Asymptotic Methods: Differential Equations

In this chapter we will apply the ideas that we developed in the previous chapter to the solution of ordinary and partial differential equations.

12.1 An Instructive Analogy: Algebraic Equations

Many of the essential ideas that we will need in order to solve differential equations using asymptotic methods can be illustrated using algebraic equations. These are much more straightforward to deal with than differential equations, and the ideas that we will use are far more transparent. We will consider two illustrative examples.

12.1.1 Example: A Regular Perturbation

Consider the cubic equation

$$x^3 - x + \epsilon = 0. \tag{12.1}$$

Although there is an explicit formula for the solution of cubic equations, it is rather cumbersome to use. Let's suppose instead that we only need to find the solutions for $\epsilon \ll 1$. If we simply set $\epsilon = 0$, we get $x^3 - x = 0$, and hence $x = -1$, 0 or 1. These are called the **leading order solutions** of the equation. These solutions are obviously not exact when ϵ is small but nonzero, so let's try to improve the accuracy of our approximation by seeking an asymptotic expansion of the solution (or more succinctly, an asymptotic solution) of the form

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3). \tag{12.2}$$

We can now substitute this into (12.1) and equate powers of ϵ. This gives us

$$\left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right)^3 - \left(x_0 + \epsilon x_1 + \epsilon^2 x_2\right) + \epsilon + O\left(\epsilon^3\right) = 0,$$

which we can rearrange into a hierarchy of powers of ϵ in the form

$$\left\{x_0^3 - x_0\right\} + \epsilon\left\{(3x_0 - 1)\, x_1 + 1\right\} + \epsilon^2\left\{3x_0 x_1^2 + \left(3x_0^2 - 1\right) x_2\right\} + O\left(\epsilon^3\right) = 0.$$

At leading order we obviously get $x_0^3 - x_0 = 0$, and hence $x_0 = -1$, 0 or 1. We will concentrate on the solution with $x_0 = 1$. At $O(\epsilon)$, $(3x_0 - 1)\, x_1 + 1 = 2x_1 + 1 = 0$, and hence $x_1 = -\frac{1}{2}$. At $O(\epsilon^2)$, $3x_0 x_1^2 + \left(3x_0^2 - 1\right) x_2 = \frac{3}{4} + 2x_2 = 0$, and hence

$x_2 = -\frac{3}{8}$. We could, of course, continue to higher order if necessary. This shows that

$$x = 1 - \frac{1}{2}\epsilon - \frac{3}{8}\epsilon^2 + O(\epsilon^3) \quad \text{for } \epsilon \ll 1.$$

Similar expansions can be found for the other two solutions of (12.1). This is a regular perturbation problem, since we have found asymptotic expansions for all three roots of the cubic equation using the simple expansion (12.2). Figure 12.1 shows that the function $x^3 - x + \epsilon$ is qualitatively similar for $\epsilon = 0$ and $0 < \epsilon \ll 1$.

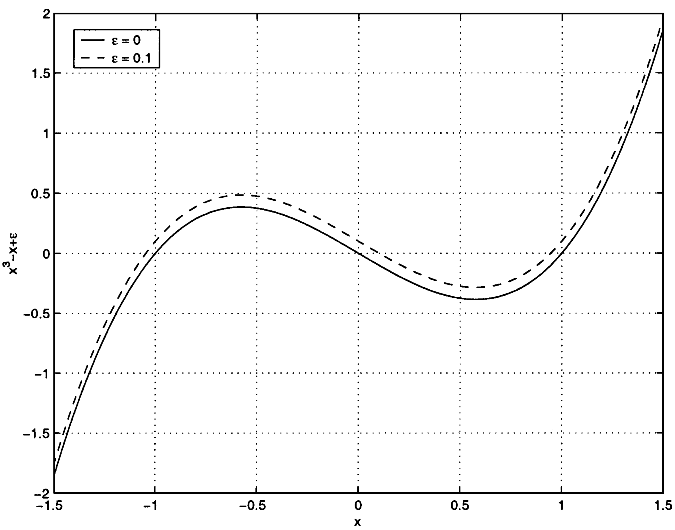

Fig. 12.1. The function $x^3 - x + \epsilon$ for $\epsilon = 0$, solid line, and $\epsilon = 0.1$, broken line.

12.1.2 Example: A Singular Perturbation

Consider the cubic equation

$$\epsilon x^3 + x^2 - 1 = 0. \tag{12.3}$$

At leading order for $\epsilon \ll 1$, $x^2 - 1 = 0$, and hence $x = \pm 1$. However, we know that a cubic equation is meant to have three solutions. What's happened to the other solution? This is an example of a singular perturbation problem, where the solution for $\epsilon = 0$ is qualitatively different to the solution when $0 < \epsilon \ll 1$. The equation changes from quadratic to cubic, and the number of solutions goes from two to three. The key point is that we have implicitly assumed that $x = O(1)$. However, the term ϵx^3, which we neglect at leading order when $x = O(1)$, becomes comparable to the term x^2 for sufficiently large x, specifically when $x = O(\epsilon^{-1})$. Figure 12.2 shows how the function $\epsilon x^3 + x^2 - 1$ changes qualitatively for $\epsilon = 0$ and $0 < \epsilon \ll 1$.

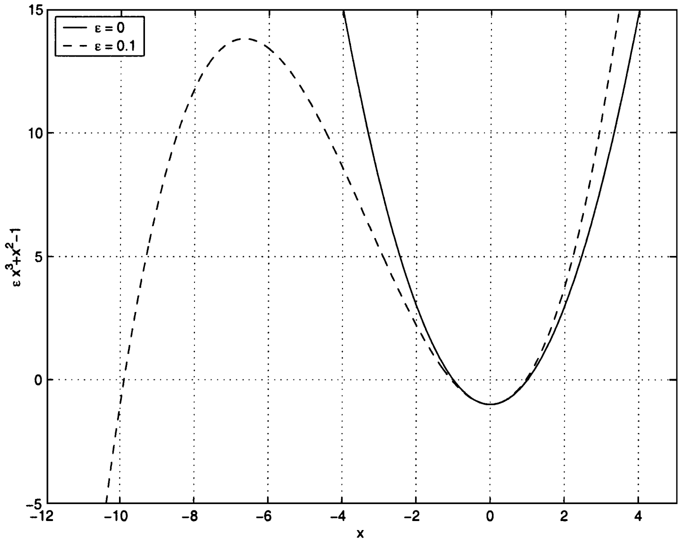

Fig. 12.2. The function $\epsilon x^3 + x^2 - 1$ for $\epsilon = 0$, solid line, and $\epsilon = 0.1$, broken line.

So, how can we proceed in a systematic way? If we expand $x = x_0 + \epsilon x_1 + O(\epsilon^2)$, we can construct the two $O(1)$ solutions, $x = \pm 1 + O(\epsilon)$, in the same manner as we did for the previous example. Since we know that there must be three solutions, we conclude that the other solution cannot have $x = O(1)$, and assume that $x = O(\epsilon^\alpha)$, with α to be determined. If we define a scaled variable, $x = \epsilon^\alpha X$, with $X = O(1)$ for $\epsilon \ll 1$, (12.3) becomes

$$\epsilon^{3\alpha+1} X^3 + \epsilon^{2\alpha} X^2 - 1 = 0. \tag{12.4}$$

We must choose α in order to obtain an **asymptotic balance** between two of the terms in (12.4). If $\alpha > 0$, the first two terms are small and cannot balance the third term, which is of $O(1)$. If $\alpha < 0$, the first two terms are large, and we can choose α so that they are of the same asymptotic order. This requires that $\epsilon^{3\alpha+1} X^3 = O(\epsilon^{2\alpha} X^2)$, and hence $\epsilon^{3\alpha+1} = O(\epsilon^{2\alpha})$. This gives $3\alpha + 1 = 2\alpha$, and hence $\alpha = -1$. This means that $x = \epsilon^{-1} X = O(\epsilon^{-1})$, as expected. Equation (12.4) now becomes

$$X^3 + X^2 - \epsilon^2 = 0. \tag{12.5}$$

Since only ϵ^2 and not ϵ appears in this rescaled equation, we expand $X = X_0 + \epsilon^2 X_1 + O(\epsilon^4)$. At leading order, $X_0^3 + X_0^2 = 0$, and hence $X_0 = -1$ or 0. Of course, $X_0 = 0$ will just give us the two solutions with $x = O(1)$ that we have already considered, so we take $X_0 = -1$. At $O(\epsilon^2)$,

$$(-1 + \epsilon^2 X_1)^3 + (-1 + \epsilon^2 X_1)^2 - \epsilon^2 + O(\epsilon^4) = 0$$

gives

$$-1 + 3\epsilon^2 X_1 + 1 - 2\epsilon^2 X_1 - \epsilon^2 + O(\epsilon^4) = 0,$$

and hence $X_1 = 1$. Therefore $X = -1+\epsilon^2+O(\epsilon^4)$, and hence $x = -1/\epsilon+\epsilon+O(\epsilon^3)$.

12.2 Ordinary Differential Equations

The solution of ordinary differential equations by asymptotic methods often proceeds in a similar way to the solution of algebraic equations, which we discussed in the previous section. We assume that an asymptotic expansion of the solution exists, substitute into the equation and boundary conditions, and equate powers of the small parameter. This determines a sequence of simpler equations and boundary conditions that we can solve. In order to introduce the main ideas, we will begin by considering some simple, constant coefficient linear ordinary differential equations before moving on to study both nonlinear ordinary differential equations and some simple partial differential equations.

12.2.1 Regular Perturbations

Consider the ordinary differential equation

$$y'' + 2\epsilon y' - y = 0, \tag{12.6}$$

to be solved for $0 \leqslant x \leqslant 1$, subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{12.7}$$

Of course, we could solve this constant coefficient ordinary differential equation analytically using the method described in Appendix 5, but it is instructive to try to construct the asymptotic solution when $\epsilon \ll 1$. We seek a solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2).$$

At leading order, $y_0'' - y_0 = 0$, subject to $y_0(0) = 0$ and $y_0(1) = 1$, which has solution

$$y_0(x) = \frac{\sinh x}{\sinh 1}.$$

If we now substitute the expansion for y into (12.6) and retain terms up to $O(\epsilon)$, we obtain

$$(y_0 + \epsilon y_1)'' + 2\epsilon(y_0 + \epsilon y_1)' - (y_0 + \epsilon y_1) + O(\epsilon^2)$$

$$= y_0'' + \epsilon y_1'' + 2\epsilon y_0' - y_0 - \epsilon y_1 + O(\epsilon^2) = 0,$$

and hence

$$y_1'' - y_1 = -2y_0' = -2\frac{\cosh x}{\sinh 1}. \tag{12.8}$$

Similarly, the boundary conditions (12.7) show that

$$y_0(0) + \epsilon y_1(0) + O(\epsilon^2) = 0, \quad y_0(1) + \epsilon y_1(1) + O(\epsilon^2) = 1,$$

and hence

$$y_1(0) = 0, \quad y_1(0) = 0. \tag{12.9}$$

By seeking a particular integral solution of (12.8) in the form $y_{1\mathrm{p}} = kx \sinh x$, and using the constants in the homogeneous solution, $y_{1\mathrm{h}} = A \sinh x + B \cosh x$, to satisfy the boundary conditions (12.9), we arrive at

$$y_1 = (1 - x)\frac{\sinh x}{\cosh 1},$$

and hence

$$y(x) = \frac{\sinh x}{\sinh 1} + \epsilon(1 - x)\frac{\sinh x}{\cosh 1} + O(\epsilon^2), \tag{12.10}$$

for $\epsilon \ll 1$. The ratio of the second to the first term in this expansion is $\epsilon(1 - x)\tanh 1$, which is uniformly small for $0 \leqslant x \leqslant 1$. This leads us to believe that the asymptotic solution (12.10) is uniformly valid in the region of solution. The situation is analogous to the example that we studied in Section 12.1.1.

One subtle point is that, although we believe that the next term in the asymptotic expansion of the solution, which we write as $O(\epsilon^2)$ in (12.10), is uniformly smaller than the two retained terms for ϵ sufficiently small, we have not proved this. We do not have a rigorous estimate for the size of the neglected term in the way that we did when we considered the exponential integral, where we were able to find an upper bound for R_N, given by (11.3). Although, for this simple, constant coefficient ordinary differential equation, we could write down the exact solution and prove that the remainder is of $O(\epsilon^2)$, in general, and in particular for nonlinear problems, this is not possible, and an act of faith is involved in trusting that our asymptotic solution provides a valid representation of the exact solution. This faith can be bolstered in a number of ways, for example, by comparing asymptotic solutions with numerical solutions, and by checking that the asymptotic solution makes sense in terms of the underlying physics of the system that we are modelling. The sensible applied mathematician always has a small, nagging doubt at the back of their mind about the validity of an asymptotic solution. For (12.6), our faith is justified, as can be seen in Figure 12.3.

12.2.2 The Method of Matched Asymptotic Expansions

Consider the ordinary differential equation

$$\epsilon y'' + 2y' - y = 0, \tag{12.11}$$

to be solved for $0 \leqslant x \leqslant 1$, subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1. \tag{12.12}$$

The observant reader will notice that this is the same as the previous example, but with the small parameter ϵ moved to the highest derivative term. We again seek an asymptotic solution of the form

$$y(x) = y_0(x) + \epsilon y_1(x) + O(\epsilon^2).$$

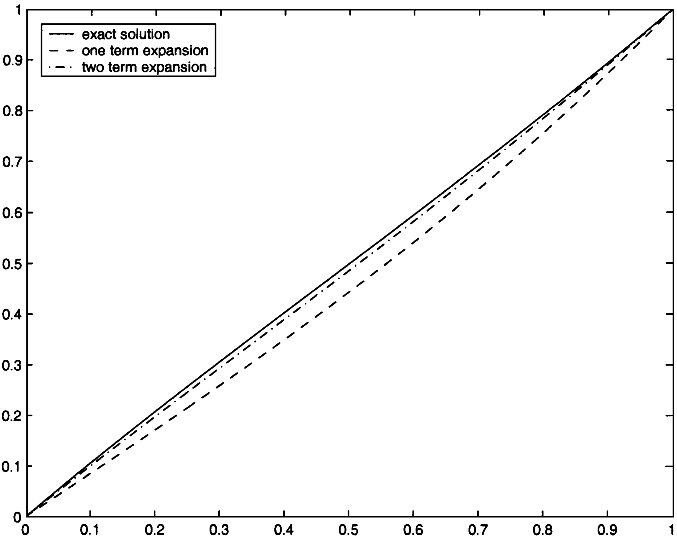

Fig. 12.3. Exact and asymptotic solutions of (12.6) when $\epsilon = 0.25$.

At leading order, $2y_0' - y_0 = 0$, which has the solution $y_0 = Ae^{x/2}$ for some constant A. However, the boundary conditions require that $y_0(0) = 0$ and $y_0(1) = 1$. How can we satisfy two boundary conditions using only one constant? Well, of course we can't. The problem is that for $\epsilon = 0$, the equation is of first order, and therefore qualitatively different from the full, second order equation. This is a singular perturbation, and is analogous to the example we studied in Section 12.1.2.

Let's proceed by satisfying the boundary condition $y_0(1) = 1$, which gives

$$y_0(x) = e^{(x-1)/2}.$$

At $O(\epsilon)$ we have

$$2y_1' - y_1 = -y_0'' = -\frac{1}{4}e^{(x-1)/2},$$

to be solved subject to $y_1(0) = 0$ and $y_1(1) = 0$. This equation can be solved using an integrating factor (see Section A5.2), which gives

$$y_1(x) = -\frac{1}{2}xe^{(x-1)/2} + ke^{(x-1)/2},$$

for some constant k. Again, we cannot satisfy both boundary conditions, and we just use $y_1(1) = 0$, which gives

$$y_1(x) = \frac{1}{2}(1-x)e^{(x-1)/2}.$$

Finally, this gives

$$y = e^{(x-1)/2}\left\{1 + \frac{1}{2}(1-x)\epsilon\right\} + O(\epsilon^2), \tag{12.13}$$

for $\epsilon \ll 1$. This suggests that $y \to e^{-1/2}(1 + \frac{1}{2}\epsilon)$ as $x \to 0$, which clearly does not satisfy the boundary condition $y(0) = 0$. We must therefore introduce a **boundary layer** at $x = 0$, across which y adjusts to satisfy the boundary condition. The idea is that, in some small neighbourhood of $x = 0$, the term $\epsilon y''$, which we neglected at leading order, becomes important because y varies rapidly.

We rescale by defining $x = \epsilon^\alpha X$, with $\alpha > 0$ (so that $x \ll 1$) and $X = O(1)$ as $\epsilon \to 0$, and write $y(x) = Y(X)$ for $X = O(1)$. Substituting this into (12.11) gives

$$\epsilon^{1-2\alpha}\frac{d^2Y}{dX^2} + 2\epsilon^{-\alpha}\frac{dY}{dX} - Y = 0.$$

Since $\alpha > 0$, the second term in this equation is large, and to obtain an asymptotic balance at leading order we must have $\epsilon^{1-2\alpha} = O(\epsilon^{-\alpha})$, which means that $1-2\alpha = -\alpha$, and hence $\alpha = 1$. So $x = \epsilon X$,

$$\frac{d^2Y}{dX^2} + 2\frac{dY}{dX} - \epsilon Y = 0, \tag{12.14}$$

and $Y(0) = 0$. It is usual to refer to the region where $\epsilon \ll x \leqslant 1$ as the **outer region**, with **outer solution** $y(x)$, and the boundary layer region where $x = O(\epsilon)$ as the **inner region** with **inner solution** $Y(X)$. The other boundary condition is to be applied at $x = 1$. However, $x = 1$ does not lie in the inner region, where $x = O(\epsilon)$. In order to fix a second boundary condition for (12.14), we will have to make sure that the solution in the inner region is consistent, in a sense that we will make clear below, with the solution in the outer region, which does satisfy $y(1) = 1$.

We now expand

$$Y(X) = Y_0(X) + \epsilon Y_1(X) + O(\epsilon^2).$$

At leading order, $Y_0'' + 2Y_0' = 0$, to be solved subject to $Y_0(0) = 0$. The solution is

$$Y_0 = A(1 - e^{-2X}),$$

for some constant A. At leading order, we now know that

$$\begin{array}{lll} y \sim e^{(x-1)/2} & \text{for } \epsilon \ll x \leqslant 1 & \text{(the outer expansion)}, \\ Y \sim A(1-e^{-2X}) & \text{for } X = O(1),\ x = O(\epsilon) & \text{(the inner expansion)}. \end{array}$$

For these two expansions to be consistent with each other, we must have

$$\lim_{X\to\infty} Y(X) = \lim_{x\to 0} y(x), \tag{12.15}$$

which gives $A = e^{-1/2}$. We will see below, where we make this vague notion of "consistency" more precise, that this is correct.

At $O(\epsilon)$ we obtain the equation for $Y_1(X)$ as

$$Y_1'' + 2Y_1' = Y_0 = A(1 - e^{-2X}).$$

Integrating this once gives

$$Y_1' + 2Y_1 = A\left(X + \frac{1}{2}e^{-2X}\right) + c_1,$$

for some constant c_1. This can now be solved using an integrating factor, and the solution that satisfies $Y_1(0) = 0$ is

$$Y_1 = \frac{1}{2}AX(1 + e^{-2X}) - c_2(1 - e^{-2X}),$$

for some constant c_2, which we need to determine. To summarize, the two-term asymptotic expansions are

$$y \sim e^{(x-1)/2} + \frac{1}{2}\epsilon(1-x)e^{(x-1)/2} \quad \text{for } \epsilon \ll x \leqslant 1,$$

$$Y \sim A(1 - e^{-2X}) + \epsilon\left\{\frac{1}{2}AX(1 + e^{-2X}) - c_2(1 - e^{-2X})\right\} \quad \text{for } X = O(1),\ x = O(\epsilon).$$

We can determine the constants A and c_2 by forcing the two expansions to be consistent in the sense that they should be equal in an **intermediate region** or **overlap region**, where $\epsilon \ll x \ll 1$. The point is that in such a region we expect *both* expansions to be valid.

We define $x = \epsilon^\beta \hat{X}$ with $0 < \beta < 1$, and write $y = \hat{Y}(\hat{X})$. In terms of the **intermediate variable**, $\hat{X}$, the outer expansion becomes

$$\hat{Y} \sim e^{-1/2}\exp\left(\frac{1}{2}\epsilon^\beta \hat{X}\right) + \frac{1}{2}\epsilon(1 - \epsilon^\beta \hat{X})e^{-1/2}\exp\left(\frac{1}{2}\epsilon^\beta \hat{X}\right).$$

When $\hat{X} = O(1)$, we can expand the exponentials as Taylor series, and find that

$$\hat{Y} = e^{-1/2}\left(1 + \frac{1}{2}\epsilon^\beta \hat{X}\right) + \frac{1}{2}e^{-1/2}\epsilon + o(\epsilon). \tag{12.16}$$

Since $x = \epsilon X = \epsilon^\beta \hat{X}$ gives $X = \epsilon^{\beta-1}\hat{X}$, the inner expansion is

$$\hat{Y} \sim A\left\{1 - \exp\left(-2\epsilon^{\beta-1}\hat{X}\right)\right\}$$

$$+\epsilon\left[\frac{1}{2}A\epsilon^{\beta-1}\hat{X}\left\{1 + \exp\left(-2\epsilon^{\beta-1}\hat{X}\right)\right\} - c_2\left\{1 - \exp\left(-2\epsilon^{\beta-1}\hat{X}\right)\right\}\right].$$

Now, since $\exp(-2\epsilon^{\beta-1}\hat{X}) = o(\epsilon^n)$ for all $n > 0$ (it is exponentially small for $\beta < 1$), we have

$$\hat{Y} = A + \frac{1}{2}A\epsilon^\beta \hat{X} - c_2\epsilon + o(\epsilon). \tag{12.17}$$

Since (12.16) and (12.17) must be identical, we need $A = e^{-1/2}$, consistent with the crude assumption, (12.15), that we made above, and also $c_2 = -\frac{1}{2}e^{-1/2}$. This process, whereby we make the inner and outer expansions consistent, is known as **asymptotic matching**, and the inner and outer expansions are known as **matched asymptotic expansions**. A comparison between the one-term inner and outer solutions and the exact solution is given in Figure 12.4. It should be clear that the inner expansion is a poor approximation in the outer region and vice versa. A little later, we will show how to construct, using the inner and outer expansions, a composite expansion that is uniformly valid for $0 \leqslant x \leqslant 1$.

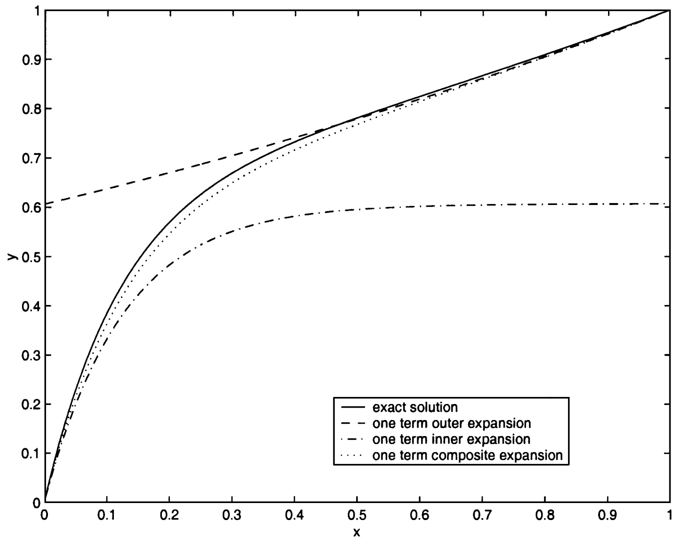

Fig. 12.4. Exact and asymptotic solutions of (12.11) when $\epsilon = 0.25$.

Van Dyke's Matching Principle

The use of an intermediate variable in an overlap region can get *very* tedious in more complicated problems. A method that works most, but not all, of the time, and is much easier to use, is **Van Dyke's matching principle**. This principle is much easier to use than to explain, but let's start with the explanation.

Let

$$y(x) \sim \sum_{n=0}^{N} \phi_n(\epsilon) y_n(x)$$

be the outer expansion and

$$Y(X) \sim \sum_{n=0}^{N} \psi_n(\epsilon) Y_n(X)$$

be the inner expansion with respect to the asymptotic sequences $\phi_n(\epsilon)$ and $\psi_n(\epsilon)$, with $x = f(\epsilon)X$. In order to analyze how the outer expansion behaves in the inner region, we can write $y(x)$ in terms of $X = x/f(\epsilon)$, and retain M terms in the resulting asymptotic expansion. We denote this by $y^{(N,M)}$, the M^{th} order inner approximation of the outer expansion. Similarly, we can write $Y(X)$ in terms of x, and retain M terms in the resulting expansion. We denote this by $Y^{(N,M)}$, the M^{th} order outer approximation of the inner expansion. Van Dyke's matching principle states that $y^{(N,M)} = Y^{(M,N)}$. Let's see how this works for our example.

In terms of the outer variable, the inner expansion is

$$Y(X) \sim A(1 - e^{-2x/\epsilon}) + \epsilon \left\{ \frac{1}{2} A \frac{x}{\epsilon} \left(1 + e^{-2x/\epsilon}\right) - c_2 \left(1 - e^{-2x/\epsilon}\right) \right\}$$

$$\sim Y^{(2,2)} = A + \frac{1}{2} A x - c_2 \epsilon,$$

for $x = O(1)$. In terms of the inner variable, the outer expansion is

$$y(x) \sim \exp\left(-\frac{1}{2} + \frac{1}{2}\epsilon X\right) + \frac{1}{2}\epsilon\,(1 - \epsilon X) \exp\left(-\frac{1}{2} + \frac{1}{2}\epsilon X\right)$$

$$\sim y^{(2,2)} = e^{-1/2}\left(1 + \frac{1}{2}\epsilon X + \frac{1}{2}\epsilon\right),$$

for $X = O(1)$. In terms of the outer variable,

$$y^{(2,2)} = e^{-1/2}\left(1 + \frac{1}{2} x + \frac{1}{2}\epsilon\right).$$

Van Dyke's matching principle states that $Y^{(2,2)} = y^{(2,2)}$, and therefore gives $A = e^{-1/2}$ and $c_2 = -\frac{1}{2} e^{-1/2}$ rather more painlessly than before.

Composite Expansions

Although we now know how to construct inner and outer solutions, valid in the inner and outer regions, it would be more useful to have an asymptotic solution valid uniformly across the whole domain of solution, $0 \leqslant x \leqslant 1$ in the example. We can construct such a uniformly valid **composite expansion** by using the inner and outer expansions. We simply need to add the two expansions and subtract the expression that gives their overlap. The overlap is just the expression that appears to the appropriate order in the intermediate region, (12.17) or (12.16), or equivalently the matched part, $y^{(2,2)}$ or $Y^{(2,2)}$. For our example problem, the one-term composite expansion is

$$y \sim y_0 + Y_0 - y^{(1,1)} = e^{(x-1)/2} + e^{-1/2}(1 - e^{-2X}) - e^{-1/2}$$

$$= e^{(x-1)/2} - e^{-1/2-2x/\epsilon} \quad \text{for } 0 \leqslant x \leqslant 1 \text{ as } \epsilon \to 0.$$

This composite expansion is shown in Figure 12.4, and shows good agreement with the exact solution across the whole domain, as expected. Note that, in terms of Van Dyke's matching principle, we can write the composite solution of any order as

$$y \sim y_c^{(M,N)} = \sum_{n=0}^{M} y_n(x) + \sum_{n=0}^{N} Y_n(X) - y^{(M,N)}.$$

The Location of the Boundary Layer

In our example, when we constructed the outer solution, we chose to satisfy the boundary condition at $x = 1$ and assume that there was a boundary layer at $x = 0$. Why is this correct? Let's see what happens if we assume that there is a boundary layer at $x = x_0$. Strictly speaking, if $x_0 \neq 0$ and $x_0 \neq 1$ this is an **interior layer**. We define scaled variables $y(x) = Y(X)$ and $x = x_0 + \epsilon^\alpha X$, with $\alpha > 0$ and Y, $X = O(1)$ for $\epsilon \ll 1$. As before, we find that we can only obtain an asymptotic balance at leading order by taking $\alpha = 1$, so that $x = x_0 + \epsilon X$ and

$$Y'' + 2Y' - \epsilon = 0.$$

At leading order, as before, $Y_0 = A_0 + B_0 e^{-2X}$. As $X \to -\infty$, Y_0 becomes exponentially large, and cannot be matched to the outer solution. This forces us to take $x_0 = 0$, since then this solution is only needed for $X \geqslant 0$, and, as we have seen, we can construct an asymptotic solution.

Interior Layers

Singular perturbations of ordinary differential equations need not always result in a boundary layer. As an example, consider

$$\epsilon y'' + 2xy' + 2x = 0, \tag{12.18}$$

to be solved for $-1 < x < 1$, subject to the boundary conditions

$$y(-1) = 2, \quad y(1) = 3. \tag{12.19}$$

We will try to construct the leading order solution for $\epsilon \ll 1$. The leading order outer solution satisfies $2x(y' + 1) = 0$, and hence $y = k - x$ for some constant k. If $y(-1) = 2$ we need $y = 1-x$, whilst if $y(1) = 3$ we need $y = 4-x$. We clearly cannot satisfy both boundary conditions with the same outer solution, so let's look for a boundary or interior layer at $x = x_0$ by defining $y(x) = Y(X)$ and $x = x_0 + \epsilon^\alpha X$, with Y, $X = O(1)$. In terms of these scaled variables, (12.18) becomes

$$\epsilon^{1-2\alpha} Y_{XX} + 2(x_0 + \epsilon^\alpha X)(\epsilon^{-\alpha} Y_X + 1) = 0.$$

If $x_0 \neq 0$, for a leading order balance we need $\epsilon^{1-2\alpha} = O(\epsilon^{-\alpha})$, and hence $\alpha = 1$. In this case, at leading order,

$$Y_{XX} + 2x_0 Y_X = 0,$$

and hence $Y = A + Be^{-2x_0 X}$. For $x_0 > 0$ this grows exponentially as $X \to -\infty$, whilst for $x_0 < 0$ this grows exponentially as $X \to \infty$. In either case, we cannot match these exponentially growing terms with the outer solution. This suggests that we need $x_0 = 0$, when

$$\epsilon^{1-2\alpha} Y_{XX} + 2XY_X + 2\epsilon^\alpha X = 0.$$

For a leading order asymptotic balance we need $\alpha = 1/2$, and hence a boundary layer with width of $O(\epsilon^{1/2})$. At leading order,

$$Y_{XX} + 2XY_X = 0,$$

which, after multiplying by the integrating factor, e^{X^2}, gives

$$\frac{d}{dX}\left(e^{X^2}Y_X\right) = 0,$$

and hence

$$Y = B + A\int_{-\infty}^{X} e^{-s^2}\,ds.$$

Now, since the interior layer is at $x = 0$, the outer solution must be

$$y = \left\{\begin{array}{ll} 1 - x & \text{for } -1 \leqslant x < O(\epsilon^{1/2}), \\ 4 - x & \text{for } O(\epsilon^{1/2}) < x \leqslant 1. \end{array}\right.$$

Since $y \to 4$ as $x \to 0^+$ and $y \to 1$ as $x \to 0^-$, we must have $Y \to 4$ as $X \to \infty$ and $Y \to 1$ as $X \to -\infty$. This allows us to fix the constants A and B and find that

$$Y(X) = 1 + \frac{3}{\sqrt{\pi}}\int_{-\infty}^{X} e^{-s^2}\,ds = \frac{1}{2}\left(5 + 3\,\mathrm{erf}\,(x)\right),$$

which leads to the one-term composite solution

$$y \sim y_c = -x + \frac{1}{2}\left\{5 + 3\,\mathrm{erf}\left(\frac{x}{\sqrt{\epsilon}}\right)\right\} \quad \text{for } -1 \leqslant x \leqslant 1 \text{ and } \epsilon \ll 1. \tag{12.20}$$

This is illustrated in Figure 12.5 for various values of ϵ. Note that the boundary conditions at $x = \pm 1$ are only satisfied by the composite expansion at leading order.

12.2.3 Nonlinear Problems

Example 1: Elliptic functions of large period

As we have already seen in Section 9.4, the Jacobian elliptic function $x = \mathrm{sn}(t;k)$ satisfies the equation

$$\frac{dx}{dt} = \sqrt{1 - x^2}\sqrt{1 - k^2x^2}, \tag{12.21}$$

subject to $x = 0$ when $t = 0$, and has periodic solutions for $k \neq 1$. When $k = 1$, the solution that corresponds to $\mathrm{sn}(t;k)$ is a heteroclinic path that connects the equilibrium points $(\pm 1, 0, 0)$ in the phase portrait shown in Figure 9.18, and hence the period tends to infinity as $k \to 1$. When k is close to unity, it seems reasonable to assume that the period of the solution is large but finite. Can we quantify this? Let's assume that $k^2 = 1 - \delta^2$, with $\delta \ll 1$, and seek an asymptotic solution for the first quarter period of $x(t)$, with $0 \leqslant x \leqslant 1$. Figure 12.6 shows $\mathrm{sn}(t;k)$ for various values of δ, and we can see that the period does increase as δ decreases and k approaches unity. The function $\mathrm{sn}(t;k)$ is available in MATLAB as `ellipj`. The quarter period is simply the value of t when $\mathrm{sn}(t;k)$ reaches its first maximum.

We seek an asymptotic solution of the form

$$x = x_0 + \delta^2 x_1 + \delta^4 x_2 + O\left(\delta^6\right).$$

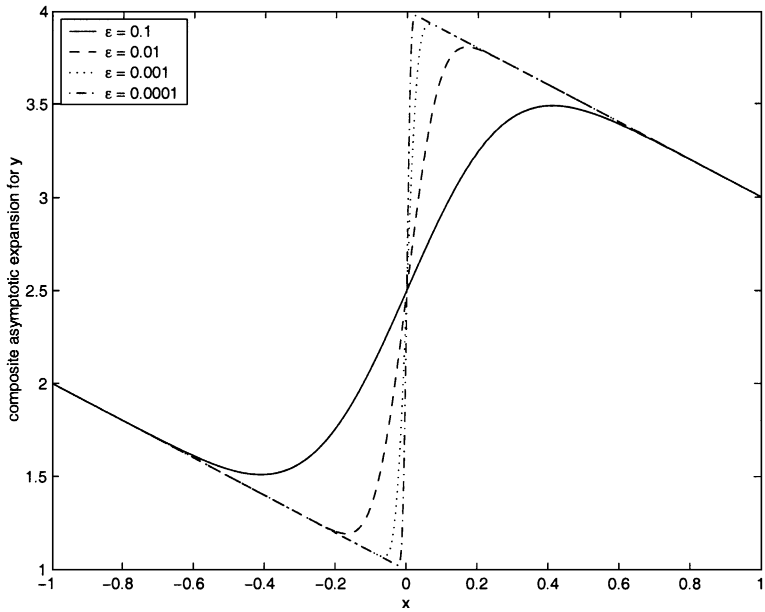

Fig. 12.5. The composite asymptotic solution, (12.20), of (12.18).

Using a binomial expansion, (12.21) is

$$\frac{dx}{dt} = \left(1 - x^2\right)\left(1 + \delta^2 \frac{x^2}{1-x^2}\right)^{1/2} = 1 - x^2 + \frac{1}{2}\delta^2 x^2 + \frac{1}{8}\delta^4 \frac{x^4}{1-x^2} + O(\delta^6).$$

This binomial expansion is only valid when x is not too close unity, so we should expect any asymptotic expansion that we develop to become nonuniform as $x \to 1$, and we treat this as the outer expansion.

At leading order,

$$\frac{dx_0}{dt} = 1 - x_0^2, \quad \text{subject to } x_0(0) = 0,$$

which has solution $x_0 = \tanh t$. At $O(\delta^2)$,

$$\frac{dx_1}{dt} = -2x_0 x_1 + \frac{1}{2}x_0^2 = -2\tanh t x_1 + \frac{1}{2}\tanh^2 t, \quad \text{subject to } x_1(0) = 0.$$

Using the integrating factor $\cosh^2 t$, we can find the solution

$$x_1 = \frac{1}{4}\left(\tanh t - t\,\mathrm{sech}^2 t\right).$$

We can now see that

$$x \sim 1 - 2e^{-2t} + \frac{1}{4}\delta^2 + O(\delta^4) \quad \text{as } t \to \infty.$$

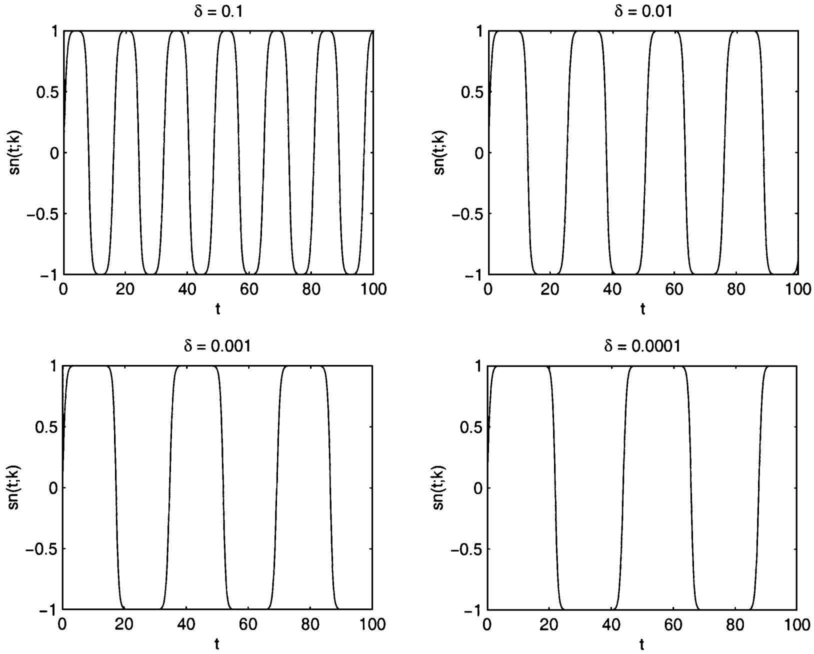

Fig. 12.6. The Jacobian elliptic function $\text{sn}(t;k)$ for various values of δ.

Although x approaches unity as $t \to \infty$, there is no nonuniformity in this expansion, so we need to go to $O(\delta^4)$. At this order,

$$\frac{dx_2}{dt} = -x_1^2 - 2x_0x_2 + x_0x_1 - \frac{1}{8}\left(\frac{x_0^4}{1-x_0^2}\right) \quad \text{subject to } x_2(0) = 0.$$

Solving this problem would be extremely tedious. Fortunately, we don't really want to know the exact expression for x_2, just its behaviour as $t \to \infty$. Using the known behaviour of the various hyperbolic functions, we find that

$$\frac{dx_2}{dt} + 2x_2 \sim -\frac{1}{32}e^{2t} \quad \text{as } t \to \infty,$$

and hence from solving this simple linear equation,

$$x_2 \sim -\frac{1}{128}e^{2t} \quad \text{as } t \to \infty.$$

This shows that

$$x \sim 1 - 2e^{-2t} + \frac{1}{4}\delta^2 - \frac{1}{128}\delta^4 e^{2t} + O(\delta^6) \quad \text{as } t \to \infty. \tag{12.22}$$

We can now see that the fourth term in this expansion becomes comparable to the third when $\delta^4 e^{2t} = O(\delta^2)$, and hence as $t \to \log(1/\delta)$, when $x = 1 + O(\delta^2)$.

We therefore define new variables for an inner region close to $x = 1$ as

$$x = 1 - \delta^2 X, \quad t = \log\left(\frac{1}{\delta}\right) + T.$$

On substituting these inner variables into (12.21), we find that, at leading order,

$$\frac{dX}{dT} = -\sqrt{2X\,(2X+1)}.$$

Using the substitution $X = \bar{X} - \frac{1}{4}$ brings this separable equation into a standard form, and the solution is

$$X = \frac{1}{4}\left\{\cosh\left(K - 2T\right) - 1\right\}. \tag{12.23}$$

We now need to determine the constant K by matching the inner solution, (12.23), with the outer solution, whose behaviour as $x \to 1$ is given by (12.22). Writing the inner expansion in terms of the outer variables and retaining terms up to $O(\delta^2)$ leads to

$$x = 1 - \frac{1}{8}e^K e^{-2t} + \frac{1}{4}\delta^2 + O(\delta^4),$$

for $t = O(1)$. Comparing this with (12.22) shows that we need $\frac{1}{8}e^K e^{-2t} = 2e^{-2t}$, and hence $K = 4\log 2$, which gives

$$x = 1 - \frac{1}{4}\delta^2\left\{\cosh\left(4\log 2 - 2T\right) - 1\right\} + O(\delta^4) \tag{12.24}$$

when $T = O(1)$. From this leading order approximation, $x = 1$ when $T = T_0 = 2\log 2 + O(\delta^2)$. This is the quarter period of the solution, so the period τ satisfies

$$\frac{1}{4}\tau = \log\left(\frac{1}{\delta}\right) + T_0,$$

and hence

$$\tau = 4\log\left(\frac{4}{\delta}\right) + O(\delta^2),$$

for $\delta \ll 1$. We conclude that the period of the solution does indeed tend to infinity as $\delta \to 0$, $k \to 1^-$, but only logarithmically fast. Figure 12.7 shows a comparison between the exact and analytical solutions. The agreement is very good for all $\delta \leqslant 1$. We produced this figure using the MATLAB script

```
Texact = []; d = 10.^(-7:0.25:0); Tasymp = 4*log(4./d);
options = optimset('Display','off','TolX', 10^-10);
for del = d
    k = sqrt(1-del^2); T2 = 2*log(4/del);
    Texact = [Texact 2*fzero(@ellipj,T2,options,k)];
end
plot(log10(d),Texact,log10(d),Tasymp, '--')
xlabel('log_{10}\delta'), ylabel('T')
legend('exact','asymptotic')
```

This uses the MATLAB function `fzero` to find where the elliptic function is zero, using the asymptotic expression as an initial estimate. Note that the function `optimset` allows us to create a variable `options` that we can pass to `fzero` as a parameter, which controls the details of its execution. In this case, we turn off the output of intermediate results, and set the convergence tolerance to 10^{-10}.

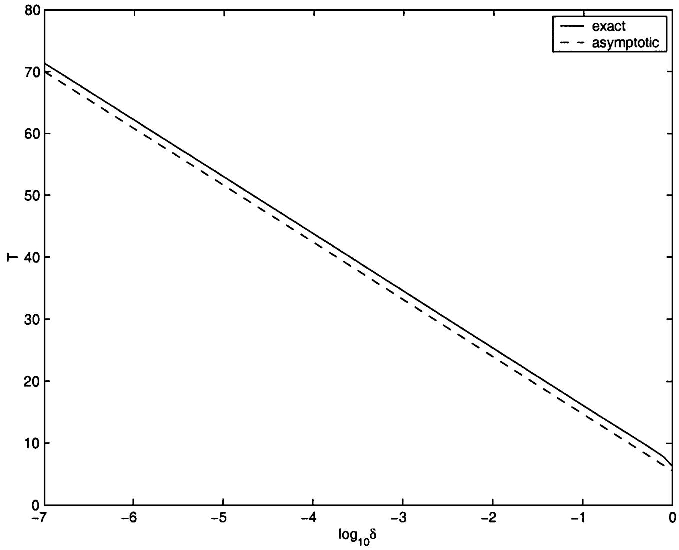

Fig. 12.7. A comparison of the exact period of the elliptic function $\mathrm{sn}(t; k)$ for $k = \sqrt{1-\delta^2}$.

Finally, by adding the solutions in the inner and outer regions and subtracting the matched part, $1 + \frac{1}{4}\delta^2 - 2e^{-2t}$, we can obtain a composite expansion, uniformly valid for $0 \leqslant t \leqslant \frac{1}{4}\tau = \log(4/\delta) + O(\delta^2)$, as

$$x = \tanh t + 2e^{-2t} + \frac{1}{4}\delta^2 \left[\tanh t - t\,\mathrm{sech}^2 t - \cosh\left\{\log\left(\frac{16}{\delta^2}\right) - 2t\right\}\right] + O(\delta^4).$$

Example 2: A thermal ignition problem

Many materials decompose to produce heat. This decomposition is usually more rapid the higher the temperature. This leads to the possibility of thermal ignition. As a material gets hotter, it releases heat more rapidly, which heats it more rapidly, and so on. This positive feedback mechanism can lead to the unwanted, and potentially disastrous, combustion of many common materials, ranging from packets of powdered milk to haystacks. The most common physical mechanism that can break this feedback loop is the diffusion of heat through the material and out through its surface. The rate of heat production due to decomposition is proportional to the volume of the material and the rate of heat loss from its surface proportional to surface area. For a sufficiently large volume of material, heat production dominates heat loss, and the material ignites. Determining the critical temperature below which it is safe to store a potentially combustible material is an important and difficult problem.†

We now want to develop a mathematical model of this problem. In Section 2.6.1, we showed how to derive the diffusion equation, (2.12), which governs the flow of heat in a body. We now need to include the effect of a chemical reaction that produces $R(x, y, z, t)$ units of heat, per unit volume, per unit time, by adding a term $R\,\delta t\,\delta x\,\delta y\,\delta z$ to the right hand side of (2.11). On taking the limit δt, δx, δy, $\delta z \to 0$, we arrive at

$$\rho c \frac{\partial T}{\partial t} = -\nabla \cdot \mathbf{Q} + R,$$

and hence, for a steady state solution ($\partial/\partial t = 0$), and since Fourier's law of heat conduction is $\mathbf{Q} = -k\nabla T$,

$$k\nabla^2 T + R = 0. \tag{12.25}$$

The rate of combustion of the material can be modelled using the **Arrhenius law**, $R = Ae^{-T_\mathrm{a}/T}$, where A is a constant and T_a is the **activation temperature**, also a constant. It is important to note that T is the **absolute temperature** here. The Arrhenius law can be derived from first principles using statistical mechanics, although we will not attempt this here (see, for example, Flowers and Mendoza, 1970). Figure 12.8 shows that the reaction rate is zero at absolute zero ($T = 0$), and remains small until T approaches the activation temperature, T_a, when it increases, with $R \to A$ as $T \to \infty$. After defining $u = T/T_\mathrm{a}$ and rescaling distances

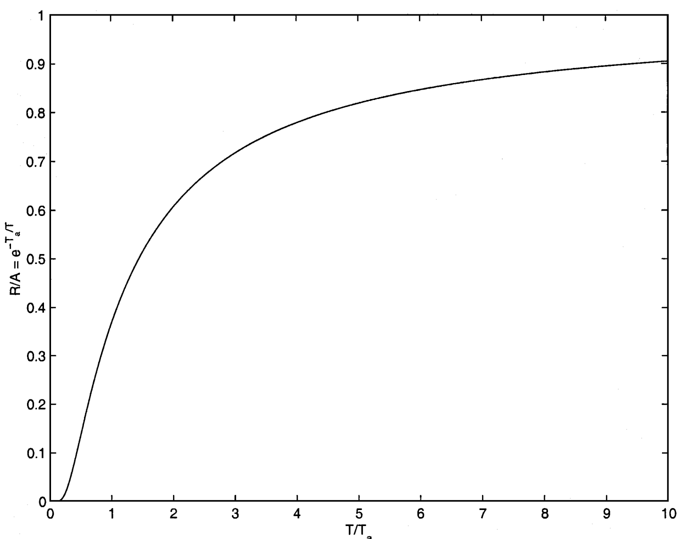

Fig. 12.8. The Arrhenius reaction rate law.

† For more background on combustion theory, see Buckmaster and Ludford (1982).

with $\sqrt{k/A}$, we arrive at the nonlinear partial differential equation

$$\nabla^2 u + e^{-1/u} = 0.$$

For a uniform sphere of combustible material with a spherically-symmetric temperature distribution, this becomes

$$\frac{d^2u}{dr^2} + \frac{2}{r}\frac{du}{dr} + e^{-1/u} = 0, \tag{12.26}$$

subject to

$$\frac{du}{dr} = 0 \;\text{ at } r = 0, \;\; u = u_\mathrm{a} \;\text{ at } r = r_\mathrm{a}. \tag{12.27}$$

Here, u_a is the dimensionless absolute temperature of the surroundings and r_a the dimensionless radius of the sphere. Note that, as we discussed earlier, we would expect that the larger r_a, the smaller u_a must be to prevent the ignition of the sphere.† A positive solution of this boundary value problem represents a possible steady state solution in which these two physical processes are in balance. If no such steady state solution exists, we can conclude that the material will ignite. A small trick that makes the study of this problem easier is to replace (12.27) with

$$\frac{du}{dr} = 0, \quad u = \epsilon \;\text{ at } r = 0. \tag{12.28}$$

We can then solve the initial value problem given by (12.26) and (12.28) for a given value of ϵ, the dimensionless temperature at the centre of the sphere, and then determine the corresponding value of $u_\mathrm{a} = u(r_\mathrm{a})$. Our task is therefore to construct an asymptotic solution of the initial value problem given by (12.26) and (12.28) when $\epsilon \ll 1$. Note that by using the integrating factor r^2, we can write (12.26) as

$$\frac{du}{dr} = -\frac{1}{r^2}\int_0^r s^2 e^{-1/u(s)}\, ds < 0,$$

and hence conclude that u is a monotone decreasing function of r. The temperature of the sphere decreases from centre to surface.

Asymptotic analysis: Region I

Since $u = \epsilon$ at $r = 0$, we define a new variable $\hat{u} = u/\epsilon$, with $\hat{u} = O(1)$ for $\epsilon \ll 1$. In terms of this variable, (12.26) and (12.28) become

$$\frac{d^2\hat{u}}{dr^2} + \frac{2}{r}\frac{d\hat{u}}{dr} + \frac{1}{\epsilon}\exp\left(-\frac{1}{\epsilon\hat{u}}\right) = 0,$$

subject to

$$\frac{d\hat{u}}{dr} = 0, \quad \hat{u} = 1 \;\text{ at } r = 0.$$

† For all the technical details of this problem, which was first studied by Gel'fand (1963), see Billingham (2000).

At leading order,

$$\frac{d^2\hat{u}}{dr^2} + \frac{2}{r}\frac{d\hat{u}}{dr} = 0, \tag{12.29}$$

which has the trivial solution $\hat{u} = 1$. We're going to need more than this to be able to proceed, so let's look for an asymptotic expansion of the form

$$\hat{u} = 1 + \phi_1(\epsilon)\hat{u}_1 + \phi_2(\epsilon)\hat{u}_2 + \cdots,$$

where $\phi_2 \ll \phi_1 \ll 1$ are to be determined. As we shall see, we need a three-term asymptotic expansion to be able to determine the scalings for the next asymptotic region.

Firstly, note that

$$\frac{1}{\epsilon}e^{-1/\epsilon\hat{u}} \sim \frac{1}{\epsilon}\exp\left\{-\frac{1}{\epsilon}\left(1+\phi_1\hat{u}_1\right)^{-1}\right\} \sim \frac{1}{\epsilon}\exp\left\{-\frac{1}{\epsilon}\left(1-\phi_1\hat{u}_1\right)\right\}$$

$$\sim \frac{1}{\epsilon}e^{-1/\epsilon}\exp\left(\frac{\phi_1}{\epsilon}\hat{u}_1\right) \sim \frac{1}{\epsilon}e^{-1/\epsilon}\left(1+\frac{\phi_1}{\epsilon}\hat{u}_1\right),$$

provided that $\phi_1 \ll \epsilon$, which we can check later. Equation (12.26) then shows that

$$\phi_1\left(\frac{d^2\hat{u}_1}{dr^2} + \frac{2}{r}\frac{d\hat{u}_1}{dr}\right) + \phi_2\left(\frac{d^2\hat{u}_2}{dr^2} + \frac{2}{r}\frac{d\hat{u}_2}{dr}\right) \sim -\frac{1}{\epsilon}e^{-1/\epsilon}\left(1+\frac{\phi_1}{\epsilon}\hat{u}_1\right). \tag{12.30}$$

In order to obtain a balance of terms, we therefore take

$$\phi_1 = \frac{1}{\epsilon}e^{-1/\epsilon} \ll \epsilon, \quad \phi_2 = \frac{1}{\epsilon^2}e^{-1/\epsilon}\phi_1 = \frac{1}{\epsilon^3}e^{-2/\epsilon},$$

and hence expand

$$\hat{u} = 1 + \frac{1}{\epsilon}e^{-1/\epsilon}\hat{u}_1 + \frac{1}{\epsilon^3}e^{-2/\epsilon}\hat{u}_2 + \cdots.$$

Now, using (12.30),

$$\frac{d^2\hat{u}_1}{dr^2} + \frac{2}{r}\frac{d\hat{u}_1}{dr} = -1, \tag{12.31}$$

subject to

$$\frac{d\hat{u}_1}{dr} = \hat{u}_1 = 0 \quad \text{at } r = 0.$$

Using the integrating factor r^2, we find that the solution is $\hat{u}_1 = -\frac{1}{6}r^2$. Similarly,

$$\frac{d^2\hat{u}_2}{dr^2} + \frac{2}{r}\frac{d\hat{u}_2}{dr} = \frac{1}{6}r^2,$$

subject to

$$\frac{d\hat{u}_2}{dr} = \hat{u}_2 = 0 \quad \text{at } r = 0,$$

and hence $\hat{u}_2 = \frac{1}{120}r^4$. This means that

$$\hat{u} \sim 1 - \frac{1}{6\epsilon}e^{-1/\epsilon}r^2 + \frac{1}{120\epsilon^3}e^{-2/\epsilon}r^4 \tag{12.32}$$

for $\epsilon \ll 1$. This expansion is not uniformly valid, since for sufficiently large r the second term is comparable to the first, when $r = O(\epsilon^{1/2}e^{1/2\epsilon})$, and the third is comparable to the second, when $r = O(\epsilon e^{1/2\epsilon}) \ll \epsilon^{1/2}e^{1/2\epsilon}$. As r increases, the first nonuniformity to occur is therefore when $r = O(\epsilon e^{1/2\epsilon})$ and $u = \epsilon + O(\epsilon^2)$. Note that this is why we needed a three-term expansion to work out the scalings for the next asymptotic region.

What would have happened if we had taken only a two-term expansion and looked for a new asymptotic region with $r = O(\epsilon^{1/2}e^{1/2\epsilon})$ and $u = O(\epsilon)$? If you try this, you will find that it is impossible to match the solutions in the new region to the solutions in region I. After some thought, you notice that the equations at $O(1)$ and $O(\frac{1}{\epsilon}e^{-1/\epsilon})$ in region I, (12.29) and (12.31), do not depend at all on the functional form of the term $e^{-1/\epsilon\hat{u}}$, which could be replaced by $e^{-1/\epsilon}$ without affecting (12.29) or (12.31). This is a sign that we need another term in the expansion in order to capture the effect of the only nonlinearity in the problem.

Asymptotic analysis: Region II

In this region we define scaled variables $u = \epsilon + \epsilon^2 U$, $r = \epsilon e^{1/2\epsilon} R$, with $U = O(1)$ and $R = O(1)$ for $\epsilon \ll 1$. At leading order, (12.26) becomes

$$\frac{d^2U}{dR^2} + \frac{2}{R}\frac{dU}{dR} + e^U = 0, \tag{12.33}$$

to be solved subject to the matching condition

$$U \sim -\frac{1}{6}R^2 \quad \text{as } R \to 0. \tag{12.34}$$

Equation (12.33) is nonlinear and nonautonomous, which usually means that we must resort to finding a numerical solution. However, we have seen in Chapter 10 that we can often make progress using group theoretical methods. Equation (12.33) is invariant under the transformation $U \mapsto U + k$, $R \mapsto e^{-k/2}R$. We can therefore make the nonlinear transformation

$$p(s) = e^{-2s}e^U, \quad q(s) = 2 + e^{-s}\frac{dU}{dR}, \quad R = e^{-s},$$

after which (12.33) and (12.34) become

$$\frac{dp}{ds} = -pq, \quad \frac{dq}{ds} = p + q - 2, \tag{12.35}$$

subject to

$$p \sim e^{-2s}, \quad q \sim 2 - \frac{1}{3}e^{-2s} \quad \text{as } s \to \infty. \tag{12.36}$$

The problem is still nonlinear, but is now autonomous, so we can use the phase plane methods that we studied in Chapter 9.

There are two finite equilibrium points, at $P_1 = (0, 2)$ and $P_2 = (2, 0)$ in the (p, q) phase plane. After determining the Jacobian at each of these points and calculating the eigenvalues in the usual way, we find that P_1 is a saddle point and P_2 is an unstable, anticlockwise spiral. Since (12.36) shows that we are interested in a solution that asymptotes to P_1 as $s \to \infty$, this solution must be represented by one of the stable separatrices of P_1. Furthermore, since $p = e^{-2s}e^U > 0$, the unique integral path that represents the solution is the stable separatrix of P_1 that lies in the half plane $p > 0$. What happens to this separatrix as $s \to -\infty$? A sensible, and correct, guess would be that it asymptotes to the unstable spiral at P_2, as shown in Figure 12.9, for which we calculated the solution numerically using MATLAB (see Section 9.3.4). The proof that this is the case is Exercise 9.17, which comes with some hints.

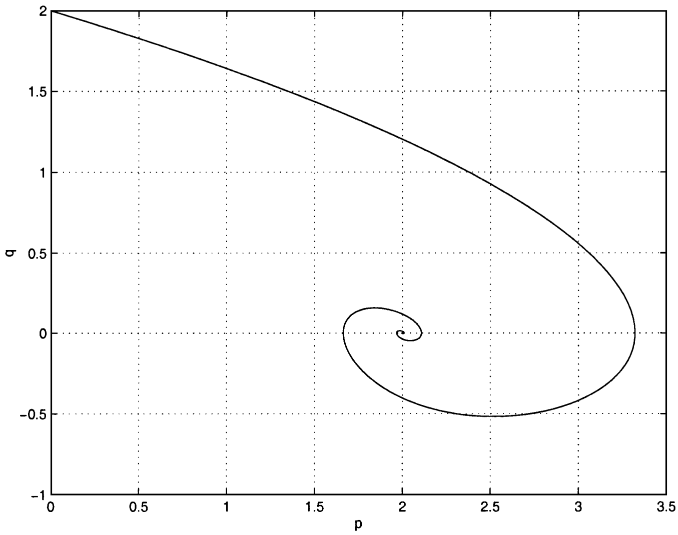

Fig. 12.9. The behaviour of the solution of (12.35) subject to (12.36) in the (p, q)-phase plane.

Since the solution asymptotes to P_2, we can determine its behaviour as $s \to -\infty$ by considering the local solution there. The eigenvalues of P_2 are $\frac{1}{2}(1 \pm i\sqrt{7})$, and therefore

$$p \sim 2 + Ae^{s/2} \sin\left(\frac{\sqrt{7}}{2}s + B\right) \quad \text{as } s \to -\infty,$$

for some constants A and B. Since $U = 2s + \log p$ and $s = -\log R$, this shows that

$$U \sim -2\log R + \log 2 - \frac{A}{2\sqrt{R}} \sin\left(\frac{\sqrt{7}}{2}\log R - B\right) \quad \text{as } R \to \infty. \tag{12.37}$$

We conclude that

$$u \sim \epsilon + \epsilon^2(\log 2 - 2\log R) \quad \text{as } R \to \infty,$$

for $\epsilon \ll 1$. When $\log R = O(\epsilon^{-1})$, the second term is comparable to the first term, and we have a further nonuniformity.

Asymptotic analysis: Region III

We define scaled variables $u = \epsilon\bar{U}$ and $S = \epsilon \log R = -\frac{1}{2} - \epsilon\log\epsilon + \epsilon\log r$, $r = \epsilon e^{1/2\epsilon} e^{S/\epsilon}$, with $U = O(1)$ and $S = O(1)$ for $\epsilon \ll 1$. In terms of these variables, (12.26) becomes

$$\epsilon\frac{d^2\bar{U}}{dS^2} + \frac{d\bar{U}}{dS} + \exp\left\{\frac{1}{\epsilon}\left(-\frac{1}{\bar{U}} + 1 + 2S\right)\right\} = 0. \tag{12.38}$$

We expand $\bar{U}$ as

$$\bar{U} = \bar{U}_0 + \epsilon\bar{U}_1 + \cdots .$$

To find the matching conditions, we write expansion (12.37) in region I in terms of the new variables and retain terms up to $O(\epsilon)$, which gives

$$\bar{U}_0 \sim 1 - 2S, \quad \bar{U}_1 \sim \log 2 \quad \text{as } S \to 0. \tag{12.39}$$

At leading order, the solution is given by the exponential term in (12.38) as

$$\bar{U}_0 = \frac{1}{1+2S},$$

which satisfies (12.39). At $O(\epsilon)$, we find that

$$\frac{d\bar{U}_0}{dS} + \exp\left\{(1+2S)^2\,\bar{U}_1\right\} = 0,$$

and hence that

$$\bar{U}_1 = \frac{1}{(1+2S)^2}\log\left\{\frac{2}{(1+2S)^2}\right\},$$

which also satisfies (12.39). We conclude that

$$u = \frac{\epsilon}{1+2S} + \frac{\epsilon^2}{(1+2S)^2}\log\left\{\frac{2}{(1+2S)^2}\right\} + O(\epsilon^3),$$

for $S = O(1)$ and $\epsilon \ll 1$. This expansion remains uniform, with $u \to 0$ as $S \to \infty$, and hence $R \to \infty$, and the solution is complete.

We can now determine $u_{\mathrm{a}} = u(r_{\mathrm{a}})$. If $r_{\mathrm{a}} \ll \epsilon e^{1/2\epsilon}$, $r = r_{\mathrm{a}}$ lies in region I, so that $u_{\mathrm{a}} \sim \epsilon - \frac{1}{6}e^{-1/\epsilon}r_{\mathrm{a}}^2 \sim \epsilon$. In other words, for u_{a} sufficiently small that $r_{\mathrm{a}} \ll u_{\mathrm{a}}e^{1/2u_{\mathrm{a}}}$, a steady state solution is always available, and we predict that the sphere will not ignite.

If $r_{\mathrm{a}} = O(\epsilon e^{1/2\epsilon})$, we need to consider the solution in region II. The oscillations in p lead to oscillations in u_{a} as a function of ϵ, as shown in Figure 12.10. In particular, the fact that $p < p_{\max} \approx 3.322$, as can be seen in Figure 12.9, leads to a maximum value of u_{a} for which a steady state solution is available, namely

$$u_{\mathrm{a}\max} \sim \epsilon + \epsilon^2\log\left(\frac{p_{\max}\epsilon^2 e^{1/\epsilon}}{r_{\mathrm{a}}^2}\right). \tag{12.40}$$

Finally, if $r_{\mathrm{a}} = O(\epsilon e^{3/2\epsilon})$, the solution in region III shows that

$$u_{\mathrm{a}} \sim \frac{\epsilon}{\frac{1}{2} + 2\epsilon\log(r_{\mathrm{a}}/\epsilon)} < u_{\mathrm{a}\max}.$$

We conclude that u_{amax} (the **critical ignition temperature** or **critical storage temperature**) gives an asymptotic approximation to the hottest ambient temperature at which a steady state solution exists, and hence at which the material will not ignite.

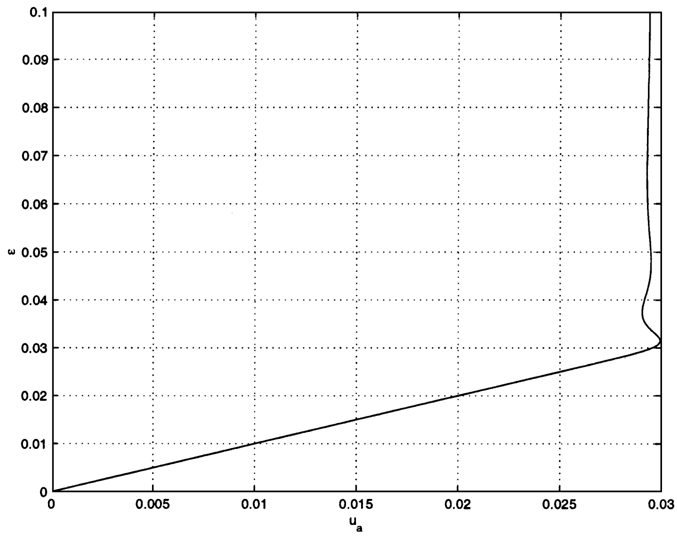

Fig. 12.10. The ambient temperature, u_{a}, as a function of ϵ in region II when $r_{\mathrm{a}} = 10^9$.

12.2.4 The Method of Multiple Scales

Let's now return to solving a simple, linear, constant coefficient ordinary differential equation that, at first sight, seems like a regular perturbation problem. Consider

$$\ddot{y} + 2\epsilon\dot{y} + y = 0, \tag{12.41}$$

to be solved subject to

$$y(0) = 1, \quad \dot{y}(0) = 0, \tag{12.42}$$

for $t \geqslant 0$, where a dot denotes d/dt. Since this is an initial value problem, we can think of y developing as a function of time, t. As usual, we expand

$$y = y_0(t) + \epsilon y_1(t) + O(\epsilon^2)$$

for $\epsilon \ll 1$. At leading order, $\ddot{y}_0 + y_0 = 0$, subject to $y_0(0) = 1$, $\dot{y}_0(0) = 0$, which has solution $y_0 = \cos t$. At $O(\epsilon)$,

$$\ddot{y}_1 + y_1 = -2\dot{y}_0 = 2\sin t,$$

subject to $y_1(0) = 0$, $\dot{y}_1(0) = 0$. After noting that the particular integral solution of this equation takes the form $y_{1\mathrm{p}} = kt\cos t$ for some constant k, we find that $y_1 = -t\cos t + \sin t$. This means that

$$y(t) \sim \cos t + \epsilon\,(-t\cos t + \sin t) \tag{12.43}$$

as $\epsilon \to 0$. As $t \to \infty$, the ratio of the second term to the first term in this expansion is asymptotic to ϵt, and is therefore no longer small when $t = O(\epsilon^{-1})$. We conclude that the asymptotic solution given by (12.43) is only valid for $t \ll \epsilon^{-1}$.

To see where the problem lies, let's examine the exact solution of (12.41) subject to the boundary conditions (12.42), which is

$$y = e^{-\epsilon t}\left(\cos\sqrt{1-\epsilon^2}t + \frac{\epsilon}{\sqrt{1-\epsilon^2}}\sin\sqrt{1-\epsilon^2}t\right). \tag{12.44}$$

As we can see from Figure 12.11, the solution is a decaying oscillation, with the decay occurring over a timescale of $O(\epsilon^{-1})$. At leading order, (12.41) is an undamped, linear oscillator. The term $2\epsilon\dot{y}$ represents the effect of weak damping, which slowly reduces the amplitude of the oscillation. The problem with the asymptotic expansion (12.43) is that, although it correctly captures the fact that $e^{-\epsilon t} \sim 1 - \epsilon t$ for $\epsilon \ll 1$ and $t \ll \epsilon^{-1}$, we need to keep the exponential rather than its Taylor series expansion if we are to construct a solution that is valid when $t = O(\epsilon^{-1})$. Figure 12.11 shows that the two-term asymptotic expansion, (12.43), rapidly becomes inaccurate once $t = O(\epsilon^{-1})$.

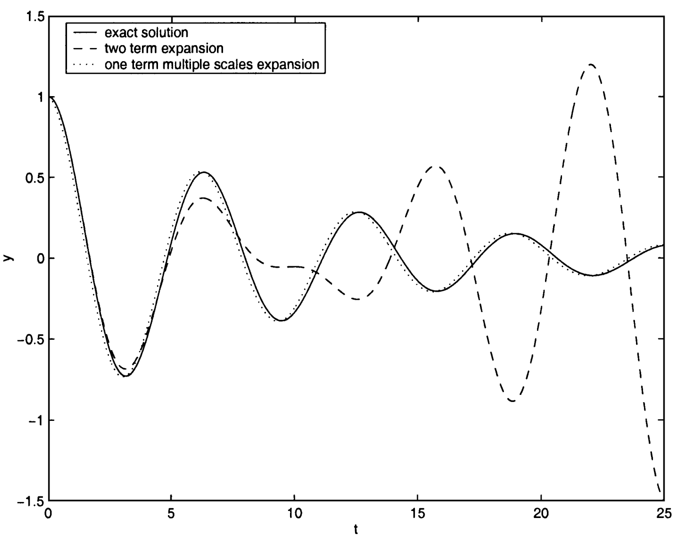

Fig. 12.11. The exact solution, (12.44), two-term asymptotic solution, (12.43), and one-term multiple scales solution (12.50) of (12.41) when $\epsilon = 0.1$.

The **method of multiple scales**, in its most basic form, consists of defining a new **slow time variable**, $T = \epsilon t$, so that when $t = O(1/\epsilon)$, $T = O(1)$, and the slow decay can therefore be accounted for. We then look for an asymptotic solution

$$y = y_0(t,T) + \epsilon y_1(t,T) + O(\epsilon^2),$$

with each term a function of both t, to capture the oscillation, and T, to capture the slow decay. After noting that

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T},$$

(12.41) becomes

$$\frac{\partial^2 y}{\partial t^2} + 2\epsilon\frac{\partial^2 y}{\partial t\partial T} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + 2\epsilon\frac{\partial y}{\partial t} + 2\epsilon^2\frac{\partial y}{\partial T} + y = 0, \tag{12.45}$$

to be solved subject to

$$y(0,0) = 1, \quad \frac{\partial y}{\partial t}(0,0) + \epsilon\frac{\partial y}{\partial T}(0,0) = 0. \tag{12.46}$$

At leading order,

$$\frac{\partial^2 y_0}{\partial t^2} + y_0 = 0,$$

subject to

$$y_0(0,0) = 1, \quad \frac{\partial y_0}{\partial t}(0,0) = 0.$$

Although this is a partial differential equation, the only derivatives are with respect to t, so we can solve as if it were an ordinary differential equation in t. However, we must remember that the 'constants' of integration can actually be functions of the other variable, T. This means that

$$y_0 = A_0(T)\cos t + B_0(T)\sin t,$$

and the boundary conditions show that

$$A_0(0) = 1, \quad B_0(0) = 0. \tag{12.47}$$

The functions $A_0(T)$ and $B_0(T)$ are, as yet, undetermined.

At $O(\epsilon)$,

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2\frac{\partial^2 y_0}{\partial t\partial T} - 2\frac{\partial y_0}{\partial t} = 2\left(\frac{dA_0}{dT} + A_0\right)\sin t - 2\left(\frac{dB_0}{dT} + B_0\right)\cos t. \tag{12.48}$$

Because of the appearance on the right hand side of the equation of the terms $\cos t$ and $\sin t$, which are themselves solutions of the homogeneous version of the equation, the particular integral solution will involve the terms $t\sin t$ and $t\cos t$. As we have seen, it is precisely terms of this type that lead to a nonuniformity in the asymptotic expansion when $t = O(\epsilon^{-1})$. The terms proportional to $\sin t$ and $\cos t$ in (12.48) are known as **secular terms** and, to keep the asymptotic expansion uniform, we must eliminate them by choosing A_0 and B_0 appropriately. In this example, we need

$$\frac{dA_0}{dT} + A_0 = 0, \quad \frac{dB_0}{dT} + B_0 = 0, \tag{12.49}$$

and hence

$$y_1 = A_1(T)\cos t + B_1(T)\sin t.$$

The initial conditions for (12.49) are given by (12.47), so the solutions are $A_0 = e^{-T}$ and $B_0 = 0$. We conclude that

$$y \sim e^{-T}\cos t = e^{-\epsilon t}\cos t \tag{12.50}$$

for $\epsilon \ll 1$. This asymptotic solution is consistent with the exact solution, (12.44), and remains valid when $t = O(\epsilon^{-1})$, as can be seen in Figure 12.11. In fact, we will see below that this solution is only valid for $t \ll \epsilon^{-2}$.

In order to proceed to find more terms in the asymptotic expansion using the method of multiple scales, we can take the exact solution, (12.44), as a guide. We know that

$$y \sim e^{-\epsilon t}\left\{\cos\left(1 - \frac{1}{2}\epsilon^2\right)t + \epsilon\sin\left(1 - \frac{1}{2}\epsilon^2\right)t\right\}, \tag{12.51}$$

for $\epsilon \ll 1$. This shows that the phase of the oscillation changes by an $O(1)$ amount when $t = O(\epsilon^{-2})$. In order to capture this, we seek a solution that is a function of the two timescales

$$T = \epsilon t, \quad \tau = \left(1 + a\epsilon^2 + b\epsilon^3 + \cdots\right)t,$$

with the constants $a, b, \ldots$ to be determined. Although this looks like a bit of a cheat, since we are only doing this because we know the exact solution, this approach works for a wide range of problems, including nonlinear differential equations.

In order to develop a one-term multiple scale expansion, we needed to consider the solution up to $O(\epsilon)$. This suggests that we will need to expand up to $O(\epsilon^2)$ to construct a two-term multiple scales expansion, with

$$y = y_0(\tau, T) + \epsilon y_1(\tau, T) + \epsilon^2 y_2(\tau, T) + O(\epsilon^3).$$

After noting that

$$\frac{d}{dt} = \epsilon\frac{\partial}{\partial T} + \left(1 + a\epsilon^2 + b\epsilon^3 + \cdots\right)\frac{\partial}{\partial \tau},$$

equation (12.41) becomes

$$\frac{\partial^2 y}{\partial \tau^2} + 2a\epsilon^2\frac{\partial^2 y}{\partial \tau^2} + 2\epsilon\frac{\partial^2 y}{\partial \tau \partial T} + \epsilon^2\frac{\partial^2 y}{\partial T^2} + 2\epsilon\frac{\partial y}{\partial \tau} + 2\epsilon^2\frac{\partial y}{\partial T} + y + O(\epsilon^3) = 0, \tag{12.52}$$

to be solved subject to

$$y(0,0) = 1, \quad \left(1 + a\epsilon^2 + b\epsilon^3 + \cdots\right)\frac{\partial y}{\partial \tau}(0,0) + \epsilon\frac{\partial y}{\partial T}(0,0) = 0. \tag{12.53}$$

We already know that

$$y_0 = e^{-T}\cos\tau, \quad y_1 = A_1(T)\cos\tau + B_1(T)\sin\tau.$$

At $O(\epsilon^2)$,

$$\frac{\partial^2 y_2}{\partial \tau^2} + y_2 = -2a\frac{\partial^2 y_0}{\partial \tau^2} - 2\frac{\partial^2 y_1}{\partial \tau \partial T} + \frac{\partial^2 y_0}{\partial T^2} - 2\frac{\partial y_1}{\partial \tau} - 2\frac{\partial y_0}{\partial T}$$

$$= 2\left(\frac{dA_1}{dt} + A_1\right)\sin\tau - 2\left\{\frac{dB_1}{dt} + B_1 - \left(a + \frac{1}{2}\right)e^{-T}\right\}\cos\tau.$$

In order to remove the secular terms we need

$$\frac{dA_1}{dt} + A_1 = 0, \quad \frac{dB_1}{dt} + B_1 = \left(a + \frac{1}{2}\right)e^{-T}.$$

At $O(\epsilon)$ the boundary conditions are

$$y_1(0,0) = A_1(0) = 0, \quad \frac{\partial y_1}{\partial \tau}(0,0) + \frac{\partial y_0}{\partial T}(0,0) = B_1(0) - 1 = 0,$$

and hence

$$A_1 = 0, \quad B_1 = \left(a + \frac{1}{2}\right)Te^{-T} + e^{-T}.$$

This gives us

$$y \sim e^{-T}\cos\tau + \epsilon\left\{\left(a + \frac{1}{2}\right)Te^{-T} + e^{-T}\right\}\sin\tau.$$

However, the part of the $O(\epsilon)$ term proportional to T will lead to a nonuniformity in the expansion when $T = O(\epsilon^{-1})$, and we must therefore remove it by choosing $a = -1/2$. We could have deduced this directly from the differential equation for B_1, since the term proportional to e^{-T} is secular. We conclude that

$$\tau = \left(1 - \frac{1}{2}\epsilon^2 + \cdots\right)t,$$

and hence obtain (12.51), as expected.

Example 1: The van der Pol Oscillator

The governing equation for the **van der Pol oscillator** is

$$\frac{d^2y}{dt^2} + \epsilon(y^2 - 1)\frac{dy}{dt} + y = 0, \tag{12.54}$$

for $t \geqslant 0$. For $\epsilon \ll 1$ this is a linear oscillator with a weak nonlinear term, $\epsilon(y^2 - 1)\dot{y}$. For $|y| < 1$ this term tends to drive the oscillations to a greater amplitude, whilst for $|y| > 1$, this term damps the oscillations. It is straightforward (at least if you're an electrical engineer!) to build an electronic circuit from resistors and capacitors whose output is governed by (12.54). It was in this context that this system was first studied extensively as a prototypical nonlinear oscillator. It is also straightforward to construct a forced van der Pol oscillator, which leads to a nonzero right hand side in (12.54), and study the chaotic response of the circuit.

Since the damping is zero when $y = 1$, a reasonable guess at the behaviour of the solution for $\epsilon \ll 1$ would be that there is an oscillatory solution with unit amplitude. Let's investigate this plausible, but incorrect, guess by considering the solution with initial conditions

$$y(0) = 1, \quad \frac{dy}{dt}(0) = 0. \tag{12.55}$$

We will use the method of multiple scales, and define a slow time scale, $T = \epsilon t$. We seek an asymptotic solution of the form

$$y = y_0(t, T) + \epsilon y_1(t, T) + O(\epsilon^2).$$

As for our linear example, we find that

$$y_0 = A_0(T) \cos\{t + \phi_0(T)\}.$$

For this problem, it is more convenient to write the solution in terms of an amplitude, $A_0(T)$, and phase, $\phi_0(T)$. The boundary conditions show that

$$A_0(0) = 1, \quad \phi_0(0) = 0. \tag{12.56}$$

At $O(\epsilon)$,

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = -2\frac{\partial^2 y_0}{\partial t \partial T} - (y_0^2 - 1)\frac{\partial y_0}{\partial t}$$

$$= 2\frac{dA_0}{dT}\sin(t + \phi_0) + 2A_0\frac{d\phi_0}{dT}\cos(t + \phi_0) + A_0 \sin(t + \phi_0)\left\{A_0^2 \cos^2(t + \phi_0) - 1\right\}.$$

In order to pick out the secular terms on the right hand side, we note that†

$$\sin\theta\cos^2\theta = \sin\theta - \sin^3\theta = \sin\theta - \frac{3}{4}\sin\theta + \frac{1}{4}\sin 3\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin 3\theta.$$

This means that

$$\frac{\partial^2 y_1}{\partial t^2} + y_1 = \left\{2\frac{dA_0}{dT} + \frac{1}{4}A_0^3 - A_0\right\}\sin(t + \phi_0) + 2A_0\frac{d\phi_0}{dT}\cos(t + \phi_0) + \frac{1}{4}A_0^3 \sin 3(t + \phi_0).$$

To suppress the secular terms we therefore need

$$\frac{d\phi_0}{dT} = 0, \quad \frac{dA_0}{dT} = \frac{1}{8}A_0(4 - A_0^2).$$

Subject to the boundary conditions (12.56), the solutions are

$$\phi_0 = 0, \quad A_0 = 2(1 + 3e^{-T})^{-1/2}.$$

Therefore

$$y \sim 2(1 + 3e^{-T})^{-1/2}\cos t$$

for $\epsilon \ll 1$, and we conclude that the amplitude of the oscillation actually tends to 2 as $t \to \infty$, as shown in Figure 12.12.

† To get $\cos^n\theta$ in terms of $\cos m\theta$ for $m = 1, 2, \dots, n$, use $e^{in\theta} = \cos n\theta + i\sin n\theta = (e^{i\theta})^n = (\cos\theta + i\sin\theta)^n$ and equate real and imaginary parts.

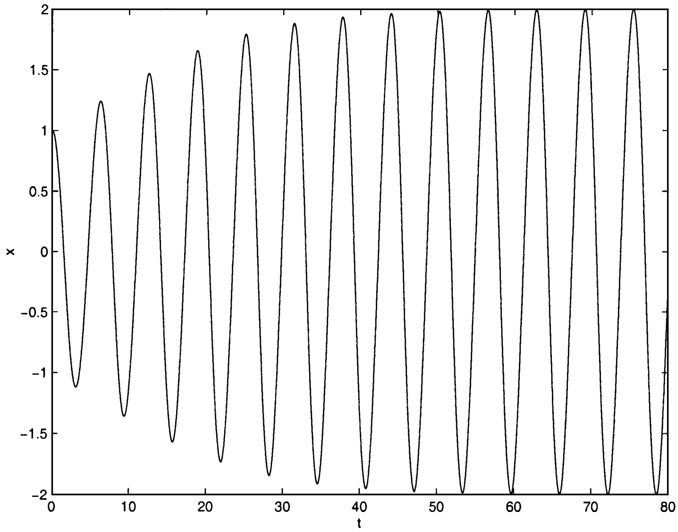

Fig. 12.12. The leading order solution of the van der Pol equation, (12.54), subject to $y(0) = 1$, $\dot{y}(0) = 0$, when $\epsilon = 0.1$.

Example 2: Jacobian elliptic functions with almost simple harmonic behaviour

Let's again turn our attention to the Jacobian elliptic function $x = \mathrm{sn}(t; k)$, which satisfies (12.21). When $k \ll 1$, this function is oscillatory and, at leading order, performs simple harmonic motion. We can see this more clearly by differentiating (12.21) and eliminating dx/dt to obtain

$$\frac{d^2x}{dt^2} + (1 + k^2)x = 2k^2x^3. \tag{12.57}$$

The initial conditions, of which there must be two for this second order equation, are $x = 0$ and, from (12.21), $dx/dt = 1$ when $t = 0$. Let's now use the method of multiple scales to see how this small perturbation affects the oscillation after a long time. As usual, we define $T = k^2t$ and $x = x(t, T)$, in terms of which (12.57) becomes

$$\frac{\partial^2 x}{\partial t^2} + 2k^2\frac{\partial^2 x}{\partial t \partial T} + k^4\frac{\partial^2 x}{\partial^2 T} + (1 + k^2)x = 2k^2x^3. \tag{12.58}$$

We seek an asymptotic solution

$$x = x_0(t, T) + k^2x_1(t, T) + O(k^4).$$

At leading order

$$\frac{\partial^2 x_0}{\partial t^2} + x_0 = 0,$$

subject to $x_0(0,0) = 0$ and $\frac{\partial x_0}{\partial t}(0,0) = 1$. This has solution

$$x_0 = A(T) \sin\{t + \phi(t)\},$$

and

$$A(0) = 1, \quad \phi(0) = 0. \tag{12.59}$$

At $O(k^2)$,

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = 2x_0^3 - x_0 - 2\frac{\partial^2 x_0}{\partial t \partial T}$$

$$= 2A^3 \sin^3(t+\phi) - A\sin(t+\phi) - 2\frac{dA}{dT}\cos(t+\phi) + 2A\frac{d\phi}{dT}\sin(t+\phi)$$

$$= \left(\frac{3}{2}A^3 - A + 2A\frac{d\phi}{dT}\right)\sin(t+\phi) - 2\frac{dA}{dT}\cos(t+\phi) - \frac{1}{2}A^3 \sin 3(t+\phi).$$

In order to remove the secular terms, we must set the coefficients of $\sin(t+\phi)$ and $\cos(t+\phi)$ to zero. This gives us two simple ordinary differential equations to be solved subject to (12.59), which gives

$$A = 1, \quad \phi = -\frac{1}{4}T,$$

and hence

$$x = \sin\left\{\left(1 - \frac{1}{4}k^2\right)t\right\} + O(k^2),$$

for $k \ll 1$. We can see that, as we would expect from the analysis given in Section 9.4, the leading order amplitude of the oscillation does not change with t, in contrast to the solution of the van der Pol equation that we studied earlier. However, the period of the oscillation changes by $O(k^2)$ even at this lowest order. If we take the analysis to $O(k^4)$, we find that the amplitude of the oscillation is also dependent on k (see King, 1988).

12.2.5 Slowly Damped Nonlinear Oscillations: Kuzmak's Method

The method of multiple scales, as we have described it above, is appropriate for weakly perturbed *linear* oscillators. Can we make any progress if the leading order problem is *nonlinear*? We will concentrate on the simple example,

$$\frac{d^2y}{dt^2} + 2\epsilon\frac{dy}{dt} + y - y^3 = 0, \tag{12.60}$$

subject to

$$y(0) = 0, \quad \frac{dy}{dt}(0) = v_0 > 0, \tag{12.61}$$

with ϵ small and positive. Let's begin by considering the leading order problem, with $\epsilon = 0$. As we saw in Chapter 9, since dy/dt does not appear in (12.60) when

$\epsilon = 0$, we can integrate once to obtain

$$\frac{dy}{dt} = \pm\sqrt{E - y^2 + \frac{1}{2}y^4}, \tag{12.62}$$

with $E = v_0^2$ from the initial conditions, (12.61). If we now assume that $v_0^2 < 1/2$†, and scale y and t using

$$y = \left(1 - \sqrt{1-2E}\right)^{1/2} \hat{y}, \quad t = \left(\frac{1-\sqrt{1-2E}}{E}\right)^{1/2} \hat{t},$$

(12.62) becomes

$$\frac{d\hat{y}}{d\hat{t}} = \pm\sqrt{1-\hat{y}^2}\sqrt{1-k^2\hat{y}^2}, \tag{12.63}$$

where

$$k = \left(\frac{1-\sqrt{1-2E}}{1+\sqrt{1-2E}}\right)^{1/2}. \tag{12.64}$$

If we compare (12.63) with the system that we studied in Section 9.4, we find that

$$y = \left(1 - \sqrt{1-2E}\right)^{1/2} \mathrm{sn}\left\{\left(\frac{E}{1-\sqrt{1-2E}}\right)^{1/2} t\,;\, k\right\}. \tag{12.65}$$

In the absence of any damping ($\epsilon = 0$), y varies periodically in a manner described by the Jacobian elliptic function sn. In addition, Example 2 in the previous section shows that $y \sim v_0 \sin t$ when $v_0 \ll 1$. This is to be expected, since the nonlinear term in (12.60) is negligible when v_0, and hence y, is small.

For ϵ small and positive, but nonzero, by analogy with what we found using the method of multiple scales in the previous section, we expect that weak damping leads to a slow decrease in the amplitude of the oscillation, and possibly some change in its phase. In order to construct an asymptotic solution valid for $t = O(\epsilon^{-1})$, when the amplitude and phase of the oscillation have changed significantly, we begin in the usual way by defining a slow time scale, $T = \epsilon t$. However, for a nonlinear oscillator, the frequency of the leading order solution depends upon the amplitude of the oscillation, so it is now convenient to define

$$\psi = \frac{\theta(T)}{\epsilon} + \phi(T), \quad \theta(0) = 0,$$

and seek a solution $y \equiv y(\psi, T)$. This was first done by Kuzmak (1959), although not in full generality.

Since

$$\frac{dy}{dt} = (\theta' + \epsilon\phi')\frac{\partial y}{\partial \psi} + \epsilon\frac{\partial y}{\partial T},$$

where a prime denotes d/dT, we can see that $\theta'(T) \equiv \omega(T)$ is the frequency of the oscillation at leading order and $\phi(T)$ the change in phase, both of which we must

† The usual graphical argument shows that this is a sufficient condition for the solution to be periodic in t.

determine as part of the solution. In terms of these new variables, (12.60) and (12.61) become

$$(\omega + \epsilon\phi')^2 \frac{\partial^2 y}{\partial \psi^2} + 2\epsilon(\omega + \epsilon\phi') \frac{\partial^2 y}{\partial T \partial \psi} + \epsilon^2 \frac{\partial^2 y}{\partial T^2} + \epsilon(\omega' + \epsilon\phi'') \frac{\partial y}{\partial \psi}$$

$$+2\epsilon(\omega + \epsilon\phi') \frac{\partial y}{\partial \psi} + 2\epsilon^2 \frac{\partial y}{\partial T} + y - y^3 = 0, \tag{12.66}$$

subject to

$$y(0,0) = 0, \quad (\omega(0) + \epsilon\phi'(0)) \frac{\partial y}{\partial \psi}(0,0) + \epsilon \frac{\partial y}{\partial T}(0,0) = v_0. \tag{12.67}$$

We now expand

$$y = y_0(\psi, T) + \epsilon y_1(\psi, T) + O(\epsilon^2),$$

and substitute into (12.66) and (12.67). At leading order, we obtain

$$\omega^2(T) \frac{\partial^2 y_0}{\partial \psi^2} + y_0 - y_0^3 = 0, \tag{12.68}$$

subject to

$$y_0(0,0) = 0, \quad \omega(0) \frac{\partial y_0}{\partial \psi}(0,0) = v_0. \tag{12.69}$$

As we have seen, this has solution

$$y_0 = \left(1 - \sqrt{1 - 2E(T)}\right)^{1/2} \mathrm{sn}\left\{\left(\frac{E(T)}{1 - \sqrt{1 - 2E(T)}}\right)^{1/2} \frac{\psi}{\omega(T)}\,;\, k(T)\right\}, \tag{12.70}$$

where k is given by (12.64). The initial conditions, (12.69), show that

$$E(0) = v_0^2, \quad \phi(0) = 0. \tag{12.71}$$

The period of the oscillation, $P(T)$, can be determined by noting that the quarter period is

$$\frac{1}{4}P(T) = \omega(T) \int_0^{\left(1 - \sqrt{1 - 2E(T)}\right)^{1/2}} \frac{ds}{\sqrt{E(T) - s^2 + \frac{1}{2}s^4}},$$

which, after a simple rescaling, shows that

$$P(T) = 4\omega(T) \left(\frac{1 - \sqrt{1 - 2E(T)}}{E(T)}\right)^{1/2} K(k(E)), \tag{12.72}$$

where

$$K(k) = \int_0^1 \frac{ds}{\sqrt{1 - s^2}\sqrt{1 - k^2 s^2}} \tag{12.73}$$

is the **complete elliptic integral of the first kind**.

As in the method of multiple scales, we need to go to $O(\epsilon)$ to determine the behaviour of the solution on the slow time scale, T. Firstly, note that we have three unknowns to determine, $E(T)$, $\phi(T)$ and $\omega(T)$, whilst for the method of multiple scales, we had just two, equivalent to $E(T)$ and $\phi(T)$. Since we introduced $\theta(T)$ simply to account for the fact that the period of the oscillation changes with the amplitude, we have the freedom to choose this new time scale so that the period of the oscillation is constant. For convenience, we take $P = 1$, so that

$$\frac{d\theta}{dT} = \omega \equiv \omega(E) = \frac{1}{4K(k(E))}\left(\frac{E}{1-\sqrt{1-2E}}\right)^{1/2}. \tag{12.74}$$

We also need to note for future reference the parity with respect to ψ of y_0 and its derivatives. Both y_0 and $\partial^2 y_0/\partial\psi^2$ are odd functions, whilst $\partial y_0/\partial\psi$ is an even function of ψ. In addition, we can now treat y_0 as a function of ψ and E, with

$$y_0(\psi, E) = \left(1-\sqrt{1-2E}\right)^{1/2} \operatorname{sn}\left\{4K(k(E))\psi\,;\, k(E)\right\}. \tag{12.75}$$

At $O(\epsilon)$ we obtain

$$\omega^2\frac{\partial^2 y_1}{\partial\psi^2} + \left(1-3y_0^2\right)y_1 = -2\omega\phi'\frac{\partial^2 y_0}{\partial\psi^2} - 2\omega\frac{\partial^2 y_0}{\partial\psi\partial E}\frac{dE}{dT} - \omega'\frac{\partial y_0}{\partial\psi} - 2\omega\frac{\partial y_0}{\partial\psi},$$

which we write as

$$L(y_1) = R_{1\mathrm{odd}} + R_{1\mathrm{even}}, \tag{12.76}$$

where

$$L = \omega^2\frac{\partial^2}{\partial\psi^2} + 1 - 3y_0^2, \tag{12.77}$$

and

$$R_{1\mathrm{odd}} = -2\omega\phi'\frac{\partial^2 y_0}{\partial\psi^2}, \quad R_{1\mathrm{even}} = -2\omega\frac{\partial^2 y_0}{\partial\psi\partial E}\frac{dE}{dT} - \omega'\frac{\partial y_0}{\partial\psi} - 2\omega\frac{\partial y_0}{\partial\psi}. \tag{12.78}$$

Now, by differentiating (12.68) with respect to ψ, we find that

$$L\left(\frac{\partial y_0}{\partial\psi}\right) = 0,$$

so that $\partial y_0/\partial\psi$ is a solution of the homogeneous version of (12.76). For the solution of (12.76) to be periodic, the right hand side must be orthogonal to the solution of the homogeneous equation, and therefore orthogonal to $\partial y_0/\partial\psi$.† This is equivalent to the elimination of secular terms in the method of multiple scales. Since $\partial y_0/\partial\psi$ is even in ψ, this means that

$$\int_0^1 \frac{\partial y_0}{\partial\psi} R_{1\mathrm{even}}\, d\psi = 0,$$

which is the equivalent of the secularity condition that determines the amplitude

† Strictly speaking, this is a result that arises from **Floquet theory**, the theory of linear ordinary differential equations with periodic coefficients.

of the oscillation in the method of multiple scales. Using (12.78), we can write this as†

$$\frac{d}{dE}\left\{\omega\int_0^1\left(\frac{\partial y_0}{\partial\psi}\right)^2 d\psi\right\}\frac{dE}{dT}+2\omega\int_0^1\left(\frac{\partial y_0}{\partial\psi}\right)^2 d\psi=0. \tag{12.79}$$

To proceed, we firstly note that

$$\int_0^1\left(\frac{\partial y_0}{\partial\psi}\right)^2 d\psi=4\int_0^{\left(1-\sqrt{1-2E}\right)^{1/2}}\frac{\partial y_0}{\partial\psi}dy_0$$

$$=\frac{4}{\omega}\int_0^{\left(1-\sqrt{1-2E}\right)^{1/2}}\sqrt{E-y_0^2+\frac{1}{2}y_0^4}\,dy_0,$$

and hence that

$$\frac{\partial}{\partial E}\left\{\omega\int_0^1\left(\frac{\partial y_0}{\partial\psi}\right)^2 d\psi\right\}=2\int_0^{\left(1-\sqrt{1-2E}\right)^{1/2}}\frac{dy_0}{\sqrt{E-y_0^2+\frac{1}{2}y_0^4}}=\frac{1}{2\omega}. \tag{12.80}$$

Secondly, using the results of Section 9.4, we find that

$$\int_0^1\left(\frac{\partial y_0}{\partial\psi}\right)^2 d\psi$$

$$=16K^2(E)\left(1-\sqrt{1-2E}\right)\int_0^1\left\{1-\mathrm{sn}^2\left(4K\psi\,;\,k\right)\right\}\left\{1-k^2\mathrm{sn}^2\left(4K\psi\,;\,k\right)\right\}d\psi$$

$$=16K(E)\left(1-\sqrt{1-2E}\right)\int_0^K\left\{1-\mathrm{sn}^2\left(\psi\,;\,k\right)\right\}\left\{1-k^2\mathrm{sn}^2\left(\psi\,;\,k\right)\right\}d\psi.$$

We now need a standard result on elliptic functions‡, namely that

$$\int_0^K\left\{1-\mathrm{sn}^2\left(\psi\,;\,k\right)\right\}\left\{1-k^2\mathrm{sn}^2\left(\psi\,;\,k\right)\right\}d\psi$$

$$=\frac{1}{3k^2}\left\{\left(1+k^2\right)L(k)-\left(1-k^2\right)K(k)\right\},$$

where

$$L(k)=\int_0^1\sqrt{\frac{1-k^2s^2}{1-s^2}}ds,$$

is the **complete elliptic integral of the second kind**§. Equation (12.79) and the definition of ω, (12.74), then show that

$$\frac{dE}{dT}=-\frac{4E}{3k^2K}\left\{\left(1+k^2\right)L\left(k\right)-\left(1-k^2\right)K(k)\right\}. \tag{12.81}$$

† Note that the quantity in the curly brackets in (12.79) is often referred to as the **action**.
‡ See, for example, Byrd (1971).
§ Although the usual notation for the complete elliptic integral of the second kind is $E(k)$, the symbol E is already spoken for in this analysis.

This equation, along with the initial condition, (12.71), determines $E(T)$. For $v_0 \ll 1$, this reduces to the multiple scales result, $dE/dT = -2E$ (see Exercise 12.10). It is straightforward to integrate (12.81) using MATLAB, since the complete elliptic integrals of the first and second kinds can be calculated using the built-in function `ellipke`. Note that we can simultaneously calculate $\theta(T)$ numerically by solving (12.74) subject to $\theta(0) = 0$. Figure 12.13 shows the function $E(T)$ calculated for $E(0) = 0.45$, and also the corresponding result using multiple scales on the linearized problem, $E = E(0)e^{-2T}$.

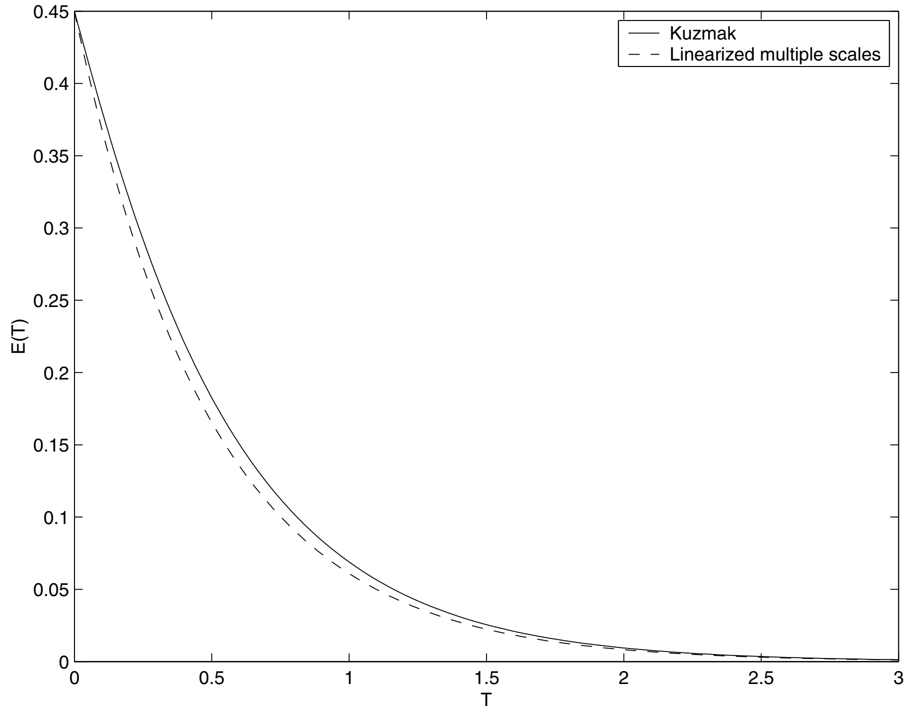

Fig. 12.13. The solution of (12.81) when $E(0) = 0.45$, and the corresponding multiple scales solution of the linearized problem, $E(T) = E(0)e^{-2T}$.

We now need to find an equation that determines the phase, $\phi(T)$. Unfortunately, unlike the method of multiple scales, we need to determine the solution at $O(\epsilon)$ in order to make progress. By differentiating (12.68) with respect to E, we find that

$$L\left(\frac{\partial y_0}{\partial E}\right) = -2\omega\frac{d\omega}{dE}\frac{\partial^2 y_0}{\partial \psi^2}.$$

We also note that

$$L\left(\psi\frac{\partial y_0}{\partial \psi}\right) = 2\omega^2\frac{\partial^2 y_0}{\partial \psi^2}, \tag{12.82}$$

and hence that

$$L\left(y_{0\text{odd}}\right) = 0,$$

where

$$y_{0\text{odd}} = \omega \frac{\partial y_0}{\partial E} + \frac{d\omega}{dE}\psi \frac{\partial y_0}{\partial \psi}. \tag{12.83}$$

The general solution of the homogeneous version of (12.76) is therefore a linear combination of the even function $\partial y_0/\partial\psi$ and the odd function $y_{0\text{odd}}$. To find the particular integral solution of (12.76), we note from (12.82) that

$$L\left(-\frac{\phi'}{\omega}\psi\frac{\partial y_0}{\partial \psi}\right) = -2\omega\phi'\frac{\partial^2 y_0}{\partial \psi^2} = R_{1\text{odd}}.$$

We therefore have

$$y_1 = A(T)\frac{\partial y_0}{\partial \psi} + B(T)\left(\omega\frac{\partial y_0}{\partial E} + \frac{d\omega}{dE}\psi\frac{\partial y_0}{\partial \psi}\right) - \frac{\phi'}{\omega}\psi\frac{\partial y_0}{\partial \psi} + y_{1\text{even}}, \tag{12.84}$$

where $y_{1\text{even}}$ is the part of the particular integral generated by $R_{1\text{even}}$, and is itself even in ψ. For y_1 to be bounded as $\psi \to \infty$, the coefficient of ψ must be zero. Noting from (12.75) that

$$\frac{\partial y_0}{\partial E} \sim 4\frac{dK}{dE}\psi\frac{\partial y_0}{\partial \psi} \quad \text{as } \psi \to \infty,$$

this means that

$$B(T) = \frac{\phi'/\omega}{4\omega dK/dE + d\omega/dE}. \tag{12.85}$$

The easiest way to proceed is to multiply (12.66) by $\partial y/\partial\psi$, and integrate over one period of the oscillation. After taking into account the parity of the components of each integrand, we find that

$$\frac{d}{dT}\left\{e^{2T}\left(\omega + \epsilon\phi'\right)\int_0^1 \left(\frac{\partial y}{\partial \psi}\right)^2 d\psi\right\} = 0.$$

At leading order, this reproduces (12.79), whilst at $O(\epsilon)$ we find that

$$\frac{d}{dT}\left\{e^{2T}\left(2\omega\int_0^1 \frac{\partial y_0}{\partial \psi}\frac{\partial y_1}{\partial \psi}d\psi + \phi'\int_0^1\left(\frac{\partial y_0}{\partial \psi}\right)^2 d\psi\right)\right\} = 0. \tag{12.86}$$

Now, using what we know about the parity of the various components of y_1, we can show that

$$\int_0^1 \frac{\partial y_0}{\partial \psi}\frac{\partial y_1}{\partial \psi}d\psi = \frac{\phi'}{d\omega/dE}\int_0^1 \frac{\partial y_0}{\partial \psi}\frac{\partial^2 y_0}{\partial \psi \partial E}d\psi = \frac{\phi'}{2d\omega/dE}\frac{\partial}{\partial E}\int_0^1\left(\frac{\partial y_0}{\partial \psi}\right)^2 d\psi,$$

and hence from (12.86) that

$$\frac{d}{dT}\left[e^{2T}\frac{\phi'}{d\omega/dE}\frac{\partial}{\partial E}\left\{\omega\int_0^1\left(\frac{\partial y_0}{\partial \psi}\right)^2 d\psi\right\}\right] = 0.$$

Using (12.80), this becomes

$$\frac{d}{dT}\left(\frac{e^{2T}\phi'}{\omega d\omega/dE}\right) = 0. \tag{12.87}$$

This is a second order equation for $\phi(T)$. Although we know that $\phi(0) = 0$, to be able to solve (12.87) we also need to know $\phi'(0)$.

At $O(\epsilon)$, the initial conditions, (12.67), are

$$y_1(0,0) = 0, \quad \omega(E(0))\frac{\partial y_1}{\partial \psi}(0,0) = -E'(0)\frac{\partial y_0}{\partial E}(0,E(0)) - \phi'(0)\frac{\partial y_0}{\partial \psi}(0,E(0)). \tag{12.88}$$

By substituting the solution (12.84) for y_1 into the second of these, we find that

$$\omega(E(0))\left\{A(0)\frac{\partial^2 y_0}{\partial \psi^2}(0,E(0)) + B(0)\omega(E(0))\frac{\partial^2 y_0}{\partial \psi \partial E}(0,E(0))\right.$$

$$\left. + \left(B(0)\frac{d\omega}{dE}(E(0)) - \frac{\phi'(0)}{\omega(E(0))}\right)\frac{\partial y_0}{\partial \psi}(0,E(0)) + \frac{\partial y_{1\text{even}}}{\partial \psi}(0,0)\right\}$$

$$= -E'(0)\frac{\partial y_0}{\partial E}(0,E(0)) - \phi'(0)\frac{\partial y_0}{\partial \psi}(0,E(0)).$$

Using (12.85), all of the terms that do not involve $\phi'(0)$ are odd in ψ, and therefore vanish when $\psi = 0$. We conclude that $\phi'(0) = 0$, and hence from (12.87) that $\phi(T) = 0$.

Figure 12.14 shows a comparison between the leading order solution computed using Kuzmak's method, the leading order multiple scales solution of the linearized problem, $y = \sqrt{v_0}e^{-T}\sin t$, and the numerical solution of the full problem when $\epsilon = 0.01$. The numerical and Kuzmak solutions are indistinguishable. Although the multiple scales solution gives a good estimate of $E(T)$, as shown in figure 12.13, it does not give an accurate solution of the full problem.

To see how the method works in general for damped nonlinear oscillators, the interested reader is referred to Bourland and Haberman (1988).

12.2.6 The Effect of Fine Scale Structure on Reaction–Diffusion Processes

Consider the two-point boundary value problem

$$\frac{d}{dx}\left\{D\left(x,\frac{x}{\epsilon}\right)\frac{d\theta}{dx}\right\} + R\left(\theta, x, \frac{x}{\epsilon}\right) = 0 \quad \text{for } 0 < x < 1, \tag{12.89}$$

subject to the boundary conditions

$$\frac{d\theta}{dx} = 0 \quad \text{at } x = 0 \text{ and } x = 1. \tag{12.90}$$

We can think of $\theta(x)$ as a dimensionless temperature, so that this boundary value problem models the steady distribution of heat in a conducting material. When $\epsilon \ll 1$, this material has a fine scale structure varying on a length scale of $O(\epsilon)$, and

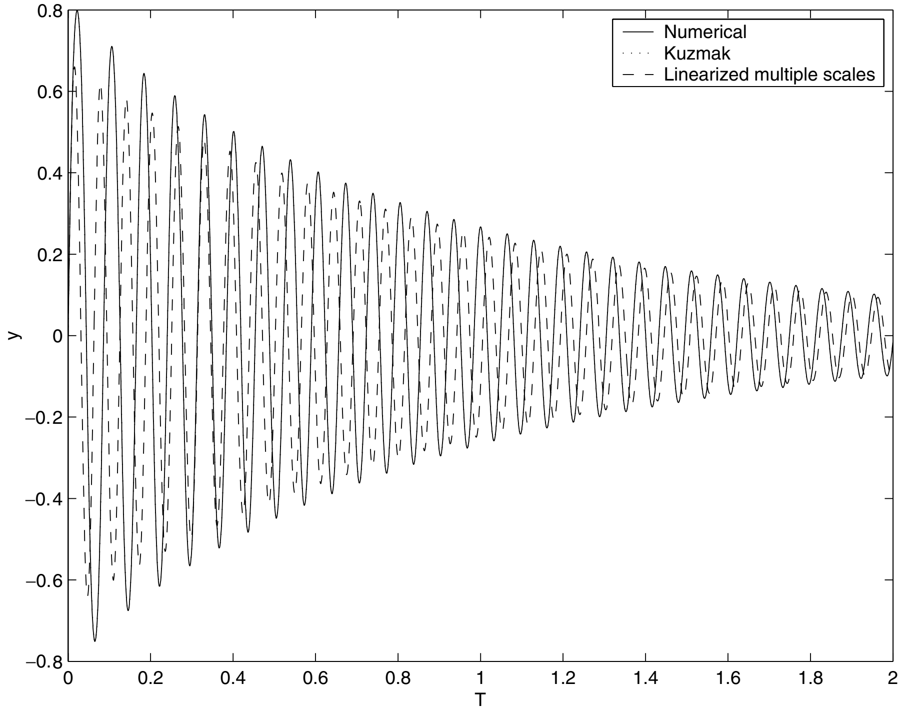

Fig. 12.14. The numerical solution of (12.60) subject to (12.61) when $v_0^2 = 0.45$ and $\epsilon = 0.01$ compared to asymptotic solutions calculated using the leading order Kuzmak solution and the leading order multiple scales solution of the linearized problem.

a coarse, background structure varying on a length scale of $O(1)$. The steady state temperature distribution represents a balance between the diffusion of heat, $(D\theta_x)_x$ (where subscripts denote derivatives), and its production by some chemical reaction, R (see Section 12.2.3, Example 2 for a derivation of this type of balance law). If we integrate (12.89) over $0 \leqslant x \leqslant 1$ and apply the boundary conditions (12.90), we find that the chemical reaction term must satisfy the solvability condition

$$\int_0^1 R\left(\theta(x), x, \frac{x}{\epsilon}\right)\, dx = 0 \tag{12.91}$$

for a solution to exist. Physically, since (12.90) states that no heat can escape from the ends of the material, (12.91) simply says that the overall rate at which heat is produced must be zero for a steady state to be possible, with sources of heat in some regions of the material balanced by heat sinks in other regions.

In order to use asymptotic methods to determine the solution at leading order when $\epsilon \ll 1$, we begin by introducing the **fast variable**, $\hat{x} = x/\epsilon$. In the usual way (see Section 12.2.4), (12.89) and (12.90) become

$$(D\theta_{\hat{x}})_{\hat{x}} + \epsilon \left\{ (D\theta_{\hat{x}})_x + (D\theta_x)_{\hat{x}} \right\} + \epsilon^2 \left\{ (D\theta_x)_x + R \right\} = 0, \tag{12.92}$$

subject to

$$\theta_{\hat{x}} + \epsilon\theta_x = 0 \quad \text{at } x = \hat{x} = 0 \text{ and } x = 1,\ \hat{x} = 1/\epsilon, \tag{12.93}$$

with $R \equiv R(\theta(x,\hat{x}), x, \hat{x})$, $D \equiv D(x,\hat{x})$. It is quite awkward to apply a boundary condition at $\hat{x} = 1/\epsilon \gg 1$, but we shall see that we can determine the equation satisfied by θ at leading order without using this, and we will not consider it below. We now expand θ as

$$\theta = \theta_0(x,\hat{x}) + \epsilon\theta_1(x,\hat{x}) + \epsilon^2\theta_2(x,\hat{x}) + O(\epsilon^3).$$

At leading order,

$$(D\theta_{0\hat{x}})_{\hat{x}} = 0,$$

subject to

$$\theta_{0\hat{x}} = 0 \quad \text{at } x = \hat{x} = 0.$$

We can integrate this once to obtain $D\theta_{0\hat{x}} = \alpha(x)$, with $\alpha(0) = 0$, and then once more, which gives

$$\theta_0 = \alpha(x)\int_0^{\hat{x}} \frac{ds}{D(x,s)} + f_0(x).$$

At $O(\epsilon)$, we find that

$$(D\theta_{1\hat{x}})_{\hat{x}} = -(D\theta_{0\hat{x}})_x - (D\theta_{0x})_{\hat{x}},$$

which we can write as

$$(D\theta_{1\hat{x}} + D\theta_{0x})_{\hat{x}} = -\alpha'(x). \tag{12.94}$$

We can integrate (12.94) to obtain

$$D\theta_{1\hat{x}} + D\theta_{0x} = -\alpha'(x)\hat{x} + \beta(x).$$

Substituting for θ_0 and integrating again shows that

$$\theta_1 = f_1(x) - f_0'(x)\hat{x} - \alpha'(x)\hat{x}\int_0^{\hat{x}} \frac{ds}{D(x,s)}$$

$$-\alpha(x)\frac{\partial}{\partial x}\int_0^{\hat{x}} \frac{\hat{x}-s}{D(x,s)}ds + \beta(x)\int_0^{\hat{x}} \frac{ds}{D(x,s)}. \tag{12.95}$$

When $\hat{x}$ is large, there are terms in this expression that are of $O(\hat{x}^2)$. These are secular, and become comparable with the leading order term in the expansion for θ when $\hat{x} = 1/\epsilon$. We must eliminate this secularity by taking

$$\lim_{\epsilon\to 0}\left\{ \alpha'(x)\epsilon\int_0^{1/\epsilon} \frac{ds}{D(x,s)} - \alpha(x)\epsilon^2\frac{\partial}{\partial x}\int_0^{1/\epsilon} \frac{1/\epsilon - s}{D(x,s)}ds \right\} = 0. \tag{12.96}$$

This is a first order ordinary differential equation for $\alpha(x)$. Since $\alpha(0) = 0$, we conclude that $\alpha \equiv 0$. Note that each of the terms in (12.96) is of $O(1)$ for $\epsilon \ll 1$. For example, $\epsilon \int_0^{1/\epsilon} \frac{ds}{D(x,s)}$ is the mean value of $1/D$ over the fine spatial scale. We now have simply $\theta_0 = f_0(x)$. This means that, at leading order, θ varies only on the coarse, $O(1)$ length scale. This does not mean that the fine scale structure has no effect, as we shall see.

Since we now know that $\theta_0 = O(1)$ when $\hat{x} = 1/\epsilon$, we must also eliminate the secular terms that are of $O(\hat{x})$ when $\hat{x}$ is large in (12.95). We therefore require that

$$f_0' = \frac{d\theta_0}{dx} = \lim_{\epsilon \to 0} \left\{ \epsilon \beta(x) \int_0^{1/\epsilon} \frac{ds}{D(x,s)} \right\}. \tag{12.97}$$

In order to determine β, and hence an equation satisfied by θ_0, we must consider one further order in the expansion.

At $O(\epsilon^2)$, (12.92) gives

$$\begin{aligned}(D\theta_{2\hat{x}} + D\theta_{1x})_{\hat{x}} &= -\left(D\theta_{1\hat{x}}\right)_x - \left(D\theta_{0x}\right)_x - R(\theta_0(x), x, \hat{x}) \\ &= -\beta'(x) - R(\theta_0(x), x, \hat{x}),\end{aligned} \tag{12.98}$$

which we can integrate twice to obtain

$$\begin{aligned}\theta_2 = f_2(x) - f_1'(x)\hat{x} + \frac{1}{2}\frac{d^2\theta_0}{dx^2}\hat{x}^2 - \frac{\partial}{\partial x}\left\{ \beta(x) \int_0^{\hat{x}} \frac{\hat{x} - s}{D(x,s)} ds \right\} + \gamma(x) \int_0^{\hat{x}} \frac{ds}{D(x,s)} \\ -\beta'(x) \int_0^{\hat{x}} \frac{s}{D(x,s)} ds - \int_0^{\hat{x}} \frac{1}{D(x,s)} \int_0^s R(\theta_0(x), x, u)\, du\, ds.\end{aligned} \tag{12.99}$$

In order to eliminate the secular terms of $O(\hat{x}^2)$, we therefore require that

$$\begin{aligned}\lim_{\epsilon \to 0} \left[\frac{1}{2}\frac{d^2\theta_0}{dx^2} - \epsilon^2 \frac{\partial}{\partial x}\left\{ \beta(x) \int_0^{1/\epsilon} \frac{1/\epsilon - s}{D(x,s)} ds \right\} \right. \\ \left. -\beta'(x)\epsilon^2 \int_0^{1/\epsilon} \frac{s}{D(x,s)} ds - \epsilon^2 \int_0^{1/\epsilon} \frac{1}{D(x,s)} \int_0^s R(\theta_0(x), x, u)\, du\, ds \right] = 0.\end{aligned}$$

If we now use (12.97) to eliminate $\beta(x)$, we arrive at

$$\begin{aligned}\lim_{\epsilon \to 0} \left\{ \frac{d^2\theta_0}{dx^2} - \frac{2\frac{d}{dx}\left(\epsilon^2 \int_0^{1/\epsilon} \frac{s}{D(x,s)} ds\right)}{\epsilon \int_0^{1/\epsilon} \frac{ds}{D(x,s)}} \frac{d\theta_0}{dx} \right. \\ \left. +2\epsilon^2 \int_0^{1/\epsilon} \frac{1}{D(x,s)} \int_0^s R(\theta_0(x), x, u)\, du\, ds \right\} = 0.\end{aligned}$$

After multiplying through by a suitable integrating factor, we can see that θ_0 satisfies the reaction–diffusion equation

$$\left\{\bar{D}(x)\theta_{0x}\right\}_x + \bar{R}(\theta_0(x), x) = 0, \tag{12.100}$$

on the coarse scale, where

$$\bar{D}(x) = \lim_{\epsilon \to 0} \left[\exp \left\{ -2 \int_0^x \frac{\frac{d}{dX}\left(\epsilon^2 \int_0^{1/\epsilon} \frac{s}{D(X,s)} ds\right)}{\epsilon \int_0^{1/\epsilon} \frac{ds}{D(X,s)}} \, dX \right\} \right], \tag{12.101}$$

$$\bar{R}(\theta_0(x), x) = \lim_{\epsilon \to 0} \left\{ 2\bar{D}(x)\epsilon^2 \int_0^{1/\epsilon} \frac{1}{D(x,s)} \int_0^s R(\theta_0(x), x, u) \, du \, ds \right\} \tag{12.102}$$

are the **fine scale averages**.

This asymptotic analysis, which is called **homogenization**, shows that the leading order temperature does not vary on the fine scale, and satisfies a standard reaction–diffusion equation on the coarse scale. However, the fine scale structure modifies the reaction term and diffusion coefficient through (12.101) and (12.102), with $\bar{D}$ the **homogenized diffusion coefficient** and $\bar{R}$ the **homogenized reaction term**. If we were to seek higher order corrections, we would find that there are variations in the temperature on the fine scale, but that these are at most of $O(\epsilon)$.

One case where $\bar{D}$ and $\bar{R}$ take a particularly simple form is when $D(x, \hat{x}) = D_0(x)\hat{D}(\hat{x})$ and $R(\theta(x), x, \hat{x}) = R_0(\theta(x), x)\hat{R}(\hat{x})$. On substituting these into (12.101) and (12.102), we find, after cancelling a constant common factor between $\bar{D}$ and $\bar{R}$, that we can use $\bar{D}(x) = D_0(x)$ and $\bar{R}(\theta, x) = KR_0(\theta, x)$, where

$$K = \lim_{\epsilon \to 0} \left\{ 2\epsilon \int_0^{1/\epsilon} \frac{1}{\hat{D}(s)} \int_0^s \hat{R}(u) \, du \, ds \right\}.$$

The homogenized diffusion coefficient and reaction term are simply given by the terms $D_0(x)$ and $R_0(\theta, x)$, modified by a measure of the mean value of the ratio of the fine scale variation of each, given by the constant K. In particular, when $\hat{D}(\hat{x}) = 1/(1 + A_1 \sin k_1\hat{x})$ and $\hat{R}(\hat{x}) = 1 + A_2 \sin k_2\hat{x}$ for some positive constants k_1, k_2, A_1 and A_2, with $A_1 < 1$ and $A_2 < 1$, we find that $K = 1$. We can illustrate this for a simple case where it is straightforward to find the exact solution of both (12.89) and (12.100) analytically. Figure 12.15 shows a comparison between the exact and asymptotic solutions for various values of ϵ when $k_1 = k_2 = 1$, $A_1 = A_2 = 1/2$, $D_0(x) = 1/(1 + x)$ and $R_0 = 2x - 1$. The analytical solution of (12.89) that vanishes at $x = 0$ (the solution would otherwise contain an arbitrary constant in this case) is

$$\theta = \frac{1}{2}x^2 - \frac{1}{4}x^4 + \epsilon^2 \left[\frac{1}{2}(1 - 2x) \left\{ \frac{1}{8} \cos\left(\frac{2x}{\epsilon}\right) - \sin\left(\frac{x}{\epsilon}\right) \right\} - \frac{1}{4}x - \frac{1}{16} \right]$$
$$+\epsilon^3 \left[2\cos\left(\frac{x}{\epsilon}\right) + \frac{3}{16}\sin\left(\frac{2x}{\epsilon}\right) - 2 \right],$$

whilst the corresponding solution of (12.100) correctly reproduces the leading order part of this.

Homogenization has been used successfully in many more challenging applications than this linear, steady state reaction diffusion equation, for example,

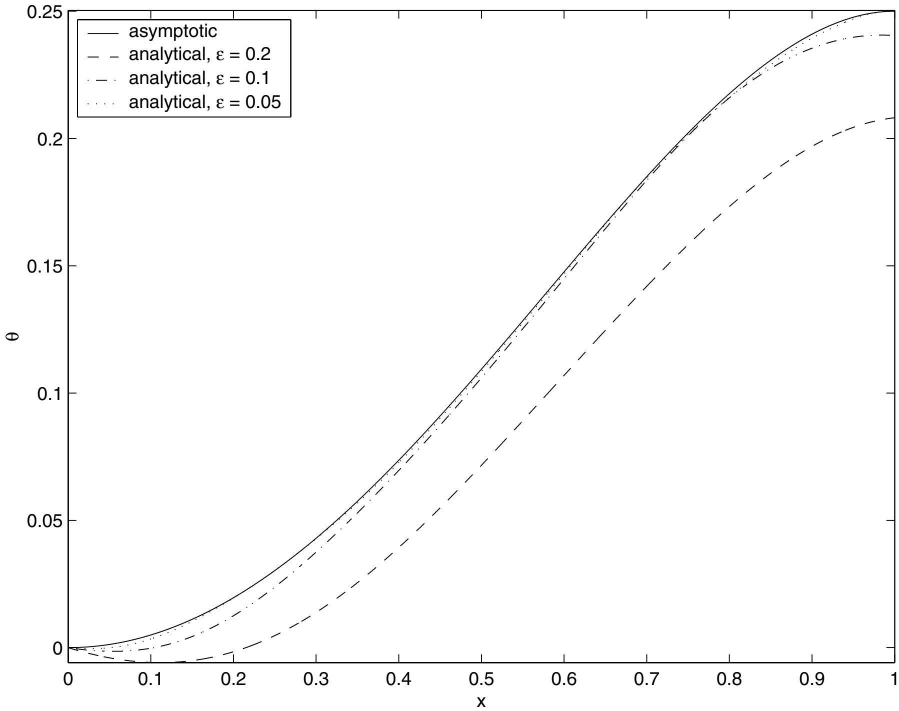

Fig. 12.15. Analytical and asymptotic solutions of (12.89) when $D = 1/(1+x)\left(1+\frac{1}{2}\sin x\right)$ and $R = (2x-1)\left(1+\frac{1}{2}\sin x\right)$.

in assessing the strength of elastic media with small carbon fibre reinforcements (Bakhvalov and Panasenko, 1989).

12.2.7 The WKB Approximation

In all of the examples that we have seen so far, we have used an expansion in algebraic powers of a small parameter to develop perturbation series solutions of differential or algebraic equations. This procedure has been at its most complex when we have needed to match a slowly varying outer solution with an exponentially-rapidly varying, or dissipative, boundary layer. This procedure doesn't always work! For example, if we consider the two-point boundary value problem

$$\epsilon^2 y''(x) + \phi(x)y(x) = 0 \text{ subject to } y(0) = 0 \text{ and } y(1) = 1, \tag{12.103}$$

the procedure fails as there are no terms to balance with the leading order term, $\phi(x)y(x)$. If there were a first derivative term in this problem, the procedure would work, although we would have a singular perturbation problem. However, a first derivative term can always be removed. Suppose that we have a differential equation of the form

$$w'' + p(x)w' + q(x)w = 0.$$

By writing $w = Wu$, we can easily show that

$$W'' + \left(\frac{2u'}{u} + p\right) W' + \left(\frac{u''}{u} + p\frac{u'}{u} + q\right) W = 0.$$

By choosing $\frac{2u'}{u} + p = 0$ and hence $u = \exp\left\{-\frac{1}{2}\int^x p(t)\,dt\right\}$, we can remove the first derivative term. Because of this, there is a sense in which $\epsilon^2 y'' + \phi y = 0$ is a generic second order ordinary differential equation, and we need to develop a perturbation method that can deal with it.

The Basic Expansion

A suitable method was proposed by Wentzel, Kramers and Brillioun (and perhaps others as well) in the 1920s. The appropriate asymptotic development is of the form

$$y = \exp\left\{\frac{\psi_0(x)}{\epsilon} + \psi_1(x) + O(\epsilon)\right\}.$$

Differentiating this gives

$$y' = \exp\left\{\frac{\psi_0}{\epsilon} + \psi_1\right\}\left\{\frac{\psi_0'}{\epsilon} + \psi_1' + O(\epsilon)\right\},$$

and

$$y'' = \exp\left\{\frac{\psi_0}{\epsilon} + \psi_1 + O(\epsilon)\right\}\left\{\frac{\psi_0''}{\epsilon} + \psi_1''(x) + O(\epsilon)\right\}$$

$$+ \exp\left\{\frac{\psi_0}{\epsilon} + \psi_1 + O(\epsilon)\right\}\left\{\frac{\psi_0'}{\epsilon} + \psi_1' + O(\epsilon)\right\}^2$$

$$= \exp\left\{\frac{\psi_0}{\epsilon} + \psi_1\right\}\left\{\frac{(\psi_0')^2}{\epsilon^2} + \frac{1}{\epsilon}\left(2\psi_0'\psi_1' + \psi_0''\right) + O(1)\right\}.$$

If we substitute these into (12.103), we obtain

$$(\psi_0')^2 + \epsilon\left(2\psi_0'\psi_1' + \psi_0''\right) + \phi(x) + O(\epsilon^2) = 0,$$

and hence

$$(\psi_0')^2 = -\phi(x), \quad \psi_1' = -\frac{\psi_0''}{2\psi_0'} = -\frac{1}{2}\frac{d}{dx}\left(\log \psi_0'\right). \tag{12.104}$$

If $\phi(x) > 0$, say for $x > 0$, we can simply integrate these equations to obtain

$$\psi_0 = \pm i\int^x \phi^{1/2}(t)\,dt + \text{constant}, \quad \psi_1 = -\frac{1}{4}\log\left(\phi(x)\right) + \text{constant}.$$

All of these constants of integration can be absorbed into a single constant, which may depend upon ϵ, in front of the exponential. The leading order solution is rapidly oscillating, or **dispersive**, in this case, and can be written as

$$y = \frac{A(\epsilon)}{\phi^{1/4}(x)} \exp\left\{ i \frac{\int^x \phi^{1/2}(t)\,dt}{\epsilon} \right\} + \frac{B(\epsilon)}{\phi^{1/4}(x)} \exp\left\{ -i \frac{\int^x \phi^{1/2}(t)\,dt}{\epsilon} \right\} + \cdots . \tag{12.105}$$

Note that this asymptotic development has assumed that $x = O(1)$, and, in order that it remains uniform, we must have $\psi_1 \ll \psi_0/\epsilon$.

If $\phi(x) < 0$, say for $x < 0$, then some minor modifications give an exponential, or **dissipative**, solution of the form,

$$y = \frac{C(\epsilon)}{|\phi(x)|^{1/4}} \exp\left\{ \frac{\int^x |\phi(t)|^{1/2}\,dt}{\epsilon} \right\} + \frac{D(\epsilon)}{|\phi(x)|^{1/4}} \exp\left\{ -\frac{\int^x |\phi(t)|^{1/2}\,dt}{\epsilon} \right\} + \cdots . \tag{12.106}$$

If $\phi(x)$ is of one sign in the domain of solution, one or other of the expansions (12.105) and (12.106) will be appropriate. If, however, $\phi(x)$ changes sign, we will have an oscillatory solution on one side of the zero and an exponentially growing or decaying solution on the other. We will consider how to deal with this combination of dispersive and dissipative behaviour after studying a couple of examples.

Example 1: Bessel functions for $x \gg 1$

We saw in Chapter 3 that Bessel's equation is

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right) y = 0.$$

If we make the transformation $y = x^{-1/2}Y$, we obtain the generic form of the equation,

$$Y'' + \left(1 + \frac{\frac{1}{2} - \nu^2}{x^2}\right) Y = 0. \tag{12.107}$$

Although this equation currently contains no small parameter, we can introduce one in a useful way by defining $\bar{x} = \delta x$. If x is large and positive, we can have $\bar{x} = O(1)$ by choosing δ sufficiently small. We have introduced this **artificial small parameter** as a device to help us determine how the Bessel function behaves for $x \gg 1$, and it *cannot* appear in the final result when we change variables back from $\bar{x}$ to x.

In terms of $\bar{x}$ and $\bar{Y}(\bar{x}) = Y(x)$, (12.107) becomes

$$\delta^2 \bar{Y}'' + \left(1 + \delta^2 \frac{\frac{1}{2} - \nu^2}{\bar{x}^2}\right) \bar{Y} = 0. \tag{12.108}$$

By direct comparison with the derivation of the WKB expansion above, in which we neglected terms of $O(\delta^2)$,

$$\psi_0 = \pm i \int^{\bar{x}} dt = \pm i\bar{x}, \quad \psi_1 = -\frac{1}{4}\log 1 = 0,$$

so that

$$\bar{Y} \sim A(\delta)\exp\left(\frac{i\bar{x}}{\delta}\right) + B(\delta)\exp\left(-\frac{i\bar{x}}{\delta}\right) + \cdots,$$

and hence

$$y \sim \frac{1}{x^{1/2}}\left(Ae^{ix} + Be^{-ix}\right)$$

is the required expansion. Note that, although we can clearly see that the Bessel functions are slowly decaying, oscillatory functions of x as $x \to \infty$, we cannot determine the constants A and B using this technique. As we have already seen, it is more appropriate to use the integral representation (11.14).

We can show that the WKB method is *not* restricted to ordinary differential equations with terms in just y'' and y by considering a further example.

Example 2: A boundary layer

Let's try to find a uniformly valid approximation to the solution of the two-point boundary value problem

$$\epsilon y'' + p(x)y' + q(x)y = 0 \text{ subject to } y(0) = \alpha,\ y(1) = \beta \tag{12.109}$$

when $\epsilon \ll 1$ and $p(x) > 0$. If we assume a WKB expansion,

$$y = \exp\left\{\frac{\psi_0(x)}{\epsilon} + \psi_1(x) + O(\epsilon)\right\},$$

and substitute into (12.109), we obtain at $O(1/\epsilon)$

$$\psi_0'\left\{\psi_0' + p(x)\right\} = 0, \tag{12.110}$$

and at $O(1)$,

$$2\psi_0'\psi_1' + \psi_0'' + p(x)\psi_1' + q(x) = 0. \tag{12.111}$$

Using the solution $\psi_0' = 0$ of (12.110) and substituting into (12.111) gives $\psi_1' = -q(x)/p(x)$, which generates a solution of the form

$$y_1 = \exp\left\{\frac{c_1}{\epsilon} - \int_0^x \frac{q(t)}{p(t)} + \cdots\right\} \sim C_1(\epsilon)\exp\left\{-\int_0^x \frac{q(t)}{p(t)}\right\}. \tag{12.112}$$

The second solution of (12.110) has $\psi_0' = -p(x)$ and hence

$$\psi_0 = -\int^x p(t)\,dt + c_2.$$

Equation (12.111) then gives

$$\psi_1 = -\log p(x) + \int_0^x \frac{q(t)}{p(t)}\,dt.$$

The second independent solution therefore takes the form

$$y_2 = \exp\left\{\frac{c_2 - \int_0^x p(t)\,dt}{\epsilon} - \log p(x) + \int_0^x \frac{q(t)}{p(t)}\,dt + \cdots\right\}$$

$$\sim \frac{C_2}{p(x)} \exp\left\{-\frac{1}{\epsilon}\int_0^x p(t)\,dt + \int_0^x \frac{q(t)}{p(t)}\,dt\right\}. \qquad (12.113)$$

Combining (12.112) and (12.113) then gives the general asymptotic solution as

$$y = C_1 \exp\left\{-\int_0^x \frac{q(t)}{p(t)}\right\} + \frac{C_2}{p(x)} \exp\left\{-\frac{1}{\epsilon}\int_0^x p(t)\,dt + \int_0^x \frac{q(t)}{p(t)}\,dt\right\}. \qquad (12.114)$$

We can now apply the boundary conditions in (12.109) to obtain

$$\alpha = C_1 + \frac{C_2}{p(0)}, \quad \beta = C_1 \exp\left\{-\int_0^1 \frac{q(t)}{p(t)}\right\} + \frac{C_2}{p(1)} \exp\left\{-\frac{1}{\epsilon}\int_0^1 p(t)\,dt + \int_0^1 \frac{q(t)}{p(t)}\,dt\right\}.$$

The term $\exp\left\{-\frac{1}{\epsilon}\int_0^1 p(t)\,dt\right\}$ is uniformly small and can be neglected, so that the asymptotic solution can be written as

$$y \sim \beta \exp\left\{\int_x^1 \frac{q(t)}{p(t)}\,dt\right\}$$

$$+\frac{p(0)}{p(x)}\left[\alpha - \beta \exp\left\{\int_0^1 \frac{q(t)}{p(t)}\,dt\right\}\right] \exp\left\{-\frac{1}{\epsilon}\int_0^x p(t)\,dt + \int_0^x \frac{q(t)}{p(t)}\,dt\right\}.$$

Finally, the last exponential in this solution is negligibly small unless $x = O(\epsilon)$ (the boundary layer), so we can write

$$y \sim \beta \exp\left\{\int_x^1 \frac{q(t)}{p(t)}\,dt\right\} + \frac{p(0)}{p(x)}\left[\alpha - \beta \exp\left\{\int_0^1 \frac{q(t)}{p(t)}\,dt\right\}\right] \exp\left\{-\frac{p(0)x}{\epsilon}\right\}.$$

This is precisely the composite expansion that we would have obtained if we had used the method of matched asymptotic expansions instead.

Connection Problems

Let's now consider the boundary value problem

$$\epsilon^2 y''(x) + \phi(x)y(x) = 0 \text{ subject to } y(0) = 1,\ y \to 0 \text{ as } x \to -\infty, \qquad (12.115)$$

with

$$\begin{array}{ll} \phi(x) > 0 & \text{for } x > 0, \\ \phi(x) \sim \phi_1 x & \text{for } |x| \ll 1,\ \phi_1 > 0, \\ \phi(x) < 0 & \text{for } x < 0. \end{array} \qquad (12.116)$$

To prevent nonuniformities as $|x| \to \infty$, we will also insist that $|\phi(x)| \gg x^{-2}$ for $|x| \gg 1$. Using the expansions (12.105) and (12.106), we can immediately write

$$y = \frac{A(\epsilon)}{\phi^{1/4}(x)} \exp\left\{i\frac{\int_0^x \phi^{1/2}(t)\,dt}{\epsilon}\right\} +$$

$$\frac{B(\epsilon)}{\phi^{1/4}(x)} \exp\left\{-i\frac{\int_0^x \phi^{1/2}(t)\,dt}{\epsilon}\right\} + \cdots \quad \text{for } x > 0, \tag{12.117}$$

$$y = C(\epsilon)\exp\left\{-\frac{1}{\epsilon}\int_x^0 |\phi(t)|^{1/2}\,dt - \frac{1}{4}\log|\phi(x)| + \cdots\right\} \quad \text{for } x < 0. \tag{12.118}$$

The problem of determining how A and B depend upon C is known as a **connection problem**, and can be solved by considering an inner solution in the neighbourhood of the origin.

For $|x| \ll 1$, $\phi \sim \phi_1 x$, so we can estimate the sizes of the terms in the WKB expansion. For $x < 0$

$$y \sim C(\epsilon)\exp\left\{-\frac{1}{\epsilon}\int_x^0 (-\phi_1 t)^{1/2}\,dt - \frac{1}{4}\log(-\phi_1 x)\right\}$$

$$\sim C(\epsilon)\exp\left\{-\frac{2}{3\epsilon}\phi_1^{1/2}(-x)^{3/2} - \frac{1}{4}\log\phi_1 - \frac{1}{4}\log(-x)\right\}.$$

We can now see that the second term becomes comparable to the first when $-x = O(\epsilon^{2/3})$. A similar estimate of the solution for $x > 0$ also gives a nonuniformity when $x = O(\epsilon^{2/3})$. The WKB solutions will therefore be valid in two outer regions with $|x| \gg \epsilon^{2/3}$. We will need a small inner region, centred on the origin, and the inner solution must match with the outer solutions. Equation (12.115) shows that the *only* rescaling possible near to the origin is in a region where $x = O(\epsilon^{2/3})$, and $\bar{x} = x/\epsilon^{2/3} = O(1)$ for $\epsilon \ll 1$. Writing (12.117) and (12.118) in terms of $\bar{x}$ leads to the matching conditions

$$\bar{y} \sim \frac{C(\epsilon)}{\phi_1^{1/4}(-\bar{x})^{1/4}\epsilon^{1/6}}\exp\left\{-\frac{2}{3}(-\bar{x})^{3/2}\phi_1^{1/2}\right\} \quad \text{as } \bar{x} \to -\infty, \tag{12.119}$$

$$\bar{y} \sim \frac{A(\epsilon)}{\phi_1^{1/4}\bar{x}^{1/4}\epsilon^{1/6}}\exp\left\{i\frac{2}{3}\bar{x}^{3/2}\phi_1^{1/2}\right\} + \frac{B(\epsilon)}{\phi_1^{1/4}\bar{x}^{1/4}\epsilon^{1/6}}\exp\left\{-i\frac{2}{3}\bar{x}^{3/2}\phi_1^{1/2}\right\} \quad \text{as } \bar{x} \to \infty. \tag{12.120}$$

If we now rewrite (12.115) in the inner region, making use of $\phi \sim \epsilon^{2/3}\phi_1\bar{x}$ at leading order, we arrive at

$$\frac{d^2\bar{y}}{d\bar{x}^2} + \phi_1\bar{x}\bar{y} = 0,$$

$$\text{subject to } \bar{y}(0) = 1, \text{ and the matching conditions (12.119) and (12.120).} \tag{12.121}$$

We can write (12.121)† in terms of a standard equation by defining $t = -\phi_1^{1/3}\bar{x}$, in terms of which (12.121) becomes

$$\frac{d^2\bar{y}}{dt^2} = t\bar{y}.$$

† Note that equation (12.121) is valid for $|x| \ll 1$, and we would expect its solution to be valid in the same domain. Hence there is an overlap of the domains for which the inner and outer solutions are valid, namely $\epsilon^{2/3} \ll |x| \ll 1$, and we can expect the asymptotic matching process to be successful. In fact the overlap domain can be refined to $\epsilon^{2/3} \ll |x| \ll \epsilon^{2/5}$ (see Exercise 12.17).

This is Airy's equation, which we met in Section 3.8, so the solution can be written as

$$\bar{y} = a\mathrm{Ai}(t) + b\mathrm{Bi}(t) = a\mathrm{Ai}\left(-\phi_1^{1/3}\bar{x}\right) + b\mathrm{Bi}\left(-\phi_1^{1/3}\bar{x}\right). \tag{12.122}$$

In Section 11.2.3 we determined the asymptotic behaviour of $\mathrm{Ai}(t)$ for $|t| \gg 1$ using the method of steepest descents. The same technique can be used for $\mathrm{Bi}(t)$, and we find that

$$\mathrm{Ai}(t) \sim \frac{1}{2}\pi^{-1/2}t^{-1/4}\exp\left(-\frac{2}{3}t^{3/2}\right),$$

$$\mathrm{Bi}(t) \sim \pi^{-1/2}t^{-1/4}\exp\left(\frac{2}{3}t^{3/2}\right) \quad \text{as } t \to \infty, \tag{12.123}$$

$$\mathrm{Ai}(t) \sim \frac{1}{\sqrt{\pi}}(-t)^{-1/4}\sin\left\{\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right\},$$

$$\mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}(-t)^{-1/4}\cos\left\{\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right\} \quad \text{as } t \to -\infty. \tag{12.124}$$

Using this known behaviour to determine the behaviour of the inner solution, (12.122), shows that

$$\bar{y} \sim a\frac{1}{2}\pi^{-1/2}\left(\phi_1^{1/3}\bar{x}\right)^{-1/4}\exp\left\{-\frac{2}{3}\left(\phi_1^{1/3}\bar{x}\right)^{3/2}\right\}$$

$$+b\pi^{-1/2}\left(\phi_1^{1/3}\bar{x}\right)^{-1/4}\exp\left\{\frac{2}{3}\left(\phi_1^{1/3}\bar{x}\right)^{3/2}\right\} \quad \text{as } \bar{x} \to -\infty.$$

In order to satisfy the matching condition (12.119), we must have $b = 0$, so that only the Airy function Ai appears in the solution, and

$$C(\epsilon) = \frac{1}{2}\pi^{-1/2}\epsilon^{1/6}\phi_1^{1/6}a. \tag{12.125}$$

The boundary condition $\bar{y}(0) = 1$ then gives $a = 1/\mathrm{Ai}(0) = \Gamma\left(\frac{2}{3}\right)3^{2/3}$, and hence determines $C(\epsilon)$ through (12.125).

As $\bar{x} \to \infty$,

$$\bar{y} \sim a\frac{1}{\sqrt{\pi}}\left(\phi_1^{1/3}\bar{x}\right)^{-1/4}\sin\left\{\frac{2}{3}\left(\phi_1^{1/3}\bar{x}\right)^{3/2} + \frac{\pi}{4}\right\}$$

$$\sim \frac{a}{\sqrt{\pi}\phi_1^{1/12}\bar{x}^{1/4}}\frac{1}{2i}\left\{\exp\left(\frac{2}{3}i\phi_1^{1/2}\bar{x}^{3/2} + \frac{\pi}{4}\right) - \exp\left(-\frac{2}{3}i\phi_1^{1/2}\bar{x}^{3/2} - \frac{\pi}{4}\right)\right\}.$$

We can therefore satisfy the matching condition (12.120) by taking

$$A(\epsilon) = B^*(\epsilon) = \frac{a\phi_1^{1/6}\epsilon^{1/6}}{2i\sqrt{\pi}}.$$

Since A and B are complex conjugate, the solution is real for $x > 0$, as of course it should be. This determines all of the unknown constants, and completes the solution. The Airy function $\mathrm{Ai}(t)$ is shown in Figure 11.12, which clearly shows the transition from dispersive oscillatory behaviour as $t \to -\infty$ to dissipative, exponential decay as $t \to \infty$.

12.3 Partial Differential Equations

Many of the asymptotic methods that we have met can also be applied to partial differential equations. As you might expect, the task is usually rather more difficult than we have found it to be for ordinary differential equations. We will proceed by considering four generic examples.

Example 1: Asymptotic solutions of the Helmholtz equation

As an example of an elliptic partial differential equation, let's consider the solution of the **Helmholtz equation**,

$$\nabla^2\phi + \epsilon^2\phi = 0 \quad \text{for } r \leqslant 1, \tag{12.126}$$

subject to $\phi(1,\theta) = \sin\theta$ for $\epsilon \ll 1$. This arises naturally as the equation that governs time-harmonic solutions of the wave equation,

$$\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = \nabla^2 z,$$

which we met in Chapter 3. If we write $z = e^{i\omega t}\phi(\mathbf{x})$, we obtain (12.126), with $\epsilon = \omega/c$. Since $\phi = O(1)$ on the boundary, we expand $\phi = \phi_0 + \epsilon^2\phi_2 + O(\epsilon^4)$, and obtain, at leading order,

$$\nabla^2\phi_0 = 0, \quad \text{subject to } \phi_0(1,\theta) = \sin\theta.$$

If we seek a separable solution of the form $\phi_0 = f(r)\sin\theta$, we obtain

$$f'' + \frac{1}{r}f' - \frac{1}{r^2}f = 0, \quad \text{subject to } f(1) = 1.$$

This has solutions of the form $f = Ar + Br^{-1}$, so the bounded solution that satisfies the boundary condition is $f(r) = r$, and hence $\phi_0 = r\sin\theta$. At $O(\epsilon^2)$, (12.126) gives

$$\nabla^2\phi_2 = -r\sin\theta, \quad \text{subject to } \phi_2(1,\theta) = 0.$$

If we again seek a separable solution, $\phi_2 = F(r)\sin\theta$, we arrive at

$$F'' + \frac{1}{r}F' - \frac{1}{r^2}F = -\frac{r}{a}, \quad \text{subject to } F(1) = 0.$$

Using the variation of parameters formula, this has the bounded solution

$$\phi_2 = \frac{1}{8}\left(r - r^3\right)\sin\theta.$$

The two-term asymptotic expansion of the solution can therefore be written as

$$\phi = r\sin\theta + \frac{1}{8}\epsilon^2\left(r - r^3\right)\sin\theta + O(\epsilon^4),$$

which is bounded and uniformly valid throughout the circle, $r \leqslant 1$.

Let's also consider a boundary value problem for the **modified Helmholtz equation**,

$$\epsilon^2\nabla\phi - \phi = 0 \text{ subject to } \phi(1,\theta) = 1 \text{ and } \phi \to 0 \text{ as } r \to \infty. \tag{12.127}$$

Note that $\phi = 0$ satisfies both the partial differential equation and the far field boundary condition, but not the boundary condition at $r = 1$. This suggests that we need a boundary layer near $r = 1$. If we define $r = 1 + \epsilon\bar{r}$ with $\bar{r} = O(1)$ in the boundary layer for $\epsilon \ll 1$, we obtain

$$\phi_{\bar{r}\bar{r}} - \phi = 0,$$

at leading order. This has solution

$$\phi = \bar{A}(\theta)e^{\bar{r}} + \bar{B}(\theta)e^{-\bar{r}},$$

which will match with the far field solution if $\bar{A} = 0$, and satisfy the boundary condition at $\bar{r} = 0$ if $\bar{B} = 1$. The inner solution, and also a composite solution valid at leading order for all $r \geqslant a$, is therefore

$$\phi = \exp\left(\frac{r-1}{\epsilon}\right).$$

Example 2: The small and large time solutions of a diffusion problem

Consider the initial value problem for the diffusion equation,

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} \text{ for } -\infty < x < \infty \text{ and } t > 0, \tag{12.128}$$

to be solved subject to the initial condition

$$c(x,0) = \begin{cases} f_0(x) & \text{for } x \leqslant 0, \\ 0 & \text{for } x > 0, \end{cases} \tag{12.129}$$

with $f_0 \in C^2(\mathbb{R})$, $f_0 \to 0$ as $x \to -\infty$, $f_0(0) \neq 0$ and

$$\int_{-\infty}^{0} f_0(x)\,dx = f_{\text{tot}}. \tag{12.130}$$

We could solve this using either Laplace or Fourier transforms . The result, however, would be in the form of a convolution integral, which does not shed much light on the structure of the solution. We can gain a lot of insight by asking how the solution behaves just after the initial state begins to diffuse (the small time solution, $t \ll 1$), and after a long time ($t \gg 1$).

The small time solution, $t \ll 1$

The general effect of diffusion is to smooth out gradients in the function $c(x,t)$. It can be helpful to think of c as a distribution of heat or a chemical concentration. This smoothing is particularly pronounced at points where c is initially discontinuous, in this case at $x = 0$ only. Diffusion will also spread the initial data into $x > 0$, where $c = 0$ initially. For this reason, we anticipate that there will be three distinct asymptotic regions.

— Region I: $x < 0$. In this region we expect a gradual smoothing out of the initial data.
— Region II: $|x| \ll 1$. The major feature in this region will be an instantaneous smoothing out of the initial discontinuity.
— Region III: $x > 0$. There will be a flux from $x < 0$ into this region, so we expect an immediate change from $c = 0$ to c nonzero.

Thinking about the physics of the problem before doing any detailed calculation is usually vital to unlocking the structure of the asymptotic solution of a partial differential equation.

Let's begin our analysis in region I by posing an asymptotic expansion valid for $t \ll 1$,

$$c(x,t) = f_0(x) + t f_1(x) + t^2 f_2(x) + O(t^3).$$

Substituting this into (12.128) gives

$$f_1 = D f_0'', \quad 2f_2 = D f_1'',$$

and hence

$$c(x,t) = f_0(x) + tDf_0''(x) + \frac{1}{2}t^2 D^2 f_0''''(x) + O(t^3). \tag{12.131}$$

Note that c increases in regions where $f_0'' > 0$, and vice versa, as physical intuition would lead us to expect. Note also that as $x \to 0^-$, $c(x,t) \sim f_0(0)$. However, in $x > 0$ we would expect c to be small, and the solutions in regions I and III will not match together without a boundary layer centred on the origin, namely region II.

Before setting up this boundary layer, it is convenient to find the solution in region III, where we have noted that c is small. If we try a WKB expansion of the form†

$$c(x,t) = \exp\left\{-\frac{A(x)}{t} + B(x)\log t + C(x) + o(1)\right\},$$

and substitute into (12.128), we obtain

$$A = DA_x^2, \quad A_x B_x = 0, \quad B = -D\left(A_{xx} + 2A_x C_x\right).$$

The solutions of these equations are

$$A = \frac{x^2}{4D} + \beta x + D\beta^2, \quad B = b, \quad C = -\left(b + \frac{1}{2}\right)\log\left(2\beta D + x\right) + d,$$

where β, b and d are constants of integration. The WKB solution in region III is therefore

$$c = \exp\left\{-\frac{1}{t}\left(\frac{x^2}{4D} + \beta x + D\beta^2\right) + b\log t - \left(b + \frac{1}{2}\right)\log\left(2\beta D + x\right) + d + o(1)\right\}. \tag{12.132}$$

† Note that we are using the small time, t, in the WKB expansion that we developed in Section 12.2.7.

In the boundary layer, region II, we define a new variable, $\eta = x/t^\alpha$. In region II, $|\eta| = O(1)$, and hence $|x| = O(t^\alpha) \ll 1$, with $\alpha > 0$ to be determined. In terms of η, (12.128) becomes

$$\frac{\partial c}{\partial t} - \frac{\alpha}{t}\eta\frac{\partial c}{\partial \eta} = \frac{D}{t^{2\alpha}}\frac{\partial^2 c}{\partial \eta^2}.$$

Since $c = O(1)$ in the boundary layer (remember, c must match with the solution in region I as $\eta \to -\infty$, where $c = O(1)$), we can balance terms in this equation to find a distinguished limit when $\alpha = 1/2$. The boundary layer therefore has thickness of $O(t^{1/2})$, which is a typical diffusive length scale. If we now expand as $c = c_0(\eta) + o(1)$, we have, at leading order,

$$-\frac{1}{2}\eta c_{0\eta} = Dc_{0\eta\eta}. \qquad (12.133)$$

If we now write the solutions in regions I and III, given by (12.131) and (12.132), in terms of η, we arrive at the matching conditions

$$c_0 \sim f_0(0) \quad \text{as } \eta \to -\infty, \qquad (12.134)$$

$$c_0 \sim \exp\left\{-\frac{D\beta^2}{t} - \frac{\beta\eta}{t^{1/2}} - \frac{\eta^2}{4D} + b\log t - \right.$$

$$\left.\left(b+\frac{1}{2}\right)\log\left(2\beta D + t^{1/2}\eta\right) + d + o(1)\right\} \quad \text{as } \eta \to \infty. \qquad (12.135)$$

The solution of (12.133) is

$$c_0(\eta) = F + G\int_{-\infty}^{\eta} e^{-s^2/4D}\,ds.$$

As $\eta \to -\infty$, $c \sim F = f_0(0)$. As $\eta \to \infty$, we can use integration by parts to show that

$$c_0 = f_0(0) + G\left\{\int_{-\infty}^{\infty} e^{-s^2/4D}\,ds - \frac{2D}{\eta}e^{-\eta^2/4D} + O\left(\frac{1}{\eta^2}e^{-\eta^2/4D}\right)\right\}.$$

In order that this is consistent with the matching condition (12.135), we need

$$f_0(0) + G\int_{-\infty}^{\infty} e^{-s^2/4D}\,ds = 0,$$

and hence $G = -f_0(0)/\sqrt{4\pi D}$. This leaves us with

$$c_0 \sim \exp\left\{-\frac{\eta^2}{4D} + \log\left(-2DG\right) - \log\eta\right\}.$$

For this to be consistent with (12.135) we need $\beta = 0$, $b = 1/2$ and $d = \log\left(-2DG\right)$.

The structure of the solution that we have constructed allows us to be rather more precise about how diffusion affects the initial data. For $|x| \gg t^{1/2}$, $x < 0$ there is a slow smoothing of the initial data that involves algebraic powers of t, given by (12.131). For $|x| \gg t^{1/2}$, $x > 0$, c is exponentially small, driven by a diffusive flux across the boundary layer. For $|x| = O(t^{1/2})$ there is a boundary layer, with the solution changing by $O(1)$ over a small length of $O(t^{1/2})$. This small time solution continues to evolve, and, when $t = O(1)$, is not calculable by asymptotic methods. When t is sufficiently large, a new asymptotic structure emerges, which we shall consider next.

The large time solution, $t \gg 1$

After a long time, diffusion will have spread out the initial data in a more or less uniform manner, and the structure of the solution is rather different from that which we discussed above for $t \ll 1$. We will start our asymptotic development where $x = O(1)$, and seek a solution of the form

$$c(x,t) = c_0(t) + c_1(x,t) + \cdots,$$

with $|c_1| \ll |c_0|$ for $t \gg 1$ to ensure that the expansion is asymptotic. If we substitute this into (12.128), we obtain

$$\dot{c}_0(t) = Dc_{1xx},$$

at leading order, which can be integrated to give

$$c_1(x,t) = \frac{\dot{c}_0(t)}{2D}x^2 + \alpha_1^{\pm}(t)x + \beta_1^{\pm}(t). \tag{12.136}$$

The distinction between α_1^+ and α_1^-, and similarly for $\beta_1^{\pm}$, is to account for differences in the solution for $x > 0$ and $x < 0$, introduced by the linear terms. As $|x| \to \infty$, c_1 grows quadratically, which causes a nonuniformity in the expansion, specifically when $x = O\left(\sqrt{c_0(t)/|\dot{c}_0(t)|}\right)$. In order to deal with this, we introduce a scaled variable, $\eta = x/\sqrt{c_0(t)/|\dot{c}_0(t)|}$, with $\eta = O(1)$ for $t \gg 1$ in this outer region. In order to match with the solution in the inner region, where $x = O(1)$, we need

$$c(\eta,t) \to c_0(t) \quad \text{as } \eta \to 0. \tag{12.137}$$

In terms of η, (12.128) becomes

$$\frac{\partial c}{\partial t} - \frac{1}{2}\eta\frac{|\dot{c}_0(t)|}{c_0(t)}\frac{d}{dt}\left(\frac{c_0(t)}{|\dot{c}_0(t)|}\right)\frac{\partial c}{\partial \eta} = \frac{|\dot{c}_0(t)|}{c_0(t)}D\frac{\partial^2 c}{\partial \eta^2}. \tag{12.138}$$

Motivated by the matching condition (12.137), we will try to solve this using the expansion $c = c_0(t)F_{\pm}(\eta) + o(c_0(t))$, subject to $F_{\pm} \to 1$ as $\eta \to 0^{\pm}$ and $F_{\pm} \to 0$ as $\eta \to \pm\infty$. The superscript $\pm$ indicates whether the solution is for $\eta > 0$ or $\eta < 0$. It is straightforward to substitute this into (12.138), but this leads to some options. The first and third terms are of $O(\dot{c}_0(t))$, whilst the second term is of $O\left(\dot{c}_0(t)\frac{d}{dt}\left(\frac{c_0(t)}{|\dot{c}_0(t)|}\right)\right)$. Should we include the second term in the leading order balance or not? Let's see what happens if we do decide to balance these terms to get the richest limit. We must then have $c_0(t)/\dot{c}_0(t) = O(t)$, and hence $c_0 = Ct^{-\alpha}$ for some constants C and α. This looks like a sensible gauge function.

If we proceed, (12.138) becomes, at leading order,

$$F_{\pm} - \frac{1}{2}\eta F_{\pm}' = DF_{\pm}''. \tag{12.139}$$

This is slightly more difficult to solve than (12.133). However, if we look for a quadratic solution, we quickly find that $F_\pm = \eta^2 + 2D$ is a solution. Using the method of reduction of order, the general solution of (12.139) is

$$F_\pm = (\eta^2 + 2D)\left\{A^\pm + B^\pm \int_0^\eta \frac{e^{-s^2/2D}}{(s^2+2D)^2}\,ds\right\}. \qquad (12.140)$$

As $\eta \to 0$, by Taylor expanding the integrand, we can show that

$$F_\pm = (\eta^2 + 2D)\left(A^\pm + B^\pm \frac{\eta}{2D}\right) + O(\eta^3).$$

To match with the inner solution, we therefore require that $A^\pm = 1/2D$. As $\eta \to \pm\infty$, using integration by parts, we find that

$$F_\pm = (\eta^2 + 2D)\left\{A^\pm \pm B^\pm \int_0^\infty \frac{e^{-s^2/2D}}{(s^2+2D)^2}\,ds + O\left(\frac{1}{\eta^5}e^{-\eta^2/4D}\right)\right\},$$

so that we require

$$A^\pm \pm B^\pm \int_0^\infty \frac{e^{-s^2/2D}}{(s^2+2D)^2}\,ds = 0.$$

The outer solution can therefore be written as

$$F_\pm = \left(1 + \frac{\eta^2}{2D}\right)\left\{1 \mp \int_0^\eta \frac{e^{-s^2/2D}}{(s^2+2D)^2}\,ds \bigg/ \int_0^\infty \frac{e^{-s^2/2D}}{(s^2+2D)^2}\,ds\right\}. \qquad (12.141)$$

In order to determine α, and hence the size of $c_0(t)$, some further work is needed.

Firstly, we can integrate (12.128) and apply the initial condition, to obtain

$$\int_{-\infty}^\infty c(x,t)\,dx = \int_{-\infty}^0 f_0(x)\,dx = f_{tot}. \qquad (12.142)$$

This just says that mass is conserved during the diffusion process. Secondly, we can write down the composite expansion

$$c = c_{\text{inner}} + c_{\text{outer}} - (c_{\text{inner}})_{\text{outer}} = c_0(t)F_\pm(\eta),$$

and use this in (12.142) to obtain

$$\int_{-\infty}^\infty c_0(t)F_\pm(\eta)\,dx = c_0(t)\sqrt{\frac{c_0(t)}{|\dot c_0(t)|}}\int_{-\infty}^\infty F_\pm(\eta)\,d\eta = f_{\text{tot}}.$$

This is now a differential equation for c_0 in the form

$$\frac{c_0^{3/2}(t)}{|\dot c_0(t)|^{1/2}} = \frac{f_{\text{tot}}}{\int_{-\infty}^\infty F_\pm(\eta)\,d\eta}.$$

In Exercise 12.18 we find that $\int_{-\infty}^\infty F_\pm(\eta)\,d\eta = \sqrt{2\pi D}$ and hence that $c_0(t) =$

$f_{\text{tot}}/\sqrt{4\pi Dt}$. We can now calculate that $\sqrt{c_0(t)/|\dot{c}_0(t)|} = O(t^{1/2})$, which is the usual diffusive length scale. Our asymptotic solution therefore takes the form

$$c(x,t) \sim \begin{cases} \dfrac{f_{\text{tot}}}{\sqrt{4\pi Dt}} & \text{for } |x| = O(t^{1/2}), \\ \dfrac{f_{\text{tot}}}{\sqrt{4\pi Dt}} F_{\pm}(\eta) & \text{for } |x| \gg t^{1/2}. \end{cases} \tag{12.143}$$

The success of this approach justifies our decision to choose $c_0(t)$ in order to obtain the richest distinguished limit. Notice that the large time solution has "forgotten" the precise details of the initial conditions. It only "remembers" the area under the initial data, at leading order.

If we consider the particular case $f_0(t) = e^x$, we find (see Exercise 12.18) that an exact solution is available, namely $c(x,t) = \frac{1}{2}e^{x+Dt}\text{erfc}\left(\frac{x}{\sqrt{4Dt}} + \sqrt{Dt}\right)$. This solution is plotted in Figure 12.16 at various times, and we can clearly see the structures that our asymptotic solutions predict emerging for both small and large times. In Figure 12.17 we plot $c(0,t)$ as a function of Dt. Our asymptotic solution predicts that $c(0,t) = \frac{1}{2} + o(1)$ for $t \ll 1$, consistent with Figure 12.17(a). In Figure 12.17(b) we can see that the asymptotic solution, $c(0,t) \sim 1/\sqrt{4\pi Dt}$ as $t \to \infty$, is in excellent agreement with the exact solution.

Example 3: The wave equation with weak damping

(i) Linear damping

Consider the equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - \epsilon\frac{\partial y}{\partial t}, \quad \text{for } t > 0 \text{ and } -\infty < x < \infty, \tag{12.144}$$

subject to the initial conditions

$$y(x,0) = Y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = 0, \tag{12.145}$$

with $\epsilon \ll 1$. The one-dimensional wave equation, (12.144) with $\epsilon = 0$, governs the small amplitude motion of an elastic string, which we met in Section 3.9.1. The additional term, ϵy_t, represents a weak, linear damping, proportional to the velocity of the string, for example due to drag on the string as it moves through the air.

The form of the initial conditions suggests that we should consider an asymptotic expansion $y = y_0 + \epsilon y_1 + O(\epsilon^2)$. On substituting this into (12.144) and (12.145) we obtain

$$\frac{\partial^2 y_0}{\partial t^2} - c^2\frac{\partial^2 y_0}{\partial x^2} = 0, \quad \text{subject to } y_0(x,0) = Y_0(x), \ \ y_{0t}(x,0) = 0, \tag{12.146}$$

$$\frac{\partial^2 y_1}{\partial t^2} - c^2\frac{\partial^2 y_1}{\partial x^2} = -\frac{\partial y_0}{\partial t}, \quad \text{subject to } y_1(x,0) = y_{1t}(x,0) = 0. \tag{12.147}$$

The initial value problem given by (12.146) is one that we studied in Section 3.9.1, and has d'Alembert's solution, (3.43),

$$y_0(x,t) = \frac{1}{2}\left\{Y_0(x-ct) + Y_0(x+ct)\right\}. \tag{12.148}$$

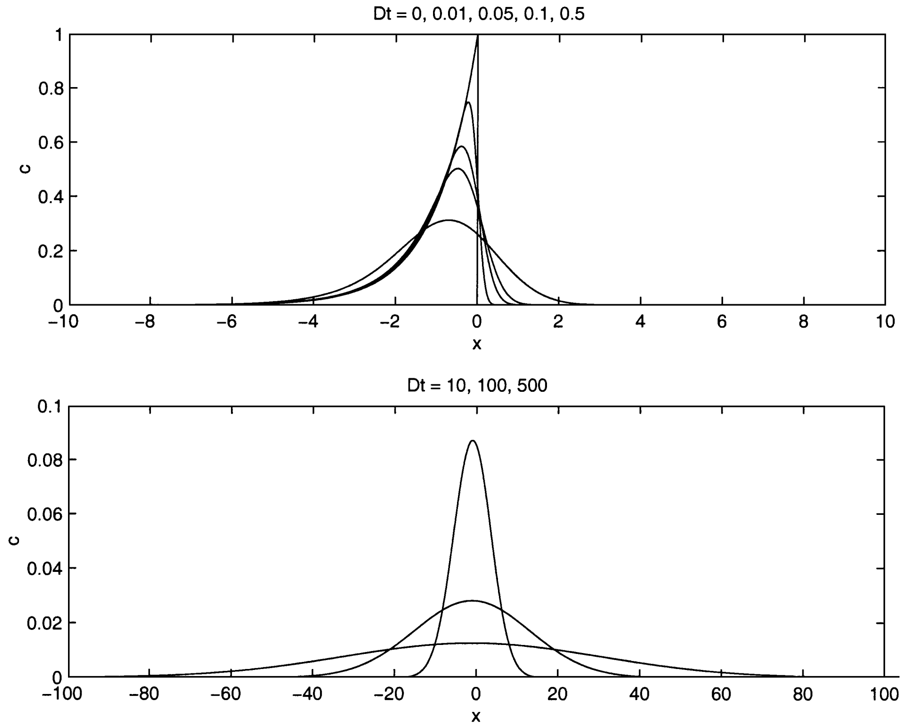

Fig. 12.16. The solution of the diffusion equation with $f_0(x) = e^x$ at various times.

This solution represents the sum of two waves, one travelling to the left and one travelling to the right, each with speed c, without change of form, and with half the initial amplitude. This is illustrated in Figure 12.18 for the initial condition $Y_0(x) = 1/(1+x^2)$. The splitting of the initial profile into left- and right-travelling waves is clearly visible.

In terms of the characteristic variables, $\xi = x - ct$, $\eta = x + ct$, (12.147) becomes

$$-4c^2 \frac{\partial^2 y_1}{\partial \xi \partial \eta} = c\left(\frac{\partial y_0}{\partial \xi} - \frac{\partial y_0}{\partial \eta}\right) = \frac{1}{2} c \left\{ Y_0'(\xi) - Y_0'(\eta) \right\}.$$

Integrating this expression twice gives the solution

$$y_1 = \frac{1}{8c} \left\{ \xi Y_0(\eta) - \eta Y_0(\xi) \right\} + F_1(\xi) + G_1(\eta). \tag{12.149}$$

The initial conditions show that

$$F_1(x) + G_1(x) = 0,$$

and

$$F_1'(x) - G_1'(x) = \frac{1}{4c} \left\{ x Y_0'(x) - Y_0(x) \right\},$$

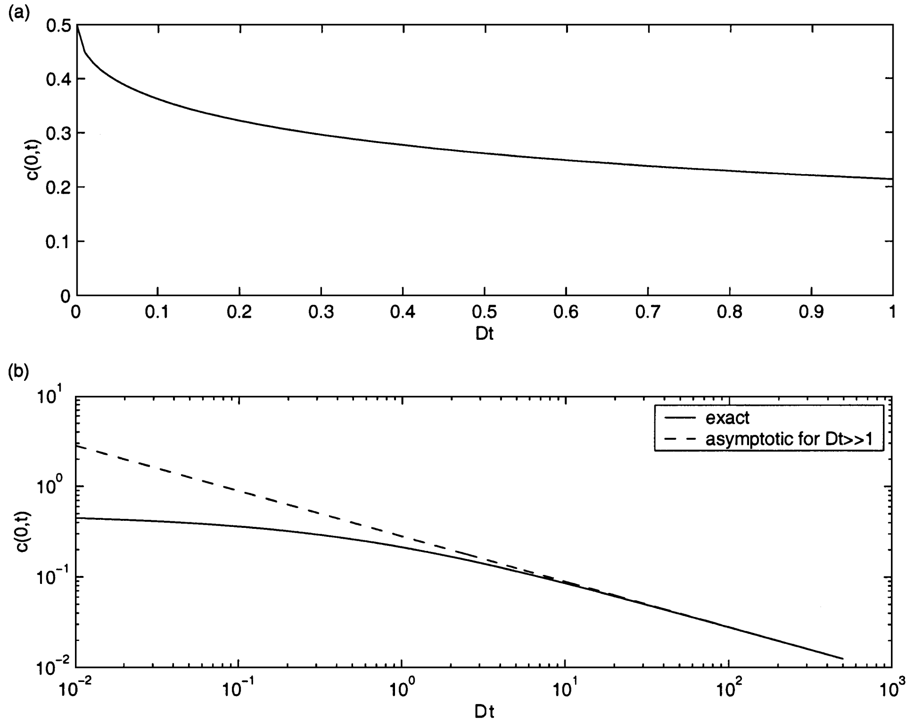

Fig. 12.17. The solution of the diffusion equation with $f_0(x) = e^x$ at $x = 0$.

which can be integrated once to give

$$F_1(x) - G_1(x) = \frac{1}{4c}\left\{xY_0(x) - 2\int_0^x Y_0(s)\,ds\right\} + b.$$

Finally,

$$F_1(x) = -G_1(x) = \frac{1}{8c}\left\{xY_0(x) - 2\int_0^x Y_0(s)\,ds\right\} + \frac{1}{2}b,$$

which, in conjunction with (12.149), shows that

$$y_1 = -\frac{1}{4}t\left\{Y_0(x+ct) + Y_0(x-ct)\right\} + \frac{1}{4c}\int_{x-ct}^{x+ct} Y_0(s)\,ds. \tag{12.150}$$

We can now see that $y_1 = O(t)$ for $t \gg 1$, and therefore that our asymptotic expansion becomes nonuniform when $t = O(\epsilon^{-1})$.

We will proceed using the method of multiple scales, defining a slow time scale $T = \epsilon t$, and looking for a solution $y = y(x,t,T)$. In terms of these new independent variables, (12.144) becomes

$$y_{tt} + 2\epsilon y_{tT} + \epsilon^2 y_{TT} = c^2 y_{xx} - \epsilon y_t - \epsilon^2 y_T. \tag{12.151}$$

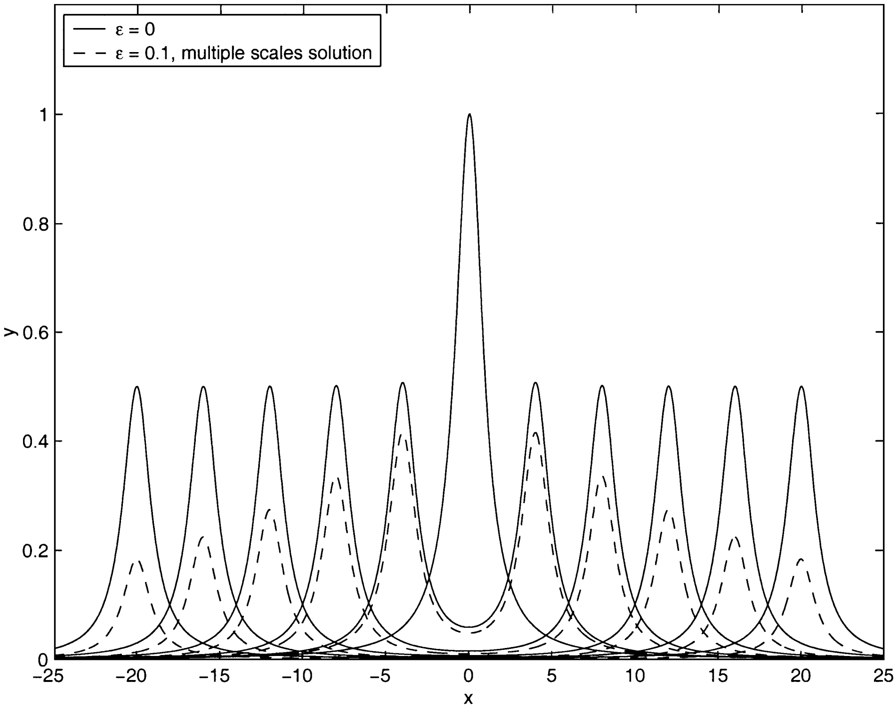

Fig. 12.18. The solution of (12.144) when $c = 1$ and $Y_0(x) = 1/(1 + x^2)$ at equal time intervals, $t = 0$, 4, 8, 12, 16, 20, when $\epsilon = 0$ and 0.1.

If we now seek an asymptotic solution of the form $y = y_0(x, t, T) + \epsilon y_1(x, t, T) + O(\epsilon^2)$, at leading order we obtain (12.146), as before, but now the solution is

$$y_0(x, t, T) = F_0(\xi, T) + G_0(\eta, T), \tag{12.152}$$

with

$$F_0(\xi, 0) = \frac{1}{2}Y_0(\xi), \quad G_0(\eta, 0) = \frac{1}{2}Y_0(\eta). \tag{12.153}$$

As usual in the method of multiple scales, we need to go to $O(\epsilon)$ to determine F_0 and G_0. We find that

$$-4cy_{1\xi\eta} = F_{0\xi} - G_{0\eta} + 2\left(F_{0\xi T} - G_{0\eta T}\right). \tag{12.154}$$

On solving this equation, the presence of the terms of the right hand side causes y_1 to grow linearly with t. In order to eliminate them, we must have

$$F_{0\xi T} = -\frac{1}{2}F_{0\xi}, \quad G_{0\eta T} = -\frac{1}{2}G_{0\eta}. \tag{12.155}$$

If we solve these equations subject to the initial conditions (12.153), we obtain

$$F_0 = \frac{1}{2}e^{-T}Y_0(\xi), \quad G_0 = \frac{1}{2}e^{-T}Y_0(\eta),$$

and hence

$$y_0 = \frac{1}{2}e^{-\epsilon t/2}\left\{Y_0(x-ct)+Y_0(x+ct)\right\}. \tag{12.156}$$

This shows that the small term, ϵy_t, in (12.144) leads to an exponential decay of the amplitude of the solution over the slow time scale, $t = O(\epsilon^{-1})$, consistent with our interpretation of this as a damping term. Figure 12.18 shows how this slow exponential decay affects the solution.

(ii) Nonlinear damping

What happens if we replace the linear damping term ϵy_t with a nonlinear damping term, $\epsilon(y_t)^3$? We must then solve

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - \epsilon\left(\frac{\partial y}{\partial t}\right)^3, \quad \text{for } t>0 \text{ and } -\infty<x<\infty, \tag{12.157}$$

subject to the initial conditions

$$y(x,0) = Y_0(x), \quad \frac{\partial y}{\partial t}(x,0) = 0. \tag{12.158}$$

We would again expect a nonuniformity when $t = O(\epsilon^{-1})$, so let's go straight to a multiple scales expansion, $y = y_0(x,t,T) + \epsilon y_1(x,t,T)$. At leading order, as before, we have (12.152) and (12.153). At $O(\epsilon)$,

$$-4cy_{1\xi\eta} = c^2(F_{0\xi} - G_{0\eta})^3 + 2\left(F_{0\xi T} - G_{0\eta T}\right). \tag{12.159}$$

In order to see clearly which terms are secular, we integrate the expression $(F_{0\xi} - G_{0\eta})^3$ twice to obtain

$$\eta\int_0^\xi F_{0\xi}^3(s)\,ds - 3G_0(\eta)\int_0^\xi F_{0\xi}^2(s)\,ds + 3F_0(\xi)\int_0^\eta G_{0\xi}^2(s)\,ds - \xi\int_0^\eta G_{0\eta}^3(s)\,ds.$$

Assuming that $F_0(s)$ and $G_0(s)$ are integrable as $s\to\pm\infty$, we can see that the terms that become unbounded as ξ and η become large are those associated with $F_{0\xi}^3$ and $G_{0\eta}^3$. We conclude that, to eliminate secular terms in (12.159), we need

$$F_{0\xi T} = -\frac{1}{2}c^2F_{0\xi}^3, \quad G_{0\eta T} = -\frac{1}{2}c^2G_{0\eta}^3,$$

to be solved subject to (12.153). The solutions are

$$F_{0\xi} = \frac{Y_0'(\xi)}{\sqrt{4+c^2T\left\{Y_0'(\xi)\right\}^2}}, \quad G_{0\eta} = \frac{Y_0'(\eta)}{\sqrt{4+c^2T\left\{Y_0'(\eta)\right\}^2}},$$

and hence

$$y_0 = -\left[\int_\xi^\infty \frac{Y_0'(s)}{\sqrt{4+c^2T\left\{Y_0'(s)\right\}^2}}\,ds + \int_\eta^\infty \frac{Y_0'(s)}{\sqrt{4+c^2T\left\{Y_0'(s)\right\}^2}}\,ds\right]. \tag{12.160}$$

In general, these integrals cannot be determined analytically, but we can see that the amplitudes of the waves do decay as T increases, and are of $O(T^{-1/2})$ for $T \gg 1$.

Example 4: The measurement of oil fractions using local electrical probes

As we remarked at the beginning of the book, many problems that arise in engineering are susceptible to mathematical modelling. We can break the modelling process down into separate steps.

(i) Identify the important physical processes that are involved.
(ii) Write down the governing equations and boundary conditions.
(iii) Define dimensionless variables and identify dimensionless constants.
(iv) Solve the governing equations using either a numerical method or an asymptotic method.

Note that, although it is possible that we can find an analytical solution, this is highly unlikely when studying real world problems. As we discussed at the start of Chapter 11, when one or more of the dimensionless parameters is small, we can use an asymptotic solution technique. Let's now discuss an example of this type of situation.

For obvious reasons, oil companies are interested in how much oil is coming out of their oilwells, and often want to make this measurement at the point where oil is entering the well as droplets, rather than at the surface. One tool that can be lowered into a producing oilwell to assist with this task is a **local probe**. This is a device with a tip that senses whether it is in oil or water. The output from the probe can be time-averaged to give the local oil fraction at the tip, and an array of probes deployed to give a measurement of how the oil fraction varies across the well. We will consider a simple device that distinguishes between oil and water by measuring electrical conductivity, which is several orders of magnitude higher in saline water than in oil.

The geometry of the electrical probe, which is made from sharpening the tip of a coaxial cable like a pencil, is shown in Figure 12.19. A voltage is applied to the core of the probe, whilst the outer layer, or cladding, is earthed. A measurement of the current between the core and the cladding is then made to determine the conductivity of the surrounding medium. Although this measurement gives a straightforward way of distinguishing between oil and water when only one liquid is present, for example when dipping the probe into a beaker containing a single liquid, the difficulty lies in interpreting the change in conductivity as a droplet of oil approaches, meets, deforms around and is penetrated by the probe. If we want to understand and model this process, there is clearly a difficult fluid mechanical problem to be solved before we can begin to relate the configuration of the oil droplet to the current through the probe (see Billingham and King, 1995). We will pre-empt all of this fluid mechanical complexity by considering what happens if, in the course of the interaction of an oil droplet with a probe, a thin layer of oil forms on the surface of the probe. How thin must this oil layer become before the current through the probe is effectively equal to that of a probe in pure water?

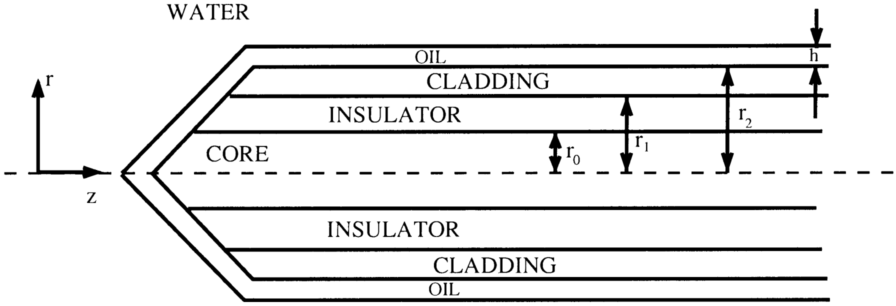

Fig. 12.19. A cross-section through an axisymmetric electrical probe.

In order to answer this question, we must solve a problem in electrostatics, since the speed at which oil–water interfaces move is much less than the speed at which electromagnetic disturbances travel (the speed of light)†. The electrostatic potential, ϕ, is an axisymmetric solution of Laplace's equation,

$$\nabla^2\phi = 0. \tag{12.161}$$

We will assume that the conducting parts of the probe are **perfect conductors**, so that

$$\phi = \begin{cases} 1 & \text{at the surface of the core,} \\ 0 & \text{at the earthed surface of the cladding.} \end{cases} \tag{12.162}$$

At interfaces between different media, for example oil and water or oil and insulator, we have the jump conditions

$$[\phi] = 0, \quad \left[\sigma\frac{\partial\phi}{\partial n}\right] = 0. \tag{12.163}$$

Square brackets indicate the change in the enclosed quantity across an interface, σ is the **conductivity**, which is different in each medium (oil, water and insulator), and $\partial/\partial n$ is the derivative in the direction normal to the interface. Equation (12.163) represents continuity of potential and continuity of current at an interface. To complete the problem, we have the far field conditions that

$$\phi \to 0 \ \text{ as } r^2 + z^2 \to \infty \text{ outside the probe,} \tag{12.164}$$

and

$$\phi \sim \phi_\infty(r) \ \text{ as } z \to \infty \text{ for } r_0 < r < r_1, \tag{12.165}$$

using cylindrical polar coordinates coaxial with the probe, and $r = 0$ at the tip. Here r_0 and r_1 are the inner and outer radii of the insulator, as shown in Figure 12.19.

† This, in itself, is an asymptotic approximation that can be made rigorous by defining a small parameter, the ratio of a typical fluid speed to the speed of light. Some approximations are, however, so obvious that justifying them rigorously is a little too pedantic. For a simple introduction to electromagnetism, see Billingham and King (2001).

The far field potential must satisfy

$$\nabla^2 \phi_\infty(r) = \frac{d^2\phi_\infty}{dr^2} + \frac{1}{r}\frac{d\phi_\infty}{dr} = 0, \quad \text{for } r_0 < r < r_1,$$

subject to

$$\phi = 1 \;\; \text{at } r = r_0, \;\; \phi = 0 \;\; \text{at } r = r_1.$$

This has solution

$$\phi_\infty = \frac{\log(r_1/r)}{\log(r_1/r_0)}. \tag{12.166}$$

Finally, we will assume that the probe is surrounded by water, except for a uniform layer of oil on its surface of thickness $h \ll r_0$. Our task is to solve the boundary value problem given by (12.161) to (12.165).

This is an example of a problem where the governing equation and boundary conditions are fairly straightforward, but the geometry is complicated. Problems like this are usually best solved numerically. However, in this case we have one region where the aspect ratio is small – the thin oil film. The potential is likely to change rapidly across this film compared with its variation elsewhere, and a numerical method will have difficulty handling this. We can, however, make some progress by looking for an asymptotic solution in the thin film. The first thing to do is to set up a local coordinate system in the oil film. The quantities h and r_0 are the natural length scales with which to measure displacements across and along the film, so we let η measure displacement across the film, with $\eta = 0$ at the surface of the probe and $\eta = 1$ at the surface of the water, and let ξ measure displacement along the film, with $\xi = 0$ at the probe tip and $\xi = 1$ a distance r_0 from the tip. Away from the tip and the edge of the probe, which we will not consider for the moment, this provides us with an orthogonal coordinate system, and (12.161) becomes

$$\frac{\partial^2\phi}{\partial\eta^2} + \delta^2\frac{\partial^2\phi}{\partial\xi^2} = 0,$$

where

$$\delta = \frac{h}{r_0} \ll 1.$$

At leading order, $\partial^2\phi/\partial\eta^2 = 0$, and hence ϕ varies linearly across the film, with

$$\phi = A(\xi)\eta + B(\xi). \tag{12.167}$$

Turning our attention now to $(12.163)_2$, since we expect variations of ϕ in the water and the insulator to take place over the geometrical length scale r_0, we have

$$\delta_j\frac{\partial\phi}{\partial\eta} = \delta\frac{\partial\phi}{\partial n} \;\; \text{at interfaces,} \tag{12.168}$$

where

$$\delta_j = \frac{\sigma_o}{\sigma_j},$$

with the subscripts o, w and i indicating oil, water and insulator respectively. We expect that $\delta_{\rm w} \ll 1$ and $\delta_{\rm i} = O(1)$, since oil and the insulator have similar conductivities, both much less than that of saline water. We conclude that, at the interface between oil and insulator, at leading order $\partial\phi/\partial\eta = 0$, and hence $A(\xi) = 0$ there. Also, from the conditions at the surface of the cladding and core, (12.162), $B(\xi) = 0$ at the cladding and $B(\xi) = 1$ at the core.

Returning now to (12.168) with $j = {\rm w}$, note that we have two small parameters, δ and $\delta_{\rm w}$. Double limiting processes like this ($\delta \to 0$, $\delta_{\rm w} \to 0$) have to be treated with care, as the final result usually depends on how fast one parameter tends to zero compared with the other. In this case, we obtain the richest asymptotic balance by assuming that

$$\frac{\delta}{\delta_{\rm w}} = K = \frac{h\sigma_{\rm w}}{r_0\sigma_{\rm o}} = O(1) \quad \text{as } \delta \to 0,$$

and

$$\frac{\partial\phi}{\partial n} = K\frac{\partial\phi}{\partial\eta} = KA(\xi).$$

We can now combine all of the information that we have, to show that at the surface of the probe, the potential in the water satisfies

$$\frac{\partial\phi}{\partial n} = \begin{cases} K(\phi - 1) & \text{at the surface of the core,} \\ 0 & \text{at the surface of the insulator,} \\ K\phi & \text{at the surface of the cladding.} \end{cases} \tag{12.169}$$

The fact that the oil film is thin allows us to apply these conditions at the surface of the probe at leading order. The key point is that this asymptotic analysis allows us to eliminate the thin film from the geometry of the problem at leading order, and instead include its effect in the boundary conditions (12.169). The solution of (12.161) subject to (12.164), (12.165) and (12.169) in the region outside the probe is geometrically simple, and easily achieved using a computer. We will not show how to do this here, as it is outside the scope of this book. We can, however, extract one vital piece of information from our analysis. We have proceeded on the basis that $K = O(1)$. What happens if $K \gg 1$ or $K \ll 1$? If $K \gg 1$, at leading order (12.169) becomes

$$\phi = \begin{cases} 1 & \text{at the surface of the core,} \\ 0 & \text{at the surface of the cladding,} \end{cases}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{at the surface of the insulator.} \tag{12.170}$$

These are precisely the boundary conditions that would apply at leading order in the absence of an oil layer. We conclude that if $K \gg 1$, and hence $h \ll r_0\sigma_{\rm o}/\sigma_{\rm w}$, the film of oil is too thin to prevent a current from passing from core to cladding through the water, and the oil cannot be detected by the probe. If $K \ll 1$, at leading order (12.169) becomes $\partial\phi/\partial n = 0$ at the surface of the probe, and hence $\phi = 0$ in the water. This then shows that $\phi = 1 - \eta$ in the oil film over the core and $\phi = 0$ in the rest of the film, from which it is straightforward to calculate the current flowing from core to cladding. In this case the oil film dominates the response of the probe, effectively insulating it from the water outside.

We conclude that there is a critical oil film thickness, $h_{\mathrm{c}} = r_0\sigma_{\mathrm{o}}/\sigma_{\mathrm{w}}$. For $h \ll h_{\mathrm{c}}$ the oil film is effectively invisible, for $h \gg h_{\mathrm{c}}$ the external fluid is effectively invisible, and the current through the probe is determined by the thickness of the film, whilst for $h = O(h_{\mathrm{c}})$ both the oil and water affect the current through the boundary conditions (12.169). For a typical oil and saline water, $h_{\mathrm{c}} \approx 10^{-9}$ m. This is such a small length that, in practice, any thin oil film coating a probe insulates it from the external fluid, and can lead to practical difficulties with this technique. In reality, local probes are used with alternating rather than direct current driving the core. One helpful effect of this is to increase the value of h_{c}, due to the way that the impedances† of oil and water change with the frequency of the driving potential.

Exercises

12.1 Determine the first two terms in the asymptotic expansion for $0 < \epsilon \ll 1$ of all the roots of each of the equations

(a) $x^3 + \epsilon x^2 - x + \epsilon = 0$,
(b) $\epsilon x^3 + x^2 - 1 = 0$,
(c) $\epsilon x^4 + (1 - 3\epsilon)x^3 - (1 + 3\epsilon)x^2 - (1 + \epsilon)x + 1 = 0$,
(d) $\epsilon x^4 + (1 - 3\epsilon)x^3 - (1 - 3\epsilon)x^2 - (1 + \epsilon)x + 1 = 0$.

In each case, sketch the left hand side of the equation for $\epsilon = 0$ and $\epsilon \ll 1$.

12.2 The function $y(x)$ satisfies the ordinary differential equation

$$\epsilon y'' + (4 + x^2)(y' + 2y) = 0, \quad \text{for } 0 \leqslant x \leqslant 1,$$

subject to $y(0) = 0$ and $y(1) = 1$, with $\epsilon \ll 1$. Show that a boundary layer is possible only at $x = 0$. Use the method of matched asymptotic expansions to determine two-term inner and outer expansions, which you should match using either Van Dyke's matching principle, or an intermediate variable. Hence show that

$$y'(0) \sim \frac{4e^2}{\epsilon} - 4e^2 + 8e^2 \tan^{-1}\left(\frac{1}{2}\right) \quad \text{as } \epsilon \to 0.$$

Construct a composite expansion, valid up to $O(\epsilon)$.

12.3 Determine the leading order outer and inner approximations to the solution of

$$\epsilon y'' + x^{1/2} y' + y = 0 \quad \text{for } 0 \leqslant x \leqslant 1,$$

subject to $y(0) = 0$ and $y(1) = 1$, when $\epsilon \ll 1$. Hence show that

$$y'(0) \sim \epsilon^{-2/3} \frac{e^2}{\Gamma\left(\frac{2}{3}\right)} \left(\frac{3}{2}\right)^{1/3}.$$

† the a.c. equivalents of the conductivities.

12.4 The function $y(x)$ satisfies the ordinary differential equation

$$\epsilon y'' + (1+x)y' - y + 1 = 0,$$

for $0 \leqslant x \leqslant 1$, subject to the boundary conditions $y(0) = y(1) = 0$, with $\epsilon \ll 1$, where a prime denotes d/dx. Determine a two-term inner expansion and a one-term outer expansion. Match the expansions using either Van Dyke's matching principle or an intermediate region. Hence show that

$$y'(0) \sim \frac{1}{2\epsilon} + 1 \quad \text{as } \epsilon \to 0.$$

12.5 Consider the boundary value problem

$$\epsilon(2y + y'') + 2xy' - 4x^2 = 0 \quad \text{for } -1 \leqslant x \leqslant 2,$$

subject to

$$y(-1) = 2, \quad y(2) = 7,$$

with $\epsilon \ll 1$. Show that it is not possible to have a boundary layer at either $x = -1$ or $x = 2$. Determine the rescaling needed for an interior layer at $x = 0$. Find the leading order outer solution away from this interior layer, and the leading order inner solution. Match these two solutions, and hence show that $y(0) \sim 2$ as $\epsilon \to 0$. Sketch the leading order solution.

Now determine the outer solutions up to $O(\epsilon)$. Show that a term of $O(\epsilon \log \epsilon)$ is required in the inner expansion. Match the two-term inner and outer expansions, and hence show that $y(0) = 2 - \frac{3}{2}\epsilon \log \epsilon + O(\epsilon)$ for $\epsilon \ll 1$.

12.6 Consider the ordinary differential equation

$$\epsilon y'' + yy' - y = 0 \quad \text{for } 0 \leqslant x \leqslant 1,$$

subject to $y(0) = \alpha$, $y(1) = \beta$, with α and β constants, and $\epsilon \ll 1$.

(a) Assuming that there is a boundary layer at $x = 0$, determine the leading order inner and outer solutions when $\alpha = 0$ and $\beta = 3$.

(b) Assuming that there is an interior layer at $x = x_0$, determine the leading order inner and outer solutions, and hence show that $x_0 = 1/2$ when $\alpha = -1$ and $\beta = 1$.

12.7 Use the method of multiple scales to determine the leading order solution, uniformly valid for $t \ll \epsilon^{-2}$, of

$$\frac{d^2y}{dt^2} + y = \epsilon y^3 \left(\frac{dy}{dt}\right)^2,$$

subject to $y = 1$, $dy/dt = 0$ when $t = 0$, for $\epsilon \ll 1$.

12.8 Consider the ordinary differential equation

$$\ddot{y} + \epsilon \dot{y} + y + \epsilon^2 y \cos^2 t = 0,$$

for $t \geqslant 0$, subject to $y(0) = 1$, $\dot{y}(0) = 0$, where a dot denotes d/dt. Use the method of multiple scales to determine a two-term asymptotic expansion, uniformly valid for all $t \ll \epsilon^{-3}$ when $\epsilon \ll 1$.

12.9 Consider the ordinary differential equation

$$\ddot{y} + \epsilon\dot{y} + y + \epsilon^2 y^3 = 0,$$

for $t \geqslant 0$, subject to $y(0) = 1$, $\dot{y}(0) = 0$, where a dot denotes d/dt. Use the method of multiple scales, with

$$T = \epsilon t, \quad \tau = t + \epsilon^2 a t, \quad y \equiv y(\tau, T),$$

to show that

$$y \sim e^{-T/2}\cos\tau \quad \text{for } \epsilon \ll 1.$$

Show further that $a = -1/8$ and determine the next term in the asymptotic expansion. (You will need to consider the first three terms in the asymptotic expansion for y.)

12.10 Show that when $v_0 \ll 1$, (12.81) becomes $dE/dT = -2E$ at leading order.

12.11 Consider the initial value problem

$$\frac{d^2y}{dt^2} - 2\epsilon\frac{dy}{dt} - y + y^3 = 0, \tag{E12.1}$$

subject to

$$y(0) = y_i > 1, \quad \frac{dy}{dt}(0) = 0. \tag{E12.2}$$

Use a graphical argument to show that when $\epsilon = 0$, y is positive, provided that $y_i < \sqrt{2}$. Use Kuzmak's method to determine the leading order approximation to the solution when $0 < \epsilon \ll 1$ and $1 < y_i < \sqrt{2}$. You should check that your solution is consistent with the linearized solution when $y_i - 1 \ll 1$. Hence show that y first becomes negative when $t = t_0 \sim T_0/\epsilon$, where

$$T_0 = \int_{\frac{1}{2}y_i^2(y_i^2-2)}^{0} \frac{3K\left(\frac{1}{k}\right)}{4\sqrt{1+2E}\left(\sqrt{1+2E}+1\right)}$$

$$\times \frac{1}{\left\{(2k^2-1)\,L\left(\frac{1}{k}\right) - 2\left(k^2-1\right)K\left(\frac{1}{k}\right)\right\}}\,dE,$$

$$k \equiv k(E) = \left(\frac{\sqrt{1+2E}+1}{2\sqrt{1+2E}}\right)^{1/2}.$$

Hints:

(a) The leading order solution can be written in terms of the Jacobian elliptic function cn (see Section 9.4).

(b)

$$\int_{\sqrt{1-\frac{1}{k^2}}}^{1} \sqrt{1-x^2}\sqrt{1-k^2+k^2x^2}\,dx$$

$$= \frac{1}{3k}\left\{\left(2k^2-1\right)L\left(\frac{1}{k}\right) - 2\left(k^2-1\right)K\left(\frac{1}{k}\right)\right\}.$$

12.12 Show that when $D(x, \hat{x}) = D_0(x)\hat{D}(\hat{x})$ and $R(\theta(x), x, \hat{x}) = R_0(\theta(x), x)\hat{R}(\hat{x})$, the homogenized diffusion coefficient and reaction term given by (12.101) and (12.102) can be written in the simple form described in Section 12.2.6.

12.13 Give a physical example that would give rise to the initial–boundary value problem

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D\left(x, \frac{x}{\epsilon}\right)\frac{\partial \theta}{\partial x}\right) \quad \text{for } 0 < x < 1,$$

subject to

$$\theta(x, 0) = \theta_{\mathrm{i}}(x) \quad \text{for } 0 \leqslant x \leqslant 1,$$

$$\frac{\partial \theta}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = 1 \text{ for } t > 0.$$

When $0 < \epsilon \ll 1$, use homogenization theory to show that, at leading order, θ is a function of x and t only, and satisfies an equation of the form

$$F(x)\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(\bar{D}(x)\frac{\partial \theta}{\partial x}\right),$$

where $\bar{D}(x)$ is given by (12.101), and $F(x)$ is a function that you should determine. If $D(x, x/\epsilon) = D_0(x)\hat{D}(x/\epsilon)$, show that, at leading order, θ satisfies the diffusion equation

$$\frac{\partial \theta}{\partial \hat{t}} = \frac{\partial}{\partial x}\left(D_0(x)\frac{\partial \theta}{\partial x}\right),$$

where

$$\hat{t} = \frac{t}{\lim_{\epsilon \to 0}\left(2\epsilon^2 \int_0^{1/\epsilon} \frac{s}{\hat{D}(s)} ds\right)}.$$

12.14 Use the WKB method to find the eigensolutions of the differential equation

$$y''(x) + (\lambda - x^2)y(x) = 0,$$

subject to $y \to 0$ as $|x| \to \infty$, when $\lambda \gg 1$.

12.15 Find the first two terms in the WKB approximation to the solution of the fourth order equation

$$\epsilon y''''(x) = \{1 - \epsilon V(x)\}\, y(x)$$

that satisfies $V(\pm\infty) = 0$ and $y(\pm\infty) = 0$, when $\epsilon \ll 1$.

12.16 Solve the connection problem

$$\epsilon^2 y''(x) + cx^2 y(x) = 0,$$

subject to $y(0) = 1$ and $y \to 0$ as $x \to -\infty$, when $\epsilon \ll 1$.

12.17 By determining the next term in the WKB expansion, verify that the overlap domain for the inner and outer solutions of the boundary value problem (12.115) is $\epsilon^{2/3} \ll |x| \ll \epsilon^{2/5}$ (see the footnote just after (12.121)).

12.18 Use Fourier transforms to solve (12.128) subject to (12.129) with $f_0(x) = e^x$, and hence show that $c(x,t) = \frac{1}{2}e^{x+Dt}\text{erfc}\left(\frac{x}{\sqrt{4Dt}} + \sqrt{Dt}\right)$. Use this expression to show that $c(0,t) \sim 1/\sqrt{4\pi Dt}$ as $t \to \infty$. By comparing this result with the large time asymptotic solution that we derived in Section 12.3, show that $\int_{-\infty}^{\infty} F_{\pm}(\eta)\,d\eta = \sqrt{2\pi D}$.

12.19 Find the small time solution of the reaction–diffusion equation

$$u_t = Du_{xx} + \alpha u,$$

subject to the initial condition

$$u(x,0) = \begin{cases} f_0(x) & \text{for } |x| < \alpha, \\ 0 & \text{for } |x| \geqslant \alpha. \end{cases}$$

What feature of the solution arises as a result of the reaction term, αu?

12.20 Find the leading order solution of the partial differential equation

$$c_t = (D + \epsilon x)^2 c_{xx} \quad \text{for } x > 0,\ t > 0,$$

when $\epsilon \ll 1$, subject to the initial condition $c(x,0) = f(x)$ and the boundary condition $c(0,t) = 0$. Your solution should remain valid for large x and t.

12.21 Find a uniformly valid solution of the hyperbolic equation

$$\epsilon(u_t + u_x) + (t-1)^2 u = 1 \quad \text{for } -\infty < x < \infty,\ t > 0,$$

when $\epsilon \ll 1$, subject to the initial condition $u(x,0) = 0$.

12.22 Find the leading order asymptotic solution of

$$\epsilon(u_{xx} + u_{yy}) + u_x + \beta u_y = 0 \quad \text{for } x > 0,\ 0 < y < L,$$

when $\epsilon \ll 1$, subject to the boundary conditions $u(x,0) = f(x)$, $u(0,y) = g(y)$ and $u(x,L) = 0$. Your solution should be uniformly valid in the domain of solution, so you will need to resolve any boundary layers that are required.

12.23 Find a uniformly valid leading order asymptotic solution of

$$\epsilon(u_{tt} - c^2 u_{xx}) + u_t + \alpha u_x = 0 \quad \text{for } -\infty < x < \infty,\ t > 0,$$

when $\epsilon \ll 1$, subject to the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = 0$. How does the solution for $c > |\alpha|$ differ from the solution when $c < |\alpha|$?

12.24 **Project: The triple deck**

Consider the innocuous looking two-point boundary value problem

$$\epsilon y'' + x^3 y' + (x^3 - \epsilon)y = 0, \quad \text{subject to } y(0) = \tfrac{1}{2},\ y(1) = 1, \tag{E12.3}$$

with $\epsilon \ll 1$.

(a) Show that the *outer* solution is $y = e^{1-x}$.

(b) Since the outer solution does not satisfy the boundary condition at $x = 0$, there must be a boundary layer. By writing $\bar{x} = x/\epsilon^{\mu}$, show that there are two possibilities, $\mu = 1$ and $\mu = \frac{1}{2}$.
As there is now a danger of confusion in our notation, we will define $\bar{x} = x/\epsilon$, and refer to the region where $\bar{x} = O(1)$ as the **lower deck**, and $x^* = x/\epsilon^{1/2}$, and refer to the region where $x^* = O(1)$ as the **middle deck**. We will refer to the region where $x = O(1)$ as the **upper deck**. This nomenclature comes from the study of boundary layers in high Reynolds number fluid flow.

(c) Show that the leading order solution in the lower deck is $\bar{y} = \frac{1}{2}e^{-\bar{x}}$, and in the middle deck $y^* = a\exp\left\{-1/2\,(x^*)^2\right\}$.

(d) Apply the simplest form of matching condition to show that $a = e$, and hence that the solution in the middle deck is

$$y^* = \exp\left\{1 - \frac{1}{2\,(x^*)^2}\right\}.$$

(e) The matching between the lower and middle decks requires that

$$\lim_{\bar{x}\to\infty} \bar{y} = \lim_{x^*\to 0} y^*,$$

which is satisfied automatically without fixing any constants. Show that the composite solution takes the form

$$y = e^{1-x} + \exp\left\{1 - \frac{1}{2\,(x^*)^2}\right\} + \frac{1}{2}e^{-\bar{x}} - e.$$

(f) Integrate (E12.3) numerically using MATLAB, and compare the numerical solution with the asymptotic solution for $\epsilon = 0.1$, 0.01 and 0.001. Can you see the structure of the triple deck?

(g) The equations for steady, high Reynolds number flow of a viscous Newtonian fluid with velocity $\mathbf{u}$ and pressure p are

$$\nabla \cdot \mathbf{u} = 0, \quad (\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \frac{1}{\mathrm{Re}}\nabla^2 \mathbf{u},$$

where Re is the Reynolds number. If these are to be solved in $-\infty < x < \infty$, $y > f(x)$ subject to $\mathbf{u} = \mathbf{0}$ on $y = f(x)$ and $\mathbf{u} \sim U\mathbf{i}$ as $x^2 + y^2 \to \infty$, where would you expect a triple or double deck structure to appear?
You can find some more references and background material in Sobey (2000). From the data in this book, estimate the maximum thickness of the boundary layer on a plate 1 m long, with water flowing past it at 10 m s^{-1}.