Computers and Mathematics with Applications 59 (2010) 1227–1235

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method

Mohammad Zurigat[a], Shaher Momani[b,*], Ahmad Alawneh[a]

[a] Department of Mathematics, University of Jordan, Amman, Jordan
[b] Department of Mathematics, Mutah University, P. O. Box 7, Al-Karak, Jordan

ARTICLE INFO

Keywords:
Systems of fractional algebraic–differential equations
Homotopy analysis method
Fractional differential equation

ABSTRACT

In this paper, we develop a framework to obtain approximate solutions to systems of algebraic–differential equations of fractional order by employing the homotopy analysis method (HAM). The study highlights the efficiency of the method and its dependence on the auxiliary parameter $\hbar$. Numerical examples are examined to highlight the significant features of the HAM method. The method shows improvements over existing analytical techniques

1. Introduction

Differential equations of fractional order have been found to be effective to describe some physical phenomena such as rheology, damping laws, fluid flow and so on [1–3]. A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [1]. Recently, many important mathematical models can be expressed in terms of systems of algebraic–differential equations of fractional order. The solution of fractional differential equations is much involved. In general, there exists no method that yields exact solutions for fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. In recent years, much research has been focused on the numerical solution of systems of ordinary differential equations and algebraic–differential equations. Some numerical methods have been developed, such as implicit Runge–Kutta method [4], Pade approximation method [5–8], homotopy perturbation method [9–11], Adomain decomposition method [12–18] and variation iteration method [19–22]. Liao [23] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method. This method has been successfully applied to solve many types of nonlinear problems [24–37]. In this paper, we further apply the homotopy analysis method to solve systems of algebraic–differential equations of fractional order. By means of introducing an auxiliary parameter $\hbar$, we can enlarge the convergence region of series solution. Furthermore, we give some examples to demonstrate the efficiency and effectiveness of the proposed method.

2. Basic definitions

For the concept of fractional derivative, we will adopt Caputo's definition which is a modification of the Riemann–Liouville definition.

Definition 2.1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x)$ is continuous in $[0, \infty)$. Clearly $C_\mu \subset C_\beta$ if $\beta < \mu$.

Definition 2.2. A function $f(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in N \cup \{0\}$, if $f^{(m)} \in C_\mu$.

* Corresponding author.
E-mail addresses: moh_zur@hotmail.com (M. Zurigat), shahermm@yahoo.com (S. Momani).

0898-1221/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.camwa.2009.07.002

Definition 2.3. The left sided Riemann–Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(x-t)^{1-\alpha}} \mathrm{d}t, \quad \alpha > 0,\ x > 0, \\ J^0 f(x) = f(x). \tag{2.1}$$

Definition 2.4. Let $f \in C_{-1}^m$, $m \in N \cup \{0\}$ then the Caputo fractional derivative of $f(x)$ is defined as

$$D_*^\alpha f(x) = \begin{cases} \left[J^{m-\alpha} f^{(m)}(x)\right], & m-1 < \alpha < m, m \in N, \\ \dfrac{\mathrm{d}^m f(x)}{\mathrm{d}x^m}, & \alpha = m. \end{cases} \tag{2.2}$$

Hence, we have the following properties [1–3]

$$\begin{aligned} &1.\ \ J^\alpha J^\nu f = J^{\alpha+\nu} f, \quad \alpha, \nu \geq 0,\ f \in C_\mu,\ \mu \geq -1. \\ &2.\ \ J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\gamma+\alpha}, \quad \alpha > 0,\ \gamma > -1,\ x > 0. \\ &3.\ \ J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0,\ m-1 < \alpha \leq m. \end{aligned} \tag{2.3}$$

3. Homotopy analysis method

In this section, we employ the homotopy analysis method [23] to the discussed problem. To show the basic idea, let us consider the following system of fractional algebraic–differential equations

$$\begin{aligned} &D_*^{\alpha_i} x_i(t) = f_i(t, x_1, \ldots, x_n, x_1', x_2', \ldots, x_n'), \quad i = 1, 2, 3, \ldots, n-1,\ t \geq 0,\ 0 < \alpha_i \leq 1, \\ &0 = g(t, x_1, \ldots, x_n), \end{aligned} \tag{1}$$

subject to the initial conditions

$$x_i(0) = a_i, \quad i = 1, 2, 3, \ldots, n. \tag{2}$$

By means of generalizing the traditional homotopy method, we can construct the so-called zero-order deformation equations

$$\begin{aligned} &(1-q)\mathcal{L}_i[\phi_i(t,q) - x_{i0}(t)] = q\,\hbar_i\, H_i(t)[D_*^{\alpha_i}\phi_i(t,q) - f_i(t, \phi_1(t,q), \phi_2(t,q), \ldots, \phi_n(t,q)), \\ &\quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t,q), \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t,q), \ldots, \frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t,q)], \quad i = 1, 2, \ldots, n-1, \\ &(1-q)\,[\phi_n(t,q) - x_{n0}(t)] = -q\,\hbar_n\, H_n(t)[g(t, \phi_1(t,q), \phi_2(t,q), \ldots, \phi_n(t,q))], \end{aligned} \tag{3}$$

subject to the initial conditions

$$\phi_i(0,q) = a_i, \quad i = 1, 2, 3, \ldots, n, \tag{4}$$

where $q \in [0, 1]$ is an embedding parameter, $\mathcal{L}_i$ are auxiliary linear operators satisfying $\mathcal{L}_i(0) = 0$, $x_{i0}(t)$ are initial guesses satisfy the initial conditions (2), $\hbar_i \neq 0$ are auxiliary parameters, $H_i(t)$ are auxiliary functions and $\phi_i(t,q)$ are unknown functions. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$, and $q = 1$, we get

$$\phi_i\,(t, 0) = x_{i0}(t), \qquad \phi_i\,(t, 1) = x_i(t), \quad i = 1, 2, 3, \ldots, n, \tag{5}$$

respectively. Thus, as q increasing from 0 to 1, the solutions $\phi_i\,(t,q)$ varies from the initial guesses $x_{i0}(t)$ to the solutions $x_i(t)$. Expanding $\phi_i\,(t,q)$ in Taylor series with respect to q, one has

$$\phi_i(t,q) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t) q^m, \quad i = 1, 2, 3, \ldots, n, \tag{6}$$

where

$$x_{im}(t) = \frac{1}{m!} \frac{\partial^m \phi_i(t,q)}{\partial q^m}\bigg|_{q=0}, \quad i = 1, 2, 3, \ldots, n. \tag{7}$$

If the auxiliary linear operators $\mathcal{L}_i$, the auxiliary parameters $\hbar_i$, the auxiliary functions $H_i(t)$ and the initial guesses $x_{i0}(t)$, are so properly chosen, then the series (6) converges at $q = 1$, one has

$$x_i(t) = x_{i0}(t) + \sum_{m=1}^{\infty} x_{im}(t), \quad i = 1, 2, 3, \ldots, n. \tag{8}$$

Define the vectors

$$\vec{x}_{im} = \{x_{i0}(t), x_{i1}(t), x_{i2}(t), \ldots, x_{im}(t)\}, \quad i = 1, 2, 3, \ldots, n. \tag{9}$$

Differentiating Eq. (3) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$ we have the so-called mth-order deformation equations

$$\begin{aligned} &\mathcal{L}_i[x_{im}(t) - \chi_m x_{i(m-1)}(t)] = \hbar_i H_i(t) R_{im}(\vec{x}_{i(m-1)}(t)), \quad i = 1, 2, 3, \ldots, n-1, \\ &x_{nm}(t) = \chi_m x_{n(m-1)}(t) + \hbar_n H_n(t) R_{nm}(\vec{x}_{n(m-1)}(t)), \end{aligned} \tag{10}$$

subject to the initial conditions

$$x_{im}(0) = 0, \quad i = 1, 2, 3, \ldots, n, \; m = 1, 2, 3, \ldots, \tag{11}$$

where

$$\begin{aligned} R_{im}(\vec{x}_{i(m-1)}) &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} \; [D_*^{\alpha_i} \phi_i(t, q) - f_i(t, \phi_1(t, q), \phi_2(t, q), \ldots, \phi_n(t, q)), \\ &\quad \frac{\mathrm{d}}{\mathrm{d}t}\phi_1(t, q), \frac{\mathrm{d}}{\mathrm{d}t}\phi_2(t, q), \ldots, \frac{\mathrm{d}}{\mathrm{d}t}\phi_n(t, q)]\Big|_{q=0}, \quad i = 1, 2, 3, \ldots, n-1, \\ R_{nm}(\vec{x}_{n(m-1)}) &= -\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} [g(t, \phi_1(t, q), \phi_2(t, q), \ldots, \phi_n(t, q))]\Big|_{q=0}, \end{aligned} \tag{12}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{13}$$

Define $\mathcal{L}_i^*$ to be an operators such that

$$\mathcal{L}_i^* \mathcal{L}_i[x_i(t)] = x_i(t) + K_i(t), \quad \mathcal{L}_i \mathcal{L}_i^* = I, \; i = 1, 2, 3, \ldots, n-1, \tag{14}$$

where I is the identity operator. For the special case, we put $H_i(t) = 1, i = 1, 2, \ldots, n$, and so we have the following mth-order deformation equations

$$\begin{aligned} &x_{im}(t) = \chi_m x_{i(m-1)}(t) + \hbar_i \mathcal{L}_i^*[R_{im}(\vec{x}_{i(m-1)}(t))] + K_i(t), \quad i = 1, 2, 3, \ldots, n-1, \\ &x_{nm}(t) = \chi_m x_{n(m-1)}(t) + \hbar_n R_{nm}(\vec{x}_{n(m-1)}(t)). \end{aligned} \tag{15}$$

4. Numerical results

To demonstrate the effectiveness of the method, we consider the following systems of fractional algebraic–differential equations

Example 4.1. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} &D_*^{\alpha} x(t) - t y'(t) + x(t) - (1+t) y(t) = 0, \quad 0 < \alpha \le 1, \\ &y(t) - \sin t = 0, \end{aligned} \tag{16}$$

subject to the initial conditions

$$x(0) = 1, \qquad y(0) = 0. \tag{17}$$

For the special case when $\alpha = 1$ the exact solution is [12]

$$x(t) = \mathrm{e}^{-t} + t \sin t, \qquad y(t) = \sin t. \tag{18}$$

We start with initial approximations

$$x_0(t) = 1, \qquad y_0(t) = 0, \tag{19}$$

and linear operator

$$\mathcal{L} = D_*^{\alpha}, \qquad \mathcal{L}^* = J^{\alpha}. \tag{20}$$

According to the formula (12), we can construct the homotopy as follows

$$\begin{aligned} &R_{1m}(\vec{x}_{m-1}(t)) = D_*^{\alpha} x_{m-1}(t) - t y'_{m-1}(t) + x_{m-1}(t) - (1+t) y_{m-1}(t), \\ &R_{2m}(\vec{y}_{m-1}(t)) = y_{m-1}(t) - (1 - \chi_m) \sin t, \end{aligned} \tag{21}$$

and the mth-order deformation equations for $m \ge 1$ become

$$\begin{aligned} &x_m(t) = \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha}[R_{1m}(\vec{x}_{m-1}(t))], \\ &y_m(t) = \chi_m y_{m-1}(t) + \hbar_2 R_{2m}(\vec{y}_{m-1}(t)). \end{aligned} \tag{22}$$

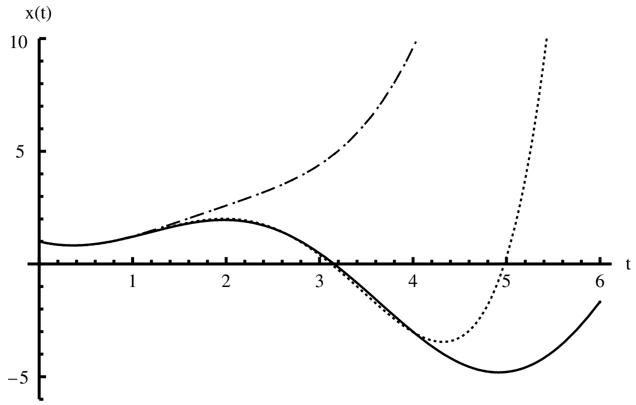

Fig. 1. Plots of solution of system (16) when $\alpha = 1$: Solid line: exact solution, dash–dotted line: $\hbar_1 = \hbar_2 = -1$, dotted line: $\hbar_1 = -0.6$, $\hbar_2 = -1$.

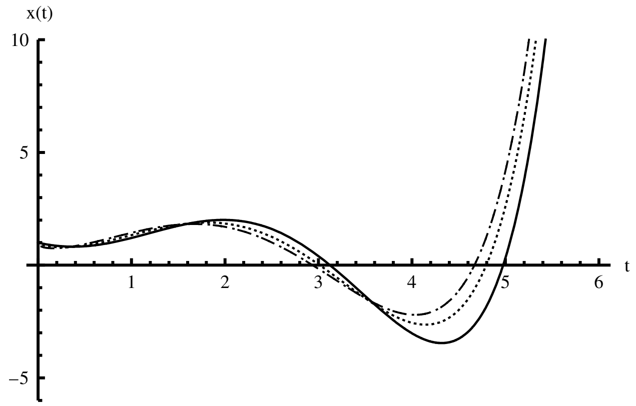

Fig. 2. Plots of solution of system (16) when $\hbar_1 = -0.6$, $\hbar_2 = -1$ Solid line: $\alpha = 1$, dotted line: $\alpha = 0.75$, dash–dotted line: $\alpha = 0.5$.

Using formula (22), we can obtain the following series solution

$$
\begin{aligned}
x(t) &= 1 + \frac{\hbar_1^4 + 4\hbar_1^3 + 6\hbar_1^2 + 4\hbar_1}{\Gamma(1+\alpha)} t^{\alpha} + \frac{2\hbar_1\hbar_2(\hbar_1^2 + \hbar_2^2 + \hbar_1\hbar_2 + 4\hbar_1 + 4\hbar_2 + 6)}{\Gamma(2+\alpha)} t^{\alpha+1} \\
&\quad + \left[\frac{\hbar_1^2(3\hbar_1^2 + 8\hbar_1 + 3)}{\Gamma(1+2\alpha)} + \frac{3\hbar_1^2 4^{-\alpha}\sqrt{\pi}}{\Gamma(1+\alpha)\Gamma(\frac{1}{2}+\alpha)} \right] t^{2\alpha} \\
&\quad + \frac{2\hbar_1\hbar_2(\hbar_1^2 + \hbar_2^2 + \hbar_1\hbar_2 + 4\hbar_1 + 4\hbar_2 + 6)}{\Gamma(3+\alpha)} t^{\alpha+2} + \frac{2\hbar_1^2\hbar_2(2\hbar_1 + \hbar_2 + 4)}{\Gamma(2+2\alpha)} t^{2\alpha+1} \\
&\quad + \frac{\hbar_1^3(3\hbar_1 + 4)}{\Gamma(1+3\alpha)} t^{3\alpha} - \frac{4\hbar_1\hbar_2(\hbar_1 + \hbar_2 + 3)}{\Gamma(4+\alpha)} t^{\alpha+3} + \frac{2\hbar_1^2\hbar_2(2\hbar_1 + \hbar_2 + 4)}{\Gamma(3+2\alpha)} t^{2\alpha+2} \\
&\quad + \frac{2\hbar_1^3\hbar_2}{\Gamma(2+3\alpha)} t^{3\alpha+1} + \frac{\hbar_1^4}{\Gamma(1+4\alpha)} t^{4\alpha} + \cdots, \\
y(t) &= -\hbar_2(\hbar_2^4 + 5\hbar_2^3 + 10\hbar_2^2 + 10\hbar_2 + 5)\sin t.
\end{aligned}
\tag{23}
$$

If we set $\hbar_1 = \hbar_2 = -1$, and $\alpha = 1$ in (23), then we obtain the same solutions given by Celik, Bayram and Yeloglu [12] using Adomian decomposition method. However, the results given by the Adomian decomposition method converge to the corresponding numerical solutions in a rather small region, as shown in Fig. 1. But different from this method, the homotopy analysis method gives a greater region of convergence with the exact solution, by choosing proper values for the auxiliary parameters $\hbar_1$ and $\hbar_2$ and by using a suitable auxiliary liner operator $\mathcal{L} = D_*^{\alpha}$. Fig. 2 shows the HAM approximate solutions for various values of α which have the same trajectories.

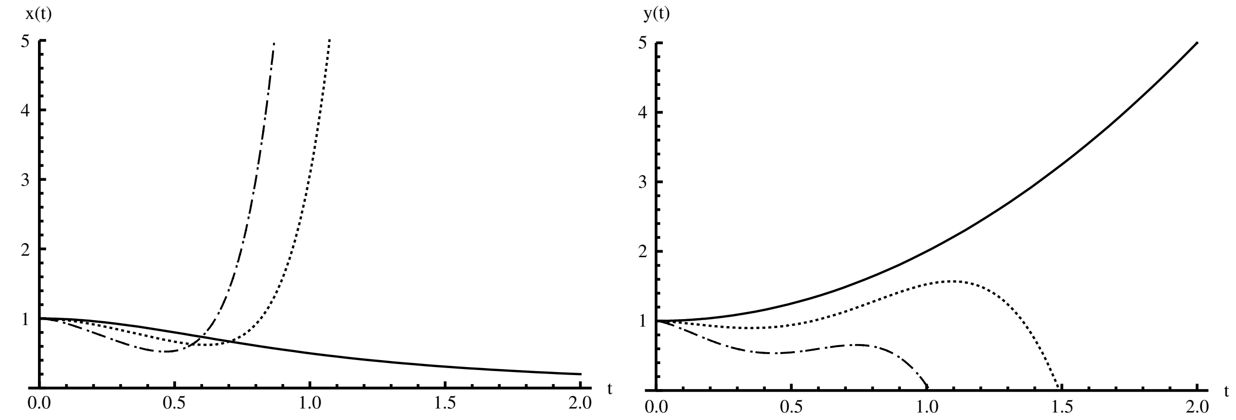

Fig. 3. Plots of solution of system (24) when $\hbar_1 = \hbar_2 = -1$ Solid line: $\alpha = 1$, dotted line: $\alpha = 0.75$, dash–dotted line: $\alpha = 0.5$, dashed line: $\alpha = 0.25$.

Example 4.2. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} &D_*^{\alpha}x(t) = -2tx^2(t) + x(t)y(t) - 1, \quad 0 < \alpha \le 1, \\ &y(t) = x(t)y(t) + t^2, \end{aligned} \tag{24}$$

subject to the initial conditions

$$x(0) = y(0) = 1. \tag{25}$$

For the special case when $\alpha = 1$ the exact solution is

$$x(t) = \frac{1}{1+t^2}, \qquad y(t) = t^2 + 1. \tag{26}$$

To solve the above problem using HAM, we choose the following initial approximations

$$x_0(t) = 1, \qquad y_0(t) = 1, \tag{27}$$

and the linear operator (20). With reference to the formula (12) we can construct the homotopy as follows

$$\begin{aligned} R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha}x_{m-1}(t) + 2t\sum_{i=0}^{m-1} x_i(t)x_{m-1-i}(t) - \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) + (1-\chi_m), \\ R_{2m}(\vec{y}_{m-1}(t)) &= y_{m-1}(t) - \sum_{i=0}^{m-1} x_i(t)y_{m-1-i}(t) - (1-\chi_m)t^2, \end{aligned} \tag{28}$$

and by using the mth-order deformation Eq. (22), we obtain the following series solution

$$\begin{aligned} x(t) &= 1 + \frac{2\,\hbar_1(\hbar_1+2)}{\Gamma(2+\alpha)}t^{\alpha+1} - \frac{2\,\hbar_1^2}{\Gamma(2+2\alpha)}t^{2\alpha+1} + \frac{2\,\hbar_1\,\hbar_2}{\Gamma(3+\alpha)}t^{\alpha+2} + \frac{8\,\hbar_1^2\,\Gamma(3+\alpha)}{\Gamma(2+\alpha)\Gamma(3+2\alpha)}t^{2\alpha+2} + \cdots, \\ y(t) &= 1 - 2\,\hbar_2\,t^2 - \frac{2\,\hbar_1\,\hbar_2}{\Gamma(2+\alpha)}t^{\alpha+1} + \cdots, \end{aligned} \tag{29}$$

Setting $\hbar_1 = \hbar_2 = -1$, and $\alpha = 1$ in (29), then we obtain the following series solution

$$\begin{aligned} x(t) &= 1 - t^2 + t^4 - \cdots = \sum_{k=0}^{\infty}(-1)^k t^{2k}, \\ y(t) &= 1 + t^2, \end{aligned} \tag{30}$$

which is the exact solution for system (24), this means that the best values for auxiliary parameters $\hbar_1$, $\hbar_1$ are $\hbar_1 = \hbar_2 = -1$. Fig. 3 shows the HAM approximate solutions for various values of α.

Example 4.3. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned} &D_*^{\alpha_1}x(t) - ty'(t) + t^2z'(t) + x(t) - (1+t)y(t) + (t^2+2t)z(t) = 0, \\ &D_*^{\alpha_2}y(t) - tz'(t) - y(t) + (t-1)z(t) = 0, \quad 0 < \alpha_1, \alpha_2 \le 1, \\ &z(t) - \sin t = 0, \end{aligned} \tag{31}$$

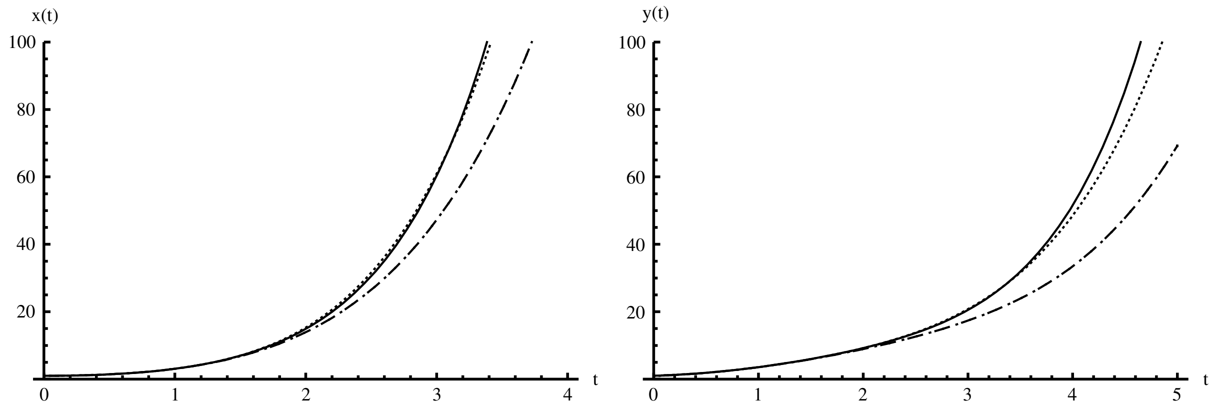

Fig. 4. Plots of solution of system (31) when $\alpha_1 = \alpha_2 = 1$ Solid line: exact solution, dash–dotted line: $\hbar_1 = \hbar_2 = \hbar_3 = -1$, dotted line: $\hbar_1 = -1, \hbar_2 = -1.35, \hbar_3 = -1$.

subject to the initial conditions

$$x(0) = 1, \qquad y(0) = 1, \qquad z(0) = 0. \tag{32}$$

When $\alpha_1 = \alpha_2 = 1$, the exact solution is [12]

$$x(t) = \mathrm{e}^{-t} + t\mathrm{e}^{t}, \qquad y(t) = \mathrm{e}^{t} + t\sin t, \qquad z(t) = \sin t. \tag{33}$$

To solve system (31) by means of homotopy analysis method, we start with initial approximations

$$x_0(t) = 1, \qquad y_0(t) = 1, \qquad z_0(t) = 0, \tag{34}$$

and linear operators

$$\mathcal{L} = D_*^{\alpha_i}, \qquad \mathcal{L}_i^* = J_i^{\alpha_i}, \quad i = 1, 2. \tag{35}$$

In view of the formula (12) we can construct the homotopy as follows

$$\begin{aligned} &R_{1m}(\vec{x}_{m-1}(t)) = D_*^{\alpha_1} x_{m-1}(t) - t y'_{m-1}(t) + t^2 z'_{m-1}(t) + x_{m-1}(t) - (t+1) y_{m-1}(t) + (t^2+2t) z_{m-1}(t),\\ &R_{2m}(\vec{y}_{m-1}(t)) = D_*^{\alpha_2} y_{m-1}(t) - t z'_{m-1}(t) - y_{m-1}(t) + (t-1) z_{m-1}(t),\\ &R_{3m}(\vec{z}_{m-1}(t)) = z_{m-1}(t) - (1-\chi_m)\sin t, \end{aligned} \tag{36}$$

and the mth-order deformation equations for $m \geq 1$ become

$$\begin{aligned} &x_m(t) = \chi_m x_{m-1}(t) + \hbar_1 J^{\alpha_1}[R_{1m}(\vec{x}_{m-1}(t))],\\ &y_m(t) = \chi_m y_{m-1}(t) + \hbar_2 J^{\alpha_2}[R_{2m}(\vec{y}_{m-1}(t))],\\ &z_m(t) = \chi_m z_{m-1}(t) + \hbar_3\, R_{3m}(\vec{z}_{m-1}(t)). \end{aligned} \tag{37}$$

Using the formula (37), we can obtain the following series solution

$$\begin{aligned} x(t) \;=\;& 1 - \frac{\hbar_1(\hbar_1^2+3\hbar_1+3)}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{\hbar_1\hbar_2(\hbar_1+\hbar_2+3)(1+\alpha_2)}{\Gamma(1+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2} - \frac{6\hbar_1\hbar_2(\hbar_1+\hbar_3+1)}{\Gamma(3+\alpha_1)} t^{\alpha_1+2}\\ &- \frac{\hbar_1^2(2\hbar_1+3)}{\Gamma(2+2\alpha_1)} t^{2\alpha_1+1} + \frac{\hbar_1\hbar_2}{\Gamma(2+\alpha_1+\alpha_2)}[(\hbar_1+\hbar_2+3)(1+\alpha_2) - 2(2+\alpha_2)] t^{\alpha_1+\alpha_2+1}\\ &+ \frac{\hbar_1^2\hbar_2(1+\alpha_2)}{\Gamma(1+2\alpha_1+\alpha_2)} t^{2\alpha_1+\alpha_2} - \frac{\hbar_1\hbar_2^2(1+2\alpha_2)}{\Gamma(1+\alpha_1+2\alpha_2)} t^{\alpha_1+2\alpha_2} + \cdots,\\ y(t) \;=\;& 1 - \frac{\hbar_2(\hbar_2^2+3\hbar_2+3)}{\Gamma(1+\alpha_2)} t^{\alpha_2} + \frac{2\hbar_2\hbar_3(\hbar_2+\hbar_3+3)}{\Gamma(2+\alpha_2)} t^{\alpha_2+1} + \hbar_2^2\left[\frac{(1+2\hbar_2)}{\Gamma(1+2\alpha_2)}\right.\\ &\left.+ \frac{2\sqrt{\pi}\,4^{-\alpha_2}}{\Gamma(1+\alpha_2)\Gamma(\frac{1}{2}+\alpha_2)}\right] t^{2\alpha_2} - \frac{2\hbar_2\hbar_3(\hbar_2+\hbar_3+3)}{\Gamma(3+\alpha_2)} t^{\alpha_2+2} - \frac{2\hbar_2^2\hbar_3}{\Gamma(2+2\alpha_2)} t^{2\alpha_2+1} - \frac{\hbar_2^3}{\Gamma(1+3\alpha_2)} t^{3\alpha_2} + \cdots,\\ z(t) \;=\;& -\hbar_3(\hbar_3^4+5\hbar_3^3+10\hbar_3^2+10\hbar_3+5)\sin t. \end{aligned} \tag{38}$$

Setting $\hbar_1 = \hbar_2 = \hbar_3 = -1$, and $\alpha_1 = \alpha_2 = 1$ in (38), then we obtain the same solutions given by Celik, Bayram and Yeloglu [12] using Adomian decomposition method. Similarly as in Example (1), Fig. 4 shows that choosing auxiliary

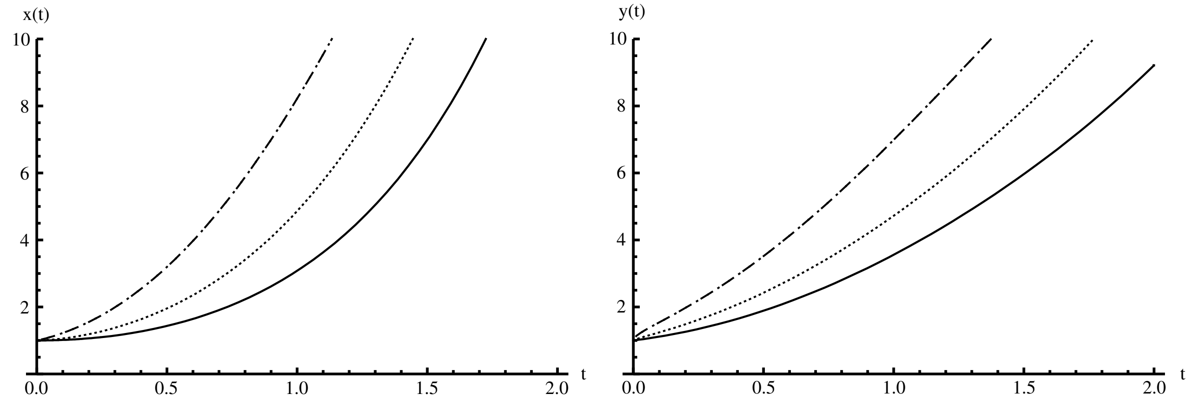

Fig. 5. Plots of solution of system (31) when $\hbar_1 = -1$, $\hbar_2 = -1.35$, $\hbar_3 = -1$ Solid line: $\alpha_1 = \alpha_2 = 1$, dotted line: $\alpha_1 = \alpha_2 = 0.75$, dash–dotted line: $\alpha_1 = \alpha_2 = 0.5$.

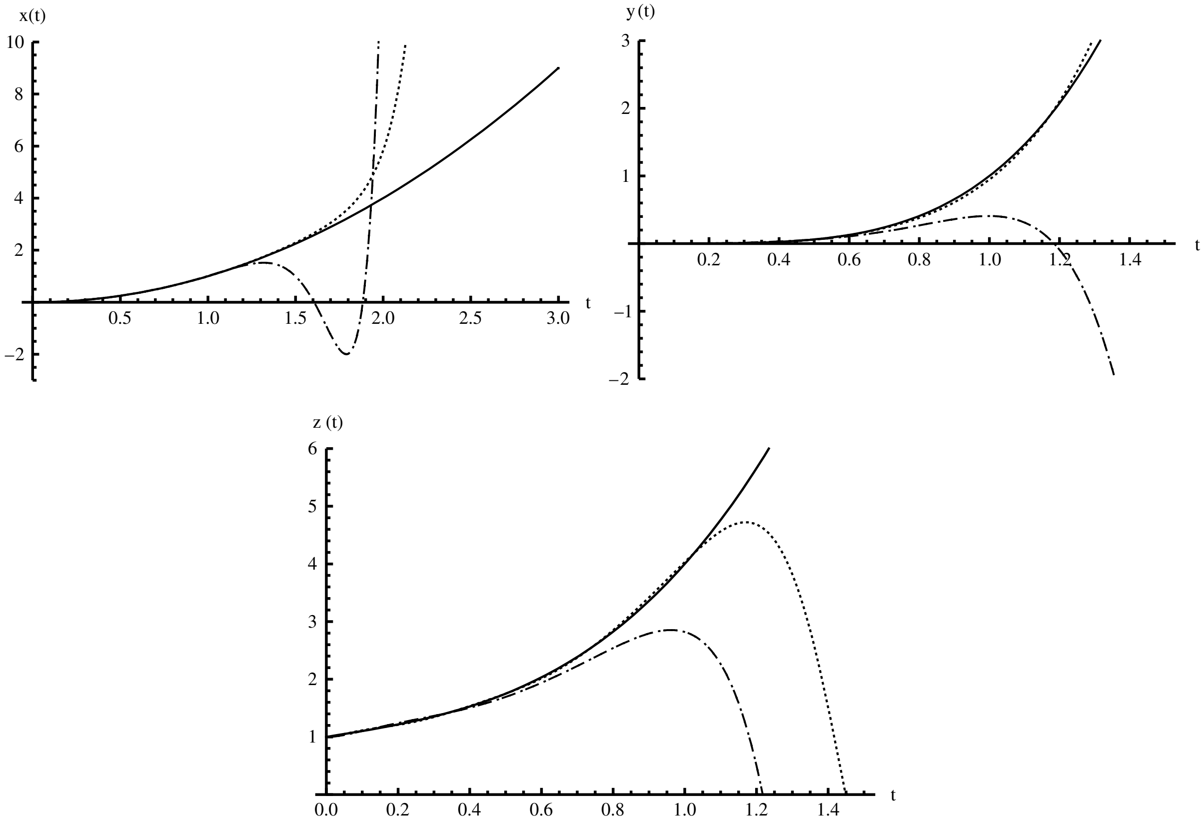

Fig. 6. Plots of solution of system (31) when $\alpha_1 = \alpha_2 = 1$. Solid line: exact solution, dash–dotted line: $\hbar_1 = \hbar_2 = \hbar_3 = -1$, dotted line: $\hbar_1 = -0.6$, $\hbar_2 = -0.9$, , $\hbar_3 = -2$.

linear operator (35) and using the values of auxiliary parameters $\hbar_1 = -1$, $\hbar_2 = -1.35$, $\hbar_3 = -1$, the HAM gives greater convergence with the exact solutions. Fig. 5 shows the HAM approximate solutions for various values of α_1, α_2.

Example 4.4. Consider the following system of fractional algebraic–differential equations

$$\begin{aligned}
&D_*^{\alpha_1} x(t) - x^2(t) + y(t) - 2t = 0, \\
&D_*^{\alpha_2} y(t) - 2z(t) + 2(t+1) = 0, \quad 0 < \alpha_1, \alpha_2 \le 1, \\
&tz(t) + y(t) - x(t) - 3x^2(t) - t = 0,
\end{aligned} \tag{39}$$

subject to the initial conditions

$$x(0) = y(0) = 0, \qquad z(0) = 1. \tag{40}$$

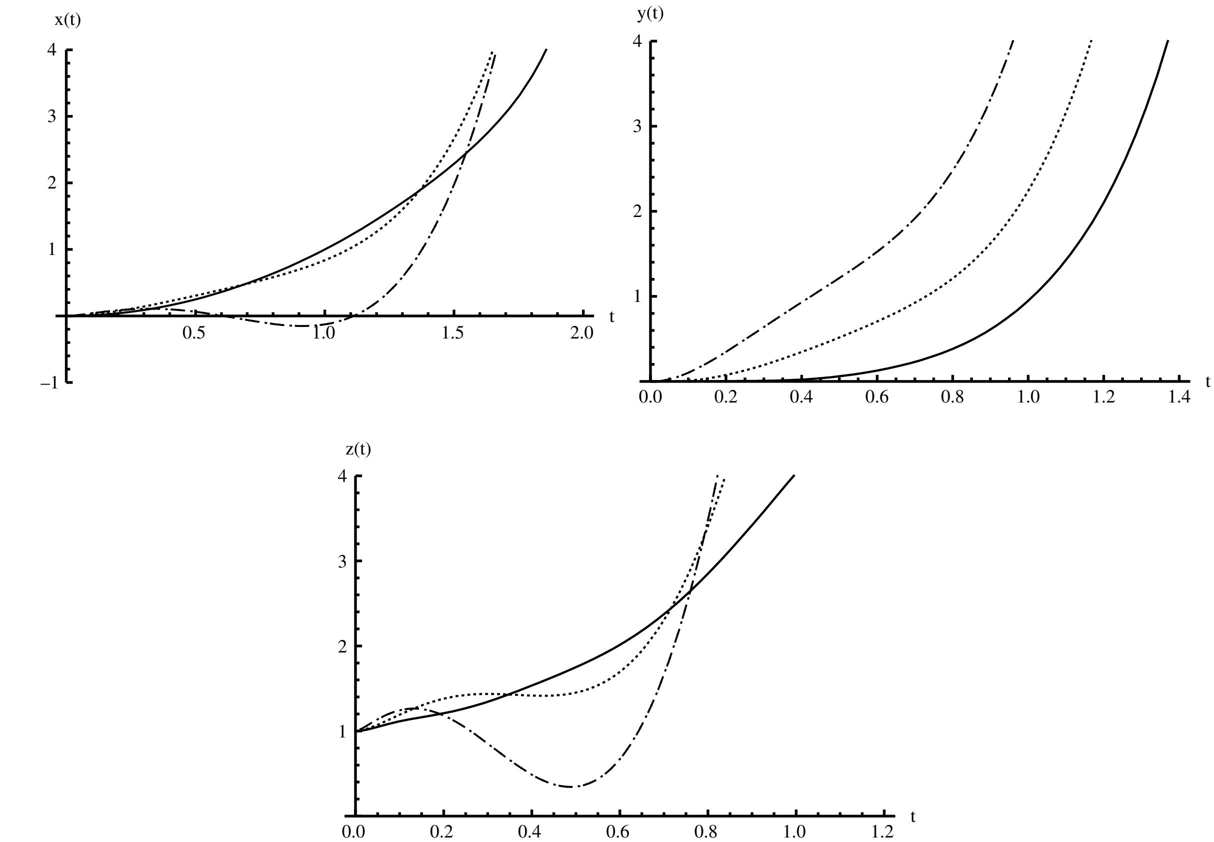

Fig. 7. Plots of solution of system (31) when $\hbar_1 = -0.6$, $\hbar_2 = -0.9$, $\hbar_3 = -2$. Solid line: $\alpha_1 = \alpha_2 = 1$, dotted line: $\alpha_1 = \alpha_2 = 0.75$, dash–dotted line: $\alpha_1 = \alpha_2 = 0.5$.

For the special case when $\alpha_1 = \alpha_2 = 1$ the exact solution is

$$x(t) = t^2, \qquad y(t) = t^4, \qquad z(t) = 2t^3 + t + 1. \tag{41}$$

If we use the linear operator (35) and the following initial approximations

$$x_0(t) = 0, \qquad y_0(t) = 0, \qquad z_0(t) = 1, \tag{42}$$

and according to the formula (12) we can construct the homotopy as follows

$$\begin{aligned}
R_{1m}(\vec{x}_{m-1}(t)) &= D_*^{\alpha_1} x_{m-1}(t) - \sum_{i=0}^{m-1} x_i(t) x_{m-1-i}(t) + y_{m-1}(t) - 2t(1-\chi_m),\\
R_{2m}(\vec{y}_{m-1}(t)) &= D_*^{\alpha_2} y_{m-1}(t) - 2z_{m-1}(t) + 2(t+1)(1-\chi_m),\\
R_{3m}(\vec{z}_{m-1}(t)) &= t z_{m-1}(t) + y_{m-1}(t) - x_{m-1}(t) - 3\sum_{i=0}^{m-1} x_i(t) x_{m-1-i}(t) - t(1-\chi_m),
\end{aligned} \tag{43}$$

In view of the mth-order deformation Eq. (37), we can obtain the following series solution

$$\begin{aligned}
x(t) &= -\frac{2\hbar_1(\hbar_1^2+3\hbar_1+3)}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{2\hbar_1\hbar_2(\hbar_1+\hbar_2+3)}{\Gamma(2+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+1} - \frac{4\hbar_1^3\,\Gamma(3+2\alpha_1)}{\Gamma(2+\alpha_1)^2\Gamma(3+3\alpha_1)} t^{3\alpha_1+2} + \cdots,\\
y(t) &= \frac{2\hbar_2(\hbar_2^2+3\hbar_2+3)}{\Gamma(2+\alpha_2)} t^{\alpha_2+1} - \frac{4\hbar_1\hbar_2\hbar_3}{\Gamma(2+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+1} - \frac{4\hbar_2^2\hbar_3}{\Gamma(3+2\alpha_2)} t^{2\alpha_2+1} + \cdots,\\
z(t) &= 1 + \frac{2\hbar_1\hbar_3(\hbar_1+3)}{\Gamma(2+\alpha_1)} t^{\alpha_1+1} + \frac{2\hbar_2\hbar_3^2}{\Gamma(2+\alpha_2)} t^{\alpha_2+2} - \frac{12\hbar_1^2\hbar_3}{\Gamma(2+\alpha_1)^2} t^{2\alpha_1+2} - \frac{2\hbar_1\hbar_2\hbar_3}{\Gamma(2+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2+1} + \cdots.
\end{aligned} \tag{44}$$

It is clear that the numerical results obtained using HAM is in good agreement with the exact solutions. This is illustrated by Fig. 6. Fig. 7 shows the HAM approximate solutions for various values of α_1 and α_2.

5. Conclusions

The purpose of this paper is to construct the homotopy analysis method to systems of algebraic–differential equations of fractional order. A comparison with the results of other numerical method such as Adomian decomposition method indicates that the HAM is accurate, fast and reliable for such problems. There are some important points to make here. First, we have great freedom to choose the auxiliary parameters $\hbar_i$ and the auxiliary linear operators $\mathcal{L}_i$. Second, the homotopy analysis method was shown to be a simple, yet powerful analytic numeric scheme for handling systems of fractional order. Finally, generally speaking; the proposed approach can be further implemented to solve other nonlinear problems in the fractional calculus field.

References

[1] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997, pp. 291–348.
[2] A. Carpinteri, F. Mainardi, Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien, New York, 1997.
[3] R. Gorenflo, F. Mainardi, Fractional calculus: Integral and differential equations of fractional order, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer, New York, 1997, pp. 223–276.
[4] U.M. Ascher, L.R. Petzold, Projected implicit Runge Kutta methods for differential–algebraic equations, SIAM J. Numer. Anal. 28 (1991) 1097–1120.
[5] N. Guzel, M. Bayram, On the numerical solution of differential–algebraic equations with index-3, Appl. Math. Comput. 175 (2) (2006) 1320–1331.
[6] N. Guzel, M. Bayram, Numerical solution of differential–algebraic equations with index-2, Appl. Math. Comput. 174 (2) (2006) 1279–1289.
[7] W. Wang, An algorithm for solving DAEs with mechanization, Appl. Math. Comput. 167 (2) (2005) 1350–1372.
[8] E. Çelik, M. Bayram, The numerical solution of physical problems modeled as a systems of differential–algebraic equations (DAEs), J. Franklin Inst. 342 (1) (2005) 1–6.
[9] S. Momani, Z. Odibat, Numerical approach to differential equations of fractional order, J. Comput. Appl. Math. 207 (2007) 96–110.
[10] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A 365 (2007) 345–350.
[11] Z. Odibat, S. Momani, Modified homotopy perturbation method: Application to quadratic Riccati differential equation of fractional order, Chaos, Solitons Fractals 36 (1) (2006) 167–174.
[12] E. Celika, M. Bayramb, T. Yeloglu, Solution of differential–algebraic equations (DAE's) by Adomian decomposition method, Internat. J. Pure Appl. Math. Sci. 3 (1) (2006) 93–100.
[13] M.M. Hosseini, Adomain decomposition method for solution of differential–algebraic equations, Appl. Math. comput. 197 (2006) 495–501.
[14] S. Momani, K. Al-Khaled, Numerical solutions for systems of fractional differential equations by the decomposition method, Appl. Math. comput. 162 (3) (2005) 1351–1365.
[15] H. Jafari, V.D. Gejji, Solving a system of nonlinear fractional differential equations using Adomaain decomposition, Appl. Math. comput. 196 (2006) 644–651.
[16] D. Lensic, The decomposition method for initial value problems, Appl. Math. Comput. 181 (2006) 206–213.
[17] D. Lensic, The decomposition method for Cauchy advection–diffusion problems, Appl. Math. Comput. 49 (4) (2005) 525–537.
[18] V. Daftardar-Gejji, H. Jafari, Adomian decomposition: A tool for solving a system of fractional differential equations, J. Math. Anal. Appl. 301 (2) (2005) 508–518.
[19] Z. Odibat, S. Momani, Application of variation iteration method to nonlinear differential equations of fractional order, Int. J. Nonlin. Sci. Numer. Simulat. 1 (7) (2006) 15–27.
[20] S. Momani, Z. Odibat, Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals 31 (2007) 1248–1255.
[21] S. Momani, Z. Odibat, Numerical approach to differential equations of fractional order, J. Comput. Appl. Math. 207 (2007) 96–110.
[22] S. Momani, Z. Odibat, Analytical approach to linear fractional partial differential equations arising in fluid mechanics, Phys. Lett. A 355 (2006) 271–279.
[23] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[24] J. Cang, Y. Tan, H. Xu, S. Liao, Series solutions of non-linear Riccati differential equations with fractional order, Chaos Solitons Fractals 40 (2009) 1–9.
[25] S.J. Liao, An approximate solution technique which does not depend upon small parameters: A special example, Int. J. Nonlinear Mech. 32 (1997) 815–822.
[26] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[27] S.J. Liao, A. Campo, Analytic solutions of the temperature distribution in Blasius viscous flow problems, J. Fluid Mech. 453 (2002) 411–425.
[28] Y.Y. Wu, S.J. Liao, Solving the one-loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method, Chaos Solitons and Fractals 23 (5) (2004) 1733–1740.
[29] S.J. Liao, Comparison between the homotopy analysis method and homotopy perturbation method, Appl. Math. Comput. 169 (2005) 1186–1194.
[30] S.J. Liao, An analytic approximate approach for free oscillations of self-excited systems, Int. J. Nonlinear Mech. 39 (2) (2004) 271–280.
[31] S.J. Liao, K.F. Cheung, Homotopy analysis of nonlinear progressive waves in deep water, J. Eng. Math. 45 (2) (2003) 105–116.
[32] S.J. Liao, An explicit analytic solution to the Thomas–Fermi equation, Appl. Math. Comput. 144 (2003) 495–506.
[33] L. Song, H. Zhang, Application of homotopy analysis method to fractional KdV—Burgers–Kuramoto equation, Phys. Lett. A 367 (2007) 88–94.
[34] S. Abbasbandy, The application of homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation, Phys. Lett. A 361 (6) (2007) 478–483.
[35] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A 360 (1) (2006) 109–113.
[36] H. Jafari, S. Seifi, Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 14 (5) (2009) 1962–1969.
[37] H. Jafari, S. Seifi, Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation, Commun. Nonlinear Sci. Numer. Simul. 14 (5) (2009) 2006–2012.