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–Brief Paper–

IMPULSIVE-MODE CONTROLLABLIZABILITY REVISITED FOR DESCRIPTOR LINEAR SYSTEMS

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ABSTRACT

New criteria for impulsive-mode controllablizability of descriptor linear systems are proposed by adopting the null space approach. The range of the possible dynamical orders of the closed-loop system with impulsive-mode controllability is characterized in terms of the original system matrices. Moreover, the parametric expressions of the impulsive-mode controllablizing controllers are also established. Since only the orthogonal transformations are involved, the design approach is numerically stable.

Key Words: Descriptor linear systems, impulsive-mode controllability, impulsive-mode controllablizability, derivative state feedback.

I. INTRODUCTION

Descriptor linear systems arise naturally in a variety of circumstances and have been extensively investigated [1, 2]. As a fundamental problem in the area of descriptor systems, elimination of impulsive behaviors via certain feedback controllers has been extensively investigated. For square descriptor systems, it was firstly pointed out in [1] that the necessary and sufficient condition for impulse elimination via state feedback is that the considered system is I-controllable. In [3, 4], the problem of impulse elimination via output feedback is considered. In [5, 6], the problem of impulse elimination via proportional-derivative (PD) output feedback has also been studied. Some conditions are established based on condensed canonical forms. Besides, the problem of impulse elimination via PD state feedback has also been investigated in [4], and the necessary and sufficient condition is established in terms of original system matrices. Recently, the concept of I-controllablizability was proposed in [7] and further investigation was made in [8]. It is shown that the condition of impulse elimination via PD state feedback is that the original system is I-controllablizable.

However, the problem of impulsive-mode elimination for general descriptor linear systems has seldom been investigated. In [9, 10], the concepts of impulsive modes and impulsive-mode controllability are proposed by the analysis of admissibility of initial conditions. In [9], the problem of impulsive-mode elimination via state feedback is considered. In order to investigate the problem of impulsive-mode elimination via PD state feedback, the concepts of impulsive-mode controllablizability are proposed in [11] by generalizing the idea in [7]. In this paper, we revisit to the impulsive-mode controllablizability for general descriptor linear systems by adopting the underlying idea of [8]. By adopting a null space approach we propose the new criteria for impulsive-mode controllablizability. In addition, the range of the dynamical orders of the resultant closed-loop system with impulsive-mode controllability is characterized and the parametric expressions of impulsive-mode controllablizing controllers are given.

Throughout this paper, $\mathcal{E}_{\text{imp}}$ denotes the class of distributions consisting of real-valued smooth functions

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denoted by $\mathcal{C}^{\infty}$, and purely impulsive δ-distribution and its derivatives denoted by $\mathcal{E}_{\text{p-imp}}$ [12]. The notation $\text{rank}(sE - A)$ is understood as the maximum rank of $sE - A$ when s varies in $\mathbb{C}$. For any matrix X, the notation S_X denotes a full column rank matrix whose columns span the null space of X.

II. PRELIMINARIES

Consider the following general descriptor linear system:

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ represent the descriptor vector and input vector, respectively; $E,\ A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times r}$ are the system matrices. The concepts of admissibility of initial conditions, impulsive modes, and impulsive-mode controllability are firstly introduced in this section. They can be found in [9, 13].

Definition 1. Consider the descriptor system $E\dot{x} = Ax$ with initial condition Ex_0.

(a) The initial condition Ex_0 is said to be admissible to the system, or simply to the pair (E, A), if the system has a solution $x \in \mathcal{E}_{\text{imp}}$ with respect to Ex_0;
(b) The system, or simply the pair (E, A), is said to have no impulsive modes if the system has a solution $x \in \mathcal{C}^{\infty}$ for every admissible Ex_0;
(c) The system, or simply the pair (E, A), is said to be free of impulse if the system has a unique solution $x \in \mathcal{C}^{\infty}$ for every admissible Ex_0.

Definition 2. Consider the descriptor system (1).

(a) The initial condition Ex_0 is said to be admissible to the system, or simply to the triple (E, A, B), if the system has a solution $(x, u) \in \mathcal{E}_{\text{imp}}$ with respect to Ex_0;
(b) The system, or simply the triple (E, A, B), is said to be impulsive-mode controllable if the system has a solution $(x, u) \in \mathcal{C}^{\infty}$ for each admissible Ex_0.

Definition 3. A feedback control $u = Kx$ is said to be admissible to (1), or simply, K is admissible to the triple (E, A, B) if for each initial condition Ex_0 admissible to the triple (E, A, B), Ex_0 is also admissible to the pair $(E, A + BK)$.

The following lemmas, which are provided in [9, 13], give conditions of initial admissibility and impulsive-mode controllability.

Lemma 1. A special initial condition Ex_0 is admissible to the triple $(E,\ A,\ B)$ if and only if

$$\text{rank}[sE - A \ \ B \ \ Ex_0] = \text{rank}[sE - A \ \ B], \tag{2}$$

and an arbitrary initial condition is admissible to the triple $(E,\ A,\ B)$ if and only if

$$\text{rank}[E \ \ A \ \ B] = \text{rank}[sE - A \ \ B].$$

Lemma 2. The triple $(E,\ A,\ B)$ is impulsive-mode controllable if and only if

$$\text{rank}[AS_E \ \ E \ \ B] = \text{rank}[sE - A \ \ B];$$

the triple $(E,\ A,\ B)$ is impulsive-mode controllable and admits arbitrary initial conditions if and only if

$$\text{rank}[AS_E \ \ E \ \ B] = \text{rank}[E \ \ A \ \ B].$$

Lemma 3. A matrix K admissible to the triple (E, A, B) exists so that the pair $(E, A + BK)$ has no impulsive modes if and only if the triple (E, A, B) is impulsive-mode controllable.

Lemma 4. A matrix K admissible to the triple (E, A, B) exists so that the pair $(E, A + BK)$ has no impulsive modes and admits arbitrary initial condition if and only if the triple (E, A, B) is impulsive-mode controllable and admits arbitrary initial condition.

For the system (1) we consider the PD state feedback

$$u = K_p x - K_d \dot{x}. \tag{3}$$

A feedback control must not impose any restrictions on admissible initial conditions through the controlled system

$$(E + BK_d)\dot{x} = (A + BK_p)x. \tag{4}$$

Similar to Definition 3, we give the following definition.

Definition 4. A feedback control $u = K_p x - K_d \dot{x}$ is said to be admissible to the system (1), or simply, (K_p, K_d) to the triple (E, A, B) if for each initial condition Ex_0 admissible to the triple (E, A, B), $(E + BK_d)x_0$ is also admissible to the pair $(E + BK_d, A + BK_p)$.

In [11], it was pointed out that the impulsive-mode controllability (with the admissibility of arbitrary initial conditions) of a descriptor linear system is preserved under state feedback. Based on this fact, the concept of impulsive mode controllablizability is proposed in [11].

Definition 5. The system (1), or the triple (E, A, B) is said to be impulsive-mode controllablizable if there exists a derivative feedback control law

$$u = -K_d \dot{x} + v \tag{5}$$

such that the closed-loop system

$$(E + BK_d)\dot{x} = Ax + Bv \tag{6}$$

is impulsive-mode controllable. In this case, the derivative state feedback control law (5) is called an impulsive-mode controllablizing controller for the system. The system (1) is said to be impulsive-mode controllablizable with admissibility for arbitrary initial conditions if and only if there exists a derivative feedback control law (5) such that the system (6) is impulsive-mode controllable and admits arbitrary initial condition. In this case, the derivative state feedback control law (5) is called an impulsive-mode controllablizing controller with the admissibility of initial conditions for the system.

For this definition, impulsive-mode controllablizability is closed related with the problem of impulsive mode elimination via PD state feedback.

III. CRITERIA FOR IMPULSIVE-MODE CONTROLLABLIZABILITY

In this section, we provide the criteria for impulsive-mode controllablizability in terms of the original system matrices.

Theorem 1. Given the matrices $E,\ A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times r}$, the system (1) is impulsive-mode controllablizable if and only if

$$\operatorname{rank}[[A\ \ 0_{m\times r}]S_{[E\ \ B]}\ \ E\ \ B] = \operatorname{rank}[sE - A\ \ B]. \tag{7}$$

Theorem 2. Given the matrices $E, A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times r}$, the system (1) is impulsive-mode controllablizable with admissibility of arbitrary initial conditions if and only if the following relation holds:

$$\operatorname{rank}[[A\ \ 0_{m\times r}]S_{[E\ \ B]}\ \ E\ \ B] = \operatorname{rank}[E\ \ A\ \ B]. \tag{8}$$

In the rest of this section, we will show Theorem 1 from Theorem 2. The proof of Theorem 2 will be given in the next section. As a preliminary, we give the following so-called controllable-uncontrollable form [9].

Lemma 5. Given the matrix triple (E, A, B), there exist two nonsingular matrices $\tilde{P}$ and $\tilde{Q}$ such that

$$E = \tilde{P}\begin{bmatrix} E_{\mathrm{c}} & 0 \\ E_{\mathrm{cr}} & E_{\mathrm{r}} \end{bmatrix}\tilde{Q},$$

$$A = \tilde{P}\begin{bmatrix} A_{\mathrm{c}} & 0 \\ A_{\mathrm{cr}} & A_{\mathrm{r}} \end{bmatrix}\tilde{Q}, \quad B = \tilde{P}\begin{bmatrix} 0 \\ B_{\mathrm{r}} \end{bmatrix}, \tag{9}$$

where E_{c} has full-column rank, and $[A_{\mathrm{r}}\ \ B_{\mathrm{r}}]$ has full row rank.

In addition, the following conclusion, which can be immediately obtained from Lemma 2, is used.

Lemma 6. The triple $(E,\ A,\ B)$ is impulsive-mode controllablizable if and only if there exists a real matrix K_d such that

$$\operatorname{rank}[AS_{(E+BK_d)}\ \ E\ \ B] = \operatorname{rank}[sE - A\ \ B]; \tag{10}$$

the triple $(E,\ A,\ B)$ is impulsive-mode controllablizable with admissibility for arbitrary initial conditions if and only there exists a real matrix K_d such that

$$\operatorname{rank}[AS_{(E+BK_d)}\ \ E\ \ B] = \operatorname{rank}[E\ \ A\ \ B]. \tag{11}$$

Now we show Theorem 1 from Theorem 2.

Proof of Theorem 1. Suppose that $\tilde{P}$ and $\tilde{Q}$ are the matrices such that (9) holds. Letting

$$K_d = [K_{d1}\ \ K_{d2}]\tilde{Q}, \tag{12}$$

where the partition is compatible, we have

$$E + BK_d = \tilde{P}\begin{bmatrix} E_{\mathrm{c}} & 0 \\ E_{\mathrm{cr}} + B_{\mathrm{r}}K_{d1} & E_{\mathrm{r}} + B_{\mathrm{r}}K_{d2} \end{bmatrix}\tilde{Q}. \tag{13}$$

In view that E_{c} has full-column rank it is easily obtained that

$$S_{(E+BK_d)} = \tilde{Q}^{-1}\begin{bmatrix} 0 \\ S_{(E_{\mathrm{r}}+B_{\mathrm{r}}K_{d2})} \end{bmatrix}.$$

According to this relation it is derived that

$$\tilde{P}^{-1}AS_{(E+BK_d)} = \tilde{P}^{-1}A\tilde{Q}^{-1}\begin{bmatrix} 0 \\ S_{(E_{\mathrm{r}}+B_{\mathrm{r}}K_{d2})} \end{bmatrix} = \begin{bmatrix} 0 \\ A_{\mathrm{r}}S_{(E_{\mathrm{r}}+B_{\mathrm{r}}K_{d2})} \end{bmatrix}.$$

Combining this with Lemma 5, we have

$$\begin{aligned} &\operatorname{rank}[AS_{(E+BK_d)}\ \ E\ \ B] \\ &= \operatorname{rank}\begin{bmatrix} 0 & E_{\mathrm{c}} & 0 & 0 \\ A_{\mathrm{r}}S_{(E_{\mathrm{r}}+B_{\mathrm{r}}K_{d2})} & E_{\mathrm{cr}} & E_{\mathrm{r}} & B_{\mathrm{r}} \end{bmatrix} \\ &= \operatorname{rank}E_{\mathrm{c}} + \operatorname{rank}[A_{\mathrm{r}}S_{(E_{\mathrm{r}}+B_{\mathrm{r}}K_{d2})}\ \ E_{\mathrm{r}}\ \ B_{\mathrm{r}}], \end{aligned}$$

and

$$\begin{aligned}
&\operatorname{rank}[sE - A \;\; B] \\
&= \operatorname{rank}\begin{bmatrix} sE_{\mathrm{c}} - A_{\mathrm{c}} & 0 & 0 \\ sE_{\mathrm{cr}} - A_{\mathrm{cr}} & sE_{\mathrm{r}} - A_{\mathrm{r}} & B_{\mathrm{r}} \end{bmatrix} \\
&= \operatorname{rank} E_{\mathrm{c}} + \operatorname{rank}[sE_{\mathrm{r}} - A_{\mathrm{r}} \;\; B_{\mathrm{r}}] \\
&= \operatorname{rank} E_{\mathrm{c}} + \operatorname{rank}[E_{\mathrm{r}} \;\; A_{\mathrm{r}} \;\; B_{\mathrm{r}}]. \qquad (14)
\end{aligned}$$

Thus the relation (10) becomes

$$\operatorname{rank}[A_{\mathrm{r}} S_{(E_{\mathrm{r}}+BK_{d2})} \;\; E_{\mathrm{r}} \;\; B_{\mathrm{r}}] = \operatorname{rank}[A_{\mathrm{r}} \;\; E_{\mathrm{r}} \;\; B_{\mathrm{r}}].$$

By applying Theorem 2, there exists a matrix K_{d2} such that the above relation holds if and only if

$$\operatorname{rank}[[A_{\mathrm{r}} \;\; 0] S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} \;\; E_{\mathrm{r}} \;\; B_{\mathrm{r}}] = \operatorname{rank}[E_{\mathrm{r}} \;\; A_{\mathrm{r}} \;\; B_{\mathrm{r}}]. \qquad (15)$$

It from (9) follows that

$$[E \;\; B] = \tilde{P} \begin{bmatrix} E_{\mathrm{c}} & 0 & 0 \\ E_{\mathrm{cr}} & E_{\mathrm{r}} & B_{\mathrm{r}} \end{bmatrix} \begin{bmatrix} \tilde{Q} & 0 \\ 0 & I_r \end{bmatrix}.$$

In view that E_{c} has full column rank, this implies that

$$S_{[E \;\; B]} = \begin{bmatrix} \tilde{Q}^{-1} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} 0 \\ S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} \end{bmatrix}.$$

So it is obtained that

$$\begin{aligned}
&\tilde{P}^{-1}[A \;\; 0_{m\times r}] S_{[E \;\; B]} \\
&= \tilde{P}^{-1}[A \;\; 0_{m\times r}] \begin{bmatrix} \tilde{Q}^{-1} & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} 0 \\ S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ [A_{\mathrm{r}} \;\; 0] S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} \end{bmatrix}.
\end{aligned}$$

With this relation we have

$$\begin{aligned}
&\operatorname{rank}[[A \;\; 0_{m\times r}] S_{[E \;\; B]} \;\; E \;\; B] \\
&= \operatorname{rank}[\tilde{P}^{-1}[A \;\; 0_{m\times r}] S_{[E \;\; B]} \;\; \tilde{P}^{-1} E \tilde{Q}^{-1} \;\; \tilde{P}^{-1} B] \\
&= \operatorname{rank}\begin{bmatrix} 0 & E_{\mathrm{c}} & 0 & 0 \\ [A_{\mathrm{r}} \;\; 0] S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} & E_{\mathrm{cr}} & E_{\mathrm{r}} & B_{\mathrm{r}} \end{bmatrix} \\
&= \operatorname{rank} E_c + \operatorname{rank}[[A_{\mathrm{r}} \;\; 0] S_{[E_{\mathrm{r}} \;\; B_{\mathrm{r}}]} \;\; E_{\mathrm{r}} \;\; B_{\mathrm{r}}].
\end{aligned}$$

By combining these relations with (14), it is easily seen that the relation (7) is equivalent to (15). So the conclusion of Theorem 1 holds.

IV. PROOF OF THEOREM 2

In this section, we give the proof of Theorem 2. In the sequel, we denote

$$\begin{aligned}
&r_e = \operatorname{rank} E, \quad r_a = \operatorname{rank} A, \quad r_b = \operatorname{rank} B, \\
&r_{eb} = \operatorname{rank}[E \;\; B], \quad r_{eab} = \operatorname{rank}[E \;\; A \;\; B].
\end{aligned}$$

The following result, which is taken from [14], plays a very vital role in the proof of Theorem 2.

Lemma 7. Let $E \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$. There exist orthogonal matrices Q, U, and V such that

$$\begin{aligned}
&UEV = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \\
&UBQ = \begin{bmatrix} 0 & 0 \\ \Sigma_B & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned} \qquad (16)$$

where $E_{22} \in \mathbb{R}^{r_b \times (r_e + r_b - r_{eb})}$ has full column rank and $\Sigma_1 \in \mathbb{R}^{(r_{eb}-r_b)\times(r_{eb}-r_b)}$ and $\Sigma_B \in \mathbb{R}^{r_b \times r_b}$ are diagonal positive definite matrices. The partitioning in UEV and UBQ is compatible.

Theorem 3. Given E, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$, suppose that orthogonal matrices Q, U and V satisfy (16). Let $UAV = [A_{ij}]_{3\times 3}$, where partitioning of UAV is compatible. The relation

$$\operatorname{rank}[[A \;\; 0_{m\times r}] S_{[E \;\; B]} \;\; E \;\; B] = \operatorname{rank}[E \;\; A \;\; B], \qquad (17)$$

holds if and only if

$$\operatorname{rank}[A_{32} \;\; A_{33}] = \operatorname{rank}[A_{31} \;\; A_{32} \;\; A_{33}]. \qquad (18)$$

Proof. According to Lemma 7, we have

$$\begin{aligned}
&\operatorname{rank}[E \;\; A \;\; B] \\
&= \operatorname{rank}\begin{bmatrix} \Sigma_1 & 0 & A_{11} & A_{12} & A_{13} & 0 \\ E_{21} & E_{22} & A_{21} & A_{22} & A_{23} & \Sigma_B \\ 0 & 0 & A_{31} & A_{32} & A_{33} & 0 \end{bmatrix} \\
&= r_{eb} + \operatorname{rank}[A_{31} \;\; A_{32} \;\; A_{33}]. \qquad (19)
\end{aligned}$$

By simple computations, it is easily obtained that

$$U(E+BK_d)V=\begin{bmatrix} \Sigma_1 & 0 & 0 \\ E_{21}+\Sigma_B K_{d11} & E_{22}+\Sigma_B K_{d12} & \Sigma_B K_{d13} \\ 0 & 0 & 0 \end{bmatrix}.$$

$$S_{[E\ \ B]}=\begin{bmatrix} V & 0 \\ 0 & Q \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ I_{r_e+r_b-r_{eb}} & 0 & 0 \\ 0 & I_{n-r_e} & 0 \\ \Sigma_B^{-1}E_{22} & 0 & 0 \\ 0 & 0 & I_{r-r_b} \end{bmatrix}.$$

Thus, we have

$$U[A\ \ 0_{m\times r}]S_{[E\ \ B]}=\begin{bmatrix} A_{12} & A_{13} & 0 \\ A_{22} & A_{23} & 0 \\ A_{32} & A_{33} & 0 \end{bmatrix}.$$

Combining this with (16), we obtain

$$\begin{aligned}&\text{rank}[[A\ \ 0_{m\times r}]S_{[E\ \ B]}\ \ E\ \ B]\\ &=\text{rank}\begin{bmatrix} A_{12} & A_{13} & \Sigma_1 & 0 & 0 \\ A_{22} & A_{23} & E_{21} & E_{22} & \Sigma_B \\ A_{32} & A_{33} & 0 & 0 & 0 \end{bmatrix}\\ &=r_{eb}+\text{rank}[A_{32}\ \ A_{33}].\end{aligned}$$

Combining this with (19), gives the conclusion. □

Lemma 8. For matrices $E, A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{m\times r}$, there holds

$$\begin{aligned}&\{\text{rank}(E+BK_d)\mid K_d\in\mathbb{R}^{r\times n}\}\\ &=\{p\mid p \text{ is an integer and } r_{eb}\\ &\quad -r_b\le p\le \min(n,r_{eb})\}.\end{aligned}$$

Now we give the proof of Theorem 2.

Proof of Theorem 2. Let $\text{rank}(E+BK_d)=p$. Suppose that orthogonal matrices $Q, U,$ and V satisfy (16), and $UAV=[A_{ij}]_{3\times 3}$, where partitioning of UAV is compatible. Let

$$Q^T K_d V=\begin{bmatrix} K_{d11} & K_{d12} & K_{d13} \\ K_{d21} & K_{d22} & K_{d23} \end{bmatrix}, \tag{20}$$

where the partition is compatible. Then,

Due to $\text{rank}(E+BK_d)=p$, it is satisfied that

$$\text{rank}[E_{22}+\Sigma_B K_{d12}\ \ \Sigma_B K_{d13}]=p+r_b-r_{eb}.$$

Obviously, a general form for all the gain matrix $[K_{d_{12}}\ \ K_{d13}]$ can be given by

$$[K_{d_{12}}\ \ K_{d13}]=[-\Sigma_B^{-1}E_{22}\ \ 0]+K, \tag{21}$$

with $\text{rank}K=p+r_b-r_{eb}$. Thus, the matrix K can be expressed by

$$K=[K_1\ \ 0]P, \tag{22}$$

where $K_1\in\mathbb{R}^{r_b\times(p+r_b-r_{eb})}$ has full column rank and $P\in\mathbb{R}^{(n-r_{eb}+r_b)\times(n-r_{eb}+r_b)}$ is nonsingular. In this case, we have

$$U(E+BK_d)V=\begin{bmatrix} \Sigma_1 & 0 & 0 \\ E_{21}+\Sigma_B K_{d11} & K_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} I_{r_{eb}-r_b} & 0 \\ 0 & P \end{bmatrix}. \tag{23}$$

This implies that

$$S_{E+BK_d}=V\begin{bmatrix} I_{r_{eb}-r_b} & 0 \\ 0 & P^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ I_{n-p} \end{bmatrix}.$$

Partitioning the matrix P^{-1} into

$$P^{-1}=[P_1\ \ P_2],\quad P_2\in\mathbb{R}^{(n+r_b-r_{eb})\times(n-p)}, \tag{24}$$

we have

$$\begin{aligned}&\text{rank}[AS_{(E+BK_d)}\ \ E\ \ B]\\ &=\text{rank}[UAS_{(E+BK_d)}\ \ UEV\ \ UBQ]\\ &=\text{rank}\begin{bmatrix} [A_{12}\ \ A_{13}]P_2 & \Sigma_1 & 0 & 0 \\ [A_{22}\ \ A_{23}]P_2 & E_{21} & E_{22} & \Sigma_B \\ [A_{32}\ \ A_{33}]P_2 & 0 & 0 & 0 \end{bmatrix}\\ &=r_{eb}+\text{rank}[A_{32}\ \ A_{33}]P_2.\end{aligned}$$

Combining this with (19), the condition (11) is converted into

$$\operatorname{rank}[A_{32} \;\; A_{33}]P_2 = \operatorname{rank}[A_{31} \;\; A_{32} \;\; A_{33}]. \tag{25}$$

Since $\operatorname{rank}[A_{32} \;\; A_{33}]P_2 \leq \operatorname{rank}[A_{32} \;\; A_{33}]$, there exists matrix P_2 satisfying (25) only if (18) holds. On the other hand, if (18) holds we can choose $P_2 = I$. In this case, $p = r_{eb} - r_b$. Moreover, it from (20), (21) and (22) that the corresponding gain matrix K_d can be given by

$$K_d = Q \begin{bmatrix} K_{d11} & -\Sigma_B^{-1} E_{22} & 0 \\ K_{d21} & K_{d22} & K_{d23} \end{bmatrix} V^T,$$

where K_{d11} and K_{d2i}, i, $= 1, 2, 3$, are the free parameter matrices. Combining the above deduction with Theorem 3, gives the conclusion of Theorem 2.

V. DYNAMIC ORDER OF THE IMPULSIVE-MODE CONTROLLABLE CLOSED-LOOP SYSTEM

According to the proof of Theorem 2, for the given descriptor linear system (E, A, B) in order to obtain the derivative gain matrix K_d making the resultant system $(E + BK_d, A, B)$ impulsive-mode controllable with the admissibility of initial conditions we need choose the matrix P_2 to satisfy

$$\operatorname{rank}[A_{32} \;\; A_{33}]P_2 = \operatorname{rank}[A_{32} \;\; A_{33}]. \tag{26}$$

For this purpose, we perform the singular value decomposition on the matrix $[A_{32} \;\; A_{33}]$. In this case, there exist two orthogonal matrices $\tilde{U}$ and $\tilde{V}$ such that

$$\tilde{U}[A_{32} \;\; A_{33}]\tilde{V} = \begin{bmatrix} \Delta & 0 \\ 0 & 0 \end{bmatrix}, \tag{27}$$

where Δ is a diagonal positive definite matrix. Let

$$\tilde{V}^T P_2 = \begin{bmatrix} P_{21} \\ P_{22} \end{bmatrix},$$

then the relation (26) becomes $\operatorname{rank} \Delta P_{21} = \operatorname{rank} \Delta$. Due to the nonsingularity of Δ, the relation $\operatorname{rank} \Delta P_{21} = \operatorname{rank} \Delta$ holds if and only if the matrix P_{21} has full row rank. Combining this fact with (24), we have the following relation

$$\operatorname{rank}[A_{32} \;\; A_{33}] \leq n - p \leq n + r_b - r_{eb}. \tag{28}$$

According to (19), we have

$$\operatorname{rank}[A_{32} \;\; A_{33}] = \operatorname{rank}[A_{31} \;\; A_{32} \;\; A_{33}] = r_{eab} - r_{eb}.$$

Thus, it from (28) follows that

$$r_{eb} - r_b \leq p \leq n + r_{eb} - r_{eab}.$$

By combining this with Lemma 8 and Theorem 2, we have the following result. For convenience, we introduce the following sets:

$$\Upsilon_{(E,A,B)} = \{K_d | K_d \in \mathbb{R}^{r \times n}, \operatorname{rank}[AS_{(E+BK_d)} \;\; E \;\; B] = \operatorname{rank}[sE - A \;\; B]\},$$

and

$$\Psi_{(E,A,B)} = \{K_d | K_d \in \mathbb{R}^{r \times n}, \operatorname{rank}[AS_{(E+BK_d)} \;\; E \;\; B] = \operatorname{rank}[E \;\; A \;\; B]\}.$$

Theorem 4. Given the matrices E, $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times r}$, the systems (1) is impulsive-mode controllablizable with the admissibility of initial conditions if and only if (8) holds. In this case, there holds

$$\begin{aligned} &\{\operatorname{rank}(E + BK_d) \mid K_d \in \Psi_{(E,A,B)}\} \\ &= \{p \mid p \text{ is an integer}, r_{eb} - r_b \\ &\leq p \leq \min(r_{eb}, n + r_{eb} - r_{eab})\}. \end{aligned}$$

Based on this theorem, with the help of the so-called controllable-uncontrollable form given in Lemma 5 the following result can also be obtained.

Theorem 5. Given the matrices $E, A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times r}$, the systems (1) is impulsive-mode controllablizable if and only if (7) holds. In this case, there holds

$$\begin{aligned} &\{\operatorname{rank}(E + BK_d) \mid K_d \in \Upsilon_{(E,A,B)}\} \\ &= \{p \mid p \text{ is an integer}, \; r_{eb} - r_b \leq p \\ &\leq \min(r_{eb}, n + r_{eb} - r_{eab})\}. \end{aligned}$$

Proof. Suppose that $\tilde{P}$ and $\tilde{Q}$ are the matrices such that (9) holds, and gain matrix K_d is expressed by (12). It from (13) follows that

$$p = \operatorname{rank}(E + BK_d) = \operatorname{rank} E_{\mathrm{c}} + \operatorname{rank}(E_{\mathrm{r}} + B_{\mathrm{r}} K_{d2}).$$

It follows from the proof of Theorem 1 that there exists $K_d \in \mathbb{R}^{r \times n}$ such that the system $(E + BK_d, A, B)$ is impulsive-mode controllable if and only if there exists K_{d2} such that the system $(E_{\mathrm{r}} + B_{\mathrm{r}} K_{d2}, A_{\mathrm{r}}, B_{\mathrm{r}})$

is impulsive-mode controllable with the admissibility of initial conditions. Let $r_1 = \text{rank}[E_{\text{r}}\ \ B_{\text{r}}]$, $r_2 = \text{rank}[E_{\text{r}}\ \ A_{\text{r}}\ \ B_{\text{r}}]$. According to Theorem 4, it is obtained that

$$r_1 - \text{rank}\, B_{\text{r}} \leq \text{rank}(E_{\text{r}} + B_{\text{r}} K_{d2}) \leq \min(r_1, n - \text{rank}\, E_{\text{c}} + r_1 - r_2).$$

According to Lemma 5, it is easily obtained that $\text{rank}\, E_{\text{c}} + r_1 = r_{eb}$, $\text{rank}\, E_{\text{c}} + r_2 = r_{eab}$, $\text{rank}\, B_{\text{r}} = r_b$. Thus we have $\text{rank}\, E_{\text{c}} + r_1 - \text{rank}\, B_{\text{r}} = r_{eb} - r_b$, and

$$\text{rank}\, E_{\text{c}} + \min(r_1, n - \text{rank}\, E_{\text{c}} + r_1 - r_2) = \min(r_{eb}, n + r_{eb} - r_{eab}).$$

Accordingly, the conclusion is true. □

VI. IMPULSIVE-MODE CONTROLLABLIZING CONTROLLER DESIGN

In this section, we consider the impulsive-mode controllablizing controller design for the system (1). With the above discussion, the problems can be precisely stated as follows.

Problem 1 (Impulsive-mode controllablizing controller design)**.** Let matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfy (7), characterize the set $\Upsilon_{(E,A,B)}$.

Problem 2 (Design of impulsive-mode controllablizing controllers with admissibility of initial conditions)**.** Let matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfy (8), characterize the set $\Psi_{(E,A,B)}$.

By applying controllable-uncontrollable form given in Lemma 5, according to the proof of Theorem 1 we have the following conclusion on the solution to Problem 1.

Theorem 6. Let matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfy (7), and the matrices $\tilde{P}$ and $\tilde{Q}$ be such that (9) holds, then

$$\Upsilon_{(E,A,B)} = \{[K_{d1}\ \ K_{d2}]\tilde{Q} \mid K_{d2} \in \Psi_{(E_{\text{r}},A_{\text{r}},B_{\text{r}})},\ K_{d1} \text{ real}\}.$$

The above theorem implies that Problem 1 can be converted into Problem 2 by applying the controllable-uncontrollable form. In the rest of this section, we focus on the solution to Problem 2. For convenience, we introduce the following notations:

$$\Psi^{p}_{(E,A,B)} = \{K_d \mid K_d \in \Psi_{(E,A,B)}, \text{rank}(E+BK_d)=p\},$$

$$\Upsilon^{p}_{(E,A,B)} = \{K_d \mid K_d \in \Psi_{(E,A,B)}, \text{rank}(E+BK_d)=p\}.$$

With these notations, from Theorems 4 and 5 the following relations follow.

Proposition 1. Let matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfy (8), then

$$\Psi_{(E,A,B)} = \bigcup_{p=r_{eb}-r_b}^{\min(r_{eb}, n+r_{eb}-r_{eab})} \Psi^{p}_{(E,A,B)}.$$

Proposition 2. Let matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfy (7), then

$$\Upsilon_{(E,A,B)} = \bigcup_{p=r_{eb}-r_b}^{\min(r_{eb}, n+r_{eb}-r_{eab})} \Upsilon^{p}_{(E,A,B)}.$$

It is seen from Proposition 1 that it suffices for characterizing the set $\Psi_{(E,A,B)}$ to characterize the set $\Psi^{p}_{(E,A,B)}$ for any integer p satisfying $r_{eb} - r_b \leq p \leq \min(r_{eb}, n + r_{eb} - r_{eab})$. By summarizing Sections IV and V, we can obtain the following theorem on the characterization of the set $\Psi^{p}_{(E,A,B)}$.

Theorem 7. Given matrices $E, A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfying (8), and an arbitrary integer p satisfying $r_{eb} - r_b \leq p \leq \min(r_{eb}, n + r_{eb} - r_{eab})$, let orthogonal matrices Q, U, and V satisfy (16), and $UAV = [A_{ij}]_{3\times 3}$, where partitioning of UAV is compatible. Further, let the orthogonal matrices $\tilde{U}$ and $\tilde{V}$ satisfy the singular value decomposition (27). Then a general form of all the matrices $K_d \in \Psi^{p}_{(E,A,B)}$ is given as follows

$$\begin{aligned} K_d &= Q \begin{bmatrix} K_{d11} & [-\Sigma_B^{-1} E_{22}\ \ 0] \\ K_{d21} & K_{d22} \end{bmatrix} V^T \\ &\quad + Q \begin{bmatrix} 0_{r_b\times(r_{eb}-r_b)} & K_1 & 0_{r_b\times(n-p)} \\ 0 & 0 & 0 \end{bmatrix} \\ &\quad \times \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \tilde{V}^T \end{bmatrix} V^T, \end{aligned} \tag{29}$$

where $K_{d11} \in \mathbb{R}^{r_b\times(r_{eb}-r_b)}$, $K_{d21} \in \mathbb{R}^{(r-r_b)\times(r_{eb}-r_b)}$, and $K_{d22} \in \mathbb{R}^{(r-r_b)\times(n+r_b-r_{eb})}$ are three arbitrarily chosen parameter matrices; $K_1 \in \mathbb{R}^{r_b\times(p+r_b-r_{eb})}$ and $P_{21} \in \mathbb{R}^{(r_{eab}-r_{eb})\times(n-p)}$ have full column and row rank, respectively; the matrices P_{11}, P_{12}, P_{22} should be chosen such that the matrix $[P_{ij}]$ is nonsingular.

For Theorem 7, we have the following corollary.

Corollary 1. Given matrices E, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times r}$ satisfying (8), let orthogonal matrices Q, U, and V satisfy (16), and $UAV = [A_{ij}]_{3\times 3}$, where partitioning of UAV is compatible. Then

(1) a general form of all the matrices $K_d \in \Psi^{r_{eb}-r_b}_{(E,A,B)}$ is given as follows

$$K_d = Q\begin{bmatrix} K_{d11} & [-\Sigma_B^{-1}E_{22} \;\; 0] \\ K_{d21} & K_{d22} \end{bmatrix} V^T,$$

where $K_{d11} \in \mathbb{R}^{r_b\times(r_{eb}-r_b)}$, $K_{d21} \in \mathbb{R}^{(r-r_b)\times(r_{eb}-r_b)}$, and $K_{d22} \in \mathbb{R}^{(r-r_b)\times(n+r_b-r_{eb})}$ are three arbitrarily chosen parameter matrices;

(2) a general form of all the matrices $K_d \in \Psi^{n+r_{eb}-r_{eab}}_{(E,A,B)}$ is given as follows

$$K_d = Q\begin{bmatrix} K_{d11} & [-\Sigma_B^{-1}E_{22} \;\; 0] \\ K_{d21} & K_{d22} \end{bmatrix} V^T + Q\begin{bmatrix} 0_{r_b\times(r_{eb}-r_b)} & K \\ 0 & 0 \end{bmatrix} V^T,$$

where $K_{d11} \in \mathbb{R}^{r_b\times(r_{eb}-r_b)}$, $K_{d21} \in \mathbb{R}^{(r-r_b)\times(r_{eb}-r_b)}$, and $K_{d22} \in \mathbb{R}^{(r-r_b)\times(n+r_b-r_{eb})}$ are three arbitrarily chosen parameter matrices; $K \in \mathbb{R}^{r_b\times(n+r_b-r_{eb})}$ is a parameter matrix satisfying rank $K = n + r_{eb} - r_{eab}$.

VII. CONCLUSIONS

New criteria for impulsive-mode controllablizability of general descriptor linear systems are developed by applying null space approach. The set of possible dynamical orders of the resulting closed-loop system with impulsive-mode controllability is characterized. Parametrization of all impulsive-mode controllablizing controllers are proposed. However, this approach can not provide the simultaneous parametrization of proportional gain matrix and derivative gain matrix for such PD state feedback. In our future work, we attempt to develop such an approach.

REFERENCES

1. Cobb, J. D., "Feedback and pole assignment in descriptor variable system," *Int. J. Control*, Vol. 33, pp. 1135–1146 (1981).
2. Xie, G. and L. Wang, "Controllability of a class of singular systems," *Asian J. Control*, Vol. 8, No. 4, pp. 424–432 (2007).
3. Lovass-Nagy, V., D. L. Powers, and R. J. Schilling, "On regularizing descriptor systems by output feedback," *IEEE Trans. Autom. Control*, Vol. 39, No. 7, pp. 1507–1509 (1994).
4. Dai, L., "Impulsive modes and causality in singular systems," *Int. J. Control*, Vol. 50, pp. 1267–1281 (1989).
5. Chu, D. L. and D. W. C. Ho, "Necessary and sufficient conditions for the output feedback regularization of descriptor systems," *IEEE Trans. Autom. Control*, Vol. 44, No. 2, pp. 405–412 (1999).
6. Bunse-Gerstner, A., V. Mehrmann, and N. K. Nichols, "Regularization of singular systems by output feedback," *IEEE Trans. Autom. Control*, Vol. 39, No. 2, pp. 1742–1748 (1994).
7. Duan, G. R. and A. G. Wu, "I-controllablizability in descriptor linear systems," *Dyn. Contin. Discrete Impulsive Syst. – Ser. A – Math. Anal.*, Vol. 13, pp. 1197–1204 (2005).
8. Wu, A. G., G. R. Duan, Y. Zhao, and H. H. Yu, "A revisit to I-controllablisability for descriptor linear systems," *IET Contr. Theory Appl.*, Vol. 1, No. 4, pp. 975–978 (2007).
9. Hou, M., "Controllability and elimination of impulsive modes in descriptor systems," *IEEE Trans. Autom. Control*, Vol. 49, No. 10, pp. 1723–1727 (2004).
10. Ishihara, J. Y. and M. H. Terra, "Impulsive controllability and observability of rectangular descriptor systems," *IEEE Trans. Autom. Control*, Vol. 46, No. 6, pp. 991–994 (2001).
11. Wu, A. G., G. R. Duan, and S. M. Zhao, "Impulsive-mode controllablisability in descriptor linear systems," *IET Contr. Theory Appl.*, Vol. 1, No. 3, pp. 558–563 (2007).
12. Geerts, T., "Solovability conditions, consistency, and weak consistent for linear different-algebra equations and time-invariant singular systems: the general case," *Linear Alg. Appl.*, Vol. 181, pp. 111–130 (1993).
13. Hou, M. and P. C. Müller, "Causal observability of descriptor systems," *IEEE Trans. Autom. Control*, Vol. 44, No. 1, pp. 158–163 (1999).
14. Chu, D. L., H. C. Chan, and D. W. C. Ho, "Regularization of singular systems by derivative and proportional output feedback," *SIAM J. Matrix Anal. Appl.*, Vol. 19, No. 1, pp. 21–38 (1998).