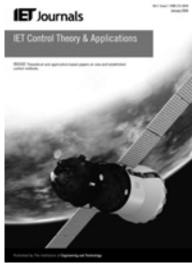
Published in IET Control Theory and Applications
Received on 24th August 2007
Revised on 28th January 2008
doi: 10.1049/iet-cta.2007.0321



ISSN 1751-8644

A delay-partitioning projection approach to stability analysis of continuous systems with multiple delay components

B. Du[1] J. Lam[1] Z. Shu[1] Z. Wang[2]

[1]Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong
[2]Department of Information Systems and Computing, Brunel University, Uxbridge, Middlesex UB8 3PH, UK
E-mail: james.lam@hku.hk

Abstract: An effective approach is introduced to study the stability of continuous systems with multiple time-varying delay components. By employing a new Lyapunov–Krasovskii functional form based on delay partitioning, delay-dependent stability criteria are established for cases with or without the information of the delay rates. The contribution of the paper is 2-fold. First, it provides an improvement, as well as generalisation, of the existing stability criteria for continuous systems with multiple time-varying delay components. Second, it is illustrated numerically that the approach can be applied to estimate the delay bound for system stability in the single delay case with reduction both in conservatism and computational complexity when compared with the existing methods.

1 Introduction

Time delay is often attributed as the major source of instability in various engineering systems. Consequently, a vast amount of effort has recently been devoted to deriving stability criteria for linear systems with constant or time-varying delays [1–3]. Many methods that aim at reducing the conservatism of these stability criteria (i.e. less conservative upper bound of the delay for the system to remain stable) have been proposed. One approach is to take an appropriate and equivalent model transformation for the original systems. Fridman and Shaked [4] summarised four main model transformations and showed different sources for conservatism under different model transformations. Under such transformations, the conservatism is mainly caused by the bounding of the cross-product terms, which appears in the derivative of the Lyapunov–Krasovskii functional. Reducing the number of such terms and employing tighter bounds on them would certainly lead to better results. In this regard, Park [5] proposed an upper bound of a vector cross-product with cross- as well as inner-products. Moon *et al.* [6] provided another inequality by reducing the limitation in [5] such that certain matrix variables are no longer required to satisfy a specific structure. Recently, a free-weighting matrix method was proposed in [7] and applied in [8, 9] to investigate the delay-dependent stability, in which the bounding techniques on some cross-product terms are not involved. This treatment produces better results, which is often associated with an increase in variables. Another approach is the construction of new Lyapunov–Krasovskii functionals with a proper distribution of the time delay (see [10]). Gu *et al.* [11] introduced LMI stability conditions via a complete and discretised Lyapunov–Krasovskii functional, which leads to results that are close to the analytical ones in some examples. In addition, augmented Lyapunov–Krasovskii functionals were proposed by Parlakçi [12], which contained a number of free weighting matrices combined with the descriptor model transformation, as well as a cross-term of the state and its derivative in the double integral.

It should be emphasised that most of the reported results on time-delay systems are based on a mathematical model with a single delay. In practical situations, signals transmitted from one point to another through a complex network may experience a few cascading signal paths, which possibly induce successive delays with different properties because of the variable network transmission conditions. When the information on the bounds and/or rates of the time-varying delays in these signal paths are known, less conservative stability criteria can be derived as opposed to lumping the magnitudes and the rates of the delays together. The advantages of such a treatment have been given in [13] with some fundamental results. One contribution of this paper is that we have provided an improvement of the results in [13] as well as a generalisation to the case with multiple delay components. Another contribution is that through the use of a Lyapunov–Krasovskii functional form based on the idea of 'delay partitioning', the results obtained have turned out to be both less conservative and less computationally demanding than the existing methods when applied to the case with a single delay.

The paper is organised as follows. In Section 2, we derive a general result based on delay partitioning for systems with multiple delay components. The result is extended to two cases, one with a single time-varying/invariant delay and the other with unknown (hence arbitrary) delay rates. In Section 3, three detailed numerical examples are used to illustrate that the proposed approach improves the existing methods and gives better (less conservative) upper bounds on the delay for stability than those reported earlier.

Notation: Throughout this paper, the notations are standard. $S_{m\times n}$ stands for matrix S of size $m \times n$ (simply abbreviated S_n when $m = n$). For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). 0 in a matrix inequality is a null matrix with an appropriate dimension. The superscript 'T' represents the transpose of the matrix. col$\{\cdot\}$ denotes a matrix column with blocks given by the matrices in $\{\cdot\}$. A block diagonal matrix with diagonal blocks $A_1, A_2, \ldots, A_r$ is denoted by diag$\{A_1, A_2, \ldots, A_r\}$. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations. For a given real matrix B, the orthogonal complement $B^{\perp}$ (possibly non-unique) is defined as the matrix with maximum column rank that satisfies $BB^{\perp} = 0$ and $B^{\perp\mathrm{T}}B^{\perp} > 0$.

2 Problem formulation

Consider the following linear system with time-varying delays

$$\Sigma : \dot{x}(t) = Ax(t) + A_{\mathrm{d}}x\left(t - \sum_{i=1}^{r} h_i(t)\right) \tag{1}$$

$$x(t) = \phi(t), \quad t \in [-\bar{h}, 0] \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state, $\phi(t)$ is the initial function and r is a positive integer. $h_i(t)$ $(i = 1, \ldots, r)$ represent the time-varying delay components in the state and are assumed to satisfy either the condition A1 or the condition A2 as follows:

A1: $h_i(t)$ is an almost everywhere differentiable function

$$0 < h_i(t) \leq \bar{h}_i < \infty, \quad \dot{h}_i(t) \leq \tau_i < \infty$$

A2: $h_i(t)$ is a measurable (e.g. piecewise-continuous) function

$$0 < h_i(t) \leq \bar{h}_i < \infty$$

Let $\bar{h}$ be an upper bound on the sum of the time delays, that is, $\sum_{i=1}^{r} h_i(t) \leq \bar{h}$. To facilitate development, define $\sigma_j(t) = \sum_{i=1}^{j} h_i(t)$ and $\bar{\sigma}_j = \sum_{i=1}^{j} \bar{h}_i$ with $\sigma_0(t) = 0$, $\bar{\sigma}_0 = 0$ in the boundary expression of the summation. Therefore $h_i(t)$ and $\bar{h}_i$ $(i = 1, \ldots, r)$ represent a partition of the lumped time-varying delay $\sigma_r(t)$ and $\bar{\sigma}_r$, respectively. Here, by employing this delay-partitioning approach, we aim at extending the methodology used in [13] for systems with two delay components via a new form of Lyapunov–Krasovskii functionals to achieve less conservative results.

Lemma 1: Let $Y \in \mathbb{R}^{n\times n}$ and the bi-diagonal upper triangular block matrix

$$J_K(Y) \triangleq \begin{pmatrix} I_n & -Y & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -Y \\ 0 & & & I_n \end{pmatrix} \in \mathbb{R}^{Kn\times Kn}$$

If $Z = \begin{pmatrix} J_K(Y) & S \end{pmatrix} \in \mathbb{R}^{Kn\times(Kn+m)}$ where $S = \begin{pmatrix} S_1 \\ \vdots \\ S_K \end{pmatrix} \in \mathbb{R}^{Kn\times m}$ with $S_i \in \mathbb{R}^{n\times m}$ $(i = 1, \ldots, K)$, then

$$Z^{\perp} = \mathrm{col}\left\{ -\sum_{i=1}^{K} Y^{i-1}S_i, \; -\sum_{i=2}^{K} Y^{i-2}S_i, \; \ldots, \; -S_K, I_m \right\}$$

Proof: First note that when a matrix W is invertible, we have $\begin{pmatrix} W & S \end{pmatrix}^{\perp} = \begin{pmatrix} -W^{-1}S \\ I_m \end{pmatrix}$. The result given in the lemma follows from the fact that $Z^{\perp} = \begin{pmatrix} J_K(Y) & S \end{pmatrix}^{\perp} = \begin{pmatrix} -J_K^{-1}(Y)S \\ I_m \end{pmatrix}$ with

$$J_K^{-1}(Y) = \begin{pmatrix} I_n & Y & \cdots & Y^{K-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Y \\ 0 & \cdots & 0 & I_n \end{pmatrix}$$

$\square$

Lemma 2 (Finsler's lemma): Consider real matrices B and M such that B has full row rank and $M = M^{\mathrm{T}}$. The following statements are equivalent:

1. There exists a scalar $\ell \in \mathbb{R}$ such that

$$\ell B^{\mathrm{T}}B - M > 0$$

2. The following condition holds

$$B^{\perp\mathrm{T}}MB^{\perp} < 0$$

IET Control Theory Appl., 2009, Vol. 3, Iss. 4, pp. 383–390
doi: 10.1049/iet-cta.2007.0321

Theorem 1: Assume that the time delay satisfies A1. System Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $X > 0$ and $Q_i \geq Q_{i+1} > 0\,(i = 1, \ldots, r-1)$ satisfying

$$B^{\perp\mathrm{T}}\begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} B^{\perp} < 0 \tag{3}$$

where $B^{\perp} \in \mathbb{R}^{(2r+3)n\times(r+2)n}$ is the orthogonal complement of $B_{(r+1)n\times(2r+3)n} = \begin{pmatrix} J_{r+1}(I_n) & S \end{pmatrix}$

$$S = \begin{pmatrix} 0 & -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & 0 \\ -I_n & 0 & \cdots & 0 & -I_n \end{pmatrix} \in \mathbb{R}^{(r+1)n\times(r+2)n}$$

and

$$\Omega_1 = \begin{pmatrix} PA + A^{\mathrm{T}}P + Q_1 + R & 0 & \cdots & 0 & PA_{\mathrm{d}} \\ * & -(1-\tau_1)(Q_1 - Q_2) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & -(1-\sum_{i=1}^{r-1}\tau_i)(Q_{r-1} - Q_r) & 0 \\ * & * & \cdots & * & -(1-\sum_{i=1}^{r}\tau_i)Q_r \end{pmatrix}$$

$$\Omega_2 = \bar{\sigma}_r\begin{pmatrix} A & 0 & \cdots & 0 & A_{\mathrm{d}} \end{pmatrix}^{\mathrm{T}} X \begin{pmatrix} A & 0 & \cdots & 0 & A_{\mathrm{d}} \end{pmatrix}$$

$$\Omega_3 = \mathrm{diag}\left\{-R,\ -\bar{h}_1^{-1}X,\ \ldots,\ -\bar{h}_r^{-1}X,\ -(\bar{\sigma}_r)^{-1}X\right\}$$

Proof: Construct the following Lyapunov–Krasovskii functionals $V = V_1 + V_2 + V_3 + V_4$ with

$$V_1 = x^{\mathrm{T}}(t)Px(t)$$

$$V_2 = \int_{t-\bar{\sigma}_r}^{t} x^{\mathrm{T}}(s)Rx(s)\,\mathrm{d}s$$

$$V_3 = \int_{-\bar{\sigma}_r}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$

$$V_4 = \sum_{i=1}^{r}\int_{t-\sigma_i(t)}^{t-\sigma_{i-1}(t)} x^{\mathrm{T}}(s)Q_i x(s)\,\mathrm{d}s$$

Now, considering the derivative of V along the solution of system Σ with respect to t, we obtain

$$\begin{aligned}\dot{V}_1 &= 2x^{\mathrm{T}}(t)P\left[Ax(t) + A_{\mathrm{d}}x(t-\sigma_r(t))\right] \\ &= x^{\mathrm{T}}[PA + A^{\mathrm{T}}P]x(t) + 2x^{\mathrm{T}}(t)PA_{\mathrm{d}}x(t-\sigma_r(t))\end{aligned} \tag{4}$$

$$\dot{V}_2 = x^{\mathrm{T}}(t)Rx(t) - x^{\mathrm{T}}(t-\bar{\sigma}_r)Rx(t-\bar{\sigma}_r) \tag{5}$$

$$\begin{aligned}\dot{V}_3 &= \int_{-\bar{\sigma}_r}^{0}\left[\dot{x}^{\mathrm{T}}(t)X\dot{x}(t) - \dot{x}^{\mathrm{T}}(t+\theta)X\dot{x}(t+\theta)\right]\mathrm{d}\theta \\ &= \bar{\sigma}_r\dot{x}^{\mathrm{T}}(t)X\dot{x}(t) - \int_{t-\bar{\sigma}_r}^{t}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\,\mathrm{d}s \\ &= \bar{\sigma}_r\dot{x}^{\mathrm{T}}(t)X\dot{x}(t) - \int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\,\mathrm{d}s \\ &\quad - \sum_{i=1}^{r}\int_{t-\sigma_i(t)}^{t-\sigma_{i-1}(t)}\dot{x}^{\mathrm{T}}(s)X\dot{x}(s)\,\mathrm{d}s \\ &\leq \bar{\sigma}_r\dot{x}^{\mathrm{T}}(t)X\dot{x}(t) \\ &\quad - \sum_{i=1}^{r}\left[\bar{h}_i^{-1}\left(\int_{t-\sigma_i(t)}^{t-\sigma_{i-1}(t)}\dot{x}(s)\,\mathrm{d}s\right)^{\mathrm{T}} X\left(\int_{t-\sigma_i(t)}^{t-\sigma_{i-1}(t)}\dot{x}(s)\,\mathrm{d}s\right)\right] \\ &\quad -(\bar{\sigma}_r)^{-1}\left(\int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)}\dot{x}(s)\,\mathrm{d}s\right)^{\mathrm{T}} X\left(\int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)}\dot{x}(s)\,\mathrm{d}s\right)\end{aligned} \tag{6}$$

and also

$$\begin{aligned}\dot{V}_4 &= x^{\mathrm{T}}(t)Q_1x(t) - \sum_{j=1}^{r-1}\left[\left(1 - \sum_{i=1}^{j}\dot{h}_i(t)\right)x^{\mathrm{T}}(t-\sigma_j(t))\right. \\ &\quad \left.\times(Q_j - Q_{j+1})x(t-\sigma_j(t))\right] - \left(1 - \sum_{i=1}^{r}\dot{h}_i(t)\right)x^{\mathrm{T}}(t-\sigma_r(t)) \\ &\quad \times Q_r x(t-\sigma_r(t)) \\ &\leq x^{\mathrm{T}}(t)Q_1x(t) \\ &\quad - \sum_{j=1}^{r-1}\left[\left(1 - \sum_{i=1}^{j}\tau_i\right)x^{\mathrm{T}}(t-\sigma_j(t))(Q_j - Q_{j+1})x(t-\sigma_j(t))\right] \\ &\quad - \left(1 - \sum_{i=1}^{r}\tau_i\right)x^{\mathrm{T}}(t-\sigma_r(t))Q_r x(t-\sigma_r(t))\end{aligned} \tag{7}$$

Note that in the above derivations, Jensen's integral inequality and the relationships $Q_i \geq Q_{i+1} > 0\,(i = 1, \ldots, r-1)$ have been used. In the sequel, we carry out the calculations by substituting (4)–(7) as follows

$$\begin{aligned}\dot{V} &\leq x^{\mathrm{T}}(t)[PA + A^{\mathrm{T}}P + Q_1 + R]x(t) \\ &\quad + (\bar{\sigma}_r)\dot{x}^{\mathrm{T}}(t)X\dot{x}(t) + 2x^{\mathrm{T}}(t)PA_{\mathrm{d}}x(t-\sigma_r(t)) \\ &\quad - \sum_{j=1}^{r-1}\left[\left(1 - \sum_{i=1}^{j}\tau_i\right)x^{\mathrm{T}}(t-\sigma_j(t))\right.\end{aligned}$$

$$\times [Q_j - Q_{j+1}] x(t - \sigma_j(t)) \Big]$$
$$- \left(1 - \sum_{i=1}^{r} \tau_i\right) x^{\mathrm{T}}(t - \sigma_r(t)) Q_r x(t - \sigma_r(t))$$
$$- x^{\mathrm{T}}(t - \bar{\sigma}_r) R x(t - \bar{\sigma}_r)$$
$$- \sum_{j=1}^{r} \left[\bar{h}_j^{-1} \left(\int_{t-\sigma_j(t)}^{t-\sigma_{j-1}(t)} \dot{x}(s)\,\mathrm{d}s \right)^{\mathrm{T}} \right.$$
$$\left. \times X \left(\int_{t-\sigma_j(t)}^{t-\sigma_{j-1}(t)} \dot{x}(s)\,\mathrm{d}s \right) \right] - (\bar{\sigma}_r)^{-1} \left(\int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)} \dot{x}(s)\,\mathrm{d}s \right)^{\mathrm{T}}$$
$$\times X \left(\int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)} \dot{x}(s)\,\mathrm{d}s \right)$$
$$= \xi^{\mathrm{T}}(t) \begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} \xi(t)$$

where

$$\zeta(t) = \mathrm{col}\{ x(t) \;\; x(t - h_1(t)) \;\; \cdots \;\; x(t - \sigma_r(t)) \}$$
$$\xi(t) = \mathrm{col}\{ \zeta(t) \;\; x(t - \bar{\sigma}_r) \;\; \textstyle\int_{t-h_1(t)}^{t} \dot{x}(s)\,\mathrm{d}s \cdots$$
$$\textstyle\int_{t-\sigma_r(t)}^{t-\sigma_{r-1}(t)} \dot{x}(s)\,\mathrm{d}s \;\; \int_{t-\bar{\sigma}_r}^{t-\bar{\sigma}_{r-1}} \dot{x}(s)\,\mathrm{d}s \}$$

By the Newton–Leibniz formula, we have

$$x(t) - x(t - \sigma_1(t)) - \int_{t-\sigma_1(t)}^{t} \dot{x}(s)\,\mathrm{d}s = 0$$
$$\vdots$$
$$x(t - \sigma_{r-1}(t)) - x(t - \sigma_r(t)) - \int_{t-\sigma_r(t)}^{t-\sigma_{r-1}(t)} \dot{x}(s)\,\mathrm{d}s = 0$$
$$x(t - \sigma_r(t)) - x(t - \bar{\sigma}_r) - \int_{t-\bar{\sigma}_r}^{t-\sigma_r(t)} \dot{x}(s)\,\mathrm{d}s = 0$$

That is

$$B\xi(t) = \big(J_{r+1}(I_n) \;\; S \big)\, \xi(t)$$
$$= \left(\begin{array}{ccccc|ccccc} I_n & -I_n & 0 & \cdots & & 0 & -I_n & 0 & \cdots & 0 \\ 0 & I_n & -I_n & \cdots & & 0 & 0 & -I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n & & -I_n & 0 & 0 & \cdots & -I_n \end{array} \right) \xi(t) = 0$$

The full-column rank matrix representation of the right orthogonal complement of B, denoted by $B^{\perp}$, can be computed via Lemma 1 with $Y = I_n$, $B^{\perp} = \mathrm{col}\left\{ -\sum_{i=1}^{K} S_i, -\sum_{i=2}^{K} S_i, \ldots, -S_K, I_m \right\}$. By Lemma 2, $\dot{V}$ is negative as long as

$$\xi^{\mathrm{T}}(t) \left(B^{\mathrm{T}} B - \begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} \right) \xi(t) > 0$$

holds, which is equivalent to inequality (3). This implies that system Σ is asymptotically stable. Hence, the proof is complete. □

Remark 1: The descriptor model transformation approach such as in [14] introduces redundant matrix variables into the Lyapunov–Krasovskii functionals, which have no contribution to improving the bound of the delay. On the other hand, it can be easily seen that if Theorem 1 holds for $\bar{h}_i$, the system is asymptotically stable for all $h_i(t) \le \bar{h}_i$. To allow comparison with the existing results, we provide the special case with $r = 2$ in the following corollary.

Corollary 1: When $r = 2$, system Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $X > 0$ and $Q_1 \ge Q_2 > 0$ satisfying

$$B^{\perp\mathrm{T}} \begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} B^{\perp} < 0$$

where

$$B^{\perp} = \begin{pmatrix} I & I & I & I \\ I & 0 & I & I \\ I & 0 & 0 & I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and

$$\Omega_1 = \begin{pmatrix} PA + A^{\mathrm{T}}P + Q_1 + R & 0 & PA_{\mathrm{d}} \\ * & -(1 - \tau_1)(Q_1 - Q_2) & 0 \\ * & * & -(1 - \tau_1 - \tau_2)Q_2 \end{pmatrix}$$
$$\Omega_2 = (\bar{h}_1 + \bar{h}_2)\big(A \;\; 0 \;\; A_{\mathrm{d}} \big)^{\mathrm{T}} X \big(A \;\; 0 \;\; A_{\mathrm{d}} \big)$$
$$\Omega_3 = \mathrm{diag}\left\{ -R, \; -\bar{h}_1^{-1} X, \; -\bar{h}_2^{-1} X, \; -(\bar{h}_1 + \bar{h}_2)^{-1} X \right\}$$

Observing the structure of the Lyapunov–Krasovskii functionals used in Theorem 1, it is obvious that the stability criteria for systems with time-invariant delay can be easily obtained by setting $\tau_i = 0\,(i = 1, \ldots, r)$. However, the detailed proof suggests that the matrix variable R, as well as $x(t - \bar{\sigma}_r)$ and $\int_{t-\bar{\sigma}_r}^{t-\bar{\sigma}_{r-1}} \dot{x}(s)\,\mathrm{d}s$ in $\xi(t)$, will turn out to be redundant. Consequently, we have the following conclusion.

doi: 10.1049/iet-cta.2007.0321

Corollary 2 (time-invariant delays): Assume that the delays in system Σ are time-invariant. Then, system Σ is asymptotically stable if there exist matrices $P > 0$, $X > 0$ and $Q_i > 0$ satisfying

$$B^{\perp T}\begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} B^{\perp} < 0$$

where $B^{\perp} \in \mathbb{R}^{(2r+1)n \times (r+1)n}$ is the orthogonal complement of

$$B_{rn\times(2r+1)n} = \begin{pmatrix} I & -I & 0 & \cdots & 0 & -I & 0 & \cdots & 0 \\ 0 & I & -I & \cdots & 0 & 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I & -I & 0 & 0 & \cdots & -I \end{pmatrix}$$

and

$$\Omega_1 = \begin{pmatrix} PA + A^{\mathrm{T}}P + Q_1 & 0 & \cdots & 0 & PA_{\mathrm{d}} \\ * & Q_2 - Q_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & Q_r - Q_{r-1} & 0 \\ * & * & \cdots & * & -Q_r \end{pmatrix}$$

$$\Omega_2 = \bar{\sigma}_r (A \quad 0 \quad \cdots \quad 0 \quad A_{\mathrm{d}})^{\mathrm{T}} X (A \quad 0 \quad \cdots \quad 0 \quad A_{\mathrm{d}})$$

$$\Omega_3 = \mathrm{diag}\left\{-\bar{h}_1^{-1}X, \ldots, -\bar{h}_r^{-1}X\right\}$$

It easily shows that when $r = 1$, Corollary 2 corresponds to Theorem 1 in [15].

Remark 2: The results given in Theorem 1 and Corollary 2 can also be employed to estimate an upper bound of a single delay for the guarantee of stability. For delay components with the same derivative ($\dot{h}_1(t) = \dot{h}_2(t) = \cdots = \dot{h}_r(t)$) for time-varying delay and equal width ($h_1 = h_2 = \cdots = h_r$) for time-invariant delay, their influence on the maximal delay bound for stability becomes identical and indistinguishable from the effects due to a single delay. Consequently, by maximising the delay bound of each component, we can compute an overall stability bound on the single effective time delay. For Σ with only a single time-varying delay $h(t)$ satisfying A1, one can partition $h(t)$ into r identical delay components, that is, $h(t) = rh_1(t)$, $\tau = r\tau_1$, then Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$, $X > 0$ and $Q_i \geq Q_{i+1} > 0$ ($i = 1, \ldots, r-1$) satisfying inequality (3) where all $\bar{h}_i$, τ_i ($i = 1, \ldots, r$) are replaced by $\bar{h}_1$ and τ_1.

For case A2, when the delay rates are not specified, a stability criterion can be obtained by employing a Lyapunov–Krasovskii functional $V = V_1 + V_2 + V_3$ with V_i ($i = 1, 2, 3$) defined in the proof of Theorem 1. The proof can then be established by letting $Q_i = 0$ ($i = 1, \ldots, r$) and following a similar line of arguments as that in Theorem 1.

Theorem 2: Assume the time delays satisfy A2. Then, system Σ is asymptotically stable if there exist matrices $P > 0$, $R > 0$ and $X > 0$ satisfying

$$B^{\perp T}\begin{pmatrix} \Omega_1 + \Omega_2 & 0 \\ 0 & \Omega_3 \end{pmatrix} B^{\perp} < 0$$

where

$$\Omega_1 = \begin{pmatrix} PA + A^{\mathrm{T}}P + R & 0 & \cdots & 0 & PA_{\mathrm{d}} \\ * & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & 0 & 0 \\ * & * & \cdots & * & 0 \end{pmatrix}$$

and $B^{\perp}$, Ω_2, Ω_3 are defined as in Theorem 1.

Remark 3: The method adopted in this paper can be extended not only to robust stability analysis but also to the general multiple-delay case (see [16]), that is

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{p} A_{d_i} x\left(t - \sum_{j=1}^{r} h_{ij}(t)\right)$$

where the time delay $h_{ij}(t)$ has the natural assumption that $0 < h_{ij}(t) \leq \bar{h}_{ij} < \infty$, $\dot{h}_{ij}(t) \leq \tau_{ij} < \infty$.

3 Numerical examples

In this section, we provide three examples to demonstrate the reduced conservatism of the proposed conditions in this paper.

Example 1: Consider the linear time-delay system Σ taken from [13] with

$$A = \begin{pmatrix} -2.0 & 0 \\ 0 & -0.9 \end{pmatrix}, \quad A_{\mathrm{d}} = \begin{pmatrix} -1.0 & 0 \\ -1.0 & -1.0 \end{pmatrix}$$

We summarise and compare the results obtained in the literature with those presented in our paper. According to Theorems 1 and 2, the 'maximal' admissible time delay $\bar{h}_{\max}$, in the sense of the feasibility of the corresponding criteria, can be computed by either specifying $\bar{h}_1$ or $\bar{h}_2$ and maximising the other. For the two cases A1 and A2, the results are summarised in Tables 1 and 2.

The numerical results shown in Table 1 illustrate that the stability condition in Corollary 1 is less conservative than that

Table 1 Comparison on Case A1 for Example 1 (two additive delays)

	$\bar{h}_{2\max}$ for given $\bar{h}_1 = 1$		$\bar{h}_{1\max}$ for given $\bar{h}_2 = 0.1$		Number of variables
	$\tau_1 = 0.1$	$\tau_1 = 0.8$	$\tau_1 = 0.1$	$\tau_1 = 0.8$	
	$\tau_2 = 0.8$	$\tau_2 = 1.6$	$\tau_2 = 0.8$	$\tau_2 = 1.6$	
Corollary 1	0.5126	0.3736	2.3007	1.3454	$2.5n^2 + 2.5n$
Lam *et al.* [13]	0.4159	0.2122	2.2638	1.1844	$12.5n^2 + 4.5n$

Table 2 Comparison on $\bar{h}_{\max}$ for Example 1 (single delay with rate τ)

	$\tau = 0.2$	$\tau = 0.999$	$\tau = 1.999$	τ unknown
Theorem 1 ($r = 5$)	3.0391	1.3930	1.3454	–
Theorem 1 ($r = 1$)	3.0391	1.3454	1.3454	–
Kao and Rantzer [17]	3.807	1.360	1.000	–
He *et al.* [18]	3.0391	1.3454	1.3454	–
Jing *et al.* [19]	3.0338	1.0018	0.9999	0.9999
Parlakçi [12]	3.0338	1.0015	0.9994	–
Theorem 2	–	–	–	1.3454

in [13], not only providing a larger upper bound on delay but also involving fewer variables.

Remark 4: Compared with the stability condition in ([18], Theorem 1), Theorem 1 in our paper when $r = 1$ gives the same upper bound on the delay but has significantly fewer variables. When $A, A_d \in \mathbb{R}^{n\times n}$, the number of variables to be determined in our result is $2n^2 + 2n$, while that in [18] is $11.5n^2 + 2.5n$. In other words, the variables in [18] are around 5.5 times more than those in Theorem 1.

Remark 5: According to Theorem 2, when r is equal to 1–20, the maximal delay has the same value 1.3454. In addition, it can be seen from Tables 1 and 2 that $\bar{h}_{\max}$ will reduce to this value gradually as minimum of τ_i $(i = 1, \ldots, r)$ increases. Hence, one may conjecture that the maximal upper bound of the time-varying delay $\bar{h}_{\max}$ can be achieved by increasing the minimum of τ_i (which corresponds to the maximum delay bound for arbitrary delay rate).

Next, we consider that the single time-invariant delay is partitioned into a number of delay components of equal width in Corollary 2. By maximising the width of the delay components, the overall stability bound on delay is the sum of all parts. The result is summarised in Table 3.

Table 3 Case with time-invariant delay components for Example 1

	$\bar{h}_{\max}$	Number of variables
He *et al.* [18]	4.4721	51
Suplin *et al.* [20]	4.4721	38
Xu and Lam [15]	4.4721	17
Gu *et al.* [11]	6.059	27
Peaucelle *et al.* [21] ($r = 9$)	6.149	177
Zhang *et al.* [22]	6.150	81
Corollary 2 ($r = 1$)	4.4721	9
Corollary 2 ($r = 5$)	6.0983	21
Corollary 2 ($r = 20$)	6.1668	66
Theoretical bound	6.1725813	

Remark 6: It follows from Table 3 that the conservatism can be reduced by introducing more delay components (and hence more matrix variables). Moreover, the number of variables to calculate in our result is $((r/2) + 1)(n^2 + n)$ and shows less conservatism for this example.

Example 2: Consider a linear system Σ taken from [23] with

$$A = \begin{pmatrix} -1.7073 & 0.6856 \\ 0.2279 & -0.6368 \end{pmatrix}, \quad A_d = \begin{pmatrix} -2.5026 & 1.0540 \\ -0.1856 & -1.5715 \end{pmatrix}$$

For the time-varying delays with $\bar{h}_1 = 0.2$, $\tau_1 = 0.3$ and $\tau_2 = 0.8$, the computed value of $\bar{h}_{2\max}$ obtained in [13] and Corollary 1 are 0.2187 and 0.2863, respectively. In addition, according to the method mentioned in Remark 2, the upper bound of delay obtained and the variables involved in the system with a single time-varying/invariant delay are listed in Table 4.

doi: 10.1049/iet-cta.2007.0321

Table 4 Case with time-varying/invariant delay components for Example 2

Time-varying	$\bar{h}_{\max}$ for $\tau = 1.1$	Number of variables	Time-invariant	$\bar{h}_{\max}$	Number of variables
Parlakçi [12]	0.4561	69	Wu *et al.* [24]	0.7163	54
Jing *et al.* [19]	0.4561	39	He *et al.* [23]	0.7918	49
He *et al.* [18]	0.5846	56	Corollary 2 ($r = 4$)	0.7959	18
Theorem 1 ($r = 5$)	0.5885	24	Corollary 2 ($r = 14$)	0.8026	48

Example 3: Consider a mechanical rotation cutting process [25] modelled by

$$\ddot{x}(t) + 2\rho\omega_n\dot{x}(t) + \omega_n^2 x(t) = \frac{\gamma}{m}(x(t - h(t)) - x(t)) \qquad (8)$$

where x represents the deflection of the machine tool and/or workpiece, ω_n the natural frequency, ρ the damping ratio, m the modal mass, and $\gamma > 0$ the cutting coefficient (which depends on the width of the cutting tool and on the nominal depth of cut). The term $\gamma(x(t - h(t)) - x(t))$ represents a delay-dependent cutting force as a result of the cutting profile of the workpiece in the previous turn. In fact, the time-delay is inversely proportional to the rotational speed of the machine.

Typical parameter values are $m = 100$, $\omega_n = 632.45$ and $\rho = 0.039585$. We consider $h(t)$ to be a differentiable function with a varying rate $\dot{h}(t) \leq 2$. Using Theorem 1, an upper bound $\bar{h}$ of the delay $h(t)$ for ensuring stability of (8) can be computed. Setting $r = 1$, for $\gamma = 10\,000$ and $\gamma = 100\,000$, we obtain that $\bar{h}$ are 0.7069 and 0.0697, respectively. These values can be improved to 0.8131 and 0.0774, respectively, if one sets $r = 5$.

4 Conclusion

This paper has provided new simplified and more efficient stability criteria for linear continuous systems with multiple delay components. The results extend and improve the existing works. A new form of Lyapunov–Krasovskii functional is constructed to improve delay-dependent stability conditions for linear continuous systems for cases with or without delay rates. New criteria with reduced conservatism are obtained and they involve less matrix parameters than the existing ones. Three numerical examples have been employed to demonstrate the effectiveness and merits of the proposed results.

5 Acknowledgments

This work is supported by RGC HKU 7031/06P, and an international joint project by the Royal Society of the UK.

6 References

[1] GAO H., CHEN T.: 'New results on stability of discrete-time systems with time-varying state delay', *IEEE Trans. Autom. Control*, 2007, **52**, (2), pp. 328–334

[2] MA S., CHENG Z., ZHANG C.: 'Delay-dependent robust stability and stabilisation for uncertain descrete singular systems with time-varying delays', *IET Control Theory Appl.*, 2007, **1**, (4), pp. 1086–1095

[3] SHIN H., CHOI H., LIM J.: 'Feedback linearisation of uncertain nonlinear systems with time delay', *IEE Proc.-Control Theory Appl.*, 2006, **153**, (6), pp. 732–736

[4] FRIDMAN E., SHAKED U.: 'Delay-dependent stability and H_∞ control: constant and time-varying delays', *Int. J. Control*, 2003, **76**, (1), pp. 48–60

[5] PARK P.: 'A delay dependent stability criterion for systems with uncertain time-invariant delays', *IEEE Trans. Autom. Control*, 1999, **44**, (4), pp. 876–877

[6] MOON Y.S., PARK P.G., KWON W.H., LEE Y.S.: 'Delay-dependent robust stabilization of uncertain state-delayed systems', *Int. J. Control*, 2001, **74**, (14), pp. 1447–1455

[7] WU M., HE Y., SHE J., LIU G.: 'Delay-dependent criteria for robust stability of time-varying delay systems', *Automatica*, 2004, **40**, pp. 1435–1439

[8] LI T., GAO L., LIN C.: 'A new criterion of delay-dependent stability for uncertain time-delay systems', *IET Control Theory Appl.*, 2007, **1**, (3), pp. 611–616

[9] XU S., LAM J., ZOU Y.: 'Further results on delay-dependent robust stability conditions of uncertain neutral systems', *Int. J. Robust Nonlinear Control*, 2005, **15**, pp. 233–246

[10] KOLMANOVSKII V.B., RICHARD J.P.: 'Stability of some linear systems with delay', *IEEE Trans. Autom. Control*, 1999, **44**, (5), pp. 984–989

[11] GU K., KHARITONOV V., CHEN J.: 'Stability of time-delay systems' (Birkhauser, Boston, MA, 2003)

doi: 10.1049/iet-cta.2007.0321

[12] PARLAKÇI M.N.A.: 'Robust stability of uncertain time-varying state-delayed systems', *IEE Proc.-Control Theory Appl.*, 2006, **153**, (4), pp. 469–477

[13] LAM J., GAO H., WANG C.: 'Stability analysis for continuous systems with two additive time-varying delay components', *Syst. Control Lett.*, 2007, **56**, (1), pp. 16–24

[14] XU S., LAM J., ZOU Y.: 'Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems', *IEE Proc.- Control Theory Appl.*, 2005, **152**, (2), pp. 147–151

[15] XU S., LAM J.: 'Improved delay-dependent stability criteria for time-delay systems', *IEEE Trans. Autom. Control*, 2005, **50**, (3), pp. 384–387

[16] GAO H., WANG C.: 'Delay-dependent robust H_∞ and $L_2 - L_\infty$ filtering for a class of uncertain nonlinear time-delay systems', *IEEE Trans. Autom. Control*, 2003, **48**, (9), pp. 1661–1666

[17] KAO C., RANTZER A.: 'Stability analysis of systems with uncertain time-varying delays', *Automatica*, 2007, **43**, pp. 959–970

[18] HE Y., WANG Q., XIE L., LIN C.: 'Further improvement of free-weighting matrices technique for systems with time-varying delay', *IEEE Trans. Autom. Control*, 2007, **52**, (2), pp. 293–299

[19] JING X., TAN D., WANG Y.: 'An LMI approach to stability of systems with severe time-delay', *IEEE Trans. Autom. Control*, 2004, **49**, (7), pp. 1192–1195

[20] SUPLIN V., FRIDMAN E., SHAKED U.: 'A projection approach to H_∞ control of time-delay systems'. IEEE Conf. Decision and Control, December 2004, pp. 4548–4553

[21] PEAUCELLE D., ARZELIER D., HENRION D., GOUAISBAUT F.: 'Quadratic separation for feedback connection of an uncertain matrix and an implicit linear transformation', *Automatica*, 2007, **43**, pp. 795–804

[22] ZHANG J., KNOSPE C., TSIOTRAS P.: 'Stability of linear time-delay systems: a delay-dependent criterion with a tight conservatism bound'. Proc. American Control Conference, June 2000, pp. 1458–1462

[23] HE Y., WANG Q., LIN C., WU M.: 'Augmented Lyapunov functional and delay-dependent stability criteria for neutral systems', *Int. J. Robust Nonlinear Control*, 2005, **15**, pp. 923–933

[24] WU M., HE Y., SHE J.: 'New delay-dependent stability criteria and stabilizing method for neutral systems', *IEEE Trans. Autom. Control*, 2004, **49**, (12), pp. 2266–2271

[25] CHIASSON J., LOISEAU J.J.: 'Applications of time delay systems' (Springer, Berlin, 2007)

doi: 10.1049/iet-cta.2007.0321