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A ONE-DIMENSIONAL TWO-BODY PROBLEM OF CLASSICAL ELECTRODYNAMICS*

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Abstract. The equations of motion for a one-dimensional two-body problem of electrodynamics were recognized by Driver [3] as a system of delay differential equations whose delays depend on the unknown trajectories of the charged particles. The following was shown to be true. Let rather arbitrary initial trajectories for a pair of particles be given on some interval $[a, t_0]$, with each particle lying in the force field of the other at time t_0. Then there exist unique trajectories satisfying the equations of motion for all $t > t_0$, unless and until the particles collide.

In this paper we extend these results to include the more general case where the particles are not necessarily in each other's force field at the instant t_0. For this more general case the equations of motion lead to an ordinary vector differential equation $x' = f(t, x)$, $x(t_0) = x_0$, with the initial point (t_0, x_0) lying on the boundary of the domain D on which the function f is well-defined. Such a problem has been referred to as the "singular Cauchy problem" or the "limit Cauchy problem". Existence and uniqueness for the two-body problem are obtained without requiring the particles to be in each others force field at any specific time.

An Appendix presents an example illustrating that the existence theorems of Cauchy–Peano and Carathéodory are not sufficient in proving existence for the two-body problem.

1. Introduction. The equations of motion for a two-body problem of classical electrodynamics have been explicitly formulated by Driver [2], [3], and recognized as a system of delay differential equations whose delays depend on the unknown trajectories. The delays arise because electrical effects are propogated at the finite speed of light. Thus, the force exerted on one particle at any given time depends on the position and velocity of the other particle at some previous time. Advanced arguments and radiation reaction are omitted.

The special case in which motion is restricted to one dimension has been considered [3], [4]. Let rather arbitrary initial trajectories for a pair of particles be given on some interval $[\alpha, t_0]$ with each particle being in the force field of the other at time t_0. Then it was shown that there exist unique trajectories satisfying the equations of motion for all $t > t_0$, unless and until the particles collide.

In the present paper we are interested in extending these results to include the case where the particles are not necessarily in each other's force field at any specific time. Also, more generality is allowed in the domain of definition for the initial trajectories. The equations of motion are identical to those in [3], except that no external electric field is considered.

2. Equations of motion, properties of trajectories and statement of the problem. Let $x_i(t)$, $i = 1, 2$, denote the positions of two point charges on the x-axis at time t, with velocities given by

$$x_i'(t) = v_i(t), \qquad i = 1, 2. \tag{1}$$

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For $j = 1, 2$ and $i \neq j$, the field, $E_j(t, x_i(t))$, at $x_i(t)$ at time t due to particle j depends on the position and velocity of particle j at some previous time $t - \tau_{ji}(t)$, where $\tau_{ji}(t)$ must satisfy the functional equation

$$c\tau_{ji}(t) = |x_i(t) - x_j(t - \tau_{ji}(t))|, \quad (i,j) = (1,2),(2,1). \tag{2}$$

The equation of motion for particle i is

$$\frac{m_i v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = q_i E_j(t, x_i(t)), \quad (i,j) = (1,2),(2,1), \tag{3}$$

where m_i is the rest mass of particle i, q_i is the magnitude of its charge, and c is the speed of light. Thus, (1), (2) and (3) form a system of six functional and functional differential equations. For convenience, equations will sometimes be written with the understanding that $i = 1$ or 2 and $(i,j) = (1,2)$ or $(2,1)$ without explicitly stating so.

Assuming any initial trajectories are such that no collisions occur, there is no loss of generality in taking $x_2(t) > x_1(t)$. If it is further assumed that each $x_i(t)$ is continuous or differentiable on appropriate intervals, with $|v_i(t)| < c$ for all t, the $E_j(t, x_i(t))$ has an explicit expression. In M.K.S. units, equation (3) becomes

$$\frac{v_i'(t)}{[1 - v_i^2(t)/c^2]^{3/2}} = \frac{(-1)^i q_1 q_2}{4\pi\varepsilon_0 m_i c^2 \tau_{ji}^2(t)} \, \frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}, \tag{4}$$
$$(i,j) = (1,2),(2,1),$$

whenever (2) has a solution. The absence of a solution to (2) means particle i is not in the force field of particle j, and hence $E_j(t, x_i(t)) = 0$ under those conditions. These equations can be derived by assuming that the fields produced by the moving charges can be calculated from the Liénard–Wiechert potentials [3].

For the functional differential system (1), (2), (3), it is natural to specify initial functions $x_i(t), i = 1, 2$, on respective intervals $[\alpha_i, t_i], i = 1, 2$, and seek continuous extensions of these functions satisfying (1), (2) and (3) on some interval to the right of t_i, $i = 1, 2$. It is also natural to assume the particles have space-like separation at t_1 and t_2. That is, $c|t_1 - t_2| \leqq |x_2(t_2) - x_1(t_1)|$. This condition insures that the field $E_j(t, x_i(t))$ for any $t \leqq t_i$ depends only on values of $x_j(t)$ for $t \leqq t_j$, and not on the unspecified part of the jth trajectory, that is, for $t > t_j$.

Before stating the problem precisely, a number of results concerning initial trajectories and solutions to (2) will be established. Throughout each of the five following lemmas it will be assumed that initial trajectories are continuously differentiable, even though this condition could often be replaced by absolute continuity, and at times by continuity. This simplification is justified by the fact that the continuity of $v_i(t)$, $i = 1, 2$, will be required for the existence and uniqueness results to be established. We begin with a lemma which follows directly from results in [3]. The proof is omitted.

LEMMA 1. *Let t_0 and γ be numbers such that $-\infty < t_0 < \gamma < \infty$, and let $x_k(t)$, $k = 1, 2$, be continuously differentiable on $(-\infty, \gamma]$ with $|x_k'(t)| < c$, where the derivatives are one-sided at γ. For fixed (i,j) and any $t \in (-\infty, \gamma]$, equation* (2) *has a solution if and only if there exists an $\alpha \in (-\infty, \gamma]$ such that*

$$|x_i(t) - x_j(\alpha)| \leqq c(t - \alpha) \tag{5}$$

and when a solution exists it is unique. Furthermore, if (2) *has a solution at* t_0, *then for all* $t \in [t_0, \gamma]$ *a unique solution* $\tau_{ji}(t)$ *to* (2) *exists,* $\tau_{ji}(t)$ *is continuous, and* $t - \tau_{ji}(t)$ *is strictly increasing. Also, if* $x_2(t) > x_1(t)$ *for all* $t \in (t_0, \gamma)$, *then* $c\tau_{ji}(t) = (-1)^i[x_i(t) - x_j(t - \tau_{ji}(t))]$, $\tau_{ji}(t)$ *is continuously differentiable on* (t_0, γ), *and* τ_{ji} *satisfies the delay differential equation*

$$\tau'_{ji}(t) = \frac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}. \tag{6}$$

Equation (6) *also shows that* $\tau'_{ji}(t) < 1$.

The next lemma characterizes initial trajectories regarding solvability of (2).

LEMMA 2. *Let* $x_k(t)$, $k = 1, 2$, *be continuously differentiable on* $(-\infty, \gamma]$, *or* $(-\infty, \gamma)$ *in case* $\gamma = \infty$, *with* $|x'_k(t)| < c$. *Then, for fixed* (i, j), *one of the following three must be true.* (i) *Equation* (2) *has no solution for any* $t \in (-\infty, \gamma]$. (ii) *Equation* (2) *has a solution for all* $t \in (-\infty, \gamma]$. (iii) *There exists a* $\hat{t} \in (-\infty, \gamma)$ *such that* (2) *has a solution for all* $t \in (\hat{t}, \gamma]$, *but none for* $t \in (-\infty, \hat{t}]$. *Also, each of the three conditions is possible.*

Proof. The following example, based on an example in [3], which in turn was adapted from Havas [6], shows that (i) is possible.

Example 1. Let $t_0 \in (-\infty, \gamma]$ and let $x_1(t) = a_1 - [c^2(t - t_0)^2 + b_1^2]^{1/2}$ and $x_2(t) = a_2 + [c^2(t - t_0)^2 + b_2^2]^{1/2}$ for all $t \in (-\infty, \gamma]$, with $a_2 - a_1 \geqq 0$, $b_1 \neq 0$ and $b_2 \neq 0$. Then, if $(i, j) = (1, 2)$ and (2) has a solution for some $t \in (-\infty, \gamma]$, it follows that

$$\begin{aligned} c\tau_{21}(t) &= -a_1 + [c^2(t - t_0)^2 + b_1^2]^{1/2} + a_2 + [c^2(t - t_0 - \tau_{21}(t))^2 + b_2^2]^{1/2} \\ &> c|t - t_0| + c|t - t_0 - \tau_{21}(t)| \\ &\geqq c\tau_{21}(t), \end{aligned}$$

a contradiction. Thus, (2) has no solution for any $t \in (-\infty, \gamma]$, even though $|x'_k(t)| < c$, $k = 1, 2$, for all $t \in (-\infty, \gamma]$.

A case where (ii) occurs is given by the following.

Example 2. Let $x_2(t)$ be such that $|x'_2(t)| \leqq \bar{c} < c$ for all $t \in (-\infty, \gamma]$, and let $x_1(t)$ be any other trajectory on $(-\infty, \gamma]$ with $|x'_1(t)| < c$. For $(i, j) = (1, 2)$ and any $t \in (-\infty, \gamma]$, define $H(\alpha)$ by $H(\alpha) = |x_1(t) - x_2(\alpha)| - c(t - \alpha)$. Hence

$$\begin{aligned} H(\alpha) &= |x_1(t) - x_2(t) + x_2(t) - x_2(\alpha)| - c(t - \alpha) \\ &\leqq |x_1(t) - x_2(t)| - (c - \bar{c})(t - \alpha). \end{aligned}$$

This implies $H(\alpha) \leqq 0$ for $\alpha \leqq t - |x_1(t) - x_2(t)|/(c - \bar{c})$ and it follows from (5) that (2) has a solution at t.

If neither (i) nor (ii) holds, define $\hat{t} = \inf\{t\colon (2) \text{ has a solution at } t\}$. Then it follows from Lemma 1 that $\hat{t} > -\infty$. It also follows from Lemma 1 that (2) has a solution for all $t > \hat{t}$. We show that (2) cannot have a solution at $\hat{t}$ by proving the set $B = \{t\colon (2) \text{ has a solution at } t\}$ is an open set, as indicated in [5]. To prove this, assume (2) has a solution at some $t_0 \in (-\infty, \gamma]$. Then there exists an α such that $|x_i(t_0) - x_j(\alpha)| \leqq c(t_0 - \alpha)$. Let β be any number such that $\beta < \alpha$, and let

$$\varepsilon = [c - \sup_{\beta \leqq s \leqq \alpha} |x'_j(s)|](\alpha - \beta) > 0\cdot$$

Take any $t \in (t_0 - \varepsilon/(2c), t_0]$. Then

$$
\begin{aligned}
|x_i(t) - x_j(\beta)| - c(t-\beta) &= \left| x_i(t_0) + \int_{t_0}^{t} x_i'(s)\,ds - x_j(\alpha) \right. \\
&\quad \left. - \int_{\alpha}^{\beta} x_j'(s)\,ds \right| - c(t_0 - \alpha) - c(t - t_0) - c(\alpha - \beta) \\
&\leqq c(t_0 - \alpha) + c(t_0 - t) + \sup_{\beta \leqq s \leqq \alpha} |x_j'(s)|(\alpha - \beta) \\
&\quad - c(t_0 - \alpha) - c(t - t_0) - c(\alpha - \beta) \\
&= 2c(t_0 - t) - \varepsilon \leqq 2c(\varepsilon/(2c)) - \varepsilon = 0.
\end{aligned}
$$

That is, $|x_i(t) - x_j(\beta)| \leqq c(t - \beta)$ and (2) has a solution at t. This along with Lemma 1 proves B is an open set, and therefore (iii) holds if neither (i) nor (ii) holds. A case where (iii) is true is given by the following.

Example 3. Let $x_1(t) = -(c^2t^2 + b^2)^{1/2}$ for all t and let $x_2(t) \equiv b > 0$. Then, with $(i, j) = (2, 1)$ it is easily shown that (2) has a solution if and only if $t > b/c$. This completes the proof of Lemma 2.

The last example in the above proof illustrates that the case which holds depends on the choice of (i, j). For if $(i, j) = (1, 2)$ in Example 3, then (ii) holds, since for any t, $|x_1(t) - x_2(\alpha)| = b + (c^2t^2 + b^2)^{1/2}$, which is less than or equal to $c(t - \alpha)$ for $\alpha \leqq t - (b + (c^2t^2 + b^2)^{1/2})/c$. That is, (2) has a solution for all t if $(i, j) = (1, 2)$.

As stated previously, we are interested in specifying initial trajectories $x_k(t)$, $k = 1, 2$, respectively on $(-\infty, t_k]$, $k = 1, 2$, and finding continuous extensions satisfying (1), (2) and (3). If the initial trajectories are such that for some fixed (i, j) equation (2) has no solution at t_i, then it is conceivable that the existence of solutions to (2) for $t > t_i$ for extended trajectories would depend on the nature of the extensions. Specifically, (2) might have a solution for all $t > t_i$ for one extension, and have no solution for values of $t > t_i$ for another extension. These possibilities must be considered because the form of $E_j(t, x_i(t))$ depends on whether or not (2) has a solution at time t. The following lemma makes it possible to determine whether $E_j(t, x_i(t))$ will be zero or nonzero for $t > t_i$, based only on the values of $x_i(t)$ for $t \leqq t_i$, and on the trajectory of particle j.

LEMMA 3. *Let $x_k(t)$, $k = 1, 2$, be continuously differentiable trajectories defined for all t, with $|x_k'(t)| < c$. For fixed (i, j) assume the equation*

$$c\tau = |x_i(t) - x_j(t - \tau)| \tag{7}$$

has a solution if and only if $t > s_i$. Then, if $y_i(t)$ is any continuously differentiable trajectory such that $y_i(t) = x_i(t)$ for $t \leqq s_i$, with $|y_i'(t)| < c$ for all t, the equation

$$c\tau = |y_i(t) - x_j(t - \tau)| \tag{8}$$

also has a solution if and only if $t > s_i$.

Proof. Clearly (8) has no solution for $t \leqq s_i$ because this would imply (7) had a solution for that t, since $x_i(t) = y_i(t)$ for $t \leqq s_i$.

Let $u > s_i$. If $y_i(u) = x_i(u)$, then (8) clearly has a solution at u because (7) does. Therefore, assume $y_i(u) \neq x_i(u)$, and consider the continuous function

$H:[s_i, u] \to R$ defined by

$$H(t) = |y_i(u) - x_i(t)| - c(u - t).$$

Then, $H(u) = |y_i(u) - x_i(u)| > 0$, and

$$H(s_i) = |y_i(u) - x_i(s_i)| - c(u - s_i)$$
$$= |y_i(u) - y_i(s_i)| - c(u - s_i) < 0.$$

Hence, there exists a $v \in (s_i, u)$ such that $H(v) = 0$, and by hypothesis, there exists an α such that $|x_i(v) - x_j(\alpha)| \leqq c(v - \alpha)$. It follows that

$$|y_i(u) - x_j(\alpha)| = |y_i(u) - x_i(v) + x_i(v) - x_j(\alpha)|$$
$$\leqq c(u - v) + c(v - \alpha) = c(u - \alpha).$$

Hence, (8) has a solution at u. Q.E.D.

The following lemma is based on a problem in [5], with the present proof being supplied by this author.

LEMMA 4. *Let continuously differentiable trajectories $x_k(t)$, $k = 1, 2$, be defined on $(-\infty, \gamma]$, with each $|x_k'(t)| < c$. Let $t_1, t_2 \in (-\infty, \gamma]$ and define $t^* = \min\{t_1, t_2\}$. Assume $x_2(t) > x_1(t)$ on $(-\infty, t^*]$, with $c|t_1 - t_2| \leqq |x_1(t_1) - x_2(t_2)|$. Further assume for fixed (i, j) that (2) has a solution if and only if $t > s_i$, for some $s_i \in [t_i, \gamma]$. Then, $\lim_{t \to s_i+} \tau_{ji}(t) = \infty$ and $\overline{\lim}_{t \to -\infty} (-1)^i x_j'(t) = c$.*

Proof. Since $t - \tau_{ji}(t)$ is monotonic by Lemma 1, $\lim_{t \to s_i+} [t - \tau_{ji}(t)]$ exists in the extended sense. That is, it is either $-\infty$ or some finite number. This implies $\lim_{t \to s_i+} \tau_{ji}(t)$ also exists, and is either $+\infty$ or some finite number. Assume $\lim_{t \to s_i+} \tau_{ji}(t) = L < \infty$, and let the sequence $\{t_n\}$ be such that $t_n \to s_i+$. For each t_n, $\tau_{ji}(t_n)$ exists and

$$|x_i(t_n) - x_j(t_n - \tau_{ji}(t_n))| = c\tau_{ji}(t_n).$$

Taking the limit as $n \to \infty$ gives $|x_i(s_i) - x_j(s_i - L)| = cL$. Hence, (2) has a solution at s_i, a contradiction.

For the second part of the lemma, we shall first show that

$$B(t) \equiv (-1)^i(x_i(s_i) - x_j(t)) = |x_i(s_i) - x_j(t)| > 0 \qquad \text{for all } t \in (-\infty, t_j].$$

From the space-like relation and the fact that $x_2(t) > x_1(t)$ on $(-\infty, t^*]$, it follows that

$$(-1)^i[x_i(t_i) - x_j(t_j)] \equiv x_2(t_2) - x_1(t_1)$$
$$\geqq x_2(t^*) - x_1(t^*) - c|t_1 - t_2| > -c|t_1 - t_2|$$
$$\geqq -|x_i(t_i) - x_j(t_j)|.$$

Hence $(-1)^i[x_i(t_i) - x_j(t_j)] = |x_i(t_i) - x_j(t_j)| > 0$. To prove $B(t) > 0$ for all $t \in (-\infty, t_j]$, we consider two cases.

Case 1. $s_i \geqq t_j$. The function $A(t) \equiv (-1)^i[x_i(t) - x_j(t_j)]$ is continuous for $t \in [t_i, s_i]$, and $A(t_i) > 0$. Also, $A(s_i) > 0$, for otherwise $A(u) = 0$ for some

$u \in [t_i, s_i]$. If $A(u) = 0$, either $u \geqq t_j$, in which case

$$|x_i(u) - x_j(t_j)| - c(u - t_j) = A(u) - c(u - t_j) \leqq 0$$

and (2) has a solution at $u \leqq s_i$, a contradiction; or, $u < t_j$, and

$$\begin{aligned}(-1)^i[x_i(t_i) - x_j(t_j)] &= (-1)^i[x_i(t_i) - x_i(u) + x_i(u) - x_j(t_j)] \\ &= (-1)^i[x_i(t_i) - x_i(u)] + A(u) \\ &\leqq c(u - t_i) < c(t_j - t_i) = c|t_j - t_i|,\end{aligned}$$

which contradicts the spacelike property. Therefore, $A(s_i) = (-1)^i[x_i(s_i) - x_j(t_j)] = B(t_j) > 0$. If $B(t) \leqq 0$ for some $t \in (-\infty, t_j)$, then $B(v) = 0$ for some $v \in (-\infty, t_j)$ by continuity. In that case $|x_i(s_i) - x_j(v)| - c(s_i - v) = -c(s_i - v) \leqq 0$ since $s_i \geqq t_j \geqq v$. Hence, (2) has a solution at s_i, a contradiction. Therefore $B(t) > 0$ for all $t \leqq t_j$ if $s_i \geqq t_j$.

Case 2. $s_i < t_j$. By hypothesis, $B(s_i) = (-1)^i[x_i(s_i) - x_j(s_i)] > 0$. Also $B(t) > 0$ for $t \leqq s_i$, for otherwise $B(\omega) = 0$ for some $\omega \in (-\infty, s_i)$. In that case

$$|x_i(s_i) - x_j(\omega)| - c(s_i - \omega) \leqq 0$$

and (2) has a solution at s_i, a contradiction. To show $B(t) > 0$ for $t \in (s_i, t_j]$, assume it is not. Then $B(z) = 0$ for some $z \in (s_i, t_j]$, which implies

$$\begin{aligned}(-1)^i[x_i(t_i) - x_j(t_j)] &= (-1)^i[x_i(t_i) - x_i(s_i) + x_i(s_i) - x_j(z) \\ &\quad + x_j(z) - x_j(t_j)] \\ &= (-1)^i[x_i(t_i) - x_i(s_i)] + B(z) \\ &\quad + (-1)^i[x_j(z) - x_j(t_j)] \\ &\leqq c(s_i - t_i) + c(t_j - z) \\ &\leqq c(s_i - t_i) + c(t_j - s_i) = c|t_i - t_j|,\end{aligned}$$

which contradicts the spacelike property. This completes the proof for Case 2. Hence $B(t) > 0$ for all $t \in (-\infty, t_j]$.

Assume $\overline{\lim}_{t \to -\infty} (-1)^i x_j'(t) = \bar{c} < c$. Then, for all $\varepsilon > 0$ there exists a t_ε such that $(-1)^i x_j'(t) \leqq \bar{c} + \varepsilon$ for all $t \leqq t_\varepsilon$. Choose $\varepsilon > 0$ such that $\bar{c} + \varepsilon < c$. For any $\alpha \in (-\infty, t_\varepsilon]$,

$$\begin{aligned}|x_i(s_i) - x_j(\alpha)| - c(s_i - \alpha) &= (-1)^i[x_i(s_i) - x_j(\alpha)] - c(s_i - \alpha) \\ &= (-1)^i\left[x_i(s_i) - x_j(t_\varepsilon) - \int_{t_\varepsilon}^{\alpha} x_j'(s)\,ds\right] - c(s_i - \alpha) \\ &\leqq (-1)^i[x_i(s_i) - x_j(t_\varepsilon)] + (\bar{c} + \varepsilon)(t_\varepsilon - \alpha) - c(s_i - \alpha) \\ &= (-1)^i[x_i(s_i) - x_j(t_\varepsilon)] - (\bar{c} + \varepsilon)(s_i - t_\varepsilon) \\ &\quad + (\bar{c} + \varepsilon - c)(s_i - \alpha).\end{aligned}$$

Since the right side of the above expression can be made $\leqq 0$ for large enough $(s_i - \alpha)$, it follows that (2) has a solution at s_i, a contradiction. Q.E.D.

The next lemma further characterizes those trajectories for which (2) does not have a solution for some values of t.

LEMMA 5. *Let continuously differentiable trajectories x_i and x_j be defined on $(-\infty, s_i]$ and $(-\infty, t_j]$ respectively, with $|x_k'(t)| < c$, $k = 1, 2$, for all t. Let $t_i \leqq s_i$, and define $t^* = \min\{t_1, t_2\}$. Assume $x_2(t) > x_1(t)$ on $(-\infty, t^*]$, with $c|t_1 - t_2| \leqq |x_1(t_1) - x_2(t_2)| \equiv x_0$. Further assume that the functional equation*

$$c\tau = |y_i(t) - x_j(t - \tau)| \tag{9}$$

has a solution if and only if $t > s_i$, for each continuously differentiable extension $y_i(t)$ of $x_i(t)$ past s_i with $|y_i'(t)| < c$. Then $\lim_{\zeta \to -\infty} \{(-1)^i[x_i(s_i) - x_j(\zeta)] - c(s_i - \zeta)\} = 0$ and for each $\zeta < t_j$, $0 < \int_{-\infty}^{\zeta} [c - (-1)^i v_j(s)]\, ds < 2x_0$, which, in particular, implies the convergence of this improper integral.

Proof. As in Lemma 4, it can be shown that $B(t) \equiv (-1)^i[x_i(s_i) - x_j(t)] > 0$ for all $t \in (-\infty, t_j]$. Consider the continuous function $M:(-\infty, t_j] \to R$ defined by $M(\zeta) = (-1)^i[x_i(s_i) - x_j(\zeta)] - c(s_i - \zeta)$. Then $M(\zeta) > 0$ for all $\zeta \in (-\infty, t_j]$, for if this were not true, (9) would have a solution at s_i by Lemma 1, a contradiction. Because $|x_j'(t)| < c$, $M(\zeta)$ decreases monotonically as $\zeta \to -\infty$. To show $\lim_{\zeta \to -\infty} M(\zeta) = 0$, assume there exists an $\varepsilon > 0$ such that $M(\zeta) > \varepsilon$ for all $\zeta \in (-\infty, t_j]$. Let $y_i(t)$ be any continuously differentiable extension of $x_i(t)$ past s_i with $|y_i'(t)| < c$. Then

$$(-1)^i[y_i(s_i) - x_j(\zeta)] - c(s_i - \zeta) > \varepsilon \quad \text{for all } \zeta \in (-\infty, t_j]. \tag{10}$$

Since the left side of (10) is continuous in s_i, there exists a $\hat{t} > s_i$ such that

$$(-1)^i[y_i(\hat{t}) - x_j(\zeta)] - c(\hat{t} - \zeta) > 0 \quad \text{for all } \zeta \in (-\infty, t_j].$$

For $\zeta < \hat{t}$, it follows that $-c(\hat{t} - \zeta) < 0$. Hence $(-1)^i[y_i(\hat{t}) - x_j(\zeta)] = |y_i(\hat{t}) - x_j(\zeta)|$ and $|y_i(\hat{t}) - x_j(\zeta)| - c(\hat{t} - \zeta) > 0$ for $\zeta \in (-\infty, \hat{t}]$. This last inequality is also true if $\zeta > \hat{t}$, and therefore it holds for all $\zeta \in (-\infty, t_j]$. But, this implies that the functional equation (9) has no solution at $\hat{t} > s_i$, a contradiction. This proves the first assertion of the lemma.

For the second assertion, let $\eta, \zeta \in (-\infty, t_j]$. Then

$$(-1)^i[x_i(s_i) - x_j(\eta)] - c(s_i - \eta) = (-1)^i[x_i(s_i) - x_j(\zeta)] - c(s_i - \zeta) - \int_{\eta}^{\zeta} [c - (-1)^i x_j'(s)]\, ds.$$

Taking the limit as $\eta \to -\infty$, and using the first part of the lemma,

$$0 = (-1)^i[x_i(s_i) - x_j(\zeta)] - c(s_i - \zeta) - \lim_{\eta \to -\infty} \int_{\eta}^{\zeta} [c - (-1)^i x_j'(s)]\, ds.$$

Hence, the improper integral converges, and

$$\int_{-\infty}^{\zeta} [c - (-1)^i x_j'(s)]\, ds = (-1)^i[x_i(s_i) - x_j(\zeta)] - c(s_i - \zeta) = M(\zeta). \tag{11}$$

Furthermore, for all $\zeta \in (-\infty, t_j)$,

$$\begin{aligned}0 < M(\zeta) &= \int_{-\infty}^{\zeta} [c - (-1)^i x_j'(s)]\, ds < \int_{-\infty}^{t_j} [c - (-1)^i x_j'(s)]\, ds \\ &= M(t_j) = (-1)^i[x_i(t_i) - x_j(t_j)] + (-1)^i[x_i(s_i) - x_i(t_i)] \\ &\quad - c(s_i - t_j) \leqq x_0 + c(s_i - t_i) - c(s_i - t_j) \\ &\leqq x_0 + c|t_i - t_j| \leqq 2x_0.\end{aligned}$$

Q.E.D.

We now make precise the problem to be considered.

Problem 1. Let $x_k(t)$, $k = 1, 2$, be continuously differentiable initial trajectories defined respectively on $(-\infty, t_k]$, $k = 1, 2$, with each $|x_k'(t)| < c$ for all t. The derivatives are one-sided at t_k, $k = 1, 2$. Let $t^* = \min\{t_1, t_2\}$ and assume $x_2(t) > x_1(t)$ for $t \in (-\infty, t^*]$. Further assume $c|t_1 - t_2| \leqq |x_1(t_1) - x_2(t_2)|$. We seek continuously differentiable extensions, $x_k(t)$, $k = 1, 2$, of the original trajectories to $(-\infty, \beta)$, for some $\beta > t_k$, $k = 1, 2$, such that

(i) $x_2(t) > x_1(t)$ for $t \in (-\infty, \beta)$,

(ii) $|x_k'(t)| < c$, $k = 1, 2$, for $t \in (-\infty, \beta)$,

(iii) for $k = 1, 2$, $x_k(t)$ and $v_k(t) = x_k'(t)$ satisfy (3) on (t_k, β), where $E_j(t, x_i(t))$ is defined by (4) whenever (2) has a solution at t, and $E_j(t, x_i(t)) = 0$ if (2) does not have a solution.

Besides the omission of the external field, there are a few differences between Problem 1 and the "one-dimensional two-body problem" considered in [3]. One is that initial trajectories extend back to $-\infty$. This is necessary in light of Lemma 4. For, if the original trajectories are such that (2) has a solution if and only if $t > t_i$ for extensions of $x_i(t)$ past $t_i (i = 1$ or $2)$, and if a solution to Problem 1 exists, then $\lim_{t \to t_i+} [t - \tau_{ji}(t)] = -\infty$. Hence (4) can be satisfied for $t \in (t_i, \beta)$ only if $x_j(t)$ is well-defined on $(-\infty, t_j]$.

But, the important difference is that the particles are not required to be within each other's force field at any specific time. It is this fact which makes the analysis significantly different from that in [3].

When the particles are within each other's influence at the end of the initial trajectories, solving the equations of motion reduces to solving a system of ordinary differential equations with the initial point lying in the interior of the set on which the equations are well-defined [3]. Hence, local existence and uniqueness follow from the assumed continuity of initial trajectories and velocities, along with the Lipschitz nature of involved functions. The extended existence and uniqueness follows from the results in [2] for delay differential systems. When the particles are not required to be within each other's force field, the equations of motion again reduce to ordinary differential equations, but the resulting initial point sometimes falls on the boundary of the set where the equations are well-defined. That is, the equations have singularities at the initial points. This requires the use of different methods in establishing existence and uniqueness.

Papers [3] and [4] contain extended existence and uniqueness results for Problem 1 for the case where $t_1 = t_2 \equiv t_0$ and (2) has a solution at t_0 for $(i, j) = (1, 2)$ and $(2, 1)$. Thus we need only extend solutions until these conditions are met.

Finally, the only type of initial trajectories not considered in Problem 1 are those for which $x_1(s) = x_2(s)$ for some $s \leqq t^*$. Since this condition implies that (2) has a solution at s for $(i, j) = (1, 2)$ and $(2, 1)$, it follows from Lemma 1 that (2) has a solution at t_k, $k = 1, 2$. As will be shown in the proof of our main result, in the latter instance one trajectory can be uniquely extended as a solution until its domain of definition coincides with that of the other. From this point the results of Driver [3] can be applied.

3. Existence and uniqueness theorem for Problem 1. The main result of the present paper is the following.

THEOREM 1. *Let $x_k(t)$, $k = 1, 2$, be initial trajectories as described in the statement of Problem 1. Then, there exists a unique solution to Problem 1 on $(-\infty, \beta)$, for some $\beta > \max\{t_1, t_2\}$, where either $\beta = \infty$ or $\lim_{t\to\beta^-} x_1(t) = \lim_{t\to\beta^-} x_2(t)$ (a collision).*

Remark. The proof of Theorem 1 is lengthy. This is in part due to the dual character of $E_j(t, x_i(t))$ which makes the proof depend on the initial trajectories, and also due to the fact that a transformation of variables is needed to apply a result from [7].

However, some simplification occurs because the six equations (1), (2), (3) are, in a sense, uncoupled. That is, the trajectories do not have to be extended simultaneously. Each trajectory can be extended separately, as will become clear in the proof.

To make the proof more readable, parts of it will be separated out as lemmas.

Proof of Theorem 1. Let (i, j) be fixed, but arbitrary. We intend to extend $x_i(t)$ until (2) is satisfied. The same method can then be used to extend $x_j(t)$ until (2) is satisfied with (i, j) reversed. If necessary, one of them can be extended until their domains of definition are equal. After this the results of Driver [3] describe further extensions of solutions.

For our fixed choice of (i, j), it follows from Lemma 2 that one of three conditions exists.

(i) Equation (2) has a solution at t_i.

(ii) Equation (2) has no solution at t_i and for any extension of $x_i(t)$ past t_i, equation (2) has no solution for some values of t greater than t_i.

(iii) Equation (2) has no solution at t_i, but would for all $t > t_i$ for any extension of $x_i(t)$ past t_i.

If (i) holds, then $x_i(t)$ need not be extended any further. Hence, we proceed with (ii). By hypothesis, if $x_i(t)$ is a solution extending past t_i, $E_j(t, x_i(t))$ will be zero for some values of $t > t_i$. That is, $x_i'(t)$ will be constant on some interval to the right of t_i. Hence, $x_i(t) = x_i(t_i) + x_i'(t_i)(t - t_i)$ for some $t > t_i$. Let

$$s_i = \sup\{t : (2) \text{ has no solution at } t\}.$$

By hypothesis, $s_i > t_i$. To show s_i is finite, observe that if

$$t \geqq t_i + \frac{|x_i(t_i) - x_j(t_j)| - c(t_i - t_j)}{c - |x_i'(t_i)|},$$

then

$$|x_i(t) - x_j(t_j)| = |x_i(t_i) + x_i'(t_i)(t - t_i) - x_j(t_j)|$$
$$\leqq |x_i(t_i) - x_j(t_j)| + |x_i'(t_i)|(t - t_i) \leqq c(t - t_j)$$

and inequality (5) is satisfied with t_j playing the role of α. That is, (2) has a solution at t. Hence $t_i < s_i < \infty$.

On $(t_i, s_i]$, $x_i(t)$ satisfies all the conditions of a solution because $v_i'(t) \equiv x_i''(t) = 0$ and $E_j(t, x_i(t)) = 0$ because (2) has no solution for $t \in (t_i, s_i]$. Furthermore, the existence of at most one solution to the system.

$$x'(t) = v(t), \qquad x(\zeta) = x_i(\zeta),$$
$$v'(t) = 0, \qquad v(\zeta) = x_i'(\zeta), \qquad \zeta \in (t_i, s_i),$$

along with Lemma 3 can be used to prove that $x_i(t)$ is the unique solution to (1), (2) and (3) on $(-\infty, s_i]$.

However, this simple extension cannot possibly be a solution for $t > s_i$, since for any extension of $x_i(t)$ past s_i, equation (2) will have a solution for all $t > s_i$, and (4) applies. That is $E_j(t, x_i(t)) \neq 0$.

Hence, $x_i(t)$ has been extended to s_i, and condition (iii) applies with s_i playing the role of t_i. The remainder of the analysis will therefore include the case where (iii) holds as well as complete the case where (ii) holds.

The following lemma is useful in extending the solution $x_i(t)$ past s_i. It establishes the existence of an $r_i > s_i$ such that any force field reaching particle i by time r_i is generated by particle j no later than t_j. That is, the extension of $x_i(t)$ for values of $t \leqq r_i$ can be made without extending $x_j(t)$ past t_j.

LEMMA 6. *Let $x_i(t)$ satisfy* (1), (2) *and* (3) *on* $(t_i, s_i]$, *and suppose* (iii) *holds with s_i playing the role of t_i. Let r_i be such that*

$$(12) \qquad 0 < r_i - s_i < (1/2c)[|x_i(s_i) - x_j(t_j)| - c(s_i - t_j)].$$

Then, for any continuously differentiable extension of $x_i(t)$ to r_i, with $|x_i'(t)| < c$, it follows that $t - \tau_{ji}(t) < t_j$ for all $t \in (s_i, r_i]$.

Remark. The right side of (12) is greater than zero, for otherwise (2) would have a solution at s_i.

Proof. The existence of $\tau_{ji}(t)$ for $t \in (s_i, r_i]$ follows from condition (iii). Assume that $t - \tau_{ji}(t) \geqq t_j$ for some $t \in (s_i, r_i]$. Then, since $t - \tau_{ji}(t)$ is continuous by Lemma 1, and $\lim_{t \to s_i+} [t - \tau_{ji}(t)] = -\infty$ by Lemma 4, there exists some $\hat{t} \in (s_i, r_i]$ such that $\hat{t} - \tau_{ji}(\hat{t}) = t_j$. But this implies

$$c\tau_{ji}(\hat{t}) = |x_i(\hat{t}) - x_j(t_j)| = |x_i(s_i) - x_j(t_j) + x_i(\hat{t})$$
$$- x_i(s_i)| > |x_i(s_i) - x_j(t_j)| - c(\hat{t} - s_i).$$

Hence,

$$\hat{t} - \tau_{ji}(\hat{t}) < \hat{t} - \frac{|x_i(s_i) - x_j(t_j)|}{c} + (\hat{t} - s_i)$$
$$= 2(\hat{t} - s_i) + (s_i - t_j) + t_j - \frac{|x_i(s_i) - x_j(t_j)|}{c}$$
$$\leqq 2(r_i - s_i) - \frac{|x_i(s_i) - x_j(t_j)|}{c} + (s_i - t_j) + t_j < t_j,$$

a contradiction. Q.E.D.

The main part of the proof of Theorem 1 can now be considered, namely the continuation of $x_i(t)$ past s_i.

Since (2) has no solution at s_i, the local existence of solutions cannot be established using standard existence results for ordinary differential equations, as was done in [3]. For, if $x_i(t)$ satisfies (1), (2) and (4) on (s_i, ζ) for some $\zeta > s_i$, then it follows from Lemma 4 that $\overline{\lim}_{t\to s_{i+}} (-1)^i v_j(t - \tau_{ij}(t)) = c$, and the right side of (4) could possibly become unbounded as $t \to s_{i+}$. In fact, the problem of continuing $x_i(t)$ past s_i can be posed as an ordinary differential equation with the initial point $(s_i, x_i(s_i), x_i'(s_i))$ lying on the boundary of the set on which the right side of the equation is well-defined (cf. Appendix A). It is also shown in Appendix A that initial trajectories exist such that the function defined by (4) cannot be continuous, or even satisfy Carathéodory conditions [1, p. 43] in any neighborhood of the initial point $(s_i, x_i(s_i), x_i'(s_i))$. This makes the existence theorems of Cauchy–Peano [1, p. 6] and Carathéodory [1, p. 43] inapplicable to the problem of continuing $x_i(t)$ past s_i.

To find an extension of $x_i(t)$ past s_i satisfying (1), (2) and (4) on (s_i, u_i), for some $u_i > s_i$, we shall first replace the functional equation (2) with the delay differential equation (6). We note that as long as $u_i \leqq r_i$, where r_i is defined in Lemma 6, any arguments of v_j will belong to $(-\infty, t_j)$, where the values of v_j are known. Hence, (1), (4) and (6) become an ordinary differential system with unknowns (x_i, v_i, τ_{ji}). It follows from Lemma 4 that the initial condition on $\tau_{ji}(t)$ is $\lim_{t\to s_{i+}} \tau_{ji}(t) = \infty$. That is, we seek a solution to (1), (4) and (6) on (s_i, u_i), for some $u_i > s_i$, with the condition that $(x_i(t), v_i(t), \tau_{ji}(t)) \to (x_i(s_i), v_i(s_i), \infty)$ as $t \to s_{i+}$. Hence, (1), (4) and (6), with initial point $(x_i(s_i), v_i(s_i), \infty)$ represent a singular Cauchy problem [7].

We shall now justify the replacement of (2) with (6). Let (x_i, v_i, τ_{ji}) satisfy (1), (2) and (4) on (s_i, u_i), for some $u_i > s_i$. Then (6) is satisfied by Lemma 1, and from Lemma 4, $\lim_{t\to s_{i+}} \tau_{ji}(t) = \infty$. Hence, (x_i, v_i, τ_{ji}) satisfy (1), (4) and (6), and $(x_i(t), v_i(t), \tau_{ji}(t)) \to (x_i(s_i), v_i(s_i), \infty)$ as $t \to s_{i+}$. Conversely, let (x_i, v_i, τ_{ji}) satisfy (1), (4) and (6) on (s_i, η), for some $\eta > s_i$, with initial point $(x_i(s_i), v_i(s_i), \infty)$. Then, (1) and (4) are satisfied and we need only show that $\tau_{ji}(t)$ satisfies (2). To this end, we rewrite (6) as

$$c\tau_{ji}'(t) = (-1)^i[v_i(t) - v_j(t - \tau_{ji}(t))(1 - \tau_{ji}'(t))].$$

Adding c to each side of the above, rearranging terms, and integrating from s_i to t, gives

$$\int_{s_i}^{t} [c - (-1)^i v_i(s)]\, ds = \int_{s_i}^{t} [c - (-1)^i v_j(s - \tau_{ji}(s))](1 - \tau_{ji}'(s))\, ds.$$

Letting $u = s - \tau_{ji}(s)$ gives

$$c(t - s_i) - (-1)^i[x_i(t) - x_i(s_i)] = \int_{-\infty}^{t - \tau_{ji}(t)} [c - (-1)^i v_j(u)]\, du.$$

Since $\lim_{t\to s_{i+}} \tau_{ji}(t) = \infty$, it can be assumed without loss of generality that $\tau_{ji}(t) > 0$ on (s_i, η) and $t - \tau_{ji}(t) < t_j$. Hence, using (11) to evaluate the integral,

$$\begin{aligned} &c(t - s_i) - (-1)^i[x_i(t) - x_i(s_i)] \\ &\quad = (-1)^i[x_i(s_i) - x_j(t - \tau_{ji}(t))] - c[s_i - t + \tau_{ji}(t)]. \end{aligned}$$

Therefore,

$$c\tau_{ji}(t) = (-1)^i[x_i(t) - x_j(t - \tau_{ji}(t)]$$
$$= |x_i(t) - x_j(t - \tau_{ji}(t))|.$$

This proves the equivalence between equations (1), (2) and (4) and equations (1), (4), (6) for the case where (2) has no solution at s_i. The equivalence for the case where (2) does have a solution at s_i can be found in [3].

By making a change of variables, we can now transform (1), (4) and (6) into an equivalent system which can be solved with the methods discussed in [7].

Let (t, x_i, v_i, τ_{ji}) be a solution to equations (1), (4) and (6) on (s_i, u_i), with $u_i \leqq r_i$. Define $\zeta = t - \tau_{ji}(t)$, $\psi(\zeta) = t$, $X(\zeta) = x_i(t)$ and $V(\zeta) = v_i(t)$. Since $\tau'_{ji}(t) < 1$ by Lemma 1, it follows that $\zeta'(t) > 0$. Hence

$$\psi'(\zeta) = \frac{dt}{d\zeta} = \frac{1}{\zeta'(\psi(\zeta))} = \frac{1}{1 - \tau'_{ji}(\psi(\zeta))}$$
$$(13) \qquad = \left[1 - \frac{(-1)^i v_i(\psi(\zeta)) - (-1)^i v_j(\zeta)}{c - (-1)^i v_j(\zeta)}\right]^{-1}$$
$$= \frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i v_i(\psi(\zeta))} = \frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i V(\zeta)}.$$

Also,

$$X'(\zeta) = x_i'(\psi(\zeta))\psi'(\zeta) = v_i(\psi(\zeta))\psi'(\zeta)$$
$$(14) \qquad = V(\zeta)\frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i V(\zeta)}.$$

Similarly, using (4),

$$V'(\zeta) = v_i'(\psi(\zeta))\psi'(\zeta)$$
$$= (1 - v_i^2(\psi(\zeta))/c^2)^{3/2}\frac{(-1)^i q_1 q_2}{4\pi\varepsilon_0 m_i c^2}\,\frac{1}{\tau_{ji}^2(\psi(\zeta))}$$
$$(15) \qquad \cdot\frac{c + (-1)^i v_j(\zeta)}{c - (-1)^i v_j(\zeta)}\,\frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i v_i(\psi(\zeta))}$$
$$= (1 - V^2(\zeta)/c^2)^{3/2}\frac{(-1)^i q_1 q_2}{4\pi\varepsilon_0 m_i c^2}\,\frac{1}{[\psi(\zeta) - \zeta]^2}\,\frac{c + (-1)^i v_j(\zeta)}{c - (-1)^i V(\zeta)}.$$

That is, ψ, X and V satisfy the ordinary differential system (13), (14), (15) on $(-\infty, u_i - \tau_{ji}(u_i))$ with the initial conditions

$$\lim_{\zeta \to -\infty} \psi(\zeta) = s_i,$$
$$(16) \qquad \lim_{\zeta \to -\infty} X(\zeta) = x_i(s_i),$$
$$\lim_{\zeta \to -\infty} V(\zeta) = v_i(s_i).$$

Conversely, let (ψ, X, V) be a solution to equations (13) through (16) on $(-\infty, \eta)$ for some $\eta \in (-\infty, t_j)$. Since $\lim_{\zeta \to -\infty} V(\zeta) = v_i(s_i) \in (-c, c)$, it can be assumed that $|V(\zeta)| < c$ on $(-\infty, \eta)$. Hence, $\psi'(\zeta) > 0$ on $(-\infty, \eta)$, and ψ^{-1}

exists with $\psi^{-1}:(s_i, u_i) \to (-\infty, \eta)$ for some $u_i > s_i$. We can assume without loss of generality that $u_i \leqq r_i$. Let $t = \psi(\zeta)$, $\tau_{ji}(t) = \psi(\zeta) - \zeta$, $x_i(t) = X(\zeta)$ and $v_i(t) = V(\zeta)$. It follows from (13), (14) and (15) that

$$x_i'(t) = X'(\zeta)/\psi'(\zeta) = V(\zeta) = v_i(t),$$

$$v_i'(t) = \frac{V'(\zeta)}{\psi'(\zeta)} = (1 - v_i^2(t)/c^2)^{3/2}\frac{(-1)^i q_1 q_2}{4\pi\varepsilon_0 m_i c^2}\,\frac{1}{\tau_{ji}^2(t)}$$

$$\cdot\frac{c + (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}$$

and

$$\tau_{ji}'(t) = \frac{\psi'(\zeta)}{\psi'(\zeta)} - \frac{1}{\psi'(\zeta)} = 1 - \frac{c - (-1)^i V(\zeta)}{c - (-1)^i v_j(\zeta)}$$

$$= \frac{(-1)^i v_i(t) - (-1)^i v_j(t - \tau_{ji}(t))}{c - (-1)^i v_j(t - \tau_{ji}(t))}.$$

That is, $x_i(t)$, $v_i(t)$ and $\tau_{ji}(t)$ satisfy equations (1), (4) and (6), with initial point $(s_i, x_i(s_i), v_i(s_i), \infty)$.

Because our transformations are uniquely defined, the system (1), (4) and (6) with initial point $(s_i, x_i(s_i), v_i(s_i), \infty)$ is equivalent to (13), (14), (15), (16). That is, one has a unique solution if and only if the other does.

The usefulness of replacing equations (1), (4) and (6) with the equivalent system (13) through (16) is shown by the following two lemmas.

LEMMA 7. *There exists a solution to equations* (13) *through* (16) *on* $(-\infty, \eta)$ *for some* $\eta \in (-\infty, t_j)$.

Remark. In proving existence we shall use a result from [7], which we state here for reference.

EXISTENCE THEOREM. *Let D be a domain in R^{1+n} and let $f: D \to R^n$. Let (ζ_0, u_0) belong to the closure of D. Assume there exists an $r > 0$, a $\delta > \zeta_0$, and a function m, integrable on $(\zeta_0, \delta]$, such that on the rectangle $R_{r\delta} = \{(\zeta, u): \zeta_0 < \zeta \leqq \delta, \|u - u_0\| \leqq r\}$, $f(\zeta, u)$ is measurable in ζ for fixed u, continuous in u for fixed ζ, and $\|f(\zeta, u)\| \leqq m(\zeta)$ for all $(\zeta, u) \in R_{r\delta}$. Then there exists an $\eta > \zeta_0$ and an absolutely continuous function $u:(\zeta_0, \eta) \to R^n$ satisfying*

(i) $(\zeta, u(\zeta)) \in D$ *for all* $\zeta \in (\zeta_0, \eta)$,

(ii) $u'(\zeta) = f(\zeta, u(\zeta))$ *for almost all* $\zeta \in (\zeta_0, \eta)$,

(iii) $\lim_{\zeta \to \zeta_{0+}} u(\zeta) = u_0$.

For $u \in R^n$, the norm of u is defined by $\|u\| = \max_{1 \leqq i \leqq n} |u_i|$.

Proof of Lemma 7. To make the existence theorem directly applicable, we make one more transformation of variables to obtain a system equivalent to equations (13) through (16). Let $H(\zeta) = V(\zeta)(1 - V^2(\zeta)/c^2)^{-1/2}$ so that

$$V(\zeta) = H(\zeta)(1 + H^2(\zeta)/c^2)^{-1/2}. \tag{17}$$

Then equations (13) through (16) are transformed into

$$\psi'(\zeta) = f_1(\zeta, \psi(\zeta), X(\zeta), H(\zeta)), \tag{18}$$

where

$$f_1(\zeta, \psi, X, H) = \frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i H(1 + H^2/c^2)^{-1/2}},$$

(19) $$X'(\zeta) = f_2(\zeta, \psi(\zeta), X(\zeta), H(\zeta)),$$

where

$$f_2(\zeta, \psi, X, H) = H(1 + H^2/c^2)^{-1/2} \frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i H(1 + H^2/c^2)^{-1/2}},$$

(20) $$H'(\zeta) = f_3(\zeta, \psi(\zeta), X(\zeta), H(\zeta)),$$

and where

$$f_3(\zeta, \psi, X, H) = \frac{(-1)^i q_1 q_2}{4\pi\varepsilon_0 m_i c^2 (\psi - \zeta)^2} \frac{c + (-1)^i v_j(\zeta)}{c - (-1)^i H(1 + H^2/c^2)^{-1/2}},$$

(21) $$\lim_{\zeta \to -\infty} \psi(\zeta) = s_i, \quad \lim_{\zeta \to -\infty} X(\zeta) = x_i(s_i), \quad \lim_{\zeta \to -\infty} H(\zeta) = v_i(s_i)(1 - v_i^2(s_i)/c^2)^{-1/2} \equiv H_0.$$

It is easily shown that the system (18) through (21) is equivalent to (13) through (16) via the transformation defined by (17). To apply the existence theorem to equations (18) through (21), let $\zeta_0 = -\infty$ and define u_0 as $u_0 = (s_i, x_i(s_i), H_0)$. Let r be any number greater than zero and define

$$\bar{c} = (|H_0| + r)(1 + (|H_0| + r)^2/c^2)^{-1/2} < c.$$

Let δ be any number less than $\lambda \equiv \min\{t_j, s_i - r\}$, and let

$$m(\zeta) = \frac{c - (-1)^i v_j(\zeta)}{c - \bar{c}} + \bar{c}\frac{c - (-1)^i v_j(\zeta)}{c - \bar{c}} + \frac{|q_1 q_2|}{4\pi\varepsilon_0 m_i c^2 [\lambda - \zeta]^2} \frac{2c}{c - \bar{c}}.$$

If $u = (\psi, X, H)$ and $R_{r\delta} = \{(t, x): -\infty < t \leqq \lambda, \|u - u_0\| \leqq r\}$, then $\psi > s_i - r \geqq \lambda$, $X \in R$, and $|H| \leqq |H_0| + r$, which implies $|V| = |H|(1 + H^2/c^2)^{-1/2} \leqq \bar{c}$ by (17). It follows easily from equations (18), (19) and (20) that $f = (f_1, f_2, f_3)$ is well-defined and continuous on $R_{r\delta}$, and

$$\|f(\zeta, \psi, X, H)\| \leqq m(\zeta) \quad \text{for all } (\zeta, \psi, X, H) \in R_{r\delta}.$$

The integrability on $(-\infty, \delta)$ of the first two terms of m follows from (11), and the integrability of the last term follows from the fact that $\delta < \lambda$.

Hence, from the "existence theorem", there exists an $\eta > -\infty$ and an absolutely continuous function $u: (-\infty, \eta) \to R^3$, say $u = (\psi, X, H)$, satisfying equations (18), (19) and (20) almost everywhere on $(-\infty, \eta)$, along with (21). However, since f_1, f_2 and f_3 are continuous, equations (18) through (20) are satisfied for all $\zeta \in (-\infty, \eta)$. Hence $(\psi, X, V) = (\psi, X, H(1 + H^2/c^2)^{-1/2})$ is a solution to equations (13) through (16) on $(-\infty, \eta)$. This completes the proof of Lemma 7.

LEMMA 8. *For any two solutions to equations* (13) *through* (16), *there exists a* $\lambda \in (-\infty, t_j)$ *such that the solutions are equal on* $(-\infty, \lambda]$.

Remark. Again we use a result from [7], which we state in terms suitable for application to our two-body problem.

UNIQUENESS THEOREM. *Let D be a domain in R^{1+n}, and let $F: D \to R^n$ be continuous on D. Let (ζ_0, w_0) belong to the closure of D. Assume for any two solutions, w and $\hat{w}$, to the problem $w'(\zeta) = F(\zeta, w(\zeta))$, $\lim_{\zeta\to\zeta_0+} w(\zeta) = w_0$, that there exists a $\lambda > \zeta_0$ and a function L, integrable on $(\zeta_0, \lambda]$, and continuous on $(\zeta_0, \lambda]$, such that*

(i) $\|F(\zeta, w(\zeta)) - F(\zeta, \hat{w}(\zeta))\| \leqq L(\zeta)\|w(\zeta) - \hat{w}(\zeta)\|$ *for all $\zeta \in (\zeta_0, \lambda]$. The λ and L depend on the solutions w and $\hat{w}$. Then any two solutions to $w' = F(\zeta, w)$, $\lim_{\zeta\to\zeta_0+} w(\zeta) = w_0$ are equal on some interval to the right of ζ_0, namely on $[\zeta_0, \lambda]$, where the λ is the one in the hypotheses.*

Proof of Lemma 8. Let (ψ, X, V) and $(\hat{\psi}, \hat{X}, \hat{V})$ be two solutions to equations (13)–(16). Choose $\lambda \in (-\infty, \min\{t_j, s_i\})$ such that $(-\infty, \lambda]$ belongs to their common domain of definition. Since $|v_i(s_i)| < c$, it can be assumed that $|V(\zeta)| \leqq \bar{c}$ and $|\hat{V}(\zeta)| \leqq \bar{c}$ for all $\zeta \in (-\infty, \lambda]$, for some $\bar{c} < c$. Consider equations (13), (14) and (15) to be in the form

$$\psi'(\zeta) = F_1(\zeta, \psi(\zeta), X(\zeta), V(\zeta)),$$
$$X'(\zeta) = F_2(\zeta, \psi(\zeta), X(\zeta), V(\zeta)),$$
$$V'(\zeta) = F_3(\zeta, \psi(\zeta), X(\zeta), V(\zeta)).$$

Then, on $(-\infty, \lambda]$, denoting $\psi(\zeta)$ by ψ, $X(\zeta)$ by X, etc., we have

$$\begin{aligned}|F_1(\zeta, \psi, X, V) - F_1(\zeta, \hat{\psi}, \hat{X}, \hat{V})| &= \left|\frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i V} - \frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i \hat{V}}\right| \\ &\leqq \frac{c - (-1)^i v_j(\zeta)}{(c - \bar{c})^2}|V - \hat{V}|.\end{aligned} \tag{22}$$

Similarly,

$$\begin{aligned}|F_2(\zeta, \psi, X, V) - F_2(\zeta, \hat{\psi}, \hat{X}, \hat{V})| &= \left|V\frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i V} - \hat{V}\frac{c - (-1)^i v_j(\zeta)}{c - (-1)^i \hat{V}}\right| \\ &\leqq c(c - \bar{c})^{-2}[c - (-1)^i v_j(\zeta)]|V - \hat{V}|.\end{aligned} \tag{23}$$

Finally, letting $K = |q_1 q_2|/(4\pi\varepsilon_0 m_i c^2)$, it follows from (15) that

$$\begin{aligned}&|F_3(\zeta, \psi, X, V) - F_3(\zeta, \hat{\psi}, \hat{X}, \hat{V})| \\ &\leqq K2c\left\{\left|\frac{(1 - V^2/c^2)^{3/2}}{(\psi - \zeta)^2[c - (-1)^i V]} - \frac{(1 - V^2/c^2)^{3/2}}{(\psi - \zeta)^2[c - (-1)^i \hat{V}]}\right|\right. \\ &\quad + \left|\frac{(1 - V^2/c^2)^{3/2}}{(\psi - \zeta)^2[c - (-1)^i \hat{V}]} - \frac{(1 - \hat{V}^2/c^2)^{3/2}}{(\psi - \zeta)^2[c - (-1)^i \hat{V}]}\right| \\ &\quad \left.+ \left|\frac{(1 - \hat{V}^2/c^2)^{3/2}}{(\psi - \zeta)^2[c - (-1)^i \hat{V}]} - \frac{(1 - \hat{V}^2/c^2)^{3/2}}{(\hat{\psi} - \zeta)^2[c - (-1)^i \hat{V}]}\right|\right\}.\end{aligned}$$

To further reduce this last inequality, consider the following. For $\zeta \in (-\infty, \lambda]$, $\psi(\zeta) > s_i$. Hence $\psi - \zeta > s_i - \zeta$. Furthermore, if $y - s > 0$, then $|(d/dy)(y - s)^{-2}| = |-2(y - s)^{-3}| \leqq 2(y - s)^{-3}$, and if $y \leqq \bar{c}$, then

$$|(d/dy)(1 - y^2)/c^2)^{3/2}| = |\tfrac{3}{2}(1 - y^2/c^2)^{1/2}(-2y/c^2)| \leqq 3\bar{c}|c^2.$$

Hence

$$
\begin{aligned}
&|F_3(\zeta, \psi, X, V) - F_3(\zeta, \hat{\psi}, \hat{X}, \hat{V})| \\
(24) \qquad &\leqq 2Kc\left\{\frac{|V - \hat{V}|}{(s_i - \zeta)^2(c - \bar{c})^2} + \frac{3\bar{c}|V - \hat{V}|}{c^2(s_i - \zeta)^2(c - \bar{c})} + \frac{|\psi - \hat{\psi}|}{(s_i - \zeta)^3(c - \bar{c})}\right\}.
\end{aligned}
$$

Let $F = (F_1, F_2, F_3)$, $u = (\psi, X, V)$ and $\hat{u} = (\hat{\psi}, \hat{X}, \hat{V})$. Then it follows from equations (22)–(24) that

$$\|F(\zeta, u(\zeta)) - F(\zeta, \hat{u}(\zeta))\| \leqq L(\zeta)\|u(\zeta) - \hat{u}(\zeta)\| \quad \text{for all } \zeta \in (-\infty, \lambda],$$

where

$$L(\zeta) = \frac{c - (-1)^i v_j(\zeta)}{(c - \bar{c})^2}(1 + c) + 2Kc\left[\frac{c^2 + 3c\bar{c} - 3\bar{c}^2}{c^2(c - \bar{c})^2(s_i - \zeta)^2} + \frac{1}{(c - \bar{c})(s_i - \zeta)^3}\right].$$

Since $\lambda < t_j$, the first term of L is integrable by (11), and the second term is also integrable, on $(-\infty, \lambda]$, because $\lambda < s_i$. Hence, $(\psi, X, V) = (\hat{\psi}, \hat{X}, \hat{V})$ on $(-\infty, \gamma]$. Q.E.D.

Because of the equivalency between systems (1), (2), (4) and (1), (4), (6) and (13), (14), (15), it follows from Lemma 7 that there exists a $w_i > s_i$ and an extension of $x_i(t)$ to $(-\infty, w_i]$ satisfying equations (1), (2) and (4) on $(s_i, w_i]$. Hence, particle i is in the force field of particle j at w_i.

However, the uniqueness of the extension to w_i has not been completely established. All that we can conclude from Lemma 8 is that any two extensions of $x_i(t)$ past s_i are equal on $(s_i, \alpha]$ for some $\alpha > s_i$, where the α will depend on the chosen solutions. To establish uniqueness on the whole interval $(s_i, w_i]$, we need the following.

LEMMA 9. *Let $x_i(t)$ and $\hat{x}_i(t)$ be extensions of $x_i(t)$ to $(-\infty, \beta]$ and $(-\infty, \hat{\beta}]$ respectively, satisfying equations* (1), (2) *and* (4) *for $t > s_i$, with $\beta \leqq r_i$ and $\hat{\beta} \leqq r_i$, where r_i is defined in Lemma* 6. *Then $x_i(t) = \hat{x}_i(t)$ on $(s_i, \min\{\beta, \hat{\beta}\}]$.*

Proof. Let $t^* = \sup\{t : x_i(t) = \hat{x}_i(t)\}$. Then it follows from Lemma 8 that $t^* > s_i$. Assume that $t^* < \min\{\beta, \hat{\beta}\}$. Then (x_i, v_i, τ_{ji}) and $(\hat{x}_i, \hat{v}_i, \hat{\tau}_{ji})$ are both solutions to equations (1), (4) and (6) and they pass through the initial point $(t^*, x_i(t^*), v_i(t^*), \tau_{ji}(t^*))$. It follows from the equivalence between equations (1), (4) and (6) with equations (13), (14) and (15) that there exist solutions, (ψ, X, V) and $(\hat{\psi}, \hat{X}, \hat{V})$, to equations (13), (14), (15) passing through the initial point $(t^* - \tau_{ji}(t^*), t^*, x_i(t^*), v_i(t^*))$. Let

$$D \equiv \{\zeta, \psi, X, V) : -\infty < \zeta < t_j, \psi > \zeta, X \in R, |V| < c\}.$$

Since $t^* < r_i$, it follows that $t^* - \tau_{ji}(t^*) < t_j$. It can also be assumed without loss of generality that $|v_i(t^*)| < c$. Hence, the above mentioned initial point belongs to D. Furthermore, the vector-valued function defined by equations (13), (14), (15) is continuous and locally Lipschitzian on D. Solutions through the initial point are therefore unique. Hence, (ψ, X, V) and $(\hat{\psi}, \hat{X}, \hat{V})$ are equal on $[t^* - \tau_{ji}(t^*), \delta)$ for some $\delta > t^* - \tau_{ji}(t^*)$. It follows from the equivalency of (1), (4), (6) with (13), (14), (15) that $x_i(t) = \hat{x}_i(t)$ on $[t^*, \alpha)$ for some $\alpha > t^*$. But, this contradicts the choice of t^*, and the proof is complete.

The existence of a unique extension of $x_i(t)$ to $(-\infty, w_i]$ satisfying equations (1), (2) and (3) on $(t_i, w_i]$ has therefore been established, for some $w_i > t_i$. And, in particular, (2) has a solution at w_i. Also, the extension depends on values of $x_j(t)$ only for $t < t_j$. In a similar manner, there exists a unique extension of $x_j(t)$ to $(-\infty, w_j]$, for some $w_j \geqq t_j$, satisfying equations (1), (2) and (3) on $(t_j, w_j]$, with (2) having a solution at w_j.

If $w_i = w_j$, then Theorem 1 will follow directly from the theorem in [4]. That theorem states that the existence and uniqueness theorem of Driver [3] remains true if the Lipschitz condition on the initial velocities, $v_i(t)$, $i = 1, 2$, is replaced by continuity.

If $w_i \neq w_j$, we can assume without loss of generality that $w_i < w_j$, and the proof of Theorem 1 will be completed by showing that $x_i(t)$ can be extended uniquely to w_j.

Let $D_1 \equiv \{(\zeta, \psi, X, V): \zeta \leqq w_j, \psi > \zeta, X \in R, |V| < c\}$. The functions defined by equations (13), (14) and (15) are continuous on D_1, and locally Lipschitzian in the interior of D_1. Since $(w_i - \tau_{ji}(w_i), w_i, x_i(w_i), v_i(w_i))$ belongs to the interior of D_1, there exists a unique solution to equations (13), (14) and (15) passing through this initial point, and therefore a unique extension of $x_i(t)$ past w_i satisfying equations (1), (2) and (4).

Assume $t^* \leqq w_j$ is such that the extensions of $x_i(t)$ cannot be extended to t^*. Hence, if (ψ, X, V) is the associated solution to equations (13), (14), (15), it follows that (ψ, X, V) cannot be extended past $\zeta^* \equiv \lim_{t \to t^{*-}} [t - \tau_{ji}(t)]$. Thus, $(\zeta, \psi(\zeta), X(\zeta), V(\zeta))$ must come arbitrarily close to the boundary of D_1 as $\zeta \to \zeta^{*-}$. That is, at least one of the following must be true.

(i) $\liminf\limits_{\zeta \to \zeta^{*-}} (\psi(\zeta) - \zeta) = 0$,

(ii) $\limsup\limits_{\zeta \to \zeta^{*-}} |V(\zeta)| = c$.

In terms of the extension of $x_i(t)$, these are equivalent to

(iii) $\liminf\limits_{t \to t^{*-}} \tau_{ji}(t) = 0$,

(iv) $\limsup\limits_{t \to t^{*-}} |v_i(t)| = c$.

We show now that neither (iii) nor (iv) can be true.

For $t \in [w_i, t^*)$, $t - \tau_{ji}(t) \in [w_i - \tau_{ji}(w_i), w_j]$, and since $|v_j(t)| < c$ on $(-\infty, w_j]$, there exists a $\hat{c} < c$ such that $|v_j(t)| \leqq \hat{c}$ on $[w_i - \tau_{ji}(w_i), w_j]$. Hence, for $t \in [w_i, t^*)$,

$$\begin{aligned} c\tau_{ji}(t) &= |x_i(t) - x_j(t - \tau_{ji}(t))| \\ &= |x_i(t) - x_j(t) + x_j(t) - x_j(t - \tau_{ji}(t))| \\ &\geqq |x_i(t) - x_j(t)| - \hat{c}\tau_{ji}(t). \end{aligned}$$

Therefore, $\tau_{ji}(t) \geqq |x_i(t) - x_j(t)|/(c + \hat{c})$ and (iii) holds only if

$$\liminf_{t \to t^{*-}} |x_i(t) - x_j(t)| = 0.$$

This, in turn, is true only if $\liminf_{t \to t^{*-}} \tau_{ij}(t) = 0$. To prove this last assertion, assume that $\liminf_{t \to t^{*-}} |x_i(t) - x_j(t)| = 0$, and $\liminf_{t \to t^{*-}} \tau_{ij}(t) = M > 0$. Then, $\tau_{ij}(t) > M/2$ on $[\alpha, t^*)$ for some $\alpha < t^*$. Thus, for all $\varepsilon > 0$, there exists a t_ε such that $t_\varepsilon \in (\min\{\alpha, t^* - \varepsilon\}, t^*)$ and $|x_i(t_\varepsilon) - x_j(t_\varepsilon)| < \varepsilon$. Since $|v_i(t)| < c$ on $(-\infty, t^*)$, it follows that $x_i(t^*) \equiv \lim_{t\to t^{*-}} x_i(t)$ exists. Hence

$$\frac{cM}{2} + c\left(\tau_{ij}(t_\varepsilon) - \frac{M}{2}\right) = |x_j(t_\varepsilon) - x_i(t_\varepsilon - \tau_{ij}(t_\varepsilon))|$$
$$\leqq |x_j(t_\varepsilon) - x_i(t_\varepsilon)| + |x_i(t_\varepsilon) - x_i(t^*) + \left|x_i(t^*) - x_i\left(t^* - \frac{M}{2}\right)\right|$$
$$+ \left|x_i\left(t^* - \frac{M}{2}\right) - x_i(t^* - \tau_{ij}(t_\varepsilon))\right|$$
$$+ |x_i(t^* - \tau_{ij}(t_\varepsilon)) - x_i(t_\varepsilon - \tau_{ij}(t_\varepsilon))|$$
$$< \varepsilon + c\varepsilon + \int_{t^*-M/2}^{t^*} |v_i(s)|\, ds + c\left(\tau_{ij}(t_\varepsilon) - \frac{M}{2}\right) + c\varepsilon.$$

It follows that $cM/2 \leqq \int_{t^*-M/2}^{t^*} |v_i(s)|\, ds$, which is impossible since $|v_i(t)| < c$ on $[t^* - M/2, t^*)$. Hence (iii) holds only if $\liminf_{t\to t^{*-}} \tau_{ij}(t) = 0$. However, the latter is impossible, for if $t \in [w_i + (w_j - w_i)/2, w_j]$, then $t - \tau_{ij}(t) < t_i$ by the manner in which $x_j(t)$ was extended to w_j. Thus,

$$\tau_{ij}(t) = t - [t - \tau_{ij}(t)] > w_i + (w_j - w_i)/2 - t_i \geqq (w_j - w_i)/2 > 0.$$

It follows that (iii) cannot hold, which implies (i) cannot hold.

Since $\liminf_{t\to t^{*-}} \tau_{ji}(t) > 0$, there exists an $M > 0$ and a $\delta < t^*$ such that $\tau_{ji}(t) > M$ on $[\delta, t^*)$. Also, on $[w_i, t^*)$, $|v_j(t - \tau_{ji}(t))| \leqq \hat{c} < c$. It follows from (4) that $v_i'(t)(1 - v_i^2(t)/c^2)^{-3/2}$ is bounded in absolute value on $[\delta, t^*)$ by some finite constant K_0. Integration of (4) then gives, for $t \in [\delta, t^*)$,

$$|v_i(t)(1 - v_i^2(t)/c^2)^{-1/2}| \leqq |v_i(\delta)(1 - v_i^2\,(\delta)/c^2)^{-1/2}| + K_0(t^* - \delta),$$

which implies $|v_i(t)| \leqq C_0$ on $[\delta, t^*)$ for some $C_0 < c$. Therefore, (iv) cannot hold, which implies (ii) cannot be true, and it follows that $x_i(t)$ can be extended to w_j. The conclusions of Theorem 1 now follow directly from the results of Driver, and the proof of Theorem 1 is complete.

Appendix A. A counterexample. It will be shown in this Appendix that generally the Cauchy–Peano existence theorem and the theorem of Carathéodory are not suitable for extending a particle's trajectory past the critical point where the particle is just about to enter the force field of the other particle. It will suffice to consider two symmetrically moving particles having equal mass m and charge q.

Letting $x_2(t) = x(t)$ and $x_1(t) = -x(t)$, equations (1), (2) and (4) reduce to the three equations

$$x'(t) = v(t), \tag{A.1}$$

$$c\tau(t) = x(t) + x(t - \tau(t)), \tag{A.2}$$

$$\frac{v'(t)}{(1 - v^2(t)/c^2)^{3/2}} = \frac{K}{\tau^2(t)}\,\frac{c - v(t - \tau(t))}{c + v(t - \tau(t))}, \tag{A.3}$$

where the constant K has units of meters, since (4) is in M.K.S. units. For the symmetric case, the problem of extending the solution past the critical point can be posed as follows.

Let $x(t)$ be continuously differentiable on $(-\infty, s]$ with $x'(t) \in (-c, c)$, and suppose (A.2) has a solution if and only if $t > s$, for any continuously differentiable extension of $x(t)$ past s with $|x'(t)| < c$. Such trajectories exist by Lemmas 2 and 3. We seek a $\beta > s$ and an extension of $x(t)$ to $(-\infty, \beta)$ satisfying equations (A.1), (A.2) and (A.3) on (s, β).

It will first be shown how this problem can be viewed as an ordinary differential equation with the right-hand side not defined at the initial point. Let $t_1 = s + x_0/c$, where $x_0 \equiv x(s)$, and let

$$D \equiv \{(t, \psi, \eta): s < t < t_1, |\psi - x_0| < c(t - s), |\eta| < c\}.$$

For $(t, \psi, \eta) \in D$, $\psi > 0$. Define $f_1: D \to R$ by

$$f_1(t, \psi, \eta) = \eta. \tag{A.4}$$

Another function f_2 is defined as follows. For each $(t, \psi, \eta) \in D$, choose any $y \in C^1((-\infty, t], (0, \infty))$ such that $|y'(\zeta)| < c$ on $(s, t]$, $y(\zeta) = x(\zeta)$ for $\zeta \leqq s$, $y(t) = \psi$, and $y'(t) = \eta$. Such a trajectory always exists because $|x'(s)| < c$. Now, by hypothesis, the functional equation $c\tau = y(\zeta) + y(\zeta - \tau)$ has a unique solution, namely $\tau_y(\zeta)$, for all $\zeta > s$, and specifically for $\zeta = t$. Furthermore, the choice of t_1 implies $t - \tau_y(t) < s$, for if $t - \tau_y(t) \geqq s$, there exists a $\hat{t} \in (s, t)$ such that $\hat{t} - \tau_y(\hat{t}) = s$. This follows from the continuity of τ_y, and the fact that $\tau_y(\zeta)$ exists on $(s, t]$ with $\lim_{\zeta \to s+} \tau_y(\zeta) = \infty$, which follows from Lemma 4. Thus

$$c\tau_y(\hat{t}) = y(\hat{t}) + y(\hat{t} - \tau_y(\hat{t})) = y(\hat{t}) + y(s) = y(\hat{t}) + x_0,$$

which implies

$$\tau_y(\hat{t}) > \frac{x_0 - c(\hat{t} - s) + x_0}{c},$$

and hence,

$$\hat{t} - \tau_y(\hat{t}) < \hat{t} - \frac{x_0 - c(\hat{t} - s) + x_0}{c} = (\hat{t} - s) + \hat{t} - \frac{2x_0}{c},$$

$$\hat{t} - \tau_y(\hat{t}) < (t_1 - s) + \left(t_1 - \frac{x_0}{c}\right) - \frac{x_0}{c} = s,$$

a contradiction. Therefore, $t - \tau_y(t) < s$, and $y(t - \tau_y(t)) = x(t - \tau_y(t))$, and $\tau_y(t)$ is the solution to $c\tau = \psi + x(t - \tau)$. That is, τ is independent of the choice of trajectory y, and depends only on t and ψ. Thus $\tau \equiv \tau(t, \psi)$ or $\tau(t, \psi, \eta)$. The function $f_2: D \to R$ can now be defined by

$$f_2(t, \psi, \eta) = \left(1 - \frac{\eta^2}{c^2}\right)^{3/2} \frac{K}{\tau^2(t, \psi)} \frac{c - x'(t - \tau(t, \psi))}{c + x'(t - \tau(t, \psi))}. \tag{A.5}$$

Both f_1 and f_2 are well-defined on D, and the problem of extending $x(t)$ past s reduces to solving the ordinary differential system $x' = f_1(t, x, v)$, $v' = f_2(t, x, v)$, with initial conditions $x(s) = x_0$, $v(s) = x'(s^-) \equiv v_0$. Furthermore, the initial point (s, x_0, v_0) lies on the boundary of D. That is, f_2 is not well-defined at (s, x_0, v_0) because $\tau(s, x_0)$ does not exist by hypothesis.

The functions f_1 and f_2 are continuous on D. The continuity of f_1 is obvious. To prove continuity for f_2, choose any $(\hat{t}, \hat{\psi}, \hat{\eta}) \in D$, and let $\hat{\tau} = \tau(\hat{t}, \hat{\psi})$. Consider the function $F(t, \psi, \eta, \tau) = c\tau - \psi - x(t - \tau)$. There exists a $p > 0$ such that F is continuous on

$$\{(t, \psi, \eta, \tau) : |t - \hat{t}| \leqq p, |\psi - \hat{\psi}| \leqq p, |\eta - \hat{\eta}| \leqq p, |\tau - \hat{\tau}| \leqq p\}.$$

Also, $F(\hat{t}, \hat{\psi}, \hat{\eta}, \hat{\tau}) = 0$ and for fixed (t, ψ, η) with $|t - \hat{t}| \leqq p$, $|\psi - \hat{\psi}| \leqq p$ and $|\eta - \hat{\eta}| \leqq p$, F is monotonically increasing in τ on $[\hat{\tau} - p, \hat{\tau} + p]$ because $|x'(t)| < c$. It follows from the implicit function theorem that $\tau(t, \psi, \eta) \equiv \tau(t, \psi)$ is continuous in a neighborhood of $(\hat{t}, \hat{\psi}, \hat{\eta})$. This proves that τ is continuous on D. It follows from the continuity of $x'(t)$ that f_2 is continuous on D.

Because $f = (f_1, f_2)$ is continuous on D, one might suspect that f could be extended to a domain $\hat{D}$ containing (s, x_0, v_0). That is, one would seek an $\hat{f} = (\hat{f}_1, \hat{f}_2)$, continuous on $\hat{D}$, such that $D \subset \hat{D}$, $(s, x_0, v_0) \in \hat{D}$, and $\hat{f} = f$ on D. It would then follow from the Cauchy–Peano existence theorem that there exists a function $x(t)$ satisfying $x' = \hat{f}_1(t, x, v)$, $v' = \hat{f}_2(t, x, v)$, and passing through (s, x_0, v_0). Then, because $|v(s)| < c$, $(t, x(t), x'(t))$ would lie in D on (s, β) for some $\beta > s$, and on that interval $x(t)$ would be the desired extension of $x(t)$ past s.

In the following counterexample we construct an initial trajectory $x(t)$ on $(-\infty, s]$ and a sequence of points $\{(t_n, \psi_n, \eta_n)\}$ lying in D with $\lim_{n\to\infty} (t_n, \psi_n, \eta_n) = (s, x_0, v_0)$ and $\lim_{n\to\infty} f_2(t_n, \psi_n, \eta_n) = \infty$. This will prove that f cannot be extended continuously to include (s, x_0, v_0).

With c given in meters/sec., choose a in meters such that $-a + c(2 + b/2 - (7/4)b^2 + (5/4)b^3) = a > 0$, where $b = (14 + \sqrt{76})/30 \in (0, 1)$. This is possible since the quantity in parentheses is positive. For $k = 1, 2, \cdots$, define $x(t)$ on $(s - (k + 1), s - k)$ by

$$\text{(A.6)} \qquad \begin{aligned} x(t) = &-a + c[k + 2^{1-k}] + c[1 - 2^{-1-k}][(s - k) - t] \\ &- \tfrac{7}{4}c2^{-k}[(s - k) - t]^2 + \tfrac{5}{4}c2^{-k}[(s - k) - t]^3. \end{aligned}$$

On $(s - 1, s - b)$ let $x(t)$ be given by (A.6) with $k = 0$, and on $[s - b, s]$ let $x(t) \equiv x(s - b - 0)$. Wherever (A.6) applies,

$$\text{(A.7)} \qquad x'(t) = -c[1 - 2^{-1-k}] + \tfrac{7}{2}c2^{-k}[(s - k) - t] - \tfrac{15}{4}c2^{-k}[(s - k) - t]^2$$

and

$$\text{(A.8)} \qquad x''(t) = -\tfrac{7}{2}c2^{-k} + \tfrac{15}{2}c2^{-k}[(s - k) - t].$$

It is straightforward to show that

$$x(s - k - 0) = x(s - k + 0) = -a + c[k + 2^{1-k}]$$

and $x'(s - k - 0) = x'(s - k + 0) = -c[1 - 2^{-1-k}]$, for $k = 1, 2, \cdots$. Also $x'(s - b - 0) = 0$. Therefore, if we set $x(s - k) = -a + c[k + 2^{1-k}]$ and $x'(s - k) = -c[1 - 2^{-1-k}]$ for $k = 1, 2, \cdots$, then $x(t) \in C^1(-\infty, s]$. For $k = 1, 2, \cdots$, $x''(t) > 0$ on $I_k = (s - (k + 1), s - k - \frac{7}{15})$, equal to zero when $t = s - k - \frac{7}{15}$, and less than zero on $J_k = (s - k - \frac{7}{15}, s - k)$. Therefore, $x'(t)$ increases on I_k, decreases on J_k, and attains a maximum value when $t = s - k - \frac{7}{15}$. Using (A.7), its value at the peak is $-c + \frac{79}{60}c2^{-k} \in (-c, 0)$. Also $x'(t)$ increases on $(s - 1, s - b)$

and is zero on $[s-b, s]$. Hence $x'(t) \in (-c, 0)$ on $(-\infty, s-b)$ and therefore $x(t)$ is positive and decreases monotonically on that interval. The minimum value for $x(t)$ therefore occurs when $t = s - b$, and at that value $x(t) = -a + c(2 + b/2 - (7/4)b^2 + (5/4)b^3) = a$, which is greater than zero by hypothesis. Therefore $x(t) > 0$ on $(-\infty, s]$, and $x'(t)$ is continuous on $(-\infty, s]$, with $|x'(t)| < c$.

Consider the sequence $\{(t_n, \psi_n, \eta_n)\} = \{(s + 2^{-n}, x_0, v_0)\}$, where $x_0 = x(s) = x(s-b) = a > 0$ and $v_0 = x'(s) = 0$. For large enough n, $(t_n, \psi_n, \eta_n) \in D$, and $(t_n, \psi_n, \eta_n) \to (s, x_0, v_0)$.

We first show that (A.2) has a solution if and only if $t > s$, for any continuously differentiable extension of $x(t)$ past s with $|x'(t)| < c$. Because of Lemma 3 we need only prove the assertion for the extension defined by $x(t) \equiv a$ for $t > s$. For any t_n, let $\tau_n = n + 1 + 2^{-n}$. Then

$$
\begin{aligned}
c\tau_n = c[n + 1 + 2^{-n}] &= a - a + c[n + 1 + 2^{1-(n+1)}] \\
&= x(t_n) + x(s - (n+1)) \\
&= x(t_n) + x(s + 2^{-n} - (n + 1 + 2^{-n})) \\
&= x(t_n) + x(t_n - \tau_n).
\end{aligned}
$$

Thus, $\tau_n = n + 1 + 2^{-n}$ satisfies the functional equation (A.2) with $t = t_n$ and by Lemma 1 it must be the unique solution, $\tau(t_n)$, to that equation. Now, for any $t > s$, there exists a $t_n \in (s, t)$, and since (A.2) has a solution at t_n, it follows from Lemma 1 that a solution also exists at t. Also, since τ is continuous in its domain of definition, and $\lim_{n\to\infty} \tau(t_n) = \infty$, it follows that $\tau(s)$ does not exist. Hence, (A.2) has a solution if and only if $t > s$, for any continuously differentiable extension of $x(t)$ past s with $|x'(t)| < c$. Finally,

$$
\begin{aligned}
f_2(t_n, \psi_n, \eta_n) &= \frac{K}{[n + 1 + 2^{-n}]^2} \frac{c - x'(s - (n+1))}{c + x'(s - (n+1))} \\
&= \frac{K}{[n + 1 + 2^{-n}]^2} \frac{c + c[1 - 2^{-1-(n+1)}]}{c - c[1 - 2^{-1-(n+1)}]} \\
&\geqq \frac{K}{[n+2]^2} \frac{c}{c2^{-2-n}} = \frac{K2^{n+2}}{(n+2)^2} \to \infty \quad \text{as } n \to \infty.
\end{aligned}
$$

Therefore, f_2 cannot be extended continuously to include (s, x_0, v_0), and the Cauchy–Peano theorem cannot be applied to the problem.

In a similar manner, one might attempt to extend f to a domain containing (s, x_0, v_0) with Carathéodory conditions being satisfied in some neighborhood $\hat{D}$ of (s, x_0, v_0). That is, have $f(t, \psi, \eta)$ measurable in t for fixed (ψ, η), and continuous in (ψ, η) for fixed t, with f bounded by a measurable function $m_u(t)$ for each compact $u \subset \hat{D}$. The same example can be used to show this is also not generally possible.

If (s, x_0, v_0) belongs to some open set $\hat{D}$, then for all t^* sufficiently near s, $(t^*, x_0 + c(t^* - s), 0)$ must lie in $\hat{D}$, since $v_0 = 0$. However, for any such t^*, a sequence of points $\{(t^*, \psi_n, 0)\}$ lying in D can be found such that $\lim_{n\to\infty}(t^*, \psi_n, 0) = (t^*, x_0 + c(t^* - s), 0)$ and $\lim_{n\to\infty} f_2(t^*, \psi_n, 0) = \infty$. This will prove that no extension of f can be continuous in (ψ, η) for fixed t in a neighborhood of (s, x_0, v_0), and the Carathéodory existence theorem cannot be applied to the problem. To prove this, let t^* be as defined above and again let $t_n = s + 2^{-n}$. Then there exists an N such that $t_n < t^*$ for all $n \geqq N$. For each $n \geqq N$ define $\psi_n = x_0 + c(t^* - t_n)$. Clearly, as $n \to \infty$, $(t^*, \psi_n, 0) \to (t^*, x_0 + c(t^* - s), 0)$, and each point lies in D. Furthermore, $\tau(t^*, \psi_n, 0) = n + 1 + t^* - s$. To prove this, note that

$$\begin{aligned}
c[n + 1 + t^* - s] &= a + c(t^* - t_n) - a + c(t_n - s) + c(n + 1) \\
&= \psi_n - a + c[2^{-n} + n + 1] \\
&= \psi_n - a + c[n + 1 + 2^{1-(n+1)}] \\
&= \psi_n + x(s - (n + 1)) = \psi_n + x(t^* - [n + 1 + t^* - s]).
\end{aligned}$$

Hence, $n + 1 + t^* - s$ satisfies the equation $c\tau = \psi_n + x(t^* - \tau)$, and must therefore be the unique solution $\tau(t^*, \psi_n, 0)$. Also,

$$\begin{aligned}
f_2(t^*, \psi_n, \eta_n) &= \frac{K}{[n + 1 + t^* - s]^2} \frac{c - x'(s - (n + 1))}{c + x'(s - (n + 1))} \\
&= \frac{K}{[n + 1 + t^* - s]^2} \frac{c + c[1 - 2^{-1-(n+1)}]}{c - c[1 - 2^{-1-(n+1)}]} \\
&\geqq \frac{K}{[n + 1 + t^* - s]^2} 2^{n+2} \to \infty \quad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof.

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