



# A class of damping models preserving eigenspaces for linear conservative port-Hamiltonian systems<sup>☆</sup>

Denis Matignon<sup>a,\*</sup>, Thomas H  lie<sup>b,1</sup>

<sup>a</sup> University of Toulouse, ISAE; 10, av. E. Belin; BP 54032. 31055 Toulouse Cedex 4, France

<sup>b</sup> IRCAM & CNRS, UMR 9912, 1, pl. Igor Stravinsky, 75004 Paris, France

## ARTICLE INFO

### Article history:

Received 1 February 2013

Accepted 2 October 2013

Recommended by A. Astolfi

Available online 18 October 2013

### Keywords:

Energy storage

Port-Hamiltonian systems

Eigenfunctions

Damping

Caughey series

Partial differential equations

Fractional Laplacian

## ABSTRACT

For conservative mechanical systems, the so-called Caughey series are known to define the class of damping matrices that preserve eigenspaces. In particular, for finite-dimensional systems, these matrices prove to be a polynomial of one reduced matrix, which depends on the mass and stiffness matrices. Damping is ensured whatever the eigenvalues of the conservative problem if and only if the polynomial is positive for positive scalar values.

This paper first recasts this result in the port-Hamiltonian framework by introducing a port variable corresponding to internal energy dissipation (resistive element). Moreover, this formalism naturally allows to cope with systems including gyroscopic effects (gyrators).

Second, generalizations to the infinite-dimensional case are considered. They consist of extending the previous polynomial class to rational functions and more general functions of operators (instead of matrices), once the appropriate functional framework has been defined. In this case, the resistive element is modelled by a given static operator, such as an elliptic PDE. These results are illustrated on several PDE examples: the Webster horn equation, the Bernoulli beam equation; the damping models under consideration are fluid, structural, rational and generalized fractional Laplacian or bi-Laplacian.

  2013 European Control Association. Published by Elsevier Ltd. All rights reserved.

## 1. Introduction

In this paper, the idea is to find and even to parametrize damping models of discrete systems (or ODEs) and continuous systems (or PDEs), which leave the eigenvectors or eigenfunctions unaffected by the damping: only the eigenvalues are shifted. To this end, in 1896, Lord Rayleigh, [26], introduced damping models named after him, which are nothing but a first order polynomial in both the mass and stiffness matrices. But the pioneering works by Caughey in 1960, shortly followed by Caughey and O'Kelly in 1965 showed a more general result: it is the structure of the *commutant* of the two matrices, or two operators, which play a central role in the theory. Hence, not only polynomials of this compound matrix prove admissible, but also series of this matrix, whence the famous *Caughey series*.

The main idea of the work is to take advantage of the port-Hamiltonian framework, see e.g. [29], and [9, Chapters 2 and 4] for a guided tour, to treat this question, and see how Caughey damping, either polynomials, rational functions, or even more general functions, can fit into it. The extension to systems of PDEs will be looked at with simple examples as well as more technically involved worked-out examples.

The outline of the paper is as follows: in Section 2, a general second order  $n$ -d.o.f mechanical system is studied, with a quite general damping matrix, we first put it into the port-Hamiltonian framework, in order to introduce both the skew-symmetric and symmetric structural matrices  $J$  and  $R$ . We first recall the definition of port-Hamiltonian systems with dissipation and the extension of the framework with resistive ports. We then concentrate on the properties for the G-part of the damping, responsible for the so-called gyroscopic effects. Then, we give the desirable properties for the C-part of damping, in order to follow the so-called *Basile hypothesis* that is the damped system still has classical normal modes. The nice sufficient condition by Caughey, back to 1960, gives rise to polynomial of matrices, is then easily put in the pH framework with external port variables linked by a closure relation. The general result, a necessary and sufficient condition, made more precise in 1965, is fully recalled, and examined in the case of rational functions and more general functions of matrices, provided that a positivity constraint is fulfilled.

<sup> </sup>A first version of this work has been presented at the IFAC Conference on Lagrangian and Hamiltonian Methods for Nonlinear Control, LHMNLC 2012, see [24].

\* Corresponding author. Tel.: +33 561338112.

E-mail addresses: [denis.matignon@isae.fr](mailto:denis.matignon@isae.fr) (D. Matignon), [thomas.helie@ircam.fr](mailto:thomas.helie@ircam.fr) (T. H  lie).

<sup>1</sup> The contribution of both authors has been done within the context of the French National Research Agency sponsored project HAMECMOPSY. Further information is available at <http://www.hamecmopsys.ens2m.fr/>.

In Section 3, we turn to the PDE case, and try to follow the same approach as before: it turns out that the commutation of operators (including the boundary conditions in their domain) happens to be the key point of the result, as first mentioned by the pioneering work by Caughey and O'Kelly in 1965: thus, we extend Rayleigh damping models to Caughey type operators, which amount to polynomials, rational functions or even more general functions (such as *fractional* powers) of a compound operator: this can be treated seriously e.g. in the case of unbounded operators with compact resolvent that are coercive and self-adjoint; a nice example of those is provided by the coupling with an elliptic PDE. In this Section, a focus is made on worked-out examples such as the Webster wave equation (that allow for space-varying coefficients), and also Bernoulli beam model.

Finally in Section 4, we give many questions that this preliminary work on damping has raised, many interesting perspectives are listed, and some ideas towards solutions are also provided, giving as broad as possible a perspective on this difficult subject.

## 2. Finite-dimensional systems: equivalent descriptions and introduction of damping models

We start with the port-Hamiltonian formulation of the  $n$ -d.o.f. finite dimensional harmonic oscillator. Following e.g. [11], the dynamic equation is usually written in the form

$$M\ddot{x} + (C + G)\dot{x} + Kx = 0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  and  $M = M^T > 0$ ,  $K = K^T \geq 0$  and the damping matrix is decomposed into its symmetric part  $C = C^T$ , and its skew-symmetric part  $G = -G^T$ .

### 2.1. Port-Hamiltonian formulation and notations

We refer to [9, Chapter 2] for the concepts recalled here.

#### 2.1.1. Port-Hamiltonian systems with dissipation

Port-Hamiltonian systems, see [30], have been widely used in modelling and control of mechanical and electromechanical systems. It has first been defined from Dirac structures (arising from the use of power conjugate variables and the skew symmetry of the interconnection structure) in the case of power preserving systems.

**Definition 1** (*port-Hamiltonian system with dissipation*). In the case of systems with dissipation, PHs are defined by

$$\frac{d}{dt}X = (J(X) - R(X))\partial_X \mathcal{H}_0(X) + g(X)u(t) \quad (2)$$

$$y(t) = g(X)^T \partial_X \mathcal{H}_0(X) \quad (3)$$

where  $X \in \mathbb{R}^n$ ,  $\mathcal{H}_0(X)$  is the Hamiltonian function usually chosen as the total energy of the system, the gradient vector  $\partial_X \mathcal{H}_0(X)$  is the driving force,  $J(X) = -J(X)^T$  and  $R(X) = R(X)^T \geq 0$  specify the interconnection matrix and the dissipation matrix of the system, respectively.

The energy balance associated to this system is

$$\begin{aligned} \frac{d\mathcal{H}_0}{dt}(t) &= (\partial_X \mathcal{H}_0(X))^T \frac{dX}{dt} \\ &= y^T u(t) - (\partial_X \mathcal{H}_0(X))^T R(\partial_X \mathcal{H}_0(X)) \\ &\leq y^T(t)u(t). \end{aligned}$$

In the case of linear systems the energy can be written as a quadratic form  $\mathcal{H}_0(X) = \frac{1}{2}X^T LX$ , where  $L$  is symmetric positive definite and is directly related to the physical parameters of the

systems; its gradient is then  $\partial_X \mathcal{H}_0(X) = LX$ , a linear operator applied to  $X$ .

**Example 1** (*Damped oscillator*). In the introductory example, by using as state variables the energy variables (i.e. the position and the momentum) and defining the Hamiltonian  $\mathcal{H}_0$  as the total energy of the system:

$$X := \begin{bmatrix} q = x, \\ p = M\dot{x} \end{bmatrix} \quad \text{and} \quad \mathcal{H}_0(X) = \frac{1}{2}p^T M^{-1}p + \frac{1}{2}q^T Kq;$$

it is possible to rewrite (1) in the form of a port-Hamiltonian system with dissipation of Definition 1: indeed, we can compute the gradient vector  $\partial_X \mathcal{H}_0(X) = \begin{bmatrix} Kq = Kx \\ M^{-1}p = \dot{x} = v \end{bmatrix}$ , and find the following matrix decomposition:

$$J := \begin{bmatrix} 0 & I \\ -I & -G \end{bmatrix} \quad \text{and} \quad R := \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}$$

Note that  $J$  is full rank  $2n$  and skew-symmetric, whereas  $R$  is symmetric positive (when  $C = C^T \geq 0$ ), with rank equal to at most  $n$ , thus not positive definite.

#### 2.1.2. About the $G$ matrix

This matrix is often not considered in modelling processes with damping, why? Because in fact it has no damping effect, of course, since simple computations show that, whatever the value of  $G$  (skew-symmetric), when  $C=0$  (which is equivalent to  $R=0$ ), the system is conservative:  $(d/dt)\mathcal{H}_0(X(t)) = 0$ .

Hence the question arises: is it a naive generalizations due to mathematicians, or do there exist mechanical examples of systems with such a matrix? Of course the dimension must be  $n \geq 2$ , otherwise  $g=0$ . Below, we cite two well-known examples.

**Coriolis force:** Let  $n=3$ , and consider the Coriolis force with rotational speed  $\omega = (p, q, r)^T$ ; then the classical term  $\omega \wedge \dot{x}$  is nothing but  $G_\omega \dot{x}$ , with

$$G_\omega := \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}.$$

**Lorentz force:** Let  $n=3$ , and consider a charged particle  $-e$  in an electromagnetic field, with  $B_0$  the induction vector, then it is subject to the Lorentz force, that is proportional to  $-e\dot{q} \wedge B_0$ , which is nothing but  $G_{B_0} \dot{q}$ , another gyroscopic term.

**Remark 1.** Finally, describing the dynamics in the *rotating axes system* will certainly simplify the dynamics, and maybe help reduce the conservative part to the canonical symplectic structure (i.e. with  $G=0$ ) thanks to a simple change of co-ordinates. In order to simplify the following, it will be assumed from now on that  $G=0$ .

#### 2.1.3. Extending the pHs framework with external port variables

We can put a port-Hamiltonian system with dissipation in a framework used, e.g. in [30].

**Definition 2** (*Extended formulation for resistive ports*). Introducing external effort  $e_p$  and flow variables  $f_p$ , which are linked by a closure relation  $e_p = Sf_p$ , with  $S = S^T \geq 0$ , we get

$$\begin{bmatrix} f \\ f_p \end{bmatrix} = \begin{bmatrix} J & G_p \\ -G_p^T & 0 \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} \quad \text{and} \quad e_p = Sf_p. \quad (4)$$

With classical flows  $f = \dot{X}$ , and efforts  $e = \partial_X \mathcal{H}_0(X)$ , the previous relation corresponds to the following dynamics:

$$\dot{X} = (J - G_p S G_p^T) \partial_X \mathcal{H}_0(X). \quad (5)$$

Hence, the structure has been extended, and we can say that  $G_p S G_p^T$  is a parametrization of the damping matrix  $R$  which is

compatible with the pH framework with external effort and flow variables.

So far, the details of the damping parametrization as  $R = G_p S G_p^T$  cannot be made more explicit on [Example 1](#), but this will be worked out later on, especially in [Section 2.2.1](#).

## 2.2. Structural damping of Caughey type

Our goal now is to parametrize those damping matrices  $C = C^T \geq 0$  which leave unchanged the normal modes of the conservative system (i.e. with  $C=0$ ) in (1). Once a condition has been found, another objective is to see to what extent these parametrized damping matrices can give rise to a more specific decomposition of  $R$  into  $R = G_p S G_p^T$ . We proceed in two steps.

### 2.2.1. Sufficient condition, [4]: the polynomial case

In [4], setting  $N := M^{1/2}$ ,  $\tilde{C} := N^{-1} C N^{-1}$  and  $\tilde{K} := N^{-1} K N^{-1}$  (which are still symmetric positive matrices), a *sufficient* condition is found for our problem, namely that  $\tilde{C}$  be a series in  $\tilde{K}$ . Finally, taking advantage of the well-known Cayley–Hamilton theorem in finite dimension, it is found to be equivalent that  $\tilde{C}$  be a polynomial in  $\tilde{K}$ . Moreover, one must not forget that  $C = C^T \geq 0$ , a positivity condition that still has to be checked.

Thus, a sufficient condition is that

$$\tilde{C} = b_0 I + \sum_{l=1}^{n-1} b_l \tilde{K}^l \quad \text{with } b_l \geq 0. \quad (6)$$

**Remark 2.** In order to use the degrees of freedom given by Caughey, some attempts have been made in e.g. [1], but the right change of variable is not performed ( $M^{-1}K$  is never a symmetric matrix, hence the results of this paper are highly questionable, at least from a mathematical point of view), even if some results seem interesting for applications.

Suppose we want to put the  $\tilde{C} := \sum_{l=0}^{n-1} b_l \tilde{K}^l$  damping model into the port-Hamiltonian framework, first we must reinterpret this relation as

$$C_n := b_0 M + \sum_{l=1}^{n-1} b_l K M^{-1} K \cdots M^{-1} K, \quad (7)$$

each term having  $l$  occurrences of  $K$  and  $l-1$  of  $M^{-1}$ . The first order development reads  $C_1 := b_0 M + b_1 K$  with  $b_0, b_1 \geq 0$ , which is nothing more than Rayleigh damping.

Second we can put it in the dissipative framework used, e.g. in [30], by introducing external effort  $e_p$  and flow variables  $f_p$ , which are linked by a closure relation  $e_p = S f_p$ , with  $S = S^T \geq 0$ .

**Lemma 1.** Let  $C_n$  defined by (7) of degree  $n$  with  $S = \text{diag}(b_l I)$ , and

$$G_p = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ M^{1/2} & K^{1/2} & K M^{-1/2} & K M^{-1} K^{1/2} & \cdots \end{bmatrix},$$

we can compute:

$$G S G^T = \begin{bmatrix} 0 & 0 \\ 0 & C_n \end{bmatrix}.$$

With this lemma in hand, system (1) can now be written as:

$$\begin{bmatrix} \dot{f} \\ \dot{f}_p \end{bmatrix} = \begin{bmatrix} J & G_p \\ -G_p^T & 0 \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} \quad \text{and} \quad e_p = S f_p.$$

The feedback form corresponds to the following dynamics:

$$\dot{X} = (J - G_p S G_p^T) \partial_X \mathcal{H}_0(X),$$

with an  $R$  matrix fully structured into  $G_p S G_p^T$ , with structure matrices  $M$  and  $K$  involved in the definition of  $G_p$ , and the  $n$  free

damping parameters  $b_l$ , to be finely tuned to represent damping, in  $S$ .

For higher order developments, i.e.  $n \geq 2$ , such as  $C_2 := b_0 M + b_1 K + b_2 K M^{-1} K$ , makes explicit use of  $M^{-1}$ , which would be preferable not to compute in many circumstances, at least from a numerical point of view. In numerical analysis though, some Finite Element Methods (FEM) make use of the so-called *mass lumping*, which consists of imposing a diagonal structure to the mass matrix  $M$ .

**Remark 3.** Another choice is possible, which circumvents this difficulty, with  $S = \text{diag}(M, K)$  and  $G$  parameterized by  $\sqrt{b_0}$ ,  $\sqrt{b_1}$ , but this somewhat nicer decomposition does not generalize easily to PDEs.

### 2.2.2. Necessary and sufficient condition, [5]: the general case

There is a more general result proved in [5], which is a *necessary and sufficient* condition; it reads

$$[\tilde{C}, \tilde{K}] = 0, \quad (8)$$

where  $[A, B] := AB - BA$  is the commutant.

**Remark 4.** Obviously we recover the previous sufficient condition (the so-called polynomial case, fully studied in [Section 2.2.1](#)) as a special case of the general condition (8).

For short, it is a good idea to write  $\tilde{C} := f(\tilde{K})$ , where function  $f$  is well defined in the *cone of symmetric positive matrices*, which readily amounts to diagonalize the transformation in an orthonormal basis, and apply  $c_i := f(k_i)$  on each coordinate, with  $k_i \geq 0$ . Now a condition for damping is that  $f(\mathbb{R}^+) \subset \mathbb{R}^+$ , so as to ensure  $\tilde{C} := f(\tilde{K}) \geq 0$ , hence  $C \geq 0$ .

As special cases, not using the Cayley–Hamilton theorem from the beginning, it can be interesting to make a distinction between

1. polynomials, defined *explicitly* by:  $\tilde{C} := Q(\tilde{K})$ , such as Rayleigh damping when  $\deg(Q) = 1$ , see [Section 2.2.1](#),
2. rational functions, which can also be defined *implicitly* by:  $P(\tilde{K})\tilde{C} := Q(\tilde{K})$ ,
3. irrational functions, such as  $\tilde{C} = \tilde{K}^\alpha$ , see e.g. [Appendix A.1](#).

In the sequel, we give some partial results on the last two interesting examples. We begin with rational functions of matrices.

**Step 1:** Inside the class of invertible matrices  $C$ , multiplying (8) by  $\tilde{C}^{-1}$ , both at the left and at the right hand sides, yields the equivalent condition  $[\tilde{C}^{-1}, \tilde{K}] = 0$ . Hence, using the previous sufficient condition, it is enough to search  $\tilde{C}^{-1}$  as another polynomial  $P(\tilde{K})$  with nonnegative coefficients, i.e.  $P(s) = \sum_{k=0}^{\deg(P)} a_k s^k$  with  $a_k \geq 0$ , so that  $\tilde{C} = [P(\tilde{K})]^{-1}$ . More generally, rational functions of  $\tilde{K}$ , that is  $R(\tilde{K}) = Q(\tilde{K}) \cdot P(\tilde{K})^{-1}$  where  $P$  and  $Q$  are polynomials with nonnegative coefficients and  $P(0) = 1$ , are well-posed. In this case,  $\tilde{C} = R(\tilde{K})$  commutes with  $\tilde{K}$  and also defines an admissible damping matrices, in the sense of (8).

**Step 2:** Such damping matrices admit extended formulation with resistive ports. Consider here the simple case where  $P$  admits simple real roots  $s_i < 0$ , for  $1 \leq i \leq \deg(P)$ , and introduce the partial fraction expansion  $R(\tilde{K}) = \sum_{i=1}^{\deg(P)} \mu_i [\tilde{K} - s_i I]^{-1} + L(\tilde{K})$  where  $L$  is a polynomial which is zero if  $\deg(P) > \deg(Q)$ . The formulation associated with the  $L$ -part is detailed in [Lemma 1](#). The complementary part in the  $C$  matrix corresponds to the terms  $\sum_{i=1}^{\deg(P)} \mu_i C_i$  where

$$C_i = M^{1/2} [M^{-1/2} K M^{-1/2} - s_i I]^{-1} M^{1/2} \quad (9)$$

$$C_i = M [K - s_i M]^{-1} M. \quad (10)$$

The contribution of one such term  $C_i$  is as in [Definition 2](#), in which the closure equation  $e_p = S f_p$  must be replaced by the implicit

equation

$$[KM^{-1} - s_i I]e_p = \mu_i f_p. \quad (11)$$

The aggregation of all components  $i$  defines an *implicit* equation involving block-diagonal matrices. Finally, the extension of this decomposition to complex poles with negative real part, and even multiple poles (either real or complex) is possible, but will not be presented here: the principle of the method is now clearly given.

**Remark 5.** Thus, recasting the previous rational family of Caughey damping into a pH framework with dissipation and external ports of Section 2.1.3 proves possible, but can be seen as quite formal so some extent: clearly,  $C$  can be decomposed into  $G_p S G_p^T$ , but almost no information is given in the  $G_p$  matrix, whereas all the structure (i.e.  $M$  and  $K$ ) and damping (i.e.  $(a_k), (b_l)$ ) information are now concentrated into the matrix  $S$  alone: this case is very different from the previous one, as detailed in Section 2.2.1.

### 3. Infinite-dimensional systems: theory and examples

We now turn to PDE models, or continuous systems. It is indeed the underlying geometric structure of PDEs which must be considered and put forward in our studies, as in [2].

Let us consider the following PDE:

$$\mathcal{M}\ddot{x} + (\mathcal{C} + \mathcal{G})\dot{x} + \mathcal{K}x = 0, \quad (12)$$

where  $\mathcal{M}$  is symmetric and coercive,  $\mathcal{K}$  is self-adjoint and positive, and the damping operator is decomposed into its self-adjoint and positive  $\mathcal{C}$  part, and its skew-adjoint part  $\mathcal{G}$ .

Our goal is to recast this PDE into an infinite-dimensional port-Hamiltonian framework, and to solve the question of preserving the eigenspaces of the conservative problem when a structured form of damping is introduced.

#### 3.1. Framework for infinite-dimensional port-Hamiltonian systems under study

We refer to [9, Chapter 4] for the concepts recalled here.

##### 3.1.1. Infinite-dimensional port-Hamiltonian systems with dissipation

Port Hamiltonian systems have been extended to the case of distributed parameter systems and more specifically in the case of linear systems defined on one dimensional spatial domain ( $z \in [0, L]$ ) by using real Hilbert spaces in [22]. In this case the associate PDE is of the form (13).

**Definition 3** (Port-Hamiltonian system with dissipation).

$$\frac{d}{dt}X(z, t) = (\mathcal{J} - \mathcal{R})\delta_X \mathcal{H}_0(X) \quad (13)$$

with quadratic Hamiltonian:

$$\mathcal{H}_0(X) = \frac{1}{2} \int_0^L X(z, t)^T \mathcal{L}X(z, t) dz,$$

and linear variational derivative:

$$\delta_X \mathcal{H}_0(X) = \mathcal{L}_z X(z, t).$$

In (13), operator  $\mathcal{J}$  is formally skew-symmetric, and operator  $\mathcal{R}$  is positive self-adjoint.

**Example 2** (Damped vibrating system). On (12), a procedure similar to that of Example 1 could be formally applied to adapt to Definition 3, but it is rather on specific cases that this work proves useful. To this end, we do not write out a general reformulation, but refer the reader to Example 3 for Euler-Bernoulli beam equation (where  $\mathcal{K} = \partial_{zz}^4$ ), and to Example 5 for Webster horn equation (where  $\mathcal{K} = -\partial_{zz}^2$  in the uniform case, and

$\mathcal{K} := -S(z)^{-1} \partial_z(S(z) \partial_z \cdot)$  in the non-uniform case). In these examples, the operators  $\mathcal{J}$  and  $\mathcal{R}$  will appear naturally.

##### 3.1.2. Example of gyroscopic effects in infinite-dimension

Here is a simple example for the  $\mathcal{G}$  operator: an ideal and incompressible fluid is governed by  $(d/dt)\mathbf{v} = -(\mathbf{v} \cdot \mathbf{grad})\mathbf{v} - (1/\rho_0) \mathbf{grad}(p)$  and  $\text{div}(\mathbf{v}) = 0$ . After some computations, we find that  $\forall \phi, \psi \in H_0^1(\Omega)$

$$\int_{\Omega} \psi \mathbf{v} \cdot \mathbf{grad} \phi dV = - \int_{\Omega} (\phi \mathbf{v} \cdot \mathbf{grad} \psi + \text{div}(\mathbf{v}) \phi \psi) dV.$$

Let  $\mathbf{V}_0$  a given divergence-free velocity field. Hence, the operator  $G : \phi \mapsto \mathbf{V}_0 \cdot \mathbf{grad} \phi$  is skew-symmetric w.r.t.  $L^2(\Omega)$ :

$$(\psi, \mathbf{V}_0 \cdot \mathbf{grad} \phi)_{L^2(\Omega)} = -(\phi, \mathbf{V}_0 \cdot \mathbf{grad} \psi)_{L^2(\Omega)};$$

this non-uniform convection term definitely plays the role of a gyroscopic term in infinite dimension.

##### 3.1.3. Extending the pHs framework with external effort and flow variables

The definition of port-Hamiltonian systems is fundamentally linked to the definition of *port variables*, usually derived from the skew symmetry of the operator in the case of open systems, and from which the Dirac structure is defined. In the case of systems of the form (13), i.e. with dissipation, the operator is no more skew symmetric. Yet, following e.g. [30], a Dirac structure can be associated with the interconnection structure defined by the extended skew symmetric operator  $\mathcal{J}_e$  as follows:

**Definition 4** (extended formulation for resistive ports).

$$\left( \frac{d}{dt} \begin{pmatrix} X(z, t) \\ f_p \end{pmatrix} \right) = \begin{pmatrix} \mathcal{J} & \mathcal{G}_p \\ -\mathcal{G}_p^* & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L}X(\xi, t) \\ e_p \end{pmatrix}, \quad (14)$$

$$\text{with } e_p = S f_p \quad (15)$$

which is equivalent to (13).

The above feedback form does correspond to the following dynamics:

$$\dot{X} = (\mathcal{J} - \mathcal{G}_p S \mathcal{G}_p^*) \delta_X \mathcal{H}_0(X).$$

Hence, we can say that  $\mathcal{G}_p S \mathcal{G}_p^*$  is a parametrization of the damping operator  $\mathcal{R}$  which is compatible with the pH framework with external effort and flow variables.

#### 3.2. Extension of structural damping of Caughey type

The problem at stake is to preserve the eigenspaces of the conservative problem  $\mathcal{C} = 0$ , when a structured form of damping  $\mathcal{C} \geq 0$  is introduced in the dynamics (12). From now on, we suppose  $\mathcal{G} = 0$ .

##### 3.2.1. The polynomial case

Note that all the operators ( $\mathcal{M}$ ,  $\mathcal{C}$ ,  $\mathcal{K}$ ) involved in (12) are supposed to be self-adjoints,  $\mathcal{M}$  being coercive, hence invertible, and  $\mathcal{K}$  positive. Letting  $\mathcal{N} := \mathcal{M}^{1/2}$ , we define  $\tilde{\mathcal{C}} := \mathcal{N}^{-1} \mathcal{C} \mathcal{N}^{-1}$  and  $\tilde{\mathcal{K}} := \mathcal{N}^{-1} \mathcal{K} \mathcal{N}^{-1}$ .

A sufficient condition for keeping the normal modes unaffected by the damping operator  $\mathcal{C}$  is that

$$\tilde{\mathcal{C}} = b_0 I + \sum_{l=1}^{n-1} b_l \tilde{\mathcal{K}}^l \quad \text{with } b_l \geq 0 \quad (16)$$

contrarily to (6), here the free parameter  $n$  stands for the degree of the polynomial  $Q(s) := \sum_{l=0}^{n-1} b_l s^l$ , and not the dimension of the state space.

This condition can be equivalently rewritten in the following format:

$$C = b_0 \mathcal{M} + \sum_{l=1}^{n-1} b_l \mathcal{K} \mathcal{M}^{-1} \mathcal{K} \dots \mathcal{M}^{-1} \mathcal{K}, \quad b_l \geq 0. \quad (17)$$

Since it is still possible to use Lemma 1, with operators instead of matrices, we can easily recast the subclass of polynomial damping (17) in the extended pH framework with external port variables presented in Section 3.1.3.

In the sequel, we choose to illustrate this result on two different examples: Euler–Bernoulli beam with Rayleigh damping, and Navier–Stokes equation for a compressible fluid. Example 3 is 1-D, linear, and involves a polynomial of degree one, but a differential operator of order 2; whereas Example 4 is 3-D, non-linear, and involves a polynomial of degree one, but with a vector-valued differential operator of order 1.

**Example 3** (Euler–Bernoulli beam with Rayleigh damping). Consider a dimensional version of the Euler–Bernoulli's beam model (see [12]), excited by the force  $f$  at  $z=0$  and with free end at  $z=L$ , which includes a fluid and a structural damping. For a constant cross-section and a homogeneous material, it corresponds to the following equations (see [15]):

$$Y l \partial_z^4 u + \rho S [b_0 + b_1 \partial_z^4] \partial_t u(t, z) + \rho S \partial_t^2 u = 0 \quad (18)$$

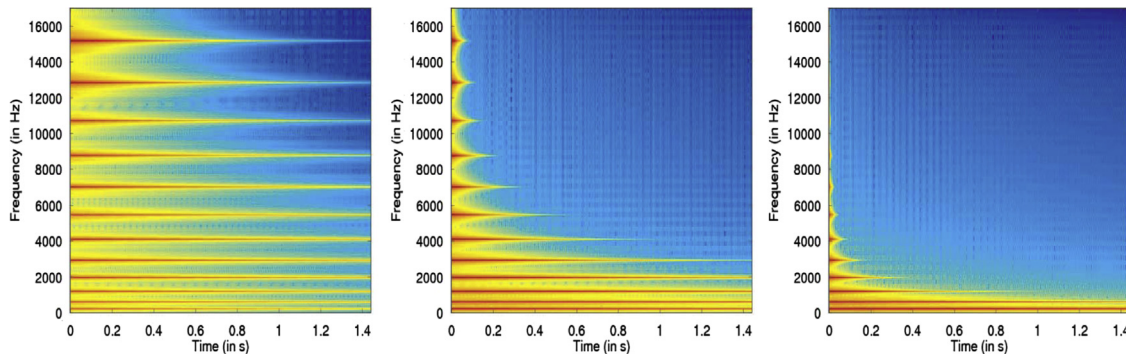
$$\partial_z^2 u(t, 0) = \partial_z^2 u(t, L) = 0 \quad (\text{no momentum}) \quad (19)$$

$$\partial_z^3 u(t, 0) = f(t) \quad (\text{force}) \quad \text{and} \quad \partial_z^3 u(t, L) = 0 \quad (\text{no force}). \quad (20)$$

In this model,  $\rho$  and  $Y$  are the density and Young's modulus of the material, respectively, and  $l = wh^3/12$  is the geometrical momentum of the bar ( $w$  is the width and  $h$  the height). Positive coefficients  $b_0$  and  $b_1$  quantify the effect of the fluid and the structural dampings, respectively.

Simulations based on a modal decomposition have been proposed in [15] for realistic sound synthesis purposes, with the following sensible physical values:  $L=0.5$  m (bar length),  $w=0.05$  m (width),  $h=0.0117$  m (height),  $Y=2.13 \times 10^{10}$  Pa (Young's modulus)  $\rho=1015$  Kg m<sup>-3</sup> (purple wood density). When no damping is present, the first and last considered modes correspond to frequencies  $f_1=220$  Hz and  $f_{12}=15\,190$  Hz, respectively.

As the damping coefficients are unknown, several physical orders of magnitude are presented: three sounds are synthesised and their respective spectrograms are presented in Fig. 1. Qualitatively, these examples show that  $b_1$  is representative of wooden bar sounds (marimba), whereas  $b_0$  is more representative of metallic bar sounds (vibraphone). It can be heard that both dampings give rise to different audible behaviours and provide a large set of sounds close to percussive bar sounds.



**Fig. 1.** Rayleigh type dampings: spectrogram of  $\partial_z^2 u(t, L)$ . (left):  $b_0 = 4e-2$  and  $b_1 = 3e-9$  (SI), sounds like a metallic bar, (center):  $b_0 = 2e-2$  and  $b_1 = 5e-8$  (SI), sounds like a glass bar, (right):  $b_0 = 1e-2$  and  $b_1 = 5e-7$  (SI) sounds like a wooden bar.

The spectrograms show how, on a practical example, such damping models can be used to improve the sound synthesis realism: both  $b_0$  and  $b_1$  are required.

For Rayleigh damping on conservative PDEs, analyzed in e.g. [18], a port-Hamiltonian formulation is available in e.g. [30]; we recall it here, for sake of clarity. In a simplified way, denoting  $v := \partial_t u$ , the dynamics now reads

$$\partial_{tt}^2 u + y(v) + \partial_z^4 u = 0,$$

with damping term (a polynomial of degree 1):

$$y(v) := b_0 v + b_1 \partial_z^4 v.$$

Classically,  $q := \partial_z^2 u$  and  $p := \partial_t u$ , with Hamiltonian  $\mathcal{H}_0 := \frac{1}{2} \int_0^L (q^2 + p^2) dz$ . We can compute the variational derivatives  $\delta_q \mathcal{H}_0 = q$  and  $\delta_p \mathcal{H}_0 = p$ , and check

$$\frac{d}{dt} \begin{bmatrix} q \\ p \end{bmatrix} = \begin{bmatrix} 0 & \partial_z^2 \\ -\partial_z^2 & 0 \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H}_0 \\ \delta_p \mathcal{H}_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} \delta_q \mathcal{H}_0 \\ \delta_p \mathcal{H}_0 \end{bmatrix};$$

(with  $C = b_0 I + b_1 \partial_z^4$ ) which has the desired  $(\mathcal{J} - \mathcal{R})$  form, with  $\mathcal{J}$  skew-symmetric and  $\mathcal{R}$  symmetric. In order to parametrize  $\mathcal{R} = \mathcal{G} S \mathcal{G}^*$ , we define next

$$\mathcal{G} := \begin{bmatrix} 0 & 0 \\ 1 & \partial_z^2 \end{bmatrix} \quad \text{and} \quad S := \text{diag}(b_0 I, b_1 I),$$

which helps to describe the whole system, using the extended efforts and flows:

$$\begin{bmatrix} f \\ f_p \end{bmatrix} = \begin{bmatrix} \mathcal{J} & \mathcal{G} \\ -\mathcal{G}^* & 0 \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} \quad \text{and} \quad e_p = S f_p.$$

The feedback form which is obtained corresponds indeed to the damped dynamics:

$$\dot{X} = (\mathcal{J} - \mathcal{G} S \mathcal{G}^*) \delta_X \mathcal{H}_0(X).$$

We now come to a non-linear system in 3 dimensions, for which the damping is fully structured: it perfectly fits into the dissipative pHs framework developed above.

**Example 4** (Navier–Stokes equations). Following [28], we consider an irrotational and isentropic fluid, in a bounded domain  $\Omega \subset \mathbb{R}^3$ . Using standard notations, the dynamical equations of the fluid can be written as

$$\frac{d}{dt} \rho = -\text{div}(\rho \mathbf{v}) \quad (21)$$

$$\frac{d}{dt} \mathbf{v} = -(\mathbf{v} \cdot \text{grad}) \mathbf{v} - \frac{1}{\rho} \text{grad } p + \frac{1}{\text{Re}} \Delta \mathbf{v}, \quad (22)$$

where pressure  $p$  is derivable from a potential energy density  $U(\rho)$ , as  $p = \rho^2 \partial U / \partial \rho$ , and where  $\mathbf{v}$  denotes the (vectorial) particle velocity. Here,  $Re$  is Reynolds number. Hence, with Hamiltonian

$$\mathcal{H}_0 := \int_{\Omega} \left( \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho U(\rho) \right) dV, \quad (23)$$

we first compute the variational derivatives

$$\delta_{\mathbf{v}} \mathcal{H}_0 = \rho \mathbf{v},$$

and

$$\delta_{\rho} \mathcal{H}_0 = \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + h(\rho),$$

with  $h(\rho) := U(\rho) + \rho \partial U / \partial \rho$  being the *enthalpy*. Then, using the identity  $(\mathbf{v} \cdot \mathbf{grad}) \mathbf{v} = \mathbf{grad}(\frac{1}{2} \mathbf{v} \cdot \mathbf{v})$  which holds since  $\mathbf{rot}(\mathbf{v}) = \mathbf{0}$ , we rewrite Eqs. (21) and (22) as

$$\frac{d}{dt} \begin{bmatrix} \rho \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} 0 & -\text{div} \\ -\mathbf{grad} & 0 \end{bmatrix} \begin{bmatrix} \delta_{\rho} \mathcal{H}_0 \\ \delta_{\mathbf{v}} \mathcal{H}_0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{C} \end{bmatrix} \begin{bmatrix} \delta_{\rho} \mathcal{H}_0 \\ \delta_{\mathbf{v}} \mathcal{H}_0 \end{bmatrix},$$

with  $\mathcal{C} = -(1/Re)\mathbf{\Delta}$ . It has the desired  $(\mathcal{J} - \mathcal{R})$  form:  $\mathcal{J}$  is skew-symmetric, since the formal adjoint of  $\text{div}$  is  $-\mathbf{grad}$ , and  $\mathcal{R}$  is symmetric and positive, since  $-\mathbf{\Delta}$  is. More important, using the identity  $\mathbf{\Delta} \mathbf{v} = \mathbf{grad}(\text{div}(\mathbf{v}))$  which holds since  $\mathbf{rot}(\mathbf{v}) = \mathbf{0}$ , the parametrization  $\mathcal{R} = \mathcal{G} S \mathcal{G}^*$  is very easily found to be

$$\mathcal{G} := \begin{bmatrix} 0 \\ \mathbf{grad} \end{bmatrix}, \quad \mathcal{G}^* = [0 \quad -\text{div}], \quad \text{and} \quad S := \frac{1}{Re} I.$$

### 3.2.2. More general cases

Once again, for models of second order in time of the form (12), a *sufficient* condition for keeping the normal modes unaffected by the damping operator  $\mathcal{C}$ , and proved in [5], is given by the commutation of the reduced operators, (including their domain). Condition (8) for finite-dimensional becomes (including the domains of these reduced operators)

$$[\tilde{\mathcal{C}}, \tilde{\mathcal{K}}] = 0. \quad (24)$$

Note that the original paper gives many counter-examples, either due to the structure of the operators, or their domains; an example is also provided.

As special cases, it proves very interesting to make a distinction between

1. polynomials, defined explicitly by:  $\tilde{\mathcal{C}} := Q(\tilde{\mathcal{K}})$ , such as Rayleigh damping when  $\deg(Q) = 1$ , already discussed in Section 3.2.1,
2. rational functions, which can also be defined implicitly by:  $P(\tilde{\mathcal{K}})\tilde{\mathcal{C}} := Q(\tilde{\mathcal{K}})$ ,
3. irrational functions, such as  $\tilde{\mathcal{C}} = \tilde{\mathcal{K}}^{\alpha}$ , see e.g. Appendix A.2.

In the sequel, we shall try to illustrate the latter two cases on worked-out examples. Let us start with a linear 1-D example with *variable coefficients* in space and rational damping.

**Example 5** (*Webster horn equation with rational damping*). This model arising in musical acoustics is a wave equation, which has coefficients  $S(z)$  variable in space, it is first put in conservative form. The horn equation [21,3], also called the Webster equation [31], is a linear 1D model of axisymmetric acoustic pipes with a varying cross-section  $z \mapsto S(z) = \pi R(z)^2$ . For acoustic bells, this equation appeared to match with measurements choosing the space variable as the curvilinear abscissa which measures the length of the wall [14,16].

Denote  $\rho_0$  and  $P_0$  as the air density and the air pressure at equilibrium, respectively. Denote  $\rho$  and  $p$  as their acoustic deviations for isentropic conditions. The wave equations which govern the acoustic pressure  $p$  and the particle velocity  $v$  are given by,

respectively,

$$\frac{1}{S(z)} \partial_z [S(z) \partial_z p(z, t)] - \frac{1}{c_0^2} \partial_t^2 p(z, t) = 0, \quad (25)$$

$$\partial_z \left[ \frac{1}{S(z)} \partial_z [S(z) v(z, t)] \right] - \frac{1}{c_0^2} \partial_t^2 v(z, t) = 0, \quad (26)$$

in the linear approximation, namely, for  $p \approx c_0^2 \rho$  where  $c_0 = \sqrt{\gamma P_0 / \rho_0}$  is the sound celerity and  $\gamma$  is the isentropic coefficient. The acoustic energy inside a pipe with length  $L$  is given by

$$H_0 := \int_0^L \left( \frac{1}{2} \rho_0 v^2 + \rho_0 U(\rho) \right) S(z) dz, \quad (27)$$

where  $U(\rho) = (c_0^2 / 2 \rho_0) \rho^2 = (\gamma P_0 / 2 \rho_0^2) \rho^2$ . Compared to (23), note that the infinitesimal volume is  $dV(z) = S(z) dz$ , that the kinetic energy is unchanged and that the potential energy is not the total internal energy of the gas, but is reduced to the acoustic part only. In acoustics, it is usually expressed as a function of  $p$ , namely,  $\rho_0 U(\rho) = p^2 / 2 \rho_0 c_0^2$ . For  $z$  in  $(0, L)$ , the corresponding port-Hamiltonian system is described by

$$\frac{d}{dt} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{S} \partial_z(S \cdot) \\ -\partial_z & 0 \end{bmatrix} \begin{bmatrix} \delta_{\rho} H_0 \\ \delta_v H_0 \end{bmatrix}, \quad (28)$$

where  $(1/S) \partial_z(S \cdot)$  stands for the divergence operator and  $\partial_z$  stands for the gradient vector projected on  $\mathbf{e}_z$ . Moreover, operator  $J$  in (28) is clearly skew-symmetric, w.r.t. the *weighted* scalar product  $(v, w) := \int_0^L v w S(z) dz$ .

**Remark 6.** In this 1D example, since the coefficients are space-varying, two distinct compound operators are to be found, depending on which variable we work on: either  $-S(z)^{-1} \partial_z [S(z) \partial_z \cdot]$  or  $-\partial_z [S(z)^{-1} \partial_z(S(z) \cdot)]$ .

Now, as far as damping is concerned, a first order rational function of operator

$$\tilde{\mathcal{K}} := -S(z)^{-1} \partial_z (S(z) \partial_z \cdot) \quad (29)$$

is being used for operator  $\tilde{\mathcal{C}}$ . In order to be self-contained, let  $v := \partial_t u$  and define  $y(v)$  as the solution to the following *static* PDE of *elliptic* type:

$$a_0 y - a_1 S^{-1} \partial_z (S \partial_z y) = b_0 v - b_1 S^{-1} \partial_z (S \partial_z v), \quad (30)$$

where  $a_1, b_1 \geq 0$ , and  $a_0, b_0 > 0$ . With appropriate boundary conditions, this problem is well-posed, thanks to Lax–Milgram theorem. The positivity condition,  $\int_0^L y(z) v(z) dz \geq 0$ , can be checked thanks to a spectral mapping theorem and  $f(\mathbb{R}^+) \subset \mathbb{R}^+$  where  $f(z) := (b_0 + b_1 z) / (a_0 + a_1 z)$ . But still, in this case, more precise results can be proved. Setting  $\delta := b_1 a_0 - a_1 b_0 \neq 0$ , two cases may occur, the so-called ARMA model is of:

1. *MA-type* when  $\delta > 0$ : First decompose the input  $v$  as  $v := (a_1 / b_1) y + w$ , then (30) implies  $(\delta / b_1) y = b_0 w + b_1 \tilde{\mathcal{K}} w$ , just like Rayleigh damping on the new input  $w$ , which guarantees positivity:

$$(y, v) = \frac{a_1}{b_1} \|y\|^2 + (y, w) \geq 0,$$

since  $(\delta / b_1)(y, w) = b_0 \|w\|^2 + b_1 (\tilde{\mathcal{K}} w, w) \geq 0$  (recall that  $\tilde{\mathcal{K}}$  is a positive self-adjoint operator).

2. *AR-type* when  $\delta < 0$ : First decompose the output  $y$  as  $y := (b_1 / a_1) v + z$ , then (30) implies that  $a_0 z + a_1 \tilde{\mathcal{K}} z = -(\delta / a_1) v$ , just like pure AR-type damping for the new output  $z$ , which

guarantees positivity:

$$(y, v) = \frac{b_1}{a_1} \|v\|^2 + (z, v) \geq 0,$$

$$\text{since } -(\delta/a_1)(z, v) = a_0 \|z\|^2 + b_1(\tilde{\mathcal{K}}z, z) \geq 0.$$

**Remark 7.** As very special case, one can consider the constant coefficient case  $S(z) = S_0$ , in which case an explicit solution can be given, namely  $y = (b_1/a_1)v - (\delta/a_1)\exp(-|z|/\ell) \star v$ , where  $\ell = \sqrt{a_1/a_0}$ , which is an integral operator of convolution type, the convolution being bilateral in space.

Some decomposition of the type of those given in Section 2.2.2 could be copied and transferred to the infinite-dimensional setting; but so far, recasting this rational model in a port-Hamiltonian setting does not prove straightforward, even using the many extensions examined in [29, Section 4]. Hence, some more works could be done in order to be able to recast these more general integro-differential systems into a Dirac structure.

Let us finally turn to a more abstract case, which is neither polynomial nor rational.

**Example 6 (Fractional Laplacian).** Also of interest is the case of fractional Laplacian or bi-Laplacian (still with ideal boundary conditions), see [13,7,8] and references therein for this specific type of fractional damping model. More recently in [10], another interesting musical application makes use of  $\mathcal{C} = (\partial_{z^4}^4)^{1/2}$ , where  $\mathcal{K} = \partial_{z^4}^4$ , for the specific damping model of piano strings.

$$\tilde{\mathcal{C}} = \tilde{\mathcal{K}}^\alpha. \quad (31)$$

**Remark 8.** We refer to Appendix A.2 for careful definitions of such non-rational functions of operators. In particular,  $(\partial_{z^4}^4)^{1/2} \neq -\partial_{z^2}^2$

even with Dirichlet boundary conditions, and this can only be seen on the domain of the operators, or the specific decomposition on eigenfunctions.

The main idea behind this somewhat quite general damping model is to see the root locus it gives rises to: explicit analytical computations can be carried out on  $y(v) := b_0 v + b_\alpha (-\Delta)^\alpha v$ , but we briefly show the root locus as a function of the  $\alpha$  parameter in Figs. 2 and 3:

- for  $0 < \alpha < 0.5$ , the dynamical system is a PDE of *hyperbolic* type, the roots are located on a curve in  $\mathbb{C}$  with a so-called parabolic branch  $\Im m(s) \propto (-\Re e(s))^\nu$  with  $\nu := 1/2\alpha > 1$ ,
- for  $\alpha = 0.5$ , the dynamical system is a PDE of *parabolic* type (the associated semigroup is analytic), the asymptote is a straight line ( $\nu = 1$ ),
- for  $0.5 < \alpha \leq 1$ , the dynamical system is a PDE of *parabolic* or *diffusive* type, the roots are eventually located on  $\mathbb{R}^-$ , with only finitely many damped oscillating roots (located on a circle when  $\alpha = 1$ , Rayleigh damping).

#### 4. Conclusion and perspectives

We have looked for a structuration of the damping models which preserve the classical normal modes of the undamped structure, the Basile hypothesis. For discrete systems, or ODEs, the Caughey series has been put in the formalism of port-Hamiltonian system, the different cases have been examined and illustrated polynomial, rational function and even more general functions satisfying the positivity constraint. For continuous systems, or PDEs, the general ideas behind Caughey series have also been put into the port-Hamiltonian setting, at least formally, and a few interesting examples have been treated.

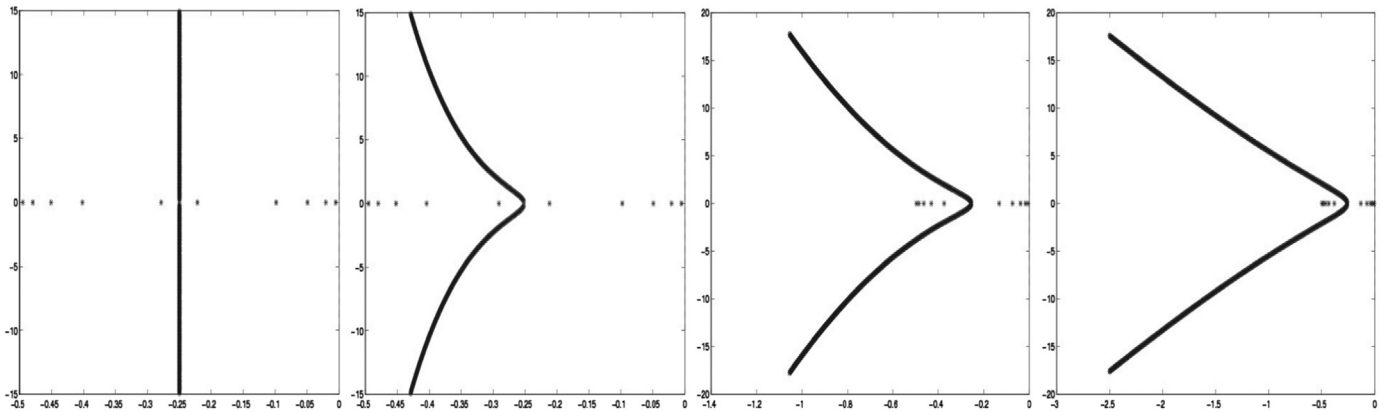


Fig. 2. From left to right:  $\alpha = 0$  fluid;  $\alpha = 0.1, 0.25, 0.4$  PDE of hyperbolic type.

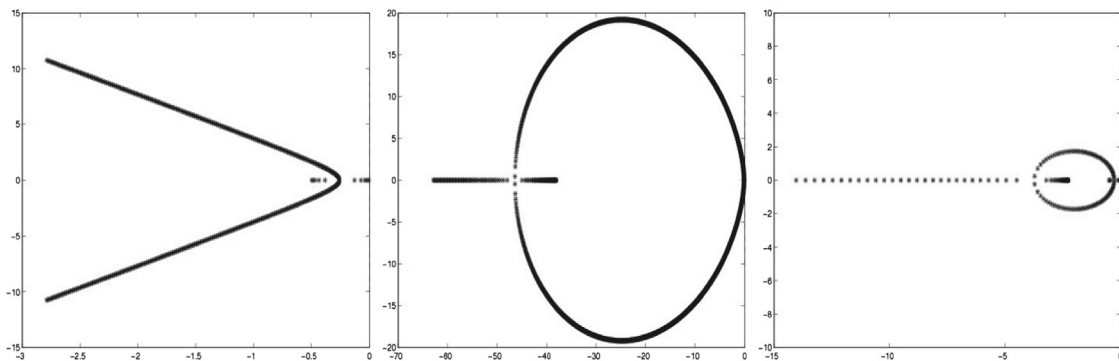


Fig. 3. From left to right:  $\alpha = 0.5$  limiting case,  $\alpha = 0.8$  PDE of parabolic type, and  $\alpha = 1$  Rayleigh damping.

Moreover, many points are to be looked at carefully, in the continuation of this preliminary work on structuration of damping, such as

- For the PDE case, ports at the boundaries of the spatial domain must definitely be taken into account, see e.g. [30] and [19, Chapters 9 and 10].
- How to use these models for the purpose of identification of damping parameters? Is an inverse problem possible, tractable in the context of structured damping?
- Possibility to adapt to nonlinear models with non-quadratic Hamiltonian? Could the Hessian functional help to define the different terms, such as  $K(q)$  and  $M(p)$ ? See e.g. [9, Section 2.3].
- Use some operational calculus on non-normal operators? Think of Riesz basis as directly related to Hilbert basis (following e.g. [20]) and then use this as a foundation for operational calculus: is that a too naive idea? Its interest is that it seems to be tractable, but to what extent, and is there a solid theory beyond that? See e.g. [17,27].
- In the previous case, how does the positivity constraint translate? Into a *positive real* condition, such as  $\Re(f(z)) \geq 0$  for  $\Re(z) > 0$ ?
- For PDEs, go to the case when the physical domain is of dimension  $d=2$  or  $3$ : things become much more intricate, new operators pop up, such as *div* and *grad*, which are adjoints one of another, but  $-\text{div}(\text{grad})$  is a scalar operator  $-\Delta$  acting on functions, whereas  $-\text{grad}(\text{div})$  is a vector-valued operators acting on vector fields (already when  $d=1$ , the non-commutativity has been noticed in Remark 6 when the coefficients are space-varying).

And, last but not least, an objection could very much be raised before going on: what is the real interest, and on what physical ground, do we look for normal modes in damped structures? Different answers are possible: one could argue that eigenvalues are affected at the first order when a slight damping is applied, whereas eigenvectors or eigenfunctions are only moved up to the second order of the damping parameter. Moreover, for many physical problems, refined damping models are not available. For instance, in applications such as in Example 3 (see e.g. [6]), an engineering approach is often used, which consists in computing the modal decomposition of the conservative problem and introducing, a posteriori, a specific damping for the dynamics of each mode according to some heuristics. Damping models that preserve the eigenspaces of the conservative problem exactly address this issues but, in an intrinsic way, that is, without having to derive the eigenstructure. This gives both a formal framework and define an equivalence class of damped models.

Finally, pHs formalism proves most useful when modelling damping for PDEs: when non-ideal boundary conditions are present, not simply Dirichlet or Neumann, such as Robin type or more general impedance boundary conditions, there is a need to clarify the underlying structure, which could very much be given, almost for free, by the port variables in the pH framework: this is, at the best of our knowledge, one of the most important reason to turn to pHs for PDEs in order to build and define coherent damping models.

## Acknowledgments

Both authors would like to thank the two anonymous reviewers, who carefully read the original submission, made useful comments, and greatly helped to enhance the final version of the paper; they are gratefully acknowledged.

The first author would like to thank Prof. J. Kergomard for fruitful discussion on the subject of gyroscopic terms for ODEs,

first giving the right name to this term, then giving the example of Coriolis effect in solid mechanics, see Section 2.1.2.

The first author would like Prof. S. De Bièvre for the nice talk on Abraham–Lorentz model and the Hamiltonian framework, see Section 2.1.2.

The first author would like to thank Prof. L. Jezequel for fruitful discussion on the subject of gyroscopic terms for PDEs, and mentioning the example of fluid mechanics in a duct with convection, see Section 3.1.2.

The first author would like to thank Prof. B. Maschke for fruitful discussion on the parametrization of implicit port-Hamiltonian systems.

## Appendix A

A part of this technical presentation on *fractional powers of matrices and operators* is borrowed from [23], see also [25] for a clear and concise course with many examples of operators and spectra.

### A.1. Fractional powers of matrices

We recall the Spectral Theorem for symmetric real-valued matrices: if  $A = A^T \in M_{n \times n}(\mathbb{R})$ , then there exist a diagonal matrix  $\Lambda$  and an orthogonal matrix  $P$ , (i.e.  $P^T P = I_n$ ), such that  $A = P^{-1} \Lambda P$ . Then, for the fractional power of a symmetric matrix, two cases may occur:

1. if  $A = A^T > 0$ , i.e.  $A$  is positive definite, then one can uniquely define  $A^{-\beta} = P^{-1} \Lambda^{-\beta} P$ , with  $\Lambda^{-\beta} = \text{diag}(\lambda_1^{-\beta}, \dots, \lambda_n^{-\beta})$ , since  $\lambda_i > 0$ .
2. if  $A = A^T \geq 0$ , i.e.  $A$  is positive, then one can uniquely define  $A^\alpha = P^{-1} \Lambda^\alpha P$ , with  $\Lambda^\alpha = \text{diag}(\lambda_1^\alpha, \dots, \lambda_n^\alpha)$ , since  $\lambda_i \geq 0$ .

### A.2. Fractional powers of operators

A key point is the *compactness* property: when it is present, this property enables to write down things into series instead of finite sums (with the celebrated sine, cosine or *Fourier series* on  $L^2(I)$ , where  $I$  is a bounded interval), and this applies both to bounded and unbounded operators in fact. When it is not present, general integrals instead of series have to be considered: we recall the celebrated *Fourier transform* on  $L^2(\mathbb{R})$ .

#### A.2.1. Fractional powers for operators with a compactness property

In an infinite-dimensional setting, things are much more complicated: we begin with the case of *bounded* operators.

**Bounded operators:** Following standard theory, if  $K$  is a *compact* and symmetric operator on a Hilbert space  $\mathcal{H}$ , using the Spectral Theorem, we get a spectral mapping theorem of the form above:

1. if  $K = K^T > 0$ , i.e.  $K$  is positive definite, then one can uniquely define  $K^{-\beta} = P^{-1} \Lambda^{-\beta} P$ , with  $\Lambda^{-\beta} = \text{diag}(k_n^{-\beta})_{n \in \mathbb{N}}$ , since  $k_n > 0$ ; this unbounded operator is defined on a domain  $D(K^{-\beta})$ , see Section A.2.1.
2. if  $K = K^T \geq 0$ , i.e.  $K$  is positive, then one can uniquely define  $K^\alpha = P^{-1} \Lambda^\alpha P$ , with  $\Lambda^\alpha = \text{diag}(k_n^\alpha)_{n \in \mathbb{N}}$ , since  $k_n \geq 0$ .

The transform  $P$  is *unitary* on  $\mathcal{H}$ , and the eigenvalues of  $K$  consist in a sequence of positive real numbers  $k_n$  which converge towards 0.

**Unbounded operators:** Now if  $A$  is *unbounded* on  $\mathcal{H}$ , with dense domain  $D(A)$  in  $\mathcal{H}$ , self-adjoint, positive, and has *compact resolvent*, then the previous setting can be applied to  $K_\rho = (\rho I - A)^{-1}$  for

$\rho \in \rho(A)$ , the resolvent set of  $A$ ; in particular the eigenvalues of  $A$  form a discrete sequence  $\lambda_n$  of positive real numbers, which grows towards infinity. When  $\lambda = 0$  is not an eigenvalue of  $K_\rho$ , the eigenvectors  $(e_n)_{n \in \mathbb{N}}$  of  $K_\rho$  form a Hilbert basis of  $\mathcal{H}$ , and we get the spectral theorem:

$$\forall \varphi \in \mathcal{H}, \quad \varphi = \sum_{n \in \mathbb{N}} (\varphi, e_n) e_n,$$

with the energy identity  $\|\varphi\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{N}} |(\varphi, e_n)|^2$ . And for any  $\gamma > 0$ , we can define the fractional power of  $A$  as follows:

$$A^\gamma \varphi = \sum_{n \in \mathbb{N}} \lambda_n^\gamma (\varphi, e_n) e_n,$$

provided  $\varphi \in D(A^\gamma)$ , where

$$D(A^\gamma) = \left\{ \varphi \in \mathcal{H}, \sum_{n \in \mathbb{N}} \lambda_n^{2\gamma} |(\varphi, e_n)|^2 < \infty \right\}.$$

This is indeed the case for  $A = -\partial_{xx}^2$  on the *bounded* interval  $I = (0, 1)$ , with Dirichlet (D) or Neumann (N) boundary conditions at each end:

- D–D case:  $\lambda_n = n^2 \pi^2$  and  $e_n \propto \sin(n\pi x)$  for  $n \geq 1$ ,
- D–N case:  $\lambda_n = (n + \frac{1}{2})^2 \pi^2$  and  $e_n \propto \sin((n + \frac{1}{2})\pi x)$  for  $n \geq 0$ ,
- N–D case:  $\lambda_n = (n + \frac{1}{2})^2 \pi^2$  and  $e_n \propto \cos((n + \frac{1}{2})\pi x)$  for  $n \geq 0$ ,
- N–N case:  $\lambda_n = n^2 \pi^2$  and  $e_n \propto \cos(n\pi x)$  for  $n \geq 0$ , (note that  $\lambda = 0$  is indeed an eigenvalue).

Also useful is the case of  $A_{\text{per}} = -\partial_{xx}^2$  with *periodic* boundary conditions on  $I$ , leading to

1.  $\lambda_0 = 0$  and  $e_0 = 1$ ,
2. for  $n \geq 1$ ,  $\lambda_n = 4\pi^2 n^2$ , the 2-dimensional eigenspace being spanned by orthogonal eigenvectors  $e_{n,1} \propto \cos(2\pi n x)$  and  $e_{n,2} \propto \sin(2\pi n x)$ ;

in which case we recover the celebrated *Fourier series* decomposition.

#### A.2.2. Fractional powers for operators without a compactness property

Consider  $A = -\partial_{xx}^2$  on the *unbounded* interval  $\mathbb{R}$ , we know from Fourier analysis in  $\mathcal{H} = L^2(\mathbb{R})$  that this operator can be diagonalized as follows, with  $P = \mathcal{F} : L^2(\mathbb{R}_x) \rightarrow L^2(\mathbb{R}_\xi)$ , the unitary Fourier transform, and  $P^{-1} = \mathcal{F}^{-1} : L^2(\mathbb{R}_\xi) \rightarrow L^2(\mathbb{R}_x)$ :

$$\begin{aligned} \hat{A} : L^2(\mathbb{R}_\xi) &\rightarrow L^2(\mathbb{R}_\xi) \\ \hat{\varphi} &\mapsto 4\pi^2 \xi^2 \hat{\varphi}, \end{aligned} \quad (32)$$

on the domain  $D(\hat{A}) = L^{2,2}(\mathbb{R}_\xi)$ , where we have set

$$L^{2,s}(\mathbb{R}_\xi) := \left\{ \hat{\varphi} \in L^2(\mathbb{R}_\xi), \int_{\mathbb{R}} (1 + 4\pi^2 \xi^2)^s |\hat{\varphi}|^2 d\xi < \infty \right\}.$$

For  $\gamma > 0$ , it is then not difficult to define the fractional power of  $A$  in the following way:  $A^\gamma = P^{-1} \hat{A}^\gamma P$ , where

$$\begin{aligned} \hat{A}^\gamma : L^2(\mathbb{R}_\xi) &\rightarrow L^2(\mathbb{R}_\xi) \\ \hat{\varphi} &\mapsto (4\pi^2 \xi^2)^\gamma \hat{\varphi}, \end{aligned} \quad (33)$$

on the domain  $D(\hat{A}^\gamma) = L^{2,2\gamma}(\mathbb{R}_\xi)$ . We can see here that since the compactness is lost, no Hilbert basis will help diagonalize the operator; even though the space  $L^2(\mathbb{R}_x)$  is separable, meaning it has a countable family which is everywhere dense, such as the Hermite functions.

## References

- [1] S. Adhikari, Damping modelling using generalized proportional damping, *J. Sound Vib.* 293 (2006) 156–170.
- [2] V. Arnold, *Lectures on Partial Differential Equations*, Universitext, Springer, 2004.
- [3] D. Bernoulli, Physical, mechanical and analytical researches on sound and on the tones of differently constructed organ pipes, *Mém. Acad. Sci. (Paris)*, 1762 (in French).
- [4] T.K. Caughey, Classical normal modes in damped linear dynamic systems, *Trans. ASME, J. Appl. Mech.* 27 (1960) 269–271.
- [5] T.K. Caughey, M.E.J. O'Kelly, Classical normal modes in damped linear dynamic systems, *Trans. ASME, J. Appl. Mech.* 32 (1965) 583–588.
- [6] R. Causse, J. Bensoam, N. Ellis, Modalys, a physical modeling synthesizer: more than twenty years of researches, developments, and musical uses, *J. Acoust. Soc. Am.* 130 (2011) 2365. (abstract).
- [7] S.P. Chen, R. Triggiani, Proof of two conjectures by G. Chen and D. L. Russell on structural damping for elastic systems, in: *Approximation and Optimization (Havana, 1987)*, vol. 1354, 1988, pp. 234–256.
- [8] S.P. Chen, R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems, *Pac. J. Math.* 136 (1) (1989) 15–55.
- [9] V. Duindam, A. Macchelli, S. Stramigioli, H. Beruvinckx (Eds.), *Modeling and Control of Complex Physical Systems. The Port-Hamiltonian Approach*, Springer Verlag, 2009.
- [10] K. Ege, *La table d'harmonie du piano - Etudes modales en basses et moyennes fréquences* (Thèse de doctorat), Ecole Polytechnique, 2009.
- [11] M. Géradin, D. Rixen, *Mechanical Vibrations: Theory and Application to Structural Dynamics*, John Wiley, 1997.
- [12] K.F. Graff, *Wave Motion in Elastic Solids*, Dover, 1975.
- [13] S. Hansen, Optimal regularity results for boundary control of elastic systems with fractional order damping, *ESAIM: Proc.* 8 (2000) 53–64.
- [14] Hélie Thomas, Mono-dimensional models of the acoustic propagation in axisymmetric waveguides, *J. Acoust. Soc. Am.* 114 (2003) 2633–2647.
- [15] T. Hélie, D. Matignon, Damping models for the sound synthesis of bar-like instruments, in: *7th International Conference on Systemics, Cybernetics and Informatics*, Orlando, Florida, 2001, pp. 541–546 (invited session).
- [16] Hélie Thomas, Hézard Thomas, Mignot Rémi, Matignon Denis, On the 1D wave propagation in wind instruments with a smooth profile, in: *Forum Acusticum*, vol. 6, Aalborg, Denmark, Juillet, 2011, pp. 1–6.
- [17] A. Intissar, *Analyse Fonctionnelle et théorie spectrale pour les opérateurs compacts non auto-adjoints*, Cépaduès, 1997.
- [18] B. Jacob, C. Trunk, M. Winklmeier, Analyticity and Riesz basis property of semigroups associated to damped vibrations, *J. Evol. Equ.* 8 (2) (2008) 263–281.
- [19] B. Jacob, H.J. Zwart, *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*, Series: Operator Theory: Advances and Applications, Subseries: Linear Operators and Linear Systems, vol. 223, Birkhäuser, 2012.
- [20] J. Kergomard, V. Debut, D. Matignon, Resonance modes in a 1-D medium with two purely resistive boundaries: calculation methods, orthogonality and completeness, *J. Acoust. Soc. Am.* 119 (2006) 1356–1367.
- [21] J.L. Lagrange, *Nouvelles recherches sur la nature et la propagation du son*, Misc. Taur. (Mélanges Phil. Math., Soc. Roy. Turin) 1 151–316.
- [22] Y. Le Gorrec, H. Zwart, B. Maschke, Dirac structures and Boundary Control Systems associated with Skew-Symmetric Differential Operators, *SIAM J. Control Optim.* 44 (5) (2005) 1864–1892.
- [23] D. Matignon, Diffusive representations for fractional Laplacian: systems theory framework and numerical issues, *Phys. Scr.* T136 (2009) 014009 (6 p.) (<http://oatao.univ-toulouse.fr/3889/>).
- [24] D. Matignon, T. Hélie, On damping models preserving the eigenfunctions of conservative systems: a port-Hamiltonian perspective. In: *IFAC Conference on Lagrangian and Hamiltonian Methods and Nonlinear Control (LHMNLC'12)* August 29–31, 2012, Bertinoro, Italy (invited session).
- [25] A.W. Naylor, G.R. Sell, *Linear Operator Theory in Engineering and Science*, Applied Mathematical Sciences Series, vol. 40, Springer Verlag, 1982.
- [26] J.W.S. Rayleigh, *Theory of Sound*, 2nd edition, Dover, New York, 1896, Reprinted 1945.
- [27] L.N. Trefethen, M. Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, 2005.
- [28] A. van der Schaft, B. Maschke, Fluid Dynamical Systems as Hamiltonian Boundary Control Systems, Memorandum 1575, University of Twente, 2001.
- [29] A. van der Schaft, B. Maschke, Port-Hamiltonian systems: network modeling and control of nonlinear physical systems, in: *Advanced Dynamics and Control of Structures*, CISM International Centre for Mechanical Sciences, vol. 444, 2004, Springer.
- [30] J. Villegas, Y. LeGorrec, H. Zwart, B. Maschke, Boundary control for a class of dissipative differential operators including diffusion systems, in: *Mathematical Theory of Networks and Systems (MTNS)*, MoP06.4. Kyoto, Japan, 2006, pp. 297–304 (invited session).
- [31] A.G. Webster, Acoustical impedance, and the theory of horns and of the phonograph, *Proc. Natl. Acad. Sci. USA* 5 (1919) 275–282. (Errata, *Proc. Natl. Acad. Sci. USA* 6 (1920) 320).