Numer. Math.
DOI 10.1007/s00211-013-0565-1

Numerische Mathematik

Approximating Lyapunov exponents and Sacker–Sell spectrum for retarded functional differential equations

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Received: 23 February 2012 / Revised: 13 December 2012

Abstract We consider Lyapunov exponents and Sacker–Sell spectrum for linear, nonautonomous retarded functional differential equations posed on an appropriate Hilbert space. A numerical method is proposed to approximate such quantities, based on the reduction to finite dimension of the evolution family associated to the system, to which a classic discrete QR method is then applied. The discretization of the evolution family is accomplished by a combination of collocation and generalized Fourier projection. A rigorous error analysis is developed to bound the difference between the computed stability spectra and the exact stability spectra. The efficacy of the results is illustrated with some numerical examples.

Mathematics Subject Classification (2000) 37M25 · 65L03 · 65L07

1 Introduction

In this paper we consider the numerical approximation of stability spectra, namely Lyapunov exponents and Sacker–Sell intervals, for a general class of linear, non-autonomous Retarded Functional Differential Equations (RFDEs). These are

The work of the first author was partially supported by the Italian National Scientific Computing Group grants "GNCS—Young Researchers" 2008 and 2009. The work of the second author was partially supported by NSF grants DMS-0812800 and DMS-1115408.

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Published online: 21 May 2013

infinite-dimensional dynamical systems so that any computational technique will make a finite-dimensional approximation. The interest is in understanding and quantifying the error in a particular finite-dimensional approximation and in how this error translates into the error in the stability spectra. The research is motivated by computational results such as those in e.g. [55], in which inferences and conjectures are made in terms of approximations to Lyapunov exponents. The goal is then to develop an efficient method and to make a rigorous error analysis.

There is a considerable literature on Lyapunov exponents in the field of dynamical systems with delays, although it should be stressed that most of the existing literature is concerned with computation of Lyapunov exponents and the consequences of the numerical results obtained from the physical and application point of view. Less attention has been paid to the mathematical and numerical analysis of the problem and the convergence and error analysis of the methods that have been employed to approximate Lyapunov exponents. The fundamental paper [35] is an excellent example with extensive computational aspects and analysis of the relevant results, while lacking rigorous theoretical insight into the adopted computational scheme. In papers such as [45,46,50,55–57] the analysis is restricted to specific problems such as the Mackey–Glass, Ikeda or Van der Pol equations, as well as to delay-coupled chaotic oscillators of Lorenz or Rössler type. In this paper we present a systematic numerical approach together with a rigorous error analysis.

The method we develop here is based on first discretizing the evolution family associated to the system of RFDEs and then applying QR based techniques borrowed from the literature on Ordinary Differential Equations (ODEs, see e.g. [27,29–32,58]) to the finite-dimensional approximation thus obtained in order to obtain approximations to (some of) the stability spectra. To this aim the Initial Value Problem (IVP) for the RFDEs is posed on an appropriate Hilbert space (see e.g. [6–9,20–24,41,42,59]) and the relevant theory about well-posedness, evolution and existence of Lyapunov exponents can be found in [10], although we briefly recall the aspects necessary to our analysis. The main contributions of the work can thus be summarized in the numerical discretization of the evolution family accompanied with its convergence analysis on the one hand and in the error analysis of the approximations to the stability spectra on the other hand.

The evolution family is first approximated by a finite-rank, infinite-dimensional operator through a combination of collocation of the IVP on Legendre-Gauss nodes and generalized Fourier projection over the state space. This part follows the lines of a similar technique developed in [14] for RFDEs posed in the classic Banach state space of continuous functions and in [13] for autonomous problems in the Hilbert context. This finite-rank operator is then reduced to a matrix, which is finally used in the discrete QR framework for computing the stability spectra.

The error analysis of the computed spectra as compared with the exact spectra is composed of two parts. The first one accounts for the error committed in applying the discrete QR method to the finite-rank evolution family instead of the exact one. The analysis benefits from the results on ODEs obtained in [32,58] and the recent extension to sequences of linear bounded operators on separable Hilbert spaces developed in [1]. The second one accounts for the relation between the dominant stability spectra associated to the finite-rank evolution and those of the final matrix approximation, which are those that can be effectively computed. They are shown to coincide. Our main result is that we have developed, for the first time, a well-defined numerical algorithm to compute part of the stability spectra for linear nonautonomous RFDEs together with a rigorous error and convergence analysis. We also note (see Sect. 6.3) many of the quantities required to apply the error analysis may be obtained from the finite dimensional approximation, i.e., from what is computed. The error in approximating the evolution operator is the main exception and much of our effort here is devoted to bounding this.

The manuscript is organized as follows. The model is presented in Sect. 2 together with basic facts taken from [10]. Section 3 is aimed at furnishing proper notation as well as collecting a series of tools to be utilized later on. Section 4 is about the theoretical aspects of the stability spectra for RFDEs and their characterization. The discretization of the evolution family is presented in Sect. 5, together with its convergence analysis in Sect. 5.1 and its matrix counterpart in Sect. 5.2. Section 6 deals with the computational aspects of the stability spectra in Sect. 6.1, their error analysis in Sect. 6.2 and the application of the latter in Sect. 6.3. Numerical tests complete the work in Sect. 7. Finally, Appendix A collects a number of technical results that are mainly used in Sect. 5.1.

2 Model and background

Let

$$X := \mathbb{R}^d \times L^2(-\tau, 0; \mathbb{R}^d), \tag{1}$$

where $\tau > 0$ and $d \geq 1$ is a given integer. The space X is called the state space and its elements will be denoted by a couple (u, φ). X is a separable Hilbert space of infinite dimension with the inner product of any two elements (u, φ) and (v, ψ) defined as

$$\langle (u, \varphi), (v, \psi) \rangle_X := v^T u + \int_{-\tau}^{0} \psi^T(t)\varphi(t)dt$$

and the induced norm $\|\cdot\|_X$ s.t.

$$\|(u, \varphi)\|_X^2 = \langle (u, \varphi), (u, \varphi) \rangle_X = |u|^2 + \|\varphi\|^2_{L^2(-\tau,0;\mathbb{R}^d)},$$

with $|\cdot|$ the euclidean norm in $\mathbb{R}^d$ and $\|\cdot\|_{L^2(-\tau,0;\mathbb{R}^d)}$ the standard induced L^2-norm s.t.

$$\|\varphi\|^2_{L^2(-\tau,0;\mathbb{R}^d)} = \int_{-\tau}^{0} |\varphi(t)|^2 dt.$$

Since a notion of orthogonality is fundamental in the context of the computation of Lyapunov exponents [3,4,27,28], we consider the class of linear and nonautonomus RFDEs described in the above Hilbert framework, hence different from the classical one where the initial data are in the Banach space of continuous functions, e.g. [33,38]. To this aim, we refer to the general linear RFDE

$$x'(t) = g(t, (x(t), x_t)), \quad \text{a.e. for } t \in \mathbb{I}, \tag{2}$$

where $\mathbb{I}$ is an interval of $\mathbb{R}$ unbounded on the right, $g : \mathbb{I} \times X \to \mathbb{R}^d$ is a continuous functional linear in its second argument and $(x(t), x_t) \in X$ is the state at time $t \in \mathbb{I}$ given by the pair consisting of the vector $x(t) \in \mathbb{R}^d$ and the function $x_t \in L^2(-\tau, 0; \mathbb{R}^d)$ defined as

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-\tau, 0], \tag{3}$$

according to the standard Hale–Krasovskii notation [38,43]. This setting is commonly used for RFDEs in product L^p-spaces [5,9,23,37,48] and it is sometimes referred to as the *extended* state [39] w.r.t. the *standard* state in the Banach framework [33,38,60], i.e. simply the (continuous) function x_t.

Notice that RFDEs (2) arising in applications have the general form

$$x'(t) = A(t)x(t) + \sum_{\ell=1}^{k} \left(B_\ell(t)x(t - \tau_\ell) + \int_{-\tau_\ell}^{-\tau_{\ell-1}} C_\ell(t, \theta)x(t + \theta)d\theta \right) \tag{4}$$

where $k \in \mathbb{N}$ is the number of delays, $0 := \tau_0 < \tau_1 < \cdots < \tau_{k-1} < \tau_k =: \tau$ and, for each $t \in \mathbb{I}$, $A(t) \in \mathbb{R}^{d \times d}$ and $B_\ell(t) \in \mathbb{R}^{d \times d}$ and $C_\ell(t, \cdot) \in L^2\left(-\tau_\ell, -\tau_{\ell-1}; \mathbb{R}^{d \times d}\right)$ for $\ell = 1, \ldots, k$.

For any $s \in \mathbb{I}$ and $(u, \varphi) \in X$, let us consider for (2) the IVP

$$\begin{cases} x'(t) = g(t, (x(t), x_t)), \text{ a.e. for } t \geq s, \\ x(s) = u, \\ x(s + \theta) = \varphi(\theta), \qquad \text{a.e. for } \theta \in [-\tau, 0). \end{cases} \tag{5}$$

Problem (5) with right-hand side given by (4) is studied in [10] for what concerns existence, uniqueness and regularity of solutions, as well as with regards to the associated evolution and abstract formulation. Moreover, existence and characterization of the relevant Lyapunov exponents is also discussed there (and recalled here in Sect. 4). We resume in the following lines the basic facts contained in [10] as adapted to (5), while we leave to its reading deeper details around the model and further implications.

According to [10, Definition 1], a *strict solution* of (5) on $[s - \tau, +\infty)$ is a function $x(\cdot) := x(\cdot; s, (u, \varphi))$ defined on $[s - \tau, +\infty)$ s.t. $x \in W^{1,2}(s, +\infty; \mathbb{R}^d)$ (see Sect. 3) and satisfies (5). Based on the assumption that g is continuous it is easy to check that (5) is well-posed [10, Theorem 2]. This allows to introduce, for any $s \in \mathbb{I}$ and $r \geq 0$, the linear bounded operator $T(s + r, s) : X \to X$ given by

$$T(s + r, s)(u, \varphi) := (x(s + r), x_{s+r}). \tag{6}$$

[10, Theorem 6] shows that the two-parameters family $\{T(s+r,s) : s \in \mathbb{I} \text{ and } r \geq 0\}$ is an *evolution family*, i.e.

(T1) $T(s+r,s)$ is bounded and linear;
(T2) $T(s,s) = I_X$, the identity operator on X;
(T3) $T(s+r,s+h)T(s+h,s) = T(s+r,s)$ for all $s \in \mathbb{I}$ and $r \geq h \geq 0$;
(T4) $\lim_{r \downarrow 0} T(s+r,s)(u,\varphi) = (u,\varphi)$ for all $(u,\varphi) \in X$;
(T5) there exist $K \geq 1$ and $\omega \geq 0$ s.t. $\|T(s+r,s)\|_{X \leftarrow X} \leq K e^{\omega r}$ (the norm is the operator induced one, see Sect. 3).

Following [10] and in view of Sect. 4 about Lyapunov exponents and Sacker–Sell intervals, as far as the sequence $\{t_n\}_{n=0}^{\infty}$ of time instants

$$t_n = s + n\tau, \quad n = 0, 1, \ldots, \tag{7}$$

is concerned, let us be licensed to write $x_n := (x(t_n), x_{t_n})$ not to overload the notation (this should not generate confusion w.r.t. (3) because of the different spaces). Then (6) becomes

$$T(t_n, t_0)x_0 = x_n$$

along the sequence (7). In view of this, the evolution of the state exiting from the given initial state $x_0 := (u,\varphi) \in X$ can be traced in discrete time by the sequence $\{x_n\}_{n=0}^{\infty} \in X$ iteratively given by

$$x_{n+1} = T_n x_n, \quad n = 0, 1, \ldots, \tag{8}$$

where we tacitly set

$$T_n := T(t_{n+1}, t_n), \quad n = 0, 1, \ldots. \tag{9}$$

Since the final goal is to construct a sequence of finite-dimensional approximations to $\{T_n\}_{n=0}^{\infty}$, we fix any $r \geq 0$ and construct in Sect. 5 an approximation to the general operator $T := T(s+r,s)$.

3 Notation and preliminaries

I_S denotes the identity of a space S and $\|\cdot\|_S$ a relevant norm. We recall that for any non-negative integer k and any reals $a < b$, the Sobolev spaces

$$\begin{aligned} H^k(a,b;\mathbb{R}^d) &:= W^{k,2}(a,b;\mathbb{R}^d) \\ &:= \left\{\psi \in L^2(a,b;\mathbb{R}^d) : \psi^{(i)} \in L^2(a,b;\mathbb{R}^d),\ i = 1,\ldots,k\right\} \end{aligned}$$

equipped with the norm $\|\cdot\|_{H^k(a,b;\mathbb{R}^d)}$ s.t.

$$\|\psi\|^2_{H^k(a,b;\mathbb{R}^d)} = \sum_{i=0}^{k} \|\psi^{(i)}\|^2_{L^2(a,b;\mathbb{R}^d)}$$

are Hilbert spaces. As a related fact, besides the state space X defined in (1) we will also make use of its subspaces

$$X^k := \mathbb{R}^d \times H^k(-\tau, 0; \mathbb{R}^d),$$

k any positive integer.

We introduce now the notation necessary for the spaces of interest in this work. Let us set

$$\begin{aligned} L &:= L^2(-\tau, 0; \mathbb{R}^d), \\ L^+ &:= L^2(0, r; \mathbb{R}^d), \end{aligned}$$

and

$$L^\pm := L^2(-\tau, r; \mathbb{R}^d)$$

together with their L^2-norms $\|\cdot\|_L$, $\|\cdot\|_{L^+}$ and $\|\cdot\|_{L^\pm}$, respectively. The relevant functions are denoted respectively as f, f^+ and $f^\pm$, the latter being tacitly intended as divided into $f := f^\pm_{|[-\tau,0]}$ and $f^+ := f^\pm_{|[0,r]}$. The same notation is adopted for spaces and functions other than L^2, e.g. X, H^k, X^k for any positive integer k and Π_K for any positive integer K, the space of algebraic polynomials of degree at most K. For $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ normed linear spaces, $\mathscr{B}(Y, Z)$ is the set of linear and bounded operators from Y to Z (simply $\mathscr{B}(Y)$ when $Z = Y$) and $\|A\|_{Z\leftarrow Y}$ is the operator norm of $A \in \mathscr{B}(Y, Z)$.

For a given positive integer M, let $\{\phi_i\}_{i=0}^\infty$ be the system of Legendre polynomials of L with $-\tau < \theta_1 < \cdots < \theta_M < 0$ the M zeros of $\phi_M \in \Pi_M$. Set $L_M := \left(\mathbb{R}^d\right)^{M+1}$ to be the discrete counterpart of L and consider the restriction operator $S_M : L \to L_M$ acting as

$$S_M f = (f_0, \ldots, f_M)^T, \quad f = \sum_{i=0}^{\infty} f_i \phi_i,$$

and the prolongation operator $P_M : L_M \to \Pi_M \subset L$ given by

$$P_M \mathbf{v} := \sum_{j=0}^{M} v_j \phi_j, \quad \mathbf{v} = (v_0, \ldots, v_M)^T.$$

Observe that $S_M P_M = I_{L_M} : L_M \to L_M$ while $P_M S_M =: F_M : L \to \Pi_M \subset L$ is the Fourier projection operator relevant to $\{\phi_i\}_{i=0}^\infty$.

With a possibly different positive integer N, exactly the same notation is adopted for L^+ and the zeros $0 < \theta_1^+ < \cdots < \theta_N^+ < r$ of the relevant Legendre polynomial $\phi_N^+ \in \Pi_N^+$. Such zeros will be used as collocation nodes for the numerical method proposed in Sect. 5.

In a similar fashion, the same notation is extended to the state space X w.r.t. the orthogonal system $\{\Phi_i\}_{i=-d}^{\infty}$ with $\Phi_{-i} := (e_i, 0)$, $i = 1, \ldots, d$, with e_i the i-th canonical vector in $\mathbb{R}^d$ and $\Phi_i = (0, \phi_i)$, $i = 0, 1, \ldots$, with $\{\phi_i\}_{i=0}^{\infty}$ the orthogonal system of Legendre polynomials of L. Then we set $X_M := \mathbb{R}^d \times L_M = \left(\mathbb{R}^d\right)^{M+2}$ for the discrete state space, $\hat{S}_M : X \to X_M$ for the restriction operator and $\hat{P}_M : X_M \to X$ for the prolongation operator. Again $\hat{S}_M \hat{P}_M = I_{X_M} : X_M \to X_M$ and $\hat{P}_M \hat{S}_M =: \hat{F}_M : X \to X$ is the Fourier projection operator relevant to $\{\Phi_i\}_{i=-d}^{\infty}$. As X is a separable Hilbert space, we shall understand without confusion the representation of any $B \in \mathscr{B}(X)$ as an infinite-dimensional matrix of entries

$$[B]_{ij} = \left\langle B\Phi_j, \Phi_i \right\rangle_X, \quad i, j = -d, \ldots, -1, 0, 1, \ldots,$$

$\mathbb{R}^d$-block indexed according to $-1, 0, 1 \ldots$ to take into account for the presence of the vector element of $\mathbb{R}^d$ (index -1) and the function coefficients (indices $0, 1, \ldots$), as well as its block-wise partition

$$B = \begin{pmatrix} (B)_{11} & (B)_{12} \\ (B)_{21} & (B)_{22} \end{pmatrix}$$

where, $(B)_{11} \in \left(\mathbb{R}^d\right)^{(M+2)\times(M+2)}$, $(B)_{12} \in \left(\mathbb{R}^d\right)^{(M+2)\times\infty}$, $(B)_{21} \in \left(\mathbb{R}^d\right)^{\infty\times(M+2)}$ and $(B)_{22} \in \left(\mathbb{R}^d\right)^{\infty\times\infty}$. To avoid misunderstandings, $(B)_{ij}$ will denote the ij-th block matrix in a block-wise partition of B, while $[B]_{ij}$ will denote the ij-th entry. It is then clear that $\hat{S}_M = (\mathrm{diag}(I_{\mathbb{R}^d}, S_M), 0)$, $\hat{P}_M = (\mathrm{diag}(I_{\mathbb{R}^d}, P_M), 0)^T$, together with $\hat{S}_M \hat{P}_M = \mathrm{diag}(I_{\mathbb{R}^d}, I_{L_M})$ and $\hat{F}_M = \mathrm{diag}(I_{\mathbb{R}^d}, F_M)$. Moreover $(B)_{11} = \hat{S}_M B \hat{P}_M$.

Eventually, and for a simpler notation, for any fixed $s \in \mathbb{I}$ and $r \geq 0$, we shift the time back by s and hence consider the IVP

$$\begin{cases} x'(t) = (G_s x)(t), \text{ a.e. for } t \in [0, r], \\ x(0) = u, \\ x(\theta) = \varphi(\theta), \qquad \text{a.e. for } \theta \in [-\tau, 0), \end{cases} \tag{10}$$

where $G_s : L^{\pm} \to L^+$ is the linear operator given by

$$(G_s x)(t) := g(s + t, (x(t), x_t)), \quad t \in [0, r]. \tag{11}$$

The associated evolution operator $T_s(r, 0)$ clearly coincides with T as defined at the end of Sect. 2.

4 Stability spectra

In [10, Section 6], Lyapunov exponents are defined based on extending to compact operators in Hilbert spaces [51, Corollary 2.2] the celebrated Oseledets's Multiplicative Ergodic Theorem [47] for finite-dimensional dynamical systems (see also [16,34]). Under the hypotheses of existence of an ergodic invariant measure and compactness of the evolution family defined through (6), it is proven that there exists a (possibly infinite) sequence of exact Lyapunov exponents defined as limit [10, Theorems 12 and 13]. Precisely, they are the logarithm of the eigenvalues of the operator $\Lambda : X \to X$ given by

$$\Lambda := \lim_{n\to\infty} [T(s+n\tau,s)^H T(s+n\tau,s)]^{1/2n}.$$

There is an extensive literature on exponential dichotomies in infinite-dimensional spaces, e.g., [17–19,52,53], which is directly related to the Sacker–Sell spectrum. Based on this, the latter is defined as the set of those values $\lambda \in \mathbb{R}$ s.t. the shifted system $(x'(t), x_t') = (\mathscr{A}(t) - \lambda I_X)(x(t), x_t)$ does not have exponential dichotomy. For the rigorous definition of $\mathscr{A}(t)$ we refer the interested reader to [10, Section 5], but basically it is the generator of the evolution family. Finally, perturbation theory for exponential dichotomies is related to the so-called Roughness Theorem (see [49]).

In Sect. 6.2 we will introduce an assumption related to integral separation, (45), that is strongly related to the continuity of Lyapunov exponents with respect to perturbations. In the case of non-distinct Lyapunov exponents a sufficient condition for the continuity of Lyapunov exponents is given in (46). The sequence of Lyapunov exponents may be characterized through an orthogonal factorization of the evolution family as detailed in the sequel. Such a characterization lays the basis for the extension to RFDEs of the standard discrete QR technique used for linear and nonautonomus ODEs [29,30,32], which is the central contribution of the present manuscript.

The main idea behind the discrete QR technique is to determine an orthogonal change of variable for the discrete time system that brings it to an upper triangular form. This transforms the sequence of operators $\{T_n\}_{n=0}^{\infty}$, for example as defined in (9), to the sequence $\{R_n\}_{n=0}^{\infty}$ obtained iteratively from a given initial (randomly chosen, [3,26]) unitary operator Q_0 by subsequent QR-factorizations as

$$Q_{n+1}R_n = T_n Q_n, \quad n = 0, 1, \ldots. \tag{12}$$

Each factorization is intended to be the unique one with non-negative diagonal elements in the upper triangular factor. This is assumed to hold throughout the paper together with the convention $\log(0) = -\infty$. According to the notation introduced in Sect. 3, $[R_k]_{ii}$ is the i-th diagonal entry of the k-th infinite-dimensional matrix.

Let

$$\Sigma_{\mathrm{LE}} = \bigcup_{i=0}^{\infty} [\alpha_i, \beta_i]$$

be the *Lyapunov spectrum* where, for $i \in \mathbb{N}$,

$$\alpha_i = \liminf_{n\to\infty} \frac{1}{t_n} \sum_{k=0}^{n-1} \log([R_k]_{ii}), \quad \beta_i = \limsup_{n\to\infty} \frac{1}{t_n} \sum_{k=0}^{n-1} \log([R_k]_{ii})$$

are the infinitely-many lower and upper Lyapunov exponents relevant to (8). Analogously, let

$$\Sigma_{\mathrm{SS}} = \bigcup_{i=0}^{\infty} [\gamma_i, \delta_i]$$

be the *Sacker–Sell spectrum* relevant to (8) where, for $i \in \mathbb{N}$,

$$\gamma_i = \liminf_{n\to\infty} \inf_{m\in\mathbb{N}} \frac{1}{t_n} \sum_{k=m}^{m+n-1} \log([R_k]_{ii}), \quad \delta_i = \limsup_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{1}{t_n} \sum_{k=m}^{m+n-1} \log([R_k]_{ii}).$$

Substitution of T_n, $n = 0, 1, \ldots$, in (12) with a suitable finite-dimensional approximation (Sect. 5) constitutes the underlying idea of the proposed approach for the numerical approximation of (part of) the stability spectra (Sect. 6).

5 Discretization of the evolution operator

Let M and N be positive integers. The goal of this section is to construct a finite-dimensional approximation of the evolution operator T introduced in Sect. 2. The way we proceed is based first in providing a suitable approximation $T_{M,N}$ still living in the exact state space X, and then deriving a matrix approximation $\mathbf{T}_{M,N}$ for the latter living in the finite-dimensional state space X_M. We show then in Sect. 6 how the Lyapunov exponents and Sacker–Sell intervals computed for the matrix approximation are exactly the nontrivial ones (i.e. those not clustered at $-\infty$) of the infinite-dimensional approximation, which in turn approximate a finite number of the exact ones associated to the true evolution operator.

5.1 The infinite-dimensional approximation

The numerical method we propose is based on first projecting the initial data (u, φ) by $\hat{F}_M$, second collocating the IVP (10) on the Legendre-Gauss zeros θ_i^+, $i = 1, \ldots, N$, of $[0, r]$ and third projecting again the result by $\hat{F}_M$. We thus define the approximation $T_{M,N} : X \to X$ of $T : X \to X$ as

$$T_{M,N} = \hat{F}_M T_N \hat{F}_M, \tag{13}$$

with $T_N : X \to X$ given by

$$T_N(u, \varphi) = \left(p_N^{\pm}(r), \left(p_N^{\pm}\right)_r\right), \tag{14}$$

where $p_N^{\pm}$ is split into φ in $[-\tau, 0)$ and $p_N^{+} \in \Pi_N^{+}$ in $[0, r]$ determined by the collocation of (10):

$$\begin{cases} \left(p_N^{+}\right)'\left(\theta_i^{+}\right) = \left(G_s p_N^{\pm}\right)\left(\theta_i^{+}\right), & i = 1, \ldots, N, \\ p_N^{+}(0) = u. \end{cases} \tag{15}$$

Notice that the definition of T_N in (14) makes sense only for an initial function φ everywhere defined, condition ensured by its use in (13). In general, instead, and in view of analyzing the convergence of $T_{M,N}$ to T in the sequel, we will assume $(u, \varphi) \in X^k$ whenever required. This ensures norm convergence, which is needed for the analysis of the error on the stability spectra as carried out in Sect. 6.

Remark 1 Let us observe that as far as $r \geq \tau$ the operator $T_{M,N}$ has finite rank since no piece of the initial function φ from the past is included in the range.

Remark 2 Let us underline that the notations T_N in (14) and T_n in (9) have different meanings, which should be clear from the context. A more specific notation would become unnecessarily cumbersome.

We thus aim at studying now the error $T - T_{M,N}$. From (13) we have

$$T - T_{M,N} = \mathscr{E}_1 + \mathscr{E}_2 + \mathscr{E}_3, \tag{16}$$

where

$$\mathscr{E}_1 = T - T_N$$

is the error due to the collocation (15), while

$$\mathscr{E}_2 = \left(I_X - \hat{F}_M\right)T_N \tag{17}$$

and

$$\mathscr{E}_3 = \hat{F}_M T_N \left(I_X - \hat{F}_M\right) \tag{18}$$

are the errors due to the projection in (13) of the collocated state at time r and of the initial state, respectively.

We separately state a bound for the norm of $\mathscr{E}_1$ in the following proposition since we retain this to be the true source of the discretization error, the other contributions being due merely to representation exigencies following projection. In the proof we use $\mathscr{L}_N^{+}$ as the interpolation operator based on the Legendre-Gauss zeros in $[0, r]$.

Proposition 1 *Assume $G_s : H^{1,\pm} \to H^{1,+}$. Then, for N sufficiently large, T_N is well-defined and there exists a positive constant c_1 independent of N s.t.*

$$\|T - T_N\|_{X^1} \le c_1 N^{-\frac{1}{2}}. \tag{19}$$

Proof Let $(u, \varphi) \in X^1$. Then

$$[T - T_N](u, \varphi) = \left(e_N^{\pm}(r), \left(e_N^{\pm}\right)_r\right),$$

where $e_N^{\pm} := x^{\pm} - p_N^{\pm} \in L^{\pm}$.

The solution x exiting from $(u, \varphi) \in X^1$ satisfies the functional equation in $L^{\pm}$

$$x^{\pm} = u_{\varphi} + V G_s x^{\pm} \tag{20}$$

as soon as we consider $u_{\varphi} \in L^{\pm}$ as the function

$$u_{\varphi}(t) := \begin{cases} u, & t \in [0, r], \\ \varphi(t), & \text{a.e. in } [-\tau, 0), \end{cases}$$

and $V : L^{+} \to L^{\pm}$ as the integral operator

$$(Vx)(t) := \begin{cases} \displaystyle\int_0^t x(\sigma) d\sigma, & t \in [0, r], \\ 0, & t \in [-\tau, 0]. \end{cases} \tag{21}$$

Similarly, we obtain from (15)

$$p_N^{\pm} = u_{\varphi} + V \mathscr{L}_N^{+} G_s p_N^{\pm} \tag{22}$$

since $\mathscr{L}_N^{+}(p_N^{+})' = (p_N^{+})'$ being $p_N^{+} \in \Pi_N^{+}$.

From (20) and (22) we obtain that $e_N^{\pm}$ satisfies the functional equation in $L^{\pm}$

$$e_N^{\pm} = V \mathscr{L}_N^{+} G_s e_N^{\pm} + V r_N^{+} \tag{23}$$

where

$$r_N^{+} := \left(I_{L^+} - \mathscr{L}_N^{+}\right) G_s x^{\pm}. \tag{24}$$

It is not difficult to verify that if $e_N^{\pm}$ solves (23), then $\varepsilon_N^{+} = \mathscr{L}_N^{+} G_s e_N^{\pm} + r_N^{+}$ solves the functional equation in L^{+}

$$\varepsilon_N^{+} = \mathscr{L}_N^{+} G_s V \varepsilon_N^{+} + r_N^{+}. \tag{25}$$

Vice-versa, if ε_N^{+} solves (25), then

$$e_N^{\pm} = V \varepsilon_N^{+} \tag{26}$$

solves (23). Lemma 7 in the Appendix ensures, for N sufficiently large, that (25) has a unique solution ε_N^+, which moreover satisfies

$$\left\|\varepsilon_N^+\right\|_{L^+} \leq 2 \left\|(I_{L^+} - G_s V)^{-1}\right\|_{L^+ \leftarrow L^+} \left\|r_N^+\right\|_{L^+}. \tag{27}$$

Then also (23) has a unique solution $e_N^\pm$ through (26), which moreover satisfies

$$\left\|e_N^\pm\right\|_{L^\pm} \leq r \left\|\varepsilon_N^+\right\|_{L^+}$$

by virtue of Lemma 4 in the Appendix. Then T_N is well-defined because the polynomial of collocation satisfying (15) is unique since so are both $x^\pm$ and $e_N^\pm$.

Now, as far as the norm of X^1 is concerned, we have

$$\begin{aligned}\|[T - T_N](u,\varphi)\|_{X^1}^2 &= |e_N^\pm(r)|^2 + \left\|(e_N^\pm)_r\right\|_{H^1}^2 \\ &= |e_N^\pm(r)|^2 + \left(\left\|(e_N^\pm)_r\right\|_L + \left\|(e_N^\pm)_r'\right\|_L\right)^2.\end{aligned} \tag{28}$$

From (26) it follows

$$e_N^\pm(r) = \int_0^r \varepsilon_N^+(\sigma)d\sigma,$$

which implies

$$|e_N^\pm(r)| \leq \sqrt{r} \left\|\varepsilon_N^+\right\|_{L^\pm}$$

by Hölder's inequality. As for the first addend in the second term of the right-hand side of (28) we have

$$\left\|(e_N^\pm)_r\right\|_L \leq \left\|e_N^\pm\right\|_{L^\pm} \leq r \left\|\varepsilon_N^+\right\|_{L^+}$$

again by virtue of Lemma 4 in the Appendix. As for the second addend it holds

$$\|(e_N^\pm)_r'\|_L^2 = \int_{-\tau}^0 \left|(e_N^\pm)'(r+\theta)\right|^2 d\theta = \int_{r-\tau}^r \left|(e_N^\pm)'(\theta)\right|^2 d\theta \leq \int_0^r \left|\varepsilon_N^+(\theta)\right|^2 d\theta = \left\|\varepsilon_N^+\right\|_{L^+}^2$$

thanks to (26). Collecting everything in (28) leads to

$$\|[T - T_N](u,\varphi)\|_{X^1} \leq \sqrt{r^2 + 3r + 1} \left\|\varepsilon_N^+\right\|_{L^+}$$

and to the existence of a positive constant c_1' independent of N s.t.

$$\|[T - T_N](u,\varphi)\|_{X^1} \leq c_1' N^{-\frac{1}{2}} \|G_s x\|_{H^{1,+}}$$

by virtue of (27), (24), the fact that $(u, \varphi) \in X^1$ implies $G_s x = x' \in H^{1,+}$ (see [10, Theorem 3]) and (50) in Theorem 4 of the Appendix. The final step is provided first by the details in the proof of [10, Theorem 3], by which

$$\|G_s x\|_{H^{1,+}} \leq \|x\|_{H^{2,+}} \leq c_1''(|u| + \|\varphi\|_L)$$

for some positive constant c_1'' independent of (u, φ), and second by observing that

$$|u| + \|\varphi\|_L \leq \sqrt{2}\,\|(u, \varphi)\|_{X^1}.$$

□

Remark 3 Observe that according to [10, Theorem 3], $(u, \varphi) \in X^1$ implies $G_s x = x' \in H^{1,+}$ only if $\varphi(0) = u$. But if one looks carefully at the proof of [10, Theorem 3] soon realizes that the lack of the latter gives raise to possible discontinuities only at time instants which are integer multiples of τ. Of course this loses of importance if $r = \tau$, but in the case $r > \tau$ the same result (19) holds true if a piecewise collocation is taken over the partition $[0, \tau] \cup [\tau, 2\tau] \cup \cdots \cup [m\tau, r]$ for some integer m. We skip on this to avoid useless technicalities.

Remark 4 It is not difficult to argue that if $(u, \varphi) \in X^k$ and a proper regularity of G_s is granted by suitable assumptions, then the error bound (19) can be refined by

$$\|T - T_N\|_{X^k} \leq c_k N^{\frac{1}{2}-k}, \tag{29}$$

with c_k constant independent of N. Nevertheless it should be noted that the smoothing nature of RFDEs gives raise to such a situation with increasing k as long as the integration time (read n according to Sect. 4) increases, which is properly the case when stability spectra have to be approximated. For an exhaustive treatment on the smoothing effect of RFDEs in the Banach setting of continuous functions and on the relevant propagation of breaking points see [2]. In the Hilbert setting similar arguments justify an increasing regularity along a step-by-step integration on $[t_0, t_1], \ldots, [t_{n-1}, t_n]$ as referred to the sequence of Sobolev spaces $H^1, \ldots, H^n$. Finally, [28, Remark 2.1] justifies the restriction from X to X^1 leading to (19) if we consider $n \geq 1$ as well as the further restrictions from X to X^k leading to (29) above if $n \geq k > 1$.

Now we take into account also for the projection errors (17) and (18) in the proof of the following theorem, which eventually provides a bound on the error $\|T - T_{M,N}\|_{X^1}$ (the similar Remark 4 still applies). The latter will serve as a basis for the error analysis in Sect. 6.2.

Theorem 1 *Assume $G_s : H^{1,\pm} \to H^{1,+}$ and $r \geq \tau$. Then, for N sufficiently large and for $M \geq N$, $T_{M,N}$ is well-defined and there exists a constant c independent of M and N s.t.*

$$\left\|T - T_{M,N}\right\|_{X^1} \leq cN^{-\frac{1}{2}}.$$

Proof The thesis follows from (16). As for $\mathscr{E}_1$, Proposition 1 applies. As for $\mathscr{E}_2$, it is zero since for $r \geq \tau$ the range of T_N is Π_N and then $M \geq N$. As for $\mathscr{E}_3$, $\hat{F}_M$ in front can be neglected for the same reasoning adopted for $\mathscr{E}_2$ above; then T_N can be replaced by T thanks to the same arguments used at the end of the proof of Proposition 1 plus (49) in the Appendix; finally the bounds for the solution of (10) exiting from $(I_X - \hat{F}_M)(u, \varphi)$ proved in [10, Theorem 2] complete the proof together again with (49). $\square$

Remark 5 In principle the constant c in Theorem 1 can be estimated. This would require to know a bound on the norm of the inverse operator in (27), the knowledge of the error constants in both (49) and (50), as well as bounds on the norm of the solution taken from [10] (which require in turn other bounds or constants whose existence is proven but whose determination might be cumbersome).

5.2 The matrix approximation

We aim at finding now a matrix representation $\mathbf{T}_{M,N}$ of the approximation $T_{M,N}$. This is useful to implement codes based on the presented algorithm if one is interested in the approximation of (part of) the stability spectra of T. As a rule, when matrix explicit entries are given, they are thought as referred to (4) rather than to the general form (2).

Let us observe that, from (13),

$$T_{M,N} = \hat{P}_M U_{M,N} \hat{S}_M \tag{30}$$

where $U_{M,N} : X_M \to X_M$ given by

$$U_{M,N} = \hat{S}_M T_N \hat{P}_M \tag{31}$$

is the finite-dimensional operator corresponding to $T_{M,N}$. For a positive integer K, we identify $(\mathbb{R}^d)^K$ with $\mathbb{R}^{dK}$. Then $\mathbf{T}_{M,N}$, and the other matrices denoted in boldface, are nothing else than the canonical representation according to the Legendre polynomials in L and L^+ as described in the following and according to Sect. 3.

We first construct matrices $\mathbf{V}_{M,N} : X_M \to X_N^+$ and $\mathbf{V}_N^+ : X_N^+ \to X_N^+$ such that

$$\mathbf{V}_N^+ \big(p_{M,N}^+(r), S_N^+ p_{M,N}^+\big) = \mathbf{V}_{M,N}(u, S_M \varphi) \tag{32}$$

where $p_{M,N}^+ \in \Pi_N^+$ is the part of $p_{M,N}^\pm$ determined by the collocation equations

$$\begin{cases} \big(p_{M,N}^+\big)'(\theta_i^+) = \big(G_s p_{M,N}^\pm\big)(\theta_i^+), & i = 1, \ldots, N, \\ p_{M,N}^+(0) = u \end{cases}$$

and $p_{M,N}^{\pm}$ is $F_M\varphi$ in $[-\tau, 0)$. It is not difficult, yet technical, to check that the above matrices have entries, respectively,

$$[\mathbf{V}_N^+]_{ij} := \begin{cases} -\phi_j^+(r) I_{\mathbb{R}^d}, & \text{if } i = -1, \\ \phi_j^+(0) I_{\mathbb{R}^d}, & \text{if } i = 0, \\ (\phi_j^+)'(\theta_i^+) I_{\mathbb{R}^d} - A\big(s+\theta_i^+\big)\phi_j^+\big(\theta_i^+\big) & \\ \quad -\sum_{\ell=1}^{k_i} \Big(B_\ell\big(s+\theta_i^+\big)\phi_j^+\big(\theta_i^+ - \tau_\ell\big) & \\ \quad + \int_{-\tau_\ell}^{-\tau_{\ell-1}} C_\ell\big(s+\theta_i^+, \theta\big)\phi_j^+\big(\theta_i^+ + \theta\big) d\theta \Big) & \\ \quad + \int_{-\theta_i^+}^{-\tau_{k_i}} C_{k_i+1}\big(s+\theta_i^+, \theta\big)\phi_j^+\big(\theta_i^+ + \theta\big) d\theta, & \text{if } i = 1, \ldots, N, \end{cases}$$

for all $j = 0, \ldots, N$, plus the first column ($j = -1$) as $(1, 0, \ldots, 0)^T \in \mathbb{R}^{d(N+2)}$, and

$$\begin{aligned} [\mathbf{V}_{M,N}]_{ij} := &\; B_{k_i+1}(s+\theta_i^+)\phi_j(\theta_i^+ - \tau_{k_i+1}) \\ &+ \int_{-\tau_{k_i+1}}^{-\theta_i^+} C_{k_i+1}(s+(\theta_i^+, \theta)\phi_j(\theta_i^+ + \theta) d\theta \\ &+ \sum_{\ell=k_i+2}^{k} \Bigg(B_\ell(s+\theta_i^+)\phi_j(\theta_i^+ - \tau_\ell) \\ &+ \int_{-\tau_\ell}^{-\tau_{\ell-1}} C_\ell(s+(\theta_i^+, \theta)\phi_j(\theta_i^+ + \theta) d\theta \Bigg) \end{aligned}$$

for all $i = 1, \ldots, N$ and $j = 0, \ldots, M$, plus $[\mathbf{V}_{M,N}]_{0,-1} = 1$ and 0 elsewhere. Above we set

$$k_i := \max\left\{\ell = 0, \ldots, k \ : \ \theta_i^+ - \tau_\ell \geq 0\right\}, \quad i = 1, \ldots, N.$$

Second, and independently of the model coefficients in (4), we construct matrices $\mathbf{W}_M, \mathbf{W}_M^- : X_M \to X_M$ and $\mathbf{W}_{M,N}^+ : X_N^+ \to X_M$ such that

$$\mathbf{W}_M\big(p_{M,N}^{\pm}(r), S_M\big(p_{M,N}^{\pm}\big)_r\big) = \mathbf{W}_{M,N}^+\big(p_{M,N}^+(r), S_N^+ p_{M,N}^+\big) + \mathbf{W}_M^-(u, S_M\varphi) \quad (33)$$

by restriction of $p_{M,N}^{\pm}$ to $[r-\tau, r]$ when $r \geq \tau$, and possibly prolongation by $F_M\varphi$ when $r < \tau$. In particular, it is sufficient to define the above matrices with entries, respectively,

$$[\mathbf{W}_M]_{ij} := \begin{cases} \phi_j(0) I_{\mathbb{R}^d}, & \text{if } i = 0, \\ \phi_j(\theta_i) I_{\mathbb{R}^d}, & \text{if } i = 1, \ldots, M, \end{cases}$$

for all $j = 0, \ldots, M$, plus $[\mathbf{W}_M]_{-1,-1} = 1$ and 0 elsewhere,

$$[\mathbf{W}^+_{M,N}]_{ij} := \begin{cases} \phi^+_j(r) I_{\mathbb{R}^d}, & \text{if } i = 0, \\ 0, & \text{if } i = 1, \ldots, M_r \\ \phi^+_j(r+\theta_i) I_{\mathbb{R}^d}, & \text{if } i = M_r + 1, \ldots, M, \end{cases}$$

for all $j = 0, \ldots, N$, plus $[\mathbf{W}^+_{M,N}]_{-1,-1} = 1$ and 0 elsewhere, and

$$[\mathbf{W}^-_M]_{ij} := \begin{cases} 0, & \text{if } i = 0, \\ \phi_j(r+\theta_i) I_{\mathbb{R}^d}, & \text{if } i = 1, \ldots, M_r, \\ 0, & \text{if } i = M_r + 1, \ldots, M, \end{cases}$$

for all $j = 0, \ldots, M$ and 0 elsewhere. Above we set

$$M_r := \begin{cases} \max\{i = 1, \ldots, M \ : \ r + \theta_i < 0\}, & \text{if defined}, \\ 0, & \text{otherwise}, \end{cases}$$

with the convention that $\mathbf{W}^+_{M,N}$ is full and $\mathbf{W}^-_M$ is empty when $M_r = 0$, i.e. for $r \geq \tau$.
Eventually, it follows from (32) and (33) that

$$\mathbf{T}_{M,N}(u, S_M\varphi) = \left(p^{\pm}_{M,N}(r), S_M\left(p^{\pm}_{M,N}\right)_r\right)$$

is the sought matrix approximation of (13) with

$$\mathbf{T}_{M,N} = (\mathbf{W}_M)^{-1}\left[\mathbf{W}^+_{M,N}\left(\mathbf{V}^+_N\right)^{-1}\mathbf{V}_{M,N} + \mathbf{W}^-_M\right].$$

Standard approximation arguments ensure that $\mathbf{W}_M$ and $\mathbf{V}^+_N$ are invertible for sufficiently large M and N, respectively.

Remark 6 Notice that for the sake of computing stability spectra according to what anticipated at the end of Sect. 4 and further developed in the forthcoming Sect. 6, it is $r = \tau$, while $M = N$ can be chosen. As a straightforward consequence, the matrices $\mathbf{W}_M$, $\mathbf{W}^+_{M,N}$ and $\mathbf{W}^-_M$ reduce to the identity and hence

$$\mathbf{T}_N := \mathbf{T}_{N,N} = \left(\mathbf{V}^+_N\right)^{-1}\mathbf{V}_N,$$

where $\mathbf{V}_N := \mathbf{V}_{N,N}$. Moreover, for models with a single discrete delay, the structure is further simplified since $k_i = 0$ for all $i = 1, \ldots, N$, i.e. the non-delayed coefficient A appears in $\mathbf{V}^+_N$ and the delayed coefficient B_1 appears only in $\mathbf{V}_N$.

6 Computing stability spectra and error analysis

Being able to approximate $T = T(s+r, s)$ for any $s \in \mathbb{I}$ and any $r \geq 0$, we go back now to $T_n = T(t_{n+1}, t_n)$ for each $n = 0, 1, \ldots$, with $t_n = s + n\tau$ as introduced in Sect. 4. What follows is referred to Lyapunov spectra, but it clearly holds unchanged w.r.t. Sacker–Sell spectra. Moreover, here and in the remaining of the section we consider M and N as being given and fixed. Then we can drop them from the notation and write

$$\overline{T}_n := (T_n)_{M,N} \tag{34}$$

and

$$\overline{\mathbf{T}}_n := (\mathbf{T}_n)_{M,N} \tag{35}$$

for simplicity.

In order to compute approximations to some of the exact Lyapunov exponents, let us compare three different levels of the same discrete QR technique:

(QR1) the exact infinite-dimensional one for T_n on X:

$$Q_{n+1}R_n = T_n Q_n, \quad n = 0, 1, \ldots, \tag{36}$$

(QR2) the approximated infinite-dimensional one for $\overline{T}_n$ on X:

$$\overline{Q}_{n+1}\overline{R}_n = \overline{T}_n \overline{Q}_n, \quad n = 0, 1, \ldots, \tag{37}$$

(QR3) the approximated finite-dimensional one for $\overline{\mathbf{T}}_n$ on X_M:

$$\overline{\mathbf{Q}}_{n+1}\overline{\mathbf{R}}_n = \overline{\mathbf{T}}_n\overline{\mathbf{Q}}_n, \quad n = 0, 1, \ldots. \tag{38}$$

The three different QRs lead to three Lyapunov spectra:

$$\begin{aligned} \Sigma_{\text{LE}} = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i], \quad \alpha_i &= \liminf_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([R_k]_{ii}), \\ \beta_i &= \limsup_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([R_k]_{ii}), \end{aligned} \tag{39}$$

$$\begin{aligned} \overline{\Sigma}_{\text{LE}} = \bigcup_{i=1}^{\infty}[\overline{\alpha}_i, \overline{\beta}_i], \quad \overline{\alpha}_i &= \liminf_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([\overline{R}_k]_{ii}), \\ \overline{\beta}_i &= \limsup_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([\overline{R}_k]_{ii}), \end{aligned} \tag{40}$$

$$\begin{aligned} \overline{\boldsymbol{\Sigma}}_{\text{LE}} = \bigcup_{i=1}^{d(M+2)}[\overline{\boldsymbol{\alpha}}_i, \overline{\boldsymbol{\beta}}_i], \quad \overline{\boldsymbol{\alpha}}_i &= \liminf_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([\overline{\mathbf{R}}_k]_{ii}), \\ \overline{\boldsymbol{\beta}}_i &= \limsup_{n\to\infty} \frac{1}{t_n}\sum_{k=0}^{n-1}\log([\overline{\mathbf{R}}_k]_{ii}), \end{aligned} \tag{41}$$

namely the exact one for T_n on X, the approximated one for $\overline{T}_n$ on X and the computable one for $\overline{\mathbf{T}}_n$ on X_M. We show in Sect. 6.2 that finitely-many exact spectral intervals (39) can be approximated by the non-degenerate ones in (40), giving a bound on the relevant errors depending on the bound $\left\|T - T_{M,N}\right\|_{X^1}$ established in Theorem 1. Before we show in Sect. 6.1 that the latter are exactly those in (41), which can thus be effectively computed via the standard finite-dimensional QR method for matrices.

6.1 The computable stability spectra

Our strategy for computing stability spectra for (36), (37) and (38) that will allow for comparison of the exact infinite-dimensional, finite-rank infinite-dimensional, and finite-dimensional approximations is as follows. Choose a random unitary $\overline{Q}_0$ such that $\overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$ is invertible, then (see Lemma 1) for $n = 0, 1, \ldots$,

$$\overline{Q}_{n+1} = \begin{pmatrix} (\overline{Q}_{n+1})_{11} & 0 \\ 0 & I \end{pmatrix}, \quad \overline{R}_{n+1} = \begin{pmatrix} (\overline{R}_{n+1})_{11} & 0 \\ 0 & 0 \end{pmatrix}. \tag{42}$$

Consider the unique sequential QR-factorization (37) in the infinite-dimensional state space X and (38) in the finite-dimensional state space X_M. We recall from Remark 1 that being $r = \tau$ each $\overline{T}_n$ in (34) has finite rank. Moreover, when $\overline{T}_n$ is intended as its infinite-dimensional matrix representation on X according to what established in Sect. 3, then (30) and (31) lead to

$$\overline{\mathbf{T}}_n = \hat{S}_M \overline{T}_n \hat{P}_M$$

and hence to

$$\overline{T}_n = \begin{pmatrix} \overline{\mathbf{T}}_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{43}$$

Lemma 1 *If for unitary $\overline{Q}_0$, $\overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$ is invertible, then for $n = 0, 1, \ldots$, $\overline{Q}_{n+1}$, $\overline{R}_{n+1}$ have the form in (42).*

Proof For general unitary $\overline{Q}_0$ we have from (43) that

$$\overline{T}_0 \overline{Q}_0 = \begin{pmatrix} \overline{\mathbf{T}}_0(\overline{Q}_0)_{11} & \overline{\mathbf{T}}_0(\overline{Q}_0)_{12} \\ 0 & 0 \end{pmatrix}.$$

Thus, from (37), $(\overline{Q}_1)_{11}(\overline{R}_0)_{11} = \overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$, $(\overline{Q}_1)_{21} = 0$, $(\overline{Q}_1)_{12} = 0$, $(\overline{Q}_1)_{22} = I$, and $(\overline{R}_0)_{12} = (\overline{Q}_1)^*_{11}\overline{\mathbf{T}}_0(\overline{Q}_0)_{12}$ (* denotes the adjoint). Subsequently,

$$\overline{Q}_1 = \begin{pmatrix} (\overline{Q}_1)_{11} & 0 \\ 0 & I \end{pmatrix},$$

and the conclusion of the lemma follows from a straightforward calculation. □

It is then clear that for arbitrary unitary $\overline{Q}_0$ with $\overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$ invertible, we obtain $(\overline{R}_{n+1})_{11} = \overline{\mathbf{R}}_{n+1}$ and $(\overline{Q}_{n+1})_{11} = \overline{\mathbf{Q}}_{n+1}$ for all $n = 0, 1, \ldots$. We can thus state the following result.

Theorem 2 *Let $m := d(M+2)$. The first m Lyapunov and Sacker–Sell intervals relevant to the finite-rank approximation $\{\overline{T}_n\}_{n=0}^{\infty}$ in (34) coincide with those relevant to the finite-dimensional approximation $\{\overline{\mathbf{T}}_n\}_{n=0}^{\infty}$ in (35), while the remaining ones are clustered at $-\infty$:*

$$[\overline{\alpha}_i, \overline{\beta}_i] = \begin{cases} [\overline{\alpha}_i, \overline{\beta}_i] & \text{for } i = 0, 1, \ldots, m-1, \\ \{-\infty\} & \text{for } i = m, m+1, \ldots, \end{cases}$$

$$[\overline{\gamma}_i, \overline{\delta}_i] = \begin{cases} [\overline{\gamma}_i, \overline{\delta}_i] & \text{for } i = 0, 1, \ldots, m-1, \\ \{-\infty\} & \text{for } i = m, m+1, \ldots. \end{cases}$$

We next state a lemma (for a proof see [29, Theorem 3.1]) that provides the starting point for our error analysis. We will assume that for all n, $\overline{T}_n = T_n + H_n$ where norm bounds on the H_n are obtained in Sect. 5.1, in particular Theorem 1.

Lemma 2 *If for unitary $\overline{Q}_0 = Q_0$, $\overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$ is invertible, then*

$$Q_{n+1} R_n \cdots R_0 = \overline{Q}_{n+1}[\overline{R}_n + E_n] \cdots [\overline{R}_0 + E_0]$$

where $E_k = -\overline{Q}_{k+1}^{*} H_k \overline{Q}_k$ for $k = 0, 1, \ldots, n$.

The error analysis in the next section involves showing that there exists a sequence $\{\tilde{Q}_n\}_{n=0}^{\infty}$ of near identity unitary matrices with $\tilde{Q}_0 = I$ such that

$$\tilde{Q}_{n+1}\tilde{R}_n = [\overline{R}_n + E_n]\tilde{Q}_n, \quad n = 0, 1, \ldots, \tag{44}$$

where the $\tilde{R}_n$ are block upper triangular depending on the integral separation structure for the particular problem.

6.2 Error in stability spectra

In this section we develop a perturbation theory for Lyapunov exponents and Sacker–Sell intervals of the infinite-dimensional approximation of the sequence $\{T_n\}_{n=0}^{\infty}$ defined in (9) with the sequence $\{\overline{T}_n\}_{n=0}^{\infty}$ defined in (34). This follows the development of the perturbation theory or error analysis begun in [29] and [30], and subsequently refined in [1,32,58]. In what follows we summarize the results obtained in [1].

We will assume that the diagonal elements of the sequence of upper triangular operators satisfy one of the following for $i > j$:

– there exists $0 < \Omega_{ij} \leq 1$ and $\alpha_{ij} > 1$ s.t.

$$\prod_{k=s}^{t} \frac{[\overline{R}_k]_{jj}}{[\overline{R}_k]_{ii}} \geq \Omega_{ij}\alpha_{ij}^{t-s+1}, \quad t \geq s \geq 0, \tag{45}$$

– for every $\alpha > 1$, there exists $M_{ij} = M_{ij}(\alpha) > 1$ s.t.

$$\frac{1}{M_{ij}}\alpha^{-(t-s+1)} \le \prod_{k=s}^{t} \frac{[\overline{R}_k]_{ii}}{[\overline{R}_k]_{jj}} \le M_{ij}\alpha^{t-s+1}, \quad t \ge s \ge 0. \tag{46}$$

We define the set of integral separation indices as

$$IS = \{i > j \ s.t. \ (45) \ holds\}.$$

and note that if $(i, j) \in IS$, then

$$\sum_{l=1}^{n}\prod_{k=l}^{n} \frac{[\overline{R}_k]_{ii}}{[\overline{R}_k]_{jj}} \le \left[\frac{1}{\Omega_{ij}}\left(\alpha_{ij}^{-1} + \cdots + \alpha_{ij}^{-n}\right)\right] \le \left[\frac{1}{\Omega_{ij}(\alpha_{ij}-1)}\right] =: \Lambda_{ij} - 1.$$

In order to obtain sharp bounds we will pose the zero finding problem (44) in a Banach space with a weighted norm. We proceed by defining a weighted norm as follows: for a sequence of operators $Z = \{Z_n\}_{n=0}^{\infty}$ define a Banach space B using the norm

$$\|Z\|_B := \sup_n \sup_{i,j} |\omega_{ij}^n [Z_n]_{ij}| \tag{47}$$

where $[Z_n]_{ij} = \langle Z_n\Phi_j, \Phi_i\rangle_X$ and ω_{ij}^n are bounded positive weights.

To obtain bounds we apply in [1] the Newton–Kantorovich Theorem to the zero finding problem $G(x) \equiv (G_1(x), G_2(x), G_3(x)) = 0$ with initial guess $Q_n = I$ for all n and

$$\begin{aligned}
\left(G_1(\{Q_k\}_{k=0}^{\infty})\right)_n &= \mathtt{low}_{IS}\left(Q_{n+1}^*[\overline{R}_n + E_n]Q_n\right),\\
\left(G_2(\{Q_k\}_{k=0}^{\infty})\right)_n &= \mathtt{upp}\left(Q_{n+1}Q_{n+1}^* - I\right),\\
\left(G_3(\{Q_k\}_{k=0}^{\infty})\right)_n &= \mathtt{rdiag}\left(Q_{n+1}Q_{n+1}^* - I\right),
\end{aligned}$$

where $(\mathtt{low}_{IS}(A))_{ij} := A_{ij}$ for $(i, j) \in IS$ and $(\mathtt{low}_{IS}(A))_{ij} := 0$ for $(i, j) \notin IS$. Similarly, $(\mathtt{upp}(A))_{ij} := A_{ij}$ for $i < j$ and $(\mathtt{upp}(A))_{ij} := 0$ for $i \ge j$. Lastly, $(\mathtt{rdiag}(A))_{ij} := \Re(A_{ii})\delta_{ij}$, where δ_{ij} is Kronecker's delta. The dependence on the integral separation structure, IS, means that we are seeking a sequence of near identity orthogonal matrices that bring the perturbed triangular system to block upper triangular where a non-trivial block corresponds to a lack of integral separation.

The exact Jacobian DG is given by

$$\begin{aligned}
\left((DG_1\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \mathtt{low}_{IS}\left(V_{n+1}^*[\overline{R}_n + E_n]Q_n + Q_{n+1}^*[\overline{R}_n + E_n]V_n\right),\\
\left((DG_2\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \mathtt{upp}\left(V_{n+1}Q_{n+1}^* + Q_{n+1}V_{n+1}^*\right),\\
\left((DG_3\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \mathtt{rdiag}\left(V_{n+1}Q_{n+1}^* + Q_{n+1}V_{n+1}^*\right),
\end{aligned}$$

where V is an arbitrary element of the space B. We approximate DG by Γ:

$$\begin{aligned}
\left((\Gamma_1\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \texttt{low}_{IS}\left(V_{n+1}^* D_n Q_n + Q_{n+1}^* D_n V_n\right),\\
\left((\Gamma_2\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \texttt{upp}\left(V_{n+1} Q_{n+1}^* + Q_{n+1} V_{n+1}^*\right),\\
\left((\Gamma_3\{Q_k\}_{k=0}^{\infty})(V)\right)_n &= \texttt{rdiag}\left(V_{n+1} Q_{n+1}^* + Q_{n+1} V_{n+1}^*\right),
\end{aligned}$$

where $D_n = \operatorname{diag}(\overline{R}_n)$ for all n. Notice that $\Gamma_2 = DG_2$ and $\Gamma_3 = DG_3$.

We next state the main error analysis result which is an application of a Newton–Kantorovich theorem with perturbed Jacobian to the system of nonlinear equations we have just defined. Subsequently in Lemma 3 we summarize the bounds, obtained in [1], on those quantities (η, δ, and K) that are necessary to apply the Newton–Kantorovich theorem and hence obtain the following result.

Theorem 3 [1] *Suppose that, for a sequence of operators $\{\overline{T}_n\}_{n=0}^{\infty}$ acting on a Hilbert space, the discrete QR process is well-defined and that the integral separation structure results in finite-dimensional blocks except possibly for an infinite-dimensional block corresponding to Lyapunov exponents of $-\infty$. Employing the bounds on η, δ, and K as in Lemma 3, we obtain the following.*

If $\delta < 1$ and $h := \frac{\eta K}{(1-\delta)^2} < \frac{1}{2}$, then the conclusion of the Newton–Kantorovich Theorem holds. In particular, there exists a sequence of operators $\{\tilde{Q}_n\}_{n=0}^{\infty}$ with $\tilde{Q}_0 = I$ s.t. (44) holds and

$$\|\{\tilde{Q}_n\}_{n=0}^{\infty} - I\|_B \le r_0 := \frac{2\eta}{(1-\delta)(1+\sqrt{1-2h})}.$$

Furthermore, if $r_0 < 1$, then the operators $\tilde{Q}_n$ will be invertible, and hence unitary for all n.

Lemma 3 [1] *We have the following bounds on η, δ, and K:*

$$\begin{aligned}
\eta &\le \sup_{(i,j)\in IS} \Lambda_{ij}\omega_{ij}\frac{|E_{ij}|}{\overline{R}_{jj}}, \quad \frac{|E_{ij}|}{\overline{R}_{jj}} := \sup_n \frac{|[E_n]_{ij}|}{[\overline{R}_n]_{jj}},\\
\delta &\le \sup_{(i,j)\in IS}\left(\Lambda_{ij}\omega_{ij}\frac{|W_{ij}|}{\overline{R}_{jj}}\right), \quad \frac{|W_{ij}|}{\overline{R}_{jj}} := \sup_n \frac{|[W_n]_{ij}|}{[\overline{R}_n]_{jj}},\\
K &\le \max\{K_1, K_2, K_3\},
\end{aligned}$$

where $[W_n]_{ij} := \left[((G_1'(x_0) - \Gamma_1)V)_n\right]_{ij}$, with V satisfying $\|V\|_B = 1$, $K_1 := \max_{i\ge -d}\omega_{ii}^n K_{ii}$, $K_2 := \max_{(i,j)\in IS}\omega_{ij}^n\Lambda_{ij}K_{ji}$, $K_3 := \max_{(i,j)\in IS}\omega_{ij}^n\Lambda_{ij}K_{ij}$, $K_{ii} := \sum_{k=-d}^{\infty}(\omega_{ik}^n)^{-2}$ for all i, and for $(i,j)\in IS$,

$$\begin{aligned}
K_{ji} := \sup\nolimits_n \frac{2}{[\overline{R}_n]_{jj}} &\sum_{k=-d}^{\infty}\left(\omega_{ik}^n\right)^{-1}\left(\omega_{jk}^n\right)^{-1}\\
&+2\sum\nolimits_{k=-d}^{\infty}\sum\nolimits_{l=-d}^{\infty}\left(\omega_{ki}^n\right)^{-1}\left(|[\overline{R}_n]_{kl}| + |[E_n]_{kl}|\right)\left(\omega_{lj}^n\right)^{-1}
\end{aligned}$$

and

$$K_{ij} := \sup_n 2 \sum_{k=-d}^{\infty} \left(\omega_{ik}^n\right)^{-1} \left(\omega_{jk}^n\right)^{-1} + 2 \sum_{k=-d}^{\infty} \sum_{l=-d}^{\infty} \left(\omega_{ki}^n\right)^{-1} \left(|[\overline{R}_n]_{kl}| + |[E_n]_{kl}|\right) \left(\omega_{lj}^n\right)^{-1}.$$

Finally, to obtain bounds on the upper Lyapunov exponents (and similarly for the lower ones and for the Sacker–Sell endpoints) we have the following, where β_i is the i-th exact exponent and $\overline{\beta}_i$ the i-th computed one.

Proposition 2 [1] *Let* $|[Q_k - I]_{ij}| < \rho_{ij}$ *for all* i, j *and for all* k. *If* $(i, i-1) \in IS$ *and* $(i+1, i) \in IS$, *then*

$$|\beta_i - \overline{\beta}_i| = \limsup_{n\to\infty} \frac{1}{t_n} \sum_{k=0}^{n-1} \log \left| \frac{[\tilde{R}_k]_{ii}}{[\overline{R}_k]_{ii}} \right| \leq \limsup_{n\to\infty} \frac{1}{t_n} \sum_{k=0}^{n-1} \log(1 + Y_{k,i}),$$

where $Y_{k,i} := \left| \frac{[E_k]_{ii}}{[\overline{R}_k]_{ii}} \right| + \sum_{r,s=-d}^{\infty} \rho_{ri}\rho_{si} \left| \frac{[\overline{R}_k + E_k]_{rs}}{[\overline{R}_k]_{ii}} \right| + \sum_{r=-d}^{\infty} \rho_{ri} \left| \frac{[\overline{R}_k + E_k]_{ri} + [\overline{R}_k + E_k]_{ir}}{[\overline{R}_k]_{ii}} \right|$.

If $(i, i-1) \notin IS$ *or* $(i+1, i) \notin IS$, *then for any* $\alpha > 1$,

$$\limsup_{n\to\infty} \frac{1}{t_n} \left(\sum_{k=0}^{n-1} \frac{d_{\alpha,u}}{\alpha} \|F_k\|_B \right) - \log(\alpha) \leq \overline{\beta}_i - \beta_i \leq \log(\alpha) + \limsup_{n\to\infty} \frac{1}{t_n} \left(\sum_{k=0}^{n-1} \frac{D_{\alpha,U}}{\alpha} \|F_k\|_B \right)$$

with

$$\|F_n\|_B \leq \left(\|\rho\|^2 + 2\|\rho\|\right) \|\overline{R}_n + E_n\|_B + \|E_n\|_B,$$

where $\|\cdot\|_B$ *is the operator supremum norm we defined in (47), and for* M_α *the maximum of the* $M_{ij}(\alpha)$ *in the non-integrally separated block,*

$$d_{\alpha,u} := \min\left\{ \frac{\alpha - 1}{M_\alpha \Omega}, \frac{1}{M_\alpha}, \alpha \right\}, \quad D_{\alpha,U} := \max\left\{ 1, M_\alpha, \Omega \frac{1 + M_\alpha}{\alpha - 1} \right\}$$

with $\frac{|[\overline{R}_n]_{ij}|}{[\overline{R}_n]_{ii}} \leq \Omega$ *for all* n *and for all* $i < j$.

Remark 7 Proposition 2 provides bounds on the approximation of Lyapunov exponents and in a similar fashion one may obtain bounds on the approximation of the endpoints of Sacker–Sell spectral intervals. Note that as the error in the approximation of the local evolution operators tends to zero uniformly, then so does the error in the approximation of the Lyapunov exponents. We cannot generally claim that we are approximating the dominant Lyapunov exponents of the infinite dimensional system. This is consistent with what occurs in finite dimensions. A simple example illustrates the difficulty. Consider the constant coefficient linear ODE

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x.$$

If we make a one-dimensional approximation with $\overline{Q}_0 \equiv (\overline{Q}_0)_{11} = 1$, then for $\overline{\mathbf{T}}_0(\overline{Q}_0)_{11}$ invertible in Lemma 1, one will approximate the non-dominant Lyapunov exponent -1. In fact, if one approximates both Lyapunov exponents for this ODE but chooses $Q_0 = \pm I$, the two-dimensional identity matrix, then one will obtain the Lyapunov exponents in reverse order which implies that the integral separation condition will not hold. Since we infer integral separation from the finite dimensional approximation, a lack of integral separation may suggest that the system in question should be considered in a different coordinate system to obtain good results. Such preprocessing of a given system may also help to obtain approximations of the dominant Lyapunov exponents.

6.3 Application of the error analysis

Due to the form of $\overline{T}_n$ and $\overline{R}_n$ in (43) and (42), respectively, the integral separation set IS takes the form $IS = IS_f \cup IS_i$ where IS_f contains the integral separation structure in the finite part as determined by $(R_n)_{11} \equiv \overline{\mathbf{R}}_n$, while IS_i denotes the degenerate integral separation between $(R_n)_{11}$ and $(R_n)_{22} \equiv 0$, i.e., $IS_i := \{(i, j) : i > m, j \leq m\}$. Thus, application of the error analysis relies upon quantities that may be computed from the finite part $\overline{\mathbf{R}}_n$ as well as bounds or estimates on the error in approximating T_n by the finite-dimensional $\overline{\mathbf{T}}_n$.

7 Numerical tests

All the tests reported in the following sections have been performed on a MacBook Pro (2.53 GHz Intel Core 2 Duo, 4 GB memory) by using Matlab codes implementing the algorithm according to Sect. 5.2 with reference to model (4). By recalling Remark 6, it is assumed that $r = \tau$ and $M = N$. Moreover, the limits (41) are truncated at $n = \bar{n}$. Tests concern both linear and nonlinear dynamics. For the latter, reference trajectories are computed in advance up to time $t_{\bar{n}}$ via the Matlab solver for delay differential equations `dde23`, [54].

7.1 Test 1: a linear autonomous equation

Due to the lack of knowledge of exact Lyapunov exponents for models of RFDEs, we first address the case of a linear autonomous equation, namely (4) with $d = 1$, $k = 1$, $A(t) = -1$, $B_1(t) = 2e$, $\tau_1 = 1$ and no distributed delay terms:

$$x'(t) = -x(t) + 2ex(t-1).$$

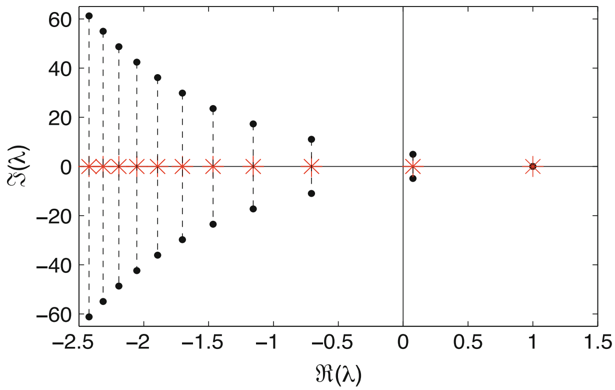

Fig. 1 Rightmost eigenvalues of Test 1, computed to machine precision with the method in [11] (*black dots*); approximated exponents for $N = 20$ and $t_{\bar{n}} = 100{,}000$ (*red stars*) (color figure online)

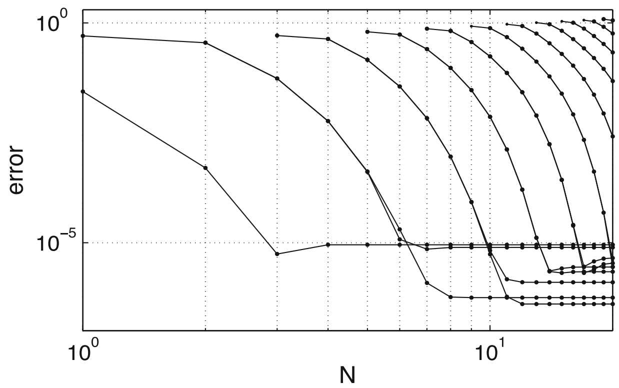

Fig. 2 Error for the first exponents of Test 1 w.r.t. N for fixed $t_{\bar{n}} = 100{,}000$

The exponents are the real part of the eigenvalues of the infinitesimal generator of the relevant solution operator semigroup. Since spectrally accurate methods to compute these eigenvalues are consolidated (see e.g. [11]), we have some "exact" (i.e. to machine precision) counterpart to compare with. Moreover, it is easy to show that the rightmost eigenvalue is $\lambda_1 = 1$.

The rightmost eigenvalues are shown in Fig. 1 (black dots), their real parts are taken as "exact" values for the exponents, whose approximations are also shown for $N = 20$ and $t_{\bar{n}} = 100{,}000$ (red stars).

Figure 2 shows the error for the first exponents for fixed $t_{\bar{n}} = 100{,}000$ and varying N from 1 to 20 with unit step. The spectral behavior w.r.t. N is evident, confirming Theorem 1, Remark 4 and the error analysis in Sect. 6.2, at least until a lower barrier of about 10^{-5}.

The barrier is due to the truncation of the limits (41) at $\bar{n}$. In fact, by considering the exponential boundedness of the evolution family (property (T5) in Sect. 2), and hence that of a (normal) fundamental solution, say $F(t)$, then from the definition of Lyapunov exponents for RFDEs as given in [10, Theorem 13], it can be proven that

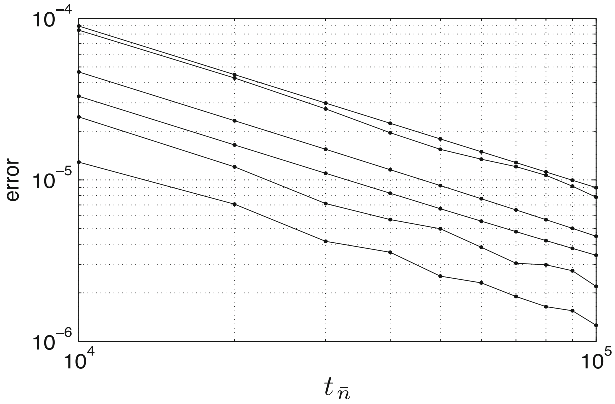

Fig. 3 Error for some of the first exponents of Test 1 w.r.t. $t_{\bar{n}}$ for fixed $N = 20$

$$\left| \frac{1}{t_{\bar{n}}} \log \|F(t_{\bar{n}})\|_X - \lambda \right| = O\left(\frac{1}{t_{\bar{n}}}\right)$$

for any $\bar{n}$ sufficiently large. Figure 3 shows the error for some of the computed exponents for fixed $N = 20$ and varying $t_{\bar{n}}$ from 10,000 to 100,000 with step 10,000, confirming what above.

As to give an idea of the computational time, running the code for Test 1 with $N = 20$ and $t_{\bar{n}} = 100{,}000$ took about 15 min. It is worthy noticing that the overall computational time is, in general, $O(\bar{n} \cdot N)$: the cubic cost of the QR step is in fact negligible w.r.t. the total time since N is very small thanks to the spectral accuracy.

7.2 Test 2: a linear time-periodic system

We consider the so-called delayed-damped Mathieu equation (see e.g. [40] and the references therein), important in analyzing tool vibrations in machining:

$$y''(t) + \kappa y'(t) + c(t)y(t) = by(t-1)$$

with

$$c(t) = c_0 + c_1 \cos\left(\frac{2\pi t}{\omega}\right).$$

W.r.t. (4) we have $d = 2$, $k = 1$,

$$A(t) = \begin{pmatrix} 0 & 1 \\ -c(t) & -\kappa \end{pmatrix},$$

$$B_1(t) = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix},$$

$\tau_1 = 1$ and absence of distributed delay terms. In the spirit of Test 1, "exact" Lyapunov exponents can be recovered similarly from the characteristic multipliers, computed to

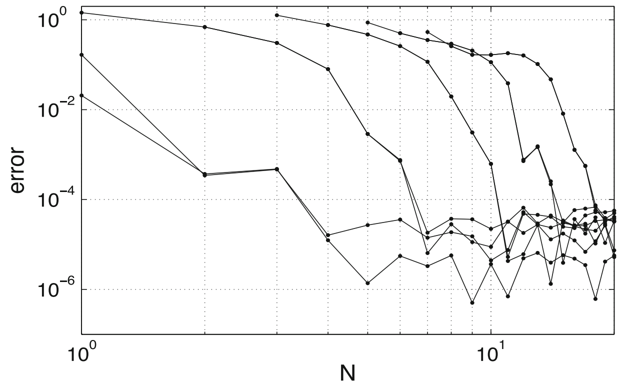

Fig. 4 Error for the first exponents of Test 2 w.r.t. N for fixed $t_n = 100{,}000$

machine precision (see e.g. the recent methodology proposed in [14]). Figure 4 shows the error for the first exponents for fixed $t_{\bar{n}} = 100{,}000$ and varying N from 1 to 20 with unit step. The following values of the model coefficients were used: $\kappa = 0.2$, $c_0 = 1$, $c_1 = 2$ and $b = -1.5$. The results further confirm the spectral convergence of the proposed technique.

7.3 Test 3: multiple distributed delays

In order to show that the method works correctly also for the case of multiple and distributed delays, we consider (4) with $d = 2$, $k = 4$,

$$A(t) = \begin{pmatrix} -3 & 1 \\ -24.646 & -35.430 \end{pmatrix},$$

$$B_1(t) = B_2(t) = B_3(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_4(t) = \begin{pmatrix} 1 & 0 \\ 2.35553 & 2.00365 \end{pmatrix},$$

$$C_1(t) = C_3(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_2(t) = \begin{pmatrix} 2 & 2.5 \\ 0 & -0.5 \end{pmatrix}, \quad C_4(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\tau_1 = 0.1$, $\tau_2 = 0.3$, $\tau_3 = 0.5$ and $\tau_4 = 1$, [12,36]:

$$\begin{cases} x_1'(t) = -3x_1(t) + x_2(t) + x_1(t-1) + \displaystyle\int_{-0.3}^{-0.1} [2x_1(t+\theta) + 2.5x_2(t+\theta)]d\theta \\ \qquad\qquad - \displaystyle\int_{-1}^{-0.5} x_1(t+\theta)d\theta \\ x_2'(t) = -24.646x_1(t) - 35.430x_2(t) + 2.35553x_1(t-1) + 2.00365x_2(t-1) \\ \qquad\qquad -0.5 \displaystyle\int_{-0.3}^{-0.1} x_2(t+\theta)d\theta - \int_{-1}^{-0.5} x_2(t+\theta)d\theta. \end{cases}$$

Table 1 First 11 exponents of Test 3 computed with $N = 15$ and varying $t_{\bar{n}} = 100$ (second column), 1,000 (third column), 10,000 (fourth column) compared with the eigenvlues computed to machine precision with the method in [11] (fifth column)

	$t_{\bar{n}} = 100$		$t_{\bar{n}} = 1{,}000$		$t_{\bar{n}} = 10{,}000$		Eigenvalues
	Value	Error	Value	Error	Value	Error	Real part
λ_1	-1.2534	7.2×10^{-3}	-1.2467	5.0×10^{-4}	-1.2464	1.1×10^{-4}	-1.2462
λ_2	-1.7124	1.4×10^{-2}	-1.6994	8.3×10^{-4}	-1.6987	9.4×10^{-5}	-1.6986
λ_3	-1.7097	1.1×10^{-2}	-1.6997	1.1×10^{-3}	-1.6987	8.0×10^{-5}	-1.6986
λ_4	-2.3598	1.1×10^{-2}	-2.3708	3.0×10^{-4}	-2.3711	3.4×10^{-5}	-2.3711
λ_5	-2.3735	2.5×10^{-3}	-2.3711	2.2×10^{-5}	-2.3711	1.3×10^{-5}	-2.3711
λ_6	-2.8346	2.1×10^{-2}	-2.8552	3.4×10^{-5}	-2.8552	7.6×10^{-5}	-2.8551
λ_7	-2.8618	6.7×10^{-3}	-2.8554	2.4×10^{-4}	-2.8553	1.3×10^{-4}	-2.8551
λ_8	-2.8757	2.1×10^{-3}	-2.8720	1.7×10^{-3}	-2.8735	1.3×10^{-4}	-2.8737
λ_9	-2.8758	2.1×10^{-3}	-2.8726	1.0×10^{-3}	-2.8736	6.7×10^{-5}	-2.8737
λ_{10}	-2.8869	6.7×10^{-3}	-2.8944	7.0×10^{-4}	-2.8935	2.0×10^{-4}	-2.8936
λ_{11}	-2.8953	1.7×10^{-3}	-2.8947	1.0×10^{-3}	-2.8938	1.5×10^{-4}	-2.8936

Similar experiments as for Test 1 and Test 2 were performed and some results are listed in Table 1, confirming the error behavior w.r.t. $t_{\bar{n}}$ for fixed N.

7.4 Test 4: the Mackey–Glass equation

As a first nonlinear case, we consider the Mackey–Glass equation for modeling blood production [44]

$$x'(t) = \frac{ax(t-\tau)}{1 + [x(t-\tau)]^c} - bx(t),$$

among the few models for which results about Lyapunov exponents exist in the literature (e.g. [35,55]). Linearization around a reference trajectory x^* leads to (4) with $d = 1$, $k = 1$,

$$\begin{aligned} A(t) &= -b, \\ B_1(t) &= \frac{a[1 + (1-c)x^*(t-\tau_1)^c]}{\{1 + [x^*(t-\tau_1)]^c\}^2}, \end{aligned}$$

$\tau_1 = \tau$ and no distributed delay terms. Experiments were run for $a = 0.2$, $b = 0.1$, $c = 10$ and $\tau = 50$, see e.g. [55]. Moreover, the reference trajectory x^* was computed by `dde23.m` with starting time $s = 0$ and for a constant initial data $\varphi(\theta) = 2$, $\theta \in [-50, 0]$, and $u = \varphi(0)$. Figure 5 shows the first six exponents obtained with $N = 20$ and $t_{\bar{n}}$ up to 100,000. The relevant values at $t_{\bar{n}} = 100{,}000$ are collected in Table 2, together with the results in [55]. Let us notice that the discrepancy between

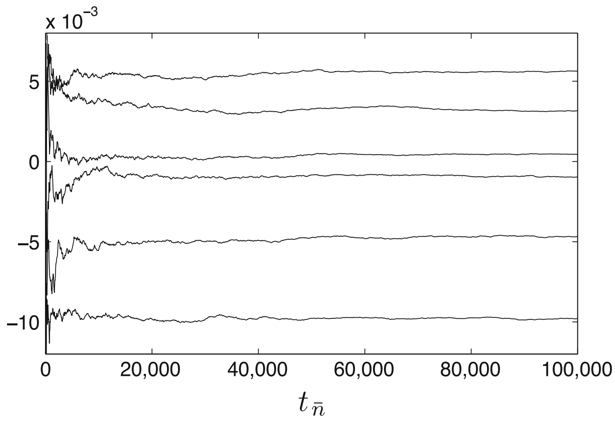

Fig. 5 First six exponents of the Mackey–Glass equation computed with $N = 20$ and $t_{\bar{n}} = 100{,}000$

Table 2 First six exponents of the Mackey–Glass equation computed with $N = 20$ and $t_{\bar{n}} = 100{,}000$ (second column) and from [55] (third column)

λ_1	5.76×10^{-3}	5.83×10^{-3}
λ_2	3.02×10^{-3}	3.15×10^{-3}
λ_3	0.65×10^{-3}	0.01×10^{-3}
λ_4	-0.85×10^{-3}	-0.29×10^{-3}
λ_5	-4.78×10^{-3}	-5.08×10^{-3}
λ_6	-9.85×10^{-3}	-9.78×10^{-3}

the different approximated values in Table 2 is of the order of 10^{-4} for all the six exponents. As explained in Test 1, this barrier is due to the truncation of the limits (41). However, it can be observed that the difference seems larger for exponents with smaller absolute value, a fact that certainly deserves further investigation.

7.5 Test 5: delay-coupled Rössler oscillators

We consider the system of two Rössler oscillators symmetrically coupled

$$\begin{cases} x_1'(t) = -x_2(t) - x_3(t), \\ x_2'(t) = x_1(t) + ax_2(t) + \epsilon[x_5(t-\tau) - x_2(t)], \\ x_3'(t) = b + x_3(t)[x_1(t) - c], \\ x_4'(t) = -x_5(t) - x_6(t), \\ x_5'(t) = x_4(t) + ax_5(t) + \epsilon[x_2(t-\tau) - x_5(t)], \\ x_6'(t) = b + x_6(t)[x_4(t) - c], \end{cases} \tag{48}$$

which has been recently studied to analyze the phenomenon of amplitude death that can occur in chaotic dynamical systems when coupled with time-delay, [50]. Linearization around a reference trajectory $x^* = (x_1^*, \ldots, x_6^*)^T$ leads to (4) with $d = 6$, $k = 1$,

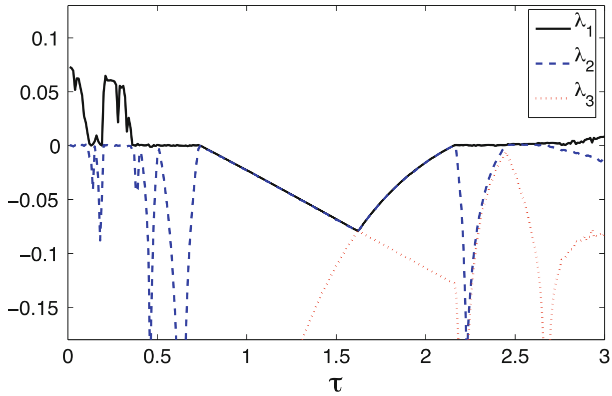

Fig. 6 First three exponents of the coupled Rössler oscillators (48) computed with $N = 15$ and $t_{\bar{n}} = 10{,}000$ for τ varying from 0.01 to 3 with step 0.01

$$A(t) = \begin{pmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 1 & a-\epsilon & 0 & 0 & 0 & 0 \\ x_3^*(t) & 0 & x_1^*(t)-c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & a-\epsilon & 0 \\ 0 & 0 & 0 & x_6^*(t) & 0 & x_4^*(t)-c \end{pmatrix},$$

$$B_1(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\tau_1 = \tau$ and no distributed delay terms. The model coefficients are given the values $a = b = 0.1$, $c = 14$ and $\epsilon = 0.5$, while the coupling delay τ is considered as a varying parameter. Moreover, the reference trajectory x^* is computed by `dde23.m` with starting time $s = 0$ and for a constant initial data $\varphi(\theta) = v \in \mathbb{R}^6, \theta \in [-\tau, 0]$, v a (pseudo)random vector, and $u = \varphi(0)$. In order to show the correctness of the method, we reproduce in Fig. 6 the results given in [50, Figure 1(b)]. The correspondence is good, although we find chaotic regime for values of τ above about 2.6: indeed, trajectories simulated with `dde23` confirmed the presence of chaotic attractors in this parameter region.

Acknowledgments The authors wish to thank Luca Dieci at GeorgiaTech (USA) for stimulating discussions on the subject of Lyapunov exponents and Wei Zou at Humboldt University (Germany) for suggesting the problem treated in Test 5.

Appendix A

Besides the notation introduced in Sect. 3, we recall that G_s is defined in (11), V in (21) and $\mathscr{L}_N^+$ is specified right before Proposition 1. We first recall in Theorem 4 basic approximation results from [15], namely (9.4.6) and (9.4.24), and then prove some technical lemmas which are used in Sect. 5.1.

Theorem 4 *Let $a < b \in \mathbb{R}$, $L := L^2(a, b; \mathbb{R}^d)$ and $H^k := H^k(a, b; \mathbb{R}^d)$, $k \geq 1$ a positive integer. Given positive integers M and N, if $\varphi \in H^k$, then there exists a constant c_F independent of M s.t.*

$$\|\varphi - F_M\varphi\|_L \leq c_F M^{-k}\|\varphi\|_{H^k} \tag{49}$$

where F_M is the Fourier projection operator w.r.t. the Legendre polynomials of L and a constant $c_{\mathscr{L}}$ independent of N s.t.

$$\|\varphi - \mathscr{L}_N\varphi\|_L \leq c_{\mathscr{L}} N^{\frac{1}{2}} N^{-k}\|\varphi\|_{H^k} \tag{50}$$

where $\mathscr{L}_N$ is the interpolation operator w.r.t. the Legendre-Gauss zeros of $[a, b]$.

Lemma 4 $\|V\|_{L^\pm \leftarrow L^+} \leq r$.

Proof Let $y \in L^+$. Then

$$\|Vy\|_{L^\pm} = \int_{-\tau}^{r} |(Vy)(t)|^2\, dt = \int_0^r \left| \int_0^t y(\sigma) d\sigma \right|^2 dt \leq \int_0^r \left(\int_0^r |y(\sigma)|^2\, d\sigma \right) dt = r\|y\|_{L^+}.$$

□

Lemma 5 $(I_{L^+} - G_s V)^{-1} \in \mathscr{B}(L^+)$.

Proof The assertion corresponds to proving that for any given $h \in L^+$ there exists a unique $f \in L^+$ solution of $(I_{L^+} - G_s V)f = h$. This in turn corresponds to the existence and uniqueness of the solution y to the IVP

$$\begin{cases} y'(t) = (G_s y)(t) + h(t), & \text{a.e. for } t \in [0, r], \\ y(0) = 0, & \\ y(\theta) = 0, & \text{a.e. for } \theta \in [-\tau, 0). \end{cases}$$

The latter follows from standard results on RFDEs, see e.g. [25]. □

Lemma 6 *Assume $G_s : H^{1,\pm} \to H^{1,+}$. Then $\left\|(I_{L^+} - \mathscr{L}_N^+)G_s V\right\|_{L^+} \to 0$ as $N \to \infty$.*

Proof Let $y \in L^+$. Then (21) implies $Vy \in H^{1,\pm}$ and, by the hypothesis, $G_s Vy \in H^{1,+}$. Now apply (50) in Theorem 4. □

Lemma 7 *Assume $G_s : H^{1,\pm} \to H^{1,+}$. Then, for sufficiently large N, $(I_{L^+} - \mathscr{L}_N^+ G_s V)^{-1} \in \mathscr{B}(L^+)$ and*

$$\left\|(I_{L^+} - \mathscr{L}_N^+ G_s V)^{-1}\right\|_{L^+} \leq 2\left\|(I_{L^+} - G_s V)^{-1}\right\|_{L^+}.$$

Proof The thesis follows by applying the Banach's Perturbation Lemma, Lemmas 5 and 6 since $I_{L^+} - \mathscr{L}_N^+ G_s V = (I_{L^+} - G_s V) + (I_{L^+} - \mathscr{L}_N^+)G_s V$. □

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