



# Stability analysis of positive descriptor systems<sup>☆</sup>

Elena Virnik

*Institut für Mathematik, TU Berlin, Strasse des 17. Juni 136, D-10623 Berlin, Germany*

Received 2 May 2007; accepted 4 March 2008

Submitted by D. Szyld

Dedicated to Richard Varga on the occasion of his 80th birthday.

---

## Abstract

In this paper, we discuss stability properties of positive descriptor systems in the continuous-time as well as in the discrete-time case. We present characterisations of positivity and establish generalised stability criteria for the case of positive descriptor systems. We show that if the spectral projector onto the finite deflating subspace of the matrix pair  $(E, A)$  is nonnegative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case. Furthermore, we provide sufficient conditions that guarantee entry-wise nonnegativity along with positive semidefiniteness of solutions of generalised projected Lyapunov equations. As an application of the framework established throughout this paper, we exemplarily generalise two criteria for the stability of two switched standard positive systems under arbitrary switching to the descriptor case.

© 2008 Elsevier Inc. All rights reserved.

*AMS classification:* 15A22; 15A24; 15A48; 93D05; 93D20

*Keywords:* Positive systems; Descriptor systems; Stability; Generalised projected Lyapunov equations; Switched positive systems

---

## 1. Introduction

We consider linear time-invariant positive descriptor systems in continuous-time

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{1.1a}$$

---

<sup>☆</sup> This research is supported by the DFG Research Center MATHEON in Berlin.

*E-mail address:* [virnik@math.tu-berlin.de](mailto:virnik@math.tu-berlin.de)

$$y(t) = Cx(t), \quad (1.1b)$$

and in discrete-time

$$Ex(t+1) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (1.2a)$$

$$y(t) = Cx(t), \quad (1.2b)$$

where  $E, A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$ ,  $C \in \mathbb{R}^{p \times n}$  are real constant coefficient matrices. In the continuous-time case the state  $x$ , input  $u$  and output  $y$  are real-valued vector functions. In the discrete-time case  $x$ ,  $u$  and  $y$  are real-valued vector sequences. Positive systems arise naturally in many applications such as pollutant transport, chemotaxis, pharmacokinetics, Leontief input–output models, population models and compartmental systems [2,6,7,9,16,22,26]. In these models, the variables represent concentrations, population numbers of bacteria or cells or, in general, measures that are per se nonnegative. Positive standard systems, i.e., where  $E$  is the identity matrix, are subject to ongoing research by many authors [1,17,18,22,26,36,37,38,40,41]. Recent advances on control theoretical issues have been made especially in the positive discrete-time case. Yet, there are still many open problems, especially for standard positive systems in continuous-time. Control theory of descriptor systems without the nonnegativity restriction is to a large extent well understood, see, e.g., [19]. Very little is known about positive descriptor systems up to now, however, some properties mainly in the discrete-time case were studied in [10,11,12,26].

It is well known that stability properties of standard systems, where  $E = I$ , are closely related to the spectral properties of the system matrix  $A$ . If the dynamics of the system, however, is described by an implicit differential or difference equation, then stability properties are determined by the eigenvalues and eigenvectors associated with the matrix pencil  $\lambda E - A$ , or just the matrix pair  $(E, A)$ .

In the case of standard positive systems, classical stability criteria take a simple form, [22,26]. In this paper we present generalisations of these stability criteria for the case of positive descriptor systems. It turns out, that if the spectral projector onto the finite deflating subspace of the matrix pair  $(E, A)$  is nonnegative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case.

Stability properties and also many other control theoretical issues such as model reduction methods or the quadratic optimal control problem are, furthermore, closely related to the solution of Lyapunov equations, see. e.g., [3,23,24,29,34]. For descriptor systems, generalised projected Lyapunov equations were presented in [39]. In the context of positive systems one is interested not only in positive (semi)definite solutions of such Lyapunov equations but rather in doubly nonnegative solutions, i.e., solutions that are both positive semidefinite and entry-wise nonnegative. Such results for standard Lyapunov equations, e.g., can easily be deduced from a more general discussion in [20]. In this paper, we provide sufficient conditions that guarantee the existence of doubly nonnegative solutions of generalised projected Lyapunov equations.

This paper is organised as follows. In Section 2 we recall fundamental properties of matrix pencils, descriptor systems, projectors and nonnegative matrices. In particular, we recall the generalised Perron-Frobenius Theorem for matrix pairs established in [35] that forms the basis for many results in this paper. In Section 3 we give characterisations of positive continuous-time and discrete-time descriptor systems. In Section 4 we generalise the special stability conditions for positive systems from the standard case, see [22], to the descriptor case. In Section 5 we establish conditions for the solutions of the continuous-time and discrete-time generalised projected Lyapunov equations, as introduced in [39], to be doubly nonnegative. Finally, in Section 6 we exemplarily show how we can use the framework established throughout this paper in order

to generalise the results on stability of two standard switched positive systems, see [32,33], to positive descriptor systems.

## 2. Preliminaries

### 2.1. Matrix pairs

Let  $E, A \in \mathbb{R}^{n \times m}$ . A matrix pair  $(E, A)$ , or a matrix pencil  $\lambda E - A$ , is called *regular* if  $E$  and  $A$  are square ( $n = m$ ) and  $\det(\lambda E - A) \neq 0$  for some  $\lambda \in \mathbb{C}$ . It is called *singular* otherwise. In this paper we only consider square and regular pencils.

A scalar  $\lambda \in \mathbb{C}$  is said to be a (*finite*) *eigenvalue* of the matrix pair  $(E, A)$  if  $\det(\lambda E - A) = 0$ . A vector  $x \in \mathbb{C}^n \setminus \{0\}$  such that  $(\lambda E - A)x = 0$  is called *eigenvector* of  $(E, A)$  corresponding to  $\lambda$ . If  $E$  is singular and  $v \in \mathbb{C}^n \setminus \{0\}$ , such that  $Ev = 0$  holds, then  $v$  is called eigenvector of  $(E, A)$  corresponding to the eigenvalue  $\infty$ . For a finite eigenvalue  $\lambda$  we denote by  $\Re(\lambda)$  its real part.

The set of all eigenvalues is called *spectrum* of  $(E, A)$  and is defined by

$$\sigma(E, A) := \begin{cases} \sigma_f(E, A), & \text{if } E \text{ is invertible,} \\ \sigma_f(E, A) \cup \{\infty\}, & \text{if } E \text{ is singular,} \end{cases}$$

where  $\sigma_f(E, A)$  is the set of all finite eigenvalues. We denote by

$$\rho_f(E, A) = \max_{\lambda \in \sigma_f(E, A)} |\lambda|,$$

the *finite spectral radius* of  $(E, A)$ . Note that for  $E = I$  we have that  $\rho_f(I, A) = \rho(A)$  is the standard spectral radius of  $A$ .

Vectors  $v_1, \dots, v_k$  form a (*right*) *Jordan chain* of the matrix pair  $(E, A)$  corresponding to the eigenvalue  $\lambda$  if

$$(\lambda E - A)v_i = -Ev_{i-1}$$

for all  $1 \leq i \leq k$  and  $v_0 = 0$ . A  $k$ -dimensional subspace  $S^{\text{def}} \subset \mathbb{C}^n$  is called (*right*) *deflating subspace* of  $(E, A)$ , if there exists a  $k$ -dimensional subspace  $\mathcal{W} \subset \mathbb{C}^n$  such that  $ES^{\text{def}} \subset \mathcal{W}$  and  $AS^{\text{def}} \subset \mathcal{W}$ . A deflating subspace  $S_\lambda^{\text{def}} \subset \mathbb{C}^n$  is called *deflating subspace* of  $(E, A)$  corresponding to the eigenvalue  $\lambda$  if it is spanned by all Jordan chains corresponding to  $\lambda$ . Let  $\lambda_1, \dots, \lambda_p$  be the pairwise distinct finite eigenvalues of  $(E, A)$  and let  $S_{\lambda_i}^{\text{def}}, i = 1, \dots, p$ , be the deflating subspaces corresponding to these eigenvalues. We call the subspace defined by

$$S_f^{\text{def}} := S_{\lambda_1}^{\text{def}} \oplus \cdots \oplus S_{\lambda_p}^{\text{def}}, \quad (2.1)$$

the *finite deflating subspace* of  $(E, A)$ .

### 2.2. Projector chains and index of $(E, A)$

A matrix  $Q \in \mathbb{R}^{n \times n}$  is called *projector* if  $Q^2 = Q$ . A projector  $Q$  is called a projector *onto* a subspace  $S \subseteq \mathbb{R}^n$  if  $\text{im } Q = S$ . It is called a projector *along* a subspace  $S \subseteq \mathbb{R}^n$  if  $\ker Q = S$ .

Let  $(E, A)$  be a regular matrix pair. As introduced in [25] we define a matrix chain by setting

$$E_0 := E, \quad A_0 := A \quad \text{and} \quad (2.2a)$$

$$E_{i+1} := E_i - A_i Q_i, \quad A_{i+1} := A_i P_i \quad \text{for } i \geq 0, \quad (2.2b)$$

where  $Q_i$  are projectors onto  $\ker E_i$  and  $P_i = I - Q_i$ . Since we have assumed  $(E, A)$  to be regular, there exists an index  $v$  such that  $E_v$  is nonsingular and all  $E_i$  are singular for  $i < v$  [30]. Note that  $v$  is independent of the specific choice of the projectors  $Q_i$ . Consequently, we say that the matrix pair  $(E, A)$  has *(tractability) index*  $v$  and denote it by  $\text{ind}(E, A) = v$ . It is well known that for regular pairs  $(E, A)$  the tractability index is equal to the differentiation index, see, e.g., [14], and it can be determined as the size of the largest Jordan block associated with the eigenvalue infinity in the Weierstraß canonical form of the pair  $(E, A)$ , see [28,30]. In the following we, therefore, only speak of the *index* of the pair  $(E, A)$ .

It is possible to construct the matrix chain in (2.2) with specific, so called *canonical*, projectors, see [31,35]. For such projectors  $Q_i$ , in particular, it holds that for all  $v \in S_f^{\text{def}}$  and for all  $i = 0, \dots, v-1$  we have  $Q_i v = 0$ . In the following, whenever we refer to the matrix chain in (2.2), we assume that it is constructed with canonical projectors. Note that

$$P_r := P_0 \dots P_{v-1} \quad (2.3)$$

is again a projector and it is the unique projector that projects onto  $S_f^{\text{def}}$  along the deflating subspace corresponding to the eigenvalue  $\infty$ , [31,35]. The deflating subspace corresponding to the eigenvalue  $\infty$  is the subspace spanned by all Jordan chains corresponding to the eigenvalue  $\infty$ , or equivalently by all Jordan chains corresponding to the eigenvalue 0 of the matrix pair  $(A, E)$ . We call  $P_r$  the *spectral projector* onto  $S_f^{\text{def}}$ .

### 2.3. Explicit solution representation

In order to formulate explicit solution representations of (1.1a) and (1.2a), respectively, we need that the matrices  $E$  and  $A$  commute. If they do not commute and the matrix pair  $(E, A)$  is regular, we can obtain commuting matrices by multiplication with a scaling factor as stated in the following Lemma [15].

**Lemma 2.1.** *Let  $(E, A)$  be a regular matrix pair. Let  $\hat{\lambda}$  be chosen such that  $\hat{\lambda}E - A$  is non-singular. Then, the matrices*

$$\hat{E} = (\hat{\lambda}E - A)^{-1}E \quad \text{and} \quad \hat{A} = (\hat{\lambda}E - A)^{-1}A$$

*commute.*

Throughout the paper, we refer to  $\hat{E}$ ,  $\hat{A}$  as defined in Lemma 2.1 independently of the specific choice of  $\hat{\lambda}$ . Furthermore, for a matrix  $B$  from system (1.1) or (1.2) we define

$$\hat{B} := (\hat{\lambda}E - A)^{-1}B. \quad (2.4)$$

Note, that for systems (1.1a) and (1.2a), respectively, the scaling by a nonsingular factor such as  $(\hat{\lambda}E - A)^{-1}$  does not change the solution.

Let  $E \in \mathbb{R}^{n \times n}$  have index  $v$ , i.e.,  $\text{ind}(E, I) = v$ . The Drazin inverse  $E^D \in \mathbb{R}^{n \times n}$  of  $E$ , see, e.g., [15,21], is uniquely defined by the properties:

$$E^D E = E E^D, \quad (2.5a)$$

$$E^D E E^D = E^D, \quad (2.5b)$$

$$E^D E^{v+1} = E^v. \quad (2.5c)$$

For the matrices  $\hat{E}$ ,  $\hat{A}$  as defined in Lemma 2.1 and their corresponding Drazin inverses, the following properties hold, see, e.g., [28]:

$$\widehat{E}\widehat{A}^D = \widehat{A}^D\widehat{E}, \quad (2.6a)$$

$$\widehat{E}^D\widehat{A} = \widehat{A}\widehat{E}^D, \quad (2.6b)$$

$$\widehat{E}^D\widehat{A}^D = \widehat{A}^D\widehat{E}^D. \quad (2.6c)$$

Note that if we form matrix products such as  $\widehat{E}^D\widehat{E}$ ,  $\widehat{E}^D\widehat{A}$ ,  $\widehat{E}\widehat{A}^D$ ,  $\widehat{E}^D\widehat{B}$ ,  $\widehat{A}^D\widehat{B}$ , the terms in  $\widehat{\lambda}$  cancel out, so that these products do not depend on the specific choice of  $\widehat{\lambda}$ , see [28, Chapter 2, Exercise 11]. This can be verified by transforming  $(E, A)$  into Weierstraß canonical form, see e.g. [13,19,28]. That is, there exist regular matrices  $W, T \in \mathbb{R}^{n \times n}$  such that

$$(E, A) = \left( W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T, W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right), \quad (2.7)$$

where  $J$  is a matrix in Jordan canonical form and  $N$  is a nilpotent matrix also in Jordan canonical form. Then, we have

$$\begin{aligned} \widehat{E} &= (\widehat{\lambda}E - A)^{-1}E = \left( \widehat{\lambda}W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T - W \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} T \right)^{-1} W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T \\ &= \left( W \begin{bmatrix} \widehat{\lambda}I - J & 0 \\ 0 & \widehat{\lambda}N - I \end{bmatrix} T \right)^{-1} W \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} T = T^{-1} \begin{bmatrix} (\widehat{\lambda}I - J)^{-1} & 0 \\ 0 & (\widehat{\lambda}N - I)^{-1}N \end{bmatrix} T, \end{aligned}$$

and similarly,

$$\widehat{A} = (\widehat{\lambda}E - A)^{-1}A = T^{-1} \begin{bmatrix} (\widehat{\lambda}I - J)^{-1}J & 0 \\ 0 & (\widehat{\lambda}N - I)^{-1} \end{bmatrix} T.$$

For the Drazin inverses of  $\widehat{E}$  and  $\widehat{A}$  we obtain

$$\widehat{E}^D = T^{-1} \begin{bmatrix} \widehat{\lambda}I - J & 0 \\ 0 & 0 \end{bmatrix} T \quad \text{and} \quad \widehat{A}^D = T^{-1} \begin{bmatrix} J^D(\widehat{\lambda}I - J) & 0 \\ 0 & \widehat{\lambda}N - I \end{bmatrix} T.$$

Here, we have used that the matrices  $J$  and  $(\widehat{\lambda}I - J)^{-1}$  commute, and for commuting matrices  $Z_1, Z_2$ , we have  $(Z_1Z_2)^D = Z_2^DZ_1^D$ , see e.g. [28]. Therefore, the products

$$\widehat{E}^D\widehat{E} = T^{-1} \begin{bmatrix} \widehat{\lambda}I - J & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\widehat{\lambda}I - J)^{-1} & 0 \\ 0 & (\widehat{\lambda}N - I)^{-1}N \end{bmatrix} T = T^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T,$$

$$\widehat{E}^D\widehat{A} = T^{-1} \begin{bmatrix} \widehat{\lambda}I - J & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\widehat{\lambda}I - J)^{-1}J & 0 \\ 0 & (\widehat{\lambda}N - I)^{-1} \end{bmatrix} T = T^{-1} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T,$$

$$\widehat{E}\widehat{A}^D = T^{-1} \begin{bmatrix} (\widehat{\lambda}I - J)^{-1} & 0 \\ 0 & (\widehat{\lambda}N - I)^{-1}N \end{bmatrix} \begin{bmatrix} (\widehat{\lambda}I - J)J^D & 0 \\ 0 & \widehat{\lambda}N - I \end{bmatrix} T = T^{-1} \begin{bmatrix} J^D & 0 \\ 0 & N \end{bmatrix} T$$

do not depend on  $\widehat{\lambda}$ . Note that  $\widehat{E}^D\widehat{E} = P_r$  is the unique spectral projector onto  $S_f^{\text{def}}$  defined in (2.3) [31,35]. Let  $\widehat{B}$  be defined as in (2.4) and  $B = W\tilde{B}$ , where  $\tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$  is partitioned according to the Weierstraß canonical form of  $(E, A)$ . Then, in a similar manner, we obtain  $\widehat{E}^D\widehat{B} = T^{-1} \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$  and  $\widehat{A}^D\widehat{B} = T^{-1} \begin{bmatrix} J^D\tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}$ , which are also independent of  $\widehat{\lambda}$ .

The following Theorem gives an explicit solution representation in terms of the Drazin inverse.

**Theorem 2.2.** *Let  $(E, A)$  be a regular matrix pair with  $E, A \in \mathbb{R}^{n \times n}$  and  $\text{ind}(E, A) = v$ . Let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1 and  $\widehat{B}$  as in (2.4). Furthermore, for the continuous-time case, let  $u \in C^v$  and denote by  $u^{(i)}$ ,  $i = 0, \dots, v - 1$ , the  $i$ th derivative of  $u$ . Then, every solution  $x \in C^1$  to Eq. (1.1a) has the form:*

$$\begin{aligned} x(t) &= e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E}v + \int_0^t e^{\widehat{E}^D \widehat{A}(t-\tau)} \widehat{E}^D \widehat{B}u(\tau) d\tau \\ &\quad - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B}u^{(i)}(t) \end{aligned} \quad (2.8)$$

for some  $v \in \mathbb{R}^n$ . In the discrete-time case, every solution sequence  $x(t)$  to Equation (1.2a) has the form:

$$\begin{aligned} x(t) &= (\widehat{E}^D \widehat{A})^t \widehat{E}^D \widehat{E}v + \sum_{\tau=0}^{t-1} (\widehat{E}^D \widehat{A})^{t-1-\tau} \widehat{E}^D \widehat{B}u(\tau) \\ &\quad - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B}u(t+i) \end{aligned} \quad (2.9)$$

for some  $v \in \mathbb{R}^n$ .

**Proof.** See, e.g., [13,28].  $\square$

**Corollary 2.3.** *Under the same assumptions as in Theorem 2.2, the continuous-time initial value problem (1.1) has a (unique) solution corresponding to the initial condition  $x_0$  and to the input  $u \in C^v$  if and only if there exists a vector  $v \in \mathbb{R}^n$  such that*

$$x_0 = \widehat{E}^D \widehat{E}v - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B}u^{(i)}(0). \quad (2.10)$$

The discrete-time initial value problem (1.2) has a (unique) solution corresponding to the initial condition  $x_0$  and to the input sequence  $u$  if and only if there exists a vector  $v \in \mathbb{R}^n$  such that

$$x_0 = \widehat{E}^D \widehat{E}v - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B}u(i). \quad (2.11)$$

**Proof.** See, e.g., [13,28].  $\square$

**Definition 2.4.** We call an initial value  $x_0$  in (1.1a) or in (1.2a) *consistent (with respect to an assigned input  $u$ )* if (2.10) or (2.11) holds, respectively.

#### 2.4. Nonnegative matrices and matrix pairs

A vector  $x \in \mathbb{R}^n$ ,  $x = [x_i]_{i=1}^n$  is called nonnegative (positive) and we write  $x \geq 0$  ( $x > 0$ ) if all entries  $x_i$  are nonnegative (positive). By  $\mathbb{R}_+^n$  we denote the space of all nonnegative vectors in  $\mathbb{R}^n$ . A matrix  $A \in \mathbb{R}^{n \times n}$ ,  $A = [a_{ij}]_{i,j=1}^n$  is called *nonnegative* and we write  $A \geq 0$  if all entries  $a_{ij}$  are nonnegative. A matrix  $A$  is called *nonnegative on a subset  $S \subset \mathbb{R}^n$*  if for all  $x \in S \cap \mathbb{R}_+^n$ , we have  $Ax \in \mathbb{R}_+^n$ , [8].

The matrix  $A$  is called *Z-matrix* if its off-diagonal entries are non-positive. In the literature, a matrix for which  $-A$  is a Z-matrix sometimes is called *L-matrix*, Metzler matrix or essentially positive matrix, see, e.g., [8,22,26,42]. Throughout this paper we will use the term  $-Z$ -matrix. For a matrix  $A$  we have that  $e^{At} \geq 0$  for all  $t \geq 0$  if and only if  $A$  is a  $-Z$ -matrix, see, e.g., [42]. Let  $B \geq 0$  with spectral radius  $\rho(B)$ . A matrix  $A$  of the form  $A = sI - B$ , with  $s > 0$ , and  $s \geq \rho(B)$  is called *M-matrix*. If  $s > \rho(B)$  then  $A$  is a *nonsingular M-matrix*, if  $s = \rho(B)$  then  $A$  is a *singular M-matrix*. The class of *M-matrices* is a subclass of the *Z-matrices*. Accordingly, a matrix for which  $-A$  is an *M-matrix* we call a  $-M$ -matrix.

We call a matrix  $A$  *c-stable* if all eigenvalues of  $A$  have negative real part. A matrix  $A$  is called *d-stable* if  $\rho(A) < 1$ .

A symmetric matrix  $A$  is called *positive (semi)definite* if for all  $x \neq 0$  we have  $(x^T Ax \geq 0)$   $x^T Ax > 0$ . If this holds for  $-A$  then  $A$  is called *negative (semi)definite*.

The following generalised Perron–Frobenius-type condition for matrix pairs is presented in [35].

**Theorem 2.5.** *Let  $(E, A)$ , with  $E, A \in \mathbb{R}^{n \times n}$ , be a regular matrix pair of  $\text{ind}(E, A) = v$ . Let a matrix chain as in (2.2) be constructed with canonical projectors  $Q_i$ ,  $i = 0, \dots, v - 1$ . If*

$$E_v^{-1} A_v \geq 0, \quad (2.12)$$

*and  $\sigma_f(E, A) \neq \emptyset$ , then the finite spectral radius  $\rho_f(E, A)$  is an eigenvalue of  $(E, A)$  and if  $\rho_f(E, A) > 0$ , then there exists a corresponding nonnegative eigenvector  $v \geq 0$ . If  $E_v^{-1} A_v$  is, in addition, irreducible, then we have that  $\rho_f(E, A)$  is simple and  $v > 0$  is unique up to a scalar multiple.*

Throughout the paper, we will frequently use the following identity, see [31,35]:

$$E_v^{-1} A_v = \widehat{E}^D \widehat{A}, \quad (2.13)$$

where  $\widehat{E}$ ,  $\widehat{A}$  are defined as in Lemma 2.1.

### 3. Positive descriptor systems

In the literature, there are many different concepts of positivity in systems theory such as internal positivity, external positivity, weak positivity, etc. [22,26]. In this paper we consider only the notion of internal positivity. Hence, whenever we refer to positivity of a system, we speak of internal positivity.

For standard systems, positivity implies that for any initial condition  $x_0 \geq 0$  and any input function  $u(t) \geq 0$  we have  $x(t) \geq 0$  and  $y(t) \geq 0$  for all  $t \geq 0$ , see [22,26]. In the case of descriptor systems, however, not every nonnegative initial value is consistent, see Corollary 2.3 and Definition 2.4. Hence, we will require consistent nonnegative initial values in the definition of positive descriptor systems.

**Definition 3.1 (Positivity).** We call the continuous-time system (1.1) with  $\text{ind}(E, A) = v$  *positive* if for all  $t \in \mathbb{R}_+$  we have  $x(t) \geq 0$  and  $y(t) \geq 0$  for any input function  $u \in C^v$  such that  $u^{(i)}(\tau) \geq 0$  for  $i = 0, \dots, v - 1$  and  $0 \leq \tau \leq t$  and any consistent initial value  $x_0 \geq 0$ .

The discrete-time system (1.2) with  $\text{ind}(E, A) = v$  is called *positive* if for all  $t \in \mathbb{N}_0$  we have  $x(t) \geq 0$  and  $y(t) \geq 0$  for any input sequence  $u(\tau) \geq 0$  for  $0 \leq \tau \leq t + v - 1$  and any consistent initial value  $x_0 \geq 0$ .

To formulate a characterisation of positivity in the continuous-time case we need the following Lemma.

**Lemma 3.2.** *For a regular matrix pair  $(E, A)$  let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1. If for all  $v \geq 0$  we have  $e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E}v \geq 0$  for all  $t \geq 0$ , then there exists  $\alpha \geq 0$  such that*

$$\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E} \geq 0.$$

**Proof.** By assumption, we obtain that

$$e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E} \geq 0 \quad \text{for all } t \geq 0. \quad (3.1)$$

We now show that from this, we obtain that  $\widehat{E}^D \widehat{E} \geq 0$  and  $[\widehat{E}^D \widehat{A}]_{ij} \geq 0$  for all pairs  $(i, j)$  such that  $[\widehat{E}^D \widehat{E}]_{ij} = 0$ . Suppose that there exists a pair  $(i, j)$  such that  $[\widehat{E}^D \widehat{E}]_{ij} < 0$  or  $[\widehat{E}^D \widehat{E}]_{ij} = 0$  and  $[\widehat{E}^D \widehat{A}]_{ij} < 0$ , then for  $t > 0$  small enough, we would obtain

$$[e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E}]_{ij} = [\widehat{E}^D \widehat{E}]_{ij} + [\widehat{E}^D \widehat{A}]_{ij}t + O(t^2) < 0,$$

which contradicts Eq. (3.1). Here, we have used the property that  $\widehat{E}^D \widehat{A} \widehat{E}^D \widehat{E} = \widehat{E}^D \widehat{A}$ , which is easily deducible from (2.5) and (2.6b). Since  $\widehat{E}^D \widehat{E} \geq 0$ , setting

$$\alpha \geq \left| \min_{(i,j):[\widehat{E}^D \widehat{E}]_{ij} \neq 0} \frac{[\widehat{E}^D \widehat{A}]_{ij}}{[\widehat{E}^D \widehat{E}]_{ij}} \right|,$$

we obtain  $\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E} \geq 0$ .  $\square$

**Remark 3.3.** The important implication of Lemma 3.2 is that we can shift the finite spectrum of the matrix pair  $(E, A)$  as in the standard case, see, e.g., [22, p. 38], so that the shifted matrix pair  $(E, A + \alpha E)$  fulfills the assumptions of Theorem 2.5 and its finite spectral radius is an eigenvalue. For any finite eigenvalue  $\mu$  of  $(E, A + \alpha E)$  we have that  $\lambda = \mu - \alpha$  is a finite eigenvalue of  $(E, A)$ . The eigenvectors and eigenspaces of  $(E, A)$  and  $(E, A + \alpha E)$  are the same. In particular, the eigenspace that corresponds to the eigenvalue  $\infty$  remains unchanged. Note that we can choose  $\alpha$  large enough such that  $\rho_f(E, A + \alpha E) > 0$  and, therefore, we always have a corresponding nonnegative eigenvector in this case.

The following theorem characterises positivity in the continuous-time case.

**Theorem 3.4.** *Let  $E, A, B, C$  be the matrices in system (1.1) with  $(E, A)$  regular of  $\text{ind}(E, A) = v$ . Let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1 and  $\widehat{B}$  as in (2.4). Furthermore, assume that*

- (i)  $(I - \widehat{E}^D \widehat{E})(\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B} \leq 0$  for  $i = 0, \dots, v - 1$ ,
- (ii)  $\widehat{E}^D \widehat{E} \geq 0$ .

*Then the continuous-time system (1.1) is positive if and only if the following three conditions hold.*

1. *There exists a scalar  $\alpha \geq 0$  such that the matrix*

$$\overline{M} := -\alpha I + (\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E})$$

*is a  $-Z$ -matrix,*

2.  $\widehat{E}^D \widehat{B} \geq 0$ ,

3.  $C$  is nonnegative on the subspace  $\mathcal{X}$  defined by

$$\mathcal{X} := \text{im}_+[\widehat{E}^D \widehat{E}, -(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}, \dots, -(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)^{v-1}\widehat{A}^D \widehat{B}], \quad (3.2)$$

where for a matrix  $W \in \mathbb{R}^{n \times q}$  we define

$$\text{im}_+ W := \{w_1 \in \mathbb{R}^n \mid \exists w_2 \in \mathbb{R}_+^q : Ww_2 = w_1\}.$$

**Proof.** “ $\Rightarrow$ ” Let the system (1.1) be positive. By definition, for all  $t \geq 0$  we have  $x(t) \geq 0$  and  $y(t) \geq 0$  for every vector function  $u \in C^v$  that satisfies  $u^{(i)}(\tau) \geq 0$  for  $i = 0, \dots, v-1$  and  $0 \leq \tau \leq t$  and for every consistent  $x_0 \geq 0$ .

1. Choose  $u \equiv 0$ , then for any  $v \geq 0$  we have that  $x_0 = \widehat{E}^D \widehat{E}v \geq 0$  is a consistent initial condition. Hence, for all  $v \geq 0$ , from (2.8) we obtain that

$$x(t) = e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E}v \geq 0 \quad \text{for all } t \geq 0. \quad (3.3)$$

Then, by Lemma 3.2, there exists a scalar  $\alpha \geq 0$  such that  $\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E} \geq 0$ . Hence, the matrix  $\widehat{M} = -\alpha I + (\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E})$  has nonnegative off-diagonal entries, i.e.  $\widehat{M}$  is a  $-Z$ -matrix.

2. Choose now  $u(\tau) = \xi \tau^v$  for some  $\xi \in \mathbb{R}_+^m$ . We have that  $u^{(i)}(\tau) \geq 0$  for  $i = 0, \dots, v-1$  and  $0 \leq \tau \leq t$ . Furthermore, we have  $u^{(i)}(0) = 0$  for  $i = 0, \dots, v-1$ . Therefore, for some  $v \in \ker \widehat{E}^D \widehat{E}$ , we have that  $x_0 = \widehat{E}^D \widehat{E}v = 0$  is a consistent initial condition. Thus, from (2.8) we obtain that for all  $t \geq 0$  we have

$$x(t) = \int_0^t e^{\widehat{E}^D \widehat{A}(t-\tau)} \widehat{E}^D \widehat{B}u(\tau) d\tau - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E}\widehat{A}^D)^i \widehat{A}^D \widehat{B}u^{(i)}(t) \geq 0. \quad (3.4)$$

Since  $\widehat{E}^D \widehat{E} \geq 0$ , we can premultiply the inequality (3.4) by  $\widehat{E}^D \widehat{E}$  and obtain

$$\widehat{E}^D \widehat{E}x(t) = \int_0^t e^{\widehat{E}^D \widehat{A}(t-\tau)} \widehat{E}^D \widehat{B}\xi \tau^v d\tau \geq 0. \quad (3.5)$$

We now show that  $\widehat{E}^D \widehat{B} \geq 0$ . Suppose that this is not the case, i.e. there exist some indices  $i, j$  with  $[\widehat{E}^D \widehat{B}]_{ij} < 0$ . Then, for  $\xi = e_j$ , the  $j$ th unit vector, and for  $t > 0$  small enough, we obtain

$$\begin{aligned} [\widehat{E}^D \widehat{E}x(t)]_i &= \int_0^t [(I + \widehat{E}^D \widehat{A}(t-\tau) + O((t-\tau)^2)) \widehat{E}^D \widehat{B}u(\tau)]_i d\tau \\ &= \int_0^t ([\widehat{E}^D \widehat{B}]_{ij} + O(t-\tau)) \tau^v d\tau < 0, \end{aligned}$$

which contradicts (3.5). Therefore, we conclude that  $\widehat{E}^D \widehat{B} \geq 0$ .

3. Note that by assumptions (i) and (ii) the subspace  $\mathcal{X}$  contains only nonnegative vectors. Let  $v \in \text{im}[\widehat{E}^D \widehat{E}]$ ,  $v \geq 0$ . For  $u \equiv 0$ , we have that  $x_0 = \widehat{E}^D \widehat{E}v \geq 0$  is consistent with  $u$ . Since the system is positive, we have

$$y(0) = Cx_0 = C\widehat{E}^D \widehat{E}v \geq 0. \quad (3.6)$$

Since  $\widehat{E}^D \widehat{E}$  is a projector, we have  $\widehat{E}^D \widehat{E}v = v$  and hence, by (3.6),  $C$  is nonnegative on  $\text{im}[\widehat{E}^D \widehat{E}]$ .

Let now  $w_0 \in \text{im}_+[-(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}]$ , then there exists  $\xi_0 \geq 0$  such that  $-(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}\xi_0 = w_0$ . Choose  $u_0(\tau) \equiv \xi_0$ . Then, we have  $u_0(0) = \xi_0$  and  $u_0^{(i)}(0) = 0$  for  $i = 1, \dots, v-1$ . The initial condition  $x_0 = -(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}\xi_0$  is nonnegative by assumption (i) and consistent with  $u_0$  for some  $v \in \ker \widehat{E}^D \widehat{E}$ . Since the system is positive, we obtain

$$y(0) = Cx_0 = -C(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}\xi_0 = Cw_0 \geq 0. \quad (3.7)$$

We have shown that for all  $w_0 \in \text{im}_+[-(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}]$  we have  $Cw_0 \geq 0$ , i.e.  $C$  is nonnegative on  $\text{im}_+[-(I - \widehat{E}^D \widehat{E})\widehat{A}^D \widehat{B}]$ .

Let  $w_1 \in \text{im}_+[-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)\widehat{A}^D \widehat{B}]$ , then there exists  $\xi_1 \geq 0$  such that  $-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)\widehat{A}^D \widehat{B}\xi_1 = w_1$ . Set  $u_1(\tau) = \xi_1 \tau$ . Then, we have  $u_1(0) = 0$ ,  $u'_1(0) = \xi_1$  and  $u_1^{(i)}(0) = 0$ ,  $i = 2, \dots, v - 1$ . The initial condition  $x_0 = -(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)\widehat{A}^D \widehat{B}\xi_1$  is nonnegative by assumption (i) and consistent with  $u_1$ . Since the system is positive, we have

$$y(0) = Cx_0 = -C(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)\widehat{A}^D \widehat{B}\xi_1 = Cw_1 \geq 0,$$

and hence,  $C$  is nonnegative on  $\text{im}_+[-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)\widehat{A}^D \widehat{B}]$ .

We now proceed in the same manner. By subsequently letting  $w_i \in \text{im}_+[-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)^{(i)}\widehat{A}^D \widehat{B}]$  for  $i = 2, \dots, v - 1$ , finding the corresponding nonnegative preimage  $\xi_i$ , setting  $u_i(\tau) = \xi_i \tau^i$  and using the same argument as above we obtain that  $C$  is nonnegative on  $\text{im}_+[-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)^i\widehat{A}^D \widehat{B}]$  for  $i = 2, \dots, v - 1$ . In total, we have shown that  $C$  is nonnegative on  $\mathcal{X}$  as in (3.2).

“ $\Leftarrow$ ” Let (i), (ii) and 1–3 hold. We have to show that system (1.1) is positive, i.e. for all  $t \geq 0$  and for every vector function  $u \in C^v$  such that  $u^{(i)}(\tau) \geq 0$  for  $i = 0, \dots, v - 1$  and  $0 \leq \tau \leq t$  and for any consistent  $x_0 \geq 0$ , we get  $x(t) \geq 0$  and  $y(t) \geq 0$ . The solution at time  $t \geq 0$  is given by

$$\begin{aligned} x(t) &= e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E}x_0 + \int_0^t e^{\widehat{E}^D \widehat{A}(t-\tau)} \widehat{E}^D \widehat{B}u(\tau) d\tau \\ &\quad - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E}\widehat{A}^D)^i \widehat{A}^D \widehat{B}u^{(i)}(t), \end{aligned} \quad (3.8)$$

and any consistent  $x_0$  satisfies  $x_0 = \widehat{E}^D \widehat{E}v - (I - \widehat{E}^D \widehat{E}) \sum_{i=0}^{v-1} (\widehat{E}\widehat{A}^D)^i \widehat{A}^D \widehat{B}u^{(i)}(0)$  for some  $v \in \mathbb{R}^n$ . We now subsequently show that the three summands in (3.8) are nonnegative.

(1) Since  $\widehat{E}^D \widehat{E} \geq 0$ , for any consistent  $x_0 \geq 0$  we get that  $\widehat{E}^D \widehat{E}x_0 \geq 0$ . Note, that for any  $v \in S_f^{\text{def}}$  we have  $\widehat{E}^D \widehat{E}v = v$  and

$$\overline{M}v = (-\alpha I + \widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E})v = \widehat{E}^D \widehat{A}v. \quad (3.9)$$

Since  $\widehat{E}^D \widehat{E}$  is a projector onto  $S_f^{\text{def}}$ , we also have

$$e^{\widehat{E}^D \widehat{A}t} \widehat{E}^D \widehat{E} = e^{\overline{M}t} \widehat{E}^D \widehat{E}. \quad (3.10)$$

and  $e^{\overline{M}t} \geq 0$ , since  $\overline{M}$  is a  $-Z$ -matrix. Hence, the first term of (3.8) is nonnegative.

(2) For the second term we have that  $\widehat{E}^D \widehat{B} \geq 0$  and therefore  $e^{\widehat{E}^D \widehat{A}(t-\tau)} \widehat{E}^D \widehat{B}u(\tau) \geq 0$  for all  $0 \leq \tau \leq t$ . Since integration is monotone, the second term is nonnegative.

(3) We have  $-(I - \widehat{E}^D \widehat{E})(\widehat{E}\widehat{A}^D)^i \widehat{A}^D \widehat{B} \geq 0$  for  $i = 0, \dots, v - 1$  and therefore the third term is also nonnegative for any vector function  $u \in C^v$  such that  $u^{(i)}(\tau) \geq 0$  for  $i = 0, \dots, v - 1$  and  $0 \leq \tau \leq t$ .

Thus,  $x(t) \geq 0$ . From  $y(t) = Cx(t)$  with  $C$  nonnegative on  $\mathcal{X}$  and  $x(t) \in \mathcal{X}$  for all  $t$ , we also conclude that  $y(t) \geq 0$ .  $\square$

**Corollary 3.5.** *Let  $E, A, B, C$  be the matrices in system (1.1) with  $(E, A)$  regular of  $\text{ind}(E, A) = v$ . Let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1 and  $\widehat{B}$  as in (2.4). Furthermore, we assume that  $(I -$*

$\widehat{E}^D \widehat{E})(\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B} \leqslant 0$  for  $i = 0, \dots, v - 1$ . If the matrix  $\widehat{E}^D \widehat{A}$  is a  $-Z$ -matrix and  $\widehat{E}^D \widehat{B}, C \geqslant 0$ , then the continuous-time system (1.1) is positive.

**Proof.** If  $\widehat{E}^D \widehat{A}$  is a  $-Z$ -matrix, this implies that  $\overline{M}$  is a  $-Z$ -matrix for  $\alpha = 0$ . Internal positivity follows from Theorem 3.4.  $\square$

The first of the following two examples demonstrates that the property that  $\widehat{E}^D \widehat{A}$  is a  $-Z$ -matrix is not necessary for the system (1.1) to be positive. The second example is a system that is not positive.

**Example 3.6.** Consider the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

Since the matrices  $E$  and  $A$  commute, we can directly compute

$$E^D A = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D E = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D B = 0.$$

Note that  $E^D A$  is not a  $-Z$ -matrix. For the state vector, we obtain

$$\begin{aligned} x(t) &= e^{E^D A t} E^D E v - (I - E^D E) A^D B u(t) \\ &= \begin{bmatrix} e^{-t} & e^{-t} - 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 + v_2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t). \end{aligned}$$

Hence, the system is positive, although  $E^D A$  is not a  $-Z$ -matrix.

**Example 3.7.** Consider the system

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

The matrices  $E$  and  $A$  commute and we can compute

$$E^D A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E^D B = 0.$$

For the solution, we obtain

$$x(t) = e^{E^D A t} E^D E v - (I - E^D E) A^D B u(t) = \begin{bmatrix} e^t & -te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u(t).$$

The system is not positive, since the first component of  $x$  may become negative.

In [11], the following characterisation of positivity in the case of discrete-time systems was given. Note, that in [11] the proof is given without the consistency requirement on  $x_0$ ,

thus, referring to a somewhat different solution concept. However, with a minor modification of the proof, the characterisation is also valid for positivity as in Definition 2.4, i.e., only for consistent initial values. Furthermore, we add the condition on the matrix  $C$  for completeness.

**Theorem 3.8.** *Let  $E, A, B, C$  be the matrices in system (1.2) with  $(E, A)$  regular of  $\text{ind}(E, A) = v$ . Let  $\widehat{E}$ ,  $\widehat{A}$  be defined as in Lemma 2.1 and  $\widehat{B}$  as in (2.4). If  $\widehat{E}^D \widehat{E} \geq 0$ , then the discrete-time system (1.2) is positive if and only if  $\widehat{E}^D \widehat{A}, \widehat{E}^D \widehat{B} \geq 0$ ,  $(I - \widehat{E}^D \widehat{E})(\widehat{E} \widehat{A}^D)^i \widehat{A}^D \widehat{B} \leq 0$  for  $i = 0, \dots, v - 1$  and  $C$  is nonnegative on  $\mathcal{X}$  as defined in (3.2).*

#### 4. Stability conditions for positive descriptor systems

In the course of this section, we consider linear homogeneous positive time-invariant systems:

- in continuous-time:

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad (4.1)$$

- or in discrete-time:

$$Ex(t+1) = Ax(t), \quad x(0) = x_0. \quad (4.2)$$

**Definition 4.1** (*c-/d-positive matrix pair*). We call a matrix pair  $(E, A)$  *c-positive* if system (4.1) is positive. We call a matrix pair  $(E, A)$  *d-positive* if system (4.2) is positive.

**Remark 4.2.** Note that by Theorem 3.4, if  $\widehat{E}^D \widehat{E} \geq 0$ , then  $(E, A)$  is c-positive if and only if there exists  $\alpha \geq 0$  such that  $\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E} \geq 0$ . By Theorem 3.8, if  $\widehat{E}^D \widehat{E} \geq 0$ , then  $(E, A)$  is d-positive if and only if  $\widehat{E}^D \widehat{A} \geq 0$ .

**Definition 4.3** (*c-/d-stable matrix pair*). A matrix pair  $(E, A)$  is called *c-stable* if all finite eigenvalues of  $(E, A)$  have negative real part. A matrix pair  $(E, A)$  is called *d-stable* if  $\rho_f(E, A) < 1$ .

Note that Definition 4.3 generalises the usual stability definition for matrices, i.e., a matrix  $A$  is called c-stable (d-stable) if  $(I, A)$  is c-stable (d-stable), see Section 2.4.

In this subsection we generalise the stability conditions for positive systems from the standard case, see [22], to the descriptor case. Stability conditions for positive systems are closely related to and can be characterised by the so called dominant eigenvalue(s) of the system.

**Definition 4.4** (*c-/d-dominant eigenvalue*). For linear continuous-time systems (4.1), we call a finite eigenvalue  $\lambda$  of the matrix pair  $(E, A)$  *c-dominant* if its real part is greater than or equal to the real part of any other eigenvalue of the matrix pair  $(E, A)$ , i.e.  $\Re(\lambda) \geq \Re(\lambda_i)$  for all  $\lambda_i \in \sigma_f(E, A)$ .

For linear discrete-time systems (4.2), we call a finite eigenvalue of the matrix pair  $(E, A)$  *d-dominant* if it is greater than or equal to any other eigenvalue of the matrix pair  $(E, A)$  in modulus, i.e.  $|\lambda| \geq |\lambda_i|$  for all  $\lambda_i \in \sigma_f(E, A)$ .

In the following theorem, we generalise the result on dominant eigenvalues in [22, Theorem 11] to descriptor systems.

**Theorem 4.5.** *Let  $(E, A)$  be a regular matrix pair. Consider the positive continuous-time system (4.1). If  $\sigma_f(E, A) \neq \emptyset$  and  $\widehat{E}^D \widehat{E} \geq 0$ , where  $\widehat{E}$  is defined as in Lemma 2.1, then the c-dominant eigenvalue  $\lambda$  of the system is real and unique. Furthermore, there exists a nonnegative eigenvector corresponding to  $\lambda$ .*

*Consider the positive discrete-time system (4.2). If  $\sigma_f(E, A) \neq \emptyset$  and  $\widehat{E}^D \widehat{E} \geq 0$ , then  $\rho_f(E, A)$  is a d-dominant eigenvalue and there exists a corresponding nonnegative eigenvector.*

**Proof.** In the continuous-time case, since  $\widehat{E}^D \widehat{E} \geq 0$ , by Remark 4.2 and Remark 3.3 we have that there exists a scalar  $\alpha > 0$  such that for the shifted matrix pair  $(E, A + \alpha E)$ , by the generalised Perron-Frobenius Theorem 2.5, the finite spectral radius  $\rho_f(E, A + \alpha E) =: \mu$  is an eigenvalue. Hence,  $\lambda = \mu - \alpha$  is an eigenvalue of  $(E, A)$  and it is the eigenvalue with the largest real part, i.e., the c-dominant eigenvalue of the positive system (4.1). Hence, the c-dominant eigenvalue  $\lambda$  is real and unique. By Remark 3.3 there exists a corresponding nonnegative eigenvector.

For a positive discrete-time system (1.2), by Remark 4.2, if  $\widehat{E}^D \widehat{E} \geq 0$ , we have that  $\widehat{E}^D \widehat{A} \geq 0$ . Hence, by the generalised Perron-Frobenius Theorem 2.5 and using the identity in (2.13), the finite spectral radius of  $(E, A)$  is an eigenvalue and, by Remark 3.3, there exists a corresponding nonnegative eigenvector.

Theorem 4.5 implies that a c-positive matrix pair is c-stable if and only if all of its real eigenvalues have negative real part. Analogously, a d-positive matrix pair is d-stable if and only if all of its real eigenvalues are in modulus less than 1.

**Example 4.6.** Let  $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ . Since  $E$  and  $A$  commute, we have  $E^D E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  and  $E^D A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ . Hence, the system (4.1) for this choice of  $(E, A)$  is positive, since  $e^{E^D A t} E^D E v = \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} e^{-t} v_1 \\ 0 \end{bmatrix} \geq 0$

for all  $v_1 \geq 0$ . Choosing  $\alpha = 1$ , we obtain

$$E^D A + \alpha E^D E = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \geq 0.$$

Hence,  $\mu := \rho(E^D A + \alpha E^D E) = 0$  is an eigenvalue and the corresponding c-dominant eigenvalue of  $(E, A)$  is  $\lambda = \mu - \alpha = -1$ . This means that  $(E, A)$  is also c-stable. Note that although  $\mu = 0$ , due to the fact that  $E^D E \geq 0$ , we have a nonnegative eigenvector corresponding to  $\mu$  and, hence, to  $\lambda$ , see Remark 3.3.

**Lemma 4.7.** *Let  $(E, A)$  be a regular c-stable matrix pair. Then, for any  $\alpha > 0$  we have that*

$$\overline{M} := -\alpha I + \widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E},$$

*is a stable (regular) matrix. If, in addition, the matrix pair  $(E, A)$  is c-positive and  $\widehat{E}^D \widehat{E} \geq 0$ , then there exists  $\alpha > 0$  such that  $\overline{M}$  is a  $-M$ -matrix.*

**Proof.** All finite eigenvalues of  $(E, A)$  are also eigenvalues of  $\widehat{E}^D \widehat{A}$  and the eigenvalue  $\infty$  of  $(E, A)$  is mapped to the eigenvalue 0 of  $\widehat{E}^D \widehat{A}$  [35]. For any finite eigenpair  $(\lambda, v)$  of  $(E, A)$ , we have

$$\overline{M}v = \widehat{E}^D \widehat{A}v = \lambda v.$$

Therefore, all stable finite eigenvalues of the pair  $(E, A)$  are stable eigenvalues of  $\overline{M}$ . For any eigenvector  $w$  corresponding to the eigenvalue  $\infty$  of  $(E, A)$ , i.e.,  $Ew = 0$ , we have by the properties of  $\widehat{E}, \widehat{A}$  in Lemma 2.1 and Eqs. (2.6) that

$$\widehat{E}^D \widehat{A}w = \widehat{E}^D \widehat{A}\widehat{E}^D \widehat{E}w = \widehat{E}^D \widehat{A}\widehat{E}^D(\lambda E - A)^{-1}Ew = 0,$$

and hence,

$$\overline{M}w = -\alpha w.$$

Thus,  $w$  is now an eigenvector corresponding to a negative eigenvalue  $-\alpha$ . Hence, all eigenvalues of  $\overline{M}$  have negative real parts and therefore  $\overline{M}$  is stable. If, in addition, the matrix pair  $(E, A)$  is c-positive and  $\widehat{E}^D \widehat{E} \geq 0$ , then by Remark 4.2 we have that there exists  $\alpha > 0$  such that  $T := \widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E} \geq 0$ . By the generalised Perron-Frobenius Theorem 2.5 we have that  $\rho(T)$  is an eigenvalue of  $T$  and  $\rho(T) - \alpha$  is the finite eigenvalue of  $(E, A)$  with the largest real part and it is negative, since  $(E, A)$  is c-stable. Therefore, we have  $\alpha > \rho(T)$  and

$$\overline{M} = -(\alpha I - T)$$

is a  $-M$ -matrix.  $\square$

In the following we generalise a Lyapunov-type stability condition from the standard case, see [22, Theorem 15], to the descriptor case.

**Theorem 4.8.** *Let the matrix pair  $(E, A)$  be regular and let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1. If  $(E, A)$  is c-positive and  $\widehat{E}^D \widehat{E} \geq 0$ , then the pair  $(E, A)$  is c-stable if and only if there exists a positive definite diagonal matrix  $Y$  such that  $(\widehat{E}^D \widehat{A})^T Y + Y(\widehat{E}^D \widehat{A})$  is negative semidefinite and negative definite on  $S_f^{\text{def}}$ .*

*If  $(E, A)$  is d-positive and  $\widehat{E}^D \widehat{E} \geq 0$ , then  $(E, A)$  is d-stable if and only if there exists a positive definite diagonal matrix  $Y$  such that  $(\widehat{E}^D \widehat{A})^T Y(\widehat{E}^D \widehat{A}) - Y$  is negative definite.*

**Proof.** Continuous-time case:

“ $\Rightarrow$ ” By Lemma 4.7, we have that there exists  $\alpha > 0$  such that the matrix

$$M := \alpha I - (\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E}),$$

is a regular  $M$ -matrix. For all  $v \in S_f^{\text{def}}$  we have by (3.9) that

$$v^T(\widehat{E}^D \widehat{A})^T Y v + v^T Y(\widehat{E}^D \widehat{A})v = v^T(-M)^T Y v + v^T Y(-M)v.$$

It is well-known that for an  $M$ -matrix  $M$  there exists a positive definite diagonal matrix  $Y$  so that the matrix  $-(M^T Y + YM)$  is negative definite, see, e.g., [4,5,8,22]. Hence,  $Y$  is a positive definite diagonal matrix such that  $(\widehat{E}^D \widehat{A})^T Y + Y(\widehat{E}^D \widehat{A})$  is negative definite on  $S_f^{\text{def}}$ . For any  $w \in \mathbb{R}^n \setminus S_f^{\text{def}}$ , we have  $\widehat{E}^D \widehat{A}w = 0$  and hence,  $(\widehat{E}^D \widehat{A})^T Y + Y(\widehat{E}^D \widehat{A})$  is negative semidefinite on  $\mathbb{R}^n$ .

“ $\Leftarrow$ ” We have to show that all finite eigenvalues of  $(E, A)$  have negative real part. If  $\sigma_f(E, A) = \emptyset$ , there is nothing to prove. Therefore, assume that  $\sigma_f(E, A) \neq \emptyset$ . Then, by Theorem 4.5, we

have that the c-dominant eigenvalue  $\lambda$  of  $(E, A)$  is real and unique. Hence, it suffices to show that  $\lambda$  is negative. Let  $v$  be an eigenvector corresponding to  $\lambda$ . Since the eigenpair  $(\lambda, v)$  is also an eigenpair of  $\widehat{E}^D \widehat{A}$ , [35], we obtain

$$\begin{aligned} v^T(\widehat{E}^D \widehat{A})^T Y v + v^T Y (\widehat{E}^D \widehat{A}) v &= v^T \lambda Y v + v^T Y \lambda v \\ &= 2\lambda v^T Y v < 0, \end{aligned}$$

whereas  $v^T Y v > 0$ . Hence,  $\lambda < 0$ .

Discrete-time case:

“ $\Rightarrow$ ” If  $\widehat{E}^D \widehat{A} \geq 0$ , for a positive system we also have  $\widehat{E}^D \widehat{A} \geq 0$ , see Remark 4.2. Since the matrix pair  $(E, A)$  is d-stable, we have  $\rho_f(E, A) < 1$  and hence, the matrix

$$M := I - \widehat{E}^D \widehat{A},$$

is a regular  $M$ -matrix. Therefore, there exists a diagonal positive definite matrix  $Y$  so that the matrix  $(\widehat{E}^D \widehat{A})^T Y (\widehat{E}^D \widehat{A}) - Y$  is negative definite, see, e.g. [4,22].

“ $\Leftarrow$ ” As in the continuous-time case, we assume that  $\sigma_f(E, A) \neq \emptyset$ . Then, by Theorem 4.5, we have that there exists a d-dominant eigenvalue  $\lambda$  of  $(E, A)$  that is nonnegative and real. Hence, it suffices to show that  $\lambda$  is smaller than 1. Let  $v$  be an eigenvector corresponding to  $\lambda$ . Since the eigenpair  $(\lambda, v)$  is also an eigenpair of  $\widehat{E}^D \widehat{A}$ , [35], we obtain

$$\begin{aligned} v^T(\widehat{E}^D \widehat{A})^T Y (\widehat{E}^D \widehat{A}) v - v^T Y v &= \lambda^2 v^T Y v - v^T Y v \\ &= (\lambda^2 - 1)v^T Y v < 0, \end{aligned}$$

whereas  $v^T Y v > 0$ . Since  $\lambda$  is nonnegative, we have  $\lambda < 1$ .  $\square$

**Corollary 4.9.** *Let the matrix pair  $(E, A)$  be regular and let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1. If  $(E, A)$  is c-positive and  $\widehat{E}^D \widehat{A} \geq 0$ , then the matrix pair  $(E, A)$  is c-stable if and only if there exists a scalar  $\alpha > 0$  such that for the matrix  $M := \alpha I - (\widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E})$  each of the following properties holds:*

1. all principal minors of  $M$  are positive;
2. the coefficients of the characteristic polynomial of  $M$  are negative.

If  $(E, A)$  is d-positive and  $\widehat{E}^D \widehat{A} \geq 0$ , then the matrix pair  $(E, A)$  is d-stable if and only if each of the properties 1–2 holds for the matrix  $M := I - \widehat{E}^D \widehat{A}$ .

**Proof.** In the continuous-time case, by Lemma 4.7 there exists  $\alpha > 0$  such that  $M$  is an  $M$ -matrix. In the discrete-time case,  $M$  is an  $M$ -matrix by Theorem 4.8. Therefore, the assertions of this Corollary follow directly from the  $M$ -matrix properties [8,22].  $\square$

## 5. Nonnegative solution of generalised Lyapunov equations

Consider the following generalised projected continuous-time Lyapunov equation [39]

$$E^T X A + A^T X E = -P_r^T G P_r, \quad (5.1)$$

where  $G \in \mathbb{R}^{n \times n}$  and  $P_r$ , as defined in (2.3), is the unique spectral projector onto the finite deflating subspace  $S_f^{\text{def}}$  of the pencil  $(E, A)$ . Note that  $P_r = \widehat{E}^D \widehat{E}$ , see Section 2.3.

**Theorem 5.1.** Let  $(E, A)$  be a regular  $c$ -stable matrix pair. Let  $\widehat{E}, \widehat{A}$  be defined as in Lemma 2.1 and assume  $\widehat{E}^D \widehat{E} \geq 0$ . Then equation (5.1) has a solution for every matrix  $G$ . A solution is given by

$$X = E_v^{-T} \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) E_v^{-1}, \quad (5.2)$$

where  $E_v$  is defined as in the matrix chain in (2.2) and  $P_r = \widehat{E}^D \widehat{E}$ . If  $G$  is symmetric positive (semi) definite, then  $X$  is symmetric positive semidefinite. If, in addition, we have that the matrix pair  $(E, A)$  is  $c$ -positive,  $G \geq 0$  and  $P_r E_v^{-1} \geq 0$ , then also  $X \geq 0$ .

**Proof.** We show that  $X$  as defined in (5.2) is solution of (5.1). Since  $(E, A)$  is  $c$ -stable, by Lemma 4.7, we have that for any  $\alpha > 0$  the matrix

$$\overline{M} := -\alpha I + \widehat{E}^D \widehat{A} + \alpha \widehat{E}^D \widehat{E}$$

is stable and  $\overline{M} P_r = P_r \overline{M} = \widehat{E}^D \widehat{A}$ . We now use the following properties that can be found in [31,35]:

$$E_v^{-1} A_i Q_i = -Q_i \quad \text{for all } i = 0, \dots, v-1, \quad (5.3)$$

where  $E_v, A_i$  are defined as in the matrix chain (2.2) with canonical projectors  $Q_i$ . By definition, we have  $E_v = E - A_0 Q_0 - \dots - A_{v-1} Q_{v-1}$  and with the identities in (5.3) we get

$$E_v^{-1} E = I - Q_0 - \dots - Q_{v-1}. \quad (5.4)$$

Since  $P_r = P_0 \cdots P_{v-1}$ , where  $P_i = I - Q_i$ , we have [31,35]

$$P_r Q_i = 0 \quad \text{for all } i = 0, \dots, v-1. \quad (5.5)$$

By using this, we obtain that

$$\begin{aligned} E^T X E &= E^T E_v^{-T} \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) E_v^{-1} E \\ &\stackrel{(5.4)}{=} E^T E_v^{-T} \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) (I - Q_0 - \dots - Q_{v-1}) \\ &\stackrel{(5.5)}{=} \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \\ &\stackrel{(3.10)}{=} \int_0^\infty e^{\overline{M}^T t} P_r^T G P_r e^{\overline{M} t} dt \end{aligned}$$

is a solution of the standard Lyapunov equation

$$(E^T X E) \overline{M} + \overline{M}^T (E^T X E) = -P_r^T G P_r.$$

On the other hand, by using the identity (2.13), we obtain

$$\begin{aligned} A^T X E &= A^T E_v^{-T} \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) E_v^{-1} E \\ &= P_r^T A^T E_v^{-T} \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) \\ &= (\widehat{E}^D \widehat{A})^T \left( \int_0^\infty e^{(\widehat{E}^D \widehat{A})^T t} P_r^T G P_r e^{(\widehat{E}^D \widehat{A})t} dt \right) \end{aligned}$$

$$\begin{aligned} &= \bar{M}^T \left( \int_0^\infty e^{(\hat{E}^D \hat{A})^T t} P_r^T G P_r e^{(\hat{E}^D \hat{A})^T t} dt \right) \\ &= \bar{M}^T (E^T X E) \end{aligned}$$

and analogously  $E^T X A = (E^T X E) \bar{M}$ . Hence, if we plug  $X$  defined in (5.2) into Eq. (5.1), then we obtain

$$\begin{aligned} E^T X A + A^T X E &= (E^T X E) \bar{M} + \bar{M}^T (E^T X E) \\ &= -P_r^T G P_r. \end{aligned}$$

If  $G$  is symmetric positive (semi)definite, then  $X$  is symmetric positive semidefinite [39]. If  $(E, A)$  is c-positive and  $P_r \geq 0$ , then  $e^{(\hat{E}^D \hat{A})^T t} P_r \geq 0$ . With  $G \geq 0$  and  $P_r E_v^{-1} \geq 0$  we obtain  $X \geq 0$ .  $\square$

In [39] a unique solution of (5.1) is obtained by introducing an additional condition

$$X = X P_l, \quad \text{where } P_l = W \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W^{-1}, \quad (5.6)$$

and  $W$  is defined as in (2.7). Note that condition (5.6) holds for the solution  $X$  as defined in (5.2), which can be verified by considering the Weierstraß canonical form.

For the discrete-time case, consider the following generalised projected discrete-time Lyapunov equation [39]

$$A^T X A - E^T X E = -P_r^T G P_r, \quad (5.7)$$

where  $G \in \mathbb{R}^{n \times n}$  and  $P_r$ , as defined in (2.3), is the unique spectral projector onto the finite deflating subspace  $S_f^{\text{def}}$  of the pair  $(E, A)$ . We state the following theorem without proof, since it is completely analogous to the proof of Theorem 5.1.

**Theorem 5.2.** *Let  $(E, A)$  be a regular d-stable matrix pair. Let  $\hat{E}, \hat{A}$  be defined as in Lemma 2.1 and assume  $\hat{E}^D \hat{A} \geq 0$ . Then Eq. (5.7) has a solution for every matrix  $G$ . A solution is given by*

$$X = E_v^{-T} \left( \sum_{t=0}^{\infty} ((\hat{E}^D \hat{A})^T)^t P_r^T G P_r (\hat{E}^D \hat{A})^t \right) E_v^{-1}, \quad (5.8)$$

where  $E_v$  is defined as in the matrix chain in (2.2) and  $P_r = \hat{E}^D \hat{A}$ . If  $G$  is symmetric positive (semi)definite, than  $X$  is symmetric positive semidefinite. If, in addition, we have that the matrix pair  $(E, A)$  is d-positive,  $G \geq 0$  and  $P_r E_v^{-1} \geq 0$ , then also  $X \geq 0$ .

In [39] a unique solution of (5.7) is obtained by introducing an additional condition  $P_l^T X = X P_l$ , where  $P_l$  is defined as in (5.6). Note that this condition holds for the solution  $X$  as defined in (5.8).

## 6. Stability of switched positive descriptor systems

The study of stability properties of switched systems is subject to ongoing research, see [27] and the references therein. Especially, in the case of standard positive systems, progress has been made on this subject due to the existence of a diagonal Lyapunov function, see, e.g., [32,33] and the references therein. The existence of a common diagonal Lyapunov function of two positive systems, i.e. a diagonal positive definite matrix  $Y$  such that

$$A_1^T Y + Y A_1 \quad \text{and}$$

$$A_2^T Y + Y A_2$$

are negative definite, guarantees the stability of the switched system under arbitrary switching. In this section, we show how we can use the framework established throughout this paper in order to generalise these results to positive descriptor systems.

The following sufficient conditions for the existence of a common diagonal Lyapunov function in the standard case can be found, e.g., in [32,33].

**Theorem 6.1.** *Let  $A_1, A_2 \in \mathbb{R}^{n \times n}$  be  $-M$ -matrices, i.e., stable  $-Z$ -matrices. Then, each of the following conditions is sufficient for the existence of a common diagonal Lyapunov function:*

1.  $A_1 A_2^{-1}$  and  $A_2^{-1} A_1$  are both  $M$ -matrices.
2.  $A_1 A_2^{-1}$  and  $A_2^{-1} A_1$  are both nonnegative.

The generalisation to positive descriptor systems uses Theorem 4.8 and is as follows.

**Theorem 6.2.** *Let  $(E_1, A_1), (E_2, A_2)$  be two  $c$ -stable matrix pairs and  $\widehat{E}_1^D \widehat{E}_1 \geq 0$  and  $\widehat{E}_2^D \widehat{E}_2 \geq 0$ . Then there exist scalars  $\alpha_1, \alpha_2 > 0$  such that*

$$M_1 := \alpha I - \widehat{E}_1^D \widehat{A}_1 - \alpha \widehat{E}_1^D \widehat{E}_1, \quad \text{and}$$

$$M_2 := \alpha I - \widehat{E}_2^D \widehat{A}_2 - \alpha \widehat{E}_2^D \widehat{E}_2$$

*are  $M$ -matrices and each of the following conditions is sufficient for the existence of a common diagonal Lyapunov function:*

1.  $M_1 M_2^{-1}$  and  $M_2^{-1} M_1$  are both  $M$ -matrices.
2.  $M_1 M_2^{-1}$  and  $M_2^{-1} M_1$  are both nonnegative.

**Proof.** By Lemma 4.7, there exist scalars  $\alpha_1, \alpha_2 > 0$  such that  $M_1, M_2$  are  $M$ -matrices. The rest follows as in the proof of the standard case in Theorem 6.1.  $\square$

## 7. Conclusions

In this paper, we have discussed positive descriptor systems in the continuous-time as well as in the discrete-time case. We have presented characterisations of positivity and generalisations of stability criteria for the case of positive descriptor systems. We have shown that if the spectral projector onto the finite deflating subspace of the matrix pair  $(E, A)$  is nonnegative, then all stability criteria for standard positive systems take a comparably simple form in the positive descriptor case. Furthermore, we have provided sufficient conditions that guarantee the existence of doubly nonnegative solutions of generalised projected Lyapunov equations. As an application of the framework established throughout this paper, we have shown how stability criteria of switched standard positive systems can be extended to the descriptor case.

## Acknowledgments

The author thanks the anonymous referees for many constructive comments which helped to significantly improve an earlier version of this paper. Also the author would like to thank V. Mehrmann for reading the manuscript with great care.

## References

- [1] D. Aeyels, P. De Leenheer, Stabilization of positive linear systems, *Systems Control Lett.*, 44 (2001) 259–271.
- [2] B.D.O. Anderson, New developments in the theory of positive systems, *Systems and Control in the Twenty-first Century* (St. Louis, MO, 1996), *Progr. Systems Control Theory*, vol. 22, Birkhäuser, Boston, MA, 1997, pp. 17–36.
- [3] A.C. Antoulas, *Approximation of Large-Scale Dynamical Systems*, *Advances in Design and Control*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.
- [4] M. Araki, Application of M-matrices to the stability problems of composite dynamical systems, *J. Math. Anal. Appl.* 52 (1975) 309–321.
- [5] G.P. Barker, A. Berman, R.J. Plemmons, Positive diagonal solutions to the Lyapunov equations, *Linear Multilinear Algebra* 5 (1977/1978) 249–256.
- [6] L. Benvenuti, A. De Santis, L. Farina (Eds.), *Positive Systems*, *Lecture Notes in Control and Information Sciences*, vol. 294, Springer-Verlag, Berlin, 2003.
- [7] L. Benvenuti, L. Farina, Positive and compartmental systems, *IEEE Trans. Automat. Control* 47 (2002) 370–373.
- [8] A. Berman, R.J. Plemmons, *Classics in Applied Mathematics*, second ed., Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1994.
- [9] G. Birkhoff, R.S. Varga, Reactor criticality and non-negative matrices, *J. Soc. Indust. Appl. Math.* 6 (1958) 354–377.
- [10] R. Bru, C. Coll, S. Romero-Vivo, E. Sánchez, Some problems about structural properties of positive descriptor systems, *Positive systems* (Rome, 2003), *Lecture Notes in Control and Information Science*, vol. 294, Springer-Verlag, Berlin, 2003, pp. 233–240.
- [11] R. Bru, C. Coll, E. Sánchez, About positively discrete-time singular systems, *System and Control: Theory and Applications*, *Electr. Comput. Eng. Ser.*, World Sci. Eng. Soc. Press, Athens, 2000, pp. 44–48.
- [12] R. Bru, C. Coll, E. Sánchez, Structural properties of positive linear time-invariant difference-algebraic equations, *Linear Algebra Appl.* 349 (2002) 1–10.
- [13] S.L. Campbell, *Singular Systems of Differential Equations*, *Research Notes in Mathematics*, Pitman (Advanced Publishing Program), Boston, MA, 1980.
- [14] S.L. Campbell, A general form for solvable linear time varying singular systems of differential equations, *SIAM J. Math. Anal.* 18 (1987) 1101–1115.
- [15] S.L. Campbell, C.D. Meyer Jr., *Generalized Inverses of Linear Transformations*, Dover Publications, New York, 1991 (Corrected reprint of the 1979 original).
- [16] C. Commault, N. Marchand (Eds.), *Positive Systems*, *Lecture Notes in Control and Information Sciences*, vol. 341, Springer-Verlag, Berlin, 2006.
- [17] P.G. Coxson, L.C. Larson, H. Schneider, Monomial patterns in the sequence  $A^k b$ , *Linear Algebra Appl.* 94 (1987) 89–101.
- [18] P.G. Coxson, H. Shapiro, Positive input reachability and controllability of positive systems, *Linear Algebra Appl.* 94 (1987) 35–53.
- [19] L. Dai, *Singular Control Systems*, *Lecture Notes in Control and Information Sciences*, vol. 118, Springer-Verlag, Berlin, 1989.
- [20] T. Damm, *Rational Matrix Equations in Stochastic Control*, *Lecture Notes in Control and Information Sciences*, vol. 297, Springer-Verlag, Berlin, 2004.
- [21] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, *Amer. Math. Monthly* 65 (1958) 506–514.
- [22] L. Farina, S. Rinaldi, *Positive Linear Systems: Theory and its Applications*, John Wiley and Sons, New York, 2000.
- [23] Z. Gajic, M.T.J. Qureshi, *Mathematics in Science and Engineering*, Academic Press, San Diego, CA, 1995.
- [24] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing, New York, 1959 (Translated by K.A. Hirsch).
- [25] E. Griepentrog, R. März, *Differential-Algebraic Equations and their Numerical Treatment*, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 88, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1986.
- [26] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

- [27] C. King, O. Mason, R.N. Shorten, F. Wirth, K. Wulff, Stability theory for switched and hybrid systems, SIAM Rev. 49 (2007) 545–592.
- [28] P. Kunkel, V. Mehrmann, Differential-algebraic equations. Analysis and Numerical Solution, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006.
- [29] P. Lancaster, M. Tismenetsky, The theory of matrices, Computer Science and Applied Mathematics, second ed., Academic Press, Orlando, FL, 1985.
- [30] R. März, Numerical methods for differential algebraic equations, Acta Numerica, Cambridge University Press, Cambridge, 1992, pp. 141–198.
- [31] R. März, Canonical projectors for linear differential algebraic equations, Comput. Math. Appl. 31 (1996) 121–135.
- [32] O. Mason, R. Shorten, Some results on the stability of positive switched linear systems, in: IEEE Conference on Decision and Control, 2004. <<http://www.hamilton.ie/ollie/Downloads/DiagonalStability.pdf>>.
- [33] O. Mason, R. Shorten, On the simultaneous diagonal stability of a pair of positive linear systems, Linear Algebra Appl. 413 (2006) 13–23.
- [34] V. Mehrmann, The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution, Lecture Notes in Control and Information Sciences, vol. 163, Springer-Verlag, Heidelberg, 1991 <<http://springerlink.com/content/110315/>>.
- [35] V. Mehrmann, R. Nabben, E. Virnik, Generalisaton of the Perron–Frobenius theory to matrix pencils, Linear Algebra Appl. 428 (2008) 20–38.
- [36] J.W. Nieuwenhuis, When to call a linear system nonnegative, Linear Algebra Appl. 281 (1998) 43–58.
- [37] Y. Ohta, H. Maeda, S. Kodama, Reachability, observability, and realizability of continuous-time positive systems, SIAM J. Control Optim. 22 (1984) 171–180.
- [38] P. Santesco, M.E. Valcher, On the zero pattern properties and asymptotic behavior of continuous-time positive system trajectories, Linear Algebra Appl. 425 (2007) 283–302.
- [39] T. Stykel, Stability and inertia theorems for generalized Lyapunov equations, Linear Algebra Appl. 355 (2002) 297–314.
- [40] M.E. Valcher, Controllability and reachability criteria for discrete time positive systems, Internat. J. Control 65 (1996) 511–536.
- [41] M.E. Valcher, Nonnegative linear systems in the behavioral approach: the autonomous case, Linear Algebra Appl. 319 (2000) 147–162.
- [42] R.S. Varga, Matrix iterative analysis, Springer Series in Computational Mathematics, vol. 27, expanded ed., Springer-Verlag, Berlin, 2000.