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On the dual linear periodic time-delay system: Spectrum and Lyapunov matrices, with application to $\mathcal{H}_2$ analysis and balancing

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Summary

We present novel theoretical concepts for linear time-periodic systems with multiple delays, which are closely related to the spectral properties and Lyapunov matrices. At the basis of the main results is the associated dual system, constructed by transposition of the systems matrices and affine transformations of their arguments. We introduce, for the first time, the concepts of the $\mathcal{H}_2$ norm and the dual Lyapunov matrix of periodic systems with delays. We show that the primal and dual system have the same $\mathcal{H}_2$ norm, characterized by primal and dual delay Lyapunov equations, which extend the well-known results for time-invariant systems with delays, and periodic systems without delays. Having at hand the pair of primal-dual Lyapunov matrices, along with some energy interpretations, allow us to generalize the concept of position balancing and explore its potential for model reduction. The obtained results are illustrated by several examples, including the delayed Mathieu equation.

KEYWORDS

$\mathcal{H}_2$ norm, model reduction, delay systems, Lyapunov matrices, periodic systems

1 | INTRODUCTION

In this paper, we consider linear time-periodic systems with multiple delays of the form

$$\begin{cases} \dot{x}(t) = \sum_{i=0}^{m} A_i(t)x(t-\tau_i) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable at time t, and functions $A_i : \mathbb{R} \to \mathbb{R}^{n\times n}, \ t \mapsto A_i(t), \ i = 0, \dots, m, B : \mathbb{R} \to \mathbb{R}^{n\times n_i}, \ t \mapsto B(t), C : \mathbb{R} \to \mathbb{R}^{n_o\times n}, \ t \mapsto C(t)$ are smooth and T-periodic. The delays are sorted in increasing order such that

$$0 = \tau_0 \leq \tau_1 < \tau_2 < \dots < \tau_m.$$

In what follows we refer to (1) as the *primal* system. Equations of the form (1) are suitable for modeling a variety of problems from different fields, such as machine tool vibrations,[1] problems related to neural networks,[2] robotics,[3] and optimal control.[4] The complexity, induced by the combination of periodicity and delays, makes them also of main interest from a theoretical perspective. The relevance has motivated several contributions to stability and robust stability analysis study in recent years, see, for example, References 5-10 and the references therein.

An important robustness measure in the analysis of linear control systems and the synthesis of robust controllers is the $\mathcal{H}_2$ norm of an appropriately defined input-output map. In this paper we introduce the concept of $\mathcal{H}_2$ norm for systems of form (1), thereby extending the definition for linear periodic delay-free systems in Reference 11, and we provide a computational expression in terms of the so-called delay Lyapunov matrix. The latter was introduced in Reference 9 for periodic delay equations within the Lyapunov-Krasovskii functionals framework for stability analysis. The expression for the norm allows to perform $\mathcal{H}_2$ analysis in a standard Lyapunov equation framework, and generalizes the well-known expressions for delay-free periodic systems, given in References 11-14, and for linear time-invariant delay systems, proposed in Reference 15.

In the context of $\mathcal{H}_2$ synthesis and eigenvalue optimization for time-invariant systems, as well as model reduction by balanced truncation, an important role is played by the dual or transposed system, described in the frequency domain by taking the point-wise transposition of the transfer matrix. As suggested by theorem 4.3 of Reference 8, where the eigenvalue problem for the monodromy operator of (1) (with zero input) is related to a finite-dimensional nonlinear eigenvalue problem for the case where delays and period are commensurate, the natural generalization of the dual system towards linear time-periodic System (2) is described by

$$\begin{cases} \dot{z}(t) = \sum_{i=0}^{m} A_i^\top(-t+\tau_i)z(t-\tau_i) + C^\top(-t)\xi(t), \\ \eta(t) = B^\top(-t)z(t), \end{cases} \tag{2}$$

which will be conformed by all properties and connections with (1) that will be derived in the subsequent sections. We note that the construction of (2) involves both taking the transpose of the coefficient matrices and affine transformations of their arguments. Surprisingly, the shifts in the arguments of A_i depend on i, hence, evaluating the right-hand side of (2) at a particular time-instant involves evaluating matrices A_i in an asynchronous matter. We further note that the dual of dual System (2) corresponds to the original System (1).

The dual system allows us to introduce, for the first time, the concept of dual Lyapunov matrix associated with periodic time-delay system (1). Besides revealing relations involving the spectra and eigenfunctions of the monodromy matrices, we show that systems (1) and (2) have the same $\mathcal{H}_2$ norm. We shall also explain how the dual characterization of this norm, in terms of both Lyapunov matrices, directly extends the corresponding results for both linear time-invariant systems in Reference 11 and periodic systems without delay in Reference 15.

Finally, the availability of a pair of primal-dual Lyapunov matrices, which can be related to observability, respectively, controllability properties of the system, allows us to generalize the concept of balancing. We provide some energy interpretations of the primal and dual Lyapunov matrix, which play a key role in generalizing the position balancing approach for time-invariant delay systems, proposed in Reference 16. The balancing on its turn provides a natural way to obtain reduced models by truncation. As these reduced models are also in the form of a periodic time-delay system, the reduction approach is structure exploiting. We use standard notations throughout the paper, and only specify the meaning of a notation if necessary.

The remainder of the paper is organized as follows. In Section 2 we analyze the spectral properties of (1) and (2). In Section 3 we introduce pairs of (primal-dual) Lyapunov matrices, and we define the $\mathcal{H}_2$ norm for system (1). Subsequently, we derive explicit expressions for this norm in terms of the primal and dual Lyapunov matrix. Energy interpretations of Lyapunov matrices are given in Section 4. Finally, Section 5 is devoted to applications and implications of the obtained theoretical results, more specifically, addressing the computation of $\mathcal{H}_2$ norms in a Lyapunov equation framework, and exploring the potential of model reduction by position balancing. Finally, some concluding remarks are given in Section 6.

2 | SPECTRAL PROPERTIES AND STABILITY

As we investigate internal stability, we consider system (1) with zero input. In order to define a forward solution, in general, a function segment over a time-interval of length τ_m is required. More precisely, for any initial function $\varphi \in X$, where $X := C([-\tau_m,\ 0], \mathbb{C}^n)$, with $C([-\tau_m,\ 0], \mathbb{C}^n)$ denoting the space of $\mathbb{C}^n$-valued continuous function on $[-\tau_m, 0]$, and

$t_0 \in \mathbb{R}$, the initial value problem

$$\begin{cases} \dot{x}(t) = \sum_{j=0}^{m} A_j(t)x(t-\tau_j), & t \in [t_0,\ \infty), \\ x(t) = \varphi(t-t_0), & t \in [t_0-\tau_m,\ t_0], \end{cases} \tag{3}$$

has a unique forward solution, which we denote by $x(t; t_0, \varphi)$. The corresponding state at time t, $t \geq t_0$, that is, the minimal information to continue the solution, is denoted by $x_t(\cdot; t_0, \varphi) \in X$, defined by

$$x_t(\vartheta; t_0, \varphi) = x(t+\vartheta; t_0, \varphi), \quad \vartheta \in [-\tau_m,\ 0].$$

The translation along the solutions is described by the solution operator $\mathcal{T}(t_1, t_0) : X \to X$, parametrized by $t_0 \in \mathbb{R},\ t_1 \in \mathbb{R}_+$ and defined through the relation

$$\mathcal{T}(t_1, t_0)\ \varphi = x_{t_0+t_1}(\cdot; t_0, \varphi), \quad \varphi \in X.$$

It can be shown that the spectrum of operator $\mathcal{T}(T, t_0)$ (recall that T is the period of functions A_j), is an at most countable compact set in the complex plane, with zero as only possible accumulation point. The spectrum is independent of the choice of t_0 and all its nonzero elements are eigenvalues.[17] Operator $\mathcal{T}(T, 0)$ is called the monodromy operator and denoted by $\mathcal{U}$ in what follows. Hence, we have

$$\mathcal{U}\varphi = x_T(\cdot; 0, \varphi), \quad \varphi \in X.$$

The nonzero eigenvalues of the monodromy operator are called Floquet multipliers of (1). By definition they satisfy the infinite-dimensional linear eigenvalue problem

$$\mathcal{U}\varphi = \mu\ \varphi, \qquad \mu \in \mathbb{C},\ \varphi \in X \setminus \{0\}. \tag{4}$$

As the Floquet multipliers determine the growth/decay of solutions of (1) in time-intervals of length T and the system is T-periodic, they are important for stability assessment. In particular, the zero solution of (1) is uniformly exponentially stable if and only if all Floquet multipliers have modulus strictly smaller than one. For more results on Floquet theory for (3), we refer to chapter 8 of Reference 18.

The following theorem relates the spectra of the monodromy operators corresponding to (1) and (2).

Theorem 1. *Let $\mathcal{U}$, respectively $\mathcal{U}_D$, be the monodromy operator corresponding to (1), respectively (2). Then their spectra satisfy $\sigma(\mathcal{U}) \setminus \{0\} = \sigma(\mathcal{U}_D) \setminus \{0\}$.*

Proof. From theorems 2.2, 4.1, and 4.3 in Reference 8, a one-to-one correspondence between the eigenvalues of $\mathcal{U}$ and $\mathcal{U}_D$ can be established for the special case where the numbers $(T, \tau_1, \dots, \tau_m)$ are commensurate. Since small delay perturbations correspond to compact perturbations on the monodromy operator and the set of nonnegative commensurate $(m+1)$-tuples is dense in $\mathbb{R}_+^{m+1}$, this result carries over to the general case. ■

In the remainder of this section we strengthen Theorem 1 for two special cases of (1) and (2). In the case of *commensurate delays and period*, more precisely, under the condition that there exist real number $\Delta > 0$, integers N and n_j, for $j = 1, \dots, m$, such that

$$T = N\Delta, \quad \tau_j = n_j\Delta, \ \ j = 1, \dots, m,$$

it is shown in Reference 8 that the Floquet multipliers of (1) coincide with the nonzero solutions of a finite-dimensional nonlinear eigenvalue problem of the form

$$M(\mu)\ v = 0,\ \ \mu \in \mathbb{C},\ \ v \in \mathbb{C}^{Nn} \setminus \{0\}, \tag{5}$$

where function $M : \mathbb{C} \to \mathbb{C}^{Nn \times Nn}$ is analytic in $\mathbb{C} \setminus \{0\}$ and, for a specified value of μ, evaluating the left-hand side of (5) involves solving an initial value problem. A pair (μ, v) satisfying (5) is called a right eigenpair of M, and v a right eigenvector corresponding to μ. A left eigenpair $(\mu, u) \in \mathbb{C} \times \mathbb{C}^{Nn} \setminus \{0\}$ satisfies $u^* M(\mu) = 0$, and u is called a left eigenvector of M corresponding to μ.

If μ is an eigenvalue of (5), then it follows from theorem 2.2 of Reference 8 that a right eigenvector can be obtained from stacking samples of the eigenfunction φ of $\mathcal{U}$, corresponding to Floquet multiplier μ. At the same time, according to theorem 4.3 of Reference 8 a left eigenvector of (5) can be obtained by stacking samples of eigenfunction ψ of operator $\mathcal{U}_D$, corresponding to Floquet multiplier $\overline{\mu}$, that is, such that

$$\mathcal{U}_D \psi = \overline{\mu} \psi.$$

We refer to Reference 8 for the details of these results.

Finally, we consider the *delay-free case*, where $\mathcal{U}$ and $\mathcal{U}_D$ are $n \times n$ matrices.

Proposition 1. *For the case $m = 0$ in (1) and (2), it holds that $\mathcal{U}_D = \mathcal{U}^*$.*

Proof. In the delay-free case, it is possible to express $\mathcal{U} = K(T, 0)$, with K the fundamental solution defined through $K(t, s) = 0$ for $t < s$, $K(s, s) = I$ and

$$\frac{\partial}{\partial t} K(t, s) = A(t) K(t, s), \quad t \geq s,$$

which induces the property

$$\frac{\partial}{\partial s} K(t, s) = -K(t, s) A(s), \quad t \geq s.$$

We observe that the fundamental solution K_D of the dual system satisfies $K_D(t, s) = K^\top(-s, -t)$, which on its turn leads to

$$\mathcal{U} = K(T, 0) = K_D^\top(0, -T) = K_D^\top(T, 0) = \mathcal{U}_D^\top,$$

from which the assertion follows. ∎

3 | LYAPUNOV MATRICES AND THE $\mathcal{H}_2$ NORM

We introduce the concept of $\mathcal{H}_2$ norm of system (1), and express it in terms of the Lyapunov matrix (Proposition 2). Next, we introduce the dual Lyapunov matrix. Finally, we show that the primal and dual system have the same $\mathcal{H}_2$ norm (Theorem 2), which allows us to characterize the latter by means of both primal and dual delay Lyapunov equations, in such a way that the generalization of existing results is emphasized (Theorem 3). Throughout the section and the remainder of the paper we assume that system (1), and thus (2), are exponentially stable, which is equivalent to the property that all Floquet multipliers are located in the open unit disk.

The definition of the Lyapunov matrix of System (1) relies on the so-called *fundamental matrix*, which generalizes the concept of fundamental solution for delay-free systems. The fundamental matrix of (1), which we denote by K, is the function $K : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$, $(t, s) \to K(t, s)$, satisfying,[19]

$$\frac{\partial}{\partial t} K(t, s) = A_0(t) K(t, s) + \sum_{i=1}^{m} A_i(t) K(t - \tau_i, s), \quad t \geq s, \tag{6}$$

with $\frac{\partial K}{\partial t}$ denoting the right-hand derivative of K with respect to t, as well as

$$K(t, s) = 0, \ \text{ for } t < s \ \text{ and } \ K(s, s) = I.$$

The fundamental matrix associated with the dual System (2) is denoted by K_D. The following lemma states it can be expressed as a function of K.

Lemma 1. *The equality $K_D(t,s) = K^\top(-s,-t)$ holds for all $(t,s) \in \mathbb{R}^2$.*

Proof. The case where $t \le s$, which implies $-s \le -t$, is trivial. Therefore, we restrict ourselves to the case where $t > s$ in the remainder of the proof.

Besides Equation (6), function K also satisfies (see Reference 9)

$$-\frac{\partial}{\partial s}K(t,s) = \sum_{i=0}^{m} K(t, s+\tau_i)A_i(s+\tau_i), \quad t \ge s, \tag{7}$$

which implies

$$\frac{\partial}{\partial t}K(-s,-t) = \sum_{i=0}^{m} K(-s,-t+\tau_i)A_i(-t+\tau_i), \quad -s \ge -t$$

and

$$\frac{\partial}{\partial t}K^\top(-s,-t) = \sum_{i=0}^{m} A_i^\top(-t+\tau_i)K^\top(-s,-(t-\tau_i)), \quad t \ge s.$$

The assertion of the lemma now follows from the definition of K_D. ■

Remark 1. In the linear time-invariant case, expressions (6) and (7) lead to the well-known commutativity like property

$$\begin{cases} \frac{\partial}{\partial t}K(t,0) = \sum_{i=0}^{m} A_i K(t-\tau_i, 0), \\ \frac{\partial}{\partial t}K(t,0) = \sum_{i=0}^{m} K(t-\tau_i, 0)A_i, \quad t \ge 0. \end{cases}$$

The Lyapunov matrix of system (1), associated with a smooth T-periodic function $W : \mathbb{R} \to \mathbb{R}^{n\times n},\ t \mapsto W(t)$ satisfying $W(t) \ge 0$ for all t, can now be defined as

$$U(s_1,s_2) := \int_{-\infty}^{\infty} K^\top(t,s_1)W(t)K(t,s_2)dt.$$

Similarly as in Reference 9, where the case of constant W is considered, it can be easily shown that U satisfies the following properties:

1. The dynamic property: for $s_2 > s_1$ we have

$$\begin{aligned} \frac{\partial}{\partial s_1}U(s_1,s_2) &= -\sum_{i=0}^{m} A_i^\top(s_1+\tau_i)U(s_1+\tau_i, s_2), \\ \frac{\partial}{\partial s_2}U(s_1,s_2) &= -\sum_{i=0}^{m} U(s_1, s_2+\tau_i)A_i(s_2+\tau_i) - K^\top(s_2,s_1)W(s_2); \end{aligned} \tag{8}$$

2. the symmetry property:

$$U^\top(s_1,s_2) = U(s_2,s_1); \tag{9}$$

3. the periodicity property:

$$U(s_1,s_2) = U(s_1+T, s_2+T); \tag{10}$$

4. the algebraic property:

$$\frac{d}{ds}U(s,s) = -\sum_{i=0}^{m} A_i^\top(s+\tau_i)U(s+\tau_i, s) - \sum_{i=0}^{m} U(s, s+\tau_i)A_i(s+\tau_i) - W(s). \tag{11}$$

Theorem 4.1 of Reference 20 states that, under the taken assumption of exponential stability, the Lyapunov matrix is uniquely determined by the first equation of (8), along with Equations (9) to (11), for the special case where W is a constant positive definite matrix. As W is the only inhomogeneous term in the linear equations, this result carries over to an arbitrary periodic function. The argument is by contradiction: if there would be distinct solutions for a particular periodic function W, then there would also be distinct solutions for any constant positive function. In the light of this comment, we do not consider the second equation of (8) anymore in what follows.

The response of System (1) to a Dirac delta function $\delta(t-s)$ at the input is given by

$$g(t,s) := C(t)K(t,s)B(s).$$

It is important to note that the system's response depends on the moment s of excitation. However, function g satisfies the periodicity property $g(t,s+T) = g(t-T,s)$. Therefore, the squared $\mathcal{H}_2$ norm of the input-output map $\mathcal{G}$ of System (1) is defined as the mean squared $\mathcal{L}_2$ norm of the impulse responses, that is,

$$\begin{aligned} ||\mathcal{G}||^2_{\mathcal{H}_2} &:= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \mathrm{tr}(g^\top(t,s)g(t,s))dt\, ds \\ &= \frac{1}{T}\int_0^T \int_{-\infty}^{\infty} \mathrm{tr}(B^\top(s)K^\top(t,s)C^\top(t)C(t)K(t,s)B(s))dt\, ds. \end{aligned}$$

From the definition of the Lyapunov matrix, we directly arrive at the following characterization.

Proposition 2. *The $\mathcal{H}_2$ norm of (1) satisfies*

$$||\mathcal{G}||_{\mathcal{H}_2} = \sqrt{\frac{1}{T}\int_0^T \mathrm{tr}\left(B^\top(s)U(s,s)B(s)\right) ds},$$

where $U(s_1,s_2)$ is the Lyapunov matrix of System (1), associated with $W(t) = C^\top(t)C(t)$.

We now turn our attention to the dual System (2). The Lyapunov matrix of System (2), associated with a T-periodic matrix valued function W, is the matrix function $V : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$, defined as

$$V(s_1,s_2) := \int_{-\infty}^{\infty} K_D^\top(t,s_1)W(t)K_D(t,s_2)dt,$$

and which we refer to as the *dual Lyapunov matrix* corresponding to (1). From (8) to (11) and a comparison of (2) and (1), it directly follows that V satisfies:

- The dynamic property:

$$\frac{\partial}{\partial s_1}V(s_1,s_2) = -\sum_{i=0}^{m} A_i(-s_1)V(s_1+\tau_i,s_2),\ s_2 > s_1,$$

- The symmetry property: $V^\top(s_1,s_2) = V(s_2,s_1)$,
- The periodicity property: $V(s_1,s_2) = V(s_1+T,s_2+T)$, and
- The algebraic property:

$$\frac{d}{ds}V(s,s) = -\sum_{i=0}^{m} A_i(-s)V(s+\tau_i,s) - \sum_{i=0}^{m} V(s,s+\tau_i)A_i^\top(-s) - W(s).$$

The $\mathcal{H}_2$ norm of the input-output map $\mathcal{G}_D$ of (2) satisfies

$$||\mathcal{G}_D||_{\mathcal{H}_2} = \sqrt{\frac{1}{T}\int_0^T \mathrm{tr}\left(C(-s)V(s,s)C^\top(-s)\right) ds}, \tag{12}$$

where V is the Lyapunov matrix of (2) associated with $W(t) = B(-t)B^\top(-t)$.

We are now ready to make further connections between (1) and (2), and the associated Lyapunov matrices U and V. Lemma 1 directly leads us to the following characterization.

Proposition 3. *Lyapunov matrix V of dual System (2), associated with function $W(t)$, satisfies*

$$V(s_1, s_2) = \int_{-\infty}^{\infty} K(-s_1, -t)W(t)K^{\top}(-s_2, -t)dt.$$

We can now state the main result of the section.

Theorem 2. *Systems (1) and (2) have the same $\mathcal{H}_2$ norm, i.e. $||\mathcal{G}||_{\mathcal{H}_2} = ||\mathcal{G}_D||_{\mathcal{H}_2}$.*

Proof. Let $l(t,s) = \mathrm{tr}\left(g^{\top}(t,s)g(t,s)\right)$. Then we have for all $N \in \mathbb{N}$,

$$\begin{aligned} ||\mathcal{G}||_{\mathcal{H}_2}^2 &= \frac{1}{T}\int_0^T \int_s^{\infty} l(t,s)dt\, ds \\ &= \frac{1}{2NT}\int_{-NT}^{NT} \int_s^{\infty} l(t,s)dt\, ds \\ &= \frac{1}{2NT}\int_{-NT}^{NT} \int_s^{NT} l(t,s)dt\, ds + R_1(N), \end{aligned} \tag{13}$$

with

$$R_1(N) = \frac{1}{2NT}\int_{-NT}^{NT} \int_{NT}^{\infty} l(t,s)dt\, ds.$$

Expression (13) can be further developed as

$$\begin{aligned} ||\mathcal{G}||_{\mathcal{H}_2}^2 &= \frac{1}{2NT}\int_{-NT}^{NT} \int_{-NT}^{t} l(t,s)ds\, dt + R_1(N) \\ &= \frac{1}{2NT}\int_{-NT}^{NT} \int_{-\infty}^{t} l(t,s)ds\, dt + R_1(N) - R_2(N), \end{aligned} \tag{14}$$

where

$$R_2(N) = \frac{1}{2NT}\int_{-NT}^{NT} \int_{-\infty}^{-NT} l(t,s)ds\, dt.$$

Note that l can also be expressed as

$$\begin{aligned} l(t,s) &= \mathrm{tr}(g(t,s)g^{\top}(t,s)) \\ &= \mathrm{tr}\left(C(t)K(t,s)B(s)\, B^{\top}(s)K^{\top}(t,s)C^{\top}(t)\right) \\ &= \mathrm{tr}\left(C(t)K_D^{\top}(-s,-t)B(s)\, B^{\top}(s)K_D(-s,-t)C^{\top}(t)\right), \end{aligned} \tag{15}$$

where we used Lemma 1. Using a substitution $s \leftarrow -s$, $t \leftarrow -t$, the first term of the final expression in (14) can rewritten as

$$\begin{aligned} &\frac{1}{2NT}\int_{-NT}^{NT} \int_t^{\infty} \mathrm{tr}\left(C(-t)K_D^{\top}(s,t)B(-s)\, B^{\top}(-s)K_D(s,t)C^{\top}(-t)\right) ds\, dt \\ &= \frac{1}{T}\int_0^T \int_t^{\infty} \mathrm{tr}\left(C(-t)K_D^{\top}(s,t)B(-s)\, B^{\top}(-s)K_D(s,t)C^{\top}(-t)\right) ds\, dt \\ &= ||\mathcal{G}_D||_{\mathcal{H}_2}^2 . \end{aligned}$$

Hence, we arrive at

$$||\mathcal{G}||_{\mathcal{H}_2}^2 = ||\mathcal{G}_D||_{\mathcal{H}_2}^2 + R_1(N) - R_2(N), \quad \forall N \in \mathbb{N}. \tag{16}$$

Since we assumed that the zero solution of (1) is exponentially stable, which implies it is uniformly exponentially stable, there exist constants $C_1 > 0$ and $\alpha > 0$ such that

$$||l(t,s)||_2 \leq C_1 e^{-\alpha(t-s)}, \quad t \geq s.$$

As a consequence, function R_1 satisfies

$$\begin{aligned} ||R_1(N)||_2 &\leq \frac{1}{2NT} \int_{-NT}^{NT} \int_{NT}^{\infty} C_1 e^{-\alpha(t-s)} dt\, ds \\ &= \frac{C_1}{2NT\alpha^2} \left(1 - e^{-2\alpha NT}\right), \end{aligned}$$

revealing the property

$$\lim_{N\to\infty} R_1(N) = 0. \tag{17}$$

From (15) it follows that $l(-t,-s) = \mathrm{tr}(g_D^\top(s,t)\, g_D(s,t))$, with $g_D(t,s)$ the response of dual System (2) to the Dirac delta function $\delta(t-s)$. Hence, there also exists a constant C_2 such that

$$||l(-t,-s)||_2 \leq C_2 e^{-\alpha(s-t)}, \quad s \geq t,$$

implying

$$||l(t,s)||_2 \leq C_2 e^{\alpha(s-t)}, \quad s \leq t.$$

Using the same arguments as for R_1, this estimates leads to

$$\lim_{N\to\infty} R_2(N) = 0. \tag{18}$$

Combining (16), (17), and (18), we arrive at the assertion of the theorem, and the proof is completed. ■

To retrieve well known characterizations of the $\mathcal{H}_2$ norm for special instances of Equation (2), we introduce function $\tilde{V} : R^2 \to \mathbb{R}^{n\times n}$ via the relation

$$\tilde{V}(s_1, s_2) := V(T - s_1, T - s_2).$$

By expressing (12) in terms of function $\tilde{V}$ and using Theorem 2, we arrive at the following statement.

Theorem 3. *The $\mathcal{H}_2$ norm of System (1) can be expressed as*

$$||\mathcal{G}||_{\mathcal{H}_2}^2 = \frac{1}{T}\int_0^T \mathrm{tr}\left(B^\top(s)U(s,s)B(s)\right) ds = \frac{1}{T}\int_0^T \mathrm{tr}\left(C(s)\tilde{V}(s,s)C^\top(s)\right) ds,$$

where U satisfies (8) to (11) for $W(t) = C^\top(t)C(t)$ and $\tilde{V}$ satisfies

$$\begin{cases} \frac{\partial}{\partial s_1}\tilde{V}(s_1,s_2) = \sum_{i=0}^{m} A_i(s_1)\tilde{V}(s_1-\tau_i, s_2), \quad s_1 \geq s_2, \\ \tilde{V}^\top(s_1,s_2) = \tilde{V}(s_2,s_1), \\ \tilde{V}(s_1,s_2) = \tilde{V}(s_1+T, s_2+T), \\ \frac{d}{ds}\tilde{V}(s,s) = \sum_{i=0}^{m} A_i(s)\tilde{V}(s-\tau_i,s) + \sum_{i=0}^{m} \tilde{V}(s,s-\tau_i)A_i^\top(s) + B(s)B^\top(s). \end{cases}$$

We conclude this section with the following observations.

- In the delay-free case only the algebraic and periodicity condition are needed to uniquely express the $\mathcal{H}_2$ norm. Setting $Q(s) = U(s,s)$ and $P(s) = \tilde{V}(s,s)$ they reduce to the standard pair of periodic Lyapunov differential equations

$$\begin{aligned}\dot{P}(s) &= A_0(s)P(s) + P(s)A_0^\top(s) + B(s)B^\top(s), \quad P(0) = P(T),\\ -\dot{Q}(s) &= A_0^\top(s)Q(s) + Q(s)A_0(s) + C^\top(s)C(s),\ Q(0) = Q(T),\end{aligned}$$

see, for example, References 11,13,14.
- For linear time-invariant (LTI) systems with delays, the expressions in Theorem 3 reduce to the results in section II of Reference 15, noting that $U(s_1, s_2) = \hat{U}(s_1 - s_2)$, $\tilde{V}(s_1, s_2) = \hat{V}(s_1 - s_2)$, with $(\hat{U}, \hat{V})$ the pair of Lyapunov matrices associated to the LTI time-delay system.

4 | ENERGY INTERPRETATIONS

In this section we give interpretations of the one-parameter families of matrices $U(s,s)$ and $V(s,s)$, associated with systems (1) and (2), in terms of energy. They will be at the basis of the reduction approach outlined in Section 5.2.

For a given input $u \in L_2([t_0,\infty), \mathbb{R}^{n_i})$, with $L_2([t_0,\infty), \mathbb{R}^{n_i})$ representing the space of $\mathbb{R}^{n_i}$-valued squared integrable functions on $[t_0,\infty)$, we denote the solution of Equation (1) with initial condition $\varphi \in X$ at time t_0 by $x(t;t_0,\varphi,u)$ and $y(t;t_0,\varphi,u)$. The variation of constants formula for (1) reads as (see Reference 21)

$$x(t;t_0,\varphi,u) = K(t,t_0)\varphi(0) + \sum_{j=1}^{m} \int_{-\tau_j}^{0} K(t, t_0+\xi+\tau_j)A_j(t_0+\xi+\tau_j)\varphi(\xi)d\xi + \int_{t_0}^{t} K(t,\xi)B(\xi)u(\xi)d\xi, \tag{19}$$

and will be key in generalizing the results in section 3 of Reference 16 from time-invariant to time-periodic delay systems.

Let us now initialize (1) at time s with

$$x(t) = \varphi(t-s) := \begin{cases} 0, & t \in [s-\tau_m,\ s), \\ q, & t = s, \end{cases} \tag{20}$$

and consider the emanating solution for zero input. We define the output energy, associated with $q \in \mathbb{R}^n$ and horizon $T_s > 0$, as

$$E_p(q,s,T_s) := \int_s^{s+T_s} ||y(t;s,\varphi,0)||^2 dt. \tag{21}$$

It follows from formula (19) that $y(t;s,\varphi,0) = C(t)x(t;s,\varphi,0) = C(t)K(t,s)q$, and

$$\lim_{T_s\to\infty} E_p(q,s,T_s) = \int_s^{\infty} ||y(t;s,\varphi,0)||^2 dt = q^\top \int_s^{\infty} K^\top(t,s)C^\top(t)C(t)K(t,s)dt\ q = q^\top U(s,s)q,$$

with $U(s_1,s_2)$ the Lyapunov matrix of (1) associated with $W(t) = C^\top(t)C(t)$.

Similarly, we can initialize dual System (2) at time s with

$$z(t) = \varphi(t-s) := \begin{cases} 0, & t \in [s-\tau_m,\ s), \\ q, & t = s, \end{cases} \tag{22}$$

and define the output energy of dual System (2), associated with $q \in \mathbb{R}^n$ and horizon T_s, as

$$E_d(q,s,T_s) := \int_s^{s+T_s} ||\eta(t;s,\varphi,0)||^2 dt,$$

with $\eta(t; s, \varphi, u)$ the output corresponding to (22). By following similar arguments as for the primal system, we obtain

$$\lim_{T_s \to \infty} E_d(q, s, T_s) = q^\top V(s, s) q,$$

where $V(s_1, s_2)$ is the Lyapunov matrix of (2) associated with $W(t) = B(-t)B^\top(t)$.

Let us now characterize the minimal energy required in the input to reach a given $q \in \mathbb{R}^n$ at time s when the system is at rest at time $s - T_s$, with $T_s > 0$:

$$E_r(q, s, T_s) := \min_{\substack{u \in L_2([s-T_s, s], \mathbb{R}^{n_i}) \\ x(s; s-T_s, 0, u) = q}} \int_{s-T_s}^{s} ||u(t)||^2 dt. \tag{23}$$

In the next lemma, we provide the control u that minimizes $E_r(q, s, T_s)$.

Lemma 2. *Let System (1) be exponentially stable and define*

$$P(\alpha, s) := \int_{s-\alpha}^{s} K(s, \xi) B(\xi) B^\top(\xi) K^\top(s, \xi) d\xi.$$

If $q \in \mathrm{Im} P(\alpha, s)$, then the unique minimizer of the right-hand side of (23) is

$$u_{\mathrm{opt}}(t) := B^\top(t) K^\top(s, t) P^\dagger(T_s, s) q,$$

and, in addition, we have

$$E_r(q, s, T_s) = q^\top P^\dagger(T_s, s) q, \tag{24}$$

where the symbol $\dagger$ denotes the Moore-Penrose inverse.

Proof. The arguments are similar as in the proof of lemma 2 in Reference 16. By the variation of constants formula (19), we have

$$x(s, s - T_s, 0, u) = \int_{s-T_s}^{s} K(s, \xi) B(\xi) u(\xi) d\xi.$$

Notice first that u_{opt} allows to reach q. Indeed, by direct substitution of u_{opt} into the previous expression we get

$$x(s, s - T_s, 0, u_{\mathrm{opt}}) = \int_{s-T_s}^{s} K(s, \xi) B(\xi) B^\top(\xi) K^\top(s, \xi) d\xi \; P^\dagger(T_s, s) q = P(T_s, s) P^\dagger(T_s, s) q = q.$$

Suppose now that there exists another control $\tilde{u}$ such that $x(s, s - T_s, 0, \tilde{u}) = q$, then

$$x(s, s - T_s, 0, \tilde{u}) - x(s, s - T_s, 0, u_{\mathrm{opt}}) = \int_{s-T_s}^{s} K(s, \xi) B(\xi) \left(\tilde{u}(\xi) - u_{\mathrm{opt}}(\xi) \right) d\xi,$$

which implies that

$$\int_{s-T_s}^{s} u_{\mathrm{opt}}^\top(\xi) \left(\tilde{u}(\xi) - u_{\mathrm{opt}}(\xi) \right) d\xi = 0.$$

It follows from the previous equality that

$$||\tilde{u}||_{L_2}^2 = \left|\left| u_{\mathrm{opt}} + \tilde{u} - u_{\mathrm{opt}} \right|\right|_{L_2}^2 = \left|\left| \tilde{u} - u_{\mathrm{opt}} \right|\right|_{L_2}^2 + \left|\left| u_{\mathrm{opt}} \right|\right|_{L_2}^2,$$

therefore $||\tilde{u}||_{L_2} \geq \left|\left| u_{\mathrm{opt}} \right|\right|_{L_2}$, with $||\cdot||_{L_2}$ denoting the norm induced by the L_2-inner product. Thus, u_{opt} is the unique global minimizer of (23).

Finally, expression (24) is deduced from

$$\int_{s-T_s}^{s} ||u_{\mathrm{opt}}(t)||^2 dt = q^\top (P^\dagger(T_s,s))^\top \int_{s-T_s}^{s} K(s,t)B(t)B^\top(t)K^\top(s,t)\ dt\ P^\dagger(T_s,s)q = q^\top P^\dagger(T_s,s)q. \qquad \blacksquare$$

We observe from Lemma 1 that

$$\begin{aligned}\lim_{\alpha\to+\infty} P(\alpha,s) &= \int_{-\infty}^{s} K(s,\xi)B(\xi)B^\top(\xi)K^\top(s,\xi)d\xi \\ &= \int_{-s}^{\infty} K_D^\top(\xi,-s)B(-\xi)B^\top(-\xi)K_D(\xi,-s)d\xi \\ &= V(-s,-s) = V(T-s,T-s) = \tilde{V}(s,s);\end{aligned}$$

hence, it is possible to characterize $E_r(q,s,T_s)$, for $T_s \to \infty$ in terms of the dual Lyapunov matrices V and $\tilde{V}$ as

$$\lim_{T_s\to\infty} E_r(q,s,T_s) = q^\top V^\dagger(T-s,T-s)q = q^\top \tilde{V}^\dagger(s,s)q,$$

for $q \in \mathrm{Im}\tilde{V}(s,s)$. The previous results are summarized in the following theorem.

Theorem 4. *Let System (1) be exponentially stable. The following expressions hold for any $q \in \mathbb{R}^n$ and $s \in \mathbb{R}$:*

$$E_{p,o}(q,s) := \lim_{T_s\to\infty} E_p(q,s,T_s) = q^\top U(s,s)q, \tag{25}$$

$$E_{d,o}(q,s) := \lim_{T_s\to\infty} E_d(q,s,T_s) = q^\top V(s,s)q \tag{26}$$

and

$$E_{p,r}(q,s) := \lim_{T_s\to\infty} E_r(q,s,T_s) = \begin{cases} q^\top \tilde{V}^\dagger(s,s)q & \text{if } q \in \mathrm{Im}\tilde{V}(s,s), \\ +\infty, & \text{otherwise} \end{cases}$$

where $U(s_1,s_2)$ and $V(s_1,s_2)$ are Lyapunov matrices of systems (1) and (2) associated with $C^\top(t)C(t)$ and $B(-t)B^\top(-t)$, respectively, and $\tilde{V}(s_1,s_2)$ is as in Theorem 3.

Note that Lyapunov matrices $U(s,s)$ and $V(s,s)$ play the role of infinite reachability and observability Grammians of periodic systems with delays. With q a unit vector, $E_{p,o}(q,s)$ can be interpreted as a measure of how well pseudo-state q, parameterizing initial condition (20) is observable in the output. At the same time the measures $E_{d,o}(q,s)$ and $E_{p,r}(q,s)$ give an indication of the reachability of vectors in direction q.

5 | IMPLICATIONS AND APPLICATIONS

5.1 | Computation of the $\mathcal{H}_2$ norm

The availability of the explicit expressions in Theorem 3 allows us to compute and optimize $\mathcal{H}_2$ norms in a Lyapunov equation framework, extending the standard approach for delay-free and nonperiodic systems.

The Lyapunov matrix can be obtained by solving Equations (8) to (11). This is a far from trivial task, as they are two-dimensional partial differential equations with periodic coefficients and delays, where the described function takes as values n-by-n matrices. In addition, function $(s_1,s_2) \mapsto U(s_1,s_2)$ is in general a nonsmooth function, inheriting this property from the fundamental matrix. However, for $m = 1$ and $T = \tau_1$ function U is smooth in the region

$$\mathcal{D} := \{(s_1,s_2) \in \mathbb{R}^2 : s_2 \in [0,\tau_1],\ s_2 - s_1 \in [0,\tau_1]\},$$

where it can also be described in terms of delay-free partial differential equations.

Proposition 4. *Let System (1) with one delay be exponentially stable and $T = \tau_1$.*[10] *Then, matrices $U_0(s_1, s_2) := U(s_1, s_2)$ and $U_1(s_1, s_2) := U(s_1 + \tau_1, s_2)$ satisfy*

$$\begin{cases} \frac{\partial}{\partial s_1} U_0(s_1, s_2) = -A_0^\top(s_1)U_0(s_1, s_2) - A_1^\top(s_1)U_1(s_1, s_2), \\ \frac{\partial}{\partial s_2} U_1(s_1, s_2) = -U_1(s_1, s_2)A_0(s_2) - U_0(s_1, s_2)A_1(s_2), \end{cases} \tag{27}$$

for $(s_1, s_2) \in \mathcal{D}$, with boundary conditions

$$\begin{cases} \frac{d(s,s)}{ds} U_0(s, s) = -A_0^\top(s)U_0(s, s) - A_1^\top(s)U_1(s, s) - U_0(s, s)A_0(s) - U_0(s - \tau_1, s)A_1(s) - W(s), \\ U_1(s - \tau_1, s) = U_0(s, s), \\ U_0(s - \tau_1, 0) = U_0(s, \tau_1), \\ U_1(s - \tau_1, 0) = U_1(s, \tau_1), \end{cases} \tag{28}$$

for $s \in [0, \tau_1]$.

The formulations (27) and (28) are at the basis of the algorithm in Reference 10 for computing the delay Lyapunov matrix. The latter can be interpreted as a *shooting method*, induced by the property that specifying U_0 and U_1 for $s_2 = 0$ and $s_2 - s_1 = 0$ uniquely determines the solution of (27), and thus the values of U_0 and U_1 for $s_2 = \tau_1$ and $s_2 - s_1 = \tau_1$. The transition map is approximated by using a Runge-Kutta scheme. In this way matrices U_0 and U_1 are computed first for $s_2 = 0$ and $s_2 - s_1 = 0$, while the corresponding solution is determined by the Runge-Kutta scheme. We refer to Reference 10 for the details of this approach, which also extends to the case where T is a multiple of τ_1. From its definition, the dual Lyapunov matrix V can be computed as the Lyapunov matrix corresponding to system (2).

A generally applicable but less efficient approach, which extends the results presented in section 2 of Reference 22 to periodic systems, consists of inferring an approximation of U from the Lyapunov matrix associated with a spectral discretization of the delay equation. This approach requires solving a standard delay-free periodic Lyapunov equation of increased dimensions.

With the following examples, we illustrate the computation of the $\mathcal{H}_2$ norm via Lyapunov matrices $U(s, s)$ and $V(s, s)$, using the numerical scheme introduced in Reference 10.

Example 1. We consider the well-known delayed Mathieu equation, whose representation in state variables is given by a system of the form (1), with matrices

$$A_0(t) = \begin{bmatrix} 0 & 1 \\ -\delta - \varepsilon \cos\left(\frac{2\pi}{T}t\right) & -\rho \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} 1 + \cos\left(\frac{2\pi}{T}t\right) \\ 1 \end{bmatrix}, \quad C(t) = \begin{bmatrix} -1 + 2\sin\left(\frac{2\pi}{T}t\right) & 1 \end{bmatrix},$$

where

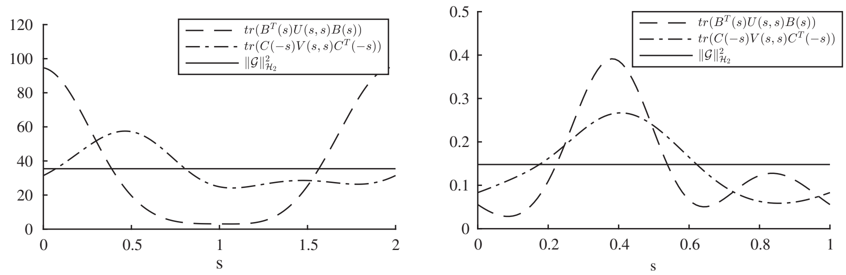
$$T = \tau_1 = 2, \ \rho = 0.1, \ \delta = 7, \ \varepsilon = 1, \ b = -1. \tag{29}$$

In Figure 1 (left), we plot

$$\operatorname{tr}\left(B^\top(s)U(s, s)B(s)\right), \quad \operatorname{tr}\left(C(-s))V(s, s)C(-s)\right), \tag{30}$$

as a function of s, where matrices $U(s, s)$ and $V(s, s)$ are Lyapunov matrices associated with $C^\top(s)C(s)$ and $B(-s)B^\top(-s)$, respectively. Their averaged value, indicated with a full line, correspond to the squared $\mathcal{H}_2$ norm,

FIGURE 1 Expressions (30) as a function of s, and the averaged value, for Example 1 (left) and Example 2 (right)



$$||\mathcal{G}||^2_{\mathcal{H}_2} = ||\mathcal{G}_D||^2_{\mathcal{H}_2} \approx 35.4432.$$

We note that, as an a posteriori indicator of accuracy, the matching of the computed $\mathcal{H}_2$ norms of primal and dual system can be used, at the price of doubling the computational cost.

Example 2. We consider System (1) with delay $\tau_1 = T = 1$ and matrices

$$A_0(t) = \begin{bmatrix} \sin(2\pi t) + \frac{1}{8}\cos(2\pi t) - 4 & \frac{5}{4}\cos(2\pi t) - \sin(2\pi t) + 3 & \sin(2\pi t) \\ \frac{3}{4}\cos(2\pi t) + 1 & \frac{1}{8}\cos(2\pi t) - 2 & -10 \\ \frac{1}{4}\cos(2\pi t) & 0 & -10 - 5\cos(2\pi t) \end{bmatrix},$$

$$A_1(t) = \begin{bmatrix} \frac{1}{2}\sin(2\pi t) & -\frac{1}{8}\cos(2\pi t)/8 & 1 \\ \frac{1}{8}\cos(2\pi t) & \frac{1}{2}\sin(2\pi t) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} \frac{1}{4}\cos(2\pi t) \\ 1 - \frac{1}{2}\sin(2\pi t) \\ 1 \end{bmatrix}, \quad C(t) = \begin{bmatrix} -1 + \frac{1}{2}\sin(2\pi t) & \frac{1}{4}\cos(2\pi t) & 1 \end{bmatrix}.$$

The right pane of Figure 1 depicts functions (30) for this example. The full line once again corresponds to the squared $\mathcal{H}_2$ norm of the system,

$$||\mathcal{G}||^2_{\mathcal{H}_2} = ||\mathcal{G}_D||^2_{\mathcal{H}_2} \approx 0.1479.$$

5.2 | Structure preserving model reduction by position balancing

The starting point is formed by the energy interpretations (25) and (26). Since both $E_{p,o}(q,s)$ and $E_{d,o}(q,s)$ are periodic in s, it is natural to look at the averaged values,

$$E_{p,\mathrm{av}}(q) := q^\top U_{\mathrm{av}} q, \tag{31}$$

and

$$E_{d,\mathrm{av}}(q) := q^\top V_{\mathrm{av}} q, \tag{32}$$

with

$$U_{\mathrm{av}} := \frac{1}{T}\int_0^T U(s,s)ds, \quad \text{and} \quad V_{\mathrm{av}} := \frac{1}{T}\int_0^T V(s,s)ds.$$

Let us now, for given nonsingular matrix $\mathcal{T} \in \mathbb{R}^{n\times n}$, apply similarity transformation

$$\overline{x}(t) = \mathcal{T}^{-1}x(t), \tag{33}$$

to (1) and, at the same time, similarity transformation

$$\overline{z}(t) = \mathcal{T}^{\top}z(t), \tag{34}$$

to (2). They lead to a transformed pair of primal-dual systems

$$\begin{cases} \dot{\overline{x}}(t) = \sum_{i=0}^{m} \bar{A}_i(t)\overline{x}(t-\tau_i) + \overline{B}(t)u(t), \\ y(t) = \overline{C}(t)x(t), \end{cases} \tag{35}$$

and

$$\begin{cases} \dot{\overline{z}}(t) = \sum_{i=0}^{m} \bar{A}_i^{\top}(-t+\tau_i)\overline{z}(t-\tau_i) + \overline{C}^{\top}(-t)\xi(t), \\ \eta(t) = \overline{B}^{\top}(-t)z(t), \end{cases} \tag{36}$$

with $\overline{B}(t) = \mathcal{T}^{-1}B(t)$, $\overline{C}(t) = C(t)\mathcal{T}$ and

$$\bar{A}_i(t) = \mathcal{T}^{-1}A_i(t)\mathcal{T}, \quad i = 0, \dots, m.$$

For systems (35) and (36) the Lyapunov matrices can be expressed as

$$\bar{U}(s_1, s_2) = \mathcal{T}^{\top}U(s_1, s_2)\mathcal{T}, \quad \overline{V}(s_1, s_2) = \mathcal{T}^{-1}V(s_1, s_2)\mathcal{T}^{-T},$$

while the kernels of (31) and (32) become

$$\bar{U}_{\rm av} = \mathcal{T}^{\top}U_{\rm av}\mathcal{T}, \quad \overline{V}_{\rm av} = \mathcal{T}^{-1}V_{\rm av}\mathcal{T}^{-T}.$$

Note that the eigenvalues of the product $\bar{U}_{\rm av}\overline{V}_{\rm av}$ are independent of $\mathcal{T}$. Inspired by the approach of Reference 16 for balanced truncation of nonperiodic delay systems, and the energy interpretations related to expressions (31) and (32), we call system (1) *position-balanced* if

$$\bar{U}_{\rm av} = \overline{V}_{\rm av} = \Sigma, \tag{37}$$

with $\Sigma \geq 0$ a diagonal matrix with diagonal elements in non-increasing order. The term position balancing stems from the property that expressions (31) and (32) only characterize partial state $p = x(t) = x_t(0) \in \mathbb{R}^n$; see[16] for an analogy with position balancing of second-order systems.

Given that $U(s,s) \geq 0$ and $V(s,s) \geq 0$ for all s, following from Theorem 4, balancing is possible under a very mild condition.

Proposition 5. *Assume that $U_{\rm av} > 0$ and $V_{\rm av} > 0$. Let factorizations $U_{\rm av} = R^{\top}R$ and $V_{\rm av} = S^{\top}S$ correspond to Cholesky factorizations. Let $\mathcal{U}\Sigma\mathcal{V}^{\top}$ be a singular value decomposition of $RS^{\top}$. Then the choice*

$$\mathcal{T} = S^{\top}\mathcal{V}\Sigma^{-\frac{1}{2}}, \quad \mathcal{T}^{-1} = \Sigma^{-\frac{1}{2}}\mathcal{U}^{\top}R,$$

in (33) and (34) induces property (37), that is, the transformed system is position-balanced.

If the system has been position-balanced, then the diagonal elements of Σ give an indication about the importance of the components of transformed state variable $\overline{x}$ (the canonical directions), with respect to the input-output behavior of the system. Partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

with the diagonal elements of Σ_2 preferably small compared to the diagonal elements of Σ_1, and making a corresponding partition of the state variable $\overline{x} = [\overline{x}_1^\top \; \overline{x}_2^\top]^\top$ and the system matrices,

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i,11} & \bar{A}_{i,12} \\ \bar{A}_{i,21} & \bar{A}_{i,22} \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}, \quad \overline{C} = [\overline{C}_1 \;\; \overline{C}_2],$$

lead us to the following reduced-order model,

$$\begin{cases} \dot{\overline{x}}_1(t) = \displaystyle\sum_{i=0}^{m} \bar{A}_{i,11}(t)\, \overline{x}_1(t-\tau_i) + \overline{B}_1(t)\, u(t), \\ y(t) = \overline{C}_1(t)\, \overline{x}_1(t). \end{cases}$$

Remark 2. The history of the state is assumed zero in deriving energies (31) and (32), whereas $E_r(p,s)$ determines the input energy to reach a point in $\mathbb{R}^n$ at time s, not considering the shape of the solution in the interval $[t-s,s)$. Hence, position balancing is based on interpretations of pseudo-states, parameterized by $p \in \mathbb{R}^n$, rather than elements of state space X. Accordingly, a transformation of the form (33) in $\mathbb{R}^n$ is constrained when it is extended to a transformation in X (for instance, it implies $\overline{\varphi}(s) = \mathcal{T}^{-1}\varphi(s),\; s \in [-\tau_m,\; 0]$, with $\mathcal{T}$ independent of s). However, a transformation in $\mathbb{R}^n$, the “physical" space in which the trajectories of (1) reside, has the advantage that the reduction approach is structure preserving, that is, the reduced model is also in the form of a periodic time-delay system.

We conclude the section with a proof-of-concept case-study for the above reduction approach.

Example 3. We consider System (1) with $n = 2$, $T = 2\pi$, $m = 1$, $\tau_1 = 1$,

$$A_0(t) = \begin{bmatrix} \sin(2\,t) + \frac{5\,\cos(t)}{8} - 4 & \frac{5\,\cos(t)}{8} - \sin(2\,t) + 1 \\ \frac{3\,\cos(t)}{8} + 1 & \frac{3\,\cos(t)}{8} - 4 \end{bmatrix}, \; A_1(t) = \begin{bmatrix} \frac{\sin(t)}{2} & -\frac{\cos(t)}{2} \\ \frac{\cos(t)}{2} & \frac{\sin(t)}{2} \end{bmatrix}, \tag{38}$$

and

$$B(t) = \begin{bmatrix} \frac{\cos(t)}{2} + \frac{\sin(t)}{8} + 1 \\ \frac{\cos(t)}{2} - \frac{\sin(t)}{8} + 1 \end{bmatrix}, \quad C(t) = [-1 + 2\cos(t) \;\; -1].$$

Note that we now have $T \neq \tau_1$. In this case we compute the Lyapunov matrices from the solution of standard delay-free periodic Lyapunov equations applied to a spectral discretization of the system (see the discussion in the previous subsection). Solving the associated periodic Lyapunov equations yields

$$U_{\rm av} = \begin{bmatrix} 1.86 & 0.769 \\ 7.68 & 0.746 \end{bmatrix}, \quad V_{\rm av} = \begin{bmatrix} 1.22 & 1.16 \\ 1.16 & 1.14 \end{bmatrix}.$$

In agreement with the statement of Proposition 5, the transformation

$$\overline{x}(t) = \mathcal{T}^{-1} x(t),$$

with

$$\mathcal{T} = \begin{bmatrix} -0.741 & -0.230 \\ -0.713 & 0.402 \end{bmatrix},$$

results in balanced (pseudo)-Grammians $\bar{U}_{\mathrm{av}}$ and $\overline{V}_{\mathrm{av}}$, equal to

$$\Sigma = \begin{bmatrix} 2.21 & 0 \\ 0 & 0.0768 \end{bmatrix}.$$

The diagonal elements of Σ indicate the potential for reduction to a first-order system. Truncating the balanced system to the first canonical direction leads us to the reduced model

$$\begin{cases} \bar{x}_1(t) &= (0.0248\sin(2t) + 1.06\cos(t) - 3.01)\,\bar{x}_1(t) \\ & \quad +\ (0.5\sin(t) - 0.126\ \cos(t))\,\bar{x}_1(t-1) \\ & \quad + (-0.681\ \cos(t) - 0.0463\sin(t) - 1.36)\ u(t), \\ y(t) &= (1.46 - 1.49\ \cos(t))\,\bar{x}_1(t). \end{cases} \tag{39}$$

Denoting by $\mathcal{G}$, respectively $\mathcal{G}_r$, the input-output map of (38) and (39) we obtain

$$||\mathcal{G}||_{\mathcal{H}_2} = 0.677,\ \ ||\mathcal{G}_r||_{\mathcal{H}_2} = 0.681,$$

while the error on the input-output map is characterized by

$$||\mathcal{G} - \mathcal{G}_r||_{\mathcal{H}_2} = 0.048.$$

Hence, the balancing procedure is able to extract the (reduced) dynamics, mainly responsible for to its input-output behavior of the system. □

6 | CONCLUSIONS

As main contributions, we presented several novel concepts and theoretical results for linear periodic systems with delays (addressing spectral properties, Lyapunov matrices and $\mathcal{H}_2$ norms), where the introduced dual system (2) played a crucial role. In addition, we illustrated the potential of the theoretical results for $\mathcal{H}_2$ analysis/synthesis and for model reduction by means of case-studies, including the well-known delayed Mathieu equation. The effectiveness, however, depends on the availability of solvers for Lyapunov matrices of large-scale systems of the form (1), which is an important direction for future research.

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SUPPORTING INFORMATION

Additional supporting information may be found online in the Supporting Information section at the end of this article.

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