Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: http://www.tandfonline.com/loi/gdea20

A topological equivalence result for a family of nonlinear difference systems having generalized exponential dichotomy

Álvaro Castañeda & Gonzalo Robledo

JOURNAL OF DIFFERENCE EQUATIONS AND APPLICATIONS, 2016
http://dx.doi.org/10.1080/10236198.2016.1192161

Taylor & Francis
Taylor & Francis Group

A topological equivalence result for a family of nonlinear difference systems having generalized exponential dichotomy

Álvaro Castañeda and Gonzalo Robledo

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Chile, Santiago, Chile

ABSTRACT

We obtain sufficient conditions ensuring the topological and strong topological equivalence of two perturbed difference systems whose linear part has a property of generalized exponential dichotomy. When the exponential dichotomy is verified, we obtain a strongly and Hölder topological equivalence.

ARTICLE HISTORY
Received 10 July 2015
Accepted 14 May 2016

KEYWORDS
Topological equivalence; generalized exponential dichotomy; difference equations

AMS SUBJECT CLASSIFICATIONS
39A06; 39A12

1. Introduction

The purpose of this article is to find sufficient conditions ensuring the topological equivalence between the difference systems

$$x_{n+1} = A_n x_n + f(n, x_n), \tag{1.1}$$

$$y_{n+1} = A_n y_n + g(n, y_n), \tag{1.2}$$

where x_n and y_n are sequences of d-dimensional column vectors, $A_n \in \mathbb{R}^d \times \mathbb{R}^d$ and the functions $f, g\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy

(A1) A_n is bounded, nonsingular and

$$||A_n - I|| \leq M \quad \text{for any} \quad n \in \mathbb{Z},$$

where $||\cdot||$ is a matrix norm.

(A2) The functions f and g are in the set $\mathcal{S}$ defined by

$$\mathcal{S} = \Big\{\mathcal{U}\colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d\colon |\mathcal{U}(n, x_1) - \mathcal{U}(n, x_2)| \leq r_n|x_1 - x_2| \quad \text{for any} \quad n \in \mathbb{Z}\Big\},$$

where $|\cdot|$ is a vector norm and the sequence r_n is nonnegative.

The concept of topological equivalence has been introduced by Palmer [7] in the non-autonomous continuous case and extended to several cases as impulsive [1,6,13,22] and discrete [11,12,15,19,20].

CONTACT Gonzalo Robledo grobledo@u.uchile.cl

Definition 1: The systems (1.1) and (1.2) are topologically equivalent if there exists a map $H\colon \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d$ with the properties

(i) For each fixed $n\in\mathbb{Z}$, the map $u\mapsto H(n,u)$ is an homeomorphism of $\mathbb{R}^d$.
(ii) $H(n,u)-u$ is bounded in $\mathbb{Z}\times\mathbb{R}^d$.
(iii) If x_n is a solution of (1.1), then $H[n,x_n]$ is a solution of (1.2).

In addition, the map $u\mapsto L(n,u)=H^{-1}(n,u)$ has properties (i)–(iii) also.

Remark 1: Notice that the notation $H[n,x_n]$ is reserved to the special case when x_n is a solution of (1.1). On the other hand, the topological equivalence between (1.1) and the linear system

$$z_{n+1}=A_n z_n \tag{1.3}$$

can be defined in a similar way.

Remark 2: Any fundamental matrix of (1.3) will be denoted by W_n.

The following definitions have been introduced by Shi and Xiong [21] in the non-autonomous continuous case and we recall its discrete version

Definition 2: If the maps $u\mapsto H(n,u)$ and $u\mapsto L(n,u)$ satisfy properties (i)–(iii) of the previous definition and for any $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that $|\xi-\xi'|<\delta$ implies $|H(n,\xi)-H(n,\xi')|<\varepsilon$ and $|L(n,\xi)-L(n,\xi')|<\varepsilon$ for any $n\in\mathbb{Z}$, then we say that the systems (1.1) and (1.2) are strongly topologically equivalent.

Definition 3: If the maps $u\mapsto H(n,u)$ and $u\mapsto L(n,u)$ are Hölder continuous for all $n\in\mathbb{Z}$ and satisfy properties (i)–(iii) of Definition 1, then we say that the systems (1.1) and (1.2) are Hölder topologically equivalent.

1.1. Hyperbolicity

The topological equivalence problem is strongly related to the *hyperbolicity* properties of the system (1.3). Indeed, it is well known that if $A_n=A$ is constant without eigenvalues having unitary module, the origin is called *hyperbolic*. This autonomous hyperbolicity condition implies that the stable and unstable manifolds cross transversely, which allows to prove (see [8] for details) the existence of a local homeomorphism with the solutions of

$$y_{n+1}=Ay_n+f(y_n)\quad \text{where } f \text{ is } C^1.$$

Nevertheless, in the non-autonomous context, there no exists an univocal definition of *hyperbolicity*, but the splitting between stable and unstable solutions of (1.3) can be recovered by the properties of *dichotomy* [3,15,18], being the exponential dichotomy the most relevant.

Definition 4: The system (1.3) has an exponential dichotomy if there exists a projection P ($P^2=P$), some constants $K\geq 1$ and $\alpha>0$ such that W_n verifies:

$$\begin{cases} ||W_nPW_m^{-1}||\leq Ke^{-\alpha(n-m)} & \text{if } n\geq m\\ ||W_n(I-P)W_m^{-1}||\leq Ke^{-\alpha(m-n)} & \text{if } n<m. \end{cases} \tag{1.4}$$

Remark 3: The notation (1.4) was taken from [16, p.165] but other equivalent notations have been introduced in [11,17].

There exists several generalizations of the exponential dichotomy. In this paper, we will be interested in the following one:

Definition 5: The system (1.3) has a generalized exponential dichotomy if there exists a projection P ($P^2 = P$), a constant $K \geq 1$ and a non-negative sequence $\{a_n\}_{n\in\mathbb{Z}}$ satisfying

$$\sum_{j=p}^{q} a_j \to +\infty \quad \text{as} \quad q \to +\infty \quad \text{for fixed } p \in \mathbb{Z}, \tag{1.5}$$

$$\sum_{j=p}^{q} a_j \to +\infty \quad \text{as} \quad p \to -\infty \quad \text{for fixed } q \in \mathbb{Z} \tag{1.6}$$

such that

$$\begin{cases} ||W_n P W_m^{-1}|| \leq K \exp\left(-\sum\limits_{j=m}^{n} a_j\right) & \text{if } n \geq m \\ ||W_n (I-P) W_m^{-1}|| \leq K \exp\left(-\sum\limits_{j=n}^{m} a_j\right) & \text{if } n < m. \end{cases} \tag{1.7}$$

The concept of generalized exponential dichotomy was introduced by Martin [14] in the continuous case, we also refer the reader to [2,9,10]. It is interesting to observe that (1.5) and (1.6) are satisfied in the case $a_j = \alpha > 0$ for any $j \in \mathbb{Z}$, which leads to the classic definition of exponential dichotomy.

Remark 4: Notice that (1.4) and (1.7) can be viewed in terms of the Green function:

$$G(n,m) = \begin{cases} W_n P W_m^{-1} & \text{if } n \geq m \\ -W_n(I-P)W_m^{-1} & \text{if } n < m. \end{cases} \tag{1.8}$$

The following example shows a linear system having a generalized exponential dichotomy but not an exponential one: let us consider (1.3) with a matrix

$$A_n = \begin{bmatrix} b_n & 0 \\ 0 & 1/b_n \end{bmatrix},$$

where $0 < b_n = b_{-n} < 1$ for any $n \in \mathbb{Z}$, $b_n \to 1$ monotonically as $n \to +\infty$ and (1.5) and (1.6) are satisfied for $a_j = |\ln(b_j)|$.

Notice that this system has the generalized exponential dichotomy since

$$W_n = \begin{bmatrix} \prod\limits_{j=0}^{n-1} b_j & 0 \\ 0 & \prod\limits_{j=0}^{n-1} \dfrac{1}{b_j} \end{bmatrix} \quad \text{with} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

leads to (1.7) with $K = 1$ and $a_j = |\ln(b_j)|$. Nevertheless, let us observe that the system has not an exponential dichotomy. Indeed, otherwise, there exists $\alpha > 0$ such that

$$\sum_{k=m}^{n} |\ln(b_k)| \geq \alpha(n-m), \quad \text{for any} \quad n \geq m,$$

then, when considering $n = m + T$ (for some $T \in \mathbb{N}$), it follows that

$$\frac{1}{T}\sum_{k=m}^{m+T} |\ln(b_k)| \geq \alpha, \quad \text{for any} \quad m \in \mathbb{Z}.$$

Now, we obtain a contradiction by letting $m \to +\infty$.

1.2. Previous results and novelty of this work

As in the continuous case, the classical approach to cope with the discrete topological equivalence problem follows the lines of Palmer's work [7], which constructs the map H by using the Green function (1.8). Indeed, the seminal work of Papaschinopoulos [16] assumes that (1.3) satisfies an exponential dichotomy. Moreover, the work of Barreira and Valls assumes a block system (1.3) having a non-uniform exponential dichotomy (see [15, p.289] for details). The result of Reinfelds and Šteinberga [19, Sec.3] considers a generic Green function with some boundedness properties. To the best of our knowledge, the case of (1.3) verifying a generalized exponential dichotomy has not been considered.

In some works as [1,21], it has been pointed out that several continuous and impulsive results only prove that the map $x \mapsto H(\cdot, x)$ is a biunivocal correspondence while its continuity property is not proved in detail. Indeed, in the discrete case, the continuity and Hölder continuity is only addressed in [15].

On the other hand, we also notice that most works in the continuous case are mainly focused in the proof of the uniform and Hölder continuity while the topological equivalence is not addressed in depth.

In this context, the novelty of our work is to consider the generalized exponential dichotomy and to prove in detail the continuity of the map $x \mapsto H(n,x)$. The reading of this proof will illustrates the technical difficulties that arises in order to obtain stronger results as uniform continuity and suggest additional conditions on the nonlinearities f and g.

It is important to state that there exist other approaches to the topological equivalence, which are complementary to the classical one described above. For example, the time-scales and measure chains [23,25].

The article is organized as follows: Section 2 states the main results. Section 3 constructs the biunivocal correspondence $x \mapsto H(n,x)$, $x \mapsto H^{-1}(n,x)$. The proof of the main results is developed in the Section 4.

2. Main results

In order to state the main results, let us recall the following notation:

Definition 6: For any sequence g_n ($n \in \mathbb{Z}$), let us define the map

$$N(n,g) = \sum_{m=-\infty}^{n-1} K \exp\left(-\sum_{j=m+1}^{n} a_j\right) g_m + \sum_{m=n}^{\infty} K \exp\left(-\sum_{j=n}^{m+1} a_j\right) g_m,$$

where K and a_j are stated in Definition 5.

Theorem 1: *Suppose that (1.3) has a generalized exponential dichotomy and the functions f and g satisfy*

(H1) $|f(n,x)| \leq F_n$ *and* $|g(n,x)| \leq G_n$ *where* F_n *and* G_n *are nonnegative sequences.*
(H2) *There exists* $B > 0$ *such that the sequences* F_n *and* G_n *verify*

$$N(n, G+F) \leq B. \tag{2.1}$$

(H3) *There exists* $\theta \in (0,1)$ *such that the sequence* r_n *stated in* (A2) *satisfies*

$$N(n, r) \leq \theta < 1, \tag{2.2}$$

then (1.1) *and* (1.2) *are topologically equivalent.*

Remark 5: Notice that:

(H1) is a technical assumption which generalizes the case studied by Papaschinopoulos [16], where it is assumed that $|f(n,x)|$ and $|g(n,x)|$ are bounded by a small enough positive constant. We emphasize that F_n and G_n are not necessarily bounded sequences.
(H2) is introduced in order to ensure that if (1.3) is perturbed by linear combinations of f and g, then the corresponding perturbed systems have a unique bounded solution. Although F_n and G_n could be unbounded sequences, (H2) says that they must be dominated by terms $\exp\left(-\sum a_n\right)$ at $\pm\infty$.
(H3) is usual in the continuous topological equivalence literature and plays a key role in several intermediate steps as the proof of the continuity of the map $u \mapsto H(n,u)$ and the use of the Banach fixed point. As before, r_n is not necessarily a bounded sequence but must be dominated by terms $\exp\left(-\sum a_n\right)$ at $\pm\infty$.

Remark 6: The proof of Theorem 1 introduces

$$A(n,j) = \sum_{k=-\infty}^{n-1-j} K \exp\left(-\sum_{p=k+1}^{n} a_p\right)\Delta_k + \sum_{k=n+j+1}^{\infty} K \exp\left(-\sum_{p=n}^{k+1} a_p\right)\Delta_k, \tag{2.3}$$

$$\overline{A}(n,j) = \sum_{k=-\infty}^{n-1-j} K \exp\left(-\sum_{p=k+1}^{n} a_p\right)\overline{\Delta}_k + \sum_{k=n+j+1}^{\infty} K \exp\left(-\sum_{p=n}^{k+1} a_p\right)\overline{\Delta}_k, \tag{2.4}$$

where

$$\Delta_k = \sup_{|u|,|u'|\leq B, x,x' \in \mathbb{R}^d} |g(k, u+x) - g(k, u'+x') + f(k,x') - f(k,x)|,$$
$$\overline{\Delta}_k = \sup_{|v|,|v'|\leq B, y,y' \in \mathbb{R}^d} |f(k, v+y) - f(k, v'+y') + g(k,y) - g(k,y')|.$$

A direct consequence of (H1) and (H2) is that $A(n,j) \to 0$ and $\overline{A}(n,j) \to 0$ as $j \to +\infty$ for any $n \in \mathbb{Z}$.

The proof of Theorem 1 will illustrate the difficulties to obtain the uniform continuity of the map $x \mapsto H(n,x)$. Indeed, we will see that the elementary properties of $A(n,j)$ and r_n only allows to prove that δ could be dependent of n.

In order to obtain sufficient conditions ensuring strongly topological equivalence, it is necessary to introduce additional properties to $A(n,j)$, $\overline{A}(n,j)$ and $\{r_n\}$ described respectively in (2.3), (2.4) and **(A2)**. Indeed, we have the following result:

Theorem 2: *Suppose that* (1.3) *has a generalized exponential dichotomy and the functions f and g satisfy* (H1)–(H3) *and*

(H4) *$A(n,j) \to 0$ as $j \to +\infty$ uniformly with respect to $n \in \mathbb{Z}$.*
(H5) *$\overline{A}(n,j) \to 0$ as $j \to +\infty$ uniformly with respect to $n \in \mathbb{Z}$.*
(H6) *For any $L \in \mathbb{N}$ with $L > 1$, there exists $M_L > 0$ such that*

$$\sup_{n\in\mathbb{Z}} \frac{1}{L} \sum_{k=n-L}^{n+L} r_k \leq M_L, \tag{2.5}$$

then (1.1) *and* (1.2) *are strongly topologically equivalent.*

Remark 7: The proof of Theorem 2 follows some ideas of Jiang [9, Theorem 2], which considers a linear system having a generalized exponential dichotomy and introduces some conditions on the nonlinearity $f(t,\cdot)$, which inspired our assumptions (H4) and (H5). In addition, in [9], it is assumed that $f(t,\cdot)$ verifies $|f(t,u) - f(t,\tilde{u})| \leq h(t)|u - \tilde{u}|$, where h is a positive and locally integrable function. Nevertheless, a careful lecture of the proof shows that local integrability of $h(\cdot)$ only ensures the continuity of the map $x \mapsto H(t,x)$ while the uniform boundedness of $t \mapsto \int_t^{t+1} h(s)\,ds$ is necessary even in the case with exponential dichotomy as shown by Xia et al. [24]. Our assumption (H6) is motivated by this uniform boundedness property.

As stated above, if $a_n = \alpha > 0$, then (1.3) has a exponential dichotomy. In addition, if F_n, G_n and r_n are also positive constants (namely, F, G and r). In this case, we have the last result

Theorem 3: *Suppose that* (1.3) *has a exponential dichotomy and the functions f and g satisfy*

(D1) *$|f(n,x)| \leq F$ and $|g(n,x)| \leq G$ where F and G are nonnegative constants.*
(D2) *The functions f and g are in the set $\mathcal{S}'$ defined by*

$$\mathcal{S}' = \Big\{\mathcal{U}\colon \mathbb{Z}\times\mathbb{R}^d \to \mathbb{R}^d\colon |\mathcal{U}(n,x_1) - \mathcal{U}(n,x_2)| \leq r|x_1 - x_2| \quad \textit{for any} \quad n \in \mathbb{Z}\Big\},$$

where $r > 0$ is such that

$$\theta = Kr\frac{1+e^{-\alpha}}{1-e^{-\alpha}} < 1, \tag{2.6}$$

then (1.1) *and* (1.2) *are strongly topologically equivalent.*
Moreover, if the constant M defined in (A1) *satisfies $M + r < \alpha$, then* (1.1) *and* (1.2) *are Hölder topologically equivalent.*

This result is strongly inspired in the work of Shi and Xiong [21] developed in the continuous case.

3. Preliminar results

In this section, we construct the map $u \mapsto H(n,u)$ and we prove that is a biunivocal correspondence that sends solutions of (1.1) into solutions of (1.2). In addition, the inverse map $u \mapsto L(n,u)$ is constructed explicitly. This section can be seen as a discrete version of the results obtained by Chen and Xia [2]. Finally, in order to make the article self contained, we introduce some definitions and results about generalized dichotomy and its corresponding perturbed linear systems.

Lemma 1: *If (1.3) has a generalized exponential dichotomy, then the unique solution of (1.3) bounded on $\mathbb{Z}$ is $y_n = 0$.*

Proof: As in [4, p.11], it is easy to verify that (1.7) implies

$$||W_n P\xi|| \leq K \exp\left(-\sum_{j=m}^{n} a_j\right) ||W_m P\xi|| \text{ if } n \geq m$$
$$||W_n(I-P)\xi|| \leq K \exp\left(-\sum_{j=n}^{m} a_j\right) ||W_m(I-P)\xi|| \text{ if } n < m.$$

for any initial condition $\xi \in \mathbb{R}^d$. In addition, let us assume that the projection P has rank $k \leq d$.

The first inequality above is equivalent to

$$\frac{1}{K}\exp\left(\sum_{j=m}^{n} a_j\right) ||W_n P\xi|| \leq ||W_m P\xi|| \quad \text{if} \quad n \geq m.$$

By using (1.6), we can see that there exists a k-dimensional subspace of initial conditions leading to solutions tending to the infinite when $m \to -\infty$.

On the other hand, the second inequality is equivalent to

$$\frac{1}{K}\exp\left(\sum_{j=n}^{m} a_j\right) ||W_n(I-P)\xi|| \leq ||W_m(I-P)\xi|| \quad \text{if} \quad n < m.$$

As before, by (1.5), we can see that there exists a $(d-k)$-dimensional subspace of initial conditions leading to solutions tending to the infinite when $m \to +\infty$. In consequence, the unique bounded solution can be the trivial one. □

Lemma 2: *If (1.3) has a generalized exponential dichotomy and a sequence q_n verifies*

(E1) $\sup_{n\in\mathbb{Z}} |N(n,|q|)| < +\infty$,

then the system

$$z_{n+1} = A_n z_n + q_n \tag{3.1}$$

has a unique bounded solution given by

$$\hat{\phi}_n = \sum_{m=-\infty}^{\infty} G(n,m+1)q_m.$$

Proof: The proof has two steps:
Boundedness of $\hat{\phi}_n$: It is straightforward (see e.g. [5]) to see that $\hat{\phi}_n$ is solution of (3.1). In order to verify that $\hat{\phi}_n$ is bounded, notice that:

$$\begin{aligned}
|\hat{\phi}_n| &\leq \sum_{m=-\infty}^{n-1} |G(n,m+1)q_m| + \sum_{m=n}^{\infty} |G(n,m+1)q_m| \\
&= \sum_{m=-\infty}^{n-1} |W_n P W_{m+1}^{-1} q_m| + \sum_{m=n}^{\infty} |W_n (I-P) W_{m+1}^{-1} q_m| \\
&\leq \sum_{m=-\infty}^{n-1} K \exp\left(-\sum_{j=m+1}^{n} a_j\right) |q_n| + \sum_{m=n}^{\infty} K \exp\left(-\sum_{j=n}^{m+1} a_j\right) |q_n| \\
&= N(n,|q|)
\end{aligned}$$

and the boundedness follows from (E1).
Uniqueness of the bounded solution: As in [2] (continuous framework), let y_n be another bounded solution of (3.1). By variation of parameters (see e.g. [5, Th. 3.17]), we know that

$$\begin{aligned}
y_n &= W_n W_0^{-1} y_0 + \sum_{r=0}^{n-1} W_n W_{r+1}^{-1} q_r \\
&= W_n W_0^{-1} y_0 + \sum_{r=0}^{n-1} W_n P W_{r+1}^{-1} q_r + \sum_{r=0}^{n-1} W_n (I-P) W_{r+1}^{-1} q_r \\
&= W_n W_0^{-1} y_0 + \sum_{r=-\infty}^{n-1} W_n P W_{r+1}^{-1} q_r - \sum_{r=-\infty}^{-1} W_n P W_{r+1}^{-1} q_r \\
&\quad + \sum_{r=0}^{\infty} W_n (I-P) W_{r+1}^{-1} q_r - \sum_{r=n}^{\infty} W_n (I-P) W_{r+1}^{-1} q_r.
\end{aligned}$$

It is important to note that the expression above is well defined because

$$\begin{aligned}
\left|\sum_{r=-\infty}^{-1} W_n P W_{r+1}^{-1} q_r\right| &= \left|W_n W_0^{-1} \sum_{r=-\infty}^{-1} W_0 P W_{r+1}^{-1} q_r\right| \\
&\leq \left|W_n W_0^{-1}\right| \sum_{r=-\infty}^{-1} |W_0 P W_{r+1}^{-1} q_r| \\
&\leq \left|W_n W_0^{-1}\right| \sum_{r=-\infty}^{-1} K \exp\left(-\sum_{j=r}^{-1} a_j\right) |q_r| \\
&\leq |W_n W_0^{-1}| N(r,|q|)
\end{aligned}$$

and let us denote

$$\sum_{r=-\infty}^{-1} W_n P W_{r+1}^{-1} q_r = W_n W_0^{-1} y_1.$$

In a similar way, we can verify that

$$\sum_{r=n}^{\infty} W_n(I-P)W_{r+1}^{-1}q_r = W_nW_0^{-1}y_2.$$

Now, we can see that

$$y_n = W_nW_0^{-1}(y_0 - y_1 + y_2) + \sum_{r=-\infty}^{n-1} W_nPW_{r+1}^{-1}q_r - \sum_{r=n}^{\infty} W_n(I-P)W_{r+1}^{-1}q_r.$$

As y_n is a bounded solution of (3.1) and (E1) implies that

$$\sum_{r=-\infty}^{n-1} W_nPW_{r+1}^{-1}q_r - \sum_{r=n}^{\infty} W_n(I-P)W_{r+1}^{-1}q_r$$

is also bounded, it follows that $x_n = W_nW_0^{-1}(y_0 - y_1 + y_2)$ is a bounded solution of (1.3). Finally, Lemma 1 implies that $y_0 = y_1 - y_2$ and the uniqueness follows. □

Lemma 3: *If* (1.3) *has a generalized exponential dichotomy and the system*

$$z_{n+1} = A_nz_n + q(n, z_n) \tag{3.2}$$

is such that

$$|q(n,z)| \le Q_n \quad and \quad |q(n,z) - q(n,\tilde{z})| \le r_n|z - \tilde{z}|, \tag{3.3}$$

where Q_n and r_n satisfy

$$N(n,Q) \le \tilde{B} \quad and \quad N(n,r) \le \theta < 1, \tag{3.4}$$

then, there exists a unique bounded solution of (3.2).

Proof: *Existence:* Let us consider the sequence $\{\varphi^{(j)}\}_j$, recursively defined by

$$\varphi_{n+1}^{(j)} = A_n\varphi_n^{(j)} + q(n, \varphi_n^{(j-1)}),$$

where $\varphi^{(0)}$ is an arbitrary sequence in $\ell_\infty(\mathbb{Z})$ satisfying $|\varphi^{(0)}|_\infty \le \tilde{B}$.

By using Lemma 2 combined with the first inequalities of (3.3) and (3.4), we can see that $\varphi^{(j)}$ is the unique solution of the above system and verifies

$$\varphi_n^{(j)} = \sum_{k=-\infty}^{+\infty} G(n, k+1)q(k, \varphi_k^{(j-1)}),$$

with $|\varphi^{(j)}|_\infty \le \tilde{B}$ for any $j \in \mathbb{N}$.

On the other hand, the second inequalities of (3.3) and (3.4) imply that

$$|\varphi^{(j)} - \varphi^{(j-1)}|_\infty \le \theta|\varphi^{(j-1)} - \varphi^{(j-2)}|_\infty$$

with $\theta \in (0,1)$, and we can see that $\varphi^{(j)}$ is a Cauchy sequence in $\ell_\infty(\mathbb{Z})$. Now, letting $j \to +\infty$ in $\varphi_n^{(j)}$, it follows that

$$\varphi_n^* = \sum_{k=-\infty}^{+\infty} G(n,k+1)q(k,\varphi_k^*),$$

is a bounded solution of (3.2).

Uniqueness: Let y_n be another bounded solution of (3.2). By following the lines of the proof of Lemma 2 combined with (3.3) and (3.4), the reader can verify that

$$y_n = \sum_{k=-\infty}^{+\infty} G(n,k+1)q(k,y_k).$$

Finally, by using the second inequalities of (3.3) and (3.4), we have that

$$|\varphi^* - y|_\infty \leq \theta|\varphi^* - y|_\infty$$

and the uniqueness follows since $0 < \theta < 1$. □

Lemma 4: *Suppose that* (1.3) *has a generalized exponential dichotomy. If the systems* (1.1)–(1.2) *satisfy* (H1)–(H3) *and $x(n,m,\xi)$ is the solution of* (1.1) *with initial condition ξ at $n = m$, then the (m,ξ)-parameter dependent system*

$$w_{n+1} = A_n w_n - f(n, x(n,m,\xi)) + g(n, w_n + x(n,m,\xi)). \tag{3.5}$$

has a unique bounded solution $n \mapsto \chi(n;(m,\xi))$ with $|\chi(n;(m,\xi))|_\infty \leq B$.

Proof: By using (H1)–(H3) and Lemma 3 with $q(n,w_n) = -f(n,x(n,m,\xi)) + g(n,w_n + x(n,m,\xi))$, we know that the unique bounded solution of (3.5) is

$$\chi(n;(m,\xi)) = \sum_{k=-\infty}^{+\infty} G(n,k+1)\{g(k,\chi(k;(m,\xi)) + x_{k,m}(\xi)) - f(k,x_{k,m}(\xi))\}, \tag{3.6}$$

where $x_{k,m}(\xi) = x(k,m,\xi)$ and the lemma follows. □

Lemma 5: *Suppose that* (1.3) *has a generalized exponential dichotomy. If the systems* (1.1) *and* (1.2) *satisfy* (H1)–(H3) *and $y(n,m,\nu)$ is the solution of* (1.2) *with initial condition ν at $n = m$, then the (m,ν)-parameter dependent system*

$$z_{n+1} = A_n z_n + f(n, z_n + y(n,m,\nu)) - g(n, y(n,m,\nu)), \tag{3.7}$$

has a unique bounded solution $n \mapsto \vartheta(n;(m,\nu))$ with $|\vartheta(n;(m,\nu))|_\infty \leq B$.

Proof: As before, by using (H1)–(H3) and Lemma 3 with $q(n,z_n) = f(n,z_n + y(n,m,\nu)) - g(n,y(n,m,\nu))$, the unique bounded solution of (3.7) is

$$\vartheta(n;(m,\nu)) = \sum_{k=-\infty}^{+\infty} G(n,k+1)\{f(k,\vartheta(k;(m,\nu)) + y_{k,m}(\nu)) - g(k,y_{k,m}(\nu))\}, \tag{3.8}$$

where $y_{k,m}(\nu) = y(k, m, \nu)$. □

Remark 8: By uniqueness of the solution of (1.1), we know that $x(n, n, x(n, m, \xi)) = x(n, m, \xi)$, which implies that (3.5) is similar to

$$w_{n+1} = A_n w_n - f(n, x(n, n, x(n, m, \xi))) + g(n, w_n + x(n, n, x(n, m, \xi)))$$

and Lemma 4 implies that

$$\chi(n; (m, \xi)) = \chi(n; (n, x(n, m, \xi))). \tag{3.9}$$

In a similar way, it can be proved that

$$\vartheta(n; (m, \nu)) = \vartheta(n; (n, y(n, m, \nu))). \tag{3.10}$$

Lemma 6: *Suppose that* (1.3) *has a generalized exponential dichotomy. If the systems* (1.1) *and* (1.2) *satisfy* (H1)–(H3)*, then there exists a unique map* $H : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$*, which verifies the following properties*

(a) *$H(n, \xi) - \xi$ is bounded for any fixed $n \in \mathbb{Z}$ and $\xi \in \mathbb{R}^d$.*
(b) *If $x_n = x(n, m, \xi)$ is solution of* (1.1)*, then $H[n, x_n]$ is solution of* (1.2).

Proof: The proof will be divided in two steps:

Step i: Existence of H. We will prove that

$$H(n, \xi) = \xi + \chi(n; (n, \xi))$$

satisfy properties (a) and (b).
Indeed, by using (3.6) combined with (H1) and (H2), we obtain that $|H(n, \xi) - \xi| \le B$. On the other hand, we replace (n, ξ) by $\big(n, x(n, m, \xi)\big)$ and (3.9) implies

$$\begin{aligned} H[n, x(n, m, \xi)] &= x(n, m, \xi) + \chi\big(n; (n, x(n, m, \xi))\big) \\ &= x(n, m, \xi) + \chi\big(n; (m, \xi)\big) \end{aligned}$$

and the reader can verify easily that $H[n, x(n, m, \xi)]$ is solution of (1.2) since $n \mapsto \chi\big(n; (m, \xi)\big)$ is solution of (3.5).
Step ii: Uniqueness of H. Let $\tilde{H}$ be another map satisfying (a) and (b). Let us observe that $u_n = \tilde{H}[n, x_n] - x_n$ is also a bounded solution of (3.5), which implies by Lemma 4 that $\tilde{H}[n, x_n] - x_n = \chi\big(n; (m, \xi)\big)$ and the uniqueness follows.

□

Lemma 7: *Suppose that* (1.3) *has a generalized exponential dichotomy. If the systems* (1.1) *and* (1.2) *satisfy* (H1)–(H3)*, then there exists a unique map* $L : \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$*, which verifies the following properties*

(a) *$L(n, \nu) - \nu$ is bounded for any fixed $n \in \mathbb{Z}$ and $\nu \in \mathbb{R}^d$.*
(b) *If $y_n = y(n, m, \nu)$ is solution of* (1.2)*, then $L[n, y_n]$ is solution of* (1.1).

Proof: It can be proved analogously as the previous result that the map

$$L(n, \nu) = \nu + \vartheta(n; (n, \nu))$$

is the unique satisfying properties (a) and (b). □

Remark 9: By Lemma 6 combined with (3.6), we know that $H[n, x(n, m, \xi)]$ can be written as follows:

$$\begin{aligned} H[n, x(n, m, \xi)] = & \sum_{k=-\infty}^{+\infty} G(n, k+1) g(k, H[k, x(k, m, \xi)]) \\ & - \sum_{k=-\infty}^{+\infty} G(n, k+1) f(k, x(k, m, \xi)) + x(n, m, \xi). \end{aligned} \tag{3.11}$$

Similarly, by Lemma 7 combined with (3.8), we know that:

$$\begin{aligned} L[n, y(n, m, \nu)] = & \sum_{k=-\infty}^{+\infty} G(n, k+1) f(k, L[k, y(k, m, \nu)]) \\ & - \sum_{k=-\infty}^{+\infty} G(n, k+1) g(k, y(k, m, \nu)) + y(n, m, \nu). \end{aligned} \tag{3.12}$$

Lemma 8: *For any solution $x(n, m, \xi)$ of (1.1) and $y(n, m, \nu)$ of (1.2) and any $n \in \mathbb{Z}$, it follows that*

$$L[n, H[n, x(n, m, \xi)]] = x(n, m, \xi) \quad \textit{and} \quad H[n, L[n, y(n, m, \nu)]] = y(n, m, \nu).$$

Proof: By Lemma 6 and Remark 9, we know that (3.11) is solution of (1.2). Now, by Lemma 7, we also know that $L[n, H[n, x_n(\xi)]]$ is a solution of (1.1) that can be written as follows:

$$\begin{aligned} L[n, H[n, (n, m, \xi)]] &= V[n, x(n, m, \xi)] \\ &= \sum_{k=-\infty}^{+\infty} G(n, k+1) f(k, V[k, x(k, m, \xi)]) \\ &\quad - \sum_{k=-\infty}^{+\infty} G(n, k+1) g(k, H[k, x(k, m, \xi)]) + H[n, x(n, m, \xi)]. \end{aligned}$$

Now, by using (3.11) combined with (A2), we can deduce that

$$\begin{aligned} |V[n, x(n, m, \xi)] - x(n, m, \xi)| \leq & \sum_{k=-\infty}^{+\infty} |G(n, k+1)| \\ & |f(k, V[k, x(k, m, \xi)]) - f(k, x(k, m, \xi))| \\ \leq & \sum_{k=-\infty}^{+\infty} |G(n, k+1)| r_k |V[k, x(k, m, \xi)] - x(k, m, \xi)| \end{aligned}$$

and (H3) implies that

$$|L[n, H[n, x(n, m, \xi)]] - x(n, m, \xi)|_\infty \leq \theta |L[n, H[n, x(n, m, \xi)]] - x(n, m, \xi)|_\infty,$$

with $\theta \in (0, 1)$, which is equivalent to

$$L[n, H[n, x(n, m, \xi)]] = x(n, m, \xi). \tag{3.13}$$

In a similar way, the reader can verify that

$$H[n, L[n, y(n, m, \nu)]] = y(n, m, \nu). \tag{3.14}$$

□

Remark 10: The maps $\xi \mapsto H(n, \xi)$ and $\nu \mapsto L(n, \nu)$ defined by

$$\begin{aligned} H(n, \xi) &= \xi + \chi(n; (n, \xi)) \\ &= \xi + \sum_{k=-\infty}^{+\infty} G(n, k+1)\{g(k, \chi(k; (n, \xi)) + x_{k,n}(\xi)) - f(k, x_{k,n}(\xi))\}, \end{aligned}$$

and

$$\begin{aligned} L(n, \nu) &= \nu + \vartheta(n; (n, \nu)) \\ &= \nu + \sum_{k=-\infty}^{+\infty} G(n, k+1)\{f(k, \vartheta(k; (n, \nu)) + y_{k,n}(\nu)) - g(k, y_{k,n}(\nu))\}, \end{aligned}$$

satisfy properties (ii) and (iii) from Definition 1, which is consequence of Lemmas 6 and 7. In order to verify property (i), notice that if $n = m$ in the identities (3.13) and (3.14), we obtain that

$$L(n, H(n, \xi)) = \xi \quad \text{and} \quad H(n, L(n, \nu)) = \nu$$

for any fixed $n \in \mathbb{Z}$. These identities ensure that $H^{-1}(n, \cdot) = L(n, \cdot)$ for any fixed n. However, the continuity of both maps must be proved.

4. Proof of main results

As stated above, we will prove the continuity properties of the maps $\xi \mapsto H(n, \xi)$ and $\nu \mapsto L(n, \nu)$, for any fixed $n \in \mathbb{Z}$. In this context, we will use the following result of continuity with respect to the initial conditions:

Lemma 9: *Let $n \mapsto x(n, k, \xi)$ (resp. $n \mapsto x(n, k, \xi')$) the solution of (1.1) passing through ξ (resp. ξ') at $n = k$. Then, it follows that*

$$|x(n, k, \xi) - x(n, k, \xi')| \leq |\xi - \xi'| \exp\left(\sum_{p=k}^{n-1} (||A_p - I|| + r_p)\right) \quad \text{if} \quad n > k \tag{4.1}$$

and

$$|x(n,k,\xi)-x(n,k,\xi')| \leq |\xi-\xi'|\exp\left(\sum_{p=n}^{k-1}(||A_p-I||+r_p)\right) \quad \textit{if} \quad n<k. \tag{4.2}$$

Proof: We will prove only the case $n > k$, the other one can be done similarly. It is straightforward to see that

$$x(n,k,\xi)=\xi+\sum_{p=k}^{n-1}(A_p-I)x(p,k,\xi)+f(p,x(p,k,\xi)).$$

By using (A2), we have

$$|x(n,k,\xi)-x(n,k,\xi')| \leq |\xi-\xi'|+\sum_{p=k}^{n-1}(||A_p-I||+r_p)|x(p,k,\xi)-x(p,k,\xi')|.$$

Finally, by the discrete Gronwall's inequality [5, Lemma 4.32], we have (4.1). □

4.1. *Proof of Theorem 1*

We will give the proof only for the map H since the other one can be done analogously.

As the identity is a continuous map, we only need to prove that the map $\xi \mapsto \chi(n;(n,\xi))$ is continuous for any fixed n. Now, let us recall that $n \mapsto \chi(n;(m,\xi))$ is the unique bounded solution of (3.5), which is given by

$$\chi(n;(m,\xi))=\sum_{k=-\infty}^{+\infty}G(n,k+1)\{g(k,\chi(k;(m,\xi))+x_{k,m}(\xi))-f(k,x_{k,m}(\xi))\}.$$

Given a fixed $n\in\mathbb{Z}$, we will prove that for any $\varepsilon>0$ and $j\in\mathbb{N}$, there exists $\delta_j(\varepsilon,n)>0$ such that

$$|\chi_j(n;(n,\xi))-\chi_j(n;(n,\xi'))|<\varepsilon \quad \text{if} \quad |\xi-\xi'|<\delta_j. \tag{4.3}$$

The proof will be made by induction by considering an initial term

$$\chi_0(n;(n,\xi))=\chi_0(n;(n,\xi'))=\phi\in\ell_\infty(\mathbb{Z}) \quad \text{with} \quad |\phi|_\infty<B,$$

and supposing that (4.3) is verified for some j as inductive hypothesis. Now, we have that

$$\begin{aligned}\chi_{j+1}(n;(n,\xi))-\chi_{j+1}(n;(n,\xi')) &= \sum_{k=-\infty}^{\infty}G(n,k+1)\Delta_k(g)-\sum_{k=-\infty}^{\infty}G(n,k+1)\Delta_k(f)\\ &= A(n,\ell)\\ &\quad+\underbrace{\sum_{k=n-\ell}^{n-1}G(n,k+1)\Delta_k(g)}_{=B_1}+\underbrace{\sum_{k=n}^{n+\ell-1}G(n,k+1)\Delta_k(g)}_{=B_2}\end{aligned}$$

$$- \underbrace{\sum_{k=n-\ell}^{n-1} G(n,k+1)\Delta_k(f)}_{=C_1} - \underbrace{\sum_{k=n}^{n+\ell-1} G(n,k+1)\Delta_k(f)}_{=C_2},$$

where $A(n,\ell)$ is defined by (2.3) while $\Delta_k(g)$ and $\Delta_k(f)$ are described by:

$$\Delta_k(g) = g(k, \chi_j(k;(n,\xi)) + x_{k,n}(\xi)) - g(k, \chi_j(k;(n,\xi')) + x_{k,n}(\xi')),$$
$$\Delta_k(f) = f(k, x_{k,n}(\xi')) - f(k, x_{k,n}(\xi)),$$

By Remark 6, $A(n,\ell)$ converges to zero when $\ell \to +\infty$ and we can choose $L(n,\varepsilon) > 0$ such that

$$|A(n,\ell)| \leq \frac{(1-\theta)}{2}\varepsilon \quad \text{for any } \ell > L(n,\varepsilon). \tag{4.4}$$

In order to estimate $|B_1| + |B_2|$, by using the properties (A2) and (1.7) combined with inductive hipothesis and Lemma 9, we can deduce:

$$\begin{aligned}
|B_1| &\leq \sum_{k=n-\ell}^{n-1} K \exp\left(-\sum_{p=k+1}^{n} a_p\right) r_k \{|\chi_j(k;(n,\xi)) - \chi_j(k;(n,\xi'))| + |x_{k,n}(\xi) - x_{k,n}(\xi')|\} \\
&\leq \sum_{k=n-\ell}^{n-1} K \exp\left(-\sum_{p=k+1}^{n} a_p\right) r_k \left\{\varepsilon + |\xi-\xi'| \exp\left(\sum_{l=k}^{n-1}\{||A_l - I|| + r_l\}\right)\right\}
\end{aligned}$$

and

$$|B_2| \leq \sum_{k=n}^{n+\ell-1} K \exp\left(-\sum_{p=n}^{k+1} a_p\right) r_k \left\{\varepsilon + |\xi-\xi'| \exp\left(\sum_{l=n}^{k-1}\{||A_l - I|| + r_l\}\right)\right\}.$$

Analogously, we can verify that

$$\begin{aligned}
|C_1| &\leq \sum_{k=n-\ell}^{n-1} K \exp\left(-\sum_{p=k+1}^{n} a_p\right) r_k |x_{k,n}(\xi) - x_{k,n}(\xi')| \\
&\leq |\xi-\xi'| \sum_{k=n-\ell}^{n-1} K \exp\left(-\sum_{p=k+1}^{n} a_p\right) r_k \exp\left(\sum_{l=k}^{n-1}\{||A_l - I|| + r_l\}\right).
\end{aligned}$$

and

$$|C_2| \leq |\xi-\xi'| \sum_{k=n}^{n+\ell-1} K \exp\left(-\sum_{p=n}^{k+1} a_p\right) r_k \exp\left(\sum_{l=n}^{k-1}\{||A_l - I|| + r_l\}\right).$$

By using (H3), we can deduce that

$$|B_1| + |B_2| \leq \varepsilon\theta + |\xi-\xi'|\Gamma(n,\ell) \quad \text{and} \quad |C_1| + |C_2| \leq |\xi-\xi'|\Gamma(n,\ell),$$

where $\Gamma(n,\ell)$ is a finite term defined by

$$\Gamma(n,\ell)=\sum_{k=n-\ell}^{n-1}K\exp\left(-\sum_{p=k+1}^{n}a_p\right)r_k\exp\left(\sum_{l=k}^{n-1}\{||A_l-I||+r_l\}\right)$$
$$+\sum_{k=n}^{n+\ell-1}K\exp\left(-\sum_{p=n}^{k+1}a_p\right)r_k\exp\left(\sum_{l=n}^{k-1}\{||A_l-I||+r_l\}\right).$$

Now, we can deduce that

$$\begin{aligned}|\chi_{j+1}(n;(n,\xi))-\chi_{j+1}(n;(n,\xi'))|&\leq|A(n,\ell)|+|B_1|+|B_2|+|C_1|+|C_2|\\&\leq\frac{(1-\theta)}{2}\varepsilon+\varepsilon\theta+2|\xi-\xi'|\Gamma(n,\ell).\end{aligned}$$

When choosing $\delta_{j+1}=\min\left\{\delta_j,\frac{(1+\theta)\varepsilon}{4\Gamma(n,\ell)}\right\}$, we can see that (4.3) is verified.

It is easy to verify that $\delta_j(n,\varepsilon)$ is a decreasing sequence, which converges to a positive constant $\delta(\varepsilon,n)$. Now, by letting $j\to+\infty$ in (4.3), we have that

$$|\chi(n;(n,\xi))-\chi(n;(n,\xi'))|<\varepsilon\quad\text{if}\quad|\xi-\xi'|<\delta(\varepsilon,n)$$

and the theorem follows.

4.2. Proof of Theorem 2

As before, we only prove that $\xi\mapsto\chi(n;(n,\xi))$ is continuous for any fixed n. Now, let us recall that $n\mapsto\chi(n;(m,\xi))$ is the unique bounded solution of (3.5) and define $\Delta_k(\xi,\xi')$ as follows

$$\Delta_k(\xi,\xi')=g\big(k,\chi(n;(n,\xi)+x_{k,n}(\xi)\big)-g\big(k,\chi(n;(n,\xi')+x_{k,n}(\xi')\big)$$
$$+f\big(k,x_{k,n}(\xi')\big)-f\big(k,x_{k,n}(\xi)\big).$$

Now, by using (A2) combined with $A(n,\ell)$ defined by (2.3), we can deduce that

$$\begin{aligned}|\chi(n;(n,\xi))-\chi(n;(n,\xi')))|&\leq\sum_{k=-\infty}^{+\infty}G(n,k)\Delta_k(\xi,\xi')\\&\leq A(n,\ell)\\&\quad+\sum_{k=n-\ell}^{n+\ell}|G(n,k+1)|r_k|\chi(n;(k,\xi))-\chi(n;(k,\xi')))|\\&\quad+2\sum_{k=n-\ell}^{n+\ell}|G(n,k+1)|r_k|x_{k,n}(\xi)-x_{k,n}(\xi')|.\end{aligned}$$

We will provide some estimations for these three terms. Firstly, by (H4) we know that there exists $L(\varepsilon) > 1$, such that

$$|A(n,\ell)| < \frac{(1-\theta)\varepsilon}{2} \quad \text{for any } \ell > L.$$

Secondly, let us define

$$||\chi(\cdot;(n,\xi)) - \chi(\cdot;(n,\xi'))||_\infty = \sup_{k\in\mathbb{Z}} |\chi(n;(k,\xi)) - \chi(n;(k,\xi'))|.$$

Now, by using consecutively Lemma 9, (A1), (H6) and (H3), we can deduce that

$$\begin{aligned}
&\sum_{k=n-L}^{n+L} |G(n,k+1)|r_k|x_{k,n}(\xi) - x_{k,n}(\xi')| \\
&\quad \le |\xi-\xi'| \sum_{k=n-L}^{n+L} |G(n,k+1)|r_k \exp\left(\sum_{p=n}^{k-1}\{M + r_p\}\right) \\
&\quad \le |\xi-\xi'|e^{ML} \sum_{k=n-L}^{n+L} |G(n,k+1)|r_k \exp\left(\sum_{p=n}^{k-1} r_p\right) \\
&\quad \le |\xi-\xi'|e^{ML} \sum_{k=n-L}^{n+L} |G(n,k+1)|r_k \exp\left(\sum_{p=n}^{n+L} r_p\right) \\
&\quad \le |\xi-\xi'|e^{ML} \sum_{k=n-L}^{n+L} |G(n,k+1)|r_k \exp\left(\frac{1}{L}\left\{\sum_{p=n}^{n+L} r_p\right\} L\right) \\
&\quad \le |\xi-\xi'|e^{(M_L+M)L} \sum_{k=n-L}^{n+L} |G(n,k+1)|r_k \\
&\quad \le \theta|\xi-\xi'|e^{(M_L+M)L}.
\end{aligned}$$

Now, by using the inequalities above, we can see that

$$\begin{aligned}
|\chi(n;(n,\xi)) - \chi(n;(n,\xi'))| \le{}& \frac{(1-\theta)\varepsilon}{2} + \theta||\chi(\cdot;(n,\xi)) - \chi(\cdot;(n,\xi'))||_\infty \\
&+ 2\theta|\xi-\xi'|e^{(M_L+M)L},
\end{aligned}$$

which implies

$$\begin{aligned}
|\chi(n;(n,\xi)) - \chi(n;(n,\xi'))| &\le ||\chi(\cdot;(n,\xi)) - \chi(\cdot;(n,\xi'))||_\infty \\
&\le \frac{\varepsilon}{2} + \frac{2\theta}{(1-\theta)}|\xi-\xi'|e^{(M_L+M)L}.
\end{aligned}$$

and the strongly topological equivalence follows by considering $\delta(\varepsilon) = \frac{(1-\theta)\varepsilon}{4\theta}e^{-(M+M_L)L}$.

4.3. Proof of Theorem 3

Firstly, note that the topological equivalence is a direct consequence of Theorem 1. Indeed, (H1) and (H3) are equivalent to (D1) and (D2). On the other hand, (H2) is always satisfied since

$$N(n, F+G) \leq K(F+G)\frac{1+e^{-\alpha}}{1-e^{-\alpha}} = B.$$

Notice also that (H4) and (H5) are a consequence of (D1) and (D2) since $|\Delta_k|$ and $|\overline{\Delta}_k|$ are bounded by $2(F+G)$ for any $k \in \mathbb{Z}$ and

$$\sum_{k=-\infty}^{n-1-J} e^{-\alpha(n-k-1)} \quad \text{and} \quad \sum_{k=n+J}^{+\infty} e^{-\alpha(k+1-n)}$$

converge to zero when $J \to +\infty$ uniformly with respect to $n \in \mathbb{Z}$. Finally, (H6) is always verified for constant sequences and all the hypotheses of Theorem 2 are satisfied, which implies strongly topological equivalence.

Now, we will prove that the map $\xi \mapsto H(n,\xi)$ is Hölder continuous for any $n \in \mathbb{Z}$. The other one can be done in a similar way. As before, we have that

$$\begin{aligned}
|\chi(n;(n,\xi)) - \chi(n;(n,\xi'))| &\leq \sum_{k=-\infty}^{\infty} G(n,k+1)|\Delta_k(g)| + \sum_{k=-\infty}^{\infty} G(n,k+1)|\Delta_k(f)| \\
&\leq \underbrace{2\sum_{k=-\infty}^{n-1-\ell} G(n,k+1)[F+G] + 2\sum_{k=n+\ell}^{\infty} G(n,k+1)[F+G]}_{=\mathcal{A}} \\
&\quad + \underbrace{\sum_{k=n-\ell}^{n-1} Ke^{-\alpha(n-k-1)}\Delta_k(g)}_{=\mathcal{B}_1} + \underbrace{\sum_{k=n}^{n+\ell-1} Ke^{-\alpha(k+1-n)}\Delta_k(g)}_{=\mathcal{B}_2} \\
&\quad - \underbrace{\sum_{k=n-\ell}^{n-1} Ke^{-\alpha(n-k-1)}\Delta_k(f)}_{=\mathcal{C}_1} - \underbrace{\sum_{k=n}^{n+\ell-1} Ke^{-\alpha(k+1-n)}\Delta_k(f)}_{=\mathcal{C}_2}.
\end{aligned}$$

The reader can deduce that

$$|\mathcal{A}| \leq \frac{2K(F+G)}{1-e^{-\alpha}}e^{-\ell\alpha}.$$

On the other hand, by using (A2) and Lemma 9, we can deduce that

$$|\mathcal{B}_1| \leq \sum_{k=n-\ell}^{n-1} Ke^{-\alpha(n-k-1)} r\Big\{||\chi(\cdot;(n,\xi)) - \chi(\cdot;(n,\xi'))||_\infty + |x_{k,n}(\xi) - x_{k,n}(\xi')|\Big\}$$

$$\leq \sum_{k=n-\ell}^{n-1} Ke^{-\alpha(n-k-1)}r\Big\{||\chi(\cdot;(n,\xi))-\chi(\cdot;(n,\xi'))||_\infty+|\xi-\xi'|e^{(M+r)(n-1-k)}\Big\}$$
$$\leq \sum_{k=n-\ell}^{n-1} Ke^{-\alpha(n-k-1)}r\Big\{||\chi(\cdot;(n,\xi))-\chi(\cdot;(n,\xi'))||_\infty+|\xi-\xi'|e^{(M+r)(\ell-1)}\Big\},$$

where

$$||\chi(\cdot;(n,\xi))-\chi(\cdot;(n,\xi'))||_\infty=\sup_{j\in\mathbb{Z}}|\chi(j;(n,\xi))-\chi(j;(n,\xi'))|.$$

Similarly, it follows that

$$|\mathcal{B}_2|\leq \sum_{k=n}^{n+\ell-1} Ke^{-\alpha(k+1-n)}r\Big\{||\chi(\cdot;(n,\xi))-\chi(\cdot;(n,\xi'))||_\infty+|\xi-\xi'|e^{(M+r)(\ell-2)}\Big\},$$

which implies that

$$|\mathcal{B}_1|+|\mathcal{B}_2|\leq\theta||\chi(\cdot;(n,\xi))-\chi(\cdot;(n,\xi'))||_\infty+\theta|\xi-\xi'|e^{(M+r)\ell},$$

where

$$\theta=\sum_{k=-\infty}^{n-1}Kre^{-\alpha(n-k-1)}+\sum_{k=n}^{\infty}Kre^{-\alpha(k+1-n)}=Kr\left\{\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\right\}<1.$$

The inequality

$$|\mathcal{C}_1|+|\mathcal{C}_2|\leq\theta|\xi-\xi'|e^{(M+r)\ell},$$

can be deduced as above. Now, it follows that

$$|\chi(n;(n,\xi))-\chi(n;(n,\xi'))|\leq\frac{2K(F+G)}{(1-e^{-\alpha})(1-\theta)}e^{-\alpha\ell}+\frac{2\theta}{1-\theta}|\xi-\xi'|e^{(M+r)\ell}$$

Let us assume that $|\xi-\xi'|<1$ and let us choose

$$\ell=\frac{1}{\alpha}\ln\left(\frac{1}{|\xi-\xi'|}\right)$$

and introduce the constants

$$D_1=1+\frac{2K(F+G)}{(1-e^{-\alpha})(1-\theta)}\quad\text{and}\quad D_2=\frac{2\theta}{1-\theta}.$$

Finally, a careful computation shows that

$$|h(n,\xi)-h(n,\xi')|\leq D_1|\xi-\xi'|+D_2|\xi-\xi'|^{1-(\frac{M+r}{\alpha})}$$
$$\leq(D_1+D_2)|\xi-\xi|^{1-(\frac{M+r}{\alpha})},$$

and the result follows.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

The first author was funded by FONDECYT Iniciación Project [11121122]. The second author was funded by FONDECYT Regular Project [1120709]. In addition, both authors were funded by MATHAMSUD 16-MATH-04 STADE.

References

[1] L. Barreira and C. Valls, *A Grobman–Hartman Theorem for nonuniformly hyperbolic dynamics*, J. Differ. Equ. 228 (2006), pp. 285–310.
[2] X. Chen and Y. Xia, *Topological conjugacy between two kinds of nonlinear differential equations via generalized exponential dichotomy*, Int. J. Differ. Equ. (2011). Art. ID 871574.
[3] ChV Coffman and J.J. Schäfer, *Dichotomies for linear difference equations*, Math. Ann. 172 (1967), pp. 139–166.
[4] W. Coppel, *Dichotomies in Stability Theory*. Vol. 629, Lecture notes in mathematics, Springer-Verlag, Berlin, 1978.
[5] S. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New-York, 2005.
[6] Y. Gao, Y. Xia, X. Yuan, and P.J.Y. Wong, *Linearization of nonautonmous impulsive systems with nonuniform exponential dichotomy*, Abstr. Appl. Anal. (2014). Art. ID 860378.
[7] Y. Gao, X. Yuan, Y. Xia, and P.J.Y. Wong, *Linearization of impulsive differential equations with ordinary dichotomy*, Abstr. Appl. Anal. (2014). Art. ID 632109.
[8] P. Hartman, *On local homeomorphisms of Euclidean spaces*, Bol. Soc. Mat. Mexicana 5 (1960), pp. 220–241.
[9] L. Jiang, *Generalized exponential dichotomy and global linearization*, J. Math. Anal. Appl. 315 (2006), pp. 474–490.
[10] L. Jiang, *Strongly topological linearization with generalized exponential dichotomy*, Nonlinear Anal. 67 (2007), pp. 1102–1110.
[11] J. Kurzweil and G. Papaschinopoulos, *Structural stability of linear discrete systems via the exponential dichotomy*, Czech. Math. J. 38 (1988), pp. 280–284.
[12] J. Kurzweil and G. Papaschinopoulos, *Topological equivalence and structural stability for linear difference equations*, J. Differ. Equ. 89 (1991), pp. 89–94.
[13] J. Lopez-Fenner and M. Pinto, *On a Hartman linearization theorem for a class of ODE with impulse effect*, Nonlinear Anal. 38 (1999), pp. 307–325.
[14] R.H. Martin Jr., *Conditional stability and separation of solutions to differential equations*, J. Differ. Equ. 13 (1973), pp. 81–105.
[15] K.J. Palmer, *A generalization of Hartman's linearization theorem*, J. Math. Anal. Appl. 41 (1973), pp. 753–758.
[16] G. Papaschinopoulos, *A linearization result for a differential equation with piecewise constant argument*, Analysis 16 (1996), pp. 161–170.
[17] G. Papaschinopoulos and J. Schinas, *Criteria for exponential dichotomy of difference equations*, Czech. Math. J. 35 (1985), pp. 295–299.
[18] M. Pinto, *Discrete dichotomies*, Comput. Math. Appl. 28 (1994), pp. 259–270.
[19] A. Reinfelds and D. Šteinberga, *Dynamical equivalence of quasilinear equations*, Int. J. Pure Appl. Math. 98 (2015), pp. 355–364.
[20] J. Schinas and G. Papaschinopoulos, *Topological equivalence for linear discrete systems via dichotomies and Lyapunov functions*, Boll. Unione Math. Ital. 4 (1985), pp. 61–70.
[21] J. Shi and K. Xiong, *On Hartman's linearization and Palmer's linearization theorem*, J. Math. Anal. Appl. 192 (1995), pp. 813–832.
[22] Y.H. Xia, X. Chen, and V. Romanowski, *On the linearization theorem of Fenner and Pinto*, J. Math. Anal. Appl. 400 (2013), pp. 439–451.

[23] Y.H. Xia, J. Li, and J.Y. Wong, *On the topological classification of dynamic equations on time scales*, Nonlinear Anal.: Real World Appl. 14 (2013), pp. 2231–2248.
[24] Y.H. Xia, R. Wang, K.I. Kou, and D. O'Regan, *On the linearization theorem for nonautonmous differential equations*, Bull. Sci. Math. 139 (2015), pp. 829–846.
[25] J. Zhang, M. Fan, and X. Chang, *Nonlinear perturbations of nonuniform exponential dichotomy on measure chains*, Nonlinear Anal. Theory, Methods Appl. 75 (2012), pp. 670–683.