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Stability analysis of arbitrarily high-index positive delay-descriptor systems

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Abstract This paper deals with the stability analysis of positive delay-descriptor systems with arbitrarily high index. First we discuss the solvability problem (i.e., about the existence and uniqueness of a solution), which is followed by the study on characterizations of the (internal) positivity. Finally, we discuss the stability analysis. Numerically verifiable conditions in terms of matrix inequality for the system's coefficients are proposed, and are examined in several examples.

Keywords Positivity · Delay · Descriptor systems · Strangeness-index .

Nomenclature

$\mathbb{N}$ ($\mathbb{N}_0$)	the set of natural numbers (including 0)
$\mathbb{R}$ ($\mathbb{C}$)	the set of real (complex) numbers
$\mathbb{C}_-$	the set $\{\lambda \in \mathbb{C} \mid Re\lambda < 0\}$
I (I_n)	the identity matrix (of size $n \times n$)
$x^{(j)}$	the j-th derivative of a function x
$C^p([-\tau, 0], \mathbb{R}^n)$	the space of p-times continuously differentiable functions from $[-\tau, 0]$ to $\mathbb{R}^n$ (for $0 \leqslant p \leqslant \infty$)
$\|\cdot\|_\infty$	the norm of the Banach space $C^0([-\tau, 0], \mathbb{R}^n)$.
$\mathrm{im}_+ W$	the space $\{W w_1$ for all $w_1 \in \mathbb{R}^n_+\}$.

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1 Introduction

{sec1}

Our focus in the present paper is on the positivity and stability analysis of linear, constant coefficients *delay-descriptor systems* of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), \quad \text{for all } t \in [t_0, t_f), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

{delay-descriptor}

where $E, A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{n,p}$, $C \in \mathbb{R}^{q,n}$, $x : [t_0 - \tau, t_f) \to \mathbb{R}^n$, $f : [t_0, t_f) \to \mathbb{R}^n$, and $\tau > 0$ is a constant delay. Together with (1), we are also concern with the associated *zero-input system*

$$E\dot{x}(t) = Ax(t) + A_d x(t-\tau), \quad \text{for all } t \in [t_0, t_f). \tag{2}$$

{zero-input system}

Systems of the form (1) can be considered as a general combination of two important classes of dynamical systems, namely *differential-algebraic equations (descriptor systems)* (DAEs)

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

{eq1.2}

where the matrix E is allowed to be singular ($\det E = 0$), and *delay-differential equations* (DDEs)

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + Bu(t). \tag{4}$$

{eq1.3}

delay-descriptor systems of the form (1) have been arisen in various applications, see Ascher and Petzold [1995], Campbell [1980], Hale and Lunel [1993], Shampine and Gahinet [2006], Zhu and Petzold [1997] and the references there in. From the theoretical viewpoint, the study for such systems is much more complicated than that for standard DDEs or DAEs. The dynamics of DDAEs has been strongly enriched, and many interesting properties, which occur neither for DAEs nor for DDEs, have been observed for DDAEs Campbell [1995], Du et al. [2013], Ha and Mehrmann [2012, 2016]. Due to these reasons, recently more and more attention has been devoted to DDAEs, Campbell and Linh [2009], Fridman [2002], Ha and Mehrmann [2012, 2016], Michiels [2011], Shampine and Gahinet [2006], Tian et al. [2014], Linh and Thuan [2015].

[....]

The short outline of this work is as follows. Firstly, in Section 2, we briefly recall the solvability analysis to system (1), which is followed by an imporant result about solution comparison for system (2) (Theorem 2). Based on the explicit solution representation in Section 2, we characterize the posivity of system (1) in Section 3. We establish there algebraic, numerically verifiable conditions in terms of the system matrix coefficients. To follow, in Section 4 we discuss further about the zero-input system (2) under biconditional requirements: stability and positivity.

2 Preliminaries

{sec2}

In this section we discuss the solvability analysis, including the solution representation and the comparison principal for the corresponding IVP to system (1), which reads in details

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-\tau) + Bu(t), \quad \text{for all } t \in [t_0, t_f), \\ x|_{[t_0-\tau, t_0]} &= \varphi : [t_0 - \tau, t_0] \to \mathbb{R}^n. \end{aligned} \tag{5}$$

{initial condition}

Here, φ is a prescribed initial trajectory (preshape function), which is necessary to achieve uniqueness of solutions. Without loss of generality, we assume that $t_0 = 0$ and $t_f = n_f \tau$, where $n_f \in \mathbb{N}$.

2.1 Existence, uniqueness and explicit solution formula

It is well-known (e.g. Du et al. [2013]) that we may consider different solution concepts for system (1). The reason is, that $E(0)\dot{x}(0^+)$ which arises from the right hand side in (1) at 0 may not be equal to $E(0)\dot{\varphi}(0^-)$. Moreover, it has been observed in Baker et al. [2002], Campbell [1980], Guglielmi and Hairer [2008] that a discontinuity of $\dot{x}$ at $t = 0$ may propagate with time, and typically $\dot{x}$ is discontinuous at every point $j\tau,\ j \in \mathbb{N}_0$ or it may not even exist. To deal with this property of DDAEs, we use the following solution concept.

{solution}

Definition 1 Let us consider a fixed input function $u(t)$.
i) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *piecewise differentiable solution* of (1), if Ex is piecewise continuously differentiable, x is continuous and satisfies (1) at every $t \in [t_0, t_f) \setminus \bigcup\limits_{j \in \mathbb{N}_0} \{j\tau\}$.
ii) A function $x : [-\tau, \infty) \to \mathbb{R}^n$ is called a *classical solution* of (1) if it is at least continuous and satisfies (1) at every $t \in [t_0, t_f)$.

Throughout this paper whenever we speak of a solution, we mean a piecewise differentiable solution. Notice that, like DAEs, DDAEs are not solvable for arbitrary initial conditions, but they have to obey certain consistency conditions.

{consistency}

Definition 2 An initial function φ is called *consistent* with (1) if the associated initial value problem (IVP) (1), (5) has at least one solution. System (1) is called *solvable* (resp. *regular*) if for every consistent initial function φ, the IVP (1), (5) has a solution (resp. has a unique solution).

Introducing sequences of matrix-valued and vector-valued functions f_j, u_j, x_j for each $j \in \mathbb{N}$, on the time interval $[0, \tau]$ via

$$\begin{aligned} f_j(t) &= f(t + (j-1)\tau),\ u_j(t) = u(t + (j-1)\tau), \\ x_j(t) &= x(t + (j-1)\tau),\ x_0(t) := \varphi(t - \tau), \end{aligned}$$

we can rewrite the IVP (1)-(5) as a sequence of non-delayed descriptor systems

$$E\dot{x}_j(t) = Ax_j(t) + A_d x_{j-1}(t) + Bu_j(t), \tag{6}$$

{j-th DAE}

for all $t \in (0, \tau)$ and for all $j = 1, 2, ..., n_f$. We notice, that for each j, the initial condition $x_j(0)$ is given due to the continuity of the solution $x(t)$ at the point $(j-1)\tau$, i.e.,

$$x_j(0) = x_{j-1}(\tau) \ . \tag{7}$$

{continuity condition}

In particular, $x_1(0) = \phi(0)$ and the function x_0 is given. Inherited from the theory of delay-different equations (Hale and Lunel [1993]), we recall the concept of *non-advancedness* as follow.

Definition 3 A regular delay-descriptor system (1) is called *non-advanced* if for any consistent and continuous initial function φ, there exists a piecewise differentiable solution $x(t)$ to the IVP (1), (5).

Obviously, the non-advancedness of system (1) is equivalent to the fact that the function x_j is at least as smooth as x_{j-1} for all $j \in \mathbb{N}$. In deed, most of systems that we have encountered in applications are non-advanced, Ascher and Petzold [1995], Shampine and Gahinet [2006], Ha [2015]. For more detailed discussions about the types of the DDAE (2), we refer the readers to Ha [2015], Ha and Mehrmann [2016], Unger [2018].

{regularity}

Definition 4 Consider the DDAE (1). The matrix triple (E, A, B) is called *regular* if the (two variable) *characteristic polynomial* $\mathfrak{P}(\lambda, \omega) := \det(\lambda E - A - \omega B)$ is not identically zero. If, in addition, $B = 0$ we say that the matrix pair (E, A) (or the pencil $\lambda E - A$) is regular. The sets $\sigma(E, A, B) := \{\lambda \in \mathbb{C} \mid \det(\lambda E - A - e^{-\lambda\tau} B) = 0\}$ and $\rho(E, A, B) = \mathbb{C} \setminus \sigma(E, A, B)$ are called the *spectrum* and the *resolvent set* of (1), respectively.

Provided that the pair (E, A) is regular, we can transform them to the Kronecker-Weierstraßcanonical form (see e.g. Dai [1989], Kunkel and Mehrmann [2006]). That is, there exist regular matrices W, $T \in \mathbb{R}^{n,n}$ such that

$$(E, A) = \left(W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, \ W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right) \ , \tag{8}$$

{KW form}

where N is a nilpotent matrix of nilpotency index ν. We also say that the pair (E, A) has a *differentiation index* ν, i.e., $\operatorname{ind}(E, A) = \nu$.

Remark 1 Two concepts non-advancedness and differentiation index are independent. In details, a non-advanced system can have arbitrarily high index, as can be seen in the following example.

{example 1}

Example 1 Consider the following systems with the parameters ε_1, ε_2.

$$\underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}}_{E} \dot{x}(t) = \underbrace{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}_{A} x(t) + \begin{bmatrix} 0 & & & \varepsilon_1 \\ & \ddots & & \\ & & 0 & \\ & & & \varepsilon_2 \end{bmatrix} x(t-h), \tag{9}$$

{eq11}

It is well-known that in this example $\mathrm{ind}(E, A) = n$. Furthermore, depending on the value of ε_2, the system will be advanced (if $\varepsilon_2 \neq 0$) and be non-advanced (if $\varepsilon_2 \neq 0$).

Let E have index $\tilde{\nu}$, i.e., $\mathrm{ind}(E, I_n) = \tilde{\nu}$, the Drazin inverse E^D of E is uniquely defined by the properties

$$E^D E = E E^D,\ E^D E E^D = E^D,\ E^D E^{\tilde{\nu}+1} = E^{\tilde{\nu}}. \tag{10}$$

{lem1}

Lemma 1 *Kunkel and Mehrmann [2006] Let (E,A) be a regular matrix pair. Then for any $\lambda \in \rho(E, A)$, two following matrices commute.*

$$\hat{E} := (\lambda E - A)^{-1} E,\ \hat{A} := (\lambda E - A)^{-1} A. \tag{11}$$ {eq20}

Furthermore, the following commutative identities hold true.

$$\hat{E}\hat{A}^D = \hat{A}^D \hat{E},\ \hat{E}^D \hat{A} = \hat{A}\hat{E}^D,\ \hat{E}^D \hat{A}^D = \hat{A}^D \hat{E}^D\ .$$

We notice that the matrix products $\hat{E}^D\hat{E}$, $\hat{E}^D\hat{A}$, $\hat{E}\hat{A}^D$, $\hat{E}^D\hat{B}$, $\hat{A}^D\hat{B}$ do not depend on the choice of λ (see e.g. Dai [1989]). Furthermore, they can be numerically computed by transforming the pair (E, A) to their Weierstrass canonical form (8) (see e.g. Gerdts [2005], Virnik [2008]).

For any $\lambda \in \rho(E, A)$, we denote

$$\hat{A}_d := (\lambda E - A)^{-1} A_d,\ \hat{B} := (\lambda E - A)^{-1} B\ . \tag{12}$$ {eq21}

Making use of the Drazin inverse, in the following theorem we present the explicit solution representation of system (1).

{sol. rep. DAE}

Theorem 1 *Consider the delay-descriptor system* (1). *Assume that* (E, A) *is a regular matrix pair with a differentiation index* $\mathrm{ind}(E, A) = \nu$. *Let* $\hat{E}$, $\hat{A}$, $\hat{A}_d$, $\hat{B}$ *be defined as in* (11), (12). *Furthermore, assume that* u *is sufficiently smooth. Then, every solution* x_j *of the DAE* (6) *has the form*

$$\begin{aligned} x_j(t) = {} & e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_j + \int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\ \left(\hat{A}_d x_{j-1}(s) + \hat{B}u_j(s)\right) ds \\ & + (\hat{E}^D\hat{E} - I)\ \sum_{i=0}^{\nu-1} (\hat{E}^D\hat{A})^i \hat{A}^D \left(\hat{A}_d x_{j-1}^{(i)}(t) + \hat{B}u_j^{(i)}(t)\right), \end{aligned} \tag{13}$$ {j-th solution}

for some vector $v_j \in \mathbb{R}^n$.

Proof. The proof is straightly followed from the explicit solution of DAEs, see [Kunkel and Mehrmann, 2006, Chap. 2]. □

Making use of (7), we directly obtain the following corollary.

Corollary 1 *The solution $x(t)$ of system (1) is continuous at the point $(j-1)\tau$ if and only if the following condition holds.*

$$(\hat{E}^D\hat{E}-I)\ x_{j-1}(\tau)=(\hat{E}^D\hat{E}-I)\ \sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\left(\hat{A}_d x_{j-1}^{(i)}(0)+\hat{B}u_j^{(i)}(0)\right)\ .$$

In particular, for the preshape function $\varphi(t)$, we must require

$$(\hat{E}^D\hat{E}-I)\ \left(\varphi(0)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\left(\hat{A}_d\varphi^{(i)}(-\tau)+\hat{B}u^{(i)}(0)\right)\right)=0\ .$$

Following from (13), we directly obtain a simpler form in case of non-advanced system as follows.

Corollary 2 *Consider system (1) and assume that it is regular and non-advanced. Then, we have*

$$\begin{aligned}x_j(t)&=e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_j+\int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\left(\hat{A}_d x_{j-1}(s)+\hat{B}u_j(s)\right)ds\\&\quad+(\hat{E}^D\hat{E}-I)\ \left(\hat{A}^D\hat{A}_d x_{j-1}(t)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u_j^{(i)}(t)\right),\end{aligned}\tag{14}$$

{sol. formula non-advanced}

Furthermore, the consistency condition at $t=0$ reads

$$(\hat{E}^D\hat{E}-I)\left(\varphi(0)+\hat{A}^D\hat{A}_d\varphi(-\tau)+\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u^{(i)}(0)\right)=0\ .\tag{15}$$

{consistency}

2.2 Comparison principal

{lem2b}

Lemma 2 *It suffices to prove that if $u_j(t)\leqslant\tilde{u}_j(t)$ and $x_{j-1}(t)\leqslant\tilde{x}_{j-1}(t)$ for all $t\in[0,\tau]$ then it follows that $x_j(t)\leqslant\tilde{x}_j(t)$ for all $t\in[0,\tau]$.*

By simple induction, making use of Lemma 2,we obtain the solution comparison for system (1).

{solution comparison 1}

Theorem 2 *Consider system (1) and assume that it is positive. Let $x(t)$ (resp. $\tilde{x}(t)$) be a state function corresponds to a reference input $u(t)$ (resp. $\tilde{u}(t)$) and a preshape function $\varphi(t)$ (resp. $\tilde{\varphi}(t)$). Furthermore, assume that the following conditions hold.*
i) $\varphi(t)\leqslant\tilde{\varphi}(t)$ for all $t\in[-\tau,0]$,
ii) $u^{(i)}(t)\leqslant\tilde{u}^{(i)}(t)$ for all $t\geq 0$ and all $i\leqslant(\nu-1)\ \lfloor t/\tau\rfloor$. Then we have $x(t)\leqslant\tilde{x}(t)$ for all $t\geq 0$.

Proof. □

Theorem 3 *Time-dependent delay will affect neither the positivity nor the stability of system (1).*

3 Characterizations of positive delay-descriptor system

{sec3}

Since most systems occur in application are non-advanced, in this section we focus on the chracterization for positivity of non-advanced delay descriptor systems. We, furthermore, notice that the non-advancedness is a necessary condition for the stability (in the Lyapunov sense) of any time-delayed system, see e.g. Hale and Lunel [1993], Du et al. [2013].

Definition 5 Consider the delay-descriptor system (1) and assume that it is non-advanced, and that the pair (E, A) is regular with $\operatorname{ind}(E, A) = \nu$. We call (1) positive if for all $t \geq 0$ we have $x(t) \geq 0$ and $y(t) \geq 0$ for any input function u and any consistent initial function $\varphi(t)$ that satisfy two following conditions.
i) $\varphi(t) \geq 0$ for all $t \in [-\tau, 0]$,
ii) $u^{(i)}(t) \geq 0$ for all $t \geq 0$ and all $i \leqslant (\nu - 1)\ \lfloor t/\tau \rfloor$.

Let us denote

$$\mathcal{K}_\nu(\hat{E}\hat{A}^D, \hat{A}^D\hat{B}) := \left[\hat{A}^D\hat{B}, \hat{E}\hat{A}^D\ \hat{A}^D\hat{B}, \ldots, (\hat{E}\hat{A}^D)^{\nu-1}\hat{A}^D\hat{B}\right] .$$

Since our systems is linear, time invariant coefficients, it would be sufficient to study the positivity on the first time interval $[0, \tau]$. Making use of (14), and let $j = 1$, we can split the solution $x_1 = x|_{[0,\tau]}$ as follows

$$\begin{aligned} x_1(t) = &\underbrace{e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v_1 + \int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{A}_d x_0(s) + (\hat{E}^D\hat{E} - I)\hat{A}^D\hat{A}_d x_0(t)}_{x_{zi}(t)} \\ &+ \underbrace{\int_0^t e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{B}u_j(s) + (\hat{E}^D\hat{E} - I)\sum_{i=0}^{\nu-1}(\hat{E}^D\hat{A})^i\hat{A}^D\hat{B}u_j^{(i)}(t)}_{x_{zs}(t)} , \end{aligned} \tag{16}$$

{eq16}

where $x_{zi}(t)$ (resp. $x_{zs}(t)$) is often called (in the theory of linear systems) the *zero input* (resp. *zero state*) solution.

{Castelan'93}

Lemma 3 *Let $F \in \mathbb{R}^{p,n}$ and $M \in \mathbb{R}^{n,n}$ and consider the linear system $\dot{z}(t) = Mz(t)$. Then, the following implication holds true:*

$$[Fz(0) \geq 0] \Rightarrow [Fz(t) \geq 0 \forall t \geq 0]$$

if and only if there exists a Metzler matrix H such that $FM = HF$.

Proposition 1 *Rami and Napp [2012] The following statements are equivalent.* {Rami12}
1) Our descriptor system is positive.
2) There exists a Metzler matrix H s.t. $\bar{A} = HP$, where $\bar{A} := \hat{E}^D\hat{A}$, $P := \hat{E}^D\hat{E}$.
3) There exists a matrix D such that $H := \bar{A} + D\ (I - P)$ is Metzler.

{zero input lemma}

Lemma 4 *Consider the delay-descriptor system* (1) *and assume that it is non-advanced, and the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = \nu$. *Let the input* $u = 0$. *Then, system* (1) *has a solution* $x(t) \geq 0$ *for all* $t \geq 0$ *and all consistent initial function* $\varphi(t) \geq 0$ *if and only if the following conditions are satisfied.*
1) $\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0$ *for some* $\alpha \geq 0$.
2) $\hat{E}^D \hat{A}_d \geq 0$, $(\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d \geq 0$.

{zero state lemma}

Lemma 5

{Thm positivity}

Theorem 4 *Consider the delay-descriptor system* (1) *and assume that it is non-advanced, and the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = \nu$. *Furthermore, assume that* $(\hat{E}^D \hat{E} - I)\ (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$ *for all* $i = 0, \ldots, \nu - 1$. *Then, system* (1) *is positive if and only if the following conditions hold.*
1) $\hat{E}^D \hat{A} = H\ \hat{E}^D \hat{E}$ *for some Metzler matrix* H.
2) $\hat{E}^D \hat{A}_d \geq 0$, $(\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d \geq 0$, $\hat{E}^D \hat{B} \geq 0$,
3) C *is non-negative on the subspace*

$$\mathcal{X} := \operatorname{im}_+ \left[\hat{E}^D \hat{E},\ (\hat{E}^D \hat{E} - I) \hat{A}^D \hat{A}_d,\ (\hat{E}^D \hat{E} - I)\ \mathcal{K}_\nu(\hat{E} \hat{A}^D, \hat{A}^D \hat{B}) \right] . \quad (17)$$

{reachable subspace}

Proof. $\Rightarrow$ We only need to prove part 3.
$\Leftarrow$ Quite simple. □

If we restrict ourself to the non-delayed case (i.e. $A_d = 0$), the direct corollary of Theorem 4 is straightforward. We, moreover, notice that this corollary has slightly improved the result [Virnik, 2008, Thm. 3.4].

{Thm positivity - DAE version}

Corollary 3 *Consider the descriptor system* (3) *and assume that the pair* (E, A) *is regular with index* $\operatorname{ind}(E, A) = \nu$. *Furthermore, assume that*
i) $(\hat{E}^D \hat{E} - I)\ (\hat{E}^D \hat{A})^i \hat{A}^D \hat{B} \geq 0$ *for* $i = 0, \ldots, \nu - 1$,
ii) $\hat{E}^D \hat{E} \geq 0$.
Then system (3) *is positive if and only if the following conditions hold.*
1) $\hat{E}^D \hat{A} + \alpha \hat{E}^D \hat{E} \geq 0$ *for some* $\alpha \geq 0$.
2) $\hat{E}^D \hat{B} \geq 0$,
3) C *is non-negative on the subspace* $\operatorname{im}_+ \left[\hat{E}^D \hat{E}, (\hat{E}^D \hat{E} - I) \mathcal{K}_\nu(\hat{E} \hat{A}^D, \hat{A}^D \hat{B}) \right]$.

4 Stability of positive delay-descriptor system

{sec4}

5 Conclusion

{sec6}

In this paper, we have discussed the positivity of strangeness-free descriptor systems in continuous time. Beside that, the characterization of positive delay-descriptor systems has been treated as well. The theoretical results are obtained mainly via an algebraic approach and a projection approach. The projection approach investigates the positivity of a given descriptor system by the positivity of an inherent ODE obtained by projecting the given system onto a subspace. On the other hand, the algebraic approach derives an underlying ODE without changing the state, input and output. Then, studying these hidden ODEs is the key point. The main difficulty here is that the derivative of the input u may occur in the new system. Despite their disadvantages, these methods can provide both necessary conditions and sufficient conditions. Beside these theoretical methods, the behaviour approach, which leads to some feasible conditions, is also implemented.

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Appendix