



Convergence within a polyhedron: controller design for time-delay systems with bounded disturbances

Phan Thanh Nam¹, Pubudu Nishantha Pathirana², Hieu Trinh²

¹Department of Mathematics, Quynhon University, Binh Dinh, Vietnam

²School of Engineering, Deakin University, Geelong, VIC 3217, Australia

E-mail: hieu.trinh@deakin.edu.au

Abstract: This study considers linear systems with state/input time-varying delays and bounded disturbances. The authors study a new problem of designing a static output feedback controller which guarantees that the state vector of the closed-loop system converges within a pre-specified polyhedron. Based on the Lyapunov–Krasovskii method combining with the free-weighting matrix technique, a new sufficient condition for the existence of a static output feedback controller is derived. The author’s condition is expressed in terms of linear matrix inequalities with two parameters need to be tuned and therefore can be efficiently solved by using a two-dimensional search method combining with convex optimisation algorithms. To be able to obtain directly an output feedback control matrix from the derived condition, they propose an appropriate combination between a state transformation with a choice of a special form of the free-weighting matrices. The feasibility and effectiveness of the derived results are illustrated through five numerical examples.

1 Introduction

Within recent years, the stabilisation problem of linear systems with time-delays in the state and the input has received much considerable attention from researchers [1–8]. Based on the Lyapunov–Krasovskii method combining with the free-weighting matrix technique, a state feedback stabilisation condition was first reported [2] for linear systems with two time-varying delays in the state and the input. This condition was given in terms of linear matrix inequalities (LMIs) with four parameters need to be tuned. For the case where there is a constant time delay in both the state and the input, by eliminating some free-weighting matrices, the authors [3–5] derived some simpler state feedback stabilisation criteria which are given in terms of linear matrix inequalities and require only one tuned parameter. In practice, the assumption of full state information is a limiting one and it is more practical if only output information is used for the controller design purpose [7–11]. To our knowledge, there are few results dealing with output feedback stabilisation problem for linear systems with time delays in the state and the input [7, 8]. By using the Lyapunov method and delay-decomposition technique, Du *et al.* [7] proposed two methods for designing static and integral output feedback controllers for linear systems with one unknown constant time delay in both the state and the input. By using the sliding mode control method, a static output feedback stabilisation condition for linear systems with state and input time-varying delays was reported in [8].

On the other hand, disturbances are unavoidable in practical control systems because of modelling errors, linearisation

approximations, unknown disturbance signals, measurement errors and so on. For systems with bounded disturbances, a central concept that has received considerable attention is the so-called reachable set, which is the set of all the states starting from the origin by inputs with peak value [12, 13]. The exact shape of reachable sets of a perturbed system is, in general, very complex and hard to obtain. Hence, it is usually approximated by outer bounding simple convex shapes like balls or ellipsoids or boxes. So far, the problem of reachable set bounding for systems with time delays and bounded disturbances has been studied extensively [13–27]. Very recently, Zhang *et al.* [27] considered a new problem which deals with the design of a state feedback controller such that reachable sets of the closed-loop system are contained in a pre-specified ellipsoid. This is an interesting and meaningful problem since the pre-specified ellipsoid can be chosen according to practical situations or special design requirements. For instance, given a set of finite points in state space $D = \{\xi_i : i = 1, \dots, r\}$ and it is required to design a controller such that reachable sets of the closed-loop system do not contain any point ξ_i . As pointed out in [27], one first finds an ellipsoid $\epsilon(P)$ (as large as possible) that does not contain any point ξ_i and then design a controller such that reachable sets of the resulting closed-loop system are contained in the ellipsoid $\epsilon(P)$. Note that, in such a situation, it is clear that a polyhedron, which is an intersection of halfspaces ([28], page 31) can express the above requirement better, that is, there exists a polyhedron Ω , which is larger than the ellipsoid $\epsilon(P)$ and does not contain any point ξ_i (for a visual illustration, see Fig. 1, where the rectangle $ABCD$ contains the ellipse $\epsilon(P)$ but does not contain any point

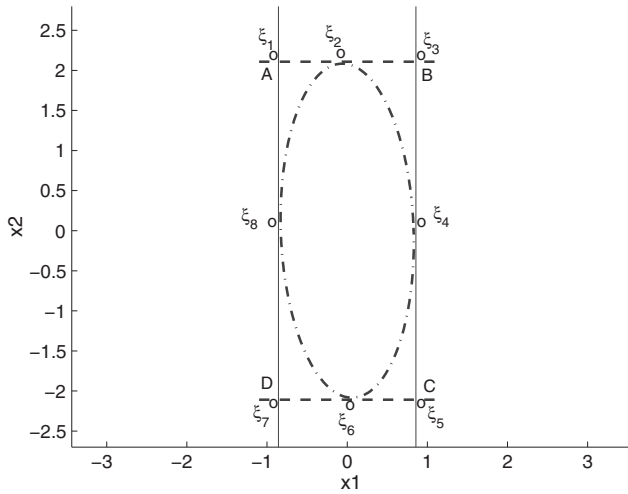


Fig. 1 Rectangle $ABCD$ and ellipse $\epsilon(P)$

$\xi_i, i = 1, \dots, 8$). Hence, the controller design problem for the case where reachable sets are contained in a polyhedron Ω will be easier than for the case where reachable sets are contained in an ellipsoid $\epsilon(P)$ which is smaller than the polyhedron Ω .

Motivated by the above, in this paper, we consider linear systems with state/input time-varying delays and bounded disturbances. We solve a new problem of designing a static output feedback controller, which guarantees that the state vector of the closed-loop system converges within a pre-specified polyhedron. To solve this problem, we employ the Lyapunov–Krasovskii method and the free-weighting matrix technique [29, 30] with a choice of a special form of the free-weighting matrices. A new sufficient condition for the existence of a static output feedback controller is derived and expressed in terms of linear matrix inequalities with two parameters need to be tuned and can be efficiently solved by using a two-dimensional (2D) search method combining with convex optimisation algorithms such as the Matlab’s LMI toolbox [31]. Furthermore, to reduce the conservatism of our derived convergence condition, we use the recent effective techniques in stability analysis for time-delay systems, that is, the Wirtinger-based integral inequality [32, 33] and the reciprocally convex combination inequality [34]. Also, for the case where disturbances are not present, the derived convergence condition is reduced to a static output feedback exponential stabilisability condition, which is shown to be less conservative than existing ones [2–5, 7, 35]. Lastly, the feasibility and effectiveness of the obtained results are illustrated through five numerical examples.

2 Problem statement and preliminaries

Consider the following linear system with state/input time-varying delays and bounded disturbances

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t - \tau_1(t)) + Bu(t) \\ &\quad + B_2u(t - \tau_2(t)) + D\omega(t), \quad t \geq 0 \\ y(t) &= Cx(t) \\ x(\theta) &= \phi(\theta), \quad \theta \in [-h, 0] \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the measured output vector and

$\omega(t) \in \mathbb{R}^k$ is the disturbance vector satisfying

$$\omega^T(t)\omega(t) \leq \bar{\omega}^2 \quad (2)$$

$\bar{\omega}$ is a given positive scalar, matrices $A \in \mathbb{R}^{n \times n}$, $A_1 \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $B_2 \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ and $D \in \mathbb{R}^{n \times k}$ are given constant matrices, C is assumed to be a full-row rank matrix, $\tau_1(t)$ and $\tau_2(t)$ are time-varying delays satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq \tau_{1M}, \dot{\tau}_1(t) \leq d_{1M} \leq 1 \\ 0 \leq \tau_2(t) \leq \tau_{2M}, \dot{\tau}_2(t) \leq d_{2M} \leq 1 \end{cases}$$

where $\tau_{1M} \geq 0$, $\tau_{2M} \geq 0$, d_{1M} and d_{2M} are known constants, $h = \max\{\tau_{1M}, \tau_{2M}\}$, $\phi(\theta) \in C^1([-h, 0], \mathbb{R}^n)$ is an initial function with its norm defined as

$$\|\phi\|_c = \max \left\{ \max_{t \in [-h, 0]} \|\phi(t)\|, \max_{t \in [-h, 0]} \|\dot{\phi}(t)\| \right\} \quad (3)$$

With the following static output feedback control law

$$u(t) = Ky(t) \quad (4)$$

where $K \in \mathbb{R}^{m \times p}$, the closed-loop system is obtained as follows

$$\begin{aligned} \dot{x}(t) &= (A + BKC)x(t) + A_1x(t - \tau_1(t)) \\ &\quad + B_2KCx(t - \tau_2(t)) + D\omega(t) \end{aligned} \quad (5)$$

Given q non-zero row matrices $L_j \in \mathbb{R}^{1 \times n}$, $j = 1, \dots, q$ and q positive scalars $b_j > 0$, $j = 1, \dots, q$. It is easy to see that $\{x \in \mathbb{R}^n : L_jx = b_j\}$ and $\{x \in \mathbb{R}^n : L_jx = -b_j\}$ are two parallel $(n-1)$ -planes in \mathbb{R}^n and the set $\Omega_j = \{x \in \mathbb{R}^n : |L_jx| \leq b_j\}$ is the area between the two parallel planes. Then, the set $\Omega = \bigcap_{j=1}^q \Omega_j$ is a polyhedron [28] and the main problem of this paper is stated as follows:

Problem: Find a static output feedback controller (4) such that every solution $x(t, \phi)$ of the closed-loop system (5) satisfies

$$\limsup_{t \rightarrow \infty} |L_jx(t, \phi)| \leq b_j, \quad j = 1, \dots, q \quad (6)$$

This means that the state vector of the closed-loop system (5) converges within the given polyhedron Ω as t tends to infinity. Note that if $\text{rank}([L_1^T \ L_2^T \ \dots \ L_q^T]) = n$ then Ω is bounded and it is called a polytope [28] in \mathbb{R}^n .

The following lemmas are useful for our main results.

Lemma 1: For a given positive scalar δ , let $V(t)$ be a Lyapunov function for system (5). If $\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\bar{\omega}^2}\omega^T(t)\omega(t) \leq 0$, $\forall t \geq 0$, then we have

$$\limsup_{t \rightarrow \infty} V(t) \leq 1$$

Proof: Putting $v(s) = e^{2\delta s} V(s)$ and taking the derivative of $v(s)$ in s , we have

$$\begin{aligned}\dot{v}(s) &= e^{2\delta s} \left(\dot{V}(s) + 2\delta V(s) - \frac{2\delta}{\omega^2} \omega^T(s) \omega(s) \right) \\ &\quad + \frac{2\delta}{\omega^2} \omega^T(s) \omega(s) e^{2\delta s} \\ &\leq 2\delta e^{2\delta s}\end{aligned}$$

Integrating from 0 to t both sides of the above inequality, we obtain

$$v(t) - v(0) \leq e^{2\delta t} - 1, \quad \forall t \geq 0$$

and hence

$$V(t) \leq 1 + e^{-2\delta t} |V(0) - 1|, \quad \forall t \geq 0$$

This implies that $\limsup_{t \rightarrow \infty} V(t) \leq 1$. The proof of Lemma 1 is completed. \square

The Wirtinger-based integral inequality [32] and the reciprocally convex combination inequality [34], which has been reformulated by Seuret and Gouaisbaut [32], are used in this paper.

Lemma 2 (the Wirtinger-based integral inequality [32]): For a given $n \times n$ -matrix $R > 0$, any differentiable function $\varphi : [a, b] \rightarrow \mathbb{R}^n$, then the following inequality holds

$$\begin{aligned}\int_a^b \dot{\varphi}(u) R \dot{\varphi}(u) du &\geq \frac{1}{b-a} (\varphi(b) - \varphi(a))^T R (\varphi(b) - \varphi(a)) \\ &\quad + \frac{12}{b-a} \Omega^T R \Omega\end{aligned}$$

where

$$\Omega = \frac{\varphi(b) + \varphi(a)}{2} - \frac{1}{b-a} \int_a^b \varphi(u) du$$

Lemma 3 (the reciprocally convex combination inequality [32, 34]): For given positive integers n, m , a scalar $\alpha \in (0, 1)$, a $n \times n$ -matrix $R > 0$, two $n \times m$ -matrices W_1, W_2 . Define, for all vector $\xi \in \mathbb{R}^m$, the function $\Theta(\alpha, R)$ given by

$$\Theta(\alpha, R) = \frac{1}{\alpha} \xi^T W_1^T R W_1 \xi + \frac{1}{1-\alpha} \xi^T W_2^T R W_2 \xi$$

If there is a matrix $X \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} R & X \\ \star & R \end{bmatrix} > 0$$

then the following inequality holds

$$\min_{\alpha \in (0,1)} \Theta(\alpha, R) \geq \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}^T \begin{bmatrix} R & X \\ \star & R \end{bmatrix} \begin{bmatrix} W_1 \xi \\ W_2 \xi \end{bmatrix}$$

Lemma 4 (Schur complement lemma [31]): Let R be a symmetric positive definite matrix. For any matrices P, S with appropriate dimensions, where $P = P^T$, then

$$\begin{bmatrix} P & S \\ S^T & R \end{bmatrix} > 0$$

if and only if $P - SR^{-1}S^T > 0$.

3 Main results

To use conveniently the output information in designing a static output feedback controller, we first take the following state transformation to re-present the output matrix in a canonical form

$$x(t) = Hz(t) \quad (7)$$

where $H = [C^+ \text{ null}(C)]$ is a non-singular matrix, C^+ denotes the Moore–Penrose inverse of C , $\text{null}(C)$ denotes an orthogonal basis for the null-space of C . Then, system (1) is transformed into the following system

$$\begin{aligned}\dot{z}(t) &= \bar{A}z(t) + \bar{A}_1 z(t - \tau_1(t)) + \bar{B}u(t) \\ &\quad + \bar{B}_2 u(t - \tau_2(t)) + \bar{D}\omega(t), \quad t \geq 0\end{aligned} \quad (8)$$

$$y(t) = \bar{C}z(t)$$

$$z(\theta) = H^{-1}\phi(\theta) := \phi(\theta), \quad \theta \in [-h, 0]$$

where $\bar{A} = H^{-1}AH$, $\bar{A}_1 = H^{-1}A_1H$, $\bar{B} = H^{-1}B$, $\bar{B}_2 = H^{-1}B_2$, $\bar{D} = H^{-1}D$ and $\bar{C} = CH$. With the choice of matrix H as above, the output matrix \bar{C} is now in a canonical form, that is, $\bar{C} = [I_p \ 0]$. Note that condition (6) is equivalent to the following condition

$$\limsup_{t \rightarrow \infty} |L_j Hz(t, \varphi)| \leq b_j, \quad j = 1, \dots, q \quad (9)$$

and the polyhedron Ω corresponds to the polyhedron $\bar{\Omega} = \{z \in \mathbb{R}^n : |L_j Hz| \leq b_j, j = 1, \dots, q\}$.

The following notations are needed in order to derive our main results. For two non-singular matrices $Z_{11} \in \mathbb{R}^{p \times p}$ and $Z_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$, matrices $Z_{21} \in \mathbb{R}^{(n-p) \times p}$, $G \in \mathbb{R}^{m \times p}$, $K \in \mathbb{R}^{m \times p}$ and two $n \times n$ positive-definite matrices R_1, R_2 , we denote the following (see equation at the bottom of the next page)

Note that, from the above notations and with some simple computations, we can verify that

$$\bar{B}K\bar{C}Z = \tilde{B}\tilde{K}\tilde{C}Z = \tilde{B} \times \begin{bmatrix} KZ_{11} & 0_{m \times (n-p)} \\ 0_{(n-m) \times p} & 0_{(n-m) \times (n-p)} \end{bmatrix} \in \mathbb{R}^{n \times n}$$

By letting $G = KZ_{11}$, then system (8) with a static output feedback controller $u(t) = K\bar{C}z(t)$ is rewritten as follows

$$[\mathcal{A}_c^T Z + \mathcal{B}_c^T G] \xi(t) = 0 \quad (10)$$

Now we are in a position to introduce the main result in the form of the following theorem.

Theorem 1: If there exist a positive scalar $\delta > 0$, a scalar λ , a positive-definite $3n \times 3n$ -matrix P , six positive-definite $n \times n$ -matrices $Q_1, Q_2, S_1, S_2, R_1, R_2$, q positive-definite $n \times n$ -matrices $P_j, j = 1, \dots, q$, two $2n \times 2n$ -matrices X_1, X_2 , two non-singular matrices $Z_{11} \in \mathbb{R}^{p \times p}$, $Z_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$, and two matrices $Z_{21} \in \mathbb{R}^{(n-p) \times p}$, $G \in \mathbb{R}^{m \times p}$ such that the following matrix inequalities hold

$$P - \begin{bmatrix} P_j & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} > 0, \quad j = 1, \dots, q \quad (11)$$

$$P_j - Z^T \Upsilon_j Z > 0, \quad j = 1, \dots, q \quad (12)$$

$$\Theta_i = \begin{bmatrix} \tilde{R}_i & X_i \\ \star & \tilde{R}_i \end{bmatrix} > 0, \quad i = 1, 2 \quad (13)$$

$$\Sigma(\tau_1, \tau_2, \delta) < 0, \quad \forall (\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\} \quad (14)$$

where

$$\begin{aligned} \Sigma(\tau_1, \tau_2, \delta) = & 2\rho(t)P[e_{10} \quad e_1 - e_3 \quad e_1 - e_5]^T \\ & + 2\delta\rho(t)P\rho^T(t) + e_1(Q_1 + S_1 + Q_2 + S_2)e_1^T \\ & - e^{-2\delta\tau_{1M}}e_3Q_1e_3^T - e^{-2\delta\tau_{1M}}(1 - d_{1M})e_2S_1e_2^T \\ & - e^{-2\delta\tau_{2M}}e_5Q_2e_5^T - e^{-2\delta\tau_{2M}}(1 - d_{2M})e_4S_2e_4^T \\ & + e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T - e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T \\ & - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T + (e_1 + \lambda e_{10}) \\ & \times (\mathcal{A}_c^T \mathcal{Z} + \mathcal{B}_c^T \mathcal{G} + \mathcal{Z}^T \mathcal{A}_c + \mathcal{G}^T \mathcal{B}_c) \\ & - 2\frac{\delta}{\omega^2}e_{11}e_{11}^T \end{aligned} \quad (15)$$

then with the static output feedback controller $u(t) = GZ_{11}^{-1}y(t)$, every solution of the closed-loop system (8) converges within the given polyhedron $\bar{\Omega}$.

Proof: Consider the following Lyapunov–Krasovskii functional

$$V = V_1 + V_2 + V_3 \quad (16)$$

where

$$\begin{aligned} V_1 &= \zeta_0^T(t)P\zeta_0(t) \\ V_2 &= \int_{t-\tau_{1M}}^t e^{2\delta(s-t)}z^T(s)F^TQ_1Fz(s)ds \\ &\quad + \int_{t-\tau_1(t)}^t e^{2\delta(s-t)}z^T(s)F^TS_1Fz(s)ds \\ &\quad + \int_{t-\tau_{2M}}^t e^{2\delta(s-t)}z^T(s)F^TQ_2Fz(s)ds \\ &\quad + \int_{t-\tau_2(t)}^t e^{2\delta(s-t)}z^T(s)F^TS_2Fz(s)ds \\ V_3 &= \tau_{1M} \int_{-\tau_{1M}}^0 \int_v^0 e^{2\delta u}z^T(t+u)F^TR_1Fz(t+u)du dv \\ &\quad + \tau_{2M} \int_{-\tau_{2M}}^0 \int_v^0 e^{2\delta u}z^T(t+u)F^TR_2Fz(t+u)du dv \end{aligned}$$

Taking the derivatives of $V_i, i = 1, 2, 3$ in t , we have

$$\begin{aligned} \dot{V}_1 + 2\delta V_1 &= 2\zeta_0^T(t)P\dot{\zeta}_0(t) + 2\delta\zeta_0^T(t)P\zeta_0(t) \\ &= \xi^T(t)\{2\rho(t)P[e_{10} \quad e_1 - e_3 \quad e_1 - e_5]^T \\ &\quad + 2\delta\rho(t)P\rho^T(t)\}\xi(t) \end{aligned} \quad (17)$$

$$\begin{aligned} Z &= \begin{bmatrix} Z_{11} & 0_{p \times (n-p)} \\ Z_{21} & Z_{22} \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G & 0_{m \times (n-p)} \\ 0_{(n-m) \times p} & 0_{(n-m) \times (n-p)} \end{bmatrix}, \quad \tilde{K} = \begin{bmatrix} K & 0_{m \times (n-p)} \\ 0_{(n-m) \times p} & 0_{(n-m) \times (n-p)} \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} \bar{C} \\ 0_{(n-p) \times n} \end{bmatrix}, \quad \mathcal{Z} = \text{diag}\{Z, \dots, Z, I_n\} \in \mathbb{R}^{11n \times 11n}, \quad \mathcal{G} = \text{diag}\{\tilde{G}, \dots, \tilde{G}\} \in \mathbb{R}^{11n \times 11n} \\ \tilde{B} &= [\bar{B} \quad 0_{n \times (n-m)}] \in \mathbb{R}^{n \times n}, \quad \tilde{B}_2 = [\bar{B}_2 \quad 0_{n \times (n-m)}] \in \mathbb{R}^{n \times n}, \quad \tilde{D} = [\bar{D} \quad 0_{n \times (n-k)}] \in \mathbb{R}^{n \times n} \\ \mathcal{A}_c^T &= [\bar{A} \quad \bar{A}_1 \quad 0_{n \times 7n} \quad -I_n \quad \tilde{D}] \in \mathbb{R}^{n \times 11n}, \quad \mathcal{B}_c^T = [\tilde{B} \quad 0_{n \times 2n} \quad \tilde{B}_2 \quad 0_{n \times 7n}] \in \mathbb{R}^{n \times 11n} \\ \mu_1(t) &= \frac{1}{\tau_1(t)} \int_{t-\tau_1(t)}^t z^T(s)ds, \quad \mu_2(t) = \frac{1}{\tau_{1M} - \tau_1(t)} \int_{t-\tau_{1M}}^{t-\tau_1(t)} z^T(s)ds \\ \mu_3(t) &= \frac{1}{\tau_2(t)} \int_{t-\tau_2(t)}^t z^T(s)ds, \quad \mu_4(t) = \frac{1}{\tau_{2M} - \tau_2(t)} \int_{t-\tau_{2M}}^{t-\tau_2(t)} z^T(s)ds \\ F &= Z^{-1}, \quad \tilde{\omega}(t) = [\omega^T(t) \quad 0_{1 \times (n-k)}]^T \\ \xi^T(t) &= [z^T(t)F^T \quad z^T(t - \tau_1(t))F^T \quad z^T(t - \tau_{1M}(t))F^T \quad z^T(t - \tau_2(t))F^T \quad z^T(t - \tau_{2M}(t))F^T \\ &\quad \mu_1(t)F^T \quad \mu_2(t)F^T \quad \mu_3(t)F^T \quad \mu_4(t)F^T \quad \dot{z}(t)F^T \quad \tilde{\omega}^T(t)] \in \mathbb{R}^{1 \times 11n} \\ \zeta_0^T(t) &= [z^T(t)F^T \quad \int_{t-\tau_{1M}}^t z^T(s)F^Tds \quad \int_{t-\tau_{2M}}^t z^T(s)F^Tds] \in \mathbb{R}^{1 \times 3n} \\ e_i &= [0_{n \times (i-1)n} \quad I_n \quad 0_{n \times (11-i)n}]^T, \quad \text{for } i = 1, \dots, 11 \\ \rho(t) &= [e_1 \quad \tau_1(t)e_6 + (\tau_{1M} - \tau_1(t))e_7 \quad \tau_2(t)e_8 + (\tau_{2M} - \tau_2(t))e_9] \in \mathbb{R}^{11n \times 3n} \\ \Gamma_1 &= [e_1 - e_2 \quad \sqrt{3}(e_1 + e_2 - 2e_6) \quad e_2 - e_3 \quad \sqrt{3}(e_2 + e_3 - 2e_7)] \in \mathbb{R}^{11n \times 4n} \\ \Gamma_2 &= [e_1 - e_4 \quad \sqrt{3}(e_1 + e_4 - 2e_8) \quad e_4 - e_5 \quad \sqrt{3}(e_4 + e_5 - 2e_9)] \in \mathbb{R}^{11n \times 4n} \\ \tilde{R}_1 &= \begin{bmatrix} R_1 & 0 \\ 0 & R_1 \end{bmatrix}, \quad \tilde{R}_2 = \begin{bmatrix} R_2 & 0 \\ 0 & R_2 \end{bmatrix} \text{ and } \Upsilon_j = \frac{1}{b_j^2}H^TL_j^TL_jH, \quad j = 1, \dots, q \end{aligned}$$

$$\begin{aligned}
\dot{V}_2 + 2\delta V_2 &= z^T(t)F^T(Q_1 + S_1 + Q_2 + S_2)Fz(t) \\
&\quad - e^{-2\delta\tau_{1M}}z^T(t - \tau_{1M})F^TQ_1Fz(t - \tau_{1M}) \\
&\quad - e^{-2\delta\tau_1(t)}(1 - \dot{\tau}_1(t))z^T(t - \tau_1(t)) \\
&\quad \times F^TS_1Fz(t - \tau_1(t)) \\
&\quad - e^{-2\delta\tau_{2M}}z^T(t - \tau_{2M})F^TQ_2Fz(t - \tau_{2M}) \\
&\quad - e^{-2\delta\tau_2(t)}(1 - \dot{\tau}_2(t))z^T(t - \tau_2(t)) \\
&\quad \times F^TS_2Fz(t - \tau_2(t)) \\
&\leq \xi^T(t)\{e_1(Q_1 + S_1 + Q_2 + S_2)e_1^T \\
&\quad - e^{-2\delta\tau_{1M}}e_3Q_1e_3^T - e^{-2\delta\tau_{2M}}e_5Q_2e_5^T \\
&\quad - e^{-2\delta\tau_{1M}}(1 - d_{1M})e_2S_1e_2^T \\
&\quad - e^{-2\delta\tau_{2M}}(1 - d_{2M})e_4S_2e_4^T\}\xi(t) \quad (18)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_3 + 2\delta V_3 &= \tau_{1M}^2\dot{z}^T(t)F^TR_1F\dot{z}(t) \\
&\quad - \tau_{1M}\int_{t-\tau_{1M}}^te^{2\delta(s-t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds \\
&\quad + \tau_{2M}^2\dot{z}^T(t)F^TR_2F\dot{z}(t) \\
&\quad - \tau_{2M}\int_{t-\tau_{2M}}^te^{2\delta(s-t)}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds \\
&\leq \xi^T(t)\{e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T \\
&\quad - \tau_{1M}e^{-2\delta\tau_{1M}}\left\{\int_{t-\tau_1(t)}^t\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\right. \\
&\quad \left.+ \int_{t-\tau_{1M}}^{t-\tau_1(t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds\right\} \\
&\quad - \tau_{2M}e^{-2\delta\tau_{2M}}\left\{\int_{t-\tau_2(t)}^t\dot{z}^T(s)F^TR_2F\dot{z}(s)ds\right. \\
&\quad \left.+ \int_{t-\tau_{2M}}^{t-\tau_2(t)}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds\right\}\} \quad (19)
\end{aligned}$$

Using Lemma 2, we obtain the following estimation

$$\begin{aligned}
&-\int_{t-\tau_1(t)}^t\dot{z}^T(s)F^TR_1F\dot{z}(s)ds \\
&\leq -\frac{1}{\tau_1(t)}(z(t) - z(t - \tau_1(t)))^T \\
&\quad \times F^TR_1F(z(t) - z(t - \tau_1(t))) \\
&\quad - \frac{12}{\tau_1(t)}\left(\frac{z(t)}{2} + \frac{z(t - \tau_1(t))}{2} - \frac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^tz(s)ds\right)^T \\
&\quad \times F^TR_1F\left(\frac{z(t)}{2} + \frac{z(t - \tau_1(t))}{2} - \frac{1}{\tau_1(t)}\int_{t-\tau_1(t)}^tz(s)ds\right) \\
&= -\xi^T(t)\frac{1}{\tau_1(t)}\{[e_1 - e_2]R_1[e_1 - e_2]^T \\
&\quad + 3[e_1 + e_2 - 2e_6]R_1[e_1 + e_2 - 2e_6]^T\}\xi(t) \quad (20)
\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}
&-\int_{t-\tau_{1M}}^{t-\tau_1(t)}\dot{z}^T(s)F^TR_1F\dot{z}(s)ds \\
&\leq -\xi^T(t)\frac{1}{\tau_{1M} - \tau_1(t)}\{[e_2 - e_3]R_1[e_2 - e_3]^T \\
&\quad + 3[e_2 + e_3 - 2e_7]R_1[e_2 + e_3 - 2e_7]^T\}\xi(t) \quad (21)
\end{aligned}$$

$$\begin{aligned}
&-\int_{t-\tau_2(t)}^t\dot{z}^T(s)F^TR_2F\dot{z}(s)ds \\
&\leq -\xi^T(t)\frac{1}{\tau_2(t)}\{[e_1 - e_4]R_2[e_1 - e_4]^T \\
&\quad + 3[e_1 + e_4 - 2e_8]R_2[e_1 + e_4 - 2e_8]^T\}\xi(t) \quad (22)
\end{aligned}$$

and

$$\begin{aligned}
&-\int_{t-\tau_{2M}}^{t-\tau_2(t)}\dot{z}^T(s)F^TR_2F\dot{z}(s)ds \\
&\leq -\xi^T(t)\frac{1}{\tau_{2M} - \tau_2(t)}\{[e_4 - e_5]R_2[e_4 - e_5]^T \\
&\quad + 3[e_4 + e_5 - 2e_9]R_2[e_4 + e_5 - 2e_9]^T\}\xi(t) \quad (23)
\end{aligned}$$

Adding (19)–(23), using (13) and Lemma 3, we obtain

$$\begin{aligned}
\dot{V}_3 + 2\delta V_3 &\leq \xi^T(t)\{e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T \\
&\quad - e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T\}\xi(t) \quad (24)
\end{aligned}$$

Combining (10) with the free-weighting matrix technique [29, 30], we have

$$2\xi^T(t)(e_1 + \lambda e_{10})[A_c^T\mathcal{Z} + B_c^T\mathcal{G}]\xi(t) = 0 \quad (25)$$

By adding (17), (18), (24) and (25), we obtain

$$\begin{aligned}
\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\omega^2}\tilde{\omega}^T(t)\tilde{\omega}(t) &\leq \xi^T(t)\Sigma(\tau_1, \tau_2, \delta)\xi(t) \\
&\quad (26)
\end{aligned}$$

By some simple computations, we can verify that

$$\left[\frac{\partial^2}{\partial\tau_i^2}\Sigma(\tau_1, \tau_2, \delta)\right] \geq 0, \quad i = 1, 2 \quad (27)$$

Consequently, $\Sigma(\tau_1, \tau_2, \delta)$ is convex with respect to τ_1 and τ_2 . Hence, if condition (14) holds then we have

$$\dot{V}(t) + 2\delta V(t) - 2\frac{\delta}{\omega^2}\tilde{\omega}^T(t)\tilde{\omega}(t) \leq 0, \quad \forall t \geq 0 \quad (28)$$

This follows that $\limsup_{t \rightarrow \infty} V(t) \leq 1$ due to Lemma 1. On the other hand, using (11) and (12), we have

$$z^T(t)\Upsilon_jz(t) \leq z^T(t)F^TP_jFz(t) \leq V(t), \quad j = 1, \dots, q$$

This implies that inequality (9) holds. The proof of Theorem 1 is completed. \square

Remark 1: Note that for each $j = 1, \dots, q$, matrix inequality (12) is a quadratic matrix inequality. By using singular value decomposition technique, we can reduce matrix inequalities (12) to linear but more conservative matrix inequalities. Indeed, for each $j = 1, \dots, q$, assuming that $[U_j, Y_j, V_j]$ is a singular value decomposition of matrix L_jH ,

then $H^T L_j^T L_j H = V_j Y_j^T Y_j V_j^T$. Since $L_j H \in \mathbb{R}^{1 \times n}$ is a non-zero matrix, matrix Y_j has a form $Y_j = [s_j \ 0 \ \dots \ 0] \in \mathbb{R}^{1 \times n}$, where s_j is a non-zero scalar. This implies that

$$\Upsilon_j = V_j \begin{bmatrix} \frac{s_j^2}{b_j^2} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} V_j^T$$

For a small positive scalar $\epsilon > 0$, we denote

$$\Upsilon_j^\epsilon = V_j \begin{bmatrix} \frac{s_j^2}{b_j^2} & 0 & 0 & \dots & 0 \\ 0 & \epsilon & 0 & \dots & 0 \\ 0 & 0 & \epsilon & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \epsilon \end{bmatrix} V_j^T$$

Then Υ_j^ϵ is positive-definite matrix and $\Upsilon_j^\epsilon \geq \Upsilon_j$. Hence, matrix inequalities (12) can be replaced by more conservative matrix inequalities $P_j - Z^T \Upsilon_j^\epsilon Z > 0, j = 1, \dots, q$. These matrix inequalities are equivalent to the following LMIs because of Lemma 4

$$\begin{bmatrix} P_j & Z^T \\ Z & (\Upsilon_j^\epsilon)^{-1} \end{bmatrix} > 0, \quad j = 1, \dots, q \quad (29)$$

Also note that matrix inequality (14) cannot be simplified into LMI. However, when λ and δ are fixed, then (14) is reduced to LMI. Therefore, we can now use a 2D search method combining with convex optimisation algorithms such as the Matlab's LMI toolbox [31] to solve matrix inequalities (11), (29), (13) and (14). Note that the two parameters λ and δ are independent, hence in practice one can use parallel computing to find the two feasible parameters. Furthermore, parameter δ is the exponential rate, therefore it is positive and finite, that is, it belongs to an interval. This helps to reduce partly the difficulty in searching for the two feasible parameters. On the other hand, the appropriate combination between a state transformation (7) with the choice of a special form of matrices \tilde{G} and Z allows us to obtain an output feedback control matrix $K = GZ_{11}^{-1}$.

Remark 2: Since the two time-varying delays considered in this paper are independent, the Lyapunov–Krasovskii functional (16) must be constructed by using different matrices for each delay. In this paper, we use the Wirtinger integral and the reciprocally convex combination inequality, which are known as recent effective techniques with moderate variables. Therefore, the number of variables in our derived conditions is moderate. However, the number of variables can be reduced for the following three special cases: (i) $\tau_1(t) \equiv \tau_2(t)$; (ii) $\tau_{1M} = \tau_{2M}$; and (iii) $\tau_1(t), \tau_2(t)$ are non-differentiable or their derivatives are unknown. For case (i), we let $Q_1 = Q_2, R_1 = R_2$ and $S_1 = S_2$. For case (ii), we let $Q_1 = Q_2, R_1 = R_2$. Finally, for case (iii), we let $S_1 = S_2 = 0$.

Remark 3: Note that $\Sigma(\tau_1, \tau_2, \delta)$ is convex with respect to τ_1 and τ_2 . It follows that if the condition (14) holds for $\forall(\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\}$ then it also holds for $\forall(\tau_1, \tau_2) \in \{0, \tilde{\tau}_{1M}\} \times \{0, \tilde{\tau}_{2M}\}$ where $\tilde{\tau}_{1M} \leq \tau_{1M}$ and $\tilde{\tau}_{2M} \leq \tau_{2M}$. This means that the condition (14) is monotonic increasing with respect to the delays' bounds τ_{1M} and

τ_{2M} . So, we can use a 2D search to calculate the maximum allowable values of the delays' bounds τ_{1M} and τ_{2M} .

Remark 4: The assumption that the derivatives of the time-varying delays are less than one is usually referred to as slow time-varying delays. For the case where the time-delays are non-differentiable or their derivatives are unknown, then this assumption is not needed and can be removed. By letting $S_1 = S_2 = 0$ and by following the same lines as in the proof of Theorem 1, we can obtain a similar result. Note that this result is more conservative than the one derived with the assumption that the derivatives of the time-varying delays are less than one.

Remark 5: For the case where the initial condition is zero, then $V(0) = 0$. Consequently, from the proof of Lemma 1, we have $V(t) \leq 1, \forall t \geq 0$. Similar to the proof of Theorem 1, we obtain

$$|L_j H z(t, 0)| \leq b_j, \forall t \geq 0, \quad \forall j = 1, \dots, q$$

Hence, the condition stated in Theorem 1 guarantees that all reachable sets of the closed-loop system (8) are bounded by the polyhedron Ω for all time.

For the case where the system (1) does not have any disturbance, by setting $\omega(t) \equiv 0$ and $D = 0$, then the convergence condition in Theorem 1 is reduced to a static output feedback exponential stabilisability condition for system (1). Here, let us recall the definition of exponential stabilisability of system (1).

Definition 1: Given a positive scalar $\delta > 0$, system (1) without any disturbance is δ -‘stabilisable’ with a static output feedback controller (4) if every solution $x(t, \phi)$ of the closed-loop system (5) satisfies

$$\exists N > 0 : \|x(t, \phi)\| \leq N \|\phi\| e^{-\delta t}, \quad \forall t \geq 0 \quad (30)$$

The positive scalars δ and N are called the convergence rate and the stability factor, respectively.

Remark 6: From the state transformation (7), it is easy to see that

$$\lambda_1 \|x(t)\|^2 \leq \|z(t)\|^2 \leq \lambda_2 \|x(t)\|^2 \quad (31)$$

where $\lambda_1 = \lambda_{\min}((H^{-1})^T (H^{-1}))$ and $\lambda_2 = \lambda_{\max}((H^{-1})^T (H^{-1}))$. This implies that if system (8) is δ -stabilisable with stability factor N then system (1) is also δ -stabilisable with stability factor $\sqrt{(\lambda_2/\lambda_1)} \|H^{-1}\| N$. Therefore, to study δ -stabilisability for system (1), we only need to study δ -stabilisability for system (8).

Let us denote that

$$\begin{aligned} \Xi(\tau_1, \tau_2, \delta) = & 2\rho(t)P[e_{10} \ e_1 - e_3 \ e_1 - e_5]^T \\ & + 2\delta\rho(t)P\rho^T(t) + e_1(Q_1 + S_1 + Q_2 + S_2)e_1^T \\ & - e^{-2\delta\tau_{1M}}e_3Q_1e_3^T - e^{-2\delta\tau_{1M}}(1 - d_{1M})e_2S_1e_2^T \\ & - e^{-2\delta\tau_{2M}}e_5Q_2e_5^T - e^{-2\delta\tau_{2M}}(1 - d_{2M})e_4S_2e_4^T \\ & + e_{10}(\tau_{1M}^2R_1 + \tau_{2M}^2R_2)e_{10}^T \\ & - e^{-2\delta\tau_{1M}}\Gamma_1\Theta_1\Gamma_1^T - e^{-2\delta\tau_{2M}}\Gamma_2\Theta_2\Gamma_2^T \\ & + (e_1 + \lambda e_{10})(\mathcal{A}_c^T Z + \mathcal{B}_c^T \mathcal{G} + Z^T \mathcal{A}_c + \mathcal{G}^T \mathcal{B}_c) \end{aligned}$$

Similarly, we also obtain a sufficient condition for δ -stabilisability of system (8) via a static output feedback controller (4) as follows.

Theorem 2: For a given positive scalar $\delta > 0$, if there exist a scalar λ , a positive-definite $3n \times 3n$ -matrix P , six positive-definite $n \times n$ -matrices $Q_1, Q_2, S_1, S_2, R_1, R_2$, two $2n \times 2n$ -matrices X_1, X_2 , two non-singular matrices $Z_{11} \in \mathbb{R}^{p \times p}, Z_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$ and two matrices $Z_{21} \in \mathbb{R}^{(n-p) \times p}, G \in \mathbb{R}^{m \times p}$ such that condition (13) and the following matrix inequality hold

$$\Xi(\tau_1, \tau_2, \delta) \leq 0, \quad \forall (\tau_1, \tau_2) \in \{0, \tau_{1M}\} \times \{0, \tau_{2M}\} \quad (32)$$

then system (8) without any disturbance is δ -stabilisable. The static output feedback controller is $u(t) = GZ_{11}^{-1}y(t)$. Moreover, every solution of the closed-loop system satisfies

$$\|z(t, \varphi)\| \leq \sqrt{\frac{\beta_2}{\beta_1}} \|\varphi\|_c e^{-\delta t}, \quad \forall t \geq 0 \quad (33)$$

where

$$\begin{aligned} \beta_1 &= \lambda_{\min}(\text{diag}\{F^T, F^T, F^T\} \times P \times \text{diag}\{F, F, F\}) \\ \beta_2 &= (1 + \tau_{1M}^2 + \tau_{2M}^2) \lambda_{\max}(\text{diag}\{F^T, F^T, F^T\} \\ &\quad \times P \times \text{diag}\{F, F, F\}) \\ &\quad + \tau_{1M} \lambda_{\max}(F^T(Q_1 + S_1)F) \\ &\quad + \tau_{2M} \lambda_{\max}(F^T(Q_2 + S_2)F) \\ &\quad + \frac{\tau_{1M}^3}{2} \lambda_{\max}(F^T R_1 F) + \frac{\tau_{2M}^3}{2} \lambda_{\max}(F^T R_2 F) \end{aligned}$$

Proof: Also consider the Lyapunov–Krasovskii functional (16) and similarly, we obtain

$$V(t) \leq V(0)e^{-2\delta t}, \quad \forall t \geq 0 \quad (34)$$

Denoting $\zeta_1^T(t) = [z^T(t) \int_{t-\tau_{1M}}^t z^T(s) ds \int_{t-\tau_{2M}}^t z^T(s) ds]$. Note that for all $s \in [-h, 0]$, we have $\|z(t+s)\|^2 \leq \|z_t\|_c^2$ and $\|\dot{z}(t+s)\|^2 \leq \|z_t\|_c^2$. By some computations, we have

$$\begin{aligned} \|\zeta_1(t)\|^2 &\leq \|z(t)\|^2 + \tau_{1M} \int_{t-\tau_{1M}}^t \|z(s)\|^2 ds \\ &\quad + \tau_{2M} \int_{t-\tau_{2M}}^t \|z(s)\|^2 ds \\ &\leq (1 + \tau_{1M}^2 + \tau_{2M}^2) \|z_t\|_c^2 \end{aligned} \quad (35)$$

and

$$\|z(t)\|^2 \leq \|\zeta_1(t)\|^2 \quad (36)$$

Combining (34), (35) with (36), we obtain the following inequality

$$\beta_1 \|z(t)\|^2 \leq \beta_1 \|\zeta_1(t)\|^2 \leq V(t) \leq \beta_2 \|z_t\|_c^2, \quad \forall t \geq 0 \quad (37)$$

which implies inequality (33). This completes the proof of Theorem 2. \square

Remark 7 (minimisation stability factor): From (37), the stability factor is $N = \sqrt{(\beta_2/\beta_1)}$. When matrices P and F are found, we can further find a scalar $\alpha_1 \geq \beta_1$ such that

$\alpha_1 \|z(t)\|^2 \leq V(t)$. To find α_1 , we can use an 1D search method for the following inequality

$$\alpha_1 \begin{bmatrix} I_n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \leq \text{diag}\{F^T, F^T, F^T\} \times P \times \text{diag}\{F, F, F\} \quad (38)$$

Hence, the stability factor N can be reduced to a minimal one $N_1 = \sqrt{(\beta_2/\alpha_1)}$. In Example 3 of the next section, we show that N_1 is smaller than N .

Remark 8: Consider an extended system of (1) as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1 x(t - \tau_1(t)) + A_2 x(t - \tau_2(t)) \\ &\quad + Bu(t) + B_2 u(t - \tau_2(t)) \end{aligned} \quad (39)$$

By re-notating $\mathcal{A}_c^T = [\bar{A} \quad \bar{A}_1 \quad 0_{n \times n} \quad \bar{A}_2 \quad 0_{n \times 5n} - I_n \quad 0_{n \times n}] \in \mathbb{R}^{n \times 11n}$, with $\bar{A}_2 = H^{-1}A_2H$, then the result in Theorem 2 also gives a δ -stabilisability criterion for system (39) via a static output feedback controller (4). Note that the authors in [3–5, 7] only considered the case where in system (39), $A_1 = 0$ and τ_2 is a constant time delay (i.e. only one constant time delay in both the state and the control input).

Remark 9: Assume that the matrix inequality $\Xi(\tau_1, \tau_2, 0) < 0$ holds. Since $\rho(t)$ is bounded, we can choose a small enough scalar $\delta_0 > 0$ such that $\Xi(\tau_1, \tau_2, \delta_0) < 0$. Hence, we have an asymptotic stabilisability criterion for system (8) via static output feedback controller (4) as given in the following corollary.

Corollary 1: System (8) without any disturbance is asymptotically stabilisable via a static output feedback controller (4) if $\Xi(\tau_1, \tau_2, 0) < 0$ and (13) hold.

4 Numerical examples

In this section, we give five examples to illustrate the feasibility and effectiveness of our results on static output feedback control for two cases: (i) in the presence of bounded disturbances; and (ii) no disturbances. For case (i), we will design a static output feedback controller, which guarantees the state vector of the closed-loop system converges within a pre-specified polyhedron Ω (Examples 1 and 2). For case (ii), we will design a static output feedback controller which guarantees δ -stability of the closed-loop system (Examples 3–5).

Example 1 (convergence condition): Consider system (1) in the presence of disturbances $\omega(t)$, which is bounded by $\bar{\omega} = 0.3$ and

$$\begin{aligned} A &= \begin{bmatrix} -1 & 0 & 0.3 & 0 \\ -0.1 & 0.2 & 1 & 0 \\ -0.3 & 0.1 & -2 & 0.2 \\ 0 & 0 & 0 & -1.2 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} -2 & -0.1 & 0 & -0.2 \\ -0.2 & 0.3 & 0.3 & 0 \\ 0.1 & 0 & -2 & -0.2 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \\ B &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

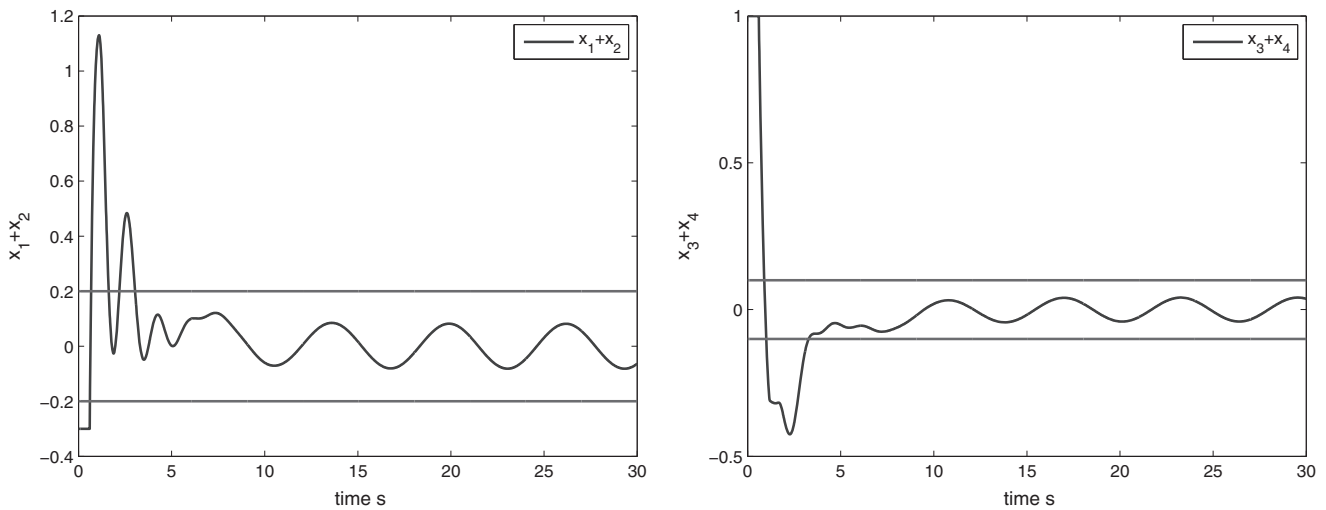


Fig. 2 Trajectories of $x_1(t) + x_2(t)$ and $x_3(t) + x_4(t)$ of the closed-loop system

$$C = \begin{bmatrix} -1 & -1 & 2 & 0.2 \\ 0.2 & 1 & 0.3 & 1 \end{bmatrix}$$

The two time-varying delays, $\tau_1(t)$ and $\tau_2(t)$ satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 0.5, & \dot{\tau}_1(t) \leq 0.05 \\ 0 \leq \tau_2(t) \leq 0.6, & \dot{\tau}_2(t) \leq 0.05 \end{cases} \quad (40)$$

Given a polyhedron $\Omega = \{x \in \mathbb{R}^4 : |L_j x| \leq b_j, j = 1, 2\}$ where $L_1 = [1 \ 1 \ 0 \ 0]$, $L_2 = [0 \ 0 \ 1 \ 1]$, $b_1 = 0.2$ and $b_2 = 0.1$. We design a static output feedback controller, which guarantees the state vector of the closed-loop system converges within the given polyhedron Ω .

By solving the LMIs (11), (13), (14) and (29) with $\epsilon = 0.01$ and two parameters need to be turned δ and λ , we obtain $\delta = 0.1$, $\lambda = 0.48$ and a static output feedback control matrix $K = [0.3581 \ -0.7808]$. For a disturbance $\omega(t) = 0.3 \sin(t)$, two time-varying delays $\tau_1(t) = 0.5 \sin^2(\frac{t}{10})$ and $\tau_2(t) = 0.6 \sin^2(\frac{t}{12})$, Fig. 2 shows that the trajectory of $L_1 x(t) = x_1(t) + x_2(t)$ of the closed-loop system converges within the specified 0.2-bound, and $L_2 x(t) = x_3(t) + x_4(t)$ converges within the specified 0.1-bound. Also, Fig. 3 shows that the vector $(L_1 x(t), L_2 x(t))$ converges within the rectangular with dimensions 0.4×0.2 .

Example 2 (convergence condition): Consider a 3D system (1) with disturbances $\omega(t)$ is bounded by $\bar{\omega} = 0.2$ and

$$\begin{aligned} A &= \begin{bmatrix} -0.8 & 0.1 & 0.5 \\ -0.1 & 0.2 & 1 \\ -0.3 & 0.1 & -2 \end{bmatrix}, \\ A_1 &= \begin{bmatrix} 0.1 & -0.1 & 0 \\ -0.2 & 0.3 & 0.3 \\ 0.1 & 0 & 1.5 \end{bmatrix} \\ B &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

The two time-varying delays $\tau_1(t)$ and $\tau_2(t)$ satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 1, & \dot{\tau}_1(t) \leq 0.1 \\ 0 \leq \tau_2(t) \leq 0.6, & \dot{\tau}_2(t) \leq 0.1 \end{cases} \quad (41)$$

Given a box $\Omega = \{x \in \mathbb{R}^3 : |x_j| \leq b_j, j = 1, 2, 3\}$ where $b_1 = b_2 = 0.2$, $b_3 = 0.05$. We design a static output feedback controller which guarantees the state vector of the closed-loop system converges within the given box Ω .

By solving the LMIs (11), (13), (14) and (29) with $\epsilon = 0.01$ and two parameters need to be turned δ and λ , we obtain $\delta = 0.05$, $\lambda = 0.23$ and a static output feedback control matrix $K = [-0.8316 \ -2.3089]$. For a disturbance $\omega(t) = 0.2 \sin(t)$, Fig. 4 shows that the trajectories of the closed-loop system converges within the given box Ω .

Example 3 (static output feedback control): Consider the system in Example 1, where there are no disturbances, that is, $\omega(t) \equiv 0$, and two time-varying delays in both the state and input satisfying

$$\begin{cases} 0 \leq \tau_1(t) \leq 0.5, & \dot{\tau}_1(t) \leq 0.1 \\ 0 \leq \tau_2(t) \leq \tau_{2M}, & \dot{\tau}_2(t) \leq 0.1 \end{cases} \quad (42)$$

In this example, we find the maximal allowable delay τ_{2M} such that the system is 0.1-stabilisable via a static output feedback controller.

By using Theorem 2 with a pre-specified convergence rate $\delta = 0.1$, the allowable value of τ_{2M} is found to be 1.47. The output feedback control matrix and parameter are $K = [0.2117 \ -0.3928]$ and $\lambda = 1.14$, respectively. By Theorem 2 and Remark 7, the stability factor is $N = 44.6578$ and the minimal value is $N_1 = 13.7191$. Moreover, we have $\sqrt{\frac{\lambda_2}{\lambda_1}} \|H^{-1}\| = 6.0805$, which implies the following estimation

$$\begin{aligned} \|x(t)\| &\leq \left(\sqrt{\frac{\lambda_2}{\lambda_1}} \|H^{-1}\| \right) N_1 \|\varphi\|_c e^{-0.1t} \\ &\leq 83.4191 \|\varphi\|_c e^{-0.1t}, \quad \forall t \geq 0 \end{aligned}$$

Fig. 5 shows trajectories of the closed-loop system where two time-varying delays are chosen as $\tau_1(t) = 0.5 \sin^2(\frac{t}{5})$ and $\tau_2(t) = 1.47 \sin^2(\frac{t}{14.7})$.

Example 4 (state feedback control): Consider system (1), which was studied in [2], with two unknown constant delays

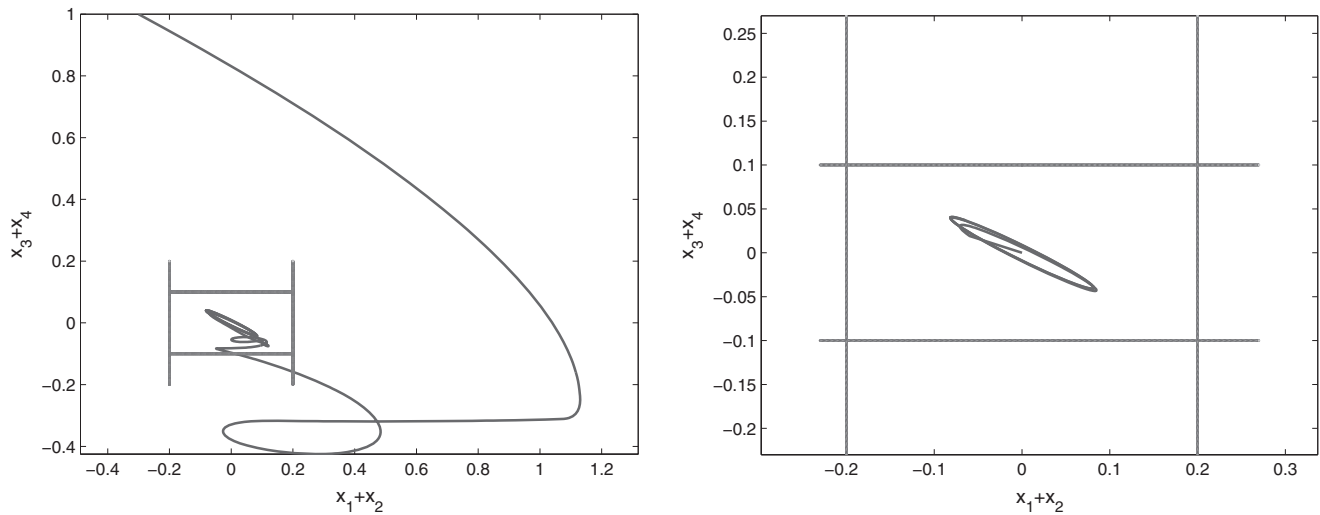


Fig. 3 Trajectory of $L_1x(t)$, $L_2x(t)$ converges within a 0.4×0.2 rectangular

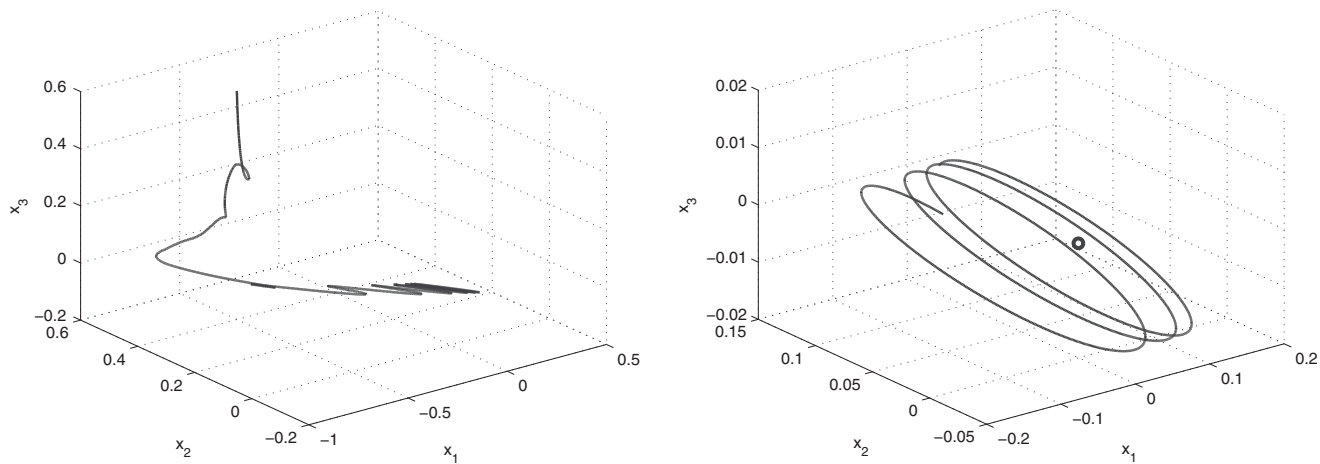


Fig. 4 Trajectory of the closed-loop system, $x(t)$, converges within the given box $\Omega = \{x \in \mathbb{R}^3 : |x_j| \leq b_j, j = 1, 2, 3\}$

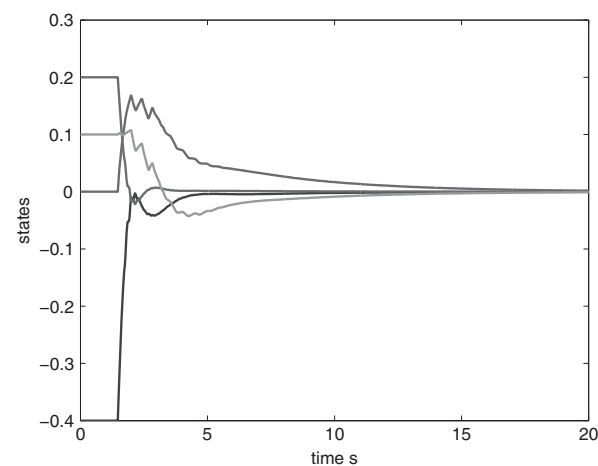


Fig. 5 Trajectories of the closed-loop system in Example 3

in the state and input and

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ -0.5 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -2 & -0.5 & 0 & 0 \\ -0.2 & -1 & 0 & 0 \\ 0.5 & 0 & -2 & -0.5 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

By using Corollary 1 with $\tau_{2M} = 0.1$, the allowable value of τ_{1M} , which ensures system is asymptotically stabilisable, is 0.77, while Theorem 4 in [2] provided a smaller value, 0.56. The state feedback control matrix and parameter are obtained as $K = [-5.0329 \ -1.9171 \ 1.5028 \ -0.4175]$ and $\lambda = 1.42$. Note that the approaches in [3–5, 7] are available for linear systems with only one delay and the approach in [8] is available for linear systems without instantaneous input (i.e. $B = 0$). Therefore, the approaches [3–5, 7, 8] cannot be applied to this example.

Example 5 (state feedback control): Consider a pendulum system (1), which was studied in [7], with time delay τ_2 is an unknown constant and $A_1 = A_2 = 0$, $B = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -21.54 & 0 & 14.96 & 0 \\ 0 & 0 & 0 & 1 \\ 65.28 & 0 & -15.59 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 8.10 \\ 0 \\ -10.31 \end{bmatrix}$$

Table 1 Computed upper bounds, τ_{2M} , for Example 5

Methods	τ_{2M}	Improvement, %
Fridman <i>et al.</i> [35]	0.0384	100
Du <i>et al.</i> [7]	0.0768	200
Theorem 2	0.2130	554

In this example, the allowable values for τ_{2M} are derived in Table 1. The state feedback control matrix and parameter are $K = [-4.9687 \ -1.4262 \ -2.7016 \ -0.7382]$ and $\lambda = 0.86$.

5 Conclusion

The paper has considered the problem of designing a static output feedback controller for linear systems with state/input time-varying delays and bounded disturbances. A new sufficient condition for the existence of a static output feedback controller, which guarantees the state vector of the closed-loop system converges within a pre-specified polyhedron, has been derived. For the case where the disturbances are not present, the derived convergence condition is reduced to a static output feedback exponential stabilisability condition. Five numerical examples have been given to illustrate the feasibility and the effectiveness of the obtained results.

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7 References

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