Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Journal of Functional Analysis 235 (2006) 330–354
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line

Nguyen Thieu Huy

Faculty of Applied Mathematics and Informatics, Hanoi University of Technology, Khoa Toan-Tin ung dung, DH Bach Khoa Hanoi, 1 Dai Co Viet, Hanoi, Viet Nam

Received 10 August 2005; accepted 10 November 2005

Available online 15 December 2005

Communicated by Paul Malliavin

Abstract

Consider an evolution family $\mathcal{U} = (U(t,s))_{t \geqslant s \geqslant 0}$ on a half-line $\mathbb{R}_+$ and an integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi) f(\xi)\, d\xi$. We characterize the exponential dichotomy of the evolution family through solvability of this integral equation in admissible function spaces which contain wide classes of function spaces like function spaces of L_p type, the Lorentz spaces $L_{p,q}$ and many other function spaces occurring in interpolation theory. We then apply our results to study the robustness of the exponential dichotomy of evolution families on a half-line under small perturbations.

Keywords: Evolution equations; Integral equations; Exponential stability of solutions; Exponential dichotomy; Admissibility of function spaces; Perturbations

1. Introduction

One of the central research interests regarding asymptotic behavior of solutions to the linear differential equation

$$\frac{dx}{dt} = A(t)x, \quad t \in [0, +\infty),\ x \in X,$$

E-mail address: huynguyen@mail.hut.edu.vn.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2005.11.002

where $A(t)$ is in general an unbounded linear operator on a Banach space X for every fixed t, is to find conditions for the solutions to be stable or to have exponential dichotomy. In the case that $A(t)$ is a matrix continuous function, Perron [22] first observed a relation between asymptotic behavior of the solutions to this equation and the properties of the differential operator $\frac{d}{dt} - A(t)$ as an operator in the space $C_b(\mathbb{R}_+, \mathbb{R}^n)$ of $\mathbb{R}^n$-valued bounded continuous functions on the half-line $\mathbb{R}_+$. These results served as a starting point for numerous works on the qualitative theory of solutions to differential equations. We refer the reader to the books by Massera and Schäffer [15] and Daleckii and Krein [6] for a characterization of the exponential dichotomy of solutions to the above equation in terms of the surjectiveness of the differential operator $\frac{d}{dt} - A(t)$ in the case of bounded $A(t)$ and by Levitan and Zhikov [13] for an extension to the infinite-dimensional case for equations defined on the whole line. Note that a similar characterization of exponential stability can be made by using the differential operator $\frac{d}{dt} - A(t)$ in suitable function spaces (see [6,7,15]). In the infinite-dimensional case, in order to characterize the exponential dichotomy of solutions to differential equations on the half-line, apart from the surjectiveness of the differential operator $\frac{d}{dt} - A(t)$ one needs additional conditions, namely the complementedness of the stable subspaces (see [6,15,17]).

Recently, there has been an increasing interest in the asymptotic behavior of solutions of differential equations in Banach spaces, in particular, in the unbounded case (see, e.g., [1,2, 8,9,11,16,17,19,27]). In this direction, we would like to mention the paper by Nguyen Van Minh, Räbiger and Schnaubelt [17] in which a characterization of exponential dichotomy was given in spaces $C_0(\mathbb{R}_+, X)$ (spaces of X-valued continuous functions vanishing at infinity) on a half-line for unbounded $A(t)$ (see also [16] for a characterization on spaces $L_p(\mathbb{R}_+, X)$).

In the present paper we will characterize exponential dichotomy in a general framework. That is, we will consider the characterization of exponential dichotomy in admissible spaces of functions defined on the half-line $\mathbb{R}_+$ (see Definitions 2.1 and 2.3). For some classes of admissible spaces of functions defined on the whole line $\mathbb{R}$ such a characterization is done by Räbiger and Schnaubelt in [24] using the theory of evolution semigroups. The situation becomes more complicated if one considers admissible spaces of functions defined only on the half-line $\mathbb{R}_+$. One cannot immediately have the corresponding left-translation evolution semigroup. One has to take into account the initial conditions to construct some kind of semigroups. However, the results can only be applied in some concrete situations (see [3,10]), because, in general, the operator $\frac{d}{dt} - A(t)$ (or its closure) defined on a half-line is not a generator. Therefore, in our strategy, we use the technique of choosing test functions related to integral equations as in [6,13,15–17] and references therein. This technique allows us to use Banach isomorphism theorem applied to an abstract differential operator to obtain explicit dichotomy estimates. Consequently, we obtain the characterization of exponential dichotomy of evolution families on the half-line $\mathbb{R}_+$ in very general admissible spaces which contain spaces of L_p-type functions defined on the half-line $\mathbb{R}_+$, and many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,q}$. Moreover, we can use our characterization of exponential dichotomy to prove the robustness of exponential dichotomy of evolution families on the half-line $\mathbb{R}_+$ under small perturbations. Our main results are contained in Theorems 4.2, 4.4, 5.1 and Corollary 5.3. These results extend those known for finite-dimensional spaces (see [1, 2,5]), for bounded $A(t)$ in Banach spaces (see [6,15]) and for concrete function spaces (see [16,17] and references therein).

In the case of unbounded $A(t)$, it is more convenient to consider an extension of the operator $\frac{d}{dt} - A(t)$, which defined by mild solutions of the inhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t)$$

(see Definition 2.7) using the evolution family arising in well-posed homogeneous Cauchy problems. We now recall the definition of an evolution family.

Definition 1.1. A family of operators $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ on a Banach space X is a (strongly continuous, exponential bounded) evolution family on the half-line if

(1) $U(t,t) = Id$ and $U(t,r)U(r,s) = U(t,s)$ for $t \geqslant r \geqslant s \geqslant 0$,
(2) the map $(t,s) \mapsto U(t,s)x$ is continuous for every $x \in X$,
(3) there are constants $K, c \geqslant 0$ such that $\|U(t,s)\| \leqslant Ke^{c(t-s)}$ for $t \geqslant s \geqslant 0$.

This notion of evolution families arises naturally from the theory of Cauchy problems for evolution equations which are well posed (see, e.g., [21, Chapter 5], [20,25]). In fact, in the terminology of [21, Chapter 5] and [20], an evolution family arises from the following well posed evolution equation

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t), \quad t \geqslant s \geqslant 0, \\ u(s) = x_s \in X, \end{cases} \tag{1}$$

where $A(t)$ are (in general unbounded) linear operators for $t \geqslant 0$. Roughly speaking, when the Cauchy problem (1) is well posed, there exists a (strongly continuous, exponential bounded) evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ solving (1), i.e., the solution of (1) is given by $u(t) := U(t,s)u(s)$. For more details on the notion of evolution families, conditions for the existence of such families and applications to partial differential equations we refer the readers to Pazy [21] (see also Nagel and Nickel [18] for a detailed discussion of well-posedness for non-autonomous abstract Cauchy problems on the whole line $\mathbb{R}$).

2. Function spaces and admissibility

We recall some notions of function spaces and admissibility. We refer the readers to Massera and Schäffer [15, Chapter 2] for wide classes of function spaces that play a fundamental role throughout the study of differential equations in the case of bounded coefficients $A(t)$ (see also Räbiger and Schnaubelt [24, Section 1] for some classes of admissible Banach function spaces of functions defined on the whole line $\mathbb{R}$).

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. As already known, the set of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) that are integrable on every compact subinterval $J \subset \mathbb{R}_+$ becomes, with the topology of convergence in the mean on every such J, a locally convex topological vector space, which we denote by $L_{1,\mathrm{loc}}(\mathbb{R}_+)$. A set of seminorms defining the topology of $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is given by $p_n(f) := \int_{J_n} |f(t)|\,dt$, $n \in \mathbb{N}$, where $\{J_n\}_{n\in\mathbb{N}} = \{[n, n+1]\}_{n\in\mathbb{N}}$ is a countable set of abutting compact intervals whose union is $\mathbb{R}_+$. With this set of seminorms one can see (see [15, Chapter 2, Section 20]) that $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a Fréchet space.

Let V be a normed space (with norm $\|\cdot\|_V$) and W be a locally convex Hausdorff topological vector space. Then, we say that V is *stronger than* W if $V \subseteq W$ and the indentity map from V into W is continuous. The latter condition is equivalent to the fact that for each continuous seminorm π of W there exists a number $\beta_\pi > 0$ such that $\pi(x) \leqslant \beta_\pi \|x\|_V$ for all $x \in V$. We write $V \hookrightarrow W$ to indicate that V is stronger than W. If, in particular, W is also a normed space (with norm $\|\cdot\|_W$) then the relation $V \hookrightarrow W$ is equivalent to the fact that $V \subseteq W$ and there is a number $\alpha > 0$ such that $\|x\|_W \leqslant \alpha\|x\|_V$ for all $x \in V$ (see [15, Chapter 2] for detailed discussions on this matter).

We can now define Banach function spaces as follows.

Definition 2.1. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called a Banach function space (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if

(1) E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$ λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,
(2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E > 0$,
(3) $E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$.

For a Banach function space E we remark that the condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}_+$ there exists a number $\beta_J \geqslant 0$ such that $\int_J |f(t)|\,dt \leqslant \beta_J \|f\|_E$ for all $f \in E$.

We state the following trivial lemma which will be frequently used in our strategy.

Lemma 2.2. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}_+$ such that they coincide with each other outside a compact interval and they are essentially bounded* (*in particular, continuous*) *on this compact interval. Then $\varphi \in E$ if and only if $\psi \in E$.*

Let now E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \bigl\{ f : \mathbb{R}_+ \to X:\ f \text{ is strongly measurable and } \bigl\|f(\cdot)\bigr\| \in E \bigr\}$$

(modulo λ-nullfunctions) endowed with the norm

$$\|f\|_{\mathcal{E}} := \bigl\| \bigl\|f(\cdot)\bigr\| \bigr\|_E.$$

One can easily see that $\mathcal{E}$ is a Banach space. We call it *the Banach space corresponding to the Banach function space E.*

Definition 2.3. The Banach function space E is called admissible if it satisfies

(1) there is a constant $M \geqslant 1$ such that for every compact interval $[a,b] \in \mathbb{R}_+$ we have

$$\int_a^b \left|\varphi(t)\right| dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \quad \text{for all } \varphi \in E, \tag{2}$$

(2) for $\varphi \in E$ the function $\Lambda_1 \varphi$ defined by $\Lambda_1\varphi(t) := \int_t^{t+1} \varphi(\tau)\,d\tau$ belongs to E,
(3) E is T_τ^+-invariant and T_τ^--invariant, where T_τ^+ and T_τ^- are defined, for $\tau \in \mathbb{R}_+$ by

$$T_\tau^+\varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \geqslant \tau \geqslant 0, \\ 0 & \text{for } 0 \leqslant t \leqslant \tau, \end{cases}$$

$$T_\tau^-\varphi(t) := \varphi(t+\tau) \quad \text{for } t \geqslant 0. \tag{3}$$

Moreover, there are constants N_1, N_2 such that $\|T_\tau^+\|_E \leqslant N_1$, $\|T_\tau^-\|_E \leqslant N_2$ for all $\tau \in \mathbb{R}_+$.

Example 2.4. Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+)\colon \sup_{t \geqslant 0} \int_t^{t+1} \left|f(t)\right| dt < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geqslant 0} \int_t^{t+1} |f(t)|\,dt$, many other function spaces occurring in interpolation theory, e.g., the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 \leqslant q < \infty$ (see [4, Theorem 3 and p. 284], [28, 1.18.6, 1.19.3]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}_+, \mathcal{B}, \lambda)$ (see [14, 2.a]) are admissible.

Remark 2.5. If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$. Indeed, put $\beta := \inf_{t\geqslant 0} \|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.1(2)). Then, from Definition 2.3(1) we derive

$$\int_t^{t+1} \left|\varphi(t)\right| dt \leqslant \frac{M}{\beta} \|\varphi\|_E \quad \text{for all } t \geqslant 0 \text{ and } \varphi \in E. \tag{4}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\varphi\|_{\mathbf{M}} \leqslant \frac{M}{\beta}\|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition.

Proposition 2.6. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $\varphi \geqslant 0$ and $\Lambda_1\varphi \in E$, where, Λ_1 is defined as in Definition* 2.3(2). *For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int_0^t e^{-\sigma(t-s)}\varphi(s)\,ds,$$

$$\Lambda''_\sigma\varphi(t) := \int_t^\infty e^{-\sigma(s-t)}\varphi(s)\,ds.$$

Then, $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geqslant 0}\int_t^{t+1}\varphi(\tau)\,d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.5)) *then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded.*

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geqslant 0$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) := e^{bt}$ for $t \geqslant 0$ and any fixed constant $b > 0$.*

Proof. (a) The proof of this assertion is essentially done in [15, 23.V.(1)]. We present it here for seek of completeness. We first prove that $\Lambda'_\sigma\varphi$ belongs to E.

Indeed, putting $a_+ := \max\{0, a\}$ for $a \in \mathbb{R}$, we remark that, by the definitions of Λ_1 and T_1^+, the equalities

$$\Lambda_1 T_1^+\varphi(t) = \int_{(t-1)_+}^{t} \varphi(s)\,ds \quad \text{and}$$

$$T_1^+\Lambda_1\varphi(t) = \begin{cases} 0 & \text{for } 0 \leqslant t \leqslant 1,\\ \int_{t-1}^t \varphi(s)\,ds & \text{for } t > 1 \end{cases}$$

hold. Since $T_1^+\Lambda_1\varphi \in E$, by Lemma 2.2, we obtain that $\Lambda_1 T_1^+\varphi$ also belongs to E. We then compute

$$\begin{aligned}\Lambda'_\sigma\varphi(t) &= \sum_{j=0}^{\infty}\int_{(t-(j+1))_+}^{(t-j)_+} e^{-\sigma(t-s)}\varphi(s)\,ds \leqslant \sum_{j=0}^{\infty} e^{-j\sigma}\int_{(t-(j+1))_+}^{(t-j)_+}\varphi(s)\,ds\\ &= \sum_{j=0}^{\infty} e^{-j\sigma}T_j^+\Lambda_1 T_1^+\varphi(t) \quad \text{for all } t \in \mathbb{R}_+.\end{aligned}$$

Moreover, $e^{-j\sigma}T_j^+\Lambda_1 T_1^+\varphi \in E$ for all j and

$$\sum_{j=0}^{\infty}\left\|e^{-j\sigma}T_j^+\Lambda_1 T_1^+\varphi\right\|_E = \sum_{j=0}^{\infty} N_1 e^{-j\sigma}\left\|\Lambda_1 T_1^+\varphi\right\|_E = \frac{N_1}{1-e^{-\sigma}}\left\|\Lambda_1 T_1^+\varphi\right\|_E.$$

Since E is a Banach function space, we obtain that $\Lambda'_\sigma \varphi \in E$ and

$$\left\|\Lambda'_\sigma \varphi\right\|_E \leqslant \frac{N_1}{1-e^{-\sigma}} \left\|\Lambda_1 T_1^+ \varphi\right\|_E.$$

We now prove that $\Lambda''_\sigma \varphi$ belongs to E. To do that we compute

$$\begin{aligned}\Lambda''_\sigma \varphi(t) &= \sum_{j=0}^{\infty} \int_{t+j}^{t+j+1} e^{-\sigma(t-s)} \varphi(s)\, ds \leqslant \sum_{j=0}^{\infty} e^{-j\sigma} \int_{t+j}^{t+j+1} \varphi(s)\, ds \\ &= \sum_{j=0}^{\infty} e^{-j\sigma} T_j^- \Lambda_1 \varphi(t) \quad \text{for all } t \in \mathbb{R}_+.\end{aligned}$$

Furthermore, $e^{-j\sigma} T_j^- \Lambda_1 \varphi \in E$ for all j and

$$\sum_{j=0}^{\infty} \left\|e^{-j\sigma} T_j^- \Lambda_1 \varphi\right\|_E \leqslant \sum_{j=0}^{\infty} N_2 e^{-j\sigma} \|\Lambda_1 \varphi\|_E = \frac{N_2}{1-e^{-\sigma}} \|\Lambda_1 \varphi\|_E.$$

Since E is a Banach function space, we obtain that $\Lambda''_\sigma \varphi \in E$ and

$$\left\|\Lambda''_\sigma \varphi\right\|_E \leqslant \frac{N_2}{1-e^{-\sigma}} \|\Lambda_1 \varphi\|_E.$$

To prove that the condition $\sup_{t \geqslant 0} \int_t^{t+1} \varphi(\tau)\, d\tau < \infty$ implies the boundedness of Λ'_σ and Λ''_σ we just apply the above result to the admissible Banach function space L_∞.

(b) Since $\chi_{[0,1]}$ belongs to E, using the above assertion (a), for any fixed constant $\alpha > 0$ we have that the function

$$v(t) := \int_0^t e^{-\alpha(t-s)} \chi_{[0,1]}(s)\, ds = \begin{cases} \frac{e^{-\alpha t}(e^{\alpha}-1)}{\alpha} & \text{for } t \geqslant 1, \\ \frac{1-e^{-\alpha t}}{\alpha} & \text{for } 0 \leqslant t < 1 \end{cases}$$

belongs to E. The assertion (b) now follows from Lemma 2.2.

(c) For the purpose of contradiction let the function $f(t) = e^{bt}$ belong to E for some $b > 0$. Then, by the inequality (4) we have that

$$\frac{1}{b} e^{bt}\left(e^b - 1\right) \leqslant \frac{M}{\beta} \|f\|_E \quad \text{for all } t \geqslant 0.$$

This is a contradiction since

$$\lim_{t \to \infty} \frac{1}{b} e^{bt}\left(e^b - 1\right) = \infty. \qquad \square$$

For a Banach space X we denote by $C_b(\mathbb{R}_+, X)$ the space of X-valued, bounded continuous functions (endowed with sup-norm $\|\cdot\|_\infty$).

Let $\mathcal{E} = \mathcal{E}(\mathbb{R}_+, X)$ be the Banach space corresponding to the admissible Banach function space E. Then, we define

$$\mathcal{E}_\infty := \mathcal{E} \cap C_b(\mathbb{R}_+, X) \quad \text{endowed with the norm} \quad \|f\|_{\mathcal{E}_\infty} := \max\{\|f\|_{\mathcal{E}}, \|f\|_\infty\}.$$

Clearly, $\mathcal{E}_\infty \hookrightarrow \mathcal{E}$.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be Banach spaces. Then, for an operator $A : V \to W$ with the domain $D(A) \subset V$, the graph norm $\|\cdot\|_A$ on $D(A)$ is defined as

$$\|x\|_A := \|x\|_V + \|Ax\|_W, \quad x \in D(A).$$

It is clear that $(D(A), \|\cdot\|_A) \hookrightarrow V$. One can easily see that, if A is closed then $(D(A), \|\cdot\|_A)$ is a Banach space.

We now consider the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)\,d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+. \tag{5}$$

We note that if the evolution family $(U(t,s))_{t\geqslant s\geqslant 0}$ arises from the well-posed Cauchy problem (1) then the function u, which satisfies (5) for some given function f, is called a mild solution of the inhomogeneous problem

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t) + f(t), & t \geqslant s \geqslant 0, \\ u(s) = x_s \in X \end{cases}$$

(see Pazy [21, Chapter 5] for more information on this matter).

We then define the operator G related to the integral equation (5) as follows.

Definition 2.7. The operator $G : D(G) \subset \mathcal{E}_\infty \to \mathcal{E}$ is defined by

$$D(G) := \{u \in \mathcal{E}_\infty : \text{there exists } f \text{ in } \mathcal{E} \text{ such that } u, f \text{ satisfy Eq. (5)}\},$$
$$Gu := f \quad \text{for } u \in D(G) \text{ and } f \in \mathcal{E} \text{ satisfying Eq. (5)}.$$

We now justify the definition of G through the following proposition.

Proposition 2.8. *The operator G is well defined, linear and closed.*

Proof. Let

$$u(t) := U(t,s)u(s) + \int_s^t U(t,\xi) f(\xi)\, d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+,$$

$$v(t) := U(t,s)v(s) + \int_s^t U(t,\xi) g(\xi)\, d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+,$$

and $u(t) = v(t)$ for a.e. $t \in \mathbb{R}_+$. Then,

$$\int_s^t U(t,\xi) g(\xi)\, d\xi = \int_s^t U(t,\xi) f(\xi)\, d\xi.$$

Hence,

$$\int_s^t U(t,\xi)\big[f(\xi) - g(\xi)\big]\, d\xi = 0.$$

Thus,

$$\frac{1}{(s-t)} \int_s^t U(t,\xi)\big[f(\xi) - g(\xi)\big]\, d\xi = 0.$$

Let $t - s \to 0$ we obtain that

$$f(t) = g(t) \quad \text{for a.e. } t \in \mathbb{R}_+.$$

Therefore, G is well defined. It is clear by definition that G is linear. We now prove that G is closed.

Let $\{v_n\}$ is a sequence in $D(G)$, such that $\lim_{n\to\infty} \|v_n - v\|_{\mathcal{E}_\infty} = 0$ for some $v \in \mathcal{E}_\infty$ and

$$\exists f \in \mathcal{E} \quad \text{such that} \quad \lim_{n\to\infty} \|Gv_n - f\|_{\mathcal{E}} = 0. \tag{6}$$

Hence,

$$\lim_{n\to\infty} \big\|v_n(t) - v(t)\big\| = 0 \quad \text{for fixed } t \in \mathbb{R}_+. \tag{7}$$

We now prove that $v \in D(G)$ and $Gv = f$. In fact, we have

$$v_n(t) = U(t,s)v_n(s) + \int_s^t U(t,\xi)Gv_n(\xi)\,d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+. \tag{8}$$

For fixed t, s a.e. in $\mathbb{R}_+$, from Definition 2.1(3) we have

$$\begin{aligned}
&\left\| \int_s^t U(t,\xi)Gv_n(\xi)\,d\xi - \int_s^t U(t,\xi)f(\xi)\,d\xi \right\| \\
&\quad \leqslant \int_s^t \|U(t,\xi)\| \|Gv_n(\xi) - f(\xi)\|\,d\xi \leqslant N_1 \int_s^t \|Gv_n(\xi) - f(\xi)\|\,d\xi \\
&\quad \leqslant \frac{N_1 M(t-s)}{\|\chi_{[s,t]}\|_E} \|Gv_n - f\|_{\mathcal{E}}.
\end{aligned}$$

From this and (6) we obtain

$$\lim_{n\to\infty} \left\| \int_s^t U(t,\xi)Gv_n(\xi)\,d\xi - \int_s^t U(t,\xi)f(\xi)\,d\xi \right\| = 0. \tag{9}$$

The equalities (7)–(9) yield

$$v(t) = U(t,s)v(s) + \int_s^t U(t,\xi)f(\xi)\,d\xi \quad \text{for a.e. } t \geqslant s \in \mathbb{R}_+.$$

Therefore, $v \in D(G)$ and $Gv = f$. □

Similarly, we define an operator G_0 related to the equation

$$u(t) = \int_0^t U(t,\xi)f(\xi)\,d\xi \tag{10}$$

as follows.

Definition 2.9. The operator $G_0 : D(G_0) \subset \mathcal{E}_\infty \to \mathcal{E}$ is defined by

$$\begin{aligned}
D(G_0) &:= \bigl\{u \in \mathcal{E}_\infty\colon \text{ there exists } f \text{ in } \mathcal{E} \text{ such that } u, f \text{ satisfy Eq. (10)}\bigr\}, \\
G_0 u &:= f \quad \text{for } u \in D(G_0) \text{ and } f \in \mathcal{E} \text{ satisfying Eq. (10)}.
\end{aligned}$$

By the same way as done in the proof of Proposition 2.8 we can prove that G_0 is a well-defined, closed and linear operator.

Remark 2.10. We have the following properties of G and G_0.

(1) $\ker G := \{u \in D(G)\colon\ u(t) = U(t,0)u(0)\}$.
(2) One can easily see that $D(G_0) := \{v \in D(G)\colon\ v(0) = 0\}$ and $Gv = G_0 v$ whenever $v \in D(G_0)$. Therefore, G is an extension of G_0.

In order to characterize the exponential stability and dichotomy of an evolution family we need the concept of $\mathcal{E}$-stable spaces defined as follows.

Definition 2.11. For an evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ on Banach space X and $t_0 \geqslant 0$ we define the $\mathcal{E}$-stable space $X_0(t_0)$ letting

$$z(t) := \begin{cases} U(t,t_0)x & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0, \end{cases}$$
$$X_0(t_0) := \big\{x \in X\colon \text{ the function } z(t) \text{ belongs to } \mathcal{E}\big\}.$$

3. Exponential stability

In this section we will give a sufficient condition for exponential stability of orbits starting from an $\mathcal{E}$-stable space $X_0(t_0) \subset X$. The obtained results will be used in the next section to characterize the exponential dichotomy. We need the following notion of correct operators.

Definition 3.1. Let V and W be Banach spaces endowed with the norm $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Then an operator $A\colon V \to W$ is said to be correct if there exists a constant $\nu > 0$ such that

$$\nu\|Av\|_W \geqslant \|v\|_V \quad \text{for } v \in D(A).$$

The following theorem connects the exponential stability of orbits starting from an $\mathcal{E}$-stable space to the correctness of the operator G_0.

Theorem 3.2. *Let $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ be an evolution family on the Banach space X such that G_0 is correct. Then, for $x \in X_0(t_0)$ we have that there exist positive constants N, η independent of x and t_0 such that*

$$\big\|U(t,t_0)x\big\| \leqslant Ne^{-\eta(t-s)}\big\|U(s,t_0)x\big\| \quad \text{for } x \in X_0(t_0),\ t \geqslant s \geqslant t_0 \geqslant 0. \tag{11}$$

Proof. Since G_0 is correct we have that there exists a constant $\nu > 0$ such that

$$\nu\|G_0 v\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}_\infty} \geqslant \|v\|_{\mathcal{E}} \quad \text{for } v \in D(G_0). \tag{12}$$

To prove (11), we first prove that there is a positive constant l such that

$$\left\|U(t,t_0)x\right\| \leqslant l\left\|U(s,t_0)x\right\| \quad \text{for } x\in X_0(t_0),\ t\geqslant s\geqslant t_0\geqslant 0. \tag{13}$$

Indeed, let ϕ be a real, continuously differentiable function such that

$$\operatorname{supp}\phi\subset(s,\infty),\qquad \phi(t)=1\quad\text{for } t\geqslant s+1,\quad\text{and}\quad \left|\phi'(t)\right|\leqslant 2.$$

Taking

$$v(t):=\begin{cases}\phi(t)U(t,t_0)x & \text{for } t\geqslant t_0,\\ 0 & \text{for } 0\leqslant t<t_0\end{cases}$$

we have $v\in\mathcal{E}_\infty$. Putting

$$f(t):=\begin{cases}\phi'(t)U(t,t_0)x & \text{for } t\geqslant t_0,\\ 0 & \text{for } 0\leqslant t<t_0\end{cases}$$

by exponential boundedness of $\mathcal{U}$ we obtain that

$$\left\|f(t)\right\|\leqslant 2Ke^c\chi_{[s,s+1]}(t)\left\|U(s,t_0)x\right\|.$$

Since $e^c\chi_{[s,s+1]}(\cdot)\|U(s,t_0)x\|$ belongs to E, by Banach lattice property, we have that $\|f(\cdot)\|$ also belongs to E. Therefore, $f\in\mathcal{E}$, and

$$\|f\|_{\mathcal{E}}\leqslant Ke^c\|\chi_{[s,s+1]}\|_E\left\|U(s,t_0)x\right\|\leqslant K_1\left\|U(s,t_0)x\right\|.$$

It can be seen that v and f satisfy (10). Therefore, $v\in D(G_0)$ and $G_0v=f$. By (12) we have

$$\sup_{t\geqslant s+1}\left\|U(t,t_0)x\right\|\leqslant\left\|\phi(\cdot)U(\cdot,t_0)x\right\|_\infty\leqslant\|v\|_{\mathcal{E}_\infty}\leqslant\nu\|f\|_{\mathcal{E}}.$$

Therefore, $\sup_{t\geqslant s+1}\|U(t,t_0)x\|\leqslant\nu K_1\|U(s,t_0)x\|$ and inequality (13) follows.

We now show that there is a number $T=T(\nu,l)>0$ such that

$$\left\|U(s+t,t_0)x\right\|\leqslant\frac{1}{2}\left\|U(s,t_0)x\right\|\quad\text{for } t\geqslant T,\ s\geqslant 0;\ x\in X_0(t_0). \tag{14}$$

Without loss of generality we can suppose that $\|x\|=1$. To prove (14), put $u(t):=U(t,t_0)x$, and let $[a,b]\subset[t_0,\infty)$ be an interval such that $\|u(b)\|>\frac{1}{2}\|u(a)\|$. From (13) we obtain that

$$l\left\|u(a)\right\|\geqslant\left\|u(t)\right\|>\frac{1}{2l}\left\|u(a)\right\|\quad\text{for } t\in[a,b]. \tag{15}$$

Put now

$$f(t) := \begin{cases} \chi_{[a,b]}(t)\frac{u(t)}{\|u(t)\|} & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0, \end{cases}$$
$$v(t) := \begin{cases} u(t)\int_{t_0}^{t} \frac{\chi_{[a,b]}(\xi)}{\|u(\xi)\|}\,d\xi & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0. \end{cases}$$

Then,

$$\|f(t)\| = \chi_{[a,b]}(t) \quad \text{and} \quad \|v(t)\| \leqslant \frac{2l(b-a)\|z(t)\|}{\|u(a)\|},$$

where $z(t)$ is defined by

$$z(t) := \begin{cases} u(t) = U(t,t_0)x & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0. \end{cases}$$

Furthermore, $\chi_{[a,b]} \in E$, and $\|z(\cdot)\| \in E$ (because $x \in X_0(t_0)$). Therefore, $f, v \in \mathcal{E}$. Also, v is continuous and bounded, hence, $v \in \mathcal{E}_\infty$. Moreover, v, f satisfy Eq. (10). Hence, $v \in D(G_0)$ and $G_0 v = f$. We thus obtain $\nu\|f\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}}$. Therefore,

$$\nu\|\chi_{[a,b]}\|_E \geqslant \nu\|f\|_{\mathcal{E}} \overset{\text{by (12)}}{\geqslant} \|v\|_{\mathcal{E}} \overset{\text{by (2)}}{\geqslant} \frac{\|\chi_{[a,b]}\|_E}{M(b-a)}\int_a^b \|v(t)\|\,dt.$$

Using now the estimates (15) we have

$$\int_a^b \|v(t)\|\,dt > \int_a^b \frac{1}{2l}\|u(a)\|\int_a^t \frac{1}{l\|u(a)\|}\,d\xi\,dt = \frac{(b-a)^2}{4l^2}.$$

This yields

$$\nu\|\chi_{[a,b]}\|_{\mathcal{E}} > \frac{M(b-a)\|\chi_{[a,b]}\|_E}{4l^2}.$$

Hence, we obtain $b - a < 4\nu l^2/M$. Putting $T := 4\nu l^2/M$, inequality (14) follows.

We finish by proving (11). Indeed, if $t \geqslant s$ writing $t - s = nT + r$ for $0 \leqslant r < T$ and some nonnegative integer n we have

$$\begin{aligned} \|U(t,0)x\| &= \|U(t-s+s,0)x\| = \|U(nT+r+s,0)x\| \\ &\overset{\text{by (14)}}{\leqslant} \frac{1}{2^n}\|U(r+s,0)x\| \overset{\text{by (13)}}{\leqslant} \frac{l}{2^n}\|U(s,0)x\| \leqslant 2le^{-(t-s)/T\ln 2}\|U(s,0)x\|. \end{aligned}$$

Taking $N := 2l$ and $\eta := \ln 2/T$, inequality (11) follows. □

As a consequence of this theorem we obtain the following corollary.

Corollary 3.3. *Under the conditions of Theorem* 3.2 *the space $X_0(t_0)$ can be expressed as*

$$X_0(t_0) = \{x \in X\colon \|U(t,t_0)x\| \leqslant Ne^{-\nu(t-t_0)}\|x\|\}, \quad t \geqslant t_0 \geqslant 0, \tag{16}$$

for certain positive constants N, ν. Hence, $X_0(t_0)$ is a closed linear subspace of X.

Proof. By inequality (11) we obtain that

$$X_0(t_0) \subseteq \{x \in X\colon \|U(t,t_0)x\| \leqslant Ne^{-\eta(t-t_0)}\|x\| \text{ for } t \geqslant t_0\}.$$

Take now $x \in X$ such that $\|U(t,t_0)x\| \leqslant Ne^{-\eta(t-t_0)}\|x\|$ for $t \geqslant t_0$ and put

$$\varphi(t) := \begin{cases} e^{-\eta(t-t_0)} & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0. \end{cases}$$

By Proposition 2.6(b) and Lemma 2.2 we have that function φ belongs to E. This yields that the function

$$z(t) := \begin{cases} U(t,t_0)x & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0 \end{cases}$$

belongs to $\mathcal{E}$. Therefore, $x \in X_0(t_0)$ and equality (16) follows. By (16) we obtain that $X_0(t_0)$ is closed. □

4. Exponential dichotomy

We will characterize the exponential dichotomy of evolution families using the operators G. Before doing so, we now make precise the notion of exponential dichotomy in the following definition.

Definition 4.1. An evolution family $\mathcal{U} = (U(t,s))_{t \geqslant s \geqslant 0}$ on the Banach space X is said to have an exponential dichotomy on $[0,\infty)$ if there exist bounded linear projections $P(t)$, $t \geqslant 0$ on X and positive constants N, η such that

(a) $U(t,s)P(s) = P(t)U(t,s)$, $t \geqslant s \geqslant 0$,
(b) the restriction $U(t,s)_| : \ker P(s) \to \ker P(t)$, $t \geqslant s \geqslant 0$, is an isomorphism (and we denote its inverse by $U(s,t)_| : \ker P(t) \to \ker P(s)$),
(c) $\|U(t,s)x\| \leqslant Ne^{-\eta(t-s)}\|x\|$ for $x \in P(s)X$, $t \geqslant s \geqslant 0$,
(d) $\|U(s,t)_| x\| \leqslant Ne^{-\eta(t-s)}\|x\|$ for $x \in \ker P(t)$, $t \geqslant s \geqslant 0$.

The constants N, ν are called dichotomy constants and the projections $P(t)$, $t \geqslant 0$, are called dichotomy projections.

We remark that properties (a)–(d) of dichotomy projections $P(t)$ already imply that

(1) $H := \sup_{t \geqslant 0} \|P(t)\| < \infty$,
(2) $t \mapsto P(t)$ is strongly continuous

(see [17, Lemma 4.2]).

We now come to our main result of this section. It characterizes exponential dichotomy of an evolution family on a half-line in terms of surjectiveness of the corresponding operator G and the complementedness of the $\mathcal{E}$-stable space $X_0(0)$.

Theorem 4.2. *Let $\mathcal{E} = \mathcal{E}(\mathbb{R}_+, X)$ be the Banach space corresponding to the admissible Banach function space E. Consider the operator G defined as in Definition* 2.7. *Then, the following assertions are equivalent.*

(i) *The evolution family $\mathcal{U} = U(t,s)_{t \geqslant s \geqslant 0}$ has an exponential dichotomy.*
(ii) *The operator $G : D(G) \subset \mathcal{E}_\infty \to \mathcal{E}$ is surjective and the $\mathcal{E}$-stable space $X_0(0)$ is closed and complemented.*

Proof. (i) $\Rightarrow$ (ii). Let the evolution family $\mathcal{U} = U(t,s)_{t \geqslant s \geqslant 0}$ has an exponential dichotomy with the corresponding dichotomy projections $P(t)$, $t \geqslant 0$. For $f \in \mathcal{E}(\mathbb{R}_+, X)$ put

$$u(t) := \int_0^t U(t,s)P(s)f(s)\,ds - \int_t^\infty U(t,s)_|\big(I - P(s)\big)f(s)\,ds. \tag{17}$$

Applying Proposition 2.6(a) for $\varphi(\cdot) = \|f(\cdot)\|$ we obtain that $u \in \mathcal{E}(\mathbb{R}_+, X)$ and u is bounded. Since u is continuous we have that $u \in \mathcal{E}_\infty$. Clearly, u and f satisfy Eq. (5). Therefore, $u \in D(G)$ and $Gu = f$. Hence, G is surjective.

We now prove that $X_0(0) = P(0)X$. Indeed, if $x \in P(0)X$ then $\|U(t,0)x\| \leqslant Ne^{-\eta t}\|x\|$ for $t \geqslant 0$ and some constants $N, \eta > 0$. By Proposition 2.6(b) we obtain that the function $z(t) := U(t,0)x$ belongs to $\mathcal{E}$. Therefore, $x \in X_0(0)$. If now $x \notin P(0)X$ then, since $X = P(0)X \oplus \ker P(0)$, we can decompose $x = x_0 + x_1$ for $x_0 \in P(0)X$ and $0 \neq x_1 \in \ker P(0)$. We hence obtain that

$$\big\|U(t,0)x\big\| = \big\|U(t,0)(x_0 + x_1)\big\| \geqslant Ne^{\eta t}\|x_1\| - Ne^{-\eta t}\|x_0\|.$$

By Proposition 2.6(c) we have that function $z(t) := U(t,0)x$ does not belong to $\mathcal{E}$. Therefore $x \notin X_0(0)$. We thus obtain $X_0(0) = P(0)X$. Therefore, $X_0(0)$ is closed and complemented in X.

(ii) $\Rightarrow$ (i). We prove this in several steps.

Step 1. Let $Z \subseteq X$ be a complement of $X_0(0)$ in X, i.e., $X = X_0(0) \oplus Z$. Set $X_1(t) = U(t,0)Z$ then

$$\begin{aligned} U(t,s)X_0(s) &\subseteq X_0(t), \quad t \geqslant s \geqslant 0, \\ U(t,s)X_1(s) &= X_1(t), \quad t \geqslant s \geqslant 0. \end{aligned} \tag{18}$$

Step 2. There are constants $N, \eta > 0$ such that

$$\|U(t,0)x\| \geqslant N e^{\eta(t-s)} \|U(s,0)x\| \quad \text{for } x \in X_1(0),\ t \geqslant s \geqslant 0. \tag{19}$$

In fact, let $Y := \{v \in D(G)\colon\ v(0) \in X_1(0)\}$ endowed with graph norm $\|v\|_G := \|v\|_{\mathcal{E}_\infty} + \|Gv\|_{\mathcal{E}}$. Then Y is a closed subspace of the Banach space $(D(G), \|\cdot\|_G)$, and hence Y is complete. By Definition 2.7 of G we have that $\ker G := \{v \in D(G)\colon\ v(t) = u(t,0)x$ for some $x \in X_0(0)\}$. Since $X = X_0(0) \oplus X_1(0)$ and G is surjective we obtain

$$G : Y \to \mathcal{E}$$

is bijective and hence an isomorphism. Thus there is a constant $\nu > 0$ such that

$$\nu \|Gv\|_{\mathcal{E}} \geqslant \|v\|_G \geqslant \|v\|_{\mathcal{E}_\infty} \geqslant \|v\|_{\mathcal{E}} \quad \text{for } v \in Y. \tag{20}$$

To prove (19) we first prove that there is a positive constant l such that

$$\|U(t,0)x\| \geqslant l \|U(s,0)x\| \quad \text{for } x \in X_1(0),\ t \geqslant s \geqslant 0. \tag{21}$$

Indeed, let ϕ be a real, continuously differentiable function such that

$$\operatorname{supp}\phi \subset (t,\infty), \qquad \phi(\xi) = 1 \quad \text{for } \xi \geqslant t+1, \quad \text{and} \quad |\phi'(\xi)| \leqslant 2.$$

Taking $v(\cdot) := (1-\phi(\cdot))U(\cdot,0)x$ we have $v \in \mathcal{E}_\infty$. Moreover, putting $f(\cdot) := \phi'(\cdot)U(\cdot,0)x$, by exponential boundedness of $\mathcal{U}$ we obtain that

$$\|f(\xi)\| = \|\phi'(\xi)U(\xi,0)x\| \leqslant 2Ke^c \chi_{[t,t+1]}(\xi) \|U(t,0)x\|.$$

Since $e^c \chi_{[t,t+1]}(\cdot)\|U(t,0)x\|$ belongs to E, by Banach lattice property, we have that $\|f(\cdot)\|$ also belongs to E. Also, f is strongly measurable, hence, $f \in \mathcal{E}$, and

$$\|f(\cdot)\|_{\mathcal{E}} = \|\phi'(\cdot)U(\cdot,0)x\|_{\mathcal{E}} \leqslant Ke^c \|\chi_{[t,t+1]}\|_E \|U(t,0)x\| \leqslant K_1 \|U(t,0)x\|.$$

It can be seen that v and f satisfy Eq. (5). Therefore, $v \in D(G)$ and $Gv = f$. Moreover, since $v(0) = x \in X_1(0)$ we obtain $v \in Y$. By (20) we have

$$\begin{aligned}\sup_{s\leqslant t}\|U(s,0)x\| &\leqslant \|(1-\phi(\cdot))U(\cdot,0)x\|_\infty \leqslant \|(1-\phi(\cdot))U(\cdot,0)x\|_{\mathcal{E}_\infty}\\ &\leqslant \nu\|\phi'(\cdot)U(\cdot,0)x\|_{\mathcal{E}}.\end{aligned}$$

Therefore, $\sup_{s\leqslant t}\|U(s,0)x\| \leqslant \nu K_1 \|U(t,0)x\|$ and inequality (21) follows.

We now show that there is a number $T = T(\nu, l) > 0$ such that

$$\|U(s+t,0)x\| \geqslant 2\|U(s,0)x\| \quad \text{for } t \geqslant T,\ s \geqslant 0;\ x \in X_1(0). \tag{22}$$

Without loss of generality we can suppose that $\|x\| = 1$. To prove (22) put $u(t) := U(t,0)x$, and let $[a,b] \subset [0,\infty)$ be an interval such that $\|u(b)\| < 2\|u(a)\|$. From (21) we obtain that

$$\frac{2}{l}\|u(a)\| > \|u(t)\| \geqslant l\|u(a)\| \quad \text{for all } t \in [a,b]. \tag{23}$$

Put now

$$f(t) := \chi_{[a,b]}(t)\frac{u(t)}{\|u(t)\|}, \qquad v(t) := -u(t)\int_t^\infty \frac{\chi_{[a,b]}(\xi)}{\|u(\xi)\|}\,d\xi.$$

Then,

$$\|f(t)\| = \chi_{[a,b]}(t) \quad \text{and} \quad \|v(t)\| \leqslant \frac{(b-a)}{l^2}\chi_{[0,b]}(t).$$

Since $\chi_{[a,b]}$ and $\chi_{[0,b]}$ belong to E we obtain that $f, v \in \mathcal{E}$. It can be seen that v is continuous and bounded, hence, $v \in \mathcal{E}_\infty$. Moreover, v, f satisfy Eq. (5) and $v(0) \in X_1(0)$. Hence, $Gv = f$, and $\nu\|f\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}}$. Therefore,

$$\nu\|\chi_{[a,b]}\|_E = \nu\|f\|_{\mathcal{E}} \overset{\text{by (20)}}{\geqslant} \|v\|_{\mathcal{E}} \overset{\text{by (2)}}{\geqslant} \frac{\|\chi_{[a,b]}\|_E}{M(b-a)}\int_a^b \|v(t)\|\,dt.$$

Using now the estimates (23) we have

$$\int_a^b \|v(t)\|\,dt > \int_a^b l\|u(a)\|\int_a^t \frac{l}{\|u(a)\|}\,d\xi\,dt = 4l^2(b-a)^2.$$

This yields $\nu\|\chi_{[a,b]}\|_{\mathcal{E}} > 4l^2M(b-a)\|\chi_{[a,b]}\|_E$. Hence, we obtain $b-a < \nu/(4Ml^2)$. Putting $T := \nu/(4Ml^2)$, inequality (22) follows.

We finish this step by proving inequality (19). Indeed, if $t \geqslant s$ writing $t - s = nT + r$ for $0 \leqslant r < T$ and some nonnegative integer n we have

$$\begin{aligned}\|U(t,0)x\| &= \|U(t-s+s,0)x\| = \|U(nT+r+s,0)x\| \\ &\overset{\text{by (22)}}{\geqslant} 2^n\|U(r+s,0)x\| \overset{\text{by (21)}}{\geqslant} l2^n\|U(s,0)x\| \geqslant \frac{l}{2}e^{(t-s)/T\ln 2}\|U(s,0)x\|.\end{aligned}$$

Taking $N := l/2$ and $\eta := \ln 2/T$, inequality (19) follows.

Step 3. $X_0(t)$ is closed and $X = X_0(t) \oplus X_1(t)$, $t \geqslant 0$.

Let $Y \subset D(G)$ be as in Step 2. Then, by Remark 2.10, $D(G_0) \subset Y$. From this and (20) we have that $\nu \|G_0 v\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}_\infty}$ for $v \in D(G_0)$. Thus, G_0 is correct. By Corollary 3.3, $X_0(t)$ is closed.

By (11), (18), (19) and the closedness of $X_1(0)$ we can easily derive that $X_1(t)$ is closed and $X_1(t) \cap X_0(t) = \{0\}$ for $t \geqslant 0$.

Finally, fix $t_0 > 0$ and $x \in X$. Set

$$v(t) = \int_t^\infty \chi_{[t_0,t_0+1]}(s)\, ds\, U(t,t_0)x, \quad t \geqslant t_0,$$
$$f(t) = -\chi_{[t_0,t_0+1]}(t) U(t,t_0)x, \quad t \geqslant t_0.$$

Then v, f solve Eq. (5) with $t \geqslant s \geqslant t_0 \geqslant 0$. Moreover, the function

$$z_v(t) := \begin{cases} v(t) & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0 \end{cases}$$

satisfies $\|z_v(t)\| \leqslant K e^c \chi_{[t_0,t_0+1]}(t)$, and hence belongs to $\mathcal{E}$. Extend f to $[0,\infty)$ by setting $f|_{[0,t_0)} = 0$, then $f \in \mathcal{E}$. By assumption there exists $w \in D(G)$ such that $Gw = f$. By the definition of G, w is a solution of Eq. (5). In particular, $w|_{[t_0,\infty)}$ also satisfies (5). Thus,

$$v(t) - w(t) = U(t,t_0)\bigl(v(t_0) - w(t_0)\bigr) = U(t,t_0)\bigl(x - w(t_0)\bigr), \quad t \geqslant t_0.$$

Furthermore, it is clear that the function

$$z_w(t) := \begin{cases} w(t) & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0 \end{cases}$$

belongs to $\mathcal{E}$. Therefore, the function

$$z(t) := z_v(t) - z_w(t) = \begin{cases} v(t) - w(t) & \text{for } t \geqslant t_0, \\ 0 & \text{for } 0 \leqslant t < t_0 \end{cases}$$

belongs to $\mathcal{E}$. This implies $x - w(t_0) \in X_0(t_0)$. On the other hand, $w(0) = w_0 + w_1$ with $w_k \in X_k(0)$, $k = 1, 2$, then $w(t_0) = U(t_0, 0)w_0 + U(t_0, 0)w_1$ and by (18) we have $U(t,t_0)w_k \in X_k(t_0)$, $k = 0, 1$. Hence $x = x - w(t_0) + w(t_0) \in X_0(t_0) + X_1(t_0)$. This proves Step 3.

Step 4. Let $P(t)$ be the projections from X onto $X_0(t)$ with kernel $X_1(t)$, $t \geqslant 0$. Then (18) implies that $P(t)U(t,s) = U(t,s)P(s)$, $t \geqslant s \geqslant 0$. From (18), (19) we obtain that $U(t,s)_{|}\ker P(s) \to \ker P(t)$, $t \geqslant s \geqslant 0$, is an isomorphism. Finally, since G_0 is correct (see the proof of Step 3, by (11) and (19) there exist constants $N, \eta > 0$ such that

$$\bigl\|U(t,s)x\bigr\| \leqslant N e^{-\eta(t-s)} \|x\| \quad \text{for } x \in P(s)X,\ t \geqslant s \geqslant 0,$$
$$\bigl\|U(s,t)x\bigr\| \leqslant N e^{-\eta(t-s)} \|x\| \quad \text{for } x \in \ker P(t),\ t \geqslant s \geqslant 0.$$

Thus $\mathcal{U}$ has an exponential dichotomy. $\square$

In our next result, we try to characterize the exponential dichotomy of evolution family on a half-line using invertibility of a certain operator derived from the operator G. The price we have to pay for such a characterization is that we have to know the $\ker P(0)$ in advance (see Theorem 4.4). Consequently, the exponential dichotomy of evolution family will be characterized by the invertibility of the restriction of G to a certain subspace of $\mathcal{E}_\infty$. This restriction will be defined as follows.

Definition 4.3. For a closed linear subspace Z of X we define

$$\mathcal{E}_Z := \big\{ f \in \mathcal{E}_\infty \colon\ f(0) \in Z \big\}.$$

Then, $\mathcal{E}_Z$ is a closed subspace of $(\mathcal{E}_\infty, \|\cdot\|_{\mathcal{E}_\infty})$. Denote by G_Z the part of G in $\mathcal{E}_Z$, i.e., $D(G_Z) = D(G) \cap \mathcal{E}_Z$ and $G_Z u = Gu$ for $u \in D(G_Z)$.

We also remark that the operator G_0 defined in Definition 2.9 is the part of G_Z in $\mathcal{E}_0 = \{f \in \mathcal{E}_\infty \colon\ f(0) = 0\}$. With these notations we obtain the following characterization of evolution families having an exponential dichotomy. This result extends those shown in [2, Theorem 1.1] for finite-dimensional spaces and in [17, Theorem 4.5] for the space $C_0(\mathbb{R}_+, X)$ of X-valued continuous functions decaying at infinity.

Theorem 4.4. *Let $\mathcal{U} = (U(t,s))_{t \geqslant s \geqslant 0}$ be an evolution family on the Banach space X and Z be a closed linear subspace of X. Then the following assertions are equivalent.*

(i) *$\mathcal{U}$ has an exponential dichotomy with* $\ker P(0) = Z$.
(ii) *$G_Z : D(G_Z) \subset \mathcal{E}_Z \to \mathcal{E}$ is invertible.*

Proof. We first note that the following proof is inspired by the proof of [17, Theorem 4.5].

(i) $\Rightarrow$ (ii). Let $P(t)$, $t \geqslant 0$, be a family of projections given by the exponential dichotomy of $\mathcal{U}$ such that $\ker P(0) = Z$. Then $P(0)X = X_0(0)$ and $X = X_0(0) \oplus Z$. Fix $f \in \mathcal{E}$. By Theorem 4.2 there is $v \in D(G)$ such that $Gv = f$. On the other hand, by definition of $X_0(0)$ the function $u : \mathbb{R} \to X$, $t \to U(t,0)P(0)v(0)$, belongs to $\mathcal{E}$ and, hence, to $\mathcal{E}_\infty$. By Definition 2.7 of G we obtain that $Gu = 0$. Moreover, $v(0) - u(0) = v(0) - P(0)v(0) \in Z$ since $X = P(0)X \oplus Z$. Therefore, $v - u \in E_Z$ and $G_Z(v-u) = G(v-u) = Gv = f$. Hence, $G_Z : D(G_Z) \to \mathcal{E}$ is surjective. If now $w \in \ker G_Z$ then, by definition of G_Z, $w(t) = U(t,0)w(0)$ with $w(0) \in Z \cap X_0(0) = \{0\}$. Thus, $w = 0$, i.e., G_Z is injective.

(ii) $\Rightarrow$ (i). Let $G_Z : D(G_Z) \to \mathcal{E}$ be invertible. Since G_Z is the restriction of G to $\mathcal{E}_Z$, this follows that G is surjective. The closedness of G_Z implies that G_Z^{-1} is bounded, and hence there is $\nu > 0$ such that $\nu\|Gv\|_{\mathcal{E}} = \nu\|G_Z v\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}_\infty}$ for all $v \in D(G_Z)$. Since G_0 is the part of G_Z in $\mathcal{E}_0 = \{f \in \mathcal{E}_\infty \colon\ f(0) = 0\}$, we obtain that $\nu\|G_0 v\|_{\mathcal{E}} \geqslant \|v\|_{\mathcal{E}_\infty}$ for all $v \in D(G_0)$. Hence, G_0 is correct. By Corollary 3.3, $X_0(0)$ is closed. We now prove that $X = X_0(0) \oplus Z$. Let now $x \in X$. If $U(t,0)x = 0$ for some $t = t_0 > 0$ then $U(t,0)x = U(t,t_0)U(t_0,0)x = 0$ for all $t \geqslant t_0$ yielding $x \in X_0(0)$. Otherwise, $U(t,0)x \neq 0$ for all $t \geqslant 0$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be continuously differentiable such that $\varphi|_{[0,1]} = 1$ and $\varphi|_{[2,\infty]} = 0$. Set $u(t) := \varphi(t)U(t,0)x$, $t \geqslant 0$, and $f(t) := \varphi'(t)U(t,0)x$, $t \geqslant 0$. Clearly, $u \in \mathcal{E}_\infty$, $f \in \mathcal{E}$, and they satisfy Eq. (5). Therefore, $Gu = f$. On the other hand, since G_Z is invertible, there exists $v \in D(G_Z) \subset D(G)$ such that $G_Z v = f = Gv$. Thus, $u - v \in \ker G$ and hence

$$(u-v)(t) = U(t,0)\big(u(0) - v(0)\big) = U(t,0)\big(x - v(0)\big), \quad t \geqslant 0.$$

Since $u - v \in \mathcal{E}$, this implies that $x - v(0) \in X_0(0)$. Thus $x = x - v(0) + v(0) \in X_0(0) + Z$.

If now $y \in X_0(0) \cap Z$ then the function w define by $w(t) := U(t,0)y$, $t \geqslant 0$, belongs to $\mathcal{E}_Z \cap \ker G$ (see definitions of $X_0(0)$ and G). Hence, $G_Z w = 0$ and by invertibility of G_Z we have that $w = 0$. Thus $y = w(0) = 0$, i.e., $X_0(0) \cap Z = \{0\}$. This yields that $X = X_0(0) \oplus Z$. The assertion now follows from Theorem 4.2. □

5. Perturbations

In this section we study the robustness of the exponential dichotomy of evolution families under small perturbations. More precisely, let B be a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(X)$. Then it is known (see [23,25] and references therein) that there exists a unique evolution family $(U_B(t,s))_{t\geqslant s\geqslant 0}$ satisfying the variation of constants formula

$$U_B(t,s)x = U(t,s)x + \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)x\,d\xi, \quad t \geqslant s \geqslant 0,\ x \in X. \tag{24}$$

We will prove that, if $(U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy and the norm $\|B\| := \sup_{t\geqslant 0}\|B(t)\|$ is sufficiently small, then $(U_B(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy as well. We note that, if we consider $(U(t,s))_{t\geqslant s\geqslant 0}$ to be "generated" by a concrete well-posed non-autonomous Cauchy problem, e.g., as in (1), then we have that $(U_B(t,s))_{t\geqslant s\geqslant 0}$ is "generated" by the perturbed problem of (1), i.e., by

$$\begin{cases} \frac{du(t)}{dt} = (A(t) + B(t))u(t), & t \geqslant s \geqslant 0, \\ u(s) = x_s \in X. \end{cases} \tag{25}$$

Therefore, our result reveals that the exponential dichotomy of the solutions to the problem (1) is robust under small perturbations by bounded operators $B(t)$. We also note that, if we consider the evolution family $(U(t,s))_{t\geqslant s}$ on the whole line $\mathbb{R}$, then the result is well known (see [8, Theorem VI.9.24] and references therein). We refer the readers to Coppel [5, Section 4] for the results on robustness of exponential dichotomy of evolution families on finite-dimensional spaces and to Daleckii and Krein [6, Section IV.5] for that of evolution families generated by bounded $A(t)$ on Banach spaces.

Theorem 5.1. *Let the evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy and let B be a strongly continuous and uniformly bounded function from $\mathbb{R}_+$ into the space $\mathcal{L}(X)$, i.e., $B \in C_b(\mathbb{R}_+, \mathcal{L}_s(X))$. Then, if the norm $\|B\| := \sup_{t\geqslant 0}\|B(t)\|$ is sufficiently small, the evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geqslant s\geqslant 0}$ defined as in* (24) *has an exponential dichotomy as well.*

Proof. Let the evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ satisfying $\ker P(0) = Z$ for some closed subspace Z of X. Let $G_{B,Z}$ be the operator corresponding to the perturbed evolution family $\mathcal{U}_B$. That is, $G_{B,Z}$ is defined as follows. If $u \in \mathcal{E}_Z$ and $f \in \mathcal{E}$ satisfy the equation

$$u(t) = U_B(t,s)u(s) + \int_s^t U_B(t,\xi)f(\xi)\,d\xi, \quad t \geqslant s \geqslant 0, \tag{26}$$

then we set $G_{B,Z}u := f$ with

$$D(G_{B,Z}) := \bigl\{u \in \mathcal{E}_Z\colon \text{there exists } f \in \mathcal{E} \text{ such that } u, f \text{ satisfy Eq. (26)}\bigr\},$$

where the space $\mathcal{E}_Z$ is defined as in Definition 4.3.

We now define the operator $\mathfrak{B}$ by $[\mathfrak{B}f](t) := B(t)f(t)$ for all $t \geqslant 0$. We then prove that $\mathfrak{B}: \mathcal{E} \to \mathcal{E}$ is a bounded linear operator and $\|\mathfrak{B}\| \leqslant \|B\|$. Indeed, take $f \in \mathcal{E}$. Then $\|[\mathfrak{B}f](t)\| = \|B(t)f(t)\| \leqslant \|B\|\|f(t)\|$ for all $t \geqslant 0$. Since $\|B\|\|f(\cdot)\|$ belongs to E, by Banach lattice property we have that the function $\|[\mathfrak{B}f](\cdot)\|$ belongs to E and $\|\|[\mathfrak{B}f](\cdot)\|\|_E \leqslant \|B\|\|\|f(\cdot)\|\|_E$. Therefore, $\mathfrak{B}f$ belongs to $\mathcal{E}$ and $\|\mathfrak{B}f\|_{\mathcal{E}} \leqslant \|B\|\|f\|_{\mathcal{E}}$. We thus obtain that $\mathfrak{B}: \mathcal{E} \to \mathcal{E}$ is a bounded linear operator and $\|\mathfrak{B}\| \leqslant \|B\|$.

We will prove that $G_{B,Z} = G_Z - \mathfrak{B}$. Indeed, let $u \in D(G_{B,Z})$ and $G_{B,Z}u = f$. Then, $u(0) \in Z$ and

$$u(t) = U_B(t,s)u(s) + \int_s^t U_B(t,\xi)f(\xi)\,d\xi, \quad t \geqslant s \geqslant 0.$$

By the definition of $\mathcal{U}_B$ we obtain that

$$\begin{aligned}
u(t) = U(t,s)u(s) &+ \int_s^t U(t,\xi)B(\xi)U_B(\xi,s)u(s)\,d\xi \\
&+ \int_s^t \left(U(t,\xi)f(\xi) + \int_\xi^t U(t,\tau)B(\tau)U_B(\tau,\xi)f(\xi)\,d\tau\right)d\xi \\
= U(t,s)u(s) &+ \int_s^t U(t,\xi)f(\xi)\,d\xi + \int_s^t U(t,\tau)B(\tau)U_B(\tau,s)u(s)\,d\tau \\
&+ \int_s^t\int_s^\tau U(t,\tau)B(\tau)U_B(\tau,\xi)f(\xi)\,d\xi\,d\tau \quad \text{(by Fubini theorem)}
\end{aligned}$$

$$= U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi)\,d\xi$$

$$+ \int_s^t U(t,\tau)B(\tau)\left(U_B(\tau,s)u(s) + \int_s^\tau U(\tau,\xi)f(\xi)\,d\xi\right)d\tau$$

$$= U(t,s)u(s) + \int_s^t U(t,\tau)\big(f(\tau) + B(\tau)u(\tau)\big)\,d\tau.$$

This is equivalent to the fact that $u \in D(G_Z - \mathfrak{B})$ and $(G_Z - \mathfrak{B})u = f$. Therefore, we obtain $G_{B,Z} = G_Z - \mathfrak{B}$. Since the evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\geqslant 0}$ satisfying $\ker P(0) = Z$, we have that G_Z is invertible. By a perturbation theorem of Kato [12, IV.1.16] we obtain that, if $\|\mathfrak{B}G_Z^{-1}\| < 1$ (this will be satisfied if $\|B\| < 1/\|G_Z^{-1}\|$), then $G_{B,Z} = G_Z - \mathfrak{B}$ is also invertible. By Theorem 4.4 we have that the evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy. □

Remark 5.2. If we take a concrete space $\mathcal{E}$, e.g., $\mathcal{E} := L_\infty(\mathbb{R}_+, X)$, then we can estimate the norm $\|G_Z^{-1}\|$ as follows.

Let the evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding dichotomy constants $N, \nu > 0$ and dichotomy projections $P(t)$, $t \geqslant 0$, satisfying $\sup_{t\geqslant 0} \|P(t)\| = H < \infty$.

We first define the Green's function

$$\mathcal{G}(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } t \geqslant \tau \geqslant 0, \\ -U(t,\tau)_|(I-P)(\tau) & \text{for } 0 \leqslant t \leqslant \tau. \end{cases}$$

Using this Green's function we can rewrite formula (17) in the form

$$u(t) = \int_0^\infty \mathcal{G}(t,\xi)f(\xi)\,d\xi, \quad t \geqslant 0. \tag{27}$$

From the proof of Theorem 4.4 (implication (i) ⇒ (ii)) we obtain that

$$\begin{aligned}\big[G_Z^{-1}f\big](t) &= u(t) - U(t,0)P(0)u(0) \\ &= \int_0^\infty \mathcal{G}(t,\xi)f(\xi)\,d\xi - U(t,0)P(0)\int_0^\infty \mathcal{G}(0,\xi)f(\xi)\,d\xi\end{aligned}$$

for $t \geqslant 0$ and $f \in L_\infty$.

We then estimate the norm

$$\left\|\left[G_Z^{-1} f\right](t)\right\| \leqslant NH \sup_{t\geqslant 0} \int_0^\infty e^{-\nu|t-\xi|}\, d\xi \|f\|_\infty + N^2 H(1+H) \int_0^\infty e^{-\nu\xi}\, d\xi \|f\|_\infty$$
$$\leqslant \frac{2}{\nu} NH(1+N+NH)\|f\|_\infty \quad \text{for } t \geqslant 0 \text{ and } f \in L_\infty.$$

Therefore, we obtain the following estimate:

$$\left\|G_Z^{-1}\right\| \leqslant \frac{2}{\nu} NH(1+N+NH).$$

Consequently, we have the following corollary.

Corollary 5.3. *Let the evolution family $\mathcal{U} = (U(t,s))_{t\geqslant s\geqslant 0}$ has an exponential dichotomy with the corresponding dichotomy constants $N, \nu > 0$ and dichotomy projections $P(t)$, $t \geqslant 0$, satisfying $\sup_{t\geqslant 0} \|P(t)\| = H < \infty$. Let $B \in C_{\mathrm{b}}(\mathbb{R}_+, \mathcal{L}_s(X))$. Then, if*

$$\|B\| < \frac{\nu}{2NH(1+N+NH)},$$

the evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geqslant s\geqslant 0}$ defined as in (24) *has an exponential dichotomy as well.*

We illustrate our results by the following example.

Example 5.4. We consider the problem

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = \sum_{k,l=1}^n D_k a_{kl}(t,x) D_l u(t,x) + \delta u(t,x) + b(t,x)u(t,x) \\ \quad \text{for } t \geqslant s \geqslant 0,\ x \in \Omega, \\ \sum_{k,l=1}^n n_k(x) a_{kl}(t,x) D_l u(t,x) = 0, \quad t \geqslant s \geqslant 0,\ x \in \partial\Omega, \\ u(s,x) = f(x), \quad x \in \Omega. \end{cases} \tag{28}$$

Here $D_k := \partial/\partial x_k$ and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ oriented by outer unit normal vectors $n(x)$. The coefficients $a_{k,l}(t,x) \in C_{\mathrm{b}}^{\mu}(\mathbb{R}_+, L_\infty(\Omega))$, $\mu > 1/2$, are supposed to be real, symmetric, and uniformly elliptic in the sense that

$$\sum_{k,l=1}^n a_{kl}(t,x) v_k v_l \geqslant \eta |v|^2 \quad \text{for a.e. } x \in \Omega \text{ and some constant } \eta > 0,$$

while the coefficient $b(t,x)$ belongs to $C_{\mathrm{b}}(\mathbb{R}_+, L_\infty(\Omega))$. Finally, the constant δ is defined by

$$\delta := -\frac{1}{2}\eta\lambda,$$

where $\lambda < 0$ denotes the largest eigenvalue of Neumann Laplacian Δ_N on Ω. We now chose the Hilbert space $X = L_2(\Omega)$ and define the operators $C(t)$ via the standard scalar product in X as

$$\big(C(t)f, g\big) = -\sum_{k,l=1}^{n} \int_{\Omega} a_{kl} D_k f(x)(t,x)\overline{D_l g(t,x)}\, dx$$

with $D(C(t)) = \{f \in W^{2,2}(\Omega)\colon \sum_{k,l}^{n} n_k(x) a_{kl}(t,x) D_l f(x) = 0,\ x \in \partial\Omega\}$. We then write the problem (28) as an abstract Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t,\cdot) = A(t)u(t,\cdot) + H(t)u(t,\cdot), & t \geqslant s \geqslant 0, \\ u(s,\cdot) = f \in X, \end{cases}$$

where $A(t) := C(t) + \delta$ and $H(t)\colon X \to X$ defined by $(H(t)f)(x) := b(t,x)f(x)$ for $f \in X$.

By Schnaubelt [26, Chapter 2, Theorem 2.8, Example 2.3], we have that the operators $A(t)$ generate an evolution family having an exponential dichotomy with the dichotomy exponent ν and dichotomy constant N provided that the Hölder constants of $a_{k,l}$ is sufficiently small. Also, the dichotomy projections $P(t)$, $t \geqslant 0$, satisfy $\sup_{t \geqslant 0} \|P(t)\| \leqslant N$. By Corollary 5.3 we now obtain that, if

$$\sup_{t \geqslant 0} \big\| b(t,\cdot) \big\|_{L_\infty(\Omega)} \leqslant \frac{\nu}{2N^2(1 + N + N^2)},$$

then the evolution family solving the problem (28) also has an exponential dichotomy.

References

[1] A. Ben-Artzi, I. Gohberg, Dichotomies of systems and invertibility of linear ordinary differential operators, Oper. Theory Adv. Appl. 56 (1992) 90–119.

[2] A. Ben-Artzi, I. Gohberg, M.A. Kaashoek, Invertibility and dichotomy of differential operators on a half-line, J. Dynam. Differential Equations 5 (1993) 1–36.

[3] S. Brendle, R. Nagel, Partial functional equations with non-autonomous past, Discrete Contin. Dyn. Syst. 8 (2002) 953–966.

[4] A.P. Calderon, Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz, Studia Math. 26 (1996) 273–299.

[5] W.A. Coppel, Dichotomy in Stability Theory, Springer-Verlag, Berlin, 1978.

[6] J.L. Daleckii, M.G. Krein, Stability of Solutions of Differential Equations in Banach Spaces, Transl. Amer. Math. Soc., Amer. Math. Soc., Providence, RI, 1974.

[7] R. Datko, Uniform asymptotic stability of evolutionary processes in a Banach space, SIAM J. Math. Anal. 3 (1972) 428–445.

[8] K.J. Engel, R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Grad. Texts in Math., vol. 194, Springer-Verlag, Berlin, 2000.

[9] N.T. Huy, N.V. Minh, Exponential dichotomy of difference equations and applications to evolution equations on the half-line, Comput. Math. Appl. 42 (2001) 301–311.

[10] N.T. Huy, Resolvents of operators and partial functional differential equations with non-autonomous past, J. Math. Anal. Appl. 289 (2004) 301–316.

[11] N.T. Huy, Exponentially dichotomous operators and exponential dichotomy of evolution equations on the half-line, Integral Equations Operator Theory 48 (2004) 497–510.
[12] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, Berlin, 1980.
[13] B.M. Levitan, V.V. Zhikov, Almost Periodic Functions and Differential Equations, Moscow Univ. Publ. House, 1978. English translation: Cambridge Univ. Press, Cambridge, 1982.
[14] J. Lindenstrauss, L. Tzafriri, Classical Banach Spaces II, Function Spaces, Springer-Verlag, Berlin, 1979.
[15] J.J. Massera, J.J. Schäffer, Linear Differential Equations and Function Spaces, Academic Press, New York, 1966.
[16] N.V. Minh, N.T. Huy, Characterizations of dichotomies of evolution equations on the half-line, J. Math. Anal. Appl. 261 (2001) 28–44.
[17] N.V. Minh, F. Räbiger, R. Schnaubelt, Exponential dichotomy exponential expansiveness and exponential dichotomy of evolution equations on the half-line, Integral Equations Operator Theory 32 (1998) 332–353.
[18] R. Nagel, G. Nickel, Well-posedness for non-autonomous abstract Cauchy problems, in: Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhäuser, Boston, MA, 2002, pp. 279–293.
[19] J. van Neerven, The Asymptotic Behaviour of Semigroups of Linear Operator, Oper. Theory Adv. Appl., vol. 88, Birkhäuser, Basel, 1996.
[20] G. Nickel, On evolution semigroups and wellposedness of non-autonomous Cauchy problems, PhD thesis, Tübingen, 1996.
[21] A. Pazy, Semigroup of Linear Operators and Application to Partial Differential Equations, Springer-Verlag, Berlin, 1983.
[22] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen, Math. Z. 32 (1930) 703–728.
[23] F. Räbiger, R. Schnaubelt, Absorption evolution families with applications to non-autonomous diffusion processes, Tübinger Ber. Funktionalanalysis 5 (1995/1996) 335–354.
[24] F. Räbiger, R. Schnaubelt, The spectral mapping theorem for evolution semigroups on spaces of vector-valued functions, Semigroup Forum 48 (1996) 225–239.
[25] R. Schnaubelt, Exponential bounds and hyperbolicity of evolution families, PhD thesis, Tübingen, 1996.
[26] R. Schnaubelt, Exponential dichotomy of non-autonomous evolution equations, Habilitationsschrift, Tübingen, 1999.
[27] R. Schnaubelt, Asymptotically autonomous parabolic evolution equations, J. Evol. Equ. 1 (2001) 19–37.
[28] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holland, Amsterdam, 1978.