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Lyapunov functions for strong exponential dichotomies

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A B S T R A C T

We give a complete characterization of a general type of exponential dichotomies in terms of Lyapunov functions, both for discrete and continuous time. This includes constructing explicitly quadratic Lyapunov functions for each exponential dichotomy. We consider the general cases of *nonautonomous* dynamics, *nonuniform* exponential dichotomies and, motivated by ergodic theory, *strong* exponential dichotomies, in the sense that there exist simultaneously lower and upper contraction and expansion bounds. As a nontrivial application, we establish the persistence of the asymptotic stability of a strong nonuniform exponential dichotomy under sufficiently small linear perturbations.

1. Introduction

The notion of an exponential dichotomy, essentially introduced by Perron in [1], plays an important role in a large part of the theory of dynamical systems, such as, for example, in invariant manifold theory. We note that the theory of exponential dichotomies and its applications are very much developed. We refer to the books [2–5] for details and further references. We particularly recommend [2] for a historical discussion. The reader may also consult the books [6–8].

On the other hand, the requirement of uniformity for the asymptotic behavior is often too stringent for the dynamics. It turns out that the notion of a nonuniform exponential dichotomy, which allows a nonuniform bound on the initial time, is much more typical. In particular, this behavior is ubiquitous in the context of ergodic theory. We refer to [9] for a systematic account of some of its consequences, in particular in connection with the existence and smoothness of invariant manifolds, the construction of topological conjugacies, and the existence of center manifolds, among other topics.

Our main objective in the present paper is to characterize completely the class of nonuniform exponential dichotomies in terms of Lyapunov functions, both for discrete and continuous time. We consider simultaneously the general cases of nonautonomous dynamics and of strong nonuniform exponential dichotomies.

The notion of Lyapunov function goes back to seminal work of Lyapunov in his 1892 thesis (see [10]). Among the first detailed accounts of the theory are the books of LaSalle and Lefschetz [11], Hahn [12] and Bhatia and Szegö [13]. Unfortunately, there exists no general method to construct Lyapunov functions for a given dynamics. Incidentally, there exists also a related approach in the context of ergodic theory, stemming essentially from work of Wojtkowski in [14]. We refer to [15] for details and references.

As a nontrivial application of our characterization of nonuniform exponential dichotomies, we establish the robustness of strong nonuniform exponential dichotomies, that is, the persistence of the asymptotic stability of a strong nonuniform exponential dichotomy under sufficiently small linear perturbations. To the best of our knowledge the corresponding robustness property was not considered earlier in the literature.

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2. Exponential dichotomies for maps

2.1. Preliminaries

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible $p \times p$ matrices. For each $m, n \in \mathbb{Z}$, we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy* if there exist projections P_m for each $m \in \mathbb{Z}$ satisfying

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n$$

for $m, n \in \mathbb{Z}$ and there exist constants

$$\overline{a} < 0 < \underline{b}, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1$$

such that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have

$$\|\mathcal{A}(m,n)P_n\| \leq De^{\overline{a}(m-n)+\varepsilon|n|}$$

and

$$\|\mathcal{A}(m,n)^{-1}Q_m\| \leq De^{-\underline{b}(m-n)+\varepsilon|m|},$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$.

Now we consider continuous functions $V_m\colon \mathbb{R}^p \to \mathbb{R}$ and we assume that there exist $C > 0$ and $\delta \geq 0$ such that

$$|V_m(x)| \leq Ce^{\delta|m|}\|x\| \tag{1}$$

for $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$. We also consider the cones

$$C^u(V_m) = \{0\} \cup V_m^{-1}(0,+\infty) \quad \text{and} \quad C^s(V_m) = \{0\} \cup V_m^{-1}(-\infty,0).$$

We say that $(V_m)_{m\in\mathbb{Z}}$ is a *Lyapunov sequence* for $(A_m)_{m\in\mathbb{Z}}$ if there exist $r_u, r_s \in \mathbb{N}$ with $r_u + r_s = p$ such that for each $m \in \mathbb{Z}$:

1. r_u and r_s are respectively the maximal dimensions of the linear subspaces inside $C^u(V_m)$ and $C^s(V_m)$;
2. for each $x \in \mathbb{R}^p$, we have

$$V_{m+1}(A_mx) \geq V_m(x). \tag{2}$$

We also consider the sets

$$E_n^u = \bigcap_{m\in\mathbb{Z}} \mathcal{A}(n,m)\overline{C^u(V_m)} \tag{3}$$

and

$$E_n^s = \bigcap_{m\in\mathbb{Z}} \mathcal{A}(n,m)\overline{C^s(V_m)} \tag{4}$$

for each $n \in \mathbb{Z}$. Clearly,

$$\mathcal{A}(m,n)E_n^u = E_m^u \quad \text{and} \quad \mathcal{A}(m,n)E_n^s = E_m^s$$

for $m, n \in \mathbb{Z}$. A Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ is said to be *strict* if there exists $\gamma \in (0,1)$ such that for every $m \in \mathbb{Z}$ and $x \in E_m^s \cup E_m^u$ we have

$$V_{m+1}(A_mx) - V_m(x) \geq \gamma|V_m(x)|$$

and

$$|V_m(x)| \geq \|x\|/C. \tag{5}$$

Finally, a Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ is said to be *quadratic* if there exists a sequence $(S_m)_{m\in\mathbb{Z}}$ of symmetric invertible $p \times p$ matrices such that

$$V_m(x) = -\mathrm{sgn} H_m(x)\sqrt{|H_m(x)|}$$

for $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$, where $H_m(x) = \langle S_mx, x\rangle$. It is shown in [16] that:

1. if there exists a strict quadratic Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ for $(A_m)_{m\in\mathbb{Z}}$ such that

   $$(1+\gamma)/(1-\gamma) > e^\delta$$

   and

   $$\limsup_{m\to\pm\infty} \frac{1}{|m|}\log\|S_m\| < \infty,$$

   then $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy;
2. if $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy, then it has a strict Lyapunov sequence.

2.2. Existence of exponential dichotomies

In this section we consider the (stronger) notion of a strong nonuniform exponential dichotomy and we obtain a criterion for its existence in terms of strongly strict Lyapunov sequences.

A strict Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ is said to be *strongly strict* if there exist $\gamma \in (0, 1)$ and $\mu_s, \mu_u \geq \gamma$ with $\mu_s < 1$ such that for each $m \in \mathbb{Z}$:

1. for $x \in E^u_m$,
$$\mu_u V_m(x) \geq V_{m+1}(A_m x) - V_m(x) \geq \gamma V_m(x); \tag{6}$$
2. for $x \in E^s_m$,
$$\mu_s |V_m(x)| \geq V_{m+1}(A_m x) - V_m(x) \geq \gamma |V_m(x)|. \tag{7}$$

This notion was introduced in [16]. We start with a preliminary result.

Theorem 1. *If there exists a strongly strict Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ for $(A_m)_{m\in\mathbb{Z}}$ satisfying*
$$(1+\gamma)/(1-\gamma) > e^{\delta}, \tag{8}$$
then:

1. *for each $n \in \mathbb{Z}$ the sets E^u_n and E^s_n in (3) and (4) are linear subspaces respectively of dimensions r_u and r_s, and*
$$\mathbb{R}^p = E^u_n \oplus E^s_n; \tag{9}$$
2. *there exist constants*
$$\underline{a} \leq \overline{a} < 0 < \underline{b} \leq \overline{b}, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1$$
such that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have
$$\begin{aligned} &\|\mathcal{A}(m,n)|E^s_n\| \leq De^{\overline{a}(m-n)+\varepsilon|n|}, \\ &\|\mathcal{A}(m,n)^{-1}|E^u_m\| \leq De^{-\underline{b}(m-n)+\varepsilon|m|}, \end{aligned} \tag{10}$$
and for every $m, n \in \mathbb{Z}$ with $m \leq n$ we have
$$\begin{aligned} &\|\mathcal{A}(m,n)|E^s_n\| \leq De^{\underline{a}(m-n)+\varepsilon|n|}, \\ &\|\mathcal{A}(m,n)^{-1}|E^u_m\| \leq De^{-\overline{b}(m-n)+\varepsilon|m|}. \end{aligned} \tag{11}$$

Proof. Since a strongly strict Lyapunov sequence is strict, the first statement follows readily from results in [17]. So we only establish the second statement. It follows from (5) that
$$E^u_n \subset C^u(V_n) \quad \text{and} \quad E^s_n \subset C^s(V_n).$$
Therefore, V_n is positive in $E^u_n \setminus \{0\}$ and negative in $E^s_n \setminus \{0\}$. For each $x \in E^s_n$, it follows from (1) and (7) that for every $m \geq n$,
$$\begin{aligned} \|\mathcal{A}(m,n)x\| &\leq C|V_m(\mathcal{A}(m,n)x)| \\ &\leq C(1-\gamma)^{m-n}|V_n(x)| \\ &\leq C^2(1-\gamma)^{m-n}e^{\delta|n|}\|x\|. \end{aligned}$$
On the other hand, for each $x \in E^u_n$, it follows from (1), (5) and (7) that for every $m \geq n$,
$$\begin{aligned} \|\mathcal{A}(m,n)x\| &\geq \frac{1}{C}e^{-\delta|m|}V_m(\mathcal{A}(m,n)x) \\ &\geq \frac{1}{C}e^{-\delta|m|}(1+\gamma)^{m-n}V_n(x) \\ &\geq \frac{1}{C^2}e^{-\delta|m|}(1+\gamma)^{m-n}\|x\|. \end{aligned}$$
Similarly, we have
$$\|\mathcal{A}(m,n)x\| \leq C^2\left(\frac{1}{1-\mu_s}\right)^{n-m} e^{\delta|n|}\|x\|$$

for $x \in E_n^s$ and $m \leq n$, and

$$\|\mathcal{A}(m,n)x\| \geq \frac{1}{C^2} e^{-\delta|m|} \left(\frac{1}{1+\mu_u}\right)^{n-m} \|x\|$$

for $x \in E_n^u$ and $m \leq n$. Hence, the inequalities in (10) and (11) hold with

$$\underline{a} = \log(1-\mu_s), \qquad \overline{a} = \log(1-\gamma), \qquad \underline{b} = \log(1+\gamma), \qquad \overline{b} = \log(1+\mu_u),$$

$\varepsilon = \delta$ and $D = C^2$. This completes the proof of the theorem. □

Now we consider strong exponential dichotomies. We say that a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible $p \times p$ matrices admits a *strong nonuniform exponential dichotomy* if there exist projections P_m for each $m \in \mathbb{Z}$ satisfying

$$P_m \mathcal{A}(m,n) = \mathcal{A}(m,n) P_n$$

for $m, n \in \mathbb{Z}$ and there exist constants

$$\underline{a} \leq \overline{a} < 0 < \underline{b} \leq \overline{b}, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1$$

such that for every $m, n \in \mathbb{Z}$ with $m \geq n$ we have

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\leq De^{\overline{a}(m-n)+\varepsilon|n|}, \\ \|\mathcal{A}(m,n)^{-1}Q_m\| &\leq De^{-\underline{b}(m-n)+\varepsilon|m|}, \end{aligned} \tag{12}$$

and for every $m, n \in \mathbb{Z}$ with $m \leq n$ we have

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\leq De^{\underline{a}(m-n)+\varepsilon|n|}, \\ \|\mathcal{A}(m,n)^{-1}Q_m\| &\leq De^{-\overline{b}(m-n)+\varepsilon|m|}, \end{aligned} \tag{13}$$

where $Q_m = \mathrm{Id} - P_m$ for each $m \in \mathbb{Z}$. For each $m \in \mathbb{Z}$, we define the *stable* and *unstable subspaces*

$$F_m^s = P_m(\mathbb{R}^p) \quad \text{and} \quad F_m^u = Q_m(\mathbb{R}^p). \tag{14}$$

Corollary 2. *If there exists a strongly strict quadratic Lyapunov sequence $(V_m)_{m\in\mathbb{Z}}$ for $(A_m)_{m\in\mathbb{Z}}$ and conditions (8) and*

$$\limsup_{m\to\pm\infty} \frac{1}{|m|} \log \|S_m\| < \infty \tag{15}$$

hold, then $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy.

Proof. Let $P_m : \mathbb{R}^p \to E_m^s$ and $Q_m : \mathbb{R}^p \to E_m^u$ be the projections associated to the decomposition $\mathbb{R}^p = E_m^s \oplus E_m^u$ in (9), with $P_m + Q_m = \mathrm{Id}$ (we emphasize that a priori P_m and Q_m might not come from a nonuniform exponential dichotomy). The following result was proved in [17].

Lemma 1. *For each $m \in \mathbb{Z}$, we have*

$$\|P_m\| \leq \sqrt{2}C^2\|S_m\| \quad \textit{and} \quad \|Q_m\| \leq \sqrt{2}C^2\|S_m\|. \tag{16}$$

Now we use condition (15). Since

$$\|\mathcal{A}(m,n)P_n\| \leq \|\mathcal{A}(m,n)|E_n^s\| \cdot \|P_n\|$$

and

$$\|\mathcal{A}(m,n)^{-1}Q_m\| \leq \|\mathcal{A}(m,n)^{-1}|E_m^s\| \cdot \|Q_m\|,$$

it follows readily from (16) and the second statement in Theorem 1 that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy. □

2.3. Existence of Lyapunov sequences

In this section we establish the converse of Corollary 2. More precisely, we show that any strong nonuniform exponential dichotomy has a strongly strict (quadratic) Lyapunov sequence.

Theorem 3. *If the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy, then it has a strongly strict quadratic Lyapunov sequence.*

Proof. Take $\varrho > 0$ such that $\varrho < \min\{-\overline{a}, \underline{b}\}$. For each $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$, let

$$U_m(x) = -V_m^s(P_m x) + V_m^u(Q_m x),$$

where

$$V_m^s(x) = \sum_{k\geq m} \|\mathcal{A}(k,m)x\|^2 e^{-2(\overline{a}+\varrho)(k-m)} + \sum_{k\leq m-1} \|\mathcal{A}(k,m)x\|^2 e^{2(\underline{a}-\varrho)(m-k)}$$

for $x \in F_m^s$, and

$$V_m^u(x) = \sum_{k\leq m-1} \|\mathcal{A}(k,m)x\|^2 e^{2(\underline{b}-\varrho)(m-k)} + \sum_{k\geq m} \|\mathcal{A}(k,m)x\|^2 e^{-2(\overline{b}+\varrho)(k-m)}$$

for $x \in F_m^u$. It follows readily from (12) and (13) that these series converge and that there exists a constant $C > 0$ such that

$$|U_m(x)| \leq Ce^{2\varepsilon|m|}\|x\|^2 \tag{17}$$

for $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$. For each $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$, let

$$V_m(x) = \operatorname{sgn} U_m(x)\sqrt{|U_m(x)|}.$$

By (17), the sequence $(V_m)_{m\in\mathbb{Z}}$ satisfies condition (1). Moreover, writing $y = P_m x$ and $z = Q_m x$, we have

$$\begin{aligned}
-V_{m+1}^s(A_m y) + V_m^s(y) &= -\sum_{k\geq m+1} \|\mathcal{A}(k,m+1)A_m y\|^2 e^{-2(\overline{a}+\varrho)(k-m-1)} + \sum_{k\geq m} \|\mathcal{A}(k,m)y\|^2 e^{-2(\overline{a}+\varrho)(k-m)} \\
&\quad - \sum_{k\leq m} \|\mathcal{A}(k,m)y\|^2 e^{2(\underline{a}-\varrho)(m+1-k)} + \sum_{k\leq m-1} \|\mathcal{A}(k,m)y\|^2 e^{2(\underline{a}-\varrho)(m-k)} \\
&= e^{2(\overline{a}+\varrho)}\|y\|^2 + (1-e^{2(\overline{a}+\varrho)}) \sum_{k\geq m} \|\mathcal{A}(k,m)y\|^2 e^{-2(\overline{a}+\varrho)(k-m)} - e^{2(\underline{a}-\varrho)}\|y\|^2 \\
&\quad + (1-e^{2(\underline{a}-\varrho)}) \sum_{k\leq m-1} \|\mathcal{A}(k,m)y\|^2 e^{2(\underline{a}-\varrho)(m-k)} \\
&= (e^{2(\overline{a}+\varrho)} - e^{2(\underline{a}-\varrho)})\|y\|^2 + (1-e^{2(\overline{a}+\varrho)})V_m^s(y) \\
&\quad + (e^{2(\overline{a}+\varrho)} - e^{2(\underline{a}-\varrho)}) \sum_{k\leq m-1} \|\mathcal{A}(k,m)y\|^2 e^{2(\underline{a}-\varrho)(m-k)} \\
&= (e^{2(\overline{a}+\varrho)} - e^{2(\underline{a}-\varrho)}) \left(\|y\|^2 + \sum_{k\leq m-1} \|\mathcal{A}(k,m)y\|^2 e^{2(\underline{a}-\varrho)(m-k)} \right) + (1-e^{2(\overline{a}+\varrho)})V_m^s(y).
\end{aligned}$$

Hence,

$$-V_{m+1}^s(A_m y) + V_m^s(y) \geq (1-e^{2(\overline{a}+\varrho)})V_m^s(y) \tag{18}$$

and

$$-V_{m+1}^s(A_m y) + V_m^s(y) \leq (1-e^{2(\underline{a}-\varrho)})V_m^s(y). \tag{19}$$

On the other hand,

$$\begin{aligned}
V_{m+1}^u(A_m z) - V_m(z) &= \sum_{k\leq m} \|\mathcal{A}(k,m+1)A_m z\|^2 e^{2(\underline{b}-\varrho)(m+1-k)} - \sum_{k\leq m-1} \|\mathcal{A}(k,m)z\|^2 e^{2(\underline{b}-\varrho)(m-k)} \\
&\quad + \sum_{k\geq m+1} \|\mathcal{A}(k,m+1)A_m z\|^2 e^{-2(\overline{b}+\varrho)(k-m-1)} - \sum_{k\geq m} \|\mathcal{A}(k,m)z\|^2 e^{-2(\overline{b}+\varrho)(k-m)} \\
&= e^{2(\underline{b}-\varrho)}\|z\|^2 + (e^{2(\underline{b}-\varrho)} - 1) \sum_{k\leq m-1} \|\mathcal{A}(k,m)z\|^2 e^{2(\underline{b}-\varrho)(m-k)} \\
&\quad - e^{2(\overline{b}+\varrho)}\|z\|^2 + (e^{2(\overline{b}+\varrho)} - 1) \sum_{k\geq m} \|\mathcal{A}(k,m)z\|^2 e^{-2(\overline{b}+\varrho)(k-m)}
\end{aligned}$$

$$\begin{aligned}
&= (e^{2(\underline{b}-\varrho)} - e^{2(\overline{b}+\varrho)})\|z\|^2 + (e^{2(\underline{b}-\varrho)} - 1)V_m^u(z) \\
&\quad + (e^{2(\overline{b}+\varrho)} - e^{2(\underline{b}-\varrho)}) \sum_{k\geq m} \|\mathcal{A}(k,m)z\|^2 e^{-2(\overline{b}+\varrho)(k-m)} \\
&= (e^{2(\underline{b}-\varrho)} - 1)V_m^u(z) + (e^{2(\overline{b}+\varrho)} - e^{2(\underline{b}-\varrho)}) \sum_{k\geq m+1} \|\mathcal{A}(k,m)z\|^2 e^{-2(\overline{b}+\varrho)(k-m)}.
\end{aligned}$$

Hence,

$$V_{m+1}^u(A_m z) - V_m^u(z) \geq (e^{2(\underline{b}-\varrho)} - 1)V_m^u(z) \tag{20}$$

and

$$V_{m+1}^u(A_m z) - V_m^u(z) \leq (e^{2(\overline{b}+\varrho)} - 1)V_m^u(z). \tag{21}$$

We first establish (2). It follows from (18) and (20) that

$$U_{m+1}(A_m x) - U_m(x) \geq \eta(V_m^s(y) + V_m^u(z)) \geq \eta|U_m(x)|, \tag{22}$$

where $\eta = \min\{1 - e^{2(\overline{a}+\varrho)}, e^{2(\underline{b}-\varrho)} - 1\}$. If $U_m(x) \geq 0$, then

$$U_{m+1}(A_m x) \geq (1+\eta)U_m(x),$$

and

$$\begin{aligned}
V_{m+1}(A_m x) &= \sqrt{U_{m+1}(A_m x)} \\
&\geq \sqrt{1+\eta}\sqrt{U_m(x)} \\
&= \sqrt{1+\eta}V_m(x).
\end{aligned}$$

If $U_m(x) < 0$, then

$$U_{m+1}(A_m x) \geq (1-\eta)U_m(x).$$

We consider two subcases. If $U_m(x) < 0$ and $U_{m+1}(A_m x) \leq 0$, then

$$V_{m+1}(A_m x) \geq \sqrt{1-\eta}V_m(x).$$

Finally, if $U_m(x) < 0$ and $U_{m+1}(A_m x) > 0$ then

$$V_{m+1}(A_m x) - V_m(x) = V_{m+1}(A_m x) + |V_m(x)| \geq |V_m(x)|.$$

We conclude that (2) holds. Furthermore, the inequalities in (6) and (7) hold with

$$\gamma < \min\{\sqrt{1+\eta} - 1, 1 - \sqrt{1-\eta}\}.$$

Now we establish (5). We note that

$$V_m^s(y) \geq \|y\|^2 \quad \text{and} \quad V_m^u(z) \geq \|z\|^2.$$

Hence, it follows from (22) that for every $x \in E_m^s$ we have

$$\begin{aligned}
|V_m(x)|^2 &= |U_m(x)| \geq |U_m(x)| - |U_{m+1}(A_m x)| \\
&= U_{m+1}(A_m x) - U_m(x) \\
&\geq \eta(\|y\|^2 + \|z\|^2) \\
&\geq \eta \max\{\|y\|^2, \|z\|^2\} \\
&\geq \eta \left(\frac{\|y\| + \|z\|}{2}\right)^2 \\
&\geq \frac{\eta}{4}\|x\|^2.
\end{aligned}$$

Therefore, $|V_m(x)| \geq (\sqrt{\eta}/2)\|x\|$. Similarly, for $x \in E_m^u$ we have

$$\begin{aligned}
|V_m(x)|^2 &= |U_m(x)| \geq U_m(x) - U_{m-1}(A_{m-1}^{-1}x) \\
&\geq \eta[V_{m-1}^s(A_{m-1}^{-1}y) + V_{m-1}^u(A_{m-1}^{-1}z)].
\end{aligned}$$

Since

$$V_{m-1}^s(A_{m-1}^{-1}y) \geq \|y\|^2 \quad \text{and} \quad V_{m-1}^u(A_{m-1}^{-1}z) \geq \|z\|^2 e^{-2(\overline{b}+\varrho)},$$

we obtain

$$|V_m(x)|^2 \geq \eta(\|y\|^2 + \|z\|^2 e^{-2(\bar{b}+\varrho)})$$

and hence,

$$|V_m(x)| \geq \frac{\sqrt{\eta}}{2} e^{-(\bar{b}+\varrho)} \|x\|.$$

We conclude that (5) holds. Since $E_m^s = F_m^s$ and $E_m^u = F_m^u$, it follows from (19) and (21) that the inequalities in (6) and (7) hold with $\mu_s = 1 - e^{\underline{a}-\varrho}$ and $\mu_u = e^{\underline{b}+\varrho} - 1$. Therefore, $(V_m)_{m\in\mathbb{Z}}$ is a strongly strict Lyapunov sequence for $(A_m)_{m\in\mathbb{Z}}$.

In order to verify that the sequence $(V_m)_{m\in\mathbb{Z}}$ is quadratic, we consider the quadratic form $H_m(x) = \langle S_m x, x\rangle$, where

$$S_m = \sum_{k\geq m} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m)P_m e^{-2(\bar{a}+\varrho)(k-m)} + \sum_{k\leq m-1} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m)P_m e^{2(\underline{a}-\varrho)(m-k)}$$
$$- \sum_{k\leq m-1} (\mathcal{A}(k,m)Q_m)^* \mathcal{A}(k,m)Q_m e^{2(\underline{b}-\varrho)(m-k)} - \sum_{k\geq m} (\mathcal{A}(k,m)Q_m)^* \mathcal{A}(k,m)Q_m e^{-2(\bar{b}+\varrho)(k-m)}.$$

We have

$$V_m(x) = -\mathrm{sgn} H_m(x) \sqrt{|H_m(x)|}.$$

Clearly, S_m is symmetric for each $m \in \mathbb{Z}$. Moreover, S_m is invertible: since

$$H_m | (F_m^s \setminus \{0\}) > 0 \quad \text{and} \quad H_m | (F_m^u \setminus \{0\}) < 0,$$

it follows from the identity $F_m^s \oplus F_m^u = \mathbb{R}^p$ that S_m is invertible for each m. Hence, $(V_m)_{m\in\mathbb{Z}}$ is a quadratic Lyapunov sequence. □

2.4. Robustness

In this section we show how the characterization of a strong nonuniform exponential dichotomy in terms of quadratic Lyapunov sequences can be used to establish their robustness.

We first establish an auxiliary result. Given $p \times p$ symmetric matrices A and B, we say that $A \geq B$ on a set X if the restriction of $A - B$ to X is positive-semidefinite.

Theorem 4. *The following properties are equivalent:*

1. $(V_m)_{m\in\mathbb{Z}}$ *is a strongly strict quadratic Lyapunov sequence for the sequence of matrices* $(A_m)_{m\in\mathbb{Z}}$*;*
2. $(V_m)_{m\in\mathbb{Z}}$ *satisfies* (1), (2) *and* (5)*, as well as*

$$(1+\gamma)^2 S_m \geq A_m^* S_{m+1} A_m \geq (1+\mu_u)^2 S_m \quad \text{on } E_m^u \tag{23}$$

and

$$(1-\gamma)^2 S_m \geq A_m^* S_{m+1} A_m \geq (1-\mu_s)^2 S_m \quad \text{on } E_m^s, \tag{24}$$

for some constants $\gamma \in (0,1)$ *and* $\mu_s, \mu_u \geq \gamma$ *with* $\mu_s < 1$.

Proof. We first assume that property 1 holds. Let $(V_m)_{m\in\mathbb{Z}}$ be a strongly strict quadratic Lyapunov sequence for $(A_m)_{m\in\mathbb{Z}}$. In particular, the sequence $(V_m)_{m\in\mathbb{Z}}$ satisfies conditions (1) and (5). If $x \in E_m^u$, then

$$V_{m+1}(A_m x) \geq 0 \quad \text{and} \quad V_m(x) \geq 0,$$

and it follows from (6) that

$$(1+\mu_u)^2 |H_m(x)| \geq |H_{m+1}(A_m x)| \geq (1+\gamma)^2 |H_m(x)|.$$

Since $H_{m+1}(A_m x) \leq 0$ and $H_m(x) \leq 0$, we obtain

$$(1+\mu_u)^2 \langle S_m x, x\rangle \leq \langle S_{m+1} A_m x, A_m x\rangle \leq (1+\gamma)^2 \langle S_m x, x\rangle,$$

and (23) holds. Similarly, if $x \in E_m^s$, then

$$V_{m+1}(A_m x) \leq 0 \quad \text{and} \quad V_m(x) \leq 0,$$

and it follows from (7) that

$$(1-\mu_s)^2 |H_m(x)| \leq |H_{m+1}(A_m x)| \leq (1-\gamma)^2 |H_m(x)|.$$

Since $H_{m+1}(A_m x) \geq 0$ and $H_m(x) \geq 0$, we obtain

$$(1-\mu_s)^2 \langle S_m x, x\rangle \leq \langle S_{m+1} A_m x, x\rangle \leq (1-\gamma)^2 \langle S_m x, x\rangle,$$

and (24) holds. This shows that property 2 holds.

Now we assume that $(V_m)_{m\in\mathbb{Z}}$ is a sequence as in property 2. In order to show that $(V_m)_{m\in\mathbb{Z}}$ is a strongly strict quadratic Lyapunov sequence for $(A_m)_{m\in\mathbb{Z}}$, it is sufficient to establish inequalities (6) and (7). If $x \in E_m^u$, then

$$H_{m+1}(A_m x) \le 0 \quad \text{and} \quad H_m(x) \le 0,$$

and it follows from (23) that

$$(1+\gamma)^2 |H_m(x)| \le |H_{m+1}(A_m x)| \le (1+\mu_u)^2 |H_m(x)|.$$

Since, $V_{m+1}(A_m x) \ge 0$ and $V_m(x) \ge 0$, we obtain

$$(1+\gamma) V_m(x) \le V_{m+1}(A_m x) \le (1+\mu_u) V_m(x),$$

and (6) holds. If $x \in E_m^s$, then

$$H_{m+1}(A_m x) \ge 0 \quad \text{and} \quad H_m(x) \ge 0,$$

and it follows from (24) that

$$(1-\gamma)^2 |H_m(x)| \ge |H_{m+1}(A_m x)| \ge (1-\mu_s)^2 |H_m(x)|.$$

Since, $V_{m+1}(A_m x) \le 0$ and $V_m(x) \le 0$, we obtain

$$(1-\gamma) V_m(x) \le V_{m+1}(A_m x) \le (1-\mu_s) V_m(x),$$

and (7) holds. This shows that $(V_m)_{m\in\mathbb{Z}}$ is a strongly strict quadratic Lyapunov sequence for $(A_m)_{m\in\mathbb{Z}}$ and property 1 holds. □

Finally, we establish the robustness of a strong nonuniform exponential dichotomy under sufficiently small linear perturbations.

Theorem 5. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be two sequences of invertible $p \times p$ matrices such that:*

1. *$(A_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy with constants $\underline{a} \le \overline{a} < 0 < \underline{b} \le \overline{b}$, $\varepsilon \ge 0$ and $D \ge 1$;*
2. *there exist $c \ge 0$ and $\beta \ge 3\varepsilon$ such that*

$$\|B_m - A_m\| \le c e^{-\beta|m|}, \quad m \in \mathbb{Z}. \tag{25}$$

If c is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy.

Proof. Let $(V_m)_{m\in\mathbb{Z}}$ be the strongly strict quadratic Lyapunov sequence for $(A_m)_{m\in\mathbb{Z}}$ constructed in the proof of Theorem 3. For each $m \in \mathbb{Z}$ and $x \in \mathbb{R}^p$, we have

$$\langle B_m^* S_{m+1} B_m x, x\rangle = \langle A_m^* S_{m+1} A_m x, x\rangle + R_m(x),$$

where

$$R_m(x) = 2\langle S_{m+1}(B_m - A_m)x, A_m x\rangle + \langle S_{m+1}(B_m - A_m)x, (B_m - A_m)x\rangle.$$

By (1), (10) and (25), we have

$$\begin{aligned} |R_m(x)| &\le 2\|S_{m+1}\| \cdot \|B_m - A_m\| \cdot \|A_m|E_m^s\| \cdot \|x\|^2 + \|S_{m+1}\| \cdot \|B_m - A_m\|^2 \cdot \|x\|^2 \\ &\le 2cC^2 D e^{2\varepsilon|m+1|} e^{-\beta|m|} e^{\overline{a}+\varepsilon|m|} \|x\|^2 + c^2 C^2 e^{2\varepsilon|m+1|} e^{-2\beta|m|} \|x\|^2 \end{aligned}$$

for $m \in \mathbb{Z}$ and $x \in E_m^s$. Since $\beta \ge 3\varepsilon$ and $\|x\|^2 \le C^2 \langle S_m x, x\rangle$, we obtain

$$|R_m(x)| \le r_1 \langle S_m x, x\rangle$$

for some constant $r_1 > 0$ that is independent of m and x. Moreover, r_1 can be made arbitrarily small taking c sufficiently small. By (24), we obtain

$$[(1-\gamma)^2 + r_1] S_m \ge B_m^* S_{m+1} B_m \ge [(1-\mu_s)^2 - r_1] S_m \quad \text{on } E_m^s.$$

Similarly, by (1), (11) and (25), we have

$$\begin{aligned} |R_m(x)| &\le 2\|S_{m+1}\| \cdot \|B_m - A_m\| \cdot \|A_m|E_m^u\| \cdot \|x\|^2 + \|S_{m+1}\| \cdot \|B_m - A_m\|^2 \cdot \|x\|^2 \\ &\le 2cC^2 D e^{2\varepsilon|m+1|} e^{-\beta|m|} e^{\overline{b}+\varepsilon|m|} \|x\|^2 + c^2 C^2 e^{2\varepsilon|m+1|} e^{-2\beta|m|} \|x\|^2 \end{aligned}$$

for $m \in \mathbb{Z}$ and $x \in E_m^u$. Since $\beta \ge 3\varepsilon$ and $\|x\|^2 \le -C^2 \langle S_m x, x\rangle$, we obtain

$$|R_m(x)| \le -r_2 \langle S_m x, x\rangle$$

for some constant $r_2 > 0$ that is independent of m and x, and by (23),

$$[(1+\gamma)^2 - r_2] S_m \ge B_m^* S_{m+1} B_m \ge [(1+\mu_u)^2 + r_2] S_m \quad \text{on } E_m^u.$$

Hence, it follows from Theorem 4 that $(V_m)_{m\in\mathbb{Z}}$ is a strongly strict quadratic Lyapunov sequence for $(B_m)_{m\in\mathbb{Z}}$. By Corollary 2, the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a strong nonuniform exponential dichotomy. □

3. Exponential dichotomies for flows

This section gives corresponding results to those in Section 2 for continuous time.

3.1. Preliminaries

We first introduce the notion of a nonuniform exponential dichotomy for continuous time. Let $A: \mathbb{R} \to M_p$ be a continuous function and consider the linear equation

$$x' = A(t)x. \tag{26}$$

Let also $T(t, \tau)$ be the associated evolution family.

We say that Eq. (26) admits a *nonuniform exponential dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ such that

$$P(t)T(t, \tau) = T(t, \tau)P(\tau), \quad t, \tau \in \mathbb{R},$$

and constants

$$\overline{a} < 0 < \underline{b}, \quad \varepsilon \geq 0 \quad \text{and} \quad D \geq 1$$

such that for every $t, \tau \in \mathbb{R}$ with $t \geq \tau$ we have

$$\|T(t, \tau)P(\tau)\| \leq De^{\overline{a}(t-\tau)+\varepsilon|\tau|}$$

and

$$\|T(t, \tau)^{-1}Q(t)\| \leq De^{-\underline{b}(t-\tau)+\varepsilon|t|},$$

where $Q(t) = \mathrm{Id} - P(t)$. For each $t \in \mathbb{R}$, we define the *stable* and *unstable subspaces* by

$$F_t^s = P(t)(\mathbb{R}^p) \quad \text{and} \quad F_t^u = Q(t)(\mathbb{R}^p).$$

Now let $V: \mathbb{R}^p \to \mathbb{R}$ be a continuous function and consider the cones

$$C^u(V) = \{0\} \cup V^{-1}(0, +\infty) \quad \text{and} \quad C^s(V) = \{0\} \cup V^{-1}(-\infty, 0).$$

We say that $V: \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ is a *Lyapunov function* for Eq. (26) if there exist $r_u, r_s \in \mathbb{N}$ with $r_s + r_u = p$ such that for each $\tau \in \mathbb{R}$ and $t \geq \tau$:

1. r_u and r_s are respectively the maximal dimensions of the linear subspaces inside $C^u(V_\tau)$ and $C^s(V_\tau)$, where $V_\tau = V(\tau, \cdot)$;
2. $$V(t, T(t, \tau)x) \geq V(\tau, x). \tag{27}$$

For each $\tau \in \mathbb{R}$, we consider the sets

$$E_\tau^u = \bigcap_{t\in\mathbb{R}} T(\tau, t)\overline{C^u(V_t)} \tag{28}$$

and

$$E_\tau^s = \bigcap_{t\in\mathbb{R}} T(\tau, t)\overline{C^s(V_t)}. \tag{29}$$

Clearly,

$$T(t, \tau)E_\tau^u = E_t^u \quad \text{and} \quad T(t, \tau)E_\tau^s = E_t^s$$

for $t, \tau \in \mathbb{R}$. Now let V be a Lyapunov function for Eq. (26) and assume that there exist $C > 0$ and $\delta \geq 0$ such that

$$|V(\tau, x)| \leq Ce^{\delta|\tau|}\|x\|$$

for $\tau \in \mathbb{R}$ and $x \in \mathbb{R}^p$. We say that V is *strict* if there exists $\theta \in (0, 1)$ such that for every $\tau \in \mathbb{R}$:

1. if $x \in E_\tau^u$, then
$$V(t, T(t, \tau)x) \geq \theta^{\tau-t}V(\tau, x), \quad t \geq \tau;$$
2. if $x \in E_\tau^s$, then
$$|V(t, T(t, \tau)x)| \leq \theta^{t-\tau}|V(\tau, x)|, \quad t \geq \tau;$$
3. if $x \in E_\tau^u \cup E_\tau^s$, then
$$|V(\tau, x)| \geq e^{-\delta|\tau|}\|x\|/C.$$

Finally, a Lyapunov function is said to be *quadratic* if for each $t \in \mathbb{R}$ there exists a symmetric invertible $p \times p$ matrix $S(t)$ such that

$$V(t,x) = -\operatorname{sgn} H(t,x)\sqrt{|H(t,x)|}$$

for $x \in \mathbb{R}^p$, where $H(t,x) = \langle S(t)x, x\rangle$. It is shown in [18] that:

1. if Eq. (26) has a strict quadratic Lyapunov function satisfying $\theta e^{\delta} < 1$ and
$$\limsup_{t\to\pm\infty} \frac{1}{|t|}\log\|S(t)\| < \infty, \tag{30}$$
then the equation admits a nonuniform exponential dichotomy;
2. if Eq. (26) admits a nonuniform exponential dichotomy with a sufficiently small ε and
$$\|T(t,s)\| \le Ke^{\alpha|t|}, \quad t \in \mathbb{R},\ s \in [t, t+c] \tag{31}$$
holds for some constants $c, \alpha, K > 0$, then there exists a strict quadratic Lyapunov function satisfying $\theta e^{\delta} < 1$ and condition (30).

3.2. Existence of exponential dichotomies

In this section we introduce the notion of a strong nonuniform exponential dichotomy for continuous time. We show that the existence of a strongly strict quadratic Lyapunov function implies the existence of a strong nonuniform exponential dichotomy.

Let V be a Lyapunov function for Eq. (26) and assume that there exist $C > 0$ and $\delta \ge 0$ such that

$$|V(\tau,x)| \le Ce^{\delta|\tau|}\|x\| \tag{32}$$

for $\tau \in \mathbb{R}$ and $x \in \mathbb{R}^p$. We say that V is *strongly strict* if there exist $\theta_1, \theta_2, \theta_3 \in (0,1)$ with $\theta_2, \theta_3 \le \theta_1$ such that for every $\tau \in \mathbb{R}$:

1. if $x \in E^u_\tau$, then
$$\theta_3^{\tau-t}V(\tau,x) \ge V(t, T(t,\tau)x) \ge \theta_1^{\tau-t}V(\tau,x), \quad t \ge \tau; \tag{33}$$
2. if $x \in E^s_\tau$, then
$$\theta_2^{t-\tau}|V(\tau,x)| \le |V(t, T(t,\tau)x)| \le \theta_1^{t-\tau}|V(\tau,x)|, \quad t \ge \tau; \tag{34}$$
3. if $x \in E^u_\tau \cup E^s_\tau$, then
$$|V(\tau,x)| \ge e^{-\delta|\tau|}\|x\|/C. \tag{35}$$

We start with a preliminary result.

Theorem 6. *If there exists a strongly strict Lyapunov function V for Eq. (26) satisfying $\theta_1 e^{\delta} < 1$, then:*

1. *for each $\tau \in \mathbb{R}$, the sets E^u_τ and E^s_τ in (28) and (29) are linear spaces, respectively, of dimensions r_u and r_s and*
$$\mathbb{R}^p = E^u_\tau \oplus E^s_\tau; \tag{36}$$
2. *there exist constants*
$$\underline{a} \le \overline{a} < 0 < \underline{b} \le \overline{b}, \quad \varepsilon \ge 0 \text{ and } D \ge 1$$
such that for every $t, \tau \in \mathbb{R}$ with $t \ge \tau$, we have
$$\begin{aligned} &\|T(t,\tau)|E^s_\tau\| \le De^{\overline{a}(t-\tau)+\varepsilon|\tau|},\\ &\|T(t,\tau)^{-1}|E^u_t\| \le De^{-\underline{b}(t-\tau)+\varepsilon|t|}, \end{aligned} \tag{37}$$
and for every $t, \tau \in \mathbb{Z}$ with $t \le \tau$, we have
$$\begin{aligned} &\|T(t,\tau)|E^s_\tau\| \le De^{\underline{a}(t-\tau)+\varepsilon|\tau|},\\ &\|T(t,\tau)^{-1}|E^u_t\| \le De^{-\overline{b}(t-\tau)+\varepsilon|t|}. \end{aligned} \tag{38}$$

Proof. The first statement is proved in [18]. It follows from (35) that

$$E^u_\tau \subset C^u(V_\tau) \quad \text{and} \quad E^s_\tau \subset C^s(V_\tau).$$

Therefore, V_τ is positive in $E^u_\tau \setminus \{0\}$ and negative in $E^s_\tau \setminus \{0\}$. For $x \in E^s_\tau$, it follows from (34) and (35) that for every $t \geq \tau$,

$$\begin{aligned}\|T(t,\tau)x\| &\leq Ce^{\delta|t|}|V(t,T(t,\tau)x)|\\ &\leq Ce^{\delta|t|}\theta_1^{t-\tau}|V(\tau,x)|\end{aligned}$$

and hence, by (32),

$$\begin{aligned}\|T(t,\tau)x\| &\leq C^2e^{\delta|\tau|}e^{\delta|t|}\theta_1^{t-\tau}\|x\|\\ &\leq C^2e^{\delta(t-\tau)}\theta_1^{t-\tau}e^{2\delta|\tau|}\|x\|.\end{aligned}$$

On the other hand, for $x \in E^u_\tau$, it follows from (32) and (33) that for every $t \geq \tau$,

$$\begin{aligned}\|T(t,\tau)x\| &\geq \frac{1}{C}e^{-\delta|t|}V(t,T(t,\tau)x)\\ &\geq \frac{1}{C}e^{-\delta|t|}\theta_1^{\tau-t}V(\tau,x)\end{aligned}$$

and hence, by (35),

$$\|T(t,\tau)x\| \geq \frac{1}{C^2}e^{-\delta|t|}\theta_1^{\tau-t}e^{-\delta|\tau|}\|x\|.$$

Similarly, we have

$$\|T(t,\tau)x\| \leq C^2e^{\delta|t|}\theta_2^{t-\tau}e^{\delta|\tau|}\|x\|$$

for $x \in E^s_\tau$ and $t \leq \tau$, and

$$\|T(t,\tau)x\| \geq \frac{1}{C^2}e^{-\delta|t|}\theta_3^{\tau-t}e^{-\delta|\tau|}$$

for $x \in E^u_\tau$ and $t \leq \tau$. Hence, the inequalities in (37) and (38) hold with

$$\underline{a} = \log\theta_2 - \delta, \qquad \overline{a} = \log\theta_1 + \delta, \qquad \underline{b} = -(\log\theta_1 + \delta), \qquad \overline{b} = -\log\theta_3 + \delta,$$

$\varepsilon = 2\delta$ and $D = C^2$. This completes the proof of the theorem. □

Now we consider strong exponential dichotomies. We say that Eq. (26) admits a *strong nonuniform exponential dichotomy* if there exist projections $P(t)$ for $t \in \mathbb{R}$ satisfying

$$P(t)T(t,\tau) = T(t,\tau)P(\tau)$$

for $t, \tau \in \mathbb{R}$ and there exist constants

$$\underline{a} \leq \overline{a} < 0 < \underline{b} \leq \overline{b}, \quad \varepsilon \geq 0 \text{ and } D \geq 1$$

such that for every $t, \tau \in \mathbb{R}$ with $t \geq \tau$ we have

$$\begin{aligned}\|T(t,\tau)P(\tau)\| &\leq De^{\overline{a}(t-\tau)+\varepsilon|\tau|},\\ \|T(t,\tau)^{-1}Q(t)\| &\leq De^{-\underline{b}(t-\tau)+\varepsilon|t|},\end{aligned} \tag{39}$$

and for every $t, \tau \in \mathbb{Z}$ with $t \leq \tau$ we have

$$\begin{aligned}\|T(t,\tau)P(\tau)\| &\leq De^{\underline{a}(t-\tau)+\varepsilon|\tau|},\\ \|T(t,\tau)^{-1}Q(t)\| &\leq De^{-\overline{b}(t-\tau)+\varepsilon|t|},\end{aligned} \tag{40}$$

where $Q(t) = \mathrm{Id} - P(t)$. For each $t \in \mathbb{R}$, we continue to define the *stable* and *unstable subspaces* by (14).

Corollary 7. *If Eq. (26) has a strongly strict quadratic Lyapunov function satisfying $\theta_1e^\delta < 1$ and*

$$\limsup_{t\to\pm\infty}\frac{1}{|t|}\log\|S(t)\| < \infty, \tag{41}$$

then the equation admits a strong nonuniform exponential dichotomy.

Proof. Let $P(t)\colon \mathbb{R}^p \to E^s_t$ and $Q(t)\colon \mathbb{R}^p \to E^u_t$ be the projections associated to the decomposition $\mathbb{R}^p = E^s_t \oplus E^u_t$ in (36), with $P(t) + Q(t) = \mathrm{Id}$. The following result was proved in [18].

Lemma 2. *For each $t \in \mathbb{R}$, we have*

$$\|P(t)\| = \|Q(t)\| \le \sqrt{2}C^2 e^{2\delta|t|}\|S(t)\|. \tag{42}$$

Now we assume that (41) holds. Since

$$\|T(t,\tau)P(\tau)\| \le \|T(t,\tau)|E^s_\tau\| \cdot \|P(\tau)\|$$

and

$$\|T(t,\tau)^{-1}Q(t)\| \le \|T(t,\tau)^{-1}|E^u_t\| \cdot \|Q(t)\|,$$

it follows readily from (42) and the second statement in Theorem 6 that Eq. (26) admits a strong nonuniform exponential dichotomy. □

3.3. Existence of Lyapunov functions

In this section we establish the converse of Corollary 7 for a class of linear equations.

Theorem 8. *If Eq. (26) admits a strong nonuniform exponential dichotomy and condition (31) holds for some $c, \alpha, K > 0$, then there exists a strongly strict quadratic Lyapunov function.*

Proof. Take $\varrho > 0$ such that $\varrho < \min\{-\overline{a}, \underline{b}\}$. For each $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$, let

$$U(t,x) = -V^s(t, P(t)x) + V^u(t, Q(t)x),$$

where

$$V^s(t,x) = \int_t^{+\infty} \|T(v,t)x\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv + \int_{-\infty}^t \|T(v,t)x\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv \tag{43}$$

for $x \in F^s_t$ and

$$V^u(t,x) = \int_{-\infty}^t \|T(v,t)x\|^2 e^{2(\underline{b}-\varrho)(t-v)}dv + \int_t^{+\infty} \|T(v,t)x\|^2 e^{-2(\overline{b}+\varrho)(v-t)}dv \tag{44}$$

for $x \in F^u_t$. It follows readily from (39) and (40) that the integrals in (43) and (44) are well defined and that there exists a constant $C > 0$ such that

$$|U(t,x)| \le Ce^{2\varepsilon|t|}\|x\|^2 \tag{45}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$. For each $t \in \mathbb{R}$ and $x \in \mathbb{R}^p$, let also

$$V(t,x) = \operatorname{sgn} U(t,x)\sqrt{|U(t,x)|}.$$

By (45), the function V satisfies (32). Moreover, writing $y = P(\tau)x$ and $z = Q(\tau)x$, for every $x \in \mathbb{R}^p$ and $t \ge \tau$, we have

$$\begin{aligned}
-V^s(t, T(t,\tau)y) + V^s(\tau,y) &= -\int_t^{+\infty} \|T(v,t)T(t,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv + \int_\tau^{+\infty} \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv \\
&\quad - \int_{-\infty}^t \|T(v,t)T(t,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv + \int_{-\infty}^\tau \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(\tau-v)}dv \\
&= \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv + (1 - e^{2(\overline{a}+\varrho)(t-\tau)}) \\
&\quad \times \int_\tau^{+\infty} \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv - \int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv \\
&\quad + (1 - e^{2(\underline{a}-\varrho)(t-\tau)}) \int_{-\infty}^\tau \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(\tau-v)}dv \\
&= (1 - e^{2(\overline{a}+\varrho)(t-\tau)})V^s(\tau,y) + (e^{2(\overline{a}+\varrho)(t-\tau)} - e^{2(\underline{a}-\varrho)(t-\tau)}) \\
&\quad \times \int_{-\infty}^\tau \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(\tau-v)}dv \\
&\quad + \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv - \int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv.
\end{aligned}$$

Lemma 3. *We have*

$$0 \le \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv - \int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv$$
$$\le (e^{2(\overline{a}+\varrho)(t-\tau)} - e^{2(\underline{a}-\varrho)(t-\tau)}) \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv.$$

Proof of the lemma. Since $\underline{a} \le \overline{a}$, we have

$$2(\underline{a}-\varrho)(t-v) \le -2(\overline{a}+\varrho)(v-t)$$

for $v \in [\tau, t]$. Hence,

$$\int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv \le \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv,$$

which yields the first inequality in the lemma. Moreover,

$$\int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-t)}dv = e^{2(\overline{a}+\varrho)(t-\tau)} \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv$$

and

$$\int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(t-v)}dv = e^{2(\underline{a}-\varrho)(t-\tau)} \int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(\tau-v)}dv.$$

Hence,

$$\int_\tau^t \|T(v,\tau)y\|^2 e^{2(\underline{a}-\varrho)(\tau-v)}dv \ge \int_\tau^t \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv$$

and the second inequality follows. □

By Lemma 3, we have

$$-V^s(t,T(t,\tau)y) + V^s(\tau,y) \ge (1 - e^{2(\overline{a}+\varrho)(t-\tau)})V^s(\tau,y) \tag{46}$$

and

$$-V^s(t,T(t,\tau)y) + V^s(\tau,y) \le (1 - e^{2(\underline{a}-\varrho)(t-\tau)})V^s(\tau,y). \tag{47}$$

On the other hand, for every $x \in \mathbb{R}^p$ and $t \ge \tau$, we have

$$\begin{aligned}
V^u(t,T(t,\tau)z) - V^u(\tau,z) &= \int_{-\infty}^t \|T(v,t)T(t,\tau)z\|^2 e^{2(\underline{b}-\varrho)(t-v)}dv - \int_{-\infty}^\tau \|T(v,\tau)z\|^2 e^{2(\underline{b}-\varrho)(\tau-v)}dv \\
&\quad + \int_t^{+\infty} \|T(v,t)T(t,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-t)}dv - \int_\tau^{+\infty} \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-\tau)}dv \\
&= \int_\tau^t \|T(v,\tau)z\|^2 e^{2(\underline{b}-\varrho)(t-v)}dv + (e^{2(\underline{b}-\varrho)(t-\tau)} - 1)\int_{-\infty}^\tau \|T(v,\tau)z\|^2 e^{2(\underline{b}-\varrho)(\tau-v)}dv \\
&\quad - \int_\tau^t \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-t)}dv + (e^{2(\overline{b}+\varrho)(t-\tau)} - 1) \\
&\quad \times \int_\tau^{+\infty} \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-\tau)}dv \\
&= (e^{2(\underline{b}-\varrho)(t-\tau)} - 1)V^u(\tau,z) + (e^{2(\overline{b}+\varrho)(t-\tau)} - e^{2(\underline{b}-\varrho)(t-\tau)}) \\
&\quad \times \int_\tau^{+\infty} \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-\tau)}dv \\
&\quad + \int_\tau^t \|T(v,\tau)z\|^2 e^{2(\underline{b}-\varrho)(t-v)}dv - \int_\tau^t \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-t)}dv.
\end{aligned}$$

The proof of the following result is analogous to the proof of Lemma 3.

Lemma 4. *We have*

$$\begin{aligned} 0 &\geq \int_\tau^t \|T(v,\tau)z\|^2 e^{2(\underline{b}-\varrho)(t-v)}dv - \int_\tau^t \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-t)}dv \\ &\geq (e^{2(\underline{b}-\varrho)(t-\tau)} - e^{2(\overline{b}+\varrho)(t-\tau)}) \int_\tau^t \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-\tau)}dv. \end{aligned}$$

By Lemma 4, we have

$$V^u(t, T(t,\tau)z) - V^u(\tau, z) \geq (e^{2(\underline{b}-\varrho)(t-\tau)} - 1)V^u(\tau, z) \tag{48}$$

and

$$V^u(t, T(t,\tau)z) - V^u(\tau, z) \leq (e^{2(\overline{b}+\varrho)(t-\tau)} - 1)V^u(\tau, z). \tag{49}$$

It follows from (46) and (48) that

$$U(t, T(t,\tau)x) - U(\tau, x) \geq \eta |U(\tau, x)|,$$

where $\eta = \min\{1 - e^{2(\overline{a}+\varrho)(t-\tau)}, e^{2(\underline{b}-\varrho)(t-\tau)} - 1\}$. Proceeding as in the case of discrete time we establish (27).

Since $\underline{b} - \varrho > \overline{a} + \varrho$, it follows from (46) and (48) that for every $x \in E^s_\tau$ we have

$$U(t, T(t,\tau)x) \geq e^{2(\overline{a}+\varrho)(t-\tau)} U(\tau, x),$$

which implies that

$$|V(t, T(t,\tau)x)| \leq e^{(\overline{a}+\varrho)(t-\tau)} |V(\tau, x)|.$$

Similarly, for every $x \in E^u_\tau$ we have

$$U(t, T(t,\tau)x) \geq e^{-2(\underline{b}-\varrho)(\tau-t)} U(\tau, x)$$

and thus,

$$V(t, T(t,\tau)x) \geq e^{-(\underline{b}-\varrho)(\tau-t)} V(\tau, x).$$

We conclude that (33) and (34) hold with $\theta_1 = \max\{e^{\overline{a}+\varrho}, e^{-(\underline{b}-\varrho)}\} \in (0, 1)$.

Now we establish (35). It follows from (46) and (48) that

$$U(\tau+1, T(\tau+1,\tau)x) - U(\tau, x) \geq \eta\big[V^s(\tau, y) + V^u(\tau, z)\big],$$

where

$$\eta = \min\{1 - e^{2(\overline{a}+\varrho)}, e^{2(\underline{b}-\varrho)} - 1\}.$$

By (31), we have

$$\begin{aligned} V^s(\tau, y) &\geq \int_\tau^{\tau+c} \|T(v,\tau)y\|^2 e^{-2(\overline{a}+\varrho)(v-\tau)}dv \\ &\geq \int_\tau^{\tau+c} \frac{\|y\|^2}{\|T(\tau,v)\|^2} e^{-2(\overline{a}+\varrho)(v-\tau)}dv \\ &\geq K_1 e^{-2\alpha|\tau|}\|y\|^2, \end{aligned}$$

for some constant $K_1 > 0$. Similarly,

$$\begin{aligned} V^u(\tau, z) &\geq \int_\tau^{\tau+c} \|T(v,\tau)z\|^2 e^{-2(\overline{b}+\varrho)(v-\tau)}dv \\ &\geq \int_\tau^{\tau+c} \frac{\|z\|^2}{\|T(\tau,v)\|^2} e^{-2(\overline{b}+\varrho)(v-\tau)}dv \\ &\geq K_2 e^{-2\alpha|\tau|}\|z\|^2 \int_\tau^{\tau+c} e^{-2(\overline{b}+\varrho)(v-\tau)}dv, \end{aligned}$$

for some constant $K_2 > 0$. We conclude that there exists $b > 0$ such that

$$U(\tau+1, T(\tau+1,\tau)x) - U(\tau, x) \geq be^{-2\alpha|\tau|}(\|y\|^2 + \|z\|^2).$$

Hence, for every $x \in E^s_\tau$, we have

$$\begin{aligned}|V(\tau,x)|^2 &= |U(\tau,x)| \ge |U(\tau,x)| - |U(\tau+1, T(\tau+1,\tau)x)| \\ &= U(\tau+1, T(\tau+1,\tau)x) - U(\tau,x) \\ &\ge be^{-2\alpha|\tau|}(\|y\|^2 + \|z\|^2) \ge be^{-2\alpha|\tau|}\max\{\|y\|^2, \|z\|^2\} \\ &\ge be^{-2\alpha|\tau|}\left(\frac{\|y\|+\|z\|}{2}\right)^2 \ge \frac{b}{4}e^{-2\alpha|\tau|}\|x\|^2.\end{aligned}$$

Similarly, for $x \in E^u_\tau$, we have

$$\begin{aligned}|V(\tau,x)|^2 &= |U(\tau,x)| \ge U(\tau,x) - U(\tau-1, T(\tau-1,\tau)x) \\ &\ge \eta[V^s(\tau-1, T(\tau-1,\tau)y) + V^u(\tau-1, T(\tau-1,\tau)z)] \\ &\ge b'e^{-2\alpha|\tau|}(\|y\|^2 + \|z\|^2) \ge \frac{b'}{4}e^{-2\alpha|\tau|}\|x\|^2\end{aligned}$$

for some constant $b' > 0$ and we conclude that condition (35) holds. Since $E^s_\tau = F^s_\tau$ and $E^u_\tau = F^u_\tau$, it follows from (47) and (49) that the inequalities in (33) and (34) hold with $\theta_2 = e^{\underline{a}-\varrho}$ and $\theta_3 = e^{-(\overline{b}+\varrho)}$.

In order to verify that the function V is quadratic, we consider the quadratic form $H(t,x) = \langle S(t)x, x\rangle$, where

$$\begin{aligned}S(t) &= \int_t^{+\infty} (T(v,t)P(t))^* T(v,t)P(t)e^{-2(\overline{a}+\varrho)(v-t)}dv + \int_{-\infty}^t (T(v,t)P(t))^* T(v,t)P(t)e^{2(\underline{a}-\varrho)(t-v)}dv \\ &\quad - \int_{-\infty}^t (T(v,t)Q(t))^* T(v,t)Q(t)e^{2(\underline{b}-\varrho)(t-v)}dv - \int_t^{+\infty} (T(v,t)Q(t))^* T(v,t)Q(t)e^{-2(\overline{b}+\varrho)(v-t)}dv.\end{aligned}$$

We have

$$V(t,x) = -\mathrm{sgn} H(t,x)\sqrt{|H(t,x)|}.$$

Clearly, $S(t)$ is symmetric for each $t \in \mathbb{R}$. Moreover, $S(t)$ is invertible: since $H(t,\cdot)$ is positive in $F^s_t \setminus \{0\}$ and negative in $F^u_t \setminus \{0\}$, it follows from the identity $F^s_t \oplus F^u_t = \mathbb{R}^p$ that $S(t)$ is invertible for each $t \in \mathbb{R}$. Hence, V is a strongly strict Lyapunov function. $\square$

3.4. Robustness

Using arguments similar to those in the case of discrete time, one can now establish the robustness of a strong nonuniform exponential dichotomy.

We start with an auxiliary result.

Proposition 9. *Let $V\colon \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}$ be a C^1 function. Then the following properties hold:*

1. *V is a Lyapunov function if and only if*

$$\dot{V}(\tau,x) \ge 0 \tag{50}$$

for every $\tau \in \mathbb{R}$ and $x \in \mathbb{R}^p$.

2. *V is a strongly strict Lyapunov function if and only if there exist $C > 0$, $\delta \ge 0$ and $\theta_1, \theta_2, \theta_3 \in (0,1)$ with $\theta_2, \theta_2 \le \theta_1$ such that for every $\tau \in \mathbb{R}$:*
   (a) *properties (32), (35) and (50) hold;*
   (b) *if $x \in E^u_\tau$, then*

$$-\log\theta_3 V(t, T(t,\tau)x) \ge \dot{V}(t, T(t,\tau)x) \ge -\log\theta_1 V(t, T(t,\tau)x) \tag{51}$$

   for $t \ge \tau$;
   (c) *if $x \in E^s_\tau$, then*

$$-\log\theta_2 |V(t, T(t,\tau)x)| \ge \dot{V}(t, T(t,\tau)x) \ge -\log\theta_1 |V(t, T(t,\tau)x)| \tag{52}$$

   for $t \ge \tau$.

Proof. The first statement of the proposition was proved in [18].

For the second statement, it is sufficient to show that the inequalities in (33) are equivalent to the ones in (51) and that the inequalities in (34) are equivalent to the ones in (52). We first assume that there exist $\theta_1, \theta_3 \in (0,1)$ with $\theta_3 \le \theta_1$ such that (33) holds. Then for every $x \in E^u_\tau$ and $t > \tau$, we have

$$\begin{aligned}\dot{V}(t, T(t,\tau)x) &= \lim_{h\to 0^+} \frac{V(t+h, T(t+h,t)T(t,\tau)x) - V(t, T(t,\tau)x)}{h} \\ &\ge \lim_{h\to 0^+} \frac{\theta_1^{-h} - 1}{h} V(t, T(t,\tau)x) \\ &= -\log\theta_1 V(t, T(t,\tau)x)\end{aligned}$$

and similarly,

$$\dot{V}(t, T(t,\tau)x) \le \lim_{h\to 0^+} \frac{\theta_3^{-h}-1}{h} V(t, T(t,\tau)x)$$
$$= -\log\theta_3 V(t, T(t,\tau)x).$$

This shows that (51) holds. Now we assume that there exist $\theta_1, \theta_3 \in (0,1)$ with $\theta_3 \le \theta_1$ such that (51) holds. By the mean value theorem, for each $x \in E_\tau^u$ and $\tau, t \in \mathbb{R}$ with $t > \tau$, there exists $t_0 \in (\tau, t)$ such that

$$\log V(t, T(t,\tau)x) - \log V(\tau, x) = \frac{\dot{V}(t_0, T(t_0,\tau)x)}{V(t_0, T(t_0,\tau)x)}(t-\tau).$$

Hence,

$$-(t-\tau)\log\theta_1 \le \log V(t, T(t,\tau)x) - \log V(\tau, x) \le -(t-\tau)\log\theta_3,$$

which implies that

$$\log\theta_1^{\tau-t} \le \log\frac{V(t, T(t,\tau)x)}{V(\tau,x)} \le \log\theta_3^{\tau-t}$$

and V satisfies the inequalities in (33). The equivalence between (52) and (34) can be obtained in a similar manner. □

We also obtain a characterization of a strongly strict quadratic Lyapunov function in terms of the matrices $S(t)$.

Theorem 10. *Let V be a quadratic Lyapunov function for Eq. (26). The following properties are equivalent:*

1. *V is a strongly strict Lyapunov function;*
2. *V satisfies (32) and (35), as well as*

$$2S(\tau)\log\theta_2 \le S'(\tau) + S(\tau)A(\tau) + A(\tau)^*S(\tau) \le 2S(\tau)\log\theta_1 \tag{53}$$

on E_τ^s and

$$-2S(\tau)\log\theta_3 \le S'(\tau) + S(\tau)A(\tau) + A(\tau)^*S(\tau) \le -2S(\tau)\log\theta_1 \tag{54}$$

on E_τ^u, for some constants $\theta_1, \theta_2, \theta_3 \in (0,1)$ with $\theta_2, \theta_3 \le \theta_1$.

Proof. It is sufficient to show that (53) is equivalent to (52) and that (54) is equivalent to (51). Take $x_0 \in \mathbb{R}^p$ and let $x(t) = T(t,\tau)x_0$. We obtain

$$\dot{H}(\tau, x(\tau)) = \langle (S'(\tau) + S(\tau)A(\tau) + A(\tau)^*S(\tau))x(\tau), x(\tau)\rangle. \tag{55}$$

On the other hand, since $V(t,x)^2 = |H(t,x)|$, we have

$$2V(t,x)\dot{V}(t,x) = \begin{cases} \dot{H}(t,x) & \text{if } H(t,x) \ge 0,\\ -\dot{H}(t,x) & \text{if } H(t,x) \le 0.\end{cases}$$

It follows from (55) that (52) holds if and only if

$$2H(t,x(t))\log\theta_2 \le \dot{H}(t,x(t)) \le 2H(t,x(t))\log\theta_1.$$

Similarly, (51) holds if and only if

$$-2H(t,x(t))\log\theta_3 \le \dot{H}(t,x(t)) \le -2H(t,x(t))\log\theta_1.$$

In view of (55) these inequalities are equivalent to (53) and (54). □

Finally, the following is our robustness result for strong nonuniform exponential dichotomies.

Theorem 11. *Let $A\colon \mathbb{R} \to M_p$ and $B\colon \mathbb{R} \to M_p$ be continuous functions such that:*

1. *Eq. (26) admits a strong nonuniform exponential dichotomy with constants $\underline{a} \le \overline{a} < 0 < \underline{b} \le \overline{b}$, $\varepsilon \ge 0$ and $D \ge 1$, and the associated evolution family satisfies (31);*
2. *there exist constants $c \ge 0$ and $\beta \ge 4\varepsilon$ such that*

$$\|B(t) - A(t)\| \le ce^{-\beta|t|}, r \quad t \in \mathbb{R}. \tag{56}$$

If c is sufficiently small, then the equation $x' = B(t)x$ admits a strong nonuniform exponential dichotomy.

Proof. Let V be the strongly strict quadratic Lyapunov function for Eq. (26) given by Theorem 8. For each $\tau \in \mathbb{R}$ and $x \in \mathbb{R}^p$, we have

$$\langle (S'(t) + S(t)B(t) + B(t)^*S(t))x, x\rangle = \langle (S'(t) + S(t)A(t) + A(t)^*S(t))x, x\rangle + R(t, x),$$

where

$$R(t, x) = 2\langle S(t)(B(t) - A(t))x, x\rangle.$$

By (32) and (56), we obtain

$$\begin{aligned} |R(t, x)| &\leq 2\|S(t)\| \cdot \|B(t) - A(t)\| \cdot \|x\|^2 \\ &\leq 2cC^2 e^{2\varepsilon|t|} e^{-\beta|t|}\|x\|^2. \end{aligned}$$

It follows from (35) that

$$|R(t, x)| \leq cC^4 e^{4\varepsilon|t|} e^{-\beta|t|}\langle S(t)x, x\rangle$$

for $x \in E^s_t$ and

$$|R(t, x)| \leq -cC^4 e^{4\varepsilon|t|} e^{-\beta|t|}\langle S(t)x, x\rangle$$

for $x \in E^u_t$. Since $\beta \geq 4\varepsilon$, there exists a constant $r > 0$ independent of x and t such that

$$|R(t, x)| \leq r\langle S(t)x, x\rangle$$

for $x \in E^s_t$ and

$$|R(t, x)| \leq -r\langle S(t)x, x\rangle$$

for $x \in E^u_t$. Moreover, r can be made arbitrarily small taking c sufficiently small. It follows from (53) and (54) that

$$[2\log\theta_2 - r]S(t) \leq S'(t) + S(t)B(t) + B(t)^*S(t) \leq [2\log\theta_1 + r]S(t)$$

on E^s_t and

$$[-2\log\theta_3 + r]S(t) \leq S'(t) + S(t)B(t) + B(t)^*S(t) \leq [-2\log\theta_1 - r]S(t)$$

on E^u_t, for some constants $\theta_1, \theta_2, \theta_3 \in (0, 1)$ with $\theta_2, \theta_3 \leq \theta_1$. Hence, it follows from Theorem 10 that V is a strongly strict Lyapunov function for the equation $x' = B(t)x$. By Corollary 7, the equation admits a strong nonuniform exponential dichotomy. $\square$

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