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Optimal control of second-order and high-order descriptor systems

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Summary

This study investigates the optimal control problem of second-order descriptor systems. The optimal control is characterized by using a new second-order generalized Riccati equation, which is directly derived in terms of the original coefficient matrices of the system. Under some assumption conditions, applying matrix's singular value decomposition and matrix transformation, a nonlinear generalized Riccati matrix equation is transformed into a linear matrix equation, and the optimal control gain can be determined by solving the linear matrix equation. Furthermore, relevant results are also extended to the optimal control problems for the high-order descriptor system. Finally, several simulation examples and the comparison with the existing linearization method are provided to illustrate the effectiveness of the developed approach.

KEYWORDS

generalized Riccati equation, optimal control, second-order singular system, singular value decomposition

1 | INTRODUCTION

Descriptor systems (or singular systems, differential-algebraic equations) are more natural mathematical descriptions than normal state space systems for the modeling of many real-world physical systems, including electrical circuit systems, economic system, mechanical systems, and other areas.[1,2] In the past decades, many significant basic theories of normal state space systems have been fluently generalized to descriptor systems, for instance, controllability and observability,[3,4] Lyapunov stability,[5,6] and H_∞ control.[7,8] Optimal control is an important branch of the modern control theory, in the last decades, optimal control problems of first-order descriptor systems have been studied extensively, many significant results have been achieved (see other works[9-14]). Thus, it can be seen that the research on the optimal control problem of first-order descriptor systems has already developed a relatively mature theory.

As the research moves along, many scholars find that in practical applications, a large number of physical systems are more suitable to model by second-order or high-order differential algebraic equations.[15-17] Second-order systems have received wide attention from different fields in mechanical vibration systems,[18-20] robotics control, and control of large flexible space structures[2,19] in the last few decades. Meanwhile, a large number of works have been published in the field of the analysis and synthesis of second-order descriptor systems, such as controllability and observability conditions,[15] stability conditions,[21,22] robust pole placement problem of second-order descriptor systems with velocity and acceleration feedback,[23-25] partial eigenstructure assignment of second-order undamped vibration systems,[26] output-feedback control,[27,28] and robust H_∞ control of uncertain mechanical systems.[29] Recently, Moysis et al[30] have determined the state response of the linear constrained mechanical system described by a high-order matrix differential equations.

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However, most of the above mentioned results mainly focus on the stability analysis of second-order systems, very few studies have been done about the optimal control design of second-order systems under a given quadratic performance index in the current works. The relevant articles published about the optimal control problem of second-order linear systems are those of Ram and Inman[31] and Zhang.[32] In the work of Ram and Inman,[31] the control input matrix is required to be an invertible square matrix, which should be impractical in applications. Zhang[32] removes the invertibility assumption on the control input matrix and obtains the optimal feedback gain matrix by solving a series of linear equations, which avoids involving the classical nonlinear Riccati equation. In addition, Kaliński and Galewski[33] propose an active optimal control method based on directly the acceleration signal for vibration surveillance in mechanical systems. Kaliński and Mazur[34] design the optimal controllers for the two-wheeled mobile robot to realize tracking of the desired trajectory. In the above works,[31-34] the system matrices are all required to be symmetric and positive definite, but actually, the system with a singular mass matrix can represent more practical mechanical systems, and some nonconservative systems usually have nonsymmetric system matrices,[35] in this case, we find that the existing methods may be not valid. Moreover, the traditional way to deal with second-order or even high-order descriptor systems is to transform them into first-order systems by introducing new variables, but when the leading coefficient matrix is singular, the classical linearization method may change the infinite zero structures and destroy some controllability properties; the solution space of the original system may be not equivalent to that of the linearized system.[15] For the large systems, the linearized system suffers from increased dimension, the computational efficiency is greatly reduced.[21] To avoid variable transformations and retain the physical meaning of the original system, in this paper, we address the optimal control problem of second-order descriptor systems directly utilizing the original system's parameters. Up to now, to the best of our knowledge, there is no unified methods to solve the optimal control problem of second-order descriptor systems without transformation into the first-order form. Inspired by the optimal control theory for linear systems via analyzing a Riccati matrix equation, the optimal control problem of second-order descriptor systems based on Riccati matrix equation is still open and has not been deeply investigated.

In the present work, we try to solve the optimal control problem for second-order systems with singular mass matrix by applying the general Riccati matrix equation method, so that the determined feedback controller can stabilize the closed-loop system and minimize a given performance index. Compared with the existing works about the optimal control problems of second-order systems, the main contributions of this paper are listed as follows: (1) without any linearization, and the restrictive assumption on the mass matrix used in the earlier works,[31,32] the solvability of the optimal control problem for second-order descriptor systems is converted into the solvability of a nonlinear Riccati matrix equation, which is derived directly by means of the coefficient matrices of the original descriptor systems; (2) to avoid the complexity in solving the nonlinear Riccati equations, by performing singular value decomposition (SVD) and matrix transformation, a linear matrix equation is derived, and then, the optimal solution is easy to be obtained; (3) the quadratic performance index is a function depending not only on the state and state derivative but also on the coefficient matrices of the system. Moreover, the constructed performance index is closely related to a candidate Lyapunov function guaranteeing the stability of the closed-loop system; (4) the above main results are extended to the case of general high-order descriptor system; (5) several numerical examples on different mechanical systems are given to demonstrate the effectiveness of the proposed method.

The remaining of this paper is organized as follows. In Section 2, some essential definitions and preliminary lemmas about matrix theory and second-order descriptor systems are elaborated. In Section 3, a Riccati matrix equation for second-order descriptor systems is derived, and the optimal control gains are determined. In Section 4, the results are extended to the general high-order descriptor system. In Section 5, several numerical examples are given to show the effectiveness of the proposed approach. Some conclusions are made in Section 6.

2 | PROBLEM FORMULATION AND PRELIMINARIES

Consider the following second-order descriptor system given by

$$\begin{cases} A_2\ddot{x}(t) + A_1\dot{x}(t) + A_0x(t) = Bu(t) \\ A_1x(0) = A_1x_0, A_2\dot{x}(0) = A_2x_1, \end{cases} \tag{1}$$

where A_2, A_1, and $A_0 \in \mathbb{R}^{n\times n}$ are the mass, damping, and stiffness constant matrices, respectively, $B \in \mathbb{R}^{n\times m}$ is control input matrix. $x(t)$ is the state vector, and $u(t)$ is control input. A_1x_0, A_2x_1 are the admissible initial conditions. The nonzero mass matrix A_2 may be singular, ie, rank $A_2 \leq n$, the symmetry and positive definiteness of matrices A_2, A_1, A_0 are not required.

The corresponding polynomial matrix is $A(s) = A_2 s^2 + A_1 s + A_0 \in \mathbb{R}^{n\times n}[s], \forall s \in \mathbb{C}$. System (1) is said to be regular, if $\det A(s) \not\equiv 0$ for some $s \in \mathbb{C}$. The concept of $\mathcal{R}_2$-controllable can be referred to the work of Losse and Mehrmann,[15] system (1) is said to be $\mathcal{R}_2$-controllable if and only if rank $[A_2 s^2 + A_1 s + A_0 \; B] = n, \forall s \in \mathbb{C}$. The fundamental assumption in this paper is that system (1) is regular and $\mathcal{R}_2$-controllable.

The main objective is to find a state-feedback control law

$$u(t) = -K_0 x(t) - K_1 \dot{x}(t) \tag{2}$$

that stabilizes the system and minimizes a desired performance index. Here, $K_0, K_1 \in \mathbb{R}^{m\times n}$ are feedback gains to be determined, then the closed-loop system is

$$A_2 \ddot{x}(t) + (A_1 + BK_1)\dot{x}(t) + (A_0 + BK_0)x(t) = 0. \tag{3}$$

Define the quadratic performance index of (1) as follows:

$$J_2(A_2x_1, A_1x_0, u) = \int_0^\infty \left[(A_2\dot{x})^T R_1 (A_2\dot{x}) + (A_2\dot{x} + A_1 x)^T R_2 (A_2\dot{x} + A_1 x) + u^T R_3 u\right] dt, \tag{4}$$

where $R_1, R_2 \in \mathbb{R}^{n\times n}$, and $R_3 \in \mathbb{R}^{m\times m}$ are all given symmetric positive definite matrices.

Remark 1. The constructed quadratic performance index J_2 is different from that of Ram and Inman[31] and Zhang,[32] in which the system matrices A_2, A_1, A_0 are assumed to be symmetric positive definite. In practical applications, the mass matrix of the second-order system may be not nonsymmetric positive definite, the method proposed in the works of Ram and Inman[31] and Zhang[32] could not deal with this problem, so the performance index J_2 in (4) is of more universality, a unified framework is developed for designing the optimal controller based on the Riccati equation method without any linearization transformations. Furthermore, the performance index J_2 is closely related to a candidate Lyapunov function, the designed optimal controller under the performance index can guarantee the stability of the closed-loop system.

Traditionally, the second-order descriptor system is analyzed via the use of linearizations, that is, the original system is turned into a first-order system by introducing new variables, the linearized first-order descriptor system is of the form

$$\begin{cases} E\dot{\omega}(t) = \mathcal{A}\omega(t) + \mathcal{B}u(t) \\ E\omega(0) = E\omega_0 \quad t \geq 0, \end{cases} \tag{5}$$

where $\omega(t) = [x(t)^T, \dot{x}(t)^T]^T \in \mathbb{R}^{2n}$, and

$$E = \begin{bmatrix} I_2 & \\ & A_2 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 0 & I_2 \\ -A_0 & -A_1 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

Let us briefly introduce some basic concepts of first-order descriptor systems (see the work of Duan[2]). The pair $(E, \mathcal{A})$ is called to be regular if $\det(sE - \mathcal{A})$ is not identically zero. The pair $(E, \mathcal{A})$ is called to be impulse free if $(E, \mathcal{A})$ is regular and $deg|sE - \mathcal{A}| = \text{rank} E$. $(E, \mathcal{A}, \mathcal{B})$ is called to be $\mathcal{R}$-controllable if and only if $\text{rank}[sE - \mathcal{A} \; \mathcal{B}] = 2n, \forall s \in \mathbb{C}$. The following lemma characterizes the optimal feedback controller for the first-order descriptor system (5).

Lemma 1 (See the work of Haddad et al[11] and Yan et al[36]).
Consider the regular, impulse-free, and $\mathcal{R}$-controllable system (5) with quadratic performance functional

$$J(E\omega_0, u(t)) = \int_0^\infty (E\omega(t))^T R_1 (E\omega(t)) + u^T(t) R_2 u(t) dt, \tag{6}$$

where $R_1, R_2 \in \mathbb{R}^{2n\times 2n}$ are given positive definite matrices. If there exists positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ satisfying the following generalized Riccati equation:

$$E^T P\mathcal{A} + \mathcal{A}^T PE + E^T R_1 E - E^T P\mathcal{B}R_2^{-1}\mathcal{B}^T PE = 0. \tag{7}$$

Then, the optimal control is $u^* = -R_2^{-1}\mathcal{B}^T PE\omega$, *and the optimal performance index is*

$$J^* = J(E\omega_0, u^*(t)) = \omega_0^T E^T PE\omega_0. \tag{8}$$

Through Lemma 1, the optimal feedback control of system (1) can be obtained by solving the generalized Riccati Equation (7). However, as we have mentioned before, when A_2 is singular, there exist some drawbacks in the process of linearization transformation, as a simple example, consider a second-order system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ddot{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u(t),$$

according to theorem 1 in the work of Henrion and Züñiga,[37] we test that polynomial matrix $A(s)$ has no infinite zero, that is, the system has no impulse solution. However, the classical first-order form yields

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

and rank $\begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = 6 \neq 4 + \text{rank} E$, so the linearized system has impulse solution, which is not consistent with the original system. Thus, the above linearization technique is not an effective approach to solve the optimal control problem of second-order descriptor systems. To obtain the main results in this paper, the following definition and lemmas are necessary.

Definition 1 (See the work of Karampetakis[38]).
Let matrix $A \in \mathbb{C}^{m\times n}$, there exists a unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the following four equations:

$$AXA = A, XAX = X, (AX)^\star = AX, (XA)^\star = XA. \tag{9}$$

X is called as Moore-Penrose pseudoinverse of A, denote as $A^\dagger$. $A^\star$ is the conjugate transpose of A, if $A \in \mathbb{R}^{m\times n}$, then $A^\star = A^T$.

Lemma 2 (See the work of Duan[2]).
For any matrix $Q \in \mathbb{R}^{m\times n}$ *with rank* $Q = r$, $\sigma_1, \ldots, \sigma_r$ *are the nonzero singular values of* Q, *there exist orthogonal matrices* $U \in \mathbb{R}^{m\times m}$ *and* $V \in \mathbb{R}^{n\times n}$, *such that*

$$Q = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T, \tag{10}$$

where $\Sigma = diag\{\sigma_1, \sigma_2, \ldots, \sigma_r\}$, $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0$.

Lemma 3 (See the work of MacDuffee[39]).
Assume that the SVD of matrix $Q \in \mathbb{R}^{m\times n}$ *defined by (10),* $P \in \mathbb{R}^{n\times n}$ *is a symmetric matrix. Then there exists a matrix* $\widetilde{P} \in \mathbb{R}^{m\times m}$ *satisfying* $QP = \widetilde{P}Q$ *if and only if* P *can be expressed as*

$$P = V \begin{bmatrix} P_{11} & \\ & P_{22} \end{bmatrix} V^T, \tag{11}$$

where $P_{11} \in \mathbb{R}^{r\times r}$, $P_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$.

3 | OPTIMAL CONTROL DESIGN FOR SECOND-ORDER DESCRIPTOR SYSTEMS

In this section, we firstly consider how to design the optimal control of the second-order descriptor system based on a Riccati matrix equation without any variable transformation.

Theorem 1. *Consider the controllable second-order descriptor system (1) under a feedback control (2) with a quadratic performance index (4). If there exist $P_1, P_2 \geq 0 \in \mathbb{R}^{n\times n}$ satisfying the following second-order generalized Riccati equation:*

$$0 = -\begin{bmatrix} A_0^T P_2 A_1 + A_1^T P_2 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_1 A_2 + A_2^T P_1 A_1 \end{bmatrix} + \begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} - \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix}^T \begin{bmatrix} P_2 \\ P_1 \end{bmatrix} B R_3^{-1} B^T \begin{bmatrix} P_2 & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix}. \tag{12}$$

Then, the minimum value of the performance index J_2 is

$$J_2^* = \min_{u^*(t)} J_2 = \dot{x}_0^T A_2^T P_1 A_2 \dot{x}_0 + (A_2\dot{x}_0 + A_1 x_0)^T P_2 (A_2 \dot{x}_0 + A_1 x_0) \tag{13}$$

and the associated optimal feedback controller is given by

$$u^* = -R_3^{-1} B^T \begin{bmatrix} P_2 & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \tag{14}$$

Moreover, the closed-loop system (3) is Lyapunov stable.

Proof. Define the integrand function in the performance index as

$$L_2 = L(A_2\dot{x}, A_1 x, u(t)) = (A_2\dot{x})^T R_1 (A_2\dot{x}) + (A_2\dot{x} + A_1 x)^T R_2 (A_2\dot{x} + A_1 x) + u^T(t) R_3 u(t), \tag{15}$$

and the Hamiltonian function for $\lambda_1(t), \lambda_2(t) \in \mathbb{R}^n$

$$H_2(A_2\dot{x}, A_1 x, u) = L(A_2\dot{x} + A_1 x, u) + \lambda_1^T(-A_1\dot{x} - A_0 x + Bu) + \lambda_2^T(-A_0 x + Bu), \tag{16}$$

where R_1 and R_2 are given positive definite matrices. Suppose that there exists an optimal control $u^*(t)$, and $J_2^* = J_2(A_2\dot{x}, A_1 x, u^*)$ is the minimal cost functional. By Bellman's principle of optimality and theorem 3.1 in the work of Haddad et al,[11] we can derive the Hamilton-Jocabi-Bellman(HJB) equation for system (1)

$$-\frac{\partial J_2^*(A_2\dot{x}(t), A_1 x(t))}{\partial t} = H_2(x, \dot{x}, u^*(t), \lambda_1(t), \lambda_2(t)). \tag{17}$$

Introduce two auxiliary variables

$$z_1 = A_2\dot{x}, \quad z_2 = A_2\dot{x} + A_1 x,$$

and define a continuously differentiable function V_2

$$V_2(z_1, z_2) = z_1^T P_1 z_1 + z_2^T P_2 z_2 = (A_2\dot{x})^T P_1 (A_2\dot{x}) + (A_2\dot{x} + A_1 x)^T P_2 (A_2\dot{x} + A_1 x) \tag{18}$$

such that $0 = \frac{\partial V_2(z_1,z_2)}{\partial t} + H_2(x, \dot{x}, u^*(t), \lambda_1(t), \lambda_2(t))$, where

$$\lambda_1(t) = \frac{\partial V_2(z_1, z_2)}{\partial z_1}, \quad \lambda_2(t) = \frac{\partial V_2(z_1, z_2)}{\partial z_2}. \tag{19}$$

Substituting (15) and (19) into (16), the Hamiltonian function in (16) can be rewritten as

$$\begin{aligned} H_2(A_2\dot{x}, A_1 x, u) &= L_2(A_2\dot{x} + A_1 x, u) + 2(A_2\dot{x})^T P_1(-A_1\dot{x} - A_0 x + Bu) \\ &\quad + 2(A_2\dot{x} + A_1 x)^T P_2 (-A_0 x + Bu) \\ &= \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \left(\begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} \begin{bmatrix} A_0^T P_2 A_1 + A_1^T P_2 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_1 A_2 + A_2^T P_1 A_1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \right. \\ &\quad + 2 \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix}^T \begin{bmatrix} P_2 \\ P_1 \end{bmatrix} Bu + u^T R_3 u. \end{aligned}$$

Based on the minimum value principle,

$$\frac{\partial H_2}{\partial u} = 2R_3 u + 2B^T \begin{bmatrix} P_2 & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = 0,$$

then

$$u^*(t) = -R_3^{-1} B^T \begin{bmatrix} P_2 & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

is the optimal feedback control, where P_1 and P_2 are symmetric positive semidefinite matrices and satisfy

$$H_2\left(A_2\dot{x}, A_1x, u^*(t)\right) = 0.$$

Therefore, if P_1 and P_2 are the solutions of the generalized Riccati Equation (12), the optimal performance index is

$$J_2^* = V_2(A_2\dot{x}_0, A_1x_0) = \dot{x}_0^T A_2^T P_1 A_2 \dot{x}_0 + (A_2\dot{x}_0 + A_1x_0)^T P_2 (A_2\dot{x}_0 + A_1x_0).$$

The associated optimal control is (14).

Next, we consider the stability of system (1) under the feedback control (14) by using Lyapunov function method. Differentiating (18) with respect to time t leads to

$$\dot{V}_2 = -\begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} A_0^T P_2 A_1 + A_1^T P_2 A_0 & A_0^T P_1 A_2 + A_0^T P_2 A_2 \\ A_2^T P_1 A_0 + A_2^T P_2 A_0 & A_1^T P_1 A_2 + A_2^T P_1 A_1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + 2\begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix}^T \begin{bmatrix} P_2 \\ P_1 \end{bmatrix} Bu(t),$$

substituting (14) into the above equation, we have

$$\dot{V}_2 = -\begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} - \begin{bmatrix} x^T & \dot{x}^T \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix}^T \begin{bmatrix} P_2 \\ P_1 \end{bmatrix} BR_3^{-1}B^T \begin{bmatrix} P_2 & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \leq 0.$$

Thereby, from Lemma 1, the closed-loop system (3) is Lyapunov stable. The proof is complete. □

Remark 2. Theorem 1 provides a unified method to determine the optimal control law based on a second-order generalized Riccati Equation (12), which is derived by directly using the original system's parameters and is completely different from the linearized form (7).

It is worth noting that the closed-loop system can achieve asymptotic stability for any admissible initial values. In addition, the solution of the optimal feedback control in Theorem 1 can be obtained by solving the generalized second-order Riccati Equation (12) with dimension $2n$; however, it is very difficult to solve directly the nonlinear Riccati equation. To overcome the difficulty, in the next analysis, applying matrix transformation and SVD theory, a nonlinear matrix Riccati equation is transformed into a linear matrix equation, the detail analysis process is as follows.

We rewrite the Riccati Equation (12) as

$$\begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \star & \Upsilon_{22} \end{bmatrix} - \begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} = 0, \tag{20}$$

where $\star$ represents the corresponding symmetric matrix, $Q = BR_3^{-1}B^T \in \mathbb{R}^{n\times n}$ and

$$\begin{aligned} \Upsilon_{11} &= A_0^T P_2 A_1 + A_1^T P_2 A_0 + A_1^T P_2 Q P_2 A_1 \\ \Upsilon_{12} &= A_0^T P_2 A_2 + A_0^T P_1 A_2 + A_1^T P_2 Q (P_1 + P_2) A_2 \\ \Upsilon_{22} &= A_1^T P_1 A_2 + A_2^T P_1 A_1 + A_2^T (P_1 + P_2) Q (P_1 + P_2) A_2. \end{aligned}$$

Due to the existence of the nonlinear terms P_2QP_2, $P_2Q(P_1 + P_2)$ and $(P_1 + P_2)Q(P_1 + P_2)$, the matrix Equation (20) is a nonlinear matrix equation, which can be transformed into the following linear matrix equation based on SVD theory in Lemma 3

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \star & \Sigma_{22} \end{bmatrix} - \begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} = 0,$$

where $\Sigma_{11}, \Sigma_{12}, \Sigma_{22}$ are defined by (21) in the following Theorem 2. Since there exist many methods for solving the linear matrix equation in the existing research studies, we give the values of symmetric positive definite matrices R_1, R_2, and R_3 respectively, and can obtain a numerical solution for the Riccati equation (12) by MATLAB Toolbox.[40] The following theorem presents the parametric representations of feedback gains K_0, K_1.

Theorem 2. *Consider the controllable second-order descriptor system (1) under a state-feedback control (2) with a quadratic performance functional (4). Assume that the SVD of the matrix* $Q \in \mathbb{R}^{n\times n}$ *is defined by (10) with* $\mathrm{rank}Q = r$, *where* $U = [\,U_1 \;\; U_2\,]$, $V = [\,V_1 \;\; V_2\,] \in \mathbb{R}^{n\times n}$, $U_1, V_1 \in \mathbb{R}^{n\times r}$, $U_2, V_2 \in \mathbb{R}^{n\times(n-r)}$, *and* $R_1, R_2, R_3 \in \mathbb{R}^{n\times n}$ *are given symmetric positive definite matrices. If there exist symmetric matrices* $P_1, P_2 \in \mathbb{R}^{n\times n} \geq 0$, $\mathcal{P}_{11}, W_{11} \in \mathbb{R}^{r\times r}$, $\mathcal{P}_{22}, W_{22} \in \mathbb{R}^{(n-r)\times(n-r)} \geq 0$ *and* $X_1, X_2, X_3 \in \mathbb{R}^{n\times n}$ *such that*

$$\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \star & \Sigma_{22} \end{bmatrix} - \begin{bmatrix} A_1^T R_2 A_1 & A_1^T R_2 A_2 \\ A_2^T R_2 A_1 & A_2^T (R_1 + R_2) A_2 \end{bmatrix} = 0, \tag{21}$$

where

$$
\begin{aligned}
\Sigma_{11} &= A_0^T P_2 A_1 + A_1^T P_2 A_0 + A_1^T X_1 Q A_1 \\
\Sigma_{12} &= A_0^T P_2 A_2 + A_0^T P_1 A_2 + A_1^T X_2 Q A_2 \\
\Sigma_{22} &= A_1^T P_1 A_2 + A_2^T P_1 A_1 + A_2^T X_3 Q A_2, \\
P_2 &= V \begin{pmatrix} \mathcal{P}_{11} & 0 \\ 0 & \mathcal{P}_{22} \end{pmatrix} V^T, P_1 + P_2 = V \begin{pmatrix} W_{11} & 0 \\ 0 & W_{22} \end{pmatrix} V^T.
\end{aligned}
$$

Then, the minimum value of the performance index J_2 is

$$
J_2^* = \dot{x}_0^T A_2^T P_1 A_2 \dot{x}_0 + (A_2 \dot{x}_0 + A_1 x_0)^T P_2 (A_2 \dot{x}_0 + A_1 x_0), \tag{22}
$$

and the associated optimal feedback control gains are as follows:

$$
\begin{aligned}
K_0 &= R_3^{-1} B^T X_1 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma \mathcal{P}_{11}^{-1} \Sigma^{-1} U_1^T A_1 \\
K_1 &= R_3^{-1} B^T X_3 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma W_{11}^{-1} \Sigma^{-1} U_1^T A_2.
\end{aligned} \tag{23}
$$

Moreover, the optimal closed-loop system is asymptotically stable.

Proof. According to Lemma 2, $Q^T Q = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} \Sigma^2 & \\ & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$, and we can derive that $V_1^T Q^T Q V_1 = \Sigma^2$. From Equation (10),

$$
Q = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma V_1^T.
$$

Let $U_1 = Q V_1 \Sigma^{-1} \in \mathbb{R}^{n \times r}$, then $U_1^T U_1 = I_r$, ie, U_1 is full column rank. Based on Definition 1, $U_1^{\dagger} = (U_1^T U_1)^{-1} U_1^T$, then U_1^T is regarded as the generalized inverse of U_1, ie, $U_1^{\dagger} = U_1^T$.

On the basis of Lemma 3, the symmetric matrix P_2 can be expressed as $P_2 = V \begin{bmatrix} \mathcal{P}_{11} & 0 \\ 0 & \mathcal{P}_{22} \end{bmatrix} V^T$, if and only if there exists $\widehat{P}_2 \in \mathbb{R}^{n \times n}$ such that $Q P_2 = \widehat{P}_2 Q$.

In fact, from Lemma 2,

$$
\begin{aligned}
Q P_2 &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T V \begin{bmatrix} \mathcal{P}_{11} & 0 \\ 0 & \mathcal{P}_{22} \end{bmatrix} V^T \\
&= \widehat{P}_2 \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = \widehat{P}_2 Q,
\end{aligned}
$$

then $([U_1 \Sigma \mathcal{P}_{11} \;\; 0\,] - [\,\widehat{P}_2 U_1 \Sigma \;\; 0\,]) V^T = 0$ because of the invertibility of matrix V, $\widehat{P}_2$ satisfied the equation

$$
U_1 \Sigma \mathcal{P}_{11} \Sigma^{-1} = \widehat{P}_2 U_1,
$$

then the general solution is

$$
\widehat{P}_2 = U_1 \Sigma \mathcal{P}_{11} \Sigma^{-1} U_1^{\dagger} + Y \left(I_n - U_1 U_1^{\dagger}\right),
$$

where Y is an arbitrary real $n \times n$ matrix. Thus, through the above analysis, there exists $\widehat{P}_2 = U_1 \Sigma \mathcal{P}_{11} \Sigma^{-1} U_1^{\dagger}$ such that $P_2 Q = Q \widehat{P}_2$.

On the other hand, denote $W = P_1 + P_2$, a similar computing process, there exists a matrix $\widehat{W} = U_1 \Sigma W_{11} \Sigma^{-1} U_1^{\dagger}$ such that $QW = \widehat{W} Q$. Next, introduce auxiliary variables $X_1 = P_2 \widehat{P}_2$, $X_2 = P_2 \widehat{W}$, and $X_3 = W \widehat{W}$. In this case, the nonlinear matrix Equation (20) is transformed into a linear matrix Equation (21). From the above analysis, we get

$$
\begin{aligned}
P_2 &= X_1 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma \mathcal{P}_{11}^{-1} \Sigma^{-1} U_1^T, \\
W &= P_1 + P_2 = X_3 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma W_{11}^{-1} \Sigma^{-1} U_1^T.
\end{aligned}
$$

By Theorem 1, the feedback control gains are determined as

$$\begin{aligned} K_0 &= R_3^{-1}B^T P_2 A_1 \\ &= R_3^{-1}B^T X_1 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma \mathcal{P}_{11}^{-1} \Sigma^{-1} U_1^T A_1, \\ K_1 &= R_3^{-1}B^T (P_2 + P_1) A_2 \\ &= R_3^{-1}B^T X_3 \left(U_1 U_1^T\right)^{\dagger} U_1 \Sigma W_{11}^{-1} \Sigma^{-1} U_1^T A_2. \end{aligned}$$

The proof is complete. □

4 | OPTIMAL CONTROL OF HIGH-ORDER DESCRIPTOR SYSTEMS

In this section, we extend the above optimal control result to the general high-order descriptor system ($k \geq 3$).

Consider the linear kth-order descriptor system given by

$$\begin{cases} A_k x^{(k)}(t) + \cdots + A_1 \dot{x}(t) + A_0 x(t) = Bu(t) \\ A_1 x(0) = A_1 x_0, \ \ldots, A_k x^{(k-1)}(0) = A_k x_{k-1}, \end{cases} \tag{24}$$

where $A_i \in \mathbb{R}^{n\times n}, i = 0, \ldots, k$, and $B \in \mathbb{R}^{n\times m}$, $x^{(i)}(t)$ represents ith derivative at time t, $(x(t), \ldots, x^{(k-1)}(t))$ is called a high-order state of system (24), with a state feedback

$$u(t) = -F_0 x(t) - F_1 \dot{x}(t) - \cdots - F_{k-1} x^{(k-1)}(t). \tag{25}$$

Lemma 4 (See the work of Son and Thuan[16]).
The high-order descriptor system (24) is $\mathcal{R}_k$-controllable if and only if

$$\operatorname{rank}\left[A_k s^k + \cdots + A_1 s + A_0 \ \ B\right] = n, \quad \forall s \in \mathbb{C}.$$

Denote the quadratic performance functional as

$$\begin{aligned} J_k(\tilde{E}\xi_0, u) = \int_0^\infty & \Big[\left(A_k x^{(k-1)}\right)^T R_1 \left(A_k x^{(k-1)}\right) \\ &+ \left(A_k x^{(k-1)} + A_{k-1}x^{(k-2)}\right)^T R_2 \left(A_k x^{(k-1)} + A_{k-1}x^{(k-2)}\right) + \cdots \\ &+ \left(A_k x^{(k-1)} + A_{k-1}x^{(k-2)} + \cdots + A_1 x\right)^T \\ &\cdot R_k \left(A_k x^{(k-1)} + A_{k-1}x^{(k-2)} + \cdots + A_1 x\right) + u^T R_{k+1} u\Big]\, dt, \end{aligned} \tag{26}$$

where $R_i \in \mathbb{R}^{n\times n}, i = 1, \ldots, k$ and $R_{k+1} \in \mathbb{R}^{m\times m}$ are all given symmetric positive definite matrices.

The next target is to design an optimal feedback controller $u^*(t)$ that guarantees closed-loop stability and minimizes a quadratic performance functional J_k over the infinite horizon.

First, for the convenience of writing, we denote that

$$\tilde{E} = \left[A_1 \ \cdots \ A_{k-1} \ A_k\right], \xi(t) = \left[x^T(t) \ \dot{x}^T(t) \ \cdots \ (x^{(k-1)})^T(t)\right]^T, \xi_0 = \left[x_0^T \ x_1^T \ \cdots \ x_{k-1}^T\right]^T.$$

Theorem 3. *Consider the controllable high-order descriptor system (24) under a state-feedback controller (25) with the quadratic performance functional (26). If there exist symmetric positive semidefinite matrices $P_1, P_2, \ldots, P_k \in \mathbb{R}^{n\times n}$ such that the high-order generalized Riccati equation*

$$\begin{aligned} 0 = &-\begin{bmatrix} \Phi_{11} & * & \cdots & * \\ \Phi_{21} & \Phi_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{k1} & \Phi_{k2} & \cdots & \Phi_{kk} \end{bmatrix} + \begin{bmatrix} \Psi_{11} & * & \cdots & * \\ \Psi_{21} & \Psi_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \Psi_{k1} & \Psi_{k2} & \cdots & \Psi_{kk} \end{bmatrix} \\ &- \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ 0 & A_2 & \cdots & A_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}^T \begin{bmatrix} P_k \\ P_{k-1} \\ \vdots \\ P_1 \end{bmatrix} B R_{k+1}^{-1} B^T \left[P_k \ P_{k-1} \ \cdots \ P_1\right] \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ 0 & A_2 & \cdots & A_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}, \end{aligned} \tag{27}$$

then the optimal performance index J_k is

$$
\begin{aligned}
J_k(\tilde{E}\xi_0, u^*(t)) &= \min_{u(t)} J_k(\tilde{E}\xi_0, u(t)) \\
&= x_{k-1}^T A_k^T P_1 A_k x_{k-1} + (A_k x_{k-1} + A_{k-1}x_{k-2})^T P_2 (A_k x_{k-1} + A_{k-1}x_{k-2}) + \cdots \\
&\quad + (A_k x_{k-1} + A_{k-1}x_{k-2} + \cdots + A_1 x_0)^T P_k (A_k x_{k-1} + A_{k-1}x_{k-2} + \cdots + A_1 x_0),
\end{aligned} \tag{28}
$$

and the optimal feedback controller is given by

$$
u^*(t) = -R_{k+1}^{-1} B^T \begin{bmatrix} P_k & P_{k-1} & \cdots & P_1 \end{bmatrix} \begin{bmatrix} A_1 & A_2 & \cdots & A_k \\ 0 & A_2 & \cdots & A_k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \\ \vdots \\ x^{(k-1)}(t) \end{bmatrix}, \tag{29}
$$

where

$$
\begin{aligned}
&\Phi_{11} = A_1^T P_k A_0 + A_0^T P_k A_1, \quad \Phi_{21} = A_2^T (P_k + P_{k-1}) A_0, \quad \Phi_{22} = A_2^T P_{k-1} A_1 + A_1^T P_{k-1} A_2, \\
&\quad \cdots \quad \cdots \quad \cdots \\
&\Phi_{k1} = A_k^T (P_1 + P_2 + \cdots + P_k) A_0, \quad \Phi_{k2} = A_k^T (P_2 + \cdots + P_k) A_1, \ \ldots, \Phi_{kk} = A_k^T P_1 A_{k-1} + A_{k-1}^T P_1 A_k,
\end{aligned}
$$

and

$$
\begin{aligned}
&\Psi_{11} = A_1^T R_k A_1, \quad \Psi_{21} = A_2^T R_k A_1, \Psi_{22} = A_2^T (R_k + R_{k-1}) A_2, \ \ldots, \\
&\Psi_{k1} = A_k^T R_k A_1, \Psi_{k2} = A_k^T (R_k + R_{k-1}) A_2, \ \ldots, \Psi_{kk} = A_k^T (R_k + R_{k-1} + \cdots + R_1) A_k.
\end{aligned}
$$

Furthermore, under the feedback control (29), then the closed-loop system (24) is stable.

Proof. Let

$$
\begin{aligned}
L_k(\tilde{E}\xi(t), u(t)) &= \left(A_k x^{(k-1)}(t)\right)^T R_1 (A_k x^{(k-1)}(t)) + \left(A_k x^{(k-1)}(t) + A_{k-1} x^{(k-2)}(t)\right)^T \\
&\quad \cdot R_2 \left(A_k x^{(k-1)}(t) + A_{k-1} x^{(k-2)}(t)\right) + \cdots \\
&\quad + \left(A_k x^{(k-1)}(t) + A_{k-1} x^{(k-2)}(t) + \cdots + A_1 x(t)\right)^T \\
&\quad \cdot R_k \left(A_k x^{(k-1)}(t) + A_{k-1} x^{(k-2)}(t) + \cdots + A_1 x(t)\right) + u^T(t) R_{k+1} u(t),
\end{aligned} \tag{30}
$$

and the high-order Hamiltonian function for $\lambda_i(t) \in \mathbb{R}^n, i = 1, \ldots, k$

$$
H_k(\tilde{E}\xi, u) = L_k(\tilde{E}\xi, u) + \lambda_1^T \left(-A_{k-1} x^{k-1} - \cdots - A_0 x + Bu\right) + \cdots + \lambda_k(-A_0 x + Bu), \tag{31}
$$

where $R_i, i = 1, \ldots, k+1$ are given positive definite matrices. Let $u^*(t)$ be the optimal feedback control for the high-order descriptor system (24), and J_k^* be the minimal performance functional. By Bellman's principle of optimality, we deduce the HJB equation for system (24) as follows:

$$
-\frac{\partial J_k^*(\tilde{E}\xi(t))}{\partial t} = H_k\left(\tilde{E}\xi(t), u^*(t), \lambda_1(t), \ldots, \lambda_k(t)\right), \tag{32}
$$

where $\lambda_k(t) = (\frac{\partial J_k^*(\tilde{E}\xi(t))}{\partial \xi})^T$. Define a continuously differentiable function V_k

$$
\begin{aligned}
V_k(\tilde{E}\xi) &= \left(A_k x^{(k-1)}\right)^T P_1 \left(A_k x^{(k-1)}\right) + \left(A_k x^{(k-1)} + A_{k-1} x^{(k-2)}\right)^T P_2 \\
&\quad \cdot \left(A_k x^{(k-1)} + A_{k-1} x^{(k-2)}\right) + \cdots \\
&\quad + \left(A_k x^{(k-1)} + A_{k-1} x^{(k-2)} + \cdots + A_1 x\right)^T P_k \left(A_k x^{(k-1)} + \cdots + A_1 x\right),
\end{aligned} \tag{33}
$$

which satisfies

$$
\frac{\partial V_k(\tilde{E}\xi, t)}{\partial t} + H_k(\tilde{E}\xi(t)), u^*(t), \frac{\partial V_k(\tilde{E}\xi(t))}{\partial \xi}\Bigg) = 0.
$$

Then, the Hamilton function in (31) is rewritten as

$$
\begin{aligned}
H_k(\tilde{E}\xi(t),u(t)) &= L_k(\tilde{E}\xi(t),u^*(t)) + V_k'(\tilde{E}\xi(t))Bu(t)\\
&= \xi^T(t)\begin{bmatrix}\Psi_{11} & * & \cdots & *\\ \Psi_{21} & \Psi_{22} & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ \Psi_{k1} & \Psi_{k2} & \cdots & \Psi_{kk}\end{bmatrix}\xi(t) - \xi^T(t)\begin{bmatrix}\Phi_{11} & * & \cdots & *\\ \Phi_{21} & \Phi_{22} & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_{k1} & \Phi_{k2} & \cdots & \Phi_{kk}\end{bmatrix}\xi(t)\\
&\quad + 2\xi^T(t)\begin{bmatrix}A_1 & A_2 & \cdots & A_k\\ 0 & A_2 & \cdots & A_k\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_k\end{bmatrix}^T\begin{bmatrix}P_k\\ P_{k-1}\\ \vdots\\ P_1\end{bmatrix}Bu(t) + u^T(t)R_{k+1}u(t).
\end{aligned}
$$

Through a simple computation, we get

$$
\frac{\partial H_k}{\partial u} = 2R_{k+1}u + 2\xi^T(t)\begin{bmatrix}A_1 & A_2 & \cdots & A_k\\ 0 & A_2 & \cdots & A_k\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_k\end{bmatrix}^T\begin{bmatrix}P_k\\ P_{k-1}\\ \vdots\\ P_1\end{bmatrix}B.
$$

Therefore, we can obtain that the associated optimal control $u^*(t)$ is of the form (29), where $P_1, \dots, P_k$ are symmetric positive semidefinite matrices and satisfy $H_k(\tilde{E}\xi, u^*(t)) = 0$, ie, (27) holds, and the optimal value function J_k is of the form (28).

Differentiating (33) with respect to time t leads to

$$
\dot{V}_k = \xi^T(t)\begin{bmatrix}\Phi_{11} & * & \cdots & *\\ \Phi_{21} & \Phi_{22} & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_{k1} & \Phi_{k2} & \cdots & \Phi_{kk}\end{bmatrix}\xi(t) + 2\xi^T(t)\begin{bmatrix}A_1 & A_2 & \cdots & A_k\\ 0 & A_2 & \cdots & A_k\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_k\end{bmatrix}^T\begin{bmatrix}P_k\\ P_{k-1}\\ \vdots\\ P_1\end{bmatrix}Bu(t).
$$

Under the feedback controller (29), we get

$$
\begin{aligned}
\dot{V}_k &= \xi^T(t)\begin{bmatrix}\Psi_{11} & * & \cdots & *\\ \Psi_{21} & \Psi_{22} & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ \Psi_{k1} & \Psi_{k2} & \cdots & \Psi_{kk}\end{bmatrix}\xi(t)\\
&\quad - \xi^T(t)\begin{bmatrix}A_1 & A_2 & \cdots & A_k\\ 0 & A_2 & \cdots & A_k\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_k\end{bmatrix}^T\begin{bmatrix}P_k\\ P_{k-1}\\ \vdots\\ P_1\end{bmatrix}BR_{k+1}^{-1}B^T\begin{bmatrix}P_k & P_{k-1} & \cdots & P_1\end{bmatrix}\begin{bmatrix}A_1 & A_2 & \cdots & A_k\\ 0 & A_2 & \cdots & A_k\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_k\end{bmatrix}\xi(t) \le 0.
\end{aligned}
$$

Thereby, from Lemma 1, with the feedback control (29), system (24) is stable. The proof is complete. □

Remark 3. Sufficient conditions for the existence of the optimal controller are developed in Theorem 3 for the high-order descriptor system (24), a high-order generalized Riccati Equation (27) is derived to characterize the optimal control gain in terms of the coefficient matrices of the high-order descriptor system, which can be viewed as a generalization of the results in Theorem 1. It should be noted that the constructed Hamilton function H_k is more complicated than that of the second-order form, and the result of Theorem 3 is applicable to $k \geq 2$ high-order descriptor systems.

The following theorem determines optimal feedback gains for the high-order descriptor system.

Theorem 4. *Consider the controllable high-order descriptor system (24) with the quadratic performance functional (26). Assume that the SVD of the matrix $BR_{k+1}^{-1}B^T = Q \in \mathbb{R}^{n\times n}$ is defined by (10). If there exist symmetric matrices $P_1, \dots, P_k \geq 0$, $P_{11}, P_{22}, W_{11}^i, W_{22}^i \geq 0, i = 1, \dots, k-1$, and matrices $X_{11}, \dots, X_{1k}, X_{22}, \dots, X_{2k}, \dots, X_{kk} \geq 0$, such that*

$$
\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1k}\\ \star & \Sigma_{22} & \cdots & \Sigma_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ \star & \star & \cdots & \Sigma_{kk}\end{bmatrix} - \begin{bmatrix}\Psi_{11} & * & \cdots & *\\ \Psi_{21} & \Psi_{22} & \cdots & *\\ \vdots & \vdots & \ddots & \vdots\\ \Psi_{k1} & \Psi_{k2} & \cdots & \Psi_{kk}\end{bmatrix} = 0, \tag{34}
$$

where

$$
\begin{aligned}
\Sigma_{11} &= A_0^T P_k A_1 + A_1^T P_k A_0 + A_1^T X_{11} Q A_1, \\
\Sigma_{12} &= A_0^T (P_k + P_{k-1}) A_2 + A_1^T X_{12} Q A_2, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
\Sigma_{1k} &= A_0^T (P_k + P_{k-1} + \cdots + P_1) A_k + A_1^T X_{1k} Q A_k, \\
\Sigma_{22} &= A_1^T P_{k-1} A_2 + A_2^T P_{k-1} A_1 + A_2^T X_{22} Q A_2, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
\Sigma_{2k} &= A_1^T (P_k + P_{k-1} + \cdots + P_1) A_k + A_2^T X_{2k} Q A_k, \\
&\cdots \quad \cdots \quad \cdots \quad \cdots \\
\Sigma_{kk} &= A_{k-1}^T P_1 A_k + A_k^T P_1 A_{k-1} + A_k X_{kk} Q A_k, \\
P_k &= V \begin{bmatrix} P_{11} & \\ & P_{22} \end{bmatrix} V^T, \\
W_i &= \sum_{j=k-i}^{k} P_j = V \begin{bmatrix} W_{11}^i & \\ & W_{22}^i \end{bmatrix} V^T, i = 1, \ldots, k-1,
\end{aligned}
$$

then the optimal performance function J_k^* is of the form (28), and the optimal feedback control $u^*(t) = -F_0 x(t) - F_1 \dot{x}(t) - \cdots - F_{k-1} x^{(k-1)}(t)$, where the feedback gains $F_i \in \mathbb{R}^{n \times n}$, $i = 0, \ldots, k-1$ is as follows:

$$
\begin{aligned}
F_0 &= R_{k+1}^{-1} B^T X_{11} \left(U_1 U_1^T\right)^\dagger U_1 \Sigma \mathcal{P}_{11}^{-1} \Sigma^{-1} U_1^T A_1, \\
F_1 &= R_{k+1}^{-1} B^T X_{22} \left(U_1 U_1^T\right)^\dagger U_1 \Sigma \left(W_{11}^1\right)^{-1} \Sigma^{-1} U_1^T A_2, \\
&\cdots \quad \cdots \quad \cdots \\
F_{k-1} &= R_{k+1}^{-1} B^T X_{kk} \left(U_1 U_1^T\right)^\dagger U_1 \Sigma \left(W_{11}^{k-1}\right)^{-1} \Sigma^{-1} U_1^T A_k.
\end{aligned}
$$

Furthermore, with the optimal control $u^*(t)$, system (24) is stable.

5 | NUMERICAL EXAMPLES

In this section, to demonstrate the results in previous sections, three numerical examples involving several different mechanical systems are considered and some comparisons with the existing linearization method to illustrate the performance of the proposed optimal controller.

Example 1. (**singular mass matrix case**): Consider a linear system consisting of two lumped masses and three spring dashpots shown in Figure 1.[27,41]

The dynamic of the system is governed by (1), where

$$
A_2 = \begin{bmatrix} m_1 & & \\ & m_2 & \\ & & 0 \end{bmatrix}, A_1 = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix}, A_0 = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

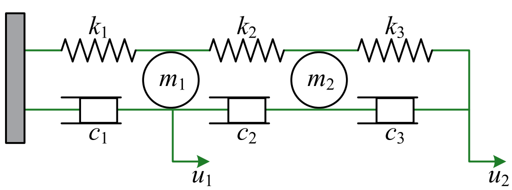

FIGURE 1 The three-spring and two-mass system [Colour figure can be viewed at wileyonlinelibrary.com]

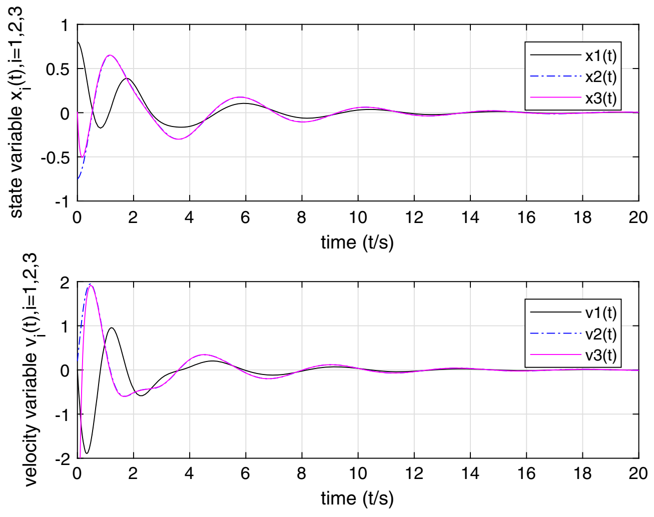

FIGURE 2 The state responses of the open-loop system in Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]

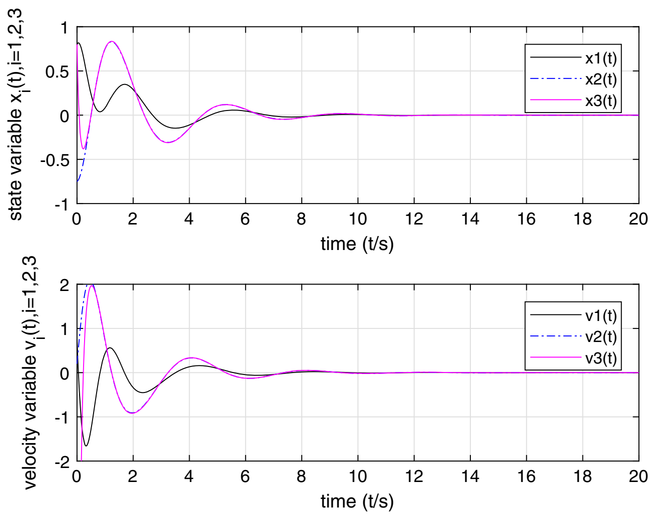

FIGURE 3 The state responses of the closed-loop system in Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]

with the parameter values $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 5$ Ns/m, $k_3 = 20$ Ns/m, $c_1 = c_3 = 2$ N/m, $c_2 = 0.5$ N/m. We can choose the initial values $x_0 = [0.8, -0.75, 0.8]^T$, $x_1 = [0.7, 0.2, 0.9]^T$. In terms of Theorem 1, and using MATLAB Toolbox, we get

$$P_1 = \begin{bmatrix} 2.7704 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 2.7704 \end{bmatrix}, P_2 = \begin{bmatrix} 2.9119 & 0.0000 & -0.4946 \\ 0.0000 & 0.0000 & 0.0000 \\ -0.4946 & 0.0000 & 1.3999 \end{bmatrix}$$

and the optimal feedback gain $K = [K_0 \ \ K_1]$, where

$$K_0 = \begin{bmatrix} -2.4266 & 0.1556 & 0.3298 \\ 0.4122 & 0.8508 & -0.9333 \end{bmatrix}, K_1 = \begin{bmatrix} -1.8941 & -0.1010 & 0 \\ 0.1649 & -0.2000 & 0 \end{bmatrix}.$$

Figure 2 and Figure 3 display the state response of the open-loop system and closed-loop system for the second-order descriptor system, respectively. One can find that the states of the closed-loop system in Figure 3 can converge to the origin within 9 seconds, which implies that the proposed controller can stabilize the system within a short time.

Example 2. (**nonsingular mass matrix case**): Considering a second-order mechanical system in the form of (1), the system matrices are the same with example 6.1 in the work of Abdelaziz[23]

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{bmatrix}, A_0 = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

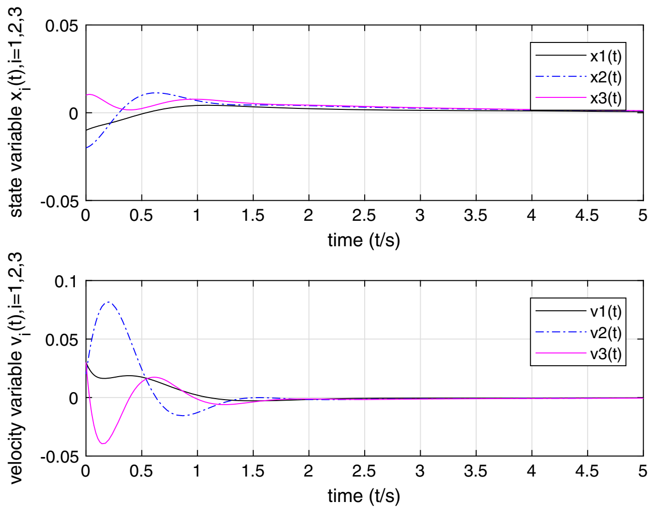

FIGURE 4 The closed-loop system responses in Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]

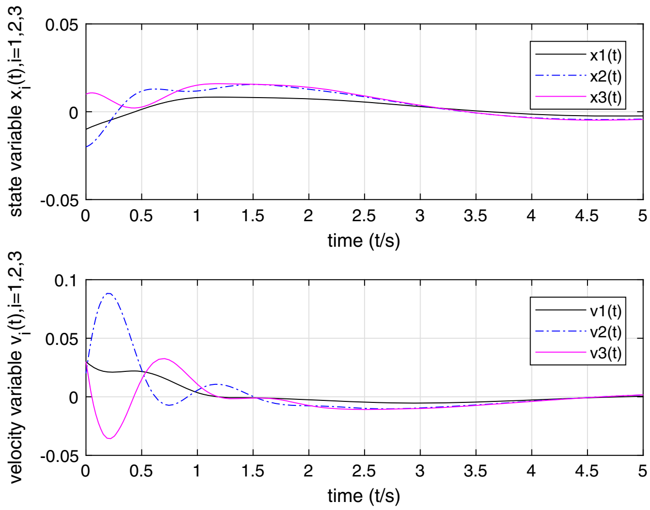

FIGURE 5 The closed-loop system responses by linearization method [Colour figure can be viewed at wileyonlinelibrary.com]

Then, the poles for the open-loop system are $\{-2.1629 \pm 6.1939i,\ -1.1859 \pm 3.0278i,\ -0.1512 \pm 1.0372i\}$. According to Theorem 1, the optimal feedback gains are

$$K_0 = \begin{bmatrix} -2.1590 & 2.5016 & -2.1205 \\ -2.5753 & 7.9253 & -7.7243 \end{bmatrix}, K_1 = \begin{bmatrix} -1.9351 & -0.3553 & -1.8140 \\ -1.8140 & -0.5151 & -8.1491 \end{bmatrix},$$

the closed-loop system's eigenvalues are located at $\{-2.3067 \pm 4.3874i,\ -8.4414,\ -2.1556 \pm 2.8362i,\ -0.4341\}$, which are different from the results in the work of Abdelaziz.[23] Select the initial conditions as $x_0 = [-0.01, -0.02, 0.01]^T$, $x_1 = [0.03, 0.02, 0.02]^T$. Figure 4 shows the simulation result of the closed-loop system, obviously, the system's states can converge faster to zero within 2 seconds. Figure 5 shows the system's response of the closed-loop system with the linearization technique, which implies the system's state can stabilize to original, but the convergence time is longer than that of our method.

Example 3. (**nonsymmetric positive definite matrix case**): Consider a second-order system in the form of (1), where the system parameters are given by

$$A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 2.5 & -0.2 & 0 \\ -0.5 & 2.5 & -2.2 \\ 0 & -2 & 1 \end{bmatrix}, A_0 = \begin{bmatrix} 5 & -5 & 0 \\ -3 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

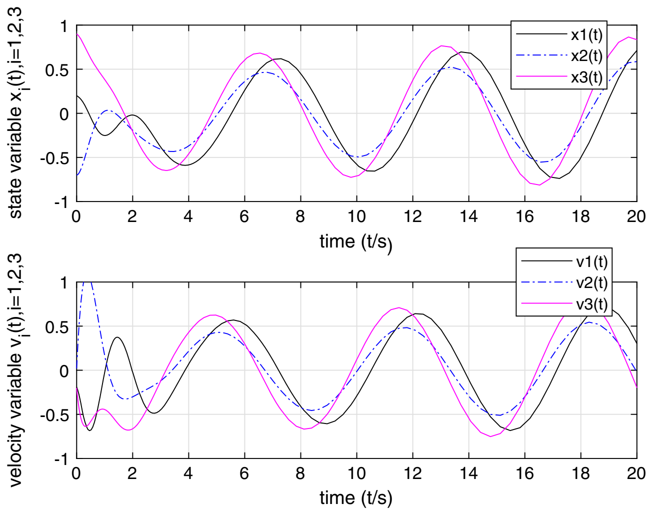

FIGURE 6 The open-loop system responses in Example 3 [Colour figure can be viewed at wileyonlinelibrary.com]

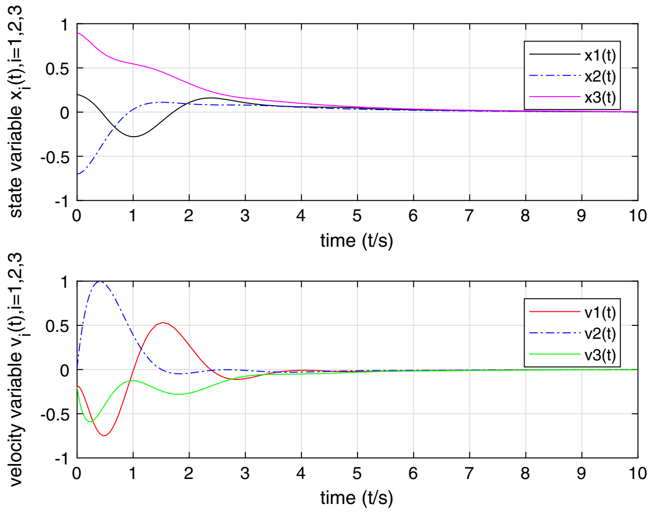

FIGURE 7 The closed-loop system responses in Example 3 [Colour figure can be viewed at wileyonlinelibrary.com]

Here, A_0, A_1, A_2 are all not symmetric, and the damping matrix A_1 is not positive definite. The characteristic roots for the open-loop system are located at $\{-1.2873 \pm 2.6950i,\ -1.7308 \pm 0.8603i, 0.0181 + 0.9486i\}$, the vibration system is not stable, and the open-loop state response of the system is shown in Figure 6. According to Theorem 1, and applying the MATLAB Toolbox, we have

$$P_1 = \begin{bmatrix} 64.7878 & 25.1402 & 81.0666 \\ 25.1402 & 58.0121 & 27.2651 \\ 81.0666 & 27.2651 & 238.3236 \end{bmatrix}, P_2 = \begin{bmatrix} 120.9274 & 50.7008 & 115.0957 \\ 50.7008 & 38.4038 & 32.1909 \\ 115.0957 & 32.1909 & 251.2319 \end{bmatrix},$$

and the optimal feedback gain $K = [K_0\ K_1]$, where

$$K_0 = \begin{bmatrix} -2.7697 & 1.2762 & -0.0355 \\ -2.7164 & 4.4501 & -1.8041 \end{bmatrix}, K_1 = \begin{bmatrix} -1.8572 & -0.7584 & -3.8188 \\ -1.9616 & -0.5946 & -6.8572 \end{bmatrix}.$$

Take the initial values as $x_0 = [0.2, -0.7, 0.9]^T$ and $x_1 = [0.01, 0.02, -0.2]^T$; the closed-loop state response is shown in Figure 7, which indicates that the proposed feedback control can effectively suppress vibration within 5 seconds. In order to show the effectiveness of the proposed control strategy further, Figure 8 shows the closed-loop state responses with the linearization technique, it is clear that our method can reduce the convergence time and energy consumption.

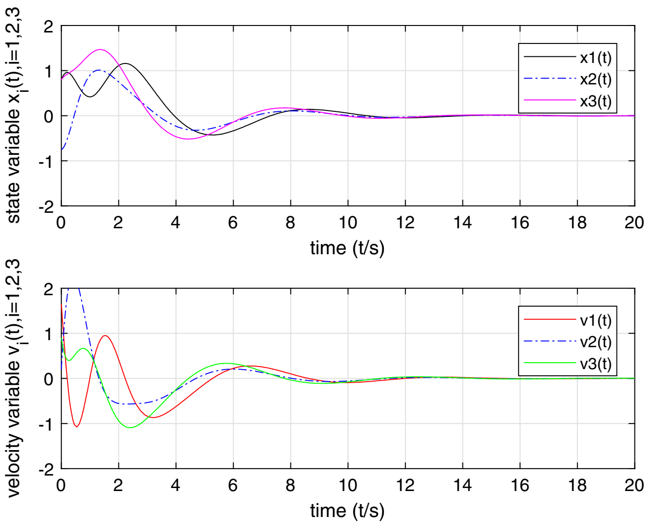

FIGURE 8 The closed-loop system responses with linearization method [Colour figure can be viewed at wileyonlinelibrary.com]

6 | CONCLUSION

In this paper, we studied the linear quadratic optimal control problem for second-order descriptor systems. Without any variable transformations, using the original system's coefficient matrices, the optimal controller is developed with a given quadratic performance functional which is more general than the existing work. By using matrix transformations and SVD theory, the derived nonlinear Riccati matrix equation was transformed into a linear matrix equation; we obtained an optimal approximate solution by solving the linear matrix equation using the MATLAB Toolbox. Moreover, we also derived the optimal feedback gains of the general high-order descriptor systems. Finally, three numerical examples are displayed to illustrate the effectiveness and feasibility of the proposed approach.

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