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Partial quadratic eigenvalue assignment in vibrating systems using acceleration and velocity feedback

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The partial quadratic eigenvalue assignment problem (PQEVAP) is to shift a few undesired eigenvalues of a damped vibrating system to suitably chosen locations, while leaving the remaining eigenvalues and corresponding eigenvectors unchanged. In this paper, an algorithm for solving PQEVAPs and the minimum norm PQEVAP (MNPQEVAP) using acceleration and velocity feedback is proposed. It is shown that solving the PQEVAP here is transformed into solving an eigenvalue assignment of a linear system of a much lower order. Furthermore, the MNPQEVAP here can be efficiently solved by a gradient-based unconstrained optimization method with the derived gradient formula. This algorithm works directly on the second-order system model, and requires the knowledge of only the open-loop eigenvalues to be replaced and their corresponding eigenvectors. Lastly, through two numerical examples, the results of solving the MNPQEVAP under two different combined feedback signals, velocity and displacement signals, and acceleration and velocity signals, are compared from two points of view, i.e. the *F*-norms of their feedback matrices and the active control energy required from the actuators.

Keywords: vibrating system; partial quadratic eigenvalue assignment; acceleration and velocity feedback; minimum norm

AMS Subject classification: 65F18; 70J50; 93B52

1. Introduction

To describe the dynamics of a structural or mechanical system, a system of second-order ordinary differential equations are usually used as follows, with structural matrices that usually have special structures such as symmetry, positive definiteness and sparsity.

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are all real constant $n \times n$ matrices, and are, respectively, mass, damping and stiffness matrices. $\mathbf{q}(t)$ and $\mathbf{f}(t)$ are real n-vectors, and represent, respectively, the system responses and external forces, and n is an integer and denotes the number of degrees of freedom (d.o.f.) of the system.

The dynamical behaviour of a vibrating system modelled by (1) is governed by the eigenvalues and eigenvectors of the corresponding quadratic matrix pencil $\mathbf{P}(\lambda)$:

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$$\mathbf{P}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K} \tag{2}$$

where scalar λ_k and the associated non-zero vector $\mathbf{x}_k$ and $\mathbf{y}_k$, which satisfy

$$\mathbf{P}(\lambda_k)\mathbf{x}_k = \mathbf{0}, \quad \mathbf{y}_k^{\mathrm{T}}\mathbf{P}(\lambda_k) = \mathbf{0} \ (k = 1, 2, \ldots, 2n) \tag{3}$$

are, respectively, the *eigenvalue* and the *right and left eigenvectors* of the quadratic matrix pencil $\mathbf{P}(\lambda)$. The eigenvalues contain natural frequencies and eigenvectors are mode shapes in vibration theory. Together, $\{\lambda_k, \mathbf{x}_k\}$ or $\{\lambda_k, \mathbf{y}_k\}$ is called a right or left *eigenpair* of (1). It is known that $\mathbf{P}(\lambda)$ has $2n$ finite eigenvalues over the complex field, provided that the mass matrix $\mathbf{M}$ is non-singular. Additionally, system (1) and the quadratic matrix pencil $\mathbf{P}(\lambda)$ are known as an open-loop system and an open-loop quadratic matrix pencil, respectively.

As some undesired dynamical behaviour of a vibrating system, such as instability and resonance, is dictated by some eigenvalues, it is desirable to reallocate these 'troublesome' eigenvalues to some suitable locations, while leaving the remaining eigenvalues and corresponding eigenvectors unchanged (this is known as the 'no spill-over' property). One way to achieve this end is to use state feedback control, which is known as *partial quadratic eigenvalue assignment problem* (PQEVAP). A practical solution of QPEVAP was first obtained by Datta et al. [1], in the single input case, which was then generalised to the multi-input case by Datta and Sarkissian in [2] and Ram and Elhay in [3]. Because of its significance in the active control of a large or complicated vibrating system, and the challenges in its theory and numerical approaches, considerable efforts have been made to solve PQEVAP, both theoretically and computationally, especially working directly on second-order dynamic system models. A partial list of published works includes those reported in [4–11]. It should be noted that another important, related and more complex research work, that is, *partial quadratic eigenstructure assignment problem*, is not studied in this paper.

To implement a multi-input state feedback control strategy, a control force of the form $\mathbf{Bu}(t)$ is applied to the structure. Here $\mathbf{B}$ is a given real $n \times m$ control matrix ($m \le n$), and for convenience is assumed to be of full column rank; and $\mathbf{u}(t)$ is a real time-dependent m-vector. Because the system is of second-order, active control using velocity and displacement feedback or acceleration and velocity feedback can be used to assign the eigenvalues, where $\mathbf{u}(t)$ takes the following special forms, respectively:

$$\mathbf{u}(t) = \mathbf{F}_1^{\mathrm{T}}\dot{\mathbf{q}} + \mathbf{G}_1^{\mathrm{T}}\mathbf{q}, \tag{4}$$

$$\mathbf{u}(t) = \mathbf{F}_2^{\mathrm{T}}\ddot{\mathbf{q}} + \mathbf{G}_2^{\mathrm{T}}\dot{\mathbf{q}}. \tag{5}$$

where $\mathbf{F}_1$ and $\mathbf{G}_1$, and $\mathbf{F}_2$ and $\mathbf{G}_2$ are unknown, constant real $n \times m$ matrices, called the feedback gain matrices. From (1) and (4) or (1) and (5), the corresponding closed-loop systems are as follows:

$$\mathbf{M}\ddot{\mathbf{q}}(t) + (\mathbf{C} - \mathbf{BF}_1^{\mathrm{T}})\dot{\mathbf{q}}(t) + (\mathbf{K} - \mathbf{BG}_1^{\mathrm{T}})\mathbf{q}(t) = \mathbf{f}(t), \tag{6}$$

$$\left(\mathbf{M} - \mathbf{BF}_2^{\mathrm{T}}\right)\ddot{\mathbf{q}}(t) + \left(\mathbf{C} - \mathbf{BG}_2^{\mathrm{T}}\right)\dot{\mathbf{q}}(t) + \mathbf{Kq}(t) = \mathbf{f}(t). \tag{7}$$

They have the corresponding closed-loop quadratic matrix pencils as follows:

$$\mathbf{P}_{c1}(\lambda) = \mathbf{M}\lambda^2 + (\mathbf{C} - \mathbf{B}\mathbf{F}_1^{\mathrm{T}})\lambda + (\mathbf{K} - \mathbf{B}\mathbf{G}_1^{\mathrm{T}}), \tag{8}$$

$$\mathbf{P}_{c2}(\lambda) = \left(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}\right)\lambda^2 + \left(\mathbf{C} - \mathbf{B}\mathbf{G}_2^{\mathrm{T}}\right)\lambda + \mathbf{K}. \tag{9}$$

Solving the PQEVAP is to find feedback matrices in (4) or (5) such that the quadratic pencil (8) or (9) has the specified eigenvalues. It should be noted that, using a variable transformation $\lambda = 1/\mu$ in (8) or (9), $\mathbf{F}_1$, $\mathbf{G}_1$ or $\mathbf{F}_2$, $\mathbf{G}_2$ solving the PQEVAP using velocity and displacement, or acceleration and velocity feedback control can be transformed to solving the PQEVAP with reciprocal eigenvalues μ under acceleration and velocity, or velocity and displacement feedback control. Thus, the corresponding closed-loop quadratic matrix pencils become, respectively,

$$\mathbf{P}_{c1}(\mu) = \left(\mathbf{K} - \mathbf{B}\mathbf{G}_1^{\mathrm{T}}\right)\mu^2 + \left(\mathbf{C} - \mathbf{B}\mathbf{F}_1^{\mathrm{T}}\right)\mu + \mathbf{M}, \tag{10}$$

$$\mathbf{P}_{c2}(\mu) = \mathbf{K}\mu^2 + \left(\mathbf{C} - \mathbf{B}\mathbf{G}_2^{\mathrm{T}}\right)\mu + \left(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}\right). \tag{11}$$

It is required for $\mathbf{M}$ and $\left(\mathbf{K} - \mathbf{B}\mathbf{G}_1^{\mathrm{T}}\right)$ to be non-singular so that the closed-loop eigenvalues of (8) or (10) are finite and non-zeros; so is for $\left(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}\right)$ and $\mathbf{K}$ of (9) or (11). Additionally, Equations (9) and (10) clearly show that the affected system matrices are different for these two solution approaches to PQEVAP using acceleration and velocity feedback control.

In the multi-input feedback control case, when the PQEVAP is solvable, the solution is not unique. Hence, it is reasonable to exploit the non-uniqueness by imposing some desirable features on the closed-loop system. Two usual additional requirements are to determine feedback matrices with small gains (or small norms) and/or to achieve a small condition number of the eigenvector matrix of the closed-loop system. These lead to the *minimum norm partial quadratic eigenvalue assignment problem* (MNPQEVAP) and the *robust partial quadratic eigenvalue assignment problem* (RPQEVAP). It is well known that small feedback gains tend to lead to smaller control signals, and thus to less energy consumption. Small gains are also beneficial in reducing noise amplification.[12] High feedback gains or high condition numbers often lead to high sensitivity of the closed-loop eigenvalues.

Recently, Qian and Xu [7,8] proposed two algorithms for the RPQEVAP, where eigenvectors are chosen in certain subspaces such that some measure of the distance between the eigenvectors and some orthogonal basis of a certain subspace are minimised. Brahma and Datta [9], and Bai et al. [10], respectively, gave parametric expressions of feedback matrices via Sylvester equations, and developed an optimization approach for the MNPQEVAP and RPQEVAP with appropriate gradient formulas. Numerical results showed that the proposed algorithm in [10] was superior in its performances in both minimised norms and the closed-loop condition number to those of [9]. Cai et al. [11] demonstrated that solving the PQEVAP is essentially solving an eigenvalue assignment of a linear system of a much lower order, and solving the MNPQEVAP and RPQEVAP is then concerned with solving a minimum norm or robust eigenvalue assignment problem associated with this linear system. They applied the technique developed in [12] to propose an algorithm for solving the MNPQEVAP and RPQEVAP. Numerical examples showed that the results of their algorithm were at least comparable with those of existing algorithms, with lower computational cost. Another advantage of their approach was the use of real-number representation of eigenvalue and eigenvector matrices, which avoids complex arithmetic. All these approaches mentioned above work directly on the second-order systems instead of transforming them into linear systems, and they only need the knowledge of a small number of eigenvalues and the corresponding eigenvectors which are to be assigned, whereas the information of the unchanged eigenpairs, which are generally unknown, is not needed. Most importantly, they all possess the no spillover property.

Another observation is that all these approaches use combined velocity and displacement feedback control to solve PQEVAP. From the open literature, works on PQEVAP using acceleration and velocity feedback are rare. Using these feedback signals is even more interesting because of the frequent use of accelerometers in practice. In [13], Datta et al. once mentioned state feedback control using acceleration and velocity to assign partial quadratic eigenstructure. Recently, Abdelaziz [14] used velocity-plus-acceleration feedback to assign the full eigenstructure of second-order systems, and extended the established results from first-order systems to second-order systems. The same author also considered the robust pole assignment problem using combined velocity and acceleration feedback for second-order linear systems in [15].

In addition to the eigenvalue assignment methods mentioned above, which can be classified as the model-based approach, it is worthwhile to point out that a new approach to eigenvalues assignment in structural vibration systems was introduced by Ram and Mottershead [16], and extended by them and their colleagues [17–20] based on measured receptances and without the need to know or evaluate system matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$. A recent paper by Ram and Mottershead [21] developed a new theory for active vibration control by pole placement using the receptance method. The formulation presented allowed for partial pole placement by multiple-input/multiple-output control using experimentally measured receptances, and it was demonstrated that the redundancy offered by multiple-input control may be used to assign not only the eigenvalues but also the eigenvectors of the system.

The algorithm proposed in [11] is adapted to solve PQEVAP and MNPQEVAP via acceleration and velocity feedback, and some differences between the present algorithm and that in [11] are discussed in this paper. The formulation presented here can also be extended to solve RPQEVAP. Based on the obtained minimum norm feedback matrices, the active control energy required from the actuators under two control strategies, i.e. velocity and displacement feedback, acceleration and velocity feedback, is compared for the same numerical examples. These results give some insight into feedback control in solving PQEVAP in terms of the magnitude of the feedback gain matrices and the amount of actuation energy. In Section 2, some notations, definitions and assumptions are presented, which will be used throughout this paper. In Section 3, the theory on the solvability of the PQEVAP is analysed and the parameterized solutions to the PQEVAP are derived. The numerical approaches for solving the MNPQEVAP are presented in Section 4, and numerical results are given in Section 5 and analysed from the active control energy point of view. Finally, some concluding remarks are given in Section 6.

2. Notation and assumptions

To avoid complex arithmetic in this paper, the eigenpairs of the original structure (i.e. the open-loop system) and the corresponding actively controlled structure (i.e. the closed-loop system) are described using *real representations*.[22,23] Without loss of generality, assume that the p eigenpairs $\{\lambda_k, \mathbf{x}_k\}_{k=1}^{p}$ of the original structure have the following forms:

$$\lambda_{2k-1} = \bar{\lambda}_{2k} = \alpha_k + i\beta_k, \quad \alpha_k \in R,\ \beta_k > 0,\ k = 1, 2, \ldots, l,$$

$$\mathbf{x}_{2k-1} = \bar{\mathbf{x}}_{2k} = \mathbf{x}_{kR} + i\mathbf{x}_{kI}, \quad \mathbf{x}_{kR},\ \mathbf{x}_{kI} \in R^n,\ k = 1, 2, \ldots, l,$$

and

$$\lambda_k \in R,\ \mathbf{x}_k \in R^n, \quad k = 2l+1, \ldots, p.$$

The real representations of $\{\lambda_k\}_{k=1}^{p}$ and $\{\mathbf{x}_k\}_{k=1}^{p}$ are defined in the following forms:

$$\mathbf{\Lambda}_1 = \mathrm{diag}\left(\begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \ldots, \begin{bmatrix} \alpha_l & \beta_l \\ -\beta_l & \alpha_l \end{bmatrix}, \lambda_{2l+1}, \ldots, \lambda_p\right) \in R^{p\times p}, \tag{12}$$

$$\mathbf{X}_1 = \left[\mathbf{x}_{1R}, \mathbf{x}_{1I}, \ldots, \mathbf{x}_{lR}, \mathbf{x}_{lI}, \mathbf{x}_{2l+1}, \ldots, \mathbf{x}_p\right] \in R^{n\times p}. \tag{13}$$

Similarly, let the real representation of the p prescribed partial eigenvalues $\left\{\tilde{\lambda}_k\right\}_{k=1}^{p}$ of the controlled structure be

$$\tilde{\mathbf{\Lambda}}_1 = \mathrm{diag}\left(\begin{bmatrix} \tilde{\alpha}_1 & \tilde{\beta}_1 \\ -\tilde{\beta}_1 & \tilde{\alpha}_1 \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{\alpha}_s & \tilde{\beta}_s \\ -\tilde{\beta}_s & \tilde{\alpha}_s \end{bmatrix},\ \lambda_{2s+1}, \ldots, \lambda_p\right) \in R^{p\times p} \tag{14}$$

and the real representation of their corresponding right eigenvectors $\{\tilde{\mathbf{x}}_k\}_{k=1}^{p}$ and left eigenvectors $\{\tilde{\mathbf{y}}_k\}_{k=1}^{p}$ be $\tilde{\mathbf{X}}_1$ and $\tilde{\mathbf{Y}}_1$, respectively, which are to be determined. Here the newly assigned eigenvalues contain s complex conjugate pairs, which is not necessarily equal to l. Also, let the remaining $2n-p$ eigenpairs of the original structure be denoted by$\{\lambda_k, \mathbf{x}_k\}_{k=p+1}^{2n}$, and let the real representation of $\{\lambda_k\}_{k=p+1}^{2n}$ and $\{\mathbf{x}_k\}_{k=p+1}^{2n}$ be $\mathbf{\Lambda}_2$ and $\mathbf{X}_2$, respectively.

The p eigenpairs $\{\lambda_k, \mathbf{x}_k\}_{k=1}^{p}$ of the original structure with the corresponding true (complex-valued) eigenvalue matrices $\mathbf{\Sigma}_1 = \mathrm{diag}\left(\lambda_1, \lambda_2, \ldots, \lambda_{2l-1}, \lambda_{2l}, \lambda_{2l+1}, \ldots, \lambda_p\right) \in R^{p\times p}$ and true (complex-valued) eigenvector matrices $\mathbf{\Phi}_1 = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{2l-1}, \mathbf{x}_{2l}, \mathbf{x}_{2l+1}, \ldots, \mathbf{x}_p] \in R^{n\times p}$ satisfy a system of algebraic equations as follows:

$$\mathbf{M\Phi}_1\mathbf{\Sigma}_1^2 + \mathbf{C\Phi}_1\mathbf{\Sigma}_1 + \mathbf{K\Phi}_1 = \mathbf{0}. \tag{15}$$

In what follows, it is shown that the real representations (12) and (13) of $\{\lambda_k, \mathbf{x}_k\}_{k=1}^{p}$ also satisfy a system of algebraic equations like (15). Indeed, let

$$\mathbf{T} = \mathrm{diag}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \ldots, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}, \mathbf{I}_{p-2l}\right), \tag{16}$$

from (12), (13) and (16), it is easy to verify that

$$\mathbf{\Sigma}_1 = \mathbf{T}^{\mathrm{H}}\mathbf{\Lambda}_1\mathbf{T}, \quad \mathbf{\Phi}_1 = \sqrt{2}\mathbf{X}_1\mathbf{T}. \tag{17}$$

Substituting (17) into (15) and considering the fact that $\mathbf{T}$ is non-singular and $\mathbf{TT}^{\mathrm{H}} = \mathbf{I}$, one gets the system of algebraic equations that $\mathbf{\Lambda}_1$ and $\mathbf{X}_1$ should satisfy as follows:

$$\mathbf{MX}_1\mathbf{\Lambda}_1^2 + \mathbf{CX}_1\mathbf{\Lambda}_1 + \mathbf{KX}_1 = \mathbf{0}. \tag{18}$$

Similar algebraic equations also hold for $\mathbf{\Lambda}_2$ and $\mathbf{X}_2$, that is

$$\mathbf{M}\mathbf{X}_2\mathbf{\Lambda}_2^2 + \mathbf{C}\mathbf{X}_2\mathbf{\Lambda}_2 + \mathbf{K}\mathbf{X}_2 = \mathbf{0} \tag{19}$$

and hold for the eigenpairs of the closed-loop system. In addition, it is known that, as indicated in [22], the p eigenvalues $\{\lambda_k\}_{k=1}^{p}$ is precisely the spectrum of its real representation matrix $\mathbf{\Lambda}_1$, which is denoted by $\sigma(\mathbf{\Lambda}_1)$. Similarly, the eigenvalues set $\{\lambda_k\}_{k=p+1}^{2n}$ and $\left\{\tilde{\lambda}_k\right\}_{k=1}^{p}$ be denoted by $\sigma(\mathbf{\Lambda}_2)$ and $\sigma\left(\tilde{\mathbf{\Lambda}}_1\right)$.

With the notation and analysis above, the PQEVAP using acceleration and velocity feedback can be stated as: given $\mathbf{\Lambda}_1$ and $\mathbf{X}_1$ satisfying (18), and $\tilde{\mathbf{\Lambda}}_1$, one is to find $\mathbf{F}_2$ and $\mathbf{G}_2 \in R^{n\times m}$ such that

$$\tilde{\mathbf{\Lambda}}_1^{2\mathrm{T}}\tilde{\mathbf{Y}}_1^{\mathrm{T}}(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}) + \tilde{\mathbf{\Lambda}}_1^{\mathrm{T}}\tilde{\mathbf{Y}}_1^{\mathrm{T}}(\mathbf{C} - \mathbf{B}\mathbf{G}_2^{\mathrm{T}}) + \tilde{\mathbf{Y}}_1^{\mathrm{T}}\mathbf{K} = \mathbf{0}, \tag{20}$$

$$(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}})\mathbf{X}_2\mathbf{\Lambda}_2^2 + (\mathbf{C} - \mathbf{B}\mathbf{G}_2^{\mathrm{T}})\mathbf{X}_2\mathbf{\Lambda}_2 + \mathbf{K}\mathbf{X}_2 = \mathbf{0}, \tag{21}$$

for some $n \times p$ real matrix $\tilde{\mathbf{Y}}_1$.

Throughout the paper, the following assumptions stand:

(A1) $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ are symmetric; $\mathbf{M}$ is positive definite ($\mathbf{M} > \mathbf{0}$), $\mathbf{K}$ is non-singular and control matrix $\mathbf{B}$ is of full column rank;

(A2) $\sigma(\mathbf{\Lambda}_1) \cap \sigma(\mathbf{\Lambda}_2) = \varnothing$ (an empty set), $\sigma\left(\tilde{\mathbf{\Lambda}}_1\right) \cap \sigma(\mathbf{\Lambda}_2) = \varnothing$, $\sigma(\mathbf{\Lambda}_1) \cap \sigma\left(\tilde{\mathbf{\Lambda}}_1\right) = \varnothing$ and $\tilde{\mathbf{\Lambda}}_1$ is non-singular;

(A3) $\lambda_i \neq \lambda_j$ and $\tilde{\lambda}_i \neq \tilde{\lambda}_j$ for $i \neq j,\ i,j = 1,2,\ldots,p$;

(A4) The original open-loop system is partially controllable for the eigenvalues $\{\lambda_k\}_{k=1}^{p}$; that is, $\mathrm{rank}(\mathbf{P}(\lambda_k), \mathbf{B}) = n, k = 1,2,\ldots,p$.

It should be noted that, in assumption (A1), stiffness matrix $\mathbf{K}$ being non-singular is required. The same requirement can be found in [1,7,8]. However, the algorithm in [11] does not require that $\mathbf{K}$ is non-singular based on a different parameter expressions for $\mathbf{F}_1$ and $\mathbf{G}_1$.

3. Parametric solutions to the PQEVAP for feedback matrices $\mathbf{F}_2$ and $\mathbf{G}_2$

Firstly, two orthogonal equalities which are of critical importance in the following deduction are presented.

Let

$$\mathbf{U} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{bmatrix}.$$

It follows from (18) and (19) that $\mathbf{U}\mathbf{\Lambda}_2 = \mathbf{\Lambda}_1^{\mathrm{T}}\mathbf{U}$, which implies $\mathbf{U} = \mathbf{0}$, since matrices $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ do not have common eigenvalues, i.e. $\sigma(\mathbf{\Lambda}_1) \cap \sigma(\mathbf{\Lambda}_2) = \varnothing$. Thus, the following orthogonality relation is obtained

$$\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2\mathbf{\Lambda}_2 \end{bmatrix} = \mathbf{0}. \tag{22}$$

Similarly, one can get another orthogonality relation

$$\begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_1 \tilde{\boldsymbol{\Lambda}}_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2 \boldsymbol{\Lambda}_2 \end{bmatrix} = \mathbf{0}, \tag{23}$$

provided that there exist $\mathbf{F}_2$ and $\mathbf{G}_2$ such that (20) and (21) hold for some matrix $\tilde{\mathbf{Y}}_1$.

Using (22), one has the following lemma concerning a necessary condition on the solvability of the PQEVAP.

Lemma 3.1. *If there exist $\boldsymbol{F}_2$ and $\boldsymbol{G}_2$ satisfying* (21)*, and the eigenvalue matrix $\boldsymbol{\Lambda}_2$ is non-singular, then they have the following forms*

$$\mathbf{F}_2 = \mathbf{M}\mathbf{X}_1 \boldsymbol{\Lambda}_1 \boldsymbol{\Phi}, \tag{24}$$

$$\mathbf{G}_2 = -\mathbf{K}\mathbf{X}_1 \boldsymbol{\Phi}, \tag{25}$$

where $\boldsymbol{\Phi} \in R^{p\times m}$ is arbitrary.

Proof

Combining (19) and (21) leads to

$$\mathbf{B}\mathbf{F}_2^{\mathrm{T}}\mathbf{X}_2\boldsymbol{\Lambda}_2^2 + \mathbf{B}\mathbf{G}_2^{\mathrm{T}}\mathbf{X}_2\boldsymbol{\Lambda}_2 = \mathbf{0},$$

which implies

$$\begin{bmatrix} \mathbf{G}_2 \\ \mathbf{F}_2 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2 \boldsymbol{\Lambda}_2 \end{bmatrix} = \mathbf{0}, \tag{26}$$

since $\mathbf{B}$ is of full column rank and eigenvalue matrix $\boldsymbol{\Lambda}_2$ is non-singular. On the other hand, from (22) it follows that

$$\boldsymbol{N}\left(\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2 \boldsymbol{\Lambda}_2 \end{bmatrix}^{\mathrm{T}} \right) = \boldsymbol{R}\left(\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1 \boldsymbol{\Lambda}_1 \end{bmatrix} \right), \tag{27}$$

since

$$\operatorname{rank}\left(\begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_2 \boldsymbol{\Lambda}_2 \end{bmatrix} \right) = 2n - p, \quad \operatorname{rank}\left(\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1 \boldsymbol{\Lambda}_1 \end{bmatrix} \right) = p.$$

Here $\boldsymbol{N}(\cdot)$ and $\boldsymbol{R}(\cdot)$ denote the null space and range of a matrix, respectively. Thus, it follows from (26) and (27) that there exists a matrix $\boldsymbol{\Phi} \in R^{p\times m}$ such that

$$\begin{bmatrix} \mathbf{G}_2 \\ \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1 \boldsymbol{\Lambda}_1 \end{bmatrix} \boldsymbol{\Phi},$$

which completes the proof. □

Remark 3.1. In lemma 3.1 of [11], the authors gave the feedback matrices $\mathbf{F}_1 = \mathbf{M}\mathbf{X}_1\boldsymbol{\Phi}$ and $\mathbf{G}_1 = (\mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1 + \mathbf{C}\mathbf{X}_1)\boldsymbol{\Phi}$ based on the displacement and velocity feedback. Again, it is worthwhile to point out that the parametric expressions (24) and (25) for solutions $\mathbf{F}_2$ and $\mathbf{G}_2$ given in this paper turn out to be in the same forms as those for $\mathbf{F}_1$ and $\mathbf{G}_1$ reported in [1,7–9], but the latter are associated with and expressed by the corresponding complex eigenvalues and eigenvectors and obtained under the displacement and velocity feedback strategy. It should be pointed out that in the approach presented in this paper and that in [11], the constructions of the feedback matrices do not involve complex arithmetic.

The following theorem gives a necessary and sufficient condition on the solvability of the PQEVAP with $\mathbf{F}_2$ and $\mathbf{G}_2$, which presents a different formula from theorem 3.1 in [11].

Theorem 3.1. *The* PQEVAP *is solvable with* $\begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1 \end{bmatrix}$ *being of full column rank if and only if there exists a matrix* $\mathbf{\Phi} \in R^{p\times m}$ *and a non-singular matrix* $\mathbf{S} \in R^{p\times p}$ *such that*

$$\mathbf{\Lambda}_1^{-1} - \mathbf{\Phi} B^{\mathrm{T}}\mathbf{X}_1 = \mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1^{-1}\mathbf{S}. \tag{28}$$

And if (28) *holds, the solutions to the* PQEVAP *are given by* (24) *and* (25).

Proof

Necessity: Note that if there exist $\mathbf{F}_2$ and $\mathbf{G}_2$ such that (20) and (21) hold for a matrix $\tilde{\mathbf{Y}}_1 \in R^{n\times p}$ making $\begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1 \end{bmatrix}$ have a full column rank, then from (23) and (27), one knows that there must exist a non-singular matrix $\mathbf{S} \in R^{p\times p}$ such that

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} - \mathbf{F}_2\mathbf{B}^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \tilde{\mathbf{Y}}_1 \\ \tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1 \end{bmatrix} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix}\mathbf{S}^{-1}. \tag{29}$$

Expanding the two sides of (29), one immediately obtains

$$\tilde{\mathbf{Y}}_1 = \mathbf{X}_1\mathbf{S}^{-1}, \tag{30}$$

since $\mathbf{K}$ is non-singular. Another equality obtained is

$$(\mathbf{M} - \mathbf{F}_2\mathbf{B}^{\mathrm{T}})\tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1 = \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{S}^{-1}. \tag{31}$$

Then substituting (24) and (30) into (31), one gets

$$\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 - \mathbf{X}_1\mathbf{\Lambda}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 = \mathbf{X}_1\mathbf{\Lambda}_1\mathbf{S}^{-1}, \tag{32}$$

since $\mathbf{M}$ is non-singular.

On the other hand, substituting (24), (25), (30) and (32) into (20) gives

$$\begin{aligned} \mathbf{0} &= (\mathbf{M} - \mathbf{F}_2\mathbf{B}^{\mathrm{T}})\tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1^2 + (\mathbf{C} - \mathbf{G}_2\mathbf{B}^{\mathrm{T}})\tilde{\mathbf{Y}}_1\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\tilde{\mathbf{Y}}_1 \\ &= (\mathbf{M} - \mathbf{F}_2\mathbf{B}^{\mathrm{T}})\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1^2 + (\mathbf{C} - \mathbf{G}_2\mathbf{B}^{\mathrm{T}})\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{S}^{-1} \\ &= \mathbf{M}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1^2 - \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1^2 + \mathbf{C}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{S}^{-1} \\ &= \mathbf{M}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1^2 - \mathbf{M}(\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 - \mathbf{X}_1\mathbf{\Lambda}_1\mathbf{S}^{-1})\tilde{\mathbf{\Lambda}}_1 + \mathbf{C}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{S}^{-1} \\ &= \mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{C}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\mathbf{X}_1\mathbf{S}^{-1}, \end{aligned}$$

where the second equality uses (30), the third equality uses (24) and (25), and the fourth equality uses (32). Substituting $\mathbf{K}\mathbf{X}_1 = -\mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1^2 - \mathbf{C}\mathbf{X}_1\mathbf{\Lambda}_1$ into $\mathbf{K}\mathbf{X}_1\mathbf{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\mathbf{\Lambda}}_1$ of the last equality, and then post-multiplying the resultant equality by $\mathbf{S}$, one gets

$$\begin{aligned}\mathbf{0} &= \mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} + \mathbf{C}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} - (\mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1 + \mathbf{C}\mathbf{X}_1)\boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} + \mathbf{K}\mathbf{X}_1 \\ &= (\mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1 + \mathbf{C}\mathbf{X}_1)(\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S}) + \mathbf{K}\mathbf{X}_1.\end{aligned} \tag{33}$$

Additionally, multiplying (32) by $\mathbf{M}$ and $\mathbf{S}$ on the left and right, respectively, gives

$$\mathbf{M}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} - \mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} = \mathbf{M}\mathbf{X}_1\boldsymbol{\Lambda}_1. \tag{34}$$

Then Equation (33), together with (34), gives rise to

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}(\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S}) = \begin{bmatrix}-\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}. \tag{35}$$

And using the facts that

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}\boldsymbol{\Lambda}_1 = \begin{bmatrix}-\mathbf{K} & \mathbf{0}\\ \mathbf{0} & \mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}, \tag{36}$$

and matrix

$$\begin{bmatrix}\mathbf{C} & \mathbf{M}\\ \mathbf{M} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}$$

is of full column rank, it follows immediately by comparing (35) with (36) that

$$\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} = \boldsymbol{\Lambda}_1, \tag{37a}$$

or

$$(\mathbf{I} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1)\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} = \boldsymbol{\Lambda}_1. \tag{37b}$$

Because $\boldsymbol{\Lambda}_1$ and $\tilde{\boldsymbol{\Lambda}}_1$ are invertible in view of the assumptions (A1), (A2) and (A3), it follows from (37b) that $\boldsymbol{\Lambda}_1^{-1}(\mathbf{I} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1)\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S} = \mathbf{I}$ or $(\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1\mathbf{S})^{-1} = \boldsymbol{\Lambda}_1^{-1}(\mathbf{I} - \boldsymbol{\Lambda}_1\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1)$. It is observed from the last formula that (28) holds, which completes the proof of necessity.

Conversely, if there exist a matrix $\boldsymbol{\Phi} \in R^{p\times m}$ and a non-singular matrix $\mathbf{S} \in R^{p\times p}$ such that (28) holds, it can easily be verified that $\begin{bmatrix}\tilde{\mathbf{Y}}_1\\ \tilde{\mathbf{Y}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}$ is of full column rank by letting $\tilde{\mathbf{Y}}_1 = \mathbf{X}_1\mathbf{S}^{-1}$, and (20) and (21) hold, where $\mathbf{F}_2$ and $\mathbf{G}_2$ are given by (24) and (25), respectively. Thus, the proof is completed. □

Theorem 3.1 above shows that the PQEVAP using the acceleration and velocity feedback is solvable with $\begin{bmatrix}\tilde{\mathbf{Y}}_1\\ \tilde{\mathbf{Y}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}$ being of full column rank if and only if (28) is solvable for $\boldsymbol{\Phi}$ and $\mathbf{S}$, which is essentially a linear eigenvalue assignment problem of dimension p. Some complementary explanations are presented below.

As is well known, for a first-order linear system of the state-space form: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{E}\mathbf{u}(t)$. Let Π be an appropriate scalar set in the complex plane. The eigenvalue assignment problem of the first-order system is that, given the controllable pair ($\mathbf{A}$, $\mathbf{E}$), determining the state feedback matrix $\mathbf{F}$ of the appropriate dimensions such that the eigenvalues of the closed-loop state matrix $\mathbf{A} + \mathbf{E}\mathbf{F}$ are at desired locations in Π. Now, taking the transpose operation of (28) gives

$$\mathbf{\Lambda}_1^{-\mathrm{T}} - \mathbf{X}_1^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}\mathbf{S}^{-\mathrm{T}}, \tag{38}$$

which means that the eigenvalues of matrix $\mathbf{\Lambda}_1^{-\mathrm{T}} - \mathbf{X}_1^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}^{\mathrm{T}}$ are the same as those of matrix $\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}$. Here, matrix $\mathbf{\Lambda}_1^{-\mathrm{T}} - \mathbf{X}_1^{\mathrm{T}}\mathbf{B}\mathbf{\Phi}^{\mathrm{T}}$ is similar to the closed-loop state matrix $\mathbf{A}+\mathbf{EF}$ of the first-order linear system above; the eigenvalues of matrix $\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}$ are eigenvalues of the closed-loop first-order model. So, solving the PQEVAP according to Theorem 3.1 is just that, given a first-order linear system $\left(\mathbf{\Lambda}_1^{-\mathrm{T}}, -\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\right)$, one has to find a feedback matrix $\mathbf{\Phi}^{\mathrm{T}}$, such that the closed-loop eigenvalues are the eigenvalues of matrix $\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}$. The Schur method is an efficient method for this problem.[24]

It is well known that this linear eigenvalue assignment problem is solvable if and only if $\left(\mathbf{\Lambda}_1^{-\mathrm{T}}, -\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\right)$ is controllable. Again, similarly as in theorem 3.2 in [11], it can be proved that the partial controllability of the original open-loop system (1) for eigenvalues $\{\lambda_k\}_{k=1}^{p}$ is equivalent to the controllability of $\left(\mathbf{\Lambda}_1^{-\mathrm{T}}, -\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\right)$. Details of the proof are omitted here.

For partial eigenvalue assignment using acceleration and velocity feedback, an important requirement that the closed-loop mass matrix $\left(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}\right)$ in the pencil $\mathbf{P}_{c2}(\lambda)$ be non-singular should be considered. With assumptions (A1), and the prescribed partial eigenvalues $\left\{\tilde{\lambda}_k\right\}_{k=1}^{p}$ are finite and non-zero, it can be guaranteed that the resultant closed-loop mass matrix $\left(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}}\right)$ in this paper is non-singular.

4. Solving MNPQEVAP for feedback matrices $\mathbf{F}_2$ and $\mathbf{G}_2$

As mentioned above, it is suitable to use the Schur method to the problem (28) to obtain matrix $\mathbf{\Phi}^{\mathrm{T}}$, and then the solution $\mathbf{F}_2$ and $\mathbf{G}_2$ to the PQEVAP is obtained by (24) and (25). However, this has not involved minimising the norms of the feedback matrices, which is denoted as the MNPQEVAP and presented below.

The MNPQEVAP is to minimise the norms of the feedback matrices $\mathbf{F}_2$ and $\mathbf{G}_2$. Let

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix} = \mathbf{Q}\begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} = [\,\mathbf{Q}_1 \quad \mathbf{Q}_2\,]\begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} = \mathbf{Q}_1\mathbf{R}, \tag{39}$$

where $\mathbf{Q}_1$ is the matrix made up from the first p columns of the orthogonal matrix $\mathbf{Q} \in R^{2n\times 2n}$, and $\mathbf{R} \in R^{p\times p}$ is a non-singular upper triangular matrix. Then using (24) and (25), the objective function to be minimised in the MNPQEVAP becomes

$$\begin{aligned} f &= \frac{1}{2}\|\mathbf{F}_2\|_F^2 + \frac{1}{2}\|\mathbf{G}_2\|_F^2 = \frac{1}{2}\|\mathbf{M}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{\Phi}\|_F^2 + \frac{1}{2}\|-\mathbf{K}\mathbf{X}_1\mathbf{\Phi}\|_F^2 \\ &= \frac{1}{2}\left\|\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix}\mathbf{\Phi}\right\|_F^2 = \frac{1}{2}\|\mathbf{R}\mathbf{\Phi}\|_F^2. \end{aligned} \tag{40}$$

On the other hand, feedback matrices $\mathbf{F}_2$ and $\mathbf{G}_2$ also need to satisfy the following algebraic equation:

$$(\mathbf{M} - \mathbf{B}\mathbf{F}_2^{\mathrm{T}})\tilde{\mathbf{X}}_1\tilde{\mathbf{\Lambda}}_1^2 + (\mathbf{C} - \mathbf{B}\mathbf{G}_2^{\mathrm{T}})\tilde{\mathbf{X}}_1\tilde{\mathbf{\Lambda}}_1 + \mathbf{K}\tilde{\mathbf{X}}_1 = \mathbf{0}, \tag{41a}$$

where $\tilde{\mathbf{X}}_1$ is the real representation of right eigenvectors associated with $\left\{\tilde{\lambda}_k\right\}_{k=1}^{p}$. Using (24) and (25), it follows from (41a) that

$$\begin{aligned}\mathbf{M}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1^2+\mathbf{C}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1+\mathbf{K}\tilde{\mathbf{X}}_1&=\mathbf{B}\mathbf{F}_2^{\mathrm{T}}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1^2+\mathbf{B}\mathbf{G}_2^{\mathrm{T}}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\\&=\mathbf{B}\begin{bmatrix}\mathbf{G}_2\\\mathbf{F}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{\mathbf{X}}_1\\\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}\tilde{\boldsymbol{\Lambda}}_1=\mathbf{B}\left(\begin{bmatrix}-\mathbf{K}&\mathbf{0}\\\mathbf{0}&\mathbf{M}\end{bmatrix}\begin{bmatrix}\mathbf{X}_1\\\mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}\boldsymbol{\Phi}\right)^{\mathrm{T}}\begin{bmatrix}\tilde{\mathbf{X}}_1\\\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}\tilde{\boldsymbol{\Lambda}}_1\\&=\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}\begin{bmatrix}\mathbf{X}_1\\\mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-\mathbf{K}&\mathbf{0}\\\mathbf{0}&\mathbf{M}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{X}}_1\\\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}\tilde{\boldsymbol{\Lambda}}_1.\end{aligned}\tag{41b}$$

Let

$$\mathbf{D}=\begin{bmatrix}\tilde{\mathbf{Y}}_1\\\tilde{\mathbf{Y}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-\mathbf{K}&\mathbf{0}\\\mathbf{0}&\mathbf{M}-\mathbf{B}\mathbf{F}_2^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{X}}_1\\\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix},\tag{42}$$

then it follows that $\mathbf{D}\tilde{\boldsymbol{\Lambda}}_1=\tilde{\boldsymbol{\Lambda}}_1^{\mathrm{T}}\mathbf{D}$, which implies that $\mathbf{D}$ is symmetric, and $\mathbf{D}$ can be proved, similarly as in [11], to be non-singular. Using (29), (42) can be rewritten as

$$\mathbf{D}=\mathbf{S}^{-\mathrm{T}}\begin{bmatrix}\mathbf{X}_1\\\mathbf{X}_1\boldsymbol{\Lambda}_1\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}-\mathbf{K}&\mathbf{0}\\\mathbf{0}&\mathbf{M}\end{bmatrix}\begin{bmatrix}\tilde{\mathbf{X}}_1\\\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1\end{bmatrix}.\tag{43}$$

Substituting (43) into (41b) gives

$$\mathbf{M}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1^2+\mathbf{C}\tilde{\mathbf{X}}_1\tilde{\boldsymbol{\Lambda}}_1+\mathbf{K}\tilde{\mathbf{X}}_1=\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\Lambda}}_1=\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1=\mathbf{B}\boldsymbol{\Gamma},\tag{44}$$

where

$$\tilde{\mathbf{S}}=\mathbf{D}\mathbf{S},\tag{45a}$$

$$\boldsymbol{\Gamma}=\Phi^{\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1.\tag{45b}$$

Additionally, in term of (45a), (45b) and $\mathbf{D}\tilde{\boldsymbol{\Lambda}}_1=\tilde{\boldsymbol{\Lambda}}_1^{\mathrm{T}}\mathbf{D}$, (28) can be rewritten as

$$\begin{aligned}&\boldsymbol{\Lambda}_1^{-1}-\boldsymbol{\Phi}\mathbf{B}^{\mathrm{T}}\mathbf{X}_1=\mathbf{S}^{-1}\tilde{\boldsymbol{\Lambda}}_1^{-1}\mathbf{S}\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\mathbf{S}^{-\mathrm{T}}\\&\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\mathbf{S}^{\mathrm{T}}-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{D}-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{D}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\mathbf{D}\\&\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}\tilde{\boldsymbol{\Lambda}}_1^{-1}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\mathbf{D}\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\mathbf{D}\tilde{\boldsymbol{\Lambda}}_1\\&\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}=\mathbf{S}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{-\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1^{\mathrm{T}}\mathbf{D}\Rightarrow\boldsymbol{\Lambda}_1^{-\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1-\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}=\tilde{\mathbf{S}}^{\mathrm{T}}.\end{aligned}$$

Pre-multiplying both sides of the last equality by$\boldsymbol{\Lambda}_1^{\mathrm{T}}$, and rearranging, gives

$$\tilde{\mathbf{S}}^{\mathrm{T}}\tilde{\boldsymbol{\Lambda}}_1-\boldsymbol{\Lambda}_1^{\mathrm{T}}\tilde{\mathbf{S}}^{\mathrm{T}}=\boldsymbol{\Lambda}_1^{\mathrm{T}}\mathbf{X}_1^{\mathrm{T}}\mathbf{B}\boldsymbol{\Gamma}.\tag{46}$$

From Sylvester Equation (46), one gets solution $\tilde{\mathbf{S}}$ in term of $\boldsymbol{\Gamma}$, showing via vec operator which vectorizes a matrix by stacking its columns, as follows:

$$\mathrm{vec}(\tilde{\mathbf{S}}^{\mathrm{T}})=(\tilde{\boldsymbol{\Lambda}}_1^{\mathrm{T}}\otimes\mathbf{I}_p-\mathbf{I}_p\otimes\boldsymbol{\Lambda}_1^{\mathrm{T}})^{-1}(\mathbf{I}_p\otimes(\boldsymbol{\Lambda}_1^{\mathrm{T}}\mathbf{X}_1^{\mathrm{T}}\mathbf{B}))\mathrm{vec}(\boldsymbol{\Gamma}).\tag{47}$$

Here $\otimes$ denotes the Kronecker product. It should be noted that Sylvester Equation (46) has a unique solution, because matrices $\boldsymbol{\Lambda}_1$ and $\tilde{\boldsymbol{\Lambda}}_1$ do not have common eigenvalues.[25] Now, using (45b), the objective function f defined in (40) can be rewritten as

$$f = \frac{1}{2}\|\mathbf{R\Phi}\|_F^2 = \frac{1}{2}\mathrm{Tr}\left(\mathbf{R\Phi\Phi}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right) = \frac{1}{2}\mathrm{Tr}\left(\mathbf{R}\tilde{\mathbf{S}}^{-1}\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}\mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Gamma}\tilde{\mathbf{\Lambda}}_1^{-1}\tilde{\mathbf{S}}^{-\mathrm{T}}\mathbf{R}^{\mathrm{T}}\right), \tag{48}$$

which, together with (47), shows that f is a function of Γ.

After some manipulations, one can obtain the gradient of f with respect to $\mathbf{\Gamma}$ as

$$\mathrm{vec}(\nabla_{\mathbf{\Gamma}} f) = \mathrm{vec}\left(\mathbf{\Gamma}\tilde{\mathbf{\Lambda}}_1^{-1}\tilde{\mathbf{S}}^{-\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\tilde{\mathbf{S}}^{-1}\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}} + \mathbf{B}^{\mathrm{T}}\mathbf{X}_1\mathbf{\Lambda}_1\mathbf{W}\right), \tag{49}$$

where $\mathbf{W}$ satisfies the Sylvester equation below

$$\mathbf{\Lambda}_1\mathbf{W} - \mathbf{W}\tilde{\mathbf{\Lambda}}_1^{\mathrm{T}} - \mathbf{\Sigma} = \mathbf{0}, \tag{50}$$

with $\mathbf{\Sigma} = \tilde{\mathbf{S}}^{-1}\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}\mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Gamma}\tilde{\mathbf{\Lambda}}_1^{-1}\tilde{\mathbf{S}}^{-\mathrm{T}}\mathbf{R}^{\mathrm{T}}\mathbf{R}\tilde{\mathbf{S}}^{-1}$.

The solution of Sylvester Equation (50) is represented by

$$\mathrm{vec}(\mathbf{W}) = (\mathbf{I}_p \otimes \mathbf{\Lambda}_1 - \tilde{\mathbf{\Lambda}}_1 \otimes \mathbf{I}_p)^{-1}\mathrm{vec}(\mathbf{\Sigma}). \tag{51}$$

So the unconstrained optimization problem (40) or (48) can be solved by any gradient-based methods, such as 'BFGS' and trust-region Newton's method, etc. As indicated in [11], this kind of optimization problems may have many local minima, thus by solving the optimization problem repeatedly with different random initial values; one is able to find the feedback matrices in the case when the cost function has a very low value.

Based on the above discussion, the following algorithm is summarised for solving the MNPQEVAP with acceleration and velocity feedback.

Algorithm 4.1:

Input: $\mathbf{M},\mathbf{C},\mathbf{K} \in R^{n\times n}$, $\mathbf{B} \in R^{n\times m}$, $\mathbf{X}_1 \in R^{n\times p}$, and $\mathbf{\Lambda}_1, \tilde{\mathbf{\Lambda}}_1 \in R^{p\times p}$.

Output: $\mathbf{F}_2$ and $\mathbf{G}_2$.

(1) *Compute the* '*economy size*' QR *decomposition*

$$\begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_1\mathbf{\Lambda}_1 \end{bmatrix} = \mathbf{Q}_1\mathbf{R}$$

as in (39).

(2) *Choose an initial matrix* $\mathbf{\Gamma}_0 \in R^{m\times p}$ *randomly, and then solve the optimization problem* (48) *by using a gradient-based method* via (47), (51), *and* (49); *find* $\mathbf{\Gamma}^*$ *such that* $f(\mathbf{\Gamma}^*) = \min f(\mathbf{\Gamma})$.

(3) *Solve Sylvester Equation* (46) *with* $\mathbf{\Gamma} = \mathbf{\Gamma}^*$. *for* $\tilde{\mathbf{S}}$, *and then compute* $\mathbf{\Phi} = \tilde{\mathbf{S}}^{-1}\tilde{\mathbf{\Lambda}}_1^{-\mathrm{T}}\mathbf{\Gamma}^{\mathrm{T}}$.

(4) *Compute* $\mathbf{F}_2$ *and* $\mathbf{G}_2$ *by* (24) *and* (25).

5. Numerical examples

In this section, two numerical results for solving the MNPQEVAP using the algorithm 4.1 above and the algorithm in [11] are presented. The resultant feedback matrices, which are obtained using the same optimization method for the same examples and assignment requirements from the same initial matrix $\mathbf{\Gamma}_0$, are explored to compute the energy required from the actuators under two control strategies. Based on the energy (as an indicator of relative merits) defined by Soong [26], the actuation energy of the two control strategies are presented as follows:

$$E_1 = \int_0^T \left[\mathbf{B}(\mathbf{F}_1^{\mathrm{T}}\dot{\mathbf{x}}(t) + \mathbf{G}_1^{\mathrm{T}}\mathbf{x}(t))\right]^{\mathrm{T}} \left[\mathbf{B}(\mathbf{F}_1^{\mathrm{T}}\dot{\mathbf{x}}(t) + \mathbf{G}_1^{\mathrm{T}}\mathbf{x}(t))\right] \mathrm{d}t, \tag{52}$$

$$E_2 = \int_0^T \left[\mathbf{B}(\mathbf{F}_2^{\mathrm{T}}\ddot{\mathbf{x}}(t) + \mathbf{G}_2^{\mathrm{T}}\dot{\mathbf{x}}(t))\right]^{\mathrm{T}} \left[\mathbf{B}(\mathbf{F}_2^{\mathrm{T}}\ddot{\mathbf{x}}(t) + \mathbf{G}_2^{\mathrm{T}}\dot{\mathbf{x}}(t))\right] \mathrm{d}t, \tag{53}$$

which are squares of the combined active control forces during a time period [0, T].

The examples are computed using MATLAB 7.11. MATLAB function *fminunc* is used to solve the optimization problem in Step 2 of the algorithm with the gradient given by (49), based on the quasi-Newton 'BFGS' method.

Example 5.1 (P 5.2 in [10]). Matrices **M**, **C**, **K** and **B** are as follows:

$$\mathbf{M} = 10\mathbf{I}_3,\ \mathbf{C} = \mathbf{0},\ \mathbf{K} = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{\pm 0.8901i, \pm 2.4940i, \pm 3.6039i\}$. This is an undamped vibrating system. In this example, two partial eigenvalue assignment schemes are considered: (1) two eigenvalues $\{\pm 3.6039i\}$ are reassigned to $\{-1, -2\}$, which is also the assignment of P 5.2 in [10]. This changes the two oscillatory frequency components into non-oscillatory decaying ones. (2) two eigenvalues $\{\pm 3.6039i\}$ are reassigned to $\{-0.5 \pm 5.0i\}$, which changes the frequency and adds damping to the two frequency components.

Case (1): The resultant velocity and displacement feedback matrices and their F-norms are as follows:

$$\mathbf{F}_1 = \begin{bmatrix} 0.4502 & -6.3323 \\ -1.0116 & 14.2285 \\ 0.8112 & -11.4104 \end{bmatrix}, \quad \|\mathbf{F}_1\|_F = 19.3554,$$

$$\mathbf{G}_1 = \begin{bmatrix} -1.6489 & 23.1929 \\ 3.7050 & -52.1140 \\ -2.9712 & 41.7922 \end{bmatrix}, \quad \|\mathbf{G}_1\|_F = 70.8918.$$

The F-norms of $\mathbf{F}_1$ and $\mathbf{G}_1$ here are comparable with those of the pure MNPQ-EVAP, which are 19.36 and 70.89 in [10]. The resultant acceleration and velocity feedback matrices and their F-norms are as follows:

$$\mathbf{F}_2 = \begin{bmatrix} 0.8244 & -11.5965 \\ -1.8525 & 26.0570 \\ 1.4856 & -20.8961 \end{bmatrix}, \quad \|\mathbf{F}_2\|_F = 35.4459,$$

$$\mathbf{G}_2 = \begin{bmatrix} 2.9235 & -41.1217 \\ -6.5691 & 92.3996 \\ 5.2680 & -74.0987 \end{bmatrix}, \quad \|\mathbf{G}_2\|_F = 125.6931.$$

Now a new harmonic excitation $f(t) = \sin(3.6t)$ is applied to the second d.o.f. This driving frequency is close to the third undamped natural frequency of the open-loop system. The time-domain responses of the open-loop system and two closed-loop

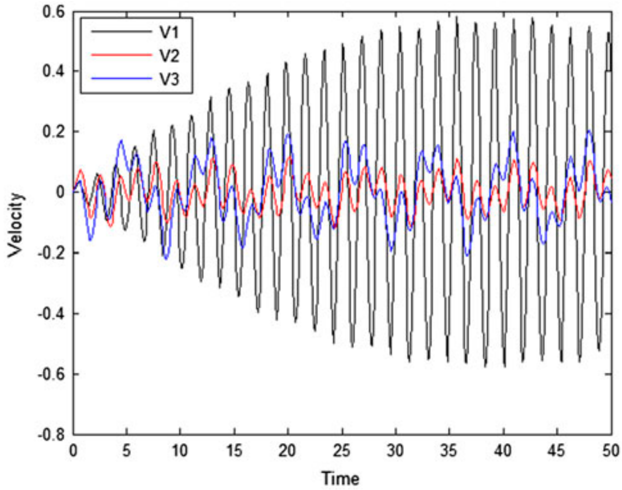

Figure 1. The velocity time history of the second d.o.f. of Example 5.1 Case (1): V1 – the open-loop system; V2 – the controlled system with the acceleration and velocity feedback; and V3 – the controlled system with the velocity and displacement feedback.

systems with gains above are shown in Figure 1. As the active control removes the undesirable undamped natural frequency value 3.6039 of the open-loop system, the original resonant response is suppressed significantly; and the steady-state response magnitude of the controlled system using the acceleration and velocity feedback is smaller than that of the controlled system using the velocity and displacement feedback, as shown in Figure 1. Additionally, one has

$$E_1 = 123.08, \quad E_2 = 120.47,$$

which indicates the two control strategies consume similar amounts of energy.

Case (2): The resultant velocity and displacement feedback matrices and their F-norms are as follows:

$$\mathbf{F}_1 = \begin{bmatrix} 0.1501 & -2.1108 \\ -0.3372 & 4.7428 \\ 0.2704 & -3.8035 \end{bmatrix}, \quad \|\mathbf{F}_1\|_F = 6.4518,$$

$$\mathbf{G}_1 = \begin{bmatrix} 1.8401 & -25.8824 \\ -4.1347 & 58.1572 \\ 3.3157 & -46.6384 \end{bmatrix}, \quad \|\mathbf{G}_1\|_F = 79.1124.$$

The resultant acceleration and velocity feedback matrices and their F-norms are as follows:

$$\mathbf{F}_2 = \begin{bmatrix} -0.0729 & 1.0250 \\ 0.1637 & -2.3033 \\ -0.1313 & 1.8471 \end{bmatrix}, \quad \|\mathbf{F}_2\|_F = 3.1332,$$

$$\mathbf{G}_2 = \begin{bmatrix} 0.0772 & -1.0857 \\ -0.1734 & 2.4396 \\ 0.1391 & -1.9564 \end{bmatrix}, \quad \|\mathbf{G}_2\|_F = 3.3186.$$

Applying the same excitation force in this case as in case (1), the time-domain responses of each system are shown in Figure 2. The steady-state response magnitude of the controlled system using the acceleration and velocity feedback is nearly the same as that of the controlled system using the velocity and displacement feedback, as shown in Figure 2. Additionally, one has

$$E_1 = 139.71, \quad E_2 = 144.98,$$

which again indicates similar amounts of energy consumption.

Example 5.2. A plane truss structure with nine bars is shown in Figure 3. Its material parameters are: elastic modulus $E = 2.1\text{e}11$ Pa, mass density $\rho = 7860\,\text{kg m}^{-3}$, cross-sectional area of each bar $A = 2.5\text{e} - 3\,\text{m}^2$. The plane truss is excited vertically by $f_1(t) = 50\sin(635t + 0.2\pi)$ N and $f_2(t) = 100\sin(332t)$ to nodes 2 and 4, respectively. These driving frequencies are close to the first and the second undamped natural frequencies of the truss structure, respectively. The damping matrix is taken as $\mathbf{C} = 1.0\text{e} - 5{*}\mathbf{K} + 1.0\text{e} - 4{*}\mathbf{M}$. This truss has open-loop eigenvalues: $-0.6 \pm 332.9i$, $-2.0 \pm 635.3i$, $-3.7 \pm 861.7i$, $-8.7 \pm 1322.7i$, $-22.0 \pm 2099.5i$, $-22.2 \pm 2109.3i$, $-24.8 \pm 2226.8i$ and $-29.0 \pm 2409.5i$. The undesirable first and second pairs of eigenvalues are reassigned to $-5.0 \pm 500.0i$ and $-6.0 \pm 800.0i$. The other unassigned eigenpairs are kept unchanged. The control matrix $\mathbf{B}$ is

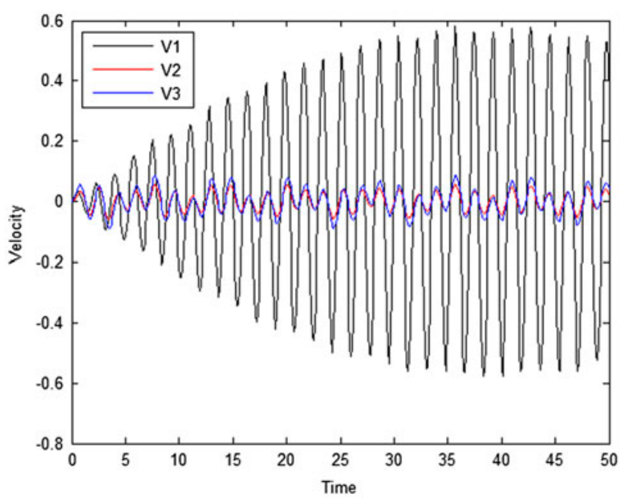

Figure 2. The velocity time history of the second d.o.f. of Example 5.1 Case (2): V1 – the open-loop system; V2 – the controlled system with the acceleration and velocity feedback; and V3 – the controlled system with the velocity and displacement feedback.

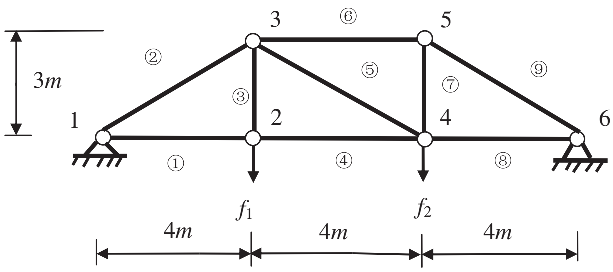

Figure 3. A plane truss with 9 bars.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The resultant velocity and displacement feedback matrices and their F-norms are as follows:

$$\mathbf{F}_1 = 10^4 \begin{bmatrix} 0.4151 & -0.5260 & -0.2098 \\ 2.5203 & -2.4424 & -0.3705 \\ 0.9663 & -1.1474 & -0.3960 \\ 3.5150 & -3.3049 & -0.3948 \\ 0.8387 & -1.0826 & -0.4480 \\ 2.1900 & -1.5056 & 0.4201 \\ 0.4440 & -0.6804 & -0.3662 \\ 1.1569 & -0.6159 & 0.4378 \end{bmatrix}, \quad \|\mathbf{F}_1\|_F = 7.1279\mathrm{e}+4,$$

$$\mathbf{G}_1 = 10^7 \begin{bmatrix} -0.1937 & -0.0180 & -0.0981 \\ -0.9436 & -0.6088 & -0.6354 \\ -0.4270 & -0.0930 & -0.2325 \\ -1.2846 & -0.9164 & -0.8914 \\ -0.3975 & -0.0230 & -0.1972 \\ -0.6291 & -0.9393 & -0.5846 \\ -0.2436 & 0.0592 & -0.0988 \\ -0.2768 & -0.6156 & -0.3183 \end{bmatrix}, \quad \|\mathbf{G}_1\|_F = 2.7728\mathrm{e}+7.$$

The resultant acceleration and velocity feedback matrices and their F-norms are as follows:

$$\mathbf{F}_2 = \begin{bmatrix} 2.6248 & 1.0908 & 1.1519 \\ 12.0474 & 21.7720 & 13.8007 \\ 5.7117 & 4.0885 & 3.3774 \\ 16.2770 & 32.4091 & 20.1659 \\ 5.4064 & 1.8002 & 2.1458 \\ 7.2747 & 31.3628 & 17.5838 \\ 3.4177 & -1.2121 & 0.1631 \\ 2.9138 & 20.1891 & 10.9161 \end{bmatrix}, \quad \|\mathbf{F}_2\|_F = 67.3273,$$

$$\mathbf{G}_2 = \begin{bmatrix} 1.5097 & 7.7582 & 3.1272 \\ -15.9399 & -145.6638 & -63.2574 \\ 0.9459 & -1.6596 & -1.1335 \\ -25.6192 & -229.1691 & -99.3230 \\ 3.7186 & 20.8079 & 8.5084 \\ -34.4727 & -284.9137 & -122.5231 \\ 5.5564 & 38.5635 & 16.2577 \\ -24.2113 & -196.5847 & -84.3825 \end{bmatrix}, \quad \|\mathbf{G}_2\|_F = 484.3763.$$

The time-domain responses of the original structure and two controlled structures with gains above are shown in Figure 4. The steady-state response magnitude of the controlled structure using the acceleration and velocity feedback is slightly larger than that of using the velocity and displacement feedback. Additionally, one has here

$$E_1 = 3100, \quad E_2 = 3190,$$

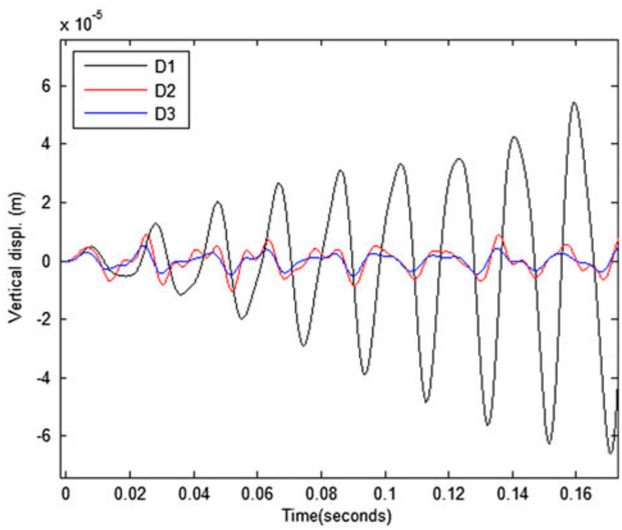

Figure 4. The vertical displacement time history of the plane truss at node 4 of Example 5.2: D1 – the original structure; D2 – the controlled structure with the acceleration and velocity feedback; and D3 – the controlled structure with the velocity and displacement feedback.

which indicates that one needs slightly larger amount of energy to achieve the partial eigenvalues assignments in this example using the acceleration and velocity feedback, than that when using the velocity and displacement feedback.

When different forms of initial matrix $\mathbf{\Gamma}_0$ are used in the computation, it is found that sometimes E_1 is greater than E_2 and sometimes it is smaller. However, F-norms of $\mathbf{F}_1$ and $\mathbf{G}_1$ are always much larger than that of $\mathbf{F}_2$ and $\mathbf{G}_2$, as shown above.

6. Conclusions

Partial quadratic eigenvalue assignment and associated minimum norm optimization (PQEVAP and MNPQEVAP) are carried out using acceleration and velocity feedback in this paper. The algorithm proposed has some attractive practical features as in [7–11], which makes it suitable in dealing with large-scale practical structures. It is found in numerical examples that the PQEVAP and MNPQEVAP using acceleration and velocity feedback normally can be solved with similar amounts of actuation energy to those of velocity and displacement feedback. Both control strategies are shown to be capable of suppressing vibration.

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