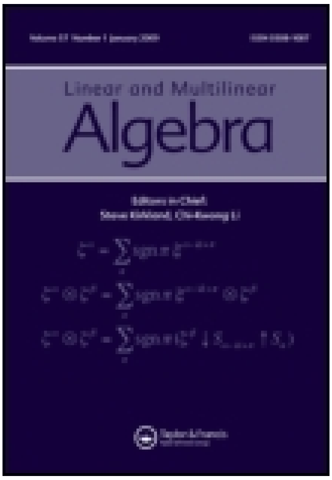

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/glma20

An algorithm to study the nonnegativity, regularity and stability via state-feedbacks of singular systems of arbitrary index

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Published online: 29 Apr 2014.

Linear and Multilinear Algebra, 2015
Vol. 63, No. 5, 882–892, http://dx.doi.org/10.1080/03081087.2014.904559

An algorithm to study the nonnegativity, regularity and stability via state-feedbacks of singular systems of arbitrary index

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Communicated by S. Kirkland

(*Received 14 March 2013; accepted 7 March 2014*)

This paper deals with singular systems of index $k \geq 1$. Our main goal is to find a state-feedback such that the closed-loop system satisfies the regularity condition and it is nonnegative and stable. In order to do that, the core-nilpotent decomposition of a square matrix is applied to the singular matrix of the system. Moreover, if the Drazin projector of this matrix is nonnegative, then the previous decomposition allows us to write the core-part of the matrix in a specific block form. In addition, an algorithm to study this kind of systems via a state-feedback is designed.

Keywords: control system; nonnegativity; state-feedback; stability

AMS Subject Classifications: 15A09; 93C05

1. Introduction

Nonnegative control systems appear in a wide range of areas such as: engineering, economical problems, electrical, mechanical and chemical processes.[1–3] The property of nonnegativity plays an important role in these applications. These systems are dynamical systems whose state variables are nonnegative at all times. In [4–6], the positive (nonnegative) singular systems have been widely developed.

Regularization of singular systems via state-feedbacks has been studied by different authors.[7–9] In general, those studies are based on the Weierstrass-Kronecker decomposition of the system, which uses two matrices P and Q that may change the information the original matrices. In this paper, we will use a different approach based on rearranging the information involved in the original matrices.

Stability of linear systems has been recently studied for autonomous descriptor systems in [10,11], for positive descriptor systems in [12,13] and for general linear systems in [14]. In both papers [14] and [11], special attention has been paid to nonnegativity of systems. Furthermore, some real problems have been treated in [15]. The design techniques proposed

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in this last case are based on the theory of externally nonnegative systems. Numerical aspects of singularity in systems can be found in [10,16–19] and those related to compute Drazin inverses in [20–22].

In some applications, the evolution of the system is represented in the state-space form and it is usual to look for feedbacks which transform it into a new system with specific properties, such as stability, symmetry, etc. In this paper, we are interested on finding feedbacks that transform the original system into a new system satisfying the regularity condition, nonnegativity and stability.

For a given matrix $A \in \mathbb{R}^{n\times n}$, a matrix $X \in \mathbb{R}^{n\times n}$ is called its Drazin inverse if the properties $XAX = X$, $AX = XA$ and $A^{q+1}X = A^q$ hold, where $q = \mathrm{ind}(A)$ is the index of A, that is, the smallest nonnegative integer such that $\mathrm{rank}(A^{q+1}) = \mathrm{rank}(A^q)$. The matrix X always exists, it is unique and denoted by A^D.[23,25,28] This generalized inverse matrix has been used to characterize the properties of nonnegative singular systems [26,27]; in both papers, the authors used the whole coefficient matrices. An important subclass corresponds to the $q = 1$ case, where the generalized inverse is called group inverse of A and denoted by $A^{\#}$. We call Drazin projector of a square matrix A to the matrix AA^D. Moreover, the Moore–Penrose inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ that satisfies $AXA = A$, $XAX = X$, $(AX)^T = AX$, and $(XA)^T = XA$. This generalized inverse always exists, it is unique and denoted by $X = A^{\dagger}$.[23,28]

Some extensions of results related to Drazin inverses on operator theory and Banach algebras have been presented, for example, in [29,30]. Also, the Drazin inverse perturbation theory has been studied from different points of view. For instance, in [31,32], algebraic approaches has been given while a setting in systems theory can be found in [33–35].

We will stand $A \geq O$ for a matrix A with nonnegative entries. The symbols $R(A)$ and $N(A)$ will denote the range and the null space of the matrix A and $\sigma(A)$ the spectrum of a square matrix A. We will use the set $\sigma(E, A) = \{\lambda \in \mathbb{C} : \det(A - \lambda E) = 0\}$ where $E, A \in \mathbb{R}^{n\times n}$. As usual, the open ball with centre $a \in \mathbb{C}$ and radius $r > 0$ is defined by $\mathcal{B}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

This paper is organized as follows. In Section 2, we will state the problem of nonnegativity for singular systems via state-feedbacks. Moreover, some preliminary results on matrices with nonnegative Drazin projector are also included. In Section 3, necessary and sufficient conditions for the existence of feedbacks such that the closed-loop system is nonnegative, regular and stable are obtained. The solution of this closed-loop system is also constructed in this section. Finally, in Section 4, we give an algorithm to construct the aforementioned feedback and examples that illustrate the results.

2. Statement of the problem

In this paper, we consider discrete-time singular control systems like:

$$\begin{cases} Ex(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $x(k) \in \mathbb{R}^{n\times 1}$, $u(k) \in \mathbb{R}^{m\times 1}$ and $y(k) \in \mathbb{R}^{p\times 1}$ with $\mathrm{rank}(E) = r < n$. In general, this system is denoted by (E, A, B, C) or by (E, A, C) when $B = O$. A wide analysis of singular systems can be found in [36] where structural properties, pole assignment and regularization of these systems are studied.

The system (E, A, B, C) is said to satisfy the regularity condition if there exists a scalar α such that $\det(\alpha E + A) \neq 0$. In this case, the system (1) has solution. For further details, we refer the reader, for example, to [37].

On the other hand, it is well known that a matrix $E \in \mathbb{R}^{n\times n}$ of positive rank and positive index can be written as $E = B_E + N_E$ where

$$B_E = S\begin{bmatrix} C & O \\ O & O \end{bmatrix}S^{-1} \qquad \text{and} \qquad N_E = S\begin{bmatrix} O & O \\ O & N \end{bmatrix}S^{-1} \tag{2}$$

with S, C nonsingular matrices and N nilpotent. Note that B_E has index 1, N_E is also nilpotent and $B_E N_E = N_E B_E = O$. Expression (2) is called the core-nilpotent decomposition of the matrix E.[28] Note that the nilpotent part disappears when the index of E equals 1.

Throughout the paper, we consider systems (E, A, B, C) which may not satisfy the regularity condition and its matrix E has nonnegative Drazin projector. In [24], under these conditions, it has been stated that the matrix E can be written by means of the core-nilpotent decomposition as $E = B_E + N_E$ where N_E is a nilpotent matrix and

$$PB_EP^T = \begin{bmatrix} I \\ O \\ S \end{bmatrix} XTY \begin{bmatrix} I & M & O \end{bmatrix} \tag{3}$$

where P is a permutation matrix, P^T its transpose, $T \in \mathbb{R}^{r\times r}$ is a nonsingular matrix, $X = \mathrm{diag}(x_1, x_2, \ldots, x_r)$, $Y = \mathrm{diag}(y_1^T, y_2^T, \ldots, y_r^T)$, x_i and y_i are unit positive vectors such that $YX = I$, and M, S are nonnegative matrices of sizes $q\times s$ and $t\times q$, respectively. Notice that expression (3) has a block structure conformable to $n \times n = (q + s + t) \times (q + s + t)$. We also remark that the decomposition (3) appeared firstly in [38]. Later, the authors studied decomposition (3) and applications in [24].

On the other hand, expression (3) allows us to define the function

$$\Psi_{M,S} : \mathbb{R}^{q\times q} \to \mathbb{R}^{n\times n}$$

as

$$\Psi_{M,S}(K) = \begin{bmatrix} I \\ O \\ S \end{bmatrix} K \begin{bmatrix} I & M & O \end{bmatrix}$$

where the matrices M, S, and K are conformable for multiplication. Clearly, $PB_EP^T = \Psi_{M,S}(XTY)$. Using block products of matrices, it is possible to deduce the following result.

LEMMA 2.1 *The function $\Psi_{M,S}$ previously defined satisfies the following properties:*

(a) $\Psi_{M,S}(K_1 + K_2) = \Psi_{M,S}(K_1) + \Psi_{M,S}(K_2)$, *for every $K_1, K_2 \in \mathbb{R}^{q\times q}$.*
(b) $\Psi_{M,S}(\mu K) = \mu\Psi_{M,S}(K)$, *for every $\mu \in \mathbb{R}$ and $K \in \mathbb{R}^{q\times q}$.*
(c) $\Psi_{M,S}(K_1K_2\ldots K_l) = \Psi_{M,S}(K_1)\Psi_{M,S}(K_2)\ldots\Psi_{M,S}(K_l)$, *for every $K_1, \ldots, K_l \in \mathbb{R}^{q\times q}$. In particular, $\Psi_{M,S}(K^s) = (\Psi_{M,S}(K))^s$, for every $K \in \mathbb{R}^{q\times q}$ and $s \in \mathbb{N}$.*

Now, using the permutation matrix P appearing in (3), the singular system (E, A, B, C) can be transformed by means of the change of state variable $z(k) = Px(k)$ into the equivalent system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ where

$$\widetilde{E} = PEP^T, \quad \widetilde{A} = PAP^T, \quad \widetilde{B} = PB, \quad \widetilde{C} = CP^T. \tag{4}$$

Clearly, some properties of the matrix E are inherited by the matrix $\widetilde{E}$. For example, E and $\widetilde{E}$ have the same index and the Drazin projector of $\widetilde{E}$ is also nonnegative. Using Lemma 2.1 and the core-nilpotent decomposition of E, the following result gives some expressions related to $\widetilde{E}$.

LEMMA 2.2 *The following properties hold:*

(a) $\widetilde{E} = \Psi_{M,S}(XTY) + PN_EP^T$ *where* $\Psi_{M,S}(XTY)PN_E = N_EP^T\Psi_{M,S}(XTY) = O$.
(b) $\widetilde{E}^D = \Psi_{M,S}(XT^{-1}Y)$ *where* $PN_EP^T\widetilde{E}^D = \widetilde{E}^DPN_EP^T = O$.
(c) $\widetilde{E}\widetilde{E}^D = \Psi_{M,S}(XY) \geq O$.

Notice that properties (a) and (b) in Lemma 2.2 can be deduced from the facts $B_EN_E = N_EB_E = O$ and $\Psi_{M,S}(XT^{-1}Y) = PB_E^{\#}P^T$. Moreover, we remark that property (a) becomes $\widetilde{E} = \Psi_{M,S}(XTY)$ when the index of E equals 1.

For the system (4), we are going to construct a state-feedback $u(k) = Fz(k)$ such that the closed-loop system $(\widetilde{E}, \widetilde{A} + \widetilde{B}F, \widetilde{C})$ which satisfies the regularity condition is nonnegative and stable. We recall that a system (E, A, B, C) is called nonnegative if $x(0) \geq 0$ and $u(k) \geq 0$ for all k imply $x(k) \geq 0$ and $y(k) \geq 0$ for all k. Furthermore, a system (E, A, B, C) is said to be stable if $\sigma(E, A) \subset \mathcal{B}(0, 1)$.

3. Constructing the feedback

As we have seen previously, the original system (E, A, B, C) given by (1) can be transformed into the equivalent system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ by means of the change of variable $z(k) = Px(k)$ as indicated in (4). In this way, the feedback $u(k) = Fz(k)$ gives the closed-loop system $(\widetilde{E}, \widetilde{A} + \widetilde{B}F, \widetilde{C})$.

In order to achieve the regularity condition, we can search for an adequate matrix F such that

$$\widetilde{A} + \widetilde{B}F = I - \beta\widetilde{E} \tag{5}$$

holds.

PROPOSITION 3.1 *For a given $\beta \in \mathbb{R}$, there exists a feedback $F \in \mathbb{R}^{m\times n}$ satisfying Equation (5) if and only if $R(I - \beta E - A) \subseteq R(B)$. In this case, the most general form for F is*

$$F = \widetilde{B}^\dagger(I - \beta\widetilde{E} - \widetilde{A}) + (I - \widetilde{B}^\dagger\widetilde{B})Z, \tag{6}$$

where Z is an arbitrary matrix of adequate size.

Proof For a given $\beta \in \mathbb{R}$, the matrix equation $\widetilde{B}F = I - \beta\widetilde{E} - \widetilde{A}$ has solution if and only if $\widetilde{B}\widetilde{B}^\dagger(I - \beta\widetilde{E} - \widetilde{A}) = I - \beta\widetilde{E} - \widetilde{A}$ (see [23]), which is equivalent to

$$(I - \widetilde{B}\widetilde{B}^\dagger)(I - \beta\widetilde{E} - \widetilde{A}) = O.$$

By using properties of the Moore–Penrose inverse, the last equation is equivalent to $R(I - \beta\widetilde{E} - \widetilde{A}) \subseteq N(I - \widetilde{B}\widetilde{B}^\dagger) = R(\widetilde{B}\widetilde{B}^\dagger) = R(\widetilde{B})$ since $\widetilde{B}\widetilde{B}^\dagger$ is a projector. This means that

$R(I - \beta PEP^T - PAP^T) \subseteq R(PB)$, which is $R(I - \beta E - A) \subseteq R(B)$, where the information has been expressed in terms of the original matrices E, A and B. Hence, by Theorem 1 in [23,p.52], we have that F has the form given in (6). $\square$

The arbitrariness of the matrix Z in expression (6) produces all the possible feedbacks for the selected value of β in (5). If we need to compute only one feedback, we can clearly take $Z = O$.

With regard to the nonnegativity of the closed-loop system, we have to study the conditions: $\widetilde{E}\widetilde{E}^D \geq O$, $\widetilde{E}^D(\widetilde{A} + \widetilde{B}F) \geq O$ and $\widetilde{C}\widetilde{E}\widetilde{E}^D \geq O$ [39, Theorem 2.1].

The first condition is satisfied by Lemma 2.2 and the analysis of the other two conditions lead to the following result.

Theorem 3.2 *Let (E, A, B, C) be the system given by (1). The closed-loop system $(\widetilde{E}, \widetilde{A} + \widetilde{B}F, \widetilde{C})$ constructed with the feedback $u(k) = Fz(k)$ and the matrices given in (4) is regular and nonnegative if there exists a scalar β such that $\widetilde{A} + \widetilde{B}F = I - \beta\widetilde{E}$ and the following conditions hold:*

(a) $T^{-1} - \beta I \geq O$,
(b) $(C_1 + C_3 S)X \geq O$,

where $\widetilde{C} = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}$ according to the blocks of $\Psi_{M,S}(XY)$ with M, S, X, Y and T as in (3).

Proof Assume that there exists a scalar β such that $\widetilde{A} + \widetilde{B}F = I - \beta\widetilde{E}$. We have to analyse the conditions $\widetilde{E}^D(\widetilde{A} + \widetilde{B}F) \geq O$ and $\widetilde{C}\widetilde{E}\widetilde{E}^D \geq O$.

Since $\widetilde{E}^D(\widetilde{A} + \widetilde{B}F) = \widetilde{E}^D(I - \beta\widetilde{E})$ and $\widetilde{E}^D = \Psi_{M,S}(XT^{-1}Y)$ by Lemma 2.2, we have that the first condition is equivalent to

$$\Psi_{M,S}(XT^{-1}Y)[I - \beta\Psi_{M,S}(XTY)] \geq O. \tag{7}$$

Thus, Lemma 2.1 allows us to write inequality (7) as:

$$\Psi_{M,S}(X(T^{-1} - \beta I)Y) \geq O,$$

where we have used that $YX = I$. This last inequality implies that the block $(1, 1)$ of $\Psi_{M,S}(X(T^{-1} - \beta I)Y)$ has to be nonnegative, which is $X(T^{-1} - \beta I)Y \geq O$. Since $X, Y \geq O$ and $YX = I$, we get $T^{-1} - \beta I \geq O$. Notice that the only block that gives information on the nonnegativity is the block $(1, 1)$ due to M and S are nonnegative matrices.

Related to the condition $\widetilde{C}\widetilde{E}\widetilde{E}^D \geq O$ to be analysed to assure the nonnegativity of the system, by Lemma 2.2, we have that it is equivalent to

$$\widetilde{C}\Psi_{M,S}(XY) \geq O.$$

Partitioning the matrix $\widetilde{C}$ according to the sizes of the blocks of $\Psi_{M,S}(XY)$ as

$$\widetilde{C} = CP^T = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \tag{8}$$

we get

$$\begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \begin{bmatrix} I \\ O \\ S \end{bmatrix} XY \begin{bmatrix} I & M & O \end{bmatrix} \geq O.$$

Again, the block (1, 1) in the last matrix contains the main information to assure the nonnegativity of the entire matrix. This condition becomes $(C_1 + C_3 S)X \geq O$ taking into account that $X \geq O$ and $YX = I$. □

Next, we analyse the stability of the closed-loop system considering that the system is regular. The following result gives sufficient conditions for the stability of the system.

THEOREM 3.3 *Let (E, A, B, C) be the system given by (1). The closed-loop system $(\widetilde{E}, \widetilde{A} + \widetilde{B}F, \widetilde{C})$ constructed with the feedback $u(k) = Fz(k)$ and the matrices given in (4) is regular and stable if there exists a scalar $\beta \in \mathcal{B}(\gamma, 1)$ such that $\widetilde{A} + \widetilde{B}F = I - \beta\widetilde{E}$ for every $\gamma \in \sigma(T^{-1})$ where T is given in (3).*

Proof Since there exists a scalar β such that $\widetilde{A} + \widetilde{B}F = I - \beta\widetilde{E}$, the regularity condition holds.

On the other hand, the closed-loop system is stable if and only if the set $\sigma(\widetilde{E}, \widetilde{A} + \widetilde{B}F) \subset \mathcal{B}(0, 1)$. By definition, the set $\sigma(\widetilde{E}, \widetilde{A} + \widetilde{B}F)$ is given by the λ's such that

$$0 = \det(\lambda\widetilde{E} - I + \beta\widetilde{E}) = \det(\gamma\widetilde{E} - I) = \gamma^n \det\left(\widetilde{E} - \frac{1}{\gamma}I\right)$$

with $\gamma = \lambda + \beta \neq 0$, that is, $1/\gamma \in \sigma(\widetilde{E})$. In order to get $\lambda \in \mathcal{B}(0, 1)$, we must have $\beta \in \mathcal{B}(\gamma, 1)$ for every γ such that $1/\gamma \in \sigma(\widetilde{E})$.

By Lemma 2.2, we have $\sigma(\widetilde{E}) = \sigma(XTY) \cup \{0\}$. Moreover, $\sigma(XTY) - \{0\} = \sigma(T)$. In fact, if $\alpha \in \sigma(XTY) - \{0\}$ then there exists $z \neq 0$ such that $XTYz = \alpha z$. Premultiplying by Y and using that $YX = I$, we get that $TYz = \alpha Yz$, so $\alpha \in \sigma(T)$ because $Yz \neq 0$. Similarly, if $\alpha \in \sigma(T)$ then there exists $z \neq 0$ such that $Tz = \alpha z$ and, furthermore, $\alpha \neq 0$ because T is nonsingular. Again, premultiplying by X and using that $YX = I$ we get that $XTYXz = \alpha Xz$, so $\alpha \in \sigma(XTY) - \{0\}$ because $Xz \neq 0$. Thus, $\sigma(XTY) - \{0\} = \sigma(T)$.

Hence, the stability of the closed-loop system is guaranteed by the condition $\beta \in \mathcal{B}(\gamma, 1)$ for every γ such that $1/\gamma \in \sigma(T)$. □

Up to now, we have analysed conditions such that the closed-loop system $(\widetilde{E}, \widetilde{A} + \widetilde{B}F, \widetilde{C})$ satisfies the regularity, nonnegativity and stability conditions. Under these assumptions, we present an explicit solution of the system where the nonnegativity of the states and outputs can be clearly checked.

The solution of the closed-loop system is $y(k) = \widetilde{C}z(k)$ where $z(k)$ is given by [37]:

$$z(k) = (\widetilde{E}^D(I - \beta\widetilde{E}))^k \widetilde{E}^D \widetilde{E} z(0)$$

with $z(0) \in R\left(\begin{bmatrix} \widetilde{E}^D\widetilde{E} & (I - \widetilde{E}^D\widetilde{E})(I - \beta\widetilde{E})^D \end{bmatrix}\right)$. This last set is the subspace of the initial admissible conditions of the system. As $\widetilde{E}^D$ and $I - \beta\widetilde{E}$ commute, the properties of the Drazin inverse allow us to write

$$\begin{aligned} z(k) &= ((I-\beta\widetilde{E})\widetilde{E}^D)^k z(0) \\ &= ((I-\beta\Psi_{M,S}(XTY)-\beta PN_E P^T)\Psi_{M,S}(XT^{-1}Y))^k z(0) \\ &= \Psi_{M,S}(X(T^{-1}-\beta I)^k Y)z(0) \end{aligned}$$

where we have used the definition of the function $\Psi_{M,S}$ and its properties given in Lemma 2.1. Hence, it is clear that the states $z(k)$ are nonnegative since $T^{-1}-\beta I \geq O$ and $z(0) \geq 0$.

Then, the outputs of the system are given by:

$$\begin{aligned} y(k) &= \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix} \Psi_{M,S}(X(T^{-1}-\beta I)^k Y)z(0) \\ &= (C_1+C_3S)X(T^{-1}-\beta I)^k Y \begin{bmatrix} I & M & O \end{bmatrix} z(0) \end{aligned}$$

where clearly $(C_1+C_3S)X \geq O$ and $T^{-1}-\beta I \geq O$ imply $y(k) \geq 0$ for a nonnegative initial admissible condition.

4. Algorithm and examples

This section gives a procedure that systematizes the reasoning presented in Section 3. In order to do that, we provide an algorithm where the existence of a feedback that guarantees the regularity, nonnegativity and stability of a system is analysed. Moreover, the construction of such a feedback is also carried out.

Algorithm:

Inputs: A singular system (E, A, B, C) that satisfies $EE^D \geq O$.

Outputs: Matrices F such that the closed-loop system $(\widetilde{E}, \widetilde{A}+\widetilde{B}F, \widetilde{C})$ is regular, nonnegative, and stable, and the solution of this system.

Step 1: Transform the original system (E, A, B, C) into the equivalent system $(\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C})$ given in (4).
Step 2: If $(C_1+C_3S)X \ngeq O$, then go to Step 10.
Step 3: Compute $\sigma(T)$ and $\Lambda = \cap\{\mathcal{B}(\gamma, 1) : \ 1/\gamma \in \sigma(T)\}$.
Step 4: If $\Lambda \cap \mathbb{R} = \emptyset$, then go to Step 10.
Step 5: Choose $\beta \in \Lambda \cap \mathbb{R}$. Note that in each step, we propose to try with a finite number of different values of β else go to Step 10.
Step 6: If $T^{-1}-\beta I \ngeq O$, then go to Step 5.
Step 7: Compute $\widetilde{B}^\dagger$.
Step 8: If $(I-\widetilde{B}\widetilde{B}^\dagger)(I-\beta\widetilde{E}-\widetilde{A}) \neq O$, then go to Step 5 or go to Step 10.
Step 9: Construct $F = \widetilde{B}^\dagger(I-\beta\widetilde{E}-\widetilde{A})+(I-\widetilde{B}^\dagger\widetilde{B})Z$ with Z arbitrary. Go to Step 11.
Step 10: *'There exists no matrix F such that the closed-loop system is nonnegative'*. Go to End.
Step 11: The closed-loop system is regular, nonnegative and stable. The outputs of the system $(\widetilde{E}, \widetilde{A}+\widetilde{B}F, \widetilde{C})$ are given by

$$y(k) = (C_1+C_3S)X(T^{-1}-\beta I)^k Y \begin{bmatrix} I & M & O \end{bmatrix} z(0).$$

End

We illustrate the obtained results with the following examples.

Example 4.1 Let (E, A, B, C) be a singular system whose matrix $E = \Psi_{M,S}(XTY) + N_E$ is given by

$$\Psi_{M,S}(XTY) = \begin{bmatrix} 3 & 0 & 0 & 0 & 3 & 6 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 6 & 12 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 7 & 7 & 8 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 42 & 0 & 6 & 6 & 42 & 84 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ O \\ S \end{bmatrix} XTY \begin{bmatrix} I & M & O \end{bmatrix}$$

with

$$S = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 2 & 4 \end{bmatrix}, M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, T = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}, X = \left[\begin{array}{c|c} 1 & 0 \\ 2 & 0 \\ \hline 0 & 1 \\ 0 & 1 \end{array}\right], Y = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1/2 & 1/2 \end{array}\right]$$

and

$$N_E = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let

$$A = \frac{1}{3} \begin{bmatrix} 6 & 1 & 0 & 0 & 3 & 6 & 0 & 0 & 0 \\ 12 & 3 & 18 & 12 & 12 & 24 & 0 & 0 & 0 \\ -1 & 0 & 4 & 2 & -1 & -2 & 0 & 0 & 0 \\ -1 & 0 & 1 & 4 & -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 8 & 0 & 7 & 7 & 8 & 16 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 42 & 0 & 6 & 6 & 42 & 84 & 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix},$$

with C_1, C_2, and C_3 arbitrary matrices satisfying $(C_1 + C_3 S)X \geq O$.

Note that Steps 1 and 2 of the algorithm hold. Now, we compute Step 3 obtaining

$$\sigma(T) = \{2, 3\} \quad \text{and} \quad \mathcal{B}(1/2, 1) \cap \mathcal{B}(1/3, 1) \cap \mathbb{R} =]-1/2, 4/3[.$$

Following with Steps 4, 5 and 6, we have to choose $\beta \in]-1/2, 4/3[$ such that

$$T^{-1} - \beta I = \begin{bmatrix} 1/3 - \beta & 0 \\ 1/6 & 1/2 - \beta \end{bmatrix} \geq O \Leftrightarrow \begin{cases} 1/3 - \beta \geq 0 \\ 1/2 - \beta \geq 0 \end{cases}.$$

So, for example, we can choose $\beta = -1/3$. Since $B^\dagger = B^T$ (Step 7), we have that $(I - BB^\dagger)(I - \beta E - A) = O$. Finally, Steps 9 and 11 provide the feedback and the solution of the closed-loop system which is regular, nonnegative and stable.

We close this paper with a real example based on the Leontief model.

Example 4.2 A dynamic Leontief model of a multisector economy has the form [36]

$$x(k) = Lx(k) + C(x(k+1) - x(k)) + Du(k)$$

where $x(k)$ is the vector of output levels, $Du(k)$ is the vector of final demands (excluding investment), L is the Leontief input-output matrix and C is the capital coefficient matrix. The matrices L and C are assumed to be known and time invariant. An essential property of these kind of models is the nonnegativity.

Considering three sector economy Leontief model with the following coefficient matrices:

$$L = \begin{bmatrix} 0.5 & 0.6667 & 0.75 \\ 0 & 0 & 0 \\ 0.5 & 1.6667 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0.3 & 0.4 & 0.45 \\ 0 & 0 & 0 \\ 0.6 & 0.8 & 0.9 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we arrive at the singular system $Cx(k+1) = (I - L + C)x(k) - Du(k)$, which is

$$\begin{bmatrix} 0.3 & 0.4 & 0.45 \\ 0 & 0 & 0 \\ 0.6 & 0.8 & 0.9 \end{bmatrix} x(k+1) = \begin{bmatrix} 0.8 & -0.2667 & -0.3 \\ 0 & 1 & 0 \\ 0.1 & -0.2667 & 0.9 \end{bmatrix} x(k) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k),$$

which is not nonnegative. We apply the Algorithm to obtain a nonnegative, regular and stable closed-loop system. To do that, let

$$\widetilde{E} = PCP^T = \begin{bmatrix} I \\ O \end{bmatrix} XTY \begin{bmatrix} I & M \end{bmatrix} = \begin{bmatrix} 0.3 & 0.45 & 0.4 \\ 0.6 & 0.9 & 0.8 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{A} = P(I - L + C)P^T = \begin{bmatrix} 0.8 & -0.3 & -0.2667 \\ 0.1 & 0.9 & -0.2667 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \widetilde{B} = P(-D) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; Y = \begin{bmatrix} 0.25 & 0.375 \end{bmatrix}, \; T = 1.2, \; M = \begin{bmatrix} 1.3333 \\ 0 \end{bmatrix}.$$

Since $\Lambda = \mathcal{B}\left(\frac{5}{6}, 1\right) \cap \mathbb{R} \neq \emptyset$, Steps 3 and 4 hold and we can choose $\beta = \frac{2}{3} \in \Lambda$ in Step 5. So, Step 6 holds and computing $\widetilde{B}^\dagger = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}$ we can check Step 8. Now, we can obtain F directly from expression in Step 9 as desired.

Funding

This paper has been partially supported by Ministry of Education of Spain [grant number DGI MTM2010-18228].

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