Comparison of Different Stability Conditions for Linear Time-Delay Systems with Incommensurate Delays

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Abstract: We compare different stability conditions for linear time-delay systems with multiple incommensurate, time-invariant delays. In total, nine sufficient stability conditions are taken from the literature and implemented in MATLAB. All of them guarantee asymptotic stability if all delays τ_k are smaller than a bound $\bar{\tau}$. The different conditions are then tested on nine examples which have served as benchmark examples in various earlier publications. The different conditions are compared with each other with respect to computational effort and maximal achievable bound $\bar{\tau}$, for which asymptotic stability is guaranteed.

Keywords: Time-delay systems, incommensurate delays, comparison of stability conditions.

1. INTRODUCTION

The stability analysis of linear, time-invariant Time-Delay Systems (TDS) with multiple, incommensurate delays τ_k is still an active field of research. One reason is that this analysis problem is considered to be $\mathcal{NP}$-hard, see Toker and Özbay [1996], Gu et al. [2003]. This $\mathcal{NP}$-hardness however does not always inhibit an efficient numerical solution within machine accuracy, cf. Jarlebring [2008], Ch. 3.7. Nonetheless, all stability conditions for TDS require a trade-off between accuracy and fast availability of the result.

In this paper, we compare nine stability conditions for TDS with incommensurate delays $\tau_k \in [0, \bar{\tau}]$ with respect to accuracy and computing time. We have chosen stability conditions based on different methods and numerical techniques, ranging from matrix norms [Niculescu et al., 1998] and polynomial eigenvalue problems [Jarlebring, 2006b] to linear matrix inequalities (LMIs) [Mehdi and Boukas, 2003], sum-of-square (SOS) techniques [Ebenbauer and Allgöwer, 2006], and linear programming [Münz et al., 2007]. All these algorithms are tested on nine benchmark examples from the literature that range from one to four states and one to three delays. To the best of the authors' knowledge, there is so far no quantitative comparison of such a broad range of different conditions.

Unfortunately, we were not able to implement all stability conditions for TDS proposed in the literature. The conditions in Breda et al. [2005], Jarlebring [2006b], Ergenc et al. [2007], Fazelinia et al. [2007], Sipahi and Olgac [2005] are based on a numerical computation of the roots of the characteristic quasi-polynomial of the TDS or the exact crossing curves where these roots cross the imaginary axis. All these algorithms require a discretization at some point. Obviously, the computational effort increases if the discretization is refined. Therefore, a fair comparison to the other algorithms concerning the computing time is hardly possible. Here, we only consider the algorithm presented in Jarlebring [2006b], which is quite easy to implement. Further interesting stability conditions have been proposed based on a discretization of the delay interval [Gu, 2003, Peaucelle et al., 2007, Gouaisbaut and Peaucelle, 2006a], complete Lyapunov functionals and SOS techniques [Peet et al., 2009, Maier et al., 2008], comparison systems [Knospe and Roozbehani, 2003], descriptor systems [Fridman and Shaked, 2001], and Taylor- and Padé-approximations [Zhang et al., 2003, Bliman and Iwasaki, 2006, Gouaisbaut and Peaucelle, 2006b]. Yet, most of these results are only published for single delays, e.g. Peaucelle et al. [2007], Gouaisbaut and Peaucelle [2006a,b], or a nondecreasing sequence of multiple delays, as in Gu [2003], Peet et al. [2009].

The paper is structured as follows: A precise problem statement is given in Section 2. The compared stability conditions are presented with more detail in Section 3. The considered examples as well as the results of the comparison are summarized in Section 4, and the paper is concluded in Section 5.

2. PROBLEM STATEMENT

We consider the following TDS

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{N} A_k x(t - \tau_k), \tag{1}$$

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Table 1. Overview of the considered stability conditions.

Label	Authors	Numerical problem	MATLAB function	Iter.
LP	Münz et al. [2007]	linear program	`linprog`	yes
SOS	Ebenbauer and Allgöwer [2006]	sum of squares	`sostools, SeDuMi`	yes
Gu	Gu et al. [2003]	matrix / system norms	`svd, normhinf, norm, eig`	no
JB	Jarlebring [2006b, 2009]	polynomial eigenvalue	`polyeig`	no
Sun	Sun et al. [1997]	matrix norm	`norm`	no
Ni	Niculescu et al. [1998]	bound on matrix exponential	`norm, expm`	no
Me	Mehdi and Boukas [2003]	LMI	LMI Lab	yes
Cao	Cao et al. [1998]	LMI	LMI Lab	yes
Xu	Xu and Liu [2003]	matrix inequality	LMI Lab	yes

with $x(t) \in \mathbb{R}^n$, $A_k \in \mathbb{R}^{n\times n}$, and $\tau_k \geq 0$ for all k. Moreover, we assume that

$$\bar{A} := \sum_{k=0}^{N} A_k \text{ is Hurwitz,}$$

i.e., the real part of all eigenvalues of $\bar{A}$ is strictly negative. We are now looking for the maximal delay bound $\bar{\tau} \in \mathbb{R}$, such that (1) is asymptotically stable if $\tau_k < \bar{\tau}$ for all k.

3. COMPARED STABILITY CONDITIONS

In this section, we review the different stability conditions used in the comparison. An overview is given in Table 1. The different approaches are labeled either by a characteristic solution principle or by a short form of the name of the first author. They can be grouped into two main branches. The first group is derived in the frequency domain (upper block of Table 1), the second in the time domain (lower block). Additionally, Table 1 gives the authors, who first published the stability condition, the numerical problem to be solved, and the MATLAB command or toolbox that is used to solve it.

Some algorithms cannot compute the maximal $\bar{\tau}$ directly but in an iterative way. These algorithms can only check stability for a given value $\bar{\tau}$. Therefore, we implemented a bisection method based search over $\bar{\tau}$ in order to determine the maximal acceptable value of $\bar{\tau}$. In order to treat all conditions equally, we used a fixed number iterations. Further investigations, which are not shown here, revealed that all algorithms "converge" after 30 iterations, meaning that the two final values diverge only marginally. If an algorithm requires such an iterative search or not is indicated in column Iter.

3.1 Frequency Domain Approaches

This subsection reviews the compared algorithms in the frequency domain. Since all frequency domain approaches result from the Laplace transformation of system (1), none of them can be extended to nonlinear TDS or TDS with time-varying delays.

LP / SOS: The approaches described in Ebenbauer and Allgöwer [2006], Münz et al. [2007], and Münz et al. [2009] use the Rekasius substitution [Rekasius, 1980] and uncertain polynomials to reformulate the characteristic equation of (1) as a multivariate polynomial. In order to guarantee stability of (1), this polynomial has to be positive in a certain domain, depending on $\bar{\tau}$. The positivity test can be performed either using a linear program (LP) or a sum-of-squares program (SOS). The numerical parameters for the LP algorithm, as described in Münz et al. [2007], are set to $\mu = \epsilon = 10^{-4}$ and $m = 6$. The numerical parameters for the SOS algorithm, as given in Ebenbauer and Allgöwer [2006], are set to $\mu = 10^{-3}$ and $\epsilon = 10^{-2}$. These parameters show a good compromise between accuracy and computing time for all studied examples. Both approaches can also be extended to uncertain characteristic equations.

Gu: Theorem 3.17 in Gu et al. [2003] is derived using μ-techniques from robust control theory. The delay terms are thereby considered as a structured uncertainty block. The theorem is given in six different formulations that guarantee stability if a matrix or system norm satisfies certain inequalities. System (1) is asymptotically stable if any of these inequalities is fulfilled. In our implementation, we check all six conditions.

JB: In Jarlebring [2006b, 2009], the characteristic equation of (1) is considered as a polynomial eigenvalue problem. This eigenvalue problem has to be tested for a parameter ϕ taking any value in the continuous interval $(-\pi, \pi]$ and another parameter $p_k \in \mathbb{Z}$. In any implementation, we have to discretize the interval $(-\pi, \pi]$ and limit the set of possible values of p_k. This implies that not all possible values of ϕ and p_k are tested and we might wrongly certify an unstable system to be stable. The risk of this error decreases by increasing the number of discretization points for $(-\pi, \pi]$ and the set of possible values for p_k. Clearly, there is a trade-off between accuracy and computing time. For our investigations, we chose a step size for the discretization of $\Delta\phi = 0.05$ and $p_k \in \{0, 1, \ldots, 1000\}$. This leads to a good compromise between a quasi-continuous solution and a short computing time. The algorithms in Jarlebring [2006b, 2009] actually build stability maps consisting of a finite number of critical points. A stability map gives a complete map of the stable and unstable areas in a parameter space, which is much more than we need to solve the problem. From this stability map, we determine $\bar{\tau}$ by considering one by one all critical points.

3.2 Time Domain Approaches

This subsection reviews the compared stability conditions in the time domain. In contrast to the solutions in the frequency domain, some of them can also deal with time-varying delays and nonlinear TDS.

Sun: In Sun et al. [1997], the stability of nonlinear TDS with bounded nonlinearities and time-varying delays is studied using matrix norms. For our implementation, we set the bound of the nonlinearities to zero. The time-variance of the delay is ignored. The algorithm requires to find a nonsingular transformation matrix T, which is a nontrivial task. As proposed by the authors, we set $T = I$, where I is the identity matrix.

Ni: In Niculescu et al. [1998], another stability condition for systems with time-varying delays and a nonlinear bounded uncertainty is given. Again, we neglect the time-varying delay and set the bound of the nonlinear uncertainty to zero. The solution is based on norm bounds on the solutions of (1). Therefore, we have to discretize the time axis in 200 steps with the same drawbacks as the algorithm JB. An additional statement about the exponential decay of solutions is neglected here.

Me: The stability of linear TDS with time-varying delays is studied in Mehdi and Boukas [2003]. The Lyapunov-based approach leads to a system of LMIs. Since we consider constant delays, we set the lower bound of the derivatives of $\tau_k(t)$ to zero and the upper bound to 10^{-10} for numerical reasons.

Cao: An uncertain TDS with time-varying delays is considered in Cao et al. [1998]. Again, the derivation is based on Lyapunov stability concepts and a LMI has to be solved in order to guarantee stability. We adapted the result for our comparison, i.e., we eliminated the uncertainties and set the bound on the derivative of τ_k to zero.

Xu: In Xu and Liu [2003], a linear TDS with time-varying bounded delays is considered. Again, we simplified the condition for fixed delays. The numerical problem to be solved is the feasibility of a matrix inequality which depends in a nonlinear fashion on a positive definite matrix P and positive scalars ρ_{kj}. The authors propose to choose $P = I$. Alternatively, we consider P that solves $P\bar{A} + \bar{A}^T P = -2I$. This shows better results than $P = I$ in some cases. Moreover, we set $\rho_{kj} = 1$ because an optimization for $\rho_{kj} = \rho$ did not lead to any improvement. The resulting LMI can be solved easily.

4. COMPARISON OF STABILITY CONDITIONS

After reviewing the compared stability conditions, we are now ready to present the main results of our investigations. Therefore, we first present the applied examples in Subsection 4.1. Then, we report on the obtained maximal delay bound $\bar{\tau}$ in Subsection 4.2 and on the computing time in Subsection 4.3. The results are summarized in Subsection 4.4.

All calculations have been performed on a Pentium 4, 2.66 GHz, 1 GB RAM, using MATLAB 7 (R14). We used SOSTOOLS Prajna et al. [2004] and SeDuMi Sturm [1999] for the solution of SOS programs and MATLAB's LMI Lab for LMI programs, respectively.

4.1 Examples

The numerical examples used in this investigation have been used in various publications as test systems. There, only very few algorithms are compared on one or two examples. The main contribution of the work at hand is the comparison of nine stability conditions on nine examples with increasing complexity with respect to number of states and number of delays. In Table 2, we summarize the system matrices of the example systems considered here and where they have been used before. All algorithms described in Section 3 are applied to all of the examples in order to find a maximum permissible boundary $\bar{\tau}$, which does not jeopardize the stability of System (1).

4.2 Boundary $\bar{\tau}$

The goal of this work is to compare different approaches to determine a maximal delay bound $\bar{\tau}$. Thus, the most important criterion of this comparison is this upper bound $\bar{\tau}$. In Figure 1, the percentage of the obtained result with respect to the exact value $\bar{\tau}_{\max}$ is given. This percentage is used because it gives a better impression on how good the determined boundaries really. In Table 3, the obtained values $\bar{\tau}$ are given additionally. The exact bound is acquired either through analytical computations or from the polynomial eigenvalue approach JB as the latter does not introduce any systematic conservatism.

In order to distinguish between the different examples, they are labeled according to the following rules: The color indicates the number of states of the example, i.e., red for scalar systems, blue for systems with two states, green for systems with three states, and black for systems with four states. Moreover, the number of delays is indicated by the symbols, a circle "$\circ$" for systems with one delay, a plus "+" for systems with two delays, and a cross "$\times$" for systems with three delays.

We see immediately that algorithms based on norms (Gu, Sun, Ni) only lead to good results for scalar systems. For systems with more states, the achieved results are quite poor. Similar results are obtained for the LMI algorithms Cao and Xu. The only exception is the LMI condition Me [Mehdi and Boukas, 2003] which performs quite well for all examples.

The stability conditions based on positive polynomials (LP, SOS) perform very well for the simpler examples. In fact, the LP approach almost achieves the exact bound for the Examples 1 to 6. SOS performes slightly worse on these examples. However, for the Examples 7 to 9, LP and SOS did not find a feasible solution of their numerical program or the solver terminated due to a lack of memory.

4.3 Computing Time

The second important criterion in assessing the different stability conditions is the required computing time. A comparison is given in Figure 2 using the same colors and symbols as in Figure 1. The time needed to determine the stability bound $\bar{\tau}$ is plotted on a logarithmic scale for the different algorithms. The exact values of the computing time can also be found in Table 4. Recall that some conditions require and iterative procedure, cf. Section 3. For these algorithms, the indicated computing time is the time needed for 30 iterations.

We see that the computing time is very low for the norm and LMI based approaches. At the same time, some algorithms take the same time for systems with more or less states and more or less delays. Others vary but without

Table 2. Dynamic matrices A_k as denoted in (1) and references of the considered examples.

Ex	A_0	A_1	A_2	A_3	REFERENCE
1	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$			Münz et al. [2007],Gu et al. [2003], Thowsen [1981]
2	$\begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$			Ebenbauer and Allgöwer [2006],Gouaisbaut and Peaucelle [2006a]
3	$\begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix}$	$\begin{bmatrix} -5.9 & 7.1 & -70.3 \\ 2 & -1 & 5 \\ 2 & 0 & 6 \end{bmatrix}$			Ebenbauer and Allgöwer [2006],Olgac and Sipahi [2002]
4	0.5	-0.9	-1.5		Jarlebring [2006b]
5	0	-1	-2		Jarlebring [2006a]
6	$\begin{bmatrix} 0 & 1 \\ -1\frac{49}{256} & -\frac{7}{8} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \frac{1}{5} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -\frac{4}{5} & 0 \end{bmatrix}$		Münz et al. [2007],Michiels and Niculescu [2006]
7	$\begin{bmatrix} -1 & 13.5 & -1 \\ -3 & -1 & -2 \\ -2 & -1 & -4 \end{bmatrix}$	$\begin{bmatrix} -5.9 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 7.1 & -70.3 \\ 0 & -1 & 5 \\ 0 & 0 & 6 \end{bmatrix}$		Ergenc et al. [2007], Sipahi and Olgac [2005]
8	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix}$	$\begin{bmatrix} -\frac{9}{200} & \frac{1.5}{200} & \frac{1}{4} & 0 \\ \frac{1}{200} & \frac{1}{200} & \frac{1}{20} & 0 \\ 0 & 0 & 0 & \frac{1}{2000} \\ -2 & -\frac{1}{2} & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{3}{80} & 0 & \frac{3}{40} & \frac{1}{8} \\ 0 & \frac{1}{20} & \frac{1}{20} & 0 \\ \frac{1}{20} & \frac{1}{20} & 0 & 0 \\ 0 & -2.5 & 0 & -1 \end{bmatrix}$		Jarlebring [2006b]
9	-2	-4	-3	-1	Sipahi [2005]

Table 3. Maximal delay $\bar{\tau}$ for all compared algorithms.

Ex.	LP	SOS	Gu	JB	Sun	Ni	Me	Cao	Xu
1	3.1054	2.86	0.5774	3.1416	0.1213	0.1015	2	0.5	0.175
2	6.1682	5.97	0.4996	6.1726	0.324	0.0876	4.3589	0.8571	0.4242
3	0.16234	0.143	0.00147	0.1624	2.25e-05	4.835e-05	0.04198	9.451e-05	2.917e-05
4	0.57577	0.38014	0.2575	0.58	0.273	0.26761	0.33726	0.2356	0.2356
5	0.5192	0.5124	0.2824	0.5239	0.3333	0.3268	0.33333	0.1936	0.2449
6	2.0952	1.9624	0.5247	2.0959	0.2611	0.1072	1.1556	0.1372	0.2112
7	0.0	0.0	0.000633	0.0559	7.45e-06	1.602e-05	0.019975	4.544e-06	9.656e-06
8	0.0	-	0.084	0.9076	0.003042	0.002214	0.71062	0.12488	0.009232
9	0.0	0.0	0.1132	0.238	0.125	0.1225	0.16137	0.02525	0.1034

Table 4. Computing time t_{comp} for $\bar{\tau}$ in seconds.

Ex	LP	SOS	Gu	JB	Sun	Ni	Me	Cao	Xu
1	142	167	10.2	6.1	3.3	0.9	3.6	2.7	16.2
2	130	139	8.4	0.7	3.3	0.9	4.1	3	15.9
3	204	1835	20.1	1.1	4.2	48.9	10.1	3.3	16.8
4	352	997	9	161.4	3.9	0.9	4.1	3	11.4
5	217	558	9.3	159.9	3.3	0.9	3.8	3	13.8
6	315	916	10.5	146.4	3.9	1.5	9.5	4.2	14.7
7	801	15750	21.3	427	3.3	49.7	39.1	9.9	15.9
8	7970	–	16.5	203	3.3	1.8	74.6	14.7	16.2
9	2803	34377	10.2	147	3.3	0.9	7.7	4.8	13.2

a clear correlation between number of states or number of delays and computing time. The polynomial eigenvalue condition JB takes roughly the same time as norm and LMI based approaches, but we see a stronger correlation between the number of delays and the computing time. In particular, only those examples with one delay (Ex. 1 to 3) are really fast. The other examples are slower than with the LMI or norm based algorithms. Note however that the computing time of the algorithm JB increases with an improved discretization, Hence, these values have to be handled with care. Compared to LP and SOS approach, norm based and LMI approaches are faster by two orders of magnitude.

4.4 Discussion

As expected, the fastest way of determining the stability for a TDS with incommensurate delays are norm and LMI based algorithms (Gu, Sun Ni, Me, Cao, Xu). However, they are also very conservative, in particular for systems with more than one state. The only exception is the condition Me given in Mehdi and Boukas [2003], which gives both a fast and accurate result.

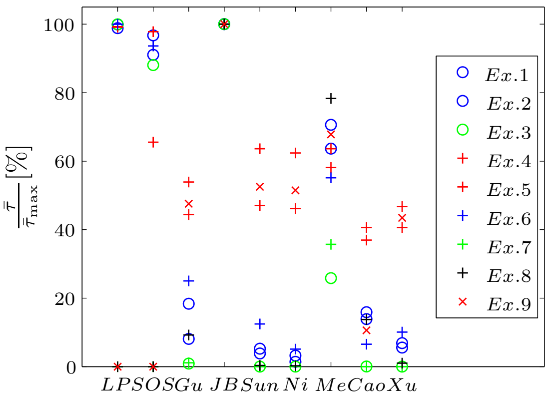

Figure 1. Normalized $\frac{\bar{\tau}}{\bar{\tau}_{\max}}$ vs. stability condition; the colors and symbols indicate: scalar systems (red), two states (blue), three states (green), four states (black), one delay ($\circ$), two delays ($+$), three delays ($\times$).

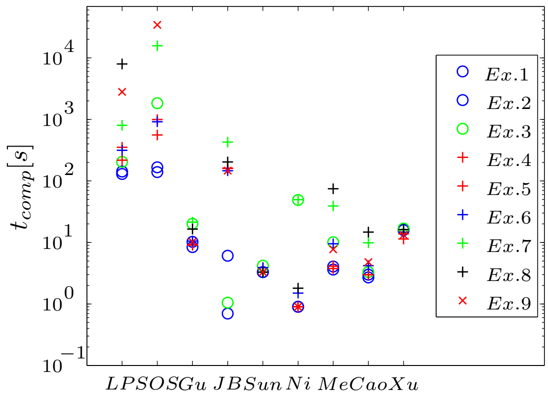

Figure 2. Computing time t_{comp} for $\bar{\tau}$ vs. stability condition; the colors and symbols indicate: scalar systems (red), two states (blue), three states (green), four states (black), one delay ($\circ$), two delays ($+$), three delays ($\times$).

If we are looking for an exact result, then the algorithm JB [Jarlebring, 2006b, 2009] gives a good condition. A stability map is computed for all examples considered here by a relatively simple code. The algorithm is not conservative except for numerical and discretization errors.

The approaches based on positive polynomials, namely LP and SOS, lead to an almost exact result. However, the numerical complexity of these programs leads to a much larger computing time than the norm and LMI based approaches. Clearly, the complexity of the positive polynomials algorithms increases significantly with the number of states and delays. For higher order systems or more delays, the algorithms were not feasible. This is probably due to the small delay bounds of the considered examples and the conservatism of the approaches.

When comparing all stability conditions, it is also important to consider possible extensions of the approaches towards time-varying delays and uncertain or nonlinear systems. Some of the norm and LMI based conditions already consider bounded nonlinearities and time-varying delays, see Section 3. The LP and SOS approach already consider uncertain polynomials for the stability test of nominal systems. Thus, it is possible to extend these approaches to uncertain system matrices as well. The polynomial eigenvalue approach JB cannot be extended to nonlinear systems and time-varying delays because the computation of eigenvalues does not allow for such perturbations.

5. CONCLUSIONS

Nine sufficient stability conditions to determine the stability bound of the delays of a linear TDS with incommensurate delays were briefly reviewed. All of them were tested on nine benchmark examples from the literature that range from one to four states and one to three delays.

Summarizing, the polynomial eigenvalues approach JB gives an exact result at reasonable computational cost. However, this condition relies on a suitable discretization. Similarly, the positive polynomial conditions LP and SOS give very accurate results without discretizations, however at higher computational cost. All other conditions using norms or LMIs are much faster but also far less accurate. A highlight is the LMI condition proposed by Mehdi, which is fast and relatively accurate.

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