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On Stabilization of Non-Uniformly Sampled Control Systems - A Survey

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Abstract. The paper presents a collection of chosen results solving the problem on stability and stabilizability of linear and linearized non-uniformly sampled systems. The results are divided onto three groups: results on systems with input delay, systems with discrete-time non-uniformly sampled control input and hybrid systems.

Introduction

Recently a non-uniform sampling attracts more and more attention and become major point of researches, since properly introduced irregularities may provide more benefits than classical sampling methods, [1-4]. Further decreasing data size and simultaneously ensuring sufficient amount of accuracy is the main purpose of using various sampling schemes. Non-uniform sampling has much more advantages, which leads to raise number of practical applications. Taking samples in not equal time spaces allow to handle the problem of reconstruction of finite dimensional signals. Therefore number of works considering signals reconstruction from non-uniform samples has been raised in last few years [5-10]. Moreover, the non-uniform sampling can bring on the suppression of aliasing [11, 12]. In chemical engineering, many variables that define quality of product are non-uniformly sampled through communication error or curtailment in sensor or actuator [13]. In medical applications, for example in the nuclear magnetic resonance, in networked control systems, in automotive application, sensing in devices etc. [14, 15].

In the literature the non-uniform sampling is called in different ways, for example the irregular sampling, the non-equidistant sampling, the staggered sampling, the uneven sampling or the non-homogeneous sampling. Non-uniform sampling can be realized either as randomized, pseudo-randomized sampling or periodic sampling with different subperiods (so called Lth order periodic sampling), see for example in [12], and [16-24].

However, for systems non-uniformly sampled, where space between samples may change, stabilization analysis is more complicated than in the classical case. One of the possible reasons is the fact that one can see on control function $u(t) = Kx(t)$ (even with constant gain matrix) on different ways. Namely, this control can be viewed as a continuous-time function with bounded input delay from the current sampling interval or as a discrete-time delay from the current sampling interval or one can consider a hybrid system, see for example [3-4], [25-29]. On the other hand, delay of a sampled-data system can be treated also as a parameter variation, see [30].

In the first part of the paper there is presented the most popular nonuniform sampling methods, i.e. randomized sampling schemes: additive random sampling, jittered random sampling, Lth order periodic sampling, recurrent sampling multi-rate sampling and time-stampless sampling. Next, the basic facts on stability of linear systems are recalled. Different approaches to the modeling of sampled-data system with non-uniform sampling and solution of stabilization problem for them are discussed in the next step. Examples of practical applications are presented on the end.

Non-uniform sampling methods

Generally sampling process converts a continuous analogue signal $x(t)$ into its discrete representation $x_s(t)$

$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-t_k) \quad (1)$$

where δ denotes the Dirac impulse and t_k is the sampling time instant. The sampling instant sequence $\{t_k: k \in N\}$ in the non-uniform sampling is defined as $t_k < t_{k+1}$ and $t_k \neq kT$, where $T > 0$ denotes the mean intersample period, while in uniform sampling $t_k = kT$, see for example in [31-33]. The non-uniform sampling can be classified in to randomized sampling modes – additive random sampling [14], [34], jittered random sampling [11] and modes of deterministic sampling, for example periodic sampling [35], recurrent sampling [36], multi-rate sampling [37] and time-stampless sampling [38], which are described in next subsections.

Jittered random sampling. The jittered sampling occurs when jitter, predictable noise, is added deliberately to the uniform sampling instants, see for example in [11], [14]. Sampling noise $r_k \in (0, \infty)$ is a random variable independently and identically distributed with expected value $E[\{r_k\}] = 0$. Sampling time instants are taken as

$$t_k = kT + r_k \quad (2)$$

with $r_k \in (-\frac{T}{2}, \frac{T}{2})$ and $E[\{r_k\}] = 0$. In (2) $T, T > 0$, denotes an intersample period. Probability distribution of the sampling noise r_k probability is given by [34]

$$p_k(t) = p_{r_k}(t - kT), \;\; k \in N \quad (3)$$

Additive random sampling. One of the most common types of non-uniform sampling is additive random sampling (ARS). This type of the sampling occurs when the lengths of sampling intervals are independent random variables, see [11]. The idea of the sampling time instants relies on adding the sampling noise to the previous sampling instants, see for example in [14], [34]. The sampling scheme is the following

$$t_k = t_{k-1} + r_k = \sum_{s=1}^{k} r_s \quad (4)$$

The probability of the density function of sampling time instants t_k is taken as [14]

$$p_k(t) = p(t_k = t) \quad (5)$$

where t_k denotes the sampling time instances for $k = 1, 2, \dots, n$, and

$$p_k(t) := (p_{r_k} * \dots * p_{r_k})(t) \quad (6)$$

where $*$ denotes the convolution operation of the sampling noise and p_{r_k} denotes probability density function of sampling noise r_k.

Simulation results showed that ARS is able to suppress aliasing better than jittered random sampling (JRS).

Periodic sampling. Another type of the non-uniform sampling, proposed in [35], is a periodic non - uniform sampling. The average sampling rate is assumed to be equal to a total bandwidth θ. It is defined as follow. Let $\{x(kT), k \in N\}$ and $\{x(kT + d_1), k \in N\}$ be two sets of samples, where $T = 2T_o$, with $T_0 = \frac{2\pi}{\theta}, d_1 < T$, is a time offset depends on the band positions of the signal, see

Fig. 1 (see also for example [39]). Lth order periodic sampling, where $L > 1$, is defined as the sampling with L different sampling periods, i.e. as the following set of time instances samplings $\{x(kT),\ k \in N\}, \{x(kT + d_1), k \in N\}, \dots, \{x(kT + d_{L-1}), k \in N\}$.

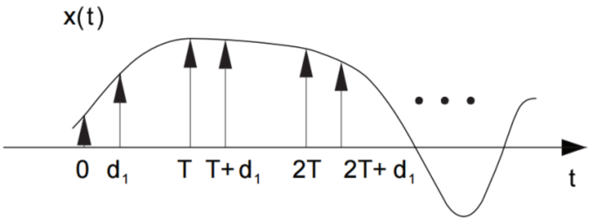

Figure 1. Periodical non-uniform sampling.

Recurrent sampling. The other sampling scheme is a recurrent non-uniform sampling, see for example [36], [40]. On this case the sampling scheme is given by:

$$t_k = \begin{cases} kT, & k = 0, 1, 2, \dots, n \\ nT + \bar{T}, & k = n + 1 \\ t_{k-(n+1)} + nT + \bar{T}, & k > n + 1 \end{cases} \tag{7}$$

where $n \in \boldsymbol{Z}^+$, $T, \bar{T} \in \boldsymbol{R}^+$, where n is nonnegative integer, $T, \bar{T}$ are nonnegative real numbers. Each sampling period consists of blocks of n intervals each of length T, followed by single interval $\bar{T}$ see Fig. 2.

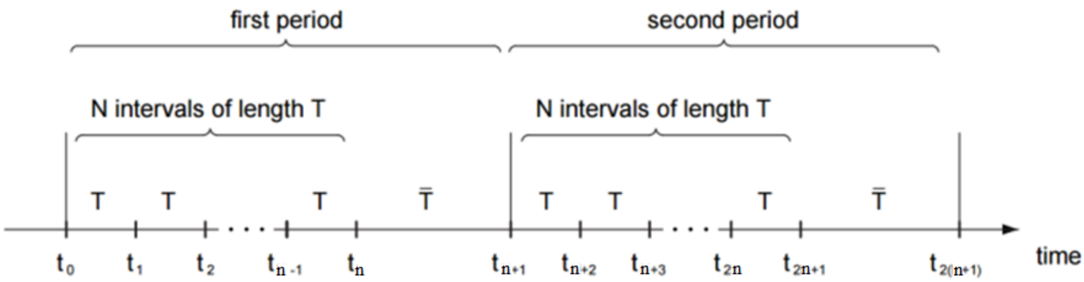

Figure 2. The exemplary recurrent non-uniform sampling scheme.

Multi-rate sampling. Generally multi-rate sampling is a periodic sampling with the overall interval periodicity T_o. Sampling pattern can be described as follow, see for example [37], [41-44]: in time period T_o there are s outputs of signal measurements with the last one measure at time instant $(k + 1)T_o$ and there are r inputs of signal updates with the first one at kT_o. At the chosen T_o there are different numbers of input signal updates $U_k = \begin{bmatrix} u(kT_o) \\ \dots \\ y(kT_o + nT - T) \end{bmatrix}$, where $nT = T_o$, output signal measurements $Y_k = \begin{bmatrix} y(kT_o + T) \\ \dots \\ y(kT_o + nT) \end{bmatrix}$, $k \in N$. The idea of multi-rate sampling is presented on Fig. 3.

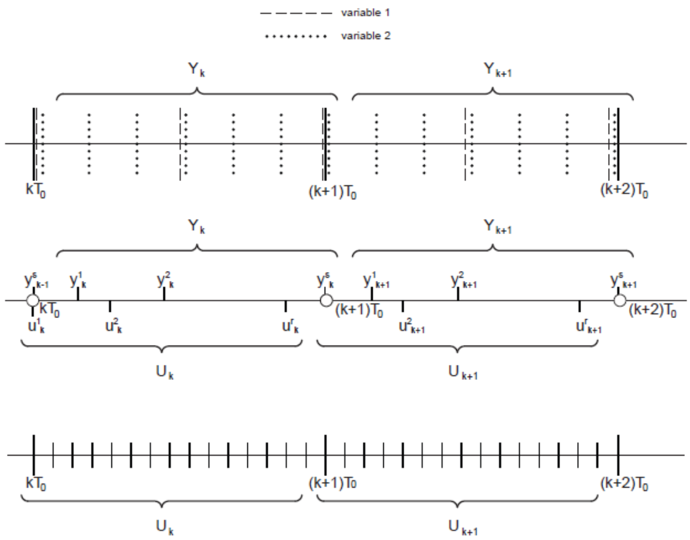

Figure 3. Multirate sampling.

Time – stampless adaptive sampling. The purpose of applying this sampling scheme is to minimize power consumption of the sampling process. The other aim is to reduce the number of sampling time instances by transmitting sampling times only if receiver is able to compute them by using a function of previous sample.

Time-stampless adaptive sampling proposed in [38], [45] is defined as follow: for a continuous time signal $x(t)$ the jth sample is taken at time instant t_j. Define $T_j := t_{j+1} - t_j$ and $\Delta_j := x(t_{j+1}) - x(t_j)$. Further, the $(n+1)$th sample is taken after a time interval with the length $T_n = f(\bigcup_{j=n-m+1}^{n-1}\{T_j, \Delta_j\})$, where $f(\cdot)$ is a sampling function. The sampling function f of m most recently taken samples has to be known (see for example [46]). Note that the sampling times can be recovered from the sampling function f and from previous samples. Therefore keeping sampling times are not necessary.

As one can see, depending on type of the application, different sampling schemes are used. In the frequency analysis additive and jittered random sampling can be used to suppress aliasing effect. Multi-rate sampling faces the problem of sampling rate which is not same for all process variables. This sampling scheme allows to introduce various sampling rates to the process variables. Other type of application is to apply non-uniform sampling to lower power consumption to extend a battery life-time, where time-stampless adaptive sampling or sampling scheme presented in [46] is beneficial. In many applications where samples needed to be pre-established, periodic and recurrent sampling can be used, due to its deterministic character [41]. Moreover most of non-uniform sampling patterns are also used to reduce size of data [45, 46].

Basic facts about stability of linear systems

The goal of this Section is to present modeling of sampled-data system with the non-uniform sampling and stability criterions for linear control systems. We will use the following notation: the superscript $'$ stands for the matrix transposition and • denotes symmetric elements of the given matrix. By $\omega(A)$ we will denote a spectrum of matrix A. By $A > 0$ we will denote the fact that matrix A is positively definite and by A$<$ 0, the fact that A is a negatively definite. By I we will denote the identity matrix.

First recall Sylvester's Theorem.
Theorem [47]: Let $Q: R^n \to R$ be a quadratic form with matrix representation $A \in R^{nxn}$. Let W_i, $i = 1, \dots, n$, denotes the $i \times i$ minor obtained from A by removing last $n-1$ rows and columns. Then Q is positive definite if and only if W_i is positive defined for $i = 1, \dots, n$. Q is negative

definite if and only if W_i is negative defined for $i = 1, 3, 5, ...,$ n and W_i is positive defined for $i = 2, 4, 6, ..., n$.

Let us consider a linear system:

$$\dot{x}(t) = Ax(t), \quad x(0) = x_o \in R^n \tag{8}$$

where $A \in R^{n\,x\,n}$. Let $\psi(t, t_0, x_0)$ denotes the trajectory of this system. It is obvious that $\psi(t, t_0, x_0) = e^{At}x_0$.

Recall that (see [48, 49]) system (8) is exponentially stable if there exists constants $M \geq 1$ and $a < 0$ such that

$$\|\psi(t, t_0, x_0)\| \leq Me^{\alpha(t-t_0)}, \ \ t \geq 0 \tag{9}$$

where $\alpha = \max\{Re\ s : s \in \omega(A)\}$. Recall also that $n \times n$ matrix A is called Schur stable if all its eigenvalues lie in the open unit disc, for discrete-time, or all eigenvalues are not smaller than 0, for continuous time, see [51].

Recall that the equilibrium point x_r of system (8) is, see [50]:

(i) uniformly stable at $t = t_0$ if for any $\varepsilon > 0$, there exist $\delta > 0$ (not dependent on t_0), such that: if $\|x_o(t_o) - x_r(t_o)\| < \delta$, then $\|\psi(t, t_o, x) - x_r(t)\| < \varepsilon$, for every $t \geq t_o$.

(ii) uniformly attractive if there is a positive constant $c = c(t_o)$ such that $x_r(t) \to 0$ as $t \to \infty$ for all $\|x_r(t_0)\| < c$.

System (8) is uniformly asymptotically stable if

(i) equilibrium point x_r is uniformly stable,

(ii) equilibrium point x_r is uniformly attractive,

(iii) there exist $\delta > 0$ (not dependent on t_0) such that, $\|x_o(t_o) - x_r(t_o)\| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

Theorem 1 [48 ÷50]: System (8) is

(i) stable if and only if all the eigenvalues of A have not positive real parts, and each eigenvalue with real part equal to zero is a singular eigenvalue.

(ii) exponentially stable if and only if all the eigenvalues of A have negative real parts.

(iii) asymptotically stable if and only if all the eigenvalues of A have negative real parts.

Stabilization of non-uniformly sampled continuous-time systems with input delay.

Let us consider linear system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{10}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input, $A \in R^{n\times n}$ and $B \in R^{n\times m}$ are constant matrices.

Problem 1. Design the state-feedback controller of the form

$$u(t) = Kx(t_k), \ t_k \leq t < t_{k+1}, \tag{11}$$

with the gain matrix $K \in R^{nxm}$ is the controller gain and t_k denotes sampling time instants.

Assumption 1: There exists a positive scalar τ_m and the sampling interval $[t_k, t_{k+1}]$ with $t_{k+1} - t_k = T_k$ such that

$$0 < T_k \leq \tau_m, \qquad \text{for all} \quad k \in N_o$$

The Problem 1 can be reformulated as follows: find a state-feedback controller of the form $u(t) = Kx(t - \tau(t))$. Then the closed-loop system is of the form

$$\dot{x}(t) = Ax(t) + BKx\big(t - \tau(t)\big), \quad (12)$$
$$\tau(t) = t - t_k,$$

for any $t \in [t_k; t_{k+1}]$. So system (12) can be considered as a linear system with uncertain and bounded delay, see [52, 53].

Lemma 1 [53] Let matrix K be given and h denote a maximal length of sampling intervals. Under Assumption 1, system (12) is stable for all the sampling periods, if there exists positive defined matrices such that $P_1, P_2, P_3, Z_1, Z_2, Z_3$ and real number $\sigma > 0$ that satisfy following linear matrix inequalities (LMIs)

$$\psi_1 < 0 \quad (13)$$

$$\begin{bmatrix} \sigma & [0\ K'B']P \\ \bullet & Z \end{bmatrix} \geq 0 \quad (14)$$

with $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$, $Z = \begin{bmatrix} Z_1 & Z_2 \\ \bullet & Z_3 \end{bmatrix}$, $\psi_1 = \psi_0 + hZ + \begin{bmatrix} 0 & 0 \\ 0 & h\sigma \end{bmatrix}$ and $\psi_0 = P' \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix}' P$.

Let $M_1 = [\, I \;\; 0 \,]$, $M_2 = \big[\, I - I \,\big]$, $M_3 = [\, A \;\; BK \,]$, where A, B, K are given by (12). Suppose that there exist symmetric positive definite matrices $P, R, S \in R^{nxn}$ and matrix $N \in R^{2nxn}$ such that $\Pi_1 := M_1'PM_3 + M_3'PM_1 - M_2'SM_2 - NM_2 - M_2'N'$, $\Pi_2 := M_2'SM_3 + M_3'SM_2 + M_3'RM_3$.

Theorem 2 [26]: Under Assumption 1, if

$$\Pi_1 + T_m\Pi_2 < 0, \quad (15)$$

$$\begin{bmatrix} \Pi_1 & T_mN \\ \bullet & -T_mR \end{bmatrix} < 0, \quad (16)$$

then system (12) is asymptotically stable for any time-varying periods of length less than T.

Let us consider the following system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (17)$$
$$u(t) = Kx(t_k),\ \text{with } t \in [t_k, t_{k+1}),$$

with state $[x'(t),\ x'(t_k)]'$. Then

$$\begin{bmatrix} \dot{x}(t)' \\ \dot{x}(t_k)' \end{bmatrix} = F \begin{bmatrix} x(t)' \\ x(t_k)' \end{bmatrix} \text{ for } t \neq t_k,\ k \in N, \quad (18)$$

where $x(t_k) = \begin{bmatrix} x(t_{\bar{k}}) \\ x(t_{\bar{k}}) \end{bmatrix}$ for $t = t_k,\ k \in N$ and $x(t_{\bar{k}}) \coloneqq \lim_{\tau \to t_k} x(\tau)$, $F \coloneqq \begin{bmatrix} A & B_u \\ 0 & 0 \end{bmatrix}$, $B_u \coloneqq BK$. At the sampling time instants t_k, the value of x remains unchanged, while the value of x is updated by $x(t_{\bar{k}})$. Let $\bar{F} = [A \quad B]$ and $M_1 = \begin{bmatrix} P \\ 0 \end{bmatrix} \bar{F} + \bar{F}'[P \quad 0] - \begin{bmatrix} I \\ -I \end{bmatrix} X_1 [I \quad -I] - N[I \quad -I]N' + \tau_{MATI}\, \bar{F}'R\bar{F}$, $M_2 = \begin{bmatrix} I \\ -I \end{bmatrix} X_1 \bar{F} + \bar{F}'X_1' [I \quad -I]$, $\bar{M}_1 = M_1 - \begin{bmatrix} 0 \\ I \end{bmatrix} X_2 [I \quad -I] - \begin{bmatrix} I \\ -I \end{bmatrix} X_2' [0 \quad I]$, $\bar{M}_2 = M_2 + \begin{bmatrix} 0 \\ I \end{bmatrix} X_2 \bar{F} + \bar{F}'X_2'[0 \quad I]$.

Theorem 3 [30]: If there exist positive defined matrices P, R and not necessarily symmetric matrices N, X_1, X_2 that satisfies the following LMIs

$$\bar{M}_1 + \tau_{MATI}\bar{M}_2 < 0, \quad \begin{bmatrix} \bar{M}_1 & \tau_{MATI}N \\ \bullet & -\tau_{MATI}R \end{bmatrix} < 0, \tag{19}$$

where τ_{MATI} denotes length of the largest sampling interval, then system (18) is globally uniformly exponentially stable.

As a particular case of system (12) one can consider the system with the state-feedback controller (11) with the following sampling scheme. Let $\eta_k = t_{k+1} - t_k$ be the length of the sampling interval $[t_k, t_{k+1})$. Suppose that there exists a scalar $\bar{\eta}$ such that $0 < \eta_k < \bar{\eta}$, $[0, \bar{\eta}) = \cup_{i=1}^{m}[\bar{\eta}_{i-1}, \bar{\eta}_i)$, with $\bar{\eta}_o = 0, \bar{\eta}_m = \bar{\eta}$. Suppose also that the probability distribution of the sampling interval η_k in subinterval $[\bar{\eta}_{i-1}, \bar{\eta}_i)$ is known. Define a piecewise right continuous indicator function

$$\pi_i(t) = \begin{cases} 1, & \eta_k \in [\bar{\eta}_{i-1}, \bar{\eta}_i), \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

for all $t \in [t_k, t_{k+1})$. Thus the closed-loop system is the following

$$\dot{x}(t) = Ax(t) + \textstyle\sum_{k=1}^{m} \pi_i(t)BK\big(t - \tau_k(k)\big), \tag{21}$$

In this case, the input delay is described as a multiple random variable $(t) \in \{\tau_k(t) \coloneqq t - t_k$, $k = 1, 2, \ldots, m$. In this case for $\alpha_i, \gamma_i \geq 0, i = 1,2, \ldots, m$, symmetric, positive defined matrices P, R_i, symmetric matrices $Q_{i,1}$, $Q_{i,2}$, matrices $N_{i,1}$, $N_{i,2}$ of appropriate dimensions $i = 1,2, \ldots, m$, and controller gain K, let define matrices $\Omega_{i,1}, \Omega_{i,2}, \Omega_{i,3}, N_i, \Pi_i$ as follows

$$\Omega_{i,1} = \begin{bmatrix} PA + A'P + 2\alpha_i P & PBK \\ \bullet & 0 \end{bmatrix}$$

$$\Omega_{i,2} - \begin{bmatrix} Q_{i,1} & -Q_{i,1} + Q_{i,2} \\ \bullet & -2Q_{i,2} + Q_{i,1} \end{bmatrix} + 2\alpha_i\big(T_i - h_i(t)\big)\begin{bmatrix} Q_{i,1} & -Q_{i,1} + Q_{i,2} \\ \bullet & -2Q_{i,2} + Q_{i,1} \end{bmatrix}$$

$$+2\alpha_i\big(T_i - \tau_i(t)\big)\begin{bmatrix} Q_{i,1}A & Q_{i,1}BK \\ \big(-Q_{i,1} + Q_{i,2}\big)A & \big(-Q_{i,1} + Q_{i,2}\big)BK \end{bmatrix}$$

$$+ 2\alpha_i\big(T_i - \tau_i(t)\big)\begin{bmatrix} A'Q_{i,1} & A'\big(-Q_{i,1} + Q_{i,2}\big) \\ K'B'Q_{i,1} & K'B'\big(-Q_{i,1} + Q_{i,2}\big) \end{bmatrix}$$

$$\Omega_{i,3} = [\, N_i \quad -N_i \,] + [\, N_i \quad -N_i \,]', \qquad \Omega_i = \Omega_{i,1} + \Omega_{i,2} + \Omega_{i,3},$$

$$N_i = col\{N_{i,1}, N_{i,2}\}, \qquad \Pi_i = \begin{bmatrix} \sqrt{T_i - \tau_i(t)}A' \\ \sqrt{T_i - \tau_i(t)}K'B' \end{bmatrix}.$$

Theorem 4 [25]: Let us consider system (21) under Assumption 1. Suppose that scalars $\alpha_i, \gamma_i \geq 0$, $i = 1,2, \ldots, m,$ and the controller gain K matrix are given. If there exist matrices $P = P' > 0$, $R_i = R_i' > 0, Q_{i,1} = Q_{i,1}'$, $Q_{i,2} = Q_{i,2}'$, $N_{i,1}$, $N_{i,2}$ of appropriate dimensions for $i = 1,2, \ldots, m$ such that

$$\alpha = \textstyle\sum_{i=1}^{m} \alpha_i\gamma_i > 0, \ \begin{bmatrix} \Omega_i & \Pi_i R_i \\ \bullet & -R_i \end{bmatrix}_{|h_i(t)=0} < 0, \ \begin{bmatrix} \Omega_i|_{\tau_i(t)=T_i} & T_i N_i \\ \bullet & -T_i e^{-2\alpha_i T_i} R_i \end{bmatrix} < 0, \tag{22}$$

then, closed loop system (21) is exponentially stable with a decay rate α.

Let $h = t - t_k$ and T is an upper bound of sampling period. Let define two families of functions

$$f_\alpha(T,h) := \begin{cases} \frac{e^{2\alpha(T-h)}-1}{2\alpha} & \text{if} \quad \alpha \neq 0 \\ T-h & \text{if} \quad \alpha = 0 \end{cases} \tag{23}$$

$$g_\alpha(T,h) := \begin{cases} \dfrac{e^{2\alpha T\left(1-e^{-2\alpha h}\right)}-1}{2\alpha} & \text{if} \quad \alpha > 0 \\ h & \text{if} \quad \alpha = 0 \\ 1-\dfrac{e^{-2\alpha h}}{2\alpha} & \text{if} \quad \alpha < 0 \end{cases} \tag{24}$$

Let $M_0 = [\,A \;\; BK\,]$, $M_1 = [\,I \;\; 0\,]$, $M_2 = [\,0 \;\; I\,]$, $M_3 = [\,I \;\; -I\,]$. Let also put $\bar{\Pi}_1 = M_1' P M_o + M_0' P M_1 - M_3' S_1 M_3 - M_2' S_2 M_3 - M_3' S_2' M_2 - N M_3 - M_3' N' + 2\alpha M_1' P M_1$ and $\bar{\Pi}_2 = M_3' S_1 M_0 + M_0' S_1 M_3 + M_0' R M_0 + M_2' S_2 M_0 + M_0' S_2' M_2$.

Theorem 5 [30, 52]: Let $\alpha \in R$. Consider families f_α, g_α be given by (23)-(24). Suppose that Assumption 1 is satisfied. Assume that there exist symmetric positive definite matrices P, R and matrices $S_1 = S_1' \in R^{nxn}, S_2 \in R^{nxn}$, $N \in R^{2nxn}$, such that

$$\bar{\Pi}_1 + f_\alpha(T,0)\Pi_2 < 0, \; \begin{bmatrix} \bar{\Pi}_1 & g_\alpha(T,T)N \\ \bullet & -g_\alpha(T,T)R \end{bmatrix} < 0, \tag{25}$$

$$\Pi_3 = \begin{bmatrix} P + f_\alpha(T,0)S_1 & f_\alpha(T,0)(S_2 - S_1) \\ \bullet & f_\alpha(T,0)(S_1 - S_2 - S_2') \end{bmatrix} > 0, \tag{26}$$

Then system (12) is exponentially stable with an exponential decay rate α for all time-varying period less than T.

Let us consider now the following control system

$$\dot{x}(t) = Ax(t) + Bu(t) + m\big(t, x(t)\big), \tag{27}$$

where $x \in R^n$, $u \in R^m$. Vector-Function $m\big(t, x(t)\big): [0,\infty) \times R^n \to R^n$ represents nonlinear uncertainties of the plant.

Assumption 2 [54]: Function $\mathrm{m}(t, x(t))$ is a piecewise-continuous nonlinear function in both arguments and for all $t \geq 0, \alpha > 0$ satisfies the following condition

$$m'\big(t, x(t)\big) m\big(t, x(t)\big) \leq \lambda^2 x'(t) M' M x(t), \tag{28}$$

where M is a constant matrix. For any given M inequality (28) defines a family M_λ of piecewise-continuous functions

$$M_\lambda := \{ m: R^{n+1} \to R^n | m'm \leq \lambda^2 x' M' M x \}, \tag{29}$$

The class M_λ is composed of functions that satisfy $M(t, 0) = 0$ in their domains of continuity, furthermore $x = 0$ is the equilibrium of (27). Under Assumption 2 the state-feedback controller is designed, as:

$$u(t) = Kx(t - h_k) \qquad \text{for all } t \in [kT + h_k, (k+1)T + h_{k+1}), \tag{30}$$

where T denotes a sampling period, h_k is network-induced delay at time instant kT, for fixed $k, k \in N$. Control system (27) with state-feedback controller (30) has the following form

$$\dot{x}(t) = Ax(t) + BKx\big(t - \eta(t)\big) + m\big(t, x(t)\big), \quad (31)$$

with $t \in [kT + h_k, (k+1)T + h_k)$ and $h(t) = t - kT,\ k \in N$.
Definition 2 [54]: System (31) is robustly asymptotically stable with decay rate α if its equilibrium $x_r = 0$ is globally asymptotically stable for all $\mathrm{m}(t, x(t)) \in M_\alpha$, where M_α is given by (29).

Let real positive defined matrices P, Q, R, S, W, the feedback gain matrix K and scalars $\eta_1 > 0, \eta_2 > 0$ be given. Let $\pi_{11}, \pi_{12}, \pi_{22}, \delta$ and γ be taken as follows

$$\Pi_{11} = \begin{bmatrix} PA + A'P - R + Q - W & PBK + R & W & P \\ \bullet & -R - S & S & 0 \\ \bullet & \bullet & -S - Q - W & 0 \\ \bullet & \bullet & \bullet & -\varepsilon I \end{bmatrix},$$

$$\Pi_{12} = \begin{bmatrix} \eta_2 A'R & (\delta - \eta_1)A'S & \delta A'W & \varepsilon H' \\ \eta_2 K'B'R & (\delta - \eta_1)K'B'S & \delta K'B'W & 0 \\ 0 & 0 & 0 & 0 \\ \eta_2 R & (\delta - \eta_1)S & \delta W & 0 \end{bmatrix}, \quad (32)$$

$$\Pi_{22} = diag\{-R, -S, -W, -\varepsilon\gamma I\},\ \delta = \tfrac{\eta_1 + \eta_2}{2},\ \gamma = \alpha^{-2}.$$

Theorem 6 [53]: Let η_1, η_2 and the feedback gain matrix K be given. If there exist a scalar $\varepsilon \geq 0$ and real positive defined matrices P, Q, R, S, W of appropriate dimensions such that

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \bullet & \Pi_{22} \end{bmatrix} < 0, \quad (33)$$

where $\Pi_{ij}, i, j = 1,2$ are defined by (32), then system (31) is robustly asymptotically stable with decay rate α.

Let us denote η_1 as a bounded delay of time. Then there exists η_2 such that $\eta_2 = sup_k[((k + 1) - k)T + h_{k+1}]$. Let $x(t) = \phi(t)$ for $t \in [t_0 - \eta_2, t_0]$.
Proposition 1 [54]: Let scalar $\eta_2 > 0$ and feedback gain K be given. If there exist real positive defined matrices $P,\ R$ such that following LMIs holds

$$\begin{bmatrix} PA + A'P - R & PBK + R & \eta_2 A'R \\ \bullet & -R & \eta_2 K'B'R \\ \bullet & \bullet & -R \end{bmatrix} < 0, \quad (34)$$

then system (31) is asymptotically stable.

Let the state space $X \in R^n$ be partitioned along a linear combination of the states as $R_i = \{x | \sigma_i < c'x < \sigma_{i+1}\}$, where $c \in R^n, \sigma_1 < \cdots < \sigma_{M+1}$ are scalars. Then each region of the state space can be represented by $\bar{R}_i = \epsilon_i = \{x: |E_i x + e_i| \leq 1\}$, where $E_i = \frac{2c'}{\sigma_{i+1} - \sigma_i}, e_i = -\frac{\sigma_{i+1} + \sigma_i}{\sigma_{i+1} - \sigma_i}$, $i \in \{1, \dots, M\}$. The set $J = \{1, \dots, M\}$ contains the indices of the regions R_i that partition the state space $X \subseteq R^n$ and $J(x) = \{i | x(t) \in \bar{R}_i\}$. The state space is given by $X = \bigcup_{i \in Z} \bar{R}_i$ where $\bar{R}_i$ is the closure of R_i.

Let us consider a sampled-data system of the form

$$\dot{x}(t) = A_i x(t) + a_i + Bu(t),\ i \in \{1, \dots, M\}, \quad (35)$$

for all $x(t) \in R_i$, where $A_i \in R^{n \times n},\ a_i \in R^n,\ B \in R^{nxm},\ u \in R^m$.

Assumption 3 [55]: The open-loop system is linear in the regions that contain the origin in their closure, i.e. $a_i,\ i \in J(0)$. It means that the origin is assumed to be an equilibrium point of the open-loop system.
Assumption 4 [55]: The state vector $x(\cdot) \in R_i$ is measured at sampling time instants t_k, where $0 < t_\varepsilon \leq t_{k+1} - t_k \leq h$ for all $k, k \in N$.

For sampled-data systems, the control input can be presented as $u(t) = K_j x(t_k)$, with $t \in [t_k, t_{k+1})$ and $x(t_k) \in R_j,\ j \in J$. or $x(t) \in R_i$ and $x(t_k) \in R_j$ system (35) can be rewritten as

$$\dot{x}(t) = A_i x(t) + a_i + BK_j x(t_k) = A_i x(t) + a_i + BK_i x(t_k) + Bw(t), \tag{36}$$

where $w(t) = (K_j - K_i)x(t_k) \in R^n$. The delay induced by state-feedback controller appears in natural way in the form $\rho(t) = t - t_k,\ t \in [t_k, t_{k+1}), k \in N$. Let $W([-h, 0], X)$ be the space of absolutely continuous functions from $[-\tau, 0]$, $X \subseteq R^n$. Denote $x_t(r) := x(t + r)$, for any $-h \leq r \leq 0$. Then for $x(t) \in R_i$ and $x(t_k) = x_t(-\rho(t)) \in R_j$ one has

$$\dot{x}(t) = A_i x(t) + a_i + BK_i x_t(-\rho(t)) + Bw(t), \tag{37}$$

with the initial condition $x_o(r) = \varphi(r),\ r \in [-h, 0]$. Let I_{n_x}denotes identity matrix of the dimension n_x and α_i denotes decay rate. Define

$$\begin{aligned}\bar{\Omega}_i = {} & [A_i\ \ BK_i\ \ B\ \ a_i]' P[I_{n_x}\ 0_{n_x}\ 0\ \ 0]' P[A_i\ \ BK_i\ \ B\ \ a_i] + \alpha[I_{n_x}\ 0_{n_x}\ 0\ \ 0]' P[I_{n_x}\ 0_{n_x}\ 0\ \ 0] \\ & - \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}' \bar{N}_i' - \bar{N}_i \begin{bmatrix} I_{n_x} & -I_{n_x} & 0 & 0 \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix} - [I_{2n_x}\ \ 0\ \ 0]' X [I_{2n_x}\ \ 0\ \ 0] \\ & + diag(\eta I_{n_x}, I_{n_x}, -\gamma I_{n_u}, 0)\end{aligned}$$

$$\begin{aligned}\bar{M}_{1i} = {} & \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}' R \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix} + \alpha[I_{2n_x}\ \ 0\ \ 0]' X [I_{2n_x}\ \ 0\ \ 0] \\ & + \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}' X [I_{2n_x}\ \ 0\ \ 0] + [I_{2n_x}\ \ 0\ \ 0] X \begin{bmatrix} A_i & BK_i & B & a_i \\ 0_{n_x} & 0_{n_x} & 0 & 0 \end{bmatrix}\end{aligned}$$

$$\bar{M}_{2i} = \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}' \bar{N}_i' - \bar{N}_i \begin{bmatrix} 0_{n_x} & 0_{n_x} & 0 & 0 \\ 0_{n_x} & I_{n_x} & 0 & 0 \end{bmatrix}$$

$$\bar{S}_i = -\alpha_i([E_i\ \ 0\ \ 0\ \ e_i]'[E_i\ \ 0\ \ 0\ \ e_i] - [0\ \ 0\ \ 0\ \ 1]'[0\ \ 0\ \ 0\ \ 1]),$$

where

$$\Omega_i = [I_{2n_{x+u}}\ \ 0] \bar{\Omega}_i [I_{2n_{x+u}}\ \ 0]', M_{1i} = [I_{2n_{x+u}}\ \ 0], \bar{M}_{1i}[I_{2n_{x+u}}\ \ 0]',$$

$$M_{2i} = [I_{2n_{x+u}}\ \ 0] \bar{M}_{2i} [I_{2n_{x+u}}\ \ 0]', N_i = [I_{2n_{x+u}}\ \ 0] \bar{N}_i, X = [I_{n_x}\ \ -I_{n_x}]' X_1 [I_{n_x}\ \ -I_{n_x}].$$

Theorem 7 [55]: Let consider system (37) under Assumptions 3 and 4. If there exist symmetric, positive defined matrices P, R, X_1, the matrix $\bar{N}_i$ and positive scalars $c_{1i}, \alpha_i, \eta, \gamma$ satisfying following conditions:

(i) $\Delta K^2 \gamma < 1$, where $\Delta K = \max_{i,j \in J} \| K_j - K_i \|$,

(ii) for all $i \in J \backslash J(0)$, matrices $\bar{\Omega}_i + \tau \bar{M}_{1i} + \bar{S}_i < 0$, $\begin{bmatrix} \bar{\Omega}_i + \tau \bar{M}_{2i} & \tau \bar{N}_i \\ \tau \bar{N}_i' & -he^{-\alpha h} \end{bmatrix} < 0$,

(iii) for all $i \in J(0)$ matrices $\Omega_i + \tau M_{1i} < 0$, $\begin{bmatrix} \Omega_i + \tau M_{1i} & hN_i \\ \tau N_i' & -he^{-\alpha \tau} R \end{bmatrix} < 0$.

then system (37) is locally uniformly exponentially stable with a decay rate larger than $\frac{\alpha}{2}$. If the state space coincide with R^{n_x} then system (35) is globally uniformly exponentially stable.

Comparison of presented results can be seen on the following simple example, see [25, 26], [30], [52, 53], [55].

Example. Let us consider linear control system example

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with $A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$. Let us take the state-feedback control of the form

$$u = Kx(t),$$

with $K = [-3.75 \quad -11.5]$. In considered sampled-data control system with non-uniform sampling, the sequence of sampling intervals is described as an independent and identically distributed process. Then the distributed interval $[0; T_{max})$ consists $k, k \in N$, subintervals i.e. $[0; T_{max}) = \bigcup_{i=1}^{k} [t_{i-1}; t_i)$. The probability distribution of sampling interval T_k in $[t_{i-1}; t_i)$ is $p_k(t_k)\{T_k \in [t_{i-1}; t_i)\} = \sigma_i$, where $\sum_{i=1}^{k} \sigma_i = 1$. The goal is to find the maximum length of the sampling interval for which the stability is still preserved.

Let $m = 3, T_1 = 0.5, T_2 = 1.0, T_3 = 2.0, \alpha_1 = 0.40,\ \alpha_2 = 0.39, \alpha_3 = -0.07$. Then conditions in (19) are feasible with:

$$P = \begin{bmatrix} 31.3206 & 57.6172 \\ 57.6172 & 137.9292 \end{bmatrix}.$$

It follows $\alpha = 0.0690 > 0$ for $\gamma_1 = 0.1, \gamma_2 = 0.2, \gamma_3 = 0.7$. From Theorem 4 it follows that given closed-loop system is exponentially stable for the maximum value T is equal to 2.00s. Under conditions of Theorem 5 maximal allowable sampling period on which the system is stable, is equal to $1.99s$. Conditions of Theorem 2 are fulfilled for the length of the maximal sampling period equals to $1.69s$. The maximum allowable length of time interval, which guarantee stability under conditions of Theorem 6 and Proposition 1 is 0.94s, the lower bound of the longest interval between two consecutive sampling times, which guarantee exponential stability by Theorem 7, is equal to 0.166s for $\alpha = 0.0001$. By Theorem 3 the upper bound of the sampling interval that guarantees stability of the system is equal to 1.11s and lower bound is equal to 0.87s.

Concluding, Theorem 4 is less restrictive than others presented above. Theorem 5 gives close results to Theorem 2. Theorem 2 is more restrictive than Theorems 4 and 5. Theorem 6 and 3 are more restrictive than previous mentioned Theorems 4, 5, 2. Theorem 7 is the most restrictive one, see also [16, 21, 52, 55].

Stabilization of linear systems with discrete-time control input.

Let us consider a continuous time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{38}$$

where $x \in R^n$ is the state vector, $u \in R^m$ is the control input, $A \in R^{nxn}, B \in R^{nxm}$. The state $x(t)$ is sampled at discrete time instances $0 = t_o < t_1 < \cdots < t_k < \cdots$. The length of kth sampling interval T_k is denoted as $T_k := t_{k+1} - t_k$, for $k \in N$. Assume that T_k is time-varying, i.e. $T_k = T_k(t)$, bounded for every k

$$0 < T_{min} \leq T_k \leq T_{max} < \infty. \tag{39}$$

Under a non-uniform sampling the control input $u(t)$ can be taken as a piecewise constant function defined as

$$u(t) = Kx(t_k) \text{ for all } t \in [t_k, t_{k+1}). \tag{40}$$

Substituting (40) into (38) one obtains the

$$x(t_{k+1}) = G(T_k)x(t_k), \tag{41}$$

where $G(T_k) \coloneqq e^{AT_k} + K\int_0^{T_k} e^{Ar}Bdr$, see for example [29], [56-58].

Let T_{nom} denotes a nominal point such that $T_{min} \leq T_{nom} \leq T_{max}$. The Schur decomposition of matrix A will be denoted as N. Let $\|N\|_2 = \sqrt{\sum_i \sum_j |n_{ij}|^2}$ denotes the Euclidean norm of matrix N. Define

$$\beta(\tau) \coloneqq \begin{cases} \sum_{k=0}^{n-1} \|N\|_2^k \left(-\frac{(-1)^k}{\alpha_1^{k+1}} + \frac{e^{\alpha_1 \tau}}{\alpha_1} \sum_{i=0}^{k} \frac{(-1)^i \tau^{k-i}}{\alpha_1^i (k-i)!}\right) & \text{for } \tau \geq 0, \alpha_1 \neq 0, \\ \sum_{k=0}^{n-1} \|N\|_2^k \left(-\frac{(-1)^k}{\alpha_2^{k+1}} + \frac{e^{\alpha_2 |\tau|}}{\alpha_2} \sum_{i=0}^{k} \frac{(-1)^i |\tau|^{k-1}}{\alpha_2^i (k-i)!}\right) & \text{for } \tau < 0, \alpha_2 \neq 0, \\ \sum_{k=0}^{n-1} \frac{\|N\|_2^k}{(k+1)!} |\tau|^{k+1}, & \text{otherwise}, \end{cases} \tag{42}$$

where $T_{min} - T_{nom} \leq \tau \leq T_{max} - T_{nom}$ and α_1, α_2 are respectively maximum eigenvalues of matrix A and $-A$. Let

$$\bar{\beta} = \beta(T_{nom}). \tag{43}$$

Theorem 8 [29]: If there exist a positive define, symmetric matrix $P \in R^{nxn}$ and a real number $\epsilon > 0$ satisfying the following

$$\begin{bmatrix} -P & \bullet & \bullet \\ G(T_{nom})P & -P + \epsilon I & \bullet \\ [A \;\; B]F(T_{nom})\begin{bmatrix} I \\ K \end{bmatrix} P & 0 & -\frac{\epsilon}{\bar{\beta}^2} I \end{bmatrix} < 0, \tag{44}$$

where $F(T_{nom}) = e^{HT_{nom}}$ and $H = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$, then system (41) with the feedback control (43) is stable for any T_k satisfying (39).

Theorem 9 [29]*:* If there exist a positive define, symmetric matrix $P \in R^{nxn}$, positive number $\epsilon \in R$ and matrix $Q \in R^{mxn}$ satisfying the following

$$\begin{bmatrix} -P & \bullet & \bullet \\ E & -P + \epsilon I & 0 \\ AE + BQ & 0 & -\frac{\epsilon}{\bar{\beta}^2} I \end{bmatrix} < 0, \tag{45}$$

with $E = e^{AT_{nom}P} + \int_0^{T_{nom}} e^{A\tau}BQd\tau$, then system (38) with a state feedback controller $K = QP^{-1}$ is stable for any T_k satisfying (39).

Let us divide interval $[T_{min}, T_{max}]$ onto N subintervals such that

$$T_{min} = T_{min,1} < T_{max,1} = T_{min,2} < T_{max,2} = T_{min,3} < \cdots < T_{max,N} = T_{max}, \tag{46}$$

then Theorem 8 conditioned by (46) can be rewritten in following way.

Theorem 10 [29]*:* If there exist a positive defined, symmetric matrix $P \in R^{nxn}$ and a real number $\epsilon_1 > 0$ satisfying

$$\begin{bmatrix} -P & \bullet & \bullet \\ G(T_{nom,i})P & -P+\epsilon_i I & 0 \\ [A \;\; B]F(T_{nom,i})\begin{bmatrix} I \\ K \end{bmatrix}P & 0 & -\frac{\epsilon_i}{\beta_i^2}I \end{bmatrix} < 0, \tag{47}$$

for $i = 1, \dots, N,\ T_{min,i} - T_{nom,i} \leq \tau \leq T_{max,i} - T_{nom,i}$, then system (41) is stable.

As a particular case of (38) one can consider system given as follows

$$\dot{x}(t) = Ax(t) + Bu(t-\tau), \tag{48}$$

Assume that the length T_k of sampling interval $[t_k; t_{k+1}]$ is bounded by $\underline{h}$ and $\overline{h}$ such that $\underline{h} \leq t_{k+1} - t_k \leq \overline{h}$. Note that states at two sequential sampling instants are related by

$$x(t_{k+1}) = \phi(t_{k+1} - t_k)x(t_k), \tag{49}$$

with $\phi(h) = e^{Ah} + BK\int_0^h e^{At}dt = I + (A+BK)\int_0^h e^{At}dt$.

Theorem 11 [56]: System (49) is exponentially stable if there exists a positively defined symmetric matrix Q such that

$$Q - \phi(h)Q\phi(h)' > 0 \;\; \text{where } \underline{h} \leq h \leq \overline{h}. \tag{50}$$

System (48) with irregular sampling and time-varying delays in the control, can be modeled by, see [57]

$$x(\bar{h}_{k+1}) = \phi(h_k)x(\bar{h}_k) + \Gamma_0(h_k, \tau_k) + \Gamma_1(h_k, \tau_k)u(\bar{h}_{k-1}), \tag{51}$$
$$u(\bar{h}_{k+1}) = -L(h_k, \tau_k)x(\bar{h}_k),\;\; \bar{h}_k = \textstyle\sum_0^k h_k, \tag{52}$$

where h_k denotes an each specific sampling interval, $k \in N$, τ_k is an each specific sampling-actuation delay, $\phi(h) = e^{Ah}$, $\Gamma_0(h,\tau) = \int_0^{h-\tau} e^{As}ds B$, $\Gamma_1(h,\tau) = e^{A(h-\tau)}\int_0^{\tau} e^{As}ds B$. In (52) $L(h_k, \tau_k)$ is the state feedback controller.

Let gain matrix ϕ in (49) be given as $\phi_{cl_k} = \begin{bmatrix} \phi(h_k) & \Gamma_1(h_k,\tau_k) \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \Gamma_0(h_k,\tau_k) \\ I \end{bmatrix} \cdot L(h_k, \tau_k)$. System evolution at the jth closed-loop completed successfully is

$$x\left(\textstyle\sum_{k=1}^{j} h_k\right) = \phi_{cl_k}\,\phi_{cl_{k-1}} \dots \phi_{cl_2}\,\phi_{cl_1}x(0). \tag{53}$$

One can distinguish three cases that depend on the type of the sequence that the closed-loop implementation originates, see [57].

Case 1: For a known constant sampling interval h_k and a known sampling-actuation delay τ_k the closed-loop system is characterized by matrix ϕ_{cl_k}. Then system (51)÷(52) is stable if and only if

$$\rho(\phi_{cl_k}) < 1,$$

where ρ is $\rho(M) = \max\{|\lambda| \lambda \text{ is an eigenvalue of } M\}$.
Case 2: For a finite sequence of known sampling intervals couples and sampling-actuation delays that repeats periodically, the closed-loop system will be described by a known finite set of matrices that repeats periodically $< \phi_{cl_1}, \phi_{cl_2}, \dots, \phi_{cl_n} >$. Then system (51)÷(52) is stable if and only if

$$\rho(\phi_{cl_1} \cdot \phi_{cl_2} \cdot \dots \cdot \phi_{cl_n}) < 1,$$

Case 3: For infinite sequence of sampling intervals and sampling-actuation delays taken randomly, the closed-loop system will be investigated by a product of an infinite number of matrices $< \phi_{cl_1}, \phi_{cl_2}, \dots, \phi_{cl_n} >$, taken randomly from a finite set of matrices, which specify as follows $\Omega = \{\phi_{cl_k} | \phi_{cl_k}\}$ is the closed-loop matrix that depends on (h_k, τ_k), for all possible combinations of (h_k, τ_k). Then Ω is asymptotically stable if and only if there exist a positive defined matrix P, such that $\phi'_{cl_k} \cdot P \cdot \phi_{cl_k} - P < 0$, for all $\phi_{cl_k} \in \Omega^k$, $k \geq 1$.

Then, see [58]
a) Ω is asymptotically stable if and only if there exist positive defined matrix P: for all $\phi_{cl_k} \in \Omega$ and $\phi'_{cl_k} \cdot P \cdot \phi_{cl_k} - P < 0$
b) Ω is asymptotically stable if and only if for all $\phi_{cl_k} \in \Omega$, $\phi'_{cl_k} \cdot \phi_{cl_k} - I < 0$

Stabilization of non-uniformly sampled hybrid systems.

Let us consider a linear continuous-time plant and discrete-time controller are given by

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B\tilde{u}(t), \\ &\tilde{u}(t) = u(t_k),\ t \in [t_k, t_{k+1}),\ k = 1,2,3,\dots \\ &u(t_{k+1}) = Ex(t_k) + Fu(t_k), \end{aligned} \tag{54}$$

where $x \in R^n$ is the state vector with $x(t_0) = x_0, u \in R^m$ is the control vector, and A, B, E, F are real matrices. Let us define matrices Θ_1 and Θ_2 in the following way

$$\Theta_1 = \begin{bmatrix} e^{AT} & \int_0^T e^{A(T-\tau)} B d\tau \\ E & F \end{bmatrix}, \Theta_2 = \begin{bmatrix} e^{A\bar{T}} & \int_0^{\bar{T}} e^{A(\bar{T}-\tau)} B d\tau \\ E & F \end{bmatrix}. \tag{55}$$

Theorem 12 [36]: The linear system (54) with recurrent sampling is uniformly asymptotically stable if and only if
(i) $|T - \bar{T}| \leq \tau_M < \infty$
(ii) matrices θ_1, θ_2 are Schur stable;
where τ_M is a permissible error, $T, \bar{T}$ are sampling periods.

Corollary 1 [36]: The equilibrium $[x', u']' = [0', 0']'$ of the linear sampled-data system (54) under non-uniform sampling periods $S = \{t_k\}_{k=1,2,3,\dots}$ is
(i) uniformly stable if $\lim_{k\to\infty} sup\|\Theta(t_k)\| = q < \infty$ and there exist positive constant $\tau > 0$ such that $sup_{k=1,2,3,\dots}\{t_{k+1} - t_k\} = \tau_0 < \tau$;
(ii) uniformly asymptotically stable in large measure if $\lim_{k\to\infty}\|\Theta(t_k)\| = 0$ and there exists a positive constant $\tau > 0$ such that $sup_{k=1,2,3,\dots}\{t_{k+1} - t_k\} = \tau_0 < \tau$.

A nonlinear sampled-data system under time-varying sampling system is described by

$$\begin{aligned} &\dot{x}(t) = f\big(x(t), \tilde{u}(t)\big),\ t \in [t_k, t_{k+1}),\ \ k = 1,2,3,\dots \\ &\tilde{u}(t) = u(t_k), \\ &u(t_{k+1}) = g\big(x(t_k), u(t_k)\big) \end{aligned} \tag{56}$$

Let linearize system (56) as follows

$$\bar{A} = \frac{\partial f(x,u)}{\partial x}_{|x=0},\ \bar{B} = \frac{\partial f(x,u)}{\partial u}_{|u=0} \tag{57}$$

$$\bar{E} = \frac{\partial g(x(t_k),u(t_k))}{\partial x(t_k)}_{|x=0},\ \bar{F} = \frac{\partial g(x(t_k),u(t_k))}{\partial u(t_k)}_{|u=0} \tag{58}$$

$$\lim_{(x,u)\to(0,0)} \frac{\|\tilde{f}(x,u)\|}{\|\Delta x\|} = 0 \quad \lim_{(x,u)\to(0,0)} \frac{\|\tilde{f}(x,u)\|}{\|\Delta u\|} = 0 \tag{59}$$

$$\lim_{(x,u)\to(0,0)} \frac{\|\tilde{g}(x,u)\|}{\|\Delta x\|} = 0 \quad \lim_{(x,u)\to(0,0)} \frac{\|\tilde{g}(x,u)\|}{\|\Delta u\|} = 0 \tag{60}$$

Then one obtains the following system

$$\begin{aligned} &\dot{x}(t) = \bar{A}x(t) + \bar{B}\tilde{u}(t) + \tilde{f}\big(x(t), u(t)\big) \\ &\tilde{u}(t) = u(t_k),\ t \in [t_k, t_{k+1}), k = 1,2,3,\dots \\ &u(t_{k+1}) = \bar{E}x(t_k) + \bar{F}u(t_k) + \tilde{g}(x(t_k), u(t_k)) \end{aligned} \tag{61}$$

Let

$$\bar{\theta}_1(t_k) = \bar{\theta}_1 = \begin{bmatrix} e^{\bar{A}T} & \int_0^T e^{\bar{A}(T-\tau)} B d\tau \\ E & F \end{bmatrix},\ \bar{\theta}_2(t_k) = \bar{\theta}_2 = \begin{bmatrix} e^{\bar{A}\bar{T}} & \int_0^{\bar{T}} e^{\bar{A}(\bar{T}-\tau)} B d\tau \\ E & F \end{bmatrix} \tag{62}$$

Theorem 13 [36]: The equilibrium $[x', u']' = [0', 0']'$ of the non-uniform sampled-data system (56) is uniformly asymptotically stable if the equilibrium $[x', u']' = [0', 0']'$ of the linearized sampled-data system (61) is exponentially stable, or
(i) $|T - \bar{T}| \le \tau_M < \infty$
(ii) matrices $\bar{\theta}_1, \bar{\theta}_2$ are Schur stable.
where τ_M denotes a permissible error, $T, \bar{T}$ are lengths of sampling intervals.

Remarks on practical application

An overview of real-life applications using non-uniform sampling, which are most popular, can be divided into following groups:

- Medical applications

There are many applications, have been studied for example in [14], [15], [38], and [59]. Few research projects establish to continuous health monitoring by using for instance custom sensors or commercial mobile phones, which records ECG signals. In the novel approach [38] sampling rate is adapted by using previously taken samples. Furthermore there is no need to keep sampling times, which significantly lower power consumption. Paper [59] handles application of non-uniform sampling in endocrine and [15] introduces advantages of using non-uniform sampling in Nuclear Magnetic Resonance (NMR), which speeding up the measurement of datasets.

- Automotive Applications

In high-tech vehicles a lot of sensors is used, which gather data and analyze it. Signals from sensors are delivered with a time stamp, but the real sampling instants can be different causing jitter sampling [11]. In [14] it is presented application of various sampling in tire pressure monitoring and non-round wheels.

• Networks applications

In Networked Control Systems sampling in asynchronous manner can be introduced in the control loop to reduce the data transmissions and to optimize computational costs [60-62]. Another case of using non-uniform sampling is adaptive network queue control. The idea presented in [11] is to consider the queue length as a continuous time function, non-uniformly sampled when packets of unequal sizes arrive. In [63] sensor networks under non-uniform sampling are described.

• Hardware

An example of where non-uniform sampling occurs is A/D converters. In literature, for instance [64-67], it is presented structures of A/D converters. In [67] power consumption is taken into consideration and there is demonstrated that random-sampling decreases power consumption by 25%. Other examples of hardware are introduced in [14], [33], and [68-70].

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