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Exponential stability and static output feedback stabilisation of singular time-delay systems with saturating actuators

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Abstract: This study deals with the class of continuous-time singular linear systems with time-varying delays. The stability and stabilisation problems of this class of systems are addressed. Delay-range-dependent sufficient conditions such that the system is regular, impulse free and α-stable are developed in the linear matrix inequality (LMI) setting and an estimate of the convergence rate of such stable systems is also presented. An iterative LMI (ILMI) algorithm to compute a static output feedback controller gains is proposed. Some numerical examples are employed to show the usefulness of the proposed results.

1 Introduction

Singular time-delay systems arise in a variety of practical systems such as networks, circuits, power systems and so on [1–3]. Since singular time-delay systems are matrix delay differential equations coupled with matrix difference equations, the study of such systems is much more complicated compared to standard state-space time-delay systems or singular systems. The existence and uniqueness of a solution to a given singular time-delay system is not always guaranteed and the system can also have undesired impulsive behaviour. Therefore for a singular time-delay system, it is important to develop conditions which guarantee that the given singular system is not only stable but also regular and impulse free.

Both delay-independent and delay-dependent stability conditions for singular time-delay systems have been derived using the time domain method, see [3–7] and references therein. However, most of the delay-dependent results in the literature tackle only the case of constant time delay where two approaches were used to prove the stability of the system. The first approach consists of decomposing the system into algebraic and differential subsystems and the stability of the differential subsystem is proved using some Lyapunov functional. Then, the algebraic variables are expressed explicitly by an iterative equation in terms of the differential variables [4]. The stability of the algebraic variables can be guaranteed if the eigenvalues of some matrix are inside a unit circle. The second approach introduced by Fridman [3] and it consists of constructing a Lyapunov–Krasovskii functional that corresponds directly to the descriptor form of the system. However, the results based on the assumption of the stability of certain operator. This assumption is shown to be satisfied if the eigenvalues of some matrix expression are inside a unit circle. Indeed, in the case of single delay, it can be shown easily that this condition is equivalent to the one used in [4] to prove the stability of the algebraic variables. The extension of these approaches to time-varying delays has not been addressed yet. In [7], where time-varying delays are considered, the response of the algebraic variables has been bounded by an exponential term using a different approach. Using this approach, it is not possible to give an estimate of the convergence rate of the states of the system.

Recently, a free-weighting matrices method is proposed in [8–10] to study the delay-dependent stability for time-delay systems with constant and time-varying delay, in which the bounding techniques on some cross product terms are not involved. The new method has been shown to be more effective in reducing conservatism entailed in previous results, especially for uncertain systems. In 2007, Zhu *et al.* adopted this technique for singular time-delay systems [5]. Also, delay-range-dependent concept was recently studied, where the delays are considered to vary in a range and thereby more applicable in practice [11].

Formally speaking, these conditions provide only the asymptotic stability of singular time-delay systems. In [12], the global exponential stability for a class of singular systems with multiple constant time delays is investigated and an estimate of the convergence rate of such systems is presented. One may ask if there exists a possibility to use the linear matrix inequality (LMI) approach for deriving exponential estimates for solutions of singular time-delay systems. In [7], exponential stability conditions in terms of LMIs are given but no estimate of the convergence rate is presented.

The problem of stabilising linear systems with saturating controls has been widely studied because of its practical interest [13]. Control saturation constraint comes from the impossibility of actuators to drive signal with unlimited amplitude or energy to the plants. However, only few works have dealt with stability analysis and the stabilisation of singular linear systems in the presence of actuator saturation, see for example [14]. It is established in [14] that a singular linear system with actuator saturation is semi-globally asymptotically stabilisable by linear state feedback if its reduced system under actuator saturation is semi-globally asymptotically stabilisable by linear feedback. To the best of the authors' knowledge, the stabilisation for singular time-delay systems in the presence of actuator saturation has not been fully addressed yet.

The static output feedback problem is probably the most important open question in control engineering. In contrast to the linear systems, there are only few papers solving the static output feedback problems for singular systems, see [15, 16]. In [16], the authors introduce an equality constraint in order to obtain an LMI sufficient conditions for admissibility of closed-loop systems. However, this equality constraint introduces conservatism. This approach has been generalised by Boukas [17] to singular time-delay systems. In [15], singular systems is assumed to have some characteristics in advance: regularity and absence of direct action of control inputs on the algebraic variables, which is not always the case.

This paper addresses two important problems that has not been fully investigated. First, delay-range-dependent exponential stability conditions for singular time-delay systems are established in terms of LMIs and an estimate of the convergence rate of the state is presented. Free weighting matrices are used in order to reduce the conservativeness of the conditions. The Lyapunov functional and some inequalities from [11] are adopted, with some modifications, in order to prove the exponential stability of the differential subsystem. The algebraic variables are expressed explicitly by an iterative method which can be seen as a generalisation of the iterative expression in [4] for constant time delay. Indeed, it is presented that the stability of the algebraic subsystem in both cases can be shown by the same condition on the eigenvalues of some matrix. This means that many of the existing results for singular systems with constant time delay can be extended easily to the systems with time-varying delay. For instance, the results in [4–6].

Second, an iterative LMI algorithm is proposed to design a stabilising static output feedback controller for singular time-delay systems in the presence of actuator saturation. The objective of the control design is 2-fold. It consists in determining both a static output feedback control law to guarantee that the system is regular, impulse-free and exponentially stable with a predefined decaying rate for the closed-loop system, and a set of safe initial conditions for which the exponential stability of the saturated closed-loop system is guaranteed. Two numerical examples are employed to show the usefulness of the proposed results.

Notation: Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript '$\top$' denotes matrix transposition and the notation $X \geq Y$ (respectively, $X > Y$) where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). $\mathbb{I}$ is the identity matrix with compatible dimensions. $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ denote, respectively, the maximal and minimal eigenvalues of matrix P. co$\{\cdot\}$ denotes a convex hull. $C_\tau = C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous vector functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. $\|\cdot\|$ refers to the Euclidean vector norm whereas $\|\phi\|_c = \sup_{-\tau \leq t \leq 0}\|\phi(t)\|$ stands for the norm of a function $\phi \in C_\tau$. C_τ^v is defined by $C_\tau^v = \{\phi \in C_\tau; \|\phi\|_c < v, v > 0\}$.

2 Problem statement and definitions

Consider the following linear singular time-delay system

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + B\text{sat}(u(t)) \quad (1a)$$
$$y(t) = Cx(t) \quad (1b)$$
$$x(t) = \phi(t), \quad t \in [-d_2, 0] \quad (1c)$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the saturating control input, $y(t) \in \mathbb{R}^q$ is the measurement, the matrix $E \in \mathbb{R}^{n\times n}$ may be singular, and we assume that

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$\mathrm{rank}(E) = r \le n$, A, A_d, B and C are known real constant matrices, $\mathrm{sat}(u(t)) = [\mathrm{sat}(u_1(t)), \ldots, \mathrm{sat}(u_m(t))]$, where $-\bar{u}_i \le \mathrm{sat}(u_i(t)) \le \bar{u}_i$, $\phi(t) \in C_\tau$ is a compatible vector valued continuous function and $d(t)$ is a time-varying continuous function that satisfies

$$0 < d_1 \le d(t) \le d_2 \quad \text{and} \quad \dot{d}(t) \le \mu < 1 \tag{2}$$

The following definitions will be used in the rest of this paper:

Definition 1:
(i) System (1) with $u(t) = 0$ is said to be regular if the characteristic polynomial, $\det(sE - A)$ is not identically zero.

(ii) System (1) with $u(t) = 0$ is said to be impulse free if $\deg(\det(sE - A)) = \mathrm{rank}(E)$.

(iii) System (1) with $u(t) = 0$ is said to be exponentially stable if there exist $\sigma > 0$ and $\gamma > 0$ such that, for any compatible initial conditions $\phi(t)$, the solution $x(t)$ to the singular time-delay system satisfies

$$\|x(t)\| \le \gamma e^{-\sigma t} \|\phi\|_c$$

(iv) System (1) with $u(t) = 0$ is said to be exponentially admissible if it is regular, impulse free and exponentially stable.

Lemma 1 ([1]): If system (1) with $u(t) = 0$ is regular and impulse free, then its solution exists and is impulse free and unique on $[0, \infty)$.

Lemma 2 ([18]): Given a matrix D, let a positive-definite matrix S and a positive scalar $\eta \in (0, 1)$ exist such that

$$D^\top S D - \eta^2 S < 0$$

then the matrix D satisfies the bound

$$\|D^i\| \le \chi e^{-\lambda i} \quad \text{with } \chi = \sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(S)}} \text{ and } \lambda = -\ln(\eta)$$

where i is a positive integer.

Now, consider the following static output feedback controller

$$u(t) = Ky(t), \quad K \in \mathbb{R}^{m \times q} \tag{3}$$

Applying this controller to system (1), we obtain the closed-loop system as follows

$$E\dot{x}(t) = Ax(t) + A_d x(t - d(t)) + B\,\mathrm{sat}(KCx(t)) \tag{4}$$

Generally, for a given stabilising static output feedback K, it is not possible to determine exactly the region of attraction of the origin with respect to system (4). Hence, a domain of initial conditions, for which the exponential stability of system (4) is ensured, has to be determined.

3 Main results

The two problems to be tackled in this section can be summarised as follows:

- Find delay-range-dependent LMI conditions that guarantees the exponential admissibility of system (1) with $u(t) = 0$, with a predefined minimum decaying rate.

- Find a static output feedback law of the form (3) and a set of initial conditions such that the closed-loop system (4) is exponentially admissible with a predefined minimum decaying rate.

Now, we present the first result.

Theorem 1: Let $0 < d_1 < d_2$, $0 \le \mu < 1$ and $\alpha > 0$ be given scalars. System (1) with $u(t) = 0$ is exponentially admissible with $\sigma = \alpha$ if there exist a non-singular matrix P, symmetric and positive-definite matrices Q_1, Q_2, Q_3, Z_1 and Z_2, and matrices M_i, N_i and S_i, $i = 1, 2$ such that the following LMI holds (5)

with the following constraint

$$E^\top P = P^\top E \ge 0 \tag{6}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & e^{\alpha d_1} M_1 E & -e^{\alpha d_2} S_1 E & \frac{e^{2\alpha d_2} - 1}{2\alpha} N_1 & cS_1 & cM_1 & \Pi_{18} \\ \star & \Pi_{22} & e^{\alpha d_1} M_2 E & -e^{\alpha d_2} S_2 E & \frac{e^{2\alpha d_2} - 1}{2\alpha} N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha d_2} - 1}{2\alpha} Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1 + Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0 \tag{5}$$

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where

$$\Pi_{11} = P^\top A + A^\top P + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top + 2\alpha E^\top P$$
$$\Pi_{12} = P^\top A_d + (N_2 E)^\top - N_1 E + S_1 E - M_1 E$$
$$\Pi_{22} = -(1-\mu)e^{-2\alpha d_2} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top$$
$$d_{12} = d_2 - d_1, \quad U = d_2 Z_1 + d_{12} Z_2, \Pi_{18} = A^\top U,$$
$$c = \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha}$$

Proof: First, we will show that the system is regular and impulse free. For this purpose, choose two non-singular matrices R, L such that

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = RAL = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{7}$$

Now, let

$$\bar{A}_d = RA_d L = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix},$$
$$\bar{P} = R^{-\top} PL = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{8}$$
$$\bar{N}_i = L^\top N_i R^{-1} = \begin{bmatrix} N_{i11} & N_{i12} \\ N_{i21} & N_{i22} \end{bmatrix},$$
$$\bar{Q}_i = L^\top Q_i L = \begin{bmatrix} Q_{i11} & Q_{i12} \\ Q_{i21} & Q_{i22} \end{bmatrix} \tag{9}$$

From (6) and (7), we conclude that $P_{12} = 0$ and $P_{11} > 0$.

Also, from (5), we obtain $\Pi_{11} < 0$ which gives $P^\top A + A^\top P + \sum_{j=1}^{3} Q_i + N_1 E + (N_1 E)^\top < 0$. Based on (7)–(9), pre- and post-multiply this inequality by $L^\top$ and L, respectively, and noting that $Q_i > 0$, we have

$$\bar{P}^\top \bar{A} + \bar{A}^\top \bar{P} + \bar{N}_1 \bar{E} + (\bar{N}_1 \bar{E})^\top < 0$$

Noting that

$$\bar{N}_1 \bar{E} = \begin{bmatrix} N_{111} & 0 \\ N_{121} & 0 \end{bmatrix}$$

which gives

$$\begin{bmatrix} \star & \star \\ \star & A_{22}^\top P_{22} + P_{22}^\top A_{22} \end{bmatrix} < 0$$

that implies in turn that $A_{22}^\top P_{22} + P_{22}^\top A_{22} < 0$.

Therefore A_{22} is non-singular, which implies in turn that system (1) is regular and impulse-free (see [19]). Next, we show the exponential stability of system (1). Since system (1) is regular and impulse free, there exist two other matrices R, L such that (see [19])

$$\bar{E} = REL = \begin{bmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = RAL = \begin{bmatrix} A_1 & 0 \\ 0 & \mathbb{I}_{n-r} \end{bmatrix} \tag{10}$$

Define $\bar{A}_d$, $\bar{P}$, $\bar{N}_i$, $\bar{Q}_i$ in a similar manner as before, $\bar{M}_i$, $\bar{S}_i$ similar to $\bar{N}_i$ and $\bar{Z}_i = R^{-\top} Z_i R^{-1}$. Using (5) and Schur complement, we obtain

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \star & \Pi_{22} \end{bmatrix} < 0$$

Substitute (10) into the previous inequality, pre- and post-multiply by $\mathrm{diag}\{L^\top, L^\top\}$, $\mathrm{diag}\{L, L\}$ and using Schur complement argument, we have

$$\begin{bmatrix} P_{22}^\top + P_{22} + \sum_{j=1}^{3} Q_{i22} & P_{22}^\top A_{d22} \\ A_{d22}^\top P_{22} & -(1-\mu)e^{-2\alpha d_2} Q_{322} \end{bmatrix} < 0$$

Pre- and post-multiplying this inequality by $[-A_{d22}^\top \; \mathbb{I}]$ and its transpose, and noting that $Q_i > 0$ and $\mu \geq 0$, we obtain

$$A_{d22}^\top Q_{322} A_{d22} - e^{-2\alpha d_2} Q_{322} < 0 \quad \text{which implies}$$
$$\rho(e^{\alpha d_2} A_{d22}) < 1 \tag{11}$$

So there exist constants $\beta > 1$ and $\gamma \in (0, 1)$ such that

$$\|e^{i\alpha d_2} A_{d22}^i\| \leq \beta \gamma^i, \quad i = 1, 2, \ldots \tag{12}$$

Let $\zeta(t) = L^{-1} x(t) = \begin{bmatrix} \zeta_1(t) \\ \zeta_2(t) \end{bmatrix}$, where $\zeta_1(t) \in \mathbb{R}^r$ and $\zeta_2(t) \in \mathbb{R}^{n-r}$. Then, system (1) becomes equivalent to the following one

$$\dot{\zeta}_1(t) = A_1 \zeta_1(t) + A_{d11} \zeta_1(t - d(t)) + A_{d12} \zeta_2(t - d(t)) \tag{13}$$
$$0 = \zeta_2(t) + A_{d21} \zeta_1(t - d(t)) + A_{d22} \zeta_2(t - d(t)) \tag{14}$$

Now, choose the Lyapunov functional as follows

$$\begin{aligned} V(t) &= \zeta^\top(t) \bar{E}^\top \bar{P} \zeta(t) + \sum_{i=1}^{2} \int_{t-d_i}^{t} \zeta^\top(s) e^{2\alpha(s-t)} \bar{Q}_i \zeta(s)\, ds \\ &+ \int_{t-d(t)}^{t} \zeta(s)^\top e^{2\alpha(s-t)} \bar{Q}_3 \zeta(s)\, ds \\ &+ \int_{-d_2}^{0} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_1 \bar{E} \dot{\zeta}(s)\, ds\, d\theta \\ &+ \int_{-d_2}^{-d_1} \int_{t+\theta}^{t} (\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)} \bar{Z}_2 \bar{E} \dot{\zeta}(s) ds\, d\theta \end{aligned}$$

Then, the time derivative of $V(t)$ along the solution of (13)

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and (14) is given by

$$
\begin{aligned}
\dot{V}(t) &= 2\zeta^\top(t)\bar{P}^\top\bar{E}\dot{\zeta}(t) + \sum_{i=1}^{2}\{\zeta^\top(t)\bar{Q}_i\zeta(t) \\
&\quad - \zeta^\top(t-d_i)e^{-2\alpha d_i}\bar{Q}_i\zeta(t-d_i)\} \\
&\quad + \zeta^\top(t)\bar{Q}_3\zeta(t) - (1-\dot{d}(t))\zeta^\top(t-d(t)) \\
&\quad \times e^{-2\alpha d(t)}\bar{Q}_3\zeta(t-d(t)) + d_2(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_1\bar{E}\dot{\zeta}(t) \\
&\quad - \int_{t-d_2}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)\,\mathrm{d}s + (d_2-d_1) \\
&\quad \times(\bar{E}\dot{\zeta}(t))^\top\bar{Z}_2\bar{E}\dot{\zeta}(t) - \int_{t-d_2}^{t-d_1}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)\,\mathrm{d}s \\
&\quad - 2\alpha\sum_{i=1}^{2}\int_{t-d_i}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_i\zeta(s)\mathrm{d}s \\
&\quad - 2\alpha\int_{t-d(t)}^{t}\zeta^\top(s)e^{2\alpha(s-t)}\bar{Q}_3\zeta(s)\,\mathrm{d}s \\
&\quad - 2\alpha\int_{-d_2}^{0}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_1\bar{E}\dot{\zeta}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&\quad - 2\alpha\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}(\bar{E}\dot{\zeta}(s))^\top e^{2\alpha(s-t)}\bar{Z}_2\bar{E}\dot{\zeta}(s)\,\mathrm{d}s\,\mathrm{d}\theta
\end{aligned} \tag{15}
$$

Adding now these terms

$$
\begin{aligned}
&+ 2\big[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\big] \\
&\times\left[\bar{E}\zeta(t) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t}\bar{E}\dot{\zeta}(s)\,\mathrm{d}s\right] \\
&+ 2\big[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\big] \\
&\times\left[\bar{E}\zeta(t-d(t)) - \bar{E}\zeta(t-d_2) - \int_{t-d_2}^{t-d(t)}\bar{E}\dot{\zeta}(s)\,\mathrm{d}s\right] \\
&+ 2\big[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\big] \\
&\times\left[\bar{E}\zeta(t-d_1) - \bar{E}\zeta(t-d(t)) - \int_{t-d(t)}^{t-d_1}\bar{E}\dot{\zeta}(s)\,\mathrm{d}s\right] \\
&+ \int_{t-d_2}^{t}\big[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\big] \\
&\times \bar{Z}_1^{-1}e^{-2\alpha(s-t)}\big[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\big]^\top\,\mathrm{d}s \\
&- \int_{t-d(t)}^{t}\big[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\big] \\
&\times \bar{Z}_1^{-1}e^{-2\alpha(s-t)}\big[\zeta^\top(t)\bar{N}_1 + \zeta^\top(t-d(t))\bar{N}_2\big]^\top\,\mathrm{d}s \\
&+ \int_{t-d_2}^{t-d_1}\big[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\big](\bar{Z}_1+\bar{Z}_2)^{-1} \\
&\times e^{-2\alpha(s-t)}\big[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\big]^\top\,\mathrm{d}s \\
&- \int_{t-d_2}^{t-d(t)}\big[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\big](\bar{Z}_1+\bar{Z}_2)^{-1} \\
&\times e^{-2\alpha(s-t)}\big[\zeta^\top(t)\bar{S}_1 + \zeta^\top(t-d(t))\bar{S}_2\big]^\top\,\mathrm{d}s \\
&+ \int_{t-d_2}^{t-d_1}\big[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\big]\bar{Z}_2^{-1}e^{-2\alpha(s-t)} \\
&\times\big[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\big]^\top\,\mathrm{d}s \\
&- \int_{t-d(t)}^{t-d_1}\big[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\big]\bar{Z}_2^{-1}e^{-2\alpha(s-t)} \\
&\times\big[\zeta^\top(t)\bar{M}_1 + \zeta^\top(t-d(t))\bar{M}_2\big]^\top\,\mathrm{d}s
\end{aligned}
$$

to (15) gives

$$
\begin{aligned}
&\dot{V}(t) + 2\alpha V(t) \\
&\quad\le \eta^\top(t)\Bigg[\Pi + \widetilde{A}^\top(d_2\bar{Z}_1 + d_{12}\bar{Z}_2)\widetilde{A} \\
&\qquad + \frac{e^{2\alpha d_2}-1}{2\alpha}\widetilde{N}\bar{Z}_1^{-1}\widetilde{N}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{S}(\bar{Z}_1+\bar{Z}_2)^{-1}\widetilde{S}^\top \\
&\qquad + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{M}\bar{Z}_2^{-1}\widetilde{M}^\top\Bigg]\eta(t) \\
&\qquad - \int_{t-d(t)}^{t}\Big[\eta^\top(t)\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\Big]e^{-2\alpha(s-t)}\bar{Z}_1^{-1} \\
&\qquad \times\Big[\eta^\top(t)\widetilde{N} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_1\Big]^\top\,\mathrm{d}s \\
&\qquad - \int_{t-d_2}^{t-d(t)}\Big[\eta^\top(t)\widetilde{S} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}(\bar{Z}_1+\bar{Z}_2)\Big] \\
&\qquad \times e^{-2\alpha(s-t)}(\bar{Z}_1+\bar{Z}_2)^{-1}\Big[\eta^\top(t)\widetilde{S} \\
&\qquad + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}(\bar{Z}_1+\bar{Z}_2)\Big]^\top\,\mathrm{d}s \\
&\qquad - \int_{t-d(t)}^{t-d_1}\Big[\eta^\top(t)\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\Big]e^{-2\alpha(s-t)}\bar{Z}_2^{-1} \\
&\qquad \times\Big[\eta^\top(t)\widetilde{M} + \bar{E}\dot{\zeta}(s)e^{2\alpha(s-t)}\bar{Z}_2\Big]^\top\,\mathrm{d}s \\
&\quad\le \eta^\top(t)\Bigg[\Pi + \widetilde{A}^\top(d_2\bar{Z}_1 + d_{12}\bar{Z}_2)\widetilde{A} \\
&\qquad + \frac{e^{2\alpha d_2}-1}{2\alpha}\widetilde{N}\bar{Z}_1^{-1}\widetilde{N}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{S}(\bar{Z}_1+\bar{Z}_2)^{-1}\widetilde{S}^\top \\
&\qquad + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{M}\bar{Z}_2^{-1}\widetilde{M}^\top\Bigg]\eta(t)
\end{aligned}
$$

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where

$$\eta(t)=\begin{bmatrix}\zeta(t)\\ \zeta(t-d(t))\\ \zeta(t-d_1)\\ \zeta(t-d_2)\end{bmatrix},$$

$$\Pi=\begin{bmatrix}\Pi_{11} & \Pi_{12} & \bar{M}_1\bar{E} & -\bar{S}_1\bar{E}\\ \star & \Pi_{22} & \bar{M}_2\bar{E} & -\bar{S}_2\bar{E}\\ \star & \star & -e^{-2\alpha d_1}\bar{Q}_1 & 0\\ \star & \star & 0 & -e^{-2\alpha d_2}\bar{Q}_2\end{bmatrix}$$

$$\widetilde{N}=\begin{bmatrix}\bar{N}_1\\ \bar{N}_2\\ 0\\ 0\end{bmatrix},\quad \widetilde{M}=\begin{bmatrix}\bar{M}_1\\ \bar{M}_2\\ 0\\ 0\end{bmatrix}$$

$$\widetilde{S}=\begin{bmatrix}\bar{S}_1\\ \bar{S}_2\\ 0\\ 0\end{bmatrix},\quad \widetilde{A}=\begin{bmatrix}\bar{A}^{\top}\\ \bar{A}_d^{\top}\\ 0\\ 0\end{bmatrix}$$

$$\Pi_{11}=\bar{P}^{\top}\bar{A}+\bar{A}^{\top}\bar{P}+\sum_{i=1}^{3}\bar{Q}_i+\bar{N}_1\bar{E}+(\bar{N}_1\bar{E})^{\top}+2\alpha\bar{E}^{\top}\bar{P}$$

$$\Pi_{12}=\bar{P}^{\top}\bar{A}_d+(\bar{N}_2\bar{E})^{\top}-\bar{N}_1\bar{E}+\bar{S}_1\bar{E}-\bar{M}_1\bar{E}$$

$$\Pi_{22}=-(1-\mu)e^{-2\alpha d_2}\bar{Q}_3+\bar{S}_2\bar{E}+(\bar{S}_2\bar{E})^{\top}-\bar{N}_2\bar{E}-(\bar{N}_2\bar{E})^{\top}-\bar{M}_2\bar{E}-(\bar{M}_2\bar{E})^{\top}$$

Pre- and post-multiply (5) by diag$\{L^{\top}, L^{\top}, e^{-\alpha d_1}L^{\top}, e^{-\alpha d_2}L^{\top}, \mathbb{I}, \mathbb{I}, \mathbb{I}, \mathbb{I}\}$ and its transpose, then using Schur complement implies

$$\dot{V}(t)+2\alpha V(t)\le 0 \quad \text{which leads to} \quad V(t)\le e^{-2\alpha t}V(\phi(t))$$

Then, the following estimation is obtained

$$\lambda_1\|\zeta_1(t)\|^2\le V(t)\le e^{-2\alpha t}V(\phi(t))\le\lambda_2 e^{-2\alpha t}\|\phi\|_c^2$$

which leads to

$$\|\zeta_1(t)\|\le\sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c e^{-\alpha t} \tag{16}$$

where $\lambda_1=\lambda_{\min}(\bar{P}_{11})>0$ and $\lambda_2>0$ is a sufficiently large scalar and can be found since $V(\phi(t))$ is a bounded quadratic functional of $\phi(t)$. In order to prove the exponential stability of the algebraic subsystem, the relation in (14) should be used. For constant time delay, an explicit equation of $\zeta_2(t)$ is found by an iterative method [4]. It can be seen that $\zeta_2(t)$ depends on $\zeta_2(t-\tau)$, where τ is the constant delay, and $\zeta_2(t-\tau)$ depends on $\zeta_2(t-2\tau)$, and so on. In the case of time-varying delay, such a direct relation cannot be found. Thus, some new variables will be defined in order to model the dependency of $\zeta_2(t)$ on past instances. Now, define

$$t_i=t_{i-1}-d(t_{i-1}),\quad i=1,2,\ldots$$
$$t_0=t$$

It can be seen that the value of $\zeta(t)$ at $t=t_i$ depends on the value of $\zeta(t)$ at $t=t_{i+1}$. From (14), we obtain

$$\begin{aligned}\zeta_2(t)&=-A_{d21}\zeta_1(t-d(t))-A_{d22}\zeta_2(t-d(t))\\ &=-A_{d21}\zeta_1(t_1)-A_{d22}\zeta_2(t_1)\end{aligned} \tag{17}$$

Now, $\zeta_2(t_1)$ can be computed from (14) as follows

$$\begin{aligned}\zeta_2(t_1)&=\zeta_2(t-d(t))\\ &=-A_{d21}\zeta_1(t-d(t)-d(t-d(t)))\\ &\quad-A_{d22}\zeta_2(t-d(t)-d(t-d(t)))\\ &=-A_{d21}\zeta_1(t_1-d(t_1))-A_{d22}\zeta_2(t_1-d(t_1))\\ &=-A_{d21}\zeta_1(t_2)-A_{d22}\zeta_2(t_2)\end{aligned}$$

Substituting this in (17), we obtain

$$\begin{aligned}\zeta_2(t)&=-A_{d21}\zeta_1(t_1)-A_{d22}[-A_{d21}\zeta_1(t_2)-A_{d22}\zeta_2(t_2)]\\ &=-A_{d21}\zeta_1(t_1)-A_{d22}[-A_{d21}\zeta_1(t_2)\\ &\quad-A_{d22}[-A_{d21}\zeta_1(t_3)-A_{d22}\zeta_2(t_3)]]\end{aligned}$$

and so on.

Note that $t_i<t_{i-1}$ therefore there exists a positive finite integer $k(t)$ such that (see Fig. 1)

$$\zeta_2(t)=(-A_{d22})^{k(t)}\zeta_2(t_{k(t)})-\sum_{i=0}^{k(t)-1}(-A_{d22})^{i}A_{d21}\zeta_1(t_{i+1}) \tag{18}$$

and $t_{k(t)}\in(-d_2,0]$. Therefore from (11), (12), (16), Lemma 2.2 and noting that

$$k(t)d_2\ge t,\quad t_i=t-\sum_{j=0}^{i-1}d(t_j)\ge t-id_2$$

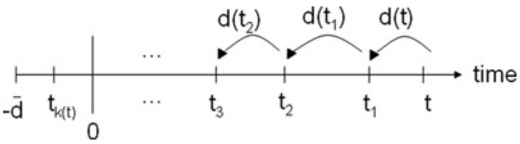

Figure 1 *The relation between different t_i, $i=1,2,\ldots$.*

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we obtain

$$\begin{aligned}\|\zeta_2(t)\| &\le \|A_{d22}{}^{k(t)}\|\,\|\phi\|_c + \|A_{d21}\| \sum_{i=0}^{k(t)-1} \|A_{d22}{}^{i}\|\,\|\zeta_1(t_{i+1})\| \\ &\le \chi e^{-\alpha d_2 k(t)}\|\phi\|_c + \|A_{d21}\|\sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|_c \\ &\quad \times \sum_{i=0}^{k(t)-1} \|A_{d22}{}^{i}\| e^{-\alpha(t-(i+1)d_2)} \\ &\le \left[\chi\|\phi\|_c + \|A_{d21}\|\sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2}\|\phi\|_c \sum_{i=0}^{k(t)-1} \|A_{d22}\|^{i} e^{i\alpha d_2}\right] e^{-\alpha t} \\ &\le \left[\chi + \|A_{d21}\|\sqrt{\frac{\lambda_2}{\lambda_1}} e^{\alpha d_2} M\right]\|\phi\|_c e^{-\alpha t}\end{aligned}$$

where

$$M = \frac{\beta}{1-\gamma}, \quad \chi = \sqrt{\frac{\lambda_{\max}(Q_{322})}{\lambda_{\min}(Q_{322})}}$$

Thus, the singular time-delay system is exponentially stable with a minimum decaying rate $= \alpha$. Finally, as we have shown that this system is also regular and impulse free, by Definition 1, we then have that the system is exponentially admissible. This completes the proof. □

Remark 1: Equation (18) can be seen as a generalisation of the iterative equation in [4] for systems with constant time delay. Also, based on (11), which is equivalent to (20) in [4] as α goes to 0, the stability of the algebraic subsystem has been shown for the case of time-varying delay. Thus, the results in [4], and in [5, 6] as well, can be extended easily to the case of time-varying delay.

Remark 2: It is noted that condition (6) is non-strict LMI, which contains equality constraints. This may result in numerical problems when checking such non-strict LMI conditions since equality constraints are fragile and usually not satisfied perfectly. Therefore strict LMI conditions are more desirable than non-strict ones from the numerical point of view [19]. Considering this, (5) and (6) can be combined into a single strict LMI. Let $P > 0$ and $S \in R^{n\times(n-r)}$ be any matrix with full column rank and satisfies $E^{\top} S = 0$. Changing P to $PE + SQ$ in (5) yields the strict LMI.

Now, the stabilisation problem of system (1) via static output feedback controller will be addressed. The technique introduced in [20] will be adopted in order to write the saturated non-linear system (4) in a linear polytopic form. Let us write the saturation term as

$$\mathrm{sat}(KCx(t)) = D(\rho(x))KCx(t), \quad D(\rho(x)) \in \mathbb{R}^{m\times m}$$

where $D(\rho(x))$ is a diagonal matrix for which the diagonal elements $\rho_i(x)$ are defined for $i = 1, \ldots, m$ as

$$\rho_i(x) = \begin{cases} -\dfrac{\bar{u}_i}{(KC)_i x} & \text{if } (KC)_i x \le -\bar{u}_i \\ 1 & \text{if } -\bar{u}_i < (KC)_i x < \bar{u}_i \\ \dfrac{\bar{u}_i}{(KC)_i x} & \text{if } (KC)_i x \ge \bar{u}_i \end{cases}$$

Using this, system (4) can be written as follows

$$E\dot{x}(t) = (A + BD(\rho(x))KC)x(t) + A_d x(t-d(t)) \tag{19}$$

The coefficient $\rho_i(x)$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, the smaller $\rho_i(x)$, the farther is the state vector from the region of linearity.

Let $0 < \underline{\rho}_i \le 1$ be a lower bound to $\rho_i(x)$, and define a vector $\underline{\rho} = [\underline{\rho}_1, \ldots, \underline{\rho}_m]$. The vector $\underline{\rho}$ is associated to the following region in the state space

$$S(K, \bar{u}^{\rho}) = \{x \in \mathbb{R}^n \mid -\bar{u}^{\rho} \le KCx \le \bar{u}^{\rho}\}$$

where every component of the vector $\bar{u}^{\rho}$ is defined by $\bar{u}_i/\underline{\rho}_i$. This vector can be viewed as a specification on the saturation tolerance. Define now matrices $A_j, j = 1, \ldots, 2^m$, as follows

$$A_j = A + BD(\gamma_j)KC$$

where $D(\gamma_j)$ is a diagonal matrix of positive scalars $\gamma_{j(i)}$ for $i = 1, \ldots, m$, which arbitrarily take the value one or $\underline{\rho}_i$. Note that the matrices A_j are the vertices of a convex polytope of matrices. If $x(t) \in S(K, \bar{u}^{\rho})$, it follows that $(A + BD(\rho(x))KC) \in \mathrm{co}\{A_1, \ldots, A_{2^m}\}$. We conclude that if $x(t) \in S(K, \bar{u}^{\rho})$, then $E\dot{x}(t)$ can be determined from the following polytopic model

$$E\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_{j,t} A_j x(t) + A_d x(t-d(t)) \tag{20}$$

with $\sum_{j=1}^{2^m} \lambda_{j,t} = 1$ and $\lambda_{j,t} \ge 0$.

Remark 3: Different control saturation models are proposed in the literature, that is regions of saturation, differential inclusion and sector modelling. In [21], a comparative analysis of these models is presented, and concluded that the differential inclusion model leads to the least conservative design. Based on that, the differential inclusion model for the actuator saturation is used here.

Then we have the following result.

Theorem 2: Let $0 < d_1 < d_2$, $\alpha > 0$, a vector $\underline{\rho}$ and $0 \le \mu < 1$ be given. If there exist symmetric and

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positive-definite matrices P, Q_1, Q_2, Q_3, Z_1 and Z_2, matrices M_i, N_i and S_i, $i = 1, 2$, matrices K and Q and a positive scalar κ such that (21)

$$\begin{bmatrix} E^\top(PE+SQ) & \underline{\rho}_i(KC)_i^\top \\ \underline{\rho}_i(KC)_i & \kappa\bar{u}_i^2 \end{bmatrix} \geq 0,\ i = 1, \ldots, m \quad (22)$$

where

$$\begin{aligned}\Pi_{j11} &= (PE+SQ)^\top A + A^\top(PE+SQ) \\ &\quad + \sum_{i=1}^{3} Q_i + N_1E + (N_1E)^\top + (PE+SQ)^\top BD(\gamma_j)KC \\ &\quad + \left((PE+SQ)^\top BD(\gamma_j)KC\right)^\top + 2\alpha E^\top(PE+SQ) \\ \Pi_{12} &= (PE+SQ)^\top A_d + (N_2E)^\top - N_1E + S_1E - M_1E \\ \Pi_{22} &= -(1-\mu)e^{-2\alpha d_2}Q_3 + S_2E + (S_2E)^\top - N_2E \\ &\quad - (N_2E)^\top - M_2E - (M_2E)^\top \\ d_{12} &= d_2 - d_1, \quad U = d_2Z_1 + d_{12}Z_2, \\ \Pi_{j18} &= A^\top U + (BD(\gamma_j)KC)^\top U, \\ c &= \frac{e^{2\alpha d_2} - e^{2\alpha d_1}}{2\alpha}\end{aligned}$$

where $S \in \mathbb{R}^{n\times(n-r)}$ is any matrix with full column rank and satisfies $E^\top S = 0$, then there exists a static output feedback controller (3) such that the closed-loop system (4) is locally exponentially admissible with $\sigma = \alpha$ for any compatible initial condition satisfying

$$\Omega(\nu_1, \nu_2) = \left\{\phi \in C_{d_2}^{v}: \frac{\|\phi\|_c^2}{\nu_1} + \frac{\|\dot{\phi}\|_c^2}{\nu_2} \leq 1\right\} \quad (23)$$

where

$$\begin{aligned}\nu_1 &= \frac{\kappa^{-1}}{\chi_1}, \quad \nu_2 = \frac{\kappa^{-1}}{\chi_2} \\ \chi_1 &= \lambda_{\max}(E^\top PE) + \sum_{i=1}^{2}\lambda_{\max}(Q_i)\frac{1-e^{-2\alpha d_i}}{2\alpha} \\ &\quad + \lambda_{\max}(Q_3)\frac{1-e^{-2\alpha d_2}}{2\alpha} \\ \chi_2 &= \lambda_{\max}(Z_1)\lambda_{\max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2} \\ &\quad + \lambda_{\max}(Z_2)\lambda_{\max}(E^\top E)\frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}\end{aligned}$$

Proof: Assume that $x(t) \in S(K, \bar{u}^\rho)$, $\forall t > 0$ (will be proved later). Therefore $E\dot{x}(t)$ can be determined from the polytopic system (20). Applying Remark 2 to (5) and (6) in Theorem 1 yields a single matrix inequality. Then, if we apply this matrix inequality 2^m times to the parameters A_j with $j = 1, \ldots, 2^m$, A_d, E, d_1, d_2 and μ, we will have (21). Now, proceeding in a similar way as for the proof of Theorem 1, yields

$$A_{j22}^\top P_{22} + P_{22}^\top A_{j22} < 0, \quad j = 1, \ldots, 2^m$$

Using the fact that $\lambda_{j,t} \geq 0$

$$\begin{gathered}\lambda_{j,t}A_{j22}^\top P_{22} + P_{22}^\top\lambda_{j,t}A_{j22} \leq 0 \\ j = 1, \ldots, 2^m, \forall t \in (0, \infty)\end{gathered}$$

adding these inequalities together and noting that $\sum_{j=1}^{2^m}\lambda_{j,t} = 1$, gives $\left[\sum_{j=1}^{2^m}\lambda_{j,t}A_{j22}\right]^\top P_{22} + P_{22}^\top\sum_{j=1}^{2^m}\lambda_{j,t}\times A_{j22} < 0$ which implies $\sum_{j=1}^{2^m}\lambda_{j,t}A_{j22}$ is non-singular $\forall t \in (0, \infty)$ which implies that system (20) is regular and impulse free. Now, choose a Lyapunov functional as in Theorem 1, and proceeding in a similar manner as

$$\begin{bmatrix} \Pi_{j11} & \Pi_{12} & e^{\alpha d_1}M_1E & -e^{\alpha d_2}S_1E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_1 & cS_1 & cM_1 & \Pi_{j18} \\ \star & \Pi_{22} & e^{\alpha d_1}M_2E & -e^{\alpha d_2}S_2E & \frac{e^{2\alpha d_2}-1}{2\alpha}N_2 & cS_2 & cM_2 & A_d^\top U \\ \star & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\frac{e^{2\alpha d_2}-1}{2\alpha}Z_1 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -c(Z_1+Z_2) & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -cZ_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -U \end{bmatrix} < 0 \quad j = 1, \ldots, 2^m \quad (21)$$

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before, then

$$\dot{V}(t) + 2\alpha V(t) \le \eta^\top(t)\Bigg[\Pi + \widetilde{A}^\top(d_2\bar{Z}_1 + d_{12}\bar{Z}_2)\widetilde{A} + \frac{e^{2\alpha d_2}-1}{2\alpha}\widetilde{N}\bar{Z}_1^{-1}\widetilde{N}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{S}(\bar{Z}_1+\bar{Z}_2)^{-1}\widetilde{S}^\top + \frac{e^{2\alpha d_2}-e^{2\alpha d_1}}{2\alpha}\widetilde{M}\bar{Z}_2^{-1}\widetilde{M}^\top\Bigg]\eta(t)$$

with all the variables as defined in Theorem 1 and A is replaced by $\sum_{j=1}^{2^m}\lambda_{j,t}A_j$. Then, by convexity and noting that $\sum_{j=1}^{2^m}\lambda_{j,t} = 1$ with $\lambda_{j,t} \ge 0$, condition (21) yields

$$\dot{V}(t) + 2\alpha V(t) \le 0$$

The rest of the proof is similar to the proof of Theorem 1, and the details are omitted.

Now, by virtue of condition (22), the ellipsoid defined by $\Gamma = \{x \in \mathbb{R}^n : x^\top E^\top(PE+SQ)\,x \le \kappa^{-1}\}$ is included in the set $S(K, \bar{u}^p)$ [20]. Suppose now that the initial condition $\phi(t)$ satisfies (23), and conditions (21) and (22) hold. Then, from the definition of $V(t)$, it follows that $x^\top(0)E^\top(PE+SQ)x(0) \le V(0) \le \chi_1\|\phi\|_c^2 + \chi_2\|\dot{\phi}\|_c^2 \le \kappa^{-1}$ and, in this case, one has $x(0) \in \Gamma \subset S$. Now, since $\dot{V}(0) < 0$, we conclude that $x^\top(t)E^\top(PE+SQ)x(t) \le V(t) \le V(0) \le \chi_1\|\phi\|_c^2 + \chi_2\|\dot{\phi}\|_c^2 \le \kappa^{-1}$, which means that $x(t) \in S(K, \bar{u}^p)$, $\forall t > 0$. This completes the proof. □

Remark 4: Being inside the set $\Omega(\nu_1, \nu_2)$, the compatibility of the initial condition is also very important especially when saturation is present. It is well known that incompatible initial conditions results in jump discontinuities because of the singular structure. Such jump discontinuities may take the system outside the set $\Omega(\nu_1, \nu_2)$, where the controller may be unable to stabilise the system.

It is obvious that (21) is a bilinear matrix inequality (BMI), and consequently its solution is very difficult. Thus, an iterative LMI (ILMI) approach similar to [22, 23] will be proposed. The derivation of the algorithm is similar to [22, 23] and will be omitted. This algorithm has the same disadvantages as those in [22, 23], that is based on a sufficient conditions. The following is the proposed algorithm and the explanation is given later.

ILMI algorithm

Step 1: OP1

$$\min_{P_0>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{p0}>0,M_p,N_p,S_p,p=1,2,\kappa} \beta_0$$
$$\text{s.t. (26) and (27)}$$
$$K = 0 \text{ and } X_0 = E$$

Set $i = 1$, $X_1 = E$, $Z_{11} = Z_{10}$ and $Z_{21} = Z_{20}$.

Step 2: OP2

$$\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,M_p,N_p,S_p,p=1,2,K,\kappa} \beta_i$$
$$\text{s.t. (26) and (27)}$$

Let β_i^* and K^* be the solution of OP2. If $\beta_i^* \le -\alpha$, where α is a prescribed decay rate, then K^* is a stabilising static output feedback gain, go to step 5, otherwise, go to step 3.

Step 3: OP3

$$\min_{P_i>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_{pi}>0,M_p,N_p,S_p,p=1,2,\kappa} \operatorname{tr}(E^\top T_i)$$
$$\text{s.t. (26) and (27)}$$
$$\beta_i = \beta_i^* \text{ and } K = K^*$$

If $\|X_iB - T_i^*B\| < \epsilon$, go to step 4, else set $i = i+1$, $X_i = T_{i-1}^*$, $Z_{1i} = Z_{1(i-1)}^*$ and $Z_{2i} = Z_{2(i-1)}^*$, then go to step 2.

Step 4: The system may not be stabilisable via static output feedback. Stop.

Step 5: OP4

$$\min_{P>0,Q,Q_1>0,Q_2>0,Q_3>0,Z_p>0,M_p,N_p,S_p,p=1,2,K,\kappa} r$$
$$\text{s.t.(26) and (27), } \beta_i = \alpha$$

$$\delta_1 \mathbb{I} \ge E^\top PE, \quad \delta_2 \mathbb{I} \ge Q_1, \quad \delta_3 \mathbb{I} \ge Q_2 \tag{24}$$

$$\delta_4 \mathbb{I} \ge Q_3, \quad \delta_5 \mathbb{I} \ge Z_1, \quad \delta_6 \mathbb{I} \ge Z_2 \tag{25}$$

where

$$r = w_1\left(\delta_1 + \frac{1-e^{-2\alpha d_1}}{2\alpha}\delta_2 + \frac{1-e^{-2\alpha d_2}}{2\alpha}\delta_3 + \frac{1-e^{-2\alpha d_2}}{2\alpha}\delta_4\right) + w_2\left(\lambda_{\max}(E^\top E)\frac{2\alpha d_2 - 1 + e^{-2\alpha d_2}}{4\alpha^2}\delta_5 + \lambda_{\max}(E^\top E)\frac{2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2}}{4\alpha^2}\delta_6\right)$$

$+ w_3\kappa$, and w_1, w_2 and w_3 are weighting factors.

We solve this problem iteratively in two steps as follows:

(a) Fix K, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_p > 0$, M_p, N_p, S_p, $p = 1, 2$, and κ.

(b) Fix Z_1 and Z_2, and solve for $P > 0$, Q, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, M_p, N_p, S_p, $p = 1, 2$, K and κ. Set $X = T$.

The set (23) is calculated from the matrices that solve OP4 (see (26))

$$\begin{bmatrix} E^\top T_i & \underline{\rho}_r (KC)_r^\top \\ \underline{\rho}_r (KC)_r & \kappa \bar{u}_r^2 \end{bmatrix} \geq 0, \quad r = 1, \ldots, m \tag{27}$$

where

$$\Pi_{11} = T_i^\top A + A^\top T_i + \sum_{i=1}^{3} Q_i + N_1 E + (N_1 E)^\top - X_i BB^\top T_i - (X_i BB^\top T_i)^\top + X_i BB^\top X_i - 2\beta_i E^\top T_i$$

$$\Pi_{j18} = A^\top U + (BD(\gamma_j)KC)^\top U$$

$$\Pi_{22} = -(1-\mu) e^{2\beta d(\beta)} Q_3 + S_2 E + (S_2 E)^\top - N_2 E - (N_2 E)^\top - M_2 E - (M_2 E)^\top$$

$$T_i = (P_i E + SQ), \quad d(\beta) = \begin{cases} d_1 & \text{if } \beta > 0 \\ d_2 & \text{if } \beta < 0 \end{cases}$$

and the other variables as defined previously.

Remark 5: The core of this algorithm is in OP2 and OP3. As shown in [22], OP2 guarantees the progressive reduction of β_i whereas OP3 guarantees the convergence of the algorithm. Yet, in [22], only X needs to be fixed in order to obtain LMIs, whereas in our case, we have also to fix either Z_1 and Z_2 or K to obtain LMIs. Thus, we will fix Z_1 and Z_2 in OP2, and K in OP3. This way of solving this problem will not affect the convergence of the algorithm. It is worth noting that although this ILMI algorithm is convergent, we may not achieve the solution because β may not always converge to its minimum. For more details on the numerical properties of the algorithm, we refer the reader to [22].

Remark 6: In order to start the algorithm, OP2 should have a solution for $i = 1$. Yet, the solution depends on the initial matrix X. In [23], some Riccati equation is proposed in order to select an initial X for the descriptor version of this algorithm. In [24], it has been proved that this Riccati equation may not have a solution and an initial value of $X = \mathbb{I}$ is proposed instead. Actually, the identity matrix may not do the job for even some simple systems, an example of such systems is

$$(A, B, C) = (\mathbb{I}, \mathbb{I}, \mathbb{I}), \quad E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Our numerical experience indicates that an initial choice of $X_0 = E$ often leads to a convergent result. With this X_0, OP1 is used here to obtain initial values for Z_1 and Z_2.

Remark 7: The minimisation of β in OP1 and OP2 should be done using the bisection method. The lower bound of the bisection method can be any value less than $-\alpha$ since we are not interested in minimising β less than $-\alpha$. The upper bound of the bisection method can be any sufficiently large number. These upper and lower bounds should be chosen only once and can be fixed throughout the algorithm.

Remark 8: OP4 is used in order to enlarge the set of initial conditions (23). The satisfaction of (24) and (25) means that $\chi_1 \leq \delta_1 + [(1 - e^{-2\alpha d_1})/2\alpha]\delta_2 + [(1 - e^{-2\alpha d_1})/2\alpha]\delta_3 + [(1 - e^{-2\alpha d_1})/2\alpha]\delta_4$ and $\chi_2 \leq \lambda_{\max}(E^\top E)[(2\alpha d_2 - 1 + e^{-2\alpha d_2})/4\alpha^2]$ $\delta_5 + \lambda_{\max}(E^\top E)[(2\alpha d_{12} - e^{-2\alpha d_1} + e^{-2\alpha d_2})/4\alpha^2]\delta_6$. Therefore because $\nu_i = \kappa^{-1}/\chi_i$, if we minimise the criterion as defined in OP4, then greater the bounds on $\|\phi\|_c^2$ and $\|\dot{\phi}\|_c^2$ tend to be. In other words, by using OP4, we orient the solutions of LMIs (21) and (22) in a sense to obtain the set $\Omega(\nu_1, \nu_2)$ as large as possible. For more discussion on this topic, we refer the reader to [20].

$$\begin{bmatrix}
\Pi_{11} & \begin{bmatrix} (B^\top T_i \\ +D(\gamma_j)KC)^\top \end{bmatrix} & \Pi_{12} & e^{-\beta_i d_1} M_1 E & -e^{-\beta_i d_2} S_1 E & \frac{e^{-2\beta_i d_2} - 1}{-2\beta_i} N_1 & \frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} S_1 & \frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} M_1 & \Pi_{j18} \\
\begin{bmatrix} (B^\top T_i \\ +D(\gamma_j)KC) \end{bmatrix} & -\mathbb{I} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & 0 & \Pi_{22} & e^{-\beta_i d_1} M_2 E & -e^{-\beta_i d_2} S_2 E & \frac{e^{-2\beta_i d_2} - 1}{-2\beta_i} N_2 & \frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} S_2 & \frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} M_2 & A_d^\top U \\
\star & 0 & \star & -Q_1 & 0 & 0 & 0 & 0 & 0 \\
\star & 0 & \star & \star & -Q_2 & 0 & 0 & 0 & 0 \\
\star & 0 & \star & \star & \star & -\frac{e^{-2\beta_i d_2} - 1}{-2\beta_i} Z_{1i} & 0 & 0 & 0 \\
\star & 0 & \star & \star & \star & \star & -\frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} (Z_{1i} + Z_{2i}) & 0 & 0 \\
\star & 0 & \star & \star & \star & \star & \star & -\frac{e^{-2\beta_i d_2} - e^{-2\beta_i d_1}}{-2\beta_i} Z_{2i} & 0 \\
\star & 0 & \star & \star & \star & \star & \star & \star & -U
\end{bmatrix}$$

$$< 0 \quad j = 1, \ldots, 2^m \tag{26}$$

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Table 1 Maximum allowable decay rates α for different d_2 with $d_1 = 0.2$ and $\mu = 0.5$

d_2	0.5	0.6	0.7	0.8	0.9	1	1.1
α	0.3239	0.3014	0.2816	0.2642	0.2411	0.1323	0.0290

4 Examples

4.1 Example 1

Consider the singular time-delay system studied in [5] with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $\beta = 0$, we know from [5] that this system is asymptotically stable for constant delay $\tau < \tau^*$ and unstable for constant delay $\tau > \tau^*$, where $\tau^* = 1.2092$. Now, allowing time-varying delay, the exponential stability of this system will be investigated using Theorem 1. For various d_2, the maximum allowable decay rates α, which guarantee the exponential stability for given lower bound d_1 and derivative bound μ, are listed in Table 1. As it is clear from the table, if we increase d_2, then we obtain smaller decay rates α. Fig. 2 gives the simulation results of x_1 and x_2 as compared to $e^{-0.3t}$ when $d(t) = 0.4 + 0.1\sin(4t)$ and the initial function is $\phi(t) = [1 \quad -1]^\top$, $t \in [-0.4, 0]$. From Fig. 2, we can see that the states x_1 and x_2 exponentially converge to zero with a decay rate greater than 0.3. Now, let $\beta = 0.5$ (i.e. the delay appear also in the algebraic constraint). For $d_1 = 0.2$, $d_2 = 0.5$ and $\mu = 0.5$, the maximum allowable decay rate is $\alpha = 0.32$.

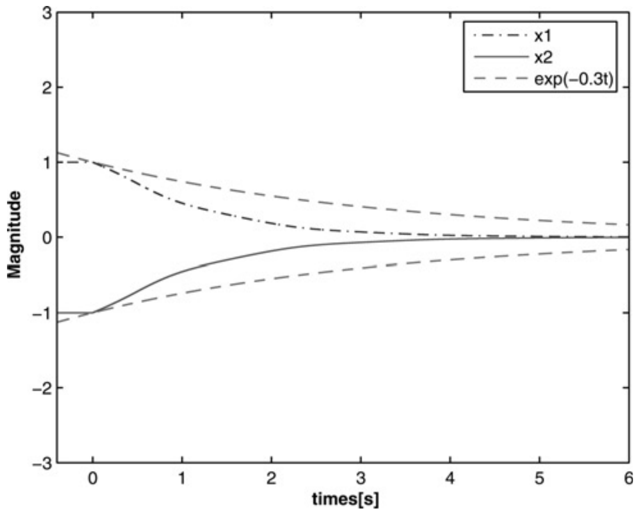

Figure 2 *Simulation results of x_1 and x_2 as compared to $e^{-0.3t}$*

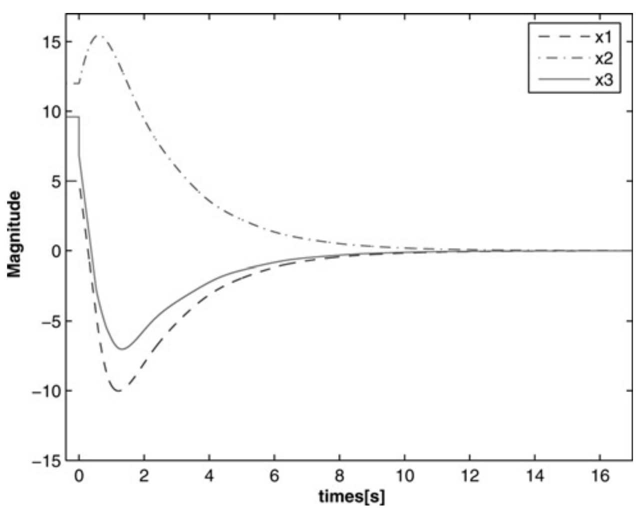

Figure 3 *Simulation results of x_1, x_2 and x_3*

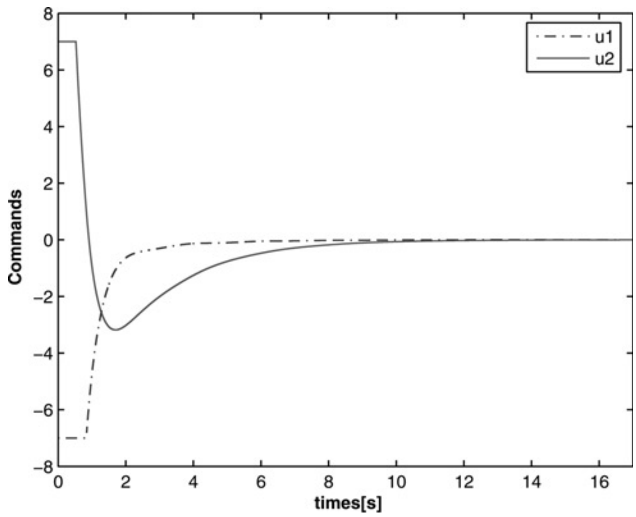

Figure 4 *Simulation results of the controllers*

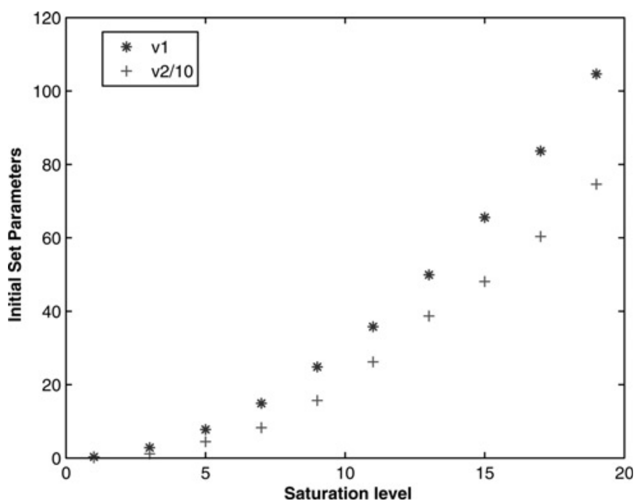

Figure 5 *v_1 and v_2 for which the exponential admissibility is guaranteed as a function of the control amplitude saturation level*

Table 2 Computation results of Example 2 with $\bar{u} = 15$

α	0.001	0.2	0.4	0.6	0.8	1	1.2
ν_1	192.1172	97.0467	48.7601	25.8165	14.0812	7.9295	5.5883
ν_2	967.1209	509.6311	268.5460	165.2845	90.6967	37.1311	28.2688
iterations	11	13	14	15	16	17	18

4.2 Example 2

Consider the singular time-delay system described by

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

$$A_d = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 0.4 & 0 \\ 0.2 & 0.3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ 0.1 & 0.3 \\ 0.1 & -0.3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

This system is originally unstable for all values of delay. Now, allowing time-varying delay, the exponential stabilisability of this system will be investigated using Theorem 3.2 and the iterative algorithm. Letting $d_1 = 0.2$, $d_2 = 0.6$, $\mu = 0.5$, $\overline{u} = 7$ and $\alpha = 0.3$, the ILMI algorithm gives after 14 iterations

$$K = \begin{bmatrix} -1.4186 & -1.2682 \\ 1.3943 & 0.8652 \end{bmatrix}, \quad \nu_1 = 14.8960$$

$$\nu_2 = 82.6586$$

Figs. 3 and 4 give the simulation results for the closed-loop system when $d(t) = |0.4 + 0.15 \sin(3t)|$ and the initial function is $\phi(t) = [5 \; 12 \; 9.6]^{\top}$, $t \in [-0.6, \; 0]$. Changing the control amplitude saturation level, Fig. 5 presents the functional dependence of ν_1 and ν_2 on the level of control saturation $\overline{u}$.

For various α, the values ν_1 and ν_2 for which we guarantee the exponential admissibility of the saturated system are listed in Table 2. The number of iterations are also listed in Table 2.

5 Conclusion

This paper has dealt with the stability and the stabilisation of the class of singular time-delay systems. A delay-range-dependent exponential stability conditions has been developed for singular time-delay systems. Also, a delay-range-dependent static output feedback controller with input saturation has been designed for singular time-delay systems and an ILMI algorithm has been proposed to compute the controller gains. The effectiveness of the results has been illustrated through examples. As a future work, the following items can be considered:

- The problem of robust stabilisation may be addressed. A similar approach to [25] can be adopted to deal with uncertainties of the polytopic type.

- In [26], two recent proposed simple modifications/generalisations of static output feedback are investigated; namely, introducing time delay in the control law and making the gain time varying. Both approaches have been shown to be complementary and existing results are brought together in a unifying framework. Motivated by this work, the generalisation of our controller should be the subject of a forthcoming publication.

- Considering the transfer delays of sensor to controller and controller to actuator that appear in many control systems, more attention has been paid to the study of stability and stabilisation of systems with control input delay. This problem has not been fully addressed for singular time-delay systems.

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