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Two parametric approaches for eigenstructure assignment in second-order linear systems

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Abstract: This paper considers eigenstructure assignment in second-order linear systems via proportional plus derivative feedback. It is shown that the problem is closely related to a type of so-called second-order Sylvester matrix equations. Through establishing two general parametric solutions to this type of matrix equations, two complete parametric methods for the proposed eigenstructure assignment problem are presented. Both methods give simple complete parametric expressions for the feedback gains and the closed-loop eigenvector matrices. The first one mainly depends on a series of singular value decompositions, and is thus numerically simple and reliable; the second one utilizes the right factorization of the system, and allows the closed-loop eigenvalues to be set undetermined and sought via certain optimization procedures. An example shows the effectiveness of the proposed approaches.

Keywords: Second-order linear systems; Eigenstructure assignment; Proportional plus derivative feedback; Parametric solutions; Singular value decomposition; Right factorization

1 Introduction

The second-order linear systems capture the dynamic behavior of many natural phenomena, and have found applications in many fields, such as vibration and structural analysis, spacecraft control and robotics control, and hence have attracted a great deal of attention ([1 ~ 12]). In this paper, we consider the control of the following second-order dynamical linear system

$$M\ddot{x} + D\dot{x} + Kx = Bu, \tag{1.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^r$ are the state vector and the control vector, respectively; $M, D, K \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are the system coefficient matrices. In certain applications, the matrices M, D and K are called the mass matrix, the structural damping matrix and the stiffness matrix, respectively. These coefficient matrices satisfy the following assumption.

Assumption 1 $\det(M) \neq 0$, $\operatorname{rank}(B) = r$.

In terms of the control of the second-order linear system (1.1), most of the results are focused on stabilization (for e.g. [2] and [3]) and pole assignment ([4 ~ 9]). Furthermore, many theoretical results for second-order systems have been developed via the corresponding extended first-order state-space model

$$\dot{z} = A_e z + B_e u, \tag{1.2}$$

where

$$A_e = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix},\ B_e = \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix}. \tag{1.3}$$

Therefore, these results eventually involve manipulations on $2n$ dimensional matrices A_e and B_e.

Eigenstructure assignment is a very important problem in linear control systems design, and has been one of the foci of attention in the last two decades (see [13, 14] and the references therein). However, for eigenstructure assignment in second-order linear systems, there have not been many results ([10 ~ 12]). Reference [10] considers eigenstructure assignment in a special class of second-order linear systems using inverse eigenvalue methods. Reference [11] proposes an algorithm for eigenstructure assignment in the second-order linear system (1.1), with M and K being symmetric positive definite, and D being symmetric positive semidefinite. This algorithm is attractive because it utilizes only the original system data M, D and K. Very recently, eigenstructure assignment in the second-order linear system (1.1) using a proportional-plus-derivative feedback controller is considered in [12]. Simple, general, and complete parametric expressions in direct closed forms for both the closed-loop eigenvector matrix and the feedback gains are established. As in [11], the approach utilises directly the original system data M, D and K, and involves manipulations on only n-dimensional matrices. However, the approach has the disadvantage that it requires the controllability of the matrix pair (D, B), which is not satisfied in some applications.

This paper considers eigenstructure assignment in the second-order linear system (1.1) via proportional plus derivative coordinate control. The only requirements on the system coefficient matrices are that M is nonsingular and B has full column rank. Based on a series of singular value decompositions and the right factorization of the system, two complete parametric approaches are proposed. Simple and complete parametric expressions for both the closed-loop eigenvector matrices and the feedback gains are established.

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These expressions contain a group of parameter vectors that represent the design degrees of freedom, which can be properly further chosen to produce a closed-loop system with some desired system specifications. As a result, the second approach, which uses the right factorization of the system, is a natural generalization of the parametric approach in [16] proposed for first-order state-space systems. With this approach, besides the group of parameter vectors, the closed-loop eigenvalues may also be treated as part of the design freedom since they appear directly in the expressions of the eigenvector matrix and the feedback gains, and hence are not necessarily chosen *a priori*.

The paper is composed of five sections. Section 2 formulates the eigenstructure assignment problem for second-order linear systems and relates it to the problem of solving a type of second-order Sylvester matrix equations. Section 3 proposes two complete parametric solutions to the type of second-order Sylvester matrix equations. Based on the solutions proposed in Section 3, two parametric methods are presented in Section 4 for the formulated eigenstructure assignment problem. An illustrative example is given in Section 5.

2 Problem formulation

For the second-order dynamical system (1.1), by choosing the following control law

$$u = F_0 x + F_1 \dot{x},\ F_0, F_1 \in \mathbb{R}^{r\times n}, \tag{2.1}$$

we obtain the closed-loop system as follows:

$$M\ddot{x} + (D - BF_1)\dot{x} + (K - BF_0)x = 0. \tag{2.2}$$

Note that $\det(M) \neq 0$, the above system (2.2) can be written in the first-order state-space form

$$\dot{z} = A_{ec} z \tag{2.3}$$

with

$$A_{ec} = \begin{bmatrix} 0 & I \\ -M^{-1}(K - BF_0) & -M^{-1}(D - BF_1) \end{bmatrix} \tag{2.4}$$

Recall the fact that a nondefective matrix possesses eigenvalues which are less sensitive to the parameter perturbations in the matrix, we here require the closed-loop matrix A_{ec} to be nondefective, that is, the Jordan form of the matrix A_{ec} possesses a diagonal form:

$$\Lambda = \mathrm{diag}(s_1, s_2, \cdots, s_{2n}), \tag{2.5}$$

where $s_i, i = 1,2,\cdots,2n$, are clearly the eigenvalues of the matrix A_{ec}.

Lemma 1 Let A_{ec} and Λ be given by (2.4) and (2.5), respectively. Then there exist matrices $V, V' \in \mathbb{C}^{n\times 2n}$ satisfying

$$A_{ec}\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} V \\ V' \end{bmatrix}\Lambda, \tag{2.6}$$

if and only if

$$MV\Lambda^2 + (D - BF_1)V\Lambda + (K - BF_0)V = 0 \tag{2.7}$$

and

$$V' = V\Lambda. \tag{2.8}$$

Proof Since

$$A_{ec}\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}(K-BF_0) & -M^{-1}(D-BF_1) \end{bmatrix}\begin{bmatrix} V \\ V' \end{bmatrix} = \begin{bmatrix} V' \\ -M^{-1}(K-BF_0)V - M^{-1}(D-BF_1)V' \end{bmatrix},$$

and

$$\begin{bmatrix} V \\ V' \end{bmatrix}\Lambda = \begin{bmatrix} V\Lambda \\ V'\Lambda \end{bmatrix},$$

the equation (2.6) is clearly equivalent to the relation (2.8) and

$$-M^{-1}(K - BF_0)V - M^{-1}(D - BF_1)V' = V'\Lambda. \tag{2.9}$$

Further, substituting (2.8) into (2.9) yields equation (2.7). □

The above lemma states that the Jordan matrix of A_{ec} is Λ if and only if there exists $V \in \mathbb{C}^{n\times 2n}$ satisfying (2.9), and in this case the corresponding eigenvector matrix of A_{ec} is given by

$$V_{ec} = \begin{bmatrix} V \\ V\Lambda \end{bmatrix}. \tag{2.10}$$

With the above understanding, the problem of eigenstructure assignment in the second-order dynamical system (1.1) via the proportional plus derivative feedback law (2.1) can be stated as follows.

Problem ESA (Eigenstructure assignment) Given system (1.1) satisfying Assumption 1, and the matrix $\Lambda = \mathrm{diag}(s_1, s_2, \cdots, s_{2n})$, with $s_i, i = 1,2,\cdots,2n$, being a group of self-conjugate complex numbers (not necessarily distinct), find a general parametric form for the matrices $F_0, F_1 \in \mathbb{R}^{r\times n}$ and $V \in \mathbb{C}^{n\times 2n}$ such that the matrix equation (2.7) and the condition

$$\det\begin{bmatrix} V \\ V\Lambda \end{bmatrix} \neq 0 \tag{2.11}$$

are satisfied.

Let

$$W = F_1 V\Lambda + F_0 V = [F_0 \quad F_1]\begin{bmatrix} V \\ V\Lambda \end{bmatrix}, \tag{2.12}$$

and then (2.7) becomes

$$MV\Lambda^2 + DV\Lambda + KV = BW. \tag{2.13}$$

Clearly, equation (2.13) becomes the type of generalized Sylvester matrix equation investigated in [15 ~ 17] when $M = 0$. In view of this fact, we call the equation (2.13) the second-order generalized Sylvester matrix equation.

It follows that, once a pair of matrices V and W that satisfy the second-order generalized Sylvester matrix equation (2.13) and condition (2.11) are obtained, a pair of control gain matrices can be easily obtained from (2.12) as follows:

$$[F_0 \quad F_1] = W\begin{bmatrix} V \\ V\Lambda \end{bmatrix}^{-1}. \tag{2.14}$$

Therefore, to solve Problem ESA, the key step is to find a solution to the following problem.

Problem SSE (Second-order Sylvester equation) Given the matrices $M, D, K \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times r}$ satisfying Assumption 1, and a diagonal matrix

$$\Lambda = \operatorname{diag}(s_1, s_2, \cdots, s_q) \in \mathbb{C}^{q\times q}, \quad (2.15)$$

find a parameterization for all the matrices $V \in \mathbb{C}^{n\times q}$ and $W \in \mathbb{C}^{r\times q}$ satisfying the second-order Sylvester matrix equation (2.13).

It should be noted that the number of the columns of the matrices V, W and Λ in the above Problem SSE are changed into q because it makes the Problem SSE more general.

3 Solution to problem SSE

Denote

$$V = [v_1 \quad v_2 \quad \cdots \quad v_q], \quad (3.1)$$

$$W = [w_1 \quad w_2 \quad \cdots \quad w_q], \quad (3.2)$$

then, in view of (2.15), we can convert the second-order Sylvester matrix equation (2.13) into the following column form

$$(s_i^2 M + s_i D + K)v_i = Bw_i, \ i = 1,2,\cdots,q. \quad (3.3)$$

3.1 Case of prescribed $s_i, i = 1,2,\cdots,q$.

The equations in (3.3) can be further written in the following form

$$\Pi_i \begin{bmatrix} v_i \\ w_i \end{bmatrix} = 0, \ i = 1,2,\cdots,q, \quad (3.4)$$

where

$$\Pi_i = [s_i^2 M + s_i D + K \quad -B], \ i = 1,2,\cdots,q. \quad (3.5)$$

This states that

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} \in \ker \Pi_i, \ i = 1,2,\cdots,q. \quad (3.6)$$

The following algorithm produces two sets of constant matrices N_i and D_i, $i = 1,2,\cdots,q$, to be used in the representation of the solution to the matrix equation (2.13).

Algorithm 1 (Solving N_i and D_i, $i = 1,2,\cdots,q$)

Step 1 Through applying SVD to the matrix Π_i, $i = 1,2,\cdots,q$, obtain two sets of matrices $P_i \in \mathbb{C}^{n\times n}$ and $Q_i \in \mathbb{C}^{(n+r)\times(n+r)}$, $i = 1,2,\cdots,q$, satisfying

$$\begin{cases} P_i \Pi_i Q_i = \begin{bmatrix} \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_{n_i}) & 0 \\ 0 & 0 \end{bmatrix}, \\ i = 1,2,\cdots,q, \end{cases} \quad (3.7)$$

where $\sigma_i > 0$, $i = 1,2,\cdots,n_i$, are the singular values of Π_i, and

$$n_i = \operatorname{rank}\,[s_i^2 M + s_i D + K \quad B], \ i = 1,2,\cdots,q. \quad (3.8)$$

Step 2 Obtain the matrices $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i = 1,2,\cdots,q$, by partitioning the matrix Q_i as follows:

$$Q_i = \begin{bmatrix} * & N_i \\ * & D_i \end{bmatrix}, \ i = 1,2,\cdots,q. \quad (3.9)$$

As a result of (3.7) and (3.9), the matrices $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i = 1,2,\cdots,q$, obtained through the above Algorithm 1 satisfy

$$\Pi_i \begin{bmatrix} N_i \\ D_i \end{bmatrix} = 0, \ i = 1,2,\cdots,q. \quad (3.10)$$

This indicates that the columns of $\begin{bmatrix} N_i \\ D_i \end{bmatrix}$ form a set of basis for $\ker \Pi_i$.

The above deduction clearly yields the following result.

Theorem 1 Let $n_i, i = 1,2,\cdots,q$, be defined by (3.8), and $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}$, $i = 1,2,\cdots,q$, be obtained via Algorithm 1. Then all the matrices V and W satisfying the second-order Sylvester matrix equation (2.13) can be parameterized by columns as follows:

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N_i \\ D_i \end{bmatrix} f_i, \ i = 1,2,\cdots,q, \quad (3.11)$$

where $f_i \in C^{n+r-n_i}$, $i = 1,2,\cdots,q$, are a set of arbitrary parameter vectors.

Definition 1 The second-order dynamical system (1.1) is called controllable if and only if the corresponding extended first-order state-space representation (1.2) and (1.3) is controllable.

Regarding the controllability of system (1.1), we have the following basic result.

Lemma 2 The second-order dynamical system (1.1) is controllable if and only if

$$\operatorname{rank}\,[s^2 M + sD + K \quad B] = n, \ \forall s \in \mathbb{C}. \quad (3.12)$$

Proof By the well-known PBH criterion, we need only to show that condition (3.12) is equivalent to

$$\operatorname{rank}\,[A_e - sI_{2n} \quad B_e] = 2n, \ \forall s \in \mathbb{C},$$

where A_e and B_e are given by (1.3).

Since

$$\begin{aligned} &\operatorname{rank}\,[A_e - sI_{2n} \quad B_e] \\ &= \operatorname{rank}\begin{bmatrix} -sI_n & I_n & 0 \\ -M^{-1}K & -M^{-1}D - sI_n & M^{-1}B \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} -sI_n & I_n & 0 \\ -K & -D - sM & B \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} 0 & I_n & 0 \\ -K - Ds - s^2 M & -D - sM & B \end{bmatrix} \\ &= n + \operatorname{rank}\,[s^2 M + sD + K \quad B], \end{aligned}$$

the conclusion is readily reached. □

Based on the above lemma, the following corollary of Theorem 1 can be immediately derived.

Corollary 1 Let system (1.1) be controllable, and Λ be given by (2.15), then the degrees of freedom existing in the general solution to the second-order Sylvester matrix equation (2.13) is q_r.

Proof Due to the controllability of system (1.1), we have from Lemma 2 $n_i = n, i = 1,2,\cdots,q$. Thus the conclusion immediately follows from Theorem 1. □

3.2 Case of undetermined $s_i, i = 1,2,\cdots,q$.

By performing the right factorization of

$$G(s) = (s^2M + sD + K)^{-1}B,$$

we can obtain a pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfying

$$(s^2M + sD + K)^{-1}B = N(s)D^{-1}(s). \quad (3.13)$$

Theorem 2 Let the system (1.1) be controllable, and $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfy the right factorization (3.13). Then

(1) The matrices V and W given by (3.1),(3.2)and

$$\begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} f_i, \ i = 1,2,\cdots,q \quad (3.14)$$

satisfy the second-order Sylvester matrix equation (2.13) for all $f_i \in \mathbb{C}^r, i = 1,2,\cdots,q$.

(2) When

$$\operatorname{rank} \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} r, \ i = 1,2,\cdots,q \quad (3.15)$$

hold, (3.14) gives all the solutions to Problem SSE.

Proof It follows from (3.13) that

$$\begin{cases} (s_i^2M + s_iD + K)N(s_i) - BD(s_i) = 0, \\ i = 1,2,\cdots,q, \end{cases} \quad (3.16)$$

Using (3.14) and (3.16), we derive

$$\begin{aligned} &(s_i^2M + s_iD + K)v_i - Bw_i \\ &= [(s_i^2M + s_iD + K)N(s_i) - BD(s_i)]f_i \\ &= 0, \ i = 1,2,\cdots,q. \end{aligned}$$

This states that the equations in (3.3) hold. Therefore, the first conclusion of the theorem also hold.

It follows from Corollary 1 that, under the controllability of system (1.1), the degrees of freedom existing in the general solution to the matrix equation (2.13), with Λ given by (2.15), is qr, while in the solution (3.14), the number of free parameters is just equal to qr. Further, it is clear that all these parameters in the solution (3.14) have contributions when condition (3.15) holds. With this we complete the proof. □

The right factorization (3.13) performs a fundamental role in the solution (3.14). When $s_i, i = 1,2,\cdots,q$, are chosen to be different from the zeros of det $(s^2M + sD + K)$, we can take

$$\begin{cases} N(s) = \operatorname{Adj}(s^2M + sD + K)B, \\ D(s) = \det(s^2M + sD + K). \end{cases}$$

For general numerical algorithms solving such right factorizations, one can refer to [18,19]. The following simple procedure can also be used.

Algorithm 2(Right coprime factorization)

Step 1 Under the R-controllability of system (1.1), find a pair of unimodular matrices $P(s)$ and $Q(s)$, of appropriate dimensions, satisfying

$$P(s)[s^2M + sD + K \quad -B]Q(s) = [I_n \quad 0].$$

Step 2 Obtain the pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ by partitioning the unimodular matrix $Q(s)$ as follows:

$$Q(s) = \begin{bmatrix} * & N(s) \\ * & D(s) \end{bmatrix}.$$

It is worth pointing out that the pair of polynomial matrices $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ satisfying the right factorization (3.13) obtained from the above Algorithm 2 are right coprime since

$$\operatorname{rank} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = r, \ \forall s \in \mathbb{C}.$$

This condition implies the condition (3.15) which ensures the completeness of the solution (3.14).

To finish this section, let us finally give a remark on the extension of the result.

Remark 1 The main results in this section can be easily extended into the case that the matrix Λ is a general Jordan form. In fact, when Λ is replaced with the following Jordan matrix $J = \text{Blockdiag}(J_1, J_2, \cdots, J_p) \in \mathbb{C}^{q\times q}$, with

$$J_i = \begin{bmatrix} s_i & 1 & & \\ & s_i & \ddots & \\ & & \ddots & 1 \\ & & & s_i \end{bmatrix} \in \mathbb{C}^{p_i\times p_i}, \ i = 1,2,\cdots,p,$$

following the development in [15,16], we can show that all matrices V and W satisfying the second-order Sylvester matrix equation (2.13) are given by

$$\begin{cases} V = [V_1 \quad V_2 \quad \cdots \quad V_p], \\ V_i = [v_{i1} \quad v_{i2} \quad \cdots \quad v_{ip_i}], \end{cases}$$

and

$$\begin{cases} W = [W_1 \quad W_2 \quad \cdots \quad W_p], \\ W_i = [w_{i1} \quad w_{i2} \quad \cdots \quad w_{ip_i}], \end{cases}$$

with

$$\begin{aligned} \begin{bmatrix} v_{ik} \\ w_{ik} \end{bmatrix} &= \begin{bmatrix} N(s_i) \\ D(s_i) \end{bmatrix} f_k + \begin{bmatrix} N^{(1)}(s_i) \\ D^{(1)}(s_i) \end{bmatrix} f_{k-1} + \cdots \\ &+ \frac{1}{(k-1)!} \begin{bmatrix} N^{(k-1)}(s_i) \\ D^{(k-1)}(s_i) \end{bmatrix} f_1, \quad (3.17) \\ &k = 1,2,\cdots,p_i, \ i = 1,2,\cdots,p, \end{aligned}$$

where $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ are a pair of polynomial matrices satisfying the right factorization (3.13).

4 Solution to problem ESA

Regarding the solutions to Problem ESA, we have the following two results which are based on the discussion in Section 2 and the results in Section 3.

Theorem 3 Let $n_i, i = 1,2,\cdots,2n$, be given by (3.8), and $N_i \in \mathbb{R}^{n\times(n+r-n_i)}$ and $D_i \in \mathbb{R}^{r\times(n+r-n_i)}, i = 1,2,\cdots,2n$, be given by Algorithm 1. Then

1) Problem ESA has solutions if and only if there exist a

group of parameters $f_i \in \mathbb{C}^{n+r-n_i}$, $i = 1,2,\cdots,2n$, satisfying the following constraints:

Constraint C1: $f_i = \bar{f}_j$ if $s_i = \bar{s}_j$.

Constraint C2$_a$: $\det V_{ca} \neq 0$ with

$$V_{ca} = \begin{bmatrix} N_1 f_1 & N_2 f_2 & \cdots & N_{2n} f_{2n} \\ s_1 N_1 f_1 & s_2 N_2 f_2 & \cdots & s_{2n} N_{2n} f_{2n} \end{bmatrix}. \tag{4.1}$$

2) When the above condition is met, all the solutions to the Problem ESA are given by

$$V = [N_1 f_1 \quad N_2 f_2 \quad \cdots \quad N_{2n} f_{2n}] \tag{4.2}$$

and

$$[F_0 \quad F_1] = [D_1 f_1 \quad D_2 f_2 \quad \cdots \quad D_{2n} f_{2n}] V_{ca}^{-1}, \tag{4.3}$$

where $f_i \in \mathbb{C}^{n+r-n_i}$, $i = 1,2,\cdots,2n$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_a$.

Theorem 4 Let system (1.1) be controllable, and $N(s) \in \mathbb{R}^{n\times r}[s]$ and $D(s) \in \mathbb{R}^{r\times r}[s]$ be a pair of polynomial matrices satisfying the right factorization (3.13). Then

(1) Problem ESA has solutions if and only if there exist a group of parameters $f_i \in \mathbb{C}^r$, $i = 1,2,\cdots,2n$, satisfying Constraint C1 and Constraint C2$_b$: $\det V_{cb} \neq 0$ with

$$V_{cb} = \begin{bmatrix} N(s_1)f_1 & N(s_2)f_2 & \cdots & N(s_{2n})f_{2n} \\ s_1 N(s_1)f_1 & s_2 N(s_2)f_2 & \cdots & s_{2n} N(s_{2n})f_{2n} \end{bmatrix}. \tag{4.4}$$

(2) When the above condition is met, all the solutions to the Problem ESA are given by

$$V = [N(s_1)f_1 \quad N(s_2)f_2 \quad \cdots \quad N(s_{2n})f_{2n}], \tag{4.5}$$

and

$$[F_0 \quad F_1] = [D(s_1)f_1 \quad D(s_2)f_2 \quad \cdots \quad D(s_{2n})f_{2n}] V_{cb}^{-1}, \tag{4.6}$$

where $f_i \in \mathbb{C}^r$, $i = 1,2,\cdots,2n$, are arbitrary parameter vectors satisfying Constraints C1 and C2$_b$.

The proof of the above two theorems can be easily carried out based on the discussion in Section 2 and the results in Section 3. The only thing that needs to be mentioned is that Constraint C1 is required because it is a necessary and sufficient condition for the matrices F_0 and F_1 given by (4.3) or (4.6) to be real.

Here are some remarks on the above results.

Remark 2 The above two theorems give complete parametric solutions to the Problem ESA. The free parameter vectors f_i, $i = 1,2,\cdots,2n$, represent the degrees of freedom in the eigenstructure assignment design, and can be sought to meet certain desired system performances. It should be noted that Constraint C1 is not a restriction at all, instead it only gives a way of selecting these design parameter vectors.

Remark 3 It follows from the well-known pole assignment result that Problem ESA has a solution when the system (1.1) is controllable and the closed-loop eigenvalues s_i, $i = 1,2,\cdots,2n$, are restricted to be distinct. In this case, there exist parameter vectors f_i, $i = 1,2,\cdots,2n$, satisfying Constraint C2$_a$ or C2$_b$. As a matter of fact, it can be reasoned that, in this case, "almost all" parameter vectors f_i, $i = 1,2,\cdots,2n$, satisfy Constraint C2$_a$ or C2$_b$. Therefore, in such applications Constraint C2$_a$ or C2$_b$ can often be ignored.

Remark 4 The solution given in Theorem 3 utilizes only a series of singular value decompositions, and hence is numerically very simple and reliable. As for the solution given in Theorem 4, it has the advantage that the closed-loop eigenvalues s_i, $i = 1,2,\cdots,2n$, can be set undetermined and used as a part of extra design degrees of freedom to be sought with f_i, $i = 1,2,\cdots,2n$, by certain optimization procedures. Furthermore, it happens that the solution given in Theorem 4 is a natural generalization of the parametric solution in [16] proposed for eigenstructure assignment in first-order state-space systems.

Remark 5 The eigenstructure assignment results can be easily extended into the defective case, that is, the case that the closed-loop system possesses a general Jordan form (refer to Remark 1). However, in terms of the control systems design, this is not desired since the eigenvalues of defective matrices are more sensitive to parameter perturbations than those of nondefective ones.

5 Example

Consider a simple linear dynamical system consisting of three lumped mass-spring dashpots, connected in series and fixed at one end as shown in Fig. 1. When $m = 1$; $k_1 = k_2 = 5$, $k_3 = 20$ and $c_1 = c_3 = 2$, $c_2 = 0.5$, the equation of motion can be written in the form of (1.1) with $M = I_3$, and

$$D = \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & 2 \\ 0 & -2 & 2 \end{bmatrix},$$

$$K = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

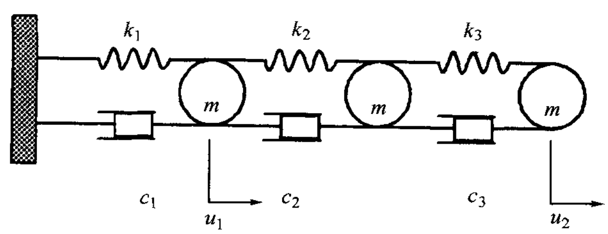

Fig. 1 Three-spring-mass-dashpot system.

Let $\Lambda = \operatorname{diag}[s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6]$, with s_i, $i = 1 \sim 6$, being a group of distinct real scalars, by applying Algorithm 2 we obtain the pair of polynomial matrices satisfying the right coprime factorization (3.13) as follows:

$$N(s) = \begin{bmatrix} 2s + 200 & 0 \\ 0 & 2s + 20 \\ -0.5s - 5 & s^2 + 2.5s + 25 \end{bmatrix},$$

$$D(s) = \begin{bmatrix} 2s^3 + 25s^2 + 70s + 200 & -s^2 - 20s - 100 \\ -0.5s^3 - 6s^2 - 20s - 100 & s^4 + 4.5s^3 + 46s^2 + 20s + 100 \end{bmatrix}.$$

By denoting $f_i = [\alpha_i \quad \beta_i]^{\mathrm{T}}$, $i = 1 \sim 6$, we have

$$v_i = \begin{bmatrix} (2s_i + 20)\alpha_i \\ (2s_i + 20)\beta_i \\ (-0.5s_i - 5)\alpha_i + (s_i^2 + 2.5s_i + 25)\beta_i \end{bmatrix}, \quad i = 1 \sim 6,$$

and

$$w_i = \begin{bmatrix} (2s_i^3 + 25s_i^2 + 70s_i + 200)\alpha_i + (-s_i^2 - 20s_i - 100)\beta_i \\ (-0.5s_i^3 - 6s_i^2 - 20s_i - 100)\alpha_i + (s_i^4 + 4.5s_i^3 + 46s_i^2 + 20s_i + 100)\beta_i \end{bmatrix}, \quad i = 1 \sim 6,$$

and the gain matrices are given by

$$[F_0 \quad F_1] = [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6] V_{ec}^{-1}$$

with

$$V_{ec} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ s_1 v_1 & s_2 v_2 & s_3 v_3 & s_4 v_4 & s_5 v_5 & s_6 v_6 \end{bmatrix}.$$

All the design parameters α_i, β_i and s_i, $i = 1 \sim 5$, can be taken as arbitrary real scalars which make the above matrix V_{ec} nonsingular (note that s_i, $i = 1 \sim 5$, should be restricted negative due to the closed-loop stability).

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