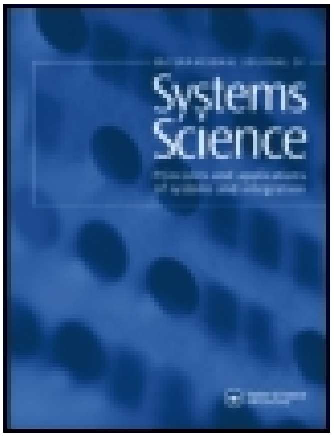


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# A second order singular linear system arising in electric power systems analysis†

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# A second order singular linear system arising in electric power systems analysis†

STEPHEN L. CAMPBELL‡ and NICHOLAS J. ROSE‡

The linear system $\mathbf{M}\ddot{\mathbf{x}}+(\alpha\mathbf{M}+\beta\mathbf{K})\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{f}$, with $\mathbf{M}$ singular, is decomposed into two linear systems, one singular and one non-singular. The singular one is solved explicitly in terms of $\mathbf{M}$, $\mathbf{K}$, $\mathbf{f}$, $\alpha$, $\beta$ and analysed.

## 1. Introduction

In their study of large-scale interconnected electric power systems Bahar and Kwatny (1980) derived a linearized equation of motion for the dynamics

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{f} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ are $n\times n$ square matrices and $\mathbf{f}$ is a vector valued function. The vector $\mathbf{x}$ represents perturbations in the generator rotor speeds and generator angle deviation from synchronous reference. The perturbations are with respect to steady-state reference values. The damping matrix $\mathbf{C}$ is assumed to have the form of Rayleigh damping, which is the usual assumption in structural dynamics. That is

$$\mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K} \tag{2}$$

for scalars $\alpha$, $\beta$. In this case (1) becomes

$$\mathbf{M}\ddot{\mathbf{x}}+(\alpha\mathbf{M}+\beta\mathbf{K})\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=\mathbf{f} \tag{3}$$

Frequently, in studying structural dynamics and vibrations the matrices $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$ in (1) are symmetric. However in Bahar and Kwatny (1980) because of restoring forces and the non-conservative nature of circulatory forces due to transfer inductances, $\mathbf{M}$, $\mathbf{K}$ are not only not necessarily symmetric, but are fairly arbitrary. Thus many of the standard results on (1) do not apply.

Equation (3) is studied in Bahar and Kwatny (1980) under the assumption that $\mathbf{M}$ is invertible. However, it is possible that $\mathbf{M}$ could be nearly singular, or depend on certain small parameters $\epsilon_i$. If a singular perturbations approach is to be used, it is necessary to understand the reduced order model ($\epsilon_i=0$).

Alternatively, (3) may be a large scale system and a reduced order model is sought. In either event, it may be necessary to consider (3) with $\mathbf{M}$ singular.

In many systems an initial description leads to a model where $\mathbf{M}$ in (1) is singular, but (1) may be reduced to a state variable form. For this reason,

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linear systems with singular coefficients are some times called descriptor systems (Luenberger 1977 a, b, Sincovec *et al.* 1979), or semi-state equations (Newcomb 1981).

In this paper explicit solutions of (3) are derived in terms of the coefficient matrices $\mathbf{M}$ and $\mathbf{K}$, where $\mathbf{M}$ is allowed to be singular. From these solutions conditions for the transient stability of (3) are derived. In the electrical power system discussed in Bahar and Kwatny (1980) this is important for system planning, operation and control.

## 2. Canonical forms

One technical assumption will be made, namely that there is a scalar $\lambda$ such that

$$\det(\lambda^2\mathbf{M}+\lambda(\alpha\mathbf{M}+\beta\mathbf{K})+\mathbf{K})\neq 0 \tag{4}$$

Assumption (4) is equivalent to assuming that functional solutions of (3), when they exist, are uniquely determined by the values of $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ (Campbell 1980).

Let $\lambda$ be as in (4) and such that $\lambda\beta+1\neq 0$. Let $\mathbf{x}=\exp(\lambda t)\mathbf{y}$; (3) becomes

$$\mathbf{M}\ddot{\mathbf{y}}+[(\alpha\mathbf{M}+\beta\mathbf{K})+2\lambda\mathbf{M}]\dot{\mathbf{y}}+[\lambda^2\mathbf{M}+\lambda(\alpha\mathbf{M}+\beta\mathbf{K})+\mathbf{K}]\mathbf{y}=\exp(-\lambda t)\mathbf{f} \tag{5}$$

This system is again in the format of (3), with different $\alpha$, $\beta$ except that the new $\mathbf{K}$, $\lambda^2\mathbf{M}+\lambda(\alpha\mathbf{M}+\beta\mathbf{K})+\mathbf{K}$, is invertible. For the remainder of this paper $\mathbf{K}$ is assumed invertible.

Multiplying (3) by $\mathbf{K}^{-1}$ gives

$$\hat{\mathbf{M}}\ddot{\mathbf{x}}+\mathbf{Q}\dot{\mathbf{x}}+\mathbf{x}=\hat{\mathbf{f}} \tag{6}$$

where

$$\hat{\mathbf{M}}=\mathbf{K}^{-1}\mathbf{M},\quad \mathbf{Q}=\alpha\mathbf{K}^{-1}\mathbf{M}+\beta=\alpha\hat{\mathbf{M}}+\beta \quad\text{and}\quad \hat{\mathbf{f}}=\mathbf{K}^{-1}\mathbf{f}$$

Now there exists an invertible matrix $\mathbf{T}$ such that

$$\mathbf{T}\hat{\mathbf{M}}\mathbf{T}^{-1}=\begin{bmatrix}\mathbf{M}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{N}_1\end{bmatrix} \tag{7}$$

where $\mathbf{M}_1$ is invertible if present and $\mathbf{N}_1$ is nilpotent of index $k$. Thus the change of variable $\mathbf{x}=\mathbf{T}^{-1}\mathbf{z}$ gives, with $\mathbf{z}=\begin{bmatrix}\mathbf{z}_1\\ \mathbf{z}_2\end{bmatrix}$,

$$\mathbf{M}_1\ddot{\mathbf{z}}_1+\mathbf{Q}_1\dot{\mathbf{z}}_1+\mathbf{z}_1=\mathbf{f}_1 \tag{8}$$

$$\mathbf{N}_1\ddot{\mathbf{z}}_2+\mathbf{Q}_2\dot{\mathbf{z}}_2+\mathbf{z}_2=\mathbf{f}_2 \tag{9}$$

$$\mathbf{Q}_1=\alpha\mathbf{M}_1+\beta,\quad \mathbf{Q}_2=\alpha\mathbf{N}_1+\beta,\quad \begin{bmatrix}\mathbf{f}_1\\ \mathbf{f}_2\end{bmatrix}=\mathbf{T}\mathbf{K}^{-1}\hat{\mathbf{f}} \tag{10}$$

Equation (8) is the type studied by Bahar and Kwatny (1980). The system (9) will be considered first.

**3. The singular subsystem**

Let $\mathbf{w}_2 = \dot{\mathbf{z}}_2$. Then (9) becomes

$$\begin{bmatrix} \mathbf{Q}_2 & \mathbf{N}_1 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{z}}_2 \\ \dot{\mathbf{w}}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_2 \\ \mathbf{0} \end{bmatrix} \tag{11}$$

Equation (11) is in the format

$$\mathbf{A}\dot{\mathbf{u}} + \mathbf{u} = \mathbf{g}, \quad \mathbf{A} \text{ singular} \tag{12}$$

The functional solutions of (12) (Campbell and Meyer 1979, Campbell 1980) are

$$\mathbf{u} = \exp(-\mathbf{A}^{\mathrm{D}} t)\mathbf{A}^{\mathrm{D}}\mathbf{A}\mathbf{u}(0) + \int_0^t \exp[\mathbf{A}^{\mathrm{D}}(s-t)]\mathbf{A}^{\mathrm{D}}\mathbf{g}(s)\, ds \tag{13}$$

$$+ [\mathbf{I} - \mathbf{A}^{\mathrm{D}}\mathbf{A}] \sum_{i=0}^{k-1} [-\mathbf{A}]^i \mathbf{g}^{(i)}(t) \tag{14}$$

The initial conditions for which functional solutions exist will be called consistent initial conditions and are characterized by (13), (14). Here $\mathbf{A}^{\mathrm{D}}$ is the Drazin inverse of $\mathbf{A}$. That is, if $\mathbf{TAT}^{-1} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{bmatrix}$ with $\mathbf{A}_1$ invertible, $\mathbf{N}$ nilpotent, then the Drazin inverse of $\mathbf{A}$, $\mathbf{A}^{\mathrm{D}}$, is given by $\mathbf{A}^{\mathrm{D}} = \mathbf{T}^{-1} \begin{bmatrix} \mathbf{A}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{T}$. An extensive treatment of the Drazin inverse may be found in Campbell and Meyer (1979). Its introduction is not essential, of course, but the Drazin inverse is helpful in summarizing certain information about projections.

If $\beta = 0$, then $\begin{bmatrix} \mathbf{Q}_2 & \mathbf{N}_1 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} = \mathbf{A}$ is nilpotent. Thus the solution of (11) is given explicitly by the term (14). If $\mathbf{f}_2 \equiv \mathbf{0}$, the only functional solution of (9) would be zero. Assume then for the remainder of this paper that $\beta \neq 0$.

In order to apply (13), (14) to the solution of (11),

$$\begin{bmatrix} \mathbf{Q}_2 & \mathbf{N}_1 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}^{\mathrm{D}} \tag{15}$$

has to be evaluated. Of course, this could be done numerically (Campbell and Meyer 1979, Wilkinson 1978). However, this note is concerned with analytical techniques. Since the Drazin inverse is well-behaved with respect to similarities, that is $(\mathbf{THT}^{-1})^{\mathrm{D}} = \mathbf{TH}^{\mathrm{D}}\mathbf{T}^{-1}$, the approach will be to try and simplify (15) by easily computed similarities. First note that

$$\begin{bmatrix} \mathbf{I} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{Q}_2 & \mathbf{N}_1 \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{L} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_2 + \mathbf{L} & \mathbf{L}^2 + \mathbf{Q}_2\mathbf{L} + \mathbf{N}_1 \\ -\mathbf{I} & -\mathbf{L} \end{bmatrix} \tag{16}$$

Formulas exist for the Drazin inverse of triangular block matrices (Campbell and Meyer 1979). If $\mathbf{L}$ is such that $\mathbf{L}$ commutes with $\mathbf{N}_1$, and $\mathbf{L}^2 + \mathbf{Q}_2\mathbf{L} + \mathbf{N}_1 = \mathbf{0}$, then (16) will be lower triangular. Now

$$\mathbf{Q}_2^2 - 4\mathbf{N}_1 = \beta^2 + (2\alpha\beta - 4)\mathbf{N}_1 + \alpha^2\mathbf{N}_1^2$$

which is invertible. Thus it has square roots which are polynomials in $\mathbf{N}_1$.

In fact, if

$$\sqrt{s}=\beta+\sum_{m=1}^{\infty}\alpha_m(s-\beta^2)^m$$

is the power series for $\sqrt{s}$ centred at $s=\beta^2$, then

$$(\mathbf{Q}_2{}^2-4\mathbf{N}_1)^{1/2}=\beta+\sum_{m=1}^{k-1}\alpha_m[(2\alpha\beta-4)\mathbf{N}_1+\alpha^2\mathbf{N}_1{}^2]$$

$$=\sum_{m=0}^{k-1}\beta_m\mathbf{N}_1{}^m \tag{17}$$

is one such square root. Let

$$\mathbf{L}=-\frac{\mathbf{Q}_2}{2}+\frac{(\mathbf{Q}_2{}^2-4\mathbf{N}_1)^{1/2}}{2}=-\frac{1}{\beta}\mathbf{N}_1+\tfrac{1}{2}\sum_{m=2}^{k-1}\beta_m\mathbf{N}_1{}^m \tag{18}$$

Then $\mathbf{L}$ has the required properties. Note that $\mathbf{L}$ is nilpotent of index $k$ and hence $\mathbf{L}^{\mathrm{D}}=\mathbf{0}$. Now notice that

$$\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{S} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{Q}_2+\mathbf{L} & \mathbf{0}\\ -\mathbf{I} & -\mathbf{L}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{S} & \mathbf{I}\end{bmatrix}=\begin{bmatrix}\mathbf{Q}_2+\mathbf{L} & \mathbf{0}\\ \mathbf{0} & -\mathbf{L}\end{bmatrix} \tag{19}$$

if

$$\mathbf{S}=-(\mathbf{Q}_2+2\mathbf{L})^{-1}=-(\mathbf{Q}_2{}^2-4\mathbf{N}_1)^{-1/2}$$

$$=\sum_{m=0}^{k-1}\gamma_m((2\alpha\beta-4)\mathbf{N}_1+\alpha^2\mathbf{N}_1{}^2)^m=\sum_{m=0}^{k-1}\delta_m\mathbf{N}_1{}^m$$

where

$$-(s)^{-1/2}=\sum_{m=0}^{\infty}\gamma_m(s-\beta^2)^m$$

Alternatively, $\mathbf{S}$ is easily found from (17). Thus

$$\begin{bmatrix}\mathbf{Q}_2 & \mathbf{N}_1\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}^{\mathrm{D}}=\begin{bmatrix}\mathbf{I} & \mathbf{L}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{S} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{Q}_2+\mathbf{L} & \mathbf{0}\\ \mathbf{0} & -\mathbf{L}\end{bmatrix}^{\mathrm{D}}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{S} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & -\mathbf{L}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}$$

$$=\begin{bmatrix}\mathbf{I} & \mathbf{L}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{S} & \mathbf{I}\end{bmatrix}\begin{bmatrix}(\mathbf{Q}_2+\mathbf{L})^{-1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{S} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{I} & -\mathbf{L}\\ \mathbf{0} & \mathbf{I}\end{bmatrix} \tag{20}$$

For notational convenience, let $\mathbf{R}=\mathbf{Q}_2+\mathbf{L}$ and note that $\mathbf{R}$ is invertible and has exactly one eigenvalue $\beta$.

Formulas (11), (20) have several consequences. If (11) is to be solved numerically in descriptor form, then it is important to have some idea of the index of $\mathbf{A}$ in (12) (Petzold 1981, Sincovec *et al.* 1979). From (20),

$$\text{Index }(\mathbf{A})=\text{Index }(\hat{\mathbf{M}}) \tag{21}$$

and the general solution of (9), written as (11), is

$$\begin{bmatrix} \mathbf{z}_2 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}+\mathbf{LS} & \mathbf{L} \\ \mathbf{S} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \exp(-\mathbf{R}^{-1}t) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{L} \\ -\mathbf{S} & \mathbf{I}+\mathbf{LS} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2(0) \\ \mathbf{w}_2(0) \end{bmatrix} \tag{22}$$

$$+ \int_0^t \begin{bmatrix} \mathbf{I}+\mathbf{LS} & \mathbf{L} \\ \mathbf{S} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \exp[-\mathbf{R}^{-1}(t-s)]\mathbf{R}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \times \begin{bmatrix} \mathbf{I} & -\mathbf{L} \\ -\mathbf{S} & \mathbf{I}+\mathbf{LS} \end{bmatrix} \begin{bmatrix} \mathbf{f}_2(s) \\ \mathbf{0} \end{bmatrix} ds \tag{23}$$

$$+ \begin{bmatrix} \mathbf{I}+\mathbf{LS} & \mathbf{L} \\ \mathbf{S} & \mathbf{I} \end{bmatrix} \left( \sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^i \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{L} \\ -\mathbf{S} & \mathbf{I}+\mathbf{LS} \end{bmatrix} \begin{bmatrix} \mathbf{f}_2^{(i)}(t) \\ \mathbf{0} \end{bmatrix} \right) \tag{24}$$

Thus for consistent initial conditions

$$\begin{bmatrix} \mathbf{I}+\mathbf{LS} & \mathbf{L} \\ \mathbf{S} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{L} \\ -\mathbf{S} & \mathbf{I}+\mathbf{LS} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2(0) \\ \mathbf{w}_2(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I}+\mathbf{LS} & -\mathbf{L}-\mathbf{L}^2\mathbf{S} \\ \mathbf{S} & -\mathbf{SL} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2(0) \\ \mathbf{w}_2(0) \end{bmatrix} \tag{25}$$

is arbitrary and

$$\begin{bmatrix} -\mathbf{LS} & \mathbf{L}+\mathbf{L}^2\mathbf{S} \\ -\mathbf{S} & \mathbf{I}+\mathbf{SL} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2(0) \\ \mathbf{w}_2(0) \end{bmatrix} = \sum_{i=0}^{k-1} \begin{bmatrix} -\mathbf{L}^{i+1}\mathbf{S} & \mathbf{L}^{i+1}+\mathbf{L}^{i+2}\mathbf{S} \\ -\mathbf{L}^i\mathbf{S} & \mathbf{L}^i+\mathbf{L}^{i+1}\mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{(i)}(0) \\ \mathbf{0} \end{bmatrix} \tag{26}$$

In particular

*Theorem* 1

All functional solutions of (9) are given by

$$\mathbf{z}_2 = (\mathbf{I}+\mathbf{LS}) \exp(-\mathbf{R}^{-1}t)(\mathbf{z}_2(0) - \mathbf{Lw}_2(0)) \tag{27}$$

$$+ \int_0^t (\mathbf{I}+\mathbf{LS}) \exp[-\mathbf{R}^{-1}(t-s)]\mathbf{R}^{-1}\mathbf{f}_2(s)\, ds \tag{28}$$

$$- \sum_{i=0}^{k-1} \mathbf{L}^{i+1}\mathbf{S}\mathbf{f}_2^{(i)}(t) \tag{29}$$

As noted earlier, the only eigenvalue of $\mathbf{R}$ is $\beta$. Thus the stability behaviour of the function solution $\mathbf{z}_2$ of the singular subsystem (9) is completely determined by the sign of $\beta$ and $\mathbf{f}^{(i)}$ for $0 \leqslant i \leqslant k$. In particular, if $\mathbf{f} = \mathbf{0}$, then $\mathbf{z}_2 \to \mathbf{0}$ as $t \to \infty$ if $\beta > 0$ and $\|\mathbf{z}_2\| \to \infty$ if $\beta < 0$.

## 4. Special cases

Two common and important special cases are when the index is one or two. It is worth pointing out the greatly simplified equations that result.

*Index One Case*

Suppose that $\hat{\mathbf{M}}$ has index one and $\beta \neq 0$. This would happen, for example, if $\hat{\mathbf{M}}$ were symmetric. Then $\mathbf{N}_1 = \mathbf{0}$ and $\mathbf{Q}_2 = \beta$, $\mathbf{L} = \mathbf{0}$, $\mathbf{R} = \beta$, $\mathbf{S} = -\beta^{-1}$ and (27)–(29) is

$$\mathbf{z}_2 = \exp(-t/\beta)\mathbf{z}_2(0) + \beta^{-1} \int_0^t \exp[(s-t)/\beta]\mathbf{f}_2(s)\, ds \tag{30}$$

*Index Two Case*

Suppose that $\hat{\mathbf{M}}$ has index two and $\beta \neq 0$. Then $(\mathbf{Q}_2{}^2 - 4\mathbf{N}_1)^{1/2} = \beta + (2\beta)^{-1}(2\alpha\beta - 4)\mathbf{N}_1 = \beta + \beta_1 \mathbf{N}_1$ and $\mathbf{L} = -\beta^{-1}\mathbf{N}_1$, $\mathbf{S} = -\beta^{-3}(\beta^2 - (\alpha\beta - 2)\mathbf{N}_1)$, $\mathbf{R} = \alpha\mathbf{N}_1 + \beta - \beta^{-1}\mathbf{N}_1 = \beta + (\alpha - \beta^{-1})\mathbf{N}_1$, $\mathbf{R}^{-1} = \beta^{-3}(\beta^2 - (\alpha\beta - 1)\mathbf{N}_1)$. Thus (30)–(32) is

$$\begin{aligned}\mathbf{z}_2 = {} & (\mathbf{I} + \beta^{-2}\mathbf{N}_1)\exp(-\beta^{-1}t)(\mathbf{I} + (\alpha\beta - 1)\beta^{-3}tN_1)(\mathbf{z}_2(0) - \beta^{-1}\mathbf{N}_1\mathbf{w}_2(0)) \\ & + \int_0^t (\mathbf{I} + \beta^{-2}\mathbf{N}_1)\exp[-\beta^{-1}(t-s)](\mathbf{I} + (\alpha\beta - 1)\beta^2\mathbf{N}_1(t-s)) \\ & \times (\beta^{-3}(\beta^2 - (\alpha\beta - 1)\mathbf{N}_1)\mathbf{f}_2(s)\, ds - \beta^{-2}\mathbf{N}_1\mathbf{f}_2(t)\end{aligned} \tag{31}$$

Since the above products are commutative and $\mathbf{N}_1{}^2 = \mathbf{0}$, many of the terms drop out if the products are multiplied out.

## 5. The non-singular subsystem

Consider the second order differential equation

$$\ddot{x} + a\dot{x} + bx = f \tag{32}$$

and assume that $ab = ba$. The solution satisfying the initial condition $x(0) = x_0$, $\dot{x}(0) = v_0$ is

$$x = C(t, a, b)x_0 + S(t, a, b)v_0 + \int_0^t S(t-a, a, b)f(s)\, ds \tag{33}$$

where

$$C(t, a, b) = \exp\left(-\frac{a}{2}t\right) s\left(\frac{a^2}{4} - b, t\right) \tag{34}$$

$$S(t, a, b) = \frac{a}{2}\exp\left(-\frac{a}{2}t\right) s\left(\frac{a^2}{4} - b, t\right) + \exp\left(-\frac{a}{2}t\right) c\left(\frac{a^2}{4} - b, t\right) \tag{35}$$

and $s(\lambda, t)$, $c(\lambda, t)$ are defined by the everywhere convergent series

$$s(\lambda, t) = \frac{\sinh(t\sqrt{\lambda})}{\sqrt{\lambda}} = t + \frac{\lambda t^3}{3!} + \frac{\lambda^2 t^5}{5!} + \dots \tag{36}$$

$$c(\lambda, t) = \cosh(t\sqrt{\lambda}) = 1 + \frac{\lambda t^2}{2!} + \frac{\lambda^2 t^4}{4!} + \dots \tag{37}$$

This can be verified by direct substitution. Now (8) can be written as

$$\ddot{\mathbf{z}}_1 + (\alpha + \beta \mathbf{M}_1^{-1})\dot{\mathbf{z}}_1 + \mathbf{M}_1^{-1}\,\mathbf{z}_1 = \tilde{\mathbf{f}}_1 \tag{38}$$

Letting $a = \alpha + \beta \mathbf{M}_1^{-1}$ and $b = \mathbf{M}_1^{-1}$ in (33) yields the solution of this equation.

It is possible to analyse the stability of the solution of the homogeneous equation corresponding to (38). For example, if $\alpha > 0$, $\beta \geqslant 0$ and the eigenvalues of $\mathbf{M}_1$ have positive real parts, then all functional solution of the homogeneous equation approach zero as $t \to \infty$.

## 6. Alternative expressions

Up to this point the calculations have been entirely in terms of the canonical form (8)–(9). The results may be translated back in terms of the original system. This is done by using the correspondences

$$\begin{array}{ll}
\mathbf{h}_1 \leftrightarrow \hat{\mathbf{M}}\hat{\mathbf{M}}^{\mathrm{D}}\mathbf{h} & (\mathbf{h} = \mathbf{w}, \mathbf{f}, \mathbf{M}) \\
\mathbf{h}_2 \leftrightarrow (\mathbf{I} - \hat{\mathbf{M}}\hat{\mathbf{M}}^{\mathrm{D}})\mathbf{h} & (\mathbf{h} = \mathbf{w}, \mathbf{f}) \\
\mathbf{N}_1 \leftrightarrow (\mathbf{I} - \hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}})\hat{\mathbf{M}} & \\
\mathbf{z}_1 \leftrightarrow \hat{\mathbf{M}}\hat{\mathbf{M}}^{\mathrm{D}}\mathbf{x} & \\
\mathbf{z}_2 \leftrightarrow (\mathbf{I} - \hat{\mathbf{M}}\hat{\mathbf{M}}^{\mathrm{D}})\mathbf{x} &
\end{array}$$

An expression from the singular system (9) is multiplied by $\mathbf{I} - \hat{\mathbf{M}}\hat{\mathbf{M}}^{\mathrm{D}}$ while anything from the system (8) is multiplied by $\hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}}$. All inverses are replaced by Drazin inverses. Thus, for example, if Index $(\hat{\mathbf{M}}) = 2$, (31) would give $(\mathbf{I} - \hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}})\mathbf{x}$ as

$$\begin{aligned}
&(\mathbf{I} + \beta^{-2}\hat{\mathbf{M}}) \exp(-\beta^{-1}t)(\mathbf{I} + (\alpha\beta - 1)\beta^{-3}t\hat{\mathbf{M}}) \\
&\qquad \times (\mathbf{I} - \hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}})(\mathbf{x}(0) - \beta^{-1}\hat{\mathbf{M}}\dot{\mathbf{x}}(0)) + \dots
\end{aligned} \tag{39}$$

and (33) could be written as

$$\begin{aligned}
\hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}}\mathbf{x} = {} & C(t, \alpha + \beta\hat{\mathbf{M}}^{\mathrm{D}}, \hat{\mathbf{M}}^{\mathrm{D}})\hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}}\mathbf{x}(0) \\
& + S(t, \alpha + \beta\hat{\mathbf{M}}^{\mathrm{D}}, \hat{\mathbf{M}}^{\mathrm{D}})\hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}}\dot{\mathbf{x}}(0) \\
& + \int_0^t S(t-s, \alpha + \beta\hat{\mathbf{M}}^{\mathrm{D}}, \hat{\mathbf{M}}^{\mathrm{D}})\hat{\mathbf{M}}^{\mathrm{D}}\hat{\mathbf{M}}\mathbf{f}(s)\, ds
\end{aligned} \tag{40}$$

Adding (39) and (40) gives the solution $\mathbf{x}$.

## 7. Conclusion

It has been shown how a singular second order system may be decomposed into a linear and a singular subsystem. The solution of the second order singular subsystem has been solved in terms of its coefficients.

Computation of Drazin inverses is discussed in Campbell and Meyer (1979) and Wilkinson (1978). The numerical solution of (12) is covered in Campbell (1980), Petzold (1981), Sincovec *et al.* (1979) and Wilkinson (1978). This paper has considered only consistent initial conditions. However the analysis and approach can easily describe the impulsive and discontinuous solutions that occur at inconsistent initial conditions. See Campbell (1980) or Cobb (1980) for details.

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