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Canonical interconnection of discrete linear port-Hamiltonian systems

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Abstract— **This paper deals with the canonical interconnection of discrete-time linear port-Hamiltonian systems. A conservative discrete linear port-Hamiltonian dynamics involving a modified conjugate port-output is introduced. It is shown that the projection yielding the discrete dynamics and the composition by canonical interconnection commute. As a by-product, symplecticity of the numerical flow is preserved by interconnection whenever input vector fields are Hamiltonian vector fields, which is analogous to the continuous case. The negative feedback interconnection of two circuits illustrates the results.**

I. INTRODUCTION

Port-Hamiltonian systems (PHSs) [13] have been derived from an energy-based mathematical representation of physical systems. A PHS actually characterizes physical phenomena from an energetic and a geometric point of view in a modular way. This way of encoding intrinsic system properties allows to consider complex system as a collection of subsystems, where each subsystem is either a physical system component or a control component. In this framework, the key idea therefore relies on the representation of energy exchanges inside subsystems and between subsystems. Thus, modeling and control issues can be addressed as a network analysis and design, where nodes are subsystems and edges are energy exchanges between nodes (hence between subsystems). A fundamental property of PHSs concerns their composition by interconnection: any power-conserving structure connects PHSs in such a way that the resulting system is again a PHS. One says that PHSs are stable under composition.

However, the discrete settings draws another picture: in general, time-discretization destroys this fundamental property and the underlying modeling and control design tools are lost. Approximation of Hamiltonian dynamics is a wide research area mainly addressed by two scientific communities with separate objectives. An extensive literature concerns the numerical approach, and more recently the discrete system dynamic approach.

On the one hand, people from numerical simulation are concerned with long time simulation [2], [9], [12], stability [15], efficiency [14] of integration methods. They study, develop and refine energetic [8], [10], [3], geometric [4], [14], [12] or multi-symplectic [1] integrators. Notice that among the huge literature available, we do only give few references since we do not precisely share the same discretization objectives. The interested reader is invited to start with [9] or [12].

On the other hand, people from automatic control, in the context of port-Hamiltonian framework, are interested in a modular representation of physical systems taking into account input and output variables. For modeling issue, discretization of the underlying geometry has been first considered, characterizing the discrete relationship between the effort and flow variables for ODEs (*e.g.*, discrete Dirac structure [17]), and PDEs (1-dimensional Stokes-Dirac structure case [5] involving boundary variables interpolation, n-dimensional case [16] by introducing a new discrete summation by parts formula on primal-dual complexes). These results strongly involve discrete geometry tools but do not consider time integration, which is an important aspect here. It is there assumed that the time-discretization scheme is chosen in such a way that the discrete energy time-variation is given by the product of discrete effort and flow variables. However, it is not the case in general (for instance, forward Euler scheme failed to fulfill this requirement). For control design issues where the numerical flow is a cornerstone, the time-discretization has been considered (for IDA-PBC design see for instance [11] where a modified port-output is introduced, and [6], [7] where a gradient-based approximation is used). The latter approaches are concerned with the discrete energetic behavior but for stability analysis purposes within dissipative PHS. Therefore, an *almost* energy balance seems sufficient to ensure asymptotic stability of the closed-loop when dissipation is added.

In this paper, we are concerned with discrete Hamiltonian dynamics and their power-conserving interconnection. The goal is to provide a discrete framework for modeling and control issues within discrete PHS. It is therefore fundamental to note that the results presented here only concern linear PHS and canonical interconnection. We shall define a class of conservative discrete dynamics which approximates linear PHS. We then process canonical interconnection of these discrete systems and show that the resulting dynamics is of the same type. Finally, we study interconnection of discrete linear PHSs and discretization of interconnected linear PHS and show that these operations commute.

This paper is organized as follows. Next section deals with continuous linear PHSs and their composition through canonical interconnection structure. Section 3 concerns the discrete linear PHS we define. It shown that the proposed discrete PHSs are conservative. In section 4, it is shown that the feedback interconnection preserves the structure, and that composition and projection maps commute. Section 5

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illustrates the results.

II. CONTINUOUS-TIME HAMILTONIAN FRAMEWORK

In this section, we recall basics on port-Hamiltonian systems, which can be found in detail in [18].

We are concerned with linear port-Hamiltonian systems on $\mathbb{R}^m$ with energy function H being a non-degenerate positive definite quadratic form. Such dynamics is generated by a triplet (J,Q,G) and the following equations

$$(\Sigma)\ :\ \left\{\begin{array}{l} \dot{x} = JQx + G(x)u \\ y = G(x)^T Qx \end{array}\right. \tag{1}$$

where J is a constant skew-symmetric $(m\times m)$-matrix, the energy $H(x) = \frac{1}{2}x^T Qx$ is a real-valued function on $\mathbb{R}^m$ with Q a non-singular symmetric positive definite $(m\times m)$-matrix, and the input vector field G is any full-column rank $(m\times p)$-matrix. $u\in\mathbb{R}^p$ is the control port-input variable and $y\in\mathbb{R}^p$ its conjugate port-output variable.

It is well-known that such dynamical systems are *conservative*. That is to say, they are passive (the supply rate is given as the input-output product) and lossless with respect to storage function H (the dissipation inequality holds with equality). Indeed, integrating the energy balance equation from t_0 to t yields to

$$H(x(t)) = H(x(t_0)) + \int_{t_0}^{t} u^T(s)y(s)\mathrm{d}s. \tag{2}$$

The previous relation translates the fact that the stored energy at time t equals the sum of the stored energy at time t_0 and the externally supplied energy during the time interval $[t_0,t]$.

Let us now consider two systems of the form (1) indexed by subscripts $_{1,2}$ with the same input dimension $p = p_1 = p_2$ and perform their (full) interconnection through the canonical power-conserving structure (*i.e.* the standard negative feedback interconnection) given as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{3}$$

where I_p stands for the $(p\times p)$-identity matrix. The resulting interconnected system Σ_{12} then writes

$$\begin{array}{rl} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = \begin{bmatrix} J_1 & -G_1 G2^T \\ G_2 G_1^T & J_2 \end{bmatrix}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \iff \quad \dot{X}_{12} & = J_{12}Q_{12}X_{12} \end{array} \tag{4}$$

The previous dynamics is again a linear Hamiltonian system (1) over $\mathbb{R}^{m_1+m_2}$. The system state is given by $X_{12} = [x_1^T\, x_2^T]^T$, the energy H_{12} is the sum of the subsystems energies $H_{12}(X_{12}) = H_1(x_1) + H_2(x_2) = \frac{1}{2}X_{12}^T Q_{12}X_{12}$ with Q_{12} a non-singular symmetric positive definite matrix (by assumptions on Q_1 and Q_2), and the structure matrix J_{12} trivially satisfies $J_{12} + J_{12}^T = 0$. One says that port-Hamiltonian systems are stable by interconnection.

Remark 2.1: Note that for simplicity, we only detail here the full interconnection case. However, partial interconnection, which means that a subset of port variables are constrained by interconnection, processes in the same way. The canonical interconnection (3) then involves $2k$ variables, with $k \leqslant \min(p_1,p_2)$, and the resulting interconnected system is of the form (1) where the Hamiltonian functions H_{12} is the sum of subsystems energies H_1 and H_2, the structure matrix J_{12} is given as in (4) (with respect to the $2k$ input vector fields involved in the interconnection), the input vector fields G_{12} is obtained by merging the remaining vector fields associated with input variables $[u_{i,k+1}\dots u_{i,p_i}]_{i=1,2}$, and the conjugate port-output directly follows from H_{12} and G_{12}.

The approach presented here aims at reproducing intrinsic PHS properties (2) and (4) at a discrete level. This is the topics of the two following sections.

III. CONSERVATIVE DISCRETE LINEAR PHS

This section investigates the time-discretization of linear PHSs (1). A discrete-time linear PHS is introduced and properties of its numerical flow are studied. It is shown to be *conservative*. This is the slight difference with the definitions of discrete-time port-Hamiltonian dynamics which can be found in the literature (*e.g.* [11], [7]) where an *almost* energy balance is satisfied. Indeed in these references, since the time-discretization is performed towards control design issues (IDA-PBC design based on model matching), energy balance is involved in the closed-loop stability analysis (Lyapounov-based) where damping is added. The objective is therefore to establish a dissipative inequality rather than a conservative property.

We shall first recall well-known basics on the approximation of linear Hamiltonian systems using the simplest geometric integrator, namely midpoint Euler. We then transfer such integrator into the port-Hamiltonian framework and define a conservative discrete linear PHS.

A. Linear Hamiltonian systems approximation

Approximation of Hamiltonian dynamics is a wide research area. A famous result concerns geometric integrators which came up when considering long time simulation issue. Relative to linear Hamiltonian systems, the time-centered Euler scheme (or midpoint Euler) is of great interest [4], [12] from an energetic and a geometric viewpoint.

Applying this scheme to (1) with $u\equiv 0$ leads to the discrete system

$$\frac{x_{n+1} - x_n}{\Delta t} = JQ\frac{x_{n+1} + x_n}{2}. \tag{5}$$

Assume that $N := -\frac{\Delta t}{2}JQ$ is non-exceptional, that is $\det(I+N) \neq 0$, which is always possible for *ad hoc* values of Δt, the numerical flow of (5) then writes

$$x_{n+1} = (I+N)^{-1}(I-N)x_n =: Ax_n. \tag{6}$$

A is said to be the Cayley transform of N. One now recalls characterizations of the properties of the numerical flow (see *e.g.* [4] for details and proofs).

We first recall the definitions of infinitesimal and symplectic matrices. Denote by $\mathbb{J}$ the matrix representation of the canonical symplectic structure over $\mathbb{R}^{2n}$, that is $\mathbb{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$.

Definition 3.1: A matrix M of order $2n$ is called *infinitesimal symplectic* if $\mathbb{J}M + M^T\mathbb{J} = 0$. Denote by $sp(n)$ the set of such matrices.

Definition 3.2: The real *symplectic group*, denoted by $Sp(n)$, is the invariant matrix group for the canonical skew-symmetric scalar product induced by $\mathbb{J}$. A matrix M belongs to $Sp(n)$ if and only if $M^T\mathbb{J}M = \mathbb{J}$.

Let us now recall two useful characterizations of energetic scheme and symplectic scheme.

Proposition 3.3: Let A and N be non-exceptional matrices connected by $A = (I+N)^{-1}(I-N)$ and $N = (I+A)^{-1}(I-A)$, and Q any arbitrary matrix. Then

$$A^TQA = Q \iff N^TQ + QN = 0 \tag{7}$$

It can be seen that this proposition characterizes energy conservation when A is the generator of the numerical flow and Q defines the energy of the system.

Proposition 3.4: If N is a non-exceptional infinitesimal symplectic matrix, then its Caylay transform belongs to $Sp(n)$.

Compute now the discrete energy of (5) at stage $n+1$ and compare it to stage n, it follows

$$\begin{aligned} H_{n+1} &= \langle x_{n+1}, Qx_{n+1}\rangle = \langle x_n, A^TQAx_n\rangle \\ &= H_n \end{aligned} \tag{8}$$

where the last equality holds since (7) is obviously satisfied. Hence, the Euler midpoint scheme applied to linear Hamiltonian systems is energy-preserving.

Moreover, when J belongs to $Sp(n)$, N lives in $sp(n)$ (by straightforward calculus) and thus A belongs to $Sp(n)$ by virtue of Proposition 3.4. That is the numerical flow (6) is symplectic. Hence, in this case, the Euler midpoint scheme preserves the geometry of the equations.

Remark 3.5: In [Ge & Marsden], it has been shown that explicit scheme that preserves energy as well as geometry leads to a numerical solution which is a shift in time of the continuous one.

B. Discrete-time linear port-Hamiltonian systems

We now apply midpoint Euler scheme to approximate port-Hamiltonian systems (1), and proposed the following definition of discrete linear PHS.

Definition 3.6: The *discrete linear port-Hamiltonian system* associated with (1) is defined by

$$\left\{ \begin{aligned} \frac{x_{n+1}-x_n}{\Delta t} &= JQ\frac{x_{n+1}+x_n}{2} + G(x_n)u_n \\ y_n &= G(x_n)^TQ\frac{x_{n+1}+x_n}{2} \end{aligned} \right. . \tag{9}$$

The trajectories of the discrete PHS (9) are given by the following numerical flow

$$\left\{ \begin{aligned} x_{n+1} &= Ax_n + B_nu_n \\ y_n &= C_nx_n + D_nu_n \end{aligned} \right. \tag{10}$$

where

$$\begin{aligned} &A = (I+N)^{-1}(I-N), \quad N = -\tfrac{\Delta t}{2}JQ \\ &B_n = \Delta t(I+N)^{-1}G(x_n) \\ &C_n = \tfrac{1}{2}G_n^TQ(I+A), \quad G_n := G(x_n) \\ &D_n = \tfrac{\Delta t}{2}G_n^TQ(I+N)^{-1}G_n \end{aligned} \tag{11}$$

This definition simply extends approximation of linear Hamiltonian dynamics (5) to linear Hamiltonian systems with port variables. The slight difference with existing definitions of discrete PHS resides in the conjugate port-output. For linear PHS, the port-output y is proportional to the state (q,p) and its discrete counterpart y_n was chosen to be proportional to (q_n,p_n), the state evaluated at stage n. Here, the discrete port-output has been derived from the discrete dynamics rather than from the the expression of the continuous one. In other words, we tried to mimic the continuous definition within the discrete setting.

We shall see that the trajectories of the proposed discrete Hamiltonian dynamics (10) satisfies an (exact) energy balance. Thus, the integrator (9) is an energetic integrator for linear Hamiltonian system with input-ouput port variables.

Proposition 3.7: The discrete linear PHS (9) is conservative w.r.t. the same storage function H.

Proof. We shall see that for any n the discrete energy balance writes $H_{n+1} - H_n = \Delta t\, u_n^T y_n$. First, expanding $H_{n+1} = \langle x_{n+1}, Qx_{n+1}\rangle$ leads to

$$\begin{aligned} H_{n+1} = \; & \tfrac{1}{2}\langle Ax_n, QAx_n\rangle + \Delta t\langle u_n, G_n^T(I+N)^{-T}QAx_n\rangle \\ & + \Delta t\langle u_n, \tfrac{\Delta t}{2}G_n^T(I+N)^{-T}Q(I+N)^{-1}G_nu_n\rangle, \end{aligned} \tag{12}$$

The first term equals H_n thanks to (7). It remains to see that the second and the third terms are associated with the contributions of C_n and D_n to conclude. Using the fact that $2(I+N)^{-1} = I+A$ and (7), it is easy to see that

$$\begin{aligned} G_n^T(I+N)^{-T}QA &= \tfrac{1}{2}G_n^T(I+A^T)QA \\ &= \tfrac{1}{2}G_n^TQ(I+A) = C_n \end{aligned} . \tag{13}$$

Finally, expanding the scalar product $\langle u_n, D_nu_n\rangle$, it follows

$$\begin{aligned} &\tfrac{\Delta t}{2}G_n^TQ(I+N)^{-1}G_n \\ &\quad = \tfrac{\Delta t}{4}G_n^TQ(I+A)G_n \\ &\quad = \tfrac{\Delta t}{4}G_n^T(I+A)^TQAG_n \\ &\quad = \tfrac{\Delta t}{2}G_n^T(I+N)^{-T}Q(I-N)(I+N)^{-1}G_n \\ &\quad = \tfrac{\Delta t}{2}G_n^T(I+N)^{-T}Q(I+N)^{-1}G_n + G_n^TSG_n \end{aligned} \tag{14}$$

where the last equality is derived by expanding $(I-N)$. The first term is precisely the second row of (12). The second term involves a skew-symmetric matrix S, thanks to (7), hence vanishes when taking the scalar product. To summarize, we have shown that for any n, the discrete energy balance takes the form $H_{n+1} - H_n = \Delta t\langle u_n, C_nx_n + D_nu_n\rangle$ which ends the proof. ■

Remark 3.8: It is worth noting that Proposition 3.7 can be proven using the relations (9) rather than using (10). Indeed, once noticed that the discrete energy balance may write $H_{n+1} - H_n = \Delta t\langle \frac{x_{n+1}-x_n}{\Delta t}, Q(x_{n+1}+x_n)\rangle$, that equals $\Delta t\langle u_n, y_n\rangle$ by (9), one recognizes a discrete power-product associated with a (discrete) Dirac structure. This mimics the continuous case.

Remark 3.9: Approximation of PHSs based on a discrete power-product is the underlying idea behind papers such as [5], [17], [16] where the characterization of the discrete relationship between effort and flow variables is addressed. However, this viewpoint of a PHS, seen as a geometric object called a (Stokes-)Dirac structure, does not care about its trajectories. As consequence, although a discrete power-product is defined, any discretization scheme does not ensure that this discrete power-product equals energy variation. In other words, the discrete energy balance as written in the previous remark is no longer true, and the discrete PHS might be no longer conservative. For instance in [11], the discrete effort-flow relationship involves the structure matrix J, thus ensuring a vanishing discrete power-product, but the latter is not directly connected to the energy balance: the discrete system is not conservative.

We shall now perform the canonical interconnection of conservative discrete port-Hamiltonian systems (9). We shall see that such dynamics are stable under composition which is analogous to the continuous case. This successful result mainly relies on Prop. 3.7.

IV. Interconnection of discrete linear PHS

In this section, the canonical interconnection of discrete linear PHS (9) is processed. The resulting interconnected system is shown to be again a discrete linear PHS. It is then shown that the projection on discrete space and composition by interconnection commutes. Moreover, the procedure preserves the symplecticity of the numerical flow whenever input vector fields are Hamiltonian vector fields.

A. Interconnection of discrete linear PHS

Consider two discrete linear PHS defined by (9) and denote them by Σ_1^d and Σ_2^d. Assume that the input vectors fields have the same (full-column) rank p, *i.e.* $G^i(x_n^i)$ belongs to $\mathbb{R}^{m_i\times p}, i=1,2$. Consider the canonical interconnection given at any stage n by (3) and denote by Σ_{12}^d the discrete system resulting from the interconnection of discrete PHSs. We claim the following.

Proposition 4.1: Σ_{12}^d is a conservative discrete linear PHS.

Proof. Denote the extended state space by $X_n^{12} = \left((x_n^1)^T\,(x_n^2)^T\right)$. Then, using the interconnection relation (3), straightforward computations leads to

$$\begin{aligned}\frac{X_{n+1}^{12}-X_n^{12}}{\Delta t} &= \begin{bmatrix} J_1Q_1 & -G_n^1(G_n^2)^TQ_2 \\ G_n^2(G_n^1)^TQ_1 & J_2Q_2\end{bmatrix}\frac{X_{n+1}^{12}+X_n^{12}}{2} \\ &= \begin{bmatrix} J_1 & -G_n^1(G_n^2)^T \\ G_n^2(G_n^1)^T & J_2\end{bmatrix}Q_{12}\frac{X_{n+1}^{12}+X_n^{12}}{2} \\ &= J_{12}^nQ_{12}\frac{X_{n+1}^{12}+X_n^{12}}{2}\end{aligned} \tag{15}$$

where Q_{12} is given as in (4). Introduce now $N_{12}^n := -\frac{\Delta t}{2}J_{12}^nQ_{12}$, it remains to see that ${N_{12}^n}^TQ_{12}+Q_{12}N_{12}^n=0$ to conclude that the global energy $H_{12}=H_1+H_2$ is conserved according to Proposition 3.3. ■

Remark 4.2: Note that, for simplicity again, only the full interconnection case has been detailed. We claim that the same holds for partial interconnection following Remark 2.1.

B. Composition of linear PHS

We shall now address the issue arising when comparing interconnection of discrete linear PHSs and discretization of interconnected linear PHS.

In the framework of linear PHS, let us introduce the (canonical) composition map $\Phi : (\Sigma_1,\Sigma_2)\mapsto\Sigma_{12}$ associated with the canonical interconnection structure defined in (3), and the projection map $\Pi : \Sigma^c\mapsto\Sigma^d$ associated with the discretization scheme defined in (9), where the upperscript ^{c} stands for *continuous* and ^{d} for *discrete*.

Consider two continuous linear PHS defined by (1) and denoted by Σ_1^c and Σ_2^c. Assume that the input vector fields G^1 and G^2 are both full-column p-rank matrices. The continuous interconnected system, denoted $\Sigma_{12}^c=\Phi(\Sigma_1^c,\Sigma_2^c)$, is of the form (4). Denote by $\Sigma_i^d=\Pi(\Sigma_i^c), i=1,2$ the corresponding discrete systems, and by $\Sigma_{12}^d=\Phi(\Sigma_1^d,\Sigma_2^d)$ the resulting discrete interconnected system.

Proposition 4.3: The projection map Π and the composition map Φ commute, that is, the following diagram

$$\begin{array}{ccc}(\Sigma_1^c,\Sigma_2^c) & \xrightarrow{\Phi} & \Sigma_{12}^c \\ \downarrow\Pi & & \downarrow\Pi \\ (\Sigma_1^d,\Sigma_2^d) & \xrightarrow{\Phi} & \Sigma_{12}^d\end{array} \tag{16}$$

is commutative.

Proof. By Proposition 3.7, one knows that $\Pi(\Sigma_{12}^c)$ is conservative. Moreover, by Proposition 4.1, Σ_{12}^d is also conservative. Notice that J_{12} in (4) evaluated at stage n is precisely J_{12}^n in (15), one concludes that these systems have the same generator. It remains to see that $\Phi(\Pi(\Sigma_2^c),\Pi(\Sigma_2^c)) = \Phi(\Sigma_1^d,\Sigma_2^d)=\Sigma_{12}^d=\Pi(\Sigma_{12}^c)=\Pi(\Phi(\Sigma_1^c,\Sigma_2^c))$. ■

Corollary 4.4: J_{12}^n defines a symplectic structure on $\mathbb{R}^{2n_1\times 2n_2}$ with respect to the induced symplectic structure whenever the input vector fields are Hamiltonian vector fields.

Proof. Consider Hamiltonian input vector fields $G^i=J_i\nabla g_i$, where g_i is a smooth real-valued function over $\mathbb{R}^{2n_i}$, and remind that the induced symplectic structure is given by $K=\begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}$. We shall check that ${J_{12}^n}^TKJ_{12}^n=K$. Straightforward calculation leads to null off-diagonal terms. To conclude, it remains to see that the diagonal terms are of the form $-J_i^T+J_i\nabla g_i[\nabla g_j^TJ_j\nabla g_j]\nabla g_i^TJ_i^T$ which equals J_i since the bracket term vanishes by skew-symmetry of J_j, where $(i,j)=(1,2)$ and $(2,1)$. ■

V. Example: interconnected LC-circuits

This example illustrates the energy-preserving interconnection of discrete linear PHSs as defined in (9). We shall first compare the discrete dynamics obtained by emulation (basically, a forward Euler scheme) with the discrete port-Hamiltonian dynamics introduced in this paper. Secondly, we derive some remarks about the energy drift with respect to the implementation of the interconnected dynamics.

Let us now consider the canonical interconnection of two LC-circuits, indexed by $_1$ and $_2$, which port-Hamiltonian representation is given by

$$(\Sigma:)\left\{\begin{array}{rcl}\begin{bmatrix}\dot q\\ \dot p\end{bmatrix} &=& \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}q/C\\ p/L\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u \\ y &=& p/L\end{array}\right. \tag{17}$$

where q is the charge and p the magnetic flux. The parameters are the capacitance and the inductance of each system $(C_1, L_1) = (0.2, 2)$ and $(C_2, L_2) = (10, 0.1)$. The initial condition is $(q_1, p_1, q_2, p_2) = (1, 0, 0, 0)$ and the timestep is $\Delta t = 0.1$

The interconnection of Σ_1^c and Σ_2^c is performed through the canonical interconnection structure (3). The resulting interconnected system Σ_{12}^c is given by (4).

A. Simulation results

In order to compare the discrete dynamics behaviors, the reference is taken as the continuous trajectories obtained with Matlab solver `ode45` with variable stepsize. The simulation time-interval is small enough to make this assumption.

Remind that *emulation* stands for a forward Euler scheme applied to the dynamical system integration, and an evaluation of the port-output at stage n. The simulation results are given in Figure 1, where trajectories of the charges q_1 and q_2 are presented.

As expected, the emulation results show a quick energy drift, and thus divergent trajectories of the interconnected system. On the contrary, the proposed discrete linear PHS (9) ensures a weak numerical energy drift, and thus similar trajectories compared with the continuous solution can be observed.

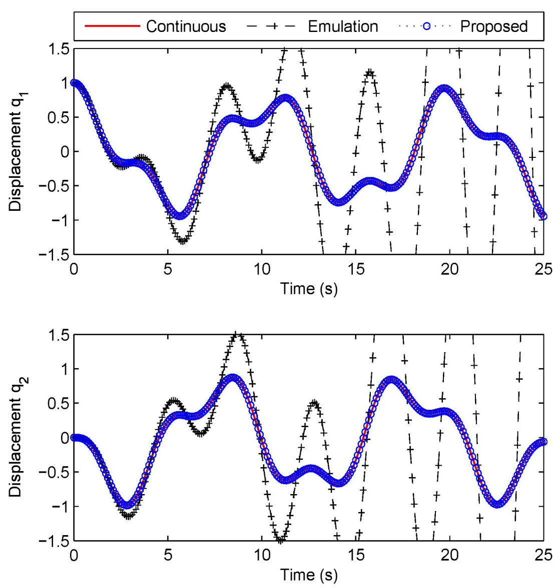

Fig. 1. Time evolution of the charges q_1 and q_2 of Σ_{12}.

Focus now on the variations of the discrete energies. As shown in Figure 2, the discrete energy balance associated with a linear PHS (9) ensures a good energetic behavior of the interconnected system Σ_{12}^d. Figure 2 actually depicts the energy variation between the initial energy and the energy at stage n. Conservative discrete dynamics are thus of fundamental importance to construct discrete complex systems as a collection of discrete subsystems.

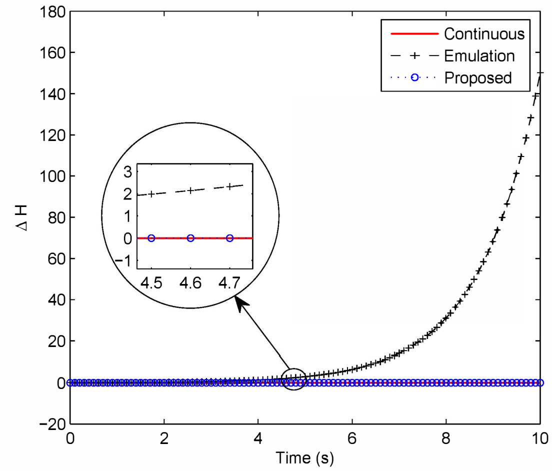

Fig. 2. Energy variation along trajectories.

B. Energy errors comparison

We now restrict our investigation to the discrete dynamics of Σ_{12} obtained by interconnection of the proposed discrete linear PHS (9).

It is worth noting that the different ways to implement the discrete dynamics of the interconnected system lead to different accuracy. Although these implementations are *a priori* identical, the associated generators might have different energetic behaviors.

Denote the numerical flow (6) of Σ_i^d by ϕ_i for $i = 1, 2$. Relative to the implemented discrete interconnected dynamics, the following energies have been computed:

1) $H[\Phi_y(\phi_1, \phi_2)]$ is associated with the generator obtained by the composition of the numerical flows ϕ_1 and ϕ_2 where $y_i(n)$ is seen as a function of $x_i(n+1)$ and $x_i(n)$.
2) $H[\Phi_u(\phi_1, \phi_2)]$ is associated with the generator obtained by the composition of the numerical flows ϕ_1 and ϕ_2 where $u_1(n)$ is seen as a function of $x_1(n)$ and $x_2(n)$.
3) $H[\Phi(\Sigma_1^d, \Sigma_2^d)]$ is associated with the direct discretization of the continuous interconnected system, *i.e.* associated with $\Sigma_{12}^d = \Pi(\Sigma_{12}^c)$.

The simulations are performed over the time-interval from 0 to 10^6s with a timestep $\Delta t = 0.1$. Simulation results are presented in Figure 3. Note that emulation can not provide such a long time simulation, and Matlab solver with variable stepsize requires much more points (which may run out of memory) without satisfactory result.

The worst case presents an energy drift about $6e-9$, twice as the best one obtained with the direct discretization. The latter numerical flow is obtained with the generator got by Cayley transform. An explanation could be that the Cayley transform involves one inverse matrix whereas the two remaining cases need more, thus limiting round-off errors propagation.

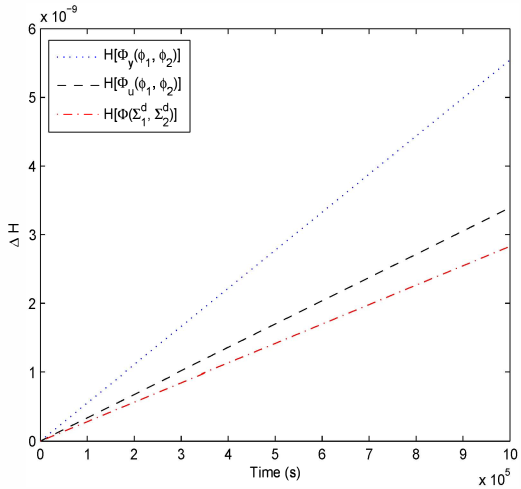

Fig. 3. Energy variation w.r.t. implementation schemes.

VI. Conclusion

In this paper, a definition of a discrete conservative linear port-Hamiltonian system is proposed. This definition overcomes those presented in the literature where an almost energy balance is satisfied. It is then shown that such discrete PHSs are stable under composition by a canonical interconnection structure. Moreover, discretization and composition are commutative operations. It is also proven that for symplectic Hamiltonian systems, the procedure preserves the geometry of the initial dynamics whenever the input vector fields are Hamiltonian vector fields. The LC-circuits example shows the performance of this approach. The implementation schemes of the discrete interconnected dynamics have been briefly compared based on the energy drift of the numerical flow.

This paper suggests the use of energetic integrators to derive discrete conservative PHSs. In this way, power-conserving interconnection of discrete PHSs is promising since the resulting dynamics is again a conservative PHS. This result mimics the continuous case and thus enables to consider discrete complex systems as a network of discrete subsystems connected through power-conserving structure. Such a discrete Hamiltonian framework would be attractive for control design, sample-data and simulation purposes.

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