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Exponential Stability of Linear Time-Invariant Systems on Time Scales

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Abstract: Several notions of exponential stability of linear time-invariant systems on arbitrary time scales are discussed. We establish a necessary and sufficient condition for the existence of uniform exponential stability. Moreover, we characterize the uniform exponential stability of a system by the spectrum of its matrix. In general, exponential stability of a system can not be characterized by the spectrum of its matrix.

Keywords: *time scale; linear dynamic equation; exponential stability; uniform exponential stability.*

Mathematics Subject Classification (2000): 34C11, 39A10, 37B55, 34D99.

1 Introduction

It is well-known that exponential decay of the solution of a linear autonomous ordinary differential equation $\dot{x}(t) = Ax(t),\ t \in \mathbb{R}$, or of an autonomous difference equation $x_{t+1} = Ax_t,\ t \in \mathbb{Z}$, can be characterized by spectral properties of A. Namely, the solutions tend to 0 exponentially as $t \to \infty$, if and only if all the eigenvalues of $A \in \mathbb{C}^{d\times d}$ have negative real parts or a modulus smaller than 1, respectively. The question, which notion of stability of a linear time-invariant dynamic equation on a time scale inherits such a property, is answered partly in Pötzsche et al [16].

The history of asymptotic stability of an equation on a general time scale goes back to the work of Aulbach and Hilger [2]. Although it unifies the time scales $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z},\ h > 0$, its assumptions are often too pessimistic since the maximal graininess is involved. For a real scalar dynamic equation, stability and instability results are obtained

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by Gard and Hoffacker [7]. Another way to approach to the asymptotic stability of linear dynamic equations using Lyapunov functions can be found in Hilger and Kloeden [10]. Pötzsche [15, Abschnitt 2.1] provides sufficient conditions for the uniform exponential stability in Banach spaces, as well as spectral stability conditions for time-varying systems on time scales. Properties of exponential stability of a time-varying dynamic equation on a time scale have been also investigated recently by Bohner and Martynyuk [3], DaCunha [5], Du and Tien [6], Hoffacker and Tisdell [11], Martynyuk [13] and Peterson and Raffoul [14].

As a thorough introduction into dynamic equations on time scales we refer to the paper by Hilger [9] or the monograph by Bohner and Peterson [4]. The paper [2] presents the theory with a focus on linear systems.

A *time scale* $\mathbb{T}$ is a non-empty, closed subset of the reals $\mathbb{R}$. For the purpose of this paper we assume from now on that $\mathbb{T}$ is unbounded from above, i.e. $\sup \mathbb{T} = \infty$. On $\mathbb{T}$ the *graininess* is defined as

$$\mu^*(t) := \inf \{s \in \mathbb{T} : t < s\} - t.$$

This paper is organized as follows. In Section 2 we introduce the class of systems we wish to study and define the concepts of exponential, uniform exponential, robust exponential and weak-uniform exponential stability. In Section 3 we first provide a necessary and sufficient condition for the existence of a uniformly exponentially stable linear time-invariant system. We show that uniform exponential stability implies robust exponential stability. An example illustrates that robust exponential stability, in general, does not imply weak-uniform exponential stability. The uniform exponential stability and the robust exponential stability of a system are characterized by the spectrum of its matrix, respectively. In Section 4 we provide an example which indicates that, in general, exponential stability of a system is not determined by the spectrum. We intend to relate the stability of a scalar system to the stability of the according Jordan system. We arrive at the statement that weak-uniform exponential stability of a system is characterized by the spectrum of its matrix.

2 Preliminaries

In the following $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. As usual, $\mathbb{K}^{d\times d}$ is the space of square matrices with d rows, I_d is the identity mapping on the d-dimensional space $\mathbb{K}^d$ over $\mathbb{K}$ and $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{d\times d}$.

Let $A \in \mathbb{K}^{d\times d}$ and consider the d-dimensional linear system of dynamic equations

$$x^\Delta = Ax. \tag{1}$$

Let $e_A : \{(t,\tau) \in \mathbb{T}\times\mathbb{T} : t \geq \tau\} \to \mathbb{K}^{d\times d}$ denote the *transition matrix* corresponding to (1), that is, $\varphi(t,\tau,\xi) = e_A(t,\tau)\xi$ solves the initial value problem (1) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^d$ and $t,\tau \in \mathbb{T}$ with $t \geq \tau$. The classical examples for this setup are the following.

Example 2.1 If $\mathbb{T} = \mathbb{R}$ we consider linear time-invariant systems of the form $\dot{x}(t) = Ax(t)$. If $\mathbb{T} = h\mathbb{Z}$, then (1) reduces to $(x(t+h) - x(t))/h = Ax(t)$ or equivalently $x(t+h) = [I_d + hA]x(t)$.

The subsequent notions of exponential stability (i), (ii), (iii) of system (1) are introduced here as in Pötzsche et al [16].

Definition 2.1 (Exponential stability) Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (1)

(i) *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \geq 1$ with

$$\|e_A(t,s)\| \leq K(s)\exp(-\alpha(t-s)) \quad \text{for } t \geq s.$$

(ii) *uniformly exponentially stable* if K can be chosen independently of s in the definition of exponential stability.

(iii) *robustly exponentially stable* if there is an $\varepsilon > 0$ such that the exponential stability of (1) implies the exponential stability of $x^\Delta = Bx$ for any $B \in \mathbb{K}^{d\times d}$ with $\|B - A\| \leq \varepsilon$.

(iv)*weak-uniformly exponentially stable* if there exists a constant $\alpha > 0$ such that for every $s \in \mathbb{T}$ there exists $K(s) \geq 1$ with

$$\|e_A(t,\tau)\| \leq K(s)e^{-\alpha(t-\tau)} \qquad \text{for all } t \geq \tau \geq s.$$

Remark 2.1 (i) The different notions of stability (i), (ii) and (iii) are partly investigated in the paper by Pötzsche et al [16], where examples are provided which show that exponential stability, in general, does neither imply uniform exponential stability nor robust exponential stability.

(ii) The notion of weak-uniform exponential stability serves as an intermediate notion between exponential stability and uniform exponential stability. Note that weak-uniform exponential stability coincides with uniform exponential stability if we can choose a bounded function $K : \mathbb{T} \to \mathbb{R}^+$ in (iv).

One of the observations in this paper is the following diagram about the relations between the stability notions:

$$\begin{tikzcd}
 & \text{uniform e. s.} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\text{weak-uniform e. s.} \arrow[dr, Rightarrow] & & \text{robust e.s.} \arrow[dl, Rightarrow] \\
 & \text{exponential stability} &
\end{tikzcd}$$

3 Uniform Exponential Stability

In this section, we deal with some fundamental properties of uniform exponential stability. More precisely, the existence and robustness of uniform exponential stability are investigated. As a consequence, we obtain a characterization of uniform exponential stability for a linear time-invariant system based on the spectrum of its matrix.

Theorem 3.1 (Existence of a uniformly exponentially stable system) *Let $\mathbb{T}$ be a time scale which is unbounded above. Then there exists a uniformly exponentially stable system on $\mathbb{R}^d$*

$$x^\Delta = Ax, \qquad A \in \mathbb{R}^{d\times d}, x \in \mathbb{R}^d,$$

if and only if the graininess of $\mathbb{T}$ is bounded above, i.e. there exists $h > 0$ such that $\mu^*(t) \leq h$ for all $t \in \mathbb{T}$.

Proof ($\Rightarrow$) Assume that there exists $A \in \mathbb{R}^{d\times d}$ such that the system

$$x^\Delta = Ax, \tag{2}$$

is uniformly exponentially stable, i.e. there exist $K > 0$, $\alpha > 0$ such that

$$\|e_A(t,s)\| \le K \exp(-\alpha(t-s)) \quad \text{for all } t \ge s. \tag{3}$$

We first show that $A \neq 0$. Indeed, suppose that $A = 0$, then $e_A(t,s) = I_d$. Hence, we have $\|e_A(t,s)\| = 1$ for all $t \ge s$ and the inequality (3) thus does not hold.

Let $t_0 \in \mathbb{T}$ be an arbitrary right scattered point, i.e. $\mu^*(t_0) > 0$. Then at the point t_0 the equation (2) becomes

$$\frac{x(t_0 + \mu^*(t_0)) - x(t_0)}{\mu^*(t_0)} = Ax(t_0).$$

This implies that $e_A(t_0 + \mu^*(t_0), t_0) = I_d + \mu^*(t_0)A$ and then by using (3) we have

$$\|I_d + \mu^*(t_0)A\| \le K \exp(-\alpha\mu^*(t_0)), \quad \text{i.e.} \ -1 + \mu^*(t_0)\|A\| \le K.$$

Therefore,

$$\mu^*(t_0) \le \frac{K+1}{\|A\|}$$

for every right scattered point $t_0 \in \mathbb{T}$, i.e. $\mathbb{T}$ has bounded graininess.

($\Leftarrow$) Assume that there exists $h > 0$ so that $\mu^*(t) \le h$ for all $t \in \mathbb{T}$. Define $A = \frac{-1}{2h} I_d$. Clearly, $I_d + \mu^*(t)A$ is invertible for all $t \in \mathbb{T}$, i.e. A is a regressive matrix. Now we will show that the system

$$x^\Delta = Ax, \tag{4}$$

is uniformly exponentially stable. Since A is a regressive diagonal matrix, Hilger [9, Theorem 7.4(iii)] implies the following explicit representation of the norm of the transition matrix of (4)

$$\begin{aligned} \|e_A(t,s)\| &= \exp \int_s^t \lim_{u \searrow \mu^*(\tau)} \frac{\log|1 - \frac{u}{2h}|}{u} \, \Delta\tau \\ &\le \exp \int_s^t \frac{-1}{2h} \, \Delta\tau = \exp\left(\frac{-1}{2h}(t-s)\right). \end{aligned}$$

This completes the proof.

From now on we only deal with a time scale with bounded graininess. In order to show the roughness of uniform exponential stability, we provide the following preparatory lemma.

Lemma 3.1 *Let $\alpha > 0$ be a positive number. Then for the corresponding scalar system $x^\Delta = \alpha x$ the following inequality holds*

$$e_\alpha(t,s) \le \exp(\alpha(t-s)) \quad \textit{for all } t \ge s.$$

Proof Since $\alpha > 0$ we have $1 + \mu^*(t)\alpha > 0$ for all $t \in \mathbb{T}$. Hence, by Hilger [9, Theorem 7.4(iii)] we have

$$\begin{aligned} \|e_\alpha(t,s)\| &= \exp \int_s^t \lim_{u \searrow \mu^*(\tau)} \frac{\log|1 + \alpha u|}{u} \, \Delta\tau \\ &\le \exp \int_s^t \alpha \, \Delta\tau = \exp\left(\alpha(t-s)\right). \end{aligned}$$

This concludes the proof.

Proposition 3.1 (Robustness of uniform exponential stability) *Let $\mathbb{T}$ be a time scale which is unbounded above and with bounded graininess. Assume that the system*

$$x^{\Delta} = Ax, \tag{5}$$

where $A \in \mathbb{C}^{d\times d}$, is uniformly exponentially stable. Then there exists $\varepsilon > 0$ such that the system

$$x^{\Delta} = Bx \tag{6}$$

is also uniformly exponentially stable for all $B \in \mathbb{C}^{d\times d}$ with $\|A - B\| \leq \varepsilon$.

Proof Let $K > 0$ and $\alpha > 0$ such that

$$\|e_A(t,s)\| \leq K\exp(-\alpha(t-s)) \qquad \text{for all } t \geq s. \tag{7}$$

The equation (6) can be rewritten as follows

$$x^{\Delta} = Ax + (B-A)x.$$

Using the variation of constants formula (see Bohner and Peterson [4, pp 195]) with the inhomogeneous part $g(t) := (B-A)e_B(t,s)$ for a fixed $s \in \mathbb{T}$, the transition matrix of (6) is determined by

$$e_B(t,s) = e_A(t,s) + \int_s^t e_A(t, u+\mu^*(u))(B-A)e_B(u,s)\,\Delta u, \quad \text{for all } t \geq s.$$

Fix $s \in \mathbb{T}$ and define $f(t) = \exp(\alpha(t-s))\|e_B(t,s)\|$. We thus obtain the following estimate

$$\begin{aligned}\exp(-\alpha(t-s))f(t) \quad &\leq \quad \|e_A(t,s)\| + \|A-B\| \cdot \\ &\qquad \int_s^t \|e_A(t,u+\mu^*(u))\| \exp(-\alpha(u-s))f(u)\,\Delta u.\end{aligned}$$

This implies with (7) that

$$f(t) \leq K + K\|A-B\| \int_s^t \exp(\alpha\mu^*(u))f(u)\,\Delta u \quad \text{for all } t \geq s. \tag{8}$$

Due to Theorem 3.1 the graininess of $\mathbb{T}$ is bounded. Fix $H > 0$ such that $\mu^*(t) \leq H$ for all $t \in \mathbb{T}$. Hence, we get from (8) that

$$f(t) \leq K + K\|A-B\|\exp(\alpha H) \int_s^t f(u)\,\Delta u \quad \text{for all } t \geq s.$$

Applying Gronwall's inequality (see Bohner and Peterson [4, Corollary 6.7]) and with $f(s) = 1$ we obtain

$$f(t) \leq Ke_M(t,s) \quad \text{for all } t \geq s,$$

where $M = K\|A-B\|\exp(\alpha H)$. By virtue of Lemma 3.1 and the definition of the function $f(t)$ we get

$$\|e_B(t,s)\| \leq K\exp((-\alpha + M)(t-s)) \quad \text{for all } t \geq s. \tag{9}$$

Choose and fix $\varepsilon > 0$ such that $K\varepsilon \exp(\alpha H) < \alpha$. Now for any $B \in \mathbb{R}^{d\times d}$ with $\|A-B\| \leq \varepsilon$, we obtain from (9) that

$$\|e_B(t,s)\| \leq K \exp((-\alpha + K\varepsilon \exp(\alpha H))(t-s)) \quad \text{for all } t \geq s.$$

Since $-\alpha + K\varepsilon \exp(\alpha H) < 0$, the claim follows.

The robustness of uniform exponential stability of a time-varying system is also investigated in DaCuhna [5, Theorem 5.1]. However, the notion of uniform exponential stability and the type of perturbation in DaCuhna [5] are different to those here. Precisely, he used the exponential functions to define uniform exponential stability. For a more details, we refer the reader to DaCuhna [5], Du and Tien [6] and the references therein.

Corollary 3.1 (Uniform implies robust exponential stability) *Let $\mathbb{T}$ be a time scale which is unbounded above and with bounded graininess. Suppose that the system*

$$x^\Delta = Ax, \quad A \in \mathbb{R}^{d\times d}, \tag{10}$$

is uniformly exponentially stable. Then system (10) *is also robustly exponentially stable.*

Next we construct an example which asserts that, in general, robust exponential stability does not imply weak-uniform exponential stability. As a consequence, robust exponential stability does not imply uniform exponential stability.

Example 3.1 Let $d = 1$. We define a sequence s_k recursively by

$$s_1 := 0, \qquad s_{k+1} = s_k + 4k, \quad k \in \mathbb{N}$$

and

$$\mathbb{T}_k := \{s_k + i | i = 0, 1, \cdots, k-1\} \cup \{s_k + k + 3i | i = 0, 1, \cdots, k-1\}, \quad k \in \mathbb{N},$$

and the time scale $\mathbb{T}$ by the discrete set

$$\mathbb{T} := \bigcup_{k=1}^{\infty} \mathbb{T}_k.$$

Clearly, $\mathbb{T}$ is unbounded above and has a bounded graininess. Consider on $\mathbb{T}$ the scalar equation

$$x^\Delta = -x. \tag{11}$$

For $k \geq 1$ an elementary calculation yields for $x_0 \in \mathbb{R}$ that

$$\varphi(s_k + 4k, s_k + k, x_0) = (-2)^k x_0.$$

This shows that the system (11) is not weak-uniformly exponentially stable, as a solution starting in $x_0 = 1$ may become arbitrarily large depending on the initial time $t_0 \in \mathbb{T}$. Now we are going to show that, on the other hand, the system (11) is robustly exponentially stable. To verify this claim we show that the perturbed system

$$x^\Delta = (-1 + \alpha)x \tag{12}$$

is exponentially stable for all $\alpha \in (-\frac{1}{10}, \frac{1}{10})$. Let $x_0 \in \mathbb{R}$ be an arbitrary initial value and $t_0 \in \mathbb{T}$. Denote by k_0 the smallest integer such that $t_0 \le s_{k_0}$. Now we are going to prove inductively the following estimate

$$|\varphi(t, s_{k_0}, x_0)| \le \left(\frac{1}{2}\right)^{\frac{t-s_{k_0}}{3}} |x_0| \quad \text{for all } s_k < t \le s_{k+1} \text{ and } k_0 \le k. \tag{13}$$

We first prove (13) in case $k = k_0$. Indeed, a straightforward computation yields that

$$\varphi(t, s_{k_0}, x_0) = \begin{cases} \alpha^m x_0 & \text{if } t = s_{k_0} + m, \text{for } m = 1, \dots k_0, \\ \alpha^{k_0}(-2+3\alpha)^m x_0 & \text{if } t = s_{k_0} + k_0 + 3m, \text{for } m = 0, 1, \dots k_0. \end{cases}$$

This implies with the inequality $|\alpha(-2+3\alpha)| \le \frac{1}{4}$ the inequality (13) in case $k = k_0$. Suppose that the inequality (13) holds for $k = n-1$. We will show that this also holds for $k = n+1$. Indeed, an elementary computation gives

$$\varphi(t, s_{k_0}, x_0) = \begin{cases} \alpha^m \varphi(s_n, s_{k_0}, x_0) & \text{if } t = s_n + m, \text{for } m = 1, \dots n, \\ \alpha^n(-2+3\alpha)^m \varphi(s_n, s_{k_0}, x_0) & \text{if } t = s_n + n + 3m, \\ & \qquad\qquad \text{for } m = 0, 1, \dots n. \end{cases}$$

This implies with the inequality $|\alpha(-2+3\alpha)| \le \frac{1}{4}$ the inequality (13) in case $k = n$ and then the claim follows. Define $K(t_0) = |\Phi(s_{k_0}, t_0, x_0)|, \beta = \frac{\log 2}{3}$ and by (13) we get

$$|\varphi(t, t_0 x_0)| \le K(t_0) \exp(-\beta(t - t_0)) \quad \text{for all } t_0 \le t.$$

Corollary 3.2 *Let T be a time scale which is unbounded above and with bounded graininess. For $\lambda \in \mathbb{C}$ consider the Jordan block $J_\lambda \in \mathbb{C}^{d\times d}$ given by*

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

The scalar equation

$$x^\Delta = \lambda x \tag{14}$$

is uniformly exponentially stable if and only if the system

$$x^\Delta = J_\lambda x \tag{15}$$

is uniformly exponentially stable.

Proof ($\Rightarrow$) Assume that (14) is uniformly stable. Hence, the equation

$$x^\Delta = \lambda I_d \, x$$

is also uniformly exponentially stable. So, by virtue of Proposition 3.1 there exists $\varepsilon > 0$ such that the system

$$x^\Delta = Bx \tag{16}$$

is uniformly exponentially stable for all $B \in \mathbb{C}^{d\times d}$ such that $\|B - \lambda I_d\| \le \varepsilon$. Define $P_\varepsilon = \mathrm{diag}(1, \varepsilon^{-1}, \dots, \varepsilon^{-d})$. A straightforward computation yields that

$$B_\varepsilon := P_\varepsilon J_\lambda P_\varepsilon^{-1} = \begin{pmatrix} \lambda & \varepsilon & 0 & \dots & 0 \\ & \lambda & \varepsilon & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

Consequently,

$$e_{J_\lambda}(t,s) = P_\varepsilon e_{B_\varepsilon}(t,s) \qquad \text{for all} \quad t \ge s. \tag{17}$$

On the other hand, $\|B_\varepsilon - \lambda I_d\| \le \varepsilon$. Hence, by (16) there exists $K, \alpha > 0$ such that

$$\|e_{B_\varepsilon}(t,s)\| \le K e^{-\alpha(t-s)} \qquad \text{for all} \quad t \ge s.$$

This implies with (17) that

$$\|e_{J_\lambda}(t,s)\| \le K\|P_\varepsilon\| e^{-\alpha(t-s)} \qquad \text{for all} \quad t \ge s.$$

Therefore, (15) is uniformly exponentially stable and it completes the proof.

($\Leftarrow$) The converse direction is trivial.

In the next theorem, we will show that uniform exponential stability of a linear system depends only on the eigenvalues of its matrix.

Theorem 3.2 *The system*

$$x^\Delta = Ax, \qquad A \in \mathbb{R}^{d\times d},$$

is uniformly exponentially stable if and only if the system

$$x^\Delta = \lambda x$$

is uniformly exponentially stable for every $\lambda \in \sigma(A)$.

Proof Without loss of generality, we deal with the norm $\|M\| = \max_{1\le i,j\le n} |m_{ij}|$ for all $M = (m_{ij}) \in \mathbb{R}^{n\times n}$. Let P be the transformation such that $P^{-1}AP = \mathrm{diag}(J_1, J_2, \dots, J_p)$, where

$$J_k = \begin{pmatrix} \lambda_k & 1 & 0 & \dots & 0 \\ & \lambda_k & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda_k \end{pmatrix}, \qquad \text{for } k = 1, 2, \dots, p,$$

are the Jordan blocks of A. Clearly, $\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_p\}$. A straightforward computation yields that

$$e_A(t,s) = P\mathrm{diag}(e_{J_1}(t,s), e_{J_2}(t,s), \dots, e_{J_p}(t,s)).$$

Therefore,

$$\frac{1}{\|P^{-1}\|} \max_{1\le k\le p} \|e_{J_k}(t,s)\| \le \|e_A(t,s)\| \le \|P\| \max_{1\le k\le p} \|e_{J_k}(t,s)\|.$$

This implies that (3.2) is exponentially stable if and only if the systems

$$x^\Delta = J_k x, \qquad k = 1, 2, \dots, p,$$

are exponentially stable. Then by virtue of Corollary 3.2 the claim follows.

Remark 3.1 The robust exponential stability depends also only on the eigenvalues of the matrix of a system.

4 Exponential Stability and Weak-uniform Exponential Stability

In view of Corollary 3.2, the question arises whether the exponential stability of a time-invariant linear system could be characterized by the spectrum of its matrix. In general, this is not the case, since in the subsequent example two systems are presented whose matrices have the same spectrum, one of them is exponentially stable and the other is not.

Example 4.1 There exists a time scale $\mathbb{T}$, which has bounded graininess such that the system

$$x^{\Delta} = -x \tag{18}$$

is exponentially stable and the system

$$x^{\Delta} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} x \tag{19}$$

is not exponentially stable.

Indeed, denote by α_n the positive number such that

$$\alpha_k(1+\alpha_k) = 2^{1-4^{k+1}}, \qquad \text{for all } k \in \mathbb{N}. \tag{20}$$

Equivalently,

$$\alpha_k := \frac{-1+\sqrt{1+2^{3-4^{k+1}}}}{2}, \qquad \text{for all } k \in \mathbb{N}.$$

We define a discrete time scale $\mathbb{T}$ as follows

$$\mathbb{T} = \bigcup_{k=1}^{\infty} \mathbb{T}_k,$$

where

$$\mathbb{T}_k := \{4^k\} \cup \{4^k + 1 - \alpha_k + 3i : i = 0, 1, \cdots, 4^k - 1\}. \tag{21}$$

To verify exponential stability of system (18) we show that

$$|e_{-1}(t,4)| \leq \left(\frac{1}{2}\right)^{t-4} \qquad \text{for all } t \in \mathbb{T}, t \geq 4. \tag{22}$$

Indeed, let t be an arbitrary but fixed element in $\mathbb{T}$. Define

$$n_0 := \max\{n : n \in \mathbb{N}, 4^n < t\}.$$

A straightforward computation together with (20) yields that

$$e_{-1}(4^{n+1}, 4^n) = \alpha_n(1+\alpha_n)2^{4^n-1} = \left(\frac{1}{2}\right)^{4^{n+1}-4^n}, \quad \text{for all } n \in \mathbb{N}. \tag{23}$$

Therefore,

$$\begin{aligned} e_{-1}(t,4) &= e_{-1}(t,4^{n_0})e_{-1}(4^{n_0},4) \\ &= e_{-1}(t,4^{n_0})\left(\frac{1}{2}\right)^{4^{n_0}-4}. \end{aligned}$$

Clearly, if $t = 4^{n_0+1}$ then (22) follows. Hence, it remains to deal with the case $t < 4^{n_0+1}$. By definition of $\mathbb{T}$, see (21), we obtain

$$|e_{-1}(t,4^{n_0})| = \alpha_{n_0} 2^k,$$

where $t = 4^{n_0} + 1 - \alpha_{n_0} + 3k$ for $k \in \{0,1,\dots,4^{n_0}-1\}$. This implies with (20) that

$$\begin{aligned} |e_{-1}(t,4^{n_0})| &\leq 2^{1-4^{n_0+1}+k} \\ &\leq \left(\frac{1}{2}\right)^{3k+1-\alpha_{n_0}} \qquad \text{for all } k \in \{0,\dots,4^{n_0}-1\}, \end{aligned}$$

proving (22). As a consequence, system (18) is exponentially stable. It remains to show that system (19) is not exponentially stable. System (19) can be represented in the following form

$$\begin{aligned} x_1^\Delta &= -x_1, \\ x_2^\Delta &= -x_2 + x_1, \end{aligned}$$

where $x = (x_1, x_2)$. Denote by $(x_1(t), x_2(t))$ the solution of this system starting at $t_0 = 4$ in $(1,1)$. A straightforward computation yields

$$\begin{aligned} \begin{pmatrix} x_1(4^{k+1}) \\ x_2(4^{k+1}) \end{pmatrix} &= \begin{pmatrix} -1-\alpha_k & 0 \\ 1 & -1-\alpha_k \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 1 & -2 \end{pmatrix}^{4^k-1} \times \\ &\qquad \times \begin{pmatrix} \alpha_k & 0 \\ 1 & \alpha_k \end{pmatrix} \begin{pmatrix} x_1(4^k) \\ x_2(4^k) \end{pmatrix}, \end{aligned}$$

which gives

$$x_2(4^{k+1}) = (2^{4^k-1} - 2^{4^k-4^{k+1}} - 1)x_1(4^k) + 2^{4^k-4^{k+1}} x_2(4^k).$$

This implies together with $x_1(4^k) = 2^{4-4^k}$ that

$$x_2(4^{k+1}) = 8 - 2^{3-4^{k+1}} + 2^{4^k-4^{k+1}} x_2(4^k) \quad \text{for all } k \in \mathbb{N}.$$

Hence, $\lim_{k\to\infty} x_2(4^k) = 8$. As a consequence, system (19) is not exponentially stable.

Proposition 4.1 *Let $\mathbb{T}$ be a time scale which is unbounded above. For $\lambda \in \mathbb{C}$ consider the Jordan block $J_\lambda \in \mathbb{C}^{d\times d}$ given by*

$$J_\lambda := \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ & \lambda & 1 & \dots & 0 \\ & & \ddots & & \vdots \\ & & & & \lambda \end{pmatrix}.$$

The scalar equation

$$x^{\Delta} = \lambda x \tag{24}$$

is weak-uniformly exponentially stable if and only if the system

$$x^{\Delta} = J_{\lambda} x \tag{25}$$

is weak-uniformly exponentially stable.

Proof Clearly, if $\lambda = 0$ then $e_0(t,\tau) = 1$ for all $t \geq \tau$. Hence, neither the system (24) nor the system (25) is weak-uniformly exponentially stable. Therefore we are only interested in the case $\lambda \neq 0$.

($\Rightarrow$) Assume that system (24) is weak-uniformly exponentially stable. Fix $s \in \mathbb{T}$. Hence, there exist $\alpha > 0$ and $K = K(s) \geq 1$ such that

$$|e_{\lambda}(t,\tau)| \leq K \exp(-\alpha(t-\tau)) \qquad \text{for all } t \geq \tau \geq s. \tag{26}$$

To verify the assertion we construct explicit bounds for the solution of (25) with initial condition $x(\tau) = \xi \in \mathbb{C}^d$ for a fixed $\tau \in \mathbb{T}$ with $\tau \geq s$. Without loss of generality we use the norm $\|x\| := \max\{|x_1|, \ldots, |x_d|\}$ for $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$ in our consideration.

Choose and fix $\varepsilon > 0$ such that $\alpha > d\varepsilon$. Define $\beta_j = \alpha + j\varepsilon - d\varepsilon$ for $j = 1, \ldots, d$. Clearly, $\beta_j > 0$ and we will prove by induction on $j = d, \ldots, 1$ that there exist constants K_j such that the j-th component of the solution of (25) is exponentially bounded by

$$|x_j(t)| \leq K_j \exp(-\beta_j(t-\tau))\|\xi\| \qquad \text{for all} \quad t \geq \tau. \tag{27}$$

For $j = d$ the assertion follows from the assumption as the d-th entry of $x(t)$ is a solution of (24) and hence by (26) we have

$$|x_d(t)| = |e_{\lambda}(t,\tau)\xi_d| \leq K_d \exp(-\beta_j(t-\tau))\|\xi\| \quad \text{for all } t \geq \tau,$$

where $K_d := K$. Assume that the assertion (27) is shown for some index $d \geq j+1 \geq 2$, i.e. there exists K_{j+1} with

$$|x_{j+1}(t)| \leq K_{j+1} \exp(-\beta_{j+1}(t-\tau))\|\xi\| \quad \text{for all } t \geq \tau. \tag{28}$$

By construction, the j-th component of the solution satisfies the equation

$$x_j^{\Delta}(t) = \lambda x_j(t) + x_{j+1}(t) \quad \text{for } t \in \mathbb{T}.$$

Using the variation of constants formula (see Bohner and Peterson [4, pp 77]) we have the representation

$$x_j(t) = e_{\lambda}(t,\tau)\xi_j + \int_{\tau}^{t} e_{\lambda}(t, u + \mu^*(u)) x_{j+1}(u)\, \Delta u.$$

Fix $t \in \mathbb{T}$. Using the exponential bound of $e_{\lambda}(t,\tau)$ and (28) we obtain

$$\begin{aligned} |x_j(t)| &\leq |e_{\lambda}(t,\tau)\xi_j| + \int_{\tau}^{t} |e_{\lambda}(t, u+\mu^*(u))||x_{j+1}(u)|\, \Delta u \\ &\leq K\|\xi\| \exp(-\alpha(t-\tau)) + K_{j+1}\|\xi\| \int_{\tau}^{t} g(u)\, \Delta u, \end{aligned} \tag{29}$$

where $g(u) := |e_\lambda(t, u + \mu^*(u))| \exp(-\beta_{j+1}(u - \tau))$. Denote by $t_1 < t_2 < \cdots < t_n$ the right scattered points in $[\tau, t]$ with

$$|1 + \lambda\mu^*(t_i)| \geq 2, \quad i = 1, 2, \ldots, n.$$

Now we are going to estimate $g(u)$ for $u \in [\tau, t]$. If $u = t_k$ for $k = 1, \ldots, n$ we get

$$\begin{aligned} e_\lambda(t, u) &= e_\lambda(t, u + \mu^*(u)) e_\lambda(u + \mu^*(u), u) \\ &= e_\lambda(t, u + \mu^*(u))(1 + \lambda\mu^*(u)). \end{aligned}$$

This implies with (26) that

$$|e_\lambda(t, u + \mu^*(u))| \leq \frac{K}{2} \exp(-\alpha(t - u)).$$

Therefore,

$$g(u) \leq \frac{K}{2} \exp(-\beta_{j+1}(t - \tau)) \qquad \text{if } u \in \{t_1, \ldots, t_n\}. \tag{30}$$

If $u \notin \{t_1, \ldots, t_n\}$, we get $\mu^*(u) \leq \frac{3}{|\lambda|}$. Applying (26) to $e_\lambda(t, u + \mu^*(u))$, we obtain

$$g(u) \leq K \exp(-\alpha(t - u)) \exp(\alpha\mu^*(u))) \exp(-\beta_{j+1}(u - \tau)).$$

Therefore,

$$g(u) \leq K \exp\left(\frac{3\alpha}{|\lambda|}\right) \exp(-\beta_{j+1}(t - \tau)) \quad \text{for } u \notin \{t_1, \ldots, t_n\}. \tag{31}$$

Combining (30) and (31), there exists $M > 0$ such that

$$g(u) \leq M \exp(-\beta_{j+1}(t - \tau)) \quad \text{for all } u \in [\tau, t].$$

This implies with (29) that

$$|x_j(t)| \leq K\|\xi\| \exp(-\alpha(t - \tau)) + MK_{j+1}\|\xi\|(t - \tau) \exp(-\beta_{j+1}(t - \tau)).$$

On the other hand, $\varepsilon(t - \tau) \leq \exp(\varepsilon(t - \tau))$ for all $t \geq \tau$. We thus obtain

$$|x_j(t)| \leq (K + \frac{MK_{j+1}}{\varepsilon})\|\xi\| \exp(-\beta_j(t - \tau)) \quad \text{for all } t \geq \tau,$$

proving (27) with $K_j := K + \frac{MK_{j+1}}{\varepsilon}$. As we have exponential decay of all components of the solution $x(t)$, we obtain the assertion.

We now construct an example which ensures that in general weak-uniform exponential stability does not imply uniform exponential stability.

Example 4.2 We define a discrete time scale $\mathbb{T}$ by

$$\mathbb{T} = \bigcup_{k=1}^{\infty} \mathbb{T}_k \cup [0, \infty),$$

where

$$\mathbb{T}_k := \{-k + \frac{-i}{k} : i = 0, 1, \ldots, k - 1\} \qquad \text{for all } k \in \mathbb{N}.$$

Consider on $\mathbb{T}$ the scalar system

$$x^{\Delta} = -x. \tag{32}$$

We first show that the system (32) is weak-uniformly exponentially stable. Obviously, for any $s \in \mathbb{T}$ with $s \geq 0$ we have

$$|e_{-1}(t, \tau)| = \exp(-(t - \tau)) \quad \text{for all } t \geq \tau \geq 0. \tag{33}$$

For an arbitrary but fixed $s \in \mathbb{T}$ with $s < 0$, we are going to estimate $|e_{-1}(t, \tau)|$ for $t \geq \tau \geq s$. A straightforward computation yields that

$$|e_{-1}(t, \tau)| = \begin{cases} 0, & \text{if } t \geq 0 > \tau, \\ \exp(-(t - \tau)), & \text{if } t \geq \tau \geq 0. \end{cases}$$

We thus obtain

$$|e_{-1}(t, \tau)| \leq K(s) \exp(-(t - \tau)) \quad \text{for all } t \geq \tau \geq s,$$

where

$$K(s) := \max_{0 \geq t \geq \tau \geq s} |e_{-1}(t, \tau)| \exp(t - \tau).$$

Hence system (32) is weak-uniformly exponentially stable. On the other hand, a direct computation gives that

$$|e_{-1}(-k, -k-1)| = (1 - \frac{1}{k})^k \quad \text{for all } k \in \mathbb{N},$$

which implies that system (32) is not uniformly exponentially stable.

Remark 4.1 Observe that Proposition 4.1 in combination with example 4.2 provides a negative answer to the question mentioned in the conclusion of Pötzsche et al [16] whether the uniform exponential stability of system (24) is a necessary condition for the exponential stability of system (25). Moreover, the time scale $\mathbb{T}$ in Pötzsche et al [16] is assumed to have bounded graininess. This assumption is dropped in Proposition 4.1.

By virtue of Proposition 4.1 in combination with an analogous argument as in the proof of Theorem 3.2 we get the following corollary to characterize weak-uniform exponential stability.

Corollary 4.1 *The system*

$$x^{\Delta} = Ax, \qquad A \in \mathbb{R}^{d \times d},$$

is weak-uniformly exponentially stable if and only if the system

$$x^{\Delta} = \lambda x$$

is weak-uniformly exponentially stable for every $\lambda \in \sigma(A)$.

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