatorname{Hom}(I)$ it holds
$$\lim_{t\to\infty} \|x(\tau(t); t_0, x_0) - x_*(t)\| = 0$$

This stability concept allows interesting criteria ensuring Zhukovsky stability, e.g. [Leo07, Theorem 5]: a bounded solution x_* of the autonomous ODE $x'(t) = g(x(t))$, $g \in C^1(\Omega \subseteq \mathbb{R}^m, \mathbb{R}^m)$ is asymptotically stable in the sense of Zhukovsky if $\gamma + \lambda_2 < 0$ whereas $0 = \lambda_1 \geq \lambda_2 \geq \dots \lambda_m$ denotes the characteristic spectrum of the variational system $z'(t) = g_x(x_*(t))\, z(t)$ and

$$\gamma := \sum_{j=1}^{m} \lambda_j - \liminf_{t\to\infty} \frac{1}{t} \int_0^t g_x(x_*(s))\, ds$$

the associated *coefficient of irregularity*. The theorem is proved using a generalization of the Poincaré map which is called a *moving Poincaré section*. If x_* is periodic then the system is known to be regular, i.e. $\gamma = 0$ holds and $\lambda_2 < 0$ is equivalent to $|\mu_2| < 1$ at which $1 = \mu_1 \geq |\mu_2| \geq \dots \geq |\mu_m|$ are the characteristic multipliers of the variational equation. In other words, the Andronov-Witt Theorem is a special case of the mentioned stability criterion.

In principle, this result can be formulated for index-2 DAEs $f\left((Dx(t))', x(t)\right) = 0$ targeting at the local state space form on a compact set containing the bounded reference solution x_*. Some background on Lyapunov exponents for linear DAEs with tractability index one is already available, e.g. in [CN04], [CN03]. An approach to numerical approximation of Lyapunov spectra of DAEs is presented in [MLV09]. In addition to considerably higher computational costs of characteristic exponents of non-periodic solutions, we lack access to a practical representation of the coefficient of irregularity belonging to the linearization of the state space form

$$\gamma = \sum_{j=1}^{m} \lambda_j - \liminf_{t\to\infty} \frac{1}{t} \int_0^t DP_1(s)\, G_2^{-1}(s)\, f_x^*\left((Dx_*(s))', x_*(s)\right) D^- ds$$

Therefore, we restrict ourselves to the case of periodic solutions mainly because under this assumption the involved quantities can be stated in terms of the given DAE.

4.4 Nonlocal existence of DAE solutions

Usually, any kind of stability property implies that solutions of the differential-algebraic system under consideration do exist in the entire future, that is on $I = [t_0, \infty)$, at least for consistent initial values nearby the reference solution. The quest for global solvability of DAEs turns out to be the crucial issue in analyzing D-component stability using generalized Lyapunov functions developed in § 6. Only few publications dealing with solvability of nonlinear DAEs on unbounded intervals are known to us. Predominantly, they rely on a dissipation inequality, e.g. the existence of a Lyapunov function, a contractivity requirement etc. or the emphasis is on structural conditions resulting in a bounded and globally Lipschitz-continuous right-hand side of the ODE representation of the inherent dynamics.

The first type is represented by [GM86, Th. 46] where asymptotic stability and even *total stability* with respect to small t-dependent perturbations of index-1 tractable DAEs (1.1) are proved under rigid conditions including contractivity of the DAE, bounded partial derivatives f_y, f_x and bounded rotation speed of a basis of $\ker f_y(y,x,t)$. Another way to ensure dissipativity of the inherent dynamics is to use Perron's Theorem for DAEs (cf. [Tis94, Theorems 3.3 and 4.2], [Mä98, Th. 3.3]) thus requiring negative real parts of the eigenvalues of the matrix pencil $\{f_y(0,x_*), f_x(0,x_*)\}$ of $f(x'(t), x(t)) = 0$ evaluated in the stationary solution x_*. Unfortunately, this approach is known to be limited to fixed points of autonomous DAEs only. In general, an appropriate definition of a Lyapunov function is necessary for non-constant solutions or non-autonomous fully implicit systems. Among recent publications on differential-algebraic systems, [CC07] presents a criterion for the existence of solutions of linear implicit index-1 systems $A(t)x'(t)+b(x(t),t) = 0$ on $I = [t_0, \infty)$. In essence, $b_x(x,t)$ is required to be bounded and $b_t(x,t)$ globally Lipschitz-continuous on the entire domain where the structural conditions

$$\begin{aligned} \mathrm{rk}\ (A(t_0)\,|b_x(x_0,t_0)) &= r = \max\{\mathrm{rk}\ A(t)\ \mid t \in I\}, \\ \det(\lambda A(t) + b_x(x,t)) &= a(x,t)\lambda^r + \ldots \end{aligned}$$

with $\|a(x,t)\| \geq c$ are valid. Furthermore [CC07, Th. 30] and [CC06] are dealing with a generalization of the Hopf Bifurcation Theorem thus providing existence of periodic solutions for parametrized linear implicit index-1 DAEs $Ax'(t)+b(x(t),\nu) = 0$ nearby a fixed point x_* such that $b(x_*,\nu) \equiv 0$.

This thesis provides stability criteria and therefore nonlocal existence for bounded solutions of a class of autonomous DAEs with index one and two and certain non-autonomous DAEs with bounded terms containing partial derivatives of f.

5 Asymptotic stability of periodic solutions

This chapter is primarily dealing with asymptotic stability of non-constant periodic solutions of autonomous index-2 DAEs. The task is much more involved compared with proving stability properties of stationary solutions of autonomous systems mainly because of two reasons. First, stability definitions in the sense of Lyapunov and of Poincaré (i.e. orbital stability) are not equivalent like in the case of fixed points. In addition, a single local parametrization of the solution manifold $\mathcal{M}_1$ nearby the fixed point is obviously sufficient for analysing its stability behaviour. In contrast, periodic solutions generally necessitate more than one local parametrization of $\mathcal{M}_1$ to construct an appropriate state space representation.

An overview

The early stability criteria for stationary solutions of autonomous differential-algebraic equations, e.g. [Tis94, Theorems 3.3 and 4.2], [Mä98, Th. 3.3] and [Rei95, Theorem 3] generalized Perron's theorem or the Local Stable and Unstable Manifold Theorem to DAEs. Among recent publications [Fra98] and [FF01] are distinguished because they deal with orbital stability of periodic solutions of autonomous Hessenberg-3 DAEs. Their main focus is on collocation methods for checking stability properties of differential-algebraic systems resulting from multibody dynamics, theoretically substantiated by a generalization of the Andronov-Witt Theorem ([FF01, Th. 3]) and the Hopf Bifurcation Theorem (Theorem 2.17, ibid.) for Hessenberg-3 systems. In the process, a transformation to a state space representation is used which is restricted to DAEs of Hessenberg structure. From the analytical point of view, the position constraints are readily available from the algebraic equations of the DAE, the velocity and acceleration constraints result from differentiation of those first-level constraints and inserting other obvious relations. This simple representation of the hidden constraints as zero-set of sufficiently smooth functions with constant rank is used excessively throughout the last-mentioned publications simplifying a construction of a suitable parametrization of the solution manifold nearby the reference trajectory. Then the Poincaré map of the index reduced DAE is restricted to the solution set of the given Hessenberg-DAE using this parametrization so the proof of the Andronov-Witt Theorem can be adapted to the state space form. In contrast, fully implicit systems do not possess such a simple representation of the constraints which is (theoretically) known a priori so the above approach is not applicable. Another interesting approach is the successful generalization of Floquet theory to τ-periodic DAEs in [LMW98] and [LMW03]. In doing so, a sufficient condition for Lyapunov stability of periodic solutions of linear implicit τ-periodic Index-2 DAEs is deduced, see [LMW03, Theorem 4.2]. Last-mentioned publications suggest the possibilities of the tractability index applied to fully implicit DAEs. However, the performed technical details could be regarded as hardly understandable in an intuitive way. This might cause an imprecise reception of the method in the recent literature, for instance the following statement in [LR06]:

> "Note that in case of periodic solutions of DAEs, two different approaches are presented in [Franke&Führer,2001] and [Lamour et al.,1998, 2003] to define monodromy operators and Floquet multipliers for DAEs."

In the present thesis, this point of view is revised. We prove that characteristic multipliers of the DAE defined by means of the tractability index have a geometric interpretation as conventional characteristic multipliers of the associated state space form. By this the mathematical background of the definitions in [LMW03] is clarified the main idea is shown to be identical with [Fra98].

We formulate and prove a criterion for orbital asymptotic stability of periodic solutions for self-oscillating (i.e. autonomous) fully implicit index-2 DAEs. The wording of this theorem almost coincides to the known Andronov-Witt theorem for ODEs except the structural assumptions for index-2 systems. As far as we know, our result is the first one considering orbital stability in the context of the tractability index. The method in use also provides an alternative proof for the stability results [LMW98, Theorem 5.1] and reveals a new view on [LMW03, Theorem 4.2]. Thereby, we do not reduce the system to the case of an asymptotically stable fixed point. Instead, we provide a τ-periodic state space form via suitable transformations and trace the stability properties of the SSF back to [Far94, Theorem 4.2.1]. In doing so, we avoid many technical details like explicit estimations using decoupling terms and devote resources into a more thorough investigation of the structure of index-2 DAEs. As a matter of fact, fully implicit autonomous and τ-periodic DAEs up to index two are supported by our approach.

In order to achieve new stability results, we had to cope with significantly higher complexity of the fundamentals of DAE analysis in Part I. With regard to stability analysis of autonomous DAEs, we strive for an autonomous state space representation assuming a constant $\operatorname{im}\begin{pmatrix} T \\ -f_y^- f_x T \end{pmatrix}(y, x)$ in a neighbourhood of the extended integral curve of x_* and DN_1 to be constant along x_*. These assumptions could be considered as questionable in the context of Hessenberg systems but we have to impose them due to higher structural complexity of general fully implicit DAEs.

Summing up, the constancy of DN_1 together with the structural assumption (2.17) and some differentiability requirements on f ensure that the configuration space $\mathcal{M}_1$ of an autonomous index-2 DAE is parametrizable in a neighbourhood of the entire solution trajectory of x_* using a single map. Per constructionem, linearization and transformation to the state space form commute so characteristic multipliers of the inherent dynamics can be formulated in terms of the given DAE. This aspect was neglected in previous publications.

We suppose that the stability criteria in this chapter could be checked automatically and quite efficiently using numerical methods. We refer to [Fra98, Ch. 3] for a detailed background on adaptation of dedicated collocation methods (and some other details mainly based on [Sey94]) to DAEs and to [Fra98, Ch. 4], [FF01] considering implementation aspects in context of DAEs stemming from multibody dynamics.

5.1 Characteristic multipliers of DAEs

Consider linear DAEs with τ-periodic coefficients and properly stated derivative term

$$A(t)(D(t)x(t))' + B(t)x(t) = 0 \tag{5.1}$$

where

$$\begin{aligned} A \in C(\mathbb{R}, \mathbb{R}^{m\times n}), \quad B \in C(\mathbb{R}, \mathbb{R}^{m\times m}), \quad D \in C(\mathbb{R}, \mathbb{R}^{n\times m}) \\ A(t+\tau) = A(t), \quad B(t+\tau) = B(t), \quad D(t+\tau) = D(t) \end{aligned}$$

We are going to define characteristic multipliers of a solution in terms of (5.1) which coincide with conventional multipliers of the associated state space form.

Denote an admissible projector on $N_1(t)$ along $K(t)$ by $Q_1(t)$.

Lemma 5.1. *Given the properly formulated index-2 DAE* (5.1)*, the matrix valued functions $DP_1D^-, R \in C^1(\mathbb{R}, \mathbb{R}^{n\times n})$ and $P_0, D^- \in C^0$ can be chosen τ-periodic if there exist continuous and τ-periodic C^1-bases of DN_1 and DK.*

Proof. Denote $r \equiv \operatorname{rk} D(t)$ and $l \equiv \dim (DK)(t)$. A continuous basis of $N_0(t)$ is required in case of a properly formulated DAE, in addition $D(t+\tau) = D(t)$ holds for (5.1). According to Lemma 8.2 there exist a continuous and τ-periodic basis $\{p_1(t), \ldots, p_{m-r}(t)\}$ of $N_0(t)$. An eligible extension to a basis $\{p_i, \tilde{p}_j\}$ of $\mathbb{R}^m$ results in the regular matrix valued C^0-function

$$\tilde{V}(t) := (\tilde{p}_1(t), \ldots, \tilde{p}_r(t), p_1(t), \ldots, p_{m-r}(t))$$

which is τ-periodic. Besides,

$$\tilde{P}_0(t) := \tilde{V}(t) \begin{pmatrix} I_r & \\ & 0_{m-r} \end{pmatrix} \tilde{V}^{-1}(t)$$

defines a continuous and τ-periodic projector on $N_0(t)$.

These properties carry over to $G_1 = AD + B\tilde{Q}_0$. Due to continuity of all functions involved, there exists a continuous and τ-periodic basis of $N_1(t) = \ker G_1(t)$ and therefore a τ-periodic basis of $(DN_1)(t)$. Same holds for an adequate complementary space $(DK)(t)$. Then again we assumed the existence of a C^1-basis of $(DN_1)(t)$ which ensures the existence of a τ-periodic C^1-basis of $(DN_1)(t)$ via Lemma 8.2. Furthermore, same argumentation applies to $(DK)(t)$ and $\ker A(t)$. Denote by

$\{D(t)s_1(t),\ldots,D(t)s_l(t)\}$	a τ-periodic C^1-basis of $(DS_1)(t)$
$\{D(t)n_1(t),\ldots,D(t)n_{r-l}(t)\}$	a τ-periodic C^1-basis of $(DN_1)(t)$
$\{\alpha_1(t),\ldots,\alpha_{n-r}(t)\}$	a τ-periodic C^1-basis of $\ker A(t)$

Analogously to Lemma 2.15 we obtain the representations

$$\left(DP_1D^-\right)(t) = M(t)\begin{pmatrix} I_l & \\ & 0_{n-l}\end{pmatrix} M^{-1}(t),\ R(t) = M(t)\begin{pmatrix} I_r & \\ & 0_{n-r}\end{pmatrix} M^{-1}(t)$$

using

$$M(t) := (Ds_1,\ldots,Ds_l,Dn_1,\ldots,Dn_{r-l},\alpha_1(t),\ldots,\alpha_{n-r}(t))$$

As a consequence, the C^1-projectors DP_1D^- and R are τ-periodic.

Obviously, $P_0(t)\,s_i(t) = D^-(t)\,D(t)\,s_i(t)$ and $P_0(t)\,n_j(t)$ are linearly independent continuous functions which can be extended to bases of $S_1(t)$ and $K(t)$, respectively. We use them and the basis functions $\{p_i(t)\}$ of $N_0(t)$ to define

$$V(t) := (P_0(t)\,s_1(t),\ldots,P_0(t)\,s_l(t),P_0(t)\,n_1(t),\ldots,P_0(t)\,n_{r-l}(t),p_1(t),\ldots,p_{m-r}(t))$$

Per constructionem,

$$P_0(t) := V(t)\begin{pmatrix} I_r & \\ & 0_{m-r}\end{pmatrix} V^{-1}(t)$$

is a continuous τ-periodic projector along N_0 and

$$\begin{aligned} D(t)V(t) &= (D(t)s_1(t),\ldots,D(t)s_l(t),D(t)n_1(t),\ldots,D(t)n_{r-l}(t),0_1,\ldots,0_{m-r}) \\ &= M(t)\begin{pmatrix} I_r & 0_{r\times m-r} \\ 0_{n-r\times r} & 0_{n-r\times m-r}\end{pmatrix}\end{aligned}$$

Therefore, $D(t)$ allows the representation

$$D(t) = M(t)\begin{pmatrix} I_r & 0_{r\times m-r} \\ 0_{n-r\times r} & 0_{n-r\times m-r}\end{pmatrix} V^{-1}(t)$$

The continuous τ-periodic function

$$D^-(t) = V(t)\begin{pmatrix} I_r & 0_{r\times m-r} \\ 0_{n-r\times r} & 0_{n-r\times m-r}\end{pmatrix}^T M^{-1}(t) \tag{5.2}$$

defines the reflexive pseudoinverse of D with respect to R and P_0,

$$DD^- = M\begin{pmatrix} I_r & \\ & 0_{n-r}\end{pmatrix} M^{-1} = R,\quad D^-D = V\begin{pmatrix} I_r & \\ & 0_{m-r}\end{pmatrix} V^{-1}(t) = P_0$$

□

Under the assumptions of Lemma 5.1 there is a unique solution $X \in C_D^1(\mathbb{R},\mathbb{R}^{m\times m})$ to the matrix valued IVP

$$A(t)\,(DX)'(t) + B(t)X(t) = 0,\quad (DP_1)(0)\,(X(0) - I_m) = 0$$

We have already proved the representation (1.27) of this fundamental system of the linear DAE reading as follows:

$$\begin{aligned} X(t) &= \mathcal{K}(t)\mathcal{U}(t)\,(DP_1)\,(t_0) \\ \mathcal{K}\,(t) &:= \begin{array}{l} D^-\,(t) - \big((UQ_0 + P_0Q_1)\,G_2^{-1}BD^-\big)\,(t) - \big(TQ_0P_1G_2^{-1}BD^-\big)\,(t) \\ -\,(Q_0Q_1D^-)\,(t)\,\big(DQ_1G_2^{-1}BD^-\big)'\,(t) + \big(Q_0Q_1G_2^{-1}BP_0P_1G_2^{-1}BD^-\big)\,(t) \end{array} \end{aligned}$$

Here, $\mathcal{U}$ denotes the fundamental system of the IRODE $u' = -DP_1G_2^{-1}BD^-u$ with $\mathcal{U}(t_0) = I$. Taking the periodicity of $(DP_1)\,(t)\,X\,(t)\,D^-\,(0)$ into consideration,

$$\begin{aligned} (DP_1)\,(0)\,X\,(\tau)\,D^-\,(0) &= \big(DP_1XD^-\big)\,(\tau) \\ &= \big(DP_1D^-\big)\,(\tau)\mathcal{U}(\tau)\,(DP_1)\,(0)D^-(\tau) \\ &= \big(DP_1D^-\big)\,(0)\mathcal{U}(\tau)\,\big(DP_1D^-\big)\,(0) \end{aligned}$$

It is evident that the eigenvalues of

$$(DP_1)\,(0)\,X\,(\tau)\,D^-\,(0)$$

correspond to the eigenvalues of the fundamental system $\mathcal{U}\,(\tau)$ of the IRODE restricted to the invariant subspace DK.

Definition 5.2. $(DP_1)\,(0)X(\tau)D^-(0) \in \mathbb{R}^{n\times n}$ is called the *monodromy matrix* of the linear τ-periodic index-2 DAE (5.1). The non-zero eigenvalues of the monodromy matrix are called *characteristic multipliers* related to (5.1).

The transition to a representation of the inherent dynamics in minimal coordinates alleviates the proof of stability criteria. We state

Lemma 5.3. *Let DN_1 and DK be constant. Then, characteristic multipliers of the linear index-2 DAE* (5.1) *coincide with conventional characteristic multipliers of the associated state space form.*

Proof. The representation (2.24) of the state space form related to a linear DAE reads

$$\xi_1'(t) = -\begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t)\left(\big(DP_1G_2^{-1}BD^-\big)\,(t)M(t) + M'(t)\right)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\xi_1(t) \quad (5.3)$$

where $\begin{pmatrix} I_l & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{l\times l} & 0_{l\times r-l} & 0_{n-r\times l} \end{pmatrix}$. This ODE is τ-periodic due to τ-periodicity of all functions involved in this representation as a consequence of the preconditions or Lemma 5.1.

Consider a fundamental system $\mathcal{U}\,(t)$ of the IRODE

$$u'(t) = -\big(DP_1G_2^{-1}BD^-\big)\,(t)u(t)$$

with $\mathcal{U}\,(0) = I_n$. Then,

$$\mathcal{U}(t)\,\big(DP_1D^-\big)\,(0) = \big(DP_1D^-\big)\,(t)\,\mathcal{U}\,(t)\,\big(DP_1D^-\big)\,(0)$$

represents a fundamental system of the IRODE on the respective invariant subspace DK. Therefore, a transformation via

$$u(t) = M(t)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\xi_1(t), \quad \xi_1(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t)\, u(t)$$

appears convenient to obtain a fundamental system of the SSF. To this purpose consider

$$\Xi(t) := \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t)\mathcal{U}(t)M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \tag{5.4}$$

Per constructionem, $\Xi(0) = I_l$ and

$$\begin{aligned}
\Xi'(t) &= \begin{pmatrix} I_l & 0 & 0 \end{pmatrix}\Big(\left(M^{-1}\right)'(t)\mathcal{U}(t) + M^{-1}(t)\mathcal{U}'(t)\Big)M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} I_l & 0 & 0 \end{pmatrix}\Big(\left(M^{-1}\right)'(t) - M^{-1}(t)\left(DP_1G_2^{-1}BD^-\right)(t)\Big)\mathcal{U}(t)M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

is valid. As a consequence of

$$\begin{aligned}
\mathcal{U}(t)\, M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} &= \mathcal{U}(t)\left(DP_1D^-\right)(0)\, M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \\
&= \left(DP_1D^-\right)(t)\,\mathcal{U}(t)\left(DP_1D^-\right)(0)\, M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \\
&= M(t)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\Xi(t)
\end{aligned}$$

and $\left(M^{-1}\right)' M = -M^{-1}M'$ we recognize that $\Xi(t)$ is a fundamental system of the state space form (5.3).

We have already proved that the monodromy matrix $\left(DP_1\right)(0)X(T)D^-(0)$ of (5.1) possesses the representation

$$\begin{aligned}
&\left(DP_1\right)(0)\, X(\tau)\, D^-(0) = \left(DP_1D^-\right)(0)\mathcal{U}(\tau)\left(DP_1D^-\right)(0) \\
&= M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(0)\mathcal{U}(\tau)M(0)\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(0)
\end{aligned}$$

Obviously, this matrix is conjugate to the embedding

$$L = \begin{pmatrix} \Xi(\tau) & & \\ & 0 & \\ & & 0 \end{pmatrix}$$

of the monodromy matrix of the SSF (5.3) in $\mathbb{R}^{n\times n}$. $\mathbb{R}^l \times \{0\} \subset \mathbb{R}^n$ is an invariant subspace belonging to L so all eigenvalues of $\Xi(\tau)$ also relate to L and there are no further non-zero eigenvalues of L. It is generally known that the eigenvalues are invariant under conjugation. Therefore the eigenvalues of the monodromy matrix of the DAE coincide with the eigenvalues of the monodromy matrix $\Xi(\tau)$ of the state space representation of the DAE. □

Characteristic multipliers belonging to periodic solutions of nonlinear τ-periodic DAEs

Consider nonlinear τ-periodic differential-algebraic equations with a properly stated derivative of the form

$$f\left((Dx)'(t), x(t), t\right) = 0 \tag{5.5}$$

with $D(t) = D(t+\tau)$, $f(y,x,t) = f(y,x,t+\tau)$ which exhibit a τ-periodic solution $x_* \in C_D^1(\mathbb{R}, \mathbb{R}^m)$. The linearization of (5.5) around x_* reads

$$f_y^*(t)\,(Dx)'(t) + f_x^*(t)\,x(t) = 0$$

with τ-periodic coefficients. Put $z_*(t) := \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}(t)\,D(t)\,x_*(t)$.

Lemma 5.4. *[Characteristic multipliers in case of nonlinear index-2 DAEs]*

If the DAE (5.5) *possesses the tractability index 2 and the structural assumptions in Lemma 2.16 are satisfied then characteristic multipliers of the linearization of* (5.5) *around x_* coincide with characteristic multipliers of the system of variational equations of the associated state space form around z_*.*

Proof. Without loss of generality, choose a matrix chain of the tractability index consisting of τ-periodic elements. According to Lemma 2.16 the system of variational equations of the SSF belonging to (5.5) around z_* corresponds to the SSF of the linearization $f_y^*(t)\,(Dx)'(t) + f_x^*(t)\,x(t) = 0$ of (5.5) around x_* restricted to DK.

Again, (5.4) defines the standardized ($\Xi(0) = I_l$) fundamental system of the linearization of the state space form associated to the given DAE (5.5). As a result, the characteristic multipliers of the linearization of the DAE around x_*, e.g. the eigenvalues of $\Xi(\tau)$, coincide with characteristic multipliers of the linearization of the respective SSF around z_*. □

Corollary 5.5. *Given the requirements of Theorem 2.22, characteristic multipliers of the linearization of an autonomous index-2 DAE* (2.28) *around a τ-periodic solution x_* are in fact the ones belonging to the linearization of the autonomous state space form around z_*.*

Proof. It is suitable to choose constant projectors of the matrix chain belonging to the autonomous properly stated DAE (2.28) like it is done in Theorem 2.22. Referring to Theorem 2.22 once again, the associated state space form exhibits the autonomous representation (2.30) and the transformation to the SSF commutes with linearization. The proposition can be proved in analogon to Lemma 5.4 because characteristic multipliers of the linearization of (2.28) around x_* and the ones belonging to the linearization of the associated SSF around z_* correspond to the eigenvalues of $\Xi(\tau)$. □

Remark 5.6. Under the assumptions of Corollary 5.5 there is at least one characteristic multiplier $\lambda = 1$ of the linearization around the τ-periodic reference solution x_*. Just consider the solution $\xi_*(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1} D x_*(t)$ of the state space form (2.30) of the initial system. Applying the chain rule,

$$\xi_*'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{g}(\xi_*(t)) = \tilde{g}_x(\xi_*(t))\,\xi_*'(t)$$

i.e. $z_* := \xi_*'$ is a τ-periodic solution of the linearization of the SSF. Besides, the representation $z_*(t) = \Xi(t)\, z_*(0)$ is true when using the standardized fundamental system (5.4). Well, τ-periodicity of the solution is equivalent to

$$z_*(0) = z_*(\tau) = \Xi(\tau)\, z_*(0)$$

Therefore $\lambda = 1$ is an eigenvalue of the monodromy matrix of the system of variational equations associated to the state space form.

5.2 The Theorem of Andronov-Witt for DAEs

In the previous chapters we stated important auxiliary results aiming at a criterion for orbital stability of periodic solutions of autonomous DAEs, namely

- the autonomous representation (2.30) of the state space form of autonomous systems
- the commutativity between linearization and transformation to the SSF
- characteristic multipliers of the DAE which are by definition those of the variational system of the associated SSF

Mentioned intermediate steps allow to check orbital stability of periodic solutions resting upon linearization and the state space form. In this case asymptotic orbital stability of DP_1-components imply stability of the entire solution vector. One essential statement of this thesis is a generalization of the well-known Andronov-Witt Theorem to differential-algebraic systems of index $k = 1, 2$.

5.2.1 Index-1 systems

Differential-algebraic systems having the tractability index one are easy to handle using the complete decoupling in Lemma 2.18. Above all, no structural assumptions except of the tractability index one are needed in order to decouple the system.

Theorem 5.7. *[Theorem of Andronov-Witt for index-1 DAEs]*

Let $x_* \in C_D^1(\mathbb{R}, \mathbb{R}^m)$ be a τ-periodic solution of the autonomous index-1 DAE

$$f\left((Dx(t))', x(t)\right) = 0$$

with a properly stated derivative term. If characteristic multipliers $\{\mu_i\}_{i=1,\dots,r=\mathrm{rk}\,D}$ *of the linearization*

$$f_y\left((Dx_*(t))', x_*(t)\right)(Dx)'(t) + f_x\left((Dx_*(t))', x_*(t)\right)x(t) = 0$$

around x_* *satisfy*

$$1 = \mu_1 > |\mu_2| \geq \ldots \geq |\mu_r|$$

then x_* *is orbitally asymptotically stable exhibiting the asymptotic phase property.*

Proof. We may choose constant bases $\{\alpha_i\}$ of $\ker f_y(y,x)$ and $\{d_i\}$ of $\mathrm{im}\, D$ to get a constant representation

$$R = V\begin{pmatrix} I_r & \\ & 0_{n-r} \end{pmatrix}V^{-1}.$$

The tractability index one of an autonomous differential-algebraic system (2.28) implies $f \in C^1(\mathcal{G}, \mathbb{R}^m)$. The implicit function theorem guarantees that the resolution function $w = w(u)$ of the equation autonomous $F(w,u) = 0$ in Lemma 2.18 is continuously differentiable. Accordingly, the state space form (2.27) simplifies to the autonomous ODE

$$\xi_1'(t) = \begin{pmatrix} I_r & 0_{r\times n-r} \end{pmatrix} V^{-1} Dw\left(V\begin{pmatrix} I_r \\ 0 \end{pmatrix}\xi_1(t)\right)$$

having a continuously differentiable right-hand side.

For index-1 tractable systems the matrix G_1 is nonsingular implying $Q_1 \equiv 0$, $P_1 \equiv I_m$ and $G_2 = G_1$. Hence the monodromy matrix of the linearization of (2.28) around x_* is $DX(\tau)D^-$, where $X(t)$ solves the matrix-valued index-1 IVP

$$f_y^*(t)(Dx(t))' + f_x^*(t)x(t) = 0, \quad D(X(0) - I_m) = 0.$$

If $\mathcal{U}(t)$ solves the linear IRODE $u'(t) = -DG_1^{-1}(t) f_x^*(t) D^- u(t)$ satisfying $\mathcal{U}(0) = I_n$ then

$$\Xi(t) := \begin{pmatrix} I_r & 0_{r\times n-r} \end{pmatrix} V^{-1}\mathcal{U}(t)V\begin{pmatrix} I_r \\ 0_{n-r\times r} \end{pmatrix}$$

solves the linearization of the SSF (2.27) around

$$\xi_1^*(t) := \begin{pmatrix} I_r & 0_{r\times n-r} \end{pmatrix} V^{-1}Dx_*(t)$$

with $\Xi(0) = I_r$ so $\Xi(\tau)$ is the monodromy matrix of the linearization of the state space form. Obviously,

$$\begin{aligned} DX(\tau)D^- &= R\mathcal{U}(\tau)R \\ &= V\begin{pmatrix} I_r \\ 0_{n-r\times r} \end{pmatrix}\Xi(\tau)\begin{pmatrix} I_r & 0_{r\times n-r} \end{pmatrix}V^{-1} \\ &= V\begin{pmatrix} \Xi(\tau) & \\ & 0_{n-r} \end{pmatrix}V^{-1} \end{aligned}$$

so characteristic multipliers of the DAE coincide with those of the linearization of the SSF. Now all requirements of the Andronov-Witt Theorem ([Far94, Theorem 5.1.2] resp. [Dem67, p. 305]) are fulfilled so the solution components

$$\xi_1(t) := \begin{pmatrix} I_r & 0_{r\times n-r} \end{pmatrix} V^{-1} Dx(t)$$

are asymptotically orbitally stable exhibiting an asymptotic phase in a neighbourhood of the trajectory of ξ_1^*. There, the function $\frac{\partial}{\partial \xi_1} w\left(V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\xi_1\right)$ is bounded and the solutions of (2.28) possess the representation

$$x(t) = P_0 x(t) + Q_0 x(t) = D^- V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\xi_1(t) + Q_0 w\left(V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\xi_1(t)\right)$$

It follows that for all $t \geq t_0$

$$\|x(t) - x_*(t)\| \leq \left(\left\|D^- V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\| + \left\|Q_0 w_u(\cdot) V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\|_\infty\right)\|\xi_1(t) - \xi_1^*(t)\|$$

Denote the trajectory of x_* by C and the one belonging to ξ_1^* by C_* and apply the τ-periodicity of the reference solution to obtain

$$\begin{aligned}
&\operatorname{dist}\ (x(t), C) = \min_{t\in[0,\tau]}\|x(t) - x_*(t)\| \\
&\leq \min_{t\in[0,\tau]}\left(\left\|D^- V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\| + \left\|Q_0 w_u(\cdot) V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\|_\infty\right)\|\xi_1(t) - \xi_1^*(t)\| \\
&= \left(\left\|D^- V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\| + \left\|Q_0 w_u(\cdot) V\begin{pmatrix} I_r \\ 0_{n-r\times r}\end{pmatrix}\right\|_\infty\right)\operatorname{dist}\ (\xi_1(t), C_*)
\end{aligned}$$

As we can see, mentioned stability properties of ξ_1^* carry over to the solutions of the original DAE. □

It is not surprising that the above formulation of the Andronov-Witt Theorem for index-1 tractable DAEs requires less differentiability of f than a mere adjustment of Theorem 5.8 because there is no need of differentiation in order to obtain hidden constraints.

5.2.2 Andronov-Witt Theorem for index-2 systems

The proof of the index-1 case suggests how to proceed with differential-algebraic systems of a more intricate structure. Indeed, linear implicit DAEs $A(Dx(t))' + b(x(t)) = 0$ with tractability index two exhibiting a constant subspace $N_0 \cap S_0(x)$ in a neighbourhood of the closed solution trajectory x_* in addition to $DN_1(x_*(t)) = \text{const.}$ and fully implicit index-2 DAEs $f\left((Dx(t))', x(t)\right) = 0$ featuring a constant image of $\begin{pmatrix} T \\ f_y^- f_x T\end{pmatrix}(y, x)$ plus $DN_1\left((Dx_*(t))', x_*(t)\right) = \text{const.}$ can be analysed in almost the same manner. Fortunately, we have already performed the somewhat lengthy detailed investigations of these systems in Part I.

Theorem 5.8. *[Theorem of Andronov-Witt for fully implicit index-2 DAEs]*
Let $x_* \in C^1_D(\mathbb{R}, \mathbb{R}^m)$ be a τ-periodic solution of the autonomous properly formulated index-2 DAE

$$f\left((Dx(t))', x(t)\right) = 0$$

Assume DN_1 to be constant along x_* and $\operatorname{im}\begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix}(y,x) = const.$ *in a neighbourhood of the extended trajectory of x_* plus the differentiability properties*[1]

$$\begin{aligned}
&(UQ_0 + P_0Q_1)\, G_2^{-1}(t) f_x^*(t)\, (P_0P_1)\,,\ \left(\left(f_y^- f_x\right)(t)\, TQ_0 + D\right) G_2^{-1}(t) f_x^*(t)\, (P_0P_1) \in C^2\\
&\left(\left(f_y^- f_x\right)(t)\, TQ_0 + D\right) G_2^{-1}(t) \left(f(y,x) - f_x^*(t)x - f_y^*(t)y\right) \in C^2\\
&(UQ_0 + P_0Q_1)\, G_2^{-1}(t) \left(f(y,x) - f_x^*(t)x - f_y^*(t)y\right) \in C^2
\end{aligned}$$

where $Q_1(y,x)$ denotes an admissible projector on $N_1(y,x)$ along $K(y,x)$ with

$$(DK)\left((Dx_*(t))', x_*(t)\right) = const.$$

If the characteristic multipliers $\{\mu_i\}_{i=1,\ldots,l=\dim DK\left((Dx_*)'(t), x_*(t)\right)}$ of the linearization of $f\left((Dx)'(t), x(t)\right) = 0$ around x_* satisfy

$$1 = \mu_1 > |\mu_2| \geq \ldots \geq |\mu_l|$$

then x_* is orbitally asymptotically stable exhibiting the asymptotic phase property.

Proof. According to Lemma 2.11 there exists a complementary subspace $K(y,x)$ to $N_1(y,x)$ such that DK is constant along the extended trajectory of x_* such that Q_1 onto N_1 along K is admissible. The requirements of Theorem 2.22 are satisfied, consequently there exists an autonomous SSF belonging to $f\left((Dx(t))', x(t)\right) = 0$. Given differentiability assumptions on f imply $\tilde{g} \in C^1$ in Representation (2.30) of the state space form.

According to Corollary 5.5 the characteristic multipliers $\mu_i \in \mathbb{C}$ of the linearization around x_* correspond to those of the variational system of the SSF (2.30) around $z_*(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1} Dx_*(t)$. Thereby $\mu_1 = 1$ is always one characteristic multiplier of the autonomous DAE around the τ-periodic solution x_* as shown in Remark 5.6.

We consider the periodic solution z_* of the autonomous state space representation (2.30) of the given DAE. Obviously, the requirements of the formulation [Far94, Theorem 5.1.2] of the Andronov-Witt theorem by M. Farkas are satisfied. For this reason z_* exhibits orbital asymptotic stability having the asymptotic phase property.

Solutions ξ of the SSF are mapped to solutions of the IRODE of the initial system on the invariant subspace $DK = (DK)\left((Dx_*)'(t), x_*(t)\right)$ by the bijection

$$u(t) = M \begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} \xi(t)\,, \quad \xi(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1} u(t)$$

[1] functions of several variables denoted t-dependent only are evaluated in $\left((Dx_*)'(t), x_*(t)\right)$

Denote the trajectory of x_* by C and observe that DP_1C is the trajectory of $u_* = DP_1x_*$. Due to stability properties of z_*, for all $\epsilon > 0$ there exists a $\delta > 0$ such that every solution u of the IVP $u(0) = u_0 \in DK$ of the IRODE with $\operatorname{dist}(u_0, DP_1C) < \delta$ satisfies

- u is defined on $[0, \infty)$
- $\forall t \geq 0 : \operatorname{dist}(u(t), DP_1C) < \epsilon$
- $\exists \alpha_0 \in \mathbb{R} : \lim_{t\to\infty} \|u(t + \alpha_0) - u_*(t)\| = 0$

Now choose $0 < \epsilon \ll 1$ such that the complete decoupling of the DAE (2.28) analyzed in Theorem 2.10 is feasible in the entire region

$$\mathcal{N}_\epsilon := \left\{ \begin{pmatrix} y \\ x \end{pmatrix} \in \mathbb{R}^{n+m} \,\middle|\, \exists t \in \mathbb{R} : \left\| \begin{pmatrix} y \\ x \end{pmatrix} - \begin{pmatrix} (Dx_*)'(t) \\ x_*(t) \end{pmatrix} \right\| < \epsilon \right\}$$

The mentioned theorem states that the components $u(t; 0, u_0) := DP_1x(t; 0, x_0)$ of a solution $x(t; 0, x_0) \in \mathcal{N}_\epsilon$, $x_0 \in \mathcal{M}_1$ of the given DAE satisfy the associated inherent regular ODE on $DK = (DK)\big((Dx_*(t))', x_*(t)\big)$. Besides, every solution u of the IRODE on DK near u_* corresponds to one solution x of $f\left((Dx(t))', x(t)\right) = 0$ via

$$\begin{aligned} x(t) &= s_2(u(t), t) \\ s_2(u, t) &:= D^-u + \tilde{k}_1(u, t) + \tilde{m}_1\left(u, \tilde{k}(u, t), t\right) \end{aligned} \tag{5.6}$$

Choose a smaller $0 < \hat{\epsilon} < \epsilon$ such that solutions of IVPs $u(t; 0, DP_1x_0)$ of the IRODE on DK with $x_0 \in \mathcal{M}_{k-1}$ and $\operatorname{dist}(DP_1x_0, DP_1C) < \delta$ exist on $[0, \infty)$ and feature $\operatorname{dist}(u(t; 0, x_0), DP_1C) < \hat{\epsilon}$. Due to the important Remark 2.21, the functions $\tilde{m}$ and $\tilde{k}$ in the solution representation (5.6) are τ-periodic and so is $s_2(u, t)$. Accordingly, $\frac{\partial}{\partial u}s_2(u, t)$ is bounded on the compact set $\mathcal{S} := \{u \in \mathbb{R}^n \mid \operatorname{dist}(u, DP_1C) \leq \hat{\epsilon}\} \times [0, \tau]$ and we obtain the estimate

$$\begin{gathered} \forall t \geq 0, x_0 \in \mathcal{M}_1, \operatorname{dist}(DP_1x_0, DP_1C) < \delta : \\ \|x(t; 0, x_0) - x_*(t)\| \leq \left\| \tfrac{\partial}{\partial u} s_2 \right\|_{\infty, \mathcal{S}} \|DP_1x(t; 0, x_0) - u_*(t)\| \end{gathered}$$

where $x(t; 0, x_0)$ defined by (5.6) is known to solve the original DAE. Hence,

$$\operatorname{dist}(x(t; 0, x_0), C) \leq \left\| \frac{\partial}{\partial u} s_2 \right\|_{\infty, \mathcal{S}} \operatorname{dist}(DP_1x(t; 0, x_0), DP_1C)$$

for all $t \geq 0$ and $x_0 \in \mathcal{M}_1$ with $\operatorname{dist}(DP_1x_0, DP_1C) < \delta$. In other words, the orbital asymptotic stability of u_* with asymptotic phase guarantees the same stability properties of x_*. □

In case of linear implicit DAEs (1.5) with $A = A(t)$, the structural condition on $\operatorname{im} \begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix}$ and the differentiability preconditions can be mitigated.

Theorem 5.9. *[Andronov-Witt Theorem for linear implicit index-2 systems]*

Let $x_* \in C^1_D(\mathbb{R}, \mathbb{R}^m)$ be a τ-periodic solution of the DAE

$$A(Dx(t))' + b(x(t)) = 0$$

with a properly stated derivative term and tractability index 2. Assume constancy of DN_1 along x_* and of $N_0 \cap S_0(x)$ in a neighbourhood of the trajectory of x_*. Let

$$(UQ_0 + P_0Q_1)\, G_2^{-1}(t) b_x(x_*(t))(P_0P_1),\ (UQ_0 + P_0Q_1)\, G_2^{-1}(t)(b(x) - b_x(x_*(t))\, x)$$

be twice continuously differentiable, where $Q_1(x)$ denotes an admissible projector on $N_1(x)$ along $K(x)$ with $(DK)(x_*(t)) = const.$ If the characteristic multipliers

$$\{\mu_i\}_{i=1,\ldots,l},\ l = \dim DN_1(x_*(t))$$

of the linearization $A(Dx(t))' + b_x(x_*(t))\, x(t) = 0$ satisfy

$$1 = \mu_1 > |\mu_2| \geq \ldots \geq |\mu_l|$$

then x_* is orbitally asymptotically stable exhibiting the asymptotic phase property.

Proof. Simple adaptation of the argument in Theorem 5.8 where Theorem 2.22 is tailored to linear implicit DAEs using the structural conditions for a complete decoupling of these systems in Lemma 2.9. □

5.2.2.1 Application to MNA equations

Theorem 5.9 can be applied to certain autonomous systems stemming from the charge-oriented Modified Nodal Analysis (MNA) of electric circuits containing nonlinear resistances, capacitances, inductances together with voltage and current sources. Consider an electrical circuit such that it is modelled with autonomous equations for each circuit element. According to [EST00], the charge-oriented MNA provides the autonomous differential-algebraic system

$$\begin{aligned} \begin{matrix} A_C q'(t) + A_R r\left(A_R^T e(t)\right) + A_L j_L(t) \\ +A_V j_V(t) + A_i i\left(\tilde{A} e(t), q'(t), j_L(t), j_V(t)\right) \end{matrix} &= 0 \\ \phi'(t) - A_L^T e(t) &= 0 \\ A_V^T e(t) - v\left(\tilde{A} e(t), q'(t), j_L(t), j_V(t)\right) &= 0 \\ q(t) - q_C\left(A_C^T e(t)\right) &= 0 \\ \phi(t) = \phi_L(j_L(t)) &= 0 \end{aligned}$$

where $e(t)$ consists of the node potentials (except the reference node), $j_*(t)$ combines the currents of some elements, q denotes the charge of capacitances and ϕ the flux of inductances. A_* denotes the element-related incidence matrices, $\tilde{A} = (A_C, A_L, A_R, A_V, A_I)$ is the reduced incidence matrix of the circuit and the index marks the element (L - inductances, V - voltage sources, C - capacitances, R - resistances and

I - current sources). The functions i and v describe the controlled current resp. voltage sources of the circuit, r the resistances, ϕ_L the inductances and q_C the capacitances.

In the following, we would like to assume v and i being independent of q'. Then the MNA equations form a linear implicit DAE of the form

$$Ax'(t) + b(x(t)) = 0$$

Using a constant projector P_0 along $\ker A$, this system is equivalent to

$$A(P_0 x(t))' + b(x(t)) = 0$$

which has a properly stated derivative term. Obviously, a matrix chain of the tractability index for the non-properly formulated version coincides with the matrix chain of the last mentioned DAE.

We rely on Theorem 4.1 from [EST00] and assume that all requirements of this theorem are fulfilled. As mentioned in the publication, the remark to [EST00, Lemma 6.2] is also applicable in case of the charged-oriented MNA ensuring constancy of the systemic subspace $N_0 \cap S_0(x)$ if all controlled current sources satisfy the conditions (2a) or (2b) of ibid., Theorem 4.1. From a representation of the canonical projector onto $N_1(x)$ along $S_1(x)$, C. Tischendorf and D. Estevez-Schwarz observe that $N_1(x)$ is constant if no controlled current sources that fulfill only the conditions (2b) or (2c) in [EST00, Th. 4.1] appear. For circuits satisfying all mentioned structural assumptions, Theorem 5.9 is applicable so the orbital asymptotic stability of a periodic reference solution $x_* \in C^1_D(\mathbb{R}, \mathbb{R}^m)$ can be checked numerically considering characteristic multipliers of the linearization $A(P_0 z(t))' + b_x(x_*(t))\, z(t) = 0$.

5.2.2.2 Self-oscillating systems: examples

One way to obtain an oscillator is to apply periodic forcing terms to a dynamical system. A simple example is an *RLC*-circuit connected to an AC voltage source. Assuming linear elements only, the inherent dynamics are given by a linear ODE $x' = Ax + f(t)$ in $\mathbb{R}^2$. The related homogeneous equation representing a damped *LC*-oscillating circuit has no periodic solutions apart from the trivial one so the forced *RLC*-circuit has a unique periodic solution for every frequency of f due to [Far94, Theorem 2.3.1]. The stability behaviour of such systems modelled by periodic differential-algebraic equations can already be analyzed by means of [LMW03, Theorem 4.2].

On the other hand, there are some well-known real systems modelled by autonomous ODEs exhibiting a *self-oscillating* behaviour. For example, the equations

$$\begin{aligned} x_1' &= x_1(a - bx_2) \\ x_2' &= x_2(-c + dx_1) \end{aligned}$$

were suggested by V. Volterra and A.J. Lotka for modelling the interaction between one predator and one pray species. The non-negative orbits of the *Lotka-Volterra* *model* are closed curves around an equilibrium point. A generalization given by Gause, Smaragdova and Freedman (cf. [Far94, p. 118] reads

$$\begin{aligned} x_1' &= x_1 f(x_1) - x_2 x_1 g(x_1) \\ x_2' &= x_2(-c + d x_1 g(x_1)) \end{aligned}$$

with adequate functions f, g. In certain configurations, this system has an asymptotic orbitally stable periodic solution.

Further notable examples include the *Brusselator equations* ([HW08, pp. 115ff.]) stemming from chemical reaction dynamics. The original ODE is in $\mathbb{R}^6$, but it can be simplified to

$$\begin{aligned} x_1' &= A + x_1^2 x_2 - (B+1)\, x_1 \\ x_2' &= B x_1 - x_1^2 x_2 \end{aligned}$$

supposed some qualities are maintained constant and the reaction rates are equal to one.

Modelling a *wheelset* running on a track is discussed in [Fra98, pp. 16ff.]. It leads to an index three DAE in Hessenberg form in $\mathbb{R}^5$. Asymptotic orbital stability of periodic motions of such Hessenberg-systems can be analyzed following the approach in [Fra98], [FF01].

The simulation of quite simple electrical circuits is able to provide nonlinear phenomena of astonishing complexity. Picking out two prominent examples out of abundant configurations, the *Van der Pol's equation* $\ddot{x} + m(x^2 - 1)\,\dot{x} + x = 0$ or equivalently

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - m(x_1^2 - 1)\, x_2 \end{aligned}$$

forms the inherent dynamics of an electrical circuit with a triode and with inductive feedback (for details, see [Far94, pp. 97 ff.]). Certain self-oscillating circuits containing vacuum tubes and RLC-oscillators are presented in [AWC65], [AWC69]. Another well-known example of chaotic behaviour exhibiting periodic solutions is *Chua's circuit.* A simplified model with linear elements only except a piecewise-linear resistor has the state space form

$$\begin{aligned} x' &= c_1(y - x - f(x)) \\ y' &= x - y + z \\ z' &= -c_2 y \end{aligned}$$

A chaotic behaviour of this system is reported as early as in [Mat84]. A survey of development in context of Chua's circuit up to the end of the last decade is given in [Mir97]. In principle, it is possible to formulate the MNA-equations of those circuits directly as an index-2 DAE instead of constructing the inherent regular ODE like it is done in the cited monographs. The resulting nonlinear DAEs do not exhibit the Hessenberg structure so the stability result [Fra98, Th. 2.7] is not applicable any longer. In such cases, our version of the Andronov-Witt Theorem (Theorem 5.8 resp. Theorem 5.9) might be helpful.

However, the mentioned examples are quite intricate because an analytical representation of the periodic reference solution x_* of the self-oscillating IRODE is usually not available and it is also possible that x_* lacks required differentiability properties. Accordingly, we are going to construct a simple, self-oscillating inherent dynamics having a known asymptotically orbitally stable periodic solution in order to illustrate our version of the Andronov-Witt stability criterion for index-2 DAEs.

Construction of the example

An autonomous self-oscillating index-2 DAE has to be formulated at least in $\mathbb{R}^4$. Obviously, the state space of an autonomous ODE has to be at least two-dimensional in order to permit non-trivial periodic solutions (otherwise f in $x' = f(x)$ is not a function). In addition, a third variable representing the algebraic constraints of the DAE and a fourth variable representing the index-1 constraints is necessary.

Actually, both Lotka-Volterra model and Van der Pol's equation can be considered as perturbations of the *harmonic oscillator*

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 \end{aligned}$$

We want to enforce solutions starting from outside of the unit circle to spiral inwards and solutions starting inside the unit circle to spiral outwards, i.e the orbit C of the reference solution $\hat{x}(t) = (\sin(t), \cos(t))$ to be an omega limit set of all trajectories nearby. Obviously, $d = d(x_1, x_2) = 1 - (x_1^2 + x_2^2)$ is a directed distance between (x_1, x_2) and C. The desirable solution behaviour originates from adding a correction term $x_i(1 - (x_1^2 + x_2^2))$ to the vector field resulting in the ODE

$$\begin{aligned} x_1' &= x_2 + x_1(1 - (x_1^2 + x_2^2)) \\ x_2' &= -x_1 + x_2(1 - (x_1^2 + x_2^2)) \end{aligned} \tag{5.7}$$

Consider a point (x_1^0, x_2^0) outside the unit circle (solid black curve) in Figure 5.1.

The dotted circle represents the corresponding solution of the harmonic oscillator and the blue vector is its tangent vector at (x_1^0, x_2^0), consisting of the components in x_i-directions. In order to enforce spiralling inwards, we decrease the speed of the movement (e.g. partial derivative) in direction of x_2 and increase it in x_1-direction. The correction terms are indicated by the red vectors. The black vector is the resulting tangent vector at (x_1^0, x_2^0), it belongs to the solution curve of (5.7). This figure clarifies the geometric idea for $x_1 \leq 0$, $x_2 \geq 0$ and $1 - (x_1^2 + x_2^2) < 0$ but the same result is also valid in the other three quadrants. The correction terms depend on the directed distance d to C. Consequently, the resulting equations (5.7) coincide with the harmonic oscillator along $\hat{x}$ guaranteeing that $\hat{x}$ is a periodic solution of the perturbed system.

Applying the classical theorem of Andronov-Witt, we show that our construction of the self-oscillating ODE works. The linearization of (5.7) along x_* reads

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'(t) = \begin{pmatrix} -2\sin^2(t) & 1 - 2\sin(t)\cos(t) \\ -1 - 2\sin(t)\cos(t) & -2\cos^2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) \tag{5.8}$$

Although the periodic solution $\bar{x} = x_*' = (-\sin, -\cos)^T$ is known, the d'Alembert

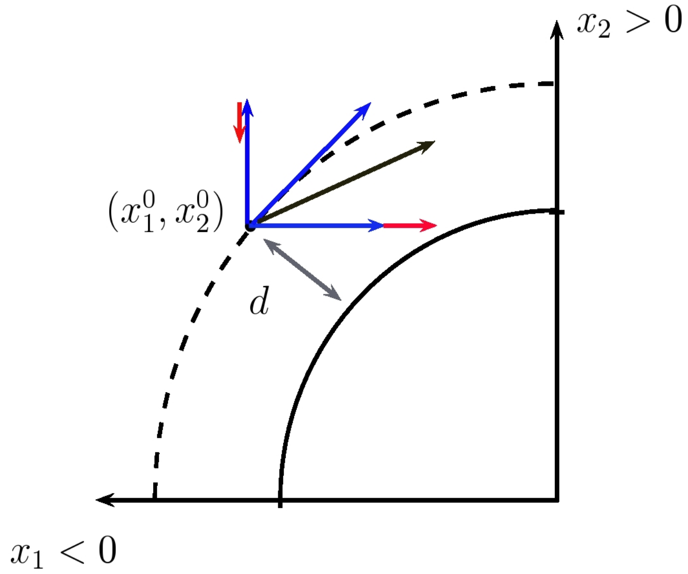

Figure 5.1: construction of a self-oscillating ODE

reduction principle is not applicable due to the zeros of sin and cos. Instead of constructing a fundamental system $X(t)$ with $X(0) = I_2$, we use the identity

$$\det X(t) = \det X(0) \exp\left(\int_0^t \mathrm{tr} A(s)\, \mathrm{d}s\right)$$

for the Wronskian $\det X(t)$. Due to $X(2\pi)\bar{x}(0) = \bar{x}(0)$, $\lambda_1 = 1$ is an eigenvalue of the monodromy matrix $X(2\pi)$ of (5.8). For 2×2 matrices, the determinant is the product of the eigenvalues, that is

$$\exp(-4\pi) = \det X(2\pi) = \lambda_1 \lambda_2 = \lambda_2$$

is the second characteristic multiplier and $|\lambda_2| < 1$. Hence, the Andronov-Witt theorem (e.g. [Far94, Th. 5.1.2]) implies that x_* is an asymptotically orbitally stable solution of (5.7) having the asymptotic phase property.

In the next step, we add some constraints in order get a differential-algebraic system with tractability index two. An algebraic constraint is given by $x_3 = x_1^2 + x_2^2 + 1$ and another one leading to an index-2 system is $x_4 = (x_3^2)' = 2x_3 x_3'$. Finally, a certain coupling is obtained by multiplying the second equation of (5.7) with x_3 and using the algebraic relation for x_3 resulting in the non-Hessenberg DAE

$$\begin{array}{rcl} x_1' - x_2 - x_1 + x_1^3 + x_1 x_2^2 & = & 0 \\ x_3 x_2' + x_1^3 + x_1 x_2 + x_1 - x_3 x_2 + x_1^2 x_2 x_3 + x_2^3 x_3 & = & 0 \\ 2x_3 x_3' - x_4 & = & 0 \\ x_1^2 + x_2^2 + 1 - x_3 & = & 0 \end{array} \tag{5.9}$$

We analyze (5.9) by means of the matrix chain of the tractability index. It holds

$$f_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3 & 0 \\ 0 & 0 & 2x_3 \\ 0 & 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; Q_0 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

$$f_x = \begin{pmatrix} -1+3x_1^2+x_2^2 & -1+2x_1x_2 & 0 & 0 \\ 3x_1^2+x_2^2+1+2x_1x_2x_3 & 2x_1x_2-x_3+x_1^2x_3+3x_2^2x_3 & x_2'-x_2+x_1^2x_2+x_2^3 & 0 \\ 0 & 0 & 2x_3' & -1 \\ 2x_1 & 2x_2 & -1 & 0 \end{pmatrix}$$

$$G_1 = \begin{pmatrix} 1 & & & \\ & x_3 & & \\ & & 2x_3 & -1 \\ & & & 0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & 2x_3 & 0 \end{pmatrix}, \; P_0Q_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

We compute $N_0 \cap S_0 = \ker D \cap \ker W_0 f_x = N_0$ and therefore $T = Q_0$. Furthermore,

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x_3 & x_2'-x_2+x_1^2x_2+x_2^3 & 0 \\ 0 & 0 & 2x_3+2x_3' & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; DP_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 0 \end{pmatrix}$$

and $\det G_2 = -x_3$. Therefore, (5.9) has index 2 on $\mathbb{R}^4 \backslash \{x_3 = 0\}$. According to construction, $x_*(t) = (\sin(t), \cos(t), 2, 0)^T$ is a periodic solution of the given DAE and im $DP_1\left((Dx_*)'(t), x_*(t)\right)$ is constant. Moreover,

$$f_y^-(x_3) = \begin{pmatrix} 1 & & & \\ & x_3^{-1} & & \\ & & (2x_3)^{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix}(x_3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(2x_3)^{-1} \end{pmatrix}$$

is a x_3-dependent function, but its image is constant, i.e the structural condition (2.17) is valid. The linearization of (5.9) along x_* reads

$$\begin{aligned} x_1' + 2\sin^2 x_1 + (2\sin\cos - 1)\, x_2 &= 0 \\ 2x_2' + \left(2\sin^2 + 2 + 4\sin\cos\right) x_1 + \left(4\cos^2 + 2\sin\cos\right) x_2 - \sin x_3 &= 0 \\ 4x_3' - x_4 &= 0 \\ 2\sin x_1 + 2\cos x_2 - x_3 &= 0 \end{aligned} \tag{5.10}$$

Inserting the constraints

$$x_3 = 2\sin(t)\, x_1 + 2\cos(t)\, x_2, \; x_4 = 4x_3'$$

into the first two equations results in the following representation of the inherent dynamics

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}'(t) = \begin{pmatrix} -2\sin^2(t) & 1-2\sin(t)\cos(t) \\ -1-2\sin(t)\cos(t) & -2\cos^2(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t)$$

which coincides with the linearization (5.8) of the state space form (5.7) of the DAE (5.9). This fact approves the commutativity between linearization and the transformation into the SSF stated in Theorem 2.22.

The monodromy matrix of (5.10) is $DP_1X(2\pi)D^-$ whereby $X(t)$ denotes the fundamental matrix of (5.10) satisfying the initial values $DP_1X(0) = DP_1$. Obviously,

$$DP_1X(2\pi)D^- = \begin{pmatrix} X_{11}(2\pi) & X_{12}(2\pi) & X_{13}(2\pi) \\ X_{21}(2\pi) & X_{22}(2\pi) & X_{23}(2\pi) \\ 0 & 0 & 0 \end{pmatrix}$$

is singular so $\lambda_3 = 0$ is an eigenvalue and if $(a_i, b_i)^T$ are eigenvectors of

$$L := \begin{pmatrix} X_{11}(2\pi) & X_{12}(2\pi) \\ X_{21}(2\pi) & X_{22}(2\pi) \end{pmatrix}$$

to the eigenvalues λ_i $(i = 1, 2)$ then $(a_i, b_i, 0)^T$ are eigenvectors of the monodromy matrix corresponding to the same eigenvalues λ_i. The initial condition $DP_1X(0) = DP_1$ implies that L equals the monodromy matrix of (5.8) having the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \exp(-4\pi)$. Therefore, the characteristic multipliers of the linear DAE (5.10) are

$$\lambda_1 = 1 > \lambda_2 = \exp(-4\pi) > \lambda_3 = 0$$

Now the asymptotic orbital stability of $x_*(t) = (\sin(t), \cos(t), 2, 0)^T$ with an asymptotic phase is confirmed by Theorem 5.8.

5.3 A stability result for periodic index-2 DAEs

We are able to state an asymptotic stability criterion for τ-periodic reference solutions of a τ-periodic DAE following the lines of Theorem 5.8.

Theorem 5.10. *[Asymptotic Lyapunov stability of periodic solutions of periodic DAEs] Let $x_* \in C_D^1(\mathbb{R}, \mathbb{R}^m)$ be a τ-periodic solution of the τ-periodic index-2 DAE*

$$f\left((Dx)'(t), x(t), t\right) = 0$$

and $Q_1(y, x, t)$ be an admissible projector on $N_1(y, x, t)$ along $K(y, x, t)$ exhibiting

$$(DN_1)\left((Dx_*)'(t), x_*(t), t\right) = const. \text{ and } (DK)\left((Dx_*)'(t), x_*(t), t\right) = const.$$

Let us suppose that

$$\operatorname{im}\begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix}(y, x, t) \text{ is dependent only on } t$$

in a neighbourhood of the extended integral curve of x_*. Furthermore, let $D \in C^1$ and

$$\begin{array}{l}(UQ_0 + P_0Q_1)(t)G_2^{-1}(t)f_x^*(t)(P_0P_1)(t),\\ \left(f_y^- f_x TQ_0 + D\right)(t)G_2^{-1}(t)f_x^*(t)(P_0P_1)(t),\\ \left(f_y^- f_x TQ_0 + D\right)(t)G_2^{-1}(t)\left(f(y,x,t) - f_x^*(t)x - f_y^*(t)y\right),\\ (UQ_0 + P_0Q_1)(t)G_2^{-1}(t)\left(f(y,x,t) - f_x^*(t)x - f_y^*(t)y\right)\end{array}$$

be twice continuously differentiable.

If the characteristic multipliers $\{\mu_i\}_{i=1,\ldots,l=\dim DK\left((Dx_*)'(t),x_*(t),t\right)}$ of the linearization $f_y^*(t)(Dx)'(t) + f_x^*(t)x(t) = 0$ satisfy

$$1 = \mu_1 > |\mu_2| \geq \ldots \geq |\mu_l|$$

then x_* is asymptotically stable in the sense of Lyapunov.

Proof. Choose an $0 < \epsilon$ such that the complete decoupling of the nonlinear DAE in Lemma 2.16 is feasible in an ϵ-tube $\mathcal{N}_\epsilon(t)$ around the extended integral curve of the reference solution x_*. We pass over to the state space representation according to Lemma 2.16. Corollary 5.4 implies that μ_i are the characteristic multipliers of the linearization of the τ-periodic SSF (2.24) around $z_*(t) = \begin{pmatrix} I_l & 0 & 0 \end{pmatrix} M^{-1}Dx_*(t)$.

Now, the τ-periodic solution z_* of the state space representation fulfills the requirements of Theorem 4.2.1 in [Far94]. Thus z_* is uniformly asymptotically stable in the sense of Lyapunov, in particular for all $0 < \hat{\epsilon} < \epsilon$ there exists a $\delta > 0$ such that solutions $z(t;t_0,z_0)$ of the SSF (2.24) with $z_0 \in \mathrm{B}_\delta(z_*(t_0))$ are subject to

$$\forall t \geq t_0 : \; \|z(t;t_0,z_0) - z_*(t)\| < \hat{\epsilon}$$

Periodicity of the linear transformation

$$M\begin{pmatrix} I_l \\ 0 \\ 0 \end{pmatrix} z(t) = \left(DP_1D^-\right)(t)\,D(t)\,x(t) = (DP_1)(t)\,x(t) = u(t)$$

of solutions $z(t;t_0,z_0)$ of the SSF to those of the inherent regular ODE on DK results in asymptotic DP_1-component stability of the DAE solution x_*.

Moreover $s_2(u,t+\tau) = s_2(u,t)$ is valid in the solution representation

$$\begin{array}{rcl} x(t) & = & s_2(u(t),t) \\ s_2(u,t) & := & D^-(t)\,u + \tilde{k}_1(u,t) + \tilde{m}_1\left(u,\tilde{k}(u,t),t\right) \end{array}$$

as stated in Remark 2.21. Therefore, $\frac{\partial}{\partial u}s_2(u,t)$ is bounded on

$$\mathcal{S} := \left\{\begin{pmatrix} u \\ t \end{pmatrix} \in \mathbb{R}^{n+1} \,\middle|\, \exists t_0 \in \mathbb{R} : \left\|\begin{pmatrix} u \\ t \end{pmatrix} - \begin{pmatrix} (DP_1)(t_0)\,x_*(t_0) \\ t_0 \end{pmatrix}\right\| \leq \hat{\epsilon}\right\}$$

so for all $x_0 \in \mathcal{M}_1(t_0)$ such that $(DP_1)(t_0)(x_0 - x_*(t_0))$ we obtain

$$\|x(t;0,x_0) - x_*(t)\| \leq \left\|\frac{\partial}{\partial u}s_2\right\|_{\infty,\mathcal{S}} \|DP_1x(t;0,x_0) - u_*(t)\|$$

Therefore asymptotic Lyapunov stability of $u_*(t) = (DP_1)(t)\,x(t)$ implies asymptotic stability in the sense of Lyapunov according to Definition 4.3 with

$$\Pi_1 = (P_0P_1)\left((Dx_*)'(t_0), x_*(t_0), t_0\right)$$

$\square$

The respective stability criterion [Far94, Th. 4.2.1] for ODEs is entrenched in Lyapunov's indirect method. It is a well-known result of Floquet theory that the linearization $z'(t) = f_x(x_*(t), t)\,z(t)$ of a τ-periodic ODE $x'(t) = f(x(t), t)$ along a τ-periodic reference solution can be transformed into a system with constant coefficients, $y'(t) = Ay(t)$ such that the eigenvalues $\mu_i \in \mathbb{C}$ of A satisfy $\operatorname{Re}\mu_i = \frac{\ln|\lambda_i|}{T}$ with the characteristic multipliers $\lambda_i \in \mathbb{C}$ of the linearization by means of a τ-periodic Lyapunov transformation, cf. [Wal00, § 18], [Far94, § 2.2]. Applying this transformation to the nonlinear ODE reduces asymptotic stability of the periodic solution x_* to stability of the trivial solution of a system $y'(t) = Ay(t) + r(y(t), t) = 0$ with a uniformly bounded nonlinearity r. Finally, the stability issue is resolved using Perron's theorem, e.g. [Far94, Th. 1.4.9]. This is exactly the way how a similar result - namely Theorem 4.2 in [LMW03] - is proved.

Theorem 5.10 presents an an alternative method to prove asymptotic stability of τ-periodic solutions for nonlinear fully implicit index-2 DAEs (5.5) exhibiting a properly stated derivative term. In comparison to the mentioned theorem, our structural assumptions differ and constancy of DN_1 and DK along x_* is required extra. We reckon that this assumption might be superfluous in case of periodic DAEs because of not being necessary for complete decoupling or preservation of stability under the transfer from an inherent regular ODE on its invariant subspace DK to a state space representation. Yet we decided to keep this precondition in order to avoid additional technical considerations which are not important for autonomous systems.

6 Lyapunov's direct method regarding DAEs

Guarding the heritage does not mean confining oneself to the heritage.

(russian proverb)

Stability properties of ordinary differential equations are often proved using the direct method of Lyapunov. To this regard, it is stated in [ERA07]:

> "Lyapunov function techniques have received constantly high interest in applied mathematics [...]. The main reasons for this interest are simplicity, intuitive appeal, and universality of these techniques. Today, there is no doubt that Lyapunov functions techniques are the main tools to be used when one is faced with a stability or stabilization problem."

The goal of this chapter is to state an appropriate definition of a Lyapunov function for fully implicit nonlinear index-2 systems in terms of the given DAE. We recapitulate the classic notion of the Lyapunov function and the main stability criterion based on it. Then we point out that properly stated DAEs can be uniquely solved with respect to $R(t)(Dx)'(t)$ and this resolution has a useful implicit representation allowing to access the right hand side of the inherent regular ODE of index-1 systems implicitly. This representation is used to motivate the definition of a Lyapunov function for index-1 tractable systems (1.2). Finally, sufficient conditions are presented which facilitate Lyapunov's direct method for DAEs having tractability index two. Strictly speaking, we are going to use the reduction of the tractability index presented in Chapter 3 in order to deduce an implicit representation of the right hand side of the IRODE of the index reduced DAE on the exact solution set $M_1(t)$. Analogously to the ODE case, the existence of a Lyapunov function on a cylindrical domain ensures nonlocal existence and stability of solutions with a slight modification that DAEs exhibit D-component stability in general. A brief survey on alternative approaches to the direct method of Lyapunov for DAEs is presented in Section 6.5. For the sake of completeness, we added a section which helps to interpret known contractivity definitions for DAEs in the context of index reduction via differentiation.

The concept of Lyapunov functions for ODEs

First, we briefly introduce the well-known idea of Lyapunov functions for ODEs. A survey of classical stability theory can be found in the recent monograph [LWY07] and in the classical ones, especially [Hah67], [Yos66] and [Dem67] and, of course, Lyapunov's masterpiece [Lja66]. To some extent less rigorous alternatives are [LL67] and [Wil73].

The question of stability of a general solution x_* of the ODE $x'(t) = f(x(t), t)$, $f \in C\left(\Omega \times \mathbb{R}^{\geq t_0}, \mathbb{R}^m\right)$ satisfying a local Lipschitz condition or having a continuous f_x can be reduced to stability of the fixed point $z_* = 0$ of the system

$$z'(t) = \tilde{f}(z(t), t) := f(z(t) + x_*(t), t) - f(x_*(t), t)$$

for $z(t) = x(t) - x_*(t)$. Furthermore we consider $\tilde{z}(t) = z(t + t_0)$ and the equivalent ODE $\tilde{z}'(t) = \tilde{f}(\tilde{z}(t), t + t_0)$ to obtain $t_0 = 0$. On this account we are going to consider Lyapunov functions for fixed points $x_* \equiv 0$ of $x'(t) = f(x(t), t)$ with $f \in C\left(\Omega \times \mathbb{R}^{\geq 0}, \mathbb{R}^m\right)$, $\Omega \subseteq \mathbb{R}^m$ a neighbourhood of the origin without loss of generality.

Definition 6.1. A function $H \in C\left(\Omega, \mathbb{R}^{\geq 0}\right)$ is called *positive definite* if $H(0) = 0$ and $H(x) > 0$ for all $x \neq 0$.

As shown in [LWY07, Th. 1.3.3], a positive definite function H is restrained by strictly increasing functions $\phi, \eta : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, i.e.

$$\phi(\|x\|) \leq H(x) \leq \eta(\|x\|) \quad \text{and} \quad \phi_i(0) = 0 = \eta_i(0)$$

Definition 6.2. A function $V \in C^1(\Omega \times \mathbb{R}^{\geq 0}, \mathbb{R})$ is called a *Lyapunov function* of the stationary solution $x_* \equiv 0$ of $x'(t) = f(x(t), t)$ if

1. $\forall t \geq 0 : V(0, t) = 0$
2. $\forall x \in \Omega, t \geq 0 : V(x, t) \geq H_1(x)$ for a positive definite $H_1 \in C(\Omega, \mathbb{R}^{\geq 0})$
3. It holds $\dot{V}(x, t) \leq 0$ on $\Omega \times \mathbb{R}^{\geq 0}$ for $\dot{V}(x, t) := V_t(x, t) + \langle f(x, t), V_x(x, t) \rangle$

The first condition ensures that $x_* \equiv 0$ is always a global minimum of the time-dependent Lyapunov function V. The second inequality means that there is a lower bound for values of V which is independent of t and strictly increasing with $\|x\|$, i.e. $V(x, t) \geq \phi(\|x\|)$. If additionally $\lim_{\|x\| \to \infty} \phi(\|x\|) = \infty$ holds then V is *weakly coercive* with respect to x according to the definition in [Zei90, p. 472], viz. $V(x, t) \to \infty$ if $\|x\| \to \infty$ independent of t. A Lyapunov function exhibiting this property is usually called *radially unbounded* in the literature on stability of ODEs. The latter restriction is required in order to prove global uniform stability of the origin, e.g [Yos66, Th. 42.5]. The third condition enforces V to be monotonically decreasing along the integral curves of $x'(t) = f(x(t), t)$ because the chain rule implies $\frac{\mathrm{d}}{\mathrm{d}t} V(x(t), t) = \dot{V}(x(t), t) \leq 0$.

In case of autonomous ODEs there is a nice geometric view on this dissipativity inequality. It is known that the gradient $\nabla V(x) = (V_x(x))^T$ is orthogonal to the level set $\{z \in \Omega \mid V(z) = v_0\}$ pointing in direction of the fastest growth of V. Condition

$$\langle \nabla V(x), f(x) \rangle = \dot{V}(x) \leq 0$$

corresponds to an obtuse angle between $f(x)$ and $\nabla V(x)$, see Figure 6.1. In [LL67, p. 37] is shown that the Taylor expansion of $V(x)$ contains terms of degree ≥ 2, e.g. the graph of V is similar to a paraboloid in a neighbourhood of the origin. Due to $\dot{V}(x) \leq 0$ a solution $x(t)$ stays on its level set of V or is "reflected" to a level set with a

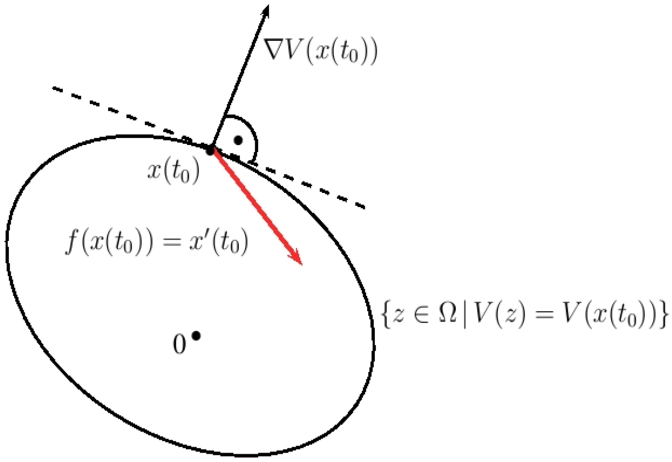

Figure 6.1: Geometric interpretation of $\dot{V}(x) \leq 0$

smaller value of V. If $V(x,t)$ is wedged between positive definite functions H_i, $i = 1, 2$ and $\dot{V}$ is negative definite then this descent towards the local minimum $V(0,t) = 0$ implies that solutions $x(t; t_0, x_0)$ are bounded and $\lim_{t\to\infty} \|x(t; t_0, x_0)\| = 0$, at least for $x_0 \in \mathrm{B}_\epsilon(0) \subseteq \Omega$. The following classic stability criterion points out this property of the above definition:

Theorem 6.3. *[Dem67, S. 237ff.]*

Let $x_* \equiv 0$ be a stationary solution of the ODE $x'(t) = f(x(t), t)$ and let the local Lipschitz condition with respect to x be satisfied.

1. *The existence of a Lyapunov function implies Lyapunov stability of x_*.*
2. *If the Lyapunov function under consideration satisfies*
   - $V(x,t) \leq H_2(x)$ *for a positive definite* $H_2 \in C\left(\Omega, \mathbb{R}^{\geq 0}\right)$
   - $\dot{V}$ *is negative definite, i.e.* $\forall x \in \Omega, t \geq 0 : \dot{V}(x,t) \leq -\tilde{H}(x)$ *with a positive definite* $\tilde{H} \in C\left(\Omega, \mathbb{R}^{\geq 0}\right)$

   then x_* is asymptotically stable in the sense of Lyapunov.

In various cases the existence of a Lyapunov function is even a necessary condition in case of a stable solution. Such statements are called converse theorems, for example [LWY07, Ch. 4], [Yos66, Ch. 5]. Furthermore Lyapunov functions allow estimates of the domain of attraction in case of asymptotic stability and they enable to prove criteria for Lagrange stability, confer the mentioned monographs.

Several research groups in applied mathematics are engaged setting up Lyapunov functions for relevant problems. For the most part, this process rests upon intuition, experience and explicit knowledge of the physical background of the respective problem. Nevertheless, the presented *second method of Lyapunov* has the pivotal theoretical advantage that no a priori knowledge about solutions of the ODE is necessary for stability investigations. This approach is also called the *direct method of Lyapunov* because it applies directly to the differential equation in question. In contrast, *Lyapunov's first* or *indirect method* uses the linearization of a system to determine the local stability of the original system.

6.1 First key note: cylindricity of the domain

The investigation of Lyapunov stability of a reference trajectory x_* of

$$x'(t) = f(x(t), t) \tag{6.1}$$

usually takes for granted that f is defined at least on an ϵ-tube around the integral curve of x_*. Indeed, the common assumption in the context of the direct method of Lyapunov is a cylindrical domain of the ODE under consideration.

According to [Wal00, § 6], there are three types of behaviour of maximal solutions $x \in C^1(I, \mathbb{R}^m)$, $I = [t_0, t_e)$ of an ODE (6.1) with $f \in C^0(\mathcal{G}, \mathbb{R}^m)$, $\mathcal{G} \subseteq \mathbb{R}^m \times \mathbb{R}$ open and f satisfying the Lipschitz condition locally:

1. $t_e = \infty$, i.e. x exists for all $t \geq t_0$
2. $t_e < \infty$ and $\limsup_{t \to t_e} \|x(t)\| = \infty$, in other words the solution "explodes" on a finite interval
3. $t_e < \infty$ and $\liminf_{t \to t_e} \operatorname{dist}(x(t), \partial\mathcal{G}) = 0$ that is x comes infinitesimally close to the boundary $\partial\mathcal{G}$ of the domain of f

A Lyapunov function $V(x,t)$ delimits the ODE solutions $x(t; t_0, x_0)$ with $\|x_0 - x_*(t_0)\| < \delta$ to

$$\mathcal{S}_\delta = \left\{ x \in \mathbb{R}^m \;\middle|\; \exists t:\ V(x,t) \leq \max_{x_0 \in \mathrm{B}_\delta(x_*(t_0))} V(x_0, t_0) \right\}$$

The existence of a positive definite lower bound $H_1 = H_1(x)$ such that $H_1(x) \leq V(x,t)$ ensures that the *"Lyapunov sack"* $\mathcal{S}_\delta$ is bounded because

$$\mathcal{S}_\delta \subseteq H_1^{-1}\left(\left[0, \max_{x_0 \in \mathrm{B}_\delta(x_*(t_0))} V(x_0, t_0) \right] \right).$$

Choosing a smaller $\delta > 0$ means "to tighten the Lyapunov sack", i.e to reduce the diameter $\sup_{x,y \in \mathcal{S}_\delta} \|x - y\|$ of $\mathcal{S}_\delta$. Now if (6.1) has a cylindrical domain $\mathcal{G} = \mathcal{G}_1 \times [t_0, \infty)$, $\mathcal{G}_1 \subseteq \mathbb{R}^m$ open set, then it is possible to choose δ such that $\mathcal{S}_\delta \subseteq \mathcal{G}_1$ has a non-vanishing distance to the boundary $\partial\mathcal{G}_1$. Consequently, the solutions are bounded and do not approach $\partial\mathcal{G}$ so there is only the first case remaining, that is they exist for all $t \geq t_0$.

The inconspicuous property of a cylindrical domain $\mathcal{G} = \mathcal{G}_1 \times [t_0, \infty)$ around the reference solution $x_* \equiv 0$ in Def. 6.2 is certainly a reasonable requirement. In general, if $\mathcal{G}$ is not cylindrical, i.e. there is no $\delta > 0$ such that

$$\mathrm{B}_\delta(0) \subseteq \{ x \in \mathbb{R}^m \mid \forall t \geq 0:\ (x,t) \in \mathcal{G} \}$$

then dist $(\partial(\mathrm{pr}_1 \mathcal{G}), \mathcal{S}_\delta) = 0$ is imaginable for every $\delta > 0$. In this case, solutions of (6.1) are still bounded due to the Lyapunov sack, but may fail to exist on the infinite interval $[t_0, \infty)$. Since the domain of an explicit ordinary differential equation is readily available, the cylindricity assumption is a mild restriction, if any. For differential-algebraic systems, the access to the domain of the inherent regular ODE in terms of the given DAE in order to ensure cylindricity turns out to be challenging.

6.2 Second key note: implicit resolution for $R(Dx)'$

Another key note behind the direct method of Lyapunov is the dissipativity equation $\dot{V}(x,t) \leq 0$ representing the total time derivative of a Lyapunov candidate function $V = V(x,t)$. It is important that this monotonicity property (i.e. the value of a Lyapunov function evaluated along the integral curve of a solution is monotonically decreasing) can be represented without knowing the solutions of (6.1). Dissipativity equations for DAEs are difficult to state because of the missing solvability of $f(x'(t), x(t), t) = 0$ with respect to the derivative $x'(t)$. This problem was noticed at an early stage of research on differential-algebraic systems, e.g. attempting to formulate the direct method of Lyapunov for linear implicit DAEs

$$P(t)\,x'(t) = f(x(t), t)$$

in a customary way in [Baj87]. Fortunately, the remedy is already presented in the same publication ([Baj87, p. 2173]):

> "The above-mentioned deficiency can be removed if an LFC[1] is constructed to be not an explicit function of x, but rather the explicit function of an auxiliary variable that depends on x in a special way."

In detail, for the above DAE it is proposed to use $y_1(t) := P(t)\,x(t)$ implying $y_1'(t) = P'(t)\,x(t) + P(t)\,f(x(t), t)$ and a Lyapunov function candidate $V = V(y_1, t)$. Assuming differentiability of the functions involved, the total time derivative of V along a solution x of the mentioned DAE is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} V(y_1(t), t) = V_t(y_1(t), t) + \left\langle V_{y_1}^T(y_1(t), t), P'(t)\,x(t) + P(t)\,f(x(t), t)\right\rangle$$

The present thesis: Lyapunov functions aiming at the inherent dynamics

The above approach was not the starting point for the methods of this thesis because there is no obvious or canonical way to extract the derivative components from a fully implicit system like nonlinear DAEs of the type (1.2) like it is done for simple linear implicit systems. In fact, our main idea was to address the inherent regular ODE of a DAE, that is to define a Lyapunov function for the IRODE strictly in terms of the original system. Starting from a complete (tractability) decoupling of index-1 DAEs, it appeared canonical to rely on D-components and therefore to define and use D-component stability. The reduction of the tractability index turned out to be fitting well into the approach so the index-2 results were stated.

We had to investigate appropriate features of the properly stated leading derivative term and index-1, 2 tractable DAEs. A distinguishing mark of a properly formulated derivative term is certainly the possibility to solve fully implicit DAEs for $R(t)\,(Dx)'(t)$ thus suggesting to formulate the Lyapunov function depending on $y = D(t)\,x$. Of course, the simplicity of the Ansatz owes heavily to the restriction on $\ker f_y(y, x, t)$ in Definition 1.2.

[1] Lyapunov Function Candidate, explanatory note

Consider properly stated DAEs $f\left((Dx)'(t), x(t), t\right) = 0$. Assuming unique solvability of the DAE, the configuration space $\mathcal{M}(t)$ is covered by solutions so that a unique solution is passing through each $x_0 \in \mathcal{M}(t_0)$. Let $x(t; t_0, x_0)$ denote the solution of the IVP $x(t_0; t_0, x_0) = x_0 \in \mathcal{M}(t_0)$ of the differential-algebraic system on its maximal interval of existence $I_{(t_0,x_0)}$. Obviously,

$$f\left(R(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)x(t; t_0, x_0)\right), x(t; t_0, x_0), t\right) = 0$$

i.e. $x(t; t_0, x_0) \in \mathcal{M}_0(t)$ and $\mathcal{M}(t) \subseteq \mathcal{M}_0(t)$.

The properly stated derivative term results in

$$\forall \xi \in \mathcal{M}_0(t)\, \exists! \mu = R(t)\mu : \ f(\mu, \xi, t) = 0$$

according to Lemma 1.19. In the particular case $x_0 \in \mathcal{M}(t_0)$ it holds

$$y = R(t_0)\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)x(t; t_0, x_0)\right)\Big|_{t=t_0}$$

for $y = R(t_0)y$ satisfying $f(y, x_0, t_0) = 0$. For this reason, the following lemma is proved:

Lemma 6.4. *[Universal implicit resolution with respect to $R(Dx)'$]*

If the differential-algebraic system (1.2) *is properly formulated and features unique solutions of the initial value problems then*

$$\begin{aligned}&\forall t_0 \in I,\ x_0 \in \mathcal{M}(t_0),\ y = R(t_0)y : \\ &f(y, x_0, t_0) = 0 \Longrightarrow y = R(t_0)\tfrac{\mathrm{d}}{\mathrm{d}t}\left(D(t)x(t; t_0, x_0)\right)\Big|_{t=t_0}\end{aligned} \tag{6.2}$$

This abstract proposition is applicable because the unique solvability can be ensured for DAEs of index $k = 1, 2$ by the means of accessible sufficient conditions. Lemma 6.4 contributes to a better understanding of the theoretical background of dissipativity inequalities in [MHT03a], [MHT03b] and [San00]. Moreover, the definition of Lyapunov functions for DAEs is motivated.

6.3 Lyapunov functions for index-1 DAEs

We wish to generalize Lyapunov functions to fully implicit index-2 DAEs because of the benefits such functions exhibit in case of ODEs. The successive sections lead to an appropriate definition. Our starting point are properly stated index-1 DAEs (1.2), that is

$$f\left((Dx)'(t), x(t), t\right) = 0$$

They admit a quite simple representation of the inherent dynamics using Lemma 2.18.

First of all notice that in case of index $k = 1, 2$ DAEs, stability of a reference solution x_* with initial value $x_*(t_0) \in \mathcal{M}_{k-1}(t_0)$ can be reduced to stability of the origin.

Transformation to the origin and $t_0 = 0$

Let $x_* : [t_0, \infty) \to \mathbb{R}^m$ be a solution of the fully implicit DAE $f\left((Dx)'(t), x(t), t\right) = 0$. The investigation of its Lyapunov stability requires to know the behaviour of the difference $z(t) = x(t) - x_*(t)$ of further solutions x of the DAE with initial value x_0 and $\|\Pi_k(x_*(t_0) - x_0)\|$ sufficiently small. Chaining the transformation to $x_* \equiv 0$ and $t_0 = 0$ presented for ODEs motivates the transition to the DAE

$$\tilde{f}\left(\left(\tilde{D}z\right)'(t), z(t), t\right) = 0 \tag{6.3}$$

with $\tilde{D}(t) := D(t + t_0)$, $\tilde{f}(y, x, t) := f\left(y + (Dx_*)'(t + t_0), x + x_*(t + t_0), t + t_0\right)$ on the domain

$$\tilde{\mathcal{G}} := \left\{(y, x, t) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{\geq 0} \quad | \quad \left(y + (Dx_*)'(t + t_0), x + x_*(t + t_0), t + t_0\right) \in \mathcal{G}\right\}$$

f is defined on a region $\mathcal{G}$ containing an open and connected neighbourhood of the extended integral curve belonging to x_*. This corresponds to a region around the extended integral curve of $z_* \equiv 0$ in $\tilde{\mathcal{G}}$.

For any solution $x \in C^1_D([t_0, \alpha), \mathbb{R}^m)$, $\alpha \in \mathbb{R} \cup \{\infty\}$ of the given DAE

$$z \in C^1_{\tilde{D}}([0, \alpha - t_0), \mathbb{R}^m), \quad z(t) := x(t + t_0) - x_*(t + t_0)$$

solves the transformed system (6.3). Hence, (asymptotic) stability in the sense of Definition 4.2 or M-component stability of the stationary solution $z_* \in C^1_{\tilde{D}}\left(\mathbb{R}^{\geq 0}, \mathbb{R}^m\right)$ of (6.3) implies the same stability property of the associated reference solution $x_* \in C^1_D\left(\mathbb{R}^{\geq t_0}, \mathbb{R}^m\right)$ of the given DAE.

Lemma 6.5. *Consider the properly stated DAE* (1.2) *having a solution x_* and the tractability index $k = 1, 2$ in a region around the extended integral curve of x_*. Then the transformed DAE* (6.3) *exhibits the same index in a region around the extended integral curve of $z_* \equiv 0$.*

Proof. The partial derivatives of $\tilde{f}$ are

$$\begin{aligned}\tilde{f}_y(y, x, t) &= f_y\left(y + (Dx_*)'(t + t_0), x + x_*(t + t_0), t + t_0\right)\\ \tilde{f}_x(y, x, t) &= f_x\left(y + (Dx_*)'(t + t_0), x + x_*(t + t_0), t + t_0\right)\end{aligned}$$

Denote the elements of the matrix chain of (1.2) by N_i, S_i, G_i for $i \in \{0, 1, 2\}$. Evaluating these elements evaluated in $\left(y + (Dx_*)'(t + t_0), x + x_*(t + t_0), t + t_0\right)$ corresponds to a matrix chain of the transformed system (6.3) evaluated in (y, x, t). □

6.3.1 An implicit representation of the index-1 IRODE

Consider the algebraic equation $f(y, x, t) = 0$, $f \in C^0(\mathcal{G}, t)$ belonging to the properly stated DAE (1.2) with tractability index one. According to Lemma 2.18 the solution components $u(t) := D(t)x(t)$ solve the ODE

$$u'(t) = f_{IRODE}(u(t), t) := R'(t)u(t) + D(t)w(u(t), t) \tag{6.4}$$

on the invariant subspace im $D(t)$. Here $w = w(u,t)$ is implicitly defined by the algebraic constraint

$$F(w(u,t), u, t) = 0, \quad F(w,u,t) := f\left(D(t)w, D^{-}(t)u + Q_0(t)w, t\right)$$

Conversely, every solution u of the IRODE with an initial value $u_0 \in \operatorname{im} D(t_0)$ corresponds to the solution

$$x(t) := D^{-}(t)\,u(t) + Q_0(t)\,w(u(t), t)$$

of the given DAE. This proves the unique solvability of the index-1 DAE supposed consistent initial values $x_0 \in \mathcal{M}_0(t_0)$ with preset $(Dx)(t_0) = u_0$, in other words locally around any point $(y,x,t) \in \mathcal{G}$ with $x \in \mathcal{M}_0(t)$ and $y = R(t)y$ such that $f(y,x,t) = 0$. The inherent regular ODE (6.4) restricted to im $D(t)$ determines the dynamics of the DAE.

Now it is self-evident to demand a Lyapunov function of the IRODE on im $D(t)$ as a stability criterion for (1.2). In doing so we try to avoid the terms stemming from the complete decoupling in the explicit representation (6.4). The definition of a Lyapunov function for index-1 systems should be formulated in terms of the initial DAE instead.

It is obvious that every solution $x \in C^1_D(I, \mathbb{R}^m)$ of the DAE satisfies the first level constraint, e.g.

$$\forall t \in I: \; x(t) \in \mathcal{M}_0(t)$$

Besides, Lemma 1.19 ensures that for every $x \in \mathcal{M}_0(t)$ there exists a unique $y = R(t)y \in \operatorname{im} D(t)$ with $f(y,x,t) = 0$. Additionally, for all $t \in I$, $x \in \mathcal{M}_0(t)$ and $y \in \mathbb{R}^n$ with $f(y,x,t) = 0$,

$$u := D(t)\,x, \quad w := D^{-}(t)\,y + Q_0(t)\,x$$

fulfill the algebraic equation $F(u,w,t) = 0$ which has a locally unique solution $w = w(u,t)$.

These considerations imply that in case of a solution x of the DAE,

$$y(t) := R(t)\,(Dx)'(t)$$

is equivalent to $D(t)w((Dx)(t), t)$. Taking the second addend in (6.4) into consideration results in the following representation of the right hand side of the IRODE:

$$\boxed{\forall t \in I,\, x \in \mathcal{M}_0(t): \; f_{IRODE}(D(t)x, t) = R'(t)D(t)x + z} \tag{6.5}$$

with the unique $z = R(t)z$ solving $f(z,x,t) = 0$.

6.3.2 Lyapunov function aiming at D-component stability

In general, the tractability index one of a fully implicit DAE could be valid only on a domain like in Figure 6.2 which gets infinitesimal close to the extended integral curve of $x_* \equiv 0$ for $t \to \infty$.

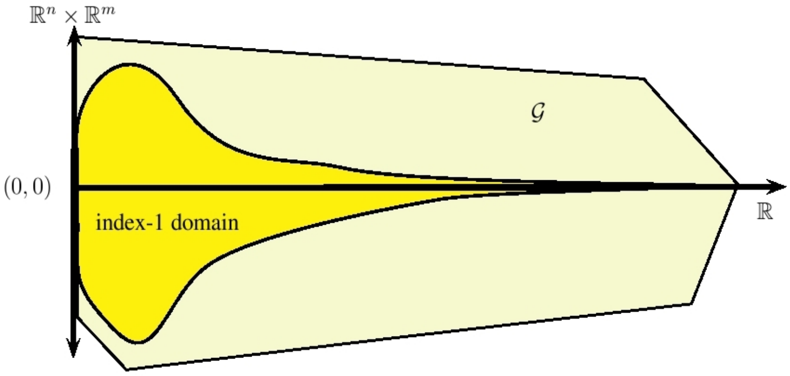

Figure 6.2: problematic index-1 domain of a DAE

This setup seems to be inappropriate for Lyapunov stability for we have to prove the existence of solutions with consistent initial values $x_0 \in \mathcal{M}_0(0)$ and $D(0)\,x_0 \in \mathrm{B}_\delta(0)$ for all $t \geq 0$. Even if these solutions proceed in an ϵ-neighbourhood of the integral curve belonging to x_* for $t \in I = [0,\infty)$, they do not converge towards the stationary solution in general. As a consequence, they could leave the subset of $\mathcal{G}$ where the index-1 condition holds thus becoming inaccessible by the means of the complete decoupling in § 6.3.1.

It is sufficient to require the feasibility of the index-1 decoupling on a cylindrical region around x_* in order to apply the algebraic relation (6.5) for an appropriate definition of a Lyapunov function of the solution $u_* := Dx_*0$ of the IRODE $u'(t) = f_{\text{IRODE}}\ (u(t),t)$ restricted to the invariant subspace im $D(t) \subseteq \mathbb{R}^n$. Let us agree to use the

Notation 6.6. In the following,

$$\mathcal{U}_\epsilon := \left\{\mathrm{B}_\epsilon\left((Dx_*)'(t)\right) \times \mathrm{B}_\epsilon\left(x_*(t)\right) \times \{t\} \mid t \geq 0\right\} \subseteq \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{\geq 0}$$

denotes the ϵ-tube around the extended integral curve of x_*,

$$\mathcal{U}_1 := \{\mathrm{B}_\epsilon\left((Dx_*)(t)\right) \times \{t\} \mid t \geq 0\} \subseteq \mathbb{R}^n \times \mathbb{R}^{\geq 0}$$

and

$$(D\mathcal{M}_0)(t) := \{D(t)\,x \in \mathbb{R}^n \mid x \in \mathcal{M}_0(t)\} = D(t)\,\mathcal{M}_0(t)$$

Furthermore, the gradient $\nabla V(u,t)$ refers to spatial arguments only, i.e. $\nabla V(u,t) := [V_u(u,t)]^T$.

Definition 6.7. [Lyapunov function for D-component stability]

$V \in C^1(\mathcal{U}_1,\mathbb{R})$ is called a *Lyapunov function* for the reference solution x_* of the properly formulated index-k, $k = 1,2$ DAE (1.2) if

1. $\forall t \geq 0 :\ V((Dx_*)(t),t) = 0$
2. There exists a positive definite function $H_1 \in C^0(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n,\mathbb{R})$ exhibiting

$$\forall t \geq 0,\ u \in (D\mathcal{M}_{k-1})(t) \cap \mathrm{B}_\epsilon((Dx_*)(t)) :\ H_1(u - (Dx_*)(t)) \leq V(u,t) \quad (6.6)$$

3. For all $t \geq 0$, $x \in \mathcal{M}_{k-1}(t)$, $\|D(t)(x - x_*(t))\| < \epsilon$, $z = R(t)z$ with $f(z, x, t) = 0$:

$$\dot{V}(D(t)x, t) := \langle R'(t) D(t) x + z, \nabla V(D(t)x, t)\rangle + V_t(D(t)x, t) \leq 0 \quad (6.7)$$

Notice that for index-1 DAEs, the simplification

$$(D\mathcal{M}_0)(t) \cap \mathrm{B}_\epsilon((Dx_*)(t)) = \mathrm{im}\ D(t) \cap \mathrm{B}_\epsilon((Dx_*)(t))$$

is possible for sufficiently small $\epsilon > 0$.

It turns out to be easier to work with a Lyapunov function without fallback to the state space representation of the inherent dynamics on $\mathbb{R}^r$. Instead, we aim directly at the inherent regular ODE on the invariant subspace im $D(t)$. The main reason is that dissipativity inequality (6.7) admits a nice expression in terms of the given DAE. The second reason to reject the state space form are additional restrictions on the bases of im $D(t)$ and ker $f_y(y, x, t)$ due to their appearance as the matrix-valued function $V(t)$ in (2.27) and in the corresponding representation of the solution vector.

Remark. The definition of $V \in C^1(\mathcal{U}_1, \mathbb{R})$ is due to convenience because the constitutive properties are required only on

$$\{\mathrm{im}\ D(t) \times \{t\} \mid t \geq 0\} \cap \mathcal{U}_1$$

If the given DAE is *numerically qualified*, that is im $D(t) =$ const. then $R'D = 0$ and $f_{IRODE}(Dx, t) = z$ are valid in (6.5) and Representation (6.7) simplifies to

$$\dot{V}(D(t)x, t) := \langle z, \nabla V(D(t)x, t)\rangle + V_t(D(t)x, t) \leq 0$$

Assumption 6.8. *Assume that the following properties hold for the DAE* (1.2) *and sufficiently small* $\epsilon > 0$ *:*

1. $\mathcal{U}_\epsilon \subseteq \mathcal{G}$ and the DAE (1.2) possesses the tractability index 1 on $\mathcal{U}_\epsilon$.
2. The domain of the maximal continuation of the resolution function $w = w(u, t)$ implicitly defined by

$$F(w, u, t) = f\left(D(t)w, D^-(t)u + Q_0(t)w, t\right)$$

contains the cylindrical region $\mathcal{U}_1$.

Theorem 6.9. *[Principal Lyapunov theorem for index-1 DAEs]*

Consider the stationary solution $x_* \equiv 0$ of *a properly formulated DAE* (1.2) *satisfying Assumption 6.8.*

1. *If a Lyapunov function* $V(u, t)$ *according to Definition 6.7 exists then* x_* *is D-component stable.*
2. *If the Lyapunov function* $V(u, t)$ *under consideration satisfies*

   · $\forall t \geq 0, u \in \mathrm{im}\ D(t) \cap \mathrm{B}_\epsilon(0) : V(u, t) \leq H_2(u)$ *for a positive definite* $H_2 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\geq 0}\right)$

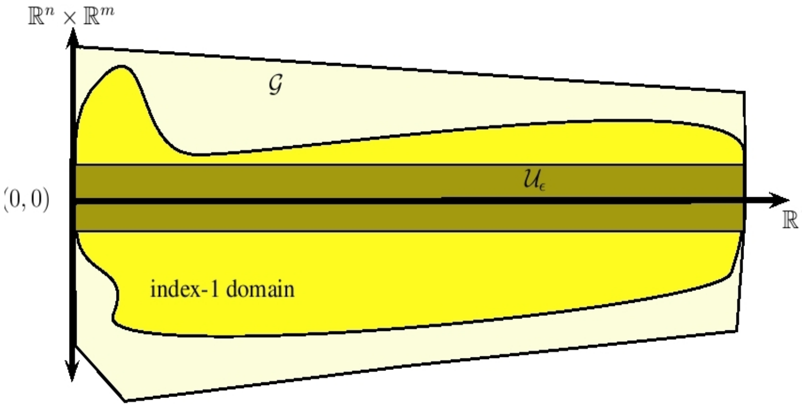

Figure 6.3: A suitable index-1 domain meeting the above assumption

- *$\dot{V}$ is negative definite, i.e. there exists a positive definite function $H_3 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\geq 0}\right)$ with*

$$\forall t \geq 0,\ u \in \operatorname{im} D(t) \cap \mathrm{B}_\epsilon(0) :\ \dot{V}(u,t) \leq -H_3(u)$$

then x_* is asymptotically D-component stable.

Proof. On D-component stability:

The basic idea of the proof is to construct a "Lyapunov sack" (cf. § 6.1) containing solutions of the inherent regular ODE. The domain of tractability index one allows a complete decoupling of the given DAE locally around each point of

$$\mathcal{U}_\epsilon \cap \{(y,x,t) \in \mathcal{G} \mid f(y,x,t) = 0\}$$

Local uniqueness of the implicitly defined functions $w = w(u,t)$ enables us to switch over to the maximal continuation. In the process, the second part of Assumption 6.8 ensures that the above set contains a cylindrical region where a global decoupling based on the maximal continuation $w(u,t)$ on $\mathcal{U}_1$ is available.

According to [LWY07, Th. 1.3.3] positive definite H_i, $i = 1,2,3$ are characterized by the existence of two strictly increasing functions $\phi_i, \eta_i : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ having the property

$$\phi_i(\|u\|) \leq H_i(u) \leq \eta_i(\|u\|) \quad \text{and} \quad \phi_i(0) = 0 = \eta_i(0)$$

Let $0 < \gamma < \epsilon$. Continuity of V at $x_* = 0$ means that

$$\exists \delta > 0 \forall \|u_0\| < \delta :\ V(u_0, 0) < \phi_1(\gamma)$$

Consider an arbitrary solution $u(t;0,u_0)$ of the IVP $u(0;0,u_0) = u_0 \in \operatorname{im} D(0) \cap \mathrm{B}_\delta(0)$ of the IRODE (6.4). Now

$$t_\gamma := \inf\{t \geq 0 \quad | \quad \|u(t;0,u_0)\| = \gamma,\ \forall 0 \leq s \leq t :\ \|u(s;0,u_0)\| < \gamma\}$$

denotes the first time when the trajectory of $u(t;0,u_0)$ hits the boundary of $\mathrm{B}_\gamma(0) \subseteq \mathbb{R}^{\mathrm{n}}$.

Suppose that $t_\gamma < \infty$. Per constuctionem, the integral curve of $u(\cdot; 0, u_0)$ restricted to $[0, t_\gamma]$ is located in the domain of the resolution $w = w(u, t)$ so we are able to use the solution representation

$$\begin{aligned} x_0 &:= D^-(0)\, u_0 + Q_0(0)\, w(u_0, 0) \\ x(t; 0, x_0) &:= D^-(t) u(t; 0, u_0) + Q_0(t) w(u(t; 0, u_0), t) \end{aligned} \tag{6.8}$$

of properly stated index-1 DAEs according to Lemma 2.18 and § 6.3.1. Obviously,

$$D(t)\, x(t; 0, x_0) = R(t)\, u(t; 0, u_0) = u(t; 0, u_0)$$

because of the invariant subspace im $D(t)$ and $u_0 \in$ im $D(0)$. The chain rule and Representation (6.5) imply

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(D(t)\, x(t; 0, x_0), t) &= V_x(D(t)\, x(t; 0, x_0), t) \tfrac{\mathrm{d}}{\mathrm{d}t}(D(t)\, x(t; 0, x_0)) \\ &\quad + V_t(D(t)\, x(t; 0, x_0), t) \\ &= V_x(D(t)\, x(t; 0, x_0), t)\, f_{\mathrm{IRODE}}(D(t)\, x(t; 0, x_0), t) \\ &\quad + V_t(D(t)\, x(t; 0, x_0), t) \\ &= \dot{V}(D(t)\, x(t; 0, x_0), t) \end{aligned}$$

The dissipation inequality (6.7) ensures monotonicity of V along solutions of the IRODE, i.e.

$$\forall 0 \le t \le t_\gamma : \frac{\mathrm{d}}{\mathrm{d}t} V(u(t; 0, x_0), t) \le 0$$

In combination with (6.6) the estimate

$$\forall 0 \le t \le t_\gamma : \phi_1(\|u(t; 0, u_0)\|) \le V(u(t; 0, u_0), t) \le V(u_0, 0) < \phi_1(\gamma) \tag{6.9}$$

holds. Strict monotonicity of ϕ_1 results in $\|u(t; 0, u_0)\| < \gamma$ on the entire interval $[0, t_\gamma]$ and this contradicts the definition of t_γ.

Above argumentation reveals that integral curves of solutions $u(t; 0, u_0)$ of the IVP $u(0; 0, u_0) = u_0 \in$ im $D(0) \cap \mathrm{B}_\delta(0)$ of the IRODE (6.4) cannot leave $\overline{\mathrm{B}_\gamma(0)} \times \mathbb{R}^{\ge 0}$. The second requirement in Assumption 6.8 ensures $\overline{\mathrm{B}_\gamma(0)} \times \mathbb{R}^{\ge 0} \subseteq \mathcal{U}_1$, hence the second and third case in § 6.1 are excluded so these solutions $u(\cdot; 0, u_0)$ of the inherent regular ODE exist on entire $\mathbb{R}^{\ge 0}$. In addition, $u(t; 0, u_0) \in$ im $D(t)$ holds due to the corresponding invariant subspace im $D(t)$. Again, we use (6.8) to construct the unique solutions $x(\cdot; 0, x_0) \in C^1_D([0, \infty), \mathbb{R}^m)$ of the given DAE which fulfill the estimate

$$\forall x_0 \in \mathcal{M}_0(0) : D(0)\, x_0 \in \mathrm{B}_\delta(0) \Longrightarrow \|D(\cdot)\, x(\cdot; 0, x_0)\|_\infty \le \gamma < \epsilon$$

implying D-component stability of x_*.

On asymptotic D-component stability:

We have already proved that $\forall 0 < \gamma \le \epsilon \exists \delta > 0$ such that solutions $x(t; 0, x_0)$ of the IVPs $x_0 \in \mathcal{M}_0(0)$, $D(0)\, x_0 \in \mathrm{B}_\delta(0) \subseteq \mathbb{R}^n$ of the DAE (1.2) exist on $\mathbb{R}^{\ge 0}$ and satisfy

$$\forall t \ge 0 : D(t)\, x(t; 0, x_0) \in \mathrm{B}_\gamma(0)$$

It remains to show that

$$\exists\tilde{\delta} > 0 \forall x_0 \in \mathcal{M}_0(0), D(0)x_0 \in \mathrm{B}_{\tilde{\delta}}(0): \lim_{t\to\infty} D(t)x(t;0,x_0) = 0$$

Again, consider $u_0 := (Dx_*)(0)$ and $u(t;0,u_0) := D(t)x(t;0,x_0)$. In this case $\dot{V}$ is dominated by $-H_3$ so we get the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u(t;0,u_0),t) \leq -\phi_3(\|u(t;0,u_0)\|) < -\phi_3(0) = 0$$

for the general solution $u(t;0,u_0)$ of the IRODE with $u_0 \in \mathrm{im}\ D(0) \cap \mathrm{B}_\delta(0) \setminus \{0\}$. Local uniqueness of solutions of the IRODE implies $u_0 \neq 0 \Rightarrow \forall t \geq 0: u(t;0,u_0) \neq 0$. Therefore $0 \leq V(u(t;0,u_0),t)$ is strictly decreasing and there exists the limit

$$\lim_{t\to\infty} V(u(t;0,u_0),t) = l(u_0) \geq 0$$

Suppose that $\exists u_0 \in \mathrm{im}\ D(0) \cap \mathrm{B}_\delta(0) \setminus \{0\}: l = l(u_0) > 0$. Then,

$$l \leq V(u(t;0,u_0),t) \leq H_2(u(t;0,u_0)) \leq \eta_2(\|u(t;0,u_0)\|)$$

resulting in

$$\|u(t;0,u_0)\| \geq \eta_2^{-1}(l) > 0$$

In other words, the solution of the inherent regular ODE $u(\cdot;0,u_0)$ belongs to the set $\left\{u \in \mathbb{R}^n \mid \eta_2^{-1}(l) \leq \|u\| \leq \gamma\right\}$ where

$$c := \inf_{\eta_2^{-1}(l)\leq\|u\|\leq\gamma} H_3(u) > 0$$

The fundamental theorem of calculus combined with the monotonicity of the integral and $\frac{\mathrm{d}}{\mathrm{d}t}V(u(t;0,u_0),t) \leq -H_3(u(t;0,u_0))$ imply

$$\begin{aligned} V(u(t;0,u_0),t) &= V(u_0,0) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}V(u(s;0,u_0),s)\,\mathrm{d}s \\ &\leq V(u_0,0) - ct \end{aligned}$$

For $t \gg 0: V(u(t;0,u_0),t) < 0$ arises in contradiction to $V(u,t) \geq H_1(u) \geq 0$. Thus,

$$\forall u_0 \in \mathrm{im}\ D(0) \cap \mathrm{B}_\delta(0): \lim_{t\to\infty} V(u(t;0,u_0),t) = 0$$

The lower bound on the Lyapunov function implies

$$0 \leq \phi_1(\|u(t;0,u_0)\|) \leq H_1(u(t;0,u_0)) \leq V(u(t;0,u_0),t)$$

The continuous function $\phi_1(\|u(\cdot;0,u_0)\|)$ is restrained by $u_* \equiv 0$ and $V(u(\cdot;0,u_0),\cdot) \in C^0$ so $\lim_{t\to\infty}\phi_1(\|u(t;0,u_0)\|) = 0$. Due to strict monotonicity of ϕ_1 and $\phi_1(0) = 0$ we deduce $\lim_{t\to\infty} u(t;0,u_0) = 0$, i.e. asymptotic D-component stability of x_*. $\square$

Notice that (6.7) in Def. 6.7 is a *local* structural condition due to $\|D(t)(x - x_*(t))\| < \epsilon$. The mere existence of a global Lyapunov function for numerically qualified DAEs (where im $D(t)$ is constant) implies that D-components of DAE solutions starting sufficiently close to x_* are bounded on their interval of existence — irrespective of the tractability index! This is based solely on the second key note, namely, the implicit resolution for $R(t)(Dx)'(t)$ and an adequate definition of the Lyapunov function for differential-algebraic systems.

Corollary 6.10. *Consider the stationary solution $x_* \equiv 0$ of the properly formulated DAE* (1.2) *with* im $D(t) = const.$ *Given a* global *Lyapunov function $V \in C^1(\mathbb{R}^n, \mathbb{R})$, in particular*

- $\exists H_1 \in C^0(\mathbb{R}^n, \mathbb{R})$ *positive definite with*

$$\forall t \geq 0,\ u \in \text{im } D:\ H_1(u - (Dx_*)(t)) \leq V(u,t)$$

- *For all $t \geq 0$, $x \in \mathcal{M}(t)$, $z = R(t)z$ with $f(z,x,t) = 0$:*

$$\dot{V}(D(t)x,t) :=\ \langle z, \nabla V(D(t)x,t)\rangle + V_t(D(t)x,t) \leq 0$$

it holds

$$\forall \epsilon > 0 \exists \delta > 0 \forall x_0 \in \mathcal{M}(0), \|D(0)x_0\| < \delta \exists x(t;0,x_0) \forall t \in I_{0,x_0} : \|D(t)x(t;0,x_0)\| < \epsilon$$

Here, $x(\cdot;0,x_0) \in C^1_D(I_{0,x_0}, \mathbb{R}^m)$ denotes a solution of the DAE on its maximal right interval $I_{0,x_0} \subseteq \mathbb{R}^{\geq 0}$ of existence and $\mathcal{M}(t)$ the configuration space of the DAE.

In addition, if $V(u,t)$ has a positive definite upper bound $H_2 \in C^0\left(\mathbb{R}^n, \mathbb{R}^{\geq 0}\right)$ and $\dot{V}$ is negative definite then

$$\forall x_0 \in \mathcal{M}(0) \text{ such that } I_{0,x_0} = \mathbb{R}^{\geq 0} :\ \lim_{t \to \infty} V(D(t)x(t;0,x_0),t) = 0.$$

Proof. Per definitionem, for all $x_0 \in \mathcal{M}(0)$ there exists at least one solution $x(t;0,x_0)$ of the DAE passing through $x(0;0,x_0) = x_0$. Due to im $D(t) = $ const. we may choose a constant projector $\hat{P} \in \mathbb{R}^{n \times n}$ onto this subspace so for all $x(\cdot) \in C^1_D$ it follows

$$\begin{aligned}
(Dx)'(t) &= (RDx)'(t) \\
&= \underbrace{R'(t)\hat{P}(Dx)(t)}_{=\left(R(t)\hat{P}\right)'(Dx)(t) = \hat{P}'(Dx)(t) = 0} + R(t)(Dx)'(t) \\
&= R(t)(Dx)'(t)
\end{aligned}$$

Applying the chain rule we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(D(t)x(t;0,x_0),t) &= V_x(D(t)x(t;0,x_0),t)\tfrac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t;0,x_0)) \\
&\quad + V_t(D(t)x(t;0,x_0),t) \\
&= V_x(D(t)x(t;0,x_0),t)R(t)\tfrac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t;0,x_0)) \\
&\quad + V_t(D(t)x(t;0,x_0),t)
\end{aligned}$$

Now $x(t;0,x_0) \in \mathcal{M}(t) \subseteq \mathcal{M}_0(t)$ so there exists a unique $z = R(t)z$ such that $f(z, x(t;0,x_0), t) = 0$. On the other hand,

$$f\left(R(t)\frac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t;0,x_0)), x(t;0,x_0), t\right) = 0$$

so we conclude $z = R(t)\frac{\mathrm{d}}{\mathrm{d}t}(D(t)x(t;0,x_0))$. Consequently,

$$\forall t \in I_{0,x_0} : \frac{\mathrm{d}}{\mathrm{d}t}V(D(t)x(t;0,x_0), t) = \dot{V}(D(t)x(t;0,x_0), t) \le 0$$

due to the global version of (6.7). In combination with above version of (6.6) the estimate

$$\phi_1(\|D(t)x(t;0,x_0)\|) \le V(D(t)x(t;0,x_0), t) \le V(D(0)x_0, 0) < \phi_1(\gamma)$$

is obtained where $\phi_1(\|u\|) \le H_1(u)$. Strict monotonicity of ϕ_1 results in

$$\|D(t)x(t;0,x_0)\| < \gamma$$

on the entire interval of existence.

The convergence of $D(\cdot)x(\cdot;0,x_0)$ towards x_* can be proved by a straightforward adaptation of the respective part of the proof of Theorem 6.9. □

Assumption 6.8 is essential in order to provide **stability** in the conventional sense, that is to attest nonlocal existence of solutions. To this purpose the access to the inherent dynamics via tractability index one on $\mathcal{U}_\epsilon$ involving Assumption 6.8 (1.) and the representation of the inherent dynamics as an IRODE on a cylindrical domain $\mathcal{U}_1 \subseteq \mathbb{R}^n \times [0,\infty)$ ensured by Assumption 6.8 (2.) are reasonable.

Stability estimates for the entire solution vector

Notice that no estimate of algebraic solution components is given in Theorem 6.9, i.e. x_* is D-component stable but, in general, the Lyapunov stability of the entire vector is not given. D-component stability is certainly a weak stability property - that is just why it can be valid even in cases where no Lyapunov stability is present at all. For example, the configuration space $\mathcal{M}(t)$ might vary with t and even expand for $t \to \infty$ but the solution can nevertheless be D-component stable as indicated in Figure 6.4.

If $Q_0(t)w_u(u,t)$ is bounded on its domain, the (asymptotic) D-component stability implies the same property for $Q_0(t)x$ components and therefore stability in the sense of Definition 4.3. This is due to the mean value theorem

$$\begin{aligned} &\|Q_0(t)w(u(t;0,u_0),t) - Q_0(t)w(u_*(t),t)\| \\ &\le \left\|\int_0^1 Q_0(t)w_u(u_*(t) - s(u(t;0,u_0) - u_*(t)))\,\mathrm{d}s\right\| \|u(t;0,u_0) - u_*(t)\| \end{aligned} \tag{6.10}$$

Observe that

$$\begin{aligned} Q_0(t)w_u(u,t) &= -Q_0(t)G_1^{-1}(\eta(u,t))f_x(\eta(u,t))D^-(t) \\ \eta(u,t) &:= \left(D(t)w(u,t), D^-(t)u + Q_0(t)w(u,t), t\right) \end{aligned}$$

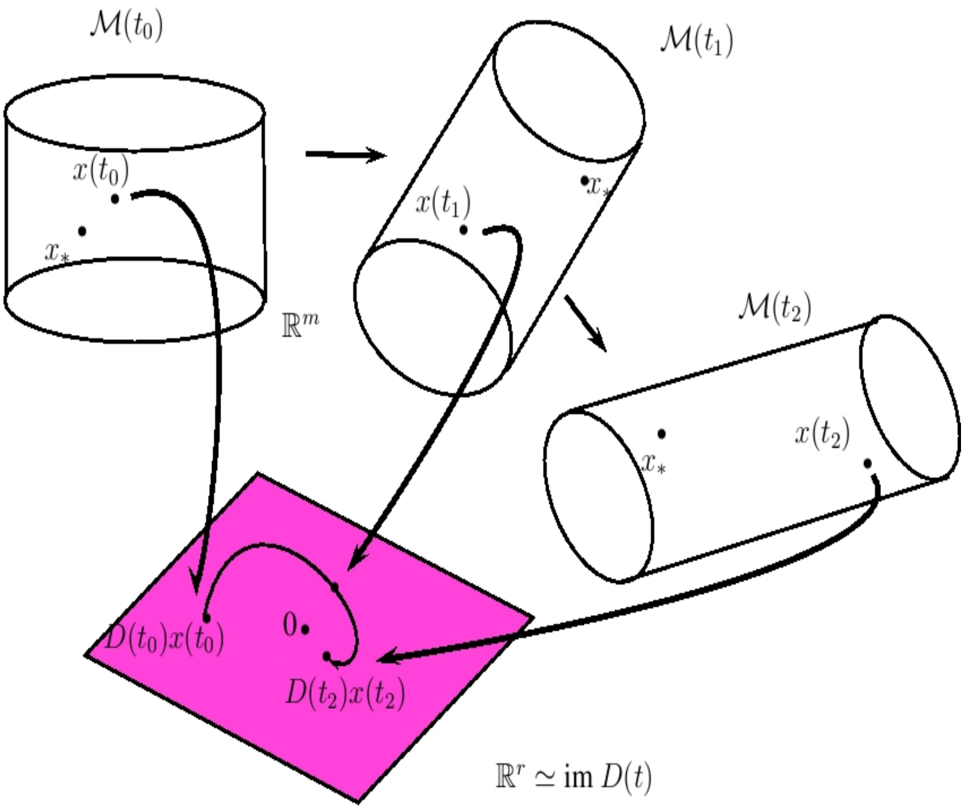

Figure 6.4: Configuration space $\mathcal{M}(t) \subseteq \mathbb{R}^m$ with certain components expanding and some components contracting for $t_0 < t_1 < t_2 \to \infty$ in a way that the stationary solution $x_* \equiv 0$ is instable in the sense of Lyapunov but D-component stable.

As in [MHT03a, p. 183], we recognize that the geometry of systemic subspaces N_0 and S_0 enters the stability estimates for the Q_0-components via the canonical projector $Q_{0,c}$ due to Representation (1.4) together with the reflexive pseudoinverse D^- , i.e.

$$Q_0\,(t)\,w_u\,(u,t) = Q_{0,c}\,(\eta\,(u,t))\,D^-\,(t)$$

On Assumption 6.8

The urgent question is how to ensure the second condition of Assumption 6.8, i.e. a global decoupling of the index-1 DAE? We would like to avoid formulations like "let $\mathcal{M}_0\,(t)$ be bounded/a closed manifold/.." and to consider only a neighbourhood of a reference solution x_* in $\mathcal{M}_0\,(t)$. Due to index one, the extended integral curve can be covered by

$$\mathrm{B}_{\epsilon_i}\left((Dx_*)'\,(t_i)\,, x_*\,(t_i)\,, t_i\right) \subseteq \mathbb{R}^{n+m+1},\ t_i \geq 0,\ i \in \mathcal{J}$$

where the complete decoupling is locally available. The problem is that $\inf_{i\in\mathcal{J}} \epsilon_i = 0$ is possible leading to a domain of the maximal continuation of w which cannot contain a cylindrical region. Heuristically, the extended integral curve of x_* cannot be compact but the extended trajectory is, if x_* is bounded with respect to the C^1_D-norm. This suggests to consider Lyapunov functions for bounded solutions of *autonomous*

DAEs. According to Lemma 2.18, the associated inherent regular ODE is autonomous therefore it sounds reasonable to require an autonomous Lyapunov function.

Theorem 6.11. *[Stability for bounded solutions of autonomous index-1 DAEs]*
A C^1_D-bounded reference solution $x_* \in C^1_D([0,\infty),\mathbb{R}^m)$ of the autonomous DAE (2.28) with a properly formulated derivative term is stable in the sense of Lyapunov if index one is valid in an ϵ-tube around the extended trajectory of x_* and given a Lyapunov function $V(u)$ according to Def. 6.7. Moreover, if $V(u)$ exhibits

- *A positive definite $H_2 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\geq 0}\right)$ such that*

$$\forall t \geq 0,\, u \in \operatorname{im} D \cap \mathrm{B}_\epsilon\left((Dx_*)(t)\right):\; V(u) \leq H_2\left(u - (Dx_*)(t)\right)$$

- *$\dot{V}$ is negative definite, i.e. there exists a positive definite $H_3 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\geq 0}\right)$ with*

$$\forall t \geq 0,\, u \in D\mathcal{M}_0 \cap \mathrm{B}_\epsilon\left((Dx_*)(t)\right):\; \dot{V}(u) \leq -H_3\left(u - (Dx_*)(t)\right)$$

then x_* is asymptotically stable according to Def. 4.3.

Proof. As already mentioned, there are no t-components in domain $\mathcal{G} \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the index-1 domain $\mathcal{U}_\epsilon$ or the cylindrical domain $\mathcal{U}_1 \subseteq \mathbb{R}^n$ of the IRODE in case of autonomous differential-algebraic systems. Due to $\|x_*\|_{C^1_D} < \infty$ and the Bolzano-Weierstraß Theorem, the extended trajectory C of x_* is a compact subset in $\mathcal{G} \subseteq \mathbb{R}^{n+m}$. Hence there exists a finite subcover of

$$\mathrm{B}_{\epsilon_i}\left((Dx_*)'(t_i), x_*(t_i)\right) \subseteq \mathcal{G},\, i \in \mathcal{J} \text{ with } C \subseteq \bigcup_{i=1}^{N} \mathrm{B}_{\epsilon_i}\left((Dx_*)'(t_i), x_*(t_i)\right)$$

such that local decoupling is possible on the respective open sets. Taking a sufficiently small $\epsilon > 0$ we obtain a global decoupling in the entire ϵ-tube around C.

Following precisely the computations in Theorem 6.9, the Lyapunov function V turns out to be a Lyapunov function of the inherent regular ODE of the DAE (2.28) restricted to $\operatorname{im} D \cap \mathcal{U}_1 \subseteq \mathbb{R}^n$. The construction of the Lyapunov sack for Dx_* together with the existence of a global decoupling in an ϵ-tube around C imply that solutions $u(t;0,u_0)$ of the IRODE on $\operatorname{im} D(t)$ with initial values u_0 sufficiently close to $Dx_*(0)$ are bounded, and therefore exist for all $t \geq 0$. Another consequence is that trajectories of these solutions do not leave $\mathcal{U}_1$ where the complete index-1 decoupling $w = w(u)$ is feasible. Using the known solution representation

$$\forall x_0 \in \mathcal{M}_0, Dx \in \mathrm{B}_\delta\left(Dx_*(0)\right):\; x(t;0,x_0) := D^- u(t;0,Dx_0) + Q_0 w\left(u(t;0,Dx_0)\right)$$

for consistent IVP $x(0;0,x_0) = x_0$ of the given DAE sufficiently close to $x_*(0)$, we obtain the (asymptotic) D-component stability property of the reference solution x_*.

Notice that

$$\forall (y,x) \in \bar{\mathcal{U}}_\epsilon:\; Q_0 w_u(Dx) \;=\; -Q_0 G_1^{-1}(y,x)\, f_x(y,x)\, D^-$$

is a continuous function on a compact set, i.e. the norm of $Q_0 w_u(u)$ is bounded on $\bar{\mathcal{U}}_1$ by $c > 0$. Using the stability estimate for the entire vector resulting from solution representation (6.8),

$$\|x(t;0,x_0) - x_*(t)\| \le \left(\left\|D^-\right\| + c\right) \|u(t;0,Dx_0) - Dx_*(t)\|$$

Due to $D = DP_0$ and sub-multiplicative induced matrix norms we are able to control the deviation of $\|x(t;0,x_0) - x_*(t)\|$ by delimiting $\|P_0(x_0 - x_*(0))\|$ thus (asymptotic) D-component stability implies (asymptotic) stability of x_* in the sense of Definition 4.3. □

Another approach to nonlocal existence of DAE solutions nearby the stationary reference solution $x_* \equiv 0$ of a non-autonomous DAE (1.2) requires the boundedness of the entire solution vector and the first requirement of Assumption (6.8) in addition to the existence of a Lyapunov function. Then, solutions $x(t;0,x_0)$ of the differential-algebraic system simply cannot seize to exist on a finite interval $[0,t_e)$, $t_e < \infty$ if the consistent initial values x_0 are chosen close enough to $x_*(0)$.

Theorem 6.12. *[Lyapunov stability for index-1 DAEs with bounded derivatives]*

Consider the stationary solution $x_* \equiv 0$ of a properly formulated DAE (1.2) satisfying the first part of Assumption (6.8) plus boundedness of

$$\begin{array}{l} R'(t)\,D(t)\,,\; D^-(t)\,,\; D(t)\,G_1^{-1}(y,x,t)\,f_x(y,x,t)\,D^-(t) \\ \text{and } Q_0(t)\,G_1^{-1}(y,x,t)\,f_x(y,x,t)\,D^-(t) \end{array} \tag{6.11}$$

on $\mathcal{U}_\epsilon$. Then, x_* is Lyapunov stable according to Def. 4.3 if there exists a Lyapunov function $V(u,t)$ according to Definition 6.7. If the Lyapunov function $V(u,t)$ under consideration satisfies

- $\forall t \ge 0, u \in \operatorname{im} D(t) \cap \mathrm{B}_\epsilon(0) : V(u,t) \le H_2(u)$ *for a positive definite* $H_2 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\ge 0}\right)$
- $\dot{V}$ *is negative definite, i.e.*

$$\forall t \ge 0,\ u \in \operatorname{im} D(t) \cap \mathrm{B}_\epsilon(0) : \dot{V}(u,t) \le -H_3(u)$$

for a positive definite $H_3 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R}^{\ge 0}\right)$

then x_* is asymptotically stable in the sense of Lyapunov.

Proof. We have already shown that the Lyapunov function $V(u,t)$ provides a bound $\|D(t)(x(t;0,x_0) - x_*(t))\| < \gamma$ if the consistent initial values x_0 belong to $\mathcal{M}_0(0)$ such that $D(0)\,x_0 \in \cap \mathrm{B}_\delta((Dx_*)(0))$ for a sufficiently small δ dependent on $\gamma > 0$. Using the solution representation (6.8) and the boundedness of D^-, $Q_0 G_1^{-1} f_x D^-$ in (6.10), we get

$$\|x(t;0,x_0) - x_*(t)\| \le \left(\left\|D^-\right\|_{\infty,\mathbb{R}^{\ge 0}} + \left\|Q_0 G_1^{-1} f_x D^-\right\|_{\infty,\mathcal{U}_\epsilon}\right) \|u(t;0,u_0) - u_*(t)\|$$

Furthermore, the right hand side of the IRODE satisfies the Lipschitz condition with respect to u due to the boundedness of

$$\frac{\partial}{\partial u} f_{\text{IRODE}}(u,t) = D(t)\,w_u(u,t) = R'(t)\,D(t) - D(t)\,G_1^{-1}(y,x,t)\,f_x(y,x,t)\,D^{-1}(t)$$

Hence $\frac{\mathrm{d}}{\mathrm{d}t} D(t)\,(x(t;0,x_0) - x_*(t))$ is bounded, i.e. the extended integral curve of $x(\cdot;0,x_0)$ proceeds in $\mathcal{U}_\epsilon \subseteq \mathcal{G}$ if $\delta > 0$ is suited to a

$$0 < \gamma < \epsilon \left(\left\|D^-\right\|_{\infty,\mathbb{R}^{\geq 0}} + \left\|Q_0 G_1^{-1} f_x D^-\right\|_{\infty,\mathcal{U}_\epsilon}\right)^{-1} \left(\left\|R'D\right\|_{\infty,\mathbb{R}^{\geq 0}} + \left\|D G_1^{-1} f_x D^-\right\|_{\infty,\mathcal{U}_\epsilon}\right)^{-1}$$

Let us assume that there exists an initial value $x_0 \in \mathcal{M}_0(0) \cap \mathrm{B}_\delta(x_*(0))$ such that $x(t;0,x_0)$ exists on the maximal right interval $[0,t_e)$ with $t_e < \infty$. Boundedness plus continuity of the extended trajectory of $x(t;0,x_0) - x_*(t)$ imply that

$$\lim_{t\to t_e} \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t;0,x_0) - (Dx_*)(t)\right), x(t;0,x_0) - x_*(t), t\right) = (\nu, \xi, t_e)$$

exists and satisfies $\|\nu\|, \|\xi\| < \epsilon$ where $\epsilon > 0$ refers to the index-1 region $\mathcal{U}_\epsilon$ in the first part of Assumption (6.8). Therefore,

$$\lim_{t\to t_e} x(t;0,x_0) = \xi + x_*(t_e)\,, \quad \lim_{t\to t_e} \frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t;0,x_0)\right) = \nu + (Dx_*)'(t_e)$$

Due to continuity of f, x and $(Dx)'$ together with

$$\forall 0 \leq t < t_e: \quad f\left((D(t)\,x(t;0,x_0))', x(t;0,x_0), t\right) = 0$$

it follows that

$$\lim_{t\to t_e} f\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t;0,x_0)\right), x(t;0,x_0), t\right) = 0$$

Accordingly, the complete index-1 decoupling is available locally around

$$\left(\nu + (Dx_*)(t_e), \xi + x_*(t_e), t_e\right) \in \mathcal{U}_\epsilon$$

thus $x(t;0,x_0)$ exists at least on $[0,t_e]$ which contradicts the assumption.

Having ensured the existence of DAE solutions on $[0,\infty)$, the asymptotic D-component stability can be proved as in Theorem 6.9. Obviously, the boundedness requirements provide that stability properties of D-components carry over to the entire solution vector, cf. the argument of Theorem 6.11. □

A similar rigid structural condition entered the setup of Theorem 46 in [GM86] where global solvability is achieved for contractive DAEs $f(x'(t), x(t), t) = 0$ with bounded G_1^{-1}, f_x and f_y. The boundedness of partial derivatives is also an important issue in requirements of [CC07, Th. 1]. Unfortunately, such a restriction becomes unacceptable for the analysis of index-2 DAEs because (6.11) has to be imposed on the index reduced system which prohibits a formulation in terms of the given DAE and implies at least boundedness of the second derivatives of f.

Example 6.13. Consider the simple linear index-1 DAE

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)' (t) + \begin{pmatrix} 1 & 0 \\ -\exp(2t) & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

defined on $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^{\geq 0}$. Assumption 6.8 is obviously fulfilled. The solution set is

$$\mathcal{M}_0(t) = \left\{ c \begin{pmatrix} \exp(-t) \\ \exp(t) \end{pmatrix} \mid c \in \mathbb{R} \right\}$$

and $D(t)x = x_1$. Here, $V(x_1) := x_1^2$ is a Lyapunov function being positive definite and $\dot{V}(x_1) = -x_1^2$ negative definite. Now the asymptotic D-component stability of x_* is guaranteed by Theorem 6.9, i.e. x_1 is asymptotically stable. In this example $f_x(t)$ is unbounded in t and the entire solution vector $x_1(0)(\exp(-t), \exp(t))^T$ is instable in the sense of Lyapunov.

Example 6.14. Consider the following, non-Hessenberg index-1 DAE

$$\begin{pmatrix} 1 \\ 2x_1(t) \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)' (t) + \begin{pmatrix} x_1(t) + x_2(t) \\ -x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This differential-algebraic systems contains the nonlinearity $2x_1(t)x_1'(t) = \frac{\mathrm{d}}{\mathrm{d}t}x_1^2(t)$. It is easy to compute $G_1(y, x_1, x_2) = \begin{pmatrix} 1 & 1 \\ 2x_1 & 1 \end{pmatrix}$ which is a nonsingular matrix for all $x_1 \neq -0.5$. Therefore, the DAE is a properly stated system with tractability index one on $\mathcal{G} = \mathbb{R} \times \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1|, |x_2| < 0.5\}$. The system exhibits the stationary solution $x_* = (0, 0)$ and the inherent regular ODE $x_1' = -\frac{x_1}{1+2x_1}$. Condition (6.7) for a Lyapunov function $V = V(x_1)$ reads

$$-\frac{x_1}{1+2x_1} V_{x_1}(x_1) \leq 0$$

On $\mathcal{G}$, $1 + 2x_1 > 0$ is true and the dissipation inequality of V is equivalent to $-x_1 V_{x_1}(x_1) \leq 0$. Again, $V(x_1) = \frac{1}{2}x_1^2$ is a Lyapunov function of the IRODE with $\dot{V}$ negative definite guaranteeing asymptotic stability of solution component x_1. Due to the autonomous structure of the given DAE, x_2 depends on x_1 only and the partial derivatives are bounded on $\mathcal{G}$ implying asymptotic stability of x_*. Considering the IRODE, separation of the variables results in

$$\exp(x_1)\, x_1 = \exp(t + c), \quad c \in \mathbb{R}$$

This equation can be solved with respect to x_1 due to the implicit function theorem, but it is difficult to obtain an explicit representation.

6.4 Lyapunov functions for index-2 systems

Having formulated Lyapunov functions for index-1 DAEs, the question arises how to define such functions appropriately to serve as a stability criterion in the more complex case of the tractability index two. Again, the challenging part is the interplay of certain dissipativity inequalities and access to the inherent dynamics of the DAE, thereby demanding both aspects to be expressed preferably in terms of the original differential-algebraic system. To cope with Lyapunov's direct method for index-2 DAEs the preliminary work on complete decoupling in Chapter 2 and on reduction of the tractability index via differentiation of constraints in Chapter 3 is deployed.

6.4.1 Lyapunov function aiming at DP_1-component stability

Like in the index-1 case, an appropriate definition of a Lyapunov function has to refer to the inherent dynamics of the differential-algebraic system near a reference solution x_*. In context of the complete decoupling of nonlinear index-2 systems we proved that the solution components

$$u(t) = (DP_1)(t)\, x(t) := (DP_1)\big((Dx_*)'(t), x_*(t), t\big)\, x(t)$$

are determined by the inherent regular ODE

$$u'(t) = f_{\mathrm{IRODE}}(u(t), t) := g(u(t), t)$$

and remaining parts of the solution vector $x(\cdot) \in C^1_D(I, \mathbb{R}^m)$ by the constraints $x(u,t) := s(u,t)$ in (2.23), supposed the preconditions of Theorem 2.10 hold. Denote $P_1(t) = P_1\big((Dx_*)'(t), x_*(t), t\big)$ and so on.

The relation $f(z_0, x_0, t_0) = 0$ for $x_0 \in \mathcal{M}_1(t_0) \subset \mathcal{M}_0(t_0)$ determines a unique $z_0 = R(t_0)\, x_0$. Under the assumption im $D(t) =$ const. it follows

$$z_0 = R(t_0)(Dx)'(t_0) = (Dx)'(t_0) = (DP_1 x)'(t_0) + (DQ_1 x)'(t_0)$$

In particular we obtain this implicit representation of the IRODE of the given index-2 DAE nearby x_*:

$$\boxed{f_{\mathrm{IRODE}}\left((DP_1)(t)x(t), t\right) = \big(DP_1 D^-\big)(t) z} \tag{6.12}$$

with the unique $z = R(t)\, z$ fulfilling $f(z, x, t) = 0$. Let us formalize the assumptions using Notation 6.6:

Definition 6.15. [Lyapunov function for DP_1-component stability]

$V \in C^1(\mathcal{U}_1, \mathbb{R})$ is called a *Lyapunov function* belonging to the reference solution x_* of the properly stated index-2 DAE (1.2) if

1. $\forall t \geq 0 : V(x_*(t), t) = 0$
2. $\exists H_1 \in C^0(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R})$ positive definite with
$$\forall t \geq 0,\ u \in \text{im } (DP_1)(t) \cap \mathrm{B}_\epsilon((DP_1 x_*)(t)) :\ H_1(u - (Dx_*)(t)) \leq V(u,t)$$
3. $\forall t \geq 0, x \in \mathcal{M}_1(t), (DP_1)(t)\, x \in \mathrm{B}_\epsilon((DP_1 x_*)(t)), z = R(t)z$ satisfying $f(z,x,t) = 0$ it holds
$$\dot{V}((DP_1)(t)\, x, t) := \begin{array}{l} \langle (DP_1 D^-)(t)\, z, \nabla V((DP_1)(t)\, x, t)\rangle \\ + V_t((DP_1)(t)\, x, t) \end{array} \leq 0 \tag{6.13}$$

Heuristically, it is clear that the presented definition expresses the conditions on a Lyapunov function of the IRODE restricted to DK. Now we have to prove that Definition 6.15 ensures DP_1-component stability of the stationary solution. To this end, we necessitate

Assumption 6.16. *Consider the DAE* (1.2) *having the stationary solution* $x_* \equiv 0$. *Let* $Q_1(y,x,t)$ *be an admissible projector on* $N_1(y,x,t)$ *along* $K(y,x,t)$. *We require*

1. The DAE (1.2) possesses the tractability index 2 on $\mathcal{U}_\epsilon \subseteq \mathcal{G}$ for sufficiently small $\epsilon > 0$
2. Additionally,

$$\forall (y,x,t) \in \mathcal{U}_\epsilon : \ \mathrm{im} \begin{pmatrix} T \\ -f_y^- f_x T \end{pmatrix} (y,x,t) = \mathrm{im} \begin{pmatrix} T \\ f_y^- f_x T \end{pmatrix} (0,0,t)$$

3. im $D(t)$, $D(t)\,N_1(0,0,t)$ and $D(t)\,K(0,0,t)$ are constant
4. The DAE (1.2) satisfies the differentiability requirements in Theorem 2.10
5. Domains of the maximal continuations of implicitly defined functions $\tilde{m}$ and $\tilde{k}$ of the complete index-2 decoupling (2.19) contain a cylindrical region around the integral curve of (u_*, w_*, t) resp. (u_*, t).

Analogously to the index-1 case, the cylindricity of domains in Assumption 6.16 5.) is very important but also quite hard to ascertain.

Theorem 6.17. *[Lyapunov function as a criterion for DP_1-component stability]*

The stationary solution $x_* \equiv 0$ of the DAE (1.2) *satisfying Assumption 6.16 is DP_1-component stable if there is a Lyapunov function $V(u,t)$ in the sense of Definition 6.15. Furthermore, x_* is asymptotically DP_1-component stable if $V(u,t)$ fulfills*

- $\forall t \geq 0, u \in DK : \ V(u,t) \leq H_2(u)$ *for* $H_2 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^\mathrm{n}, \mathbb{R}^{\geq 0}\right)$ *positive definite*
- $\dot{V}$ *is negative definite, i.e.*

$$\forall t \geq 0,\ u \in (DP_1\mathcal{M}_1)(t) \cap \mathcal{U}_1 : \ \dot{V}(u,t) \leq -H_3(u)$$

for a positive definite $H_3 \in C^0\left(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^\mathrm{n}, \mathbb{R}^{\geq 0}\right)$

Proof. Assumption 6.16 1.)—4.) enclose the requirements of the local complete decoupling approach in Theorem 2.10 on the entire $\mathcal{U}_\epsilon \subseteq \mathcal{G}$. Consider the maximal extension of the implicitly defined functions $s(u,t)$ and $g(u,t)$ in Representation (2.19) of the inherent regular ODE. At the same time Assumption 6.16 5.) expresses that domains of these functions contain the $\hat{\epsilon}$-tube around the respective components of the stationary solution x_* for a sufficiently small $\hat{\epsilon} > 0$. In other words, the IRODE is defined at least on the $\hat{\epsilon}$-tube around the integral curve of $DP_1 x_*$. W.l.o.g. assume $\epsilon = \hat{\epsilon}$.

From here on every step of Theorem 6.9 is repeated with some minor changes because we focus on solution components $u_0 = (DP_1)(0)\,x_0$, $u(t;0,u_0) = (DP_1)(t)\,x(t;0,x_0)$

and consider the associated inherent regular ODE. Precisely, the solution representation

$$x\left(t;0,x_0\right)=s\left(u\left(t;0,\left(DP_1\right)(0)\,x_0\right),t\right)$$

and the IRODE (2.19) replace the index-1 counterparts. As stated in (6.12),

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}V\left(\left(DP_1\right)(t)\,x\left(t;0,x_0\right),t\right)=\\
&=\begin{cases} V_x\left(\left(DP_1\right)(t)\,x\left(t;0,x_0\right),t\right)f_{\mathrm{IRODE}}\left(\left(DP_1\right)(t)\,x\left(t;0,x_0\right),t\right)\\ +V_t\left(\left(DP_1\right)(t)\,x\left(t;0,x_0\right),t\right)\end{cases}\\
&=\dot{V}\left(\left(DP_1\right)(t)\,x\left(t;0,x_0\right),t\right).
\end{aligned}$$

Therefore trajectories of solutions $u\left(t;0,u_0\right)$ of the inherent regular ODE (2.19) with initial values $u\left(0;0,u_0\right)=u_0\in DK\cap\mathrm{B}_\delta\left(0\right)$ stay in the compact set $\overline{\mathrm{B}_\gamma\left(0\right)}$. Consequently, they exist on $\mathbb{R}^{\geq 0}$. It holds $u\left(t;0,u_0\right)\in\mathrm{im}\ \left(DP_1\right)(t)=DK$ because DK is an invariant subspace of the IRODE and the integral curves proceed in the domain of the solution representation so we construct $C_D^1\left([0,\infty),\mathbb{R}^m\right)$-solutions of the DAE fulfilling

$$\forall x_0\in\mathcal{M}_1\left(0\right):\ \left(DP_1\right)(0)\,x_0\in\mathrm{B}_\delta\left(0\right)\subseteq\mathbb{R}^n\Rightarrow\left\|\left(DP_1\right)(\cdot)\,x\left(\cdot;0,x_0\right)\right\|_\infty\leq\gamma<\epsilon$$

i.e. x_* is DP_1-stable.

Asymptotic DP_1-stability is proved using the same approach as in Theorem 6.9. □

In general, Definition 6.15 is not stated in terms of the given DAE according to § 6.4.3 by reason of the projector $\left(DP_1\right)(t)=\left(DP_1\right)(0,0,t)$. Using the canonical projector,

$$\mathrm{im}\ \left(DP_{1,\mathrm{c}}\right)(y,x,t)=D\left(t\right)S_1\left(y,x,t\right)=D\left(t\right)\mathrm{pr}_2\left(\mathrm{T}_{(y,x)}\mathcal{N}_1\left(t\right)\right)$$

where

$$\begin{aligned}
&\mathcal{N}_1\left(t\right):=\left\{(z,x)\in\mathbb{R}^m\times\mathbb{R}^m\,|\,\tilde{f}\left(z,x,t\right):=f\left(D\left(t\right)z,P_0\left(t\right)x+Q_0\left(t\right)z,t\right)=0\right\}\\
&\mathrm{T}_{(y,x)}\mathcal{N}_1\left(t\right)=\ker\mathrm{D}_{(y,x)}\tilde{f}\left(D^-\left(t\right)y+Q_0\left(t\right)x,P_0\left(t\right)x,t\right).
\end{aligned}$$

6.4.2 A criterion for D-component stability

Another possibility to define an appropriate Lyapunov function is to make use of the index-reduced system associated to $f\left((Dx)'\left(t\right),x(t),t\right)=0$ with a properly formulated derivative term. The reduction of the tractability index via differentiation requires $N_0(t)$ to be constant and im $G_1(y,x,t)$ dependent on (P_0x,t) only, cf. Theorem 3.4. Using a projector $W_1(P_0x,t)$ along im $G_1(y,x,t)$, the following representation of the index-reduced DAE is obtained:

$$\begin{aligned}
0\ &=\ \tilde{f}\left((Dx)'\left(t\right),x(t),t\right),\\
\tilde{f}\left(y,x,t\right)\ &:=\ \begin{array}{l}\left(I-W_1\left(P_0x,t\right)\right)f\left(y,x,t\right)+W_1\left(P_0x,t\right)\left(W_1f\right)_x\left(P_0x,t\right)D^-(t)y\\ +W_1\left(P_0x,t\right)\left(W_1f\right)_x\left(P_0x,t\right)\left(D^-\right)'(t)D(t)x+W_1\left(P_0x,t\right)\left(W_1f\right)_t\left(P_0x,t\right)\end{array}
\end{aligned}$$

Due to Theorem 3.6, $\mathcal{M}_1(t) = \tilde{\mathcal{M}}_0(t) \cap \mathcal{M}_0(t)$ is an invariant set of the IR-DAE, in which $\tilde{\mathcal{M}}_0(t)$ denotes the first-level constraint of the IR-DAE. Inequality (6.7) for a Lyapunov function of the index reduced system restricted to $\mathcal{M}_1(t)$ reads

$$\dot{V}(D(t)x, t) \leq 0 \text{ for all } t \geq 0, x \in \mathcal{M}_1(t), z = R(t)z \text{ with } \tilde{f}(z, x, t) = 0$$

Under the assumptions of Theorem 3.4, solutions $x(t; t_0, x_0)$ of the given DAE (1.2) with consistent initial values $x_0 \in \mathcal{M}_1(0)$ correspond to the unique solutions of the associated index-reduced DAE $\tilde{f}\left((Dx)'(t), x(t), t\right) = 0$ restricted to $\mathcal{M}_1(t)$. Thus,

$$f\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t; t_0, x_0)\right)\Big|_{t=t_0}, x_0, t_0\right) = 0 = \tilde{f}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t; t_0, x_0)\right)\Big|_{t=t_0}, x_0, t_0\right) \tag{6.14}$$

For $x_0 \in \mathcal{M}_1(0) \subseteq \mathcal{M}_1(0) \cap \tilde{\mathcal{M}}_0(0)$, the terms $z = R(0)z$ and $\tilde{z} = R(0)\tilde{z}$ are unique in $f(z, x_0, 0) = 0$ respectively $\tilde{f}(\tilde{z}, x_0, 0) = 0$. Taking (6.14) into consideration, this means

$$z = R(0)\frac{\mathrm{d}}{\mathrm{d}t}\left(D(t)\,x(t; t_0, x_0)\right)_{|t=t_0} = \tilde{z}$$

In particular, the monotonicity condition (6.7) on $V(u, t)$ along solutions of the index reduced DAE restricted to $\mathcal{M}_1(t)$ can be written in the following implicit form:

$$\langle R'(t)D(t)x + z, V_x(D(t)\,x, t)\rangle + V_t(D(t)\,x, t) \leq 0$$

for all $t \geq 0, x \in \mathcal{M}_1(t), z = R(t)z$ with $f(z, x, t) = 0$. In other words, Definition 6.7 turns out to be adequate. Here, the Lyapunov function applies to the D-components of solutions of the associated index-reduced DAE on $\mathcal{M}_1(t)$ which are known to coincide with solutions of the given DAE. Again, structural conditions using Notation 6.6 are required to perform the index reduction via differentiation plus Assumption (6.8) addressing the IR-DAE.

Assumption 6.18. *Consider the reference solution $x_* \in C_D^1\left(\mathbb{R}^{\geq 0}, \mathbb{R}^m\right)$ of a properly formulated DAE (1.2). For a sufficiently small $\epsilon > 0$ require*

1. *Tractability index 2 on $\mathcal{U}_\epsilon \subseteq \mathcal{G}$*
2. $\ker D(t) = $ *const. and* $\forall\,(y, x, t) \in \mathcal{U}_\epsilon :$ $\operatorname{im} G_1(y, x, t)$ *dependent on* $(P_0 x, t)$
3. $\operatorname{rk}\, f_y(y, x, t)\, D(t) + W_1(P_0 x, t)\,(W_1 f)_x(P_0 x, t) = $ *const. on* $\mathcal{U}_\epsilon$
4. *D^-, W_1 continuously differentiable and $W_1 f \in C^2$*

We are going to state an analogon to Theorem 6.9 for index-2 systems.

Theorem 6.19. *[Principal D-stability criterion for index-2 systems]*

The stationary solution $x_* \equiv 0$ of the properly formulated DAE $f\left((Dx)'(t), x(t), t\right) = 0$ satisfying Assumption 6.18 and Assumption 6.8 2.) for its index-reduced system is D-component stable, if there exists a Lyapunov function $V(u, t)$ according to Def. 6.7. Additionally, if $V(u, t)$ exhibits a positive definite upper bound $H_2 \in C\left(\mathrm{B}_\epsilon(0), \mathbb{R}^{\geq 0}\right)$ and $\dot{V}$ is negative definite then x_* is asymptotically D-component stable.

Proof. Assumption (6.18) comprises the requirements of Theorem 3.4 on entire $\mathcal{U}_\epsilon$ resulting in the existence of the index-reduced DAE (3.2) associated to (1.2). The IR-DAE features the tractability index one in a neighbourhood of any point in $\mathcal{U}_\epsilon \cap \{(y,x,t) \in \mathcal{G} \mid f(y,x,t) = 0\}$. Assumption 6.8 2.) guarantees that there is an $\epsilon > 0$ such that index one is valid on entire $\mathcal{U}_\epsilon \subseteq \mathcal{G}$ and the domain of the maximal continuation of the resolution $w = w(u,t)$ belonging to the index-reduced DAE contains the cylindrical region $\mathcal{U}_1$.

By means of Theorem 6.9, we are able to attest the property

$$\forall 0 < \gamma < \epsilon \exists \delta > 0 \forall u_0 \in (D\mathcal{M}_1)(0) \cap \mathrm{B}_\delta(0), \, t \geq 0: \, \|u(t;0,u_0)\| < \gamma$$

for the general solution $u(t;0,u_0)$ of the initial value problem $u(0;0,u_0) = u_0$ of the IRODE of the associated index-reduced DAE. V matches relevant properties of a Lyapunov function of the stationary solution x_* of the IR-DAE restricted to $\mathcal{M}_1(t) \subseteq \tilde{\mathcal{M}}_0(t)$. Trajectories of solutions of the inherent regular ODE belonging to the index-reduced DAE with initial values satisfying $u_0 \in (D\mathcal{M}_1)(0) \cap \mathrm{B}_\epsilon(0)$ proceed in the compact set $\mathrm{B}_\gamma(0)$ and therefore exist for all $t \geq 0$. It is possible to apply the solution representation (6.8) of the index-1 IR-DAE due to the second part of Assumption 6.8 aiming at the IR-DAE. Hence, the existence of solutions of the IR-DAE with initial value problems $x_0 \in \mathcal{M}_1(0)$, $\|D(0)x_0\| < \delta$ is proved. Keep in mind that $\mathcal{M}_1(t) \subseteq \tilde{\mathcal{M}}_0(t)$ is an invariant subset of the index-reduced system and that solutions of the IR-DAE on $\mathcal{M}_1(t)$ are exactly the solutions $x(t;0,x_0)$ of the original DAE. As a by-product we get the estimate $\forall t \geq 0: \|D(t)x(t;0,x_0)\| \leq \gamma$, i.e D-component stability of x_*.

Once again, repeat the argument of Theorem 6.11 in order to provide asymptotic D-component stability of x_* in case of $V(u,t)$ having a positive definite upper bound and a negative definite $\dot{V}(u,t)$. □

The proved stability criterion obviously holds if $\dot{V}(u,t)$ is required for all $t \geq 0, x \in \mathcal{M}_0(t) \cap \mathrm{B}_\epsilon(0)$ with $\|D(t)x\| \leq \epsilon$. Then, only first-level constraints are involved in the definition of the Lyapunov function. Generally, the coupling of differential equations and constraints usually results in involved dissipation inequalities which has to be simplified using relations valid for elements of the configuration space $\mathcal{M}_1(t)$ only. So we cannot expect that neglecting properties of the configuration set $\mathcal{M}_1(t)$ always works, cf. Example 6.21.

Assumption (6.8) for the IR-DAE is too abstract and, again, it disturbs the formulation in terms of the original DAE. As discussed in Section 6.3, this is problem can be avoided considering bounded solutions of autonomous DAEs.

Theorem 6.20. *[Stability for bounded solutions of autonomous index-2 DAEs]*

The C^1_D-bounded solution $x_* \in C^1_D(\mathbb{R}^{\geq 0}, \mathbb{R}^m)$ of the properly formulated autonomous DAE (2.28) satisfying Assumption 6.18 is stable[2], if there exists a Lyapunov function $V(u)$ according to Def. 6.7. Additionally, if $V(u)$ exhibits a positive definite upper

[2] Here, a slightly modified notion of Lyapunov stability is aspired. Precisely, the projector $\Pi_2 = P_0 P_1 \left((Dx_*)'(0), x_*(0)\right)$ in Definition 4.3 is replaced by the rectangular matrix $\Pi_2 = D$.

bound $H_2 \in C\left(\mathrm{B}_\epsilon(0), \mathbb{R}^{\geq 0}\right)$ *and* $\dot{V}(u)$ *is negative definite for all* $t \geq, u \in D\mathcal{M}_1 \cap \mathrm{B}_\epsilon\left(\left(Dx_*\right)(t)\right)$ *then* x_* *is asymptotically stable.*

Proof. Assumption 6.18 enables to use Theorem 3.4. As a consequence, the tractability index one of the autonomous IR-DAE (3.2), i.e.

$$(I - W_1(P_0 x)) f\left((Dx(t))', x(t)\right) + W_1(P_0 x)(W_1 f)_x(P_0 x) D^- (Dx(t))' = 0$$

is valid locally around any point of the extended trajectory $\hat{C}$ of x_* because of

$$f\left((Dx_*(t))', x_*(t)\right) = 0 \text{ and } \left((Dx_*(t))', x_*(t)\right) \in \mathcal{U}_\epsilon$$

where the index of the original DAE is 2. Remember that $\hat{C}$ is compact in accordance with $\|x_*\|_{C^1_D} < \infty$ and the Bolzano-Weierstraß Theorem so there is an $0 < \hat{\epsilon} < \epsilon$ such that the index-reduced DAE has the tractability index one on the entire

$$\mathcal{U}_{\hat{\epsilon}} = \left\{\mathrm{B}_{\hat{\epsilon}}\left((Dx_*)'(t)\right) \times \mathrm{B}_{\hat{\epsilon}}(x_*(t)) \subseteq \mathcal{G} \subseteq \mathbb{R}^n \times \mathbb{R}^m \mid t \in \mathbb{R}^{\geq 0}\right\}$$

It is straightforward to apply the argument of Theorem 6.11 to the index-reduced system with a slight adaptation that only solutions with consistent initial values on the invariant set $\mathcal{M}_1$ of the IR-DAE have to be considered. This restriction is necessary because the dissipativity inequality (6.7) does not apply in case of IR-DAE solutions with general initial values $x_0 \in \tilde{\mathcal{M}}_0$.

Consequently, for all $\gamma > 0$ there exists a $\delta > 0$ such that IVPs $x_0 \in \mathcal{M}_1$, $Dx_0 \in \mathrm{B}_\delta(x_*(0))$ of the autonomous DAE (2.28) are uniquely solvable on $\mathbb{R}^{\geq 0}$ and

$$\|x(t; 0, x_0) - x_*(t)\| < \gamma$$

If $\dot{V}(u)$ is negative definite and $V(u)$ has a positive definite upper bound then

$$\lim_{t \to \infty} \|x(t; 0, x_0) - x_*(t)\| = 0$$

results from asymptotic D-component stability of x_*. $\square$

Example 6.21. Consider the following autonomous differential-algebraic system

$$\begin{array}{lcl} 2x_1 x_1' + (1 + 2x_2) x_2' - x_1 - x_4 & = & 0 \\ x_1' + x_2 & = & 0 \\ x_3' - x_4 & = & 0 \\ x_3 - x_1^2 - x_2^2 & = & 0 \end{array} \tag{6.15}$$

which is non-Hessenberg and (weakly) coupled. First, we construct the matrix chain of the tractability index:

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Q_0 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}$$

$$f_y = \begin{pmatrix} 2x_1 & 1+2x_2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ f_x = \begin{pmatrix} 2x_1'-1 & 2x_2' & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ -2x_1 & -2x_2 & 1 & 0 \end{pmatrix}$$

It holds $\dim N_0 \cap S_0 = \text{rk } G_1 \equiv 3$ with

$$G_1 = \begin{pmatrix} 2x_1 & 1+2x_2 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ N_1 = \text{span} \begin{pmatrix} 0 \\ 1 \\ 1+2x_2 \\ 1+2x_2 \end{pmatrix}$$

Choosing Q_1 in an admissible way ($Q_1 Q_0 = 0$), we obtain

$$Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1+2x_2 & 0 & 0 \\ 0 & 1+2x_2 & 0 & 0 \end{pmatrix},\ G_2 = \begin{pmatrix} 2x_1 & 1+2x_2+2x_2' & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Laplace extension results in

$$\det G_2 (x_1, x_2, x_2') \equiv -1$$

Therefore the tractability index 2 of (6.15) is ensured in entire $\mathcal{U}_\epsilon$. Moreover,

$$\text{im } G_1 (x_1, x_2) = \text{span} \left(\begin{pmatrix} 2x_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1+2x_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

$W_1 = Q_0$ projects along $\text{im } G_1 (x_1, x_2)$ and

$$\text{rk } f_y D + W_1 f_x = \text{rk} \begin{pmatrix} 2x_1 & 1+2x_2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2x_1 & -2x_2 & 1 & 0 \end{pmatrix} \equiv 3.$$

As a result, Assumption 6.18 is valid. In order to apply Theorem 6.20 we have to check the dissipation inequality (6.7) for a Lyapunov candidate function $V(x_1, x_2, x_3)$. Solving $f(z, x) = 0$ for z, we get

$$z_1 = -x_2,\ z_2 = \frac{x_1 + x_4 + 2x_1 x_2}{1+2x_2},\ z_3 = x_4$$

due to $1 + 2x_2 \neq 0$ on $\mathcal{U}_\epsilon$, $\epsilon < 0.5$. Therefore, (6.7) reads as follows:

$$-x_2 (1+2x_2) V_{x_1} + (x_1 + x_4 + 2x_1 x_2) V_{x_2} + x_4 (1+2x_2) V_{x_3} \leq 0$$

This inequality can be simplified using the property $\forall (x_1, \ldots, x_4) \in \mathcal{M}_1 : x_4 = 0$. It follows

$$-x_2 (1+2x_2) V_{x_1} + (x_1 + 2x_1 x_2) V_{x_2} \leq 0$$

and V_{x_3} does not influence the inequality so we assume $V = V(x_1, x_2)$. Now

$$V(x_1, x_2) := \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

turns out to be a Lyapunov function for the stationary solution x_* of the DAE (6.15) exhibiting $\dot{V}(x_1, x_2) \equiv 0$ and Theorem 6.20 ensures stability of x_*. This is already the best we can achieve because the general solution of the DAE (6.15) is

$$\left(c_1 \cos(t) - c_2 \sin(t), c_1 \sin(t) + c_2 \cos(t), c_1^2 + c_2^2, 0\right)^T$$

so the stationary solution does not exhibit asymptotic stability. Besides, Laplace expansion reveals that the eigenvalues of the matrix pencil $\{f_y(0,0)D, f_x(0,0)\}$ of the linearization in x_* are $\lambda_{1,2} = \pm i$ thus $\mathrm{Re}(\lambda_i) = 0$ so Perron's Theorem for index-2 DAEs (e.g. [Mä98, Th. 3.3]) is not applicable.

The above calculations exemplify that our decoupling procedure is not unique. For example, the dynamics are expressed by means of

$$DP_1(x_*(t), t)\, x = \left(x_1, 0, x_3 - (1 + 2(x_*)_1(t))\, x_2\right)^T$$

instead of $(x_1, x_2)^T$.

Our investigations in Part I reveal that the coupling between differential equations and constraints is an important aspect in DAE theory. To this purpose, consider the influence of the coupling in (6.15). Regarding the third and fourth equations, (6.15) turns out to be equivalent to the decoupled Hessenberg-2 DAE

$$\begin{array}{rcl} x_1'(t) & = & -x_2(t) \\ x_2'(t) & = & x_1(t) \\ x_3'(t) & = & x_4(t) \\ 0 & = & x_1^2(t) + x_2^2(t) - x_3(t) \end{array} \tag{6.16}$$

Obviously, (6.16) satisfies Assumption 6.18. A Lyapunov function $V = V(x_1, x_2, x_3)$ for the stationary solution $x_* \equiv 0$ is given, if condition (6.7) is valid, that is

$$\forall \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{M}_1 : \; -x_2 V_{x_1}(x_1, x_2, x_3) + x_1 V_{x_2}(x_1, x_2, x_3) + x_4 V_{x_3}(x_1, x_2, x_3) \leq 0$$

This condition is simplified to $-x_2 V_{x_1}(x_1, x_2) + x_1 V_{x_2}(x_1, x_2) \leq 0$ by the ansatz $V = V(x_1, x_2)$. If $V_{x_1}(x_1, x_2) = x_1$ and $V_{x_2}(x_1, x_2) = x_2$, e.g. $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, we obtain a Lyapunov function according to Def. 6.7. Again, Theorem 6.20 implies stability of the stationary solution. We do not have to exploit the structure of $\mathcal{M}_1(t)$ because the DAE is in a decoupled form.

6.4.3 Excursus: formulation in terms of the initial system

The expression “in terms of the given DAE” occurring in the thesis refers to the formulation of stability criteria for the state space form of the DAE avoiding explicit access to the tangent bundle of the configuration space. It is meant that no parametrization or projection onto the tangent space of a subspace of the first level constraint $\mathcal{M}_0(t)$ is applied and that the state space representation is not accessed directly. Furthermore, we would like to exclude direct access to the state space form or to quantities defined by the implicit function theorem.

In the context of Lyapunov functions, we allow the configuration space $\mathcal{M}_1(t)$ as the geometric location of solutions of the given DAE at time t to be involved in the definition. Remember that the configuration space of an ODE nearby the stationary solution x_* is simply a neighbourhood $\Omega \subseteq \mathbb{R}^m$ of the origin. In case of differential-algebraic equations we have to incorporate the constraints together with the focus on $D(t)\,x \in \mathbb{R}^n$ solution components. This is reflected in Definition 6.7. Conditions (6.6) and (6.7) are originally aiming at the Representation (6.5) of the inherent dynamics via Lemma 1.19. Notice that only a local section of $\mathcal{M}_1(t)$ around the reference solution is involved there. For higher-index DAEs ($k \geq 2$) the hidden constraints have to be respected by using the configuration space $\mathcal{M}_{k-1}(t)$. One could try to replace $\mathcal{M}_{k-1}(t)$ in Def. 6.7 by $\mathcal{M}_0(t) \supset \mathcal{M}_{k-1}(t)$. If feasible, explicit knowledge of $\mathcal{M}_1(t)$ is avoided but the boundedness and dissipativity inequality become more restrictive.

6.5 Further approaches to Lyapunov functions for DAEs

The theory of Lyapunov functions for differential-algebraic systems is in a steady process of development. One important issue is an appropriate substitution of the resolution of $f(x'(t), x(t), t) = 0$ for the derivative $x'(t)$ in the classic definition of a Lyapunov function because such a resolution is a priori not available for DAEs. Equivalently, the inherent dynamics together with the hidden constraints of the DAE have to be incorporated in some way in order to get practical stability criteria. In this thesis the appropriate definition of a Lyapunov function as stability criterion for index $k = 1, 2$ DAEs is obtained by means of Lemma 6.4, the complete decoupling and reduction of the tractability index via differentiation. After deriving these results we came across the preprint [TL10] published at the end of January 2010. This preprint is dealing with an alternative approach to define Lyapunov functions for differential-algebraic systems. The emphasis in [TL10] is on switched DAEs but we outline the relevant considerations addressing non-switched systems.

The approach of D. Liberzon and S. Trenn

Autonomous linear-implicit systems

$$E(x(t))\,x'(t) = f(x(t))$$

with sufficiently smooth functions E, f are considered. Due to the standard formulation of the DAE the classical solvability concept, that is $x \in C^1$ is used. With regard to switched DAEs the authors demand even piecewise C^∞-solutions. The goal is to ensure asymptotic stability of the stationary solution $x_* \equiv 0$ by a suitable Lyapunov function. To this purpose the configuration space $\mathcal{M}$ of the DAE is supposed to be a closed manifold, optionally with a boundary. In particular a finite maximal interval of existence of solutions is excluded. Additionally, the unique solvability of the DAE is required. According to [TL10, Def. 2.6], a Lyapunov function is a continuously differentiable function $V : \mathcal{M} \to \mathbb{R}^{\geq 0}$ featuring

- V is positive definite and the preimage of $[0, V(x)]$ under V is bounded for all $x \in \mathcal{M}$
- there exists a continuous function $F : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{\geq 0}$ with
$$\forall x \in \mathcal{M},\ z \in \mathrm{T}_x\mathcal{M} :\ \nabla V(x)\, z = F(x, E(x)\, z)$$
- $\forall x \in \mathcal{M} :\ \dot{V}(x) := F(x, f(x)) < 0$

Under the mentioned requirements it is proved that the existence of such a Lyapunov function implies stability of x_*. Thereto, we quote [TL10, p. 3]:

> "[...] we [...] generalize Lyapunov's Direct Method to the DAE case in Theorem 2.7. This result is based on a presumably new definition of a Lyapunov function for the DAE (2) as formulated in Definition 2.6."

There are some reasons why the described approach to define Lyapunov functions in case of DAEs is not suitable for the focus of this thesis. First of all, to prove the existence of solutions on the interval $[t_0, \infty)$ is a challenging task for ODEs and even more for differential-algebraic systems. Here, this property is simply *assumed* by the restrictive postulation of a closed solution manifold of the DAE. Besides, the feature of the entire configuration space to be a C^1-manifold is superfluous because stability considerations focus on a segment of $\mathcal{M}$ in vicinity of the reference solution. This is reflected by the Lyapunov function according to Definitions 6.7 and 6.15, but not in [TL10]. If we had required compactness of every solution of (1.2) a priori, then we would be done with Corollary 6.10, providing (asymptotic) D-component stability in the general case of *non-autonomous* fully implicit DAEs! Instead, Assumption 6.8 and 6.16 subsume structural conditions on the given DAE which are probably easier to check. In essence, we demand the tractability index $k = 1, 2$ on a cylindrical region around x_* and the assumptions necessary for the reduction of the tractability index 2 via differentiation of constraints. They already ensure unique solvability of the DAE. Notice that we are interested in C_D^1-solutions resulting in lower smoothness assumptions on the system. Finally, we get along without the explicit usage of the tangent space $\mathrm{T}_x\mathcal{M}$ which should be avoided in formulation of Lyapunov functions in the original problem setting.

The approach of F. Allgöwer and C. Ebenbauer

An alternative to the above procedure is introduced in [AE07]. There, fully implicit *controlled* DAEs
$$f(x'(t), x(t), u(t)) = 0$$

with a certain admissible control u and the stationary solution $x_* \equiv 0$ are considered. The main tool is the *derivative array*

$$F_\mu(\xi, \omega) := \begin{bmatrix} f(x', x, u) \\ \frac{\mathrm{d}}{\mathrm{d}t} f(x', x, u) \\ \vdots \\ \frac{\mathrm{d}^\mu}{\mathrm{d}t^\mu} f(x', x, u) \end{bmatrix} \tag{6.17}$$

for the algebraic variables $x \in \mathbb{R}^m$, $u \in \mathbb{R}^n$

$$\xi := \left(x, x', \ldots, x^{(\mu+1)}\right)^T, \quad \omega := \left(u, u', \ldots, u^{(\mu+1)}\right)^T \tag{6.18}$$

For the sake of the derivative array (6.17) f, x and u are supposed to be sufficiently smooth. Then, the expression $\frac{\mathrm{d}^k}{\mathrm{d}t^k} f(x'(t), x(t), u(t))$ is evaluated using the chain rule thereby introducing formal variables $x^{(k)}$, $u^{(k)}$ for the k-th derivative of the respective function. A positive definite and radially unbounded (i.e. $\lim_{\|x\| \to \infty} V(x) = \infty$) function $V \in C^1(\mathbb{R}^m, \mathbb{R})$ is called a Lyapunov function for the stationary solution of $f(x'(t), x(t), u(t)) = 0$ if $\mu \in \mathbb{N}$ and $\rho : \mathbb{R}^{(\mu+2)(m+n)} \to \mathbb{R} \cup \{\infty\}$ exist with

$$\nabla V(x)\, x' \leq \|F_\mu(\xi, \omega)\|^2 \rho(\xi, \omega) \tag{6.19}$$

on $\left\{\left(\left(x, x', \ldots, x^{(\mu+1)}\right)^T, \left(u, u', \ldots, u^{(\mu+1)}\right)^T\right) \in \mathbb{R}^{(\mu+2)(m+n)} \mid x \in \mathrm{B}_\epsilon(0) \subset \mathbb{R}^m\right\}$. A sufficiently smooth solution $(x, u)^T$ of the given DAE also satisfies the extended system $F_\mu(\xi(t), \omega(t)) = 0$ if the derivatives of x, u are used in the definition of $(\xi, \omega)^T(t)$. For this reason Condition (6.19) evaluated along the extended solution vector $(\xi(t), \omega(t))^T$ is equivalent to $\nabla V(x)\, x' \leq 0$, i.e. V is decreasing along solutions of the DAE. Thus stability of x_* in the sense of Lyapunov is obtained for all admissible controls u.

This method is clearly arranged and it works without assuming an index of the DAE a priori. On the other hand, it is known that the entire information of the differential-algebraic system with differentiation index μ is already contained in $F_\mu(\xi, \omega) = 0$. From this point of view the tangent bundle of the configuration space is nevertheless required which is roughly the same as in the Liberzon/Trenn-approach. Certainly, Inequality (6.19) enables numerical computations due to the derivative array. According to [AE07], it is convenient to verify Inequality (6.19) for polynomial f, V and ρ although the practicability is seriously affected by the high dimension of the resulting system.

The approach of V. Bajic

Vladimir Bajic's method to construct a Lyapunov candidate function $V(y)$ depending on an auxiliary variable $y = y(x)$ in a way that the inequality $\dot{V}(y) \leq 0$ mimics the total time derivative of a conventional Lyapunov function along solutions of an ODE is outlined in § 6.2.

We present a feasible choice $y = D(t)\, x$ resp. $y = (DP_1)(t)\, x$ of the auxiliary transformation for fully implicit DAEs with a properly formulated derivative term. At the same time the *connection between the transformation and the inner structure of the*

DAE is clarified: our Definition 6.7 has the reasonable interpretation to refer to the inherent regular ODE of an index-1 system or to the IRODE of the index reduced DAE, respectively. This feature also suggests how to prove nonlocal existence of solutions.

The main body of literature on DAEs in the bibliography pays almost no attention to V. Bajic's considerations on stability of DAEs, although they originate from the late 1980s—we found some of them as a cross reference in the (engineering) context of [KD99]. Certainly, the reasoning is fragmentary and driven by practice rather than mathematical theory, e.g. the important questions of existence and uniqueness of solutions of DAEs are simply skipped in [Baj86, Baj87, Baj90, MB89]. Of course, without using a sound index concept for characterizing and bounding structural complexity of differential-algebraic systems, there is no way to succeed with these issues. Nevertheless these publications help to classify the results of the present thesis. We believe that [Baj86, Baj87] could have had an influence on the development of Lyapunov-like stability criteria in the DAE community with a predominantly mathematical inclination.

The approach of Pham Van Viet

In the recent past some efforts to define Lyapunov functions for nonlinear DAEs with the tractability index $k = 1, 2$ are made by Vietnamese mathematicians. For the purpose of comparison, we briefly present the results from [VT04] and [Vie05]. Linear-implicit DAEs of the form

$$A(t)\,x'(t) + B(t)\,x(t) = f(x(t), t) \tag{6.20}$$

with $A \in C^1([t_0,\infty), \mathbb{R}^{m\times m})$ bounded, $B \in C([t_0,\infty), \mathbb{R}^{m\times m})$, $f \in C^0([t_0,\infty)\times\mathbb{R}^m, \mathbb{R}^m)$ with $f_t \in C^0$ and $f_x \in C^1$ are considered. The tractability index 2 is required together with the structural conditions

$$\begin{aligned} &\operatorname{im} A(t)\,P_0(t)\,P_1(t) \equiv \text{const.}\\ &Q_1(t)\,G_2^{-1}(t)\,f(x(t),t) \equiv 0\\ &Q_0(t)\,G_2^{-1}(t)\,f(x(t),t) \equiv Q_0(t)\,\tilde{D}(t)\,P_0(t)\,P_1(t)\,x(t) \end{aligned} \tag{6.21}$$

for all solutions $x(t)$ of (6.20). At this, the existence of a suitable function $\tilde{D} \in C^0([t_0,\infty), \mathbb{R}^{m\times m})$ is demanded. Denote the matrix chain of the tractability index for the DAE $A(t)\,x'(t) + B(t)\,x(t) = 0$ by $P_i(t)$, $G_i(t)$ and the corresponding elements for (6.20) by $P_i(x,t)$ etc. These matrix chains and the tractability index $k = 1, 2$ are similar to [GM86] and thus a prequel to the tractability index for properly formulated DAEs used in this thesis. Now, $(Ax)' = A'x + Ax'$ reveals that the DAE (6.20) on its geometric solution space is equivalent to the ODE

$$y'(t) = (A'(t) - B(t))\,T(t)\,y(t) + f(T(t)\,y, t) \tag{6.22}$$

on im AP_0P_1 having the initial value $y(t_0) = A(t_0)\,P_0(t_0)\,P_1(t_0)\,x_0$,

$$T(t) := \left(I - (Q_0Q_1)' - QP_1G_2^{-1}B + Q_0\tilde{D}\right)(t)\,G_2^{-1}(t)$$

In publication [VT04] as well as in [Vie05, Assumption (B)] dealing with index-1 DAEs it is merely *assumed* that the ordinary differential equations involved do possess unique solutions of the respective initial value problems on $[t_0, \infty)$. In order to define a Lyapunov function for index-2 DAEs (6.20), the monotonicity condition

$$\dot{V}(y,t) := \limsup_{s \to 0+} \frac{1}{s} \left(V\left(y + s\left(A' - B\right)(t)\, T(t)\, y + f\left(T(t)\, y, t\right)\right) - V(y,t) \right) \leq 0 \tag{6.23}$$

on a suitable domain is required. Consequently, $V(x(t), t)$ is decreasing along solutions of the inherent regular ODE (6.22). According to [VT04, Th. 2.1] the existence of such a Lyapunov function ensures stability in the sense of Lyapunov of $x_* \equiv 0$. Several classic results considering Lyapunov functions for ODEs (the majority stemming from [Yos66]) are carried over to the representation (6.22) of the inherent dynamics and hence to the given DAE.

From our the point of view the above approach exhibits several drawbacks. The existence of a classic Lyapunov function is sufficient to ensure solvability of the ODE under consideration on entire $[t_0, \infty)$, at least in case of initial values nearby the reference solution. Therefore it appears reasonable to demand this property from an appropriate generalization of the Lyapunov concept to DAEs. In [Vie05] and [VT04], no importance is attached to this fundamental feature of Lyapunov functions, the nontrivial question of existence of solutions of index-2 DAEs on unbounded intervals remains open. Moreover it is not obvious how to check Assumption (6.21) efficiently. It is cumbersome that (6.23) is not formulated in terms of the given DAE and that the analysis does not include fully implicit systems, possibly with $\|f_y\|_\infty = \infty$ so practicability as well as mathematical elegance of the above approach could be questioned.

Summing up, there exist several different mathematical approaches to adapt the direct method of Lyapunov to differential-algebraic equations. The connection to differentiation index marked in [AE07], the usage of tangential space of the solution manifold in [TL10] and the implicit representation of the inherent dynamics in Lemma 6.4 potentially hint at a common geometric origin of the discussed concepts of Lyapunov functions for DAEs. In our honest opinion, the overall understanding of differential-algebraic systems (e.g. all-embracing knowledge of connections between the index concepts) is not advanced enough to claim the "right" definition of a Lyapunov function for DAEs at the moment. After balancing pros and cons of the presented approaches and new insights provided in § 6.6 and § 7.3, it seems to be of theoretical and practical interest to take the definitions of Lyapunov functions - as stated in this thesis - into consideration.

6.6 Understanding contractivity definitions for DAEs

Dissipativity inequalities represent a natural generalization of Lyapunov functions. They provide evidence for global stability of solutions on a certain region. Considering ODEs $x'(t) = f(x(t), t)$ with $f \in C(\mathcal{G} \subseteq \mathbb{R}^m \times I \subseteq \mathbb{R}, \mathbb{R}^m)$ it is decisive to replace the Lipschitz condition $\|f(x,t) - f(\tilde{x},t)\| \leq L_f \|x - \tilde{x}\|$ by the so called *one-sided Lipschitz condition* (cf. [DV84, § 1.2])

$$\forall t \in I,\ x, \tilde{x} \in \mathcal{U}_t : \ \langle f(x,t) - f(\tilde{x},t), x - \tilde{x} \rangle \leq \beta(t) \|x - \tilde{x}\|^2 \tag{6.24}$$

with a piecewise continuous function $\beta : I \to \mathbb{R}$ and $\mathcal{U}_t \subseteq \{x \in \mathbb{R}^m \mid (x,t) \in \mathcal{G}\}$. Here $\langle \cdot, \cdot \rangle$ denotes a scalar product and $\|\cdot\|$ the induced vector norm.

Remark 6.22. The one-sided Lipschitz condition

$$\forall x, \tilde{x} \in \mathcal{U} : \langle f(x) - f(\tilde{x}), x - \tilde{x} \rangle \leq 0$$

is called a *monotonicity condition* on the right hand side of an autonomous ODE $x'(t) = f(x(t))$ in [SH96, p. 178]. For one dimensional systems the scalar product corresponds to multiplication of the arguments times a constant, thus the one-sided Lipschitz condition reads $f(x) \leq f(\tilde{x})$ for $x \geq \tilde{x}$, i.e. f is monotonically decreasing.

An ODE is called *(strictly) contractive* if the one-sided Lipschitz condition (6.24) with $\beta(t) \leq 0$ $(\beta(t) < 0)$ is valid. The notation results from the (strict) contractivity of the flow associated to the differential equation on $\{\mathcal{U}_t \times \{t\} \mid t \in I\} \subseteq \mathcal{G}$. In other words,

$$\forall t, t_0 \in I, t \geq t_0 \text{ and } x_0, x_1 \in \mathcal{U}_{t_0} : \|x(t; t_0, x_0) - x(t; t_0, x_1)\| \leq \exp(\beta(t)) \|x_0 - x_1\| \tag{6.25}$$

holds.

Contractivity for index-1 DAEs

One of the most essential theoretic results in [MHT03a] is the generalization of the notion of contractivity to properly formulated differential-algebraic systems (1.2). A fully implicit index-1 DAE is called *contractive* if the associated inherent regular ODE is contractive on im $D(t)$. To this end the one-sided Lipschitz condition is formulated for the IRODE in [MHT03a, § 3.1], that is

$$\begin{array}{l}\langle D(t)(w(u(t),t) - w(\tilde{u}(t),t)), D(t)(u(t) - \tilde{u}(t))\rangle \\ + \langle R'(t)(u(t) - \tilde{u}(t)), D(t)(u(t) - \tilde{u}(t))\rangle\end{array} \leq -\beta \|D(t)(u(t) - \tilde{u}(t))\|^2$$

where $x(t)$ denotes the solutions of the DAE, $u(t) := D(t)x(t)$ and $w(u,t)$ the implicitly defined resolution function in the complete decoupling as per § 6.5. The laconic statement

> "The following definition takes up this idea, but in terms of the original DAE."

introduces the important contractivity definition

$$\begin{array}{l}\exists \beta \geq 0 \forall t \geq t_0,\ x, \tilde{x} \in \mathcal{M}_0(t),\ z, \tilde{z} \in \text{im } D(t) \text{ with } f(z,x,t) = 0 = f(\tilde{z}, \tilde{x}, t) : \\ \langle z - \tilde{z} + R'(t) D(t)(x - \tilde{x}), D(t)(x - \tilde{x})\rangle \leq -\beta \|D(t)(x - \tilde{x})\|^2\end{array} \tag{6.26}$$

for index-1 systems in [MHT03a, p. 183]. It is shown (Proposition 7, ibidem) that the DAE is contractive in the above sense if and only if the associated IRODE is contractive on im $D(t)$. Formulation (6.26) appears canonical if the implicit resolution of the DAE for $R(Dx)'$ stemming from Lemma 6.4 is applied. According to [Tis09], the dependency on expressions in terms of the complete decoupling is the reason why the subsequent publication [MHT03b] restricts to contractivity of linear index-2 DAEs only. Then, by virtue of the superposition principle, it is sufficient to investigate the trivial solution of the homogeneous linear DAE, i.e. $\tilde{x} = 0$ is used. The contractivity definition for linear index-2 DAEs from [MHT03b, Def. 24] reads

$$\begin{array}{l} \exists \beta \geq 0 \forall t \geq 0,\ x \in \mathcal{M}_1(t),\ z = R(t)\, z \text{ with } A(t)\, z + B(t)\, x = 0: \\ \langle z + R'(t)\, D(t)\, x, D(t)\, x \rangle \leq -\beta \left\| D(t)\, x \right\|^2 \end{array} \tag{6.27}$$

Given (6.27), Theorem 26 in [MHT03b] proves the contractivity of the inherent regular ODE given, i.e an estimate of the type (6.25) for the DP_1-components of the solution vector. An estimate for the entire vector is achieved using the solution representation for linear differential-algebraic systems, yet including the factor $\left\| P_{1,\mathrm{c}}(t) \right\|_{(DS_1)(t)}$ which reflects the geometry of the time-varying systemic subspaces.

This result can be improved formulating the one-sided Lipschitz condition directly for nonlinear index-2 DAEs such that the stronger estimate

$$\left\| D(t)\, x(t; t_0, x_0) - D(t)\, x(t; t_0, x_1) \right\| \leq \exp\left(-\beta (t - t_0)\right) \left\| D(t_0)(x_0 - x_1) \right\|$$

affecting the differentiable solution components Dx instead of DP_1x is achieved.

Contractivity notion for index-2 DAEs

The P_0-contractivity of the DAE $f(x'(t), x(t), t) = 0$ on a manifold $\Gamma(t) \subseteq \mathcal{M}_0(t)$ is defined in [San00, §2.2.2] by the means of

$$\begin{array}{c} \exists \beta \geq 0 \forall t \geq 0,\ (z, x, t), (\tilde{z}, \tilde{x}, t) \in \mathbb{R}^m \times \Gamma(t) \times I, \\ f(z, x, t) = 0 = f(\tilde{z}, \tilde{x}, t),\ Q_0 z = Q_0 \tilde{z} = 0: \\ \langle z - \tilde{z}, P_0(x - \tilde{x}) \rangle \leq -\beta \left\| P_0(x - \tilde{x}) \right\|^2 \end{array} \tag{6.28}$$

Here, Q_0 is a constant projector onto $\ker f_y(y, x, t)$ and $\langle \cdot, \cdot \rangle$ is a suitable scalar product on $\mathbb{R}^m$ together with the induced norm $\|\cdot\|$. The definition is acknowledged to be based on a former definition of contractivity given by R. März. Assuming this definition, the estimate

$$\left\| P_0 x(t, t_0, x_0) - P_0 x(t; t_0, x_1) \right\| \leq \exp\left(-\beta (t - t_0)\right) \left\| P_0(x_0 - x_1) \right\|$$

for solutions $x(t; t_0, x_i) \in \Gamma(t)$ of the original DAE can be proved easily. The techniques presented in the current chapter of this thesis make a simple explanation possible: *Condition* (6.28) *for P_0-contractivity states the contractivity of the IRODE of the associated index-reduced system on* $\Gamma(t) \cap \mathcal{M}_1(t)$.

We are going to specify this result modifying the notion of P_0-contractivity a little.

Definition 6.23. A fully implicit nonlinear DAE (1.2) with index $k = 1, 2$ and a properly formulated derivative term is called *D-component contractive* if there is a scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ and a piecewise continuous function $\beta : I \to \mathbb{R}^{\leq 0}$ such that

$$\begin{array}{l} \forall t \in I,\ x_i \in \mathcal{M}_{k-1}(t),\ z_i = R(t)\, z_i \text{ with } f(z_i, x_i, t) = 0: \\ \begin{array}{l} \langle z_1 - z_2, D(t)(x_1 - x_2) \rangle \\ + \langle R'(t) D(t)(x_1 - x_2), D(t)(x_1 - x_2) \rangle \end{array} \leq \beta(t) \left\| D(t)(x_1 - x_2) \right\|^2 \end{array} \tag{6.29}$$

is satisfied.

The above definition coincides with [MHT03a, Def. 6] in case of index-1 DAEs .

Theorem 6.24. *[Interpretation of D-component contractivity]*

If the nonlinear DAE 1.2 exhibits tractability index $k = 1, 2$ on its entire domain and the system is D-contractive according to Definition 6.23 then solutions $x\left(t; t_0, x_i\right)$ of initial value problems $x_1, x_2 \in \mathcal{M}_{k-1}\left(t_0\right)$ of the DAE (1.2) on their respective maximal intervals of existence I_{t_0,x_i} satisfy the following estimate

$$\begin{gathered}\forall t \in I_{t_0,x_1} \cap I_{t_0,x_2}, t \geq t_0 : \\ \left\|D\left(t\right)x\left(t, t_0, x_1\right) - D\left(t\right)x\left(t; t_0, x_2\right)\right\| \leq \exp\left(\int_{t_0}^{t} \beta\left(s\right)\mathrm{d}s\right)\left\|D\left(t\right)\left(x_1 - x_2\right)\right\|\end{gathered} \tag{6.30}$$

In addition,

1. *D-component contractivity of an index-1 system implies the one-sided Lipschitz condition of the associated inherent regular ODE on its invariant subspace* im $D\left(t\right)$.
2. *If the requirements of Theorem 3.4 are fulfilled then D-component contractivity of an index-2 DAE implies the same property of the associated index-reduced DAE restricted to $\mathcal{M}_1\left(t\right)$. In other words, the one-sided Lipschitz condition of the IRODE of the IR-DAE restricted to $\left(D\mathcal{M}_1\right)\left(t\right)$ holds.*

Proof. Estimate (6.30) for D-components of the solution vector is obtained by minor adjustments of the proof of [San00, Th. 2.2.15]. Choose $x_1, x_2 \in \mathcal{M}_{k-1}\left(t_0\right)$ and set

$$\begin{aligned}\alpha\left(t\right) &:= \tfrac{1}{2}\left\|D\left(t\right)\left(x\left(t; t_0, x_1\right) - x\left(t; t_0, x_2\right)\right)\right\|^2, \\ y_i\left(t\right) &:= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(D\left(t\right)x\left(t; t_0, x_i\right)\right)\end{aligned}$$

on the common maximal interval of existence $I_{t_0,x_1} \cap I_{t_0,x_2}$. The scalar product is a symmetric bilinear form so

$$\alpha'\left(t\right) = < y_1\left(t\right) - y_2\left(t\right), D\left(t\right)\left(x\left(t; t_0, x_1\right) - x\left(t; t_0, x_2\right)\right) >$$

Due to $D = RD$ we obtain $y_i\left(t\right) = R'\left(t\right)D\left(t\right)x\left(t; t_0, x_i\right) + R\left(t\right)y_i\left(t\right)$ and

$$\alpha'\left(t\right) = \begin{array}{l}\left\langle R'\left(t\right)D\left(t\right)\left(x\left(t; t_0, x_1\right) - x\left(t; t_0, x_2\right)\right), D\left(t\right)\left(x\left(t; t_0, x_1\right) - x\left(t; t_0, x_2\right)\right)\right\rangle \\ + \left\langle R\left(t\right)\left(y_1\left(t\right) - y_2\left(t\right)\right), D\left(t\right)\left(x\left(t; t_0, x_1\right) - x\left(t; t_0, x_2\right)\right)\right\rangle\end{array}$$

Inequality (6.29) implies

$$\alpha'\left(t\right) \leq 2\beta\left(t\right)\alpha\left(t\right)$$

because of $f\left(R\left(t\right)y_i\left(t\right), x\left(t; t_0, x_i\right), t\right) = 0$. Applying Grönwall's Lemma (cf. [Wal00, p. 330]) results in

$$\alpha\left(t\right) \leq \alpha\left(t_0\right)\exp\left(2\int_{t_0}^{t} \beta\left(s\right)\mathrm{d}s\right)$$

which is equivalent to Estimate (6.30). Obviously, this is the reason why (6.29) is called D-component contractivity.

The second proposition is the important one, it helps to interpret the dissipativity inequality (6.29) for index-k DAEs or to state this inequality without knowing it in advance.

With regard to index-1 DAEs, it is known that the complete decoupling of the system is available locally around any point $(y, x, t) \in \mathcal{G}$ featuring $f(y, x, t) = 0$ and the tractability index one, at least if rk $f_y(y, x, t) D(t)$ is assumed to be locally constant. As we have shown in § 6.3.1, the implicit representation (6.5) of the right hand side of the IRODE is valid. Consequently we identify (6.29) as an implicit representation of the one-sided Lipschitz condition (6.24) for the IRODE restricted to its invariant subspace im $D(t)$.

For index-2 systems, Theorem 3.4 ensures the existence and tractability index 1 of the index reduced DAE (3.2) belonging to the given differential-algebraic system. Inequality (6.29) addressing the IR-DAE $\tilde{f}\left((Dx)'(t), x(t), t\right) = 0$ and its first level constraint $\tilde{M}_0(t)$ reads $(i = 1, 2)$:

$$\begin{array}{l} \forall t \in I,\; x_i \in \tilde{\mathcal{M}}_0(t),\; z_i = R(t) z_i \text{ with } \tilde{f}(z_i, x_i, t) = 0 : \\ \begin{array}{l} \langle z_1 - z_2, D(t)(x_1 - x_2)\rangle \\ + \langle R'(t)D(t)(x_1 - x_2), D(t)(x_1 - x_2)\rangle \end{array} \leq \beta(t) \left\| D(t)(x_1 - x_2)\right\|^2 \end{array}$$

Now the argument in § 6.4.2 readily applies. Hence

$$\forall x_i \in \mathcal{M}_1(t) \subseteq \tilde{\mathcal{M}}_0(t)\, \exists! y_i = R(t) y_i : \; f(y_i, x_i, t) = 0 \text{ and } \tilde{f}(y_i, x_i, t) = 0$$

Lemma 1.19 implies $y_i = z_i$ for the z_i in the above inequality. It follows that (6.29) constitutes a formulation of the D-component contractivity condition for the index-reduced system restricted to $\mathcal{M}_1(t)$ in terms of the original DAE. □

7 Outlook

Ars longa, vita brevis

(Hippocrates)

Investigating stability properties of nonlinear differential-algebraic systems, we came across many interesting publications, gathered and tried out several approaches. Understandably enough, only the most successful paths were selected to form the main body of the present thesis. We would like to comprise some more promising ideas in this outlook, which have not found their way into stand-alone theorems so far, partly because (our academic) life is short, but art is long. Of course, nearly every question we answered motivates new topics to work on so this chapter also includes open questions, thus corresponding to Georg Cantor's conception - *in re mathematica ars proponendi questionem pluris facienda est quam solvendi.*

7.1 Perspectives for index-3 tractable differential-algebraic systems

Modelling of mechanical systems by second-order ODEs with constraints often results in differential-algebraic systems of the type

$$\begin{aligned} x_1'(t) &= f(x_1(t), x_2(t), x_3(t), t) \\ x_2'(t) &= g(x_1(t), x_2(t), t) \\ 0 &= h(x_2(t), t) \end{aligned}$$

with a regular matrix $h_{x_2} g_{x_1} f_{x_3}$, cf. [AP97, § 9.1.1]. These DAEs are called index-3 systems in Hessenberg form. Hessenberg-3 systems can be analysed without applying the concepts of the tractability index because differentiating the first level constraints $0 = h(x_2(t), t)$ twice reveals the hidden constraints leading to a quite simple construction of a state space form of the original DAE. Dealing with nonlinear DAEs in a general unstructured (i.e. fully implicit) form usually requires subtle methods of analysis. The efforts we have undergone are justified because circuit simulation provides a class of nonlinear MNA equations with tractability index lower or equal to 2 for many relevant configurations (cf. [EST00]) which are not in Hessenberg form. However, the structural condition $N_0(t) \cap S_0(x, t)$ independent of x is known to be valid and some information on N_1 and S_1 is available so our tools for fully implicit index-2 systems assert themselves in this case.

The stability results in the context of Lyapunov's second or direct method for index-2 DAEs presented in this thesis are based either on reduction of the tractability index via differentiation of constraints or on a complete decoupling of the system. The first or indirect method of Lyapunov for DAEs is vitally dependent on commutativity between linearization and transformation of a DAE into the associated state space form. As a matter of principle, mentioned techniques can be applied to differential-algebraic systems with tractability index 3 as well. However, more involved structural assumptions are required in order to succeed with the calculations which is a less critical issue for index-2 DAEs. At the moment, there is a lack of clarity concerning an area of application for orbital stability of periodic solutions of autonomous index-3 DAEs which are not in Hessenberg form. That is why an elaborate construction of the state space representation for general index-3 DAEs is waived and solely an outline of the procedure is presented. Same holds for the gradual reduction of the tractability index from 3 to 1.

First of all, we have to define the tractability index tree. To this end, continue the matrix chain of the tractability index as follows:

$$\begin{aligned} N_2(y,x,t) &= \ker G_2(y,x,t) \\ S_2(y,x,t) &= \{z \in \mathbb{R}^m \mid f_x(y,x,t)P_0(t)P_1(y,x,t)\, z \in \operatorname{im} G_2(y,x,t)\} \\ G_3(y,x,t) &= G_2(y,x,t) + f_x(y,x,t)P_0(t)P_1(y,x,t)\, Q_2(y,x,t) \end{aligned}$$

We require admissible projectors Q_1 and Q_2 in the process, i.e.

$$Q_2(y,x,t)\, Q_1(y,x,t) \equiv 0,\; Q_2(y,x,t)\, Q_0(t) \equiv 0,\; Q_1(y,x,t)Q_0(t) \equiv 0.$$

Definition 7.1. The properly stated differential-algebraic equation (1.2) possesses the tractability index 3 on a subset $\mathcal{U}$ of its domain $\mathcal{G}$ if

$$\forall\,(y,x,t) \in \mathcal{U}, i = 0,1,2 :\ \operatorname{rk}\, G_i(y,x,t) \equiv r_i < m \text{ and } G_3(y,x,t) \in \mathrm{GL}_m(\mathbb{R})$$

Resembling the index-2 case, nonsingularity of $G_3(y,x,t)$ is equivalent to

$$N_2(y,x,t) \cap S_2(y,x,t) = \{0\}$$

Using a projector W_2 along im G_2 it holds $S_2 = \ker W_2 f_x P_0 P_1$.

Remark 7.2. The above definition of tractability index 3 adapts the corresponding definition in [Mä05] to the simplified matrix chain. These minor adjustments are also used in order to decouple linear index-3 DAEs in [Sch01].

7.1.1 Index reduction

Consider fully implicit properly formulated DAEs (1.2) and suppose im $G_2(y,x,t)$ to depend on $(P_0(t)\,x,t)$ only. Applying a projector $W_2(P_0(t)\,x,t)$ along im $G_2(y,x,t)$ similarly to Lemma 3.3, we obtain the property

$$(W_2 f)(y,x,t) = (W_2 f)(P_0(t)\,x,t)$$

This is why the constraint $(W_2 f)(P_0(t)\,x,t) = 0$ can be replaced by

$$W_2(P_0(t)\,x,t)\,\frac{\mathrm{d}}{\mathrm{d}t}(W_2 f)(P_0(t)\,x,t) = 0$$

Using the chain rule we arrive at the system $\tilde{f}\left((Dx)'(t), x(t), t\right) = 0$,

$$\tilde{f}(y,x,t) := \begin{array}{l} (I - W_2)(P_0(t)x,t) f(y,x,t) + W_2(P_0(t)x,t)(W_2 f)_t(P_0(t)x,t) \\ +W_2(P_0(t)x,t)(W_2 f)_x(P_0(t)x,t)\left(D^-(t)y + (D^-)'(t)D(t)x\right) \end{array} \tag{7.1}$$

It follows

$$\tilde{f}_y(y,x,t) = f_y(y,x,t) + W_2(P_0(t)x,t)(W_2 f)_x(P_0(t)x,t)D^-(t)$$

and $\ker \tilde{f}_y = \ker f_y$ i.e. the differential-algebraic system (7.1) has a properly formulated derivative term.

We restrict ourselves to (y,x,t) in the domain of f exhibiting $f(y,x,t) = 0$ and consider the matrix chain $\tilde{G}_i$, $\tilde{Q}_i$ of (7.1). It follows

$$\begin{array}{rcl} \tilde{G}_1(y,x,t) & = & \tilde{G}_0(y,x,t) + f_x(y,x,t)Q_0(t) \\ & = & G_1(y,x,t) + W_2(P_0(t)x,t) f_x(y,x,t)P_0(t) \end{array}$$

because the terms depending on $(P_0(t)x,t)$ vanish if the partial derivative with respect to x is multiplied by $Q_0(t)$ and

$$(W_2 f)_x(P_0(t)x,t)z = W_2(P_0(t)x,t) f_x(y,x,t)z + \underbrace{[W_2(P_0(t)x,t)]_x\left(z, \underbrace{f(y,x,t)}_{=0}\right)}_{=0}$$

It follows that $\tilde{N}_1 = N_1 \cap \ker W_2(W_1 f_x P_0)$. Notice that a projector $\tilde{Q}_1$ on $\tilde{N}_1$ fulfills $Q_1\tilde{Q}_1 = \tilde{Q}_1$, therefore

$$\tilde{G}_2 z = \begin{array}{l} G_1 z + G_2\tilde{Q}_1 z + [W_2]_x\left(P_0\tilde{Q}_1 z, W_2\left((W_2 f)_x D^- y + (W_2 f)_t\right)\right) \\ +W_2(W_2 f)_x P_0 z + W_2[W_2 f]_{xx}\left(P_0\tilde{Q}_1 z, D^- y\right) + W_2(W_2 f)_{tx}P_0\tilde{Q}_1 z \end{array}$$

In case of Hessenberg-3 systems P_0P_1 is constant and W_2, $W_2 f$ dependent on (P_0P_1x,t). Consequently $\tilde{Q}_1 = Q_1$ and condition $\tilde{G}_2(y,x,t)z = 0$ is equivalent to

$$G_2(y,x,t)z = 0, \quad W_2(P_0x,t) f_x(y,x,t)P_0P_1 z = 0$$

i.e. $z \in (N_2 \cap S_2)(y,x,t) = \{0\}$. As a result, the method reduces the tractability index from 3 to 2 so it is adequate to call (7.1) a representation of the index-reduced DAE.

We do not go into details with regard to suitable structural assumptions for the index reduction of general index-3 DAEs. If we succeed to set up clearly arranged criteria for the index reduction, then a Lyapunov function aiming at the IR-DAE on $\mathcal{M}_2$ (t) can be defined in terms of the original DAE. To this end, simply put $k = 3$ in Definition 6.7. Essentially, the existence of a Lyapunov function for the inherent dynamics of such systems obtained after two steps of index reduction—and restricted to the configuration space $\mathcal{M}_2(t)$—is demanded. Again, the following requirements are sufficient to guarantee nonlocal existence:

1. Criteria ensuring feasibility of the index reduction and the tractability index 3 on $\mathcal{U}_\epsilon$ (Notation 6.6)
2. A cylindrical domain of the inherent dynamics.

Motivated by Theorem 6.20, it is reasonable to expect that the second requirement is already fulfilled for C^1_D-bounded solutions of autonomous fully implicit index-3 DAEs.

Remark 7.3. Definition 6.23 of D-contractivity also supports $k = 3$. Obviously, the exponential estimate (6.30) is valid for fully implicit index-3 DAEs and we conjecture that the notion of D-contractivity for such DAEs also has an interpretation as D-contractivity of the index-reduced DAE exhibiting index 2, i.e. contractivity of the inherent dynamics of the DAE obtained after two index reduction steps.

7.1.2 State space form of index-3 DAEs

From a complete decoupling of properly formulated linear index-3 DAEs (e.g. [Sch01]) it is known that DP_1P_2-components of the solution are determined by an inherent regular ODE. Hence we suppose that the dynamical solution components of index-3 DAEs (1.2) can be identified by $(DP_1P_2)\left((Dx_*)'(t), x_*(t), t\right)$, at least in a neighbourhood of the extended integral curve of the reference solution x_*. Following the lines of § 2.1, we consider the modified Taylor expansion (2.3)

$$f_y^*(t)\,(Dx)'(t) + f_x^*(t)x(t) + \tilde{h}\left((Dx)'(t), x(t), t\right) - r(t) = 0$$

At this point an adequate generalization of the index-2 identities

$$\begin{aligned} I &= P_0P_1 + (P_0Q_1 + UQ_0) + TQ_0 \\ I &= P_0P_1 + (UQ_0 + Q_1)(P_0Q_1 + UQ_0) + TQ_0P_1 \end{aligned}$$

has to be found in order to transform the above Taylor expansion of the index-3 DAE into an equivalent system of four equations. It is difficult to choose an appropriate splitting $I = P_0P_1P_2 + \ldots$ such that the resulting structural conditions for the complete decoupling are as simple and reasonable as

$$N_0(t) \cap S_0(y, x, t) = (N_0 \cap S_0)(t)$$

Anyway, the first equation of the splitted DAE resulting from multiplication of (2.3) by $DP_1P_2G_3^{-1}$ has the form (cf. [Sch01, p. 12])

$$\begin{array}{l} u'(t) - (DP_1P_2D^-)'(t)\,u(t) + \left(DP_1P_2G_3^{-1}BD^-\right)(t)\,u(t) \\ + \left(DP_1P_2G_3^{-1}\right)(t)\,(h(u'(t), v'(t), w'(t), u(t), v(t), w(t), z(t), t) - r(t)) \end{array} = 0$$

Here, $u(t) := (DP_1P_2)(t)\,x(t)$ and v, w, z denote the remaining solution components such that $x(t) = D^-(t)\,u(t) + v(t) + w(t) + z(t)$. The terms u', v' and w' stand for the differentiable components, i.e. $(Dx)'(t) = u'(t) + v'(t) + w'(t)$ and the auxiliary function reads $h(u', v', w', u, v, w, z, t) = \tilde{h}\left((Dx)', x, t\right)$. W.l.o.g. we consider the extended DAE (1.15) belonging to the original system and suppose $A = A(t)$. Hence the inherent dynamics for $u(t)$ are implicitly given by

$$\begin{array}{l} u'(t) - (DP_1P_2D^-)'(t)\,u(t) + \left(DP_1P_2G_3^{-1}BD^-\right)(t)\,u(t) \\ + \left(DP_1P_2G_3^{-1}\right)(t)\,(h(u(t), v(t), w(t), z(t), t) - r(t)) \end{array} = 0$$

Solving the remaining three equations of the splitted DAE for v, w, z such that $v = v(u,t)$, $w = w(u,t)$ etc. results in the inherent regular ODE

$$\begin{aligned}&u'(t) - (DP_1P_2D^-)'(t)\,u(t) + \left(DP_1P_2G_3^{-1}BD^-\right)(t)\,u(t)\\ &+ \left(DP_1P_2G_3^{-1}\right)(t)\,(h(u(t), v(u(t),t), w(u(t),t), z(u(t),t), t) - r(t))\end{aligned} = 0 \quad (7.2)$$

of the index-3 DAE

$$A(t)\,(Dx)'(t) + b(x(t),t) = 0$$

on its invariant subspace im $(DP_1P_2)(t)$. Represent the projector $DP_1P_2D^-$ with respect to bases of systemic subspaces like in it is done in Lemma 2.15. Then, im DP_1P_2 is parametrized and the IRODE (7.2) is transformed into a state space form. Per constructionem, the function h in the modified Taylor expansion vanishes together with its partial derivatives h_u, h_v, h_w, h_z evaluated along the extended integral curve of x_*. In other words, the complete decoupling and linearization along x_* commute. This property carries over to the state space form.

With regard to Theorem 2.22, it is reasonable to conjecture that the state space representation of autonomous DAEs exhibiting

$$\text{im } (DP_1P_2)(t) = \text{const.}$$

turns out to be autonomous. Furthermore, we think that characteristic multipliers of linear index-3 DAEs are likely to coincide with the characteristic multipliers of the state space form like in the index-2 case. Summarizing, if above considerations are valid, then the linearization principle will apply and the Andronov-Witt Theorem will be proved for fully implicit index-3 DAEs as demonstrated for index-1,2 systems.

7.2 Regularization of fully implicit systems

Another interesting property of the presented index reduction approach is to suggest a regularization of fully implicit DAEs which obey Assumption 3.1. In this case, the equivalent representation

$$\begin{array}{rcl}(I - W_1(P_0x,t))\,f\left((Dx)'(t), x(t), t\right) & = & 0\\ (W_1f)(P_0x,t) & = & 0\end{array} \quad (7.3)$$

of the differential-algebraic system (1.2) is valid. Apparently, this structure looks similar to a Hessenberg-2 DAE (1.6)! There are reasons to interpret fully implicit DAEs (1.2) with im $G_1(y,x,t)$ dependent on (P_0x,t) as *generalized Hessenberg-2 systems*, that is the most important structural properties of a Hessenberg system are abstracted from its semi-explicit form. For example, it is possible to adapt some known regularization techniques for Hessenberg systems to (7.3). Let $\epsilon \in \mathbb{R}\setminus\{0\}$.

1. A special case of the regularization approach by Michael Knorrenschild ([Kno88, Ch. 2]) for Hessenberg-DAEs (1.6) reads

$$\begin{array}{rcl}x_1'(t) & = & h(x_1(t), x_2(t), t)\\ 0 & = & g(x_1(t) + \epsilon x_1'(t), t)\end{array}$$

Naturally, we suggest the approach

$$\begin{aligned}(I - W_1(P_0 x, t))\, f\big((Dx)'(t), x(t), t\big) &= 0\\ (W_1 f)\big(P_0 x + \epsilon D^-(t)(Dx)'(t), t\big) &= 0\end{aligned} \tag{7.4}$$

for generalized Hessenberg-2 DAEs.

2. The second approach is called *März parametrization* in [Han92], its formulation for nonlinear DAEs (1.1) is as follows:

$$f\big(x'(t), x(t) + \epsilon P_0 (P_0 x)'(t), t\big) = 0$$

where $P_0(t)$ denotes a projector on $\ker f_y(y, x, t)$. Actually,

$$P_0 (P_0 x)' = P_0 \big(D^- D x\big)' = D^- (Dx)' + P_0 \big(D^-\big)' Dx \tag{7.5}$$

but we would like to formulate the März parametrization for DAEs with a properly stated derivative term like this:

$$f\big((Dx)'(t), x(t) + \epsilon D^-(t)(Dx)'(t), t\big) = 0 \tag{7.6}$$

The aim of the game

There are two key aspects behind any regularization approach - first, the regularized DAE should be easier to handle, which usually means a lower index. Secondly, it is necessary to show that solutions of the regularization are connected to those of the original DAE in a suitable manner, for example to state some asymptotic expansion in terms of parameter ϵ. Thorough results are known mainly for Hessenberg systems, cf. [HE95, HHS92] and the references therein. The estimate

$$\|x_\epsilon(t) - x(t)\| \le C_1 \exp\left(-\frac{\sigma t}{\epsilon}\right) + C_2 \epsilon$$

is achieved in [Han95] regarding solutions $x(t)$ of the linear implicit DAE

$$A(x(t))\, x'(t) = g(x(t))$$

with $\ker A(t) = $ const. and solutions $x_\epsilon(t)$ belonging to the März regularization. It is based on a result of the differential geometric singular perturbation theory and requires *geometric* index 2 of the DAE under consideration. Unfortunately, there are some difficulties to attest the geometric index 1 of the regularized DAE and this fact has let us be intrigued by the possibility to access fully implicit systems using (7.4) and (7.6).

Consider the constitutive function of the März regularization,

$$\tilde{f}(y, x, t) := f\big(y, x + \epsilon D^-(t)\, y, t\big)$$

Then,

$$\begin{aligned}\tilde{f}_y(y, x, t) &= f_y(y, x + \epsilon D^-(t)\, y, t) + \epsilon f_x(y, x + \epsilon D^-(t)\, y, t)\, D^-(t)\\ \tilde{f}_x(y, x, t)\, Q_0(t) &= f_x(y, x + \epsilon D^-(t)\, y, t)\, Q_0(t)\\ \tilde{G}_1(y, x, t) &= G_1(y, x + \epsilon D^-(t)\, y, t) + \epsilon \underbrace{f_x\big(y, x + \epsilon D^-(t)\, y, t\big)\, P_0(t)}_{=W_1 f_x P_0 + (I - W_1) f_x P_0}\end{aligned}$$

Obviously, März parametrization of index-2 tractable DAEs leads to systems of index one if

$$N_1 \cap (I - W_1) f_x P_0 = \{0\} \text{ and rk } G_0 + \epsilon f_x P_0 = \text{const.}$$

In case of Hessenberg-2 systems, $\ker f_x(x,t) P_0 \subseteq \ker W_1 f_x(x,t) P_0 = S_1(x,t)$ is valid. Hence,

$$\ker \tilde{G}_1(y,x,t) = (N_1 \cap S_1)\big(y, x + \epsilon D^-(t) y, t\big) = \{0\}$$

We have to ensure a constant rk $\tilde{G}_0(x,t)$ in order to obtain the tractability index 1 of the regularized DAE (7.6). Using Notation (1.6), the latter condition is equivalent to a constant rank rk $I_n - h_{x_1}$.

Remark. Taking the addend $P_0(D^-)' Dx$ in (7.5) into consideration does not influence the computations leading to the tractability index one of the regularized DAE.

The constitutive function of the Knorrenschild approach reads

$$\hat{f}(y,x,t) := f(y,x,t) + (W_1 f)\big(P_0(t) x + \epsilon D^-(t)(Dx)'(t)\big) - (W_1 f)(P_0(t) x, t)$$

We get

$$\begin{aligned} \hat{f}_y(y,x,t) &= f_y(y, x + \epsilon D^-(t) y, t) + \epsilon (W_1 f)_x (y, x + \epsilon D^-(t) y, t) D^-(t) \\ \hat{f}_x(y,x,t) Q_0(t) &= f_x(y, x + \epsilon D^-(t) y, t) Q_0(t) \end{aligned}$$

because $W_1 f$ does not depend on $Q_0(t) x$. Accordingly,

$$\hat{G}_1(y,x,t) = G_1\big(y, x + \epsilon D^-(t) y, t\big) + \epsilon (W_1 f)_x \big(y, x + \epsilon D^-(t) y, t\big) P_0(t)$$

In contrast to the setup of Theorem 3.4, the general case $(W_1 f)_x P_0 \neq W_1 f_x P_0$ has to be respected because

$$\begin{aligned} &(W_1 f)_x (y, x + \epsilon D^-(t) y, t) P_0(t) z = \\ &[W_1(P_0(t) x + \epsilon D^-(t) y, t)]_x (z, f(y, x + \epsilon D^-(t) y, t)) \\ &+ W_1(P_0(t) x + \epsilon D^-(t) y, t) f_x(y, x + \epsilon D^-(t) y, t) P_0(t) z \end{aligned}$$

and $f(y, x + \epsilon D^-(t) y, t) \neq 0$, even if $f(y,x,t) = 0$. Let us assume that im $G_1(y,x,t)$ is t-dependent only and rk $W_1(t) f_x(y,x,t) = \text{const}$. Then,

$$\ker \hat{G}_1(y,x,t) = (N_1 \cap S_1)\big(y, x + \epsilon D^-(t) y, t\big) = \{0\}$$

and rk $\hat{G}_0(y,x,t) = \text{rk } G_0 + \epsilon W_1 f_x$ is constant, in other words (7.4) exhibits index 1.

Lemma 7.4. *The regularization approach* (7.4) *applied to a properly formulated index-2 DAE* (1.2) *where* im $G_1(y,x,t)$ *has a basis* $\{\beta_i(t)\}_{i=1,\ldots,r=\text{rk } G_1}$ *and* $W_1 f_x$ *has a constant rank results in tractability index one of the regularized system* (7.4).

A physical interpretation of the regularization process for some DAEs resulting from circuit simulation is also possible, cf. [Kno88] and the references therein.

7.3 A link to logarithmic norms for DAEs

The concept of a logarithmic norm introduced by S.M. Lozinskij and G. Dahlquist is a useful tool in the perturbation analysis of ordinary differential equations. According to [DV84, pp. 17-35], the logarithmic norm

$$\mu[A] := \lim_{h\to^+0} \frac{\|I_n + hA\| - 1}{h}$$

of a matrix $A \in \mathbb{R}^{n\times n}$ is the smallest possible one-sided Lipschitz constant of the linear ODE $x' = Ax$, if an inner product norm is used. A logarithmic norm enables stability estimates for *arbitrary* norms, thereby the one-sided Lipschitz condition (6.24) for nonlinear ODEs $x'(t) = f(x(t), t)$ is replaced by

$$\forall t \in I, x \in \mathcal{U}_t : \ \mu[f_x(x,t)] \le \beta(t) \tag{7.7}$$

The generalization of classical properties (as in [Str75, DV84]) of logarithmic norms for matrix pencils with an emphasis on asymptotic stability estimates for linear DAEs is obtained in [Cel98, GCH99]. In the context of tractability index, [Win00] provides an appropriate definition of a logarithmic matrix norm referring to the IRODE of a linear DAE, the resulting estimates being closely related to those of [GCH99] for index-1 systems. Moreover, nonlinear DAEs with tractability index 1 are treated using a generalization of (7.7), i.e. the logarithmic norm of the t-dependent linearization with respect to the subspace im P_0. The aim of [HS01] is to develop a conceptual functional-analytical framework for analysing asymptotic stability of DAEs such that linearizations and logarithmic norm notions applicable only to bounded operators are avoided [HS01, p. 825]. Their main tool is a *least upper bound logarithmic Lipschitz constant* of an operator $f : X \to X$ defined by

$$M^+[f] := \sup_{u,v\in X, u\neq v} \frac{(u-v, f(u) - f(v))_+}{\|u-v\|^2} \tag{7.8}$$

with respect to the semi-inner product

$$(u,v)_+ := \|u\| \lim_{h\to^+0} \frac{\|u+hv\| - \|u\|}{h}$$

and $(X, \|\cdot\|)$ being a Banach space. Hereafter, a monotonicity inequality for the propagation of perturbations in the differential solution components P_0x (where P_0 is a projector and $Q_0 = I - P_0$) of linear-implicit index-1 DAEs

$$P_0x'(t) = f(x(t))$$

are proved if the Q_0-restricted logarithmic Lipschitz constant

$$M^+_{Q_0}[Q_0f] := \sup_{\substack{u,\delta v\in X,\, Q_0\delta v\neq 0,\\ u+Q_0\delta v\in \mathrm{Dom}(f)}} \frac{(Q_0\delta v, Q_0f(u+Q_0\delta v) - Q_0f(u))_+}{\|Q_0\delta v\|^2}$$

is negative, cf. [HS01, Th. 3.5].

We have proved that a fully implicit index-2 system satisfying assumptions of Theorem 3.4 exhibits the following representation of its inherent dynamics on $(D\mathcal{M}_1)(t)$:

$$\begin{aligned} u'(t) &= f_{\text{dyn}}(u(t), t) \\ f_{\text{dyn}}(u, t) &= z + R'(t)\, u \end{aligned}$$

with $u = D(t)\, x$, $x \in \mathcal{M}_1(t)$ and $z = R(t)\, z$ such that $f(z, x, t) = 0$. Actually, f_{dyn} denotes the right hand side of the IRODE of the corresponding index reduced system on $(D\mathcal{M}_1)(t)$. The least upper bound logarithmic Lipschitz constant for f_{dyn} at a time $t \in I$ admits the representation

$$\begin{aligned} M^{+}[f_{\text{dyn}}(\cdot, t)] &= \sup_{u,v \in (D\mathcal{M}_1)(t), u \neq v} \frac{(u - v, f_{\text{dyn}}(u, t) - f_{dyn}(v, t))_{+}}{\|u - v\|^2} \\ &= \sup_{\substack{x_i \in \mathcal{M}_1(t),\, x_1 \neq x_2 \\ z_i = R(t) z_i:\, f(z_i, x_i, t) = 0}} \frac{(D(t)(x_1 - x_2), z_1 - z_2 + R'(t)\, D(t)(x_1 - x_2))_{+}}{\|D(t)(x_1 - x_2)\|^2} \end{aligned}$$

Now the dissipativity resp. monotonicity condition on the DAE reads

$$\forall t \geq 0: \; M^{+}[f_{\text{dyn}}(\cdot, t)] \leq \beta(t) \leq 0 \tag{7.9}$$

If an inner product norm is considered, then (7.9) is equivalent to D-component contractivity condition (6.29) for the DAE. According to Theorem 6.24, we get the asymptotic estimate (6.30), i.e. a non-expansive flow of the differential-algebraic system with respect to D-components of the solution vector. Notice that Criterion (7.9) neither builds on linearization nor requires the Lipschitz condition for f_{dyn}.

7.4 On practical computation of a Lyapunov function

> In theory, there is no difference between theory and practice. But, in practice, there is.
>
> *(Jan L. A. van de Snepscheut)*

Although Chapter 6 deals with Lyapunov functions for differential-algebraic systems of index 1 and 2 from a purely analytical point of view, we cannot deny the importance of a numerical method to obtain an approximation of a Lyapunov function, e.g. in order to check stability or even to estimate the domain of attraction of an asymptotically stable solution. To this end, the results from [Mar02][1] seem very promising to us. A continuous, piecewise linear Lyapunov function for the zero solution of an autonomous ODE $x'(t) = f(x(t))$ is constructed using convex optimization techniques, precisely Linear Programming. This approach contains involved details, but the simplified main idea is the following: a suitable grid (i.e. a simplicial partition of the domain) is constructed and adequate constraints for the linear program are stated such that a

[1] the contemporary name of the author is Sigurður Freyr Hafstein

piecewise linear, continuous Lyapunov candidate function is uniquely defined on this grid and satisfies the Lyapunov monotonicity condition in every grid point. Certainly, a piecewise linear Lyapunov function is not differentiable any longer, but this is not a problem because such a function fulfills a local Lipschitz condition.

On differentiability of a Lyapunov function

Actually, there is no need to require differentiability of the Lyapunov function, we can perfectly do with a continuous function $V(u,t)$ satisfying a local Lipschitz condition with respect to u. The property of $V(x(t),t)$ being non-increasing along solutions of the ODE $x'(t) = f(x(t),t)$ is ensured by its non-increasing total time derivative which can be expressed by the dissipativity/monotonicity inequality

$$\dot{V}(x,t) = V_t(x,t) + V_x(x,t) f(x,t) \leq 0$$

without actually solving the system. Another well-known approach is to require non-increasing right-hand upper *Dini derivative* D^+ of $V(x(t),t)$. The upper right hand Dini derivative of a function $g : I \subseteq \mathbb{R} \to \mathbb{R}^m$ at $t \in I$ is defined by

$$\mathrm{D}^+ g(t) := \limsup_{h \to^+ 0} \frac{g(t+h) - g(t)}{h}$$

According to [Yos66, § 1] and [Hah67, p. 196],

$$\mathrm{D}^+ V(x(t),t) = \limsup_{h \to^+ 0} \frac{V(x(t) + hf(x(t),t), t+h) - V(x(t),t)}{h}$$

along a solution x of $x'(t) = f(x(t),t)$ if $V(u,t) \geq 0$ is continuous and fulfills a local Lipschitz condition with respect to u. This is easily seen using continuity of involved functions, Taylor expansion of $x(t+h) = x(t) + x'(t+\xi_h)$, $\xi_h \in (t, t+h)$ and

$$V(x+y) - V(x) = \|V(x+y)\| - \|V(x)\| \leq \|V(x+y) - V(x)\| \leq L\|y\|$$

as well as $V(x) - V(x+y) \leq L\|y\|$. Consequently, the following definition of a non-differentiable Lyapunov function for DAEs arises:

Definition 7.5. [Non-differentiable Lyapunov function for D-component stability]
$V = V(u,t) \in C^0(\mathcal{U}_1, \mathbb{R})$ satisfying a local Lipschitz condition with respect to u is called a *(non-differentiable) Lyapunov function* for the reference solution x_* of the properly formulated DAE (1.2) with tractability index $k = 1, 2$ and $\operatorname{im} D(t) = \text{const.}$ if

1. $\forall t \geq 0 : V((Dx_*)(t), t) = 0$
2. There exists a positive definite function $H_1 \in C^0(\mathrm{B}_\epsilon(0) \subseteq \mathbb{R}^n, \mathbb{R})$ exhibiting
$$\forall t \geq 0,\ u \in (D\mathcal{M}_{k-1})(t) \cap \mathrm{B}_\epsilon((Dx_*)(t)) :\ H_1(u - (Dx_*)(t)) \leq V(u,t)$$
3. For all $t \geq 0$, $x \in \mathcal{M}_{k-1}(t)$, $\|D(t)(x - x_*(t))\| < \epsilon$, $z = R(t)z$ with $f(z,x,t) = 0$:
$$\dot{V}(D(t)x, t) := \limsup_{h \to^+ 0} \frac{V(D(t)x + hz, t+h) - V(D(t)x, t)}{h} \leq 0 \qquad (7.10)$$

Evaluation of the implicit monotonicity condition

The problem with Definition 7.5 is how to access the implicit monotonicity condition resp. dissipation inequality (7.10) in numerical computations? In case of autonomous index-1 DAEs (2.28), we are able to obtain the quantities necessary for Hafstein's algorithm such that it applies to the inherent regular ODE on im D:

1. Consider a suitable grid on the invariant subspace im D
2. Compute the consistent initialization for every grid point $u_i \in \operatorname{im} D$

$$(y_i, x_i) \in \mathbb{R}^n \times \mathbb{R}^m \text{ with } y_i = Ry_i,\ f(y_i, x_i) = 0$$

   of the DAE by the means of the *tractability index-1 projection* onto the constraint manifold $\mathcal{M}_0$. Precisely,

   a) The zero solution $(y_0, x_0) = (0, 0)$ is known in advance (starting point)
   b) The initial guess to a consistent initialization (y_i, x_i) corresponding to a u_i is $(y_j, D^- u_i + Q_0 x_j)$ where (y_j, x_j) denotes the consistent initialization belonging to an adjacent grid point u_j. Compute a Lagrange multiplier $\lambda \in \mathbb{R}^m$ by solving

$$F_i(\lambda) := f\left(y_j + D\lambda, \left(D^- u_i + Q_0 x_j\right) + Q_0 \lambda\right) = 0$$

   with Newton's method which is feasible because

$$\frac{\partial}{\partial \lambda} F_i = f_y D + f_x Q_0 = G_1$$

   is nonsingular and (y_i, x_i) is located nearby (y_j, x_j). Due to continuous dependence of index-1 solutions on initial values, last property applies if the grid is sufficiently fine. Then,

$$(y_i, x_i) := \left(Ry_j + D\lambda, \left(D^- u_i + Q_0 x_j\right) + Q_0 \lambda\right)$$

   is a consistent initialization of the DAE with $Dx_i = u_i$ and $y_i = Ry_i$.

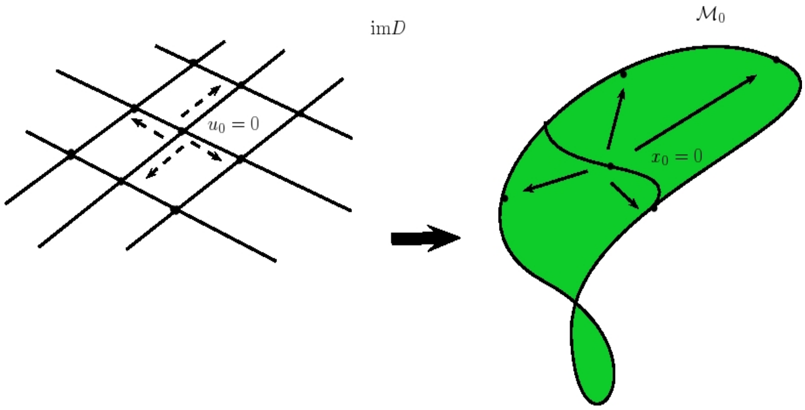

Figure 7.1: Order of processing the consistent initializations

In case of tractability index 2, a consistent initialization is more expensive, but nevertheless possible for DAEs allowing the index reduction, cf. [ES00, ESL01].

7.5 Open questions

Concerning the previous sections, we have to point out that practical sufficient conditions ensuring the decoupling procedure or the reduction of the tractability index 3 to 2 are still to be investigated. We believe that a reasonable way to generate applicable results is to identify a class of nonlinear, *fully implicit* index-3 DAEs where asymptotic stability results are of vital importance, stick to and exploit the structural properties of such system class.

Furthermore, it is an open question, how solutions of the regularized DAE relate to solutions of the original system, in the best case on unbounded common intervals of existence.

We have shown that cylindricity of the domain of the inherent dynamics is a key property in Lyapunov's direct method for DAEs. It is an issue of both theoretical and practical reason to derive further structural assumptions in order to obtain a cylindrical domain of the IRODE of the IR-DAE thus making Lyapunov's direct method in Chapter 6 applicable to C_D^1-unbounded solutions of differential-algebraic systems.

The interconnection between the functional-analytic concepts of least upper bound logarithmic Lipschitz constants applied to differential-algebraic systems à la [HS01] and dissipativity inequalities (monotonicity of a Lyapunov function via (6.7) resp. (6.13), D-component contractivity (6.29)) stated in § 7.3 certainly deserves further refinements. We plan to investigate this issue in a forthcoming paper.

Last but not least, the numerical treatment of the new qualitative resp. theoretical aspects presented in this thesis is also desirable. As indicated in the course of Chapter 5, certain background on numerical methods for the computation of characteristic multipliers and related questions like solving boundary value problems for DAEs is available. In our personal opinion, situation becomes less comfortable when it comes to Lyapunov functions. We hope that considerations in Section 7.4 might be helpful to obtain a piecewise linear approximation to a Lyapunov function for a zero-solution of an autonomous DAE. For smaller index-1 problems the computational costs are conjectured to be reasonable. The practical realization of the approach and further development aiming at tractability index two DAEs need to be performed. One additional interesting issue in geometric numerical integration is preservation of a Lyapunov function by numerical methods. Intuitively, there is some similarity to the concepts around B-stability and contractivity of numerical methods applied to contractive differential equations, the latter questions already studied for DAEs with index 1,2 in [MHT03a, MHT03b, San00]. As far as we know, some results are available for ODEs (e.g. [GQ05, MQR98, MQR99]) but the DAE case is still unrevealed.

8 Appendix

8.1 Auxiliary results

Lemma 8.1. *[Basic properties of projectors]*

Let $V, W \subseteq \mathbb{R}^n$ be linear subspaces.

1. *If Q is a projector on V along W then its complementary projector $P = I - Q$ projects along V on W*
2. *It holds $\ker P \oplus \operatorname{im} P = \mathbb{R}^n$ for a projector P onto a subspace of $\mathbb{R}^n$*
3. *If Q_1, Q_2 both project on V then $Q_1 Q_2 = Q_2$*
4. *If P_1, P_2 are projectors along W then $P_1 P_2 = P_1$*
5. *Assuming $V \oplus W = \mathbb{R}^n$, there exists a unique projector P realizing this decomposition of $\mathbb{R}^n$, i.e. projecting on V along W.*
6. *A projector P is self-adjoint iff it is an orthogonal projector, i.e. P projects on V along $V^\perp$.*

1)-4) follow immediately from the definition of complementary projectors and the idempotence $P^2 = P$ of a projector.

On 5) It is assumed that $V \oplus W = \mathbb{R}^n$ so consider bases $\{v_1, \ldots, v_s\}$ of V and $\{w_1, \ldots, w_{n-s}\}$ of W. Then, $\{v_1, \ldots, v_s, w_1, \ldots, w_j\}$ are linearly independent and form the columns of a nonsingular matrix

$$M := (v_1, \ldots, v_s, w_1, \ldots, w_{n-s})$$

Then, the following basis representation

$$[P]^{\{v_i, w_j\}}_{\{v_i, w_j\}} = \begin{pmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{pmatrix}$$

constitutes a projector on V along W. At this, $[T]^{\{a_i\}}_{\{b_i\}}$ denotes the matrix representation of a linear mapping T with respect to the basis $\{a_i\}$ of input vectors and the basis $\{b_i\}$ of output vectors. The change of bases results in

$$[P]^{\{e_i\}}_{\{e_i\}} = M [P]^{\{v_i, w_j\}}_{\{v_i, w_j\}} M^{-1} = M \begin{pmatrix} I_s & 0 \\ 0 & 0_{n-s} \end{pmatrix} M^{-1} \tag{8.1}$$

because $M = [I]_{\{e_i\}}^{\{v_i,w_j\}}$ is the transformation matrix of the basis $\{v_i, w_j\}$ into the canonical basis $\{e_i\}$ of $\mathbb{R}^n$. For two projectors P_1, P_2 onto V along W,

$$P_1 x = P_1 (v + w) = P_1 v = v = P_2 v = P_2 x$$

that is

$$P_1 = P_2$$

holds, whereas $x = v + w$ is the unique decomposition with $v \in V$, $w \in W$ due to $V \oplus W = \mathbb{R}^n$.

On 6) With P being a projector, P^* is also idempotent with $\mathrm{im}\, P^* = (\ker P)^\perp$, $\ker P^* = (\mathrm{im}\, P)^\perp$. The property $P = P^*$ implies $\mathrm{im}\, P = (\ker P)^\perp$ and $\ker P = (\mathrm{im}\, P)^\perp$ thus fixing the orthogonal projector. Conversely, the orthogonal projector satisfies $P = P^*$.

Lemma 8.2. *Let $V(t) \subseteq \mathbb{R}^n$ be a parameter dependent linear subspace. If $V(t)$ is T-periodic and has a basis consisting of continuously differentiable functions then there exists a T-periodic C^1-basis of $V(t)$.*

Proof. The proof follows the lines of [LMW03, § 3]. Consider the basis $\{c_1(t), \ldots, c_r(t)\}$ of $V(t)$ consisting of C^1-functions and build a matrix valued function

$$C(t) := (c_1(t), \ldots c_r(t))$$

of these column vectors. According to Theorem 6 from [GM86, p. 195] the expression

$$\mathcal{P}(t) := C(t) \left(C^T(t) C(t)\right)^{-1} C^T(t)$$

defines an orthogonal C^1-projector on $V(t)$.

We have assumed $V(t+T) = V(t)$ so for all $y \in V(t+T) = V(t)$:

$$\mathcal{P}(t)y = y = \mathcal{P}(t+T)y,$$

in addition to $\mathcal{P}(t)y = 0 = \mathcal{P}(t+T)y$ for $y \in (V(t+T))^\perp = (V(t))^\perp$. It follows $\mathcal{P}(t+T) = \mathcal{P}(t)$.

Now it is to prove that solutions $\{\alpha_i\}_{i=1,\ldots,r}$ of the initial value problems

$$\alpha_i'(t) = \mathcal{P}'(t)\alpha_i(t), \qquad \alpha_i(0) = c_i(0)$$

constitute a $T-$periodic C^1-Basis of $V(t)$.

Obviously, the above C^1-solutions α_i possess the representation

$$\alpha_i(t) = \exp\left(\int_0^t \mathcal{P}'(s) ds\right) c_i(0)$$

The $T-$periodicity of α_i results from the fundamental theorem of calculus and $T-$periodicity of $\mathcal{P}(t)$ via

$$\begin{aligned} \alpha_i(t+T) &= \exp\left(\int_0^{t+T} \mathcal{P}'(s) ds\right) c_i(0) = \exp\left(\mathcal{P}(t+T) - \mathcal{P}(0)\right) c_i(0) \\ &= \exp\left(\mathcal{P}(t) - \mathcal{P}(0)\right) c_i(0) = \alpha_i(t) \end{aligned}$$

α_i are linearly independent because same holds for the initial values $c_i(0)$. With $\mathcal{Q} = I - \mathcal{P}$ we obtain

$$\begin{aligned}(\mathcal{Q}\alpha_i)'(t) &= \mathcal{Q}'(t)\alpha_i(t) + \mathcal{Q}(t)\alpha_i'(t) \\ &= \mathcal{Q}'(t)\,(I - \mathcal{P}(t))\,\alpha_i(t) \\ &= \mathcal{Q}'(t)\,(\mathcal{Q}\alpha_i)\,(t)\end{aligned}$$

Due to $\mathcal{Q}(0)\alpha_i(0) = 0$ of this linear ODE we obtain $\mathcal{Q}\alpha_i \equiv 0$ resp. $\mathcal{P}\alpha_i = \alpha_i$, i.e. $\{a_1(t), \ldots, a_r(t)\}$ constitute a T-periodic C^1-basis of $V(t)$. □

Lemma 8.3. *Consider an open set $\tilde{\mathcal{G}} \subseteq \mathbb{R}^m$ and a matrix valued function $M \in C^1(\tilde{\mathcal{G}}, \mathbb{R}^{k\times s})$ together with $b \in C^1(\tilde{\mathcal{G}}, \mathbb{R}^s)$. The Jacobian $[M(x_*)\,b(x_*)]_x$ has the following representation*

$$[M(x_*)\,b(x_*)]_x\, z = M(x_*)\,b_x(x_*)\,z + [M(x_*)]_x\,(z, b(x_*))$$

with $z \in \mathbb{R}^m$ and the bilinear form $[M(x_*)]_x$ defined by

$$[M(x_*)]_x\,(z, v) := \sum_{j=1}^{m} z_j \left[\frac{\partial}{\partial x_j} M(x_*)\right] v \tag{8.2}$$

Moreover, the i-th component of $[M(x_*)]_x$ features the representation

$$\left([M(x_*)]_x\,(z, v)\right)_i = \sum_{l=1}^{s} v_l\,(\mathrm{D}M_{il}(x_*)\,z) = \sum_{l=1}^{s} v_l \frac{\partial}{\partial z} M_{il}(x_*) \tag{8.3}$$

Proof. We validate the above representations for the i-th component, $i = 1, \ldots, k$.

For the j-th unit vector $e_j \in \mathbb{R}^m$ it holds $(e_j)_i = \delta_{ij}$. The product rule for real functions implies

$$\begin{aligned}([Mb]_x e_j)_i &= ([Mb]_x)_{ij} = \frac{\partial}{\partial x_j}(Mb)_i = \sum_{l=1}^{s} \frac{\partial}{\partial x_j}(M_{il}b_l) = \sum_{l=1}^{s} b_l \frac{\partial}{\partial x_j} M_{il} + \sum_{l=1}^{s} M_{il} \frac{\partial}{\partial x_j}(b_l) \\ &= \left(\left[\frac{\partial}{\partial x_j} M(x)\right] b(x)\right)_i + \sum_{l=1}^{s} M_{il}(x)\left(b_x(x)\right)_{lj} \\ &= \left(\left[\frac{\partial}{\partial x_j} M(x)\right] b(x)\right)_i + \left(M(x)\,b_x\right)_{ij} \\ &= \left(\left[\frac{\partial}{\partial x_j} M(x)\right] b(x)\right)_i + \left(M(x)\,b_x(x)\,e_j\right)_i\end{aligned}$$

Use the linear combination $z = \sum_{j=1}^{m} z_j e_j$ in order to obtain Representation (8.2) for $[M(x)]_x\,(z, b(x))$.

Considering the i-th component of $[M(x)]_x(z,b)$ and (8.2),

$$\begin{aligned}([M(x)]_x(z,v))_i &= \left(\sum_{j=1}^m z_j\left[\frac{\partial}{\partial x_j}M(x)\right]v\right)_i\\ &= \sum_{j=1}^m z_j\left(\left[\frac{\partial}{\partial x_j}M(x)\right]v\right)_i\\ &= \sum_{j=1}^m\sum_{k=1}^s z_j\left[\frac{\partial}{\partial x_j}M(x)\right]_{ik}v_k\end{aligned}$$

Now $\frac{\partial}{\partial x_j}M(x)$ is defined componentwise so $\left[\frac{\partial}{\partial x_j}M(x)\right]_{ik} = \frac{\partial}{\partial x_j}\left[M(x)\right]_{ik}$ hence

$$\begin{aligned}\left([M(x)]_x(z,b)\right)_i &= \sum_{j=1}^m\sum_{k=1}^s z_j\frac{\partial}{\partial x_j}\left[M(x)\right]_{ik}b_k\\ &= \sum_{k=1}^s b_k\left(\sum_{j=1}^m\frac{\partial}{\partial x_j}\left[M(x)\right]_{ik}z_j\right) = \sum_{k=1}^s b_k\left[\mathrm{D}M_{ik}z\right]\end{aligned}$$

$\mathrm{D}M_{ik}z$ is the directional derivative $\frac{\partial}{\partial z}M_{il}(x)$ for continuously differentiable $M_{ik}(x)$. This proves the lemma. □

Lemma 8.4. *(Characterization of autonomous ODEs)*
Let $D \subseteq \mathbb{R}^m$ be a region and $f \in C^1(D,\mathbb{R})$. The explicit ordinary differential equation

$$x'(t) = f(x(t),t) \tag{8.4}$$

is autonomous if and only if all solutions x are invariant under translation, i.e. $\forall c \in \mathbb{R}:\ \tilde{x}(t) = x(t+c)$ also solves (8.4).

Proof. Solutions of an autonomous ODE are obviously invariant under translations.

Require that all solutions of (8.4) are translation-invariant. Let us assume $f_t \neq 0$, i.e. there exists a $x_0 \in D$ and $t_0, t_1 \in \mathbb{R}$ with $f(x_0,t_0) \neq f(x_0,t_1)$. Here, D is an open set and f continuously differentiable. According to the theorem of Picard-Lindelöf, there exists a unique solution $x \in C^1(I,D)$ of the IVP $x(t_0) = x_0$ of (8.4) on the interval $I := (t_0-\epsilon, t_0+\epsilon)$ for a sufficiently small $\epsilon > 0$.

Define $c := t_1 - t_0$ and the function $\tilde{x}(t) := x(t-c)$ in $C^1(I+c,D)$. By virtue of translation invariance of solutions associated to (8.4) and the choice of the t-domain, $\tilde{x}$ is a solution of the ODE (8.4) satisfying the initial values

$$\tilde{x}(t_1) = x(t_1-c) = x(t_0) = x_0$$

Per constructionem $\tilde{x}'(t_1) = x'(t_0)$ is valid which contradicts the assumptions because

$$\tilde{x}'(t_1) = f(\tilde{x}(t_1),t_1) = f(x_0,t_1) \neq f(x_0,t_0) = x'(t_0)$$

□

8.2 Decoupling of nonlinear DAEs using a projector $\tilde{P}_1$ on S_1

§ 2.1 covers in detail the nonlinear decoupling using a projector Q_1 on N_1 along $K \supseteq N_0$. Supplementary, we decouple DAEs applying a projector $\tilde{P}$ on S_1 leading to an inherent regular ODE on the associated invariant subspace DS_1. Due to its structural complexity, the mentioned representation seems to be hardly suitable for further investigations.

Consider linear-implicit DAEs (2.5) with $A = A(t)$ possessing a solution $x_* \in C_D^1(I, \mathbb{R}^m)$. Again, we exploit the equivalent representation (2.3) of the form

$$A(t)(Dx)'(t) + b_x^*(t)x(t) + \tilde{h}(x(t), t) - r(t) = 0$$

Assume that the subspaces $(DS_1)(x_*(t), t)$ and $(DK)(x_*(t), t)$ are constant. Following the lines of § 1.5.2, the steps of the linear decoupling of

$$A(t)(Dx)'(t) + b_x^*(t)x(t) = 0$$

are applied to the entire DAE. Thereby we use the canonical index-2 projector $P_1(x,t) = P_{1,can}(x,t)$ on $S_1(x,t)$ along $N_1(x,t)$ and a projector $\tilde{P}_1(x,t)$ on $S_1(x,t)$ along $N_1(x,t)$ resulting in the equivalent system

$$\begin{array}{r} u'(t) + \left(DP_1\tilde{Q}_1D^-\right)(t)(Dy)'(t) \\ + \left(DP_1G_2^{-1}\right)(t)\left(h\left(u(t), y(t), w(t), t\right) - r(t)\right) \\ + \left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) + \left(DP_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)y(t) \end{array} = 0 \tag{8.5}$$

$$\begin{array}{r} \left(I + \left(UQ_0 + P_0\tilde{P}_1\right)G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)y(t) \\ + \left(\left(PQ_1 + UQ_0\right)G_2^{-1}\right)(t)\left(h\left(u(t), y(t), w(t), t\right) - r(t)\right) \\ + \left(UQ_0G_2^{-1}b_x^*D^-\right)(t)u(t) \end{array} = 0 \tag{8.6}$$

$$\begin{array}{r} \left(TQ_0P_1G_2^{-1}b_x^*D^-\right)(t)u(t) + w(t) \\ + \left(TQ_0P_1G_2^{-1}\right)(t)\left(h\left(u(t), y(t), w(t), t\right) - r(t)\right) \\ - \left(Q_0Q_1D^-\right)(t)(Dy)'(t) + \left(TQ_0P_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)y(t) \end{array} = 0 \tag{8.7}$$

Here, $P_1(t) := P_1(x_*(t), t)$ etc. and u, y, w defined according to (1.32).

Equation (8.6) constitutes

$$\tilde{M}\left(u(t), w(t), \tilde{Z}(t)y(t), t\right) = 0$$

with $\tilde{Z}(t) = \left(UQ_0 + P_0\tilde{Q}_1\right)(t)$ and

$$\tilde{M}(u, w, y, t) = \begin{array}{l} \left(I + \left(UQ_0 + P_0\tilde{P}_1\right)G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)y + \left(UQ_0G_2^{-1}b_x^*D^-\right)(t)u \\ + \left(\left(PQ_1 + UQ_0\right)G_2^{-1}\right)(t)\left(h(u, y, w, t) - r(t)\right) \end{array}$$

The partial derivatives of this function are

$$\begin{array}{rcl} \tilde{M}_y(u,w,y,t) & = & \left(I+\left(UQ_0+P_0\tilde{P}_1\right)G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t) \\ & & +\left(\left(PQ_1+UQ_0\right)G_2^{-1}\right)(t)h_y(u,y,w,t) \\ \tilde{M}_u(u,w,y,t) & = & \left(UQ_0G_2^{-1}b_x^*D^-\right)(t)+\left(\left(PQ_1+UQ_0\right)G_2^{-1}\right)(t)h_u(u,y,w,t) \\ \tilde{M}_w(u,w,y,t) & = & \left(\left(PQ_1+UQ_0\right)G_2^{-1}\right)(t)h_w(u,y,w,t) \end{array}$$

Take

$$h_y(u_*(t),w_*(t),y_*(t),t)=\tilde{h}_x(x_*(t),t)\left(UQ_0+P_0\tilde{Q}_1\right)(t)\equiv 0$$

and $\tilde{Q}_1(t)\tilde{Z}(t)=\tilde{Q}_1(t)$ into consideration resulting in

$$\tilde{M}_z(u_*(t),w_*(t),Z(t)y_*(t),t)=\left(I+\left(UQ_0+P_0\tilde{P}_1\right)G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)$$

with $z=\tilde{Z}(t)\,y$. This matrix is regular for all $t\in I$ so there exists a unique implicitly defined function $\tilde{m}=\tilde{m}(u,w,t)$ exhibiting

$$\begin{array}{lcl} \tilde{m}(u_*(t_0),w_*(t_0),t_0) & = & y_*(t_0) \\ \tilde{M}\left(u,w,\tilde{Z}(t)\tilde{m}(u,w,t),t\right) & = & 0 \\ \tilde{m}(u,w,t) & = & \left(UQ_0+P_0\tilde{Q}_1\right)(t)\tilde{m}(u,w,t) \\ \tilde{m}_w(u,w,t) & = & -\left(\tilde{M}_y^{-1}\tilde{M}_w\right)(u,w,\tilde{m}(u,w,t),t) \\ \tilde{m}_u(u,w,t) & = & -\left(\tilde{M}_y^{-1}\tilde{M}_u\right)(u,w,\tilde{m}(u,w,t),t) \end{array}$$

A complete decoupling of the DAE can be obtained under the assumption

$$Q_1(t)\tilde{m}_w(u,w,t)=0\Leftrightarrow Q_1(t)\tilde{m}=Q_1(t)\tilde{m}(u,t) \tag{8.8}$$

Then,

$$\frac{d}{dt}\left(D(t)\tilde{m}(u(t),t)\right)=\left(D\tilde{m}\right)_u(u(t),t)\,u'(t)+\left(D\tilde{m}\right)_t(u(t),t)$$

Inserting this into (8.5) provides the constraint

$$\left.\begin{array}{l} \left(I+\left(DP_1\tilde{Q}_1\tilde{m}\right)_u(u(t),t)\right)u'(t)+\left(DP_1\tilde{Q}_1D^-\right)(t)\left(D\tilde{m}\right)_t(u(t),t) \\ +\left(DP_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)\tilde{m}(u(t),t)+\left(DP_1G_2^{-1}f_x^*D^-\right)(t)u(t) \\ +\left(DP_1G_2^{-1}\right)(t)\left(h\left(u(t),\tilde{m}\left(u(t),w(t),t\right),w(t),t\right)-r(t)\right) \end{array}\right. =0$$

Again, this equation can be solved for u' locally around the integral curve of x_* because $\tilde{h}_x(x_*(t),t)=0$ implies

$$D(t)\tilde{m}_u(u_*(t),w_*(t),t)=-\left(D(t)+D\tilde{P}_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)\left(UQ_0G_2^{-1}b_x^*D^-\right)(t)=0$$

According to a perturbation lemma, $I+\left(DP_1\tilde{Q}_1D^-D\tilde{m}_u\right)(u,t)$ is regular in a neighbourhood of the integral curve of x_* where $\left\|\left(DP_1\tilde{Q}_1D^-D\tilde{m}_u\right)(u,t)\right\|<1$. Therefore,

$$u'(t)=v\left(u(t),w(t),t\right) \tag{8.9}$$

with

$$v(u,w,t) = -\left(I + \left(DP_1\tilde{Q}_1\tilde{m}_u\right)(u,t)\right)^{-1} \cdot$$
$$\cdot \begin{cases} \left(DP_1\tilde{Q}_1D^-\right)(t)\,(D\tilde{m})_t\,(u,t) + \left(DP_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)\tilde{m}\,(u,t) \\ + \left(DP_1G_2^{-1}b_x^*D^-\right)(t)u + \left(DP_1G_2^{-1}\right)(t)\,(h\,(u,\tilde{m}\,(u,w,t)\,,w,t) - r(t)) \end{cases}$$

Summing up, (8.7) can be written as

$$\tilde{K}\,(u,(TQ_0)\,(t)w,t) = 0$$

with

$$\tilde{K}\,(u,w,t) = \begin{cases} \left(TQ_0P_1G_2^{-1}b_x^*D^-\right)(t)u + \left(TQ_0P_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)\tilde{m}\,(u,t) + w(t) \\ + \left(TQ_0P_1G_2^{-1}\right)(t)\,(h\,(u(t),\tilde{m}\,(u,w,t)\,,w(t),t) - r(t)) \\ - \left(Q_0Q_1D^-\right)(t)\,(D\tilde{m})_t\,(u,t) - \left(Q_0Q_1D^-\right)(t)\,(D\tilde{m})_u\,(u,t)\,v\,(u,w,t) \end{cases}$$

With $\xi = (TQ_0)\,(t)\,w$ it holds

$$\begin{aligned} &\tilde{K}_\xi\,(u,(TQ_0)\,(t)w,t) = \\ &\left(TQ_0P_1G_2^{-1}\right)(t)\,(h_w + h_y\tilde{m}_w)\begin{pmatrix} u,\tilde{m}\,(u,(TQ_0)\,(t)w,t)\,, \\ (TQ_0)\,(t)w,t \end{pmatrix}(TQ_0)\,(t) \\ &+I - \left(Q_0Q_1D^-\right)(t)\,(D\tilde{m})_u\,(u,t)\,v_w\,(u,(TQ_0)\,(t)w,t)\,(TQ_0)\,(t) \end{aligned}$$

and

$$v_w\,(u,(TQ_0)\,(t)w,t) = \begin{aligned} &-\left(I + \left(DP_1\tilde{Q}_1\tilde{m}_u\right)(u,t)\right)^{-1} \cdot \\ &\cdot\left(DP_1G_2^{-1}\,(h_y\tilde{m}_w + h_w)\,TQ_0\right)(u,(TQ_0)\,(t)w,t) \end{aligned}$$

The partial derivatives h_y, h_w vanish in $(u_*(t), y_*(t), w_*(t), t)$ so

$$\tilde{K}_\xi\,(u_*(t),(TQ_0)\,(t)w_*(t),t) \equiv I_m$$

Consequently a local resolution function $\tilde{k} = \tilde{k}\,(u,t)$ satisfying

$$\begin{aligned} \tilde{k}\,(u_*(t_0),t_0) &= w_*(t_0) \\ \tilde{K}\left(u,(TQ_0)\,(t)\tilde{k}\,(u,t)\,,t\right) &= 0 \\ \tilde{k}\,(u,t) &= (TQ_0)\,(t)\tilde{k}\,(u,t) \end{aligned}$$

exists. Inserting this function in (8.9) we obtain the explicit differential equation $u'(t) = v\left(u(t),\tilde{k}\,(u(t),t)\,,t\right)$, more precisely

$$\begin{aligned} u'(t) = &-\left(I + \left(DP_1\tilde{Q}_1\tilde{m}_u\right)(u(t),t)\right)^{-1} \cdot \\ &\cdot \begin{cases} \left(DP_1\tilde{Q}_1D^-\right)(t)\,(D\tilde{m})_t\,(u(t),t) + \left(DP_1G_2^{-1}b_x^*P_0P_1\tilde{Q}_1\right)(t)\tilde{m}\,(u(t),t) \\ + \left(DP_1G_2^{-1}\right)(t)\left(h\left(u,\tilde{m}\left(u,\tilde{k}\,(u(t),t)\,,t\right),\tilde{k}\,(u(t),t)\,,t\right) - r(t)\right) \\ + \left(DP_1G_2^{-1}b_x^*D^-\right)(t)u(t) \end{cases} \end{aligned} \tag{8.10}$$

This ODE is called the *inherent regular ODE* of the original system (2.5) with respect to the decoupling with $\tilde{P}_1$ on S_1.

Lemma 8.5. *DS_1 is an invariant subspace of the IRODE* (8.10)

Proof. Notice that for (u,t) sufficiently close to $(u_*(t),t)$ the estimate

$$\left\| \left(DP_1\tilde{Q}_1\tilde{m}_u \right)(u(t),t) \right\| < 1$$

holds. For this reason the inverse of $I + DP_1\tilde{Q}_1\tilde{m}_u$ can be represented by the Neumann series,

$$\left(I + DP_1\tilde{Q}_1\tilde{m}_u \right)^{-1} = \sum_{n\in\mathbb{N}} (-1)^n \left(DP_1\tilde{Q}_1\tilde{m}_u \right)^n$$

It follows that

$$\left(I - D\tilde{P}_1D^- \right)\left(I + \left(DP_1\tilde{Q}_1\tilde{m}_u \right) \right)^{-1} DP_1 = \begin{array}{c} \left(I - D\tilde{P}_1D^- \right) \cdot \\ \cdot \left(I + DP_1 \left(\tilde{Q}_1\tilde{m}_u + \ldots \right) \right) DP_1 \end{array} = 0$$

because $\left(I - D\tilde{P}_1D^- \right)$ projects along $DS_1 = \text{im } DP_1$. Therefore,

$$\left(I - D\tilde{P}_1D^- \right)(t)u'(t) = 0$$

Using a constant projector $\hat{P}$ on DS_1,

$$\left(D\tilde{P}_1D^- \right)' D\tilde{P}_1D^- = \left(D\tilde{P}_1D^-\hat{P} \right)' D\tilde{P}_1D^- = 0$$

so $v(t) := \left(I - D\tilde{P}_1D^- \right)(t)u(t)$ solves the homogeneous ODE

$$\begin{aligned} v'(t) &= \left(I - D\tilde{P}_1D^- \right)'(t)u(t) + \left(I - D\tilde{P}_1D^- \right)(t)u'(t) \\ &= -\left(D\tilde{P}_1D^- \right)'(t)v(t) \end{aligned}$$

Consequently, $u(t_0) \in DS_1 \Leftrightarrow v(t_0) = 0 \Rightarrow v(t) \equiv 0 \Leftrightarrow u(t) \in DS_1$. □

The structural condition (8.8) reads

$$0 = (Q_1\tilde{m}_w)(u,w,t) = -Q_1 \left(\tilde{M}_y^{-1}\tilde{M}_w \right)(u,w,\tilde{m}(u,w,t),t)$$

with $\tilde{M}_w(u,w,y,t) = \left((PQ_1 + UQ_0) G_2^{-1} \right)(t)\tilde{h}_x(D^-u + y + w, t)(TQ_0)(t)$. Similarly to the proof of Lemma 2.9 the simplified assumption (2.15) implies $\tilde{M}_w \equiv 0$ and the feasibility of the decoupling of linear implicit DAEs having $DS_1 = \text{const.}$ without the need for $Q_0Q_1D^- \in C^1$. Summarizing the assumptions,

Theorem 8.6. *Consider the index-2 DAE*

$$A(t)(Dx)'(t) + b(x(t),t) = 0$$

together with a solution $x_* \in C_D^1(I, \mathbb{R}^m)$ *on a compact interval* I. *Assume that*

$$\begin{gathered}\left(\left(UQ_0 + P_0 Q_{1,can}\right) G_{2,can}^{-1} b_x^* P_0 P_{1,can}\right)(t),\\ \left(\left(UQ_0 + P_0 Q_{1,can}\right) G_{2,can}^{-1}\right)(t)\left(b\left(x,t\right) - b_x^*(t)x\right) \in C^2\end{gathered}$$

is independent of x *in a neighbourhood of the integral curve of* x_* *and*

$$\left(DS_1\right)\left(x_*(t),t\right) = const.$$

is valid. Let $\tilde{P}_1(t)$ *be a projector on* $N_1\left(x_*(t),t\right)$ *along* $K(t)$ *such that* DK *is constant. Then the inherent dynamics of the DAE for the* $D\tilde{P}_1$*-components nearby* x_* *is determined by the IRODE* (8.10) *on its invariant subspace* DS_1.

Proof. The smoothness assumptions imply $\tilde{M}$ to be twice continuously differentiable with respect to u, w, y. This property carries over to $\tilde{m}$ due to the implicit function theorem. Because of $N_0(t) \cap S_0(x,t) = \left(N_0 \cap S_0\right)(t)$ there exists an implicitly defined function $\tilde{k}$ like above which is continuously differentiable w.r.t. u, w. Consequently, $D\tilde{P}_1$-components of a solution $x \in B_\epsilon(x_*) \in C_D^1(I, \mathbb{R}^m)$ of the original DAE satisfy necessarily the inherent regular ODE (8.10) on DS_1. Vice versa, every solution of the IRODE (8.10) on the invariant subspace DS_1 induces a solution $x = D^- u + w + y$ of the original DAE. $\square$

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Index

Eigenständigkeitserklärung

Ich versichere, dass ich die von mir vorgelegte Dissertation selbständig angefertigt, die benutzten Quellen und Hilfsmittel vollständig angegeben und die Stellen der Arbeit – einschließlich Tabellen, Karten und Abbildungen –, die anderen Werken im Wortlaut oder dem Sinn nach entnommen sind, in jedem Einzelfall als Entlehnung kenntlich gemacht habe; dass diese Dissertation noch keiner anderen Fakultät oder Universität zur Prüfung vorgelegen hat; dass sie – abgesehen von unten angegebenen Teilpublikationen – noch nicht veröffentlicht worden ist sowie, dass ich eine solche Veröffentlichung vor Abschluss des Promotionsverfahrens nicht vornehmen werde. Die Bestimmungen der Promotionsordnung sind mir bekannt. Die von mir vorgelegte Dissertation ist von Frau Prof. Dr. Caren Tischendorf betreut worden.

Hürth, den 25.10.2010

Michael Menrath