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ATTRACTION DOMAINS OF DEGENERATE SINGULAR EQUILIBRIA IN QUASI-LINEAR ODES*

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Abstract. The present paper addresses stability properties of singular equilibria arising in a given family of quasi-linear ODEs. These ODEs model continuous-time methods for root-finding problems and their singular equilibria are *degenerate* in the sense that the linearization of the system at equilibrium yields a singular matrix pencil. The analysis is based upon a normal form defined by a codimension-one regular system coupled with a singular scalar equation. The key step is the formulation of certain codimension-one Lyapunov matrix equations which incorporate the relevant singular information and allow for the construction of Lyapunov functions supporting the stability analysis. This approach makes it possible to state precisely the asymptotic stability of such degenerate equilibria, and provides a local estimation of the corresponding attraction domains. An application to the computation of singular DC operating points in nonlinear circuits is discussed.

Key words. implicit ODE, singularity, normal form, stability of equilibria, matrix equation, Lyapunov function, root-finding, nonlinear circuit

AMS subject classifications. 15A24, 34A09, 34C20, 37B25, 65H20, 94C05

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1. Introduction.

Consider the implicit differential system

$$y' = \xi(y,z) \tag{1a}$$

$$zz' = \zeta(y,z), \tag{1b}$$

where $y \in \mathbb{R}^{n-1}$, $z \in \mathbb{R}$, and the functions $\xi(y,z)$, $\zeta(y,z)$ are sufficiently smooth.

System (1) arises as a local normal form for quasi-linear ODEs $A(u)u' = f(u)$ (where A and f are sufficiently smooth matrix- and vector-valued functions) around a *singular point* satisfying $\det A(u^*) = 0$ and $(\det A)'(u^*)v \neq 0$, $\forall v \in \mathrm{Ker} A(u^*) - \{0\}$ [11, 16, 19, 22]. It is also closely related to certain singular differential-algebraic equations (DAEs) [3, 4, 17]. These singular equations are relevant in problems arising in magnetohydrodynamics, electrical circuits, or power system theory, to name a few [5, 6, 15, 23].

Singularities of the normal form (1) are located in the hyperplane $z = 0$, and can be classified (see [3, 4, 6, 12, 13, 17, 19, 23] and references therein) into *pseudo-equilibria* (defined by the condition $\zeta(y,0) = 0$), *forward impasse points* ($\zeta(y,0) < 0$), or *backward impasse points* ($\zeta(y,0) > 0$). Smooth solutions may be defined through pseudoequilibrium points, whereas a pair of trajectories collapse when reaching an impasse point, either in forward or backward time direction.

The present work is focused on stability issues concerning *singular equilibria* of (1), characterized by the pair of conditions $\xi(y,0) = 0$, $\zeta(y,0) = 0$. Within the context of semiexplicit DAEs, these singular equilibria have been analyzed in [3] under an assumption of regularity on the matrix pencil (see [18] and references therein for

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background on this topic) describing the linearization of the problem. For system (1), this matrix pencil would read $\mu C - D$, μ being a complex parameter, whereas $C = \mathrm{diag}\{I_{n-1}, 0\}$ and D is the Jacobian matrix of the right-hand side of (1) evaluated at equilibrium; regularity of the matrix pencil means that there exists a μ_0 such that $\mu_0 C - D$ is invertible, allowing for the definition of a Kronecker index for the pencil.

In contrast, the present paper will address *degenerate* singular equilibria, displaying a singular pencil in the linearization. Specifically, we will consider ODEs of the form

$$y' = Hy + \beta(y,z), \tag{2a}$$

$$zz' = \lambda z^2 + y^T G y + \gamma(y,z), \tag{2b}$$

where $H,\ G \in \mathbb{R}^{(n-1)\times(n-1)}$, G being symmetric, and $\lambda \in \mathbb{R}$. The functions $\beta(y,z)$ and $\gamma(y,z)$ are $O(\|(y,z)\|)^2$ and $O(\|(y,z)\|)^3$, respectively. Singularity of the matrix pencil arising in the linearization at the origin would in this situation follow from the vanishing of the last row of $\mu C - D$ for any μ, with the above-explained notation.

System (2), with $H = -I_{n-1},\ \lambda = -1/2$, is proved in [19] to describe a normal form around a singular equilibrium of the so-called continuous Newton method (see [18, 20, 21] and the bibliography therein)

$$-J(u)u' = f(u) \tag{3}$$

for sufficiently smooth $f : \mathbb{R}^n \to \mathbb{R}^n$, J being the Jacobian matrix of f. Euler discretization of (3) yields the classical Newton iteration for root-finding and optimization problems. The interest of a continuous-time scheme in this setting stems from its better properties regarding global issues and singular problems, together with the fact that a unique continuous system may lead to different iterative techniques, including damped and accelerated versions of basic methods, through the use of different integration schemes. Therefore, the same continuous-time study may be of interest for a wide family of discrete-time techniques [21].

A singular equilibrium of (3) is defined by the pair of conditions $f(u^*) = 0$ and $\mathrm{rk} J(u^*) < n$, together with the assumption that u^* is a limit point of the set where J is invertible. Singular roots arise for instance in predictor-corrector continuation methods [1], and are mapped into the origin in the normal form (2) [19]. In the discrete-time context of the classical Newton method, several results concerning the existence of locally cone-shaped regions of attraction for these singular zeroes were proved in [8, 10, 14] and references therein. Continuous-time extensions have been addressed in [18, 20], and applications of this approach to the original discrete-time setting can be found in [21].

Nevertheless, several issues in this direction remain open. In section 2, we analyze the actual local shape of attraction domains of degenerate equilibria of (2) and the dynamic phenomena which are responsible for these local shapes. Under certain assumptions, the attraction domain comprises a cone-shaped region, but the actual domain may be larger, which may have important implications regarding the use of (3) and related discretizations in singular root-finding problems. This local shape will be shown to be intimately linked with the nature of singularities surrounding the equilibrium and, in particular, with the backward or forward nature of nearby impasse points. The analysis will be supported on the use of several Lyapunov functions, based in turn on certain $(n-1)$-dimensional Lyapunov matrix equations constructed from the "regular" part (2a) but incorporating the relevant information from the singular equation (2b). An application to the computation of singular DC operating points in nonlinear circuits is discussed in section 3.

The reader is referred to [2] for background on semiflows, invariance, and Lyapunov functions. Useful facts coming from matrix analysis and involving, in particular, the Lyapunov matrix equation can be found in [9].

2. Dynamics around degenerate equilibria. System (2) and, in particular, the continuous Newton method, define a (possibly not complete) flow Φ on the set of regular points

$$\mathcal{X} = \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \; / z \neq 0\}. \tag{4}$$

Nevertheless, in order to analyze the behavior of this flow near the origin, the reader should avoid simply considering an "extension" of this flow to $\mathcal{X} \cup \{0\}$ by adding $\Phi(t, 0) = 0$ for $t \in \mathbb{R}$, since the resulting extension of Φ may not be well defined as a flow. Therefore, our present approach will be based on the fact that well-defined, complete semiflows are induced by (2) on certain positively invariant subsets of $\mathcal{X}$; asymptotic stability of the origin may then be precisely addressed for such semiflows.

Inspired on convergence results for singular equilibria of Newton's method [8, 10, 14, 18, 20], one may conjecture if these asymptotic stability results for (2) may follow from the assumption that λ and the spectral abscissa

$$\alpha = \max_{\mu \in \sigma(H)} \operatorname{Re} \mu \tag{5}$$

are negative. Note that, for the continuous Newton method, it is $\alpha = -1, \; \lambda = -1/2$. System (7) in section 2.1 will prove that this conjecture is false in general. It is shown in section 2.2 that, at least, the conditions $\alpha < 0, \; \lambda < 0$ make it possible to prove that nearby trajectories remain on a given neighborhood of the origin as long as they are defined.

If the matrix G is positive definite, then the assumptions $\alpha < 0, \; \lambda < 0$ suffice indeed to guarantee the asymptotic convergence to the origin of all trajectories emanating from *regular* points within a given neighborhood of the equilibrium. This is illustrated by the case $\eta > 0$ in section 2.1, and proved in general in section 2.3. The local attraction domains of such singular roots are therefore significantly larger than a cone-shaped region, making those solutions more easily computable through Newton-based techniques.

Nevertheless, without the positive definiteness of G, this nice behavior will not be displayed. It will be shown in section 2.4 that, under the dominance condition $\alpha < \lambda < 0$, there exists a cone-shaped region with vertex in the origin which is positively invariant and asymptotically convergent to the degenerate equilibrium. Compared with previous approaches in this direction (see [18] and references therein), the Lyapunov function method used to prove this result will additionally give a hint for the estimation of the actual local shape of the attraction domain. An example is provided by the case $\eta < 0$ in section 2.1.

Particularization of these results for the continuous Newton method and an application in circuit theory can be found in sections 2.5 and 3, respectively.

2.1. A glimpse. Consider the vector field $f(y, z) = (y, z^2 + 2\eta y^2)$, with $(y, z) \in \mathbb{R}^2$, $\eta \in \mathbb{R}$. The continuous Newton method (3) reads $y' = -y, \; 4\eta y y' + 2zz' = -z^2 - 2\eta y^2$ or, equivalently,

$$y' = -y, \tag{6a}$$

$$zz' = -(1/2)z^2 + \eta y^2, \tag{6b}$$

which has the form depicted in (2), with $Hy = -y,\ \lambda = -1/2,\ y^T Gy = \eta y^2, \beta = \gamma = 0$. The origin may be easily shown to be a degenerate singular equilibrium regardless of the value of $\eta \in \mathbb{R}$. Nevertheless, this parameter strongly influences the dynamic behavior around the origin, as discussed below.

Note that the particular value $\eta = 0$ yields a removable singularity, since (6) would amount in this case to $y' = -y,\ z' = -z/2$, leading to an asymptotically stable equilibrium in the classical sense. This nongeneric phenomenon has been analyzed in [18, 20, 21] and will not be considered further here.

Case $\eta > 0$. Solutions of (6) read

$$\begin{aligned} y(t) &= y_0 e^{-t}, \\ z(t) &= \mathrm{sg}(z_0)\sqrt{z_0^2 e^{-t} + 2\eta y_0^2(e^{-t} - e^{-2t})}. \end{aligned}$$

Trajectories with $z_0 \neq 0$ are well defined for all $t \geq 0$, since the radicand is always positive. Furthermore, every initial point with $z_0 \neq 0$ converges to the origin. This means that the domain of attraction of the origin is the set of regular points, showing that of singular roots in Newton-based techniques may be significantly larger than a locally cone-shaped set.

Concerning general systems of the form (2), this behavior will be shown in section 2.3 to follow from the positive definiteness of G, together with the assumptions $\lambda < 0$, $\alpha < 0$. The first condition forces all singularities in a neighborhood of the equilibrium to behave as backward impasse points, and amounts in this example to $\eta > 0$. Trajectories are then repelled by the singular manifold and must evolve towards the equilibrium.

Case $\eta < 0$. In this case, if we rewrite the radicand as $(z_0^2 + 2\eta y_0^2)e^{-t} - 2\eta y_0^2 e^{-2t}$, it is not difficult to check that initial points (y_0, z_0) in the cone-shaped region

$$\{(y_0, z_0) \in \mathbb{R}^2 : z_0^2 + 2\eta y_0^2 \geq 0\} \ \equiv\ \{(y_0, z_0) \in \mathbb{R}^2 : |y_0| \leq |z_0|/\sqrt{2|\eta|}\}$$

guarantee that the radicand remains positive and, therefore, trajectories are well defined for all positive t. It is also easy to check that all these solutions converge to the origin.

On the contrary, the condition $z_0^2 + 2\eta y_0^2 < 0 \equiv |y_0| > |z_0|/\sqrt{2|\eta|}$ yields a positive collapse-time

$$t^* = \ln\left(\frac{2\eta y_0^2}{z_0^2 + 2\eta y_0^2}\right),$$

beyond which trajectories are not defined. Note that $2\eta y_0^2 < z_0^2 + 2\eta y_0^2 < 0$ and, therefore, $t^* > 0$. This means that trajectories outside the cone evolve towards a forward impasse point, where they cease to exist. This directional convergence phenomenon will be shown in section 2.4 to be a general property of systems of the form (2) with $\alpha < \lambda < 0$ and, in particular, of the continuous Newton method. The actual local shape of the attraction domain can be estimated with the Lyapunov function approach discussed there.

A generalization of (6). If we finally consider the system

$$y' = \alpha y, \tag{7a}$$
$$zz' = \lambda z^2 + \eta y^2, \tag{7b}$$

with $\eta < 0$, $\lambda < \alpha < 0$, it may be shown that solutions are

$$y(t) = y_0 e^{\alpha t},$$
$$z(t) = \mathrm{sg}(z_0)\sqrt{\left(z_0^2 + \frac{\eta}{\lambda - \alpha} y_0^2\right) e^{2\lambda t} - \frac{\eta}{\lambda - \alpha} y_0^2 e^{2\alpha t}}.$$

Now, any $y_0 \neq 0$ yields a positive escape time

$$t^* = \frac{\ln\left(1 + \frac{(\lambda-\alpha)z_0^2}{\eta y_0^2}\right)}{2(\alpha - \lambda)},$$

where it is to be noted that both $(\lambda - \alpha)z_0^2/\eta y_0^2$ and $\alpha - \lambda$ are positive. This means that only the z-coordinate curve is convergent to the origin and suggests that the dominance condition $\alpha < \lambda < 0$ is a key requirement in the phenomenon of directional stability.

2.2. Positive invariance with $\alpha < 0$, $\lambda < 0$. As a preliminary result let us recall that, writing as $\mu_1, \ldots, \mu_n$ the n (not necessarily distinct) real eigenvalues of an $n \times n$ symmetric matrix A, and denoting

$$\eta_A = \min\{\mu_1, \ldots, \mu_n\}, \tag{8a}$$
$$\kappa_A = \max\{\mu_1, \ldots, \mu_n\}, \tag{8b}$$

then $\eta_A|x|^2 \leq x^T A x \leq \kappa_A |x|^2$ for any vector $x \in \mathbb{R}^n$, $|\ |$ standing for the Euclidean norm.

PROPOSITION 1. *Consider a quasi-linear ODE of the form* (2) *with $\lambda < 0$. Assume that the spectral abscissa defined in* (5) *verifies $\alpha < 0$, and denote $\tilde{\kappa} = \max\{\kappa_G, 0\} \geq 0$, where κ is defined as in* (8b). *Let P be the positive definite solution of the $(n-1)$-dimensional Lyapunov matrix equation*

$$PH + H^T P = 2(-\tilde{\kappa} + \lambda) I_{n-1}. \tag{9}$$

Then, there exists $r_0 > 0$ such that

$$V(y, z) = y^T P y + z^2 \tag{10}$$

satisfies $V' \leq 0$ on $\{x = (y, z) \in \mathcal{X}\ /\ |x| \leq r_0\}$. Hence, for $0 < V_0 < \min_{|x|=r_0} V(y, z)$, the level sets $\{x = (y, z) \in \mathcal{X}\ /\ V(y, z) \leq V_0\}$ are positively invariant.

Proof. The derivative of V along trajectories of (2) reads

$$V' = y^T(PH + H^T P)y + 2[y^T P\beta(y, z) + \lambda z^2 + y^T G y + \gamma(y, z)].$$

Now, using (9) and the fact that $y^T G y \leq \tilde{\kappa}|y|^2$, we have

$$\begin{aligned} V' &\leq 2[(-\tilde{\kappa} + \lambda)|y|^2 + y^T P\beta(y, z) + \lambda z^2 + \tilde{\kappa}|y|^2 + \gamma(y, z)] \\ &= 2[\lambda(|y|^2 + z^2) + y^T P\beta(y, z) + \gamma(y, z)]. \end{aligned}$$

Let $r_\beta,\ c_\beta,\ r_\gamma,\ c_\gamma$ be positive constants such that

$$|x| \leq r_\beta \Rightarrow |\beta(x)| \leq c_\beta |x|^2,$$
$$|x| \leq r_\gamma \Rightarrow |\gamma(x)| \leq c_\gamma |x|^3,$$

and define

$$r_0 = \min\left\{r_\beta,\ r_\gamma,\ \frac{|\lambda|}{\kappa_P c_\beta + c_\gamma}\right\}.$$

Then, $|x| \leq r_0$ implies that

$$|y^T P\beta(y,z) + \gamma(y,z)| \leq |y|\kappa_P c_\beta |x|^2 + c_\gamma |x|^3 \leq r_0(\kappa_P c_\beta + c_\gamma)|x|^2 \leq |\lambda||x|^2$$

and, therefore, $V' \leq 0$. Positive invariance of the level sets $\{x = (y,z) \in \mathcal{X} \ / \ V(y,z) \leq V_0\}$ follows from [2, Theorem 18.2]. ◻

Denoting

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix},$$

the level sets $\{x = (y,z) \in \mathcal{X} \ / \ V(y,z) \leq V_0\}$ can be alternatively described in terms of the Hilbert norm

$$\|x\| = \sqrt{x^T \tilde{P} x} \equiv \sqrt{y^T P y + z^2} = \sqrt{V(y,z)}. \tag{11}$$

To this end, let us define

$$\mathcal{B}(0,\rho) \ = \{x \in \mathbb{R}^n \ / \ \|x\| \leq \rho\}, \tag{12a}$$
$$\mathcal{B}^+(0,\rho) = \{x = (y,z) \in \mathbb{R}^n \ / \ \|x\| \leq \rho,\ z > 0\}, \tag{12b}$$
$$\mathcal{B}^-(0,\rho) = \{x = (y,z) \in \mathbb{R}^n \ / \ \|x\| \leq \rho,\ z < 0\}, \tag{12c}$$
$$\mathcal{B}^\pm(0,\rho) = \mathcal{B}^+(0,\rho) \cup \mathcal{B}^-(0,\rho) = \mathcal{B}(0,\rho) \cap \mathcal{X}. \tag{12d}$$

From the relations $\eta_{\tilde{P}}|x|^2 \leq \|x\|^2 \leq \kappa_{\tilde{P}}|x|^2$, it is easy to check that $\min_{|x|=r_0} V(y,z) = \eta_{\tilde{P}} r_0^2$. Defining $\rho_0 = \sqrt{\eta_{\tilde{P}}}r_0$, the following result can be immediately derived from Proposition 1.

Corollary 1. *If $\alpha < 0$ and $\lambda < 0$, system* (2) *induces a semiflow Φ on $\mathcal{B}^\pm(0,\rho)$ for all positive $\rho < \rho_0$.*

Note that $\mathcal{B}^\pm(0,\rho)$ excludes points in the hyperplane $z = 0$. From the result above the reader should not conclude that, in general, trajectories evolve towards the origin; note that there might be forward impasse points on $z = 0$ attracting solutions in finite time. In this situation, the additional dominance condition $\alpha < \lambda < 0$ (verified, in particular, by the continuous Newton method), allows one to prove the existence of a locally cone-shaped region which is positively invariant and asymptotically convergent to the origin, as shown in section 2.4. Nevertheless, in the particular case in which the origin is entirely surrounded by backward impasse points, then the origin is actually a stable attractor for the dynamics on $\mathcal{B}^\pm(0,\rho)$, without the need for the above-mentioned dominance condition. This simpler case, associated with the positive definiteness of G, is considered in section 2.3.

2.3. Completeness and asymptotic stability with positive definite G.

Theorem 1. *Consider the quasi-linear ODE* (2)*. Assume that $\alpha < 0$, $\lambda < 0$, and that G is positive definite. Then, there exists a positive $\rho_1 < \rho_0$ such that the semiflow Φ induced by* (2) *on $\mathcal{B}^\pm(0,\rho_1)$ is complete; that is, all solutions are defined on the time interval* $[0,\infty)$*. Furthermore,* $\lim_{t\to\infty}\Phi(t,x_0) = 0$ *for all x_0 in $\mathcal{B}^\pm(0,\rho_1)$.*

Proof. The key aspect here is that the smallness of ρ_1 must guarantee that all singularities in $\mathcal{B}(0,\rho_1) - \{0\}$ are backward impasse points. To achieve this, choose any positive r_1 satisfying

$$r_1 < \min\left\{r_0, \frac{\eta_G}{c_\gamma}\right\}, \tag{13}$$

where $\eta_G > 0$ since G is positive definite, and define $\rho_1 = \sqrt{\eta_{\tilde{P}}}r_1$. Hence, if $\|(y,0)\| \leq \rho_1 = \sqrt{\eta_{\tilde{P}}}r_1$, then $|y| \leq \|(y,0)\|/\sqrt{\eta_{\tilde{P}}} \leq r_1$ and $|\gamma(y,0)| \leq c_\gamma|y|^3 \leq c_\gamma r_1|y|^2 < \eta_G|y|^2$ (note that $y = 0$ is excluded). This implies that $\gamma(y,0) > -\eta_G|y|^2$ and, since $y^TGy \geq \eta_G|y|^2$,

$$y^TGy + \gamma(y,0) > 0,\ (y,0) \in \mathcal{B}(0,\rho_1) - \{0\},$$

showing that all singularities in $\mathcal{B}(0,\rho_1) - \{0\}$ are backward impasse points. This fact will suffice to prove that all trajectories are well defined in $[0,\infty)$.

Suppose that $x_0 = (y_0,z_0) \in \mathcal{B}^+(0,\rho_1)$ is such that the trajectory emanating from this point is defined for a maximal positive time $t^+(x_0) < \infty$ (the reasoning in $\mathcal{B}^-(0,\rho_1)$ would be entirely analogous). In this situation, [2, Proposition 10.12] shows that for every compact set $\mathcal{C}$ on $\mathcal{B}^+(0,\rho_1)$ there would exist a time $t_0 < t^+(x_0)$ such that $\Phi(t,x_0) \notin \mathcal{C}$ for $t > t_0$. If we define $\mathcal{C}_\varepsilon = \{(y,z) \in \mathcal{B}^+(0,\rho_1)\ /\ z \geq \varepsilon\}$, for every $\varepsilon > 0$ there would exist a $t_0(\varepsilon)$ such that $z(t) < \varepsilon$ if $t > t_0(\varepsilon)$. This shows that, under the assumption $t^+(x_0) < \infty$, it would be $\lim_{t\to t^+(x_0)} z(t) = 0$.

Nevertheless, the positive definiteness of G precludes $z(t)$ from reaching the set $z = 0$ in finite time, as shown below. Since $|\gamma(y,z)| \leq c_\gamma r_1|x|^2 \leq \eta_G|x|^2$, we have $\gamma \geq -\eta_G|x|^2$ and, therefore,

$$\lambda z^2 + y^TGy + \gamma(y,z) \geq \lambda z^2 + \eta_G|y|^2 - \eta_G(|y|^2 + z^2) = (\lambda - \eta_G)z^2.$$

This means that the real-valued function $(\lambda z^2 + y^TGy + \gamma(y,z))/z$ is bounded below by $(\lambda - \eta_G)z$ on $\mathcal{B}^+(0,\rho_1)$. Since orbits do not leave $\mathcal{B}^+(0,\rho_1)$ due to Proposition 1, the z-component of the trajectory emanating from (y_0,z_0), with $z_0 > 0$, satisfies $z(t) \geq z_0e^{(\lambda-\eta_G)t} > 0$ for all finite t, in contradiction with $\lim_{t\to t^+(x_0)} z(t) = 0$. This proves that $t^+(x_0) = \infty$ and, since x_0 is arbitrary, the semiflow is complete on $\mathcal{B}^\pm(0,\rho_1)$.

In this situation, the function V defined in (10) behaves as a classical Lyapunov function, since limit points verifying $z = 0$, $y \neq 0$ are ruled out by the backward nature of impasse points in $\mathcal{B}(0,\rho_1)$. Hence, $\Phi(t,x_0) \to 0$ as $t \to \infty$, for all x_0 in $\mathcal{B}^\pm(0,\rho_1)$. ∎

Under the hypotheses of Theorem 1, the semiflow Φ may be safely extended to $\mathcal{B}^\pm(0,\rho_1) \cup \{0\}$ by adding $\Phi(t,0) = 0$ for $t \in [0,\infty)$, the origin being an asymptotically stable equilibrium of the resulting semiflow. In applications concerning the continuous Newton method, this situation yields a domain of attraction for a singular root which comprises all regular points within a usual ball about the degenerate equilibrium, as compiled in item 1 of Theorem 3 in section 2.5. An example of this nice behavior is given by the case $V_0 = 0$ in the nonlinear circuit presented in section 3.

2.4. Directional convergence with $\alpha < \lambda < 0$ and arbitrary G. If G is not positive definite, forward impasse points of system (2) may (and actually will, if G is not positive semidefinite) be displayed around the origin, precluding the application of the results discussed in section 2.3. Nevertheless, Proposition 1 and Corollary 1 can be strengthened under the additional dominance condition $\alpha < \lambda < 0$, which is satisfied in particular by the continuous Newton method, for which $\alpha = -1$, $\lambda = -1/2$.

In this direction, note that the condition $\alpha < \lambda$ implies that $H - \lambda I_{n-1}$ is an asymptotically stable matrix, since $\sigma(H - \lambda I_{n-1}) = \{\mu - \lambda \,/\, \mu \in \sigma(H)\}$, and, therefore, the condition $\alpha = \max_{\mu\in\sigma(H)} \mathrm{Re}\, \mu < \lambda$ yields $\max_{\mu\in\sigma(H)} \mathrm{Re}\, (\mu - \lambda) < 0$. This fact guarantees that the matrix equation (14) below has indeed a positive definite solution.

THEOREM 2. *Consider a quasi-linear ODE* (2) *with $\alpha < \lambda < 0$, and let Q be the positive definite solution of the $(n-1)$-dimensional Lyapunov matrix equation*

$$Q(H - \lambda I_{n-1}) + (H - \lambda I_{n-1})^T Q = -I_{n-1}. \tag{14}$$

Then, for every $\theta > 0$ (satisfying additionally $\theta < 1/\sqrt{2|\eta_G|}$ if $\eta_G < 0$, where η_G is defined as in (8a)*), there exists a positive $\rho(\theta) < \rho_0$ such that*

$$U(y, z) = y^T Q y - \theta^2 z^2 \tag{15}$$

satisfies $U' \leq 0$ on $\partial\mathcal{K}(0, \theta) \cap \mathcal{B}^{\pm}(0, \rho(\theta))$, where

$$\mathcal{K}(0, \theta) = \{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : y^T Q y \leq \theta^2 z^2\}, \tag{16}$$

and $\partial\mathcal{K}(0, \theta)$ stands for the boundary $\{(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : y^T Q y = \theta^2 z^2\}$. Hence, the set $\mathcal{K}(0, \theta) \cap \mathcal{B}^{\pm}(0, \rho(\theta))$ is positively invariant for the semiflow Φ defined in Corollary 1*. Furthermore, the restriction of Φ to $\mathcal{K}(0, \theta) \cap \mathcal{B}^{\pm}(0, \rho(\theta))$ is complete, and* $\lim_{t\to\infty} \Phi(t, x_0) = 0$ *for all x_0 in $\mathcal{K}(0, \theta) \cap \mathcal{B}^{\pm}(0, \rho(\theta))$.*

Proof. The derivative of U along trajectories of (2) reads

$$U' = y^T(QH + H^T Q)y + 2[y^T Q\beta(y, z) - \theta^2(\lambda z^2 + y^T G y + \gamma(y, z))].$$

In the boundary $\partial\mathcal{K}(0, \theta)$, it is $U = 0$ or, equivalently, $\theta^2 z^2 = y^T Q y$. Therefore, in $\partial\mathcal{K}(0, \theta)$ we have

$$\begin{aligned} y^T(QH + H^T Q)y - 2\lambda\theta^2 z^2 &= y^T(QH + H^T Q)y - 2\lambda y^T Q y \\ &= y^T[Q(H - \lambda I_{n-1}) + (H - \lambda I_{n-1})^T Q]y = -|y|^2, \end{aligned}$$

yielding

$$\begin{aligned} U' &= -|y|^2 - 2\theta^2 y^T G y + 2[y^T Q\beta(y, z) - \theta^2\gamma(y, z)] \\ &\leq -|y|^2 - 2\theta^2\eta_G|y|^2 + 2[y^T Q\beta(y, z) - \theta^2\gamma(y, z)]. \end{aligned}$$

In light of the restriction imposed on θ when $\eta_G < 0$, we always have $-1 - 2\theta^2\eta_G < 0$. The property $U' \leq 0$, together with the other claims in the theorem, follows then from the choice of $\rho(\theta)$ in a way such that $2|y^T Q\beta(y, z) - \theta^2\gamma(y, z)| \leq (1 + 2\theta^2\eta_G)|y|^2$ whenever $U = 0$, $\|x\| \leq \rho(\theta)$. Details are straightforward and are left to the reader. ∎

2.5. The continuous Newton method at singular roots: Attraction domains.

THEOREM 3. *Consider the normal form* (2) *for the continuous Newton method around a singular equilibrium, for which $H = -I_{n-1}$, $\lambda = -1/2$. Then, the solution of the Lyapunov matrix equation* (14) *reads $Q = I_{n-1}$. Depending on the inertia of G in* (2b)*, the following sets are included in the attraction domain of the origin, $\mathcal{A}(0)$, according to Theorems* 1 *and* 2*:*

1. *If $\eta_G > 0$, that is, if G is positive definite, then there exists a positive ρ_1 such that*

$$\mathcal{B}^{\pm}(0,\rho_1) \subset \mathcal{A}(0),$$

$\mathcal{B}^{\pm}$ being defined in (12b).

2. *If $\eta_G \leq 0$, that is, if G is not positive definite, then for every positive $\theta < 1/\sqrt{2|\eta_G|}$ there exists a positive $\rho(\theta)$ such that the Euclidean cone $\mathcal{K}(0,\theta) = \{(y,z) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y|^2 \leq \theta^2 z^2\}$ verifies that*

$$\mathcal{K}(0,\theta) \cap \mathcal{B}^{\pm}(0,\rho(\theta)) \subset \mathcal{A}(0).$$

In the particular case $\eta_G = 0$, the result holds for any $\theta > 0$.

Note that the cone-shaped regions described in the second item of Theorem 3 are also included in the attraction domain of cases with positive definite G, but in this situation the first item yields a wider estimation of the local domain. It is also worth remarking that weak problems, for which these results may be improved, are not considered here (see [18, 20] and references therein). In cases with nonpositive definite G, the limit value $\theta = 1/\sqrt{2|\eta_G|}$ may provide a rather nice estimation of the actual extension of the cone-shaped region convergent to a singular root in the continuous Newton method. This value will also be shown to play a role in discrete-time counterparts of this method, particularly in the classical Newton iteration. This is illustrated, together with several additional features of these techniques, in a nonlinear circuit example addressed in section 3.

3. Bifurcation points of nonlinear circuits. The circuit displayed in Figure 1 includes an independent current source I_0, a linear capacitor C, two linear resistors R_1 and R_2, a nonlinear voltage-controlled current source (VCCS) with a quadratic characteristic $i = v^2$ (v being the voltage drop across the resistor R_1), a Josephson junction, and an independent voltage source V_0. The Josephson junction consists of two superconductors separated by an oxide barrier [7], and can be considered as a nonlinear inductor characterized by the differential relation $\phi' = v_J$, where ϕ denotes the magnetic flux in the junction, together with a (simplified) sinusoidal current-flux relation $i_J = \sin\phi$. As shown in Figure 1, most parameters in the circuit have been normalized for simplicity.

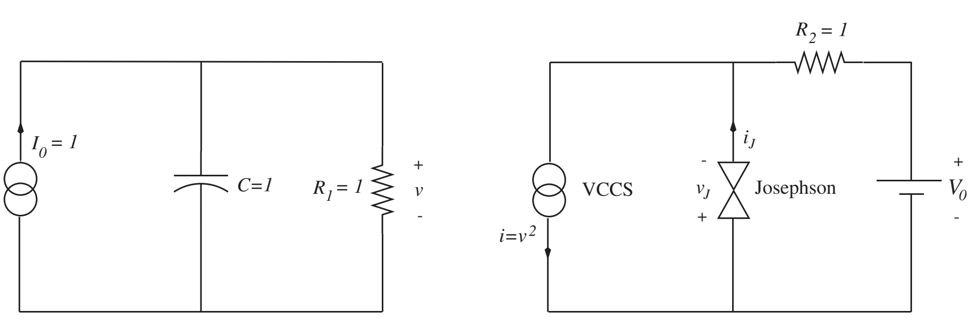


FIG. 1. *Nonlinear circuit.*

The dynamics of this circuit can be described in terms of the capacitor charge q and the flux ϕ in the Josephson junction through the ODE

$$q' = -q + 1, \tag{17a}$$
$$\phi' = q^2 - \sin\phi - V_0, \tag{17b}$$

whereas the continuous Newton method for the right-hand side of (17) can be written as

$$q' = -q + 1, \tag{18a}$$
$$\cos\phi\ \phi' = -q^2 + 2q - \sin\phi - V_0. \tag{18b}$$

Singularities of this quasi-linear ODE are defined by the condition $\cos\phi = 0$, which yields $\phi = \pi/2 + k\pi,\ k \in \mathbb{Z}$. Note that the location of singularities does not depend on the value of V_0. Singular equilibria will be displayed only if $\sin\phi = \pm 1$ at equilibrium (defined by $q = 1$, $\sin\phi = 1 - V_0$), that is, if $V_0 = 0$ or $V_0 = 2$. It can be checked that these values yield saddle-node bifurcations for (17) at $q = 1$, $\phi = \pi/2 + 2k\pi,\ k \in \mathbb{Z}$, for $V_0 = 0$, and $q = 1$, $\phi = -\pi/2 + 2k\pi,\ k \in \mathbb{Z}$, for $V_0 = 2$.

Let us consider the behavior of the continuous Newton method (18) regarding these singular operating points, focusing on

(a) $q = 1$, $\phi = \pi/2$ for $V_0 = 0$;

(b) $q = 1$, $\phi = -\pi/2$ for $V_0 = 2$.

In both cases, the normal form (2) can be easily computed through the coordinate change

$$y = q - 1, \tag{19a}$$
$$z = \cos\phi, \tag{19b}$$

with $\phi \in (0, \pi)$ for (a) and $\phi \in (-\pi, 0)$ for (b). Some simple computations lead to

$$y' = -y, \tag{20a}$$
$$zz' = -(1/2)z^2 \pm y^2 + \text{h.o.t.} \tag{20b}$$

The "+" sign in (20b) corresponds to (a), whereas the "−" case is obtained in (b). Note that the quadratic terms of (20b) are those of (6b) with $\eta = 1$ and $\eta = -1$, respectively.

Case (a). Singularities near the equilibrium $q = 1$, $\phi = \pi/2$, for $V_0 = 0$, are backward impasse points and, in light of the results in section 2.3 (see also item 1 in Theorem 3), every regular point sufficiently close to the singular equilibrium must evolve towards the origin. Computer simulations indicate that this is actually the case: Figure 2(a) displays an estimation of the attraction domain of this singular operating point. Note that the boundary of the attraction domain is partially defined by the straight lines $\phi = -\pi/2$ and $\phi = 3\pi/2$, which correspond to singularities of the quasi-linear ODE (18).

Case (b). Concerning the equilibrium $q = 1$, $\phi = -\pi/2$, for $V_0 = 2$, it follows from the results discussed in section 2.4 and compiled in item 2 of Theorem 3 that, for all $\theta < 1/\sqrt{2}$, there must exist a $\rho(\theta)$ defining a region of the form $y^2 \leq \theta^2 z^2$ positively invariant and convergent to the origin. In the limit case $\theta = 1/\sqrt{2}$, the set

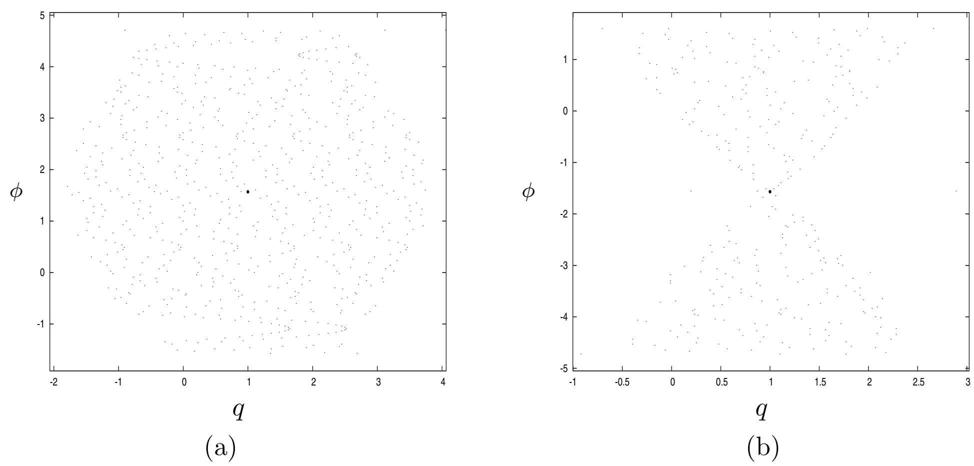


FIG. 2. *Domains of attraction of the continuous Newton method:* (a) *equilibrium at* $(1, \pi/2)$*,* $V_0 = 0$*;* (b) *equilibrium at* $(1, -\pi/2)$*,* $V_0 = 2$*.*

$y^2 = (1/2)z^2$ reads, in the original coordinates q, ϕ and using (19),

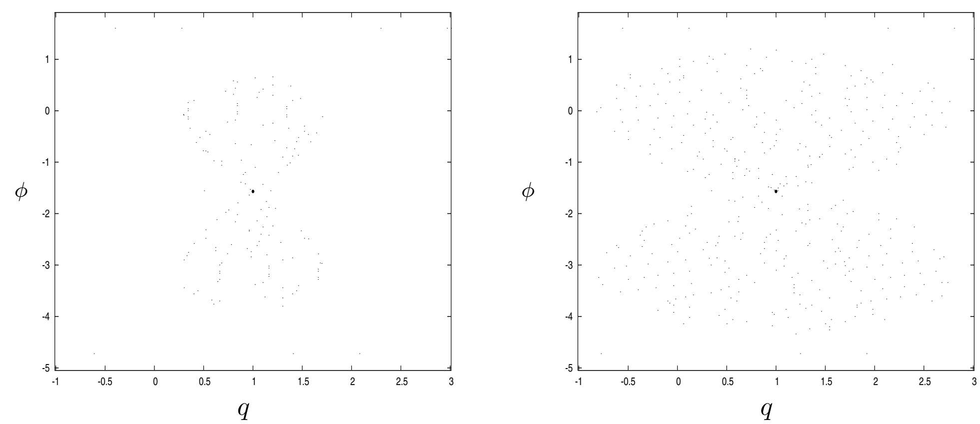
$$(q-1)^2 = (1/2)\cos^2\phi, \quad -\pi < \phi < 0. \tag{21}$$

This curve is plotted in Figure 2(b), together with a computer estimation of the attraction domain of the singular equilibrium. The figure clearly indicates that (21) provides, in this case, an accurate estimation of the boundary of the attraction domain near the degenerate solution. It is worth remarking that, as we move away from the singularity, the incidence of higher order terms becomes more significant and the divergence between the curve representing (21) and the boundary of the attraction domain is more apparent. Equation (21) is not plotted for $\phi < -\pi$ and $\phi > 0$ since the coordinate change (19) remains valid only for $-\pi < \phi < 0$. attraction domain is clearly delimited by the straight lines

With illustrative purposes, let us briefly address this directional convergence phenomenon in the discrete-time setting. Figure 3(a) displays the set of points which converge in the classical Newton iteration (obtained as the Euler discretization of (18) with stepsize 1) to the degenerate equilibrium without crossing the singular manifold; that is, the iteration is truncated for any orbit which jumps from one side of the singular manifold to the other (a jump which is allowed by the discrete-time nature of the method). The curve (21) again provides an accurate estimation of the boundary of the attraction domain for this truncated iteration.

The computer estimation of the actual local domain of attraction of the singular equilibrium for the discrete-time method is shown in Figure 3(b). This domain comprises not only the one shown in (a), but also those points which converge to the solution after crossing the singular manifold. For instance, the initial point $q = 2.5,\ \phi = -0.5$ may be shown to "jump over" the singular manifold in the first iteration step, reaching $q = 1.0,\ \phi = -3.6571$ and then evolving towards the singular root without additional jumps. Note that the first iteration step is exact in the q-component due to the decoupled and linear nature of (18a).

Although additional details in this direction are beyond the purposes of the present paper, let us finally remark here that this approach provides a linearly con-



(a) (b)

FIG. 3. *Local domains of attraction of* $(1, -\pi/2)$*:* (a) *truncated and* (b) *standard discrete-time Newton method,* $V_0 = 2$.

vergent iteration to the singular root. This follows from the value $\lambda = -1/2$ in the normal form (2b): Euler discretization with stepsize 1 places an eigenvalue at $1/2$ in the linearized discrete-time system. Quadratic convergence to singular roots may be recovered through the use of certain Runge–Kutta discretizations; see [21].

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