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STABILITY LOSS IN QUASILINEAR DAES BY DIVERGENCE OF A PENCIL EIGENVALUE*

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Abstract. The divergence through infinity of certain eigenvalues of a linearized differential-algebraic equation (DAE) may result in a stability change along an equilibrium branch. This behavior cannot be exhibited by explicit ODEs, and its analysis in index one contexts has been so far unduly restricted to semiexplicit systems. By means of a geometric reduction framework we extend the characterization of this phenomenon to quasilinear DAEs, which comprise semiexplicit problems as a particular case. Our approach clarifies the nature of the singularities which are responsible for the stability change and also accommodates rank deficiencies in the leading system matrix. We show how to address this problem in a matrix pencil setting, an issue which leads to certain results of independent interest involving the geometric index of a quasilinear DAE and the Kronecker index of its linearization. The results are shown to be of interest in electrical circuit theory, since the differential-algebraic network models actually used in circuit simulation are not in semiexplicit but in quasilinear form.

Key words. differential-algebraic equation, matrix pencil, geometric index, Kronecker index, stability change, singularity, electrical circuit

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1. Introduction.

Differential-algebraic equations (DAEs) nowadays play an important role in dynamical system modeling. *Quasilinear* DAEs are found in most applications [21, 22, 36, 48] and have the form

$$A(x,\mu)x' = f(x,\mu) \tag{1}$$

in an autonomous, parametrized context. Here $A \in C^k(W_0 \times \mathcal{I}, \mathbb{R}^{m\times m})$, $\mathbb{R}^{m\times m}$ denoting the set of real $m \times m$ matrices, and $f \in C^k(W_0 \times \mathcal{I}, \mathbb{R}^m)$; the set W_0 is open in $\mathbb{R}^m$, $\mathcal{I}$ is a real interval where the parameter μ takes values, and k is large enough as to accommodate eventually needed differentiations. From both the analytical and numerical perspectives, different approaches have been developed in the last decades for the study of so-called *regular* DAEs; cf. [9, 15, 19, 22, 26, 30, 37, 45] and references therein. Most of these frameworks are based on different index notions (which include the differentiation, geometric, tractability, perturbation, and strangeness indices, among others) and, roughly speaking, unveil the DAE behavior in terms of some type of related explicit ODE.

However, in the presence of *singularities*, it is not even possible to describe the local behavior of a DAE in terms of an explicit ODE. New dynamic phenomena are exhibited by singular DAEs, impasse points being a paradigmatic instance [10, 11, 31, 34, 35, 41, 46]. In parametrized problems, a stability change phenomenon due to the divergence of an eigenvalue was first analyzed by Venkatasubramanian [49, 50] and later addressed by several authors [4, 5, 6, 42, 43, 52]. This arises in different

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application fields [3, 24, 44, 51]; related problems are discussed in [8, 23, 53]. The change of stability occurs when an equilibrium branch intersects a singular manifold, and this results in the divergence of one (or more) eigenvalue(s) through infinity. This is a consequence of the fact that the leading coefficient $a_r(\mu)$ of the polynomial which defines these eigenvalues (cf. (25) in subsection 4.2) vanishes at a critical parameter value. When the remaining eigenvalues lie on $\mathbb{C}^-$, the equilibrium branch experiences a stability loss. This is the case in the voltage collapse phenomenon examined by Venkatasubramanian in his seminal work [49].

This change of stability was called in [49] a "singularity-induced bifurcation," an expression which is often used in the literature. Here the term "singularity" refers to the singular manifold mentioned above, and "bifurcation" is used to refer to a system which is not structurally stable in the sense that there exist arbitrarily close systems which are not topologically equivalent to it [2, 17]. In a parametrized family, a change in the qualitative properties of the system occurs when the parameter undergoes a (so-called) bifurcation value. In our case there is a change of stability and hence a change in the topological properties of the local phase portrait, but the reader should not expect any splitting of equilibria, in contrast to what happens in explicit ODEs when an eigenvalue of the linearization along an equilibrium locus vanishes at a given parameter value.

Such a stability change cannot happen in the context of explicit ODEs, for which the aforementioned coefficient $a_r(\mu)$ never vanishes. This phenomenon is therefore specific to implicit ODEs and, in particular, DAEs. However, the working setting of the references cited above is unduly restricted to the context of *semiexplicit* DAEs, which have the structure

$$y' = h(y, z, \mu), \tag{2a}$$

$$0 = g(y, z, \mu), \tag{2b}$$

the algebraic (nondifferential) constraints being given explicitly. This semiexplicit form can be seen as a particular case of (1) in which A has the (constant) block-diagonal form block-diag$(I, 0)$. However, many DAEs in applications are *not* in semiexplicit form, an important instance being defined by the models resulting from modified nodal analysis (MNA) of electrical circuits (cf. section 5).

The present paper undertakes the goal of extending the characterization of this phenomenon to the quasilinear index one DAE (1), without assuming any special structure on it. Notice that [42, Theorem 1] also characterizes related phenomena in quasilinear problems, but in an index zero context, that is, in a setting which implies that A is nonsingular except on the singular hypersurface (this precludes, in particular, semiexplicit systems). By contrast, the present analysis is directed to systems with an everywhere rank-deficient matrix A, found in most applications and including semiexplicit DAEs.

Our approach will be based on reduction methods and the geometric index notion. Besides driving the analysis beyond the semiexplicit context, the present approach provides additional insight into the geometric structure of the singularities from which the stability change stems. The local DAE trajectories will lie on a set $W_1(\mu)$ which is not explicitly given but which by certain hypotheses can be guaranteed to be a manifold. The singularity will be due to the lack of transversality of this set with the space $\ker A(x, \mu)$. These underlying geometric properties are somehow hidden in semiexplicit problems due to the very particular structure of these systems; previous statements for semiexplicit DAEs can be derived as a corollary of the more general result here presented as detailed at the end of subsection 4.2. Moreover, it will be shown that this stability loss phenomenon may also result from rank changes in the leading matrix $A(x, \mu)$, a behavior which cannot be depicted by semiexplicit DAEs simply because the leading matrix is constant for them.

Some results of independent interest will show up in the analysis. The main issue is how to guarantee that the results can be addressed via linearization in a matrix pencil setting, an approach which is well-suited for semiexplicit DAEs [6, 42] but whose extension to quasilinear problems displays several difficulties. Extending previous results from [32, 37, 40, 45] we will prove that at equilibrium points where a quasilinear DAE has a well-defined geometric index, the local pencil is a regular one with the same index. This property, of general interest in DAE theory, will be based on the fact that the reduction process supporting the geometric index notion conveys a Kronecker index reduction in the local pencil. The fact that the linear stability properties (e.g., exponential stability or hyperbolicity) of equilibria are characterized by the pencil eigenvalues follows as a byproduct of this analysis. In particular, this property will be used in section 4 to show that the number of diverging eigenvalues is determined by the index of the local pencil at the singularity.

The paper is organized as follows. After presenting background material in section 2, we discuss in section 3 the aforementioned issues involving the relation between the geometric index of a quasilinear DAE and the Kronecker index of its linearization. Supported on these results, the stability change phenomenon described above is characterized for quasilinear DAEs in section 4. Finally, section 5 applies this framework to MNA-modeled electrical circuits.

2. Background.

2.1. The geometric index and reduction methods for regular DAEs.

Fixing in (1) the parameter μ at a given value, we are led to an autonomous, nonparametrized quasilinear DAE which, with notational abuse, can be written in the form

$$A(x)x' = f(x), \tag{3}$$

where $A \in C^k(W_0, \mathbb{R}^{m\times m})$, $f \in C^k(W_0, \mathbb{R}^m)$, W_0 being an open subset of $\mathbb{R}^m$. In order to undertake the analysis of the parametrized quasilinear DAE (1), it is essential to understand how reduction techniques and the geometric index notion unveil the local behavior of (3). Geometric reduction methods were developed in the 1990s mainly by Reich [38, 39] and Rabier and Rheinboldt [33]. For the sake of completeness we present below a brief introduction to this approach; detailed discussions can be found in [33, 37].

Our attention will be focused on problems in which A is rank-deficient in the whole of W_0. This covers, in particular, the case of semiexplicit DAEs, which can be written in the form

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} h(y,z) \\ g(y,z) \end{pmatrix}, \tag{4}$$

with $h \in C^k(W_0, \mathbb{R}^r)$, $g \in C^k(W_0, \mathbb{R}^p)$, and W_0 open in $\mathbb{R}^{r+p}$.

To begin with, note that any C^1 solution of (3) must lie on the set

$$W_1 = \{x \in W_0 \ / \ f(x) \in \operatorname{im} A(x)\}. \tag{5}$$

Fix a point $x^* \in W_1$, and assume that the leading matrix $A(x)$ has constant rank $r < m$ on some neighborhood of x^*. Then, on a possibly smaller neighborhood U, there exist matrix-valued maps $H \in C^k(U, \mathbb{R}^{(m-r)\times m})$ and $P_1 \in C^k(U, \mathbb{R}^{r\times m})$ such that, for all x in U, the identity $\ker H(x) = \operatorname{im} A(x)$ holds and the restriction $P_1(x)|_{\operatorname{im} A(x)}$ yields an isomorphism $\operatorname{im} A(x) \to \mathbb{R}^r$ (see, e.g., [1]). Note that both $H(x)$ and $P_1(x)$ have maximal rank ($m-r$ and r, respectively).

By construction, given $x \in U$ it is true that $v \in \operatorname{im} A(x) \Leftrightarrow H(x)v = 0$, and therefore the set W_1 can be locally described as $W_1 \cap U = \{x \in U \ / \ H(x)f(x) = 0\}$. We say that $x^* \in W_1$ is a *0-regular* point if the aforementioned local constant rank condition on $A(x)$ holds and, additionally, $H(x)f(x)$ is a submersion at x^* (i.e., the derivative $(Hf)'$ has maximal rank $m-r$). It is worth remarking that the submersion condition does not depend on the choice of $H(x)$ and is actually equivalent to the requirement that the map $F : W_0 \times \mathbb{R}^m \to \mathbb{R}^m$ defined by $F(x,p) = A(x)p - f(x)$ is a submersion at (x^*, p^*) for any choice of p^* satisfying $F(x^*, p^*) = 0$ (cf. Lemma 3.1 in [45]). The assumption that Hf is a submersion at x^* together with the local description of W_1 as the zero set of $H(x)f(x)$ locally make W_1 an r-dimensional C^k-manifold around x^*.

Now, if $\varphi_1 : \Omega_1 \to W_1 \cap U_0$ is a local parametrization of the set W_1 around a 0-regular point x^*, with Ω_1 open in $\mathbb{R}^r$ and $U_0 \subseteq U$, it is not difficult to check that $x(t)$ is a solution of (3) within U_0 if and only if $x(t) \in W_1$ for all t and $\xi(t) = \varphi_1^{-1}(x(t))$ is a solution of

$$A_1(\xi)\xi' = f_1(\xi), \quad \xi \in \Omega_1 \subseteq \mathbb{R}^r, \tag{6}$$

with

$$A_1(\xi) = P_1(\varphi_1(\xi))A(\varphi_1(\xi))\varphi_1'(\xi), \tag{7a}$$

$$f_1(\xi) = P_1(\varphi_1(\xi))f(\varphi_1(\xi)). \tag{7b}$$

Details can be found in [45, Theorem 3.1]. The system (6) is called a one-step *local reduction* of the original DAE (3). Notice the quasilinear form of the reduction (6).

Let ξ^* stand for $\varphi_1^{-1}(x^*)$; the point x^* is said to be *regular with geometric index one* if $A_1(\xi^*) \in \mathbb{R}^{r\times r}$ is nonsingular. Again, this notion is not dependent on the choice of the reduction operators P_1, φ_1. In this situation, the reduction (6) can be obviously rewritten as an explicit ODE on some neighborhood of ξ^* within Ω_1. This provides a local coordinate description of the flow defined by the DAE on the manifold W_1, which is locally filled by solutions of the equation.

If A_1 is a singular matrix at ξ^*, we may check if the constant rank and submersion conditions discussed above hold for the reduction (6). If this is the case, we may repeat the procedure to compute a two-step local reduction (A_2, f_2) via reduction operators P_2, φ_2, and so on. If the procedure can be carried over until a nonsingular A_ν is met, then x^* is said to be *regular with geometric index* ν, provided that A_ν is the first nonsingular matrix in the chain. Again, the ν-step reduction

$$A_\nu(u)u' = f_\nu(u) \tag{8}$$

can be locally rewritten in an explicit manner and can be understood to describe in coordinates the flow induced by the DAE on the so-called solution manifold.

The following characterization of index one points will be useful later (cf. [45, Proposition 3.1]).

PROPOSITION 1. *Assume that $A(x)$ has constant rank $r < m$ around a given point $x^* \in W_1$. Then x^* is regular with index one for* (3) *if and only if the matrix*

$$\mathcal{S}(x^*) = \begin{pmatrix} P_1(x^*)A(x^*) \\ (Hf)'(x^*) \end{pmatrix} \tag{9}$$

is nonsingular.

The idea supporting this result is that $A_1(\xi^*) = P_1(\varphi_1(\xi^*))A(\varphi_1(\xi^*))\varphi_1'(\xi^*)$ is a nonsingular matrix if and only if $\ker P_1(\varphi_1(\xi^*))A(\varphi_1(\xi^*)) \cap \operatorname{im} \varphi_1'(\xi^*) = \{0\}$ (note that $\ker \varphi_1'(\xi^*) = \{0\}$ since φ_1 is a local parametrization); the subspace $\operatorname{im} \varphi_1'(\xi^*)$ describes the tangent space to W_1 at x^* and can be written as $\ker (Hf)'(x^*)$. Since $(Hf)'(x^*)$ has maximal rank, it follows that the index one condition amounts to $\ker P_1(\varphi_1(\xi^*))A(\varphi_1(\xi^*)) \cap \ker (Hf)'(x^*) = \{0\}$, a condition which is indeed equivalent to the nonsingularity of $\mathcal{S}(x^*)$.

Provided that A has constant rank around x^* and that Hf is a submersion at x^*, the identity $\ker P_1(\varphi_1(\xi^*))A(\varphi_1(\xi^*)) = \ker A(\varphi_1(\xi^*))$ holds by the construction of P_1. From the remarks above it follows that, in this situation, the failing of the index one requirement is due to the loss of transversality of the space $\ker A$ and the manifold W_1 at x^*.

2.2. Linearization, matrix pencils, and the Kronecker index. Let us now focus on a given equilibrium point of (3), that is, a point $x^* \in W_0$ for which $f(x^*) = 0$. If x^* is a regular point with geometric index ν, we may ask about the local qualitative behavior of the flow defined by the DAE on the solution manifold (W_1 in index one problems) or, equivalently, about the qualitative properties of the equilibrium u^* for the reduction $A_\nu(u)u' = f_\nu(u)$, with $x^* = \varphi_1 \circ \cdots \circ \varphi_\nu(u^*)$. The goal is to figure out sufficient conditions guaranteeing that the linear stability properties of this equilibrium can be addressed in *matrix pencil* terms; cf. section 3 below.

Given a pair of matrices $A,\ B$ in $\mathbb{R}^{m\times m}$, the matrix pencil $\{A, B\}$ is defined as the one-parameter family $\{\lambda A + B : \lambda \in \mathbb{C}\}$. The *spectrum* $\sigma(A, B)$ of the pencil is the set $\{\lambda \in \mathbb{C} \ /\ \det(\lambda A + B) = 0\}$. If there exists some $\lambda \in \mathbb{C}$ such that $\lambda A + B$ is nonsingular, the matrix pencil is called *regular* [13]. For a regular pencil, there exist nonsingular matrices $E, F \in \mathbb{R}^{m\times m}$ such that [13],

$$EAF = \begin{pmatrix} I_s & 0 \\ 0 & N \end{pmatrix},\ EBF = \begin{pmatrix} W & 0 \\ 0 & I_{m-s} \end{pmatrix}, \tag{10}$$

where $W \in \mathbb{R}^{s\times s}$ for some $s \le m$, and $N \in \mathbb{R}^{(m-s)\times(m-s)}$ is a nilpotent matrix with index $\nu \le m-s$, that is, a matrix verifying $N^\nu = 0$, $N^{\nu-1} \neq 0$. The multiplication by nonsingular matrices E, F defines an equivalence relation on the set of matrix pencils known as *strict equivalence*.

The matrices in (10) define the *Kronecker canonical form* of the pencil, and the nilpotency index ν is called the *Kronecker index* of the matrix pencil; even though we are restricting the analysis to regular pencils, this can be considered as a particularization of a more general form accommodating also singular pencils [13]. The pencil is said to have index zero if $s = m$, which amounts to requiring that A is nonsingular. The index one case is characterized by a null matrix N of dimension $m - s > 0$: in this situation, it is easy to check that $s = \operatorname{rk} A$. On the contrary, in higher index cases ($\nu \ge 2$) we have $s < \operatorname{rk} A$.

Note that $\det(\lambda A + B)$ is a polynomial in λ with degree no greater than m. Regardless of the index, using the Kronecker canonical form it is easy to check that for a regular pencil $\{A, B\}$ the identity $\sigma(A, B) = \sigma(-W)$ holds. This means that the spectrum has exactly s eigenvalues (counted with multiplicity) or, equivalently, that the *characteristic polynomial* $\det(\lambda A + B)$ has degree s. A regular pencil with index $\nu \ge 1$ is said to have an infinite eigenvalue of multiplicity $m - s$. For later use, note that the characteristic polynomial is preserved (up to a nonvanishing multiplicative constant) by the aforementioned strict equivalence relation.

3. Geometric index vs. Kronecker index. When addressing the study of equilibria for a quasilinear DAE of the form (3), a natural question arises, namely, under which conditions the linear stability properties of the equilibrium for the flow of the DAE can be characterized in terms of the *linearized* problem, that is, in terms of the pencil $\{A(x^*), -f'(x^*)\}$. Different sufficient conditions have been given in the framework of the tractability index [27, 28, 29, 47] and the geometric index [32, 37, 40].

In particular, in [32, Lemma 5.1] it is shown that the matrix pencil spectrum is preserved, disregarding multiplicities, in a one-step reduction of the DAE by proving that the spaces $\ker(\lambda A(x^*) - f'(x^*))$ and $\ker(\lambda A_1(\xi^*) - {f_1}'(\xi^*))$ are isomorphic. Lemma 1 below comprises a slightly more general statement, namely, the result that not only the spectrum but actually the characteristic polynomial of the pencil is preserved (up to a nonvanishing multiplicative constant) in the reduction, the proof just relying on elementary matrix pencil properties.

We also show that the reduction process which supports the geometric index conveys a Kronecker index reduction in the linearized problem. The regularity of the pencil at points with a well-defined geometric index, as well as the coincidence of the geometric and the Kronecker indices, will follow as direct consequences.

LEMMA 1. *Assume that x^* is a 0-regular equilibrium point of the DAE* (3)*. If the matrix pencil $\{A(x^*), -f'(x^*)\}$ is regular with Kronecker index ν, then the pencil $\{A_1(\xi^*), -{f_1}'(\xi^*)\}$ coming from the one-step reduction* (6) *is regular with Kronecker index $\nu - 1$. Additionally, both pencils have the same characteristic polynomial (up to multiplication by a nonvanishing constant) and therefore the same spectrum.*

Proof. By the construction of P_1 and H in subsection 2.1 above, the matrix

$$\begin{pmatrix} P_1(x^*) \\ H(x^*) \end{pmatrix} \tag{11}$$

is easily checked to be nonsingular. Premultiply both matrices in the pencil $\{A(x^*), -f'(x^*)\}$ by (11) to get the strictly equivalent one

$$\left\{ \begin{pmatrix} P_1(x^*)A(x^*) \\ 0 \end{pmatrix}, \begin{pmatrix} -P_1(x^*)f'(x^*) \\ -H(x^*)f'(x^*) \end{pmatrix} \right\}, \tag{12}$$

where we have made use of the identity $H(x^*)A(x^*)=0$.

Now, let $\tilde{E}$ be any rectangular matrix making

$$E = \begin{pmatrix} \varphi_1'(\xi^*) & \tilde{E} \end{pmatrix}$$

nonsingular. By multiplying both matrices in (12) by E we derive the pencil

$$\left\{ \begin{pmatrix} P_1(x^*)A(x^*)\varphi_1'(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -P_1(x^*)f'(x^*)\varphi_1'(\xi^*) & K_2 \\ 0 & K_3 \end{pmatrix} \right\}, \tag{13}$$

with $K_1 = P_1(x^*)A(x^*)\tilde{E}$, $K_2 = -P_1(x^*)f'(x^*)\tilde{E}$, and $K_3 = -H(x^*)f'(x^*)\tilde{E}$. We have used the relation $-H(x^*)f'(x^*)\varphi_1'(\xi^*) = 0$, which results from the fact that the kernel of $(Hf)'(x^*) = H(x^*)f'(x^*)$ defines the tangent space to W_1 at x^* and coincides with the subspace $\operatorname{im}\varphi_1'(\xi^*)$. By construction, the pencil (13) is strictly equivalent to $\{A(x^*), -f'(x^*)\}$.

For later use, note that the nonvanishing rows at the top of the first matrix in (13) have full rank because $P_1(x^*)A(x^*)$ has full row rank and E is nonsingular. Similarly, K_3 is nonsingular since $H(x^*)f'(x^*)$ has full row rank and E is nonsingular. The latter makes the pencil (13) strictly equivalent to

$$\left\{ \begin{pmatrix} A_1(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -f_1'(\xi^*) & K_2 \\ 0 & I \end{pmatrix} \right\}, \tag{14}$$

where $A_1(\xi^*) = P_1(x^*)A(x^*)\varphi_1'(\xi^*)$ and $f_1'(\xi^*) = -P_1(x^*)f'(x^*)\varphi_1'(\xi^*)$.

Premultiply (14) by $\left(\begin{smallmatrix} I & -K_2 \\ 0 & I \end{smallmatrix}\right)$ to obtain the strictly equivalent pencil

$$\left\{ \begin{pmatrix} A_1(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -f_1'(\xi^*) & 0 \\ 0 & I \end{pmatrix} \right\}, \tag{15}$$

which is regular with index ν since the sequence of matrix multiplications performed so far preserves regularity and the Kronecker index. Notice that the characteristic polynomial of this pencil (which, up to a nonvanishing multiplicative constant, coincides with that of $\{A(x^*), -f'(x^*)\}$) reads

$$\det\left[\lambda \begin{pmatrix} A_1(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -f_1'(\xi^*) & 0 \\ 0 & I \end{pmatrix}\right] = \det[\lambda A_1(\xi^*) - f_1'(\xi^*)]. \tag{16}$$

This means that the characteristic polynomials of $\{A(x^*), -f'(x^*)\}$ and $\{A_1(\xi^*), -f_1'(\xi^*)\}$ differ only by a nonvanishing multiplicative constant. In particular, from the assumption of regularity on $\{A(x^*), -f'(x^*)\}$ it follows that the pencil $\{A_1(\xi^*), -f_1'(\xi^*)\}$ is regular as well and has the same spectrum as $\{A(x^*), -f'(x^*)\}$.

Additionally, the regularity of the pencil (15) implies that its Kronecker index ν equals the index of the null eigenvalue of the matrix

$$\left(\lambda \begin{pmatrix} A_1(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -f_1'(\xi^*) & 0 \\ 0 & I \end{pmatrix}\right)^{-1} \begin{pmatrix} A_1(\xi^*) & K_1 \\ 0 & 0 \end{pmatrix} \tag{17}$$

for any λ not being a pencil eigenvalue (cf. [15, Theorem A.12]). By means of elementary matrix computations, the matrix (17) can be rewritten as

$$\begin{pmatrix} (\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} A_1(\xi^*) & (\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} K_1 \\ 0 & 0 \end{pmatrix}, \tag{18}$$

and its jth power reads

$$\begin{pmatrix} [(\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} A_1(\xi^*)]^{j-1} & 0 \\ 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} (\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} A_1(\xi^*) & (\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} K_1 \\ 0 & 0 \end{pmatrix}.$$

Since the nonvanishing rows of the second matrix have full row rank, the rank of this product equals that of $[(\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} A_1(\xi^*)]^{j-1}$. This makes it clear that the index of the null eigenvalue of the matrix (17) (that is, ν) exceeds by one that of $(\lambda A_1(\xi^*) - f_1'(\xi^*))^{-1} A_1(\xi^*)$, which in turn equals the Kronecker index of the pencil $\{A_1(\xi^*), -f_1'(\xi^*)\}$. The matrix pencil $\{A_1(\xi^*), -f_1'(\xi^*)\}$ has therefore Kronecker index $\nu - 1$, and this completes the proof. ∎

Lemma 1 comprises the key property needed in the proof of the following result.

THEOREM 1. *Let x^* be an equilibrium of the quasilinear DAE* (3) *with geometric index ν. Then the local pencil $\{A(x^*), -f'(x^*)\}$ is regular with Kronecker index ν.*

Additionally, the characteristic polynomial $\det[\lambda A(x^*) - f'(x^*)]$ *coincides, up to a nonvanishing multiplicative constant, with that of any ν-step local reduction of the*

DAE. Hence, the linear stability properties of x^* are characterized by the spectrum of the matrix pencil $\{A(x^*), -f'(x^*)\}$.

Proof. Let $\{(A_1, f_1), \ldots, (A_\nu, f_\nu)\}$ be a local reduction sequence for the DAE (3) around the equilibrium point x^*. As shown in Lemma 1 above, the characteristic polynomial is preserved in the reduction process (up to multiplication by a nonvanishing constant), and therefore the characteristic polynomials $\det[\lambda A(x^*) - f'(x^*)]$ and $\det[\lambda A_\nu(u^*) - f'_\nu(u^*)]$ differ only by a nonzero multiplicative constant.

Since x^* has geometric index ν, the matrix $A_\nu(u^*)$ is nonsingular. The nonsingularity of $A_\nu(u^*)$ shows that the characteristic polynomial $\det[\lambda A_\nu(u^*) - f'_\nu(u^*)]$ does not vanish identically and therefore neither does $\det[\lambda A(x^*) - f'(x^*)]$ so that the pencil $\{A(x^*), -f'(x^*)\}$ is a regular one.

Moreover, the matrix A_ν is the first nonsingular one (at the equilibrium point) within the sequence $\{A, A_1, \ldots, A_\nu\}$. This means that the Kronecker index of $\{A(x^*), -f'(x^*)\}$ must equal the geometric index ν. Indeed, if the Kronecker index (say, $\tilde{\nu}$) exceeds ν, the matrix A_ν should be singular since, according to Lemma 1, every reduction step decreases the Kronecker index by one, and therefore the pencil $\{A_\nu(u^*), -f'_\nu(u^*)\}$ would have index $\tilde{\nu} - \nu > 0$. On the contrary, if the Kronecker index $\tilde{\nu}$ is less than ν, for the same reason the pencil $\{A_{\tilde{\nu}}(\eta^*), -f'_{\tilde{\nu}}(\eta^*)\}$ would have index zero, meaning that $A_{\tilde{\nu}}(\eta^*)$ should be a nonsingular matrix; this contradicts the fact that A_ν is the first nonsingular matrix at the equilibrium point. $\square$

4. Eigenvalue divergence and change of stability.

Let us drive our attention back to the parametrized quasilinear DAE (1). Assume that we are given an equilibrium branch $(x^f(\mu), \mu)$. If these equilibrium points are regular (that is, if they have a well-defined geometric index), then as discussed in the previous section their linear stability properties are defined by the matrix pencil $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$. Suppose now that the regularity requirement does not hold at a certain parameter value μ^*. The matrix pencil spectrum may well display an eigenvalue diverging from $\mathbb{R}^-$ to $\mathbb{R}^+$ (or vice-versa) through infinity, resulting in a change in the linear stability properties of the equilibrium. If the remaining eigenvalues lie on $\mathbb{C}^-$, the transition from $\mathbb{R}^-$ to $\mathbb{R}^+$ leads to a stability loss in the equilibrium branch. In this section we provide a detailed characterization of this phenomenon.

4.1. Parametrized problems and singularities.

In a parametrized setting, the set W_1 introduced in subsection 2.1 will depend on the values of the parameter so that we can write

$$W_1(\mu) = \{x \in W_0 \ / \ f(x, \mu) \in \operatorname{im} A(x, \mu)\}. \tag{19}$$

Suppose that x^* is an equilibrium point for a given parameter value μ^*; namely, let $f(x^*, \mu^*) = 0$, and assume that $A(x, \mu)$ has constant rank $r < m$ on some neighborhood of (x^*, μ^*). This implies that there exists an open neighborhood U of (x^*, μ^*) which accommodates C^k matrix-valued mappings $P_1 : U \to \mathbb{R}^{r \times m}$ and $H : U \to \mathbb{R}^{(m-r) \times m}$ such that, for all $(x, \mu) \in U$, the restriction $P_1(x, \mu)\big|_{\operatorname{im} A(x,\mu)}$ is an isomorphism $\operatorname{im} A(x, \mu) \to \mathbb{R}^r$, and the identity $\ker H(x, \mu) = \operatorname{im} A(x, \mu)$ holds. The latter makes it possible to describe $W_1(\mu)$ as $H(x, \mu) f(x, \mu) = 0$. Assume also that the matrix of partial derivatives $(Hf)_x$ has maximal rank $m - r$ at (x^*, μ^*).

In this situation, the matrix

$$\mathcal{S}(x, \mu) = \begin{pmatrix} P_1(x, \mu) A(x, \mu) \\ (Hf)_x(x, \mu) \end{pmatrix} \tag{20}$$

characterizes index one points on U according to Proposition 1. Roughly speaking, we will work in an index one context with some exceptional points where the index one requirements fail; in this setting, these *singularities* are reflected by the fact that the matrix $\mathcal{S}$ becomes singular. It is worth mentioning that, in a higher index context, $\mathcal{S}$ would be singular everywhere; the reader is referred to [45] for a detailed discussion of singular points in arbitrary quasilinear DAEs. In the framework of subsection 4.2, since P_1A and $(Hf)_x$ have maximal rank, the singularities of the problem will be locally defined by the lack of transversality of $\ker A(x,\mu)$ and $W_1(\mu)$, since the tangent space to the latter is given by $\ker (Hf)_x(x,\mu)$. Subsection 4.3 will accommodate singularities which arise from rank deficiencies in A.

4.2. Constant rank in $A(x,\mu)$. In the following discussion we make use of the operators P_1, H, and $\mathcal{S}$ introduced above. For the sake of simplicity we also assume (often without explicit mention) that the statements for parameter values $\mu \neq \mu^*$ do hold in a sufficiently small neighborhood of μ^*.

THEOREM 2. *Assume that $A \in C^2(W_0 \times \mathcal{I}, \mathbb{R}^{m\times m})$ and $f \in C^2(W_0 \times \mathcal{I}, \mathbb{R}^m)$ in* (1). *Let x^* be an equilibrium point for a given μ^*; that is, assume that $f(x^*,\mu^*) = 0$, and suppose that $A(x,\mu)$ has constant rank r on some neighborhood of (x^*,μ^*). Assume additionally that*

(i) *the matrices f_x and*

$$\begin{pmatrix} f_x & f_\mu \\ (\det\mathcal{S})_x & (\det\mathcal{S})_\mu \end{pmatrix} \tag{21}$$

are nonsingular at (x^*,μ^*) and

(ii) *the matrix pencil $\{A(x^*,\mu^*), -f_x(x^*,\mu^*)\}$ is regular with Kronecker index two.*

Then there exists a C^2 curve of equilibria $(x^f(\mu),\mu)$ in $\mathbb{R}^{m+1}$ which passes through (x^*,μ^*). Locally around $(x^f(\mu),\mu)$, the set $W_1(\mu)$ is an r-dimensional manifold; for $\mu \neq \mu^*$ this manifold is locally foliated by solutions of (1), *and the linear stability properties of the equilibrium $(x^f(\mu),\mu)$ for the flow of* (1) *on $W_1(\mu)$ are defined by the spectrum of the pencil $\{A(x^f(\mu),\mu), -f_x(x^f(\mu),\mu)\}$, which is a regular one. When μ increases through μ^*, the equilibrium undergoes a stability change owing to the transition of one pencil eigenvalue from $\mathbb{R}^-$ to $\mathbb{R}^+$ or from $\mathbb{R}^+$ to $\mathbb{R}^-$ by divergence through $\pm\infty$.*

Proof. The local existence of the equilibrium branch $(x^f(\mu),\mu)$ is an immediate consequence of the implicit function theorem, since $f_x(x^*,\mu^*)$ is assumed to be nonsingular in item (i) above.

As indicated above, the set $W_1(\mu)$ (which accommodates, in particular, the curve of equilibria since $0 = f(x^f(\mu),\mu) \in \operatorname{im} A(x^f(\mu),\mu)$) can be locally described as the zero set of $H(x,\mu)f(x,\mu)$. Because of the fact that both H and f_x have maximal rank and using $f(x^f(\mu),\mu) = 0$, the derivative $(Hf)_x(x^f(\mu),\mu) = H(x^f(\mu),\mu)f_x(x^f(\mu),\mu)$ has itself maximal rank, thus showing that the mapping Hf is a submersion at $(x^f(\mu),\mu)$. This yields a local r-dimensional manifold structure on $W_1(\mu)$ for every fixed μ in a sufficiently small neighborhood of μ^*.

Since f_x is nonsingular at (x^*,μ^*), the nonsingularity of the matrix depicted in (21) implies that the Schur complement of f_x, that is,

$$(\det\mathcal{S})_\mu - (\det\mathcal{S})_x f_x^{-1} f_\mu, \tag{22}$$

is nonsingular as well [20]. From the implicit function theorem, it is easy to see that the expression displayed in (22) equals the derivative

$$\frac{d}{d\mu}\det\mathcal{S}(x^f(\mu),\mu). \tag{23}$$

The nonvanishing of the derivative in (23), together with the fact—proved below—that $\det \mathcal{S}(x^*, \mu^*) = 0$, implies that $\det \mathcal{S}(x^f(\mu), \mu) \neq 0$ for $\mu \neq \mu^*$, thus showing that the equilibrium points $x^f(\mu)$ have geometric index one for parameter values $\mu \neq \mu^*$. According to the results discussed in subsection 2.1, this means that $W_1(\mu)$ is completely filled by solutions of the DAE in a neighborhood of $x^f(\mu)$ for $\mu \neq \mu^*$.

Now, since for $\mu \neq \mu^*$ the equilibrium $x^f(\mu)$ has geometric index one, according to Theorem 1 its linear stability properties can be assessed in terms of the matrix pencil $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$, which is regular with Kronecker index one. The characteristic polynomial of this matrix pencil can be easily examined using a specific parametrization of $W_1(\mu)$ as detailed in what follows.

Indeed, since $H(x^*, \mu^*)$ has maximal rank and $f_x(x^*, \mu^*)$ is nonsingular, as indicated above the product $Hf_x = (Hf)_x$ (keep in mind that $f(x^*, \mu^*) = 0$) has itself maximal rank $m - r$ at (x^*, μ^*). From the implicit function theorem it follows that $m-r$ variables z from within x (say, without loss of generality, the last $m-r$ ones) can be locally written in terms of μ and the r remaining (i.e., the first) variables in x, to be denoted by y, as $z = \psi(y, \mu)$. This yields, for every fixed μ, a local parametrization of the manifold $W_1(\mu)$ of the form

$$\xi \to \varphi_1(\xi, \mu) = \begin{pmatrix} \xi \\ \psi(\xi, \mu) \end{pmatrix}. \tag{24}$$

From the $P_1(x, \mu)$ operator introduced in subsection 4.1 above and the local parametrization $\varphi_1(\xi, \mu)$, we derive a one-step local reduction $A_1(\xi, \mu)\xi' = f_1(\xi, \mu)$ in which the leading matrix A_1 has the expression

$$\begin{aligned} A_1(\xi, \mu) &= P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu) \varphi_{1\xi}(\xi, \mu) \\ &= \begin{pmatrix} (P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_1 & (P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_2 \end{pmatrix} \\ &\quad \times \begin{pmatrix} I \\ \psi_\xi(\xi, \mu) \end{pmatrix} \\ &= (P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_1 \\ &\quad -(P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_2 ((Hf)_z)^{-1} (\varphi_1(\xi, \mu), \mu) (Hf)_y (\varphi_1(\xi, \mu), \mu), \end{aligned}$$

where we have used the identity $\psi_\xi(\xi, \mu) = -((Hf)_z(\varphi_1(\xi, \mu), \mu))^{-1} (Hf)_y(\varphi_1(\xi, \mu), \mu)$ resulting from the implicit function theorem; note also that the subindex 1 (resp., 2) signals the first r (resp., last $m - r$) columns of the matrix $P_1 A$. The last expression depicted above for $A_1(\xi, \mu)$ shows that this matrix is the Schur complement of $(Hf)_z$ in

$$\mathcal{S}(\varphi_1(\xi, \mu), \mu) = \begin{pmatrix} (P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_1 & (P_1(\varphi_1(\xi, \mu), \mu) A(\varphi_1(\xi, \mu), \mu))_2 \\ (Hf)_y(\varphi_1(\xi, \mu), \mu) & (Hf)_z(\varphi_1(\xi, \mu), \mu) \end{pmatrix}.$$

The fact that the matrix pencil $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$ has Kronecker index one if $\mu \neq \mu^*$ implies that its spectrum has $\operatorname{rk} A = r$ eigenvalues. We know the characteristic polynomials $\det[\lambda A(x^f(\mu), \mu) - f_x(x^f(\mu), \mu)]$ and $\det[\lambda A_1(\xi^f(\mu), \mu) - (f_1)_\xi(\xi^f(\mu), \mu)]$ to differ only by a nonvanishing multiplicative constant, and therefore we can write

$$\det[\lambda A_1(\xi^f(\mu), \mu) - (f_1)_\xi(\xi^f(\mu), \mu)] = a_r(\mu)\lambda^r + \cdots + a_1(\mu)\lambda + a_0(\mu) \tag{25}$$

in some neighborhood of μ^* for certain coefficients $a_i(\mu)$. Here $\xi^f(\mu)$ stands for the description of $x^f(\mu)$ in the coordinates ξ. Note, in particular, that the leading coefficient $a_r(\mu)$ equals $\det A_1(\xi^f(\mu),\mu)$. Since, as detailed above, A_1 is a Schur reduction of $\mathcal{S}$, we have

$$a_r(\mu) = \det A_1(\xi^f(\mu),\mu) = \alpha(\mu)\det \mathcal{S}(x^f(\mu),\mu)$$

for some nonzero factor $\alpha(\mu)$. The nonvanishing of the derivative in (23), together with the identity $\det \mathcal{S}(x^f(\mu^*),\mu^*) = 0$ (see below), implies that $a_r(\mu^*) = 0$, $a_r'(\mu^*) \neq 0$.

Finally, the index two condition on the matrix pencil $\{A(x^*,\mu^*), -f_x(x^*,\mu^*)\}$ implies that $\{A_1(\xi^*,\mu^*), -(f_1)_\xi(\xi^*,\mu^*)\}$ is an index one pencil, following Lemma 1. Note incidentally that, in particular, this implies that $A_1(\xi^*,\mu^*)$ is singular or, equivalently, that $\det \mathcal{S}(x^*,\mu^*) = 0$. Additionally, the fact that A_1 is a Schur reduction of $\mathcal{S}$ means that the rank deficiency is the same in both matrices; the nonvanishing of the derivative in (23) implies that $\mathcal{S}$ is rank-deficient by one, and hence so it is A_1, meaning that $\operatorname{rk} A_1(\xi^*,\mu^*) = r-1$. Since the pencil $\{A_1(\xi^*,\mu^*), -(f_1)_\xi(\xi^*,\mu^*)\}$ is index one, the number of eigenvalues of its spectrum equals the rank $r-1$ of the leading matrix A_1 so that $a_{r-1}(\mu^*) \neq 0$. Together with the conditions $a_r(\mu^*) = 0$, $a_r'(\mu^*) \neq 0$, this implies (see, e.g., Theorem 2.1 in [52]) that one root of the polynomial (25) (that is, one eigenvalue of the pencil $\{A_1(\xi^f(\mu),\mu), -(f_1)_\xi(\xi^f(\mu),\mu)\}$ and, accordingly, of $\{A(x^f(\mu),\mu), -f_x(x^f(\mu),\mu)\}$ which is known to have the same spectrum) changes sign by divergence through $\pm\infty$ as μ increases through μ^*. The transition from $\mathbb{R}^-$ to $\mathbb{R}^+$ or from $\mathbb{R}^+$ to $\mathbb{R}^-$ hence yields a stability change in the equilibrium. $\square$

It is worth mentioning that in the statement of Theorem 2 we do not need to assume explicitly that $\det \mathcal{S}(x^*,\mu^*) = 0$ because this follows from the requirement that the pencil $\{A(x^*,\mu^*), -f_x(x^*,\mu^*)\}$ is index two, as explained above.

Remark. The assumption that the matrix in (21) is nonsingular at (x^*,μ^*) means that the branch of equilibria is transverse to the set $\det \mathcal{S} = 0$ at (x^*,μ^*), since the nonsingularity of (21) is equivalent to the nonvanishing of (23). This nonvanishing requirement, together with $\det \mathcal{S}(x^*,\mu^*) = 0$, can be equivalently expressed by the pair of conditions

$$\ker \mathcal{S}(x^f(\mu^*),\mu^*) = \operatorname{span}[v] \neq \{0\}, \tag{26}$$

$$\frac{d\mathcal{S}}{d\mu}(x^f(\mu^*),\mu^*)v \notin \operatorname{im} \mathcal{S}(x^f(\mu^*),\mu^*). \tag{27}$$

This is a consequence of Jacobi's formula

$$\frac{d}{d\mu}(\det M(\mu)) = \operatorname{tr}\left((\operatorname{Adj} M(\mu))\frac{d}{d\mu}M(\mu)\right),$$

where tr stands for the trace and Adj for the adjoint, that is, the transpose of the matrix of cofactors; cf. [14]. We just need to write $M(\mu) = \mathcal{S}(x^f(\mu),\mu)$ and use the property $(\operatorname{tr} P)v = Pv$, which holds when P has a unique nonnull eigenvalue and v is an associated eigenvector (this is the case for $P = (\operatorname{Adj} M)M'$ and $v \in \ker M - \{0\}$ in the present setting). Alternatively, the reader can differentiate the identity $\det M(\mu)\, I = \operatorname{Adj} M(\mu)\ M(\mu)$ at μ^* and multiply by $v \in \ker M(\mu^*) = \ker \mathcal{S}(x^f(\mu^*),\mu^*)$ to get

$$\frac{d \det \mathcal{S}}{d\mu}(x^f(\mu^*),\mu^*)\, v = \operatorname{Adj} \mathcal{S}(x^f(\mu^*),\mu^*)\ \frac{d\mathcal{S}}{d\mu}(x^f(\mu^*),\mu^*)\, v.$$

The equivalence follows easily from the fact that a singular $n\times n$ matrix $\mathcal{S}$ verifies $\operatorname{Adj} \mathcal{S} \neq 0$ if and only if $\operatorname{rk} \mathcal{S} = n-1$, together with the property $\ker \operatorname{Adj} \mathcal{S} = \operatorname{im} \mathcal{S}$ for any corank-one matrix $\mathcal{S}$.

Local dynamics for $\boldsymbol{\mu = \mu^*}$. As detailed in the proof of Theorem 2, the set $W_1(\mu)$ locally has an r-dimensional manifold structure for parameter values μ close to (and including) μ^*. The dynamics on this manifold are defined by the reduced quasilinear system $A_1(\xi, \mu)\xi' = f_1(\xi, \mu)$, and this is also valid for $\mu = \mu^*$. However, only for $\mu \neq \mu^*$ it is true that there is a regular flow filling (locally) this manifold and that the linear stability properties of this flow are defined by the spectrum of the pencil $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$; one eigenvalue of this spectrum diverges through $\pm\infty$ as μ increases through μ^*, and this is responsible for the stability change.

Nevertheless, some information can be given about the dynamics of the DAE at the critical parameter value $\mu = \mu^*$. Note first that the remaining $r-1$ eigenvalues of the linearization depend continuously on μ and their values at μ^* are given by the $r-1$ eigenvalues of the pencil $\{A(x^*, \mu^*), -f_x(x^*, \mu^*)\}$; because of the assumption of nonsingularity on $f_x(x^*, \mu^*)$, $\lambda = 0$ is not an eigenvalue of this pencil, and therefore these $r-1$ eigenvalues stay away from the origin.

These eigenvalues can be shown to characterize the linear stability properties of a restricted regular flow which is defined by the DAE on a codimension-one submanifold of $W_1(\mu^*)$. This is a consequence of Lemma 3 and Corollary 5 of [8] when applied to the enlargement $\xi' = p, 0 = A_1(\xi, \mu)p - f_1(\xi, \mu)$; note, in particular, that the requirement $CBk \notin \operatorname{im} D$ in Lemma 3 of [8] holds by the property $\operatorname{ind}\{A_1(\xi^*, \mu^*), -(f_1)_\xi(\xi^*, \mu^*)\} = 1$ derived in the proof of Theorem 2 above. Additionally, provided that $\ker A_1(\xi^*, \mu^*)$ intersects transversally the singular set $\det A_1(\xi, \mu^*) = 0$, it can be shown that there exists an invariant curve on $W_1(\mu^*)$ accommodating a solution which crosses smoothly the singularity in finite time, akin to the results in [7].

Semiexplicit problems. Applying Theorem 2 to the DAE (2), one obtains the corresponding statement for semiexplicit systems proved in [42] (cf. Theorem 2 there). Indeed, in the light of (4), we may set $P_1 = (I_r \;\; 0)$ and $H = (0 \;\; I_p)$. This yields

$$\mathcal{S}(y, z, \mu) = \begin{pmatrix} I_r & 0 \\ g_y(y, z, \mu) & g_z(y, z, \mu) \end{pmatrix},$$

and therefore $\det \mathcal{S}(y, z, \mu) = \det g_z(y, z, \mu)$. The matrices in item (i) of Theorem 2 above then read

$$\begin{pmatrix} h_y & h_z \\ g_y & g_z \end{pmatrix}, \quad \begin{pmatrix} h_y & h_z & h_\mu \\ g_y & g_z & g_\mu \\ (\det g_z)_y & (\det g_z)_z & (\det g_z)_\mu \end{pmatrix}, \tag{28}$$

whereas the matrix pencil in item (ii) has the form

$$\left\{ \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \; -\begin{pmatrix} h_y & h_z \\ g_y & g_z \end{pmatrix} \right\}. \tag{29}$$

The eigenvalue transition stated in Theorem 2 above therefore holds if, at a given equilibrium (y^*, z^*, μ^*), both matrices in (28) are nonsingular and the pencil (29) is regular with Kronecker index two.

For the sake of comparison with Theorem 2 in [42], note that zero is not a pencil eigenvalue if and only if the first matrix in (28) is nonsingular. Notice also that this nonsingularity requirement implies that $(g_y \;\; g_z)$ has maximal rank at (y^*, z^*, μ^*) and hence makes $W_1(\mu)$ (which is explicitly defined by the condition $g(y, z, \mu) = 0$) an r-dimensional manifold for all μ in a neighborhood of μ^*.

4.3. Rank deficiencies in $A(x,\mu)$. The stability change phenomenon discussed above may also result from a rank drop in the leading matrix $A(x,\mu)$ of the quasilinear DAE (1) at a singularity as detailed in this subsection. It is worth noting that this phenomenon cannot be displayed by the semiexplicit system (2), in which the leading matrix is the constant one block-diag$(I,0)$.

Focus the attention on an equilibrium point x^* for (1) at a given parameter value μ^*. Even though the matrix $A(x,\mu)$ will undergo a rank drop at (x^*,μ^*), we will make the assumption that there exists a neighborhood $\tilde{U}$ of (x^*,μ^*) and an r-dimensional linear space $L(x,\mu)$ varying smoothly with (x,μ) and such that $\operatorname{im} A(x,\mu) = L(x,\mu)$ on a dense subset of $\tilde{U}$. The space $L(x,\mu)$ can be thought of as a *continuation* of $\operatorname{im} A(x,\mu)$ which is assumed to exist. Note, in particular, that this implies that the identity $\operatorname{rk} A(x,\mu) = r$ holds on the aforementioned dense subset of $\tilde{U}$. This way, we may still assert the existence of a (possibly smaller) neighborhood U and smoothly varying operators $P_1 : U \to \mathbb{R}^{r\times m}$ and $H : U \to \mathbb{R}^{(m-r)\times m}$ such that, for all $(x,\mu) \in U$, the restriction $P_1(x,\mu)|_{L(x,\mu)}$ is an isomorphism $L(x,\mu) \to \mathbb{R}^r$, and the identity $\ker H(x,\mu) = L(x,\mu)$ holds.

Defining the matrix $\mathcal{S}(x,\mu)$ as in (20), we are then allowed to tackle the stability change phenomenon in the present framework as follows. As before, without explicit mention we assume that the claims for $\mu \neq \mu^*$ hold in a sufficiently small neighborhood of μ^*.

THEOREM 3. *Assume that $A \in C^2(W_0 \times \mathcal{I}, \mathbb{R}^{m\times m})$, $f \in C^2(W_0 \times \mathcal{I}, \mathbb{R}^m)$ in* (1). *Let x^* be an equilibrium of* (1) *for a given μ^*. Suppose that there exists a local r-dimensional continuation $L(x,\mu)$ of $\operatorname{im} A(x,\mu)$ which depends on (x,μ) in a C^2 manner. Assume additionally that*

(i) *the matrices f_x and*

$$\begin{pmatrix} f_x & f_\mu \\ (\det \mathcal{S})_x & (\det \mathcal{S})_\mu \end{pmatrix} \tag{30}$$

are nonsingular at (x^*,μ^*);

(ii) $\operatorname{rk} A(x^*,\mu^*) = r-1$*;*

(iii) *the matrix pencil $\{A(x^*,\mu^*), -f_x(x^*,\mu^*)\}$ is regular with Kronecker index one.*

Then there exists a C^2 curve of equilibria $(x^f(\mu),\mu)$ in $\mathbb{R}^{m+1}$ which passes through (x^*,μ^*). For $\mu \neq \mu^*$, locally around $(x^f(\mu),\mu)$ the set $W_1(\mu)$ is an r-dimensional manifold which is locally foliated by solutions of (1)*, and the linear stability properties of the equilibrium $(x^f(\mu),\mu)$ for the flow of* (1) *on $W_1(\mu)$ are defined by the spectrum of the matrix pencil $\{A(x^f(\mu),\mu), -f_x(x^f(\mu),\mu)\}$, which is a regular one. When μ increases through μ^*, the equilibrium undergoes a stability change owing to the transition of one pencil eigenvalue from $\mathbb{R}^-$ to $\mathbb{R}^+$ or from $\mathbb{R}^+$ to $\mathbb{R}^-$ by divergence through $\pm\infty$.*

Proof. Many steps in the proof parallel those in Theorem 2. Indeed, the existence of a C^2-curve of equilibria passing through (x^*,μ^*) follows from the implicit function theorem as in Theorem 2. The rank deficiency of $A(x^*,\mu^*)$ assumed in item (ii) above implies that $\det \mathcal{S}(x^*,\mu^*) = 0$, since the first r rows of the matrix $\mathcal{S}$ depicted in (20) cannot have full row rank. Additionally, the nonsingularity of (30) guarantees, again as in Theorem 2, that $\det \mathcal{S}(x^f(\mu),\mu) \neq 0$ for $\mu \neq \mu^*$, and then at those points $\operatorname{rk} A(x,\mu) = r$. Moreover, since $\operatorname{rk} A = r$ on a dense subset of U, the relation $\operatorname{rk} A(x,\mu) = r$ holds on a neighborhood of $(x^f(\mu),\mu)$, always for $\mu \neq \mu^*$.

Now, $H(x^*,\mu^*)$ has maximal rank $m-r$ and since so it has $f_x(x^*,\mu^*)$ the product $Hf_x(x^*,\mu^*) = (Hf)_x(x^*,\mu^*)$ (remember that $f(x^*,\mu^*) = 0$) has itself maximal rank. This is then the case on some neighborhood of (x^*,μ^*), and therefore the set $Hf = 0$

has locally a manifold structure. Because of the fact that $\operatorname{rk} A(x, \mu) = r$, as explained above, the set $Hf = 0$ is locally coincident with $W_1(\mu)$ near $(x^f(\mu), \mu)$ for $\mu \neq \mu^*$, and therefore $W_1(\mu)$ has locally an r-dimensional manifold structure for $\mu \neq \mu^*$.

Moreover, the property $\det \mathcal{S}(x^f(\mu), \mu) \neq 0$ if $\mu \neq \mu^*$ proved above makes $(x^f(\mu), \mu)$ a regular point with geometric index one for $\mu \neq \mu^*$. This means that the (index one) pencil $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$ characterizes the linear stability properties of the equilibrium $x^f(\mu)$ for $\mu \neq \mu^*$. Its behavior when μ increases through μ^* can be assessed in terms of a one-step reduction for $\mu \neq \mu^*$ exactly as in Theorem 2. The key remark is that φ_1 in (24) now defines a local parametrization of the set $Hf = 0$ which, for $\mu \neq \mu^*$, coincides locally with $W_1(\mu)$. Together with the operator P_1 constructed from the assumed existence of the continuation $L(x, \mu)$, we obtain a one-step reduction of the DAE in which the leading matrix $A_1(\xi, \mu)$ has the same expression as in Theorem 2, arising again as a Schur reduction of $\mathcal{S}$. This means that the characteristic polynomial of $\{A_1(\xi^f(\mu), \mu), -(f_1)_\xi(\xi^f(\mu), \mu)\}$ (which differs from that of $\{A(x^f(\mu), \mu), -f_x(x^f(\mu), \mu)\}$ only by a nonzero multiplicative constant) still has the form displayed in (25) with $a_r(\mu^*) = 0$, $a_r'(\mu^*) \neq 0$ as before.

The condition that remains to be checked is $a_{r-1}(\mu^*) \neq 0$, which cannot be derived as in Theorem 2. Now, this requirement follows from the index one assumption on the matrix pencil at the singularity stated in item (iii); this implies that the number of eigenvalues equals the rank of the A matrix, which by item (ii) is $r-1$. The spectrum at (x^*, μ^*) then has $r - 1$ eigenvalues, meaning that $a_{r-1}(\mu^*) \neq 0$. The result then follows as in Theorem 2. $\square$

Note that, for the rank deficiency to characterize the stability change phenomenon, now we need to assume that the index does *not* change at the singularity (cf. item (iii) in the statement of Theorem 3). In this situation a minimal rank drop at (x^*, μ^*) guarantees that exactly one eigenvalue diverges through $\pm\infty$ as stated above. This index one requirement cannot be neglected; see, specifically, the case $G_2 = 0$, $\mu_2 = 1$ in subsection 5.4 below.

5. Stability loss in MNA-modeled electrical circuits. Qualitative properties of electrical circuits are often addressed in terms of state space models, which unfortunately are not available in many practical situations. It is therefore of interest to address them using *semistate* models based on DAEs. In this section we illustrate the stability loss phenomenon characterized above by discussing it in the framework of certain quasilinear DAEs actually used in electrical circuit simulation. The quasilinear form of these differential-algebraic systems drives the analysis beyond the semiexplicit context of [4, 6, 43, 44, 49, 50, 52].

5.1. MNA. DAEs are widely used in electrical circuit modeling and analysis. In particular, MNA models are used by current circuit simulation programs such as SPICE or TITAN in order to set up the network equations [12, 18, 45, 48]. These models have the quasilinear form

$$A_c C(A_c^T e, \mu) A_c^T e' = -A_r \gamma(A_r^T e, \mu) - A_l i_l - A_v i_v - A_i i_s(t), \tag{31a}$$

$$L(i_l, \mu) i_l' = A_l^T e, \tag{31b}$$

$$0 = v_s(t) - A_v^T e. \tag{31c}$$

Here A_c, A_r, A_l, A_v, and A_i are reduced incidence matrices describing how the branches accommodating capacitors, resistors, inductors, and voltage and current sources, respectively, are related to the circuit nodes. The model variables x are defined by the node potentials e, the inductor currents i_l, and the voltage source currents i_v; in turn, $i_s(t)$ and $v_s(t)$ are excitation terms coming from the current and voltage sources, respectively. Resistors are assumed to be voltage-controlled, with a current-voltage characteristic $i_r = \gamma(v_r, \mu)$. The matrices C and L stand for the incremental capacitance and inductance, respectively.

The symbol μ is used to distinguish certain circuit parameters which may be responsible for stability changes. Notice that these changes may stem from the presence of μ in the right-hand side of (31) but also from its appearance in the leading matrix of this system, namely,

$$A(x,\mu) = \begin{pmatrix} A_c C(A_c^T e, \mu) A_c^T & 0 & 0 \\ 0 & L(i_l,\mu) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{32}$$

which may result in rank deficiencies. We will illustrate these phenomena by addressing a specific example.

5.2. A circuit example. Consider the circuit depicted in Figure 1.

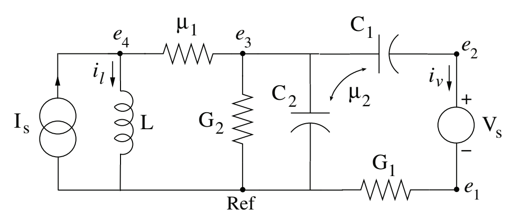

FIG. 1. *A coupled circuit with parameters μ_1, μ_2.*

The circuit consists of a DC current source I_s, a DC voltage source V_s, three linear resistors with conductances G_1, G_2, and μ_1, a linear inductor with inductance L, and two coupled capacitors, with capacitance matrix

$$C = \begin{pmatrix} C_1 & \mu_2 \\ \mu_2 & C_2 \end{pmatrix}. \tag{33}$$

The case $\mu_2 = 0$ would stand for a pair of uncoupled linear capacitors with capacitances C_1, C_2.

The MNA model (31) has the form $A(\mu)x' + B(\mu)x = g$, where the semistate vector $x = (e_1, e_2, e_3, e_4, i_l, i_v)$ is defined by the potentials at the circuit nodes (except at the reference one) and the currents through the inductor and the voltage source. The matrices $A(\mu)$, $B(\mu)$ can be shown to read as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_1 & -C_1+\mu_2 & 0 & 0 & 0 \\ 0 & -C_1+\mu_2 & C_1+C_2-2\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} G_1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & G_2+\mu_1 & -\mu_1 & 0 & 0 \\ 0 & 0 & -\mu_1 & \mu_1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, the contribution of the sources is comprised in the excitation term $g = (0, 0, 0, I_s, 0, -V_s)$, which is constant because of the DC nature of both sources.

The mapping $f(x, \mu)$ in (1) is given by $-B(\mu)x + g$ and hence verifies $f_x = -B$; from the fact that $\det B = G_1(G_2 + \mu_1)$ it follows that the conditions $G_1 \neq 0$, $G_2 + \mu_1 \neq 0$ make B nonsingular. Both requirements will be assumed to hold in what follows. The circuit then has a unique equilibrium which is defined by the conditions $e_1 = e_3 = e_4 = i_v = 0$, $e_2 = V_s$, and $i_l = I_s$. Below we analyze stability changes in this equilibrium due to the fact that certain values of the parameters μ_1 and μ_2 make the DAE a singular one.

Note that the leading matrix A has rank three except if $L = 0$ or $C_1C_2 - \mu_2^2 = 0$, the latter making the capacitance matrix C in (33) a singular one. We will assume throughout that $L \neq 0$ and consider first some cases in which $C_1C_2 - \mu_2^2 \neq 0$, yielding a locally constant rank in A, to address later rank deficiencies in A owing to values of μ_2 for which $C_1C_2 - \mu_2^2 = 0$.

5.3. Constant rank in A. Let us fix the parameter μ_2 with a value such that $C_1C_2 - \mu_2^2 \neq 0$, and assume that $L \neq 0$, $G_1 \neq 0$, and $G_2 \neq 0$ (the latter will be needed to guarantee that $G_2 + \mu_1 \neq 0$ at the critical parameter value). As indicated above, the conditions $C_1C_2 - \mu_2^2 \neq 0$, $L \neq 0$ imply that the leading matrix A has constant rank $r = 3$. Set

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{34}$$

so that the matrix $\mathcal{S}$ in (20) reads

$$\mathcal{S} = \begin{pmatrix} 0 & C_1 & -C_1 + \mu_2 & 0 & 0 & 0 \\ 0 & -C_1 + \mu_2 & C_1 + C_2 - 2\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ -G_1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \mu_1 & -\mu_1 & -1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{35}$$

The determinant of this matrix is $-L\mu_1(C_1C_2 - \mu_2^2)$. Since we have assumed that $L \neq 0$ and $C_1C_2 - \mu_2^2 \neq 0$, the MNA model becomes singular when $\mu_1 = 0$, which corresponds to open-circuiting the resistor with conductance μ_1; otherwise, the DAE is index one.

If the hypotheses stated in Theorem 2 are met, the equilibrium point computed above is expected to undergo a stability change when μ_1 decreases through zero. As indicated above, the matrix $f_x = -B$ is nonsingular, whereas the nonsingularity of the matrix (21) in item (i) of Theorem 2 amounts to the condition $(\det \mathcal{S})_{\mu_1} \neq 0$, which does actually hold since $(\det \mathcal{S})_{\mu_1} = -L(C_1C_2 - \mu_2^2) \neq 0$.

It remains to check the regularity and the Kronecker index of the pencil $\{A, B\}$ at the singularity. To this aim we can make use of the projector-based characterization of matrix pencils originally introduced in [25] and [16]. Provided that Q_0 is a projector onto $\ker A$, the pencil $\{A, B\}$ is regular with Kronecker index one if and only if $A_1 = A + BQ_0$ is a nonsingular matrix. If this is not the case, letting in turn Q_1 be a projector onto $\ker A_1$, the pencil is regular with Kronecker index two if and only if $A_2 = A_1 + B(I - Q_0)Q_1$ is nonsingular. Using the fact that $\mu_1 = 0$ at the singularity, we compute

$$A_1 = \begin{pmatrix} G_1 & 0 & 0 & 0 & 0 & -1 \\ 0 & C_1 & -C_1+\mu_2 & 0 & 0 & 1 \\ 0 & -C_1+\mu_2 & C_1+C_2-2\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & L & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

which is singular, and

$$A_2 = \begin{pmatrix} G_1 & 0 & 0 & 0 & 0 & -1 \\ 0 & C_1 & -C_1+\mu_2 & 0 & 0 & 1 \\ 0 & -C_1+\mu_2 & C_1+C_2-2\mu_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & L & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

by means of the projectors

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix A_2 is nonsingular because of the assumption $C_1C_2 - \mu_2^2 \neq 0$, and this means that the matrix pencil at the singularity is indeed regular with Kronecker index two.

According to Theorem 2, the equilibrium is then expected to undergo a change of stability at $\mu_1 = 0$. As a matter of example, fixing the parameter values $L = C_1 = C_2 = G_1 = G_2 = 1$, $\mu_2 = 0$, computer calculations show that one eigenvalue diverges from $-\infty$ to $+\infty$ as μ_1 decreases through zero; the other two pencil eigenvalues are located at -0.382 and -2.618 when the system takes on the value $\mu_1 = 0$. The asymptotic stability of the equilibrium is lost when μ_1 becomes negative.

5.4. Rank deficiencies in *A*. Let us assume now that the parameter μ_2 takes on a nonvanishing value for which $C_1C_2 - \mu_2^2 = 0$, and suppose that $L \neq 0$, $G_1 \neq 0$ and that μ_1 is fixed at a nonzero value such that $G_2 + \mu_1 \neq 0$. The assumption $\mu_2 \neq 0$ implies $C_1 \neq 0 \neq C_2$ because of the identity $C_1C_2 - \mu_2^2 = 0$.

With these parameter values, the matrix A experiences a rank drop from three to two. However, its image space admits the smooth continuation

$$\operatorname{span}\left[(0,1,0,0,0,0),\ (0,0,1,0,0,0),\ (0,0,0,0,1,0)\right],$$

and therefore the operators P_1, H introduced in (34), as well as the matrix $\mathcal{S}$ in (35), are still valid in the present setting.

Since we are facing a rank-deficient problem, a stability change is expected to occur if the hypotheses of Theorem 3 do hold. The matrix $f_x = -B$ is nonsingular as before, and the nonsingularity of (30) now relies on the condition $2L\mu_1\mu_2 \neq 0$, which is met because of the assumptions specified above. Item (i) in Theorem 3 is therefore satisfied, as well as the minimal rank deficiency assumed in item (ii). It remains to check that the matrix pencil is regular with Kronecker index one at the singularity. Noting that $C_1C_2 - \mu_2^2 = 0$, $C_1 \neq 0$, we can take

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{C_1-\mu_2}{C_1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

which yields

$$A_1 = \begin{pmatrix} G_1 & 0 & 0 & 0 & 0 & -1 \\ 0 & C_1 & -C_1+\mu_2 & 0 & 0 & 1 \\ 0 & -C_1+\mu_2 & C_1+C_2-2\mu_2+G_2+\mu_1 & -\mu_1 & 0 & 0 \\ 0 & 0 & -\mu_1 & \mu_1 & 0 & 0 \\ 0 & 0 & 0 & -1 & L & 0 \\ 1 & 0 & -\frac{C_1-\mu_2}{C_1} & 0 & 0 & 0 \end{pmatrix}.$$

Using the identity $\mu_2^2 = C_1C_2$ the determinant of this matrix can be simplified to

$$L\mu_1(G_1(C_1 + C_2 - 2\mu_2) + G_2C_1). \tag{36}$$

If the parameter values prevent this expression from vanishing, then the pencil is index one at the singularity, item (iii) of Theorem 3 is met, and a change of stability is expected to occur.

This is the case if we fix, for instance, the values $L = C_1 = C_2 = G_1 = G_2 = \mu_1 = 1$; as μ_2 increases through 1 (a value which makes $C_1C_2 - \mu^2 = 0$) one eigenvalue diverges from $-\infty$ to $+\infty$, the other two being located at -1 and -2, respectively. Again, the asymptotic stability of the equilibrium is lost when the parameter increases through the critical value $\mu_2 = 1$.

The index one requirement coming from item (iii) cannot be dropped. If we set $L = C_1 = C_2 = G_1 = \mu_1 = 1$ and make $G_2 = 0$, then *a pair* of eigenvalues diverge from $\pm i\infty$ to $\pm\infty$ when μ_2 increases through 1, yielding a spiral-saddle transition. Notice that these parameter values annihilate the expression depicted in (36). This behavior is reminiscent of the phenomena addressed for semiexplicit DAEs in [5, 43].

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